Homework 5 - Quantum Mechanics I

NAME: ____ **SCORE:** _____________

Deadline: Thursday 14th December 2023 by 11:00am
Credits: 20 points **Number of problems:** 5
Type of evaluation: Formative Evaluation

- This homework includes problems on units 4 and 5 of the QM course.
- This assignment should be submitted individually by the deadline.
- Please justify all calculations and highlight your answers.

1. **(4 points) Angular momentum**

   (a) Let L_x, L_y, and L_z be the components of the orbital angular momentum operator $\vec{\boldsymbol{L}}$. Calculate the value of the commutator: $[L_x L_y, L_z]$.

   (b) Suppose we have a spin-$\frac{1}{2}$ particle, whose quantum state can be represented by the eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ of the S_z operator. Using the Pauli matrix σ_x, calculate a normalised eigenstate of the S_x operator with an eigenvalue $-\frac{\hbar}{2}$.

   (c) Calculate the normalised spin eigenfunctions of a system with two spin-$\frac{1}{2}$ particles. These are called singlet and triplet states, why?

   (d) Calculate the orbital angular momentum eigenfunction $Y_\ell^m(\theta, \phi)$ in a quantum state for which the operator $\boldsymbol{L}^2$ has an eigenvalue $6\hbar^2$ and the operator $\boldsymbol{L}_z$ has an eigenvalue $-\hbar$.

2. **(4 points) Radial wave function of atoms**

   The wave function of an electron in a hydrogenic atom with an atomic number $Z = 70$ and mass number $A = 173.05$ is given by:

   $$\psi(r) = Be^{-r/a}$$

   where $a = \frac{a_0}{Z}$ and $a_0 = 0.53$ Å is the Bohr radius. This (Ytterbium, Yb) atom contains only one electron with charge e. The charge and radius of its nucleus are eZ and $R = 1.2(A^{1/3})$ fm, respectively.

   (a) Normalise the wave function.

   (b) Calculate the probability that the electron is found in the nucleus.

   (c) What is the probability that the electron is in the region $y < 0$?

   (d) What is the probability that the electron is in the region $x, y, z > 0$?

3. **(4 points) Spinor**

   Consider an electron in the spin state:

   $\chi = C \begin{bmatrix} 1-3i \\ 2 \end{bmatrix}$

   (a) Determine the normalisation constant C.

   (b) Find the expectation values of S_x, S_y, and S_z.

   (c) Find the uncertainties σ_{S_x}, σ_{S_y}, and σ_{S_z}.

   (d) Are the results consistent with all three uncertainty principles?

4. **(4 points) Electron in a Magnetic Field**
   Consider an electron (at rest) embedded in an oscillating magnetic field:

   $$\vec{B} = B_0 \cos(\omega t)\, \vec{k},$$

   where B_0 and ω are constants.

   (a) Construct the Hamiltonian matrix for this system. Is this Hamiltonian time-dependent or time-independent?

   (b) The electron starts out (at $t = 0$) in the spin-up state with respect to the x axis (i.e., $\chi(0) = \chi_+^{(x)}$). Determine $\chi(t)$ at any subsequent time by solving the Schrödinger equation.

   (c) Find the probability of getting $-\frac{\hbar}{2}$, if you measure S_x.

   (d) What is the minimum field strength (B_0) required to force a complete flip in S_x?

5. **(4 points) Two-particle systems**
   Consider a system of two non-interacting quantum particles (both of mass m) inside a 1D infinite square well potential of width, a.

   (a) Based on the analysis carried out in unit 2 of the course for a particle trapped in this potential, write down the one-particle wave function and the respective energy for each particle in the system.

   (b) Write down the composite wave function of the two-particle system assuming the particles are: distinguishable (system D), identical bosons (system B), and identical fermions (system F).

   (c) Based on the previous results, find the ground state of each two-particle system (D, B, and F), jointly with the respective energy. Briefly explain your findings.

   (d) Find the first three excited states for each two-particle system (D, B, and F), jointly with their respective energies.