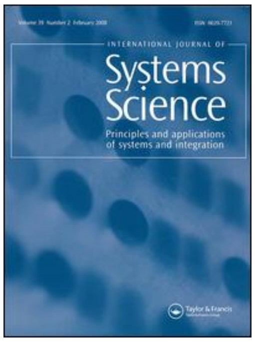

INTERNATIONAL JOURNAL OF SYSTEMS SCIENCE

ISSN: 0020-7721 (Print) 1464-5319 (Online) Journal homepage: https://www.tandfonline.com/loi/tsys20

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HONGZHOU WANG & HOANG PHAM

International Journal of Systems Science, 1996, volume 27, number 10, pages 1055–1062

A quasi renewal process and its applications in imperfect maintenance

HONGZHOU WANG† and HOANG PHAM†

This paper proposes a quasi renewal process, and its application in maintenance theory is discussed. The properties of this quasi renewal process are studied and its renewal function is derived. Three imperfect maintenance models are proposed and they model imperfect maintenance in a way that, after maintenance, the lifetime of a unit will decrease to a fraction of its immediate previous one. According to the theory of the quasi-renewal process developed in this paper, the expected maintenance cost rate and availability are obtained and optimum maintenance policies are discussed for these three models. Finally, a class of related optimization problems is discussed and a numerical example is presented.

1. Introduction

Renewal theory had its origin in the study of strategies for replacement of technical components. In the renewal process, the times between successive events are supposed to be independent and identically distributed. Most maintenance models using renewal theory were actually based on this assumption, that is, 'as good as new'. It is well-known that after repair the system may not be as good as new in maintenance practice. Barlow and Hunter (1960) introduced the notation of 'minimal repair' (as bad as old, see the Appendix) in which the unit failure rate remains undisturbed by repair and they used non-homogeneous Poisson processes to model it. Clearly, 'as good as new' and 'as bad as old' represent two extreme types of repair results. Most repair actions are, however, somewhere between these extremes, and are often called imperfect repair in recent maintenance literature. Imperfect maintenance problems have received more and more attention in the maintenance field (for example, Bhattacharjee 1987, Brown and Proschan 1983, Hollander *et al.* 1992, Nakagawa 1981, Nakagawa and Yasui 1987, Fontenot and Proschan 1984, Kijimma 1989, Makis and Jardine 1992, Whitater and Samanieg 1989). A basic treatment method on imperfect repair is that after repair the unit is 'as good as new' with probability p and 'as bad as old' with probability $q = 1 - p$ (see Bhattacharjee 1987, Brown and Proschan 1983, Nakagawa, 1981, Nakagawa and Yasui 1987). Each repair is thus a weighted average of replacement by a new one and a minimal repair. In this paper, a general renewal process is proposed and its application in imperfect maintenance is discussed.

Received 15 October 1995. Revised 3 January 1996. Accepted 12 January 1996.

† Department of Industrial Engineering, Rutgers University, Piscataway, NJ 08855-0909, U.S.A.

2. A quasi renewal process

Let $\{N(t), t > 0\}$ be a counting process and let X_n denote the time between the $(n-1)$st and the nth event of this process, $n \geq 1$.

Definition 1: If the sequence of non-negative random variables $\{X_1, X_2, X_3, \ldots\}$ is independent and $X_i = \alpha X_{i-1}$ for $i \geq 2$ where $\alpha > 0$ is a constant, then the counting process $\{N(t), t \geq 0\}$ is said to be a *quasi renewal process with parameter* α *and the first interarrival time* X_1. □

When $\alpha = 1$ this process becomes the ordinary renewal process. Later, we will see that this quasi renewal process can be used to model a maintenance process when $\alpha \leq 1$ and a reliability growth process in product developing and burn-in for $\alpha > 1$.

Assuming that the PDF, CDF, survival function and failure rates of random variable X_1 are $f_1(x)$, $F_1(x)$, $s_1(x)$ and $r_1(x)$ respectively, then the PDF, CDF, survival function, failure rate, mean and variance of X_n for

$n = 1, 2, 3, 4, \ldots$ are

$$\left.\begin{aligned} f_n(x) &= \frac{1}{\alpha^{n-1}} f_1\left(\frac{1}{\alpha^{n-1}} x\right) & F_n(x) &= F_1\left(\frac{1}{\alpha^{n-1}} x\right) \\ s_n(x) &= s_1\left(\frac{1}{\alpha^{n-1}} x\right) & r_n(x) &= \frac{1}{\alpha^{n-1}} r_1\left(\frac{1}{\alpha^{n-1}} x\right) \\ E(X_n) &= \alpha^{n-1} E(X_1) & \operatorname{Var}(X_n) &= \alpha^{2n-2} \operatorname{Var}(X_1). \end{aligned}\right\}$$

Because of the non-negativity of X_1 and the fact that X_1 is not identically 0 we conclude that $E(X_1) = \mu_1 \neq 0$.

