IDENTICAL PARTICLES

Exchange Forces

Suppose one particle is in state $\psi_a(x)$, and the other is in state $\psi_b(x)$, and these two states are orthogonal and normalised.

If the two particles are distinguishable, and number 1 is the one in state ψ_a, then the combined wave function is:

$$\psi(x_1, x_2) = \psi_a(x_1)\psi_b(x_2);$$

Identical bosons: $$\psi_+(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_a(x_1)\psi_b(x_2) + \psi_b(x_1)\psi_a(x_2)\right];$$

Identical fermions: $$\psi_-(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_a(x_1)\psi_b(x_2) - \psi_b(x_1)\psi_a(x_2)\right].$$

The expectation value of the square of the separation distance between the two particles:

$$\left\langle (x_1 - x_2)^2 \right\rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2\langle x_1 x_2 \rangle.$$

IDENTICAL PARTICLES

Exchange Forces:

Case 1: Distinguishable particles.

$$\left\langle x_1^2\right\rangle = \int x_1^2\left|\psi_a(x_1)\right|^2 dx_1 \int \left|\psi_b(x_2)\right|^2 dx_2 = \left\langle x^2\right\rangle_a$$

(the expectation value of x^2 in the one-particle state ψ_a),

$$\left\langle x_2^2\right\rangle = \int \left|\psi_a(x_1)\right|^2 dx_1 \int x_2^2\left|\psi_b(x_2)\right|^2 dx_2 = \left\langle x^2\right\rangle_b,$$

and

$$\langle x_1 x_2\rangle = \int x_1\left|\psi_a(x_1)\right|^2 dx_1 \int x_2\left|\psi_b(x_2)\right|^2 dx_2 = \langle x\rangle_a \langle x\rangle_b.$$

In this case, then,

$$\left\langle (x_1 - x_2)^2 \right\rangle_d = \left\langle x^2\right\rangle_a + \left\langle x^2\right\rangle_b - 2\langle x\rangle_a \langle x\rangle_b.$$

(Incidentally, the answer would—of course—be the same if particle 1 had been in state ψ_b, and particle 2 in state ψ_a.)

IDENTICAL PARTICLES

Exchange Forces:

Case 2: Identical particles.

$$\begin{aligned}\langle x_1^2\rangle = \frac{1}{2}\Bigg[&\int x_1^2|\psi_a(x_1)|^2\,dx_1\int|\psi_b(x_2)|^2\,dx_2\\&+\int x_1^2|\psi_b(x_1)|^2\,dx_1\int|\psi_a(x_2)|^2\,dx_2\\&\pm\int x_1^2\psi_a(x_1)^*\psi_b(x_1)\,dx_1\int\psi_b(x_2)^*\psi_a(x_2)\,dx_2\\&\pm\int x_1^2\psi_b(x_1)^*\psi_a(x_1)\,dx_1\int\psi_a(x_2)^*\psi_b(x_2)\,dx_2\Bigg]\\=\frac{1}{2}\Big[&\langle x^2\rangle_a+\langle x^2\rangle_b\pm 0\pm 0\Big]=\frac{1}{2}\left(\langle x^2\rangle_a+\langle x^2\rangle_b\right).\end{aligned}$$

Similarly,

$$\langle x_2^2\rangle=\frac{1}{2}\left(\langle x^2\rangle_b+\langle x^2\rangle_a\right).$$

(Naturally, $\langle x_2^2\rangle=\langle x_1^2\rangle$, since you can't tell them apart.) But

IDENTICAL PARTICLES

Exchange Forces:

Case 2: Identical particles.

$$\begin{aligned}
\langle x_1 x_2 \rangle = \frac{1}{2}\Big[&\int x_1 |\psi_a(x_1)|^2 \, dx_1 \int x_2 |\psi_b(x_2)|^2 \, dx_2 \\
&+ \int x_1 |\psi_b(x_1)|^2 \, dx_1 \int x_2 |\psi_a(x_2)|^2 \, dx_2 \\
&\pm \int x_1 \psi_a(x_1)^* \psi_b(x_1) \, dx_1 \int x_2 \psi_b(x_2)^* \psi_a(x_2) \, dx_2 \\
&\pm \int x_1 \psi_b(x_1)^* \psi_a(x_1) \, dx_1 \int x_2 \psi_a(x_2)^* \psi_b(x_2) \, dx_2 \Big] \\
= \frac{1}{2} &\left(\langle x \rangle_a \langle x \rangle_b + \langle x \rangle_b \langle x \rangle_a \pm \langle x \rangle_{ab} \langle x \rangle_{ba} \pm \langle x \rangle_{ba} \langle x \rangle_{ab} \right) \\
= \langle x &\rangle_a \langle x \rangle_b \pm |\langle x \rangle_{ab}|^2,
\end{aligned}$$

where

$$\langle x \rangle_{ab} \equiv \int x \psi_a(x)^* \psi_b(x) \, dx.$$

Thus

$$\left\langle (x_1 - x_2)^2 \right\rangle_\pm = \left\langle x^2 \right\rangle_a + \left\langle x^2 \right\rangle_b - 2\langle x \rangle_a \langle x \rangle_b \mp 2|\langle x \rangle_{ab}|^2.$$

IDENTICAL PARTICLES

Exchange Forces:

Case 2: Identical particles.

Distinguishable particles:
$$\left\langle (x_1 - x_2)^2 \right\rangle_d = \left\langle x^2 \right\rangle_a + \left\langle x^2 \right\rangle_b - 2 \langle x \rangle_a \langle x \rangle_b .$$

Identical particles.:
$$\left\langle (x_1 - x_2)^2 \right\rangle_\pm = \left\langle x^2 \right\rangle_a + \left\langle x^2 \right\rangle_b - 2\langle x \rangle_a \langle x \rangle_b \mp 2|\langle x \rangle_{ab}|^2 .$$

Difference:
$$\left\langle (\Delta x)^2 \right\rangle_\pm = \left\langle (\Delta x)^2 \right\rangle_d \mp 2 |\langle x \rangle_{ab}|^2 ;$$

Identical bosons (the upper signs) tend to be somewhat closer together, and **identical fermions** (the lower signs) somewhat farther apart, than distinguishable particles in the same two states.

$\langle x \rangle_{ab}$ *vanishes* unless the two wave functions actually *overlap.*

IDENTICAL PARTICLES

Exchange Forces:

Case 2: Identical particles.

The *interesting* case is when the overlap integral is *not* zero.

The system behaves as though there were:

a **“force of attraction”** between identical bosons, **pulling them closer together**, and

a **“force of repulsion”** between identical fermions, **pushing them apart**.

We call it an **exchange force**, although it’s not really a force at all—no physical agency is pushing on the particles. **What is it then?**

It is a purely *geometrical* consequence of the symmetrisation requirement. It is also a strictly quantum mechanical phenomenon, with no classical counterpart.