The hydrogen atom

Angular momentum: Ladder operator technique

Let: $L_\pm \equiv L_x \pm i L_y.$

Its commutator with L_z is:

$$\left[L_z, L_\pm\right] = \left[L_z, L_x\right] \pm i \left[L_z, L_y\right] = i\hbar L_y \pm i \left(-i\hbar L_x\right) = \pm\hbar \left(L_x \pm i L_y\right),$$

$$\Rightarrow \quad \left[L_z, L_\pm\right] = \pm\hbar L_\pm.$$

Remember: $\left[L^2, \mathbf{L}\right] = 0. \quad \Rightarrow \quad \left[L^2, L_\pm\right] = 0.$

Then, f is a common eigenfunction:

$$\left[L^2, L_\pm\right] = 0. \quad \Rightarrow \quad L^2 \left(L_\pm f\right) = L_\pm\left(L^2 f\right) = L_\pm \left(\lambda f\right) = \lambda \left(L_\pm f\right),$$

Therefore, $L_\pm f$ is also an eigenfunction of L^2 with the same eigenvalue λ.

The hydrogen atom

Angular momentum: Ladder operator technique

$L_{\pm}f$ is also an eigenfunction of L^2 with the same eigenvalue λ.

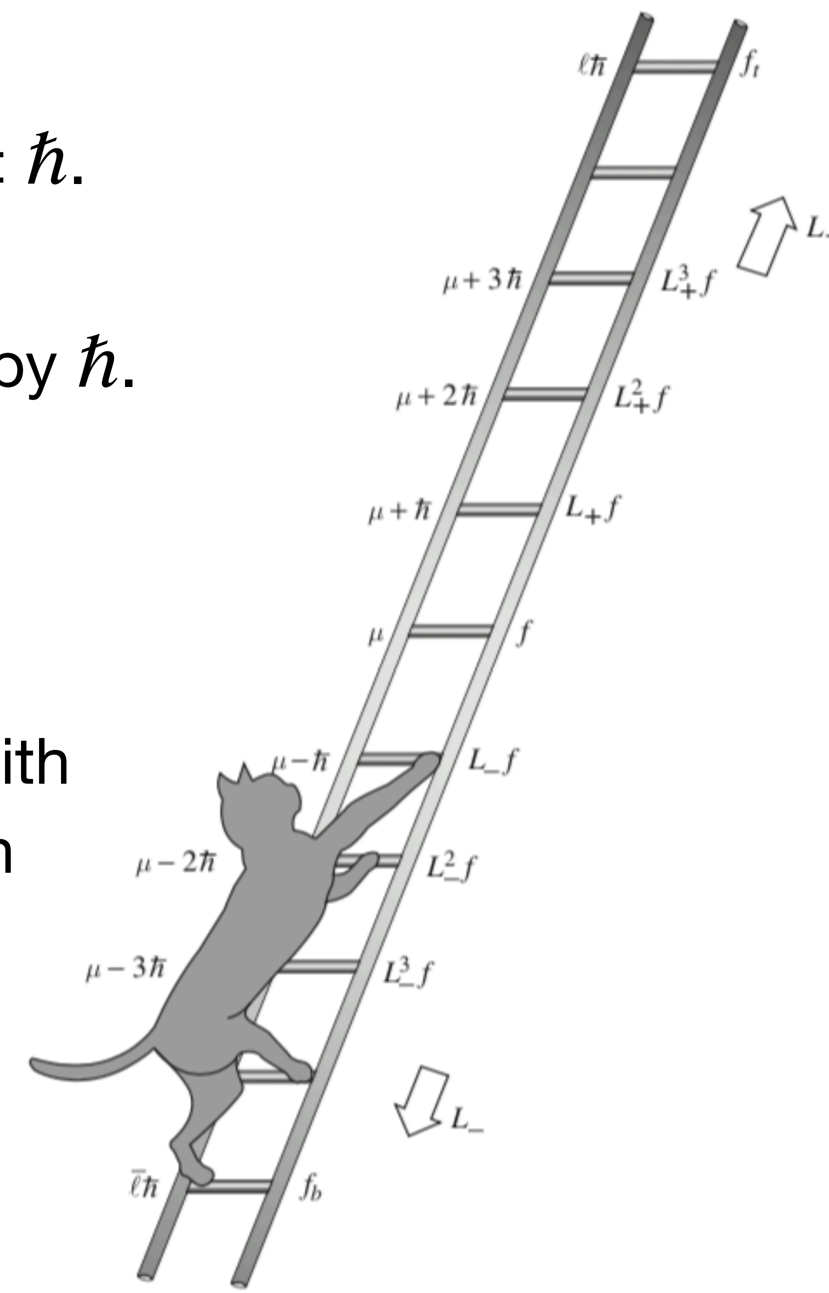
$$[L_z, L_{\pm}] = \pm\hbar L_{\pm} \quad \Rightarrow \quad \begin{aligned} L_z(L_{\pm}f) &= (L_zL_{\pm} - L_{\pm}L_z)f + L_{\pm}L_zf = \pm\hbar L_{\pm}f + L_{\pm}(\mu f) \\ &= (\mu \pm \hbar)(L_{\pm}f), \end{aligned}$$

