Homework 3 - Quantum Mechanics I

NAME: ____ **SCORE:** ____________

Deadline: Thursday 20th July 2023 by 10:00am (submission only on paper)
Credits: 20 points **Number of problems:** 5
Type of evaluation: Formative Evaluation

- This homework includes problems on units 2 and 3 of the QM course.
- This assignment should be submitted individually by the deadline.
- Please justify all calculations and highlight your answers.

1. **(3 points) Probability current**
As we reviewed in class, the wave function of a particle is a complex-valued probability amplitude that depends on position, x, and time, t. As time progresses, the wave function changes and the probability of finding a particle in certain position also changes with it. Since the sum of all probabilities should always be 1, this means that the probability 'flows' from one region to another one, akin to a fluid or a current. This 'flow' can be described mathematically by the so-called probability current j, which for the wave function Ψ of a non-relativistic particle of mass m in 1D is defined as:

$$j(x,t) = \frac{\hbar}{2\,m\,i}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x}\right)$$

Find the probability current, j, of a superposition of 2 currents of particles of mass m, momentum p, and energy $\frac{p^2}{2\,m}$, moving in opposite directions. The amplitudes of the particle currents are α and β, respectively. Hint: Write the wave function for the superposition first.

2. **(5 points) The time-independent Schrödinger equation**
Consider a current of particles with energies $E > V_0$ moving from $x = -\infty$ to the right, under the influence of a Heaviside potential $V(x)$ given by:

$$V(x) = \begin{cases} V_0, & x \geq 0 \\ 0, & x < 0, \end{cases}$$

where A, n, and x_0 are constants.

(a) Sketch the potential and write down the time-independent Schrödinger equation.
(b) Find the stationary state solutions for each region of interest.
(c) Express the transmitted and reflected amplitudes in terms of the incident amplitude.
(d) Find the probability current, $j(x)$, in each region of interest.
(e) Use the results from part (d) to find and plot (using your favourite programming tool) the reflection and transmission coefficients, and check that $T + R = 1$. Note that the transmission coefficient is not simply $\frac{|F|^2}{|A|^2}$ (with A the incident amplitude and F the transmitted amplitude), because the transmitted wave travels at a different speed.

3. **(5 points) Mathematical formalism of quantum mechanics**

   (a) Consider the orthonormal states $|1\rangle$, $|2\rangle$, and $|3\rangle$. For which value of x are the following states, $|\Psi_1\rangle = 5|1\rangle - 3|2\rangle + 2|3\rangle$ and $|\Psi_2\rangle = |1\rangle - 5|2\rangle + x|3\rangle$, orthogonal?

   (b) Assuming that $\gamma \in \mathbb{R}$ (but not necessarily positive), for what range of γ is the function $f(x) = x^{\gamma-1}$ in Hilbert space, on the interval $(0,1)$? What about $x\,f(x)$ and $\frac{d}{dx}f(x)$?

   (c) Let $|n\rangle$ be the normalised n-th energy eigenstate of the 1D harmonic oscillator. We know that $\hat{H}|n\rangle = \hbar\omega(n+\frac{1}{2})|n\rangle$. If $|\psi\rangle$ is a normalised ensemble state that can be expressed as a linear combination of the eigenstates as follows: $|\psi\rangle = \frac{1}{\sqrt{14}}|1\rangle - \frac{2}{\sqrt{14}}|2\rangle + \frac{3}{\sqrt{14}}|3\rangle$, what is the expectation value of the energy operator in this ensemble state?

   (d) Consider the state $\Psi = \frac{1}{\sqrt{5}}\Psi_{-1} + \frac{1}{\sqrt{4}}\Psi_{+1} + \frac{1}{\sqrt{20}}\Psi_{+2} + \frac{1}{\sqrt{2}}\Psi_{+3}$, which is a linear combination of four orthonormal eigenstates of the operator $\hat{Q}$ corresponding to eigenvalues -1, $+1$, $+2$, and $+3$. Calculate the expectation value of the operator $\hat{Q}$ for this state.

4. **(3 points) Hamiltonian, eigenvalues and eigenvectors**

   Consider a quantum system in a state, $|\Psi\rangle$:

   $$\Psi = \frac{1}{\sqrt{3}}\begin{bmatrix} i \\ -i \\ i \end{bmatrix}$$

   The Hamiltonian is represented by the matrix shown below:

   $$\hat{H} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

   (a) Determine the eigenvalues and eigenvectors of $\hat{H}$. What do the eigenvalues represent?
   (b) Which eigenvalue of $\hat{H}$ is most likely to emerge from a measurement?
   (c) Find $\langle H\rangle$, $\langle H^2\rangle$, and σ_H.

5. **(4 points) Dirac notation: brakets and dual basis**

   Consider a 3D vector space spanned by an orthonormal basis $|1\rangle$, $|2\rangle$, $|3\rangle$. In this basis, let the $|\Psi_0\rangle$ and $|\Psi_1\rangle$ kets be:

   $$|\Psi_0\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{i}{2}|2\rangle + \frac{1}{2}|3\rangle$$

   $$|\Psi_1\rangle = \frac{1}{\sqrt{3}}|1\rangle + \frac{i}{\sqrt{3}}|3\rangle$$

   (a) Are these kets normalised? If not, normalise them.
   (b) Write $\langle\Psi_0|$ and $\langle\Psi_1|$ in terms of the dual basis $\langle 1|$, $\langle 2|$, $\langle 3|$.
   (c) Find $\langle\Psi_0|\Psi_1\rangle$ and $\langle\Psi_1|\Psi_0\rangle$, and confirm that $\langle\Psi_1|\Psi_0\rangle = \langle\Psi_0|\Psi_1\rangle^*$.
   (d) Find all the matrix elements of the operators $\hat{M}_{01} = |\Psi_0\rangle\langle\Psi_1|$, $\hat{M}_{00} = |\Psi_0\rangle\langle\Psi_0|$, and $\hat{M}_{11} = |\Psi_1\rangle\langle\Psi_1|$ in this basis, and construct their respective matrices, are they hermitian?