Theorem 1: *If $f_1(x)$ belongs to IFR, DFR, $IFRA$, $DFRA$, NBU, NWU (for definitions see the Appendix), then $f_n(x)$ is in the same category for $n = 2, 3, \ldots$.*

Proof: Suppose that the failure rate of X_n is differentiable. From (1) the derivative of the failure rate of X_n is given by

$$r_n'(x) = \frac{1}{\alpha^{2n-2}} r_1'\left(\frac{1}{\alpha^{n-1}} x\right).$$

From the above equation we can see that if $r_1(x)$ is increasing or decreasing then $r_n(x)$ is also increasing or decreasing, respectively. Therefore, for the first two categories the conclusion follows.

Next assume that

$$s_1(x + y) \leq (\geq) s_1(x) s_1(y).$$

Then it follows that

$$\begin{aligned} s_n(x + y) &= s_1\left(\frac{x + y}{\alpha^{n-1}}\right) \leq (\geq) s_1\left(\frac{x}{\alpha^{n-1}}\right) s_1\left(\frac{y}{\alpha^{n-1}}\right) \\ &= s_n(x) s_n(y). \end{aligned}$$

Therefore, if $s_1(x)$ is NBU (NWU) then $s_n(x)$ is also in the same category.

Finally, note that the derivatives with respect to x

$$\begin{aligned} [s_n^{1/x}(x)]_x' &= \left[\exp\left(\frac{1}{x} \ln s_1\left(\frac{x}{\alpha^{n-1}}\right)\right)\right]_x' \\ &= -\frac{1}{\alpha^{2n-2}} s_1^{1/x}\left(\frac{x}{\alpha^{n-1}}\right)\left[\left(\frac{x}{\alpha^{n-1}}\right)^{-2} \ln s_1\left(\frac{x}{\alpha^{n-1}}\right)\right. \\ &\qquad \left. + \left(\frac{x}{\alpha^{n-1}}\right)^{-1} \frac{f_1\left(\frac{x}{\alpha^{n-1}}\right)}{s_1\left(\frac{x}{\alpha^{n-1}}\right)}\right] \end{aligned}$$

$$[s_1^{1/x}(x)]_x' = -s_1^{1/x}(x)\left[x^{-2} \ln s_1(x) + x^{-1} \frac{f_1(x)}{s_1(x)}\right]$$

and

$$s_1^{1/x}(x) \geq 0 \quad s_1^{1/x}\left(\frac{x}{\alpha^{n-1}}\right) \geq 0 \quad \text{for } x \geq 0.$$

From them it follows that if $[s_1^{1/x}(x)]_x'$ is increasing or decreasing respectively, $[s_n^{1/x}(x)]_x'$ is also increasing or decreasing respectively. Thus, for the last two categories conclusion holds. This completes the proof of Theorem 1. □

It is worthwhile noting that if $f_1(x)$ is NBUE or NWUE, then $f_n(x)$ may not be in the same category for $n = 2, 3, \ldots$

Theorem 2: *The shape parameters of X_n are the same for $n = 1, 2, 3, 4, \ldots$ for a quasi renewal process if X_1 follows the gamma, weibull or lognormal distribution.*

Proof: The proof is straightforward. □

Remark: This means that after 'renewal' the shape parameters of the interarrival time will not change. In reliability theory, the shape parameters of a lifetime of a product tend to relate to its failure mechanism. Usually, if it possesses the same failure mechanism then a product will have the same shape parameters of its lifetimes at different usage conditions. Because most maintenance does not change the failure mechanism we can expect that the lifetime of a system will have the same shape parameters. Thus, in this sense, a quasi renewal process will be suitable to model the maintenance process. □

It is worth noting that

$$\lim_{n \to \infty} \frac{E(X_1 + X_2 + \cdots + X_n)}{n} = \lim_{n \to \infty} \frac{\mu_1(1 - \alpha^n)}{(1 - \alpha)n} = \begin{cases} 0 & \text{when } \alpha < 1 \\ +\infty & \text{when } \alpha > 1. \end{cases}$$

Therefore, for the reliability growth process the average lifetime goes to infinity and for the maintenance process the mean lifetime goes to 0 when the process goes on for a very long time.

3. Distribution of $N(t)$

Consider a quasi renewal process with parameter α and the first interarrival time X_1. Clearly, the total number $N(t)$ of 'renewals' that has occurred up to time t and the arrival time of the nth renewal, SS_n, has the following relationship

$$N(t) \geq n \Leftrightarrow SS_n \leq t.$$

That is, $N(t)$ is at least n if and only if the nth renewal occurs prior to or at time t. It is easily seen that

$$SS_n = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \alpha^{i-1} X_1, \quad n \geq 1.$$

Take

$$SS_0 = 0.$$

Thus, we have

$$\begin{aligned} P\{N(t) = n\} &= P\{N(t) \geq n\} - P\{N(t) \geq n+1\} \\ &= P\{SS_n \leq t\} - P\{SS_{n+1} \leq t\} \\ &= G_n(t) - G_{n+1}(t), \end{aligned}$$

where $G_n(t)$ is the convolution of the interarrival times $F_1, F_2, \ldots, F_n$.

