Thin Airfoil Theory using Discrete Vortex Method
by Daamanjyot Barara (Adapted from Katz and Plotkin)

The solver to derive the solution to thin airfoil problem is based on the discrete vortex method. The solver's discrete vortex method uses lumped-vortex elements to provide an approximate solution to the fundamental equation derived from thin airfoil theory i.e., integral equation for vortex distribution. The procedure for arriving at the numerical solution through the solver is briefly explained in the following steps along with the key equations as discussed by Katz and Plotkin [1]-

1. User Input- As a first step, the solver accepts the following inputs from the user-

a) NACA 4-digit airfoil- A prompt asks the user to input the NACA 4-digit designation of the airfoil to be analyzed. Based on the nomenclature of the NACA 4-digit series, the 4-digit airfoil designation entered by the user is used to calculate the coordinates of mean camber line.

b) Number of panels, M- The discretization of the airfoil's camber line depends on the number of panels it must be divided into.

c) Angle of attack- The angle of attack at which the computation is performed.

2. Geometry Discretization- Based on the inputs from the user, the airfoil's camber line is discretized into M panels i.e., lumped-vortex element. As the solver uses the 'Uniform Distribution Method', the panels are of equal lengths spanning over the chord of the airfoil. After the discretization of the camber line, the solver proceeds with computation of the geometrical data for each panel that includes – coordinates of vortex points and collocation points, length of the panel, and normal vectors.

For a lumped vortex element, the vortex points are placed at the quarter chord position of each of the corresponding panels as the lift acts at the center of pressure point i.e., present at the quarter chord of the panel. The zero normal flow boundary condition is imposed at the collocation points located at the three-quarter chord of each panel. The normal vector n_i for each of the panels is computed using Eq. (1) below,

$$n_i = (\sin\alpha_i\,, \cos\alpha_i) = (\frac{-\Delta z}{c_j}, \frac{\Delta x}{c_j}) \tag{1}$$

with,

$$\Delta x = x_{i+1} - x_i \ \ , \ \Delta z = z_{i+1} - z_i \ , \ c_j = \sqrt{\Delta x^2 + \Delta z^2} \tag{2}$$

where Δx and Δz represents the difference the corner points of each of the panels, which is used to calculate the length of each panel, c_j. The normal vector n_i, on each panel is computed using the aforementioned geometrical data.

3. Computation of influence coefficients- In this phase, a set of linear algebraic equations are derived for each panel that satisfy the imposed streamline boundary condition at the collocation point. This achieved by setting the normal component of velocity to zero.

$$q.n = 0 \tag{3}$$

Since the normal velocity component is a combination of self-induced velocity and freestream velocity, Eq. can be rewritten as,

$$(u, w).n \ + (U_\infty, W_\infty).n \ = 0 \tag{4}$$

Where, the first term represents self-induced velocity i.e., the velocity induced by the distribution of lumped vortex elements on itself. While, the second term represents freestream velocity.

The self-induced part of the normal velocity is calculated by accounting for the velocities induced at each collocation point by the vortex of unit intensity located at the panel j. The induced velocities by the vortex can be calculated as,

$$u(x,z) = \frac{\Gamma_j}{2\pi}\frac{z - z_0}{r^2}\ ;\ w(x,z) = -\frac{\Gamma_j}{2\pi}\frac{x - x_0}{r^2} \tag{5}$$

with,
$$r^2 = (x - x_0)^2 + (z - z_0)^2$$

Where, (x, z) are the coordinates of the collocation point and (x_0, z_0) is the coordinate of vortex. Rewriting the above eq. for obtaining velocities induced by a vortex of unit strength i.e., $\Gamma_j = 1$,

$$u(x,z) = \frac{1}{2\pi}\frac{z - z_0}{r^2}\ ;\ w(x,z) = -\frac{1}{2\pi}\frac{x - x_0}{r^2} \tag{6}$$

