Fourier Series

The general form of Fourier series in the interval $(0, 2l)$ is given by

$$f(x) = a_0 + \Sigma a_n \cos\left(\frac{n\pi x}{l}\right) + \Sigma b_n \sin\left(\frac{n\pi x}{l}\right)$$

where

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x) \cdot \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x) \cdot \sin\left(\frac{n\pi x}{l}\right) dx$$

Suppose the interval is $(0, 2\pi)$ compare it with $(0, 2l)$ $\therefore l = \pi$
substitute in above integral

If interval is $(-l, l)$, then general form of fourier series is

$$f(x) = a_0 + \Sigma a_n \cos\left(\frac{n\pi x}{l}\right) + \Sigma b_n \sin\left(\frac{n\pi x}{l}\right)$$

$9 \times 9 = 81 \to$ even)

$f(x) = E / O$
$b_n = 0$ / $a_0, a_n = 0$

where

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cdot \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \cdot \sin\left(\frac{n\pi x}{l}\right) dx$$

$$\int_{-a}^{a} f(x)\,dx = 2 \cdot \int_0^a f(x)\,dx \text{ if even}$$
$$= 0 \text{ if odd.}$$

if $f(x)$ is an even function then

$\boxed{b_n = 0}$

if $f(x)$ is odd function then

$a_0 = 0$
$a_n = 0$

$\frac{n}{\pi(1+n^2)} \quad (1 - e^{-2\pi})$

$$f(x) = \frac{1}{2\pi}(1-e^{-2\pi}) + \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty} \frac{1}{1+n^2}\cos nx + \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty} \frac{n}{1+n^2}\sin nx$$

Q2] Obtain fourier series for function

$f(x) = x^2$ in interval $(0, 2\pi)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)$$

$l = \pi$

$$f(x) = a_0 + \sum a_n\cos(nx) + \sum b_n\sin(nx)$$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x^2\,dx$$

$$= \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{(2\pi)^3}{3} - 0\right]$$

$$\frac{8\pi^3}{18\pi} = \frac{4\pi^2}{9} \quad \frac{1}{2\pi}\left[\frac{8\pi^3}{3}\right]$$

$$= \frac{4\pi^3}{3}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cdot\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\cdot\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) + \frac{2x\cos nx}{n^2} + \frac{2\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{2(2\pi)\cdot\cos 2n\pi}{n^2}\right]$$

$$= \frac{4}{n^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\cdot\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\cdot\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) - 2x\left(\frac{\sin nx}{n^2}\right) + 2\left(+\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{(2\pi)^2(-1)}{n} + \frac{2(+1)}{n^3} - \frac{2}{n^3}\right] \qquad \frac{2}{n^3}$$

$$= \frac{1}{\pi}\left[\frac{-4\pi^2}{n}\right] = \frac{-4\pi}{n}$$

$\cos n\pi = (-1)^n \qquad \cos 2n\pi = 1$

$\frac{4}{9}\,(uv_1 - u'v_2 + u''v_3)$

$f(x) = a_0 + \sum a_n \cos nx + \sum b_n \sin nx$

$$x^2 = \frac{4\pi^3}{3} + \sum \frac{4}{n^2} \cos nx - \sum \frac{4\pi}{n} \sin nx$$

Q3] Obtain fourier series of the function

$f(x) = e^x$ in $(-\pi, \pi)$

$f(x) = a_0 + \sum a_n \cos nx + \sum b_n \sin nx.$

e^x is neither even nor odd

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x\,dx$$

$$= \frac{1}{2\pi}\left[e^x\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[e^{\pi} - e^{-\pi}\right] \qquad \sinh\pi = \frac{e^{\pi} - e^{-\pi}}{2}$$

$$= \frac{\sinh\pi}{\pi}$$

$\cos \pi \quad -1 \quad \cos 2\pi \quad (-1)^n$

$\boxed{\cos 2n\pi = 1}$

$\boxed{\cos n\pi = (-1)^n}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cdot \cos nx\,dx.$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left[\cos nx + n \sin nx\right]\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}(\cos n\pi + n\sin n\pi) - \frac{e^{-\pi}}{1+n^2}(\cos n(-\pi) + n(\sin n(-\pi))\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}(-1)^n}{1+n^2} - \frac{e^{-\pi}(-1)^n}{1+n^2}\right]$$

