Convexity adjustments à la Malliavin

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Abstract

In this paper, we develop a novel method based on Malliavin calculus to find an approximation for the convexity adjustment for various classical interest rate products. Malliavin calculus provides a simple way to get a template for the convexity adjustment. We find the approximation for Futures, OIS Futures, FRAs, and CMSs under a general family of the one-factor Cheyette model. We have also seen the excellent quality of the numerical accuracy of the formulas obtained.

1 Introduction

Mathematical finance aims to find a methodology to price consistently all the instruments quoted in the market. When working with fixed income derivatives, a classic research topic is the introduction of a price adjustment to achieve this. This adjustment is called convexity adjustment. It is non-linear and depends on the interest rate model.

There are several reasons to include this type of adjustment. One of them is to incorporate futures on the yield curve construction. Futures and other fixed-income instruments are quoted differently. The firsts are linear against the yield, but the others are not. Therefore, the changes in value and yield of different contracts are different. This difference will depend on the volatility and correlation of the yield curve.

But it is not the only one. The fixed-income market has several features changing the schedule of payments. For example, in a swap in arrears, the floating coupon fixing and payment are on the same date. Or in a CMS swap, the floating rate is linked to a rate longer than the floating length. Any customization of an interest rate product based on changing time, currency, margin, or collateral will require a convexity adjustment. Deep down, by making these changes, we are mixing the martingale measures.

Convexity adjustments have become popular again. Not only by the increase in volatility in the markets. In addition, as a consequence of the transition in risk-free rates from the IBOR (InterBank Offered Rates) indices to the ARR (Alternative Reference Rates) indices, also called RFR. Both indices try to represent the same thing, the risk-free rate, but they are fundamentally different. While the former represents the average rate at which Panel Banks believe they could borrow money, the latter is calculated backward based on transactions. Therefore, these new products need their corresponding convexity adjustment.

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The first references on the convexity adjustment were Ritchken and Sankarasubramanian (1993), Flesaker (1993) and Brotherton-Ratcliffe and Iben (1993), published almost simultaneously. A convexity formula for averaging contracts was found in Ritchken and Sankarasubramanian (1993). Flesaker derived a convexity adjustment for computing the expected Libor rate under the Ho-Lee model in a continuous and discrete setting in Flesaker (1993). Brotherton-Ratcliffe and Iben (1993) used the Taylor expansion on the inverse function for calculating the convexity adjustment. In the following years, several improvements were made. For example, the convexity adjustment was extended to other payoffs in Hull (2006). Hart (1997) improved the Taylor expansion. Kirikos and Novak (1997) derived the convexity adjustment for the Hull-White model. Afterwards, we can find papers that extend the convexity adjustment to different payoffs, see Benhamou (2000b) or Hagan (2003). Or by applying alternative techniques such as the change of measure in Pelsser (2001), a martingale approach in Benhamou (2000a) or the effects of stochastic volatility in Piterbarg and Renedo (2006) and Hagan and Woodward (2020).

In the present paper, we find an alternative way to calculate the convexity adjustment for a general interest rate model. The idea is to use the Itô's representation theorem. Unfortunately, the theorem does not give an insight into how to calculate the elements therein. Therefore, it is necessary to introduce basic concepts of Malliavin calculus to apply the Clark-Ocone representation formula.

The structure of the paper is as follows. In Section 2, we give a brief introduction to Malliavin calculus. In Section 3, we provide preliminary information and discuss the notation used in the article. The notation introduced will not be repeated unless necessary to guide the reader through the results. In Section 4, we compute the convexity adjustment for several payoffs commonly negotiated in the interest rate trading desks. We also present some numerical experiments to check the accuracy of the analytical results obtained. We provide the code for these examples at https://github.com/Dagalon/GeneralConvexityAdjustment. The conclusions can be found in Section 5, as well as future lines of research to explore.

2 Basic introduction to Malliavin calculus

Malliavin calculus is an infinite-dimensional calculus in a Gaussian space. In other words, this is a theory that provides a way to calculate the derivatives of random variables defined in a Gaussian probability space with respect to the underlying noise. The initial objective of Malliavin was the study of the existence of densities of Wiener functionals such as solutions of stochastic differential equations. But, nowadays, it has become an important tool in stochastic analysis due to the increase in its applications. Some of these applications include stochastic calculus for fractional Brownian motion, central limit theorems for multiple stochastic integrals, and an extension of the Itô formula for anticipative processes, but especially mathematical finance. For example, we can apply Malliavin calculus for computing hedging strategies, Greeks, or obtain price approximations. See, for example, Alòs and García Lorite (2021) or Nualart (1995) for more general content.

In our case, we are interested in using the Malliavin calculus to apply the Clark–Ocone representation theorem. But, first of all, let's introduce some basic concepts.

Now, we introduce the derivative operator in the Malliavin calculus sense and the divergence operator to establish the notation that we use in the remainder of the paper.

Consider $W = \{W_t, t \in [0, T]\}$ a Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathcal{F}$ is generated by W, equipped with its Brownian filtration $(\mathcal{F}_t)_{t \geq 0}$. Let $H = L^2([0, T])$ and denote by

$$W(t) := \int_0^T h(s) \, \mathrm{d}W_s,$$

the Itô integral of a deterministic function $h \in H$, also known as Wiener integral. Let $\mathcal{S}$ be the

set of smooth random variables of the form

$$F = f(W(t_1), \ldots, W(t_n))$$

with $t_1, \ldots, t_n \in [0, T]$ and f is a infinitely differentiable bounded function.

The derivative of a random variable F, $D_s F$, is defined as the stochastic process given by

$$D_s F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} (W(t_1), \ldots, W(t_n)) \, 1_{[0, t_j]}(s), \quad s \in [0, T].$$

Nualart (1995) stated that these operators are closable from $L^p(\Omega)$ into $L^p(\Omega; L^2[0, T])$ for any $p \geq 1$, and we denote by $\mathbb{D}^{n,p}$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{n,p} := \left(E[F]^p + \sum_{i=1}^{n} E \left\| D^i F \right\|_{L^2([0, T]^i)}^p \right)^{\frac{1}{p}}.$$

We define δ as the adjoint of derivative operator D, also referred to as the Skorohod integral. The domain of δ, denoted by $Dom \, \delta$, is the set of elements $u \in L^2([0, T] \times \Omega)$ such that there exists $\delta(u) \in L^2(\Omega)$ satisfying the duality relation

$$E[\delta(u)F] = E \left[\int_0^T D_s F u_s \, \mathrm{d}s \right].$$

The operator δ is an extension of the Itô integral in the sense that the set $L_a^2([0, T] \times \Omega)$ of square integrable and adapted processes is included in $Dom \, \delta$ and the operator δ restricted to $L_a^2([0, T] \times \Omega)$ coincides with the Itô stochastic integral.

For any $u \in Dom \, \delta$, we will use the following notation

$$\delta(u) = \int_0^T u_s \, \mathrm{d}W_s.$$

The representation of functionals of Brownian motion by stochastic integrals, also known as martingale representation, has been widely studied over the years. It states that if F is a square-integrable random variable, there exists a unique adapted process φ in $L^2(\Omega \times [0, T]; \mathbb{R})$ such that

$$F = E[F] + \int_0^T \varphi_s^i dW_s^i.$$

In other words, there exists a unique martingale representation or, more precisely, the integrand φ in the representation exists and is unique in $L^2(\Omega \times [0, T]; \mathbb{R})$.

