

Wireless Network Capacity versus Ollivier-Ricci Curvature under Heat-Diffusion (HD) Protocol

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Abstract—Negative curvature in the relatively new Ollivier-Ricci sense of a wireless network graph is shown to be the culprit behind large queue occupancy, large routing energy, and restricted capacity region under any throughput optimal protocol. This is the wireless counterpart of the congestion phenomenon occurring in a Gromov negatively curved wired network under least cost path routing. Significantly different protocols call for significantly different curvature concepts to explain the “congestion” phenomenon. The rationale for the Ollivier-Ricci curvature is that it is defined in terms of a transportation cost—formalized by the Wasserstein distance—under a diffusion process. The Heat-Diffusion protocol used in this paper is driven by the queue differential, so that interpreting packets as calories, it lends itself to a genuine heat diffusion, yet retaining the 3-stage process of weighting-scheduling-forwarding of well-known Back-Pressure protocol. The main result is that the transportation definition of the Ollivier-Ricci curvature allows for the direct connection—without resorting to the Laplacian operator of heat calculus—between curvature and queue occupancy, routing energy, and capacity region.

I. INTRODUCTION

One of the great successes of the geometric approach to networks has been the explanation of the congestion phenomenon as resulting from least cost path routing on a Gromov hyperbolic graph. Many of the classical graph generators, e.g., scale-free generator, representative of real networks have indeed been shown to have negative curvature properties. It is a basic fact of differential geometry that on a negatively curved manifold the geodesics between uniformly distributed pairs of points have a tendency to pass through a “core” where the congestion is observed. Observe that we use a manifold argument to explain a graph phenomenon; this is an argument that has become popular over the past 10 years and justified by the common “coarse” geometric properties of Gromov negatively curved graphs and hyperbolic manifolds.

In wireless networks, the link interference adds extra difficulty relative to wireline networks. Under interference and link capacity constraints, the throughput optimality is crucially important in the context of limited wireless resources. Then the relevant protocol to address these issues is the Back-Pressure (BP) or a variation thereof. Different protocols call for different curvature concepts to explain the broader “congestion” concept of wireless networks: queue occupancy, routing energy, or more specifically capacity region. The routing energy is the space-discrete version of the Dirichlet

energy of heat diffusion on a manifold. By the Dirichlet principle, a protocol that minimizes routing energy would follow heat diffusion. In a set of papers [1]–[3], we designed a protocol, referred to as *Heat-Diffusion (HD)*, which follows the heat diffusion process on a graph, yet keeping *weighting-scheduling-forwarding* structure of Back-Pressure.

This paper identifies the relevant graph topological features that impact queue occupancy, routing energy, and capacity region under HD protocol. It is a mathematical fact that heat diffusion is regulated by the Ricci curvature or a coarse version thereof, the Ollivier-Ricci curvature. As main result, we show that the more negative the Ollivier-Ricci curvature, the higher the queue occupancy, the smaller the capacity region.

A. Ollivier-Ricci curvature and heat diffusion

Given a heat equation solution $T(x, t)$ over a Riemannian manifold M endowed with a heat capacity measure dC , the heat measure $dQ = TdC$ is the solution of a continuity equation over the space $(\mathcal{P}_2(M), W_2)$ of probability measures defined over the Riemannian manifold M and endowed with the 2nd Wasserstein metric W_2 . In a bit the same way as sectional curvature regulates the divergence/convergence of the geodesics on the Riemannian manifold M , the (Ollivier-)Ricci curvature (coarsely) regulates the divergence/convergence of the heat diffusion paths in $(\mathcal{P}_2(M), W_2)$ (see [14]). For example, if P_t is the heat kernel over M and P_t^* the dual over $\mathcal{P}_2(M)$, we have $W_2(P_t^*Q_1, P_t^*Q_2) \leq e^{-Kt}W_2(Q_1, Q_2)$, for $K \leq \text{Ric}(M)$, where $\text{Ric}(\cdot)$ denotes the Ricci curvature. It suffices to replace manifold by graph and think the probability measure defined on the vertices of a graph as the packet distribution to understand the role of the Ollivier-Ricci curvature.

