# Midterm Exam

NAME: ______________________________________________ SCORE: ______________

Subject: Quantum Mechanics I
Date: Monday 8 August 2022
Duration: 120 minutes
Credits: 20 points
Number of problems: 10 in total (8 in part I and 2 in part II)
Type of evaluation: Midterm Exam

**- This exam consists of two parts. Part I is closed-book and contains concept questions and short-answer problems. Part II is open-book and contains long-answer problems.**

**- Unless stated otherwise, write your answers in SI units, and consider all bolded quantities as vector quantities. Please highlight the answers.**

**PART I:**

**I.1. Choose the correct answer to each question or statement given below, and briefly justify your choice in the white space assigned to each of them.**

1. **(1 point) Compton scattering**
   For which scattering angle ($\theta$ in °) is the photon wavelength shift twice the Compton wavelength of the electron?

   A. $\theta = 0°$
   B. $\theta = 45°$
   C. $\theta = 90°$
   D. $\theta = 180°$ because $(1-\cos\hat{\theta}) = 2$ in $\Delta\lambda = \underbrace{\frac{h}{m_e c}}_{\text{Compton wavelength of electron}}(1-\cos\hat{\theta}) = 2\frac{h}{m_e c}$

2. **(1 point) Photoelectric effect**
   The work function, $\Phi$, of lithium is 2.5 eV. What is the maximum wavelength of light that can cause the photoelectric effect in lithium?

   A. 398 nm
   B. 498 nm
   C. 598 nm
   D. 698 nm

   We know $Ke_{max} = h\frac{c}{\lambda} - \overbrace{h\frac{c}{\lambda_{max}}}^{\phi \equiv \text{work function}}$

   $\Rightarrow \phi = h\frac{c}{\lambda_{max}} \Rightarrow \lambda_{max} = \frac{hc}{\phi} = \frac{1.99\times10^{-25}\,Jm}{4\times10^{-19}\,J}$

   $\Rightarrow \lambda_{max} = 4.98\times10^{-7}m = 498\,nm$

3. **(1 point) de Broglie wavelength**
   What is the de Broglie wavelength of a baseball of mass 0.125 kg and size 0.08 m moving at $28\,\text{m}\,\text{s}^{-1}$?

   A. $1.89 \times 10^{-34}$ m
   B. $2.32 \times 10^{-33}$ m
   C. $3.00 \times 10^{-35}$ m
   D. $1.89 \times 10^{34}$ m

   de Broglie wavelength:

   $\lambda_{dB} = \frac{h}{p} = \frac{h}{mv} = \frac{6.63\times10^{-34}\,J\cdot s}{3.5\,kg\,\frac{m}{s}}$

   $\Rightarrow \lambda_{dB} = 1.89\times10^{-34}\,m$

4. **(1 point) Relevance of quantum mechanics**
Based on the result above, is quantum mechanics relevant when studying the physics of a baseball? Why? Let $D = 0.08\,m$ the diameter of the ball (see problem 3)

A. Yes.
B. No, because: $\frac{\lambda_{dB}}{D} = 2.4 \times 10^{-33}$ which is $\ll 1$ (quantum effects are negligible)

5. **(1 point) Expectation values**
In quantum mechanics, the expectation value of the position of a particle represents:

A. the most probable value of its position.
B. the average value of the position measured in repeated experiments on the same particle.
C. the average value of the position measured on identical particles in the same state.
D. the only possible value of its position.
because it is not the mode, repeated experiments should agree, and $\hat{x}$ does not have determinate eigenvalues.

6. **(1 point) Quantum harmonic oscillator**
The ground energy value, $E_0$, of a quantum harmonic oscillator is:

A. $\frac{3}{2}\hbar\omega$
B. $\hbar\omega$
C. $\frac{1}{2}\hbar\omega$ because the energy levels for the harmonic oscillator are given by $E_n = (n + \frac{1}{2})\hbar\omega$, so $E_0 = \frac{1}{2}\hbar\omega$
D. 0

**I.2. Provide answers/solutions to the following items.**

7. **(2 points) Bound states and scattering states**
(a) Write down two differences between bound states and scattering states.