Unfortunately, it is not easy to find an analytic representation of the process φ. Here, the Malliavin calculus helps us to find a solution. When the random variable F is Malliavin differentiable, the process φ appearing in Itô's representation theorem, is given by

$$\varphi^i = E \left[D_s^W F | \mathcal{F}_s \right].$$

In fact,

$$F = E[F] + \int_0^T E \left[D_s^W F | \mathcal{F}_s \right] dW_s \tag{1}$$

is the Clark-Ocone representation formula.

3 Preliminaries and notation

In this section, we give the basic preliminaries and notation necessary throughout the paper.

3.1 A tale of two curves

Consider a continuous-time economy where zero-coupon bonds are traded for all maturities. The price at time t of a zero-coupon bond with maturity T is denoted by $P(t,T)$ where $0 \leq t \leq T$. Clearly, $P(T,T) = 1$. The compounded instantaneous forward rate is defined as:

$$f(t,T) = -\partial_T \ln P(t,T)$$

and the spot interest rates as:

$$r(t) = \lim_{T \longrightarrow t} -\partial_T \ln P(t,T).$$

Therefore, the zero-coupon bond price is given by

$$P(t,T) = \exp\left(-\int_t^T f(t,u)du\right).$$

Before the financial crisis, there was a single curve framework based on the same curve for discounting and forecasting. Since then, the market has adopted a multi-curve approach with two different curves: the discount curve and the estimation curve. We will use the following notation:

- The **discount forward curve** is built with OIS instruments which are considered the best approximation for the risk-free rate. We will denote the forward discount rate curve by $f_{ois}(t,T)$ and the discount curve by $P_{ois}(t,T)$.

- The **estimation forward curve** is chosen based on the maturity of the underlying rate. Until the crisis, the spread between the OIS and the Ibor was negligible. For example, the OIS 6M and Ibor 6M. Nowadays, due to credit and liquidity reasons, there is a spread between them. As a consequence, the estimation curve is tenor-dependent. We will denote the forward estimation rate curve by $f_E(t,T)$ and the estimation discount curve by $P_E(t,T)$.

- The **basis forward curve** is the difference between the estimation forward curve and the discount forward curve, i.e. $s(t,T) = f_E(t,T) - f_{ois}(t,T)$. In this paper, we will assume that the basis are not stochastic. Therefore, it can be obtained directly from the market at time $t = 0$ i.e $s(t, t + u) = s(0, u)$ for $u \geq 0$.

Consequently, the estimation forward curve $f_E(t,T)$ is given by

$$f_E(t,T) = f_{ois}(t,T) + s(t,T). \tag{2}$$

Given the discount curve $P_{ois}(t,T)$ and using the representation (2), it is possible to find the discount curve for the estimation curve using the relation

$$P_E(t,T) = H(t,T)P_{ois}(t,T) \tag{3}$$

where $H(t,T) = \exp\left(-\int_t^T s(t,u)du\right)$.

3.2 The model

We will assume that the f_{ois} dynamics follows a single factor Heath-Jarrow-Morton (HJM) model under the $\mathbb{Q}$-measure. Therefore, let $T > 0$ a fixed time horizon, $t > 0$ the starting time, and W

a Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, the HJM model is defined by

$$df_{ois}(t, T) = \sigma(t, T)\nu(t, T)dt + \sigma(t, T)dW_t^{\mathbb{Q}} \tag{4}$$

where $\nu(t, T) = \int_t^T \sigma(t, s)ds$ and $\sigma(t, T)$ are $\mathcal{F}_t$-adapted process that are positive functions for all t, T. In particular, we have that

$$f_{ois}(t, T) = -\partial_T \ln P_{ois}(t, T).$$

To have a Markovian representation of the HJM, we will assume that the volatility is separable, i.e.

$$\sigma(t, T) = h(t)g(T) \tag{5}$$

with g a positive time-dependent function and h a non-negative process. This version of the HJM is also known as the Cheyette model, Cheyette (2001).

In particular, following Andersen and Piterbarg (2010a), we will define

$$\eta_t = g(t)h(t, x_t, y_t)$$
$$k_t = -\frac{\partial_t g(t)}{g(t)}.$$

Then, we have the following proposition.

Proposition 3.1. *Consider the HJM model* (4) *with the separable volatility condition* (5). *Define the stochastic processes* $x(t)$ *and* $y(t)$ *by*

$$dx_t = (-k(t)x(t) + y(t)) \, dt + \eta \, (t, x(t), y(t)) \, dW_t^{\mathbb{Q}}$$
$$dy_t = \left(\eta^2(t) - 2k(t)y(t)\right) \, dt,$$
$$x(0) = y(0) = 0. \tag{6}$$

All zero-discount bonds are deterministic functions of the processes $x(t)$ *and* $y(t)$,

$$P_{ois}(t, T) = P(t, T, x(t), y(t)),$$

where

$$P_{ois}(t, T, x, y) = \frac{P_{ois}(0, T)}{P_{ois}(0, t)} \exp\left(-G(t, T)x - \frac{1}{2}G^2(t, T)y\right), \tag{7}$$

where $G(t, T) = \int_t^T \exp\left(-\int_t^u k(s)ds\right) du$ *and the short rate is*

$$r_{ois}(t) = f_{ois}(t, t) = f_{ois}(0, t) + x(t). \tag{8}$$

The whole interest rate curve can be reduced to the evolution of the two-state variables $x(t)$ and $y(t)$. The variable $x(t)$ constitutes the main yield curve driver, whereas $y(t)$ is an auxiliary 'convexity' variable. Note that the function $y(t)$ is not deterministic, however, it does not have a diffusion term. We call such processes locally deterministic.

We can see from (6) that

$$x(t_a) = \int_0^{t_a} \exp\left(-\int_u^{t_a} k(w)dw\right) y(u)du + \int_0^{t_a} \exp\left(-\int_u^{t_a} k(w)dw\right) \eta(u, x(u), y(u))dW_u^{\mathbb{Q}}.$$

In order to have a more manageable model, we will follow the ideas of Andersen and Piterbarg (2010a) where the state variables are approximated. So, we can approximate $y(t)$ as

$$\bar{y}(t) := \int_0^t \exp\left(-2\int_u^t k(w)dw\right) \eta^2(u, x(0), y(0))du \tag{9}$$

and

$$\bar{x}_{t_a} := \bar{x}_0(t_a) \quad + \quad \int_0^{t_a} \exp\left(-\int_u^{t_a} k(w)dw \right) \bar{y}_u du$$

$$+ \quad \int_0^{t_a} \exp\left(-\int_u^{t_a} k(w)dw \right) \eta(u, \bar{x}(u), \bar{y}_u)dW_u^{\mathbb{Q}} \qquad (10)$$

with initial condition $\bar{x}_0(t_a)$ will be chosen appropriately depending on the case. In the appendix A.1.1 we can find the estimation for $\mathbb{E}\left[(x_t - \bar{x}_t)^2 \right]$ and the approximation order.

