# STRATEGIC NETWORK FORMATION WITH MANY AGENTS

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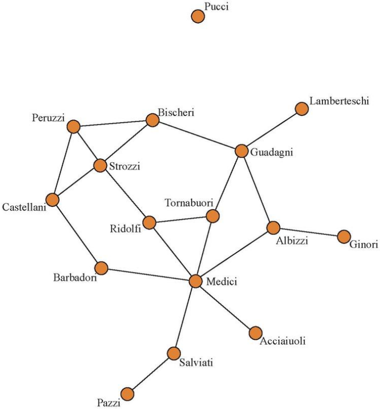
- Introduction
- Model Description
- Asymptotic Representation of Network
- Convergence to the Limiting Distribution

***Network Formation Model***



# Introduction

This author derives a tractable approximation to the distribution of network links using many-player asymptotics. And he demonstrates convergence of the link frequency distribution from finite pairwise stable networks to the (many-player) limiting distribution.

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- Player i's payoffs are of the form

$$\Pi_i(\boldsymbol{L}) = B_i(\boldsymbol{L}) - C_i(\boldsymbol{L}) \tag{2}$$

# Model Description

- the incremental benefit of adding a link ij

$$U_{ij}(\boldsymbol{L}) := B_i(\boldsymbol{L} + \{ij\}) - B_i(\boldsymbol{L} - \{ij\}) \tag{3}$$

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- this paper specifies the marginal benefit function as

$$U_{ij}(\boldsymbol{L}) = U^*_{ij}(\boldsymbol{L}) + \sigma\eta_{ij} \tag{5}$$

- and marginal costs as

$$MC_{ij}(\boldsymbol{L}) := \max_{k=1,\ldots,J} \sigma\eta_{i0,k} \tag{6}$$

# Model Description

- Node-specific network statistics

$$S_1(\boldsymbol{L}, \boldsymbol{X}; i) := \sum_{j \neq i} L_{ij} \tag{7}$$

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$$S_2(\boldsymbol{L}, \boldsymbol{X}; i) := \frac{\sum_{j \neq i} L_{ij} \mathbf{1}\left\{x_{jk} = \bar{x}_k\right\}}{\sum_{j \neq i} L_{ij}} \tag{8}$$

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- Edge-specific network statistics

$$T_1(\boldsymbol{L}, \boldsymbol{X}; i, j) = \sum_{k \neq i,j} L_{ik} L_{jk} \tag{9}$$

-

$$T_2(\boldsymbol{L}, \boldsymbol{X}; i, j) = \max\{L_{ik} L_{jk} : k \neq i, j\} \tag{10}$$

# Model Description

The marginal benefit function can be defined as

$$U^*_{ij}(\boldsymbol{L}) \equiv U^*(x_i, x_j; S_i, S_j, T_{ij}) \tag{11}$$

# Pairwise Stable Network

***Definition* 1 (*Pairwise Stable Network*)** The undirected graph $\boldsymbol{L}$ is a pairwise stable network (PSN) if for any link ij with $L_{ij} = 1$,

$$\Pi_i(\boldsymbol{L}) \geq \Pi_i(\boldsymbol{L} - \{ij\}), and\ \Pi_j(\boldsymbol{L}) \geq \Pi_j(\boldsymbol{L} - \{ij\})$$

and any link ij with $L_{ij} = 0$,

$$\Pi_i(\boldsymbol{L} + \{ij\}) < \Pi_i(\boldsymbol{L}), or\ \Pi_j(\boldsymbol{L} + \{ij\}) < \Pi_j(\boldsymbol{L})$$

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# Link Frequency Distribution

$$F_n(x_1, x_2; s_1, s_2, t_{12}) :=$$
$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i} P(L_{ij} = 1, x_i \leq x_1, x_j \leq x_2, s_i \leq s_1, s_j \leq s_2, t_{ij} \leq t_{12}) \quad (12)$$

# Limiting Model $\mathcal{F}_0^*$

The limiting model $\mathcal{F}_0^*$ can be described in terms of pairwise stable subnetworks on finite network neighborhoods $\mathcal{N}_{ij}$ around a pair of nodes $i, j$.

