General QM problem

Time-dependent Schrödinger equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi,$$

Separation of variables, assuming V=V(x).

$$\Psi(x,t) = \psi(x)\,\varphi(t),$$

We obtain 2 ODEs:

(wiggle factor)

1. $$\frac{d\varphi}{dt} = -\frac{iE}{\hbar}\varphi,$$

$$\varphi(t) = e^{-iEt/\hbar}.$$

Time-independent Schrödinger equation:

2. $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi.$$

To solve it we need V(x).

Separable Solutions:

- They are stationary states.

- Every expectation value is constant in time

- They are states of definite total energy, i.e., every measurement of the total energy is certain to return the value E.

$$\sigma_H^2 = \langle H^2 \rangle - \langle H \rangle^2 = E^2 - E^2 = 0.$$

- The general solution is a linear combination of separable solutions, i.e., there is a different wave function for each allowed energy:

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}.$$

General Solution:

- The strategy is first to solve the time-*in*dependent Schrödinger equation.

- This yields, in general, an infinite set of solutions, $\{\psi_n(x)\}$, each with its own associated energy, $\{E_n\}$.

- To fit $\psi(x, 0)$ you write down the general linear combination of these solutions:

$$\Psi(x, 0) = \sum_{n=1}^{\infty} c_n \, \psi_n(x);$$

- Construct global solution from the stationary states:

$$\boxed{\Psi(x, t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar} = \sum_{n=1}^{\infty} c_n \Psi_n(x, t).}$$

- Coefficients:

$\lvert c_n \rvert^2$	is the *probability* that a measurement of the energy would return the value E_n.

Bound states versus scattering states

Bound states are *normalisable*, and labeled by a *discrete index n*

Scattering states are *non-normalisable*, and labeled by a *continuous variable k*.

$$\begin{cases} E < V(-\infty) \text{ and } V(+\infty) \Rightarrow & \text{bound state,} \\ E > V(-\infty) \text{ or } V(+\infty) \Rightarrow & \text{scattering state.} \end{cases}$$

$$\begin{cases} E < 0 \Rightarrow & \text{bound state,} \\ E > 0 \Rightarrow & \text{scattering state.} \end{cases}$$

Time-independent Schrödinger equation

Time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi.$$

To solve it we need V(x).

Steps:

1. Define and sketch V(x).
2. Divide the problem into regions of interest.
3. Analyse the expected type of solutions (e.g. bound states or scattering states)
4. Divide the problem based on the energy of the particle.
5. Re-write Schrödinger equation for each region.
6. Define an appropriate and real wavenumber.
7. Solve the resulting ODE.
8. Analyse the asymptotic behaviour of the ODE solutions, remove diverging terms.
9. Analyse boundary conditions (usually two: ψ(x) and ψ'(x) have to be continuous).
10. Find energies and normalise the solution.
11. Append the wiggle factor and construct a general solution.