Final Exam - Quantum Mechanics I

NAME: ____ **SCORE:** ______________

Date: Tuesday 14 March 2023
Duration: 120 minutes Credits: 20 points
Number of problems: 14 in total (12 in part I, and 2 in part II)

- This exam consists of two parts. Part I is closed-book and contains concept questions and short-answer problems. Part II is open-book and contains long-answer problems.

- Constants:
Speed of light: $c = 3 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Planck constant: $h = 6.63 \times 10^{-34}\,\mathrm{J\,s}$
Rydberg constant: $\mathcal{R} = 1.097 \times 10^{7}\,\mathrm{m^{-1}}$

PART I:
I.1. Choose the correct answer to each question or statement given below, and briefly justify your choice in the white space assigned to each of them.

1. **(1 point) Planck's law**
   What is the frequency of UV light that has an energy of 2.39×10^{-18} J?

   A. 2.32×10^{9} Hz
   B. 3.60×10^{15} Hz
   C. 1.58×10^{-51} Hz
   D. $3 \times 10^{8}\,\mathrm{m\,s^{-1}}$

2. **(1 point) Photoelectric effect**
   The energy of photoelectrons emitted from a metal surface can be increased by:

   A. using light of higher frequency.
   B. using light of longer wavelength.
   C. using light of higher intensity.
   D. using monochromatic, polarised light.

3. **(1 point) Wave functions and operators**
   If Ψ is a solution of the Schrödinger equation and $\hat{\mathrm{Q}}$ is the operator corresponding to a physical observable x, the quantity $\Psi^{*}\,\hat{\mathrm{Q}}\,\Psi$ may be integrated in order to obtain the:

   A. normalisation constant for Ψ.
   B. time derivative of x.
   C. expectation value of x.
   D. spatial overlap of $\hat{\mathrm{Q}}$ with Ψ.

4. **(1 point) Normalisation**
The eigenfunctions of a rigid dumbbell rotating about its centre have a dependence on the azimuthal angle, ϕ, of the form $\psi(\phi) = Ae^{im\phi}$, where m is a quantum number and A is a constant. Which of the following values of A will properly normalise the eigenfunction?

A. 1
B. 2π
C. $\sqrt{2\pi}$
D. $\frac{1}{\sqrt{2\pi}}$

5. **(1 point) Quantum harmonic oscillator**
Characteristics of the quantum harmonic oscillator include which of the following?
I. A spectrum of evenly spaced energy states.
II. A potential energy function that is linear in the position coordinate.
III. A ground state that is characterised by zero kinetic energy.
IV. A nonzero probability of finding the oscillator outside the classical turning points.

A. I only
B. IV only
C. I and IV only
D. II and III only
E. I, II, III, and IV

6. **(1 point) Spectrum of hydrogen**
Every series of the hydrogen spectrum has an upper and a lower limit in wavelength. The spectral series which has an upper limit of wavelength equal to 1875.2 nm is:

A. Lyman series.
B. Paschen series.
C. Balmer series.

7. **(1 point) Pauli matrices and commutator**

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Consider the Pauli spin matrices σ_x, σ_y, and σ_z and the identity matrix I given above. The commutator $[\sigma_x, \sigma_y]$ is equal to which of the following?

A. $2\,i\,\sigma_x$
B. $2\,i\,\sigma_y$
C. $2\,i\,\sigma_z$
D. 0

8. **(1 point) Orbital angular momentum**
Consider a particle with orbital angular momentum $L = \sqrt{6}\,\hbar$. Which of the following gives the possible values of a measurement of L_x, the x-component of L?

A. $-3\hbar$, $-2\hbar$, $-\hbar$, 0, $\hbar$, $2\hbar$, $3\hbar$
B. $-2\hbar$, $-\hbar$, 0, $\hbar$, $2\hbar$
C. $-\hbar$, 0, $\hbar$
D. $-\frac{\hbar}{2}$, $\frac{\hbar}{2}$

9. **(1 point) Bosons and fermions**
Which of the following statements about bosons and/or fermions is true?

