Report for

A proposal to demonstrate non-abelian anyons on a NISQ device

This paper provides a proposal to realize non-Abelian topological states on NISQ devices. The topological order being simulated is the Kitaev's quantum double model associated with the dihedral group $G = D_4$, the symmetry group of the square. The model is defined on a 2-dimensional lattice with each edge supporting a Hilbert space with a basis consisting of the group elements in G. The Hamiltonian is a sum of pairwise commuting local projectors and hence is frustration-free. The quasi-particle excitations/anyons are described by the irreducible representations of the quantum double $D(G)$ of G.

The work in this papers follows from a series of recent efforts to 'engineer' topologically ordered states on quantum processors. In 2021, Google Quantum AI group and collaborators first realized toric code states (the quantum double model based on $\mathbb{Z}_{\nvDash}$) on its superconducting quantum processor; later in 2023, they realized certain lattice defects in a stabilizer codes which behave like the non-Abelian Ising anyon. Also in 2023, the Quantinuum group and collaborators realized non-Abelian topological states by gauging an Abelian symmetry-protected state on its trapped-ion quantum processor. The non-Abelian topological state corresponds to a twisted version of the quantum double model based on $\mathbb{Z}_2^3$ which in an appropriate sense is equivalent to the untwisted quantum double model based on D_4.

The current paper is closest to the last of the three mentioned efforts since they realized essentially the same anyons corresponding to the topological order $D(D_4)$. However, their differences are also clear. The current paper implements the model on the nose, in the sense that it simulates precisely the gauge lattice model defined by the Hamiltonian in the Kitaev framework, while in the Quantinuum paper, the states are obtained by gauging an Abelian order and only Morita equivalent to the $D(D_4)$ order. The current paper also proposes explicit methods to implement the ground state, the ribbon operators, the fusion of anyons, and the braiding of anyons. Finally, it also provides a way to measure the modular data. The authors in this paper also performed numerical simulations using Google's 'cirq google' python package on Google's cloud computing platform. The numerical results agree with theoretical predictions to a good precision.

The work in this paper is complementary to that of Quantinuum. One significant advantage of this paper is that the circuits are not adaptive, that is, it requires measurements only at the end of the process; furthermore, the methods proposed here can be straightforwardly generalized to larger lattices and to other groups when the quantum hardware supports more qubits. The paper is well organized and written. I recommend publication, as long as the following issues are addressed.

1. (pg. 5, lower right of the page) I think you cannot define the model on **arbitrary** directed graphs. A graph consists of vertices and edges, but has no information on plaquettes/faces. It doesn't even make sense to speak of faces of a graph. Rather, to define the model, you choose an oriented compact boundaryless surface and a cell decomposition (or polygon decomposition) of the surface. The 1-cells should be oriented. The collection of 0- and 1-cells form a directed graph on the surface, but you do need the faces as well.

2. (pg. 7, top right below Figure 1) It seems to me the map $g \mapsto q_c^{-1} g q_{c'}$ is not a projection.

Rather, it's just a map from G to $Z(r)$. A projection requires it to be identity on $Z(r)$.

3. (pg. 9, Step 1 in the **Algorithm**) The (left) parenthesis is not paired.

4. (pg. 10, left column) The statements about two-way implementation and constant-depth implementation of the ribbon operator seem problematic to me. Note that, in each application of Step 2, you update the ancilla $|c, i\rangle$, and in each application of Step 1, you need to use the **updated** ancilla to do the multiplication. Hence, even if you prepare an ancilla for each triangle initialized at certain state, you still need to modify each of them according to the location of the triangle in the ribbon which cannot be done in parallel.

5. (pg. 10, bottom of left column) A typo. 'Dor type I the ...'

6. (pg. 11, Step 3) Isn't U_a just the identity operator (up to a scalar)? U_a is a unitary operator that acts on the ancilla qudit $\mathbb{C}[H]$. Have you defined the states $|\chi_H; i', j'\rangle_a$? Which Hilbert space do they live in? One way to define them is simply,
$$|\chi; i, j\rangle = \sum_{h \in H} \Gamma^{\chi}_{ij}(h)^* |h\rangle, \quad \text{up to a scalar,}$$
In such a case, U_a is proportional to the Identity operator Schur's orthogonality lemma.

7. (pg. 11, Eqn 15) Summation over j is missing.

8. (pg. 11, Eqn 16) To apply the Schur's orthogonality lemma, one of the two Γ terms needs to be conjugated. You probably need to change U_a accordingly. Also, the notation $|\chi, p; j\rangle$ seems to be used without definition.

9. (pg. 12, Table 3) In the first table of Table 3, the character of r for the irrep $-i$ should be $-i$, instead of i.

10. (pg. 13, top left) One of the χ should be conjugated in the summation to define an inner product, which is a standard one.

11. (pg. 14, top left) "their representations one-dimensional". Missing 'are'.

12. (pg. 14, Eqn 23, 24) I'm confused. Aren't irreps of H_r, H_m, H_{mr} are 1-dimensional? Eqn 23 seems to say a representation is encoded by 2 qubits? some clarification is useful here.

13. (pg. 16, Figure 6) Since the Kitaev model is defined on a lattice of surface. It might be useful to comment which surface the quasi one-dimensional lattice in Figure 6 lives. I assume it's the sphere, but do you count the big outer face (the complement of the lattice) as one of the faces in the model?

14. (pg. 34, below Table 6) Can you explain what $\tilde{X}$ is for a given anyon X? And what do you mean by "let $X \to \tilde{X}$"?