Physics Advanced Topics Daniel Castillo Castro

Homework 1

Contents

1 Preliminaries

1.1 2 Level System (with pulses)

1.1.1 Hamiltonian under unitary transformation

The matrix H written in the $\{|e\rangle, |g\rangle\}$ basis (naming $c_x = b_x cos(\omega t)$ and $c_y = b_y cos(\omega t)$, $= 1$)

$$H = \begin{pmatrix} b_x cos(\omega t) \\ b_y cos(\omega t) \\ \gamma B \end{pmatrix} \cdot \vec{\sigma} = \begin{pmatrix} \gamma B & c_x - ic_y \\ c_x + ic_y & -\gamma B \end{pmatrix} \tag{1}$$

Is called Hamiltonian because is present in the Schrodinger equation:

$$\begin{pmatrix} |\dot{e}\rangle \\ |\dot{g}\rangle \end{pmatrix} = \begin{pmatrix} -i\gamma B & -c_y - ic_x \\ c_y - ic_x & i\gamma B \end{pmatrix} \begin{pmatrix} |e\rangle \\ |g\rangle \end{pmatrix} \tag{2}$$

Now, for the change of basis:

$$\begin{pmatrix} |\tilde{e}\rangle \\ |\tilde{g}\rangle \end{pmatrix} = \begin{pmatrix} e^{i\omega t} & 0 \\ 0 & e^{i\omega t} \end{pmatrix} \begin{pmatrix} |e\rangle \\ |g\rangle \end{pmatrix} \tag{3}$$

The time derivative is, by rule of chain:

$$\frac{d}{dt} \begin{pmatrix} |\tilde{e}\rangle \\ |\tilde{g}\rangle \end{pmatrix} = (i\omega) \begin{pmatrix} |\tilde{e}\rangle \\ |\tilde{g}\rangle \end{pmatrix} + e^{i\omega t} \begin{pmatrix} |\dot{e}\rangle \\ |\dot{g}\rangle \end{pmatrix} \tag{4}$$

And using the equation 2

$$\frac{d}{dt} \begin{pmatrix} |\tilde{e}\rangle \\ |\tilde{g}\rangle \end{pmatrix} = (i\omega\mathbb{I} + e^{i\omega t} \begin{pmatrix} -i\gamma B & -c_y - ic_x \\ c_y - ic_x & i\gamma B \end{pmatrix} e^{-i\omega t}) \begin{pmatrix} |\tilde{e}\rangle \\ |\tilde{g}\rangle \end{pmatrix} \tag{5}$$

By the definition of c_x and c_y it appears a product between $cos(\omega t)$ and $e^{-i\omega}$ that can be approximated if rapidly oscillating terms are neglected:

$$cos(\omega t)e^{-i\omega t} = \frac{(e^{i\omega t} + e^{-i\omega t})e^{-i\omega t}}{2} = \frac{1 + e^{-2i\omega t}}{2} \simeq \frac{1}{2} \tag{6}$$

Defining $c_\pm = c_x \pm ic_y$, the approximation of 6 makes it:

$$c_\pm e^{-i\omega t} = (c_x \pm ic_y)e^{-i\omega t} = (b_x \pm ib_y)cos(\omega t)e^{-i\omega t} \simeq \frac{b_x \pm ib_y}{2} \tag{7}$$

Using 7, 5 can be simplified:

$$\frac{d}{dt}\begin{pmatrix}|\tilde{e}\rangle\\|\tilde{g}\rangle\end{pmatrix} = \begin{pmatrix}-i(\gamma B-\omega) & -ie^{i\omega t}\frac{b_-}{2}\\ -ie^{i\omega t}\frac{b_+}{2} & i(\gamma B-\omega)\end{pmatrix}\begin{pmatrix}|\tilde{e}\rangle\\|\tilde{g}\rangle\end{pmatrix} \tag{8}$$