① Bound states are normalisable, scattering states are not.
② Bound are labeled by a discrete index "n", scattering states by a continuous variable K.

Bonus:
③ Bound states are physically realisable states on their own.
④ $E<0$ for bound states, while $E>0$ for scattering states.

(b) Provide two examples of potentials that allow only bound states, only scattering states, and both bound and scattering states.

Examples of:
- Only bound states: infinite square well potential and harmonic oscillator.
- Only scattering states: free particles and a potential hill with no dips.
- Both bound and scattering states: delta function well and finite square well potential

8. **(2 points) Normalisation**
A wave function for a particle in a box of length $L$ is given by:

$$\psi(x) = C\,x(L-x)$$

(a) Find the normalisation constant, $C$.
(b) Why do wave functions need to be normalised?

a) We know that:

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1 \Rightarrow \int_0^L |\psi(x)|^2 dx = 1$$

because the particle is trapped in a box of length $L$.

$$\int_0^L (Cx(L-x))(Cx(L-x))\,dx = C^2 \int_0^L (xL-x^2)^2 dx =$$

$$= C^2 \int_0^L (x^2L^2 - 2x^3L + x^4)\,dx = C^2\left(\frac{1}{3}x^3L^2 - \frac{1}{2}x^4L + \frac{1}{5}x^5\right)\Big|_0^L =$$

$$= C^2\left(\frac{1}{3}L^5 - \frac{1}{2}L^5 + \frac{1}{5}L^5\right) = C^2\frac{L^5}{30} = 1$$

$$\Rightarrow \boxed{C = \sqrt{\frac{30}{L^5}}} \qquad \Rightarrow \psi(x) = \sqrt{\frac{30}{L^5}}\,x(L-x)$$

b) They need to be normalised to represent real particles. Besides, the statistical interpretation of QM states that $|\psi|^2 dx$ is the probability of finding the particle between $x \wedge x+dx$, so the sum of all probabilities ($\int$) has to be 1.

9. **(2 points) The time-independent Schrödinger equation**
Consider the following 1D wave function:

$$\psi(x) = A\left(\frac{x}{x_0}\right)^n e^{-\frac{x}{x_0}}$$

where $A$, $n$, and $x_0$ are constants.

(a) Write down the time-independent Schrödinger equation.
(b) Find the potential $V(x)$ and energy $E$, for which this wave function is a solution to the Schrödinger equation. Assume that as $x \to +\infty$, $V(x) \to 0$.

a) The time-independent Schrödinger equation reads:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \qquad ①$$

b) We know that: $\psi(x) = A\left(\frac{x}{x_0}\right)^n e^{-\frac{x}{x_0}}$

Let us calculate its 2nd derivative so that we can plug it into the Sch. eq above:

$$\frac{d\psi}{dx} = A\left(\frac{n}{x_0}\right)\left(\frac{x}{x_0}\right)^{n-1} e^{-\frac{x}{x_0}} - A\left(\frac{1}{x_0}\right)\left(\frac{x}{x_0}\right)^n e^{-\frac{x}{x_0}}$$

$$\frac{d^2\psi}{dx^2} = A\left(\frac{n(n-1)}{x_0^2}\right)\left(\frac{x}{x_0}\right)^{n-2} e^{-\frac{x}{x_0}} - A\left(\frac{n}{x_0^2}\right)\left(\frac{x}{x_0}\right)^{n-1} e^{-\frac{x}{x_0}} -$$

$$- A\left(\frac{n}{x_0^2}\right)\left(\frac{x}{x_0}\right)^{n-1} e^{-\frac{x}{x_0}} + A\left(\frac{1}{x_0^2}\right)\left(\frac{x}{x_0}\right)^n e^{-\frac{x}{x_0}}$$

$$\Rightarrow \frac{d^2\psi}{dx^2} = A\left(\frac{n(n-1)}{x_0^2}\right)\left(\frac{x}{x_0}\right)^{n-2} e^{-\frac{x}{x_0}} - 2A\left(\frac{n}{x_0^2}\right)\left(\frac{x}{x_0}\right)^{n-1} e^{-\frac{x}{x_0}} +$$

$$+ A\left(\frac{n}{x_0^2}\right)\left(\frac{x}{x_0}\right)^{n-1} e^{-\frac{x}{x_0}} =$$

$$= \left[\frac{n(n-1)}{x^2} - 2\frac{n}{x_0 x} + \frac{1}{x_0^2}\right]\underbrace{A\left(\frac{x}{x_0}\right)^n e^{-\frac{x}{x_0}}}_{\psi(x)} \qquad ②$$