3.3 Model constraints

To calculate the convergence order of the convexity adjustment approximation, we use the following hypotheses on $\eta(t, x, y)$.

Hypothesis 3.2. *The process η_t is global Lipschitz and differentiable a.s. In addition, we will suppose that*

$$\alpha_1 \leq \eta(t, x, y) \leq \alpha_2 \quad \forall (t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \quad and \quad \alpha_1, \alpha_2 > 0.$$

$$|\eta(t, x_2, y_2) - \eta(t, x_1, y_1)| \leq C_{x,y} \|(x_2 - x_1, y_2 - y_1)\| \quad \forall (t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$$

with $\|\cdot\|$ euclidean norm in $\mathbb{R}^2$.

The mean reversion function $k(\cdot)$ influences the range and flexibility of the volatility structure. The function is always positive and, in practice, it is usually low.

Hypothesis 3.3. *The mean reversion function $k(\cdot)$ is a continuous and positive a.s such that*

$$m_k < k(t) \leq M_k \quad \forall t \geq 0.$$

As a consequence.

Remark 3.4. *Under these assumptions on $k(\cdot)$, we have that*

$$\lim_{t \to \infty} I(\alpha, 0, t) := \lim_{t \to \infty} \int_0^t \exp\left(-\alpha \int_u^t k(s)ds \right) du \leq \frac{1}{\alpha m_k} \quad with \quad \alpha > 0.$$

On other hand,

$$\lim_{t \to \infty} J(\alpha, 0, t) := \lim_{t \to \infty} \int_0^t G^\alpha(u, t) \exp\left(-\alpha \int_u^t k(s)ds \right) du \leq \frac{1}{\alpha m_k^{\alpha+1}}$$

The hypotheses have been chosen for simplicity, but they can be replaced by suitable integrability conditions. We should also note that under the hypothesis (3.2), $\partial_x \eta(t, x, y)$ and $\partial_y \eta(t, x, y)$ are bounded.

4 Convexity Adjustment

In this section, we derive the convexity adjustment for different products. The advantage of using the Malliavin calculus is that it allows us to derive a general representation formula for the convexity adjustment. Furthermore, as we will see later, it is possible to obtain closed formulas for the convexity adjustment when the volatility of the Cheyette model is time-dependent.

We are going to introduce a general idea of the method. Let us define the process $Z_t = f(x_t)$. Suppose that Z_t is a martingale under a measure $\mathbb{Q}_1$. However, we are interested in calculate

$\mathbb{E}^{\mathbb{Q}_2}[Z_T]$ where $\mathbb{Q}_2$ is a measure under which Z_t is not martingale and such that $dW_t^{\mathbb{Q}_1} = dW_t^{\mathbb{Q}_2} + \lambda_t dt$. Then, if we use the Clark-Ocone representation, we have that

$$f(x_t) = \mathbb{E}^{\mathbb{Q}_1}[f(x_t)] + \int_0^t \mathbb{E}_s^{\mathbb{Q}_1}[f'(x_t)D_s x_t] \, dW_s^{\mathbb{Q}_1}$$

Now, taking $\mathbb{E}^{\mathbb{Q}_2}(\cdot)$ in the previous expression and using Girsanov's theorem, we get that

$$\mathbb{E}^{\mathbb{Q}_2}[f(x_t)] = f(x_0) + \mathbb{E}^{\mathbb{Q}_2}\left[\int_0^t \mathbb{E}_s^{\mathbb{Q}_1}[f'(x_t)D_s x_t] \lambda_s ds\right]. \tag{11}$$

The second term is the convexity adjustment due to the change of measure from $\mathbb{Q}_1$ to $\mathbb{Q}_2$. The different choices of f, $\mathbb{Q}_1$, and $\mathbb{Q}_2$ will allow us to obtain a convexity adjustment approximation for the different cases of interest.

4.1 FRAs Vs futures

The cash flows in FRAs and futures are computed under different measures. Consequently, we need to adjust the futures price quote to transform them into FRAs price quotes. As usual, we will define the forward rate at time t_0 between t_1 and t_2 under the forward curve E as:

$$L_E(t_0, t_1, t_2) = \frac{1}{\delta_{t_1,t_2}} \left(\frac{P_E(t_0,t_1)}{P_E(t_0,t_2)} - 1\right) \tag{12}$$

where $P_E(t,T)$ is the discount factor for the curve E from t to T, and δ_{t_1,t_2} is the year fraction between t_1 and t_2. Note that $L_E(t_0, t_1, t_2)$ is a martingale under the forward measure $\mathbb{Q}^{t_2}$ associated with the numeraire $P_{ois}(t, t_2)$.

Theorem 4.1. *[Convexity Adjustment approximation for Futures] Given the Cheyette model in (6), the hypotheses 3.2 and 3.3, and considering the approximations in (9) and (10). Then, the convexity adjustment approximation for futures is*

$$CA(t, t_0, t_1) = \frac{P_E(0, t_1)}{\delta_{t_1,t_2} P_E(0, t_2)} \left(G(t_0, t_2) - G(t_0, t_1)\right) \int_0^{t_0} \beta(s, t_0, \bar{x}_0(t_0), \bar{y}_s)\nu(s, t_2)ds + E(t_0) \tag{13}$$

with

$$\beta(u, t_0, x, y) = \exp\left(-\int_u^{t_0} k(w)dw\right) \eta(u, x, y).$$

The error $E(t_0)$ is given by (A.1.1) with $f(x, y) = \frac{\frac{P_{ois}(t_0,t_1,x,y)}{P_{ois}(t_0,t_2,x,y)}-1}{\delta_{t_0,t_1}}$ and behaves as $\mathcal{O}(t_0)$ when $t_0 \to 0$ and $\|E(t_0)\|_2^2 < \infty$ when $t_0 \to \infty$.

Proof. See appendix B.2.1. $\qquad\square$

Example 4.2 (Convexity adjustment for futures under the Hull-White model)**.** *The Cheyette model can be reduced to Hull-White model using the following parameters*

$$g(T) = \exp(-kT),$$
$$h(t) = \sigma.$$

Moreover, from the definition of $g(\cdot)$ and $h(\cdot)$, we have that

$$\eta_s = \sigma \exp(-ks),$$
$$\beta(s, u, x_0, \bar{y}_s) = \sigma \exp(-ku),$$
$$\nu(s, t_2) = \sigma \frac{\exp(-ks) - \exp(-kt_2)}{k}.$$

Then the convexity adjustment (13) *is*

$$CA(t_0, t_1) \approx \frac{\sigma^2 \exp(-kt_0) P_E(0, t_1)}{\delta_{t_1, t_2} P_E(0, t_2)} \left(\frac{1 - \exp(-kt_0)}{k^2} - \frac{t_0 \exp(-kt_2)}{k} \right).$$

