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Beyond the Poisson renewal process: A tutorial survey

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Abstract

After sketching the basic principles of renewal theory and recalling the classical Poisson process, we discuss two renewal processes characterized by waiting time laws with the same power asymptotics defined by special functions of Mittag–Leffler and of Wright type. We compare these three processes with each other.

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1. Introduction

The Poisson process is well known to play a fundamental role in renewal theory. In the present paper, by using special functions of Mittag–Leffler and Wright type in the definitions of waiting time distributions, we provide a generalization and a variant to this classical process. These distributions, characterized by power-law asymptotics in contrast to the exponential law of the Poisson process, have been studied by several authors both from mathematical and physical point of view, see e.g., [2,10–12,14,21] and references therein.

The structure of our paper is as follows. In Section 2, we recall the basic renewal theory including its fundamental concepts like waiting time between events, the survival probability, the renewal function. If the waiting time is exponentially distributed we have the classical Poisson process, which is Markovian: this is the topic of Section 3. However, other waiting time distributions are also relevant in applications, in particular such ones having a power-law decay of their density. In this context we analyse, respectively, in Sections 4 and 5, two non-Markovian renewal processes with waiting time distributions described by functions of Mittag–Leffler and Wright type: both depend on a parameter $\beta \in (0, 1)$ related to the common exponent in the power law. In the limit $\beta = 1$ the first becomes the Poisson process whereas the second goes over into the deterministic process producing its events at equidistant instants of time. In Section 6, after sketching the differences between the renewal processes of Mittag–Leffler and Wright type, we compare numerically their survival functions and their probability densities in the special case $\beta = \frac{1}{2}$ with respect to the corresponding functions of the classical Poisson process. Concluding remarks are given in Section 7.

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2. Essentials of renewal theory

We present a brief introduction to the renewal theory by using our notation. For more details see e.g., the classical treatises by Cox [3], Feller [5], and the more recent book by Ross [23]. We begin to recall that a stochastic process $\{N(t),\ t\geqslant 0\}$ is called a *counting process* if $N(t)$ represents the total number of "events" that have occurred up to time t. It is called a *renewal process* if the times between successive events, $T_1, T_2, \ldots,$ are independent identically distributed (*iid*) non-negative random variables, obeying a given probability law. We call these times *waiting times* (or inter-arrival times) and the instants $t_0 = 0$, $t_k = \sum_{j=1}^{k} T_j\ (k = 1, 2, \ldots)$ *renewal times*. Let the *waiting times* be distributed like T and let

$$\Phi(t) := P(T \leqslant t) \tag{2.1}$$

be the common probability distribution function, that we assume to be absolutely continuous. Then the corresponding probability density function[1] $\phi(t)$ and the probability distribution function $\Phi(t)$ are related by

$$\phi(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t), \quad \Phi(t) = \int_0^t \phi(t')\,\mathrm{d}t'. \tag{2.2}$$

We recall that $\phi(t)\geqslant 0$ with $\int_0^\infty \phi(t)\,\mathrm{d}t = 1$ and $\Phi(t)$ is a non-decreasing function in $\mathbf{R}^+$ with $\Phi(0) = 0$, $\Phi(+\infty) = 1$. Often, especially in Physics, the *probability density function* is abbreviated by *pdf*, so that, in order to avoid confusion, the probability distribution function is called the *probability cumulative function* and abbreviated by *pcf*. When the non-negative random variable represents the lifetime of a technical system, it is common practice to call $\Phi(t)$ the *failure probability* and

$$\Psi(t) := P(T > t) = \int_t^\infty \phi(t')\,\mathrm{d}t' = 1 - \Phi(t), \tag{2.3}$$

the *survival probability*, because $\Phi(t)$ and $\Psi(t)$ are the respective probabilities that the system does or does not fail in $(0, t]$. These terms, however, are commonly adopted for any renewal process.

As a matter of fact, the *renewal process* is defined by the *counting process*

$$N(t) := \begin{cases} 0 & \text{for } 0\leqslant t < t_1, \\ \max\{k | t_k \leqslant t,\ k = 1, 2, \ldots\} & \text{for } t \geqslant t_1. \end{cases} \tag{2.4}$$

$N(t)$ is thus the random number of renewals occurring in $(0, t]$. We easily recognize that $\Psi(t) = P(N(t)=0)$. Continuing in the general theory, we set $F_1(t) = \Phi(t)$, $f_1(t) = \phi(t)$, and in general

$$F_k(t) := P(t_k = T_1 + \cdots + T_k \leqslant t), \quad f_k(t) = \frac{\mathrm{d}}{\mathrm{d}t}F_k(t), \quad k\geqslant 1. \tag{2.5}$$

$F_k(t)$ is the probability that the sum of the first k waiting times does not exceed t, and $f_k(t)$ is the corresponding density. $F_k(t)$ is normalized because $\lim_{t\to\infty} F_k(t) = P(t_k = T_1 + \cdots + T_k < \infty) = \Phi(+\infty) = 1$. In fact, the sum of k random variables each of which is finite with probability 1 is finite with probability 1 itself. We set for consistency $F_0(t) = \Theta(t)$, the Heaviside unit step function (with $\Theta(0) := \Theta(0^+)$) so that $F_0(t) \equiv 1$ for $t\geqslant 0$, and $f_0(t) = \delta(t)$, the Dirac delta generalized function.