And finally, defining $B_x = \frac{e^{i\omega t}b_-}{2}$ and $B_y = \frac{e^{i\omega}b_+}{2}$, the hamiltonian for the $\{|\tilde{e}\rangle, |\tilde{g}\rangle\}$ basis is ($\Delta = \gamma B - \omega$, $B_\pm = B_x \pm iB_y$)

$$H = \begin{pmatrix}B_x\\B_y\\\Delta\end{pmatrix}\cdot\vec{\sigma} = \begin{pmatrix}\Delta & B_-\\ B_+ & -\Delta\end{pmatrix} \tag{9}$$

1.1.2 Time evolution for the excited state

The matrix of 9 can be splitted into 2 operators ($r^2 = \Delta^2 + B_x^2 + B_y^2$):

$$H = \frac{1}{2}\begin{pmatrix}\Delta+r & B_-\\ B_+ & r-\Delta\end{pmatrix} - \frac{1}{2}\begin{pmatrix}r-\Delta & -B_-\\ -B_+ & \Delta+r\end{pmatrix} \tag{10}$$

The both operators are projectors for orthogonal states:

$$\begin{aligned}\frac{1}{2}\begin{pmatrix}\Delta+r & B_-\\ B_+ & r-\Delta\end{pmatrix} &\Rightarrow v_+ = \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\Delta+r}\\ \frac{B_+}{\sqrt{\Delta+r}}\end{pmatrix}\\ \frac{1}{2}\begin{pmatrix}r-\Delta & -B_-\\ -B_+ & \Delta+r\end{pmatrix} &\Rightarrow v_- = \frac{1}{\sqrt{2}}\begin{pmatrix}\frac{-B_-}{\sqrt{\Delta+r}}\\ \sqrt{\Delta+r}\end{pmatrix}\end{aligned} \tag{11}$$

Therefore, it was obtained a Spectral Decomposition for H

$$H = v_+v_+^\dagger - v_-v_-^\dagger \tag{12}$$

So, the evolution operator e^{iHt} for the system is ($\hbar = 1$)

$$e^{-it}v_+v_+^\dagger + e^{it}v_-v_-^\dagger = \begin{pmatrix}rcos(t) - i\Delta sin(t) & -iB_-sin(t)\\ iB_+sin(t) & rcos(t) + i\Delta sin(t)\end{pmatrix} \tag{13}$$

If the initial condition is:

$$|\phi(0)\rangle = |\tilde{e}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{14}$$

The state after time evolution is:

$$|\phi(t)\rangle = e^{-iHt} |\phi(0)\rangle = \begin{pmatrix} rcos(t) - i\Delta sin(t) \\ iB_+ sin(t) \end{pmatrix} \tag{15}$$

1.1.3 Can the evolution to reach the ground state?

With the definition given in 15, it can be answered this question. If in a given time $|\tilde{\phi(t)}\rangle = |g\rangle$ this implies that $\langle g| |\phi(t)\rangle = 1$:

$$B_+B_- sin^2(t) = 1 \Rightarrow sin^2(t) = \frac{1}{B_+B_-} \tag{16}$$

Using that $cos^2(t) = 1 - sin^2(t)$ and 16, it can be write down the another condition: $\langle e| |\phi(t)\rangle = 0$

$$r^2 cos^2(t) + \Delta^2 sin^2(t) = 0 \Rightarrow r^2(1 - \frac{1}{B_+B_-}) + \frac{\Delta^2}{B_+B_-} = 0 \tag{17}$$

If $r^2 = \Delta^2 + B_+B_-$, the numerator 17 gives the condition:

$$r^2 B_+B_- - r^2 + \Delta^2 = 0 \Rightarrow (r^2 - 1)B_+B_- = 0 \Rightarrow r^2 = 1 \tag{18}$$

So, for the ground state can be reached, it must needed

$$r^2 = \Delta^2 + B_+B_- = (\gamma B - \omega) + \frac{b_x^2 + b_y^2}{4} = 1 \tag{19}$$

