Lab Assignments Computational Finance

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Submission guidelines

These assignments can be done in groups of three students. Reports with a *clear and concise description of the assignment, the methods, the results and discussion* should be submitted before the deadlines. You are free to choose the programming language/environment in which you would like to write your computer programs. If you have questions about the assignments do not hesitate to contact the teaching assistant or the lecturer.

Grading scheme

- Each of the three assignments carries equal weight of 20% and the exam is worth 40%;
- The score of the exam should be 5 points (on the scale of 1 to 10) and higher for passing the course;

Assignment 1	Assignment 2	Assignment 3	Exam
20%	20%	20%	40%

Assignment 2: Monte Carlo (MC) Methods in Finance

Part I

Basic Option Valuation

As we derived in the class, for the purpose of pricing options, we can assume that the stock price S evolves in the risk neutral world:

$$dS = rSdt + \sigma SdZ \tag{1}$$

where r is the risk free return, σ is the volatility, and dZ is the increment of a Wiener process. Let the expiry time of an option be T, and let

$$N = \frac{T}{\Delta t}$$

$$S^n = S(n\Delta t)$$

Then, given an initial price S^0, M realizations of the path of a risky asset are generated using the algorithm (Euler method)

$$S^{n+1} = S^n + S^n(r\Delta t + \sigma\varphi\sqrt{\Delta t})$$

where φ is a normally distributed random variable with mean zero and unit variance.
For special cases of constant coefficients, we can avoid time stepping errors for geometric Brownian motion, since we can integrate Equation 1 exactly to get

$$S^T = S^0 e^{(r-0.5\sigma^2)T+\sigma\sqrt{T}Z} \tag{2}$$

The price of an option can be calculated by computing the discounted value of the average pay-off, i.e.

$$V(S^0, t=0) = e^{-rT}\frac{\sum_{m=1}^{M} payoff^m(S^N)}{M}$$

Write a computer program for the Monte Carlo method. Price European put option with (T = 1 year, K =€99, r = 6%, S = €100 and σ = 20%). Carry out convergence studies by increasing the number of trials. How do your results compare with the results obtained in assignment 1? Perform numerical tests for varying values for the strike and the volatility parameter. What is the standard error of your estimate and what does this tell you about the accuracy?

Part II

Estimation of Sensitivities in MC

1. The hedge parameter δ in Monte Carlo can be estimated by the bump-and-revalue method. Calculate the δ by applying the following methods:

    - Use different seeds for the bumped and unbumped estimate of the value;

- Use the same seed for the bumped and unbumped estimate of the value;

Compare your results which the values obtained in Assignment I.

2. Consider a digital option which pays 1 euro if the stock price at expiry is higher than the strike and otherwise nothing. Calculate the hedge parameter δ using the method used in 1. Explain your results and use the **sophisticated methods** discussed in the lectures, i.e. the pathwise and likelihood ratio methods with the application of smoothing if necessary, to improve your results.

Part III

Variance Reduction

A major drawback of Monte Carlo simulation is that a large number of realizations are typically required to obtain accurate results. Therefore techniques to speed-up the simulations are quite useful. In this assignment students will work on different variance reduction techniques and quasi-Monte Carlo methods to accelerate numerical valuation of financial derivatives.

Variance Reduction by Control Variates. For the control variates technique an accurate estimate of the value of an option that is similar to the one that you would like to price is required. For valuation of an Asian option based on *arithmetic averages* one can use the value of an Asian option based on *geometric averages*. This case can be solved analytically.

1. Write a program for the price of an Asian option based on geometric averages using the analytical expression derived in A. Compare the values you obtain with the analytical expression to those obtained by using Monte-Carlo simulations.

2. Explain how the strategy of using this as a control variate works.

3. Apply the control variates technique for the calculation of the value of the Asian option based on arithmetic averages. Study the performance of this technique for different parameter settings (number of paths, strike, number of time points used in the average etc.)

Appendices: Theory

The section below treats the derivation of the analytical solution for the Asian option based on geometric averages.

