Lectures on the Poisson Process

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List of symbols

$\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$	set of integers
$\mathbb{N} = \{1, 2, 3, 4, \dots\}$	set of positive integers
$\mathbb{N}_0 = \{0, 1, 2, \dots\}$	set of non-negative integers
$\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$	extended set of positive integers
$\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$	extended set of non-negative integers
$\mathbb{R} = (-\infty, \infty), \mathbb{R}_+ = [0, \infty)$	real line, non-negative real line
$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$	extended real line
$\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$	extended half-line
$\mathbb{R}(\mathbb{X}), \mathbb{R}_+(\mathbb{X})$	$\mathbb{R}$-valued (resp. $\mathbb{R}_+$-valued) measurable functions on $\mathbb{X}$
$\overline{\mathbb{R}}(\mathbb{X}), \overline{\mathbb{R}}_+(\mathbb{X})$	$\overline{\mathbb{R}}$-valued (resp. $\overline{\mathbb{R}}_+$-valued) measurable functions on $\mathbb{X}$
u^+, u^-	positive and negative part of an $\mathbb{R}$-valued function u
$\operatorname{card} A = \vert A\vert$	number of elements of a set A
$[n]$	$\{1, \dots, n\}$
Σ_n	group of permutations of $[n]$
$(n)_k = n(n-1)\cdots(n-k+1)$	descending factorial
$(\Omega, \mathcal{F}, \mathbb{P})$	probability space
$\mathbb{E}[X]$	expectation of a random variable X
$\mathbb{V}\mathrm{ar}[X]$	variance of a random variable X
$\mathbb{C}\mathrm{ov}[X, Y]$	covariance between random variables X and Y
L_η	Laplace functional of a point process η
$\overset{d}{=}, \overset{d}{\to}$	equality and convergence in distribution
δ_x	Dirac measure at the point x
$\mathbf{N}(\mathbb{X}) \equiv \mathbf{N}$	set of all s-finite counting measures on $\mathbb{X}$
$\mathbf{N}_{<\infty}(\mathbb{X}) \equiv \mathbf{N}_{<\infty}$	set of all finite counting measures on $\mathbb{X}$
$\mathbf{N}_l(\mathbb{X}) \equiv \mathbf{N}_l$	set of all locally finite counting measures on $\mathbb{X}$
$\mathbf{N}_s(\mathbb{X}) \equiv \mathbf{N}_s$	set of all simple counting measures in $\mathbf{N}_l$

$a \wedge b, a \vee b$	minimum and maximum of $a, b \in \overline{\mathbb{R}}$
$\mathbf{1}\{\cdot\}$	indicator function
$a^{\oplus} := \mathbf{1}\{a \neq 0\}a^{-1}$	generalized inverse of $a \in \mathbb{R}$
$\mathcal{B}(\mathbb{X}) \equiv \mathcal{X}$	Borel σ-field on the metric space $\mathbb{X}$
$\mathcal{X}_b$	bounded Borel subsets of a metric space $\mathbb{X}$
$\mathbb{R}^d$	Euclidean space of dimension $d \in \mathbb{N}$
$\mathcal{B}^d := \mathcal{B}(\mathbb{R}^d)$	Borel σ-field on $\mathbb{R}^d$
λ_d	Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}^d)$
$\lVert \cdot \rVert$	Euclidean norm on $\mathbb{R}^d$
$\langle \cdot, \cdot \rangle$	scalar product on $\mathbb{R}^d$
$\mathcal{C}^d$	compact subsets of $\mathbb{R}^d$
$\mathcal{C}^{(d)}$	non-empty compact subsets of $\mathbb{R}^d$
$\mathcal{K}^d$	compact convex subsets of $\mathbb{R}^d$
$\mathcal{K}^{(d)}$	non-empty compact convex subsets of $\mathbb{R}^d$
$\mathcal{R}^d$	convex ring in $\mathbb{R}^d$ (finite unions of convex sets)
$V_0, \dots, V_d$	intrinsic volumes
$x \in \mu$	short for $\mu\{x\} := \mu(\{x\}) > 0$, $\mu \in \mathbf{N}$
$B(x, r)$	closed ball with centre x and radius $r \geq 0$
$\kappa_d = \lambda_d(B^d)$	volume of the unit ball in $\mathbb{R}^d$

Preface

The Poisson process generates point patterns in a purely random manner. It plays a fundamental role in probability theory and its applications, and enjoys a rich and beautiful theory. While many of the applications involve point processes on the line, or more generally in Euclidean space, many others do not. Fortunately, one can develop much of the theory in the abstract setting of a general measurable space.

The present introduction to Poisson processes on a general (measurable) phase space requires a sound knowledge of measure-theoretic probability theory. However, specific knowledge of stochastic processes is not assumed. Since the focus is always on the probabilistic structure, technical issues of measure theory are kept in the background, whenever possible. Some basic facts from measure and probability theory are collected in the appendices.

This book is intended to be a basis for graduate courses or seminars on the Poisson process. It might also serve as an introduction to point process theory. Each chapter is supposed to cover material that can be presented (at least in principle) in a single lecture. In practice, it may not always be possible to get through an entire chapter in one lecture; however, in most chapters the most essential material is presented in the early part of the chapter (which definitely can be covered in the lecture), and the later part could feasibly be left as background reading if necessary. While it is recommended to read the earlier chapters in a linear order, there is some scope to pick and choose from the later chapters.

When treating a classical and central subject of probability theory, a certain overlap with other textbooks is inevitable. Much of the material of the earlier chapters, for instance, can also be found (in a slightly more restricted form) in the highly recommended book [54] by J.F.C. Kingman. Further results on Poisson processes, as well as on general random measures and point processes, are presented in the monographs [4, 16, 18, 38, 44, 45, 48, 61, 74, 101]. Comments on the history of the main results and on the literature are given at the end of the book.

Stochastic geometry is concerned with mathematical models for random geometric structures [3, 16, 30, 90, 109]. The Poisson process is fundamental to stochastic geometry and the applications areas discussed in this book lie largely in this direction, reflecting the tastes and expertise of the authors. In particular, we discuss Voronoi tessellations, stable allocations, hyperplane processes, the Boolean model, and the Gilbert graph.

Besides stochastic geometry, there are many other fields of application of the Poisson process, which are not covered in this volume. These include Lévy processes [6], Brownian excursion theory [102], queueing networks [4, 111], and Poisson limits in extreme value theory [101].

The book divides loosely into three parts. In the first part we develop basic results on the Poisson process in the general setting. In the second part we introduce models and results of stochastic geometry, most but not all of which are based on the Poisson process, and which are most naturally developed in the Euclidean setting. Chapters 9,10,16,17,22 are devoted exclusively to stochastic geometry while other chapters use stochastic geometry models for illustrating the theory. In the third part we return to the general setting and describe more advanced results on the stochastic analysis of the Poisson process.

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1

The Poisson distribution

The Poisson distribution arises as a limit of the Binomial distribution. This chapter contains a brief discussion of some of its fundamental properties as well as the Poisson limit theorem for null arrays of integer-valued random variables.

1.1 Definition and basic properties

A random variable X is said to have a *binomial distribution* $\mathrm{Bi}(n, p)$ with parameters $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ and $p \in [0, 1]$ if

$$\mathbb{P}(X = k) = \mathrm{Bi}(n, p; k) := \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, \dots, n. \tag{1.1}$$

In the case $n = 1$ this is the *Bernoulli distribution* with parameter $p \in [0, 1]$. If $X_1, \dots, X_n$ are independent random variables with such a Bernoulli distribution, then their sum has a binomial distribution, that is

$$X_1 + \cdots + X_n \overset{d}{=} X, \tag{1.2}$$

where X has the distribution $\mathrm{Bi}(n, p)$ and where $\overset{d}{=}$ denotes equality in distribution. It follows that the expectation and variance of X are given by

$$\mathbb{E}[X] = np, \qquad \mathbb{V}\mathrm{ar}[X] = np(1 - p). \tag{1.3}$$

A random variable X has a *Poisson distribution* $\mathrm{Po}(\gamma)$ with parameter $\gamma \geq 0$ if

$$\mathbb{P}(X = k) = \mathrm{Po}(\gamma; k) := \frac{\gamma^k}{k!} e^{-\gamma}, \quad k \in \mathbb{N}_0. \tag{1.4}$$

If $\gamma = 0$, then $\mathbb{P}(X = 0) = 1$, since we take $0^0 := 1$. Also we allow $\gamma = \infty$; in this case we put $\mathbb{P}(X = \infty) = 1$ so $\mathrm{Po}(\infty; k) = 0$ for $k \in \mathbb{N}_0$.

The Poisson distribution arises as a limit of binomial distributions as follows. Let the sequence $p_n \in [0,1]$, $n \in \mathbb{N}$, satisfy $np_n \to \gamma$ as $n \to \infty$, with $\gamma \in (0,\infty)$. Then, for $k \in \{0,\dots,n\}$,

$$\binom{n}{k} p_n^k (1-p_n)^{n-k} = \frac{(np_n)^k}{k!} \cdot \frac{(n)_k}{n^k} \cdot (1-p_n)^{-k} \cdot \left(1 - \frac{np_n}{n}\right)^n \to \frac{\gamma^k}{k!} e^{-\gamma}, \quad (1.5)$$

as $n \to \infty$, where

$$(n)_k := n(n-1)\cdots(n-k+1) \quad (1.6)$$

is the k-th *descending factorial* (of n) with $(n)_0$ interpreted as 1.

Suppose X is a Poisson random variable with finite parameter γ. Then its expectation is given by

$$\mathbb{E}[X] = e^{-\gamma} \sum_{k=0}^{\infty} k \frac{\gamma^k}{k!} = e^{-\gamma} \gamma \sum_{k=1}^{\infty} \frac{\gamma^{k-1}}{(k-1)!} = \gamma. \quad (1.7)$$

The *probability generating function* of X (or of $\mathrm{Po}(\gamma)$) is given by

$$\mathbb{E}[s^X] = e^{-\gamma} \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} s^k = e^{-\gamma} \sum_{k=0}^{\infty} \frac{(\gamma s)^k}{k!} = e^{\gamma(s-1)}, \quad s \in [0,1]. \quad (1.8)$$

It follows that the *Laplace transform* of X (or of $\mathrm{Po}(\gamma)$) is given by

$$\mathbb{E}[e^{-tX}] = \exp[-\gamma(1-e^{-t})], \quad t \geq 0. \quad (1.9)$$

A similar calculation to (1.8) shows that the *factorial moments* of X are given by

$$\mathbb{E}[(X)_k] = \gamma^k, \quad k \in \mathbb{N}_0. \quad (1.10)$$

where $(0)_0 := 1$ and $(0)_k := 0$ for $k \geq 1$. Equation (1.10) implies that

$$\mathbb{V}\mathrm{ar}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[(X)_2] + \mathbb{E}[X] - \mathbb{E}[X]^2 = \gamma. \quad (1.11)$$

We continue with a characterization of the Poisson distribution.

Proposition 1.1 *An $\mathbb{N}_0$-valued random variable X has distribution* $\mathrm{Po}(\gamma)$ *if and only if for any function $f\colon \mathbb{N}_0 \to \mathbb{R}_+$, we have*

$$\mathbb{E}[Xf(X)] = \gamma\,\mathbb{E}[f(X+1)]. \quad (1.12)$$

Proof By a similar calculation to (1.7) and (1.8) we obtain for any function $f\colon \mathbb{N}_0 \to \mathbb{R}_+$ that (1.12) holds. Conversely, if (1.12) holds for all such functions f, then we can make the particular choice $f := \mathbf{1}_{\{k\}}$ for $k \in \mathbb{N}$, to obtain the recursion

$$k\,\mathbb{P}(X = k) = \gamma\,\mathbb{P}(X = k-1).$$

This recursion has (1.4) as its only (probability) solution, so the result follows. □

1.2 Relationships with the binomial and multinomial distribution

The next result says that if X and Y are independent Poisson random variables, then $X+Y$ is also Poisson and the conditional distribution of X given $X+Y$ is binomial:

Proposition 1.2 *Let X and Y be independent with distributions* $\mathrm{Po}(\gamma)$ *and* $\mathrm{Po}(\delta)$ *respectively, with* $0 < \gamma+\delta < \infty$. *Then $X+Y$ has distribution* $\mathrm{Po}(\gamma+\delta)$ *and*

$$\mathbb{P}(X = k \mid X + Y = n) = \mathrm{Bi}(n, \gamma/(\gamma + \delta); k), \qquad n \in \mathbb{N}_0,\ k = 0, \dots, n.$$

Proof For $n \in \mathbb{N}_0$ and $k \in \{0, \dots, n\}$,

$$\begin{aligned}\mathbb{P}(X = k, X + Y = n) = \mathbb{P}(X = k, Y = n - k) &= \frac{\gamma^k}{k!} e^{-\gamma} \frac{\delta^{n-k}}{(n-k)!} e^{-\delta} \\ &= e^{-(\gamma+\delta)} \Big(\frac{(\gamma+\delta)^n}{n!}\Big) \binom{n}{k} \Big(\frac{\gamma}{\gamma+\delta}\Big)^k \Big(\frac{\delta}{\gamma+\delta}\Big)^{n-k} \\ &= \mathrm{Po}(\gamma+\delta; n)\, \mathrm{Bi}(n, \gamma/(\gamma+\delta); k),\end{aligned}$$

and the assertions follow. □

Let Z be an $\mathbb{N}_0$-valued random variable and let $Z_1, Z_2, \dots$ be a sequence of independent random variables that have a Bernoulli distribution with parameter $p \in [0, 1]$. If Z and $(Z_n)_{n\ge1}$ are independent, then the random variable

$$X := \sum_{j=1}^{Z} Z_j \tag{1.13}$$

is called a *p-thinning* of Z, where we set $X := 0$ if $Z = 0$. This means that the conditional distribution of X given $Z = n$ is binomial with parameters n and p.

The following partial converse of Proposition 1.2 is a noteworthy property of the Poisson distribution.

Proposition 1.3 *Let $p \in [0, 1]$. Let Z have a Poisson distribution with parameter $\gamma \ge 0$ and let X be a p-thinning of Z. Then X and $Z - X$ are independent and Poisson distributed with parameters $p\gamma$ and* $(1 - p)\gamma$, *respectively.*

Proof The result follows once we have shown that

$$\mathbb{P}(X = m, Z - X = n) = \mathrm{Po}(p\gamma; m)\,\mathrm{Po}((1-p)\gamma; n), \quad m, n \in \mathbb{N}_0. \tag{1.14}$$

Since the conditional distribution of X given $Z = m + n$ is binomial with parameters $m + n$ and p, we have

$$\begin{aligned}\mathbb{P}(X = m, Z - X = n) &= \mathbb{P}(Z = m + n)\,\mathbb{P}(X = m \mid Z = m + n) \\ &= \left(\frac{e^{-\gamma}\gamma^{m+n}}{(m+n)!}\right)\binom{m+n}{m} p^m (1-p)^n \\ &= \left(\frac{p^m\gamma^m}{m!}\right) e^{-p\gamma} \left(\frac{(1-p)^n\gamma^n}{n!}\right) e^{-(1-p)\gamma},\end{aligned}$$

and (1.14) follows. □

1.3 The Poisson limit theorem

The next result generalizes (1.5) and stresses the relevance of the Poisson distribution.

Proposition 1.4 *Suppose for $n \in \mathbb{N}$ that $m_n \in \mathbb{N}$ and that $X_{n,1}, \dots, X_{n,m_n}$ are independent random variables taking values in $\mathbb{N}_0$. Let $p_{n,i} := \mathbb{P}(X_{n,i} \geq 1)$ and assume that*

$$\lim_{n\to\infty} \max_{1 \leq i \leq m_n} p_{n,i} = 0. \tag{1.15}$$

Assume further that $\lambda_n := \sum_{i=1}^{m_n} p_{n,i} \to \gamma$ as $n \to \infty$, where $\gamma > 0$, and that

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} \mathbb{P}(X_{n,i} \geq 2) = 0. \tag{1.16}$$

Let $X_n := \sum_{i=1}^{m_n} X_{n,i}$. Then for $k \in \mathbb{N}_0$ we have

$$\lim_{n\to\infty} \mathbb{P}(X_n = k) = \mathrm{Po}(\gamma; k). \tag{1.17}$$

Proof Let $X'_{n,i} := \mathbf{1}\{X_{n,i} \geq 1\} = \min\{X_{n,i}, 1\}$ and $X'_n := \sum_{i=1}^{m_n} X'_{n,i}$. Since $X'_{n,i} \neq X_{n,i}$ if and only if $X_{n,i} \geq 2$, we have

$$\mathbb{P}(X'_n \neq X_n) \leq \sum_{i=1}^{m_n} \mathbb{P}(X_{n,i} \geq 2).$$

By assumption (1.16) we can assume without loss of generality that $X'_{n,i} =$

$X_{n,i}$ for all $n \in \mathbb{N}$ and $i \in \{1, \dots, m_n\}$. Moreover it is no loss of generality to assume that $p_{n,i} < 1$. We then have

$$\mathbb{P}(X_n = k) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m_n} p_{n,i_1} p_{n,i_2} \cdots p_{n,i_k} \frac{\prod_{j=1}^{m_n}(1 - p_{n,j})}{(1 - p_{n,i_1}) \cdots (1 - p_{n,i_k})}. \quad (1.18)$$

Let $\mu_n := \max_{1 \le i \le m_n} p_{n,i}$. Since $\sum_{j=1}^{m_n} p_{n,j}^2 \le \lambda_n \mu_n \to 0$ as $n \to \infty$, we have

$$\begin{aligned} \log\Big(\prod_{j=1}^{m_n}(1 - p_{n,j})\Big) &= \sum_{j=1}^{m_n}(-p_{n,j} + O(p_{n,j}^2)) \\ &\to -\gamma \text{ as } n \to \infty. \end{aligned} \quad (1.19)$$

Also,

$$\inf_{1 \le i_1 < i_2 < \cdots < i_k \le m_n} (1 - p_{n,i_1}) \cdots (1 - p_{n,i_k}) \ge (1 - \mu_n)^k \to 1 \text{ as } n \to \infty. \quad (1.20)$$

Finally, with $\sum^{\neq}_{i_1,\dots,i_k \in \{1,2,\dots,m_n\}}$ denoting summation over all ordered k-tuples of distinct elements of $\{1, 2, \dots, m_n\}$, we have

$$k! \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m_n} p_{n,i_1} p_{n,i_2} \cdots p_{n,i_k} = \sum^{\neq}_{i_1,\dots,i_k \in \{1,2,\dots,m_n\}} p_{n,i_1} p_{n,i_2} \cdots p_{n,i_k},$$

and

$$\begin{aligned} 0 &\le \left(\sum_{i=1}^{m_n} p_{n,i}\right)^k - \sum^{\neq}_{i_1,\dots,i_k \in \{1,2,\dots,m_n\}} p_{n,i_1} p_{n,i_2} \cdots p_{n,i_k} \\ &\le \binom{k}{2} \sum_{i=1}^{m_n} p_{n,i}^2 \left(\sum_{j=1}^{m_n} p_{n,j}\right)^{k-2} \end{aligned}$$

which tends to zero as $n \to \infty$. Therefore

$$k! \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m_n} p_{n,i_1} p_{n,i_2} \cdots p_{n,i_k} \to \gamma^k \text{ as } n \to \infty. \quad (1.21)$$

The result follows from (1.18) by using (1.19), (1.20), and (1.21). □

Random variables $X_{n,i}$ as in Proposition 1.4 are said to constitute a *null array* if (1.15) holds. In this case the Poisson convergence (1.17) is even equivalent to $\lambda_n \to \gamma$ and (1.16); see e.g. [45].

1.4 Exercises

Exercise 1.1 Prove equation (1.10).

Exercise 1.2 Let X be a random variable taking values in $\mathbb{N}_0$. Assume that there is a $\gamma \geq 0$ such that $\mathbb{E}[(X)_k] = \gamma^k$ for all $k \in \mathbb{N}_0$. Show that X has a Poisson distribution. (Hint: Derive the Taylor series for $g(s) := \mathbb{E}[s^X]$ at $s_0 = 1$.)

Exercise 1.3 Confirm Proposition 1.3 by showing that

$$\mathbb{E}[s^X t^{Z-X}] = e^{p\gamma(s-1)} e^{(1-p)\gamma(t-1)}, \quad s, t \in [0, 1],$$

using a direct computation.

Exercise 1.4 (Generalization of Proposition 1.2) Let $m \in \mathbb{N}$ and suppose that $X_1, \dots, X_m$ are independent random variables with Poisson distributions $\mathrm{Po}(\gamma_1), \dots, \mathrm{Po}(\gamma_m)$, respectively. Show that $X := X_1 + \cdots + X_m$ is Poisson distributed with parameter $\gamma := \gamma_1 + \cdots + \gamma_m$. Assuming $\gamma > 0$, show moreover for any $k \in \mathbb{N}$ that

$$\mathbb{P}(X_1 = k_1, \dots, X_m = k_m \mid X = k) = \frac{k!}{k_1! \cdots k_m!} \left(\frac{\gamma_1}{\gamma}\right)^{k_1} \cdots \left(\frac{\gamma_m}{\gamma}\right)^{k_m} \tag{1.22}$$

for $k_1 + \cdots + k_m = k$. This is a *multinomial distribution* with parameters k and $\gamma_1/\gamma, \dots, \gamma_m/\gamma$.

Exercise 1.5 (Generalization of Proposition 1.3) Let $m \in \mathbb{N}$ and let $Z_1, Z_2, \dots$ be a sequence of independent random vectors in $\mathbb{R}^m$ with common distribution $\mathbb{P}(Z_1 = e_i) = p_i$, $i \in \{1, \dots, m\}$, where e_i is the ith unit vector in $\mathbb{R}^m$ and $p_1 + \cdots + p_m = 1$. Let Z have a Poisson distribution with parameter γ, independent of $(Z_1, Z_2, \dots)$. Show that the components of the random vector $X := \sum_{j=1}^{Z} Z_j$ are independent and Poisson distributed with parameters $p_1\gamma, \dots, p_m\gamma$.

Exercise 1.6 (Bivariate extension of Proposition 1.4) Let $\gamma > 0, \delta \geq 0$. Suppose for $n \in \mathbb{N}$ that $m_n \in \mathbb{N}$ and for $1 \leq i \leq m_n$ that $p_{n,i}, q_{n,i}, \in [0, 1)$ with $\sum_{i=1}^{m_n} p_{n,i} \to \gamma$ and $\sum_{i=1}^{m_n} q_{n,i} \to \delta$ and $\max_{1 \leq i \leq m_n} \max(p_{n,i}, q_{n,i}) \to 0$ as $n \to \infty$. Suppose for $n \in \mathbb{N}$ that $(X_n, Y_n) = \sum_{i=1}^{m_n} (X_{n,i}, Y_{n,i})$ where each $(X_{n,i}, Y_{n,i})$ is a random 2-vector whose components are Bernoulli distributed with parameters $p_{n,i}, q_{n,i}$ respectively and satisfy $X_{n,i} Y_{n,i} = 0$ almost surely. Assume the random vectors $(X_{n,i}, Y_{n,i})$, $1 \leq i \leq m_n$, are independent. Prove that X_n, Y_n are asymptotically (as $n \to \infty$) distributed as a pair of independent Poisson variables with parameters γ, δ, i.e. for $k, \ell \in \mathbb{N}_0$,

$$\lim_{n \to \infty} \mathbb{P}(X_n = k, Y_n = \ell) = e^{-(\gamma+\delta)} \frac{\gamma^k \delta^\ell}{k! \ell!}.$$

Exercise 1.7 (Probability of Poisson variable being even) Suppose X is Poisson distributed with parameter $\gamma > 0$. Using the fact that the probability generating function (1.8) extends to $s = -1$, show that $\mathbb{P}(X/2 \in \mathbb{Z}) = (1 + e^{-2\gamma})/2$. For $k \in \mathbb{N}$ with $k \geq 3$, using the fact that the probability generating function (1.8) extends to a k-th complex root of unity, find a closed-form formula for $\mathbb{P}(X/k \in \mathbb{Z})$.

Exercise 1.8 Let $\gamma > 0$, and suppose X is Poisson with parameter γ. Suppose $f : \mathbb{N} \to \mathbb{R}_+$ is such that $\mathbb{E}[f(X)^{1+\varepsilon}] < \infty$ for some $\varepsilon > 0$. Show that $\mathbb{E}[f(X + k)] < \infty$ for any $k \in \mathbb{N}$.

Exercise 1.9 Let $0 < \gamma < \gamma'$. Give an example of two random variables X, Y with X Poisson distributed with parameter γ and Y Poisson with parameter γ', such that $Y - X$ *is not* Poisson distributed. (Hint: First consider a pair X', Y' such that $Y' - X'$ is Poisson distributed, and then modify finitely many of the values of their joint probability mass function.)

2

Point processes

A point process is a random collection of at most countably many points, possibly with multiplicities. This chapter defines this concept for an arbitrary measurable space and provides several criteria for equality in distribution.

2.1 Fundamentals

The idea of a point process is that of a random, at most countable, collection Z of points in some space $\mathbb{X}$. A good example to think of is the d-dimensional Euclidean space $\mathbb{R}^d$. Ignoring measurability issues for the moment, we might think of Z as a mapping $\omega \mapsto Z(\omega)$ from Ω into the system of countable subsets of $\mathbb{X}$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is an underlying probability space. Then Z can be identified with the family of mappings

$$\omega \mapsto \eta(\omega, B) := \mathrm{card}(Z(\omega) \cap B), \quad B \subset \mathbb{X},$$

counting the number of points that Z has in B. (We write $\mathrm{card}\, A$ for the number of elements of a set A.) Clearly, for any fixed $\omega \in \Omega$ the mapping $\eta(\omega, \cdot)$ is a measure, namely the *counting measure* supported by $Z(\omega)$. It turns out to be a mathematically fruitful idea to define point processes as random counting measures.

To give the general definition of a point process we let $(\mathbb{X}, \mathcal{X})$ be a measurable space. Let $\mathbf{N}_{<\infty}(\mathbb{X}) \equiv \mathbf{N}_{<\infty}$ denote the space of all measures μ on $\mathbb{X}$ such that $\mu(B) \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ for all $B \in \mathcal{X}$, and let $\mathbf{N}(\mathbb{X}) \equiv \mathbf{N}$ be the space of all measures that can be written as a countable sum of measures from $\mathbf{N}_{<\infty}$. A trivial example of an element of $\mathbf{N}$ is the *zero measure* 0 that is identically zero on $\mathcal{X}$. A less trivial example is the *Dirac measure* δ_x at a point $x \in \mathbb{X}$ given by $\delta_x(B) := \mathbf{1}_B(x)$. More generally, any (finite or infinite) sequence $(x_n)_{n=1}^k$ of elements of $\mathbb{X}$, where $k \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ is the number of terms in the sequence, can be used to define a measure

$$\mu = \sum_{n=1}^{k} \delta_{x_n}. \tag{2.1}$$

Then $\mu \in \mathbf{N}$ and

$$\mu(B) = \sum_{n=1}^{k} \mathbf{1}_B(x_n), \quad B \in \mathcal{X}.$$

More generally we have for any measurable $f\colon \mathbb{X} \to [0, \infty]$ that

$$\int f \, d\mu = \sum_{n=1}^{k} f(x_n). \tag{2.2}$$

We can allow for $k = 0$ in (2.1). In this case μ is the zero measure. The points $x_1, x_2, \dots$ are not assumed to be pairwise distinct. If $x_i = x_j$ for some $i, j \leq k$ with $i \neq j$, then μ is said to have *multiplicities*. In fact, the multiplicity of x_i is the number $\mathrm{card}\{j \leq k : x_j = x_i\}$.

In general one cannot guarantee that any $\mu \in \mathbf{N}$ can be written in the form (2.1); see Exercise 2.5. Fortunately, only weak assumptions on $(\mathbb{X}, \mathcal{X})$ and μ are required to achieve this; see e.g. Corollary 6.3. Moreover, large parts of the theory can be developed without imposing further assumptions on $(\mathbb{X}, \mathcal{X})$, other than to be a measurable space.

A measure ν on $\mathbb{X}$ is said to be *s-finite*, if ν is a countable sum of finite measures. Any element of $\mathbf{N}$ is s-finite. We recall that a measure ν on $\mathbb{X}$ is σ-finite if and only if there is a sequence $B_m \in \mathcal{X}$, $m \in \mathbb{N}$, such that $\cup_m B_m = \mathbb{X}$ and $\nu(B_m) < \infty$ for all $m \in \mathbb{N}$. Clearly every σ-finite measure is s-finite. In contrast to σ-finite measures, any countable sum of s-finite measures is again s-finite. If the points x_n in (2.1) are all the same, then this measure μ is not σ-finite.

Let $\mathcal{N}(\mathbb{X}) \equiv \mathcal{N}$ denote the σ-field generated by the collection of subsets of $\mathbf{N}$ containing all the sets

$$\{\mu \in \mathbf{N} : \mu(B) = k\}, \quad B \in \mathcal{X},\ k \in \mathbb{N}_0.$$

This means that $\mathcal{N}$ is the smallest σ-field on $\mathbf{N}$ such that $\mu \mapsto \mu(B)$ is measurable for all $B \in \mathcal{X}$.

Definition 2.1 A *point process* on $\mathbb{X}$ is a random element η in $(\mathbf{N}, \mathcal{N})$, that is, a measurable mapping $\eta\colon \Omega \to \mathbf{N}$.

If η is a point process on $\mathbb{X}$ and $B \in \mathcal{X}$, then we denote by $\eta(B)$ the mapping $\omega \mapsto \eta(\omega, B) := \eta(\omega)(B)$. By the definitions of η and the σ-field $\mathcal{N}$ these are random variables taking values in $\overline{\mathbb{N}}_0 := \overline{\mathbb{N}} \cup \{0\}$, that is

$$\{\eta(B) = k\} \equiv \{\omega \in \Omega : \eta(\omega, B) = k\} \in \mathcal{F}, \quad B \in \mathcal{X},\ k \in \overline{\mathbb{N}}_0. \tag{2.3}$$

Conversely, a mapping $\eta \colon \Omega \to \mathbf{N}$ is a point process if (2.3) holds. In this case we call $\eta(B)$ the *number of points* of η in B.

Example 2.2 Let X be a random element in $\mathbb{X}$. Then

$$\eta := \delta_X,$$

is a point process. Indeed, the required measurability property follows from

$$\{\eta(B) = k\} = \begin{cases} \{X \in B\}, & \text{if } k = 1, \\ \{X \notin B\}, & \text{if } k = 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

The above one-point process can be generalized as follows.

Example 2.3 Let $\mathbb{Q}$ be a probability measure on $\mathbb{X}$ and suppose that $X_1, \dots, X_m$ are independent random elements in $\mathbb{X}$ with distribution $\mathbb{Q}$. Then

$$\eta := \delta_{X_1} + \cdots + \delta_{X_m}$$

is a point process on $\mathbb{X}$. Because

$$\mathbb{P}(\eta(B) = k) = \binom{m}{k} \mathbb{Q}(B)^k (1 - \mathbb{Q}(B))^{m-k}, \quad k = 0, \dots, m,$$

η is referred to as a *binomial process* with *sample size* m and *sampling distribution* $\mathbb{Q}$.

In this example, the random measure η can be written as a sum of Dirac measures, and we formalize the class of point processes having this property as follows:

Definition 2.4 We shall refer to a point process η on $\mathbb{X}$ as a *proper point process* if there exist random elements $X_1, X_2, \dots$ in $\mathbb{X}$ and an $\overline{\mathbb{N}}_0$-valued random variable κ such that almost surely

$$\eta = \sum_{n=1}^{\kappa} \delta_{X_n}. \tag{2.4}$$

The motivation for this terminology is that the intuitive notion of a point process is as a (random) set of points, rather than as an integer-valued measure. A proper point process is one which can be interpreted as a countable (random) set of points in $\mathbb{X}$ (possibly with repetitions), thereby better fitting this intuition.

The class of proper point processes is very large. Indeed, we shall see later that if $\mathbb{X}$ is a Borel subspace of a complete separable metric space, then *any* locally finite point process on $\mathbb{X}$ (see Definition 2.13) is proper, and that for general $(\mathbb{X}, \mathcal{X})$, if η is a *Poisson* point process on $\mathbb{X}$ then there is a proper point process on $\mathbb{X}$ having the same distribution as η (these concepts will be defined in due course); see Corollary 6.3 and Corollary 3.7. Exercise 2.5 shows, however, that not all point processes are proper.

2.2 Campbell's formula

A first characteristic of a point process are the mean numbers of points lying in measurable sets:

Definition 2.5 The *intensity measure* of a point process η on $\mathbb{X}$ is the measure λ defined by

$$\lambda(B) := \mathbb{E}[\eta(B)], \quad B \in \mathcal{X}. \tag{2.5}$$

It follows from basic properties of expectation that the intensity measure of a point process is indeed a measure.

Example 2.6 The intensity measure of a binomial process with sample size m and sampling distribution $\mathbb{Q}$ is given by

$$\lambda(B) = \mathbb{E}\Big[\sum_{k=1}^{m} \mathbf{1}\{X_k \in B\}\Big] = \sum_{k=1}^{m} \mathbb{P}(X_k \in B) = m\,\mathbb{Q}(B).$$

Independence of the random variables $X_1, \dots, X_m$ is not required for this calculation.

Let us denote by $\mathbb{R}(\mathbb{X})$ the set of all measurable functions $u\colon \mathbb{X} \to \mathbb{R}$. Let $\mathbb{R}_+(\mathbb{X})$ be the set of all those $u \in \mathbb{R}(\mathbb{X})$ with $u \geq 0$. Any $u \in \mathbb{R}(\mathbb{X})$ is the difference $u^+ - u^-$ of two functions in $\mathbb{R}_+(\mathbb{X})$, where $u^+(x) := \max\{u(x), 0\}$ and $u^-(x) := \max\{-u(x), 0\}$. We recall from measure theory (cf. [45] and Appendix A) that for any measure ν on $\mathbb{X}$, the integral $\int u\,d\nu \equiv \int u(x)\,\nu(dx)$ of $u \in \mathbb{R}(\mathbb{X})$ with respect to ν is defined as

$$\int u(x)\,\nu(dx) \equiv \int u\,d\nu := \int u^+\,d\nu - \int u^-\,d\nu$$

whenever this expression is not of the form $\infty - \infty$. Otherwise we here use the convention $\int u(x)\,\nu(dx) := 0$. We often write

$$\nu(u) := \int u(x)\,\nu(dx),$$

so that $\nu(B) = \nu(\mathbf{1}_B)$ for any $B \in \mathcal{X}$. All these definitions apply equally well in the case of a measurable function $u\colon \mathbb{X} \to \overline{\mathbb{R}} := [-\infty, \infty]$. If η is a random measure, then $\eta(u) \equiv \int u\,d\eta$ denotes the mapping $\omega \mapsto \int u(x)\,\eta(\omega, dx)$.

Proposition 2.7 (Campbell's Formula) *Let η be a point process on $(\mathbb{X}, \mathcal{X})$ with intensity measure λ. Let $u \in \mathbb{R}(\mathbb{X})$. Then $\int u(x)\,\eta(dx)$ is a random variable. Moreover,*

$$\mathbb{E}\Big[\int u(x)\,\eta(dx)\Big] = \int u(x)\,\lambda(dx), \tag{2.6}$$

whenever $u \geq 0$ or $\int |u(x)|\,\lambda(dx) < \infty$.

Proof If $u(x) = \mathbf{1}_B(x)$ for some $B \in \mathcal{X}$ then $\int u(x)\,\eta(dx) = \eta(B)$ and both assertions are true by definition. By standard techniques of measure theory (linearity and monotone convergence) this can be extended, first to measurable simple functions and then to arbitrary $u \in \mathbb{R}_+(\mathbb{X})$. Assume now that $u \in \mathbb{R}(\mathbb{X})$ satisfies $\int |u(x)|\,\lambda(dx) < \infty$. Then the first part of the proof shows that $\eta(u^+)$ and $\eta(u^-)$ both have a finite expectation and that

$$\mathbb{E}[\eta(u)] = \mathbb{E}[\eta(u^+)] - \mathbb{E}[\eta(u^-)] = \lambda(u^+) - \lambda(u^-) = \lambda(u).$$

This concludes the proof. $\square$

It is sometimes useful to note that Campbell's formula (2.6) continues to hold for all elements u in the space $\overline{\mathbb{R}}_+(\mathbb{X})$ of all measurable functions from $\mathbb{X}$ to $\overline{\mathbb{R}}_+ := [0, \infty]$. To see this we can apply the formula to $\min\{u, n\}$ and take the limit as $n \to \infty$.

2.3 Distribution of a point process

In accordance with the terminology of probability theory (see Appendix B), the *distribution* of a point process η on $\mathbb{X}$ is the probability measure $\mathbb{P}_\eta$ on $(\mathbf{N}, \mathcal{N})$, given by $A \mapsto \mathbb{P}(\eta \in A)$. If η' is another point process with the same distribution, we write $\eta \overset{d}{=} \eta'$.

The following device is a powerful tool for analyzing point processes. We use the convention $e^{-\infty} := 0$.

Definition 2.8 The *Laplace* (or *characteristic*) *functional* of a point process η on $\mathbb{X}$ is the mapping $L_\eta\colon \mathbb{R}_+(\mathbb{X}) \to [0,1]$ defined by

$$L_\eta(u) := \mathbb{E}\Big[\exp\Big(-\int u(x)\,\eta(dx)\Big)\Big].$$

Example 2.9 Let η be the binomial process of Example 2.3. Then, for $u \in \mathbb{R}_+(\mathbb{X})$,

$$\begin{aligned} L_\eta(u) &= \mathbb{E}\Big[\exp\Big(-\sum_{k=1}^m u(X_k)\Big)\Big] = \mathbb{E}\Big[\prod_{k=1}^m \exp[-u(X_k)]\Big] \\ &= \prod_{k=1}^m \mathbb{E}[\exp[-u(X_k)]] = \Big[\int \exp[-u(x)]\,\mathbb{Q}(dx)\Big]^m. \end{aligned}$$

The following proposition characterizes equality in distribution for point processes.

Proposition 2.10 *For point processes η and η' on $\mathbb{X}$ the following assertions are equivalent.*

(i) $\eta \overset{d}{=} \eta'$.
(ii) $(\eta(B_1),\dots,\eta(B_m)) \overset{d}{=} (\eta'(B_1),\dots,\eta'(B_m))$ *for all* $m \in \mathbb{N}$ *and for all* $B_1,\dots,B_m \in \mathcal{X}$.
(iii) $L_\eta(u) = L_{\eta'}(u)$ *for all* $u \in \mathbb{R}_+(\mathbb{X})$.
(iv) *For all* $u \in \mathbb{R}_+(\mathbb{X})$, $\eta(u) \overset{d}{=} \eta'(u)$ *as random variables in* $\overline{\mathbb{R}}$.

In particular, the Laplace functional of a point process determines its distribution.

Proof First we prove that (i) implies (iv). Given $u \in \mathbb{R}_+(\mathbb{X})$, define the function $g_u\colon \mathbf{N} \to \overline{\mathbb{R}}_+$ by $\mu \mapsto \int u\,d\mu$. By Proposition 2.7 (or a direct check based on first principles), g_u is a measurable function. Also,

$$\mathbb{P}_{\eta(u)}(\cdot) = \mathbb{P}(\eta(u) \in \cdot) = \mathbb{P}(\eta \in g_u^{-1}(\cdot)),$$

and likewise for η'. So if $\eta \overset{d}{=} \eta'$ then also $\eta(u) \overset{d}{=} \eta'(u)$.

Next we show that (iv) implies (iii). For any $\overline{\mathbb{R}}_+$-valued random variable Y we have $\mathbb{E}[\exp(-Y)] = \int e^{-y}\,\mathbb{P}_Y(dy)$, which is determined by the distribution $\mathbb{P}_Y$. Hence if (iv) holds then

$$L_\eta(u) = \mathbb{E}[\exp(-\eta(u))] = \mathbb{E}[\exp(-\eta'(u))] = L_{\eta'}(u)$$

for all $u \in \mathbb{R}_+(\mathbb{X})$, so (iii) holds.

Assume now that (iii) holds and consider a *simple function* of the form

$u = c_1\mathbf{1}_{B_1} + \cdots + c_m\mathbf{1}_{B_m}$, where $m \in \mathbb{N}$, $B_1, \ldots, B_m \in \mathcal{X}$, and $c_1, \ldots, c_m \in (0, \infty)$. Then

$$L_\eta(u) = \mathbb{E}\Big[\exp\Big(-\sum_{j=1}^{m} c_j\eta(B_j)\Big)\Big] = \hat{\mathbb{P}}_{(\eta(B_1),\ldots,\eta(B_m))}(c_1, \ldots, c_m) \tag{2.7}$$

where for any measure μ on $\mathbb{R}_+^m$ we write $\hat{\mu}$ for its multivariate Laplace transform. Since a finite measure on $\mathbb{R}_+^m$ is determined by its Laplace transform (this follows from Proposition B.4), we can conclude that the restriction of $\mathbb{P}_{(\eta(B_1),\ldots,\eta(B_m))}$ (a measure on $[0, \infty]^m$) to $(0, \infty)^m$ is the same as the restriction of $\mathbb{P}_{(\eta'(B_1),\ldots,\eta'(B_m))}$ to $(0, \infty)^m$. Then using the fact that $\mathbb{P}_{(\eta(B_1),\ldots,\eta(B_m))}$ and $\mathbb{P}_{(\eta'(B_1),\ldots,\eta'(B_m))}$ are probability measures on $[0, \infty]^m$, by forming suitable complements we obtain that $\mathbb{P}_{(\eta(B_1),\ldots,\eta(B_m))} = \mathbb{P}_{(\eta'(B_1),\ldots,\eta'(B_m))}$ (these details are left to the reader). In other words, (iii) implies (ii).

Finally we assume (ii) and prove (i). By (ii), $\mathbb{P}_\eta$ and $\mathbb{P}_{\eta'}$ coincide on the system $\mathcal{H}$ consisting of all sets of the form

$$\{\mu \in \mathbf{N} : \mu(B_1) = k_1, \ldots, \mu(B_m) = k_m\},$$

where $m \in \mathbb{N}$, $B_1, \ldots, B_m \in \mathcal{X}$, and $k_1, \ldots, k_m \in \mathbb{N}_0$. Clearly $\mathcal{H}$ is a π-system, that is, closed under pairwise intersections. Moreover, the smallest σ-field $\sigma(\mathcal{H})$ containing $\mathcal{H}$ is the full σ-field $\mathcal{N}$. Hence (i) follows from the fact that a probability measure is determined by its values on a generating π-system; see Theorem A.4. $\square$

2.4 Point processes on metric spaces

Let us now assume that $\mathbb{X}$ is a metric space with metric ρ; see Appendix A.2. Then it is always to be understood that $\mathcal{X}$ is the Borel σ-field $\mathcal{B}(\mathbb{X})$ of $\mathbb{X}$. In particular the singleton $\{x\}$ is in $\mathcal{X}$ for all $x \in \mathbb{X}$. If ν is a measure on $\mathbb{X}$ then we often write $\nu\{x\} := \nu(\{x\})$. If $\nu\{x\} = 0$ for all $x \in \mathbb{X}$, then ν is said to be *diffuse*. Moreover, if $\mu \in \mathbf{N}(\mathbb{X})$ then we write $x \in \mu$ if $\mu(\{x\}) > 0$.

A set $B \subset \mathbb{X}$ is said to be *bounded* if it is empty or its *diameter*

$$d(B) := \sup\{\rho(x, y) : x, y \in B\}$$

is finite.

Definition 2.11 Suppose that $\mathbb{X}$ is a metric space. The system of bounded measurable subsets of $\mathbb{X}$ is denoted by $\mathcal{X}_b$. A measure ν on $\mathbb{X}$ is said to be *locally finite* if $\nu(B) < \infty$ for every $B \in \mathcal{X}_b$. Let $\mathbf{N}_l(\mathbb{X})$ denote the set of all locally finite elements of $\mathbf{N}(\mathbb{X})$ and let $\mathcal{N}_l(\mathbb{X}) := \{A \cap \mathbf{N}_l(\mathbb{X}) : A \in \mathcal{N}(\mathbb{X})\}$.

Fix some $x_0 \in \mathbb{X}$. Then any bounded set B is contained in the closed *ball* $B(x_0, r) = \{x \in \mathbb{X} : \rho(x, x_0) \le r\}$ for sufficiently large $r > 0$. In fact, if $B \ne \emptyset$, then we can take for instance $r := d(B) + \rho(x_1, x_0)$ for some $x_1 \in B$. Note that $B(x_0, n) \uparrow \mathbb{X}$ as $n \to \infty$. Hence a measure ν on $\mathbb{X}$ is locally finite if and only if $\nu(B(x_0, n)) < \infty$ for each $n \in \mathbb{N}$. In particular the set $\mathbf{N}_l(\mathbb{X})$ is measurable, that is $\mathbf{N}_l(\mathbb{X}) \in \mathcal{N}(\mathbb{X})$. Moreover, any locally finite measure is σ-finite.

Proposition 2.12 *Let η and η' be point processes on a metric space $\mathbb{X}$. Suppose $\eta(u) \overset{d}{=} \eta'(u)$ for all $u \in \mathbb{R}_+(\mathbb{X})$ such that $\{u > 0\}$ is bounded. Then $\eta \overset{d}{=} \eta'$.*

Proof Suppose that

$$\eta(u) \overset{d}{=} \eta'(u), \quad u \in \mathbb{R}_+(\mathbb{X}),\ \{u > 0\} \text{ bounded}. \tag{2.8}$$

Then $L_\eta(u) = L_{\eta'}(u)$ for any $u \in \mathbb{R}_+(\mathbb{X})$ such that $\{u > 0\}$ is bounded. Given any $v \in \mathbb{R}_+(\mathbb{X})$, we can choose a sequence u_n, $n \in \mathbb{N}$, of functions in $\mathbb{R}_+(\mathbb{X})$ such that $\{u_n > 0\}$ is bounded for each n, and $u_n \uparrow v$ pointwise. Then by dominated convergence and (2.8),

$$L_\eta(v) = \lim_{n\to\infty} L_\eta(u_n) = \lim_{n\to\infty} L_{\eta'}(u_n) = L_{\eta'}(v),$$

so $\eta \overset{d}{=} \eta'$ by Proposition 2.10. □

Definition 2.13 A point process η on a metric space $\mathbb{X}$ is said to be *locally finite*, if $\mathbb{P}(\eta(B) < \infty) = 1$ for every bounded $B \in \mathcal{X}$.

If required, we could interpret a locally finite point process η as a random element of the space $(\mathbf{N}_l(\mathbb{X}), \mathcal{N}_l(\mathbb{X}))$, introduced in Definition 2.11. Indeed, we can define another point process $\tilde{\eta}$ by $\tilde{\eta}(\omega, \cdot) := \eta(\omega, \cdot)$ if the latter is locally finite and by $\tilde{\eta}(\omega, \cdot) := 0$ (the zero measure) otherwise. Then $\tilde{\eta}$ is a random element of $(\mathbf{N}_l(\mathbb{X}), \mathcal{N}_l(\mathbb{X}))$ that coincides $\mathbb{P}$-almost surely with η.

The reader might have noticed that the proof of Proposition 2.12 has not really used the metric on $\mathbb{X}$. The proof of the following refinement of this proposition uses the metric in an essential way.

Proposition 2.14 *Let η and η' be locally finite point processes on a metric space $\mathbb{X}$. Suppose $\eta(u) \overset{d}{=} \eta'(u)$ for all continuous $u\colon \mathbb{X} \to \mathbb{R}_+$ such that $\{u > 0\}$ is bounded. Then $\eta \overset{d}{=} \eta'$.*

Proof Assume that $\eta(u) \overset{d}{=} \eta'(u)$ for all u in the space $\mathbf{G}$ of continuous functions $u\colon \mathbb{X} \to \mathbb{R}_+$ such that $\{u > 0\}$ is bounded. Since $\mathbf{G}$ is closed under non-negative linear combinations, it follows as in the proof that (iii) implies (ii) in Proposition 2.10 that

$$(\eta(u_1), \eta(u_2), \dots) \stackrel{d}{=} (\eta'(u_1), \eta'(u_2), \dots),$$

first for a finite sequence and then (by Theorem A.4 in the Appendix) for an infinite sequence $u_n \in \mathbf{G}$, $n \in \mathbb{N}$. Take a bounded closed set $C \subset \mathbb{X}$ and, for $n \in \mathbb{N}$ define,

$$u_n(x) := \max\{1 - n\rho(x, C), 0\}, \quad x \in \mathbb{X},$$

where $\rho(x, C) := \inf\{\rho(x, y) : y \in C\}$. It is easy to check that $u_n \in \mathbf{G}$. Moreover, $u_n \downarrow \mathbf{1}_C$ as $n \to \infty$, and since η is locally finite we obtain that $\eta(u_n) \to \eta(C)$ $\mathbb{P}$-a.s. The same relation holds for η'. It follows that statement (ii) of Proposition 2.10 holds, whenever $B_1, \dots, B_m$ are closed and bounded. By the monotone class theorem, $\mathbb{P}_\eta$ and $\mathbb{P}_{\eta'}$ coincide on the σ-field generated by the π-system consisting of all sets of the form

$$\{\mu \in \mathbf{N}_l : \mu(B_1) = k_1, \dots, \mu(B_m) = k_m\}, \tag{2.9}$$

where $m \in \mathbb{N}$, $B_1, \dots, B_m \subset \mathbb{X}$ are closed and $k_1, \dots, k_m \in \mathbb{N}_0$. By Exercise 2.6 this σ-field coincides with $\mathcal{N}_l$, concluding the proof. □

2.5 Exercises

Exercise 2.1 Give an example of a point process η on a measurable space $(\mathbb{X}, \mathcal{X})$ with intensity measure λ, and $u \in \mathbb{R}(\mathbb{X})$ (violating the condition that $u \geq 0$ or $\int |u(x)|\lambda(dx) < \infty$), such that the Campbell formula (2.6) fails.

Exercise 2.2 Let $\mathcal{X}^* \subset \mathcal{X}$ be a π-system generating $\mathcal{X}$. Let η be a point process on $\mathbb{X}$ that is σ*-finite* on $\mathcal{X}^*$, meaning that there is a sequence $C_n \in \mathcal{X}^*$, $n \in \mathbb{N}$, such that $\cup_{n=1}^\infty C_n = \mathbb{X}$ and $\mathbb{P}(\eta(C_n) < \infty) = 1$ for all $n \in \mathbb{N}$. Let η' be another point process on $\mathbb{X}$ and suppose that the equality in Proposition 2.10 (ii) holds for all $B_1, \dots, B_m \in \mathcal{X}^*$ and $m \in \mathbb{N}$. Show that $\eta \stackrel{d}{=} \eta'$.

Exercise 2.3 Show that Proposition 2.10 remains valid if $B_1, \dots, B_m \in \mathcal{X}$ in (ii) are assumed to be pairwise disjoint.

Exercise 2.4 Let $\eta_1, \eta_2, \dots$ be a sequence of point processes and define $\eta := \eta_1 + \eta_2 + \cdots$, that is $\eta(\omega, B) := \eta_1(\omega, B) + \eta_2(\omega, B) + \cdots$ for all $\omega \in \Omega$ and $B \in \mathcal{X}$. Show that η is a point process. (Hint: Prove first that $\mathbf{N}(\mathbb{X})$ is closed under countable summation.)

Exercise 2.5 Suppose that $\mathbb{X} = [0,1]$. Find a σ-field $\mathcal{X}$ and a measure μ on $(\mathbb{X}, \mathcal{X})$ such that $\mu(\mathbb{X}) = 1$ and $\mu(B) \in \{0,1\}$ for all $B \in \mathcal{X}$, which is not of the form $\mu = \delta_x$ for some $x \in \mathbb{X}$. (Hint: Take the system of all finite subsets of $\mathbb{X}$ as a generator of $\mathcal{X}$.)

Exercise 2.6 Suppose that $\mathbb{X}$ is a metric space $\mathbb{X}$. Let $C \subset \mathbb{X}$ be closed and bounded and let $\mathbf{N}_C := \{\mu \in \mathbf{N}(\mathbb{X}) : \mu(C) < \infty\}$. Let $\mathcal{N}_C$ be the σ-field on $\mathbf{N}_C$ generated by the system of sets $\{\mu \in \mathbf{N}_C : \mu(B) = k\}$, where $B \subset C$ is measurable and $k \in \mathbb{N}_0$. Use the monotone class theorem to show that $\mathcal{N}_C$ is also generated by the system of sets $\{\mu \in \mathbf{N}_C : \mu(B) = k\}$, where $B \subset C$ is closed and $k \in \mathbb{N}_0$. Use this to prove that $\mathcal{N}_l(\mathbb{X})$ is generated by the system of sets $\{\mu \in \mathbf{N}_l(\mathbb{X}) : \mu(B) = k\}$, where $B \subset C$ is closed and bounded and $k \in \mathbb{N}_0$.

Exercise 2.7 Let η be a point process on $\mathbb{X}$ with intensity measure λ and let $B \in \mathcal{X}$ such that $\lambda(B) < \infty$. Show that

$$\lambda(B) = -\frac{d}{dt} L_\eta(t\mathbf{1}_B)\Big|_{t=0}.$$

Exercise 2.8 Let η be a point process on $\mathbb{X}$. Show for each $B \in \mathcal{X}$ that

$$\mathbb{P}(\eta(B) = 0) = \lim_{t\to\infty} L_\eta(t\mathbf{1}_B).$$

3

Poisson processes

For a Poisson point process the number of points in a given set has a Poisson distribution. Moreover, the number of points in disjoint sets are stochastically independent. A Poisson process exists on a general s-finite measure space. Its distribution is characterized by a specific exponential form of the Laplace functional.

3.1 Definition of a Poisson process

In this chapter we fix an arbitrary measurable space $(\mathbb{X}, \mathcal{X})$. We are now ready for the definition of the main subject of this volume. Recall that for $\gamma \in [0, \infty]$, the Poisson distribution $\mathrm{Po}(\gamma)$ was defined at (1.4).

Definition 3.1 Let λ be an s-finite measure on $\mathbb{X}$. A *Poisson process* with intensity measure λ is a point process η on $\mathbb{X}$ with the following two properties:

(i) For any $B \in \mathcal{X}$ the distribution of $\eta(B)$ is $\mathrm{Po}(\lambda(B))$, that is to say $\mathbb{P}(\eta(B) = k) = \mathrm{Po}(\lambda(B); k)$ for all $k \in \mathbb{N}_0$.
(ii) The random variables $\eta(B_1), \dots, \eta(B_m)$ are independent whenever the sets $B_1, \dots, B_m \in \mathcal{X}$ are pairwise disjoint.

Property (i) of Definition 3.1 is responsible for the name of the Poisson process. A point process with property (ii) is said to be *completely independent*. (One also says that η has *independent increments* or is *completely random*.) For a (locally finite) point process on a complete separable metric space without multiplicities and a diffuse intensity measure, we will see in Chapter 6 that the two defining properties of a Poisson process are equivalent.

If η is a Poisson process with intensity measure λ then $\mathbb{E}[\eta(B)] = \lambda(B)$, so that Definition 3.1 is consistent with Definition 2.5. In particular, if $\lambda = 0$ is the zero measure, then $\mathbb{P}(\eta(\mathbb{X}) = 0) = 1$.

Let us first record that for each s-finite λ there is at most one Poisson process with intensity measure λ, up to equality in distribution.

Proposition 3.2 *Let η and η' be two Poisson processes on $\mathbb{X}$ with the same s-finite intensity measure. Then $\eta \overset{d}{=} \eta'$.*

Proof The result follows from Proposition 2.10 and Exercise 2.3. □

3.2 Existence of Poisson processes

We shall show below by means of an explicit construction that Poisson processes exist. Before we can do this, we need to deal with the *superposition* of independent Poisson processes.

Theorem 3.3 (Superposition Theorem) *Let η_i, $i \in \mathbb{N}$, be a sequence of independent Poisson processes on $\mathbb{X}$ with intensity measures λ_i. Then*

$$\eta := \sum_{i=1}^{\infty} \eta_i \tag{3.1}$$

is a Poisson process with intensity measure $\lambda := \lambda_1 + \lambda_2 + \cdots$.

Proof Exercise 2.4 shows that η is a point process.

For $n \in \mathbb{N}$ and $B \in \mathcal{X}$, we have by Exercise 1.4 that $\xi_n(B) := \sum_{i=1}^n \eta_i(B)$ has a Poisson distribution with parameter $\sum_{i=1}^n \lambda_i(B)$. Also $\xi_n(B)$ converges monotonically to $\eta(B)$ so by continuity of probability, and the fact that $\mathrm{Po}(\gamma; j)$ is continuous in γ for $j \in \mathbb{N}_0$, for all $k \in \mathbb{N}_0$ we have

$$\begin{aligned}\mathbb{P}(\eta(B) \le k) &= \lim_{n\to\infty} \mathbb{P}(\xi_n(B) \le k) \\ &= \lim_{n\to\infty} \sum_{j=0}^{k} \mathrm{Po}\left(\sum_{i=1}^{n} \lambda_i(B); j\right) = \sum_{j=0}^{k} \mathrm{Po}\left(\sum_{i=1}^{\infty} \lambda_i(B); j\right)\end{aligned}$$

so that $\eta(B)$ has the $\mathrm{Po}(\lambda(B))$ distribution.

Let $B_1, \dots, B_m \in \mathcal{X}$ be pairwise disjoint. Then $(\eta_i(B_j), 1 \le j \le m, i \in \mathbb{N})$ is a family of independent random variables, so that by the grouping property of independence the random variables $\sum_i \eta_i(B_1), \dots, \sum_i \eta_i(B_m)$, are independent. Thus η is completely independent. □

Now we construct a Poisson process on $(\mathbb{X}, \mathcal{X})$ with arbitrary s-finite intensity measure. We start by generalizing Example 2.3.

Definition 3.4 Let $\mathbb{V}$ and $\mathbb{Q}$ be probability measures on $\mathbb{N}_0$ and $\mathbb{X}$, respectively. Suppose that $X_1, X_2, \dots$ are independent random elements in $\mathbb{X}$

with distribution $\mathbb{Q}$, and let κ be a random variable with distribution $\mathbb{V}$, independent of (X_n). Then

$$\eta := \sum_{k=1}^{\kappa} \delta_{X_k} \tag{3.2}$$

is called a *mixed binomial process* with *mixing distribution* $\mathbb{V}$ and *sampling distribution* $\mathbb{Q}$.

The following result provides the key for the construction of Poisson processes.

Proposition 3.5 *Let $\mathbb{Q}$ be a probability measure on $\mathbb{X}$ and let $\gamma \geq 0$. Suppose that η is a mixed binomial process with mixing distribution* $\mathrm{Po}(\gamma)$ *and sampling distribution $\mathbb{Q}$. Then η is a Poisson process with intensity measure $\gamma\,\mathbb{Q}$.*

Proof Let κ and (X_n) be given as in Definition 3.4. To prove property (ii) of Definition 3.1 it is no loss of generality to assume that $B_1, \dots, B_m$ are pairwise disjoint measurable subsets of $\mathbb{X}$ satisfying $\cup_{i=1}^m B_i = \mathbb{X}$. (Otherwise we can add the complement of this union.) Define

$$Z_j := (\mathbf{1}_{B_1}(X_j), \dots, \mathbf{1}_{B_m}(X_j)), \quad j \in \mathbb{N}.$$

Then, by (3.2),

$$\sum_{j=1}^{\kappa} Z_j = (\eta(B_1), \dots, \eta(B_m)),$$

and Exercise 1.5 shows that $\eta(B_1), \dots, \eta(B_m)$ are independent random variables and, moreover, that $\eta(B_k)$ has a Poisson distribution with parameter $\gamma\,\mathbb{P}(X_1 \in B_k) = \gamma\,\mathbb{Q}(B_k)$. $\square$

Theorem 3.6 (Existence Theorem) *Let λ be an s-finite measure on $\mathbb{X}$. Then there exists a Poisson process on $\mathbb{X}$ with intensity measure λ.*

Proof The result is trivial if $\lambda(\mathbb{X}) = 0$.

Suppose for now that $0 < \lambda(\mathbb{X}) < \infty$. On a suitable probability space, assume that $\kappa, X_1, X_2, \dots$ are independent random elements with κ taking values in $\mathbb{N}_0$ and each X_i taking values in $\mathbb{X}$, with κ having the $\mathrm{Po}(\lambda(\mathbb{X}))$ distribution and each X_i having $\lambda(\cdot)/\lambda(\mathbb{X})$ as its distribution. Here the probability space can be taken to be a suitable product space; see the proof of Corollary 3.7 below. Let η be the mixed binomial process given by (3.2). Then by Proposition 3.5, η is a Poisson process with intensity measure λ, as required.

Now suppose that $\lambda(\mathbb{X}) = \infty$. There is a sequence λ_i, $i \in \mathbb{N}$, of measures on $(\mathbb{X}, \mathcal{X})$ with strictly positive and finite total measure, such that $\lambda = \sum_{i=1}^{\infty} \lambda_i$. On a suitable (product) probability space, let η_i, $i \in \mathbb{N}$, be a sequence of independent Poisson processes with η_i having intensity measure λ_i. This is possible by the preceding part of the proof. Set $\eta = \sum_{i=1}^{\infty} \eta_i$. By the superposition theorem (Theorem 3.3), η is a Poisson process with intensity measure λ, and the proof is complete. □

A corollary of the preceding proof is that on arbitrary $(\mathbb{X}, \mathcal{X})$, *every Poisson point process is proper* (see Definition 2.4), up to equality in distribution.

Corollary 3.7 *Let λ be an s-finite measure on $\mathbb{X}$. Then there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting random elements $X_1, X_2, \dots$ in $\mathbb{X}$ and κ in $\overline{\mathbb{N}}_0$, such that*

$$\eta := \sum_{n=1}^{\kappa} \delta_{X_n} \tag{3.3}$$

is a Poisson process with intensity measure λ.

Proof For simplicity we consider only the case with $\lambda(\mathbb{X}) = \infty$ and take the measures λ_i, $i \in \mathbb{N}$, as in the last part of the proof of Theorem 3.6. Let $\gamma_i := \lambda_i(\mathbb{X})$ and $\mathbb{Q}_i := \gamma_i^{-1}\lambda_i$. We shall take $(\Omega, \mathcal{F}, \mathbb{P})$ to be the product of spaces $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$, $i \in \mathbb{N}$, where each $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$ is again an infinite product of probability spaces $(\Omega_{ij}, \mathcal{F}_{ij}, \mathbb{P}_{ij})$, $j \in \mathbb{N}_0$, with $\Omega_{i0} := \mathbb{N}_0$, $\mathbb{P}_{i0} := \mathrm{Po}(\gamma_i)$, and $(\Omega_{ij}, \mathcal{F}_{ij}, \mathbb{P}_{ij}) := (\mathbb{X}, \mathcal{X}, \mathbb{Q}_i)$ for $j \geq 1$. On this space we can define independent random elements κ_i, $i \in \mathbb{N}$, and X_{ij}, $i, j \in \mathbb{N}$, such that κ_i has distribution $\mathrm{Po}(\gamma_i)$ and X_{ij} has distribution $\mathbb{Q}_i$; see Theorem B.2. The proof of Theorem 3.6 shows how to define $\kappa, X_1, X_2, \dots$ in terms of these random variables in a measurable (algorithmic) way. The details are left to the reader. □

As a consequence of Corollary 3.7, when checking a statement involving only the distribution of a Poisson process η, it is no restriction of generality to assume that η is proper. Exercise 3.8 shows that there are Poisson processes which are not proper. On the other hand, Corollary 6.3 will show that any suitably regular point process on a Borel subset of a complete separable metric space is proper.

The next result is a converse to Proposition 3.5.

Proposition 3.8 *Let η be a Poisson process on $\mathbb{X}$ with intensity measure λ satisfying $0 < \lambda(\mathbb{X}) < \infty$. Then η has the distribution of a mixed binomial* *process with mixing distribution* $\mathrm{Po}(\lambda(\mathbb{X}))$ *and sampling distribution* $\mathbb{Q} := \lambda(\mathbb{X})^{-1}\lambda$. *The conditional distribution* $\mathbb{P}(\eta \in \cdot \mid \eta(\mathbb{X}) = m)$, $m \in \mathbb{N}$, *is that of a binomial process with sample size* m *and sampling distribution* $\mathbb{Q}$.

Proof Let η' be a mixed binomial process that has mixing distribution $\mathrm{Po}(\lambda(\mathbb{X}))$ and sampling distribution $\mathbb{Q}$. Then $\eta' \overset{d}{=} \eta$ by Propositions 3.5 and 3.2. This is our first assertion. Also by definition $\mathbb{P}(\eta' \in \cdot \mid \eta'(X) = m)$ has the distribution of a binomial process with sample size m and sampling distribution $\mathbb{Q}$, and by the first assertion so does $\mathbb{P}(\eta \in \cdot \mid \eta(X) = m)$, yielding the second assertion. □

3.3 The Laplace functional of a Poisson process

The following characterization of Poisson processes is of great value for both theory and applications.

Theorem 3.9 (Laplace Functional of the Poisson Process) *Let* λ *be an s-finite measure on* $\mathbb{X}$ *and let* η *be a point process on* $\mathbb{X}$. *Then* η *is a Poisson process with intensity measure* λ *if and only if*

$$L_\eta(u) = \exp\Big[-\int (1 - e^{-u(x)})\,\lambda(dx)\Big], \quad u \in \mathbb{R}_+(\mathbb{X}). \tag{3.4}$$

Proof Assume first that η is a Poisson process with intensity measure λ. Consider the simple function $u := c_1\mathbf{1}_{B_1} + \cdots + c_m\mathbf{1}_{B_m}$, where $m \in \mathbb{N}$, $c_1, \dots, c_m \in (0, \infty)$ and $B_1, \dots, B_m \in \mathcal{X}$ are pairwise disjoint. Then

$$\mathbb{E}[\exp[-\eta(u)]] = \mathbb{E}\Big[\exp\Big(-\sum_{i=1}^m c_i\eta(B_i)\Big)\Big] = \mathbb{E}\Big[\prod_{i=1}^m \exp[-c_i\eta(B_i)]\Big].$$

Using the complete independence and the formula (1.9) for the Laplace transform of the Poisson distribution (this also holds for $\mathrm{Po}(\infty)$) yields

$$\begin{aligned} L_\eta(u) &= \prod_{i=1}^m \mathbb{E}\Big[\exp[-c_i\eta(B_i)]\Big] = \prod_{i=1}^m \exp[-\lambda(B_i)(1 - e^{-c_i})] \\ &= \exp\Big[-\sum_{i=1}^m \lambda(B_i)(1 - e^{-c_i})\Big] = \exp\Big[-\sum_{i=1}^m \int_{B_i} (1 - e^{-u})\,d\lambda\Big]. \end{aligned}$$

Since $1 - e^{-u(x)} = 0$ for $x \notin B_1 \cup \cdots \cup B_m$, this is the right-hand side of (3.4). For general $u \in \mathbb{R}_+(\mathbb{X})$, choose simple functions u_n with $u_n \uparrow u$ as $n \to \infty$. Then by the monotone convergence theorem $\eta(u_n) \uparrow \eta(u)$ as $n \to \infty$ and by dominated convergence for expectations the left-hand side of

$$\mathbb{E}[\exp[-\eta(u_n)]] = \exp\Big[-\int(1 - e^{-u_n(x)})\,\lambda(dx)\Big]$$

tends to $L_\eta(u)$. By monotone convergence again (this time for the integral with respect to λ) the right-hand side tends to the right-hand side of (3.4).

Assume now that (3.4) holds. Let η' be a Poisson process with intensity measure λ. (By Theorem 3.6, such an η' exists.) By the preceding argument $L_{\eta'}(u) = L_\eta(u)$ for all $u \in \mathbb{R}_+(\mathbb{X})$. Therefore by Proposition 2.10, $\eta \overset{d}{=} \eta'$; that is, η is a Poisson process with intensity measure λ. □

3.4 Exercises

Exercise 3.1 Let $\gamma > 0$ and $\delta > 0$. Find a random vector (X, Y) such that X, Y and $X + Y$ are Poisson distributed with parameter γ, δ and $\gamma + \delta$ respectively, but X and Y are not independent. Deduce that there exist a measure space $(\mathbb{X}, \mathcal{X}, \lambda)$ and a point process on $\mathbb{X}$ satisfying part (i) but not part (ii) of the definition of a Poisson process (Definition 3.1).

Exercise 3.2 Show that there exist a measure space $(\mathbb{X}, \mathcal{X}, \lambda)$ and a point process η on $\mathbb{X}$ satisfying (i) of Definition 3.1, and (ii) of that definition with 'independent' replaced by 'pairwise independent', but which is not a Poisson point process. In other words, show that we can have $\eta(B)$ Poisson distributed for all $B \in \mathcal{X}$, and $\eta(A)$ independent of $\eta(B)$ for all disjoint pairs $A, B \in \mathcal{X}$, but $\eta(A_1), \dots, \eta(A_k)$ not mutually independent for all disjoint $A_1, \dots, A_k \in \mathcal{X}$.

Exercise 3.3 Let η be a Poisson process on $\mathbb{X}$ with intensity measure λ and let $B \in \mathcal{X}$ with $0 < \lambda(B) < \infty$. Suppose $B_1, \dots, B_n$ are sets in $\mathcal{X}$ forming a partition of B. Show for all $k_1, \dots, k_n \in \mathbb{N}_0$ and $m := \sum_i k_i$ that

$$\mathbb{P}\big(\cap_{i=1}^n \{\eta(B_i) = k_i\} \mid \eta(B) = m\big) = \left(\frac{m!}{k_1!k_2!\cdots k_n!}\right)\prod_{i=1}^n \left(\frac{\lambda(B_i)}{\lambda(B)}\right)^{k_i}.$$

(Hint: Use Exercise 1.4.)

Exercise 3.4 Let $\mathbb{V}$ be a probability measure on $\mathbb{N}_0$ with generating function

$$G_{\mathbb{V}}(s) := \sum_{n=0}^{\infty} \mathbb{V}(\{n\})s^n, \quad s \in [0, 1].$$

Let η be a mixed binomial process with mixing distribution $\mathbb{V}$ and sampling distribution $\mathbb{Q}$. Show that

$$L_\eta(u) = G_{\mathbb{V}}\Big(\int e^{-u}\, d\mathbb{Q}\Big), \quad u \in \mathbb{R}_+(\mathbb{X}).$$

Assume now that $\mathbb{V}$ is a Poisson distribution; show that the above formula is consistent with Theorem 3.9.

Exercise 3.5 Let η be a point process on $\mathbb{X}$. Using the convention $e^{-\infty} := 0$, the Laplace functional $L_\eta(u)$ can be defined for any $u \in \overline{\mathbb{R}}_+(\mathbb{X})$. Assume now that η is a Poisson process with intensity measure λ. Use Theorem 3.9 to show that

$$\mathbb{E}\Big[\prod_{n=1}^{\kappa} u(X_n)\Big] = \exp\Big[-\int (1 - u(x))\, \lambda(dx)\Big], \tag{3.5}$$

for any measurable $u \colon \mathbb{X} \to [0, 1]$, where η is assumed to be given by (3.3).

The left-hand side of (3.5) is called the *probability generating functional* of η. It can be defined for any point process (proper or not) by taking the expectation of $\exp\big[\int \ln u(x)\, \eta(dx)\big]$.

Exercise 3.6 Let η be a Poisson process with finite intensity measure λ. Show for all $f \in \mathbb{R}_+(\mathbf{N})$ that

$$\mathbb{E}[f(\eta)] = e^{-\lambda(\mathbb{X})} f(0) + e^{-\lambda(\mathbb{X})} \sum_{n=1}^{\infty} \frac{1}{n!} \int f(\delta_{x_1} + \cdots + \delta_{x_n})\, \lambda^n(d(x_1, \ldots, x_n)).$$

Exercise 3.7 Let η be a Poisson process with s-finite intensity measure λ and let $f \in \mathbb{R}_+(\mathbf{N})$ such that $\mathbb{E}[f(\eta)] < \infty$. Suppose that η' is a Poisson process with intensity measure λ' such that $\lambda = \lambda' + \nu$ for some finite measure ν. Show that $\mathbb{E}[f(\eta')] < \infty$. (Hint: Use the superposition theorem.)

Exercise 3.8 In the setting of Exercise 2.5, show that there is a probability measure λ on $(\mathbb{X}, \mathcal{X})$ and a Poisson process η with intensity measure λ such that η is not proper. (Hint: The measure λ can be taken as the solution to Exercise 2.5. The point process η can be defined as a random multiple of λ.)

Exercise 3.9 Let $0 < \gamma < \gamma'$. Give an example of two Poisson processes η, η' on $(0, 1)$ with intensity measures $\gamma\lambda_1$ and $\gamma'\lambda_1$ respectively (λ_1 denoting Lebesgue measure), such that $\eta \le \eta'$ but $\eta' - \eta$ is *not* a Poisson process. (Hint: Use Exercise 1.9.)

4

The Mecke equation and factorial measures

The Mecke equation provides a way to compute the expectation of integrals, i.e. sums with respect to a Poisson process, where the integrand can depend on both the point process and the point in the state space. This functional equation characterises a Poisson process. The Mecke identity can be extended to integration with respect to factorial measures, i.e. to multiple sums. Factorial measures can also be used to define the Janossy measures, thus providing a local description of a general point process. The factorial moment measures of a point process are defined as the expected factorial measures. They describe the probability of the occurrence of points in a finite number of infinitesimally small sets.

4.1 The Mecke equation

In this chapter we let $(\mathbb{X}, \mathcal{X})$ be an arbitrary measurable space. The following characterizing equation (4.1) is a fundamental tool for analyzing the Poisson process and can also be used in many specific calculations. In the special case where $\mathbb{X}$ has just a single element, Theorem 4.1 reduces to an earlier result about the Poisson distribution, namely Proposition 1.1.

Theorem 4.1 (Mecke Equation) *Let λ be an s-finite measure on $\mathbb{X}$ and η a point process on $\mathbb{X}$. Then η is a Poisson process with intensity measure λ if and only if*

$$\mathbb{E}\Big[\int f(x,\eta)\,\eta(dx)\Big] = \int \mathbb{E}[f(x,\eta+\delta_x)]\,\lambda(dx) \tag{4.1}$$

for all $f \in \mathbb{R}_+(\mathbb{X}\times\mathbf{N})$. In this case (4.1) *also holds for $f \in \mathbb{R}(\mathbb{X}\times\mathbf{N})$, provided that*

$$\int \mathbb{E}[|f(x,\eta+\delta_x)|]\,\lambda(dx) < \infty. \tag{4.2}$$

Proof Let us start by noting that the mapping $(x, \mu) \mapsto \mu + \delta_x$ (adding a point x to the counting measure μ) from $\mathbb{X} \times \mathbf{N}$ to $\mathbf{N}$ is measurable. Indeed, the mapping $(x, \mu) \mapsto \mu(B) + \mathbf{1}_B(x)$ is measurable for all $B \in \mathcal{X}$.

If η is a Poisson process, then (4.1) is a special case of (4.11) to be proved below.

Assume now that (4.1) holds for all measurable $f \geq 0$. Let $B_1, \dots, B_m$ be disjoint sets in $\mathcal{X}$ with $\lambda(B_i) < \infty$ for each i. For $k_1, \dots, k_m \in \mathbb{N}_0$ with $k_1 \geq 1$ we define

$$f(x, \mu) = \mathbf{1}_{B_1}(x) \prod_{i=1}^{m} \mathbf{1}\{\mu(B_i) = k_i\}, \quad (x, \mu) \in \mathbb{X} \times \mathbf{N}.$$

Then

$$\mathbb{E}\left[\int f(x, \eta)\, \eta(dx)\right] = \mathbb{E}\left[\eta(B_1) \prod_{i=1}^{m} \mathbf{1}\{\eta(B_i) = k_i\}\right] = k_1 \mathbb{P}\left(\cap_{i=1}^{m} \{\eta(B_i) = k_i\}\right),$$

while for $x \in \mathbb{X}$,

$$\mathbb{E}[f(x, \eta + \delta_x)] = \mathbf{1}_{B_1}(x)\, \mathbb{P}(\eta(B_1) = k_1 - 1, \eta(B_2) = k_2, \dots, \eta(B_m) = k_m)$$

so that by (4.1),

$$k_1\, \mathbb{P}\left(\cap_{i=1}^{m} \{\eta(B_i) = k_i\}\right) = \lambda(B_1)\, \mathbb{P}\left(\{\eta(B_1) = k_1 - 1\} \cap \cap_{i=2}^{m} \{\eta(B_i) = k_i\}\right).$$

Assume that $\mathbb{P}(\cap_{i=2}^{m} \{\eta(B_i) = k_i\}) > 0$ and note that otherwise $\eta(B_1)$ and the event $\cap_{i=2}^{m} \{\eta(B_i) = k_i\}$ are independent. Putting

$$\pi_k = \mathbb{P}(\eta(B_1) = k \mid \cap_{i=2}^{m} \{\eta(B_i) = k_i\}),$$

we have

$$\pi_k = \lambda(B_1) \pi_{k-1} / k, \quad k \in \mathbb{N}.$$

The only probability distribution satisfying this recursion is given by $\pi_k = \mathrm{Po}(\lambda(B_1); k)$, regardless of $k_2, \dots, k_m$; hence $\eta(B_1)$ is $\mathrm{Po}(\lambda(B_1))$ distributed, and independent of $\cap_{i=2}^{m} \{\eta(B_i) = k_i\}$. Hence by an induction on m, the variables $\eta(B_1), \dots, \eta(B_m)$ are independent.

For general $B \in \mathcal{X}$ we still get for all $k \in \mathbb{N}$ that

$$k\, \mathbb{P}(\eta(B) = k) = \lambda(B)\, \mathbb{P}(\eta(B) = k - 1),$$

with the (measure theory) convention $\infty \cdot 0 := 0$. Assuming $\lambda(B) = \infty$ we obtain $\mathbb{P}(\eta(B) = k - 1) = 0$ and hence $\mathbb{P}(\eta(B) = \infty) = 1$.

It follows that η has the defining properties of the Poisson process. □

4.2 Factorial measures and the multivariate Mecke equation

Equation (4.1) admits a useful generalization involving multiple integration. To formulate this version we consider, for $m \in \mathbb{N}$, the m-th power $(\mathbb{X}^m, \mathcal{X}^m)$ of $(\mathbb{X}, \mathcal{X})$; see Appendix A.1. Suppose $\mu \in \mathbf{N}$ is given by

$$\mu = \sum_{j=1}^{k} \delta_{x_j} \tag{4.3}$$

for some $k \in \overline{\mathbb{N}}_0$ and some $x_1, x_2, \ldots \in \mathbb{X}$ (not necessarily distinct) as in (2.1). Then we define another measure $\mu^{(m)} \in \mathbf{N}(\mathbb{X}^m)$ by

$$\mu^{(m)}(C) = \sum_{i_1,\ldots,i_m \le k}^{\neq} \mathbf{1}\{(x_{i_1}, \ldots, x_{i_m}) \in C\}, \quad C \in \mathcal{X}^m, \tag{4.4}$$

where the superscript $\neq$ indicates summation over m-tuples with pairwise different entries and where an empty sum is defined as zero. (In the case $k = \infty$ this involves only integer-valued indices.) In other words this means that

$$\mu^{(m)} = \sum_{i_1,\ldots,i_m \le k}^{\neq} \delta_{(x_{i_1},\ldots,x_{i_m})}. \tag{4.5}$$

To aid understanding, it is helpful to consider in (4.4) a set C of the special product form $B_1 \times \cdots \times B_m$. If these sets are pairwise disjoint, then the right-hand side of (4.4) factorizes, yielding

$$\mu^{(m)}(B_1 \times \cdots \times B_m) = \prod_{j=1}^{m} \mu(B_j). \tag{4.6}$$

If, on the other hand, $B_j = B$ for all $j \in \{1, \ldots, m\}$ then, clearly,

$$\mu^{(m)}(B^m) = \mu(B)(\mu(B) - 1) \cdots (\mu(B) - m + 1) = (\mu(B))_m. \tag{4.7}$$

Therefore $\mu^{(m)}$ is called m-th *factorial measure* of μ. For $m = 2$ and arbitrary $B_1, B_2 \in \mathcal{X}$ we obtain from (4.6) and (4.8) (or a direct counting argument) that

$$\mu^{(2)}(B_1 \times B_2) = \mu(B_1)\mu(B_2) - \mu(B_1 \cap B_2), \tag{4.8}$$

provided that $\mu(B_1 \cap B_2) < \infty$. Otherwise $\mu^{(2)}(B_1 \times B_2) = \infty$.

Factorial measures satisfy the following useful recursion:

Lemma 4.2 *Let $\mu \in \mathbf{N}$ be given by* (4.3) *and define $\mu^{(1)} := \mu$. Then, for* *all* $m \in \mathbb{N}$,

$$\mu^{(m+1)} = \int \Big[\int \mathbf{1}\{(x_1, \dots, x_{m+1}) \in \cdot\} \mu(dx_{m+1}) - \sum_{j=1}^{m} \mathbf{1}\{(x_1, \dots, x_m, x_j) \in \cdot\} \Big] \mu^{(m)}(d(x_1, \dots, x_m)). \tag{4.9}$$

Proof Let $m \in \mathbb{N}$ and $C \in \mathcal{X}^m$. Then

$$\mu^{(m+1)}(C) = \sum_{i_1,\dots,i_m \le k}^{\neq} \sum_{\substack{j=1 \\ j \notin \{i_1,\dots,i_m\}}}^{k} \mathbf{1}\{(x_{i_1}, \dots, x_{i_m}, x_j) \in C\}$$

Here the inner sum equals

$$\sum_{j=1}^{k} \mathbf{1}\{(x_{i_1}, \dots, x_{i_m}, x_j) \in C\} - \sum_{l=1}^{m} \mathbf{1}\{(x_{i_1}, \dots, x_{i_m}, x_{i_l}) \in C\},$$

where the latter difference is either a non-negative integer (if the first sum is finite) or ∞ (if the first sum is infinite). This proves the result. □

For a general space $(\mathbb{X}, \mathcal{X})$ there is no guarantee that a measure $\mu \in \mathbf{N}$ can be represented as in (4.3); see Exercise 2.5. Equation (4.9) suggests a recursive definition of the factorial measures of a general $\mu \in \mathbf{N}$, without using a representation as a sum of Dirac measures. The next proposition confirms this idea.

Proposition 4.3 *For any* $\mu \in \mathbf{N}$ *there is a unique sequence* $\mu^{(m)} \in \mathbf{N}(\mathbb{X}^m)$, $m \in \mathbb{N}$, *satisfying* $\mu^{(1)} := \mu$ *and the recursion* (4.9). *The mappings* $\mu \mapsto \mu^{(m)}$ *are measurable.*

The proof of Proposition 4.3 is given in the Appendix (see Proposition A.16) and can be skipped without too much loss. It is enough to remember that $\mu^{(m)}$ can be defined by (4.4), whenever μ is given by (4.3). This follows from Lemma 4.2 and the fact that the solution of (4.9) must be unique. It follows by induction that (4.6) and (4.7) remain valid for general $\mu \in \mathbf{N}$; see Exercise 4.3.

Let η be a point process on $\mathbb{X}$ and let $m \in \mathbb{N}$. Proposition 4.3 shows that $\eta^{(m)}$ is a point process on $\mathbb{X}^m$. If η is proper and given as at (2.4), then

$$\eta^{(m)} = \sum_{i_1,\dots,i_m \in \{1,\dots,\kappa\}}^{\neq} \delta_{(X_{i_1},\dots,X_{i_m})}. \tag{4.10}$$

We continue with the multivariate version of the Mecke equation (4.1).

Theorem 4.4 (Multivariate Mecke Equation) *Let η be a Poisson process on $\mathbb{X}$ with s-finite intensity measure λ and let $m \in \mathbb{N}$. Then, for any $f \in \overline{\mathbb{R}}_+(\mathbb{X}^m \times \mathbf{N})$,*

$$\mathbb{E}\Big[\int f(x_1,\dots,x_m,\eta)\,\eta^{(m)}(d(x_1,\dots,x_m))\Big] \\ = \int \mathbb{E}[f(x_1,\dots,x_m,\eta+\delta_{x_1}+\cdots+\delta_{x_m})]\,\lambda^m(d(x_1,\dots,x_m)). \quad (4.11)$$

This formula remains true for $f \in \mathbb{R}(\mathbb{X}^m \times \mathbf{N})$, provided that

$$\int \mathbb{E}[|f(x_1,\dots,x_m,\eta+\delta_{x_1}+\cdots+\delta_{x_m})|]\,\lambda^m(d(x_1,\dots,x_m))\Big] < \infty. \quad (4.12)$$

Proof By Proposition 4.3, the map $\mu \mapsto \mu^{(m)}$ is measurable, so that (4.11) involves only the distribution of η. By Corollary 3.7 we can hence assume that η is proper and given by (2.4). Let us first assume that $\lambda(\mathbb{X}) < \infty$. Then $\lambda = \gamma\,\mathbb{Q}$ for some $\gamma \ge 0$ and some probability measure $\mathbb{Q}$ on $\mathbb{X}$. By Proposition 3.5, we can then assume that η is a mixed binomial process as in Definition 3.4, with κ having the $\mathrm{Po}(\gamma)$ distribution. Let $f \in \overline{\mathbb{R}}_+(\mathbb{X}^m \times \mathbf{N})$. Then we obtain from (4.10) and (2.2) that the left-hand side of (4.11) equals

$$e^{-\gamma}\sum_{k=m}^{\infty}\frac{\gamma^k}{k!}\mathbb{E}\Big[\sum_{i_1,\dots,i_m\in\{1,\dots,k\}}^{\neq} f(X_{i_1},\dots,X_{i_m},\delta_{X_1}+\cdots+\delta_{X_k})\Big] \\ =e^{-\gamma}\sum_{k=m}^{\infty}\frac{\gamma^k}{k!}\sum_{i_1,\dots,i_m\in\{1,\dots,k\}}^{\neq}\mathbb{E}[f(X_{i_1},\dots,X_{i_m},\delta_{X_1}+\cdots+\delta_{X_k})],$$

where we have first used independence of κ and (X_n) and then the fact that we can perform integration and summation in any order we want (since $f \ge 0$). Let us denote by $\mathbf{y} = (y_1,\dots,y_m)$ a generic element of $\mathbb{X}^m$. Since the X_i are independent with distribution $\mathbb{Q}$, the above equals

$$e^{-\gamma}\sum_{k=m}^{\infty}\frac{\gamma^k (k)_m}{k!}\mathbb{E}\Big[\int f\Big(\mathbf{y},\sum_{i=1}^{k-m}\delta_{X_i}+\sum_{j=1}^{m}\delta_{y_j}\Big)\,\mathbb{Q}^m(d\mathbf{y})\Big] \\ =e^{-\gamma}\gamma^m\sum_{k=m}^{\infty}\frac{\gamma^{k-m}}{(k-m)!}\int \mathbb{E}\Big[f\Big(\mathbf{y},\sum_{i=1}^{k-m}\delta_{X_i}+\sum_{j=1}^{m}\delta_{y_j}\Big)\Big]\,\mathbb{Q}^m(d\mathbf{y}) \\ =\int \mathbb{E}[f(y_1,\dots,y_m,\eta+\delta_{y_1}+\cdots+\delta_{y_m})]\,\lambda^m(d(y_1,\dots,y_m)),$$

where we have again used the mixed binomial representation. This proves (4.11) for finite λ.

Now suppose $\lambda(\mathbb{X}) = \infty$. As in the proof of Theorem 3.6 we can then assume that $\eta = \sum_i \eta_i$, where η_i are independent Poisson processes with intensity measures λ_i each having finite total measure. By the grouping property of independence the point processes

$$\xi_i := \sum_{j \le i} \eta_j, \quad \chi_i := \sum_{j \ge i+1} \eta_j,$$

are independent for each $i \in \mathbb{N}$. By Proposition A.16 we have $\xi_i^{(m)} \uparrow \eta^{(m)}$ as $i \to \infty$. Hence we can apply monotone convergence (Theorem A.11) to see that the left-hand side of (4.11) is given by

$$\lim_{i\to\infty} \mathbb{E}\Big[\int f(x_1, \dots, x_m, \xi_i + \chi_i)\, \xi_i^{(m)}(d(x_1, \dots, x_m))\Big]$$
$$= \lim_{i\to\infty} \mathbb{E}\Big[\int f_i(x_1, \dots, x_m, \xi_i)\, \xi_i^{(m)}(d(x_1, \dots, x_m))\Big],$$

where $f_i(x_1, \dots, x_m, \mu) := \mathbb{E}[f(x_1, \dots, x_m, \mu + \chi_i)]$, $(x_1, \dots, x_m, \mu) \in \mathbb{X}^m \times \mathbf{N}$. Setting $\lambda'_i := \sum_{j=1}^{i} \lambda_j$, we can now apply the previous result to obtain from Fubini's theorem that the above equals

$$\lim_{i\to\infty} \int \mathbb{E}[f_i(x_1, \dots, x_m, \xi_i + \delta_{x_1} + \dots + \delta_{x_m})]\, (\lambda'_i)^m(d(x_1, \dots, x_m))$$
$$= \lim_{i\to\infty} \int \mathbb{E}[f(x_1, \dots, x_m, \eta + \delta_{x_1} + \dots + \delta_{x_m})]\, (\lambda'_i)^m(d(x_1, \dots, x_m)).$$

By monotone convergence (Theorem A.11) this is the right-hand side of (4.11). For functions f that may take negative values but satisfy the integrability assumption (4.12), formula (4.11) can be derived by taking positive and negative parts of f. □

Next we formulate another useful version of the multivariate Mecke equation. For $\mu \in \mathbf{N}$ and $x \in \mathbb{X}$ we define the measure $\mu \setminus \delta_x \in \mathbf{N}$ by

$$\mu \setminus \delta_x := \begin{cases} \mu - \delta_x, & \text{if } \mu \ge \delta_x, \\ \mu, & \text{otherwise.} \end{cases} \tag{4.13}$$

For $x_1, \dots, x_m \in \mathbb{X}$, the measure $\mu \setminus \delta_{x_1} \setminus \dots \setminus \delta_{x_m} \in \mathbf{N}$ is defined inductively.

Theorem 4.5 (Multivariate Mecke Equation) *Let η be a proper Poisson process on $\mathbb{X}$ with s-finite intensity measure λ and let $m \in \mathbb{N}$. Then, for any $f \in \mathbb{R}_+(\mathbb{X}^m \times \mathbf{N})$,*

$$\mathbb{E}\Big[\int f(x_1, \dots, x_m, \eta \setminus \delta_{x_1} \setminus \dots \setminus \delta_{x_m})\, \eta^{(m)}(d(x_1, \dots, x_m))\Big]$$
$$= \int \mathbb{E}[f(x_1, \dots, x_m, \eta)]\, \lambda^m(d(x_1, \dots, x_m)). \tag{4.14}$$

Proof If $\mathbb{X}$ is a subspace of a complete separable metric space as in Proposition 6.2, then it is easy to show that $(x_1, \dots, x_m, \mu) \mapsto \mu \setminus \delta_{x_1} \setminus \cdots \setminus \delta_{x_m}$ is a measurable mapping from $\mathbb{X}^m \times \mathbf{N}_l(\mathbb{X})$ to $\mathbf{N}_l(\mathbb{X})$. In that case, and if λ is locally finite, (4.14) follows upon applying (4.11) to the function $(x_1, \dots, x_m, \mu) \mapsto f(x_1, \dots, x_m, \mu \setminus \delta_{x_1} \setminus \cdots \setminus \delta_{x_m})$. In the general case we use that η is proper. Therefore the mapping $(\omega, x_1, \dots, x_m) \mapsto \eta(\omega) \setminus \delta_{x_1} \setminus \cdots \setminus \delta_{x_m}$ is measurable, which is enough to make (4.14) a meaningful statement. The proof can proceed in exactly the same way as the proof of Theorem 4.4. □

4.3 Janossy measures

Factorial measures can be used to describe the restriction of point processes as follows.

Definition 4.6 *Let η be a point process on $\mathbb{X}$, let $B \in \mathcal{X}$ and $m \in \mathbb{N}$. The* Janossy measure of order *m of η restricted to B is the measure on $\mathbb{X}^m$ defined by*

$$J_{\eta,B,m} := \frac{1}{m!} \mathbb{E}[\mathbf{1}\{\eta(B) = m\} \eta_B^{(m)}(\cdot)]. \tag{4.15}$$

The number $J_{\eta,B,0} := \mathbb{P}(\eta(B) = 0)$ is called the Janossy measure of order 0.

Note that the Janossy measures $J_{\eta,B,m}$ are symmetric (see (A.17))) and

$$J_{\eta,B,m}(\mathbb{X}^m) = \mathbb{P}(\eta(B) = m), \quad m \in \mathbb{N}. \tag{4.16}$$

If $\mathbb{P}(\eta(B) < \infty) = 1$, then the Janossy measures determine the distribution of the restriction η_B of η to B:

Theorem 4.7 *Let η and η' be point processes on $\mathbb{X}$. Let $B \in \mathcal{X}$ and assume that $J_{\eta,B,m} = J_{\eta',B,m}$ for each $m \in \mathbb{N}_0$. Then*

$$\mathbb{P}(\eta(B) < \infty, \eta_B \in \cdot) = \mathbb{P}(\eta'(B) < \infty, \eta'_B \in \cdot).$$

Proof For notational convenience we may assume that $B = \mathbb{X}$. Let $m \in \mathbb{N}$ and suppose that $\mu \in \mathbf{N}$ satisfies $\mu(\mathbb{X}) = m$. We assert for each $A \in \mathcal{N}$ that

$$\mathbf{1}\{\mu \in A\} = \frac{1}{m!} \int \mathbf{1}\{\delta_{x_1} + \cdots + \delta_{x_m} \in A\} \mu^{(m)}(d(x_1, \dots, x_m)). \tag{4.17}$$

Since both sides of (4.17) are finite measures in A it suffices to prove this identity for each set A of the form

$$A = \{\mu \in \mathbf{N} : \mu(B_1) = i_1, \dots, \mu(B_n) = i_n\},$$

where $n \in \mathbb{N}$, $B_1, \dots, B_n \in \mathcal{X}$ and $i_1, \dots, i_n \in \mathbb{N}_0$. Given such a set, let ν

be defined as in Lemma A.17. Then $\mu \in A$ if and only if $\nu \in A$ and the right-hand side of (4.17) does not change upon replacing μ by ν. Hence it suffices to check (4.17) for finite sums of Dirac measures. This is obvious from (4.4).

