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Uncertain Alternating Renewal Process and Its Application

Kai Yao and Xiang Li

***Abstract*—Uncertain process is a sequence of uncertain variables indexed by time and space. First, this paper presents a kind of uncertain process, known as the uncertain alternating renewal process, whose alternating interarrival times are uncertain variables. Then, it proves an uncertain alternating renewal theorem on the limit value of average working rate. Finally, an application of the alternating renewal theorem is discussed.**

***Index Terms*—Alternating renewal process, renewal process, uncertain process, uncertainty theory.**

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I. Introduction

The alternating renewal process, which is one of the most popular processes in renewal theory, is used to model systems ON and OFF alternately for some time. In probability theory, the process is assumed to behave randomly, and parameters, such as interarrival times or system lifetimes, are usually considered to be random variables.

In 1965, Zadeh [1] proposed the concept of fuzzy set via membership function. As for fuzzy optimization methods, Yang *et al.* [2] handled the fuzziness in the train timetable problem through a fuzzy expected value model and proposed an effective branch-and-bound algorithm to obtain a robust solution. In order to study a renewal process behaving fuzzily, fuzzy set theory has been introduced to renewal theory, bringing about fuzzy renewal process, where the interarrival times and other parameters are regarded as fuzzy variables. Zhao and Liu [3] discussed a fuzzy renewal process and proved a fuzzy elementary renewal theorem, as well as renewal reward theorem. Hong [4] discussed a renewal process in which interarrival times and rewards were depicted by L-R fuzzy variables under triangular norm. Fuzzy random variable was first proposed by Kwakernaak [5], [6] and then developed by Puri and Ralescu [7] in 1986. Random fuzzy variable was first proposed by Liu [8] in 2002. In fuzzy random theory and random fuzzy theory, those parameters are characterized as fuzzy random variables and random fuzzy variables, respectively. Researchers have done a lot of work in these areas, such as Hwang [9], Dozzi *et al.* [10], Wang *et al.* [11], and Zhao and Tang [12].

As we know, a fundamental premise of applying probability theory is that the estimated probability is close enough to the real frequency, no matter the probability is subjective or objective. In our daily life, we often lack observed data due to economic reasons or technical difficulties. In this case, we have to invite some domain experts to evaluate the belief degree. However, human beings tend to put too much weight on unlikely events (see [13] and [14]). Thus, subjective probability sometimes fails to model the belief degree, unless some observed data are obtained to revise the belief degree. So far, some theories have been proposed to deal with the belief degree such as possibility theory (see [15]) and Dempster–Shafer theory (see [16] and [17]). An application of belief degree to predict system's behavior is shown in [18].

In 2007, an uncertainty theory was founded by Liu [19] to deal with belief degree based on uncertain measure when the belief degree has a much wider range than the real frequency. Similar to possibility measure, uncertain measure is also a type of nonprobabilistic measure, and any phenomenon that satisfies uncertain measure is said to be uncertain. In order to model the evolution of uncertain phenomenon, Liu [20] proposed uncertain process in 2008. Meanwhile, Liu [20] proposed uncertain renewal process as a special but important case, where the interarrival times are regarded as uncertain variables. After that, Liu [21] proposed uncertain renewal reward process whose interarrival times and rewards are considered to be uncertain variables. In 2011, Yao [22] founded a theory of uncertain calculus with respect to uncertain renewal process.

In everyday life, some uncertain systems are usually in two states: ON and OFF. They are initially ON and remain ON for some uncertain time; then, they go OFF and remain OFF for some uncertain time alternately. These systems cannot be described by uncertain renewal process or uncertain renewal reward process. Thus, we have to introduce other kinds of uncertain processes. This paper aims to give an important process in renewal theory based on uncertainty theory, named uncertain alternating renewal process. Throughout this paper, the emphasis is put on the uncertainty distribution of the availability of the system, i.e., the ratio of system working time to the total time. The rest of this paper is organized as follows. In Section II, we review some concepts and properties about uncertain renewal process. After that, a concept of uncertain alternating renewal process will be presented in Section III, and an elementary alternating renewal theorem will be proved in Section IV. Finally, we give an application of the alternating renewal process in Section V.

II. Preliminary

Uncertainty theory is a branch of axiomatic mathematics to deal with the experts' belief degree. It has been applied to uncertain programming (see [23]), uncertain risk analysis (see [24]), uncertain finance (see [25]), uncertain logic (see [26]), and uncertain inference (see [27] and [28]). This section will introduce some basic definitions about uncertain theory and review the main results obtained in renewal process with uncertain interarrival times and rewards.

Definition 1 [19]: Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms.

Axiom 1 (Normality Axiom): $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2 (Duality Axiom): $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom 3 (Subadditivity Axiom): For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\left\{\Lambda_i\right\}.$$

Besides, the product uncertain measure on the product σ-algebra $\mathcal{L}$ is defined by Liu [25] as follows.