Since the heat equation involves the Laplacian, one could think of establishing the curvature-diffusion connection via the Laplacian, as shown in Figure 1, and indeed Bauer [4] derived some bounds on the spectrum of the graph Laplacian in terms of the Ollivier-Ricci curvature. Another connection is the usual “second smallest eigenvalue” of the Laplacian related via Cheeger’s theorem to the expansion coefficient. There is the folk statement that “high expansion coefficient” means negative curvature; however, this statement has recently been challenged by Ollivier [10] where in his Problem T, he proposes to find

“... a family of expanders (i.e., a family of graphs of bounded degree, spectral gap bounded away from 0 and diameter tending to ∞) with non-negative Ricci curvature.”

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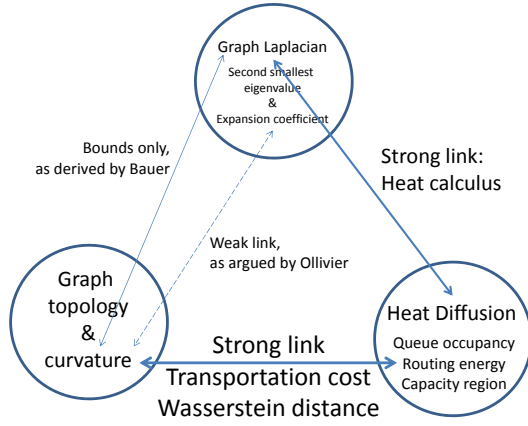


Fig. 1. Heat diffusion versus curvature versus graph Laplacian

In the same Figure 1, “Heat Diffusion” refers to a large body of mathematical work on how heat flows on a manifold (even a Finsler manifold) or a graph (even a directed graph) along with its application to wireless networking. Naturally this leads to defining generalized Laplacian concepts, e.g, Laplacian on Finsler manifolds.

In the present paper, we develop the *direct* link between the performance of the Heat-Diffusion protocol and the curvature, as shown at the bottom of Figure 1. This connection is important because the Heat-Diffusion protocol is throughput optimal, so any relationship between Heat-Diffusion protocol and curvature means a connection between capacity region and curvature.

II. OVERVIEW OF THERMODYNAMICAL HEAT-DIFFUSION

At timeslot n , let $Q_i^{(d)}(n)$ denote the number of packets queued at the network layer of node i , with destination node d , with link ij constrained by the capacity $\mu_{ij}(n)$ with link cost factor ρ_{ij} .

- **HD Weighting:** The algorithm first finds the d -packets to transmit as

$$Q_{ij}^{(d)}(n) = \max \left\{ 0, Q_i^{(d)}(n) - Q_j^{(d)}(n) \right\}, \quad (1)$$

$$d_{ij}^*(n) = \arg \max_d Q_{ij}^{(d)}(n).$$

To attribute a weight to each link, the HD algorithm performs the following:

$$\widehat{f}_{ij}(n) = \min \left\{ \left\lceil 1/2 Q_{ij}^{(d^*)}(n) \right\rceil, Q_i^{(d^*)}(n), \mu_{ij}(n) \right\}, \quad (2)$$

$$w_{ij}(n) = \left(\widehat{f}_{ij}(n) \right) \left(Q_{ij}^{(d^*)}(n) \right). \quad (3)$$

- **HD Scheduling** A scheduling matrix $\mathbf{S}(n)$ is chosen in a scheduling set \mathcal{S} by solving a centralized optimization problem as

$$\mathbf{S}(n) = \arg \max_{\mathbf{S} \in \mathcal{S}} \sum_{(i,j) \in \mathcal{E}} \mathbf{S}_{(i,j)} w_{ij}(n) / \rho_{ij}(n), \quad (4)$$

- **HD Forwarding:** Subsequent to the scheduling stage, each activated link transmits $\widehat{f}_{ij}(n)$ number of packets in accordance with (2).