$$= \frac{(-1)^n}{\pi(1+n^2)}\left[e^{\pi} - e^{-\pi}\right]$$

$$= \frac{(-1)^n}{\pi(1+n^2)} \cdot 2\sinh\pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cdot \sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left[\sin nx - n\cos nx\right]\right]_{-\pi}^{\pi}$$

$$\frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}(\sin n\pi - n\cos n\pi) - \frac{e^{-\pi}}{1+n^2}(\sin n(-\pi) - n\cos n(-\pi))\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}(-n(-1)^n) - \frac{e^{-\pi}}{1+n^2}(-n(-1)^n)\right]$$

$$= \frac{-n(-1)^n}{1+n^2}\left[e^{\pi} - e^{-\pi}\right]$$

$$= \frac{n(-1)^n}{1+n^2}\ \cancel{2\sinh\pi}\ \cancel{2\cosh\pi}\ 2\sinh\pi$$

$$f(x) = a_0 + \sum a_n\cos nx + \sum b_n\sin nx$$

$$= \frac{\sinh\pi}{\pi} + \frac{\sum(-1)^n 2\sinh\pi}{\pi(1+n^2)} - \frac{n(-1)^n}{1+n^2}\ 2\sinh\pi,$$

Q Find Fourier series of $f(x)$ in interval $(0, 2\pi)$ when

$$f(x) = x, \quad 0 < x \le \pi$$
$$= 2\pi - x, \quad \pi \le x < 2\pi.$$

$$\rightarrow f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right].$$

$$= \frac{1}{2\pi}\left[\left[\frac{x^2}{2}\right]_0^{\pi} + \left[2\pi x - \frac{x^2}{2}\right]_{\pi}^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi^2}{2} + \left[4\pi^2 - 2\pi^2 - 2\pi^2 + \frac{\pi^2}{2}\right]\right]$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|x\frac{\sin nx}{n} + (1)\frac{\cos nx}{n^2}\right|_0^{\pi} + \left|(2\pi - x)\frac{\sin nx}{n} + 1\cdot\left(-\frac{\cos nx}{n^2}\right)\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi\sin n\pi}{n} + \frac{\cos n\pi}{n^2} - \frac{1}{n^2} - \frac{-1}{n^2} + \frac{(-1)^n}{n^2}\right]$$

$\frac{1}{n}\int -\frac{\cos nx}{n} - \frac{\sin nx}{n^2}$

$f(-x)$ odd

$(-\pi, \pi)$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot \sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cdot \sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cdot \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left| x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right|_0^{\pi}\right.$$

$$\left. + \left|(2\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \pi\frac{(-1)^n}{n}\right]$$

$$= 0$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left[(-1)^n - 1\right]\cos nx$$

Q Obtain Fourier series of $f(x)$

$$f(x) = x + \frac{\pi}{2}, \quad -\pi < x < 0$$

$$= \frac{\pi}{2} - x, \quad 0 < x < \pi.$$

→ To check odd/even function.

$$f(-x) = -x + \frac{\pi}{2}, \quad -\pi < -x < 0$$

$$f(-x) = \frac{\pi}{2} + x, \quad 0 < -x < \pi,$$

multiply by -1 & sign reverse.

$$\therefore f(-x) = \frac{\pi}{2} - x, \quad \pi > x > 0 \Rightarrow 0 < x < \pi$$

$$= \frac{\pi}{2} + x, \quad 0 > x > -\pi \Rightarrow -\pi < x < 0$$

$\therefore f(-x) = f(x)$

$\therefore f(x)$ is an even function.

$\therefore b_n = 0$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)dx$$

$$= \frac{1}{\pi}\left[\frac{\pi x}{2} - \frac{x^2}{2}\right]_0^{\pi} = \frac{1}{2\pi}\left[\pi^2 - \pi^2\right]$$

$$= 0$$

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$

for even $= \frac{1}{2\pi}\cdot 2\int_0^{\pi} f(x)\,dx$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x).\cos nx\,dx$

$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)\cos nx\,dx$

$= \frac{2}{\pi}\left[\left(\frac{\pi}{2} - x\right)\frac{\sin nx}{n} - (-1)\left(\frac{-\cos nx}{n^2}\right)\Big|_0^{\pi}\right]$