A Analytical expression for the price of an Asian call option based on geometric averages

Recall that the Black-Scholes model describes the evolution of a stock price in the risk-neutral world through the stochastic differential equation (SDE)

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \tag{3}$$

with r the risk-free rate of return, σ the volatility of the stock price, and W a standard Wiener process. The solution of the stochastic differential equation (3) is

$$S(T) = S(0)\exp\left([r - \frac{1}{2}\sigma^2]T + \sigma\sqrt{T}Z\right), \tag{4}$$

where T is the maturity, $S(0)$ is the current price of the stock and Z is a standard normal random variable. The logarithm of the stock price is therefore normally distributed, and the stock price itself has a lognormal distribution.
As the product of lognormally distributed random variables is also lognormally distributed, deriving an analytical expression for the price of an Asian option based on geometric averages is a rather straightforward task. Suppose that we observe the value of the stock N times before the maturity time T. The payoff of an Asian call option based on geometric averages is given by

$$\left(\tilde{A}_N - K\right)^+, \tag{5}$$

where $\tilde{A}_N = \left(\prod_{i=0}^{N} S\left(\frac{iT}{N}\right)\right)^{\frac{1}{N+1}}$ and $S\left(\frac{iT}{N}\right)$ the stock price at time $\frac{iT}{N}$. Furthermore, we know

$$\prod_{i=0}^{N} S(t_i) = \frac{S(t_N)}{S(t_{N-1})}\left(\frac{S(t_{N-1})}{S(t_{N-2})}\right)^2\left(\frac{S(t_{N-2})}{S(t_{N-3})}\right)^3\cdots\left(\frac{S(t_2)}{S(t_1)}\right)^{N-1}\left(\frac{S(t_1)}{S(t_0)}\right)^{N} S(0)^{(N+1)}, \tag{6}$$

where $t_i = \frac{iT}{N}$ for $i = 1, \ldots, N$.
From (4) it follows that

$$\begin{aligned}
\frac{S(t_N)}{S(t_{N-1})} &= \exp\left([r - \frac{1}{2}\sigma^2]\frac{T}{N} + \sigma\sqrt{\frac{T}{N}}X_1\right),\\
\frac{S(t_{N-1})}{S(t_{N-2})} &= \exp\left([r - \frac{1}{2}\sigma^2]\frac{T}{N} + \sigma\sqrt{\frac{T}{N}}X_2\right),\\
&\vdots\\
\frac{S(t_2)}{S(t_1)} &= \exp\left([r - \frac{1}{2}\sigma^2]\frac{T}{N} + \sigma\sqrt{\frac{T}{N}}X_{N-1}\right),\\
\frac{S(t_1)}{S(t_0)} &= \exp\left([r - \frac{1}{2}\sigma^2]\frac{T}{N} + \sigma\sqrt{\frac{T}{N}}X_N\right),
\end{aligned}$$

where $X_i, i = 1, \ldots, N$ are independent, standard-normally distributed random variables. In addition, we have

$$\log\left(\frac{\left(\prod_{i=0}^{N} S(t_i)\right)^{\frac{1}{N+1}}}{S(0)}\right) = \frac{1}{N+1}\log\left(\frac{\prod_{i=1}^{N} S(t_i)}{S(0)^N}\right)$$

$$= \frac{1}{N+1}\log\left(\frac{S(t_N)}{S(t_{N-1})}\left(\frac{S(t_{N-1})}{S(t_{N-2})}\right)^2\left(\frac{S(t_{N-2})}{S(t_{N-3})}\right)^3\cdots\left(\frac{S(t_2)}{S(t_1)}\right)^{N-1}\left(\frac{S(t_1)}{S(t_0)}\right)^N\right)$$

$$= \frac{1}{N+1}\left(\log\frac{S(t_N)}{S(t_{N-1})} + 2\log\left(\frac{S(t_{N-1})}{S(t_{N-2})}\right) + \cdots + N\log\left(\frac{S(t_1)}{S(t_0)}\right)\right)$$

$$= \frac{1}{N+1}\left(\left([r - \frac{1}{2}\sigma^2]\frac{T}{N} + \sigma\sqrt{\frac{T}{N}}X_1\right) + 2\left([r - \frac{1}{2}\sigma^2]\frac{T}{N} + \sigma\sqrt{\frac{T}{N}}X_2\right) + \right.$$

$$\left.\cdots + N\left([r - \frac{1}{2}\sigma^2]\frac{T}{N} + \sigma\sqrt{\frac{T}{N}}X_N\right)\right)$$