② in ①: $$-\frac{\hbar^2}{2m}\left[\frac{n(n-1)}{x^2} - 2\frac{n}{x_0 x} + \frac{1}{x_0^2}\right]\psi = (E - V)\psi \qquad ③$$

When $x \to +\infty$; $V(x) \to 0$, so from ③ we get:

$$\Rightarrow \boxed{E = -\frac{\hbar^2}{2m x_0^2}} \qquad ④$$

Finally, ④ in ③:

$$V = \cancel{-\frac{\hbar^2}{2m x_0^2}} + \frac{\hbar^2}{2m}\left[\frac{n(n-1)}{x^2} - 2\frac{n}{x_0 x}\right] + \cancel{\frac{\hbar^2}{2m x_0^2}}$$

$$\Rightarrow \boxed{V = V(x) = \frac{\hbar^2}{2m}\left[\frac{n(n-1)}{x^2} - 2\frac{n}{x_0 x}\right]}$$

PART II:
Solve the following problems and highlight the answers.

10. **(4 points) Particle in a infinite square well potential**
Consider a 1D problem for a particle of mass $m$ that is free to move in the interval $x \in [0, a]$. The potential $V(x)$ is zero in this interval and infinite elsewhere. For that system consider a solution of the Schrödinger equation of the form:

$$\Psi_n(x,t) = \begin{cases} 0, & x < 0 \\ B \sin\left(\frac{n\pi}{a}x\right) e^{-i\phi_n(t)}, & 0 \le x \le a \\ 0, & x > a \end{cases}$$

Here $n \geq 1$.

(a) Find the expression for the (real) phase $\phi_n(t)$ so that the above wave function solves the Schrödinger equation.
(b) Find the normalisation constant $B$.
(c) Use $\Psi_n(x,0)$ to calculate $\langle x \rangle$, $\langle x^2 \rangle$, and $\sigma_x$.
(d) Use $\Psi_n(x,0)$ to calculate $\langle p \rangle$, $\langle p^2 \rangle$, and $\sigma_p$.
(e) Does the uncertainty principle hold?

11. **(4 points) Square potential barrier: transmission and reflection coefficients**
Let us consider a step potential, $V(x)$, given by the following piecewise function:

$$V(x) = \begin{cases} 0, & x < 0 \\ V_0, & x > 0, \end{cases}$$

(a) Calculate the reflection coefficient, for the case $E < V_0$.
(b) Calculate the reflection coefficient for the case $E > V_0$.
(c) For a potential (such as this one) that does not go back to zero to the right of the barrier, the transmission coefficient is not simply $\frac{|F|^2}{|A|^2}$ (with $A$ the incident amplitude and $F$ the transmitted amplitude), because the transmitted wave travels at a different speed. Show that for $E > V_0$:

$$T = \sqrt{\frac{E - V_0}{E}} \frac{|F|^2}{|A|^2}$$

What is $T$, for $E < V_0$?
(d) For $E > V_0$, calculate the transmission coefficient for the step potential, and check that $T + R = 1$.