$= \frac{2}{\pi}\left[0 - \frac{(-1)^n}{n^2} + \frac{1}{n^2}\right]$

$= \frac{2}{\pi n^2}\left(1 - (-1)^n\right)$

$f(x) = 0 + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left(1-(-1)^n\right).\cos nx\,dx$

$\int_{-a}^{a} f(x)\,dx = 2\cdot\int_0^{a} f(x)\,dx$ if f is even

$= 0$ if odd

Q $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$

$f(-x) = 1 - \frac{2x}{\pi} \quad -\pi \le -x \le 0$

$f(-x) = 1 + \frac{2x}{\pi} \quad 0 \le -x \le \pi$

$f(-x) = 1 - \frac{2x}{\pi} \quad \pi \ge x \ge 0$

$= 1 + \frac{2x}{\pi} \quad 0 \ge x \ge -\pi$

$\therefore f(-x) = f(x)$

$f(x)$ is an even function,

$b_n = 0$

$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$

$= \frac{1}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx$

$= \frac{1}{\pi}\left(x - \frac{x^2}{\pi}\right)_0^{\pi}$

$= \frac{1}{\pi}\left[\pi - \pi\right]$

$= 0$

$\frac{\cancel{2}}{\pi}\frac{x^2}{\cancel{2}}$

$n\pi \quad \frac{(-1)^n}{} \quad 1 \quad -\frac{2}{\pi}$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\left(\frac{+\sin nx}{n}\right) - \left(\frac{-2}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n} - \frac{2}{\pi}\frac{\cos nx}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\left(1-2\right)\frac{\sin n\pi}{n} - \frac{2}{\pi}\frac{\cos n\pi}{n^2} + \left[(1-0)\left(\frac{\sin 0}{n}\right) + \left(\frac{-2}{\pi}\right)\left(\frac{-\cos 0}{n^2}\right)\right]\right]$$

$$= \frac{2}{\pi}\left[\frac{-2(-1)^n}{\pi n^2} + \frac{2}{\pi n^2}\right]$$

$$= \frac{4}{\pi^2 n^2}\left(1-(-1)^n\right)$$

$$f(x) = a_0 + \sum a_n\cdot\cos nx$$

$$= \frac{4}{\pi^2}\sum \frac{1-(-1)^n}{n^2}\cdot\cos nx$$

$-x\,(0)$

$\sin x \cos x$

$2\cos x\cdot\sin = \sin(A+B) - \sin(A-B)$

Q2] $f(x) = x\cdot\cos(x)$ in $(-\pi, \pi)$

$$f(-x) = -x\cos(-x)$$
$$= -x\cos x$$

is odd function

$a_0 = 0$, $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cdot\cos(x)\cdot\sin(nx)\,dx$$

$$= \frac{2}{2\pi}\int_0^{\pi} x\left[\sin(1+n)x - \sin(1-n)x\right]dx$$

$$= \frac{2}{2\pi}\int_0^{\pi}\left(x\cdot\sin(1+n)x - x\sin(1-n)x\right)dx$$

$$= \frac{2}{2\pi}\left[x\left(\frac{-\cos x(1+n)}{(1+n)}\right) - (1)\left(\frac{-\sin x(1+n)}{(1+n)^2}\right) - \left[x\left(\frac{-\cos x(1-n)}{(1-n)}\right) - (1)\left(\frac{-\sin x(1-n)}{(1-n)^2}\right)\right]\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[\pi\left(\frac{-\cos\pi(1+n)}{1+n}\right) + \frac{\sin\pi(1+n)}{(1+n)^2} + \frac{\pi\cos\pi(1-n)}{(1-n)} - \frac{\sin\pi(1-n)}{(1-n)^2}\right]\right\}$$

$$\cos(\pi+\theta) = -\cos\theta$$
$$\cos(\pi+n\pi) = -\cos n\pi$$
$$= -(-1)^n$$

$\cos(1-n)\pi = \cos(\pi - n\pi)$ | $\sin(1+n)\pi = \sin(\pi + n\pi)$

$= -\cos n\pi$ | $= -\sin n\pi$

$-(-1)^n$ | $= 0$

$\sin(1-n)\pi = \sin(\pi - n\pi)$

$= \sin\pi = 0$

$$= \frac{1}{\pi}\left[\frac{\pi(-1)^n}{1+n} - \frac{\pi(-1)^n}{1-n}\right]$$

$$= (-1)^n\left[\frac{1}{1+n} - \frac{1}{1-n}\right]$$

$$= (-1)^n\left[\frac{\cancel{1}-n-\cancel{1}-n}{1-n^2}\right]$$

$$= \frac{(-1)^n \times -2n}{1-n^2}$$

$$b_n = \frac{-2(-1)^n n}{1-n^2} \quad \text{if } n \neq 1$$

To find ~~bn~~ b_1 ...