If the mean value of $N(t)$ is defined as the renewal function $M(t)$ then

$$\begin{aligned} M(t) &= E[N(t)] \\ &= \sum_{n=1}^{\infty} P\{N(t) \geq n\} \\ &= \sum_{n=1}^{\infty} P\{SS_n \leq t\} \\ &= \sum_{n=1}^{\infty} G_n(t). \end{aligned}$$

The derivative of $M(t)$ is known as the renewal density

$$m(t) = M'(t).$$

In renewal theory, random variables representing the interarrival distributions only assume non-negative values, the Laplace transform of its distribution $F_1(t)$ is defined by

$$\tilde{F}_1(s) = \int_0^{\infty} e^{-sx}\, dF_1(x).$$

Thus

$$\tilde{F}_n(s) = \int_0^{\infty} e^{-\alpha^{n-1} st}\, dF_1(t) = \tilde{F}_1(\alpha^{n-1} s)$$

$$\tilde{M}_n(s) = \sum_{n=1}^{\infty} \tilde{G}_n(s) = \sum_{n=1}^{\infty} \tilde{F}_1(s) \cdot \tilde{F}_1(\alpha s) \cdots \tilde{F}_1(\alpha^{n-1} s).$$

Because there is a one-to-one correspondence between distribution functions and its Laplace transforms, the first interarrival distribution of a quasi renewal process uniquely determines its renewal function.

4. Application in maintenance models

General assumptions

(*a*) The planning horizon is infinite.

(*b*) The failure rate $r_1(t)$ is continuous and monotonously increasing and differentiable.

(*c*) $F_1(t)$ is absolutely continuous and $F(0) = 0$.

(*d*) The unit begins to operate at time 0.

(*e*) Repairs at failures and preventive maintenance (PM) take negligible time for Models 1 and 2, but not for Model 3.

Next we discuss the optimal maintenance policies for the following models.

4.1. *Periodic preventive maintenance (Model 1)*

Suppose that a unit is preventively maintained at times kT, $k = 1, 2, \ldots$, at a cost c_p, independently of the unit's failure history where the constant $T > 0$ and the PM is perfect. The unit undergoes imperfect repair at failures between PMs at cost c_f in the sense that, after repair, the lifetime will reduce to a fraction α of the immediately previous one. Thus, we can apply the quasi renewal theory.

Result 1: The long-run expected maintenance cost per unit time, or maintenance cost rate is,

$$L(T) = \frac{c_p + c_f M(T)}{T}, \tag{2}$$

where $M(T)$ is the renewal function of a quasi renewal process with parameter α. □

The claimed expression in Result 1 is straightforward. We can see the form of $L(T)$ is the same as the result by the ordinary renewal theory based on the perfect failure repair assumption (Barlow and Proschan 1965). However, the renewal functions are different.

Theorem 3: *There exists an optimum T^* which minimizes $L(T)$ where $0 < T^* \leq \infty$ and the resulting minimum value of $L(T)$ is $c_f m(T^*)$.*

Proof: Note that $L(T)$ is continuous for $0 < T < \infty$ because we assume that $F_1(t)$ is continuous. It is easy to see that $L(T) \to \infty$ when $T \to 0$ from (2). If we explain PM at interval $T = \infty$ as maintenance only at failure, that is, no PM, it follows that $L(T)$ has a minimum for $0 < T \leq \infty$. A necessary condition that a finite value T^* minimizes $L(T)$ is that it satisfies the following equation, obtained by differentiating $L(T)$ with respect to T and setting the derivative equal to 0

$$T^* m(T^*) - M(T^*) = \frac{c_p}{c_f},$$

where $m(\cdot)$ is the renewal density. Substituting this equation into (2) it follows that the minimum value of $L(T)$ is $c_f m(T^*)$. □

4.2. *Periodic preventive maintenance (Model 2)*

This model is exactly like Model 1 in § 4.1 except that the unit is preventively maintained at times kT ($k = 1, 2, \ldots$) at a cost c_p where the constant $T > 0$ and the PM is imperfect in the sense that after PM the unit is as good as new with probability p and restored to its condition just prior to failure (minimal repair) with probability $q = 1 - p$.