10.

a). Since $\Psi_n(x,t)$ should be a solution to Schrödinger's equation, we can substitute it into it:

$$i\hbar \frac{\partial \Psi_n}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi_n}{\partial x^2} + V\Psi_n$$

Here $V = V(x) = 0$ in the interval $x \in [0, a]$, so:

$$i\hbar \frac{\partial}{\partial t}\left(B\sin\left(\frac{n\pi}{a}x\right)e^{-i\phi_n}\right) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(B\sin\left(\frac{n\pi}{a}x\right)e^{-i\phi_n}\right)$$

$$\Rightarrow i\hbar\left(-i\dot{\phi}_n B\sin\left(\frac{n\pi}{a}x\right)e^{-i\phi_n}\right) = +\frac{\hbar^2}{2m}\left(\frac{n^2\pi^2}{a^2}Be^{-i\phi_n}\sin\left(\frac{n\pi}{a}x\right)\right)$$

Simplifying:

$$\dot{\phi}_n = \frac{\hbar n^2\pi^2}{2ma^2} \quad \Rightarrow \quad \boxed{\phi_n = \phi_n(t) = \frac{\hbar n^2\pi^2}{2ma^2}t}$$

b) Now we know that:

$$\Psi_n = B\sin\left(\frac{n\pi}{a}x\right)e^{-i\frac{\hbar n^2\pi^2}{2ma^2}t}$$

We normalise it:

$$\int_0^a \Psi_n^* \Psi_n \, dx = 1 \Rightarrow \int_0^a B^2\sin^2\left(\frac{n\pi}{a}x\right)dx = 1$$

$$\Rightarrow B^2\int_0^a \frac{1}{2}\left(1-\cos\left(\frac{2n\pi}{a}x\right)\right)dx = 1$$

$$\Rightarrow B^2\left[x - \sin\left(\frac{2n\pi}{a}x\right)\cdot\frac{a}{2n\pi}\right]_0^a = 2 \Rightarrow B^2 a = 2 \Rightarrow \boxed{B = \sqrt{\frac{2}{a}}}$$

c) At $t=0$, we have:

$$\Psi_{n,0} = B\sin\left(\frac{n\pi}{a}x\right)$$

The expectation values are:

$$\langle x\rangle = \int_0^a x\,|\Psi_{n,0}|^2\,dx = B^2\int_0^a x\sin^2\left(\frac{n\pi}{a}x\right)dx \Rightarrow \text{integration by parts:}$$

$$\langle x\rangle = \frac{2}{a}\left[\frac{1}{4}x^2 - \underbrace{\frac{a}{4n\pi}x\sin\left(\frac{2n\pi}{a}x\right)}_{\text{is zero}} + \underbrace{\frac{a}{2n\pi}\cos\left(\frac{2n\pi}{a}x\right)}_{\text{cancels out}}\right]_0^a$$

$$\boxed{\langle x\rangle = \frac{a}{2}}$$

$$\langle x^2\rangle = \int_0^a x^2 |\Psi_{n,0}|^2 dx = B^2 \int_0^a x^2 \sin^2\left(\frac{n\pi}{a}x\right) dx \rightarrow \text{integration by parts}$$

$$\langle x^2\rangle = \frac{2}{a}\left[\frac{x^3}{6} - \frac{a}{4n\pi}x^2 \sin\left(\frac{2n\pi}{a}x\right) - \frac{a^2}{4n^2\pi^2}x\cos\left(\frac{2n\pi}{a}x\right) + \right.$$

$$\left. + \frac{a^3}{8n^3\pi^3}\sin\left(\frac{2n\pi}{a}x\right)\right]_0^a$$

Simplifying: (all sin terms become zero)

$$\langle x^2\rangle = \frac{2}{a}\left[\frac{a^3}{6} - \frac{a^3}{4n^2\pi^2}\right] \Rightarrow \boxed{\langle x^2\rangle = a^2\left[\frac{1}{3} - \frac{1}{2n^2\pi^2}\right]}$$

Therefore:

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{a^2}{3} - \frac{a^2}{2n^2\pi^2} - \frac{a^2}{4}$$

$$\Rightarrow \boxed{\sigma_x = a\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}}$$

d) For momentum:

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} \Rightarrow \boxed{\langle p\rangle = 0}$$

$$\langle p^2\rangle = \int_0^a \Psi_{n,0}^* \left(-i\hbar\frac{\partial}{\partial x}\right)^2 \Psi_{n,0}\, dx = +\hbar^2 \int_0^a B^2 \sin^2\left(\frac{n\pi}{a}x\right)\left(\frac{n^2\pi^2}{a^2}\right)dx$$

$$\langle p^2\rangle = \frac{2\hbar^2 n^2\pi^2}{2a^3}\left[x - \underbrace{\sin\left(\frac{2n\pi}{a}x\right)\frac{a}{2n\pi}}_{\text{becomes zero}}\right]_0^a \Rightarrow \boxed{\langle p^2\rangle = \frac{\hbar^2 n^2\pi^2}{a^2}}$$