$$b_1 = \frac{2}{\pi}\int_0^\pi f(x)\sin x\,dx$$

$$= \frac{2}{\pi}\int_0^\pi x\cdot\cos x\cdot\sin x\,dx$$

$$= \frac{1}{\pi}\int_0^\pi x\cdot\sin 2x\,dx$$

$(-1)^n$

$$= \frac{1}{\pi}\left[\frac{x(-\cos 2x)}{2} - (1)\left(-\frac{\sin 2x}{4}\right)\right]_0^\pi$$

$$= \frac{1}{\cancel{\pi}}\left[\frac{-\cancel{\pi}}{2} + 0\right]$$

$$b_1 = -\frac{1}{2}$$

$$\therefore f(x) = \sum b_n \sin nx$$

$$= -\frac{1}{2} - 2\sum_{n=2}^{\infty}\frac{(-1)^n\, n}{1-n^2}\cdot\sin nx$$

26/4/…

Q* Half Range Sine Series. . $(-\pi, \pi)$

$$f(x) = \sum b_n \sin\left(\frac{n\pi x}{l}\right)$$

where

$$b_n = \frac{2}{l}\int_0^l f(x)\cdot \sin\left(\frac{n\pi x}{l}\right) dx$$

& $a_0 = 0$, $a_n = 0$

* Half range cosine-series $(0, \pi)$

$$f(x) = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cdot \cos\left(\frac{n\pi x}{l}\right) dx$$

& $b_n = 0$

Q Find half range cosine series

$f(x) = x$ in $(0, \pi)$

$\rightarrow$ $f(x) = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right)$

$$a_0 = \frac{1}{\pi}\int_0^\pi x\,dx$$

$$= \frac{1}{\pi}\cdot\left[\frac{x^2}{2}\right]_0^\pi$$

$$= \frac{\pi^2}{2\pi} = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^\pi x\cdot \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\cdot\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\pi(0) + (1)\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

Q Find half range cosine series for

$f(x) = x\cdot(\pi - x)$ in $(0, \pi)$

$f(x) = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right)$

$a_0 = \frac{1}{\pi}\int_0^{\pi} \cancel{f x}\, x(\pi - x)\,dx$

$= \frac{1}{\pi}\int_0^{\pi} (\pi x - x^2)\,dx$

$= \frac{1}{\pi}\left[\pi\cdot\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}$

$= \frac{1}{\pi}\left[\pi\cdot\frac{\pi^2}{2} - \frac{\pi^3}{3}\right]$

$= \frac{\pi^{\cancel{3}}}{\cancel{\pi}}\cdot\left[\frac{1}{2} - \frac{1}{3}\right]$

$= \frac{\pi^2}{6}$

$a_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\cdot\cos(nx)\,dx$

$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\cdot\cos(nx)\,dx$

$= \frac{2}{\pi}\left[(\pi x - x^2)\cdot\frac{\sin(nx)}{n} - (\pi - 2x)\left(-\frac{\cos(nx)}{n^2}\right)\right]_0^{\pi}$

~ + ~

$= \frac{2}{\pi}\left[\left((\pi^2 - \pi^2)\cdot\frac{\sin n\pi}{n} + (\pi - 2\pi)\frac{\cos(n\pi)}{n^2}\right) - (\pi)\frac{\cos \pi n}{n^2}\right]$

$-\pi$

$-\pi + 2\pi$

$-\pi \quad -\pi$

$= \frac{2}{\pi}\left[\cancel{-\pi\cdot\frac{(-1)^n}{n^2}} - \frac{\pi(-1)^n}{n^2}\right]$

$= \frac{2}{\pi}\left[(\pi x - x^2)\cdot\frac{\sin nx}{n} - (\pi - 2x)\left(-\frac{\cos nx}{n^2}\right) - 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$

