Quiz 3

NAME: ____ SCORE:

Subject: Quantum Mechanics I
Date: Monday 5 September 2022
Duration: 60 minutes
Credits: 20 points, Number of questions: 10
Type of evaluation: Laboratory (LAB)

Part A. **Choose the correct answer to each question or statement given below, and briefly justify your choice in the white space assigned to each of them.**

1. **(2 points) Hermitian operators**
   Which of the following statements is correct?

   A. Hermitian operators can have complex eigenvalues.
   B. All Hermitian operators represent observables.
   C. A Hermitian operator must be real.
   D. All observables can be represented by a Hermitian operator.

   Because their expectation values $\in \mathbb{R}$ and can be measured.

2. **(2 points) Eigenfunctions and eigenvalues**
   Which of the following pairs represent eigenfunctions and corresponding eigenvalues of the differential operator $\frac{d}{dx}$?

   A. e^{x^2} and x^2 ✗ $\rightarrow \frac{d}{dx} e^{x^2} = 2x e^{x^2}$
   B. e^{x^2} and $2x$ ✗ $\rightarrow$ this is a function, not a number
   C. $\sin ax$ and a ✗
   D. e^{ix} and i ✓

   $\frac{d}{dx} \sin(ax) = ax \cos ax$

   $\frac{d}{dx} e^{ix} = i e^{ix}$ ✓

3. **(2 points) Generalised uncertainty principle**
   If two operators, A and B, have a commutator equal to $3x + 4ix$, the uncertainty relation between their corresponding observables will be:

   A. $\sigma_A \sigma_B \geq 5x$
   B. $\sigma_A \sigma_B \geq 7x$
   C. $\sigma_A \sigma_B \geq 3x + 4ix$
   D. $\sigma_A \sigma_B \geq \frac{5x}{2}$

   $$[\hat{A}, \hat{B}] = 3x + 4ix$$
   $$\sigma_A^2 \sigma_B^2 \geqslant \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle\right)^2$$
   $$\Rightarrow \sigma_A^2 \sigma_B^2 \geqslant \left(\frac{1}{2i} (3x + 4ix)\right)^2$$
   $$\Rightarrow \sigma_A^2 \sigma_B^2 \geqslant \left(\frac{9x^2 + 16x^2}{4}\right) = \frac{25}{4} x^2$$
   $$\Rightarrow \sigma_A \sigma_B \geqslant \frac{5}{2} x$$

4. **(2 points) Schrödinger equation**
   What is the correct Dirac formulation for the Schrödinger Equation?

   A. $\hat{H}|\Psi\rangle = i\hbar \frac{\partial |\Psi\rangle}{\partial t}$
   B. $\hat{H}|\Psi\rangle = \frac{d|\Psi\rangle}{dt}$
   C. $-\frac{\hbar^2}{2m} \hat{H}|\Psi\rangle = i\hbar \frac{d|\Psi\rangle}{dt}$
   D. $\hat{H}|\Psi\rangle = i\hbar |\frac{d\Psi}{dt}\rangle$

   $$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$
   In Dirac notation: $\Psi \rightarrow |\Psi\rangle$
   $$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V \Rightarrow i\hbar \frac{\partial |\Psi\rangle}{\partial t} = \hat{H}|\Psi\rangle$$

5. **(2 points) The hydrogen atom**
   The energy required to knock out the electron in the third orbit of a hydrogen atom is equal to:

   A. $+13.6\,\text{eV}$
   B. $+\frac{13.6}{9}\,\text{eV}$
   C. $-13.6\,\text{eV}$
   D. $+\frac{13.6}{3}\,\text{eV}$

   Bohr formula: $E_n = \frac{E_1}{n^2}$; $E_1 = -13.6\,\text{e-V}$

   when $n = 3$: $E_3 = -\frac{13.6}{9}\,\text{e-V}$

   To remove the e^- we need $-E_3$.

6. **(2 points) Spectrum of hydrogen**
   Which of the following statements is true?