Axiom 4 (Product Axiom): Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$ Then, the product uncertain measure $\mathcal{M}$ on the product σ-algebra satisfies

$$\mathcal{M}\left\{\prod_{i=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}.$$

An uncertain variable is a measurable function from an uncertain space to the set of real numbers, and a random variable is a measurable function from a probability space to the set of real numbers.

Definition 2 [19]: Let ξ be an uncertain variable. Then, its uncertainty distribution is defined by

$$\Phi(x) = \mathcal{M}\{\xi \le x\}$$

for any real number x.

Definition 3 [19]: The uncertain variables $\xi_1, \xi_2, \ldots, \xi_m$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{m} (\xi_i \in B_i)\right\} = \bigwedge_{k=1}^{m} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_m$ of real numbers.

Uncertain process is a sequence of uncertain variables driven by time or space. Uncertain renewal process is one of the most important kind of renewal process, where the interarrival times are regarded as uncertain variable instead of random variable.

Definition 4 [20]: Let $\xi_1, \xi_2, \ldots$ be independent identically distributed (iid) positive uncertain variables. Define $S_0 = 0$ and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for $n \ge 1$. Then, the uncertain process

$$N_t = \max_{n \ge 0}\{n | S_n \le t\}$$

is called an uncertain renewal process.

For an uncertain renewal process, Liu [21] proved that N_t/t converges in distribution to $1/\xi_1$. Based on this, Liu [21] proved the elementary renewal theorem, i.e.,

$$\lim_{t\to\infty} \frac{E[N_t]}{t} = E\left[\frac{1}{\xi_1}\right]$$

under the assumption that $E[1/\xi_1]$ exists.

Definition 5 [21]: Let $\xi_1, \xi_2, \ldots$ be iid uncertain interarrival times, and let $\eta_1, \eta_2, \ldots$ be iid uncertain rewards. It is also assumed that $\xi_1, \eta_1, \xi_2, \eta_2, \ldots$ are independent. Then, the uncertain process

$$R_t = \sum_{i=0}^{N_t} \eta_i$$

is called an uncertain renewal reward process, where N_t is an uncertain renewal process.

For an uncertain renewal reward process, Liu [21] proved that R_t/t converges in distribution to η_1/ξ_1. Based on this, Liu [21] proved the renewal reward theorem, i.e.,

$$\lim_{t\to\infty} \frac{E[R_t]}{t} = E\left[\frac{\eta_1}{\xi_1}\right]$$

under the assumption that $E[\eta_1/\xi_1]$ exists.

III. Uncertain Alternating Renewal Process

In this section, we shall discuss uncertain alternating renewal process and prove the uncertain alternating renewal theorem.

Definition 6: Let $\xi_1, \xi_2, \ldots$ be a sequence of iid positive uncertain variables, and let $\eta_1, \eta_2, \ldots$ be another sequence of iid positive uncertain variables. It is also assumed that $\xi_1, \eta_1, \xi_2, \eta_2, \ldots$ are independent. Then, the uncertain process

$$A_t = \begin{cases} t - \sum_{i=1}^{N_t} \eta_i & \\ \qquad \text{if } \sum_{i=1}^{N_t} (\xi_i + \eta_i) \le t < \sum_{i=1}^{N_t} (\xi_i + \eta_i) + \xi_{N_t+1} \\ \sum_{i=1}^{N_t+1} \xi_i & \\ \qquad \text{if } \sum_{i=1}^{N_t} (\xi_i + \eta_i) + \xi_{N_t+1} < t < \sum_{i=1}^{N_t+1} (\xi_i + \eta_i) \end{cases}$$

is called an uncertain alternating renewal process, where N_t is an uncertain renewal process with uncertain interarrival times $\xi_1 + \eta_1, \xi_2 + \eta_2, \ldots$

Consider a system that can be in one of two states: ON and OFF. Initially, it is ON, and it remains ON for an uncertain time ξ_1; it then goes OFF and remains OFF for an uncertain time η_1; it then goes ON for an uncertain time ξ_2 and then OFF for an uncertain time η_2 and then ON, and so forth. In this case, the uncertain alternating renewal process A_t denotes the total time that the system is ON before some time t.

Theorem 1: Let A_t be an uncertain alternating renewal process with alternating interarrival times $\xi_1, \eta_1, \xi_2, \eta_2, \ldots$ Then

$$\sum_{i=1}^{N_t} \xi_i \le A_t \le \sum_{i=1}^{N_t+1} \xi_i. \tag{1}$$

Proof: By the definition of the uncertain alternating renewal process A_t, we will verify the inequality in two cases. Case I: Assume that

$$\sum_{i=1}^{N_t} (\xi_i + \eta_i) \le t < \sum_{i=1}^{N_t} (\xi_i + \eta_i) + \xi_{N_t+1}.$$

