Homework 4 - Quantum Mechanics I

NAME: ____ **SCORE:** ____________

Deadline: Thursday 27th July 2023 by 10:00am (submission only on paper)
Credits: 20 points **Number of problems:** 5
Type of evaluation: Formative Evaluation

- This homework includes problems on units 3 and 4 of the QM course.
- This assignment should be submitted individually by the deadline.
- Please justify all calculations and highlight your answers.

1. **(3 points) Formalism**

   Consider a quantum particle in the ground state of the harmonic oscillator.
   (a) Compute its momentum-space wave function, $\Phi(p, t)$.

   (b) Calculate the probability that a measurement of momentum, p, returns a value outside the classical range for the same energy, E.

2. **(4 points) Angular momentum**

   (a) Let L_x, L_y, and L_z be the components of the orbital angular momentum operator $\vec{\boldsymbol{L}}$. Calculate the value of the commutator: $[L_x L_y, L_z]$.

   (b) Suppose we have a spin-$\frac{1}{2}$ particle, whose quantum state can be represented by the eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ of the S_z operator. Using the Pauli matrix σ_x, calculate a normalised eigenstate of the S_x operator with an eigenvalue $-\frac{\hbar}{2}$.

   (c) Calculate the normalised spin eigenfunctions of a system with two spin-$\frac{1}{2}$ particles. These are called singlet and triplet states, why?

   (d) Calculate the orbital angular momentum eigenfunction $Y_\ell^m(\theta, \phi)$ in a quantum state for which the operator $\boldsymbol{L}^2$ has an eigenvalue $6\hbar^2$ and the operator $\boldsymbol{L}_z$ has an eigenvalue $-\hbar$.

3. **(4 points) Spin**

   Consider an electron in the spin state:
   $$\chi = C \begin{bmatrix} 1 - 3i \\ 2 \end{bmatrix}$$
   (a) Determine the normalisation constant C.

   (b) Find the expectation values of S_x, S_y, and S_z.

   (c) Find the uncertainties σ_{S_x}, σ_{S_y}, and σ_{S_z}.

   (d) Are the results consistent with all three uncertainty principles?

4. **(4 points) Spherical harmonics**

(a) A quantum system is known to be in the (unnormalised) state described by the wave function $\psi(\theta, \phi) = 5Y_4^3 + Y_6^3 - 2Y_6^0$, where the $Y_\ell^m(\theta, \phi)$ are the spherical harmonics. What is the probability of finding the system in a state with quantum number $m = 3$?

(b) Construct all the possible spherical harmonics, $Y_\ell^m(\theta, \phi)$, for $\ell = 3$.

(c) Using your favourite programming language, make 3D plots of all of them.

(d) Choose two of the spherical harmonics constructed in part (b), and prove that they are normalised and orthogonal.

5. **(5 points) Hydrogen atom**

(a) Construct all the possible spatial wave functions, $\psi_{nlm}(r, \theta, \phi) = R_{n\ell}(r)Y_\ell^m(\theta, \phi)$, of the hydrogen atom for $(n, \ell, m) = (4, 2, m)$.

(b) Using your favourite programming language, make density plots of all of these states.

(c) Calculate the energy level of these states in units of eV.

(d) In terms of the Bohr radius, find $\langle r \rangle$, $\langle x \rangle$, $\langle r^2 \rangle$ and $\langle x^2 \rangle$ for an electron in the ground state of hydrogen.

(e) Find $\langle x^2 \rangle$ in the state $(n, \ell, m) = (4, 2, m)$ with the highest possible value of m that is allowed. How different is the result with respect to that calculated in part (d) for the ground state of hydrogen?