The term of self-induced velocities consists of the coefficients called influence coefficients a_{ij}, that is defined as the velocity component normal to the collocation point i due to the vortex of unit strength at panel j,

$$\mathrm{a}_{ij} = (u,w)_{ij} \,.\, n_i \tag{7}$$

As the freestream contribution to the normal velocity component is known, the equation (7) can be rewritten for the collocation point i by transferring the freestream component to the right-hand side,

$$\sum_{i=1}^{M} \Gamma_j\,(u,w)_{ij}. n_i = -\,(U_\infty, W_\infty). n_i \tag{8}$$

Rewriting the above equation for M collocation points in terms of influence coefficient,

$$\sum_{i=1}^{M} a_{ij}\Gamma_j = RHS_i \tag{9}$$

In the code, a nested DO loop is used to solve the system of equations and compute influence coefficients,

DO $\boldsymbol{i = 1, M}$ (Loop over collocation points)
DO $\boldsymbol{j = 1, M}$ (Loop over vortex points)
Compute the induced velocities $(u, w)_{ij}$ at the collocation point i due to a vortex at panel j (Eq. 6)
Compute the influence coefficient (Eq. 7) $\mathrm{a}_{ij} = (u,w)_{ij} \,.\, n_i$
END DO LOOP
Solve for RHS, $RHS_i = -\,(U_\infty, W_\infty). n_i$
END DO LOOP

4. Establish RHS- The right-hand side vector of the equation is solved for each of the collocation point i and computed in the outer loop of the nested DO loop stated above,

$$RHS_i = -\,(U_\infty, W_\infty). n_i \tag{10}$$

Where,
$$(U_\infty, W_\infty) = Q_\infty(\cos\alpha\,, \sin\alpha)$$

5. Solving system of equations – In this phase, once the influence coefficient a_{ij} for each of the collocation points i and the right-hand side equation RHS_i is computed, the linear set of equations are solved to calculate the circulation strength Γ_j for each of the M panels.

6. Computation of Pressure Distribution and Loads – Once the circulation strength for each of the panel is computed, it is further used to compute the pressure distribution and lift over the airfoil using the equation (11) and (12) respectively,

$$\Delta C_p = \frac{2}{U_\infty}\frac{\Gamma_j}{c_j} \tag{11}$$

$$C_l = \frac{2}{U_\infty\ c_j}\sum_{j=1}^{M} \Gamma_j \tag{12}$$

In the current code, U_∞ has been set to unity. The code has been attached in the Appendix A of the assignment.

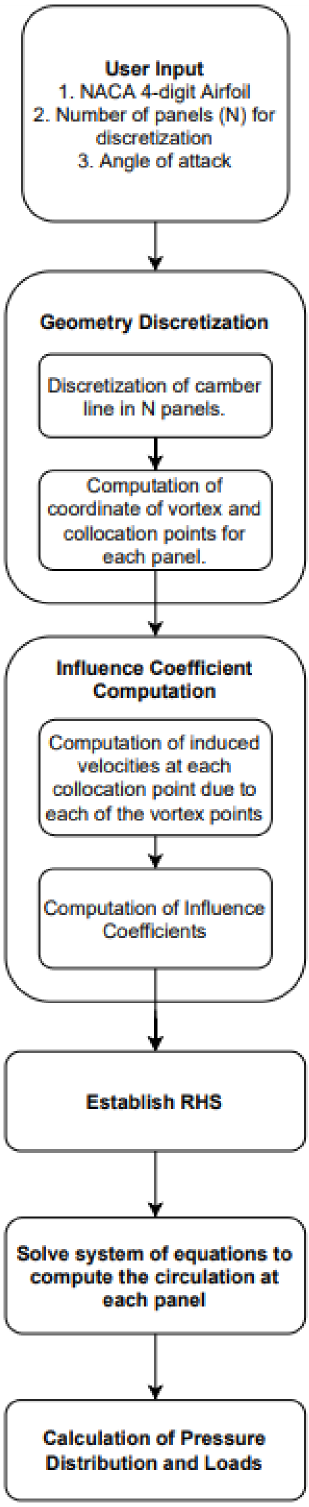

Figure 1 Flowchart of the Solver