A relevant quantity is the function $v_k(t)$ that represents the probability that k events occur in the interval $(0, t]$. We get, for any $k\geqslant 0$,

$$v_k(t) := P(N(t) = k) = P(t_k \leqslant t, t_{k+1} > t) = \int_0^t f_k(t')\Psi(t - t')\,\mathrm{d}t'. \tag{2.6}$$

For $k = 0$ we recover $v_0(t) = \Psi(t)$.

[1] Let us remark that, as it is popular in Physics, we use the word density also for generalized functions that can be interpreted as probability measures. In these cases the function $\Phi(t)$ may lose its absolute continuity.

Another relevant quantity is the *renewal function* $m(t)$ defined as the expected value of the process $N(t)$, that is

$$m(t) := E(N(t)) = \langle N(t)\rangle = \sum_{k=1}^{\infty} P(t_k \leqslant t). \tag{2.7}$$

Thus this function represents the average number of events in the interval $(0, t]$ and can be shown to uniquely determine the renewal process, see e.g., [23]. It is related to the waiting time distribution by the so-called *renewal equation*,

$$m(t) = \Phi(t) + \int_0^t m(t-t')\phi(t')\,\mathrm{d}t' = \int_0^t \left[1 + m(t-t')\right]\phi(t')\,\mathrm{d}t'. \tag{2.8}$$

If the mean waiting time (the first moment) is finite, namely

$$\rho := \langle T\rangle = \int_0^{\infty} t\phi(t)\,\mathrm{d}t < \infty, \tag{2.9}$$

it is known that, with probability 1, $t_k/k \to \rho$ as $k \to \infty$, and $N(t)/t \to 1/\rho$ as $t \to \infty$. These facts imply the *elementary renewal theorem*,

$$\frac{m(t)}{t} \to \frac{1}{\rho} \quad \text{as } t \to \infty. \tag{2.10}$$

We shall also consider renewal processes with infinite mean waiting time for which fat tails with power-law asymptotics are responsible:

$$\phi(t) \sim \frac{A_\infty}{t^{(1+\beta)}}, \quad \Psi(t) \sim \frac{A_\infty}{\beta t^{\beta}} \quad \text{for } t \to \infty,\ \ 0<\beta<1,\ \ A_\infty > 0. \tag{2.11}$$

It is convenient to use the common $*$, notation for the time convolution of two causal well-behaved (generalized) functions $f(t)$ and $g(t)$,

$$\int_0^t f(t')g(t-t')\,\mathrm{d}t' = (f*g)(t) = (g*f)(t) = \int_0^t f(t-t')g(t')\,\mathrm{d}t',$$

and treat the renewal processes also by the Laplace transform. Throughout we will denote by $\div$ the juxtaposition of a sufficiently well-behaved (generalized) function $f(t)$ with its Laplace transform $\widetilde{f}(s)$ according to

$$f(t) \div \mathscr{L}\{f(t); s\} = \widetilde{f}(s) = \int_0^{+\infty} \mathrm{e}^{-st} f(t)\,\mathrm{d}t, \quad \Re s > s_0.$$

For the generalized Dirac function $\delta(t)$ consistently we will have $\widetilde{\delta}(s) \equiv 1$.

Because of (2.5) we recognize $f_k(t)$ as the k-fold auto-convolution of $\phi(t)$,

$$f_k(t) = (\phi^{*k})(t) \div \widetilde{f}_k(s) = [\widetilde{\phi}(s)]^k, \tag{2.12}$$

so that Eq. (2.6) simply reads:

$$v_k(t) = (\Psi * \phi^{*k})(t) \div \widetilde{v}_k(s) = [\widetilde{\phi}(s)]^k\,\widetilde{\Psi}(s) \quad \text{with } \widetilde{\Psi}(s) = \frac{1-\widetilde{\phi}(s)}{s}. \tag{2.13}$$

With $\widetilde{\Phi}(s) = \widetilde{\phi}(s)/s$, we obtain from (2.8) the useful relations

$$m(t) = \Phi(t) + (m*\phi)(t) \div \widetilde{m}(s) = \widetilde{\Phi}(s) + \widetilde{m}(s)\,\widetilde{\phi}(s), \tag{2.14}$$

$$\widetilde{m}(s) = \frac{\widetilde{\phi}(s)}{s[1-\widetilde{\phi}(s)]}, \quad \widetilde{\phi}(s) = \frac{s\widetilde{m}(s)}{1+s\widetilde{m}(s)}. \tag{2.15}$$

3. The Poisson process as a renewal process

The most celebrated renewal process is the *Poisson process* (with parameter $\lambda > 0$). It is characterized by a survival function of exponential type:

$$\Psi(t) = \mathrm{e}^{-\lambda t} \div \widetilde{\Psi}(s) = \frac{1}{\lambda + s}, \tag{3.1}$$

so that the corresponding density for the waiting times is exponential

$$\phi(t) = \lambda \mathrm{e}^{-\lambda t} \div \widetilde{\phi}(s) = \frac{\lambda}{\lambda + s}, \tag{3.2}$$

and the renewal function is linear in time:

$$m(t) = \lambda t \div \widetilde{m}(s) = \frac{\widetilde{\phi}(s)}{s[1 - \widetilde{\phi}(s)]} = \frac{\lambda}{s^2}. \tag{3.3}$$