Then, we have

$$A_t = t - \sum_{i=1}^{N_t} \eta_i \geq \sum_{i=1}^{N_t} (\xi_i + \eta_i) - \sum_{i=1}^{N_t} \eta_i = \sum_{i=1}^{N_t} \xi_i$$

and

$$A_t = t - \sum_{i=1}^{N_t} \eta_i \leq \sum_{i=1}^{N_t} (\xi_i + \eta_i) + \xi_{N_t+1} - \sum_{i=1}^{N_t} \eta_i = \sum_{i=1}^{N_t+1} \xi_i.$$

Case II: Assume that

$$\sum_{i=1}^{N_t} (\xi_i + \eta_i) + \xi_{N_t+1} < t < \sum_{i=1}^{N_t+1} (\xi_i + \eta_i).$$

Then

$$A_t = \sum_{i=1}^{N_t+1} \xi_i$$

and the inequality

$$\sum_{i=1}^{N_t} \xi_i \leq A_t \leq \sum_{i=1}^{N_t+1} \xi_i$$

holds obviously. The theorem is thus verified.

IV. Alternating Renewal Theorem

Lemma 1: Let A, B be two uncertain events in uncertain space $(\Gamma, \mathcal{L}, \mathcal{M})$. Then

$$\mathcal{M}\{A\} \leq \mathcal{M}\{A \cap B\} + \mathcal{M}\{B^c\}.$$

Proof: It follows from the subadditivity and monotonicity of uncertain measure that

$$\begin{aligned}\mathcal{M}\{A\} &= \mathcal{M}\{A \cap (B \cup B^c)\} = \mathcal{M}\{(A \cap B) \cup (A \cap B^c)\} \\ &\leq \mathcal{M}\{A \cap B\} + \mathcal{M}\{A \cap B^c\} \leq \mathcal{M}\{A \cap B\} + \mathcal{M}\{B^c\}.\end{aligned}$$

The lemma is thus verified.

Theorem 2: Let A_t be an uncertain alternating renewal process with regular alternating uncertain interarrival times $\xi_1, \eta_1, \xi_2, \eta_2, \ldots$ Assume ξ_1 and η_1 have uncertainty distributions Φ and Ψ, respectively. Then

$$\lim_{t \to \infty} \mathcal{M}\left\{\sum_{i=1}^{N_t} \xi_i / t \leq x\right\} \leq \sup_{z \geq 0} \Phi(zx) \wedge (1 - \Psi(z - zx)).$$

Proof: Write

$$B_m = \bigcap_{n=1}^{\infty} \{\xi_n \leq \Phi^{-1}(1 - m^{-1})\}.$$

Then, we have

$$\mathcal{M}\{B_m\} = \min_{n \geq 1} \mathcal{M}\{\xi_n \leq \Phi^{-1}(1 - m^{-1})\} = 1 - m^{-1}$$

by the independence of ξ_n's and

$$\mathcal{M}\{B_m^c\} = 1 - \mathcal{M}\{B_m\} = m^{-1}$$

by the self-duality of uncertain measure. It follows from Lemma 1 that

$$\begin{aligned}
&\mathcal{M}\left\{\sum_{i=1}^{N_t} \xi_i \leq x\right\} \\
&= \mathcal{M}\left\{\bigcup_{k=0}^{\infty} (N_t = k) \cap \left(\sum_{i=1}^{k} \xi_i \leq x\right)\right\} \\
&= \mathcal{M}\left\{\bigcup_{k=0}^{\infty} \left(\sum_{i=1}^{k} (\xi_i + \eta_i) \leq t < \sum_{i=1}^{k+1} (\xi_i + \eta_i)\right)\right. \\
&\qquad\qquad \left. \cap \left(\sum_{i=1}^{k} \xi_i \leq x\right)\right\} \\
&\leq \mathcal{M}\left\{\bigcup_{k=0}^{\infty} \left(\sum_{i=1}^{k+1} (\xi_i + \eta_i) > t\right) \cap \left(\sum_{i=1}^{k} \xi_i \leq x\right)\right\} \\
&\leq \mathcal{M}\left\{\bigcup_{k=0}^{\infty} \left(\xi_{k+1} + \sum_{i=1}^{k+1} \eta_i > t - x\right) \cap \left(\sum_{i=1}^{k} \xi_i \leq x\right)\right\} \\
&\leq \mathcal{M}\left\{\bigcup_{k=0}^{\infty} \left(\xi_{k+1} + \sum_{i=1}^{k+1} \eta_i > t - x\right)\right. \\
&\qquad\qquad \left. \cap \left(\sum_{i=1}^{k} \xi_i \leq x\right) \cap B_m\right\} + \mathcal{M}\{B_m^c\} \\
&\leq \mathcal{M}\left\{\bigcup_{k=0}^{\infty} \left(\sum_{i=1}^{k+1} \eta_i > t - x - \Phi^{-1}(1 - m^{-1})\right)\right. \\
&\qquad\qquad \left. \cap \left(\sum_{i=1}^{k} \xi_i \leq x\right) \cap B_m\right\} + \mathcal{M}\{B_m^c\} \\
&\leq \mathcal{M}\left\{\bigcup_{k=0}^{\infty} \left(\sum_{i=1}^{k+1} \eta_i > t - x - \Phi^{-1}(1 - m^{-1})\right)\right. \\
&\qquad\qquad \left. \cap \left(\sum_{i=1}^{k} \xi_i \leq x\right)\right\} \wedge \mathcal{M}\{B_m\} + \mathcal{M}\{B_m^c\} \\
&= \max_{k \geq 0} \left(1 - \Psi\left(\frac{t - x}{k + 1} - \frac{\Phi^{-1}(1 - m^{-1})}{k + 1}\right)\right) \\
&\qquad\qquad \wedge \Phi\left(\frac{x}{k}\right) \wedge (1 - m^{-1}) + m^{-1}
\end{aligned}$$