In the Figure 1, we can check the accuracy of the last formula versus Monte Carlo. The parameters used are $\sigma = 0.015$, $k = 0.003$, and flat curve with level $r = 0.01$.
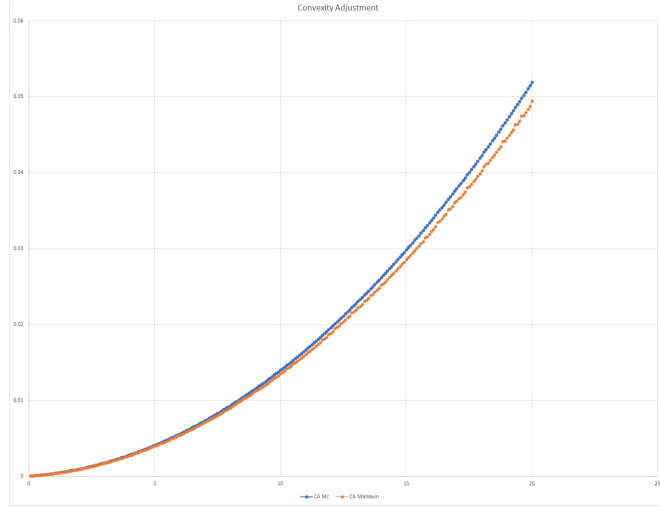

Figure 1: Futures: MonteCarlo Simulation vs Malliavin

4.2 OIS futures

In this section, we will derive the convexity adjustment for short-term interest rate swaps, also known as STIRs. It is common in the market to find two versions of these futures, depending on how the fixings of the reference index are used. Given $t_0 < t_1$, we will define the overnight compounding rate as

$$R(t_0, t_1) := \frac{1}{\delta_{t_0, t_1}} \left(\exp \left(\int_{t_0}^{t_1} r_{ois}(u) du \right) - 1 \right)$$

and the overnight average rate as

$$R_{avg}(t_0, t_1) := \frac{1}{\delta_{t_0, t_1}} \int_{t_0}^{t_1} r_{ois}(u) du.$$

We observe that both, $R(\cdot, t_0, t_1)$ and $R_{avg}(\cdot, t_0, t_1)$ are not predictable and are only observable in t_1. However, $R(\cdot, t_0, t_1)$ and $R_{avg}(\cdot, t_0, t_1)$ are flows that will be payed in t_1. Therefore, we can consider that the expected value under the measure $\mathbb{Q}$ is observable during the entire period $[t_0, t_1]$. Let us define the next $\mathbb{Q}$ martingales:

$$\bar{R}(t, t_0, t_1) := \mathbb{E}_t^{\mathbb{Q}} \left[R(t_0, t_1) \right],$$
$$\bar{R}_{avg}(t, t_0, t_1) := \mathbb{E}_t^{\mathbb{Q}} \left[R_{avg}(t_0, t_1) \right].$$

Before continuing, we will do several observations. The first observation is that if we define $F(t, t_0, t_1) = \mathbb{E}^{\mathbb{Q}^{t_1}} \left[R(t_0, t_1) \right]$, then we have that when $t \in [0, t_0]$

$$F(t, t_0, t_1) = \frac{1}{P_{ois}(t, t_1)} \mathbb{E}_t^{\mathbb{Q}} \left[\exp \left(- \int_t^{t_1} r_{ois}(u) du \right) R(t_0, t_1) \right]$$
$$= \frac{1}{\delta_{t_0, t_1}} \left(\frac{P_{ois}(t, t_0)}{P_{ois}(t, t_1)} - 1 \right),$$

and when $t \in [t_0, t_1]$, we have

$$F(t, t_0, t_1) = \frac{1}{P_{ois}(t, t_1)} \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(-\int_t^{t_1} r_{ois}(u)du \right) R(t_0, t_1) \right]$$

$$= \frac{1}{\delta_{t_0, t_1}} \left(\frac{\exp\left(\int_t^{t_1} r_{ois}(u)du \right)}{P_{ois}(t_0, t)} - 1 \right).$$

Then, the convexity adjustment for $R(t_0, t_1)$ is

$$CA_{ois}(t, t_0, t_1) = F(t, t_0, t_1) - \bar{R}(t, t_0, t_1). \tag{14}$$

The second observation is that we have the following equivalence

$$\mathbb{E}^{\mathbb{Q}} \left[R_{avg}(t_0, t_1) \right] = \frac{1}{\delta_{t_0, t_1}} \mathbb{E}^{\mathbb{Q}} \left[\log\left(1 + \delta_{t_0, t_1} R(t_0, t_1) \right) \right] \tag{15}$$

To clarify the convexity expression and the calculations, we define

$$I(t_0, t_1) := \int_{t_0}^{t_1} r_{ois}(s)ds.$$

Theorem 4.3. *[Convexity Adjustment approximation for OIS Futures] Given the Cheyette model in (6), the hypotheses 3.2 and 3.3, and considering the approximations in (9) and (10). Then, the convexity adjustment approximation for OIS futures is*

$$\bar{R}(t_0, t_1) = \frac{1}{\delta_{t_0, t_1}} \left(\exp\left(\mathbb{E}_t^{\mathbb{Q}} \left[I(t_0, t_1) \right] \right) \exp\left(-\frac{1}{2} \int_t^{t_1} \Gamma^2(s, t_0, t_1)ds \right) - 1 \right) + E(t_0), \tag{16}$$

and for the average OIS future is

$$\bar{R}_{avg}(t_0, t_1) = \frac{\mathbb{E}_t^{\mathbb{Q}} \left[I(t_0, t_1) \right]}{\delta_{t_0, t_1}} \approx \frac{1}{\delta_{t_0, t_1}} \left(\log\left(1 + \delta_{t_0, t_1} \mathbb{E}_t^{\mathbb{Q}} \left[R(t_0, t_1) \right] \right) - \frac{1}{2} \int_0^{t_1} \Gamma^2(s, t_0, t_1)ds \right), \tag{17}$$

where

$$\Gamma(s, t_0, t_1) = \eta(s, x_0, y_0) \int_{\max(s, t_0)}^{t_1} \exp\left(-\int_s^u k(w)dw \right) du. \tag{18}$$

The error $E(t_0)$ is given by (A.1.1) with $f(x, y) = \frac{\frac{1}{P_{ois}(t_0, t_1, x, y)} - 1}{\delta_{t_0, t_1}}$ and behaves as $\mathcal{O}(t_0)$ when $t_0 \to 0$ and $\|E(t_0)\|_2^2 < \infty$ when $t_0 \to \infty$.

Proof. See appendix B.2.2. $\qquad \square$

Remark 4.4. *We must note that (16) and (17) are exact when $\eta(t, x_t, y_t)$ is only a time-dependent function, as in the case of the Hull-White model.*

Remark 4.5. *We can calculate the convexity adjustment for the case $t_0 < t < t_1$ similarly to when $t < t_0$. For this, we will define*

$$I(t, t_1) := \int_t^{t_1} r_{ois}(s)ds,$$

$$R(t_0, t_1) := \frac{1}{\delta_{t_0, t_1}} \left(\frac{\exp\left(\int_t^{t_1} r_{ois}(s)ds \right)}{P_{ois}(t_0, t)} - 1 \right),$$

and

$$R_{avg}(t_0, t_1) := \frac{1}{\delta_{t_0, t_1}} \left(\int_{t_0}^t r_{ois}(s)ds + \int_t^{t_1} r_{ois}(s)ds \right).$$