$= \frac{2}{\pi}\left[-(\pi - 2\pi)\left(\frac{-(-1)^n}{n^2}\right) + \frac{\pi\cdot(-1)^0}{n^2}\right]$

$= \frac{2}{\pi}\left[\frac{-\pi(-1)^n}{n^2} - \frac{\pi}{n^2}\right]$

$= \frac{-2}{n^2}\left((-1)^n + 1\right)$

$f(x) = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty} \frac{(-1)^n + 1}{n^2}\cdot\cos nx$

Q Obtain half range sine series for

$f(x) = \frac{\pi}{4}$ in interval $(0, \pi)$

$\rightarrow \quad f(x) = \sum b_n \sin\left(\frac{n\pi x}{4}\right)$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \frac{\pi}{4} \cdot \sin(nx)\,dx$$

$$= \frac{2}{\pi} \cdot \frac{\pi}{4}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi}$$

$$= \frac{1}{2} \left[\frac{-(-1)^n}{n} + \frac{1}{n}\right]$$

$$= \frac{1}{2n}\left[1 - (-1)^n\right]$$

$$f(x) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}\left[1 - (-1)^n\right] \cdot \sin nx$$

Q Find half range cosine series for

$$f(x) = \begin{cases} x & 0 < x < \pi/2 \\ \pi - x & , \pi/2 < x < \pi \end{cases}$$

$b_n = 0$

$f(x) = a_0 + \sum a_n \cos nx.$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi/2} + \left[\pi x - \frac{x^2}{2}\right]_{\pi/2}^{\pi}\right]$$

$$\frac{1}{\pi}\left[\cancel{\frac{\pi^2}{8}} + \cancel{\frac{\pi^2}{2}} - \cancel{\frac{\pi^2}{2}} - \cancel{\frac{\pi^2}{8}}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{8} + \pi^2 - \frac{\pi^2}{2} - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{8} + \cancel{\pi^2} - \cancel{\frac{\pi^2}{2}} + \frac{\pi^2}{8}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{4}\pi^2\right]$$

$$= \frac{\pi}{4}$$

$\pi^2 - \frac{\pi^2}{2}$

$\frac{\pi^2}{2}$

$\frac{\pi^2}{2} - \frac{\pi^2}{8}$

$\pi^2 - \frac{\pi^2}{2}$

$\frac{2\pi^2}{8}$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cdot \cos(nx)\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x \cdot \cos(nx)\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos(nx)\,dx\right]$$

u · v

$$= \frac{2}{\pi}\left[\left(x \cdot \frac{\sin nx}{n} - (1)\left(\frac{-\cos nx}{n^2}\right)\right)_0^{\pi/2} + \left((\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\cdot\left(\frac{-\cos nx}{n^2}\right)\right)_{\pi/2}^{\pi}\right]$$

$a(x)$

$$= \frac{2}{\pi}\left[\frac{\pi}{2}\frac{\sin n\pi/2}{n} + \frac{\cos n\pi/2}{n^2} - \frac{1}{n^2} - \frac{2(-1)^n}{n^2} - \frac{\pi}{2}\frac{\sin n\pi/2}{n} + \frac{\cos n\pi/2}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2\cos n\pi/2}{n^2} - \frac{1}{n^2} - \frac{(-1)^n}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\left[2\cos n\pi/2 - (-1)^n - 1\right]$$

$$a_1 = \frac{2}{\pi}\left[-1 + 1\right] = 0$$

$$a_2 = \frac{2}{\pi 4}\left[2\cos\pi - (-1)^2 - 1\right]$$

$$= \frac{1}{2\pi}\left[-2 - 1 - 1\right]$$

$$= \frac{-4}{2\pi} = \frac{-2}{\pi}$$

$$a_3 = \frac{2}{\pi 9}\left[2\cos\frac{3\pi}{2} - (-1)^3 - 1\right]$$

$$= \frac{2}{9\pi}\left[0 - (-1) - 1\right]$$

$$= 0$$

$$a_4 = \frac{2}{\pi 16}\left[2\cos 2\pi - (-1)^4 - 1\right]$$

$$= \frac{1}{8\pi}\left[2 - 1 - 1\right]$$

$$= 0$$

$(-1)^n$

$\cos n\frac{\pi}{2}$

$$a_5 = \frac{2}{\pi 25}\left[2\cos\frac{5\pi}{2} - (-1)^5 - 1\right]$$

$$= \frac{2}{25\pi}\left[0 + 1 - 1\right]$$

$$= 0$$

$$a_6 = \frac{2}{\pi 36}\left[2\cos 3\pi - (-1)^6 - 1\right]$$

$$= \frac{1}{18\pi}\left[-2 - 1 - 1\right]$$

$$= \frac{-4}{18\pi} = \frac{-2}{9\pi}$$

$$f(x) = a_0 + \sum a_n \cos nx$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\cos 2x - \frac{2}{9\pi}\cos 6x + \cdots$$

