# Solution for Θ:

$$\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \left[\ell(\ell+1)\sin^2\theta - m^2\right]\Theta = 0,$$

The solution reads:

$$\Theta(\theta) = A P_\ell^m(\cos\theta)$$

where $P_l^m$ is the **associated Legendre function**, defined by:

$$P_\ell^m(x) \equiv (-1)^m \left(1 - x^2\right)^{m/2} \left(\frac{d}{dx}\right)^m P_\ell(x),$$ for $m \geq 0$

and $P_l(x)$ is the $\ell$th **Legendre polynomial**, defined by the **Rodrigues formula**:

$$P_\ell(x) \equiv \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell \left(x^2 - 1\right)^\ell.$$

# Solution for ϴ:

For negative values of *m*:

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell - m)!}{(\ell + m)!} P_\ell^m(x).$$

$P_\ell(x)$ is a polynomial (of degree $\ell$) in $x$, and is even or odd according to the parity of $\ell$.

$P_\ell^m(x)$ is not, in general, a polynomial — if $m$ is odd it carries a factor of $(1-x^2)^{0.5}$

$\ell$ must be a non-negative *integer.*

If $m > \ell$, $P_\ell^m = 0$. For any given $\ell$, then, there are $(2\ell + 1)$ possible values of $m$:

$$\ell = 0,\ 1,\ 2, \ldots \quad \Rightarrow \quad m = -\ell,\ -\ell + 1,\ \ldots,\ -1,\ 0,\ 1, \ldots,\ \ell - 1,\ \ell.$$

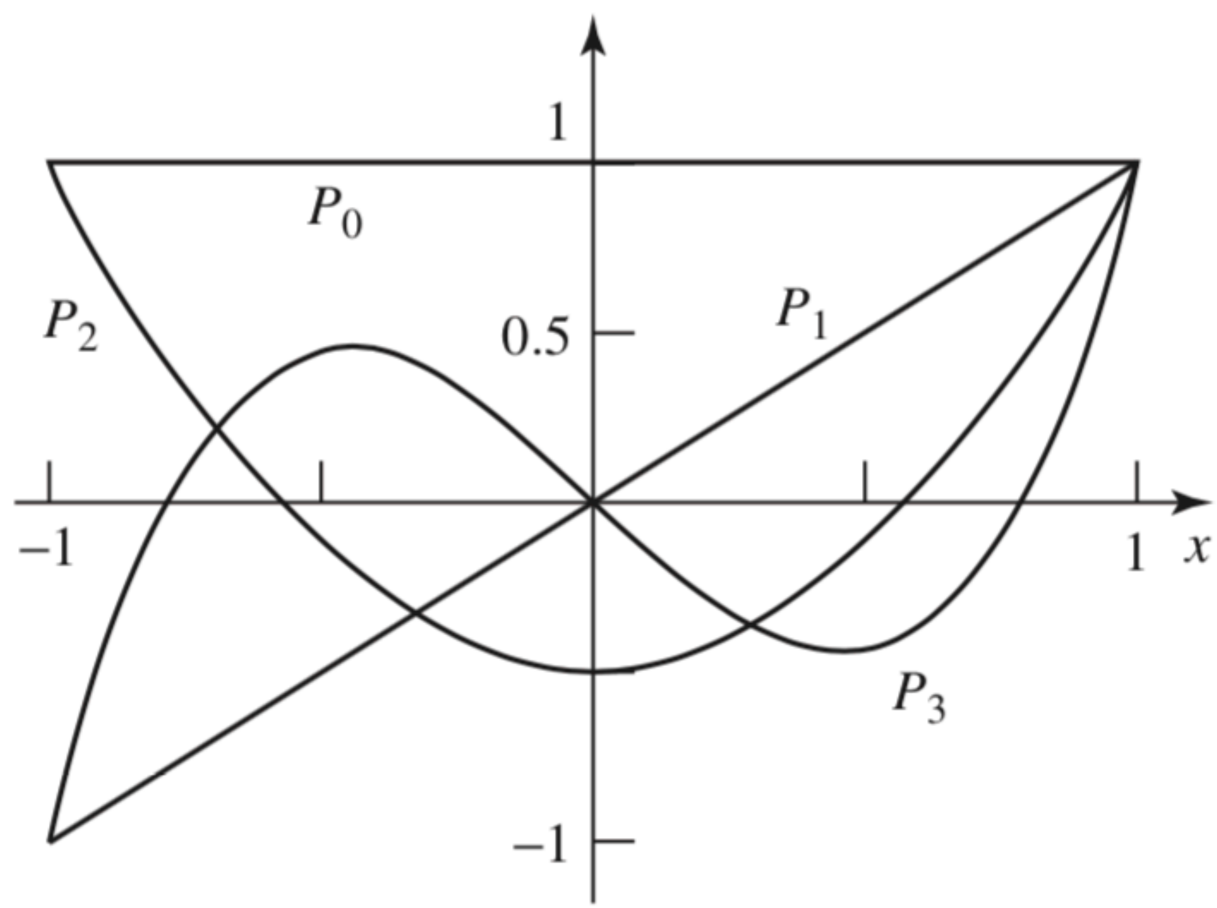
# Solution for $\Theta$:

For negative values of *m*:

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell - m)!}{(\ell + m)!} P_\ell^m(x).$$

## First Legendre polynomials:

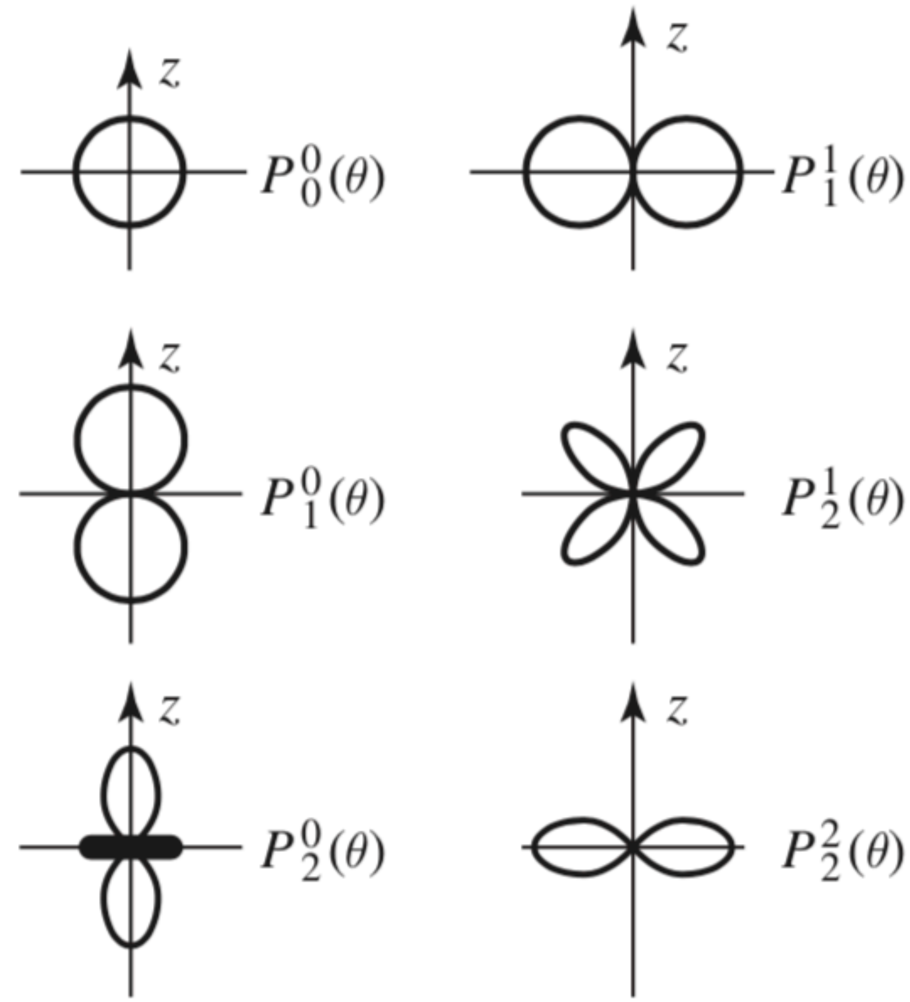
$$P_0 = 1$$
$$P_1 = x$$
$$P_2 = \frac{1}{2}(3x^2 - 1)$$
$$P_3 = \frac{1}{2}(5x^3 - 3x)$$
$$P_4 = \frac{1}{8}(35x^4 - 30x^2 + 3)$$
$$P_5 = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$