The model $\mathcal{F}_0^*$ describes the distribution generating the network neighborhoods $\mathcal{N}_{ij}$ in the many-player limit as well as the distribution of network outcomes on these neighborhoods.

# Limiting Model $\mathcal{F}_0^*$

The distribution can be described in terms of three components:

- the $\boldsymbol{reference\ distribution}$ $M^*$ which is a cross-sectional p.d.f. of potential outcomes for the endogenous node characteristics $s_l$ and $t_l$. given exogenous attributes in the relevant subnetwork.
- the $\boldsymbol{inclusive\ value\ function}$ $H^*(x, s)$ which gives a sufficient statistic for the link opportunity set $W_i(\boldsymbol{L}^*)$ of a node with characteristics $x_i = x$ and $s_i = s$ with respect to her link formation decisions.
- the $\boldsymbol{edge - level\ response}$ $Q^*(l_{12}, s_1, s_2, t_{12}|x_1, x_2) := Q^*(l_{12}, s_1, s_2, t_{12}|x_1, x_2; H^*, M^*)$ which corresponds to a conditional probability of a link $ij$ forming together with the resulting values of the endogenous network variables $s_1, s_2, t_{12}$.

The resulting limiting link frequency distribution has p.d.f.

$$f_0^*(x_1, x_2; s_1, s_2) = \frac{s_{11}s_{12} \exp\{U^*(x_1, x_2; s_1, s_2) + U^*(x_2, x_1; s_2, s_1)\}}{(1 + H^*(x_1, s_1))(1 + H^*(x_2, s_2))} \times M^*(s_1|x_1, x_2)M^*(s_2|x_2, x_1)w(x_1)w(x_2) \tag{13}$$

# Limiting Model $\mathcal{F}_0^*$

The inclusive value function $H^*(x_1, s_1)$ is a nonnegative function satisfying the fixed-point condition

$$H^*(x; s) = \Psi_0[H^*, M^*](x; s) \tag{14}$$

where the fixed-point operator $\Psi_0$

$$\Psi_0[H, M](x; s) := \int \frac{s_{12} \exp\left\{U^*(x, x_2; s, s_2) + U^*(x_2, x; s_2, s)\right\}}{1 + H(x_2; s_2)} \times M^*(s_2|x_2, x_1) w(x_2) ds_2 dx_2 \tag{15}$$

# Limiting Model $\mathcal{F}_0^*$

The reference distribution

$$M^*(s_1|x_1, x_2) = \Omega_0[H^*, M^*](x_1, x_2; s_1) \tag{16}$$

where the operator $\Omega_0$ maps $H, M$ to the conditional distribution for the network statistic $s_i$ given $x_i$ resulting from the edge-level response in the cross section.

In the case of no endogenous interaction effects,

$$\Omega_0[H, M](x_1, x_2; s_{11}) := \frac{H(x_1)^{s_{11}}}{(1 + H^*(x_1))^{s_{11}+1}} \tag{17}$$

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# Convergence to the Limiting Distribution

***Assumption* 1 (*Systematic Part of Payoffs*)**(i) The systematic parts of payoffs are uniformly bounded in absolute value for some value of $t = t_0, |U^*(x, x', s, s', t_0)| \leq \bar{U} < \infty$. Furthermore, (ii) at all values of $s, s'$, the function $U^*(x, x', s, s', t_0)$ is $p \geq 1$ times differentiable in $x$ with uniformly bounded partial derivatives. (iii) The supports of the payoff-relevant network statistics, $\mathcal{S}$ and $\mathcal{T}$ , and the type space $\mathcal{X}$ are compact sets.

