The hydrogen atom

Angular momentum:

The principal quantum number (*n*) determines the energy of the state:

$$E_n = -\left[\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} = \frac{E_1}{n^2}, \quad n = 1, 2, 3, \ldots.$$

ℓ and *m* are related to the orbital angular momentum:

In the classical theory of central forces, energy and angular momentum are the fundamental conserved quantities. The angular momentum of a particle (with respect to the origin) is given by:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \qquad L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x.$$

The corresponding quantum operators are obtained by the standard prescription:

$$p_x \rightarrow -i\hbar\partial/\partial x, \; p_y \rightarrow -i\hbar\partial/\partial y, \; p_z \rightarrow -i\hbar\partial/\partial z.$$

We need to obtain the eigenvalues and the eigenfunctions of the angular momentum operators.

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Angular momentum: Eigenvalues

The operators L_x and L_y do not commute:

$$\begin{aligned}\left[L_x, L_y\right] &= \left[yp_z - zp_y, zp_x - xp_z\right] \\ &= \left[yp_z, zp_x\right] - \left[yp_z, xp_z\right] - \left[zp_y, zp_x\right] + \left[zp_y, xp_z\right].\end{aligned}$$

x with p_x, y with p_y, and z with p_z *fail* to commute:

$$\left[L_x, L_y\right] = yp_x\left[p_z, z\right] + xp_y\left[z, p_z\right] = i\hbar\left(xp_y - yp_x\right) = i\hbar L_z.$$

We can get the others by cyclic permutation of the indices ($x \rightarrow y$, $y \rightarrow z$, $z \rightarrow x$):

$$\boxed{\left[L_x, L_y\right] = i\hbar L_z; \quad \left[L_y, L_z\right] = i\hbar L_x; \quad \left[L_z, L_x\right] = i\hbar L_y.}$$

These are the **fundamental commutation relations for angular momentum.**

Notice that L_x , L_y , and L_z are *incompatible* observables.

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Angular momentum: Eigenvalues

According to the generalised uncertainty principle:

$$\boxed{\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i}\left\langle\left[\hat{A}, \hat{B}\right]\right\rangle\right)^2 .}$$

$$\Rightarrow$$

$$\sigma_{L_x}^2 \sigma_{L_y}^2 \geq \left(\frac{1}{2i}\left\langle i\hbar L_z\right\rangle\right)^2 = \frac{\hbar^2}{4}\left\langle L_z\right\rangle^2$$

$$\sigma_{L_x}^2 \sigma_{L_y}^2 \geq \left(\frac{1}{2i}\left\langle i\hbar L_z\right\rangle\right)^2 = \frac{\hbar^2}{4}\left\langle L_z\right\rangle^2$$

$$\sigma_{L_x} \sigma_{L_y} \geq \frac{\hbar}{2}\left|\left\langle L_z\right\rangle\right| .$$

There are no states that are simultaneously eigenfunctions of L_x and Ly.

The *square* of the *total* angular momentum: $L^2 \equiv L_x^2 + L_y^2 + L_z^2,$

does commute with L_x:

$$\begin{aligned}
\left[L^2, L_x\right] &= \left[L_x^2, L_x\right] + \left[L_y^2, L_x\right] + \left[L_z^2, L_x\right] \\
&= L_y\left[L_y, L_x\right] + \left[L_y, L_x\right] L_y + L_z\left[L_z, L_x\right] + \left[L_z, L_x\right] L_z \\
&= L_y\left(-i\hbar L_z\right) + \left(-i\hbar L_z\right) L_y + L_z\left(i\hbar L_y\right) + \left(i\hbar L_y\right) L_z \\
&= 0.
\end{aligned}$$

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Angular momentum: Eigenvalues

$$\begin{aligned}\left[L^2, L_x\right] &= \left[L_x^2, L_x\right] + \left[L_y^2, L_x\right] + \left[L_z^2, L_x\right] \\ &= L_y\left[L_y, L_x\right] + \left[L_y, L_x\right]L_y + L_z\left[L_z, L_x\right] + \left[L_z, L_x\right]L_z \\ &= L_y(-i\hbar L_z) + (-i\hbar L_z)L_y + L_z\left(i\hbar L_y\right) + \left(i\hbar L_y\right)L_z \\ &= 0.\end{aligned}$$

L^2 also commutes with L_y and L_z:

$$\left[L^2, L_x\right] = 0, \quad \left[L^2, L_y\right] = 0, \quad \left[L^2, L_z\right] = 0,$$

$$\Rightarrow \quad \left[L^2, \mathbf{L}\right] = 0.$$

We can hope to find simultaneous eigenstates of L^2 and (say) L_z:

$$L^2 f = \lambda f \quad \text{and} \quad L_z f = \mu f.$$