Result 2: The long-run expected maintenance cost per unit time, or cost rate, is

$$L(T) = \frac{c_p + c_f p^2[M(T) + \sum_{i=2}^{\infty} q^{i-1}M(iT)]}{T}, \tag{3}$$

where $M(iT)$ is the renewal function of a quasi renewal process with parameter α. □

Proof: The times between consecutive perfect preventive maintenance constitute a renewal cycle. From the classical renewal reward theory (Cox 1962, Feller 1968, Ross 1983) we have

$$L(T) = \frac{C(T)}{D(T)},$$

where $C(T)$ is the expected maintenance cost per renewal cycle and $D(T)$ is the expected duration of a renewal cycle. It is easy to verify that

$$C(T) = \sum_{i=1}^{\infty} q^{i-1}p[ic_p + c_f M(iT)]$$

$$D(T) = \sum_{i=1}^{\infty} q^{i-1}p(iT).$$

Therefore, (3) follows. □

Note that if $p = 1$, corresponding to perfect PM, the above equation coincides with the result in Model 1.

Theorem 4: *There exists an optimum T^* which minimizes $L(T)$ where $0 < T^* \leq \infty$ and the resulting minimum value of $L(T)$ is $c_f p^2 \sum_{i=1}^{n} q^{i-1}[iT^*m(iT^*)]$.*

Proof: Note that $L(T)$ is continuous for $0 < T < \infty$ because we assume that $F_1(t)$ is continuous and that $L(T) \to \infty$ as $T \to \infty$ from (3). If we explain PM at an interval $T = \infty$ as maintenance only at failure, that is, no PM, it follows that $L(T)$ has a minimum for $0 < T \leq \infty$. □

A necessary condition that a finite value T^* minimizes $L(T)$ is that it satisfies the following equation, obtained by differentiating $L(T)$ with respect to T and setting the derivative equal to 0

$$\sum_{i=1}^{n} q^{i-1}[iT^*m(iT^*) - M(iT^*)] = c_p/(c_f p^2),$$

from which it follows that the optimum cost rate $L(T)$ is $c_f p^2 \sum_{i=1}^{n} q^{i-1}[iT^*m(iT^*)]$ where $m(\cdot)$ is the renewal density.

4.3. *Periodic maintenance (Model 3)*

Assume that a unit is repaired at failure i at a cost $c_f + (i - 1)c_v$ if and only if $i \leq k - 1$ where $i = 1, 2, \ldots$, and the repair is imperfect in a sense that after repair the lifetime will reduce to a fraction α of the immediately previous one. Notice that the repair cost increase by c_v for each next imperfect repair. Suppose that the first imperfect repair time is random variable Y_1 with mean η_1, the second imperfect repair time is βY_1 with mean $\beta\eta_1$ and the $(k - 1)$th repair time is $\beta^{k-2}Y_1$ with mean $\beta^{k-2}\eta_1$ where the constant $\beta \geq 1$ means the repair times are increasing as the number of imperfect repairs increases. After the $(k - 1)$th imperfect repair at failure the unit is preventively maintained at times kT ($k = 1, 2, \ldots$) at a cost c_p where the constant $T > 0$ and the PM is imperfect in a sense that after PM the unit is as good as 'new' with probability p and restored to its condition just prior to failure (minimal repair) with probability $q = 1 - p$. Here, suppose that the PM time is a random variable W with mean w. If there is a failure between kT, an imperfect repair is performed at a cost c_{fr} with negligible repair time in a sense that after repair the lifetime of this unit will be reduced to a fraction λ of its immediately previous one where $0 < \lambda < 1$.

One possible interpretation of this model is: when a new unit is put into use, the first $k - 1$ times of failure repairs, because the unit is young at that time, will be performed at a low cost $c_f + (i - 1)c_v$ for $i = 0, 1, 2, \ldots, k-2$, which results in imperfect repairs. After the $(k - 1)$th imperfect repair at failure the unit will be in a bad condition and then a better or perfect maintenance is necessary at a higher cost c_p or c_{fr}.

Result 3: The long-run expected maintenance cost per unit time, or cost rate, is

$$L(T, k) = \frac{(k-1)c_f + \frac{(k-1)(k-2)}{2}c_v + c_p p^{-1} + pc_{fr}\sum_{i=1}^{\infty} q^{i-1}M(iT)}{\frac{\mu_1(1-\alpha^{k-1})}{1-\alpha} + \frac{\eta_1(1-\beta^{k-1})}{1-\beta} + \frac{T}{p} + w}, \tag{4}$$

where $M(t)$ is the renewal function of a quasi renewal process with parameter λ and the first interarrival time distribution $F_1(\alpha^{1-k}t)$. □

Proof: The times between consecutive perfect PM constitutes a renewal cycle. From the classical renewal reward theory we have

$$L(T, k) = \frac{C(T, k)}{D(T, k)},$$

where $C(T, k)$ is the expected maintenance cost per renewal cycle and $D(T, k)$ is the expected duration of a renewal cycle. It is easy to obtain that

$$C(T, k) = (k-1)c_f + \frac{(k-1)(k-2)}{2}c_v + \sum_{i=1}^{\infty} q^{i-1}p[ic_p + c_{fr}M_k(iT)]$$

$$D(T, k) = \frac{\mu_1(1-\alpha^{k-1})}{1-\alpha} + \frac{\eta_1(1-\beta^{k-1})}{1-\beta} + \sum_{i=1}^{\infty}(w + iT)pq^{i-1}.$$

