Multiview sensing with unknown permutations: An optimal transport approach

Yanting Ma   Petros T. Boufounos   Hassan Mansour   Shuchin Aeron

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- Recovery using Permutation Regularization
- Efficient Recovery using Optimal Transport Based Relaxation
- Experiments on Synthetic Data

Introduction

This paper explores the problem of recovering a signal measured through a linear system while undergoing a partially known permutation.

Figure 1: Imaging of deformable object in motion.

Introduction

This paper has the following contributions:

- It proposes a formulation that regularizes the permutation to be estimated.
- It proposes a method of relaxation to introduce the optimal transport theory.
- It generalizes the unlabeled sensing problem, which introduces an optional linear operator measuring the permuted data.

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Acquisition Model

$$\boldsymbol{y}_i = \boldsymbol{A}_i \boldsymbol{P}_i \boldsymbol{F}_i \boldsymbol{x} + \boldsymbol{n}_i,\ i = 1, 2, \ldots, K \tag{1}$$

where,

- $\boldsymbol{A}_i$ are known linear measurement operators;
- $\boldsymbol{F}_i$ are known operators that partly predict the deformation of $\boldsymbol{x}$;
- $\boldsymbol{P}_i$ are **unknown** permutation matrices;

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where,

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- $\boldsymbol{F}_i$ are known operators that partly predict the deformation of $\boldsymbol{x}$;
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The objective is to recover $\boldsymbol{x} \in \mathbb{R}^N$ from the measurements $\boldsymbol{y}_i$.

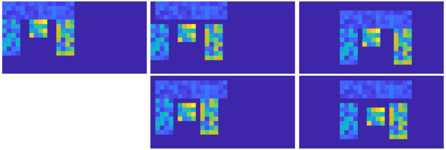

Figure 2: Illustration of the goal.

Here, the authors make the following mild assumptions:

- The support of x is known;
- $\boldsymbol{P}_i$ that move pixels far from their estimated position in the 2D image domain are less likely.

Recovery using Permutation Regularization

To estimate $\boldsymbol{P}_i$ and $\boldsymbol{x}$ from (1),

$$\min_{\boldsymbol{P}_i \in \mathcal{P}, \boldsymbol{x}} \sum_{i=1}^{K} \left(\frac{1}{2} \|\boldsymbol{y}_i - \boldsymbol{A}_i \boldsymbol{P}_i \boldsymbol{F}_i \boldsymbol{x}\|_2^2 + \beta R(\boldsymbol{P}_i) \right) \tag{2}$$

where $\mathcal{P}$ is a set of $N \times N$ permutation matrices and $R(\boldsymbol{P}_i)$ is a regularization for $\boldsymbol{P}_i$.

Recovery using Permutation Regularization

$$R(\boldsymbol{P}_i) := \sum_{n,n'=1}^{N} \|l[n] - l[n']\|_2^2 \boldsymbol{P}_i[n, n'] \tag{3}$$

- $l[n']$ true position, $l[n]$ transfered position.
- n, n' indices of position.

Relaxation

$$\min_{\boldsymbol{P}_i \in \mathcal{P}, \boldsymbol{x}} \sum_{i=1}^{K} \left(\frac{1}{2} \|\boldsymbol{y}_i - \boldsymbol{A}_i \boldsymbol{x}_i\|_2^2 + \beta R(\boldsymbol{P}_i) \right) \tag{4}$$
$$\text{subject to} \;\; \boldsymbol{x}_i - \boldsymbol{P}_i \boldsymbol{F}_i \boldsymbol{x} = 0, \; \forall i$$

Relaxation

$$\begin{aligned} &\min_{\boldsymbol{P}_i \in \mathcal{P}, \boldsymbol{x}} \sum_{i=1}^{K} \left(\frac{1}{2} \|\boldsymbol{y}_i - \boldsymbol{A}_i \boldsymbol{x}_i\|_2^2 + \beta R(\boldsymbol{P}_i) \right) \\ &\text{subject to } \ \boldsymbol{x}_i - \boldsymbol{P}_i \boldsymbol{F}_i \boldsymbol{x} = 0, \ \forall i \end{aligned} \tag{4}$$

