LECTURE NOTES

Convexity, Duality, and Lagrange Multipliers

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with

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Preface

These lecture notes were developed for the needs of a graduate course at the Electrical Engineering and Computer Science Department at M.I.T. They focus selectively on a number of fundamental analytical and computational topics in (deterministic) optimization that span a broad range from continuous to discrete optimization, but are connected through the recurring theme of convexity, Lagrange multipliers, and duality. These topics include Lagrange multiplier theory, Lagrangian and conjugate duality, and nondifferentiable optimization. The notes contain substantial portions that are adapted from my textbook "Nonlinear Programming: 2nd Edition," Athena Scientific, 1999. However, the notes are more advanced, more mathematical, and more research-oriented.

As part of the course I have also decided to develop in detail those aspects of the theory of convex sets and functions that are essential for an in-depth coverage of Lagrange multipliers and duality. I have long thought that convexity, aside from being an eminently useful subject in engineering and operations research, is also an excellent vehicle for assimilating some of the basic concepts of analysis within an intuitive geometrical setting. Unfortunately, the subject's coverage in mathematics and engineering curricula is scant and incidental. I believe that at least part of the reason is that while there are a number of excellent books on convexity, as well as a true classic (Rockafellar's 1970 book), none of them is well suited for teaching nonmathematicians who form the largest part of the potential audience.

I have therefore tried in these notes to make convex analysis accessible by limiting somewhat its scope and by emphasizing its geometrical character, while at the same time maintaining mathematical rigor. The coverage of the theory is significantly extended in the exercises, whose detailed solutions will be eventually posted on the internet. I have included as many insightful illustrations as I could come up with, and I have tried to use geometric visualization as a principal tool for maintaining the students' interest in mathematical proofs.

An important part of my approach has been to maintain a close link between the theoretical treatment of convexity concepts with their application to optimization. For example, in Chapter 1, soon after the development of some of the basic facts about convexity, I discuss some of their applications to optimization and saddle point theory; soon after the discussion of hyperplanes and cones, I discuss conical approximations and necessary conditions for optimality; soon after the discussion of polyhedral convexity, I discuss its application in linear and integer programming; and soon after the discussion of subgradients, I discuss their use in optimality conditions. I follow consistently this style in the remaining chapters, although having developed in Chapter 1 most of the needed convexity theory, the discussion in the subsequent chapters is more heavily weighted towards optimization.

In addition to their educational purpose, these notes aim to develop two topics that I have recently researched with two of my students, and to integrate them into the overall landscape of convexity, duality, and optimization. These topics are:

(a) A new approach to Lagrange multiplier theory, based on a set of enhanced Fritz-John conditions and the notion of constraint pseudonormality. This work, joint with my Ph.D. student Asuman Ozdaglar, aims to generalize, unify, and streamline the theory of constraint qualifications. It allows for an abstract set constraint (in addition to equalities and inequalities), it highlights the essential structure of constraints using the new notion of pseudonormality, and it develops the connection between Lagrange multipliers and exact penalty functions.

(b) A new approach to the computational solution of (nondifferentiable) dual problems via incremental subgradient methods. These methods, developed jointly with my Ph.D. student Angelia Nedić, include some interesting randomized variants, which according to both analysis and experiment, perform substantially better than the standard subgradient methods for large scale problems that typically arise in the context of duality.

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Convex Analysis and Optimization

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In this chapter we provide the mathematical foundations for Lagrange multiplier theory and duality, which are developed in the subsequent chapters. After an introductory first section, we focus on convex analysis with an emphasis on optimization-related topics. We assume no prior knowledge of the subject, and we develop the subject in detail.

As we embark on the study of convexity, it is worth listing some of the properties of convex sets and functions that make them so special in optimization.

(a) *Convex functions have no local minima that are not global*. Thus the difficulties associated with multiple disconnected local minima, whose global optimality is hard to verify in practice, are avoided.

(b) *Convex sets are connected and have feasible directions at any point* (assuming they consist of more than one point). By this we mean that given any point x in a convex set X, it is possible to move from x along some directions and stay within X for at least a nontrivial interval. In fact a stronger property holds: given any two distinct points x and $\overline{x}$ in X, the direction $\overline{x} - x$ is a feasible direction at x, and all feasible directions can be characterized this way. For optimization purposes, this is important because it allows a calculus-based comparison of the cost of x with the cost of its close neighbors, and forms the basis for some important algorithms. Furthermore, much of the difficulty commonly associated with discrete constraint sets (arising for example in combinatorial optimization), is not encountered under convexity.

(c) *Convex sets have a nonempty relative interior*. In other words, when viewed within the smallest affine set containing it, a convex set has a nonempty interior. Thus convex sets avoid the analytical and computational optimization difficulties associated with "thin" and "curved" constraint surfaces.

(d) *A nonconvex function can be "convexified" while maintaining the optimality of its global minima*, by forming the convex hull of the epigraph of the function.

(e) *The existence of a global minimum of a convex function over a convex set is conveniently characterized in terms of directions of recession* (see Section 1.3).

(f) *Polyhedral convex sets (those specified by linear equality and inequality constraints) are characterized in terms of a finite set of extreme points and extreme directions*. This is the basis for finitely terminating methods for linear programming, including the celebrated simplex method (see Section 1.6).

(g) *Convex functions are continuous and have nice differentiability properties*. In particular, a real-valued convex function is directionally differentiable at any point. Furthermore, while a convex function need not be differentiable, it possesses subgradients, which are nice and geometrically intuitive substitutes for a gradient (see Section 1.7). Just like gradients, subgradients figure prominently in optimality conditions and computational algorithms.

(h) *Convex functions are central in duality theory*. Indeed, the dual problem of a given optimization problem (discussed in Chapters 3 and 4) consists of minimization of a convex function over a convex set, even if the original problem is not convex.

(i) *Closed convex cones are self-dual with respect to polarity*. In words, if X^* denotes the set of vectors that form a nonpositive inner product with all vectors in a set X, we have $C = (C^*)^*$ for any closed and convex cone C. This simple and geometrically intuitive property (discussed in Section 1.5) underlies important aspects of Lagrange multiplier theory.

(j) *Convex, lower semicontinuous functions are self-dual with respect to conjugacy*. It will be seen in Chapter 4 that a certain geometrically motivated conjugacy operation on a given convex, lower semicontinuous function generates a convex, lower semicontinuous function, and when applied for a second time regenerates the original function. The conjugacy operation is central in duality theory, and has a nice interpretation that can be used to visualize and understand some of the most profound aspects of optimization.

Our approach in this chapter is to maintain a close link between the theoretical treatment of convexity concepts with their application to optimization. For example, soon after the development for some of the basic facts about convexity in Section 1.2, we discuss some of their applications to optimization in Section 1.3; and soon after the discussion of hyperplanes and cones in Sections 1.4 and 1.5, we discuss conditions for optimality. We follow this approach throughout the book, although given the extensive development of convexity theory in Chapter 1, the discussion in the subsequent chapters is more heavily weighted towards optimization.

1.1 LINEAR ALGEBRA AND ANALYSIS

In this section, we list some basic definitions, notational conventions, and results from linear algebra and analysis. We assume that the reader is familiar with this material, so no proofs are given. For related and additional material, we recommend the books by Hoffman and Kunze [HoK71], Lancaster and Tismenetsky [LaT85], and Strang [Str76] (linear algebra), and the books by Ash [Ash72], Ortega and Rheinboldt [OrR70], and Rudin [Rud76] (analysis).

Notation

If X is a set and x is an element of X, we write $x \in X$. A set can be specified in the form $X = \{x \mid x \text{ satisfies } P\}$, as the set of all elements satisfying property P. The union of two sets X_1 and X_2 is denoted by $X_1 \cup X_2$ and their intersection by $X_1 \cap X_2$. The symbols $\exists$ and $\forall$ have the meanings "there exists" and "for all," respectively. The empty set is denoted by $\emptyset$.

The set of real numbers (also referred to as scalars) is denoted by $\Re$. The set $\Re$ augmented with $+\infty$ and $-\infty$ is called the *set of extended real numbers*. We denote by $[a, b]$ the set of (possibly extended) real numbers x satisfying $a \leq x \leq b$. A rounded, instead of square, bracket denotes strict inequality in the definition. Thus $(a, b]$, $[a, b)$, and (a, b) denote the set of all x satisfying $a < x \leq b$, $a \leq x < b$, and $a < x < b$, respectively. When working with extended real numbers, we use the natural extensions of the rules of arithmetic: $x \cdot 0 = 0$ for every extended real number x, $x \cdot \infty = \infty$ if $x > 0$, $x \cdot \infty = -\infty$ if $x < 0$, and $x + \infty = \infty$ and $x - \infty = -\infty$ for every scalar x. The expression $\infty - \infty$ is meaningless and is never allowed to occur.

If f is a function, we use the notation $f : X \mapsto Y$ to indicate the fact that f is defined on a set X (its *domain*) and takes values in a set Y (its *range*). If $f : X \mapsto Y$ is a function, and U and V are subsets of X and Y, respectively, the set $\big\{f(x) \mid x \in U\big\}$ is called the *image* or *forward image of* U, and the set $\big\{x \in X \mid f(x) \in V\big\}$ is called the *inverse image of* V.

1.1.1 Vectors and Matrices

We denote by $\Re^n$ the set of n-dimensional real vectors. For any $x \in \Re^n$, we use x_i to indicate its ith *coordinate*, also called its ith *component*.

Vectors in $\Re^n$ will be viewed as column vectors, unless the contrary is explicitly stated. For any $x \in \Re^n$, x' denotes the transpose of x, which is an n-dimensional row vector. The *inner product* of two vectors $x, y \in \Re^n$ is defined by $x'y = \sum_{i=1}^n x_i y_i$. Any two vectors $x, y \in \Re^n$ satisfying $x'y = 0$ are called *orthogonal*.

If x is a vector in $\Re^n$, the notations $x > 0$ and $x \geq 0$ indicate that all coordinates of x are positive and nonnegative, respectively. For any two vectors x and y, the notation $x > y$ means that $x - y > 0$. The notations $x \geq y$, $x < y$, etc., are to be interpreted accordingly.

If X is a set and λ is a scalar we denote by λX the set $\{\lambda x \mid x \in X\}$. If X_1 and X_2 are two subsets of $\Re^n$, we denote by $X_1 + X_2$ the *vector sum*

$$\{x_1 + x_2 \mid x_1 \in X_1,\ x_2 \in X_2\}.$$

We use a similar notation for the sum of any finite number of subsets. In the case where one of the subsets consists of a single vector $\overline{x}$, we simplify this notation as follows:

$$\overline{x} + X = \{\overline{x} + x \mid x \in X\}.$$

Given sets $X_i \subset \Re^{n_i}$, $i = 1, \ldots, m$, the *Cartesian product* of the X_i, denoted by $X_1 \times \cdots \times X_m$, is the subset

$$\big\{(x_1, \ldots, x_m) \mid x_i \in X_i,\ i = 1, \ldots, m\big\}$$

of $\Re^{n_1 + \cdots + n_m}$.

Subspaces and Linear Independence

A subset S of $\Re^n$ is called a *subspace* if $ax + by \in S$ for every $x, y \in S$ and every $a, b \in \Re$. An *affine set* in $\Re^n$ is a translated subspace, i.e., a set of the form $\overline{x} + S = \{\overline{x} + x \mid x \in S\}$, where $\overline{x}$ is a vector in $\Re^n$ and S is a subspace of $\Re^n$. The *span* of a finite collection $\{x_1, \ldots, x_m\}$ of elements of $\Re^n$ (also called the *subspace generated* by the collection) is the subspace consisting of all vectors y of the form $y = \sum_{k=1}^m a_k x_k$, where each a_k is a scalar.

The vectors $x_1, \ldots, x_m \in \Re^n$ are called *linearly independent* if there exists no set of scalars $a_1, \ldots, a_m$ such that $\sum_{k=1}^m a_k x_k = 0$, unless $a_k = 0$ for each k. An equivalent definition is that $x_1 \neq 0$, and for every $k > 1$, the vector x_k does not belong to the span of $x_1, \ldots, x_{k-1}$.

If S is a subspace of $\Re^n$ containing at least one nonzero vector, a *basis* for S is a collection of vectors that are linearly independent and whose span is equal to S. Every basis of a given subspace has the same number of vectors. This number is called the *dimension* of S. By convention, the subspace $\{0\}$ is said to have dimension zero. The *dimension of an affine set* $\overline{x} + S$ is the dimension of the corresponding subspace S. Every subspace of nonzero dimension has an *orthogonal basis*, i.e., a basis consisting of mutually orthogonal vectors.

Given any set X, the set of vectors that are orthogonal to all elements of X is a subspace denoted by $X^\perp$:

$$X^\perp = \{y \mid y'x = 0,\ \forall\ x \in X\}.$$

If S is a subspace, $S^\perp$ is called the *orthogonal complement* of S. It can be shown that $(S^\perp)^\perp = S$ (see the Polar Cone Theorem in Section 1.5). Furthermore, any vector x can be uniquely decomposed as the sum of a vector from S and a vector from $S^\perp$ (see the Projection Theorem in Section 1.3.2).

Matrices

For any matrix A, we use A_{ij}, $[A]_{ij}$, or a_{ij} to denote its ijth element. The *transpose* of A, denoted by A', is defined by $[A']_{ij} = a_{ji}$. For any two matrices A and B of compatible dimensions, the transpose of the product matrix AB satisfies $(AB)' = B'A'$.

If X is a subset of $\Re^n$ and A is an $m \times n$ matrix, then the image of X under A is denoted by AX (or $A \cdot X$ if this enhances notational clarity):

$$AX = \{Ax \mid x \in X\}.$$

If X is subspace, then AX is also a subspace.

Let A be a square matrix. We say that A is *symmetric* if $A' = A$. We say that A is *diagonal* if $[A]_{ij} = 0$ whenever $i \neq j$. We use I to denote the identity matrix. The *determinant* of A is denoted by $\det(A)$.

Let A be an $m \times n$ matrix. The *range space* of A, denoted by $R(A)$, is the set of all vectors $y \in \Re^m$ such that $y = Ax$ for some $x \in \Re^n$. The *null space* of A, denoted by $N(A)$, is the set of all vectors $x \in \Re^n$ such that $Ax = 0$. It is seen that the range space and the null space of A are subspaces. The *rank* of A is the dimension of the range space of A. The rank of A is equal to the maximal number of linearly independent columns of A, and is also equal to the maximal number of linearly independent rows of A. The matrix A and its transpose A' have the same rank. We say that A has *full rank*, if its rank is equal to $\min\{m, n\}$. This is true if and only if either all the rows of A are linearly independent, or all the columns of A are linearly independent.

The range of an $m \times n$ matrix A and the orthogonal complement of the nullspace of its transpose are equal, i.e.,

$$R(A) = N(A')^{\perp}.$$

Another way to state this result is that given vectors $a_1, \ldots, a_n \in \Re^m$ (the columns of A) and a vector $x \in \Re^m$, we have $x'y = 0$ for all y such that $a_i'y = 0$ for all i if and only if $x = \lambda_1 a_1 + \cdots + \lambda_n a_n$ for some scalars $\lambda_1, \ldots, \lambda_n$. This is a special case of Farkas' Lemma, an important result for constrained optimization, which will be discussed later in Section 1.6. A useful application of this result is that if S_1 and S_2 are two subspaces of $\Re^n$, then

$$S_1^{\perp} + S_2^{\perp} = (S_1 \cap S_2)^{\perp}.$$

This follows by introducing matrices B_1 and B_2 such that $S_1 = \{x \mid B_1 x = 0\} = N(B_1)$ and $S_2 = \{x \mid B_2 x = 0\} = N(B_2)$, and writing

$$S_1^{\perp} + S_2^{\perp} = R\big([\, B_1' \quad B_2' \,]\big) = N\left(\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\right)^{\perp} = \big(N(B_1) \cap N(B_2)\big)^{\perp} = (S_1 \cap S_2)^{\perp}$$

A function $f : \Re^n \mapsto \Re$ is said to be *affine* if it has the form $f(x) = a'x + b$ for some $a \in \Re^n$ and $b \in \Re$. Similarly, a function $f : \Re^n \mapsto \Re^m$ is said to be *affine* if it has the form $f(x) = Ax + b$ for some $m \times n$ matrix A and some $b \in \Re^m$. If $b = 0$, f is said to be a *linear function* or *linear transformation*.

1.1.2 Topological Properties

Definition 1.1.1: A *norm* $\|\cdot\|$ on $\Re^n$ is a function that assigns a scalar $\|x\|$ to every $x \in \Re^n$ and that has the following properties:

(a) $\|x\| \geq 0$ for all $x \in \Re^n$.

(b) $\|\alpha x\| = |\alpha| \cdot \|x\|$ for every scalar α and every $x \in \Re^n$.

(c) $\|x\| = 0$ if and only if $x = 0$.

(d) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \Re^n$ (this is referred to as the *triangle inequality*).

The *Euclidean norm* of a vector $x = (x_1, \ldots, x_n)$ is defined by

$$\|x\| = (x'x)^{1/2} = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}.$$

The space $\Re^n$, equipped with this norm, is called a *Euclidean space*. We will use the Euclidean norm almost exclusively in this book. In particular, *in the absence of a clear indication to the contrary, $\|\cdot\|$ will denote the Euclidean norm.* Two important results for the Euclidean norm are:

Proposition 1.1.1: (Pythagorean Theorem) For any two vectors x and y that are orthogonal, we have

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

Proposition 1.1.2: (Schwartz inequality) For any two vectors x and y, we have

$$|x'y| \leq \|x\| \cdot \|y\|,$$

with equality holding if and only if $x = \alpha y$ for some scalar α.

Two other important norms are the *maximum norm* $\|\cdot\|_\infty$ (also called *sup-norm* or *ℓ_∞-norm*), defined by

$$\|x\|_\infty = \max_i |x_i|,$$

and the *ℓ_1-norm* $\|\cdot\|_1$, defined by

$$\|x\|_1 = \sum_{i=1}^n |x_i|.$$

Sequences

We use both subscripts and superscripts in sequence notation. Generally, we prefer subscripts, but we use superscripts whenever we need to reserve the subscript notation for indexing coordinates or components of vectors and functions. The meaning of the subscripts and superscripts should be clear from the context in which they are used.

A sequence $\{x_k \mid k = 1, 2, \ldots\}$ (or $\{x_k\}$ for short) of scalars is said to *converge* if there exists a scalar x such that for every $\epsilon > 0$ we have $|x_k - x| < \epsilon$ for every k greater than some integer K (depending on ϵ). We call the scalar x the *limit* of $\{x_k\}$, and we also say that $\{x_k\}$ *converges to* x; symbolically, $x_k \to x$ or $\lim_{k\to\infty} x_k = x$. If for every scalar b there exists some K (depending on b) such that $x_k \geq b$ for all $k \geq K$, we write $x_k \to \infty$ and $\lim_{k\to\infty} x_k = \infty$. Similarly, if for every scalar b there exists some K such that $x_k \leq b$ for all $k \geq K$, we write $x_k \to -\infty$ and $\lim_{k\to\infty} x_k = -\infty$.

A sequence $\{x_k\}$ is called a *Cauchy sequence* if for every $\epsilon > 0$, there exists some K (depending on ϵ) such that $|x_k - x_m| < \epsilon$ for all $k \geq K$ and $m \geq K$.

A sequence $\{x_k\}$ is said to be *bounded above* (respectively, *below*) if there exists some scalar b such that $x_k \leq b$ (respectively, $x_k \geq b$) for all k. It is said to be *bounded* if it is bounded above and bounded below. The sequence $\{x_k\}$ is said to be monotonically *nonincreasing* (respectively, *nondecreasing*) if $x_{k+1} \leq x_k$ (respectively, $x_{k+1} \geq x_k$) for all k. If $\{x_k\}$ converges to x and is nonincreasing (nondecreasing), we also use the notation $x_k \downarrow x$ ($x_k \uparrow x$, respectively).

Proposition 1.1.3: Every bounded and monotonically nonincreasing or nondecreasing scalar sequence converges.

Note that a monotonically nondecreasing sequence $\{x_k\}$ is either bounded, in which case it converges to some scalar x by the above proposition, or else it is unbounded, in which case $x_k \to \infty$. Similarly, a monotonically nonincreasing sequence $\{x_k\}$ is either bounded and converges, or it is unbounded, in which case $x_k \to -\infty$.

The *supremum* of a nonempty set X of scalars, denoted by $\sup X$, is defined as the smallest scalar x such that $x \geq y$ for all $y \in X$. If no such scalar exists, we say that the supremum of X is ∞. Similarly, the *infimum* of X, denoted by $\inf X$, is defined as the largest scalar x such that $x \leq y$ for all $y \in X$, and is equal to $-\infty$ if no such scalar exists. For the empty set, we use the convention

$$\sup(\varnothing) = -\infty, \qquad \inf(\varnothing) = \infty.$$

(This is somewhat paradoxical, since we have that the sup of a set is less than its inf, but works well for our analysis.) If $\sup X$ is equal to a scalar $\overline{x}$ that belongs to the set X, we say that $\overline{x}$ is the *maximum point* of X and we often write

$$\overline{x} = \sup X = \max X.$$

Similarly, if $\inf X$ is equal to a scalar $\overline{x}$ that belongs to the set X, we often write

$$\overline{x} = \inf X = \min X.$$

Thus, when we write $\max X$ (or $\min X$) in place of $\sup X$ (or $\inf X$, respectively) we do so just for emphasis: we indicate that it is either evident, or it is known through earlier analysis, or it is about to be shown that the maximum (or minimum, respectively) of the set X is attained at one of its points.

Given a scalar sequence $\{x_k\}$, the supremum of the sequence, denoted by $\sup_k x_k$, is defined as $\sup\{x_k \mid k = 1, 2, \ldots\}$. The infimum of a sequence is similarly defined. Given a sequence $\{x_k\}$, let $y_m = \sup\{x_k \mid k \geq m\}$, $z_m = \inf\{x_k \mid k \geq m\}$. The sequences $\{y_m\}$ and $\{z_m\}$ are nonincreasing and nondecreasing, respectively, and therefore have a limit whenever $\{x_k\}$ is bounded above or is bounded below, respectively (Prop. 1.1.3). The limit of y_m is denoted by $\limsup_{k\to\infty} x_k$, and is referred to as the *limit superior* of $\{x_k\}$. The limit of z_m is denoted by $\liminf_{k\to\infty} x_k$, and is referred to as the *limit inferior* of $\{x_k\}$. If $\{x_k\}$ is unbounded above, we write $\limsup_{k\to\infty} x_k = \infty$, and if it is unbounded below, we write $\liminf_{k\to\infty} x_k = -\infty$.

Proposition 1.1.4: Let $\{x_k\}$ and $\{y_k\}$ be scalar sequences.

(a) There holds

$$\inf_k x_k \leq \liminf_{k\to\infty} x_k \leq \limsup_{k\to\infty} x_k \leq \sup_k x_k.$$

(b) $\{x_k\}$ converges if and only if $\liminf_{k\to\infty} x_k = \limsup_{k\to\infty} x_k$ and, in that case, both of these quantities are equal to the limit of x_k.

(c) If $x_k \leq y_k$ for all k, then

$$\liminf_{k\to\infty} x_k \leq \liminf_{k\to\infty} y_k, \qquad \limsup_{k\to\infty} x_k \leq \limsup_{k\to\infty} y_k.$$

(d) We have

$$\liminf_{k\to\infty} x_k + \liminf_{k\to\infty} y_k \leq \liminf_{k\to\infty}(x_k + y_k),$$

$$\limsup_{k\to\infty} x_k + \limsup_{k\to\infty} y_k \geq \limsup_{k\to\infty}(x_k + y_k).$$

A sequence $\{x_k\}$ of vectors in $\Re^n$ is said to converge to some $x \in \Re^n$ if the ith coordinate of x_k converges to the ith coordinate of x for every i. We use the notations $x_k \to x$ and $\lim_{k\to\infty} x_k = x$ to indicate convergence for vector sequences as well. The sequence $\{x_k\}$ is called bounded (respectively, a Cauchy sequence) if each of its corresponding coordinate sequences is bounded (respectively, a Cauchy sequence). It can be seen that $\{x_k\}$ is bounded if and only if there exists a scalar c such that $\|x_k\| \leq c$ for all k.

Definition 1.1.2: We say that a vector $x \in \Re^n$ is a *limit point* of a sequence $\{x_k\}$ in $\Re^n$ if there exists a subsequence of $\{x_k\}$ that converges to x.

Proposition 1.1.5: Let $\{x_k\}$ be a sequence in $\Re^n$.

(a) If $\{x_k\}$ is bounded, it has at least one limit point.

(b) $\{x_k\}$ converges if and only if it is bounded and it has a unique limit point.

(c) $\{x_k\}$ converges if and only if it is a Cauchy sequence.

$o(\cdot)$ Notation

If p is a positive integer and $h : \Re^n \mapsto \Re^m$, then we write

$$h(x) = o\big(\|x\|^p\big)$$

if

$$\lim_{k\to\infty} \frac{h(x_k)}{\|x_k\|^p} = 0,$$

for all sequences $\{x_k\}$, with $x_k \neq 0$ for all k, that converge to 0.

Closed and Open Sets

We say that x is a *closure point* or *limit point* of a set $X \subset \Re^n$ if there exists a sequence $\{x_k\}$, consisting of elements of X, that converges to x. The *closure* of X, denoted $\mathrm{cl}(X)$, is the set of all limit points of X.

Definition 1.1.3: A set $X \subset \Re^n$ is called *closed* if it is equal to its closure. It is called *open* if its complement (the set $\{x \mid x \notin X\}$) is closed. It is called *bounded* if there exists a scalar c such that the magnitude of any coordinate of any element of X is less than c. It is called *compact* if it is closed and bounded.

Definition 1.1.4: A *neighborhood* of a vector x is an open set containing x. We say that x is an *interior* point of a set $X \subset \Re^n$ if there exists a neighborhood of x that is contained in X. A vector $x \in \mathrm{cl}(X)$ which is not an interior point of X is said to be a *boundary* point of X.

Let $\|\cdot\|$ be a given norm in $\Re^n$. For any $\epsilon > 0$ and $x^* \in \Re^n$, consider the sets

$$\big\{x \mid \|x - x^*\| < \epsilon\big\}, \qquad \big\{x \mid \|x - x^*\| \le \epsilon\big\}.$$

The first set is open and is called an *open sphere* centered at x^*, while the second set is closed and is called a *closed sphere* centered at x^*. Sometimes the terms *open ball* and *closed ball* are used, respectively.

Proposition 1.1.6:

(a) The union of finitely many closed sets is closed.

(b) The intersection of closed sets is closed.

(c) The union of open sets is open.

(d) The intersection of finitely many open sets is open.

(e) A set is open if and only if all of its elements are interior points.

(f) Every subspace of $\Re^n$ is closed.

(g) A set X is compact if and only if every sequence of elements of X has a subsequence that converges to an element of X.

(h) If $\{X_k\}$ is a sequence of nonempty and compact sets such that $X_k \supset X_{k+1}$ for all k, then the intersection $\cap_{k=0}^{\infty} X_k$ is nonempty and compact.

The topological properties of subsets of $\Re^n$, such as being open, closed, or compact, do not depend on the norm being used. This is a consequence of the following proposition, referred to as the *norm equivalence property in* $\Re^n$, which shows that if a sequence converges with respect to one norm, it converges with respect to all other norms.

Proposition 1.1.7: For any two norms $\|\cdot\|$ and $\|\cdot\|'$ on $\Re^n$, there exists a scalar c such that $\|x\| \leq c\|x\|'$ for all $x \in \Re^n$.

Using the preceding proposition, we obtain the following.

Proposition 1.1.8: If a subset of $\Re^n$ is open (respectively, closed, bounded, or compact) with respect to some norm, it is open (respectively, closed, bounded, or compact) with respect to all other norms.

Sequences of Sets

Let $\{X_k\}$ be a sequence of nonempty subsets of $\Re^n$. The *outer limit* of $\{X_k\}$, denoted $\limsup_{k\to\infty} X_k$, is the set of all $x \in \Re^n$ such that every neighborhood of x has a nonempty intersection with infinitely many of the sets X_k, $k = 1, 2, \ldots$. Equivalently, $\limsup_{k\to\infty} X_k$ is the set of all limits of subsequences $\{x_k\}_{\mathcal{K}}$ such that $x_k \in X_k$, for all $k \in \mathcal{K}$.

The *inner limit* of $\{X_k\}$, denoted $\liminf_{k\to\infty} X_k$, is the set of all $x \in \Re^n$ such that every neighborhood of x has a nonempty intersection with all except finitely many of the sets X_k, $k = 1, 2, \ldots$. Equivalently, $\liminf_{k\to\infty} X_k$ is the set of all limits of sequences $\{x_k\}$ such that $x_k \in X_k$, for all $k = 1, 2, \ldots$.

The sequence $\{X_k\}$ is said to converge to a set X if

$$X = \liminf_{k\to\infty} X_k = \limsup_{k\to\infty} X_k,$$

in which case X is said to be the limit of $\{X_k\}$. The inner and outer limits are closed (possibly empty) sets. If each set X_k consists of a single point x_k, $\limsup_{k\to\infty} X_k$ is the set of limit points of $\{x_k\}$, while $\liminf_{k\to\infty} X_k$ is just the limit of $\{x_k\}$ if $\{x_k\}$ converges, and otherwise it is empty.

Continuity

Let $f : X \mapsto \Re^m$ be a function, where X is a subset of $\Re^n$, and let x be a point in X. If there exists a vector $y \in \Re^m$ such that the sequence $\big\{f(x_k)\big\}$ converges to y for every sequence $\{x_k\} \subset X$ such that $\lim_{k\to\infty} x_k = x$, we write $\lim_{z\to x} f(z) = y$. If there exists a vector $y \in \Re^m$ such that the sequence $\big\{f(x_k)\big\}$ converges to y for every sequence $\{x_k\} \subset X$ such that $\lim_{k\to\infty} x_k = x$ and $x_k \leq x$ (respectively, $x_k \geq x$) for all k, we write $\lim_{z\uparrow x} f(z) = y$ [respectively, $\lim_{z\downarrow x} f(z)$].

Definition 1.1.5: Let X be a subset of $\Re^n$.

(a) A function $f : X \mapsto \Re^m$ is called *continuous* at a point $x \in X$ if $\lim_{z\to x} f(z) = f(x)$.

(b) A function $f : X \mapsto \Re^m$ is called *right-continuous* (respectively, *left-continuous*) at a point $x \in X$ if $\lim_{z\downarrow x} f(z) = f(x)$ [respectively, $\lim_{z\uparrow x} f(z) = f(x)$].

(c) A real-valued function $f : X \mapsto \Re$ is called *upper semicontinuous* (respectively, *lower semicontinuous*) at a point $x \in X$ if $f(x) \geq \limsup_{k\to\infty} f(x_k)$ [respectively, $f(x) \leq \liminf_{k\to\infty} f(x_k)$] for every sequence $\{x_k\}$ of elements of X converging to x.

If $f : X \mapsto \Re^m$ is continuous at every point of a subset of its domain X, we say that f *is continuous over that subset*. If $f : X \mapsto \Re^m$ is continuous at every point of its domain X, we say that f *is continuous*. We use similar terminology for right-continuous, left-continuous, upper semicontinuous, and lower semicontinuous functions.

Proposition 1.1.9:

(a) The composition of two continuous functions is continuous.

(b) Any vector norm on $\Re^n$ is a continuous function.

(c) Let $f : \Re^n \mapsto \Re^m$ be continuous, and let $Y \subset \Re^m$ be open (respectively, closed). Then the inverse image of Y, $\big\{x \in \Re^n \mid f(x) \in Y\big\}$, is open (respectively, closed).

(d) Let $f : \Re^n \mapsto \Re^m$ be continuous, and let $X \subset \Re^n$ be compact. Then the forward image of X, $\big\{f(x) \mid x \in X\big\}$, is compact.

Matrix Norms

A norm $\|\cdot\|$ on the set of $n \times n$ matrices is a real-valued function that has the same properties as vector norms do when the matrix is viewed as an element of $\Re^{n^2}$. The norm of an $n \times n$ matrix A is denoted by $\|A\|$.

We are mainly interested in *induced norms*, which are constructed as follows. Given any vector norm $\|\cdot\|$, the corresponding induced matrix norm, also denoted by $\|\cdot\|$, is defined by

$$\|A\| = \sup_{\left\{x \in \Re^n \mid \|x\|=1\right\}} \|Ax\|.$$

It is easily verified that for any vector norm, the above equation defines a bona fide matrix norm having all the required properties.

Note that by the Schwartz inequality (Prop. 1.1.2), we have

$$\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|y\|=\|x\|=1} |y'Ax|.$$

By reversing the roles of x and y in the above relation and by using the equality $y'Ax = x'A'y$, it follows that $\|A\| = \|A'\|$.

1.1.3 Square Matrices

Definition 1.1.6: A square matrix A is called *singular* if its determinant is zero. Otherwise it is called *nonsingular* or *invertible*.

Proposition 1.1.10:

(a) Let A be an $n \times n$ matrix. The following are equivalent:

(i) The matrix A is nonsingular.

(ii) The matrix A' is nonsingular.

(iii) For every nonzero $x \in \Re^n$, we have $Ax \neq 0$.

(iv) For every $y \in \Re^n$, there is a unique $x \in \Re^n$ such that $Ax = y$.

(v) There is an $n \times n$ matrix B such that $AB = I = BA$.

(vi) The columns of A are linearly independent.

(vii) The rows of A are linearly independent.

(b) Assuming that A is nonsingular, the matrix B of statement (v) (called the *inverse* of A and denoted by A^{-1}) is unique.

(c) For any two square invertible matrices A and B of the same dimensions, we have $(AB)^{-1} = B^{-1}A^{-1}$.

Definition 1.1.7: The *characteristic polynomial* ϕ of an $n \times n$ matrix A is defined by $\phi(\lambda) = \det(\lambda I - A)$, where I is the identity matrix of the same size as A. The n (possibly repeated or complex) roots of ϕ are called the *eigenvalues* of A. A nonzero vector x (with possibly complex coordinates) such that $Ax = \lambda x$, where λ is an eigenvalue of A, is called an *eigenvector* of A associated with λ.

Proposition 1.1.11: Let A be a square matrix.

(a) A complex number λ is an eigenvalue of A if and only if there exists a nonzero eigenvector associated with λ.

(b) A is singular if and only if it has an eigenvalue that is equal to zero.

Note that the only use of complex numbers in this book is in relation to eigenvalues and eigenvectors. All other matrices or vectors are implicitly assumed to have real components.

Proposition 1.1.12: Let A be an $n \times n$ matrix.

(a) If T is a nonsingular matrix and $B = TAT^{-1}$, then the eigenvalues of A and B coincide.

(b) For any scalar c, the eigenvalues of $cI + A$ are equal to $c + \lambda_1, \ldots, c + \lambda_n$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A.

(c) The eigenvalues of A^k are equal to $\lambda_1^k, \ldots, \lambda_n^k$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A.

(d) If A is nonsingular, then the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A.

(e) The eigenvalues of A and A' coincide.

Symmetric and Positive Definite Matrices

Symmetric matrices have several special properties, particularly regarding their eigenvalues and eigenvectors. In what follows in this section, $\|\cdot\|$ denotes the Euclidean norm.

Proposition 1.1.13: Let A be a symmetric $n \times n$ matrix. Then:

(a) The eigenvalues of A are real.

(b) The matrix A has a set of n mutually orthogonal, real, and nonzero eigenvectors $x_1, \ldots, x_n$.

(c) Suppose that the eigenvectors in part (b) have been normalized so that $\|x_i\| = 1$ for each i. Then

$$A = \sum_{i=1}^{n} \lambda_i x_i x_i',$$

where λ_i is the eigenvalue corresponding to x_i.

Proposition 1.1.14: Let A be a symmetric $n \times n$ matrix, and let $\lambda_1 \leq \cdots \leq \lambda_n$ be its (real) eigenvalues. Then:

(a) $\|A\| = \max\{|\lambda_1|, |\lambda_n|\}$, where $\|\cdot\|$ is the matrix norm induced by the Euclidean norm.

(b) $\lambda_1\|y\|^2 \leq y'Ay \leq \lambda_n\|y\|^2$ for all $y \in \Re^n$.

(c) If A is nonsingular, then

$$\|A^{-1}\| = \frac{1}{\min\{|\lambda_1|, |\lambda_n|\}}.$$

Proposition 1.1.15: Let A be a square matrix, and let $\|\cdot\|$ be the matrix norm induced by the Euclidean norm. Then:

(a) If A is symmetric, then $\|A^k\| = \|A\|^k$ for any positive integer k.

(b) $\|A\|^2 = \|A'A\| = \|AA'\|$.

Definition 1.1.8: A symmetric $n \times n$ matrix A is called *positive definite* if $x'Ax > 0$ for all $x \in \Re^n$, $x \neq 0$. It is called *positive semidefinite* if $x'Ax \geq 0$ for all $x \in \Re^n$.

Throughout this book, the notion of positive definiteness applies exclusively to symmetric matrices. Thus *whenever we say that a matrix is positive (semi)definite, we implicitly assume that the matrix is symmetric*.

Proposition 1.1.16:

(a) The sum of two positive semidefinite matrices is positive semidefinite. If one of the two matrices is positive definite, the sum is positive definite.

(b) If A is a positive semidefinite $n \times n$ matrix and T is an $m \times n$ matrix, then the matrix TAT' is positive semidefinite. If A is positive definite and T is invertible, then TAT' is positive definite.

Proposition 1.1.17:

(a) For any $m \times n$ matrix A, the matrix $A'A$ is symmetric and positive semidefinite. $A'A$ is positive definite if and only if A has rank n. In particular, if $m = n$, $A'A$ is positive definite if and only if A is nonsingular.

(b) A square symmetric matrix is positive semidefinite (respectively, positive definite) if and only if all of its eigenvalues are nonnegative (respectively, positive).

(c) The inverse of a symmetric positive definite matrix is symmetric and positive definite.

Proposition 1.1.18: Let A be a symmetric positive semidefinite $n \times n$ matrix of rank m. There exists an $n \times m$ matrix C of rank m such that

$$A = CC'.$$

Furthermore, for any such matrix C:

(a) A and C' have the same null space: $N(A) = N(C')$.

(b) A and C have the same range space: $R(A) = R(C)$.

Proposition 1.1.19: Let A be a square symmetric positive semidefinite matrix.

(a) There exists a symmetric matrix Q with the property $Q^2 = A$. Such a matrix is called a *symmetric square root* of A and is denoted by $A^{1/2}$.

(b) There is a unique symmetric square root if and only if A is positive definite.

(c) A symmetric square root $A^{1/2}$ is invertible if and only if A is invertible. Its inverse is denoted by $A^{-1/2}$.

(d) There holds $A^{-1/2}A^{-1/2} = A^{-1}$.

(e) There holds $AA^{1/2} = A^{1/2}A$.

1.1.4 Derivatives

Let $f : \Re^n \mapsto \Re$ be some function, fix some $x \in \Re^n$, and consider the expression

$$\lim_{\alpha \to 0} \frac{f(x + \alpha e_i) - f(x)}{\alpha},$$

where e_i is the ith unit vector (all components are 0 except for the ith component which is 1). If the above limit exists, it is called the ith *partial derivative* of f at the point x and it is denoted by $(\partial f / \partial x_i)(x)$ or $\partial f(x)/\partial x_i$ (x_i in this section will denote the ith coordinate of the vector x). Assuming all of these partial derivatives exist, the *gradient* of f at x is defined as the column vector

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}.$$

For any $y \in \Re^n$, we define the one-sided *directional derivative* of f in the direction y, to be

$$f'(x; y) = \lim_{\alpha \downarrow 0} \frac{f(x + \alpha y) - f(x)}{\alpha},$$

provided that the limit exists. We note from the definitions that

$$f'(x; e_i) = -f'(x; -e_i) \qquad \Rightarrow \qquad f'(x; e_i) = (\partial f / \partial x_i)(x).$$

If the directional derivative of f at a vector x exists in all directions y and $f'(x; y)$ is a linear function of y, we say that f is *differentiable* at x. This type of differentiability is also called *Gateaux differentiability*. It is seen that f is differentiable at x if and only if the gradient $\nabla f(x)$ exists and satisfies $\nabla f(x)'y = f'(x; y)$ for every $y \in \Re^n$. The function f is called *differentiable over a given subset U of $\Re^n$* if it is differentiable at every $x \in U$. The function f is called *differentiable* (without qualification) if it is differentiable at all $x \in \Re^n$.

If f is differentiable over an open set U and the gradient $\nabla f(x)$ is continuous at all $x \in U$, f is said to be *continuously differentiable over U*. Such a function has the property

$$\lim_{y \to 0} \frac{f(x + y) - f(x) - \nabla f(x)'y}{\|y\|} = 0, \qquad \forall\ x \in U, \tag{1.1}$$

where $\|\cdot\|$ is an arbitrary vector norm. If f is continuously differentiable over $\Re^n$, then f is also called a *smooth* function.

The preceding equation can also be used as an alternative definition of differentiability. In particular, f is called *Frechet differentiable* at x

if there exists a vector g satisfying Eq. (1.1) with $\nabla f(x)$ replaced by g. If such a vector g exists, it can be seen that all the partial derivatives $(\partial f/\partial x_i)(x)$ exist and that $g = \nabla f(x)$. Frechet differentiability implies (Gateaux) differentiability but not conversely (see for example [OrR70] for a detailed discussion). In this book, when dealing with a differentiable function f, we will always assume that f is continuously differentiable (smooth) over a given open set [$\nabla f(x)$ is a continuous function of x over that set], in which case f is both Gateaux and Frechet differentiable, and the distinctions made above are of no consequence.

The definitions of differentiability of f at a point x only involve the values of f in a neighborhood of x. Thus, these definitions can be used for functions f that are not defined on all of $\Re^n$, but are defined instead in a neighborhood of the point at which the derivative is computed. In particular, for functions $f : X \mapsto \Re$ that are defined over a strict subset X of $\Re^n$, we use the above definition of differentiability of f at a vector x *provided x is an interior point of the domain X*. Similarly, we use the above definition of differentiability or continuous differentiability of f over a subset U, *provided U is an open subset of the domain X*. Thus any mention of differentiability of a function f over a subset implicitly assumes that this subset is open.

If $f : \Re^n \mapsto \Re^m$ is a vector-valued function, it is called differentiable (or smooth) if each component f_i of f is differentiable (or smooth, respectively). The *gradient matrix* of f, denoted $\nabla f(x)$, is the $n \times m$ matrix whose ith column is the gradient $\nabla f_i(x)$ of f_i. Thus,

$$\nabla f(x) = \Big[\nabla f_1(x) \cdots \nabla f_m(x)\Big].$$

The transpose of ∇f is called the *Jacobian* of f and is a matrix whose ijth entry is equal to the partial derivative $\partial f_i/\partial x_j$.

Now suppose that each one of the partial derivatives of a function $f : \Re^n \mapsto \Re$ is a smooth function of x. We use the notation $(\partial^2 f/\partial x_i \partial x_j)(x)$ to indicate the ith partial derivative of $\partial f/\partial x_j$ at a point $x \in \Re^n$. The *Hessian* of f is the matrix whose ijth entry is equal to $(\partial^2 f/\partial x_i \partial x_j)(x)$, and is denoted by $\nabla^2 f(x)$. We have $(\partial^2 f/\partial x_i \partial x_j)(x) = (\partial^2 f/\partial x_j \partial x_i)(x)$ for every x, which implies that $\nabla^2 f(x)$ is symmetric.

If $f : \Re^{m+n} \mapsto \Re$ is a function of (x, y), where $x = (x_1, \ldots, x_m) \in \Re^m$ and $y = (y_1, \ldots, y_n) \in \Re^n$, we write

$$\nabla_x f(x,y) = \begin{bmatrix} \frac{\partial f(x,y)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x,y)}{\partial x_m} \end{bmatrix}, \qquad \nabla_y f(x,y) = \begin{bmatrix} \frac{\partial f(x,y)}{\partial y_1} \\ \vdots \\ \frac{\partial f(x,y)}{\partial y_n} \end{bmatrix}.$$

We denote by $\nabla^2_{xx} f(x,y)$, $\nabla^2_{xy} f(x,y)$, and $\nabla^2_{yy} f(x,y)$ the matrices with components

$$\big[\nabla^2_{xx} f(x,y)\big]_{ij} = \frac{\partial^2 f(x,y)}{\partial x_i \partial x_j}, \qquad \big[\nabla^2_{xy} f(x,y)\big]_{ij} = \frac{\partial^2 f(x,y)}{\partial x_i \partial y_j},$$

$$\big[\nabla^2_{yy} f(x,y)\big]_{ij} = \frac{\partial^2 f(x,y)}{\partial y_i \partial y_j}.$$

If $f : \Re^{m+n} \mapsto \Re^r$, $f = (f_1, f_2, \ldots, f_r)$, we write

$$\nabla_x f(x,y) = \big[\nabla_x f_1(x,y) \cdots \nabla_x f_r(x,y)\big],$$

$$\nabla_y f(x,y) = \big[\nabla_y f_1(x,y) \cdots \nabla_y f_r(x,y)\big].$$

Let $f : \Re^k \mapsto \Re^m$ and $g : \Re^m \mapsto \Re^n$ be smooth functions, and let h be their composition, i.e.,

$$h(x) = g\big(f(x)\big).$$

Then, the *chain rule* for differentiation states that

$$\nabla h(x) = \nabla f(x) \nabla g\big(f(x)\big), \qquad \forall\ x \in \Re^k.$$

Some examples of useful relations that follow from the chain rule are:

$$\nabla\big(f(Ax)\big) = A' \nabla f(Ax), \qquad \nabla^2\big(f(Ax)\big) = A' \nabla^2 f(Ax) A,$$

where A is a matrix,

$$\nabla_x\Big(f\big(h(x), y\big)\Big) = \nabla h(x) \nabla_h f\big(h(x), y\big),$$

$$\nabla_x\Big(f\big(h(x), g(x)\big)\Big) = \nabla h(x) \nabla_h f\big(h(x), g(x)\big) + \nabla g(x) \nabla_g f\big(h(x), g(x)\big).$$

We now state some theorems relating to differentiable functions that will be useful for our purposes.

Proposition 1.1.20: (Mean Value Theorem) Let $f : \Re^n \mapsto \Re$ be continuously differentiable over an open sphere S, and let x be a vector in S. Then for all y such that $x + y \in S$, there exists an $\alpha \in [0, 1]$ such that

$$f(x + y) = f(x) + \nabla f(x + \alpha y)' y.$$

Proposition 1.1.21: (Second Order Expansions) Let $f : \Re^n \mapsto \Re$ be twice continuously differentiable over an open sphere S, and let x be a vector in S. Then for all y such that $x + y \in S$:

(a) There holds

$$f(x+y) = f(x) + y'\nabla f(x) + \tfrac{1}{2} y' \left(\int_0^1 \left(\int_0^t \nabla^2 f(x+\tau y) d\tau \right) dt \right) y.$$

(b) There exists an $\alpha \in [0,1]$ such that

$$f(x+y) = f(x) + y'\nabla f(x) + \tfrac{1}{2} y'\nabla^2 f(x+\alpha y) y.$$

(c) There holds

$$f(x+y) = f(x) + y'\nabla f(x) + \tfrac{1}{2} y'\nabla^2 f(x) y + o\big(\|y\|^2\big).$$

Proposition 1.1.22: (Implicit Function Theorem) Consider a function $f : \Re^{n+m} \mapsto \Re^m$ of $x \in \Re^n$ and $y \in \Re^m$ such that:

(1) $f(\overline{x}, \overline{y}) = 0$.

(2) f is continuous, and has a continuous and nonsingular gradient matrix $\nabla_y f(x,y)$ in an open set containing $(\overline{x}, \overline{y})$.

Then there exist open sets $S_{\overline{x}} \subset \Re^n$ and $S_{\overline{y}} \subset \Re^m$ containing $\overline{x}$ and $\overline{y}$, respectively, and a continuous function $\phi : S_{\overline{x}} \mapsto S_{\overline{y}}$ such that $\overline{y} = \phi(\overline{x})$ and $f\big(x, \phi(x)\big) = 0$ for all $x \in S_{\overline{x}}$. The function ϕ is unique in the sense that if $x \in S_{\overline{x}}$, $y \in S_{\overline{y}}$, and $f(x,y) = 0$, then $y = \phi(x)$. Furthermore, if for some integer $p > 0$, f is p times continuously differentiable the same is true for ϕ, and we have

$$\nabla \phi(x) = -\nabla_x f\big(x, \phi(x)\big) \Big(\nabla_y f\big(x, \phi(x)\big) \Big)^{-1}, \qquad \forall\, x \in S_{\overline{x}}.$$

As a final word of caution to the reader, let us mention that one can easily get confused with gradient notation and its use in various formulas, such as for example the order of multiplication of various gradients in the chain rule and the Implicit Function Theorem. Perhaps the safest guideline to minimize errors is to remember our conventions:

(a) A vector is viewed as a column vector (an $n \times 1$ matrix).

(b) The gradient ∇f of a scalar function $f : \Re^n \mapsto \Re$ is also viewed as a column vector.

(c) The gradient matrix ∇f of a vector function $f : \Re^n \mapsto \Re^m$ with components $f_1, \ldots, f_m$ is the $n \times m$ matrix whose columns are the (column) vectors $\nabla f_1, \ldots, \nabla f_m$.

With these rules in mind one can use "dimension matching" as an effective guide to writing correct formulas quickly.

1.2 CONVEX SETS AND FUNCTIONS

We now introduce some of the basic notions relating to convex sets and functions. The material of this section permeates all subsequent developments in this book, and will be used in the next section for the discussion of important issues in optimization, such as the existence of optimal solutions.

1.2.1 Basic Properties

The notion of a convex set is defined below and is illustrated in Fig. 1.2.1.

Definition 1.2.1: Let C be a subset of $\Re^n$. We say that C is *convex* if

$$\alpha x + (1 - \alpha) y \in C, \qquad \forall\, x, y \in C,\ \forall\, \alpha \in [0, 1]. \tag{1.2}$$

Note that the empty set is by convention considered to be convex. Generally, when referring to a convex set, it will usually be apparent from the context whether this set can be empty, but we will often be specific in order to minimize ambiguities.

The following proposition lists some operations that preserve convexity of a set.

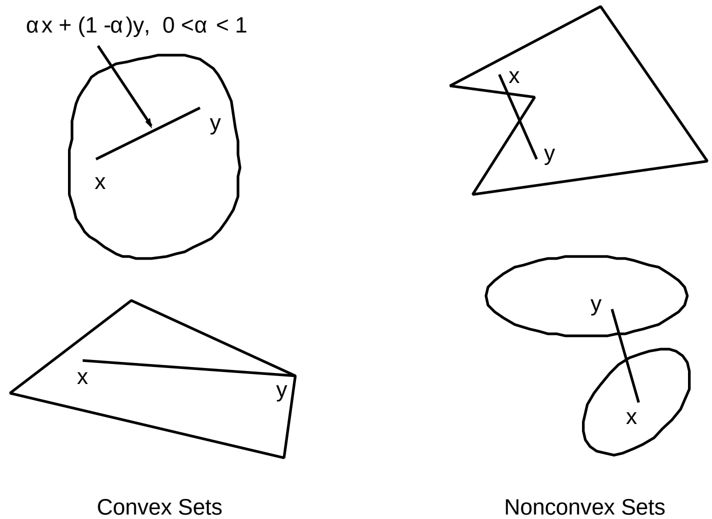

Figure 1.2.1. Illustration of the definition of a convex set. For convexity, linear interpolation between any two points of the set must yield points that lie within the set.

Proposition 1.2.1:

(a) The intersection $\cap_{i\in I} C_i$ of any collection $\{C_i \mid i \in I\}$ of convex sets is convex.

(b) The vector sum $C_1 + C_2$ of two convex sets C_1 and C_2 is convex.

(c) The set λC is convex for any convex set C and scalar λ. Furthermore, if C is a convex set and λ_1, λ_2 are positive scalars, we have

$$(\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C.$$

(d) The closure and the interior of a convex set are convex.

(e) The image and the inverse image of a convex set under an affine function are convex.

Proof: The proof is straightforward using the definition of convexity, cf. Eq. (1.2). For example, to prove part (a), we take two points x and y from $\cap_{i\in I} C_i$, and we use the convexity of C_i to argue that the line segment connecting x and y belongs to all the sets C_i, and hence, to their intersection. The proofs of parts (b)-(e) are similar and are left as exercises for the reader. **Q.E.D.**

A set C is said to be a *cone* if for all $x \in C$ and $\lambda > 0$, we have $\lambda x \in C$. A cone need not be convex and need not contain the origin (although the origin always lies in the closure of a nonempty cone). Several of the results of the above proposition have analogs for cones (see the exercises).

Convex Functions

The notion of a convex function is defined below and is illustrated in Fig. 1.2.2.

Definition 1.2.2: Let C be a convex subset of $\Re^n$. A function $f : C \mapsto \Re$ is called *convex* if

$$f\big(\alpha x + (1-\alpha)y\big) \le \alpha f(x) + (1-\alpha) f(y), \qquad \forall\ x, y \in C,\ \forall\ \alpha \in [0,1]. \tag{1.3}$$

The function f is called *concave* if $-f$ is convex. The function f is called *strictly convex* if the above inequality is strict for all $x, y \in C$ with $x \neq y$, and all $\alpha \in (0,1)$. For a function $f : X \mapsto \Re$, we also say that f is *convex over the convex set* C if the domain X of f contains C and Eq. (1.3) holds, i.e., when the domain of f is restricted to C, f becomes convex.

