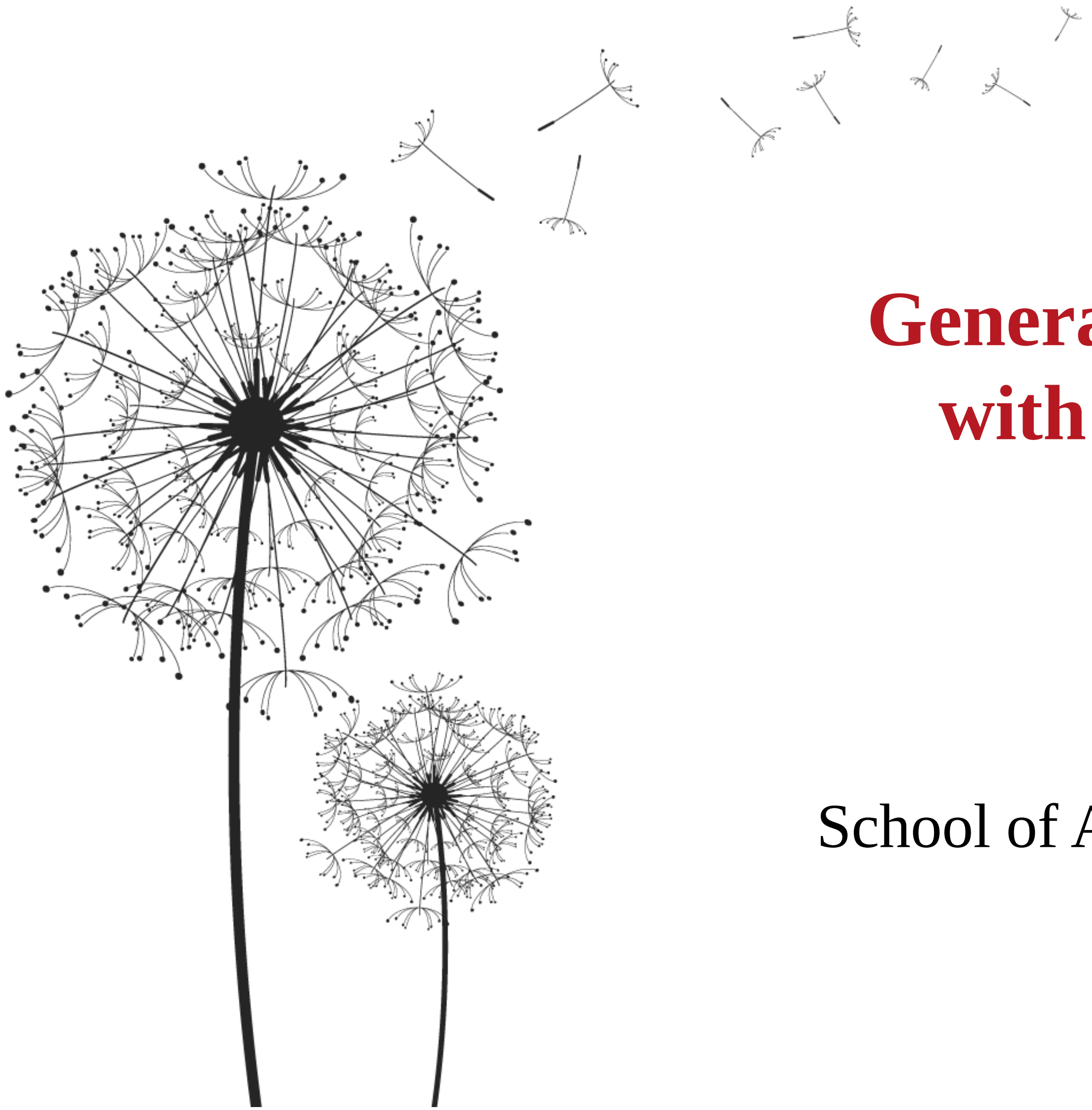

Generalized Optimal Guidance Law with a Terminal Intercept Angle

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Content of Paper

- Introduction
- Problem Formulation
- Design of Optimal Guidance Law
- Simulation
- Conclusion

Abstract

The general meaning of the paper

Abstract

The intercept angle frame is defined which axis is in the direction of desired impact angle, and the engagement kinematics is established in the interception angle frame. Generalized weight optimal guidance laws with interception angle constraints are studied for lag-free control systems. For the systems with elementary function weighting, the analytical forms of weighted optimal guidance laws can be obtained if the integrations of the inverse of the weighting function up to triple can be analytical given. The results can applied to guidance law designs for accomplishing different guidance objectives. For some specific weighted function, the proposed guidance law has extended the results in references.

Keywords*—interception angle constraints, weighted function, optimal guidance law*

Introduction

Introduce the background of guidance law

Introduction

Factors: Miss distance, Impact angle, Convergence, Target information

Method	Advantage	Disadvantage

Introduction

Generalized Optimal Guidance Law (WOGL)