Therefore:

$$\sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2$$

$$\Rightarrow \boxed{\sigma_p = \frac{\hbar n\pi}{a}}$$

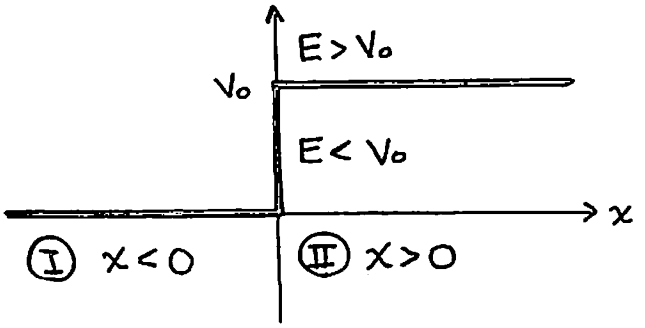
e) The uncertainty principle reads: $\sigma_x\sigma_p \geq \frac{\hbar}{2}$

In this case:

$$\sigma_x\sigma_p = \sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}\,\hbar n\pi > \frac{\hbar}{2} \quad \text{(always, for all "}n\text{")}$$

$\therefore$ the uncertainty principle holds.

11.




Step potential

a) Let's find solutions to the time-independent Sch. eq.:

Ⓘ $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \Rightarrow \frac{d^2\psi}{dx^2} = -\underbrace{\frac{2mE}{\hbar^2}}_{k}\psi \Rightarrow \frac{d^2\psi}{dx^2} = -k^2\psi$

$\Rightarrow \boxed{\psi_I = Ae^{ikx} + Be^{-ikx}}$ is the solution for $x<0$, in both cases.

Ⓘⅰ $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi \Rightarrow \frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E-V_0)\psi$

When $E<V_0$, we take: $l_1 = \sqrt{\frac{2m}{\hbar^2}(V_0-E)}$

blows up

$\Rightarrow \frac{d^2\psi}{dx^2} = l_1^2\psi \Rightarrow \psi_{II}^{(1)} = \cancel{Ce^{+lx}} + De^{-lx}$

$\Rightarrow \boxed{\psi_{II}^{(1)} = De^{-l_1x}}$ → the wavefunction is not zero in II!

We apply the boundary conditions at $x=0$:

① $\psi$ should be continuous $\Rightarrow$ $A+B=D$

② $\frac{d\psi}{dx}$ should be continuous $\Rightarrow$ $ikA - ikB = -l_1D$

② + $l$ × ①: $(l_1+ik)A + (l_1-ik)B = 0$

$\Rightarrow \left|\frac{B}{A}\right| = \frac{|l_1+ik|}{|l_1-ik|}$

The reflection coefficient is:

$R^{(1)} = \left|\frac{B}{A}\right|^2 = \frac{l_1^2+k^2}{l_1^2+k^2} \Rightarrow \boxed{R^{(1)} = 1}$ → full reflection.

b) When $E>V_0$, we take: $l_2 = \sqrt{\frac{2m}{\hbar^2}(E-V_0)}$

wave moving →

$\Rightarrow \frac{d^2\psi}{dx^2} = -l_2^2\psi \Rightarrow \psi_{II}^{(2)} = Fe^{+il_2x} + Ge^{-il_2x}$

wave moving ←

$\Rightarrow \boxed{\psi_{II}^{(2)} = Fe^{il_2x}}$

We keep the wave moving right.