III. OLLIVIER-RICCI CURVATURE

In the wake of the newly formulated concept of *Ollivier-Ricci curvature* [4], [9], heat calculus was redirected towards the connection between the heat kernel (related to graph Laplacian) and the new curvature concept (see, e.g., [6] and the work of Gregoryan). However, here, our interest in the Ollivier-Ricci curvature rather stems from its very definition in terms of a “transportation cost,” which can be linked to queue occupancy, routing cost, even time to reach steady-state.

Consider a weighted graph $((\mathcal{V}, \mathcal{E}), \rho)$. On this graph, over each vertex i , we define a probability measure m_i on $\mathcal{N}(i) := \{j \in \mathcal{V} : ij \in \mathcal{E}\}$ as follows:

$$m_i(j) = \frac{\rho_{ij}}{\sum_{j \in \mathcal{N}(i)} \rho_{ij}}, \quad \text{if } ij \in \mathcal{E},$$

$$= 0, \quad \text{otherwise.}$$

In full agreement with the recent work on network traffic versus graph curvature [7], [8], the Ollivier-Ricci curvature is defined in terms of the transport properties of the graph:

Definition 1: The Ollivier-Ricci curvature of the graph $((\mathcal{V}, \mathcal{E}), \rho)$ endowed with the set of probability measures $\{m_i : i \in \mathcal{V}\}$ is defined along the geodesic path $[i, j]$ as

$$\kappa([i, j]) = 1 - \frac{W_1(m_i, m_j)}{d(i, j)}, \quad (5)$$

where $W_1(m_i, m_j)$ is the first Wasserstein distance [12], [13] between the probability measures m_i and m_j defined on $\mathcal{N}(i)$ and $\mathcal{N}(j)$, resp.,

$$W_1(m_i, m_j) = \inf_{\xi^{ij}} \sum_{k, \ell \in \mathcal{N}(i) \times \mathcal{N}(j)} d(k, \ell) \xi^{ij}(k, \ell),$$

where the infimum is extended over all “coupling” measures ξ^{ij} defined on $\mathcal{N}(i) \times \mathcal{N}(j)$ and projecting on the first (second) factor as m_i (m_j), that is,

$$\sum_{\ell \in \mathcal{N}(j)} \xi^{ij}(k, \ell) = m_i(k), \quad \left(\sum_{k \in \mathcal{N}(i)} \xi^{ij}(k, \ell) = m_j(\ell) \right),$$

where $d(i, j)$ is the usual metric emanating from the edge weight ρ .

More intuitively, $\xi^{ij}(k, \ell)$ is called *transference plan*. It tells us how much of the mass of $k \in \mathcal{N}(i)$ is transferred to $\ell \in \mathcal{N}(j)$, but it does not tell us anything about the actual path that the mass has to follow. Ollivier [9] showed that (5) is a coarse version of the usual Ricci curvature on manifolds.

From here on, we shall restrict the definition to edges instead of paths. The specialization of (5) to edges is trivial and left to the reader. The first result of the edge-specialized concept is that W_1 satisfies the triangle inequality,

$$W_1(ij) + W_1(jk) \geq W_1([i, k]).$$

It follows that, if we take $\rho_{ij} = 1$,

$$\frac{1}{2} (\kappa(ij) + \kappa(jk)) \leq \kappa([i, k]).$$

Therefore, it is enough to evaluate the curvature between two neighboring points [9, Proposition 19] to derive lower bounds on the Ollivier-Ricci curvature.

Bauer [4] developed a general sharp inequality for undirected, weighted, connected, finite (multi)graph of N vertices $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as follows:

$$\begin{aligned} \kappa(ij) \geq & - \left(1 - \frac{\rho_{ij}}{d_x} - \frac{\rho_{ij}}{d_j} - \sum_{i_1, i_1 \sim i, i_1 \sim j} \frac{\rho_{i_1 i}}{d_i} \vee \frac{\rho_{i_1 j}}{d_i} \right) + \\ & - \left(1 - \frac{\rho_{ij}}{d_x} - \frac{\rho_{ij}}{d_j} - \sum_{i_1, i_1 \sim i, i_1 \sim j} \frac{\rho_{i_1 i}}{d_i} \wedge \frac{\rho_{i_1 j}}{d_i} \right) + \\ & + \sum_{i_1, i_1 \sim i, i_1 \sim j} \frac{\rho_{i_1 i}}{d_i} \wedge \frac{\rho_{i_1 j}}{d_i} + \frac{\rho_{ii}}{d_i} + \frac{\rho_{jj}}{d_j}. \end{aligned} \quad (6)$$

In the above, $a_+ = \max(a, 0)$; $a \wedge b = \min(a, b)$; $a \vee b = \max(a, b)$; $d_i = \sum_{j \sim i} \rho_{ij}$, ρ_{ii} is the weight of self-loops; $j \sim i$ denotes the existence of an edge between j and i .