The Poisson process is Markovian because of its exponentially distributed waiting time. We know that the probability that k events occur in an interval of length t is given by the celebrated *Poisson distribution* with

$$v_k(t) = \frac{(\lambda t)^k}{k!}\,\mathrm{e}^{-\lambda t} \div \widetilde{v}_k(s) = [\widetilde{\phi}(s)]^k\,\widetilde{\Psi}(s) = \frac{\lambda^k}{(\lambda + s)^{k+1}}, \quad k = 0, 1, 2, \ldots\ . \tag{3.4}$$

The probability distribution related to the sum of k *iid* exponential random variables is known as the so-called *Erlang distribution* (of order $k \geqslant 1$). The corresponding density (the *Erlang pdf*) and distribution (the *Erlang pcf*) are

$$f_k(t) = \lambda\,\frac{(\lambda t)^{k-1}}{(k-1)!}\,\mathrm{e}^{-\lambda t} \div \widetilde{f}_k(s) = [\widetilde{\phi}(s)]^k = \frac{\lambda^k}{(\lambda + s)^k}, \quad k = 1, 2, \ldots, \tag{3.5}$$

$$\begin{aligned} F_k(t) &= \int_0^t f_k(t')\,\mathrm{d}t' = 1 - \sum_{n=0}^{k-1} \frac{(\lambda t)^n}{n!}\,\mathrm{e}^{-\lambda t} = \sum_{n=k}^{\infty} \frac{(\lambda t)^n}{n!}\,\mathrm{e}^{-\lambda t} \\ &\div \widetilde{F}_k(s) = \frac{[\widetilde{\phi}(s)]^k}{s} = \frac{\lambda^k}{s(\lambda + s)^k}. \end{aligned} \tag{3.6}$$

In the limiting case $k = 0$ we recover $f_0(t) = \delta(t)$ and $F_0(t) = \Theta(t)$.

4. The renewal process of Mittag–Leffler type

A "fractional" generalization of the Poisson process has been recently proposed by Mainardi et al. [17]. Noting that the survival probability for the Poisson renewal process (with parameter $\lambda > 0$) obeys the ordinary differential equation (of relaxation type)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) = -\lambda \Psi(t), \quad t \geqslant 0; \quad \Psi(0^+) = 1, \tag{4.1}$$

the required generalization is obtained by replacing in (4.1) the first derivative by the fractional derivative in Caputo's sense[2] of order $\beta \in (0, 1]$. We thus obtain, taking for simplicity $\lambda = 1$, the equation

$$_tD_*^{\beta}\Psi(t) = -\Psi(t), \quad t \geqslant 0, \ \ 0 < \beta \leqslant 1; \quad \Psi(0^+) = 1, \tag{4.2}$$

[2] The Caputo derivative of order $\beta \in (0, 1]$ of a well-behaved function $f(t)$ in $\mathbf{R}^+$

$$_tD_*^{\beta} f(t) := \begin{cases} \dfrac{1}{\Gamma(1-\beta)} \int_0^t \dfrac{f^{(1)}(\tau)}{(t-\tau)^{\beta}}\,\mathrm{d}\tau, & 0 < \beta < 1, \\ \dfrac{\mathrm{d}}{\mathrm{d}t} f(t), & \beta = 1 \end{cases}$$

can be defined through its Laplace transform $\mathscr{L}\{_tD_*^{\beta} f(t); s\} = s^{\beta}\,\widetilde{f}(s) - s^{\beta-1} f(0^+)$. For more information on the theory and the applications of the Caputo fractional derivative (of any order $\beta > 0$), see e.g., [8,13,15,22].

whose solution can be obtained by using the technique of the Laplace transforms. We have

$$\Psi(t) = E_\beta(-t^\beta) \quad \text{from } \widetilde{\Psi}(s) = \frac{s^{\beta-1}}{1+s^\beta}, \quad 0<\beta\leqslant 1, \tag{4.3}$$

hence

$$\phi(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) = -\frac{\mathrm{d}}{\mathrm{d}t}E_\beta(-t^\beta) \quad \text{corresponding to } \widetilde{\phi}(s) = \frac{1}{1+s^\beta}. \tag{4.4}$$

Here E_β denotes the Mittag–Leffler function[3] of order β. We call this process the *renewal process of Mittag–Leffler type* (of order β).

Hereafter, we find it convenient to summarize the most relevant features of the functions $\Psi(t)$ and $\phi(t)$ when $0<\beta<1$. We begin to quote their expansions in power series of t^β (convergent for $t\geqslant 0$) and their $t\to\infty$ asymptotics

$$\Psi(t) = \sum_{n=0}^{\infty}(-1)^n\frac{t^{\beta n}}{\Gamma(\beta n+1)} \sim \frac{\sin(\beta\pi)}{\pi}\,\frac{\Gamma(\beta)}{t^\beta}, \tag{4.5}$$

$$\phi(t) = \frac{1}{t^{1-\beta}}\sum_{n=0}^{\infty}(-1)^n\frac{t^{\beta n}}{\Gamma(\beta n+\beta)} \sim \frac{\sin(\beta\pi)}{\pi}\,\frac{\Gamma(\beta+1)}{t^{\beta+1}}. \tag{4.6}$$

We note that for $0<\beta<1$ both functions $\Psi(t)$, $\phi(t)$ keep the complete monotonicity,[4] characteristic for the Poissonian case $\beta=1$. In contrast to the Poissonian case, in the case $0<\beta<1$ the mean waiting time is infinite because the waiting time laws instead of exponential exhibit power-law decay according to (2.11) with the constant A_∞ derived from (4.5)–(4.6) as

$$A_\infty = \Gamma(\beta+1)\sin(\beta\pi)/\pi = \beta\Gamma(\beta)\sin(\beta\pi)/\pi. \tag{4.7}$$

Hence the process is no longer Markovian but of long-memory type.

