Research Report: Path following & variable actuation system

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Introduction

The addressed problem involved path following. Intuitively, following a path means joining a curve in a 3D or 2D space, described by an equation.
Mathematically, joining expresses that the center of mass of the robot and its orientation have to tend asymptotically to a particular value. The particular value is such that the robot is on the path and its orientation is aligned with the path (or has some arbitrarily chosen value).

The path following problem has been studied before my arrival by researchers around the world and in particular by those present at the LIRMM such as my tutor Mr. Lapierre. In other words, there are solutions to the problem. Nevertheless, what there is not, is a general solution that takes into account the temporal evolution of the number of actuators of the system, but also how to use them when "there are more actuators than what the system needs" for the mission. An actuator is a motorized part of the system. The number of actuators is important because it expresses whether or not the system can go in all the directions or rotate in any orientation. In a 3D space, there are 6 degrees of freedom (3 translations and 3 rotations). In a 2D space it is only 3 (2 translations and one rotation). An iso-actuated system is a system carrying as many actuators as degrees of freedom. Under-actuated systems are those with less and over actuated systems are those with more. Systems can be designed to be under-actuated or over-actuated, in order to have more fail-safes, robustness, or simply save energy.
Thus, what is the link between an over actuated, iso-actuated and under-actuated system in terms of control? In some cases, the 3 types of systems can follow a path.
For example, a system can be designed to be over actuated in order to have more robustness. One motor could be lost or be turned off in order to save energy.
Can the system still follow the path in that case? Which directions can the system follow and which can it not?
One approach to cope with the loss of a motor would be to design different controllers for the different situations. Nevertheless, the problem is that the controllers would only apply to a particular system. Not to mention, that there would be a discontinuity in the system when switching from one controller to another. And even if this approach were to be adopted, there is still the question of proving that such an approach guarantees mathematically that the system converges to the path and does not have singularities or diverges.
That is the subject on which I will work. My internship thus revolves around taking an existing solution to the path following problem and generalize it in order to take into account under, iso and over-actuation.

1 Preliminary results

This is a section whose purpose is to present the work that has been done by the researchers at the LIRMM, but also to introduce the notations and equations that will be used throughout this entire paper.

1.1 The unicycle Robot

The unicycle type robot is the default example of robotics. It is introduced to every student studying robotics. It is studied for it has interesting properties and is (in appearance) a simple example to control.

1.1.1 The kinematic equations of the error

Let a Robot evolving in the $z = 0$ plane be described by its 2D position $\mathbf{P}$ and orientation θ_m such as in Figure (1):

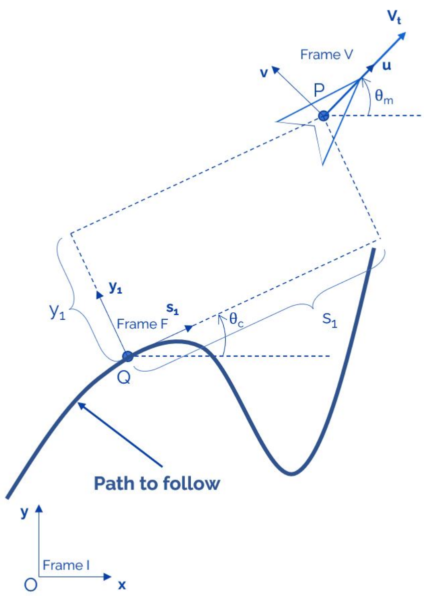

Figure 1: Unicycle robot

If A is a point, let $\mathbf{a}$ denote the vector $\overrightarrow{OA}$. $R(\theta)$ will denote the 2D rotation matrix of angle θ and $R_3(\theta)$ will denote the 3D rotation matrix of angle θ around the $\hat{z}$ axis:

$$R(\theta) = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} \text{ and } R_3(\theta) = \begin{pmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then, as shown in Figure (1), we have the following equalities, which are: $\mathbf{q} = \mathbf{p} + \mathbf{r}, \mathbf{p} = \begin{pmatrix} X \\ Y \end{pmatrix}_I$ and $\mathbf{r} = \begin{pmatrix} s_1 \\ y_1 \end{pmatrix}_F$

$\mathbf{q}$ can be expressed by its curvilinear abscissa and be thus described in the Frenet Frame (F) as $\mathbf{q} = \begin{pmatrix} s \\ 0 \end{pmatrix}_F$.

Thus:

$$\frac{d\mathbf{q}}{dt} = \frac{d\mathbf{p}}{dt} + \frac{d(R(\theta_c) \times \mathbf{r})}{dt}_I \tag{1}$$

We will take the following notations: $\theta = \theta_m - \theta_c,\ \nu = \|\mathbf{V_t}\|$ and $\mathbf{V_t} = \begin{pmatrix} u \\ 0 \end{pmatrix}_F$

With:

$$\frac{d(R(\theta_c) \times \mathbf{r})}{dt} = \dot{\theta}_c R(\pi/2) R(\theta_c)\mathbf{r} + R(\theta_c)\dot{\mathbf{r}} \text{ and } \dot{\theta}_c = c_c \dot{s} \tag{2}$$

Where $c_c = c_c(s) = \frac{d\theta_c}{ds}(s)$ is the curvature of the path at point s.

This yields,

$$\begin{aligned}
\frac{d\mathbf{q}}{dt} &= \frac{d\mathbf{p}}{dt} + \dot{\theta}_c R(\theta_c) R(\pi/2)\mathbf{r} + R(\theta_c)\dot{\mathbf{r}} \\
\iff R(\theta_c)^T \frac{d\mathbf{q}}{dt} &= R(\theta_c)^T \frac{d\mathbf{p}}{dt} + c_c \dot{s} R(\pi/2)\mathbf{r} + \dot{\mathbf{r}} \\
\iff R(\theta_c)^T R(\theta_m)\mathbf{V_t} &= \begin{pmatrix} \dot{s} \\ 0 \end{pmatrix}_F + c_c \dot{s} \begin{pmatrix} -y_1 \\ s_1 \end{pmatrix}_F + \begin{pmatrix} \dot{s}_1 \\ \dot{y}_1 \end{pmatrix}_F \\
\iff \begin{pmatrix} \dot{s}_1 \\ \dot{y}_1 \end{pmatrix}_F &= R(\theta_m - \theta_c)\mathbf{V_t} - \dot{s} \begin{pmatrix} 1 - c_c y_1 \\ c_c s_1 \end{pmatrix}_F
\end{aligned} \tag{3}$$

$$\iff \begin{pmatrix} \dot{s}_1 \\ \dot{y}_1 \end{pmatrix}_F = R(\theta)\mathbf{V_t} - \dot{s} \begin{pmatrix} 1 - c_c y_1 \\ c_c s_1 \end{pmatrix}_F \tag{4}$$

At this point it is important to note that Q is an arbitrarily chosen point on the path. Q could be for example chosen such as the orthogonal projection on the path (*i.e.* the nearest point from the robot to the path). This would imply $s_1 = 0$, constantly. This will put a constraint on the system and force us to solve for $\dot{s}$. Indeed equation (3) shows that if $s_1 = 0$, then $0 = cos(\theta)u - \dot{s}(1 - c_c y_1)$ and solving for $\dot{s}$, gives us $\dot{s} = \frac{cos(\theta)u}{(1-c_c y_1)}$, thus engendering a singularity at $y_1(s) = \frac{1}{c_c(s)}$. This means that the robot is coincident with the center of the curvature circle of the path at point s, where the closest point to the path is not unique. This constraint could be solved by constantly having $|y_1| < 1/c_{c,max}$. As pointed in [1], this constraint imposes that the initiation position of the robot is included in a tube surrounding the path of width equal to $1/c_{c,max}$).

1.1.2 Kinematic Controller

Equation (4) is the differential equation of the error between the unicycle position and the defined point Q on the path. Paper [2] proposes a kinematic controller of the type $(u, \dot{s}, \dot{\theta})$ for an unicycle type robot, such that $(s_1, y_1, \theta - \delta) \underset{t\to\infty}{\longrightarrow} (0, 0, 0)$, where$\delta$ is an angle of approach and will be defined later in this paper. Indeed, usually Q is defined as the orthogonal projection of point P on the path and thus as stated in the previous section, induces a constraint of the type $|y_1| < 1/c_{c,max}$. In order to get rid of this constraint, [2] had the idea of controlling $\dot{s}$ as a virtual control input. This relaxes the constraint on y_1 and the singularity at $y_1(s) = c_c(s)$ vanishes. Here is the controller that they propose:

$$\begin{pmatrix} \dot{s} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} ucos(\theta) + k_1 s_1 \\ \dot{\delta} - \gamma y_1 usin(\delta)\frac{sin(\theta)-sin(\delta)}{\theta-\delta} - k_2(\theta - \delta) \end{pmatrix} \tag{5}$$

With k_1 and k_2 being arbitrarily chosen positive gains.
Equation (5) was obtained using a Lyapunov-based method. This method relies on the following theorems:

1.1.2.1 Theorems

Barbalat's Lemma
If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a double differentiable, f' is uniformly continous function and $\lim_{t\to\infty} f(t) \in \mathbb{R}$,
Then $\lim_{t\to\infty} f'(t) = 0$.
Uniform continuity sufficient condition
If $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable and f' bounded, then f uniformly continuous on I.
Corollary of Barbalat's Lemma
If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a double differentiable function, $\lim_{t\to\infty} f(t) \in \mathbb{R}$ and f'' is bounded,
Then $\lim_{t\to\infty} f'(t) = 0$.
LaSalle's invariance principle
Let Ω be a positively invariant set of the autonomous system. Suppose that every solution starting in Ω converges to a set $E \subset \Omega$ and let M be the largest invariant set contained in E. Then every bounded solution starting in Ω converges to M as $t \longrightarrow \infty$.