so $L_{\pm}f$ is an eigenfunction of *Lz* with the *new* eigenvalue $\mu \pm \hbar$.

L_+ is the **raising operator**: it *increases* the eigenvalue of L_Z by $\hbar$.

L_- is the **lowering operator**: it *lowers* the eigenvalue by $\hbar$.

For a given value of λ, then, we obtain a "ladder" of states, with each "rung" separated from its neighbours by one unit of $\hbar$ in the eigenvalue of *Lz.*



The hydrogen atom

Angular momentum: Ladder operator technique

There must exist a "top rung", *ft* , such that: $L_+ f_t = 0.$

Let $\hbar\ell$ be the eigenvalue of L_z at the top rung: $L_z f_t = \hbar\ell f_t; \quad L^2 f_t = \lambda f_t.$

Now,

$$L_\pm L_\mp = (L_x \pm i L_y)(L_x \mp i L_y) = L_x^2 + L_y^2 \mp i (L_x L_y - L_y L_x)$$
$$= L^2 - L_z^2 \mp i (i\hbar L_z),$$

We have:

$$L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z.$$

Thus:

$$L^2 f_t = \left(L_- L_+ + L_z^2 + \hbar L_z\right) f_t = \left(0 + \hbar^2\ell^2 + \hbar^2\ell\right) f_t = \hbar^2 \ell (\ell + 1) f_t,$$

$$\lambda = \hbar^2\ell(\ell+1).$$

This tells us the eigenvalue of L^2 in terms of the *maximum* eigenvalue of L_z.

The hydrogen atom

Angular momentum: Ladder operator technique

There must also exist a "bottom rung", *fb* , such that: $L_- f_b = 0.$

Let $\hbar\bar{\ell}$ be the eigenvalue of L_z at the bottom rung: $L_z f_b = \hbar\bar{\ell} f_b; \quad L^2 f_b = \lambda f_b.$

Remember: $L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z.$

$$\Rightarrow \quad L^2 f_b = \left(L_+ L_- + L_z^2 - \hbar L_z\right) f_b = \left(0 + \hbar^2\bar{\ell}^2 - \hbar^2\bar{\ell}\right) f_b = \hbar^2\bar{\ell}(\bar{\ell} - 1) f_b,$$

$$\Rightarrow \quad \lambda = \hbar^2\bar{\ell}\,(\bar{\ell} - 1)$$

Comparing with: $\lambda = \hbar^2\ell\,(\ell + 1)$

$$\ell\,(\ell + 1) = \bar{\ell}\,(\bar{\ell} - 1) \quad \Rightarrow \quad \begin{array}{ll} \bar{\ell} = \ell + 1 & ✗ \\ \bar{\ell} = -\ell. & ✓ \end{array}$$

So the eigenvalues of *Lz* are *m*$\boxed{?}$, where *m* goes from $-\ell$ to $+\ell$, in *N* integer steps.

It follows that $\ell = -\ell + N$, and hence $\ell = N/2$, so l must be an integer or a half-integer.