From (4.17) we obtain for each $A \in \mathcal{N}$ that

$$\begin{aligned}
&\mathbb{P}(\eta(\mathbb{X}) < \infty, \eta \in A)\\
&\quad = J_{\eta,\mathbb{X},0}\mathbf{1}\{0 \in A\} + \sum_{m=1}^{\infty} \int \mathbf{1}\{\delta_{x_1} + \cdots + \delta_{x_m} \in A\}\, J_{\eta,\mathbb{X},m}(d(x_1, \dots, x_m))
\end{aligned}$$

and hence the assertion. □

4.4 Factorial moment measures

Definition 4.8 For $m \in \mathbb{N}$ the m-th *factorial moment measure* of a point process η is the measure α_m on $\mathbb{X}^m$ defined by

$$\alpha_m(C) := \mathbb{E}[\eta^{(m)}(C)], \quad C \in \mathcal{X}^m.$$

If the point process η is proper, i.e. given by (2.4), then

$$\alpha_m(C) = \mathbb{E}\Big[\sum\nolimits^{\neq}_{i_1,\dots,i_m \le \kappa} \mathbf{1}\{(X_{i_1}, \dots, X_{i_m}) \in C\}\Big] \tag{4.18}$$

and hence for $f \in \mathbb{R}_+(\mathbb{X}^m)$

$$\int_{X^m} f(x_1, \dots, x_m)\, \alpha_m(d(x_1, \dots, x_m)) = \mathbb{E}\Big[\sum\nolimits^{\neq}_{i_1,\dots,i_m \le \kappa} f(X_{i_1}, \dots, X_{i_m})\Big].$$

The first factorial moment measure of a point process η is just the intensity measure of Definition 2.5, while the second describes the second order properties of η. For instance it follows from (4.8) (and Exercise 4.3 if η is not proper) that

$$\alpha_2(B_1 \times B_2) = \mathbb{E}[\eta(B_1)\eta(B_2)] - \mathbb{E}[\eta(B_1 \cap B_2)], \tag{4.19}$$

provided that $\mathbb{E}[\eta(B_1 \cap B_2)] < \infty$.

Theorem 4.4 has the following immediate consequence:

Corollary 4.9 *Given $m \in \mathbb{N}$ the m-th factorial moment measure of a Poisson process with s-finite intensity measure λ is λ^m.*

Proof Apply (4.11) to the function $f(x_1, \dots, x_m, \eta) = \mathbf{1}\{(x_1, \dots, x_m) \in C\}$ for $C \in \mathcal{X}^m$. □

If η is a Poisson process with intensity measure λ, then by Corollary 4.9 and (4.19), for all $B_1, B_2 \in \mathcal{X}$ with $\lambda(B_1) < \infty$ and $\lambda(B_2) < \infty$ the covariance between $\eta(B_1)$ and $\eta(B_2)$ is given by

$$\mathbb{C}\mathrm{ov}[\eta(B_1), \eta(B_2)] = \lambda(B_1 \cap B_2). \tag{4.20}$$

Under certain assumptions the factorial moment measures of a point process determine its distribution. To derive this result we need the following lemma. We use the conventions $e^{-\infty} := 0$ and $\log 0 := -\infty$.

Lemma 4.10 *Let η be a point process on $\mathbb{X}$. Let $B \in \mathcal{X}$ and assume that there exists $c \geq 1$ such that the factorial moment measures α_n of η satisfy*

$$\alpha_n(B^n) \leq n!c^n, \quad n \geq 1. \tag{4.21}$$

Let $u \in \mathbb{R}_+(\mathbb{X})$ and $a < c^{-1}$ be such that $u(x) < a$ for $x \in B$ and $u(x) = 0$ for $x \notin B$. Then

$$\mathbb{E}\Big[\exp\Big(\int \log(1 - u(x))\,\eta(dx)\Big)\Big]$$
$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int u(x_1)\cdots u(x_n)\,\alpha_n(d(x_1, \ldots, x_n)). \tag{4.22}$$

Proof Since u vanishes outside B, we have

$$P := \exp\Big(\int \log(1 - u(x))\,\eta(dx)\Big) = \exp\Big(\int \log(1 - u(x))\,\eta_B(dx)\Big).$$

Hence we can assume that $\eta(\mathbb{X} \setminus B) = 0$. Since $\alpha_1(B) = \mathbb{E}[\eta(B)] < \infty$, we can also assume that $\eta(B) < \infty$. But then we obtain from Exercise 4.5 that

$$P = \sum_{n=0}^{\infty} (-1)^n P_n,$$

where $P_0 := 1$ and

$$P_n := \frac{1}{n!} \int u(x_1)\cdots u(x_n)\,\eta^{(n)}(d(x_1, \ldots, x_n)),$$

and where we note that $\eta^{(n)} = 0$ if $n > \eta(\mathbb{X})$; see (4.7). Exercise 4.8 asks the reader to prove that

$$\sum_{n=0}^{2m-1} (-1)^n P_n \leq P \leq \sum_{n=0}^{2m} (-1)^n P_n, \quad m \geq 1. \tag{4.23}$$

These inequalities show that

$$\Big|P - \sum_{n=0}^{k}(-1)^n P_n\Big| \le P_k, \quad k \ge 1.$$

It follows that

$$\Big|\mathbb{E}[P] - \mathbb{E}\Big[\sum_{n=0}^{k}(-1)^n P_n\Big]\Big| \le \mathbb{E}[P_k] = \frac{1}{k!}\int u(x_1)\cdots u(x_k)\,\alpha_k(d(x_1,\dots,x_k)),$$

where we have used the definition of the factorial moment measures. The last term can be bounded by

$$\frac{a^k}{k!}\alpha_k(B^k) \le a^k c^k \to 0$$

as $k \to \infty$. This finishes the proof. □

Proposition 4.11 *Let η and η' be two point processes on $\mathbb{X}$ with the same factorial moment measures α_n, $n \ge 1$. Assume that there is an increasing sequence $B_k \in \mathcal{X}$, $k \in \mathbb{N}$, with union $\mathbb{X}$ and a sequence $c_k > 0$, $k \in \mathbb{N}$, such that*

$$\alpha_n(B_k^n) \le n!c_k^n, \quad k, n \in \mathbb{N}. \tag{4.24}$$

Then $\eta \overset{d}{=} \eta'$.

Proof By Proposition 2.10 and monotone convergence it suffices to prove that $L_\eta(\upsilon) = L_{\eta'}(\upsilon)$ for each $\upsilon \in \mathbb{R}_+(\mathbb{X})$ such that $\upsilon(x) = 0$ for all $x \notin B$, for some $B \in \{B_k : k \in \mathbb{N}\}$. This puts us into the setting of Lemma 4.10. Let $\upsilon \in \mathbb{R}_+(\mathbb{X})$ have the upper bound $a > 0$. For each $t \in [0, -a^{-1}\log(1 - c^{-1}))$ we can apply the lemma with $u := 1 - e^{-t\upsilon}$, where we can assume that $c > 1$. This gives us $L_\eta(t\upsilon) = L_{\eta'}(t\upsilon)$. Since $t \mapsto L_\eta(t\upsilon)$ is analytic on $(0, \infty)$, we obtain $L_\eta(t\upsilon) = L_{\eta'}(t\upsilon)$ for all $t \ge 0$ and, in particular, $L_\eta(\upsilon) = L_{\eta'}(\upsilon)$. □

4.5 Exercises

Exercise 4.1 Let $\mu \in \mathbf{N}$ be given by (4.3) and let $m \in \mathbb{N}$. Show that

$$\mu^{(m)}(C) = \int\cdots\int \mathbf{1}_C(x_1,\dots,x_m)\Big(\mu - \sum_{j=1}^{m-1}\delta_{x_j}\Big)(dx_m)\Big(\mu - \sum_{j=1}^{m-2}\delta_{x_j}\Big)(dx_{m-1}) \cdots(\mu - \delta_{x_1})(dx_2)\,\mu(dx_1), \quad C \in \mathcal{X}^m. \tag{4.25}$$

This formula involves integrals with respect to *signed measures* of the form $\mu - \nu$, where $\mu, \nu \in \mathbf{N}$ and ν is finite. These integrals are defined as a difference of integrals in the natural way.

Exercise 4.2 Let $\mu \in \mathbf{N}$ and $x \in \mathbb{X}$. Show for all $m \in \mathbb{N}$ with $m \geq 2$ and all $B \in \mathcal{X}^m$ that

$$\begin{aligned}(\mu + \delta_x)^{(m)}(B) \\ = \mu^{(m)}(B) + \int [\mathbf{1}\{(x, x_1, \dots, x_{m-1}) \in B\} + \cdots + \mathbf{1}\{(x_1, \dots, x_{m-1}, x) \in B\}] \\ \times \mu^{(m-1)}(d(x_1, \dots, x_{m-1})).\end{aligned}$$

(Hint: The above right-hand side defines a measure μ_m. Let $\mu_1 := \mu + \delta_x$ and show by induction that the sequence (μ_m) satisfies the recursion (4.9).)

Exercise 4.3 Let $\mu \in \mathbf{N}$. Use the recursion (4.9) to show that (4.6), (4.7), and (4.8) hold.

Exercise 4.4 Let $\mu \in \mathbf{N}$ be given by $\mu := \sum_{j=1}^{k} \delta_{x_j}$ for some $k \in \mathbb{N}_0$ and some $x_1, \dots, x_k \in \mathbb{X}$. Let $u \colon \mathbb{X} \to \mathbb{R}$ be measurable. Show that

$$\prod_{j=1}^{k}(1 - u(x_j)) = 1 + \sum_{n=1}^{k} \frac{(-1)^n}{n!} \int u(x_1) \cdots u(x_n)\, \mu^{(n)}(d(x_1, \dots, x_n)).$$

Exercise 4.5 Let $\mu \in \mathbf{N}$ such that $\mu(\mathbb{X}) < \infty$ and let $u \in \mathbb{R}_+(\mathbb{X})$. Show that

$$\begin{aligned}\exp\Big(\int \log(1 - u(x))\, \mu(dx)\Big) \\ = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int \prod_{j=1}^{n} u(x_j)\, \mu^{(n)}(d(x_1, \dots, x_n)).\end{aligned}$$

(Hint: If u takes only a finite number of values, then the result follows from Lemma A.17 and Exercise 4.4. The general case follows from monotone convergence.)

Exercise 4.6 (Converse to Theorem 4.4) Let $m \in \mathbb{N}$ with $m > 1$. Prove or disprove that for any σ-finite measure space $(\mathbb{X}, \mathcal{X}, \lambda)$, if η is a point process on $\mathbb{X}$ satisfying (4.11) for all $f \in \mathbb{R}_+(\mathbb{X}^m \times \mathbf{N})$, then η is a Poisson process with intensity measure λ. (For $m = 1$, this statement is true by Theorem 4.1.)

Exercise 4.7 Give an alternative (inductive) proof of the multivariate Mecke identity (4.11) using the univariate version (4.1) and the recursion (4.9).

Exercise 4.8 Prove the inequalities (4.23). (Hint: Use induction.)

Exercise 4.9 Let η be a Poisson process on $\mathbb{X}$ with intensity measure λ and let $B \in \mathcal{X}$ with $0 < \lambda(B) < \infty$. Let $U_1, \dots, U_n$ be independent random elements of $\mathbb{X}$ with distribution $\lambda(B)^{-1}\lambda(B \cap \cdot)$ and assume that $(U_1, \dots, U_n)$ and η are independent. Show that the distribution of $\eta + \delta_{U_1} + \cdots + \delta_{U_n}$ is absolutely continuous with respect to $\mathbb{P}(\eta \in \cdot)$ and that $\mu \mapsto \lambda(B)^{-n}\mu^{(n)}(B)$ is a version of the density.

5

Mappings, markings and thinnings

It has been in shown in Chapter 3 that an independent superposition of Poisson processes is again Poisson. The properties of a Poisson process are preserved under other operations. A mapping from the state space to another space induces a Poisson process on the new state space. A more intriguing persistence property is the Poisson nature of position dependent markings and thinnings of a Poisson process.

5.1 Mappings and restrictions

Consider two measurable spaces $(\mathbb{X}, \mathcal{X})$ and $(\mathbb{Y}, \mathcal{Y})$ along with a measurable mapping $T : \mathbb{X} \to \mathbb{Y}$. For any measure μ on $(\mathbb{X}, \mathcal{X})$ we define the *image* of μ under T (also known as the *push-forward* of μ), to be the measure $T(\mu)$ defined by $T(\mu) = \mu \circ T^{-1}$, i.e.

$$T(\mu)(C) := \mu(T^{-1}C), \quad C \in \mathcal{Y}. \tag{5.1}$$

In particular, if η is a point process on $\mathbb{X}$, then for any $\omega \in \Omega$, $T(\eta(\omega))$ is a measure on $\mathbb{Y}$ given by

$$T(\eta(\omega))(C) = \eta(\omega, T^{-1}(C)), \quad C \in \mathcal{Y}. \tag{5.2}$$

If η is a proper point process, i.e. one given by $\eta = \sum_{n=1}^{\kappa} \delta_{X_n}$ as in (2.4), the definition of $T(\eta)$ implies that

$$T(\eta) = \sum_{n=1}^{\kappa} \delta_{T(X_n)}. \tag{5.3}$$

Theorem 5.1 (Mapping Theorem) *Let η be a point process on $\mathbb{X}$ with intensity measure λ and let $T : \mathbb{X} \to \mathbb{Y}$ be measurable. Then $T(\eta)$ is a point process with intensity measure $T(\lambda)$. If η is a Poisson process, then $T(\eta)$ is a Poisson process too.*

Proof We first note that $T(\mu) \in \mathbf{N}$ for any $\mu \in \mathbf{N}$. Indeed, if $\mu = \sum_{j=1}^{\infty} \mu_j$, then $T(\mu) = \sum_{j=1}^{\infty} T(\mu_j)$. Moreover, if the μ_j are $\mathbb{N}_0$-valued, so are the $T(\mu_j)$.

For any $C \in \mathcal{Y}$, $T(\eta)(C)$ is a random variable and by the definition of the intensity measure its expectation is

$$\mathbb{E}[T(\eta)(C)] = \mathbb{E}[\eta(T^{-1}C)] = \lambda(T^{-1}C) = T(\lambda)(C). \tag{5.4}$$

If η is a Poisson process, then it can be checked directly that $T(\eta)$ is completely independent (property (ii) of Definition 3.1), and that $T(\eta)(C)$ has a Poisson distribution with parameter $T(\lambda)(C)$ (property (i) of Definition 3.1). □

If η is a Poisson process on $\mathbb{X}$ then we may discard all of its points outside a set $B \in \mathcal{X}$ to obtain another Poisson process. The *restriction* ν_B of a measure ν on $\mathbb{X}$ to a set $B \in \mathcal{X}$ is defined by

$$\nu_B(B') := \nu(B \cap B'), \quad B' \in \mathcal{X}. \tag{5.5}$$

If η is a point process on $\mathbb{X}$, then so is its restriction η_B.

Theorem 5.2 (Restriction Theorem) *Let η be a Poisson process on $\mathbb{X}$ with s-finite intensity measure λ and let $C_1, C_2, \ldots \in \mathcal{X}$ be pairwise disjoint. Then $\eta_{C_1}, \eta_{C_2}, \ldots$ are independent Poisson processes with intensity measures $\lambda_{C_1}, \lambda_{C_2}, \ldots$ respectively.*

Proof As in the proof of Proposition 3.5, it is no restriction of generality to assume that the union of the sets C_i is all of $\mathbb{X}$. (If not, add the complement of this union to the sequence (C_i).) First note that for each $i \in \mathbb{N}$, η_{C_i} has intensity measure λ_{C_i} and satisfies the two defining properties of a Poisson process. By the existence theorem (Theorem 3.6) we can find a sequence η_i, $i \in \mathbb{N}$, of independent Poisson processes on a suitable (product) probability space, with η_i having intensity measure λ_{C_i} for each i.

By the superposition theorem (Theorem 3.3), the point process $\eta' := \sum_{i=1}^{\infty} \eta_i$ is a Poisson process with intensity measure λ. Then $\eta' \overset{d}{=} \eta$ by Proposition 3.2. Hence for any k and any $f_1, \ldots, f_k \in \mathbb{R}_+(\mathbf{N})$ we have

$$\mathbb{E}\Big[\prod_{i=1}^{k} f_i(\eta_{C_i})\Big] = \mathbb{E}\Big[\prod_{i=1}^{k} f_i(\eta'_{C_i})\Big] = \mathbb{E}\Big[\prod_{i=1}^{k} f_i(\eta_i)\Big] = \prod_{i=1}^{k} \mathbb{E}[f_i(\eta_i)].$$

Taking into account that $\eta_{C_i} \overset{d}{=} \eta_i$ for all $i \in \mathbb{N}$ (Proposition 3.2), we get the result. □

5.2 The marking theorem

Suppose that η is a proper point process, i.e. one that can be represented as in (2.4). Suppose that one wishes to give each of the points X_n a random *mark* Y_n with values in some measurable space $(\mathbb{Y}, \mathcal{Y})$, called the *mark space*. Given η, these marks are assumed to be independent while their conditional distribution is only allowed to depend on the value of X_n but not on any other information contained in η. This marking procedure yields a point process ξ on the product space $\mathbb{X} \times \mathbb{Y}$. Theorem 5.6 will show the remarkable fact that ξ is a Poisson process whenever η is.

To make the above marking idea precise, let K be a *probability kernel* from $\mathbb{X}$ to $\mathbb{Y}$, that is a mapping $K\colon \mathbb{X} \times \mathcal{Y} \to [0, 1]$ such that $K(x, \cdot)$ is a probability measure for each $x \in \mathbb{X}$ and $K(\cdot, C)$ is measurable for each $C \in \mathcal{Y}$.

Definition 5.3 Let $\eta = \sum_{n=1}^{\kappa} \delta_{X_n}$ be a proper point process on $\mathbb{X}$. Let K be a probability kernel from $\mathbb{X}$ to $\mathbb{Y}$. Let $Y_1, Y_2, \dots$ be random elements in $\mathbb{Y}$ and assume that the conditional distribution of $(Y_n)_{n \le m}$ given $\kappa = m \in \overline{\mathbb{N}}$ and $(X_n)_{n \le m}$ is that of independent random variables with distributions $K(X_n, \cdot)$, $n \le m$. Then the point process

$$\xi := \sum_{n=1}^{\kappa} \delta_{(X_n, Y_n)} \tag{5.6}$$

is called a *K-marking* of η. If there is a probability measure $\mathbb{Q}$ on $\mathbb{Y}$ such that $K(x, \cdot) = \mathbb{Q}$ for all $x \in \mathbb{X}$, then ξ is called an *independent $\mathbb{Q}$-marking* of η.

For the rest of this section we fix a probability kernel from $\mathbb{X}$ to $\mathbb{Y}$. Since the reader might harbour doubts about whether the ξ in (5.6) is well defined, owing to the conditioning on events of probability zero (namely the values taken by each X_n), we give more details on the construction of ξ. Given $m \in \overline{\mathbb{N}}$ with $\mathbb{P}(\kappa = m) > 0$, we let $\mathbb{P}_m$ be a choice of the conditional distribution of $(X_1, X_2, \dots, X_m)$ given $\kappa = m$ (if $m < \infty$) or of $(X_1, X_2, X_3, \dots)$ (if $m = \infty$). There are many possible choices of $\mathbb{P}_m$, since the order of the points X_n is immaterial to the point process η; we shall check later that the distribution of ξ is independent of the choice of $\mathbb{P}_m$. Define a measure $\tilde{\mathbb{P}}_m$ on $(\mathbb{X} \times \mathbb{Y})^m$, for finite m by

$$\tilde{\mathbb{P}}_m(C) = \int \cdots \int \mathbf{1}_C(x_1, y_1, \dots, x_m, y_m)\, K(x_1, dy_1) \dots K(x_m, dy_m) \times \\ \times \mathbb{P}_m(d(x_1, \dots, x_m)),$$

and for $m = \infty$ by

$$\tilde{\mathbb{P}}_\infty(C) = \int \cdots \int \mathbf{1}_C(x_1, y_1, x_2, y_2, \dots)\Bigg(\prod_{i=1}^{\infty} K(x_i, dy_i)\Bigg)\mathbb{P}_\infty(d(x_1, x_2, \dots)).$$

Then we define ξ by (5.6), where for finite m we set

$$\mathbb{P}_{X_1,Y_1,\dots,X_m,Y_m}(d(x_1, y_1, \dots, x_m, y_m) \mid \kappa = m) = \tilde{\mathbb{P}}_m(d(x_1, y_1, \dots, x_m, y_m))$$

and for $m = \infty$ we set

$$\mathbb{P}_{X_1,Y_1,X_2,Y_2,\dots}(d(x_1, y_1, x_2, y_2, \dots) \mid \kappa = \infty) = \tilde{\mathbb{P}}_\infty(d(x_1, y_1, x_2, y_2, \dots)).$$

The proposition shows in particular, that the distribution of a K-marking of η is uniquely determined by the distribution of η and K.

Proposition 5.4 *Let ξ be a K-marking of a proper point process η on $\mathbb{X}$ as in Definition 5.3. Then the Laplace functional of ξ is given by*

$$L_\xi(u) = L_\eta(u^*), \quad u \in \mathbb{R}_+(\mathbb{X} \times \mathbb{Y}), \tag{5.7}$$

where

$$u^*(x) := -\log\Big[\int e^{-u(x,y)}\, K(x, dy)\Big], \quad x \in \mathbb{X}. \tag{5.8}$$

Proof Recall that $\overline{\mathbb{N}}_0 := \mathbb{N}_0 \cup \{\infty\}$. For $u \in \mathbb{R}_+(\mathbb{X} \times \mathbb{Y})$ we have by the discussion after Definition 5.3 that

$$\begin{aligned} L_\xi(u) &= \sum_{m \in \overline{\mathbb{N}}_0} \mathbb{E}\Big[\mathbf{1}\{\kappa = m\} \exp\Big[-\sum_{k=1}^{m} u(X_k, Y_k)\Big]\Big] \\ &= \sum_{m \in \overline{\mathbb{N}}_0} \mathbb{P}(\kappa = m) \int \cdots \int \exp\Big[-\sum_{k=1}^{m} u(x_k, y_k)\Big] \prod_{k=1}^{m} K(x_k, dy_k)\, \mathbb{P}_m(d\mathbf{x}^{(m)}) \end{aligned}$$

where $\mathbf{x}^{(m)}$ stands for $(x_1, \dots, x_m)$ for $m \in \mathbb{N}$ and $\mathbf{x}^{(\infty)}$ stands for $(x_1, x_2, \dots)$. In the case $m = 0$ empty sums are set to 0 while empty products are set to 1. Therefore

$$L_\xi(u) = \sum_{m \in \overline{\mathbb{N}}_0} \mathbb{P}(\kappa = m) \int \Big(\prod_{k=1}^{m} \int \exp\left[-u(x_k, y_k)\right] K(x_k, dy_k)\Big)\mathbb{P}_m(d\mathbf{x}^{(m)}).$$

Using the function u^* defined by (5.8) this means that

$$L_\xi(u) = \sum_{m\in\overline{\mathbb{N}}_0} \mathbb{P}(\kappa = m) \int \Big[\prod_{k=1}^{m} \exp[-u^*(x_k)] \Big] \mathbb{P}_m(d\mathbf{x}^{(m)})$$
$$= \sum_{m\in\overline{\mathbb{N}}_0} \mathbb{E}\Big[\mathbf{1}\{\kappa = m\} \exp\Big(- \sum_{k=1}^{m} u^*(X_k)\Big)\Big]$$

which is the right-hand side of the asserted identity (5.7). □

The next result says that the intensity measure of a K-marking of a point process with intensity measure λ is given by $\lambda \otimes K$, where

$$(\lambda \otimes K)(C) := \iint \mathbf{1}_C(x, y)\, K(x, dy)\, \lambda(dx), \quad C \in \mathcal{X} \otimes \mathcal{Y}. \tag{5.9}$$

In the case of an independent $\mathbb{Q}$-marking this is the product measure $\lambda \otimes \mathbb{Q}$.

Proposition 5.5 *Let η be a proper point process on $\mathbb{X}$ with s-finite intensity measure λ and let ξ be a K-marking of η. Then ξ is a point process on $\mathbb{X} \times \mathbb{Y}$ with s-finite intensity measure $\lambda \otimes K$.*

Proof One way to proceed is via a direct calculation as in the proof of Proposition 5.4.

Alternatively we can use the measure $\mathbb{P}_k$ introduced after Definition 5.3. For $j \in \mathbb{N}$ and $k \in \overline{\mathbb{N}}$ with $j \le k$, let $\mathbb{P}_{k,j}$ be the image of $\mathbb{P}_k$ under projection from $\mathbb{X}^k$ onto the jth component. Then

$$\lambda = \sum_{k\in\overline{\mathbb{N}}} \mathbb{P}(\kappa = k) \sum_{j=1}^{k} \mathbb{P}_{k,j} \tag{5.10}$$

while if λ_ξ denotes the intensity measure of ξ then

$$\lambda_\xi = \sum_{k\in\overline{\mathbb{N}}} \mathbb{P}(\kappa = k) \sum_{j=1}^{k} \mathbb{P}_{k,j} \otimes K.$$

The result follows because we see from (5.9) that $(\lambda \otimes K)(C)$ is countably additive in λ and hence by (5.10) $\lambda_\xi = \lambda \otimes K$. □

Now we formulate the previously announced behaviour of Poisson processes under marking.

Theorem 5.6 (Marking Theorem) *Let ξ be a K-marking of a proper Poisson process η with intensity measure λ. Then ξ is a Poisson process with intensity measure $\lambda \otimes K$.*

Proof Let $u \in \mathbb{R}_+(\mathbb{X} \times \mathbb{Y})$. By Proposition 5.4 and Theorem 3.9,

$$\begin{aligned} L_\xi(u) &= \exp\Big[- \int (1 - e^{-u^*(x)})\, \lambda(dx) \Big] \\ &= \exp\Big[- \iint (1 - e^{-u(x,y)})\, K(x, dy)\, \lambda(dx) \Big]. \end{aligned}$$

Another application of Theorem 3.9 shows that ξ is a Poisson process. □

Under some technical assumptions we shall see in Proposition 6.16 that any Poisson process on a product space is a K-marking for some kernel K, determined by the intensity measure.

5.3 Thinnings

A thinning keeps the points of a point process η with a probability that may depend on the location and removes them otherwise. Given η, the thinning decisions are independent for different points. The formal definition can be based on a special K-marking:

Definition 5.7 Let $p\colon \mathbb{X} \to [0, 1]$ be measurable and consider the probability kernel K from $\mathbb{X}$ to $\{0, 1\}$ defined by

$$K_p(x, \cdot) := (1 - p(x))\delta_0 + p(x)\delta_1, \quad x \in \mathbb{X}.$$

If ξ is a K_p-marking of a proper point process η, then $\xi(\cdot \times \{1\})$ is called a *p-thinning* of η.

More generally, let p_i, $i \in \mathbb{N}$, be a sequence of measurable functions from $\mathbb{X}$ to $[0, 1]$ such that

$$\sum_{i=1}^{\infty} p_i(x) = 1, \quad x \in \mathbb{X}. \tag{5.11}$$

Define a probability kernel K from $\mathbb{X}$ to $\mathbb{N}$ by

$$K(x, \{i\}) := p_i(x), \quad x \in \mathbb{X},\ i \in \mathbb{N}. \tag{5.12}$$

If ξ is a K-marking of a point process η, then $\eta_i := \xi(\cdot \times \{i\})$ is a p_i-thinning of η for every $i \in \mathbb{N}$. By Proposition 5.5, η_i has intensity measure $p_i(x)\, \lambda(dx)$, where λ is the intensity measure of η. The following generalization of Proposition 1.3 is consistent with the superposition theorem (Theorem 3.3).

Theorem 5.8 *Let ξ be a K-marking of a proper Poisson process η, where K is given as in* (5.12). *Then $\eta_i := \xi(\cdot \times \{i\})$, $i \in \mathbb{N}$, are independent Poisson processes.*

Proof By Theorem 5.6, ξ is a Poisson process. Hence we can apply Theorem 5.2 with $C_i := \mathbb{X} \times \{i\}$ to obtain the result. □

We record the following important special case. If η and η' are two proper point processes with $\eta' \leq \eta$, then $\eta - \eta'$ (given by $\eta(\cdot) - \eta'(\cdot)$) is again a proper point process.

Corollary 5.9 (Thinning Theorem) *Let $p\colon \mathbb{X} \to [0, 1]$ be measurable and let η_p be a p-thinning of a proper Poisson process η. Then η_p and $\eta - \eta_p$ are independent Poisson processes.*

5.4 Exercises

Exercise 5.1 (Displacement Theorem) Let λ be an s-finite measure on the Euclidean space $\mathbb{R}^d$, let $\mathbb{Q}$ be a probability measure on $\mathbb{R}^d$ and let the *convolution* $\lambda * \mathbb{Q}$ be the measure on $\mathbb{R}^d$, defined by

$$(\lambda * \mathbb{Q})(B) := \iint \mathbf{1}_B(x + y)\, \lambda(dx)\, \mathbb{Q}(dy), \quad B \in \mathcal{B}(\mathbb{R}^d).$$

Show that $\lambda * \mathbb{Q}$ is s-finite. Let $\eta = \sum_{n=1}^{\kappa} \delta_{X_n}$ be a Poisson process with intensity measure λ and let (Y_n) be a sequence of independent random vectors with distribution $\mathbb{Q}$ that is independent of η. Show that $\eta' := \sum_{n=1}^{\kappa} \delta_{X_n+Y_n}$ is a Poisson process with intensity measure $\lambda * \mathbb{Q}$.

Exercise 5.2 Let η_1 and η_2 be independent Poisson processes with intensity measure λ_1 and λ_2 respectively. Let p be a Radon-Nikodým derivative of λ_1 with respect to $\lambda := \lambda_1 + \lambda_2$. Show that η_1 has the same distribution as a p-thinning of $\eta_1 + \eta_2$.

Exercise 5.3 Let $\xi_1, \dots, \xi_n$ be identically distributed point processes and let $\xi^{(n)}$ be an n^{-1}-thinning of $\xi := \xi_1 + \cdots + \xi_n$. Show that $\xi^{(n)}$ has the same intensity measure as ξ_1. Give examples where $\xi_1, \dots, \xi_n$ are independent and where $\xi^{(n)}$ and ξ_1 have (resp. do not have) the same distribution.

Exercise 5.4 Let $p\colon \mathbb{X} \to [0, 1]$ be measurable and η_p be a p-thinning of a proper point process η. Using Proposition 5.4 or otherwise, show that

$$L_{\eta_p}(u) = \mathbb{E}\Big[\exp\Big(\int \log\big(1 - p(x) + p(x)e^{-u(x)}\big)\, \eta(dx)\Big)\Big], \quad u \in \mathbb{R}_+(\mathbb{X}).$$

Exercise 5.5 Let η be a proper Poisson process with σ-finite intensity measure λ. Let λ' be another σ-finite measure on $\mathbb{X}$ and let $\rho := \lambda + \lambda'$. Let $h := d\lambda/d\rho$ (resp. $h' := d\lambda'/d\rho$) be the Radon-Nikodým derivative of λ (resp. λ') w.r.t. ρ; see Theorem A.9. Let $B := \{x \in \mathbb{X} : h(x) > h'(x)\}$ and define $p\colon \mathbb{X} \to [0, 1]$ by $p(x) := h'(x)/h(x)$ for $x \in B$ and by $p(x) := 1$, otherwise. Let η' be a p-thinning of η and let η'' be a Poisson process with intensity measure $\mathbf{1}_{\mathbb{X}\setminus B}(x)(h'(x) - h(x))\,\rho(dx)$, independent of η'. Show that $\eta' + \eta''$ is a Poisson process with intensity measure λ'.

Exercise 5.6 (Poisson cluster process) Let K be a probability kernel from $\mathbb{X}$ to $\mathbf{N}$. Let η be a proper Poisson process on $\mathbb{X}$ with intensity measure λ and let ξ be a K-marking of η. Show that

$$\chi(B) := \int \mu(B)\,\xi(d(x, \mu)), \quad B \in \mathcal{X}, \tag{5.13}$$

defines a point process χ on $\mathbb{X}$ with intensity measure

$$\lambda'(B) = \iint \mu(B)\,K(x, d\mu)\,\lambda(dx), \quad B \in \mathcal{X}.$$

Show also that the Laplace functional of χ is given by

$$L_\chi(v) = \exp\Big[-\int (1 - e^{-\mu(v)})\,\tilde{\lambda}(d\mu)\Big], \quad v \in \mathbb{R}_+(\mathbb{X}), \tag{5.14}$$

where $\tilde{\lambda} := \int K(x, \cdot)\,\lambda(dx)$.

Exercise 5.7 Let χ be a Poisson cluster process as in Exercise 5.6 and let $B \in \mathcal{X}$. Show that

$$\mathbb{P}(\chi(B) = 0) = \exp\Big[-\int \mathbf{1}\{\mu(B) > 0\}\,\tilde{\lambda}(d\mu)\Big].$$

(Hint: Combine Exercise 2.8 and (5.14).)

Exercise 5.8 Let χ be as in Exercise 5.6 and let $B \in \mathcal{X}$. Show that $\mathbb{P}(\chi(B) < \infty) = 1$ if and only if $\tilde{\lambda}(\{\mu \in \mathbf{N} : \mu(B) = \infty\}) = 0$ and $\tilde{\lambda}(\{\mu \in \mathbf{N} : \mu(B) > 0\}) < \infty$. (Hint: Use $\mathbb{P}(\chi(B) < \infty) = \lim_{t\downarrow 0} \mathbb{E}[e^{-t\chi(B)}]$.)

6

Characterizations of the Poisson process

A point process without multiplicities is said to be simple. For simple point processes on a metric space without fixed atoms the two defining properties of a Poisson process are equivalent. In fact Rényi's theorem says that in this case already the appropriate form of the empty space probabilities imply that the point process is Poisson. On the other hand a weak (pairwise) version of the complete independence property leads to the same conclusion. A related criterion can be based on the factorial moment measures.

6.1 Simple point processes

In this chapter we assume that the state space $(\mathbb{X}, \mathcal{X})$ is a Borel subspace of a complete separable metric space (CSMS). That is, $\mathbb{X}$ is a Borel subset of a CSMS and $\mathcal{X}$ is the σ-field on $\mathbb{X}$ generated by the open sets in the inherited metric; see Appendix A.2. In particular $\mathbb{X}$ is a metric space in its own right. Recall from Definition 2.11 that $\mathbf{N}_l(\mathbb{X})$ denotes the class of all measures from $\mathbf{N}(\mathbb{X})$ that are locally finite, that is finite on the system $\mathcal{X}_b$ of bounded measurable sets.

We now prove the result mentioned just after Definition 2.4, that all locally finite point processes on $\mathbb{X}$ are proper. The following concept together with Theorem A.18 in the Appendix is key to the proofs of results in this section.

Definition 6.1 A *Borel space* is a measurable space $(\mathbb{Y}, \mathcal{Y})$ such that there is a Borel-measurable bijection φ from $\mathbb{Y}$ to a Borel subset of the unit interval $[0, 1]$ with measurable inverse.

Proposition 6.2 *There are measurable mappings $\pi_n \colon \mathbf{N} \to \mathbb{X}$, $n \in \mathbb{N}$, such that for all $\mu \in \mathbf{N}_l(\mathbb{X})$ we have*

$$\mu = \sum_{n=1}^{\mu(\mathbb{X})} \delta_{\pi_n(\mu)}. \tag{6.1}$$

Proof Let $\mathbf{N}_{<\infty}$ denote the set of all $\mu \in \mathbf{N}$ such that $\mu(\mathbb{X}) < \infty$. This is a measurable subset of $\mathbf{N}$. By Theorem A.18, $(\mathbb{X}, \mathcal{X})$ is a Borel space, so there is a measurable bijection φ from $\mathbb{X}$ onto a Borel subset U of $[0, 1]$ such that the inverse of φ is measurable. For $\mu \in \mathbf{N}_{<\infty}$ we define a finite measure $\varphi(\mu)$ on $\mathbb{R}$ by $\varphi(\mu) := \mu \circ \varphi^{-1}$, that is,

$$\varphi(\mu)(B) = \mu(\{x : \varphi(x) \in B\}), \quad B \in \mathcal{B}(\mathbb{R}).$$

Here we interpret φ as a mapping from $\mathbb{X}$ to $\mathbb{R}$, so that $\varphi^{-1}(B) = \emptyset$ whenever $B \cap U = \emptyset$. Hence $\varphi(\mu)$ is concentrated on U, that is $\varphi(\mu)(\mathbb{R} \setminus U) = 0$. For $n \in \mathbb{N}$, set

$$Y_n(\mu) := \inf\{x \in \mathbb{R} : \varphi(\mu)((-\infty, x]) \geq n\}, \quad \mu \in \mathbf{N}_{<\infty},$$

where $\inf \emptyset := \infty$. For $n > \mu(\mathbb{X})$ we have $Y_n(\mu) = \infty$. For $n \leq \mu(\mathbb{X})$ we have $Y_n(\mu) \in U$. Indeed, in this case $\varphi(\mu)\{Y_n(\mu)\} > 0$.

For $x \in \mathbb{R}$ we have

$$\begin{aligned}\{\mu \in \mathbf{N}_{<\infty} : Y_n(\mu) \leq x\} &= \{\mu \in \mathbf{N}_{<\infty} : \varphi(\mu)((-\infty, x]) \geq n\} \\ &= \{\mu \in \mathbf{N}_{<\infty} : \mu(\varphi^{-1}((-\infty, x])) \geq n\},\end{aligned}$$

so Y_n is a measurable mapping on $\mathbf{N}_{<\infty}$. Also

$$\varphi(\mu)(B) = \sum_{n=1}^{\mu(\mathbb{X})} \delta_{Y_n(\mu)}(B), \quad \mu \in \mathbf{N}_{<\infty}, \tag{6.2}$$

for all B of the form $B = (-\infty, x]$ with $x \in \mathbb{R}$ (a π-system of sets), and hence for all Borel sets $B \subset \mathbb{R}$ (by Theorem A.4). Define

$$X_n(\mu) := \begin{cases} \varphi^{-1}(Y_n(\mu)), & \text{if } n \leq \mu(\mathbb{X}), \\ x_0, & \text{otherwise}, \end{cases}$$

where $x_0 \in \mathbb{X}$ is some fixed point. By (6.2) we have for all $B \in \mathcal{B}(\mathbb{R})$ that

$$\mu(B) = \mu(\varphi^{-1}(\varphi(B)) = \sum_{n=1}^{\mu(\mathbb{X})} \mathbf{1}\{Y_n(\mu) \in \varphi(B)\} = \sum_{n=1}^{\mu(\mathbb{X})} \mathbf{1}\{X_n(\mu) \in B\}$$

and hence $\mu = \sum_{n=1}^{\mu(\mathbb{X})} \delta_{X_n(\mu)}$. Then (6.1) holds with $\pi_n(\mu) = X_n(\mu)$.

To derive (6.1) for a general $\mu \in \mathbf{N}_l := \mathbf{N}_l(\mathbb{X})$, let $B_1, B_2, \dots$ be a sequence of disjoint bounded sets in $\mathcal{X}$, forming a partition of $\mathbb{X}$. Recall from (5.5) the definition of the restriction μ_{B_i} of μ to B_i. From the first part of the proof we obtain for each $i \in \mathbb{N}$ measurable mappings $\pi_{i,j} \colon \mathbf{N}_l \to B_i$, $j \in \mathbb{N}$, such that

$$\mu_{B_i} = \sum_{j=1}^{\mu(B_i)} \delta_{\pi_{i,j}(\mu)}, \quad \mu \in \mathbf{N}_l.$$

(If $\mu(B_i) = 0$ we can set $\pi_{i,j}(\mu) := x_0$.) Let $\mu \in \mathbf{N}_l$. If $\mu = 0$ is the zero measure, for all $n \in \mathbb{N}$ we set $\pi_n(\mu) := x_0$. Otherwise we let $k_1 = k_1(\mu)$ be the smallest $i \in \mathbb{N}$ such that $\mu(B_i) > 0$ and define $\pi_n(\mu) := \pi_{k_1,n}(\mu)$ for $1 \le n \le \mu(B_{k_1})$. If $\mu(\mathbb{X}) = \mu_{B_{k_1}}(\mathbb{X})$ we let $\pi_n(\mu) := x_0$ for $n > \mu(B_1)$. Otherwise we define $\pi_{k_1+m}(\mu) := \pi_{k_2,m}(\mu)$ for $1 \le m \le \mu(B_{k_2})$, where $k_2 \equiv k_2(\mu)$ is the smallest $i \in \mathbb{N}$ such that $\mu(B_i) > \mu_{B_{k_1}}(B_i)$. It is now clear how to construct a sequence $\pi_n : \mathbf{N}_l \to \mathbb{X}$, $n \in \mathbb{N}$, inductively, such that (6.1) holds. Measurability can be proved by induction, using the fact that the $\pi_{i,j}$ are measurable. Since $\mathbf{N}_l$ is a measurable subset of $\mathbf{N}$ (see the discussion after Definition 2.11) the mappings π_n can be extended to measurable mappings on $\mathbf{N}$. □

As a corollary we obtain that any locally finite point process on $\mathbb{X}$ (see Definition 2.13) is proper:

Corollary 6.3 *Let η be a locally finite point process on $\mathbb{X}$. Then η is a proper point process. That is, there exist random elements $X_1, X_2, \dots$ in $\mathbb{X}$ and an $\overline{\mathbb{N}}_0$-valued random variable κ such that almost surely*

$$\eta = \sum_{n=1}^{\kappa} \delta_{X_n}. \tag{6.3}$$

Proof Let $\kappa := \eta(\mathbb{X})$. Whenever η is locally finite we use Proposition 6.2 to define $X_n := \pi_n(\eta)$ for $n \in \mathbb{N}$. Otherwise let the X_n equal some fixed point in $\mathbb{X}$. □

Definition 6.4 A measure $\mu \in \mathbf{N}$ is said to be *simple* if $\mu\{x\} \le 1$ for all $x \in \mathbb{X}$. Let $\mathbf{N}_s(\mathbb{X})$ denote the set of all simple locally finite measures in $\mathbf{N}(\mathbb{X})$.

Corollary 6.5 *The set $\mathbf{N}_s(\mathbb{X})$ is measurable, i.e. $\mathbf{N}_s(\mathbb{X}) \in \mathcal{N}(\mathbb{X})$.*

Proof By Proposition 6.2, a measure $\mu \in \mathbf{N}(\mathbb{X})$ is in $\mathbf{N}_s(\mathbb{X})$ if and only if $\mu \in \mathbf{N}_l(\mathbb{X})$ (a measurable subset of $\mathbf{N}(\mathbb{X})$) and $\pi_m(\mu) \neq \pi_n(\mu)$ for all $m, n \in \mathbb{N}$ such that $m \neq n$ and $m, n \le \mu(\mathbb{X})$. It remains to note that the diagonal $\{(x, y) \in \mathbb{X}^2 : x = y\}$ is measurable. Indeed, this holds if $\mathbb{X}$ is a Borel subset of $[0, 1]$. Using Theorem A.18, the measurability can be extended to the general case. □

Point processes without simplicities deserve a special name:

Definition 6.6 A point process η is said to be *simple* if $\mathbb{P}(\eta \in \mathbf{N}_s(\mathbb{X})) = 1$.

If η is a simple point process on $\mathbb{X}$ and $\eta' \overset{d}{=} \eta$, then η' is also simple.

Proposition 6.7 *Let η be a Poisson process on $\mathbb{X}$ with locally finite intensity measure λ. Then η is simple if and only if λ is diffuse.*

Proof Assume first that λ is not diffuse. Choosing an $x \in \mathbb{X}$ with $c := \lambda\{x\} > 0$ we obtain that

$$\mathbb{P}(\eta\{x\} \geq 2) = 1 - e^{-c} - ce^{-c} > 0,$$

so that η is not simple.

Assume, conversely, that λ is diffuse. We need to show that η is simple. Since the distribution of a Poisson process equals that of the sum of countably many mixed binomial processes on pairwise disjoint sets (note that λ is σ-finite and check the proof of Theorem 3.6), it is enough to show that a mixed binomial process with a diffuse sampling distribution $\mathbb{Q}$ is simple. By the form (3.2) of mixed sample processes we can further reduce to the case of a binomial process as in Example 2.3. Such a process is simple if $\mathbb{P}(X_i = X_j) = 0$ for $i \neq j$. This probability is given by

$$\mathbb{P}(X_i = X_j) = \iint \mathbf{1}_D(x, y)\, \mathbb{Q}(dx)\, \mathbb{Q}(dy) = \int \mathbb{Q}\{y\}\, \mathbb{Q}(dy) = 0,$$

where $D := \{(x, y) \in \mathbb{X}^2 : x = y\}$ is the diagonal in $\mathbb{X}^2$. □

6.2 Rényi's Theorem

The following (at first glance surprising) result shows that the two defining properties of a Poisson process are not independent of each other. In fact this result, together with Theorem 6.10, shows that under certain extra conditions, either one of the defining properties of the Poisson process implies the other one. We base the proof on a more general result for simple point processes.

Theorem 6.8 (Rényi's Theorem) *Suppose that λ is a diffuse locally finite measure on $\mathbb{X}$, and that η is a simple point process on $\mathbb{X}$ satisfying*

$$\mathbb{P}(\eta(B) = 0) = \exp[-\lambda(B)], \quad B \in \mathcal{X}_b. \tag{6.4}$$

Then η is a Poisson process with intensity measure λ.

Proof Let η' be a Poisson process with intensity measure λ. Then assumption (6.4) implies (6.5) below. Proposition 6.7 shows that η' is simple. Theorem 6.9 shows that η and η' have the same distribution. □

Theorem 6.9 *Let η and η' be simple point processes on $\mathbb{X}$ such that*

$$\mathbb{P}(\eta(B) = 0) = \mathbb{P}(\eta'(B) = 0), \quad B \in \mathcal{X}_b. \tag{6.5}$$

Then $\eta \overset{d}{=} \eta'$.

Proof By Proposition 2.12 it is sufficient to prove that $\eta_B \overset{d}{=} \eta'_B$ for every bounded Borel set $B \subset \mathbb{X}$. Therefore we can assume that

$$\mathbb{P}(\eta(\mathbb{X}) < \infty) = \mathbb{P}(\eta'(\mathbb{X}) < \infty) = 1. \tag{6.6}$$

By Theorem A.18, $\mathbb{X}$ is a Borel space. Take a measurable bijection φ from $\mathbb{X}$ onto a Borel subset of $I := [1/4, 3/4]$ such that the inverse of φ is measurable. The point process ξ on I defined by

$$\xi(B) := \eta \circ \varphi^{-1}(B) = \eta(\{x \in \mathbb{X} : \varphi(x) \in B\}), \quad B \in \mathcal{B}(I), \tag{6.7}$$

is again simple. The same holds for $\xi' := \eta' \circ \varphi^{-1}$. Since φ is one-to-one we have $\eta = \xi \circ \varphi$ and $\eta' = \xi' \circ \varphi$. Furthermore, since $\mu \mapsto \mu(\varphi(B))$ is measurable for all $B \in \mathcal{X}$, $\mu \mapsto \mu \circ \varphi$ is a measurable mapping from $\mathbf{N}(I)$ to $\mathbf{N}(\mathbb{X})$. Since equality in distribution is preserved under measurable mappings, it is now enough to prove that $\xi \overset{d}{=} \xi'$.

Let $\mathcal{N}^*$ denote the sub-σ-field of $\mathcal{N}(I)$ generated by the system

$$\mathcal{H} := \{\{\mu \in \mathbf{N}(I) : \mu(B) = 0\} : B \in \mathcal{B}(I)\}.$$

Since, for any measure μ on I and any $B, B' \in \mathcal{B}(I)$, the equation $\mu(B \cup B') = 0$ is equivalent to $\mu(B) = \mu(B') = 0$, $\mathcal{H}$ is a π-system. By assumption (6.5), $\mathbb{P}_\xi$ agrees with $\mathbb{P}_{\xi'}$ on $\mathcal{H}$, and therefore by Theorem A.4, $\mathbb{P}_\xi$ agrees with $\mathbb{P}_{\xi'}$ on $\mathcal{N}^*$.

For any $n \in \mathbb{N}$ and $j \in \{1, \dots, n\}$ let $I_{n,j} := ((j-1)/n, j/n]$. Given $B \in \mathcal{B}(I)$, define

$$g_{n,B}(\mu) = \sum_{j=1}^{n} \mu(I_{n,j} \cap B) \wedge 1, \quad \mu \in \mathbf{N}(I),\ n \in \mathbb{N}, \tag{6.8}$$

where $a \wedge b := \min\{a, b\}$ denotes the minimum of $a, b \in \overline{\mathbb{R}}$. Define the function $g_B : \mathbf{N}(I) \to \overline{\mathbb{R}}_+$ by

$$g_B(\mu) := \liminf_{n \to \infty} g_{n,B}(\mu), \quad \mu \in \mathbf{N}(I). \tag{6.9}$$

Then g_B is an $\mathcal{N}^*$-measurable function on $\mathbf{N}(I)$. Also ξ is a simple finite point process on I, so that $\xi(B) = g_B(\xi)$, almost surely, and therefore for any $m \in \mathbb{N}$, $B_1, \dots, B_m \in \mathcal{B}(I)$ and $k_1, \dots, k_m \in \mathbb{N}_0$, we have

$$\mathbb{P}(\cap_{i=1}^m \{\xi(B_i) = k_i\}) = \mathbb{P}(\xi \in \cap_{i=1}^m g_{B_i}^{-1}(\{k_i\})),$$

and since $\cap_{i=1}^m g_{B_i}^{-1}(\{k_i\}) \in \mathcal{N}^*$, the corresponding probability for ξ' is the same. Therefore by Proposition 2.10, $\xi' \overset{d}{=} \xi$. □

A point process η on $\mathbb{X}$ satisfies

$$\eta\{x\} = 0, \quad \mathbb{P}\text{-a.s.}, \ x \in \mathbb{X} \tag{6.10}$$

if and only if its intensity measure is diffuse. If, in addition, η is simple, then the following result shows that only a weak version of the complete independence property is enough to ensure that η is a Poisson process. This complements Theorem 6.8.

Theorem 6.10 *Suppose that η is a simple point process on $\mathbb{X}$ satisfying* (6.10). *Assume also that the events $\{\eta(B) = 0\}$ and $\{\eta(B') = 0\}$ are independent whenever $B, B' \in \mathcal{X}$ are disjoint. Then η is a Poisson process.*

Proof By Proposition 2.12 it is sufficient to prove that η_B is a Poisson process for each bounded Borel set $B \subset \mathbb{X}$. As in the proof of Theorem 6.9 we can assume that $\mathbb{X} = \mathbb{R}$, $\eta(\mathbb{R} \setminus I) = 0$ and $\mathbb{P}(\eta(\mathbb{R}) < \infty) = 1$, where $I := [0, 1]$.

For $t \in \mathbb{R}$ set $f(t) := \mathbb{P}(\eta((-\infty, t]) = 0)$ which is clearly nonincreasing. Clearly $f(-1) = 1$. Suppose $f(1) = 0$. Let $t_0 := \inf\{t \in \mathbb{R} : f(t) = 0\}$. By continuity of $\mathbb{P}$, (6.10), and the assumption $\mathbb{P}(\eta(\mathbb{R}) < \infty) = 1$, we have $\mathbb{P}(\eta((t_0 - 1/n, t_0 + 1/n)) = 0) \to 1$ as $n \to \infty$. Hence we can choose n with

$$c := \mathbb{P}(\eta((t_0 - 1/n, t_0 + 1/n]) = 0) > 0.$$

Then by our assumption we have

$$f(t_0 + 1/n) = c f(t_0 - 1/n) > 0$$

which is a contradiction, so $f(1) > 0$.

Define

$$\lambda(B) := -\log \mathbb{P}(\eta(B) = 0), \quad B \in \mathcal{B}(\mathbb{R}). \tag{6.11}$$

Then $\lambda(\emptyset) = 0$ and $\lambda(\mathbb{R}) < \infty$. We show that λ is a measure. By our assumption λ is additive and hence also finitely additive. Let C_n, $n \in \mathbb{N}$, be an increasing sequence of Borel sets with union C. Then the events $\{\eta(C_n) = 0\}$ are decreasing and have intersection $\{\eta(C) = 0\}$. Therefore $\lambda(C_n) \to \lambda(C)$ as $n \to \infty$, showing that λ is indeed a measure. Furthermore, (6.10) implies for any $x \in I$ that $\lambda\{x\} = -\log \mathbb{P}(\eta\{x\} = 0) = 0$, so that λ is diffuse. Now we can apply Rényi's theorem (Theorem 6.8) to conclude that η is a Poisson process. □

6.3 Completely orthogonal point processes

For simple point processes satisfying (4.24) the assumptions of Proposition 4.11 can be relaxed as follows.

Theorem 6.11 *Suppose that η and η' are simple point processes on $\mathbb{X}$ such that for any collection $B_1, \dots, B_m$ of disjoint measurable sets*

$$\mathbb{E}[\eta(B_1) \cdots \eta(B_m)] = \mathbb{E}[\eta'(B_1) \cdots \eta'(B_m)]. \tag{6.12}$$

Suppose also that the factorial moment measures of η satisfy (4.24). *Then $\eta \overset{d}{=} \eta'$.*

Proof As in the proof of Theorem 6.9 we can assume that $\mathbb{X} = \mathbb{R}$.

We wish to apply Proposition 4.11. Let $m \in \mathbb{N}$ with $m \geq 2$ and let

$$D_m := \{(x_1, \dots, x_m) \in \mathbb{X}^m : \text{there exist } i < j \text{ with } x_i = x_j\} \tag{6.13}$$

denote the *generalized diagonal* in $\mathbb{X}^m$. Let $\mathcal{H}$ be the class of all measurable sets in $\mathbb{R}^m$ which are either of the form $B_1 \times \cdots \times B_m$ with the $B_1, \dots, B_m$ measurable and disjoint, or are contained in the generalized diagonal D_m. Then $\mathcal{H}$ is a π-system and the m-th factorial moment measure of η agrees with that of η' on all sets in $\mathcal{H}$. Indeed, this is true by assumption for the first kind of set in $\mathcal{H}$, and by Exercise 6.11 and our assumption both factorial moment measures are zero on the diagonal D_m. Then by Theorem A.4 and Proposition 4.11 we are done if we can show $\mathcal{H}$ generates the product σ-field $\mathcal{B}(\mathbb{R})^m = \mathcal{B}(\mathbb{R}^m)$; see Lemma A.23. The latter is generated by all sets of the form $B_1 \times \cdots \times B_m$, where $B_1, \dots, B_m$ are open intervals. Let us fix such intervals. For any $n \in \mathbb{N}$ and $j \in \mathbb{Z}$ let $I_{n,j} := ((j-1)/n, j/n]$. Define

$$J_{n,i} := \{j \in \mathbb{Z} : I_{n,j} \subset B_i\}, \quad i \in \{1, \dots, m\},$$

and $J_n := J_{n,1} \times \cdots \times J_{n,m}$. Let Δ_m denote the generalized diagonal in $\mathbb{Z}^m$. We leave it to the reader to check that

$$B_1 \times \cdots \times B_m \setminus D_m = \bigcup_{n=1}^{\infty} \bigcup_{(i_1,\dots,i_m) \in J_n \setminus \Delta_m} I_{n,i_1} \times \cdots \times I_{n,i_m}$$

It therefore follows that $B_1 \times \cdots \times B_m \in \sigma(\mathcal{H})$, finishing the proof. □

We say that a point process η on $\mathbb{X}$ is *completely orthogonal* if

$$\mathbb{E}[\eta(B_1)\cdots\eta(B_m)] = \prod_{j=1}^{m} \mathbb{E}[\eta(B_j)] \tag{6.14}$$

for all $m \in \mathbb{N}$ and all pairwise disjoint $B_1, \dots, B_m \in \mathcal{X}$.

Theorem 6.11 implies the following characterization of simple Poisson processes.

Theorem 6.12 *Let η be a simple, completely orthogonal point process on $\mathbb{X}$ with a σ-finite diffuse intensity measure. Then η is a Poisson process.*

Proof Let λ denote the intensity measure of η and let η' be a Poisson process with intensity measure λ. By Proposition 6.7, η' is simple. Corollary 4.9, (4.6) and assumption (6.14) show that the hypothesis (6.12) of Theorem 6.11 is satisfied. It remains to note that η' satisfies (4.24). □

6.4 Uniformly σ-finite point processes

For later use we next show that the preceding results can be derived in the more general setting where $(\mathbb{X}, \mathcal{X})$ is a Borel space. Extending our earlier definitions we say a measure $\mu \in \mathbf{N}(\mathbb{X})$ is *simple* if $\mu\{x\} \le 1$ for all $x \in \mathbb{X}$ and that a general measure ν on $(\mathbb{X}, \mathcal{X})$ is *diffuse* if $\nu\{x\} = 0$ for all $x \in \mathbb{X}$.

Definition 6.13 A point process η on an arbitrary measurable space $(\mathbb{X}, \mathcal{X})$ is said to be *uniformly σ-finite* if there exist $B_n \in \mathcal{X}$, $n \in \mathbb{N}$, such that $\cup_{n=1}^{\infty} B_n = \mathbb{X}$ and

$$\mathbb{P}(\eta(B_n) < \infty) = 1, \quad n \in \mathbb{N}. \tag{6.15}$$

A uniformly σ-finite point process η on a Borel space $\mathbb{X}$ is called *simple* if there is an $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 1$ and $\eta(\omega, \{x\}) \le 1$ for all $x \in \mathbb{X}$ and all $\omega \in A$.

Exercise 6.9 justifies the second part of Definition 6.13. We note that Poisson processes with σ-finite intensity measure and locally finite point processes on a metric space are uniformly σ-finite. Theorems 6.8, 6.9 and 6.10 remain valid for uniformly σ-finite point processes on a Borel space. For instance we have:

Theorem 6.14 *Suppose that λ is a σ-finite diffuse measure on a Borel space $\mathbb{X}$, and that η is a uniformly σ-finite simple point process on $\mathbb{X}$ satisfying*

$$\mathbb{P}(\eta(B) = 0) = \exp[-\lambda(B)], \tag{6.16}$$

for all $B \in \mathcal{X}$ with $\lambda(B) < \infty$ and $\mathbb{P}(\eta(B) < \infty) = 1$. Then η is a Poisson process.

6.5 Turning distributional to almost sure identities

In this section we prove a converse of Theorem 5.6. Consider a Poisson process ξ on $\mathbb{X} \times \mathbb{Y}$ with intensity measure λ_ξ, where $(\mathbb{X}, \mathcal{X})$ and $(\mathbb{Y}, \mathcal{Y})$ are Borel subspaces of a CSMS. Assuming that $\lambda := \lambda_\xi(\cdot \times \mathbb{Y})$ is σ-finite, we can apply Theorem A.13 to obtain that $\lambda_\xi = \lambda \otimes K$, where K is a probability kernel from $\mathbb{X}$ to $\mathbb{Y}$. Since $\xi(\cdot \times \mathbb{Y})$ is a Poisson process with intensity measure λ (Theorem 5.1) Theorem 5.6 shows that ξ has the same distribution as a K-marking of $\xi(\cdot \times \mathbb{Y})$. Moreover, if λ is locally finite, then it turns out that the second coordinates of the points of ξ have the conditional independence properties of Definition 5.3.

First we refine Proposition 6.2 in a special case. Let $\mathbf{N}^*$ denote the measurable set of all $\mu \in \mathbf{N}(\mathbb{X} \times \mathbb{Y})$ such that $\mu(\cdot \times \mathbb{Y}) \in \mathbf{N}_s(\mathbb{X})$.

Lemma 6.15 *There is a measurable mapping $T : \mathbb{X} \times \mathbf{N}^* \to \mathbb{Y}$ such that*

$$\mu = \sum_{n=1}^{\bar{\mu}(\mathbb{X})} \delta_{(\pi_n(\bar{\mu}), T(\pi_n(\bar{\mu}), \mu))}, \quad \mu \in \mathbf{N}^*, \tag{6.17}$$

where $\bar{\mu} := \mu(\cdot \times \mathbb{Y})$.

Proof Let $\mu \in \mathbf{N}^*$. If $\bar{\mu}\{x\} = 0$ we set $T(x, \mu) := y_0$ for some fixed value $y_0 \in \mathbb{Y}$. If $\bar{\mu}\{x\} > 0$ then $\nu := \mu(\{x\} \times \cdot)$ is an integer-valued measure on $\mathbb{Y}$ with $\nu(\mathbb{X}) = 1$. By Proposition 6.2 there exists a unique $y \in \mathbb{Y}$ such that $\nu\{y\} = 1$, so that we can define $T(x, \mu) := y$. Then (6.17) holds.

It remains to show that the mapping T is measurable. Let $C \in \mathcal{Y}$. Then we have for all $(x, \mu) \in \mathbb{X} \times \mathbf{N}^*$ that

$$\mathbf{1}\{T(x, \mu) \in C\} = \mathbf{1}\{\bar{\mu}\{x\} = 0, y_0 \in C\} + \mathbf{1}\{\mu(\{x\} \times C) > 0\}.$$

Since $\mathbb{X} \times \mathbb{Y}$ is a Borel subspace of the CSMS $\mathbb{X} \times \mathbb{Y}$, Proposition 6.2 implies that $(x, \mu) \mapsto (\mathbf{1}\{\bar{\mu}\{x\} = 0\}, \mathbf{1}\{\mu(\{x\} \times C) > 0\})$ is measurable. This shows that T is measurable. □

Proposition 6.16 *Let ξ be a Poisson process on $\mathbb{X} \times \mathbb{Y}$, where $(\mathbb{X}, \mathcal{X})$ and $(\mathbb{Y}, \mathcal{Y})$ are Borel subspaces of a CSMS. Suppose that the intensity measure of ξ is given by $\lambda \otimes K$, where λ is a locally finite measure on $\mathbb{X}$ and K is a probability kernel from $\mathbb{X}$ to $\mathbb{Y}$. Then ξ is a K-marking of $\eta := \xi(\cdot \times \mathbb{Y})$.*

Proof We first assume that λ is diffuse. Since λ is locally finite and diffuse, we can apply Proposition 6.7 to the Poisson process η to obtain that $\mathbb{P}(\xi \in \mathbf{N}^*) = 1$. It is then no restriction of generality to assume that $\xi \in \mathbf{N}^*$ everywhere on Ω. Lemma 6.15 implies the representation

$$\xi = \sum_{n=1}^{\kappa} \delta_{(X_n, Y_n)},$$

where $\kappa := \xi(\mathbb{X} \times \mathbb{Y})$, and for each $n \in \mathbb{N}$, $X_n := \pi_n(\bar{\xi})$, $Y_n := T(X_n, \xi)$. (Recall that $\bar{\xi} = \xi(\cdot \times \mathbb{Y})$.) We wish to show that the sequence (Y_n) has the properties required in Definition 5.3. Let

$$\tilde{\xi} = \sum_{n=1}^{\kappa} \delta_{(X_n, \tilde{Y}_n)}$$

be a K-marking of η, where the sequence (Y_n) in Definition 5.3 is replaced by $(\tilde{Y}_n)$. From Theorem 5.6 we have that $\xi \overset{d}{=} \tilde{\xi}$. In particular we can again assume that $\tilde{\xi} \in \mathbf{N}^*$ everywhere on Ω. Lemma 6.15 shows that $\tilde{Y}_n := T(X_n, \tilde{\xi})$. Since $X_n = \pi_n(\xi(\cdot \times \mathbb{Y})) = \pi_n(\tilde{\xi}(\cdot \times \mathbb{Y}))$ for $n \le \kappa$, the distributional identity $\xi \overset{d}{=} \tilde{\xi}$ implies for all $m \in \bar{\mathbb{N}}$ that

$$\mathbb{P}((X_n, Y_n)_{n \le m} \in \cdot \mid \kappa = m) = \mathbb{P}((X_n, \tilde{Y}_n)_{n \le m} \in \cdot \mid \kappa = m).$$

This finishes the proof in the case of a diffuse λ.

Now we skip the assumption that λ be diffuse. Proposition 6.2 implies the almost sure representation

$$\xi = \sum_{n=1}^{\kappa} \delta_{(X_n, Y_n)},$$

where $\kappa := \xi(\mathbb{X} \times \mathbb{Y})$ and $(X_n, Y_n) := \pi_n(\xi)$, $n \in \mathbb{N}$. Let $U_1, U_2, \dots$ be independent random variables uniformly distributed on $(0, 1)$, independent of ξ. Then by the marking theorem (Theorem 5.6), the point process

$$\xi' := \sum_{n=1}^{\kappa} \delta_{(U_n, X_n, Y_n)}$$

is Poisson with intensity measure $\lambda_1 \otimes (\lambda \otimes K) = (\lambda_1 \otimes \lambda) \otimes K'$, where λ_1 denotes Lebesgue measure on $(0, 1)$ and $K'((u, x), dy) := K(x, dy)$. Since $\lambda_1 \otimes \lambda$ is diffuse (and locally finite), the first part of the proof shows that ξ' is a K'-marking of

$$\eta' := \sum_{n=1}^{\kappa} \delta_{(U_n, X_n)}.$$

It is now straightforward to check that ξ is a K-marking of η. □

6.6 Exercises

Exercise 6.1 Suppose that $\mathbb{X}$ is a Borel subspace of a CSMS. Show that the mapping $(x,\mu) \mapsto \mu\{x\}$ from $\mathbb{X} \times \mathbf{N}_l(\mathbb{X})$ to $\mathbb{N}_0$ is $\mathcal{B}(\mathbb{X}) \otimes \mathcal{N}_l(\mathbb{X})$-measurable. (Hint: Use Proposition 6.2.)

Exercise 6.2 Give an example to show that if the word 'simple' is omitted from the hypothesis of Rényi's theorem, then the conclusion need not be true. (This can be done by taking a simple point process and modifying it to make it 'complicated', i.e. not simple.)

Exercise 6.3 Give an example to show that if the word 'diffuse' is omitted from the hypothesis of Rényi's theorem, then the conclusion need not be true. (This can be done by taking a 'complicated' point process and modifying it to make it simple.)

Exercise 6.4 Let λ be a locally finite measure on a CSMS $\mathbb{X}$ and let $A := \{x \in \mathbb{X} : \lambda\{x\} = 0\}$. Show that λ_A is diffuse and that $\lambda_{\mathbb{X}\setminus A}$ is purely discrete. Generalize this result to σ-finite measures on a Borel space. (Hint: Show that $\{x \in \mathbb{X} : \lambda\{x\} > 0\}$ is at most countably infinite.)

Exercise 6.5 Give an example to show that if we drop the assumption (6.10) from the conditions of Theorem 6.10, then we cannot always conclude that η is a Poisson process.

Exercise 6.6 Suppose that $(\mathbb{X}, \mathcal{X})$ is a Borel subspace of a CSMS. Define for each $\mu \in \mathbf{N}_l(\mathbb{X})$ the measure $\mu^* \in \mathbf{N}_s(\mathbb{X})$ by

$$\mu^* := \int \mu\{x\}^{\oplus} \mathbf{1}\{x \in \cdot\}\, \mu(dx), \tag{6.18}$$

where $a^{\oplus} := \mathbf{1}\{a \neq 0\} a^{-1}$ is the generalized inverse of $a \in \mathbb{R}$. Prove that the mapping $\mu \mapsto \mu^*$ from $\mathbf{N}_l(\mathbb{X})$ to $\mathbf{N}_l(\mathbb{X})$ is measurable. Prove also that the system of all sets $\{\mu \in \mathbf{N}_l(\mathbb{X}) : \mu(B) = 0\}$, where B is a bounded Borel set, is a π-system generating the σ-field

$$\mathcal{N}^* = \{\{\mu \in \mathbf{N}_l(\mathbb{X}) : \mu^* \in A\} : A \in \mathcal{N}_l(\mathbb{X})\}.$$

(Hint: Check the proof of Theorem 6.9.)

Exercise 6.7 Suppose that $(\mathbb{X}, \mathcal{X})$ is a Borel subspace of a CSMS. Let $\mathbf{N}_s(\mathbb{X})$ denote the space of all locally finite simple measures in $\mathbf{N}(\mathbb{X})$ and let $\mathcal{N}_s(\mathbb{X}) := \{A \cap \mathbf{N}_s(\mathbb{X}) : A \in \mathcal{N}(\mathbb{X})\}$. Show that the system of all sets

$\{\mu \in \mathbf{N}_s(\mathbb{X}) : \mu(B) = 0\}$, where B is a bounded Borel set, is a π-system generating $\mathcal{N}_s(\mathbb{X})$.

Exercise 6.8 Let $\mathbf{N}_{<\infty} := \{\mu \in \mathbf{N}(\mathbb{X}) : \mu(\mathbb{X}) < \infty\}$, where $(\mathbb{X}, \mathcal{X})$ is a Borel space. Show that there are measurable mappings $\pi_n \colon \mathbf{N}_{<\infty} \to \mathbb{X}$ such that (6.1) holds for all $\mu \in \mathbf{N}_{<\infty}$. Show also that these mappings are uniquely determined up to enumeration.

Exercise 6.9 Let η be a uniformly σ-finite point process on a Borel space $(\mathbb{X}, \mathcal{X})$. Show that η is proper.

Exercise 6.10 Let η and η' be point processes on an arbitrary measure space $(\mathbb{X}, \mathcal{X})$. Assume that there are $B_n \in \mathcal{X}$, $n \in \mathbb{N}$, such that $\cup_{n=1}^{\infty} B_n = \mathbb{X}$ and such that (6.15) holds for both η and η'. Prove that $\eta \overset{d}{=} \eta'$ if and only if $\eta_{B_n} \overset{d}{=} \eta'_{B_n}$ for each $n \in \mathbb{N}$.

Exercise 6.11 Let $(\mathbb{X}, \mathcal{X})$ be a Borel subspace of a CSMS, $\mu \in \mathbf{N}_s(\mathbb{X})$ and $m \in \mathbb{N}$ with $m \geq 2$. Show that $\mu^{(m)}(D_m) = 0$, where the generalized diagonal D_m is given by (6.13). Why is D_m a measurable set? (Hint: Use Proposition 6.2 and (4.5).)

7

Poisson processes on the real line

A Poisson process on the real half-line is said to be homogeneous if its intensity measure is a multiple of Lebesgue measure. Such a process is characterized by the fact that the distances between consecutive points are independent and identically exponentially distributed. Using conditional distributions this result can be generalized to position-dependent markings of non-homogeneous Poisson processes. An interesting example of a non-homogeneous Poisson process is given by the consecutive record values in a sequence of independent and identically distributed non-negative random variables.

7.1 The interval theorem

In this chapter we study point processes η on the real half-line $\mathbb{R}_+ := [0, \infty)$. We shall assume that η is simply but will allow for at most one accumulation point of its atoms.

If μ is a measure on $\mathbb{R}_+$ (or on $\mathbb{R}$) and $I \subset \mathbb{R}_+$ is an interval, then we write $\mu I := \nu(I)$. For $\mu \in \mathbf{N}(\mathbb{R}_+)$ set

$$T_n(\mu) := \inf\{t \geq 0 : \mu[0, t] \geq n\}, \quad n \in \overline{\mathbb{N}},$$

where $\inf \emptyset := \infty$. Let $\mathbf{N}^+$ be the space of all measures $\mu \in \mathbf{N}(\mathbb{R}_+)$ such that $\mu[T_\infty(\mu), \infty) = 0$ and $T_n(\mu) < T_{n+1}(\mu)$ whenever $T_n(\mu) < \infty$. In Exercise 7.1 it is asked to show that $\mathbf{N}^+ \in \mathcal{N}(\mathbb{R}_+)$.

Definition 7.1 A point process η on $\mathbb{R}_+$ is said to be *ordinary* if it satisfies $\mathbb{P}(\eta \in \mathbf{N}^+) = 1$.

If η is an ordinary point process we can almost surely write

$$\eta = \sum_{n=1}^{\infty} \mathbf{1}\{T_n < \infty\}\delta_{T_n}, \tag{7.1}$$

where $T_n := T_n(\eta)$ for $n \in \overline{\mathbb{N}}$. Sometimes T_n is called n-th *arrival time* of η. In the case $T_n = \infty$ the measure $\mathbf{1}\{T_n < \infty\}\delta_{T_n}$ has to be interpreted as the zero measure on $\mathbb{R}_+$. The case of *explosion*, that is $T_\infty < \infty$, might occur with positive probability or even with probability one.

An important example of an ordinary point process is a *homogeneous Poisson process* of *rate* (or *intensity*) $\gamma > 0$. This is a Poisson process η on $\mathbb{R}_+$ with intensity measure $\gamma\lambda_+$, where λ_+ is the Lebesgue measure on $\mathbb{R}_+$. (More generally, for $d \in \mathbb{N}$ and $B \in \mathcal{B}(\mathbb{R}^d)$ a *homogeneous Poisson process* on B is a Poisson process η on B whose intensity measure is a multiple of Lebesgue measure on B. This multiple is called *rate* of η.) A homogeneous Poisson process η on $\mathbb{R}_+$ is *stationary* in the sense that

$$\theta_t^+\eta \overset{d}{=} \eta, \quad t \in \mathbb{R}_+, \tag{7.2}$$

where, for any measure μ on $\mathbb{R}_+$ and $t \in \mathbb{R}_+$ the measure $\theta_t^+\mu$ on $\mathbb{R}_+$ is defined by

$$\theta_t^+\mu(B) := \mu(B+t) = \int \mathbf{1}\{s - t \in B\}\,\mu(ds), \quad B \in \mathcal{B}(\mathbb{R}_+). \tag{7.3}$$

Note that the integration in (7.3) can be restricted to $s \geq t$.

Our first aim in this chapter is to characterize homogeneous Poisson processes in terms of the inter-point distances $T_n - T_{n-1}$, where $T_0 := 0$.

Theorem 7.2 (Interval Theorem) *Let η be a point process on $\mathbb{R}_+$. Then η is a homogeneous Poisson process with rate $\gamma > 0$ if and only if the $T_n - T_{n-1}$, $n \geq 1$, are independent and exponentially distributed with parameter γ.*

Proof Suppose first that η is a Poisson process as stated. Let $n \in \mathbb{N}$. Since η is locally finite we have

$$\{T_n \leq t\} = \{\eta[0,t] \geq n\}, \quad \mathbb{P}\text{-a.s.} \tag{7.4}$$

Since $\mathbb{P}(\eta(\mathbb{R}_+) = \infty) = 1$ we have $\mathbb{P}(T_n < \infty) = 1$. Let $f \in \mathbb{R}_+(\mathbb{R}_+^n)$. Then

$$\begin{aligned}\mathbb{E}[f(T_1, T_2 - T_1 \dots, T_n - T_{n-1})] = \mathbb{E}\Big[\int & \mathbf{1}\{t_1 < \cdots < t_n\} \\ \times f(t_1, t_2 - t_1, \dots, t_n - t_{n-1})\mathbf{1}\{\eta[0,t_n) = n-1\}\,&\eta^{(n)}(d(t_1,\dots,t_n))\Big]. \end{aligned}\tag{7.5}$$

Now we use the multivariate Mecke theorem (Theorem 4.4). Since, for $0 \leq t_1 < \cdots < t_n$,

$$\{(\eta + \delta_{t_1} + \cdots + \delta_{t_n})[0,t_n) = n-1\} = \{\eta[0,t_n) = 0\},$$

the right-hand side of (7.5) equals

$$\gamma^n \int \mathbf{1}\{0 < t_1 < \cdots < t_n\} f(t_1, t_2 - t_1, \dots, t_n - t_{n-1}) \exp[-\gamma t_n]\, d(t_1, \dots, t_n),$$

where the integration is with respect to Lebesgue measure on $\mathbb{R}^n$. After the change of variables $s_1 := t_1, s_2 := t_2 - t_1, \dots, s_n := t_n - t_{n-1}$ this yields

$$\begin{aligned}
&\mathbb{E}[f(T_1, T_2 - T_1 \dots, T_n - T_{n-1})] \\
&\quad = \int_0^\infty \dots \int_0^\infty f(s_1, \dots, s_n)\gamma^n \exp[-\gamma(s_1 + \cdots + s_n)]\, ds_1 \cdots ds_n.
\end{aligned}$$

Therefore $T_1, T_2 - T_1, \dots, T_n - T_{n-1}$ are independent and exponentially distributed with parameter γ. Since $n \in \mathbb{N}$ is arbitrary, the asserted properties of the sequence (T_n) follow.

Assume, conversely, that (T_n) has the stated properties. Let η' be a homogeneous Poisson process of intensity $\gamma > 0$. Then η' has a representation as in (7.1) with random variables T'_n instead of T_n. We have just proved that $(T_n) \overset{d}{=} (T'_n)$. Since, for any $B \in \mathcal{B}(\mathbb{R}_+)$,

$$\eta(B) = \sum_{n=1}^\infty \mathbf{1}\{T_n \in B\}$$

is a measurable function of the sequence (T_n), we can use Proposition 2.10 ((ii) implies (i)) to conclude that $\eta \overset{d}{=} \eta'$ and hence η is a homogeneous Poisson process. □

A Poisson process on $\mathbb{R}_+$ whose intensity measure is not a multiple of λ_+ is called *non-homogeneous*. Such a process can be constructed from a homogeneous Poisson process by a suitable time transform. This procedure is a special case of the mapping theorem (Theorem 5.1). Let ν be a locally finite measure on $\mathbb{R}_+$ and define a function $\nu^{\leftarrow} : \mathbb{R}_+ \to [0, \infty]$ by

$$\nu^{\leftarrow}(t) := \inf\{s \geq 0 : \nu[0, s] \geq t\}, \quad t \geq 0, \tag{7.6}$$

where $\inf \emptyset := \infty$. This function is increasing left-continuous and, in particular, measurable.

Proposition 7.3 *Let ν be a locally finite measure on $\mathbb{R}_+$, let η be a homogeneous Poisson process on $\mathbb{R}_+$ with rate 1 and let (T_n) be given by (7.1). Then*

$$\eta' := \sum_{n=1}^\infty \mathbf{1}\{\nu^{\leftarrow}(T_n) < \infty\}\delta_{\nu^{\leftarrow}(T_n)} \tag{7.7}$$

is a Poisson process on $\mathbb{R}_+$ with intensity measure ν.

Proof By the mapping theorem (Theorem 5.1) $\sum_{n=1}^{\infty} \delta_{\nu^{\leftarrow}(T_n)}$ is a Poisson process on $\overline{\mathbb{R}}_+$ with intensity measure

$$\lambda = \int \mathbf{1}\{\nu^{\leftarrow}(t) \in \cdot\}\, dt.$$

Proposition A.25 shows that $\lambda = \nu$, and the assertion follows. □

7.2 Marked Poisson processes

In this section we consider Poisson processes on $\mathbb{R}_+ \times \mathbb{Y}$, where (the mark space) $(\mathbb{Y}, \mathcal{Y})$ is a Borel subspace of a CSMS. Consider a point process ξ on $\mathbb{R}_+ \times \mathbb{Y}$ such that $\eta := \xi(\cdot \times \mathbb{Y})$ is ordinary. Then we have almost surely that

$$\xi = \sum_{n=1}^{\infty} \mathbf{1}\{T_n < \infty\}\delta_{(T_n, Y_n)}, \tag{7.8}$$

where $T_n := T_n(\eta)$, $n \in \mathbb{N}$, and where the Y_n are defined by $\xi\{(T_n, Y_n)\} = 1$ for $T_n < \infty$. For $T_n = \infty$ we let Y_n equal some fixed value in $\mathbb{Y}$. Then the marks Y_n are random elements of $\mathbb{Y}$. In fact, Exercise 7.3 shows that Lemma 6.15 holds with $(\mathbb{X}, \mathbf{N}^*)$ replaced by $(\mathbb{R}_+, \mathbf{N}^+(\mathbb{Y}))$, where $\mathbf{N}^+(\mathbb{Y})$ is the space of all $\mu \in \mathbf{N}(\mathbb{R}_+ \times \mathbb{Y})$ such that $\mu(\cdot \times \mathbb{Y}) \in \mathbf{N}^+$.

If ξ is a Poisson process, then our next result (Theorem 7.4) provides a formula for the distribution of $(T_1, Y_1, \dots, T_n, Y_n)$ in terms of the intensity measure of ξ. Corollary 7.5 will then extend Theorem 7.2 by allowing both for marks and for non-homogeneity of the Poisson process.

For a measure μ on $\mathbb{R}_+ \times \mathbb{Y}$ and $t \geq 0$ we define another measure $\vartheta_t^+ \mu$ on $\mathbb{R}_+ \times \mathbb{Y}$ by

$$\vartheta_t^+ \mu(B) := \int \mathbf{1}\{(s - t, y) \in B\}\, \mu(d(s, y)), \quad B \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Y}.$$

This definition generalizes (7.3). It can be shown that $(t, \mu) \mapsto \vartheta_t^+ \mu$ is measurable on $\mathbb{R}_+ \times \mathbf{N}^+(\mathbb{Y})$; see Lemma 8.6 for a closely related statement.

Theorem 7.4 *Let ξ be a Poisson process on $\mathbb{R}_+ \times \mathbb{Y}$ with a σ-finite intensity measure λ such that $\mathbb{P}(\xi(\cdot \times \mathbb{Y}) \in \mathbf{N}^+) = 1$ and let (T_n, Y_n) be given by* (7.8). *Then the following hold for all $n \in \mathbb{N}$.*

(i) *For any $f \in \mathbb{R}_+((\mathbb{R}_+ \times \mathbb{Y})^n)$,*

$$\mathbb{E}[\mathbf{1}\{T_n < \infty\} f(T_1, Y_1, \dots, T_n, Y_n)] = \int \mathbf{1}\{0 < t_1 < \cdots < t_n\}$$
$$\times f(t_1, y_1, \dots, t_n, y_n) \exp[-\lambda((0, t_n] \times \mathbb{Y})]\, \lambda^n(d(t_1, y_1, \dots, t_n, y_n)).$$

(ii) *The conditional distribution of* $\vartheta^+_{T_n}\xi$ *given* $(T_1, Y_1, \dots, T_n, Y_n)$ *and* $T_n < \infty$ *is almost surely that of a Poisson process with intensity measure* $\vartheta^+_{T_n}\lambda$.

Proof We interpret ξ as a random element of the space $\mathbf{N}^+(\mathbb{Y})$ introduced above. The assumptions and Proposition 6.7 imply that the measure $\lambda(\cdot\times\mathbb{Y})$ is diffuse. We now use the same idea as in the proof of Theorem 7.2. Let f be as in (i) and let $g \in \mathbb{R}_+(\mathbf{N}^+(\mathbb{Y}))$. Then

$$\mathbb{E}[\mathbf{1}\{T_n < \infty\}f(T_1, Y_1, \dots, T_n, Y_n)g(\vartheta^+_{T_n}\xi)] = \mathbb{E}\Big[\int \mathbf{1}\{t_1 < \cdots < t_n\} \\ \times f(t_1, y_1, \dots, t_n, y_n)g(\vartheta^+_{t_n}\xi)\mathbf{1}\{\eta[0, t_n) = n-1\}\,\xi^{(n)}(d(t_1, y_1, \dots, t_n, y_n))\Big].$$

By the Mecke equation this equals

$$\int \mathbf{1}\{t_1 < \cdots < t_n\}f(t_1, y_1, \dots, t_n, y_n) \\ \times \mathbb{E}[g(\vartheta^+_{t_n}\xi)\mathbf{1}\{\eta[0, t_n) = 0\}]\,\lambda^n(d(t_1, y_1, \dots, t_n, y_n)).$$

By Theorem 5.2, $\mathbf{1}\{\eta[0, t_n) = 0\}$ and $g(\vartheta^+_{t_n}\xi)$ are independent for any fixed t_n. Moreover, $\vartheta^+_{t_n}\xi$ is a Poisson process with intensity measure $\vartheta^+_{t_n}\lambda$. Therefore we obtain both (i) and (ii). □

If, in the situation of Theorem 7.4, $\lambda(\mathbb{R}_+ \times \mathbb{Y}) < \infty$, then ξ has only finitely many points and $T_n = \infty$ for $n > \xi(\mathbb{R}_+ \times \mathbb{Y})$. In fact, the theorem implies that, $\mathbb{P}$-a.s. on the event $\{T_n < \infty\}$,

$$\mathbb{P}(T_{n+1} = \infty \mid T_0, Y_0, \dots, T_n, Y_n) = \exp[-\lambda([T_n, \infty) \times \mathbb{Y})].$$

If ν, ν' are measures on a measurable space $(\mathbb{X}, \mathcal{X})$ and $f \in \mathbb{R}_+(\mathbb{X})$, we write $\nu'(dx) = f(x)\,\nu(dx)$ if f is a density of ν' with respect to ν, that is $\nu'(B) = \nu(\mathbf{1}_B f)$ for all $B \in \mathcal{X}$.

Corollary 7.5 *Under the hypotheses of Theorem 7.4, we have for all* $n \in \mathbb{N}_0$ *that*

$$\mathbf{1}\{t < \infty\}\mathbb{P}((T_{n+1} - T_n, Y_{n+1}) \in d(t, y) \mid T_0, Y_0, \dots, T_n, Y_n) \\ = \exp[-\lambda([T_n, T_n + t] \times \mathbb{Y})]\lambda(d(t, y)), \quad \mathbb{P}\text{-a.s. on } \{T_n < \infty\},$$

where $T_0 = 0$ *and* Y_0 *is chosen as a constant function.*

Independent markings of homogeneous Poisson processes can be characterized as follows.

Theorem 7.6 *Let the point process ξ on $\mathbb{R}_+ \times \mathbb{Y}$ be given by* (7.8) *and define η by* (7.1). *Let $\gamma > 0$ and let $\mathbb{Q}$ be a probability measure on $\mathbb{Y}$. Then ξ is an independent $\mathbb{Q}$-marking of a homogeneous Poisson process with rate $\gamma > 0$ if and only if $T_1, Y_1, T_2 - T_1, Y_2 \dots$ are independent, the $T_n - T_{n-1}$ have an exponential distribution with parameter γ, and the Y_n have distribution $\mathbb{Q}$.*

Proof If η is a homogeneous Poisson process and ξ is an independent $\mathbb{Q}$-marking of η, then by Theorem 5.6, ξ is a Poisson process with intensity measure $\gamma\lambda_+ \otimes \mathbb{Q}$. Hence the properties of the sequence $((T_n, Y_n))$ follow from Corollary 7.5 (or from Theorem 7.4). The converse is an immediate consequence of the interval theorem (Theorem 7.2). □

7.3 Record processes

Here we discuss how non-homogeneous Poisson processes describe the occurrence of *record levels* in a sequence $X_1, X_2, \dots$ of independent random variables with values in $\mathbb{R}_+$ and common distribution $\mathbb{Q}$. Let $N_1 := 1$ be the first record time and $R_1 := X_1$ the first record. The further record times $N_2, N_3, \dots$ are defined inductively by

$$N_{k+1} := \inf\{n > N_k : X_n > X_{N_k}\}, \quad k \in \mathbb{N},$$

where $\inf \emptyset := \infty$. The k-th record level is $R_k := X_{N_k}$. We consider the following point process on $\mathbb{R}_+ \times \mathbb{N}$:

$$\chi := \sum_{n=1}^{\infty} \mathbf{1}\{N_{n+1} < \infty\}\delta_{(R_n, N_{n+1}-N_n)}. \tag{7.9}$$

Proposition 7.7 *Let $(X_n)_{n\geq 1}$ be a sequence of independent $\mathbb{R}_+$-valued random variables with common distribution $\mathbb{Q}$ and assume that $\mathbb{Q}$ is diffuse. Then the point process χ on $\mathbb{R}_+ \times \mathbb{N}$ defined by* (7.9) *is a Poisson process whose intensity measure λ is given by*

$$\lambda(dt \times \{k\}) = \mathbb{Q}(0, t]^{k-1}\mathbb{Q}(dt), \quad k \in \mathbb{N}.$$

Proof Let $n \in \mathbb{N}$, $k_1, \dots, k_n \in \mathbb{N}$, and $f \in \mathbb{R}_+(\mathbb{R}_+^n)$. We assert that

$$\begin{aligned}
&\mathbb{E}[\mathbf{1}\{N_2 - N_1 = k_1, \dots, N_{n+1} - N_n = k_n\} f(R_1, \dots, R_n)] \\
&\quad = \int \mathbf{1}\{t_1 < \cdots < t_n\} f(t_1, \dots, t_n) \\
&\qquad\qquad \times \mathbb{Q}[0, t_1]^{k_1-1} \cdots \mathbb{Q}[0, t_n]^{k_n-1}\mathbb{Q}(t_n, \infty)\, \mathbb{Q}^n(d(t_1, \dots, t_n)).
\end{aligned} \tag{7.10}$$

To prove this let A denote the event inside the indicator in the above left-hand side. Set $Y_i = X_{k_1+\cdots+k_i}$ for $1 \le i \le n$, and let $B := \{Y_1 < \cdots < Y_n\}$. Then the left-hand side of (7.10) equals

$$\mathbb{E}[\mathbf{1}_A \mathbf{1}_B f(Y_1, ..., Y_n)] = \mathbb{E}[f(Y_1, \dots, Y_n)\mathbf{1}_B\, \mathbb{P}(A \mid Y_1, \dots, Y_n)],$$

where the identity follows from independence and Fubini's theorem (or, equivalently, by conditioning on $Y_1, \dots, Y_n$). This equals the right-hand side of (7.10).

Effectively, the range of integration in (7.10) can be restricted to $t_n < t_\infty$, where

$$t_\infty := \sup\{t \in \mathbb{R}_+ : \mathbb{Q}[0,t] < 1\}.$$

Indeed, since $\mathbb{Q}$ is diffuse, we have $\mathbb{Q}[t_\infty, \infty) = 0$. Summing in (7.10) over $k_1, \dots, k_n \in \mathbb{N}$, we obtain that

$$\begin{aligned}\mathbb{E}[\mathbf{1}\{N_{n+1} < \infty\} f(R_1, \dots, R_n)]&\\ = \int \mathbf{1}\{t_1 < \cdots < t_n\} f(t_1, \dots, t_n)&\\ \times \mathbb{Q}(t_1,\infty)^{-1} \cdots \mathbb{Q}(t_{n-1},\infty)^{-1}\, \mathbb{Q}^n(d(t_1, \dots, t_n)).& \qquad (7.11)\end{aligned}$$

Taking $f \equiv 1$ and performing the integration yields $\mathbb{P}(N_{n+1} < \infty) = 1$.

Next we note that

$$\lambda(dt \times \mathbb{N}) = (\mathbb{Q}[t,\infty))^{\oplus} \mathbb{Q}(dt), \qquad (7.12)$$

is the *hazard measure* of $\mathbb{Q}$, where $a^{\oplus} := \mathbf{1}\{a \ne 0\} a^{-1}$ is the generalized inverse of $a \in \mathbb{R}$. Therefore Proposition A.26 implies that

$$\mathbb{Q}[t,\infty) = \exp[-\lambda([0,t] \times \mathbb{N})]. \qquad (7.13)$$

Hence we obtain for all $n \in \mathbb{N}$ and $g \in \mathbb{R}_+((\mathbb{R}_+ \times \mathbb{N})^n)$ from (7.10) that

$$\begin{aligned}\mathbb{E}[g(R_1, N_2 - N_1, \dots, R_n, N_{n+1} - N_n)]&\\ = \int \mathbf{1}\{t_1 < \cdots < t_n\} g(t_1, k_1, \dots, t_n, k_n)&\\ \times \exp[-\lambda([0,t] \times \mathbb{N})]\, \lambda^n(d(t_1, k_1, \dots, t_n, k_n)).& \qquad (7.14)\end{aligned}$$

Now let ξ be a Poisson process as in Theorem 7.4 (with $\mathbb{Y} = \mathbb{N}$) with intensity measure λ. The identity (7.13) implies that $\lambda([0,\infty) \times \mathbb{N}) = \infty$, so that $\mathbb{P}(T_n < \infty) = 1$ holds for all $n \in \mathbb{N}$. Comparing (7.14) and Theorem 7.4 (i) yields

$$(R_1, N_2 - N_1, \dots, R_n, N_{n+1} - N_n) \stackrel{d}{=} (T_1, Y_1, \dots, T_n, Y_n), \quad n \in \mathbb{N}.$$

As in the final part of the proof of Theorem 7.2 we obtain $\xi \overset{d}{=} \chi$ and hence the assertion of the theorem. □

Proposition 7.7, (7.12) and the mapping theorem (Theorem 5.1) imply that the point process $\chi(\cdot \times \mathbb{N})$ of successive record levels is a Poisson process with the hazard measure of $\mathbb{Q}$ as intensity measure. Further consequences of the proposition are discussed in Exercise 7.12.

7.4 Polar representation of homogeneous Poisson processes

In this section we discuss how homogeneous Poisson processes naturally show up in a spatial setting. For $d \in \mathbb{N}$ let ν_{d-1} denote the uniform distribution on the *unit sphere* $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$. This normalized *spherical Lebesgue measure* on $\mathbb{S}^{d-1}$ is the probability measure defined by

$$\nu_{d-1}(C) := \kappa_d^{-1} \int_{B^d} \mathbf{1}\{x/\|x\| \in C\}\, dx, \quad C \in \mathcal{B}(\mathbb{S}^{d-1}), \tag{7.15}$$

where $B^d := \{x \in \mathbb{R}^d : \|x\| \le 1\}$ is the *unit ball* and $\kappa_d := \lambda_d(B^d)$ its volume. For $x = 0$ we let $x/\|x\|$ equal some fixed point in $\mathbb{S}^{d-1}$.

Proposition 7.8 (Polar representation of a homogeneous Poisson process) *Let ζ be a homogeneous Poisson process on $\mathbb{R}^d$ with intensity $\gamma > 0$. Then the point process ξ on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ defined by*

$$\xi(A) := \int \mathbf{1}\{(\kappa_d\|x\|^d, x/\|x\|) \in A\}\, \zeta(dx), \quad A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{S}^{d-1}), \tag{7.16}$$

is an independent ν_{d-1}-marking of a homogeneous Poisson process with intensity γ.

Proof By Theorem 5.1 (mapping theorem) and Proposition 6.16 it is sufficient to show that

$$\mathbb{E}[\xi(A)] = \gamma \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbf{1}\{(r, u) \in A\}\, dr\, \nu_{d-1}(du), \quad A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{S}^{d-1}). \tag{7.17}$$

To this end, we need the *polar representation* of Lebesgue measure, which says that

$$\int g(x)\, dx = d\kappa_d \int_{\mathbb{S}^{d-1}} \int_0^\infty r^{d-1} g(ru)\, dr\, \nu_{d-1}(du), \tag{7.18}$$

for all $g \in \mathbb{R}_+(\mathbb{R}^d)$. Indeed, if $g(x) = \mathbf{1}\{\|x\| \le s, x/\|x\| \in C\}$, for $s \ge 0$

and $C \in \mathcal{B}(\mathbb{S}^{d-1})$, then (7.18) follows from definition (7.15) and the scaling properties of Lebesgue measure. Using first Campbell's formula for ζ and then (7.18) yields

$$\begin{aligned}\mathbb{E}[\xi(A)] &= \gamma \int \mathbf{1}\{(\kappa_d\|x\|^d, x/\|x\|) \in A\}\, dx \\ &= \gamma d\kappa_d \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbf{1}\{(\kappa_d r^d, u) \in A\} r^{d-1}\, dr\, \nu_{d-1}(du).\end{aligned}$$

Hence (7.17) follows upon a change of variables. □

Proposition 7.8 can be used along with the interval theorem to simulate the points of a homogeneous Poisson process in order of increasing distance from the origin.