Let us consider the following scalar quantity which depends on time:

$$\mathcal{L}_1(t) = \frac{1}{2}(s_1^2 + y_1^2 + \frac{1}{\gamma}(\theta - \delta)^2) \geq 0 \tag{6}$$

δ is a function of the variable (y_1, u). It is the "angle of approach", and is part of the guidance strategy. It is the angle with which the vehicle will approach the path. δ that verifies the following properties:

1. $\delta(y_1 = 0, u) = 0$
2. $\forall(y_1, u) \in \mathbb{R}^2,\ y_1 usin(\delta(y_1, u)) \leq 0$

(One example of such a function is $\delta = -\pi/2 \times tanh(K_{y_1} u y_1)$)

Then,

$$\dot{\mathcal{L}_1} = s_1\dot{s_1} + y_1\dot{y_1} + \frac{1}{\gamma}(\theta - \delta)(\dot{\theta} - \dot{\delta}) \tag{7}$$

And using the equations that $\dot{s_1}, \dot{y_1}$ and $\dot{\theta}$ satisfy, implies that:

$$\dot{\mathcal{L}_1} = s_1(ucos(\theta) - \dot{s}) + y_1 usin(\delta) + \frac{1}{\gamma}(\theta - \delta)(\dot{\theta} - \dot{\delta} + \gamma y_1 u\frac{sin(\theta) - sin(\delta)}{\theta - \delta})$$

The choice $\dot{s} = ucos(\theta) + k_1 s_1$ and $\dot{\theta} = \dot{\delta} - \gamma y_1 usin(\delta)\frac{sin(\theta)-sin(\delta)}{\theta-\delta} - k_2(\theta - \delta)$, implies:

$$\dot{\mathcal{L}_1} = -k_1 s_1^2 + y_1 usin(\delta) - \frac{1}{\gamma}(\theta - \delta)^2 \leq 0$$

Moreover, $\ddot{\mathcal{L}}_1$ is bounded because of its mathematical expression. Thus $\dot{\mathcal{L}}_1$ is uniformly continuous. Furthermore, $\lim_{t\to\infty} \mathcal{L}_1$ exists and is finite because $\mathcal{L}_1 \geq 0$ and $\mathcal{L}_1$ is a decreasing function. Using Barbalat's Lemma implies $\lim_{t\to\infty} \dot{\mathcal{L}}_1 = 0$. $\dot{\mathcal{L}}_1$ being a sum of negatives terms and tending to zero implies that each term tends to 0: $(s_1, y_1, \psi - \delta) \to 0$.

1.2 General 2D Robot

The unicycle robot is an under actuated system. It can not produce any sway. A more general type of robot will now be considered. This system which will be able to have a speed along $\hat{u}$ and $\hat{v}$. Such a robot will have a total speed $\mathbf{V_t} = \begin{pmatrix} u \\ v \end{pmatrix}_V$ of whose angle with the axis $\hat{u}$ will be denoted β as shown in Figure (2):

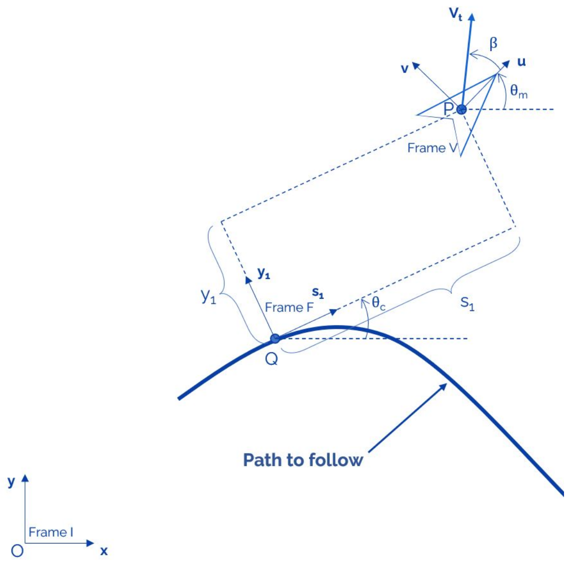

Figure 2: General 2D Robot

1.2.1 Kinematic Equations of the error space

The kinematic equations of the error space (4) still hold. The main difference is now that $\mathbf{V_t}$ is not longer equal to $\begin{pmatrix} u \\ 0 \end{pmatrix}_V$, but to $\begin{pmatrix} u \\ v \end{pmatrix}_V$. Thus, the equation becomes:

$$\begin{pmatrix} \dot{s}_1 \\ \dot{y}_1 \end{pmatrix}_F = R(\theta) \begin{pmatrix} u \\ v \end{pmatrix} - \dot{s} \begin{pmatrix} 1 - c_c y_1 \\ c_c s_1 \end{pmatrix}_F \tag{8}$$

1.3 Actuators and dynamics

Actuators produce forces which then induce accelerations. In order to take actuation into account we need to consider a dynamical model. That is, considering forces, torques and geometrical parameters of the system. In a kinematic model, we control the system using speeds and angular velocities, which is to say that we can control the system's position using first order derivatives. In a dynamical model, we use accelerations and angular accelerations in order to control the robot's position. In other terms, we use 2^{nd} order derivatives to influence the system's position.

1.3.1 The dynamic model of the system

In order to cope with the actuation problem, a dynamic model is required. Let us consider a robot with n actuators such as in Figure (3) :

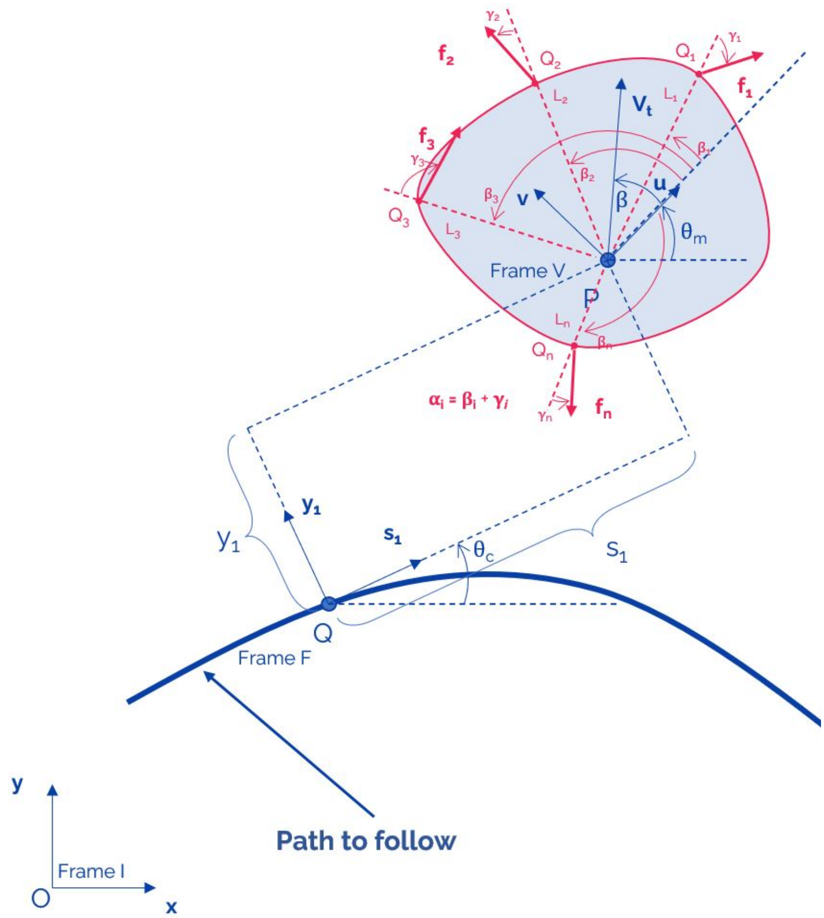

Figure 3: Robot with n actuators

Each f_i is a force that is produced by an actuator. Each force is placed at distance L_i between point P and the point Q_i where the force is applied. The vector $\overrightarrow{PQ_i}$ has an angle with the $\hat{u}$ axis of β_i. The force f_i has an angle γ_i with the vector $\overrightarrow{PQ_i}$. When $\gamma_i = 0$ the force points outwards (in the same direction as the vector $\overrightarrow{PQ_i}$). We will take the following notation: $\alpha_i = \gamma_i + \beta_i$. The dynamic equation will be written in the body frame in the next paragraph.