The renewal function of this process can be deduced from the Laplace transforms in (2.15) and (4.4); we find

$$\widetilde{m}(s) = \frac{1}{s^{1+\beta}} \quad \text{hence } m(t) = \frac{t^\beta}{\Gamma(1+\beta)}, \quad t\geqslant 0, \quad 0<\beta\leqslant 1. \tag{4.8}$$

For the generalization of Eqs. (3.4)–(3.6), concerning the Poisson and the Erlang distributions, we give the Laplace transform formula

$$\mathscr{L}\{t^{\beta k}E_\beta^{(k)}(-t^\beta); s\} = \frac{k!\,s^{\beta-1}}{(1+s^\beta)^{k+1}}, \quad \beta>0, \quad k=0,1,2,\ldots \tag{4.9}$$

[3] The Mittag–Leffler function with parameter β, defined as

$$E_\beta(z) := \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\beta n+1)}, \quad \beta>0, \quad z\in\mathbf{C},$$

is an entire function of order $1/\beta$, reducing for $\beta=1$ to $\exp(z)$. For detailed information on the functions of Mittag–Leffler type the reader may consult e.g., [4,8,13,16,22,24] and references therein.

[4] Complete monotonicity of a function $f(t)$, $t\geqslant 0$, is equivalent to its representability as (real) Laplace transform of a non-negative function or measure. Recalling the theory of the Mittag–Leffler functions of order less than 1, we obtain for $0<\beta<1$ the following representations, see e.g., [8],

$$\Psi(t) = \frac{\sin(\beta\pi)}{\pi}\int_0^\infty \frac{r^{\beta-1}\,\mathrm{e}^{-rt}}{r^{2\beta}+2r^\beta\cos(\beta\pi)+1}\,\mathrm{d}r, \quad t\geqslant 0,$$

$$\phi(t) = \frac{\sin(\beta\pi)}{\pi}\int_0^\infty \frac{r^{\beta}\,\mathrm{e}^{-rt}}{r^{2\beta}+2r^\beta\cos(\beta\pi)+1}\,\mathrm{d}r, \quad t\geqslant 0.$$

with $E_\beta^{(k)}(z) := (\mathrm{d}^k/\mathrm{d}z^k)E_\beta(z)$, that can be deduced from the book by Podlubny, see (1.80) in [22]. Then we get, with $0<\beta<1$ and $k=0,1,2,\ldots,$

$$v_k(t) = \frac{t^{k\beta}}{k!} E_\beta^{(k)}(-t^\beta) \quad \text{from } \widetilde{v}_k(s) = \widetilde{\Psi}(s)[\widetilde{\psi}(s)]^k = \frac{s^{\beta-1}}{(1+s^\beta)^{k+1}}, \tag{4.10}$$

as generalization of the Poisson distribution (with parameter t), what we call the *β-fractional Poisson distribution*. Similarly, with $0<\beta<1$ and $k=1,2,\ldots,$ we obtain the *β-fractional Erlang pdf* (of order $k\geqslant 1$):

$$f_k(t) = \beta \frac{t^{k\beta-1}}{(k-1)!} E_\beta^{(k)}(-t^\beta) \quad \text{from } \widetilde{\phi}_k(s) = [\widetilde{\psi}(s)]^k = \frac{1}{(1+s^\beta)^k}, \tag{4.11}$$

and the corresponding *β-fractional Erlang pcf*:

$$F_k(t) = \int_0^t f_k(t')\,\mathrm{d}t' = 1 - \sum_{n=0}^{k-1} \frac{t^{n\beta}}{n!} E_\beta^{(n)}(-t^\beta) = \sum_{n=k}^{\infty} \frac{t^{n\beta}}{n!} E_\beta^{(n)}(-t^\beta). \tag{4.12}$$

5. The renewal process of Wright type

Another possible choice for obtaining an analytically treatable variant to the Poisson process, suggested by Mainardi et al. [20], is based on the assumption that the *waiting-time pdf* $\phi(t)$ is the density of an extremal, unilateral, Lévy stable distribution with index $\beta\in(0,1)$. This density exhibits, as we shall show, the same power-law asymptotics as the previous renewal process, but the transition to the limit $\beta=1$ is singular, and the Poisson process is no longer obtained. Now for $t\geqslant 0$,

$$\Psi(t) = \begin{cases} 1-\Phi_{-\beta,1}(-1/t^\beta), & 0<\beta<1, \\ \Theta(t)-\Theta(t-1), & \beta=1, \end{cases} \quad \text{from } \widetilde{\Psi}(s) = \frac{1-\mathrm{e}^{-s^\beta}}{s}, \tag{5.1}$$

