The hydrogen atom

Spin 1/2:

$s = 1/2$ is the spin of the particles that make up ordinary matter (protons, neutrons, and electrons), as well as all quarks and all leptons.

There are just *two* eigenstates: $|s\,m\rangle$;

1. **spin up** (informally, ↑): $\left|\frac{1}{2}\,\frac{1}{2}\right\rangle$

2. **spin down** (informally, ↓): $\left|\frac{1}{2}\left(-\frac{1}{2}\right)\right\rangle$

Using these as basis vectors, the general state of a spin-1/2 particle can be represented by a two-element column matrix (or **spinor**):

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-,$$

where:

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The hydrogen atom

Spin 1/2:

The spin operators become matrices: $S^2|s\ m\rangle = \hbar^2 s\,(s+1)\,|s\ m\rangle; \quad S_z|s\ m\rangle = \hbar m|s\ m\rangle;$

$$\mathsf{S}^2\chi_+ = \frac{3}{4}\hbar^2\chi_+ \quad \text{and} \quad \mathsf{S}^2\chi_- = \frac{3}{4}\hbar^2\chi_-.$$

If we write S^2 as a matrix with undetermined elements:

$$\mathsf{S}^2 = \begin{pmatrix} c & d \\ e & f \end{pmatrix}$$

$$\mathsf{S}^2\chi_+ = \frac{3}{4}\hbar^2\chi_+ \Rightarrow \begin{pmatrix} c & d \\ e & f \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3}{4}\hbar^2\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} c \\ e \end{pmatrix} = \begin{pmatrix} \frac{3}{4}\hbar^2 \\ 0 \end{pmatrix}, \quad c = (3/4)\hbar^2 \text{ and } e = 0.$$

$$\mathsf{S}^2\chi_- = \frac{3}{4}\hbar^2\chi_- \Rightarrow \begin{pmatrix} c & d \\ e & f \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{3}{4}\hbar^2\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} d \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{3}{4}\hbar^2 \end{pmatrix}, \quad d = 0 \text{ and } f = (3/4)\,\hbar^2.$$

Conclusion: $$\mathsf{S}^2 = \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The hydrogen atom

Spin 1/2:

Similarly:

$$\mathsf{S}_z \chi_+ = \frac{\hbar}{2}\chi_+, \qquad \mathsf{S}_z \chi_- = -\frac{\hbar}{2}\chi_-,$$

For which:

$$\mathsf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Remember:

$$S_\pm \left| s\ m \right\rangle = \hbar\sqrt{s\,(s+1) - m\,(m \pm 1)}\ \left| s\ \ (m \pm 1) \right\rangle,$$

$$\mathsf{S}_+ \chi_- = \hbar\chi_+, \quad \mathsf{S}_- \chi_+ = \hbar\chi_-, \quad \mathsf{S}_+ \chi_+ = \mathsf{S}_- \chi_- = 0,$$

$$\mathsf{S}_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathsf{S}_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Now $S_\pm = S_x \pm i\,S_y$, so $S_x = (1/2)\,(S_+ + S_-)$ and $S_y = (1/2i)\,(S_+ - S_-)$, and hence

$$\mathsf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathsf{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The hydrogen atom

Spin 1/2 (Pauli spin matrices).

Since S_x, S_y, and S_z all carry a factor of $\hbar/2$, it is tidier to write $\mathbf{S} = (\hbar/2)\boldsymbol{\sigma}$, where

$$\boxed{\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.}$$

These are the famous **Pauli spin matrices.**

S_x, S_y, S_z, and S^2 are all *hermitian* matrices (as they *should* be, since they represent observables).

On the other hand, S_+ and S_- are *not* hermitian—evidently they are not observable.

The eigenspinors of S_z are:

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \left(\text{eigenvalue } +\frac{\hbar}{2}\right); \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \left(\text{eigenvalue } -\frac{\hbar}{2}\right).$$

The hydrogen atom

Spin 1/2 (Pauli spin matrices).

Remember: $$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-,$$

If you measure S_z on a particle in the general state χ, you could get $+\hbar/2$, with probability $|a|^2$, or $-\hbar/2$, with probability $|b|^2$. Since these are the *only* possibilities:

$$|a|^2 + |b|^2 = 1$$

(i.e. the spinor must be *normalised*: $\chi^\dagger\chi = 1$.

But what if, instead, we chose to measure S_x? What are the possible results and probabilities?

We need to know the eigenvalues and eigenspinors of S_x

The characteristic equation is:

$$\begin{vmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 = \left(\frac{\hbar}{2}\right)^2 \Rightarrow \lambda = \pm\frac{\hbar}{2}.$$

The hydrogen atom

Spin 1/2 (Pauli spin matrices).

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The possible values for S_X are the same as those for S_Z. The eigenspinors are obtained via:

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \pm\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \qquad \beta = \pm\alpha.$$

The (normalised) eigenspinors of S_X are:

$$\chi_+^{(x)} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \left(\text{eigenvalue } +\frac{\hbar}{2}\right); \; \chi_-^{(x)} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \left(\text{eigenvalue } -\frac{\hbar}{2}\right).$$

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Spin 1/2 (Pauli spin matrices).

The (normalised) eigenspinors of S_x are:

$$\chi_+^{(x)} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \left(\text{eigenvalue } + \frac{\hbar}{2}\right); \; \chi_-^{(x)} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \left(\text{eigenvalue } - \frac{\hbar}{2}\right).$$

The generic spinor χ can be expressed as a linear combination of them:

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_- \qquad\qquad \chi = \left(\frac{a+b}{\sqrt{2}}\right)\chi_+^{(x)} + \left(\frac{a-b}{\sqrt{2}}\right)\chi_-^{(x)}$$

If you measure S_x, the probability of getting $+\hbar/2$ is $(1/2)|a+b|^2$, and the probability of getting $-\hbar/2$ is $(1/2)|a-b|^2$.

The hydrogen atom

Spin 1/2 (implications):

For a particle in the state χ_+, what is the *z*-component of a particle's spin angular momentum?

We can answer unambiguously: $+\hbar/2$.

What is the *x*-component of that particle's spin angular momentum?

If you measure S_x, the chances are fifty-fifty of getting either $+\hbar/2$ or $-\hbar/2$.

It simply *does not have* a particular *x*-component of spin.