$$= \frac{1}{N+1}\left((1 + 2 + \cdots + N)\,[r - \frac{1}{2}\sigma^2]\frac{T}{N} + \sum_{i=1}^{N}\sigma\sqrt{\frac{T}{N}}iX_i\right)$$

$$= \frac{1}{N+1}\left(\frac{(N+1)N}{2}\frac{T}{N}[r - \frac{1}{2}\sigma^2] + \sigma\sqrt{\frac{T}{N}}\sum_{i=1}^{N} iX_i\right)$$

$$= \frac{[r - \frac{1}{2}\sigma^2]T}{2} + \frac{\sigma\sqrt{\frac{T}{N}}\sum_{i=1}^{N} iX_i}{N+1}. \tag{7}$$

Since $X_i, i = 1, \ldots N$ are i.i.d. we have

$$\frac{\sigma\sqrt{\frac{T}{N}}\sum_{i=1}^{N} iX_i}{N+1} \sim N\left(0, \frac{\sigma^2 T\left(1^2 + 2^2 + \cdots + N^2\right)}{N(N+1)^2}\right), \tag{8}$$

or

$$\frac{\sigma\sqrt{\frac{T}{N}}\sum_{i=1}^{N} iX_i}{N+1} \sim N\left(0, \frac{(2N+1)\,\sigma^2 T}{6(N+1)}\right), \tag{9}$$

and thus

$$\frac{\sigma\sqrt{\frac{T}{N}}\sum_{i=1}^{N} iX_i}{N+1} = \sigma\sqrt{\frac{(2N+1)\,T}{6(N+1)}}Z, \tag{10}$$

where $Z \sim N(0, 1)$.
From (7) and (10) we obtain

$$\log\left(\frac{\left(\prod_{i=0}^{N} S(t_i)\right)^{\frac{1}{N+1}}}{S(0)}\right) = [\tilde{r} - \frac{1}{2}\tilde{\sigma}^2]T + \tilde{\sigma}\sqrt{T}Z \tag{11}$$

where

$$\tilde{\sigma} = \sigma\sqrt{\frac{2N+1}{6(N+1)}} \tag{12}$$

and

$$\tilde{r} = \frac{[r - \frac{1}{2}\sigma^2] + \tilde{\sigma}^2}{2}. \tag{13}$$

Hence, the price $C_g^A(S(0), T)$ can be obtained by the risk-neutral method

$$\begin{aligned}
C_g^A(S(0), T) &= \exp(-rT)\mathbb{E}\left(\left(\prod_{i=0}^{N} S\left(\frac{iT}{N}\right)\right)^{\frac{1}{N+1}} - K\right)^+ \\
&= \exp\left((\tilde{r} - r)T\right)\exp(-\tilde{r}T)\mathbb{E}\left(S(0)\exp\left([\tilde{r} - \frac{1}{2}\tilde{\sigma}^2]T + \tilde{\sigma}\sqrt{T}Z\right) - K\right)^+ \\
&= \exp\left((\tilde{r} - r)T\right)C(S(0), T),
\end{aligned}$$

where $C(S(0), T)$ is the price of a European call option with risk-free interest rate $\tilde{r}$ and volatility $\tilde{\sigma}$, which can be easily calculated by using the Black-Scholes formula. Therefore we get

$$\begin{aligned}
C_g^A(S(0), T) &= \exp\left((\tilde{r} - r)T\right)\left(S(0)\Phi(\tilde{d}_1) - K\exp(-\tilde{r}T)\Phi(\tilde{d}_2)\right) \\
&= \exp\left(-rT\right)\left(S(0)\exp(\tilde{r}T)\Phi(\tilde{d}_1) - K\Phi(\tilde{d}_2)\right),
\end{aligned}$$

where Φ is the cumulative distribution function of the standard normal distribution and

$$\begin{aligned}
\tilde{d}_1 &= \frac{\log\frac{S(0)}{K} + \left(\tilde{r} + \frac{1}{2}\tilde{\sigma}^2\right)T}{\sqrt{T}\tilde{\sigma}}, \\
\tilde{d}_2 &= \frac{\log\frac{S(0)}{K} + \left(\tilde{r} - \frac{1}{2}\tilde{\sigma}^2\right)T}{\sqrt{T}\tilde{\sigma}}.
\end{aligned}$$