Lecture 14: Portfolio Theory

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Fall 2013

Outline

Markowitz Mean-Variance Analysis (MVA)

Single-Period Analyisis

- m risky assets: $i = 1, 2, \ldots, m$
- Single-Period Returns: $m-$variate random vector
$$\mathbf{R} = [R_1, R_2, \ldots, R_m]'$$
- Mean and Variance/Covariance of Returns:
$$E[\mathbf{R}] = \boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}, Cov[\mathbf{R}] = \boldsymbol{\Sigma} = \begin{bmatrix} \Sigma_{1,1} & \cdots & \Sigma_{1,m} \\ \vdots & \ddots & \vdots \\ \Sigma_{m,1} & \cdots & \Sigma_{m,m} \end{bmatrix}$$
- Portfolio: $m-$vector of weights indicating the fraction of portfolio wealth held in each asset
$$\mathbf{w} = (w_1, \ldots, w_m) : \sum_{i=1}^m w_i = 1.$$
- Portfolio Return: $R_{\mathbf{w}} = \mathbf{w}'\mathbf{R} = \sum_{i=1}^m w_i R_i$ a r.v. with
$$\begin{array}{lllll} \alpha_{\mathbf{w}} & = & E[R_{\mathbf{w}}] & = & \mathbf{w}'\boldsymbol{\alpha} \\ \sigma^2_{\mathbf{w}} & = & var[R_{\mathbf{w}}] & = & \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \end{array}$$

Expected return
" variance

Markowitz Mean Variance Analysis

Evaluate different portfolios **w** using the mean-variance pair of the portfolio: $(\alpha_{\mathbf{w}}, \sigma^2_{\mathbf{w}})$ with preferences for

- Higher expected returns $\alpha_{\mathbf{w}}$
- Lower variance $var_{\mathbf{w}}$

Problem I: Risk Minimization: For a given choice of target mean return α_0, choose the portfolio **w** to

$$\begin{array}{ll} \text{Minimize:} & \frac{1}{2}\mathbf{w}'\mathbf{\Sigma}\mathbf{w} \\ \text{Subject to:} & \mathbf{w}'\boldsymbol{\alpha} = \alpha_0 \\ & \mathbf{w}'\mathbf{1}_m = 1 \end{array}$$

min variance for expected return

Solution: Apply the method of Lagrange multipliers to the convex optimization (minimization) problem subject to linear constraints:

Risk Minimization Problem

- Define the Lagrangian
$$L(\mathbf{w}, \lambda_1, \lambda_2) = \frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} + \lambda_1(\alpha_0 - \mathbf{w}'\boldsymbol{\alpha}) + \lambda_2(1 - \mathbf{w}'\mathbf{1}_m)$$
- Derive the first-order conditions
$$\begin{array}{rclcl} \frac{\partial L}{\partial \mathbf{w}} & = & \mathbf{0}_m & = & \boldsymbol{\Sigma}\mathbf{w} - \lambda_1\boldsymbol{\alpha} - \lambda_2\mathbf{1}_m \\ \frac{\partial L}{\partial \lambda_1} & = & 0 & = & \alpha_0 - \mathbf{w}'\boldsymbol{\alpha} \\ \frac{\partial L}{\partial \lambda_2} & = & 0 & = & 1 - \mathbf{w}'\mathbf{1}_m \end{array}$$
- Solve for $\mathbf{w}$ in terms of λ_1, λ_2:
$$\mathbf{w}_0 = \lambda_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha} + \lambda_2\boldsymbol{\Sigma}^{-1}\mathbf{1}_m$$
- Solve for λ_1, λ_2 by substituting for $\mathbf{w}$:
$$\begin{array}{rclclcl} \alpha_0 & = & \mathbf{w}_0'\boldsymbol{\alpha} & = & \lambda_1(\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}) & + & \lambda_2(\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\mathbf{1}_m) \\ 1 & = & \mathbf{w}_0'\mathbf{1}_m & = & \lambda_1(\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\mathbf{1}_m) & + & \lambda_2(\mathbf{1}_m'\boldsymbol{\Sigma}^{-1}\mathbf{1}_m) \end{array}$$
$$\Longrightarrow \begin{bmatrix} \alpha_0 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \text{ with}$$
$a = (\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha})$, $b = (\boldsymbol{\alpha}'\boldsymbol{\Sigma}^{-1}\mathbf{1}_m)$, and $c = (\mathbf{1}_m'\boldsymbol{\Sigma}^{-1}\mathbf{1}_m)$