1.3.2 Dynamic Equations

Foremost, let $P = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_I$ be the position of the robot in the reference frame I. Let $\mathbf{V_r} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}_V$ be the speed of the robot expressed in the robot frame V. Thus, $\dot{P} = \mathbf{R}\mathbf{V_t}$ with $\mathbf{R} = R_3(\theta_m)$. Therefore, $\ddot{P} = \dot{\mathbf{R}}\mathbf{V_r} + \mathbf{R}\dot{\mathbf{V}}_\mathbf{r}$, which amounts to $\dot{\mathbf{V}}_\mathbf{r} = \mathbf{R}^T\ddot{P} - \mathbf{R}^T\dot{\mathbf{R}}\mathbf{V_r}$. The term $\mathbf{R}^T\dot{\mathbf{R}}\mathbf{V_r}$ is equal to $\omega_\mathbf{r} \wedge \mathbf{V_r}$, where $\omega_\mathbf{r}$ is the rotation vector of the robot expressed in the body frame.

The second law of Newton tells us that $R^T\ddot{P} = \frac{1}{m}F_r$, where F_r are the external forces expressed in the robot frame. We can write $\mathbf{F_r} = \begin{pmatrix} F_u \\ F_v \\ F_z \end{pmatrix}$. $F_z = 0$ in our case. Thus: $\dot{\mathbf{V}}_\mathbf{r} = \frac{1}{m}\mathbf{F_r} - \omega_\mathbf{r} \wedge \mathbf{V_r}$. As for $\omega_\mathbf{r}$, since we are in the plane, $\omega_\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_m \end{pmatrix}$ and Newton law in rotation tells us $\ddot{\theta}_m =$

$\frac{1}{J}\begin{pmatrix}L_1 sin(\gamma_1) & L_2 sin(\gamma_2) & . & . & L_n sin(\gamma_n)\end{pmatrix}\begin{pmatrix}f_1\\ f_2\\ .\\ .\\ f_n\end{pmatrix}$, where the $L_i = \|\overrightarrow{PQ}_i\|$, J the inertia moment of the robot around the $\hat{z}$ axis and as stated n is the number of actuators.

The product $\begin{pmatrix}L_1 sin(\gamma_1) & L_2 sin(\gamma_2) & . & . & L_n sin(\gamma_n)\end{pmatrix}\begin{pmatrix}f_1\\ f_2\\ .\\ .\\ f_n\end{pmatrix}$ can be denoted Γ and is the torque along the $\hat{z}$ axis.

Since we are working the in plane, the third component of $\omega_\mathbf{r} \wedge \mathbf{V_r}$ is equal to 0. We can express the first 2 lines of the vectorial product as such: $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\omega_\mathbf{r} \wedge \mathbf{V_r} = -\dot{\theta}_m R(\pi/2)\mathbf{V_t}$
On the other hand, the force f_i can be expressed by a vector in the body frame as such: $F_{i,r} = R(\alpha_i)\begin{pmatrix}f_i\\ 0\end{pmatrix} = f_i\begin{pmatrix}cos(\alpha_i)\\ sin(\alpha_i)\end{pmatrix}$
If we let $F = \begin{pmatrix}f_1\\ f_2\\ .\\ .\\ f_n\end{pmatrix}$, the equation can be rewritten as:

$$\begin{aligned}\dot{V}_t &= \frac{1}{m}\begin{pmatrix}cos(\alpha_1) & cos(\alpha_2) & . & . & cos(\alpha_n)\\ sin(\alpha_1) & sin(\alpha_2) & . & . & sin(\alpha_n)\end{pmatrix}F - \dot{\theta}_m R(\pi/2)V_t\\ \ddot{\theta}_m &= \frac{1}{J}\begin{pmatrix}L_1 sin(\gamma_1) & L_2 sin(\gamma_2) & . & . & L_n sin(\gamma_n)\end{pmatrix}F\end{aligned} \tag{9}$$

The matrix $\begin{pmatrix}cos(\alpha_1) & cos(\alpha_2) & . & . & cos(\alpha_n)\\ sin(\alpha_1) & sin(\alpha_2) & . & . & sin(\alpha_n)\\ L_1 sin(\gamma_1) & L_2 sin(\gamma_2) & . & . & L_n sin(\gamma_n)\end{pmatrix}$ has a significant role in our study.
It will therefore be denoted **A'**. **A** will be reserved for another Matrix which will be called **Actuation Matrix**. This matrix is formed using **A'** and will be introduced in the next paragraph.

1.3.3 Modeling the loss of a motor

Losing a motor is such that the force that is produced is equal to 0 independently of the input f_i given for that motor. It is as setting a column in the Matrix of system (9) that ties the force F to V_t and $\ddot{\theta}_m$ to the null column. This could be modeled by replacing the force F by $\mathbf{D}F$, where $\mathbf{D} = diag(d_1, d_2.., d_n)$. If motor number i stopped working or is half as efficient then d_i could be set to 0 or 1/2 respectively. F is then the desired force to be produced by the motor and $\mathbf{D} \times F$ the actual resulting force. Notice that when all the motors work correctly, $\mathbf{D} = I_n$ with I_n being the identity matrix of size n.
With the matrix **D** in mind, let us introduce the **Actuation Matrix A** defined as **A** = **A'** × **D**:

$$\mathbf{A} = \begin{pmatrix}d_1 cos(\alpha_1) & d_2 cos(\alpha_2) & . & . & d_n cos(\alpha_n)\\ d_1 sin(\alpha_1) & d_2 sin(\alpha_2) & . & . & d_n sin(\alpha_n)\\ d_1 L_1 sin(\gamma_1) & d_2 L_2 sin(\gamma_2) & . & . & d_n L_n sin(\gamma_n)\end{pmatrix}$$

1.3.4 Particular case, n=3

Here is an example of the particular case in which $n = 3, \gamma_i = \pi/2, \beta_1 = 0, L_i = L$. By taking into account the formalism that replaces F by DF, the dynamic equations can be written as the following:

$$\begin{aligned}\dot{V}_t &= \frac{1}{m}\begin{pmatrix}0 & -d_2 sin(\beta_2) & -d_3 sin(\beta_3)\\ d_1 & d_2 cos(\beta_2) & d_3 cos(\beta_3)\end{pmatrix}F - \dot{\theta}_m R(\pi/2)V_t\\ \ddot{\theta}_m &= \frac{L}{J}\begin{pmatrix}d_1 & d_2 & d_3\end{pmatrix}F\end{aligned} \tag{10}$$

An illustration of the situation is given in Figure (4):

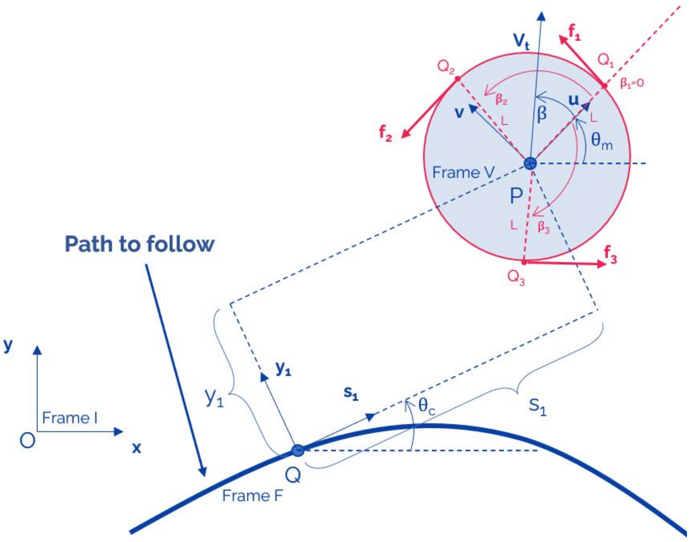

Figure 4: Particular case: $n = 3, \gamma_i = \pi/2, \beta_1 = 0, L_i = L$

1.4 Polar Coordinates

The equations (9) can be written in a polar form. The polar form would involve the norm of the speed $\nu = \|\mathbf{V_t}\|$, the angle between $\mathbf{V_t}$ and the $\hat{u}$ axis, denoted β, and at last, the angle of the speed relative to the path $\psi = \theta + \beta$. The polar coordinates can be deduced by changing coordinates. The following relation $\mathbf{V_t} = R(\beta)\begin{pmatrix}\nu\\0\end{pmatrix}$, implies:

$$\begin{aligned}
\dot{V}_t &= \dot{\beta}R(\beta)R(\pi/2)\begin{pmatrix}\nu\\0\end{pmatrix} + R(\beta)\begin{pmatrix}\dot{\nu}\\0\end{pmatrix}\\
\implies \dot{V}_t &= R(\beta)\begin{pmatrix}\dot{\nu}\\ \dot{\beta}\nu\end{pmatrix}\\
\implies \begin{pmatrix}\dot{\nu}\\ \dot{\beta}\end{pmatrix} &= \begin{pmatrix}1 & 0\\ 0 & \frac{1}{\nu}\end{pmatrix} R(\beta)^T\dot{V}_t
\end{aligned} \tag{11}$$