# Convergence to the Limiting Distribution

***Assumption* 2 (*Idiosyncratic Part of Payoffs*)** $\eta_{ij}$ and $\eta_{i0,k}$ are i.i.d. draws from the distribution $G(s)$, and are independent of $x_i, x_j$, where (i) the c.d.f. $G(s)$ is absolutely continuous with density $g(s)$, and (ii) the upper tail of the distribution $G(s)$ is of type I with auxiliary function $a(s) := \frac{1-G(s)}{g(s)}$ .

# Convergence to the Limiting Distribution

***Assumption* 3 (*Network Size*)**(i) The number $n$ of agents in the network grows to infinity, and (ii) the random draws for marginal costs $MC_i$ are governed by the sequence $J = [n^{1/2}]$, where $[x]$ denotes the value of $x$ rounded to the closest integer. (iii) The scale parameter for the taste shifters $\sigma \equiv \sigma_n = \frac{1}{a(b_n)}$, where $b_n = G^{-1}(1 - \frac{1}{\sqrt{n}})$, and $a(s)$ is the auxiliary function specified in Assumption 2 (ii). Furthermore, (iv) for any values $t_1 \neq t_2 \in \mathcal{T}, |U(x, x', s, s', t_1) - U(x, x', s, s', t_2)|$ may increase with $n$, and there exists a constant
$B_T < \infty$ such that for any sequence of pairwise stable networks $(\boldsymbol{L}_n^*)_{n \geq 2}$,
$\sup_{x,x',s,s'} (\mathbb{E}[\exp\{2|U(x, x', s, s', T(\boldsymbol{L}_n^*, x, x', i, j)) - U(x, x', s, s', t_0)|\}])^( 1/$
$\exp\{B_T\}$ for $n$ sufficiently large.

# Convergence to the Limiting Distribution

***Assumption* 4 (*Pairwise Stability*)** Let $\mathcal{N}_s \subset \{1, \ldots, n\}$ be the subset of nodes for which the network $L^*$ satisfies the payoff conditions for pairwise stability in Definition 1. Then for any $\epsilon > 0, |\mathcal{N}_s|/n > 1 - \epsilon$ with probability approaching 1 as $n$ increases.

# Convergence to the Limiting Distribution

***Assumption* 5** (i) The mapping $\Omega_0$ is compact and upper hemi-continuous in $H,\ M$ for all $x \in \mathcal{X}$ and $S \subset \mathcal{R}$, and (ii) the core of $\Omega_0[H, M]$ is nonempty, where the boundary of the core of $\Omega_0[H, M]$ is in some compact subset $\mathcal{U} \subset \Delta(\mathcal{X} \times \mathcal{R})$ for all values of $H,\ M$. (iii) $\sup_{x, Z \subset \mathcal{R}} |\hat{\Omega}_n[H, M](Z) - \Omega_0[H, M](Z)| \to 0$ uniformly in $H \in \mathcal{G}$ and distributions $M \in \mathcal{U}$.

# Convergence to the Limiting Distribution

***Theorem* 1 (*Fixed Point Existence*)** Suppose that Assumptions 1 and 5 (i)-(ii) hold. Then the mapping $(H, M) \to (\Psi_0, \Omega_0)$ has a fixed point.

***Theorem* 2 (*Convergence*)** Suppose that Assumptions 1-5 hold, and let $\mathcal{F}_0^*$ be the set of distributions characterized by (13) (14) and (16). Then for any pairwise or cyclically stable network there exists a distribution $F_0^*(x_1, x_2; s_1, s_2) \in \mathcal{F}_0^*$ such that the link frequency distribution

$$\sup_{x_1, x_2, s_1, s_2, t_{12}} |\hat{F}_n(x_1, x_2; s_1, s_2, t_{12}) - F_0^*(x_1, x_2; s_1, s_2, t_{12})| = o_p(1)$$

Furthermore, convergence is uniform with respect to selection among pairwise stable networks.