Homework 4 - Quantum Mechanics I

NAME/S: ____ **SCORE:** ________

Deadline: Thursday 2nd March 2023 by 10:00am (submission only on paper)
Credits: 20 points $\rightarrow$ 20 credits **Number of problems:** 4
Type of evaluation: Formative Evaluation

- This homework consists of problems related to the formalism of quantum mechanics.

- You may submit this assignment either individually or in pairs. If you work in pairs, only 1 copy is needed. Submitted assignments should have maximum two authors.

- Please solve the following problems and highlight the answers.

1. **(7 points) Quantum mechanical operators**

   (a) Show that projection operators are idempotent, i.e., that $\hat{P}^2 = \hat{P}$.

   (b) Calculate the eigenvalues of the Hermitian matrix: $\begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}$

   (c) The raising and lowering operators for the quantum harmonic oscillator satisfy the following: $\hat{a}_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$, $\hat{a}_-|n\rangle = \sqrt{n}\,|n-1\rangle$ for energy eigenstates $|n\rangle$ with energy E_n. Calculate the first-order shift in the $n = 2$ energy level due to the perturbation: $\Delta H = V(\hat{a}_- + \hat{a}_+)^2$, where V is constant.

   (d) Show that the position and momentum operators are hermitian. Using these results, construct the hermitian conjugate of the raising ladder operator that we used to study the quantum harmonic oscillator, $\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega x)$.

2. **(6 points) Mathematical formalism of quantum mechanics**

   (a) Consider the orthonormal states $|1\rangle$, $|2\rangle$, and $|3\rangle$. For which value of x are the following states, $|\Psi_1\rangle = 5\,|1\rangle - 3\,|2\rangle + 2\,|3\rangle$ and $|\Psi_2\rangle = |1\rangle - 5\,|2\rangle + x\,|3\rangle$, orthogonal?

   (b) Assuming that $\gamma \in \mathbb{R}$ (but not necessarily positive), for what range of γ is the function $f(x) = x^{\gamma-1}$ in Hilbert space, on the interval $(0,1)$? What about $x\,f(x)$ and $\frac{d}{dx}f(x)$?

   (c) Let $|n\rangle$ be the normalised n-th energy eigenstate of the 1D harmonic oscillator. We know that $\hat{H}\,|n\rangle = \hbar\omega(n+\frac{1}{2})\,|n\rangle$. If $|\psi\rangle$ is a normalised ensemble state that can be expressed as a linear combination of the eigenstates as follows: $|\psi\rangle = \frac{1}{\sqrt{14}}\,|1\rangle - \frac{2}{\sqrt{14}}\,|2\rangle + \frac{3}{\sqrt{14}}\,|3\rangle$, what is the expectation value of the energy operator in this ensemble state?

3. **(3 points) Hamiltonian, eigenvalues and eigenvectors**

Consider a quantum system in a state, $|\Psi\rangle$:

$$\Psi = \frac{1}{\sqrt{3}} \begin{bmatrix} i \\ -i \\ i \end{bmatrix}$$

The Hamiltonian is represented by the matrix shown below:

$$\hat{H} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) Determine the eigenvalues and eigenvectors of $\hat{H}$. What do the eigenvalues represent?
(b) Which eigenvalue of $\hat{H}$ is most likely to emerge from a measurement?
(c) Find $\langle H \rangle$, $\langle H^2 \rangle$, and σ_H.

4. **(4 points) Dirac notation: brakets and dual basis**

Consider a 3D vector space spanned by an orthonormal basis $|1\rangle$, $|2\rangle$, $|3\rangle$. In this basis, let the $|\Psi_0\rangle$ and $|\Psi_1\rangle$ kets be:

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}} |1\rangle + \frac{i}{2} |2\rangle + \frac{1}{2} |3\rangle$$

$$|\Psi_1\rangle = \frac{1}{\sqrt{3}} |1\rangle + \frac{i}{\sqrt{3}} |3\rangle$$

(a) Are these kets normalised? If not, normalise them.
(b) Write $\langle \Psi_0|$ and $\langle \Psi_1|$ in terms of the dual basis $\langle 1|$, $\langle 2|$, $\langle 3|$.
(c) Find $\langle \Psi_0|\Psi_1\rangle$ and $\langle \Psi_1|\Psi_0\rangle$, and confirm that $\langle \Psi_1|\Psi_0\rangle = \langle \Psi_0|\Psi_1\rangle^*$.
(d) Find all the matrix elements of the operators $\hat{M}_{01} = |\Psi_0\rangle \langle \Psi_1|$, $\hat{M}_{00} = |\Psi_0\rangle \langle \Psi_0|$, and $\hat{M}_{11} = |\Psi_1\rangle \langle \Psi_1|$ in this basis, and construct their respective matrices, are they hermitian?