7.5 Exercises

Exercise 7.1 Let $\mu \in \mathbf{N}(\mathbb{R}_+)$ and define the function $f_\mu \in \overline{\mathbb{R}}_+(\mathbb{R}_+)$ as the right-continuous version of $t \mapsto \mu[0,t]$, that is

$$f_\mu(t) := \lim_{s \downarrow t} \mu[0,s], \quad t \geq 0.$$

(If μ is locally finite, then $\mu[0,t] = f_\mu(t)$.) Let $n \in \overline{\mathbb{N}}$ and $t \in \mathbb{R}_+$. Show that $T_n(\mu) \leq t$ if and only if $f_\mu(t) \geq n$. Show that this implies that the T_n are measurable mappings on $\mathbf{N}(\mathbb{R}_+)$ and that $\mathbf{N}^+$ (see Definition 7.1) is a measurable subset of $\mathbf{N}(\mathbb{R}_+)$.

Exercise 7.2 Let T_n be the n-th point of a homogeneous Poisson process η on $\mathbb{R}_+$ with intensity γ. Use (7.4) to show that

$$\mathbb{P}(T_n \in dt) = \frac{\gamma^n}{(n-1)!} t^{n-1} e^{-\gamma t}\, dt.$$

This is a Gamma distribution with scale parameter γ and shape parameter n; see also Example 15.4.

Exercise 7.3 Let $\mathbb{Y}$ be a Borel subspace of a CSMS and let $C \in \mathcal{B}(\mathbb{Y})$. Show that $(t,\mu) \mapsto \mu(\{t\} \times C)$ is a measurable mapping on $\mathbb{R}_+ \times \mathbf{N}^+(\mathbb{Y})$. Show that Lemma 6.15 remains true when replacing $(\mathbb{X}, \mathbf{N}^*)$ by $(\mathbb{R}_+, \mathbf{N}^+(\mathbb{Y}))$. (Hint: It suffices to prove $(t,\mu) \mapsto \mathbf{1}\{t \neq T_\infty(\mu)\}\mu(\{t\} \times C)$ is measurable, which can be done by a limit procedure. Check the proof of Lemma 6.15 to see the second assertion.)

Exercise 7.4 Assume that the hypotheses of Theorem 7.4 apply, and that there is a probability kernel J from $\mathbb{R}_+$ to $(\mathbb{Y}, \mathcal{Y})$ such that

$$\lambda(d(t,y)) = J(t,dy)\,\lambda(dt \times \mathbb{Y}).$$

(By Theorem A.13 this is no restriction of generality.) Show for all $n \in \mathbb{N}$ that

$$\mathbb{P}(Y_n \in dy \mid T_1, Y_1, \dots, Y_{n-1}, T_n) = J(T_n, dy), \quad \mathbb{P}\text{-a.s. on } \{T_n < \infty\}.$$

Exercise 7.5 Let η be a homogeneous Poisson process of intensity $\gamma > 0$. Prove the following law of large numbers:

$$\lim_{t\to\infty} \frac{\eta[0,t]}{t} = \gamma, \quad \mathbb{P}\text{-a.s.} \tag{7.19}$$

(Hint: Use the fact that $\eta[0,n]/n$ satisfies a law of large numbers; see Theorem B.5.)

Exercise 7.6 Let η be a Poisson process on $\mathbb{R}_+$, whose intensity measure ν satisfies $0 < \nu(t) < \infty$ for all sufficiently large t and $\nu(\infty) = \infty$. Use Exercise 7.5 to prove that

$$\lim_{t\to\infty} \frac{\eta[0,t]}{\nu[0,t]} = 1, \quad \mathbb{P}\text{-a.s.}$$

Exercise 7.7 Let η_+ and η_- be two independent homogeneous Poisson processes on $\mathbb{R}_+$ with intensity γ. Define a point process η on $\mathbb{R}$ by

$$\eta(B) := \eta_+(B \cap \mathbb{R}_+) + \eta_-(B^* \cap \mathbb{R}_+), \quad B \in \mathcal{B}^1,$$

where $B^* := \{-t : t \in B\}$, and show that η is a homogeneous Poisson process on $\mathbb{R}$.

Exercise 7.8 Let η be a point process on $\mathbb{R}_+$ with intensity measure λ_+ and let ν be a locally finite measure on $\mathbb{R}_+$. Show that the point process η' defined by (7.7) has intensity measure ν. (Hint: Use Theorem 5.1 and the properties of generalized inverses; see the proof of Proposition 7.3.)

Exercise 7.9 Show that Proposition 7.7 remains valid for a sequence $(X_n)_{n\geq 1}$ of independent and identically distributed random elements of $\overline{\mathbb{R}}_+$, provided that the distribution of X_1 is diffuse on $\mathbb{R}_+$.

Exercise 7.10 Let T be a random element of $\overline{\mathbb{R}}_+$. Show that there is a Poisson process η on $\mathbb{R}_+$ such that $T \overset{d}{=} T_1(\eta)$. (Hint: Use Exercise 7.9.)

Exercise 7.11 Suppose the assumptions of Proposition 7.7 hold. For $n \in \mathbb{N}$ let $M_n := \max\{X_1, \dots, X_n\}$ (running maximum) and for $t \in \mathbb{R}_+$ let

$$L_t = \int \mathbf{1}\{s \le t\} k \chi(d(s, k)).$$

Show that $\mathbb{P}$-a.s. $\inf\{n \in \mathbb{N} : M_n > t\} = 1 + L_t$, provided $\mathbb{Q}[0, t] > 0$. Hence $L_t + 1$ is the first time the running maximum exceeds the level t.

Exercise 7.12 Suppose the assumptions of Proposition 7.7 hold and define L_t as in Exercise 7.11. Show that L_a and $L_b - L_a$ are independent whenever $0 \le a < b$. Show also that

$$\mathbb{E}[w^{L_b - L_a}] = \frac{\mathbb{Q}(b, \infty)(1 - w\,\mathbb{Q}(b, \infty))}{\mathbb{Q}(a, \infty)(1 - w\,\mathbb{Q}(a, \infty))}, \quad w \in [0, 1], \tag{7.20}$$

whenever $\mathbb{Q}[0, b] < 1$. Use this formula to prove that

$$\mathbb{P}(L_b - L_a = n) = \frac{\mathbb{Q}(b, \infty)}{\mathbb{Q}(a, \infty)} \mathbb{Q}(a, b] \mathbb{Q}(b)^{n-1}, \quad n \in \mathbb{N}. \tag{7.21}$$

(Hint: Use Theorem 3.9, Proposition A.25 and the logarithmic series to prove (7.20). Then compare the result with the probability generating function of the right-hand side of (7.21).)

Exercise 7.13 Let the assumptions of Proposition 7.7 hold. For $j \in \mathbb{N}$ let I_j be the indicator of the event that X_j is a record. Use a direct combinatorial argument to show that $I_1, I_2, \dots$ are independent with $\mathbb{P}(I_j = 1) = 1/j$.

Exercise 7.14 By setting $g(x) := e^{-\|x\|^2/2}$ in (7.18), show that the volume of the unit ball $B^d \subset \mathbb{R}^d$ is given by $\kappa_d = 2\pi^{d/2}/\Gamma(1 + d/2)$, where the Gamma function $\Gamma(\cdot)$ is defined by (B.9).

8

Stationary point processes

A point process $\mathbb{R}^d$ is said to be stationary, if it looks statistically the same from all sites of $\mathbb{R}^d$. In this case the intensity measure is a multiple of Lebesgue measure. The Palm distribution of a stationary point process can be introduced via a refined Campbell theorem. It describes the behaviour of the point process as seen from a randomly chosen point of the process, shifted to the origin. For ergodic point processes this will be made precise in Chapter 10. The Mecke characterization implies that a stationary point process is Poisson if and only if its Palm distribution is the distribution of the original process with an extra point added at the origin.

8.1 Stationarity

In this chapter we fix $d \in \mathbb{N}$ and consider point processes η on the Euclidean space $\mathbb{X} = \mathbb{R}^d$ for some $d \in \mathbb{N}$. To distinguish between points of η and elements of $\mathbb{R}^d$ we call the latter *sites*. Stationarity is an important invariance concept in probability theory. Our aim here is to discuss a few basic properties of stationary point processes, using the Poisson process as illustration.

The formal definition of stationarity is based on the family of *shifts* $\theta_y \colon \mathbf{N} \to \mathbf{N}$, $y \in \mathbb{R}^d$, defined by

$$\theta_y \mu(B) := \mu(B + y), \quad \mu \in \mathbf{N},\ B \in \mathcal{B}^d, \tag{8.1}$$

where $B + y := \{x + y : x \in B\}$ and $\mathcal{B}^d := \mathcal{B}(\mathbb{R}^d)$ is the Borel σ-field on $\mathbb{R}^d$. We write $B - y := B + (-y)$. A good way to memorize (8.1) is the formula $\theta_y \delta_y = \delta_0$, where 0 is the origin in $\mathbb{R}^d$. Definition (8.1) is equivalent to

$$\int g(x)\,(\theta_y \mu)(dx) = \int g(x - y)\,\mu(dx), \quad \mu \in \mathbf{N},\ g \in \mathbb{R}_+(\mathbb{R}^d). \tag{8.2}$$

We note that θ_0 is the identity on $\mathbf{N}$ and the *flow property* $\theta_y \circ \theta_x = \theta_{x+y}$ for all $x, y \in \mathbb{R}^d$. For any fixed $y \in \mathbb{R}^d$, the mapping θ_y is measurable.

Definition 8.1 A point process η on $\mathbb{R}^d$ is said to be *stationary* if $\theta_x \eta \overset{d}{=} \eta$ for all $x \in \mathbb{R}^d$.

Let λ^d denote Lebesgue measure on $\mathbb{R}^d$. Under a natural integrability assumption the intensity measure of a stationary point process is a multiple of λ_d.

Proposition 8.2 *Let η be a stationary point process on $\mathbb{R}^d$ such that the quantity*

$$\gamma := \mathbb{E}[\eta([0,1]^d)] \tag{8.3}$$

is finite. Then the intensity measure of η equals $\gamma\lambda_d$.

Proof Stationarity of η implies that its intensity measure is translation invariant, that is $\lambda(B + x) = \lambda(B)$ for all $B \in \mathcal{B}^d$ and all $x \in \mathbb{R}^d$. Moreover, $\lambda([0, 1]^d) = \gamma < \infty$. It is a fundamental result from measure theory that $\gamma\lambda_d$ is the only measure with these two properties. □

The number γ given by (8.3) is called the *intensity* of η. For a stationary Poisson process this intensity determines the distribution:

Proposition 8.3 *Let η be a Poisson process on $\mathbb{R}^d$ such that the quantity γ defined by* (8.3) *is finite. Then η is stationary if and only if the intensity measure λ of η equals $\gamma\lambda_d$.*

Proof In view of Proposition 8.2 we need only to show that $\lambda = \gamma\lambda_d$ implies that $\theta_x \eta$ has the same distribution as η for all $x \in \mathbb{R}^d$. Since θ_x preserves Lebesgue measure, this follows from Theorem 5.1 (mapping theorem) or by a direct check of Definition 3.1. □

Our next result shows that a stationary point process cannot have a positive but finite number of points. For any $\mu \in \mathbf{N}$ we denote the *support* of μ by

$$\operatorname{supp}\mu := \{x : \mu\{x\} > 0\}. \tag{8.4}$$

Proposition 8.4 *Suppose that η is a stationary point process on $\mathbb{R}^d$. Then* $\mathbb{P}(0 < \eta(\mathbb{R}^d) < \infty) = 0$.

Proof Assume on the contrary that $\mathbb{P}(0 < \eta(\mathbb{R}^d) < \infty) > 0$ and consider the conditional probability measure $\mathbb{P}' := \mathbb{P}(\cdot \mid 0 < \eta(\mathbb{R}^d) < \infty)$. Since $\eta(\mathbb{R}^d) = \theta_x \eta(\mathbb{R}^d)$, η is stationary under $\mathbb{P}'$. For each non-empty finite set $B \subset \mathbb{R}^d$ let $l(B)$ denote the lexicographic minimum of B and note that $l(B + x) = l(B) + x$, for all $x \in \mathbb{R}^d$. For all other sets $B \subset \mathbb{R}^d$ let $l(B) := 0$.

By Proposition 6.2, $\mu \mapsto l(\operatorname{supp}\mu)$ is a measurable mapping on $\mathbf{N}_{<\infty}$. For $x \in \mathbb{R}^d$ the random variable $l(\operatorname{supp}\eta)$ has under $\mathbb{P}'$ the same distribution as

$$l(\operatorname{supp}\theta_x\eta) = l((\operatorname{supp}\eta) - x) = l(\operatorname{supp}\eta) - x,$$

where the second identity holds $\mathbb{P}'$-a.s. This contradicts the fact that there is no translation invariant probability measure on $\mathbb{R}^d$. □

In what follows we shall consider only locally finite point processes η. It is then convenient (and no loss of generality) to assume that $\eta(\omega) \in \mathbf{N}_l$ for all $\omega \in \Omega$, where $\mathbf{N}_l := \mathbf{N}_l(\mathbb{R}^d) \in \mathcal{N}$ is the space of locally finite measures from $\mathbf{N}$ as in Definition 2.11. The advantage is that Lemma 8.6 below shows that the mapping $(x, \mu) \mapsto \theta_x\mu$ is measurable on $\mathbb{R}^d \times \mathbf{N}_l$.

8.2 The Palm distribution

If η is stationary, then the distribution of η does not change under a shift of the origin. We also say that $\mathbb{P}_\eta = \mathbb{P}(\eta \in \cdot)$ is the *stationary distribution* of η. We now introduce another distribution that describes η as seen from a *typical point* of η located at the origin. In Chapter 10 we shall make this precise under an additional *ergodicity* hypothesis. (The concept of ergodicity is defined before Proposition 8.10.)

Theorem 8.5 (Refined Campbell Theorem) *Suppose that η is a stationary point process on $\mathbb{R}^d$ with finite positive intensity γ. Then there exists a unique probability measure $\mathbb{P}^0_\eta$ on $\mathbf{N}_l$ such that*

$$\mathbb{E}\Big[\int f(x, \theta_x\eta)\,\eta(dx)\Big] = \gamma \iint f(x, \mu)\,\mathbb{P}^0_\eta(d\mu)\,dx, \quad f \in \mathbb{R}_+(\mathbb{R}^d \times \mathbf{N}_l). \tag{8.5}$$

Proof Since the intensity of η is finite, Proposition 8.2 shows that η is locally finite. For each $A \in \mathcal{N}_l$ we define a measure ν_A on $\mathbb{R}^d$ by

$$\nu_A(B) := \mathbb{E}\Big[\int \mathbf{1}_B(x)\mathbf{1}_A(\theta_x\eta)\,\eta(dx)\Big], \quad B \in \mathcal{B}^d,$$

where the integrations are justified by Lemma 8.6. By definition of $\nu_A(\cdot)$ and (8.2) we have for all $y \in \mathbb{R}^d$ that

$$\begin{aligned}\nu_A(B + y) &= \mathbb{E}\Big[\int \mathbf{1}_B(x - y)\mathbf{1}_A(\theta_x\eta)\,\eta(dx)\Big]\\ &= \mathbb{E}\Big[\int \mathbf{1}_B(x)\mathbf{1}_A(\theta_{x+y}\eta)\,(\theta_y\eta)(dx)\Big].\end{aligned}$$

Because of the flow property $\theta_{x+y}\eta = \theta_x(\theta_y\eta)$ we can use stationarity to conclude that

$$\nu_A(B+y) = \mathbb{E}\Big[\int \mathbf{1}_B(x)\mathbf{1}_A(\theta_x\eta)\,\eta(dx)\Big] = \nu_A(B),$$

so that ν_A is translation invariant. Furthermore, $\nu_A(B) \le \mathbb{E}[\eta(B)] = \gamma\lambda_d(B)$, so that ν_A is locally finite. Hence there is a number $\gamma_A \ge 0$ such that

$$\nu_A(B) = \gamma_A\lambda_d(B), \quad B \in \mathcal{B}^d. \tag{8.6}$$

Choosing $B = [0,1]^d$ shows that $\gamma_A = \nu_A([0,1]^d)$ is a measure in A. Since $\gamma_{\mathbf{N}_l} = \gamma$, the definition $\mathbb{P}^0_\eta(A) := \gamma_A/\gamma$ yields a probability measure $\mathbb{P}^0_\eta$ and (8.6) implies that

$$\mathbb{E}\Big[\int \mathbf{1}_B(x)\mathbf{1}_A(\theta_x\eta)\,\eta(dx)\Big] = \gamma\,\mathbb{P}^0_\eta(A)\lambda_d(B), \quad A \in \mathcal{N}_l,\ B \in \mathcal{B}^d. \tag{8.7}$$

Hence (8.5) holds for functions of the form $f(x,\mu) = \mathbf{1}_B(x)\mathbf{1}_A(\mu)$ and then also for general measurable indicator functions by the monotone class theorem (Theorem A.1). Linearity of the integral and monotone convergence yield (8.5) for general $f \in \mathbb{R}_+(\mathbb{R}^d \times \mathbf{N}_l)$.

Conversely, (8.5) yields (8.7) and therefore

$$\mathbb{P}^0_\eta(A) = \frac{1}{\gamma\lambda_d(B)}\mathbb{E}\Big[\int \mathbf{1}_B(x)\mathbf{1}_A(\theta_x\eta)\,\eta(dx)\Big], \tag{8.8}$$

provided that $0 < \lambda_d(B) < \infty$. □

Clearly (8.8) extends to

$$\int f(\mu)\,\mathbb{P}^0_\eta(d\mu) = \frac{1}{\gamma\lambda_d(B)}\mathbb{E}\Big[\int \mathbf{1}_B(x) f(\theta_x\eta)\,\eta(dx)\Big], \tag{8.9}$$

whenever $0 < \lambda_d(B) < \infty$ and $f \in \mathbb{R}_+(\mathbf{N}_l)$. Multiplying this identity by $\gamma\lambda_d(B)$ yields a special case of (8.5).

We still need to prove the following measurability assertion.

Lemma 8.6 *The mapping $(x,\mu) \mapsto \theta_x\mu$ from $\mathbb{R}^d \times \mathbf{N}_l$ to $\mathbf{N}_l$ is measurable.*

Proof Since the system of all compact subsets of $\mathbb{R}^d$ is a π-system generating $\mathcal{B}^d$, the monotone class theorem shows that $\mathcal{N}_l$ is the smallest σ-field such that $\mu \mapsto \mu(B)$ is measurable for all compact sets B. Hence it is sufficient to show that $(x,\mu) \mapsto \theta_x\mu(B)$ is measurable for each compact $B \subset \mathbb{R}^d$. Pick such a B. For each $n \in \mathbb{N}$, let $C_n := [0, 2^{-n})^d$ denote the half-open cube of side length 2^{-n}. $A_n := \{2^{-n}y : y \in \mathbb{Z}^d\}$ denote the grid of size 2^{-n}. Let $B_n \subset \mathbb{R}^d$ be the set of all points of the form $y + w$, where $y \in B$ and $\|w\| \le d^{1/2}2^{-n}$, where $\|\cdot\|$ denotes the Euclidean norm. For a given $x \in \mathbb{R}^d$

and any $n \in \mathbb{N}$ there is a unique $z_n \in A_n$ such that $x \in C_n + z_n$. It is easily seen that $B_{n+1} + z_{n+1} \subset B_n + z_n$ and $\cap_n(B_n + z_n) = B + x$. Hence it follows that

$$\mu(B + x) = \lim_{n\to\infty} \sum_{z\in A_n} \mathbf{1}_{C_n+z}(x)\mu(B_n + z)$$

is the monotone limit of measurable functions (of (x, μ)) and hence measurable. $\square$

Definition 8.7 Under the assumptions of Theorem 8.5 the measure $\mathbb{P}^0_\eta$ is called the *Palm distribution* of η.

Sometimes we shall use the refined Campbell theorem in the equivalent form

$$\mathbb{E}\Big[\int f(x, \eta)\, \eta(dx)\Big] = \gamma \iint f(x, \theta_{-x}\mu)\, \mathbb{P}^0_\eta(d\mu)\, dx, \quad f \in \mathbb{R}_+(\mathbb{R}^d \times \mathbf{N}_l). \tag{8.10}$$

Indeed, for any $f \in \mathbb{R}_+(\mathbb{R}^d \times \mathbf{N}_l)$ we can apply (8.5) with $\tilde{f} \in \mathbb{R}_+(\mathbb{R}^d \times \mathbf{N}_l)$ defined by $\tilde{f}(x, \mu) := f(x, \theta_{-x}\mu)$.

It follows from Proposition 6.2 that the mapping $(x, \mu) \mapsto \mathbf{1}\{\mu\{x\} > 0\}$ is measurable on $\mathbb{R}^d \times \mathbf{N}_l$. Indeed, we have

$$\mathbf{1}\{\mu\{x\} = 0\} = \prod_{n=1}^{\mu(\mathbb{X})} \mathbf{1}\{\pi_n(\mu) \neq x\}.$$

By (8.8) we have for a stationary point process η that

$$\mathbb{P}^0_\eta(\{\mu \in \mathbf{N}_l : \mu\{0\} > 0\}) = \gamma^{-1}\mathbb{E}\Big[\int \mathbf{1}_{[0,1]^d}(x)\mathbf{1}\{\eta\{x\} > 0\}\, \eta(dx)\Big] = 1, \tag{8.11}$$

so that $\mathbb{P}^0_\eta$ is concentrated on those $\mu \in \mathbf{N}_l$ having an atom at the origin. If η is simple, we shall see in Proposition 9.4 that $\mathbb{P}^0_\eta$ can be interpreted as the conditional distribution of η given that $\eta\{0\} > 0$.

We finish this section with a result on deterministic translation-invariant thinnings of stationary point processes, needed in Chapter 10.

Lemma 8.8 *Let ξ be a stationary point process on $\mathbb{R}^d$ with a finite positive intensity γ. Let $g\colon \mathbb{R}^d \times \mathbf{N}_l \to \{0, 1\}$ be a measurable function with the invariance property*

$$g(x - y, \theta_y\mu) = g(x, \mu), \quad x, y \in \mathbb{R}^d,\ \mu \in \mathbf{N}_l. \tag{8.12}$$

Then

$$\xi' := \int \mathbf{1}\{x \in \cdot\} g(x,\xi)\,\xi(dx) \tag{8.13}$$

is a stationary point process with intensity

$$\mathbb{E}[\xi'([0,1]^d)] = \gamma \int g(0,\mu)\,\mathbb{P}^0_\xi(d\mu).$$

Proof Define a mapping $T\colon \mathbf{N}_l \to \mathbf{N}_l$ by

$$T(\mu)(B) := \int_B g(y,\mu)\,\mu(dy), \quad B \in \mathcal{B}^d,\ \mu \in \mathbf{N}_l.$$

Then $\xi' = T(\xi)$ and Proposition 2.7 (Campbell's formula) shows that T is measurable. For $\mu \in \mathbf{N}$, $x \in \mathbb{R}^d$ and $B \in \mathcal{B}^d$ we have

$$\begin{aligned}(\theta_x T(\mu))(B) = T(\mu)(B+x) &= \int \mathbf{1}\{y - x \in B\} g(y,\mu)\,\mu(dy)\\ &= \int_B g(y+x,\mu)\,(\theta_x\mu)(dy) = \int_B g(y,\theta_x\mu)\,(\theta_x\mu)(dy),\end{aligned}$$

where we have used the invariance assumption (8.12) for the final identity. Therefore $\theta_x \circ T = T \circ \theta_x$ and the first assertion follows because $\theta_x\xi \overset{d}{=} \xi$.

The second assertion follows from the identities

$$\mathbb{E}[\xi([0,1]^d)] = \mathbb{E}\Big[\int_{[0,1]^d} g(y,\xi)\,\xi(dy)\Big] = \mathbb{E}\Big[\int_{[0,1]^d} g(0,\theta_y\xi)\,\xi(dy)\Big]$$

and the refined Campbell theorem (Theorem 8.5). □

8.3 The Mecke-Slivnyak theorem

Our next result is a stationary version of Mecke's theorem (Theorem 4.1).

Theorem 8.9 (Mecke-Slivnyak Theorem) *Let η be a stationary point process on $\mathbb{R}^d$ with positive finite intensity. Then η is a Poisson process if and only if*

$$\mathbb{P}^0_\eta = \mathbb{P}(\eta + \delta_0 \in \cdot). \tag{8.14}$$

Proof Let γ denote the intensity of η. Assume first that η is a Poisson process. For any $A \in \mathcal{N}_l$ we then obtain from the Mecke equation (4.1) that

$$\begin{aligned}\mathbb{E}\Big[\int \mathbf{1}_{[0,1]^d}(x)\mathbf{1}_A(\theta_x\eta)\,\eta(dx)\Big] &= \gamma\,\mathbb{E}\Big[\int \mathbf{1}_{[0,1]^d}(x)\mathbf{1}_A(\theta_x(\eta+\delta_x))\,dx\Big]\\ = \gamma \int \mathbf{1}_{[0,1]^d}(x)\mathbb{P}(\eta+\delta_0 \in A)\,dx &= \gamma\,\mathbb{P}(\eta+\delta_0 \in A),\end{aligned}$$

where we have used Fubini's theorem and stationarity for the second identity. Hence (8.14) follows from (8.8) with $B = [0,1]^d$.

Conversely, if (8.14) holds, we take $f \in \mathbb{R}_+(\mathbb{R}^d \times \mathbf{N}_l)$ to obtain from (8.10) that

$$\mathbb{E}\Big[\int f(x,\eta)\,\eta(dx)\Big] = \gamma \int \mathbb{E}[f(x,\theta_{-x}(\eta+\delta_0))]\,dx.$$

Stationarity yields that the Mecke equation (4.1) holds with $\lambda = \gamma\lambda_d$. Theorem 4.1 then implies that η is a Poisson process. □

8.4 Ergodicity

Sometimes stationary point processes satisfy a useful zero-one law. The *invariant σ-field* is defined by

$$\mathcal{I}_l := \{A \in \mathcal{N}_l : \theta_x A = A \text{ for all } x \in \mathbb{R}^d\}, \tag{8.15}$$

where $\theta_x A := \{\theta_x\mu : \mu \in A\}$. The point process η is said to be *ergodic* if $\mathbb{P}(\eta \in A) \in \{0,1\}$ for all $A \in \mathcal{I}_l$. Recall that a function $h\colon \mathbb{R}^d \to \mathbb{R}$ satisfies $\lim_{\|x\|\to\infty} h(x) = c$ for some $c \in \mathbb{R}$ if for each $\varepsilon > 0$ there is an $c > 0$ such that each $x \in \mathbb{R}^d$ with $\|x\| > c$ satisfies $|f(x) - c| \le \varepsilon$.

Proposition 8.10 *Suppose that η is a stationary point process on $\mathbb{R}^d$. Assume that there exists a π-system $\mathcal{H}$ generating $\mathcal{N}_l$ and that*

$$\lim_{\|x\|\to\infty} \mathbb{P}(\eta \in A, \theta_x\eta \in A') = \mathbb{P}(\eta \in A)\mathbb{P}(\eta \in A') \tag{8.16}$$

for all $A, A' \in \mathcal{H}$. Then (8.16) *holds for all $A, A' \in \mathcal{N}_l$.*

Proof We shall use the monotone class theorem (Theorem A.1). First fix $A' \in \mathcal{N}_l$. Let $\mathcal{D}$ be the class of sets $A \in \mathcal{N}_l$ satisfying (8.16). Then $\mathbf{N}_l \in \mathcal{D}$ and $\mathcal{D}$ is closed with respect to proper differences. Let $A_n \in \mathcal{H}$, $n \in \mathbb{N}$, be such that $A_n \uparrow A$ for some $A \in \mathcal{N}_l$. Then we have for all $x \in \mathbb{R}^d$ that

$$\begin{aligned}
&|\mathbb{P}(\eta \in A)\mathbb{P}(\eta \in A') - \mathbb{P}(\eta \in A, \theta_x\eta \in A')|\\
&\quad\le |\mathbb{P}(\eta \in A)\mathbb{P}(\eta \in A') - \mathbb{P}(\eta \in A_n)\mathbb{P}(\eta \in A')|\\
&\qquad + |\mathbb{P}(\eta \in A_n)\mathbb{P}(\eta \in A') - \mathbb{P}(\eta \in A_n, \theta_x\eta \in A')|\\
&\qquad + |\mathbb{P}(\eta \in A_n, \theta_x\eta \in A') - \mathbb{P}(\eta \in A, \theta_x\eta \in A')|.
\end{aligned}$$

For large n the first and third terms are small uniformly in $x \in \mathbb{R}^d$. For any fixed $n \in \mathbb{N}$ the second term tends to 0 as $\|x\| \to \infty$. It follows that $A \in \mathcal{D}$. Hence $\mathcal{D}$ is a Dynkin system and Theorem A.1 shows that $\sigma(\mathcal{H}) = \mathcal{N}_l$. Therefore (8.16) holds for all $A \in \mathcal{N}_l$ and $A' \in \mathcal{H}$.

Now fix $A \in \mathcal{N}_l$ and let $\mathcal{D}'$ be the class of sets $A' \in \mathcal{N}_l$ satisfying (8.16). It follows as before that $\mathcal{D}'$ is a Dynkin system. When checking that $\mathcal{D}'$ contains any monotone union A' of sets $A'_n \in \mathcal{D}'$ one has to use the fact that

$$\begin{aligned}|\mathbb{P}(\eta \in A, \theta_x\eta \in A'_n) - \mathbb{P}(\eta \in A, \theta_x\eta \in A')| &\le \mathbb{P}(\theta_x\eta \in A') - \mathbb{P}(\theta_x\eta \in A'_n) \\ &= \mathbb{P}(\eta \in A') - \mathbb{P}(\eta \in A'_n),\end{aligned}$$

where the second equality comes from the stationarity of η. Theorem A.1 shows that $\mathcal{D}'$ contains all $A' \in \mathcal{N}_l$, implying the assertion. □

Any stationary point process η satisfying (8.16) for all $A, A' \in \mathcal{N}_l$ is called *mixing*. A point process with this property is ergodic. Indeed, if $A \in \mathcal{N}_l$ satisfies $\theta_x A = A$ for all $x \in \mathbb{R}^d$, then we can take $A = A'$ in (8.16) and conclude that $\mathbb{P}(\eta \in A) = (\mathbb{P}(\eta \in A))^2$. The reader will see later (see Exercise 10.1) that Proposition 8.10 is based on a generic principle that applies in more general settings.

Proposition 8.11 *Let η be a stationary Poisson process with finite intensity. Then η is mixing and in particular is ergodic.*

Proof Proposition 6.7 shows that η is simple, that is $\mathbb{P}(\eta \in \mathbf{N}_s) = 1$, where $\mathbf{N}_s := \mathbf{N}_s(\mathbb{R}^d)$. Define $\mathcal{N}_s := \{A \cap \mathbf{N}_s : A \in \mathcal{N}_l\}$. Let $\mathcal{H}$ denote the system of all sets of the form $\{\mu \in \mathbf{N}_s : \mu(B) = 0\}$ for some bounded $B \in \mathcal{B}^d$. By Exercise 6.7, $\mathcal{H}$ is a π-system and generates $\mathcal{N}_s := \{A \cap \mathbf{N}_s : A \in \mathcal{N}_l\}$. Let $B, B' \in \mathcal{B}^d$ be bounded. Let $A = \{\mu \in \mathbf{N}_s : \mu(B) = 0\}$ and define A' similarly in terms of B'. For $x \in \mathbb{R}^d$ we have by (8.1)

$$\{\theta_x\eta \in A'\} = \{\eta(B' + x) = 0\}.$$

If $\|x\|$ is sufficiently large, then $B \cap (B' + x) = \emptyset$, so that the events $\{\eta \in A\}$ and $\{\theta_x\eta \in A'\}$ are independent. Therefore,

$$\mathbb{P}(\eta \in A, \theta_x\eta \in A') = \mathbb{P}(\eta \in A)\,\mathbb{P}(\theta_x\eta \in A') = \mathbb{P}(\eta \in A)\,\mathbb{P}(\eta \in A'),$$

implying (8.16). Since $\mathbb{P}(\eta \in \mathbf{N}_s) = \mathbb{P}(\theta_x\eta \in \mathbf{N}_s) = 1$ for all $x \in \mathbb{R}^d$, the proof of Proposition 8.10 shows that (8.16) holds for all $A, A' \in \mathcal{N}_l$. □

8.5 Exercises

Exercise 8.1 Let η be a stationary point process on $\mathbb{R}$. Prove that

$$\mathbb{P}(\eta \ne 0, \eta((-\infty, 0]) < \infty) = \mathbb{P}(\eta \ne 0, \eta([0, \infty)) < \infty) = 0.$$

Exercise 8.2 Let η be a stationary simple point process on $\mathbb{R}^d$ with positive and finite intensity. Show that $\mathbb{P}^0_\eta(\mathbf{N}_s(\mathbb{R}^d)) = 1$.

Exercise 8.3 Let $\mathbb{Q}$ be a probability measure on $\mathbf{N}(\mathbb{R}^d)$ and let K be the probability kernel from $\mathbb{R}^d$ to $\mathbf{N}(\mathbb{R}^d)$ defined by $K(x, A) := \mathbb{Q}(\theta_x A)$. Let η be a stationary Poisson process on $\mathbb{R}^d$ with a (finite) intensity $\gamma \geq 0$ and let χ be a Poisson cluster process as in Exercise 5.6. Show that χ is stationary with intensity $\gamma \int \mu([0, 1]^d)\, \mathbb{Q}(d\mu)$. (Hint: Use the Laplace functional in Exercise 5.6 to prove the stationarity.)

Exercise 8.4 Let $C := [0, n)^d + a$ be a half-open cube of side length $n \in \mathbb{N}$. Let $\mu \in \mathbf{N}_l(\mathbb{R}^d)$ such that $\mu(\mathbb{R}^d \setminus C) = 0$ and let X be uniformly distributed on C. Show that

$$\eta := \sum_{y \in n\mathbb{Z}^d} \theta_{y+X}\mu$$

is a stationary point process with intensity $\mu(C)/\lambda_d(C)$. Is this point process ergodic?

Exercise 8.5 (Palm-Khinchin equations) Let η be a stationary simple point process on $\mathbb{R}$ with finite intensity γ such that $\mathbb{P}(\eta = 0) = 0$. Let $x \geq 0$ and $j \in \mathbb{N}_0$. Show that $\mathbb{P}$-almost surely

$$\mathbf{1}\{\eta(0, x] \leq j\} = \int \mathbf{1}\{t \leq 0\}\mathbf{1}\{\eta(t, x] = j\}\, \eta(dt).$$

Then use the refined Campbell theorem to prove that

$$\mathbb{P}(\eta(0, x] \leq j) = \gamma \int_x^\infty \mathbb{P}(\eta^0(0, t] = j)\, dt,$$

where η^0 has distribution $\mathbb{P}^0_\eta$.

Exercise 8.6 Let η be a stationary, locally finite point process with finite positive intensity γ, and let $A \in \mathbf{N}_l$. Show that $\mathbb{P}^0_\eta(A) = 1$ if and only if

$$\eta(\{x \in \mathbb{R}^d : \theta_x\eta \notin A\}) = 0, \quad \mathbb{P}\text{-a.s.}$$

(Hint: Use (8.8) to prove in the case $\mathbb{P}^0_\eta(A) = 1$ that

$$\eta(\{x \in B : \theta_x\eta \in A\}) = \eta(B), \quad \mathbb{P}\text{-a.s.},$$

for any $B \in \mathcal{B}^d$ with $0 < \lambda_d(B) < \infty$.)

Exercise 8.7 Let $T\colon \mathbf{N}_l \to \mathbf{N}_l$ be a measurable mapping such that

$$T(\theta_x\mu) = \theta_x T(\mu), \quad (x, \mu) \in \mathbb{R}^d \times \mathbf{N}_l.$$

Show that if η is stationary (resp. stationary and ergodic) point process on $\mathbb{R}^d$, then so is $T(\eta)$.

Exercise 8.8 Give an example of a stationary locally finite point process η with $\mathbb{E}[\eta([0,1]^d] = \infty$.

Exercise 8.9 Let χ be the stationary Poisson cluster process defined in Exercise 5.6. Assume that $\gamma_{\mathbb{Q}} := \int \mu([0,1]^d)\, \mathbb{Q}(d\mu) \in (0,\infty)$. Show that the Palm distribution of χ is given by

$$\mathbb{P}^0_\chi = \int \mathbb{P}(\eta + \mu \in \cdot)\, \mathbb{Q}^0(d\mu),$$

where the probability measure $\mathbb{Q}^0$ is given by

$$\mathbb{Q}^0 := \gamma_{\mathbb{Q}}^{-1} \iint \mathbf{1}\{\theta_x \mu \in \cdot\}\, \mu(dx)\, \mathbb{Q}(d\mu).$$

(Hint: Use the definition (5.13) of χ and the Mecke equation for ξ.)

9

Palm distributions of simple point processes

For a given simple point process, Voronoi tessellations partition the space into regions based on the nearest neighbour principle. They are a fundamental model of stochastic geometry but also provide a convenient setting for formulating the close relationship between the stationary and the Palm distribution of a stationary simple point process. The latter is a volume-debiased version of the first while, conversely, the former is a volume-biased version of the latter. The Palm distribution of a stationary simple point process can be interpreted as the conditional distribution given that there is a point at the origin.

9.1 Voronoi tessellations and the inversion formula

In this chapter we assume that $d \in \mathbb{N}$ and η is a stationary simple point process on $\mathbb{R}^d$ with finite intensity γ. We shall discuss some basic relationships between the stationary distribution and the Palm distribution of η.

For simplicity we assume for all $\omega \in \Omega$ that $\eta(\omega)$ is a simple and locally finite counting measure, that is, $\eta(\omega) \in \mathbf{N}_s := \mathbf{N}_s(\mathbb{R}^d)$. We also assume that $\mathbb{P}(\eta(\mathbb{R}^d) = 0) = 0$, so that $\mathbb{P}(\eta(\mathbb{R}^d) = \infty) = 1$ by Proposition 8.4. In particular $\gamma > 0$. Finally, let η^0 denote a *Palm version* of η, that is a point process (defined on the basic probability space $(\Omega, \mathcal{F}, \mathbb{P})$) with distribution $\mathbb{P}^0_\eta$. By Exercise 8.2 η^0 is simple, while (8.11) implies that $\mathbb{P}(0 \in \eta^0) = 1$.

In what follows we take advantage of the geometric idea of a *Voronoi tessellation*. For $\mu \in \mathbf{N}_s$ with $\mu(\mathbb{R}^d) > 0$ and for $x \in \mathbb{R}^d$ let $\tau(x, \mu) \in \mu$ be the nearest neighbour of x in $\operatorname{supp} \mu$, i.e. the point in μ of minimal Euclidean distance from x. If there is more than one such point we take the lexicographically smallest. In the (exceptional) case $\mu(\mathbb{R}^d) = 0$ we put $\tau(x, \mu) := x$ for all $x \in \mathbb{R}^d$. The mapping $\tau \colon \mathbb{R}^d \times \mathbf{N}_s \to \mathbb{R}^d$ is *covariant* under translations, that is,

$$\tau(x - y, \theta_y \mu) = \tau(x, \mu) - y, \quad x, y \in \mathbb{R}^d,\ \mu \in \mathbf{N}_s. \tag{9.1}$$

For $x \in \mu \in \mathbf{N}_s$ the *Voronoi cell* of x (with respect to μ) is defined by

$$C(x, \mu) := \{y \in \mathbb{R}^d : \tau(y, \mu) = x\}. \tag{9.2}$$

If $\mu(\mathbb{R}^d) \neq 0$ these cells are pairwise disjoint and cover $\mathbb{R}^d$. In the following formulas we shall frequently use the abbreviation

$$C_0 := C(0, \eta^0).$$

This random set is also called the *typical cell* of the Voronoi tessellation. Further we denote by

$$X := \tau(0, \eta) \tag{9.3}$$

the point of η closest to the origin and set $X := 0$ if $\eta(\mathbb{R}^d) = 0$.

Theorem 9.1 *For all $h \in \mathbb{R}_+(\mathbb{R}^d \times \mathbf{N}_s)$ it is the case that*

$$\mathbb{E}[h(X, \eta)] = \gamma\, \mathbb{E}\Big[\int_{C_0} h(-x, \theta_x \eta^0)\, dx\Big]. \tag{9.4}$$

Proof Equation (8.10) (refined Campbell theorem) and a change of variables yield

$$\mathbb{E}\Big[\int f(x, \eta)\, \eta(dx)\Big] = \gamma\, \mathbb{E}\Big[\int f(-x, \theta_x \eta^0)\, dx\Big], \tag{9.5}$$

for all $f \in \mathbb{R}_+(\mathbb{R}^d \times \mathbf{N}_s)$. We apply this formula with

$$f(x, \mu) := h(x, \mu)\mathbf{1}\{\tau(0, \mu) = x\}. \tag{9.6}$$

Then the left-hand side of (9.5) reduces to the left-hand side of (9.4). Since, by the covariance property (9.1), $\tau(0, \theta_x \eta^0) = -x$ if and only if $\tau(x, \eta^0) = 0$ (that is, $x \in C_0$), the right-hand side of (9.5) coincides with the right-hand side of (9.4). □

Let $f \in \mathbb{R}_+(\mathbf{N}_s)$. Taking $h(x, \mu) := f(\mu)$ in (9.4) yields the *inversion formula*

$$\mathbb{E}[f(\eta)] = \gamma\, \mathbb{E}\Big[\int_{C_0} f(\theta_x \eta^0)\, dx\Big], \quad f \in \mathbb{R}_+(\mathbf{N}_s), \tag{9.7}$$

expressing the stationary distribution in terms of the Palm distribution. The choice $f \equiv 1$ yields the intuitively obvious formula

$$\mathbb{E}[\lambda_d(C_0)] = \gamma^{-1}. \tag{9.8}$$

Let $g \in \mathbb{R}_+(\mathbf{N}_s)$. Taking $f(x, \mu) := g(\theta_x \mu)$ in (9.4) yields

$$\gamma\, \mathbb{E}[\lambda_d(C_0) g(\eta^0)] = \mathbb{E}[g(\theta_X \eta)], \tag{9.9}$$

showing that the distribution of $\theta_X\eta$ is absolutely continuous with respect to the Palm distribution. The formula says that the stationary distribution is (up to a shift) a *volume-biased* version of the Palm distribution.

We define the (stationary) *zero-cell* of η by

$$V_0 := C(X, \eta) = \{x \in \mathbb{R}^d : \tau(x, \eta) = \tau(0, \eta)\}.$$

In the exceptional case where $\eta(\mathbb{R}^d) = 0$, we have defined $\tau(x, \mu) := x$ for all $x \in \mathbb{R}^d$ so that $V_0 = \{0\}$. The next result implies that the Palm distribution can be derived from the stationary distribution by *volume-debiasing* and shifting X to 0.

Proposition 9.2 *We have for all $f \in \mathbb{R}_+(\mathbf{N}_s)$ that*

$$\gamma\, \mathbb{E}[f(\eta^0)] = \mathbb{E}[\lambda_d(V_0)^{-1} f(\theta_X\eta)]. \tag{9.10}$$

Proof We apply (9.9) with $g(\mu) := f(\mu) \cdot \lambda_d(C(0, \mu))^{-1}$, to obtain that

$$\gamma\, \mathbb{E}[f(\eta^0)] = \mathbb{E}[\lambda_d(C(0, \theta_X\eta))^{-1} f(\theta_X\eta)].$$

Since $C(0, \theta_X\eta) = C(X, \eta) - X = V_0 - X$ the result follows. □

Given $\alpha \in \mathbb{R}$, putting $f(\mu) := \lambda_d(C(0, \mu))^{\alpha+1}$ in equation (9.10) yields

$$\gamma\, \mathbb{E}[\lambda_d(C_0)^{\alpha+1}] = \mathbb{E}[\lambda_d(V_0)^{\alpha}]. \tag{9.11}$$

In particular

$$\mathbb{E}[\lambda_d(V_0)^{-1}] = \gamma. \tag{9.12}$$

By Jensen's inequality (Proposition B.1), $\mathbb{E}[\lambda_d(V_0)^{-1}] \geq (\mathbb{E}[\lambda_d(V_0)])^{-1}$. Hence, by (9.12) and (9.8) we obtain that

$$\mathbb{E}[\lambda_d(C_0)] \leq \mathbb{E}[\lambda_d(V_0)]. \tag{9.13}$$

9.2 Local interpretation of Palm distributions

Since η is stationary and simple we might expect that both $\mathbb{P}(\eta(B) = 1)$ and $\mathbb{P}(\eta(B) \geq 1)$ behave like $\gamma\lambda_d(B)$ for small $B \in \mathcal{B}^d$. This is made precise by the following result.

Proposition 9.3 *For $n \in \mathbb{N}$ let $B_n \in \mathcal{B}^d$ with $\lambda_d(B_n) > 0$ and $0 \in B_n$. Assume that the diameter of B_n tends to 0 as $n \to \infty$. Then*

$$\lim_{n\to\infty} \frac{\mathbb{P}(\eta(B_n) \geq 1)}{\lambda_d(B_n)} = \lim_{n\to\infty} \frac{\mathbb{P}(\eta(B_n) = 1)}{\lambda_d(B_n)} = \gamma. \tag{9.14}$$

Proof The inversion formula (9.7) yields

$$\mathbb{P}(\eta(B_n) \geq 1) = \gamma\,\mathbb{E}\Big[\int_{C_0} \mathbf{1}\{\eta^0(B_n + x) \geq 1\}\,dx\Big] \geq \gamma\,\mathbb{E}\Big[\int_{C_0} \mathbf{1}_{B_n}(-x)\,dx\Big]$$
$$= \gamma\,\mathbb{E}[\lambda_d(C_0 \cap (-B_n))],$$

where $-B_n := \{-x : x \in B_n\}$ and we have used the fact that $\eta^0\{0\} \geq 1$ a.s. Fatou's lemma (Lemma A.6) implies that

$$\liminf_{n\to\infty} \frac{\mathbb{P}(\eta(B_n) \geq 1)}{\lambda_d(B_n)} \geq \gamma\,\mathbb{E}\left[\liminf_{n\to\infty} \frac{\lambda_d(C_0 \cap (-B_n))}{\lambda_d(B_n)}\right] = \gamma.$$

Since

$$2\,\mathbb{P}(\eta(B_n) > 0) - \gamma\lambda_d(B_n) = 2\,\mathbb{P}(\eta(B_n) > 0) - \mathbb{E}[\eta(B_n)]$$
$$\leq \mathbb{P}(\eta(B_n) = 1) + \mathbb{P}(\eta(B_n) = 1) + 2\,\mathbb{P}(\eta(B_n) = 2) - \mathbb{E}[\eta(B_n)]$$
$$\leq \mathbb{P}(\eta(B_n) = 1),$$

it also follows that

$$\liminf_{n\to\infty} \frac{\mathbb{P}(\eta(B_n) = 1)}{\lambda_d(B_n)} \geq 2\liminf_{n\to\infty} \frac{\mathbb{P}(\eta(B_n) > 0)}{\lambda_d(B_n)} - \gamma \geq \gamma.$$

Since $\mathbb{P}(\eta(B_n) = 1) \leq \mathbb{P}(\eta(B_n) \geq 1) \leq \mathbb{E}[\eta(B_n)] = \gamma\lambda_d(B)$, (9.14) follows. □

Here now is a local interpretation of the Palm distribution.

Proposition 9.4 *For $n \in \mathbb{N}$ let $r_n > 0$ and let B_n be the closed ball with centre 0 and radius r_n. Assume that $r_n \to 0$ as $n \to \infty$. Let $g \in \mathbb{R}(\mathbf{N}_s)$ be bounded. Then*

$$\lim_{n\to\infty} \mathbb{E}[g(\theta_X\eta) \mid \eta(B_n) \geq 1] = \mathbb{E}[g(\eta^0)]. \tag{9.15}$$

If, moreover, $x \mapsto g(\theta_x\mu)$ is continuous for all $\mu \in \mathbf{N}_s$, then

$$\lim_{n\to\infty} \mathbb{E}[g(\eta) \mid \eta(B_n) \geq 1] = \mathbb{E}[g(\eta^0)]. \tag{9.16}$$

Proof By the triangle inequality

$$\Big|\mathbb{E}[g(\theta_X\eta) \mid \eta(B_n) \geq 1] - \mathbb{E}[g(\eta^0)]\Big| \leq I_{1,n} + I_{2,n},$$

where

$$I_{1,n} := \gamma^{-1}\lambda_d(B_n)^{-1}\Big|\mathbb{E}[\mathbf{1}\{\eta(B_n) \geq 1\}g(\theta_X\eta)] - \gamma\lambda_d(B_n)\mathbb{E}[g(\eta^0)]\Big|$$

and

$$\begin{aligned}I_{2,n} &:= \big|(\mathbb{P}(\eta(B_n) \geq 1)^{-1} - \gamma^{-1}\lambda_d(B_n)^{-1})\mathbb{E}[\mathbf{1}\{\eta(B_n) \geq 1\}g(\theta_X\eta)]\big| \\ &\leq \frac{\mathbb{E}[\mathbf{1}\{\eta(B_n) \geq 1\}|g(\theta_X\eta)|]}{\gamma\lambda_d(B_n)}\left|\frac{\gamma\lambda_d(B_n)}{\mathbb{P}(\eta(B_n) \geq 1)} - 1\right|.\end{aligned}$$

Since g is bounded, Proposition 9.3 implies that $I_{2,n} \to 0$ as $n \to \infty$.

By the refined Campbell theorem (Theorem 8.5),

$$I_{1,n} = \gamma^{-1}\lambda_d(B_n)^{-1}\left|\mathbb{E}\left[\mathbf{1}\{\eta(B_n) \geq 1\}g(\theta_X\eta) - \int_{B_n} g(\theta_x\eta)\,\eta(dx)\right]\right|.$$

Since B_n is a ball, $X \in B_n$ whenever $\eta(B_n) \geq 1$. In the last expectation, distinguish the cases $\eta(B_n) = 1$ and $\eta(B_n) \geq 2$; then we obtain

$$\begin{aligned}I_{1,n} &\leq \gamma^{-1}\lambda_d(B_n)^{-1}\mathbb{E}\left[\mathbf{1}\{\eta(B_n) \geq 2\}\int \mathbf{1}_{B_n}(x)|g(\theta_x\eta)|\,\eta(dx)\right] \\ &\leq c\gamma^{-1}\lambda_d(B_n)^{-1}\mathbb{E}[\mathbf{1}\{\eta(B_n) \geq 2\}\eta(B_n)],\end{aligned}$$

where c is an upper bound of $|g|$. Therefore

$$I_{1,n} \leq \frac{c(\mathbb{E}[\eta(B_n)] - \mathbb{P}(\eta(B_n) = 1))}{\gamma\lambda_d(B_n)} = c\left(1 - \frac{\mathbb{P}(\eta(B_n) = 1)}{\gamma\lambda_d(B_n)}\right),$$

which tends to zero by Proposition 9.3.

To prove (9.16) it is now sufficient to show that

$$\lim_{n\to\infty} \mathbb{E}[|g(\eta) - g(\theta_X\eta)| \mid \eta(B_n) \geq 1] = 0.$$

Define, for any $\varepsilon > 0$,

$$g_\varepsilon(\mu) := \sup\{|g(\mu) - g(\theta_x\mu)| : \|x\| \leq \varepsilon\}, \quad \mu \in \mathbf{N}_s.$$

Assuming that g has the stated additional continuity property, the supremum can be taken over a countable dense subset of $B(0, \varepsilon)$. Since $\mu \mapsto \theta_x\mu$ is measurable for each $x \in \mathbb{R}^d$ (Lemma 8.6), we see that g_ε is a measurable function. Fixing $\varepsilon > 0$ we note that

$$\mathbb{E}[|g(\eta) - g(\theta_X\eta)| \mid \eta(B_n) \geq 1] \leq \mathbb{E}[g_\varepsilon(\theta_X\eta) \mid \eta(B_n) \geq 1]$$

for sufficiently large n, where we have again used the fact that $X \in B_n$ if $\eta(B_n) \geq 1$. Applying (9.15) to g_ε we therefore obtain that

$$\limsup_{n\to\infty} \mathbb{E}[|g(\eta) - g(\theta_X\eta)| \mid \eta(B_n) \geq 1] \leq \mathbb{E}[g_\varepsilon(\eta^0)].$$

The assumption on g implies that $g_\varepsilon(\eta^0) \to 0$ as $\varepsilon \to 0$, so that $\mathbb{E}[g_\varepsilon(\eta^0)] \to 0$ by dominated convergence. This concludes the proof. □

9.3 Exercises

Exercise 9.1 Let η be a stationary simple point process with finite intensity γ and $\mathbb{P}(\eta(\mathbb{R}^d) = \infty) = 1$. Let X be given by (9.3). Show that the conditional distribution of $-X$ given $\theta_X\eta$ is the uniform distribution on $V_0 - X$. (Hint: Use Theorem 9.1 then Proposition 9.2 and then again Theorem 9.1.)

Exercise 9.2 Let $h \in \mathbb{R}(\mathbb{R}^d)$ be continuous with compact support. Show that $\mu \mapsto \int f\, d(\theta_x\mu)$ is continuous for all $\mu \in \mathbf{N}_s$.

Exercise 9.3 Let η be a locally finite stationary point process on $\mathbb{R}^d$. Show that

$$\eta^* := \int \eta\{x\}^{\oplus}\mathbf{1}\{x \in \cdot\}\,\eta(dx) \tag{9.17}$$

is a stationary point process; see also Exercise 6.6. (Hint: Use Exercise 8.7.)

Exercise 9.4 Let η be a stationary locally finite point process on $\mathbb{R}^d$ with finite intensity but do not assume that η is simple. Show that the first limit in (9.14) exists and equals $\mathbb{E}[\eta^*([0,1]^d)]$, where η^* is given by (9.17). Show by means of an example that the second identity in (9.14) can fail if η is not simple.

Exercise 9.5 Let η be a stationary simple point process with finite intensity γ. Let the sequence (B_n) be as in Proposition 9.3 and let X_n be a random vector such that X_n is the point of η in B_n whenever $\eta(B_n) = 1$. Let $g \in \mathbb{R}(\mathbf{N}_l)$ be bounded. Show that

$$\lim_{n\to\infty} \mathbb{E}[g(\theta_{X_n}\eta) \mid \eta(B_n) = 1] = \mathbb{E}[g(\eta^0)].$$

Exercise 9.6 In the setting of Exercise 9.5, assume that $x \mapsto g(\theta_x\mu)$ is continuous for all $\mu \in \mathbf{N}_s$. Show that

$$\lim_{n\to\infty} \mathbb{E}[g(\eta) \mid \eta(B_n) = 1] = \mathbb{E}[g(\eta^0)].$$

(Hint: Use Exercise 9.5 and the proof of (9.16).)

10

The extra head problem

Is it possible to choose a point of a stationary Poisson process such that after removing this point and centering the process around its position, the resulting process is still Poisson? More generally one can ask whether it is possible to choose a point of a stationary point process such that the re-centered process has the Palm distribution. This question can be answered using balanced allocations, which partition the space into regions of equal volume in a translation invariant way, such that each point is associated with exactly one region. Under an ergodicity assumption a spatial version of the Gale-Shapley algorithm provides an important example of such an allocation.

10.1 The problem

Let $d \in \mathbb{N}$ and suppose η is a stationary process on $\mathbb{R}^d$ with finite and positive intensity $\gamma > 0$. Let T be a $\sigma(\eta)$-measurable random element of $\mathbb{R}^d$ such that $\mathbb{P}(\eta\{T\} = 1) = 1$, where

$$\sigma(\eta) := \{\{\eta \in A\} : A \in \mathcal{N}(\mathbb{R}^d)\}$$

is the σ-field generated by η. Thus T picks one of the points of η using only the information contained in η. The shifted point process $\theta_T\eta$ (that is, the point process $\omega \mapsto \theta_{T(\omega)}\eta(\omega)$) has a point at the origin and we might ask whether

$$\theta_T\eta \overset{d}{=} \eta^0, \tag{10.1}$$

where η^0 is the Palm version of η. If η is a Poisson process, then the Mecke-Slivnyak theorem (Theorem 8.9) implies that (10.1) is equivalent to

$$\theta_T\eta \setminus \delta_0 \overset{d}{=} \eta, \tag{10.2}$$

where we recall definition (4.13). This (as well as the more general version (10.1)) is known as the *extra head problem*. The terminology comes from the analogous discrete problem, given a doubly infinite sequence of independent and identically distributed coin tosses, of picking out a 'head' in the sequence such that the distribution of the remaining coin tosses (centred around the picked coin) is still that of the original sequence. It turns out that the extra head problem can be solved using *transport properties* of point processes.

Before solving the extra head problem we need to introduce a purely deterministic concept. It is convenient to add the point ∞ to $\mathbb{R}^d$ and to define $\mathbb{R}^d_\infty := \mathbb{R}^d \cup \{\infty\}$. We equip this space with the σ-field generated by $\mathcal{B}^d \cup \{\{\infty\}\}$. We define $\infty + x = \infty - x := \infty$ for all $x \in \mathbb{R}^d$. Every $\mu \in \mathbf{N}_s$ is identified with its support $\operatorname{supp} \mu$; see (8.4).

Definition 10.1 An *allocation* is a measurable mapping $\tau\colon \mathbb{R}^d \times \mathbf{N}_s \to \mathbb{R}^d_\infty$ such that

$$\tau(x, \mu) \in \mu \cup \{\infty\}, \quad \mu \in \mathbf{N}_s,$$

and such that it is *covariant* under shifts, i.e.

$$\tau(x - y, \theta_y \mu) = \tau(x, \mu) - y, \quad \mu \in \mathbf{N}_s,\ x, y \in \mathbb{R}^d, \tag{10.3}$$

where the shift operator θ_y was defined at (8.1).

Given an allocation τ, define

$$C^\tau(x, \mu) = \{y \in \mathbb{R}^d \colon \tau(y, \mu) = x\}, \quad \mu \in \mathbf{N}_s,\ x \in \mathbb{R}^d. \tag{10.4}$$

Note that $C^\tau(x, \mu) = \emptyset$ whenever $x \notin \mu$. The system $\{C^\tau(x, \mu) : x \in \mu\}$ forms a partition of $\{x \in \mathbb{R}^d : \tau(x, \mu) \neq \infty\}$ into measurable sets. The covariance property (10.3) implies

$$C^\tau(x - y, \theta_y \mu) = C^\tau(x, \mu) - y, \quad \mu \in \mathbf{N}_s,\ x, y \in \mathbb{R}^d. \tag{10.5}$$

An important example is the Voronoi tessellation discussed in Chapter 9. We do not assume that $x \in C^\tau(x, \mu)$ or that $C^\tau(x, \mu) \neq \emptyset$ for $x \in \mu$.

Now fix a stationary simple point process η on $\mathbb{R}^d$ with intensity $\gamma \in (0, \infty)$ and assume that $\mathbb{P}(\eta(\mathbb{R}^d) \neq 0) = 1$. We may then consider η as a random element of the space $\mathbf{N}_s := \mathbf{N}_s(\mathbb{R}^d)$ of all locally finite and simple counting measures, introduced in Chapter 6. As in Chapter 9, let η^0 be a point process with the Palm distribution $\mathbb{P}^0_\eta$. We can assume that η^0 is a random element of $\mathbf{N}_s$.

Theorem 10.2 *Let τ be an allocation and $f, g \in \mathbb{R}_+(\mathbf{N}_s)$. Then*

$$\mathbb{E}[\mathbf{1}\{\tau(0, \eta) \neq \infty\} f(\eta) g(\theta_{\tau(0,\eta)} \eta)] = \gamma\, \mathbb{E}\Big[g(\eta^0) \int_{C^\tau(0,\eta^0)} f(\theta_x \eta^0)\, dx\Big]. \tag{10.6}$$

Proof The proof is similar to the that of Theorem 9.1. Apply (9.5) (a direct consequence of the refined Campbell theorem) to the function $(x, \mu) \mapsto f(\mu)g(\theta_x\mu)\mathbf{1}\{\tau(0,\mu) = x\}$. □

Definition 10.3 Let $\alpha > 0$. An allocation τ is *α-balanced for η* if

$$\mathbb{P}(\lambda_d(C^\tau(x,\eta)) = \alpha \text{ for all } x \in \eta) = 1. \tag{10.7}$$

Lemma 10.4 *An allocation τ is α-balanced for η if and only if*

$$\mathbb{P}(\lambda_d(C^\tau(0,\eta^0)) = \alpha) = 1. \tag{10.8}$$

Proof By (10.5) we have $C^\tau(0,\theta_x\eta) = C^\tau(x,\eta) - x$ for all $x \in \mathbb{X}$. Then we obtain from translation invariance of Lebesgue measure and Theorem 8.5 that

$$\begin{aligned}\mathbb{E}\Big[\int \mathbf{1}\{\lambda_d(C^\tau(x,\eta)) \neq \alpha\}\,\eta(dx)\Big] &= \mathbb{E}\Big[\int \mathbf{1}\{\lambda_d(C^\tau(0,\theta_x\eta)) \neq \alpha\}\,\eta(dx)\Big] \\ &= \gamma \int \mathbb{P}(\lambda_d(C^\tau(0,\eta^0)) \neq \alpha)\,dx,\end{aligned}$$

and the result follows. □

The following theorem clarifies the relevance of balanced allocations for the extra head problem.

Theorem 10.5 *Let $\alpha > 0$. Let τ be an allocation and put $T := \tau(0,\eta)$. Then τ is α-balanced for η if and only if*

$$\mathbb{P}(T \neq \infty) = \alpha\gamma \tag{10.9}$$

and

$$\mathbb{P}(\theta_T\eta \in \cdot \mid T \neq \infty) = \mathbb{P}^0_\eta. \tag{10.10}$$

Proof If τ is α-balanced for η then (10.6) with $f \equiv 1$ yields that

$$\mathbb{E}[\mathbf{1}\{T \neq \infty\}g(\theta_T\eta)] = \gamma\alpha\,\mathbb{E}[g(\eta^0)].$$

This yields both (10.9) and (10.10). Assume, conversely, that (10.9) and (10.10) hold. Let $C_0 := C^\tau(0,\eta^0)$. Using (10.6) with $f \equiv 1$ followed by a multiplication with $\mathbb{P}(T < \infty)^{-1}$ gives us

$$\mathbb{E}[g(\eta^0)] = \mathbb{P}(T \neq \infty)^{-1}\gamma\,\mathbb{E}[g(\eta^0)\lambda_d(C_0)] = \alpha^{-1}\mathbb{E}[g(\eta^0)\lambda_d(C_0)].$$

In particular, $\mathbb{E}[\lambda_d(C_0)] = \alpha$. Choosing $g(\mu) = \lambda_d(C^\tau(0,\mu))$ yields

$$\mathbb{E}[\lambda_d(C_0)^2] = \alpha\,\mathbb{E}[\lambda_d(C_0)] = \alpha^2.$$

Since this implies (10.8), τ is α-balanced for η. □

10.2 The point-optimal Gale-Shapley algorithm

The identity (10.9) shows that α-balanced allocations can exist only if $\alpha \leq \gamma^{-1}$. The (unconditional) extra head problem arises in the case $\alpha = \gamma^{-1}$.

We now describe one way to construct α-balanced allocations. Suppose that each point of $\mu \in \mathbf{N}_s$ starts growing at time 0 at unit speed, trying to capture a region of volume α. In the absence of any interaction with other growing points, for $t \leq \alpha^{1/d}\kappa_d^{-1/d}$ a point $x \in \mu$ grows to a ball $B(x, t)$ by time t, where (in accordance with (A.18))

$$B(x, t) := \{y \in \mathbb{R}^d : \|y - x\| \leq t\}. \tag{10.11}$$

However, a growing point can only capture sites that have not been claimed by some other point before. Once a region reaches volume α, it stops growing. This idea can be formalized as follows.

Algorithm 10.6 Let $\alpha > 0$ and $\mu \in \mathbf{N}_s$. For $n \in \mathbb{N}$, $x \in \mu$ and $z \in \mathbb{R}^d$, define the set $C_n(x) \subset \mathbb{R}^d$ as the set of sites *claimed* by x at stage n, the set $R_n(x) \subset \mathbb{R}^d$ of sites *rejecting* x during the first n stages and $A_n(z) \subset \mu$ as the set of points of μ claiming site z at stage n, via the following recursion. Define $R_0(x) := \emptyset$ for all $x \in \mu$ and for $n \in \mathbb{N}$:

(i) For $x \in \mu$, define

$$r_n(x) := \inf\{r \geq 0 : \lambda_d(B(x, r) \setminus R_{n-1}(x)) \geq \alpha\},$$
$$C_n(x) := B(x, r_n(x)) \setminus R_{n-1}(x).$$

(ii) For $z \in \mathbb{R}^d$, define

$$A_n(z) := \{x \in \mu : z \in C_n(x)\}.$$

If $A_n(z) \neq \emptyset$ then define

$$\tau_n(z) := l(\{x \in A_n(z) : \|z - x\| = d(z, A_n(z))\})$$

as the point *shorlisted* by site z at stage n, where $l(B)$ denotes the lexicographic minimum of a finite non-empty set $B \subset \mathbb{R}^d$. If $A_n(z) = \emptyset$ then define $\tau_n(z) := \infty$.

(iii) For $x \in \mu$, define

$$R_n(x) := R_{n-1}(x) \cup \{z \in C_n(x) : \tau_n(z) \neq x\}.$$

The *point-optimal Gale-Shapley allocation* with *appetite* α is defined as follows. (The superscript p stands for “point-optimal”). Consider Algorithm 10.6 for $\mu \in \mathbf{N}_s$ and let $z \in \mathbb{R}^d$. If $\tau_n(z) = \infty$ (that is $A_n(z) = \emptyset$) for all $n \in \mathbb{N}$ we put $\tau^{\alpha,p}(\mu, z) := \infty$. Otherwise, set $n_1 = \min\{n : A_n(z) \neq \emptyset\}$. We assert that $A_n(z) \neq \emptyset$ for $n \geq n_1$, and $\|z - \tau_n(z)\|$ is nonincreasing in n for $n \geq n_1$. This is proved by induction: given n, if $A_n(z) \neq \emptyset$ then $\tau_n(z) \in A_n(z)$ so $z \in C_n(\tau_n(z))$. Hence $z \notin R_{n-1}(\tau_n(z))$ so $z \notin R_n(\tau_n(z))$ by (iii) . Hence $z \in C_{n+1}(\tau_n(z))$ so $\tau_n(z) \in A_{n+1}(z)$ and hence $\|\tau_{n+1}(z) - z\| \leq \|\tau_n(z) - z\|$, completing the induction. Moreover, this argument shows that for $n \geq n_1$, if $\tau_n(z) \neq \tau_{n+1}(z)$ then $z \in R_{n+1}(\tau_n(z))$ (by (iii)) and hence $z \in R_m(\tau_n(z))$ for all $m > n$ so $\tau_m(x) \neq \tau_n(x)$ for all $m > n$. Therefore, since μ is locally finite, there exist $x \in \mu$ and $n_0 \in \mathbb{N}$ such that $\tau_n(z) = x$ for all $n \geq n_0$. In this case we define $\tau^{\alpha,\mathrm{p}}(z, \mu) := x$.

We have used the lexicographic minimum in (ii) to break ties in a shift-covariant way. An alternative is to leave $\tau_n(z)$ undefined, whenever z has the same distance from two different points of μ. We shall prove that $\tau^{\alpha,\mathrm{p}}$ has the following properties.

Definition 10.7 An allocation τ has *appetite* $\alpha > 0$ if both

$$\lambda_d(C^\tau(x, \mu)) \leq \alpha, \quad x \in \mu,\ \mu \in \mathbf{N}_s, \tag{10.12}$$

and the zero measure $\mu = 0$ is the only $\mu \in \mathbf{N}_s$ satisfying

$$\{z \in \mathbb{R}^d : \tau(z, \mu) = \infty\} \neq \emptyset \quad \text{and} \quad \{x \in \mu : \lambda_d(C^\tau(x, \mu)) < \alpha\} \neq \emptyset. \tag{10.13}$$

Lemma 10.8 *The point-optimal Gale-Shapley allocation $\tau^{\alpha,\mathrm{p}}$ is an allocation with appetite α.*

Proof It follows by induction over the stages of Algorithm 10.6 that the mappings τ_n are measurable as a function of both z and μ. (The proof of this fact is left to the reader.) Hence $\tau^{\alpha,\mathrm{p}}$ is measurable. Moreover it is clear that $\tau^{\alpha,\mathrm{p}}$ has the covariance property (10.3). Upon defining $\tau^{\alpha,\mathrm{p}}$ we noted that for any $z \in \mathbb{R}^d$, either $\tau^{\alpha,\mathrm{p}}(z, \mu) = \infty$ or $\tau_n(z) = x$ for some $x \in \mu$ and all sufficiently large $n \in \mathbb{N}$. Therefore

$$\mathbf{1}\{\tau^{\alpha,\mathrm{p}}(z, \mu) = x\} = \lim_{n\to\infty} \mathbf{1}\{z \in C_n(x)\}, \quad z \in \mathbb{R}^d. \tag{10.14}$$

On the other hand, by step (i) of the algorithm we have $\lambda_d(C_n(x)) \leq \alpha$, so that (10.12) follows from Fatou's lemma (Lemma A.6).

Finally assume the strict inequality $\lambda_d(C^\tau(x, \mu)) < \alpha$ for some $x \in \mu$. We assert that the radii $r_n(x)$ defined in step (i) of the algorithm diverge. To see this, we assume on the contrary that $r(x) := \lim_{n\to\infty} r_n(x) < \infty$. By (10.14) there exist $n_0 \in \mathbb{N}$ and $\alpha_1 < \alpha$ such that $\lambda_d(B(x, r_n(x)) \setminus R_{n-1}(x)) \leq \alpha_1$ for $n \geq n_0$. Hence there is an $\alpha_2 \in (\alpha_1, \alpha)$ such that $\lambda_d(B(x, r(x)) \setminus R_{n-1}(x)) \leq \alpha_2$ for $n \geq n_0$, implying the contradiction $r_{n+1}(x) > r(x)$ for $n \geq n_0$. Now taking $z \in \mathbb{R}^d$, we hence have $z \in C_n(x)$ for some $n \geq 1$, so that z shortlists either x or some closer point of μ. In either case, $\tau(\mu, z) \neq \infty$. □

10.3 Existence of balanced allocations

We now return to the stationary point process η which was fixed before Theorem 10.2. Under an additional hypothesis on η we shall prove that an allocation with appetite $\alpha \le \gamma^{-1}$ (and in particular the Gale-Shapley allocation) is α-balanced for η. To formulate this condition, recall the definition (8.15) of the invariant σ-field $\mathcal{I}_l$ and define

$$\mathcal{I}_\eta := \{\eta^{-1}(A) : A \in \mathcal{I}_l\}.$$

The condition we need is

$$\mathbb{E}[\eta([0,1]^d) \mid \mathcal{I}_\eta] = \gamma, \quad \mathbb{P}\text{-a.s.} \tag{10.15}$$

In particular this holds if η is ergodic. If (10.15) holds then Exercise 10.4 and the definition of the conditional expectation imply that

$$\mathbb{E}[\eta(B) \mid \mathcal{I}_\eta] = \gamma\lambda_d(B), \quad \mathbb{P}\text{-a.s., } B \in \mathcal{B}^d. \tag{10.16}$$

Theorem 10.9 *Assume that* (10.15) *holds and let τ be an allocation with appetite $\alpha \in (0, \gamma^{-1}]$. Then τ is α-balanced for η.*

Proof Let A be the set of all $\mu \in \mathbf{N}_s$ with $\{x \in \mu : \lambda_d(C^\tau(x,\mu)) < \alpha\} \neq \emptyset$. The covariance property (10.5) implies that $A \in \mathcal{I}_l$. In view of (10.13) we obtain from Theorem 10.2 that

$$\mathbb{P}(\eta \in A) = \mathbb{P}(\tau(0,\eta) \neq \infty, \eta \in A) = \gamma\,\mathbb{E}[\mathbf{1}\{\eta^0 \in A\}\lambda_d(C^\tau(0,\eta^0))].$$

Therefore by (8.9), for all $B \in \mathcal{B}^d$ with $0 < \lambda_d(B) < \infty$ we have

$$\begin{aligned}\mathbb{P}(\eta \in A) &= \lambda_d(B)^{-1}\mathbb{E}\Big[\int_B \mathbf{1}\{\theta_x\eta \in A\}\lambda_d(C^\tau(0,\theta_x\eta))\,\eta(dx)\Big]\\ &= \lambda_d(B)^{-1}\mathbb{E}\Big[\mathbf{1}\{\eta \in A\}\int_B \lambda_d(C^\tau(x,\eta))\,\eta(dx)\Big], \end{aligned}\tag{10.17}$$

where we have used invariance of A and (10.5). Taking (10.12), this yields

$$\begin{aligned}\mathbb{P}(\eta \in A) &\le \lambda_d(B)^{-1}\alpha\,\mathbb{E}[\mathbf{1}\{\eta \in A\}\eta(B)] \qquad (10.18)\\ &= \lambda_d(B)^{-1}\alpha\,\mathbb{E}[\mathbf{1}\{\eta \in A\}\mathbb{E}[\eta(B) \mid \mathcal{I}_\eta]]\\ &= \alpha\gamma\,\mathbb{P}(\eta \in A) \le \mathbb{P}(\eta \in A),\end{aligned}$$

where we have used (10.16) (a consequence of assumption (10.15)) and the assumption $\alpha \le \gamma^{-1}$. Therefore inequality (10.18) is in fact an equality, so that by (10.18) and (10.17)

$$\mathbb{E}\Big[\mathbf{1}\{\eta \in A\}\int_B (\alpha - \lambda_d(C^\tau(x,\eta)))\,\eta(dx)\Big] = 0.$$

As $B \uparrow \mathbb{R}^d$ this yields

$$\mathbf{1}\{\eta \in A\} \int (\alpha - \lambda_d(C^\tau(x, \eta)))\, \eta(dx) = 0, \quad \mathbb{P}\text{-a.s.}$$

Hence $\lambda_d(C^\tau(x, \eta(\omega))) = \alpha$ for all $x \in \eta(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \{\eta \in A\}$. By definition of A this is possible only if $\mathbb{P}(\eta \in A) = 0$. Hence τ is α-balanced for η. □

Corollary 10.10 *There is an allocation that is γ^{-1}-balanced for η if and only if* (10.15) *holds. In this case the point-optimal Gale-Shapley allocation $\tau^{\gamma^{-1},\mathrm{p}}$ is one possible choice.*

Proof In view of Theorem 10.9 and Lemma 10.8 we need to prove only one implication and assume that τ is an allocation that is γ^{-1}-balanced for η. Then we have almost surely for any $B \in \mathcal{B}^d$,

$$\gamma \int \mathbf{1}\{\tau(z, \eta) \in B\}\, dz = \gamma \sum_{x \in \eta} \int \mathbf{1}\{\tau(z, \eta) = x, x \in B\}\, dz = \eta(B).$$

Taking $A \in \mathcal{I}_I$ (so that $\{\theta_z \eta \in A\} = \{\eta \in A\}$) and using the shift-covariance (10.3), we obtain

$$\begin{aligned} \mathbb{E}[\mathbf{1}\{\eta \in A\}\eta(B)] &= \gamma\, \mathbb{E}\Big[\int \mathbf{1}\{\theta_z \eta \in A, \tau(0, \theta_z \eta) + z \in B\}\, dz\Big] \\ &= \gamma\, \mathbb{E}\Big[\int \mathbf{1}\{\eta \in A, \tau(0, \eta) + z \in B\}\, dz\Big] = \gamma\, \mathbb{P}(A)\lambda_d(B), \end{aligned}$$

where we have used Fubini's theorem and stationarity to get the two final identities. This proves (10.15). □

By Proposition 8.11 the next corollary applies in particular to a stationary Poisson process.

Corollary 10.11 *Suppose that the point process η is ergodic. Then the point-optimal Gale-Shapley allocation $\tau^{\gamma^{-1},p}$ provides a solution of the general extra head problem* (10.1).

Proof Condition (10.15) follows by ergodicity. Then by Lemma 10.8 and Theorem 10.9 the allocation $\tau^{\gamma^{-1},p}$ is γ^{-1}-balanced, and then by Theorem 10.5 we have (10.1) for $T = \tau(0, \eta)$. □

10.4 Some properties of balanced allocations

If τ is an allocation with appetite $\alpha = \gamma^{-1}$ then Theorem 10.9 and (10.9) imply that $\mathbb{P}(\tau(0,\eta) \neq \infty) = 1$. The following result shows that this remains true for $\alpha > \gamma^{-1}$.

Proposition 10.12 *Assume that* (10.15) *holds and let τ be an allocation with appetite $\alpha \geq \gamma^{-1}$. Then $\mathbb{P}(\tau(0,\eta) \neq \infty) = 1$ and*

$$\mathbb{E}[\lambda_d(C^\tau(0,\eta^0))] = \gamma^{-1}. \tag{10.19}$$

If $\alpha > \gamma^{-1}$, then there are $\mathbb{P}$-a.s. infinitely many points $x \in \eta$ with the property $\lambda_d(C^\tau(x,\eta)) < \alpha$.

Proof Applying (10.6) with $f \equiv 1$ and $g = \mathbf{1}_{A'}$ with $A' \in \mathcal{I}_l$ yields

$$\mathbb{P}(\eta \in A', \tau(0,\eta) \neq \infty) = \gamma\,\mathbb{E}[\mathbf{1}\{\eta^0 \in A'\}\lambda_d(C^\tau(0,\eta^0))]. \tag{10.20}$$

Among other things this implies (10.19) once we have proved the first assertion. Since A' is flow invariant it follows from the definition (8.7) of $\mathbb{P}^0_\eta$ that

$$\mathbb{P}(\eta^0 \in A') = \gamma^{-1}\mathbb{E}[\mathbf{1}_{A'}\eta([0,1]^d)] = \gamma^{-1}\mathbb{E}[\mathbf{1}_{A'}\mathbb{E}[\eta([0,1]^d) \mid \mathcal{I}_\eta]] = \mathbb{P}(\eta \in A'),$$

where we have used assumption (10.15). The identity (10.20) can hence be written as

$$\begin{aligned}\gamma\,\mathbb{E}[\mathbf{1}\{\eta^0 \in A'\}(\alpha - \lambda_d(C^\tau(0,\eta^0)))] - \mathbb{P}(\eta \in A', \tau(0,\eta) = \infty)\\ = (\gamma\alpha - 1)\,\mathbb{P}(\eta \in A').\end{aligned} \tag{10.21}$$

We now choose A' as the set of all $\mu \in \mathbf{N}_s$ with $\int \mathbf{1}\{\tau(x,\mu) = \infty\}\,dx > 0$. It is easy to check that $A' \in \mathcal{I}_l$. By Fubini's theorem, invariance of A', stationarity, and the covariance property (10.3),

$$\begin{aligned}\mathbb{E}\Big[\mathbf{1}\{\eta \in A'\}\int \mathbf{1}\{\tau(x,\eta) = \infty\}\,dx\Big] &= \int \mathbb{P}(\theta_x\eta \in A', \tau(x,\eta) = \infty)\,dx\\ &= \int \mathbb{P}(\eta \in A', \tau(0,\eta) = \infty)\,dx.\end{aligned}$$

Assuming $\mathbb{P}(\eta \in A') > 0$, this yields $\mathbb{P}(\eta \in A', \tau(0,\eta) = \infty) > 0$. Let $A \in \mathcal{I}_l$ be defined as in the proof of Theorem 10.9. Definition 10.7 implies that $A' \subset \mathbf{N}_s \setminus A$, so that the first term on the left-hand side of (10.21) vanishes. As this contradicts our assumption $\gamma\alpha - 1 \geq 0$ we must have $\mathbb{P}(\eta \in A') = 0$, and hence

$$0 = \mathbb{E}\Big[\int \mathbf{1}\{\tau(x,\eta) = \infty\}\,dx\Big] = \int \mathbb{P}(\tau(x,\eta) = \infty)\,dx$$
$$= \int \mathbb{P}(\tau(0,\eta) = \infty)\,dx,$$

where the last line comes from the covariance property and stationarity. Hence we have the first assertion of the proposition.

Assume, finally, that $\alpha > \gamma^{-1}$ and consider the point process

$$\eta' := \int \mathbf{1}\{x \in \cdot, \lambda_d(C^\tau(x,\eta)) < \alpha\}\,\eta(dx).$$

Since λ_d is translation invariant it follows from Lemma 8.8 that η' is stationary and has intensity $\gamma\,\mathbb{P}(\lambda_d(C^\tau(\eta^0, 0)) < \alpha)$. On the other hand, since τ has appetite α, equation (10.19) shows that $\mathbb{P}(\lambda_d(C^\tau(\eta^0, 0)) < \alpha) > 0$. Hence Proposition 8.4 yields the final assertion. □

10.5 The modified Palm distribution

Corollary 10.10 and Theorem 10.5 show that the extra head problem (10.1) can only be solved under the assumption (10.15). To indicate what happens without this assumption we introduce the *sample intensity*

$$\hat{\eta} := \mathbb{E}[\eta([0,1]^d) \mid \mathcal{I}_\eta] \tag{10.22}$$

and call an allocation τ *balanced for* η if

$$\mathbb{P}(\lambda_d(C^\tau(x,\eta)) = \hat{\eta}^{-1} \text{ for all } x \in \eta) = 1. \tag{10.23}$$

It is possible to generalize the proof of Theorem 10.9, so as to show that balanced allocations exist without further assumptions on η. (The idea is to use allocations with a random appetite $\hat{\eta}^{-1}$.) If τ is such a balanced allocation and $T := \tau(\eta, 0)$, then one can show that

$$\mathbb{P}(\theta_T \in \cdot) = \mathbb{P}^*_\eta, \tag{10.24}$$

where

$$\mathbb{P}^*_\eta := \mathbb{E}\Big[\hat{\eta}^{-1} \int \mathbf{1}\{x \in [0,1]^d, \theta_x\eta \in \cdot\}\,\eta(dx)\Big] \tag{10.25}$$

is the *modified Palm distribution* of η. This distribution describes the statistical behaviour of η as seen from a *randomly chosen* point of η. If (10.15) holds, then it coincides with the Palm distribution $\mathbb{P}^0_\eta$. We do not give further details.

10.6 Exercises

Exercise 10.1 Let $(\mathbb{Y}, \mathcal{Y}, \mathbb{Q})$ be an s-finite measure space and suppose that ξ is a Poisson process on $\mathbb{R}^d$ with intensity measure $\gamma\lambda_d \otimes \mathbb{Q}$. Show that $\vartheta_x \xi \overset{d}{=} \xi$ for all $x \in \mathbb{R}^d$, where $\vartheta_x \colon \mathbf{N}(\mathbb{R}^d \times \mathbb{Y}) \to \mathbf{N}(\mathbb{R}^d \times \mathbb{Y})$ is the measurable mapping (shift) defined by

$$\vartheta_x \mu := \int \mathbf{1}\{(x' - x, y) \in \cdot\}\, \mu(d(x', y)).$$

Show also that ξ has the mixing property

$$\lim_{\|x\| \to \infty} \mathbb{P}(\xi \in A, \vartheta_x \xi \in A') = \mathbb{P}(\xi \in A)\mathbb{P}(\xi \in A')$$

for all $A, A' \in \mathcal{N}(\mathbb{R}^d \times \mathbb{Y})$.

Exercise 10.2 Let η be a stationary Poisson process on $\mathbb{R}$ and $X \in \eta$ the point of η closest to the origin. Show that $\theta_X \eta \setminus \delta_0$ is not a Poisson process. (Hint: Use Exercise 7.7 and Theorem 7.2 (interval theorem).)

Exercise 10.3 Extend the assertion of Exercise 10.2 to arbitrary dimensions.

Exercise 10.4 Let η be a stationary point process satisfying (10.15). Let $A \in \mathcal{I}_\eta$. Use the proof of Proposition 8.2 to show that

$$\mathbb{E}[\mathbf{1}_A \eta(B)] = \gamma\, \mathbb{P}(A) \lambda_d(B), \quad B \in \mathcal{B}^d.$$

Exercise 10.5 Formulate and prove a generalization of Proposition 9.2 and its consequence (9.11) to an arbitrary allocation τ.

Exercise 10.6 Let η be a stationary locally finite point process on $\mathbb{R}^d$ such that $\mathbb{P}(0 < \hat{\eta} < \infty) = 1$, where $\hat{\eta}$ is given by (10.22). Define the modified Palm distribution $\mathbb{P}^*_\eta$ of η by (10.25). Show that $\mathbb{P}_\eta$ and $\mathbb{P}^*_\eta$ coincide on the invariant σ-field $\mathcal{I}_\eta$, defined by (8.15).

11

Stable allocations

In the Gale-Shapley allocation introduced in the preceding chapter, the idea is that points and sites both prefer to be allocated as close as possible. As a result there is no point and no site that prefer each other over their current partners. This property is called stability. Stable allocations are essentially unique. To prove this it is useful to introduce a point-optimal version of the Gale-Shapley algorithm.

11.1 Stability

Let $d \in \mathbb{N}$. We shall consider a stationary point process η on $\mathbb{R}^d$ such that $\mathbb{P}(\eta(\mathbb{R}^d) \neq 0) = 1$. As before we interpret η as a random element in the space $\mathbf{N}_s = \mathbf{N}_s(\mathbb{R}^d)$ of all locally finite simple counting measures.

We start with the following definition.

Definition 11.1 Let τ be an allocation with appetite $\alpha > 0$. Let $\mu \in \mathbf{N}_s$, $x \in \mu$, and $z \in \mathbb{R}^d$. We say the site z *desires* x if $\|z - x\| < \|\tau(z, \mu) - z\|$, where $\|\infty\| := \infty$. We say the point x *covets* z if

$$\|x - z\| < \|x - z'\| \text{ for some } z' \in C^\tau(x, \mu), \text{ or } \lambda_d(C^\tau(x, \mu)) < \alpha.$$

The pair (z, x) is called *unstable* (for μ and with respect to τ) if z desires x and x covets z. The allocation τ is *stable* if there is no μ with an unstable pair.

Lemma 11.2 *The point-optimal Gale-Shapley allocation with appetite $\alpha > 0$ is stable.*

Proof Take $\mu \in \mathbf{N}_s$, $x \in \mu$, and $z \in \mathbb{R}^d$. Consider Algorithm 10.6. If z desires x, then $z \notin C_n(x)$ for all $n \geq 1$. But if x covets z, then $z \in C_n(x)$ for some $n \geq 1$. Therefore (z, x) cannot be an unstable pair. □

11.2 The site-optimal Gale-Shapley allocation

Our goal is to prove that stable allocations are essentially uniquely determined. To achieve this goal, it is helpful to introduce the *site-optimal* Gale-Shapley allocation.

Algorithm 11.3 Let $\alpha > 0$ and $\mu \in \mathbf{N}_s$. For $n \in \mathbb{N}$, $x \in \mu$ and $z \in \mathbb{R}^d$, define $\tau_n(z) \in \mu \cup \{\infty\}$ as the point *claimed* by z at stage n, the set $R_n(z) \subset \mu$ of points *rejecting* z during the first n stages, the set $A_n(x) \subset \mathbb{R}^d$ of sites claiming point x at stage n, and the set $S_n(x) \subset \mathbb{R}^d$ of sites *shortlisted* by x at stage n via the following recursion. Define $R_0(z) := \emptyset$ for all $z \in \mathbb{R}^d$ and take $n \in \mathbb{N}$.

(i) For $z \in \mathbb{R}^d$, define

$$\tau_n(z) := \begin{cases} l(\{x \in \mu : \|z - x\| = d(z, \mu \setminus R_{n-1}(z))\}), & \text{if } \mu \setminus R_{n-1}(z) \neq \emptyset, \\ \infty, & \text{otherwise.} \end{cases}$$

(ii) For $x \in \mu$, define

$$\begin{aligned} A_n(x) &:= \{z \in \mathbb{R}^d : \tau_n(z) = x\}, \\ r_n(x) &:= \inf\{r \geq 0 : \lambda_d(B(x, r) \cap A_n(x)) \geq \alpha\}, \\ S_n(x) &:= A_n(x) \cap B(x, r_n(x)). \end{aligned}$$

(iii) For $z \in \mathbb{R}^d$ let $R_n(z) := R_{n-1}(z) \cup \{x\}$ if $\tau_n(z) = x \in \mu$ and $z \notin B(x, r_n(x))$. Otherwise define $R_n(z) := R_{n-1}(z)$.

Given $\alpha > 0$ the *site-optimal Gale-Shapley allocation* $\tau^{\alpha,\mathrm{s}}$ with appetite α is defined as follows. (The superscript s stands for "site-optimal".) Let $\mu \in \mathbf{N}_s$ and consider Algorithm 11.3. For $z \in \mathbb{R}^d$ there are two cases. In the first case z is rejected by every point, that is $\mu \in \cup_{n=1}^{\infty} R_n(z)$. Then we define $\tau^{\alpha,\mathrm{s}}(z) := \infty$. In the second case there exist $x \in \mu$ and $n_0 \in \mathbb{N}$ such that $z \in S_n(x)$ for all $n \geq n_0$. In this case we define $\tau^{\alpha,\mathrm{s}}(z, \mu) := x$.

The following lemma shows that $\tau^{\alpha,\mathrm{s}}$ has properties similar to those of the point-optimal version. The proof can be given as before and is left to the reader.

Lemma 11.4 *The site-optimal Gale-Shapley allocation $\tau^{\alpha,\mathrm{s}}$ is a stable allocation with appetite α.*

11.3 Auxiliary properties of stable allocations

For an allocation τ we introduce functions $g_\tau, h_\tau \colon \mathbf{N}_s \times \mathbb{R}^d \times \overline{\mathbb{R}}_+ \to \mathbb{R}_+$ by

$$g_\tau(z, \mu, r) := \mathbf{1}\{\|\tau(z, \mu) - z\| \le r\},$$
$$h_\tau(x, \mu, r) := \lambda_d(C^\tau(x, \mu) \cap B(x, r)),$$

where $B(x, \infty) := \mathbb{R}^d$ and $\|\infty\| := \infty$. In a sense these functions describe the quality of the allocation for a site $z \in \mathbb{R}^d$ or a point $x \in \mu$ respectively. The following optimality properties of the Gale-Shapley allocations are key to the proof of the uniqueness of stable allocations.

Proposition 11.5 *Let τ be a stable allocation with appetite $\alpha > 0$. Let $\mu \in \mathbf{N}_s$ and $r \in \overline{\mathbb{R}}_+$. Then*

$$g_{\tau^{\alpha,s}}(z, \mu, r) \ge g_\tau(z, \mu, r) \ge g_{\tau^{\alpha,p}}(z, \mu, r), \quad \lambda_d\text{-a.e. } z, \tag{11.1}$$
$$h_{\tau^{\alpha,p}}(x, \mu, r) \ge h_\tau(x, \mu, r) \ge h_{\tau^{\alpha,s}}(x, \mu, r), \quad x \in \mu. \tag{11.2}$$

For the proof of Proposition 11.5 we shall need two lemmas. The first one is also used later on.

Lemma 11.6 *Let τ be a stable allocation with appetite $\alpha > 0$. Let $\mu_0, \mu \in \mathbf{N}_s$ such that $\mu_0 \le \mu$. Then for λ_d-a.e. z with $\tau(z, \mu_0) \ne \infty$, the point $\tau(z, \mu_0)$ never rejects z in the site-optimal Gale-Shapley algorithm for μ. In particular,*

$$\|\tau^{\alpha,s}(z, \mu) - z\| \le \|\tau^{\alpha,s}(z, \mu_0) - z\|, \quad \lambda_d\text{-a.e. } z. \tag{11.3}$$

Proof For each $n \in \mathbb{N}$ we need to show that for λ_d-a.e. z with $\tau(z, \mu_0) \ne \infty$, the point $\tau(z, \mu_0)$ never rejects z in the first n stages of the site-optimal Gale-Shapley algorithm. We do this by induction on n. For $n = 1$ we have to show that for all $x \in \mu_0$ and λ_d-a.e. z with $\tau(z, \mu_0) = x$ the site z is not rejected by x in the first stage of the site-optimal Gale-Shapley algorithm for μ. Let us assume the opposite. Then there exist $x \in \mu_0$ and a measurable set

$$R_1 \subset C^\tau(x, \mu_0) \cap (A_1(x) \setminus B(x, r_1(x)))$$

with $\lambda_d(R_1) > 0$. Then $S_1 := S_1(x) = A_1(x) \cap B(x, r_1(x))$ satisfies $\lambda_d(S_1) = \alpha$ and every site in S_1 is closer to x than some site in R_1 is to x. Since $R_1 \cap S_1 = \emptyset$ we obtain for the set $T_1 := S_1 \setminus C^\tau(x, \mu_0)$ that

$$\begin{aligned} \lambda_d(T_1) &= \lambda_d(S_1 \setminus (C^\tau(x, \mu_0) \setminus R_1)) \ge \lambda_d(S_1) - \lambda_d(C^\tau(x, \mu_0) \setminus R_1) \\ &= \alpha - \lambda_d(C^\tau(x, \mu_0)) + \lambda_d(R_1) > 0, \end{aligned}$$

where we have used that $\lambda_d(C^\tau(x,\mu_0)) \leq \alpha$. In the first step of the algorithm every site applies to the nearest point from μ. Moreover, since $T_1 \cap C^\tau(x,\mu_0) = \emptyset$ we have $\tau(z,\mu_0) \neq x$ for $z \in T_1$. Therefore (and since the set of sites which have the same distance from two different points of μ has Lebesgue measure 0) we obtain that $\|\tau(z,\mu_0) - z\| > \|x - z\|$ for λ_d-a.e. $z \in T_1$. But since every site in T_1 is closer to x than some site in $R_1 \subset C^\tau(x,\mu_0)$ is to x, the pair (z,x) is unstable for μ_0 with respect to τ, whenever $\|\tau(z,\mu_0) - z\| > \|x - z\|$ and $z \in T_1$. This contradicts the assumed stability of τ.