And injecting $\dot{\mathbf{V}}_\mathbf{t}$ into (11) yields the dynamic equations for ν and β and ψ:

$$\begin{aligned}
\begin{pmatrix}\dot{\nu}\\ \dot{\beta}\end{pmatrix} &= \frac{1}{m}\begin{pmatrix}1 & 0\\ 0 & \frac{1}{\nu}\end{pmatrix} R(\beta)^T \begin{pmatrix} cos(\alpha_1) & cos(\alpha_2) & . & . & cos(\alpha_n)\\ sin(\alpha_1) & sin(\alpha_2) & . & . & sin(\alpha_n)\end{pmatrix} F - \begin{pmatrix}0\\ \dot{\theta}_m\end{pmatrix}\\
\dot{\psi} &= \dot{\theta}_m - \dot{\theta}_c + \dot{\beta}\\
\ddot{\theta}_m &= \frac{1}{J}\begin{pmatrix} d_1L_1sin(\gamma_1) & d_2L_2sin(\gamma_2) & . & . & d_nL_nsin(\gamma_n)\end{pmatrix} F
\end{aligned} \tag{12}$$

2 The employed strategy and the statement of the problem

2.1 Analysis

The goal of the path following algorithm is to bring the center of mass of the robot to the path. Moving a point in 2D space requires 2 degrees of freedom. Thus, in order to follow the path, we could control ν and ψ. It is needed that the robot has a speed ν strictly greater than 0 and is able to control the direction ψ of that speed accordingly. Indeed, if the speed is 0, the robot can not follow the path.
Two options are available to control ψ: θ_m and β. They are related to the forces and torques that are applied on the system but not for the same derivative order. As shown by the relation in equation (11), $\dot{\beta}$ is directly related to the forces applied on the system whereas $\dot{\theta}_m$ is not, though, $\ddot{\theta}_m$ is.
Two strategies then emerge in order to drive ψ to a desired value (value being the guidance reference function $\delta = -\theta_a tanh(Ky_1), K > 0, \theta_a \in [-\frac{\pi}{2}, \frac{\pi}{2}]$).
In [3] it was shown that as long as $\psi - \delta \underset{t\to\infty}{\longrightarrow} 0$ and $\nu > 0$ then $(s_1, y_1) \underset{t\to\infty}{\longrightarrow} (0, 0)$. This was proved using

Lasalle's principle. Moreover, in the proof, it did not matter how $\psi - \delta \underset{t\to\infty}{\longrightarrow} 0$. The meaning of this is that ψ can satisfy any differential equation and as long as that equation is such that $\psi - \delta \underset{t\to\infty}{\longrightarrow} 0$, then $(s_1, y_1) \underset{t\to\infty}{\longrightarrow} (0,0)$. That will come in handy later.
Since $\psi = \theta_m + \beta - \theta_c$, we can either control $\dot{\psi}$ using $\dot{\beta}$ or control $\ddot{\psi}$ using $\ddot{\theta}_m$. Controlling $\dot{\psi}$ or $\ddot{\psi}$ means that we will find a differential equation of the first or second order (not necessarily linear) that if ψ satisfies, then $\psi - \delta \underset{t\to\infty}{\longrightarrow} 0$. Choosing a variable among $\dot{\beta}$ and $\ddot{\theta}_m$ to control $\dot{\psi}$ or $\ddot{\psi}$ will necessarily require that the other variable becomes a measurement. In other words, it needs to become a state variable. For example, if we control $\ddot{\psi}$ using $\ddot{\theta}_m$ then we need to be able to measure $\ddot{\beta}$.
Therefore, let us state the requirements that the robot has to satisfy:
Foremost, the robot has to prioritize following the path, as long as the actuation allows it. In fact, we will see that, as long as 2 motors work, it is possible to follow the path.
Secondly, if the robot has 3 or more motors which are well positioned (we will see what well positioned means further down), then we have to use them to accomplish some optional task. Here are some examples: orient the vehicle to a desired orientation (for example, always face a stationary chosen point), be reactive (this might come in handy in obstacle avoidance), avoid motor dead-zones (it is the minimum command required in order for the motor to start produce a force/torque) etc.
The controller has to take actuation into account. Indeed, if we have 3 motors, and chose as an optional task the orientation of the vehicle, then if one motor is lost, the control must drop the orientation of the vehicle and prioritize following the path. As stated, we have at hand two strategies to ensure $\psi - \delta \underset{t\to\infty}{\longrightarrow} 0$.
We will try the two methods and see that each has "singularities". Because these singularities exist, 2 controllers had to be developed in order to cope with them. The goal would be to be to have one controller that unites the 2 cases.

2.2 A has a rank equal to 3

When the Actuation Matrix has full rank, the path can be followed and an optional task can be accomplished as well. I chose to start with the orientation of the robot.

2.2.1 Following the path

Let's begin with trying to control $\dot{\psi}$ through the means of $\dot{\beta}$. If $\mathbf{A}$ is the Actuation Matrix, then its first two lines must form a matrix whose rank is 2 to insure that we can control $\dot{\beta}$ and $\dot{\nu}$ independently:

$$\mathbf{A} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} \implies (rank \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = 2 \implies \text{following the path is possible })$$

This could also be expressed otherwise. Indeed, if we write $\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & . & . & c_n \end{pmatrix}$, then we need that $det(c_i, c_j) \neq 0$ for one pair of $(i,j), i \neq j$ to satisfy the condition. We have $c_i = \begin{pmatrix} d_i cos(\alpha_i) \\ d_i sin(\alpha_i) \end{pmatrix}$. Thus $det(c_i, c_j) = \begin{vmatrix} d_i cos(\alpha_i) & d_j cos(\alpha_j) \\ d_i sin(\alpha_i) & d_j sin(\alpha_j) \end{vmatrix} = d_i d_j \begin{vmatrix} cos(\alpha_i) & cos(\alpha_j) \\ sin(\alpha_i) & sin(\alpha_j) \end{vmatrix} = d_i d_j sin(\alpha_j - \alpha_i)$.
Thus we have the following equivalence:

$$rank \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = 2 \iff \exists (i,j), i \neq j \ / \ d_i d_j sin(\alpha_j - \alpha_i) \neq 0 \tag{13}$$

This condition means the system has at least two non-colinear motors that are still working.
With this condition verified, we now need to find a differential equation that if ψ verifies, then $\psi - \delta \underset{t\to\infty}{\longrightarrow} 0$.
A proposition is the following equation:

$$\dot{\psi} = \dot{\delta} + k(\delta - \psi), k > 0 \tag{14}$$

For more information on why this implies $\psi - \delta \underset{t\to\infty}{\longrightarrow} 0$ please refer to the Appendix. From the equality $\dot{\psi} = \dot{\beta} + \dot{\theta}_m - \dot{\theta}_c$, we can deduce that if $\dot{\beta} = \dot{\delta} + k(\delta - \psi) - \dot{\theta}_m + \dot{\theta}_c$, then ψ would satisfy (14). This can be done because $\dot{\beta}$ is controllable. Driving ν to a desired reference ν_d speed is fairly easy. If we let $\dot{\nu} = \dot{\nu}_d + k'(\nu_d - \nu)$,$k' > 0$, then $\nu - \nu_d \underset{t\to\infty}{\longrightarrow} 0$.

2.2.2 Optional task: Orienting the system

Moreover, since A is full rank, we can try to orient the vehicle to a desired orientation θ_d. Following the path is assured by the variables $\dot{\beta}$ and $\dot{\nu}$ which are chosen to drive $\psi - \delta \underset{t\to\infty}{\longrightarrow} 0$ and assure $\nu > 0$. As shown in [3], this implies that $(s_1, y_1) \underset{t\to\infty}{\longrightarrow} (0, 0)$. To orient the robot, we can use $\ddot{\theta}_m$. Let θ_d be the desired orientation of the robot.
Let us set $\ddot{\theta}_m = \ddot{\theta}_d + k_1(\dot{\theta}_d - \dot{\theta}_m) + k_0(\theta_d - \theta_m)$ with k_1, k_0 some positive well chosen coefficients.
Then $\theta_m - \theta_d \underset{t\to\infty}{\longrightarrow} 0$. For more information on this, please refer to the Appendix.

