Quiz 4 - Computational Physics II

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Excellent!

Date: Thursday 29 May 2025 (17h00) **Duration:** 45 minutes
Credits: 20 points (4 questions) **Type of evaluation:** LAB

Provide concise answers to the following items:

1. **(4 points) Partial differential equations (PDEs) in Fourier space**
   (a) Write down the 1D heat equation and the 1D one-way wave equation in Fourier space.
   (b) Explain the difference between diffusion and advection processes.

a) let $\tilde{u}(k,t)$ be the Fourier trans. of $u(x,t)$

① Heat Eqn.

Real → Fourier

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \longrightarrow \frac{\partial \tilde{u}}{\partial t} = -\alpha k^2 \tilde{u}$$

② One way wave Eqn.

Real → Fourier

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \longrightarrow \frac{\partial \tilde{u}}{\partial t} = -ick\tilde{u}$$

Diffusion:

→ Heat eqn. k^2 dependence
→ Makes spreading of solutions
→ High frequency modes decay exponentially
→ Energy dissipative process

Advection

→ wave Equation, k dependence
→ Makes transport of solutions
→ All frequencies move at the same speed
→ Energy conservation process

2. **(6 points) Numerical Stability**
   Explain 3 different methods by which we can determine the stability of a numerical scheme.

① Von Neumann Stability

We put a Fourier mode into

$$u_j^n = \xi^n e^{ikj\Delta x}$$

then we must derive the amplification factor ξ and stability requires that

$$|\xi| \leq 1$$

for all k

② Truncation error Analysis

We keep the higher-order terms on taylor series expansions to understand how they modify the PDE.
we need to examine the coefficients of artificial diffusion that appear. we use the upwind scheme

$$\frac{u\Delta x}{2}(1-c) \; ; \; c = \frac{u\Delta t}{\Delta x}$$

so if $c > 1$ we get negative diffusion which is unphysical or inestability

③ Domain of dependency Analysis

we compare the numerical domain of dependence with the physical domain of dependence where physical domain is traced backwards in time, and the numerical domain are grid points influenced by finite differences. For stability the numerical domain must contain and include the physical domain. This method is necessary but not sufficient sometimes

3. **(5 points) Finite-difference methods for PDEs**
Write down the 3D Poisson equation and its central-difference approximation including errors.

Poisson: $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f(x,y,z)$ ✓

C.D. $\frac{u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}}{(\Delta x)^2}$ ✓ $+ \frac{u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k}}{(\Delta y)^2}$ ✓ $+ \frac{u_{i,j,k+1} - 2u_{i,j,k} + u_{i,j,k-1}}{(\Delta z)^2}$ ✓ $= f_{i,j,k}$

Truncation Error: $O((\Delta x)^2, (\Delta y)^2, (\Delta z)^2)$ ✓

Global Error : $O(h^2)$ ✓

This is for central differences. ✓

4. **(5 points) Boundary Conditions for advection problems**
Consider the advection equation, $u_t + c\,u_x = 0$ (with $c > 0$) on the domain $x \in [0, L]$ with N_x physical grid points plus 2 ghost zones (one at each end). The array `u` has size $N_x + 2$, and the physical points have indices 1 to N_x. The CFL number is defined as CFL $= c\,\Delta t/\Delta x$. The upwind scheme for interior points is: `u_new[1:Nx+1] = u[1:Nx+1] - CFL * (u[1:Nx+1] - u[0:Nx])`. Provide the Python code lines to set the ghost zone values of `u_new` for each boundary condition type below:

1. **Periodic boundaries:** The domain wraps around, so $u(0, t) = u(L, t)$.
2. **Dirichlet boundaries:** The boundaries have fixed values, so $u(0, t) = 0.5$ and $u(L, t) = 0.0$.
3. **Neumann boundaries:** The boundary gradients are set, so $u_x(0, t) = 0.0$ and $u_x(L, t) = 1.0$.

①

u_new[0] = u_new[Nx] ✓

u_new[Nx+1] = u_new[1] ✓

②

u_new[0] = 0.5 ✓

u_new[Nx+1] = 0. ✓

③ If $\frac{\partial u}{\partial x}(0,t) = 0$ ✓, $\frac{\partial u}{\partial x}(L,t) = 1$ ✓, $dx = \frac{L}{Nx}$ ✓

u_new[0] = u_new[1] ✓

u_new[Nx+1] = u_new[Nx] + 1.0 * dx ✓