1.2 Time Evolution in Two Level System

For the next Hamiltonian ($|n|^2 = n_x^2 + n_y^2 + n_z^2 = 1$):

$$H = \gamma B \hat{n} \cdot \vec{\sigma} = \begin{pmatrix} \gamma B n_z & \gamma B(n_x - in_y) \\ \gamma B(n_x + in_y) & -\gamma B n_z \end{pmatrix} \tag{20}$$

The procedure done in 10 can be done two, finding H as a sum of 2 proyectors, and by the way, it spectral decomposition:

$$H = \frac{\gamma B}{2}[\begin{pmatrix} 1+n_z & n_x - in_y \\ n_x + in_y & 1-n_z \end{pmatrix} - \begin{pmatrix} 1-n_z & -n_x + in_y \\ -n_x - in_y & 1+n_z \end{pmatrix}] \quad (21)$$

Using it, it can be found the evolution operator, that has the same form of the found in 13 ($n_\pm = n_x \pm in_y$):

$$e^{-iH\tau} = \begin{pmatrix} cos(\frac{\gamma B\tau}{2}) - in_z sin(\frac{\gamma B\tau}{2}) & -in_- sin(\frac{\gamma B\tau}{2}) \\ in_+ sin(\frac{\gamma B\tau}{2}) & cos(\frac{\gamma B\tau}{2}) + in_z sin(\frac{\gamma B\tau}{2}) \end{pmatrix} \quad (22)$$

Defining $\theta = \gamma b\tau$, the evolution operator can be understood as a rotating operator:

$$e^{-iH\tau} = \begin{pmatrix} cos(\frac{\theta}{2}) - in_z sin(\frac{\theta}{2}) & -in_- sin(\frac{\theta}{2}) \\ in_+ sin(\frac{\theta}{2}) & cos(\frac{\theta}{2}) + in_z sin(\frac{\theta}{2}) \end{pmatrix} \quad (23)$$

The operator of 23 will be analised the next special cases:

- If $\hat{n} = \hat{x}$, $n_x = 1, n_y = n_z = 0$, 23 becomes

$$e^{-iH\tau} = \begin{pmatrix} cos(\frac{\theta}{2}) & -isin(\frac{\theta}{2}) \\ isin(\frac{\theta}{2}) & cos(\frac{\theta}{2}) \end{pmatrix} \quad (24)$$

- If $\hat{n} = \hat{y}$, $n_y = 1, n_z = n_x = 0$, 23 becomes

$$e^{-iH\tau} = \begin{pmatrix} cos(\frac{\theta}{2}) & -sin(\frac{\theta}{2}) \\ sin(\frac{\theta}{2}) & cos(\frac{\theta}{2}) \end{pmatrix} \quad (25)$$

- If $\hat{n} = \hat{z}$, $n_z = 1, n_x = n_y = 0$, 23 becomes

$$e^{-iH\tau} = \begin{pmatrix} cos(\frac{\theta}{2}) - isin(\frac{\theta}{2}) & 0 \\ 0 & cos(\frac{\theta}{2}) + isin(\frac{\theta}{2}) \end{pmatrix} \quad (26)$$

$$
\begin{array}{llll}
|\phi(0)\rangle = |e\rangle & \hat{n}=\hat{x} & \hat{n}=\hat{y} & \hat{n}=\hat{z} \\
\theta = \frac{\pi}{2} & |\phi(t)\rangle = \frac{|e\rangle + i|g\rangle}{\sqrt{2}} & |\phi(t)\rangle = \frac{|e\rangle + |g\rangle}{\sqrt{2}} & |\phi(t)\rangle = \frac{1-i}{\sqrt{2}}|e\rangle \\
\theta = \pi & |\phi(t)\rangle = i|g\rangle & |\phi(t)\rangle = |g\rangle & |\phi(t)\rangle = -i|e\rangle
\end{array}
$$

$$
\begin{array}{llll}
|\phi(0)\rangle = |g\rangle & \hat{n}=\hat{x} & \hat{n}=\hat{y} & \hat{n}=\hat{z} \\
\theta = \frac{\pi}{2} & |\phi(t)\rangle = \frac{|g\rangle - i|e\rangle}{\sqrt{2}} & |\phi(t)\rangle = \frac{|g\rangle - |e\rangle}{\sqrt{2}} & |\phi(t)\rangle = \frac{1+i}{\sqrt{2}}|g\rangle \\
\theta = \pi & |\phi(t)\rangle = -i|e\rangle & |\phi(t)\rangle = -|e\rangle & |\phi(t)\rangle = i|g\rangle
\end{array}
$$