Risk Minimization Problem

Variance of Optimal Portfolio with Return α_0

With the given values of λ_1 and λ_2, the solution portfolio

$$\mathbf{w}_0 = \lambda_1 \mathbf{\Sigma}^{-1}\boldsymbol{\alpha} + \lambda_2 \mathbf{\Sigma}^{-1}\mathbf{1}_m$$

has minimum variance equal to

$$\begin{aligned} \sigma_0^2 &= \mathbf{w}_0'\mathbf{\Sigma}\mathbf{w}_0 \\ &= \lambda_1^2(\boldsymbol{\alpha}'\mathbf{\Sigma}^{-1}\boldsymbol{\alpha}) + 2\lambda_1\lambda_2(\boldsymbol{\alpha}'\mathbf{\Sigma}^{-1}\mathbf{1}_m) + \lambda_2^2(\mathbf{1}_m'\mathbf{\Sigma}^{-1}\mathbf{1}_m) \\ &= \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}' \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \end{aligned}$$

Substituting $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1} \begin{bmatrix} \alpha_0 \\ 1 \end{bmatrix}$ gives

$$\sigma_0^2 = \begin{bmatrix} \alpha_0 \\ 1 \end{bmatrix}' \begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1} \begin{bmatrix} \alpha_0 \\ 1 \end{bmatrix} = \frac{1}{ac-b^2}\left(c\alpha_0^2 - 2b\alpha_0 + a\right)$$

- Optimal portfolio has variance σ_0^2: parabolic in the mean α_0

Equivalent Optimization Problems

Problem II: Expected Return Maximization: For a given choice of target return variance σ_0^2, choose the portfolio $\mathbf{w}$ to

$$\begin{aligned} \text{Maximize:} \quad & E(R_{\mathbf{w}}) = \mathbf{w}'\boldsymbol{\alpha} \\ \text{Subject to:} \quad & \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} = \sigma_0^2 \\ & \mathbf{w}'\mathbf{1}_m = 1 \end{aligned}$$

max return for a given variance

Problem III: Risk Aversion Optimization: Let $\lambda \geq 0$ denote the *Arrow-Pratt* risk aversion index gauging the trade-ff between risk and return. Choose the portfolio $\mathbf{w}$ to

$$\begin{aligned} \text{Maximize:} \quad & \left[E(R_{\mathbf{w}}) - \tfrac{1}{2}\lambda var(R_{\mathbf{w}})\right] = \mathbf{w}'\boldsymbol{\alpha} - \tfrac{1}{2}\lambda \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \\ \text{Subject to:} \quad & \mathbf{w}'\mathbf{1}_m = 1 \end{aligned}$$

N.B

- Problems I,II, and III solved by equivalent Lagrangians
- **Efficient Frontier**:$\{(\alpha_0, \sigma_0^2) = (E(R_{\mathbf{w}_0}), var(R_{\mathbf{w}_0})) | \mathbf{w}_0 \text{ optimal}\}$
- Efficient Frontier: traces of α_0 (I), σ_0^2 (II), or λ (III)

Outline

Mean-Variance Optimization with Risk-Free Asset

Risk-Free Asset: In addition to the risky assets $(i = 1, \ldots, m)$ assume there is a risk-free asset $(i = 0)$ for which

$$R_0 \equiv r_0, \text{ i.e., } E(R_0) = r_0, \text{ and } var(R_0) = 0.$$

Treasuries or other high rate national bonds (10y bund for EU?)