For the induction step we let $n \in \mathbb{N}$ and assume for λ_d-a.e. z with $\tau(z,\mu_0) \neq \infty$, that the point $\tau(z,\mu_0)$ has not rejected z in the first n stages of the site-optimal Gale-Shapley algorithm. To show that this is also true for $n+1$ in place of n we assume the opposite. Then there exist $x \in \mu_0$ and a measurable set

$$R \subset C^\tau(x,\mu_0) \cap (A_{n+1}(x) \setminus B(x, r_{n+1}(x)))$$

with $\lambda_d(R) > 0$. Then $S := S_{n+1}(x) = A_{n+1}(x) \cap B(x, r_{n+1}(x))$ satisfies $\lambda_d(S) = \alpha$ and every site in S is closer to x than some site in R is to x. As before we obtain that $T := S \setminus C^\tau(x,\mu_0)$ satisfies $\lambda_d(T) > 0$. A site $z \in S \supset T$ applies to x in stage $n+1$ of the algorithm. Therefore z must have been rejected by all closer points of μ and in particular by all closer points of μ_0 in one of the first n stages. Combining this with $\tau(z,\mu_0) \neq x$ for $z \in T$ and with the induction hypothesis shows that $\|\tau(z,\mu_0) - z\| > \|x - z\|$ for λ_d-a.e. $z \in T$. As seen above this contradicts stability of τ.

The final assertion follows upon taking $\tau = \tau^{\alpha,s}$ and noting that a site z that does not have the same distance from two different points of μ is allocated to the closest point in μ which does not reject it. □

The proof of the following lemma is similar to the preceding one and left as an exercise; see Exercise 11.4.

Lemma 11.7 *Let τ be a stable allocation with appetite $\alpha > 0$. Let $\mu \in \mathbf{N}_s$. Then for λ_d-a.e. z with $\tau(z,\mu) \neq \infty$, the site z never rejects $\tau(z,\mu)$ in the point-optimal Gale-Shapley algorithm for μ.*

Proof of Proposition 11.5 As in the proof of (11.3) it follows that

$$\|\tau^{\alpha,s}(z,\mu) - z\| \geq \|\tau(z,\mu) - z\|, \quad z \in B,$$

where $B \in \mathcal{B}^d$ satisfies $\lambda(\mathbb{R}^d \setminus B) = 0$. The first inequality (11.1) then holds for all $z \in B$.

Now we prove the first inequality in (11.2). Assume the contrary, so that there exists $x \in \mu$ such that $h_{\tau^{\alpha,p}}(x,\mu,r) < h_\tau(x,\mu,r)$. Then,

$$T := B(x,r) \cap (C^\tau(x,\mu) \setminus C^{\tau^{\alpha,p}}(x,\mu))$$

is a set with positive Lebesgue measure. Moreover, since $\tau^{\alpha,p}$ and τ both have appetite α, either $\lambda_d(C^{\tau^{\alpha,p}}(x,\mu)) < \alpha$ or $C^{\tau^{\alpha,p}}(x,\mu) \setminus B(x,r) \neq \emptyset$. In the first case x has claimed every site and must therefore have been rejected by the sites in T. In the second case x has claimed all sites in $B(x,r)$. Again it must have been rejected by all sites in T. This contradicts Lemma 11.7.

The other inequalities are proved similarly. □

We shall also need the following consequence of Theorem 10.2.

Lemma 11.8 *Let τ be an allocation. Then*

$$\mathbb{E}[g_\tau(0,\eta,r)] = \gamma\,\mathbb{E}[h_\tau(0,\eta^0,r)], \quad r \in [0,\infty].$$

Proof This time we use Theorem 10.2 with $g \equiv 1$. Let $r \in \overline{\mathbb{R}}_+$ and define $f(\mu) := \mathbf{1}\{\|\tau(0,\mu)\| \le r\}$, $\mu \in \mathbf{N}_s$. Then

$$\begin{aligned}\mathbb{E}[g_\tau(0,\eta,r)] = \mathbb{E}[f(\eta)] &= \gamma\,\mathbb{E}\Big[\int_{C^\tau(0,\eta^0)} \mathbf{1}\{\|\tau(0,\theta_z\eta^0)\| \le r\}\,dz\Big]\\ &= \gamma\,\mathbb{E}\Big[\int_{C^\tau(0,\eta^0)} \mathbf{1}\{\|\tau(z,\eta^0) - z\| \le r\}\,dz\Big],\end{aligned}$$

where we have used (10.3) to get the last identity. But if $z \in C^\tau(0,\eta^0)$ then $\tau(z,\eta^0) = 0$ provided that $0 \in \eta^0$, an event of probability 1. The result follows. □

11.4 Uniqueness of stable allocations

We are now able to prove the following uniqueness result.

Theorem 11.9 *Let τ be stable allocation with appetite $\alpha > 0$. Then*

$$\lambda_d(\{z \in \mathbb{R}^d : \tau(z,\eta) \neq \tau^{\alpha,p}(z,\eta)\}) = 0, \quad \mathbb{P}\text{-a.s.} \tag{11.4}$$

Proof Let $r \in [0,\infty]$. By (10.3) and stationarity we have $\mathbb{E}[g_\tau(z,\eta,r)] = \mathbb{E}[g_\tau(0,\eta,r)]$ for all $z \in \mathbb{R}^d$. Applying Proposition 11.5 and Lemma 11.8 yields

$$\begin{aligned}\mathbb{E}[g_{\tau^{\alpha,s}}(0,\eta,r)] \ge \mathbb{E}[g_{\tau^{\alpha,p}}(0,\eta,r)] &= \gamma\,\mathbb{E}[h_{\tau^{\alpha,p}}(0,\eta^0,r)]\\ &\ge \gamma\,\mathbb{E}[h_{\tau^{\alpha,s}}(0,\eta^0,r)] = \mathbb{E}[g_{\tau^{\alpha,s}}(0,\eta^0,r)].\end{aligned}$$

Therefore,

$$\mathbb{E}[g_{\tau^{\alpha,s}}(z,\eta,r)] = \mathbb{E}[g_{\tau^{\alpha,p}}(z,\eta,r)], \quad x \in \mathbb{R}^d.$$

By Lemma 11.8,

$$g_{\tau^{\alpha,s}}(z,\eta,r) \geq g_{\tau^{\alpha,p}}(z,\eta,r), \quad \lambda_d\text{-a.e. } z,$$

so that

$$g_{\tau^{\alpha,s}}(z,\eta,r) = g_{\tau^{\alpha,p}}(z,\eta,r), \quad \mathbb{P}\text{-a.s, } \lambda_d\text{-a.e. } z. \tag{11.5}$$

Now we take $\mu \in \mathbf{N}_s$ and $z \in \mathbb{R}^d$ such that

$$\mathbf{1}\{\|\tau^{\alpha,s}(z,\mu) - z\| \leq r\} = \mathbf{1}\{\|\tau^{\alpha,p}(z,\mu) - z\| \leq r\}, \quad r \in D,$$

where $D \subset \mathbb{R}_+$ is countable and dense. Then $\|\tau^{\alpha,s}(z,\mu)-z\| = \|\tau^{\alpha,p}(z,\mu)-z\|$ and hence $\tau^{\alpha,s}(z,\mu) = \tau^{\alpha,p}(z,\mu)$, provided that z has not the same distance from two different points of μ. Since the set of such z has Lebesgue measure 0, (11.5) implies that

$$\tau^{\alpha,s}(z,\eta) = \tau^{\alpha,p}(z,\eta), \quad \mathbb{P}\text{-a.s, } \lambda_d\text{-a.e. } z. \tag{11.6}$$

Another application of Lemma 11.8 then gives for all $r \geq 0$

$$g_{\tau}(z,\eta,r) = g_{\tau^{\alpha,p}}(z,\eta,r), \quad \mathbb{P}\text{-a.s, } \lambda_d\text{-a.e. } z,$$

which is (11.5) with τ instead of $\tau^{\alpha,s}$. By the same argument as before, this concludes the proof. □

11.5 Moment properties

Finally in this chapter we show that a stable allocation with appetite α has poor moment properties. In view of Theorem 10.5 we consider only the case $\alpha = \gamma^{-1}$.

Theorem 11.10 *Let η be a stationary Poisson process on $\mathbb{R}^d$ and let τ be a stable allocation with appetite γ^{-1}. Then $\mathbb{E}[\|\tau(0,\eta)\|^d] = \infty$.*

Proof Let $\alpha := \gamma^{-1}$. By Theorem 11.9 there is (essentially) only one stable allocation with appetite α. By Lemma 11.4 we can therefore assume that $\tau = \tau^{\alpha,s}$. We first prove that

$$\lambda_d(\{z \in \mathbb{R}^d : \|\tau^{\alpha,s}(z,\eta) - z\| \geq \|z\| - 1\} = \infty, \quad \mathbb{P}\text{-a.s.} \tag{11.7}$$

Let $m \in \mathbb{N}$ and let $U_1, \dots, U_m$ be independent and uniformly distributed on the unit ball B^d, independent of η. Define the point process

$$\eta' := \eta + \sum_{i=1}^{m} \delta_{U_i}.$$

We can assume that η and η' are random elements of $\mathbf{N}_s$. Let

$$A := \{\mu \in \mathbf{N}_s : \lambda_d(C^\tau(x,\mu)) = \alpha \text{ for all } x \in \mu\}.$$

Theorem 10.9 and Lemma 11.4 show that $\mathbb{P}(\eta \in A) = 1$. From Exercise 4.9 we conclude that also $\mathbb{P}(\eta' \in A) = 1$. Therefore

$$\lambda_d(\{z \in \mathbb{R}^d : \tau(z,\eta') \in B^d\}) \geq m\alpha, \quad \mathbb{P}\text{-a.s.}$$

But Lemma 11.6 shows, for λ_d-a.e. z with $\tau(z,\eta') \in B^d$, that

$$\|\tau(z,\eta) - z\| \geq \|\tau(z,\eta') - z\| \geq \|z\| - 1.$$

Since m is arbitrary, (11.7) follows.

By Fubini's theorem, (10.3) and stationarity,

$$\begin{aligned}\mathbb{E}\Big[\int \mathbf{1}\{\|\tau(z,\eta) - z\| \geq \|z\| - 1\}\, dz\Big] &= \int \mathbb{P}(\|\tau(z,\eta) - z\| \geq \|z\| - 1)\, dz \\ &= \int \mathbb{P}(\|\tau(0,\eta)\| \geq \|z\| - 1)\, dz.\end{aligned}$$

By the polar representation (7.18) of Lebesgue measure the last integral equals

$$d\kappa_d \int_0^\infty \mathbb{P}(\|\tau(0,\eta)\| + 1 \geq r) r^{d-1}\, dr = \kappa_d\, \mathbb{E}[(\|\tau(0,\eta)\| + 1)^d],$$

where the identity is a well-known consequence of Fubini's theorem (Theorem A.12). The relationship (11.7) implies $\mathbb{E}[(\|\tau(0,\eta)\| + 1)^d] = \infty$ and hence the assertion. □

11.6 Exercises

Exercise 11.1 Let $d = 1$ and $\mu := \delta_0 + \delta_1$. Compute the point-optimal Gale-Shapley allocation $\tau^{\alpha,\mathrm{p}}(\cdot,\mu)$ for $\alpha = 1$ and $\alpha = 2$. Does it coincide with the site-optimal allocation $\tau^{\alpha,\mathrm{s}}(\cdot,\mu)$?

Exercise 11.2 Let $\mu \in \mathbf{N}_s(\mathbb{R}^d)$ such that $\mu(\mathbb{R}^d) < \infty$. Let τ be a stable allocation with appetite $\alpha > 0$. Show that

$$\tau(x,\mu) = \tau^{\alpha,\mathrm{p}}(x,\mu), \quad \lambda_d\text{-a.e. } x.$$

(Hint: There is an $n \in \mathbb{N}$ and a cuboid $C \subset \mathbb{R}^d$ with side length n such that $\mu(\mathbb{R}^d \setminus C) = 0$. Apply Theorem 11.9 to the stationary point process η defined in Exercise 8.1.)

Exercise 11.3 Give an example of an allocation with appetite α that is not stable. (Hint: Use a version of the point-optimal Gale-Shapley Algorithm 11.3 with *impatient* sites.)

Exercise 11.4 Prove Lemma 11.7. (Hint: Proceed similar to the proof of Lemma 11.6).

Exercise 11.5 A point process η on $\mathbb{R}^d$ is called *insertion tolerant* if for each Borel set $B \subset \mathbb{R}^d$ with $0 < \lambda_d(B) < \infty$ and each random vector X independent of η and with the uniform distribution on B the distribution of $\eta + \delta_U$ is absolutely continuous with respect to $\mathbb{P}(\eta \in \cdot)$. Show that Theorem 11.10 remains valid for a stationary insertion tolerant point process η.

12

Poisson integrals

The Wiener-Itô integral is the centered Poisson process integral. By means of a basic isometry equation it can be defined for any function that is square integrable with respect to the intensity measure. Wiener-Itô integrals of higher order are defined in terms of factorial measures of the appropriate order. Joint moments of such integrals can be expressed in terms of combinatorial diagram formulae. This implies moment formulae and central limit theorems for Poisson U-statistics. The theory is illustrated with a Poisson process of hyperplanes.

12.1 The Wiener-Itô integral

In this chapter we fix a σ-finite measure space $(\mathbb{X}, \mathcal{X}, \lambda)$. Let η denote a Poisson process on $\mathbb{X}$ with intensity measure λ. Let $f \in \mathbb{R}(\mathbb{X}^m)$ for some $m \in \mathbb{N}$. Corollary 4.9 shows that

$$\mathbb{E}\Big[\int f\,d\eta^{(m)}\Big] = \int f\,d\lambda^m, \tag{12.1}$$

where the factorial measures $\eta^{(m)}$ are given by (4.10). Our aim is to compute joint moments of random variables of the type $\int f\,d\eta^{(m)}$.

We start with a necessary and sufficient condition on $f \in \mathbb{R}_+(\mathbb{X})$ for the integral $\eta(f) = \int f\,d\eta$ to be almost surely finite.

Proposition 12.1 *Let $f \in \mathbb{R}_+(\mathbb{X})$. Then $\mathbb{P}(\eta(f) < \infty) = 1$ if and only if*

$$\int (f \wedge 1)\,d\lambda < \infty. \tag{12.2}$$

Proof Assume that (12.2) holds. Then $\lambda(\{f \geq 1\}) < \infty$ and therefore $\eta(\{f \geq 1\}) < \infty$ a.s. Hence $\eta(\mathbf{1}\{f \geq 1\}f) < \infty$ a.s., where we can assume without loss of generality that η is proper. Further we have from Proposition

2.7 that

$$\mathbb{E}[\eta(\mathbf{1}\{f < 1\}f)] = \lambda(\mathbf{1}\{f < 1\}f) < \infty,$$

so that $\eta(\mathbf{1}\{f < 1\}f) < \infty$ a.s.

Assume, conversely, that (12.2) fails. By Theorem 3.9,

$$\mathbb{E}[e^{-\eta(f)}] = \exp\big[-\lambda(1 - e^{-f})\big]. \tag{12.3}$$

The inequality $(1 - e^{-t}) \geq (1 - e^{-1})(t \wedge 1)$, $t \geq 0$, implies $\lambda(1 - e^{-f}) = \infty$. and hence $\mathbb{E}[e^{-\eta(f)}] = 0$. Therefore $\mathbb{P}(\eta(f) = \infty) = 1$. □

Recall that $L^p(\lambda) = \{f \in \mathbb{R}(\mathbb{X}) : \lambda(|f|^p) < \infty\}$; see Appendix A.1. For $f \in L^1(\lambda)$ the *compensated integral* of f with respect to η is defined by

$$I(f) := \eta(f) - \lambda(f). \tag{12.4}$$

It follows from Campbell's formula (Proposition 2.7) that

$$\mathbb{E}[I(f)] = 0. \tag{12.5}$$

The random variable $I(f)$ is also denoted $\int f\, d(\eta - \lambda)$ or $\int f\, d\hat{\eta}$, where $\hat{\eta} := \eta - \lambda$. However, the reader should keep in mind that $\hat{\eta}$ is not defined on the whole σ-field $\mathcal{X}$ but only on $\{B \in \mathcal{X} : \lambda(B) < \infty\}$. Let $L_0(\lambda)$ denote the set of all $f \in \mathbb{R}(\mathbb{X})$ such that f is bounded and $\lambda(\{f \neq 0\}) < \infty$. Note that $L_0(\lambda) \subset L^p(\lambda)$ for all $p \geq 1$. The compensated integral has the following very useful isometry property.

Lemma 12.2 *Suppose $f \in L_0(\lambda)$. Then*

$$\mathbb{E}[I(f)^2] = \int f^2\, d\lambda. \tag{12.6}$$

Proof It is easy to check (see (4.10)) that

$$\eta(f)^2 = \int f(x)f(y)\, \eta^{(2)}(d(x, y)) + \int f(x)^2\, \eta(dx).$$

Taking expectations we obtain from Corollary 4.9 that

$$\mathbb{E}[\eta(f)^2] = \int f(x)f(y)\, \lambda^2(d(x, y)) + \int f(x)^2\, \lambda(dx) = (\lambda(f))^2 + \lambda(f^2),$$

where we note that both integrals are well-defined and finite. Therefore

$$\mathbb{E}[I(f)^2] = (\lambda(f))^2 - 2\mathbb{E}[\eta(f)\lambda(f)] + (\lambda(f))^2 + \lambda(f^2),$$

and the result follows. □

The set $L_0(\lambda)$ is dense in $L^p(\lambda)$ for any $p \geq 1$. Indeed, take a sequence $B_n \uparrow \mathbb{X}$ with $\lambda(B_n) < \infty$ for all $n \in \mathbb{N}$. For $f \in L^p(\lambda)$ the sequence

$$f_n := \mathbf{1}_{\{|f| \leq n\}} \mathbf{1}_{B_n} f, \quad n \in \mathbb{N}, \tag{12.7}$$

converges pointwise towards f. Moreover, $|f - f_n| \leq 2|f|$, so by dominated convergence $\lambda(|f - f_n|^p) \to 0$ as $n \to \infty$. This fact can be used to extend I to a mapping from $L^2(\lambda)$ to $L^2(\mathbb{P})$ as follows.

Proposition 12.3 *The mapping $I\colon L_0(\lambda) \to L^2(\mathbb{P})$ defined by* (12.4) *can be uniquely extended to a linear mapping $I\colon L^2(\lambda) \to L^2(\mathbb{P})$ such that* (12.5) *and* (12.6) *hold for all $f \in L^2(\lambda)$. If $f \in L^1(\lambda) \cap L^2(\lambda)$, then $I(f)$ is almost surely given by* (12.4).

Proof The proof is based on basic Hilbert space arguments. For $f \in L^2(\lambda)$ we define (f_n) by (12.7) and then obtain from (12.6) that

$$\mathbb{E}[(I(f_m) - I(f_n))^2] = \mathbb{E}[(I(f_m - f_n))^2] = \lambda((f_m - f_n)^2),$$

which tends to 0 as $m, n \to \infty$. Since $L^2(\mathbb{P})$ is complete, the sequence $(I(f_n))$ converges in $L^2(\mathbb{P})$ to some element of $L^2(\mathbb{P})$; we define $I(f)$ to be this limit. If in addition $f \in L^1(\lambda)$, then dominated convergence implies that $\lambda(f_n) \to \lambda(f)$, while dominated convergence and Proposition 12.1 imply that $\eta(f_n) \to \eta(f)$ a.s. Hence our new definition is consistent with (12.4). Since $\mathbb{E}[I(f_n)] = 0$ for all $n \in \mathbb{N}$ the L^2-convergence yields that $\mathbb{E}[I(f)] = 0$. Furthermore,

$$\mathbb{E}[I(f)^2] = \lim_{n\to\infty} \mathbb{E}[I(f_n)^2] = \lim_{n\to\infty} \lambda(f_n^2) = \lambda(f^2),$$

where we have used Lemma 12.2, and dominated convergence for the final identity. The linearity

$$I(af + bg) = aI(f) + bI(g) \quad \mathbb{P}\text{-a.s.}, \; f, g \in L^2(\lambda), \; a, b \in \mathbb{R}, \tag{12.8}$$

follows from the linearity of I on $L^1(\lambda)$.

If $f, g \in L^2(\lambda)$ coincide λ-a.e., then (12.6) implies that

$$\mathbb{E}[(I(f) - I(g))^2] = \mathbb{E}[(I(f - g))^2] = \lambda((f - g)^2) = 0,$$

so that $I(f) = I(g)$ a.s. Hence $I\colon L^2(\mathbb{P}) \to L^2(\mathbb{P})$ is a well-defined mapping.

If I' is another extension with the same properties as I then we can use the Minkowski inequality and $I(f_n) = I'(f_n)$ to conclude that

$$(\mathbb{E}[(I(f) - I'(f))^2])^{1/2} \leq (\mathbb{E}[(I(f) - I(f_n))^2])^{1/2} + (\mathbb{E}[(I'(f) - I'(f_n))^2])^{1/2}$$

for all $n \in \mathbb{N}$. The isometry (12.6) implies that both terms in the above right-hand side tend to 0 as $n \to \infty$. $\square$

Definition 12.4 For $f \in L^2(\lambda)$ the random variable $I(f) \in L^2(\mathbb{P})$ is called the (stochastic) *Wiener-Itô integral* of f.

Let $f \in \mathbb{R}(\mathbb{X})$ and define (f_n) by (12.7). If $f \in L^1(\lambda)$, then the sequence $I(f_n)$ converges almost surely towards $I(f)$, defined pathwise (that is for every $\omega \in \Omega$) by (12.4). If, however, $f \in L^2(\lambda) \setminus L^1(\lambda)$, then $I(f_n)$ converges to $I(f)$ in $L^2(\mathbb{P})$ and hence only in probability.

Note that $L^1(\lambda)$ is not contained in $L^2(\lambda)$ unless $\lambda(\mathbb{X}) < \infty$. It is possible to extend I to the set of all $f \in \mathbb{R}(\mathbb{X})$ satisfying

$$\int |f| \wedge f^2 \, d\lambda < \infty; \tag{12.9}$$

see Exercise 12.1.

12.2 Higher order Wiener-Itô integrals

In this section we turn to Poisson integrals of higher order. For $m \in \mathbb{N}$ and $f \in L^1(\lambda^m)$ define

$$I_m(f) := \sum_{J \subset [m]} (-1)^{m-|J|} \iint f(x_1, \dots, x_m)\, \eta^{(|J|)}(dx_J)\, \lambda^{m-|J|}(dx_{J^c}), \tag{12.10}$$

where $[m] := \{1, \dots, m\}$, $J^c := [m] \setminus J$, $x_J := (x_j)_{j \in J}$ and where $|J| := \operatorname{card} J$ denotes the cardinality of J. The inner integral in (12.10) is interpreted as $f(x_1, \dots, x_m)$ when $J = \emptyset$. This means that we set $\eta^{(0)}(c) := c$ for all $c \in \mathbb{R}$. Similarly, when $J = [m]$ the outer integration is performed according to the rule $\lambda^0(c) := c$ for all $c \in \mathbb{R}$. It follows from Theorem 4.4 and Fubini's theorem that

$$\mathbb{E}[I_m(f)] = 0. \tag{12.11}$$

A function $f \colon \mathbb{X}^m \to \mathbb{R}$ is said to be *symmetric* if

$$f(x_1, \dots, x_m) = f(x_{\pi(1)}, \dots, x_{\pi(m)}), \quad (x_1, \dots, x_m) \in \mathbb{X}^m,\ \pi \in \Sigma_m, \tag{12.12}$$

where Σ_m is the set of permutations of $[m]$, that is the set of all bijective mappings from $[m]$ to $[m]$. If $f \in L^1(\lambda^m)$ is symmetric, then the symmetry of $\eta^{(m)}$ (see (A.17)) and λ^m imply that

$$I_m(f) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \eta^{(k)} \otimes \lambda^{m-k}(f), \tag{12.13}$$

where $\eta^{(k)} \otimes \lambda^{m-k}(f)$ is the integral of f with respect to the product measure $\eta^{(k)} \otimes \lambda^{m-k}$. In accordance with our convention $\eta^{(0)}(c) = \lambda^0(c) = c$, we have set $\eta^{(0)} \otimes \lambda^m := \lambda^m$ and $\eta^{(m)} \otimes \lambda^0 := \eta^{(m)}$.

For $m = 1$ the definition (12.10) reduces to (12.4). For $m = 2$ we have

$$
\begin{aligned}
I_2(f) = \int f(x_1, x_2)\, \eta^{(2)}(d(x_1, x_2)) - \iint f(x_1, x_2)\, \eta(dx_1)\, \lambda(dx_2) \\
- \iint f(x_1, x_2)\, \lambda(dx_1)\, \eta(dx_2) + \int f(x_1, x_2)\, \lambda^2(d(x_1, x_2)).
\end{aligned}
\tag{12.14}
$$

In general, $I_m(f)$ is a linear combination of $\eta^{(m)}(f)$ and integrals of the type $\eta^{(k)}(f_k)$ for $k \leq m-1$, where f_k is obtained from f by integrating $m-k$ variables with respect to λ^{m-k}. There are some advantages in dealing with $I_m(f)$ rather than with $\eta^{(m)}(f)$.

Our goal is to find formulas for the mixed moments $\mathbb{E}[\prod_{i=1}^{\ell} I_{n_i}(f_i)]$, where $\ell, n_1, \dots, n_\ell \in \mathbb{N}$ and $f_i \in L(\lambda^{n_i})$ for $i \in \{1, \dots, \ell\}$. To do so we shall assume that the f_i are symmetric and exploit symmetry and combinatorial arguments.

Let $n \in \mathbb{N}$. A *subpartition* of $[n]$ is a family of disjoint non-empty subsets of $[n]$, which we call *blocks*. A *partition* of $[n] := \{1, \dots, n\}$ is a subpartition σ of $[n]$ such that $\cup_{J\in\sigma} J = [n]$. We denote by Π_n (respectively Π^*_n) the system of all partitions (respectively subpartitions) of $[n]$. The cardinality of $\sigma \in \Pi^*_n$ (i.e. the number of blocks of σ) is denoted by $|\sigma|$, while the cardinality of $\cup_{J\in\sigma} J$ is denoted by $\|\sigma\|$. If σ is a partition, then $\|\sigma\| = n$.

Consider $\ell, n_1, \dots, n_\ell \in \mathbb{N}$. Define $n := n_1 + \cdots + n_\ell$ and

$$
J_i := \{j : n_1 + \cdots + n_{i-1} < j \leq n_1 + \cdots + n_i\}, \quad i = 1, \dots, \ell. \tag{12.15}
$$

Let $\pi := \{J_i : 1 \leq i \leq \ell\}$ and let $\Pi(n_1, \dots, n_\ell) \subset \Pi_n$ (respectively $\Pi^*(n_1, \dots, n_\ell) \subset \Pi^*_n$) denote the set of all $\sigma \in \Pi_n$ (respectively $\sigma \in \Pi^*_n$) with $|J \cap J'| \leq 1$ for all $J \in \pi$ and all $J' \in \sigma$. Let $\Pi_{\geq 2}(n_1, \dots, n_\ell)$ (respectively $\Pi_{=2}(n_1, \dots, n_\ell)$) denote the set of all $\sigma \in \Pi(n_1, \dots, n_\ell)$ with $|J| \geq 2$ (respectively $|J| = 2$) for all $J \in \sigma$. It is instructive to visualize the pair (π, σ) as a *diagram* with rows $J_1, \dots, J_\ell$, where the elements in each block $J \in \sigma$ are encircled by a closed curve. Since the blocks of a $\sigma \in \Pi(n_1, \dots, n_\ell)$ are not allowed to contain more than one entry from each row, one might call the diagram (π, σ) *non-flat*. Any $\sigma \in \Pi_{\geq 2}(n_1, \dots, n_\ell)$ induces a partition $\sigma^* \in \Pi_\ell$: σ^* is the finest partition of $[\ell]$ such that two numbers $i, j \in [\ell]$ lie in the same block of σ^* if there exists a block of σ that intersects both J_i and J_j. (For $i, j \in [\ell]$ connect i and j by an edge, whenever such a block exists. Then σ^* is the set of components of the resulting graph on $[\ell]$.) Let $\tilde{\Pi}_{\geq 2}(n_1, \dots, n_\ell)$ be the set of all $\sigma \in \Pi_{\geq 2}(n_1, \dots, n_\ell)$ such that $|\sigma^*| = 1$.

The *tensor product* $\otimes_{i=1}^{\ell} f_i$ of functions $f_i \colon \mathbb{X}^{n_i} \to \mathbb{R}$, $i \in \{1, \dots, \ell\}$, is the function from $\mathbb{X}^n$ to $\mathbb{R}$ which maps each $(x_1, \dots, x_n)$ to $\prod_{i=1}^{\ell} f_i(x_{J_i})$.

In the case that $f_1 = \cdots = f_\ell = f$ we write $f^{\otimes \ell}$ instead of $\otimes_{i=1}^{\ell} f_i$. For any $n \geq 1$ and $p \geq 1$ let $L_s^p(\lambda^n)$ denote the set of all $f \in L^p(\lambda^n)$ that are symmetric. For any function $f \colon \mathbb{X}^n \to \mathbb{R}$ and $\sigma \in \Pi_n^*$ we define $f_\sigma \colon \mathbb{X}^{n+|\sigma|-\|\sigma\|} \to \mathbb{R}$ by identifying the arguments belonging to the same $J \in \sigma$. (The arguments $x_1, \dots, x_{n+|\sigma|-\|\sigma\|}$ have to be inserted in the order of occurrence.) In the case $n = 4$ and $\sigma = \{\{2,3\},\{4\}\}$, for instance, we have $f_\sigma(x_1, x_2, x_3) = f(x_1, x_2, x_2, x_3)$. The partition $\{\{1\},\{2,3\},\{4\}\}$ and the subpartition $\{\{2,3\}\}$ lead to the same function.

Theorem 12.5 *Let $f_i \in L_s^1(\lambda^{n_i})$, $i = 1, \dots, \ell$, where $\ell, n_1, \dots, n_\ell \in \mathbb{N}$. Assume that*

$$\int (\otimes_{i=1}^{\ell} |f_i|)_\sigma \, d\lambda^{|\sigma|} < \infty, \quad \sigma \in \Pi(n_1, \dots, n_\ell). \tag{12.16}$$

Then

$$\mathbb{E}\Big[\prod_{i=1}^{\ell} I_{n_i}(f_i)\Big] = \sum_{\sigma \in \Pi_{\geq 2}(n_1,\dots,n_\ell)} \int (\otimes_{i=1}^{\ell} f_i)_\sigma \, d\lambda^{|\sigma|}. \tag{12.17}$$

Proof We abbreviate $f := \otimes_{i=1}^{\ell} f_i$. The definition (12.10) and Fubini's theorem imply that

$$\prod_{i=1}^{\ell} I_{n_i}(f_i) = \sum_{I \subset [n]} (-1)^{n-|I|} \int \cdots \int f(x_1, \dots, x_n) \tag{12.18}$$
$$\eta^{(|I \cap J_1|)}(dx_{I \cap J_1}) \cdots \eta^{(|I \cap J_\ell|)}(dx_{I \cap J_\ell}) \, \lambda^{n-|I|}(dx_{I^c}),$$

where $I^c := [n] \setminus I$, and where we use definition (12.15) of J_i. For fixed $I \subset [n]$ we may split the region of integration into regions indexed by the set of $\sigma \in \Pi^*(n_1, \dots, n_\ell)$ such that $\cup_{J \in \sigma} J = I$. For such σ we integrate over those $(x_1, \dots, x_n)$ satisfying $x_i = x_j$ whenever i and j belong to the same block of σ but not otherwise. By Corollary 4.9 applied with $h(y_1, \dots, y_{|\sigma|}) = f(z_1, \dots, z_m)$ taking $z_i = y_j$ for i in the j-th block of σ, the contribution of σ to the expectation of the right-hand side of (12.18) equals $(-1)^{n-\|\sigma\|} \int f_\sigma \, d\lambda^{|\sigma|+n-\|\sigma\|}$. Therefore,

$$\mathbb{E}\Big[\prod_{i=1}^{\ell} I_{n_i}(f_i)\Big] = \sum_{\sigma \in \Pi^*(n_1,\dots,n_\ell)} (-1)^{n-\|\sigma\|} \int f_\sigma \, d\lambda^{|\sigma|+n-\|\sigma\|}. \tag{12.19}$$

Assumption (12.16) implies that all of these integrals are finite. Given $\sigma \in \Pi^*(n_1, \dots, n_\ell)$ with $|J| \geq 2$ for all $J \in \sigma$, let $\Pi_1(\sigma)$ denote the set of all $\sigma_1 \in \Pi^*(n_1, \dots, n_\ell)$ such that $\sigma \subset \sigma_1$ and $|J| \leq 1$ for all $J \in \sigma_1 \setminus \sigma$. (Note that $\sigma \in \Pi_1(\sigma)$.) Observe that

$$\int f_{\sigma_1}\, d\lambda^{|\sigma_1|+n-\|\sigma_1\|} = \int f_\sigma\, d\lambda^{|\sigma|+n-\|\sigma\|}$$

for all $\sigma_1 \in \Pi_1(\sigma)$. Moreover, for $n - \|\sigma\| \geq 1$ we have that

$$\sum_{\sigma_1 \in \Pi_1(\sigma)} (-1)^{n-\|\sigma_1\|} = 0.$$

Since every $\tau \in \Pi^*(n_1, \dots, n_\ell)$ has a unique $\sigma \in \Pi^*(n_1, \dots, n_\ell)$ with $|J| \geq 2$ for all $J \in \sigma$ such that $\tau \in \Pi_1(\sigma)$, we can partition $\Pi^*(n_1, \dots, n_\ell)$ into the sets $\Pi_1(\sigma)$, $\sigma \in \Pi^*(n_1, \dots, n_\ell)$ with $|J| \geq 2$ for all $J \in \sigma$. As shown above the sums over all $\sigma_1 \in \Pi_1(\sigma)$ with $\|\sigma\| < n$ vanish and only the integrals related to partitions $\sigma \in \Pi_{\geq 2}(n_1, \dots, n_\ell)$ remain. Therefore, (12.19) implies the asserted identity (12.17). □

Let $n \in \mathbb{N}$ and let $L_{0,s}(\lambda^n)$ denote the set of all symmetric $f \in \mathbb{R}(\mathbb{X}^n)$ such that f is bounded and $\lambda^n(\{f \neq 0\}) < \infty$. The following result generalizes Lemma 12.2.

Corollary 12.6 *Let $m, n \in \mathbb{N}$, $f \in L_{0,s}(\lambda^m)$ and $g \in L_{0,s}(\lambda^n)$. Then*

$$\mathbb{E}[I_m(f)I_n(g)] = \mathbf{1}\{m = n\}m! \int fg\, d\lambda^m. \tag{12.20}$$

Proof The assumptions allow us to apply Theorem 12.5 with $\ell = 2$, $f_1 = f$, and $g_2 = g$. First assume $m = n$. Then each element of $\Pi_{\geq 2}(m, n)$ is a partition of $[2m]$ with m blocks each having two elements, one from $\{1, \dots, m\}$ and one from $\{m + 1, \dots, 2m\}$. We identify each element of $\Pi_{\geq 2}(m, m)$ with a permutation of $[m]$ in the obvious manner. With Σ_m denoting the set of all such permutations, (12.17) gives us

$$\mathbb{E}[I_m(f)I_n(g)] = \sum_{\sigma \in \Sigma_m} \int f(x_1, \dots, x_m) g(x_{\sigma(1)}, \dots, x_{\sigma(m)})\, \lambda^m(d(x_1, \dots, x_m)).$$

By symmetry each term in the right-hand side above equals $\int fg\, d\lambda^m$. Hence (12.20) holds for $m = n$. If $m \neq n$ then $\Pi_{\geq 2}(m, n) = \emptyset$, so that (12.20) holds in this case too. □

The following result can be proved in the same manner as Proposition 12.3; see Exercise 12.2.

Proposition 12.7 *The mappings $I_m \colon L_{0,s}(\lambda^m) \to L^2(\mathbb{P})$, $m \in \mathbb{N}$, can be uniquely extended to linear mappings $I_m \colon L^2_s(\lambda^m) \to L^2(\mathbb{P})$ so that* (12.11) *and* (12.20) *hold for all $f \in L_s(\lambda^m)$ and $g \in L_s(\lambda^n)$. If $f \in L^1(\lambda^m) \cap L^2_s(\lambda^m)$, then $I_m(f)$ is almost surely given by* (12.10).

Definition 12.8 For $m \in \mathbb{N}$ and $f \in L^2_s(\lambda^m)$ the random variable $I_m(f) \in L^2(\mathbb{P})$ is called the (stochastic, multiple) *Wiener-Itô integral* of f.

12.3 Poisson U-statistics

In the rest of this chapter we apply the preceding results to *U-statistics*. Let $m \in \mathbb{N}$ and $h \in L^1_s(\lambda^m)$ and set

$$U := \int h(x_1, \dots, x_m)\, \eta^{(m)}(d(x_1, \dots, x_m)). \tag{12.21}$$

Then U is known as a *Poisson U-statistic* with *kernel function h*. For $n \in \{0, \dots, m\}$, define $h_n \in L^1_s(\lambda^n)$ by

$$h_n(x_1, \dots, x_n) := \binom{m}{n} \int h(x_1, \dots, x_n, y_1, \dots, y_{m-n})\, \lambda^{m-n}(d(y_1, \dots, y_{m-n})), \tag{12.22}$$

where $h_0 := \lambda^m(h)$. With our earlier convention $\lambda^0(c) = c$, $c \in \mathbb{R}$, this means that $h_m = h$. A Poisson U-statistic can be rewritten as follows.

Lemma 12.9 *Let U be the Poisson U-statistic given by* (12.21) *and define the functions* h_n *by* (12.22). *Then*

$$U = \mathbb{E}[U] + \sum_{n=1}^{m} I_n(h_n), \quad \mathbb{P}\text{-a.s.} \tag{12.23}$$

Proof We note that $\mathbb{E}[U] = \lambda^m(h) = h_0$ and set $\eta^{(0)} \otimes \lambda^0(h_0) := h_0$. Then we obtain from (12.13) that

$$\begin{aligned}
\mathbb{E}[U] + \sum_{n=1}^{m} I_n(h_n) &= \sum_{n=0}^{m} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \eta^{(k)} \otimes \lambda^{n-k}(h_n) \\
&= \sum_{k=0}^{m} \sum_{n=k}^{m} (-1)^{n-k} \binom{n}{k} \binom{m}{n} \eta^{(k)} \otimes \lambda^{m-k}(h) \\
&= \sum_{k=0}^{m} \binom{m}{k} \eta^{(k)} \otimes \lambda^{m-k}(h) \sum_{r=0}^{m-k} (-1)^r \binom{m-k}{r},
\end{aligned} \tag{12.24}$$

where we have used the substitution $r := n - k$ and the combinatorial identity

$$\binom{k+r}{k} \binom{m}{k+r} = \binom{m}{k} \binom{m-k}{r}.$$

The inner sum at (12.24) vanishes for $m > k$ and equals 1 otherwise. The result follows. □

Together with Proposition 12.7 the preceding lemma yields the following result.

Proposition 12.10 *Let the Poisson U-statistic U be given by* (12.21) *and assume that the functions h_n defined by* (12.22) *are square integrable with respect to λ^n for all $n \in [m]$. Then U is square integrable with variance*

$$\mathbb{V}\mathrm{ar}[U] = \sum_{n=1}^{m} n! \int h_n^2 \, d\lambda^n. \tag{12.25}$$

Proof Lemma 12.9, the assumption $h_n \in L^2(\lambda^n)$ and Proposition 12.7 imply that U is square integrable. Moreover, the isometry relation (12.20) (see Proposition 12.7) implies (12.25). □

If, in the situation of Proposition 12.10, $\lambda(h_1^2) = 0$ then we call U *degenerate*. This happens if and only if

$$\lambda\Big\{x_1 : \int h(x_1, \dots, x_m)\, \lambda^{m-1}(d(x_2, \dots, x_m)) \neq 0\Big\} = 0. \tag{12.26}$$

Therefore $\lambda^m(h) \neq 0$ is sufficient for U not to be degenerate.

In the remainder of this chapter we extend our setting by considering for $t > 0$ a Poisson process η_t with intensity measure $\lambda_t := t\lambda$. We study Poisson U-statistics of the form

$$U_t := b(t) \int h(x_1, \dots, x_m)\, \eta_t^{(m)}(d(x_1, \dots, x_m)), \tag{12.27}$$

where $m \in \mathbb{N}$, $b(t) > 0$ for all $t > 0$, and the kernel function $h \in L_s^1(\lambda^m)$ does not depend on t. For any integer $\ell \geq 2$ we abbreviate $\Pi^\ell(m) := \Pi(m, \dots, m) \subset \Pi_{m\ell}$. Recall the definition (B.5) of the double factorial $(\ell - 1)!!$ for even integer $l \geq 2$. This is the number of (perfect) *matchings* π of $[\ell]$. Such a matching is a partition of $[\ell]$ whose sets are all of size 2. If ℓ is odd, then no such matching exists.

Theorem 12.11 *Let $m \in \mathbb{N}$ and $h \in L_s^1(\lambda^m)$. Let U_t be the Poisson U-statistic given by* (12.27) *and let $\ell \geq 2$ be an integer. Assume that $\int (|h|^{\otimes \ell})_\sigma \, d\lambda^{|\sigma|} < \infty$ for all $\sigma \in \Pi^\ell(m)$. Assume also that $\lambda(h_1^2) > 0$, where h_1 is given by* (12.22) *for $n = 1$. Then*

$$\lim_{t\to\infty} \frac{\mathbb{E}[(U_t - \mathbb{E}U_t)^\ell]}{(\mathbb{V}\mathrm{ar}[U_t])^{1/2}} = \begin{cases} (\ell - 1)!!, & \text{if } \ell \text{ is even}, \\ 0, & \text{if } \ell \text{ is odd}. \end{cases} \tag{12.28}$$

Proof Let $n \in \{1, \dots, m\}$. Our assumptions on h imply that the mapping that sends the vector $(x_1, \dots, x_n, y_1, \dots, y_{m-n}, z_1, \dots, z_{m-n})$ to the product of $|h(x_1, \dots, x_n, y_1, \dots, y_{m-n})|$ and $|h(x_1, \dots, x_n, z_1, \dots, z_{m-n})|$ is integrable with respect to λ^{2m-n}. (To see this we might take a σ containing the blocks $\{1, 2n+1\}, \dots, \{n, 3n\}$ while the other blocks are all singletons.) By Fubini's theorem $h_n \in L^2(\lambda^n)$, where h_n is given by (12.22). For $t > 0$, define the function $h_{t,n}$ by (12.22) with (h, λ) replaced by $(b(t)h, \lambda_t)$. Then $h_{t,n} = b(t)t^{m-n}h_n$, so that Proposition 12.10 yields that U_t is square integrable and

$$\mathbb{V}\mathrm{ar}[U_t] = b(t)^2 \sum_{n=1}^{m} n!t^{2m-n} \int h_n^2 \, d\lambda^n. \tag{12.29}$$

In the remainder of the proof we assume without loss of generality that $b(t) = 1$ for all $t > 0$. Since $\lambda(h_1^2) > 0$, we obtain from (12.29) that

$$(\mathbb{V}\mathrm{ar}[U_t])^{\ell/2} = (\lambda(h_1^2))^{\ell/2} t^{\ell(m-1/2)} + p(t), \tag{12.30}$$

where the remainder term $p(\cdot)$ is a polynomial of degree strictly less than $\ell(m-1/2)$. Defining $I_{t,m}$ by (12.10) with (λ, η) replaced by (λ_t, η_t) and using (12.23), linearity of $I_{t,n}$ and the definition $\lambda_t := t\lambda$, we have

$$\begin{aligned}(U_t - \mathbb{E}U_t)^\ell &= \Big(\sum_{n=1}^{m} I_{t,n}(t^{m-n}h_n)\Big)^\ell \\ &= \sum_{(n_1,\dots,n_\ell)\in[m]^\ell} \prod_{i=1}^{\ell} t^{m-n_i} I_{t,n_i}(h_{n_i}).\end{aligned}$$

For $(n_1, \dots, n_\ell) \in [m]^\ell$ it follows as in the first part of the proof that $h_{n_1}, \dots, h_{n_\ell}$ satisfy the assumptions (12.16) of Theorem 12.5 (applied to η_t). Therefore,

$$\mathbb{E}[(U_t - \mathbb{E}U_t)^\ell] = \sum_{(i_1,\dots,i_\ell)\in[m]^\ell} \sum_{\sigma\in\Pi_{\geq 2}(n_1,\dots,n_\ell)} \int (\otimes_{i=1}^{\ell} t^{m-n_i} h_{n_i})_\sigma \, d\lambda_t^{|\sigma|}. \tag{12.31}$$

For $(n_1, \dots, n_\ell) \in [m]^\ell$ and $\sigma \in \Pi_{\geq 2}(n_1, \dots, n_\ell)$ consider a term in the right-hand side of (12.31). The exponent of t is given by $|\sigma| + \sum(m - n_i)$. Because $|\sigma| \leq \lfloor(\sum n_i)/2\rfloor$ and $\sum n_i \geq \ell$ we have

$$|\sigma| + \sum_{i=1}^{\ell}(m - n_i) \leq \ell m - \Big(\sum_{i=1}^{m} n_i\Big)/2 \leq \ell m - \ell/2,$$

so that the exponent of t is at most $\ell m - \ell/2$. For even ℓ this is obtained if and only if $n_1 = \dots = n_\ell = 1$, and the partition σ satisfies $|J| = 2$ for all $J \in \sigma$. The corresponding term in the right-hand side of (12.31) is then given by $(\lambda(h_1^2))^{\ell/2} t^{\ell(m-1/2)}$, matching (12.30). Moreover, there are exactly

$(\ell-1)!!$ partitions σ with the preceding property. This proves (12.28) for even ℓ.

If ℓ is odd, the maximal exponent in the numerator in (12.28) is not larger than $\ell m-(\ell+1)/2$ (in fact the order is attained). Comparing this with (12.30) shows that the limit (12.28) vanishes. □

The preceding proof implies the following result on the asymptotic variance of a Poisson U-statistic.

Proposition 12.12 *Let U_t be the Poisson U-statistic given by* (12.27)*. Let the functions h_n be given by* (12.22) *and assume that $h_n \in L^2(\lambda^n)$ for each $n \in [m]$. Then*

$$\lim_{t\to\infty} b(t)^{-2} t^{1-2m}\, \mathbb{V}\mathrm{ar}[U_t] = \int h_1^2\, d\lambda. \tag{12.32}$$

Proof It is easy to see (and explained at the beginning of the proof of Theorem 12.11) that our assumption that $h_n \in L^2(\lambda^n)$ for each $n \in [m]$ is equivalent to the integrability assumption of Theorem 12.11 in the case $\ell = 2$. Hence we have (12.29) and the result follows. □

Theorem 12.11 leads to the following central limit theorem.

Theorem 12.13 (Central limit theorem for Poisson U-statistics) *Suppose that $m \in \mathbb{N}$ and $h \in L^1_s(\lambda^m)$ and that U_t is given by* (12.27)*. Assume that $\int (|h|^{\otimes \ell})_\sigma\, d\lambda^{|\sigma|} < \infty$ for all $\ell \in \mathbb{N}$ and all $\sigma \in \Pi^\ell(m)$. Assume also that $\lambda(h_1^2) > 0$, where h_1 is given by* (12.22) *for $n = 1$. Then*

$$(\mathbb{V}\mathrm{ar}[U_t])^{-1/2}(U_t - \mathbb{E}U_t) \xrightarrow{d} N \text{ as } t \to \infty, \tag{12.33}$$

where $\xrightarrow{d}$ denotes convergence in distribution and N is a standard normal random variable.

Proof By (B.4) we have $\mathbb{E}[N^\ell] = (\ell-1)!!$ if ℓ is even and $\mathbb{E}[N^\ell] = 0$ otherwise. Moreover, with the help of Proposition B.4 one can show that the distribution of N is determined by these moments. Theorem 12.11 says that the moments of the random variable $(\mathbb{V}\mathrm{ar}[U_t])^{-1/2}(U_t - \mathbb{E}U_t)$ converge towards the moments of N. The method of moments (see Proposition B.13) gives the asserted convergence in distribution. □

The following lemma is helpful for checking the integrability assumptions of Theorem 12.13.

Lemma 12.14 *Let $\ell \in \mathbb{N}$ such that $h \in L^\ell_s(\lambda^m)$. Assume that $\{h \neq 0\} \subset B^m$,* *where $B \in \mathcal{X}$ satisfies $\lambda(B) < \infty$. Then we have $\int (|h|^{\otimes \ell})_\sigma \, d\lambda^{|\sigma|} < \infty$ for all $\sigma \in \Pi^\ell(m)$.*

Proof Apply Exercise 12.5 in the case $f_1 = \cdots = f_\ell = h$. □

12.4 Poisson hyperplane processes

Finally in this chapter we discuss a model from stochastic geometry. Let $d \in \mathbb{N}$ and let $\mathbb{H}_{d-1}$ denote the space of all hyperplanes in $\mathbb{R}^d$. Any such hyperplane H is of the form

$$H_{u,r} := \{y \in \mathbb{R}^d : \langle y, u \rangle = r\}, \tag{12.34}$$

where u is an element of the unit sphere $\mathbb{S}^{d-1}$, $r \geq 0$, and $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product. We can make $\mathbb{H}_{d-1}$ a measurable space by introducing the smallest σ-field $\mathcal{H}$ containing the sets

$$[K] := \{H \in \mathbb{H}_{d-1} : H \cap K \neq \emptyset\}, \quad K \in \mathcal{C}^d,$$

where $\mathcal{C}^d$ denotes the system of all compact subsets of $\mathbb{R}^d$. In fact, $\mathbb{H}_{d-1} \cup \{\emptyset\}$ is a closed subset of the space $\mathcal{F}^d$ of all closed subsets of $\mathbb{R}^d$, equipped with the Fell topology, as defined in Appendix A.3. In particular, $\mathbb{H}_{d-1}$ is a Borel subspace of a CSMS; see Appendix A.3.

We fix a measure λ on $\mathbb{H}_{d-1}$ satisfying

$$\lambda([K]) < \infty, \quad K \in \mathcal{C}^d. \tag{12.35}$$

In particular, λ is σ-finite. As before, for $t > 0$ let η_t be a Poisson process with intensity measure $\lambda_t := t\lambda$. The point process η_t is called a *Poisson hyperplane process* while the union

$$Z := \bigcup_{H \in \eta} H \tag{12.36}$$

of all hyperplanes in η is called a *Poisson hyperplane tessellation*. The cells of this tessellation are the (closures of the) connected components of the complement $\mathbb{R}^d \setminus Z$.

Recall from Appendix A.3 that $\mathcal{K}^d$ denotes the space of all convex compact subsets of $\mathbb{R}^d$. Let $W \in \mathcal{K}^d$ and $m \in \mathbb{N}$. Let $\psi : \mathcal{K}^d \to \mathbb{R}$ be a function such that $(H_1, \dots, H_m) \mapsto \psi(H_1 \cap \cdots \cap H_m \cap W)$ is a measurable mapping from $(H_{d-1})^m$ to $\mathbb{R}$. We also assume that $\psi(\emptyset) = 0$ and that $\psi(H_1 \cap \cdots \cap H_m \cap W) \leq c_W$ for λ^m-a.e. $(H_1, \dots, H_m)$, where c_W depends on W but not on K. We consider the Poisson U-statistic given by

$$U_t := \frac{1}{m!} \int \psi(H_1 \cap \cdots \cap H_m \cap W) \, \eta_t^{(m)}(d(H_1, \dots, H_m)), \quad t > 0. \tag{12.37}$$

Under rather weak assumptions on λ the point process η_t is simple and the intersection of m different hyperplanes from η_t is almost surely either empty or an affine space of dimension $m-n$; see Exercise 12.9. If we choose $\psi(K) = V_{m-n}(K)$ as the $(m-n)$-th intrinsic volume of $K \in \mathcal{K}^d$, then U_t is the total volume of the intersections of the $(m-n)$-dimensional faces of the hyperplane tessellation with W. This ψ satisfies the assumptions above. Another possible choice is $\psi(K) = \mathbf{1}\{K \neq \emptyset\}$. In that case U_t is the number of $(m-n)$-dimensional faces intersecting W.

To formulate a corollary to Theorem 12.13 we define, for $n \in [m]$, a function $\psi_n \in \mathbb{R}((\mathbb{H}_{d-1})^n)$ by

$$\psi_n(H_1, \dots, H_n) := \int \psi(H_1 \cap \cdots \cap H_m \cap W)\, \lambda^{m-n}(d(H_{n+1}, \dots, H_m)). \tag{12.38}$$

Corollary 12.15 *Let ψ satisfy the assumptions formulated above and define U_t by* (12.37). *Then*

$$\mathbb{E}[U_t] = \frac{t^m}{m!} \int \psi(H_1 \cap \cdots \cap H_m \cap W)\, \lambda^m(d(H_1, \dots, H_m)),$$
$$\mathbb{V}\mathrm{ar}[U_t] = \sum_{n=1}^{m} \frac{1}{n!((m-n)!)^2} t^{2m-n} \int \psi_n^2 \, d\lambda^n.$$

If, moreover,

$$\int \psi(H_1 \cap \cdots \cap H_m \cap W)\, \lambda^m(d(H_1, \dots, H_m)) \neq 0, \tag{12.39}$$

then the central limit theorem (12.33) *holds.*

Proof It follows from (12.35) and Lemma 12.14 that U_t satisfies the integrability assumptions of Proposition 12.10 and Theorem 12.13. Hence we can apply Theorem 12.13 and (12.29) with

$$h(H_1, \dots, H_m) := \frac{1}{m!} \psi(H_1 \cap \cdots \cap H_m \cap W)$$

and $h_n = \binom{m}{n}(m!)^{-1}\psi_n$. As noted at (12.26), the inequality (12.39) implies the non-degeneracy assumption of Theorem 12.13. □

12.5 Exercises

Exercise 12.1 Let $f \in \mathbb{R}(\mathbb{X})$ satisfy (12.9); define $g := \mathbf{1}\{|f| \le 1\}f$ and $h := \mathbf{1}\{|f| > 1\}f$. Show that

$$\int g^2\, d\lambda + \int |h|\, d\lambda = \int |f| \wedge |f|^2\, d\lambda.$$

This result justifies the definition

$$I(f) := I(g) + I(h), \tag{12.40}$$

where $I(g)$ is given by Definition 12.4 and $I(h) := \eta(h) - \lambda(h)$. Let $f_n \in \mathbb{R}(\mathbb{X})$, $n \in \mathbb{N}$, be bounded such that $\lambda(\{f_n \neq 0\}) < \infty$, $|f_n| \le |f|$ and $f_n \to f$. Show that $I(f_n) \to I(f)$ in probability.

Exercise 12.2 Prove Proposition 12.7.

Exercise 12.3 Let $f, g \in L^1(\lambda) \cap L^2(\lambda)$ and assume moreover that the functions fg^2, f^2g, f^2g^2 are all in $L^1(\lambda)$. Show that

$$\mathbb{E}[I(f)^2 I(g)^2] = \lambda(f^2)\lambda(g^2) + 2[\lambda(fg)]^2 + \lambda(f^2g^2).$$

In particular

$$\mathbb{E}[I(f)^4] = 3[\lambda(f^2)]^2 + \lambda(f^4),$$

provided that $f \in L^1(\lambda) \cap L^4(\lambda)$.

Exercise 12.4 Let $f \in L^1_s(\lambda^2)$ and $g \in L^1(\lambda)$. Show that

$$\mathbb{E}[I_2(f)^2 I_1(g)] = 4 \int f(x_1, x_2)^2 g(x_1)\, \lambda^2(d(x_1, x_2)).$$

holds under suitable integrability assumptions on f and g.

Exercise 12.5 Let $f_i \in L^1_s(\lambda^{n_i})$, $i = 1, \dots, \ell$, where $\ell, n_1, \dots, n_\ell \in \mathbb{N}$. Assume for any $i \in [\ell]$ that $f_i \in L^\ell(\lambda^{n_i})$ and $\{f_i \neq 0\} \subset B^{n_i}$, where $B \in \mathcal{X}$ satisfies $\lambda(B) < \infty$. Show for any $\sigma \in \Pi(n_1, \dots, n_\ell)$ that

$$\left(\int (\otimes_{i=1}^{\ell} |f_i|)_\sigma\, d\lambda^{|\sigma|}\right)^{\ell} \le \lambda(B)^{|\sigma| - n_1} \int |f_1|^\ell\, d\lambda^{n_1} \cdots \lambda(B)^{|\sigma| - n_\ell} \int |f_\ell|^\ell\, d\lambda^{n_\ell}.$$

Note that this implies (12.16). (Hint: Apply the multivariate Hölder inequality (A.3) in the case $m = \ell$ and $p_1 = \cdots = p_m = m$.)

Exercise 12.6 (Moment and factorial moment measures) Suppose that η

is a proper point process with factorial moment measures α_j, $j \in \mathbb{N}$. Show that

$$\mathbb{E}\Big[\int f\, d\eta^m\Big] = \sum_{\sigma\in\Pi_m} \int f_\sigma\, d\alpha_\sigma.$$

Exercise 12.7 Show that the mapping $(u, r) \mapsto H(u, r)$ from $\mathbb{S}^{d-1} \times \mathbb{R}$ to $\mathbb{H}_{d-1}$ is measurable; see (12.34). Show also that $(H, x) \mapsto H + x := \{y + x : y \in H\}$ is a measurable mapping from $\mathbb{H}_{d-1} \times \mathbb{R}^d$ to $\mathbb{H}_{d-1}$. (Hint: For any compact set $K \subset \mathbb{R}^d$ the set $\{(u, r) : H(u, r) \cap K = \emptyset\}$ is open in $\mathbb{S}^{d-1} \times \mathbb{R}$.)

Exercise 12.8 Let the measure λ_1 on $\mathbb{H}_{d-1}$ be given by

$$\lambda_1(\cdot) := \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} \mathbf{1}\{H(u, r) \in \cdot\}\, dr\, \mathbb{Q}(du), \tag{12.41}$$

where $\gamma > 0$ and the *directional distribution* $\mathbb{Q}$ is a probability measure on the unit sphere. For $t > 0$ let η_t be a Poisson process with intensity measure $\lambda_t := t\lambda$. Show that λ_t satisfies (12.35). Show also that η_t is *stationary* in the sense that the distribution of $\theta_x\eta_t$ does not depend on $x \in \mathbb{R}^d$. Here, for any point $x \in \mathbb{R}^d$ and any measure μ on $\mathbb{H}_{d-1}$, $\theta_x\mu$ denotes the measure $\mu(\{H : H - x \in \cdot\})$.

Exercise 12.9 Let λ be a locally finite measure on $\mathbb{H}_{d-1}$ that is absolutely continuous with respect to the measure λ_1 defined in Exercise 12.8. Let η be a Poisson process with intensity measure λ. Show that η is simple and that the intersection of m different hyperplanes from η is almost surely either empty or an affine space of dimension $m - n$. (Hint: Use Corollary 4.9.)

13

Random measures and Cox processes

A Cox process is a Poisson process with a random intensity measure and hence the result of a doubly stochastic procedure. A careful study of Cox processes requires the concept of a random measure, a natural and important generalization of a point process. The distribution of a Cox process determines that of its random intensity measure. Mecke's characterization of the Poisson process via a functional integral equation extends to Cox processes. An independent thinning of a Cox process is again Cox. This property is even characteristic for Cox processes, a fact that is proved here in the case of a discrete state space.

13.1 Random measures

A Cox process (here denoted η) can be interpreted as the result of a *doubly stochastic* procedure, that first generates a random measure ξ and then a Poisson process with intensity measure ξ. Before making this idea precise we need to introduce the concept of a random measure. Fortunately, the basic definitions and results are natural extensions of what we have seen before.

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space and let $\mathbf{M}(\mathbb{X}) \equiv \mathbf{M}$ denote the set of all s-finite measures μ on $\mathbb{X}$. Let $\mathcal{M}(\mathbb{X}) \equiv \mathcal{M}$ denote the smallest σ-field of subsets of $\mathbf{M}$ such that $\mu \mapsto \mu(B)$ is a measurable mapping on $\mathbf{M}$ for all $B \in \mathcal{X}$. For the following and later definitions we recall that all random elements are defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 13.1 A *random measure* on $\mathbb{X}$ is a random element ξ in the space $(\mathbf{M}, \mathcal{M})$, that is, a measurable mapping $\xi \colon \Omega \to \mathbf{M}$.

As in the case of point processes, if ξ is a random measure we denote by $\xi(B)$ the random variable $\omega \mapsto \xi(\omega, B) := \xi(B)(\omega)$. The distribution of a random measure ξ on $\mathbb{X}$ is the probability measure $\mathbb{P}_\xi$ on $(\mathbf{M}, \mathcal{M})$, given by $A \mapsto \mathbb{P}(\xi \in A)$. As in the point process case this distribution is determined by the family of random vectors $(\xi(B_1), \dots, \xi(B_m))$ for $B_1, \dots, B_m \in \mathcal{X}$ and $m \in \mathbb{N}$. It is also determined by the *Laplace functional* $L_\xi : \mathbb{R}_+(\mathbb{X}) \to [0, 1]$ defined by

$$L_\xi(u) := \mathbb{E}\Big[\exp\Big(-\int u(x)\,\xi(dx)\Big)\Big].$$

For ease of reference we summarize these facts with the following Proposition.

Proposition 13.2 *Let η and η' be random measures on $\mathbb{X}$. Then the assertions (i)-(iii) of Proposition 2.10 are equivalent.*

Proof The proof is essentially the same as that of Proposition 2.10. Only the proof of the fact that (ii) implies (i) requires some modification. Instead of the sets (2.9) we can use the sets

$$\{\mu \in \mathbf{M} : \mu(B_1) \in C_1, \dots, \mu(B_m) \in C_m\},$$

where $m \in \mathbb{N}$, $B_1, \dots, B_m \in \mathcal{X}$, and $C_1, \dots, C_m \in \mathcal{B}(\overline{\mathbb{R}}_+)$. The class of all such sets is a π-system generating $\mathcal{M}$ and (i) follows as before. □

As in the point process case the *intensity measure* of a random measure ξ is the measure ν on $\mathbb{X}$ defined by $\nu(B) := \mathbb{E}[\xi(B)]$, $B \in \mathcal{X}$. It satisfies *Campbell's formula*

$$\mathbb{E}\Big[\int \upsilon(x)\,\xi(dx)\Big] = \int \upsilon(x)\,\nu(dx), \quad \upsilon \in \mathbb{R}_+(\mathbb{X}), \tag{13.1}$$

which can be proved in the same manner as Proposition 2.7. For $m \in \mathbb{N}$ we can form the m-th power ξ^m of ξ. This is a random measure on $\mathbb{X}^m$; see Exercise 13.2. The m-th *moment measure* of a random measure ξ is the measure β_m on $\mathbb{X}^m$ defined by

$$\beta_m(B) := \mathbb{E}[\xi^m(B)], \quad B \in \mathcal{X}^m. \tag{13.2}$$

By definition, any point process is a random measure, even though, in the present generality we cannot prove that $\mathbf{N}$ is a measurable subset of $\mathbf{M}$. Here is another general example of a random measure. Later in this book we shall encounter further examples.

Example 13.3 Let $(Y(x))_{x \in \mathbb{X}}$ be a non-negative *random field* on $\mathbb{X}$, that is a family of $\mathbb{R}_+$-valued variables $\omega \mapsto Y(\omega, x)$. Assume that the random field is *measurable*, meaning that $(\omega, x) \mapsto Y(\omega, x)$ is a measurable mapping on $\Omega \times \mathbb{X}$. Let ν be a σ-finite measure on $\mathbb{X}$. Then

$$\xi(B) := \int \mathbf{1}_B(x) Y(x)\,\nu(dx), \quad B \in \mathcal{X},$$

defines a random measure ξ. Indeed, that $\xi(\omega, \cdot)$ is s-finite is a general fact from measure theory and easy to prove. The same is true for the measurablity of $\omega \mapsto \xi(\omega, B)$; see also the paragraph preceding Fubini's theorem (Theorem A.12). The intensity measure of ξ is the measure with density $\mathbb{E}[Y(x)]$ with respect to ν.

13.2 Cox processes

For any $\lambda \in \mathbf{M}(\mathbb{X})$ let Π_λ denote the distribution of a Poisson process with intensity measure λ. The existence of this object is guaranteed by Theorem 3.6.

Lemma 13.4 *Let $f \in \overline{\mathbb{R}}_+(\mathbf{N})$. Then $\lambda \mapsto \Pi_\lambda(f) = \int f\, d\Pi_\lambda$ is a measurable mapping from $\mathbf{M}$ to $\overline{\mathbb{R}}_+$.*

Proof Assume first that $f(\mu) = \mathbf{1}\{\mu(B_1) = k_1, \dots, \mu(B_m) = k_m\}$ for $m \in \mathbb{N}$, pairwise disjoint sets $B_1, \dots, B_m \in \mathcal{X}$ and $k_1, \dots, k_m \in \mathbb{N}_0$. Then

$$\Pi_\lambda(f) = \prod_{j=1}^{m} \frac{\lambda(B_j)^{k_j}}{k_j!} \exp[-\lambda(B_j)],$$

which is clearly a measurable function of λ. By the monotone class theorem (Theorem A.1) the measurability property extends to $f = \mathbf{1}_A$ for arbitrary $A \in \mathcal{N}$. The general case follows from monotone convergence. □

Definition 13.5 Let ξ be a random measure on $\mathbb{X}$. A point process η on $\mathbb{X}$ is a *Cox process directed by* ξ if

$$\mathbb{P}(\eta \in A \mid \xi) = \Pi_\xi(A), \quad \mathbb{P}\text{-a.s., } A \in \mathcal{N}. \tag{13.3}$$

Then ξ is called a *directing random measure* of η.

Let ξ be a random measure on $\mathbb{X}$. By Lemma 13.4 the right-hand side of (13.3) is a random variable for any fixed $A \in \mathcal{N}$. The left-hand side is a conditional probability as defined in the Appendix at (B.16). Equation (13.3) is equivalent to

$$\mathbb{E}[h(\xi)\mathbf{1}\{\eta \in A\}] = \mathbb{E}[h(\xi)\Pi_\xi(A)], \quad A \in \mathcal{N},\ h \in \mathbb{R}_+(\mathbf{M}). \tag{13.4}$$

Given any probability measure $\mathbb{Q}$ on $(\mathbf{M}, \mathcal{M})$, it is always the case that a Cox process directed by a random measure with distribution $\mathbb{Q}$ exists. For instance, $(\Omega, \mathcal{F}, \mathbb{P})$ can be taken as $(\Omega, \mathcal{F}) = (\mathbf{M} \times \mathbf{N}, \mathcal{M} \otimes \mathcal{N})$ and

$$\mathbb{P}(\cdot) := \iint \mathbf{1}\{(\lambda, \mu) \in \cdot\}\, \Pi_\lambda(d\mu)\, \mathbb{Q}(d\lambda).$$

Taking ξ and η as the first and second projections from Ω to $\mathbf{M}$ and $\mathbf{N}$ respectively, it is easy to check that $\mathbb{P}_\xi = \mathbb{Q}$ and that (13.3) holds.

Suppose η is a Cox process on $\mathbb{X}$ directed by a random measure ξ. By the monotone class theorem (Theorem A.1) and monotone convergence, equation (13.4) can be extended to

$$\mathbb{E}[h(\xi)f(\eta)] = \mathbb{E}\Big[h(\xi)\int f\, d\Pi_\xi\Big],$$

for $f \in \mathbb{R}_+(\mathbf{N})$ and $h \in \mathbb{R}_+(\mathbf{M})$. Therefore,

$$\mathbb{E}[f(\eta) \mid \xi] = \int f(\mu)\, \Pi_\xi(d\mu), \quad \mathbb{P}\text{-a.s., } f \in \mathbb{R}_+(\mathbf{N}). \tag{13.5}$$

As a first consequence we obtain for any $B \in \mathcal{X}$ that

$$\mathbb{E}[\eta(A)] = \mathbb{E}[\mathbb{E}[\eta(A) \mid \xi]] = \mathbb{E}\Big[\int \mu(A)\, \Pi_\xi(d\mu)\Big] = \mathbb{E}[\xi(A)],$$

where we have used that $\int \mu(A)\, \Pi_\lambda(d\mu) = \lambda(A)$ for all $\lambda \in \mathbf{M}$. Hence ξ and η have the same intensity measure. The next result deals with the second order moment measure.

Proposition 13.6 *Let η be a Cox process on $\mathbb{X}$ with directing random measure ξ. Let $v \in L^1(\nu)$, where ν is the intensity measure of ξ. Then $\mathbb{E}[\eta(v)^2] < \infty$ if and only if $\mathbb{E}[\xi(v)^2] < \infty$ and $v \in L^2(\nu)$. In this case*

$$\mathbb{V}\mathrm{ar}[\eta(v)] = \nu(v^2) + \mathbb{V}\mathrm{ar}[\xi(v)]. \tag{13.6}$$

Proof Let $\lambda \in \mathbf{M}$ and assume that $v \in L^1(\lambda)$. Proposition 12.3 implies that

$$\int (\mu(v))^2\, \Pi_\lambda(d\mu) = \lambda(v^2) + (\lambda(v))^2 \tag{13.7}$$

first under the additional assumption $v \in L^2(\lambda)$ but then, allowing for the value ∞ on both sides of (13.6), for general $v \in L^1(\lambda)$. From Campbell's formula (13.1) we have $\mathbb{E}[\eta(|v|)] = \mathbb{E}[\xi(|v|)] = \nu(|v|) < \infty$, so that we can apply (13.7) for $\mathbb{P}_\xi$-a.e. λ. It follows that

$$\mathbb{E}[(\eta(v))^2] = \mathbb{E}[\mathbb{E}[(\eta(v))^2 \mid \xi]] = \mathbb{E}[\xi(v^2)] + \mathbb{E}[(\xi(v))^2] = \nu(v^2) + \mathbb{E}[(\xi(v))^2].$$

This shows the asserted equivalence. The formula (13.6) for the variance follows upon subtracting $(\mathbb{E}[\eta(v)])^2 = (\mathbb{E}[\xi(v)])^2$ from both sides. □

If η' is a Poisson process with intensity measure ν, then $\nu(v^2)$ is the variance of $\eta'(v)$. If η is a Cox process with the same intensity measure, then (13.6) shows that the variance of $\eta(v)$ is at least the variance of $\eta'(v)$. Thus a Cox process is Poisson only if the directing measure is deterministic.

Corollary 4.9 shows for $m \in \mathbb{N}$ that the m-th factorial moment measure of a Cox process η directed by ξ is given by

$$\mathbb{E}[\eta^{(m)}(\cdot)] = \mathbb{E}[\xi^m(\cdot)], \tag{13.8}$$

where $\mathbb{E}[\xi^m(\cdot)]$ is the m-th moment measure of ξ; see (13.2).

The next result shows that the distribution of a Cox process determines that of the directing random measure.

Theorem 13.7 *Let η and η' be Cox processes on $\mathbb{X}$ with directing random measures ξ and ξ', respectively. Then $\eta \overset{d}{=} \eta'$ if and only if $\xi \overset{d}{=} \xi'$.*

Proof We obtain from (13.5) and Theorem 3.9 that

$$\mathbb{E}[\exp[-\eta(u)]] = \mathbb{E}\Big[\exp\Big(-\int (1 - e^{-u(x)})\,\xi(dx)\Big)\Big], \quad u \in \mathbb{R}_+(\mathbb{X}). \tag{13.9}$$

A similar equation holds for the pair (η', ξ'). Therefore $\xi \overset{d}{=} \xi'$ implies $L_\eta = L_{\eta'}$ and hence $\eta \overset{d}{=} \eta'$.

Assume, conversely, that $\eta \overset{d}{=} \eta'$. Let $v\colon \mathbb{X} \to [0,1)$ be measurable. Taking $u := -\log(1-v)$ in (13.9) implies that

$$\mathbb{E}[\exp[-\xi(v)]] = \mathbb{E}[\exp[-\xi'(v)]].$$

For any such v we then also have

$$\mathbb{E}[\exp[-t\xi(v)]] = \mathbb{E}[\exp[-t\xi'(v)]], \quad t \in [0,1].$$

Since a probability measure on $\mathbb{R}_+$ is determined by its Laplace transform (see Proposition B.4), $\xi(v) \overset{d}{=} \xi'(v)$. The latter identity can be extended to arbitrary bounded v and then to any $v \in \mathbb{R}_+(\mathbb{X})$ using monotone convergence. An application of Proposition 13.2 concludes the proof. □

13.3 The Mecke equation for Cox processes

In this section we extend Theorem 4.1.

Theorem 13.8 (Mecke Equation for Cox Processes) *Let ξ be a random measure on $\mathbb{X}$ and η a point process on $\mathbb{X}$. Then η is a Cox process directed by ξ if and only if*

$$\mathbb{E}\Big[\int f(x, \eta, \xi)\,\eta(dx)\Big] = \mathbb{E}\Big[\int f(x, \eta + \delta_x, \xi)\,\xi(dx)\Big], \tag{13.10}$$

for every $f \in \mathbb{R}_+(\mathbb{X} \times \mathbf{N} \times \mathbf{M})$.

Proof Suppose that η is a Cox process directed by ξ. It is sufficient to prove equation (13.10) for functions f of the form $f(x,\mu,\lambda) = g(x,\mu)h(\lambda)$ for $g \in \mathbb{R}_+(\mathbb{X}\times\mathbf{N})$ and $h \in \mathbb{R}_+(\mathbf{M})$. Then, after conditioning,

$$\mathbb{E}\Big[\int f(x,\eta,\xi)\,\eta(dx)\Big] = \mathbb{E}\Big[h(\xi)\iint g(x,\mu)\,\mu(dx)\,\Pi_\xi(d\mu)\Big].$$

By Theorem 4.1 the right-hand side equals

$$\begin{aligned}\mathbb{E}\Big[h(\xi)&\iint g(x,\mu+\delta_x)\,\xi(dx)\,\Pi_\xi(d\mu)\Big]\\ &= \mathbb{E}\Big[h(\xi)\iint g(x,\mu+\delta_x)\,\Pi_\xi(d\mu)\,\xi(dx)\Big]\\ &= \mathbb{E}\Big[h(\xi)\int \mathbb{E}[g(x,\eta+\delta_x)\mid\xi]\,\xi(dx)\Big].\end{aligned}$$

Using Exercise 13.1, we see that (13.10) holds.

Assume, conversely, that

$$\mathbb{E}\Big[h(\xi)\int g(x,\eta)\,\eta(dx)\Big] = \int \mathbb{E}[h(\xi)g(x,\eta+\delta_x)]\,\xi(dx), \tag{13.11}$$

for all $g \in \mathbb{R}_+(\mathbb{X}\times\mathbf{N})$ and $h \in \mathbb{R}_+(\mathbf{M})$. This implies that

$$\mathbb{E}\Big[\int g(x,\eta)\,\eta(dx)\,\Big|\,\xi\Big] = \mathbb{E}\Big[\int g(x,\eta+\delta_x)\,\xi(dx)\,\Big|\,\xi\Big], \quad \mathbb{P}\text{-a.s.}$$

If there was a regular conditional probability distribution of η given ξ, we could again appeal to Theorem 4.1, to conclude that η is a Cox process. As this cannot be guaranteed in the present generality, we have to resort to the proof of Theorem 4.1 and take disjoint sets $A_1,\dots,A_m$ in $\mathcal{X}$ and $k_1,\dots,k_m \in \mathbb{N}_0$. Then (13.11) implies, as in the proof of Theorem 4.1, that

$$\begin{aligned}&k_1\mathbb{P}(\eta(A_1)=k_1,\dots,\eta(A_m)=k_m\mid\xi)\\ &\quad = \xi(A_1)\mathbb{P}(\eta(A_1)=k_1-1,\eta(A_2)=k_2,\dots,\eta(A_m)=k_m\mid\xi), \quad \mathbb{P}\text{-a.s.},\end{aligned}$$

with the measure theory convention $\infty\cdot 0 := 0$. This implies that

$$\mathbb{P}(\eta(A_1)=k_1,\dots,\eta(A_m)=k_m\mid\xi) = \prod_{j=1}^m \frac{\xi(A_j)^{k_j}}{k_j!}\exp[-\xi(A_j)], \quad \mathbb{P}\text{-a.s.},$$

where we recall our convention that $(\infty^k)\exp[-\infty] = 0$ for all $k\in\mathbb{N}_0$. This is enough to imply (13.3). □

13.4 Cox processes on metric spaces

In this section we assume that $\mathbb{X}$ is a Borel subspace of a complete separable metric space. Let $\mathbf{M}_l$ denote the set of all locally finite measures on $\mathbb{X}$. Also let $\mathbf{M}_d$ denote the set of all locally finite measures on $\mathbb{X}$, that are also diffuse.

Lemma 13.9 *The sets $\mathbf{M}_l$ and $\mathbf{M}_d$ are measurable subsets of $\mathbf{M}$.*

Proof Just as in the case of $\mathbf{N}_l$ the measurability of $\mathbf{M}_l$ follows from the fact that a measure μ on $\mathbb{X}$ is locally finite if and only if $\mu(B_n) < \infty$ for all $n \in \mathbb{N}$, where B_n denotes the ball $B(x_0, n)$ for some fixed $x_0 \in \mathbb{X}$.

To prove the second assertion, we note that a measure $\mu \in \mathbf{M}$ is in $\mathbf{M}_d$ if and only if $\mu_{B_n} \in \mathbf{M}_{<\infty} \cap \mathbf{M}_d$ for each $n \in \mathbb{N}$, where $\mathbf{M}_{<\infty}$ is the set of all finite measures on $\mathbb{X}$. Hence it suffices to prove that $\mathbf{M}_{<\infty} \cap \mathbf{M}_d$ is measurable. This follows from Exercise 13.9. Indeed, a measure $\mu \in \mathbf{M}_{<\infty}$ is diffuse if and only if $\mu = \mu_0$. □

Definition 13.10 A random measure ξ on a Borel subspace $\mathbb{X}$ of a complete separable metric space is said to be *locally finite*, if $\mathbb{P}(\xi(B) < \infty) = 1$ for each bounded $B \in \mathcal{X}$. A locally finite random measure ξ is said to be *diffuse* if $\mathbb{P}(\xi \in \mathbf{M}_d) = 1$.

Theorem 13.11 *Let ξ and ξ' be locally finite random measures on $\mathbb{X}$ and assume that ξ is diffuse. Then $\xi \overset{d}{=} \xi'$ if and only if $\xi(B) \overset{d}{=} \xi'(B)$ for all $B \in \mathcal{X}_b$.*

Proof Let η and η' be Cox processes directed by ξ and ξ', respectively. Then η and η' are locally finite. Moreover, by Proposition 6.7 and Lemma 13.9 the point process η is simple. Assuming $\xi(B) \overset{d}{=} \xi'(B)$ for all $B \in \mathcal{X}_b$ and applying equation (13.9) with $u = c\mathbf{1}_B$ for $B \in \mathcal{X}_b$ and $c \geq 0$, shows that $\eta(B) \overset{d}{=} \eta'(B)$. Let

$$\eta^* := \int \eta'\{x\}^{\oplus} \mathbf{1}\{x \in \cdot\}\, \eta'(dx)$$

be the simple point process with the same support as η', where we recall from Exercise 9.3 that $a^{\oplus} := \mathbf{1}\{a \neq 0\}a^{-1}$ is the generalized inverse of $a \in \mathbb{R}$. Now Theorem 6.9 implies that $\eta \overset{d}{=} \eta^*$. In particular $\mathbb{E}[\eta'(B)] = \mathbb{E}[\eta^*(B)]$, and since $\eta'(B) \geq \eta^*(B)$ we obtain the relation $\mathbb{P}(\eta'(B) = \eta^*(B)) = 1$. Since $\mathcal{X}$ is countably generated we deduce that $\mathbb{P}(\eta' = \eta^*) = 1$ and hence that $\eta \overset{d}{=} \eta'$. Now we can conclude the assertion from Theorem 13.7. □

13.5 Thinning characterization in case of a discrete state space

By Exercise 13.7 a thinning of a Cox process is again a Cox process. In fact, this property characterizes the class of Cox processes. In Corollary 13.13 below we prove this result for an at most countable state space $\mathbb{X}$. For convenience we take $\mathbb{X} = \mathbb{N}$. In this case a random measure ξ is locally finite if $\mathbb{P}(\xi\{k\} < \infty) = 1$ for all $k \in \mathbb{N}$. A sequence ξ_n, $n \in \mathbb{N}$, of locally finite random measures on $\mathbb{N}$ *converges in distribution* to a locally finite random measure ξ if $(\xi_n\{k\})_{k\geq 1} \xrightarrow{d} (\xi\{k\})_{k\geq 1}$; see Appendix B. By Propositions B.12 and B.11 we have that $\xi_n \xrightarrow{d} \xi$ if and only if $\mathbb{E}[\exp[-\xi_n(u)]] \to \mathbb{E}[\exp[-\xi(u)]]$ for all $u \in \mathbb{R}_+(\mathbb{N})$ with finite support, that is, with $u(k) = 0$ for all but finitely many $k \in \mathbb{N}$.

Theorem 13.12 *Let η_n, $n \in \mathbb{N}$, be locally finite point processes on $\mathbb{N}$ such that η_n is a p_n-thinning of a locally finite point process χ_n, where the $p_n \in \mathbb{R}(\mathbb{N})$ take values in $[0, 1)$ and satisfy $p_n(k) \to 0$ as $n \to \infty$ for each $k \in \mathbb{N}$. Then there is a locally finite point process η on $\mathbb{N}$ such that $\eta_n \xrightarrow{d} \eta$ as $n \to \infty$ if and only if*

$$p_n\chi_n \xrightarrow{d} \xi, \quad \text{as } n \to \infty \tag{13.12}$$

for some locally finite random measure ξ. In this case η has the distribution of a Cox process directed by ξ.

Proof From Exercise 5.4 we know that for every $u \in \mathbb{R}_+(\mathbb{N})$,

$$\mathbb{E}[\exp[-\eta_n(u)]] = \mathbb{E}\Big[\exp\Big[\int \log\big(1 - p_n(x)(1 - e^{-u(x)})\big)\,\chi_n(dx)\Big]\Big]. \tag{13.13}$$

Using the inequalities $pr \leq -\log(1 - pr) \leq -r\log(1 - p)$ for all $r \in [0, 1]$ and $p \in [0, 1)$, it follows that

$$\begin{aligned}\mathbb{E}[\exp[-p'_n\chi_n(1 - e^{-u})]] &\leq \mathbb{E}[\exp[-\eta_n(u)]]\\ &\leq \mathbb{E}[\exp[-p_n\chi_n(1 - e^{-u})]],\end{aligned} \tag{13.14}$$

where $p'_n := -\log(1 - p_n)$.

Assume now that (13.12) holds and let $u \in \mathbb{R}_+(\mathbb{N})$ have finite support. Then $1-e^{-u}$ also has finite support. Since $|p_n - p'_n| \leq p_n O(p_n)$ with $O(r) \to 0$ as $r \to 0$, we obtain from (13.14) that

$$\lim_{n\to\infty} \mathbb{E}[\exp[-\eta_n(u)]] = \mathbb{E}[\exp[-\xi(1 - e^{-u})]].$$

Hence (13.9) implies that $\eta_n \xrightarrow{d} \eta$, where η has the distribution of a Cox process directed by ξ.

Assume, conversely, that $\eta_n \xrightarrow{d} \eta$ for some locally finite point process η. Let $m \in \mathbb{N}$. For any measure μ on $\mathbb{N}$ let $\mu^{[m]}$ be the vector $(\mu\{1\}, \dots, \mu\{m\})$. By Lemma B.10 and the second inequality in (13.14) (applied to $-\log(1 - tv)$ for $v \in \mathbb{R}(\mathbb{N})$ and sufficiently small $t > 0$), the sequence $(p_n\chi_n)^{[m]}$, $n \in \mathbb{N}$, of random vectors is tight. If $I \subset \mathbb{N}$ is a sequence, then Proposition B.9 implies that there is another sequence $J \subset I$ such that $(p_n\chi_n)^{[m]} \xrightarrow{d} \xi_m$ along J for some random vector ξ_m. In the first part of the proof we have shown that the distribution of $\eta^{[m]}$ is that of a Cox process directed by ξ_m, where we interpret both $\eta^{[m]}$ and ξ_m as random measures on $\mathbb{N}$ by setting $\eta^{[m]}\{k\} = \xi_m\{k\} := 0$ for $k > m$. It is clear that the distributions of ξ_m are consistent in the sense of Theorem B.2. Hence there is a random measure ξ on $\mathbb{N}$ (defined on a suitable probability space) such that $\xi^{[m]} \overset{d}{=} \xi_m$, $m \in \mathbb{N}$. Let η' be a Cox process directed by ξ. By Exercise 13.4, $(\eta')^{[m]}$ is a Cox process directed by $\xi^{[m]}$. Since $\xi^{[m]} \overset{d}{=} \xi_m$ we conclude from the definition of a Cox process that $\eta^{[m]} \overset{d}{=} (\eta')^{[m]}$ for any $m \in \mathbb{N}$. Hence $\eta \overset{d}{=} \eta'$ and the proof is complete. □

Corollary 13.13 *Suppose that η is a locally finite point process on an at most countable state space $\mathbb{X}$. Then η is a Cox process if and only if for every $p \in (0, 1)$, η has the same distribution as a p-thinning of some locally finite point process.*

Proof Exercise 13.7 gives the only if. The implication the other way follows upon applying Theorem 13.12 with $\eta_n := \eta$ and $p_n := 1/n$. □

13.6 Exercises

Exercise 13.1 Let η be a Cox process directed by ξ and let $f \in \mathbb{R}_+(\mathbb{X}\times\mathbf{N})$. Show that

$$\mathbb{E}\Big[\int f(x,\eta)\,\xi(dx)\,\Big|\,\xi\Big] = \int \mathbb{E}[f(x,\eta) \mid \xi]\,\xi(dx), \quad \mathbb{P}\text{-a.s.} \tag{13.15}$$

(Hint: Start with the choice $f(x,\mu) := \mathbf{1}\{x \in B, \mu \in A\}$ for $B \in \mathcal{X}$ and $A \in \mathcal{N}$. Then use the monotone class theorem.)

Exercise 13.2 Let ξ be a random measure on $\mathbb{X}$ and let $m \in \mathbb{N}$. Show that ξ^m is a random measure on $\mathbb{X}^m$.

Exercise 13.3 Let η be a Cox process directed by a random measure of the form $Y\rho$, where $Y \geq 0$ is a random variable and ρ a fixed s-finite measure on $\mathbb{X}$. Then η is called a *mixed Poisson proces*. Assume now in particular, that Y has a *Gamma distribution* with shape parameter a and scale parameter b; see (B.8). Let $B \in \mathcal{X}$ with $0 < \rho(B) < \infty$; show that

$$\mathbb{P}(\eta(B) = n) = \frac{\Gamma(n+a)}{\Gamma(n+1)\Gamma(a)} \left[\frac{\rho(B)}{b+\rho(B)}\right]^n \left[\frac{b}{b+\rho(B)}\right]^a, \quad n \in \mathbb{N}_0.$$

This is a negative binomial distribution with parameters $p = \frac{b}{b+\rho(B)}$ and a; see (B.11).

Exercise 13.4 Let η be a Cox process directed by ξ and $B \in \mathcal{X}$. Show that η_B is a Cox process directed by ξ_B.

Exercise 13.5 Let ξ be a random measure on $\mathbb{X}$ with $\mathbb{P}(0 < \xi(B) < \infty) = 1$ for some $B \in \mathcal{X}$. For $m \in \mathbb{N}$, define a probability measure $\mathbb{Q}_m$ on $\mathbf{M} \times \mathbb{X}^m$ by

$$\iint \mathbf{1}\{(\lambda, x_1, \dots, x_m) \in \cdot\} \, (\lambda(B)^m)^{\oplus} (\lambda_B)^m (d(x_1, \dots, x_m)) \, \mathbb{V}(d\lambda),$$

where $\mathbb{V}$ is the distribution of ξ. Show that there is a unique probability measure $\mathbb{Q}$ on $\mathbf{M} \times \mathbb{X}^\infty$ satisfying $\mathbb{Q}(A \times B \times \mathbb{X}^\infty) = \mathbb{Q}_m(A \times B)$ for all $m \in \mathbb{N}$, $A \in \mathcal{M}$ and $B \in \mathcal{X}^m$. What is the probabilistic interpretation of $\mathbb{Q}$?

Exercise 13.6 Let ξ be a random measure on $\mathbb{X}$ such that $\mathbb{P}(\xi(B_n) < \infty) = 1$ for a measurable partition $\{B_n : n \in \mathbb{N}\}$ of $\mathbb{X}$. Construct a suitable probability space supporting a proper Cox process directed by ξ. (Hint: Use Exercise 13.5.)

Exercise 13.7 Let η be a Cox process directed by a random measure ξ satisfying the assumption of Exercise 13.6. Assume η to be proper and let χ be a K-marking of η, where K is a probability kernel from $\mathbb{X}$ to some measurable space $(\mathbb{Y}, \mathcal{Y})$. Show that χ has the distribution of a Cox process directed by the random measure $\tilde{\xi} := \iint \mathbf{1}\{(x, y) \in \cdot\} \, K(x, dy) \, \xi(dx)$. Show in particular that a p-thinning of η has the distribution of a Cox process directed by the random measure $p(x) \, \xi(dx)$. (Hint: Use Proposition 5.4 and (13.9).)

Exercise 13.8 Let the assumptions of Exercise 13.7 be satisfied. Assume in addition that the sequence $(Y_n)_{n \ge 1}$ used in Definition 5.3 to define the K-marking of η is conditionally independent of ξ given $(\kappa, (X_n)_{n \le \kappa})$. Show that χ is a Cox process directed by the random measure $\tilde{\xi}$.

Exercise 13.9 Let $\mathbb{X}$ be a Borel space and let $\mathbf{M}_{<\infty}$ denote the space of all finite measures on $\mathbb{X}$. Show that there are measurable mappings $\tau_n \colon \mathbf{M}_{<\infty} \to (0, \infty)$, and $\pi_n \colon \mathbf{M}_{<\infty} \to \mathbb{X}$, $n \in \mathbb{N}$, along with measurable mappings $k\colon \mathbf{M}_{<\infty} \to \mathbb{N}_0$, and $D\colon \mathbf{M}_{<\infty} \to \mathbf{M}_{<\infty}$ such that $\sum_{n=1}^{k(\mu)} \delta_{\pi_n(\mu)}$ is simple, $D(\mu)$ is diffuse for each $\mu \in \mathbf{M}_{<\infty}$ and

$$\mu = D(\mu) + \sum_{n=1}^{k(\mu)} \tau_n(\mu)\delta_{\pi_n(\mu)}, \quad \mu \in \mathbf{M}_{<\infty}. \tag{13.16}$$

(Hint: Extend the method used in the proof of Proposition 6.2.)

14

Permanental processes

By definition, the factorial moment measures of an α-permanental point process have densities of an explicit algebraic form. These densities are the α-permanents arising from a given non-negative definite kernel function K and determine the distribution. If 2α is an integer, then an α-permanental process can be constructed as a Cox process, whose directing random measure is determined by independent Gaussian random fields with covariance function K. The proof of this fact is based on moment formulae for Gaussian random variables. The local Janossy measures of a permanental Cox process are given as the α-permanent of a suitably modified kernel function. The number of points in a bounded region is the sum of independent geometric random variables.

14.1 Definition and uniqueness

In this chapter the state space $(\mathbb{X}, \mathcal{X})$ is assumed to be a separable and locally compact metric space equipped with its Borel σ-field; see Section A.2. A set $B \subset \mathbb{X}$ is said to be *relatively compact* if its closure is compact. Let $\mathcal{X}_{rc}$ denote the system of all relatively compact $B \in \mathcal{X}$. We fix a measure ν on $\mathbb{X}$ such that $\nu(B) < \infty$ for every $B \in \mathcal{X}_{rc}$. Two important examples are $\mathbb{X} = \mathbb{R}^d$ with ν being Lebesgue measure and $\mathbb{X} = \mathbb{N}$ with ν being counting measure.

Let $m \in \mathbb{N}$ and $\sigma \in \Sigma_m$ a permutation of $[m]$. A *cycle* of σ is a k-tuple $(i_1 \dots i_k) \in [m]^k$ with distinct entries (written without commas), where $k \in [m]$, $\sigma(i_j) = i_{j+1}$ for $j \in [k-1]$ and $\sigma(i_k) = i_1$. In this case $(i_2 \dots i_k\, i_1)$ denotes the same cycle, that is cyclic permutations of a cycle are identified. The number k is called the length of the cycle.

Definition 14.1 Let $m \in \mathbb{N}$. Let $A = (a_{i,j})$ be an $m \times m$-matrix of real numbers. For $r \in [m]$ let

$$\mathrm{per}^{(r)}(A) := \sum_{\sigma \in \Sigma_m^{(r)}} \prod_{i=1}^{m} a_{i,\sigma(i)},$$

where $\Sigma_m^{(r)}$is the set of permutations of $[m] = \{1, \dots, m\}$ with exactly r cycles. For $\alpha \in \mathbb{R}$ the *α-permanent* of A is defined by

$$\mathrm{per}_\alpha(A) := \sum_{\sigma \in \Sigma_m} \alpha^{\#\sigma} \prod_{i=1}^{m} a_{i,\sigma(i)} = \sum_{r=1}^{m} \alpha^m \,\mathrm{per}^{(r)}(A),$$

where Σ_m is the group of permutations of $[m]$ and $\#\sigma$ is the number of cycles of $\sigma \in \Sigma_m$. The number $\mathrm{per}_1(A)$ is called the *permanent* of A.

Let $\mathbb{X}^*$ denote the support of ν. In this chapter we fix a symmetric jointly continuous function (sometimes called a *kernel*) $K \colon \mathbb{X}^* \times \mathbb{X}^* \to \mathbb{R}$. We assume that K is non-negative definite; see (B.17). We extend K to $\mathbb{X} \times \mathbb{X}$ by setting $K(x, y) := 0$ for $(x, y) \notin \mathbb{X}^* \times \mathbb{X}^*$.

Definition 14.2 Let $\alpha > 0$. Let η be a point process on $\mathbb{X}$ such that for every $m \in \mathbb{N}$ the m-th factorial moment measure of η is given by

$$\alpha_m(d(x_1, \dots, x_m)) = \mathrm{per}_\alpha([K](x_1, \dots, x_m))\, \nu^m(d(x_1, \dots, x_m)), \qquad (14.1)$$

where $[K](x_1, \dots, x_m)$ is the $m \times m$-matrix with entries $K(x_i, x_j)$. Then η is said to be an *α-permanental* process with kernel K (with respect to ν).