for any positive integer m. Then, we have

$$\begin{aligned}\mathcal{M}\left\{\sum_{i=1}^{N_t} \xi_i / t \leq x\right\} &\leq \sup_{k \geq 0} \Phi\left(\frac{tx}{k}\right) \wedge (1 - m^{-1}) \\ &\wedge \left(1 - \Psi\left(\frac{t - tx}{k + 1} - \frac{\Phi^{-1}(1 - m^{-1})}{k + 1}\right)\right) + m^{-1}.\end{aligned}$$

It is easy to verify that the optimal k tends to ∞ as $t \to \infty$. Thus

$$\begin{aligned}\lim_{t \to \infty} \mathcal{M}\left\{\sum_{i=1}^{N_t} \xi_i / t \leq x\right\} &\leq \sup_{z \geq 0} \Phi(zx) \wedge (1 - m^{-1}) \\ &\wedge (1 - \Psi(z - zx)) + m^{-1}.\end{aligned}$$

Since the equality holds for any positive integer m, letting $m \to \infty$, we have

$$\lim_{t \to \infty} \mathcal{M}\left\{\sum_{i=1}^{N_t} \xi_i / t \leq x\right\} \leq \sup_{z \geq 0} \Phi(zx) \wedge (1 - \Psi(z - zx)).$$

The theorem is thus verified.

Theorem 3: Let A_t be an uncertain alternating renewal process with regular alternating uncertain interarrival times $\xi_1, \eta_1, \xi_2, \eta_2, \ldots$

Assume ξ_1 and η_1 have uncertainty distributions Φ and Ψ, respectively. Then

$$\lim_{t\to\infty}\mathcal{M}\left\{\sum_{i=1}^{N_t+1}\xi_i/t\le x\right\}\ge\inf_{z\ge0}\Phi(zx)\vee(1-\Psi(z-zx)).$$

Proof: Write

$$B_m=\bigcap_{n=1}^{\infty}\{\xi_n\le\Phi^{-1}(1-m^{-1})\}.$$

Then, we have

$$\mathcal{M}\{B_m\}=\min_{n\ge1}\mathcal{M}\{\xi_n\le\Phi^{-1}(1-m^{-1})\}=1-m^{-1}$$

by the independence of ξ_n's and

$$\mathcal{M}\{B_m^c\}=1-\mathcal{M}\{B_m\}=m^{-1}$$

by the self-duality of uncertain measure. It follows from Lemma 1 that

$$\begin{aligned}
&\mathcal{M}\left\{\sum_{i=1}^{N_t+1}\xi_i>x\right\}\\
&=\mathcal{M}\left\{\bigcup_{k=0}^{\infty}(N_t=k)\cap\left(\sum_{i=1}^{k+1}\xi_i>x\right)\right\}\\
&=\mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\sum_{i=1}^{k}(\xi_i+\eta_i)\le t<\sum_{i=1}^{k+1}(\xi_i+\eta_i)\right)\right.\\
&\qquad\left.\cap\left(\sum_{i=1}^{k+1}\xi_i>x\right)\right\}\\
&\le\mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\sum_{i=1}^{k}(\xi_i+\eta_i)\le t\right)\cap\left(\sum_{i=1}^{k+1}\xi_i>x\right)\right\}\\
&\le\mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\sum_{i=1}^{k}\eta_i\le t-x+\xi_{k+1}\right)\cap\left(\sum_{i=1}^{k+1}\xi_i>x\right)\right\}\\
&\le\mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\sum_{i=1}^{k}\eta_i\le t-x+\xi_{k+1}\right)\right.\\
&\qquad\left.\cap\left(\sum_{i=1}^{k+1}\xi_i>x\right)\cap B_m\right\}+\mathcal{M}\{B_m^c\}\\
&\le\mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\sum_{i=1}^{k}\eta_i\le t-x+\Phi^{-1}(1-m^{-1})\right)\right.\\
&\qquad\left.\cap\left(\sum_{i=1}^{k+1}\xi_i>x\right)\cap B_m\right\}+\mathcal{M}\{B_m^c\}\\
&\le\mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\sum_{i=1}^{k}\eta_i\le t-x+\Phi^{-1}(1-m^{-1})\right)\right.\\
&\qquad\left.\cap\left(\sum_{i=1}^{k+1}\xi_i>x\right)\right\}\wedge\mathcal{M}\{B_m\}+\mathcal{M}\{B_m^c\}\\
&=\max_{k\ge0}\Psi\left(\frac{t-x}{k}+\frac{\Phi^{-1}(1-m^{-1})}{k}\right)\\
&\qquad\wedge\left(1-\Phi\left(\frac{x}{k+1}\right)\right)\wedge(1-m^{-1})+m^{-1}
\end{aligned}$$