The connection between the Ollivier-Ricci curvature and the graph Laplacian is developed in [4] and not reproduced here.

IV. OLLIVIER-RICCI CURVATURE OF STANDARD GRAPH GENERATORS VERSUS NETWORK STABILITY: SINGLE CLASS NETWORK AND CURVATURE LOWER BOUND

The HD protocol simulation is setup as follows:

- **Graphs used.** A total of 9 different graphs of the same order ($|\mathcal{V}| = 50$) are used to relate the Ollivier-Ricci curvature and the performance of the HD protocol that runs on the graphs. Those graphs are
 - a complete graph,
 - three scale-free random graphs with different growth parameter,
 - three Erdős-Rényi random graphs with different attachment probabilities, generated by MIT Matlab Network Analysis toolbox [5],
 - two Small-World random graphs generated by Watts-Strogatz model (k -nearest neighbors connection with $k = 3, 4$, respectively, rewiring probability is 0.2).
- **Link capacity.** For the queue occupancy and the routing energy, the link capacity is set to infinity. For the capacity region in the next section, the link capacity is set to 1500 packets for every link.
- **Packet arrival rate.** Except for the simulation related to the capacity region, in order to avoid random effects and to single out the specific impact of the network topology, the packet arrival rate is set to 1 Packet per Timeslot (PpT) from every node to one single sink (single-class) or to every other node (multi-class).
- **Scheduling.** It is known that for K -hop interference model for $K > 1$, the maximum weighted matching

problem is NP-hard. For $K = 1$, however, the problem is polynomial-time solvable. One algorithm one would mention is Edmonds' blossom algorithm. The weighting in our simulation is provided by the quadratic weight along edges specified in the HD protocol.

- **Position of the sink.** In single-class HD protocol, the position of the sink is chosen at random. However, different positions of the sink should result in different simulation results. Therefore, in the next section, an unbiased model using *multi-class* HD protocol is used to rule out the effect brought by the position of the sink.
- **Outline of simulation.** Essentially, we link congestion to curvature. The congestion of the network is reflected in two measurements:

- the average queue occupancy over all nodes, $\frac{1}{|\mathcal{V}|} \sum_{i \in \mathcal{V}} Q_i(n)$;
- the average routing energy over all links, $E_K(i, j) := \frac{1}{|\mathcal{E}|} \sum_{i, j \in \mathcal{E}} \frac{1}{2} (Q_i(n) - Q_j(n))^2$.

Curvature is measured in two different ways:

- as the average of the lower bound on the Ollivier-Ricci curvature over all edges;
- as the exact Ollivier-Ricci curvature averaged over all edges (next section).

Two different setups are considered on all nine different graph models:

- the single-class (5,000 timeslots in total are run);
- the multi-class (20,000 timeslots in total are run in the next section). The multi-class has been proved to be the superposition of single-class Heat-Diffusions in steady-state.

A. Queue occupancy versus Ollivier-Ricci curvature

The Ollivier-Ricci curvature estimate used in the simulations is the lower bound (6) on the curvature averaged over all edges. The steady-state queue occupancy of the network increases as Ollivier-Ricci curvature of the graph decreases as shown in Figure 2, although not monotonically, yet the slope of the interpolation line is -0.28.

B. Average routing energy versus Ollivier-Ricci curvature

Since the routing energy is averaged over all edges, it provides a global metric, which should be related to the Ollivier-Ricci curvature of the graph, as Figure 3 indeed shows. The routing energy increases with decreasing curvature with a slope of -0.23, as expected.

