

The hydrogen atom

Angular momentum:

The principal quantum number (n) determines the energy of the state:

$$E_n = - \left[\frac{m_e}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \right] \frac{1}{n^2} = \frac{E_1}{n^2}, \quad n = 1, 2, 3, \dots$$

ℓ and m are related to the orbital angular momentum:

In the classical theory of central forces, energy and angular momentum are the fundamental conserved quantities. The angular momentum of a particle (with respect to the origin) is given by:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \quad L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x.$$

The corresponding quantum operators are obtained by the standard prescription:

$$p_x \rightarrow -i\hbar\partial/\partial x, \quad p_y \rightarrow -i\hbar\partial/\partial y, \quad p_z \rightarrow -i\hbar\partial/\partial z.$$

We need to obtain the eigenvalues and the eigenfunctions of the angular momentum operators.

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Angular momentum: Eigenvalues

The operators L_x and L_y do not commute:

$$\begin{aligned}[L_x, L_y] &= [yp_z - zp_y, zp_x - xp_z] \\ &= [yp_z, zp_x] - [yp_z, xp_z] - [zp_y, zp_x] + [zp_y, xp_z].\end{aligned}$$

x with p_x , y with p_y , and z with p_z *fail* to commute:

$$[L_x, L_y] = yp_x [p_z, z] + xp_y [z, p_z] = i\hbar (xp_y - yp_x) = i\hbar L_z.$$

We can get the others by cyclic permutation of the indices ($x \rightarrow y$, $y \rightarrow z$, $z \rightarrow x$):

$$[L_x, L_y] = i\hbar L_z; \quad [L_y, L_z] = i\hbar L_x; \quad [L_z, L_x] = i\hbar L_y.$$

These are the **fundamental commutation relations for angular momentum**.

Notice that L_x , L_y , and L_z are *incompatible* observables.

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Angular momentum: Eigenvalues

According to the generalised uncertainty principle:

$$\boxed{\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2} \quad \rightarrow \quad \begin{aligned} \sigma_{L_x}^2 \sigma_{L_y}^2 &\geq \left(\frac{1}{2i} \langle i\hbar L_z \rangle \right)^2 = \frac{\hbar^2}{4} \langle L_z \rangle^2 \\ \sigma_{L_x}^2 \sigma_{L_y}^2 &\geq \left(\frac{1}{2i} \langle i\hbar L_z \rangle \right)^2 = \frac{\hbar^2}{4} \langle L_z \rangle^2 \\ \sigma_{L_x} \sigma_{L_y} &\geq \frac{\hbar}{2} |\langle L_z \rangle|. \end{aligned}$$

There are no states that are simultaneously eigenfunctions of L_x and L_y .

The *square* of the *total* angular momentum: $L^2 \equiv L_x^2 + L_y^2 + L_z^2$,

does commute with L_x :

$$\begin{aligned} [L^2, L_x] &= [L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x] \\ &= L_y [L_y, L_x] + [L_y, L_x] L_y + L_z [L_z, L_x] + [L_z, L_x] L_z \\ &= L_y (-i\hbar L_z) + (-i\hbar L_z) L_y + L_z (i\hbar L_y) + (i\hbar L_y) L_z \\ &= 0. \end{aligned}$$

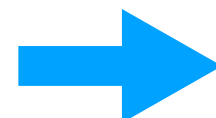
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Angular momentum: Eigenvalues

$$\begin{aligned}[L^2, L_x] &= [L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x] \\&= L_y [L_y, L_x] + [L_y, L_x] L_y + L_z [L_z, L_x] + [L_z, L_x] L_z \\&= L_y (-i\hbar L_z) + (-i\hbar L_z) L_y + L_z (i\hbar L_y) + (i\hbar L_y) L_z \\&= 0.\end{aligned}$$

L^2 also commutes with L_y and L_z :

$$[L^2, L_x] = 0, \quad [L^2, L_y] = 0, \quad [L^2, L_z] = 0,$$


$$[L^2, \mathbf{L}] = 0.$$

We can hope to find simultaneous eigenstates of L^2 and (say) L_z :

$$L^2 f = \lambda f \quad \text{and} \quad L_z f = \mu f.$$