$$\phi(t) = \begin{cases} \dfrac{1}{t}\Phi_{-\beta,0}\left(-\dfrac{1}{t^\beta}\right), & 0<\beta<1 \\ \delta(t-1), & \beta=1 \end{cases} \quad \text{from } \widetilde{\phi}(s) = \mathrm{e}^{-s^\beta}, \tag{5.2}$$

where $\Phi_{-\beta,1}$ and $\Phi_{-\beta,0}$ denote Wright functions.[5] In view of their presence we call this process the *renewal process of Wright type*. Let us quote for $0<\beta<1$ their for $t\geqslant 0$ convergent expansions

$$\Psi(t) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\Gamma(\beta n)}{n!}\frac{\sin(\pi\beta n)}{t^{\beta n}}, \quad 0<\beta<1, \tag{5.3}$$

$$\phi(t) = \frac{1}{\pi t}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\Gamma(\beta n+1)}{n!}\frac{\sin(\pi\beta n)}{t^{\beta n}}, \quad 0<\beta<1. \tag{5.4}$$

We see that the first term of the series is identical to the asymptotic representation of the corresponding functions for the renewal process of Mittag–Leffler type, see (4.5)–(4.7). Their behaviour near 0 is provided by the first term of their asymptotic $t\to 0$ expansions, namely from [20],

$$\Psi(t) \sim 1 - At^{b/2}\exp(-Bt^{-b}), \quad \phi(t) \sim Ct^{-c}\exp(-Bt^{-b}), \tag{5.5}$$

[5] The Wright function with parameters λ, μ, defined as

$$\Phi_{\lambda,\mu}(z) := \sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\lambda n+\mu)}, \quad \lambda>-1, \quad \mu\in\mathbf{C}, \quad z\in\mathbf{C}$$

is an entire function of order $\rho=1/(1+\lambda)$. For detailed information on functions of Wright type the reader may consult e.g., [7,15,19], and references therein. We note that in the handbook [4, vol. 3, Chapter 18], presumably for a misprint, the Wright function is considered only for $\lambda\geqslant 0$.

where

$$A=\left[\frac{1}{2\pi(1-\beta)\beta^{1/(1-\beta)}}\right]^{1/2},\quad B=(1-\beta)\beta^{b},\quad C=\left[\frac{\beta^{1/(1-\beta)}}{2\pi(1-\beta)}\right]^{1/2},$$
$$b=\frac{\beta}{1-\beta},\quad c=\frac{2-\beta}{2(1-\beta)}. \tag{5.6}$$

As far as the functions $v_k(t)$, see (2.13), are concerned, we have

$$\widetilde{v_0}(s)=\widetilde{\Psi}(s)=\frac{1-\mathrm{e}^{-s^{\beta}}}{s}, \tag{5.7}$$

so,

$$v_0(t)=\Psi(t)=\begin{cases}1-\Phi_{-\beta,1}\left(-\dfrac{1}{t^{\beta}}\right), & 0<\beta<1,\\ \Theta(t)-\Theta(t-1), & \beta=1,\end{cases} \tag{5.8}$$

and

$$\widetilde{v_k}(s)=\widetilde{\Psi}(s)[\widetilde{\psi}(s)]^k=\frac{\mathrm{e}^{-ks^{\beta}}}{s}-\frac{\mathrm{e}^{-(k+1)s^{\beta}}}{s},\quad k=1,2,\ldots, \tag{5.9}$$

from which, in view of the scaling property of the Laplace transform,

$$v_k(t)=\begin{cases}\Phi_{-\beta,1}\left(-\dfrac{k}{t^{\beta}}\right)-\Phi_{-\beta,1}\left(-\dfrac{k+1}{t^{\beta}}\right), & 0<\beta<1,\\ \Theta(t-k)-\Theta(t-k-1), & \beta=1.\end{cases} \tag{5.10}$$

Let us close this section with a discussion on the renewal function $m(t)$. We have for $0<\beta\leqslant 1$ from Eqs. (2.15) and (5.1) the Laplace transforms:

$$\widetilde{m}(s)=\frac{\widetilde{\phi}(s)}{s[1-\widetilde{\phi}(s)]}=\frac{1}{s}\,\frac{\mathrm{e}^{-s^{\beta}}}{1-\mathrm{e}^{-s^{\beta}}}=\frac{1}{s}\sum_{k=1}^{\infty}\mathrm{e}^{-ks^{\beta}}. \tag{5.11}$$

Then, since the term by term inversion is allowed, we have

$$m(t)=\begin{cases}\sum\limits_{k=1}^{\infty}\Phi_{-\beta,1}\left(\dfrac{k}{t^{\beta}}\right), & 0<\beta<1,\\ \sum\limits_{k=1}^{\infty}\Theta(t-k)=[t], & \beta=1,\end{cases} \tag{5.12}$$

where $[t]$ denotes the greatest integer less than or equal to t. We observe that, whereas for the process of Mittag–Leffler type we have the explicit expression (4.8) for the renewal function, for the process of Wright type we cannot find the corresponding expression since we do not know how to sum the series of Wright functions in (5.12). We can, however, see the asymptotics near zero and near infinity and apply Tauber theory: $s\to 0$ gives $\widetilde{m}(s)=1/s^{1+\beta}$, hence,

$$m(t)\sim t^{\beta}/\Gamma(1+\beta)\quad \text{for } t\to\infty, \tag{5.13}$$

$s\to\infty$ gives $\widetilde{m}(s)=\exp(-s^{\beta})/s=\widetilde{\Phi}(s)$, hence, in view of (5.5)–(5.6),

$$m(t)\sim\Phi(t)\sim At^{b/2}\exp(-Bt^{-b})\quad \text{for } t\to 0. \tag{5.14}$$

6. The Mittag–Leffler and Wright processes in comparison

In this section, we intend to compare the renewal processes of Mittag–Leffler and Wright type introduced in the Sections 4 and 5, using the upper indices a and b to distinguish the relevant functions characterizing these processes.

