Homework 2 - Quantum Mechanics I

NAME: ____ **SCORE:** _____________

Deadline: Monday 7$^{\text{th}}$ October 2023 by 14:00
Credits: 20 points **Number of problems:** 4
Type of evaluation: Formative Evaluation

- This homework includes problems on units 1 and 2 of the QM course.
- This assignment should be submitted individually by the deadline.
- Please justify all calculations and highlight your answers.

1. **(5 points) Probability densities**
Consider the probability density function (PDF) of the Beta distribution is defined as:

$$\rho(x; \alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}$$

over the domain $x \in [0, 1]$, where $\alpha > 0$ and $\beta > 0$ are shape parameters, $B(\alpha, \beta)$ is the Beta function, which serves as the normalisation constant and is defined as:

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

where Γ refers to Gamma functions.

(a) Find the expectation value $\langle x \rangle$.
(b) Find the expectation value $\langle x^2 \rangle$.
(c) Find σ_x.
(d) If $\rho(x; \alpha, \beta) \equiv |\psi(x)|^2$ for a particle in an infinite box over $x \in [0, 1]$, find the conditions (for α and β) for which $\psi(0) = \psi(1) = 0$. What do the α and β constants represent?
(e) Plug some fiducial numbers for these constants, and sketch the graph of $\rho(x; \alpha, \beta)$ using your favourite programming language.

2. **(5 points) Infinite square well potential and expectation values**
In class we solved the Schrödinger equation for an infinite square well potential of width L. Such potential allows for bound solutions only as the particle cannot escape from the well. The solutions we found had the following functional form:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\, \sin\left(\frac{n\,\pi}{L}\, x\right) \tag{1}$$

For the n-th state given by the function above, calculate:

(a) The expectation values associated with the position x: $\langle x \rangle$, $\langle x^2 \rangle$.
(b) The expectation values associated with the momentum p: $\langle p \rangle$, $\langle p^2 \rangle$.
(c) The dispersions σ_x and σ_p, and their product $\sigma_x\, \sigma_p$.
(d) Use programming tools to make a plot of $(\sigma_x\, \sigma_p)$ vs. n.
(e) Is the uncertainty principle satisfied? Which of the $\psi_n(x)$ states comes closest to the uncertainty limit?

3. **(5 points) Plane waves for matter particles.**
Consider that the plane wave for a matter particle moving in the x-direction with momentum $p = \hbar k$ is:

$$\Psi(x,t) = \cos(kx - \omega t) + \gamma \sin(kx - \omega t),$$

where γ is a constant. A physical requirement is that an arbitrary displacement of x or an arbitrary shift of t should not alter the character of the wave. We therefore demand that after a phase shift by some constant ϵ, we have the following relationship:

$$\cos(kx - \omega t + \epsilon) + \gamma \sin(kx - \omega t + \epsilon) = a\left[\cos(kx - \omega t) + \gamma \sin(kx - \omega t)\right],$$

where a is some constant that may depend on ϵ.

(a) Use trigonometric identities to write the equations for γ, ϵ and a that follow from the above requirement.
(b) Find the possible solutions for γ and the associated a.
(c) Which one is the solution that corresponds to our conventional description of a matter wave?
(d) Take the matter wave solution. Plug some fiducial numbers (with realistic physical units) for the constants k and ω, and use your favourite programming tool to sketch $|\Psi(x,t)|^2$ versus x for $t = 0$ and two later times.
(e) Use several matter wave solutions localised around k_0 to construct a wave packet, report the phase and group velocities, and use your favourite programming tool to sketch your wave packet.

4. **(5 points) Finite square well potential**

In class we studied the finite square well potential and found that this potential admits both scattering states (when $E > 0$) and bound states (when $E < 0$). For the latter, we derived the even solutions and numerically solved a transcendental equation for the allowed energies.

(a) Normalise the even states found in class (there is no need to solve the problem again, you can use the solutions we already found).
(b) Following the same approach we followed in class, find from scratch the odd bound state wave functions, $\psi(x)$, for the finite square well.
(c) Derive the transcendental equation for the allowed energies of these odd bound states.
(d) Solve it graphically and numerically (using your favourite programming tool). Sketch the solutions for different well depths, well widths, and particle masses. Briefly discuss the results.
(e) Study and discuss the two limiting cases and how the energy levels compare to those found for the even bound state wave functions studied in class. Is there always an odd bound state?