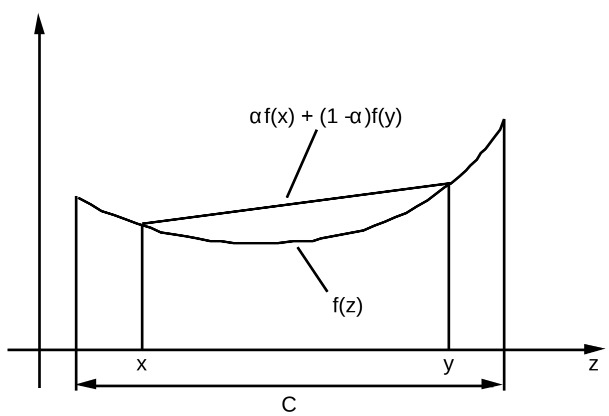

Figure 1.2.2. Illustration of the definition of a function that is convex over a convex set C. The linear interpolation $\alpha f(x) + (1-\alpha) f(y)$ overestimates the function value $f(\alpha x + (1-\alpha)y)$ for all $\alpha \in [0,1]$.

If C is a convex set and $f : C \mapsto \Re$ is a convex function, the level sets $\{x \in C \mid f(x) \le \gamma\}$ and $\{x \in C \mid f(x) < \gamma\}$ are convex for all scalars

γ. To see this, note that if $x, y \in C$ are such that $f(x) \leq \gamma$ and $f(y) \leq \gamma$, then for any $\alpha \in [0,1]$, we have $\alpha x + (1-\alpha)y \in C$, by the convexity of C, and $f\big(\alpha x + (1-\alpha)y\big) \leq \alpha f(x) + (1-\alpha)f(y) \leq \gamma$, by the convexity of f. However, the converse is not true; for example, the function $f(x) = \sqrt{|x|}$ has convex level sets but is not convex.

We generally prefer to deal with convex functions that are real-valued and are defined over the entire space $\Re^n$ (rather than over just a convex subset). However, in some situations, prominently arising in the context of duality theory, we will encounter functions f that are convex over a convex subset C and cannot be extended to functions that are convex over $\Re^n$. In such situations, it may be convenient, instead of restricting the domain of f to the subset C where f takes real values, to extend the domain to the entire space $\Re^n$, but allow f to take the value ∞.

We are thus motivated to introduce *extended real-valued* convex functions that can take the value of ∞ at some points. In particular, if $C \subset \Re^n$ is a convex set, a function $f : C \mapsto (-\infty, \infty]$ is called *convex* over C (or simply convex when $C = \Re^n$) if the condition

$$f\big(\alpha x + (1-\alpha)y\big) \leq \alpha f(x) + (1-\alpha)f(y), \qquad \forall\ x, y \in C,\ \forall\ \alpha \in [0,1]$$

holds. The function f is called *strictly convex* if the above inequality is strict for all x and y in C such that $f(x) < \infty$ and $f(y) < \infty$. It can be seen that if f is convex, the level sets $\{x \in C \mid f(x) \leq \gamma\}$ and $\{x \in C \mid f(x) < \gamma\}$ are convex for all scalars γ.

One complication when dealing with extended real-valued functions is that we sometimes need to be careful to exclude the unusual case where $f(x) = \infty$ for all $x \in C$, in which case f is still convex, since it trivially satisfies the above inequality (such functions are sometimes called *improper* in the literature of convex functions, but we will not use this terminology here). Another area of concern is working with functions that can take both values $-\infty$ and ∞, because of the possibility of the forbidden expression $\infty - \infty$ arising when sums of function values are considered. For this reason, we will try to minimize the occurances of such functions. On occasion we will deal with extended real-valued functions that can take the value $-\infty$ but not the value ∞. In particular, a function $f : C \mapsto [-\infty, \infty)$, where C is a convex set, is called *concave* if the function $-f : C \mapsto (-\infty, \infty]$ is convex as per the preceding definition.

We define the *effective domain* of an extended real-valued function $f : X \mapsto (-\infty, \infty]$ to be the set

$$\mathrm{dom}(f) = \big\{x \in X \mid f(x) < \infty\big\}.$$

Note that if X is convex and f is convex over X, then $\mathrm{dom}(f)$ is a convex set. Similarly, we define the effective domain of an extended real-valued function $f : X \mapsto [-\infty, \infty)$ to be the set $\mathrm{dom}(f) = \big\{x \in X \mid f(x) > -\infty\big\}$.

Note that by replacing the domain of an extended real-valued convex function with its effective domain, we can convert it to a real-valued function. In this way, we can use results stated in terms of real-valued functions, and we can also avoid calculations with ∞. Thus, the entire subject of convex functions can be developed without resorting to extended real-valued functions. The reverse is also true, namely that extended real-valued functions can be adopted as the norm; for example, the classical treatment of Rockafellar [Roc70] uses this approach.

Generally, functions that are real-valued over the entire space $\Re^n$ are more convenient (and even essential) in numerical algorithms and also in optimization analyses where a calculus-oriented approach based on differentiability is adopted. This is typically the case in nonconvex optimization, where nonlinear equality and nonconvex inequality constraints are involved (see Chapter 2). On the other hand, extended real-valued functions offer notational advantages in a convex programming setting (see Chapters 3 and 4), and in fact may be more natural because some basic constructions around duality involve extended real-valued functions. Since we plan to deal with nonconvex as well as convex problems, and with analysis as well as numerical methods, we will adopt a flexible approach and use both real-valued and extended real-valued functions. However, consistently with prevailing optimization practice, we will generally prefer to avoid using extended real-valued functions, unless there are strong notational or other advantages for doing so.

Epigraphs and Semicontinuity

An extended real-valued function $f : X \mapsto (-\infty, \infty]$ is called *lower semicontinuous* at a vector $x \in X$ if $f(x) \le \liminf_{k\to\infty} f(x_k)$ for every sequence $\{x_k\}$ converging to x. This definition is consistent with the corresponding definition for real-valued functions [cf. Def. 1.1.5(c)]. If f is lower semicontinuous at every x in a subset $U \subset X$, we say that f is *lower semicontinuous over* U.

The *epigraph* of a function $f : X \mapsto (-\infty, \infty]$, where $X \subset \Re^n$, is the subset of $\Re^{n+1}$ given by

$$\mathrm{epi}(f) = \big\{(x, w) \mid x \in X,\ w \in \Re,\ f(x) \le w\big\};$$

(see Fig. 1.2.3). Note that if we restrict f to its effective domain $\big\{x \in X \mid f(x) < \infty\big\}$, so that it becomes real-valued, the epigraph remains unaffected. Epigraphs are very useful for our purposes because questions about convexity and lower semicontinuity of functions can be reduced to corresponding questions about convexity and closure of their epigraphs, as shown in the following proposition.

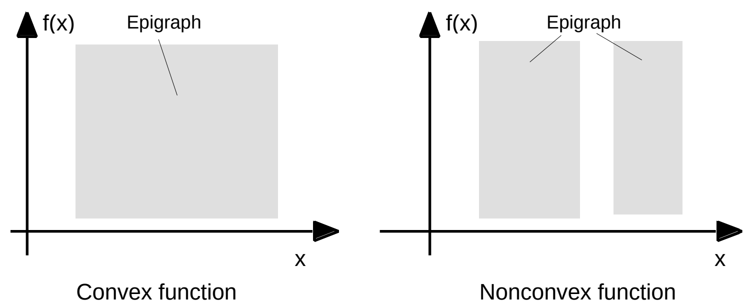

Figure 1.2.3. Illustration of the epigraphs of extended real-valued convex and nonconvex functions.

Proposition 1.2.2: Let $f : X \mapsto (-\infty, \infty]$ be a function. Then:

(a) $\text{epi}(f)$ is convex if and only if $\text{dom}(f)$ is convex and f is convex over $\text{dom}(f)$.

(b) Assuming $X = \Re^n$, the following are equivalent:

 (i) The level set $\{x \mid f(x) \le \gamma\}$ is closed for all scalars γ.

 (ii) f is lower semicontinuous over $\Re^n$.

 (iii) $\text{epi}(f)$ is closed.

Proof: (a) Assume that $\text{dom}(f)$ is convex and f is convex over $\text{dom}(f)$. If (x_1, w_1) and (x_2, w_2) belong to $\text{epi}(f)$ and $\alpha \in [0,1]$, we have $x_1, x_2 \in \text{dom}(f)$ and

$$f(x_1) \le w_1, \qquad f(x_2) \le w_2,$$

and by multiplying these inequalities with α and $(1-\alpha)$, respectively, by adding, and by using the convexity of f, we obtain

$$f\big(\alpha x_1 + (1-\alpha)x_2\big) \le \alpha f(x_1) + (1-\alpha) f(x_2) \le \alpha w_1 + (1-\alpha) w_2.$$

Hence the vector $\big(\alpha x_1 + (1-\alpha)x_2,\, \alpha w_1 + (1-\alpha)w_2\big)$, which is equal to $\alpha(x_1, w_1) + (1-\alpha)(x_2, w_2)$, belongs to $\text{epi}(f)$, showing the convexity of $\text{epi}(f)$.

Conversely, assume that $\text{epi}(f)$ is convex, and let $x_1, x_2 \in \text{dom}(f)$ and $\alpha \in [0,1]$. The pairs $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$ belong to $\text{epi}(f)$, so by convexity, we have

$$\big(\alpha x_1 + (1-\alpha)x_2,\, \alpha f(x_1) + (1-\alpha) f(x_2)\big) \in \text{epi}(f).$$

Therefore, by the definition of epi(f), it follows that $\alpha x_1 + (1-\alpha)x_2 \in$ dom(f), so dom(f) is convex, while

$$f\big(\alpha x_1 + (1-\alpha)x_2\big) \leq \alpha f(x_1) + (1-\alpha) f(x_2),$$

so f is convex over dom(f).

(b) We first show that (i) implies (ii). Assume that the level set $\big\{x \mid f(x) \leq \gamma\big\}$ is closed for all scalars γ, fix a vector $\overline{x} \in \Re^n$, and let $\{x_k\}$ be a sequence converging to $\overline{x}$. If $\liminf_{k\to\infty} f(x_k) = \infty$, then we are done, so assume that $\liminf_{k\to\infty} f(x_k) < \infty$. Let $\{x_k\}_{\mathcal{K}} \subset \{x_k\}$ be a subsequence along which the limit inferior of $\{f(x_k)\}$ is attained. Then for γ such that $\liminf_{k\to\infty} f(x_k) < \gamma$ and all sufficiently large k with $k \in \mathcal{K}$, we have $f(x_k) \leq \gamma$. Since each level set $\big\{x \mid f(x) \leq \gamma\big\}$ is closed, it follows that $\overline{x}$ belongs to all the level sets with $\liminf_{k\to\infty} f(x_k) < \gamma$, implying that $f(\overline{x}) \leq \liminf_{k\to\infty} f(x_k)$, and that f is lower semicontinuous at $\overline{x}$. Therefore, (i) implies (ii).

We next show that (ii) implies (iii). Assume that f is lower semicontinuous over $\Re^n$, and let $(\overline{x}, \overline{w})$ be the limit of a sequence $\{(x_k, w_k)\} \subset$ epi(f). Then we have $f(x_k) \leq w_k$, and by taking the limit as $k \to \infty$ and by using the lower semicontinuity of f at $\overline{x}$, we obtain $f(\overline{x}) \leq \liminf_{k\to\infty} f(x_k) \leq \overline{w}$. Hence, $(\overline{x}, \overline{w}) \in$ epi(f) and epi(f) is closed. Thus, (ii) implies (iii).

We finally show that (iii) implies (i). Assume that epi(f) is closed, and let $\{x_k\}$ be a sequence that converges to some $\overline{x}$ and belongs to the level set $\big\{x \mid f(x) \leq \gamma\big\}$ for some γ. Then $(x_k, \gamma) \in$ epi(f) for all k and $(x_k, \gamma) \to (\overline{x}, \gamma)$, so since epi$(f)$ is closed, we have $(\overline{x}, \gamma) \in$ epi(f). Hence, $\overline{x}$ belongs to the level set $\big\{x \mid f(x) \leq \gamma\big\}$, implying that this set is closed. Therefore, (iii) implies (i). **Q.E.D.**

If the epigraph of a function $f : X \mapsto (-\infty, \infty]$ is a closed set, we say that f is a *closed* function. Thus, if we extend the domain of f to $\Re^n$ and consider the function $\tilde{f}$ given by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X, \\ \infty & \text{if } x \notin X, \end{cases}$$

we see that according to the preceding proposition, *f is closed if and only $\tilde{f}$ is lower semicontinuous over $\Re^n$*. Note that if f is lower semicontinuous over dom(f), it is not necessarily closed; take for example f to be constant for x in some nonclosed set and ∞ otherwise. Furthermore, if f is closed it is not necessarily true that dom(f) is closed; for example, the function

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0, \\ \infty & \text{otherwise}, \end{cases}$$

is closed but dom(f) is the open half-line of positive numbers. On the other hand, *if* dom(f) *is closed and f is lower semicontinuous over* dom(f)*, then*

f *is closed* because epi(f) is closed, as can be seen by reviewing the relevant part of the proof of Prop. 1.2.2(b).

Common examples of convex functions are affine functions and norms; this is straightforward to verify using the definition of convexity. For example, for any $x, y \in \Re^n$ and any $\alpha \in [0, 1]$, by using the triangle inequality, we have

$$\|\alpha x + (1 - \alpha)y\| \leq \|\alpha x\| + \|(1 - \alpha)y\| = \alpha\|x\| + (1 - \alpha)\|y\|,$$

so the norm function $\|\cdot\|$ is convex. The following proposition provides some means for recognizing convex functions, and gives some algebraic operations that preserve convexity of a function.

Proposition 1.2.3:

(a) Let $f_1, \ldots, f_m : \Re^n \mapsto (-\infty, \infty]$ be given functions, let $\lambda_1, \ldots, \lambda_m$ be positive scalars, and consider the function $g : \Re^n \mapsto (-\infty, \infty]$ given by

$$g(x) = \lambda_1 f_1(x) + \cdots + \lambda_m f_m(x).$$

If $f_1, \ldots, f_m$ are convex, then g is also convex, while if $f_1, \ldots, f_m$ are closed, then g is also closed.

(b) Let $f : \Re^n \mapsto (-\infty, \infty]$ be a given function, let A be an $m \times n$ matrix, and consider the function $g : \Re^n \mapsto (-\infty, \infty]$ given by

$$g(x) = f(Ax).$$

If f is convex, then g is also convex, while if f is closed, then g is also closed.

(c) Let $f_i : \Re^n \mapsto (-\infty, \infty]$ be given functions for $i \in I$, where I is an arbitrary index set, and consider the function $g : \Re^n \mapsto (-\infty, \infty]$ given by

$$g(x) = \sup_{i \in I} f_i(x).$$

If f_i, $i \in I$, are convex, then g is also convex, while if f_i, $i \in I$, are closed, then g is also closed.

Proof: (a) Let $f_1, \ldots, f_m$ be convex. We use the definition of convexity to write for any $x, y \in \Re^n$ and $\alpha \in [0, 1]$,

$$\begin{aligned} g\big(\alpha x + (1-\alpha)y\big) &= \sum_{i=1}^m \lambda_i f_i\big(\alpha x + (1-\alpha)y\big) \\ &\le \sum_{i=1}^m \lambda_i \big(\alpha f_i(x) + (1-\alpha) f_i(y)\big) \\ &= \alpha \sum_{i=1}^m \lambda_i f_i(x) + (1-\alpha) \sum_{i=1}^m \lambda_i f_i(y) \\ &= \alpha g(x) + (1-\alpha) g(y). \end{aligned}$$

Hence g is convex.

Let the functions $f_1, \ldots, f_m$ be closed. Then they are lower semicontinuous at every $x \in \Re^n$ [cf. Prop. 1.2.2(b)], so for every sequence $\{x_k\}$ converging to x, we have $f_i(x) \le \liminf_{k\to\infty} f_i(x_k)$ for all i. Hence

$$g(x) \le \sum_{i=1}^m \lambda_i \liminf_{k\to\infty} f_i(x_k) \le \liminf_{k\to\infty} \sum_{i=1}^m \lambda_i f_i(x_k) = \liminf_{k\to\infty} g(x_k).$$

where we have used Prop. 1.1.4(d) (the sum of the limit inferiors of sequences is less or equal to the limit inferior of the sum sequence). Therefore, g is lower semicontinuous at all $x \in \Re^n$, so by Prop. 1.2.2(b), it is closed.

(b) This is straightforward, along the lines of the proof of part (a).

(c) A pair (x, w) belongs to the epigraph

$$\text{epi}(g) = \big\{(x, w) \mid g(x) \le w\big\}$$

if and only if $f_i(x) \le w$ for all $i \in I$, or $(x, w) \in \cap_{i\in I}\text{epi}(f_i)$. Therefore,

$$\text{epi}(g) = \cap_{i\in I}\text{epi}(f_i).$$

If the f_i are convex, the epigraphs $\text{epi}(f_i)$ are convex, so $\text{epi}(g)$ is convex, and g is convex by Prop. 1.2.2(a). If the f_i are closed, then the epigraphs $\text{epi}(f_i)$ are closed, so $\text{epi}(g)$ is closed, and g is closed. **Q.E.D.**

Characterizations of Differentiable Convex Functions

For differentiable functions, there is an alternative characterization of convexity, given in the following proposition and illustrated in Fig. 1.2.4.

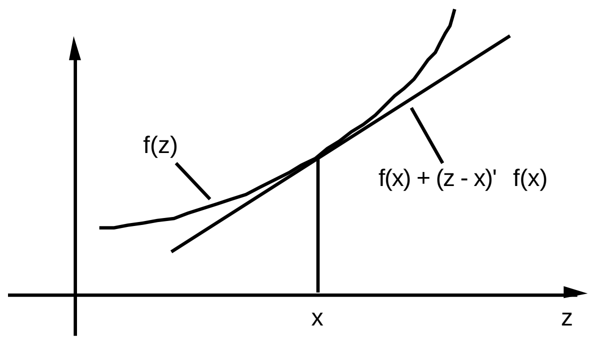

Figure 1.2.4. Characterization of convexity in terms of first derivatives. The condition $f(z) \geq f(x) + (z-x)'\nabla f(x)$ states that a linear approximation, based on the first order Taylor series expansion, underestimates a convex function.

Proposition 1.2.4: Let $C \subset \Re^n$ be a convex set and let $f : \Re^n \mapsto \Re$ be differentiable over $\Re^n$.

(a) f is convex over C if and only if

$$f(z) \geq f(x) + (z-x)'\nabla f(x), \qquad \forall\ x, z \in C. \tag{1.4}$$

(b) f is strictly convex over C if and only if the above inequality is strict whenever $x \neq z$.

Proof: We prove (a) and (b) simultaneously. Assume that the inequality (1.4) holds. Choose any $x, y \in C$ and $\alpha \in [0,1]$, and let $z = \alpha x + (1-\alpha)y$. Using the inequality (1.4) twice, we obtain

$$f(x) \geq f(z) + (x-z)'\nabla f(z),$$
$$f(y) \geq f(z) + (y-z)'\nabla f(z).$$

We multiply the first inequality by α, the second by $(1-\alpha)$, and add them to obtain

$$\alpha f(x) + (1-\alpha)f(y) \geq f(z) + \big(\alpha x + (1-\alpha)y - z\big)'\nabla f(z) = f(z),$$

which proves that f is convex. If the inequality (1.4) is strict as stated in part (b), then if we take $x \neq y$ and $\alpha \in (0,1)$ above, the three preceding inequalities become strict, thus showing the strict convexity of f.

Conversely, assume that f is convex, let x and z be any vectors in C with $x \neq z$, and for $\alpha \in (0,1)$, consider the function

$$g(\alpha) = \frac{f\big(x + \alpha(z-x)\big) - f(x)}{\alpha}, \qquad \alpha \in (0,1].$$

We will show that $g(\alpha)$ is monotonically increasing with α, and is strictly monotonically increasing if f is strictly convex. This will imply that

$$(z-x)'\nabla f(x) = \lim_{\alpha \downarrow 0} g(\alpha) \leq g(1) = f(z) - f(x),$$

with strict inequality if g is strictly monotonically increasing, thereby showing that the desired inequality (1.4) holds (and holds strictly if f is strictly convex). Indeed, consider any α_1, α_2, with $0 < \alpha_1 < \alpha_2 < 1$, and let

$$\overline{\alpha} = \frac{\alpha_1}{\alpha_2}, \qquad \overline{z} = x + \alpha_2(z - x). \tag{1.5}$$

We have

$$f\big(x + \overline{\alpha}(\overline{z} - x)\big) \leq \overline{\alpha} f(\overline{z}) + (1 - \overline{\alpha}) f(x),$$

or

$$\frac{f\big(x + \overline{\alpha}(\overline{z} - x)\big) - f(x)}{\overline{\alpha}} \leq f(\overline{z}) - f(x), \tag{1.6}$$

and the above inequalities are strict if f is strictly convex. Substituting the definitions (1.5) in Eq. (1.6), we obtain after a straightforward calculation

$$\frac{f\big(x + \alpha_1(z - x)\big) - f(x)}{\alpha_1} \leq \frac{f\big(x + \alpha_2(z - x)\big) - f(x)}{\alpha_2},$$

or

$$g(\alpha_1) \leq g(\alpha_2),$$

with strict inequality if f is strictly convex. Hence g is monotonically increasing with α, and strictly so if f is strictly convex. **Q.E.D.**

Note a simple consequence of Prop. 1.2.4(a): if $f : \Re^n \mapsto \Re$ is a convex function and $\nabla f(x^*) = 0$, then x^* minimizes f over $\Re^n$. This is a classical sufficient condition for unconstrained optimality, originally formulated (in one dimension) by Fermat in 1637.

For twice differentiable convex functions, there is another characterization of convexity as shown by the following proposition.

Proposition 1.2.5: Let $C \subset \Re^n$ be a convex set and let $f : \Re^n \mapsto \Re$ be twice continuously differentiable over $\Re^n$.

(a) If $\nabla^2 f(x)$ is positive semidefinite for all $x \in C$, then f is convex over C.

(b) If $\nabla^2 f(x)$ is positive definite for all $x \in C$, then f is strictly convex over C.

(c) If $C = \Re^n$ and f is convex, then $\nabla^2 f(x)$ is positive semidefinite for all x.

Proof: (a) By Prop. 1.1.21(b), for all $x, y \in C$ we have

$$f(y) = f(x) + (y-x)'\nabla f(x) + \tfrac{1}{2}(y-x)'\nabla^2 f\big(x + \alpha(y-x)\big)(y-x)$$

for some $\alpha \in [0,1]$. Therefore, using the positive semidefiniteness of $\nabla^2 f$, we obtain

$$f(y) \geq f(x) + (y-x)'\nabla f(x), \qquad \forall\, x, y \in C.$$

From Prop. 1.2.4(a), we conclude that f is convex.

(b) Similar to the proof of part (a), we have $f(y) > f(x) + (y-x)'\nabla f(x)$ for all $x, y \in C$ with $x \neq y$, and the result follows from Prop. 1.2.4(b).

(c) Suppose that $f : \Re^n \mapsto \Re$ is convex and suppose, to obtain a contradiction, that there exist some $x \in \Re^n$ and some $z \in \Re^n$ such that $z'\nabla^2 f(x) z < 0$. Using the continuity of $\nabla^2 f$, we see that we can choose z with small enough norm so that $z'\nabla^2 f(x + \alpha z) z < 0$ for every $\alpha \in [0,1]$. Then, using again Prop. 1.1.21(b), we obtain $f(x+z) < f(x) + z'\nabla f(x)$, which, in view of Prop. 1.2.4(a), contradicts the convexity of f. **Q.E.D.**

If f is convex over a strict subset $C \subset \Re^n$, it is not necessarily true that $\nabla^2 f(x)$ is positive semidefinite at any point of C [take for example $n = 2$, $C = \{(x_1, 0) \mid x_1 \in \Re\}$, and $f(x) = x_1^2 - x_2^2$]. A strengthened version of Prop. 1.2.5 is given in the exercises. It can be shown that the conclusion of Prop. 1.2.5(c) also holds if C is assumed to have nonempty interior instead of being equal to $\Re^n$.

The following proposition considers a strengthened form of strict convexity characterized by the following equation:

$$\big(\nabla f(x) - \nabla f(y)\big)'(x-y) \geq \alpha \|x-y\|^2, \qquad \forall\, x, y \in \Re^n, \tag{1.7}$$

where α is some positive number. Convex functions with this property are called *strongly convex with coefficient* α.

Proposition 1.2.6: (Strong Convexity) Let $f : \Re^n \mapsto \Re$ be smooth. If f is strongly convex with coefficient α, then f is strictly convex. Furthermore, if f is twice continuously differentiable, then strong convexity of f with coefficient α is equivalent to the positive semidefiniteness of $\nabla^2 f(x) - \alpha I$ for every $x \in \Re^n$, where I is the identity matrix.

Proof: Fix some $x, y \in \Re^n$ such that $x \neq y$, and define the function $h : \Re \mapsto \Re$ by $h(t) = f\big(x + t(y-x)\big)$. Consider some $t, t' \in \Re$ such that

$t < t'$. Using the chain rule and Eq. (1.7), we have

$$\begin{aligned}\Big(\frac{dh}{dt}(t') - \frac{dh}{dt}(t)\Big)(t'-t) \\ &= \Big(\nabla f\big(x + t'(y-x)\big) - \nabla f\big(x + t(y-x)\big)\Big)'(y-x)(t'-t) \\ &\geq \alpha(t'-t)^2\|x-y\|^2 > 0.\end{aligned}$$

Thus, dh/dt is strictly increasing and for any $t \in (0,1)$, we have

$$\frac{h(t)-h(0)}{t} = \frac{1}{t}\int_0^t \frac{dh}{d\tau}(\tau)\,d\tau < \frac{1}{1-t}\int_t^1 \frac{dh}{d\tau}(\tau)\,d\tau = \frac{h(1)-h(t)}{1-t}.$$

Equivalently, $th(1) + (1-t)h(0) > h(t)$. The definition of h yields $tf(y) + (1-t)f(x) > f\big(ty + (1-t)x\big)$. Since this inequality has been proved for arbitrary $t \in (0,1)$ and $x \neq y$, we conclude that f is strictly convex.

Suppose now that f is twice continuously differentiable and Eq. (1.7) holds. Let c be a scalar. We use Prop. 1.1.21(b) twice to obtain

$$f(x+cy) = f(x) + cy'\nabla f(x) + \frac{c^2}{2}y'\nabla^2 f(x+tcy)y,$$

and

$$f(x) = f(x+cy) - cy'\nabla f(x+cy) + \frac{c^2}{2}y'\nabla^2 f(x+scy)y,$$

for some t and s belonging to $[0,1]$. Adding these two equations and using Eq. (1.7), we obtain

$$\frac{c^2}{2}y'\big(\nabla^2 f(x+scy)+\nabla^2 f(x+tcy)\big)y = \big(\nabla f(x+cy)-\nabla f(x)\big)'(cy) \geq \alpha c^2\|y\|^2.$$

We divide both sides by c^2 and then take the limit as $c \to 0$ to conclude that $y'\nabla^2 f(x)y \geq \alpha\|y\|^2$. Since this inequality is valid for every $y \in \Re^n$, it follows that $\nabla^2 f(x) - \alpha I$ is positive semidefinite.

For the converse, assume that $\nabla^2 f(x) - \alpha I$ is positive semidefinite for all $x \in \Re^n$. Consider the function $g : \Re \mapsto \Re$ defined by

$$g(t) = \nabla f\big(tx + (1-t)y\big)'(x-y).$$

Using the Mean Value Theorem (Prop. 1.1.20), we have $\big(\nabla f(x)-\nabla f(y)\big)'(x-y) = g(1) - g(0) = (dg/dt)(t)$ for some $t \in [0,1]$. The result follows because

$$\frac{dg}{dt}(t) = (x-y)'\nabla^2 f\big(tx + (1-t)y\big)(x-y) \geq \alpha\|x-y\|^2,$$

where the last inequality is a consequence of the positive semidefiniteness of $\nabla^2 f\big(tx + (1-t)y\big) - \alpha I$. **Q.E.D.**

As an example, consider the quadratic function

$$f(x) = x'Qx,$$

where Q is a symmetric matrix. By Prop. 1.2.5, the function f is convex if and only if Q is positive semidefinite. Furthermore, by Prop. 1.2.6, f is strongly convex with coefficient α if and only if $\nabla^2 f(x) - \alpha I = 2Q - \alpha I$ is positive semidefinite for some $\alpha > 0$. Thus f is strongly convex with some positive coefficient (as well as strictly convex) if and only if Q is positive definite.

1.2.2 Convex and Affine Hulls

Let X be a subset of $\Re^n$. A *convex combination* of elements of X is a vector of the form $\sum_{i=1}^m \alpha_i x_i$, where m is a positive integer, $x_1, \ldots, x_m$ belong to X, and $\alpha_1, \ldots, \alpha_m$ are scalars such that

$$\alpha_i \geq 0, \quad i = 1, \ldots, m, \qquad \sum_{i=1}^m \alpha_i = 1.$$

Note that if X is convex, then the convex combination $\sum_{i=1}^m \alpha_i x_i$ belongs to X (this is easily shown by induction; see the exercises), and for any function $f : \Re^n \mapsto \Re$ that is convex over X, we have

$$f\left(\sum_{i=1}^m \alpha_i x_i\right) \leq \sum_{i=1}^m \alpha_i f(x_i). \tag{1.8}$$

This follows by using repeatedly the definition of convexity. The preceding relation is a special case of *Jensen's inequality* and can be used to prove a number of interesting inequalities in applied mathematics and probability theory.

The *convex hull* of a set X, denoted $\text{conv}(X)$, is the intersection of all convex sets containing X, and is a convex set by Prop. 1.2.1(a). It is straightforward to verify that the set of all convex combinations of elements of X is convex, and is equal to the convex hull $\text{conv}(X)$ (see the exercises). In particular, if X consists of a finite number of vectors $x_1, \ldots, x_m$, its convex hull is

$$\text{conv}\big(\{x_1, \ldots, x_m\}\big) = \left\{ \sum_{i=1}^m \alpha_i x_i \;\middle|\; \alpha_i \geq 0,\, i = 1, \ldots, m,\, \sum_{i=1}^m \alpha_i = 1 \right\}.$$

We recall that an affine set M is a set of the form $x + S$, where S is a subspace, called the *subspace parallel to* M. If X is a subset of $\Re^n$, the *affine hull* of X, denoted $\text{aff}(X)$, is the intersection of all affine sets containing X. Note that $\text{aff}(X)$ is itself an affine set and that it contains conv(X). It can be seen that the affine hull of X, the affine hull of the convex hull conv(X), and the affine hull of the closure cl(X) coincide (see the exercises). For a convex set C, the *dimension* of C is defined to be the dimension of aff(C).

Given a subset $X \subset \Re^n$, a *nonnegative combination* of elements of X is a vector of the form $\sum_{i=1}^m \alpha_i x_i$, where m is a positive integer, $x_1, \ldots, x_m$ belong to X, and $\alpha_1, \ldots, \alpha_m$ are nonnegative scalars. If the scalars α_i are all positive, the combination $\sum_{i=1}^m \alpha_i x_i$ is said to be *positive*. The *cone generated by* X, denoted by cone(X), is the set of all nonnegative combinations of elements of X. It is easily seen that cone(X) is a convex cone, although it need not be closed [cone(X) can be shown to be closed in special cases, such as when X is a finite set – this is one of the central results of polyhedral convexity and will be shown in Section 1.6].

The following is a fundamental characterization of convex hulls.

Proposition 1.2.7: (Caratheodory's Theorem) Let X be a subset of $\Re^n$.

(a) Every x in cone(X) can be represented as a positive combination of vectors $x_1, \ldots, x_m \in X$ that are linearly independent, where m is a positive integer with $m \leq n$.

(b) Every x in conv(X) can be represented as a convex combination of vectors $x_1, \ldots, x_m \in X$ such that $x_2 - x_1, \ldots, x_m - x_1$ are linearly independent, where m is a positive integer with $m \leq n + 1$.

Proof: (a) Let x be a nonzero vector in the cone(X), and let m be the smallest integer such that x has the form $\sum_{i=1}^m \alpha_i x_i$, where $\alpha_i > 0$ and $x_i \in X$ for all $i = 1, \ldots, m$. If the vectors x_i were linearly dependent, there would exist scalars $\lambda_1, \ldots, \lambda_m$, with $\sum_{i=1}^m \lambda_i x_i = 0$ and at least one of the λ_i is positive. Consider the linear combination $\sum_{i=1}^m (\alpha_i - \overline{\gamma}\lambda_i) x_i$, where $\overline{\gamma}$ is the largest γ such that $\alpha_i - \gamma\lambda_i \geq 0$ for all i. This combination provides a representation of x as a positive combination of fewer than m vectors of X – a contradiction. Therefore, $x_1, \ldots, x_m$, are linearly independent, and since any linearly independent set of vectors contains at most n elements, we must have $m \leq n$.

(b) The proof will be obtained by applying part (a) to the subset of $\Re^{n+1}$ given by

$$Y = \big\{(x, 1) \mid x \in X\big\}.$$

If $x \in \text{conv}(X)$, then $x = \sum_{i=1}^m \gamma_i x_i$ for some positive γ_i with $1 = \sum_{i=1}^m \gamma_i$, i.e., $(x, 1) \in \text{cone}(Y)$. By part (a), we have $(x, 1) = \sum_{i=1}^m \alpha_i (x_i, 1)$, for some positive $\alpha_1, \ldots, \alpha_m$ and vectors $(x_1, 1), \ldots, (x_m, 1)$, which are linearly independent (implying that $m \le n+1$) or equivalently

$$x = \sum_{i=1}^{m} \alpha_i x_i, \qquad 1 = \sum_{i=1}^{m} \alpha_i.$$

Finally, to show that $x_2 - x_1, \ldots, x_m - x_1$ are linearly independent, assume to arrive at a contradiction, that there exist $\lambda_2, \ldots, \lambda_m$, not all 0, such that

$$\sum_{i=2}^{m} \lambda_i (x_i - x_1) = 0.$$

Equivalently, defining $\lambda_1 = -(\lambda_2 + \cdots + \lambda_m)$, we have

$$\sum_{i=1}^{m} \lambda_i (x_i, 1) = 0,$$

which contradicts the linear independence of $(x_1, 1), \ldots, (x_m, 1)$. **Q.E.D.**

It is not generally true that the convex hull of a closed set is closed [take for instance the convex hull of the set consisting of the origin and the subset $\{(x_1, x_2) \mid x_1 x_2 = 1,\ x_1 \ge 0,\ x_2 \ge 0\}$ of $\Re^2$]. We have, however, the following.

Proposition 1.2.8: The convex hull of a compact set is compact.

Proof: Let X be a compact subset of $\Re^n$. By Caratheodory's Theorem, a sequence in conv(X) can be expressed as $\left\{\sum_{i=1}^{n+1} \alpha_i^k x_i^k\right\}$, where for all k and i, $\alpha_i^k \ge 0$, $x_i^k \in X$, and $\sum_{i=1}^{n+1} \alpha_i^k = 1$. Since the sequence

$$\big\{(\alpha_1^k, \ldots, \alpha_{n+1}^k, x_1^k, \ldots, x_{n+1}^k)\big\}$$

belongs to a compact set, it has a limit point $\big\{(\alpha_1, \ldots, \alpha_{n+1}, x_1, \ldots, x_{n+1})\big\}$ such that $\sum_{i=1}^{n+1} \alpha_i = 1$, and for all i, $\alpha_i \ge 0$, and $x_i \in X$. Thus, the vector $\sum_{i=1}^{n+1} \alpha_i x_i$, which belongs to conv$(X)$, is a limit point of the sequence $\left\{\sum_{i=1}^{n+1} \alpha_i^k x_i^k\right\}$, showing that conv$(X)$ is compact. **Q.E.D.**

1.2.3 Closure, Relative Interior, and Continuity

We now consider some generic topological properties of convex sets and functions. Let C be a nonempty convex subset of $\Re^n$. The closure cl(C) of C is also a nonempty convex set (Prop. 1.2.1). While the interior of C

may be empty, it turns out that convexity implies the existence of interior points relative to the affine hull of C. This is an important property, which we now formalize.

We say that x is a *relative interior point* of C, if $x \in C$ and there exists a neighborhood N of x such that $N \cap \text{aff}(C) \subset C$, i.e., x is an interior point of C relative to aff(C). The *relative interior of* C, denoted ri(C), is the set of all relative interior points of C. For example, if C is a line segment connecting two distinct points in the plane, then ri(C) consists of all points of C except for the end points. The set C is said to be *relatively open* if $\text{ri}(C) = C$. A point $x \in \text{cl}(C)$ such that $x \notin \text{ri}(C)$ is said to be a *relative boundary point* of C. The set of all relative boundary points of C is called the *relative boundary* of C.

The following proposition gives some basic facts about relative interior points.

Proposition 1.2.9: Let C be a nonempty convex set.

(a) (*Line Segment Principle*) If $x \in \text{ri}(C)$ and $\overline{x} \in \text{cl}(C)$, then all points on the line segment connecting x and $\overline{x}$, except possibly $\overline{x}$, belong to ri(C).

(b) (*Nonemptiness of Relative Interior*) ri(C) is a nonempty and convex set, and has the same affine hull as C. In fact, if m is the dimension of aff(C) and $m > 0$, there exist vectors $x_0, x_1, \ldots, x_m \in \text{ri}(C)$ such that $x_1 - x_0, \ldots, x_m - x_0$ span the subspace parallel to aff(C).

(c) $x \in \text{ri}(C)$ if and only if every line segment in C having x as one endpoint can be prolonged beyond x without leaving C [i.e., for every $\overline{x} \in C$, there exists a $\gamma > 1$ such that $x + (\gamma - 1)(x - \overline{x}) \in C$].

Proof: (a) In the case where $\overline{x} \in C$, see Fig. 1.2.5. In the case where $\overline{x} \notin C$, to show that for any $\alpha \in (0, 1]$ we have $x_\alpha = \alpha x + (1 - \alpha)\overline{x} \in \text{ri}(C)$, consider a sequence $\{x_k\} \subset C$ that converges to $\overline{x}$, and let $x_{k,\alpha} = \alpha x + (1 - \alpha)x_k$. Then as in Fig. 1.2.5, we see that $\{z \mid \|z - x_{k,\alpha}\| < \alpha\epsilon\} \cap \text{aff}(C) \subset C$ for all k. Since for large enough k, we have

$$\{z \mid \|z - x_\alpha\| < \alpha\epsilon/2\} \subset \{z \mid \|z - x_{k,\alpha}\| < \alpha\epsilon\},$$

it follows that $\{z \mid \|z - x_\alpha\| < \alpha\epsilon/2\} \cap \text{aff}(C) \subset C$, which shows that $x_\alpha \in \text{ri}(C)$.

(b) Convexity of ri(C) follows from the Line Segment Principle of part (a). By using a translation argument if necessary, we assume without loss of generality that $0 \in C$. Then, the affine hull of C is a subspace of dimension m. If $m = 0$, then C and aff(C) consist of a single point, which is a unique

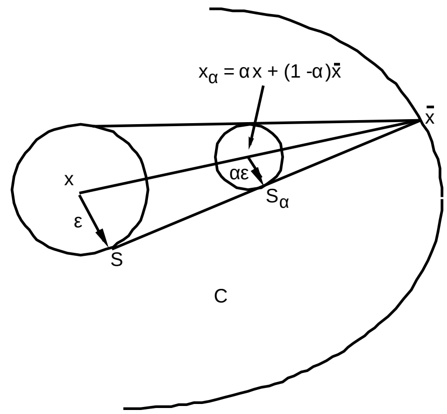

Figure 1.2.5. Proof of the Line Segment Principle for the case where $\overline{x} \in C$. Since $x \in \mathrm{ri}(C)$, there exists a sphere $S = \{z \mid \|z - x\| < \epsilon\}$ such that $S \cap \mathrm{aff}(C) \subset C$. For all $\alpha \in (0, 1]$, let $x_\alpha = \alpha x + (1 - \alpha)\overline{x}$ and let $S_\alpha = \{z \mid \|z - x_\alpha\| < \alpha\epsilon\}$. It can be seen that each point of $S_\alpha \cap \mathrm{aff}(C)$ is a convex combination of $\overline{x}$ and some point of $S \cap \mathrm{aff}(C)$. Therefore, $S_\alpha \cap \mathrm{aff}(C) \subset C$, implying that $x_\alpha \in \mathrm{ri}(C)$.

relative interior point. If $m > 0$, we can find m linearly independent vectors $z_1, \ldots, z_m$ in C that span $\mathrm{aff}(C)$; otherwise there would exist $r < m$ linearly independent vectors in C whose span contains C, contradicting the fact that the dimension of $\mathrm{aff}(C)$ is m. Thus $z_1, \ldots, z_m$ form a basis for $\mathrm{aff}(C)$.

Consider the set

$$X = \left\{ x \;\middle|\; x = \sum_{i=1}^{m} \alpha_i z_i,\ \sum_{i=1}^{m} \alpha_i < 1,\ \alpha_i > 0,\ i = 1, \ldots, m \right\}$$

(see Fig. 1.2.6). This set is open relative to $\mathrm{aff}(C)$, i.e., for every $x \in X$, there exists an open set N such that $x \in N$ and $N \cap \mathrm{aff}(C) \subset X$. [To see this, note that X is the inverse image of the open set in $\Re^m$

$$\left\{ (\alpha_1, \ldots, \alpha_m) \;\middle|\; \sum_{i=1}^{m} \alpha_i < 1,\ \alpha_i > 0,\ i = 1, \ldots, m \right\}$$

under the linear transformation from $\mathrm{aff}(C)$ to $\Re^m$ that maps $\sum_{i=1}^{m} \alpha_i z_i$ into $(\alpha_1, \ldots, \alpha_m)$; openness of the above set follows by continuity of linear transformation.] Therefore all points of X are relative interior points of C, and $\mathrm{ri}(C)$ is nonempty. Since by construction, $\mathrm{aff}(X) = \mathrm{aff}(C)$ and $X \subset \mathrm{ri}(C)$, it follows that $\mathrm{ri}(C)$ and C have the same affine hull.

To show the last assertion of part (b), consider vectors

$$x_0 = \alpha \sum_{i=1}^{m} z_i, \qquad x_i = x_0 + \alpha z_i, \quad i = 1, \ldots, m,$$

where α is a positive scalar such that $\alpha(m+1) < 1$. The vectors $x_0, \ldots, x_m$ are in the set X and in the relative interior of C, since $X \subset \mathrm{ri}(C)$. Furthermore, because $x_i - x_0 = \alpha z_i$ for all i and vectors $z_1, \ldots, z_m$ span $\mathrm{aff}(C)$, the vectors $x_1 - x_0, \ldots, x_m - x_0$ also span $\mathrm{aff}(C)$.

(c) If $x \in \mathrm{ri}(C)$, the given condition holds by the Line Segment Principle. Conversely, let x satisfy the given condition. We will show that $x \in \mathrm{ri}(C)$. By part (b), there exists a vector $\overline{x} \in \mathrm{ri}(C)$. We may assume that $\overline{x} \neq x$, since otherwise we are done. By the given condition, since $\overline{x}$ is in C, there is a $\gamma > 1$ such that $y = x + (\gamma - 1)(x - \overline{x}) \in C$. Then we have $x = (1 - \alpha)\overline{x} + \alpha y$, where $\alpha = 1/\gamma \in (0, 1)$, so by part (a), we obtain $x \in \mathrm{ri}(C)$. **Q.E.D.**

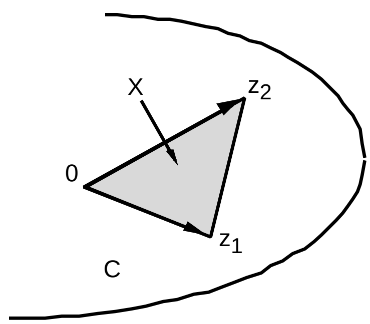

Figure 1.2.6. Construction of the relatively open set X in the proof of nonemptiness of the relative interior of a convex set C that contains the origin. We choose m linearly independent vectors $z_1, \ldots, z_m \in C$, where m is the dimension of $\mathrm{aff}(C)$, and let

$$X = \left\{ \sum_{i=1}^{m} \alpha_i z_i \;\middle|\; \sum_{i=1}^{m} \alpha_i < 1,\ \alpha_i > 0,\ i = 1, \ldots, m \right\}.$$

In view of Prop. 1.2.9(b), C and $\mathrm{ri}(C)$ all have the same dimension. It can also be shown that C and $\mathrm{cl}(C)$ have the same dimension (see the exercises). The next proposition gives several properties of closures and relative interiors of convex sets.

Proposition 1.2.10: Let C be a nonempty convex set.

(a) $\mathrm{cl}(C) = \mathrm{cl}\big(\mathrm{ri}(C)\big)$.

(b) $\mathrm{ri}(C) = \mathrm{ri}\big(\mathrm{cl}(C)\big)$.

(c) Let $\overline{C}$ be another nonempty convex set. Then the following three conditions are equivalent:

(i) C and $\overline{C}$ have the same relative interior.

(ii) C and $\overline{C}$ have the same closure.

(iii) $\mathrm{ri}(C) \subset \overline{C} \subset \mathrm{cl}(C)$.

(d) $A \cdot \mathrm{ri}(C) = \mathrm{ri}(A \cdot C)$ for all $m \times n$ matrices A.

(e) If C is bounded, then $A \cdot \mathrm{cl}(C) = \mathrm{cl}(A \cdot C)$ for all $m \times n$ matrices A.

Proof: (a) Since $\mathrm{ri}(C) \subset C$, we have $\mathrm{cl}\big(\mathrm{ri}(C)\big) \subset \mathrm{cl}(C)$. Conversely, let $\overline{x} \in \mathrm{cl}(C)$. We will show that $\overline{x} \in \mathrm{cl}\big(\mathrm{ri}(C)\big)$. Let x be any point in $\mathrm{ri}(C)$ [there exists such a point by Prop. 1.2.9(b)], and assume that that $\overline{x} \neq x$ (otherwise we are done). By the Line Segment Principle [Prop. 1.2.9(a)], we have $\alpha x + (1-\alpha)\overline{x} \in \mathrm{ri}(C)$ for all $\alpha \in (0, 1]$. Thus, $\overline{x}$ is the limit of the sequence $\big\{(1/k)x + (1 - 1/k)\overline{x} \mid k \geq 1\big\}$ that lies in $\mathrm{ri}(C)$, so $\overline{x} \in \mathrm{cl}\big(\mathrm{ri}(C)\big)$.

(b) The inclusion $C \subset \mathrm{cl}(C)$ follows from the definition of a relative interior point and the fact $\mathrm{aff}(C) = \mathrm{aff}\big(\mathrm{cl}(C)\big)$ (see the exercises). To prove the reverse inclusion, let $z \in \mathrm{ri}\big(\mathrm{cl}(C)\big)$. We will show that $z \in \mathrm{ri}(C)$. By Prop. 1.2.9(b), there exists an $x \in \mathrm{ri}(C)$. We may assume that $x \neq z$ (otherwise we are done). We choose $\gamma > 1$, with γ sufficiently close to 1 so that the vector $y = z + (\gamma - 1)(z - x)$ belongs to $\mathrm{ri}\big(\mathrm{cl}(C)\big)$ [cf. Prop. 1.2.9(c)], and hence also to $\mathrm{cl}(C)$. Then we have $z = (1-\alpha)x + \alpha y$ where $\alpha = 1/\gamma \in (0, 1)$, so by the Line Segment Principle [Prop. 1.2.9(a)], we obtain $z \in \mathrm{ri}(C)$.

(c) If $\mathrm{ri}(C) = \mathrm{ri}(\overline{C})$, part (a) implies that $\mathrm{cl}(C) = \mathrm{cl}(\overline{C})$. Similarly, if $\mathrm{cl}(C) = \mathrm{cl}(\overline{C})$, part (b) implies that $\mathrm{ri}(C) = \mathrm{ri}(\overline{C})$. Furthermore, if any of these conditions hold the relation $\mathrm{ri}(\overline{C}) \subset \overline{C} \subset \mathrm{cl}(\overline{C})$ implies condition (iii). Finally, assume that condition (iii) holds. Then by taking closures, we have $\mathrm{cl}\big(\mathrm{ri}(C)\big) \subset \mathrm{cl}(\overline{C}) \subset \mathrm{cl}(C)$, and by using part (a), we obtain $\mathrm{cl}(C) \subset \mathrm{cl}(\overline{C}) \subset \mathrm{cl}(C)$. Hence C and $\overline{C}$ have the same closure.

(d) For any set X, we have $A \cdot \mathrm{cl}(X) \subset \mathrm{cl}(A \cdot X)$, since if a sequence $\{x_k\} \subset X$ converges to some $x \in \mathrm{cl}(X)$ then the sequence $\{Ax_k\} \subset A \cdot X$ converges to Ax, implying that $Ax \in \mathrm{cl}(A \cdot X)$. We use this fact and part (a) to write

$$A \cdot \mathrm{ri}(C) \subset A \cdot C \subset A \cdot \mathrm{cl}(C) = A \cdot \mathrm{cl}\big(\mathrm{ri}(C)\big) \subset \mathrm{cl}\big(A \cdot \mathrm{ri}(C)\big).$$

Thus $A \cdot C$ lies between the set $A \cdot \mathrm{ri}(C)$ and the closure of that set, implying that the relative interiors of the sets $A \cdot C$ and $A \cdot \mathrm{ri}(C)$ are equal [part (c)]. Hence $\mathrm{ri}(A \cdot C) \subset A \cdot \mathrm{ri}(C)$. We will show the reverse inclusion by taking any $z \in A \cdot \mathrm{ri}(C)$ and showing that $z \in \mathrm{ri}(A \cdot C)$. Let x be any vector in $A \cdot C$, and let $\overline{z} \in \mathrm{ri}(C)$ and $\overline{x} \in C$ be such that $A\overline{z} = z$ and $A\overline{x} = x$. By part Prop. 1.2.9(c), there exists $\gamma > 1$ such that the vector $\overline{y} = \overline{z} + (\gamma - 1)(\overline{z} - \overline{x})$ belongs to C. Thus we have $A\overline{y} \in A \cdot C$ and $A\overline{y} = z + (\gamma - 1)(z - x)$, so by Prop. 1.2.9(c) it follows that $z \in \mathrm{ri}(A \cdot C)$.

(e) By the argument given in part (d), we have $A \cdot \mathrm{cl}(C) \subset \mathrm{cl}(A \cdot C)$. To show the converse, choose any $x \in \mathrm{cl}(A \cdot C)$. Then, there exists a sequence $\{\overline{x}_k\} \subset C$ such that $A\overline{x}_k \to x$. Since C is bounded, $\{\overline{x}_k\}$ has a subsequence that converges to some $\overline{x} \in \mathrm{cl}(C)$, and we must have $A\overline{x} = x$. It follows that $x \in A \cdot \mathrm{cl}(C)$. **Q.E.D.**

Note that if C is closed but unbounded, the set $A \cdot C$ need not be closed [cf. part (e) of the above proposition]. For example take the closed set $C = \big\{(x_1, x_2) \mid x_1 x_2 \geq 1,\, x_1 \geq 0,\, x_2 \geq 0\big\}$ and let A have the effect of projecting the typical vector x on the horizontal axis, i.e., $A \cdot (x_1, x_2) = (x_1, 0)$. Then $A \cdot C$ is the (nonclosed) halfline $\big\{(x_1, x_2) \mid x_1 > 0,\, x_2 = 0\big\}$.

Proposition 1.2.11: Let C_1 and C_2 be nonempty convex sets:

(a) Assume that the sets $\mathrm{ri}(C_1)$ and $\mathrm{ri}(C_2)$ have a nonempty intersection. Then

$$\mathrm{cl}(C_1 \cap C_2) = \mathrm{cl}(C_1) \cap \mathrm{cl}(C_2), \qquad \mathrm{ri}(C_1 \cap C_2) = \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2).$$

(b) $\mathrm{ri}(C_1 + C_2) = \mathrm{ri}(C_1) + \mathrm{ri}(C_2)$.

Proof: (a) Let $y \in \mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)$. Let x be a vector in the intersection $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$ (which is nonempty by assumption). By the Line Segment Principle [Prop. 1.2.9(a)], the vector $\alpha x + (1 - \alpha) y$ belongs to $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$ for all $\alpha \in (0, 1]$. Hence y is the limit of a sequence $\alpha_k x + (1 - \alpha_k) y \subset \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$ with $\alpha_k \to 0$, implying that $y \in \mathrm{cl}\big(\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)\big)$. Hence, we have

$$\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2) \subset \mathrm{cl}\big(\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)\big) \subset \mathrm{cl}(C_1 \cap C_2).$$

Also $C_1 \cap C_2$ is contained in $\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)$, which is a closed set, so we have

$$\mathrm{cl}(C_1 \cap C_2) \subset \mathrm{cl}(C_1) \cap \mathrm{cl}(C_2).$$

Thus, equality holds throughout in the preceding two relations, so that $\mathrm{cl}(C_1 \cap C_2) = \mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)$. Furthermore, the sets $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$ and

$C_1 \cap C_2$ have the same closure. Therefore, by Prop. 1.2.10(c), they have the same relative interior, implying that

$$\mathrm{ri}(C_1 \cap C_2) \subset \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2).$$

To show the converse, take any $x \in \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$ and any $y \in C_1 \cap C_2$. By Prop. 1.2.9(c), the line segment connecting x and y can be prolonged beyond x by a small amount without leaving $C_1 \cap C_2$. By the same proposition, it follows that $x \in \mathrm{ri}(C_1 \cap C_2)$.