If η is α-permanental with kernel K, then the case $m = 1$ of (14.1) implies that the intensity measure of η is given by

$$\mathbb{E}[\eta(B)] = \int_B \alpha K(x, x)\, \nu(dx), \qquad B \in \mathcal{X}. \qquad (14.2)$$

If B is compact, then $K(x, x)$ is bounded on B (being a continuous function). Since we have assumed that $\nu(B) < \infty$, we have $\mathbb{E}[\eta(B)] < \infty$ and in particular $\mathbb{P}(\eta(B) < \infty) = 1$.

We first note that the distribution of a permanental process is uniquely determined by its kernel.

Proposition 14.3 *Let $\alpha > 0$. Suppose that η and η' are α-permanental processes with kernel K. Then $\eta \overset{d}{=} \eta'$.*

Proof Let $B \subset \mathbb{X}$ be compact. Since K is continuous there exists $c > 0$ such that $|K(x, y)| \le c$ for all $x, y \in B$. Therefore

$$\int_{B^m} |\,\mathrm{per}_\alpha([K](x_1, \dots, x_m))|\, \nu^m(d(x_1, \dots, x_m)) \le m!(\max\{\alpha, 1\})^m c^m \nu(B)^m.$$

Since $\mathbb{X}$ is σ-compact (by Lemma A.19), the assertion follows from Proposition 4.11. □

We continue with a simple example.

Example 14.4 Suppose that $\mathbb{X} = \{1\}$ is a singleton and that $\nu\{1\} = 1$. (A point process η on $\mathbb{X}$ can be identified with the random variable $\eta\{1\}$.) Set $\gamma := K(1,1) \geq 0$. Let $\alpha > 0$ and $m \in \mathbb{N}$. For $(x_1, \dots, x_m) := (1, \dots, 1)$ and with E_m denoting the $m \times m$-matrix with all entries equal to 1, we have that

$$\operatorname{per}_\alpha([K](x_1, \dots, x_m)) = \gamma^m \operatorname{per}_\alpha(E_m) = \gamma^m \alpha(\alpha+1)\cdots(\alpha+m-1), \tag{14.3}$$

where the second identity follows from Exercise 14.1. By Definition 14.2, (4.7) and (14.3), a point process η on $\mathbb{X}$ is α-permanental with kernel K if

$$\mathbb{E}[(\eta(\mathbb{X}))_m] = \gamma^m \alpha(\alpha+1)\cdots(\alpha+m-1), \quad m \geq 1. \tag{14.4}$$

Exercise 14.2 shows that these are the factorial moments of a negative binomial distribution with parameters $1/(1+\gamma)$ and α. Hence an α-permanental process with kernel K exists. Proposition 4.11 shows that its distribution is uniquely determined.

14.2 Moments of Gaussian random variables

In order to show later on the existence in general of permanental processes, we need to establish a combinatorial formula for mixed moments of normal random variables. For $\ell \in \mathbb{N}$ let $M(\ell)$ denote the set of matchings of $[\ell]$ and note that $M(\ell)$ is empty if ℓ is odd. For any $\pi \in \Pi_\ell$ (in particular for π a matching) we denote the blocks of π (in some arbitrary order) as $J_1(\pi), \dots, J_{|\pi|}(\pi)$. In the case that π is a matching we write $J_r(\pi) = \{k_r(\pi), k'_r(\pi)\}$.

Lemma 14.5 (Wick formula) *Let $\ell \in \mathbb{N}$ and let $f_1, \dots, f_\ell$ be functions on $\mathbb{N}$ such that $\sum_{m=1}^\infty f_i(m)^2 < \infty$ for all $i \in \{1, \dots, \ell\}$. Let $Y_1, Y_2, \dots$ be independent standard normal random variables and define*

$$X_i := \sum_{m=1}^\infty Y_m f_i(m), \quad i = 1, \dots, \ell. \tag{14.5}$$

Then

$$\mathbb{E}\Big[\prod_{i=1}^\ell X_i\Big] = \sum_{\pi \in M(\ell)} \prod_{i=1}^{\ell/2} \bigg(\sum_{m=1}^\infty f_{k_i(\pi)}(m) f_{k'_i(\pi)}(m)\bigg). \tag{14.6}$$

Proof Let λ_0 denote the counting measure on $\mathbb{N}$. We first show that both sides of (14.6) depend on each of the individual functions $f_1, \dots, f_\ell$ in an $L^2(\lambda_0)$-continuous way. To see this we let $f_1^{(n)} \in L^2(\lambda_0)$ be such that $f_1^{(n)} \to f_1$ in $L^2(\lambda_0)$ as $n \to \infty$. For each $n \in \mathbb{N}$ define $X_1^{(n)}$ by (14.5) with $i = 1$ and $f_1^{(n)}$ in place of f_1. It is easy to check that $\mathbb{E}[(X_1 - X_1^{(n)})^2] \to 0$ as $n \to \infty$. Since normal random variables have moments of all orders, so does $\prod_{i=2}^{\ell} X_i$. By the Cauchy-Schwarz inequality

$$\lim_{n\to\infty} \mathbb{E}\Big[\big|X - X_1^{(n)}\big| \prod_{i=2}^{\ell} X_i\Big] = 0,$$

so that $\mathbb{E}[X_1^{(n)} \prod_{i=2}^{\ell} X_i] \to \mathbb{E}[\prod_{i=1}^{\ell} X_i]$ as $n \to \infty$. By another application of the Cauchy-Schwarz inequality, the right hand side of (14.6) also depends on f_1 in an $L^2(\lambda_0)$-continuous manner.

To prove (14.6) we can now assume that $f_i(m) = 0$ for all $i \in [\ell]$ and all sufficiently large m. We then obtain that

$$\mathbb{E}\Big[\prod_{i=1}^{\ell} X_i\Big] = \sum_{m_1,\dots,m_\ell=1}^{\infty} f_1(m_1)\cdots f_\ell(m_\ell)\,\mathbb{E}[Y_{m_1}\cdots Y_{m_\ell}].$$

Each $(m_1, \dots, m_\ell)$ in the sum determines a partition $\sigma \in \Pi_\ell$ by letting $i, k \in [\ell]$ be in the same block of σ if and only if $m_i = m_k$. Writing the distinct values of $m_1, \dots, m_k$ as $n_1, \dots, n_{|\sigma|}$ and using (B.4) and (B.6), we may deduce that

$$\begin{aligned}\mathbb{E}\Big[\prod_{i=1}^{\ell} X_i\Big] &= \sum_{\sigma\in\Pi_\ell} \sideset{}{^{\neq}}\sum_{n_1,\dots,n_{|\sigma|}\in\mathbb{N}} \prod_{r=1}^{|\sigma|} \Big(|M(|J_r(\sigma)|)| \prod_{i\in J_r(\sigma)} f_i(n_r)\Big) \\ &= \sum_{\sigma\in\Pi_\ell} c_\sigma \sideset{}{^{\neq}}\sum_{n_1,\dots,n_{|\sigma|}\in\mathbb{N}} \prod_{r=1}^{|\sigma|} \prod_{i\in J_r(\sigma)} f_i(n_r), \qquad (14.7)\end{aligned}$$

where $c_\sigma := \prod_{r=1}^{|\sigma|} |M(|J_r(\sigma)|)|$ is the number of matchings $\pi \in M(\ell)$ such that each block of π is contained in a block of σ (that is, π is a refinement of σ).

Now consider the right hand side of (14.6). By a similar partitioning argument to the above, this equals

$$\sum_{\pi\in M(\ell)} \sum_{m_1,\dots,m_{\ell/2}=1}^{\infty} \prod_{r=1}^{\ell/2} \prod_{i\in J_r(\pi)} f_i(m_r) = \sum_{\pi\in M(\ell)} \sum_{\sigma\in\Pi_{\ell/2}} \sideset{}{^{\neq}}\sum_{n_1,\dots,n_{|\sigma|}\in\mathbb{N}} \prod_{r=1}^{|\sigma|} \prod_{i\in J_r(\sigma,\pi)} f_i(n_r),$$

where $J_r(\sigma, \pi) := \cup_{j\in J_r(\sigma)} J_j(\pi)$. Each pair (π, σ) in the sum determines a partition $\pi \in \Pi_\ell$ by $\pi := \{J_r(\pi, \sigma) : 1 \le r \le |\sigma|\}$, and each $\pi \in \Pi_\ell$ is obtained from c_π such pairs. Hence, the last display equals the expression (14.7) so the result is proved. □

14.3 Construction of permanental processes

In this section we construct α-permanental processes in the case $2\alpha \in \mathbb{N}$. In fact, under certain assumptions on K such a process exists for other values of α (as already shown by Example 14.4) but proving this is beyond the scope of this volume. By Theorem B.19 and Proposition B.21 there exists a measurable centred *Gaussian random field* $Z = (Z(x))_{x\in\mathbb{X}}$ with *covariance function* $K/2$, that is $(Z(x_1), \dots, Z(x_m))$ has a multivariate normal distribution for all $m \in \mathbb{N}$ and $(x_1, \dots, x_m) \in \mathbb{X}^m$, $\mathbb{E}[Z(x)] = 0$ for all $x \in \mathbb{X}$, and

$$\mathbb{E}[Z(x)Z(y)] = \frac{K(x,y)}{2}, \quad x, y \in \mathbb{X}. \tag{14.8}$$

Theorem 14.6 *Let $k \in \mathbb{N}$ and let $Z_1, \dots, Z_k$ be independent measurable random fields with the same distribution as Z. Define a random measure ξ on $\mathbb{X}$ by*

$$\xi(B) := \int_B (Z_1(x)^2 + \cdots + Z_k(x)^2)\, \nu(dx), \quad B \in \mathcal{X}, \tag{14.9}$$

and let η be a Cox process directed by ξ. Then η is $(k/2)$-permanental.

The proof of Theorem 14.6 is based on an explicit (spectral) representation of the Gaussian random field Z. For $B \in \mathcal{X}_{rc}$ let $B^* \subset \mathbb{X}$ be the support of ν_B. Then B^* is a closed subset of the closure of B (assumed to be compact) and therefore compact. By Lemma A.22,

$$\nu(B \setminus B^*) = 0. \tag{14.10}$$

Applying Mercer's theorem (Theorem B.20) to the metric space B^*, we see that there exist $\gamma_{B,j} \geq 0$ and $v_{B,j} \in L^2(\nu_{B^*})$, $j \in \mathbb{N}$, such that

$$\int_B v_{B,i}(x) v_{B,j}(x)\, \nu(dx) = \mathbf{1}\{i = j\}, \quad i, j \in \mathbb{N}, \tag{14.11}$$

and

$$K(x,y) = \sum_{j=1}^{\infty} \gamma_{B,j} v_{B,j}(x) v_{B,j}(y), \quad x, y \in B^*, \tag{14.12}$$

where the convergence is uniform and absolute. In (14.11) (and also later) we interpret $v_{B,j}$ as functions on $B \cup B^*$ by setting $v_{B,j}(x) := 0$ for $x \in$

$B \setminus B^*$. In view of (14.10) this modification has no effect on our subsequent calculations. A consequence of (14.12) is

$$\sum_{j=1}^{\infty} \gamma_{B,j} v_{B,j}(x)^2 = K(x,x) < \infty, \quad x \in B^* \cup B. \tag{14.13}$$

Combining this with (14.11), we obtain from monotone convergence that

$$\int_B K(x,x)\,\nu(dx) = \sum_{j=1}^{\infty} \gamma_{B,j}, \tag{14.14}$$

a finite number.

Now let $k \in \mathbb{N}$ and let $Y_{i,j}$, $i = 1,\dots,k$, $j \in \mathbb{N}$, be a family of independent random variables with the standard normal distribution. Define independent measurable random fields $Z_{B,i} = (Z_{B,i}(x))_{x \in B^* \cup B}$, $i \in \{1,\dots,k\}$, by

$$Z_{B,i}(x) := \frac{1}{\sqrt{2}} \sum_{j=1}^{\infty} \sqrt{\gamma_{B,j}}\, Y_{i,j} v_{B,j}(x), \quad x \in B^* \cup B, \tag{14.15}$$

making the convention that $Z_{B,i}(x) := 0$, whenever the series diverges. By (14.13) and Proposition B.6, (14.15) converges almost surely and in $L^2(\mathbb{P})$. Since componentwise almost sure convergence of random vectors implies convergence in distribution, it follows that $Z_{B,1},\dots,Z_{B,k}$ are Gaussian random fields. By the $L^2(\mathbb{P})$-convergence of (14.15),

$$\mathbb{E}[Z_{B,i}(x)Z_{B,i}(y)] = \frac{1}{2}\sum_{j=1}^{\infty} \gamma_{B,j} v_{B,j}(x) v_{B,j}(y) = \frac{K(x,y)}{2}, \quad x,y \in B^*.$$

It follows that

$$((Z_{B,1}(x))_{x\in B^*},\dots,(Z_{B,n}(x))_{x\in B^*}) \overset{d}{=} ((Z_1(x))_{x\in B^*},\dots,(Z_k(x))_{x\in B^*}). \tag{14.16}$$

Therefore, when dealing with the restriction of $(Z_1,\dots,Z_k)$ to a given set $B \in \mathcal{X}_{rc}$, we can work with the explicit representation (14.15). Later we shall need the following fact.

Lemma 14.7 *Let $B \in \mathcal{X}_{rc}$ and $W_B(x) := Z_{B,1}(x)^2 + \cdots + Z_{B,k}(x)^2$, $x \in B$. Then we have $\mathbb{P}$-a.s. that*

$$\int_B W_B(x)\,\nu(dx) = \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{\infty} \gamma_{B,j} Y_{i,j}^2. \tag{14.17}$$

Proof Let $i \in \{1, \dots, k\}$ and $n \in \mathbb{N}$. Then by (14.15),

$$\mathbb{E}\Big[\int_{B^*}\Big(Z_{B,i}(x) - \frac{1}{\sqrt{2}}\sum_{j=1}^{n}\sqrt{\gamma_{B,j}}Y_{i,j}v_{B,j}(x)\Big)^2\,\nu(dx)\Big]$$
$$= \frac{1}{2}\int_B \mathbb{E}\Big[\Big(\sum_{j=n+1}^{\infty}\sqrt{\gamma_{B,j}}Y_{i,j}v_{B,j}(x)\Big)^2\Big]\,\nu(dx)$$
$$= \frac{1}{2}\int_B\Big(\sum_{j=n+1}^{\infty}\gamma_{B,j}v_{B,j}(x)^2\Big)\nu(dx) = \frac{1}{2}\sum_{j=n+1}^{\infty}\gamma_{B,j} \to 0 \quad \text{as } n \to \infty,$$

where we have used (14.14) to get the convergence. By Proposition B.7,

$$\int_B\Big(Z_{B,i}(x) - \frac{1}{\sqrt{2}}\sum_{j=1}^{n}\sqrt{\gamma_{B,j}}Y_{i,j}v_{B,j}(x)\Big)^2\,\nu(dx) \to 0$$

in probability. Hence we obtain from the Minkowski inequality that

$$\int_B\Big(\frac{1}{\sqrt{2}}\sum_{j=1}^{n}\sqrt{\gamma_{B,j}}Y_{i,j}v_{B,j}(x)\Big)^2\,\nu(dx) \to \int_B Z_{B,i}(x)^2\,\nu(dx)$$

in probability. By the orthogonality relation (14.11),

$$\int_B\Big(\frac{1}{\sqrt{2}}\sum_{j=1}^{n}\sqrt{\gamma_{B,j}}Y_{i,j}v_{B,j}(x)\Big)^2\,\nu(dx) = \frac{1}{2}\sum_{j=1}^{n}\gamma_{B,j}Y_{i,j}^2 \to \frac{1}{2}\sum_{j=1}^{\infty}\gamma_{B,j}Y_{i,j}^2,$$

with almost sure convergence. Summing both limits over $i \in \{1, \dots, k\}$ proves the result. □

Proof of Theorem 14.6 Let $\alpha := k/2$. Let

$$W_k(x) := Z_1(x)^2 + \cdots + Z_k(x)^2, \quad x \in \mathbb{X}.$$

In view of (14.9) and (13.8) we have to show that

$$\mathbb{E}[W_k(x_1)\cdots W_k(x_m)] = \mathrm{per}_{k/2}([K](x_1, \dots, x_m)), \quad \nu^m\text{-a.e. } (x_1, \dots, x_m). \tag{14.18}$$

For the rest of the proof we fix $m \in \mathbb{N}$ and $x_1, \dots, x_m \in \mathbb{X}$. By (14.10) and (14.16) we can assume that $Z_i = Z_{B,i}$ for every $i \in \{1, \dots, k\}$, where $B \in \mathcal{X}_{rc}$ satisfies $\{x_1, \dots, x_m\} \subset B^*$.

We now prove (14.18) for $k = 1$. Set $(z_1, \dots, z_{2m}) := (x_1, x_1, \dots, x_m, x_m)$ and for $i \in [2m], n \in \mathbb{N}$ set $f_i(n) = \sqrt{\gamma_{B,n}/2}\,v_{B,n}(z_i)$. Then by (14.15),

$$W_1(x_1)\cdots W_1(x_m) = \prod_{r=1}^{2m}\Big(\sum_{n=1}^{\infty}\sqrt{\frac{\gamma_{B,n}}{2}}v_{B,n}(z_r)Y_{1,n}\Big) = \prod_{r=1}^{2m}\Big(\sum_{j=1}^{\infty}f_r(j)Y_{1,j}\Big)$$

so by (14.6),

$$\begin{aligned}\mathbb{E}[W_1(x_1)\cdots W_1(x_m)] &= \sum_{\pi\in M(2m)}\prod_{i=1}^{m}\left(\sum_{n=1}^{\infty} f_{k_i(\pi)}(n) f_{k_i'(\pi)}(n)\right)\\ &= \sum_{\pi\in M(2m)}\prod_{i=1}^{m}\left(\sum_{n=1}^{\infty}\frac{\gamma_{B,n}}{2}v_{B,n}(z_{k_i(\pi)})v_{B,n}(z_{k_i'(\pi)})\right)\end{aligned}$$

so by (14.12),

$$\mathbb{E}[W_1(x_1)\cdots W_1(x_m)] = 2^{-m}\sum_{\pi\in M(2m)}\prod_{i=1}^{m} K(z_{k_i(\pi),k_i'(\pi)}) \tag{14.19}$$

Any matching $\pi \in M(2m)$ defines a permutation σ of $[m]$. The cycles of this permutation are defined as follows. Partition $[2m]$ into m blocks $J_i := \{2i-1, 2i\}$, $i \in [m]$. Let $j_2 \in [2m]$ such that $\{1, j_2\} \in \pi$. Then $j_2 \in J_{i_2}$ for some $i_2 \in [m]$ and we define $\sigma(1) := i_2$. If $i_2 = 1$, then (1) is the first cycle of σ. Otherwise there is a $j_2' \in J_i \setminus \{j_2\}$ and a $j_3 \in [2m]$ such that $\{j_2, j_3\} \in \pi$. Then $j_3 \in J_{i_3}$ for some $i_3 \in [m]$; we let $\sigma(i_2) := i_3$. If $i_3 = 1$, then $(1\, i_2)$ is the first cycle. Otherwise we continue with this procedure. After a finite number of recursions we obtain the first cycle $(i_1 \dots i_k)$ for some $k \in [m]$, where $i_1 := 1$. To get the second cycle we remove the blocks $J_{i_1}, \dots, J_{i_k}$ and proceed as before (starting with the first available block). This procedure yields the cycles of σ after a finite number of steps. (In the case $m = 3$ for instance, the matching $\{\{1, 2\}, \{3, 5\}, \{4, 6\}\}$ gives the permutation $\sigma(1) = 1$, $\sigma(2) = 3$, and $\sigma(3) = 2$. This permutation has two cycles.) The corresponding contribution to the right-hand side of (14.19) is

$$2^{-m}\prod_{i=1}^{m} K(x_i, x_{\sigma(i)})$$

and depends only on the permutation and not on the matching. Since any permutation σ of $[m]$ with r cycles of lengths $k_1, \dots, k_r$ corresponds to $2^{k_1-1}\cdots 2^{k_r-1} = 2^{m-r}$ matchings, the case $k = 1$ of (14.18) follows.

Finally consider a general $k \in \mathbb{N}$. Let $(x_1, \dots, x_m) \in \mathbb{X}^m$ such that (14.18) holds. Since $(Z_i(x_1)^2, \dots, Z_i(x_m)^2)$, $i \in [k]$, are independent vectors,

$$
\begin{aligned}
\mathbb{E}[W_k(x_1)\cdots W_k(x_m)] &= \mathbb{E}\Bigg[\sum_{i_1=1}^{k}\cdots\sum_{i_m=1}^{k} Z_{i_1}(x_1)^2\cdots Z_{i_m}(x_m)^2\Bigg] \\
&= \sum_{\pi\in\Pi_m}\ \sum_{j_1,\dots,j_{|\pi|}\in[k]}^{\neq} \mathbb{E}\Bigg[\prod_{s=1}^{|\pi|}\prod_{r\in J_s(\pi)} Z_{j_r}(x_r)^2\Bigg] \\
&= \sum_{\pi\in\Pi_m}(k)_{|\pi|}\prod_{s=1}^{|\pi|}\mathbb{E}\Bigg[\prod_{r\in J_s(\pi)} Z(x_r)^2\Bigg] \qquad (14.20)
\end{aligned}
$$

and hence by the case of (14.18) already proved,

$$
\begin{aligned}
\mathbb{E}[W_k(x_1)\cdots W_k(x_m)] &= \sum_{\pi\in\Pi_m}(k)_{|\pi|}\prod_{J\in\pi}\mathrm{per}_{1/2}([K]((x_j)_{j\in J})) \\
&= \sum_{\pi\in\Pi_m}(k)_{|\pi|}\sum_{\sigma\in\Sigma_m:\pi\geq\sigma}(1/2)^{\#\sigma}\prod_{i=1}^{m}K(x_i, x_{\sigma(i)}),
\end{aligned}
$$

where $\pi \geq \sigma$ here means that for each cycle of σ, all entries in the cycle lie in the same block of π. Hence

$$
\mathbb{E}[W_k(x_1)\cdots W_k(x_m)] = \sum_{\sigma\in\Sigma_m}\sum_{\pi\in\Pi_m:\pi\geq\sigma}(k)_{|\pi|}(1/2)^{\#\sigma}\prod_{i=1}^{m}K(x_i, x_{\sigma(i)})
$$

and therefore the general case of (14.18) follows from the algebraic identity

$$
\sum_{\pi\in\Pi_n}(k)_{|\pi|} = k^n, \quad k, n \in \mathbb{N}.
$$

This identity may be proved by the same argument as in (14.20) (viz. decomposition of multi-indices according to the induced partition), but now with each of the variables $Z_i(x_j)$ replaced by the unit constant. □

14.4 Distributional properties of permanental Cox processes

In this section we provide a more detailed analysis of the probabilistic properties of the permanental Cox process in Theorem 14.6.

Let $B \in \mathcal{X}_{rc}$ and define a symmetric function $\tilde{K}_B\colon B \times B \to \mathbb{R}$ by

$$
\tilde{K}_B(x, y) = \sum_{j=1}^{\infty}\tilde{\gamma}_{B,j}v_{B,j}(x)v_{B,j}(y), \quad x, y \in B, \qquad (14.21)
$$

where

$$\tilde{\gamma}_{B,j} := \frac{\gamma_{B,j}}{1 + \gamma_{B,j}}, \quad j \in \mathbb{N}. \tag{14.22}$$

Since $0 \le \tilde{\gamma}_{B,j} \le \gamma_{B,j}$ this series converges absolutely. (For $x \in B \setminus B^*$ or $y \in B \setminus B^*$ we have $\tilde{K}_B(x, y) = 0$.) Exercise 14.5 asks the reader to show that $\tilde{K}_B$ is non-negative definite. Given $\alpha > 0$ we define

$$\delta_{B,\alpha} := \prod_{j=1}^{\infty} \frac{1}{(1 + \gamma_{B,j})^{\alpha}} = \prod_{j=1}^{\infty} (1 - \tilde{\gamma}_{B,j})^{\alpha}. \tag{14.23}$$

For $B \in \mathcal{X}$ and $m \in \mathbb{N}$, recall from Definition 4.6 that $J_{\eta,B,m}$ denotes the Janossy measure of order m of a point process η restricted to B.

Theorem 14.8 *Let $k \in \mathbb{N}$ and set $\alpha := k/2$. Let η be a α-permanental process with kernel K and let $B \in \mathcal{X}_{rc}$. Define $\tilde{K}_B$ and $\delta_{B,\alpha}$ by* (14.21) *and* (14.23)*, respectively. Then we have for each $m \in \mathbb{N}$ that*

$$J_{\eta,B,m}(d(x_1, \dots, x_m)) = \frac{\delta_{B,\alpha}}{m!} \operatorname{per}_{\alpha}([\tilde{K}_B](x_1, \dots, x_m))\, \nu^m(d(x_1, \dots, x_m)). \tag{14.24}$$

Proof By Theorem 14.6 and Proposition 14.3 we can assume that η is a Cox process directed by the random measure ξ given by (14.9). Let $m \in \mathbb{N}$ and let $C \in \mathcal{X}^m$. By conditioning with respect to ξ and then using the multivariate version of the Mecke equation (see also the proof of (13.10)), we obtain that

$$\begin{aligned} J_{\eta,B,m}(C) &= \frac{1}{m!} \mathbb{E}[\mathbf{1}\{\eta(B) = m\} \eta_B^{(m)}(C)] \\ &= \frac{1}{m!} \mathbb{E}\Big[\int_C \mathbf{1}\{(\eta + \delta_{x_1} + \cdots + \delta_{x_m})(B) = m\}\, \xi_B^m(d(x_1, \dots, x_m)) \Big]. \end{aligned}$$

For $x_1, \dots, x_m \in B$ we have $(\eta + \delta_{x_1} + \cdots + \delta_{x_m})(B) = m$ if and only if $\eta(B) = 0$. Therefore we obtain via conditioning with respect to ξ that

$$J_{\eta,B,m}(C) = \frac{1}{m!} \mathbb{E}\Big[\exp[-\xi(B)] \int_{B^m} \mathbf{1}\{(x_1, \dots, x_m) \in C\}\, \xi^m(d(x_1, \dots, x_m)) \Big],$$

a formula that is valid for general Cox processes. Using the definition (14.9) of ξ together with (14.16) we obtain that

$$\begin{aligned} J_{\eta,B,m}(C) = \frac{1}{m!} \mathbb{E}\Big[&\exp\Big(- \int W_B(x)\, \nu(dx) \Big) \\ &\times \int_{B^m} \mathbf{1}\{(x_1, \dots, x_m) \in C\} W_B(x_1) \cdots W_B(x_m)\, \nu^m(d(x_1, \dots, x_m)) \Big], \end{aligned}$$

where $W_B(x) := Z_{B,1}(x)^2 + \cdots + Z_{B,k}(x)^2$ has been defined in Lemma 14.7. Hence we have to show that for ν^m-a.e. $(x_1, \dots, x_m) \in (B^*)^m$ we have

$$\mathbb{E}\Big[\exp\Big(-\int_B W_B(x)\,\nu(dx)\Big)W_B(x_1)\cdots W_B(x_m)\Big] = \delta_{B,\alpha}\,\mathrm{per}_\alpha([\tilde{K}_B](x_1,\dots,x_m)). \tag{14.25}$$

Let $Y'_{i,j} := \frac{\sqrt{\gamma_{B,j}}}{\sqrt{2}} Y_{i,j}$. By Lemma 14.7 the left-hand side of (14.25) can be written as

$$\mathbb{E}\Big[f(Y')\prod_{i=1}^{k}\prod_{j=1}^{\infty}\exp\big[-(Y'_{i,j})^2\big]\Big], \tag{14.26}$$

where Y' denotes the double array $(Y'_{i,j})$ and f is a well-defined function on $(\mathbb{R}^\infty)^k$. Now let $Y''_{i,j} := \frac{\sqrt{\tilde{\gamma}_{B,j}}}{\sqrt{2}} Y_{i,j}$. If $\gamma_{B,j} > 0$, the densities φ_1 and φ_2 of $Y'_{i,j}$ and $Y''_{i,j}$ respectively are related by

$$\varphi_2(t) = \sqrt{1+\gamma_{B,j}}\,e^{-t^2}\varphi_1(t), \quad t \in \mathbb{R}.$$

Therefore (14.26) equals

$$\mathbb{E}\Big[f(Y'')\prod_{i=1}^{k}\prod_{j=1}^{\infty}(1+\gamma_{B,j})^{-1/2}\Big] = \delta_{B,\alpha}\,\mathbb{E}[f(Y'')], \tag{14.27}$$

where $Y'' := (Y''_{i,j})$. By (14.15) we have

$$f(Y'') = \sum_{i=1}^{k}\Big(\sum_{j=1}^{\infty}\frac{\sqrt{\tilde{\gamma}_{B,j}}}{\sqrt{2}}Y_{i,j}v_{B,j}(x_1)\Big)^2 \cdots \sum_{i=1}^{k}\Big(\sum_{j=1}^{\infty}\frac{\sqrt{\tilde{\gamma}_{B,j}}}{\sqrt{2}}Y_{i,j}v_{B,j}(x_m)\Big)^2,$$

so we can apply (14.18) (with $\tilde{K}_B$ in place of K) to obtain (14.25) and hence the assertion (14.24). □

With the correct interpretation, (14.24) remains true for $m = 0$. To see this, let ξ be a random measure and η be a Cox process as in Theorem 14.6. Then by (14.17)

$$\mathbb{P}(\eta(B) = 0) = \mathbb{E}[\exp[-\xi(B)]] = \prod_{i=1}^{k}\prod_{j=1}^{\infty}\mathbb{E}\Big[\exp\Big(-\frac{\gamma_{B,j}}{2}Y_{i,j}^2\Big)\Big].$$

Using (B.7) we get

$$\mathbb{P}(\eta(B) = 0) = \prod_{i=1}^{k}\prod_{j=1}^{\infty}(1+\gamma_{B,j})^{-1/2},$$

that is

$$J_{\eta,B,0} \equiv \mathbb{P}(\eta(B) = 0) = \delta_{B,\alpha}. \tag{14.28}$$

Combining (14.24) with (4.16) yields

$$\mathbb{P}(\eta(B) = m) = \frac{\delta_{B,\alpha}}{m!} \int_{B^m} \mathrm{per}_\alpha([\tilde{K}_B](x_1, \dots, x_m))\, \nu^m(d(x_1, \dots, x_m)). \tag{14.29}$$

To give an explicit probabilistic interpretation of the above right-hand side, we need a combinatorial lemma.

Lemma 14.9 *Let $\alpha > 0$ and $B \in \mathcal{X}_{rc}$. Define*

$$c := \max\{\alpha, 1\}\nu(B) \sup\{K(x, y) : x, y \in B\}. \tag{14.30}$$

Then we have for $s \in [0, c^{-1} \wedge 1)$ that

$$1 + \sum_{m=1}^{\infty} \frac{s^m}{m!} \int_{B^m} \mathrm{per}_\alpha([\tilde{K}_B](x_1, \dots, x_m))\nu^m(d(x_1, \dots, x_m)) = \prod_{j=1}^{\infty} (1 - s\tilde{\gamma}_{B,j})^{-\alpha}.$$

Proof We abbreviate $\tilde{K} := \tilde{K}_B$, $\tilde{\gamma}_j := \tilde{\gamma}_{B,j}$, and $v_j := v_{B,j}$, $j \in \mathbb{N}$. Let $m \in \mathbb{N}$ and $x_1, \dots, x_m \in B$. For $r \in [m]$ the number $\mathrm{per}^{(r)}([\tilde{K}](x_1, \dots, x_m))$ is given in Definition 14.2. For $r = 1$ we have

$$\int_{B^m} \mathrm{per}^{(1)}([\tilde{K}](x_1, \dots, x_m))\, \nu^m(d(x_1, \dots, x_m))$$
$$= \sum_{\sigma \in \Sigma_m^{(1)}} \sum_{r_1, \dots, r_m = 1}^{\infty} \int_{B^m} \prod_{j=1}^{m} \tilde{\gamma}_{r_j} v_{r_j}(x_i) v_{r_j}(x_{\sigma(i)})\, \nu^m(d(x_1, \dots, x_m)), \tag{14.31}$$

where we have used the uniform convergence in (14.12) to interchange summation and integration. A symmetry argument shows that each $\sigma \in \Sigma_m^{(1)}$ contributes the same to the above right-hand side. Moreover, there are $(m-1)!$ permutations with exactly one cycle. Therefore (14.31) equals

$$(m-1)! \sum_{r_1, \dots, r_m = 1}^{\infty} \int_{B^m} \prod_{j=1}^{m} \tilde{\gamma}_{r_j} v_{r_j}(x_i) v_{r_j}(x_{i+1})\, \nu^m(d(x_1, \dots, x_m)),$$

where x_{m+1} has to be read as x_1. By (14.11) and Fubini's theorem the integral above vanishes unless $r_1 = \cdots = r_m$. Therefore

$$\int_{B^m} \mathrm{per}^{(1)}([\tilde{K}](x_1, \dots, x_m))\, \nu^m(d(x_1, \dots, x_m)) = (m-1)! \sum_{r=1}^{\infty} \tilde{\gamma}_r^m.$$

Using the logarithmic series $-\log(1-x) = x + x^2/2 + x^3/3 + \cdots$, $x \in [0,1)$, it follows that

$$\sum_{m=1}^{\infty} \frac{s^m}{m!} \int_{B^m} \mathrm{per}^{(1)}([\tilde{K}](x_1,\ldots,x_m))\, \nu^m(d(x_1,\ldots,x_m)) = \sum_{m=1}^{\infty} \frac{s^m}{m} \sum_{r=1}^{\infty} \tilde{\gamma}_r^m = D_s,$$

where

$$D_s := -\sum_{j=1}^{\infty} \log(1 - s\tilde{\gamma}_j). \tag{14.32}$$

Since $-\log(1-x) = xR(x)$, $x \in [0,1)$, where $R(\cdot)$ is bounded on $[0,1)$, and since $\sum_{j=1}^{\infty} \tilde{\gamma}_j < \infty$, the series (14.32) converges.

Now let $r \in \mathbb{N}$ be arbitrary. Then

$$D_s^r = \sum_{m_1,\ldots,m_r=1}^{\infty} \frac{s^{m_1+\cdots+m_r}}{m_1!\cdots m_r!} \prod_{j=1}^{r} \int \mathrm{per}^{(1)}([\tilde{K}](\mathbf{x}_j))\, \nu^{m_j}(d\mathbf{x}_j),$$

where we identify ν with its restriction to B. Therefore

$$\frac{D_s^r}{r!} = \sum_{m=r}^{\infty} \frac{s^m}{m!} \sum_{\substack{m_1,\ldots,m_r=1 \\ m_1+\cdots+m_r=m}}^{m} \frac{m!}{r!m_1!\cdots m_r!} \prod_{j=1}^{r} \int \mathrm{per}^{(1)}([\tilde{K}](\mathbf{x}))\, \nu^{m_j}(d\mathbf{x}).$$

We assert for all $m \geq r$ that

$$\frac{1}{r!} \sum_{\substack{m_1,\ldots,m_r=1 \\ m_1+\cdots+m_r=m}}^{m} \frac{m!}{m_1!\cdots m_r!} \prod_{j=1}^{r} \int \mathrm{per}^{(1)}([\tilde{K}](\mathbf{x}))\, \nu^{m_j}(d\mathbf{x})$$
$$= \int \mathrm{per}^{(r)}([\tilde{K}](\mathbf{x}))\, \nu^m(d\mathbf{x}). \tag{14.33}$$

Indeed, if $\sigma \in \Sigma_m$ has r cycles with lengths $m_1,\ldots,m_r$, then

$$\int \prod_{i=1}^{m} \tilde{K}(x_i, x_{\sigma(i)})\, \nu^m(d(x_1,\ldots,x_m))$$
$$= \prod_{j=1}^{r} \frac{1}{(m_j-1)!} \int \mathrm{per}^{(1)}([\tilde{K}](\mathbf{x}))\, \nu^{m_j}(d\mathbf{x}).$$

The factor $\frac{m!}{m_1\cdots m_r}$ in the left-hand side of (14.33) is the number of ways to create an ordered sequence of r cycles with of lengths $m_1,\ldots,m_r$ that partition $[m]$. The factor $1/r!$ reflects the fact that any permutation of cycles leads to the same permutation of $[m]$. Exercise 14.6 asks the reader to give a complete proof of (14.33).

By (14.33),

$$\frac{D_s^r}{r!} = \sum_{m=r}^{\infty} \frac{s^m}{m!} \int \mathrm{per}^{(r)}([\tilde{K}](\mathbf{x}))\, \nu^m(d\mathbf{x}).$$

It follows that

$$\begin{aligned} e^{\alpha D_s} &= \sum_{r=0}^{\infty} \frac{\alpha^r D_s^r}{r!} = 1 + \sum_{m=1}^{\infty} \frac{s^m}{m!} \sum_{r=1}^{m} \int \alpha^r\, \mathrm{per}^{(r)}([\tilde{K}](\mathbf{x}))\, \nu^m(d\mathbf{x}) \\ &= 1 + \sum_{m=1}^{\infty} \frac{s^m}{m!} \int \mathrm{per}_\alpha([\tilde{K}](\mathbf{x}))\, \nu^m(d\mathbf{x}). \end{aligned}$$

Since $\int |\,\mathrm{per}_\alpha([\tilde{K}](\mathbf{x}))|\, \nu^m(d\mathbf{x}) \le m!c^m$ and $sc < 1$, this series converges absolutely. In view of the definition (14.32) of D_s, this finishes the proof. □

The following theorem should be compared with Example 14.4.

Theorem 14.10 *Let the assumptions of Theorem 14.8 be satisfied. Then*

$$\eta(B) \overset{d}{=} \sum_{j=1}^{\infty} \zeta_j, \tag{14.34}$$

where ζ_j, $j \in \mathbb{N}$, are independent, and ζ_j has for each $j \in \mathbb{N}$ a negative binomial distribution with parameters $1/(1 + \gamma_{B,j})$ and α.

Proof Define c by (14.30) and let and let $s \in [0, c^{-1} \wedge 1)$. By (14.28) and (14.29),

$$\mathbb{E}[s^{\eta(B)}] = \delta_{B,\alpha} + \delta_{B,\alpha} \sum_{m=1}^{\infty} \frac{s^m}{m!} \int_{B^m} \mathrm{per}_\alpha([\tilde{K}_B](x_1, \dots, x_m))\, \nu^m(d(x_1, \dots, x_m)).$$

Using Lemma 14.9 and the definition (14.23) of $\delta_{B,\alpha}$ gives

$$\mathbb{E}[s^{\eta(B)}] = \prod_{j=1}^{\infty} (1 - \tilde{\gamma}_{B,j})^{\alpha} (1 - s\tilde{\gamma}_{B,j})^{-\alpha} = \prod_{j=1}^{\infty} \mathbb{E}[s^{\zeta_j}], \tag{14.35}$$

where we have used (B.14) (with $p = 1-\tilde{\gamma}_{B,j} = 1/(1+\gamma_{B,j})$ and $a = \alpha$) to get the second identity. The assertion now follows from Proposition B.4. □

Equation (14.34) implies that

$$\mathbb{E}[\eta(B)] = \sum_{j=1}^{\infty} \mathbb{E}[\zeta_j] = \alpha \sum_{j=1}^{\infty} \gamma_{B,j},$$

where the expectation of a negative binomial random variable can be obtained by differentiating (B.14). This identity is in accordance with (14.2) and (14.14). Since $\sum_{j=1}^{\infty} \gamma_{B,j} < \infty$ we have $\sum_{j=1}^{\infty} \mathbb{E}[\zeta_j] < \infty$ a.s., so that a.s. only finitely many of the ζ_j are not zero (even though all $\gamma_{B,j}$ might be positive). Exercise 14.7 has a formula for the variance of $\zeta(B)$.

14.5 Exercises

Exercise 14.1 Let $m \in \mathbb{N}$ and $k \in [m]$. Let $s(m,k)$ be the number of permutations in Σ_m having exactly k cycles. (These are called the *Stirling numbers* of the first kind.) Show that

$$s(m+1,k) = ms(m,k) + s(m,k-1), \quad k,m \in \mathbb{N},$$

where $s(m,0) := 0$. Use this to prove by induction that

$$\sum_{k=1}^{m} s(m,k)x^k = x(x+1)\cdots(x+m-1), \quad x \in \mathbb{R}.$$

Exercise 14.2 Let Z be random variable having a negative binomial distribution with parameters $p \in (0,1]$ and $a > 0$. Show that the factorial moments of Z are given by

$$\mathbb{E}[(Z)_m] = (1-p)^m p^{-m} a(a+1)\cdots(a+m-1), \quad m \geq 1.$$

Exercise 14.3 Prove

$$\mathrm{per}_{k\beta} A = \sum_{i=1}^{k} \sum_{\pi \in \Pi_m^i} (k)_i \prod_{J \in \pi} \mathrm{per}_\beta A_J \tag{14.36}$$

Exercise 14.4 Let ξ be an α-permanental process and $B_1, B_2 \in \mathcal{X}_{rc}$. Show that $\mathbb{C}\mathrm{ov}[\eta(B_1), \eta(B_2)] \geq 0$.

Exercise 14.5 Show that $\tilde{K}_B$ defined by (14.21) is non-negative definite function.

Exercise 14.6 Give a complete argument for the identity (14.33).

Exercise 14.7 Let $\alpha := k/2$ for some $k \in \mathbb{N}$ and let η be α-permanental with kernel K. Let $B \subset \mathcal{X}_{rc}$ and show that $\eta(B)$ has the finite variance

$$\mathbb{V}\mathrm{ar}[\eta(B)] = \alpha \sum_{j=1}^{\infty} \gamma_{B,j}(1+\gamma_{B,j}),$$

where the $\gamma_{B,j}$ are as in (14.12).

Exercise 14.8 Let $B \in \mathcal{X}_{rc}$ and assume that there exists $\gamma > 0$ such that $\gamma_{B,j} \in \{0, \gamma\}$ for each $j \in \mathbb{N}$, where the $\gamma_{B,j}$ are as in (14.12). Let $k \in \mathbb{N}$ and let η be $(k/2)$-permanental with kernel K. Show that $\eta(B)$ has a negative binomial distribution and identify the parameters.

Exercise 14.9 Let $m \in \mathbb{N}$, $p \in (0, 1]$ and $a > 0$. Let $(\zeta_1, \dots, \zeta_m)$ be a random element of $\mathbb{N}_0^m$ such that $\zeta := \zeta_1 + \cdots + \zeta_m$ has a negative binomial distribution with parameters p and a. Moreover, assume for all $\ell \geq 1$ that the conditional distribution of $(\zeta_1, \dots, \zeta_m)$ given $\zeta = \ell$ is multinomial with parameters ℓ and $q_1, \dots, q_m \geq 0$, where $q_1 + \cdots + q_m = 1$. Show that

$$\mathbb{E}[s_1^{\zeta_1} \cdots s_m^{\zeta_m}] = \left(\frac{1}{p} - \frac{1-p}{p} \sum_{r=1}^{m} s_r q_r \right)^{-a}, \quad s_1, \dots, s_m \in [0, 1].$$

(Hint: At some stage you have to use the identity

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+a)}{\Gamma(n+1)\Gamma(a)} q^n = (1-q)^{-a}, \quad q \in [0, 1),$$

which follows from the fact that (B.11) is a probability distribution.)

Exercise 14.10 Let $k \in \mathbb{N}$ and suppose that $\eta_1, \dots, \eta_k$ are independent $1/2$-permanental processes with kernel K. Show that $\eta_1 + \cdots + \eta_k$ is $(k/2)$-permanental. (Hint: Assume that $\eta_1, \dots, \eta_k$ are Cox processes and use the identity (13.9).)

Exercise 14.11 For each $k \in \mathbb{N}$ let η_k be a $(k/2)$-permanental process with kernel $(2/k)K$. Let $B \in \mathcal{X}_{rc}$ and show that $\eta_k(B) \xrightarrow{d} \zeta_B$ as $k \to \infty$, where ζ_B has a Poisson distribution with mean $\int_B K(x, x)\, \nu(dx)$. (Hint: Use (14.35), Proposition B.11 and (14.14).)

15

Compound Poisson processes

A compound Poisson process ξ is a purely discrete random measure that is given as an integral with respect to a Poisson process η on a product space. The first coordinates of the Poisson points describe the positions and the second the weights of the atoms of ξ. A compound Poisson process is completely independent. Of particular interest is the case where the intensity measure of ξ is of product form. The second factor is then known as the Lévy measure of ξ. The central result of this chapter asserts that a completely random measure without fixed atoms is the sum of a compound Poisson process and a deterministic diffuse measure. The chapter concludes with a brief discussion of linear functionals of ξ and the shot noise Cox process.

15.1 Definition and basic properties

Let $(\mathbb{Y}, \mathcal{Y})$ be a measurable space and let η be a Poisson process on $\mathbb{Y} \times (0, \infty)$ with s-finite intensity measure λ. A *compound Poisson process* is a random measure ξ on $\mathbb{Y}$ of the form

$$\xi(B) = \int_B r\,\eta(d(y, r)), \quad B \in \mathcal{Y}. \tag{15.1}$$

We might think of a point of η as being a point in $\mathbb{Y}$ with the second coordinate representing its weight. Then the integral (15.1) is the weighted sum of all points lying in B. Proposition 12.1 implies for any $B \in \mathcal{Y}$ that $\mathbb{P}(\xi(B) < \infty) = 1$ if and only if

$$\int_B (r \wedge 1)\,\lambda(d(y, r)) < \infty. \tag{15.2}$$

We need to check that ξ really is a random measure in the sense of Definition 13.10.

Proposition 15.1 *Let η be a point process on $\mathbb{Y} \times (0, \infty)$. Then ξ given by* (15.1) *is a random measure.*

Proof For $\omega \in \Omega$, the measure property of $\xi(\omega, \cdot)$ follows from monotone convergence. The property that $\xi \colon \Omega \to \mathbf{M}$ is measurable follows from Proposition 2.7. It remains to show that ξ is s-finite everywhere on Ω. To this end we write $\eta = \sum_{j=1}^{\infty} \eta_j$, where $\eta_j(\omega)$ is for each $\omega \in \Omega$ a finite measure on $\mathbb{X}$. Then $\xi = \sum_{i,j=1}^{\infty} \xi_{i,j}$, where

$$\xi_{i,j}(B) := \int_B \mathbf{1}\{r < j \le r + 1\} r \, \eta_i(d(y, r)), \quad B \in \mathcal{Y},$$

and the proof is complete. □

If ξ is a compound Poisson process, Theorem 5.2 shows that it has the following remarkable property.

Definition 15.2 A random measure ξ on a measurable space $(\mathbb{X}, \mathcal{X})$ is *completely independent*, if $\xi(B_1), \dots, \xi(B_m)$ are stochastically independent whenever $B_1, \dots, B_m \in \mathcal{X}$ are pairwise disjoint.

Proposition 15.3 *Let the compound Poisson process ξ be given by* (15.1). *Then*

$$L_\xi(u) = \exp\Big[- \int (1 - e^{-ru(y)}) \, \lambda(d(y, r)) \Big], \quad u \in \mathbb{R}_+(\mathbb{Y}). \tag{15.3}$$

Proof Let $u \in \mathbb{R}_+(\mathbb{Y})$. It follows from (15.1) that

$$\exp[-\xi(u)] = \exp\Big(- \int r u(y) \, \eta(d(y, r)) \Big).$$

Theorem 3.9 implies the assertion. □

We can apply (15.3) with $u = t\mathbf{1}_B$ for $t \ge 0$ and $B \in \mathcal{Y}$. Since $1 - e^{-tr\mathbf{1}_B(y)} = 0$ for $y \notin B$, we then obtain

$$\mathbb{E}[\exp[-t\xi(B)]] = \exp\Big[- \int (1 - e^{-tr}) \, \lambda(B, dr) \Big], \quad t \ge 0, \tag{15.4}$$

where $\lambda(B, \cdot) := \lambda(B \times \cdot)$.

Of particular interest is the case where $\lambda = \rho_0 \otimes \nu$, where ν is a measure on $(0, \infty)$ satisfying

$$\int (r \wedge 1) \, \nu(dr) < \infty, \tag{15.5}$$

and ρ_0 is an s-finite measure on $\mathbb{Y}$. Then (15.4) simplifies to

$$\mathbb{E}[\exp[-t\xi(B)]] = \exp\left[-\rho_0(B)\int (1-e^{-tr})\,\nu(dr)\right], \quad t \geq 0, \tag{15.6}$$

where we note that (15.5) is equivalent to $\int (1-e^{-tr})\,\nu(dr) < \infty$ for one (and then for all) $t > 0$. In particular, ξ is ρ_0-*symmetric* in the sense that $\xi(B) \overset{d}{=} \xi(B')$ whenever $\rho_0(B) = \rho_0(B')$. Therefore we call ξ a ρ_0-*symmetric compound Poisson process* with *Lévy measure* ν. Since $\varepsilon\mathbf{1}\{r \geq \varepsilon\} \leq r \wedge 1$ for all $r \geq 0$ and $\varepsilon \in (0,1)$ we obtain from (15.5) that ν is finite on any closed interval not containing 0, that is,

$$\nu([\varepsilon,\infty)) < \infty, \quad \varepsilon > 0. \tag{15.7}$$

Example 15.4 Consider a ρ_0-symmetric compound Poisson process with Lévy measure

$$\nu(dr) := r^{-1}e^{-br}dr, \tag{15.8}$$

where $b > 0$ is a fixed parameter. Then we obtain from (15.6) and the identity

$$\int_0^\infty (1-e^{-ur})r^{-1}e^{-r}\,dr = \log(1+u), \quad u \geq 0 \tag{15.9}$$

(easily checked by differentiation) that

$$\mathbb{E}[\exp(-t\xi(B))] = (1+t/b)^{-\rho_0(B)}, \quad t \geq 0, \tag{15.10}$$

provided that $0 < \rho_0(B) < \infty$. Hence $\xi(B)$ has a Gamma distribution (see (B.10)) with shape parameter $a := \rho_0(B)$ and scale parameter b. Therefore ξ is called a *Gamma random measure* with *shape measure* ρ_0 and scale parameter b. An interesting feature of ξ comes from the fact that $\int_0^\infty r^{-1}e^{-br}\,dr = \infty$, which implies that if $\mathbb{Y}$ is a Borel space, ρ_0 is diffuse and $B \in \mathcal{Y}$ satisfies $\rho_0(B) > 0$, then almost surely $\xi\{y\} > 0$ for infinitely many $y \in B$.

Example 15.5 Let λ_+ denote Lebesgue measure on $(0,\infty)$ and consider a λ_+-symmetric compound Poisson process with Lévy measure ν. The stochastic process $Y := (Y_t)_{t\geq 0} := (\xi[0,t])_{t\geq 0}$ is called *subordinator* with Lévy measure ν. This process has independent increments in the sense that $Y_{t_1}, Y_{t_2} - Y_{t_1}, \dots, Y_{t_n} - Y_{t_{n-1}}$ are independent whenever $n \geq 2$ and $0 \leq t_1 < \cdots < t_n$. Moreover, the increments are *homogeneous*, that is, for any $h > 0$, the distribution of $Y_{t+h} - Y_t = \xi(t, t+h]$ does not depend on $t \geq 0$. If ν is given as in Example 15.4, then Y is called a *Gamma process*.

15.2 Moments of symmetric compound Poisson processes

The moments of a symmetric compound Poisson process can be expressed in terms of the moments of the associated Lévy measure. Before formulating the result, we introduce for $n \in \mathbb{N}$ and $k_1, \dots, k_n \in \mathbb{N}_0$ the notation

$$\begin{bmatrix} n \\ k_1, \dots, k_n \end{bmatrix} := \frac{n!}{(1!)^{k_1} k_1! (2!)^{k_2} k_2! \cdots (n!)^{k_n} k_n!}, \tag{15.11}$$

whenever $1k_1 + 2k_2 + \cdots + nk_n = n$. In all other cases this number is defined as 0. This is the number of ways to partition $\{1, \dots, n\}$ into k_i blocks of size i for $i \in \{1, \dots, n\}$.

Proposition 15.6 *Let ρ_0 be an s-finite measure on $\mathbb{Y}$ and let ξ be a ρ_0-symmetric compound Poisson process with Lévy measure ν. Let $B \in \mathcal{Y}$ and $n \in \mathbb{N}$. Then*

$$\mathbb{E}[\xi(B)^n] = \sum_{r_1,\dots,r_n \in \mathbb{N}_0} \begin{bmatrix} n \\ r_1, \dots, r_n \end{bmatrix} \rho_0(B)^{r_1+\cdots+r_n} \prod_{i=1}^{n} \alpha_i^{r_i}, \tag{15.12}$$

where $\alpha_i := \int r^i \, \nu(dr)$.

Proof The proof is similar to that of Theorem 12.5 and can in fact be derived from this result. We prefer to give a direct argument. First we use Fubini's theorem to obtain that

$$\mathbb{E}[\xi(B)^n] = \int_{(B\times\mathbb{R}_+)^n} r_1 \cdots r_n \, \eta^n(d((y_1, r_1), \dots, (y_n, r_n))).$$

Consider now a partition of $\{1, \dots, n\}$. In each of the blocks the indices of the integration variables are taken to be equal while they are supposed to be distinct in different blocks. Summing over all partitions and using the symmetry properties of factorial measures, we obtain that

$$\begin{aligned} \mathbb{E}[\xi(B)^n] = \sum_{k_1,\dots,k_n \in \mathbb{N}_0} \begin{bmatrix} n \\ k_1, \dots, k_n \end{bmatrix} \int_{(B\times\mathbb{R}_+)^{k_1+\cdots+k_n}} \prod_{i=1}^{n} \prod_{j_i=k_1+\cdots+k_{i-1}+1}^{k_1+\cdots+k_i} r_{j_i}^i \\ \times \eta^{(k_1+\cdots+k_n)}(d((y_1, r_1), \dots, (y_{k_1+\cdots+k_n}, r_{k_1+\cdots+k_n}))), \end{aligned} \tag{15.13}$$

where $k_0 := 0$. Since the integrand in the above right-hand side is non-negative, we can use the multivariate Mecke theorem (Theorem 4.4) and Fubini's theorem to derive (15.12). □

Corollary 15.7 *Let ξ be as in Proposition 15.6. Suppose that $B \in \mathcal{Y}$ satisfies $0 < \rho_0(B) < \infty$ and let $n \in \mathbb{N}$. Then $\mathbb{E}[\xi(B)^n] < \infty$ if and only if $\int r^n \, \nu(dr) < \infty$.*

Proof Assume that $\mathbb{E}[\xi(B)^n] < \infty$. Observe that the summand with $k_n = 1$ (and $k_1 = \cdots = k_{n-1} = 0$) in the right-hand side of (15.12) equals $\rho_0(B)^n \alpha_n$. Since $\rho_0(B) > 0$ we get $\alpha_n < \infty$. Assume, conversely that $\alpha_n < \infty$. Then (15.5) implies $\alpha_i < \infty$ for any $i \in \{1, \dots, n\}$. Since $\rho_0(B) < \infty$ we hence obtain from (15.12) that $\mathbb{E}[\xi(B)^n] < \infty$. □

For $n = 1$ obtain from (15.12) that

$$\mathbb{E}[\xi(B)] = \rho_0(B) \int r\, \nu(dr),$$

which is nothing but Campbell's formula (13.1) for this random measure. In the case $n = 2$ we have

$$\mathbb{E}[\xi(B)^2] = \rho_0(B)^2 \Big(\int r\, \nu(dr) \Big)^2 + \rho_0(B) \int r^2\, \nu(dr)$$

and therefore

$$\mathbb{V}\mathrm{ar}[\xi(B)] = \rho_0(B) \int r^2\, \nu(dr).$$

Exercises 15.3 and 15.4 give two generalizations of (15.12).

We continue with a counterpart of Definition 6.13.

15.3 Poisson representation of completely random measures

Definition 15.8 A random measure ξ on $\mathbb{Y}$ is said to be *uniformly σ-finite* if there exists $B_n \in \mathcal{Y}$, $n \in \mathbb{N}$, such that $B_n \uparrow \mathbb{Y}$ as $n \to \infty$ and

$$\mathbb{P}(\xi(B_n) < \infty) = 1, \quad n \in \mathbb{N}.$$

In this case and if $\mathbb{Y}$ is a Borel space, ξ is said to be *diffuse* if there is an $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 1$ and $\xi(\omega, \{x\}) = 0$ for all $x \in \mathbb{X}$ and all $\omega \in A$.

The second part of Definition 15.8 is justified by Exercise 13.9. The following converse of Proposition 15.1 reveals the significance of Poisson processes for completely independent random measures. A random measure ξ on $(\mathbb{Y}, \mathcal{Y})$ is said to be *almost surely deterministic* if there is a measure ν on $\mathbb{Y}$ such that $\mathbb{P}(\xi(B) = \nu(B)) = 1$ for all $B \in \mathcal{Y}$.

Theorem 15.9 *Suppose that $(\mathbb{Y}, \mathcal{Y})$ is a Borel space and let ξ be a uniformly σ-finite completely independent random measure on $\mathbb{Y}$. Assume that*

$$\xi\{y\} = 0, \quad \mathbb{P}\text{-a.s.}, \ y \in \mathbb{Y}. \tag{15.14}$$

Then there is a Poisson process η on $\mathbb{Y} \times \mathbb{R}_+$ with diffuse σ-finite intensity measure, and a (deterministic) diffuse measure ν on $\mathbb{Y}$, such that

$$\xi(B) = \nu(B) + \int_B r\,\eta(d(y,r)), \quad B \in \mathcal{Y},\ \mathbb{P}\text{-a.s.} \tag{15.15}$$

Proof By Exercise 13.9 (a version of Proposition 6.2 that applies to general measures) we have

$$\xi = \chi + \sum_{n=1}^{\kappa} Y_n \delta_{X_n}, \quad \mathbb{P}\text{-a.s.}, \tag{15.16}$$

where $\kappa := k(\xi)$, $Y_n := \tau_n(\xi)$, $X_n := \pi_n(\xi)$, $\sum_{n=1}^{\kappa} \delta_{X_n}$ is a simple point process, and χ is a diffuse random measure. Since ξ is uniformly σ-finite, χ has the same property. Moreover, $\eta := \sum_{n=1}^{\kappa} \delta_{(X_n,Y_n)}$ is a uniformly σ-finite point process on $\mathbb{Y} \times (0,\infty)$. Then $\xi(B) = \chi(B) + \int_B r\,\eta(d(y,r))$ (a.s. uniformly in $B \in \mathcal{Y}$) and we need to show that χ is a.s. deterministic and that η is a Poisson process. Let $C \in \mathcal{Y} \otimes \mathcal{B}((0,\infty))$ be such that $\mathbb{P}(\eta(C) < \infty) = 1$ and define the simple point process $\eta' := \eta_C(\cdot \times (0,\infty))$. Then $\mathbb{P}(\eta'(\mathbb{Y}) < \infty) = 1$ and equation (15.16) implies that

$$\eta'(B) = \int \mathbf{1}\{y \in B, (y, \xi\{y\}) \in C\}\,\xi(dy), \quad B \in \mathcal{Y}.$$

In particular $\eta'(B)$ is a measurable function of ξ_B. The proof of Theorem 5.2 (restriction theorem) shows that $\xi_{B_1}, \dots, \xi_{B_m}$ are independent whenever $B_1, \dots, B_m \in \mathcal{Y}$ are pairwise disjoint. It follows that η' is completely independent. Moreover (15.14) implies that η' (and also η) has a diffuse intensity measure. By Theorem 6.10, η' is a Poisson process. In particular

$$\mathbb{P}(\eta(C) = 0) = \mathbb{P}(\eta' = 0) = \exp[-\lambda(C)],$$

where λ is the intensity measure of η and where we have used the identity $\eta(C) = \eta'(\mathbb{Y})$. Theorem 6.14 yields that η is a Poisson process. The decomposition (15.16) shows for any $B \in \mathcal{Y}$ that $\chi(B)$ depends only on the restriction ξ_B. Therefore χ is completely independent, so that Proposition 15.10 (to be proved below) shows that χ is almost surely deterministic. □

The proof of the preceding theorem has used the following fact in a crucial way.

Proposition 15.10 *Let ξ be a diffuse, uniformly σ-finite random measure on a Borel space $(\mathbb{Y}, \mathcal{Y})$. If ξ is completely independent, then ξ is almost surely deterministic.*

Proof Given $B \in \mathcal{Y}$, define

$$\nu(B) := -\log \mathbb{E}[\exp[-\xi(B)]],$$

using the convention $-\log 0 := \infty$. Since ξ is completely independent, the function ν is finitely additive. Moreover, if $C_n \uparrow C$ with $C_n, C \in \mathcal{Y}$, then monotone and dominated convergence show that $\nu(C_n) \uparrow \nu(C)$. Hence ν is a measure. Since ξ is diffuse, ν is diffuse. Moreover, since ξ is uniformly σ-finite, ν is σ-finite. Let η be a Cox process directed by ξ and let η' be a Poisson process with intensity measure ν. By Proposition 6.7 (which applies in our present slightly more general setting) η and η' are both simple and we have

$$\mathbb{P}(\eta(B) = 0) = \mathbb{E}[\exp[-\xi(B)]] = \exp[-\nu(B)] = \mathbb{P}(\eta'(B) = 0), \quad B \in \mathcal{Y}.$$

By Theorem 6.8 (again this result applies) $\eta \overset{d}{=} \eta'$, so that Theorem 13.7 yields $\xi \overset{d}{=} \nu$. Now let $\mathcal{H}$ be a countable π-system generating $\mathcal{Y}$. Then the event $A := \{\xi(B) = \nu(B) \text{ for all } B \in \mathcal{H}\}$ has full probability. As it is no restriction of generality to assume that the sets B_n from Definition 15.8 are in $\mathcal{H}$, we can apply Theorem A.4 to conclude that $\xi = \nu$ on A. □

If ξ is a random measure on a Borel space $(\mathbb{Y}, \mathcal{Y})$ and $y \in \mathbb{Y}$ is such that

$$\mathbb{P}(\xi\{y\} > 0) > 0, \tag{15.17}$$

then y is called a *fixed atom* of ξ. Theorem 15.9 does not apply to a completely independent random measure ξ with fixed atoms. Exercise 15.7 shows, however, that any such ξ is the independent superposition of a random measure with countably many fixed atoms and a completely independent random measure satisfying (15.14).

15.4 First order integrals

Let ξ be a compound Poisson process on a measurable space $(\mathbb{Y}, \mathcal{Y})$ as defined by (15.1). For $f \in \mathbb{R}(\mathbb{Y})$ we may then try to form the integral $\int f \, d\xi$. Expressing this as

$$\int f(z)\, \xi(dz) = \int r f(y)\, \eta(d(y, r)),$$

we can apply Proposition 12.1 to see that the integral converges almost surely if $\int r|f(z)|\, \lambda(d(r, z)) < \infty$. If g is another function with this property, then we have for all $a, b \in \mathbb{R}$ that

$$\int (af(z) + bg(z))\,\xi(dz) = a \int f(z)\,\xi(dz) + b \int g(z)\,\xi(dz), \quad \mathbb{P}\text{-a.s.},$$

so that $\int f(z)\,\xi(dz)$ can be interpreted as first order integral with respect to ξ. For applications it is useful to make f dependent on a parameter $x \in \mathbb{X}$, where $(\mathbb{X}, \mathcal{X})$ is another measurable space. To do so, we take a measurable function $k \in \mathbb{R}(\mathbb{X} \times \mathbb{Y})$ (known as kernel) and define a random field $(Y(x))_{x\in\mathbb{X}}$ by

$$Y(x) := \int rk(x, y)\,\eta(d(y, r)), \quad x \in \mathbb{X}. \tag{15.18}$$

Here we can drop the assumption that the weights be positive and assume that η is a Poisson process on $\mathbb{Y} \times \mathbb{R}$ such that

$$\int |rk(x, y)|\,\lambda(d(y, r)) < \infty, \quad x \in \mathbb{X}, \tag{15.19}$$

where λ is the intensity measure of η. Then for all $x \in \mathbb{X}$ the right-hand side of (15.18) is a.s. finite and we make the convention $Y(x) := 0$ whenever it is not. Using the monotone class theorem it can be shown that the random field is measurable; see Example 13.3. By Campbell's formula (Proposition 2.7),

$$\mathbb{E}[Y(x)] = \int rk(x, y)\,\lambda(d(y, r)), \quad x \in \mathbb{X}. \tag{15.20}$$

Proposition 15.11 *Let* $(Y(x))_{x\in\mathbb{X}}$ *be the random field given by* (15.18). *Assume that* (15.19) *holds and that*

$$\int r^2 k(x, y)^2\,\lambda(d(y, r)) < \infty, \quad x \in \mathbb{X}. \tag{15.21}$$

Then

$$\mathbb{C}\mathrm{ov}[Y(x), Y(z)] = \int r^2 k(x, y)k(z, y)\,\lambda(d(y, r)), \quad x, z \in \mathbb{X}. \tag{15.22}$$

Proof Up to a centering (see (15.20)), $Y(x)$ is a first order Wiener-Itô integral. Hence the result follows from Proposition 12.3. □

Example 15.12 Let $d \in \mathbb{N}$. Consider the random field $(Y(x))_{x\in\mathbb{R}^d}$ given by (15.18) in the case $\mathbb{X} = \mathbb{Y} = \mathbb{R}^d$ with a kernel of the form $k(x, y) = \tilde{k}(x - y)$ for some $\tilde{k} \in L^1(\lambda_d)$. Assume also that the intensity measure λ of η is the product $\lambda_d \otimes \nu$ for some measure ν on $\mathbb{R}$ satisfying $\int |r|\,\nu(dr) < \infty$. Then

$$Y(x) - \mathbb{E}[Y(x)] = \int r\tilde{k}(x - y)\,\eta(d(y, r)) - \iint r\tilde{k}(x - y)\,dy\,\nu(dr).$$

The random field $(Y(x))_{x\in\mathbb{R}^d}$ is known as Poisson driven *shot noise*, while $\{Y(x) - \mathbb{E}[Y(x)]\}_{x\in\mathbb{R}^d}$ is known as Poisson driven *moving average field*. A specific example is

$$\tilde{k}(x) = \mathbf{1}\{x_1 \geq 0, \ldots, x_d \geq 0\} \exp[-\langle v, x\rangle],$$

where $v \in \mathbb{R}^d$ is a fixed parameter.

Example 15.13 Let λ be a σ-finite measure on $\mathbb{Y} \times (0, \infty)$ and let η be a Poisson process with intensity measure λ. Let the random field $(Y(x))_{x\in\mathbb{X}}$ be given by (15.18), where the kernel k is assumed to be non-negative. Let ρ be a σ-finite measure on $\mathbb{X}$ such that

$$\iint \mathbf{1}\{x \in \cdot\} r k(x, y)\, \lambda(d(y, r))\, \rho(dx) \tag{15.23}$$

is a σ-finite measure on $\mathbb{X}$. Then (15.19) holds for ρ-a.e. $x \in \mathbb{X}$. A Cox process χ driven by the random measure $Y(x)\rho(dx)$ is called a *shot noise Cox process* (SNCP). It has the intensity measure (15.23).

Proposition 15.14 *Let χ be a SNCP as in Example 15.13. The Laplace functional of χ is given by*

$$L_\chi(u) = \exp\Big[- \int (1 - e^{-ru^*(y)})\, \lambda(d(y, r))\Big], \quad u \in \mathbb{R}_+(\mathbb{X}), \tag{15.24}$$

where $u^*(y) := \int (1 - e^{-u(x)}) k(x, y)\, \rho(dx)$, $y \in \mathbb{Y}$.

Proof By (13.9) and Fubini's theorem, we have for any $u \in \mathbb{R}_+(\mathbb{X})$ that

$$L_\chi(u) = \mathbb{E}\Big[\exp\Big(- \iint (1 - e^{-u(x)}) r k(x, y)\, \rho(dx)\, \eta(d(y, r))\Big)\Big].$$

Theorem 3.9 yields the assertion. □

Example 15.15 Let χ be a SNCP as in Example 15.13 and assume that $\lambda(d(y, r)) = r^{-1} e^{-br} \rho_0(dy) dr$ as in Example 15.4. Proposition 15.14 and (15.10) yield

$$L_\chi(u) = \exp\Big[- \int \log\Big(1 + \frac{u^*(y)}{b}\Big) \rho_0(dy)\Big]. \tag{15.25}$$

15.5 Exercises

Exercise 15.1 Let $n \geq 2$ and let

$$\Delta_n := \{(p_1, \ldots, p_n) \in [0, 1]^n : p_1 + \cdots + p_n = 1\}$$

denote the simplex of all n-dimensional probability vectors. Lebesgue measure on Δ_n is given by the formula

$$\mu_n(C) := \int_{[0,1]^n} \mathbf{1}\{(x_1,\dots,x_{n-1},1-x_1-\cdots-x_{n-1}) \in C\}\,\lambda_{n-1}(d(x_1,\dots,x_{n-1}))$$

for $C \in \mathcal{B}(\Delta_n)$. This is the Hausdorff measure $\mathcal{H}_{n-1}$ (introduced in Appendix A.3) restricted to Δ_n. The *Dirichlet distribution* with parameters $\alpha_1,\dots,\alpha_n \in (0,\infty)$ is the distribution on Δ_n with density

$$\frac{\Gamma(\alpha_1+\cdots+\alpha_n)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)} x_1^{\alpha_1-1}\cdots x_n^{\alpha_n-1} \tag{15.26}$$

with respect to μ_n.

Let ξ be a Gamma random measure as in Example 15.4 and assume that the shape measure satisfies $0 < \rho_0(\mathbb{Y}) < \infty$. Then the random measure $\zeta := \xi(\cdot)/\xi(\mathbb{Y})$ can be considered as a random probability measure. Let $B_1,\dots,B_n$ be a measurable partition of $\mathbb{Y}$ such that $\rho_0(B_i) > 0$ for all $i \in \{1,\dots,n\}$. Show that $(\zeta(B_1),\dots,\zeta(B_n))$ has a Dirichlet distribution with parameters $\rho_0(B_1),\dots,\rho_0(B_n)$ and is independent of $\xi(\mathbb{Y})$. (Hint: Use Example 15.4 to express the expectation of a function of $(\zeta(B_1),\dots,\zeta(B_n),\xi(\mathbb{Y}))$ as a Lebesgue integral with respect to $(x_1\dots,x_n) \in \mathbb{R}_+^n$ and change variables in an appropriate way.)

Exercise 15.2 Let ξ be a compound Poisson process as in Proposition 15.6 and let $B \in \mathcal{Y}$. Show that

$$\begin{aligned}\mathbb{E}[\xi(B)^3] =& \rho_0(B)^3\Big(\int r\,\nu(dr)\Big)^3 + 3\rho_0(B)^2\Big(\int r\,\nu(dr)\Big)\Big(\int r^2\,\nu(dr)\Big)\\ &+\rho_0(B)\int r^3\,\nu(dr).\end{aligned}$$

Exercise 15.3 Let ξ be a compound Poisson process as in Proposition 15.6. Let $n \in \mathbb{N}$ and $f \in \mathbb{R}_+(\mathbb{Y})$. Show that

$$\mathbb{E}[\xi(f)^n] = \sum_{r_1,\dots,r_n\in\mathbb{N}_0} \begin{bmatrix} n \\ r_1,\dots,r_n\end{bmatrix} \rho_0(B)^{r_1+\cdots+r_n}\prod_{i=1}^n \Big(\int f^i\,d\rho_0\Big)^{r_i}\Big(\int r^i\,\nu(dr)\Big)^{r_i}.$$

(Hint: Generalize (15.13). You may assume that η is proper.)

Exercise 15.4 Let η be a Poisson process on $\mathbb{Y}\times\mathbb{R}$ with intensity measure $\lambda = \rho_0 \otimes \nu$ where ρ_0 is s-finite and ν is a measure on $\mathbb{R}$ with $\nu\{0\} = 0$ and

$$\int (|r| \wedge 1)\,\nu(dr) < \infty. \tag{15.27}$$

Let $B \in \mathcal{Y}$ satisfy $\rho_0(B) < \infty$. Show that (15.1) converges almost surely.

Take $n \in \mathbb{N}$ and prove that $\mathbb{E}[\xi(B)^n] < \infty$ if and only if $\int |r|^n \, \nu(dr) < \infty$ and, moreover, that (15.12) remains true in this case.

Exercise 15.5 Let $p \in (0, 1)$. Show that the measure ν on $(0, \infty)$ defined by $\nu(dr) := r^{-1-p} dr$ satisfies (15.5). Let ξ be a ρ_0-symmetric compound Poisson process on $\mathbb{Y}$ with Lévy measure ν for some s-finite measure ρ_0 on $\mathbb{Y}$. Show for all $B \in \mathcal{Y}$ that

$$\mathbb{E}[\exp[-t\xi(B)]] = \exp\left[-p^{-1}\Gamma(1-p)\rho_0(B)t^p\right], \quad t \geq 0.$$

Exercise 15.6 (Self-similar random measure) Let ξ be as in Exercise 15.5 and assume in addition that $\mathbb{Y} = \mathbb{R}^d$ for some $d \in \mathbb{N}$ and that ρ_0 is Lebesgue measure. Show that ξ is *self-similar* in the sense that $\xi_c \overset{d}{=} \xi$ for any $c > 0$, where $\xi_c(B) := c^{-d/p}\xi(cB)$, $B \in \mathcal{B}^d$. (Hint: Use Exercise 15.5.)

Exercise 15.7 Let ξ be a uniformly σ-finite random measure on a Borel space $\mathbb{Y}$. Show that that the set $B \subset \mathbb{Y}$ of fixed atoms of ξ is at most countable. Assume in addition that ξ is completely independent. Show that there are independent random variables Z_x, $x \in B$, and a completely independent random measure ξ_0 without fixed atoms such that

$$\xi = \xi_0 + \sum_{x \in B} Z_x \delta_{Z_x}, \quad \mathbb{P}\text{-a.s.}$$

Exercise 15.8 Let the random field $(Y(x))_{x \in \mathbb{R}^d}$ be as in Example 15.12. Show that this field is stationary, that is, for any $m \in \mathbb{N}$ and $x_1, \dots, x_m \in \mathbb{R}^d$, the distribution of $(Y(x_1 + x), \dots, Y(x_m + x))$ does not depend on $x \in \mathbb{R}^d$.

Exercise 15.9 Let χ be a SNCP as in Example 15.13 and assume moreover, that

$$\iint \mathbf{1}\{x \in \cdot\} r^2 k(x, y)^2 \, \lambda(d(y, r)) \, \rho(dx)$$

is a σ-finite measure on $\mathbb{X}$. Show that the second factorial moment measure α_2 of χ is given by $\alpha_2(d(x_1, x_2)) = g_2(x_1, x_2) \, \rho^2(d(x_1, x_2))$, where

$$\begin{aligned} g_2(x_1, x_2) := &\int r^2 k(x_1, y) k(x_2, y) \, \lambda(d(y, r)) \\ &+ \left(\int r k(x_1, y) \, \lambda(d(y, r))\right)\left(\int r k(x_2, y) \, \lambda(d(y, r))\right). \end{aligned}$$

Compare this with the case $m = 2$ of Theorem 14.6. (Hint: Use Proposition 15.11.)

Exercise 15.10 Let χ be a SNCP as in Example 15.12 and assume that $\mathbb{Y}$ is at most countable. Show that χ is a countable superposition of independent mixed Poisson processes; see Exercise 13.3.

Exercise 15.11 A random measure ξ is *infinitely divisible* if for every integer $m \in \mathbb{N}$ there are independent random measures $\xi_1, \dots, \xi_m$ such that $\xi \overset{d}{=} \xi_1 + \cdots + \xi_m$. Show that a compound Poisson process has this property.

Exercise 15.12 Let χ be a shot noise Cox process and $m \in \mathbb{N}$. Show that there are independent point processes $\chi_1, \dots, \chi_m$ such that $\chi \overset{d}{=} \chi_1 + \cdots + \chi_m$. This shows that χ is an *infinitely divisible* point process.

16

The Boolean model and the Gilbert graph

The spherical Boolean model Z is the union of random balls, where the centres are forming a stationary Poisson process η on $\mathbb{R}^d$ and where the radii are obtained from independent marking of η. The capacity functional of Z associates with each compact set $C \subset \mathbb{R}^d$ the probability that Z intersects C. It can be explicitly expressed in terms of the intensity of η and the radius distribution. A consequence are formulae for contact distributions of Z. The Gilbert graph, a close relative of the Boolean model, is a random graph with vertex set η and an edge between two vertices if the associated balls overlap. A cluster is a maximally connected set of of vertices. The point process of clusters isomorphic to a given finite connected graph is stationary with an intensity, that can in principle be computed as an integral with respect to a suitable power of the radius distribution.

16.1 Capacity functional

Let $d \in \mathbb{N}$ and let η be a stationary Poisson process on $\mathbb{R}^d$ with intensity $\gamma > 0$. By Corollary 6.5 there exists a sequence $X_1, X_2, \dots$ of random vectors in $\mathbb{R}^d$ such that almost surely

$$\eta = \sum_{n=1}^{\infty} \delta_{X_n}. \tag{16.1}$$

Suppose further that $(R_n)_{n\geq 1}$ is a sequence of independent and identically distributed $\mathbb{R}_+$-valued random variables, independent of η. Recall from (10.11) that the closed ball with centre $x \in \mathbb{R}^d$ and radius $r \geq 0$ is denoted by $B(x,r) := \{y \in \mathbb{R}^d : \|y - x\| \leq r\}$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$. The union

$$Z := \bigcup_{n=1}^{\infty} B(X_n, R_n) \tag{16.2}$$

of the balls with centres X_n and radii $R_n \geq 0$ is a (random) subset of $\mathbb{R}^d$. This is an important model of stochastic geometry:

Definition 16.1 Let η be a stationary Poisson process on $\mathbb{R}^d$ with intensity $\gamma > 0$, given as in (16.1). Let $\mathbb{Q}$ be a probability measure on $\mathbb{R}_+$ and let

$$\xi = \sum_{n=1}^{\infty} \delta_{(X_n, R_n)}, \tag{16.3}$$

be an independent $\mathbb{Q}$-marking of η. Then the random set (16.2) is called the (*spherical* Poisson) *Boolean model* with intensity γ and radius distribution $\mathbb{Q}$.

It is helpful to note that by the marking theorem (Theorem 5.6) the point process ξ defined by (16.3) is a Poisson process with intensity measure $\gamma\lambda_d \otimes \mathbb{Q}$.

Formally, a Boolean model Z is the mapping $\omega \mapsto Z(\omega)$ from Ω into the space of all subsets of $\mathbb{R}^d$. We shall show below that

$$\{Z \cap C = \emptyset\} = \{\omega \in \Omega : Z(\omega) \cap C = \emptyset\} \in \mathcal{F}, \quad C \in \mathcal{C}^d, \tag{16.4}$$

where $\mathcal{C}^d$ denotes the system of all compact subsets of $\mathbb{R}^d$. The mapping $C \mapsto \mathbb{P}(Z \cap C = \emptyset)$ is known as the *capacity functional* of Z. It is given by (16.8). Before proving these facts we need to introduce some notation.

The *Minkowski sum* $K \oplus L$ of sets $K, L \subset \mathbb{R}^d$ is given by

$$K \oplus L := \{x + y : x \in K, y \in L\}. \tag{16.5}$$

The Minkowski sum of K and the ball $B(0, r)$ centred at the origin with radius r is called the *parallel set* of K at distance r. If $K \subset \mathbb{R}^d$ is closed, then

$$K \oplus B(0, r) = \{x \in \mathbb{R}^d : d(x, K) \leq r\} = \{x \in \mathbb{R}^d : B(x, r) \cap K \neq \emptyset\}, \tag{16.6}$$

where

$$d(x, K) := \inf\{\|y - x\| : y \in K\} \tag{16.7}$$

is the Euclidean distance of $x \in \mathbb{R}^d$ from a set $K \subset \mathbb{R}^d$ and $\inf \emptyset := \infty$.

Theorem 16.2 *Let Z be a Boolean model with intensity γ and radius distribution $\mathbb{Q}$. Then* (16.4) *holds and moreover*

$$\mathbb{P}(Z \cap C = \emptyset) = \exp\Big[-\gamma \int \lambda_d(C \oplus B(0, r))\, \mathbb{Q}(dr)\Big], \quad C \in \mathcal{C}^d. \tag{16.8}$$

Proof Let $C \in \mathcal{C}^d$. We may assume that $C \neq \emptyset$. It is easy to see that the distance (16.7) has the Lipschitz property

$$|d(x,C) - d(y,C)| \leq \|x - y\|, \quad x, y \in \mathbb{R}^d.$$

Hence the mapping $(x,r) \mapsto d(x,C) - r$ is continuous. Together with (16.6) this implies that

$$A := \{(x,r) \in \mathbb{R}^d \times \mathbb{R}_+ : B(x,r) \cap C \neq \emptyset\}$$

is a Borel set. With ξ given by (16.3) we have

$$\{Z \cap C = \emptyset\} = \{\xi(A) = 0\}, \tag{16.9}$$

so we obtain (16.4). Since ξ is a Poisson process with intensity measure $\gamma\lambda_d \otimes \mathbb{Q}$, we have that

$$\mathbb{P}(Z \cap C = \emptyset) = \exp[-\gamma(\lambda_d \otimes \mathbb{Q})(A)]. \tag{16.10}$$

Using (16.6) we obtain that

$$\begin{aligned}(\lambda_d \otimes \mathbb{Q})(A) &= \iint \mathbf{1}\{B(x,r) \cap C \neq \emptyset\}\, dx\, \mathbb{Q}(dr) \\ &= \int \lambda_d(C \oplus B(0,r))\, \mathbb{Q}(dr), \end{aligned} \tag{16.11}$$

and hence (16.8). □

16.2 Volume fraction and covering property

In what follows we consider a Boolean model Z with fixed intensity γ and radius distribution $\mathbb{Q}$. It is then convenient to introduce a random variable R_0 with distribution $\mathbb{Q}$ and to rewrite (16.8) as

$$\mathbb{P}(Z \cap C \neq \emptyset) = 1 - \exp(-\gamma\, \mathbb{E}[\lambda_d(C \oplus B(0,R_0))]), \quad C \in \mathcal{C}^d. \tag{16.12}$$

Taking $C = \{x\}$ we obtain that

$$p(x) := \mathbb{P}(x \in Z) = 1 - \exp\big(-\gamma\kappa_d \mathbb{E}[R_0^d]\big), \quad x \in \mathbb{R}^d, \tag{16.13}$$

where $\kappa_d := \lambda_d(B(0,1))$ is the volume of the unit ball in $\mathbb{R}^d$. Because of the next proposition, the number $p := p(0)$ is called the *volume fraction* of Z.

Proposition 16.3 *The mapping $(\omega, x) \mapsto \mathbf{1}_{Z(\omega)}(x)$ is measurable and*

$$\mathbb{E}[\lambda_d(Z \cap B)] = p\lambda_d(B), \quad B \in \mathcal{B}(\mathbb{R}^d). \tag{16.14}$$

Proof The asserted measurability follows from the identity

$$1 - \mathbf{1}_{Z(\omega)}(x) = \prod_{n=1}^{\infty} \mathbf{1}\{\|x - X_n(\omega)\| > R_n(\omega)\}, \quad (\omega, x) \in \Omega \times \mathbb{R}^d.$$

Take $B \in \mathcal{B}(\mathbb{R}^d)$. Since $p(x)$ does not depend on $x \in \mathbb{R}^d$, Fubini's theorem yields

$$\mathbb{E}[\lambda_d(Z \cap B)] = \mathbb{E}\Big[\int \mathbf{1}_Z(x)\mathbf{1}_B(x)\,dx\Big] = \int_B \mathbb{E}[\mathbf{1}\{x \in Z\}]\,dx = p\lambda_d(B),$$

and hence the assertion. □

The next result gives a necessary and sufficient condition for Z to cover all of $\mathbb{R}^d$. For $A' \subset \Omega$ we write $\mathbb{P}(A') = 1$ if there is an $A \in \mathcal{F}$ with $A \subset A'$ and $\mathbb{P}(A) = 1$.

Theorem 16.4 *We have* $\mathbb{P}(Z = \mathbb{R}^d) = 1$ *if and only if* $\mathbb{E}[R_0^d] = \infty$.

Proof Assume that $A \in \mathcal{F}$ satisfies $A \subset \{Z = \mathbb{R}^d\}$ and $\mathbb{P}(A) = 1$. Then $\mathbb{P}(0 \in Z) = 1$ so that (16.13) implies $\mathbb{E}[R_0^d] = \infty$.

Assume, conversely, that $\mathbb{E}[R_0^d] = \infty$. As a preliminary result we first show for any $n \in \mathbb{N}$ that

$$\lambda_d \otimes \mathbb{Q}(\{(x, r) \in \mathbb{R}^d \times \mathbb{R}_+ : B(0, n) \subset B(x, r)\}) = \infty. \tag{16.15}$$

Since $B(0, n) \subset B(x, r)$ if and only if $r \geq \|x\| + n$ the left side of (16.15) equals

$$\int \mathbf{1}\{r \geq \|x\| + n)\,dx\,\mathbb{Q}(dr) = \kappa_d \int_0^{\infty} \mathbf{1}\{r \geq n\}(r - n)^d\,\mathbb{Q}(dr).$$

This is bounded below by

$$\kappa_d \int_0^{\infty} \mathbf{1}\{r \geq 2n\}\Big(\frac{r}{2}\Big)^d\,\mathbb{Q}(dr) = \kappa_d 2^{-d}\,\mathbb{E}[\mathbf{1}\{R_0 \geq 2n\}R_0^d],$$

proving the claim (16.15). But then it follows that the ball $B(0, n)$ is almost surely covered by even infinitely many of the balls $B(x, r)$, $(x, r) \in \xi$. Since n is arbitrary, we obtain that $\mathbb{P}(Z = \mathbb{R}^d) = 1$. □

16.3 Contact distribution functions

In what follows we assume that

$$\mathbb{E}[R_0^d] < \infty. \tag{16.16}$$

Our next aim is to derive *contact distribution functions* of Z, using Theorem 16.2. Define

$$X_\circ := \inf\{r \geq 0 : Z \cap B(0,r) \neq \emptyset\}.$$

By (16.4), $X_\circ$ is a random variable. Since $\mathbb{P}(X_\circ = 0) = \mathbb{P}(0 \in Z) = p$, the distribution of $X_\circ$ has an atom at 0. Therefore it is convenient to consider the conditional distribution of $X_\circ$ given that the origin is not covered by Z. The function

$$H_\circ(t) := \mathbb{P}(X_\circ \leq t \mid 0 \notin Z), \quad t \geq 0, \tag{16.17}$$

is called the *spherical contact distribution function* of Z.

Proposition 16.5 *The spherical contact distribution function of Z is given by*

$$H_\circ(t) = 1 - \exp\Big(- \gamma\kappa_d \sum_{j=1}^{d} \binom{d}{j} t^j\, \mathbb{E}[R_0^{d-j}]\Big), \quad t \geq 0. \tag{16.18}$$

Proof Take $t \geq 0$. Equation (16.12) implies that

$$\begin{aligned}\mathbb{P}(X_\circ > t) &= \mathbb{P}(Z \cap B(0,t) = \emptyset) = \exp\big(- \gamma\, \mathbb{E}[\lambda_d(B(0,R_0+t))]\big) \\ &= \exp\big(- \gamma\kappa_d\, \mathbb{E}[(R_0+t)^d]\big).\end{aligned}$$

Since

$$1 - H_\circ(t) = \mathbb{P}(X_\circ > 0)^{-1}\mathbb{P}(X_\circ > t),$$

we can use the binomial formula to derive the result. □

For $x, y \in \mathbb{R}^d$ let $[x,y] := \{x + s(y-x) : 0 \leq s \leq 1\}$ denote the line segment between x and y. Let $u \in \mathbb{R}^d$ with $\|u\| = 1$ and let

$$X_{[u]} := \inf\{r \geq 0 : Z \cap [0, ru] \neq \emptyset\}$$

denote the linear distance of the origin from Z in direction u. The function

$$H_{[u]}(t) := \mathbb{P}(X_{[u]} \leq t \mid 0 \notin Z), \quad t \geq 0, \tag{16.19}$$

is called the *linear contact distribution function* of Z in direction u. The next proposition shows that if $\mathbb{P}(R_0 = 0) < 1$ then $H_{[u]}$ is the distribution function of an exponential distribution with a mean that does not depend on the direction u.

Proposition 16.6 *Let $u \in \mathbb{R}^d$ with $\|u\| = 1$. The linear contact distribution function of Z in direction u is given by*

$$H_{[u]}(t) = 1 - \exp\big(- \gamma t\kappa_d\, \mathbb{E}[R_0^{d-1}]\big), \quad t \geq 0. \tag{16.20}$$

Proof This time (16.12) implies for any $t \geq 0$ that

$$\mathbb{P}(X_{[u]} > t) = \exp\big(-\gamma\,\mathbb{E}[\lambda_d(B(0, R_0) \oplus [0, tu])]\big).$$

It is a well-known geometric fact (elementary in the cases $d = 2, 3$) that

$$\lambda_d(B(0, R_0) \oplus [0, tu]) = \lambda_d(B(0, R_0)) + t\kappa_{d-1}R_0^{d-1}.$$

Since

$$1 - H_{[u]}(t) = (1 - p)^{-1}\mathbb{P}(X_{[u]} > t)$$

the result follows from (16.13). □

16.4 The Gilbert graph

We need to introduce some graph terminology. An (undirected) *graph* is a pair $G = (V, E)$, where V is a set of *vertices* and $E \subset \{\{x, y\} : x, y \in V, x \neq y\}$ is the set of *edges*. The number $\operatorname{card} V$, if finite, is known as the *order* of G. An edge $\{x, y\} \in E$ is thought of as connecting its endpoints x and y. Two distinct points $x, y \in V$ are said to be *connected* if there exist $m \in \mathbb{N}$ and $x_0, \dots, x_m \in V$ such that $x_0 = x$, $x_m = y$ and $\{x_{i-1}, x_i\} \in E$ for all $i \in [m]$. The graph G is *connected* if any two of its vertices are connected. Two graphs $G = (V, E)$ and $G' = (V', E')$ are said to be *isomorphic* if there is a bijection $T : V \to V'$ such that $\{x, y\} \in E$ if and only if $\{T(x), T(y)\} \in E'$ for all $x, y \in V$ with $x \neq y$. In this case we write $G \simeq G'$.

The remainder of this chapter is concerned with the *Gilbert graph*, a close relative of the spherical Boolean model. We continue to work with the Poisson process (16.1) and the independent marking

$$\xi := \sum_{n=1}^{\infty} \delta_{(X_n, R_n)}$$

based on a sequence $(R_n)_{n\geq 1}$ of independent $\mathbb{R}_+$-valued random variables with common distribution $\mathbb{Q}$. Suppose any two points $X_m, X_n \in \eta$, $m \neq n$, are connected by an *edge* whenever the associated balls overlap, that is, $B(X_m, R_m) \cap B(X_n, R_n) \neq \emptyset$. This yields an undirected *random graph* with vertex set η.

For a formal definition of the Gilbert graph we introduce the space $\mathbb{R}^{[2d]}$ of all sets $e \subset \mathbb{R}^d$ containing exactly two elements. Any $e \in \mathbb{R}^{[2d]}$ is a potential edge of the graph. When equipped with a suitable metric, $\mathbb{R}^{[2d]}$ becomes a separable metric space; see Exercise 17.5.

Definition 16.7 Let ξ be an independent marking of a stationary Poisson process on $\mathbb{R}^d$ as introduced above. Define the point process χ on $\mathbb{R}^{[2d]}$ by

$$\chi := \int \mathbf{1}\{\{x,y\} \in \cdot, x < y\}\mathbf{1}\{B(x,r) \cap B(y,s) \neq \emptyset\}\, \xi^2(d((x,r),(y,s))),$$

where $x < y$ means that x is lexicographically strictly smaller than y. Then we call the pair (η, χ) the *Gilbert graph* (based on η) with radius distribution $\mathbb{Q}$. In the special case where $\mathbb{Q}$ is concentrated on a single positive value (all balls have a fixed radius), it is also known as the *random geometric graph*.

Given distinct points $x_1, \dots, x_k \in \eta$ we let $G(x_1, \dots, x_k, \chi)$ denote the graph with vertex set $\{x_1, \dots, x_k\}$ and edges induced by χ, that is such that $\{x_i, x_j\}$ is an edge if and only if $\{x_i, x_j\} \in \chi$. This graph is a *cluster* if is connected and if none of the x_i is connected to a point in $\eta - \delta_{x_1} - \cdots - \delta_{x_k}$. Let G be a connected graph with $k \geq 2$ vertices. The point process η_G of all clusters isomorphic to G is then defined by

$$\begin{aligned} \eta_G := \int & \mathbf{1}\{x_1 \in \cdot, x_1 < \cdots < x_k\} \\ & \mathbf{1}\{G(x_1, \dots, x_k, \chi) \text{ is a cluster isomorphic to } G\}\, \eta^k(d(x_1, \dots, x_k)). \end{aligned} \tag{16.21}$$

Hence a cluster isomorphic to G contributes to $\eta_G(C)$ if its lexicographic minimum lies in C. The indicator $\mathbf{1}\{x_1 < \cdots < x_k\}$ ensures that each cluster is counted only once. Given distinct $x_1, \dots, x_k \in \mathbb{R}^d$ and $r_1, \dots, r_k \in \mathbb{R}_+$ we define a graph $\Gamma_k(x_1, r_1, \dots, x_k, r_k)$ with vertex set $\{x_1, \dots, x_k\}$ by taking $\{x_i, x_j\}$ as an edge if $B(x_i, r_i) \cap B(x_j, r_j) \neq \emptyset$.

The following theorem shows that η_G is stationary and yields a formula for its intensity. To ease notation, for each $k \in \mathbb{N}$ we define a function $h_k \in \mathbb{R}_+((\mathbb{R}^d \times \mathbb{R}_+)^k)$ by

$$h_k(x_1, r_1, \dots, x_k, r_k) := \mathbb{E}\Big[\lambda_d\Big(\bigcup_{j=1}^{k} B(x_j, R_0 + r_j)\Big)\Big],$$

where R_0 has distribution $\mathbb{Q}$.

Theorem 16.8 *Let $k \geq 2$ and suppose that G is a connected graph with k vertices. Then the point process η_G is stationary and has intensity*

$$\begin{aligned} \gamma_G := \gamma^k \iint & \mathbf{1}\{0 < y_2 < \cdots < y_k\}\mathbf{1}\{\Gamma_k(y_1, r_1, \dots, y_k, r_k) \simeq G) \\ & \times \exp[-\gamma h_k(y_1, r_1, \dots, y_k, r_k)]\, d(y_2, \dots, y_k)\, \mathbb{Q}^k(d(r_1, \dots, r_k)), \end{aligned} \tag{16.22}$$

where $y_1 := 0$.