2.3 A has a rank equal to 2

When the rank of **A** is equal to 2, it means physically that we have 2 motors. We will not consider the case where two motors are place identically in the same place with the same orientation as such a case is physically impossible.
We will distinguish 2 cases: The 2 motors are not colinear & the 2 motors are colinear. That is because of condition (13). We will see that, even if conditon (13) is not satisfied, it is still possible to follow the path, though the structure of the control will have to change.

2.3.1 Following the path: Two non-colinear motors are available

Let's assume that condition (13) is verified. In that case we can use the same equations as above which are:

$$(C) : \begin{cases} \dot{\nu} = \dot{\nu}_d + k'(\nu_d - \nu) \\ \dot{\beta} = \dot{\delta} + k(\delta - \psi) - \dot{\theta}_m + \dot{\theta}_c \\ \ddot{\theta}_m = \ddot{\theta}_d + k_1(\dot{\theta}_d - \dot{\theta}_m) + k_0(\theta_d - \theta_m) \end{cases} \tag{15}$$

All the k-coefficients are strictly positive.
In this scenario, we have to prioritize following the path and therefore can not orient the vehicle. To model the discontinuity that happens when a motor is lost will require the introduction of the following function whose input is the Actuation Matrix:

$$\zeta(\mathbf{A}) = \begin{cases} 0_3 & \text{if } Rank(\mathbf{A}) \leq 1 \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & \text{if } Rank(\mathbf{A}) = 2 \\ I_3 & \text{if } Rank(\mathbf{A}) \geq 3 \end{cases} \tag{16}$$

Equation (12) shows that $\dot{\nu}, \dot{\beta}, \ddot{\theta}_m$ are related to the forces F by a affine relation such as:

$$\begin{pmatrix} \dot{\nu} \\ \dot{\beta} \\ \ddot{\theta}_m \end{pmatrix} = \mathbf{BAD}F + b \tag{17}$$

Where $b = \begin{pmatrix} 0 \\ -\dot{\theta}_m \\ 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} \frac{cos(\beta)}{m} & \frac{sin(\beta)}{m} & 0 \\ -\frac{sin(\beta)}{m\nu} & \frac{cos(\beta)}{m\nu} & 0 \\ 0 & 0 & \frac{1}{J} \end{pmatrix}$. **B** is invertible. In the case where $Rank(\mathbf{A}) = 3$ we inverted as such:

$$F = (\mathbf{BA})^{-1} \left(\begin{pmatrix} \dot{\nu} \\ \dot{\beta} \\ \ddot{\theta}_m \end{pmatrix} - b \right) \tag{18}$$

We have to "invert" only the first two lines of equation (20) because we have a rank of two and want to prioritize following the path. In order to do so, we need to use the Penrose inverse on the first two lines of **BA** Matrix as such:

$$F = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \mathbf{BA} \right)^+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \left(\begin{pmatrix} \dot{\nu} \\ \dot{\beta} \\ \ddot{\theta}_m \end{pmatrix} - b \right) \tag{19}$$

This can be rewritten with the ζ function in a very compact form and generalizes both cases ($Rank(\mathbf{A}) =$ 2 or 3):

$$F = (\zeta(\mathbf{A})\mathbf{BA})^+ \zeta(\mathbf{A}) \left(\begin{pmatrix} \dot{\nu} \\ \dot{\beta} \\ \ddot{\theta}_m \end{pmatrix} - b \right) \tag{20}$$

The ζ function can be further improved using the work done in [4]. Indeed, the ζ function is using conditions based only on the rank of $\mathbf{A}$, but in [4], it was pointed that in practice $\mathbf{A}$ is usually full rank but that does not necessarily mean that we have to consider it as such. One example is when we have 3 motors, 2 parallel and one orthogonal to the others. If we had to consider an AUV that moves at high speeds under water along the direction of the first 2 colinear motors then the third motor would become inefficient because in order for the flow of water to be aspired by the third motor, it would have to make a 90° turn. The third motor should thus be considered as "lost" in this case. The ζ function could be improved by taking this into account. In our example, it could consider that it should only use 2 motors if ($Rank(\mathbf{A}) > 2$ **AND** Motor 3 produces enough force per unit of power) **OR** $Rank(\mathbf{A}) = 2$.

2.3.1.1 What happens to $\dot{\theta}_m$?

An important matter to study is what happens to $\dot{\theta}_m$. As it is not controlled, it will be free. It is rather difficult to find the asymptotic limits of our system. Thus I resorted to simulations. Paths can be in practice approximated by the concatenation of pieces of circles and straight lines. Thus, I created multiple simulations with different actuation properties in which the robot follows circles of different Radius and lines of different slopes. The circles have diameters of 10m, 15m,.. 35m and 40m. The simulation was done using the particular case and setting $d_1 = 0, d_2 = d_3 = 1$ and varying β_2 and β_3. Here are the simulation results for the following actuation configurations:

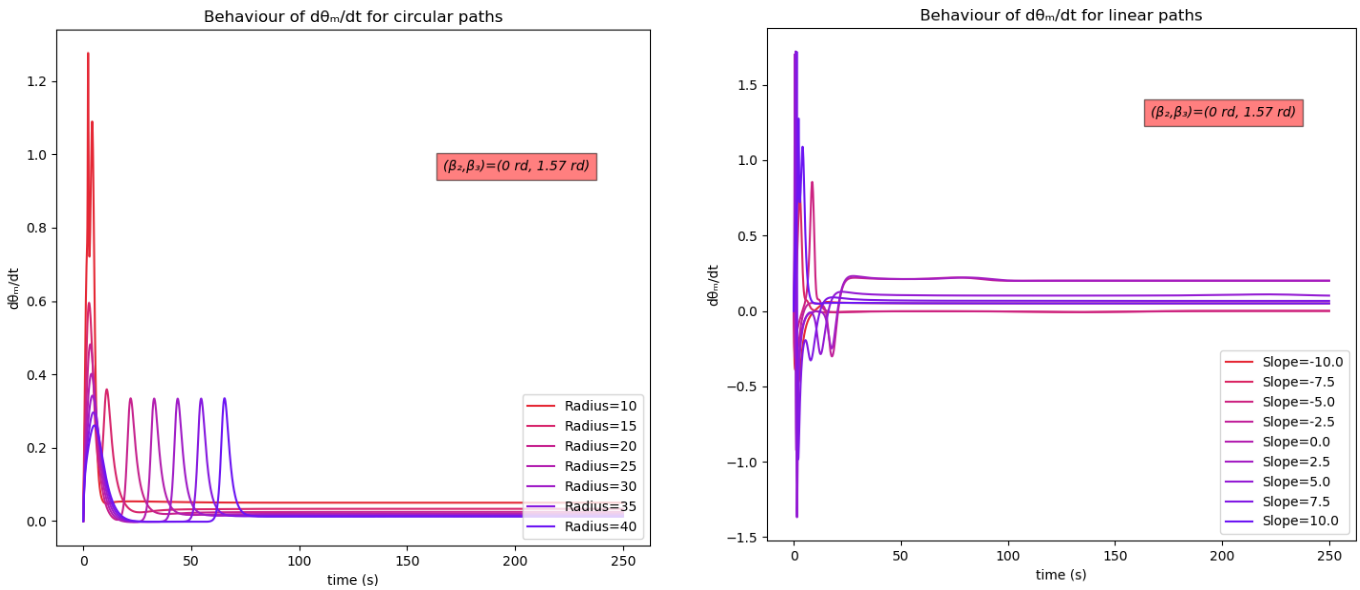

Figure 5: Simulation for circular and linear paths

The rest of the simulations are found in the Appendix. The curves are very similar which is why only 2 are shown here. We can notice using those curves that $\dot{\theta}_m$ seems to have a real limit as $t \longrightarrow \infty$. Although this is no proof by any means, it suggests nevertheless that the system does not spin infinitely fast as time goes on. The simulations were all done using the following starting condition :

$$X_0 = (x_0, y_0, u_0, v_0, s_0, \theta_{m,0}, \dot{\theta}_{m,0}) = (-5, 10, 0.5, 0, 0, 0, 0)$$

2.3.1.2 Two colinear motors are available: a change in the controller structure is required

Condition (13) seems reasonable. The problem comes when we consider a robot with 2 motors which are colinear. Then, our controller has singularities for some values of β. We intuitively know that we can still follow the path and our intuition is proven in [3] because a controller for such a case in provided. Although, this is not a complete proof, because in [3], fluid friction is taken into account and our model does not take that into account. Not a proof, but more of a demonstration is the following simulation where the vehicle (without friction) which has 2 colinear motors was driven using the keyboard (humanly operated):