From these the above theorem follows. □

A necessary condition that finite values (T^*, k^*) minimize $L(T, k)$ is that they satisfy the following equation, obtained by differentiating $L(T, k)$ with respect to T and k, and then setting the derivatives equal to 0

$$\left[\frac{\mu_1(1-\alpha^{k-1})}{1-\alpha} + \frac{\eta_1(1-\beta^{k-1})}{1-\beta} + \frac{T}{p} + w\right] \times pc_{fr}\sum_{i=1}^{\infty} iq^{i-1}m(iT) - c_{fr}\sum_{i=1}^{\infty} q^{i-1}M(iT) = (k-1)c_f p^{-1} + \frac{(k-1)(k-2)}{2}c_v p^{-1} + c_p p^{-2}$$

$$\left[\frac{\mu_1(1-\alpha^{k-1})}{1-\alpha} + \frac{\eta_1(1-\beta^{k-1})}{1-\beta} + \frac{T}{p} + w\right]\left[c_f + \frac{(2k-3)}{2}c_v\right] = \left[\frac{-\mu_1\alpha^{k-1}\ln\alpha}{1-\alpha} + \frac{-\eta_1\beta^{k-1}\ln\beta}{1-\beta}\right] \times \left[(k-1)c_f + \frac{(k-1)(k-2)}{2}c_v + c_p p^{-1} + pc_{fr}\sum_{i=1}^{\infty} q^{i-1}M(iT)\right].$$

5. Availability and optimization problem

If we assume that the repair time is not negligible: the first imperfect repair time is random variable Y_1 with mean η_1, the second imperfect repair time is βY_1 with mean $\beta\eta_1$ and the kth repair time is $\beta^{k-1}Y_1$ with mean $\beta^{k-1}\eta_1$ where the constant $\beta \geq 1$ means the repair times are increasing as the number of repairs increases. If we suppose that the kth repair is perfect it is easy to verify that the limiting average availability is, by the ordinary renewal reward theorem

$$A(k) = \frac{\dfrac{\mu_1(1-\alpha^k)}{1-\alpha}}{\dfrac{\mu_1(1-\alpha^k)}{1-\alpha} + \dfrac{\eta_1(1-\beta^k)}{1-\beta}}, \tag{5}$$

Taking k as a real number and differentiating $A(k)$ with respect to k

$$\frac{dA(k)}{dk} = -\left[1 + \frac{\eta_1(1-\alpha)(1-\beta^k)}{\mu_1(1-\beta)(1-\alpha^k)}\right]^{-2}\frac{\eta_1(1-\alpha)}{\mu_1(1-\beta)} \times \frac{-\beta^k(1-\alpha^k)\ln\beta - \alpha^k(\beta^k-1)\ln\alpha}{(1-\alpha^k)^2}.$$

Let

$$W(x) = \beta^x(1-\alpha^x)\ln\beta + \alpha^x(\beta^x - 1)\ln\alpha.$$

Noting that

$$W(0) = 0$$

and

$$\frac{dW(x)}{dx} = \beta^x(1-\alpha^x)(\ln\beta)^2 + \alpha^x(\beta^x-1)(\ln\alpha)^2 > 0,$$

it follows that $W(x) > 0$ for $x > 0$, and

$$\frac{dA(k)}{dk} < 0.$$

Therefore, the optimum availability $\mu_1/(\mu_1 + \eta_1)$ is obtained when $k = 1$ if the maintenance cost factor is ignored.

From § 4.3 it follows that for the maintenance Model 3 the asymptotic average availability is, by the ordinary renewal reward theory

$$A(T, k) = \frac{\dfrac{\mu_1(1-\alpha^{k-1})}{1-\alpha} + \dfrac{T}{p}}{\dfrac{\mu_1(1-\alpha^{k-1})}{1-\alpha} + \dfrac{\eta_1(1-\beta^{k-1})}{1-\beta} + \dfrac{T}{p} + w}. \tag{6}$$

If a system consists of n independent and separately-maintained components then the limiting system availability is $A_s = h(A_1, A_2, \ldots, A_m)$, where A_i is the components' limiting availability (see Barlow and Proschan 1975). Therefore, we can determine the limiting average system availability in terms of its subsystems' limiting average availabilities supposing that all subsystems are separately maintained and operate independently.

In the periodic maintenance Model 3, if we further suppose that after $(k-1)$ imperfect repairs the unit will be subject to preventive maintenance at times kT $(k = 1, 2, \ldots)$ and repairs at failure, then the repair at failure is perfect and the corresponding repair time is random variable Q with mean η_2. The PM is imperfect in the sense that after PM the unit will be as good as new with probability p_1 and as bad as old with probability p_2 and will fail (worst repair) and need repair with probability p_3 where $p_1 + p_2 + p_3 = 1$. Let us further assume that the perfect and worse PM times have means η_4 and η_5 respectively and the repair time upon failure caused by PM is a random variable V with mean η_3. The reason that PM results in the unit failure is given by Nakagawa and Yasui (1987). Yak *et al.* (1984) and Nakagawa and Yasui (1987) first studied this class of problems.