And replacing for $\theta = \frac{\pi}{2}$ and $\theta = \pi$, it will be obtained the next tables for $|\phi_0\rangle = |e\rangle$ and $|\phi_0\rangle = |g\rangle$:

1.3 Final State after Sequences

For an initial state ($|\alpha|^2 + |\beta|^2 = 1$):

$$
|\varphi_0\rangle = \alpha|e\rangle + \beta|g\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{27}
$$

It can be done a Ramsey Sequence ($\varphi = \frac{\gamma B \tau}{2}$)

$$
\begin{aligned}
(\frac{\pi}{2}_x) : \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &\rightarrow \begin{pmatrix} \frac{\alpha+i\beta}{\sqrt{2}} \\ \frac{\beta-i\alpha}{\sqrt{2}} \end{pmatrix} \\
(\tau) : \begin{pmatrix} \frac{\alpha+i\beta}{\sqrt{2}} \\ \frac{\beta-i\alpha}{\sqrt{2}} \end{pmatrix} &\rightarrow \begin{pmatrix} \frac{(cos\varphi - in_z sin\varphi)(\alpha+i\beta) - in_- sin\varphi(\beta-i\alpha)}{\sqrt{2}} \\ \frac{in_+ sin\varphi(\alpha+i\beta) + (cos\varphi + in_z sin\varphi)(\beta-i\alpha)}{\sqrt{2}} \end{pmatrix} \\
(\frac{\pi}{2}_x) : \begin{pmatrix} \frac{(cos\varphi - isin\varphi)(\alpha+i\beta)}{\sqrt{2}} \\ \frac{(cos\varphi + isin\varphi)(\beta-i\alpha)}{\sqrt{2}} \end{pmatrix} &\rightarrow \begin{pmatrix} \frac{e^{-i\varphi}(\alpha+i\beta) + ie^{i\varphi}(\beta-i\alpha)}{2} \\ \frac{e^{i\varphi}(\beta-i\alpha) - ie^{-i\varphi}(\alpha+i\beta)}{2} \end{pmatrix}
\end{aligned} \tag{28}
$$

It can be done a kind of Echo Sequence ($\varphi = \frac{\gamma B\tau}{2}$)

$$
\begin{aligned}
(\frac{\pi}{2}_x) : \begin{pmatrix}\alpha\\ \beta\end{pmatrix} &\rightarrow \begin{pmatrix}\frac{\alpha+i\beta}{\sqrt{2}}\\ \frac{\beta-i\alpha}{\sqrt{2}}\end{pmatrix}\\
(\tau) : \begin{pmatrix}\frac{\alpha+i\beta}{\sqrt{2}}\\ \frac{\beta-i\alpha}{\sqrt{2}}\end{pmatrix} &\rightarrow \begin{pmatrix}\frac{(cos\varphi-in_z sin\varphi)(\alpha+i\beta)-in_- sin\varphi(\beta-i\alpha)}{\sqrt{2}}\\ \frac{in_+ sin\varphi(\alpha+i\beta)+(cos\varphi+in_z sin\varphi)(\beta-i\alpha)}{\sqrt{2}}\end{pmatrix}\\
(\pi_x) : \begin{pmatrix}\frac{(cos\varphi-isin\varphi)(\alpha+i\beta)}{\sqrt{2}}\\ \frac{(cos\varphi+isin\varphi)(\beta-i\alpha)}{\sqrt{2}}\end{pmatrix} &\rightarrow \begin{pmatrix}-ie^{i\varphi}(\frac{\beta-i\alpha}{\sqrt{2}})\\ ie^{-i\varphi}(\frac{\alpha+i\beta}{\sqrt{2}})\end{pmatrix}\\
(\tau) : \begin{pmatrix}-e^{i\varphi}(\frac{\alpha+i\beta}{\sqrt{2}})\\ -e^{-i\varphi}(\frac{\beta-i\alpha}{\sqrt{2}})\end{pmatrix} &\rightarrow \begin{pmatrix}-e^{-i\varphi}e^{i\varphi}\frac{\alpha+i\beta}{\sqrt{2}}\\ -e^{i\varphi}e^{-i\varphi}\frac{\beta-i\alpha}{\sqrt{2}}\end{pmatrix}\\
(\frac{\pi}{2}_x) : \begin{pmatrix}-\frac{\alpha+i\beta}{\sqrt{2}}\\ -\frac{\beta-i\alpha}{\sqrt{2}}\end{pmatrix} &\rightarrow \begin{pmatrix}\frac{-(\alpha+i\beta+i(\beta-i\alpha))}{2}\\ \frac{-(\beta-i\alpha-i(\alpha+i\beta))}{2}\end{pmatrix}
\end{aligned}
\tag{29}
$$