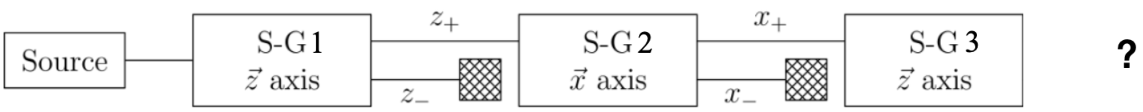
A. Bosons have symmetric wave functions and obey the Pauli exclusion principle.
B. Bosons have antisymmetric wave functions and do not obey the Pauli exclusion principle.
C. Fermions have symmetric wave functions and obey the Pauli exclusion principle.
D. Fermions have antisymmetric wave functions and obey the Pauli exclusion principle.
E. Bosons and fermions obey the Pauli exclusion principle.

I.2. Provide answers/solutions to the following items.

10. **(1.5 point) Angular momentum, and identical particles**
(i) Explain the difference between orbital angular momentum and spin angular momentum.

(ii) Explain the connection between the symmetrisation requirement for identical particles and the Pauli exclusion principle. Which particles does it apply to?

(iii) What is the output of the third Stern-Gerlach (S-G3) apparatus in the scheme below? S-G1 and S-G3 measure the deflection on the z axis of the spin state of a neutron beam. S-G2 measures the deflection on the x axis. The x-z-plane is orthogonal to the neutron beam and the cross-hatched squares denote the blocking of a given output.



11. **(1.5 point) Commutators, spins, and quantum numbers**
Write down the correct answers for the following operations (c and d require proof):

(a) If $\hat{x}$ and $\hat{p}$ are the position and momentum operators, $[\hat{x}, \hat{p}] =$

(b) If $\hat{L}_x$ and $\hat{L}_z$ are the x- and z-components of $\hat{\mathbf{L}}$, $[\hat{L}_x, \hat{L}_z] =$

(c) The net spin/s if we combine two quarks (spin-$\frac{1}{2}$ particles) is/are:

(d) All the allowed combinations of quantum numbers (n, l, m) for an electron in the $n = 3$ shell of the hydrogen atom:

12. **(2 points) Spinor**
An electron is in the spin state: $\chi = A \begin{pmatrix} 3i \\ 4 \end{pmatrix}$, where A is a normalisation constant.

(a) Determine the normalisation constant A.
(b) Find the expectation values of S_x, S_y, and S_z.
(c) Find the uncertainties (standard deviations) for S_x, S_y, and S_z.

PART II:
Solve the following problems and highlight the answers.

13. **(3 points) Particle in a 3D infinite potential well**
Suppose we have a particle in a 3D spherical and infinite potential well:

$$V(r) = \begin{cases} 0 & 0 \leq r \leq a \\ \infty & r > a \end{cases}$$

where (r, θ, ϕ) are the radial and angular (spherical) coordinates.
(a) Write down the time-dependent Schrödinger equation for the particle in spherical coordinates. Use variable separation to split the spatial and temporal terms.
(b) Apply variable separation to the time-independent Schrödinger equation and write the differential equations of the radial and angular parts. Solve the angular equation.
(c) Compute the energy levels and the stationary wave function, $\psi(r, \theta, \phi)$ for $\ell = 0$.

14. **(3 points) Radial wave function of atoms**
The wave function of an electron in a hydrogenic atom with an atomic number $Z = 25$ and mass number $A = 54.94$ is given by:

$$\psi(r) = Be^{-r/a}$$

where $a = \frac{a_0}{Z}$ and $a_0 = 0.53$ Å is the Bohr radius. This (manganese, Mn) atom contains only one electron with charge e. The charge and radius of its nucleus are eZ and $R = 1.2(A^{1/3})$ fm, respectively.

(a) Normalise the wave function.
(b) Calculate the probability that the electron is found in the nucleus.
(c) What is the probability that the electron is in the region $y < 0$?