According to the ordinary renewal reward theory, the limiting average availability A is

$$A(T,k) = \frac{U(T,k)}{U(T,k) + D(T,k)},$$

where $U(T,k)$ and $D(T,k)$ are respectively the accumulating failure-free time and the repair time in one renewal cycle. It is easy to obtain

$$\begin{aligned} U(T,k) &= \frac{\mu_1(1-\alpha^{k-1})}{1-\alpha} \\ &\quad + \left[\int_0^T t\,\mathrm{d}F_1(\alpha^{1-k}t) + p_2\int_T^{2T} t\,\mathrm{d}F_1(\alpha^{1-k}t) \right. \\ &\quad \left. + p_2^2\int_{2T}^{3T} t\,\mathrm{d}F_1(\alpha^{1-k}t) + \cdots\right] \\ &= \frac{\mu_1(1-\alpha^{k-1})}{1-\alpha} + (1-p_2)\sum_{i=1}^{\infty} p_2^{i-1}\int_0^{iT} s_1(\alpha^{1-k}t)\,\mathrm{d}t \end{aligned}$$

$$\begin{aligned} D(T,k) &= \frac{\eta_1(1-\beta^{k-1})}{1-\beta} + (\eta_3+\eta_5)p_3\sum_{i=1}^{\infty} p_2^{i-1}s_1(\alpha^{1-k}iT) \\ &\quad + \eta_2(1-p_2)\sum_{i=1}^{\infty} p_2^{i-1}F_1(\alpha^{1-k}iT) \\ &\quad + \eta_4 p_1\sum_{i=1}^{\infty} p_2^{i-1}s_1(\alpha^{1-k}iT). \end{aligned}$$

Let

$$\begin{aligned} CL(T,k) &= U(T,k) + D(T,k) \\ &= \frac{\mu_1(1-\alpha^{k-1})}{1-\alpha} + \frac{\eta_1(1-\beta^{k-1})}{1-\beta} \\ &\quad + (1-p_2)\sum_{i=1}^{\infty} p_2^{i-1}\int_0^{iT} s_1(\alpha^{1-k}t)\,\mathrm{d}t \\ &\quad + (\eta_3+\eta_5)p_3\sum_{i=1}^{\infty} p_2^{i-1}\,s_1(\alpha^{1-k}iT) \\ &\quad + \eta_2(1-p_2)\sum_{i=1}^{\infty} p_2^{i-1}F_1(\alpha^{1-k}iT) \\ &\quad + \eta_4 p_1\sum_{i=1}^{\infty} p_2^{i-1}s_1(\alpha^{1-k}iT). \end{aligned}$$

Then the unit's availability is

$$A(T,k) = \frac{\dfrac{\mu_1(1-\alpha^{k-1})}{1-\alpha} + (1-p_2)\displaystyle\sum_{i=1}^{\infty} p_2^{i-1}\int_0^{iT} s_1(\alpha^{1-k}t)\,\mathrm{d}t}{CL(T,k)}. \tag{7}$$

The optimal T and k, which maximize $A(T,k)$ satisfy the following simultaneous equations if they exist

$$\begin{aligned} &\left[(1-p_2)\sum_{i=1}^{\infty} ip_2^{i-1}s_1(\alpha^{1-k}iT)\right] \\ &\quad\times\left[\frac{\mu_1(1-\alpha^{k-1})}{1-\alpha} + \frac{\eta_1(1-\beta^{k-1})}{1-\beta}\right. \\ &\qquad + (1-p_2)\sum_{i=1}^{\infty} p_2^{i-1}\int_0^{iT} s_1(\alpha^{1-k}t)\,\mathrm{d}t \\ &\qquad + (\eta_3+\eta_5)p_3\sum_{i=1}^{\infty} p_2^{i-1}s_1(\alpha^{1-k}iT) \\ &\qquad + \eta_2(1-p_2)\sum_{i=1}^{\infty} p_2^{i-1}F_1(\alpha^{1-k}iT) \\ &\qquad \left. + \eta_4 p_1\sum_{i=1}^{\infty} p_2^{i-1}s_1(\alpha^{1-k}iT)\right] \\ &\quad - \left[\frac{\mu_1(1-\alpha^{k-1})}{1-\alpha} + (1-p_2)\sum_{i=1}^{\infty} p_2^{i-1}\int_0^{iT} s_1(\alpha^{1-k}t)\,\mathrm{d}t\right] \\ &\quad\times\left\{\sum_{i=1}^{\infty} ip_2^{i-1}[(1-p_2)s_1(\alpha^{1-k}iT)\right. \\ &\quad + \alpha^{1-k}f_1(\alpha^{1-k}iT) \\ &\quad \left.\times(\eta_2(1-p_2) - (\eta_3+\eta_5)p_3 - \eta_4 p_1)]\right\} = 0 \end{aligned}$$