Problem Formulation

Set up the intercepting problem in a planar engagement

Problem Formulation

- Nonlinear Kinematics

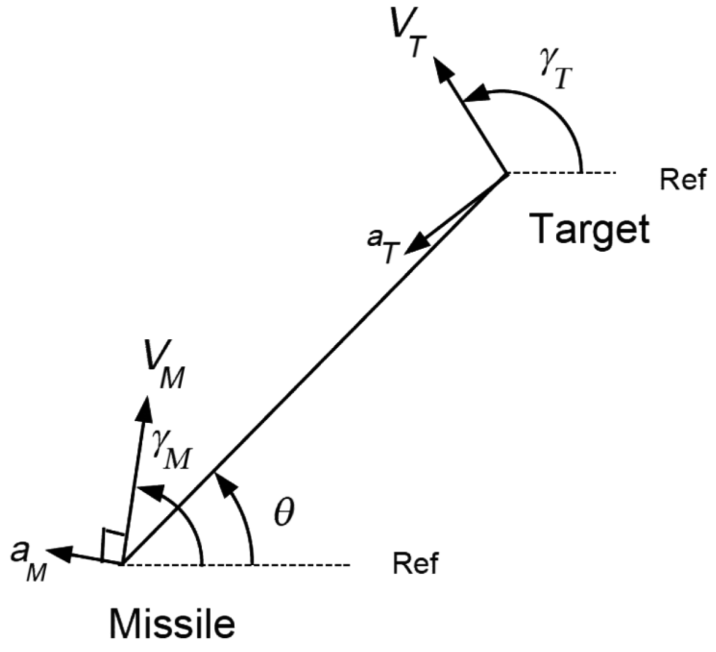
Without losing generality, the engagement between a missile and a target is shown in Fig. 1. V_M and V_T are represented as the missile velocity and target, a_M and a_T donate their lateral accelerations, γ_M and γ_T define their flight-path angles respectively. In this paper, V_M and V_T are assumed as constants. Except these, the relative distance between the missile and the target is defined as r and the line of-sight (LOS) angle as θ. So, the equations of kinematic engagement were described as follows:

$$\dot{r} = V_r = V_T \cos\theta_T - V_M \cos\theta_M$$

$$\dot{\theta} = V_\theta = \left(V_T \sin\theta_T - V_M \sin\theta_M\right)/r$$

$$\dot{\gamma}_M = \frac{a_M}{V_M} \qquad \dot{\gamma}_T = \frac{a_T}{V_T}$$

$$\dot{a}_T = \left(\omega_T - a_T\right)/\tau_T$$



Where $\theta_T = \gamma_T - \theta$ and $\theta_M = \gamma_M - \theta$

We denote the required intercept angle $x_4^c = \gamma_M + \gamma_T$

Problem Formulation

- Linearized Kinematics for Guidance Law Derivation

The state vector of the linearized problem is $\boldsymbol{x} = \begin{bmatrix} z & \dot{z} & a_T & (\gamma_T + \gamma_M) \end{bmatrix}$

The equations of motion are

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = a_T \cos(\gamma_{T0} + \theta_0) - a_M \cos(\gamma_{M0} - \theta_0) \\ \dot{x}_3 = (\omega_T - a_T)/\tau_T \\ \dot{x}_4 = a_T/V_T + a_M/V_M \end{cases}$$

The matrix form of the equation set is therefore $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u_M + \boldsymbol{C}\omega_T$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\gamma_{T0} + \theta_0) & 0 \\ 0 & 0 & -1/\tau_T & 0 \\ 0 & 0 & 1/V_T & 0 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ -d_M \cos(\gamma_{M0} - \theta_0) \\ 0 \\ -d_M/V_M \end{bmatrix} \quad \boldsymbol{C} = \begin{bmatrix} 0 \\ 0 \\ 1/\tau_T \\ 0 \end{bmatrix}$$

Design of Generalized Optimal Guidance Laws

Design generalized optimal guidance law

Design of Optimal Guidance Laws

- A. Quadratic Cost Function

$$J = \frac{a}{2} x_1^2(t_f) + \frac{b}{2}\left[x_4(t_f) - x_4^c\right]^2 + \frac{1}{2}\int_0^{t_f} W(\tau) u^2(\tau) d\tau \qquad W(\tau) > 0, \tau \in [t_0, t_f]$$

Miss Distance | Terminal Angle | Control Command | Weight Function

If $a \to \infty$ yields a perfect intercept guidance law

If $b \to \infty$ yields a perfect intercept angle guidance law

Weight function $W(\tau) > 0, \tau \in [t_0, t_f]$ can adjust control command, such as

$$W(\tau) = 1$$

$$W(\tau) = e^{N \cdot t_{go}}$$

$$W(\tau) = \frac{1}{t_{go}^2}$$

$t_{go} = -r/V_r$: approximate time to go by

Design of Optimal Guidance Laws

- A. Quadratic Cost Function

$$J=\frac{a}{2}x_1^{2}\left(t_f\right)+\frac{b}{2}\left[x_4\left(t_f\right)-x_4^{c}\right]^2+\frac{1}{2}\int_0^{t_f}W\left(\tau\right)u^2\left(\tau\right)d\tau \qquad W\left(\tau\right)>0,\tau\in\left[t_0,t_f\right]$$

Define a new state vector Zt that satisfies

$$\boldsymbol{Z}(t)=\boldsymbol{D\Phi}\left(t_f,t\right)x\left(t\right)+\boldsymbol{D}\int_t^{t_f}\boldsymbol{\Phi}\left(t_f,t\right)\boldsymbol{C}a_T d\tau$$

where

$$\boldsymbol{D}=\begin{bmatrix}1&0&0&0\\0&0&0&1\end{bmatrix}$$

$$\boldsymbol{\Phi}\left(t\right)=e^{\boldsymbol{A}t}=L^{-1}\left[\left(s\boldsymbol{I}-\boldsymbol{A}\right)^{-1}\right]=L^{-1}\begin{bmatrix}s&-1&0\\0&s&0\\0&0&s\end{bmatrix}^{-1}=\begin{bmatrix}1&t&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\boldsymbol{\Phi}\left(t_f,t\right)=\begin{bmatrix}1&t_f-t&0\\0&1&0\\0&0&1\end{bmatrix} \qquad \dot{\boldsymbol{\Phi}}\left(t_f,t\right)=-e^{-\boldsymbol{A}\left(t_f-t\right)}\boldsymbol{A}=-\boldsymbol{\Phi}\left(t_f,t\right)\boldsymbol{A}$$

Design of Optimal Guidance Laws

A. Quadratic Cost Function

$$J=\frac{a}{2}x_1^2\left(t_f\right)+\frac{b}{2}\left[x_4\left(t_f\right)-x_4^c\right]^2+\frac{1}{2}\int_0^{t_f}W\left(\tau\right)u^2\left(\tau\right)d\tau \qquad W\left(\tau\right)>0,\tau\in\left[t_0,t_f\right]$$

