Homework 3 - Quantum Mechanics I

NAME: ____ **SCORE:** ____________

Deadline: Thursday 16 May 2024 by 19:00 (via email to: wbanda@yachaytech.edu.ec)
Credits: 20 points **Number of problems:** 4 **Type of evaluation:** Formative Evaluation

- This homework includes problems on units 2 and 3 of the QM course programme.
- This assignment should be submitted individually by the deadline.
- If you work with classmates, please state their names and write your own reports.
- Please submit a professional homework report and clearly highlight all answers.
- All calculation steps should be justified to get full credits.
- Any code used should be attached to get full credits.

1. **(5 points) Finite square well potential**
   In class we studied the finite square well potential and found that this potential admits both scattering states (when $E > 0$) and bound states (when $E < 0$). For the latter, we derived the even solutions and numerically solved a transcendental equation for the allowed energies.

   (a) Following the same approach we followed in class, find the odd bound state wave functions, $\psi(x)$, for the finite square well.

   (b) Derive the transcendental equation for the allowed energies of these odd bound states.

   (c) Solve it graphically and numerically (using your favourite programming tool).

   (d) Study and discuss the two limiting cases and how the energy levels compare to those found for the even bound state wave functions studied in class. Is there always an odd bound state?

   (e) Normalise the even and odd bound state wave functions.

2. **(5 points) Vectors and operators in QM formalism**
   (a) Compute the momentum-space wave function, $\Phi(p, t)$, of a quantum particle in the ground state of the harmonic oscillator.

   (b) For the same particle considered in (a), calculate the probability that a measurement of momentum, p, returns a value outside the classical range for the same energy, E.

   (c) Find the normalised eigenvectors and the corresponding eigenvalues of a quantum mechanical observable described by the matrix below:

   $$\hat{Q} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

   (d) For the same observable considered in (c), is there any degeneracy? Could you provide a physical example where these results would be relevant?

3. **(5 points) Dirac notation: brakets and dual basis**
Consider that $|e_1\rangle$, $|e_2\rangle$, $|e_3\rangle$ is an orthonormal basis. In this basis, let the $|\Psi_\alpha\rangle$ and $|\Psi_\beta\rangle$ kets be:

$$|\Psi_\alpha\rangle = 2\,i\;|e_1\rangle - 3\;|e_2\rangle + i\;|e_3\rangle$$

$$|\Psi_\beta\rangle = 3\;|e_1\rangle - 2\;|e_2\rangle + 4\;|e_3\rangle$$

(a) Are these kets normalised? If not, normalise them.

(b) Write $\langle\Psi_\alpha|$ and $\langle\Psi_\beta|$ in terms of the dual basis $\langle e_1|$, $\langle e_2|$, $\langle e_3|$.

(c) Compute the inner products $\langle\Psi_\alpha|\Psi_\beta\rangle$ and $\langle\Psi_\beta|\Psi_\alpha\rangle$, and confirm that $\langle\Psi_\beta|\Psi_\alpha\rangle = \langle\Psi_\alpha|\Psi_\beta\rangle^*$.

(d) Let $c = 4 + 7\,i$, and compute $|c\,\Psi_\alpha\rangle$ and $|\Psi_\alpha - c\,\Psi_\beta\rangle$.

(e) Find all the matrix elements of the operators $\hat{M}_{\alpha\beta} = |\Psi_\alpha\rangle\,\langle\Psi_\beta|$, $\hat{M}_{\alpha\alpha} = |\Psi_\alpha\rangle\,\langle\Psi_\alpha|$, and $\hat{M}_{\beta\beta} = |\Psi_\beta\rangle\,\langle\Psi_\beta|$ in this basis, and construct their respective matrices, are they hermitian?

4. **(5 points) Hamiltonian, eigenvalues and eigenvectors**
Consider a two-state quantum mechanical system in the basis:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

whose Hamiltonian is represented by the matrix shown below:

$$\hat{H} = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_1 \end{bmatrix}$$

(a) Find the eigenvalues of $\hat{H}$. What do the eigenvalues represent?

(b) Find the eigenvectors of $\hat{H}$, and express them in terms of $|0\rangle$ and $|1\rangle$.

(c) Find $\langle H\rangle$, $\langle H^2\rangle$, and σ_H for $|\Psi(t=0)\rangle = |0\rangle$.

(d) If $|\Psi(t=0)\rangle = |0\rangle$, find the state of the system at any time t, $|\Psi(t)\rangle$, described by the Schrödinger equation: $i\,\hbar\,\frac{\partial|\Psi\rangle}{\partial t} = \hat{H}\;|\Psi\rangle$

(e) Based on the results obtained in (d), what physical quantities do a_1 and a_2 represent in the quantum system and why?