Relax the equality constraint to

$$\begin{aligned} &\min_{\boldsymbol{P}_i \in \mathcal{P}, \boldsymbol{x}} \sum_{i=1}^{K} \left(\frac{1}{2} \|\boldsymbol{y}_i - \boldsymbol{A}_i \boldsymbol{x}_i\|_2^2 + \beta R(\boldsymbol{P}_i) \right) \\ &\text{subject to } \ \|\boldsymbol{x}_i - \boldsymbol{P}_i \boldsymbol{F}_i \boldsymbol{x}\|_2^2 \le t, \ \forall i \end{aligned} \tag{5}$$

Relaxation

Rewrite (5) in Lagrangian form,

$$\min_{\boldsymbol{P}_i \in \mathcal{P}, \boldsymbol{x}, \boldsymbol{x}_i} \sum_{i=1}^{K} \left(\frac{1}{2} \|\boldsymbol{y}_i - \boldsymbol{A}_i \boldsymbol{x}_i\|_2^2 + \beta R(\boldsymbol{P}_i) + \frac{\lambda}{2} \|\boldsymbol{x}_i - \boldsymbol{P}_i \boldsymbol{F}_i \boldsymbol{x}\|_2^2 \right) \tag{6}$$

where t and λ are inversely related.

Relaxation

Since $\boldsymbol{P}_i$ is a permutation matrix, the final term in (6) can be expressed as

$$\|\boldsymbol{x}_i - \boldsymbol{P}_i\boldsymbol{F}_i\boldsymbol{x}\|_2^2 = \sum_{n,n'=1}^{K} (\boldsymbol{x}_i[n] - (\boldsymbol{F}_i\boldsymbol{x}[n']))^2 \boldsymbol{P}[n, n'] \tag{7}$$

Relaxation

Since $\boldsymbol{P}_i$ is a permutation matrix, the final term in (6) can be expressed as

$$\|\boldsymbol{x}_i - \boldsymbol{P}_i\boldsymbol{F}_i\boldsymbol{x}\|_2^2 = \sum_{n,n'=1}^{K} (\boldsymbol{x}_i[n] - (\boldsymbol{F}_i\boldsymbol{x}[n']))^2 \boldsymbol{P}[n,n'] \tag{7}$$

$$\begin{array}{cccc} \boldsymbol{x} & \boldsymbol{F}_i\boldsymbol{x} & \boldsymbol{P}_i & \boldsymbol{P}_i\boldsymbol{F}_i\boldsymbol{x} \\ \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} & \begin{pmatrix} e'_1 \\ e'_2 \\ e'_3 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & \begin{pmatrix} e'_3 \\ e'_1 \\ e'_2 \end{pmatrix} \end{array}$$

Figure 3: Illustration of this equation.

Relaxation

$$R(\boldsymbol{P}_i) := \sum_{n,n'=1}^{N} \|l[n] - l[n']\|_2^2 \boldsymbol{P}_i[n,n']$$

$$\|\boldsymbol{x}_i - \boldsymbol{P}_i\boldsymbol{F}_i\boldsymbol{x}\|_2^2 = \sum_{n,n'=1}^{K} (\boldsymbol{x}_i[n] - (\boldsymbol{F}_i\boldsymbol{x}[n']))^2 \boldsymbol{P}[n,n']$$

Use (3) and (7) to define the cost matrix as

$$\boldsymbol{C}(\boldsymbol{x}_i, \boldsymbol{F}_i\boldsymbol{x})[n,n'] := \|l[n] - l[n']\|_2^2 + \frac{\lambda}{2\beta}(\boldsymbol{x}_i[n] - (\boldsymbol{F}_i\boldsymbol{x})[n'])^2 \qquad (8)$$