Although the Ollivier-Ricci curvature and the network congestion are not monotonically related, they still show a correlation. The reason of the low correlation value mainly comes from two sources. The first is that the result of single-class Heat-Diffusion depends on the position of the sink, which is chosen at random in the simulation. The second is that the lower bound on Ollivier-Ricci curvature is still an estimate. (This situation will be rectified in the next section by computing the *exact* Ollivier-Ricci curvature.)

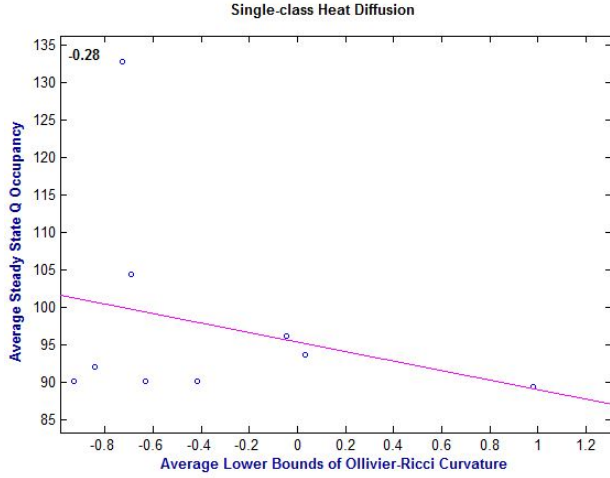


Fig. 2. Single class: queue occupancy averaged over all nodes in steady state versus lower bounds of Ollivier-Ricci curvature averaged over all links in each of the 9 graphs. (The red line is the least square error linear interpolation of the queue occupancy versus the curvature.)

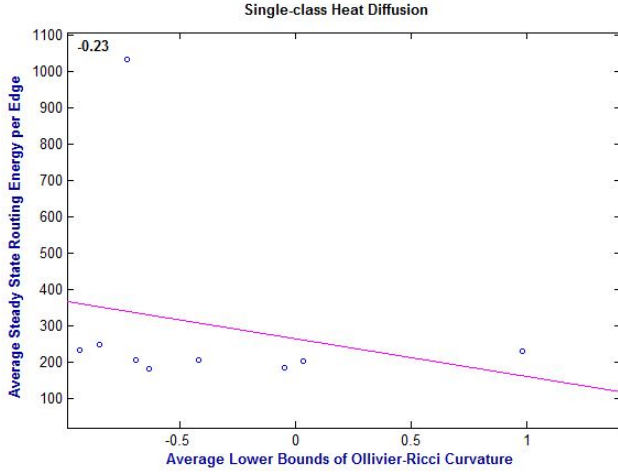


Fig. 3. Single class: average routing energy in steady state versus lower bounds of Ollivier-Ricci curvature averaged over all links.

V. OLLIVIER RICCI CURVATURE OF STANDARD GRAPH GENERATORS VERSUS NETWORK STABILITY: SINGLE & MULTIPLE CLASS NETWORK AND EXACT CURVATURE

The results based on the lower bound (6) can be inaccurate in some cases. For example, for a Euclidean square lattice, for two neighboring nodes that are not on the boundary of the graph, the lower bound calculated by Bauer's estimate is -1. However, the *exact* value of the transportation cost is 1, which results in an *exactly* vanishing Ollivier-Ricci curvature, as expected.

Based on the above concern, we need to calculate the *exact* Ollivier-Ricci curvature instead of a lower bound to further develop the congestion versus curvature relationship. One way to do the calculation is to utilize the Earth Mover's Distance (EMD). EMD is a measure of the distance between two probability distributions, usually used in image processing [11]. It is