(b) Consider the linear transformation $A : \Re^{2n} \mapsto \Re^n$ given by $A(x_1, x_2) = x_1 + x_2$ for all $x_1, x_2 \in \Re^n$. The relative interior of the Cartesian product $C_1 \times C_2$ (viewed as a subset of $\Re^{2n}$) is easily seen to be $\mathrm{ri}(C_1) \times \mathrm{ri}(C_2)$ (see the exercises). Since $A(C_1 \times C_2) = C_1 + C_2$, the result follows from Prop. 1.2.10(d). **Q.E.D.**

The requirement that $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) \neq \emptyset$ is essential in part (a) of the above proposition. As an example, consider the subsets of the real line $C_1 = \{x \mid x > 0\}$ and $C_2 = \{x \mid x < 0\}$. Then we have $\mathrm{cl}(C_1 \cap C_2) = \emptyset \neq \{0\} = \mathrm{cl}(C_1) \cap \mathrm{cl}(C_2)$. Also, consider $C_1 = \{x \mid x \geq 0\}$ and $C_2 = \{x \mid x \leq 0\}$. Then we have $\mathrm{ri}(C_1 \cap C_2) = \{0\} \neq \emptyset = \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$.

Continuity of Convex Functions

We close this section with a basic result on the continuity properties of convex functions.

Proposition 1.2.12: If $f : \Re^n \mapsto \Re$ is convex, then it is continuous. More generally, if $f : \Re^n \mapsto (-\infty, \infty]$ is convex, then f when restricted to $\mathrm{dom}(f)$, is continuous over the relative interior of $\mathrm{dom}(f)$.

Proof: Restricting attention to the affine hull of $\mathrm{dom}(f)$ and using a transformation argument if necessary, we assume without loss of generality, that the origin is an interior point of $\mathrm{dom}(f)$ and that the unit cube $X = \{x \mid \|x\|_\infty \leq 1\}$ is contained in $\mathrm{dom}(f)$. It will suffice to show that f is continuous at 0, i.e, that for any sequence $\{x_k\} \subset \Re^n$ that converges to 0, we have $f(x_k) \to f(0)$.

Let e_i, $i = 1, \ldots, 2^n$, be the corners of X, i.e., each e_i is a vector whose entries are either 1 or -1. It is not difficult to see that any $x \in X$ can be expressed in the form $x = \sum_{i=1}^{2^n} \alpha_i e_i$, where each α_i is a nonnegative scalar and $\sum_{i=1}^{2^n} \alpha_i = 1$. Let $A = \max_i f(e_i)$. From Jensen's inequality [Eq. (1.8)], it follows that $f(x) \leq A$ for every $x \in X$.

For the purpose of proving continuity at zero, we can assume that $x_k \in X$ and $x_k \neq 0$ for all k. Consider the sequences $\{y_k\}$ and $\{z_k\}$ given by

$$y_k = \frac{x_k}{\|x_k\|_\infty}, \qquad z_k = -\frac{x_k}{\|x_k\|_\infty};$$

(cf. Fig. 1.2.7). Using the definition of a convex function for the line segment that connects y_k, x_k, and 0, we have

$$f(x_k) \leq \big(1 - \|x_k\|_\infty\big) f(0) + \|x_k\|_\infty f(y_k).$$

Since $\|x_k\|_\infty \to 0$ and $f(y_k) \leq A$ for all k, by taking the limit as $k \to \infty$, we obtain

$$\limsup_{k\to\infty} f(x_k) \leq f(0).$$

Using the definition of a convex function for the line segment that connects x_k, 0, and z_k, we have

$$f(0) \leq \frac{\|x_k\|_\infty}{\|x_k\|_\infty + 1} f(z_k) + \frac{1}{\|x_k\|_\infty + 1} f(x_k)$$

and letting $k \to \infty$, we obtain

$$f(0) \leq \liminf_{k\to\infty} f(x_k).$$

Thus, $\lim_{k\to\infty} f(x_k) = f(0)$ and f is continuous at zero. **Q.E.D.**

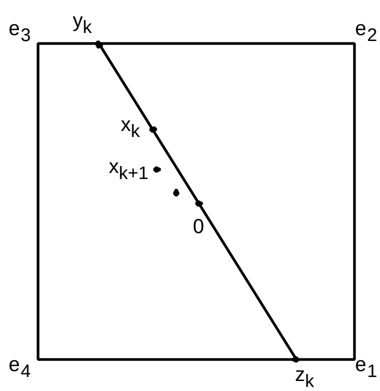

Figure 1.2.7. Construction for proving continuity of a convex function (cf. Prop. 1.2.12).

A straightforward consequence of the continuity of a real-valued function f that is convex over $\Re^n$ is that its epigraph as well as the level sets $\big\{x \mid f(x) \leq \gamma\big\}$ are closed and convex (cf. Prop. 1.2.2).

1.2.4 Recession Cones

Some of the preceding results [Props. 1.2.8, 1.2.10(e)] have illustrated how boundedness affects the topological properties of sets obtained through various operations on convex sets. In this section we take a closer look at this issue.

Given a convex set C, we say that a vector y is a *direction of recession* of C if $x + \alpha y \in C$ for all $x \in C$ and $\alpha \geq 0$. In words, y is a direction of recession of C if starting at any x in C and going indefinitely along y, we never cross the relative boundary of C to points outside C. The set of all directions of recession is a cone containing the origin. It is called the *recession cone* of C and it is denoted by R_C (see Fig. 1.2.8). The following proposition gives some properties of recession cones.

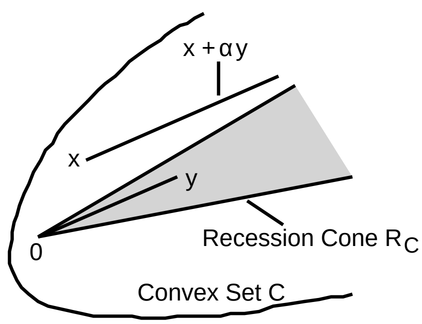

Figure 1.2.8. Illustration of the recession cone R_C of a convex set C. A direction of recession y has the property that $x + \alpha y \in C$ for all $x \in C$ and $\alpha \geq 0$.

Proposition 1.2.13: (Recession Cone Theorem) Let C be a nonempty closed convex set.

(a) The recession cone R_C is a closed convex cone.

(b) A vector y belongs to R_C if and only if there exists a vector $x \in C$ such that $x + \alpha y \in C$ for all $\alpha \geq 0$.

(c) R_C contains a nonzero direction if and only if C is unbounded.

(d) The recession cones of C and $\mathrm{ri}(C)$ are equal.

(e) If D is another closed convex set such that $C \cap D \neq \emptyset$, we have

$$R_{C\cap D} = R_C \cap R_D.$$

Proof: (a) If y_1, y_2 belong to R_C and λ_1, λ_2 are positive scalars such that $\lambda_1 + \lambda_2 = 1$, we have for any $x \in C$ and $\alpha \geq 0$

$$x + \alpha(\lambda_1 y_1 + \lambda_2 y_2) = \lambda_1(x + \alpha y_1) + \lambda_2(x + \alpha y_2) \in C,$$

where the last inclusion holds because $x + \alpha y_1$ and $x + \alpha y_2$ belong to C by the definition of R_C. Hence $\lambda_1 y_1 + \lambda_2 y_2 \in R_C$, implying that R_C is convex.

Let y be in the closure of R_C, and let $\{y_k\} \subset R_C$ be a sequence converging to y. For any $x \in C$ and $\alpha \geq 0$ we have $x + \alpha y_k \in C$ for all k, and because C is closed, we have $x + \alpha y \in C$. This implies that $y \in R_C$ and that R_C is closed.

(b) If $y \in R_C$, every vector $x \in C$ has the required property by the definition of R_C. Conversely, let y be such that there exists a vector $x \in C$ with $x + \alpha y \in C$ for all $\alpha \geq 0$. We fix $\overline{x} \in C$ and $\alpha > 0$, and we show that $\overline{x} + \alpha y \in C$. We may assume that $y \neq 0$ (otherwise we are done) and without loss of generality, we may assume that $\|y\| = 1$. Let

$$z_k = x + k\alpha y, \qquad k = 1, 2, \ldots.$$

If $\overline{x} = z_k$ for some k, then $\overline{x} + \alpha y = x + \alpha(k+1)y$ which belongs to C and we are done. We thus assume that $\overline{x} \neq z_k$ for all k, and we define

$$y_k = \frac{z_k - \overline{x}}{\|z_k - \overline{x}\|}, \qquad k = 1, 2, \ldots$$

(see the construction of Fig. 1.2.9).

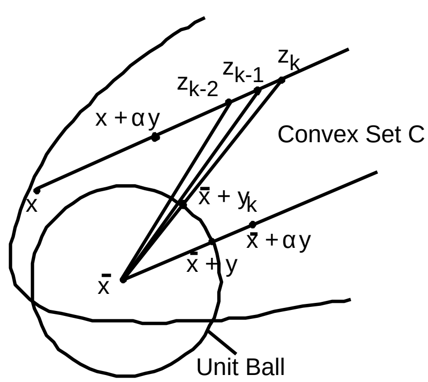

Figure 1.2.9. Construction for the proof of Prop. 1.2.13(b).

We have

$$y_k = \frac{\|z_k - x\|}{\|z_k - \overline{x}\|} \cdot \frac{z_k - x}{\|z_k - x\|} + \frac{x - \overline{x}}{\|z_k - \overline{x}\|} = \frac{\|z_k - x\|}{\|z_k - \overline{x}\|} \cdot y + \frac{x - \overline{x}}{\|z_k - \overline{x}\|}.$$

Because z_k is an unbounded sequence,

$$\frac{\|z_k - x\|}{\|z_k - \overline{x}\|} \to 1, \qquad \frac{x - \overline{x}}{\|z_k - \overline{x}\|} \to 0,$$

so by combining the preceding relations, we have $y_k \to y$. Thus $\overline{x} + \alpha y$ is the limit of $\{\overline{x} + \alpha y_k\}$. The vector $\overline{x} + \alpha y_k$ lies between $\overline{x}$ and z_k in the line segment connecting $\overline{x}$ and z_k for all k such that $\|z_k - \overline{x}\| \geq \alpha$, so by convexity of C, we have $\overline{x} + \alpha y_k \in C$ for all sufficiently large k. Since $\overline{x} + \alpha y_k \to \overline{x} + \alpha y$ and C is closed, it follows that $\overline{x} + \alpha y$ must belong to C.

(c) Assuming that C is unbounded, we will show that R_C contains a nonzero direction (the reverse is clear). Choose any $\overline{x} \in C$ and any unbounded sequence $\{z_k\} \subset C$. Consider the sequence $\{y_k\}$, where

$$y_k = \frac{z_k - \overline{x}}{\|z_k - \overline{x}\|},$$

and let y be a limit point of $\{y_k\}$ (compare with the construction of Fig. 1.2.9). For any fixed $\alpha \geq 0$, the vector $\overline{x} + \alpha y_k$ lies between $\overline{x}$ and z_k in the line segment connecting $\overline{x}$ and z_k for all k such that $\|z_k - \overline{x}\| \geq \alpha$. Hence by convexity of C, we have $\overline{x} + \alpha y_k \in C$ for all sufficiently large k. Since $\overline{x} + \alpha y$ is a limit point of $\{\overline{x} + \alpha y_k\}$, and C is closed, we have $\overline{x} + \alpha y \in C$. Hence the nonzero vector y is a direction of recession.

(d) If $y \in R_{\mathrm{ri}(C)}$, then for a fixed $x \in \mathrm{ri}(C)$ and all $\alpha \geq 0$, we have $x + \alpha y \in \mathrm{ri}(C) \subset C$. Hence, by part (b), we have $y \in R_C$. Conversely, if $y \in R_C$, for any $x \in \mathrm{ri}(C)$, we have $x + \alpha y \in C$ for all $\alpha \geq 0$. It follows from the Line Segment Principle [cf. Prop. 1.2.9(a)] that $x + \alpha y \in \mathrm{ri}(C)$ for all $\alpha \geq 0$, so that y belongs to $R_{\mathrm{ri}(C)}$.

(e) By the definition of direction of recession, $y \in R_{C\cap D}$ implies that $x + \alpha y \in C \cap D$ for all $x \in C \cap D$ and all $\alpha > 0$. By part (b), this in turn implies that $y \in R_C$ and $y \in R_D$, so that $R_{C\cap D} \subset R_C \cap R_D$. Conversely, by the definition of direction of recession, if $y \in R_C \cap R_D$ and $x \in C \cap D$, we have $x + \alpha y \in C \cap D$ for all $\alpha > 0$, so $y \in R_{C\cap D}$. Thus, $R_C \cap R_D \subset R_{C\cap D}$. **Q.E.D.**

Note that part (c) of the above proposition yields a characterization of compact and convex sets, namely that a closed convex set is bounded if and only if $R_C = \{0\}$. A useful generalization is that for a compact set $W \subset \Re^m$ and an $m \times n$ matrix A, the set

$$V = \{x \in C \mid Ax \in W\}$$

is compact if and only if $R_C \cap N(A) = \{0\}$. To see this, note that the recession cone of the set

$$\overline{V} = \{x \in \Re^n \mid Ax \in W\}$$

is $N(A)$ [clearly $N(A) \subset R_{\overline{V}}$; if $x \notin N(A)$ but $x \in R_{\overline{V}}$ we must have $\alpha Ax \in W$ for all $\alpha > 0$, which contradicts the boundedness of W]. Hence, the recession cone of V is $R_C \cap N(A)$, so by Prop. 1.2.13(c), V is compact if and only if $R_C \cap N(A) = \{0\}$.

One possible use of recession cones is to obtain conditions guaranteeing the closure of linear transformations and vector sums of convex sets in the absence of boundedness, as in the following two propositions (some refinements are given in the exercises).

Proposition 1.2.14: Let C be a nonempty closed convex subset of $\Re^n$ and let A be an $m \times n$ matrix with nullspace $N(A)$ such that $R_C \cap N(A) = \{0\}$. Then AC is closed.

Proof: For any $y \in \mathrm{cl}(AC)$, the set

$$C_\epsilon = \big\{x \in C \mid \|y - Ax\| \le \epsilon\big\}$$

is nonempty for all $\epsilon > 0$. Furthermore, by the discussion following the proof of Prop. 1.2.13, the assumption $R_C \cap N(A) = \{0\}$ implies that C_ϵ is compact. It follows that the set $\cap_{\epsilon>0} C_\epsilon$ is nonempty and any $x \in \cap_{\epsilon>0} C_\epsilon$ satisfies $Ax = y$, so $y \in AC$. **Q.E.D.**

Proposition 1.2.15: Let $C_1, \ldots, C_m$ be nonempty closed convex subsets of $\Re^n$ such that the equality $y_1 + \cdots + y_m = 0$ for some vectors $y_i \in R_{C_i}$ implies that $y_i = 0$ for all $i = 1, \ldots, m$. Then the vector sum $C_1 + \cdots + C_m$ is a closed set.

Proof: Let C be the Cartesian product $C_1 \times \cdots \times C_m$ viewed as a subset of $\Re^{mn}$ and let A be the linear transformation that maps a vector $(x_1, \ldots, x_m) \in \Re^{mn}$ into $x_1 + \cdots + x_m$. We have

$$R_C = R_{C_1} + \cdots + R_{C_m}$$

(see the exercises) and

$$N(A) = \big\{(y_1, \ldots, y_m) \mid y_1 + \cdots + y_m = 0,\, y_i \in \Re^n\big\},$$

so under the given condition, we obtain $R_C \cap N(A) = \{0\}$. Since $AC = C_1 + \cdots + C_m$, the result follows by applying Prop. 1.2.14. **Q.E.D.**

When specialized to just two sets C_1 and C_2, the above proposition says that if there is no nonzero direction of recession of C_1 that is the opposite of a direction of recession of C_2, then $C_1 + C_2$ is closed. This is true in particular if $R_{C_1} = \{0\}$ which is equivalent to C_1 being compact [cf. Prop. 1.2.13(c)]. We thus obtain the following proposition.

Proposition 1.2.16: Let C_1 and C_2 be closed, convex sets. If C_1 is bounded, then $C_1 + C_2$ is a closed and convex set. If both C_1 and C_2 are bounded, then $C_1 + C_2$ is a compact and convex set.

Proof: Closedness of $C_1 + C_2$ follows from the preceding discussion. If both C_1 and C_2 are bounded, then $C_1 + C_2$ is also bounded and hence also compact. **Q.E.D.**

Note that if C_1 and C_2 are both closed and unbounded, the vector sum $C_1 + C_2$ need not be closed. For example consider the closed sets of $\Re^2$ given by $C_1 = \big\{(x_1, x_2) \mid x_1 x_2 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0\big\}$ and $C_2 = \big\{(x_1, x_2) \mid x_1 = 0\big\}$. Then $C_1 + C_2$ is the open halfspace $\{(x_1, x_2) \mid x_1 > 0\}$.

E X E R C I S E S

1.2.1

(a) Show that a set is convex if and only if it contains all the convex combinations of its elements.

(b) Show that the convex hull of a set coincides with the set of all the convex combinations of its elements.

1.2.2

Let C be a nonempty set in $\Re^n$, and let λ_1 and λ_2 be positive scalars. Show by example that the sets $(\lambda_1 + \lambda_2)C$ and $\lambda_1 C + \lambda_2 C$ may differ when C is not convex [cf. Prop. 1.2.1].

1.2.3 (Properties of Cones)

(a) For any collection $\{C_i \mid i \in I\}$ of cones, the intersection $\cap_{i \in I} C_i$ is a cone.

(b) The Cartesian product $C_1 \times C_2$ of two cones C_1 and C_2 is a cone.

(c) The vector sum $C_1 + C_2$ of two cones C_1 and C_2 is a cone.

(d) The closure of a cone is a cone.

(e) The image and the inverse image of a cone under a linear transformation is a cone.

1.2.4 (Convex Cones)

(a) For any collection of vectors $\{a_i \mid i \in I\}$, the set $C = \{x \mid a_i'x \leq 0,\, i \in I\}$ is a closed convex cone.

(b) Show that a cone C is convex if and only if $C + C \subset C$.

(c) Let C_1 and C_2 be convex cones containing the origin. Show that

$$C_1 + C_2 = \text{conv}(C_1 \cup C_2),$$

$$C_1 \cap C_2 = \bigcup_{\alpha \in [0,1]} \big((1-\alpha)C_1 \cap \alpha C_2\big).$$

1.2.5 (Properties of Cartesian Product)

Given sets $X_i \subset \Re^{n_i}$, $i = 1, \ldots, m$, let $X = X_1 \times \cdots \times X_m$ be their Cartesian product.

(a) Show that the convex hull (closure, affine hull) of X is equal to the Cartesian product of the convex hulls (closures, affine hulls, respectively) of the X_i's.

(b) Show that cone(X) is equal to the Cartesian product of cone(X_i).

(c) Assuming $X_1, \ldots, X_m$ are convex, show that the relative interior (recession cone) of X is equal to the Cartesian product of the relative interiors (recession cones) of the X_i's.

1.2.6

Let $\{C_i \mid i \in I\}$ be an arbitrary collection of convex sets in $\Re^n$, and let C be the convex hull of the union of the collection. Show that

$$C = \bigcup \left(\sum_{i \in I} \alpha_i C_i \right),$$

where the union is taken over all convex combinations such that only finitely many coefficients α_i are nonzero.

1.2.7

Let X be a nonempty set.

(a) Show that X, $\mathrm{conv}(X)$, and $\mathrm{cl}(X)$ have the same dimension.

(b) Show that $\mathrm{cone}(X) = \mathrm{cone}\big(\mathrm{conv}(X)\big)$.

(c) Show that the dimension of $\mathrm{conv}(X)$ is at most as large as the dimension of $\mathrm{cone}(X)$. Give an example where the dimension of $\mathrm{conv}(X)$ is smaller than the dimension of $\mathrm{cone}(X)$.

(d) Assuming that the origin belongs to $\mathrm{conv}(X)$, show that $\mathrm{conv}(X)$ and $\mathrm{cone}(X)$ have the same dimension.

1.2.8 (Lower Semicontinuity under Composition)

(a) Let $f : \Re^n \to \Re^m$ be a continuous function and $g : \Re^m \to \Re$ be a lower semicontinuous function. Show that the function h defined by $h(x) = g\big(f(x)\big)$ is lower semicontinuous.

(b) Let $f : \Re^n \to \Re$ be a lower semicontinuous function, and $g : \Re \to \Re$ be a lower semicontinuous and monotonically nondecreasing function. Show that the function h defined by $h(x) = g\big(f(x)\big)$ is lower semicontinuous.

1.2.9 (Convexity under Composition)

(a) Let f be a convex function defined on a convex set $C \subset \Re^n$, and let g be a convex monotonically nondecreasing function of a single variable [i.e., $g(y) \le g(\overline{y})$ for $y < \overline{y}$]. Show that the function h defined by $h(x) = g\big(f(x)\big)$ is convex over C. In addition, if g is monotonically increasing and f is strictly convex, then h is strictly convex.

(b) Let $f = (f_1, \ldots, f_m)$ where each $f_i : \Re^n \to \Re$ is a convex function, and let $g : \Re^m \to \Re$ be a function such that $g(u_1, \ldots, u_m)$ is convex and nondecreasing in u_i for each i. Show that the function h defined by $h(x) = g\big(f(x)\big)$ is convex.

1.2.10 (Convex Functions)

Show that the following functions are convex:

(a) $f_1 : X \to \Re$ is given by

$$f_1(x_1, \ldots, x_n) = -(x_1 x_2 \cdots x_n)^{\frac{1}{n}}$$

where $X = \{(x_1, \ldots, x_n) \mid x_1 \ge, \ldots, x_n \ge 0\}$.

(b) $f_2(x) = \ln\big(e^{x_1} + \cdots + e^{x_n}\big)$ with $(x_1, \ldots, x_n) \in \Re^n$.

(c) $f_3(x) = \|x\|^p$ with $p \ge 1$ and $x \in \Re^n$.

(d) $f_4(x) = \frac{1}{f(x)}$ with f a concave function over $\Re^n$ and $f(x) > 0$ for all x.

(e) $f_5(x) = \alpha f(x) + \beta$ with f a convex function over $\Re^n$, and α and β scalars such that $\alpha \geq 0$.

(f) $f_6(x) = \max\{0, f(x)\}$ with f a convex function over $\Re^n$.

(g) $f_7(x) = ||Ax - b||$ with A an $m \times n$ matrix and b a vector in $\Re^m$.

(h) $f_8(x) = x'Ax + b'x + \beta$ with A an $n \times n$ positive semidefinite symmetric matrix, b a vector in $\Re^n$, and β a scalar.

(i) $f_9(x) = e^{\beta x' Ax}$ with A an $n \times n$ positive semidefinite symmetric matrix and β a positive scalar.

(j) $f_{10}(x) = f(Ax + b)$ with f a convex function over $\Re^m$, A an $m \times n$ matrix, and b a vector in $\Re^m$.

1.2.11

Use the Line Segment Principle and the method of proof of Prop. 1.2.5(c) to show that if C is a convex set with nonempty interior, and $f : \Re^n \mapsto \Re$ is convex and twice continuously differentiable over C, then $\nabla^2 f(x)$ is positive semidefinite for all $x \in C$.

1.2.12

Let $C \subset \Re^n$ be a convex set and let $f : \Re^n \mapsto \Re$ be twice continuously differentiable over C. Let S be the subspace that is parallel to the affine hull of C. Show that f is convex over C if and only if $y'\nabla^2 f(x) y \geq 0$ for all $x \in C$ and $y \in S$.

1.2.13

Let $f : \Re^n \mapsto \Re$ be a differentiable function. Show that f is convex over a convex set C if and only if

$$\big(\nabla f(x) - \nabla f(y)\big)'(x - y) \geq 0, \qquad \forall\ x, y \in C.$$

Hint: The condition above says that the function f, restricted to the line segment connecting x and y, has monotonically nondecreasing gradient; see also the proof of Prop. 1.2.6.

1.2.14 (Ascent/Descent Behavior of a Convex Function)

Let $f : \Re \mapsto \Re$ be a convex function of a single variable.

(a) *(Monotropic Property)* Use the definition of convexity to show that f is "turning upwards" in the sense that if x_1, x_2, x_3 are three scalars such that $x_1 < x_2 < x_3$, then

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

(b) Use part (a) to show that there are four possibilities as x increases to ∞: (1) $f(x)$ decreases monotonically to $-\infty$, (2) $f(x)$ decreases monotonically to a finite value, (3) $f(x)$ reaches some value and stays at that value, (4) $f(x)$ increases monotonically to ∞ when $x \geq \overline{x}$ for some $\overline{x} \in \Re$.

1.2.15 (Posynomials)

A *posynomial* of scalar variables $y_1, \ldots, y_n$ is a function of the form

$$g(y_1, \ldots, y_n) = \sum_{i=1}^{m} \beta_i y_1^{a_{i1}} \cdots y_n^{a_{in}},$$

where the scalars y_i and β_i are positive, and the exponents a_{ij} are real numbers. Show the following:

(a) A posynomial need not be convex.

(b) By a logarithmic change of variables, where we set

$$f(x) = \ln\bigl(g(y_1, \ldots, y_n)\bigr), \qquad b_i = \ln \beta_i, \qquad x_i = \ln y_i, \quad \forall\, i = 1, \ldots, m,$$

we obtain a convex function

$$f(x) = \ln \exp(Ax + b)$$

defined for all $x \in \Re^n$, where $\ln \exp(x) = \ln\bigl(e^{x_1} + \cdots + e^{x_n}\bigr)$, A is an $m \times n$ matrix with entries a_{ij}, and $b \in \Re^m$ is a vector with components b_i.

(c) In general, every function $g : \Re^n \to \Re$ of the form

$$g(y) = g_1(y)^{\gamma_1} \cdots g_r(y)^{\gamma_r},$$

where g_k is a posynomial and $\gamma_k > 0$ for all k, can be transformed by the logarithmic change of variables into a convex function f given by

$$f(x) = \sum_{k=1}^{r} \gamma_k \ln \exp(A_k x + b_k),$$

with the matrix A_k and the vector b_k being associated with the posynomial g_k for each k.

1.2.16 (Arithmetic-Geometric Mean Inequality)

Show that if $\alpha_1, \ldots, \alpha_n$ are positive scalars with $\sum_{i=1}^{n} \alpha_i = 1$, then for every set of positive scalars $x_1, \ldots, x_n$, we have

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \leq \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n,$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$. *Hint*: Show that $-\ln x$ is a strictly convex function on $(0, \infty)$.

1.2.17

Use the result of Exercise 1.2.16 to verify Young's inequality

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q},$$

where $p > 0$, $q > 0$, $1/p + 1/q = 1$, $x \geq 0$, and $y \geq 0$. Then, use Young's inequality to verify Holder's inequality

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q \right)^{1/q}.$$

1.2.18 (Convexity under Minimization I)

Let $h : \Re^{n+m} \mapsto \Re$ be a convex function. Consider the function $f : \Re^n \mapsto \Re$ given by

$$f(x) = \inf_{u \in U} h(x, u),$$

where we assume that U is a nonempty and convex subset of $\Re^m$, and that $f(x) > -\infty$ for all x. Show that f is convex, and for each x, the set $\{u \in U \mid h(x, u) = f(x)\}$ is convex. *Hint*: There cannot exist $\alpha \in [0, 1]$, x_1, x_2, $u_1 \in U$, $u_2 \in U$ such that $f\big(\alpha x_1 + (1 - \alpha) x_2\big) > \alpha h(x_1, u_1) + (1 - \alpha) h(x_2, u_2)$.

1.2.19 (Convexity under Minimization II)

(a) Let C be a convex set in $\Re^{n+1}$ and let

$$f(x) = \inf\{w \mid (x, w) \in C\}.$$

Show that f is convex over $\Re^n$.

(b) Let $f_1, \ldots, f_m$ be convex functions over $\Re^n$ and let

$$f(x) = \inf \left\{ \sum_{i=1}^{m} f_i(x_i) \;\middle|\; \sum_{i=1}^{m} x_i = x \right\}.$$

Assuming that $f(x) > -\infty$ for all x, show that f is convex over $\Re^n$.

(c) Let $h : \Re^m \mapsto \Re$ be a convex function and let

$$f(x) = \inf_{By = x} h(y),$$

where B is an $n \times m$ matrix. Assuming that $f(x) > -\infty$ for all x, show that f is convex over the range space of B.

(d) In parts (b) and (c), show by example that if the assumption $f(x) > -\infty$ for all x is violated, then the set $\big\{x \mid f(x) > -\infty\big\}$ need not be convex.

1.2.20 (Convexity under Minimization III)

Let $\{f_i \mid i \in I\}$ be an arbitrary collection of convex functions on $\Re^n$. Define the convex hull of these functions $f : \Re^n \to \Re$ as the pointwise infimum of the collection, i.e.,

$$f(x) = \inf\big\{w \mid (x,w) \in \text{conv}\big(\cup_{i\in I}\text{epi}(f_i)\big)\big\}.$$

Show that $f(x)$ is given by

$$f(x) = \inf\left\{\sum_{i\in I}\alpha_i f_i(x_i) \;\Big|\; \sum_{i\in I}\alpha_i x_i = x\right\},$$

where the infimum is taken over all representations of x as a convex combination of elements x_i, such that only finitely many coefficients α_i are nonzero.

1.2.21 (Convexification of Nonconvex Functions)

Let X be a nonempty subset of $\Re^n$, let $f : X \mapsto \Re$ be a function that is bounded below over X. Define the function $F : \text{conv}(X) \mapsto \Re$ by

$$F(x) = \inf\big\{w \mid (x,w) \in \text{conv}\big(\text{epi}(f)\big)\big\}.$$

(a) Show that F is convex over $\text{conv}(X)$ and it is given by

$$F(x) = \inf\left\{\sum_{i=1}^m \alpha_i f(x_i) \;\Big|\; \sum_{i=1}^m \alpha_i x_i = x,\, x_i \in X,\, \sum_{i=1}^m \alpha_i = 1,\, \alpha_i \geq 0,\, m \geq 1\right\}$$

(b) Show that

$$\inf_{x\in\text{conv}(X)} F(x) = \inf_{x\in X} f(x).$$

(c) Show that the set of global minima of F over $\text{conv}(X)$ includes all global minima of f over X.

1.2.22 (Minimization of Linear Functions)

Show that minimization of a linear function over a set is equivalent to minimization over its convex hull. In particular, if $X \subset \Re^n$ and $c \in \Re^n$, then

$$\inf_{x\in X} c'x = \inf_{\text{conv}(X)} c'x.$$

Furthermore, the infimum in the left-hand side above is attained if and only if the infimum in the right-hand side is attained.

1.2.23 (Extension of Caratheodory's Theorem)

Let X_1 and X_2 be subsets of $\Re^n$, and let $X = \mathrm{conv}(X_1) + \mathrm{cone}(X_2)$. Show that every vector x in X can be represented in the form

$$x = \sum_{i=1}^{k} \alpha_i x_i + \sum_{i=k+1}^{m} \alpha_i x_i,$$

where m is a positive integer with $m \leq n+1$, the vectors $x_1, \ldots, x_k$ belong to X_1, the vectors $x_{k+1}, \ldots, x_m$ belong to X_2, and the scalars $\alpha_1, \ldots, \alpha_m$ are nonnegative with $\alpha_1 + \cdots + \alpha_k = 1$. Furthermore, the vectors $x_2 - x_1, \ldots, x_k - x_1, x_{k+1}, \ldots, x_m$ are linearly independent.

1.2.24

Let $x_0, \ldots, x_m$ be vectors in $\Re^n$ such that $x_1 - x_0, \ldots, x_m - x_0$ are linearly independent. The convex hull of $x_0, \ldots, x_m$ is called an *m-dimensional simplex*, and $x_0, \ldots, x_m$ are called the *vertices* of the simplex.

(a) Show that the dimension of a convex set C is the maximum of the dimensions of the simplices included in C.

(b) Use part (a) to show that a nonempty convex set has a nonempty relative interior.

1.2.25

Let X be a bounded subset of $\Re^n$. Show that

$$\mathrm{cl}\big(\mathrm{conv}(X)\big) = \mathrm{conv}\big(\mathrm{cl}(X)\big).$$

In particular, if X is closed and bounded, then $\mathrm{conv}(X)$ is closed and bounded (cf. Prop. 1.2.8).

1.2.26

Let C_1 and C_2 be two nonempty convex sets such that $C_1 \subset C_2$.

(a) Give an example showing that $\mathrm{ri}(C_1)$ need not be a subset of $\mathrm{ri}(C_2)$.

(b) Assuming that the sets $\mathrm{ri}(C_1)$ and $\mathrm{ri}(C_2)$ have nonempty intersection, show that $\mathrm{ri}(C_1) \subset \mathrm{ri}(C_2)$.

(c) Assuming that the sets C_1 and $\mathrm{ri}(C_2)$ have nonempty intersection, show that the set $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$ is nonempty.

1.2.27

Let C be a nonempty convex set.

(a) Show the following refinement of the Line Segment Principle [Prop. 1.2.9(c)]: $x \in \mathrm{ri}(C)$ if and only if for every $\overline{x} \in \mathrm{aff}(C)$, there exists $\gamma > 1$ such that $x + (\gamma - 1)(x - \overline{x}) \in C$.

(b) Assuming that the origin lies in $\mathrm{ri}(C)$, show that $\mathrm{cone}(C)$ coincides with $\mathrm{aff}(C)$.

(c) Show the following extension of part (b) to a nonconvex set: If X is a nonempty set such that the origin lies in the relative interior of $\mathrm{conv}(X)$, then $\mathrm{cone}(X)$ coincides with $\mathrm{aff}(X)$.

1.2.28

Let C be a compact set.

(a) Assuming that C is a convex set not containing the origin in its relative boundary, show that $\mathrm{cone}(C)$ is closed.

(b) Give examples showing that the assertion of part (a) fails if C is unbounded or C contains the origin in its relative boundary.

(c) The convexity assumption in part (a) can be relaxed as follows: assuming that $\mathrm{conv}(C)$ does not contain the origin in its boundary, show that $\mathrm{cone}(C)$ is closed. *Hint*: Use part (a) and Exercise 1.2.7(b).

1.2.29

(a) Let C be a convex cone. Show that $\mathrm{ri}(C)$ is also a convex cone.

(b) Let $C = \mathrm{cone}\big(\{x_1, \ldots, x_m\}\big)$. Show that

$$\mathrm{ri}(C) = \left\{ \sum_{i=1}^{m} \alpha_i x_i \;\middle|\; \alpha_i > 0,\ i = 1, \ldots, m \right\}.$$

1.2.30

Let A be an $m \times n$ matrix and let C be a nonempty convex set in $\Re^m$. Assuming that the inverse image $A^{-1} \cdot C$ is nonempty, show that

$$\mathrm{ri}(A^{-1} \cdot C) = A^{-1} \cdot \mathrm{ri}(C), \qquad \mathrm{cl}(A^{-1} \cdot C) = A^{-1} \cdot \mathrm{cl}(C).$$

[Compare these relations with those of Prop. 1.2.10(d) and (e), respectively.]

1.2.31 (Lipschitz Property of Convex Functions)

Let $f : \Re^n \to \Re$ be a convex function and X be a bounded set in $\Re^n$. Then f has the Lipschitz property over X, i.e., there exists a positive scalar c such that

$$|f(x) - f(y)| \leq c \cdot ||x - y||, \qquad \forall\ x, y \in X.$$

1.2.32

Let C be a closed convex set and let M be an affine set such that the intersection $C \cap M$ is nonempty and bounded. Show that for every affine set $\overline{M}$ that is parallel to M, the intersection $C \cap \overline{M}$ is bounded when nonempty.

1.2.33 (Recession Cones of Nonclosed Sets)

Let C be a nonempty convex set.

(a) Show by counterexample that part (b) of the Recession Cone Theorem need not hold when C is not closed.

(b) Show that

$$R_C \subset R_{\mathrm{cl}(C)}, \qquad \mathrm{cl}(R_C) \subset R_{\mathrm{cl}(C)}.$$

Give an example where the inclusion $\mathrm{cl}(R_C) \subset R_{\mathrm{cl}(C)}$ is strict, and another example where $\mathrm{cl}(R_C) = R_{\mathrm{cl}(C)}$. Also, give an example showing that $R_{\mathrm{ri}(C)}$ need not be a subset of R_C.

(c) Let $\overline{C}$ be a closed convex set such that $C \subset \overline{C}$. Show that $R_C \subset R_{\overline{C}}$. Give an example showing that the inclusion can fail if $\overline{C}$ is not closed.

1.2.34 (Recession Cones of Relative Interiors)

Let C be a nonempty convex set.

(a) Show that a vector y belongs to $R_{\mathrm{ri}(C)}$ if and only if there exists a vector $x \in \mathrm{ri}(C)$ such that $x + \alpha y \in C$ for every $\alpha \geq 0$.

(b) Show that $R_{\mathrm{ri}(C)} = R_{\mathrm{cl}(C)}$.

(c) Let $\overline{C}$ be a relatively open convex set such that $C \subset \overline{C}$. Show that $R_C \subset R_{\overline{C}}$. Give an example showing that the inclusion can fail if $\overline{C}$ is not relatively open. [Compare with Exercise 1.2.33(b).]

Hint: In part (a), follow the proof of part (b) of the Recession Cone Theorem. In parts (b) and (c), use the result of part (a).

1.2.35

Let C be a nonempty convex set in $\Re^n$ and let A be an $m \times n$ matrix.

(a) Show the following refinement of Prop. 1.2.14: if $R_{\mathrm{cl}(C)} \cap N(A) = \{0\}$, then

$$\mathrm{cl}(A \cdot C) = A \cdot \mathrm{cl}(C), \qquad A \cdot R_{\mathrm{cl}(C)} = R_{A \cdot \mathrm{cl}(C)}.$$

(b) Give an example showing that $A \cdot R_{\mathrm{cl}(C)}$ and $R_{A \cdot \mathrm{cl}(C)}$ can differ when $R_{\mathrm{cl}(C)} \cap N(A) \neq \{0\}$.

1.2.36 (Lineality Space and Recession Cone)

Let C be a nonempty convex set in $\Re^n$. Define the *lineality space* of C, denoted by L, to be a subspace of vectors y such that simultaneously $y \in R_C$ and $-y \in R_C$.

(a) Show that for every subspace $S \subset L$

$$C = (C \cap S^{\perp}) + S.$$

(b) Show the following refinement of Prop. 1.2.14 and Exercise 1.2.35: if A is an $m \times n$ matrix and $R_{\mathrm{cl}(C)} \cap N(A)$ is a subspace of L, then

$$\mathrm{cl}(A \cdot C) = A \cdot \mathrm{cl}(C), \qquad R_{A \cdot \mathrm{cl}(C)} = A \cdot R_{\mathrm{cl}(C)}.$$

1.2.37

This exercise is a refinement of Prop. 1.2.15.

(a) Let $C_1, \ldots, C_m$ be nonempty closed convex sets in $\Re^n$ such that the equality $y_1 + \cdots + y_m = 0$ with $y_i \in R_{C_i}$ implies that each y_i belongs to the lineality space of C_i. Then the vector sum $C_1 + \cdots + C_m$ is a closed set and

$$R_{C_1 + \cdots + C_m} = R_{C_1} + \cdots + R_{C_m}.$$

(b) Show the following extension of part (a) to nonclosed sets: Let $C_1, \ldots, C_m$ be nonempty convex sets in $\Re^n$ such that the equality $y_1 + \cdots + y_m = 0$ with $y_i \in R_{\mathrm{cl}(C_i)}$ implies that each y_i belongs to the lineality space of $\mathrm{cl}(C_i)$. Then we have

$$\mathrm{cl}(C_1 + \cdots + C_m) = \mathrm{cl}(C_1) + \cdots + \mathrm{cl}(C_m),$$

$$R_{\mathrm{cl}(C_1 + \cdots + C_m)} = R_{\mathrm{cl}(C_1)} + \cdots + R_{\mathrm{cl}(C_m)}.$$

1.3 CONVEXITY AND OPTIMIZATION

In this section we discuss applications of convexity to some basic optimization issues, such as the existence and uniqueness of global minima. Several other applications, relating to optimality conditions and polyhedral convexity, will be discussed in subsequent sections.

1.3.1 Global and Local Minima

Let X be a nonempty subset of $\Re^n$ and let $f : \Re^n \mapsto (-\infty, \infty]$ be a function. We say that a vector $x^* \in X$ is a *minimum of f over X* if $f(x^*) = \inf_{x \in X} f(x)$. We also call x^* a *minimizing point* or *minimizer* or *global minimum over X*. Alternatively, we say that f *attains a minimum over X at x^**, and we indicate this by writing

$$x^* \in \arg\min_{x \in X} f(x).$$

We use similar terminology for maxima, i.e., a vector $x^* \in X$ such that $f(x^*) = \sup_{x \in X} f(x)$ is said to be a *maximum of f over X*, and we indicate this by writing

$$x^* \in \arg\max_{x \in X} f(x).$$

If the domain of f is the set X (instead of $\Re^n$), we also call x^* a (global) minimum or (global) maximum of f (without the qualifier "over X").

A basic question in minimization problems is whether an optimal solution exists. This question can often be resolved with the aid of the classical theorem of Weierstrass, which states that a continuous function attains a minimum over a compact set. We will provide a more general version of this theorem, and to this end, we introduce some terminology. We say that a function $f : \Re^n \mapsto (-\infty, \infty]$ is *coercive* if

$$\lim_{k \to \infty} f(x_k) = \infty$$

for every sequence $\{x_k\}$ such that $\|x_k\| \to \infty$ for some norm $\|\cdot\|$. Note that as a consequence of the definition, the level sets $\{x \mid f(x) \leq \gamma\}$ of a coercive function f are bounded whenever they are nonempty.

Proposition 1.3.1: (Weierstrass' Theorem) Let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed function. Assume that one of the following three conditions holds:

(1) $\text{dom}(f)$ is compact.

(2) $\text{dom}(f)$ is closed and f is coercive.

(3) There exists a scalar γ such that the set

$$\{x \mid f(x) \le \gamma\}$$

is nonempty and compact.

Then, f attains a minimum over $\Re^n$.

Proof: If $f(x) = \infty$ for all $x \in \Re^n$, then every $x \in \Re^n$ attains the minimum of f over $\Re^n$. Thus, with no loss of generality, we assume that $\inf_{x\in\Re^n} f(x) < \infty$. Assume condition (1). Let $\{x_k\} \subset \text{dom}(f)$ be a sequence such that

$$\lim_{k\to\infty} f(x_k) = \inf_{x\in\Re^n} f(x).$$

Since $\text{dom}(f)$ is bounded, this sequence has at least one limit point x^* [Prop. 1.1.5(a)]. Since f is closed, f is lower semicontinuous at x^* [cf. Prop. 1.2.2(b)], so that $f(x^*) \le \lim_{k\to\infty} f(x_k) = \inf_{x\in\Re^n} f(x)$, and x^* is a minimum of f.

Assume condition (2). Consider a sequence $\{x_k\}$ as in the proof under condition (1). Since f is coercive, $\{x_k\}$ must be bounded and the proof proceeds similar to the proof under condition (1).

Assume condition (3). If the given γ is equal to $\inf_{x\in\Re^n} f(x)$, the set of minima of f over $\Re^n$ is $\{x \mid f(x) \le \gamma\}$, and since by assumption this set is nonempty, we are done. If $\inf_{x\in\Re^n} f(x) < \gamma$, consider a sequence $\{x_k\}$ as in the proof under condition (1). Then, for all k sufficiently large, x_k must belong to the set $\{x \mid f(x) \le \gamma\}$. Since this set is compact, $\{x_k\}$ must be bounded and the proof proceeds similar to the proof under condition (1). **Q.E.D.**

The most common application of the above proposition is when we want to minimize a real-valued function $f : \Re^n \mapsto \Re$ over a nonempty set X. Then by applying Prop. 1.3.1 to the extended real-valued function

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X, \\ \infty & \text{otherwise,} \end{cases} \tag{1.9}$$

we see that f attains a minimum over X if X is closed and f is lower semicontinuous over X (implying that $\tilde{f}$ is closed), and either (1) X is bounded or (2) f is coercive or (3) some level set $\big\{x \in X \mid f(x) \leq \gamma\big\}$ is nonempty and compact.

For another example, suppose that we want to minimize a closed convex function $f : \Re^n \mapsto (-\infty, \infty]$ over a closed convex set X. Then, by using the function $\tilde{f}$ of Eq. (1.9), we see that f attains a minimum over X if the set $X \cap \mathrm{dom}(f)$ is compact or if f is coercive or if some level set $\big\{x \in X \mid f(x) \leq \gamma\big\}$ is nonempty and compact.

Note that with appropriate adjustments, the preceding analysis applies to the existence of maxima of f over X. For example, if a real-valued function f is upper semicontinuous at all points of a compact set X, then f attains a maximum over X.

We say that a vector $x^* \in X$ is a *local minimum of f over X* if there exists some $\epsilon > 0$ such that $f(x^*) \leq f(x)$ for every $x \in X$ satisfying $\|x - x^*\| \leq \epsilon$, where $\|\cdot\|$ is some vector norm. If the domain of f is the set X (instead of $\Re^n$), we also call x^* a local minimum of f (without the qualifier "over X"). Local maxima are defined similarly.

An important implication of convexity of f and X is that all local minima are also global, as shown in the following proposition and in Fig. 1.3.1.

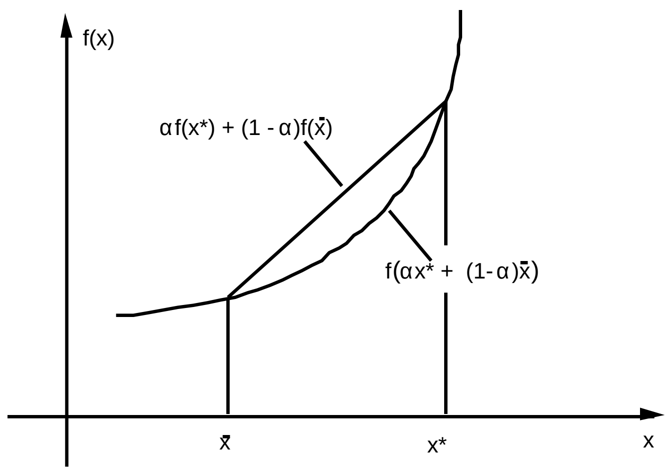

Figure 1.3.1. Proof of why local minima of convex functions are also global. Suppose that f is convex, and assume to arrive at a contradiction, that x^* is a local minimum that is not global. Then there must exist an $\overline{x} \in X$ such that $f(\overline{x}) < f(x^*)$. By convexity, for all $\alpha \in (0,1)$,

$$f\big(\alpha x^* + (1-\alpha)\overline{x}\big) \leq \alpha f(x^*) + (1-\alpha) f(\overline{x}) < f(x^*).$$

Thus, f has strictly lower value than $f(x^*)$ at every point on the line segment connecting x^* with $\overline{x}$, except at x^*. This contradicts the local minimality of x^* over X.

Proposition 1.3.2: If $X \subset \Re^n$ is a convex set and $f : X \mapsto (-\infty, \infty]$ is a convex function, then a local minimum of f is also a global minimum. If in addition f is strictly convex, then there exists at most one global minimum of f.

Proof: See Fig. 1.3.1 for a proof that a local minimum of f is also global. Let f be strictly convex, and to obtain a contradiction, assume that two distinct global minima x and y exist. Then the midpoint $(x + y)/2$ must belong to X, since X is convex. Furthermore, the value of f must be smaller at the midpoint than at x and y by the strict convexity of f. Since x and y are global minima, we obtain a contradiction. **Q.E.D.**

1.3.2 The Projection Theorem

In this section we develop a basic result of analysis and optimization.

Proposition 1.3.3: (Projection Theorem) Let C be a nonempty closed convex set and let $\|\cdot\|$ be the Euclidean norm.

(a) For every $x \in \Re^n$, there exists a unique vector $z \in C$ that minimizes $\|z - x\|$ over all $z \in C$. This vector is called the *projection of x on C*, and is denoted by $P_C(x)$, i.e.,

$$P_C(x) = \arg\min_{z \in C} \|z - x\|.$$

(b) For every $x \in \Re^n$, a vector $z \in C$ is equal to $P_C(x)$ if and only if

$$(y - z)'(x - z) \leq 0, \qquad \forall\ y \in C.$$

(c) The function $f : \Re^n \mapsto C$ defined by $f(x) = P_C(x)$ is continuous and nonexpansive, i.e.,

$$\big\|P_C(x) - P_C(y)\big\| \leq \|x - y\|, \qquad \forall\ x, y \in \Re^n.$$

(d) The distance function $d : \Re^n \mapsto \Re$, defined by

$$d(x, C) = \min_{z \in C} \|z - x\|,$$

is convex.

Proof: (a) Fix x and let w be some element of C. Minimizing $\|x-z\|$ over all $z \in C$ is equivalent to minimizing the same function over all $z \in C$ such that $\|x-z\| \leq \|x-w\|$, which is a compact set. Furthermore, the function g defined by $g(z) = \|z-x\|^2$ is continuous. Existence of a minimizing vector follows by Weierstrass' Theorem (Prop. 1.3.1).

To prove uniqueness, notice that the square of the Euclidean norm is a strictly convex function of its argument because its Hessian matrix is the identity matrix, which is positive definite [Prop. 1.2.5(b)]. Therefore, g is strictly convex and it follows that its minimum is attained at a unique point (Prop. 1.3.2).

(b) For all y and z in C we have

$$\|y-x\|^2 = \|y-z\|^2 + \|z-x\|^2 - 2(y-z)'(x-z) \geq \|z-x\|^2 - 2(y-z)'(x-z).$$

Therefore, if z is such that $(y-z)'(x-z) \leq 0$ for all $y \in C$, we have $\|y-x\|^2 \geq \|z-x\|^2$ for all $y \in C$, implying that $z = P_C(x)$.

Conversely, let $z = P_C(x)$, consider any $y \in C$, and for $\alpha > 0$, define $y_\alpha = \alpha y + (1-\alpha)z$. We have

$$\begin{aligned}\|x-y_\alpha\|^2 &= \|(1-\alpha)(x-z) + \alpha(x-y)\|^2 \\ &= (1-\alpha)^2\|x-z\|^2 + \alpha^2\|x-y\|^2 + 2(1-\alpha)\alpha(x-z)'(x-y).\end{aligned}$$

Viewing $\|x-y_\alpha\|^2$ as a function of α, we have

$$\frac{\partial}{\partial\alpha}\big\{\|x-y_\alpha\|^2\big\}\Big|_{\alpha=0} = -2\|x-z\|^2 + 2(x-z)'(x-y) = -2(y-z)'(x-z).$$

Therefore, if $(y-z)'(x-z) > 0$ for some $y \in C$, then

$$\frac{\partial}{\partial\alpha}\big\{\|x-y_\alpha\|^2\big\}\Big|_{\alpha=0} < 0$$

and for positive but small enough α, we obtain $\|x-y_\alpha\| < \|x-z\|$. This contradicts the fact $z = P_C(x)$ and shows that $(y-z)'(x-z) \leq 0$ for all $y \in C$.

(c) Let x and y be elements of $\Re^n$. From part (b), we have $\big(w-P_C(x)\big)'\big(x-P_C(x)\big) \leq 0$ for all $w \in C$. Since $P_C(y) \in C$, we obtain

$$\big(P_C(y) - P_C(x)\big)'\big(x - P_C(x)\big) \leq 0.$$

Similarly,

$$\big(P_C(x) - P_C(y)\big)'\big(y - P_C(y)\big) \leq 0.$$

Adding these two inequalities, we obtain

$$\big(P_C(y) - P_C(x)\big)'\big(x - P_C(x) - y + P_C(y)\big) \leq 0.$$

By rearranging and by using the Schwartz inequality, we have

$$\big\|P_C(y)-P_C(x)\big\|^2 \le \big(P_C(y)-P_C(x)\big)'(y-x) \le \big\|P_C(y)-P_C(x)\big\|\cdot\|y-x\|,$$

showing that $P_C(\cdot)$ is nonexpansive and *a fortiori* continuous.

(d) Assume, to arrive at a contradiction, that there exist $x_1, x_2 \in \Re^n$ and an $\alpha \in (0,1)$ such that

$$d\big(\alpha x_1 + (1-\alpha)x_2, C\big) > \alpha d(x_1, C) + (1-\alpha)d(x_2, C).$$

Then there must exist $z_1, z_2 \in C$ such that

$$d\big(\alpha x_1 + (1-\alpha)x_2, C\big) > \alpha\|z_1 - x_1\| + (1-\alpha)\|z_2 - x_2\|,$$

which implies that

$$\big\|\alpha z_1 + (1-\alpha)z_2 - \alpha x_1 - (1-\alpha)x_2\big\| > \alpha\|x_1 - z_1\| + (1-\alpha)\|x_2 - z_2\|.$$

This contradicts the triangle inequality in the definition of norm. **Q.E.D.**

Figure 1.3.2 illustrates the necessary and sufficient condition of part (b) of the Projection Theorem.

