Spectral Analysis of Signals

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1 Introduction and Basic Concepts

As the book stated, the essence of the spectral estimation is to estimate how the total power of the signal is distributed over frequency (spatial frequency) from a finite number of a stationary data sequence. In the following context, the stationary data sequence, or namely the second-order stationary sequence, is defined as follow.

Definition 1.1 (Second-order Stationary Sequence)**.** The discrete-time signal $\{y(t); t = 0, \pm 1, \pm 2, \ldots\}$ is assumed to be a second-order stationary sequence if the following assumptions are satisfied:

$$E\{y(t)\} = 0, \tag{1}$$

$$r(k) = E\left\{y(t)y^*(t-k)\right\}. \tag{2}$$

There are two definitions correspond to the power spectral density (PSD) of random signals. In the first cases, the PSD is defined as the DTFT of the covariance sequence

$$\phi(\omega) = \sum_{k=-\infty}^{\infty} r[k]e^{-i\omega k}. \tag{3}$$

The second definition of PSD is

$$\phi(\omega) = \lim_{N\to\infty} E\left\{\frac{1}{N}\left|\sum_{t=1}^{N} y(t)e^{-i\omega t}\right|^2\right\}. \tag{4}$$

This definition is equivalent to Eq.(3) under the mild assumption that the covariance sequence $\{r(k)\}$ decays sufficiently rapidly.

With the definition of PSD, the spectral estimation problem can be readily defined as: From a finite-length record $\{y[1], \ldots, y[N])\}$ of a second-order stationary random process, determine an estimate $\hat{\phi}(\omega)$ of its power spectral density $\phi(\omega)$, for $\omega \in [-\pi, \pi]$.

There are two main approaches to the PSD estimation problem. The nonparametric approach proceeds to estimate the PSD by relying essentially only on the basic definitions and on some properties that directly follow from these definitions. In particular, these methods do not make any assumption on the functional form of $\phi(\omega)$. This is in contrast with the parametric approach. That approach makes assumptions on the signal under study, which lead to a parameterized functional form of the PSD, and then proceeds by estimating the parameters in the PSD model.

2 Nonparametric Methods

2.1 Periodogram

The periodogram method relies on the Eq.(1) of the PSD.

$$\hat{\phi}_p(\omega) = \frac{1}{N} \left| \sum_{t=1}^{N} y(t) e^{-i\omega t} \right|^2 \quad (\text{ Periodogram }). \tag{5}$$

2.2 Correlogram

The correlation-based PSD leads to the correlogram spectral estimator.

$$\hat{\phi}_c(\omega) = \sum_{k=-(N-1)}^{N-1} \hat{r}(k) e^{-i\omega k} \quad (\text{ Correlogram }). \tag{6}$$

where $\hat{r}(k)$ denotes an estimate of the covariance lag $r(k)$, obtained from the available sample $\{y(1), \ldots, y(N)\}$.

An important result that reveals the relationship between periodogram and correlogram is that

$$\boxed{\hat{\phi}_c(\omega) \text{ evaluated using the standard biased ACS estimates coincides with } \phi_p(\omega)}. \tag{7}$$

The equivalence of the periodogram and correlogram spectral estimators can be used to derive their properties simultaneously.

2.3 Properties of the Periodogram Method

2.3.1 Bias Analysis of the Periodogram

$$E\left\{\hat{\phi}_p(\omega)\right\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(\psi) W_B(\omega - \psi) d\psi. \tag{8}$$

where $W_B(\omega)$ is the DTFT of the triangular window.

It follows from Eq.(8) that $W_B(\omega)$ should be a close approximation to a Dirac impulse. The half-power ($3dB$) width of the main lobe of $W_B(\omega)$ can be shown to be approximately $2\pi/N$ radians.

$$\text{main lobe width in frequency } f \simeq 1/N. \tag{9}$$

In the following context, $\frac{1}{N}$ is referred as the spectral resolution limit of the periodogram method. As N goes to infinity, we can see that

$$\lim_{N\to\infty} E\left\{\hat{\phi}_p(\omega)\right\} = \phi(\omega). \tag{10}$$

Hence, the periodogram is an asymptotically unbiased spectral estimator.