$\|V_i\| = 1 \rightarrow$ vector is orthonormal

* Orthogonality, Orthonormality

The set of function $f_1(x), f_2(x) \dots f_n(x)$ is said to be orthogonal on interval (a, b) if

$$\int_a^b f_m(x)\, f_n(x)\,dx = 0 \quad \text{if } m \neq n$$

$$\neq 0 \quad \text{if } m = n$$

i.e

$$\int_a^b f_m(x) \cdot f_n(x)\,dx = 0, \quad m \neq n$$

$$\int_a^b [f_m(x)]^2\,dx \neq 0, \quad m = n$$

The set of function $f_1(x), f_2(x) \dots f_n(x)$ is said to be orthonormal in interval (a, b)

if $\int_a^b f_m(x)\, f_n(x) = 0 \Rightarrow$ if $m \neq n$

$= 1 \Rightarrow$ if $m = n$

Q Show that set of function $\cos nx$, $n = 1, 2, 3, \dots$ is orthogonal on int $(0, 2\pi)$

$\rightarrow$ $f_n(x) = \cos nx$

(i) for $m \neq n$

$$\int_0^{2\pi} \cos mx \cdot \cos nx$$

$$\frac{1}{2}\int_0^{2\pi} 2\cos mx \cdot \cos nx\,dx$$

$$\frac{1}{2}\int_0^{2\pi} [\cos(m+n)x + \cos(m-n)x]\,dx$$

$$\frac{1}{2}\left[\frac{\sin(m+n)x}{(m+n)} + \frac{\sin(m-n)x}{(m-n)}\right]_0^{2\pi}$$

$$= 0$$

(ii) for $m = n$

$$\int_0^{2\pi} \cos^2 mx\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} (1 + \cos 2mx)\,dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2mx}{2m}\right]_0^{2\pi}$$

$$= \frac{1}{2}[2\pi]$$

$$= \pi \neq 0$$

$\therefore$ The given set of func ~~[illegible]~~ is orthogonal on interval $(0, 2\pi)$

Q. Prove that $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = \frac{3x^2-1}{2}$ are orthogonal over $(-1, 1)$

$\longrightarrow$ $\int_{-1}^{1} f_1(x) \cdot f_2(x)\,dx = \int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1}$

$= \frac{1}{2} - \frac{1}{2}$

$= 0$

$\int_{-1}^{1} f_1(x) \cdot f_3(x)\,dx = \int_{-1}^{1} \frac{3x^2-1}{2} = \cancel{\frac{1}{2} \int 3}$

$= \frac{\cancel{3}}{2} \cdot \left[\frac{x^3}{\cancel{3}}\right]_{-1}^{1} - \frac{1}{2}[x]_{-1}^{1}$

$= \frac{1}{2} + \frac{1}{2} - \frac{1}{2}\cancel{[1+1]}$

$= 0$

$\int_{-1}^{1} f_2(x) \cdot f_3(x)\,dx = \int_{-1}^{1} x \cdot \left(\frac{3x^2-1}{2}\right) dx$

$= \int_{-1}^{1} \left(\frac{3x^3}{2} - \frac{1}{2}x\right) dx$

$= \frac{3}{2}\left[\frac{x^4}{4}\right]_{-1}^{1} - \frac{1}{2}\left[\frac{x^2}{2}\right]_{-1}^{1}$

$\frac{1}{2} + \frac{1}{2}$

$= \frac{3}{8}[1-1] - \frac{1}{2}\left[\frac{1}{2} - \left(\frac{-1}{2}\right)\right]$