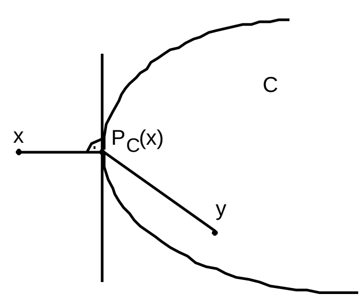

Figure 1.3.2. Illustration of the condition satisfied by the projection $P_C(x)$. For each vector $y \in C$, the vectors $x - P_C(x)$ and $y - P_C(x)$ form an angle greater than or equal to $\pi/2$ or, equivalently,

$$\big(y - P_C(x)\big)'\big(x - P_C(x)\big) \le 0.$$

1.3.3 Directions of Recession and Existence of Optimal Solutions

The recession cone, discussed in Section 1.2.4, is also useful for analyzing the existence of optimal solutions of convex optimization problems. A key idea here is that a convex function can be described in terms of its epigraph, which is a convex set. The recession cone of the epigraph can be used to obtain the directions along which the function decreases monotonically. This is the idea underlying the following proposition.

Proposition 1.3.4: Let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed convex function and consider the level sets $L_a = \big\{x \mid f(x) \leq a\big\}$.

(a) All the level sets L_a that are nonempty have the same recession cone, given by

$$R_{L_a} = \big\{y \mid (y, 0) \in R_{\mathrm{epi}(f)}\big\},$$

where $R_{\mathrm{epi}(f)}$ is the recession cone of the epigraph of f.

(b) If one nonempty level set L_a is compact, then all nonempty level sets are compact.

Proof: From the formula for the epigraph

$$\mathrm{epi}(f) = \big\{(x, w) \mid f(x) \leq w\big\},$$

it can be seen that for all a for which L_a is nonempty, we have

$$\big\{(x, a) \mid x \in L_a\big\} = \mathrm{epi}(f) \cap \big\{(x, a) \mid x \in \Re^n\big\}.$$

The recession cone of the set in the left-hand side above is $\big\{(y, 0) \mid y \in R_{L_a}\big\}$. By Prop. 1.2.13(e), the recession cone of the set in the right-hand side is equal to the intersection of the recession cone of epi(f) and the recession cone of $\big\{(x, a) \mid x \in \Re^n\big\}$, which is equal to $\big\{(y, 0) \mid y \in \Re^n\big\}$, the horizontal subspace that passes through the origin. Thus we have

$$\big\{(y, 0) \mid y \in R_{L_a}\big\} = \big\{(y, 0) \mid (y, 0) \in R_{\mathrm{epi}(f)}\big\},$$

from which it follows that

$$R_{L_a} = \big\{y \mid (y, 0) \in R_{\mathrm{epi}(f)}\big\}.$$

This proves part (a). Part (b) follows by applying Prop. 1.2.13(c) to the recession cone of epi(f). **Q.E.D.**

For a closed convex function $f : \Re^n \mapsto (-\infty, \infty]$, the (common) recession cone of the nonempty level sets of f is referred to as the *recession cone of* f, and is denoted by R_f (see Fig. 1.3.3). Thus

$$R_f = \big\{y \mid f(x) \geq f(x + \alpha y),\ \forall\ x \in \Re^n,\ \alpha \geq 0\big\}.$$

Each $y \in R_f$ is called a *direction of recession of* f. Since f is closed, its level sets are closed, so by the Recession Cone Theorem (cf. Prop. 1.2.13), R_f is also closed.

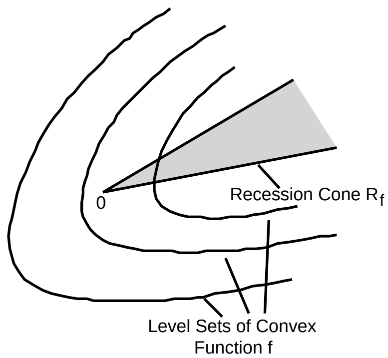

Figure 1.3.3. Recession cone R_f of a closed convex function f. It is the (common) recession cone of the nonempty level sets of f.

The most intuitive way to look at directions of recession is from a descent viewpoint: if we start at any $x \in \Re^n$ and move indefinitely along a direction of recession y, we must stay within each level set that contains x, or equivalently we must encounter exclusively points z with $f(z) \leq f(x)$. In words, *a direction of recession of f is a direction of uninterrupted nonascent for f*. Conversely, if we start at some $x \in \Re^n$ and while moving along a direction y, we encounter a point z with $f(z) > f(x)$, then y cannot be a direction of recession. It is easily seen via a convexity argument that once we cross the relative boundary of a level set of f we never cross it back again, and with a little thought (see Fig. 1.3.4 and the exercises), it follows that *a direction that is not a direction of recession of f is a direction of eventual uninterrupted ascent of f*. In view of these observations, it is not surprising that directions of recession play a prominent role in characterizing the existence of solutions of convex optimization problems, as shown in the following proposition.

Proposition 1.3.5: (Existence of Solutions of Convex Programs) Let X be a closed convex subset of $\Re^n$, and let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed convex function such that $f(x) < \infty$ for at least one vector $x \in X$. Let X^* be the set of minimizing points of f over X. Then X^* is nonempty and compact if and only if X and f have no common nonzero direction of recession.

Proof: Let $f^* = \inf_{x \in X} f(x)$, and note that

$$X^* = X \cap \{x \mid f(x) \leq f^*\}.$$

Since by assumption $f^* < \infty$, and X and f are closed, the two sets in the above intersection are closed, so X^* is closed as well as convex. If X^* is nonempty and compact, it has no nonzero direction of recession [Prop.

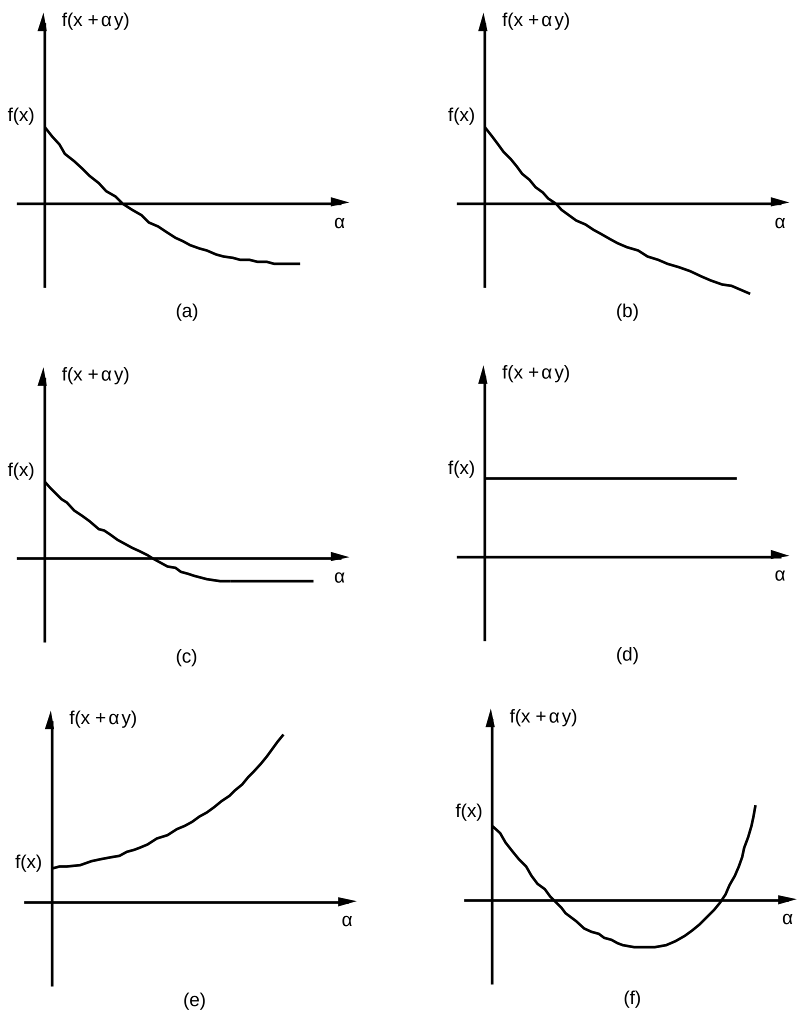

Figure 1.3.4. Ascent/descent behavior of a closed convex function starting at some $x \in \Re^n$ and moving along a direction y. If y is a direction of recession of f, there are two possibilities: either f decreases monotonically to a finite value or $-\infty$ [figures (a) and (b), respectively], or f reaches a value that is less or equal to $f(x)$ and stays at that value [figures (c) and (d)]. If y is *not* a direction of recession of f, then eventually f increases monotonically to ∞ [figures (e) and (f)]; i.e., for some $\overline{\alpha} \geq 0$ and all $\alpha_1, \alpha_2 \geq \overline{\alpha}$ with $\alpha_1 < \alpha_2$ we have $f(x + \alpha_1 y) < f(x + \alpha_2 y)$.

1.2.13(c)]. Therefore, there is no nonzero vector in the intersection of the recession cones of X and $\{x \mid f(x) \leq f^*\}$. This is equivalent to X and f having no common nonzero direction of recession.

Conversely, let a be a scalar such that the set

$$X_a = X \cap \big\{x \mid f(x) \leq a\big\}$$

is nonempty and has no nonzero direction of recession. Then, X_a is closed [since X is closed and $\big\{x \mid f(x) \leq a\big\}$ is closed by the closedness of f], and by Prop. 1.2.13(c), X_a is compact. Since minimization of f over X and over X_a yields the same set of minima, X^*, by Weierstrass' Theorem (Prop. 1.3.1) X^* is nonempty, and since $X^* \subset X_a$, we see that X^* is bounded. Since X^* is closed it is compact. **Q.E.D.**

If the closed convex set X and the closed convex function f of the above proposition have a common direction of recession, then either X^* is empty [take for example, $X = (-\infty, 0]$ and $f(x) = e^x$] or else X^* is nonempty and unbounded [take for example, $X = (-\infty, 0]$ and $f(x) = \max\{0, x\}$].

Another interesting question is what happens when X and f have a common direction of recession, call it y, but f is bounded below over X:

$$f^* = \inf_{x \in X} f(x) > -\infty.$$

Then for any $x \in X$, we have $x + \alpha y \in X$ (since y is a direction of recession of X), and $f(x + \alpha y)$ is monotonically nondecreasing to a finite value as $\alpha \to \infty$ (since y is a direction of recession of f and $f^* > -\infty$). Generally, the minimum of f over X need not be attained. However, it turns out that the minimum is attained in an important special case: when f is quadratic and X is polyhedral (i.e., is defined by linear equality and inequality constraints).

To understand the main idea, consider the problem

$$\begin{aligned} &\text{minimize} \quad f(x) = c'x + \frac{1}{2}x'Qx \\ &\text{subject to} \quad Ax = 0, \end{aligned} \tag{1.10}$$

where Q is a positive semidefinite symmetric $n \times n$ matrix, $c \in \Re^n$ is a given vector, and A is an $m \times n$ matrix. Let $N(Q)$ and $N(A)$ denote the nullspaces of Q and A, respectively. There are two possibilities:

(a) For some $x \in N(A) \cap N(Q)$, we have $c'x \neq 0$. Then, since $f(\alpha x) = \alpha c'x$ for all $\alpha \in \Re$, it follows that f becomes unbounded from below either along x or along $-x$.

(b) For all $x \in N(A) \cap N(Q)$, we have $c'x = 0$. In this case, we have $f(x) = 0$ for all $x \in N(A) \cap N(Q)$. For $x \in N(A)$ such that $x \notin N(Q)$, since $N(Q)$ and $R(Q)$, the range of Q, are orthogonal subspaces, x can be uniquely decomposed as $x_R + x_N$, where $x_N \in N(Q)$ and

$x_R \in R(Q)$, and we have $f(x) = c'x + (1/2)x_R' Q x_R$, where x_R is the (nonzero) component of x along $R(Q)$. Hence $f(\alpha x) = \alpha c'x + (1/2)\alpha^2 x_R' Q x_R$ for all $\alpha > 0$, with $x_R' Q x_R > 0$. It follows that f is bounded below along all feasible directions $x \in N(A)$.

We thus conclude that for f to be bounded from below along all directions in $N(A)$ it is necessary and sufficient that $c'x = 0$ for all $x \in N(A) \cap N(Q)$. However, *boundedness from below of a convex cost function f along all directions of recession of a constraint set does not guarantee existence of an optimal solution, or even boundedness from below over the constraint set* (see the exercises). On the other hand, since the constraint set $N(A)$ is a subspace, it is possible to use a transformation $x = Bz$ where the columns of the matrix B are basis vectors for $N(A)$, and view the problem as an unconstrained minimization over z of the cost function $h(z) = f(Bz)$, which is positive semidefinite quadratic. We can then argue that boundedness from below of this function along all directions z is necessary and sufficient for existence of an optimal solution. This argument indicates that problem (1.10) has an optimal solution if and only if $c'x = 0$ for all $x \in N(A) \cap N(Q)$. By using a translation argument, this result can also be extended to the case where the constraint set is a general affine set of the form $\{x \mid Ax = b\}$ rather than the subspace $\{x \mid Ax = 0\}$.

In part (a) of the following proposition we state the result just described (equality constraints only). While we can prove the result by formalizing the argument outlined above, we will use instead a more elementary variant of this argument, whereby the constraints are eliminated via a penalty function; this will give us the opportunity to introduce a line of proof that we will frequently employ in other contexts as well. In part (b) of the proposition, we allow linear inequality constraints, and we show that a convex quadratic program has an optimal solution if and only if its optimal value is bounded below. Note that the cost function may be linear, so the proposition applies to linear programs as well.

Proposition 1.3.6: (Existence of Solutions of Quadratic Programs) Let $f : \Re^n \mapsto \Re$ be a quadratic function of the form

$$f(x) = c'x + \frac{1}{2}x'Qx,$$

where Q is a positive semidefinite symmetric $n \times n$ matrix and $c \in \Re^n$ is a given vector. Let also A be an $m \times n$ matrix and $b \in \Re^m$ be a vector. Denote by $N(A)$ and $N(Q)$, the nullspaces of A and Q, respectively.

(a) Let $X = \{x \mid Ax = b\}$ and assume that X is nonempty. The following are equivalent:

  (i) f attains a minimum over X.

  (ii) $f^* = \inf_{x \in X} f(x) > -\infty$.

  (iii) $c'y = 0$ for all $y \in N(A) \cap N(Q)$.

(b) Let $X = \{x \mid Ax \leq b\}$ and assume that X is nonempty. The following are equivalent:

  (i) f attains a minimum over X.

  (ii) $f^* = \inf_{x \in X} f(x) > -\infty$.

  (iii) $c'y \geq 0$ for all $y \in N(Q)$ such that $Ay \leq 0$.

Proof: (a) (i) clearly implies (ii).

We next show that (ii) implies (iii). For all $x \in X$, $y \in N(A) \cap N(Q)$, and $\alpha \in \Re$, we have $x + \alpha y \in X$ and

$$f(x + \alpha y) = c'(x + \alpha y) + \frac{1}{2}(x + \alpha y)'Q(x + \alpha y) = f(x) + \alpha c'y.$$

If $c'y \neq 0$, then either $\lim_{\alpha \to \infty} f(x + \alpha y) = -\infty$ or $\lim_{\alpha \to -\infty} f(x + \alpha y) = -\infty$, and we must have $f^* = -\infty$. Hence (ii) implies that $c'y = 0$ for all $y \in N(A) \cap N(Q)$.

We finally show that (iii) implies (i) by first using a translation argument and then using a penalty function argument. Choose any $\overline{x} \in X$, so that $X = \overline{x} + N(A)$. Then minimizing f over X is equivalent to minimizing $f(\overline{x} + y)$ over $y \in N(A)$, or

$$\begin{aligned} &\text{minimize} \;\; h(y) \\ &\text{subject to} \;\; Ay = 0, \end{aligned}$$

where

$$h(y) = f(\overline{x} + y) = f(\overline{x}) + \nabla f(\overline{x})'y + \frac{1}{2}y'Qy.$$

For any integer $k > 0$, let

$$h_k(y) = h(y) + \frac{k}{2}\|Ay\|^2 = f(\overline{x}) + \nabla f(\overline{x})'y + \frac{1}{2}y'Qy + \frac{k}{2}\|Ay\|^2. \quad (1.11)$$

Note that for all k

$$h_k(y) \leq h_{k+1}(y), \qquad \forall\, y \in \Re^n,$$

and

$$\inf_{y\in\Re^n} h_k(y) \leq \inf_{y\in\Re^n} h_{k+1}(y) \leq \inf_{Ay=0} h(y) \leq h(0) = f(\overline{x}). \quad (1.12)$$

Denote

$$S = \big(N(A) \cap N(Q)\big)^\perp$$

and write any $y \in \Re^n$ as $y = z + w$, where

$$z \in S, \qquad w \in S^\perp = N(A) \cap N(Q).$$

Then, by using the assumption $c'w = 0$ [implying that $\nabla f(\overline{x})'w = (c + Q\overline{x})'w = 0$], we see from Eq. (1.11) that

$$h_k(y) = h_k(z + w) = h_k(z), \quad (1.13)$$

i.e., h_k is determined in terms of its restriction to the subspace S. It can be seen from Eq. (1.11) that the function h_k has no nonzero direction of recession in common with S, so $h_k(z)$ attains a minimum over S, call it y_k, and in view of Eq. (1.13), y_k also attains the minimum of $h_k(y)$ over $\Re^n$.

From Eq. (1.12), we have

$$h_k(y_k) \leq h_{k+1}(y_{k+1}) \leq \inf_{Ay=0} h(y) \leq f(\overline{x}), \quad (1.14)$$

and we will use this relation to show that $\{y_k\}$ is bounded and each of its limit points minimizes $h(y)$ subject to $Ay = 0$. Indeed, from Eq. (1.14), the sequence $\{h_k(y_k)\}$ is bounded, so if $\{y_k\}$ were unbounded, then assuming without loss of generality that $y_k \neq 0$, we would have $h_k(y_k)/\|y_k\| \to 0$, or

$$\lim_{k\to\infty} \left(\frac{1}{\|y_k\|} f(\overline{x}) + \nabla f(\overline{x})'\hat{y}_k + \|y_k\| \left(\frac{1}{2}\hat{y}_k' Q\hat{y}_k + \frac{k}{2}\|A\hat{y}_k\|^2 \right) \right) = 0,$$

where $\hat{y}_k = y_k/\|y_k\|$. For this to be true, all limit points $\hat{y}$ of the bounded sequence $\{\hat{y}_k\}$ must be such that $\hat{y}'Q\hat{y} = 0$ and $A\hat{y} = 0$, which is impossible since $\|\hat{y}\| = 1$ and $\hat{y} \in S$. Thus $\{y_k\}$ is bounded and for any one of its limit points, call it $\overline{y}$, we have $\overline{y} \in S$ and

$$\limsup_{k\to\infty} h_k(y_k) = f(\overline{x}) + \nabla f(\overline{x})'\overline{y} + \frac{1}{2}\overline{y}'Q\overline{y} + \limsup_{k\to\infty} \frac{k}{2}\|Ay_k\|^2 \leq \inf_{Ay=0} h(y).$$

It follows that $A\overline{y} = 0$ and that $\overline{y}$ minimizes $h(y)$ over $Ay = 0$. This implies that the vector $y = \overline{x} + \overline{y}$ minimizes $f(x)$ subject to $Ax = b$.

(b) Clearly (i) implies (ii), and similar to the proof of part (a), (ii) implies that $c'y \geq 0$ for all $y \in N(Q)$ with $Ay \leq 0$.

Finally, we show that (iii) implies (i) by using the corresponding result of part (a). For any $x \in X$, let $J(x)$ denote the index set of active constraints at x, i.e., $J(x) = \{j \mid a_j'x = b_j\}$, where the a_j' are the rows of A.

For any sequence $\{x_k\} \subset X$ with $f(x_k) \to f^*$, we can extract a subsequence such that $J(x_k)$ is constant and equal to some J. Accordingly, we select a sequence $\{\overline{x}_k\} \subset X$ such that $f(\overline{x}_k) \to f^*$, and the index set $J(\overline{x}_k)$ is equal for all k to a set $\overline{J}$ that is *maximal* over all such sequences [for any other sequence $\{x_k\} \subset X$ with $f(x_k) \to f^*$ and such that $J(x_k) = J$ for all k, we cannot have $\overline{J} \subset J$ unless $\overline{J} = J$].

Consider the problem

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad a_j'x = b_j, \quad j \in \overline{J}. \end{aligned} \tag{1.15}$$

Assume, to come to a contradiction, that this problem does not have a solution. Then, by part (a), we have $c'y < 0$ for some $y \in N(\overline{A}) \cap N(Q)$, where $\overline{A}$ is the matrix having as rows the a_j', $j \in \overline{J}$. Consider the line $\{\overline{x}_k + \gamma y \mid \gamma > 0\}$. Since $y \in N(Q)$, we have

$$f(\overline{x}_k + \gamma y) = f(\overline{x}_k) + \gamma c'y,$$

so that

$$f(\overline{x}_k + \gamma y) < f(\overline{x}_k), \qquad \forall\, \gamma > 0.$$

Furthermore, since $y \in N(\overline{A})$, we have

$$a_j'(\overline{x}_k + \gamma y) = b_j, \qquad \forall\, j \in \overline{J},\ \gamma > 0.$$

We must also have $a_j'y > 0$ for at least one $j \notin \overline{A}$ [otherwise (iii) would be violated], so the line $\{\overline{x}_k + \gamma y \mid \gamma > 0\}$ crosses the relative boundary of X for some $\overline{\gamma}_k > 0$. The sequence $\{x_k\}$, where $x_k = \overline{x}_k + \overline{\gamma}_k y$, satisfies $\{x_k\} \subset X$, $f(x_k) \to f^*$ [since $f(x_k) \leq f(\overline{x}_k)$], and the active index set $J(x_k)$ strictly contains $\overline{J}$ for all k. This contradicts the maximality of $\overline{J}$, and shows that problem (1.15) has an optimal solution, call it $\overline{x}$.

Since $\overline{x}_k$ is a feasible solution of problem (1.15), we have

$$f(\overline{x}) \leq f(\overline{x}_k), \qquad \forall\, k,$$

so that

$$f(\overline{x}) \leq f^*.$$

We will now show that $\overline{x}$ minimizes f over X, by showing that $\overline{x} \in X$, thereby completing the proof. Assume, to arrive at a contradiction, that $\overline{x} \notin X$. Let $\hat{x}_k$ be a point in the interval connecting $\overline{x}_k$ and $\overline{x}$ that belongs to X and is closest to $\overline{x}$. We have that $J(\hat{x}_k)$ strictly contains $\overline{J}$ for all k. Since $f(\overline{x}) \leq f(\overline{x}_k)$ and f is convex over the interval $[\overline{x}_k, \overline{x}]$, it follows that

$$f(\hat{x}_k) \leq \max\big\{f(\overline{x}_k), f(\overline{x})\big\} = f(\overline{x}_k).$$

Thus $f(\hat{x}_k) \to f^*$, which contradicts the maximality of $\overline{J}$. **Q.E.D.**

1.3.4 Existence of Saddle Points

Suppose that we are given a function $\phi : X \times Z \mapsto \Re$, where $X \subset \Re^n$, $Z \subset \Re^m$, and we wish to either

$$\begin{aligned} &\text{minimize} \quad \sup_{z \in Z} \phi(x, z) \\ &\text{subject to} \quad x \in X \end{aligned}$$

or

$$\begin{aligned} &\text{maximize} \quad \inf_{x \in X} \phi(x, z) \\ &\text{subject to} \quad z \in Z. \end{aligned}$$

These problems are encountered in at least three major optimization contexts:

(1) *Worst-case design*, whereby we view z as a parameter and we wish to minimize over x a cost function, assuming the worst possible value of x. A special case of this is the *discrete minimax problem*, where we want to minimize over $x \in X$

$$\max\big\{f_1(x), \dots, f_m(x)\big\},$$

where the f_i are some given functions. Here, Z is the finite set $\{1, \dots, m\}$. Within this context, it is important to provide characterizations of the max function

$$\max_{z \in Z} \phi(x, z),$$

particularly its directional derivative. We do this in Section 1.7, where we discuss the differentiability properties of convex functions.

(2) *Exact penalty functions*, which can be used for example to convert constrained optimization problems of the form

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in X, \qquad g_j(x) \leq 0, \quad j = 1, \dots, r \end{aligned} \tag{1.16}$$

to (less constrained) minimax problems of the form

$$\begin{aligned}&\text{minimize} \quad f(x) + c\max\{0, g_1(x), \ldots, g_r(x)\}\\&\text{subject to} \quad x \in X,\end{aligned}$$

where c is a large positive penalty parameter. This conversion is useful for both analytical and computational purposes, and will be discussed in Chapters 2 and 4.

(3) *Duality theory*, where using problem (1.16) as an example, we introduce the, so called, Lagrangian function

$$L(x, \mu) = f(x) + \sum_{j=1}^{r} \mu_j g_j(x)$$

involving the vector $\mu = (\mu_1, \ldots, \mu_r) \in \Re^r$, and the dual problem

$$\begin{aligned}&\text{maximize} \quad \inf_{x \in X} L(x, \mu)\\&\text{subject to} \quad \mu \geq 0.\end{aligned} \tag{1.17}$$

The original (primal) problem (1.16) can also be written as

$$\begin{aligned}&\text{minimize} \quad \sup_{\mu \geq 0} L(x, \mu)\\&\text{subject to} \quad x \in X\end{aligned}$$

[if x violates any of the constraints $g_j(x) \leq 0$, we have $\sup_{\mu \geq 0} L(x, \mu) = \infty$, and if it does not, we have $\sup_{\mu \geq 0} L(x, \mu) = f(x)$]. Thus the primal and the dual problems (1.16) and (1.17) can be viewed in terms of a minimax problem.

We will now derive conditions guaranteeing that

$$\sup_{z \in Z} \inf_{x \in X} \phi(x, z) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z), \tag{1.18}$$

and that the inf and the sup above are attained. This is a major issue in duality theory because it connects the primal and the dual problems [cf. Eqs. (1.16) and (1.17)] through their optimal values and optimal solutions. In particular, when we discuss duality in Chapter 3, we will see that a major question is whether there is no duality gap, i.e., whether the optimal primal and dual values are equal. This is so if and only if

$$\sup_{\mu \geq 0} \inf_{x \in X} L(x, \mu) = \inf_{x \in X} \sup_{\mu \geq 0} L(x, \mu). \tag{1.19}$$

We will prove in this section one major result, the Saddle Point Theorem, which guarantees the equality (1.18), as well as the attainment of the inf and sup, assuming convexity/concavity assumptions on ϕ and (essentially) compactness assumptions on X and Z.† Unfortunately, the Saddle Point Theorem is only partially adequate for the development of duality theory, because compactness of Z and, to some extent, compactness of X turn out to be restrictive assumptions [for example Z corresponds to the set $\{\mu \mid \mu \geq 0\}$ in Eq. (1.19), which is not compact]. We will state another major result, the Minimax Theorem, which is more relevant to duality theory and gives conditions guaranteing the minimax equality (1.18), although it does not guarantee the attainment of the inf and sup. We will also give additional theorems of the minimax type in Chapter 3, when we discuss duality and make a closer connection with the theory of Lagrange multipliers.

A first observation regarding the potential validity of the minimax equality (1.18) is that we always have the inequality

$$\sup_{z \in Z} \inf_{x \in X} \phi(x, z) \leq \inf_{x \in X} \sup_{z \in Z} \phi(x, z), \tag{1.20}$$

[for every $\overline{z} \in Z$, write $\inf_{x \in X} \phi(x, \overline{z}) \leq \inf_{x \in X} \sup_{z \in Z} \phi(x, z)$ and take the supremum over $\overline{z} \in Z$ of the left-hand side]. However, special conditions are required to guarantee equality.

Suppose that x^* is an optimal solution of the problem

$$\begin{aligned} &\text{minimize} \quad \sup_{z \in Z} \phi(x, z) \\ &\text{subject to} \quad x \in X \end{aligned} \tag{1.21}$$

and z^* is an optimal solution of the problem

$$\begin{aligned} &\text{maximize} \quad \inf_{x \in X} \phi(x, z) \\ &\text{subject to} \quad z \in Z. \end{aligned} \tag{1.22}$$

---

† The Saddle Point Theorem is also central in game theory, as we now briefly explain. In the simplest type of zero sum game, there are two players: the first may choose one out of n moves and the second may choose one out of m moves. If moves i and j are selected by the first and the second player, respectively, the first player gives a specified amount a_{ij} to the second. The objective of the first player is to minimize the amount given to the other player, and the objective of the second player is to maximize this amount. The players use mixed strategies, whereby the first player selects a probability distribution $x = (x_1, \ldots, x_n)$ over his n possible moves and the second player selects a probability distribution $z = (z_1, \ldots, z_m)$ over his m possible moves. Since the probability of selecting i and j is $x_i z_j$, the expected amount to be paid by the first player to the second is $\sum_{i,j} a_{ij} x_i z_j$ or $x'Az$, where A is the $n \times m$ matrix with elements a_{ij}.

If each player adopts a worst case viewpoint, whereby he optimizes his choice against the worst possible selection by the other player, the first player must minimize $\max_z x'Az$ and the second player must maximize $\min_x x'Az$. The main result, a special case of the existence result we will prove shortly, is that these two optimal values are equal, implying that there is an amount that can be meaningfully viewed as the value of the game for its participants.

Then we have

$$\sup_{z\in Z}\inf_{x\in X}\phi(x,z)=\inf_{x\in X}\phi(x,z^*)\le\phi(x^*,z^*)\le\sup_{z\in Z}\phi(x^*,z)=\inf_{x\in X}\sup_{z\in Z}\phi(x,z). \tag{1.23}$$

If the minimax equality [cf. Eq. (1.18)] holds, then equality holds throughout above, so that

$$\sup_{z\in Z}\phi(x^*,z)=\phi(x^*,z^*)=\inf_{x\in X}\phi(x,z^*), \tag{1.24}$$

or equivalently

$$\phi(x^*,z)\le\phi(x^*,z^*)\le\phi(x,z^*),\qquad\forall\ x\in X,\ z\in Z. \tag{1.25}$$

A pair of vectors $x^*\in X$ and $z^*\in Z$ satisfying the two above (equivalent) relations is called a *saddle point* of ϕ (cf. Fig. 1.3.5).

We have thus shown that if the minimax equality (1.18) holds, any vectors x^* and z^* that are optimal solutions of problems (1.21) and (1.22), respectively, form a saddle point. Conversely, if (x^*,z^*) is a saddle point, then the definition (1.24) implies that

$$\inf_{x\in X}\sup_{z\in Z}\phi(x,z)\le\sup_{z\in Z}\inf_{x\in X}\phi(x,z).$$

This, together with the minimax inequality (1.20) guarantee that the minimax equality (1.18) holds and from Eq. (1.23), x^* and z^* are optimal solutions of problems (1.21) and (1.22), respectively.

We summarize the above discussion in the following proposition.

Proposition 1.3.7: A pair (x^*,z^*) is a saddle point of ϕ if and only if the minimax equality (1.18) holds, and x^* and z^* are optimal solutions of problems (1.21) and (1.22), respectively.

Note a simple consequence of the above proposition: the set of saddle points, *when nonempty*, is the Cartesian product $X^*\times Z^*$, where X^* and Z^* are the sets of optimal solutions of problems (1.21) and (1.22), respectively. In other words x^* and z^* can be *independently* chosen within the sets X^* and Z^*, respectively, to form a saddle point. Note also that if the minimax equality (1.18) does not hold, there is no saddle point, even if the sets X^* and Z^* are nonempty.

One can visualize saddle points in terms of the sets of minimizing points over X for fixed $z\in Z$ and maximizing points over Z for fixed $x\in X$:

$$\hat X(z)=\big\{\hat x\mid\hat x\text{ minimizes }\phi(x,z)\text{ over }X\big\},$$

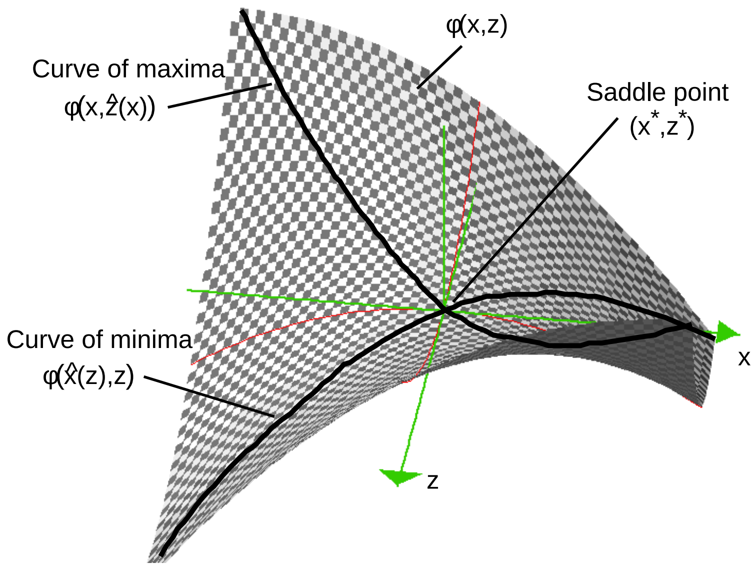

Figure 1.3.5. Illustration of a saddle point of a function $\phi(x,z)$ over $x \in X$ and $z \in Z$; the function plotted here is $\phi(x,z) = \frac{1}{2}(x^2 + 2xz - z^2)$. Let

$$\hat{x}(z) = \arg\min_{x \in X} \phi(x,z), \qquad \hat{z}(x) = \arg\max_{z \in Z} \phi(x,z).$$

In the case illustrated, $\hat{x}(z)$ and $\hat{z}(x)$ consist of unique minimizing and maximizing points, respectively, so we view $\hat{x}(z)$ and $\hat{z}(x)$ as (single-valued) functions; otherwise $\hat{x}(z)$ and $\hat{z}(x)$ should be viewed as set-valued mappings. We consider the corresponding curves $\phi\big(\hat{x}(z),z\big)$ and $\phi\big(x,\hat{z}(x)\big)$. By definition, a pair (x^*,z^*) is a saddle point if and only if

$$\max_{z \in Z} \phi(x^*,z) = \phi(x^*,z^*) = \min_{x \in X} \phi(x,z^*),$$

or equivalently, if (x^*,z^*) lies on both curves [$x^* = \hat{x}(z^*)$ and $z^* = \hat{z}(x^*)$]. At such a pair, we also have

$$\max_{z \in Z} \phi\big(\hat{x}(z),z\big) = \max_{z \in Z} \min_{x \in X} \phi(x,z) = \phi(x^*,z^*) = \min_{x \in X} \max_{z \in Z} \phi(x,z) = \min_{x \in X} \phi\big(x,\hat{z}(x)\big),$$

so that

$$\phi\big(\hat{x}(z),z\big) \le \phi(x^*,z^*) \le \phi\big(x,\hat{z}(x)\big), \qquad \forall\, x \in X,\ z \in Z$$

(see Prop. 1.3.7). Visually, the curve of maxima $\phi\big(x,\hat{z}(x)\big)$ must lie "above" the curve of minima $\phi\big(\hat{x}(z),z\big)$ (completely, i.e., for all $x \in X$ and $z \in Z$), but the two curves should meet at (x^*,z^*).

$$\hat{Z}(x) = \big\{\hat{z} \mid \hat{z} \text{ maximizes } \phi(x,z) \text{ over } Z\big\}.$$

The definition implies that the pair (x^*, z^*) is a saddle point if and only if it is a "point of intersection" of $\hat{X}(\cdot)$ and $\hat{Z}(\cdot)$ in the sense that

$$x^* \in \hat{X}(z^*), \qquad z^* \in \hat{Z}(x^*);$$

see Fig. 1.3.5.

We now consider conditions that guarantee the existence of a saddle point. We have the following classical result.

Proposition 1.3.8: (Saddle Point Theorem) Let X and Z be closed, convex subsets of $\Re^n$ and $\Re^m$, respectively, and let $\phi : X \times Z \mapsto \Re$ be a function. Assume that for each $z \in Z$, the function $\phi(\cdot, z) : X \mapsto \Re$ is convex and lower semicontinuous, and for each $x \in X$, the function $\phi(x, \cdot) : Z \mapsto \Re$ is concave and upper semicontinuous. Then there exists a saddle point of ϕ under any one of the following four conditions:

(1) X and Z are compact.

(2) Z is compact and there exists a vector $\overline{z} \in Z$ such that $\phi(\cdot, \overline{z})$ is coercive.

(3) X is compact and there exists a vector $\overline{x} \in X$ such that $-\phi(\overline{x}, \cdot)$ is coercive.

(4) There exist vectors $\overline{x} \in X$ and $\overline{z} \in Z$ such that $\phi(\cdot, \overline{z})$ and $-\phi(\overline{x}, \cdot)$ are coercive.

Proof: We first prove the result under the assumption that X and Z are compact [condition (1)], and the additional assumption that $\phi(x, \cdot)$ is strictly concave for each $x \in X$.

By Weierstrass' Theorem (Prop. 1.3.1), the function

$$f(x) = \max_{z \in Z} \phi(x, z), \qquad x \in X,$$

is real-valued and the maximum above is attained for each $x \in X$ at a point denoted $\hat{z}(x)$, which is unique by the strict concavity assumption. Furthermore, by Prop. 1.2.3(c), f is lower semicontinuous, so again by Weierstrass' Theorem, f attains a minimum over $x \in X$ at some point x^*. Let $z^* = \hat{z}(x^*)$, so that

$$\phi(x^*, z) \le \phi(x^*, z^*) = f(x^*), \qquad \forall\ z \in Z. \tag{1.26}$$

We will show that (x^*, z^*) is a saddle point of ϕ, and in view of the above relation, it will suffice to show that $\phi(x^*, z^*) \le \phi(x, z^*)$ for all $x \in X$.

Choose any $x \in X$ and let

$$x_k = \frac{1}{k}x + \left(1 - \frac{1}{k}\right)x^*, \qquad z_k = \hat{z}(x_k), \qquad k = 1, 2, \ldots. \tag{1.27}$$

Let $\overline{z}$ be any limit point of $\{z_k\}$ corresponding to a subsequence $\mathcal{K}$ of positive integers. Using the convexity of $\phi(\cdot, z_k)$, we have

$$f(x^*) \le f(x_k) = \phi(x_k, z_k) \le \frac{1}{k}\phi(x, z_k) + \left(1 - \frac{1}{k}\right)\phi(x^*, z_k). \tag{1.28}$$

Taking the limit as $k \to \infty$ and $k \in \mathcal{K}$, and using the upper semicontinuity of $\phi(x^*, \cdot)$, we obtain

$$f(x^*) \le \limsup_{k\to\infty,\, k\in\mathcal{K}} \phi(x^*, z_k) \le \phi(x^*, \overline{z}) \le \max_{z\in Z} \phi(x^*, z) = f(x^*).$$

Hence equality holds throughout above, and it follows that $\phi(x^*, z) \le \phi(x^*, \overline{z}) = f(x^*)$ for all $z \in Z$. Since z^* is the unique maximizer of $\phi(x^*, \cdot)$ over Z, we see that $\overline{z} = \hat{z}(x^*) = z^*$, so that $\{z_k\}$ has z^* as its unique limit point, *independently of the choice of the vector x within X* (this is the fine point in the argument where the strict concavity assumption is needed).

We have shown that for any $x \in X$, the corresponding sequence $\{z_k\}$ [cf. Eq. (1.27)], converges to the vector $z^* = \hat{z}(x^*)$, satisfies Eq. (1.28), and also, in view of Eq. (1.26), satisfies

$$\phi(x^*, z_k) \le \phi(x^*, z^*) = f(x^*), \qquad k = 1, 2, \ldots$$

Combining this last relation with Eq. (1.28), we have

$$\phi(x^*, z^*) \le \frac{1}{k}\phi(x, z_k) + \left(1 - \frac{1}{k}\right)\phi(x^*, z^*), \qquad \forall\, x \in X,$$

or

$$\phi(x^*, z^*) \le \phi(x, z_k), \qquad \forall\, x \in X.$$

Taking the limit as $z_k \to z^*$, and using the upper semicontinuity of $\phi(x, \cdot)$, we obtain

$$\phi(x^*, z^*) \le \phi(x, z^*), \qquad \forall\, x \in X.$$

Combining this relation with Eq. (1.26), we see that (x^*, z^*) is a saddle point of ϕ.

Next, we remove the assumption of strict concavity of $\phi(x^*, \cdot)$. We introduce the functions $\phi_k : X \times Z \mapsto \Re$ given by

$$\phi_k(x, z) = \phi(x, z) - \frac{1}{k}\|z\|^2, \qquad k = 1, 2, \ldots$$

Since $\phi_k(x^*, \cdot)$ is strictly concave, there exists (based on what has already been proved) a saddle point (x_k^*, z_k^*) of ϕ_k, satisfying

$$\phi(x_k^*, z) - \frac{1}{k}\|z\|^2 \le \phi(x_k^*, z_k^*) - \frac{1}{k}\|z_k^*\|^2 \le \phi(x, z_k^*) - \frac{1}{k}\|z_k^*\|^2, \ \ \forall\, x \in X,\ z \in Z.$$

Let (x^*, z^*) be a limit point of (x_k^*, z_k^*). By taking limit as $k \to \infty$ and by using the semicontinuity assumptions on ϕ, it follows that

$$\phi(x^*, z) \le \liminf_{k\to\infty} \phi(x_k^*, z) \le \limsup_{k\to\infty} \phi(x, z_k^*) \le \phi(x, z^*), \quad \forall\, x \in X,\ z \in Z. \tag{1.29}$$

By setting alternately $x = x^*$ and $z = z^*$ in the above relation, we obtain

$$\phi(x^*, z) \le \phi(x^*, z^*) \le \phi(x, z^*), \qquad \forall\, x \in X,\ z \in Z,$$

so (x^*, z^*) is a saddle point of ϕ.

We now prove the existence of a saddle point under conditions (2), (3), or (4). For each integer k, we introduce the convex and compact sets

$$X_k = \big\{x \in X \mid \|x\| \le k\big\}, \qquad Z_k = \big\{z \in Z \mid \|z\| \le k\big\}.$$

Choose $\overline{k}$ large enough so that

$$\overline{k} \ge \begin{cases} \max_{z\in Z} \|z\| & \text{if condition (2) holds,} \\ \max_{x\in X} \|x\| & \text{if condition (3) holds,} \\ \max\{\|\overline{x}\|,\ \|\overline{z}\|\} & \text{if condition (4) holds,} \end{cases}$$

and $X_{\overline{k}}$ is nonempty under condition (2) while $Z_{\overline{k}}$ is nonempty under condition (3). Note that for $k \ge \overline{k}$, the sets X_k and Z_k are nonempty. Furthermore, $Z = Z_k$ under condition (2) and $X = X_k$ under condition (3). Using the result already shown under condition (1), for each $k \ge \overline{k}$, there exists a saddle point over $X_k \times Z_k$, i.e., a pair (x_k, z_k) such that

$$\phi(x_k, z) \le \phi(x_k, z_k) \le \phi(x, z_k), \qquad \forall\, x \in X_k,\ z \in Z_k. \tag{1.30}$$

Assume that condition (2) holds. Then, since Z is compact, $\{z_k\}$ is bounded. If $\{x_k\}$ were unbounded, the coercivity of $\phi(\cdot, \overline{z})$ would imply that $\phi(x_k, \overline{z}) \to \infty$ and from Eq. (1.30) it would follow that $\phi(x, z_k) \to \infty$ for all $x \in X$. By the upper semicontinuity of $\phi(x, \cdot)$, this contradicts the boundedness of $\{z_k\}$. Hence $\{x_k\}$ must be bounded, and (x_k, z_k) must have a limit point (x^*, z^*). Taking limit as $k \to \infty$ in Eq. (1.30), and using the semicontinuity assumptions on ϕ, it follows that

$$\phi(x^*, z) \le \liminf_{k\to\infty} \phi(x_k, z) \le \limsup_{k\to\infty} \phi(x, z_k) \le \phi(x, z^*), \quad \forall\, x \in X,\ z \in Z. \tag{1.31}$$

By alternately setting $x = x^*$ and $z = z^*$ in the above relation, we see that (x^*, z^*) is a saddle point of ϕ.

A symmetric argument with the obvious modifications, shows the result under condition (3). Finally, under condition (4), note that Eq. (1.30) yields for all k,

$$\phi(x_k, \overline{z}) \leq \phi(x_k, z_k) \leq \phi(\overline{x}, z_k).$$

If $\{x_k\}$ were unbounded, the coercivity of $\phi(\cdot, \overline{z})$ would imply that $\phi(x_k, \overline{z}) \to \infty$ and hence $\phi(\overline{x}, z_k) \to \infty$, which together with the upper semicontinuity of $\phi(x, \cdot)$ violates the coercivity of $-\phi(\overline{x}, \cdot)$. Hence $\{x_k\}$ must be bounded, and a symmetric argument shows that $\{z_k\}$ must be bounded. Thus (x_k, z_k) must have a limit point (x^*, z^*). The result then follows from Eq. (1.31), similar to the case where condition (2) holds. **Q.E.D.**

It is easy to construct examples showing that the convexity of X and Z are essential assumptions for the above proposition (this is also evident from Fig. 1.3.5). The assumptions of compactness/coercivity and lower/upper semicontinuity of ϕ are essential for existence of a saddle point (just as they are essential in Weierstrass' Theorem). An interesting question is whether convexity/concavity and lower/upper semicontinuity of ϕ are sufficient to guarantee the minimax equality (1.18). Unfortunately this is not so for reasons that also touch upon some of the deeper aspects of duality theory (see Chapter 3). Here is an example:

Example 1.3.1

Let

$$X = \{x \in \Re^2 \mid x \geq 0\}, \qquad Z = \{z \in \Re \mid z \geq 0\},$$

and let

$$\phi(x, z) = e^{-\sqrt{x_1 x_2}} + z x_1,$$

which satisfy the convexity/concavity and lower/upper semicontinuity assumptions of Prop. 1.3.8. For all $z \geq 0$, we have

$$\inf_{x \geq 0} \left\{ e^{-\sqrt{x_1 x_2}} + z x_1 \right\} = 0,$$

since the expression in braces is nonnegative for $x \geq 0$ and can approach zero by taking $x_1 \to 0$ and $x_1 x_2 \to \infty$. Hence

$$\sup_{z \geq 0} \inf_{x \geq 0} \phi(x, z) = 0.$$

We also have for all $x \geq 0$

$$\sup_{z \geq 0} \left\{ e^{-\sqrt{x_1 x_2}} + z x_1 \right\} = \begin{cases} 1 & \text{if } x_1 = 0, \\ \infty & \text{if } x_1 > 0. \end{cases}$$

Hence

$$\inf_{x \geq 0} \sup_{z \geq 0} \phi(x, z) = 1,$$

so $\inf_{x \geq 0} \sup_{z \geq 0} \phi(x, z) > \sup_{z \geq 0} \inf_{x \geq 0} \phi(x, z)$. The difficulty here is that the compactness/coercivity assumptions of Prop. 1.3.8 are violated.

On the other hand, with convexity/concavity and lower/upper semicontinuity of ϕ, together with one additional assumption, given in the following proposition, it is possible to guarantee the minimax equality (1.18) without guaranteeing the existence of a saddle point. The proof of the proposition requires the machinery of hyperplanes, and will be given at the end of the next section.

Proposition 1.3.9: (Minimax Theorem) Let X and Z be nonempty convex subsets of $\Re^n$ and $\Re^m$, respectively, and let $\phi : X \times Z \mapsto \Re$ be a function. Assume that for each $z \in Z$, the function $\phi(\cdot, z) : X \mapsto \Re$ is convex and lower semicontinuous, and for each $x \in X$, the function $\phi(x, \cdot) : Z \mapsto \Re$ is concave and upper semicontinuous. Let the function $p : \Re^m \mapsto [-\infty, \infty]$ be defined by

$$p(u) = \inf_{x \in X} \sup_{z \in Z} \{\phi(x, z) - u'z\}. \tag{1.32}$$

Assume that

$$-\infty < p(0) < \infty,$$

and that $p(0) \leq \liminf_{k \to \infty} p(u_k)$ for every sequence $\{u_k\}$ with $u_k \to 0$. Then, the minimax equality holds, i.e.,

$$\sup_{z \in Z} \inf_{x \in X} \phi(x, z) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z).$$

Within the duality context discussed in the beginning of this subsection, the minimax equality implies that there is no duality gap. The above theorem indicates that, aside from convexity and semicontinuity assumptions, the properties of the function p around $u = 0$ are critical. As an illustration, note that in Example 1.3.1, p is given by

$$\begin{aligned} p(u) &= \inf_{x \geq 0} \sup_{z \geq 0} \left\{ e^{-\sqrt{x_1 x_2}} + z(x_1 - u) \right\} \\ &= \begin{cases} \infty & \text{if } u < 0, \\ 1 & \text{if } u = 0, \\ 0 & \text{if } u > 0. \end{cases} \end{aligned}$$

Thus it can be seen that even though $p(0)$ is finite, p is not lower semicontinuous at 0. As a result the assumptions of Prop. 1.3.9 are violated and the minimax equality does not hold.

The significance of the function p within the context of convex programming and duality will be elaborated on and clarified further in Chapter 4. Furthermore, the proof of the above theorem, given in Section 1.4.2, will indicate some common circumstances where p has the desired properties around $u = 0$.

EXERCISES

1.3.1

Let $f : \Re^n \mapsto \Re$ be a convex function, let X be a closed convex set, and assume that f and X have no common direction of recession. Let X^* be the optimal solution set (nonempty and compact by Prop. 1.3.5) and let $f^* = \inf_{x \in X} f(x)$. Show that:

(a) For every $\epsilon > 0$ there exists a $\delta > 0$ such that every vector $x \in X$ with $f(x) \leq f^* + \delta$ satisfies $\min_{x^* \in X^*} \|x - x^*\| \leq \epsilon$.

(b) Every sequence $\{x_k\} \subset X$ satisfying $\lim_{k \to \infty} f(x_k) \to f^*$ is bounded and all its limit points belong to X^*.

1.3.2

Let C be a convex set and S be a subspace. Show that projection on S is a linear transformation and use this to show that the projection of the set C on S is a convex set, which is compact if C is compact. Is the projection of C always closed if C is closed?

1.3.3 (Existence of Solution of Quadratic Programs)

This exercise deals with an extension of Prop. 1.3.6 to the case where the quadratic cost may not be convex. Consider a problem of the form

$$\begin{aligned} &\text{minimize} \quad c'x + \frac{1}{2}x'Qx \\ &\text{subject to} \quad Ax \leq b, \end{aligned}$$

where Q is a symmetric (not necessarily positive semidefinite) matrix, c and b are given vectors, and A is a matrix. Show that the following are equivalent:

(a) There exists at least one optimal solution.

(b) The cost function is bounded below over the constraint set.

(c) The problem has at least one feasible solution, and for any feasible $\overline{x}$, there is no $y \in \Re^n$ such that $Ay \leq 0$ and either $y'Qy < 0$ or $y'Qy = 0$ and $(c + Q\overline{x})'y < 0$.

1.3.4

Let $C \subset \Re^n$ be a nonempty convex set, and let $f : \Re^n \mapsto \Re$ be an affine function such that $\inf_{x \in C} f(x)$ is attained at some $x^* \in \text{ri}(C)$. Show that $f(x) = f(x^*)$ for all $x \in C$.

1.3.5 (Existence of Optimal Solutions)

Let $f : \Re^n \mapsto \Re$ be a convex function, and consider the problem of minimizing f over a closed and convex set X. Suppose that f attains a minimum along all half lines of the form $\{x + \alpha y \mid \alpha \geq 0\}$ where $x \in X$ and y is in the recession cone of X. Show that we may have $\inf_{x \in X} f(x) = -\infty$. *Hint*: Use the case $n = 2$, $X = \Re^2$, $f(x) = \min_{z \in C} \|z - x\|^2 - x_1$, where $C = \big\{(x_1, x_2) \mid x_1^2 \leq x_2\big\}$.

1.3.6 (Saddle Points in two Dimensions)

Consider a function ϕ of two real variables x and z taking values in compact intervals X and Z, respectively. Assume that for each $z \in Z$, the function $\phi(\cdot, z)$ is minimized over X at a unique point denoted $\hat{x}(z)$. Similarly, assume that for each $x \in X$, the function $\phi(x, \cdot)$ is maximized over Z at a unique point denoted $\hat{z}(x)$. Assume further that the functions $\hat{x}(z)$ and $\hat{z}(x)$ are continuous over Z and X, respectively. Show that ϕ has a unique saddle point (x^*, z^*). Use this to investigate the existence of saddle points of $\phi(x, z) = x^2 + z^2$ over $X = [0, 1]$ and $Z = [0, 1]$.

1.4 HYPERPLANES

Some of the most important principles in convexity and optimization, including duality, revolve around the use of hyperplanes, i.e., $(n-1)$-dimensional affine sets. A hyperplane has the property that it divides the space into two halfspaces. We will see shortly that a distinguishing feature of a closed convex set is that it is the intersection of all the halfspaces that contain it. Thus any closed convex set can be described in dual fashion: (a) as the union of all points contained in the set, and (b) as the intersection of all halfspaces containing the set. This fundamental principle carries over to a closed convex function, once the function is specified in terms of its (closed and convex) epigraph. In this section, we develop the principle just outlined, and we apply it to a construction that is central in duality theory. We then use this construction to prove the minimax theorem given at the end of the preceding section.

