Linköpings tekniska högskola
IEI
Produktionsekonomi

Finansiell optimering
Laboration 4
November 15, 2018

Computer exercise 4 - Conditional Value at Risk (CVaR)

Aim: You will build an optimization model that determines the portfolio with minimum CVaR.

Background: Value at Risk (VaR), is a common risk measure on financial markets, with some inherent drawbacks. The risk measure lack control of the risk above the VaR value. It is not sub-additive, a property which is important in risk measures, since increased diversification should lead to decreased risk. If the investor in spite of these drawbacks would like to find an optimal portfolio with respect to VaR he would face a non-convex optimization problem which is very difficult to solve. A closely related risk measure is to measure the expected loss in excess of VaR, CVaR. Rockafellar and Uryasev[1] realized that CVaR in the objectve or as a constraint can be formulated with a linear programming formulation. The risk measure has now become common on financial markets.

An investor has an initial portfolio where the investment in each asset is given by the vector $b \in \mathbb{R}^{n\times 1}$ and $c \in \mathbb{R}$ is the investment in the risk-free asset. The initial asset prices is $P_1 \in \mathbb{R}^{n\times 1}$, which gives the initial wealth $P_1^T b + c$. The investor can buy and sell assets, $x \in \mathbb{R}^{n\times 1}$, which gives the investment in each asset, $b + x$, and the investment in the risk-free asset, $c - P_1^T x$. Given scenario $i \in \mathcal{L}$ with probability p_i the prices of assets are $P_i \in \mathbb{R}^{n\times 1}$ which gives the loss $L(x, P_i) = P_1^T b + c - (P_i^T(b + x) + e^{rt}(c - P_1^T x))$, where r is the continuously compounded risk-free interest rate for time t.
α-VaR defines the loss that will not be exceeded with probability α (e.g. 95%).

$$VaR(x, \alpha) = \min\{\zeta \in \mathbb{R} | \Psi(x, \zeta) \geq \alpha\}, \tag{1}$$

where $\Psi(x, \zeta)$ describes the probability that the loss is smaller than ζ, e.g.

$$\Psi(x, \zeta) = \sum_{i \in \mathcal{L} | L(x,P_i) \leq \zeta} p_i. \tag{2}$$

CVaR is the expected loss that exceed VaR,

$$CVaR(x, \alpha) = E[L(x, P_i) | L(x, P_i) > \zeta] = \frac{\displaystyle\sum_{i \in \mathcal{L} | L(x,P_i) > \zeta} p_i L(x, P_i)}{\displaystyle\sum_{i \in \mathcal{L} | L(x,P_i) > \zeta} p_i} = \frac{\displaystyle\sum_{i \in \mathcal{L} | L(x,P_i) > \zeta} p_i L(x, P_i)}{1 - \alpha}. \tag{3}$$

[1]Rockafellar, R.T. and Uryasev, S., Optimization of Conditional Value-At-Risk, The Journal of Risk, vol 2, page 21-41, 2000.

To realize that the problem can be solved as a linear optimization problem the following definition is made $y_i = \max[0, L(x, P_i) - \zeta]$, $i \in \mathcal{L}$, which allows gives the following expression

$$\begin{aligned}
\sum_{i\in\mathcal{L}} p_i y_i &= \sum_{i\in\mathcal{L}|L(x,P_i)\leq\zeta} p_i y_i + \sum_{i\in\mathcal{L}|L(x,P_i)>\zeta} p_i y_i && (4)\\
&= 0 + \sum_{i\in\mathcal{L}|L(x,P_i)>\zeta} p_i L(x, P_i) - \zeta \sum_{i\in\mathcal{L}|L(x,P_i)>\zeta} p_i && (5)\\
&= \sum_{i\in\mathcal{L}|L(x,P_i)>\zeta} p_i L(x, P_i) - \zeta(1-\alpha) \Leftrightarrow && (6)
\end{aligned}$$

$$\Leftrightarrow \frac{\displaystyle\sum_{i\in\mathcal{L}|L(x,P_i)>\zeta} p_i L(x, P_i)}{1-\alpha} = \zeta + \frac{\displaystyle\sum_{i\in\mathcal{L}} p_i y_i}{1-\alpha} \qquad (7)$$

The left hand side is the definition of CVaR, which can be calculated with the right hand side. The optimization model that minimize CVaR can be formulated as

$$\begin{array}{lll}
\min\limits_{x,y,\zeta} & \zeta + \dfrac{\sum_{i\in\mathcal{L}} p_i y_i}{1-\alpha} & \\
\text{s.t.} & L(x, P_i) = P_1^T b + c - (P_i^T(b+x) + e^{rt}(c - P_1^T x)) & i \in \mathcal{L} \\
& y_i = \max[0, L(x, P_i) - \zeta] & i \in \mathcal{L} \\
& x \in X &
\end{array} . \qquad (8)$$

The second constraint is non-linear, but since y_i is minimized it can be reformulated as $y_i \geq 0, y_i \geq L(x, P_i) - \zeta$ which gives the linear optimization model

$$\begin{array}{lll}
\min\limits_{x,y,\zeta} & \zeta + \dfrac{\sum_{i\in\mathcal{L}} p_i y_i}{1-\alpha} & \\
\text{s.t.} & y_i \geq P_1^T b + c - (P_i^T(b+x) + e^{rt}(c - P_1^T x)) - \zeta & i \in \mathcal{L} \\
& y_i \geq 0 & i \in \mathcal{L} \\
& x \in X &
\end{array} . \qquad (9)$$

The set X can e.g. define that a certain expected return is required, $\bar{\mu}$, that short selling is not allowed and that money may not be borrowed,

$$\left.\begin{array}{l}
\sum_{i\in\mathcal{L}} p_i \left(P_i^T(x+b) + e^{rt}(c - P_1^T x)\right) = (1+\bar{\mu})(P_1^T b + c) \\
b + x \geq 0 \\
c - P_1^T x \geq 0
\end{array}\right\} = X. \qquad (10)$$

To assist you there is a program which generates scenarios in Matlab and write them to an AMPL file, whereafter a call to AMPL is made from Matlab. The file CVaR.zip can be downloaded from lisam.

Exercise: Determine the portfolio with minimal CVaR in one year given the expected return $\bar{\mu} = 0.1$!

Exercise: Relax the constraints for shorting and borrowing. What happens with CVaR? What is the explanation?

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Exercise: Compute Value-at-Risk in Matlab for the optimal solution using round(alpha*nSamples) to determine the quantile. Compare with the solution from AMPL!

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Exercise: Compute Conditional Value-at-Risk in Matlab for the optimal solution using 1+round(alpha*nSamples) to determine the quantile. Compare with the solution from AMPL!

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Below follows an example of an AMPL formulation of a model.

inventory.mod

```
param T;             # number of time periods
param initInventory; # initial inventory
param c{1..T};       # productions cost
param u{1..T};       # production capacity
param l{1..T};       # inventory cost
param d{1..T};       # demand

var x{1..T} >= 0;    # number of manufactured units
var y{0..T} >= 0;    # number of stored units

minimize totalcost: sum {t in 1..T} (c[t]*x[t] + l[t]*y[t]);

subject to inventory {t in 1..T} : y[t-1] + x[t] = d[t] + y[t];

subject to initialinventory : y[0] = initInventory;

subject to capacitycon {t in 1..T} : x[t] <= u[t];
```

inventory.mod