Define a new state vector Zt that satisfies

$$\boldsymbol{Z}(t)=\boldsymbol{D\Phi}\left(t_f,t\right)x\left(t\right)+\boldsymbol{D}\int_t^{t_f}\boldsymbol{\Phi}\left(t_f,t\right)\boldsymbol{C}a_T d\tau$$

The derivative with respect to time of the new state vector Z*t* is

$$\dot{\boldsymbol{Z}}(t)=\boldsymbol{D}\left[\dot{\boldsymbol{\Phi}}\left(t_f,t\right)x+\boldsymbol{\Phi}\left(t_f,t\right)\dot{x}\left(t\right)\right]-\boldsymbol{D\Phi}\left(t_f,t\right)\boldsymbol{C}a_T=\boldsymbol{D}\Phi\left(t_f,t\right)\boldsymbol{B}u_M$$

which is state independent. Z*tf* can be expressed using Eq

$$\boldsymbol{Z}\left(t_f\right)=\boldsymbol{Dx}\left(t_f\right)=\left[x_1\left(t_f\right),x_4\left(t_f\right)\right]^T$$

Order Reduction

Using these new variables, the cost function can also be expressed using only the new state vector Z*t* as

$$J=\frac{a}{2}Z_1^2\left(t_f\right)+\frac{b}{2}\left[Z_4\left(t_f\right)-x_4^c\right]^2+\frac{1}{2}\int_0^{t_f}W\left(\tau\right)u^2\left(\tau\right)d\tau$$

Zero-effort miss (ZEM)

Zero-effort angle error(ZEAE)

Design of Optimal Guidance Laws

- *B. Optimal Guidance Law*

  - The Hamiltonian of the problem is

$$H = \frac{1}{2} W(\tau) u^2(\tau) + \dot{Z}_1 \lambda_1 + \dot{Z}_2 \lambda_2$$

The adjoint equations and solutions are

$$\begin{cases} \dot{\lambda}_1 = -\dfrac{\partial H}{\partial Z_1} = 0; & \lambda_1(t_f) = aZ_1(t_f) \\ \dot{\lambda}_2 = -\dfrac{\partial H}{\partial Z_2} = 0; & \lambda_2(t_f) = b\left[Z_2(t_f) - x_4^c\right] \end{cases} \Longrightarrow \begin{cases} \lambda_1(t) = aZ_1(t_f) \\ \lambda_2(t) = b\left[Z_2(t_f) - x_4^c\right] \end{cases}$$

The optimal controller for the missile satisfies $u^* = \arg_u \min H$

For obtaining an analytic solution for the guidance law, we will assume the missile having ideal dynamics,

$$\begin{cases} \dot{Z}_1 = -(t_f - t)\cos(\gamma_{M0} - \theta_0) u_M \\ \dot{Z}_2 = u_M / V_M \end{cases} \Rightarrow \begin{cases} \dot{Z}_1 = -(t_f - t) u \\ \dot{Z}_2 = u / v'_M \end{cases} \qquad v'_M = V_M \cos(\gamma_{M0} - \theta_0)$$

So, the Hamiltonian can be rewritten as $H = \dfrac{1}{2} W(\tau) u^2(\tau) - \lambda_1 (t_f - t) u + \lambda_2 \dfrac{u}{v'_M}$

Design of Optimal Guidance Laws

- *B. Optimal Guidance Law*
  - The Hamiltonian of the problem is

$$H = \frac{1}{2} W(\tau) u^2(\tau) - \lambda_1 (t_f - t) u + \lambda_2 \frac{u}{v'_M}$$

For the considered ideal dynamics, the optimal guidance law simplifies to

$$\frac{\partial H}{\partial u} = 0 \Rightarrow u^*(t) = \frac{\lambda_1 (t_f - t) - \lambda_2 / v'_M}{W(\tau)} = \frac{a Z_1(t_f)(t_f - t) - \frac{b}{v'_M}\left[Z_2(t_f) - x_4^c\right]}{W(\tau)} = \frac{a Z_1(t_f) t_{go} - \frac{b}{v'_M}\left[Z_2(t_f) - x_4^c\right]}{W(t_{go})}$$

Substituting u(t) into $\dot{Z}_1$, $\dot{Z}_2$ and integrating from *t* to *tf*, yields the following two coupled algebraic equations:

$$\begin{cases} Z_1(t_f) = \frac{1}{\Delta}\left(\left(1 + \frac{b}{v_M'^2} f_3\right) \cdot Z_1(t) + \frac{b}{v'_M} f_2 \cdot \left[Z_2(t) - x_4^c\right]\right) \\ Z_2(t_f) = \frac{1}{\Delta}\left(\frac{a}{v'_M} f_2 \cdot Z_1(t) + (1 + a f_1) \cdot \left[Z_2(t) - x_4^c\right]\right) \end{cases}$$

$$\Delta = (1 + a f_1)\left(1 + \frac{b}{v_M'^2} f_3\right) - \frac{ab}{v_M'^2} f_2^2$$

$$f_1 \triangleq \int \frac{t_{go}^2}{W(t_{go})} dt_{go} \qquad f_2 \triangleq \int \frac{t_{go}}{W(t_{go})} dt_{go}$$

$$f_3 \triangleq \int \frac{1}{W(t_{go})} dt_{go}$$

Design of Optimal Guidance Laws

B. Optimal Guidance Law

- The optimal controller

$$u^*(t)=\frac{aZ_1(t_f)t_{go}-\frac{b}{v_M'}\left[Z_2(t_f)-x_4^c\right]}{W(t_{go})}$$

$$\Longrightarrow$$

$$u(t)=\frac{N_{ZEM}}{t_{go}^2}Z_1(t)-N_{ZEAE}\frac{v_M'}{t_{go}}\left[Z_2(t)-x_4^c\right]$$

$$N_{ZEAE}=\frac{at_{go}^3}{W(t_{go})\cdot(1+af_1)}+\frac{bt_{go}^2\cdot k_2\cdot k_3}{v_M'^2+k_1}$$

