The hydrogen atom

Addition of Angular Momenta

Suppose now that we have *two* particles, with spins s_1 and s_2.

$s_1 \rightarrow |s_1\, m_1\rangle$

$s_2 \rightarrow |s_2\, m_2\rangle$

we denote the composite state by: $|s_1\, s_2\, m_1\, m_2\rangle$

$$S^{(1)^2}\,|s_1\, s_2\, m_1\, m_2\rangle = s_1\,(s_1+1)\,\hbar^2\,|s_1\, s_2\, m_1\, m_2\rangle\,,$$
$$S^{(2)^2}\,|s_1\, s_2\, m_1\, m_2\rangle = s_2\,(s_2+1)\,\hbar^2\,|s_1\, s_2\, m_1\, m_2\rangle\,,$$
$$S_z^{(1)}\,|s_1\, s_2\, m_1\, m_2\rangle = m_1\hbar\,|s_1\, s_2\, m_1\, m_2\rangle\,,$$
$$S_z^{(2)}\,|s_1\, s_2\, m_1\, m_2\rangle = m_2\hbar\,|s_1\, s_2\, m_1\, m_2\rangle\,.$$

What is the *total* angular momentum of the system?

$$\mathbf{S} = \mathbf{S}^{(1)} + \mathbf{S}^{(2)}$$

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Addition of Angular Momenta

What is the *total* angular momentum of the system? $\mathbf{S} = \mathbf{S}^{(1)} + \mathbf{S}^{(2)}$

What is the net spin, *s*, of the combination, and what is the *z* component, *m*?

The *z* component is easy:

$$\begin{aligned} S_z \left| s_1\, s_2\, m_1\, m_2 \right\rangle &= S_z^{(1)} \left| s_1\, s_2\, m_1\, m_2 \right\rangle + S_z^{(2)} \left| s_1\, s_2\, m_1\, m_2 \right\rangle , \\ &= \hbar \left(m_1 + m_2 \right) \left| s_1\, s_2\, m_1\, m_2 \right\rangle = \hbar m \left| s_1\, s_2\, m_1\, m_2 \right\rangle \end{aligned}$$

Thus: $m = m_1 + m_2$

The net spin, *s,* is much more subtle. If you combine spin s_1 with spin s_2, what total spins *s* can we get?

The answer is that you get every spin from ($s_1 + s_2$) down to ($s_1 - s_2$)

or ($s_2 - s_1$), if $s_2 > s_1$ in integer steps:

$$\boxed{s = (s_1 + s_2),\ (s_1 + s_2 - 1),\ (s_1 + s_2 - 2),\ \ldots,\ |s_1 - s_2|.}$$

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Roughly speaking, the highest total spin occurs when the individual spins are aligned parallel to one another, and the lowest occurs when they are antiparallel.

Example 1: if you package together a particle of spin 3/2 with a particle of spin 2, you could get a total spin of 7/2, 5/2, 3/2, or 1/2, depending on the configuration.

Example 2: If a hydrogen atom is in the state ψ_{nlm}, the net angular momentum of the electron (spin plus orbital) is ℓ +1/2 or ℓ -1/2.

If you now throw in spin of the *proton*, the atom's *total* angular momentum quantum number is ℓ + 1, ℓ, or ℓ - 1 (and ℓ can be achieved in two distinct ways, depending on whether the electron alone is in the ℓ + 1/2 configuration or the ℓ - 1/2 configuration).

The hydrogen atom

Addition of Angular Momenta

The combined state $|sm\rangle$ with total spin s and z-component m will be some linear combination of the composite states $|s_1 s_2 m_1 m_2\rangle$:

$$|s\, m\rangle = \sum_{m_1+m_2=m} C^{s_1 s_2 s}_{m_1 m_2 m} |s_1\, s_2\, m_1\, m_2\rangle$$

Because the z components add, the only composite states that contribute are those for which:

$$m_1 + m_2 = m$$

The constants: $C^{s_1 s_2 s}_{m_1 m_2 m}$ are called **Clebsch-Gordan coefficients.**

Example 1: The shaded column of the 2 × 1

$$|3\,0\rangle = \frac{1}{\sqrt{5}}|2\,1\rangle\,|1\,-1\rangle + \sqrt{\frac{3}{5}}|2\,0\rangle\,|1\,0\rangle + \frac{1}{\sqrt{5}}|2\,-1\rangle\,|1\,1\rangle$$

If two particles (of spin 2 and spin 1) are at rest in a box, and the *total* spin is 3, and its z component is 0, then a measurement of $S_z^{(1)}$ could return the value $\hbar$ (with probability 1/5), or 0 (with probability 3/5), or $-\hbar$ (with probability 1/5). Notice that the probabilities add up to 1 (the sum of the squares of any column on the Clebsch–Gordan table is 1).

The hydrogen atom

Clebsch-Gordan coefficients $C^{s_1 s_2 s}_{m_1 m_2 m}$

Table 4.8: *Clebsch–Gordan coefficients. (A square root sign is understood for every entry; the minus sign, if present, goes* outside *the radical.)*

1/2 × 1/2

		1
		+1
+1/2	+1/2	1

		1	0
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		1
		−1
−1/2	−1/2	1

1 × 1/2

		3/2
		+3/2
+1	+1/2	1

		3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

		3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

		3/2
		−3/2
−1	−1/2	1

2 × 1

		3
		+3
+2	+1	1

		3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

		3	2	1
		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	6/15	−1/2	1/10

		3	2	1
		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

		3	2	1
		−1	−1	−1
0	−1	6/15	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

		3	2
		−2	−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

		3
		−3
−2	−1	1

1 × 1

		2
		+2
+1	+1	1

		2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

		2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

		2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

		2
		−2
−1	−1	1

3/2 × 1

		5/2
		+5/2
+3/2	+1	1

		5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

		5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

		5/2	3/2	1/2
		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

		5/2	3/2
		−3/2	−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

		5/2
		−5/2
−3/2	−1	1

2 × 1/2

		5/2
		+5/2
+2	1/2	1

		5/2	3/2
		3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

		5/2	3/2
		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

		5/2	3/2
		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

		5/2	3/2
		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

		5/2
		−5/2
−2	−1/2	1

3/2 × 1/2

		2
		+2
+3/2	+1/2	1

		2	1
		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

		2	1
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		2	1
		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

		2
		−2
−3/2	−1/2	1