13.5

Torsion of a curve
Tangential and Normal Components of Acceleration

Recall:

Length of a curve $= \int_a^b |\mathbf{r}'(t)|\, dt$

Arc length function $s(t) = \int_a^t \left|\mathbf{r}'(u)\right| du \qquad \frac{ds}{dt} = |\mathbf{r}'(t)|$

Arc length parametrization $\mathbf{r}(s)$ with $|\mathbf{r}'(s)| = 1$

Unit tangent vector $\mathbf{T} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \mathbf{r}'(\mathrm{s})$

Curvature: $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \left|\mathbf{r}''(s)\right| = \dfrac{\left|\mathbf{T}'(t)\right|}{\left|\mathbf{r}'(t)\right|} = \dfrac{\left|\mathbf{r}'(t)\times\mathbf{r}''(t)\right|}{\left|\mathbf{r}'(t)\right|^3}$

Arc length function $s(t) = \int_a^t \left|\mathbf{r}'(u)\right| du$ $\quad s$ measures distance traveled starting at $t = a$

$\frac{ds}{dt} = |\mathbf{r}'(t)|$ measures speed of motion $\quad \mathbf{r}(s) = \mathbf{r}(t(s))$ $\quad$ "arc length parametrization"

if s is arc length parameter, then $\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \frac{\frac{d\mathbf{r}}{dt}}{\frac{ds}{dt}}$ hence $|\mathbf{r}'(s)| = \left|\frac{\frac{d\mathbf{r}}{dt}}{|\mathbf{r}'(t)|}\right| = \left|\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right| = 1$ "you travel with speed 1"

$$\boxed{\text{If } s \text{ is arc length parameter, then } \quad |\mathbf{r}'(s)| = 1}$$

Assume that t is a parameter with $|\mathbf{r}'(t)| = 1$:

If your basepoint is $t = 0$, then $s(t) = \int_0^t \left|\mathbf{r}'(u)\right| du = \int_0^t 1 du = t$

So $s = t$, which means t is already the arclength parameter.

If I assume your basepoint is $t = a$, then $\quad s(t) = \int_a^t \left|\mathbf{r}'(u)\right| du = \int_a^t 1 du = t - a$

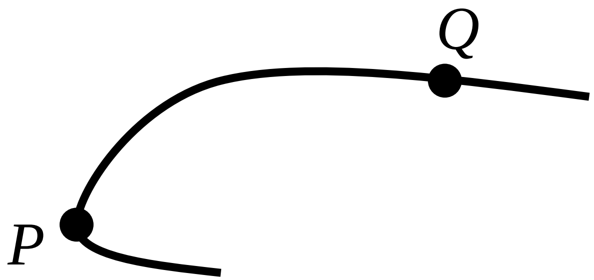
t is still an arc length parameter, it just measures distance starting at a

in either case, distance traveled from $s = \alpha$ to $s = \beta$ is simply $\beta - \alpha$

Examples:

a) arc length parametrization of a straight line: $\mathbf{r}(\mathrm{s}) = \mathbf{r}_0 + s\mathbf{v}$ $\quad$ with $|\mathbf{v}| = 1$

b) arc length parametrization of a circle $x^2 + \mathrm{y}^2 = \mathrm{r}^2$: $\mathbf{r}(\mathrm{s}) = \langle r\cos(\frac{s}{r}), r\sin(\frac{s}{r})\rangle \quad 0 \le s \le 2\pi r$



curvature at P > curvature at Q

Unit tangent vector $\mathbf{T} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \mathbf{r}'(s)$

$$\boxed{\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\mathbf{T}'(s)\right|}$$

$$\boxed{\kappa = \left|\mathbf{r}''(s)\right|}$$

curvature measure how quickly we turn if we travel at speed 1

Frenet Frame:

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| \qquad \frac{d\mathbf{T}}{ds}$$ is also called the *curvature vector*