It can be done another kind of Echo Sequence ($\varphi = \frac{\gamma B\tau}{2}$)

$$
\begin{aligned}
(\frac{\pi}{2}_x) : \begin{pmatrix}\alpha\\ \beta\end{pmatrix} &\rightarrow \begin{pmatrix}\frac{\alpha+i\beta}{\sqrt{2}}\\ \frac{\beta-i\alpha}{\sqrt{2}}\end{pmatrix}\\
(\tau) : \begin{pmatrix}\frac{\alpha+i\beta}{\sqrt{2}}\\ \frac{\beta-i\alpha}{\sqrt{2}}\end{pmatrix} &\rightarrow \begin{pmatrix}\frac{(cos\varphi-in_z sin\varphi)(\alpha+i\beta)-in_- sin\varphi(\beta-i\alpha)}{\sqrt{2}}\\ \frac{in_+ sin\varphi(\alpha+i\beta)+(cos\varphi+in_z sin\varphi)(\beta-i\alpha)}{\sqrt{2}}\end{pmatrix}\\
(\pi_x) : \begin{pmatrix}\frac{(cos\varphi-isin\varphi)(\alpha+i\beta)}{\sqrt{2}}\\ \frac{(cos\varphi+isin\varphi)(\beta-i\alpha)}{\sqrt{2}}\end{pmatrix} &\rightarrow \begin{pmatrix}-ie^{i\varphi}(\frac{\beta-i\alpha}{\sqrt{2}})\\ ie^{-i\varphi}(\frac{\alpha+i\beta}{\sqrt{2}})\end{pmatrix}\\
(\tau) : \begin{pmatrix}-e^{i\varphi}(\frac{\alpha+i\beta}{\sqrt{2}})\\ -e^{-i\varphi}(\frac{\beta-i\alpha}{\sqrt{2}})\end{pmatrix} &\rightarrow \begin{pmatrix}-e^{-i\varphi}e^{i\varphi}\frac{\alpha+i\beta}{\sqrt{2}}\\ -e^{i\varphi}e^{-i\varphi}\frac{\beta-i\alpha}{\sqrt{2}}\end{pmatrix}\\
(\frac{\pi}{2}_y) : \begin{pmatrix}-\frac{\alpha+i\beta}{\sqrt{2}}\\ -\frac{\beta-i\alpha}{\sqrt{2}}\end{pmatrix} &\rightarrow \begin{pmatrix}\frac{-(\alpha+i\beta+\beta-i\alpha)}{2}\\ \frac{-(\beta-i\alpha-\alpha-i\beta)}{2}\end{pmatrix}
\end{aligned}
\tag{30}
$$

