The hydrogen atom

Electron in a Magnetic Field

A spinning charged particle constitutes a magnetic dipole. Its **magnetic dipole moment**, μ, is proportional to its spin angular momentum, **S.**

$$\boldsymbol{\mu} = \gamma \mathbf{S}$$

The proportionality constant, γ , is called the **gyromagnetic ratio**.

The gyromagnetic ratio of an object whose charge and mass are identically distributed is ***q*/2*m***, where ***q*** is the charge and ***m*** is the mass.

For reasons that are fully explained only in relativistic quantum theory, the gyromagnetic ratio of the electron is (almost) exactly *twice* the classical value: **γ = –*e*/*m*.**

When a magnetic dipole is placed in a magnetic field **B**, it experiences a torque, **μ × B**, which tends to line it up parallel to the field (just like a compass needle).

The hydrogen atom

Electron in a Magnetic Field

The energy associated with this torque is:

$$H = -\boldsymbol{\mu} \cdot \mathbf{B},$$

so the Hamiltonian matrix for a spinning charged particle, at rest in a magnetic field **B**, is:

$$\mathsf{H} = -\gamma \mathbf{B} \cdot \mathsf{S},$$

where S is the appropriate spin matrix:

$$\mathsf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathsf{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

ELECTROMAGNETIC INTERACTIONS

Minimal Coupling

In classical electrodynamics the force on a particle of charge *q* moving with velocity **v** through electric and magnetic fields **E** and **B** is given by the **Lorentz force law**:

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

This force cannot be expressed as the gradient of a scalar potential energy function, and therefore the Schrödinger equation in its original form cannot accommodate it. But in the more sophisticated form there is no problem:

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$$

The classical Hamiltonian for a particle of charge *q* and momentum **p**, in the presence of electromagnetic fields is

$$H = \frac{1}{2m}\left(\mathbf{p} - q\mathbf{A}\right)^2 + q\varphi,$$

where **A** is the vector potential and ϕ is the scalar potential:

$$\mathbf{E} = -\nabla\varphi - \partial\mathbf{A}/\partial t, \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

ELECTROMAGNETIC INTERACTIONS

Minimal Coupling

Making the standard substitution **p** → $-i\hbar\nabla$, we obtain the Hamiltonian operator

$$\hat{H} = \frac{1}{2m}\left(-i\hbar\boldsymbol{\nabla} - q\mathbf{A}\right)^2 + q\varphi,$$

and the Schrödinger equation becomes:

$$\boxed{i\hbar\frac{\partial\Psi}{\partial t} = \left[\frac{1}{2m}\left(-i\hbar\boldsymbol{\nabla} - q\mathbf{A}\right)^2 + q\varphi\right]\Psi.}$$

This is the quantum implementation of the Lorentz force law; it is sometimes called the **minimal coupling rule.**