Homework 4 - Quantum Mechanics I

NAME: ____ **SCORE:** ____________

Deadline: Friday 6 December 2024 by 5pm
Credits: 20 points **Number of problems:** 5
Type of evaluation: Formative Evaluation

- **This homework includes problems on unit 4 of the QM course programme.**
- **This assignment should be submitted individually by the deadline.**
- **If you work with classmates, please state their names and write your own reports.**
- **Please submit a professional homework report and clearly highlight all answers.**
- **All calculation steps should be justified to get full credits.**
- **Any code used should be attached to get full credits.**

1. **(4 points) Spherical harmonics**

   (a) A quantum system is known to be in the (unnormalised) state described by the wave function $\psi(\theta, \phi) = 4\, Y_3^2 + 5\, Y_4^1 + 2\, Y_4^1$, where the $Y_\ell^m(\theta, \phi)$ are the spherical harmonics. What is the probability of finding the system in a state with quantum number $m = 1$?

   (b) Analytically construct all the possible spherical harmonics, $Y_\ell^m(\theta, \phi)$, for $\ell = 2$.

   (c) Using your favourite programming language, make 3D plots of all of them.

   (d) Choose two of the spherical harmonics constructed in part (b), and prove that they are normalised and orthogonal.

2. **(4 points) Hydrogen atom**

   (a) Construct all the possible spatial wave functions, $\psi_{nlm}(r, \theta, \phi) = R_{n\ell}(r) Y_\ell^m(\theta, \phi)$, of the hydrogen atom for $(n, \ell, m) = (4, 1, m)$.

   (b) Using your favourite programming language, make density plots of all of these states.

   (c) Calculate the energy level of these states in units of eV.

   (d) In terms of the Bohr radius, find $\langle r \rangle$, $\langle x \rangle$, $\langle r^2 \rangle$ and $\langle x^2 \rangle$ for an electron in the ground state of hydrogen.

3. **(4 points) Angular momentum and Spin**

   (a) Let L_x, L_y, and L_z be the components of the orbital angular momentum operator $\vec{\boldsymbol{L}}$. Calculate the value of the commutator: $[L_x L_y, L_z]$.

   (b) Suppose we have a spin-$\frac{1}{2}$ particle, whose quantum state can be represented by the eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ of the S_z operator. Using the Pauli matrix σ_x, calculate a normalised eigenstate of the S_x operator with an eigenvalue $+\frac{\hbar}{2}$.

(c) Consider an electron in the spin state: $\chi = C \begin{bmatrix} 1+i \\ 2-i \end{bmatrix}$. Normalise χ, and find the expectation values of S_x, S_y, and S_z.

(d) Find the uncertainties σ_{S_x}, σ_{S_y}, and σ_{S_z} for χ in (c). Are the results consistent with all three uncertainty principles?

4. **(4 points) Radial wave function of atoms**

The wave function of an electron in a hydrogenic atom with an atomic number $Z = 70$ and mass number $A = 173.05$ is given by:

$$\psi(r) = Be^{-r/a}$$

where $a = \frac{a_0}{Z}$ and $a_0 = 0.53$ Å is the Bohr radius. This (Ytterbium, Yb) atom contains only one electron with charge e. The charge and radius of its nucleus are eZ and $R = 1.2(A^{1/3})$ fm, respectively.

(a) Normalise the wave function.

(b) Calculate the probability that the electron is found in the nucleus.

(c) What is the probability that the electron is in the region $y < 0$?

(d) Using your favourite programming language, make high-quality 3D and density plots of $\psi(r)$. Label the plots appropriately.

5. **(4 points) Electron in a Magnetic Field**

Consider an electron (at rest) embedded in an oscillating magnetic field:

$$\vec{B} = B_0 \cos(\omega t)\, \vec{k},$$

where B_0 and ω are constants.

(a) Construct the Hamiltonian matrix for this system. Is this Hamiltonian time-dependent or time-independent?

(b) The electron starts out (at $t = 0$) in the spin-up state with respect to the x axis (i.e., $\chi(0) = \chi_+^{(x)}$). Determine $\chi(t)$ at any subsequent time by solving the Schrödinger equation.

(c) Find the probability of getting $-\frac{\hbar}{2}$, if you measure S_x.

(d) What is the minimum field strength (B_0) required to force a complete flip in S_x?