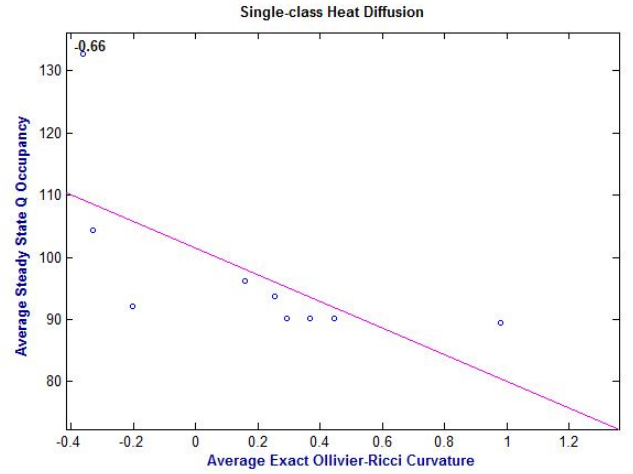


Fig. 4. Single-class: average queue occupancy in steady-state versus exact Ollivier-Ricci curvature averaged over all links.

based on the *exact* minimum cost to transform one distribution to the other. For accurate simulations, this minimal cost should be calculated by linear programming [2] as follows:

In the set-up of the original Monge-Kantorovich problem, let S denote the vector of supply and S_i denote the amount of supply at location i . Let D denote the vector of demand and D_j denotes the amount of demand at location j . We also have a cost matrix C , where C_{ij} is the transportation cost of moving a unit mass from supply location i to demand location j . Let $x(i, j)$ denotes the amount of mass transported from i to j . In linear programming, we are minimizing the total cost

$$\text{Optimal Cost} = \min_{x(i,j) \geq 0} \sum_i \sum_j x(i, j) C(i, j)$$

subject to the constraints

$$\sum_j x(i, j) \leq S(i), \quad \sum_i x(i, j) \leq D(j).$$

Linear programming allows the extension of this computational method for symmetric directed graphs to general directed graphs.

A. Congestion of HD protocol versus exact Ollivier-Ricci curvature with improvements

Based on the same simulation setup as in the previous section, we now relate the congestion of single-class HD protocol with the *exact* Ollivier-Ricci curvature.

From the results (Figures 4-5), we see that the correlation between exact Ollivier-Ricci curvature and congestion is significantly improved compared with the correlation between lower bound of Ollivier-Ricci curvature and congestion. This can be further improved by going to multi-class HD protocol (Figures 6-7). The simulation setup is basically the same. The only difference is that for each timeslot, every node is sending 1 packet to every other node instead of only one specific sink.

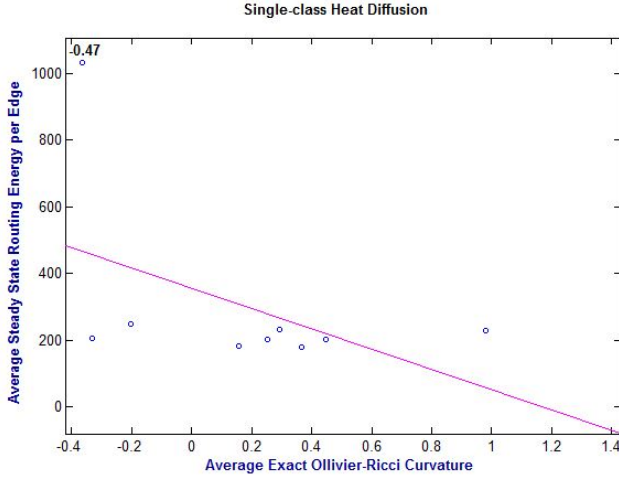


Fig. 5. Single-class: average routing energy in steady-state versus exact Ollivier-Ricci curvature averaged over all links.

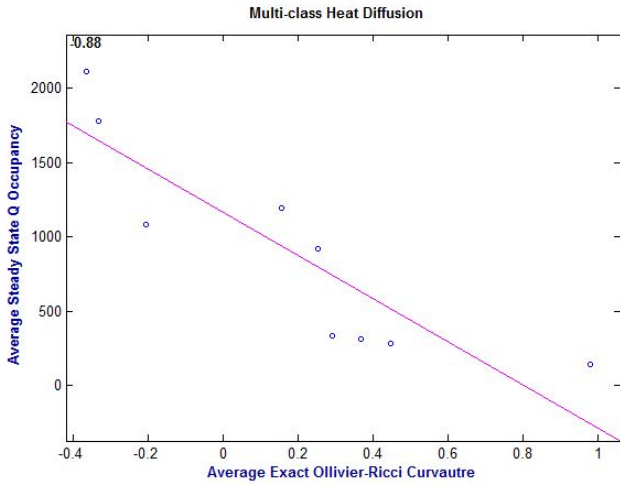


Fig. 6. Multi-class: average queue occupancy in steady-state versus exact Ollivier-Ricci curvature averaged over all links.