Relaxation

$$\min_{\boldsymbol{P}_i \in \mathcal{P}, \boldsymbol{x}, \boldsymbol{x}_i} \sum_{i=1}^{K} \left(\frac{1}{2} \|\boldsymbol{y}_i - \boldsymbol{A}_i \boldsymbol{x}_i\|_2^2 + \beta R(\boldsymbol{P}_i) + \frac{\lambda}{2} \|\boldsymbol{x}_i - \boldsymbol{P}_i \boldsymbol{F}_i \boldsymbol{x}\|_2^2 \right)$$

Rewrite (6) as

$$\min_{\boldsymbol{x}, \boldsymbol{x}_i} \sum_{i=1}^{K} \left(\|\boldsymbol{y}_i - \boldsymbol{A}_i \boldsymbol{x}_i\|_2^2 + \beta \min_{\boldsymbol{P}_i \in \mathcal{P}} \langle \boldsymbol{C}(\boldsymbol{x}_i, \boldsymbol{F}_i \boldsymbol{x}), \boldsymbol{P}_i \rangle \right) \tag{9}$$

Here, this paper uses a joint distribution $\boldsymbol{P} \in \prod(\boldsymbol{u}, \boldsymbol{v})$ to replace the $\boldsymbol{P} \in \mathcal{P}$, where $\prod(\boldsymbol{u}, \boldsymbol{v}) = \{\boldsymbol{P} : 0 \leq \boldsymbol{P}_{i,j} \leq 1, \boldsymbol{P}\mathbf{1} = \boldsymbol{u}_i, \boldsymbol{P}^T\mathbf{1} = \boldsymbol{v}_i\}, \boldsymbol{u}_i \in [0, 1]^N, \boldsymbol{v}_i \in [0, 1]^N$

$$\min_{\boldsymbol{x}, \boldsymbol{x}_i} \sum_{i=1}^{K} \left(\|\boldsymbol{y}_i - \boldsymbol{A}_i\boldsymbol{x}_i\|_2^2 + \beta \min_{\boldsymbol{P}_i \in \prod(\boldsymbol{u}_i, \boldsymbol{v}_i)} \langle \boldsymbol{C}(\boldsymbol{x}_i, \boldsymbol{F}_i\boldsymbol{x}), \boldsymbol{P}_i \rangle \right) \tag{10}$$

When $\boldsymbol{u}_i = \boldsymbol{v}_i = \mathbf{1}$, (9) is equivalent to (10).

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Choice of marginals

Given a signal $x \in \mathbb{R}^N$, let $a : \mathbb{R}^N \to [0,1]^N$ be a function that maps reflectivity values to a probability distribution, defined as

$$a(\boldsymbol{x})[n] := \frac{\boldsymbol{I}\{\boldsymbol{x}[n] > T\}}{\sum_{k=1}^{N} \boldsymbol{I}\{\boldsymbol{x}[k] > T\}}, \quad n = 1, \ldots, N, \tag{11}$$

where $\boldsymbol{I}$ is an indicator function and $T > 0$ is some predefined threshold.

Recovery using Optimal Transport

Define the marginals as $\boldsymbol{u} = a(\boldsymbol{x}), \boldsymbol{v} = a(\boldsymbol{F}_i\boldsymbol{x})$ and define an optimal transport distance between $a(\boldsymbol{x}_i)$ and $a(\boldsymbol{F}_i\boldsymbol{x})$ as

$$\mathrm{OT}(a(\boldsymbol{x}_i), a(\boldsymbol{F}_i\boldsymbol{x})) = \min_{\boldsymbol{P}_i \in \prod(a(\boldsymbol{x}_i), a(\boldsymbol{F}_i\boldsymbol{x}))} \langle \boldsymbol{C}(\boldsymbol{x}_i, \boldsymbol{F}_i\boldsymbol{x}), \boldsymbol{P}_i \rangle \tag{12}$$