$$N_{ZEM}=\frac{bt_{go}k_3}{v_M'^2+k_1}$$

$$k_1=\frac{\left(bf_3+abf_1f_3-abf_2^2\right)}{(1+af_1)}$$

$$k_2=\frac{af_2}{(1+af_1)}$$

$$k_3=\frac{\left(at_{go}\cdot f_2-1-af_1\right)}{W(t_{go})\cdot(1+af_1)}$$

$$f_1\triangleq\int\frac{t_{go}^2}{W(t_{go})}dt_{go}$$

$$f_2\triangleq\int\frac{t_{go}}{W(t_{go})}dt_{go}$$

$$f_3\triangleq\int\frac{1}{W(t_{go})}dt_{go}$$

Other condition

$$Z_1(t)=-V_rt_{go}^2\dot{\theta}+a_T\cos(\gamma_{T0}+\theta_0)t_{go}^2/2$$

$$Z_2(t)=t_{go}\,a_T/V_T+\gamma_T+\gamma_M$$

Design of Optimal Guidance Laws

- *C. Generalized Weight Optimal Guidance Law*

$$J=\frac{a}{2}x_1^2\left(t_f\right)+\frac{b}{2}\left[x_4\left(t_f\right)-x_4^c\right]^2+\frac{1}{2}\int_0^{t_f}W\left(\tau\right)u^2\left(\tau\right)d\tau \qquad W\left(\tau\right)>0,\tau\in\left[t_0,t_f\right]$$

$$u\left(t\right)=\frac{N_{ZEM}}{t_{go}^2}Z_1\left(t\right)-N_{ZEAE}\frac{v_M'}{t_{go}}\left[Z_2\left(t\right)-x_4^c\right]$$

$$N_{ZEAE}=\frac{at_{go}^3}{W\left(t_{go}\right)\cdot\left(1+af_1\right)}+\frac{bt_{go}^2\cdot k_2\cdot k_3}{v_M'^2+k_1}$$

$$N_{ZEM}=\frac{bt_{go}k_3}{v_M'^2+k_1}$$

$$k_1=\frac{\left(bf_3+abf_1f_3-abf_2^2\right)}{\left(1+af_1\right)}$$

$$k_2=\frac{af_2}{\left(1+af_1\right)}$$

$$k_3=\frac{\left(at_{go}\cdot f_2-1-af_1\right)}{W\left(t_{go}\right)\cdot\left(1+af_1\right)}$$

$$\xrightarrow{W\left(t_{go}\right)=1}$$

$$\begin{cases} f_1=\int t_{go}^2\cdot dt_{go}=\frac{1}{3}t_{go}^3 \\ f_2=\int t_{go}^1\cdot dt_{go}=\frac{1}{2}t_{go}^2 \\ f_3=\int 1\cdot dt_{go}=t_{go} \end{cases}$$

$$u\left(t\right)=\frac{N_{ZEM}}{t_{go}^2}Z_1\left(t\right)-N_{ZEAE}\frac{v_M'}{t_{go}}\left[Z_2\left(t\right)-x_4^c\right]$$

$$N_{ZEAE}=\frac{at_{go}^3}{W\left(t_{go}\right)\cdot\left(1+af_1\right)}+\frac{bt_{go}^2\cdot k_2\cdot k_3}{v_M'^2+k_1}$$

$$N_{ZEM}=\frac{bt_{go}k_3}{v_M'^2+k_1}$$

$$k_1=\frac{b\left(at_{go}^4+12t_{go}\right)}{4\left(3+at_{go}^3\right)}$$

$$k_2=\frac{3at_{go}^2}{2\left(3+at_{go}^3\right)}$$

$$k_3=\frac{at_{go}^3-6}{2\left(3+at_{go}^3\right)}$$

Design of Optimal Guidance Laws

C. Generalized Weight Optimal Guidance Law

$$J=\frac{a}{2}x_1^2\left(t_f\right)+\frac{b}{2}\left[x_4\left(t_f\right)-x_4^c\right]^2+\frac{1}{2}\int_0^{t_f}W\left(\tau\right)u^2\left(\tau\right)d\tau \qquad W\left(\tau\right)>0,\tau\in\left[t_0,t_f\right]$$

$$u\left(t\right)=\frac{N_{ZEM}}{t_{go}^2}Z_1\left(t\right)-N_{ZEAE}\frac{v_M'}{t_{go}}\left[Z_2\left(t\right)-x_4^c\right]$$

$$N_{ZEAE}=\frac{at_{go}^3}{W\left(t_{go}\right)\cdot\left(1+af_1\right)}+\frac{bt_{go}^2\cdot k_2\cdot k_3}{v_M'^2+k_1}$$

$$N_{ZEM}=\frac{bt_{go}k_3}{v_M'^2+k_1}$$

$$k_1=\frac{\left(bf_3+abf_1f_3-abf_2^2\right)}{\left(1+af_1\right)}$$

$$k_2=\frac{af_2}{\left(1+af_1\right)}$$

$$k_3=\frac{\left(at_{go}\cdot f_2-1-af_1\right)}{W\left(t_{go}\right)\cdot\left(1+af_1\right)}$$

$$\xrightarrow{W\left(t_{go}\right)=t_{go}^{-N}}$$

$$\begin{cases} f_1=\int\frac{t_{go}^2}{W\left(t_{go}\right)}dt_{go}=\int t_{go}^{N+2}dt_{go}=\frac{1}{N+3}t_{go}^{N+3} \\ f_2=\int\frac{t_{go}^2}{W\left(t_{go}\right)}dt_{go}=\int t_{go}^{N+1}dt_{go}=\frac{1}{N+2}t_{go}^{N+2} \\ f_3=\int\frac{1}{W\left(t_{go}\right)}dt_{go}=\int t_{go}^{N}dt_{go}=\frac{1}{N+1}t_{go}^{N+1} \end{cases}$$