2.4 Variance Analysis of the Periodogram

For the asymptotic variance/covariance of $\hat{\phi}_p(\omega)$ in the case of Gaussian complex/circular white noise, the following result holds.

$$\lim_{N\to\infty} E\left\{\left[\hat{\phi}_p(\omega_1) - \phi(\omega_1)\right]\left[\hat{\phi}_p(\omega_2) - \phi(\omega_2)\right]\right\} = \begin{cases} \phi^2(\omega_1), & \omega_1 = \omega_2 \\ 0, & \omega_1 \neq \omega_2 \end{cases} \tag{11}$$

For a much more general signal obtained by linearly filtering the Gaussian white noise sequence $\{e(t)\}$

$$y(t) = \sum_{k=1}^{\infty} h_k e(t-k), \tag{12}$$

whose PSD is given by

$$\phi_y(\omega) = |H(\omega)|^2 \phi_e(\omega). \tag{13}$$

Then the asymptotic variance/covariance result (11) is also valid for a general linear signal as defined in (13).

The above conclusion shows that the periodogram is an inconsistent spectral estimator which continues to fluctuate around the true PSD, with a nonzero variance, even if the length of the processed sample increases without bound.

2.5 The Black-Tukey Method

The poor statistical quality of the periodogram PSD estimator has been intuitively explained as arising from both the poor accuracy of $\hat{r}(k)$ and the large number of (even if small) covariance estimation errors that are cumulatively summed up in $\hat{\phi}_c(\omega)$. Both these effects may be reduced by truncating the sum in the definition formula of $\hat{\phi}_c(\omega)$, (6). Following this idea leads to the Blackman-Tukey estimator, which is given by

$$\hat{\phi}_{BT}(\omega) = \sum_{k=-(M-1)}^{M-1} w(k)\hat{r}(k)e^{-i\omega k} = \hat{\phi}_p(\omega) * W(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{\phi}_p(\psi)W(\omega-\psi)d\psi. \tag{14}$$

The variance of the periodogram is decreased by reducing the variance of the ACF estimate by calculating more robust ACF estimates over fewer data points ($M < N$).

2.5.1 Variance and Bias Analysis of the Black-Tukey Method

- Bias:

$$E\left\{\hat{\phi}_{BT}\left(\omega_m\right)\right\} \approx \frac{1}{2\pi} P_{xx}(\omega) * W(\omega). \tag{15}$$

- Variance:

$$\operatorname{Var}\left\{\hat{\phi}_{BT}\left(\omega_m\right)\right\} \approx P_{xx}^2(\omega)\frac{1}{N}\sum_{k=-M}^{M} w^2[k]. \tag{16}$$

From the above equation, for a small bias M needs to be large to minimize the width of the mainlobe of $W\left(\omega_m\right)$, whereas M should be small in order to minimize the variance.

2.5.2 Tradeoffs in Window Design

As we can see from the Eq.(14), the design of $W(\omega)$ is critical. The following result are useful in selecting window to use in the Black-Tukey method.

- The choice of window's length should be based on a tradeoff between spectral resolution (bias) and statistical variance.
- The selection of window's shape should be based on a tradeoff between smearing and leakage effects.

2.6 Filter Bank Methods

The essence of the filter bank methods is the assumption that the PSD is constant over the band $[\omega - \beta\pi, \omega + \beta\pi]$, for some given $\beta \ll 1$. The interpretation of the filter bank method can be considered as first feeding the signal $y(t)$ into a bandpass filter with desired bandwidth and center frequency. Then calculate its output power and divided it by the filter bandwidth to get the PSD estimation.

2.6.1 Slepian Filter Bank Method

The key of the slepian filter bank method is its design criterion of the bandpass filter. While maximizing the bandpass output, it sets a limit of the output power at all frequency. The derivation is introduced as follows.

