Unit 3. Introduction to Computational Fluid Dynamics (CFD)

Lecture 309: Lax-Wendroff scheme

Reference book:

"FUNDAMENTALS OF NUMERICAL METHODS FOR FLUID FLOWS" by Volpiani (CFD03)
https://www.psvolpiani.com/courses

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Introduction to CFD: Lax-Wendroff method

Scheme comparison:

Advection equation: $$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

Upwind scheme: $$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} = 0$$

Central scheme with a diffusion term: $$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\underbrace{\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}}_{\text{centered term}} - \frac{\theta}{2\Delta t}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) = 0$$

Introduction to CFD: Lax-Wendroff method

Problem statement:

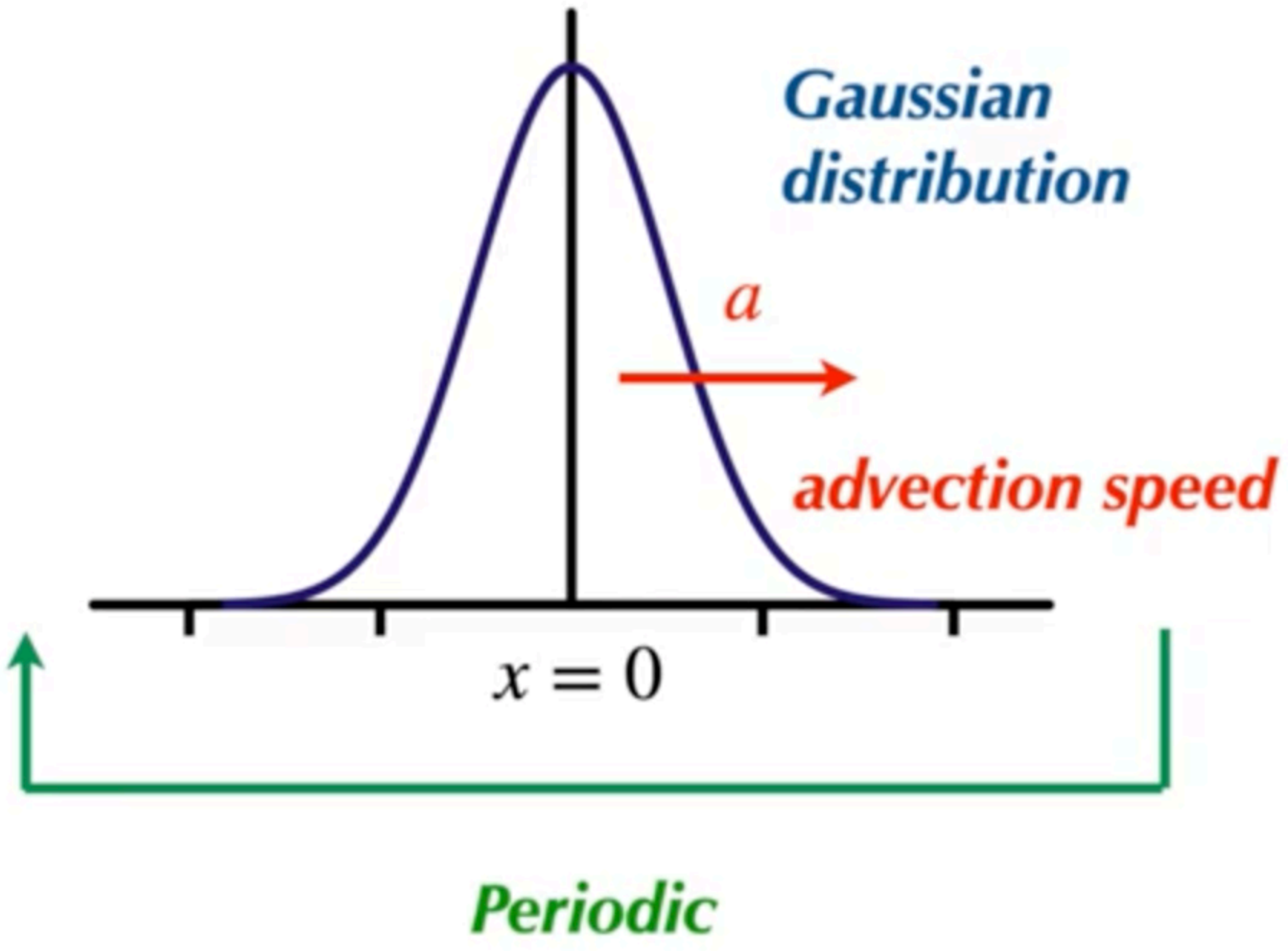
Transport equation: $$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad (1)$$

Initial condition: $$u_0(x) = \exp\left(-\frac{1}{2}\left(\frac{x}{0.4}\right)^2\right) \qquad (2)$$

Boundary conditions: $$u(-2,t) = u(2,t) \qquad (3)$$

$$\Omega = \{x \in \mathbb{R} / -2 \leqslant x \leqslant 2\}$$

$$\tau = \{t \in \mathbb{R}^* / t > 0\}$$



Introduction to CFD: Lax-Wendroff method

Truncation error:

$$\varepsilon_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \frac{\theta}{2\Delta t}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