$$u\left(t\right)=\frac{N_{ZEM}}{t_{go}^2}Z_1\left(t\right)-N_{ZEAE}\frac{v_M'}{t_{go}}\left[Z_2\left(t\right)-x_4^c\right]$$

$$N_{ZEAE}=\frac{at_{go}^3}{W\left(t_{go}\right)\cdot\left(1+af_1\right)}+\frac{bt_{go}^2\cdot k_2\cdot k_3}{v_M'^2+k_1}$$

$$N_{ZEM}=\frac{bt_{go}k_3}{v_M'^2+k_1}$$

$$k_1=\frac{b\left[\left(N+2\right)^2\left(N+3\right)t_{go}^{N+1}+at_{go}^{2N+4}\right]}{\left(N+1\right)\left(N+2\right)^2\left(N+3+at_{go}^{N+3}\right)}$$

$$k_2=\frac{\left(N+3\right)at_{go}^{N+2}}{\left(N+2\right)\left(N+3+at_{go}^{N+3}\right)}$$

$$k_3=\frac{\left[\left(N+1\right)at_{go}^{N+3}-\left(N+1\right)\left(N+2\right)\left(N+3\right)\right]}{\left(N+1\right)\left(N+2\right)\left(N+3+at_{go}^{N+3}\right)}t_{go}^{N}$$

Simulation

Validate the performance of the optimal guidance law

Simulation

- In this section, through several numerical simulations, the performance of proposed guidance law can be validated.

Missile info

Target info

- Scenario 1:

$$(x_M, y_M) = (0,0)$$
$$V_M = 500 m/s$$
$$\theta_{M0} = 0$$

$$(x_T, y_T) = (2000,0)$$
$$V_T = 300 m/s$$
$$\theta_{T0} = 0$$
$$a_T = 5g \quad m/s^2$$

- Scenario 2:

$$(x_M, y_M) = (1k, 2k)$$
$$V_M = 1.8 km/s$$
$$\theta_{M0} = 0$$

$$(x_T, y_T) = (80k, 90k)$$
$$V_T = 3 km/s$$
$$\theta_{T0} = -135^\circ$$
$$a_T = 5g \quad m/s^2$$

- Some basic parameters

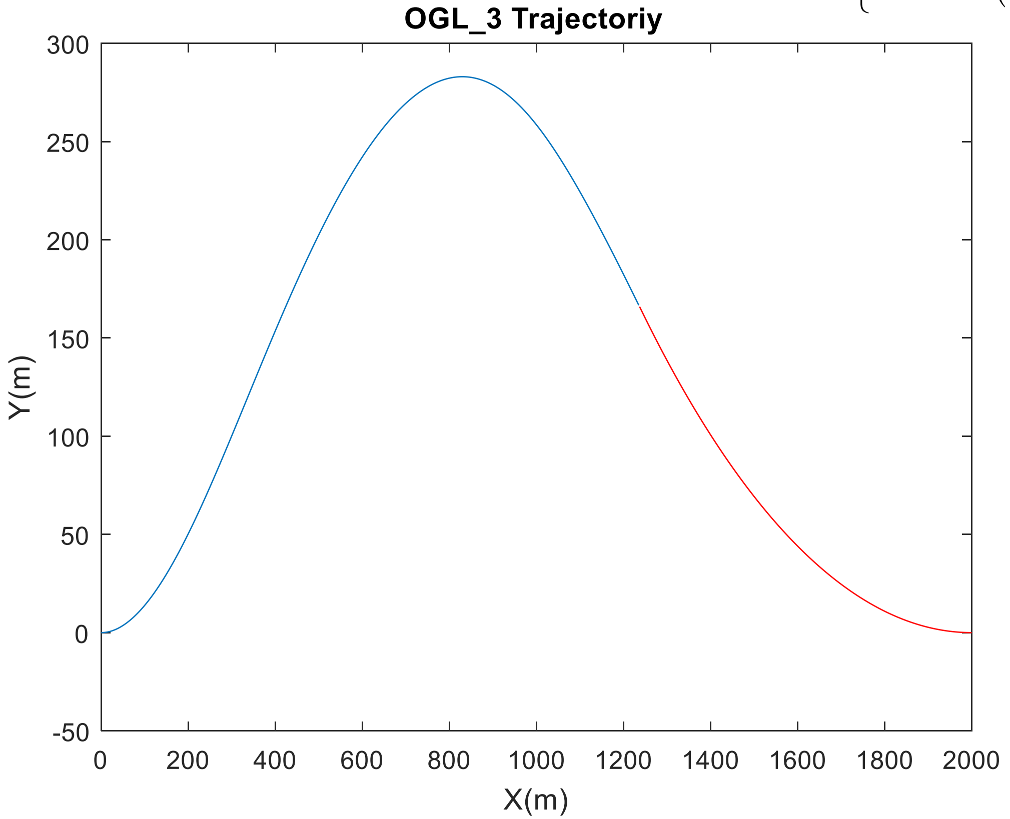
$$a = 10^3, b = 10^8, x_c^4 = 180^\circ$$