Proof Let $\mathbf{N}^*$ denote the measurable set of all $\mu \in \mathbf{N}(\mathbb{R}^d \times \mathbb{R}_+)$ such that $\mu(\cdot \times \mathbb{Y}) \in \mathbf{N}_s$ (the space of all locally finite and simple counting measures on $\mathbb{R}^d$). It is no restriction of generality to assume that ξ is a random element in $\mathbf{N}^*$ and that η is a random element in $\mathbf{N}_s$. We construct a measurable mapping $T_G\colon \mathbf{N}^* \to \mathbf{N}_s$ as follows. Let $\mu \in \mathbf{N}^*$ and $(x_1, r_1) \in \mathbb{R}^d \times \mathbb{R}_+$ and define $f_G(x_1, r_1, \mu) := 1$ if and only if $(x_1, r_1) \in \mu$, there are $(x_2, r_2), \dots, (x_k, r_k) \in \mu$ such that $G(x_1, r_1, \dots, x_k, r_k) \simeq G$, $x_1 < \cdots < x_k$, and $B(x_i, r_i) \cap B(x, r) = \emptyset$ for all $i \in \{1, \dots, k\}$ and all $(x, r) \in \mu \setminus \delta_{(x_1,r_1)} \setminus \cdots \setminus \delta_{(x_k,r_k)}$. In all other cases we set $f_G(x_1, r_1, \mu) := 0$. Define

$$T_G\mu := \int \mathbf{1}\{x_1 \in \cdot\} f_G(x_1, r_1, \mu)\, \mu(d(x_1, r_1))$$

and note that $\eta_G = T_G(\xi)$. The mapping T_G has the (covariance) property

$$T_G(\theta_x^*\mu) = \theta_x T_G(\mu), \quad (x, \mu) \in \mathbb{R}^d \times \mathbf{N}^*, \tag{16.23}$$

where θ_x is given by (8.1) (see also (8.2)) and $\theta_x^*\mu \in \mathbf{N}^*$ is defined by

$$\theta_x^*\mu := \int \mathbf{1}\{(y - x, r) \in \cdot\}\, \mu(d(x, r)).$$

By Exercise 16.2 we have

$$\theta_x^*\xi \overset{d}{=} \xi, \quad x \in \mathbb{R}^d. \tag{16.24}$$

Combining this fact with (16.23) shows that for all $x \in \mathbb{R}^d$

$$\theta_x\eta_G = \theta_x T_G(\xi) = T_G(\theta_x^*\xi) \overset{d}{=} T_G(\xi) = \eta_G. \tag{16.25}$$

Thus, η_G is stationary.

Let $(X_{i_1}, R_{i_1}), \dots, (X_{i_k}, R_{i_k})$ be distinct points of ξ. The graph

$$\Gamma' := G(X_{i_1}, \dots, X_{i_k}, \chi) = \Gamma_k(X_{i_1}, R_{i_1}, \dots, X_{i_k}, R_{i_k})$$

is a cluster isomorphic to G if and only if $\Gamma' \simeq G$ and none of the X_{i_j} is connected to a point in $\eta - \delta_{X_{i_1}} - \cdots - \delta_{X_{i_k}}$. Given η and $(R_{i_1}, \dots, R_{i_k})$ the probability of this second event is given by

$$\prod_{y \in \eta - \delta_{X_{i_1}} - \cdots - \delta_{X_{i_k}}} \mathbb{P}\Big(B(y, R_0) \cap \bigcup_{j=1}^{k} B(X_{i_j}, R_{i_j}) = \emptyset\Big),$$

where R_0 has the distribution $\mathbb{Q}$ and is independent of ξ. Let $C \in \mathcal{B}^d$ with $\lambda_d(C) = 1$. By the multivariate Mecke equation (Theorem 4.5) and Exercise

3.5,

$$\mathbb{E}[\eta_G(C)] = \gamma^k \iint \mathbf{1}\{x_1 \in C, x_1 < \cdots < x_k\}\mathbf{1}\{\Gamma_k(x_1, r_1, \dots, x_k, r_k) \simeq G\}$$
$$\times \exp\Big[-\gamma \int h'_k(y, x_1, r_1, \dots, x_k, r_k)\, dy\Big] d(x_1, \dots, x_k)\, \mathbb{Q}^k(d(r_1, \dots, r_k)),$$

where

$$h'_k(y, x_1, r_1, \dots, x_k, r_k) := \mathbb{P}\Big(B(y, R_0) \cap \bigcup_{j=1}^k B(x_j, r_j) \neq \emptyset\Big).$$

Since $B(y, R_0) \cap \bigcup_{j=1}^k B(x_j, r_j) \neq \emptyset$ if and only if $y \in \bigcup_{j=1}^k B(x_j, R_0 + r_j)$ we obtain from Fubini's theorem that

$$\int h'_k(y, x_1, r_1, \dots, x_k, r_k)\, dy = h_k(x_1, r_1, \dots, x_k, r_k).$$

Note that

$$\Gamma_k(x_1, r_1, \dots, x_k, r_k) \simeq \Gamma_k(0, r_1, x_2 - x_1, r_2, \dots, x_k - x_1, r_k)$$

and that $x_1 < \cdots < x_k$ if and only if $0 < x_2 - x_1 < \cdots < x_k - x_1$. Moreover,

$$h_k(x_1, r_1, \dots, x_k, r_k) = h_k(0, r_1, x_2 - x_1, r_2, \dots, x_k - x_1, r_k).$$

Performing the change of variables $y_i := x_i - x_1$ for $i \in \{2, \dots, k\}$ and using that $\lambda_d(C) = 1$ gives the asserted formula (16.22). □

If the graph G has only one vertex, then $\eta_1 := \eta_G$ is the point process of *isolated points* of the Gilbert graph (η, χ). In this case (a simplified version of) the proof of Theorem 16.8 yields that η_1 is a stationary point process with intensity

$$\begin{aligned}\gamma_1 &:= \gamma \int \exp[-\gamma\, \mathbb{E}[\lambda_d(B(0, R_0 + r))]\, \mathbb{Q}(dr) \\ &= \gamma \int \exp\big[-\gamma\kappa_d\, \mathbb{E}[(R_0 + r)^d]\big]\, \mathbb{Q}(dr),\end{aligned}$$

where $\kappa_d := \lambda_d(B(0, 1))$ is the volume of the unit ball.

For $k \in \mathbb{N}$ let $\mathbf{G}_k$ denote a set of connected graphs with k vertices containing exactly one member of each isomorphism equivalence class. Thus for any connected graph G with k vertices there is exactly one $G' \in \mathbf{G}_k$ such that $G \simeq G'$. Then $\sum_{G \in \mathbf{G}_k} \eta_G$ is a stationary point process counting the *k-clusters* of (η, χ), that is, the clusters with k vertices.

Example 16.9 The set $\mathbf{G}_2$ contains one graph, namely one with two vertices and one edge connecting the two vertices. Theorem 16.8 shows that the intensity of 2-clusters is given by

$$\gamma_2 := \frac{\gamma^2}{2} \int \mathbf{1}\{\|z\| \le r_1 + r_2\}$$
$$\times \exp\big[-\gamma\,\mathbb{E}[\lambda_d(B(0,R_0+r_1) \cup B(z,R_0+r_2))]\,dz\,\mathbb{Q}^2(d(r_1,r_2)).$$

For $x \in \eta$ denote by $C(x)$ the set of vertices in the cluster containing x. This set consists of x and all vertices $y \in \eta$ connected to x in the Gilbert graph. For $k \in \mathbb{N}$ we let

$$\eta_k := \int \mathbf{1}\{x \in \cdot, \operatorname{card} C(x) = k\}\,\eta(dx) \tag{16.26}$$

denote the point process of all points of η that belong to a cluster of size k. Note that η_1 is the point process of isolated points introduced before.

Theorem 16.10 *Let $k \ge 2$. Then η_k is a stationary point process with intensity*

$$\gamma_k := \frac{\gamma^k}{(k-1)!} \iint \mathbf{1}\{\Gamma(y_1,r_1,\dots,y_k,r_k) \text{ is connected}\} \tag{16.27}$$
$$\times \exp[-\gamma h_k(y_1,r_1,\dots,y_k,r_k)]\,d(y_2,\dots,y_k)\,\mathbb{Q}^k(d(r_1,\dots,r_k)),$$

where $y_1 := 0$.

Proof Stationarity of η_k follows as at (16.25).

Let $G \in \mathbf{G}_k$ and $j \in \{1,\dots,k\}$. In the definition of η_G we used the lexicographically smallest point to label a cluster. Using instead the j-smallest point yields a stationary point process $\eta_G^{(j)}$. Exactly as in the proof of Theorem 16.8 it follows that $\eta_G^{(j)}$ has intensity

$$\gamma_G^{(j)} := \gamma^k \int_{B_j} \mathbf{1}\{\Gamma_k(y_1,r_1,\dots,y_k,r_k) \simeq G\}$$
$$\times \exp[-\gamma h_k(y_1,r_1,\dots,y_k,r_k)]\,d(y_2,\dots,y_k)\,\mathbb{Q}^k(d(r_1,\dots,r_k)),$$

where $y_1 := 0$ and B_j denotes the set of all $(y_2,\dots,y_k) \in (\mathbb{R}^d)^{k-1}$ such that $y_2 < \cdots < y_k$ and $0 < y_2$ for $j = 1$, $y_{j-1} < 0 < y_j$ for $j \in \{2,\dots,k-1\}$ and $y_k < 0$ for $j = k$. Since clearly $\gamma_k = \sum_{G \in \mathbf{G}_k} \sum_{j=1}^k \gamma_G^{(j)}$ we obtain that

$$\gamma_k = \gamma^k \int \mathbf{1}\{y_2 < \cdots < y_k\}\mathbf{1}\{\Gamma_k(y_1,r_1,\dots,y_k,r_k) \text{ is connected}\}$$
$$\times \exp[-\gamma h_k(y_1,r_1,\dots,y_k,r_k)]\,d(y_2,\dots,y_k)\,\mathbb{Q}^k(d(r_1,\dots,r_k)),$$

so that the symmetry of the integrand (without the first indicator) implies the asserted identity (16.27). □

The quantity $k\gamma_k/\gamma$ is the fraction of Poisson points that belong to a cluster of size k. Hence it can be interpreted as probability that a typical point of η belongs to a cluster of size k. This interpretation can be deepened by introducing the point process $\eta^0 := \eta + \delta_0$ and the Gilbert graph (η^0, χ^0), where $\eta^0 := \eta + \delta_0$ and χ^0 is a point process on $\mathbb{R}^{[2d]}$ that is defined (in terms of an independent marking of η^0) as before. Then

$$\mathbb{P}(\operatorname{card} C^0(0) = k) = k\gamma_k/\gamma, \quad k \in \mathbb{N}, \tag{16.28}$$

where $C^0(x)$ is the cluster of $x \in \eta^0$ in the Gilbert graph (η^0, χ^0); see Exercise 16.3.

16.5 Exercises

Exercise 16.1 Consider the Gilbert graph under the assumption that R_0^d has an infinite mean. Show that $\gamma_G = 0$ for all connected graphs (with a finite number of vertices).

Exercise 16.2 Prove the stationarity relation (16.24).

Exercise 16.3 Prove (16.28).

Exercise 16.4 Consider a Boolean model based on the Poisson process η and call a point $x \in \eta$ *visible* if x is contained in exactly one of the balls $B(X_n, R_n)$, $n \in \mathbb{N}$. Show that the point process η_v of visible points is stationary with intensity

$$\gamma_v := \gamma\, \mathbb{E}[\exp(-\gamma\kappa_d\, \mathbb{E}[R_0^d])].$$

Now let $c > 0$ and consider the class of all Boolean models with the same intensity γ and radius distributions $\mathbb{Q}$ satisfying $\int r^d\, \mathbb{Q}(dr) = c$. Show that the intensity of visible points is then minimized by the distribution $\mathbb{Q}$ concentrated on a single point. (Hint: Use Jensen's inequality to prove the second part.)

17

The Boolean model with general grains

The spherical Boolean model Z is generalized so as to accommodate arbitrary random compact grains. The capacity functional of Z can again be written in an exponential form involving the intensity and the grain distribution. This implies an explicit formula for the covariance of Z. Moreover, in the case of convex grains, the Steiner formula of convex geometry leads to a formula for the spherical contact distribution function involving the mean intrinsic volumes of a typical grain. In the general case the capacity functional determines the intensity and the grain distribution up to a centering.

17.1 Capacity functional

Let $\mathcal{C}^{(d)}$ denote the space of non-empty compact subsets of $\mathbb{R}^d$ and define the *Hausdorff distance* between sets $K, L \in \mathcal{C}^{(d)}$ by

$$\delta(K, L) := \inf\{\varepsilon \geq 0 : K \subset L \oplus B(0, \varepsilon), L \subset K \oplus B(0, \varepsilon)\}. \tag{17.1}$$

It is easy to check that $\delta(\cdot, \cdot)$ is a metric on the space $\mathcal{C}^{(d)}$. We equip $\mathcal{C}^{(d)}$ with the associated Borel σ-field $\mathcal{B}(\mathcal{C}^{(d)})$. For $B \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$ we recall the notation $B + x := \{y + x : y \in B\}$.

Definition 17.1 Let η be a stationary Poisson process on $\mathbb{R}^d$ with intensity $\gamma > 0$, given as in (16.1). Let $\mathbb{Q}$ be a probability measure on $\mathcal{C}^{(d)}$ and let

$$\xi = \sum_{n=1}^{\infty} \delta_{(X_n, Z_n)}, \tag{17.2}$$

be an independent $\mathbb{Q}$-marking of η. Then

$$Z := \bigcup_{n=1}^{\infty} (Z_n + X_n) \tag{17.3}$$

is called the *Boolean model* with intensity γ and *grain distribution* $\mathbb{Q}$ (or the *Boolean model induced by* ξ for short).

As in the previous chapter, Z is a short-hand notation for the mapping $\omega \mapsto Z(\omega)$. We wish to generalize Theorem 16.2 for the spherical Boolean model and give a formula for the *capacity functional* $C \mapsto \mathbb{P}(Z \cap C \neq \emptyset)$ of Z. As preparation we need to identify useful generators of the Borel σ-field $\mathcal{B}(C^{(d)})$. For $B \subset \mathbb{R}^d$ define

$$C_B := \{K \in C^{(d)} : K \cap B \neq \emptyset\}; \quad C^B := \{K \in C^{(d)} : K \cap B = \emptyset\}. \tag{17.4}$$

Lemma 17.2 *The σ-field $\mathcal{B}(C^{(d)})$ is generated by $\{C_B : B \in C^{(d)}\}$.*

Proof It is a quick consequence of the definition of the Hausdorff distance that C^B is open, whenever $B \in C^d$. Hence the σ-field $\mathcal{H}$ generated by $\{C_B : B \in C^d\}$ is contained in $\mathcal{B}(C^{(d)})$.

To prove $\mathcal{B}(C^{(d)}) \subset \mathcal{H}$ we first note that the Hausdorff distance is *separable*, that is, there is a countable dense subset of $C^{(d)}$; see Exercise 17.3. It follows from elementary properties of separable metric spaces that any open set in $C^{(d)}$ is either empty or a countable union of closed balls. Hence it is sufficient to show that for any $K \in C^{(d)}$ and $\varepsilon > 0$ the closed ball

$$B(K, \varepsilon) = \{L \in C^{(d)} : L \subset K \oplus B(0, \varepsilon), K \subset L \oplus B(0, \varepsilon)\}$$

is in $\mathcal{H}$. Since $L \subset K \oplus B(0, \varepsilon)$ is equivalent to $L \cap (\mathbb{R}^d \setminus (K \oplus B(0, \varepsilon))) = \emptyset$ we have

$$\{L \in C^{(d)} : L \subset K \oplus B(0, \varepsilon)\} = \bigcap_{n \in \mathbb{N}} \{L \in C^{(d)} : L \cap (B_n \setminus (K \oplus B(0, \varepsilon))) = \emptyset\}, \tag{17.5}$$

where B_n is the interior of the ball $B(0, n)$. Since $A_n := B_n \setminus K \oplus B(0, \varepsilon)$ is open, it is easy to prove that $C^{A_n} \in \mathcal{H}$, so that the right-hand side of (17.5) is in $\mathcal{H}$ as well. It remains to show that $C_{K,\varepsilon} := \{L \in C^{(d)} : K \subset L \oplus B(0, \varepsilon)\}$ is in $\mathcal{H}$. To this end we take a countable dense set $D \subset K$ (see Lemma A.21) and note that $K \not\subset L \oplus B(0, \varepsilon)$ if and only if there is some $x \in D$ such that $B(x, \varepsilon) \cap L = \emptyset$. (Use (16.6) and a continuity argument.) Therefore $C^{(d)} \setminus C_{K,\varepsilon}$ is a countable union of sets of the form C^B, where B is a closed ball. Hence $C_{K,\varepsilon} \in \mathcal{H}$ and the proof is complete. □

For $C \subset \mathbb{R}^d$ let $C^* := \{-x : x \in C\}$ denote the *reflection* of C in the origin.

Theorem 17.3 *Let Z be a Boolean model with intensity γ and grain distribution $\mathbb{Q}$. Then* (16.4) *holds and, moreover,*

$$\mathbb{P}(Z \cap C = \emptyset) = \exp\Big[-\gamma \int \lambda_d(K \oplus C^*)\,\mathbb{Q}(dK)\Big], \quad C \in \mathcal{C}^d. \tag{17.6}$$

Proof We can follow the proof of Theorem 16.2. It is suffices to consider the case with $C \in \mathcal{C}^{(d)}$. By Exercise 17.6, the mapping $(x, K) \mapsto K + x$ from $\mathbb{R}^d \times \mathcal{C}^{(d)}$ to $\mathcal{C}^{(d)}$ is continuous and hence measurable. By Lemma 17.2,

$$A := \{(x, K) \in \mathbb{R}^d \times \mathcal{C}^{(d)} : (K + x) \cap C \neq \emptyset\}$$

is a measurable set. Since (16.9) holds, (16.4) follows. Moreover, since ξ is a Poisson process with intensity measure $\gamma\lambda_d \otimes \mathbb{Q}$ we again obtain (16.10). Using the fact that

$$\{x \in \mathbb{R}^d : (K + x) \cap C \neq \emptyset\} = C \oplus K^*, \tag{17.7}$$

together with the reflection invariance of Lebesgue measure, we obtain the assertion (17.6). □

Taking $C = \{x\}$ in (17.6) yields, as in (16.13), that

$$p(x) := \mathbb{P}(x \in Z) = 1 - \exp\Big(-\gamma\,\mathbb{E}\Big[\int \lambda_d(K)\,\mathbb{Q}(dK)\Big]\Big), \quad x \in \mathbb{R}^d. \tag{17.8}$$

The quantity $p := p(0)$ is the *volume fraction* of Z.

Proposition 17.4 *The mapping $(\omega, x) \mapsto \mathbf{1}_{Z(\omega)}(x)$ is measurable and*

$$\mathbb{E}[\lambda_d(Z \cap B)] = p\lambda_d(B), \quad B \in \mathcal{B}(\mathbb{R}^d). \tag{17.9}$$

Proof By (17.2),

$$1 - \mathbf{1}_Z(x) = \prod_{n=1}^{\infty} \mathbf{1}\{x \notin Z_n + X_n\}, \quad x \in \mathbb{R}^d.$$

Therefore the asserted measurability follows from the fact that the mappings $(x, K) \mapsto \mathbf{1}\{x \notin K\}$ and $(x, K) \mapsto K + x$ are measurable on $\mathbb{R}^d \times \mathcal{C}^{(d)}$, see Exercises 17.7 and 17.6. Equation (17.9) can then be proved in the same manner as (16.14). □

17.2 Spherical contact distribution function and covariance

In this section we first give a more general version of Proposition 16.5 under an additional assumption on $\mathbb{Q}$. Let $\mathcal{K}^{(d)}$ denote the system of all convex $K \in \mathcal{C}^{(d)}$. It can be easily shown that $\mathcal{K}^{(d)}$ is a closed and hence measurable subset of $C^{(d)}$. Recall from Appendix A.3 the definition of the intrinsic volumes $V_0, \dots, V_d$ as non-negative continuous functions on $\mathcal{K}^{(d)}$. If the grain distribution $\mathbb{Q}$ is concentrated on $\mathcal{K}^{(d)}$ (i.e. $\mathbb{Q}(\mathcal{K}^{(d)}) = 1$), then we can define

$$\phi_i := \int V_i(K)\,\mathbb{Q}(dK), \quad i = 0, \dots, d. \tag{17.10}$$

Note that $\phi_0 = 1$. The Steiner formula (A.22) implies that $\phi_i < \infty$ for all $i \in \{0, \dots, d\}$ if and only if

$$\int \lambda_d(K \oplus B(0, r))\,\mathbb{Q}(dK) < \infty, \quad r \geq 0. \tag{17.11}$$

We shall always assume that (17.11) is satisfied. The *spherical contact distribution function* $H_\circ$ of Z is defined by (16.17).

Proposition 17.5 *Suppose that $\mathbb{Q}$ is concentrated on $\mathcal{K}^{(d)}$. Then the spherical contact distribution function of the Boolean model Z is given by*

$$H_\circ(t) = 1 - \exp\Big[-\sum_{j=0}^{d-1} t^{d-j}\kappa_{d-j}\gamma\phi_j\Big], \quad t \geq 0, \tag{17.12}$$

where $\phi_0, \dots, \phi_d$ are defined by (17.10).

Proof Let $t \geq 0$. Similarly to the proof of Proposition 16.5 we obtain from (17.6) that

$$1 - H_\circ(t) = \exp\Big[-\gamma\int(\lambda_d(K \oplus B(0, t)) - \lambda_d(K))\,\mathbb{Q}(dK)\Big].$$

By assumption on $\mathbb{Q}$ we can use the Steiner formula (A.22) (recall that $V_d = \lambda_d$) to simplify the exponent and to conclude the proof of (17.12). □

Next we deal with second order properties of the Boolean model. The function $(x, y) \mapsto \mathbb{P}(x \in Z, y \in Z)$ is called *covariance* (or *two point correlation function*) of Z. It can be expressed in terms of the function

$$\beta_d(x) := \int \lambda_d(K \cap (K + x))\,\mathbb{Q}(dK), \quad x \in \mathbb{R}^d, \tag{17.13}$$

as follows.

Theorem 17.6 *The covariance of Z is given by*

$$\mathbb{P}(x \in Z, y \in Z) = p^2 + (1 - p)^2(e^{\gamma\beta_d(y-x)} - 1), \quad x, y \in \mathbb{R}^d.$$

Proof Let Z_0 have distribution $\mathbb{Q}$ and let $x, y \in \mathbb{R}^d$. By (17.6),

$$\begin{aligned}\mathbb{P}(Z \cap \{x, y\} = \emptyset) &= \exp(-\gamma\, \mathbb{E}[\lambda_d((Z_0 + x) \cup (Z_0 + y))]) \\ &= \exp(-\gamma\, \mathbb{E}[\lambda_d(Z_0 \cup (Z_0 + y - x))]).\end{aligned}$$

By additivity of λ_d and linearity of expectation we obtain that

$$\begin{aligned}\mathbb{P}(Z \cap \{x, y\} = \emptyset) &= \exp(-\gamma\, \mathbb{E}[\lambda_d(Z_0)]) \exp(-\gamma\, \mathbb{E}[\lambda_d(Z_0 + y - x)]) \\ &\quad \times \exp(\gamma\, \mathbb{E}[\lambda_d(Z_0 \cap (Z_0 + y - x))]) \\ &= (1 - p)^2 \exp[\gamma \beta_d(y - x)],\end{aligned}$$

where we have used (17.8). By the additivity of probability

$$\mathbb{P}(Z \cap \{x, y\} = \emptyset) = \mathbb{P}(x \notin Z, y \notin Z) = \mathbb{P}(x \in Z, y \in Z) + 1 - 2p,$$

and the result follows. □

17.3 Identifiability of intensity and grain distribution

In this section we shall prove that the capacity functional a Boolean model Z determines intensity and centred grain distribution of the underlying marked Poisson process. To this end we need the following lemma, which is of some independent interest.

Lemma 17.7 *Let ν be a measure on $\mathcal{C}^{(d)}$ satisfying*

$$\nu(C_B) < \infty, \quad B \in \mathcal{C}^d. \tag{17.14}$$

Then ν is determined by its values on $\{C_B : B \in \mathcal{C}^d\}$.

Proof For $m \in \mathbb{N}$ and $B_0, \dots, B_m \in \mathcal{C}^d$ let

$$C^{B_0}_{B_1,\dots,B_m} := C^{B_0} \cap C_{B_1} \cap \cdots \cap C_{B_m}. \tag{17.15}$$

Since $C^{B_0} \cap C^{B_0'} = C^{B_0 \cup B_0'}$ for all $B_0, B_0' \in \mathcal{C}^d$, the sets of the form (17.15) form a π-system. By Lemma 17.2 this is a generator of $\mathcal{B}(\mathcal{C}^{(d)})$. Since (17.14) easily implies that ν is σ-finite, the assertion follows from Theorem A.4 once we have shown that

$$\nu(C^{B_0}_{B_1,\dots,B_m}) = \sum_{j=0}^{m} (-1)^{j+1} \sum_{1 \le i_1 < \cdots < i_j \le m} \nu(C_{B_0 \cup B_{i_1} \cup \cdots \cup B_{i_j}}). \tag{17.16}$$

In fact, we only need the case with $B_0 = \emptyset$ but it is simpler to prove the more general case. Moreover, the identity (17.16) is of some independent interest. For $m = 1$ the identity means that

$$\nu(C^{B_0}_{B_1}) = \nu(C_{B_0 \cup B_1}) - \nu(C_{B_0}),$$

a direct consequence of the equality $C^{B_0}_{B_1} = C_{B_0\cup B_1} \setminus C_{B_0}$. In the general case we can use the equality

$$C^{B_0}_{B_1,\dots,B_m} = C^{B_0}_{B_1,\dots,B_{m-1}} \setminus C^{B_0\cup B_m}_{B_1,\dots,B_{m-1}}$$

and induction. □

We need to fix a *centre function* $c\colon \mathcal{C}^{(d)} \to \mathbb{R}^d$. This is a measurable function satisfying

$$c(K + x) = c(K) + x, \quad (x, K) \in \mathbb{R}^d \times \mathcal{C}^{(d)}. \tag{17.17}$$

An example is the centre of the (uniquely determined) *circumball* of $K \in \mathcal{C}^{(d)}$, that is, of the smallest ball containing K.

Theorem 17.8 *Let Z and Z' be Boolean models with respective intensities γ and γ' and grain distribution $\mathbb{Q}$ and $\mathbb{Q}'$. Assume that $\mathbb{Q}$ satisfies* (17.11). *If*

$$\mathbb{P}(Z \cap C = \emptyset) = \mathbb{P}(Z' \cap C = \emptyset), \quad C \in \mathcal{C}^d, \tag{17.18}$$

then $\gamma = \gamma'$ and $\mathbb{Q}(\{K : K - c(K) \in \cdot\}) = \mathbb{Q}'(\{K : K - c(K) \in \cdot\})$.

Proof Define a measure ν on $(\mathcal{C}^{(d)}, \mathcal{B}(\mathcal{C}^{(d)}))$ by

$$\nu(\cdot) := \gamma \iint \mathbf{1}\{K + x \in \cdot\}\, \mathbb{Q}(dK)\, dx.$$

Similarly define a measure ν' by replacing $(\gamma, \mathbb{Q})$ with $(\gamma', \mathbb{Q}')$. Assume that (17.18) holds. Taking into account (17.7), Theorem 17.3 and assumption (17.11) imply that

$$\nu(\mathcal{C}_B) = \nu'(\mathcal{C}_B) < \infty, \quad C \in \mathcal{C}^d, \tag{17.19}$$

where we recall (17.4). In particular both measures satisfy (17.14). By Lemma 17.7 we conclude that $\nu = \nu'$.

Take a measurable set $A \subset \mathcal{C}^{(d)}$ and $B \in \mathcal{B}^d$ with $\lambda_d(B) = 1$. Then, using property (17.17) of a centre function,

$$\begin{aligned}
&\int \mathbf{1}\{K - c(K) \in A, c(K) \in B\}\, \nu(dK) \\
&\quad = \gamma \iint \mathbf{1}\{K + x - c(K + x) \in A, c(K + x) \in B\}\, \mathbb{Q}(dK)\, dx \\
&\quad = \gamma \iint \mathbf{1}\{K - c(K) \in A, c(K) + x \in B\}\, dx\, \mathbb{Q}(dK) \\
&\quad = \gamma\, \mathbb{Q}(\{K : K - c(K) \in A\}).
\end{aligned}$$

Since ν' satisfies a similar equation and $\nu = \nu'$, the assertion follows. □

17.4 Exercises

Exercise 17.1 Let Z be a Boolean model whose grain distribution $\mathbb{Q}$ satisfies (17.11). Show that Z is almost surely a closed set. (Hint: First prove that any compact set is intersected by only a finite number of the grains $Z_n + X_n$.)

Exercise 17.2 Suppose that $\int \lambda_d(K \oplus B(0, \varepsilon))\, \mathbb{Q}(dK) < \infty$ for some (fixed) $\varepsilon > 0$. Prove that (17.11) then holds.

Exercise 17.3 Show that the space $\mathcal{C}^{(d)}$ equipped with the Hausdorff distance (17.1) is a separable metric space. (Hint: Use a dense countable subset of $\mathbb{R}^d$.)

Exercise 17.4 Let $m \in \mathbb{N}$ and let $\mathcal{C}_m \subset \mathcal{C}^{(d)}$ be the space of all compact non-empty subsets of $\mathbb{R}^d$ with at most m points. Show that $\mathcal{C}_m$ is closed (with respect to the Hausdorff-distance).

Exercise 17.5 For $i \in \{1, 2\}$, let $e_i := \{x_i, y_i\}$, where $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$ satisfy $x_1 \neq x_2$ and $y_1 \neq y_2$. Show that the Hausdorff distance between e_1 and e_2 is given by

$$\delta(e_1, e_2) = (\|x_1 - y_1\| \vee \|x_2 - y_2\|) \wedge (\|x_1 - y_2\| \vee \|x_2 - y_1\|).$$

Show that $\mathbb{R}^{[2d]}$ is not closed in $\mathcal{C}^{(d)}$. Use Exercise 17.4 to show that $\mathbb{R}^{[2d]}$ is a measurable subset of $\mathcal{C}^{(d)}$. Show finally that a set $C \subset \mathbb{R}^{[2d]}$ is bounded if and only if $U(C)$ is bounded, where $U(C)$ is the union of all $e \in C$.

Exercise 17.6 Prove that the mapping $(x, K) \mapsto K + x$ from $\mathbb{R}^d \times \mathcal{C}^{(d)}$ to $\mathcal{C}^{(d)}$ is continuous. Prove also that the mapping $(K, L) \mapsto K \oplus L$ is continuous on $\mathcal{C}^{(d)} \times \mathcal{C}^{(d)}$. Why is this a more general statement?

Exercise 17.7 Prove that the mapping $(x, K) \mapsto \mathbf{1}_K(x)$ from $\mathbb{R}^d \times \mathcal{C}^{(d)}$ to $\mathbb{R}$ is measurable. (Hint: Show that $\{(x, K) \in \mathbb{R}^d \times \mathcal{C}^{(d)} : x \in K\}$ is closed.)

Exercise 17.8 Consider a Boolean model whose grain distribution $\mathbb{Q}$ satisfies (17.11) as well as the equation $\mathbb{Q}(\{K \in \mathcal{C}^{(d)} : \lambda_d(\partial K) = 0\}) = 1$, where ∂K is the boundary of a set K. Show that

$$\lim_{x \to 0} \mathbb{P}(0 \in Z, x \in Z) = \mathbb{P}(0 \in Z).$$

Exercise 17.9 Let ν be a measure on $\mathcal{C}^{(d)}$ satisfying (17.14). Show that ν is locally finite. Show also that the measure

$$\nu := \int_0^\infty \mathbf{1}\{B(0, r) \in \cdot\}\, dr$$

is locally finite but does not satisfy (17.14).

Exercise 17.10 Let Z be a Boolean model and let $W \in \mathcal{B}^d$. Show that

$$\mathbb{V}\mathrm{ar}[\lambda_d(Z \cap W)] = (1-p)^2 \int \lambda_d(W \cap (W+x))(e^{\gamma\beta_d(x)} - 1)\, dx,$$

where the function β_d is given by (17.13).

Exercise 17.11 Consider a Boolean model whose grain distribution $\mathbb{Q}$ satisfies

$$\int \lambda_d(K)^2\, \mathbb{Q}(dK) < \infty. \tag{17.20}$$

Prove that

$$\int (e^{\gamma\beta_d(x)} - 1)\, dx < \infty.$$

(Hint: Use the inequality $e^t - 1 \le te^t$, valid for all $t \ge 0$.)

Exercise 17.12 Let $W \subset \mathbb{R}^d$ be a Borel set with $0 < \lambda_d(W) < \infty$ such that the boundary of W has Lebesgue measure 0. Show that

$$\lim_{r\to\infty} \lambda_d(rW)^{-1}\lambda_d(rW \cap (rW + x)) = 1$$

for all $x \in \mathbb{R}^d$, where $rW := \{rx : x \in W\}$. (Hint: Decompose W into its interior and $W \cap \partial W$; see Section A.2.)

Exercise 17.13 Under the assumptions of Exercise 17.11, let $W \subset \mathbb{R}^d$ be as in Exercise 17.12. Use Exercises 17.10 and 17.12 to prove that

$$\lim_{r\to\infty} \lambda_d(rW)^{-1}\, \mathbb{V}\mathrm{ar}[\lambda_d(Z \cap rW)] = (1-p)^2 \int (e^{\gamma\beta_d(x)} - 1)\, dx.$$

Exercise 17.14 (Poisson particle processes) Let the Poisson process ξ be as in Definition 17.1 and assume that (17.20) holds. Show that

$$\xi' := \sum_{n=1}^{\infty} \delta_{Z_n + X_n}$$

is a Poisson process on $\mathcal{C}^{(d)}$ with intensity measure

$$\nu := \iint \mathbf{1}\{K + x \in \cdot\}\, dx\, \mathbb{Q}(dK). \tag{17.21}$$

Show also that ν satisfies (17.14).

Assume, conversely, that ξ' is a Poisson process with an intensity measure ν satisfying (17.14). Show that the point process ξ^* defined by

$$\xi^*(C) := \int \mathbf{1}\{(c(K), K - c(K)) \in C\}\, \xi'(dK), \quad C \in \mathcal{B}(\mathbb{R}^d \times \mathcal{C}^{(d)}),$$

is a Poisson process whose intensity measure equals $\gamma^* \lambda_d \otimes \mathbb{Q}^*$ for some $\gamma^* > 0$ and a probability measure $\mathbb{Q}^*$ satisfying (17.20).

18

Fock space representation

A Poisson functional F is a measurable function of a Poisson process η. The difference operator $D_x F$ is the increment of F, when adding an extra point x to η. It can be iterated to yield difference operators of higher orders. If F is square integrable, then the infinite sequence of expected difference operators is an element of a direct sum of Hilbert spaces, namely the Fock space associated with the intensity measure of η. The second moment of F coincides with the squared norm of this Fock space representation. A first consequence is the Poincaré inequality, bounding the variance of F in terms of the difference operator. The Fock space representation plays a crucial role in the final three chapters of this book.

18.1 Difference operators

In this chapter we consider a Poisson process η on arbitrary measurable space $(\mathbb{X}, \mathcal{X})$ with σ-finite intensity measure λ and distribution $\mathbb{P}_\eta$ on $\mathbf{N} := \mathbf{N}(\mathbb{X})$. Let $f \in \mathbb{R}(\mathbf{N})$. For $x \in \mathbb{X}$ the function $D_x f \in \mathbb{R}(\mathbf{N})$ is defined by

$$D_x f(\mu) := f(\mu + \delta_x) - f(\mu), \quad \mu \in \mathbf{N}. \tag{18.1}$$

Iterating this definition, we define $D^n_{x_1,\dots,x_n} f \in \mathbb{R}(\mathbf{N})$ for any $n \geq 2$ and $(x_1, \dots, x_n) \in \mathbb{X}^n$ inductively by

$$D^n_{x_1,\dots,x_n} f := D^1_{x_1} D^{n-1}_{x_2,\dots,x_n} f, \tag{18.2}$$

where $D^1 := D$ and $D^0 f = f$. Note that

$$D^n_{x_1,\dots,x_n} f(\mu) = \sum_{J \subset \{1,2,\dots,n\}} (-1)^{n-|J|} f\Big(\mu + \sum_{j \in J} \delta_{x_j}\Big), \tag{18.3}$$

where $|J|$ denotes the number of elements of J. This shows that $D^n_{x_1,\dots,x_n} f$ is symmetric in $x_1, \dots, x_n$ and that $(\mu, x_1, \dots, x_n) \mapsto D^n_{x_1,\dots,x_n} f(\mu)$ is measurable.

Example 18.1 Assume that $\mathbb{X} = C^{(d)}$ is the space of all non-empty compact subsets of $\mathbb{R}^d$, as in Definition 17.1. For $\mu \in \mathbf{N}$ let

$$Z(\mu) := \bigcup_{K \in \mu} K, \tag{18.4}$$

whenever μ is locally finite (with respect to the Hausdorff distance); otherwise let $Z(\mu) := \emptyset$. Let ν be a finite measure on $\mathbb{R}^d$ and define $f : \mathbf{N} \to \mathbb{R}_+$ by

$$f(\mu) := \nu(Z(\mu)), \tag{18.5}$$

By Proposition 6.2, we have for all $x \in \mathbb{R}^d$ and all locally finite $\mu \in \mathbf{N}$ that

$$1 - \mathbf{1}_{Z(\mu)}(x) = \prod_{n=1}^{\mu(C^{(d)})} \mathbf{1}\{x \notin \pi_n(\mu)\}.$$

Therefore Exercise 17.7 implies that $(x, \mu) \mapsto \mathbf{1}_{Z(\mu)}(x)$ is measurable on $\mathbb{R}^d \times \mathbf{N}$. In particular, f is a measurable mapping. For any locally finite $\mu \in \mathbf{N}$ and any $K \in C^{(d)}$, we have

$$f(\mu + \delta_K) = \nu(Z(\mu) \cup K) = \nu(Z(\mu)) + \nu(K) - \nu(Z(\mu) \cap K),$$

that is

$$D_K f(\mu) = \nu(K) - \nu(Z(\mu) \cap K) = \nu(K \cap Z(\mu)^c).$$

This identity can be generalized to

$$D^n_{K_1,\dots,K_n} f(\mu) = (-1)^{n+1} \nu(K_1 \cap \cdots \cap K_n \cap Z(\mu)^c) \quad \mu \in \mathbf{N},$$

for all $n \in \mathbb{N}$ and $K_1, \dots, K_n \in C^{(d)}$.

The next lemma yields further insight into the difference operators. Recall from Chapter 12 that for $h \in \mathbb{R}(\mathbb{X})$ and $n \in \mathbb{N}$ the function $h^{\otimes n} \in \mathbb{R}(\mathbb{X}^n)$ is defined by

$$h^{\otimes n}(x_1, \dots, x_n) := \prod_{i=1}^{n} h(x_i). \tag{18.6}$$

Lemma 18.2 *Let $\upsilon \in \mathbb{R}_+(\mathbb{X})$ and define $f \in \mathbb{R}_+(\mathbf{N})$ by $f(\mu) = \exp[-\mu(\upsilon)]$, $\mu \in \mathbf{N}$. Let $n \in \mathbb{N}$. Then*

$$D^n_{x_1,\dots,x_n} f(\mu) = \exp[-\mu(\upsilon)](e^{-\upsilon} - 1)^{\otimes n}(x_1, \dots, x_n), \quad x_1, \dots, x_n \in \mathbb{X}. \tag{18.7}$$

Proof For each $\mu \in \mathbf{N}$ and $x \in \mathbb{X}$ we have

$$f(\mu + \delta_x) = \exp\Big[- \int v(y)\,(\mu + \delta_x)(dy) \Big] = \exp[-\mu(v)]\exp[-v(x)],$$

so that

$$D_x f(\mu) = \exp[-\mu(v)](\exp[-v(x)] - 1).$$

Iterating this identity yields for all $n \in \mathbb{N}$ and all $x_1, \dots, x_n \in \mathbb{X}$ that

$$D^n_{x_1,\dots,x_n} f(\mu) = \exp[-\mu(v)] \prod_{i=1}^{n} (\exp[-v(x_i)] - 1)$$

and hence the assertion. □

18.2 Fock space representation

To formulate the main result of this chapter we need to introduce some notation. For $n \in \mathbb{N}$ and $f \in \mathbb{R}(\mathbf{N})$ we define the symmetric measurable function $T_n f \colon \mathbb{X}^n \to \mathbb{R}$ by

$$T_n f(x_1, \dots, x_n) := \mathbb{E}[D^n_{x_1,\dots,x_n} f(\eta)], \tag{18.8}$$

and set $T_0 f := \mathbb{E}[f(\eta)]$, whenever these expectations are finite. Otherwise we set $T_n f(x_1, \dots, x_n) := 0$.

The scalar product of $f, g \in L^2(\lambda^n)$ is denoted by

$$\langle f, g \rangle_n := \int f g \, d\lambda^n.$$

Denote by $\| \cdot \|_n := \langle \cdot, \cdot \rangle_n^{1/2}$ the associated norm. For $n \in \mathbb{N}$ let $\mathbf{H}_n$ be the space of symmetric functions in $L^2(\lambda^n)$, and let $\mathbf{H}_0 := \mathbb{R}$. In this chapter we prove that the mapping $f \mapsto (T_n(f))_{n \geq 0}$ is an isometry from $L^2(\mathbb{P}_\eta)$ to the *Fock space* $\mathbf{H}$ given by the direct sum of the spaces $\mathbf{H}_n$, $n \geq 0$, with L^2 norms scaled by $1/\sqrt{n!}$.

Theorem 18.3 (Fock Space Representation) *Let $f, g \in L^2(\mathbb{P}_\eta)$. Then*

$$\mathbb{E}[f(\eta)g(\eta)] = (\mathbb{E}[f(\eta)])(\mathbb{E}[g(\eta)]) + \sum_{n=1}^{\infty} \frac{1}{n!} \langle T_n f, T_n g \rangle_n. \tag{18.9}$$

In particular,

$$\mathbb{E}[f(\eta)^2] = (\mathbb{E}[f(\eta)])^2 + \sum_{n=1}^{\infty} \frac{1}{n!} \|T_n f\|_n^2. \tag{18.10}$$

For any $f \in L^2(\mathbb{P}_\eta)$ we define $Tf := (T_n f)_{n\geq 0}$, where $T_n f$ is given at (18.8). Theorem 18.3 asserts that $Tf \in \mathbf{H}$ for $f \in L^2(\mathbb{P}_\eta)$ and

$$\mathbb{E}[f(\eta)g(\eta)] = \langle Tf, Tg\rangle_{\mathbf{H}}, \quad f, g \in L^2(\mathbb{P}_\eta). \tag{18.11}$$

We prove the theorem in stages.

Let $\mathcal{X}_0$ be the system of all measurable $B \in \mathcal{X}$ having $\lambda(B) < \infty$. Let $\mathbb{R}_0(\mathbb{X})$ be the space of all bounded functions $v \in \mathbb{R}_+(\mathbb{X})$ vanishing outside some $B \in \mathcal{X}_0$ (allowed to depend on v). Let $\mathbf{G}$ denote the space of all (bounded and measurable) functions $g : \mathbf{N} \to \mathbb{R}$ of the form

$$g(\mu) = a_1 e^{-\mu(v_1)} + \cdots + a_n e^{-\mu(v_n)}, \tag{18.12}$$

where $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{R}$ and $v_1, \ldots, v_n \in \mathbb{R}_0(\mathbb{X})$.

Lemma 18.4 *Equation* (18.11) *holds for* $f, g \in \mathbf{G}$.

Proof By linearity it suffices to consider functions f and g of the form

$$f(\mu) = \exp[-\mu(v)], \quad g(\mu) = \exp[-\mu(w)]$$

for $v, w \in \mathbb{R}_0(\mathbb{X})$. From (18.7) and Theorem 3.9 we obtain that

$$T_n f = \exp[-\lambda(1 - e^{-v})](e^{-v} - 1)^{\otimes n}. \tag{18.13}$$

Since $v \in \mathbb{R}_0(\mathbb{X})$ it follows that $T_n f \in \mathbf{H}_n$, $n \geq 0$. Using Theorem 3.9 again, we obtain that

$$\mathbb{E}[f(\eta)g(\eta)] = \exp[-\lambda(1 - e^{-(v+w)})]. \tag{18.14}$$

On the other hand we have from (18.13) (putting $\lambda^0(c) := c$ for all $c \in \mathbb{R}$) that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{n!}\langle T_n f, T_n g\rangle_n \\
&\quad = \exp[-\lambda(1 - e^{-v})] \exp[-\lambda(1 - e^{-w})] \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n(((e^{-v} - 1)(e^{-w} - 1))^{\otimes n}) \\
&\quad = \exp[-\lambda(2 - e^{-v} - e^{-w})] \exp[\lambda((e^{-v} - 1)(e^{-w} - 1))].
\end{aligned}$$

This equals the right-hand side of (18.14). □

To extend (18.11) to general $f, g \in L^2(\mathbb{P}_\eta)$ we need two lemmas.

Lemma 18.5 The set $\mathbf{G}$ is dense in $L^2(\mathbb{P}_\eta)$.

Proof Let $\mathbf{W}$ be the space of all bounded measurable $g : \mathbf{N} \to \mathbb{R}$ that can be approximated in $L^2(\mathbb{P}_\eta)$ by functions in $\mathbf{G}$. This space is closed under monotone uniformly bounded convergence.Also it contains the constant functions. The space $\mathbf{G}$ is closed under multiplication. Denote by $\mathcal{N}'$ the smallest σ-field on $\mathbf{N}$ such that $\mu \mapsto h(\mu)$ is measurable for all $h \in \mathbf{G}$. A functional version of the monotone class theorem (Theorem A.3) implies that $\mathbf{W}$ contains any bounded $\mathcal{N}'$-measurable g. On the other hand we have for any $C \in \mathcal{X}_0$ that

$$\mu(C) = \lim_{t \to 0+} t^{-1}(1 - e^{-t\mu(C)}), \quad \mu \in \mathbf{N},$$

such that $\mu \mapsto \mu(C)$ is $\mathcal{N}'$-measurable. Since λ is σ-finite, for any $C \in \mathcal{X}$ there is a monotone sequence $C_k \in \mathcal{X}_0$, $k \in \mathbb{N}$, with union C, so that $\mu \mapsto \mu(C)$ is $\mathcal{N}'$-measurable. Hence $\mathcal{N}' = \mathcal{N}$ and it follows that $\mathbf{W}$ contains all bounded measurable functions. Hence $\mathbf{W}$ is dense in $L^2(\mathbb{P}_\eta)$ and the proof is complete. □

Lemma 18.6 Suppose that $f, f^1, f^2, \ldots \in L^2(\mathbb{P}_\eta)$ satisfy $f^k \to f$ in $L^2(\mathbb{P}_\eta)$ as $k \to \infty$, and that $h\colon \mathbf{N} \to [0, 1]$ is measurable. Let $n \in \mathbb{N}$ and $C \in \mathcal{X}_0$. Then

$$\lim_{k\to\infty} \int_{C^n} \mathbb{E}[h(\eta)|D^n_{x_1,\ldots,x_n} f(\eta) - D^n_{x_1,\ldots,x_n} f^k(\eta)|]]\, \lambda^n(d(x_1, \ldots, x_n)) = 0. \tag{18.15}$$

Proof By (18.3), it suffices to prove that

$$\lim_{n\to\infty} \int_{C^n} \mathbb{E}\Big[h(\eta)\Big|f\Big(\eta + \sum_{i=1}^m \delta_{x_i}\Big) - f^k\Big(\eta + \sum_{i=1}^m \delta_{x_i}\Big)\Big|\Big]\, \lambda^n(d(x_1, \ldots, x_n)) = 0 \tag{18.16}$$

for all $m \in \{0, \ldots, n\}$. For $m = 0$ this is obvious. Assume $m \in \{1, \ldots, n\}$. By the multivariate Mecke equation (see (4.11)), the integral in (18.16) equals

$$\begin{aligned}
&\lambda(C)^{n-m}\mathbb{E}\Big[\int_{C^m} h(\eta)\Big|f\Big(\eta + \sum_{i=1}^m \delta_{x_i}\Big) - f^k\Big(\eta + \sum_{i=1}^m \delta_{x_i}\Big)\Big|\, \lambda^m(d(x_1, \ldots, x_m))\Big] \\
&\quad = \lambda(C)^{n-m}\mathbb{E}\Big[\int_{C^m} h\Big(\eta - \sum_{i=1}^n \delta_{x_i}\Big)|f(\eta) - f^k(\eta)|\, \eta^{(m)}(d(x_1, \ldots, x_m))\Big] \\
&\quad \le \lambda(C)^{n-m}\mathbb{E}[|f(\eta) - f^k(\eta)|\eta^{(m)}(C^m)].
\end{aligned}$$

By the Cauchy-Schwarz inequality the last expression is bounded above by

$$\lambda(C)^{n-m}(\mathbb{E}[(f(\eta) - f^k(\eta))^2])^{1/2}(\mathbb{E}[(\eta^{(m)}(C^m))^2])^{1/2}.$$

Since the Poisson distribution has moments of all orders, we obtain (18.16) and hence the lemma. □

Proof of Theorem 18.3 By linearity and the polarization identity

$$4\langle f, g\rangle_{\mathbf{H}} = \langle f+g, f+g\rangle_{\mathbf{H}} - \langle f-g, f-g\rangle_{\mathbf{H}}$$

it suffices to prove (18.10). By Lemma 18.5 there exist $f^k \in \mathbf{G}$, defined for $k \in \mathbb{N}$, satisfying $f^k \to f$ in $L^2(\mathbb{P}_\eta)$ as $k \to \infty$. By Lemma 18.4, Tf^k, $k \in \mathbb{N}$, is a Cauchy sequence in $\mathbf{H}$. Let $\tilde{f} = (\tilde{f}_n) \in \mathbf{H}$ be the limit, meaning that

$$\lim_{k\to\infty} \sum_{n=0}^{\infty} \frac{1}{n!} \|T_n f^k - \tilde{f}_n\|_n^2 = 0. \tag{18.17}$$

Taking the limit in the identity $\mathbb{E}[f^k(\eta)^2] = \langle Tf^k, Tf^k\rangle_{\mathbf{H}}$ yields $\mathbb{E}[f(\eta)^2] = \langle \tilde{f}, \tilde{f}\rangle_{\mathbf{H}}$. Equation (18.17) implies that $\tilde{f}_0 = \mathbb{E}[f(\eta)] = T_0 f$. It remains to show that for any $n \geq 1$,

$$\tilde{f}_n = T_n f, \quad \lambda^n\text{-a.e.} \tag{18.18}$$

Let $C \in \mathcal{X}_0$ and let $B := C^n$. Recall from (5.5) that $(\lambda^n)_B$ is the restriction of the measure λ^n to B. By (18.17) $T_n f^k$ converges in $L^2((\lambda^n)_B)$ (and hence in $L^1((\lambda^n)_B)$) to $\tilde{f}_n$, while by the definition (18.8) of T_n, and the case $h \equiv 1$ of (18.15), $T_n f^k$ converges in $L^1((\lambda^n)_B)$ to $T_n f$. Hence these $L^1(\mathbb{P})$ limits must be the same almost everywhere, so that $\tilde{f}_n = T_n f$ λ^n-a.e. on B. Since λ is assumed to be σ-finite, this implies (18.18) and hence the theorem. □

18.3 Poincaré inequality

As a first consequence of the Fock space representation we derive an upper bound for the variance of a Poisson functional in terms of the expected squared difference operator.

Theorem 18.7 (Poincaré Inequality) *For any $f \in L^2(\mathbb{P}_\eta)$,*

$$\mathbb{V}\mathrm{ar}[f(\eta)] \leq \mathbb{E}\Big[\int (f(\eta+\delta_x) - f(\eta))^2\, \lambda(dx)\Big]. \tag{18.19}$$

Proof We can assume that the right-hand side of (18.19) is finite. In particular $D_x f \in L^2(\mathbb{P}_\eta)$ for λ-a.e. x. By (18.10),

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}[f(\eta)] &= \int (\mathbb{E}[D_x f(\eta)])^2 \,\lambda(dx) \\
&\quad + \sum_{n=2}^{\infty} \frac{1}{n!} \iint (\mathbb{E}[D^{n-1}_{x_1,\dots,x_{n-1}} D_x f(\eta)])^2 \,\lambda^{n-1}(d(x_1,\dots,x_{n-1}))\,\lambda(dx) \\
&\le \int (\mathbb{E}[D_x f(\eta)])^2 \,\lambda(dx) \\
&\quad + \sum_{m=1}^{\infty} \frac{1}{m!} \iint (\mathbb{E}[D^{m}_{x_1,\dots,x_m} D_x f(\eta)])^2 \,\lambda^{m}(d(x_1,\dots,x_m))\,\lambda(dx).
\end{aligned}
$$

Applying (18.9) to $D_x f$ yields the result. □

The Poincaré inequality is sharp. Indeed, let $F := \eta(B)$ for some $B \in \mathcal{X}$ with $\lambda(B) < \infty$. Then $D_x(F) = \mathbf{1}\{x \in B\}$ for all $x \in \mathbb{X}$ and the right-hand side of (18.19) equals $\lambda(B)$, the variance of F.

18.4 Exercises

Exercise 18.1 Let $v \in \mathbb{R}(\mathbb{X})$ and define $f \in \mathbb{R}_+(\mathbf{N})$ by $f(\mu) := \int v\,d\mu$ if $\int |v|\,d\mu < \infty$ and by $f(\mu) := 0$ otherwise. Show for all $x \in \mathbb{X}$ and all $\mu \in \mathbf{N}$ with $\mu(|v|) < \infty$ that $D_x f(\mu) = f(x)$.

Exercise 18.2 Let $f, g \in \mathbb{R}(\mathbf{N})$ and $x \in \mathbb{X}$. Show that

$$D_x(fg) = (D_x f)g + f(D_x g) + (D_x f)(D_x g).$$

Exercise 18.3 Let $f, \tilde{f} \colon \mathbf{N} \to \mathbb{R}$ be measurable functions such that $f(\eta) = \tilde{f}(\eta)$ $\mathbb{P}$-a.s. Show that for all $n \in \mathbb{N}$,

$$D^n_{x_1,\dots,x_n} f(\eta) = D^n_{x_1,\dots,x_n} \tilde{f}(\eta), \quad \lambda^n\text{-a.e. } (x_1,\dots,x_n),\ \mathbb{P}\text{-a.s.}$$

(Hint: Use the multivariate Mecke equation (4.11).)

Exercise 18.4 Let $f \in \mathbb{R}(\mathbf{N})$ and $r \ge 0$. Define $f_r \in \mathbb{R}(\mathbf{N})$ by

$$f_r := (f \wedge r) \vee (-r) = \mathbf{1}\{f > r\}r + \mathbf{1}\{-r \le f \le r\}f - \mathbf{1}\{f < -r\}r.$$

Show for all $x \in \mathbb{X}$ and $\mu \in \mathbf{N}(\mathbb{X})$ that $|D_x f_r| \le |D_x f|$ for all $x \in \mathbb{X}$.

Exercise 18.5 (L^1-version of the Poincaré inequality) Let $f \in L^1(\mathbb{P}_\eta)$. Show that

$$\mathbb{E}[f(\eta)^2] \le \mathbb{E}[f(\eta)] + \mathbb{E}\Big[\int (f(\eta + \delta_x) - f(\eta))^2 \,\lambda(dx)\Big],$$

where the second term on the right might be infinite. (Hint: Apply Theorem 18.7 to the functions f_r defined in Exercise 18.4.)

Exercise 18.6 Let $\mathbb{X} = C^{(d)}$ as in Example 18.1 and define the function f by (18.5). Let the measure λ be given by the right-hand side of (17.21), where $\mathbb{Q}$ is assumed to satisfy (17.11) and

$$\int (\nu(K+z))^2 \, dz \, \mathbb{Q}(dK) < \infty. \tag{18.20}$$

Show that $f \in L^2(\mathbb{P}_\eta)$ and, moreover, that

$$T_n f(K_1, \dots, K_n) = (-1)^{n+1}(1-p)\nu(K_1 \cap \dots \cap K_n),$$

where $p = \mathbb{P}(0 \in Z(\eta))$ is the volume fraction of the Boolean model $Z(\eta)$. Also show that (18.20) is implied by (17.20), whenever $\nu(dx) = \mathbf{1}_W(x)dx$ for some $W \in \mathcal{B}^d$ with $\lambda_d(W) < \infty$.

Exercise 18.7 Let $\mathbb{X} = C^{(d)}$ and let λ be as in Exercise 18.7. Let ν_1, ν_2 be two finite measures on $\mathbb{R}^d$ satisfying (18.20) and define, for $i \in \{1, 2\}$, $f_i(\mu) := \nu_i(Z(\mu))$, $\mu \in \mathbf{N}$; see (18.5). Use Fubini's theorem and Theorem 17.6 to prove that

$$\mathbb{C}\mathrm{ov}(f_1(\eta), f_2(\eta)) = (1-p)^2 \iint (e^{\beta_d(x_1 - x_2)} - 1) \, \nu_1(dx_1) \, \nu_2(dx_2),$$

where β_d is given by (17.13). Confirm this result using Theorem 18.3 and Exercise 18.6.

Exercise 18.8 Let $v \in L^1(\lambda) \cap L^2(\lambda)$ and define the function $f \in \mathbb{R}(\mathbf{N})$ as in Exercise 18.1. Show that (18.19) is an equality in this case.

Exercise 18.9 Let $f \in L^2(\mathbb{P}_\eta)$. Show that

$$\mathbb{V}\mathrm{ar}[f(\eta)] \geq \int (\mathbb{E}[f(\eta + \delta_x) - f(\eta)])^2 \, \lambda(dx).$$

Exercise 18.10 Let $f \in \mathbb{R}_+(\mathbf{N})$ such that $\mathbb{E}[f(\eta)^{1+\varepsilon}] < \infty$ for some $\varepsilon > 0$. Show for $k \in \mathbb{N}$ that $\mathbb{E}[f(\eta + \delta_{x_1} + \dots + \delta_{x_k})] < \infty$ for λ^k-a.e. $(x_1, \dots, x_k) \in \mathbb{X}^k$. (Hint: Use either Exercise 1.8 or the multivariate Mecke equation together with Hölder's inequality.)

Exercise 18.11 Let $f(\mu) := e^{-\mu(v)}$, $\mu \in \mathbf{N}(\mathbb{X})$, where $v : \mathbb{X} \to [0, \infty)$ is a measurable function vanishing outside a set $B \in \mathcal{X}$ with $\lambda(B) < \infty$. Assume that the Poisson process η is proper. Use equations (12.13) and (18.13) to show that $\mathbb{P}$-a.s.

$$\sum_{n=0}^{\infty} \frac{1}{n!} I_n(T_n f) = \sum_{k=0}^{\eta(B)} \frac{1}{k!} \eta^{(k)}((e^{-v} - 1)^{\otimes k}).$$

Identify the right-hand side of this equation to show that

$$f(\eta) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(T_n f), \tag{18.21}$$

where the series converges $\mathbb{P}$-a.s. and in $L^2(\mathbb{P})$.

Exercise 18.12 Let $f \in L^2(\mathbb{P}_\eta)$. Use Theorem 18.3 and (12.20) to show that the infinite series

$$S_f := \sum_{n=0}^{\infty} \frac{1}{n!} I_n(T_n f) \tag{18.22}$$

converges in $L^2(\mathbb{P})$.

Exercise 18.13 (Chaos expansion) Let $f \in L^2(\mathbb{P}_\eta)$. Show that

$$f(\eta) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(T_n f), \tag{18.23}$$

where the series converges in $L^2(\mathbb{P})$. (Hint: Show first that (18.23) holds for $h \in \mathbf{G}$ and that

$$\mathbb{E}[(S_f - h(\eta))^2] = \mathbb{E}[(f(\eta) - h(\eta))^2],$$

where S_f is given by (18.22). Use Lemma 18.5 to conclude the assertion.)

19

Perturbation analysis

The expectation of a Poisson functional F can be viewed as a function of the intensity measure λ. Under first moment assumptions on F the (suitable defined) directional derivatives of this function can be expressed in terms of the difference operator of F. This can be applied to geometric functionals of a Boolean model Z with convex grains, driven by a Poisson process with intensity $t \geq 0$. If F is an additive functional of the restriction of Z to a convex observation window (viewed as a function of t), then its expectation satisfies a linear differential equation. An interesting example is an explicit formula for the expected Euler characteristic of a planar Boolean model with an isotropic grain distribution.

19.1 A perturbation formula

In this chapter we consider an arbitrary measurable space $(\mathbb{X}, \mathcal{X})$ and a Poisson process η_λ on $\mathbb{X}$ with s-finite intensity measure λ. We study the effect of a *perturbation* of the intensity measure λ on the expectation of a fixed functional of η.

To explain the idea we take a finite measure ν on $\mathbb{X}$, along with a bounded measurable function $f\colon \mathbf{N}(\mathbb{X}) \to \mathbb{R}$ and study the behaviour of $\mathbb{E}[f(\eta_{\lambda+t\nu})]$ as $t \downarrow 0$. Here and later, given any s-finite measure ρ on X, we let η_ρ denote a Poisson process with this intensity measure. By the superposition theorem (Theorem 3.3) we can write $\mathbb{E}[f(\eta_{\lambda+t\nu})] = \mathbb{E}[f(\eta_\lambda + \eta'_{t\nu})]$, where $\eta'_{t\nu}$ is a Poisson process with intensity measure $t\nu$, independent of η_λ. Then

$$\begin{aligned}
&\mathbb{E}[f(\eta_{\lambda+t\nu})]\\
&\quad= \mathbb{E}[f(\eta_\lambda)]\mathbb{P}(\eta'_{t\nu}(\mathbb{X}) = 0) + \mathbb{E}[f(\eta_\lambda + \eta'_{t\nu}) \mid \eta'_{t\nu}(\mathbb{X}) = 1]\mathbb{P}(\eta'_{t\nu}(\mathbb{X}) = 1)\\
&\qquad + \mathbb{E}[f(\eta_\lambda + \eta'_{t\nu}) \mid \eta'_{t\nu}(\mathbb{X}) \geq 2]\mathbb{P}(\eta'_{t\nu}(\mathbb{X}) \geq 2).
\end{aligned}$$

The measure ν can be written as $\nu = \gamma\,\mathbb{Q}$, where $\gamma \in \mathbb{R}_+$ and $\mathbb{Q}$ is a probability measure on $\mathbb{X}$. Using Proposition 3.5 (and the independence of η_λ

and η'_{tv}) to rewrite the second term in the right-hand side above, we obtain that

$$\mathbb{E}[f(\eta_{\lambda+tv})] = e^{-t\gamma}\,\mathbb{E}[f(\eta_\lambda)] + \gamma t e^{-t\gamma} \int \mathbb{E}[f(\eta_\lambda + \delta_x)]\,\mathbb{Q}(dx) + R_t, \quad (19.1)$$

where

$$R_t := (1 - e^{-t\gamma} - \gamma t e^{-t\gamma})\mathbb{E}[f(\eta_\lambda + \eta'_{tv}) \mid \eta'_{tv}(\mathbb{X}) \geq 2].$$

Since f is bounded, $|R_t| \leq ct^2$ for some $c > 0$ and it follows that

$$\lim_{t\downarrow 0} t^{-1}(\mathbb{E}[f(\eta_{\lambda+tv})] - \mathbb{E}[f(\eta_\lambda)]) = -\gamma\,\mathbb{E}[f(\eta_\lambda)] + \gamma \int \mathbb{E}[f(\eta_\lambda + \delta_x)]\,\mathbb{Q}(dx).$$

Therefore, the right-hand derivative of $\mathbb{E}[f(\eta_{\lambda+tv})]$ at $t = 0$ is given by

$$\frac{d^+}{dt}\mathbb{E}[f(\eta_{\lambda+tv})]\bigg|_{t=0} = \int \mathbb{E}[D_x f(\eta_\lambda)]\,\nu(dx). \quad (19.2)$$

The following results elaborate on (19.2). Recall from Theorem A.8 the Hahn-Jordan decomposition $\nu = \nu_+ - \nu_-$ of a finite signed measure ν on $\mathbb{X}$. We also recall from the Appendix that the integral $\int f\,d\nu$ is defined as $\int f\,d\nu_+ - \int f\,d\nu_-$, whenever this makes sense. If ν is a finite signed measure, then we denote by $I(\lambda, \nu)$ the set of all $t \in \mathbb{R}$ such that $\lambda + t\nu$ is a measure. Then $0 \in I(\lambda, \nu)$ and it is easy to see that $I(\lambda, \nu)$ is a (possibly infinite) closed interval. We abbreviate $\mathbf{N} := \mathbf{N}(\mathbb{X})$.

Theorem 19.1 (Perturbation Formula) *Let ν be a finite signed measure on $\mathbb{X}$ and suppose that $f \in \mathbb{R}(\mathbf{N})$ is bounded. Then*

$$\frac{d}{dt}\mathbb{E}[f(\eta_{\lambda+t\nu})] = \int \mathbb{E}[D_x f(\eta_{\lambda+t\nu})]\,\nu(dx), \quad t \in I(\lambda, \nu). \quad (19.3)$$

(For t in the boundary of $I(\lambda, \nu)$ this is a one-sided derivative.) Furthermore we have for all $t \in I(\lambda, \nu)$ that

$$\mathbb{E}[f(\eta_{\lambda+t\nu})] = \mathbb{E}[f(\eta_\lambda)] + \int_0^t \int_{\mathbb{X}} \mathbb{E}[D_x f(\eta_{\lambda+s\nu})]\,\nu(dx)\,ds, \quad (19.4)$$

where we use the convention $\int_0^t := -\int_t^0$ for $t < 0$.

Proof We first assume that ν is a measure. It is enough to prove (19.3) for $t = 0$ since then for general $t \in I(\lambda, \nu)$ we can apply this formula with λ replaced by $\lambda + t\nu$. Assume that $-s \in I(\lambda, \nu)$ for all sufficiently small $s > 0$. For such s we let $\eta'_{s\nu}$ be a Poisson process with intensity measure $s\nu$, independent of $\eta_{\lambda-s\nu}$. Then it follows exactly as at (19.1) that

$$\mathbb{E}[f(\eta_\lambda)] = e^{-s\gamma}\mathbb{E}[f(\eta_{\lambda-s\nu})] + \gamma s e^{-s\gamma} \int \mathbb{E}[f(\eta_{\lambda-s\nu} + \delta_x)]\,\mathbb{Q}(dx) + R_s,$$

where $|R_s| \le cs^2$ for some $c > 0$. Therefore

$$\begin{aligned} -s^{-1}(\mathbb{E}[f(\eta_{\lambda-s\nu})] - \mathbb{E}[f(\eta_\lambda)]) =& s^{-1}(e^{-s\gamma} - 1)\mathbb{E}[f(\eta_{\lambda-s\nu})] \\ &+ \gamma e^{-s\gamma} \int \mathbb{E}[f(\eta_{\lambda-s\nu} + \delta_x)]\,\mathbb{Q}(dx) + s^{-1}R_s. \end{aligned} \tag{19.5}$$

Since ν is a finite measure

$$\mathbb{P}(\eta_\lambda \neq \eta_{\lambda-s\nu}) = \mathbb{P}(\eta'_{s\nu} \neq 0) \to 0,$$

as $s \downarrow 0$. Since f is bounded it follows that $\mathbb{E}[f(\eta_{\lambda-s\nu})] \to \mathbb{E}[f(\eta_\lambda)]$ as $s \downarrow 0$. Similarly $\mathbb{E}[f(\eta_{\lambda-s\nu} + \delta_x)]$ tends to $\mathbb{E}[f(\eta_\lambda + \delta_x)]$ for all $x \in \mathbb{X}$. By dominated convergence, even the integral with respect to $\mathbb{Q}$ tends to 0. Therefore (19.5) implies that the left hand derivative of $\mathbb{E}[f(\eta_{\lambda+t\nu})]$ at $t = 0$ coincides with the right-hand side of (19.2). Hence (19.3) follows.

By dominated convergence the right-hand side of (19.3) is a continuous function of $t \in I$. Therefore (19.4) follows upon integration.

We now consider the case where $\nu = \nu_+ - \nu_-$ is a general finite signed measure. Suppose first that $a \in I(\lambda, \nu)$ for some $a > 0$. Then by (19.4) for $0 \le t \le a$ we have

$$\mathbb{E}[f(\eta_\lambda)] - \mathbb{E}[f(\eta_{\lambda-t\nu_-})] = \int_0^t \int_{\mathbb{X}} \mathbb{E}[D_x f(\eta_{\lambda+(u-t)\nu_-})]\,\nu_-(dx)\,du, \tag{19.6}$$

and

$$\mathbb{E}[f(\eta_{\lambda-t\nu_-+t\nu_+})] - \mathbb{E}[f(\eta_{\lambda-t\nu_-})] = \int_0^t \int_{\mathbb{X}} \mathbb{E}[D_x f(\eta_{\lambda-t\nu_-+u\nu_+})]\,\nu_+(dx)\,du. \tag{19.7}$$

For $s \ge 0$, let η_s^- be a Poisson process with intensity measure $s\nu_-$ independent of $\eta_{\lambda-s\nu}$. By the superposition theorem (Theorem 3.3) we can assume for all $s \ge 0$ that $\eta_\lambda = \eta_{\lambda-s\nu} + \eta_s^-$. Then it follows as before that

$$\mathbb{P}(\eta_\lambda \neq \eta_{\lambda-s\nu}) = \mathbb{P}(\eta_s^- \neq 0) \to 0,$$

as $s \downarrow 0$, since ν_- is a finite measure. Since also f is bounded we have $\mathbb{E}[D_x f(\eta_{\lambda-s\nu_-})] \to \mathbb{E}[D_x f(\eta_\lambda)]$ as $s \downarrow 0$, so the right-hand side of (19.6) is asymptotic to $t \int \mathbb{E}[D_x f(\eta_\lambda)]\,\nu_-(dx)$ as $t \downarrow 0$. Similarly $\mathbb{E}[f(\eta_{\lambda-t\nu_-+u\nu_+}) - f(\eta_{\lambda-t\nu_-})] \to 0$ as $t, u \downarrow 0$, so the right-hand side of (19.7) is asymptotic to $t \int \mathbb{E} D_x f(\eta_\lambda)\,\nu_+(dx)$ as $t \downarrow 0$. Then we can deduce (19.2) from (19.6) and (19.7).

If $\lambda - a\nu$ is a measure for some $a > 0$, then applying the same argument with $-\nu$ instead of ν gives the differentiability at $t = 0$ of $\mathbb{E}[f(\eta_{\lambda+t\nu})]$. For an arbitrary $t \in I(\lambda, \nu)$ we can apply this result to the measure $\lambda + t\nu$ (instead of λ) to obtain (19.3).

To prove (19.4) we need to show that the right-hand side of (19.3) is a continuous function of t. To this end, we take $s, t \in I(\lambda, \nu)$ such that $s < t$ and set $h := t - s$. We let $\eta'_{h\nu_+}$ and $\eta'_{h\nu_-}$ be independent Poisson processes with intensity measures $h\nu_+$ and $h\nu_-$ respectively, such that the pair $(\eta'_{h\nu^+}, \eta'_{h\nu_-})$ is independent of $\eta_{\lambda-(t+h)\nu_-+t\nu_+}$. We can then assume that

$$\eta_{\lambda+s\nu} = \eta_{\lambda-t\nu_-+s\nu_+} + \eta'_{h\nu_-}, \qquad \eta_{\lambda+t\nu} = \eta_{\lambda-t\nu_-+s\nu_+} + \eta'_{h\nu_+}$$

and it follows that

$$\mathbb{P}(\eta_{\lambda+s\nu} \neq \eta_{\lambda+t\nu}) \leq \mathbb{P}(\eta'_{h\nu_-} + \eta'_{h\nu_+} \neq 0) \to 0$$

as $h \downarrow 0$. Using bounded convergence we may conclude that the right-hand side of (19.3) is uniformly continuous. □

19.2 Power series representation

Next we derive a power series representation of $\mathbb{E}[f(\eta_{\lambda+t\nu})]$.

Theorem 19.2 *Suppose that the assumptions of Theorem 19.1 are satisfied. Then*

$$\mathbb{E}[f(\eta_{\lambda+t\nu})] = \sum_{n=0}^{\infty} \frac{t^n}{n!} \int \mathbb{E}[D^n_{x_1,\dots,x_n} f(\eta_\lambda)]\, \nu^n(d(x_1, \dots, x_n)) \tag{19.8}$$

for all $t \in I(\lambda, \nu)$, where the series converges absolutely and where for $n = 0$ the summand is interpreted as $\mathbb{E}[f(\eta_\lambda)]$.

Proof First suppose that $t \in I(\lambda, \nu)$ is positive. Iterating (19.4) gives

$$\begin{aligned}\mathbb{E}[f(\eta_{\lambda+t\nu})] = \mathbb{E}[f(\eta_\lambda)] + t \int_{\mathbb{X}} \mathbb{E}[D_{x_1} f(\eta_\lambda)]\, \nu(dx_1) \\ + \int_{\mathbb{X}^2} \int_0^t \int_0^{s_1} \mathbb{E}[D^2_{x_1,x_2} f(\eta_{\lambda+s_2\nu})]\, ds_1\, ds_2\, \nu^2(d(x_1, x_2)).\end{aligned}$$

More generally, we have for all $m \in \mathbb{N}$ that

$$\begin{aligned}\mathbb{E}[f(\eta_{\lambda+t\nu})] &= \sum_{n=0}^{m} \frac{t^n}{n!} \int_{\mathbb{X}^n} \mathbb{E}[D^n_{x_1,\dots,x_n} f(\eta_\lambda)]\, \nu^n(d(x_1, \dots, x_n)) \\ &+ \int_{\mathbb{X}^{m+1}} \int_{A_{t,m+1}} \mathbb{E}[D^{m+1}_{x_1,\dots,x_{m+1}} f(\eta_{\lambda+s_{m+1}\nu})]\, d(s_1, \dots, s_{m+1})\, \nu^{m+1}(d(x_1, \dots, x_{m+1})),\end{aligned}$$

where $A_{t,m+1} := \{(s_1, \dots, s_{m+1}) \in \mathbb{R}^{m+1} : 0 \le s_{m+1} \le \cdots \le s_1 \le t\}$. Since f is bounded, the absolute value of the last term is bounded by $c(2t)^{m+1}/(m+1)!$ for some $c > 0$ and (19.8) follows.

For general $t \in I(\lambda, \nu)$ we can apply the preceding result to the measure $t\nu$. □

The previous result required the function f to be bounded. For applications in stochastic geometry this assumption is too strong. The next results apply to more general functions. For a finite signed measure ν with Hahn-Jordan decomposition $\nu = \nu_+ - \nu_-$ we denote by $|\nu| = \nu_+ + \nu_-$ the *total variation measure* of ν.

Theorem 19.3 *Let ν be a finite signed measure on $\mathbb{X}$. Suppose that $f \in \mathbb{R}(\mathbf{N})$ and $t \in I(\lambda, \nu)$ satisfy $\mathbb{E}[|f(\eta_{\lambda+|t||\nu|})|] < \infty$. Then* (19.8) *holds, where all of the expectations in the sum exist and the series converges absolutely.*

Proof It suffices to treat the case $t = 1$, since then one could replace ν with $t\nu$. For all $k \in \mathbb{N}$ we define a bounded function $f_k = (f \wedge k) \vee (-k) \in \mathbb{R}(\mathbf{N})$ as in Exercise 18.4. By Theorem 19.2,

$$\mathbb{E}[f_k(\eta_{\lambda+\nu})] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathbb{E}[D^n f_k(\eta_\lambda)]\, (h_+ - h_-)^{\otimes n}\, d|\nu|^n, \tag{19.9}$$

where h_- (resp. h_+) is a Radon-Nikodým derivative of ν_- (resp. ν_+) with respect to $|\nu| = \nu_- + \nu_+$ (see Theorem A.9) and where we recall the definition (18.6) of $(h_+ - h_-)^{\otimes n}$. Since $\nu_- \le |\nu|$ and $\nu_+ \le |\nu|$ we have that $h_-(x) \le 1$ and $h_+(x) \le 1$ for $|\nu|$-a.e. x. Since ν_- and ν_+ are mutually singular we also have $h_-(x)h_+(x) = 0$ for $|\nu|$-a.e. x. Therefore,

$$|(h_+(x) - h_-(x))| = h_+(x) + h_-(x) \le 1, \quad |\nu|\text{-a.e. } x \in \mathbb{X}.$$

Now let $k \to \infty$ in (19.9). By Exercise 3.7 we have $\mathbb{E}[|f(\eta_\lambda)|] < \infty$. Dominated convergence shows that the left-hand side of (19.9) tends to $\mathbb{E}[f(\eta_{\lambda+\nu})]$. Also $D^n_{x_1,\dots,x_n} f_k(\eta_\lambda)$ tends to $D^n_{x_1,\dots,x_n} f(\eta_\lambda)$ for all $n \in \mathbb{N}_0$, all $x_1, \dots, x_n \in \mathbb{X}$ and everywhere on Ω. Furthermore

$$|D^n_{x_1,\dots,x_n} f_k(\eta_\lambda)| \le \sum_{J \subset \{1,\dots,n\}} \Big| f\Big(\eta_\lambda + \sum_{j \in J} \delta_{x_j}\Big)\Big|.$$

We shall show that

$$I := \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathbb{E}\Big[\sum_{J \subset \{1,\dots,n\}} \Big| f\Big(\eta_\lambda + \sum_{j \in J} \delta_{x_j}\Big)\Big|\Big]\, |\nu|^n(d(x_1, \dots, x_n)) < \infty;$$

we can then deduce (19.8) from (19.9) and dominated convergence.

By symmetry I equals

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} |\nu|(\mathbb{X})^{n-m} \int \mathbb{E}[|f(\eta_\lambda + \delta_{x_1} + \cdots + \delta_{x_m})|] \, |\nu|^m(d(x_1, \ldots, x_m)).$$

Swapping the order of summation yields that I equals

$$\exp[|\nu|(\mathbb{X})] \sum_{m=0}^{\infty} \frac{1}{m!} \int \mathbb{E}[|f(\eta_\lambda + \delta_{x_1} + \cdots + \delta_{x_m})|] \, |\nu|^m(d(x_1, \ldots, x_m))$$
$$= \exp[2|\nu|(\mathbb{X})]] \int \mathbb{E}[|f(\eta_\lambda + \eta'_{|\nu|})|],$$

where $\eta'_{|\nu|}$ is a Poisson process with intensity measure $|\nu|$, independent of η_λ, and where we have used Exercise 3.6 (or Proposition 3.5) to achieve the equality. Thanks to the superposition theorem (Theorem 3.3) we obtain that

$$I = \exp[2|\nu|(\mathbb{X})]]\mathbb{E}[|f(\eta_{\lambda+|\nu|})|],$$

which is finite by assumption. □

For $f \in \mathbb{R}(\mathbf{N})$ and ν a finite signed measure on $\mathbb{X}$ we let $I_f(\lambda, \nu)$ denote the set of all $t \in I(\lambda, \nu)$ such that $\mathbb{E}[|f(\eta_{\lambda+|t||\nu|})|] < \infty$. By Exercise 3.7 this is an interval. As a consequence of Theorem 19.3 we can generalize (19.3) to derivatives of higher order and to potentially unbounded functions f.

Theorem 19.4 *Let ν be a finite signed measure on $\mathbb{X}$ and let $f \in \mathbb{R}(\mathbf{N})$. Then the function $t \mapsto \mathbb{E}[f(\eta_{\lambda+t\nu})]$ is infinitely differentiable on the interior I^0 of $I_f(\lambda, \nu)$ and, for all $n \in \mathbb{N}$ and $t \in I^0$,*

$$\frac{d^n}{dt^n}\mathbb{E}[f(\eta_{\lambda+t\nu})] = \int \mathbb{E}[D^n_{x_1,\ldots,x_n} f(\eta_{\lambda+t\nu})] \, \nu^n(d(x_1, \ldots, x_n)). \tag{19.10}$$

Proof The asserted differentiability is a consequence of (19.8) and well-known properties of power series. The same is true for (19.10) in the case $t = 0$. For general $t \in I^0$ we can apply this formula with λ replaced by $\lambda + t\nu$. □

19.3 Geometric densities of the Boolean model

As an application, consider the Boolean model in $\mathbb{R}^d$, as in Definition 17.1. Let η_t be a stationary Poisson process on $\mathbb{R}^d$ with intensity $t \geq 0$ and let $\mathbb{Q}$ be a grain distribution satisfying (17.11). Recall that $\mathcal{C}^{(d)} = \mathcal{C}^d \setminus \{\emptyset\}$ is the system of all non-empty compact subsets of $\mathbb{R}^d$ equipped with the Hausdorff distance (17.1). We assume that $\mathbb{Q}$ is concentrated on the system

$\mathcal{K}^{(d)}$ of all convex $K \in \mathcal{C}^{(d)}$. We let ξ_t be an independent $\mathbb{Q}$-marking of η_t and define

$$Z_t := \bigcup_{(K,x)\in\xi_t} (K + x). \tag{19.11}$$

By Exercise 17.1 we can assume without loss of generality that for all $\omega \in \Omega$ and all compact convex $W \subset \mathbb{R}^d$ the set $Z_t(\omega) \cap W$ is a finite (possibly empty) union of convex sets. We define the *convex ring* $\mathcal{R}^d$ to be the system of all such sets. As mentioned in Appendix A.1, $\mathcal{R}^d \setminus \{\emptyset\}$ is a Borel subset of $\mathcal{C}^{(d)}$. Given compact $W \subset \mathbb{R}^d$ we claim that $Z_t \cap W$ is a random variable in $\mathcal{C}^d$ (equipped with the σ-field generated by $\mathcal{B}(\mathcal{C}^{(d)})$ and $\{\emptyset\}$). Indeed, $\{Z_t \cap W = \emptyset\}$ is obviously measurable. By Lemma 17.2 we hence need to check that the set $\{Z_t \cap W \cap C = \emptyset\}$ is measurable for all $C \in \mathcal{C}^d$. This follows from relation (16.4); see Theorem 17.3.

Now we consider the intrinsic volumes $V_i : \mathcal{R}^d \to \mathbb{R}$, $i \in \{0, \dots, d\}$, as defined in Appendix A.2. First we focus on V_{d-1} and refer to (A.23) for the geometric interpretation. We are interested in the *surface density*

$$S_W(t) := \mathbb{E}[V_{d-1}(Z_t \cap W)], \quad t \geq 0,$$

where $W \in \mathcal{K}^d$. It turns out that $S_W(t)$ can be easily expressed in terms of ϕ_d and ϕ_{d-1}, where we recall from definition (17.10) that $\phi_i := \int V_i(K)\, \mathbb{Q}(dK)$ for $i \in \{0, \dots, d\}$.

To avoid repetition we first work in a more general setting. A measurable additive function $\varphi\colon \mathcal{R}^d \to \mathbb{R}$ is said to be *locally finite* if

$$\sup\{|\varphi(K)| : K \in \mathcal{K}^d, K \subset W\} \leq c_W, \quad W \in \mathcal{K}^d, \tag{19.12}$$

where c_W is a finite constant, depending on $W \in \mathcal{K}^d$. Being monotone on $\mathcal{K}^d$, the intrinsic volumes are locally finite.

Lemma 19.5 *Let $\varphi\colon \mathcal{R}^d \to \mathbb{R}$ be measurable, additive and locally finite. For $W \in \mathcal{K}^d$ let $S_{\varphi,W}(t) := \mathbb{E}[\varphi(Z_t \cap W)]$. Then $S_{\varphi,W}$ is analytic and the derivative is given by*

$$\begin{aligned} S'_{\varphi,W}(t) = &\iint \varphi(W \cap (K + x))\, dx\, \mathbb{Q}(dK) \\ &- \iint \mathbb{E}[\varphi(Z_t \cap W \cap (K + x))]\, dx\, \mathbb{Q}(dK). \end{aligned} \tag{19.13}$$

Proof We aim to apply Theorem 19.4 with

$$\mathbb{X} := \{(x, K) \in \mathbb{R}^d \times \mathcal{K}^{(d)} : (K + x) \cap W \neq \emptyset\},$$

$\lambda = 0$, and ν the restriction of $\lambda_d \otimes \mathbb{Q}$ to $\mathbb{X}$. By assumption (17.11) and (16.11), this measure is finite. Define the function $f\colon \mathbf{N}(\mathbb{X}) \to \mathbb{R}_+$ by

$$f(\mu) := \varphi(Z(\mu) \cap W),$$

where $Z(\mu) := \bigcup_{(x,K)\in\mu}(K + x)$ if $\mu(\mathbb{X}) < \infty$ and $Z(\mu) := \emptyset$, otherwise. As at (19.11) it follows that $\mu \mapsto Z(\mu)$ is a measurable mapping taking values in the convex ring $\mathcal{R}^d$. Since φ is a measurable function, so is f. To show that $\mathbb{E}[f(\eta_{t\nu})] < \infty$, we write $\eta_{t\nu}$ in the form

$$\eta_{t\nu} = \sum_{n=1}^{\kappa} \delta_{(X_n, Z'_n)},$$

where κ is a random element of $\mathbb{N}_0$, (X_n) is a sequence of random vectors in $\mathbb{R}^d$ and (Z'_n) is a sequence of random elements of $\mathcal{K}^{(d)}$. Let $Y_n := Z'_n + X_n$. By the inclusion-exclusion principle (A.29),

$$\begin{aligned} f(\eta_{t\nu}) &= f\Big(\bigcup_{n=1}^{\kappa} Y_n \cap W\Big) \\ &= \sum_{n=1}^{\kappa}(-1)^{n-1} \sum_{1\le i_1<\cdots<i_n\le\kappa} \varphi_j(W \cap Y_{i_1} \cdots \cap Y_{i_n}). \end{aligned} \tag{19.14}$$

Using (19.12) we get

$$f(\eta_{t\nu}) \le \sum_{n=1}^{\kappa} \binom{\kappa}{n} c_W \le 2^{\kappa} c_W.$$

It follows that $\mathbb{E}[f(\eta_{t\nu})] < \infty$.

By Theorem 19.3, the function $S_{\varphi,W}$ is analytic on $\mathbb{R}_+$. By Theorem 19.4 the derivative is given by

$$\begin{aligned} S'_{\varphi,W}(t) &= \iint \mathbb{E}[D_{(x,K)} f(\eta_{t\nu})]\, dx\, \mathbb{Q}(dK) \\ &= \iint \mathbb{E}[\varphi((Z_t \cup (K + x)) \cap W) - \varphi(Z_t \cap W)]\, dx\, \mathbb{Q}(dK). \end{aligned}$$

From the additivity property (A.28) and linearity of integrals we obtain (19.13), provided that

$$\iint (|\varphi(W \cap (K + x))| + \mathbb{E}[|\varphi(Z_t \cap W \cap (K + x))|])\, dx\, \mathbb{Q}(dK) < \infty.$$

Since $\varphi(\emptyset) = 0$, we have

$$\iint |\varphi(W \cap (K + x))|\, dx\, \mathbb{Q}(dK) \le c_W \iint \mathbf{1}\{W \cap (K + x) \ne \emptyset\}\, dx\, \mathbb{Q}(dK),$$

which is finite by (17.7) and assumption (17.11). Using (19.14) with W replaced by $W \cap (K + x)$ gives $\mathbb{E}[|\varphi(Z_t \cap W \cap (K + x))|] \leq c_W \mathbb{E}[2^\kappa]$, uniformly in $(K, x) \in \mathcal{K}^d \times \mathbb{R}^d$. Hence we obtain as above that

$$\iint \mathbb{E}[|\varphi(Z_t \cap W \cap (K + x))|]\, dx\, \mathbb{Q}(dK)$$
$$\leq c_W \mathbb{E}[2^\kappa] \iint \mathbf{1}\{W \cap (K + x) \neq \emptyset\}\, dx\, \mathbb{Q}(dK) < \infty$$

and the proposition is proved. □

We shall also need the following Lemma.

Lemma 19.6 *Let $m \in \mathbb{N}$. For $j \in \{0, \dots, m\}$, let $\varphi_j : \mathcal{R}^d \to \mathbb{R}$ be a measurable, additive and locally finite function. For $W \in \mathcal{K}^d$ let $S_{j,W}(t) := \mathbb{E}[\varphi_j(Z_t \cap W)]$. Suppose that for some fixed $i \in \{0, \dots, m\}$,*

$$\iint \varphi_i(A \cap (K + x))\, dx\, \mathbb{Q}(dK) = \sum_{j=0}^{m} c_{i,j} \varphi_j(A), \quad A \in \mathcal{R}^d, \tag{19.15}$$

for certain constants $c_{i,j} \in \mathbb{R}$ depending on $\mathbb{Q}$ but not on A. Then $S_{i,W}$ is a differentiable function satisfying

$$S'_{i,W}(t) = \sum_{j=0}^{m} c_{i,j} \varphi_j(W) - \sum_{j=0}^{m} c_{i,j} S_{j,W}(t). \tag{19.16}$$

Proof Applying (19.13) with $\varphi = \varphi_i$ and using (19.15) twice (the second time with $Z_t \cap W$ in place of W), we obtain that

$$S'_{i,W}(t) = \sum_{j=0}^{m} c_{i,j} \varphi_j(W) - \mathbb{E}\Big[\sum_{j=0}^{m} c_{i,j} \varphi_j(Z_t \cap W)\Big],$$

where we have also used Fubini's theorem. This yields the assertion. □

Proposition 19.7 *Let Z_t be as in* (19.11) *and let $W \in \mathcal{K}^d$. Then*

$$S_W(t) = \phi_{d-1} t e^{-t\phi_d} V_d(W) + (1 - e^{-t\phi_d}) V_{d-1}(W), \quad t \geq 0. \tag{19.17}$$

The right-hand side of (19.17) admits a clear geometric interpretation. The first term is the mean surface content in W of all grains that are not covered by other grains. Indeed, $\phi_{d-1} t V_d(W)$ is the mean surface content of all grains ignoring overlapping, while $e^{-t\phi_d}$ can be interpreted as the probability that a point on the boundary of a conributing grain is not covered by other grains; see (19.18). The second term is the contribution of that part of the boundary of W which is covered by the Boolean model.

Proof of Proposition 19.7 By Lemma 19.6 and Lemma 19.8 below,

$$\begin{aligned} S'_W(t) &= \int V_d(W)V_{d-1}(K)\,\mathbb{Q}(dK) + \int V_{d-1}(W)V_d(K)\,\mathbb{Q}(dK) \\ &\quad - \int \mathbb{E}[V_d(Z_t \cap W)]V_{d-1}(K)\,\mathbb{Q}(dK) \\ &\quad - \int \mathbb{E}[V_{d-1}(Z_t \cap W)]V_d(K)\,\mathbb{Q}(dK). \end{aligned}$$

By Proposition 17.4 we have for all $t \geq 0$ that

$$\mathbb{E}[V_d(Z_t \cap W)] = (1 - e^{-t\phi_d})V_d(W), \tag{19.18}$$

and therefore

$$S'_W(t) = V_d(W)\phi_{d-1} + V_{d-1}(W)\phi_d - (1 - e^{-t\phi_d})V_d(W)v_{d-1} - S_W(t)\phi_d.$$

Note that $S_W(0) = 0$. It is easily checked that this linear differential equation is (uniquely) solved by the right-hand side of the asserted formula (19.17). □

The following integral equation was crucial for the proof of Proposition 19.7.

Lemma 19.8 *Let $A, B \in \mathcal{R}^d$. Then*

$$\int V_{d-1}(A \cap (B + x))\,dx = V_d(A)V_{d-1}(B) + V_{d-1}(A)V_d(B). \tag{19.19}$$

Proof Recall from Section A.3 that $\mathcal{H}_{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$. We first prove a more general result, namely

$$\int \mathcal{H}_{d-1}(\partial(A \cap (B + x)))\,dx = \lambda_d(A)\mathcal{H}_{d-1}(\partial B) + \mathcal{H}_{d-1}(\partial A)\lambda_d(B), \tag{19.20}$$

for all closed sets A, B such that $\mathcal{H}_{d-1}(\partial A) < \infty$ and $\mathcal{H}_{d-1}(\partial B) < \infty$. Since $\partial(A \cap B) \subset \partial A \cup \partial B$ it is not hard to show that

$$\partial(A \cap B) = (\partial A \cap \operatorname{int} B) \cup (\operatorname{int} A \cap \partial B) \cup (\partial A \cap \partial B), \tag{19.21}$$

where $\operatorname{int} A$ denotes the interior of the set A. Note that this is a disjoint union. Since $\operatorname{int}(B + x) = \operatorname{int} B + x$ we have

$$\begin{aligned} \int \mathcal{H}_{d-1}(\partial A \cap \operatorname{int}(B + x))\,dx &= \iint \mathbf{1}\{y \in \partial A, y \in \operatorname{int} B + x\}\,\mathcal{H}_{d-1}(dy)\,dx \\ &= \iint \mathbf{1}\{y \in \partial A, -x \in \operatorname{int} B - y\}\,dx\,\mathcal{H}_{d-1}(dy) \\ &= \lambda_d(\operatorname{int} B)\mathcal{H}_{d-1}(\partial A), \end{aligned}$$

where $B - y := B + (-y)$ and where the use of Fubini's theorem is justified by the fact that the restriction of $\mathcal{H}_{d-1}$ to ∂A is a finite measure. The assumption $\mathcal{H}_{d-1}(\partial B) < \infty$ implies $\mathcal{H}_d(\partial B) = \lambda_d(\partial B) = 0$. (We leave the reader to verify this fact directly using the definitions (A.19) and (A.20) of Hausdorff measure.) Therefore, $\lambda_d(\operatorname{int} B) = \lambda_d(B)$.

Using the translation invariance of $\mathcal{H}_{d-1}$ we obtain

$$\begin{aligned}\int \mathcal{H}_{d-1}(\operatorname{int} A \cap \partial(B + x))\, dx &= \int \mathcal{H}_{d-1}((\operatorname{int} A - x) \cap \partial B)\, dx \\ &= \lambda_d(A)\mathcal{H}_{d-1}(\partial B).\end{aligned}$$

Finally we have

$$\int \mathcal{H}_{d-1}(\partial A \cap \partial(B + x))\, dx = \mathcal{H}_{d-1}(\partial A)\lambda_d(\partial B) = 0,$$

concluding the proof of (19.20).