In the processes 28, 29 and 30, the operators were applicated and in the next step, is simplified the result state. Therefore, The final and normalised vectors after the processes are:

$$\begin{aligned}
|\phi\rangle_{Ramsey} &= \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha+i\beta\\ \beta-i\alpha\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} + \frac{i}{\sqrt{2}}\begin{pmatrix}\beta\\ -\alpha\end{pmatrix}\\
|\phi\rangle_{Echo1} &= \begin{pmatrix}\frac{-\alpha-i\beta}{\sqrt{2}}\\ \frac{-\beta+i\alpha}{\sqrt{2}}\end{pmatrix} = \frac{-1}{\sqrt{2}}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} + \frac{-i}{\sqrt{2}}\begin{pmatrix}\beta\\ -\alpha\end{pmatrix}\\
|\phi\rangle_{Echo2} &= \begin{pmatrix}\frac{(\alpha+i\beta)(i-1)}{2}\\ \frac{(\beta-i\alpha)(i-1)}{2}\end{pmatrix} = \frac{i-1}{2}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} + \frac{-1-i}{2}\begin{pmatrix}\beta\\ -\alpha\end{pmatrix}
\end{aligned} \tag{31}$$

For the 3 vectors, the probability to obtain again the initial state of 27 can be obtained by the splitting made in 31 for each one of them. The probabilities will be:

$$\begin{aligned}
P_{0,Ramsey} &= (\frac{1}{\sqrt{2}})^2 = \frac{1}{2}\\
P_{0,Echo1} &= (\frac{-1}{\sqrt{2}})^2 = \frac{1}{2}\\
P_{0,Echo2} &= |\frac{i-1}{2}|^2 = (\frac{\sqrt{2}}{2})^2 = \frac{1}{2}
\end{aligned} \tag{32}$$

2 Central Spin and Random Fluctuating Field

In the suggested problem (spin $\frac{1}{2}$ under a magnetic field B_0, is valid a Hamiltonian similar to the defined in 20. If the rotating frame is along the $\hat{z}$ axis, $n_z = 1$ and $n_x = n_y = 0$:

$$H = \gamma B\hat{\sigma}_z = \begin{pmatrix}\gamma B & 0\\ 0 & -\gamma B\end{pmatrix} \tag{33}$$

The evolution operator derived by 33 is

$$e^{-iH\tau} = \begin{pmatrix}cos(\frac{\gamma B\tau}{2}) - isin(\frac{\gamma B\tau}{2}) & 0\\ 0 & cos(\frac{\gamma B\tau}{2}) + isin(\frac{\gamma B\tau}{2})\end{pmatrix} \tag{34}$$

2.1 Static Magnetic Field

The both analysis are made an initial state given by 27 and the tables developed in a previous section:

2.1.1 State after Ramsey sequence

As it has done in 28, it is made a Ramsey Sequence, now with the Hamiltonian of 34:

$$
\begin{aligned}
(\frac{\pi}{2}_x) : \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &\rightarrow \begin{pmatrix} \frac{\alpha+i\beta}{\sqrt{2}} \\ \frac{\beta-i\alpha}{\sqrt{2}} \end{pmatrix} \\
(\tau) : \begin{pmatrix} \frac{\alpha+i\beta}{\sqrt{2}} \\ \frac{\beta-i\alpha}{\sqrt{2}} \end{pmatrix} &\rightarrow \begin{pmatrix} \frac{(cos\varphi-isin\varphi)(\alpha+i\beta)}{\sqrt{2}} \\ \frac{(cos\varphi+isin\varphi)(\beta-i\alpha)}{\sqrt{2}} \end{pmatrix} \\
(\frac{\pi}{2}_x) : \begin{pmatrix} \frac{e^{-i\varphi}(\alpha+i\beta)}{\sqrt{2}} \\ \frac{e^{i\varphi}(\beta-i\alpha)}{\sqrt{2}} \end{pmatrix} &\rightarrow \begin{pmatrix} \frac{e^{-i\varphi}(\alpha+i\beta)+ie^{i\varphi}(\beta-i\alpha)}{2} \\ \frac{e^{i\varphi}(\beta-i\alpha)-ie^{-i\varphi}(\alpha+i\beta)}{2} \end{pmatrix}
\end{aligned}
\tag{35}
$$

