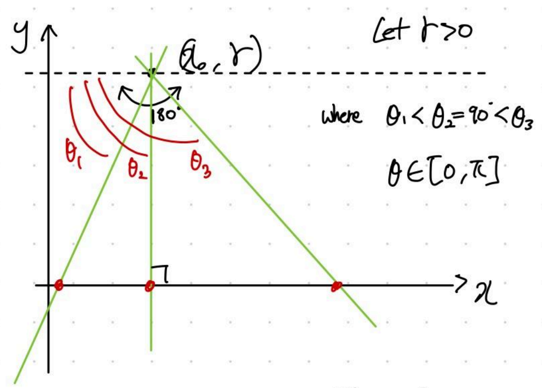
Cauchy distribution

David Hwang

Definition: Let's consider a point (x_0, γ) in the x-y plane, and select a line passing through the point, with its direction (angle with the x-axis) chosen uniformly (between $-180°$ and $0°$) at random. The intersection of the line with the x-axis follows a Cauchy distribution with location x_0 and scale γ.

We can mathematically formalize the Cauchy distribution:



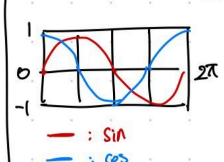
I) θ_1 with $0 \le \theta_1 < \frac{\pi}{2} = 90°$

Intersection : $x_0 - \gamma \times \tan(\frac{\pi}{2} - \theta_1)$

$= x_0 - \frac{\gamma}{\tan\theta_1}$ where $\tan\theta_1 \ge 0$

remark

$$\tan(\tfrac{\pi}{2} - \theta) = \frac{\sin(\frac{\pi}{2} - \theta)}{\cos(\frac{\pi}{2} - \theta)}$$



$$= \frac{\sin(\frac{\pi}{2})\cos\theta - \cos\frac{\pi}{2}\sin\theta}{\cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta}$$

$$= \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$

II) $\theta_2 = \frac{\pi}{2} = 90°$

Intersection : $x_0 - \gamma \tan(\frac{\pi}{2} - \theta_2)$

$= x_0 - \frac{\gamma}{\tan\theta_2} = x_0$

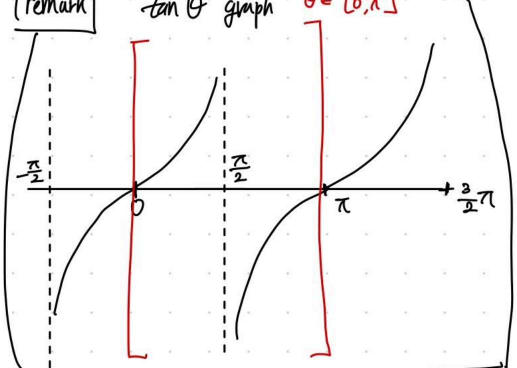
III) $\theta_3 > \frac{\pi}{2}$

Intersection : $x_0 - \gamma \tan(\frac{\pi}{2} - \theta_3)$

$= x_0 - \frac{\gamma}{\tan\theta_3}$

Where $\tan\theta_3 \le 0$

remark $\tan\theta$ graph $\theta \in [0, \pi]$



Then we can conclude that the Cauchy distribution (i.e. the intersection) follow the equation,

$$f(\theta; x_0, \gamma) = x_0 - \frac{\gamma}{\tan\theta}$$

$$(\theta \in [0, \pi])$$

We can modify this more known-shape

since
$$\frac{-1}{\tan\theta} = \frac{-\cos\theta}{\sin\theta} = \frac{\sin(\theta)\cos(\frac{\pi}{2}) - \cos\theta\sin\frac{\pi}{2}}{\cos\theta\cos\frac{\pi}{2} + \sin\theta\sin\frac{\pi}{2}}$$

$$= \frac{\sin(\theta - \frac{\pi}{2})}{\cos(\theta - \frac{\pi}{2})} = \tan(\theta - \tfrac{\pi}{2})$$

Let $u \in [0, 1]$ such that $u\pi = \theta$

Then

$$f(\theta; x_0, \gamma) = x_0 - \frac{\gamma}{\tan\theta} = x_0 + \gamma \cdot \tan(\theta - \tfrac{\pi}{2})$$

$$\Rightarrow \boxed{f(u; x_0, \gamma) = x_0 + \gamma \cdot \tan(\pi(u - \tfrac{1}{2}))}$$

where $u \in [0, 1]$ ∎

Thus, we get the distribution equation: $f(u; x_0, \gamma) = x_0 + \gamma * tan(\pi(u - \frac{1}{2}))$, where u is a sample from a uniform distribution from [0,1].

