IDENTICAL PARTICLES

Two-particle systems:

For a *single* particle, $\Psi(\mathbf{r},t)$ is a function of the spatial coordinates, **r**, and the time, *t* (we'll ignore spin, for the moment).

The state of a *two*-particle system is a function of the coordinates of particle one ($\mathbf{r}_1$), the coordinates of particle two ($\mathbf{r}_2$), and the time:

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, t).$$

Its time evolution is determined by the Schrödinger equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi,$$

where *H* is the Hamiltonian:

$$\hat{H} = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(\mathbf{r}_1, \mathbf{r}_2, t)$$

the subscript on ∇ indicates differentiation with respect to the coordinates of particle 1 or particle 2, as the case may be.

IDENTICAL PARTICLES

Two-particle systems:

The statistical interpretation indicates: the probability of finding particle 1 in the volume $d^3\mathbf{r}_1$ *and* particle 2 in the volume $d^3\mathbf{r}_2$

$$|\Psi(\mathbf{r}_1, \mathbf{r}_2, t)|^2 \, d^3\mathbf{r}_1 \, d^3\mathbf{r}_2$$

Ψ must be normalised:

$$\int |\Psi(\mathbf{r}_1, \mathbf{r}_2, t)|^2 \, d^3\mathbf{r}_1 \, d^3\mathbf{r}_2 = 1.$$

For time-independent potentials, we obtain a complete set of solutions by separation of variables:

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, t) = \psi(\mathbf{r}_1, \mathbf{r}_2) e^{-iEt/\hbar},$$

where the spatial wave function (ψ) satisfies the time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m_1}\nabla_1^2\psi - \frac{\hbar^2}{2m_2}\nabla_2^2\psi + V\psi = E\psi,$$

where E is the total energy of the system.

IDENTICAL PARTICLES

Two-particle systems:

Solving this is difficult, but two special cases can be reduced to one-particle problems:

1. Noninteracting particles: Suppose the particles do not interact with one another, but each is subject to some external force. For example, they might be attached to two different springs. In that case the total potential energy is the *sum* of the two:

$$V(\mathbf{r}_1, \mathbf{r}_2) = V_1(\mathbf{r}_1) + V_2(\mathbf{r}_2),$$

Using separation of variables: $$\psi(\mathbf{r}_1, \mathbf{r}_2) = \psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2).$$

we find that $\psi_a(\mathbf{r}_1)$ and $\psi_b(\mathbf{r}_2)$ each satisfy the one-particle Schrödinger equation:

$$-\frac{\hbar^2}{2m_1}\nabla_1^2\psi_a(\mathbf{r}_1) + V_1(\mathbf{r}_1)\psi_a(\mathbf{r}_1) = E_a\psi_a(\mathbf{r}_1),$$
$$-\frac{\hbar^2}{2m_2}\nabla_2^2\psi_b(\mathbf{r}_2) + V_2(\mathbf{r}_2)\psi_b(\mathbf{r}_2) = E_b\psi_b(\mathbf{r}_2),$$

and $E = E_a + E_b$.

IDENTICAL PARTICLES

Two-particle systems:

In this case the two-particle wave function is a simple *product* of one-particle wave functions:

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, t) = \psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2)e^{-i(E_a+E_b)t/\hbar}$$
$$= \left(\psi_a(\mathbf{r}_1)e^{-iE_at/\hbar}\right)\left(\psi_b(\mathbf{r}_2)e^{-iE_bt/\hbar}\right) = \Psi_a(\mathbf{r}_1, t)\Psi_b(\mathbf{r}_2, t),$$

It makes sense to say that particle 1 is in state *a*, and particle 2 is in state *b*.

Any linear combination of such solutions will still satisfy the (time-dependent) Schrödinger:

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, t) = \frac{3}{5}\Psi_a(\mathbf{r}_1, t)\,\Psi_b(\mathbf{r}_2, t) + \frac{4}{5}\Psi_c(\mathbf{r}_1, t)\,\Psi_d(\mathbf{r}_2, t).$$

The state of particle 1 depends on that of particle 2, and vice versa. If we measured the energy of particle 1, we might get E_a (with probability 9/25), then the energy of particle 2 is definitely E_b, or you might get E_c (probability 16/25), in which case the energy of particle 2 is E_d.

We say that the two particles are **entangled** (Schrödinger's lovely term). An entangled state is one that *cannot* be written as a product of single-particle states.

IDENTICAL PARTICLES

Two-particle systems:

2. Central potentials: Suppose the particles interact *only* with one another, via a potential that depends on their separation:

$$V(\mathbf{r}_1, \mathbf{r}_2) \to V(|\mathbf{r}_1 - \mathbf{r}_2|).$$

The hydrogen atom would be an example, if you include the motion of the proton. In this case the two-body problem reduces to an equivalent one-body problem.

In general, though, the two particles will be subject both to external forces *and* to mutual interactions, and this makes the analysis more complicated.

Example: potential of the two electrons in a helium atom: each feels the Coulomb attraction of the nucleus (charge 2*e*), and at the same time they repel one another:

$$V(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{4\pi\epsilon_0}\left(-\frac{2e^2}{|\mathbf{r}_1|} - \frac{2e^2}{|\mathbf{r}_2|} + \frac{e^2}{|\mathbf{r}_1 - \mathbf{r}_2|}\right).$$