   A. The Lyman series is a continuous spectrum. ✗ we have discrete spectra
   B. The Paschen series is a discrete spectrum in the infrared. ✓ transitions to $n_f = 3$
   C. The Balmer series is a discrete spectrum in the ultraviolet. ✗ → optical
   D. The Lyman series is a discrete spectrum in the optical. ✗ → ultraviolet

7. **(2 points) Spectrum of hydrogen**
   An electron jumps from the 4th orbit to the 2nd orbit of the hydrogen atom. The frequency in Hz of the emitted radiation will be: (Recall that: $\mathcal{R} = 10^7\,\mathrm{m}^{-1}$, $c = 3 \times 10^8\,\mathrm{m\,s}^{-1}$)

   A. $\frac{3}{16} \times 10^5$
   B. $\frac{3}{16} \times 10^{15}$
   C. $\frac{9}{16} \times 10^{15}$
   D. $\frac{3}{4} \times 10^{15}$

   Rydberg formula:

   $$\frac{1}{\lambda} = R\left(\frac{1}{n_f^2} - \frac{1}{n_0^2}\right) = 10^7 m^{-1}\left(\frac{1}{2^2} - \frac{1}{4^2}\right) = \frac{\nu}{c}$$

   $$\Rightarrow \nu = \frac{3}{16} \times 10^{15}\ \mathrm{Hz}$$

8. **(2 points) Harmonic oscillator eigenstates**
   Let $|n\rangle$ represent the normalised nth energy eigenstate of the one dimensional harmonic oscillator, $H|n\rangle = \hbar\omega(n + \frac{1}{2})|n\rangle$. If $|\Psi\rangle$ is a normalised ensemble state that can be expanded as a linear combination, $|\Psi\rangle = \frac{1}{\sqrt{14}}|1\rangle - \frac{2}{\sqrt{14}}|2\rangle + \frac{3}{\sqrt{14}}|3\rangle$ of the eigenstates, what is the expectation value of the energy operator in this ensemble state?

   A. $\frac{102}{14}\hbar\omega$
   B. $\frac{43}{14}\hbar\omega$
   C. $\frac{23}{14}\hbar\omega$
   D. $\frac{17}{\sqrt{14}}\hbar\omega$

   $$\langle\Psi|H|\Psi\rangle = \frac{1}{14}\langle 1|H|1\rangle + \frac{4}{14}\langle 2|H|2\rangle + \frac{9}{14}\langle 3|H|3\rangle$$

   $$\langle\Psi|H|\Psi\rangle = \frac{1}{14}\left(\frac{3}{2}\hbar\omega\right) + \frac{4}{14}\left(\frac{5}{2}\hbar\omega\right) + \frac{9}{14}\left(\frac{7}{2}\hbar\omega\right)$$

   $$\Rightarrow \langle\Psi|H|\Psi\rangle = \frac{86}{28}\hbar\omega \Rightarrow \langle\Psi|H\Psi\rangle = \frac{43}{14}\hbar\omega$$

   $$\uparrow$$

   $$\langle 1|1\rangle = \langle 2|2\rangle = \langle 3|3\rangle = 1$$

Part B. Provide concise answers to the following items:

9. **(2 points) Hilbert space**
   (a) Briefly explain: why should wave functions belong to the Hilbert space?
   Wave functions should be square integrable, so that they can be normalised (i.e. they represent real particles) and have inner products defined.
   (b) Write down the mathematical condition that defines Hermitian operators.
   A Hermitian operator can be applied to either term of an inner product:
   $\langle f|\hat{Q}g\rangle = \langle\hat{Q}f|g\rangle$ for all $f(x) \wedge g(x)$

10. **(2 points) Solutions for the hydrogen atom**
    Briefly explain: (a) the terms of the potential of the hydrogen atom,
    $V_{eff} = V + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}$, where the first term is the actual potential and the second term is the centrifugal potential associated with a pseudo-force.
    (b) how we solved the Schrödinger equation, and
    We used separation of variables: $\Psi = \psi_{n\ell m}(r,\theta,\phi)\varphi(t)$.
    (c) what solutions we found.
    $\varphi(t) = e^{-i\frac{E_n}{\hbar}t}$ is the wiggle factor
    $\psi_{n\ell m}(r,\theta,\phi) = R_{n\ell}(r)Y_\ell^m(\theta,\phi)$ → angular wavefunction associated with spherical harmonics
    ↓
    radial wavefunction associated with Laguerre polynomials