The hydrogen atom

Angular momentum: Eigenvalues

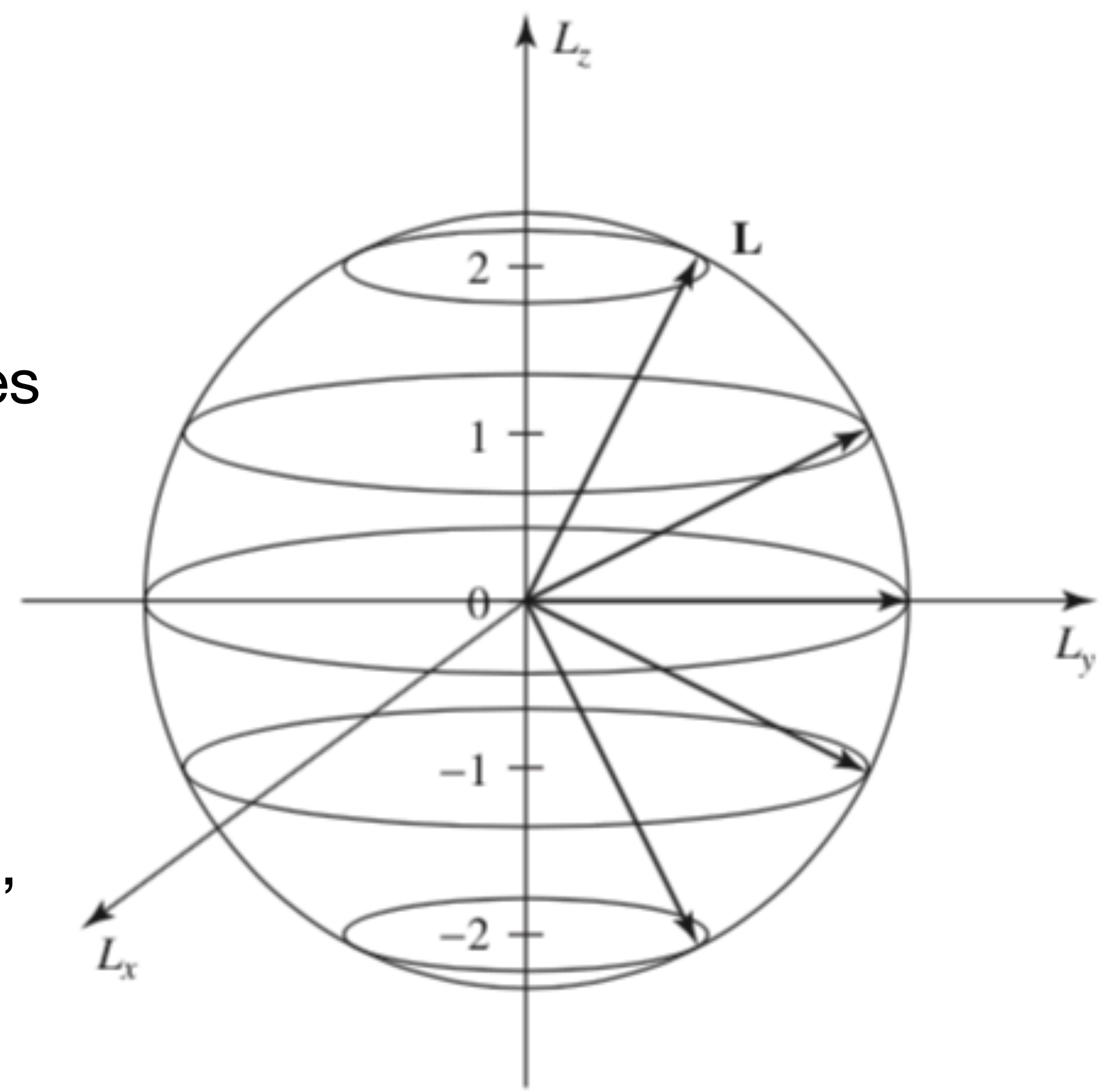
The eigenfunctions are characterised by the numbers ℓ and m:

$$\boxed{L^2 f_\ell^m = \hbar^2 \ell(\ell+1) f_\ell^m; \quad L_z f_\ell^m = \hbar m f_\ell^m,}$$

where: $\ell = 0,\ 1/2,\ 1,\ 3/2, \ldots;$

$$m = -\ell,\ -\ell+1,\ \ldots,\ \ell-1,\ \ell.$$

For a given value of ℓ, there are $2\ell+1$ different values of m (i.e. $2\ell+1$ "rungs" on the "ladder").



Angular momentum states (for $\ell = 2$).

Arrows are possible angular momenta (in units of $\hbar$), they all have the same length.

Their z components are the allowed values of m (-2,-1, 0, 1, 2).