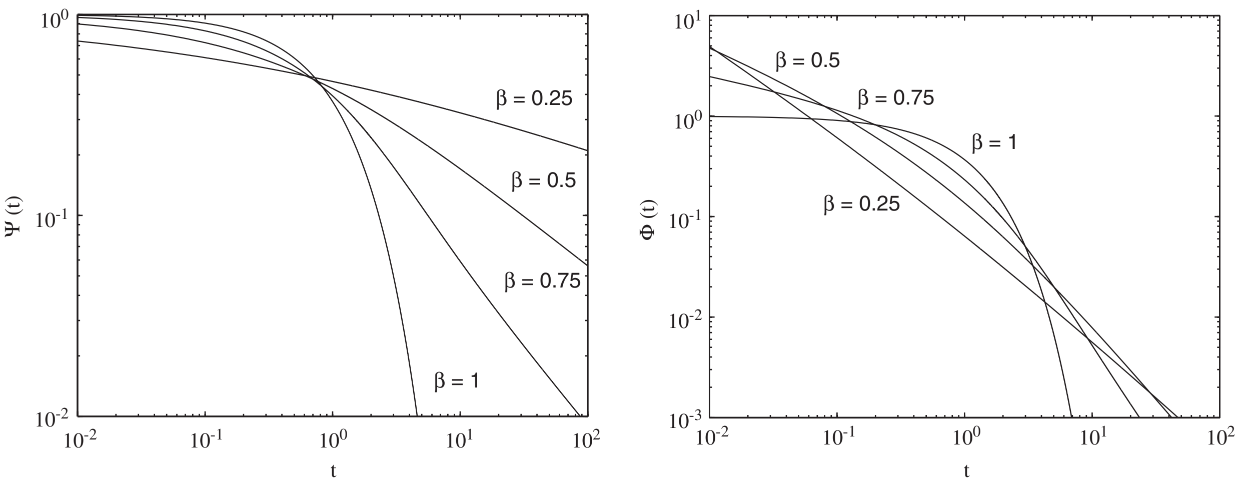

Fig. 1. The functions $\Psi(t)$ (left) and $\phi(t)$ (right) versus t $(10^{-2} < t < 10^{2})$ for the renewal process of Mittag–Leffler type with $\beta = 0.25, 0.50, 0.75, 1$.

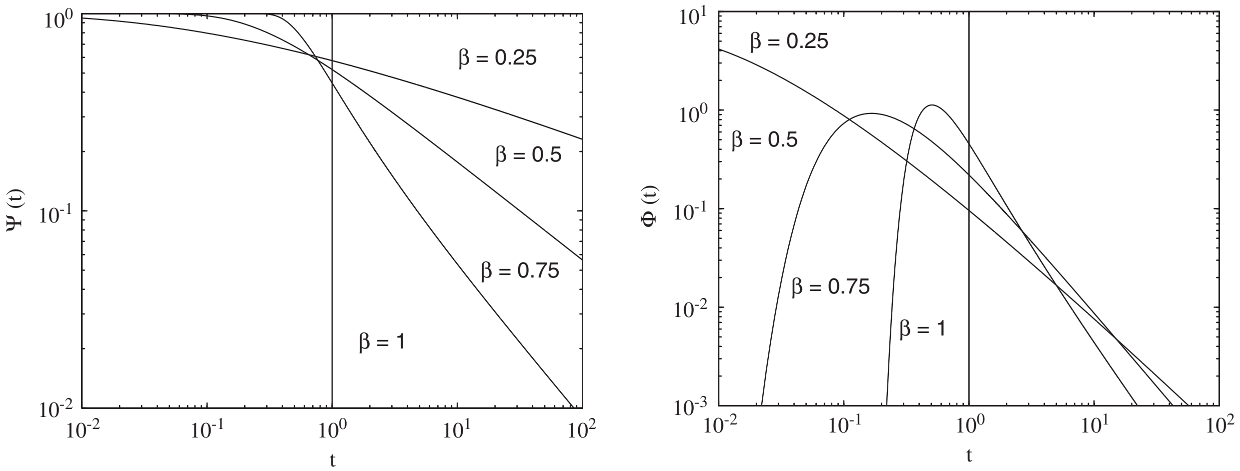

Fig. 2. The functions $\Psi(t)$ (left) and $\phi(t)$ (right) versus t $(10^{-2} < t < 10^{2})$ for the renewal process of Wright type with $\beta = 0.25, 0.50, 0.75, 1$.

