Definition 1. *Given a partial order $(D, \sqsubseteq)$, a non-empty subset $\Delta \subseteq D$ is called directed if*

$$\forall x, y \in \Delta . \exists z.\ x \sqsubseteq z \text{ and } y \sqsubseteq z$$

In the sequel $\Delta \subseteq_{dir}$ stands for: "Δ is a directed subset of D" (when clear from the context, the subscript is omitted). A partial order $(D, \sqsubseteq)$ is called a directed complete partial order (dcpo) if every $\Delta \subseteq D$ has a least upperbound (lub) denoted $\bigsqcup \Delta$. If moreover $(\Delta, \sqsubseteq)$ has a least element (written $\perp$), then it is called a complete partial order (cpo).

Definition 2. *Let $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ be partial orders. A function $f : D_1 \to D_2$ is called monotonic if*

$$\forall x, y \in D . x \sqsubseteq_1 y \Rightarrow f(x) \sqsubseteq_2 f(y)$$

If D_1 and D_2 are dcpo's, a function $f : D_1 \to D_2$ is called continuous if it is monotonic and

$$\forall \Delta \subseteq_{dir} X . f(\bigsqcup_1 \Delta) = \bigsqcup_2 f(\Delta)$$

(Notice that a monotonic function maps directed sets to directed sets). A fixpoint of $f : D \to D$ is an element x such that $f(x) = x$. A prefixpoint of $f : D \to D$ is an element x such that $f(x) \sqsubseteq x$. If f has a least fixpoint, we denote if by $fix(f)$.

Theorem 3. *If D is a cpo and $f : D \to D$ is continuous then $\bigsqcup_{n \in \omega} f^n(\perp)$ is a fixpoint of f, and is the least prefixpoint of f (hence it is the least fixpoint of f).*

Proof. From $\perp \sqsubseteq f(\perp)$, we get by monotonicity that $\perp, f(\perp), \ldots, f^n(\perp), \ldots$ is a increasing chain, thus is directed. By continuity of f, we have

$$f(\bigsqcup_{n \in \omega} f^n(\perp)) = \bigsqcup_{n \in \omega} f^{n+1}(\perp) = \bigsqcup_{n \in \omega} f^n(\perp)$$

Suppose $f(x) \sqsubseteq x$. We show $f^n(\perp) \sqsubseteq x$ by induction on n. The base case is clear by minimality of $\perp$. Suppose $f^n(\perp) \sqsubseteq x$: by monotonicity $f^{n+1}(\perp) \sqsubseteq f(x)$ and we conclude by transitivity. $\square$