Let the input to the filter be white noise of unit variance. Then the power of the output is:

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi}|H(\omega)|^2 d\omega &= \sum_{k=0}^{N-1}\sum_{p=0}^{N-1} h_k h_p^* \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega(p-k)} d\omega\right] \\ &= \sum_{k=0}^{N-1}\sum_{p=0}^{N-1} h_k h_p^* \delta_{k,p} = h^* h. \end{aligned} \tag{17}$$

The part of the power that lies in the baseband is given by

$$\frac{1}{2\pi}\int_{-\beta\pi}^{\beta\pi}|H(\omega)|^2 d\omega = h^*\left\{\frac{1}{2\pi}\int_{-\beta\pi}^{\beta\pi} a(\omega)a^*(\omega)d\omega\right\} h \triangleq h^*\Gamma h. \tag{18}$$

Since the filter h must be such that the power of the filtered signal in the baseband is as large as possible relative to the total power, we are led to the following optimization problem:

$$\max_h h^*\Gamma h \quad \text{subject to } h^* h = 1 \tag{19}$$

2.6.2 Capon Method

The slepian filter bank method uses a bandpass filter specifically designed in case of a white noise input. Presumably, it might be valuable to take the data properties into consideration when designing the bandpass. This is the basic idea behind the Capon method.

The idea that lies behind the derivation of Capon method is that the filter is designed for minimizing the total power subject to the constraint that the filter passes the frequency band undistorted. This idea leads to the following optimization problem:

$$\min_h h^* R h \quad \text{subject to } h^* a(\omega) = 1. \tag{20}$$

3 Parametric Methods

3.1 Rational Spectra

3.1.1 Signals with Rational Spectra

Our analysis begins by using the following result:

- The arbitrary rational PSD can be associated with a signal obtained by filtering white noise of power σ^2 through the rational filter with transfer function $H(\omega) = B(\omega)/A(\omega)$.

The filtering referred to the above result can be written in the time domain as

$$A(z)y(t) = B(z)e(t), \tag{21}$$

where $y(t)$ is the filter output, and commonly refers as autoregressive moving average (ARMA) signal.

3.1.2 Covariance Structure of ARMA Process

Equation (21) can be written as

$$y(t) + \sum_{i=1}^{n} a_i y(t-i) = \sum_{j=0}^{m} b_j e(t-j), \quad (b_0 = 1)\,. \tag{22}$$

Multiplying (22) by $y^*(t-k)$, taking expectation, and then afterwards assuming the filter $H(z) = B(z)/A(z)$ is asymptotically (as the order of n and m increases) stable and causal, we have:

$$r(k) + \sum_{i=1}^{n} a_i r(k-i) = \sigma^2 \sum_{j=0}^{m} b_j h^*_{j-k}. \tag{23}$$

3.2 AR Signals

3.2.1 Yule-Walker Method

For AR signals, $m = 0$ and $B(z) = 1$. We have from equation (23) that

$$r(0) + \sum_{i=1}^{n} a_i r(-i) = \sigma^2 \sum_{j=0}^{0} b_j h^*_j = \sigma^2 \tag{24}$$

The above equation gives the Yule-Walker equation:

$$\begin{bmatrix} r(0) & r(-1) & \dots & r(-n) \\ r(1) & r(0) & & \vdots \\ \vdots & & \ddots & r(-1) \\ r(n) & \dots & & r(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sigma^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{25}$$

3.2.2 Least-Square Method

Let $y(t)$ be an AR process of order n. Then it satisfies

$$\begin{aligned} e(t) &= y(t) + \sum_{i=1}^{n} a_i y(t-i) = y(t) + \varphi^T(t)\theta \\ &\triangleq y(t) + \hat{y}(t). \end{aligned} \tag{26}$$

where $\varphi(t) = [y(t-1), \dots, y(t-n)]^T$. We interpret $\hat{y}(t)$ as a linear prediction of $y(t)$ from the n previous samples $y(t-1), \dots, y(t-n)$, and we interpret $e(t)$ as the corresponding prediction error. The vector θ that minimizes the prediction error variance $\sigma_n^2 \triangleq E\left\{|e(t)|^2\right\}$ is the AR coefficient vector. The vector θ that minimizes error variance is given by

$$\theta = -R_n^{-1} r_n. \tag{27}$$

By substituting the R_n and r_n by standard biased ACS estimate, we can draw a conclusion that:

- The autocorrelation method of least squares AR estimation is equivalent to the Yule-Walker method.

