Homework 4 - Quantum Mechanics I

NAME: ____ **SCORE:** ____________

Deadline: Friday 31 May 2024 by 17:00
Credits: 20 points **Number of problems:** 4
Type of evaluation: Formative Evaluation

- **This homework is individual and includes problems on unit 4.**
- **Please send a single PDF file via email to:** wbanda@yachaytech.edu.ec
- **If you work with classmates, please state their names and write your own reports.**
- **Please submit a professional homework report and clearly highlight all answers.**
- **All calculation steps should be justified to get full credits.**
- **Any code used should be attached to get full credits.**

1. **(5 points) Spherical harmonics**

   (a) A quantum system is known to be in the (unnormalised) state described by the wave function $\psi(\theta, \phi) = 2\, Y_5^2 + 7\, Y_4^0 - 4\, Y_6^2$, where the $Y_\ell^m(\theta, \phi)$ are the spherical harmonics. What is the probability of finding the system in a state with quantum number $m = 2$?

   (b) What is the orbital angular momentum eigenfunction Y_ℓ^m in a state for which the operators L^2 and L_z have eigenvalues $6\hbar^2$ and $-\hbar$, respectively?

   (c) Analytically construct all the possible spherical harmonics, $Y_\ell^m(\theta, \phi)$, for $\ell = 3$.

   (d) Using your favourite programming language, make 3D plots of all of them.

   (e) Choose two of the spherical harmonics constructed in part (b), and prove that they are normalised and orthogonal.

2. **(5 points) Hydrogen atom**

   (a) Construct all the possible spatial wave functions, $\psi_{nlm}(r, \theta, \phi) = R_{n\ell}(r) Y_\ell^m(\theta, \phi)$, of the hydrogen atom for $(n, \ell, m) = (4, 1, m)$.

   (b) Using your favourite programming language, make density plots of all of these states.

   (c) Calculate the energy level of these states in units of eV.

   (d) In terms of the Bohr radius, find $\langle r \rangle$, $\langle x \rangle$, $\langle r^2 \rangle$ and $\langle x^2 \rangle$ for an electron in the ground state of hydrogen.

   (e) Find $\langle x^2 \rangle$ in the state $(n, \ell, m) = (4, 1, m)$ with the highest possible value of m that is allowed. How different is the result with respect to that calculated in part (d) for the ground state of hydrogen?

3. **(5 points) Spin**

(a) Suppose we have a spin-$\frac{1}{2}$ particle, whose quantum state can be represented by the eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ of the S_z operator. Using the Pauli matrix σ_y, calculate a normalised eigenstate of the S_y operator with an eigenvalue $+\frac{\hbar}{2}$.

(b) Consider the Pauli spin matrices σ_x, σ_y, and σ_z and the identity matrix, I. Calculate the commutators: $[\sigma_x, \sigma_y]$ and $[\sigma_z, \sigma_y]$.

(c) Calculate the normalised spin eigenfunctions of a system with two spin-$\frac{1}{2}$ particles. These are called singlet and triplet states, why?

(d) Consider an electron in the spin state: $\chi = C \begin{bmatrix} 1-3i \\ 2 \end{bmatrix}$. Normalise χ, and find the expectation values of S_x, S_y, and S_z.

(e) Find the uncertainties σ_{S_x}, σ_{S_y}, and σ_{S_z} for χ in (d). Are the results consistent with all three uncertainty principles?

4. **(5 points) Particle in a 3D infinite potential well**

Suppose we have a particle in a 3D spherical and infinite potential well:

$$V(r) = \begin{cases} 0 & 0 \leq r \leq a \\ \infty & r > a \end{cases}$$

where (r, θ, ϕ) are the radial and angular (spherical) coordinates.

(a) Write down the time-dependent Schrödinger equation for the particle in spherical coordinates.

(b) Use variable separation to split the spatial and temporal terms. What is the solution for the temporal term?

(c) Apply variable separation to the time-independent Schrödinger equation and write the differential equations of the radial and angular parts. Solve the angular equation.

(d) Compute the energy levels and the stationary wave function, $\psi(r, \theta, \phi)$ for $\ell = 0$.