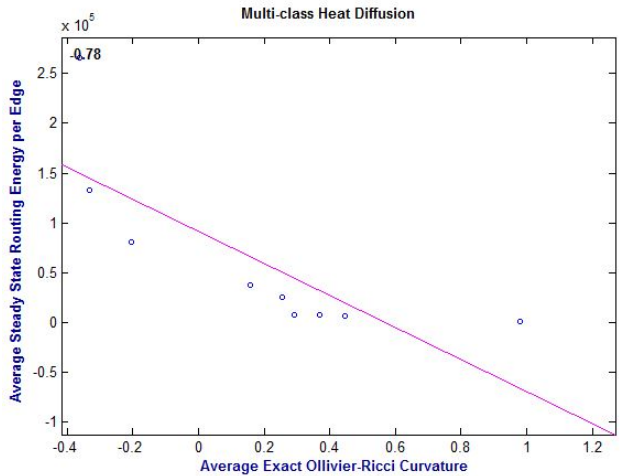


Fig. 7. Multi-class: average routing energy in steady-state versus exact Ollivier-Ricci curvature averaged over all links.

B. Network capacity region versus Ollivier-Ricci curvature

In this simulation, every graph is subject to different arrival rates. Previously, all nodes had an arrival rate of 1 PpT (packet per timeslot). Here, in this simulation, the arrival rate is uniformly increased to 2 PpT, 3 PpT, 5 PpT and 10 PpT. The change of arrival rate is applied to every node. The link capacity is uniformly set to 1500 packet for every link.

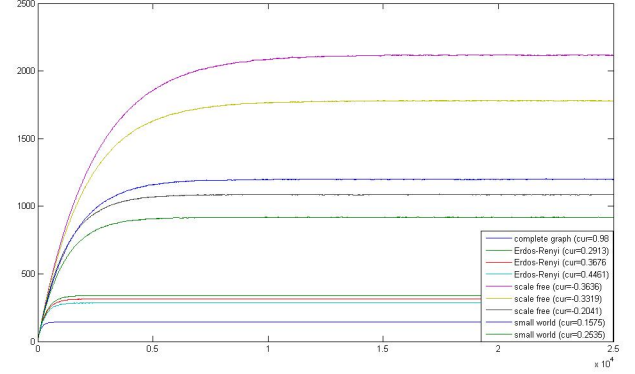


Fig. 8. Evolution of average queue occupancy with time under uniform arrival rate.

In Figure 8, the link capacity is still infinity and the arrival rate is still 1 PpT. Figure 8 shows the initial curve of the average queue occupancy versus timeslot. It can be seen that for graphs with different curvatures, it takes longer time for more negatively curved networks to converge to steady-state. Also, the steady-state queue occupancy is higher in more negatively curved networks, as also shown in Figure 6. However, no matter how long it takes them to converge, the networks still converge within a finite amount of timeslots because the link capacity is infinity.

Figure 9 show the performance with 1500 link capacity under arrival rates of 2 PpT, 3PpT, 5PpT, and 10 PpT, respectively. We can make several observations from the Figure 9.

- As the arrival rate goes higher, the graphs with more negative curvature tend to lose convergence more rapidly than those with more positive curvature.
- Under specific arrival rate, if the graph with more positive curvature cannot converge to steady-state, then graphs with curvature less than that graph would not converge either. This demonstrates, although not a proof, that as the curvature of a graph is decreasing, the capacity region is also becoming smaller.
- No matter what the arrival rate is, the time needed to reach steady-state and the steady-state average queue occupancy follow the same pattern as before, that is, they increase with decreasing curvature.
- For any particular graph, if the arrival rate is increased, so will the time needed to reach steady-state.