The hydrogen atom

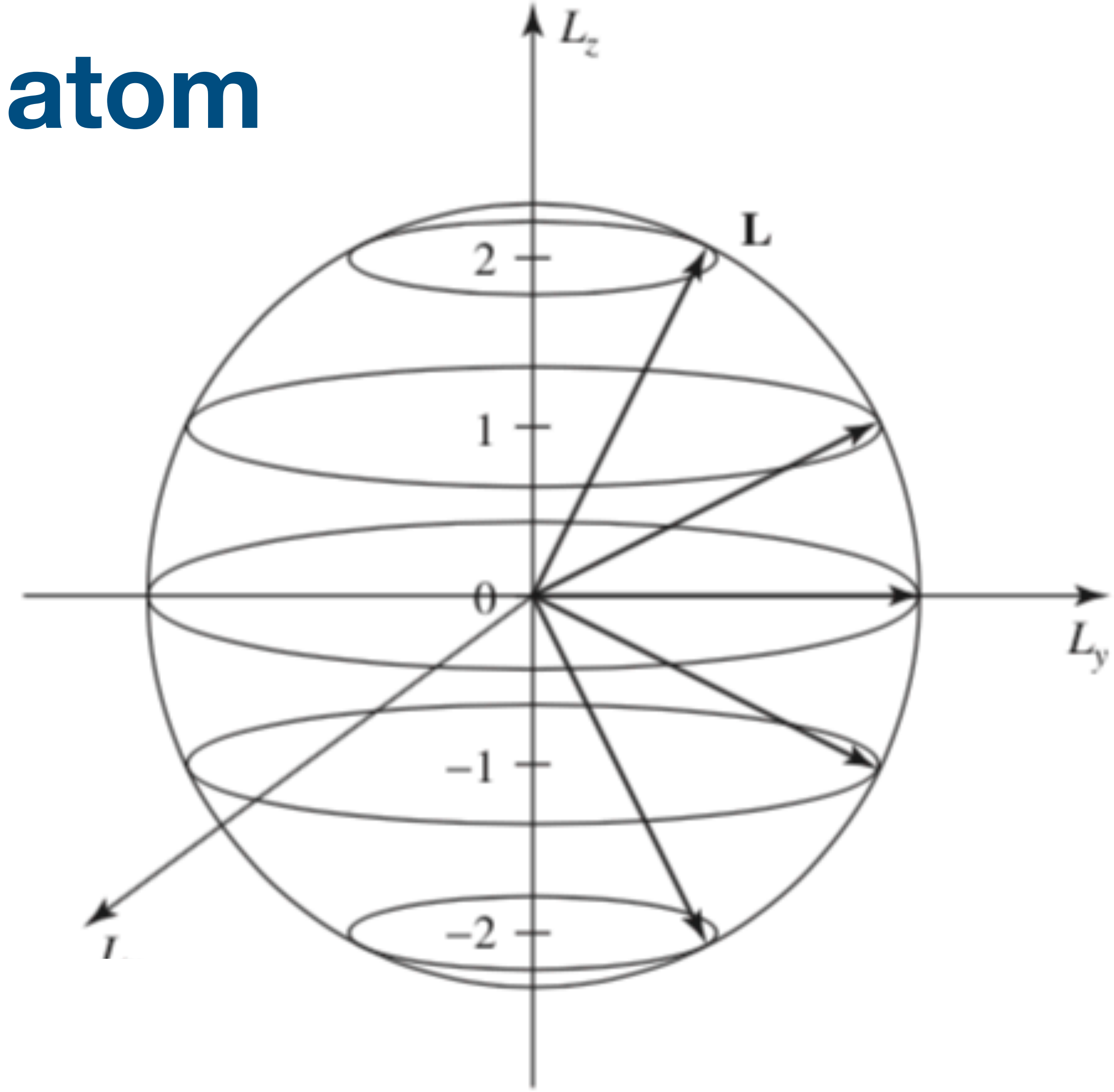
Angular momentum: Eigenvalues

Arrows are possible angular momenta (in units of $\hbar$), they all have the same length: $\sqrt{\ell\,(\ell+1)}$

Their *z* components are the allowed values of *m* (-2,-1, 0, 1, 2).

The magnitude of the vectors (the radius of the sphere) is *greater* than the maximum *z* component:

$$\sqrt{\ell\,(\ell+1)} > \ell,$$



ngular momentum states (for $\ell = 2$).

The uncertainty principle implies that we cannot *know* all three components of L.

Actually, there *aren't* three components — a particle simply cannot *have* a determinate angular momentum vector.

If L_z has a well-defined value, then L_x and L_y do *not*.