Stability

Monday, 4 April 2022 14:09

System: $m(C\ddot{\theta} + R^{-1}\dot{\theta} + I_c \sin\theta) + \begin{bmatrix} -M I_s \\ A\theta + \delta f \end{bmatrix} = 0$ where: $\delta f \equiv 2\pi(\bar{\Phi}_{ext} - Z)$, $m \equiv \begin{bmatrix} M \\ AL \end{bmatrix}$

Main Theory A: This system is stable at stationairy point θ obeying $m I_c \sin\theta + \begin{bmatrix} -M I_s \\ A\theta + \delta f \end{bmatrix} = 0$

IF: $m I_c \cos\theta\, m^T + \begin{bmatrix} 0 & 0 \\ 0 & ALA^T \end{bmatrix}$ is positive semidefinite

proof: Combine theorems B, C, D, E, F

Definition: Any system $A\dot{x} = f(x)$ is stable at stationairy point x_p if:

- A invertible
- $f(x_p) = 0$
- $J \equiv A^{-1} \nabla_x f(x)\big|_{x=x_p}$ has eigenvalues ≤ 0

Theory B: The system $P_1 \ddot{x} + P_2 \dot{x} + F(x) = 0$ is dynamically stable at stationairy point x_p if:

- P_1, P_2 invertible
- $F(x_p) = 0$
- quadratic eigenvalue problem $\left[\lambda^2 P_1 + \lambda P_2 + \nabla_x F(x)\right] v = 0$ has negative eigenvalues

proof: defines $p = \dot{x}$, then:

$$\begin{bmatrix} P_1 & P_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -F(x) \\ p \end{bmatrix} \rightarrow \begin{bmatrix} \dot{p} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -P_1^{-1}(F(x) + P_2 p) \\ p \end{bmatrix} \equiv f(p, x)$$

stable if eig of $J \equiv \nabla_{px} f(p,x) < 0$ where: $J = \begin{bmatrix} -P_1^{-1}P_2 & -P_1^{-1}\nabla_x F(x) \\ I & 0 \end{bmatrix}$

$$J v = \lambda v \rightarrow \begin{bmatrix} & \cdots & \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \lambda \leq 0$$

$$\rightarrow \begin{cases} -P_1^{-1}P_2 v_1 - P_1^{-1}\nabla_x F(x) v_2 = \lambda v_1 \\ v_1 = \lambda v_2 \end{cases}$$

$$\rightarrow \left[\lambda^2 P_1 + \lambda P_2 + \nabla_x F(x)\right] v_2 = 0 \quad \text{QED}$$

Theory C: $\left[\lambda^2 P_1 + \lambda P_2 + \nabla_x F(x)\right] v = 0$ has $\mathrm{Re}[\lambda] \leq 0$ if $P_1, P_2, \nabla_x F(x)$ are all positive semidefinite (and thus symmetric)

proof: $\forall v \in V$ holds:

$$\lambda^2 v^T P_1 v + \lambda v^T P_2 v + v^T \nabla_x F(x) v = 0$$

If $P_1, P_2, \nabla_x F(x)$ are all positive semidefinite then:

$\alpha\lambda^2 + \beta\lambda + \gamma = 0$ where $\alpha, \beta, \gamma \geq 0$

$$\lambda = -\frac{\beta}{2\alpha} \pm \frac{\sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

$$\mathrm{Re}[\lambda] = \begin{cases} -\frac{\beta}{2\alpha} \leq 0 & \text{if } \beta^2 - 4\alpha\gamma < 0 \\ -\frac{\beta}{2\alpha} \pm \frac{\sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \leq -\frac{\beta}{2\alpha} + \frac{\beta}{2\alpha} \leq 0 & \text{if } \beta^2 - 4\alpha\gamma \geq 0 \end{cases}$$

≤ 0

If it holds for all $v \in V$, it also holds for all eigenvectors v

so for all eigenvalues λ holds $\mathrm{Re}[\lambda] \leq 0$ QED

Definition A is split into $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ such that $A_2 L = 0$ and $A_1 L$ has only non-zero rows.

Definition Transformation $T = [M^T, LA_1^T, A_2^T]$

Theorem D: T is invertible so that any θ uniquely decomposes into $T\begin{bmatrix} \varphi \\ \gamma \\ \varepsilon \end{bmatrix}$ (using cut space M orthogonal complement of cycle space A so $MA^T = 0$)

proof:

$$A_2\theta = A_2 A_2^T \varepsilon \rightarrow \varepsilon = (A_2 A_2^T)^{-1} A_2 \theta$$

$$A_1\theta = A_1 L A_1^T \gamma + A_1 A_2^T \varepsilon \rightarrow \gamma = [A_1 L A_1^T]^{-1}(A_1\theta - A_1 A_2^T \varepsilon)$$

$$M\theta = MM^T\varphi + MLA_1^T\gamma \rightarrow \varphi = [MM^T]^{-1}(M\theta - MLA_1^T\gamma)$$

so T is invertible if $A_2 A_2^T$, $A_1 L A_1^T$, MM^T are invertible

#1, #3 are true because the rows of A are linearly independent and the rows of M are linearly indep.

#2 L is psd so ALA^T is psd. By construction $A_1 L A_1^T$ is pd and thus invertible.

Theorem E: Under the decomposition $\theta \rightarrow T\begin{bmatrix} \varphi \\ \gamma \\ \varepsilon \end{bmatrix}$, system transforms to:

$P_1 \ddot{x} + P_2 \dot{x} + F(\theta) = 0$ where

$P_1 = \bar{m} C \bar{m}^T$

note: $\bar{m} = \begin{bmatrix} M \\ A_1 L \end{bmatrix}$, $x \equiv \begin{bmatrix} \varphi \\ \gamma \end{bmatrix}$

$\varepsilon = -(A_2 A_2^T)^{-1} \delta f$

$P_1 \ddot{x} + P_2 \dot{x} + F(\theta) = 0$ where

$P_1 = \bar{m} C \bar{m}^T$

$P_2 = \bar{m} \tilde{R}^{-1} \bar{m}^T$

$F(\theta) = \bar{m} I_c \sin(\theta) + \begin{bmatrix} -M I_s \\ A_1 \theta + \delta f_1 \end{bmatrix}$

note: $\bar{m} = \begin{bmatrix} M \\ A_1 L \end{bmatrix}$, $x \equiv \begin{bmatrix} \varphi \\ \gamma \end{bmatrix}$

$\varepsilon = -(A_2 A_2^T)^{-1} \delta f_2$

$\dot{\varepsilon} = \ddot{\varepsilon} = 0$

P_1, P_2 are positive semidefinite because C and $\tilde{R}^{-1}$ are psd.

Theorem F: $\nabla_{\varphi,\gamma} F(\varphi, \gamma) = \bar{m} I_c \cos(\theta) \bar{m}^T + \begin{bmatrix} 0 & 0 \\ 0 & A_1 L A_1^T \end{bmatrix}$

This equals $m I_c \cos(\theta) m^T + \begin{bmatrix} 0 & 0 \\ 0 & A L A^T \end{bmatrix}$ with rows/cols of zero's removed,

so the first is psd iff the second one is.