Careful inspection of the various graphs being generated shows that the number of edges increases with the curvature, and trivially the capacity region increases. Simulations at

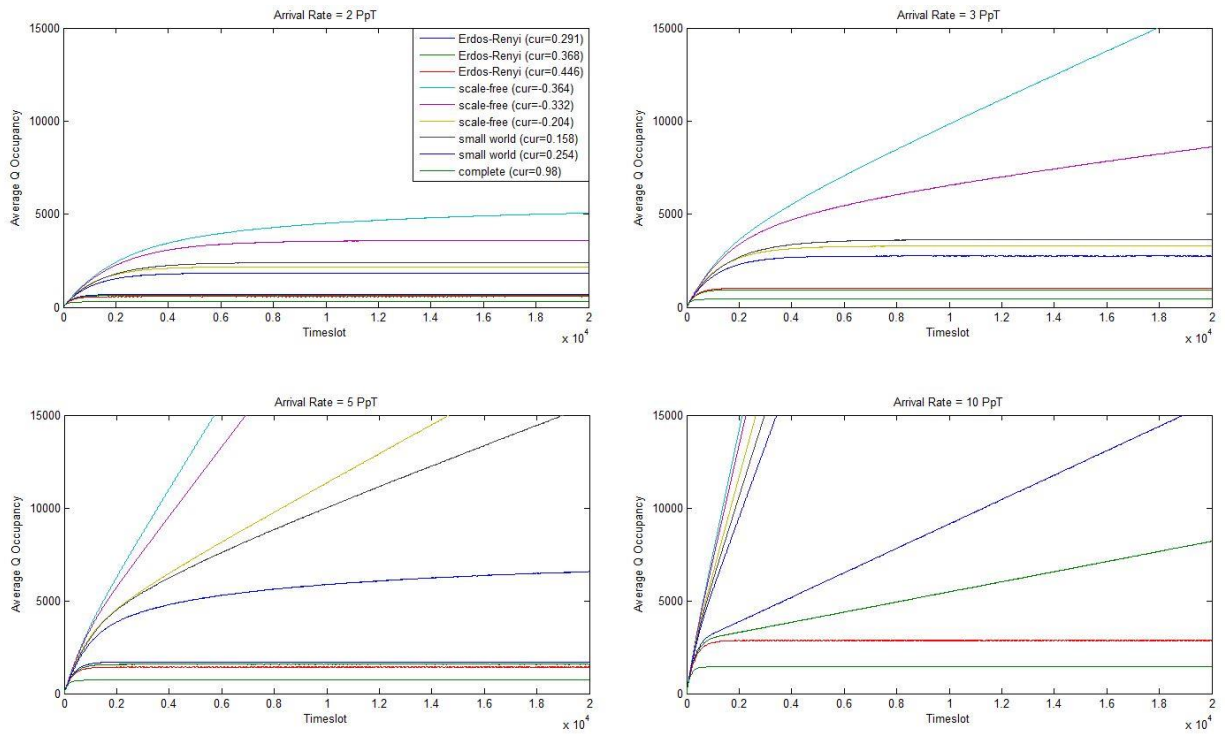


Fig. 9. Evolution of average queue occupancy with time under different arrival rates

constant number of edges still show that the capacity region increases with the curvature. However, the important feature is that the number of edges is only one of the factors contributing to the curvature; among other factors, one will mention the combinatorics of the edges and the link cost factors ρ_{ij} .

VI. CONCLUSION: TOWARDS OLLIVIER-RICCI CURVATURE CONTROL

In case the Ollivier-Ricci curvature is too negative for acceptable queue occupancy, one could think of changing the wireless parameters—within the constraints on the resources—in such a way as to increase the curvature, hence removing some of the overloaded queues and/or improving the capacity region. Such an approach has been very successful in wired networks, where the link weights are adjusted by the so-called Ricci flow to make the network of maximum curvature, under a given network combinatorics. Translated into wireless networking, this would mean changing the link cost factors ρ_{ij} , subject to the constrained resources, so as to maximize κ . However, in wireless networking, another variable to be manipulated is the combinatorics of the wireless channels. While this optimization is considerably more complicated than the wireline one, it is more flexible than the wireline one and hence would allow more curvature control authority. This is left for further research.

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