

The hydrogen atom

Spin 1/2:

$s = 1/2$ is the spin of the particles that make up ordinary matter (protons, neutrons, and electrons), as well as all quarks and all leptons.

There are just *two* eigenstates: $|s\ m\rangle$;

1. **spin up** (informally, \uparrow): $\left| \frac{1}{2} \ \frac{1}{2} \right\rangle$

2. **spin down** (informally, \downarrow): $\left| \frac{1}{2} \ \left(-\frac{1}{2}\right) \right\rangle$

Using these as basis vectors, the general state of a spin-1/2 particle can be represented by a two-element column matrix (or **spinor**):

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-, \quad \text{where:}$$

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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Spin 1/2:

The spin operators become matrices: $S^2 |s m\rangle = \hbar^2 s(s+1) |s m\rangle$; $S_z |s m\rangle = \hbar m |s m\rangle$;

$$S^2 \chi_+ = \frac{3}{4} \hbar^2 \chi_+ \quad \text{and} \quad S^2 \chi_- = \frac{3}{4} \hbar^2 \chi_-.$$

If we write S^2 as a matrix with undetermined elements:

$$S^2 = \begin{pmatrix} c & d \\ e & f \end{pmatrix}$$

$$S^2 \chi_+ = \frac{3}{4} \hbar^2 \chi_+ \quad \Rightarrow \quad \begin{pmatrix} c & d \\ e & f \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3}{4} \hbar^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} c \\ e \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \hbar^2 \\ 0 \end{pmatrix}, \quad c = (3/4) \hbar^2 \text{ and } e = 0.$$

$$S^2 \chi_- = \frac{3}{4} \hbar^2 \chi_- \quad \Rightarrow \quad \begin{pmatrix} c & d \\ e & f \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{3}{4} \hbar^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} d \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{3}{4} \hbar^2 \end{pmatrix}, \quad d = 0 \text{ and } f = (3/4) \hbar^2.$$

Conclusion:
$$S^2 = \frac{3}{4} \hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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Spin 1/2:

Similarly:

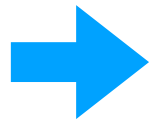
$$S_z \chi_+ = \frac{\hbar}{2} \chi_+, \quad S_z \chi_- = -\frac{\hbar}{2} \chi_-,$$

For which:

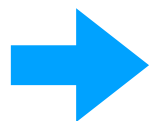
$$S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Remember:

$$S_{\pm} |s m\rangle = \hbar \sqrt{s(s+1) - m(m \pm 1)} |s (m \pm 1)\rangle,$$



$$S_+ \chi_- = \hbar \chi_+, \quad S_- \chi_+ = \hbar \chi_-, \quad S_+ \chi_+ = S_- \chi_- = 0,$$



$$S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Now $S_{\pm} = S_x \pm i S_y$, so $S_x = (1/2) (S_+ + S_-)$ and $S_y = (1/2i) (S_+ - S_-)$, and hence

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

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Spin 1/2 (Pauli spin matrices).

Since S_x , S_y , and S_z all carry a factor of $\hbar/2$, it is tidier to write $\mathbf{S} = (\hbar/2)\boldsymbol{\sigma}$, where

$$\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These are the famous **Pauli spin matrices**.

S_x , S_y , S_z , and S^2 are all *hermitian* matrices (as they *should* be, since they represent observables).

On the other hand, S_+ and S_- are *not* hermitian—evidently they are not observable.

The eigenspinors of S_z are:

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \left(\text{eigenvalue} + \frac{\hbar}{2} \right); \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \left(\text{eigenvalue} - \frac{\hbar}{2} \right).$$

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Spin 1/2 (Pauli spin matrices).

Remember: $\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-$,

If you measure S_z on a particle in the general state χ , you could get $+\hbar/2$, with probability $|a|^2$, or $-\hbar/2$, with probability $|b|^2$. Since these are the *only* possibilities:

$$|a|^2 + |b|^2 = 1$$

(i.e. the spinor must be *normalised*: $\chi^\dagger \chi = 1$).

But what if, instead, we chose to measure S_x ? What are the possible results and probabilities?

We need to know the eigenvalues and eigenspinors of S_x

The characteristic equation is:

$$\begin{vmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 = \left(\frac{\hbar}{2}\right)^2 \Rightarrow \lambda = \pm \frac{\hbar}{2}.$$

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The possible values for S_x are the same as those for S_z . The eigenspinors are obtained via:

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm \frac{\hbar}{2} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \pm \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \beta = \pm \alpha.$$

The (normalised) eigenspinors of S_x are:

$$\chi_+^{(x)} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \quad \left(\text{eigenvalue} + \frac{\hbar}{2} \right); \quad \chi_-^{(x)} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \quad \left(\text{eigenvalue} - \frac{\hbar}{2} \right).$$

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The generic spinor χ can be expressed as a linear combination of them:

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_- \qquad \chi = \left(\frac{a+b}{\sqrt{2}} \right) \chi_+^{(x)} + \left(\frac{a-b}{\sqrt{2}} \right) \chi_-^{(x)}$$

If you measure S_x , the probability of getting $+\hbar/2$ is $(1/2)|a+b|^2$, and the probability of getting $-\hbar/2$ is $(1/2)|a-b|^2$.

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Spin 1/2 (implications):

For a particle in the state χ_+ , what is the z-component of a particle's spin angular momentum?

We can answer unambiguously: $+\hbar/2$.

What is the x-component of that particle's spin angular momentum?

If you measure S_x , the chances are fifty-fifty of getting either $+\hbar/2$ or $-\hbar/2$.

It simply *does not have* a particular x-component of spin.