9

Example 4.6 (Convexity adjustment for OIS futures under the Hull-White model). *Similarly to the Example 4.2, we can find the equivalent parameters for the Hull-White model:*

$$\Gamma(s, t_0, t_1) = \frac{\sigma \exp(-ks)}{k}\left(\exp(-k(\max(s, t_0) - s)) - \exp(-k(t_1 - s))\right),$$

$$\mathbb{E}^{\mathbb{Q}}\left[I(t_0, t_1)\right] = -\log\left(\frac{P_{ois}(0, t_1)}{P_{ois}(0, t_0)}\right)$$
$$+ \frac{\sigma^2}{2k^2}\left(\delta_{t_0, t_1} - 2\frac{\exp(-kt_0) - \exp(-kt_1)}{k} + \frac{\exp(-2kt_0) - \exp(-2kt_1)}{2k}\right).$$

Therefore, we have that

$$\frac{1}{2}\int_0^{t_1}\Gamma^2(s, t_0, t_1)ds = \frac{\sigma^2 t_0}{2k^2}\left(\exp(-kt_0) + \exp(-2kt_1) - 2\exp(-k(t_1 + t_0))\right)$$
$$+ \frac{\sigma^2}{2k^2}\left(\frac{\exp(-2kt_0) - \exp(-2kt_1)}{2k} + \exp(-kt_0)t_0\right.$$
$$\left. - 2\frac{\exp(-2kt_0) - \exp(-k(t_0 + t_1))}{k}\right).$$

Then, if we substitute the last equalities in (16), we get an approximation for OIS future at $t = 0$.

The following figure shows the accuracy of (16) and (17). The parameters used to run the Monte Carlo have been $k = 0.003$, $\sigma = 0.01$, and flat curve $r = 0.01$.
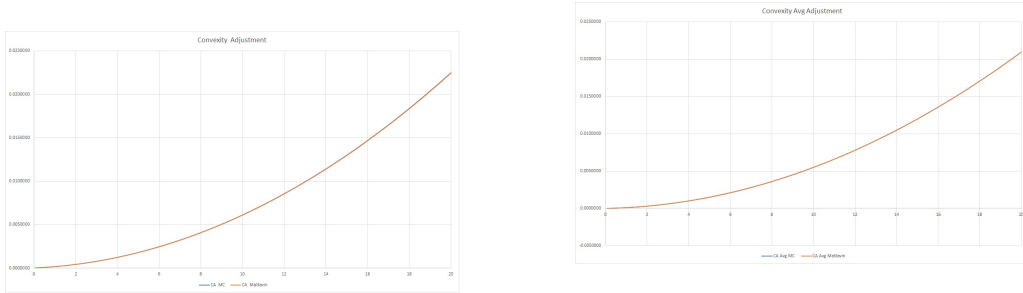
(a) OIS Future: MonteCarlo Simulation vs Malliavin

(b) Average OIS Future: MonteCarlo Simulation vs Malliavin

Figure 2: Convexity adjustment for compounding and average OIS

4.3 FRAs in arrears

A FRA in arrears is the most classic example among convexity adjustment products. The price is given by

$$P_E(0, t_1)\mathbb{E}^{\mathbb{Q}^{t_1}}\left[L_E(t_1, t_1, t_2)\right], \tag{19}$$

i.e. the cash flow associated with a FRA in arrears is $L_E(t_1, t_1, t_2)$ in t_1.

Theorem 4.7. *[Convexity Adjustment approximation for FRAs in Arrears] Given the Cheyette model in* (6), *the hypotheses* 3.2 *and* 3.3, *and considering the approximations in* (9) *and* (10). *Then, the convexity adjustment approximation for FRAs in Arrears is*

$$CA(t_0, t_1) = \frac{G(t_1, t_2)}{\delta_{t_1,t_2} P_E(0, t_1, t_2)} \beta(s, t_1, \bar{x}_0(t_1), \bar{y}_s)$$
$$\cdot \int_0^{t_1} \overline{DM}(s, t_1) \Big(\bar{\nu}(s, t_2, \bar{x}_0(t_1)) - \bar{\nu}(s, t_1, \bar{x}_0(t_1)) \Big) ds + E(t_1). \tag{20}$$

The error $E(t_1)$ is given by (A.1.1) *with $f(x,y) = \frac{1}{\delta_{t_1,t_2}} \left(\frac{1}{P_E(t_1,t_2,x,y)} - 1 \right)$ and behaves as $\mathcal{O}(t_1)$ when $t_1 \to 0$ and $\|E(t_1)\|_2^2 < \infty$ when $t_1 \to \infty$.*

Proof. See appendix B.2.3. $\qquad\square$

Example 4.8 (Convexity adjustment for FRAs in Arrears under the Hull-White model)**.** *The model can be restricted to a Hull-White model with constant parameters. The analytical approximation obtained from* (20) *is*

$$CA(t_0, t_1) \approx \frac{G(t_1, t_2)}{\delta_{t_1,t_2} P_E(0, t_1, t_2)} \frac{\sigma^2}{k} \int_0^{t_1} \Big(\exp(-k(t_1 + u)) - \exp(-k(t_2 + u)) \Big) du$$

In Figure 3, *we compare the approximation with a Monte Carlo method when the parameters are $\sigma = 0.1$ and $k = 0.007$.*
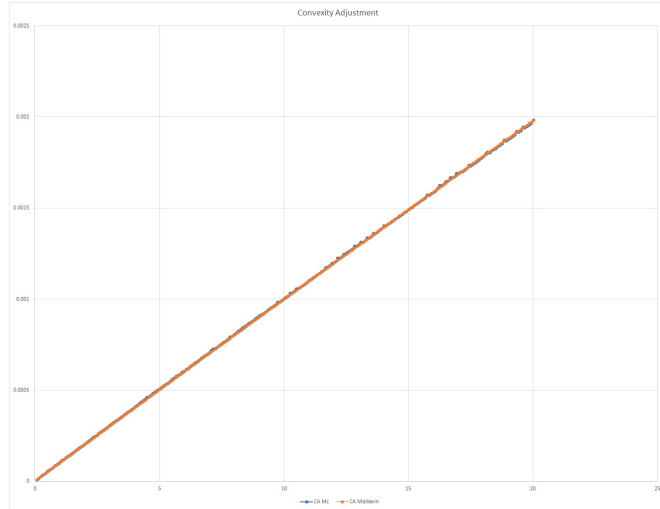

Figure 3: FRA in Arrears: MonteCarlo Simulation vs Malliavin

4.4 CMSs

The last product we will approximate the convexity adjustment are CMS. We will introduce some notation that we will use throughout the section. We define the swap rate from t_a to T_b at time t as

$$S_{a,b}(t) := \frac{\sum_{i=1}^{n_E} \delta_{t_{i-1}^E, t_i^E} L^E(t, t_{i-1}^E, t_i^E) P_{ois}(t, t_i^E)}{01(t, t_a, T_b)}$$

where

$$01(t, t_a, t_b) = \sum_{j=1}^{n_f} \delta_{t_{i-1}^f, t_i^f} P_{ois}(t, t_j^f)$$

$$t_a = t_0^E < t_i^E < \cdots < t_{n_E}^E = t_b \quad i = 0, \cdots, n_E$$

$$t_a = t_0^f < t_j^f < \cdots < t_{n_f}^f = t_b \quad j = 0, \cdots, n_f$$

The same way, we will define the OIS swap rate as

$$S_{a,b}^{ois}(t) = \frac{P_{ois}(t, T_a^E) - P_{ois}(t, T_b^E)}{01(t, t_a, t_b)}.$$

