Schrödinger equation in spherical coordinates:

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2\psi}{\partial\phi^2}\right)\right]+V\psi=E\psi.$$

Variable separation:

$$\psi(r,\theta,\phi)=R(r)\,Y(\theta,\phi).$$

$$-\frac{\hbar^2}{2m}\left[\frac{Y}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{R}{r^2\sin^2\theta}\frac{\partial^2Y}{\partial\phi^2}\right]+VRY=ERY.$$
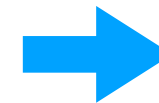
Dividing by YR and multiplying by $-2mr^2/\hbar^2$:

$$\left\{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-\frac{2mr^2}{\hbar^2}[V(r)-E]\right\}+\frac{1}{Y}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2Y}{\partial\phi^2}\right\}=0.$$

Schrödinger equation in spherical coordinates:

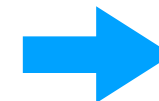$$\left\{ \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}[V(r) - E] \right\} + \frac{1}{Y}\left\{ \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} \right\} = 0.$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}[V(r) - E] = \ell(\ell+1);$$

➡️ **The Radial Equation**

$$\frac{1}{Y}\left\{ \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} \right\} = -\ell(\ell+1).$$

➡️ **The Angular Equation**

The Angular Equation

$$\frac{1}{Y} \left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial Y}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 Y}{\partial\phi^2} \right\} = -\ell(\ell+1).$$
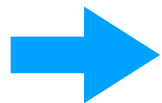
Multiplying by: $Y \sin^2\theta$,

$$\sin\theta \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial Y}{\partial\theta} \right) + \frac{\partial^2 Y}{\partial\phi^2} = -\ell(\ell+1) \sin^2\theta \, Y.$$

Separation of variables: $Y(\theta,\phi) = \Theta(\theta)\,\Phi(\phi)$.

Dividing by: $\Theta\Phi$

$$\left\{ \frac{1}{\Theta} \left[\sin\theta \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) \right] + \ell(\ell+1) \sin^2\theta \right\} + \frac{1}{\Phi} \frac{d^2\Phi}{d\phi^2} = 0.$$

$$\frac{1}{\Theta} \left[\sin\theta \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) \right] + \ell(\ell+1) \sin^2\theta = m^2$$
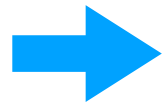
$$\frac{1}{\Phi} \frac{d^2\Phi}{d\phi^2} = -m^2.$$

Solution for Φ:

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi \;\Rightarrow\; \boxed{\Phi(\phi) = e^{im\phi}.}$$

where *m* can be positive or negative.

when φ advances by 2π, we return to the same point in space.

$$\Phi(\phi + 2\pi) = \Phi(\phi).$$

$$\exp[im(\phi + 2\pi)] = \exp(im\phi), \text{ or } \exp(2\pi im) = 1.$$

m must be an integer: $\quad m = 0, \pm 1, \pm 2, \ldots.$