We apply the boundary conditions at $x=0$:

① $\psi$ should be continuous $\Rightarrow$ $A+B=F$

② $\frac{d\psi}{dx}$ should be continuous $\Rightarrow$ $ikA - ikB = il_2 F$

①$\times(-il_2)+$②: $(-il_2+ik)A-(il_2+ik)B=0$

$$\Rightarrow \left|\frac{B}{A}\right| = \frac{|k-l_2|}{|k+l_2|}$$

The reflection coefficient is:

$$R^{(2)} = \left|\frac{B}{A}\right|^2 = \frac{(k-l_2)^2}{(k+l_2)^2} = \frac{(k-l_2)^4}{(k^2-l_2^2)^2}$$

In terms of the initial variables:

$$\boxed{R^{(2)} = \frac{(\sqrt{E}-\sqrt{E-V_0})^4}{V_0^2}}$$

c) To calculate the transmitted coefficient, we first need to calculate the probability currents in I ∧ II.

Recall:
$$J(x) = \frac{\hbar}{2mi}\left[\psi^* \frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right]$$

In I; we use $\psi_I$:

$$J_I(x) = \frac{\hbar}{2mi}\left[(Ae^{-ikx}+Be^{ikx})(ikAe^{ikx}-ikBe^{ikx}) - (Ae^{ikx}+Be^{-ikx})(-ikAe^{-ikx}+ikBe^{ikx})\right]$$

$$\Rightarrow J_I(x) = \frac{\hbar k}{2m}\left[2|A|^2-2|B|^2\right] \Rightarrow J_I(x) = \frac{\hbar k}{m}(\underbrace{|A|^2}_{\text{incident}} - \underbrace{|B|^2}_{\text{reflected}})$$

In II; we use $\psi_{II}^{(2)}$ for $E>V_0$:

$$J_{II}(x) = \frac{\hbar}{2mi}\left[Fe^{-il_2x}(il_2Fe^{il_2x}) - Fe^{il_2x}(-il_2Fe^{-il_2x})\right]$$

$$\Rightarrow J_{II}(x) = \frac{\hbar l_2}{2m}\left[2|F|^2\right] \Rightarrow J_{II}(x) = \frac{\hbar l_2}{m}\underbrace{|F|^2}_{\text{transmitted}}$$

The transmitted coefficient relates the incident and transmitted amplitudes:

$$T = \frac{\frac{\hbar l_2}{m}|F|^2}{\frac{\hbar k}{m}|A|^2} = \frac{l_2}{k}\frac{|F|^2}{|A|^2} \underset{\text{In terms of the initial variables}}{\Rightarrow} \boxed{T = \frac{\sqrt{E-V_0}}{\sqrt{E}}\frac{|F|^2}{|A|^2}}$$

Now, for $E < V_0$; we take $\psi_{II}^{(1)}$:

$$J_{II}(x) = \frac{\hbar}{2mi}\left[De^{-l_1 x}\left(-Dl_1 e^{-l_1 x}\right) - De^{-l_1 x}\left(-De^{-l_1 x}\right)\right]$$

$$\Rightarrow J_{II}(x) = 0$$

Thus, the transmission coefficient reads:

$$T = \frac{0}{\frac{\hbar k}{m}|A|^2} \Rightarrow \boxed{T = 0}$$

d) For $E > V_0$, we have

① $A + B = F$

② $ikA - ikB = il_2 F$

①$\times ik +$②: $\quad 2ikA = i(k + l_2)F$

$$\Rightarrow \frac{|F|}{|A|} = \frac{2k}{k + l_2} \Rightarrow \frac{|F|^2}{|A|^2} = \frac{4k^2}{(k+l_2)^2} = \frac{4k^2(k-l_2)^2}{(k^2 - l_2^2)^2}$$

The transmitted coefficient becomes:

$$T = \frac{\sqrt{E - V_0}}{\sqrt{E}} \frac{4E\left(\sqrt{E} - \sqrt{E - V_0}\right)^2}{V_0^2}$$

$$\Rightarrow \boxed{T = \frac{4\sqrt{E}\sqrt{E - V_0}\left(\sqrt{E} - \sqrt{E - V_0}\right)^2}{V_0^2}}$$

Let's check:

$$T + R^{(2)} = \frac{4kl_2 + k^2 - 2kl_2 + l_2^2}{(k + l_2)^2} = \frac{(k+l_2)^2}{(k+l_2)^2} \Rightarrow \boxed{T + R^{(2)} = 1}$$