Homework 4 - Quantum Mechanics I

NAME: ____ **SCORE:** _____________

Deadline: Friday 1 December 2023 by 16:00
Credits: 20 points **Number of problems:** 5
Type of evaluation: Formative Evaluation

- This homework includes problems on units 3 and 4 of the QM course.
- This assignment should be submitted individually by the deadline.
- Please justify all calculations and highlight your answers.

1. **(4 points) Mathematical formalism of quantum mechanics**

   (a) Consider the orthonormal states: $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$. For which value of x are the following states, $|\Psi_1\rangle = 4|1\rangle - 3|2\rangle + 7|3\rangle + |4\rangle$ and $|\Psi_2\rangle = 2|1\rangle + 5|2\rangle - x|3\rangle - 2|4\rangle$, orthogonal?

   (b) Let $|n\rangle$ be the normalised n-th energy eigenstate of the 1D harmonic oscillator. We know that $\hat{H}|n\rangle = \hbar\omega(n + \frac{1}{2})|n\rangle$. If $|\psi\rangle$ is a normalised ensemble state that can be expressed as a linear combination of the eigenstates as follows: $|\psi\rangle = \frac{1}{\sqrt{14}}|1\rangle - \frac{2}{\sqrt{14}}|2\rangle + \frac{3}{\sqrt{14}}|3\rangle$, what is the expectation value of the energy operator in this ensemble state?

   (c) Consider the state $\Psi = \frac{1}{\sqrt{5}}\Psi_{-1} + \frac{1}{\sqrt{4}}\Psi_{+1} + \frac{1}{\sqrt{20}}\Psi_{+2} + \frac{1}{\sqrt{2}}\Psi_{+3}$, which is a linear combination of four orthonormal eigenstates of the operator $\hat{Q}$ corresponding to eigenvalues -1, $+1$, $+2$, and $+3$. Calculate the expectation value of the operator $\hat{Q}$ for this state.

   (d) Considering the derivative operator, $\hat{Q} = \frac{d}{dx}$, find: $\left(\sin\hat{Q}\right)x^5$.

2. **(4 points) Dirac notation: brakets and dual basis**

   Consider a 3D vector space spanned by an orthonormal basis $|1\rangle$, $|2\rangle$, $|3\rangle$. In this basis, let the $|\Psi_0\rangle$ and $|\Psi_1\rangle$ kets be:

   $|\Psi_0\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{i}{2}|2\rangle + \frac{1}{2}|3\rangle$

   $|\Psi_1\rangle = \frac{1}{\sqrt{3}}|1\rangle + \frac{i}{\sqrt{3}}|3\rangle$

   (a) Are these kets normalised? If not, normalise them.

   (b) Write $\langle\Psi_0|$ and $\langle\Psi_1|$ in terms of the dual basis $\langle 1|$, $\langle 2|$, $\langle 3|$.

   (c) Find $\langle\Psi_0|\Psi_1\rangle$ and $\langle\Psi_1|\Psi_0\rangle$, and confirm that $\langle\Psi_1|\Psi_0\rangle = \langle\Psi_0|\Psi_1\rangle^*$.

   (d) Find all the matrix elements of the operators $\hat{M}_{01} = |\Psi_0\rangle\langle\Psi_1|$, $\hat{M}_{00} = |\Psi_0\rangle\langle\Psi_0|$, and $\hat{M}_{11} = |\Psi_1\rangle\langle\Psi_1|$ in this basis, and construct their respective matrices, are they hermitian?

3. **(3 points) Wave function formalism**

Consider a quantum particle in the ground state of the harmonic oscillator.
(a) Compute its momentum-space wave function, $\Phi(p,t)$.

(b) Calculate the probability that a measurement of momentum, p, returns a value outside the classical range for the same energy, E.

4. **(4 points) Spherical harmonics**

(a) A quantum system is known to be in the (unnormalised) state described by the wave function $\psi(\theta,\phi) = 5Y_4^3 + Y_6^3 - 2Y_6^0$, where the $Y_\ell^m(\theta,\phi)$ are the spherical harmonics. What is the probability of finding the system in a state with quantum number $m = 3$?

(b) Construct all the possible spherical harmonics, $Y_\ell^m(\theta,\phi)$, for $\ell = 2$.

(c) Using your favourite programming language, make 3D plots of all of them.

(d) Choose two of the spherical harmonics constructed in part (b), and prove that they are normalised and orthogonal.

5. **(5 points) Hydrogen atom**

(a) Construct all the possible spatial wave functions, $\psi_{nlm}(r,\theta,\phi) = R_{n\ell}(r)Y_\ell^m(\theta,\phi)$, of the hydrogen atom for $(n,\ell,m) = (3,2,m)$.

(b) Using your favourite programming language, make density plots of all of these states.

(c) Calculate the energy level of these states in units of eV.

(d) In terms of the Bohr radius, find $\langle r\rangle$, $\langle x\rangle$, $\langle r^2\rangle$ and $\langle x^2\rangle$ for an electron in the ground state of hydrogen.

(e) Find $\langle x^2\rangle$ in the state $(n,\ell,m) = (3,2,m)$ with the lowest possible value of m that is allowed. How different is the result with respect to that calculated in part (d) for the ground state of hydrogen?