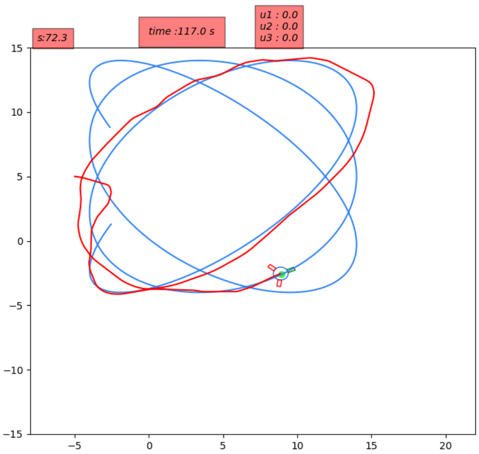

Figure 6: The path was followed using the keyboard

This suggests that a controller should exist.
The physical explanation of the singularities for some values of β is that when two motors are colinear and $\mathbf{V_t}$ is aligned with the two motors, accelerating does not change the angle β.
Now, let's assume that only two motors are working and that they are colinear with each other. This can be achieved by setting $n = 2, d_1 = d_2 = 1, \alpha_1 = \alpha_2\ (\pi)$ in the Actuation Matrix. This implies that $cos(\alpha_1) = \pm cos(\alpha_2)$ and $sin(\alpha_1) = \pm sin(\alpha_2)$. Let's let $\alpha_1 = \alpha$. The previous method does not work anymore because $\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} cos(\alpha) & \pm cos(\alpha) \\ sin(\alpha) & \pm sin(\alpha) \end{pmatrix}$ is of rank 1. Equation (12) expresses the singularity:

$$\dot{\beta} = \frac{sin(\alpha - \beta)}{m\nu}(f_1 \pm f_2) - \dot{\theta}_m$$

$\dot{\beta}$ can not be controlled by the f_i when $\beta = \alpha\ (\pi)$. We need to deal with this singularity, and to do so, we will use [3] to find a controller for this situation.
When we consider a robot with only 2 motors that are colinear, $\dot{\beta}$ can not be controlled for $\beta = \alpha$. $\ddot{\theta}_m$ is now a potential candidate in order to drive ψ to δ.
In the previous section, it was absurd to talk about "directions in which the robot can go". Now, not only does it make sense, but it's necessary. Since two motors are colinear, there is a "forward" direction. We need to find that direction. Once it is found, we can make a change of coordinates which will simplify the equation and bring us to a case familiar to [3].
As stated previously, we have $\alpha = \alpha_1 = \alpha_2\ (\pi)$. Two cases are then to be distinguished: we either have $\alpha_1 = \alpha_2$ or $\alpha_1 = \alpha_2 + \pi$. Those two cases are in fact the same. It is only a change in sign in the input of the force applied. We can thus choose one of the cases. We will thus make the assumption that $\alpha_1 = \alpha_2 = \alpha$.
Let's begin by making the change in coordinates which will simplify equation (9). In our case, it can be written as such:

$$\begin{aligned} \dot{V}_t &= \frac{1}{m}\begin{pmatrix} cos(\alpha) & cos(\alpha) \\ sin(\alpha) & sin(\alpha) \end{pmatrix} F - \dot{\theta}_m R(\pi/2) V_t = \frac{1}{m}(f_1 + f_2)\begin{pmatrix} cos(\alpha) \\ sin(\alpha) \end{pmatrix} - \dot{\theta}_m R(\pi/2) V_t \\ \ddot{\theta}_m &= \frac{1}{J}\begin{pmatrix} L_1 sin(\gamma_1) & L_2 sin(\gamma_2) \end{pmatrix} F \end{aligned} \tag{21}$$

The "forward" direction is clearly $\begin{pmatrix} cos(\alpha) & sin(\alpha) \end{pmatrix}^T$. We will name $V_{t,\rho} = \begin{pmatrix} u_\rho & v_\rho \end{pmatrix}^T$ the speed of the robot in that frame. Therefore, we have $V_t = R(\alpha)V_{t,\rho}$ and thus $\dot{V}_t = R(\alpha)\dot{V}_{t,\rho}$. Therefore:

$$\begin{aligned} \dot{V}_{t,\rho} &= R(\alpha)^T \dot{V}_t \\ \dot{V}_{t,\rho} &= R(\alpha)^T (\frac{1}{m}(f_1 + f_2)\begin{pmatrix} cos(\alpha) \\ sin(\alpha) \end{pmatrix} - \dot{\theta}_m R(\pi/2) V_t) \\ \dot{V}_{t,\rho} &= \frac{1}{m}(f_1 + f_2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \dot{\theta}_m R(\pi/2) V_{t,\rho} \end{aligned}$$

With this done, we have:

$$\begin{pmatrix} u_\rho \\ \ddot{\theta}_m \end{pmatrix} = \begin{pmatrix} \frac{1}{m} & \frac{1}{m} \\ \frac{1}{J} L_1 sin(\gamma_1) & \frac{1}{J} L_2 sin(\gamma_2) \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \begin{pmatrix} \dot{\theta}_m v_\rho \\ 0 \end{pmatrix} = A_\rho F + b \tag{22}$$

We thus need that $rank(A_\rho) = 2$. Which is to say that:

$$\frac{1}{Jm}(L_1 sin(\gamma_1) - L_2 sin(\gamma_2)) \neq 0 \tag{23}$$
$$\iff \quad (L_1 sin(\gamma_1) - L_2 sin(\gamma_2)) \neq 0 \tag{24}$$

The physical explanation is that there must not be a line that passes through the 2 forces. Here is an example where this condition is not satisfied:

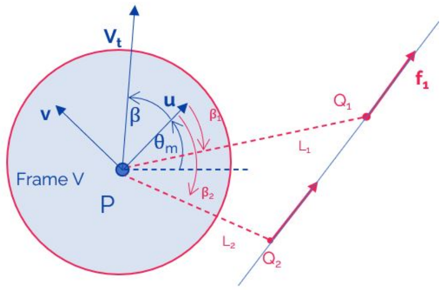

Figure 7: $L_1 sin(\gamma_1) - L_2 sin(\gamma_2) = 0$

It can clearly be seen that in such a case, the system is of rank 1 and can not turn and go forward independently. Such a case is to be dismissed. We will assume going forward that (24) is satisfied.
The goal is now to that insure $\psi - \delta \underset{t\to\infty}{\longrightarrow} 0$ and $u_\rho > 0$. The difference with the previous paragraph is that ψ will not satisfy a first order differential equation like (14) but another one. As stated in the introductory paragraph, when [3] proved that their controller worked, they used Lasalle's Invariance Principle.
Foremost, they showed that their controller drove $\psi - \delta$ to 0 as $t \longrightarrow \infty$. Then, they considered the state space where ψ was exactly equal to δ. Being in that state space implied that $(s_1, y_1) \longrightarrow (0, 0)$.
Lasalle's principle tells us that as long as $\psi - \delta \longrightarrow 0$ and $\nu > 0$, the system converges to a state where $(s_1, y_1) = (0, 0)$. We can use this here. Please refer to the Appendix for the theorem on differential equation with constant coefficients in order to understand the following.
With the result stated in the Appendix in mind, if we let $y = \psi - \delta$ and choose the coefficients of a polynomial accordingly, then $y = \psi - \delta \underset{t\to\infty}{\longrightarrow} 0$.
In previous paragraph, we tried to control $\dot{\psi}$ through $\dot{\beta}$. We have $\dot{\psi} = \dot{\theta}_m + \dot{\beta} - \dot{\theta}_c$. $\dot{\beta}$ is related to the first derivative of ψ. We thus exhibited the polynomial $P = X + k, k > 0$ and set $P(\delta - \psi) = 0$, *i.e.* $(\dot{\delta} - \dot{\psi}) + k(\delta - \psi) = 0$ to be the differential equation that we wanted ψ to satisfy. We then let $\dot{\beta} = (\dot{\delta} + k(\delta - \psi)) - \dot{\theta}_m + \dot{\theta}_c$. The choice of this $\dot{\beta}$ insured that ψ satisfied the desired differential equation.
Now, if we try to control ψ using $\ddot{\theta}_m$, we will have to exhibit a differential equation of ψ of the second order because $\ddot{\theta}_m$ is related to $\ddot{\psi}$. Let us choose the polynomial $P = X^2 + k_1 X + k_0$ such that its roots are strictly negative. Then ψ has to verify $(\ddot{\delta} - \ddot{\psi}) + k_1(\dot{\delta} - \dot{\psi}) + k_0(\delta - \psi) = 0$ which is equivalent to saying that $\ddot{\psi} = \ddot{\delta} + k_1(\dot{\delta} - \dot{\psi}) + k_0(\delta - \psi)$. Since $\ddot{\psi} = \ddot{\theta}_m + \ddot{\beta} - \ddot{\theta}_c$, setting $\ddot{\theta}_m = \ddot{\theta}_c - \ddot{\beta} + \ddot{\delta} + k_1(\dot{\delta} - \dot{\psi}) + k_0(\delta - \psi)$, will insure that ψ satisfies the desired differential equation which then implies $\psi - \delta \longrightarrow 0$.
In [3], the authors used a Lyapunov method and backstepping techniques, but they landed on the same result.
To insure $\nu > 0$, we can set $u_\rho = \dot{u}_{d,\rho} + k(u_{d,\rho} - u_\rho)$, which implies that $u_{d,\rho} - u_\rho \underset{t\to\infty}{\longrightarrow} 0$, which implies $\nu > 0$ because $\nu = \sqrt{u_\rho^2 + v_\rho^2}$. $u_{d,\rho} > 0$ is a reference forward velocity profile.

