Homework 3 - Quantum Mechanics I

NAME: ____ **SCORE:** _____________

Deadline: Tuesday 31st October 2023 by 13:00
Credits: 20 points **Number of problems:** 4
Type of evaluation: Formative Evaluation

This assignment is individual and consists of 4 problems related to unit 2 of quantum mechanics. Please justify all calculations and highlight the answers.

1. **(5 points) Infinite square well potential and expectation values**

   In class we solved the Schrödinger equation for an infinite square well potential of width L. Such potential allows for bound solutions only as the particle cannot escape from the well. The solutions we found had the following functional form:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\,\sin\left(\frac{n\,\pi}{L}\,x\right) \tag{1}$$

   For the n-th state given by the function above, calculate:

   (a) The expectation values associated with the position x: $\langle x \rangle$, $\langle x^2 \rangle$.
   (b) The expectation values associated with the momentum p: $\langle p \rangle$, $\langle p^2 \rangle$.
   (c) The dispersions σ_x and σ_p, and their product $\sigma_x\,\sigma_p$.
   (d) Use programming tools to make a plot of $(\sigma_x\,\sigma_p)$ vs. n.
   (e) Is the uncertainty principle satisfied? Which of the $\psi_n(x)$ states comes closest to the uncertainty limit?

2. **(5 points) Finite square well potential**

   In class we studied the finite square well potential and found that this potential admits both scattering states (when $E > 0$) and bound states (when $E < 0$). For the latter, we derived the even solutions and numerically solved a transcendental equation for the allowed energies.

   (a) Following the same approach we followed in class, find the odd bound state wave functions, $\psi(x)$, for the finite square well.
   (b) Derive the transcendental equation for the allowed energies of these odd bound states.
   (c) Solve it graphically and numerically (using your favourite programming tool).
   (d) Study and discuss the two limiting cases and how the energy levels compare to those found for the even bound state wave functions studied in class. Is there always an odd bound state?
   (e) Normalise the even and odd bound state wave functions.

3. **(5 points) Probability current**

As we reviewed in class, the wave function of a particle is a complex-valued probability amplitude that depends on position, x, and time, t. As time progresses, the wave function changes and the probability of finding a particle in certain position also changes with it. Since the sum of all probabilities should always be 1, this means that the probability 'flows' from one region to another one, akin to a fluid or a current. This 'flow' can be described mathematically by the so-called probability current j, which for the wave function Ψ of a non-relativistic particle of mass m in 1D is defined as:

$$j(x,t) = \frac{\hbar}{2\,m\,i}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x}\right)$$

(a) What are the units of $j(x,t)$?
(b) Find the probability current, j, of a superposition of 2 currents of particles of mass m, momentum p, and energy $\frac{p^2}{2m}$, moving in opposite directions. The amplitudes of the particle currents are α and β, respectively. Hint: Write the wave function for the superposition first.
(c) If $P_{ab}(t)$ is the probability of finding the particle in the range $(a < x < b)$, at time t, show that:

$$\frac{dP_{ab}}{dt} = j(a,t) - j(b,t)$$

(d) Find the probability current for the wave function, $\Psi(x,t) = C\,e^{-c\left[\frac{m\,x^2}{\hbar} + i\,t\right]}$, of a particle of mass m, where C and c are positive real constants.

4. **(5 points) The time-independent Schrödinger equation**

Consider a current of particles with energies $E > V_0$ moving from $x = -\infty$ to the right, under the influence of a Heaviside potential $V(x)$ given by:

$$V(x) = \begin{cases} V_0, & x \geq 0 \\ 0, & x < 0, \end{cases}$$

where A, n, and x_0 are constants.

(a) Sketch the potential and write down the time-independent Schrödinger equation.
(b) Find the stationary state solutions for each region of interest.
(c) Express the transmitted and reflected amplitudes in terms of the incident amplitude.
(d) Find the probability current, $j(x)$, in each region of interest.
(e) Use the results from part (d) to find and plot (using your favourite programming tool) the reflection and transmission coefficients, and check that $T + R = 1$. Note that the transmission coefficient is not simply $\frac{|F|^2}{|A|^2}$ (with A the incident amplitude and F the transmitted amplitude), because the transmitted wave travels at a different speed.