The Cauchy distribution has the following probability density function:

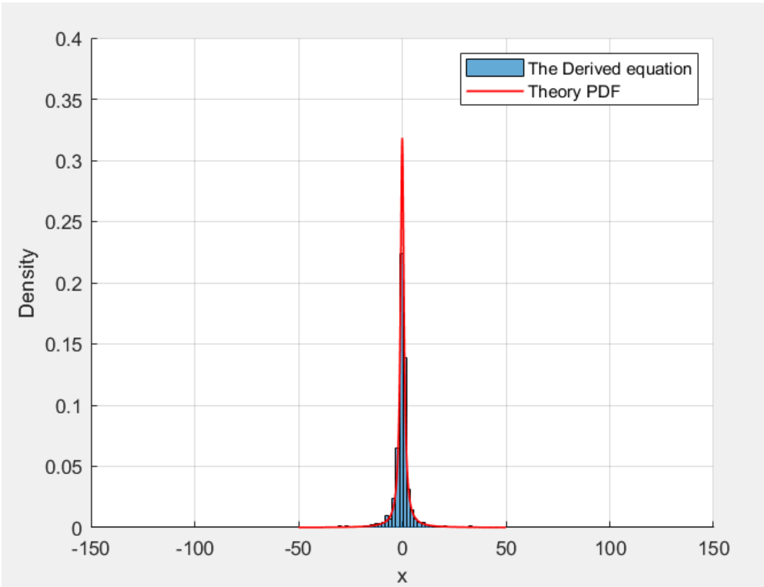
$$f(x; x_0, \gamma) = \frac{1}{\pi}[\frac{\gamma}{(x-x_0)^2+\gamma^2}]$$

If we substitute the variables γ to Δ_v and x_0 to $\overline{v_\theta}$, we can get the same equation for the PDF of Lorentzian probability in the paper:

$$p(v_\theta) = \frac{1}{\pi}[\frac{\Delta_v}{(v_\theta-\overline{v_\theta})^2+\Delta_v^2}]$$

Now, we create a histogram using the above distribution equation and put it in the same figure with the theoretical PDF of the Cauchy distribution to check whether it follows the theoretical probability density shape. Here is the Matlab code:

```
% Cauchy distribution & PDF
%
% We create a histogram using the Cauchy distribution and put it in the
% figure of the PDF equation of Cauchy distribution to check whether
% it follows the theoretical PDF shape.


%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Main parameter & Equations
%%%%%%%%%%%%%%%%%%%%%%%%%%%

x_0=0; % mean of all x; arbitrary setting
gamma=1; % scale-factor; arbitrary setting

x=-50:0.1:50; % x-axis vector
u=rand(length(x),1); % sample from unifrom distribution from [0,1]

Cauchy_D=x_0+gamma*(tan(pi*(u-1/2))); % Cauchy distribution
Cauchy_PDF=1/pi*(gamma./((x-x_0).^2+gamma^2)); % PDF of Cauchy distribution


%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% plots
%%%%%%%%%%%%%%%%%%%%%%%%%%%
clf; hold on;
histogram(Cauchy_D,1000,'Normalization','pdf') %
plot(x,Cauchy_PDF,'r','linewidth',1)
xlabel('x'); ylabel('Density')
legend('Derived Equation','Theory PDF')
xlim([-150,150])
ylim([0,0.4])
grid on;hold off;
```



From the simulation figure above, we can see that the distribution equation that we derived matches well with the theoretical PDF equation.