Remark 4.9. *Note from* (3) *that*

$$S_{a,b}(t) = S_{a,b}^{ois}(t) + \frac{\sum_{i=1}^{n_E} \delta_{t_{i-1}^E, t_i^E} \alpha(t, t_{i-1}^E, t_i^E) P_{ois}(t, t_i^E)}{01(t, t_a, t_b)}$$

where

$$\alpha(t, t_{i-1}^E, t_i^E) = \frac{1}{\delta_{t_{i-1}^E, t_i^E}} \left(\frac{H(t, t_{i-1}^E)}{H(t, t_i^E)} - 1 \right).$$

We will suppose that variability of spread term structure $\alpha(t, t_{i-1}^E, t_i^E)$ is low. Therefore, it is reasonable to freeze it at time $t = 0$. Then, we have that

$$S_{a,b}(t) \approx S_{a,b}^{ois}(t) + \frac{\sum_{i=1}^{n_E} \delta_{t_{i-1}^E, t_i^E} \alpha(0, t_{i-1}^E, t_i^E) P_{ois}(0, t_i^E)}{01(0, t_a, t_b)}. \tag{21}$$

Theorem 4.10. *[Convexity Adjustment approximation for CMS] Given the Cheyette model in* (6), *the hypotheses 3.2 and 3.3, and considering the approximations in* (9) *and* (10), *and*

$$M(t, t_p) = \frac{P_{ois}(t, t_p)}{01(t, t_a, t_p)}. \tag{22}$$

Then, we have the temporal convexity adjustment for a CMS rate is approximated by

$$CA(t_p) \approx \frac{\partial_x S_{a,b}(t_a, \bar{x}_0(t_a), \bar{y}_0) \partial_x M(t_a, t_p, \bar{x}_0(t_a), \bar{y}_0)}{M(0, t_p)} \int_0^{t_a} \beta^2(s, t_a, \bar{x}_0(t_a), \bar{y}_s) \tag{23}$$

$$\cdot \exp\left(-2 \int_s^{t_a} \partial_x (\beta(u, t_a, \bar{x}_u, \bar{y}_u) \mu(u, \bar{x}_u, \bar{y}_u, t_a, t_b))|_{\bar{x}_u = \bar{x}_0(t_a)} du \right) ds + E(t_a)$$

with $\beta(u, t_a, x, y) = \exp\left(-\int_u^{t_a} k_w dw \right) \eta(u, x, y)$ and the error $E(t_a)$ is given by (A.1.1) with $f(x, y) = M(t_a, t_p, x, y) S_{a,b}(t_a, x, y)$ and behaves as $\mathcal{O}(t_a)$ when $t_a \to 0$ and $\|E(t_a)\|_2^2 < \infty$ when $t_a \to \infty$.

Proof. See appendix B.2.4. $\square$

Remark 4.11. *The key point is to calculate an approximation of $\mathbb{E}_s^{0,1}[D_s \bar{x}_{t_a}]$. The simplest cases are on the Hull-White or Ho-Lee model. The general case is treated in (A.1.4).*

Example 4.12 (Convexity adjustment for CMS under the Hull-White model)**.** *To check the accuracy of the last approximation, we compute with a Monte Carlo simulation the exact value of $\mathbb{E}^{t_p}\left[S_{a,b}^{ois}(t_a) \right]$ under spot measure $\mathbb{Q}$, i.e we will compute $\frac{1}{P(0,t_p)} \mathbb{E}^{\mathbb{Q}}\left[\frac{S_{a,b}(t_a)}{\beta_{t_a}} \right]$. For the Hull-White model case, we have that*

$$D_s x_{t_a} = \sigma \exp(-(t_a - s)),$$

and the volatility is only time-dependent i.e

$$\partial_x(\beta(u, t_a, \bar{x}_u, \bar{y}_u)\mu(u, \bar{x}_u, \bar{y}_u, t_a, t_b)) = 0.$$

Therefore, (23) is equal to

$$\mathbb{E}^{t_p}(S_{a,b}(t_a)) \approx S_{a,b}^{ois}(0) + \frac{\partial_x S_{a,b}^{ois}(t_a, \bar{x}_0(t_a), \bar{y}(t_a))\partial_x M(t_a, t_p, \bar{x}_0(t_a), \bar{y}(t_a))}{M(0, t_p)} \frac{\sigma^2(1 - \exp(-2kt_a))}{2k}.$$

In Figure 4, we can observe the CMS convexity adjustment when the tenor of the underlying swap is 5Y. We have compared the above approximation and a Monte Carlo simulation for a Hull-White model with parameters $\sigma = 0.01$ and $k = 0.0007$
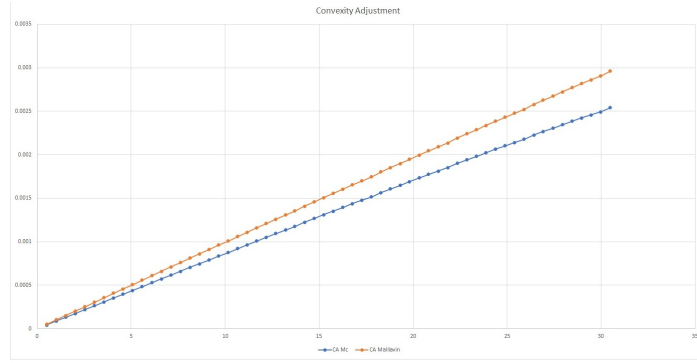

Figure 4: Convexity Mc Vs Convexity Malliavin

5   Conclusions

In the present paper, we develop a novel method based on the Malliavin calculus to find an approximation for the convexity adjustment for various classical interest rate products. Although the convexity adjustment could be calculated using other techniques, Malliavin calculus provides a simple way to get a template for the convexity adjustment. We find the approximation for Futures, OIS Futures, FRAs, and CMSs under a general family of the one-factor Cheyette model. We have also seen the excellent quality of the numerical accuracy of the formulas obtained.

In future work, the methodology could be extrapolated to a Cheyette model with stochastic volatility or even to a model with several factors.

Appendix

A.1   Auxiliary lemmas

A.1.1   Estimation of $\mathbb{E}_s^{\mathbb{Q}}[(x_t - \bar{x}_t)^2]$

To obtain the order of the approximation, we will use the ideas of the paper Benhamou et al. (2010). Basically, we will define a parametrization of the process $x_{t,\epsilon}$ and $y_{t,\epsilon}$ with $\epsilon \in [0, 1]$. The role of ϵ is only as an interpolation parameter between the process. We will suppose the next

Cheyette model parametrization

$$y_{\epsilon,t} = \int_0^t \exp\left(-2\int_u^t k(u')du'\right) \epsilon^2 \eta^2(u, x_{\epsilon,u}, y_{\epsilon,u}) du \tag{24}$$

$$x_{\epsilon,t} = \int_0^t \exp\left(-\int_u^t k(u')du'\right) \bar{y}_{\epsilon,u} du + \int_0^t \exp\left(-\int_u^t k(u')du'\right) \epsilon\eta(u, x_{\epsilon,u}, y_{\epsilon,u})\bar{y}_u du \tag{25}$$

The reason of ϵ^2 in $y_{\epsilon,t}$ is because $\mathbb{E}^{\mathbb{Q}}[y_t] = \mathbb{V}ar(x_t)$. Therefore, if the scale of $x_{\epsilon,t}$ is ϵ, then the scale of $y_{\epsilon,t}$ must be ϵ^2.