We begin by pointing out the major differences between the survival functions $\Psi^{a}(t)$ and $\Psi^{b}(t)$, provided by Eqs. (4.3) and (5.1), respectively, for a common index β when $0 < \beta \leqslant 1$. These differences, visible from a comparison of the plots (with logarithmic scales) in the left plates of Figs. 1 and 2, can be easily inferred by analytical arguments, as previously pointed out (in a preliminary way) in the paper by Mainardi et al. [20]. Again we stress the different behaviour of the two processes in the limit $\beta \to 1$, for $t \geqslant 0$. Whereas $\Psi^{a}(t)$ and $\phi^{a}(t)$ tend to the exponential $\exp(-t)$, $\Psi^{b}(t)$ tends to the box function $\Theta(t) - \Theta(t-1)$ and the corresponding *waiting-time pdf* $\phi^{b}(t)$ tends to the shifted Dirac delta function $\delta(t-1)$. The first of these processes is thus a direct generalization of the Poisson process since, for the limiting value $\beta = 1$, the Poisson process is recovered. In distinct contrast, the second process changes its character from stochastic to deterministic: for $\beta = 1$ at every instant $t = n$, n a natural number, an event happens (and never at other instants) so that $N(t) = [t]$, hence trivially $m(t) := E(N(t)) = [t]$ as we had already found in (5.12) by summation. We refer to this peculiar counting process as the *clock process* because of its similarity with the tick-tick of a (perfect) clock. We also note that in the limit $\beta = 1$ the densities of the Mittag–Leffler and Wright processes have an identical finite first moment since $\rho = \int_0^{\infty} t \exp(-t)\, \mathrm{d}t = \int_0^{\infty} t\,\delta(t-1)\, \mathrm{d}t = 1$.

It is instructive to consider the special value $\beta = \frac{1}{2}$ because in cases (a) and (b) we have an explicit representation of the corresponding survival functions and waiting-time densities in terms of well known functions. For the renewal process of Mittag–Leffler type with $\beta = \frac{1}{2}$ we have

$$\Psi^a(t) = E_{1/2}(-\sqrt{t}) = \mathrm{e}^t \,\mathrm{erfc}(\sqrt{t}) = \mathrm{e}^t \,\frac{2}{\sqrt{\pi}} \int_{\sqrt{t}}^{\infty} \mathrm{e}^{-u^2}\,\mathrm{d}u, \quad t \geqslant 0, \tag{6.1}$$

$$\phi^a(t) = -\frac{\mathrm{d}}{\mathrm{d}t} E_{1/2}(-\sqrt{t}) = \frac{1}{\sqrt{\pi t}} - \mathrm{e}^t \,\mathrm{erfc}(\sqrt{t}), \quad t \geqslant 0, \tag{6.2}$$

where erfc denotes the *complementary error* function.

For the renewal process of Wright type with $\beta = \frac{1}{2}$ we obtain for $t \geqslant 0$,

$$\Psi^b(t) = 1 - \Phi_{-1/2,1}(-1/\sqrt{t}) = 1 - \mathrm{erfc}\left(\frac{1}{2\sqrt{t}}\right) = \mathrm{erf}\left(\frac{1}{2\sqrt{t}}\right), \tag{6.3}$$

$$\phi^b(t) = \frac{1}{t}\, \Phi_{-1/2,0}(-1/\sqrt{t}) = \frac{1}{2\sqrt{\pi}}\, t^{-3/2} \exp\left(-\frac{1}{4t}\right). \tag{6.4}$$

We observe that for this particular value of β the expression for the density[6] is obtained not only from the sum of the convergent series (5.4) but also exactly from its asymptotic representation for $t \to 0$, see (5.5)–(5.6).

We easily note the common asymptotic (power-law) behaviour of the survival and density functions as $t \to \infty$ in the cases (a) and (b), that is, indicating by the index ∞ that we mean the leading asymptotic term:

$$\Psi_\infty(t) = \frac{t^{-1/2}}{\sqrt{\pi}}, \quad \phi_\infty(t) = \frac{t^{-3/2}}{2\sqrt{\pi}}. \tag{6.5}$$

It is now interesting to compare numerically the survival functions and the density functions of the two processes, that is (6.1)–(6.2) with (6.3)–(6.4), respectively. We find it worthwhile to add in the comparison their asymptotic expressions in (6.5) and also the corresponding functions of the Poisson process, namely (3.1)–(3.2). In Fig. 3, we display the plots concerning the above comparison for the survival functions (left plate) and for the density functions (right plate) adopting the continuous line for the exact functions of the two long-memory processes and the dashed line both for the asymptotic power-law functions and for the exponential functions of the Poisson process. The comparison is made by using logarithmic scales for $10^{-1} \leqslant t \leqslant 10^1$, just before the onset of the power-law regime; note that, at least for the shown case $\beta = \frac{1}{2}$, the Wright process reaches the asymptotic power-law regime a little bit earlier than the corresponding Mittag–Leffler process.