Taylor's series:

$$u_j^{n+1} = u_j^n + \Delta t\left(\frac{\partial u}{\partial t}\right)_j^n + \frac{\Delta t^2}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_j^n + \mathcal{O}(\Delta t^3)$$

$$u_{j+1}^n = u_j^n + \Delta x\left(\frac{\partial u}{\partial x}\right)_j^n + \frac{\Delta x^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n + \frac{\Delta x^3}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_j^n + \mathcal{O}(\Delta x^4)$$

$$u_{j-1}^n = u_j^n - \Delta x\left(\frac{\partial u}{\partial x}\right)_j^n + \frac{\Delta x^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n - \frac{\Delta x^3}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_j^n + \mathcal{O}(\Delta x^4)$$

Introduction to CFD: Lax-Wendroff method

Truncation error:

$$\varepsilon_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \frac{\theta}{2\Delta t}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \left(\frac{\partial u}{\partial t}\right)_j^n + \mathcal{O}(\Delta t^1)$$

Taylor's series:

$$\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = \left(\frac{\partial u}{\partial x}\right)_j^n + \mathcal{O}(\Delta x^2)$$

$$\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} = \left(\frac{\partial^2 u}{\partial x^2}\right)_j^n + \mathcal{O}(\Delta x^2)$$

Introduction to CFD: Lax-Wendroff method

Truncation error:

$$\varepsilon_j^n = \boxed{\frac{u_j^{n+1} - u_j^n}{\Delta t}} + a \boxed{\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}} - \frac{\theta}{2\Delta t}\left(\boxed{u_{j+1}^n - 2u_j^n + u_{j-1}^n}\right)$$

Taylor's series:

$$\boxed{\frac{u_j^{n+1} - u_j^n}{\Delta t}} = \left(\frac{\partial u}{\partial t}\right)_j^n + \mathcal{O}(\Delta t^1)$$

$$\boxed{\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}} = \left(\frac{\partial u}{\partial x}\right)_j^n + \mathcal{O}(\Delta x^2)$$

$$\frac{\boxed{u_{j+1}^n - 2u_j^n + u_{j-1}^n}}{\Delta x^2} = \left(\frac{\partial^2 u}{\partial x^2}\right)_j^n + \mathcal{O}(\Delta x^2)$$

The error becomes:

$$\varepsilon_j^n = \left(\frac{\partial u}{\partial t} + \mathcal{O}(\Delta t)\right)_j^n + a\left(\frac{\partial u}{\partial x} + \mathcal{O}(\Delta x^2)\right)_j^n - \frac{\theta \Delta x^2}{2\Delta t}\left(\frac{\partial^2 u}{\partial x^2} + \mathcal{O}(\Delta x^2)\right)_j^n$$

Introduction to CFD: Lax-Wendroff method

Truncation error:

$$\varepsilon_j^n = \boxed{\frac{u_j^{n+1} - u_j^n}{\Delta t}} + a\boxed{\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}} - \frac{\theta}{2\Delta t}\left(\boxed{u_{j+1}^n - 2u_j^n + u_{j-1}^n}\right)$$

Taylor's series:

$$\boxed{\frac{u_j^{n+1} - u_j^n}{\Delta t}} = \left(\frac{\partial u}{\partial t}\right)_j^n + \frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_j^n + \mathcal{O}(\Delta t^2)$$

$$\boxed{\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}} = \left(\frac{\partial u}{\partial x}\right)_j^n + \mathcal{O}(\Delta x^2)$$

$$\frac{\boxed{u_{j+1}^n - 2u_j^n + u_{j-1}^n}}{\Delta x^2} = \left(\frac{\partial^2 u}{\partial x^2}\right)_j^n + \mathcal{O}(\Delta x^2)$$

The error becomes:

$$\varepsilon_j^n = \left(\frac{\partial u}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + \mathcal{O}(\Delta t^2)\right)_j^n + a\left(\frac{\partial u}{\partial x} + \mathcal{O}(\Delta x^2)\right)_j^n - \frac{\theta \Delta x^2}{2\Delta t}\left(\frac{\partial^2 u}{\partial x^2} + \mathcal{O}(\Delta x^2)\right)_j^n$$

Introduction to CFD: Lax-Wendroff method

Truncation error:

$$\varepsilon_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \frac{\theta}{2\Delta t}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

$$\varepsilon_j^n = \left(\frac{\partial u}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + \mathcal{O}(\Delta t^2)\right)_j^n + a\left(\frac{\partial u}{\partial x} + \mathcal{O}(\Delta x^2)\right)_j^n - \frac{\theta \Delta x^2}{2\Delta t}\left(\frac{\partial^2 u}{\partial x^2} + \mathcal{O}(\Delta x^2)\right)_j^n$$

Clue:

$$\frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_j^n - \frac{\theta \Delta x^2}{2\Delta t}\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n = 0$$