for any positive integer m. Then, we have

$$\begin{aligned}\mathcal{M}\left\{\sum_{i=1}^{N_t+1}\xi_i/t>x\right\}&\le\max_{k\ge0}\Psi\left(\frac{t-tx}{k}+\frac{\Phi^{-1}(1-m^{-1})}{k}\right)\\
&\quad\wedge\left(1-\Phi\left(\frac{tx}{k+1}\right)\right)\wedge(1-m^{-1})+m^{-1}.\end{aligned}$$

It is easy to verify that the optimal k tends to ∞ as $t\to\infty$. Thus

$$\begin{aligned}\lim_{t\to\infty}\mathcal{M}\left\{\sum_{i=1}^{N_t+1}\xi_i/t>x\right\}&\le\sup_{z\ge0}(1-\Phi(zx))\\
&\quad\wedge\Psi(z-zx)\wedge(1-m^{-1})+m^{-1}.\end{aligned}$$

Since the equality holds for any positive integer m, letting $m\to\infty$, we have

$$\lim_{t\to\infty}\mathcal{M}\left\{\sum_{i=1}^{N_t+1}\xi_i/t>x\right\}\le\sup_{z\ge0}(1-\Phi(zx))\wedge\Psi(z-zx).$$

By the self-duality of uncertain measure, we have

$$\begin{aligned}&\lim_{t\to\infty}\mathcal{M}\left\{\sum_{i=1}^{N_t+1}\xi_i/t\le x\right\}\\
&\ge1-\sup_{z\ge0}(1-\Phi(zx))\wedge\Psi(z-zx)\\
&=\inf_{z\ge0}\Phi(zx)\vee(1-\Psi(z-zx)).\end{aligned}$$

The theorem is thus verified.

Theorem 4: Let A_t be an uncertain alternating renewal process with regular alternating uncertain interarrival times $\xi_1,\eta_1,\xi_2,\eta_2,\ldots$ Then, A_t/t converges in distribution to $\xi_1/(\xi_1+\eta_1)$ as $t\to\infty$.

Proof: It follows from inequality (1) that

$$\left\{\sum_{i=1}^{N_t+1}\xi_i/t\le x\right\}\subset\left\{\frac{A_t}{t}\le x\right\}\subset\left\{\sum_{i=1}^{N_t}\xi_i/t\le x\right\}.$$

Therefore, A_t/t has an uncertainty distribution Υ_t satisfying

$$\mathcal{M}\left\{\sum_{i=1}^{N_t+1}\xi_i/t\le x\right\}\le\Upsilon_t(x)\le\mathcal{M}\left\{\sum_{i=1}^{N_t}\xi_i/t\le x\right\}.$$

Assume that ξ_1 and η_1 have uncertainty distributions Φ and Ψ, respectively. Then, by Theorems 2 and 3, we have

$$\begin{aligned}&\inf_{z\ge0}\Phi(zx)\vee(1-\Psi(z(1-x)))\le\lim_{t\to\infty}\Upsilon_t(x)\\
&\le\sup_{z\ge0}\Phi(zx)\wedge(1-\Psi(z(1-x))).\end{aligned}$$

Since

$$\begin{aligned}&\inf_{z\ge0}\Phi(zx)\vee(1-\Psi(z(1-x)))\\
&=\sup_{z\ge0}\Phi(zx)\wedge(1-\Psi(z(1-x)))\end{aligned}$$

we obtain

$$\lim_{t\to\infty}\Upsilon_t(x)=\sup_{z\ge0}\Phi(zx)\wedge(1-\Psi(z-xz))$$

which is just the uncertainty distribution of $\xi_1/(\xi_1+\eta_1)$. The theorem is thus verified.

Theorem 5: Assume that A_t is an uncertain alternating renewal process with regular alternating uncertain interarrival times $\xi_1, \eta_1, \xi_2, \eta_2, \ldots$ If $E[\xi_1/(\xi_1 + \eta_1)]$ exists, then

$$\lim_{t\to\infty} \frac{E[A_t]}{t} = E\left[\frac{\xi_1}{\xi_1 + \eta_1}\right].$$

If ξ_1 and η_1 have regular uncertainty distributions Φ and Ψ, respectively, then

$$\lim_{t\to\infty} \frac{E[A_t]}{t} = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(\alpha) + \Psi^{-1}(1-\alpha)} d\alpha.$$