1.4.1 Support and Separation Theorems

A *hyperplane* is a set of the form $\{x \mid a'x = b\}$, where $a \in \Re^n$, $a \neq 0$, and $b \in \Re$, as illustrated in Fig. 1.4.1. An equivalent definition is that a hyperplane in $\Re^n$ is an affine set of dimension $n-1$. The vector a is called the *normal* vector of the hyperplane (it is orthogonal to the difference $x-y$ of any two vectors x and y that belong to the hyperplane). The two sets

$$\{x \mid a'x \geq b\}, \qquad \{x \mid a'x \leq b\},$$

are called the *halfspaces* associated with the hyperplane (also referred to as the *positive and negative halfspaces*, respectively). We have the following result, which is also illustrated in Fig. 1.4.1. The proof is based on the Projection Theorem and is illustrated in Fig. 1.4.2.

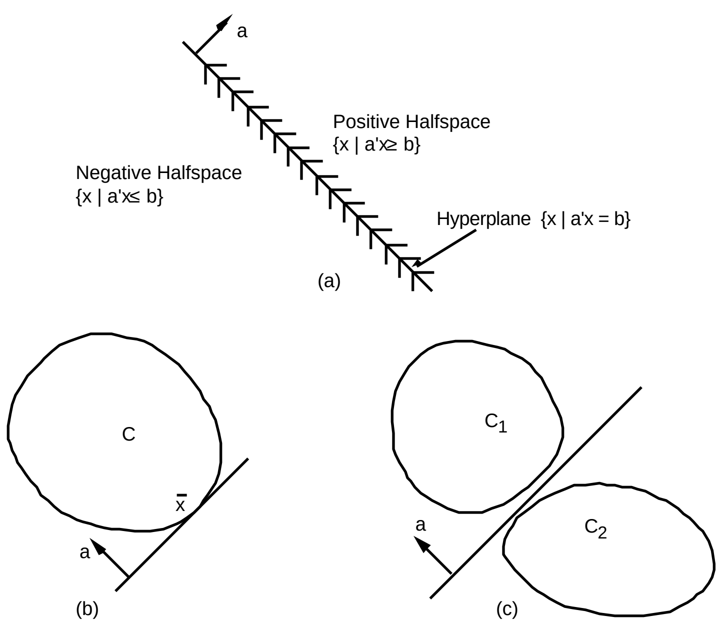

Figure 1.4.1. (a) A hyperplane $\{x \mid a'x = b\}$ divides the space into two halfspaces as illustrated. (b) Geometric interpretation of the Supporting Hyperplane Theorem. (c) Geometric interpretation of the Separating Hyperplane Theorem.

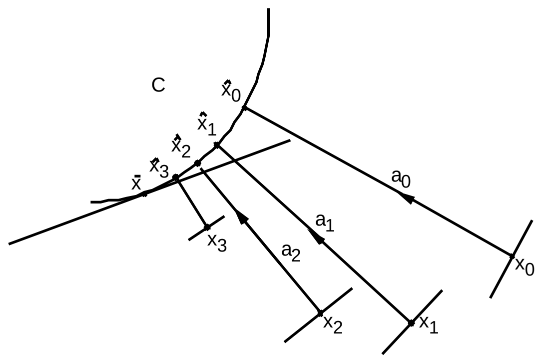

Figure 1.4.2. Illustration of the proof of the Supporting Hyperplane Theorem for the case where the vector $\overline{x}$ belongs to the closure of C. We choose a sequence $\{x_k\}$ of vectors not belonging to the closure of C which converges to $\overline{x}$, and we project x_k on the closure of C. We then consider, for each k, the hyperplane that is orthogonal to the line segment connecting x_k and its projection, and passes through x_k. These hyperplanes "converge" to a hyperplane that supports C at $\overline{x}$.

Proposition 1.4.1: (Supporting Hyperplane Theorem) If $C \subset \Re^n$ is a convex set and $\overline{x}$ is a point that does not belong to the interior of C, there exists a vector $a \neq 0$ such that

$$a'x \geq a'\overline{x}, \qquad \forall\ x \in C. \tag{1.33}$$

Proof: Consider the closure $\mathrm{cl}(C)$ of C, which is a convex set by Prop. 1.2.1(d). Let $\{x_k\}$ be a sequence of vectors not belonging to $\mathrm{cl}(C)$, which converges to $\overline{x}$; such a sequence exists because $\overline{x}$ does not belong to the interior of C. If $\hat{x}_k$ is the projection of x_k on $\mathrm{cl}(C)$, we have by part (b) of the Projection Theorem (Prop. 1.3.3)

$$(\hat{x}_k - x_k)'(x - \hat{x}_k) \geq 0, \qquad \forall\ x \in \mathrm{cl}(C).$$

Hence we obtain for all $x \in \mathrm{cl}(C)$ and k,

$$(\hat{x}_k-x_k)'x \geq (\hat{x}_k-x_k)'\hat{x}_k = (\hat{x}_k-x_k)'(\hat{x}_k-x_k)+(\hat{x}_k-x_k)'x_k \geq (\hat{x}_k-x_k)'x_k.$$

We can write this inequality as

$$a_k'x \geq a_k'x_k, \qquad \forall\ x \in \mathrm{cl}(C),\ k = 0, 1, \ldots, \tag{1.34}$$

where

$$a_k = \frac{\hat{x}_k - x_k}{\|\hat{x}_k - x_k\|}.$$

We have $\|a_k\| = 1$ for all k, and hence the sequence $\{a_k\}$ has a subsequence that converges to a nonzero limit a. By considering Eq. (1.34) for all a_k belonging to this subsequence and by taking the limit as $k \to \infty$, we obtain Eq. (1.33). **Q.E.D.**

Proposition 1.4.2: (Separating Hyperplane Theorem) If C_1 and C_2 are two nonempty, disjoint, and convex subsets of $\Re^n$, there exists a hyperplane that separates them, i.e., a vector $a \neq 0$ such that

$$a'x_1 \leq a'x_2, \qquad \forall\ x_1 \in C_1,\ x_2 \in C_2. \tag{1.35}$$

Proof: Consider the convex set

$$C = \{x \mid x = x_2 - x_1,\, x_1 \in C_1,\, x_2 \in C_2\}.$$

Since C_1 and C_2 are disjoint, the origin does not belong to C, so by the Supporting Hyperplane Theorem there exists a vector $a \neq 0$ such that

$$0 \leq a'x, \qquad \forall\ x \in C,$$

which is equivalent to Eq. (1.35). **Q.E.D.**

Proposition 1.4.3: (Strict Separation Theorem) If C_1 and C_2 are two nonempty, disjoint, and convex sets such that C_1 is closed and C_2 is compact, there exists a hyperplane that strictly separates them, i.e., a vector $a \neq 0$ and a scalar b such that

$$a'x_1 < b < a'x_2, \qquad \forall\ x_1 \in C_1,\ x_2 \in C_2. \tag{1.36}$$

Proof: Consider the problem

$$\begin{array}{ll} \text{minimize} & \|x_1 - x_2\| \\ \text{subject to} & x_1 \in C_1,\ x_2 \in C_2. \end{array}$$

This problem is equivalent to minimizing the distance $d(x_2, C_1)$ over $x_2 \in C_2$. Since C_2 is compact, and the distance function is convex (cf. Prop. 1.3.3) and hence continuous (cf. Prop. 1.2.12), the problem has at least one solution $(\overline{x}_1, \overline{x}_2)$ by Weierstrass' Theorem (cf. Prop. 1.3.1). Let

$$a = \frac{\overline{x}_2 - \overline{x}_1}{2}, \qquad \overline{x} = \frac{\overline{x}_1 + \overline{x}_2}{2}, \qquad b = a'\overline{x}.$$

Then, $a \neq 0$, since $\overline{x}_1 \in C_1$, $\overline{x}_2 \in C_2$, and C_1 and C_2 are disjoint. The hyperplane

$$\{x \mid a'x = b\}$$

contains $\overline{x}$, and it can be seen from the preceding discussion that $\overline{x}_1$ is the projection of $\overline{x}$ on C_1, and $\overline{x}_2$ is the projection of $\overline{x}$ on C_2 (see Fig. 1.4.3). By Prop. 1.3.3(b), we have

$$(\overline{x} - \overline{x}_1)'(x_1 - \overline{x}_1) \leq 0, \qquad \forall\, x_1 \in C_1$$

or equivalently, since $\overline{x} - \overline{x}_1 = a$,

$$a'x_1 \leq a'\overline{x}_1 = a'\overline{x} + a'(\overline{x}_1 - \overline{x}) = b - \|a\|^2 < b, \qquad \forall\, x_1 \in C_1.$$

Thus, the left-hand side of Eq. (1.36) is proved. The right-hand side is proved similarly. **Q.E.D.**

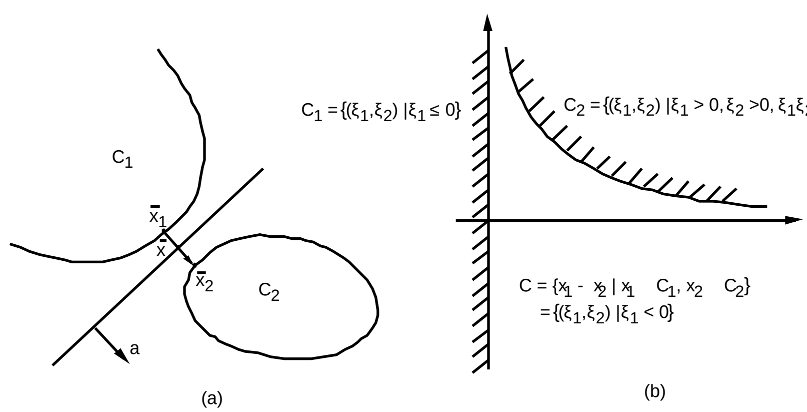

Figure 1.4.3. (a) Illustration of the construction of a strictly separating hyperplane of two disjoint closed convex sets C_1 and C_2 one of which is also bounded (cf. Prop. 1.4.3). (b) An example showing that if none of the two sets is compact, there may not exist a strictly separating hyperplane. This is due to the fact that the distance $d(x_2, C_1)$ can approach 0 arbitrarily closely as x_2 ranges over C_2, but can never reach 0.

The preceding proposition may be used to provide a fundamental characterization of closed convex sets.

Proposition 1.4.4: The closure of the convex hull of a set $C \subset \Re^n$ is the intersection of the halfspaces that contain C. In particular, a closed and convex set is the intersection of the halfspaces that contain it.

Proof: Assume first that C is closed and convex. Then C is contained in the intersection of the halfspaces that contain C, so we focus on proving the reverse inclusion. Let $x \notin C$. Applying the Strict Separation Theorem (Prop. 1.4.3) to the sets C and $\{x\}$, we see that there exists a halfspace containing C but not containing x. Hence, if $x \notin C$, then x cannot belong to the intersection of the halfspaces containing C, proving that C contains that intersection. Thus the result is proved for the case where C is closed and convex.

Consider the case of a general set C. Each halfspace H that contains C must also contain $\mathrm{conv}(C)$ (since H is convex), and also $\mathrm{cl}\big(\mathrm{conv}(C)\big)$ (since H is closed). Hence the intersection of all halfspaces containing C and the intersection of all halfspaces containing $\mathrm{cl}\big(\mathrm{conv}(C)\big)$ coincide. From what has been proved for the case of a closed convex set, the latter intersection is equal to $\mathrm{cl}\big(\mathrm{conv}(C)\big)$. **Q.E.D.**

1.4.2 Nonvertical Hyperplanes and the Minimax Theorem

In the context of optimization theory, supporting hyperplanes are often used in conjunction with epigraphs of functions f on $\Re^n$. Since $\mathrm{epi}(f) = \{(x,w) \mid w \geq f(x)\}$ is a subset of $\Re^{n+1}$, the normal vector of a hyperplane is a nonzero vector of the form (μ, β), where $\mu \in \Re^n$ and $\beta \in \Re$. We say that the hyperplane is *horizontal* if $\mu = 0$ and we say that it is *vertical* if $\beta = 0$.

Note that f is bounded if and only if $\mathrm{epi}(f)$ is contained in a halfspace corresponding to a horizontal hyperplane. Note also that if a hyperplane with normal (μ, β) is nonvertical (i.e., $\beta \neq 0$), then it crosses the $(n+1)$st axis (the axis associated with w) at a unique point. If $(\overline{x}, \overline{w})$ is any vector on the hyperplane, the crossing point has the form $(0, \xi)$, where

$$\xi = \frac{\mu'}{\beta}\overline{x} + \overline{w}, \tag{1.37}$$

since from the hyperplane equation, we have $(0,\xi)'(\mu,\beta) = (\overline{x}, \overline{w})'(\mu, \beta)$. On the other hand, it can be seen that if the hyperplane is vertical, it either contains the entire $(n+1)$st axis, or else it does not cross it at all; see Fig. 1.4.4.

Vertical lines in $\Re^{n+1}$ are sets of the form $\{(\overline{x}, w) \mid w \in \Re\}$, where $\overline{x}$ is a fixed vector in $\Re^n$. It can be seen that vertical hyperplanes, as well as the corresponding halfspaces, consist of the union of the vertical lines that pass through their points. If $f(x) > -\infty$ for all x, then $\mathrm{epi}(f)$ cannot contain a vertical line, and it appears plausible that $\mathrm{epi}(f)$ is contained in some halfspace corresponding to a nonvertical hyperplane. We prove this fact in greater generality in the following proposition, which will also be useful as a first step in the subsequent development.

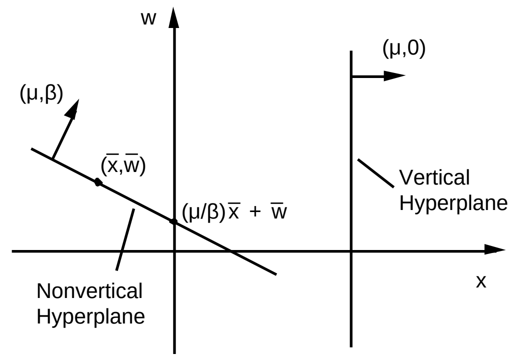

Figure 1.4.4. Illustration of vertical and nonvertical hyperplanes in $\Re^{n+1}$.

Proposition 1.4.5: Let C be a nonempty convex subset of $\Re^{n+1}$, and let the points of $\Re^{n+1}$ be denoted by (u, w), where $u \in \Re^n$ and $w \in \Re$.

(a) If C contains no vertical lines, then C is contained in a halfspace corresponding to a nonvertical hyperplane; that is, there exists (μ, β) such that $\mu \in \Re^n$, $\beta \in \Re$, $\beta \neq 0$, and $\gamma \in \Re$ such that

$$\mu' u + \beta w \geq \gamma, \qquad \forall\ (u, w) \in C.$$

(b) If C contains no vertical lines and $(\overline{u}, \overline{w})$ does not belong to the closure of C, there exists a nonvertical hyperplane strictly separating $(\overline{u}, \overline{w})$ from C.

Proof: We first note that if C contains no vertical lines, then $\text{ri}(C)$ contains no vertical lines, which implies that $\text{cl}(C)$ contains no vertical lines, since the recession cones of $\text{cl}(C)$ and $\text{ri}(C)$ coincide, as shown in the exercises to Section 1.2. Thus if we prove the result assuming that C is closed, the proof for the case where C is not closed will readily follow by replacing C with $\text{cl}(C)$. Hence we assume without loss of generality that C is closed.

(a) By Prop. 1.4.4, C is the intersection of all halfspaces that contain it. If every hyperplane containing C in one of its halfspaces were vertical, we would have

$$C = \cap_{i \in I} \big\{ (u, w) \mid \mu_i' u \geq \gamma_i \big\}$$

for a collection of nonzero vectors μ_i, $i \in I$, and scalars γ_i, $i \in I$. Then for every $(\overline{u}, \overline{w}) \in C$, the vertical line $\big\{ (\overline{u}, w) \mid w \in \Re \big\}$ also belongs to C.

It follows that if no vertical line belongs to C, there exists a nonvertical hyperplane containing C.

(b) Since $(\overline{u}, \overline{w})$ does not belong to C, there exists a hyperplane strictly separating $(\overline{u}, \overline{w})$ from C (cf. Prop. 1.4.3). If this hyperplane is nonvertical, we are done, so assume otherwise. Then we have a nonzero vector $\overline{\mu}$ and a scalar $\overline{\gamma}$ such that

$$\overline{\mu}' u > \overline{\gamma} > \overline{\mu}' \, \overline{u}, \qquad \forall \ (u, w) \in C.$$

Consider a nonvertical hyperplane containing C in one of its subspaces, so that for some (μ, β) and γ, with $\beta \neq 0$, we have

$$\mu' u + \beta w \geq \gamma, \qquad \forall \ (u, w) \in C.$$

By multiplying this relation with any $\epsilon > 0$ and adding it to the preceding relation, we obtain

$$(\overline{\mu} + \epsilon \mu)' u + \epsilon \beta w > \overline{\gamma} + \epsilon \gamma, \qquad \forall \ (u, w) \in C.$$

Since $\overline{\gamma} > ol\mu' \, \overline{u}$, there is a small enough ϵ such that

$$\overline{\gamma} + \epsilon \gamma > (\overline{\mu} + \epsilon \mu)' \overline{u} + \epsilon \beta \overline{w}.$$

From the above two relations, we obtain

$$(\overline{\mu} + \epsilon \mu)' u + \epsilon \beta w > (\overline{\mu} + \epsilon \mu)' \overline{u} + \epsilon \beta \overline{w}, \qquad \forall \ (u, w) \in C,$$

implying that there is a nonvertical hyperplane with normal $(\overline{\mu} + \epsilon \mu, \epsilon \beta)$ that strictly separates $(\overline{u}, \overline{w})$ from C. **Q.E.D.**

Min Common/Max Crossing Duality

Hyperplanes allow insightful visualization of duality concepts. This is particularly so in a construction involving two simple optimization problems, which we now discuss. These problems will prove particularly relevant in the context of duality, and will also form the basis for the proof of the Minimax Theorem, stated at the end of the preceding section.

Let S be a subset of $\Re^{n+1}$ and consider the following two problems.

(a) *Min Common Point Problem*: Among all points that are common to both S and the $(n+1)$st axis, we want to find one whose $(n+1)$st component is minimum.

(b) *Max Crossing Point Problem*: Consider nonvertical hyperplanes that contain S in their corresponding "upper" halfspace, i.e., the halfspace that contains the vertical halfline $\{(0, w) \mid w \geq 0\}$ in its recession cone

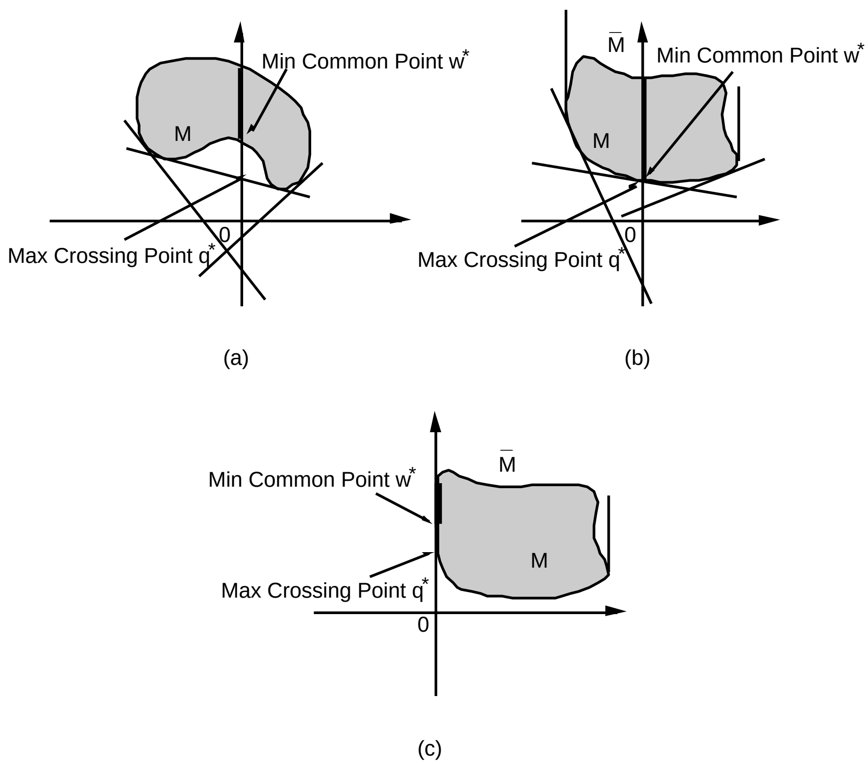

Figure 1.4.5. Illustration of the optimal values of the min common and max crossing problems. In (a), the two optimal values are not equal. In (b), when S is "extended upwards" along the $(n+1)$st axis it yields the set

$$\overline{S} = \Big\{(u,w) \mid \text{there exists } \overline{w} \text{ with } \overline{w} \le w \text{ and } (u,\overline{w}) \in S\Big\},$$

which is closed and convex. As a result, the two optimal values are equal. In (c), the set $\overline{S}$ is convex but not closed, and there are points $(0,\overline{w})$ on the vertical axis with $\overline{w} < w^*$ that lie in the closure of $\overline{S}$. Here q^* is equal to the minimum such value of $\overline{w}$, and we have $q^* < w^*$.

(see Fig. 1.4.5). We want to find the maximum crossing point of the $(n+1)$st axis with such a hyperplane.

Figure 1.4.5 suggests that the optimal value of the max crossing point problem is no larger than the optimal value of the min common point problem, and that under favorable circumstances the two optimal values are equal.

We now formalize the analysis of the two above problems and provide conditions that guarantee equality of their optimal values. The min common point problem is

$$\begin{array}{ll} \text{minimize} & w \\ \text{subject to} & (0,w) \in S, \end{array} \tag{1.38}$$

and its optimal value is denoted by w^*, i.e.,

$$w^* = \inf_{(0,w)\in S} w.$$

Given a nonvertical hyperplane in $\Re^{n+1}$, multiplication of its normal vector (μ, β), $\beta \neq 0$, by a nonzero scalar maintains the normality property. Hence, the set of nonvertical hyperplanes can be equivalently described as the set of all hyperplanes with normals of the form $(\mu, 1)$. A hyperplane of this type crosses the $(n+1)$st axis at

$$\xi = \overline{w} + \mu'\overline{u},$$

where $(\overline{u}, \overline{w})$ is any vector that lies on the hyperplane [cf. Eq. (1.37)]. In order for the upper halfspace corresponding to the hyperplane to contain S, we must have

$$\xi = \overline{w} + \mu'\overline{u} \le w + \mu'u, \qquad \forall\ (u,w) \in S.$$

Combining the two above equations, we see that the max crossing point problem can be expressed as

$$\begin{array}{ll} \text{maximize} & \xi \\ \text{subject to} & \xi \le w + \mu'u, \quad \forall\ (u,w) \in S, \\ & \mu \in \Re^n, \end{array}$$

or equivalently,

$$\begin{array}{ll} \text{maximize} & \xi \\ \text{subject to} & \xi \le \inf_{(u,w)\in S}\{w + \mu'u\}, \quad \mu \in \Re^n. \end{array}$$

Thus, the max crossing point problem is given by

$$\begin{array}{ll} \text{maximize} & q(\mu) \\ \text{subject to} & \mu \in \Re^n, \end{array} \tag{1.39}$$

where

$$q(\mu) = \inf_{(u,w)\in S}\{w + \mu'u\}. \tag{1.40}$$

We denote by q^* the corresponding optimal value,

$$q^* = \sup_{\mu \in \Re^n} q(\mu).$$

Note that for every $(u, w) \in S$ and every $\mu \in \Re^n$, we have

$$q(\mu) = \inf_{(u,w) \in S} \{w + \mu' u\} \leq \inf_{(0,w) \in S} w = w^*,$$

so by taking the supremum of the left-hand side over $\mu \in \Re^n$, we obtain

$$q^* \leq w^*, \tag{1.41}$$

i.e., the max crossing point is no higher than the min common point, as suggested by Fig. 1.4.5. Note also that if $-\infty < w^* < \infty$, then $(0, w^*)$ is seen to be a closure point of the set S, so if in addition S is closed and convex, and admits a nonvertical supporting hyperplane at $(0, w^*)$, then we have $q^* = w^*$ and the optimal values q^* and w^* are attained. Between the "unfavorable" case where $q^* < w^*$, and the "most favorable" case where $q^* = w^*$ while the optimal values q^* and w^* are attained, there are several intermediate cases. The following proposition provides assumptions that guarantee the equality $q^* = w^*$, but not necessarily the attainment of the optimal values.

Proposition 1.4.6 (Min Common/Max Crossing Theorem): Consider the min common point and max crossing point problems, and assume the following:

(1) $-\infty < w^* < \infty$.

(2) The set

$$\overline{S} = \big\{(u, w) \mid \text{there exists } \overline{w} \text{ with } \overline{w} \leq w \text{ and } (u, \overline{w}) \in S\big\}$$

is convex.

(3) For every sequence $\big\{(u_k, w_k)\big\} \subset S$ with $u_k \to 0$,

$$w^* \leq \liminf_{k \to \infty} w_k.$$

Then we have $q^* = w^*$.

Proof: We first note that $(0, w^*)$ is a closure point of $\overline{S}$, since by the definition of w^*, there exists a sequence $\big\{(0, w_k)\big\} \subset S \subset \overline{S}$ such that $w_k \to w^*$.

We next show by contradiction that $\overline{S}$ does not contain any vertical lines. If this were not so, since $\overline{S}$ is convex, the direction $\big\{(0,-\gamma) \mid \gamma \geq 0\big\}$ would be a direction of recession of $\mathrm{cl}(\overline{S})$ and hence also a direction of recession of $\mathrm{ri}(\overline{S})$ (see the exercises in Section 1.2). Since $(0,w^*)$ is a closure point of $\overline{S}$, there exists a sequence $\big\{(u_k,w_k)\big\} \subset \mathrm{ri}(\overline{S})$ converging to $(0,w^*)$, so the sequence $\big\{(u_k,w_k-\gamma)\big\}$ belongs to $\mathrm{ri}(\overline{S})$ for each fixed $\gamma > 0$. In view of the definition of $\overline{S}$, for any $\gamma > 0$, there is a sequence $\big\{(u_k,\overline{w}_k)\big\} \subset S$ with $\overline{w}_k \leq w_k - \gamma$ for all k, so that $\liminf_{k\to\infty} \overline{w}_k \leq w^* - \gamma$. Since $u_k \to 0$, this contradicts assumption (3).

As shown above, $\overline{S}$ does not contain any vertical lines. Furthermore, using condition (3) and the fact that w^* is the optimal value of the min common point problem, it follows that for any $\epsilon > 0$, the vector $(0,w^*-\epsilon)$ does not belong to the closure of $\overline{S}$. It follows, by using Prop. 1.4.5(b), that there exists a hyperplane separating $(0,w^*-\epsilon)$ from $\overline{S}$. This hyperplane crosses the $(n+1)$st axis at a unique point, which must lie between $w^*-\epsilon$ and w^*, and must also be less or equal to the optimal value q^* of the max crossing point problem. Since ϵ can be arbitrarily small, it follows that $q^* \geq w^*$. In view of the fact $q^* \leq w^*$, which holds always [cf. Eq. (1.41)], the result follows. **Q.E.D.**

Proof of the Minimax Theorem

We will now prove the Minimax Theorem stated at the end of last section (cf. Prop. 1.3.9). In particular, assume that X and Z are nonempty convex subsets of $\Re^n$ and $\Re^m$, respectively, $\phi : X \times Z \mapsto \Re$ is a function such that for each $z \in Z$, the function $\phi(\cdot,z) : X \mapsto \Re$ is convex and lower semicontinuous, and for each $x \in X$, the function $\phi(x,\cdot) : Z \mapsto \Re$ is concave and upper semicontinuous. Then if the additional assumptions of Prop. 1.3.9 regarding the function

$$p(u) = \inf_{x\in X} \sup_{z\in Z} \{\phi(x,z) - u'z\}$$

hold, we will show that

$$\sup_{z\in Z} \inf_{x\in X} \phi(x,z) = \inf_{x\in X} \sup_{z\in Z} \phi(x,z).$$

We first prove the following lemma.

Lemma 1.4.1: Let X and Z be nonempty convex subsets of $\Re^n$ and $\Re^m$, respectively, and let $\phi : X \times Z \mapsto \Re$ be a function. Assume that for each $z \in Z$, the function $\phi(\cdot, z) : X \mapsto \Re$ is convex, and consider the function p defined by

$$p(u) = \inf_{x \in X} \sup_{z \in Z} \{\phi(x, z) - u'z\}, \qquad \forall\ u \in \Re^m.$$

Then the set
$$P = \big\{u \mid p(u) < \infty\big\}$$
is convex and p satisfies

$$p\big(\alpha u^1 + (1-\alpha)u^2\big) \le \alpha p(u^1) + (1-\alpha)p(u^2) \tag{1.42}$$

for all $\alpha \in [0, 1]$ and all $u_1, u_2 \in P$.

Proof: Let $u^1 \in P$ and $u^2 \in P$. Rewriting $p(u)$ as

$$p(u) = \inf_{x \in X} l(x, u),$$

where $l(x, u) = \sup_{z \in Z} \big\{\phi(x, z) - u'z\big\}$, we have that there exist sequences $\{x_k^1\} \subset X$ and $\{x_k^2\} \subset X$ such that

$$l(x_k^1, u^1) \to p(u^1), \qquad l(x_k^2, u^2) \to p(u^2).$$

By convexity of X, we have $\alpha x_k^1 + (1-\alpha)x_k^2 \in X$ for all $\alpha \in [0, 1]$. Using also the convexity of $\phi(\cdot, z)$ for each $z \in Z$, we have

$$\begin{aligned}
p\big(\alpha u^1 + (1-\alpha)u^2\big) &\le l\big(\alpha x_k^1 + (1-\alpha)x_k^2, \alpha u^1 + (1-\alpha)u^2\big) \\
&= \sup_{z \in Z} \Big\{\phi\big(\alpha x_k^1 + (1-\alpha)x_k^2, z\big) - \big(\alpha u^1 + (1-\alpha)u^2\big)' z\Big\} \\
&\le \sup_{z \in Z} \Big\{\alpha\phi(x_k^1, z) + (1-\alpha)\phi(x_k^2, z) - \big(\alpha u^1 + (1-\alpha)u^2\big)' z\Big\} \\
&\le \alpha \sup_{z \in Z} \big\{\phi(x_k^1, z) - u^{1\prime} z\big\} + (1-\alpha) \sup_{z \in Z} \big\{\phi(x_k^2, z) - u^{2\prime} z\big\} \\
&= \alpha l(x_k^1, u^1) + (1-\alpha) l(x_k^2, u^2).
\end{aligned}$$

Since
$$\alpha l(x_k^1, u^1) + (1-\alpha) l(x_k^2, u^2) \to \alpha p(u^1) + (1-\alpha)p(u^2),$$
it follows that

$$p\big(\alpha u^1 + (1-\alpha)u^2\big) \le \alpha p(u^1) + (1-\alpha)p(u^2).$$

Furthermore, since u^1 and u^2 belong to P, the left-hand side of the above relation is less than ∞, so $\alpha u^1 + (1-\alpha)u^2 \in P$ for all $\alpha \in [0,1]$, implying that P is convex. **Q.E.D.**

Note that Lemma 1.4.1 allows the possibility that the function p not only takes finite values and the value ∞, but also the value $-\infty$. However, *under the additional assumption of Prop. 1.3.9 that* $p(0) \leq \liminf_{k\to\infty} p(u_k)$ *for all sequences* $\{u_k\}$ *with* $u_k \to 0$*, we must have* $p(u) > -\infty$ *for all* $u \in \Re^n$*. Furthermore, p is is an extended-real valued convex function, which is lower semicontinuous at* $u = 0$. The reason is that if $p(\overline{u}) = -\infty$ for some $\overline{u} \in \Re^m$, then from Eq. (1.42), we must have $p(\alpha\overline{u}) = -\infty$ for all $\alpha \in (0,1]$, contradicting the assumption that $p(0) \leq \liminf_{k\to\infty} p(u_k)$ for all sequences $\{u_k\}$ with $u_k \to 0$. Thus, under the assumptions of Prop. 1.3.9, p maps $\Re^n$ into $(-\infty,\infty]$, and in view of Lemma 1.4.1, p is an extended-real valued convex function. Furthermore, p is lower semicontinuous at $u = 0$ in view again of the assumption that $p(0) \leq \liminf_{k\to\infty} p(u_k)$ for all sequences $\{u_k\}$ with $u_k \to 0$.

We now prove Prop. 1.3.9 by reducing it to an application of the Min Common/Max Crossing Theorem (cf. Prop. 1.4.6) of the preceding section. In particular, we show that with an appropriate selection of the set S, the assumptions of Prop. 1.3.9 are essentially equivalent to the corresponding assumptions of the Min Common/Max Crossing Theorem. Furthermore, the optimal values of the corresponding min common and max crossing point problems are

$$q^* = \sup_{z\in Z} \inf_{x\in X} \phi(x,z), \qquad w^* = \inf_{x\in X} \sup_{z\in Z} \phi(x,z).$$

We choose the set S (as well the set $\overline{S}$) in that theorem to be the epigraph of p, i.e.,

$$S = \overline{S} = \big\{(u,w) \mid u \in P,\ p(u) \leq w\big\},$$

which is convex in view of the convexity of the function p shown above. Thus assumption (2) of the Min Common/Max Crossing Theorem is satisfied. We clearly have

$$w^* = p(0),$$

so the assumption $-\infty < p(0) < \infty$ of Prop. 1.3.9 is equivalent to assumption (1) of the Min Common/Max Crossing Theorem. Furthermore, the assumption that $p(0) \leq \liminf_{k\to\infty} p(u_k)$ for all sequences $\{u_k\}$ with $u_k \to 0$ is equivalent to the last remaining assumption (3) of the Min Common/Max Crossing Theorem. Thus the conclusion $q^* = w^*$ of the theorem holds.

From the definition of p, it follows that

$$w^* = p(0) = \inf_{x\in X} \sup_{z\in Z} \phi(x,z).$$

Thus, to complete the proof of Prop. 1.3.9, all that remains is to show that for the max crossing point problem corresponding to the epigraph S of p, we have

$$q^* = \sup_{\mu \in \Re^m} \inf_{(u,w)\in S} \{w + \mu' u\} = \sup_{z\in Z} \inf_{x\in X} \phi(x, z).$$

To prove this, let us write for every $\mu \in \Re^m$,

$$\begin{aligned} q(\mu) &= \inf_{(u,w)\in S} \{w + \mu' u\} \\ &= \inf_{\{(u,w)\mid u\in P,\ p(u)\le w\}} \{w + \mu' u\} \\ &= \inf_{u\in P} \big\{p(u) + \mu' u\big\} \end{aligned}$$

From the definition of p, we have for every $\mu \in \Re^m$,

$$\begin{aligned} q(\mu) &= \inf_{u\in P} \big\{p(u) + u'\mu\big\}, \\ &= \inf_{u\in P} \inf_{x\in X} \sup_{z\in Z} \big\{\phi(x, z) + u'(\mu - z)\big\} \\ &= \inf_{x\in X} \inf_{u\in P} \sup_{z\in Z} \big\{\phi(x, z) + u'(\mu - z)\big\}. \end{aligned}$$

We now show that $q(\mu)$ is given by

$$q(\mu) = \begin{cases} \inf_{x\in X} \phi(x, \mu) & \text{if } \mu \in Z, \\ -\infty & \text{if } \mu \notin Z. \end{cases} \tag{1.43}$$

Indeed, if $\mu \in Z$, then

$$\sup_{z\in Z} \big\{\phi(x, z) + u'(\mu - z)\big\} \ge \phi(x, \mu), \qquad \forall\ x \in X,\ \forall\ u \in P,$$

so

$$\inf_{u\in P} \sup_{z\in Z} \big\{\phi(x, z) + u'(\mu - z)\big\} \ge \phi(x, \mu), \qquad \forall\ x \in X. \tag{1.44}$$

To show that we have equality in Eq. (1.44), we define the function $r_x(z) : \Re^m \mapsto (-\infty, \infty]$ as

$$r_x(z) = \begin{cases} -\phi(x, z) & \text{if } z \in Z, \\ +\infty & \text{otherwise,} \end{cases}$$

which is closed and convex for all $x \in X$ by the given concavity/upper semicontinuity assumptions, so the epigraph of $r_x(z)$, $\mathrm{epi}\big(r_x\big)$, is a closed and convex set. Since $\mu \in Z$, the point $\big(\mu, r_x(\mu)\big)$ belongs to $\mathrm{epi}(r_x)$. For some $\epsilon > 0$, we consider the point $\big(\mu, r_x(\mu) - \epsilon\big)$, which does not belong to $\mathrm{epi}(r_x)$. Since $r_x(z) > -\infty$ for all z, the set $\mathrm{epi}(r_x)$ does not contain any vertical lines. Therefore, by Prop. 1.4.5(b), for all $x \in X$, there exists a nonvertical hyperplane with normal $(\overline{u}, \zeta)$ with $\zeta \neq 0$, and a scalar c such that

$$\overline{u}'\mu + \zeta\big(r_x(\mu) - \epsilon\big) < c < \overline{u}'z + \zeta w, \qquad \forall\ (z, w) \in \mathrm{epi}(r_x).$$

Since w can be made arbitrarily large, we have $\zeta > 0$, and we can take $\zeta = 1$. In particular for $w = r_x(z)$, with $z \in Z$, we have

$$\overline{u}'\mu + \big(r_x(\mu) - \epsilon\big) < c < \overline{u}'z + r_x(z), \qquad \forall\ z \in Z,$$

or equivalently,

$$\phi(x, z) + \overline{u}'(\mu - z) < \phi(x, \mu) + \epsilon, \qquad \forall\ z \in Z.$$

Letting $\epsilon \to 0$, we obtain

$$\inf_{u \in \Re^m} \sup_{z \in Z} \big\{\phi(x, z) + u'(\mu - z)\big\} \leq \sup_{z \in Z} \big\{\phi(x, z) + \overline{u}'(\mu - z)\big\} \leq \phi(x, \mu),\ \forall\ x \in X$$

which together with Eq. (1.44) implies that

$$\inf_{u \in \Re^m} \sup_{z \in Z} \big\{\phi(x, z) + u'(\mu - z)\big\} = \phi(x, \mu), \qquad \forall\ x \in X.$$

Therefore, if $\mu \in Z$, $q(\mu)$ has the form of Eq. (1.43).

If $\mu \notin Z$, we consider a sequence $\{w_k\}$ that goes to $-\infty$. Since $\mu \notin Z$, the sequence of points (μ, w_k) does not belong to the $\mathrm{epi}(r_x)$. Therefore, similar to the argument above, there exists a sequence of nonvertical hyperplanes with normals $(u_k, 1)$ such that

$$\phi(x, z) + u_k'(\mu - z) < w_k, \qquad \forall\ z \in Z,\ \forall\ k,$$

so that

$$\inf_{u \in \Re^m} \sup_{z \in Z} \big\{\phi(x, z) + u'(\mu - z)\big\} \leq \sup_{z \in Z} \big\{\phi(x, z) + u_k'(\mu - z)\big\} \leq w_k,\ \forall\ k.$$

Taking the limit in the above equation along the relevant subsequence, we obtain

$$\inf_{u \in \Re^m} \sup_{z \in Z} \big\{\phi(x, z) + u'(\mu - z)\big\} = -\infty, \qquad \forall\ x \in X,$$

which proves that $q(\mu)$ has the form given in Eq. (1.43) and it follows that

$$q^* = \sup_{z \in Z} \inf_{x \in X} \phi(x, z).$$

E X E R C I S E S

1.4.1 (Strong Separation)

Let C_1 and C_2 be two nonempty, convex subsets of $\Re^n$, and let B denote the unit ball in $\Re^n$. A hyperplane H is said to separate strongly C_1 and C_2 if there exists an $\epsilon > 0$ such that $C_1 + \epsilon B$ is contained in one of the open halfspaces associated with H and $C_2 + \epsilon B$ is contained in the other. Show that:

(a) The following three conditions are equivalent:

(i) There exists a hyperplane strongly separating C_1 and C_2.

(ii) There exists a vector $b \in \Re^n$ such that $\inf_{x \in C_1} b'x > \sup_{x \in C_2} b'x$.

(iii) $\inf_{x_1 \in C_1,\, x_2 \in C_2} \|x_1 - x_2\| > 0$.

(b) If C_1 and C_2 are closed and have no common directions of recession, there exists a hyperplane strongly separating C_1 and C_2.

(c) If the two sets C_1 and C_2 have disjoint closures, and at least one of the two is bounded, there exists a hyperplane strongly separating them.

1.4.2 (Proper Separation)

Let C_1 and C_2 be two nonempty, convex subsets of $\Re^n$. A hyperplane H is said to separate properly C_1 and C_2 if C_1 and C_2 are not both contained in H. Show that the following three conditions are equivalent:

(i) There exists a hyperplane properly separating C_1 and C_2.

(ii) There exists a vector $b \in \Re^n$ such that

$$\inf_{x \in C_1} b'x \geq \sup_{x \in C_2} b'x, \qquad \sup_{x \in C_1} b'x > \inf_{x \in C_2} b'x,$$

(iii) The relative interiors $\mathrm{ri}(C_1)$ and $\mathrm{ri}(C_2)$ have no point in common.

1.5 CONICAL APPROXIMATIONS AND CONSTRAINED OPTIMIZATION

Optimality conditions for constrained optimization problems revolve around the behavior of the cost function at points of the contraint set around a candidate for optimality x^*. Within this context, approximating the constraint set locally around x^* by a conical set turns out to be particularly useful. In this section, we develop this approach and we derive necessary conditions for constrained optimality. We first introduce some basic notions relating to cones.

Given a set C, the cone given by

$$C^* = \{y \mid y'x \leq 0,\ \forall\, x \in C\},$$

is called the *polar cone* of C (see Fig. 1.5.1). Clearly, the polar cone C^*, being the intersection of a collection of closed halfspaces, is closed and convex (regardless of whether C is closed and/or convex). If C is a subspace, it can be seen that the polar cone C^* is equal to the orthogonal subspace $C^\perp$. The following basic result generalizes the equality $C = (C^\perp)^\perp$, which holds in the case where C is a subspace.

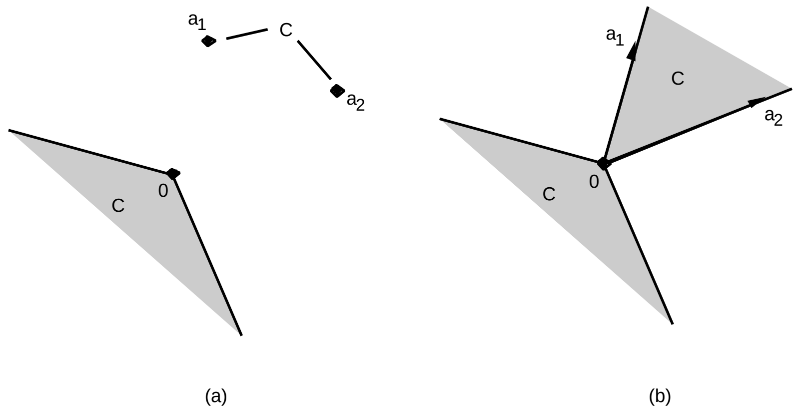

Figure 1.5.1. Illustration of the polar cone C^* of a set $C \subset \Re^2$. In (a) C consists of just two points, a_1 and a_2, while in (b) C is the convex cone $\{x \mid x = \mu_1 a_1 + \mu_2 a_2,\ \mu_1 \geq 0,\ \mu_2 \geq 0\}$. The polar cone C^* is the same in both cases.

Proposition 1.5.1: (Polar Cone Theorem)

(a) For any set C, we have

$$C^* = \big(\mathrm{cl}(C)\big)^* = \big(\mathrm{conv}(C)\big)^* = \big(\mathrm{cone}(C)\big)^*.$$

(b) For any cone C, we have

$$(C^*)^* = \mathrm{cl}\big(\mathrm{conv}(C)\big).$$

In particular, if C is closed and convex, then $(C^*)^* = C$.

Proof: (a) Clearly, we have $\big(\mathrm{cl}(C)\big)^* \subset C^*$. Conversely, if $y \in C^*$, then $y'x_k \leq 0$ for all k and all sequences $\{x_k\} \subset C$, so that $y'x \leq 0$ for all limits x of such sequences. Hence, $y \in \big(\mathrm{cl}(C)\big)^*$ and $C^* \subset \big(\mathrm{cl}(C)\big)^*$.

Similarly, we have $\big(\mathrm{conv}(C)\big)^* \subset C^*$. Conversely, if $y \in C^*$, then

$y'x \leq 0$ for all $x \in C$ so that $y'z \leq 0$ for all z that are convex combinations of vectors $x \in C$. Hence $y \in \big(\mathrm{conv}(C)\big)^*$ and $C^* \subset \big(\mathrm{conv}(C)\big)^*$. A nearly identical argument also shows that $C^* = \big(\mathrm{cone}(C)\big)^*$.

(b) Figure 1.5.2 shows that if C is closed and convex, then $(C^*)^* = C$. From this it follows that

$$\Big(\big(\mathrm{cl}\big(\mathrm{conv}(C)\big)\big)^*\Big)^* = \mathrm{cl}\big(\mathrm{conv}(C)\big),$$

and by using part (a) in the left-hand side above, we obtain $(C^*)^* = \mathrm{cl}\big(\mathrm{conv}(C)\big)$. **Q.E.D.**

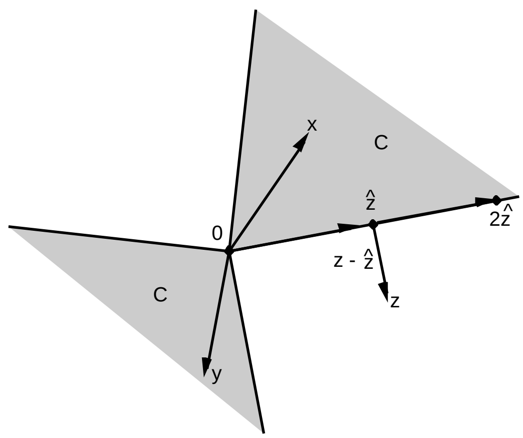

Figure 1.5.2. Proof of the Polar Cone Theorem for the case where C is a closed and convex cone. If $x \in C$, then for all $y \in C^*$, we have $x'y \leq 0$, which implies that $x \in (C^*)^*$. Hence, $C \subset (C^*)^*$. To prove the reverse inclusion, take $z \in (C^*)^*$, and let $\hat{z}$ be the unique projection of z on C, as shown in the figure. Since C is closed, the projection exists by the Projection Theorem (Prop. 1.3.3), which also implies that

$$(z - \hat{z})'(x - \hat{z}) \leq 0, \qquad \forall\ x \in C.$$

By taking in the preceding relation $x = 0$ and $x = 2\hat{z}$ (which belong to C since C is a closed cone), it is seen that

$$(z - \hat{z})'\hat{z} = 0.$$

Combining the last two relations, we obtain $(z-\hat{z})'x \leq 0$ for all $x \in C$. Therefore, $(z - \hat{z}) \in C^*$, and since $z \in (C^*)^*$, we obtain $(z - \hat{z})'z \leq 0$, which when added to $(z - \hat{z})'\hat{z} = 0$ yields $\|z - \hat{z}\|^2 \leq 0$. Therefore, $z = \hat{z}$ and $z \in C$, implying that $(C^*)^* \subset C$.

The analysis of a constrained optimization problem often centers on how the cost function behaves along directions leading away from a local minimum to some neighboring feasible points. The sets of the relevant directions constitute cones that can be viewed as approximations to the constraint set, locally near a point of interest. Let us introduce two such cones, which are important in connection with optimality conditions.

Definition 1.5.1: Given a subset X of $\Re^n$ and a vector $x \in X$, a *feasible direction* of X at x is a vector $y \in \Re^n$ such that there exists an $\overline{\alpha} > 0$ with $x + \alpha y \in X$ for all $\alpha \in [0, \overline{\alpha}]$. The set of all feasible directions of X at x is a cone denoted by $F_X(x)$.

It can be seen that if X is convex, the feasible directions at x are the vectors of the form $\alpha(\overline{x} - x)$ with $\alpha > 0$ and $\overline{x} \in X$. However, when X is nonconvex, the cone of feasible directions need not provide interesting information about the local structure of the set X near the point x. For example, often there is no nonzero feasible direction at x when X is nonconvex [think of the set $X = \{x \mid h(x) = 0\}$, where $h : \Re^n \mapsto \Re$ is a nonlinear function]. The next definition introduces a cone that provides information on the local structure of X even when there is no nonzero feasible direction.

Definition 1.5.2: Given a subset X of $\Re^n$ and a vector $x \in X$, a vector y is said to be a *tangent* of X at x if either $y = 0$ or there exists a sequence $\{x_k\} \subset X$ such that $x_k \neq x$ for all k and

$$x_k \to x, \qquad \frac{x_k - x}{\|x_k - x\|} \to \frac{y}{\|y\|}.$$

The set of all tangents of X at x is a cone called the *tangent cone* of X at x, and is denoted by $T_X(x)$.

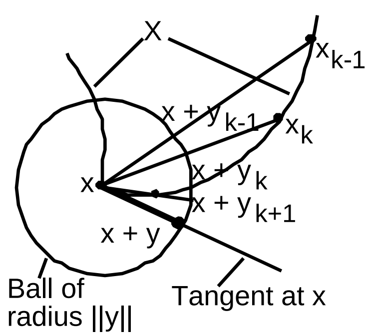

Figure 1.5.3. Illustration of a tangent y at a vector $x \in X$. There is a sequence $\{x_k\} \subset X$ that converges to x and is such that the normalized direction sequence $(x_k - x)/\|x_k - x\|$ converges to $y/\|y\|$, the normalized direction of y, or equivalently, the sequence

$$y_k = \frac{\|y\|(x_k - x)}{\|x_k - x\|}$$

illustrated in the figure converges to y.

Thus a nonzero vector y is a tangent at x if it is possible to approach x with a feasible sequence $\{x_k\}$ such that the normalized direction sequence $(x_k - x)/\|x_k - x\|$ converges to $y/\|y\|$, the normalized direction of y (see Fig. 1.5.3). The following proposition provides an equivalent definition of a tangent, which is occasionally more convenient.

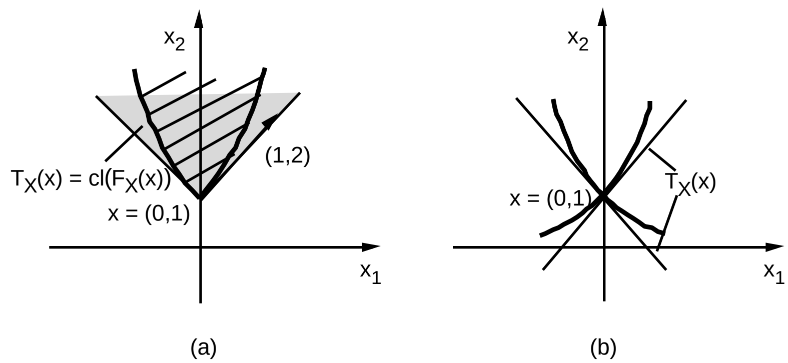

Figure 1.5.4. Examples of the cones $F_X(x)$ and $T_X(x)$ of a set X at the vector $x = (0, 1)$. In (a), we have

$$X = \Big\{(x_1, x_2) \mid (x_1 + 1)^2 - x_2 \le 0,\ (x_1 - 1)^2 - x_2 \le 0\Big\}.$$

Here X is convex and the tangent cone $T_X(x)$ is equal to the closure of the cone of feasible directions $F_X(x)$ (which is an open set in this example). Note, however, that the vectors $(1, 2)$ and $(-1, 2)$ (as well the origin) belong to $T_X(x)$ and also to the closure of $F_X(x)$, but are not feasible directions. In (b), we have

$$X = \Big\{(x_1, x_2) \mid \big((x_1 + 1)^2 - x_2\big)\big((x_1 - 1)^2 - x_2\big) = 0\Big\}.$$

Here the set X is nonconvex, and $T_X(x)$ is closed but not convex. Furthermore, $F_X(x)$ consists of just the zero vector.

Proposition 1.5.2: A vector y is a tangent of a set X at a vector $x \in X$ if and only if there exists a sequence $\{x_k\} \subset X$ with $x_k \to x$, and a positive sequence $\{\alpha_k\}$ such that $\alpha_k \to 0$ and $(x_k - x)/\alpha_k \to y$.

Proof: If $\{x_k\}$ is the sequence in the definition of a tangent, take $\alpha_k = \|x_k - x\|/\|y\|$. Conversely, if $\alpha_k \to 0$, $(x_k - x)/\alpha_k \to y$, and $y \ne 0$, clearly $x_k \to x$ and

$$\frac{x_k - x}{\|x_k - x\|} = \frac{(x_k - x)/\alpha_k}{\|(x_k - x)/\alpha_k\|} \to \frac{y}{\|y\|},$$

so $\{x_k\}$ satisfies the definition of a tangent. **Q.E.D.**

Figure 1.5.4 illustrates the cones $T_X(x)$ and $F_X(x)$ with examples. The following proposition gives some of the properties of the cones $F_X(x)$ and $T_X(x)$.

Proposition 1.5.3: Let X be a nonempty subset of $\Re^n$ and let x be a vector in X. The following hold regarding the cone of feasible directions $F_X(x)$ and the tangent cone $T_X(x)$.

(a) $T_X(x)$ is a closed cone.

(b) $\mathrm{cl}\big(F_X(x)\big) \subset T_X(x)$.

