Subtask - 1.

$$l(\theta) = \sum_{i=1}^{m} \left[y_i \log\left(h_\theta(x_i)\right) + (1-y_i) \log\left(1 - h_\theta(x_i)\right) \right]$$

$$H = \frac{\delta^2 l(\theta)}{\delta\theta\, \delta\theta^T}$$

$$h_\theta(x_i) = g(\theta^T x) = g(z) = \frac{1}{1+e^{-z}}.$$

- $$\frac{d\,g(z)}{dz} = \frac{d}{dz}(1+e^{-z})^{-1} = -e^{-z}(1+e^{-z})^{-2}$$

$$= \frac{-1}{1+e^{-z}} \cdot \frac{e^{-z}}{1+e^{-z}}$$

$$= -g(z) \cdot (1-g(z))$$

- $$\frac{d\,g(z)}{d\theta} = \frac{x^T\theta}{d\theta} = x^T$$
- $$\frac{dz}{d\theta^T} = \frac{d\,\theta^T x}{d\theta^T} = x.$$

- $$\frac{\delta \log(g(z))}{d\theta^T} = \frac{1}{g(z)} \cdot \frac{d\,g(z)}{d\theta^T}$$

$$= \frac{1}{g(z)} \cdot \frac{d\,g(z)}{dz} \cdot \frac{dz}{d\theta^T}$$

$$= -(1-g(z)) \cdot x$$

Similarly,

$$\frac{d \log(1-g(z))}{d\theta^T} = -g(z)\,x$$

$$\therefore \frac{d\, l(\theta)}{d\theta^T} = -y_i x_i (1-g(z_i)) + (1-y_i) x_i\, g(z_i)$$

$$= x_i (g(z_i) - y_i)$$

$$H = \frac{d^2 l(\theta)}{d\theta\, d\theta^T} = (x_i x_i^T)\, g(z_i)(1-g(z_i))$$

Say, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad x^T = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$

$x . x^T = \begin{bmatrix} x_1^2 & x_1 x_2 \\ x_2 x_1 & x_2^2 \end{bmatrix} = A$ (say)

For H to be positive semi-definite, it has to be symmetric with eigenvalues non-negative.

$\therefore H = H^T$

$H = x x^T g(z)(1-g(z))$

$H^T = \cancel{x^T} \left(x x^T g(z)(1-g(z))\right)^T$

$= x x^T g(z)(1-g(z)).$

$\therefore H = H^T$, proved.

We know $g(z)(1-g(z)) \geq 0$., since $g(z) \geq 0$;
$(1-g(z)) \geq 0$

Characteristic eqn of $x . x^T$ (A) is:

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} x_1^2 - \lambda & x_1 x_2 \\ x_1 x_2 & x_2^2 - \lambda \end{vmatrix} = 0.$$

$\therefore (x_1^2 - \lambda)(x_2^2 - \lambda) - (x_1 x_2)^2 = 0$

$\therefore \cancel{x_1^2 x_2^2} + \lambda^2 - \lambda(x_1^2 + x_2^2) - \cancel{(x_1 x_2)^2} = 0.$

$\therefore \lambda(\lambda - x_1^2 + x_2^2) = 0$

$\therefore \lambda = 0, \ (x_1^2 + x_2^2)$

Clearly, $x_1^2 + x_2^2 > 0$

$\therefore \lambda > 0$ or $\lambda = 0$.

$\therefore$ The eigenvalues of $A \geq 0$.

This can be generalised for any $n \times n$ $x \cdot x^T$ matrix.

Since $x \cdot x^T$ is semi positive definite.

$K(x, x^T)$ will also be semipositive definite

$\therefore H = g(\theta^T x)(1 - g(\theta^T x))$. $x x^T$ is semi positive definite

— proved.