Proof: Let Υ_t denote an uncertainty distribution of A_t/t. Then, we have

$$\frac{E[A_t]}{t} = \int_0^1 1 - \Upsilon_t(x) dx$$

by the definition of expected value. On the other hand, $\xi_1/(\xi_1 + \eta_1)$ has an uncertainty distribution

$$\sup_{z\ge 0} \Phi(zx) \wedge (1 - \Psi(z - xz))$$

whose expected value is

$$E\left[\frac{\xi_1}{\xi_1 + \eta_1}\right] = \int_0^1 1 - \sup_{z\ge 0} \Phi(zx) \wedge (1 - \Psi(z - xz))\, dx.$$

Note that

$$0 \le 1 - \Upsilon_t(x) \le 1 \quad \forall t, x$$

and

$$\lim_{t\to\infty} 1 - \Upsilon_t(x) = 1 - \sup_{z\ge 0} \Phi(zx) \wedge (1 - \Psi(z - xz)) \quad \forall x$$

by Theorem 4. It follows from the Lebesgue dominated convergence theorem that

$$\begin{aligned}\lim_{t\to\infty} \frac{E[A_t]}{t} &= \lim_{t\to\infty} \int_0^1 1 - \Upsilon_t(x) dx \\ &= \int_0^1 1 - \sup_{z\ge 0} \Phi(zx) \wedge (1 - \Psi(z - xz))\, dx \\ &= E\left[\frac{\xi_1}{\xi_1 + \eta_1}\right].\end{aligned}$$

In addition, since the inverse uncertainty distribution of $\xi_1/(\xi_1 + \eta_1)$ is

$$\frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(\alpha) + \Psi^{-1}(1-\alpha)}$$

we get

$$\lim_{t\to\infty} \frac{E[A_t]}{t} = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(\alpha) + \Psi^{-1}(1-\alpha)} d\alpha.$$

Corollary 1: Let A_t be an uncertain alternating renewal process with regular alternating uncertain interarrival times $\xi_1, \eta_1, \xi_2, \eta_2, \ldots$ If the alternating interarrival times ξ_1 and η_1 are identically distributed, then

$$\lim_{t\to\infty} \frac{E[A_t]}{t} = \frac{1}{2}.$$

Proof: Note that

$$\begin{aligned}\lim_{t\to\infty} \frac{E[A_t]}{t} &= \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(\alpha) + \Phi^{-1}(1-\alpha)} d\alpha \\ &= \int_0^1 \frac{\Phi^{-1}(1-\alpha)}{\Phi^{-1}(\alpha) + \Phi^{-1}(1-\alpha)} d\alpha\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 \frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(\alpha) + \Phi^{-1}(1-\alpha)} d\alpha \\ &\quad + \int_0^1 \frac{\Phi^{-1}(1-\alpha)}{\Phi^{-1}(\alpha) + \Phi^{-1}(1-\alpha)} d\alpha \\ &= \int_0^1 \frac{\Phi^{-1}(\alpha) + \Phi^{-1}(1-\alpha)}{\Phi^{-1}(\alpha) + \Phi^{-1}(1-\alpha)} d\alpha = 1.\end{aligned}$$

The corollary follows immediately.

Example 1: Consider a linear uncertain alternating renewal process, where ξ_1 and η_1 are positive linear uncertain variables with uncertainty distributions $\mathcal{L}(a_1, b_1)$ and $\mathcal{L}(a_2, b_2)$, respectively. It follows from the uncertain alternating renewal theorem that

$$\lim_{t\to\infty} \frac{E[A_t]}{t} = \begin{cases} \dfrac{1}{2}\dfrac{a_1 + b_1}{a_1 + b_2}, & \text{if } b_1 + a_2 = a_1 + b_2 \\ \dfrac{b_1 - a_1}{b_1 + a_2 - a_1 - b_2} + \dfrac{a_1 a_2 - b_1 b_2}{(b_1 + a_2 - a_1 - b_2)^2} \ln \dfrac{b_1 + a_2}{a_1 + b_2} & \text{otherwise.} \end{cases}$$

If the uncertain variables ξ_1 and η_1 are identically distributed, i.e., $a_1 = a_2$ and $b_1 = b_2$, then we have

$$\lim_{t\to\infty} \frac{E[A_t]}{t} = \frac{1}{2}.$$

V. Application

Consider a new system that can be in one of two states at any time: working or under repair. However, data of the working time and the repairing time are unavailable if the system is completely new. In this case, we have to invite some experts to evaluate the possible working and repairing time. Thus, the model is stated as follows.

Initially, the system is working. After an uncertain time ξ_1, the system fails and undergoes repair for an uncertain time η_1 and at an uncertain cost δ_1. When the repair is completed, the system becomes as good as new and begins to work again. After an uncertain time ξ_2, the system fails and undergoes repair for an uncertain time η_2 and at an uncertain cost δ_2. The process continues infinitely. There are usually three quantities to describe the system: system failure rate, repair cost rate, and availability.