We will start with the estimation of $\mathbb{E}^{\mathbb{Q}}\left[(y_t - \bar{y}_t)^2\right]$. We will define the next parametrization for $\nu \in [0,1]$

$$z_u^\nu = \eta^2(u, \nu x_u, \nu y_u).$$

Then, we have that $z_u^1 = \eta(u, x_u, y_u)$ y $z_u^0 = \eta(u, 0, 0)$. Therefore,

$$\eta^2(u, x_u, y_u) = \eta^2(u, 0, 0) + \int_0^1 \partial_\nu z_u^\nu d\nu$$

$$= \eta^2(u, 0, 0) + 2\int_0^1 \eta(u, \nu x_u, \nu y_u)\nabla_{x,y}\eta(u, \nu x_u, \nu y_u) \cdot (x_u, y_u) d\nu.$$

Moreover, we have that

$$y_t = \bar{y}_t + E_y[t]$$

with

$$E_y[t] = 2\int_0^t \exp\left(-2\int_u^t k(u')du'\right) \int_0^1 \eta(u, \nu x_u, \nu y_u)\nabla_{x,y}\eta(u, \nu x_u, \nu y_u) \cdot (x_u, y_u) d\nu du$$

Now, from Cauchy-Schwarz inequality, we obtain that

$$\mathbb{E}^{\mathbb{Q}}\left[(E_y[t])^2\right] \le 4\mathbb{E}^{\mathbb{Q}}\left[\int_0^t \exp\left(-4\int_u^t k(u')du'\right) \left(\int_0^1 \eta(u, \nu x_u, \nu y_u)\|\nabla_{x,y}\eta(u, \nu x_u, \nu y_u)\|^2\|(x_u, y_u)\|d\nu\right)^2 du\right]^2$$

$$\le 4\|\eta\|_\infty^2 \max(\|\partial_x\eta\|_\infty^2, \|\partial_y\eta\|_\infty^2) \int_0^t \mathbb{E}^{\mathbb{Q}}\left[\|(x_u, y_u)\|^2\right] du$$

$$= M_k^y \int_0^t \mathbb{E}^{\mathbb{Q}}\left[\|(x_u, y_u)\|^2\right] du. \tag{26}$$

The next step is to estimate $\mathbb{E}^{\mathbb{Q}}\left[(x_t - \bar{x}_t)^2\right]$. From the definition of x_t, we have that

$$x_t = \int_0^t \exp\left(-\int_u^t k(u')du'\right) \bar{y}_u du + \int_0^t \exp\left(-\int_u^t k(u')du'\right) E_y[u] du + \int_0^t \eta(t, x_u, y_u) dW_u.$$

To find a expansion of $\eta(t, x_u, y_u)$ centered in $\bar{y}_u$, we will define $z_u^\nu = (1-\nu)\bar{y}_u + \nu y_u$. Then

$$\eta(t, x_u, y_u) = \eta(t, x_u, \bar{y}_u) + \int_0^1 \partial_y\eta(t, x_u, z_u^\nu)E_y[u] d\nu.$$

We will define

$$E_x[u] = \int_0^1 \partial_y\eta(t, x_u, z_u^\nu)E_y[u] d\nu.$$

Therefore, we have the next representation of x_t:

$$x_t = \bar{x}_t + \int_0^t \exp\left(-\int_u^t k(u')du'\right) E_y[u] du + \int_0^t \exp\left(-\int_u^t k(u')du'\right) E_x[u] dW_u.$$

Note that
$$\mathbb{E}^{\mathbb{Q}}[(E_x[u])^2] \leq \|\partial_y \eta\|_\infty^2 \mathbb{E}^{\mathbb{Q}}[(E_y[u])^2].$$

Using (26), we get that

$$\mathbb{E}^{\mathbb{Q}}[(E_x[u])^2] \leq 4\|\eta\|_\infty^2 \|\partial_y \eta\|_\infty^2 \max(\|\partial_x \eta\|_\infty^2, \|\partial_y \eta\|_\infty^2) \int_0^u \mathbb{E}^{\mathbb{Q}}\left[\|(x_s, y_s)\|^2\right] ds. \qquad (27)$$

By using (27) and (26), we have that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[(x_t - \bar{x}_t)^2] &\leq 2\mathbb{E}^{\mathbb{Q}}\left[\int_0^t \exp\left(-\int_u^t k(u')du'\right) E_y(u)du\right]^2 \\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[\int_0^t \exp\left(-\int_u^t k(u')du'\right) E_x(u)dW_u\right]^2 \\
&\leq 2\mathbb{E}^{\mathbb{Q}}\left[\int_0^t \exp\left(-2\int_u^t k(u')du'\right) \mathbb{E}^{\mathbb{Q}}(E_y^2(u))du\right] \\
&\quad + \int_0^t \exp\left(-2\int_u^t k(u')du'\right) \mathbb{E}^{\mathbb{Q}}[E_x^2[u]]du.
\end{aligned}$$

Join each part and for an adequate constant M_k^x, we have the following inequality

$$\mathbb{E}^{\mathbb{Q}}[(x_t - \bar{x}_t)^2] \leq M_k^x \int_0^t \exp\left(-2\int_u^t k(u')du'\right) \mathbb{E}^{\mathbb{Q}}\left[\|(x_u, y_u)\|^2\right] du.$$

Finally, we only have to estimate $\mathbb{E}^{\mathbb{Q}}\left[\|(x_s, y_s)\|^2\right]$. From the definition of y_t and given that $\|\eta\|_\infty < \infty$, we have that

$$\mathbb{E}^{\mathbb{Q}}[y_u^2] \leq \|\eta\|_\infty^2 \int_0^u \exp\left(-2\int_s^u k(u')du'\right) ds.$$

We must note that at the short term and under assumptions (3.3), we get that

$$\lim_{t \to 0} \mathbb{E}^{\mathbb{Q}}[y_u^2] = \|\eta\|_\infty^2 t$$

and

$$\lim_{t \to \infty} \mathbb{E}^{\mathbb{Q}}[y_u^2] < \infty.$$

Now, we will estimate $\mathbb{E}^{\mathbb{Q}}[x_u^2]$. From the definition of x_t and using $(a+b)^2 \leq 2(a^2 + b^2)$, we have that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[x_u^2] &\leq 2\mathbb{E}^{\mathbb{Q}}\left[\int_0^u \exp\left(-\int_s^u k(u')du'\right) y_s ds\right]^2 + 2\mathbb{E}^{\mathbb{Q}}\left[\int_0^u \exp\left(-\int_s^u k(u')du'\right) \eta(s, x_s, y_s)dW_s\right]^2 \\
&\leq 2\int_0^u \exp\left(-2\int_s^u k(u')du'\right) \mathbb{E}^{\mathbb{Q}}[y_s]^2 ds + 2\int_0^u \exp\left(-2\int_s^u k(u')du'\right) \mathbb{E}^{\mathbb{Q}}\left[\eta^2(s, x_s, y_s)\right] ds \\
&\leq 2\|\eta\|_\infty \int_0^u \exp\left(-2\int_s^u k(u')du'\right) \int_0^s \exp\left(-2\int_{s_1}^s k(u')du'\right) ds_1 ds \\
&\quad + 2\|\eta\|_\infty \int_0^u \exp\left(-2\int_s^u k(u')du'\right) ds
\end{aligned}$$