3.2.3 ARMA Signals

3.2.4 Modified Yule-Walker Method

The modified Yule-Walker method is a two-stage procedure for estimating the ARMA spectral density. In the first stage we estimate the AR coefficients using equation (25). In the second stage, we use the AR coefficient and ACS estimates in equation to estimate the γ_k coefficients. We describe the two steps below.

Writing equation (23) for $k = m+1, m+2, \dots, m+M$ in a matrix form gives

$$\begin{bmatrix} r(m) & r(m-1) & \dots & r(m-n+1) \\ r(m+1) & r(m) & & r(m-n+2) \\ \vdots & & \ddots & \vdots \\ r(m+M-1) & \dots & \dots & r(m-n+M) \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = - \begin{bmatrix} r(m+1) \\ r(m+2) \\ \vdots \\ r(m+M) \end{bmatrix}. \tag{28}$$

Once the AR estimates are obtained, we turn to the problem of estimating the MA part of the ARMA spectrum. Let

$$\gamma_k = E\left\{[B(z)e(t)][B(z)e(t-k)]^*\right\} \tag{29}$$

denote the covariances of the MA part. Since the PSD of this part of the ARMA signal model is given by:

$$\sigma^2|B(\omega)|^2 = \sum_{k=-m}^{m} \gamma_k e^{-i\omega k} \tag{30}$$

it suffices to estimate $\{\gamma_k\}$ in order to characterize the spectrum of the MA part. From (3.2.7) and (3.7.5), we obtain

$$\begin{aligned}
\gamma_k &= E\left\{[A(z)y(t)][A(z)y(t-k)]^*\right\} \\
&= \sum_{j=0}^{n}\sum_{p=0}^{n} a_j a_p^* E\left\{y(t-j)y^*(t-k-p)\right\} \\
&= \sum_{j=0}^{n}\sum_{p=0}^{n} a_j a_p^* r(k+p-j) \qquad \left(a_0 \triangleq 1\right),
\end{aligned} \tag{31}$$

for $k = 0, \ldots, m$.

Finally, the ARMA spectrum is estimated as follows:

$$\hat{\phi}(\omega) = \frac{\sum_{k=-m}^{m} \hat{\gamma}_k e^{-i\omega k}}{|\hat{A}(\omega)|^2}. \tag{32}$$

4 Parametric Method for Line Spectra

4.1 Nonlinear Least Squares (NLS) Method

We first described the signal by the following sinusoidal model:

$$y(t) = x(t) + e(t); \quad x(t) = \sum_{k=1}^{n} \alpha_k e^{i(\omega_k t + \varphi_k)}, \tag{33}$$

where $x(t)$ denotes the noise-free complex-valued sinusoidal signal; $\{\alpha_k\}, \{\omega_k\}$, $\{\varphi_k\}$ are its amplitudes, (angular) frequencies and initial phases, respectively; and $e(t)$ is an additive observation noise.

The nonlinear regression method determining the unknown parameters as the minimizers of the following criterion:

$$f(\omega, \alpha, \varphi) = \sum_{t=1}^{N} \left| y(t) - \sum_{k=1}^{n} \alpha_k e^{i(\omega_k t + \varphi_k)} \right|^2. \tag{34}$$