Principal unit normal : $\mathbf{N} = \dfrac{\frac{d\mathbf{T}}{ds}}{\left|\frac{d\mathbf{T}}{ds}\right|} = \dfrac{\frac{d\mathbf{T}}{dt}}{\left|\frac{d\mathbf{T}}{dt}\right|}$ since $\dfrac{d\mathbf{T}}{ds} = \dfrac{d\mathbf{T}}{dt}\cdot\dfrac{dt}{ds}$

($\mathbf{N}$ is only defined when $\kappa \neq 0$!) and $\dfrac{dt}{ds} > 0$ is a scalar

since $\mathbf{T}\cdot\mathbf{T} = 1$, we have $\mathbf{T}\cdot\mathbf{T}' = 0$ or $\mathbf{T}\cdot\mathbf{N} = 0$ $\quad \mathbf{N}$ is orthogonal to $\mathbf{T}$

a third vector is the **binormal** $\boxed{\mathbf{B} = \mathbf{T}\times\mathbf{N}}$

$\mathbf{B}$ is orthogonal to $\mathbf{T}$ and $\mathbf{N}$ and of unit length: $|\mathbf{B}| = |\mathbf{T}||\mathbf{N}|\sin(\frac{\pi}{2}) = 1$

Altogether, we have *Frenet frame* (or TNB frame) $\mathbf{T}, \mathbf{N}, \mathbf{B}$

They are all of unit length and orthogonal to each other (like $\mathbf{i}, \mathbf{j}, \mathbf{k}$)

they form a moving frame: http://en.wikipedia.org/wiki/Frenet_frame

Torsion:

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| \qquad \mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left|\frac{d\mathbf{T}}{ds}\right|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} \qquad \text{or} \quad \frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

$\mathbf{Claim:}$ $\dfrac{d\mathbf{B}}{ds}$ is parallel to $\mathbf{N}$:

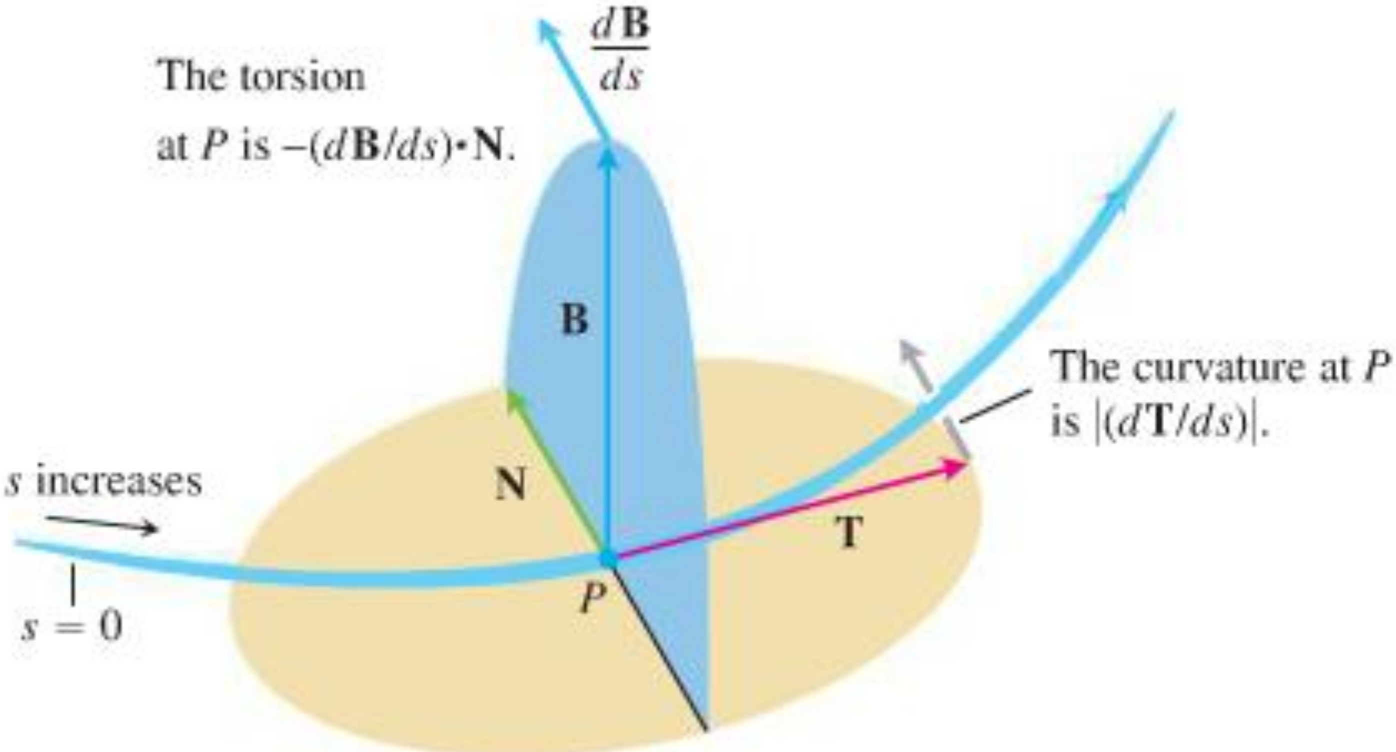
$$\mathbf{B}\cdot\mathbf{B} = 1 \quad \Rightarrow \quad 2\frac{d\mathbf{B}}{ds}\cdot\mathbf{B} = 0$$