Portfolio With Investment in Risk-Free Asset

- Suppose the investor can invest in the m risky investment as well as in the risk-free asset.

  $\mathbf{w}'\mathbf{1}_m = \sum_{i=1}^m w_i$ is invested in risky assets and
  $1 - \mathbf{w}\mathbf{1}_m$ is invested in the risk-free asset.
- If borrowing allowed, $(1 - \mathbf{w}\mathbf{1}_m)$ can be negative.
- Portfolio: $R_{\mathbf{w}} = \mathbf{w}'\mathbf{R} + (1 - \mathbf{w}'\mathbf{1}_m)R_0$, where $\mathbf{R} = (R_1, \ldots, R_m)$, has expected return and variance:

$$\begin{aligned} \alpha_{\mathbf{w}} &= \mathbf{w}'\boldsymbol{\alpha} + (1 - \mathbf{w}'\mathbf{1}_m)r_0 \\ \sigma^2_{\mathbf{w}} &= \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \end{aligned}$$

Note: R_0 has zero variance and is uncorrelated with $\mathbf{R}$

Mean-Variance Optimization with Risk-Free Asset

Problem I': Risk Minimization with Risk-Free Asset

For a given choice of target mean return α_0, choose the portfolio $\mathbf{w}$ to

$$\begin{array}{ll} \text{Minimize:} & \frac{1}{2}\mathbf{w}'\mathbf{\Sigma}\mathbf{w} \\ \text{Subject to:} & \mathbf{w}'\boldsymbol{\alpha} + (1 - \mathbf{w}'\mathbf{1}_m)r_0 = \alpha_0 \end{array}$$

Solution: Apply the method of Lagrange multipliers to the convex optimization (minimization):

- Define the Lagrangian
$$L(\mathbf{w}, \lambda_1) = \tfrac{1}{2}\mathbf{w}'\mathbf{\Sigma}\mathbf{w} + \lambda_1[(\alpha_0 - r_0) - \mathbf{w}'(\boldsymbol{\alpha} - \mathbf{1}_m r_0)]$$
- Derive the first-order conditions
$$\begin{array}{ccccc} \frac{\partial L}{\partial \mathbf{w}} & = & \mathbf{0}_m & = & \mathbf{\Sigma}\mathbf{w} - \lambda_1[\boldsymbol{\alpha} - \mathbf{1}_m r_0] \\ \frac{\partial L}{\partial \lambda_1} & = & 0 & = & (\alpha_0 - r_0) - \mathbf{w}'(\boldsymbol{\alpha} - \mathbf{1}_m r_0) \end{array}$$
- Solve for $\mathbf{w}$ in terms of λ_1: $\mathbf{w}_0 = \lambda_1 \mathbf{\Sigma}^{-1}[\boldsymbol{\alpha} - \mathbf{1}_m r_0]$
and $\lambda_1 = (\alpha_0 - r_0)/[(\boldsymbol{\alpha} - \mathbf{1}_m r_0)'\mathbf{\Sigma}^{-1}(\boldsymbol{\alpha} - \mathbf{1}_m r_0)]$

Mean-Variance Optimization with Risk-Free Asset

Available Assets for Investment:

- Risky Assets ($i = 1, \ldots, m$) with returns: $\mathbf{R} = (R_1, \ldots, R_m)$ with
$$E[\mathbf{R}] = \boldsymbol{\alpha} \text{ and } Cov[\mathbf{R}] = \Sigma$$
- Risk-Free Asset with return R_0 : $R_0 \equiv r_0$, a constant.