$$\begin{aligned} &\left[\frac{-\mu_1\alpha^{k-1}\ln\alpha}{1-\alpha} - (1-p_2)\alpha^{1-k}\ln\alpha\right. \\ &\quad\left.\times\sum_{i=1}^{\infty} p_2^{i-1}\int_0^{iT} tf_1(\alpha^{1-k}t)\,\mathrm{d}t\right] \\ &\quad\times\left[\frac{\mu_1(1-\alpha^{k-1})}{1-\alpha} + \frac{\eta_1(1-\beta^{k-1})}{1-\beta}\right. \\ &\qquad + (1-p_2)\sum_{i=1}^{\infty} p_2^{i-1}\int_0^{iT} s_1(\alpha^{1-k}t)\,\mathrm{d}t \\ &\qquad + ((\eta_3+\eta_5)p_3 + \eta_4 p)\sum_{i=1}^{\infty} p_2^{i-1}s_1(\alpha^{1-k}iT) \\ &\qquad \left. + \eta_2(1-p_2)\sum_{i=1}^{\infty} p_2^{i-1}F_1(\alpha^{1-k}iT)\right] \\ &\quad - \left[\frac{\mu_1(1-\alpha^{k-1})}{1-\alpha} + (1-p_2)\sum_{i=1}^{\infty} p_2^{i-1}\int_0^{iT} s_1(\alpha^{1-k}t)\,\mathrm{d}t\right] \end{aligned}$$

$$\times \left\{ -\frac{\mu_1 \alpha^{k-1} \ln \alpha}{1-\alpha} - \frac{\eta_1 \beta^{k-1} \ln \beta}{1-\beta} + \sum_{i=1}^{\infty} p_2^{i-1} \alpha^{1-k} \ln \alpha \left[-(1-p_2) \int_0^{iT} t f_1(\alpha^{1-k} t)\, dt + iT\alpha^{1-k} f_1(\alpha^{1-k} iT) \times (\eta_2(1-p_2) - (\eta_3+\eta_5)p_3 - \eta_4 p_1) \right] \right\} = 0.$$

The above two equations are obtained by differentiating $A(T, k)$ with respect to T and k respectively and setting them equal to 0.

Sometimes it may be required that when some availability requirements are satisfied the optimum maintenance policy is obtained. For the maintenance Model 3, noting the asymptotic average availability in (6), the following optimization problem can be formulated in terms of decision variables T and k

minimize $L(T, k) =$

$$\frac{(k-1)c_f + \dfrac{(k-1)(k-2)}{2} c_v + c_p p^{-1} + p c_{f_r} \sum_{i=1}^{\infty} q^{i-1} M(iT)}{\dfrac{\mu_1(1-\alpha^{k-1})}{1-\alpha} + \dfrac{\eta_1(1-\beta^{k-1})}{1-\beta} + \dfrac{T}{p} + w}, \tag{8}$$

subject to

$$\begin{cases} A(T,k) = \dfrac{\dfrac{\mu_1(1-\alpha^{k-1})}{1-\alpha} + \dfrac{T}{p}}{\dfrac{\mu_1(1-\alpha^{k-1})}{1-\alpha} + \dfrac{\eta_1(1-\beta^{k-1})}{1-\beta} + \dfrac{T}{p} + w} \geq A_0 \\ k = 2, 3, \ldots \\ T > 0, \end{cases}$$

where constant A_0 is the availability requirement.

Similarly, some other models can be formulated and these models can easily be solved by using any nonlinear programming software.

6. A numerical example

To illustrate the optimization model (8) we now present a numerical example. Assume that the lifetime, X_1, of a new system follows the normal distribution with mean μ and variance σ^2, that is

$$f_1(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\{(x-\mu)^2/2\sigma^2\}.$$

Note that the normal distribution is IFR. For periodic maintenance Model 3, after the $(k-1)$th imperfect repair at failure, the PDF of the lifetime of this system is

$$f_n(x) = \frac{1}{\alpha^{n-1}} f_1\left(\frac{1}{\alpha^{n-1}} x\right).$$

According to § 2, the renewal function of a quasi renewal process with parameter λ and the first interarrival time distribution

$$f_n(x) = \frac{1}{\alpha^{n-1}} f_1\left(\frac{1}{\alpha^{n-1}} x\right)$$

is

$$M(t) = \sum_{n=1}^{\infty} G_n(t).$$

It is easy to obtain that

$$G_n(t) = N\left(\frac{\mu\alpha^{k-1}(1-\lambda^n)}{1-\lambda}, \frac{\sigma^2\alpha^{2k-2}(1-\lambda^{2n})}{1-\lambda^2}\right),$$