Turning to the proof of (19.19) we first note that both sides of this formula are additive (in the sense of (A.28)) in A and B. Therefore we can assume that A and B are convex. If both A and B have non-empty interior, then $A \cap (B + x)$ either has non-empty interior or is empty for λ_d-a.e. $x \in \mathbb{R}^d$ and (19.19) follows from (19.20) and (A.23). If A has an empty interior and B not (or vice versa), then the left-hand sides of (19.20) and (19.19) coincide. Since $\lambda_d(A) = 0$ (resp. $\lambda_d(B) = 0$), the right-hand sides coincide as well. If both A and B have empty interior, then $\mathcal{H}_{d-1}(\partial(A \cap (B + x))) = 0$ for λ_d-a.e. $x \in \mathbb{R}^d$ and (19.19) degenerates to the trivial identity $0 = 0$. □

Finally in this chapter we deal with the *Euler characteristic* V_0 in the case $d = 2$; see Appendix A.3 for the definition and a geometric interpretation. The grain distribution $\mathbb{Q}$ is *isotropic* if $\mathbb{Q}(\{\rho K : K \in A\}) = \mathbb{Q}(A)$ for all measurable $A \subset \mathcal{K}^{(d)}$ and (proper) rotations $\rho\colon \mathbb{R}^d \to \mathbb{R}^d$, where $\rho K := \{\rho(x) : x \in K\}$.

Theorem 19.9 *Let Z_t be a Boolean model in $\mathbb{R}^2$ with an isotropic grain distribution $\mathbb{Q}$. Then, for all $W \in \mathcal{K}^d$,*

$$\begin{aligned}&\mathbb{E}[V_0(Z_t \cap W)] \qquad (19.22)\\ &\quad = (1 - e^{-t\phi_2})V_0(W) + \frac{2}{\pi} t e^{-t\phi_2}\phi_1 V_1(W) + t e^{-t\phi_2} V_2(W) - \frac{1}{\pi} t^2 e^{-t\phi_2}\gamma_1^2 V_2(W).\end{aligned}$$

Proof In the first part of the proof we work in general dimensions. We would like to apply Lemma 19.6 to the measurable and additive functions

$V_0, \dots, V_d$ and with $i = 0$. To do so, we need to establish (19.15), that is

$$\iint V_0(A \cap (K + x))\, dx\, \mathbb{Q}(dK) = \sum_{j=0}^{d} c_j V_j(A), \quad A \in \mathcal{R}^d, \tag{19.23}$$

for certain constants $c_0, \dots, c_d \in \mathbb{R}$. Since both sides of this equation are additive in A, we can assume that $A \in \mathcal{K}^d$. Then the left-hand side of (19.23) simplifies to

$$\varphi(A) := \iint \mathbf{1}\{A \cap (K + x) \neq \emptyset\}\, dx\, \mathbb{Q}(dK) = \int V_d(K \oplus A^*)\, \mathbb{Q}(dK), \tag{19.24}$$

where $A^* := -A$. By Exercise 19.7 the function φ is invariant under translations and rotations. We now prove that φ is continuous on $\mathcal{K}^{(d)}$ (with respect to Hausdorff distance). If $A_n \in \mathcal{K}^{(d)}$ converge to some $A \in \mathcal{K}^{(d)}$, then $(A_n)^*$ converges to A^*. Exercise 17.6 and the continuity of V_d on $\mathcal{K}^{(d)}$ (mentioned in Appendix A.3) show that $V_d(K \oplus (A_n)^*)$ tends to $V_d(K \oplus A^*)$ for any $K \in \mathcal{K}^{(d)}$. Moreover, the definition of the Hausdorff distance implies that the A_n are all contained in some (sufficiently large) ball. By (17.11) we can apply dominated onvergence to conclude that $\varphi(A_n) \to \varphi(A)$ as $n \to \infty$.

Theorem A.24 shows that (19.23) holds. To determine the coefficients $c_0, \dots, c_d$ we take $A = B(0, r)$ for $r \geq 0$. The Steiner formula (A.22) implies that

$$\varphi(B(0, r)) = \sum_{j=0}^{d} r^j \kappa_j \phi_{d-j}. \tag{19.25}$$

On the other hand, by (A.25) and (A.24), for $A = B(0, r)$ the right-hand side of (19.23) equals

$$\sum_{j=0}^{d} c_j r^j \binom{d}{j} \frac{\kappa_d}{\kappa_{d-j}}.$$

It follows that

$$c_j = \phi_{d-j} \frac{j!\kappa_j (d-j)!\kappa_{d-j}}{d!\kappa_d}, \quad j = 0, \dots, d.$$

In the remainder of the proof we assume that $d = 2$. Then $c_0 = \phi_2$,

$c_1 = (2\phi_1)/\pi$, and $c_2 = 1$. Inserting (19.17) and (19.18) into (19.16) yields

$$\begin{aligned}\frac{d}{dt}\mathbb{E}[V_0(Z_t \cap W)] = \sum_{j=0}^{2} c_j V_j(W) - c_0\,\mathbb{E}[V_0(Z_t \cap W)] - c_1\phi_1 t e^{-t\phi_2} V_2(W)\\ - c_1(1 - e^{-t\phi_2})V_1(W) - c_2(1 - e^{-t\phi_2})V_2(W).\end{aligned}$$

A simple calculation shows that this differential equation is indeed solved by (19.22). □

19.4 Exercises

Exercise 19.1 Let λ be an s-finite measure on $\mathbb{X}$ and ν a finite measure on $\mathbb{X}$. Suppose that $f \in \mathbb{R}(\mathbf{N})$ satisfies $\mathbb{E}[|f(\eta_{\lambda+t|\nu|})|] < \infty$ for some $t > 0$. Prove by a direct calculation that (19.8) holds. (Hint: Use the calculation in the proof of Theorem 19.3.)

Exercise 19.2 Let ν be a finite measure on $\mathbb{X}$ and let $f \in \mathbb{R}(\mathbf{N})$ satisfy $\mathbb{E}[|f(\eta_{a\nu})|] < \infty$ for some $a > 0$. Show that

$$\frac{d}{dt}\mathbb{E}[f(\eta_{t\nu})] = t^{-1}\mathbb{E}\int (f(\eta_{t\nu}) - f(\eta_{t\nu} \setminus \delta_x))\,\eta_{t\nu}(dx), \quad t \in (0, a).$$

Exercise 19.3 Let ν be a finite measure on $\mathbb{X}$ and let $A \in \mathcal{N}$ be increasing, that is, $\mu \in A$ implies $\mu + \delta_x \in A$ for all $x \in \mathbb{X}$. Let

$$N_A(t) := \int \mathbf{1}\{\eta_{t\nu} \in A, \eta_{t\nu} \setminus \delta_x \notin A\}\,\eta_{t\nu}(dx)$$

denote the number of points of $\eta_{t\nu}$ that are *pivotal* for A. Show that

$$\frac{d}{dt}\mathbb{P}(\eta_{t\nu} \in A) = t^{-1}\mathbb{E}[N_A(t)], \quad t > 0.$$

Exercise 19.4 Let ν be a finite signed measure on $\mathbb{X}$ and let $t > 0$ be such that $\lambda + t\nu$ is a measure. Let $f \in \mathbb{R}(\mathbf{N})$ satisfy $\mathbb{E}[|f(\eta_{\lambda+t\nu})|] < \infty$, $\mathbb{E}[|f(\eta_\lambda)|] < \infty$, and

$$\int_0^t \int_{\mathbb{X}} \mathbb{E}[|D_x f(\eta_{\lambda+s|\nu|})|]\,|\nu|(dx)\,ds < \infty. \tag{19.26}$$

Prove that then (19.4) holds. (Hint: Apply Theorem 19.3 to a suitably truncated function f and apply dominated convergence.)

Exercise 19.5 Let ν be a finite signed measure and $t > 0$ such that $\lambda + t\nu$ is a measure. Suppose that $f \in \mathbb{R}_+(\mathbf{N})$ satisfies (19.26). Show that $\mathbb{E}[f(\eta_{\lambda+t\nu})] < \infty$ if and only if $\mathbb{E}[f(\eta_\lambda)] < \infty$. (Hint: Use Exercise 19.4 and Fatou's Lemma A.6.)

Exercise 19.6 Let $S_W(t)$ be as in Proposition 19.7. Show that

$$\lim_{r\to\infty} \frac{S_{B(0,r)}(t)}{\lambda_d(B(0,r))} = \phi_{d-1} t e^{-t\phi_d}.$$

Formulate and prove an analogous result in the setting of Theorem 19.9.

Exercise 19.7 Let $\mathbb{Q}$ be a distribution on $\mathcal{K}^d$ satisfying (17.11). Show that the function φ defined by (19.24) is invariant under translations. Assume in addition that $\mathbb{Q}$ is invariant under rotations; show that then φ has the same property.

20

Covariance identities

Given a square integrable Poisson functional F and $t \in [0, 1]$, the Poisson functional $P_t F$ is defined by a combination of t-thinning and independent superposition. The family $P_t F$ interpolates between the expectation of F and F. The Fock space series representation of the covariance between two Poisson functionals can be rewritten as an integral equation involving only the first order difference operator and the operator P_t. This identity will play a key role in the next chapter on normal approximation. A corollary is the Harris-FKG correlation inequality.

20.1 Mehler's formula

In this chapter we consider a proper Poisson process η on a measurable space $(\mathbb{X}, \mathcal{X})$ with σ-finite intensity measure λ and distribution $\mathbb{P}_\eta$. Let L^0_η be the space of all random variables (Poisson functionals) F such that $F = f(\eta)$ $\mathbb{P}$-a.s. for some measurable $f \colon \mathbf{N} \to \mathbb{R}$. This f is called a *representative* of F. If f is a (fixed) representative of F we define

$$D^n_{x_1,\dots,x_n} F := D^n_{x_1,\dots,x_n} f(\eta), \quad n \in \mathbb{N},\ x_1, \dots, x_n \in \mathbb{X}.$$

This definition is justified by Exercise 18.3.

For $q > 0$ let L^q_η denote the space of all $F \in L^0_\eta$ such that $\mathbb{E}[|F|^q] < \infty$. For $F \in L^1_\eta$ with representative f we define

$$P_t F := \mathbb{E}\Big[\int f(\eta_t + \mu)\, \Pi_{(1-t)\lambda}(d\mu) \,\Big|\, \eta \Big], \quad t \in [0, 1],$$

where η_t is a t-thinning of η and where we recall that $\Pi_{\lambda'}$ denotes the distribution of a Poisson process with intensity measure λ'. Since Theorem 3.3 and the thinning theorem (Corollary 5.9) imply that

$$\Pi_\lambda = \mathbb{E}\Big[\int \mathbf{1}\{\eta_t + \mu \in \cdot\}\, \Pi_{(1-t)\lambda}(d\mu) \Big], \tag{20.1}$$

it again follows that the definition of P_tF does not depend on the representative of F up to almost sure equality. Moreover, Lemma B.18 shows that

$$P_tF = \int \mathbb{E}[f(\eta_t + \mu) \mid \eta]\, \Pi_{(1-t)\lambda}(d\mu), \quad t \in [0, 1], \tag{20.2}$$

We also note that

$$P_tF = \mathbb{E}[f(\eta_t + \eta'_{1-t}) \mid \eta],$$

where η'_{1-t} is a Poisson process with intensity measure $(1-t)\lambda$, independent of the pair (η, η_t). Exercise 20.1 yields further insight into the properties of the operator P_t.

Equation (20.1) implies that

$$\mathbb{E}[P_tF] = \mathbb{E}[F], \quad F \in L^1_\eta, \tag{20.3}$$

while the (conditional) Jensen inequality (Proposition B.1) implies for any $p \geq 1$ the contractivity property

$$\mathbb{E}[|P_tF|^p] \leq \mathbb{E}[|F|^p], \quad t \in [0, 1],\ F \in L^p_\eta. \tag{20.4}$$

The proof of the covariance identity in Theorem 20.2 below is based on the following result of independent interest.

Lemma 20.1 (Mehler's Formula) *Let $F \in L^2_\eta$, $n \in \mathbb{N}$ and $t \in [0, 1]$. Then*

$$D^n_{x_1,\dots,x_n}(P_tF) = t^n P_t D^n_{x_1,\dots,x_n}F, \quad \lambda^n\text{-a.e. } (x_1, \dots, x_n) \in \mathbb{X}^n,\ \mathbb{P}\text{-a.s.} \tag{20.5}$$

In particular

$$\mathbb{E}[D^n_{x_1,\dots,x_n}P_tF] = t^n \mathbb{E}[D^n_{x_1,\dots,x_n}F], \quad \lambda^n\text{-a.e. } (x_1, \dots, x_n) \in \mathbb{X}^n. \tag{20.6}$$

Proof Let f be a representative of F. We first assume that $f(\mu) = e^{-\mu(\upsilon)}$ for some $\upsilon \in \mathbb{R}_0(\mathbb{X})$, where $\mathbb{R}_0(\mathbb{X})$ is as before (18.12). It follows directly from the Definition 5.7 of a t-thinning (see also Exercise 5.4), that

$$\mathbb{E}[e^{-\eta_t(\upsilon)} \mid \eta] = \exp\Big[\int \log\big(1 - t + te^{-\upsilon(y)}\big)\, \eta(dy)\Big]. \tag{20.7}$$

Hence the following function f_t is a representative of P_tF:

$$f_t(\mu) := \exp\Big[-(1-t)\int \big(1 - e^{-\upsilon}\big)\, d\lambda\Big] \exp\Big[\int \log\big((1-t) + te^{-\upsilon(y)}\big)\, \mu(dy)\Big]. \tag{20.8}$$

Let $x \in \mathbb{X}$. Since

$$\exp\Big[\int \log\left(1 - t + te^{-v(y)}\right)(\mu + \delta_x)(dy)\Big]$$
$$= \exp\Big[\int \log\left(1 - t + te^{-v(y)}\right)\mu(dy)\Big](1 - t + te^{-v(x)}),$$

we obtain that

$$D_x P_t F = f_t(\eta + \delta_x) - f_t(\eta) = t(e^{-v(x)} - 1)f_t(\eta) = t(e^{-v(x)} - 1)P_t F.$$

This identity can be iterated to yield for all $n \in \mathbb{N}$ and all $(x_1, \dots, x_n) \in \mathbb{X}^n$ that

$$D^n_{x_1,\dots,x_n} P_t F = t^n \prod_{i=1}^n (e^{-v(x_i)} - 1)P_t F.$$

On the other hand we have from 18.7 that $\mathbb{P}$-a.s.

$$P_t D^n_{x_1,\dots,x_n} F = P_t \prod_{i=1}^n (e^{-v(x_i)} - 1)F = \prod_{i=1}^n (e^{-v(x_i)} - 1)P_t F,$$

so that (20.5) holds for Poisson functionals of the given form.

By linearity (20.5) extends to all F with a representative in the set $\mathbf{G}$ defined at (18.12). By Lemma 18.5 there are $f^k \in \mathbf{G}$, $k \in \mathbb{N}$, satisfying $F^k := f^k(\eta) \to F = f(\eta)$ in $L^2(\mathbb{P})$ as $k \to \infty$. Therefore we obtain from the contractivity property (20.4) that

$$\mathbb{E}[(P_t F^k - P_t F)^2] = \mathbb{E}[(P_t(F^k - F))^2] \le \mathbb{E}[(F^k - F)^2] \to 0,$$

as $k \to \infty$. Taking $B \in \mathcal{X}$ with $\lambda(B) < \infty$, it therefore follows from Lemma 18.6 that

$$\mathbb{E}\Big[\int_{B^n} |D^n_{x_1,\dots,x_n} P_t F_k - D^n_{x_1,\dots,x_n} P_t F| \, \lambda^n(d(x_1, \dots, x_n))\Big] \to 0,$$

as $k \to \infty$. On the other hand we obtain from the Fock space representation (18.10) that $\mathbb{E}[|D^n_{x_1,\dots,x_n} F|] < \infty$ for λ^n-a.e. $(x_1, \dots, x_n) \in \mathbb{X}^n$, so that linearity of P_t and (20.4) imply

$$\mathbb{E}\Big[\int_{B^n} |P_t D^n_{x_1,\dots,x_n} F_k - P_t D^n_{x_1,\dots,x_n} F| \, \lambda^n(d(x_1, \dots, x_n))\Big]$$
$$\le \int_{B^n} \mathbb{E}[|D^n_{x_1,\dots,x_n}(F_k - F)|] \, \lambda^n(d(x_1, \dots, x_n)).$$

Again, this latter integral tends to 0 as $k \to \infty$. Since (20.5) holds for any

F_k we obtain from the triangle inequality that

$$\mathbb{E}\Big[\int_{B^n} |D^n_{x_1,\dots,x_n} P_t F - t^n P_t D^n_{x_1,\dots,x_n} F|\,\lambda^n(d(x_1,\dots,x_n))\Big] = 0.$$

Therefore (20.5) holds $\mathbb{P} \otimes (\lambda_B)^n$-a.e., and hence also $\mathbb{P} \otimes \lambda^n$-a.e.

Taking the expectation in (20.5) and using (20.3) proves (20.6). □

20.2 Two covariance identities

For $F \in L^2_\eta$ we denote by DF the mapping $(\omega, x) \mapsto (D_x F)(\omega)$. The theorem below requires the additional assumption $DF \in L^2(\mathbb{P} \otimes \lambda)$, that is,

$$\mathbb{E}\Big[\int (D_x F)^2\,\lambda(dx)\Big] < \infty. \tag{20.9}$$

Theorem 20.2 *For any $F, G \in L^2_\eta$ such that $DF, DG \in L^2(\mathbb{P} \otimes \lambda)$,*

$$\mathbb{E}[FG] - \mathbb{E}[F]\mathbb{E}[G] = \mathbb{E}\Big[\iint_0^1 (D_x F)(P_t D_x G)\,dt\,\lambda(dx)\Big]. \tag{20.10}$$

Proof Exercise 20.6 shows that the integrand in the right-hand side of (20.10) can be assumed to be measurable. Using the Cauchy-Schwarz inequality and then the contractivity property (20.4) implies

$$\Big(\mathbb{E}\Big[\iint_0^1 |D_x F||P_t D_x G|\,dt\,\lambda(dx)\Big]\Big)^2 \le \mathbb{E}\Big[\int (D_x F)^2\,\lambda(dx)\Big]\mathbb{E}\Big[\int (D_x G)^2\,\lambda(dx)\Big], \tag{20.11}$$

which is finite by assumption. Therefore we can use Fubini's theorem and (20.5) to obtain that the right-hand side of (20.10) equals

$$\iint_0^1 t^{-1}\mathbb{E}[(D_x F)(D_x P_t G)]\,dt\,\lambda(dx). \tag{20.12}$$

For $t \in [0, 1]$ and λ-a.e. $x \in \mathbb{X}$ we can apply the Fock space isometry (18.9) to $D_x F$ and $D_x P_t G$. Taking into account Lemma 20.1 this gives

$$\mathbb{E}[(D_x F)(D_x P_t G)] = t\,\mathbb{E}[D_x F]\mathbb{E}[D_x G] + \sum_{n=1}^{\infty} \frac{t^{n+1}}{n!} \int \mathbb{E}[D^{n+1}_{x_1,\dots,x_n,x} F]\,\mathbb{E}[D^{n+1}_{x_1,\dots,x_n,x} G]\,\lambda^n(d(x_1,\dots,x_n)).$$

Inserting this into (20.12), applying Fubini's theorem (to be justified below) and performing the integration over $[0,1]$ shows that (20.12) equals

$$\int \mathbb{E}[D_xF]\mathbb{E}[D_xG]\,\lambda(dx)$$
$$+\sum_{n=1}^{\infty}\frac{1}{(n+1)!}\iint \mathbb{E}[D^{n+1}_{x_1,\dots,x_n,x}F]\,\mathbb{E}[D^{n+1}_{x_1,\dots,x_n,x}G]\,\lambda^n(d(x_1,\dots,x_n))\,\lambda(dx)$$
$$=\sum_{m=1}^{\infty}\frac{1}{m!}\int \mathbb{E}[D^m_{x_1,\dots,x_m}F]\,\mathbb{E}[D^m_{x_1,\dots,x_m}G]\,\lambda^m(d(x_1,\dots,x_m)).$$

By Theorem 18.3 this equals $\mathbb{E}[FG]-\mathbb{E}[F]\mathbb{E}[G]$, which yields the asserted formula (20.10). The use of Fubini's theorem is justified by (18.10) and the Cauchy-Schwarz inequality. □

Next we prove a symmetric version of Theorem 20.2 that avoids additional integrability assumptions.

Theorem 20.3 *For any $F \in L^2_\eta$,*

$$\mathbb{E}\Big[\iint_0^1 (\mathbb{E}[D_xF \mid \eta_t])^2\,dt\,\lambda(dx)\Big] < \infty, \tag{20.13}$$

and for any $F, G \in L^2_\eta$,

$$\mathbb{E}[FG]-\mathbb{E}[F]\mathbb{E}[G] = \mathbb{E}\Big[\iint_0^1 \mathbb{E}[D_xF \mid \eta_t]\mathbb{E}[D_xG \mid \eta_t]\,dt\,\lambda(dx)\Big]. \tag{20.14}$$

Proof By the thinning theorem (Corollary 5.9), η_t and $\eta-\eta_t$ are independent Poisson processes with intensity measures $t\lambda$ and $(1-t)\lambda$, respectively. Therefore we have for $F \in L^2_\eta$ with representative f that

$$\mathbb{E}[D_xF \mid \eta_t] = \int D_xf(\eta_t+\mu)\,\Pi_{(1-t)\lambda}(d\mu) \tag{20.15}$$

holds a.s. It is easy to see that the right-hand side of (20.15) is a jointly measurable function of (the suppressed) $\omega \in \Omega$, $x \in \mathbb{X}$, and $t \in [0,1]$.

Now we take $F, G \in L^2_\eta$ with representatives f and g. Let us first assume that $DF, DG \in L^2(\mathbb{P}\otimes\lambda)$. Then (20.13) follows from the (conditional) Jensen inequality while (20.15) implies for all $t \in [0,1]$ and $x \in \mathbb{X}$, that

$$\begin{aligned}\mathbb{E}[(D_xF)(P_tD_xG)] &= \mathbb{E}\Big[D_xF\int D_xg(\eta_t+\mu)\,\Pi_{(1-t)\lambda}(d\mu)\Big]\\ &= \mathbb{E}[D_xF\,\mathbb{E}[D_xG \mid \eta_t]] = \mathbb{E}[\mathbb{E}[D_xF \mid \eta_t]\mathbb{E}[D_xG \mid \eta_t]].\end{aligned}$$

Therefore (20.14) is just another version of (20.10).

In this second step of the proof we consider general $F, G \in L^2_\eta$. By

Lemma 18.5 there is a sequence $F^k \in \mathbf{G}$, $k \in \mathbb{N}$, such that $\mathbb{E}[(F-F^k)^2] \to 0$ as $k \to \infty$. Equation 18.7 shows for each $k \geq 1$ that $DF^k \in L^2(\mathbb{P} \otimes \lambda)$. We have just proved that

$$\mathbb{V}\mathrm{ar}[F^k - F^l] = \mathbb{E}\Big[\int (\mathbb{E}[D_x F^k \mid \eta_t] - \mathbb{E}[D_x F^l \mid \eta_t])^2 \, \lambda^*(d(x,t))\Big],$$

holds for all $k, l \in \mathbb{N}$, where λ^* is the product of λ and Lebesgue measure on $[0,1]$. Since the space $L^2(\mathbb{P} \otimes \lambda^*)$ is complete, there exists $h \in L^2(\mathbb{P} \otimes \lambda^*)$ satisfying

$$\lim_{k\to\infty} \mathbb{E}\Big[\int (h(x,t) - \mathbb{E}[D_x F^k \mid \eta_t])^2\Big] \lambda^*(d(x,t)) = 0. \tag{20.16}$$

On the other hand it follows from Lemma 18.6 that for any $C \in \mathcal{X}_0$ with $\lambda(C) < \infty$

$$\begin{aligned}\int_{C\times[0,1]} &\mathbb{E}[|\mathbb{E}[D_x F^k \mid \eta_t] - \mathbb{E}[D_x F \mid \eta_t]|] \, \lambda^*(d(x,t)) \\ &\leq \int_{C\times[0,1]} \mathbb{E}[|D_x F^k - D_x F|] \, \lambda^*(d(x,t)) \to 0, \quad \text{as } k \to \infty.\end{aligned}$$

Comparing this with (20.16) shows that $h(\omega, x, t) = \mathbb{E}[D_x F \mid \eta_t](\omega)$ for $\mathbb{P} \otimes \lambda^*$-a.e. $(\omega, x, t) \in \Omega \times C \times [0,1]$ and hence also for $\mathbb{P} \otimes \lambda^*$-a.e. $(\omega, x, t) \in \Omega \times \mathbb{X} \times [0,1]$. Therefore the fact that $h \in L^2(\mathbb{P} \otimes \lambda^*)$ implies (20.13). Now let G^k, $k \in \mathbb{N}$, be a sequence approximating G similarly. Then equation (20.14) holds with (F^k, G^k) instead of (F, G). But the right-hand side is just a scalar product in $L^2(\mathbb{P} \otimes \lambda^*)$. Taking the limit as $k \to \infty$ and using the L^2-convergence proved above, yields the general result. □

20.3 The Harris-FKG inequality

As an application of the preceding theorem we obtain a useful correlation inequality for increasing functions of η. Given $B \in \mathcal{X}$, a function $f \in \mathbb{R}(\mathbf{N})$ is *increasing on B* if $f(\mu + \delta_x) \geq f(\mu)$ for all $\mu \in \mathbf{N}$ and all $x \in B$. It is *decreasing on B* if $(-f)$ is increasing on B.

Theorem 20.4 (Harris-FKG Inequality) *Suppose $B \in \mathcal{X}$. Let $f, g \in L^2(\mathbb{P}_\eta)$ be increasing on B and decreasing on $\mathbb{X} \setminus B$. Then*

$$\mathbb{E}[f(\eta)g(\eta)] \geq (\mathbb{E}[f(\eta)])(\mathbb{E}[g(\eta)]). \tag{20.17}$$

Proof The assumptions imply that the right-hand side of (20.14) is non-negative. Hence the result follows. □

20.4 Exercises

Exercise 20.1 Suppose that $\mathbb{X}$ is a Borel subspace of a CSMS and that λ is locally finite. Let $U_1, U_2, \dots$ be independent random variables, uniformly distributed on $[0, 1]$ and let η'_{1-t} be a Poisson process with intensity measure $(1-t)\lambda$, independent of the sequence (U_n). Let $\mu = \sum_{n=1}^{k} \delta_{x_n} \in \mathbf{N}$ be locally finite (for $k \in \overline{\mathbb{N}}_0$ and $x_1, x_2, \dots \in \mathbb{X}$) and define

$$\xi_t(\mu) := \eta'_{1-t} + \sum_{n=1}^{k} \mathbf{1}\{U_n \le t\}\delta_{x_n}.$$

If $\mu \in \mathbf{N}$ is not locally finite, let $\xi_t(\mu) := 0$. Let $F \in L^1_\eta$ have representative f. Show that $\mu \mapsto \mathbb{E}[f(\xi_t(\mu))]$ is a representative of $P_t F$. (Hint: You need to show that $f_t(\mu) := \mathbb{E}[f(\xi_t(\mu))]$ is a measurable function of μ and that $\mathbb{E}[g(\eta)f_t(\eta)] = \mathbb{E}[g(\eta)P_t F]$ for all $g \in \mathbb{R}_+(\mathbf{N})$.)

Exercise 20.2 Let $v \in L^1(\lambda)$ and $F := I(v)$. Let $t \in [0, 1]$ and show that $P_t F = tF$, $\mathbb{P}$-a.s. (Hint: Take $f(\mu) := \mu(v) - \lambda(v)$ as a representative of F.)

Exercise 20.3 Let $u \in L^2(\lambda)$ and $F \in L^2_\eta$ such that $DF \in L^2(\mathbb{P} \otimes \lambda)$. Use Theorem 20.2 to show that

$$\mathbb{C}\mathrm{ov}[I_1(u), F] = \mathbb{E}\Big[\int u(x) D_x F\, \lambda(dx)\Big].$$

Give an alternative proof using the Mecke equation making the additional assumption $u \in L^1(\lambda)$.

Exercise 20.4 Let $h \in L^1_s(\lambda^2)$ and let $F := I_2(h)$. Show that $D_x F = 2I_1(h_x)$ for λ-a.e. $x \in \mathbb{X}$, where $h_x(y) := u(x, y)$, $y \in \mathbb{X}$. (Hint: Use Exercise 4.2.)

Exercise 20.5 Let $h \in L^1(\lambda) \cap L^2(\lambda)$ and let $F := I_2(h)$. Use Exercise 20.4 to show that $DF \in L^2(\mathbb{P} \otimes \lambda)$.

Exercise 20.6 Let $f \in \mathbb{R}(\mathbb{X} \times \mathbf{N})$ such that $f_x \in L^1(\mathbb{P}_\eta)$ for each $x \in \mathbb{R}$, where $f_x := f(x, \cdot)$. Let $F_x := f_x(\eta)$ and show that there is a jointly measurable version of $P_t F_x$, that is a $\tilde{f} \in \mathbb{R}(\mathbb{X} \times [0, 1] \times \Omega)$ satisfying

$$\tilde{f}(x, t, \cdot) = \mathbb{E}\Big[\int f_x(\eta_t + \mu)\, \Pi_{(1-t)\lambda}(d\mu) \,\Big|\, \eta\Big], \quad \mathbb{P}\text{-a.s.}, \ x \in \mathbb{X},\ t \in [0, 1].$$

(Hint: Assume first that f does not depend on $x \in \mathbb{X}$. Then (20.8) shows the assertion in the case $f \in \mathbf{G}$, where $\mathbf{G}$ is defined before Lemma 18.4. The proof of this Lemma shows $\sigma(\mathbf{G}) = \mathcal{N}$, so that Theorem A.3 can be used to derive the assertion for a general bounded $f \in \mathbb{R}(\mathbf{N})$.)

Exercise 20.7 Let $h \in L^1(\lambda) \cap L^2(\lambda)$ and $F := I_2(h)$. Show that

$$Z := \iint_0^1 (P_t D_x F)(D_x F)\, dt\, \lambda(dx)$$

is in $L^2(\mathbb{P})$. Prove further that

$$Z = 2 \int I(h_x)^2\, \lambda(dx), \quad \mathbb{P}\text{-a.s.}$$

and

$$D_y Z = 4 \int h(x,y) I(h_x)\, \lambda(dx) + 2 \int h(x,y)^2\, \lambda(dx), \quad \lambda\text{-a.e. } y \in \mathbb{X},\ \mathbb{P}\text{-a.s.}$$

(Hint: Use Exercises 20.2 and 20.4).

Exercise 20.8 Let Z be a Boolean model as in Definition 17.1 and let $K_1, \dots, K_n$ be compact subsets of $\mathbb{R}^d$. Use Theorem 20.4 to show that

$$\mathbb{P}(Z \cap K_1 \neq \emptyset, \dots, Z \cap K_n \neq \emptyset) \geq \prod_{j=1}^{n} \mathbb{P}(Z \cap K_j \neq \emptyset).$$

Give an alternative proof based on Theorem 17.3.

21

Normal approximation of Poisson functionals

The Wasserstein distance quantifies the distance between two distributions. Stein's method is a general method for obtaining bounds on this distance. Combining this method with the covariance identity of the previous chapter yields upper bounds on the Wasserstein distance between the distribution of a standardized Poisson functional and the standard normal distribution. In their most explicit form these bounds depend only on the first and second order difference operators.

21.1 Stein's method

In this chapter we consider a Poisson process η on an arbitrary measurable space $(\mathbb{X}, \mathcal{X})$ with σ-finite intensity measure λ. For a given Poisson functional $F \in L^2_\eta$ (that is, F is a square-integrable and $\sigma(\eta)$-measurable random variable) we are interested in the distance between the distribution of F and the standard normal distribution. We use here the *Wasserstein distance* to quantify the discrepancy between the laws of two random variables X_0, X_1. This distance is defined by

$$d_1(X_0, X_1) = \sup_{h \in \mathbf{Lip}(1)} |\mathbb{E}[h(X_0)] - \mathbb{E}[h(X_1)]|,$$

where $\mathbf{Lip}(1)$ denotes the space of all Lipschitz functions $h : \mathbb{R} \to \mathbb{R}$ with a Lipschitz constant less than or equal to one; see Appendix B. If a sequence (X_n) of random variables satisfies $\lim_{n\to\infty} d_1(X_n, X_0) = 0$, then Proposition B.8 shows that X_n converges to X_0 in distribution. Here we are interested in the central limit theorem, that is, in the case where X_0 has a standard normal distribution.

Let $\mathbf{AC}_{1,2}$ be the set of all differentiable functions $g\colon \mathbb{R} \to \mathbb{R}$ such that the derivative g' is absolutely continuous and satisfies $\sup\{|g'(x)| : x \in \mathbb{R}\} \le 1$ and $\sup\{|g''(x)| : x \in \mathbb{R}\} \le 2$, for some version g'' of the Radon-Nikodým derivative of g'. The following theorem is the key to the results of this chapter. Throughout this chapter we let N denote a standard normal random variable.

Theorem 21.1 (Stein's Method) *Let $F \in L^1(\mathbb{P})$. Then*

$$d_1(F, N) \le \sup_{g \in \mathbf{AC}_{1,2}} |\mathbb{E}[g'(F) - Fg(F)]|. \tag{21.1}$$

Proof Let $h \in \mathbf{Lip}(1)$. Proposition B.14 in the Appendix shows that there exists $g \in \mathbf{AC}_{1,2}$ such that

$$h(x) - \mathbb{E}[h(N)] = g'(x) - xg(x), \quad x \in \mathbb{R}. \tag{21.2}$$

It follows that

$$|\mathbb{E}[h(F)] - \mathbb{E}[h(N)]| = |\mathbb{E}[g'(F) - Fg(F)]|.$$

Taking the supremum yields the assertion. □

Next we use the covariance identity of Theorem 20.2 to turn the general bound (21.1) into a result for Poisson functionals.

Theorem 21.2 *Assume that the Poisson process η is proper and suppose that $F \in L^2_\eta$ satisfies $DF \in L^2(\mathbb{P} \otimes \lambda)$ and $\mathbb{E}[F] = 0$. Then*

$$\begin{aligned} d_1(F, N) \le \mathbb{E}\Big[\Big|1 - \iint_0^1 (P_t D_x F)(D_x F)\, dt\, \lambda(dx)\Big|\Big] \\ + \mathbb{E}\Big[\iint_0^1 |P_t D_x F|(D_x F)^2\, dt\, \lambda(dx)\Big]. \end{aligned} \tag{21.3}$$

Proof Let f be a representative of F and let $g \in \mathbf{AC}_{1,2}$. Then we have for all $x \in \mathbb{X}$ that

$$D_x g(F) = g(f(\eta + \delta_x)) - g(f(\eta)) = g(F + D_x F) - g(F). \tag{21.4}$$

Since g is Lipschitz (by the boundedness of its first derivative) it follows that $|D_x g(F)| \le |D_x F|$ and therefore $Dg(F) \in L^2(\mathbb{P} \otimes \lambda)$. Moreover, since

$$|g(F)| \le |g(F) - g(0)| + |g(0)| \le |F| + |g(0)|,$$

also $g(F) \in L^2_\eta$. Then Theorem 20.2 yields

$$\mathbb{E}[Fg(F)] = \mathbb{E}\Big[\iint_0^1 (P_t D_x F)(D_x g(F))\, dt\, \lambda(dx)\Big]$$

and it follows that

$$|\mathbb{E}[g'(F) - Fg(F)]| \le \mathbb{E}\Big[\Big|g'(F) - \iint_0^1 (P_t D_x F)(D_x g(F))\, dt\, \lambda(dx)\Big|\Big]. \tag{21.5}$$

We assert that there exists a measurable function $R\colon \mathbb{X} \times \mathbf{N} \to \mathbb{R}$ such that

$$D_x g(F) = g'(F)D_x F + R(x,\eta)(D_x F)^2, \quad x \in \mathbb{X}. \tag{21.6}$$

Indeed, if $D_x f(\mu) \neq 0$ for $x \in \mathbb{X}$ and $\mu \in \mathbf{N}$, we can define

$$R(x,\mu) := (D_x f(\mu))^{-2}(D_x g(f(\mu)) - g'(f(\mu))D_x f(\mu)).$$

Otherwise we set $R(x,\mu) := 0$. Since $D_x F = 0$ implies that $D_x g(F) = 0$, we obtain (21.6). Using (21.6) in (21.5) gives

$$\begin{aligned}|\mathbb{E}[g'(F) - Fg(F)]| \le \mathbb{E}\Big[|g'(F)|\Big|1 - \iint_0^1 (P_t D_x F)(D_x F)\,dt\,\lambda(dx)\Big|\Big] \\ + \mathbb{E}\Big[\iint_0^1 |P_t D_x F||R(x,\eta)|(D_x F)^2\,dt\,\lambda(dx)\Big]. \end{aligned}\tag{21.7}$$

By assumption $|g'(F)| \le 1$. Moreover, (21.4), (21.6), Proposition A.29, and the assumption $|g''(y)| \le 2$ for all λ-a.e. $y \in \mathbb{R}$ imply that $|R(x,\eta)| \le 1$. Using these facts in (21.7) and applying Theorem 21.1 gives the bound (21.3). □

21.2 Second order Poincaré inequality

Since the bound (21.3) involves the operators P_t it is often not easy to apply. The bound provided by the following main result of this chapter involves only the first and second order difference operators. In fact it can be represented in terms of the following three constants:

$$\alpha_{F,1} := 2\Big[\int (\mathbb{E}[(D_{x_1}F)^2(D_{x_2}F)^2])^{1/2}(\mathbb{E}[\Delta_{x_1,x_2,x_3}(F)])^{1/2}\lambda^3(d(x_1,x_2,x_3))\Big]^{1/2},$$

$$\alpha_{F,2} := \Big[\int \mathbb{E}[\Delta_{x_1,x_2,x_3}(F)]\,\lambda^3(d(x_1,x_2,x_3))\Big]^{1/2},$$

$$\alpha_{F,3} := \int \mathbb{E}[|D_x F|^3]\,\lambda(dx),$$

where

$$\Delta_{x_1,x_2,x_3}(F) := (D^2_{x_1,x_3}F)^2(D^2_{x_2,x_3}F)^2, \quad x_1,x_2,x_3 \in \mathbb{X}. \tag{21.8}$$

Theorem 21.3 *Suppose that $F \in L^2_\eta$ satisfies $DF \in L^2(\mathbb{P} \otimes \lambda)$, $\mathbb{E}[F] = 0$ and $\mathbb{V}\mathrm{ar}[F] = 1$. Then,*

$$d_1(F,N) \le \alpha_{F,1} + \alpha_{F,2} + \alpha_{F,3}. \tag{21.9}$$

Proof We can assume that $\alpha_{F,1}, \alpha_{F,2}, \alpha_{F,3}$ are finite. Otherwise the result is trivial.

Since (21.9) concerns only the distribution of η, it is no restriction of generality to assume that η is a proper point process. Our starting point is the inequality (21.3). By Hölder's inequality the second term in the right-hand side can be bounded from above by

$$\iint_0^1 (\mathbb{E}[|P_t D_x F|^3])^{1/3} (\mathbb{E}[|D_x F|^3])^{2/3} \, dt \, \lambda(dx) \le \alpha_{F,3},$$

where the inequality comes from the contractivity property (20.4). Applying Jensen's inequality to the first term in (21.3), we see that it is enough to show that

$$\left(\mathbb{E}\left[\left(1 - \iint_0^1 (P_t D_x F)(D_x F) \, dt \, \lambda(dx)\right)^2\right]\right)^{1/2} \le \alpha_{F,1} + \alpha_{F,2}. \qquad (21.10)$$

Let $Z := \iint_0^1 (P_t D_x F)(D_x F) \, dt \, \lambda(dx)$. Theorem 20.2 and our assumptions on F imply that $\mathbb{E}[Z] = \mathbb{V}\mathrm{ar}[F] = 1$. Hence the left-hand side of (21.10) equals $(\mathbb{E}[Z^2] - 1)^{1/2}$. The L^1-version of the Poincaré inequality (see Exercise 18.5) implies that

$$\mathbb{V}\mathrm{ar}[Z] = \mathbb{E}[Z^2] - 1 \le \mathbb{E}\left[\int (D_y Z)^2 \, \lambda(dy)\right]. \qquad (21.11)$$

We assert next that

$$\mathbb{E}[Z^2] - 1 \le \mathbb{E}\left[\int \left(\iint_0^1 |D_y[(P_t D_x F)(D_x F)]| \, dt \, \lambda(dx)\right)^2 \lambda(dy)\right]. \qquad (21.12)$$

To see this, assume first that

$$\iint_0^1 |D_y[(P_t D_x F)(D_x F)]| \, dt \, \lambda(dx) < \infty, \quad \mathbb{P}\text{-a.s.}, \ \lambda\text{-a.e. } y \in \mathbb{X}. \qquad (21.13)$$

Since our assumption $DF \in L^2(\mathbb{P} \otimes \lambda)$ and inequality (20.11) imply that $\iint_0^1 |(P_t D_x F)(D_x F)| \, dt \, \lambda(dx) < \infty$ holds $\mathbb{P}$-almost surely, it follows that

$$D_y Z = \iint_0^1 D_y[(P_t D_x F)(D_x F)] \, dt \, \lambda(dx),$$

again $\mathbb{P}$-a.s. and for λ-a.e. $y \in \mathbb{X}$. Hence (21.12) is a consequence of (21.11). If (21.13) fails, then (21.12) holds for trivial reasons.

Comparison of (21.12) and (21.10) now shows that the inequality

$$\left(\mathbb{E}\left[\int\left(\iint_0^1 |D_y((P_tD_xF)(D_xF))|\,dt\,\lambda(dx)\right)^2\lambda(dy)\right]\right)^{1/2} \le \alpha_{F,1}+\alpha_{F,2} \tag{21.14}$$

would imply (21.10).

We now verify (21.14). To begin with we apply Exercise 18.2 and the inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ for any $a,b,c \in \mathbb{R}$ (a consequence of Jensen's inequality) to obtain

$$\mathbb{E}\left[\int\left(\iint_0^1 |D_y((P_tD_xF)(D_xF))|\,dt\,\lambda(dx)\right)^2\lambda(dy)\right] \le 3(I_1+I_2+I_3), \tag{21.15}$$

where

$$\begin{aligned}
I_1 &:= \mathbb{E}\left[\int\left(\iint_0^1 |D_yP_tD_xF||D_xF|\,dt\,\lambda(dx)\right)^2\lambda(dy)\right],\\
I_2 &:= \mathbb{E}\left[\int\left(\iint_0^1 |P_tD_xF||D^2_{x,y}F|\,dt\,\lambda(dx)\right)^2\lambda(dy)\right],\\
I_3 &:= \mathbb{E}\left[\int\left(\iint_0^1 |D_yP_tD_xF||D^2_{x,y}F|\,dt\,\lambda(dx)\right)^2\lambda(dy)\right].
\end{aligned}$$

We shall bound I_1, I_2, I_3 by a repeated application of the Cauchy-Schwarz and (sometimes conditional) Jensen inequalities. Often we will apply Fubini's theorem (also the conditional version in Lemma B.18) with no explicit mention of it. For notational convenience integration with respect to dt always refers to Lebesgue measure on the interval $[0,1]$. Let f denote a representative of F.

We start with I_2. Take $y \in \mathbb{X}$ and define the random variable

$$I_2(y) := \left(\iint |P_tD_xF||D^2_{x,y}F|\,dt\,\lambda(dx)\right)^2.$$

By the representation (20.2) of the operator P_t and the triangle inequality for conditional expectations we have a.s. that

$$\begin{aligned}
I_2(y) &\le \left(\iiint \mathbb{E}[|D_xf(\eta_t+\mu)|\mid\eta]|D^2_{x,y}F|\,\lambda(dx)\,\Pi_{(1-t)\lambda}(d\mu)\,dt\right)^2\\
&= \left(\iiint \mathbb{E}[|D_xf(\eta_t+\mu)|\mid\eta]|D^2_{x,y}F|\,\Pi_{(1-t)\lambda}(d\mu)\,dt\,\lambda(dx)\right)^2\\
&= \left(\iint \mathbb{E}\left[\int |D_xf(\eta_t+\mu)||D^2_{x,y}F|\,\lambda(dx)\,\Big|\,\eta\right]\Pi_{(1-t)\lambda}(d\mu)\,dt\right)^2,
\end{aligned}$$

where we have used the pull out property of conditional expectations and Fubini's theorem. Next we apply Jensen's inequality to the two outer integrations and the conditional expectation to obtain that

$$I_2(y) \le \iint \mathbb{E}\Big[\Big(\int |D_x f(\eta_t+\mu)||D^2_{x,y}F|\,\lambda(dx)\Big)^2 \,\Big|\, \eta\Big]\,\Pi_{(1-t)\lambda}(d\mu)\,dt.$$

By Fubini's theorem (and the pull out property) it follows that a.s.

$$\begin{aligned}
I_2(y) \le& \int |D^2_{x_1,y}F||D^2_{x_2,y}F| \iint \mathbb{E}[|D_{x_1}f(\eta_t+\mu)||D_{x_2}f(\eta_t+\mu)| \mid \eta]\\
&\qquad\times \Pi_{(1-t)\lambda}(d\mu)\,dt\,\lambda^2(d(x_1,x_2))\\
\le& \int |D^2_{x_1,y}F||D^2_{x_2,y}F|\Big(\iint \mathbb{E}[(D_{x_1}f(\eta_t+\mu))^2(D_{x_2}f(\eta_t+\mu))^2 \mid \eta]\\
&\qquad\times \Pi_{(1-t)\lambda}(d\mu)\,dt\Big)^{1/2}\lambda^2(d(x_1,x_2)),
\end{aligned}$$

where we have applied Jensen's inequality to the conditional expectation and the integration $\Pi_{(1-t)\lambda}(d\mu)\,dt$. Using the Cauchy-Schwarz inequality and the law of the total expectation it follows that

$$\begin{aligned}
I_2 =& \mathbb{E}\Big[\int I_2(y)\,\lambda(dy)\Big]\\
\le& \int \Big(\iint \mathbb{E}[(D_{x_1}f(\eta_t+\mu))^2(D_{x_2}f(\eta_t+\mu))^2]\,\Pi_{(1-t)\lambda}(d\mu)\,dt\Big)^{1/2}\\
&\qquad[\mathbb{E}[\Delta_{x_1,x_2,y}(F)]]^{1/2}\,\lambda^3(d(x_1,x_2,y)).
\end{aligned}$$

Now we can use (20.1) to conclude that

$$\begin{aligned}
I_2 \le& \int (\mathbb{E}[(D_{x_1}F)^2(D_{x_2}F)^2])^{1/2}(\mathbb{E}[\Delta_{x_1,x_2,y}(F)])^{1/2}\,\lambda^3(d(x_1,x_2,y))\\
=& \frac{1}{4}\alpha^2_{F,1}.
\end{aligned} \tag{21.16}$$

Next we consider I_1. Take $y \in \mathbb{X}$ and define the random variable

$$I_1(y) := \Big(\iint_0^1 |D_y P_t D_x F||D_x F|\,dt\,\lambda(dx)\Big)^2.$$

Then, similarly to the above,

$$I_1(y) \le \frac{1}{4}\Big(\iint 2t\,\mathbb{E}\Big[\int |D^2_{x,y}f(\eta_t+\mu)||D_xF|\,\lambda(dx)\,\Big|\,\eta\Big]\,\Pi_{(1-t)\lambda}(d\mu)\,dt\Big)^2,$$

where we have also used Lemma 20.1. Arguing as before, and using the probability measure $2tdt$ instead of dt (on $[0,1]$), we obtain that

$$I_1 \leq \frac{1}{4}\int (\mathbb{E}[(D_{x_1}F)^2(D_{x_2}F)^2])^{1/2}(\mathbb{E}[\Delta_{x_1,x_2,y}(F)])^{1/2}\,\lambda^3(d(x_1,x_2,y)) \\ = \frac{1}{16}\alpha_{F,1}^2. \tag{21.17}$$

We leave it as an exercise to show that

$$I_3 \leq \frac{1}{4}\int \mathbb{E}[\Delta_{x_1,x_2,x_3}(F)]\,\lambda^3(d(x_1,x_2,y)) = \frac{1}{4}\alpha_{F,2}^2. \tag{21.18}$$

Combining the bounds (21.16), (21.17) and (21.18) with the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ (valid for all $a, b \geq 0$) yields

$$(3(I_1 + I_2 + I_3))^{1/2} \leq \sqrt{3}\sqrt{I_1 + I_2} + \sqrt{3}\sqrt{I_3} \\ \leq \sqrt{3}\frac{\sqrt{5}}{\sqrt{16}}\alpha_{F,1} + \frac{\sqrt{3}}{2}\alpha_{F,2} \leq \alpha_{F,1} + \alpha_{F,2}.$$

Inserting this into (21.15) implies (21.14) and hence the theorem. □

21.3 Normal approximation of linear functionals

We continue with a simple example in the general setting.

Example 21.4 Let $g \in L^1(\lambda) \cap L^3(\lambda)$ such that $\int g^2\,d\lambda = 1$. Let $F := I(g)$ as defined by (12.4). Then $\mathbb{E}[F] = 0$ while Proposition 12.3 implies that $\mathbb{V}\mathrm{ar}[F] = 1$. The definition (12.4) and Proposition 12.1 show for all (x_1, x_2) that $D_{x_1}F = g(x_1)$ and $D^2_{x_1,x_2}F = 0$, $\mathbb{P}$-a.s. Hence Theorem 21.3 implies that

$$d_1(I(g), N) \leq \int |g|^3\,d\lambda. \tag{21.19}$$

Our next proposition shows that the bound (21.19) is optimal up to a multiplicative constant.

Proposition 21.5 *Let X_t be a Poisson distributed random variable with parameter $t > 0$ and define $\hat{X}_t := t^{-1/2}(X_t - t)$. Then $d_1(\hat{X}_t, N) \leq t^{-1/2}$ and*

$$\liminf_{t\to\infty} \sqrt{t}\,d_1(\hat{X}_t, N) \geq 1/4.$$

Proof The upper bound follows from (21.19) when applied to a homogeneous Poisson process on $\mathbb{R}_+$ of unit intensity with $g(t) := t^{-1/2}\mathbf{1}_{[0,t]}$. (Alternatively we could apply (21.19) to a Poisson process on a state space with just one point.)

To derive the lower bound we construct a special Lipschitz function.

Let $h\colon \mathbb{R} \to \mathbb{R}$ be the continuous 1-periodic function which is zero on the integers and increases (resp. decreases) with rate 1 on the interval $[0, 1/2]$ (resp. $[1/2, 1]$). This function has Lipschitz constant 1. The same is then true for the function $h_t\colon \mathbb{R} \to \mathbb{R}$ defined by $h_t(x) := h(\sqrt{t}x + t)/\sqrt{t}$. By definition, $h_t(\hat{X}_t) \equiv 0$. On the other hand

$$\lim_{t\to\infty} \sqrt{t}\,\mathbb{E}[h_t(N)] = 1/4; \tag{21.20}$$

see Exercise 21.2. Since $d_1(\hat{X}_t, N) \geq \mathbb{E}[h_t(N)]$, the desired inequality follows. □

21.4 Exercises

Exercise 21.1 Let $f \in L^{2+\varepsilon}_\eta$ for some $\varepsilon > 0$ and assume that $\lambda(\mathbb{X}) < \infty$. Show that $Df(\eta) \in L^2(\mathbb{P} \otimes \lambda)$.

Exercise 21.2 Prove (21.20). Is the result true for other random variables?

Exercise 21.3 (Normal approximation of the second chaos) Suppose that $h \in L^1_s(\lambda^2) \cap L^4(\lambda^2)$ such that $2\lambda^2(h^2) = 1$ and let $F := I_2(h)$. (Note that (12.20) implies $\mathbb{E}[F^2] = 1$.) Define $h_1 \in \mathbb{R}(\mathbb{X}^2)$ and $h_2 \in \mathbb{R}(\mathbb{X})$ by

$$h_1(x_1, x_2) := \int h(x_1, y)h(x_2, y)\,\lambda(dy),$$
$$h_2(x) := \int h(x, y)^2\,\lambda(dy),$$

and prove that

$$d_1(F, N) \leq 4[\lambda^2(h_1^2)]^{1/2} + (2 + 4\sqrt{6})[\lambda(h_2^2)]^{1/2} + 4\sqrt{6}[\lambda^2(h^4)]^{1/2}.$$

(Hint: As in the proof of Theorem 21.3, it follows that

$$d_1(F, N) \leq \left(\int \mathbb{E}[(D_y Z)^2]\,\lambda(dy)\right)^{1/2} + \alpha_{F,3}.$$

Use Exercise 20.7 and the isometry (12.20) to bound the first term on the right. Use the Cauchy-Schwarz inequality and the moment formula from Exercise 12.3 to bound the second term.)

Exercise 21.4 (CLT for non-homogeneous Poisson processes) Let η be a Poisson process on $\mathbb{R}_+$ with intensity measure ν as in Exercise 7.6. Show that $\nu(t)^{-1/2}(\eta(t) - \nu(t)) \xrightarrow{d} N$ as $t \to \infty$. (Hint: Use Example 21.4.)

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Normal approximation in the Boolean model

The intersection of a Boolean model Z with convex grains with a convex observation window W is a finite union of convex set and hence amenable to additive translation invariant functionals φ, as the intrinsic volumes. The general results of the previous chapter yield bounds on the Wasserstein distance between the distribution of the standardized random variable $\varphi(Z \cap W)$ and a standard normal distribution. These bounds depend on the variance of $\varphi(Z \cap W)$ and are of the presumably optimal order $\lambda_d(W)^{-1/2}$ whenever this variance grows like the volume $\lambda_d(W)$ of W.

22.1 Normal approximation of the volume

In this chapter we shall study the normal approximation of geometric functionals of a Boolean model with general grains. Our aim is to apply the second order Poincaré inequality of Theorem 21.3. As in Definition 17.1, let $d \in \mathbb{N}$, let $\mathbb{Q}$ be a probability measure on the space $C^{(d)}$ of non-empty compact sets in R^d, let ξ be an independent $\mathbb{Q}$-marking of a stationary Poisson process η in $\mathbb{R}^d$ with intensity $\gamma \in (0, \infty)$ and let Z be the Boolean model induced by ξ. Recall from Theorem 5.6 that ξ is a Poisson process on $\mathbb{R}^d \times C^{(d)}$ with intensity measure $\lambda = \gamma\lambda_d \otimes \mathbb{Q}$. We assume that (17.11) is satisfied.

Let $W \subset \mathbb{R}^d$ be a compact set satisfying $\lambda_d(W) > 0$. As in Chapter 19, we are interested in Poisson functionals of the form $\varphi(Z \cap W)$, where φ is a suitable (geometric) function defined on compact sets. We start with the case $\varphi = \lambda_d$. Later we shall assume that $\mathbb{Q}$ is concentrated on the system $\mathcal{K}^{(d)}$ of non-empty convex sets and study general translation invariant additive functions φ.

As announced, we consider the random variable given by the Poisson functional

$$F_{W,d} := \lambda_d(Z \cap W). \tag{22.1}$$

Recall from Proposition 17.4 that $\mathbb{E}[F_{W,d}] = p\lambda_d(W)$, where p is given by (17.8), and from Exercise 17.10 that

$$\mathbb{V}\mathrm{ar}[F_{W,d}] = (1-p)^2 \int \lambda_d(W \cap (W+x))(e^{\gamma\beta_d(x)} - 1)\, dx, \tag{22.2}$$

where β_d is given by (17.13). We need to assume that $\phi_{d,3} < \infty$, where

$$\phi_{j,k} := \int (V_j(K))^k\, \mathbb{Q}(dK), \quad j \in \{0,\dots,d\},\ k \in \mathbb{N}, \tag{22.3}$$

and $V_0,\dots,V_d$ are the intrinsic volumes; see Appendix A.3. Further we use the notation

$$c_W := \lambda_d(W)(1-p)^{-2}\Big[\int \lambda_d(W \cap (W+x))(e^{\gamma\beta_d(x)} - 1)\, dx\Big]^{-1}. \tag{22.4}$$

Theorem 22.1 *Define $F_{W,d}$ by* (22.1)*, where Z is the Boolean model as above. Assume that $\phi_{d,3} < \infty$. Let $\hat{F}_{W,d} := (\mathbb{V}\mathrm{ar}[F_{W,d}])^{-1/2}(F_{W,d} - \mathbb{E}[F_{W,d}])$. Then*

$$d_1(\hat{F}_{W,d}, N) \le [2(\gamma\phi_{d,2})^{3/2}c_W + \gamma^{3/2}\phi_{d,2}c_W + \gamma\phi_{d,3}(c_W)^{3/2}](\lambda_d(W))^{-1/2}. \tag{22.5}$$

Proof We apply Theorem 21.3. To simplify notation, assume $\gamma = 1$ and let $\sigma := \mathbb{V}\mathrm{ar}[F_{W,d}]^{1/2}$.

By the additivity of Lebesgue measure, we have for all $(x,K) \in \mathbb{R}^d \times \mathcal{C}^{(d)}$ (similarly to Example 18.1) that

$$D_{(x,K)}F_{W,d} = \lambda_d((K+x) \cap W) - \lambda_d(Z \cap (K+x) \cap W). \tag{22.6}$$

(We leave it to the reader to construct a suitable representative of $F_{W,d}$.) Iterating this identity yields for all $(x_1,K_1),(x_2,K_2) \in \mathbb{R}^d \times \mathcal{C}^{(d)}$ that

$$\begin{aligned} D^2_{(x_1,K_1),(x_2,K_2)}F_{W,d} = {}& \lambda_d(Z \cap (K_1+x_x) \cap (K_2+x_2) \cap W) \\ & - \lambda_d((K_1+x_1) \cap (K_2+x_2) \cap W). \end{aligned} \tag{22.7}$$

In particular

$$|D_{(x,K)}F_{W,d}| \le \lambda_d((K+x) \cap W), \tag{22.8}$$

$$|D^2_{(x_1,K_1),(x_2,K_2)}F_{W,d}| \le \lambda_d((K_1+x_1) \cap (K_2+x_2) \cap W). \tag{22.9}$$

The following calculations rely on the monotonicity of Lebesgue measure and the following direct consequence of Fubini's theorem:

$$\int \lambda_d(A \cap (B+x))\, dx = \lambda_d(A)\lambda_d(B), \quad A, B \in \mathcal{B}^d. \tag{22.10}$$

By (22.8),

$$\mathbb{E}\Big[\int (D_{(x,K)}F_{W,d})^2\,\lambda(d(x,K))\Big] \le \iint \lambda_d(K_x\cap W)\lambda_d(W)\,dx\,\mathbb{Q}(dK)$$
$$= (\lambda(W))^2\int \lambda_d(K)\,\mathbb{Q}(dK),$$

where $K_x := K + x$. Hence $DF_{W,d} \in L^2(\mathbb{P}\otimes\lambda)$ and Theorem 21.3 applies. Let $\hat{F} := \hat{F}_{W,d}$. Using the bounds (22.8) and (22.9) in the definition of $\alpha_{\hat{F},1}$ yields

$$(\alpha_{\hat{F},1})^2 \le \frac{4}{\sigma^4}\iiiint \lambda_d(K_x\cap W)\lambda_d(L_y\cap W)\lambda_d(K_x\cap M_z\cap W)$$
$$\times\,\lambda_d(L_y\cap M_z\cap W)\,dx\,dy\,dz\,\mathbb{Q}^3(d(K,L,M)).$$

Since $\lambda_d(K_x) = \lambda(K)$ we get

$$(\alpha_{\hat{F},1})^2 \le \frac{4}{\sigma^4}\iiiint \lambda_d(K)\lambda_d(L)\lambda_d(K_x\cap M_z\cap W)\lambda_d(L_y\cap M_z\cap W)$$
$$\times\,dx\,dy\,dz\,\mathbb{Q}^3(d(K,L,M))$$
$$= \frac{4}{\sigma^4}\iiint \lambda_d(K)^2\lambda_d(L)\lambda_d(M_z\cap W)\lambda_d(L_y\cap M_z\cap W)$$
$$\times\,dy\,dz\,\mathbb{Q}^3(d(K,L,M)),$$

where we have used (22.10) to get the equality. Using $\lambda_d(M_z\cap W)\le\lambda_d(M)$ and applying (22.10) twice gives

$$(\alpha_{\hat{F},1})^2 \le \frac{4}{\sigma^4}\int \lambda_d(K)^2\lambda_d(L)^2\lambda_d(M)^2\lambda_d(W)\,\mathbb{Q}^3(d(K,L,M)),$$

that is,

$$\alpha_{\hat{F},1} \le 2(\phi_{d,2})^{3/2}\frac{(\lambda_d(W))^{1/2}}{\sigma^2} = 2(\phi_{d,2})^{3/2}c_W(\lambda_d(W))^{-1/2}. \tag{22.11}$$

Similarly we obtain that

$$\alpha_{\hat{F},2} \le \phi_{d,2}\frac{(\lambda_d(W))^{1/2}}{\sigma^2} = \phi_{d,2}c_W(\lambda_d(W))^{-1/2} \tag{22.12}$$

and

$$\alpha_{\hat{F},3} \le \phi_{d,3}\frac{\lambda_d(W)}{\sigma^3} = \phi_{d,3}(c_W)^{3/2}(\lambda_d(W))^{-1/2}. \tag{22.13}$$

Inserting these bounds into (21.9) gives the result (22.5). □

From now on we assume that $W \in \mathcal{K}^{(d)}$ is a compact, convex and non-empty set. In particular, the boundary of W has Lebesgue measure 0. Let $W_r := r^{1/d}W$ for $r > 0$. Then $\lambda_d(W_r) = r\lambda_d(W)$ and Exercise 17.13 shows that $c_{W_r} \to 1$ as $r \to \infty$. Hence (22.5) implies the central limit theorem $\hat{F}_{W_r,d} \xrightarrow{d} N$ as $r \to \infty$. The rate of convergence (with respect to the Wasserstein distance) is $r^{-1/2}$. Proposition 21.5 suggests that this rate is presumably optimal. Indeed, if $\mathbb{Q}$ is concentrated on a single grain with small volume $v > 0$ then $\lambda_d(Z \cap W)$ approximately equals $v\eta(W)$.

22.2 Normal approximation of additive functionals

In the remainder of this chapter we assume that $\mathbb{Q}$ is concentrated on the system $\mathcal{K}^{(d)}$ of all convex $K \in C^{(d)}$. As in Chapter 16 we can then assume without loss of generality that $Z(\omega) \cap W$ is a member of the convex ring $\mathcal{R}^d$ for all $\omega \in \Omega$. In fact, $Z \cap W$ is a random element of the convex ring; see the discussion after (19.11).

Let $\varphi \colon \mathcal{R}^d \to \mathbb{R}$ be a translation invariant measurable additive function, also satisfying

$$M(\varphi) := \sup\{|\varphi(K)| : K \in \mathcal{K}^d, K \subset Q_1\} < \infty, \tag{22.14}$$

where $Q_1 := [-1/2, 1/2]^d$ denotes the unit cube centred at the origin. The intrinsic volumes have all these properties. We wish to apply Theorem 21.3 to the Poisson functional

$$F_{W,\varphi} := \varphi(Z \cap W). \tag{22.15}$$

We define the *Wills functional* $\overline{V} \colon \mathcal{R}^d \to \mathbb{R}$ by $\overline{V} := \sum_{j=0}^{d} V_j$, and assume that

$$\int \overline{V}(K)^3 \, \mathbb{Q}(dK) < \infty. \tag{22.16}$$

Theorem 22.2 *Suppose that φ satisfies the above assumptions and that* (22.16) *holds. Assume that $\sigma^2_{W,\varphi} := \mathbb{V}\mathrm{ar}[F_{W,\varphi}] > 0$, where $F_{W,\varphi}$ is given by* (22.15). *Let $\hat{F}_{W,\varphi} := (\sigma_{W,\varphi})^{-1}(F_{W,\varphi} - \mathbb{E}[F_{W,\varphi}])$. Then*

$$d_1(\hat{F}_{W,\varphi}, N) \leq c_1\sigma^{-2}_{W,\varphi}\overline{V}(W)^{1/2} + c_2\sigma^{-3}_{W,\varphi}\overline{V}(W), \tag{22.17}$$

where c_1, c_2 do not depend on W.

We base the proof of (22.17) on Theorem 21.3 and some preliminary geometric facts. In what follows we let the assumptions of Theorem 22.2 be satisfied. It is convenient to write

$$\varphi_Z(K) := |\varphi(K)| + |\varphi(Z \cap K)|, \quad K \in \mathcal{R}^d. \tag{22.18}$$

Proposition 22.3 *There is a constant $c > 0$ such that for any $K, L \in \mathcal{K}^d$,*

$$\mathbb{E}[\varphi_Z(K)^2\varphi_Z(L)^2] \le c\overline{V}(K)^2\overline{V}(L)^2, \tag{22.19}$$

$$\mathbb{E}[\varphi_Z(K)^2] \le c\overline{V}(K)^2, \quad \mathbb{E}[\varphi_Z(K)^3] \le c\overline{V}(K)^3. \tag{22.20}$$

Proof Let $Q_1 := [-1/2, 1/2]^d$ denote the unit cube in $\mathbb{R}^d$. Let $K \in \mathcal{K}^d$. It is no loss of generality to assume that $K \neq \emptyset$. We define

$$\mathcal{Q}(K) := \{Q_1 + z : z \in \mathbb{Z}^d, (Q_1 + z) \cap K \neq \emptyset\}.$$

By the inclusion-exclusion principle (A.29) we have

$$|\varphi(Z \cap K)| = \Big|\varphi\Big(Z \cap K \cap \bigcup_{Q \in \mathcal{Q}(K)} Q\Big)\Big| \le \sum_{\mathcal{I} \subset \mathcal{Q}(K):\mathcal{I} \neq \emptyset} \Big|\varphi\Big(Z \cap K \cap \bigcap_{Q \in \mathcal{I}} Q\Big)\Big|.$$

For each compact set $C \subset \mathbb{R}^d$ let

$$N(C) := \int \mathbf{1}\{(M + x) \cap C \neq \emptyset\}\,\xi(d(x, M)) \tag{22.21}$$

denote the number of grains in $\{M + x : (x, M) \in \xi\}$ hitting C. For each non-empty subset $\mathcal{I} \subset \mathcal{Q}(K)$, fix some cube $Q_{\mathcal{I}} \in \mathcal{I}$ and let $Z_1, \dots, Z_{N(Q_{\mathcal{I}})}$ denote the grains hitting $Q_{\mathcal{I}}$. Then, for $\emptyset \neq J \subset \{1, \dots, N(Q_{\mathcal{I}})\}$, assumption (22.14) and the translation invariance of φ yield that

$$\Big|\varphi\Big(\bigcap_{j \in J} Z_j \cap K \cap \bigcap_{Q \in \mathcal{I}} Q\Big)\Big| \le M(\varphi).$$

Using the inclusion-exclusion formula again and taking into account the fact that $\varphi(\emptyset) = 0$, we get

$$\begin{aligned}
|\varphi(Z \cap K)| &\le \sum_{\mathcal{I} \subset \mathcal{Q}(K):\mathcal{I} \neq \emptyset} \Big|\varphi\Big(\bigcup_{j=1}^{N(Q_{\mathcal{I}})} Z_j \cap K \cap \bigcap_{Q \in \mathcal{I}} Q\Big)\Big| \\
&\le \sum_{\mathcal{I} \subset \mathcal{Q}(K):\mathcal{I} \neq \emptyset} \; \sum_{J \subset \{1,\dots,N(Q_{\mathcal{I}})\}:J \neq \emptyset} \Big|\varphi\Big(\bigcap_{j \in J} Z_j \cap K \cap \bigcap_{Q \in \mathcal{I}} Q\Big)\Big| \\
&\le \sum_{\mathcal{I} \subset \mathcal{Q}(K):\mathcal{I} \neq \emptyset} \mathbf{1}\Big\{\bigcap_{Q \in \mathcal{I}} Q \neq \emptyset\Big\} 2^{N(Q_{\mathcal{I}})} M(\varphi).
\end{aligned} \tag{22.22}$$

Similarly,

$$|\varphi(K)| \leq M(\varphi) \sum_{I \subset Q(K): I \neq \emptyset} \mathbf{1}\Big\{ \bigcap_{Q \in I} Q \neq \emptyset \Big\}. \tag{22.23}$$

A combinatorial argument (left to the reader) shows that

$$\operatorname{card}\Big\{ I \subset Q(K) : I \neq \emptyset \bigcap_{Q \in I} Q \neq \emptyset \Big\} \leq 2^{2^d} \operatorname{card} Q(K).$$

Since $\operatorname{card} Q(K) \leq V_d(K + B(0, \sqrt{d}))$, Steiner's formula (A.22) yields

$$\operatorname{card} Q(K) \leq \sum_{i=0}^{d} \kappa_{d-i} d^{(d-i)/2} V_i(K) \leq c_3 \overline{V}(K) \tag{22.24}$$

for some $c_3 > 0$.

Let $L \in \mathcal{K}^d$. To bound the expectation of the product $\varphi(Z \cap K)^2 \varphi(Z \cap L)^2$ we need to bound

$$\mathbb{E}[2^{N(Q_1+x_1)} 2^{N(Q_1+x_2)} 2^{N(Q_1+x_3)} 2^{N(Q_1+x_4)}],$$

where $x_1, \dots, x_4 \in \mathbb{R}^d$, see (22.22). Exercise 22.1 shows that these expectations are bounded uniformly in $x_1, \dots, x_4$. Combining this fact with (22.22), (22.23), and (22.24) yields inequality (22.19). The inequalities (22.20) follow in the same way. □

Proposition 22.4 *There exists $c_4 > 0$ such that for all $K, L \in \mathcal{K}^d$,*

$$\int \overline{V}(K \cap (L + x))\, dx \leq c_4 \overline{V}(K) \overline{V}(L). \tag{22.25}$$

Proof Using the same notation as in the proof of Proposition 22.3 and the fact that $\overline{V}$ is increasing and translation invariant, we obtain that

$$\begin{aligned} \int \overline{V}(K \cap (L + x))\, dx &\leq \sum_{I \subset Q(K): I \neq \emptyset} \int \overline{V}\Big(K \cap \bigcap_{Q \in I} Q \cap (L + x) \Big)\, dx \\ &\leq \sum_{I \subset Q(K): I \neq \emptyset} \mathbf{1}\Big\{ \bigcap_{Q \in I} Q \neq \emptyset \Big\} \int \overline{V}(L \cap (Q_1 + x))\, dx. \end{aligned}$$

Let B' denote a ball of radius $\sqrt{d}/2$. Then

$$\int \overline{V}(L \cap (Q_1 + x))\, dx \leq \int \overline{V}(L \cap (B' + x))\, dx =: \psi(L).$$

The function ψ is additive, continuous on $\mathcal{K}^{(d)}$, and translation invariant. Since B' and $\overline{V}$ are rotation invariant the same applies to ψ. Theorem A.24

implies that $\psi(L) \leq c_5\overline{V}(L)$ for some $c_5 > 0$. On the other hand, it was shown in the proof of Proposition 22.3 that

$$\operatorname{card}\left\{\mathcal{I} \subset Q(K) : \mathcal{I} \neq \emptyset, \bigcap_{Q\in\mathcal{I}} Q \neq \emptyset\right\} \leq c_5\overline{V}(K)$$

for some $c_5 > 0$. Combining the preceding inequalities, yields the assertion. □

Proof of Theorem 22.2 We intend to apply Theorem 21.3. Similarly to (22.6) and (22.7) we have for all $(x_1, K_1), (x_2, K_2) \in \mathbb{R}^d \times \mathcal{K}^{(d)}$ that

$$|D_{(x_1,K_1)}F_{W,\varphi}| \leq \varphi_Z((K_1 + x_1) \cap W), \tag{22.26}$$

$$|D^2_{(x_1,K_1),(x_2,K_2)}F_{W,\varphi}| \leq \varphi_Z((K_1 + x_x) \cap (K_2 + x_2) \cap W). \tag{22.27}$$

Let $\sigma := \sigma_{W,\varphi}$ and $F := \hat{F}_{W,\varphi}$. Using (22.26), (22.27) and Proposition 22.3 (and assuming $\gamma = 1$) yields

$$\alpha_{F,1}^2 \leq \frac{4c}{\sigma^4} \iiiint \overline{V}(K_x \cap W)\overline{V}(L_y \cap W)\overline{V}(K_x \cap M_z \cap W) \\ \times \overline{V}(L_y \cap M_z \cap W)\, dx\, dy\, dz\, \mathbb{Q}^3(d(K, L, M)).$$

Since the Wills functional is the sum of intrinsic volumes, it is monotone increasing on $\mathcal{K}^d$ and translation invariant. Combining these facts with Proposition 22.4 we can argue as in the proof of Theorem 22.1 to conclude that

$$\alpha_{F,1}^2 \leq \frac{4cc_4^3}{\sigma^4}\lambda_d(W) \int \overline{V}(K)^2\overline{V}(L)^2\overline{V}(M)^2\, \mathbb{Q}^3(d(K, L, M)).$$

Assumption (22.16) implies that the above integral is finite. The constants $\alpha_{F,2}$ and $\alpha_{F,3}$ can be treated in the same way.

Since the first inequality in (22.20) and Proposition 22.4 imply that $DF \in L^2(\mathbb{P} \otimes \lambda)$, Theorem 21.3 now yields the assertion. □

22.3 Central limit theorems

Recall that W is a convex compact set such that $\lambda_d(W) > 0$. As before we define $W_r := r^{1/d}W$ for $r > 0$. Then Theorem 22.2 yields a central limit theorem, provided that

$$\liminf_{r\to\infty} r^{-1}\sigma_{W_r,\varphi} > 0. \tag{22.28}$$

Theorem 22.5 *Let the assumptions of Theorem 22.2 be satisfied and suppose in addition that* (22.28) *holds. Then there exists* $\bar{c} > 0$ *such that*

$$d_1(\hat{F}_{W_r,\varphi}, N) \le \bar{c} r^{-1/2}, \quad r \ge 1. \tag{22.29}$$

In particular $\hat{F}_{W_r,\varphi} \xrightarrow{d} N$ *as* $r \to \infty$.

Proof Since

$$\overline{V}(W_r) = \sum_{j=0}^{d} r^{j/d} V_j(W)$$

we have $r^{-1}\overline{V}(W_r) \to 1$ as $r \to \infty$. Therefore (22.29) is an immediate consequence of Theorem 22.2 and assumption (22.28). □

We have seen in the first section of this chapter that (22.28) holds for the volume function $\varphi = V_d$. It can be expected that this asymptotic relation holds for general φ, at least under rather weak assumptions. Here we restrict ourselves to just two further intrinsic volumes and refer to the historical comments in the Appendix for further discussion. Recall the definition (22.3) of the second moments $\phi_{d,2}$ and $\phi_{d-1,2}$.

Proposition 22.6 *Assume that* $\phi_{d-1,2} + \phi_{d,2} < \infty$ *and that*

$$\mathbb{Q}(\{K \in \mathcal{K}^{(d)} : \gamma\phi_{d-1} V_d(K) \neq V_{d-1}(K)\}) > 0. \tag{22.30}$$

Then (22.28) *holds for the intrinsic volume* $\varphi = V_{d-1}$.

Proof The idea is to apply Exercise 18.9. It follows from (22.26) and Proposition 19.7 that

$$\mathbb{E}[D_{(x,K)}F_{W_r,\varphi}] = e^{-\gamma\phi_d} V_{d-1}((K+x) \cap W_r) - \gamma\phi_{d-1} e^{-\gamma\phi_d} V_d((K+x) \cap W_r)$$

for any $(x, K) \in \mathbb{R}^d \times \mathcal{K}^{(d)}$ and $r > 0$. Therefore,

$$\iint (\mathbb{E}[D_{(x,K)}F_{W_r,\varphi}])^2 \, dx \, \mathbb{Q}(dK) \ge \iint \mathbf{1}\{K + x \subset W_r\} \psi(K) \, dx \, \mathbb{Q}(dK),$$

where

$$\psi(K) := e^{-2\gamma\phi_d}(V_{d-1}(K) - \gamma\phi_{d-1} V_d(K))^2.$$

A change of variables gives us, for any fixed $K \in \mathcal{K}^{(d)}$, that

$$r \int \mathbf{1}\{K + x \subset W_r\} \, dx = \int \mathbf{1}\{r^{-1/d} K + y \subset W\} \, dy.$$

Note that for any y in the interior of W the inclusion $r^{-1/d}K + y \subset W$ holds for all sufficiently large r. Fatou's lemma (Lemma A.6) shows that

$$\liminf_{r\to\infty} r \iint (\mathbb{E}[D_{(x,K)}F_{W_r,\varphi}])^2 \, dx \, \mathbb{Q}(dK) \geq V_d(W) \int \psi(K) \, \mathbb{Q}(dK),$$

where we have used the fact that the boundary of a convex set has Lebesgue measure 0. The integrability assumptions apply that $F_{W_r,\varphi}$ is square integrable; see the proof of Theorem 22.2. Taking into account our assumption $\int \psi \, d\mathbb{Q} > 0$ the assertion now follows from Exercise 18.9. □

Finally we consider the Euler characteristic for an isotropic Boolean model in the plane.

Proposition 22.7 *Let Z be a Boolean model in $\mathbb{R}^2$ with an isotropic grain distribution satisfying $\phi_{1,2} + \phi_{2,2} < \infty$. Assume also that*

$$\mathbb{Q}\left(\left\{K \in \mathcal{K}^{(d)} : 1 - \frac{2\gamma\phi_1}{\pi} V_1(K) + \frac{\gamma^2\gamma_1^2}{\pi} V_2(K) \neq 0\right\}\right) > 0. \tag{22.31}$$

Then (22.28) *holds for the Euler characteristic $\varphi = V_0$.*

Proof The proof is the same as that of Proposition 22.6, using Theorem 19.9 in place of Proposition 19.7. □

22.4 Exercises

Exercise 22.1 Let ξ be a Poisson process on $\mathbb{R}^d \times C^{(d)}$ with intensity measure $\lambda = \gamma\lambda_d \otimes \mathbb{Q}$ and suppose that (17.11) holds. Let $C \subset \mathbb{R}^d$ be compact and $C_i := C + x_i$ for $i \in \{1, \dots, m\}$, where $m \in \mathbb{N}$ and $x_1, \dots, x_m \in \mathbb{R}^d$. For $I \subset [m]$ let N_I denote the number of points $(x, K) \in \xi$ such that $(K + x) \cap C_i \neq \emptyset$ for $i \in I$ and $(K + x) \cap C_i = \emptyset$ for $i \notin I$. Show that the N_I are independent Poisson random variables. Use this fact to show that

$$\mathbb{E}[2^{N(C_1)} \cdots 2^{N(C_m)}] \leq \exp\left[2^m(2^m - 1)\gamma \int \lambda_d(K \oplus C^*) \, \mathbb{Q}(dK)\right],$$

where $C^* := -C$ and the random variables $N(C_i)$ are defined by (22.21).

Appendix A

Some measure theory

A.1 General measure theory

We assume that the reader is familiar with measure theory but provide here the basic concepts and results. More detail can be found in [8, 21, 45].

If f is a function from a set $\mathbb{X}$ to a set $\mathbb{Y}$, then we write $f\colon \mathbb{X} \to \mathbb{Y}$ and denote by $f(x)$ the value of f at x. However, to avoid cumbersome notation and to save space we will often speak of the function $f(x)$ if there is no risk of ambiguity. Let f and g be functions from $\mathbb{X}$ to the extended real line $\overline{\mathbb{R}} := [-\infty, +\infty]$. Using the convention $0\infty = \infty 0 = 0(-\infty) = (-\infty)0 = 0$ we may define the product fg pointwise by $(fg)(x) := f(x)g(x)$. Similarly, we define the function $f + g$, whenever the sets $\{x : f(x) = -\infty, g(x) = \infty\}$ and $\{x : f(x) = \infty, g(x) = -\infty\}$ are empty. Let $\mathbf{1}_A$ denote the indicator function of A on $\mathbb{X}$ taking the value one on A and zero on $\mathbb{X} \setminus A$. If $f\colon A \to \overline{\mathbb{R}}$ we do not hesitate to interpret $\mathbf{1}_A f$ as a function on $\mathbb{X}$ with the obvious definition. Then the equality $\mathbf{1}_A f = g$ means that f and g agree on A (i.e. $f(x) = g(x)$ for all $x \in A$) and g vanishes outside A. Sometimes it is convenient to write $\mathbf{1}\{x : x \in A\}$ instead of $\mathbf{1}_A$ and $\mathbf{1}\{x \in A\}$ instead of $\mathbf{1}_A(x)$.

In what follows all sets will be subsets of a fixed set Ω. A class $\mathcal{H}$ of sets is said to be *closed* with respect to finite intersections if $A \cap B \in \mathcal{H}$ whenever $A, B \in \mathcal{H}$. In this case one also says that $\mathcal{H}$ is a *π-system*. One defines similarly the notion of $\mathcal{H}$ being closed with respect to countable intersections, or closed with respect to finite unions, and so on. A class $\mathcal{A}$ of sets is called a *field* (on Ω) if firstly, $\Omega \in \mathcal{A}$ and secondly, $A, B \in \mathcal{A}$ implies that $A \setminus B \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$, that is, $\mathcal{A}$ is closed with respect to finite unions, and set differences. A field $\mathcal{A}$ that is closed with respect to countable unions (or, equivalently, countable intersections) is called a *σ-field*. The symbol $\sigma(\mathcal{H})$ denotes the σ-field generated by $\mathcal{H}$, i.e. the smallest σ-field containing $\mathcal{H}$. In this case $\mathcal{H}$ is called a *generator* of $\sigma(\mathcal{H})$. A class $\mathcal{D}$ of sets is called a *monotone system* if it is closed with respect to countable increasing unions and with respect to countable decreasing intersections, i.e. $A_n \subset A_{n+1}$, $A_n \in \mathcal{D}$, implies $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$ and $A_n \supset A_{n+1}$, $A_n \in \mathcal{D}$, implies $\bigcap_{n=1}^{\infty} A_n \in \mathcal{D}$. Thus, a field $\mathcal{D}$ is a σ-field if it is monotone. A class $\mathcal{D}$ of sets is called a *Dynkin system* (also known as a λ-system) if $\Omega \in \mathcal{D}$ and if it is closed with respect to countable increasing unions and it is closed with respect to proper differences, i.e. if $A, B \in \mathcal{D}$ with $A \subset B$ implies $B \setminus A \in \mathcal{D}$. In this case $\mathcal{D}$ is a monotone system. The following theorem is a well-known version of a so-called *monotone class theorem*. If nothing else is stated then all definitions and theorems in this chapter can be found in [45], which is our basic reference for measure and probability theory.

Theorem A.1 (Monotone Class Theorem) *Let $\mathcal{H}$ and $\mathcal{D}$ be classes of subsets of Ω satisfying $\mathcal{H} \subset \mathcal{D}$. Suppose that $\mathcal{H}$ is a π-system and that $\mathcal{D}$ is a Dynkin system. Then $\sigma(\mathcal{H}) \subset \mathcal{D}$.*

A *measurable space* is a pair $(\mathbb{X}, \mathcal{X})$ where $\mathbb{X}$ is a set and $\mathcal{X}$ is a σ-field of subsets of $\mathbb{X}$. Let $(\mathbb{X}, \mathcal{X})$ be a measurable space and let f be a mapping from Ω into $\mathbb{X}$. If $\mathcal{F}$ is a σ-field on Ω, then f is said to be $\mathcal{F}$-$\mathcal{X}$-measurable if $f^{-1}(\mathcal{X}) \subset \mathcal{F}$, where $f^{-1}(\mathcal{X}) := \{f^{-1}(B) : B \in \mathcal{X}\}$. If there is no risk of ambiguity, we will also speak of $\mathcal{F}$-measurability or, simply, of measurability. The σ-field $\sigma(f)$ generated by f is the smallest σ-field $\mathcal{G}$ such that f is $\mathcal{G}$-$\mathcal{X}$-measurable; it is given by $f^{-1}(\mathcal{X})$. If $\mathbb{X} = \overline{\mathbb{R}}$ and nothing else is said, then $\mathcal{X}$ will always be given by the σ-field $\mathcal{B}(\overline{\mathbb{R}})$ on $\overline{\mathbb{R}}$, which is generated by the system of open sets in $\mathbb{R}$ along with $\{-\infty\}$ and $\{+\infty\}$. More generally, if $\mathbb{X} \in \mathcal{B}(\overline{\mathbb{R}})$, then we shall take $\mathcal{X}$ as the *trace σ-field* $\{B \cap \mathbb{X} : B \in \mathcal{B}(\overline{\mathbb{R}})\}$; see also Subsection A.2. Now let $\mathcal{F}$ be fixed. Then we denote by $\overline{\mathbb{R}}(\Omega)$ the set of all measurable functions from Ω to $\overline{\mathbb{R}}$. The symbols $\overline{\mathbb{R}}_+(\Omega)$, (resp. $\mathbb{R}_+(\Omega)$) denote the non-negative (resp. non-negative and finite) elements of $\overline{\mathbb{R}}(\Omega)$.

Theorem A.2 *Let f be a mapping from Ω into the measurable space $(\mathbb{X}, \mathcal{X})$ and let g be an $\overline{\mathbb{R}}$-valued function on Ω. Then g is $\sigma(f)$-measurable if and only if there exists an $\overline{\mathbb{R}}$-valued measurable function h from $\mathbb{X}$ into $\overline{\mathbb{R}}$ such that $g = h \circ f$.*

If $\mathbf{G}$ is a class of functions from Ω into $\mathbb{X}$, then we denote by $\sigma(\mathbf{G})$ the smallest σ-field $\mathcal{A}$ on Ω such that f is $\mathcal{A}$-$\mathcal{X}$-measurable for all $f \in \mathbf{G}$. It is given by $\sigma\{f^{-1}(B) : f \in \mathbf{G}, B \in \mathcal{X}\}$. The next theorem is a functional version of the monotone class theorem; see Theorem 2.12.9 in [9].

Theorem A.3 *Let $\mathbf{W}$ be a vector space of $\mathbb{R}$-valued, bounded functions on Ω that contains the constant functions. Further, suppose that for every increasing, uniformly bounded sequence of non-negative functions $f_n \in \mathbf{W}$,*

$n \in \mathbb{N}$, the function $f = \lim_{n\to\infty} f_n$ belongs to $\mathbf{W}$. *Let* $\mathbf{G}$ *be a subset of* $\mathbf{W}$ *that is closed with respect to multiplication. Then* $\mathbf{W}$ *contains all bounded $\sigma(\mathbf{G})$-measurable functions on Ω.*

Let $(\mathbb{X}, \mathcal{X})$ and $(\mathbb{Y}, \mathcal{Y})$ be two measurable spaces. When $\mathcal{X}$ and $\mathcal{Y}$ are fixed and nothing else is said, measurability on $\mathbb{X} \times \mathbb{Y}$ always refers to the *product σ-field* $\mathcal{X} \otimes \mathcal{Y}$ generated by all sets $A \times B$ with $A \in \mathcal{X}$ and $B \in \mathcal{Y}$. The measurable space $(\mathbb{X} \times \mathbb{Y}, \mathcal{X} \otimes \mathcal{Y})$ is called the *product* of $(\mathbb{X}, \mathcal{X})$ and $(\mathbb{Y}, \mathcal{Y})$. Given a finite number $(\mathbb{X}_1, \mathcal{X}_1), \dots, (\mathbb{X}_n, \mathcal{X}_n)$ of measurable spaces we can define the product $(\mathbb{X}_1 \times \cdots \times \mathbb{X}_n, \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n)$ in a similar way. In the case where $(\mathbb{X}_i, \mathcal{X}_i) = (\mathbb{X}, \mathcal{X})$ for every $i \in \{1, \dots, n\}$ we abbreviate this product as $(\mathbb{X}^n, \mathcal{X}^n)$ and refer to it as the n-th power of $(\mathbb{X}, \mathcal{X})$. Let $(\mathbb{X}_i, \mathcal{X}_i)$, $i \in \mathbb{N}$, be a countable collection of measurable spaces. The infinite product $\otimes_{i=1}^\infty \mathcal{X}_i$ is the σ-field on $\times_{i=1}^\infty \mathbb{X}_i$ generated by the sets

$$B_1 \times \cdots \times B_n \times \bigtimes\nolimits_{i=n+1}^\infty \mathbb{X}_i, \tag{A.1}$$

where $B_i \in \mathcal{X}_i$ for $i \in \{1, \dots, n\}$ and $n \in \mathbb{N}$.

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space. A function $\lambda \colon \mathcal{X} \to [0, \infty]$ is said to be *additive* if $\lambda(B \cup B') = \lambda(B) + \lambda(B')$ for all disjoint $B, B' \in \mathcal{X}$. In this case λ is finitely additive in the obvious sense. A *measure* on a measurable space $(\mathbb{X}, \mathcal{X})$ is a function $\lambda \colon \mathcal{X} \to [0, \infty]$ such that $\lambda(\emptyset) = 0$ and such that λ is σ-additive (countably additive), that is

$$\lambda\Big(\bigcup_{n=1}^\infty B_n\Big) = \sum_{n=1}^\infty \lambda(B_n),$$

whenever $B_1, B_2, \dots$ are pairwise disjoint sets in $\mathcal{X}$. In this case the triple $(\mathbb{X}, \mathcal{X}, \lambda)$ is called a *measure space*. For simplicity we sometimes speak of a measure on $\mathbb{X}$. A measure λ on $(\mathbb{X}, \mathcal{X})$ is said to be *σ-finite* on a class $\mathcal{H} \subset \mathcal{X}$ if there are sets $B_n \in \mathcal{H}$, $n \in \mathbb{N}$, such that $\cup_{n=1}^\infty B_n = \mathbb{X}$ and $\lambda(B_n) < \infty$ for all $n \in \mathbb{N}$. In the case $\mathcal{H} = \mathcal{X}$ we say just that λ is σ-finite and that $(\mathbb{X}, \mathcal{X}, \lambda)$ is a σ-finite measure space. The following result can easily be proved using Theorem A.1.

Theorem A.4 *Let μ, ν be measures on $(\mathbb{X}, \mathcal{X})$. Assume that μ and ν agree on a π-system $\mathcal{H}$ with $\sigma(\mathcal{H}) = \mathcal{X}$. Assume also that there is an increasing sequence $B_n \in \mathcal{H}$, $n \in \mathbb{N}$, such that $\mu(B_n) = \nu(B_n) < \infty$ for all $n \in \mathbb{N}$ and $\cup_{n=1}^\infty B_n = \mathbb{X}$. Then $\mu = \nu$.*

Let $(\mathbb{X}, \mathcal{X}, \lambda)$ be a measure space. The *integral* $\int f \, d\lambda$ of $f \in \overline{\mathbb{R}}_+(\mathbb{X})$ with respect to λ is defined as follows. If f is *simple*, that is of the form

$$f = \sum_{i=1}^{m} c_i \mathbf{1}_{B_i}$$

for some $m \in \mathbb{N}$, $c_1, \dots, c_m \in \mathbb{R}_+$ and $B_1, \dots, B_m \in \mathcal{X}$, then

$$\int f \, d\lambda := \sum_{i=1}^{m} c_i \lambda(B_i).$$

Any $f \in \overline{\mathbb{R}}_+(\mathbb{X})$ is the limit of simple functions f_n given by

$$f_n(x) := n\mathbf{1}\{n < f(x)\} + \sum_{j=1}^{n2^n-1} j2^{-n}\mathbf{1}\{j2^{-n} < f(x) \le (j+1)2^{-n}\}$$

and one defines $\int f \, d\lambda$ as the finite or infinite limit of $\int f_n \, d\lambda$. To extend the integral to $f \in \overline{\mathbb{R}}(\mathbb{X})$ we define

$$\int f \, d\lambda = \int f^+ \, d\lambda - \int f^- \, d\lambda,$$

whenever one of the integrals on the right-hand side is finite. Here

$$f^+(x) := f(x) \vee 0, \quad f^-(x) := -(f(x) \wedge 0),$$

and $a \vee b$ (respectively $a \wedge b$) denote the maximum (minimum) of two numbers $a, b \in \mathbb{R}$. For definiteness we put $\int f \, d\lambda := 0$ in the case $\int f^+ \, d\lambda = \int f^- \, d\lambda = \infty$. Sometimes we abbreviate $\lambda(f) := \int f \, d\lambda$. One also writes $\int_B f \, d\lambda := \lambda(\mathbf{1}_B f)$.

For any $p > 0$ let $L^p(\lambda) = \{f \in \mathbb{R}(\mathbb{X}) : \lambda(|f|^p) < \infty\}$. The mapping $f \mapsto \lambda(f)$ is linear on $L^1(\lambda)$ and satisfies the *triangle inequality* $|\lambda(f)| \le \lambda(|f|)$. If $f \ge 0$, λ-a.e. (that is $\lambda(\{x \in \mathbb{X} : f(x) < 0\}) = 0$) then $\lambda(f) = 0$ implies that $f = 0$ λ-a.e.

The next frequently used results show that the integral has nice continuity properties.

Theorem A.5 (Monotone Convergence) *Let $f_n \in \overline{\mathbb{R}}_+(\mathbb{X})$, $n \in \mathbb{N}$, be such that $f_n \uparrow f$ for some $f \in \overline{\mathbb{R}}_+(\mathbb{X})$. Then $\lambda(f_n) \uparrow \lambda(f)$.*

Lemma A.6 (Fatou) *Let $f_n \in \overline{\mathbb{R}}_+(\mathbb{X})$, $n \in \mathbb{N}$. Then*

$$\liminf_{n\to\infty} \lambda(f_n) \ge \lambda(\liminf_{n\to\infty} f_n).$$

Theorem A.7 (Dominated Convergence) *Let $f_n \in \overline{\mathbb{R}}_+(\mathbb{X})$, $n \in \mathbb{N}$, be such that $f_n \to f$ for some $f \in \overline{\mathbb{R}}_+(\mathbb{X})$ and $|f_n| \le g$ for some $g \in L^1(\lambda)$. Then $\lambda(f_n) \to \lambda(f)$.*

Suppose that $p, q > 0$ with $1/p+1/q = 1$ and let $f \in L^p(\lambda)$ and $g \in L^q(\lambda)$. *Hölder's inequality* says that then

$$\int |fg|\, d\lambda \le \Big(\int |f|^p\, d\lambda\Big)^{1/p}\Big(\int |f|^p\, d\lambda\Big)^{1/q}. \tag{A.2}$$

In the special case $p = q = 2$ this is known as the *Cauchy-Schwarz inequality*. Hölder's inequality can be generalized to

$$\int |f_1| \cdots |f_m|\, d\lambda \le \prod_{i=1}^{m}\Big(\int |f_i|^{p_i}\, d\lambda\Big)^{1/p_i}, \tag{A.3}$$

where $m \in \mathbb{N}$, $p_1, \dots, p_m$ are positive numbers with $1/p_1 + \cdots + 1/p_m = 1$ and $f_i \in L^{p_i}(\lambda)$ for $i \in \{1, \dots, m\}$.