Assume that the successive working times ξ_i's are iid uncertain variables with an uncertainty distribution Φ, the successive repair times η_i's are iid uncertain variables with an uncertainty distribution Ψ and independent of ξ_i's, and the successive repair costs δ_i's are uncertain variables with an uncertainty distribution Υ and independent of ξ_i's and η_i's. Then, $\xi_1 + \eta_1$ has an uncertainty distribution

$$\begin{aligned}\Gamma(x) &= \mathcal{M}\{\xi_1 + \eta_1 \le x\} \\ &= \mathcal{M}\left\{\bigcup_{y\in\Re} (\xi_1 \le y) \cap (\eta_1 \le x - y)\right\} \\ &= \sup_{y\in\Re} \mathcal{M}\{\xi_1 \le y\} \wedge \mathcal{M}\{\eta_1 \le x - y\} \\ &= \sup_{y\in\Re} \Phi(y) \wedge \Psi(x - y)\end{aligned}$$

by the operational law of uncertain variables.

Consider a cycle from the time that the system begins working to the time that the repair is completed. Such cycles are iid and form an uncertain renewal process N_t with uncertain interarrival times $\xi_1 + \eta_1, \xi_2 + \eta_2, \ldots$ In fact, the uncertain renewal process N_t is just the system failure times before given time t, and N_t/t is the system failure rate. Since N_t takes only nonnegative integer values, we have

$$\begin{aligned}\mathcal{M}\{N_t \le x\} &= \mathcal{M}\{N_t \le \lfloor x \rfloor\} \\ &= \mathcal{M}\left\{\sum_{i=1}^{\lfloor x \rfloor}(\xi_i + \eta_i) \ge t\right\} \\ &= 1 - \mathcal{M}\left\{\sum_{i=1}^{\lfloor x \rfloor + 1}(\xi_i + \eta_i) \le t\right\} \\ &= 1 - \Gamma\left(\frac{t}{\lfloor x \rfloor + 1}\right)\end{aligned}$$

where $\lfloor x \rfloor$ represents the maximal integer less than or equal to x. It follows from the definition of expected value that

$$\begin{aligned}E[N_t] &= \int_0^\infty \mathcal{M}\{N_t \ge x\}dx \\ &= \sum_{n=0}^{\infty}\Gamma\left(\frac{t}{n+1}\right) \\ &= \sum_{n=1}^{\infty}\Gamma\left(\frac{t}{n}\right).\end{aligned}$$

Furthermore, Liu [21] proved the elementary renewal theorem, i.e.,

$$\lim_{t\to\infty} E\left[\frac{N_t}{t}\right] = E\left[\frac{1}{\xi_1 + \eta_1}\right].$$

Thus, the expected system failure rate is $E\left[1/(\xi_1 + \eta_1)\right]$ by experts' estimation.

Let R_t denotes the total repair cost of the system before some time t. Then, R_t is an uncertain renewal reward process with uncertain interarrival times $\xi_1 + \eta_1, \xi_2 + \eta_2, \ldots$ and uncertain rewards $\delta_1, \delta_2, \ldots$ i.e.,

$$R_t = \sum_{i=1}^{N_t} \delta_i.$$

The system repair cost rate can be represented by an uncertain variable R_t/t. By the renewal reward theorem by Liu [21], we have

$$\lim_{t\to\infty} \frac{E\left[R_t\right]}{t} = E\left[\frac{\delta_1}{\xi_1 + \eta_1}\right].$$

Thus, the expected repair cost rate is $E[\delta_1/(\xi_1 + \eta_1)]$ by experts' estimation.

Let A_t denote the total working time of the system before some time t. Then, A_t is just an uncertain alternating renewal process with alternating interarrival times $\xi_1, \eta_1, \xi_2, \eta_2, \ldots$ The availability of the system can be represented by an uncertain variable A_t/t. By the alternating renewal theorem, we have

$$\lim_{t\to\infty} \frac{E\left[A_t\right]}{t} = E\left[\frac{\xi_1}{\xi_1 + \eta_1}\right].$$

Thus, the expected availability of the system is $E[\xi_1/(\xi_1 + \eta_1)]$ by experts' estimation.

Example 2: Suppose that the uncertain failure times $\xi_1, \xi_2, \ldots$ are iid linear uncertain variables $\mathcal{L}(a, b)$ and that the repair times $\eta_1, \eta_2, \ldots$ are constant c. Then, the expected time to failure is $(a + b)/2$, and the time for a cycle is an linear uncertain variable $\mathcal{L}(a + c, b + c)$. Thus, the expected system failure times are

$$E[N_t] = \sum_{n=1}^{\lfloor t/(a+c) \rfloor} \frac{t - n(a + c)}{n(b - a)}$$

and system failure rate is

$$\lim_{t\to\infty} \frac{E[N_t]}{t} = \frac{\ln(b + c) - \ln(a + c)}{b - a}.$$

The expected availability of the system is

$$\lim_{t\to\infty} \frac{E\left[A_t\right]}{t} = 1 - \frac{c}{b - a}\ln\frac{b + c}{a + c}.$$

VI. Conclusion

It is well known that probability theory provides a mathematical foundation for stochastic renewal theory, while possibility theory provides a mathematical foundation for fuzzy renewal theory. In this paper, uncertainty theory is first introduced to alternating renewal process, and uncertain renewal theory is further developed to model repairable systems in uncertain environment. This paper has proved an uncertain alternating renewal theorem based on an estimation of the uncertainty distribution. It also gave a simple application of the uncertain alternating renewal process.