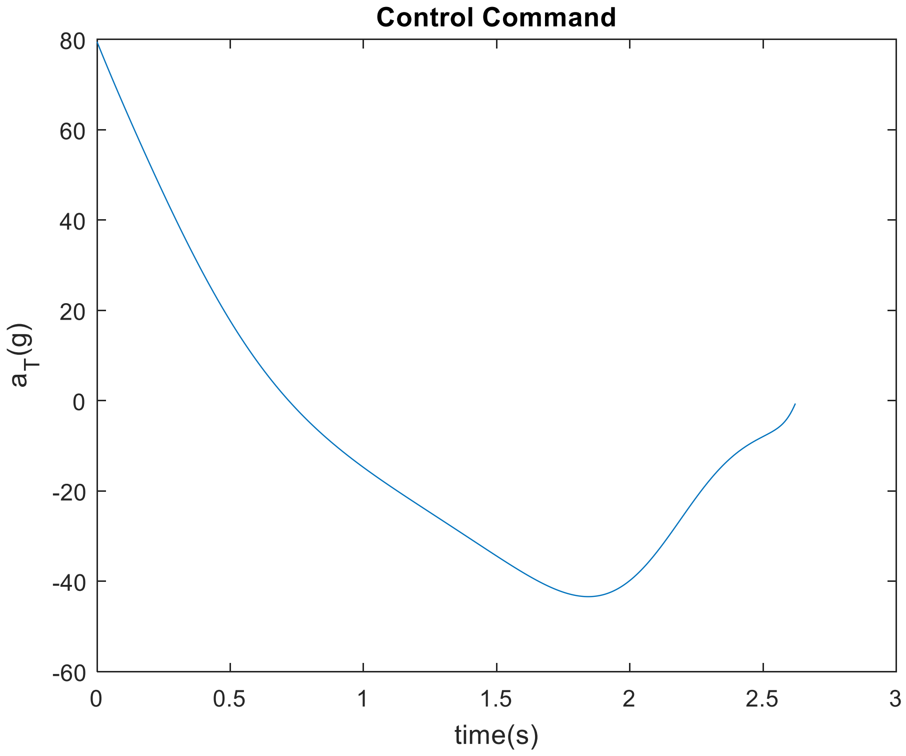
Simulation

➢ **1) *Scenario 1***

$$W\left(t_{go}\right) = e^{N \cdot t_{go}}$$

$$\begin{cases} f_1 = \int \dfrac{t_{go}^2}{W\left(t_{go}\right)} dt_{go} = \int t_{go}^{N+2} dt_{go} = \dfrac{1}{N+3} t_{go}^{N+3} \\ f_2 = \int \dfrac{t_{go}^2}{W\left(t_{go}\right)} dt_{go} = \int t_{go}^{N+1} dt_{go} = \dfrac{1}{N+2} t_{go}^{N+2} \\ f_3 = \int \dfrac{1}{W\left(t_{go}\right)} dt_{go} = \int t_{go}^{N} dt_{go} = \dfrac{1}{N+1} t_{go}^{N+1} \end{cases}$$