We finally consider the functions $v_k(t)$ for the two long-memory processes. For the functions $v_k^a(t)$ of the Mittag–Leffler process, see (4.10), we can take profit of the recurrence relations for repeated integrals of the error functions, see e.g., [1, Section 7.2, pp. 299–300], to compute the derivatives of the Mittag–Leffler functions in Eqs. (4.10). We recall for $n = 0, 1, 2, \ldots,$

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}(\mathrm{e}^{z^2}\,\mathrm{erfc}(z)) = (-1)^n 2^n n!\,\mathrm{e}^{z^2} I^n\,\mathrm{erfc}(z), \tag{6.7}$$

where $I^n\,\mathrm{erfc}(z) = \int_z^\infty I^{n-1}\,\mathrm{erfc}(\zeta)\,\mathrm{d}\zeta$ and $I^{-1}\,\mathrm{erfc}(z) = 2\exp(-z^2)/\sqrt{\pi}$.

[6] We point out

$$\mathscr{L}\left\{\phi^b(t) = \frac{t^{-3/2}}{2\sqrt{\pi}} \exp\left(-\frac{1}{4t}\right); s\right\} = \exp(-s^{1/2}).$$

This Laplace transform pair was noted by Lévy with respect to the unilateral stable density of order $\frac{1}{2}$ and later, independently, by Smirnov. In the probability literature, the distribution corresponding to this *pdf* is known as the *Lévy–Smirnov* stable distribution. On this occasion, we point out a misprint in our paper [20]. There, in Eq. (3.25) for the Lévy–Smirnov density, the factor 2 in front of $\sqrt{\pi}$ was missed.

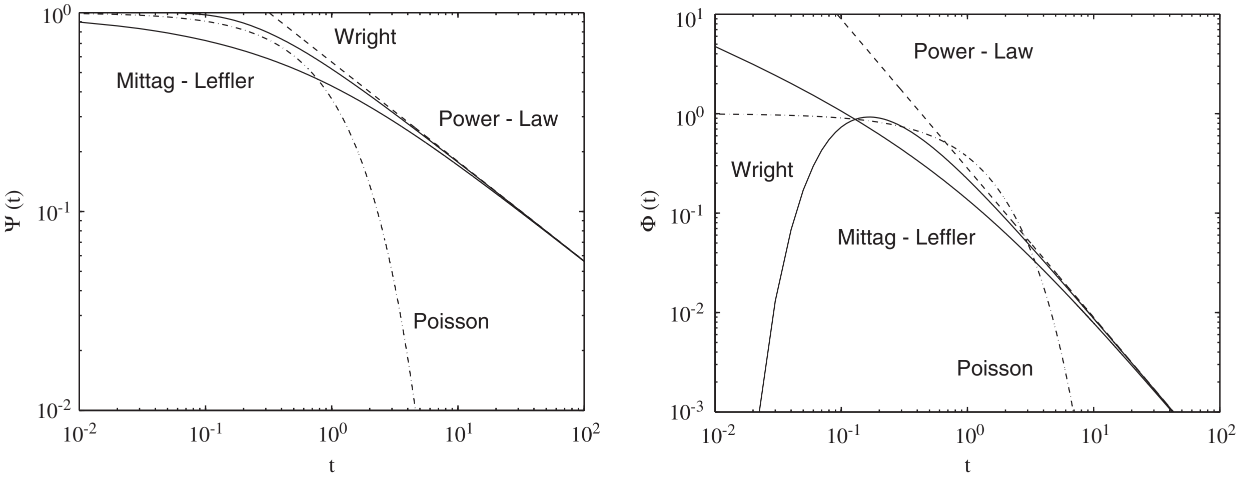

Fig. 3. Comparison versus time of the survival functions $\Psi(t)$ (left plate) and of the corresponding probability densities $\phi(t)$ (right plate) in the case $\beta = \frac{1}{2}$.

For the Wright process, see (5.10), we have $v_0^b(t) = \Psi^b(t)$ as in (6.3) and

$$v_k^b(t) = \left[\operatorname{erfc}\left(\frac{k}{2\sqrt{t}}\right) - \operatorname{erfc}\left(\frac{k+1}{2\sqrt{t}}\right)\right], \quad k = 1, 2, \dots \, . \tag{6.8}$$

7. Conclusions and comments

We have discussed the basic theory of three types of renewal processes and compared these with each other in analytical and numerical aspects, namely the *Poisson* process, the process of *Mittag–Leffler* type, and the process of *Wright* type. Whereas the Poisson process is characterized by an exponentially distributed waiting time between events and hence is *Markovian*, the other two processes are characterized by power-law decay of their waiting time density, which is defined via a function of Mittag–Leffler or Wright type, respectively. The corresponding processes clearly are no longer Markovian; they are labelled by a parameter β, $0 < \beta < 1$. The Mittag–Leffler process directly generalizes the Poisson process which is recovered for the limiting value $\beta = 1$. Contrastingly, the Wright process changes for $\beta = 1$ its character drastically: it becomes deterministic, an event now happening when and only when time t attains as value a natural number.

In other papers, see [18], we have treated *compound renewal processes*, often called *continuous time random walks* that happen in time and space, their temporal structure being determined by a renewal process. In their theory and the analysis of fractional diffusion processes, functions of Mittag–Leffler and Wright types play celebrated roles, not yet all of them exhaustingly highlighted and made intuitively understandable in the literature. The Mittag–Leffler process appears as a universal limiting process of certain processes of power-law type, see e.g., [6]. The Wright process plays an essential role in subordinative simulation of sample paths in space-time fractional diffusion, by parametric subordination, see [9].

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