4.1.1 Pisarenko and MUSIC Methods

The MUSIC method make use of the structure and the information contained in the covariance matrix. Assuming that the noise signal is white Gaussian, we computing the sample covariance matrix

$$\hat{R} = \frac{1}{N} \sum_{t=m}^{N} \tilde{y}(t)\tilde{y}^*(t) \tag{35}$$

, and decompose it as

$$APA^* + \sigma^2 I, \tag{36}$$

where

$$P = \begin{bmatrix} \alpha_1^2 & & 0 \\ & \ddots & \\ 0 & & \alpha_n^2 \end{bmatrix} \tag{37}$$

$$\begin{aligned} a(\omega) &\triangleq \left[1 \; e^{-i\omega} \ldots e^{-i(m-1)\omega}\right]^T \quad (m \times 1) \\ A &= [a(\omega_1) \ldots a(\omega_n)] \quad (m \times n). \end{aligned} \tag{38}$$

Let $\hat{S}$ and $\hat{G}$ denote the corresponding matrix made by the arrangement of the eigenvectors of the non-zero eigenvalues and zero eigenvalues, respectively. Then the true frequency values are the only solutions of the equation

$$a^*(\omega)GG^*a(\omega) = 0 \text{ for any } m > n. \tag{39}$$

The main features of the MUSIC method can be summarized as follows:

- Their statistical performance is close to the ultimate performance corresponding to the NLS method.
- It may generally have a lower resolution threshold than that of the periodogram.
- The chief drawback of MUSIC method, as compared with the NLS method, is that their performance significantly degrades if the measurement noise cannot be assumed to be white.

5 Spatial Methods

In this section, the author consider the problem of locating n radiating sources by using an array of m passive sensors. The development of the spatial model is based on a number of simplifying assumptions. So the only parameter that characterizes the source locations is the direction of arrival (DOA).

5.1 Array Model

$$y(t) = [a(\theta_1) \dots a(\theta_n)] \begin{bmatrix} s_1(t) \\ \vdots \\ s_n(t) \end{bmatrix} + e(t) \triangleq As(t) + e(t), \tag{40}$$

$$\theta_k = \text{ the DOA of the } k \text{ th source}$$
$$s_k(t) = \text{ the signal corresponding to the } k \text{ th source}$$

where

$$a(\theta) = \begin{bmatrix} 1 & e^{-i\omega_c\tau_2} \dots e^{-i\omega_c\tau_m} \end{bmatrix}^T. \tag{41}$$

For uniform linear array,

$$\tau_k = (k-1)\frac{d\sin\theta}{c} \qquad \text{for } \theta \in [-90°, 90°]. \tag{42}$$

Analog to the frequency derivation of the PSD estimation, we define w_s as the spatial frequency, which can be written as:

$$\omega_s = 2\pi f_s = \omega_c \frac{d\sin\theta}{c}. \tag{43}$$

Therefore, for $a(\theta)$ we have

$$a(\theta) = \left[1 \quad e^{-i\omega_s} \dots e^{-i(m-1)\omega_s}\right]^T. \tag{44}$$

Equipped with the array model, we can reduced the problem of DOA finding to that of estimating the parameters θ_k in equation (40). For parametric method, as there is a direct analogy between equation (40) and the model for Line Spectra, most of the methods developed in line spectra can be used for DOA estimation.

5.2 Nonparamaetric Methods

In the time series case, a FIR filter is defined by the relation

$$y_F(t) = \sum_{k=0}^{m-1} h_k u(t-k) \triangleq h^* y(t) \tag{45}$$

where $\{h_k\}$ are the filter weights, $u(t)$ is the input to the filter and

$$\begin{aligned} h &= [h_0 \dots h_{m-1}]^* \\ y(t) &= [u(t) \dots u(t-m+1)]^T. \end{aligned} \tag{46}$$

Similarly, we can use the spatial samples $\{y_k(t)\}_{k=1}^{m}$ obtained with a sensor array to define a spatial filter:

$$y_F(t) = h^* y(t) \tag{47}$$

We can rewrite the above equation into another form:

$$y_F(t) = [h^* a(\theta)]\, s(t). \tag{48}$$

This equation clearly shows that the spatial filter can be selected to enhance (attenuate) the signals coming from a given direction θ, by making $h^* a(\theta)$ large (small). This observation lies at the basis of the DOA estimation methods. All of these methods can be derived by using the filter bank approaches in section 2.6.