$$\mathbf{B}\cdot\mathbf{T} = 0 \quad \Rightarrow \mathbf{0} = \frac{d\mathbf{B}}{ds}\cdot\mathbf{T} + \mathbf{B}\cdot\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{B}}{ds}\cdot\mathbf{T} + \mathbf{B}\cdot\kappa\mathbf{N} = \frac{d\mathbf{B}}{ds}\cdot\mathbf{T}$$

Since $\dfrac{d\mathbf{B}}{ds}\cdot\mathbf{B} = 0$ and $\dfrac{d\mathbf{B}}{ds}\cdot\mathbf{T} = 0$ we see $\dfrac{d\mathbf{B}}{ds}$ is a multiple of $\mathbf{N}$

This multiple (up to sign) is called torsion:

$$\boxed{\frac{d\mathbf{B}}{ds} = -\tau\ \mathbf{N}} \qquad \text{or} \qquad \boxed{\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}}$$



B is the normal vector to the plane spanned by **T** and **N**

$\dfrac{d\mathbf{B}}{ds}$ measure the "tilt" of this plane since $\dfrac{d\mathbf{B}}{ds} = -\tau\,\mathbf{N}$ we also have $\left|\dfrac{d\mathbf{B}}{ds}\right| = |\tau|$

τ (up to sign) measures the magnitude of the tilt

Example: a circle of radius r: $\mathbf{r}(t) = \langle r\cos(t), r\sin(t), 0\rangle$

arc length parametrization: $\mathbf{r}(s) = \langle r\cos(\frac{s}{r}), r\sin(\frac{s}{r}), 0\rangle$

$$\mathbf{T} = \mathbf{r}'(s) = \langle -\sin(\frac{s}{r}), \cos(\frac{s}{r}), 0\rangle \qquad \frac{d\mathbf{T}}{ds} = \langle -\frac{1}{r}\cos(\frac{s}{r}), -\frac{1}{r}\sin(\frac{s}{r}), 0\rangle$$

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left|\frac{d\mathbf{T}}{ds}\right|} = \langle -\cos(\frac{s}{r}), -\sin(\frac{s}{r}), 0\rangle \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{r}$$

$$B = T \times N = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin(\frac{s}{r}) & \cos(\frac{s}{r}) & 0 \\ -\cos(\frac{s}{r}) & -\sin(\frac{s}{r}) & 0 \end{vmatrix} = \left(\sin^2(\frac{s}{r}) + \cos^2(\frac{s}{r})\right)\mathbf{k} = \mathbf{k}$$

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = 0$$

for every plane curve $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \mathbf{k}$ and torsion $\tau = 0$!