(c) If X is convex, then $F_X(x)$ and $T_X(x)$ are convex, and we have

$$\mathrm{cl}\big(F_X(x)\big) = T_X(x).$$

Proof: (a) Let $\{y_k\}$ be a sequence in $T_X(x)$ that converges to y. We will show that $y \in T_X(x)$. If $y = 0$, then $y \in T_X(x)$, so assume that $y \neq 0$. By the definition of a tangent, for every y_k there is a sequence $\{x_k^i\} \subset X$ with $x_k^i \neq x$ such that

$$\lim_{i\to\infty} x_k^i = x, \qquad \lim_{i\to\infty} \frac{x_k^i - x}{||x_k^i - x||} = \frac{y_k}{||y_k||}.$$

For $k = 1, 2, \ldots$, choose an index i_k such that $i_1 < i_2 < \ldots < i_k$ and

$$\lim_{k\to\infty} ||x_k^{i_k} - x|| = 0, \qquad \lim_{k\to\infty} \left\| \frac{x_k^{i_k} - x}{||x_k^{i_k} - x||} - \frac{y_k}{||y_k||} \right\| = 0.$$

We also have for all k

$$\left\| \frac{x_k^{i_k} - x}{||x_k^{i_k} - x||} - \frac{y}{||y||} \right\| \le \left\| \frac{x_k^{i_k} - x}{||x_k^{i_k} - x||} - \frac{y_k}{||y_k||} \right\| + \left\| \frac{y_k}{||y_k||} - \frac{y}{||y||} \right\|,$$

so the fact $y_k \to y$, and the preceding two relations imply that

$$\lim_{k\to\infty} \|x_k^{i_k} - x\| = 0, \qquad \lim_{k\to\infty} \left\| \frac{x_k^{i_k} - x}{||x_k^{i_k} - x||} - \frac{y}{||y||} \right\| = 0.$$

Hence $y \in T_X(x)$ and $T_X(x)$ is closed.

(b) Clearly every feasible direction is also a tangent, so $F_X(x) \subset T_X(x)$. Since by part (a), $T_X(x)$ is closed, the result follows.

(c) If X is convex, the feasible directions at x are the vectors of the form $\alpha(\overline{x} - x)$ with $\alpha > 0$ and $\overline{x} \in X$. From this it can be seen that $F_X(x)$ is convex. Convexity of $T_X(x)$ will follow from the convexity of $F_X(x)$ once we show that $\mathrm{cl}\big(F_X(x)\big) = T_X(x)$.

In view of part (b), it will suffice to show that $T_X(x) \subset \mathrm{cl}\big(F_X(x)\big)$. Let $y \in T_X(x)$ and, using Prop. 1.5.2, let $\{x_k\}$ be a sequence in X and $\{\alpha_k\}$ be a positive sequence such that $\alpha_k \to 0$ and $(x_k - x)/\alpha_k \to y$. Since X is a convex set, the direction $(x_k - x)/\alpha_k$ is feasible at x for all k. Hence $y \in \mathrm{cl}\big(F_X(x)\big)$, and it follows that $T_X(x) \subset \mathrm{cl}\big(F_X(x)\big)$. **Q.E.D.**

The tangent cone finds an important application in the following basic necessary condition for local optimality:

Proposition 1.5.4: Let $f : \Re^n \mapsto \Re$ be a smooth function, and let x^* be a local minimum of f over a set $X \subset \Re^n$. Then

$$\nabla f(x^*)'y \geq 0, \qquad \forall\ y \in T_X(x^*).$$

If X is convex, this condition can be equivalently written as

$$\nabla f(x^*)'(x - x^*) \geq 0, \qquad \forall\ x \in X,$$

and in the case where $X = \Re^n$, reduces to $\nabla f(x^*) = 0$.

Proof: Let y be a nonzero tangent of X at x^*. Then there exists a sequence $\{\xi_k\}$ and a sequence $\{x_k\} \subset X$ such that $x_k \neq x^*$ for all k,

$$\xi_k \to 0, \qquad x_k \to x^*,$$

and

$$\frac{x_k - x^*}{\|x_k - x^*\|} = \frac{y}{\|y\|} + \xi_k.$$

By the Mean Value Theorem, we have for all k

$$f(x_k) = f(x^*) + \nabla f(\tilde{x}_k)'(x_k - x^*),$$

where $\tilde{x}_k$ is a vector that lies on the line segment joining x_k and x^*. Combining the last two equations, we obtain

$$f(x_k) = f(x^*) + \frac{\|x_k - x^*\|}{\|y\|}\nabla f(\tilde{x}_k)'y_k, \tag{1.45}$$

where

$$y_k = y + \|y\|\xi_k.$$

If $\nabla f(x^*)'y < 0$, since $\tilde{x}_k \to x^*$ and $y_k \to y$, it follows that for all sufficiently large k, $\nabla f(\tilde{x}_k)'y_k < 0$ and [from Eq. (1.45)] $f(x_k) < f(x^*)$. This contradicts the local optimality of x^*.

When X is convex, we have $\mathrm{cl}\big(F_X(x)\big) = T_X(x)$ (cf. Prop. 1.5.3). Thus the condition shown can be written as

$$\nabla f(x^*)'y \geq 0, \qquad \forall\, y \in \mathrm{cl}\big(F_X(x)\big),$$

which in turn is equivalent to

$$\nabla f(x^*)'(x - x^*) \geq 0, \qquad \forall\, x \in X.$$

If $X = \Re^n$, by setting $x = x^* + e_i$ and $x = x^* - e_i$, where e_i is the ith unit vector (all components are 0 except for the ith, which is 1), we obtain $\partial f(x^*)/\partial x_i = 0$ for all $i = 1, \ldots, n$, so $\nabla f(x^*) = 0$. **Q.E.D.**

A direction y for which $\nabla f(x^*)'y < 0$ may be viewed as a *descent direction* of f at x^*, in the sense that we have (by Taylor's theorem)

$$f(x^* + \alpha y) = f(x^*) + \alpha \nabla f(x^*)'y + o(\alpha) < f(x^*)$$

for sufficiently small but positive α. Thus Prop. 1.5.4 says that *if x^* is a local minimum, there is no descent direction within the tangent cone $T_X(x^*)$.*

Note that the necessary condition of Prop. 1.5.4 can equivalently be written as

$$-\nabla f(x^*) \in T_X(x^*)^*$$

(see Fig. 1.5.5). There is an interesting converse of this result, namely that given any vector $z \in T_X(x^*)^*$, there exists a smooth function f such that $-\nabla f(x^*) = z$ and x^* is a local minimum of f over X. We will return to this result and to the subject of conical approximations when we discuss Lagrange multipliers in Chapter 2.

The Normal Cone

In addition to the cone of feasible directions and the tangent cone, there is one more conical approximation that is of special interest for the optimization topics covered in this book. This is the *normal cone* of X at a vector $x \in X$, denoted by $N_X(x)$, and obtained from the polar cone $T_X(x)^*$ by means of a closure operation. In particular, we have $z \in N_X(x)$ if there exist sequences $\{x_k\} \subset X$ and $\{z_k\}$ such that $x_k \to x$, $z_k \to z$, and $z_k \in T_X(x_k)^*$ for all k. Equivalently, the graph of $N_X(\cdot)$, viewed as a point-to-set mapping, is the intersection of the closure of the graph of $T_X(\cdot)^*$ with the set $\big\{(x, z) \mid x \in X\big\}$. In the case where X is a closed set,

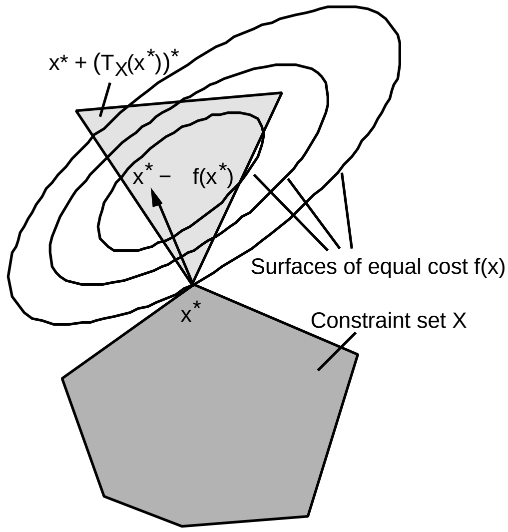

Figure 1.5.5. Illustration of the necessary optimality condition

$$-\nabla f(x^*) \in T_X(x^*)^*$$

for x^* to be a local minimum of f over X.

the set $\big\{(x,z) \mid x \in X\big\}$ contains the closure of the graph of $T_X(\cdot)^*$, so the graph of $N_X(\cdot)$ is equal to the closure of the graph of $T_X(\cdot)^*$:

$$\big\{(x,z) \mid x \in X,\, z \in N_X(x)\big\} = \mathrm{cl}\Big(\big\{(x,z) \mid x \in X,\, z \in T_X(x)^*\big\}\Big) \qquad \text{if } X \text{ is closed.}$$

It can be seen that $N_X(x)$ is a closed cone containing $T_X(x)^*$, but it need not be convex like $T_X(x)^*$ (see the examples of Fig. 1.5.6). In the case where $T_X(x)^* = N_X(x)$, we say that X is *regular* at x. An important consequence of convexity of X is that it implies regularity, as shown in the following proposition.

Proposition 1.5.5: Let X be a convex set. Then for all $x \in X$, we have

$$z \in T_X(x)^* \qquad \text{if and only if} \qquad z'(\overline{x} - x) \le 0, \quad \forall\, \overline{x} \in X. \quad (1.46)$$

Furthermore, X is regular at all $x \in X$.

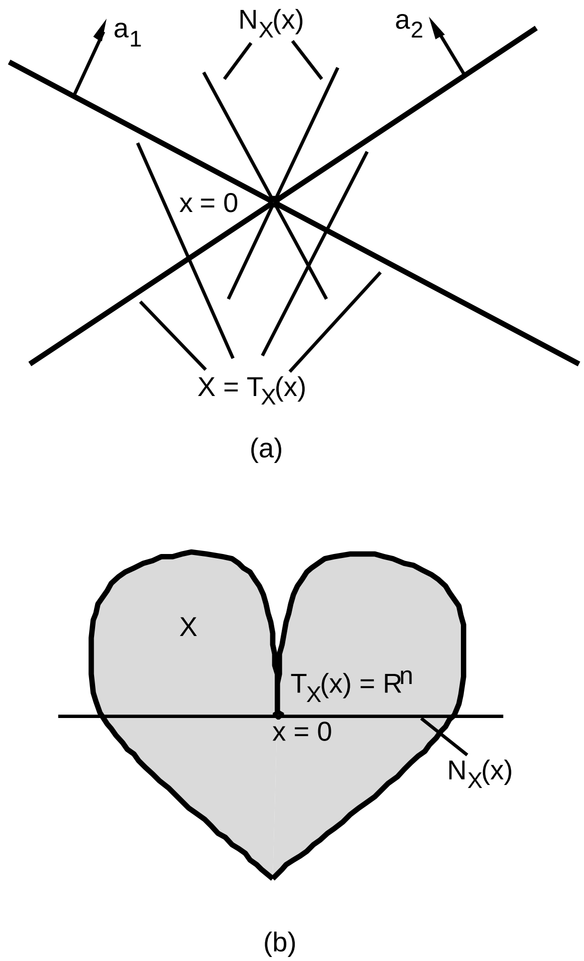

Figure 1.5.6. Examples of normal cones. In the case of figure (a), X is the union of two lines passing through the origin:

$$X = \{x \mid (a_1'x)(a_2'x) = 0\}.$$

For $x = 0$ we have $T_X(x) = X$, $T_X(x)^* = \{0\}$, while $N_X(x)$ is the nonconvex set consisting of the two lines of vectors that are collinear to either a_1 or a_2. Thus X is not regular at $x = 0$. At all other vectors $x \in X$, we have regularity with $T_X(x)^*$ and $N_X(x)$ equal to either the line of vectors that are collinear to a_1 or the line of vectors that are collinear to a_2.

In the case of figure (b), X is regular at all points except at $x = 0$, where we have $T_X(x) = \Re^n$, $T_X(x)^* = \{0\}$, while $N_X(x)$ is equal to the horizontal axis.

Proof: Since $(\overline{x} - x) \in F_X(x) \subset T_X(x)$ for all $\overline{x} \in X$, it follows that if $z \in T_X(x)^*$, then $z'(\overline{x} - x) \leq 0$ for all $\overline{x} \in X$. Conversely, let x be such that $z'(\overline{x} - x) \leq 0$ for all $\overline{x} \in X$, and to arrive at a contradiction, assume that $z \notin T_X(x)^*$. Then there exists some $y \in T_X(x)$ such that $z'y > 0$. Since $\mathrm{cl}\big(F_X(x)\big) = T_X(x)$ [cf. Prop. 1.5.3(c)], there exists a sequence $\{y_k\} \subset F_X(x)$ such that $y_k \to y$, so that $y_k = \alpha_k(x_k - x)$ for some $\alpha_k > 0$ and some $x_k \in X$. Since $z'y > 0$, we have $\alpha_k z'(x_k - x) > 0$ for large enough k, which is a contradiction.

If $x \in X$ and $z \in N_X(x)$, there must exist sequences $\{x_k\} \subset X$ and $\{z_k\}$ such that $x_k \to x$, $z_k \to z$, and $z_k \in T_X(x_k)^*$. By Eq. (1.46) (which critically depends on the convexity of X), we must have $z_k'(\overline{x} - x_k) \leq 0$ for all $\overline{x} \in X$. Taking the limit as $k \to \infty$, we obtain $z'(\overline{x} - x) \leq 0$ for all $\overline{x} \in X$, which by Eq. (1.46), implies that $z \in T_X(x)^*$. Thus, we have $N_X(x) \subset T_X(x)^*$. Since the reverse inclusion always holds, it follows that $N_X(x) = T_X(x)^*$, so that X is regular at x. **Q.E.D.**

Note that convexity of $T_X(x)$ does not imply regularity of X at x, as the example of Fig. 1.5.6(b) shows. However, a converse can be shown, namely that if X is closed and is regular at x, then $T_X(x)$ is equal to the polar of $N_X(x)$:

$$T_X(x) = N_X(x)^* \qquad \text{if } X \text{ is closed and is regular at } x$$

(see Rockafellar and Wets [RoW98], p. 221). Thus, for a closed X, regularity at x implies that the closed cone $T_X(x)$ is convex. This result, which is central in nonsmooth analysis, will not be needed for our development in this book.

E X E R C I S E S

1.5.1 (Fermat's Principle in Optics)

Let $C \subset \Re^n$ be a closed convex set, and let y and z be given vectors in $\Re^n$ such that the line segment connecting y and z does not intersect with C. Consider the problem of minimizing the sum of distances $\|y - x\| + \|z - x\|$ over $x \in C$. Derive a necessary and sufficient optimality condition. Does an optimal solution exist and if so, is it unique? Discuss the case where C is closed but not convex.

1.5.2

Let C_1, C_2, and C_3 be three closed subsets of $\Re^n$. Consider the problem of finding a triangle with minimum perimeter that has one vertex on each of the three sets, i.e., the problem of minimizing $\|x_1 - x_2\| + \|x_2 - x_3\| + \|x_3 - x_1\|$ subject to $x_i \in C_i$, $i = 1, 2, 3$, and the additional condition that x_1, x_2, and x_3 do not lie on the same line. Show that if (x_1^*, x_2^*, x_3^*) defines an optimal triangle, there exists a vector z^* in the triangle such that

$$(z^* - x_i^*) \in T_{C_i}(x_i^*)^*, \qquad i = 1, 2, 3.$$

1.5.3 (Cone Decomposition Theorem)

Let $C \subset \Re^n$ be a closed convex cone and let x be a given vector in $\Re^n$. Show that:

(a) $\hat{x}$ is the projection of x on C if and only if

$$\hat{x} \in C, \qquad x - \hat{x} \in C^*, \qquad (x - \hat{x})'\hat{x} = 0.$$

(b) The following two properties are equivalent:

   (i) x_1 and x_2 are the projections of x on C and C^*, respectively.

   (ii) $x = x_1 + x_2$ with $x_1 \in C$, $x_2 \in C^*$, and $x_1'x_2 = 0$.

1.5.4

Let $C \subset \Re^n$ be a closed convex cone and let a be a given vector in $\Re^n$. Show that for any positive scalars β and γ, we have

$$\max_{\|x\| \le \beta,\, x \in C} a'x \le \gamma \qquad \text{if and only if} \qquad a \in C^* + \{x \mid \|x\| \le \gamma/\beta\}.$$

1.5.5 (Quasiregularity)

Consider the set

$$X = \big\{x \mid h_i(x) = 0,\ i = 1, \ldots, m,\ g_j(x) \le 0,\ j = 1, \ldots, r\big\},$$

where the functions $h_i : \Re^n \mapsto \Re$ and $g_j : \Re^n \mapsto \Re$ are smooth. For any $x \in X$ let $A(x) = \big\{j \mid g_j(x) \le 0\big\}$ and consider the subspace

$$V(x) = \big\{y \mid \nabla h_i(x)'y = 0,\ i = 1, \ldots, m,\ \nabla g_j(x)'y \le 0,\ j \in A(x)\big\}.$$

Show that:

(a) $T_X(x) \subset V(x)$.

(b) $T_X(x) = V(x)$ if either the gradients $\nabla h_i(x)$, $i = 1, \ldots, m$, $\nabla g_i(x)$, $j \in A(x)$, are linearly independent, or all the functions h_i and g_j are linear. *Note*: The property $T_X(x) = V(x)$ is called *quasiregularity*, and will be significant in the Lagrange multiplier theory of Chapter 2.

1.5.6 (Regularity of Constraint Systems)

Consider the set X of Exercise 1.5.5. Show that if at a given $x \in X$, the quasiregularity condition $T_X(x) = V(x)$ of that exercise holds, then X is regular at x.

1.5.7 (Necessary and Sufficient Optimality Condition)

Let $f : \Re^n \mapsto \Re$ be a smooth convex function and let X be a nonempty subset of $\Re^n$. Show that the condition

$$\nabla f(x^*)'(x - x^*) \geq 0, \qquad \forall\, x \in X,$$

is necessary and sufficient for a vector $x^* \in X$ to be a global minimum of f over X.

1.6 POLYHEDRAL CONVEXITY

In this section, we develop some basic results regarding the geometry of polyhedral sets. We start with properties of cones that have a polyhedral structure, and proceed to discuss characterizations of more general polyhedral sets. We then apply the results obtained to linear and integer programming.

1.6.1 Polyhedral Cones

We introduce two different ways to view cones with polyhedral structure, and our first objective in this section is to show that these two views are related through the polarity relation and are in some sense equivalent.

We say that a cone $C \subset \Re^n$ is *polyhedral*, if it has the form

$$C = \{x \mid a_j'x \leq 0,\ j = 1, \ldots, r\},$$

where $a_1, \ldots, a_r$ are some vectors in $\Re^n$. We say that a cone $C \subset \Re^n$ is *finitely generated*, if it is generated by a finite set of vectors, i.e., if it has the form

$$C = \text{cone}\big(\{a_1, \ldots, a_r\}\big) = \left\{ x \;\middle|\; x = \sum_{j=1}^{r} \mu_j a_j,\ \mu_j \geq 0,\ j = 1, \ldots, r \right\},$$

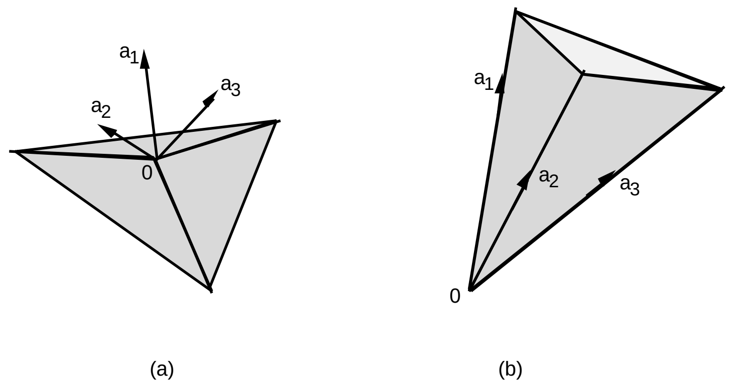

Figure 1.6.1. (a) Polyhedral cone defined by the inequality constraints $a_j'x \le 0$, $j = 1, 2, 3$. (b) Cone generated by the vectors a_1, a_2, a_3.

where $a_1, \ldots, a_r$ are some vectors in $\Re^n$. These definitions are illustrated in Fig. 1.6.1.

Note that sets defined by linear equality constraints (as well as linear inequality constraints) are also polyhedral cones, since a linear equality constraint may be converted into two inequality constraints. In particular, if $e_1, \ldots, e_m$ and $a_1, \ldots, a_r$ are given vectors, the set

$$\{x \mid e_i'x = 0,\ i = 1, \ldots, m,\ a_j'x \le 0,\ j = 1, \ldots, r\}$$

is a polyhedral cone since it can be written as

$$\{x \mid e_i'x \le 0,\ -e_i'x \le 0,\ i = 1, \ldots, m,\ a_j'x \le 0,\ j = 1, \ldots, r\}.$$

Using a related conversion, it can be seen that the cone

$$\left\{x \;\middle|\; x = \sum_{i=1}^{m} \lambda_i e_i + \sum_{j=1}^{r} \mu_j a_j,\ \lambda_i \in \Re,\ i = 1, \ldots, m,\ \mu_j \ge 0,\ j = 1, \ldots, r\right\}$$

is finitely generated, since it can be written as

$$\left\{x \;\middle|\; x = \sum_{i=1}^{m} \lambda_i^+ e_i + \sum_{i=1}^{m} \lambda_i^- (-e_i) + \sum_{j=1}^{r} \mu_j a_j,\right.$$
$$\left.\lambda_i^+ \ge 0,\ \lambda_i^- \ge 0,\ i = 1, \ldots, m,\ \mu_j \ge 0,\ j = 1, \ldots, r\right\}.$$

A polyhedral cone is closed, since it is the intersection of closed half-spaces. A finitely generated cone is also closed and is in fact polyhedral, but this is a fairly deep fact, which is shown in the following proposition.

Proposition 1.6.1:

(a) Let $a_1, \ldots, a_r$ be vectors of $\Re^n$. Then, the finitely generated cone

$$C = \left\{ x \;\middle|\; x = \sum_{j=1}^{r} \mu_j a_j,\ \mu_j \geq 0,\ j = 1, \ldots, r \right\} \tag{1.47}$$

is closed and its polar cone is the polyhedral cone given by

$$C^* = \big\{ x \mid a_j' x \leq 0,\ j = 1, \ldots, r \big\}. \tag{1.48}$$

(b) (*Farkas' Lemma*) Let x, $e_1, \ldots, e_m$, and $a_1, \ldots, a_r$ be vectors of $\Re^n$. We have $x'y \leq 0$ for all vectors $y \in \Re^n$ such that

$$y'e_i = 0, \quad \forall\, i = 1, \ldots, m, \qquad y'a_j \leq 0, \quad \forall\, j = 1, \ldots, r,$$

if and only if x can be expressed as

$$x = \sum_{i=1}^{m} \lambda_i e_i + \sum_{j=1}^{r} \mu_j a_j,$$

where λ_i and μ_j are some scalars with $\mu_j \geq 0$ for all j.

(c) (*Minkowski-Weyl Theorem*) A cone is polyhedral if and only if it is finitely generated.

Proof: (a) We first show that the polar cone of C has the desired form (1.48). If y satisfies $y'a_j \leq 0$ for all j, then $y'x \leq 0$ for all $x \in C$, so the set in the right-hand side of Eq. (1.48) is a subset of C^*. Conversely, if $y \in C^*$, i.e., if $y'x \leq 0$ for all $x \in C$, then (since a_j belong to C) we have $y'a_j \leq 0$, for all j. Thus, C^* is a subset of the set in the right-hand side of Eq. (1.48).

There remains to show that C is closed. We will give two proofs for this. The first proof is simpler and suffices for the purpose of showing Farkas' Lemma [part (b)]. The second proof shows a stronger result, namely that C is polyhedral. This not only shows that C is closed, but also proves half of the Minkowski-Weyl Theorem [part (c)].

The first proof that C is closed is based on induction on the number of vectors r. When $r = 1$, C is either $\{0\}$ (if $a_1 = 0$) or a halfline, and is therefore closed. Suppose, for some $r \geq 1$, all cones of the form

$$\left\{ x \;\middle|\; x = \sum_{j=1}^{r} \mu_j a_j,\ \mu_j \geq 0,\ j = 1, \ldots, r \right\}$$

are closed. Then, we will show that a cone of the form

$$C_{r+1} = \left\{ x \;\middle|\; x = \sum_{j=1}^{r+1} \mu_j a_j,\ \mu_j \geq 0,\ j = 1, \ldots, r+1 \right\}$$

is also closed. Without loss of generality, assume that $\|a_j\| = 1$ for all j. If the vectors $-a_1, \ldots, -a_{r+1}$ belong to C_{r+1}, then C_{r+1} is the subspace spanned by $a_1, \ldots, a_{r+1}$ and is therefore closed. If the negative of one of the vectors, say $-a_{r+1}$, does not belong to C_{r+1}, consider the cone

$$C_r = \left\{ x \;\middle|\; x = \sum_{j=1}^{r} \mu_j a_j,\ \mu_j \geq 0,\ j = 1, \ldots, r \right\},$$

which is closed by the induction hypothesis. Let

$$\gamma = \min_{x \in C_r,\, \|x\|=1} a_{r+1}' x.$$

Since the set $\{x \in C_r \mid \|x\| = 1\}$ is nonempty and compact, the minimum above is attained at some $x^* \in C_r$ with $\|x^*\| = 1$ by Weierstrass' Theorem. Using the Schwartz Inequality, we have

$$\gamma = a_{r+1}' x^* \geq -\|a_{r+1}\| \cdot \|x^*\| = -1,$$

with equality if and only if $x^* = -a_{r+1}$. Because $x^* \in C_r$ and $-a_{r+1} \notin C_r$, equality cannot hold above, so that $\gamma > -1$. Let $\{x_k\} \subset C_{r+1}$ be a convergent sequence. We will prove that its limit belongs to C_{r+1}, thereby showing that C_{r+1} is closed. Indeed, for all k, we have $x_k = \xi_k a_{r+1} + y_k$, where $\xi_k \geq 0$ and $y_k \in C_r$. Using the fact $\|a_{r+1}\| = 1$ and the definition of γ, we obtain

$$\begin{aligned} \|x_k\|^2 &= \xi_k^2 + \|y_k\|^2 + 2\xi_k a_{r+1}' y_k \\ &\geq \xi_k^2 + \|y_k\|^2 + 2\gamma \xi_k \|y_k\| \\ &= \big(\xi_k - \|y_k\|\big)^2 + 2(1+\gamma)\xi_k \|y_k\|. \end{aligned}$$

Since $\{x_k\}$ converges, $\xi_k \geq 0$, and $1 + \gamma > 0$, it follows that the sequences $\{\xi_k\}$ and $\{y_k\}$ are bounded and hence, they have limit points denoted by ξ and y, respectively. The limit of $\{x_k\}$ is

$$\lim_{k \to \infty} (\xi_k a_{r+1} + y_k) = \xi a_{r+1} + y,$$

which belongs to C_{r+1}, since $\xi \geq 0$ and $y \in C_r$ (by the closedness hypothesis on C_r). We conclude that C_{r+1} is closed, completing the proof.

We now give an alternative proof that C is closed, by showing that it is polyhedral. The proof is constructive and uses induction on the number of vectors r. When $r = 1$, there are two possibilities: (a) $a_1 = 0$, in which case $C = \{0\}$, and

$$C = \{x \mid u_i'x \leq 0,\ -u_i'x \leq 0,\ i = 1, \ldots, n\},$$

where u_i is the ith unit coordinate vector; (b) $a_1 \neq 0$, in which case C is a closed halfline, which using the Polar Cone Theorem, is characterized as the set of vectors that are orthogonal to the subspace orthogonal to a_1 and also make a nonnegative inner product with a_1, i.e.,

$$C = \{x \mid b_i'x \leq 0,\ -b_i'x \leq 0,\ i = 1, \ldots, n-1,\ -a_1'x \leq 0\},$$

where $b_1, \ldots, b_{n-1}$ are basis vectors for the $(n-1)$-dimensional subspace that is orthogonal to the vector a_1. In both cases (a) and (b), C is polyhedral.

Assume that for some $r \geq 1$, a set of the form

$$C_r = \left\{ x \;\middle|\; x = \sum_{j=1}^{r} \mu_j a_j,\ \mu_j \geq 0,\ j = 1, \ldots, r \right\}$$

has a polyhedral representation

$$P_r = \big\{x \mid b_j'x \leq 0,\ j = 1, \ldots, m\big\}.$$

Let a_{r+1} be a given vector in $\Re^n$, and consider the set

$$C_{r+1} = \left\{ x \;\middle|\; x = \sum_{j=1}^{r+1} \mu_j a_j,\ \mu_j \geq 0,\ j = 1, \ldots, r+1 \right\}.$$

We will show that C_{r+1} has a polyhedral representation.

Let

$$\beta_j = a_{r+1}'b_j, \qquad j = 1, \ldots, m,$$

and define the index sets

$$J^- = \{j \mid \beta_j < 0\}, \qquad J^0 = \{j \mid \beta_j = 0\}, \qquad J^+ = \{j \mid \beta_j > 0\}.$$

If $J^- \cup J^0 = \emptyset$, or equivalently $a_{r+1}'b_j > 0$ for all j, we see that $-a_{r+1}$ is an interior point of P_r. Hence there is an $\epsilon > 0$ such that for each $\mu_{r+1} > 0$, the open sphere centered at $-\mu_{r+1}a_{r+1}$ of radius $\mu_{r+1}\epsilon$ is contained in P_r. It follows that for any $x \in \Re^n$ we have that $-\mu_{r+1}a_{r+1} + x \in P_r$

for all sufficiently large μ_{r+1}, implying that x belongs to C_{r+1}. Therefore $C_{r+1} = \Re^n$, so that C_{r+1} is a polyhedral set.

We may thus assume that $J^- \cup J^0 \neq \emptyset$. We will show that if $J^+ = \emptyset$ or $J^- = \emptyset$, the set C_{r+1} has the polyhedral representation

$$P_{r+1} = \{x \mid b_j' x \leq 0,\ j \in J^- \cup J^0\},$$

and if $J^+ \neq \emptyset$ and $J^- \neq \emptyset$, it has the polyhedral representation

$$P_{r+1} = \{x \mid b_j' x \leq 0,\ j \in J^- \cup J^0,\ b_{l,k}' x \leq 0,\ l \in J^+,\ k \in J^-\},$$

where

$$b_{l,k} = b_l - \frac{\beta_l}{\beta_k} b_k, \qquad \forall\ l \in J^+,\ \ \forall\ k \in J^-.$$

This will complete the induction.

Indeed, we have $C_{r+1} \subset P_{r+1}$ since by construction, the vectors $a_1, \ldots, a_{r+1}$ satisfy the inequalities defining P_{r+1}. To show the reverse inclusion, we fix a vector $x \in P_{r+1}$ and we verify that there exist $\mu_1, \ldots, \mu_{r+1} \geq 0$ such that $x = \sum_{j=1}^{r+1} \mu_j a_j$, or equivalently that there exists $\mu_{r+1} \geq 0$ such that

$$x - \mu_{r+1} a_{r+1} \in P_r.$$

We consider three cases.

(1) $J^+ = \emptyset$. Then, since $x \in P_{r+1}$, we have $b_j' x \leq 0$ for all $j \in J^- \cup J^0$, implying $x \in P_r$, so that $x - \mu_{r+1} a_{r+1} \in P_r$ for $\mu_{r+1} = 0$.

(2) $J^+ \neq \emptyset$, $J^0 \neq \emptyset$, and $J^- = \emptyset$. Then, since $x \in P_{r+1}$, we have $b_j' x \leq 0$ for all $j \in J^0$, so that

$$b_j'(x - \mu a_{r+1}) \leq 0, \qquad \forall\ j \in J^0,\ \forall\ \mu \geq 0,$$

$$b_j'(x - \mu a_{r+1}) \leq 0, \qquad \forall\ j \in J^+,\ \forall\ \mu \geq \max_{j \in J^+} \frac{b_j' x}{\beta_j}.$$

Hence, for μ_{r+1} sufficiently large, we have $x - \mu_{r+1} a_{r+1} \in P_r$.

(3) $J^+ \neq \emptyset$ and $J^- \neq \emptyset$. Then since $x \in P_{r+1}$, we have $b_{l,k}' x \leq 0$ for all $l \in J^+$, $k \in J^-$, implying that

$$\frac{b_l' x}{\beta_l} \leq \frac{b_k' x}{\beta_k}, \qquad \forall\ l \in J^+,\ \forall\ k \in J^-. \tag{1.49}$$

Furthermore, for $x \in P_{r+1}$, there holds $b_k' x / \beta_k \geq 0$ for all $k \in J^-$, so that for μ_{r+1} satisfying

$$\max\left\{0, \max_{l \in J^+} \frac{b_l' x}{\beta_l}\right\} \leq \mu_{r+1} \leq \min_{k \in J^-} \frac{b_k' x}{\beta_k},$$

we have

$$b_j'x - \mu_{r+1}b_j'a_{r+1} \le 0, \qquad \forall\ j \in J^0,$$

$$b_k'x - \mu_{r+1}b_k'a_{r+1} = \beta_k\left(\frac{b_k'x}{\beta_k} - \mu_{r+1}\right) \le 0, \qquad \forall\ k \in J^-,$$

$$b_l'x - \mu_{r+1}b_l'a_{r+1} = \beta_l\left(\frac{b_l'x}{\beta_l} - \mu_{r+1}\right) \le 0, \qquad \forall\ l \in J^+.$$

Hence $b_j'x - \mu_{r+1}a_{r+1} \le 0$ for all j, implying that $x - \mu_{r+1}a_{r+1} \in P_r$, and completing the proof.

(b) Define $a_{r+i} = e_i$ and $a_{r+m+i} = -e_i$, $i = 1, \ldots, m$. The result to be shown translates to

$$x \in P^* \qquad \Longleftrightarrow \qquad x \in C,$$

where

$$P = \{y \mid y'a_j \le 0,\ j = 1, \ldots, r + 2m\},$$

$$C = \left\{x \;\middle|\; x = \sum_{j=1}^{r+2m} \mu_j a_j,\ \mu_j \ge 0\right\}.$$

Since by part (a), $P = C^*$ and C is closed and convex, we have $P^* = \big(C^*\big)^* = C$ by the Polar Cone Theorem (Prop. 1.5.1).

(c) We have already shown in the alternative proof of part (a) that a finitely generated cone is polyhedral, so there remains to show the converse. Let P be a polyhedral cone given by

$$P = \{x \mid a_j'x \le 0,\ j = 1, \ldots, r\}.$$

By part (a),

$$P = \left\{x \;\middle|\; x = \sum_{j=1}^{r} \mu_j a_j,\ \mu_j \ge 0,\ j = 1, \ldots, r\right\}^*,$$

and by using also the closedness of finitely generated cones proved in part (a) and the Polar Cone Theorem, we have

$$P^* = \left\{x \;\middle|\; x = \sum_{j=1}^{r} \mu_j a_j,\ \mu_j \ge 0,\ j = 1, \ldots, r\right\}.$$

Thus, P^* is finitely generated and therefore polyhedral, so that by part (a), its polar $(P^*)^*$ is finitely generated. By the Polar Cone Theorem, $(P^*)^* = P$, implying that P is finitely generated. **Q.E.D.**

1.6.2 Polyhedral Sets

A nonempty set $P \subset \Re^n$ is said to be a *polyhedral set* (or *polyhedron*) if it has the form

$$P = \big\{x \mid a_j'x \le b_j,\ j = 1, \ldots, r\big\},$$

where a_j are some vectors in $\Re^n$ and b_j are some scalars. A polyhedron may also involve linear equalities, which may be converted into two linear inequalities. In particular, if $e_1, \ldots, e_m$ and $a_1, \ldots, a_r$ are given vectors, and $d_1, \ldots, d_m$ and $b_1, \ldots, b_r$ are given scalars, the set

$$\{x \mid e_i'x = d_i,\ i = 1, \ldots, m,\ a_j'x \le b_j,\ j = 1, \ldots, r\}$$

is polyhedral, since it can be written as

$$\{x \mid e_i'x \le d_i,\ -e_i'x \le -d_i,\ i = 1, \ldots, m,\ a_j'x \le b_j,\ j = 1, \ldots, r\}.$$

The following is a fundamental result, showing that a polyhedral set can be represented as the sum of the convex hull of a finite set of points and a finitely generated cone.

Proposition 1.6.2: A set P is polyhedral if and only if there exist a nonempty finite set of vectors $\{v_1, \ldots, v_m\}$ and a finitely generated cone C such that $P = \mathrm{conv}\big(\{v_1, \ldots, v_m\}\big) + C$, i.e.,

$$P = \left\{x \;\middle|\; x = \sum_{j=1}^m \mu_j v_j + y,\ \sum_{j=1}^m \mu_j = 1,\ \mu_j \ge 0,\ j = 1, \ldots, m,\ y \in C\right\}.$$

Proof: Assume that P is polyhedral. Then, it has the form

$$P = \big\{x \mid a_j'x \le b_j,\ j = 1, \ldots, r\big\},$$

for some vectors a_j and scalars b_j. Consider the polyhedral cone in $\Re^{n+1}$ given by

$$\hat{P} = \big\{(x, w) \mid 0 \le w,\ a_j'x \le b_j w,\ j = 1, \ldots, r\big\}$$

and note that

$$P = \big\{x \mid (x, 1) \in \hat{P}\big\}.$$

By the Minkowski-Weyl Theorem [Prop. 1.6.1(c)], $\hat{P}$ is finitely generated, so it has the form

$$\hat{P} = \left\{(x, w) \;\middle|\; x = \sum_{j=1}^m \mu_j v_j,\ w = \sum_{j=1}^m \mu_j d_j,\ \mu_j \ge 0,\ j = 1, \ldots, m\right\},$$

for some vectors v_j and scalars d_j. Since $w \geq 0$ for all vectors $(x, w) \in \hat{P}$, we see that $d_j \geq 0$ for all j. Let

$$J^+ = \{j \mid d_j > 0\}, \qquad J^0 = \{j \mid d_j = 0\}.$$

By replacing μ_j by μ_j/d_j for all $j \in J^+$, we obtain the equivalent description

$$\hat{P} = \left\{ (x, w) \;\middle|\; x = \sum_{j=1}^{m} \mu_j v_j,\ w = \sum_{j \in J^+} \mu_j,\ \mu_j \geq 0,\ j = 1, \ldots, m \right\}.$$

Since $P = \{x \mid (x, 1) \in \hat{P}\}$, we obtain

$$P = \left\{ x \;\middle|\; x = \sum_{j \in J^+} \mu_j v_j + \sum_{j \in J^0} \mu_j v_j,\ \sum_{j \in J^+} \mu_j = 1,\ \mu_j \geq 0,\ j = 1, \ldots, m \right\}.$$

Thus, P is the vector sum of the convex hull of the vectors v_j, $j \in J^+$, and the finitely generated cone

$$\left\{ \sum_{j \in J^0} \mu_j v_j \;\middle|\; \mu_j \geq 0,\ j \in J^0 \right\}.$$

To prove that the vector sum of $\mathrm{conv}\big(\{v_1, \ldots, v_m\}\big)$ and a finitely generated cone is a polyhedral set, we use a reverse argument; we pass to a finitely generated cone description, we use the Minkowski-Weyl Theorem to assert that this cone is polyhedral, and we finally construct a polyhedral set description. The details are left as an exercise for the reader. **Q.E.D.**

1.6.3 Extreme Points

Given a convex set C, a vector $x \in C$ is said to be an *extreme point* of C if there do not exist vectors $y \in C$ and $z \in C$, with $y \neq x$ and $z \neq x$, and a scalar $\alpha \in (0, 1)$ such that $x = \alpha y + (1 - \alpha)z$. An equivalent definition is that x cannot be expressed as a convex combination of some vectors of C, all of which are different from x.

Thus an extreme point of a set cannot lie strictly between the endpoints of any line segment contained in the set. It follows that an extreme point of a set must lie on the relative boundary of the set. As a result, an open set has no extreme points, and more generally a convex set that coincides with its relative interior, has no extreme points, except in the special case where the set consists of a single point. As another consequence of the definition, a convex cone may have at most one extreme point, the origin.

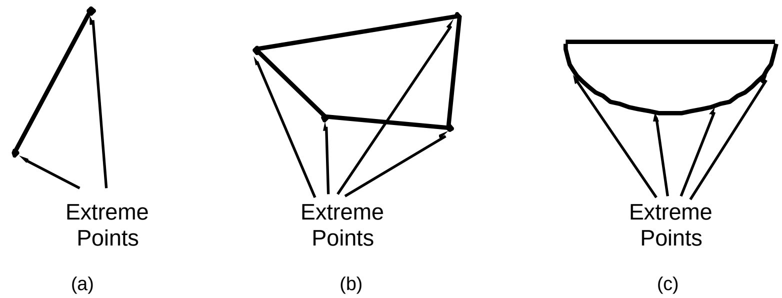

Figure 1.6.2. Illustration of extreme points of various convex sets. In the set of (c), the extreme points are the ones that lie on the circular arc.

We also show in what follows that a bounded polyhedral set has a finite number of extreme points and in fact it is equal to the convex hull of its extreme points. Figure 1.6.2 illustrates the extreme points of various types of sets.

The following proposition provides some intuition into the nature of extreme points.

Proposition 1.6.3: Let C be a nonempty closed convex set in $\Re^n$.

(a) If H is a hyperplane that passes through a relative boundary point of C and contains C in one of its halfspaces, then every extreme point of $C \cap H$ is also an extreme point of C.

(b) C has at least one extreme point if and only if it does not contain a line, i.e., a set L of the form $L = \{x + \alpha d \mid \alpha \in \Re\}$ where d is a nonzero vector in $\Re^n$ and x is some vector in C.

(c) (*Krein - Milman Theorem*) If C is bounded (in addition to being convex and closed), then C is equal to the convex hull of its extreme points.

Proof: (a) Let $\overline{x}$ be an element of $C \cap H$ which is not an extreme point of C. Then we have $\overline{x} = \alpha y + (1 - \alpha)z$ for some $\alpha \in (0, 1)$, and some $y \in C$ and $z \in C$, with $y \neq x$ and $z \neq x$. Since $\overline{x} \in H$, the halfspace containing C is of the form $\{x \mid a'x \geq a'\overline{x}\}$, where $a \neq 0$. Then $a'y \geq a'\overline{x}$ and $a'z \geq a'\overline{x}$, which in view of $\overline{x} = \alpha y + (1 - \alpha)z$, implies that $a'y = a'\overline{x}$ and $a'z = a'\overline{x}$. Therefore, $y \in C \cap H$ and $z \in C \cap H$, showing that $\overline{x}$ is not an extreme point of $C \cap H$.

(b) To arrive at a contradiction, assume that C has an extreme point x and contains a line $\{\overline{x} + \alpha d \mid \alpha \in \Re\}$, where $\overline{x} \in C$ and $d \neq 0$. Then by the

Recession Cone Theorem [Prop. 1.2.13(b)] and the closedness of C, both d and $-d$ are directions of recession of C, so the line $\{x + \alpha d \mid \alpha \in \Re\}$ belongs to C. This contradicts the fact that x is an extreme point.

Conversely, we use induction on the dimension of the space to show that if C does not contain a line, it must have an extreme point. This is true in the real line $\Re$, so assume it is true in $\Re^{n-1}$. Since C contains no line, we must have $C \neq \Re^n$, so that C has a relative boundary point. Take any $\overline{x}$ on the relative boundary of C, and any hyperplane H passing through $\overline{x}$ and containing C in one of its halfspaces. By using a translation argument if necessary, we may assume without loss of generality that $\overline{x} = 0$. Then H is an $(n-1)$-dimensional subspace, so the set $C \cap H$ lies in an $(n-1)$-dimensional space and contains no line. Hence, by the induction hypothesis, it must have an extreme point. By part (a), this extreme point must also be an extreme point of C.

(c) The proof is based on induction on the dimension of the space. On the real line, every compact convex set C is a line segment whose endpoints are the extreme points of C, so that C is the convex hull of its extreme points. Suppose now that every compact convex set in $\Re^{n-1}$ is the convex hull of its extreme points. Let C be a compact convex set in $\Re^n$. Since C is bounded, it contains no line and by part (b), it has at least one extreme point. By convexity of C, the convex hull of all extreme points of C is contained in C. To show the inverse inclusion, choose any $x \in C$, and consider a line that lies in $\text{aff}(C)$ and passes through x. Since C is compact, the intersection of this line and C is a line segment whose endpoints, say x_1 and x_2, belong to the relative boundary of C. Let H_1 be a hyperplane that passes through x_1 and contains C in one of its halfspaces. Similarly, let H_2 be a hyperplane that passes through x_2 and contains C in one of its halfspaces. The intersections $C \cap H_1$ and $C \cap H_2$ are compact convex sets that lie in the hyperplanes H_1 and H_2, respectively. By viewing H_1 and H_2 as $(n-1)$-dimensional spaces, and by using the inductive hypothesis, we see that each of the sets $C \cap H_1$ and $C \cap H_2$ is the convex hull of its extreme points. Hence x_1 is a convex combination of some extreme points of $C \cap H_1$, and x_2 is a convex combination of some extreme points of $C \cap H_2$. By part (a), every extreme point of $C \cap H_1$ and every extreme point of $C \cap H_2$ is also an extreme point of C, so that each of x_1 and x_2 is a convex combination of some extreme points of C. Since x lies in the line segment connecting x_1 and x_2, it follows that x is a convex combination of some extreme points of C, showing that C is contained in the convex hull of all extreme points of C. **Q.E.D.**

The boundedness assumption is essential in the Krein - Milman Theorem. For example, the only extreme point of the ray $C = \{\lambda y \mid \lambda \geq 0\}$, where is y is a given nonzero vector, is the origin, but none of its other points can be generated as a convex combination of extreme points of C.

As an example of application of the preceding proposition, consider a nonempty polyhedral set of the form

$$\{x \mid Ax = b,\, x \geq 0\},$$

or of the form

$$\{x \mid Ax \leq b,\, x \geq 0\},$$

where A is an $m \times n$ matrix and $b \in \Re^n$. Such polyhedra arise commonly in linear programming formulations, and clearly do not contain a line. Hence by Prop. 1.6.3(b) they always possess at least one extreme point.

One of the fundamental linear programming results is that if a linear function f attains a minimum over a polyhedral set C having at least one extreme point, then f attains a minimum at some extreme point of C (as well as possibly at some other nonextreme points). We will come to this fact after considering the more general case where f is concave, and C is closed and convex.

Proposition 1.6.4: Let $C \subset \Re^n$ be a closed convex set that does not contain a line, and let $f : C \mapsto \Re$ be a concave function attaining a minimum over C. Then f attains a minimum at some extreme point of C.

Proof: If C consists of just a single point, the result holds trivially. We may thus assume that C contains at least two distinct points. We first show that f attains a minimum at some vector that is not in $\mathrm{ri}(C)$. Let x^* be a vector where f attains a minimum over C. If $x^* \notin \mathrm{ri}(C)$ we are done, so assume that $x^* \in \mathrm{ri}(C)$. By Prop. 1.2.9(c), every line segment having x^* as one endpoint can be prolonged beyond x^* without leaving C, i.e., for any $\overline{x} \in C$, there exists $\gamma > 1$ such that

$$\hat{x} = x^* + (\gamma - 1)(x^* - \overline{x}) \in C.$$

Therefore

$$x^* = \frac{\gamma - 1}{\gamma}\overline{x} + \frac{1}{\gamma}\hat{x},$$

and by the concavity of f and the optimality of x^*, we have

$$f(x^*) \geq \frac{\gamma - 1}{\gamma} f(\overline{x}) + \frac{1}{\gamma} f(\hat{x}) \geq f(x^*),$$

implying that $f(\overline{x}) = f(x^*)$ for any $\overline{x} \in C$. Furthermore, there exists an $\overline{x} \in C$ with $\overline{x} \notin \mathrm{ri}(C)$, since C is closed, contains at least two points, and does not contain a line.

We have shown so far that the minimum of f is attained at some $\overline{x} \notin \mathrm{ri}(C)$. If $\overline{x}$ is an extreme point of C, we are done. If it is not an extreme point, consider a hyperplane H passing through $\overline{x}$ and containing C in one of its halfspaces. The intersection $C_1 = C \cap H$ is closed, convex, does not contain a line, and lies in an affine set M_1 of dimension $n-1$. Furthermore, f attains its minimum over C_1 at $\overline{x}$. Thus, by the preceding argument, it also attains its minimum at some $x_1 \in C_1$ with $x_1 \notin \mathrm{ri}(C_1)$. If x_1 is an extreme point of C_1, then by Prop. 1.6.3, it is also an extreme point of C and the result follows. If x_1 is not an extreme point of C_1, then we may view M_1 as a space of dimension $n-1$ and we form C_2, the intersection of C_1 with a hyperplane in M_1 that passes through x_1 and contains C_1 in one of its halfspaces. This hyperplane will be of dimension $n-2$. We can continue this process for at most n times, when a set C_n consisting of a single point is obtained. This point is an extreme point of C_n and, by repeated application of Prop. 1.6.3, an extreme point of C. **Q.E.D.**

The following is an important special case of the preceding proposition.

Proposition 1.6.5: Let C be a closed convex set and let $f : C \mapsto \Re$ be a concave function. Assume that for some $m \times n$ matrix A of rank n and some $b \in \Re^n$ we have

$$Ax \geq b, \qquad \forall\ x \in C.$$

Then if f attains a minimum over C, it attains a minimum at some extreme point of C.

Proof: In view of Prop. 1.6.4, it suffices to show that C does not contain a line. Assume, to obtain a contradiction, that C contains the line

$$L = \{\overline{x} + \lambda d \mid \lambda \in \Re^n\},$$

where $\overline{x} \in C$ and d is a nonzero vector. Since A has rank n, the vector Ad is nonzero, implying that the image

$$AL = \{A\overline{x} + \lambda Ad \mid \lambda \in \Re\}$$

is also a line. This contradicts the assumption $Ax \geq b$ for all $x \in C$. **Q.E.D.**

1.6.4 Extreme Points and Linear Programming

We now consider a polyhedral set P and we characterize the set of its extreme points (also called *vertices*). By Prop. 1.6.2, P can be represented as

$$P = \hat{P} + C,$$

where $\hat{P}$ is the convex hull of some vectors $v_1, \ldots, v_m$ and C is a finitely generated cone:

$$\hat{P} = \left\{ x \;\middle|\; x = \sum_{j=1}^{m} \mu_j v_j,\ \sum_{j=1}^{m} \mu_j = 1,\ \mu_j \geq 0,\ j = 1, \ldots, m \right\}.$$

We note that an extreme point $\overline{x}$ of P cannot be of the form $\overline{x} = \hat{x} + y$, where $\hat{x} \in \hat{P}$ and $y \neq 0$, $y \in C$, since in this case $\overline{x}$ would be the midpoint of the line segment connecting the distinct vectors $\hat{x}$ and $\hat{x} + 2y$. Therefore, an extreme point of P must belong to $\hat{P}$, and since $\hat{P} \subset P$, it must also be an extreme point of $\hat{P}$. An extreme point of $\hat{P}$ must be one of the vectors $v_1, \ldots, v_m$, since otherwise this point would be expressible as a convex combination of $v_1, \ldots, v_m$. Thus the set of extreme points of P is either empty, or nonempty and finite. Using Prop. 1.6.3(b), it follows that *the set of extreme points of P is nonempty and finite if and only if P contains no line*.

If P is bounded, then we must have $P = \hat{P}$, and it follows from the Krein - Milman Theorem that P *is equal to the convex hull of its extreme points* (not just the convex hull of the vectors $v_1, \ldots, v_m$). The following proposition gives another and more specific characterization of extreme points of polyhedral sets, which is central in the theory of linear programming.

Proposition 1.6.6: Let P be a polyhedral set in $\Re^n$.

(a) If P has the form

$$P = \{x \mid a_j' x \le b_j,\ j = 1, \dots, r\},$$

where a_j and b_j are given vectors and scalars, respectively, then a vector $v \in P$ is an extreme point of P if and only if the set

$$A_v = \{a_j \mid a_j' v = b_j,\ j \in \{1, \dots, r\}\}$$

contains n linearly independent vectors.

(b) If P has the form

$$P = \{x \mid Ax = b,\ x \ge 0\},$$

where A is a given $m \times n$ matrix and b is a given vector in $\Re^m$, then a vector $v \in P$ is an extreme point of P if and only if the columns of A corresponding to the nonzero coordinates of v are linearly independent.

(c) If P has the form

$$P = \{x \mid Ax = b,\ c \le x \le d\},$$

where A is a given $m \times n$ matrix, b is a given vector in $\Re^m$, and c, d are given vectors in $\Re^n$, then a vector $v \in P$ is an extreme point of P if and only if the columns of A corresponding to the coordinates of v that lie strictly between the corresponding coordinates of c and d are linearly independent.

Proof: (a) If the set A_v contains fewer than n linearly independent vectors, then the system of equations

$$a_j' w = 0, \qquad \forall\ a_j \in A_v$$

has a nonzero solution $\overline{w}$. For sufficiently small $\gamma > 0$, we have $v + \gamma \overline{w} \in P$ and $v - \gamma \overline{w} \in P$, thus showing that v is not an extreme point. Thus, if v is an extreme point, A_v must contain n linearly independent vectors.