Such that the final state is:

$$
|\phi(\tau)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha+i\beta \\ \beta-i\alpha \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \frac{i}{\sqrt{2}}\begin{pmatrix} \beta \\ -\alpha \end{pmatrix} \tag{36}
$$

And the probability to be in the initial state is:

$$
P_0 = (\frac{1}{\sqrt{2}})^2 = \frac{1}{2} \tag{37}
$$

2.1.2 State after Echo sequence

As it has done in 29, it is made a Echo Sequence, now with the Hamiltonian of 34:

$$\begin{gathered}(\frac{\pi}{2}_x):\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\rightarrow\begin{pmatrix}\frac{\alpha+i\beta}{\sqrt{2}}\\ \frac{\beta-i\alpha}{\sqrt{2}}\end{pmatrix}\\ (\tau):\begin{pmatrix}\frac{\alpha+i\beta}{\sqrt{2}}\\ \frac{\beta-i\alpha}{\sqrt{2}}\end{pmatrix}\rightarrow\begin{pmatrix}\frac{(cos\varphi-isin\varphi)(\alpha+i\beta)}{\sqrt{2}}\\ \frac{(cos\varphi+isin\varphi)(\beta-i\alpha)}{\sqrt{2}}\end{pmatrix}\\ (\pi_x):\begin{pmatrix}\frac{e^{-i\varphi}(\alpha+i\beta)}{\sqrt{2}}\\ \frac{e^{i\varphi}(\beta-i\alpha)}{\sqrt{2}}\end{pmatrix}\rightarrow\begin{pmatrix}-ie^{i\varphi}(\frac{\beta-i\alpha}{\sqrt{2}})\\ ie^{-i\varphi}(\frac{\alpha+i\beta}{\sqrt{2}})\end{pmatrix}\\ (\tau):\begin{pmatrix}-e^{i\varphi}(\frac{\alpha+i\beta}{\sqrt{2}})\\ -e^{-i\varphi}(\frac{\beta-i\alpha}{\sqrt{2}})\end{pmatrix}\rightarrow\begin{pmatrix}-e^{-i\varphi}e^{i\varphi}\frac{\alpha+i\beta}{\sqrt{2}}\\ -e^{i\varphi}e^{-i\varphi}\frac{\beta-i\alpha}{\sqrt{2}}\end{pmatrix}\\ (\frac{\pi}{2}_x):\begin{pmatrix}-\frac{\alpha+i\beta}{\sqrt{2}}\\ -\frac{\beta-i\alpha}{\sqrt{2}}\end{pmatrix}\rightarrow\begin{pmatrix}\frac{-(\alpha+i\beta+i(\beta-i\alpha))}{2}\\ \frac{-(\beta-i\alpha-i(\alpha+i\beta))}{2}\end{pmatrix}\end{gathered} \tag{38}$$

Such that the final state is:

$$|\phi(\tau)\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}-\alpha-i\beta\\ -\beta+i\alpha\end{pmatrix}=\frac{-1}{\sqrt{2}}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}+\frac{-i}{\sqrt{2}}\begin{pmatrix}\beta\\ -\alpha\end{pmatrix} \tag{39}$$

And the probability to be in the initial state is the same that the obtained in 37, that is $P_0=\frac{1}{2}$.

2.2 Static and Random Magnetic Fields

If it is added a random element in the Magnetic Field, it will be observed more effects. Making a Taylor expansion for the exponential function:

$$e^{i\phi}\simeq 1+i\phi-\phi^2+... \tag{40}$$

It can be obtained by this the expectation value, if as usual $<\phi>=0$

$$<e^{i\phi}>=1-<\phi>^2 \tag{41}$$

2.2.1 Expectation Value of Random Function

2.2.2 Spectral Densities for Different Noises

2.2.3 Ramsey sequence for noisey system

2.2.4 Echo sequence for noisey system