Optimal Portfolio P**: Target Return** $= \alpha_0$

- Invests in risky assets according to fractional weights vector:
$$\mathbf{w}_0 = \lambda_1 \boldsymbol{\Sigma}^{-1}[\boldsymbol{\alpha} - \mathbf{1}_m r_0], \text{ where}$$
$$\lambda_1 = \lambda_1(P) = \frac{(\alpha_0 - r_0)}{(\boldsymbol{\alpha} - \mathbf{1}_m r_0)' \boldsymbol{\Sigma}^{-1} (\boldsymbol{\alpha} - \mathbf{1}_m r_0)}$$
- Invests in the risk-free asset with weight $(1 - \mathbf{w}_0' \mathbf{1}_m)$
- Portfolio return: $R_P = \mathbf{w}_0' \mathbf{R} + (1 - \mathbf{w}_0' \mathbf{1}_m) r_0$

Mean-Variance Optimization with Risk-Free Asset

- Portfolio return: $R_P = \mathbf{w}_0'\mathbf{R} + (1 - \mathbf{w}_0'\mathbf{1}_m)r_0$
- Portfolio variance:

$$\begin{aligned} Var(R_P) &= Var(\mathbf{w}_0'\mathbf{R} + (1 - \mathbf{w}_0'\mathbf{1}_m)r_0) = Var(\mathbf{w}_0'\mathbf{R}) \\ &= \mathbf{w}_0'\boldsymbol{\Sigma}\mathbf{w}_0 = (\alpha_0 - r_0)^2/[(\boldsymbol{\alpha} - \mathbf{1}_m r_0)'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha} - \mathbf{1}_m r_0)] \end{aligned}$$

Market Portfolio M

asset allocation proportional to overall market allocation

- The fully-invested optimal portfolio with

$$\mathbf{w}_M : \mathbf{w}_M' \mathbf{1}_m = 1.$$

I.e.

$$\mathbf{w}_M = \lambda_1 \mathbf{\Sigma}^{-1}[\boldsymbol{\alpha} - \mathbf{1}_m r_0], \text{ where}$$
$$\lambda_1 = \lambda_1(M) = \left(\mathbf{1}_m' \mathbf{\Sigma}^{-1}[\boldsymbol{\alpha} - \mathbf{1}_m r_0]\right)^{-1}$$

- Market Portfolio Return: $R_M = \mathbf{w}_M' \mathbf{R} + 0 \cdot R_0$

$$\begin{aligned}
E(R_M) &= E(\mathbf{w}_M' \mathbf{R}) = \mathbf{w}_M' \boldsymbol{\alpha} = \frac{(\boldsymbol{\alpha}' \mathbf{\Sigma}^{-1}[\boldsymbol{\alpha} - \mathbf{1}_m r_0])}{(\mathbf{1}_m' \mathbf{\Sigma}^{-1}[\boldsymbol{\alpha} - \mathbf{1}_m r_0])} \\
&= r_0 + \frac{[\boldsymbol{\alpha} - \mathbf{1}_m r_0]' \mathbf{\Sigma}^{-1}[\boldsymbol{\alpha} - \mathbf{1}_m r_0])}{(\mathbf{1}_m' \mathbf{\Sigma}^{-1}[\boldsymbol{\alpha} - \mathbf{1}_m r_0])} \\
Var(R_M) &= \mathbf{w}_M' \mathbf{\Sigma} \mathbf{w}_M \\
&= \frac{(E(R_M) - r_0)^2}{[(\boldsymbol{\alpha} - \mathbf{1}_m r_0)' \mathbf{\Sigma}^{-1}(\boldsymbol{\alpha} - \mathbf{1}_m r_0)]} = \frac{[\boldsymbol{\alpha} - \mathbf{1}_m r_0]' \mathbf{\Sigma}^{-1}[\boldsymbol{\alpha} - \mathbf{1}_m r_0])}{(\mathbf{1}_m' \mathbf{\Sigma}^{-1}[\boldsymbol{\alpha} - \mathbf{1}_m r_0])^2}
\end{aligned}$$

Tobin's Separation Theorem: *Every optimal portfolio invests in a combination of the risk-free asset and the Market Portfolio.*

Let P be the optimal portfolio for target expected return α_0 with risky-investment weights $\mathbf{w}_P$, as specified above.