# Solution for Θ:

We need $P_\ell{}^m(\cos\theta)$, and $(1 - \cos^2\theta)^{0.5} = \sin\theta$, so $P_\ell{}^m(\cos\theta)$ is always a polynomial in $\cos\theta$, multiplied — if $m$ is odd—by $\sin\theta$ .

| | |
|---|---|
| $P_0^0 = 1$ | $P_2^0 = \frac{1}{2}(3\cos^2\theta - 1)$ |
| $P_1^1 = -\sin\theta$ | $P_3^3 = -15\sin\theta(1-\cos^2\theta)$ |
| $P_1^0 = \cos\theta$ | $P_3^2 = 15\sin^2\theta\cos\theta$ |
| $P_2^2 = 3\sin^2\theta$ | $P_3^1 = -\frac{3}{2}\sin\theta(5\cos^2\theta - 1)$ |
| $P_2^1 = -3\sin\theta\cos\theta$ | $P_3^0 = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$ |




*graphs of* $r = \left|P_\ell^m(\cos\theta)\right|$ *(in these plots r tells you the magnitude of the function in the direction θ; each figure should be rotated about the z axis).*

# Normalisation condition: solution for Θ:

The volume element in spherical coordinates:

$$d^3\mathbf{r} = r^2 \sin\theta \, dr \, d\theta \, d\phi = r^2 \, dr \, d\Omega, \quad \text{where} \quad d\Omega \equiv \sin\theta \, d\theta \, d\phi,$$

Normalisation condition:

$$\int |\Psi|^2 \, d^3\mathbf{r} = 1, \quad \Rightarrow \quad \int |\psi|^2 r^2 \sin\theta \, dr \, d\theta \, d\phi = \int |R|^2 r^2 \, dr \int |Y|^2 \, d\Omega = 1.$$

It is convenient to normalise *R* and *Y* separately:

$$\int_0^\infty |R|^2 r^2 \, dr = 1$$

$$\int_0^\pi \int_0^{2\pi} |Y|^2 \sin\theta \, d\theta \, d\phi = 1.$$

# Normalisation condition: solution for ϴ:

The normalised angular wave functions are called **spherical harmonics**:

$$\boxed{Y_\ell^m(\theta,\phi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, e^{im\phi}\, P_\ell^m(\cos\theta)\,,}$$

They are orthogonal: $$\int_0^\pi \int_0^{2\pi} \left[Y_\ell^m(\theta,\phi)\right]^* \left[Y_{\ell'}^{m'}(\theta,\phi)\right] \sin\theta\, d\theta\, d\phi = \delta_{\ell\ell'}\delta_{mm'}$$

## Spherical Harmonics:

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2} \qquad Y_2^{\pm 2} = \left(\frac{15}{32\pi}\right)^{1/2} \sin^2\theta e^{\pm 2i\phi}$$

$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta \qquad Y_3^0 = \left(\frac{7}{16\pi}\right)^{1/2} (5\cos^3\theta - 3\cos\theta)$$

$$Y_1^{\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta e^{\pm i\phi} \qquad Y_3^{\pm 1} = \mp\left(\frac{21}{64\pi}\right)^{1/2} \sin\theta\,(5\cos^2\theta - 1)e^{\pm i\phi}$$

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2} (3\cos^2\theta - 1) \qquad Y_3^{\pm 2} = \left(\frac{105}{32\pi}\right)^{1/2} \sin^2\theta\cos\theta e^{\pm 2i\phi}$$

$$Y_2^{\pm 1} = \mp\left(\frac{15}{8\pi}\right)^{1/2} \sin\theta\cos\theta e^{\pm i\phi} \qquad Y_3^{\pm 3} = \mp\left(\frac{35}{64\pi}\right)^{1/2} \sin^3\theta e^{\pm 3i\phi}$$