Von Neumann Stability Analysis for the Explicit Method

1D Heat Equation

We consider the 1D heat equation:

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}$$

where c is the thermal diffusivity.

Explicit Finite Difference Scheme (FTCS)

Discretize time and space:

$$\frac{u_i^{j+1} - u_i^j}{k} = c\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2}$$

Rewriting:

$$u_i^{j+1} = ru_{i+1}^j + (1-2r)u_i^j + ru_{i-1}^j$$

where $r = \frac{ck}{h^2}$.

Von Neumann Stability Analysis

Assume a Fourier mode solution of the form:

$$u_i^j = G^j e^{i\theta ih}$$

Substitute into the update formula:

$$G^j e^{i\theta ih} = rG^{j-1}e^{i\theta(i+1)h} + (1-2r)G^{j-1}e^{i\theta ih} + rG^{j-1}e^{i\theta(i-1)h}$$

Divide both sides by $G^{j-1}e^{i\theta ih}$:

$$G = re^{i\theta h} + (1-2r) + re^{-i\theta h}$$

Use Euler's formula:

$$G = (1-2r) + r(e^{i\theta h} + e^{-i\theta h}) = (1-2r) + 2r\cos(\theta h)$$

Stability Condition

The magnitude of the amplification factor G must satisfy:

$$|G| \leq 1$$

Since G is real in this case:

$$-1 \leq G \leq 1$$

We have:

$$G = 1 - 2r(1 - \cos(\theta h))$$

The maximum value of $(1 - \cos(\theta h))$ is 2, so the minimum value of G is:

$$G_{\min} = 1 - 4r$$

For stability, we require:

$$1 - 4r \geq -1 \Rightarrow r \leq \frac{1}{2}$$

Also, $r \geq 0$. Hence the stability condition is:

$$0 \leq r \leq \frac{1}{2}$$