Example: Compute $\mathbf{T},\mathbf{N},\mathbf{B}$ of the circular helix: $\mathbf{r}(t)=\langle a\cos(t),a\sin(\mathrm{t}),bt\rangle$

$$\mathbf{r}'(t)=\langle -a\sin(t),a\cos(t),b\rangle$$

hence $\mathbf{T}=\dfrac{\langle -a\sin(t),a\cos(t),b\rangle}{\sqrt{a^2+b^2}}$

$$\frac{d\mathbf{T}}{dt}=\frac{\langle -a\cos(t),-a\sin(t),0\rangle}{\sqrt{a^2+b^2}}$$

$$\left|\frac{d\mathbf{T}}{dt}\right|=\frac{1}{\sqrt{a^2+b^2}}\sqrt{\left(a^2\cos^2(t)+a^2\sin^2(t)\right)}=\frac{a}{\sqrt{a^2+b^2}}$$

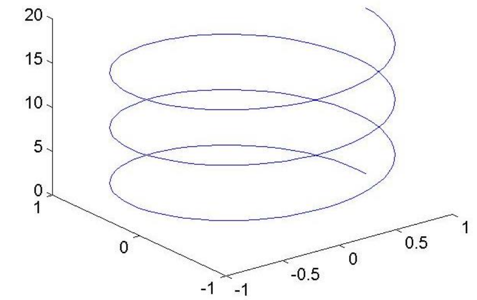
$$\kappa=\left|\frac{d\mathbf{T}}{ds}\right|=\frac{\left|\mathbf{T}'(t)\right|}{\left|\mathbf{r}'(t)\right|}=\frac{a}{\sqrt{a^2+b^2}}\cdot\frac{1}{\sqrt{a^2+b^2}}$$

curvature $\kappa=\dfrac{a}{a^2+b^2}$

principle unit normal $\mathbf{N}=\dfrac{\frac{d\mathbf{T}}{dt}}{\left|\frac{d\mathbf{T}}{dt}\right|}=\langle -\cos(t),-\sin(t),0\rangle$

binormal $\mathbf{B}=\mathbf{T}\times\mathbf{N}=\dfrac{1}{\sqrt{a^2+b^2}}\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -a\sin(t) & a\cos(t) & b\\ -\cos(t) & -\sin(t) & 0\end{vmatrix}=\dfrac{1}{\sqrt{a^2+b^2}}\left(b\sin(t)\mathbf{i}-b\cos(\mathrm{t})\mathbf{j}+a\mathbf{k}\right)$

What is the torsion of the circular helix?



circular helix: $\mathbf{r}(t) = \langle a\cos(t), a\sin(\mathrm{t}), bt\rangle$

$$\mathbf{T} = \frac{\langle -a\sin(t), a\cos(t), b\rangle}{\sqrt{a^2+b^2}} \qquad \mathbf{N} = \langle -\cos(t), -\sin(t), 0\rangle$$

$$\mathbf{B} = \frac{1}{\sqrt{a^2+b^2}}\langle -b\sin(t), -b\cos(t), a\rangle \qquad \kappa = \frac{a}{a^2+b^2}$$

$\tau = -\dfrac{d\mathbf{B}}{ds}\cdot\mathbf{N}$ but t is not arc length parameter s !

we need a formula for the torsion in a general parameter t

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{\mathbf{r}'(t)\cdot\left(\mathbf{r}''(t)\times\mathbf{r}'''(t)\right)}{\left|\mathbf{r}'(t)\times\mathbf{r}''(t)\right|^2}$$

where $\mathbf{r}(t) = \langle x(t), y(\mathrm{t}), z(t)\rangle$

and $\mathbf{v} = \mathbf{r}', \mathbf{a} = \mathbf{r}''$

a computation shows

that for the helix we have:

$$\tau = \frac{b}{a^2+b^2}$$

http://en.wikipedia.org/wiki/Frenet_frame

Decompose the acceleration vector $\mathbf{a} = \mathbf{r}''(t)$

use $\mathbf{v} = \mathbf{r}'$ and $\mathbf{a} = \mathbf{r}''$

$$\boxed{\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}}$$

$$\mathbf{v} = |\mathbf{r}'|\mathbf{T} = \frac{ds}{dt}\mathbf{T}$$

Recall: $|\mathbf{r}'| = \dfrac{ds}{dt}$ $\qquad \mathbf{T} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$

