IDENTICAL PARTICLES

Bosons and Fermions

Suppose we have two noninteracting particles, number 1 in the (one-particle) state $\psi_a(\mathbf{r})$, and number 2 in the state $\psi_b(\mathbf{r})$. In that case $\psi(\mathbf{r}_1,\mathbf{r}_2)$ is the product:

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2).$$

This assumes that we can tell the particles apart—otherwise it wouldn't make any sense to claim that number 1 is in state ψ_a and number 2 is in state ψ_b; all we could say is that *one* of them is in the state ψ_a and the other is in state ψ_b, but we wouldn't know which is which.

All electrons are *utterly identical*, in a way that no two classical objects can ever be. It's not just that *we* don't know which electron is which.

Quantum mechanics neatly accommodates the existence of particles that are *indistinguishable in principle*: We simply construct a wave function that is *noncommittal* as to which particle is in which state. There are actually two ways to do it:

$$\psi_{\pm}(\mathbf{r}_1, \mathbf{r}_2) = A\left[\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) \pm \psi_b(\mathbf{r}_1)\psi_a(\mathbf{r}_2)\right];$$

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(Symmetrisation requirement)

The theory admits two kinds of identical particles:

- **bosons** (the plus sign), and
- **fermions** (the minus sign).

Boson states are **symmetric** under interchange, $\psi_+(\mathbf{r}_2, \mathbf{r}_1) = \psi_+(\mathbf{r}_1, \mathbf{r}_2)$; fermion states are **antisymmetric** under interchange, $\psi_-(\mathbf{r}_2, \mathbf{r}_1) = -\psi_-(\mathbf{r}_1, \mathbf{r}_2)$. It so happens that:

$$\begin{cases} \text{all particles with } \textit{integer} \text{ spin are bosons, and} \\ \text{all particles with } \textit{half integer} \text{ spin are fermions.} \end{cases}$$

This **connection between spin and statistics** (bosons and fermions have quite different statistical properties) can be *proved* in *relativistic* quantum mechanics.

IDENTICAL PARTICLES

Bosons and Fermions

T*wo identical fermions* (for example, two electrons) *cannot occupy the same state*.

For if $\psi_a = \psi_b$, we are left with no wave function at all:

$$\psi_{-}(\mathbf{r}_1, \mathbf{r}_2) = A\left[\psi_a(\mathbf{r}_1)\psi_a(\mathbf{r}_2) - \psi_a(\mathbf{r}_1)\psi_a(\mathbf{r}_2)\right] = 0,$$

This is the **Pauli exclusion principle,** which is a consequence of the rules for constructing two-particle wave functions, applying to *all* identical fermions.