Conversely, suppose that A_v contains a subset $\bar{A}_v$ consisting of n linearly independent vectors. Suppose that for some $y \in P$, $z \in P$, and $\alpha \in (0, 1)$, we have $v = \alpha y + (1 - \alpha) z$. Then for all $a_j \in \bar{A}_v$, we have

$$b_j = a_j' v = \alpha a_j' y + (1 - \alpha) a_j' z \le \alpha b_j + (1 - \alpha) b_j = b_j.$$

Thus v, y, and z are all solutions of the system of n linearly independent equations

$$a_j' w = b_j, \qquad \forall\ a_j \in \bar{A}_v.$$

Hence $v = y = z$, implying that v is an extreme point of P.

(b) Let k be the number of zero coordinates of v, and consider the matrix $\bar{A}$, which is the same as A except that all the columns corresponding to the zero coordinates of v are set to zero. We write P in the form

$$P = \{x \mid Ax \leq b,\ -Ax \leq -b,\ -x \leq 0\},$$

and apply the result of part (a). We obtain that v is an extreme point if and only if $\bar{A}$ contains $n - k$ linearly independent rows, which is equivalent to the $n - k$ nonzero columns of A (corresponding to the nonzero coordinates of v) being linearly independent.

(c) The proof is essentially the same as the proof of part (b). **Q.E.D.**

The following theorem illustrates the importance of extreme points in linear programming. In addition to its analytical significance, it forms the basis for algorithms, such as the celebrated simplex method (see any linear programming book, e.g., Dantzig [Dan63], Chvatal [Chv83], or Bertsimas and Tsitsiklis [BeT97]), which systematically search the set of extreme points of the constraint polyhedron for an optimal extreme point.

Proposition 1.6.7: (Fundamental Theorem of Linear Programming) Let P be a polyhedral set that has at least one extreme point. Then if a linear function attains a minimum over P, it attains a minimum at some extreme point of P.

Proof: Since P is polyhedral, it has a representation

$$P = \{x \mid Ax \geq b\},$$

for some $m \times n$ matrix A and some $b \in \Re^m$. If A had rank less than n, then its nullspace would contain some nonzero vector d, so P would contain a line parallel to d, contradicting the existence of an extreme point [cf. Prop. 1.6.3(b)]. Thus A has rank n and the result follows from Prop. 1.6.5. **Q.E.D.**

Extreme Points and Integer Programming

Many important optimization problems, in addition to the usual equality and inequality constraints, include the requirement that the optimization variables take integer values, such as 0 or 1. We refer to such problems as *integer programming* problems, and we note that they arise in a broad variety of practical settings, such as for example in scheduling, resource allocation, and engineering design, as well in many combinatorial optimization contexts, such as matching, and traveling salesman problems.

The methodology for solving an integer programming problem is very diverse, but an important subset of this methodology relies on the solution of a continuous optimization problem, called the *relaxed problem*, which is derived from the original by neglecting the integer constraints while maintaining all the other constraints. In many important situations the relaxed problem is a linear program that can be solved for an extreme point optimal solution, by a variety of algorithms, including the simplex method. A particularly fortuitous situation arises if this extreme point solution happens to have integer components, since it will then solve optimally not just the relaxed problem, but also the original integer programming problem. Thus polyhedra whose extreme points have integer components are of special significance. We will now use Prop. 1.6.6 to characterize an important class of polyhedra with this property.

Consider a polyhedral set P of the form

$$P = \{x \mid Ax = b,\ c \le x \le d\}, \tag{1.50}$$

where A is a given $m \times n$ matrix, b is a given vector in $\Re^m$, and c and d are given vectors in $\Re^n$. We assume that all the components of the matrix A and the vectors b, c, and d are integer. Given an extreme point v of P, we want to determine conditions under which the components of v are integer.

Let us say that a square matrix with integer components is *unimodular* if its determinant is 0, 1, or -1, and let us say that a rectangular matrix with integer components is *totally unimodular* if each of its square submatrices is unimodular. We have the following proposition.

Proposition 1.6.8: Let P be a polyhedral set

$$P = \{x \mid Ax = b,\ c \le x \le d\},$$

where A is a given $m \times n$ matrix, b is a given vector in $\Re^m$, and c and d are given vectors in $\Re^n$. Assume that all the components of the matrix A and the vectors b, c, and d are integer, and that the matrix A is totally unimodular. Then all the extreme points of P have integer components.

Proof: Let v be an extreme point of P. Consider the subset of indexes

$$I = \{i \mid c_i < v_i < d_i\},$$

and without loss of generality, assume that

$$I = \{1, \ldots, \overline{m}\}$$

for some integer $\overline{m}$. Let $\overline{A}$ be the matrix consisting of the first $\overline{m}$ columns of A and let $\overline{v}$ be the vector consisting of the first $\overline{m}$ components of v. Note that each of the last $n - \overline{m}$ components of v is equal to either the corresponding component of c or to the corresponding component of d, which are integer. Thus the extreme point v has integer components if and only if the subvector $\overline{v}$ does.

From Prop. 1.6.6, $\overline{A}$ has linearly independent columns, so $\overline{v}$ is the unique solution of the system of equations

$$\overline{A}y = \overline{b},$$

where $\overline{b}$ is equal to b minus the last $n - \overline{m}$ columns of A multiplied with the corresponding components of v (each of which is equal to either the corresponding component of c or the corresponding component of d, so that $\overline{b}$ has integer components). Equivalently, there exists an invertible $\overline{m} \times \overline{m}$ submatrix $\tilde{A}$ of $\overline{A}$ and a subvector $\tilde{b}$ of $\overline{b}$ with $\overline{m}$ components such that

$$\overline{v} = (\tilde{A})^{-1}\tilde{b}.$$

The components of $\overline{v}$ will be integer if we can guarantee that the components of the inverse $(\tilde{A})^{-1}$ are integer. By Cramer's formula, each of the components of the inverse of a matrix is a fraction with a sum of products of the components of the matrix in the numerator and the determinant of the matrix in the denominator. Since by hypothesis, A is totally unimodular, the invertible submatrix $\tilde{A}$ is unimodular, and its determinant is either equal to 1 or is equal to -1. Hence $(\tilde{A})^{-1}$ has integer components, and it follows that $\overline{v}$ (and hence also the extreme point v) has integer components. **Q.E.D.**

The total unimodularity of the matrix A in the above proposition can be verified in a number of important special cases, some of which are discussed in the exercises. The most important such case arises in network optimization problems, where A is the, so-called, *arc incidence matrix* of a given directed graph (we refer to network optimization books such as Rockafellar [Roc84] or Bertsekas [Ber98] for a detailed discussion of these problems and their broad applications). Here A has a row for each node and a column for each arc of the graph. The component corresponding to the ith row and a given arc is a 1 if the arc is outgoing from node i, is a -1 if the arc is incoming to i, and is a 0 otherwise. Then we can show that the determinant of each square submatrix of A is 0, 1, or -1 by induction on the dimension of the submatrix. In particular, the submatrices of dimension 1 of A are the scalar components of A, which are 0, 1, or -1. Suppose that the determinant of each square submatrix of dimension $n \geq 1$ is 0, 1, or -1. Consider a square submatrix of dimension $n+1$. If this matrix has a column with all components 0, the matrix is singular, and its determinant is 0. If the matrix has a column with a single nonzero component (a 1 or a -1), by expanding its determinant along that component and using the induction hypothesis, we see that the determinant is 0, 1, or -1. Finally, if each column of the matrix has two components (a 1 and a -1), the sum of its rows is 0, so the matrix is singular, and its determinant is 0. Thus, in linear network optimization where the only constraints, other than upper and lower bounds on the variables, are equality constraints corresponding to an arc incidence matrix, integer extreme point optimal solutions can generically be found.

E X E R C I S E S

1.6.1

Let V be the convex hull of a finite set of points $\{v_1, \ldots, v_m\}$ in $\Re^n$, and let $C \subset \Re^n$ be a finitely generated cone. Show that $V + C$ is a polyhedral set. *Hint*: Use the Minkowski-Weyl Theorem to construct a polyhedral description of $V + C$.

1.6.2

Let P be a polyhedral set represented as

$$P = \left\{ x \;\middle|\; x = \sum_{j=1}^{m} \mu_j v_j + y,\ \sum_{j=1}^{m} \mu_j = 1,\ \mu_j \geq 0,\ j = 1, \ldots, m,\ y \in C \right\},$$

where $v_1, \ldots, v_m$ are some vectors and C is a finitely generated cone (cf. Prop. 1.6.2). Show that the recession cone of P is equal to C.

1.6.3

Show that the image and the inverse image of a polyhedral set under a linear transformation is a polyhedral set.

1.6.4

Let $C_1 \subset \Re^n$ and $C_2 \subset \Re^m$ be polyhedral sets. Show that the Cartesian product $C_1 \times C_2$ is a polyhedral set. Show also that if C_1 and C_2 are polyhedral cones, then $C_1 \times C_2$ is a polyhedral cone.

1.6.5

Let C_1 and C_2 be two polyhedral sets in $\Re^n$. Show that $C_1 + C_2$ is a polyhedral set. Show also that if C_1 and C_2 are polyhedral cones, then $C_1 + C_2$ is a polyhedral cone.

1.6.6 (Polyhedral Strong Separation)

Let C_1 and C_2 be two nonempty disjoint polyhedral sets in $\Re^n$. Show that C_1 and C_2 can be strongly separated, i.e., there is a nonzero vector $a \in \Re^n$ such that

$$\sup_{x \in C_1} a'x < \inf_{z \in C_2} a'z.$$

(Compare this with Exercise 1.4.1.)

1.6.7

Let C be a polyhedral set containing the origin.

(a) Show that $\mathrm{cone}(C)$ is a polyhedral cone.

(b) Show by counterexample that if C does not contain the origin, $\mathrm{cone}(C)$ may not be a polyhedral cone.

1.6.8 (Polyhedral Functions)

A function $f : \Re^n \mapsto (-\infty, \infty]$ is *polyhedral* if its epigraph is a polyhedral set in $\Re^{n+1}$ not containing vertical lines [i.e., lines of the form $\{(x, w) \mid w \in \Re\}$ for some $x \in \Re^n$]. Show the following:

(a) A polyhedral function is convex over $\Re^n$.

(b) f is polyhedral if and only if $\mathrm{dom}(f)$ is a polyhedral set and

$$f(x) = \max\{a_1'x + b_1, \ldots, a_m'x + b_m\}, \qquad \forall\ x \in \mathrm{dom}(f)$$

for some vectors $a_i \in \Re^n$ and scalars b_i.

(c) The function given by

$$F(y) = \sup_{\sum_{i=1}^m \mu_i = 1,\ \mu_i \geq 0} \left\{ \left(\sum_{i=1}^m \mu_i a_i \right)' y + \sum_{i=1}^m \mu_i b_i \right\}$$

is polyhedral.

1.6.9

Let A be an $m \times n$ matrix and let $g : \Re^m \mapsto (-\infty, \infty]$ be a polyhedral function. Show that the function f given by $f(x) = g(Ax)$ is polyhedral on $\Re^n$.

1.6.10

Let P be a polyhedral set represented as

$$P = \left\{ x \;\middle|\; x = \sum_{j=1}^{m} \mu_j v_j + y,\ \sum_{j=1}^{m} \mu_j = 1,\ \mu_j \geq 0,\ j = 1, \ldots, m,\ y \in C \right\},$$

where $v_1, \ldots, v_m$ are some vectors and C is a finitely generated cone (cf. Prop. 1.6.2). Show that each extreme point of P is equal to some vector v_i that cannot be represented as a convex combination of the remaining vectors v_j, $j \neq i$.

1.6.11

Consider a nonempty polyhedral set of the form

$$P = \big\{ x \mid a_j' x \leq b_j,\ j = 1, \ldots, r \big\},$$

where a_j are some vectors and b_j are some scalars. Show that P has an extreme point if and only if the set of vectors $\{a_j \mid j = 1, \ldots, r\}$ contains a subset of n linearly independent vectors.

1.6.12 (Isomorphic Polyhedral Sets)

It is easily seen that if a linear transformation is applied to a polyhedron, the result is a polyhedron whose extreme points are the images of some (but not necessarily all) of the extreme points of the original polyhedron. This exercise clarifies the circumstances where the extreme points of the original and the transformed polyhedra are in one-to-one correspondence. Let P and Q be two polyhedra of $\Re^n$ and $\Re^m$, respectively.

(a) Show that there is an affine function $f : \Re^n \mapsto \Re^m$ that maps each extreme point of P into a distinct extreme point of Q if and only if there exists an affine function $g : \Re^n \mapsto \Re^m$ that maps each extreme point of Q into a distinct extreme point of P. *Note*: If there exist such affine mappings f and g, we say that P and Q are *isomorphic*.

(b) Show that P and Q are isomorphic if and only if there exist affine functions $f : \Re^n \mapsto \Re^m$ and $g : \Re^n \mapsto \Re^m$ such that $x = g\big(f(x)\big)$ for all $x \in P$ and $y = f\big(g(y)\big)$ for all $y \in Q$.

(c) Show that if P and Q are isomorphic then P and Q have the same dimension.

(d) Let A be an $k \times n$ matrix and let

$$P = \{x \in \Re^n \mid Ax \leq b,\ x \geq 0\},$$

$$Q = \big\{(x, z) \in \Re^{n+k} \mid Ax + z = b,\ x \geq 0,\ z \geq 0\big\}.$$

Show that P and Q are isomorphic.

1.6.13

Show by an example that the set of extreme points of a compact set need not be closed. *Hint* : Consider a line segment $C_1 = \{(x_1, x_2, x_3) \mid x_1 = 0, x_2 = 0, -1 \leq x_3 \leq 1\}$ and a circular disk $C_2 = \{(x_1, x_2, x_3) \mid (x_1 - 1)^2 + x_2^2 \leq 1,\ x_3 = 0\}$, and verify that $\text{conv}(C_1 \cup C_2)$ is compact, while its set of the extreme points is not closed.

1.6.14

Show that a compact convex set is polyhedral if and only if it has a finite number of extreme points. Give an example showing that the assertion fails if boundedness of the set is replaced by a weaker assumption that the set contains no lines.

1.6.15 (Gordan's Theorem of the Alternative)

Let $a_1, \ldots, a_r$ be vectors in $\Re^n$.

(a) Then exactly one of the following two conditions holds:

(i) There exists a vector $x \in \Re^n$ such that

$$a_1'x < 0, \ldots, a_r'x < 0.$$

(ii) There exists a vector $\mu \in \Re^r$ such that $\mu \neq 0$, $\mu \geq 0$, and

$$\mu_1 a_1 + \cdots + \mu_r a_r = 0.$$

(b) Show that an alternative and equivalent statement of part (a) is the following: a polyhedral cone has nonempty interior if and only if its polar cone does not contain a line, i.e., a set of the form $\{\alpha z \mid \alpha \in \Re\}$, where z is a nonzero vector.

1.6.16

Let $a_1, \ldots, a_r$ be vectors in $\Re^n$ and let $b_1, \ldots b_r$ be scalars. Then exactly one of the following two conditions holds:

(i) There exists a vector $x \in \Re^n$ such that

$$a_1'x \leq b_1, \ldots, a_r'x \leq b_r.$$

(ii) There exists a vector $\mu \in \Re^r$ such that $\mu \geq 0$ and

$$\mu_1 a_1 + \cdots + \mu_r a_r = 0, \qquad \mu_1 b_1 + \cdots + \mu_r b_r < 0.$$

1.6.17 (Convex System Alternatives)

Let C be a nonempty convex set in $\Re^n$ and let $f_i : C \mapsto \Re$, $i = 1, \ldots, r$ be convex functions. Then exactly one of the following two conditions holds:

(i) There exists a vector $x \in C$ such that

$$f_1(x) < 0, \ldots, f_r(x) < 0.$$

(ii) There exists a vector $\mu \in \Re^r$ such that $\mu \neq 0$, $\mu \geq 0$, and

$$\mu_1 f_1(x) + \cdots + \mu_r f_r(x) \geq 0, \qquad \forall\, x \in C.$$

1.6.18 (Convex-Affine System Alternatives)

Let C be a nonempty convex set in $\Re^n$. Let $f_i : C \mapsto \Re$, $i = 1, \ldots, \overline{r}$ be convex functions, and let $f_i : C \mapsto \Re$, $i = \overline{r} + 1, \ldots, r$ be affine functions such that the system

$$f_{\overline{r}+1}(x) \leq 0, \ldots, f_r(x) \leq 0$$

has a solution $\overline{x} \in \mathrm{ri}(C)$. Then exactly one of the following two conditions holds:

(i) There exists a vector $x \in C$ such that

$$f_1(x) < 0, \ldots, f_{\overline{r}}(x) < 0, \qquad f_{\overline{r}+1}(x) \leq 0, \ldots, f_r(x) \leq 0.$$

(ii) There exists a vector $\mu \in \Re^r$ such that $\mu \neq 0$, $\mu \geq 0$, and

$$\mu_1 f_1(x) + \cdots + \mu_r f_r(x) \geq 0, \qquad \forall\, x \in C.$$

1.6.19 (Facets)

Let P be a polyhedral set and let H be a hyperplane that passes through a relative boundary point of P and contains P in one of its halfspaces (cf., Prop. 1.6.3). The set $F = P \cap H$ is called a *facet* of P.

(a) Show that each facet is a polyhedral set.

(b) Show that the number of distinct facets of P is finite.

(c) Show that each extreme point of C, viewed as a singleton set, is a facet.

(d) Show that if $\dim(P) > 0$, there exists a facet of P whose dimension is $\dim(P) - 1$.

1.6.20

Let $f : \Re^n \to (-\infty, \infty]$ be a polyhedral function and let $P \subset \Re^n$ be a polyhedral set. Show that if $\inf_{x \in C} f(x)$ is finite, then the set of minimizers of f over C is nonempty. *Hint* : Use Exercise 1.6.8(b), and replace the problem of minimizing f over C by an equivalent linear program.

1.6.21

Show that an $m \times n$ matrix A is totally unimodular if and only if every subset J of $\{1, \ldots, n\}$ can be partitioned into two subsets J_1 and J_2 such that

$$\left| \sum_{j \in J_1} a_{ij} - \sum_{j \in J_2} a_{ij} \right| \leq 1, \qquad \forall\, i = 1, \ldots, m.$$

1.6.22

Let A be a matrix with components that are either 1 or -1 or 0. Suppose that A has at most two nonzero components in each column. Show that A is totally unimodular.

1.6.23

Let A be a matrix with components that are either 0 or 1. Suppose that in each column, all the components that are equal to 1 appear consecutively. Show that A is totally unimodular.

1.6.24

Let A be a square invertible matrix such that all the components of A are integer. Show that A is unimodular if and only if the solution of the system $Ax = b$ has integer components for every vector b that has integer components. *Hint*: To prove that A is unimodular, use the system $Ax = u_i$, where u_i is the ith unit vector, to show that A^{-1} has integer components. Then use the equality $\det(A) \cdot \det(A^{-1}) = 1$.

1.7 SUBGRADIENTS

Much of optimization theory revolves around comparing the value of the cost function at a given point with its values at neighboring points. This calls for an analysis approach that uses derivatives, as we have seen in Section 1.5. When the cost function is nondifferentiable, this approach breaks down, but fortunately, it turns out that in the case of a convex cost function there is a convenient substitute, the notion of directional differentiabity and the related notion of a subgradient, which are the subject of this section.

1.7.1 Directional Derivatives

Convex sets and functions can be characterized in many ways by their behavior along lines. For example, a set is convex if and only if its intersection with any line is convex, a convex set is bounded if and only if its intersection with every line is bounded, a function is convex if and only if it is convex along any line, and a convex function is coercive if and only if it is coercive along any line. Similarly, it turns out that the differentiability properties of a convex function are determined by the corresponding properties along lines. With this in mind, we first consider convex functions of a single variable.

Let I be an interval of real numbers, and let $f : I \mapsto \Re$ be convex. If $x, y, z \in I$ and $x < y < z$, then we can show the relation

$$\frac{f(y) - f(x)}{y - x} \le \frac{f(z) - f(x)}{z - x} \le \frac{f(z) - f(y)}{z - y}, \tag{1.51}$$

which is illustrated in Fig. 1.7.1. For a formal proof, note that, using the definition of a convex function [cf. Eq. (1.3)], we obtain

$$f(y) \le \left(\frac{y - x}{z - x}\right) f(z) + \left(\frac{z - y}{z - x}\right) f(x)$$

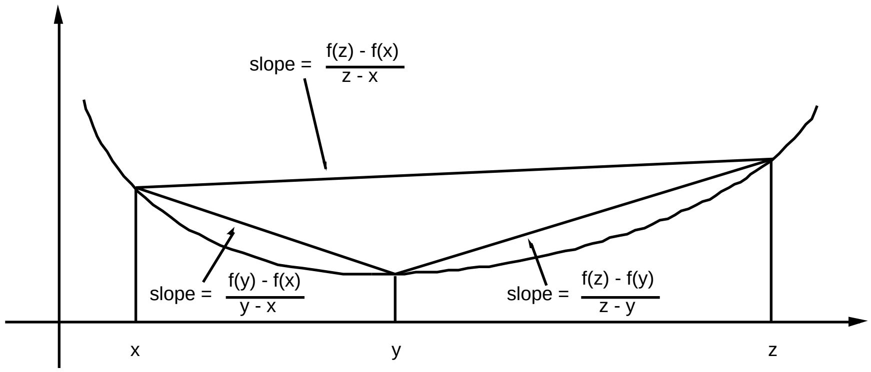

Figure 1.7.1. Illustration of the inequalities (1.51). The rate of change of the function f is nondecreasing with its argument.

and either of the desired inequalities follows by appropriately rearranging terms.

Let a and b be the infimum and the supremum, respectively, of I, also referred to as the *left and right end points of* I, respectively. For any $x \in I$ that is not equal to the right end point, and for any $\alpha > 0$ such that $x + \alpha \in I$, we define

$$s^+(x, \alpha) = \frac{f(x + \alpha) - f(x)}{\alpha}.$$

Let $0 < \alpha \leq \alpha'$. We use the first inequality in Eq. (1.51) with $y = x + \alpha$ and $z = x + \alpha'$ to obtain $s^+(x, \alpha) \leq s^+(x, \alpha')$. Therefore, $s^+(x, \alpha)$ is a nondecreasing function of α and, as α decreases to zero, it converges either to a finite number or to $-\infty$. Let $f^+(x)$ be the value of the limit, which we call the *right derivative* of f at the point x. Similarly, for any $x \in I$ that is not equal to the left end point, and for any $\alpha > 0$ such that $x - \alpha \in I$, we define

$$s^-(x, \alpha) = \frac{f(x) - f(x - \alpha)}{\alpha}.$$

By a symmetrical argument, $s^-(x, \alpha)$ is a nonincreasing function of α. Its limit as α decreases to zero, denoted by $f^-(x)$, is called the *left derivative* of f at the point x, and is either finite or equal to ∞.

In the case where the end points a and b belong to the domain I of f, we define for completeness $f^-(a) = -\infty$ and $f^+(b) = \infty$. The basic facts about the differentiability properties of one-dimensional convex functions can be easily visualized, and are given in the following proposition.

Proposition 1.7.1: Let $I \subset \Re$ be an interval with end points a and b, and let $f : I \mapsto \Re$ be a convex function.

(a) We have $f^-(x) \le f^+(x)$ for every $x \in I$.

(b) If x belongs to the interior of I, then $f^+(x)$ and $f^-(x)$ are finite.

(c) If $x, z \in I$ and $x < z$, then $f^+(x) \le f^-(z)$.

(d) The functions $f^-, f^+ : I \mapsto [-\infty, +\infty]$ are nondecreasing.

(e) The function f^+ (respectively, f^-) is right– (respectively, left–) continuous at every interior point of I. Also, if $a \in I$ (respectively, $b \in I$) and f is continuous at a (respectively, b), then f^+ (respectively, f^-) is right– (respectively, left–) continuous at a (respectively, b).

(f) If f is differentiable at a point x belonging to the interior of I, then $f^+(x) = f^-(x) = (df/dx)(x)$.

(g) For any $x, z \in I$ and any d satisfying $f^-(x) \le d \le f^+(x)$, we have

$$f(z) \ge f(x) + d(z - x).$$

(h) The function $f^+ : I \mapsto (-\infty, \infty]$ [respectively, $f^- : I \mapsto [-\infty, \infty)$] is upper (respectively, lower) semicontinuous at every $x \in I$.

Proof: (a) If x is an end point of I, the result is trivial because $f^-(a) = -\infty$ and $f^+(b) = \infty$. We assume that x is an interior point, we let $\alpha > 0$, and use Eq. (1.51), with $y = x + \alpha$ and $z = y - \alpha$, to obtain $s^-(x, \alpha) \le s^+(x, \alpha)$. Taking the limit as α decreases to zero, we obtain $f^-(x) \le f^+(x)$.

(b) Let x belong to the interior of I and let $\alpha > 0$ be such that $x - \alpha \in I$. Then $f^-(x) \ge s^-(x, \alpha) > -\infty$. For similar reasons, we obtain $f^+(x) < \infty$. Part (a) then implies that $f^-(x) < \infty$ and $f^+(x) > -\infty$.

(c) We use Eq. (1.51), with $y = (z + x)/2$, to obtain $s^+\big(x, (z - x)/2\big) \le s^-\big(z, (z - x)/2\big)$. The result then follows because $f^+(x) \le s^+\big(x, (z - x)/2\big)$ and $s^-\big(z, (z - x)/2\big) \le f^-(z)$.

(d) This follows by combining parts (a) and (c).

(e) Fix some $x \in I$, $x \ne b$, and some positive δ and α such that $x + \delta + \alpha < b$. We allow x to be equal to a, in which case f is assumed to be continuous at a. We have $f^+(x + \delta) \le s^+(x + \delta, \alpha)$. We take the limit, as δ decreases to zero, to obtain $\lim_{\delta \downarrow 0} f^+(x + \delta) \le s^+(x, \alpha)$. We have used here the fact that $s^+(x, \alpha)$ is a continuous function of x, which is a consequence of the continuity of f (Prop. 1.2.12). We now let α decrease to zero to obtain $\lim_{\delta \downarrow 0} f^+(x + \delta) \le f^+(x)$. The reverse inequality is also true because f^+

is nondecreasing and this proves the right–continuity of f^+. The proof for f^- is similar.

(f) This is immediate from the definition of f^+ and f^-.

(g) Fix some $x, z \in I$. The result is trivially true for $x = z$. We only consider the case $x < z$; the proof for the case $x > z$ is similar. Since $s^+(x, \alpha)$ is nondecreasing in α, we have $\big(f(z) - f(x)\big)/(z - x) \geq s^+(x, \alpha)$ for α belonging to $(0, z - x)$. Letting α decrease to zero, we obtain $\big(f(z) - f(x)\big)/(z - x) \geq f^+(x) \geq d$ and the result follows.

(h) This follows from parts (a), (d), (e), and the defining property of semicontinuity (Definition 1.1.5). **Q.E.D.**

We will now discuss notions of directional differentiablity of multidimensional real-valued functions (the extended real-valued case will be discussed later). Given a function $f : \Re^n \mapsto \Re$, a point $x \in \Re^n$, and a vector $y \in \Re^n$, we say that f is *directionally differentiable at x in the direction y* if there is a scalar $f'(x; y)$ such that

$$f'(x; y) = \lim_{\alpha \downarrow 0} \frac{f(x + \alpha y) - f(x)}{\alpha}.$$

We call $f'(x; y)$ the *directional derivative of f at x in the direction y*. We say that f is *directionally differentiable at a point x* if it is directionally differentiable at x in all directions. As in Section 1.1.4, we say that f is *differentiable at x* if it is directionally differentiable at x and $f'(x; y)$ is a linear function of y denoted by

$$f'(x; y) = \nabla f(x)' y$$

where $\nabla f(x)$ is the *gradient* of f at x.

An interesting property of real-valued convex functions is that they are directionally differentiable at all points x. This is a consequence of the directional differentiability of scalar convex functions, as can be seen from the relation

$$f'(x; y) = \lim_{\alpha \downarrow 0} \frac{f(x + \alpha y) - f(x)}{\alpha} = \lim_{\alpha \downarrow 0} \frac{F_y(\alpha) - F_y(0)}{\alpha} = F_y^+(0), \qquad (1.52)$$

where $F_y^+(0)$ is the right derivative of the convex scalar function

$$F_y(\alpha) = f(x + \alpha y)$$

at $\alpha = 0$. Note that the above calculation also shows that the left derivative $F_y^-(0)$ of F_y is equal to $-f'(x; -y)$ and, by using Prop. 1.7.1(a), we obtain $F_y^-(0) \leq F_y^+(0)$, or equivalently,

$$-f'(x; -y) \leq f'(x; y), \qquad \forall\, y \in \Re^n. \qquad (1.53)$$

Note also that for a convex function, the difference quotient $\big(f(x+\alpha y)-f(x)\big)/\alpha$ is a monotonically nondecreasing function of α, so an equivalent definition of the directional derivative is

$$f'(x;y)=\inf_{\alpha>0}\frac{f(x+\alpha y)-f(x)}{\alpha}.$$

The following proposition generalizes the upper semicontinuity property of right derivatives of scalar convex functions [Prop. 1.7.1(h)], and shows that if f is differentiable, then its gradient is continuous.

Proposition 1.7.2: Let $f:\Re^n\mapsto\Re$ be convex, and let $\{f_k\}$ be a sequence of convex functions $f_k:\Re^n\mapsto\Re$ with the property that $\lim_{k\to\infty}f_k(x_k)=f(x)$ for every $x\in\Re^n$ and every sequence $\{x_k\}$ that converges to x. Then for any $x\in\Re^n$ and $y\in\Re^n$, and any sequences $\{x_k\}$ and $\{y_k\}$ converging to x and y, respectively, we have

$$\limsup_{k\to\infty}f_k'(x_k;y_k)\le f'(x;y). \tag{1.54}$$

Furthermore, if f is differentiable over $\Re^n$, then it is continuously differentiable over $\Re^n$.

Proof: For any $\mu>f'(x;y)$, there exists an $\overline{\alpha}>0$ such that

$$\frac{f(x+\alpha y)-f(x)}{\alpha}<\mu,\qquad\forall\ \alpha\le\overline{\alpha}.$$

Hence, for $\alpha\le\overline{\alpha}$, we have

$$\frac{f_k(x_k+\alpha y_k)-f_k(x_k)}{\alpha}\le\mu$$

for all sufficiently large k, and using Eq. (1.52), we obtain

$$\limsup_{k\to\infty}f_k'(x_k;y_k)<\mu.$$

Since this is true for all $\mu>f'(x;y)$, inequality (1.54) follows.

If f is differentiable at all $x\in\Re^n$, then using the continuity of f and the part of the proposition just proved, we have for every sequence $\{x_k\}$ converging to x and every $y\in\Re^n$,

$$\limsup_{k\to\infty}\nabla f(x_k)'y=\limsup_{k\to\infty}f'(x_k;y)\le f'(x;y)=\nabla f(x)'y.$$

By replacing y by $-y$ in the preceding argument, we obtain

$$-\liminf_{k\to\infty}\nabla f(x_k)'y=\limsup_{k\to\infty}\big(-\nabla f(x_k)'y\big)\le-\nabla f(x)'y.$$

Therefore, we have $\nabla f(x_k)'y\to\nabla f(x)'y$ for every y, which implies that $\nabla f(x_k)\to\nabla f(x)$. Hence, the gradient is continuous. **Q.E.D.**

1.7.2 Subgradients and Subdifferentials

Given a convex function $f : \Re^n \mapsto \Re$, we say that a vector $d \in \Re^n$ is a *subgradient* of f at a point $x \in \Re^n$ if

$$f(z) \geq f(x) + (z - x)'d, \qquad \forall\, z \in \Re^n. \tag{1.55}$$

If instead f is a concave function, we say that d is a subgradient of f at x if $-d$ is a subgradient of the convex function $-f$ at x. The set of all subgradients of a convex (or concave) function f at $x \in \Re^n$ is called the *subdifferential* of f at x, and is denoted by $\partial f(x)$.

A subgradient admits an intuitive geometrical interpretation: it can be identified with a nonvertical supporting hyperplane to the graph of the function, as illustrated in Fig. 1.7.2. Such a hyperplane provides a linear approximation to the function, which is an underestimate in the case of a convex function and an overestimate in the case of a concave function. Figure 1.7.3 provides some examples of subdifferentials.

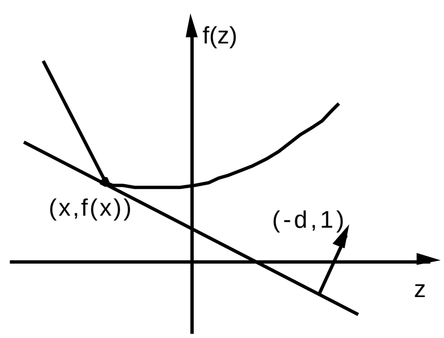

Figure 1.7.2. Illustration of a subgradient of a convex function f. The defining relation (1.55) can be written as

$$f(z) - z'd \geq f(x) - x'd, \qquad \forall\, z \in \Re^n.$$

Equivalently, the hyperplane in $\Re^{n+1}$ that has normal $(-d, 1)$ and passes through $\big(x, f(x)\big)$ supports the epigraph of f, as shown in the figure.

The directional derivative and the subdifferential of a convex function are closely linked, as shown in the following proposition.

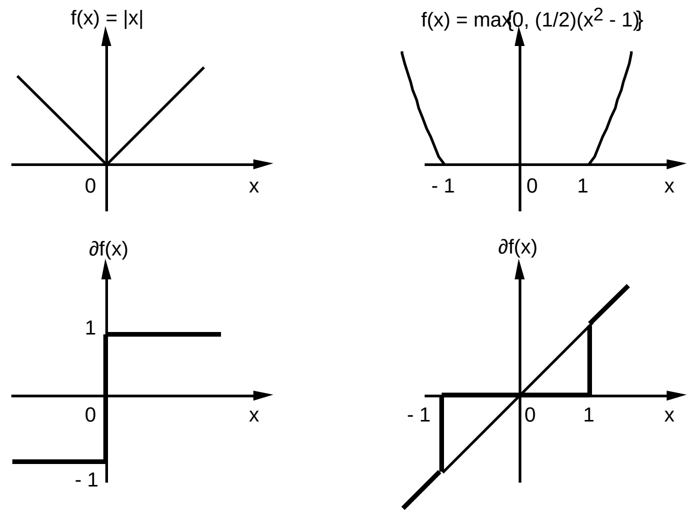

Figure 1.7.3. The subdifferential of some scalar convex functions as a function of the argument x.

Proposition 1.7.3: Let $f : \Re^n \mapsto \Re$ be convex. For every $x \in \Re^n$, the following hold:

(a) A vector d is a subgradient of f at x if and only if

$$f'(x; y) \geq y'd, \qquad \forall\ y \in \Re^n.$$

(b) The subdifferential $\partial f(x)$ is a nonempty, convex, and compact set, and there holds

$$f'(x; y) = \max_{d \in \partial f(x)} y'd, \qquad \forall\ y \in \Re^n. \tag{1.56}$$

In particular, f is differentiable at x with gradient $\nabla f(x)$, if and only if it has $\nabla f(x)$ as its unique subgradient at x.

(c) If X is a bounded set, the set $\cup_{x \in X} \partial f(x)$ is bounded.

Proof: (a) The subgradient inequality (1.55) is equivalent to

$$\frac{f(x + \alpha y) - f(x)}{\alpha} \geq y'd, \qquad \forall\ y \in \Re^n,\ \alpha > 0.$$

Since the quotient on the left above decreases monotonically to $f'(x; y)$ as $\alpha \downarrow 0$ [Eq. (1.51)], we conclude that the subgradient inequality (1.55) is equivalent to $f'(x;y) \geq y'd$ for all $y \in \Re^n$. Therefore we obtain

$$d \in \partial f(x) \qquad \Longleftrightarrow \qquad f'(x;y) \geq y'd, \qquad \forall\ y \in \Re^n. \tag{1.57}$$

(b) From Eq. (1.57), we see that $\partial f(x)$ is the intersection of the closed halfspaces $\big\{d \mid y'd \leq f'(x;y)\big\}$, where y ranges over the nonzero vectors of $\Re^n$. It follows that $\partial f(x)$ is closed and convex.

To show that $\partial f(x)$ is also bounded, suppose to arrive at a contradiction that there is a sequence $\{d_k\} \subset \partial f(x)$ with $||d_k|| \to \infty$. Let $y_k = \frac{d_k}{||d_k||}$. Then, from the subgradient inequality (1.55), we have

$$f(x + y_k) \geq f(x) + y_k' d_k = f(x) + ||d_k||, \qquad \forall\ k,$$

so it follows that $f(x + y_k) \to \infty$. This is a contradiction since f is convex and hence continuous, so it is bounded on any bounded set. Thus $\partial f(x)$ is bounded.

To show that $\partial f(x)$ is nonempty and that Eq. (1.56) holds, we first observe that Eq. (1.57) implies that $f'(x;y) \geq \max_{d \in \partial f(x)} y'd$ [where the maximum is $-\infty$ if $\partial f(x)$ is empty]. To show the reverse inequality, take any x and y in $\Re^n$, and consider the subset of $\Re^{n+1}$

$$C_1 = \big\{(\mu, z) \mid \mu > f(z)\big\},$$

and the half-line

$$C_2 = \big\{(\mu, z) \mid \mu = f(x) + \alpha f'(x;y),\ z = x + \alpha y,\ \alpha \geq 0\big\};$$

see Fig. 1.7.4. Using the definition of directional derivative and the convexity of f, it follows that these two sets are nonempty, convex, and disjoint. By applying the Separating Hyperplane Theorem (Prop. 1.4.2), we see that there exists a nonzero vector $(\gamma, w) \in \Re^{n+1}$ such that

$$\gamma\mu + w'z \geq \gamma\big(f(x) + \alpha f'(x;y)\big) + w'(x + \alpha y), \qquad \forall\ \alpha \geq 0,\ z \in \Re^n,\ \mu > f(z). \tag{1.58}$$

We cannot have $\gamma < 0$ since then the left-hand side above could be made arbitrarily small by choosing μ sufficiently large. Also if $\gamma = 0$, then Eq. (1.58) implies that $w = 0$, which is a contradiction. Therefore, $\gamma > 0$ and by dividing with γ in Eq. (1.58), we obtain

$$\mu + (z - x)'(w/\gamma) \geq f(x) + \alpha f'(x;y) + \alpha y'(w/\gamma),\ \forall\ \alpha \geq 0,\ z \in \Re^n,\ \mu > f(z). \tag{1.59}$$

By setting $\alpha = 0$ in the above relation and by taking the limit as $\mu \downarrow f(z)$, we obtain $f(z) \geq f(x) + (z - x)'(-w/\gamma)$ for all $z \in \Re^n$, implying that $(-w/\gamma) \in \partial f(x)$. Hence $\partial f(x)$ is nonempty. By setting $z = x$ and $\alpha = 1$ in Eq. (1.59), and by taking the limit as $\mu \downarrow f(x)$, we obtain $y'(-w/\gamma) \geq$

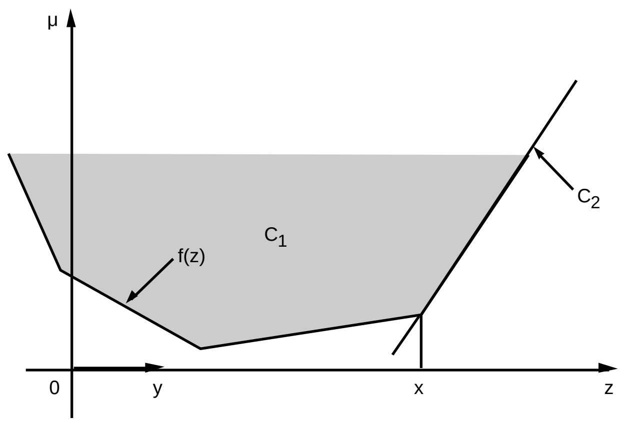

Figure 1.7.4. Illustration of the sets C_1 and C_2 used in the hyperplane separation argument of the proof of Prop. 1.7.3(b).

$f'(x;y)$, which implies that $\max_{d\in\partial f(x)} y'd \geq f'(x;y)$. The proof of Eq. (1.56) is complete.

From the definition of directional derivative, we see that f is differentiable at x with gradient $\nabla f(x)$ if and only if the directional derivative $f'(x;y)$ is a linear function of the form $f'(x;y) = \nabla f(x)'y$. Thus, from Eq. (1.56), f is differentiable at x with gradient $\nabla f(x)$, if and only if it has $\nabla f(x)$ as its unique subgradient at x.

(c) Assume the contrary, i.e. that there exists a sequence $\{x_k\} \subset X$, and a sequence $\{d_k\}$ with $d_k \in \partial f(x_k)$ for all k and $\|d_k\| \to \infty$. Without loss of generality, we assume that $d_k \neq 0$ for all k, and we denote $y_k = d_k/\|d_k\|$. Since both $\{x_k\}$ and $\{y_k\}$ are bounded, they must contain convergent subsequences. We assume without loss of generality that x_k converges to some x and y_k converges to some y with $\|y\| = 1$. By Eq. (1.56), we have

$$f'(x_k;y_k) \geq d_k'y_k = \|d_k\|,$$

so it follows that $f'(x_k;y_k) \to \infty$. This contradicts, however, Eq. (1.54), which requires that $\limsup_{k\to\infty} f'(x_k;y_k) \leq f'(x;y)$. **Q.E.D.**

The next proposition shows some basic properties of subgradients.

Proposition 1.7.4: Let $f : \Re^n \mapsto \Re$ be convex.

(a) If a sequence $\{x_k\}$ converges to x and $d_k \in \partial f(x_k)$ for all k, the sequence $\{d_k\}$ is bounded and each of its limit points is a subgradient of f at x.

(b) If f is equal to the sum $f_1 + \cdots + f_m$ of convex functions $f_j : \Re^n \mapsto \Re$, $j = 1, \ldots, m$, then $\partial f(x)$ is equal to the vector sum $\partial f_1(x) + \cdots + \partial f_m(x)$.

Proof: (a) By Prop. 1.7.3(c), the sequence $\{d_k\}$ is bounded, and by Prop. 1.7.3(a), we have

$$y'd_k \leq f'(x_k; y), \qquad \forall\ y \in \Re^n.$$

If d is a limit point of $\{d_k\}$, we have by taking limit in the above relation and by using Prop. 1.7.2

$$y'd \leq \limsup_{k\to\infty} f'(x_k; y) \leq f'(x; y), \qquad \forall\ y \in \Re^n.$$

Therefore, by Prop. 1.7.3(a), we have $d \in \partial f(x)$.

(b) It will suffice to prove the result for the case where $f = f_1 + f_2$. If $d_1 \in \partial f_1(x)$ and $d_2 \in \partial f_2(x)$, then from the subgradient inequality (1.55), we have

$$f_1(z) \geq f_1(x) + (z - x)'d_1, \qquad \forall\ z \in \Re^n,$$

$$f_2(z) \geq f_2(x) + (z - x)'d_2, \qquad \forall\ z \in \Re^n,$$

so by adding, we obtain

$$f(z) \geq f(x) + (z - x)'(d_1 + d_2), \qquad \forall\ z \in \Re^n.$$

Hence $d_1 + d_2 \in \partial f(x)$, implying that $\partial f_1(x) + \partial f_2(x) \subset \partial f(x)$.

To prove the reverse inclusion, suppose to arrive at a contradiction, that there exists a $d \in \partial f(x)$ such that $d \notin \partial f_1(x) + \partial f_2(x)$. Since by Prop. 1.7.3(b), the sets $\partial f_1(x)$ and $\partial f_2(x)$ are compact, the set $\partial f_1(x) + \partial f_2(x)$ is compact (cf. Prop. 1.2.16), and by Prop. 1.4.3, there exists a hyperplane strictly separating d from $\partial f_1(x) + \partial f_2(x)$, i.e., a vector y and a scalar b such that

$$y'(d_1 + d_2) < b < y'd, \qquad \forall\ d_1 \in \partial f_1(x),\ \forall\ d_2 \in \partial f_2(x).$$

Therefore,

$$\max_{d_1 \in \partial f_1(x)} y'd_1 + \max_{d_2 \in \partial f_2(x)} y'd_2 < y'd,$$

and by Prop. 1.7.3(b),

$$f_1'(x; y) + f_2'(x; y) < y'd.$$

By using the definition of directional derivative, we have $f_1'(x;y)+f_2'(x;y) = f'(x;y)$, so that

$$f'(x;y) < y'd,$$

which is a contradiction in view of Prop. 1.7.3(a). **Q.E.D.**

We close this section with some versions of the chain rule for directional derivatives and subgradients.

Proposition 1.7.5: (Chain Rule)

(a) Let $f : \Re^n \mapsto \Re$ be a convex function and let A be an $m \times n$ matrix. Then the subdifferential of the function F defined by

$$F(x) = f(Ax),$$

is given by

$$\partial F(x) = A'\partial f(Ax) = \big\{A'g \mid g \in \partial f(Ax)\big\}.$$

(b) Let $f : \Re^n \mapsto \Re$ be a convex function and let $g : \Re \mapsto \Re$ be a smooth scalar function. Then the function F defined by

$$F(x) = g\big(f(x)\big)$$

is directionally differentiable at all x, and its directional derivative is given by

$$F'(x;y) = \nabla g\big(f(x)\big)f'(x;y), \qquad \forall\ x \in \Re^n,\ y \in \Re^n. \tag{1.60}$$

Furthermore, if g is convex and monotonically nondecreasing, then F is convex and its subdifferential is given by

$$\partial F(x) = \nabla g\big(f(x)\big)\partial f(x), \qquad \forall\ x \in \Re^n. \tag{1.61}$$

Proof: (a) It is seen using the definition of directional derivative that

$$F'(x;y) = f'(Ax;Ay), \qquad \forall\ y \in \Re^n.$$

Let $g \in \partial f(Ax)$ and $d = A'g$. Then by Prop. 1.7.3(a), we have

$$g'z \le f'(Ax;z) \qquad \forall\ z \in \Re^m,$$

and in particular,

$$g' Ay \leq f'(Ax; Ay) \qquad \forall\, y \in \Re^n,$$

or

$$(A'g)'y \leq F'(x; y), \qquad \forall\, y \in \Re^n.$$

Hence, by Prop. 1.7.3(a), we have $A'g \in \partial F(x)$, so that $A'\partial f(Ax) \subset \partial F(x)$.

To prove the reverse inclusion, suppose to come to a contradiction, that there exists a $d \in \partial F(x)$ such that $d \notin A'\partial f(Ax)$. Since by Prop. 1.7.3(b), the set $\partial f(Ax)$ is compact, the set $A'\partial f(Ax)$ is also compact [cf. Prop. 1.1.9(d)], and by Prop. 1.4.3, there exists a hyperplane strictly separating d from $A'\partial f(Ax)$, i.e., a vector y and a scalar b such that

$$y'(A'g) < b < y'd, \qquad \forall\, g \in \partial f(Ax).$$

From this we obtain

$$\max_{g \in \partial f(Ax)} (Ay)'g < y'd,$$

or, by using Prop. 1.7.3(b),

$$f'(Ax; Ay) < y'd.$$

Since $f'(Ax; Ay) = F'(x; y)$, it follows that

$$F'(x; y) < y'd,$$

which is a contradiction in view of Prop. 1.7.3(a).

(b) We have

$$F'(x; y) = \lim_{\alpha \downarrow 0} \frac{F(x + \alpha y) - F(x)}{\alpha} = \lim_{\alpha \downarrow 0} \frac{g\big(f(x + \alpha y)\big) - g\big(f(x)\big)}{\alpha}. \quad (1.62)$$

From the convexity of f it follows that there are three possibilities: (1) for some $\overline{\alpha} > 0$, $f(x + \alpha y) = f(x)$ for all $\alpha \in (0, \overline{\alpha}]$, (2) for some $\overline{\alpha} > 0$, $f(x + \alpha y) > f(x)$ for all $\alpha \in (0, \overline{\alpha}]$, (3) for some $\overline{\alpha} > 0$, $f(x + \alpha y) < f(x)$ for all $\alpha \in (0, \overline{\alpha}]$.

In case (1), from Eq. (1.62), we have $F'(x; y) = f'(x; y) = 0$ and the given formula (1.60) holds. In case (2), from Eq. (1.62), we have for all $\alpha \in (0, \overline{\alpha}]$

$$F'(x; y) = \lim_{\alpha \downarrow 0} \frac{f(x + \alpha y) - f(x)}{\alpha} \cdot \frac{g\big(f(x + \alpha y)\big) - g\big(f(x)\big)}{f(x + \alpha y) - f(x)}.$$

As $\alpha \downarrow 0$, we have $f(x + \alpha y) \to f(x)$, so the preceding equation yields $F'(x; y) = \nabla g\big(f(x)\big) f'(x; y)$. The proof for case (3) is similar.

If g is convex and monotonically nondecreasing, then F is convex since for any $x_1, x_2 \in \Re^n$ and $\alpha \in [0,1]$, we have

$$\begin{aligned}\alpha F(x_1) + (1-\alpha)F(x_2) &= \alpha g\big(f(x_1)\big) + (1-\alpha)g\big(f(x_2)\big)\\ &\geq g\big(\alpha f(x_1) + (1-\alpha)f(x_2)\big)\\ &\geq g\Big(f\big(\alpha x_1 + (1-\alpha)x_2\big)\Big)\\ &= F\big(\alpha x_1 + (1-\alpha)x_2\big),\end{aligned}$$

where for the first inequality we use the convexity of g, and for the second inequality we use the convexity of f and the monotonicity of g. To obtain the formula for the subdifferential of F, we note that by Prop. 1.7.3(a), $d \in \partial F(x)$ if and only if $y'd \leq F'(x;y)$ for all $y \in \Re^n$, or equivalently (from what has been already shown)

$$y'd \leq \nabla g\big(f(x)\big) f'(x;y), \qquad \forall\ y \in \Re^n.$$

If $\nabla g\big(f(x)\big) = 0$, this relation yields $d = 0$, so $\partial F(x) = \{0\}$ and the desired formula (1.61) holds. If $\nabla g\big(f(x)\big) \neq 0$, we have $\nabla g\big(f(x)\big) > 0$ by the monotonicity of g, so we obtain

$$y' \frac{d}{\nabla g\big(f(x)\big)} \leq f'(x;y), \qquad \forall\ y \in \Re^n,$$

which, by Prop. 1.7.3(a), is equivalent to $d/\nabla g\big(f(x)\big) \in \partial f(x)$. Thus we have shown that $d \in \partial F(x)$ if and only if $d/\nabla g\big(f(x)\big) \in \partial f(x)$, which proves the desired formula (1.61). **Q.E.D.**

1.7.3 ϵ-Subgradients

We now consider a notion of approximate subgradient. Given a convex function $f : \Re^n \mapsto \Re$ and a scalar $\epsilon > 0$, we say that a vector $d \in \Re^n$ is an *ϵ-subgradient* of f at a point $x \in \Re^n$ if

$$f(z) \geq f(x) + (z-x)'d - \epsilon, \qquad \forall\ z \in \Re^n. \tag{1.63}$$

If instead f is a concave function, we say that d is an ϵ-subgradient of f at x if $-d$ is a subgradient of the convex function $-f$ at x. The set of all ϵ-subgradients of a convex (or concave) function f at $x \in \Re^n$ is called the *ϵ-subdifferential* of f at x, and is denoted by $\partial f_\epsilon(x)$. The ϵ-subdifferential is illustrated geometrically in Fig. 1.7.5.

ϵ-subgradients find several applications in nondifferentiable optimization, particularly in the context of computational methods. Some of their important properties are given in the following proposition.

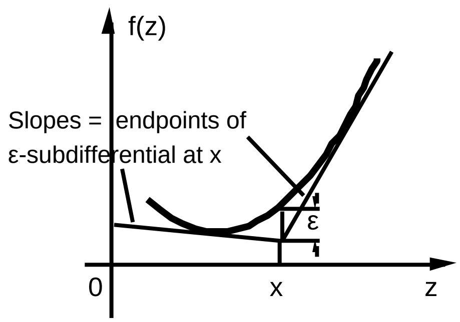

Figure 1.7.5. Illustration of the ϵ-subdifferential $\partial_\epsilon f(x)$ of a convex one-dimensional function $f : \Re \mapsto \Re$. The ϵ-subdifferential is a bounded interval, and corresponds to the set of slopes indicated in the figure. Its left endpoint is

$$f_\epsilon^-(x) = \sup_{\delta<0} \frac{f(x+\delta) - f(x) + \epsilon}{\delta},$$

and its right endpoint is

$$f_{j,\epsilon}^+(x) = \inf_{\delta>0} \frac{f(x+\delta) - f(x) + \epsilon}{\delta}.$$

Proposition 1.7.6: Let $f : \Re^n \mapsto \Re$ be convex and ϵ be a positive scalar. For every $x \in \Re^n$, the following hold:

(a) The ϵ-subdifferential $\partial f_\epsilon(x)$ is a nonempty, convex, and compact set, and for all $y \in \Re^n$ there holds

$$\inf_{\alpha>0} \frac{f(x+\alpha y) - f(x) + \epsilon}{\alpha} = \max_{d \in \partial_\epsilon f(x)} y'd.$$

(b) We have $0 \in \partial_\epsilon f(x)$ if and only if

$$f(x) \leq \inf_{z \in \Re^n} f(z) + \epsilon.$$

(c) If a direction y is such that $y'd < 0$ for all $d \in \partial_\epsilon f(x)$, then

$$\inf_{\alpha>0} f(x+\alpha y) < f(x) - \epsilon.$$

(d) If $0 \notin \partial_\epsilon f(x)$, then the direction $y = -\overline{d}$, where

$$\overline{d} = \arg \min_{d \in \partial_\epsilon f(x)} \|d\|,$$

satisfies $y'd < 0$ for all $d \in \partial_\epsilon f(x)$.