- P invests in the same risky assets as the Market Portfolio and in the same proportions! The only difference is the total weight, $w_M = \mathbf{w}'_P \mathbf{1}_m$:

$$\begin{aligned} w_M = \frac{\lambda_1(P)}{\lambda_1(M)} &= \frac{(\alpha_0 - r_0)/[(\boldsymbol{\alpha} - \mathbf{1}_m r_0)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha} - \mathbf{1}_m r_0)]}{\left(\mathbf{1}'_m \boldsymbol{\Sigma}^{-1}[\boldsymbol{\alpha} - \mathbf{1}_m r_0]\right)^{-1}} \\ &= (\alpha_0 - r_0)\frac{\left(\mathbf{1}'_m \boldsymbol{\Sigma}^{-1}[\boldsymbol{\alpha} - \mathbf{1}_m r_0]\right)}{[(\boldsymbol{\alpha} - \mathbf{1}_m r_0)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha} - \mathbf{1}_m r_0)]} \\ &= (\alpha_0 - r_0)/(E(R_M) - r_0) \end{aligned}$$

- $R_P = (1 - w_M) r_0 + w_M R_M$
- $\sigma_P^2 = var(R_P) = var(w_M R_M) = w_M^2 Var(R_M) = w_M^2 \sigma_M^2$.
- $E(R_P) = r_0 + w_M (E(R_M) - r_0)$

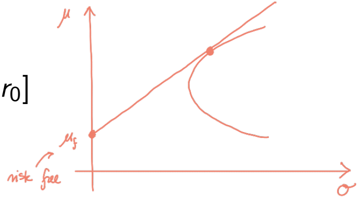
Mean Variance Optimization with Risk-Free Asset

Capital Market Line (CML): The efficient frontier of optimal portfolios as represented on the (σ_P, μ_P)-plane of return expectation (μ_P) vs standard-deviation (σ_P) for all portfolios.

$$\begin{aligned} \textbf{CML} &= \{(\sigma_P, E(R_P)) : \text{ P optimal with } w_M = \mathbf{w}'_P\mathbf{1}_m > 0\} \\ &= \{(\sigma_P, \mu_P) = (\sigma_P, r_0 + w_M(\mu_M - r_0)), w_M \geq 0\} \end{aligned}$$

Risk Premium/Market Price of Risk

$$\begin{aligned} E(R_P) &= r_0 + w_M[E(R_M) - r_0] \\ &= r_0 + \left(\frac{\sigma_P}{\sigma_M}\right)[E(R_M) - r_0] \\ &= r_0 + \sigma_P\left[\frac{E(R_M) - r_0}{\sigma_M}\right] \end{aligned}$$



- $\left[\frac{E(R_M) - r_0}{\sigma_M}\right]$ is the 'Market Price of Risk'
- Portfolio P's expected return increases linearly with risk (σ_P).

Mean Variance Optimization

Key Papers

- Markowitz, H. (1952), "Portfolio Selection", *Journal of Finance,* **7** (1): 77-91.
- Tobin, J. (1958) " Liquidity Preference as a Behavior Towards Risk,", *Review of Economic Studies,* 67: 65-86.
- Sharpe, W.F. (1964), "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk," *Journal of Finance,* 19: 425-442.
- Lintner, J. (1965), "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review of Economics and Statistics,* 47: 13-37.
- Fama, E.F. (1970), "Efficient Capital Markets: A Review of Theory and Empirical Work," *Journal of Finance,*" 25: 383-417.

