Homework 3 - Quantum Mechanics I

NAME/S: ____ **SCORE:** _________

Deadline: Friday 20th January 2023 by 08:00am (submission only on paper)
Credits: 20 points $\rightarrow$ 20 credits **Number of problems:** 4
Type of evaluation: Formative Evaluation

- This homework consists of problems related to the concepts reviewed in class about quantum mechanics and solutions to Schrödinger's equation for different potentials.

- You may submit this assignment either individually or in pairs. If you work in pairs, only 1 copy is needed. Submitted assignments should have maximum two authors.

- Unless stated otherwise, write your answers in SI units, and consider all bolded quantities as vector quantities. Please also highlight the answers.

1. **(5 points) Free particles: Gaussian wave packets**
We studied free particles in class and showed that they are represented by wave packets. Consider the case of a free particle whose initial wave function is given by:

$$\Psi(x,0) = \alpha\, e^{-\beta\, x^2}, \tag{1}$$

where α and β are real and positive constants.

(a) Find α by normalising the initial wave function, $\Psi(x,0)$.
(b) Find $\Psi(x,t)$. Hint: compute $\phi(k)$ via Fourier analysis first, and then plug it into the wave packet function.
(c) Find $|\Psi(x,t)|^2$. Then, plug some fiducial numbers, and sketch $|\Psi(x,t)|^2$ versus x for $t=0$ and two later times. Qualitatively, what happens to $|\Psi(x,t)|^2$ as time progresses?
(d) Calculate $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$, σ_x, and σ_p.
(e) Does the uncertainty principle hold? At what time t does the system come closest to the uncertainty limit?

2. **(4 points) Infinite square well potential and expectation values**
In class we solved the Schrödinger equation for an infinite square well potential of width a. Such potential allows for bound solutions only as the particle cannot escape from the well. The solutions we found had the following functional form:

$$\psi_n(x) = \sqrt{\frac{2}{a}}\,\sin\left(\frac{n\,\pi}{a}\,x\right) \tag{2}$$

For the n-th state given by the function above, calculate:

(a) The expectation values associated with the position x: $\langle x\rangle$, $\langle x^2\rangle$.
(b) The expectation values associated with the momentum p: $\langle p\rangle$, $\langle p^2\rangle$.
(c) The dispersions σ_x and σ_p, and their product $\sigma_x\,\sigma_p$.
(d) Use programming tools to make a plot of $(\sigma_x\,\sigma_p)$ vs. n.
(e) Is the uncertainty principle satisfied? Which of the $\psi_n(x)$ states comes closest to the uncertainty limit?

3. **(5 points) Finite square well potential**

   In class we studied the finite square well potential and found that this potential admits both scattering states (when $E > 0$) and bound states (when $E < 0$). For the latter, we derived the even solutions and numerically solved a transcendental equation for the allowed energies.

   (a) Following the same approach we followed in class, find the odd bound state wave functions, $\psi(x)$, for the finite square well.
   (b) Derive the transcendental equation for the allowed energies of these odd bound states.
   (c) Solve it graphically and numerically (using your favourite programming tool).
   (d) Study and discuss the two limiting cases and how the energy levels compare to those found for the even bound state wave functions studied in class. Is there always an odd bound state?
   (e) Normalise the even and odd bound state wave functions.

4. **(5 points) Square potential barrier: transmission and reflection coefficients**
   Let us consider a time-independent square potential barrier, $V(x)$, given by the following piecewise function:

$$V(x) = \begin{cases} 0, & x < 0 \\ V_0, & 0 \leq x \leq L \\ 0, & x > L, \end{cases}$$

   (a) Sketch $V(x)$, labelling the three regions of interest as I, II, and III.
   (b) Find the stationary states that describe particles arriving from $x = -\infty$ with energy $E > V_0$, and sketch the solutions using programming tools.
   (c) Analyse the boundary conditions to compute the transmission and reflection coefficients. Sketch these coefficients versus the barrier width, L, and briefly discuss the results.
   (d) Find the stationary states that describe particles arriving from $x = -\infty$ with energy $E < V_0$, and sketch the solutions using programming tools.
   (e) Compute the transmission coefficient, and discuss how this quantum result differs with respect to classical expectations.