(e) If f is equal to the sum $f_1 + \cdots + f_m$ of convex functions $f_j : \Re^n \mapsto \Re$, $j = 1, \ldots, m$, then

$$\partial_\epsilon f(x) \subset \partial_\epsilon f_1(x) + \cdots + \partial_\epsilon f_m(x) \subset \partial_{m\epsilon} f(x).$$

Proof: (a) We have

$$d \in \partial_\epsilon f(x) \quad \Longleftrightarrow \quad f(x+\alpha y) \geq f(x) + \alpha y'd - \epsilon, \quad \forall\ \alpha > 0,\ \forall\ y \in \Re^n. \tag{1.64}$$

Hence

$$d \in \partial_\epsilon f(x) \quad \Longleftrightarrow \quad \inf_{\alpha>0} \frac{f(x+\alpha y) - f(x) + \epsilon}{\alpha} \geq y'd, \quad \forall\ y \in \Re^n. \tag{1.65}$$

It follows that $\partial_\epsilon f(x)$ is the intersection of the closed subspaces

$$\left\{ d \;\middle|\; \inf_{\alpha>0} \frac{f(x+\alpha y) - f(x) + \epsilon}{\alpha} \geq y'd \right\}$$

as y ranges over $\Re^n$. Hence $\partial_\epsilon f(x)$ is closed and convex. To show that $\partial_\epsilon f(x)$ is also bounded, suppose to arrive at a contradiction that there is a sequence $\{d_k\} \subset \partial_\epsilon f(x)$ with $||d_k|| \to \infty$. Let $y_k = \frac{d_k}{||d_k||}$. Then, from Eq. (1.64), we have for $\alpha = 1$

$$f(x + y_k) \geq f(x) + ||d_k|| - \epsilon, \qquad \forall\ k,$$

so it follows that $f(x + y_k) \to \infty$. This is a contradiction since f is convex and hence continuous, so it is bounded on any bounded set. Thus $\partial_\epsilon f(x)$ is bounded.

To show that $\partial_\epsilon f(x)$ is nonempty and satisfies

$$\inf_{\alpha>0} \frac{f(x + \alpha y) - f(x) + \epsilon}{\alpha} = \max_{d\in\partial_\epsilon f(x)} y'd, \qquad \forall\ y \in \Re^n,$$

we argue similar to the proof of Prop. 1.7.3(b). Consider the subset of $\Re^{n+1}$

$$C_1 = \big\{(\mu, z) \mid \mu > f(z)\big\},$$

and the half-line

$$C_2 = \{(\mu, z) \mid \mu = f(x)-\epsilon+\beta \inf_{\alpha>0} \frac{f(x + \alpha y) - f(x) + \epsilon}{\alpha}, z = x+\beta y, \beta \geq 0\}.$$

These sets are nonempty and convex. They are also disjoint, since we have for all $(\mu, z) \in C_2$

$$\begin{aligned}\mu &= f(x) - \epsilon + \beta \inf_{\alpha>0} \frac{f(x + \alpha y) - f(x) + \epsilon}{\alpha} \\ &\leq f(x) - \epsilon + \beta\frac{f(x + \beta y) - f(x) + \epsilon}{\beta} \\ &= f(x + \beta y) \\ &= f(z).\end{aligned}$$

Hence there exists a hyperplane separating them, i.e., a vector $(\gamma, w) \neq (0, 0)$ such that for all $\beta \geq 0$, $z \in \Re^n$, and $\mu > f(z)$,

$$\gamma\mu + w'z \leq \gamma \left(f(x) - \epsilon + \beta \inf_{\alpha>0} \frac{f(x + \alpha y) - f(x) + \epsilon}{\alpha} \right) + w'(x + \beta y). \tag{1.66}$$

We cannot have $\gamma > 0$, since then the left-hand side above could be made arbitrarily large by choosing μ to be sufficiently large. Also, if $\gamma = 0$, Eq. (1.66) implies that $w = 0$, contradicting the fact that $(\gamma, w) \neq (0, 0)$. Therefore, $\gamma < 0$ and after dividing Eq. (1.66) by γ, we obtain for all $\beta \geq 0$, $z \in \Re^n$, and $\mu > f(z)$,

$$\mu + \left(\frac{w}{\gamma}\right)'(z - x) \geq f(x) - \epsilon + \beta \inf_{\alpha>0} \frac{f(x + \alpha y) - f(x) + \epsilon}{\alpha} + \beta \left(\frac{w}{\gamma}\right)' y. \tag{1.67}$$

Taking the limit above as $\mu \downarrow f(z)$ and setting $\beta = 0$, we obtain

$$f(z) \geq f(x) - \epsilon + \left(-\frac{w}{\gamma}\right)'(z - x), \qquad \forall\ z \in \Re^n.$$

Hence $-w/\gamma$ belongs to $\partial_\epsilon f(x)$, showing that $\partial_\epsilon f(x)$ is nonempty. Also by taking $z = x$ in Eq. (1.67), and by letting $\mu \downarrow f(x)$ and by dividing with β, we obtain

$$-\frac{w}{\gamma}' y \geq -\frac{\epsilon}{\beta} + \inf_{\alpha > 0} \frac{f(x + \alpha y) - f(x) + \epsilon}{\alpha}.$$

Since β can be chosen as large as desired, we see that

$$-\frac{w}{\gamma}' y \geq \inf_{\alpha > 0} \frac{f(x + \alpha y) - f(x) + \epsilon}{\alpha}.$$

Combining this relation with Eq. (1.65), we obtain

$$\max_{d \in \partial_\epsilon f(x)} d'y = \inf_{\alpha > 0} \frac{f(x + \alpha y) - f(x) - \epsilon}{\alpha}.$$

(b) By definition, $0 \in \partial_\epsilon f(x)$ if and only if $f(z) \geq f(x) - \epsilon$ for all $z \in \Re^n$, which is equivalent to $\inf_{z \in \Re^n} f(z) \geq f(x) - \epsilon$.

(c) Assume that a direction y is such that

$$\max_{d \in \partial_\epsilon f(x)} y'd < 0, \tag{1.68}$$

while $\inf_{\alpha > 0} f(x + \alpha y) \geq f(x) - \epsilon$. Then $f(x + \alpha y) - f(x) \geq -\epsilon$ for all $\alpha > 0$, or equivalently

$$\frac{f(x + \alpha y) - f(x) + \epsilon}{\alpha} \geq 0, \qquad \forall\ \alpha > 0.$$

Consequently, using part (a), we have

$$\max_{d \in \partial_\epsilon f(x)} y'd = \inf_{\alpha > 0} \frac{f(x + \alpha y) - f(x) + \epsilon}{\alpha} \geq 0.$$

which contradicts Eq. (1.68).

(d) The vector $\overline{d}$ is the projection of the zero vector on the convex and compact set $\partial_\epsilon f(x)$. If $0 \notin \partial_\epsilon f(x)$, we have $\|\overline{d}\| > 0$, while by Prop. 1.3.3,

$$(d - \overline{d})'\overline{d} \geq 0, \qquad \forall\ d \in \partial_\epsilon f(x).$$

Hence

$$d'\overline{d} \geq ||\overline{d}||^2 > 0, \qquad \forall\ d \in \partial_\epsilon f(x).$$

(e) It will suffice to prove the result for the case where $f = f_1 + f_2$. If $d_1 \in \partial_\epsilon f_1(x)$ and $d_2 \in \partial_\epsilon f_2(x)$, then from Eq. (1.64), we have

$$f_1(x + \alpha y) \geq f_1(x) + \alpha y' d_1 - \epsilon, \qquad \forall\ \alpha > 0,\ \forall\ y \in \Re^n,$$

$$f_2(x + \alpha y) \geq f_2(x) + \alpha y' d_2 - \epsilon, \qquad \forall\ \alpha > 0,\ \forall\ y \in \Re^n,$$

so by adding, we obtain

$$f(x + \alpha y) \geq f(x) + \alpha y'(d_1 + d_2) - 2\epsilon, \qquad \forall\ \alpha > 0,\ \forall\ y \in \Re^n.$$

Hence from Eq. (1.64), we have $d_1 + d_2 \in \partial_{2\epsilon} f(x)$, implying that $\partial_\epsilon f_1(x) + \partial_\epsilon f_2(x) \subset \partial_{2\epsilon} f(x)$.

To prove that $\partial_\epsilon f(x) \subset \partial_\epsilon f_1(x) + \partial_\epsilon f_2(x)$, we use an argument similar to the one of the proof of Prop. 1.7.4(b). Suppose to come to a contradiction, that there exists a $d \in \partial_\epsilon f(x)$ such that $d \notin \partial_\epsilon f_1(x) + \partial_\epsilon f_2(x)$. Since by part (a), the sets $\partial_\epsilon f_1(x)$ and $\partial_\epsilon f_2(x)$ are compact, the set $\partial_\epsilon f_1(x) + \partial_\epsilon f_2(x)$ is compact (cf. Prop. 1.2.16), and by Prop. 1.4.3, there exists a hyperplane strictly separating d from $\partial_\epsilon f_1(x) + \partial_\epsilon f_2(x)$, i.e., a vector y and a scalar b such that

$$y'(d_1 + d_2) < b < y'd, \qquad \forall\ d_1 \in \partial_\epsilon f_1(x),\ \forall\ d_2 \in \partial_\epsilon f_2(x).$$

From this we obtain

$$\max_{d_1 \in \partial_\epsilon f_1(x)} y' d_1 + \max_{d_2 \in \partial_\epsilon f_2(x)} y' d_2 < y'd,$$

and by part (a),

$$\inf_{\alpha > 0} \frac{f_1(x + \alpha y) - f_1(x) + \epsilon}{\alpha} + \inf_{\alpha > 0} \frac{f_2(x + \alpha y) - f_2(x) + \epsilon}{\alpha} < y'd.$$

Let α_j, $j = 1, 2$, be positive scalars such that

$$\frac{f_1(x + \alpha_1 y) - f_1(x) + \epsilon}{\alpha_1} + \frac{f_2(x + \alpha_2 y) - f_2(x) + \epsilon}{\alpha_2} < y'd. \tag{1.69}$$

Define

$$\overline{\alpha} = \frac{1}{1/\alpha_1 + 1/\alpha_2}.$$

As a consequence of the convexity of f_j, $j = 1, 2$, the ratio $\big(f_j(x + \alpha y) - f_j(x)\big)/\alpha$ is monotonically nondecreasing in α. Thus, since $\alpha_j \geq \overline{\alpha}$, we have

$$\frac{f_j(x + \alpha_j y) - f_j(x)}{\alpha_j} \geq \frac{f_j(x + \overline{\alpha} y) - f_j(x)}{\overline{\alpha}}, \qquad j = 1, 2,$$

and Eq. (1.69) together with the definition of $\overline{\alpha}$ yields

$$\begin{aligned} y'd &> \frac{f_1(x+\alpha_1 y) - f_1(x) + \epsilon}{\alpha_1} + \frac{f_1(x+\alpha_1 y) - f_1(x) + \epsilon}{\alpha_1} \\ &\geq \frac{\epsilon}{\overline{\alpha}} + \frac{f_1(x+\overline{\alpha} y) - f_1(x)}{\overline{\alpha}} + \frac{f_2(x+\overline{\alpha} y) - f_2(x)}{\overline{\alpha}} \\ &\geq \inf_{\alpha>0} \frac{f(x+\alpha y) - f(x) + \epsilon}{\alpha}. \end{aligned}$$

This contradicts Eq. (1.65), thus implying that $\partial_\epsilon f(x) \subset \partial_\epsilon f_1(x) + \partial_\epsilon f_2(x)$. **Q.E.D.**

Parts (b)-(d) of Prop. 1.7.6 contain the elements of an important algorithm (called *ϵ-descent algorithm* and introduced by Bertsekas and Mitter [BeM71], [BeM73]) for minimizing convex functions to within a tolerance of ϵ. At a given point x, we check whether $0 \in \partial_\epsilon f(x)$. If this is so, then by Prop. 1.7.6(a), x is an ϵ-optimal solution. If not, then by going along the direction opposite to the vector of minimum norm on $\partial_\epsilon f(x)$, we are guaranteed a cost improvement of at least ϵ. An implementation of this algorithm due to Lemarechal [Lem74], [Lem75] is given in the exercises.

1.7.4 Subgradients of Extended Real-Valued Functions

We have focused so far on real-valued convex functions $f : \Re^n \mapsto \Re$, which are defined over the entire space $\Re^n$ and are convex over $\Re^n$. The notion of a subdifferential and a subgradient of an extended real-valued convex function $f : \Re^n \mapsto (-\infty, \infty]$ can be developed along similar lines. In particular, a vector d is a subgradient of f at a vector $x \in \mathrm{dom}(f)$ [i.e., $f(x) < \infty$] if the subgradient inequality holds, i.e.,

$$f(z) \geq f(x) + (z-x)'d, \qquad \forall\, z \in \Re^n.$$

If the extended real-valued function $f : \Re^n \mapsto [-\infty, \infty)$ is concave, d is said to be a subgradient of f at a vector x with $f(x) > -\infty$ if $-d$ is a subgradient of the convex function $-f$ at x.

The subdifferential $\partial f(x)$ is the set of all subgradients of the convex (or concave) function f. By convention, $\partial f(x)$ is considered empty for all x with $f(x) = \infty$. Note that contrary to the case of real-valued functions, $\partial f(x)$ may be empty, or closed but unbounded. For example, the extended real-valued convex function given by

$$f(x) = \begin{cases} -\sqrt{x} & \text{if } 0 \leq x \leq 1, \\ \infty & \text{otherwise,} \end{cases}$$

has the subdifferential

$$\partial f(x) = \begin{cases} -\frac{1}{2\sqrt{x}} & \text{if } 0 < x < 1, \\ [-1/2, \infty) & \text{if } x = 1, \\ \emptyset & \text{if } x \leq 0 \text{ or } 1 < x. \end{cases}$$

Thus, $\partial f(x)$ can be empty and can be unbounded at points x that belong to the effective domain of f (as in the cases $x = 0$ and $x = 1$, respectively, of the above example). However, it can be shown that $\partial f(x)$ is nonempty and compact at points x that are *interior* points of the effective domain of f, as also illustrated by the above example.

Similarly, a vector d is an ϵ-subgradient of f at a vector x such that $f(x) < \infty$ if

$$f(z) \geq f(x) + (z - x)'d - \epsilon, \qquad \forall\ z \in \Re^n.$$

The ϵ-subdifferential $\partial_\epsilon f(x)$ is the set of all ϵ-subgradients of f. Figure 1.7.6 illustrates the definition of $\partial_\epsilon f(x)$ for the case of a one-dimensional function f. The figure illustrates that even when f is extended real-valued, the ϵ-subdifferential $\partial_\epsilon f(x)$ is nonempty at all points of its effective domain. This can be shown for multi-dimensional functions f as well.

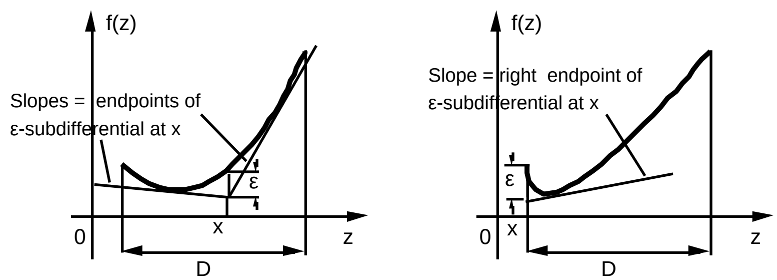

Figure 1.7.6. Illustration of the ϵ-subdifferential $\partial_\epsilon f(x)$ of a one-dimensional function $f : \Re \mapsto (-\infty, \infty]$, which is convex and has as effective domain an interval D. The ϵ-subdifferential corresponds to the set of slopes indicated in the figure. Note that $\partial_\epsilon f(x)$ is nonempty at all $x \in D$. Its left endpoint is

$$f_\epsilon^-(x) = \begin{cases} \sup_{\delta<0,\, x+\delta\in D} \frac{f(x+\delta)-f(x)+\epsilon}{\delta} & \text{if } \inf D < x, \\ -\infty & \text{if } \inf D = x, \end{cases}$$

and its right endpoint is

$$f_{j,\epsilon}^+(x) = \begin{cases} \inf_{\delta>0,\, x+\delta\in D} \frac{f(x+\delta)-f(x)+\epsilon}{\delta} & \text{if } x < \sup D, \\ \infty & \text{if } x = \sup D. \end{cases}$$

Note that these endpoints can be $-\infty$ (as in the figure on the right) or ∞. For $\epsilon = 0$, the above formulas also give the endpoints of the subdifferential $\partial f(x)$. Note that while $\partial f(x)$ is nonempty for all x in the interior of D, it may be empty for x at the relative boundary of D (as in the figure on the right).

One can provide generalized versions of the results of Props. 1.7.3-1.7.6 within the context of extended real-valued convex functions, but with appropriate adjustments and additional assumptions to deal with cases where $\partial f(x)$ may be empty or noncompact. Some of these generalizations are discussed in the exercises.

1.7.5 Directional Derivative of the Max Function

As mentioned in Section 1.3.4, the max function $f(x) = \max_{z \in Z} \phi(x, z)$ arises in a variety of interesting contexts, including duality. It is therefore important to characterize this function, and the following proposition gives the directional derivative and the subdifferential of f for the case where $\phi(\cdot, z)$ is convex for all $z \in Z$.

Proposition 1.7.7: (Danskin's Theorem) Let $Z \subset \Re^m$ be a compact set, and let $\phi : \Re^n \times Z \mapsto \Re$ be continuous and such that $\phi(\cdot, z) : \Re^n \mapsto \Re$ is convex for each $z \in Z$.

(a) The function $f : \Re^n \mapsto \Re$ given by

$$f(x) = \max_{z \in Z} \phi(x, z) \tag{1.70}$$

is convex and has directional derivative given by

$$f'(x; y) = \max_{z \in Z(x)} \phi'(x, z; y), \tag{1.71}$$

where $\phi'(x, z; y)$ is the directional derivative of the function $\phi(\cdot, z)$ at x in the direction y, and $Z(x)$ is the set of maximizing points in Eq. (1.70)

$$Z(x) = \left\{ \overline{z} \;\middle|\; \phi(x, \overline{z}) = \max_{z \in Z} \phi(x, z) \right\}.$$

In particular, if $Z(x)$ consists of a unique point $\overline{z}$ and $\phi(\cdot, \overline{z})$ is differentiable at x, then f is differentiable at x, and $\nabla f(x) = \nabla_x \phi(x, \overline{z})$, where $\nabla_x \phi(x, \overline{z})$ is the vector with coordinates

$$\frac{\partial \phi(x, \overline{z})}{\partial x_i}, \qquad i = 1, \ldots, n.$$

(b) If $\phi(\cdot, z)$ is differentiable for all $z \in Z$ and $\nabla_x \phi(x, \cdot)$ is continuous on Z for each x, then

$$\partial f(x) = \text{conv}\big\{ \nabla_x \phi(x, z) \mid z \in Z(x) \big\}, \qquad \forall\, x \in \Re^n.$$

(c) The conclusion of part (a) also holds if, instead of assuming the Z is compact, we assume that $Z(x)$ is nonempty for all $x \in \Re^n$, and that ϕ and Z are such that for every convergent sequence $\{x_k\}$, there exists a bounded sequence $\{z_k\}$ with $z_k \in Z(x_k)$ for all k.

Proof: (a) The convexity of f has been established in Prop. 1.2.3(c). We note that since ϕ is continuous and Z is compact, the set $Z(x)$ is nonempty by Weierstrass' Theorem (Prop. 1.3.1) and f takes real values. For any

$z \in Z(x)$, $y \in \Re^n$, and $\alpha > 0$, we use the definition of f to obtain

$$\frac{f(x + \alpha y) - f(x)}{\alpha} \geq \frac{\phi(x + \alpha y, z) - \phi(x, z)}{\alpha}.$$

Taking the limit as α decreases to zero, we obtain $f'(x; y) \geq \phi'(x, z; y)$. Since this is true for every $z \in Z(x)$, we conclude that

$$f'(x; y) \geq \sup_{z \in Z(x)} \phi'(x, z; y), \qquad \forall\ y \in \Re^n. \tag{1.72}$$

To prove the reverse inequality and that the supremum in the right-hand side of the above inequality is attained, consider a sequence $\{\alpha_k\}$ with $\alpha_k \downarrow 0$ and let $x_k = x + \alpha_k y$. For each k, let z_k be a vector in $Z(x_k)$. Since $\{z_k\}$ belongs to the compact set Z, it has a subsequence converging to some $\overline{z} \in Z$. Without loss of generality, we assume that the entire sequence $\{z_k\}$ converges to $\overline{z}$. We have

$$\phi(x_k, z_k) \geq \phi(x_k, z), \qquad \forall\ z \in Z,$$

so by taking the limit as $k \to \infty$ and by using the continuity of ϕ, we obtain

$$\phi(x, \overline{z}) \geq \phi(x, z), \qquad \forall\ z \in Z.$$

Therefore, $\overline{z} \in Z(x)$. We now have

$$\begin{aligned} f'(x; y) &\leq \frac{f(x + \alpha_k y) - f(x)}{\alpha_k} \\ &= \frac{\phi(x + \alpha_k y, z_k) - \phi(x, \overline{z})}{\alpha_k} \\ &\leq \frac{\phi(x + \alpha_k y, z_k) - \phi(x, z_k)}{\alpha_k} \\ &= -\frac{\phi(x + \alpha_k y + \alpha_k(-y), z_k) - \phi(x + \alpha_k y, z_k)}{\alpha_k} \\ &\leq -\phi'(x + \alpha_k y, z_k; -y) \\ &\leq \phi'(x + \alpha_k y, z_k; y), \end{aligned} \tag{1.73}$$

where the last inequality follows from inequality (1.53). By letting $k \to \infty$, we obtain

$$f'(x; y) \leq \limsup_{k \to \infty} \phi'(x + \alpha_k y, z_k; y) \leq \phi'(x, \overline{z}; y),$$

where the right inequality in the preceding relation follows from Prop. 1.4.2 with $f_k(\cdot) = \phi(\cdot, z_k)$, $x_k = x + \alpha_k y$, and $y_k = y$. This relation together with inequality (1.72) proves Eq. (1.71).

For the last statement of part (a), if $Z(x)$ consists of the unique point $\overline{z}$, Eq. (1.71) and the differentiability assumption on ϕ yield

$$f'(x;y) = \phi'(x,\overline{z};y) = y'\nabla_x\phi(x,\overline{z}), \qquad \forall\ y \in \Re^n,$$

which implies that $\nabla f(x) = \nabla_x\phi(x,\overline{z})$.

(b) By part (a), we have

$$f'(x;y) = \max_{z\in Z(x)} \nabla_x\phi(x,z)'y,$$

while by Prop. 1.7.3, we have

$$f'(x;y) = \max_{z\in \partial f(x)} d'y.$$

For all $\overline{z} \in Z(x)$ and $y \in \Re^n$, we have

$$\begin{aligned} f(y) &= \max_{z\in Z} \phi(y,z) \\ &\geq \phi(y,\overline{z}) \\ &\geq \phi(x,\overline{z}) + \nabla_x\phi(x,\overline{z})'(y-x) \\ &= f(x) + \nabla_x\phi(x,\overline{z})'(y-x). \end{aligned}$$

Therefore, $\nabla_x\phi(x,\overline{z})$ is a subgradient of f at x, implying that

$$\text{conv}\big\{\nabla_x\phi(x,z) \mid z \in Z(x)\big\} \subset \partial f(x).$$

To prove the reverse inclusion, we use a hyperplane separation argument. By the continuity of $\phi(x,\cdot)$ and the compactness of Z, we see that $Z(x)$ is compact, so by the continuity of $\nabla_x\phi(x,\cdot)$, the set $\big\{\nabla_x\phi(x,z) \mid z \in Z(x)\big\}$ is compact. By Prop. 1.2.8, it follows that $\text{conv}\big\{\nabla_x\phi(x,z) \mid z \in Z(x)\big\}$ is compact. If $d \in \partial f(x)$ while $d \notin \text{conv}\big\{\nabla_x\phi(x,z) \mid z \in Z(x)\big\}$, by the Strict Separation Theorem (Prop. 1.4.3), there exist $y \neq 0$ and $\gamma \in \Re$, such that

$$d'y > \gamma > \nabla_x\phi(x,z)'y, \qquad \forall\ z \in Z(x).$$

Therefore, we have

$$d'y > \max_{z\in Z(x)} \nabla_x\phi(x,z)'y = f'(x;y),$$

contradicting Prop. 1.7.3. Thus $\partial f(x) \subset \text{conv}\big\{\nabla_x\phi(x,z) \mid z \in Z(x)\big\}$ and the proof is complete.

(c) The proof of this part is nearly identical to the proof of part (a). **Q.E.D.**

A simple but important application of the above proposition is when Z is a finite set and ϕ is linear in x for all z. In this case, $f(x)$ can be represented as

$$f(x) = \max\{a_1'x + b_1, \ldots, a_m'x + b_m\},$$

where $a_1, \ldots, a_m$ are given vectors in $\Re^n$ and $b_1, \ldots, b_m$ are given scalars. Then $\partial f(x)$ is the convex hull of the set of all vectors a_i such that $a_i'x + b_i = \max\{a_1'x + b_1, \ldots, a_m'x + b_m\}$.

E X E R C I S E S

1.7.1 (Properties of Directional Derivative)

Let $f : \Re^n \mapsto \Re$ be a convex function and let $x \in \Re^n$ be a fixed vector. Then for the directional derivative function $f'(x; \cdot) : \Re^n \mapsto \Re$ the following hold.

(a) $f'(x; \lambda y) = \lambda f(x; y)$ for all $\lambda \geq 0$ and $y \in \Re^n$.

(b) $f'(x; \cdot)$ is convex over $\Re^n$.

(c) The level set $\{y \mid f'(x; y) \leq 0\}$ is a closed convex cone.

1.7.2 (Chain Rule for Directional Derivatives)

Let $f : \Re^n \to \Re^m$ and $g : \Re^m \mapsto \Re^k$ be given functions, and let $x \in \Re^n$ be a given point. Suppose that f and g are directionally differentiable at x, and g is such that for all $w \in \Re^m$

$$g'(y; w) = \lim_{\alpha \downarrow 0,\ z \to w} \frac{g(y + \alpha z) - g(y)}{\alpha}.$$

Then the composite function $F(x) = g\big(f(x)\big)$ is directionally differentiable at x and the following chain rule applies

$$F'(x; d) = g'\big(f(x); f'(x; d)\big), \qquad \forall\ d \in \Re^n.$$

1.7.3

Let $f : \Re^n \mapsto \Re$ be a convex function. Show that a vector $d \in \Re^n$ is a subgradient of f at x if and only if the function $d'y - f(y)$ achieves its maximum at $y = x$.

1.7.4

Show the following:

(a) For $f(x) = ||x||$, we have

$$\partial f(x) = \begin{cases} \{x/||x||\} & \text{if } x \neq 0, \\ \{x \mid ||x|| \leq 1\} & \text{if } x = 0. \end{cases}$$

(b) For $f(x) = \max_{1 \leq i \leq n} |x_i|$, we have

$$\partial f(x) = \begin{cases} \text{conv}\{\text{sign}(x_i)e_i \mid i \in J_x\} & \text{if } x \neq 0, \\ \text{conv}\{\pm e_1, \ldots, \pm e_n\} & \text{if } x = 0, \end{cases}$$

where $\text{sign}(t)$ denotes the sign of a scalar t, e_i is the i-th basis vector in $\Re^n$, and J_x is the set of all i such that $f(x) = |x_i|$.

(c) For the function

$$f(x) = \begin{cases} 0 & \text{if } x \in C, \\ \infty & \text{if } x \notin C, \end{cases}$$

where C is a nonempty convex set, we have

$$\partial f(x) = \begin{cases} N_C(x) & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C. \end{cases}$$

1.7.5

Let $f : \Re^n \mapsto \Re$ be a convex function and $X \subset \Re^n$ be a bounded set. Show that f is Lipschitz continuous over X, i.e., that there exists a scalar L such that

$$\big|f(x) - f(y)\big| \leq L\ \|x - y\|, \qquad \forall\ x, y \in X.$$

Show also that

$$f'(x; y) \leq L\ ||y||, \qquad \forall\ x \in X, \quad \forall\ y \in \Re^n.$$

Hint: Use the boundedness property of the subdifferential (Prop. 1.7.3).

1.7.6 (Polar Cones of Level Sets)

Let $f : \Re^n \mapsto \Re$ be a convex function and let $x \in \Re^n$ be a fixed vector.

(a) Show that

$$\{y \mid f'(x; y) \leq 0\} = \Big(\text{cone}\big(\partial f(x)\big)\Big)^*.$$

(b) Assuming that the level set $\{z \mid f(z) \leq f(x)\}$ is nonempty, show that the normal cone of $\{z \mid f(z) \leq f(x)\}$ at the point x coincides with $\text{cone}\big(\partial f(x)\big)$.

1.7.7

Let $f : \Re^n \mapsto (-\infty, \infty]$ be a convex function. Show that $\partial f(x)$ is nonempty if x belongs to the relative interior of $\mathrm{dom}(f)$. Show also that $\partial f(x)$ is nonempty and bounded if and only if x belongs to the interior of $\mathrm{dom}(f)$.

1.7.8

Let $f_i : \Re^n \mapsto (-\infty, \infty]$, $i = 1, \ldots, m$ be convex functions, and let $f = f_1 + \cdots + f_m$. Show that

$$\partial f_1(x) + \cdots + \partial f_m(x) \subset \partial f(x), \qquad \forall x \in \Re^n.$$

Furthermore, assuming that the relative interiors of $\mathrm{dom}(f_i)$ have a nonempty intersection, show that

$$\partial f_1(x) + \cdots + \partial f_m(x) = \partial f(x).$$

Hint: Follow the line of proof of Prop. 1.7.8.

1.7.9

Let $C_1, \ldots, C_m$ be convex sets in $\Re^n$ such that $\mathrm{ri}(C_1) \cap \cdots \cap \mathrm{ri}(C_m)$ is nonempty. Show that

$$N_{C_1 \cap \cdots \cap C_m}(x) = N_{C_1}(x) + \cdots + N_{C_m}(x).$$

Hint: For each i, let $f_i(x) = 0$ when $x \in C$ and $f_i(x) = \infty$ otherwise. Then use Exercises 1.7.4(c) and 1.7.8.

1.7.10

Let $f : \Re^n \mapsto \Re$ be a convex function. Show that

$$\cap_{\epsilon > 0} \partial_\epsilon f(x) = \partial f(x), \qquad \forall\, x \in \Re^n.$$

1.7.11

Let $f : \Re^n \mapsto \Re$ be a convex function and $X \subset \Re^n$ a bounded set. Show that for every $\epsilon > 0$, the set $\cup_{x \in X} \partial_\epsilon f(x)$ is bounded.

1.7.12 Continuity of ϵ-Subdifferential [Nur77]

Let $f : \Re^n \mapsto \Re$ be a convex function and $\epsilon > 0$ be a scalar. For every $x \in \Re^n$, the following hold:

(a) If a sequence $\{x_k\}$ converges to x and $d_k \in \partial_\epsilon f(x_k)$ for all k, then the sequence $\{d_k\}$ is bounded and each of its limit points is an ϵ-subgradient of f at x.

(b) If $d \in \partial_\epsilon f(x)$, then for every sequence $\{x_k\}$ converging to x, there is a sequence $\{d_k\}$ such that $d_k \in \partial_\epsilon f(x_k)$ and $d_k \to d$.

Note: This exercise shows that a point-to-set mapping $x \to \partial_\epsilon f(x)$ is continuous. In particular, part (a) shows that the mapping $x \to \partial_\epsilon f(x)$ is upper semicontinuous, while part (b) shows that this mapping is lower semicontinuous.

1.7.13 (Steepest Descent Directions of Convex Functions)

Let $f : \Re^n \mapsto \Re$ be a convex function and fix a vector $x \in \Re^n$. Show that a vector $\overline{d}$ is the vector of minimum norm in $\partial f(x)$ if and only if either $\overline{d} = 0$ or else $\overline{d}/\|\overline{d}\|$ minimizes $f'(x; d)$ over all d with $\|d\| \leq 1$.

1.7.14 (Generating Descent Directions of Convex Functions)

Let $f : \Re^n \mapsto \Re$ be a convex function and fix a vector $x \in \Re^n$. A vector $d \in \Re^n$ is said to be a descent direction of f at x if the corresponding directional derivative of f satisfies

$$f'(x; d) < 0.$$

This exercise provides a method for generating a descent direction in circumstances where obtaining a single subgradient at x is relatively easy.

Assume that x does not minimize f and let g_1 be some subgradient of f at x. For $k = 2, 3, \ldots$, let w_k be the vector of minimum norm in the convex hull of $g_1, \ldots, g_{k-1}$,

$$w_k = \arg \min_{g \in conv\{g_1, \ldots, g_{k-1}\}} \|g\|.$$

If w_k is a descent direction stop; else let g_k be an element of $\partial f(x)$ such that

$$g_k{}' w_k = \min_{g \in \partial f(x)} g' w_k = f'(x; w_k).$$

Show that this process terminates in a finite number of steps with a descent direction. *Hint*: If w_k is not a descent direction, then $g_i{}' w_k \geq \|w_k\|^2 \geq \|g^*\|^2 > 0$ for all $i = 1, \ldots, k-1$, where g^* is the subgradient of minimum norm, while at the same time $g_k{}' w_k \leq 0$. Consider a limit point of $\{(w_k, g_k)\}$.

1.7.15 (Implementing the ϵ-Descent Method [Lem74])

Let $f : \Re^n \mapsto \Re$ be a convex function. This exercise shows how the procedure of Exercise 1.7.14 can be modified so that it generates an ϵ-descent direction at a given vector x. At the typical step of this procedure, we have $g_1, \ldots, g_{k-1} \in \partial_\epsilon f(x)$. Let w_k be the vector of minimum norm in the convex hull of $g_1, \ldots, g_{k-1}$,

$$w_k = \arg \min_{g \in conv\{g_1,\ldots,g_{k-1}\}} \|g\|.$$

If $w_k = 0$, stop; we have $0 \in \partial_\epsilon f(x)$, so by Prop. 1.7.6(b), $f(x) \leq \inf_{z\in\Re^n} f(z) + \epsilon$ and x is ϵ-optimal. Otherwise, by a search along the line $\{x - \alpha w_k \mid \alpha \geq 0\}$, determine whether there exists a scalar $\overline{\alpha}$ such that $f(x - \overline{\alpha} w_k) < f(x) - \epsilon$. If such a $\overline{\alpha}$ can be found, stop and replace x with $x - \overline{\alpha} w_k$ (the value of f has been improved by at least ϵ). Otherwise let g_k be an element of $\partial_\epsilon f(x)$ such that

$${g_k}' w_k = \min_{g \in \partial_\epsilon f(x)} g' w_k.$$

Show that this process will terminate in a finite number of steps with either an improvement of the value of f by at least ϵ, or by confirmation that x is an ϵ-optimal solution.

1.8 OPTIMALITY CONDITIONS

In this section we generalize the constrained optimality conditions of Section 1.5 to problems involving a nondifferentiable but convex cost function. We will first use directional derivatives to provide a necessary and sufficient condition for optimality in the problem of minimizing a convex function $f : \Re^n \mapsto \Re$ over a convex set $X \subset \Re^n$. It can be seen that x^* is a global minimum of f over X if and only if

$$f'(x^*; x - x^*) \geq 0, \qquad \forall\, x \in X.$$

This follows using the definition (1.52) of directional derivative and the fact that the difference quotient

$$\frac{f\big(x^* + \alpha(x - x^*)\big) - f(x^*)}{\alpha}$$

is a monotonically nondecreasing function of α. This leads to the following optimality condition.

Proposition 1.8.1: Let $f : \Re^n \mapsto \Re$ be a convex function. A vector x^* minimizes f over a convex set $X \subset \Re^n$ if and only if there exists a subgradient $d \in \partial f(x^*)$ such that

$$d'(x - x^*) \geq 0, \qquad \forall\ x \in X.$$

Equivalently, x^* minimizes f over a convex set $X \subset \Re^n$ if and only if

$$0 \in \partial f(x^*) + T_X(x^*)^*,$$

where $T_X(x^*)^*$ is the polar of the tangent cone of X at x^*.

Proof: Suppose that for some $d \in \partial f(x^*)$ and all $x \in X$, we have $d'(x - x^*) \geq 0$. Then, since from the definition of a subgradient we have $f(x) - f(x^*) \geq d'(x - x^*)$ for all $x \in X$, we obtain $f(x) - f(x^*) \geq 0$ for all $x \in X$, so x^* minimizes f over X. Conversely, suppose that x^* minimizes f over X. Consider the set of feasible directions of X at x^*

$$F_X(x^*) = \{w \neq 0 \mid x^* + \alpha w \in X \text{ for some } \alpha > 0\},$$

and the cone

$$W = -F_X(x^*)^* = \big\{d \mid d'w \geq 0, \forall\ w \in F_X(x^*)\big\}.$$

If $\partial f(x^*)$ and W have a point in common, we are done, so to arrive at a contradiction, assume the opposite, i.e., $\partial f(x^*) \cap W = \emptyset$. Since $\partial f(x^*)$ is compact and W is closed, by Prop. 1.4.3 there exists a hyperplane strictly separating $\partial f(x^*)$ and W, i.e., a vector y and a scalar c such that

$$g'y < c < d'y, \qquad \forall\ g \in \partial f(x^*),\ \forall\ d \in W.$$

Using the fact that W is a closed cone, it follows that

$$c < 0 \leq d'y, \qquad \forall\ d \in W, \tag{1.74}$$

which when combined with the preceding inequality, also yields

$$\max_{g \in \partial f(x^*)} g'y < c < 0.$$

Thus, using part (b), we have $f'(x^*; y) < 0$, while from Eq. (1.74), we see that y belongs to the polar cone of $F_X(x^*)^*$, which by the Polar Cone theorem (Prop. 1.5.1), implies that y is in the closure of the set of feasible directions $F_X(x^*)$. Hence for a sequence y_k of feasible directions converging to y we have $f'(x^*; y_k) < 0$, which contradicts the optimality of x^*.

The last statement follows from the convexity of X which implies that $T_X(x^*)^*$ is the set of all z such that $z'(x-x^*) \leq 0$ for all $x \in X$ (cf. Props. 1.5.3 and 1.5.5). **Q.E.D.**

Note that the above proposition generalizes the optimality condition of Prop. 1.6.2 for the case where f is convex and smooth:

$$\nabla f(x^*)'(x-x^*) \geq 0, \qquad \forall\, x \in X.$$

In the special case where $X = \Re^n$, we obtain a basic necessary and sufficient condition for unconstrained optimality of x^*:

$$0 \in \partial f(x^*).$$

This optimality condition is also evident from the subgradient inequality (1.55).

We finally extend the optimality conditions of Props. 1.6.2 and 1.8.1 to the case where the cost function involves a (possibly nonconvex) smooth component and a convex (possibly nondifferentiable) component.

Proposition 1.8.2: Let x^* be a local minimum of a function $f : \Re^n \mapsto \Re$ over a subset X of $\Re^n$. Assume that the tangent cone $T_X(x^*)$ is convex, and that f has the form

$$f(x) = f_1(x) + f_2(x),$$

where f_1 is convex and f_2 is smooth. Then

$$-\nabla f_2(x^*) \in \partial f_1(x^*) + T_X(x^*)^*.$$

Proof: The proof is analogous to the one of Prop. 1.6.2. Let y be a nonzero tangent of X at x^*. Then there exists a sequence $\{\xi_k\}$ and a sequence $\{x_k\} \subset X$ such that $x_k \neq x^*$ for all k,

$$\xi_k \to 0, \qquad x_k \to x^*,$$

and

$$\frac{x_k - x^*}{\|x_k - x^*\|} = \frac{y}{\|y\|} + \xi_k.$$

We write this equation as

$$x_k - x^* = \frac{\|x_k - x^*\|}{\|y\|} y_k, \tag{1.75}$$

where

$$y_k = y + \|y\|\xi_k.$$

By the convexity of f_1, we have

$$-f_1'(x_k; x_k - x^*) \le f_1'(x_k; x^* - x_k)$$

and

$$f_1(x_k) + f_1'(x_k; x^* - x_k) \le f_1(x^*),$$

and by adding these inequalities, we obtain

$$f_1(x_k) \le f_1(x^*) + f_1'(x_k; x_k - x^*), \qquad \forall\ k.$$

Also, by the Mean Value Theorem, we have

$$f_2(x_k) = f_2(x^*) + \nabla f_2(\tilde{x}_k)'(x_k - x^*), \qquad \forall\ k,$$

where $\tilde{x}_k$ is a vector on the line segment joining x_k and x^*. By adding the last two relations,

$$f(x_k) \le f(x^*) + f_1'(x_k; x_k - x^*) + \nabla f_2(\tilde{x}_k)'(x_k - x^*), \qquad \forall\ k,$$

so using Eq. (1.75), we obtain

$$f(x_k) \le f(x^*) + \frac{\|x_k - x^*\|}{\|y\|}\big(f_1'(x_k; y_k) + \nabla f_2(\tilde{x}_k)'y_k\big), \qquad \forall\ k. \tag{1.76}$$

We now show by contradiction that $f_1'(x^*; y) + \nabla f_2(x^*)'y \ge 0$. Indeed, if $f_1'(x^*; y) + \nabla f_2(x^*)'y < 0$, then since $\tilde{x}_k \to x^*$ and $y_k \to y$, it follows, using also Prop. 1.7.2, that for all sufficiently large k,

$$f_1'(x_k; y_k) + \nabla f_2(\tilde{x}_k)'y_k < 0$$

and [from Eq. (1.76)] $f(x_k) < f(x^*)$. This contradicts the local optimality of x^*.

We have thus shown that $f_1'(x^*; y) + \nabla f_2(x^*)'y \ge 0$ for all $y \in T_X(x^*)$, or equivalently, by Prop. 1.7.3(b),

$$\max_{d \in \partial f_1(x^*)} d'y + \nabla f_2(x^*)'y \ge 0,$$

or equivalently [since $T_X(x^*)$ is a cone]

$$\min_{\substack{\|y\| \le 1 \\ y \in T_X(x^*)}} \max_{d \in \partial f_1(x^*)} \big(d + \nabla f_2(x^*)\big)'y = 0.$$

We can now apply the Saddle Point Theorem (Prop. 1.3.8 – the convexity/concavity and compactness assumptions of that proposition are satisfied) to assert that there exists a $\overline{d} \in \partial f_1(x^*)$ such that

$$\min_{\substack{\|y\| \leq 1 \\ y \in T_X(x^*)}} \big(\overline{d} + \nabla f_2(x^*)\big)' y = 0.$$

This implies that

$$-\big(\overline{d} + \nabla f_2(x^*)\big) \in T_X(x^*)^*,$$

which in turn is equivalent to $-\nabla f_2(x^*) \in \partial f_1(x^*) + T_X(x^*)^*$. **Q.E.D.**

Note that in the special case where $f_1(x) \equiv 0$, we obtain Prop. 1.6.2. The convexity assumption on $T_X(x^*)$ is unnecessary in this case, but it is essential in general. [Consider the subset $X = \big\{(x_1, x_2) \mid x_1 x_2 = 0\big\}$ of $\Re^2$; it is easy to construct a convex nondifferentiable function that has a global minimum at $x^* = 0$ without satisfying the necessary condition of Prop. 1.8.2.]

In the special case where $f_2(x) \equiv 0$ and X is convex, Prop. 1.8.2 yields the necessity part of Prop. 1.8.1. More generally, when X is convex, an equivalent statement of Prop. 1.8.2 is that if x^* is a local minimum of f over X, there exists a subgradient $\overline{d} \in \partial f_1(x^*)$ such that

$$\big(\overline{d} + \nabla f_2(x^*)\big)'(x - x^*) \geq 0, \qquad \forall\, x \in X.$$

This is because for a convex set X, we have $z \in T_X(x^*)^*$ if and only if $z'(x - x^*) \leq 0$ for all $x \in X$.

E X E R C I S E S

1.8.1

Consider the problem of minimizing a convex function $f : \Re^n \mapsto \Re$ over the polyhedral set

$$X = \{x \mid a_j' x \leq b_j,\ j = 1, \ldots, r\}.$$

Show that x^* is an optimal solution if and only if there exist scalars $\mu_1^*, \ldots, \mu_r^*$ such that

(i) $\mu_j^* \geq 0$ for all j, and $\mu_j^* = 0$ for all j such that $a_j' x^* < b_j$.

(ii) $0 \in \partial f(x^*) + \sum_{j=1}^{r} \mu_j^* a_j$.

Hint: Characterize the cone $T_X(x^*)^*$, and use Prop. 1.8.1 and Farkas' Lemma.

1.9 NOTES AND SOURCES

The material in this chapter is classical and is developed in various books. Most of these books relate to both convex analysis and optimization, but differ in their scope and emphasis.

The book by Rockafellar [Roc70], is widely viewed as the classic convex analysis text, and contains a more extensive development of convexity that the one given here, although it does not cross over into nonconvex optimization. The book by Rockafellar and Wets [RoW98] is a deep and detailed treatment of "variational analysis," a broad spectrum of topics that integrate classical analysis, convexity, and optimization of both convex and nonconvex (possibly nonsmooth) functions. The normal cone, introduced by Mordukhovich [Mor76], and the work of Clarke on nonsmooth analysis [Cla83] play a central role in this subject. The book by Rockafellar and Wets contains a wealth of material beyond what is covered here, and is strongly recommended for the advanced reader who wishes to obtain a comprehensive view of the connecting links between convex and classical analysis.

Among other books with detailed accounts of convexity, Stoer and Witzgall [StW70] discuss similar topics as Rockafellar Roc70] but less comprehensively. Ekeland and Temam [EkT76] develop the subject in infinite dimensional spaces. Hiriart-Urruty and Lemarechal [HiL93] emphasize algorithms for dual and nondifferentiable optimization. Rockafellar [Roc84] focuses on convexity and duality in network optimization and an important generalization, known as monotropic programming. Bertsekas [Ber98] also gives a detailed coverage of this material, which owes much to the early work of Minty [Min60] on network optimization. Bonnans and Shapiro [BoS00] emphasize sensitivity analysis and discuss infinite dimensional problems as well. Borwein and Lewis [BoL00] develop many of the concepts in Rockafellar and Wets [RoW98], but more succinctly. Schrijver [Sch86] provides an extensive account of polyhedral convexity with applications to integer programming and combinatorial optimization, and gives many historical references.

Among the early contributors to convexity theory, we note Steinitz [Ste13], [Ste14], [Ste16], who developed the theory of relative interiors, recession cones, and polar cones; Minkowski [Min11], who first investigated supporting hyperplane theorems; Caratheodory [Car11], who also investigated hyperplane separation and gave the theorem on convex hulls that carries his name. The Minkowski-Weyl Theorem given here is due to Weyl [Wey35]. The notion of a tangent and the tangent cone originated with Bouligand [Bou30], [Bou32]. Subdifferential theory owes much to the work of Fenchel, who in his 1951 lecture notes [Fen51] laid the foundations for the subsequent work on convexity and its connection with optimization.

REFERENCES

[BeM71] Bertsekas, D. P, and Mitter, S. K., 1971. "Steepest Descent for Optimization Problems with Nondifferentiable Cost Functionals," Proc. 5th Annual Princeton Confer. Inform. Sci. Systems, Princeton, N. J., pp. 347-351.

[BeM73] Bertsekas, D. P., and Mitter, S. K., 1973. "A Descent Numerical Method for Optimization Problems with Nondifferentiable Cost Functionals," SIAM J. on Control, Vol. 11, pp. 637-652.

[BeT97] Bertsimas, D., and Tsitsiklis, J. N., 1997. Introduction to Linear Optimization, Athena Scientific, Belmont, MA.

[Ber98] Bertsekas, D. P., 1998. Network Optimization: Continuous and Discrete Models, Athena Scientific, Belmont, MA.

[Ber99] Bertsekas, D. P., 1999. Nonlinear Programming: 2nd Edition, Athena Scientific, Belmont, MA.

[BoS00] Bonnans, J. F., and Shapiro, A., 2000. Perturbation Analysis of Optimization Problems, Springer-Verlag, Berlin and N. Y.

[BoL00] Borwein, J. M., and Lewis, A. S., Convex Analysis and Nonlinear Optimization, Springer-Verlag, N. Y.

[Bou30] Bouligand, G., 1930. "Sur les Surfaces Depourvues de Points Hyperlimites," Annales de la Societe Polonaise de Mathematique, Vol. 9, pp. 32-41.

[Bou32] Bouligand, G., 1932. Introduction a la Geometrie Infinitesimale Directe, Gauthiers-Villars, Paris.

[Car11] Caratheodory, C., 1911. "Uber den Variabilitatsbereich der Fourierschen Konstanten von Positiven Harmonischen Funktionen," Rendiconto del Circolo Matematico di Palermo, Vol. 32, pp. 193-217; reprinted in Constantin Caratheodory, Gesammelte Mathematische Schriften, Band III (H. Tietze, ed.), C. H. Beck'sche Verlagsbuchhandlung, Munchen, 1955, pp. 78-110.

[Chv83] Chvatal, V., 1983. Linear Programming, W. H. Freeman and Co., N. Y.

[Cla83] Clarke, F. H., 1983. Optimization and Nonsmooth Analysis, Wiley, N. Y.

[Dan63] Dantzig, G. B., 1963. Linear Programming and Extensions, Princeton Univ. Press, Princeton, N. J.

[EkT76] Ekeland, I., and Temam, R., 1976. Convex Analysis and Variational Problems, North-Holland Publ., Amsterdam.

[Fen51] Fenchel, W., 1951. "Convex Cones, Sets, and Functions," Mimeographed Notes, Princeton Univ.

[HiL93] Hiriart-Urruty, J.-B., and Lemarechal, C., 1993. Convex Analysis and Minimization Algorithms, Vols. I and II, Springer-Verlag, Berlin and N. Y.

[HoK71] Hoffman, K., and Kunze, R., 1971. Linear Algebra, Prentice-Hall, Englewood Cliffs, N. J.

[LaT85] Lancaster, P., and Tismenetsky, M., 1985. The Theory of Matrices, Academic Press, N. Y.

[Lem74] Lemarechal, C., 1974. "An Algorithm for Minimizing Convex Functions," in Information Processing '74, Rosenfeld, J. L., (Ed.), pp. 552-556, North-Holland, Amsterdam.

[Lem75] Lemarechal, C., 1975. "An Extension of Davidon Methods to Nondifferentiable Problems," Math. Programming Study 3, Balinski, M., and Wolfe, P., (Eds.), North-Holland, Amsterdam, pp. 95-109.

[Min11] Minkowski, H., 1911. "Theorie der Konvexen Korper, Insbesondere Begrundung Ihres Ober Flachenbegriffs," Gesammelte Abhandlungen, II, Teubner, Leipsig.

[Min60] Minty, G. J., 1960. "Monotone Networks," Proc. Roy. Soc. London, A, Vol. 257, pp. 194-212.

[Mor76] Mordukhovich, B. S., 1976. "Maximum Principle in the Problem of Time Optimal Response with Nonsmooth Constraints," J. of Applied Mathematics and Mechanics, Vol. 40, pp. 960-969.

[Nur77] Nurminskii, E. A., "The Continuity of ϵ-Subgradient Mappings," (Russian) Kibernetika (Kiev) 1977, No. 5, pp. 148–149.

[OrR70] Ortega, J. M., and Rheinboldt, W. C., 1970. Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, N. Y.

[RoW98] Rockafellar, R. T., and Wets, R. J.-B., 1998. Variational Analysis, Springer-Verlag, Berlin.

[Roc70] Rockafellar, R. T., 1970. Convex Analysis, Princeton Univ. Press, Princeton, N. J

[Roc84] Rockafellar, R. T., 1984. Network Flows and Monotropic Optimization, Wiley, N. Y.; republished by Athena Scientific, Belmont, MA, 1998.

[Rud76] Rudin, W., 1976. Principles of Mathematical Analysis, McGraw-Hill, N. Y.

[Scr86] Shrijver, A., 1986. Theory of Linear and Integer Programming, Wiley, N. Y.

[Ste13] Steinitz, H., 1913. "Bedingt Konvergente Reihen und Konvexe System, I, J. of Math., Vol. 143, pp. 128-175.

[Ste14] Steinitz, H., 1914. "Bedingt Konvergente Reihen und Konvexe System, II, J. of Math., Vol. 144, pp. 1-40.

[Ste16] Steinitz, H., 1916. "Bedingt Konvergente Reihen und Konvexe System, III, J. of Math., Vol. 146, pp. 1-52.

[StW70] Stoer, J., and Witzgall, C., 1970. Convexity and Optimization in Finite Dimensions, Springer-Verlag, Berlin.

[Str76] Strang, G., 1976. Linear Algebra and Its Applications, Academic Press, N. Y.

[Wey35] Weyl, H., 1935. "Elementare Theorie der Konvexen Polyeder," Commentarii Mathematici Helvetici, Vol. 7, pp. 290-306.