QM applications

Larmor precession

Larmor precession: Imagine a particle of spin 1/2 at rest in a uniform magnetic field, which points in the z-direction:

$$\mathbf{B} = B_0\hat{k}. \tag{4.159}$$

The Hamiltonian (Equation 4.158) is

$$\mathsf{H} = -\gamma B_0 \mathsf{S}_z = -\frac{\gamma B_0 \hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.160}$$

The eigenstates of H are the same as those of S_z:

$$\begin{cases} \chi_+, & \text{with energy } E_+ = -\left(\gamma B_0 \hbar\right)/2, \\ \chi_-, & \text{with energy } E_- = +\left(\gamma B_0 \hbar\right)/2. \end{cases} \tag{4.161}$$

Larmor precession

The energy is lowest when the dipole moment is parallel to the field—just as it would be classically.

Since the Hamiltonian is time independent, the general solution to the time-dependent Schrödinger equation,

$$i\hbar\frac{\partial\chi}{\partial t} = \mathsf{H}\chi, \tag{4.162}$$

can be expressed in terms of the stationary states:

$$\chi(t) = a\chi_+ e^{-iE_+t/\hbar} + b\chi_- e^{-iE_-t/\hbar} = \begin{pmatrix} ae^{i\gamma B_0 t/2} \\ be^{-i\gamma B_0 t/2} \end{pmatrix}.$$

The constants a and b are determined by the initial conditions:

$$\chi(0) = \begin{pmatrix} a \\ b \end{pmatrix},$$

(of course, $|a|^2 + |b|^2 = 1$). With no essential loss of generality[46] I'll write $a = \cos(\alpha/2)$ and $b = \sin(\alpha/2)$, where α is a fixed angle whose physical significance will appear in a moment. Thus

$$\chi(t) = \begin{pmatrix} \cos(\alpha/2)\ e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)\ e^{-i\gamma B_0 t/2} \end{pmatrix}. \tag{4.163}$$

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To get a feel for what is happening here, let's calculate the expectation value of **S**, as a function of time:

$$\langle S_x \rangle = \chi(t)^\dagger \mathsf{S}_x \chi(t)$$

$$= \begin{pmatrix} \cos(\alpha/2) e^{-i\gamma B_0 t/2} & \sin(\alpha/2) e^{i\gamma B_0 t/2} \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\alpha/2) e^{i\gamma B_0 t/2} \\ \sin(\alpha/2) e^{-i\gamma B_0 t/2} \end{pmatrix}$$

$$= \frac{\hbar}{2} \sin\alpha \cos(\gamma B_0 t). \tag{4.164}$$

Similarly,

$$\langle S_y \rangle = \chi(t)^\dagger \mathsf{S}_y \chi(t) = -\frac{\hbar}{2} \sin\alpha \sin(\gamma B_0 t), \tag{4.165}$$

and

$$\langle S_z \rangle = \chi(t)^\dagger \mathsf{S}_z \chi(t) = \frac{\hbar}{2} \cos\alpha. \tag{4.166}$$

Thus $\langle \mathbf{S} \rangle$ is tilted at a constant angle α to the z axis, and precesses about the field at the **Larmor frequency**

$$\omega = \gamma B_0, \tag{4.167}$$

Larmor precession

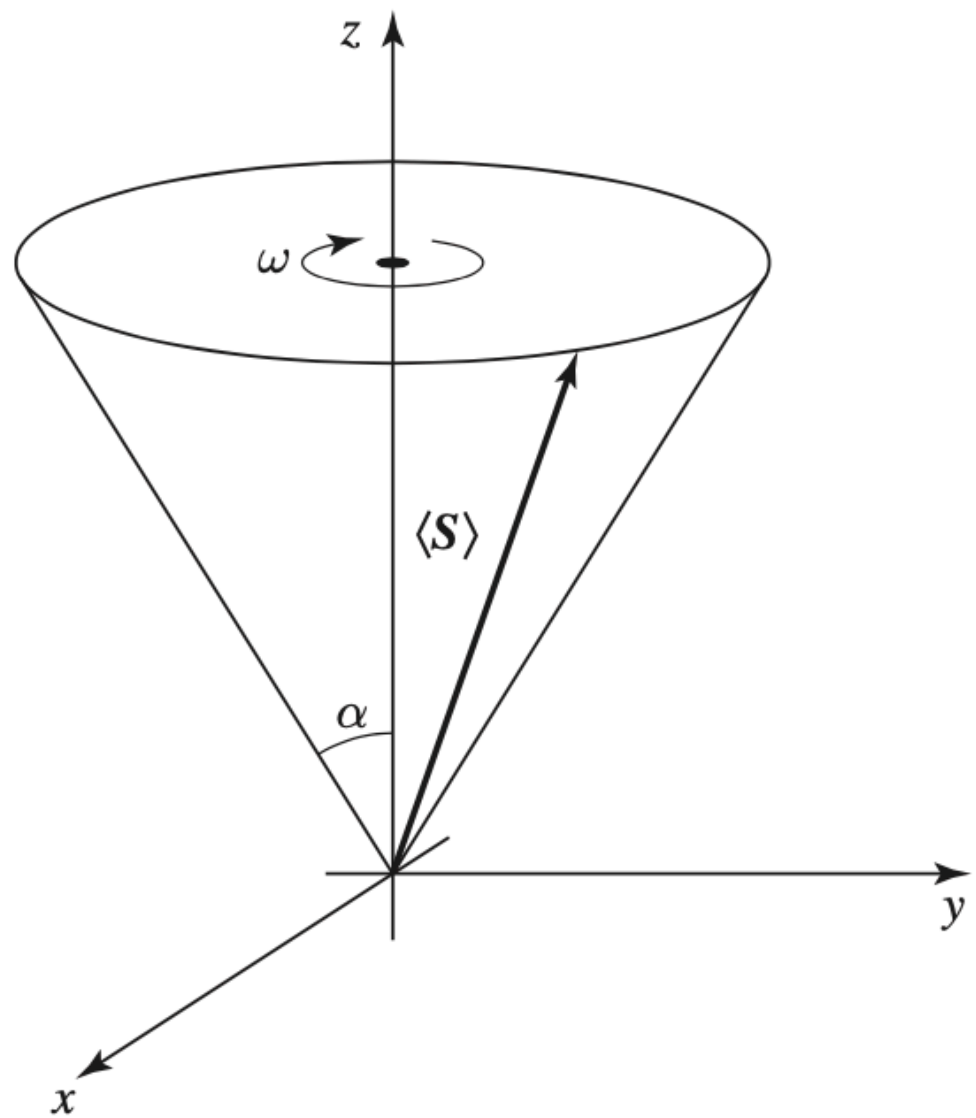

Figure 4.14: Precession of $\langle \mathbf{S} \rangle$ in a uniform magnetic field.

just as it would classically[47] (see Figure 4.14). No surprise here—Ehrenfest's theorem (in the form derived in Problem 4.23) guarantees that $\langle \mathbf{S} \rangle$ evolves according to the classical laws. But it's nice to see how this works out in a specific context.