Outline

Von Neumann-Morgenstern Utility Theory

- Rational portfolio choice must apply preferences based on
  Expected Utility
- The optimal portfolio solves the
  Expected Utility Maximization Problem

Investor: Initial wealth W_0

Action: Portfolio choice P (investment weights-vector $\mathbf{w}_P$)

Outcome: Wealth after one period $W = W_0[1 + R_P]$.

Utility Function: $u(W) : [0, \infty) \longrightarrow \Re$

Quantitative measure of outcome value to investor.

Expected Utility: $E[u(W)] = E[u(W_0[1 + R_p])]$

Utility Theory

Utility Functions

- Basic properties: ↑ Wealth ↑ utility
  - $u'(W) > 0$: increasing (always)
  - $u''(W) < 0$: decreasing marginal utility (typically)
- Defnitions of risk aversion:
  - **Absolute Risk Aversion**: $\lambda_A(W) = -\frac{u''(W)}{u'(W)}$
  - **Relative Risk Aversion**: $\lambda_R(W) = -\frac{Wu''(W)}{u'(W)}$
- If $u(W)$ is smooth (bounded derivatives of sufficient order),

$$\begin{aligned} u(W) &\approx u(w_*) + u'(w_*)(W - w_*) + \tfrac{1}{2}u''(w_*)(W - w_*)^2 + \cdots \\ &= \underbrace{(constants)}_{u_* - u'_* w_*} + u'(w_*)[W - \tfrac{1}{2}\lambda_A(w_*)(W - w_*)^2] + \cdots \end{aligned}$$

  Taking expectations

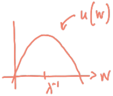
$$E[u(W)] \propto E[W - \tfrac{1}{2}\lambda(W - w_*)^2] \approx E[W] - \tfrac{1}{2}\lambda Var[W]$$

(setting $w_* = E[W]$)

Utility Functions

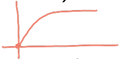
Linear Utility: $u(W) = a + bW, \quad b > 0$

Quadratic Utility: $u(W) = W - \frac{1}{2}\lambda W^2, \quad \lambda > 0,$ $(\text{ and } W < \lambda^{-1})$



Exponential Utility: $u(W) = 1 - e^{-\lambda W}, \lambda > 0$

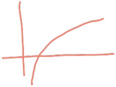
Constant Absolute Risk Aversion (CARA)



Power Utility: $u(W) = W^{(1-\lambda)}, \quad 0 < \lambda < 1$

Constant Relative Risk Aversion (CRRA)

Logarithmic Utility: $u(W) = ln(W)$



Outline

Portfolio Optimization Constraints

Long Only:

$$\mathbf{w} : w_j \geq 0, \forall j$$

Holding Constraints:

$$L_i \leq w_i \leq U_i$$

where $\mathbf{U} = (U_1, \ldots, U_m)$ and $\mathbf{L} = (L_1, \ldots, L_m)$ are upper and lower bounds for the m holdings.

Turnover Constraints:

$$\Delta\mathbf{w} = (\Delta w_1, \ldots, \Delta w_m)$$

The change vector of portfolio holdings satisfies

$|\Delta w_j| \leq U_i$, for individual asset limits $\mathbf{U}$

$\sum_{i=1}^{m} |\Delta w_j| \leq U_*$, for portfolio limit U_*

Portfolio Optimization Constraints

Benchmark Exposure Constraints:

$\mathbf{w}_B$ the fractional weights of a Benchmark portfolio

$R_B = \mathbf{w}_B \mathbf{R}$, return of Benchmark portfolio

(e.g., S&P 500 Index, NASDAQ 100, Russell 1000/2000)

$$|\mathbf{w} - \mathbf{w}_B| = \sum_{i=1}^{m} |[\mathbf{w} - \mathbf{w}_B]_i| < U_B$$

Tracking Error Constraints:

For a given Benchmark portfolio B with fractional weights $\mathbf{w}_B$, compute the variance of the Tracking Error

$$\begin{aligned} TE_P &= (R_P - R_B) = [\mathbf{w} - \mathbf{w}_B]\mathbf{R} \\ var(TE_P) &= var([\mathbf{w} - \mathbf{w}_B]\mathbf{R}) \\ &= [\mathbf{w} - \mathbf{w}_B]' Cov(\mathbf{R}) [\mathbf{w} - \mathbf{w}_B] \\ &= [\mathbf{w} - \mathbf{w}_B]' \Sigma [\mathbf{w} - \mathbf{w}_B] \end{aligned}$$

Apply the constraint:

$$var(TE_P) = [\mathbf{w} - \mathbf{w}_B]' \Sigma [\mathbf{w} - \mathbf{w}_B] \leq \bar{\sigma}^2_{TE}.$$

Portfolio Optimization Constraints

Risk Factor Constraints:

For Factor Model

$$R_{i,t} = \alpha_i + \sum_{k=1}^{K} \beta_{i,k} f_{j,t} + \epsilon_{i,t}$$

- Constrain Exposure to Factor k
  $$|\sum_{i=1}^{m} \beta_{i,k} w_i| < U_k,$$
- Neutralize exposure to all risk factors:
  $$|\sum_{i=1}^{m} \beta_{i,k} w_i| = 0, \; k = 1, \ldots, K$$

Other constraints:

- Minimum Transaction Size
- Minimum Holding Size
- Integer Constraints

General Linear and Quadratic Constraints

For

- $\mathbf{w}$: target portfolio
- $\mathbf{x} = \mathbf{w} - \mathbf{w}_0$: transactions given current portfolio $\mathbf{w}_0$
- $\mathbf{w}_B$: benchmark portfolio

Linear Constraints: Specify m-column matrices A_w, A_x, A_B and m-vectors u_w, u_x, u_B and constrain

$$\begin{array}{rcl} A_w \mathbf{w} & \leq & u_w \\ A_x \mathbf{x} & \leq & u_x \\ A_B(\mathbf{w} - \mathbf{w}_B) & \leq & u_B \end{array}$$

Quadratic Constraints: Specify $m \times m$-matrices Q_w, Q_x, Q_B and m-vectors q_w, q_x, q_B and constrain

$$\begin{array}{rcl} \mathbf{w}' Q_w \mathbf{w} & \leq & q_w \\ \mathbf{x}' Q_x \mathbf{x} & \leq & q_x \\ (\mathbf{w} - \mathbf{w}_B)' Q_B (\mathbf{w} - \mathbf{w}_B) & \leq & q_B \end{array}$$

Outline

Estimating Return Expectations and Covariance

How to estimate

$\bar{r}_i = E[r_i]$

$R_{ij} = E[r_i r_j]$ $\sigma_{ij} = E[(r_i - \bar{r}_i)(r_j - \bar{r}_j)]$

Sample Means and Covariance

- Motivation
  - Least squares estimates
  - Unbiased estimates
  - Maximum likelihood estimates under certain Gaussian assumptions

  Issues:
  - Choice of estimation period
  - Impact of estimation error (!!)

Alternatives

- Apply exponential moving averages
- Apply dynamic factor models
- Conduct optimization with alternative simple models
  - Single-Index Factor Model (Sharpe)
  - Constant correlation model

Outline

Alternative Risk Measures

When specifying a portfolio P by $\mathbf{w}_P$, such that

$$R_P = \mathbf{w}'_P\mathbf{R}, \text{ with asset returns } \mathbf{R} \sim (\alpha, \mathbf{\Sigma}).$$

consider optimization problems replacing the portfolio variance with alternatives

Mean Absolute Deviation:

$$\begin{aligned} MAD(R_P) &= E\left(|\mathbf{w}'(R_p - \alpha)|\right) \\ &= E\left(|\textstyle\sum_{i=1}^m w_i(R_i - \alpha_i)|\right) \end{aligned}$$