2.3.1.2.1 Singularity

As stated before, using one variable for control means that the other variable becomes a measurement. One problem with driving ψ towards δ using $\ddot{\theta}_m$ is that the term $\ddot{\beta}$ (which has to be a measurement now) appears and hides terms such as $\dot{u}, \ddot{u}, \dot{v}, \ddot{v}$ which depend on the forces F_u, F_v and their time derivatives. The equations used in [3] are the equations of an AUV, which takes into account fluid friction:

$$\begin{aligned} \dot{u} &= \tfrac{1}{m_u}(F_u - d_u) & \qquad d_u &= -X_{uu}u^2 - X_{vv}v^2 \\ \dot{v} &= \tfrac{1}{m_v}(F_v - d_v - m_{ur}ur) & \qquad d_v &= -Y_v uv - Y_{v|v|}v|v| \\ \ddot{\theta}_m &= \tfrac{1}{m_r}(\Gamma - d_r) & \qquad d_r &= -N_{uv}uv - N_{v|v|}v|v| - N_r ur \end{aligned}$$

X_{ij}, Y_{ij}, N_{ij} and m_i are constant coefficients.
In [3], this issue was overcome because the force F_u was set to $F_u = d_u + \frac{1}{m_u}(\dot{u_d} + k(u_d - u))$ and F_v was equal to 0 because the AUV did not have sideways motors mounted. These forces are therefore functions of the state of the system. Thus $\dot{F_u}$ and $\dot{F_v}$ can also be expressed in terms of the state of the system. With those forces being chosen, one could find the value of $\ddot{\beta}$ using the dynamical equations and then find what Torque Γ to apply.
In summary, the issue was overcome because F_u and F_v were chosen before hand to be state functions, which then let the authors find the measurement $\ddot{\beta}$, which was then used to calculate the torque Γ.
Usually we have the needed measurements without having to impose the forces. We then inject this information into a guidance system. This guidance system gives the accelerations necessary in order to insure that we go where we want. Then, those accelerations go into the controller which gives the commands or forces that we need to provide in order to follow the chosen path. Our case is a bit unique because we were forced to choose the forces F_u and F_v before hand, and that allowed us to find the Torque Γ.
In doing so, $\ddot{\beta}$ can be expressed in terms of the state of the system and the issue is gone (almost).
In solving for $\ddot{\beta}$, the authors of [3] arrived at an equation of the form $\ddot{\beta}(1 - \frac{m_{ur}}{m_v} cos(\beta)^2) = g(X)$ where X is the state of the system. To be able to isolate $\ddot{\beta}$ and have no division by 0, it was necessary that $\frac{m_{ur}}{m_v} < 1$. This condition means that there is a preferential movement direction and when we turn, we do not slip sideways completely but get slowed, a bit like a unicycle-type robot. This factor makes the AUV look similar to the Unicycle. The unicycle is in fact a case where we get slowed infinitely because we do not slip sideways when we turn. The fluid friction is what allowed the authors to drive the vehicle a bit similarly to a unicycle.
In fact, if we tried to apply a similar method on our circle-type-robot system, we would arrive at a equation of the type $\ddot{\beta}(1 - cos(\alpha - \beta)^2) = g(X)$. This equation can not be solved for $\ddot{\beta}$ when $\beta = \alpha\ (\pi)$. The friction is necessary in order to not have singularities with this method. Thus we are stuck. The method used in [3] can not be applied to our system because we do not have fluid friction and also we do not have a preferential movement direction. Our equations are symmetrical in the u and v directions whereas in [3] they are not. Let us thus consider the equations of the AUV stated above. They can be written and such:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \ddot{\theta}_m \end{pmatrix} = M^{-1} \left(\begin{pmatrix} F_u \\ F_v \\ \Gamma \end{pmatrix} - d \right) = M^{-1} \left(\mathbf{A}F - d \right) \tag{25}$$

With $M^{-1} = diag(\frac{1}{m_u}, \frac{1}{m_v}, \frac{1}{m_r})$ and $d = (d_u, d_v + m_{ur}ur, d_r)^T$. From this we can deduce that:

$$\begin{pmatrix} \dot{\nu} \\ \dot{\beta} \\ \ddot{\theta}_m \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\nu} & 0 \\ 0 & 0 & 1 \end{pmatrix} R_3(\beta)^T M^{-1} \left(\mathbf{A}F - d \right) \tag{26}$$

Thus using the same method as above we can inverse using the ζ function:

$$F = (\zeta(\mathbf{A}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\nu} & 0 \\ 0 & 0 & 1 \end{pmatrix} R_3(\beta)^T M^{-1}\mathbf{A})^+ \zeta(\mathbf{A}) \left(\begin{pmatrix} \dot{\nu} \\ \dot{\beta} \\ \ddot{\theta}_m \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\nu} & 0 \\ 0 & 0 & 1 \end{pmatrix} R_3(\beta)^T M^{-1} d \right) \tag{27}$$

This only works when condition (13) is satisfied. For colinear motors, in our non friction system, we first found the "forward direction" and then made a change in coordinates. Our system was symmetrical. What happens when we take the equations of an AUV and try to find the "forward direction" ? Let's assume that there is a forward direction ρ of angle α with the $\hat{u}$ axis. Let $M' = M[0:2], d' = d[0:2]$.As stated above we can write:

$$\begin{aligned} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} &= M'^{-1}((f_1 + f_2) \begin{pmatrix} cos(\alpha) \\ sin(\alpha) \end{pmatrix} - d') \\ &= M'^{-1}(R(\alpha) \begin{pmatrix} f_1 + f_2 \\ 0 \end{pmatrix} - d') \\ \implies \begin{pmatrix} \dot{u}_\rho \\ \dot{v}_\rho \end{pmatrix} &= R(\alpha)^T M'^{-1}(R(\alpha) \begin{pmatrix} f_1 + f_2 \\ 0 \end{pmatrix} - d') \\ &= R(\alpha)^T M'^{-1} R(\alpha)(\begin{pmatrix} f_1 + f_2 \\ 0 \end{pmatrix} - R(\alpha)^T d'). \end{aligned} \tag{28}$$

The matrix $R(\alpha)^T M'^{-1} R(\alpha)$ would be diagonal for every α if $m_u = m_v$. This implies a symmetry in the system. d' is replaced by $R(\alpha)^T d'$. In the following paragraphs we will indenfity d to $R_3(\alpha)^T d$ and M^{-1}

to $R_3(\alpha)^T M^{-1} R_3(\alpha)$.
For colinear motors, we will use the second controller stated previously:

$$(C') : \begin{cases} u_\rho &= \dot{u}_{d,\rho} + k(u_{d,\rho} - u_\rho) \\ \ddot{\theta}_m &= \ddot{\theta}_c - \ddot{\beta} + \ddot{\delta} + k_1(\dot{\delta} - \dot{\psi}) + k_0(\delta - \psi) \end{cases} \tag{29}$$

In this case we have, using the same logic, the following expression for F:

$$\begin{aligned} F &= \left(diag(\frac{1}{m_u}, \frac{1}{m_r})\mathbf{A}\right)^+ \left(\begin{pmatrix} \dot{u}_\rho \\ \ddot{\theta}_m \end{pmatrix} + diag(\frac{1}{m_u}, \frac{1}{m_r})d[0:2]\right) \\ &= \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} M^{-1}\mathbf{A}\right)^+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \left(\begin{pmatrix} \dot{u}_\rho \\ \ddot{\theta}_m \\ 0 \end{pmatrix} + M^{-1}d\right) \\ &= \left(\zeta(\mathbf{A})M^{-1}\mathbf{A}\right)^+ \zeta(\mathbf{A}) \left(\begin{pmatrix} \dot{u}_\rho \\ \ddot{\theta}_m \\ 0 \end{pmatrix} + M^{-1}d\right) \end{aligned} \tag{30}$$

Both cases can be unified using the following functions of the state of the system $\mathbf{X}$:

$$\begin{aligned} \zeta_1(\mathbf{X}) &= \begin{cases} diag(1, \frac{1}{\nu}, 1)R_3(\beta)^T M^{-1} & \text{if (13) is satisfied} \\ M^{-1} & \text{else} \end{cases} \\ \zeta_2(\mathbf{X}) &= \begin{cases} \begin{pmatrix} \dot{\nu}_d + k'(\nu_d - \nu) \\ \dot{\delta} + k(\delta - \psi) - \dot{\theta}_m + \dot{\theta}_c \\ \ddot{\theta}_d + k_1(\dot{\theta}_d - \dot{\theta}_m) + k_0(\theta_d - \theta_m) \end{pmatrix} & \text{if (13) is satisfied} \\ \begin{pmatrix} \dot{u}_{d,\rho} + k(u_{d,\rho} - u_\rho) \\ \ddot{\theta}_c - \ddot{\beta} + \ddot{\delta} + k_1(\dot{\delta} - \dot{\psi}) + k_0(\delta - \psi) \end{pmatrix} & \text{if (24) is satisfied and (13) is not} \end{cases} \end{aligned} \tag{31}$$

Finally, F can be written in a general form including all cases as such:

$$F = (\zeta(\mathbf{A})\zeta_1(X, \mathbf{A})\mathbf{A})^+ \zeta(\mathbf{A}) \left(\zeta_2(X, \mathbf{A}) + \zeta_1(X, \mathbf{A})d\right) \tag{32}$$

Note: to find $\ddot{\beta}$ you shall use the method in [3].