$$\mathbf{v}' = \frac{d^2 s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}'$$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'}{|\mathbf{T}'|}$ $\qquad \kappa = \dfrac{|\mathbf{T}'|}{|\mathbf{r}'|}$

$$\mathbf{a} = \frac{d^2 s}{dt^2}\mathbf{T} + \frac{ds}{dt}(\quad)$$

hence $\mathbf{T}' = |\mathbf{T}'|\mathbf{N} = \kappa|\mathbf{r}'|\mathbf{N} = \kappa\dfrac{ds}{dt}\mathbf{N}$

$$\mathbf{a} = \frac{d^2 s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\kappa\frac{ds}{dt}\mathbf{N}\right)$$

$$\mathbf{a} = \frac{d^2 s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}$$

$$a_{\mathbf{T}} = \frac{d^2 s}{dt^2} \qquad a_{\mathbf{N}} = \kappa\left(\frac{ds}{dt}\right)^2$$

$$\boxed{a_{\mathbf{T}} = \frac{d}{dt}\left(|\mathbf{r}'|\right)} \qquad \boxed{a_{\mathbf{N}} = \kappa|\mathbf{r}'|^2}$$

$$\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$$

tangential acceleration: $a_{\mathbf{T}} = \dfrac{d^2 s}{dt^2} = \dfrac{d}{dt}(|\mathbf{r}'|)$

normal acceleration: $a_{\mathbf{N}} = \kappa\left(\dfrac{ds}{dt}\right)^2 = \kappa|\mathbf{r}'|^2$

if a car travels along a curve, it feels an internal acceleration of $\dfrac{d^2 s}{dt^2}$

and a force of magnitude $ma_{\mathbf{N}} = m\kappa|\mathbf{r}'|^2$ (centrifugal force)

large curvature (tight curve) and large speed^2 = problems !

if you travel at unit speed, then $a_{\mathbf{T}} = 0$, and force $= m\kappa$

other formulas:

$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \dfrac{\mathbf{a}\cdot\mathbf{v}}{|\mathbf{v}|} = \dfrac{\mathbf{r}'\cdot\mathbf{r}''}{|\mathbf{r}'|}$ $\qquad a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \dfrac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|} = \dfrac{|\mathbf{r}'\times\mathbf{r}''|}{|\mathbf{r}'|}$ (try to show this....)

also useful: $\mathbf{a}\cdot\mathbf{a} = a_{\mathbf{T}}^2 + a_{\mathbf{N}}^2$ $\qquad \boxed{a_{\mathbf{N}} = \sqrt{|\mathbf{a}|^2 - a_{\mathbf{T}}^2}}$

Example: A car travels along a track of radius r with velocity a

$a_{\mathbf{T}} = \dfrac{d}{dt}(|\mathbf{r}'|) = 0 \qquad a_{\mathbf{N}} = \kappa|\mathbf{r}'|^2 = \dfrac{1}{r}a^2$

13.6

Acceleration in Polar Coordinates

Newton's law of gravitation (1687):

$$\mathbf{F} = \frac{GmM}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|}$$

Inverse square law

$\mathbf{r}$ is the vector from the center of the sun to the planet

M is the mass of the sun

m is the mass of the planet

G is the gravitational constant

$G = 6.674\times10^{-11}\ N\ m^2\ kg^{-2}$ (from 1798)

$$\mathbf{F} = m\mathbf{a} \quad \Rightarrow \quad \mathbf{a} = \mathbf{r}'' = -\frac{GM}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|}$$

$$\frac{d}{dt}\left(\mathbf{r}\times\mathbf{r}'\right) = \mathbf{r}'\times\mathbf{r}' + \mathbf{r}\times\mathbf{r}'' = \mathbf{r}\times\mathbf{r}'' = 0$$

since $\mathbf{r}''$ is parallel to $\mathbf{r}$ by Newton's law

hence $\mathbf{r}\times\mathbf{r}'$ is a constant vector $\mathbf{C}$

in particular $\mathbf{r}\cdot\mathbf{C} = 0$

$\Rightarrow$ the planet moves in a plane orthogonal to $\mathbf{C}$!