Homework 2 - Quantum Mechanics I

NAME: ____ **SCORE:** _________

Deadline: Thursday 4 April 2024 by 14:00 (2pm)
Credits: 20 points $\rightarrow$ 20 credits **Number of problems:** 4
Type of evaluation: Formative Evaluation

- This homework is individual and includes problems on units 1 and 2.
- Please send a single PDF file via email to: wbanda@yachaytech.edu.ec
- If you work with classmates, please state their names and write your own reports.
- Please submit a professional homework report and clearly highlight all answers.
- All calculation steps should be justified to get full credits.
- Any code used should be attached to get full credits.

1. **(5 points) Probability densities**
   Consider the log-normal distribution:

   $\rho(x) = \frac{C}{x}\, e^{-\frac{(\ln(x)-\mu)^2}{2\,\sigma^2}}$,

   where C, μ, and σ are positive real constants.

   (a) Determine C.
   (b) Find $\langle x \rangle$.
   (c) Find $\langle x^2 \rangle$, and σ_x.
   (d) What do C, μ, and σ represent?
   (e) Plug some fiducial numbers for these constants, and sketch the graph of $\rho(x)$ using your favourite programming language for 3 different cases. Make sure both tails of the distributions can be viewed.

2. **(4 points) The Schrödinger equation, potentials and general solutions**
   Consider the following 1D wave function:

   $$\psi(x) = B\left(\frac{x}{l_0}\right)^k e^{-\frac{x}{l_0}}$$

   where B, k, and l_0 are constants.

   (a) Write down the time-independent Schrödinger equation.
   (b) Find the potential $V(x)$ and energy E, for which this wave function is a solution to the Schrödinger equation. Assume that as $x \rightarrow +\infty$, $V(x) \rightarrow 0$.
   (c) Plug some fiducial values and sketch the potential, $V(x)$, versus x, using your favourite programming tool.
   (d) Write down an expression for a more general time-dependent solution.

3. **(6 points) Wave functions, normalisation, and expectation values**
The wave function of a particle at time $t = 0$ is given by the following piecewise function:

$$\Psi(x, t = 0) = \begin{cases} C\,\frac{x^2}{a^2}, & 0 \le x \le a \\ C\,\frac{b-x}{b-a}, & a \le x \le b \\ 0, & \text{everywhere else,} \end{cases}$$

where C, a, and b are positive constants.

(a) Find an expression for C.
(b) Plug some fiducial numbers for the constants, and use programming tools to sketch $\Psi(x, t = 0)$ as a function of x.
(c) Where is the particle most likely to be found at $t = 0$?
(d) What is the probability of finding the particle to the left of $x = a$?
(e) What is the probability of finding the particle to the right of $x = a$?
(f) For which value/s of a (in terms of b) are the above probabilities (i.e. the probabilities of finding the particle to the left and right of $x = a$) the same?
(g) What is the expectation value of x?

4. **(5 points) Free particles: Gaussian wave packets**
We studied free particles in class and showed that they are represented by wave packets. Consider the case of a free particle whose initial wave function is given by:

$$\Psi(x, 0) = A\, e^{-2\,x^2},$$

where A is a real and positive constant.

(a) Find A by normalising the initial wave function, $\Psi(x, 0)$.
(b) Find $\Psi(x, t)$. Hint: compute $\phi(k)$ via Fourier analysis first, and then plug it into the wave packet function.
(c) Find $|\Psi(x, t)|^2$. Then, plug some fiducial numbers, and use your favourite programming tools to sketch $|\Psi(x, t)|^2$ versus x for $t = 0$ and two later times. Qualitatively, what happens to $|\Psi(x, t)|^2$ as time progresses?
(d) Calculate $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, $\langle p^2 \rangle$, σ_x, and σ_p.
(e) Does the uncertainty principle hold? At what time t does the system come closest to the uncertainty limit?