where N represents the normal cumulative function with mean $\mu\alpha^{k-1}(1-\lambda^n)/(1-\lambda)$ and variance

$$\sigma^2\alpha^{2k-2}(1-\lambda^{2n})/(1-\lambda^2).$$

Therefore, the renewal function

$$M(t) = \sum_{n=1}^{\infty} N\left(\frac{\mu\alpha^{k-1}(1-\lambda^n)}{1-\lambda}, \frac{\sigma^2\alpha^{2k-2}(1-\lambda^{2n})}{1-\lambda^2}\right) = \sum_{n=1}^{\infty} \Phi\left(\left[t - \frac{\mu\alpha^{k-1}(1-\lambda^n)}{1-\lambda}\right] \bigg/ \left(\frac{\sigma^2\alpha^{2k-2}(1-\lambda^{2n})}{1-\lambda^2}\right)^{1/2}\right),$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function.

Now assume that

$$\mu = 10 \quad \sigma = 1 \quad \eta_1 = 0{\cdot}9 \quad c_f = \$1 \quad c_v = \$0{\cdot}06 \quad c_p = \$3$$
$$c_{f_r} = \$4 \quad \alpha = 0{\cdot}95 \quad \beta = 1{\cdot}05 \quad p = 0{\cdot}95 \quad w = 0{\cdot}2$$
$$\lambda = 0{\cdot}95 \quad A_0 = 0{\cdot}94.$$

Substituting the above parameters and the renewal function into the optimization model (8) we obtain

minimize
$L(T, k) =$

$$\frac{k + 0{\cdot}03(k-1)(k-2) + \frac{41}{19} + 3{\cdot}8 \sum_{i=1}^{\infty} 0{\cdot}05^{i-1} \sum_{n=1}^{\infty} \times \Phi\left([iT - 200 \times 0{\cdot}95^{k-1}(1-0{\cdot}95^n)] \bigg/ \left(\dfrac{0{\cdot}95^{2k-2}(1-0{\cdot}95^{2n})}{0{\cdot}0975}\right)^{1/2}\right)}{200(1-0{\cdot}95^{k-1}) + 18(1{\cdot}05^{k-1}-1) + \dfrac{T}{0{\cdot}95} + 0{\cdot}2}$$

subject to

$$\begin{cases} A(T,k) = \dfrac{200(1-0{\cdot}95^{k-1}) + \dfrac{T}{0{\cdot}95}}{200(1-0{\cdot}95^{k-1}) + 18(1{\cdot}05^{k-1}-1) + \dfrac{T}{0{\cdot}95} + 0.2} \geq 0{\cdot}94 \\ k = 2, 3, \ldots \\ T > 0 \end{cases}$$

Using nonlinear integer programming software we can find the optimal solution (T^*, k^*) that minimizes the maintenance cost rate given that the availability is at least 0·94

$$T^* = 11{\cdot}9114 \qquad k^* = 3$$

and the corresponding minimum cost rate and availability are respectively

$$L(T^*, k^*) = 0{\cdot}2741 \qquad A(T^*, k^*) = 0{\cdot}94.$$

The results show that the optimal maintenance policy is to perform repair at the first two failures of the system at a cost of \$1, and then do PM every 11·9114 time units at a cost of \$3 and repair the system upon failure between PMs at a cost \$4.

Appendix

Acronyms

NBU	new better than used
NWU	new worse than used
NBUE	new better than used in expectation
NWUE	new worse than used in expectation
IFRA	increasing failure rate in average
DFRA	decreasing failure rate in average
IFR	increasing failure rate
DFR	decreasing failure rate

Assume that lifetime has a distribution function $F(t)$ and survival function $s(t) = 1 - F(t)$ with mean μ. We have the following definitions (for details see Barlow and Proschan 1965):

(1) s is NBU if $s(x+y) \leq s(x) \cdot s(y)$ for all $x, y \geq 0$.
s is NWU if $s(x+y) \geq s(x) \cdot s(y)$ for all $x, y \geq 0$.

(2) s is NBUE if $\int_t^\infty s(x)\,dx \leq \mu s(t)$ for all $t \geq 0$.
s is NWUE if $\int_t^\infty s(x)\,dx \geq \mu s(t)$ for all $t \geq 0$.

(3) s is IFRA if $[s(x)]^{1/x}$ is decreasing in x for $x > 0$.
s is DFRA if $[s(x)]^{1/x}$ is increasing in x for $x > 0$.

(4) s is IFR if and only if $[F(t+x) - F(t)]/s(t)$ is increasing in t for all $x > 0$.
s if DFR if and only if $[F(t+x) - F(t)]/s(t)$ is decreasing in t for all $x > 0$.

(5) *Minimal repair.* Restore the system to its condition just prior to failure, i.e. if a system fails at age t and undergoes minimal repair, then the repaired system has survival function $s(x+t)/s(t)$.

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