Recovery using Optimal Transport

$$\min_{\boldsymbol{x},\boldsymbol{x}_i} \sum_{i=1}^{K} \left(\|\boldsymbol{y}_i - \boldsymbol{A}_i \boldsymbol{x}_i\|_2^2 + \beta \min_{\boldsymbol{P}_i \in \mathcal{P}} \langle \boldsymbol{C}(\boldsymbol{x}_i, \boldsymbol{F}_i \boldsymbol{x}), \boldsymbol{P}_i \rangle \right)$$

Rewrite the relaxed version as

$$\begin{aligned} &\min_{x,x_i} \sum_{i=1}^{K} f(\boldsymbol{x}, \boldsymbol{x}_i), \text{where} \\ &f(\boldsymbol{x}, \boldsymbol{x}_i) = \|\boldsymbol{y}_i - \boldsymbol{A}_i \boldsymbol{x}_i\|_2^2 + \beta \mathrm{OT}(a(\boldsymbol{x}_i), a(\boldsymbol{F}_i \boldsymbol{x})) \end{aligned} \tag{13}$$

Gradient Descent

Let $\boldsymbol{f}$ and $\boldsymbol{g}$ be Lagrangian multipliers, the Lagrangian form of optimal transport distance is

$$\begin{aligned} L(\boldsymbol{P}_i, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{x}_i, \boldsymbol{x}) = \langle \boldsymbol{C}(\boldsymbol{x}_i, \boldsymbol{F}_i\boldsymbol{x}), \boldsymbol{P}_i \rangle + \langle \boldsymbol{f}, \boldsymbol{P}_i\mathbf{1} - a(\boldsymbol{x}_i) \rangle \\ + \left\langle \boldsymbol{g}, \boldsymbol{P}_i^T\mathbf{1} - a(\boldsymbol{F}_i\boldsymbol{x}) \right\rangle \end{aligned} \tag{14}$$

$$\begin{aligned} \nabla_{x_i} \mathrm{OT}(a(\boldsymbol{x}_i), a(\boldsymbol{F}_i\boldsymbol{x})) &= \nabla_{x_i} L(\boldsymbol{P}_i^*, \boldsymbol{f}^*, \boldsymbol{g}^*, \boldsymbol{x}_i, \boldsymbol{x}) \\ &= \nabla_{x_i} \langle \boldsymbol{C}(\boldsymbol{x}_i, \boldsymbol{F}_i\boldsymbol{x}), \boldsymbol{P}_i^* \rangle - \nabla_{x_i} \langle \boldsymbol{f}^*, a(\boldsymbol{x}_i) \rangle \\ &= \nabla_{x_i} \langle \boldsymbol{C}(\boldsymbol{x}_i, \boldsymbol{F}_i\boldsymbol{x}), \boldsymbol{P}_i^* \rangle \end{aligned} \tag{15}$$

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Experiments on Synthetic Data

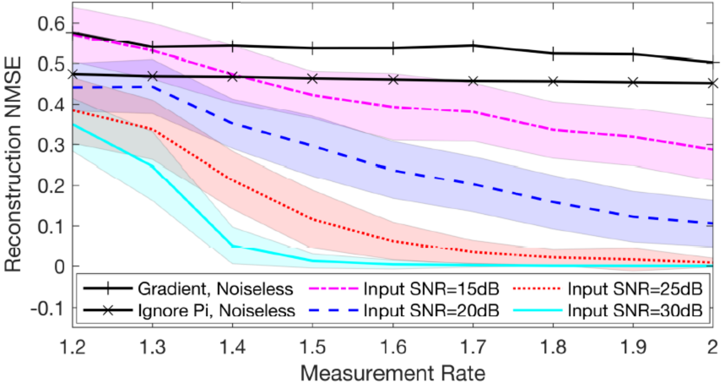

Figure 4: NMSE as a function of measurement rate at various input SNR.