$= 0 - \frac{1}{2}[1]$

$= \cancel{-1/2}\ 0$

$[x]_{-1}^{1}$ $\quad 1 - (-1)$ over 2

$\int_{-1}^{1} f_1^2(x)\,dx = 2 \neq 0$

$\int_{-1}^{1} f_2^2(x)\,dx = \frac{2}{3}$

$\int_{-1}^{1} f_3^2(x)\,dx = \int_{-1}^{1} \frac{(3x^2-1)^2}{4} = \int_{-1}^{1} \frac{9x^4 - 6x^2 + 1}{4}$

$= \frac{2}{5} \neq 0$

$\therefore$ The set is orthogonal

Q3] Is $S = \left\{ \sin\left(\frac{\pi x}{4}\right), \sin\left(\frac{3\pi x}{4}\right), \sin\left(\frac{5\pi x}{4}\right), \cdots \right\}$ orthogonal over $(0, 1)$

$\longrightarrow$ $f_n(x) = \sin\left(\frac{(2n+1)\pi x}{4}\right)$; $n = 0, 1, 2, 3, \ldots$

(1) for $m \neq n$

$\int_0^1 f_m(x) \cdot f_n(x)\,dx = \int_0^1 \sin\left(\frac{(2m+1)\pi x}{4}\right) \cdot \sin\left(\frac{(2n+1)\pi x}{4}\right)$

$= \frac{1}{2}\int_0^1 \left\{\cos\left(\frac{2m+\cancel{1}-2n-\cancel{1}}{4}\right)\pi x - \cos\left(\frac{2m+1+2n+1}{4}\right)\pi x\right\}$

$= \frac{1}{2}\int_0^1 \left[\cos\left(\frac{(m-n)}{2}\right) - \cos\left(\frac{(m+n+1)\pi x}{2}\right)\right] dx$

$\sin\frac{3\pi}{2}$

$$= \frac{1}{2}\left[\frac{\sin\left(\frac{(m-n)\pi x}{2}\right)}{\frac{(m-n)\pi}{2}} - \frac{\sin\left(\frac{(m+n+1)\pi x}{2}\right)}{\frac{(m+n+1)\pi}{2}}\right]_0^1$$

$\neq 0$

$\therefore$ Given f_n is not orthogonal

Q4] Show that the set of function

$\sin x, \sin 3x, \sin 5x, \ldots$ is orthogonal over int $[0, \pi/2]$, Hence construct orthonormal set of functions

$f_n(x) = \sin(2n+1)x \quad n = 0, 1, 2, \ldots$

(1) for $m \neq n$

$$\int_0^1 f_m(x) \cdot f_n(x)\,dx = \int_0^{\pi/2} \sin(2m+1)x \cdot \sin(2n+1)x\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos(2m+1-2n-1)x - \cos(2m+1+2n+1)x\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2} [\cos 2(m-n)x - \cos 2(m+n+1)x]\,dx$$

$$= \frac{1}{2}\left[\cancel{-\cos}\ \frac{\sin 2(m-n)x}{2(m-n)} - \frac{\sin 2(m+n+1)x}{2(m+n+1)}\right]_0^{\pi/2}$$

$= 0$

(ii) for $m = n$

$$\int_0^{\pi/2} f_n^2(x)\,dx$$

$$= \int_0^{\pi/2} \sin^2(2n+1)x\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2} [1 - \cos 2(2n+1)x]\,dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2(2n+1)x}{2(2n+1)}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2}\right]_0 = \frac{\pi}{4}$$

$\neq 0$

$\therefore$ The given set of func is orthogonal

$$\int_0^{\pi/2} f_n^2(x)\,dx = \frac{\pi}{4} \quad \text{--- (1)}$$

To construct set of orthonormal func divide eqn (1) by $\frac{\pi}{4}$

$$\therefore \frac{4}{\pi}\int_0^{\pi/2} f_n^2(x)\,dx = 1$$

$$= \int_0^{\pi/2} \frac{4}{\pi} \cdot f_n(x) \cdot f_n(x) = 1$$

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$$\int_0^{\pi/2} \frac{2}{\sqrt{\pi}} \cdot f_n(x) \cdot \frac{2}{\sqrt{\pi}} f_n(x)\, dx = 1$$

$\therefore \frac{2\sin x}{\sqrt{\pi}}, \frac{2\sin 3x}{\sqrt{\pi}}, \frac{2\sin 5x}{\sqrt{\pi}}$ is set of orthonormal function.