Linear programming with linear/quadratic constraints

Semi-Variance:

$$SemiVar(R_p) = E\left[min(R_p - E[R_p], 0)^2\right]$$

Down-side variance (probability-weighted)

Alternative Risk Measures

Value-at-Risk(VAR): RiskMetrics methodology developed by JP Morgan. *VaR* is the magnitude of the percentile loss which occurs rarely, i.e., with probability ϵ $(= 0.05, 0.01, or\ 0.001)$

$$VaR_{1-\epsilon}(R_p) = min\{r : Pr(R_p \leq -r) \leq \epsilon\}$$

max loss with probability ε

- Tracking and reporting of risk exposures in trading portfolios
- VaR is not convex, or sub-additive, i.e,

$$VaR(R_{P_1} + R_{P_2}) \leq VaR(R_{P_1}) + VaR(R_{P_2})$$

why?

may not hold (*VaR* does not improve with diversification).

Conditional Value-at-Risk (CVar): Expected shortfall, expected tail loss, tail *VaR* given by

$$CVaR_{1-\epsilon}(R_p) = E\left[-R_P \mid -R_P \geq VaR_{1-\epsilon}(R_p)\right]$$

See Rockafellar and Uryasev (2000) for optimization of CVaR

Alternative Risk Measures

Coherent Risk Measures A risk measure $s(\cdot)$ for portfolio return distributions is coherent if it has the following properties:

Monotonicity: If $R_P \leq R_{P'}$, $w.p.1$, then $s(R_P) \geq s(R_{P'})$

Subadditivity: $s(R_P + R_{P'}) \leq s(R_P) + s(R_{P'})$

Positive homogeneity: $s(cR_P) = cs(R_P)$ for any real $c > 0$

Translational invariance: $s(R_P + a) \leq s(R_P) - a$, for any real a.

N.B.

- $Var(R_p)$ is not coherent (not monotonic)
- VAR is not coherent (not subadditive)
- $CVaR$ is coherent.

Risk Measures with Skewness/Kurtosis

Consider the Taylor Series expansion of the $u(W)$ about $w_* = E(W)$, where $W = W_0(1 + R_P)$ is the wealth after one period when initial wealth W_0 is invested in portfolio P.

$$\begin{aligned} u(W) &= u(w_*) + u'(w_*)(W - w_*) + \tfrac{1}{2}u''(w_*)(W - w_*)^2 \\ &\quad + \tfrac{1}{3!}u^{(3)}(w_*)(W - w_*)^3 + \tfrac{1}{4!}u^{(4)}(w_*)(W - w_*)^4 \\ &\quad + O[(W - w_*)^5] \end{aligned}$$

Taking expectations

$$\begin{aligned} E[u(W)] &= u(w_*) + 0 + \tfrac{1}{2}u''(w_*)var(W) \\ &\quad + \tfrac{1}{3!}u^{(3)}(w_*)Skew(W) + \tfrac{1}{4!}u^{(4)}(w_*)Kurtosis(W) \\ &\quad + O[(W - w_*)^5] \end{aligned}$$

Portfolio Optimization with Higher Moments

Max: $E(R_P) - \lambda_1 Var(R_P) + \lambda_2 Skew(R_P) - \lambda_3 Kurtosis(R_P)$

Subject to: $\mathbf{w}'\mathbf{1}_m = 1$, where $R_P = \mathbf{w}'R_P$

Portfolio Optimization with Higher Moments

Notes:

- Higher positive Skew is preferred.
- Lower even moments may be preferred (less dispersion)
- Estimation of Skew and Kurtosis complex: outlier sensitivity; requires large sample sizes.
- Optimization approaches
    - Multi-objective optimization methods.
    - Polynomial Goal Programming (PGP).

18.S096 Topics in Mathematics with Applications in Finance
Fall 2013