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = -a\frac{\partial}{\partial x}\left(-a\frac{\partial u}{\partial x}\right) = a^2\frac{\partial^2 u}{\partial x^2}$$

$$\left[\frac{\theta \Delta x^2}{2\Delta t} - \frac{a^2 \Delta t}{2}\right]\frac{\partial^2 u}{\partial x^2} = 0$$

Introduction to CFD: Lax-Wendroff method

Truncation error:

$$\varepsilon_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \frac{\theta}{2\Delta t}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

$$\varepsilon_j^n = \left(\frac{\partial u}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + \mathcal{O}(\Delta t^2)\right)_j^n + a\left(\frac{\partial u}{\partial x} + \mathcal{O}(\Delta x^2)\right)_j^n - \frac{\theta \Delta x^2}{2\Delta t}\left(\frac{\partial^2 u}{\partial x^2} + \mathcal{O}(\Delta x^2)\right)_j^n$$

$$\frac{\theta \Delta x^2}{2\Delta t} - \frac{a^2 \Delta t}{2} = 0$$

If $\theta = (a\Delta t/\Delta x)^2$ ***the numerical scheme is therefore consistent with a precision of order 2 in both time and space.***

Introduction to CFD: Stability of the Lax-Wendroff method

Defining $\quad v^n(x) = u_j^n \quad$ **if** $\quad x \in [(j-1/2)\Delta x, (j+1/2)\Delta x]$

Fourier transform $\quad \hat{v}^n(k) = \dfrac{1}{\sqrt{2\pi}} \int_\Omega v^n(x) \exp(-ixk) \mathrm{d}x$

Clue $\quad \hat{v}_{j+m} = e^{(ikm\Delta x)} \hat{v}_i$

$$\frac{\hat{v}_j^{n+1} - \hat{v}_j^n}{\Delta t} + \frac{a}{2\Delta x} \hat{v}_j^n \left[\mathrm{e}^{ik\Delta x} - \mathrm{e}^{-ik\Delta x} \right] - \frac{\theta}{2\Delta t} \hat{v}_j^n \left[\mathrm{e}^{ik\Delta x} - 2 + \mathrm{e}^{-ik\Delta x} \right] = 0$$

Introduction to CFD: Stability of the Lax-Wendroff method

$$\frac{\hat{v}_j^{n+1} - \hat{v}_j^n}{\Delta t} + \frac{a}{2\Delta x}\hat{v}_j^n\left[e^{ik\Delta x} - e^{-ik\Delta x}\right] - \frac{\theta}{2\Delta t}\hat{v}_j^n\left[e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right] = 0$$

$$\frac{\hat{v}_j^{n+1}}{\hat{v}_j^n} = 1 - \frac{a\Delta t}{2\Delta x}\left[e^{ik\Delta x} - e^{-ik\Delta x}\right] + \frac{\theta}{2}\left[e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right]$$

$$\boxed{\sigma = a\Delta t/\Delta x}$$

$$\frac{\hat{v}_j^{n+1}}{\hat{v}_j^n} = 1 - \sigma i \sin(k\Delta x) + \theta(\cos(k\Delta x) - 1)$$

Amplification coefficient: $$\left\|\frac{\hat{v}_j^{n+1}}{\hat{v}_j^n}\right\|^2 = [1 + \theta(\cos(k\Delta x) - 1)]^2 + \sigma^2 \sin^2(k\Delta x)$$

Introduction to CFD: Stability of the Lax-Wendroff method

Case 1, $\theta = 0$ **:** $$\left\| \frac{\hat{v}_j^{n+1}}{\hat{v}_j^n} \right\|^2 = 1 + \sigma^2 \sin^2(k\Delta x) \geq 1 \quad \forall\, k\Delta x$$

Case 2, $\theta = (a\Delta t/\Delta x)^2$ **:** $$\left\| \frac{\hat{v}_j^{n+1}}{\hat{v}_j^n} \right\|^2 = 1 + 2\theta(\cos(k\Delta x) - 1) + \theta^2(\cos(k\Delta x) - 1)^2 + \sigma^2(1 - \cos(k\Delta x))(1 + \cos(k\Delta x))$$

$$1 - 2\theta(1 - \cos(k\Delta x)) + \theta^2(1 - \cos(k\Delta x))^2 + \sigma^2(1 - \cos(k\Delta x))(1 + \cos(k\Delta x)) \leq 1$$

$$\theta^2(1 - \cos(k\Delta x)) - 2\theta + \sigma^2(1 + \cos(k\Delta x)) \leq 0$$

Stable if: $\sigma \leq 1$

$$(1 - \cos(k\Delta x))(\sigma^4 - \sigma^2) \leq 0$$

Introduction to CFD: Stability of the Lax-Wendroff method

Case 1, $\theta = 0$ **:**

unconditionally unstable !

Case 2, $\theta = (a\Delta t/\Delta x)^2$ **:**

Stable if: $\sigma \leq 1$

Case 3 :

Stable if: $\sigma^2 \leq \theta \leq 1$

$$\uparrow$$

$$\theta^2(1 - \cos(k\Delta x)) - 2\theta + \sigma^2(1 + \cos(k\Delta x)) = 0$$