2.3.2 Summary

Here is a summary of this entire section:
Everything worked as long as we had 2 non colinear motors in our system. The condition expressing this is (13). When we have 2 motors that are colinear, they must not be align on a same line. That is because otherwise the system is of rank 1 and can not follow the path. The condition expressing this is (24). When we consider such a system, we use $\ddot{\theta}_m$ to impose $\psi - \delta \underset{t \to \infty}{\longrightarrow} 0$. In doing so, we need to measure $\ddot{\beta}$. $\ddot{\beta}$ can be obtained using the dynamic equations and expressed in terms of the state of the system (almost). In [3], they managed to do such thing because of fluid friction. Because fluid friction exists, an AUV could be constructed such that it has a preferential movement direction ($\frac{m_{ur}}{m_v} < 1$) and thus be more and more similar to a unicycle-type robot. This allowed the authors to isolate $\ddot{\beta}$. Our system is symmetrical and does not have such a constraint to allow us to isolate $\ddot{\beta}$. In conclusion, we needed to consider system with friction. In the case of the AUV, our controller would have the following structure which is a concatenation of the controller developed above and that in [3]:

$$F = (\zeta(\mathbf{A})\zeta_1(X, \mathbf{A})\mathbf{A})^+ \zeta(\mathbf{A}) \left(\zeta_2(X, \mathbf{A}) + \zeta_1(X, \mathbf{A})d\right) \tag{33}$$

The ζ and ζ_i functions being stated above.

2.3.3 Conclusion

To sum up, losing actuation creates discontinuities in the first controller that was developed. This comes when we only have two motors and they are colinear. To cope with this discontinuity, we used [3]. It would be preferable to have one controller that unifies both. In [3], the authors could develop their

controller because of a preferential movement direction that the system had. This direction could be given to the system because of fluid friction. The system they considered was a torpedo and thus moving forward is more advantageous than moving sideways. The proof needed to show that by unifying the two controllers the system remains stable is simple. Both controller drive $\psi - \delta \underset{t\to\infty}{\longrightarrow} 0$. Since we have a finite number of switches, after all the switches have been done, one control structure will remain which insures $\psi - \delta \underset{t\to\infty}{\longrightarrow} 0$. Thus the system is stable.
The ζ and ζ_i functions can be improved using the work done in [4]. Indeed, sometimes a robot will be able to go in all direction, but some of the direction will be very costly. The ζ and ζ_i could take this into account by finding if forward and sway costs are roughly the same, or if one is much more costly than the other. In the first case, the robot could use the first control structure (C) developed. In the second, it would have to use a "go forward and turn" approach that the second control structure provides (C').

3 Appendix

3.1 Theorem: Differential equation with constant coefficients

Let us state a general property about differential equations:

Let y be a real number function. Let $m > 0$ be an integer and $k_0, k_0, .., k_{m-1}$ be the coefficients of the polynomial $P = X^m + \sum_{i=0}^{m-1} k_i X_i$
Let P be written in prime factors $P = (X - \alpha_1)^{\gamma_1}(X - \alpha_2)^{\gamma_2}..(X - \alpha_n)^{\gamma_n}$, with α_i being the complex roots of P and γ_i being the respective multiplicity.

Then the solution to the differential equation $P(y) = 0$ is exactly $y(t) = \sum_{i=1}^{n} P_i(t) \times e^{\alpha_i t}$, with P_i being a polynomial such that $deg(P) < \gamma_i$. The polynomials P_i can be found using the initial conditions.
Moreover, If the roots of P have only strictly negative real parts, then $y \underset{t \to \infty}{\longrightarrow} 0$.

Comment: The coefficients of P_i are unknown, but can be found using the initial conditions. Indeed, $P(y) = 0$ is a differential equation of order m. Every P_i has a degree strictly inferior to γ_i. The γ_i verify $\sum_{i=1}^{n} \gamma_i = m$. Thus to find all the polynomials, we need m values for the initial conditions. This is measurable in practice by having the knowledge of the different values at $t = 0$: $y(0) = y_0, \dot{y}(0) = \dot{y}_0, .., y^{m-1}(0) = y_0^{m-1}$.

Comment: y is a sum of products of polynomial and exponential functions. Therefore, as $t \longrightarrow \infty$, the limit is imposed by the exponential functions. Since, all of the exponential functions are of the form $e^{\alpha_i t}$ with $Re(\alpha_i) < 0$, then $P_i(t) \times e^{\alpha_i t} \underset{t \to \infty}{\longrightarrow} 0$ and thus $y(t) \underset{t \to \infty}{\longrightarrow} 0$.

Example: Let $\ddot{y} + 2\dot{y} + y = 0$ be a differential equation. We have $P = X^2 + 2X + 1$ and $P(y) = 0$. P can factored such as $P = (X + 1)^2$. Then the solution is then $y = P_1 e^{(-1) \times t}$ with $deg(P_1) < 2$. Otherwise, $y = (at + b)e^{-t}$ with a, b being factors imposed by the initial conditions. We can see that $y(t) \underset{t \to \infty}{\longrightarrow} 0$, independently of a and b.
For a proof of this theorem, please refer to pages 361 and 366 of [5].

3.2 Simulations

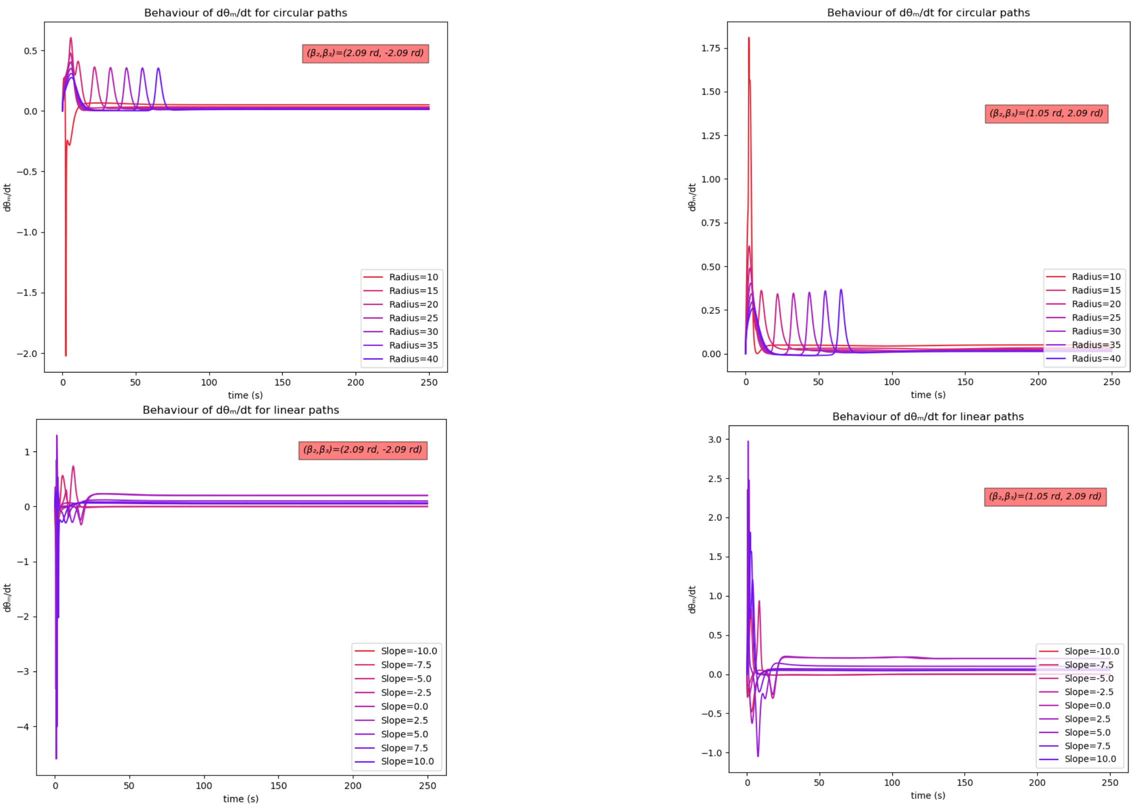

Figure 8: Simulation for circular and linear paths

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