Homework 2 - Quantum Mechanics I

NAME: ____ **SCORE:** _________

Deadline: Wednesday 11 October 2023 by 13:00 (submission only on paper)
Credits: 20 points → 20 credits **Number of problems:** 4
Type of evaluation: Formative Evaluation

This assignment is individual and consists of 4 problems related to unit 2 of quantum mechanics. Please justify all calculations and highlight the answers.

1. **(5 points) Probability densities**
   Consider the log-normal distribution:

   $\rho(x) = \frac{A}{x}\, e^{-\frac{(\ln(x)-\mu)^2}{2\,\sigma^2}}$,

   where A, μ, and σ are positive real constants.
   (a) Determine A.
   (b) Find $\langle x \rangle$.
   (c) Find $\langle x^2 \rangle$, and σ_x.
   (d) What do A, μ, and σ represent?
   (e) Plug some fiducial numbers for these constants, and sketch the graph of $\rho(x)$ using your favourite programming language.

2. **(6 points) Wave functions, normalisation, and expectation values**
   The wave function of a particle at time $t = 0$ is given by the following piecewise function:

$$\Psi(x, t = 0) = \begin{cases} C\,\frac{x}{\alpha}, & 0 \leq x \leq \alpha \\ C\,\frac{\beta - x}{\beta - \alpha}, & \alpha \leq x \leq \beta \\ 0, & \text{everywhere else,} \end{cases}$$

   where C, α, and β are positive constants.

   (a) Find an expression for C.
   (b) Plug some fiducial numbers for the constants, and use programming tools to sketch $\Psi(x, t = 0)$ as a function of x.
   (c) Where is the particle most likely to be found at $t = 0$?
   (d) What is the probability of finding the particle to the left of α? Check your result in the limiting cases $\beta = \alpha$ and $\beta = 2\,\alpha$.
   (e) What is the probability of finding the particle between α and β?
   (f) What is the expectation value of x?

3. **(5 points) Free particles: Gaussian wave packets**
We studied free particles in class and showed that they are represented by wave packets. Consider the case of a free particle whose initial wave function is given by:

$$\Psi(x,0) = \alpha\, e^{-\beta\, x^2}, \tag{1}$$

where α and β are real and positive constants.

(a) Find α by normalising the initial wave function, $\Psi(x,0)$.
(b) Find $\Psi(x,t)$. Hint: compute $\phi(k)$ via Fourier analysis first, and then plug it into the wave packet function.
(c) Find $|\Psi(x,t)|^2$. Then, plug some fiducial numbers, and use your favourite programming tools to sketch $|\Psi(x,t)|^2$ versus x for $t = 0$ and two later times. Qualitatively, what happens to $|\Psi(x,t)|^2$ as time progresses?
(d) Calculate $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$, σ_x, and σ_p.
(e) Does the uncertainty principle hold? At what time t does the system come closest to the uncertainty limit?

4. **(4 points) The Schrödinger equation, potentials and general solutions**
Consider the following 1D wave function:

$$\psi(x) = A\left(\frac{x}{x_0}\right)^n e^{-\frac{x}{x_0}}$$

where A, n, and x_0 are constants.

(a) Write down the time-independent Schrödinger equation.
(b) Find the potential $V(x)$ and energy E, for which this wave function is a solution to the Schrödinger equation. Assume that as $x \to +\infty$, $V(x) \to 0$.
(c) Plug some fiducial values and sketch the potential, $V(x)$, versus x, using your favourite programming tool.
(d) Write down an expression for a more general time-dependent solution.