A quick consequence of (A.2) is the *Minkowski inequality*

$$\Big(\int |f+g|^p\, d\lambda\Big)^{1/p} \le \Big(\int |f|^p\, d\lambda\Big)^{1/p} + \Big(\int |g|^p\, d\lambda\Big)^{1/p}, \quad f, g \in L^p(\lambda).$$

Identifying $f, \tilde{f} \in L^p(\lambda)$ whenever $\lambda(f \ne \tilde{f}) = 0$, and giving f the norm $\big(\int |f|^p\, d\lambda\big)^{1/p}$, $L^p(\lambda)$ becomes a normed vector space. A remarkable feature of this space is its *completeness*. This means that if (f_n) is a Cauchy sequence in $L^p(\lambda)$, that is $\lim_{m,n\to\infty} \int |f_m - f_n|^p\, d\lambda = 0$, then there is a $f \in L^p(\lambda)$ such that $\lim_{n\to\infty} f_n = f$ in $L^p(\lambda)$.

Let λ, ν be measures on $(\mathbb{X}, \mathcal{X})$. If $\nu(B) = 0$ for all $B \in \mathcal{X}$ with $\lambda(B) = 0$, then ν is said to be *absolutely continuous* with respect to λ and one writes $\nu \ll \lambda$. The two measures ν and λ are said to be *mutually singular* if there exists some $A \in \mathcal{X}$ such that $\nu(A) = \lambda(\mathbb{X} \setminus A) = 0$. A *finite signed measure* (on $\mathbb{X}$ or $(\mathbb{X}, \mathcal{X})$) is a σ-additive bounded function $\nu\colon \mathcal{X} \to \mathbb{R}$ such that $\nu(\emptyset) = 0$.

Theorem A.8 (Hahn-Jordan Decomposition) *Let ν be a finite signed measure on $\mathbb{X}$. Then there exist uniquely determined mutually singular finite measures ν_+ and ν_- such that $\nu = \nu_+ - \nu_-$.*

Theorem A.9 (Radon-Nikodým) *Let λ, ν be σ-finite measures on $(\mathbb{X}, \mathcal{X})$ such that $\nu \ll \lambda$. Then there exists $f \in \mathbb{R}_+(\mathbb{X})$ such that*

$$\nu(B) = \int_B f\, d\lambda, \quad B \in \mathcal{X}. \tag{A.4}$$

The function f in (A.4) is called the *Radon-Nikodým derivative* (or *density*) of ν with respect to λ. We write $\nu = f\lambda$. If g is another such function then $f = g$, λ-a.e., that is $\lambda(\{f \ne g\}) = 0$, where $\{f \ne g\}$ stands for $\{x \in \mathbb{X} : f(x) \ne g(x)\}$.

We need to integrate with respect to a finite signed measure ν on $(\mathbb{X}, \mathcal{X})$. For $f \in \overline{\mathbb{R}}(\mathbb{X})$ we define

$$\int f \, d\nu := \int f \, d\nu_+ - \int f \, d\nu_-$$

whenever this expression is well defined. This can be written as an ordinary integral as follows. Let ρ be a finite measure such that $\nu_- \ll \rho$ and $\nu_+ \ll \rho$, and let h_- and h_+ denote the corresponding Radon-Nikodým derivatives. Then

$$\int f \, d\nu = \int f(h_+ - h_-) \, d\rho, \tag{A.5}$$

where the values $-\infty$ and ∞ are allowed. A natural choice is $\rho = \nu_+ + \nu_-$. This is the *total variation measure* of ν.

Any countable sum of measures is a measure. A measure λ is said to be *s-finite* if

$$\lambda = \sum_{n=1}^{\infty} \lambda_n \tag{A.6}$$

is a countable sum of finite measures λ_n. For any $f \in \overline{\mathbb{R}}(\mathbb{X})$ we then have

$$\int f \, d\lambda = \sum_{n=1}^{\infty} \int f \, d\lambda_n. \tag{A.7}$$

This remains true for $f \in L^1(\lambda)$. Any σ-finite measure is s-finite. The converse is not true. If μ is a measure on $(\mathbb{X}, \mathcal{X})$ such that $\mu(\mathbb{X}) < \infty$ we can define a measure ν by multiplying μ by infinity. (Recall that $0 \cdot \infty = 0$). Then $\nu(B) = 0$ if $\mu(B) = 0$ and $\nu(B) = \infty$ otherwise. If $\mu(\mathbb{X}) > 0$ then ν is s-finite but not σ-finite. If the measure ν is of this form, i.e. if $\nu = \infty \cdot \mu$ for some finite measure μ on $\mathbb{X}$ with $\mu(X) > 0$, then we say that ν is *totally infinite*. The sum of a σ-finite and a totally infinite measure is s-finite. The converse is also true:

Theorem A.10 *Let λ be an s-finite measure. Then there exist a σ-finite measure λ' and a totally infinite measure λ'' such that λ' and λ'' are mutually singular and $\lambda = \lambda' + \lambda''$.*

Proof Assume that λ is given as in (A.6) and let ν be a finite measure such that $\lambda_n \ll \nu$ for all $n \in \mathbb{N}$. (The construction of ν is left as an exercise.) By Theorem A.9 there are $f_n \in \mathbb{R}_+(\mathbb{X})$ such that $\lambda_n = f_n \nu$, i.e. $\lambda_n(B) = \nu(\mathbf{1}_B f_n)$ for all $B \in \mathcal{X}$. Define $f := \sum_{n=1}^{\infty} f_n$. Then f is a measurable function from $\mathbb{X}$ to $[0, \infty]$ and by Theorem A.5 (monotone convergence), $\lambda = f\nu$. It is easy to see that the restriction of λ to $A := \{f < \infty\}$ (defined by $B \mapsto \lambda(A \cap B)$) is σ-finite. Moreover, by the definition of integrals the restriction of λ to $\mathbb{X} \setminus A$ is totally infinite. □

If the measure λ is given by (A.6), then $\nu_n := \sum_{j=1}^{n} \lambda_j \uparrow \lambda$, in the sense that $\nu_n(B) \le \nu_{n+1}(B)$ for all $n \in \mathbb{N}$ and all $B \in \mathcal{X}$ and $\nu_n(B) \to \lambda(B)$ as $n \to \infty$. We use this notation also for general measures. Theorem A.5 on monotone convergence can be generalized as follows.

Theorem A.11 *Let ν_n, $n \in \mathbb{N}$, be σ-finite measures such that $\nu_n \uparrow \nu$ for some measure ν. Assume also that $f_n \in \overline{\mathbb{R}}(\mathbb{X})$, $n \in \mathbb{N}$, satisfy $\nu_n(f_n < 0) = 0$, $n \in \mathbb{N}$, and $f_n \uparrow f$ for some $f \in \overline{\mathbb{R}}(\mathbb{X})$. Then $\nu_n(f_n) \uparrow \nu(f)$.*

Proof Let ν_0 be a finite measure such that $\nu_n \ll \nu_0$ for all $n \in \mathbb{N}$ and let $g_n \in \mathbb{R}_+(\mathbb{X})$ be the corresponding Radon-Nikodým derivatives. Since $\nu_n \le \nu_{n+1}$ we have that $g_n \le g_{n+1}$, ν_0-a.e. Hence there exists $g \in \overline{\mathbb{R}}_+(\mathbb{X})$ such that $g_n \uparrow g$ holds ν_0-a.e. It follows that

$$\nu(f) = \lim_{n\to\infty} \nu_n(f) = \lim_{n\to\infty} \nu_0(g_n f) = \nu_0(gf),$$

where we have used monotone convergence to obtain the final identity. The measurable functions $f_n' := \mathbf{1}\{f_n \ge 0\} f_n$, are again increasing in $n \in \mathbb{N}$. By assumption we have $\nu_n(f_n) = \nu_n(f_n')$ for each $n \in \mathbb{N}$ and it follows from monotone convergence that

$$\nu_n(f_n) = \nu_n(f_n') = \nu_0(g_n f_n') \uparrow \nu_0(gf') = \nu(f'),$$

where $f' := \lim_{n\to\infty} f_n$. Let $k \in \mathbb{N}$. By assumption $\nu_k(f_n < 0) = 0$ for all $n \ge k$ and hence $\nu_k(f_n < 0 \text{ for all } n \ge k) = 0$. Therefore $\nu_k(f \ne f') = 0$ and then also $\nu(f \ne f') = 0$. This concludes the proof. □

Let λ be an s-finite measure on $(\mathbb{X}, \mathcal{X})$. Then $(\mathbb{X}, \mathcal{X}, \lambda)$ is said to be an s-finite measure space. Let $(\mathbb{Y}, \mathcal{Y})$ be an additional measurable space. If $f \in \overline{\mathbb{R}}(\mathbb{X} \times \mathbb{Y})$, then $y \mapsto \int f(x,y)\, \lambda(dx)$ is a measurable function on $\mathbb{Y}$. Hence, if ν is a measure on $(\mathbb{Y}, \mathcal{Y})$ we can form the double integral $\iint f(x,y)\, \lambda(dx)\, \nu(dy)$. In particular we can define the *product* $\lambda \otimes \nu$ as the measure on $(\mathbb{X} \times \mathbb{Y}, \mathcal{X} \otimes \mathcal{Y})$ given by

$$(\lambda \otimes \nu)(A) := \iint \mathbf{1}_A(x,y)\, \lambda(dx)\, \nu(dy), \quad A \in \mathcal{X} \otimes \mathcal{Y}. \tag{A.8}$$

If ν is also s-finite, then so is $\lambda \otimes \nu$. The product is linear with respect to countable sums and therefore also associative.

Theorem A.12 (Fubini's Theorem) *Let $(\mathbb{X}, \mathcal{X}, \lambda)$ and $(\mathbb{Y}, \mathcal{Y}, \nu)$ be two s-finite measure spaces and $f \in \mathbb{R}_+(\mathbb{X} \times \mathbb{Y})$. Then*

$$\iint f(x,y)\,\lambda(dx)\,\nu(dy) = \iint f(x,y)\,\nu(dy)\,\lambda(dx) \tag{A.9}$$

and both integrals coincide with $(\lambda \otimes \nu)(f)$. These assertions remain true for all $f \in L^1(\lambda \otimes \nu)$.

If λ and ν are σ-finite, then $\lambda \otimes \nu$ is σ-finite and uniquely determined by

$$(\lambda \otimes \nu)(B \times C) = \lambda(B)\nu(C), \quad B \in \mathcal{X}, C \in \mathcal{Y}. \tag{A.10}$$

In this case the proof of the above theorem can be found in the textbooks. The slightly more general case can be derived by using the formula (A.7) for both λ and ν and then applying Fubini's theorem in the case of two finite measures.

Let us now consider s-finite measure spaces $(\mathbb{X}_i, \mathcal{X}_i, \lambda_i)$, $i \in \{1, \dots, n\}$, for some $n \geq 2$. Then the product $\otimes_{i=1}^n \lambda_i$ of $\lambda_1, \dots, \lambda_n$ is an s-finite measure on $(\times_{i=1}^n \mathbb{X}_i, \otimes_{i=1}^n \mathcal{X}_i)$, defined inductively in the obvious way. Of particular importance is the case $(\mathbb{X}_i, \mathcal{X}_i, \lambda_i) = (\mathbb{X}, \mathcal{X}, \lambda)$ for all $i \in \{1, \dots, n\}$. Then we write $\lambda^n := \otimes_{i=1}^n \lambda_i$.

The power ν^n can also be defined for a finite signed measure ν. Similarly to (A.5) we have for $f \in \mathbb{R}(\mathbb{X}^n)$ that

$$\int f\,d\nu^n = \int f(h_+ - h_-)^{\otimes n}\,d|\nu|, \tag{A.11}$$

where the tensor product $(h_+ - h_-)^{\otimes n}$ is defined by (18.6).

A *kernel* from $(\mathbb{X}, \mathcal{X})$ to $(\mathbb{Y}, \mathcal{Y})$ (or abbreviated: from $\mathbb{X}$ to $\mathbb{Y}$) is a mapping K from $\mathbb{X} \times \mathcal{Y}$ to $\overline{\mathbb{R}}_+$ such that $K(\cdot, A)$ is measurable for all $A \in \mathcal{Y}$ and such that $K(x, \cdot)$ is a measure on $\mathbb{Y}$ for all $x \in \mathbb{X}$. It is called a *probability kernel* (respectively *sub-probability kernel*) if $K(x, \mathbb{Y}) = 1$ (≤ 1) for all $x \in \mathbb{X}$. A countable sum of kernels is again a kernel. A countable sum of sub-probability kernels is called an s-finite kernel. If K is an s-finite kernel and $f \in \overline{\mathbb{R}}(\mathbb{X} \times \mathbb{Y})$ then $y \mapsto \int f(x,y)\,K(x,dy)$ is a measurable function. If, in addition, λ is an s-finite measure on $(\mathbb{X}, \mathcal{X})$, then

$$(\lambda \otimes K)(A) := \iint \mathbf{1}_A(x,y)\,K(x,dy)\,\lambda(dx), \quad A \in \mathcal{X} \otimes \mathcal{Y},$$

defines an s-finite measure $\lambda \otimes K$ on $(\mathbb{X} \times \mathbb{Y}, \mathcal{X} \otimes \mathcal{Y})$.

Theorem A.13 (Disintegration) *Let $(\mathbb{X}, \mathcal{X})$ be a measurable space and let $(\mathbb{Y}, \mathcal{Y})$ be a Borel space. Let ν be a measure on $(\mathbb{X} \times \mathbb{Y}, \mathcal{X} \otimes \mathcal{Y})$ such that*

$\lambda := \nu(\cdot \times \mathbb{Y})$ is σ-finite. Then there exists a stochastic kernel K from $\mathbb{X}$ to $\mathbb{Y}$ such that $\nu = \lambda \otimes K$.

In the remainder of this subsection we prove Proposition 4.3. To this end we need some notation and auxiliary results. Let $\mathbf{N}_{<\infty}$ denote the set of all $\mu \in \mathbf{N} := \mathbf{N}(\mathbb{X})$ with $\mu(\mathbb{X}) < \infty$. For $\mu \in \mathbf{N}_{<\infty}$ the recursion (4.9) is solved by

$$\mu^{(m)} = \int \cdots \int \mathbf{1}\{(x_1, \dots, x_m) \in \cdot\} \Big(\mu - \sum_{j=1}^{m-1} \delta_{x_j}\Big)(dx_m) \cdots \mu(dx_1), \quad \text{(A.12)}$$

where the integrations are with respect to finite signed measures. Note that $\mu^{(m)}$ is a signed measure such that $\mu^{(m)}(C) \in \mathbb{Z}$ for all $C \in \mathcal{X}^m$. At this stage it might not be obvious that $\mu^{(m)}(C) \geq 0$. If, however, μ is given by (4.3) with $k \in \mathbb{N}$, then Lemma 4.2 shows that (A.12) coincides with (4.4). Hence $\mu^{(m)}$ is a measure in this case. For each $\mu \in \mathbf{N}_{<\infty}$ we denote by $\mu^{(m)}$ the signed measure (A.12). This is in accordance with the recursion (4.9). The next lemma shows that $\mu^{(m)}$ is a measure.

Lemma A.14 *Let $\mu \in \mathbf{N}_{<\infty}$ and $m \in \mathbb{N}$. Then $\mu^{(m)}(C) \geq 0$ for all $C \in \mathcal{X}^m$.*

Proof Let $B_1, \dots, B_m \in \mathcal{X}$ and let Π_m denote the set of partitions of $\{1, \dots, m\}$. The definition (A.12) implies that

$$\mu^{(m)}(B_1 \times \cdots \times B_m) = \sum_{\pi \in \Pi_m} c_\pi \prod_{J \in \pi} \mu\Big(\bigcap_{i \in J} B_i\Big), \quad \text{(A.13)}$$

where the coefficients $c_\pi \in \mathbb{R}$ do not depend on $B_1, \dots, B_m$ and μ. For instance

$$\begin{aligned}\mu^{(3)}(B_1 \times B_2 \times B_3) &= \mu(B_1)\mu(B_2)\mu(B_3) - \mu(B_1)\mu(B_2 \cap B_3) \\ &\quad - \mu(B_2)\mu(B_1 \cap B_3) - \mu(B_3)\mu(B_1 \cap B_2) + 2\mu(B_1 \cap B_2 \cap B_3).\end{aligned}$$

It follows that the left-hand side of (A.13) is determined by the values of μ on the field generated by $B_1, \dots, B_m$. This is the smallest field containing all of the non-empty sets (*atoms*) of the form

$$B = B_1^{i_1} \cap \cdots \cap B_m^{i_m},$$

where $i_1, \dots, i_m \in \{0, 1\}$ and, for $B \subset \mathbb{X}$, $B^1 := B$ and $B^0 := \mathbb{X} \setminus B$. Let $\mathcal{A}$ denote the set of all these atoms. For $B \in \mathcal{A}$ we take $x \in B$ and let $\mu_B := \mu(B)\delta_x$. Then the measure

$$\mu' := \sum_{B \in \mathcal{A}} \mu_B$$

is a finite sum of Dirac measures and (A.13) implies that

$$(\mu')^{(m)}(B_1 \times \cdots \times B_m) = \mu^{(m)}(B_1 \times \cdots \times B_m).$$

Therefore it follows from (4.4) (applied to μ') that $\mu^{(m)}(B_1 \times \cdots \times B_m) \geq 0$.

Let $\mathcal{A}_m$ be the system of all finite disjoint unions of sets $C_1 \times \cdots \times C_m$, with $C_1, \dots, C_m \in \mathcal{X}$. This is a field; see Proposition 3.2.3 in [21]. From the first step of the proof and additivity of $\mu^{(m)}$ we deduce that $\mu^{(m)}(A) \geq 0$ holds for all $A \in \mathcal{A}_m$. The system $\mathcal{M}$ of all sets $A \in \mathcal{X}^m$ with the property that $\mu^{(m)}(A) \geq 0$, is closed with respect to (countable) monotone unions and intersections. Hence a monotone class theorem (see e.g. Theorem 4.4.2 in [21]) implies that $\mathcal{M} = \mathcal{X}^m$. Therefore $\mu^{(m)}$ is non-negative. □

Lemma A.15 *Let $\mu, \nu \in \mathbf{N}_{<\infty}$ with $\mu \leq \nu$. Let $m \in \mathbb{N}$. Then $\mu^{(m)} \leq \nu^{(m)}$.*

Proof By a monotone class argument it suffices to show that

$$\mu^{(m)}(B_1 \times \cdots \times B_m) \leq \nu^{(m)}(B_1 \times \cdots \times B_m) \tag{A.14}$$

for all $B_1, \dots, B_m \in \mathcal{X}$. Fixing the latter sets we define the system $\mathcal{A}$ of atoms of the generated field as in the proof of Lemma A.14. For $B \in \mathcal{A}$ we choose $x \in B$ and define $\mu_B := \mu(B)\delta_x$ and $\nu_B := \nu(B)\delta_x$. Then

$$\mu' := \sum_{B \in \mathcal{A}} \mu_B, \quad \nu' := \sum_{B \in \mathcal{A}} \nu_B$$

are finite sums of Dirac measures satisfying $\mu' \leq \nu'$. By (A.13) we have

$$\mu^{(m)}(B_1 \times \cdots \times B_m) = (\mu')^{(m)}(B_1 \times \cdots \times B_m).$$

A similar identity holds for $\nu^{(m)}$ and $(\nu')^{(m)}$. Therefore (4.4) (applied to μ' and ν') implies the asserted inequality (A.14). □

We are now in a position to prove a slightly more detailed version of Proposition 4.3.

Proposition A.16 *For each $\mu \in \mathbf{N}$ there is a sequence $\mu^{(m)}$, $m \in \mathbb{N}$, of measures on $(\mathbb{X}^m, \mathcal{X}^m)$ satisfying $\mu^{(1)} := \mu$ and the recursion* (4.9). *Moreover, the mapping $\mu \mapsto \mu^{(m)}$ is measurable. Finally, if $\mu_n \uparrow \mu$ for a sequence (μ_n) of finite measures in $\mathbf{N}$, then $(\mu_n)^{(m)} \uparrow \mu^{(m)}$.*

Proof For $\mu \in \mathbf{N}_{<\infty}$ the functions defined by (A.12) satisfy the recursion (4.9) and are measures by Lemma A.14.

For general $\mu \in \mathbf{N}$ we proceed by induction. For $m = 1$ we have $\mu^{(1)} = \mu$ and there is nothing to prove. Assume now that $m \geq 1$ and that the measures

$\mu^{(1)}, \dots, \mu^{(m)}$ satisfy the first $m-1$ recursions and have the properties stated in the proposition. Then (4.9) enforces the definition

$$\mu^{(m+1)}(C) := \int K(x_1, \dots, x_m, \mu, C)\, \mu^{(m)}(d(x_1, \dots, x_m)) \tag{A.15}$$

for $C \in \mathcal{X}^{m+1}$, where

$$\begin{aligned} &K(x_1, \dots, x_m, \mu, C) \\ &\quad := \int \mathbf{1}\{(x_1, \dots, x_{m+1}) \in C\}\, \mu(dx_{m+1}) - \sum_{j=1}^{m} \mathbf{1}\{(x_1, \dots, x_m, x_j) \in C\}. \end{aligned}$$

The function $K \colon \mathbb{X}^m \times \mathbf{N} \times \mathcal{X}^m \to (-\infty, \infty]$ is a *signed kernel* in the following sense. The mapping $(x_1, \dots, x_m, \mu) \mapsto K(x_1, \dots, x_m, \mu, C)$ is measurable for all $C \in \mathcal{X}^{m+1}$, while $K(x_1, \dots, x_m, \mu, \cdot)$ is σ-additive for all $(x_1, \dots, x_m, \mu) \in \mathbb{X}^m \times \mathbf{N}$. Hence it follows from (A.15) and the measurability properties of $\mu^{(m)}$ (which are part of the induction hypothesis) that $\mu^{(m+1)}(C)$ is a measurable function of μ.

Next we show that

$$K(x_1, \dots, x_m, \mu, C) \geq 0, \quad \mu^{(m)}\text{-a.e. } (x_1, \dots, x_m) \in \mathbb{X}^m \tag{A.16}$$

holds for all $\mu \in \mathbf{N}$ and all $C \in \mathcal{X}^{m+1}$. Since $\mu^{(m)}$ is a measure (by induction hypothesis), (A.15), (A.16) and monotone convergence then imply that $\mu^{(m+1)}$ is a measure. Fix $\mu \in \mathbf{N}$ and choose a sequence (μ_n) of finite measures in $\mathbf{N}$ such that $\mu_n \uparrow \mu$. Lemma A.14 (applied to μ_n and $m + 1$) implies that

$$K(x_1, \dots, x_m, \mu_n, C) \geq 0, \quad (\mu_n)^{(m)}\text{-a.e. } (x_1, \dots, x_m) \in \mathbb{X}^m,\ n \in \mathbb{N}.$$

Indeed, we have for all $B \in \mathcal{X}^m$ that

$$\int_B K(x_1, \dots, x_m, \mu_n, C)\, (\mu_n)^{(m)}(d(x_1, \dots, x_m)) = (\mu_n)^{(m+1)}((B \times \mathbb{X}) \cap C) \geq 0.$$

Since $K(x_1, \dots, x_m, \cdot, C)$ is increasing, this implies that

$$K(x_1, \dots, x_m, \mu, C) \geq 0, \quad (\mu_n)^{(m)}\text{-a.e. } (x_1, \dots, x_m) \in \mathbb{X}^m,\ n \in \mathbb{N}.$$

By the induction hypothesis we have that $(\mu_n)^{(m)} \uparrow \mu^{(m)}$ so that (A.16) follows.

To finish the induction it remains to show that $(\mu_n)^{(m+1)}(C) \uparrow \mu^{(m+1)}(C)$ for each $C \in \mathcal{X}^{n+1}$. For each $n \in \mathbb{N}$, let us define a measurable function $f_n \colon \mathbb{X}^m \to (-\infty, \infty]$ by $f_n(x_1, \dots, x_m) := K(x_1, \dots, x_m, \mu_n, C)$. Then $f_n \uparrow$

f, where the function f is given by $f(x_1, \dots, x_m) := K(x_1, \dots, x_m, \mu, C)$. Hence we can apply Theorem A.11 (and (A.15)) to obtain that

$$(\mu_n)^{(m+1)}(C) = (\mu_n)^{(m)}(f_n) \uparrow \mu^{(m)}(f) = \mu^{(m+1)}(C).$$

This finishes the proof. □

The following property of the factorial measures is useful.

Lemma A.17 *Let $n \in \mathbb{N}$ and $B_1, \dots, B_n \in \mathcal{X}$ be pairwise disjoint. Let $\mathcal{A}$ be the field generated by these sets and let $\mu \in \mathbf{N}$. Then there is a $\nu \in \mathbf{N}$ of the form* (4.3) *such that $\mu^{(m)}(B) = \nu^{(m)}(B)$ for each $m \in \mathbb{N}$ and each $B \in \mathcal{A}^m$.*

Proof Let $\mathcal{A}_0$ be the system of all intersections $B_1^{i_1} \cap \dots \cap B_n^{i_n}$, where $i_1, \dots, i_m \in \{0, 1\}$. For each $B \in \mathcal{A}_0$ with $B \neq \emptyset$ we take some $x_B \in B$ and define

$$\nu := \sum_{B \in \mathcal{A}_0 : B \neq \emptyset} \mu(B) \delta_{x_B}.$$

Then $\mu(C) = \nu(C)$ for each $C \in \mathcal{A}$. The assertion now follows by induction based on the recursion (4.9). □

For any $\mu \in \mathbb{N}$ and $m \in \mathbb{N}$ the measure $\mu^{(m)}$ is *symmetric*, that is

$$\int f(x_1, \dots, x_m)\, \mu^{(m)}(d(x_1, \dots, x_m)) \\ = \int f(x_{\pi(1)}, \dots, x_{\pi(m)})\, \mu^{(m)}(d(x_1, \dots, x_m)) \qquad \text{(A.17)}$$

for each $f \in \mathbb{R}_+(\mathbb{X}^m)$ and all bijective mappings π from $[m] := \{1, \dots, m\}$ to $[m]$. To see this, we may first assume that $\mu(\mathbb{X}) < \infty$. If f is the product of indicator functions, then (A.17) is implied by (A.13). The case of a general $f \in \mathbb{R}_+(\mathbb{X}^m)$ follows by a monotone class argument. For a general $\mu \in \mathbf{N}$ we can use the final assertion of Proposition A.16. Product measures λ^m yield other examples of measures satisfying (A.17).

A.2 Metric spaces

A *metric* on a set $\mathbb{X}$ is a symmetric function $\rho\colon \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+$ satisfying $\rho(x, y) = 0$ if and only if $x = y$ and the *triangle inequality*

$$\rho(x, y) \leq \rho(x, z) + \rho(z, y), \quad x, y, z \in \mathbb{X}.$$

Then the pair $(\mathbb{X}, \rho)$ is called a *metric space*. A sequence $x_n \in \mathbb{X}$, $n \in \mathbb{N}$, *converges* to $x \in \mathbb{X}$ if $\lim_{n\to\infty} \rho(x_n, x) = 0$. The closed *ball* with centre

$x_0 \in \mathbb{X}$ and radius $r \geq 0$ is defined by

$$B(x_0, r) := \{x \in \mathbb{X} : \rho(x, x_0) \leq r\}. \tag{A.18}$$

A set $U \subset \mathbb{X}$ is said to be *open*, if for each $x_0 \in U$ there is some $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subset U$. A set $U \subset \mathbb{X}$ is said to be *closed* if its complement $\mathbb{X} \setminus B$ is open. The *closure* of a set $B \subset \mathbb{X}$ is the smallest closed set containing B. The *interior* $\operatorname{int} B$ of $B \subset \mathbb{X}$ is the largest open subset of B. The *boundary* ∂B of B is the set theoretic difference of its closure and interior.

The *Borel* σ-field $\mathcal{B}(\mathbb{X})$ on a metric space $\mathbb{X}$ is the σ-field generated by the open sets; see [21]. Another generator of $\mathcal{B}(\mathbb{X})$ is the system of closed sets. If $C \subset \mathbb{X}$ then we can restrict the metric to $C \times C$ to obtain a *subspace* of $\mathbb{X}$. With respect to this restricted metric the open (resp. closed) sets are of the form $C \cap U$, where U is open (resp. closed) in $\mathbb{X}$. Therefore the σ-field $\mathcal{B}(C)$ generated by the open sets equals $\mathcal{B}(C) = \{B \cap C : B \in \mathcal{B}(\mathbb{X})\}$, the trace of $\mathcal{B}(\mathbb{X})$ on C. If $C \in \mathcal{B}(\mathbb{X})$, then we call $(C, \mathcal{B}(C))$ a *Borel subspace* of $\mathbb{X}$.

A metric space is said to be *complete* if every Cauchy sequence converges in $\mathbb{X}$, and *separable* if it has a countable dense subset. A complete separable metric space is abbreviated as CSMS. The following result on Borel spaces (see Definition 6.1) is quite useful.

Theorem A.18 *Let $(C, \mathcal{B}(C))$ be a Borel subspace of a CSMS $\mathbb{X}$. Then $(C, \mathcal{B}(C))$ is a Borel space.*

A metric space is said to be *σ-compact*, if it is a countable union of compact sets. A metric space is said to be *locally compact*, if every $x \in \mathbb{X}$ has a compact neighborhood U, that is a compact set containing x in its interior. It is easy to see that any σ-compact metric space is separable. Here is a partial converse of this assertion:

Lemma A.19 *Let $\mathbb{X}$ be a locally compact separable metric space. Then $\mathbb{X}$ is σ-compact.*

Proof Let $C \subset \mathbb{X}$ be an at most countable dense subset of $\mathbb{X}$ and let $\mathcal{U}$ be the collection of all open sets $\{z \in \mathbb{X} : \rho(y, z) < 1/n\}$, where $y \in C$ and $n \in \mathbb{N}$. For each $x \in \mathbb{X}$ there exists an open set U_x with compact closure and with $x \in U_x$. There exists $V_x \in \mathcal{U}$ such that $x \in V_x \subset U_x$. The closures of the sets V_x, $x \in \mathbb{X}$, are compact and cover $\mathbb{X}$. □

The next fact is easy to prove:

Lemma A.20 *Each closed subset of a locally compact metric space is locally compact.*

Lemma A.21 *Each subspace B of a separable metric space $\mathbb{X}$ is separable.*

Proof Let $C \subset \mathbb{X}$ be an at most countable dense subset of $\mathbb{X}$. For each $x \in C$ and each $n \in \mathbb{N}$ we take a point $y(x,n) \in B(x,1/n) \cap B$, provided this intersection is not empty. The set of all such points $y(x,n)$ is dense in B. □

Let ν be a measure on a metric space $\mathbb{X}$. The *support* $\operatorname{supp}\nu$ of ν is the intersection of all closed sets $F \subset \mathbb{X}$ such that $\nu(\mathbb{X} \setminus F) = 0$.

Lemma A.22 *Let ν be a measure on a separable metric space $\mathbb{X}$. Then $\nu(\mathbb{X} \setminus \operatorname{supp}\nu) = 0$.*

Proof By definition, the set $\mathbb{X} \setminus \operatorname{supp}\nu$ is the union of all open sets $U \subset \mathbb{X}$ with $\nu(U) = 0$. In a separable metric space every open cover of an open set has a countable subcover. Hence there exists an at most countable system $\mathcal{U}$ of open sets U with $\nu(U) = 0$ such that $\cup_{U\in\mathcal{U}} U = \mathbb{X} \setminus \operatorname{supp}\nu$. Hence the result follows from the subadditivity of ν. □

If $(\mathbb{X},\rho)$ and $(\mathbb{X}',\rho')$ are metric spaces then $\mathbb{X} \times \mathbb{X}'$ is a metric space in its own right. One natural choice of a *product metric* is

$$((x,x'),(y,y')) \mapsto (\rho(x,y)^2 + \rho(x',y')^2)^{1/2}.$$

In particular this defines the Borel σ-field $\mathcal{B}(\mathbb{X} \times \mathbb{Y})$. We have:

Lemma A.23 *Suppose that $\mathbb{X}$ and $\mathbb{Y}$ are separable metric spaces. Then $\mathcal{B}(\mathbb{X} \times \mathbb{Y}) = \mathcal{B}(\mathbb{X}) \otimes \mathcal{B}(\mathbb{Y})$.*

A.3 Hausdorff measure and intrinsic volumes

We fix a natural number $d \in \mathbb{N}$ and consider the Euclidean space $\mathbb{R}^d$ with its Borel σ-field $\mathcal{B}^d := \mathcal{B}(\mathbb{R}^d)$, scalar product $\langle\cdot,\cdot\rangle$ and Euclidean norm $\|\cdot\|$. We discuss a few basic properties of Lebesgue and Hausdorff measure, referring to [45] for more detail on the first and to [25] for more information on the second. We also introduce the intrinsic volumes of convex bodies, referring to [109] and [108] for further detail.

The *diameter* of a non-empty set $B \subset \mathbb{R}^d$ is the possibly infinite number $d(B) := \sup\{\|x-y\| : x,y \in B\}$. The *Lebesgue measure* (or *volume* function) λ_d on $(\mathbb{R}^d,\mathcal{B}^d)$ is the unique measure satisfying $\lambda_d([0,1]^d) = 1$ and the translation invariance $\lambda_d(B) = \lambda_d(B+x)$ for all $(B,x) \in \mathcal{B}^d \times \mathbb{R}^d$. In particular λ_d is *locally finite*, that is $\lambda_d(B) < \infty$ for all bounded Borel sets $B \subset \mathbb{R}^d$. We also have $\lambda_d = (\lambda_1)^d$ and therefore $\lambda_d(rB) = r^d\lambda_d(B)$ for all $r \geq 0$ and

$B \in \mathcal{B}^d$, where $rB := \{rx : x \in B\}$. The Lebesgue measure is also invariant under rotations, that is $\lambda_d(\rho B) = \lambda_d(B)$ for all $B \in \mathcal{B}^d$ and all rotations $\rho\colon \mathbb{R}^d \to \mathbb{R}^d$. (Here we write $\rho B := \{\rho x : x \in B\}$.) Recall that a rotation is a linear isometry (called proper if it preserves the orientation). For $f \in \overline{\mathbb{R}}(\mathbb{R}^d)$ one usually writes $\int f(x)\,dx$ instead of $\int f(x)\,\lambda_d(dx)$.

The volume of the *unit ball* $B^d := \{x \in \mathbb{R}^d : \|x\| \le 1\}$ is denoted by $\kappa_d := \lambda_d(B^d)$. This volume can be expressed with the help of the Gamma function; see Exercise (7.14). We mention the special cases $\kappa_1 = 2$, $\kappa_2 = \pi$, and $\kappa_3 = (4\pi)/3$. It is convenient to define $\kappa_0 := 1$. Note that the ball

$$B(x, r) := rB^d + x = \{y \in \mathbb{R}^d : \|y - x\| \le r\}$$

centred at $x \in \mathbb{R}^d$ with radius $r \ge 0$ has volume $\kappa_d r^d$.

For $k \in \{0, \dots, d\}$ and $\delta > 0$ we set

$$\mathcal{H}_{k,\delta}(B) := \frac{\kappa_k}{2^k} \inf\Big\{ \sum_{j=1}^{\infty} d(B_j)^k : B \subset \bigcup_{j=1}^{\infty} B_j, d(B_j) \le \delta \Big\}, \quad B \subset \mathbb{R}^d, \tag{A.19}$$

where the infimum is taken over all countable collections $B_1, B_2, \dots$ of subsets of $\mathbb{R}^d$ and where $d(\emptyset) := 0$. Note that $\mathcal{H}_{k,\delta}(B) = \infty$ is possible for $k < d$ even for bounded sets B. Define

$$\mathcal{H}_k(B) := \lim_{\delta \downarrow 0} \mathcal{H}_{k,\delta}(B). \tag{A.20}$$

The restriction of $\mathcal{H}_k$ to $\mathcal{B}^d$ is a measure, the k-dimensional *Hausdorff measure*. For $k = d$ we obtain the Lebesgue measure while $\mathcal{H}_0$ is the counting measure. If $B \subset \mathbb{R}^d$ is a k-dimensional smooth manifold then $\mathcal{H}_k(B)$ is known as the *volume* of B.

For $K, L \subset \mathbb{R}^d$ we define the *Minkowski sum* $K \oplus L$ by

$$K \oplus L := \{x + y : x \in K, y \in L\}.$$

The Minkowski sum of K and the ball $B(0, r)$ centred at the origin with radius r is called the *parallel set* of K at distance r. If $K \subset \mathbb{R}^d$ is closed, then

$$K \oplus rB^d = \{x \in \mathbb{R}^d : d(x, K) \le r\} = \{x \in \mathbb{R}^d : B(x, r) \cap K \ne \emptyset\},$$

where

$$d(x, A) := \inf\{\|x - y\| : y \in A\} \tag{A.21}$$

is the distance of x from a set $A \subset \mathbb{R}^d$ and $\inf \emptyset := \infty$. A *convex body* is a non-empty, compact and convex subset of $\mathbb{R}^d$. The system of all convex bodies is denoted by $\mathcal{K}^{(d)}$. We let $\mathcal{K}^d := \mathcal{K}^{(d)} \cup \{\emptyset\}$. It turns out that the parallel volume of a convex body is a polynomial of degree d:

$$\lambda_d(K \oplus rB^d) = \sum_{j=0}^{d} r^{d-j}\kappa_{d-j}V_j(K), \quad K \in \mathcal{K}^d. \tag{A.22}$$

This is known as the *Steiner formula* and determines the *intrinsic volumes* $V_0(K), \dots, V_d(K)$ of K. Clearly $V_i(\emptyset) = 0$ for all $i \in \{0, \dots, d\}$. Further it is easy to see that $V_d(K) = \lambda_d(K)$ and $V_0(K) = 1$ if $K \neq \emptyset$. More generally, if the dimension of the affine hull of K equals j, then $V_j(K)$ equals the j-dimensional Haussdorff measure $\mathcal{H}^j(K)$ of K. If K has non-empty interior, then

$$V_{d-1}(K) = \frac{1}{2}\mathcal{H}_{d-1}(\partial K), \tag{A.23}$$

where ∂K denotes the boundary of K. If the interior of K is empty, then $V_{d-1}(K) = \mathcal{H}_{d-1}(\partial K) = \mathcal{H}_{d-1}(K)$. These facts are suggested by the limit relation

$$2V_{d-1}(K) = \lim_{r\downarrow 0} r^{-1}(\lambda_d(K \oplus rB^d) - \lambda_d(K)).$$

Taking $K = B^d$ in (A.22) and comparing the coefficients in the resulting identity between polynomials, yields

$$V_i(B^d) = \binom{d}{i}\frac{\kappa_d}{\kappa_{d-i}}, \quad i = 0, \dots, d. \tag{A.24}$$

The intrinsic volumes inherit from Lebesgue measure the properties of invariance under translations and rotations. Moreover, the scaling property of Lebesgue measure implies for any $i \in \{0, \dots, d\}$ that the function V_i is *homogeneous* of degree i, that is,

$$V_i(rK) = r^i V_i(K), \quad K \in \mathcal{K}^d,\ r \geq 0. \tag{A.25}$$

A less obvious property of the intrinsic volumes is that they are monotone increasing with respect to set inclusion. In particular, since $V_i(\emptyset) = 0$, the intrinsic volumes are non-negative. The restrictions of the intrinsic volumes to $\mathcal{K}^{(d)}$ are continuous with respect to the *Hausdorff distance*, defined by

$$\delta(K, L) := \min\{\varepsilon \geq 0 : K \subset L \oplus \varepsilon B^d, L \subset K \oplus \varepsilon B^d\}, \quad K, L \in \mathcal{K}^{(d)}. \tag{A.26}$$

A further important property is *additivity*, that is

$$V_i(K \cup L) = V_i(K) + V_i(L) - V_i(K \cap L), \tag{A.27}$$

whenever $K, L, (K \cup L) \in \mathcal{K}^d$. The following result highlights the relevance of intrinsic volumes for convex geometry.

Theorem A.24 (Hadwiger's Characterization) *Suppose that $\varphi : \mathcal{K}^d \to \mathbb{R}$ is additive, continuous on $\mathcal{K}^d \setminus \{\emptyset\}$, and invariant under translations and proper rotations. Then there exist $c_0, \dots, c_d \in \mathbb{R}$ such that*

$$\varphi(K) = \sum_{i=0}^{d} c_i V_i(K), \quad K \in \mathcal{K}^d.$$

For applications in stochastic geometry it is necessary to extend the intrinsic volumes to the *convex ring* $\mathcal{R}^d$. A set $K \subset \mathbb{R}^d$ belongs to $\mathcal{R}^d$ if it can be represented as a finite (possibly empty) union of compact convex sets. (Note that $\emptyset \in \mathcal{R}^d$.) The space $\mathcal{R}^d \setminus \{\emptyset\}$ is a subset of the space $C^{(d)}$ of all non-empty compact subsets of $\mathbb{R}^d$. The latter can be equipped with the Hausdorff distance (defined again by (A.26)) and the associated Borel σ-field. Equipped with this metric, $C^{(d)}$ becomes a complete and separable metric space (CSMS). For the separability we refer also to Exercise 17.3. Both $\mathcal{K}^{(d)}$ and $\mathcal{R}^d \setminus \{\emptyset\}$ are measurable subsets of $C^{(d)}$. (The first is even closed.) In particular, upon extending the Borel σ-field from $C^{(d)}$ to C^d in the usual minimal way (all elements of $\mathcal{B}(C^{(d)})$ and the singleton $\{\emptyset\}$ should be measurable), $\mathcal{R}^d$ is a measurable subset of C^d.

A function $\varphi\colon \mathcal{R}^d \to \mathbb{R}$ is said to be *additive* if $\varphi(\emptyset) = 0$ and

$$\varphi(K \cup L) = \varphi(K) + \varphi(L) - \varphi(K \cap L), \quad K, L \in \mathcal{R}^d. \tag{A.28}$$

Such an additive function satisfies the *inclusion-exclusion principle*

$$\varphi(K_1 \cup \cdots \cup K_m) = \sum_{n=1}^{m} (-1)^{n-1} \sum_{1 \le i_1 < \cdots < i_n \le m} \varphi(K_{i_1} \cap \cdots \cap K_{i_n}) \tag{A.29}$$

for all $K_1, \dots, K_m \in \mathcal{R}^d$ and all $m \in \mathbb{N}$. The intrinsic volumes V_i can be extended from $\mathcal{K}^d$ to $\mathcal{R}^d$ such that this extended function (still denoted by V_i) is additive. By (A.29) this extension must be unique. It is the existence that requires a (non-trivial) proof. Then V_d is still the volume, while (A.23) holds whenever $K \in \mathcal{R}^d$ is the closure of its interior. Moreover, $V_{d-1}(K) \ge 0$ for all $K \in \mathcal{R}^d$. The function V_0 is known as the *Euler characteristic* and takes on integer values. In particular, $V_0(K) = 1$ for all $K \in \mathcal{K}^{(d)}$. In two dimensions the number $V_0(K)$ can be interpreted as the number of connected components minus the number of holes of $K \in \mathcal{R}^2$. The intrinsic volumes are measurable functions on $\mathcal{R}^d$.

Let $\mathcal{F}^d$ denote the space of all closed subsets of $\mathbb{R}^d$. The *Fell topology* on this space is the smallest topology such that the sets $\{F \in \mathcal{F}^d : F \cap G \neq \emptyset\}$

and $\{F \in \mathcal{F}^d : F \cap K = \emptyset\}$ are open for all open sets $G \subset \mathbb{R}^d$ and all compact sets $K \subset \mathbb{R}^d$. Let $\{x_n : n \in \mathbb{N}\}$ be a countable dense subset of $\mathbb{R}^d$. Then

$$\rho(F, F') := \sum_{n=1}^{\infty} 2^{-n}(|d(x_n, F) - d(x_n, F')| \wedge 1), \quad F, F' \in \mathcal{F}^d, \tag{A.30}$$

defines a metric on $\mathcal{F}^d$ that is compatible with the Fell topology and such that $\mathcal{F}^d$ is a compact and separable metric space; see [45, Theorem A2.5]. The mapping $(F, F') \mapsto F \cap F'$ from $\mathcal{F}^d \times \mathcal{F}^d \to \mathcal{F}^d$ is measurable, while for any compact $K \subset \mathbb{R}^d$ the mapping from $\mathcal{F}^d$ to C^d given by $F \mapsto F \cap K$ is measurable. We refer to [109] for these and many more facts on the Fell topology.

A.4 Measures on the real half-line

In this subsection we consider a locally finite measure ν on $\mathbb{R}_+ = [0, \infty)$ with the Borel σ-field. We abbreviate $\nu(t) := \nu([0, t])$, $t \in \mathbb{R}_+$, and note that ν can be identified with the (right-continuous) mapping $t \mapsto \nu(t)$. We define a function $\nu^{\leftarrow} : \mathbb{R}_+ \to [0, \infty]$ by

$$\nu^{\leftarrow}(t) := \inf\{s \geq 0 : \nu(s) \geq t\}, \quad t \geq 0, \tag{A.31}$$

where $\inf \emptyset := \infty$. This function is increasing and left-continuous and, in particular, measurable; see e.g. [101]. We also define $\nu(\infty) := \lim_{t\to\infty} \nu(t)$.

Proposition A.25 *Let $f \in \mathbb{R}_+(\mathbb{R}_+)$. Then*

$$\int f(t)\, \nu(dt) = \int_0^{\nu(\infty)} f(\nu^{\leftarrow}(t))\, dt. \tag{A.32}$$

If ν is diffuse, then we have for all $g \in \mathbb{R}_+(\mathbb{R}_+)$ that

$$\int g(\nu(t))\, \nu(dt) = \int_0^{\nu(\infty)} g(t)\, dt. \tag{A.33}$$

Proof For all $s, t \in \mathbb{R}_+$ the inequalities $\nu^{\leftarrow}(t) \leq s$ and $t \leq \nu(s)$ are equivalent; see [101]. For $0 \leq a < b < \infty$ and $f := \mathbf{1}_{(a,b]}$ we therefore obtain that

$$\begin{aligned}\int_0^{\nu(\infty)} f(\nu^{\leftarrow}(t))\, dt &= \int_0^{\nu(\infty)} \mathbf{1}\{a < \nu^{\leftarrow}(t) \leq b\}\, dt \\ &= \int_0^{\nu(\infty)} \mathbf{1}\{\nu(a) < t \leq \nu(b)\}\, dt = \nu(b) - \nu(a),\end{aligned}$$

so that (A.32) follows for this choice of f. Also (A.32) holds for $f = \mathbf{1}_{\{0\}}$, since $\nu^{\leftarrow}(t) = 0$ if and only if $t \le \nu(\{0\})$. We leave it to the reader to prove the case of general f using the tools from measure theory presented in Subsection A.1.

Assume now that ν is diffuse and let $g \in \mathbb{R}_+(\mathbb{R}_+)$. Applying (A.32) with $f(t) := g(\nu(t))$ yields

$$\int g(\nu(t))\,\nu(dt) = \int_0^{\nu(\infty)} g(\nu(\nu^{\leftarrow}(t)))\,dt = \int_0^{\nu(\infty)} g(t)\,dt,$$

since $\nu(\nu^{\leftarrow}(t)) = t$; see [101]. □

Assume now that ν is a measure on $\mathbb{R}_+$ with $\nu(\mathbb{R}_+) \le 1$. With the definition $\nu(\{\infty\}) := 1 - \nu(\mathbb{R}_+)$ we might then interpret ν as a probability measure on $\overline{\mathbb{R}}$. The *hazard measure* of ν, is the measure R_ν on $\mathbb{R}_+$ given by

$$R_\nu(dt) := (\nu[t,\infty))^{\oplus}\nu(dt), \tag{A.34}$$

where $a^{\oplus} := \mathbf{1}\{a \ne 0\}a^{-1}$ is the generalized inverse of $a \in \mathbb{R}$. The following result is a consequence of the exponential formula of Lebesgue-Stieltjes calculus (see [11, 61]) and a special case of Theorem A5.10 in [61].

Proposition A.26 *If ν is a diffuse measure on $\mathbb{R}_+$ with $\nu(\mathbb{R}_+) \le 1$ and hazard measure R_ν, then $\nu((t,\infty]) = \exp[-R_\nu([0,t])]$ for all $t \in \mathbb{R}_+$.*

A.5 Absolutely continuous functions

Let $I \subset \mathbb{R}$ be a non-empty *interval*. This means that the relations $a, b \in I$ and $a < b$ imply that $[a,b] \subset I$. A function $f\colon I \to \mathbb{R}$ is *absolutely continuous* if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(y_i) - f(x_i)| \le \varepsilon$$

whenever $n \in \mathbb{N}$ and $x_1, \dots, x_n, y_1, \dots, y_n \in I$ satisfy $x_i \le y_i$ for all $i \in \{1, \dots, n\}$, $y_i < x_{i+1}$ for all $i \in \{1, \dots, n-1\}$ and $\sum_{i=1}^{n} |y_i - x_i| \le \delta$.

For $f \in \mathbb{R}(I)$ and $a, b \in I$ we write

$$\int_a^b f(t)\,dt := \int_{[a,b]} f(t)\,\lambda_1(dt)$$

if $a \le b$ and

$$\int_a^b f(t)\,dt := -\int_{[b,a]} f(t)\,\lambda_1(dt)$$

if $a > b$. Absolutely continuous functions can be characterized as follows. Recall that λ_1 denotes the Lebesgue measure on $\mathbb{R}$.

Theorem A.27 *Let $a < b$ and suppose that $f\colon [a,b] \to \mathbb{R}$ is a function. Then f is absolutely continuous if and only if there is a function $f' \in L^1((\lambda_1)_{[a,b]})$ such that*

$$f(x) = \int_a^x f'(t)\,dt, \quad x \in [a,b]. \tag{A.35}$$

The function f' in (A.35) is the *Radon-Nikodým derivative* of f. It is uniquely determined almost everywhere with respect to Lebesgue measure on $[a,b]$.

Proposition A.28 (Product Rule) *Suppose that $f, g \in \mathbb{R}(I)$ are absolutely continuous with Radon-Nikodým derivatives f', g'. Then the product fg is absolutely continuous with Radon-Nikodým derivative $f'g + fg'$.*

Let $\mathbf{AC}^2$ denote the space of all absolutely continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ with a Radon-Nikodým derivative f' that is absolutely continuous. The following result can be proved using the preceding product rule.

Proposition A.29 *Let $f \in \mathbf{AC}^2$. Then*

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x f''(t)(x-t)\,dt, \quad x \in \mathbb{R},$$

where f'' is a Radon-Nikodým derivative of f'.

Appendix B

Some probability theory

For the reader's convenience we provide here terminology and some basic results of measure-theoretic probability theory. More detail can be found, for instance, in [8, 21] or the first chapters of [45].

A *probability space* is a measure space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}(\Omega) = 1$. Then $\mathbb{P}$ is called a *probability measure* (sometimes also a *distribution*), the sets $A \in \mathcal{F}$ are called *events*, while $\mathbb{P}(A)$ is known as the *probability* of the event A. In this book the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ will be fixed.

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space. A *random element* of $\mathbb{X}$ (or of $(\mathbb{X}, \mathcal{X})$) is a measurable mapping $X \colon \Omega \to \mathbb{X}$. The *distribution* $\mathbb{P}_X$ of X is the image of $\mathbb{P}$ under X, that is $\mathbb{P}_X := \mathbb{P} \circ X^{-1}$ or, written more explicitly, $\mathbb{P}_X(A) = \mathbb{P}(X \in A)$, $A \in \mathcal{X}$. Here we use the common abbreviation

$$\mathbb{P}(X \in A) := \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\}).$$

It is easy to prove that for $f \in \overline{\mathbb{R}}(\mathbb{X})$ we have $\mathbb{E}[f(X)] = \int_{\mathbb{X}} f \, d\,\mathbb{P}_X$. We write $X \overset{d}{=} Y$ to express the fact that two random elements X, Y of $\mathbb{X}$ have the same distribution.

Of particular importance is the case $(\mathbb{X}, \mathcal{X}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, or, more generally, $(\mathbb{X}, \mathcal{X}) = (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$. A random element X of this space is called *random variable* while the integral $\int X \, d\mathbb{P}$ is the *expectation* (or *mean*) $\mathbb{E}[X]$ of X. If $\mathbb{E}[|X|^a] < \infty$ for some $a > 0$ then $\mathbb{E}[|X|^b] < \infty$ for all $b \in [0, a]$. In the case $a = 1$ we say that X is *integrable* while in the case $a = 2$ we say that X is *square integrable*. In the latter case the *variance* of X is defined as

$$\mathbb{V}\mathrm{ar}[X] := \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

We have $\mathbb{P}(X = \mathbb{E}[X]) = 1$ if and only if $\mathbb{V}\mathrm{ar}[X] = 0$. For random variables X, Y we write $X \le Y$, $\mathbb{P}$-almost surely (shorter: $\mathbb{P}$-a.s. or just a.s.) if $\mathbb{P}(X \le Y) = 1$. For $A \in \mathcal{F}$ we write $X \le Y$, $\mathbb{P}$-a.s. on A if $\mathbb{P}(A \setminus \{X \le Y\}) = 0$. The *covariance* between two square integrable random variables X and Y

is defined by

$$\mathbb{C}\mathrm{ov}[X,Y] := \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] = \mathbb{E}[XY] - (\mathbb{E}[X])(\mathbb{E}[Y]).$$

Moreover, the Cauchy-Schwarz inequality (see (A.2)) says that

$$|\mathbb{E}[XY]| \le (\mathbb{E}[X^2])^{1/2}(\mathbb{E}[Y^2])^{1/2},$$

or $|\mathbb{C}\mathrm{ov}[X,Y]| \le (\mathbb{V}\mathrm{ar}[X])^{1/2}(\mathbb{V}\mathrm{ar}[Y])^{1/2}$. Here is another useful inequality for the expectation of convex functions of a *random vector* X in $\mathbb{R}^d$, that is of a random element of $\mathbb{R}^d$.

Proposition B.1 (Jensen's Inequality) *Let $X = (X_1,\dots,X_d)$ be a random vector in $\mathbb{R}^d$ whose components are in $L^1(\mathbb{P})$ and let $f\colon \mathbb{R}^d \to \mathbb{R}$ be convex such that $\mathbb{E}[|f(X)|] < \infty$. Then $\mathbb{E}[f(X)] \ge f(\mathbb{E}[X_1],\dots,\mathbb{E}[X_d])$.*

Let $T \ne \emptyset$ be an (index) set. A family $\{\mathcal{F}_t : t \in T\}$ of σ-fields contained in $\mathcal{F}$ is said to be *independent* if

$$\mathbb{P}(A_{t_1} \cap \cdots \cap A_{t_k}) = \mathbb{P}(A_{t_1}) \cdots \mathbb{P}(A_{t_k})$$

for any distinct $t_1,\dots,t_k \in T$ and any $A_{t_1} \in \mathcal{F}_{t_1},\dots,A_{t_k} \in \mathcal{F}_{t_k}$. A family $\{X_t \in T\}$ of random variables with values in some measurable spaces $(\mathbb{X}_t, \mathcal{X}_t)$ is independent if the family $\{\sigma(X_t) : t \in T\}$ of generated σ-fields is independent.

The following result guarantees the existence of infinite sequences of independent random variables in a general setting.

Theorem B.2 *Let $(\Omega_n, \mathcal{F}_n, \mathbb{Q}_n)$, $n \in \mathbb{N}$, be probability spaces. Then there exists a unique probability measure $\mathbb{Q}$ on the space $(\times_{n=1}^\infty \Omega_n, \otimes_{n=1}^\infty \mathcal{F}_n)$ such that $\mathbb{Q}(A \times \times_{m=n+1}^\infty \Omega_m) = \otimes_{i=1}^m \mathbb{Q}_i(A)$ for all $n \in \mathbb{N}$ and $A \in \otimes_{m=1}^n \mathcal{F}_m$.*

Under a Borel assumption, Theorem B.2 extends to general probability measures on infinite products.

Theorem B.3 *Let $(\Omega_n, \mathcal{F}_n)$, $n \in \mathbb{N}$, be a sequence of Borel spaces and let $\mathbb{Q}_n$ be probability measures on $(\Omega_1 \times \cdots \times \Omega_n, \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n)$ such that $\mathbb{Q}_{n+1}(\cdot \times \Omega_{n+1}) = \mathbb{Q}_n$ for all $n \in \mathbb{N}$. Then there is a unique probability measure $\mathbb{Q}$ on $(\times_{n=1}^\infty \Omega_n, \otimes_{n=1}^\infty \mathcal{F}_n)$ such that $\mathbb{Q}(A \times \times_{m=n+1}^\infty \Omega_m) = \mathbb{Q}_n(A)$ for all $n \in \mathbb{N}$ and $A \in \otimes_{m=1}^n \mathcal{F}_m$.*

The *characteristic function* of a random vector $X = (X_1,\dots,X_d)$ in $\mathbb{R}^d$ is the function $\varphi_X\colon \mathbb{R}^d \to \mathbb{C}$ defined by

$$\varphi_X(t) := \mathbb{E}[\exp[-\mathbf{i}\langle X, t\rangle]], \quad t = (t_1,\dots,t_d) \in \mathbb{R}^d, \tag{B.1}$$

where $\mathbb{C}$ denotes the complex numbers and $\mathbf{i} := \sqrt{-1}$ is the imaginary unit. The *Laplace transform* of a random vector $X = (X_1, \dots, X_d)$ in $\mathbb{R}^d_+$ (a random element of $\mathbb{R}^d_+$) is the function L_X on $\mathbb{R}^d_+$ defined by

$$L_X(t) := \mathbb{E}[\exp(-\langle X, t\rangle)], \quad t = (t_1, \dots, t_d) \in \mathbb{R}^d_+. \tag{B.2}$$

Proposition B.4 (Uniqueness Theorem) *Two random vectors in $\mathbb{R}^d$ (resp. in $\mathbb{R}^d_+$) have the same distribution if and only if their characteristic functions (resp. Laplace transforms) coincide.*

A sequence (X_n) of random variables *converges $\mathbb{P}$-almost surely* (shorter: $\mathbb{P}$-a.s. or just a.s.) to a random variable X if the event

$$\Big\{\lim_{n\to\infty} X_n = X\Big\} := \Big\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\Big\}$$

has probability 1. A similar notation is used for infinite series of random variables.

Theorem B.5 (Law of Large Numbers) *Let $X_1, X_2, \dots$ be independent and identically distributed random variables such that $\mathbb{E}[|X_1|] < \infty$. Then $1/n(X_1 + \dots + X_n)$ converges almost surely to $\mathbb{E}[X_1]$.*

The following criterion for the convergence of a series with independent summands is useful.

Proposition B.6 *Let $X_n \in L^2(\mathbb{P})$, $n \in \mathbb{N}$, be independent random variables satisfying $\sum_{n=1}^\infty \mathbb{V}\mathrm{ar}[X_n] < \infty$. Then the series $\sum_{n=1}^\infty (X_n - \mathbb{E}[X_n])$ converges $\mathbb{P}$-almost surely and in $L^2(\mathbb{P})$.*

A sequence (X_n) of random variables *converges!in probability* to a random variable X if $\mathbb{P}(|X_n - X|) \geq \varepsilon) \to 0$ as $n \to \infty$ for each $\varepsilon > 0$. Each almost surely converging sequence converges in probability. *Markov's inequality*

$$\mathbb{P}(Z \geq \varepsilon) \leq \frac{\mathbb{E}[Z]}{\varepsilon}, \quad \varepsilon > 0,$$

valid for non-negative random variables Z, implies the following fact.

Proposition B.7 *Let $p \geq 1$ and suppose that the random variables X_n, $n \in \mathbb{N}$, converge in $L^p(\mathbb{P})$ to X. Then $X_n \to X$ in probability.*

Let $X, X_1, X_2, \dots$ be random elements in a metric space $\mathbb{X}$ (equipped with its Borel σ-field). The sequence (X_n) is said to *converge in distribution* to X if $\lim_{n\to\infty} \mathbb{E}[f(X_n)] = \mathbb{E}[f(X)]$ for every continuous bounded function $f : \mathbb{X} \to \mathbb{R}$. One writes $X_n \xrightarrow{d} X$ as $n \to \infty$. Let ρ denote the metric on $\mathbb{X}$.

Then a function $f\colon \mathbb{X} \to \mathbb{R}$ is said to be *Lipschitz* if there is some $c \geq 0$ such that

$$|f(x) - f(y)| \leq c\rho(x, y), \quad x, y \in \mathbb{X}.$$

The smallest of such c is the *Lipschitz constant* of f. The following result is proved (but not stated) in [7].

Proposition B.8 *A sequence (X_n) of random elements in a metric space $\mathbb{X}$ converges in distribution to X if and only if $\lim_{n\to\infty} \mathbb{E}[f(X_n)] = \mathbb{E}[f(X)]$ for any bounded Lipschitz function $f\colon \mathbb{X} \to \mathbb{R}$.*

A sequence (X_n) of random vectors in $\mathbb{R}^d$ is said to be *tight*, if

$$\lim_{c\to\infty} \sup\{\mathbb{P}(\|X_n\| > c) : n \in \mathbb{N}\} = 0.$$

Proposition B.9 *A sequence $(X_n)_{n\geq 1}$ of random vectors in $\mathbb{R}^d$ is tight if and only if for any infinite sequence $I \subset \mathbb{N}$ there is an infinite subsequence $J \subset I$ and a random vector X such that $X_n \xrightarrow{d} X$ as $n \to \infty$ along J.*

For the next result we refer to [45, Lemma 5.2].

Lemma B.10 *A sequence $(X_n)_{n\geq 1}$ of random vectors in $\mathbb{R}^d_+$ is tight if and only if*

$$\lim_{t\to 0} \liminf_{n\to\infty} L_{X_n}(t) = 1, \quad t \in \mathbb{R}^d_+. \tag{B.3}$$

Proposition B.11 *A sequence $(X_n)_{n\geq 1}$ of random vectors in $\mathbb{R}^d$ converges in distribution to a random vector X if and only if $\lim_{n\to\infty} \varphi_{X_n}(t) = \varphi_X(t)$ for all $t \in \mathbb{R}^d$. A sequence $(X_n)_{n\geq 1}$ of random vectors in $\mathbb{R}^d_+$ converges in distribution to a random vector X if and only if $\lim_{n\to\infty} L_{X_n}(t) = L_X(t)$ for all $t \in \mathbb{R}^d_+$.*

The space $\mathbb{R}^\infty$ of all sequences of real numbers can be equipped with the metric

$$\rho(x, y) := \sum_{k=1}^{\infty} 2^{-k}(|x^k - y^k| \wedge 1), \quad x = (x^k)_{k\geq 1}, y = (y^k)_{k\geq 1} \in \mathbb{R}^\infty.$$

In this case we have the following result.

Proposition B.12 *Let $X_n = (X_n^k)_{k\geq 1}$, $n \in \mathbb{N}$, and $X = (X^k)_{k\geq 1}$ be random elements in $\mathbb{R}^\infty$. Then $X_n \xrightarrow{d} X$ if and only if $(X_n^1, \dots, X_n^k) \xrightarrow{d} (X^1, \dots, X^k)$ for any $k \in \mathbb{N}$.*

A random variable N is said to be *standard normal* if its distribution has density $x \mapsto (2\pi)^{-1/2} \exp(-x^2/2)$ with respect to Lebesgue measure on $\mathbb{R}$. Its characteristic function is given by $t \mapsto \exp(-t^2/2)$ while its moments are given by

$$\mathbb{E}[N^k] = \begin{cases} (k-1)!!, & \text{if } k \text{ is even}, \\ 0, & \text{otherwise}, \end{cases} \tag{B.4}$$

where for an even integer $k \geq 2$ we define the *double factorial* of $k-1$ by

$$(k-1)!! := (k-1) \cdot (k-3) \cdots 3 \cdot 1. \tag{B.5}$$

Note that this is the same as the number of matchings of $[k] := \{1, ..., k\}$ (a matching of $[k]$ is a partition of $[k]$ into disjoint blocks of size 2). Indeed, it can be easily checked by induction that

$$\operatorname{card} M(k) = \begin{cases} (k-1)!!, & \text{if } k \text{ is even}, \\ 0, & \text{otherwise}, \end{cases} \tag{B.6}$$

where $M(k)$ denotes the set of matchings of $[k]$. The moment formula (B.4) can be proved by partial integration or by writing the characteristic function of N as a power series. Taking $c > 0$ and using a change of variables we can derive from (B.4) that

$$\mathbb{E}[\exp(-cN^2)N^{2m}] = (1+2c)^{-m-1/2}(2m-1)!!, \quad m \in \mathbb{N}_0, \tag{B.7}$$

where $(-1)!! := 1$. A random variable X is said to have a *normal distribution* with mean $a \in \mathbb{R}$ and variance $b \geq 0$, if $X \overset{d}{=} bN + a$, where N is standard normal. A sequence $(X_n)_{n \geq 1}$ of random variables satisfies the *central limit theorem* if $X_n \xrightarrow{d} N$ as $n \to \infty$.

A random vector $X = (X_1, \ldots, X_d)$ has a *multivariate normal distribution* if $\langle X, t \rangle$ has for all $t \in \mathbb{R}^d$ a normal distribution. In this case the distribution of X is determined by the means $\mathbb{E}[X_i]$ and covariances $\mathbb{E}[X_i X_j]$, $i, j \in \{1, \ldots, d\}$. Moreover, if a sequence $(X^{(n)})_{n \geq 1}$ of random vectors with a multivariate normal distribution converges in distribution to a random vector X, then X has a multivariate normal distribution.

Let $a, b > 0$. A random variable $Y \geq 0$ has a *Gamma distribution* with shape parameter a and scale parameter b if its distribution has Lebesgue density

$$x \mapsto b^a \Gamma(a)^{-1} x^{a-1} e^{-bx} \tag{B.8}$$

on $\mathbb{R}_+$, where

$$\Gamma(a) := \int_0^\infty t^{a-1} e^{-t}\, dt, \quad a > 0, \tag{B.9}$$

is the *Gamma function*. (In particular $\Gamma(a) = (a-1)!$ for $a \in \mathbb{N}$.) Then Y has Laplace transform

$$\mathbb{E}[e^{-tY}] = \left(\frac{b}{b+t}\right)^a, \quad t \geq 0. \tag{B.10}$$

A random element Z of $\mathbb{N}_0$ has a *negative binomial distribution* with parameters $p \in (0,1]$ and $a > 0$ if

$$\mathbb{P}(Z = n) = \frac{\Gamma(n+a)}{\Gamma(n+1)\Gamma(a)}(1-p)^n p^a, \quad n \in \mathbb{N}_0. \tag{B.11}$$

This can be seen to be a probability distribution by Taylor expansion of $(1-x)^{-a}$ evaluated at $x = 1-p$. Exercise 13.3 shows that this distribution can be interpreted as a Gamma mixture of Poisson distributions. For $a \in \mathbb{N}$, Z may be interpreted as the number of failures before the a-th success in a sequence of Bernoulli trials. In the special case $a = 1$ we get the *geometric distribution*

$$\mathbb{P}(Z = n) = (1-p)^n p, \quad n \in \mathbb{N}_0. \tag{B.12}$$

Another interesting special case is $a = 1/2$. In this case

$$\mathbb{P}(Z = n) = \frac{(2n-1)!!}{2^n n!}(1-p)^n p^{1/2}, \quad n \in \mathbb{N}_0. \tag{B.13}$$

This follows from the fact that $\Gamma(n + 1/2) = (2n-1)!!2^{-n}\sqrt{\pi}$, $n \in \mathbb{N}_0$. The probability generating function of Z is given by

$$\mathbb{E}[s^Z] = p^a(1 - s + sp)^{-a}, \quad s \in [0,1]. \tag{B.14}$$

Proposition B.13 *Let $X, X_1, X_2, \dots$ be random variables and assume that $\mathbb{E}[|X|^k] < \infty$ for all $k \in \mathbb{N}$. Suppose that*

$$\lim_{n\to\infty} \mathbb{E}[X_n^k] = \mathbb{E}[X^k], \quad k \in \mathbb{N},$$

and that the distribution of X is uniquely determined by the moments $\mathbb{E}[X^k]$, $k \in \mathbb{N}$. Then $X_n \xrightarrow{d} X$ as $n \to \infty$.

Let $\mathbf{Lip}(1)$ denote the space of Lipschitz functions $h\colon \mathbb{X} \to \mathbb{R}$ with Lipschitz constant less than or equal to 1. For a given $h \in \mathbf{Lip}(1)$ a function $g\colon \mathbb{R} \to \mathbb{R}$ is said to satisfy *Stein's equation* for h if

$$h(x) - \mathbb{E}[h(N)] = g'(x) - xg(x), \quad x \in \mathbb{R}, \tag{B.15}$$

where N is a standard normal random variable.

Proposition B.14 (Stein's Equation) *Suppose that $h \in \mathbf{Lip}(1)$. Then there exists a differentiable solution g of* (B.15) *such that g' is absolutely continuous and such that $g'(x) \leq \sqrt{2/\pi}$ and $g''(x) \leq 2$ for λ_1-a.e. $x \in \mathbb{R}$, where g'' is a Radon-Nikodým derivative of g'.*

Proof We assert that the function

$$g(x) := e^{x^2/2} \int_{-\infty}^{x} e^{-y^2/2}(h(y) - \mathbb{E}[h(N)])\, dy, \quad x \in \mathbb{R}$$

is a solution. Indeed, the product rule (Proposition A.28) implies that g is absolutely continuous. Moreover, one version of the Radon-Nikodým derivative is given by

$$\begin{aligned} g'(x) &= xe^{x^2/2} \int_{-\infty}^{x} e^{-y^2/2}(h(y) - \mathbb{E}[h(N)])\, dy + e^{x^2/2}e^{-x^2/2}(h(x) - \mathbb{E}[h(N)]) \\ &= h(x) - \mathbb{E}[h(N)] + xg(x). \end{aligned}$$

Hence (B.15) holds. Since a Lipschitz function is absolutely continuous (this can be checked directly) it follows from the product rule that g' is absolutely continuous. The bounds for g' and g'' follow from some lines of calculus which we omit; see [13, Lemma 4.2] for the details. □

Let X be a random variable and $\mathcal{G} \subset \mathcal{F}$ a σ-field. If there exists a $\mathcal{G}$-measurable random variable Y such that

$$\mathbb{E}[\mathbf{1}_C X] = \mathbb{E}[\mathbf{1}_C Y], \quad C \in \mathcal{G}, \tag{B.16}$$

then Y is said to be a version of the *conditional expectation* of X given $\mathcal{G}$. If Y' is another version, then it follows that $Y = Y'$, $\mathbb{P}$-a.s. If, on the other hand, Y is a version of the conditional expectation of X given $\mathcal{G}$, and Y' is another $\mathcal{G}$-measurable random variable satisfying $Y = Y'$, $\mathbb{P}$-a.s., then Y' is also a version of the conditional expectation of X, given $\mathcal{G}$. We use the notation

$$Y = \mathbb{E}[X \mid \mathcal{G}]$$

to denote one fixed version of the conditional expectation, if it exists. If the σ-field $\mathcal{G}$ is generated by an at most countable family of disjoint sets, then the conditional distribution can be expressed in terms of (ordinary) conditional expectation

$$\mathbb{E}[X \mid A] := \mathbb{P}(A)^{-1}\mathbb{E}[\mathbf{1}_A X]$$

of X with respect to an event $A \in \mathcal{F}$. Here we set $0/0 := 0$.

Proposition B.15 (i) *Let X be an integrable random variable and $\mathcal{G} \subset \mathcal{F}$ a σ-field generated by an at most countable family of disjoint elements $A_1, A_2, \ldots$ of $\mathcal{F}$ whose union is Ω. Then,*

$$\mathbb{E}[X \mid \mathcal{G}] = \sum_{n \geq 1} \mathbf{1}_{A_n} \mathbb{E}[X \mid A_n], \quad \mathbb{P}\text{-a.s.}$$

(ii) *If X is as in* (i) *and $A \in \mathcal{G}$ is an atom of a σ-field $\mathcal{G}$ (i.e., for all $B \in \mathcal{G}$ with $B \subset A$, either $B = A$ or $B = \emptyset$), then*

$$\mathbf{1}_A \mathbb{E}[X \mid \mathcal{G}] = \mathbf{1}_A \mathbb{E}[X \mid A], \quad \mathbb{P}\text{-a.s.}$$

In the general case one has the following result, that can be proved with the aid of the Radon-Nikodým theorem (Theorem A.9).

Proposition B.16 *Let X be a random variable and $\mathcal{G} \subset \mathcal{F}$ a σ-field.*

(i) *If X is non-negative, then $\mathbb{E}[X \mid \mathcal{G}]$ exists and may be chosen finite if and only if the measure*

$$C \mapsto \mathbb{E}[\mathbf{1}_C X]$$

is σ-finite on $\mathcal{G}$.

(ii) *If X is integrable, then $\mathbb{E}[X \mid \mathcal{G}]$ exists and may be chosen finite.*

For $A \in \mathcal{F}$ the random variable

$$\mathbb{P}(A \mid \mathcal{G}) := \mathbb{E}[\mathbf{1}_A \mid \mathcal{G}]$$

is called (a version of the) *conditional probability* of A given $\mathcal{G}$. Let Y be a random element in a measurable space $(\mathbb{Y}, \mathcal{Y})$ and X a random variable. We write

$$\mathbb{E}[X \mid Y] := \mathbb{E}[X \mid \sigma(Y)]$$

if the latter expression is defined.

The conditional expectation is linear, monotone and satisfies the triangle and Jensen inequalities. The following properties can be verified immediately from the definition. Property (iii) is called the *law of total expectation* while (vi) is called the *pull out property*. If nothing else is said, then all relations concerning conditional expectations hold $\mathbb{P}$-almost surely.

Theorem B.17 *Consider $\overline{\mathbb{R}}_+$-valued random variables X and Y and σ-fields $\mathcal{G}$, $\mathcal{G}_1$, $\mathcal{G}_2 \subset \mathcal{F}$. Then:*

(i) *If $\mathcal{G} = \{\emptyset, \Omega\}$, then $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X]$.*
(ii) *If X is $\mathcal{G}$-measurable, then $\mathbb{E}[X \mid \mathcal{G}] = X$.*
(iii) $\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}]] = \mathbb{E}[X]$.

(iv) *Assume that* $\mathcal{G}_1 \subset \mathcal{G}_2$ $\mathbb{P}$*-a.s., i.e., assume that for every* $A \in \mathcal{G}_1$ *there is a set* $B \in \mathcal{G}_2$ *with* $\mathbb{P}((A \setminus B) \cup (B \setminus A)) = 0$*. Then*

$$\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}_2] \mid \mathcal{G}_1] = \mathbb{E}[X \mid \mathcal{G}_1].$$

(v) *Suppose that* X *is independent of* $\mathcal{G}$*, i.e. that* X *is independent of* $\mathbf{1}_A$ *for all* $A \in \mathcal{G}$*. Then* $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X]$*.*

(vi) *Suppose that* X *is* $\mathcal{G}$*-measurable. Then* $\mathbb{E}[XY \mid \mathcal{G}] = X\,\mathbb{E}[Y \mid \mathcal{G}]$*.*

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space and $f : \Omega \times \mathbb{X} \to \overline{\mathbb{R}}_+$ a measurable function. Then, for any $x \in \mathbb{X}$, $f(x) := f(\cdot, x)$ (this is the mapping $\omega \mapsto f(\omega, x)$) is a random variable. Hence, if $\mathcal{G} \subset \mathcal{F}$ is a σ-field, we can form the conditional expectation $\mathbb{E}[f(x) \mid \mathcal{G}]$. A *measurable version* of this conditional expectation is a measurable function $\tilde{f} : \Omega \times \mathbb{X} \to \overline{\mathbb{R}}_+$ such that $\tilde{f}(x) = \mathbb{E}[f(x) \mid \mathcal{G}]$ holds $\mathbb{P}$-almost surely for any $x \in \mathbb{X}$. Using the monotone class theorem the linearity of the conditional expectation can be extended as follows.

Lemma B.18 *Let* $(\mathbb{X}, \mathcal{X}, \lambda)$ *be an s-finite measure space and* $f : \Omega \times \mathbb{X} \to \overline{\mathbb{R}}_+$ *a measurable function. Let* $\mathcal{G} \subset \mathcal{F}$ *be a* σ*-field. Then there is a measurable version of* $\mathbb{E}[f(x) \mid \mathcal{G}]$ *satisfying*

$$\mathbb{E}\Big[\int f(x)\,\lambda(dx) \,\Big|\, \mathcal{G}\Big] = \int \mathbb{E}[f(x) \mid \mathcal{G}]\,\lambda(dx), \quad \mathbb{P}\text{-a.s.}$$

The assertion remains true for each $f \in L^1(\mathbb{P} \otimes \lambda)$*.*

Let $\mathbb{X}$ be a non-empty set, for instance a Borel subset of $\mathbb{R}^d$. A *random field* (on $\mathbb{X}$) is a family $Z = (Z(x))_{x \in \mathbb{X}}$ of real-valued random variables. Equivalently, Z is a random element of the space $\mathbb{R}^{\mathbb{X}}$ of all functions from $\mathbb{X}$ to $\mathbb{R}$, equipped with the smallest σ-field making all projection mappings $f \mapsto f(t)$, $t \in \mathbb{X}$, measurable. It is customary to write $Z(\omega, x) := Z(\omega)(x)$ for $\omega \in \Omega$ and $x \in \mathbb{X}$. A random field $Z' = (Z'(x))_{x \in \mathbb{X}}$ (defined on the same probability space as the random field Z) is said to be a *version* of Z if $\mathbb{P}(Z(x) = Z'(x)) = 1$ for each $x \in \mathbb{X}$. In this case $Z \overset{d}{=} Z'$.

A random field $Z = (Z(x))_{x \in \mathbb{X}}$ is *square integrable* if $\mathbb{E}[Z(x)^2] < \infty$ for each $x \in \mathbb{X}$. In this case the *covariance function* K of Z is defined by

$$K(x, y) := \mathbb{E}[(Z(x) - \mathbb{E}[Z(x)])(Z(y) - \mathbb{E}[Z(y)])], \quad x, y \in \mathbb{X}.$$

This function is *non-negative definite*, that is

$$\sum_{i,j=1}^{m} c_i c_j K(x_i, x_j) \geq 0, \quad c_1, \ldots, c_m \in \mathbb{R},\ x_1, \ldots, x_m \in \mathbb{X},\ m \in \mathbb{N}. \tag{B.17}$$

A random field Z is *Gaussian* if for each $k \in \mathbb{N}$ and all $x_1, \dots, x_k \in \mathbb{X}$, the random vector $(Z(x_1), \dots, Z(x_k))$ has a multivariate normal distribution. In this case the distribution of Z is determined by the mean values $\mathbb{E}[Z(x)]$, $x \in \mathbb{X}$, and the covariance function of Z. A random field $Z = (Z(x))_{x \in \mathbb{X}}$ is said to be *centred* if $\mathbb{E}[Z(x)] = 0$ for each $x \in \mathbb{X}$. The following theorem can be proved using the well-known Kolmogorov consistency theorem; see [45, Theorem 6.16].

Theorem B.19 *Let $K\colon \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ be symmetric and non-negative definite. Then there exists a centred Gaussian random field with covariance function K.*

The following result (see e.g. [42]) is an extension of the spectral theorem for symmetric non-negative matrices. It is helpful for the explicit construction of Gaussian random fields. Recall from Section A.2 that $\operatorname{supp} \nu$ denotes the support of a measure ν on a metric space.

Theorem B.20 (Mercer's Theorem) *Suppose that $\mathbb{X}$ is a compact metric space. Let $K\colon \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ be a symmetric, non-negative definite and continuous function. Let ν be a finite measure on $\mathbb{X}$. Then there exist $\gamma_j \geq 0$ and $v_j \in L^2(\nu)$, $j \in \mathbb{N}$, such that*

$$\int v_i(x) v_j(x)\, \nu(dx) = \mathbf{1}\{i = j\}, \quad i, j \in \mathbb{N}, \tag{B.18}$$

and

$$K(x, y) = \sum_{j=1}^{\infty} \gamma_j v_j(x) v_j(y), \quad x, y \in \operatorname{supp} \nu, \tag{B.19}$$

where the convergence is absolute and uniform.

A random field $(Z(x))_{x \in \mathbb{X}}$ is said to be *measurable*, if $(\omega, x) \mapsto Z(\omega, x)$ is a measurable function.

Proposition B.21 *Let $\mathbb{X}$ be a locally compact separable metric space and let ν be a measure on $\mathbb{X}$ which is finite on compact sets. Let $\mathbb{X}^*$ denote the support of ν. Let $Z = (Z(x))_{x \in \mathbb{X}}$ be a centred Gaussian random field with a continuous covariance function. Then $(Z(x))_{x \in \mathbb{X}^*}$ has a measurable version.*

Proof In principle the result can be derived from [20, Theorem II.2.6]. We give here another argument based on the Gaussian nature of the random field.

The set $\mathbb{X}^*$ is closed and therefore a locally compact separable metric space in its own right; see Lemmas A.20 and A.21. Let ν^* be the measure on $\mathbb{X}^*$ defined as the restriction of ν to the measurable subsets of $\mathbb{X}^*$. It is easy to see that $\operatorname{supp}\nu^* = \mathbb{X}^*$. Therefore it is no restriction of generality to assume that $\mathbb{X} = \mathbb{X}^*$.

Let us first assume that $\mathbb{X}$ is compact. Then the assertion can be deduced from a more fundamental property of Z, namely the *Karhunen-Lòeve expansion*; see [2]. Let K be the covariance function of Z. With γ_j and v_j given as in Mercer's theorem (Theorem B.20), this expansion reads

$$Z(x) = \sum_{j=1}^{\infty} \sqrt{\gamma_j} Y_j v_j(x), \quad x \in \mathbb{X}, \tag{B.20}$$

where the convergence is in $L^2(\mathbb{P})$. Since (B.19) implies $\sum_{j=1}^{\infty} \gamma_j v_j(x)^2 < \infty$, Proposition B.6 shows that the series in (B.20) converges almost surely. Let $Z'(x)$ denote the right-hand side of (B.20), whenever the series converges. Otherwise set $Z'(x) := 0$. Then Z' is a measurable version of Z.

In the general case we find a monotone increasing sequence U_n, $n \geq 1$, of open sets with compact closures B_n and $\cup U_n = \mathbb{X}$. For $n \in \mathbb{N}$ let the measure ν_n on B_n be given as the restriction of ν to B_n and let $C_n \subset B_n$ be the support of ν_n. Let ν_n' be the measure on C_n given as the restriction of ν to C_n. Then it is easy to see that $\operatorname{supp}\nu_n' = C_n$. From the first part of the proof we know that there is a measurable version $(Z(x))_{x \in C_n}$. Since U_n is open it follows from the definition of the support of ν_n that $U_n \subset C_n$. Hence there is a measurable version of $(Z(x))_{x \in U_n}$. Since $\cup U_n = \mathbb{X}$ it is now clear how to construct a measurable version of Z. $\square$

Appendix C

Historical notes

Chapter 1

The Poisson distribution was derived by Poisson [95] as the limit of binomial probabilities. Proposition 1.4 is a modern version of this limit theorem. The Poisson distribution was also briefly mentioned by De Moivre [82]. The early applications of the Poisson distribution were mostly directed to "law of small numbers"; see von Bortkiewicz [10]. However, the fundamental work by de Finetti [26], Kolmogorov [55], Lévy [66], and Khinchin [50] clarified the role of the Poisson distribution as the basic building block of a pure jump type stochastic process with independent increments. Khinchin wrote in [49]: "... genau so, wie die Gauss-Laplacesche Verteilung die Struktur der stetigen stochastischen Prozesse beherrscht ..., erweist sich die Poissonsche Verteilung als elementarer Baustein des allgemeinen unstetigen (sprungweise erfolgenden) stochastischen Prozesses, was zweifellos den wahren Grund ihrer großen Anwendungsfähigkeit klarlegt."

Chapter 2

The first systematic treatment of point processes on a general measurable phase space using the modern approach via random counting measures was given by Moyal [83]. The results of this chapter along with historical comments can be found (in a slightly more restricted generality) in the textbooks [48, 44, 18, 45]. Proposition 2.7 goes back to Campbell [12].

Chapter 3

Motivated by practical applications, the Poisson process on the real half-line was introduced by Lundberg [68], Rutherford and Geiger [105], and Erlang [23]. The first rigorous derivation and definition of a spatial Poisson process (discrete chaos) was given by Wiener [118]. The great generality of Poisson processes had been anticipated by Abbe [1]; see [110] for a translation. Comments on the earlier history can be found in [18, 29, 38, 46]. The construction of Poisson processes as an infinite sum of independent mixed binomial processes (implicitly in [118]) as well as Theorem 3.9 is due to Moyal [83]; cf. also [53] and [75]. The reader might also wish to check the relevant historical comments in [18, 45, 46, 48].

Chapter 4

Theorem 4.1 was proved by Mecke [75], who used a different (and very elegant) argument to prove that equation (4.1) implies the properties of a Poisson process. Moment measures of point processes and factorial moment measures of simple point processes were thoroughly studied by Krickeberg [57]. An extensive discussion of factorial moment measures is given in [18]. Lemma 4.10 can be found in [83]. Proposition 4.11 can be derived from [120, Corollary 2.1].

Chapter 5

Mappings and markings are very special cases of so-called cluster fields, extensively studied in [48]. The persistence properties of the Poisson process can be found in the latter monograph. The invariance of the Poisson process (on the line) under independent thinnings was observed by Rényi [98]. The general marking theorem (Theorem 5.6; see also Proposition 6.16) is due to Prékopa [96]. A special case was proved by Doob [20].

Chapter 6

Theorem 6.8 was proved by Rényi [100], while the general point process version in Theorem 6.9 is due to Mönch [78]; see also Kallenberg [43]. Since a simple point process can be identified with its support, Theorem 6.8 is closely related to Choquet capacities; see [80, Theorem 8.3] and [45, Theorem 24.22]. The fact that complete independence on the line implies the Poisson property (Theorem 6.10) was already noted by Erlang [23]. In a more general Euclidean setting the result was proved in Doob [20] (in the homogeneous case) and Ryll-Nardzewski [106] in the non-homogeneous case. For a general phase space the theorem is in Moyal [83]. The general (and quite elegant) setting of a Borel state space was propagated in Kallenberg [45].

Chapter 7

Some textbooks use the properties of Theorem 7.2 to define (homogeneous) Poisson processes on the real line. The theorem was proved by Doob [20], but might have been common wisdom ever since the Poisson process was introduced in [68, 105, 23]. We refer here to Khinchin [49, 51]. Another short proof of the fact that a Poisson process has independent and exponentially distributed interarrival times can be based on the strong Markov property; see e.g. [45]. The argument given here might be new. Doob [20] discussed non-homogeneous Poisson processes in the more general context of stochastic processes with independent increments. More details on a dynamic (martingale) approach to marked point processes on the real line can be found in the monographs [11, 61]. The Poisson properties of the process of record levels (see Proposition 7.7) was observed by Dwass [22]. The result of Exercise 7.13 was derived by Rényi [99]. A nice introduction to extreme value theory is given in [101].

Chapter 8

Palm distributions were introduced by Palm [85]. For stationary point processes on the line the skew factorization of Theorem 8.5 is due to Matthes [72]. His elegant approach was extended by Mecke [75] to accommodate point processes on a locally compact Abelian group. Theorem 8.9 was proved by Mecke [75]. A preliminary version for stationary Poisson processes on the line was obtained by Slivnyak [113]. The formulae of Exercise 8.5 are due to Palm [85] and Khinchin [51]. Stationary point processes are extensively studied in [18, 48, 59, 116].

Chapter 9

Theorem 9.1 can be seen as a special case of the inversion formula proved in Mecke [75]. An extensive discussion of stationary Voronoi tessellations can be found in [16, 109]. Khinchin [51] attributes Proposition 9.3 to Korolyuk. For stationary point processes on the line Proposition 9.4 was proved by Ryll-Nardzewski [107], while the general case is treated in [48].

Chapter10

The extra head problem for an i.i.d. Bernoulli sequence was formulated and solved by Liggett [67]. Stable allocations balancing Lebesgue measure and the stationary Poisson process were introduced and studied by Holroyd and Peres [33]. Theorem 10.2 is taken from Last [58].

Chapter 11

The chapter is based on the article [34] by Hoffman, Holroyd, and Peres.

Chapter 12

Multiple stochastic integrals were introduced by Wiener [118] and Itô [39, 40]. The pathwise identity (12.10) was noted by Surgailis [115]. Multiple Poisson integrals of more general integrands were studied by Kallenberg and Szulga [47]. Theorem 12.5 can be found in [115, 87, 65]. Theorems 12.11 and 12.13 as well as Corollary 12.15 are taken from [65]. The seminal paper on moment formulae for point processes is Campbell [12].

Chapter 13

Doubly stochastic Poisson processes were introduced by Cox [17] and systematically studied in [48, 28]. Kallenberg [44] gives an introduction to the general theory of random measures. Theorem 13.7 was proved by Krickeberg [56]. Corollary 13.13 is due to Mecke [76] and holds for a general phase space. Theorem 13.12 (which also holds for a general phase space) was proved in Kallenberg [44]. Theorem 13.11 was proved by Kallenberg [43]; see also Grandell [28].

Chapter 14

Permanental processes were introduced into the mathematics literature by Macchi [69]. In the case of a finite state space these processes were introduced and studied by Vere-Jones [117]. Theorems 14.6 and 14.8 are from Shirai and Takahashi [112], a key paper on permanental and determinantal point processes. The same applies to Lemma 14.9, although the present proof was inspired by [73]. Under a different assumption on the kernel it was proved in [112] that α-permanental processes exist for any $\alpha > 0$; see also [117]. A survey of the probabilistic properties of permanental and determinantal point processes can be found in [36]. Theorem 14.10 is taken from this source. The Wick formula of Lemma 14.5 can e.g. be found in [87].

We have assumed continuity of the covariance kernel to apply the classical Mercer theorem and to guarantee the existence of a measurable version of the associated Gaussian random field. However, it is enough to assume that the associated integral operator is locally of trace class; see [112].

Chapter 15

Proposition 15.3 can be found in Moyal [83]. Proposition 15.6 can be seen as a specific version of a general combinatorial relationship between the moments and cumulants of random variables; see e.g. [18, Chapter 5] or [87]. The explicit Lévy-Khinchin representation in Theorem 15.9 was derived by Khinchin [52] for processes on the line and by Kingman [53] in the general case. This representation also holds for Lévy processes (processes with homogeneous and independent increments), a generalization of the subordinators discussed in Example 15.5. In these cases the representation was obtained in [26, 55, 66, 50]; see the comments on Chapter 1. The reader might wish to read the textbook [45] for a modern derivation of this fundamental result. The present proof of Proposition 15.10 (a classical result) is taken from the forthcoming monograph [46]. Exercise 15.11 indicates the close relationhip between infinite divisibility and complete independence. We refer here to [48, 44] and to [45] for the case of random variables and Lévy processes.

Chapter 16

The spherical Boolean model has already many features of the Boolean model with general grains (treated in Chapter 17) while avoiding the technicalities of working with probability measures on the space of compact (convex) sets. In the case of deterministic radii the Gilbert graph was introduced by Gilbert in his seminal work [27] and extensively studied in Penrose [90]. Theorem 16.4 on complete coverage can be found in Hall [30].

Chapter 17

The first systematic treatment of the Boolean model was given by Matheron [71]. Theorem 17.8 is essentially from [71]. We refer to [30, 109, 16] for an extensive treatment of the Boolean model and to [71, 80, 109] for the theory of general random closed sets. Percolation properties of the Boolean model are studied in [30, 74].

Chapter 18

This chapter is based on [63]. The chaos expansion of square-integrable Poisson functionals (Exercise 18.13) as a series of orthogonal multiple Wiener-Itô integrals was derived in [118, 40]. The explicit version (based on the difference operators) was proved by Ito [41] for homogenous Poisson processes on the line, and in [63] for general Poisson processes. The general Poincaré inequality of Theorem 18.7 was proved in Wu [119] using the Clark-Ocone representation of Poisson martingales. Chen [15] established this inequality for infinitely divisible random vectors with independent components.

Chapter 19

In the context of a finite number of independent Bernoulli random variables Theorem 19.4 can be found in Esary and Proschan [24]. Later it was rediscovered by Margulis [70] and then again by Russo [104]. The Poisson version (19.10) is due to Zuyev [121] (for a bounded function f). In fact, this is nothing but Kolmogorov's forward equation for a pure birth process. Theorem 19.3 was proved (under stronger assumptions) by Molchanov and Zuyev [81]. For square-integrable random variables it can be extended to certain (signed) σ-finite perturbations; see [60]. The present treatment of integrable random variables and finite signed perturbations might be new. Proposition 19.7 and Theorem 19.9 are classical results of stochastic geometry and were discovered by Miles [77] and Davy [19]. While the first result is easy to guess, Theorem 19.9 might come as a surprise. The result can be generalized to all intrinsic volumes of an isotropic Boolean model in $\mathbb{R}^d$; see [109]. The present approach via a perturbation formula is new and can be extended so as to cover the general case.

Chapter 20

Mehler's formula from Lemma 20.1 was originally devised for Gaussian processes; see e.g. [84]. The present version for Poisson processes (Lemma 20.1) as well as Theorem 20.2 are taken from [62]. Other versions of the covariance identity of Theorem 20.2 were derived in [15, 35, 63, 92, 119]. Theorem 20.3 is closely related to the *Clark-Ocone martingale representation*; see [64]. The Harris-FKG inequality of Theorem 20.4 was proved by Roy [103] by reduction to the discrete version for Bernoulli random fields; see [31]. An elegant direct argument (close to the one presented here) was given by Wu [119].

Chapter 21

The fundamental Theorem 21.1 was proved by Stein [114]. Theorem 21.2 stems from Peccati, Solé, Taqqu and Utzet [86]. The *second order Poincaré inequality* of Theorem 21.3 was proved in [62] after Chatterjee [14] proved a corresponding result for Gaussian vectors. Exercise 21.3 is essentially Example 4.4 in [86]. Abbe [1] derived a quantitative version of the normal approximation of the Poisson distribution; see Example 21.4. The normal approximation of higher order stochastic integrals and Poisson U-statistics was treated in [86] and Reitzner and Schulte [97]. Many Poisson functionals arising in stochastic geometry have a property of *stabilization* (local dependence); central limit and normal approximation theorems based on this property have been established in [93, 94, 91].

Chapter 22

Central limit theorems for intrinsic volumes and more general additive functions of the Boolean model (see Theorem 22.5) were proved in [37]. The volume was already studied in Baddeley [5]. The surface content was treated in Molchanov [79] before Heinrich and Molchanov [32] treated more general curvature-based non-negative functionals. A central limit theorem for the number of components in the Boolean model was established in [89].

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