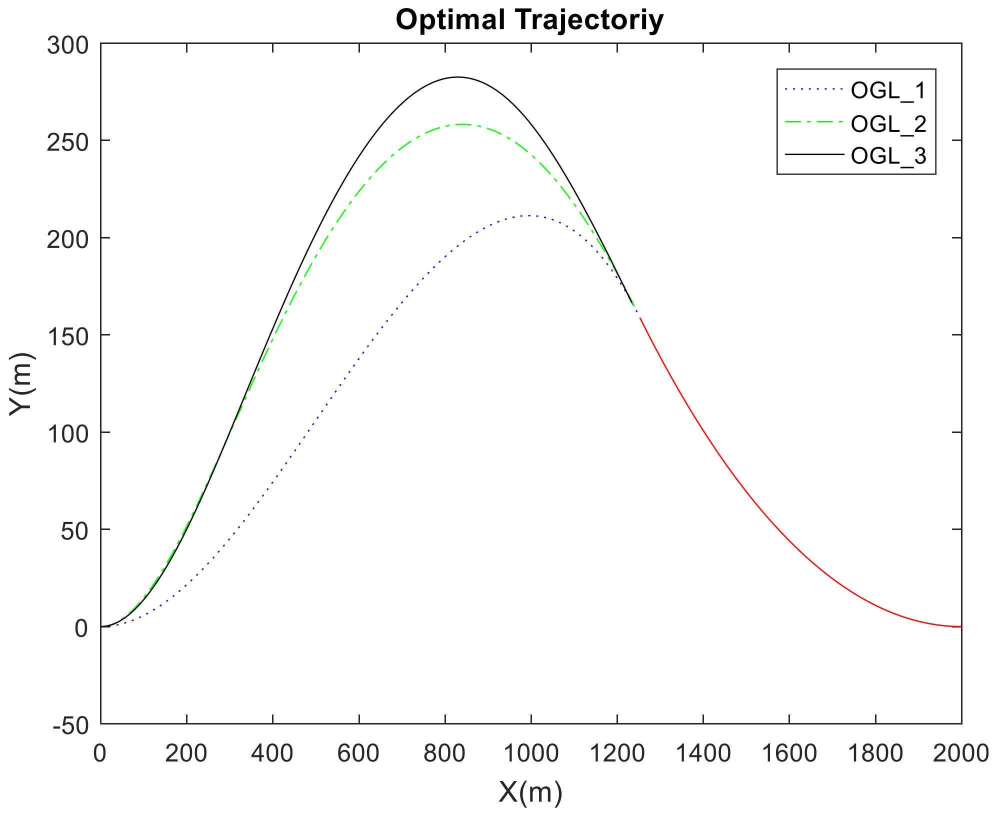
Simulation

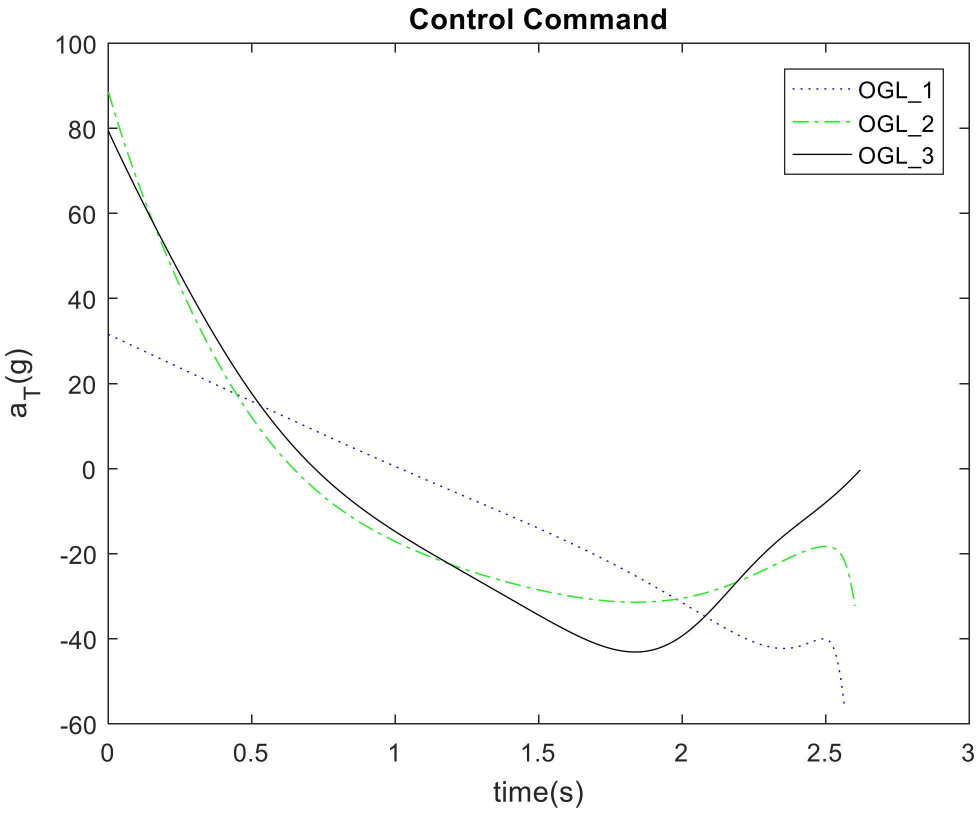
- **1) *Scenario 1***

OGL_1 $W\left(t_{go}\right)=1$

OGL_2 $W\left(t_{go}\right)=e^{N\cdot t_{go}}$

OGL_3 $W\left(t_{go}\right)=t_{go}^{-N}$

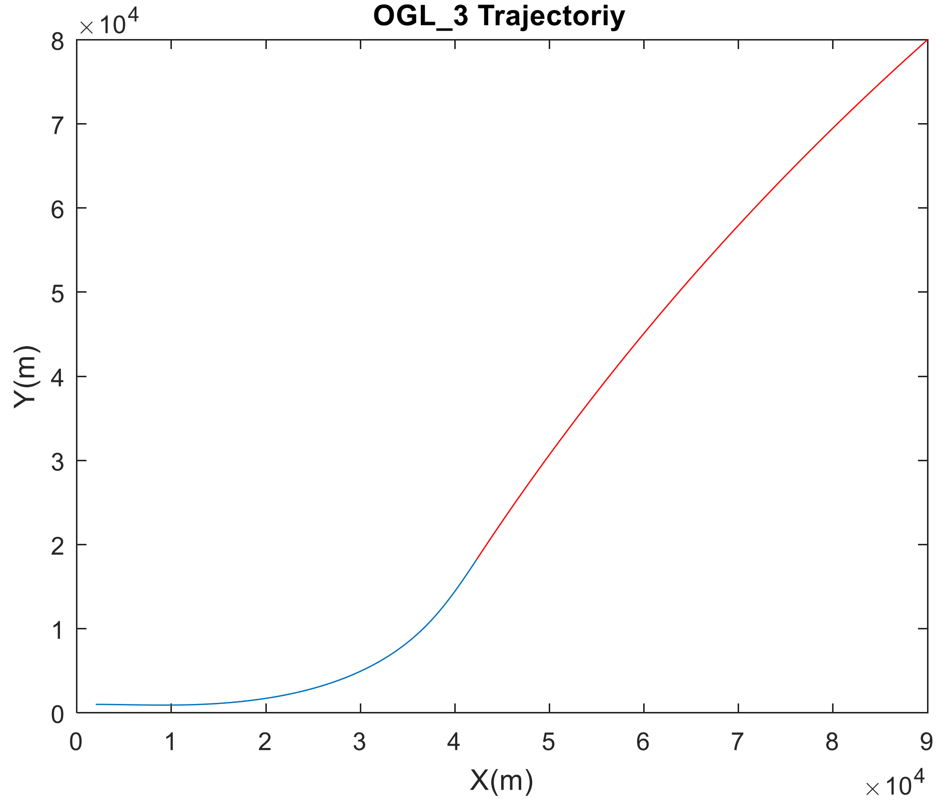


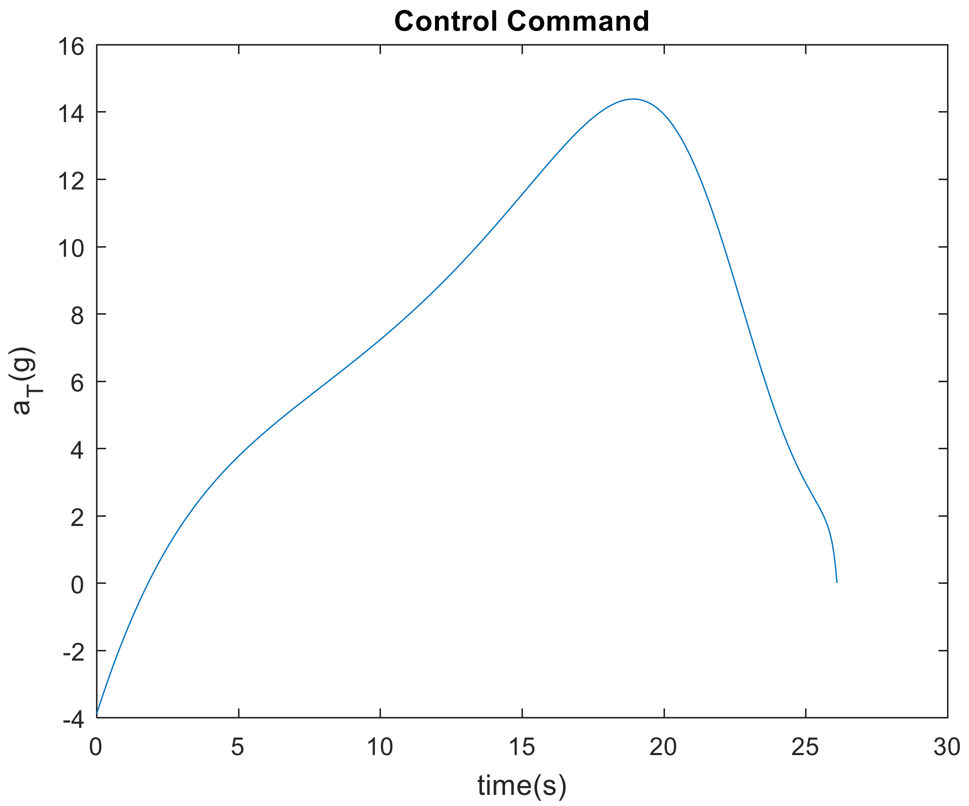


Simulation

- **2) *Scenario 2***

