Quantum Mechanics I

Lecture Notes by Wladimir E. Banda Barragán

School of Physical Sciences and Nanotechnology
Yachay Tech University

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UC3
Mathematical formalism of Quantum Mechanics

UC3 contents:

- Linear algebra, Hermitian operators, and Hilbert space.
- Eigenfunctions, eigenvectors, and eigenvalues for discrete and continuous spectra.
- Dirac notation and the generalised statistical interpretation.
- Operators of position and momentum and the uncertainty principle.

Mathematical Formalism of QM:

QM theory is based on linear algebra:

i) Wavefunctions : states → abstract vectors (functions in ∞-dim. spaces)

ii) Operators : observables → linear transformations

Vectors in QM: $|\alpha\rangle$

They are represented by the N-tuple of its components $\{a_n\}$ with respect to a specified orthonormal basis:

$$|\alpha\rangle = a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_N \end{pmatrix}$$

Inner product: $\langle\alpha|\beta\rangle$

It's a complex number:

$$\langle\alpha|\beta\rangle = a_1^* b_1 + a_2^* b_2 + \cdots + a_N^* b_N$$

Linear transformations: T

They are represented by matrices (wrt the specified basis)

$$|\beta\rangle = \hat{T}|\alpha\rangle \to b = Ta = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1N} \\ t_{21} & t_{22} & & \\ \vdots & \vdots & \ddots & \\ t_{N1} & & & t_{NN} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$$

Time-independent Schrödinger equation

<u>What is new / different in QM?</u>

- Vectors live in ∞-dim. spaces
- manipulations that work in N-dim, may not work in ∞-dim.

<u>Vector space:</u>

It is the collection of all functions of x.

Wave functions must be normalized to represent possible physical states.

$$\int |\psi|^2 dx = 1$$

$\Rightarrow$ they must be square-integrable functions, on an interval.

<u>Hilbert space:</u> $L^2(a, b)$

It is a vector space that contains the set of all square-integrable functions

Wave functions in QM $\in$ Hilbert space.

$$f(x) \rightarrow \int_a^b |f(x)|^2 dx < \infty$$

Time-independent Schrödinger equation

Inner product of 2 functions: $f(x)$ ^ $g(x)$

$$\langle f|g\rangle \equiv \int_a^b f(x)^* g(x)\, dx$$

if $f, g \in H \Rightarrow \langle f|g\rangle$ is guaranteed to exist.

converges to a finite number.

Schwarz inequality:

$$\left|\int_a^b f(x)^* g(x)\, dx\right| \leq \sqrt{\int_a^b |f(x)|^2 dx \int_a^b |g(x)|^2 dx}$$

Properties:

1) $\langle g|f\rangle = \langle f|g\rangle^*$

2) $\langle f|f\rangle = \int_a^b |f(x)|^2 dx. \Rightarrow \langle f|f\rangle \in \mathbb{R}$, >0 (unless $f(x)=0$)

3) A function is normalised if $\langle f|f\rangle = 1$

Time-independent Schrödinger equation

4) Two functions are orthogonal if $\langle f|g\rangle = 0$

5) $\{f_n\}$ (a set of functions) is orthonormal if normalised and mutually orthogonal.

$$\langle f_m|f_n\rangle = \delta_{mn}$$

6) A set of functions is complete if any other function, $\in H$, can be expressed as a linear combination:

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

If $\{f_n\}$ are orthonormal, $c_n = \langle f_n|f\rangle$ (Fourier's trick)