Homework 2 - Quantum Mechanics I

NAME: ____ **SCORE:** ____________

Deadline: Thursday 15th June 2023 by 10:00am (submission only on paper)
Credits: 20 points **Number of problems:** 4
Type of evaluation: Formative Evaluation

- This homework includes problems on units 1 and 2 of the QM course.
- This assignment should be submitted individually by the deadline.
- Please justify all calculations and highlight your answers.

1. **(5 points) Probability densities**
   Consider the log-normal distribution:

   $\rho(x) = \frac{\alpha}{x}\, e^{-\frac{(\ln(x)-\mu)^2}{2\,\sigma^2}}$,

   where α, μ, and σ are positive real constants.
   (a) Determine α.
   (b) Find $\langle x\rangle$.
   (c) Find $\langle x^2\rangle$, and σ_x.
   (d) What do α, μ, and σ represent?
   (e) Plug some fiducial numbers for these constants, and sketch the graph of $\rho(x)$ using your favourite programming language.

2. **(5 points) Infinite square well potential and expectation values**
   In class we solved the Schrödinger equation for an infinite square well potential of width L. Such potential allows for bound solutions only as the particle cannot escape from the well. The solutions we found had the following functional form:

   $$\psi_n(x) = \sqrt{\frac{2}{L}}\,\sin\left(\frac{n\,\pi}{L}\,x\right) \tag{1}$$

   For the n-th state given by the function above, calculate:

   (a) The expectation values associated with the position x: $\langle x\rangle$, $\langle x^2\rangle$.
   (b) The expectation values associated with the momentum p: $\langle p\rangle$, $\langle p^2\rangle$.
   (c) The dispersions σ_x and σ_p, and their product $\sigma_x\,\sigma_p$.
   (d) Use programming tools to make a plot of $(\sigma_x\,\sigma_p)$ vs. n.
   (e) Is the uncertainty principle satisfied? Which of the $\psi_n(x)$ states comes closest to the uncertainty limit?

3. **(5 points) Free particles: Gaussian wave packets**
We studied free particles in class and showed that they are represented by wave packets. Consider the case of a free particle whose initial wave function is given by:

$$\Psi(x,0) = \alpha\, e^{-\beta\, x^2}, \tag{2}$$

where α and β are real and positive constants.

(a) Find α by normalising the initial wave function, $\Psi(x,0)$.
(b) Find $\Psi(x,t)$. Hint: compute $\phi(k)$ via Fourier analysis first, and then plug it into the wave packet function.
(c) Find $|\Psi(x,t)|^2$. Then, plug some fiducial numbers, and sketch $|\Psi(x,t)|^2$ versus x for $t = 0$ and two later times. Qualitatively, what happens to $|\Psi(x,t)|^2$ as time progresses?
(d) Calculate $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, $\langle p^2 \rangle$, σ_x, and σ_p.
(e) Does the uncertainty principle hold? At what time t does the system come closest to the uncertainty limit?

4. **(5 points) Finite square well potential**

In class we studied the finite square well potential and found that this potential admits both scattering states (when $E > 0$) and bound states (when $E < 0$). For the latter, we derived the even solutions and numerically solved a transcendental equation for the allowed energies.

(a) Following the same approach we followed in class, find the odd bound state wave functions, $\psi(x)$, for the finite square well.
(b) Derive the transcendental equation for the allowed energies of these odd bound states.
(c) Solve it graphically and numerically (using your favourite programming tool).
(d) Study and discuss the two limiting cases and how the energy levels compare to those found for the even bound state wave functions studied in class. Is there always an odd bound state?
(e) Normalise the even and odd bound state wave functions.