$$W\left(t_{go}\right) = e^{N \cdot t_{go}}$$





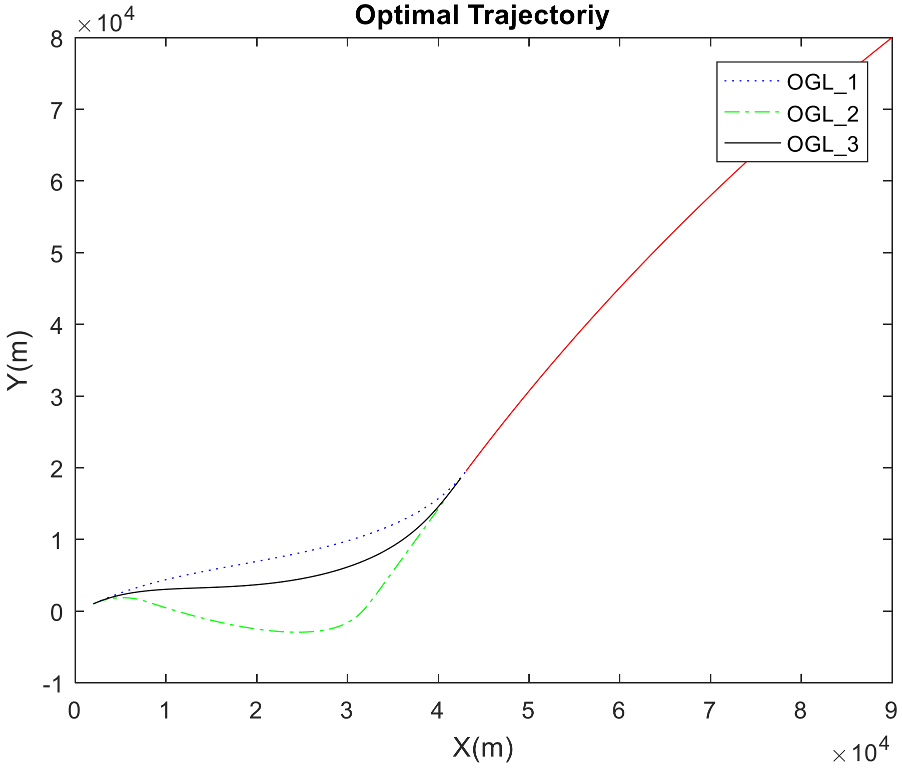
Simulation

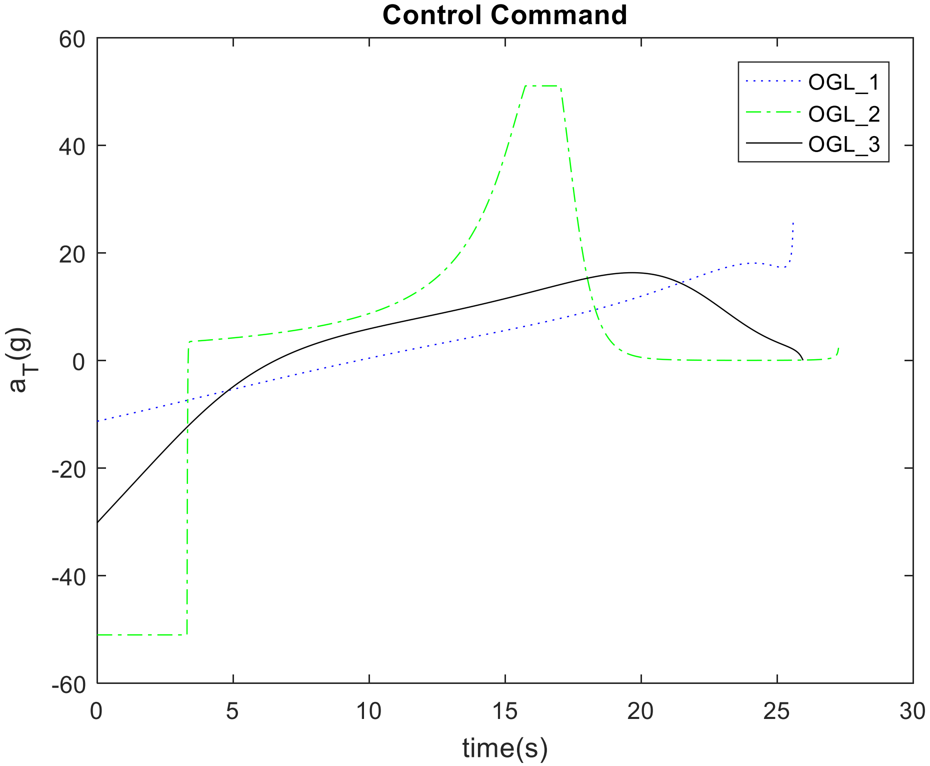
- **2) *Scenario 2***

OGL_1 $W\left(t_{go}\right)=1$

OGL_2 $W\left(t_{go}\right)=e^{N\cdot t_{go}}$

OGL_3 $W\left(t_{go}\right)=t_{go}^{-N}$





Conclusion

Make a conclusion about the paper

Conclusion

01

02

03

04

Thank you!