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LMI Solution for Robust Static Output Feedback Control of Discrete Takagi–Sugeno Fuzzy Models

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***Abstract*—This paper deals with the stabilization problem of discrete-time Takagi–Sugeno (T–S) fuzzy systems via static output controller (SOFC). The proposed method uses the descriptor approach to study this problem and leads to strict linear matrix inequality ($\mathcal{LMI}$) formulation. In contrast with the existing results, the method allows coping with multiple output matrices, as well as uncertainties. Moreover, the new proposed method can lead to less conservative results by introducing slack variables and considering multiple Lyapunov matrices. A robust SOFC for uncertain T–S fuzzy models is also derived in strict $\mathcal{LMI}$ terms. Numerical examples are given to illustrate the effectiveness of the proposed design results.**

***Index Terms*—Static output control, strict linear matrix inequalities ($\mathcal{LMI}s$), Takagi–Sugeno (T–S) fuzzy models, uncertainties.**

I. Introduction

Over the past two decades, the successful Takagi–Sugeno (T-S) approach has been extensively studied to deal with nonlinearities and uncertainties in industrial plants [24]. Indeed, due to the complexity and nonlinearity of real systems, T–S fuzzy models were proposed for stability analysis and control design [4]– [12], [30], [32], [33]. Moreover, the capability of T–S models to represent uncertainties, coming from modeling errors and/or faults, has already been shown [11], [17], [33]. In this framework, many papers that deal with robust stability and stabilization of uncertain T–S fuzzy systems are available in the literature [15], [18]–[21], [25]–[28], [46], [47]. These works concern the design of controllers based on either state feedback control, observer-based control, dynamic output feedback control, or static output feedback control (SOFC) [1]–[3], [14], [16], [17], [22], [23], [29]–[31], [34], [45]. However, few results deal with the last two problems and particulary for SOFC implying different output matrices with uncertainties. For example, in [13] and [31], a robust controller via SOFC is studied for common output matrices with a single Lypunov matix. The derived result is in linear matrix inequalities ($\mathcal{LMI}s$) [40] with additional equality constraint. Strict $\mathcal{LMI}$ conditions are obtained by using invertible matrix T transforming output matrix to $CT = [I, 0]$. The proposed results are conservative since they lead to a diagonal Lyapunov matrix. Note that the used coordinate transformation (T) is almost impossible for different output matrices. The case of multiple output matrices is also studied in [1] and [13]. However, the drawbacks of the proposed results are that 1) the given conditions use a single Lyapunov matrix; 2) the result is not strictly $\mathcal{LMI}s$; and 3) the design conditions involve N (where N is the number of local models) equality constraints leading to a particular structure of the Lyapunov matrix and are not easy to solve; indeed they imply constraint on the rank of the matrix composed by the N output matrices. In addition, the proposed approaches could not deal with uncertain output matrices.

In this paper, we propose to design SOFC for discrete-time T–S fuzzy systems in a general framework. First, instead of a common quadratic Lyapunov function, general Lyapunov functions are considered. Second, in order to derive $\mathcal{LMI}$ conditions, we take profit of the redundancy induced by a descriptor formulation. Third, the use of Finsler's lemma introduces slack variables that give extra freedom degrees. This very generic formulation allows dealing with new relaxed $\mathcal{LMI}$ conditions for several problems: the more classical one, i.e., a constant output matrix, or more general ones, different output matrices with or without uncertainties. Notice in these last cases that no satisfactory result in an $\mathcal{LMI}$ form is available in the literature.

The outline of this paper is as follows. First, the T–S system description and preliminary result are stated in Section II. In Section III, the proposed approach is given, and the main result is proposed in $\mathcal{LMI}$ formulation. Robustness conditions to design SOFC for uncertain T–S fuzzy models are then given using strict $\mathcal{LMI}$ constraints. In Section IV, examples to show the effectiveness of the proposed design $\mathcal{LMI}$ conditions are proposed. Conclusion completes this paper.

Notation. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. Superscript "$^{\mathsf{T}}$" denotes matrix transposition, notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is a positive semidefinite (respectively, positive definite) matrix, symbol $(*)$ denotes the transpose elements in the symmetric position, $\mathbf{I}$ is the identity matrix with compatible dimensions, and $I_N = \{1, 2, \ldots, N\}$.

II. Problem Position and Preliminary Results

Let us consider the following discrete-time T–S fuzzy model described by [24]:

$$x(t+1) = \sum_{i=1}^{N} \xi_i(z(t))(A_i x(t) + B_i u(t)) \tag{1a}$$

$$y(t) = \sum_{i=1}^{N} \xi_i(z(t)) C_i x(t) \tag{1b}$$

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