So as before at the short term

$$\lim_{t \to 0} \mathbb{E}^{\mathbb{Q}}[x_u^2] = 2\|\eta\|_\infty^2 t$$

and when $(u \to \infty)$ we have that $\mathbb{E}^{\mathbb{Q}}[x_u^2]$ remains bounded. Then, if we use the above inequalities

$$\mathbb{E}^{\mathbb{Q}}[(y_t - \bar{y}_t)^2] \leq M_k^y \int_0^t \exp\left(-2\int_u^t k(u')du'\right) \int_0^u \exp\left(-2\int_s^u k(u')du'\right) ds du$$

and

$$\mathbb{E}^{\mathbb{Q}}[(x_t - \bar{x}_t)^2] \leq 2M_k^x \|\eta\|_\infty \int_0^t \exp\left(-2\int_u^t k(u')du'\right) \int_0^u \exp\left(-2\int_s^u k(u')du'\right) \cdot$$

$$\cdot \int_0^s \exp\left(-2\int_{s_1}^s k(u')du'\right) ds_1 ds du$$

$$+ (2M_K^x \|\eta\|_\infty + 1) \int_0^t \exp\left(-2\int_u^t k(u')du'\right) \int_0^u \exp\left(-2\int_s^u k(u')du'\right) ds du.$$

Therefore, we have that

$$\mathbb{E}^{\mathbb{Q}}[(x_t - \bar{x}_t)^2] \leq \frac{2}{3} M_k^x \|\eta\|_\infty \left(\int_0^t \exp\left(-2\int_u^t k(u')\right) du\right)^3$$

$$+ \frac{(2M_K^x \|\eta\|_\infty + 1)}{2} \left(\int_0^t \exp\left(-2\int_u^t k(u')\right) du\right)^2.$$

Lemma A.1.1. *[Estimation $f(x_t, y_t)$] Given f continuous and derivable, with $\|\partial_x f\|_\infty < \infty$ and $\|\partial_y f\|_\infty < \infty$. Then*

$$f(x_t, y_t) = f(\bar{x}_t, \bar{y}_t) + E(t)$$

where $E(t) = \int_0^1 \nabla_{x,y} (u, x_{\epsilon,t}, y_{\epsilon,t}) \cdot (x_t - \bar{x}_t, y_t - \bar{y}_t) d\epsilon$ and

$$x_{\epsilon,t} = \epsilon x_t + (1 - \epsilon)\bar{x}_t$$
$$y_{\epsilon,t} = \epsilon y_t + (1 - \epsilon)\bar{y}_t.$$

In addition, we have that

$$\mathbb{E}[E^2(t)] \leq \max(\|\partial_x f\|_\infty, \|\partial_x f\|_\infty)\mathbb{E}^{\mathbb{Q}}[\|(x_t - \bar{x}_t, y_t - \bar{y}_t)\|^2].$$

Proof. From the definition of $x_{\epsilon,t}$ and $y_{\epsilon,t}$ we have by the fundamental theorem calculus that

$$f(x_t, y_t) - f(\bar{x}_t, \bar{y}_t) = \int_0^1 \nabla_{x,y} f(u, x_{\epsilon,t}, y_{\epsilon,t}) \cdot (x_t - \bar{x}_t, y_t - \bar{y}_t) d\epsilon$$

Now, if we use Cauchy-Schwarz and boundness of partial derivatives of f we have that

$$\mathbb{E}^{\mathbb{Q}}[(f(x_t, y_t) - f(\bar{x}_t, \bar{y}_t))^2] \leq \max(\|\partial_x f\|_\infty, \|\partial_x f\|_\infty)\mathbb{E}^{\mathbb{Q}}[\|(x_t - \bar{x}_t, y_t - \bar{y}_t)\|^2].$$

$\square$

Lemma A.1.2. *[Approximation $D_s \bar{x}_{t_a}$] Given the Cheyette model in (6), the hypotheses 3.2 and 3.3, and considering the approximations in (9) and (10). Then,*

$$D_s \bar{x}_{t_a} = \beta(s, t_a, \bar{x}_s, \bar{y}_s)\bar{M}(s, t_a, t_p) \quad \text{under the measure } \mathbb{Q}^{t_p}$$
$$D_s \bar{x}_{t_a} = \beta(s, t_a, \bar{x}_s, \bar{y}_s)\bar{M}(s, t_a) \quad \text{under the measure } \mathbb{Q} \tag{28}$$

where

$$\bar{M}(s, t_a, t_p) = \exp\left(-\int_s^{t_a} \left(\frac{(\partial_x \beta(u, t_a, \bar{x}_0, \bar{y}_{t_a}))^2}{2} - \exp\left(-\int_u^{t_a} k_u du\right) \partial_x(\eta(u, \bar{x}_u, \bar{y}_u)\bar{\nu}(u, t_p))\right) du\right)$$

$$\cdot \exp\left(\int_s^{t_0} \partial_x \beta(u, t_a, \bar{x}_0, \bar{y}_{t_a}) dW_u^{\mathbb{Q}}\right)$$

and

$$\bar{M}(s, t_a) = \exp\left(-\int_s^{t_a} \left(\frac{(\partial_x \beta(u, t_a, \bar{x}_0, \bar{y}_{t_a}))^2}{2} - \right) du\right) \exp\left(\int_s^{t_0} \partial_x \beta(u, t_a, \bar{x}_0, \bar{y}_{t_a}) dW_u^{\mathbb{Q}}\right)$$

with $\beta(u, t_a, x, y) = \exp\left(-\int_u^{t_a} k'_u du'\right) \partial_x \eta(u, x, y)$.

We have also that

$$\mathbb{E}_s^{t_p}[D_s \bar{x}_{t_a}] \approx \beta(s, t_a, \bar{x}_s, \bar{y}_s)\overline{DM}(s, t_a),$$
$$\mathbb{E}_s^{\mathbb{Q}}[D_s \bar{x}_{t_a}] \approx \beta(s, t_a, \bar{x}_s, \bar{y}_s) \tag{29}$$

where

$$\overline{DM}(s, t_a) = \mathbb{E}_s^{t_p}\left[\exp\left(-\int_s^{t_a} \exp\left(-\int_u^{t_a} k_u du\right)\left(\partial_x \eta(u, \bar{x}_0(t_a), \bar{y}_u)\bar{\nu}(u, t_p, \bar{x}_0(t_a)) + \eta(u, \bar{x}_0(t_a), \bar{y}_u)\right.\right.\right.$$
$$\left.\left.\left. \cdot \partial_x \bar{\nu}(t, t_p, \bar{x}_0(t_a))\right)du\right)\right]$$

with

$$\bar{\nu}(t, t_p) = \int_t^{t_p} \eta(s, \bar{x}_s, \bar{y}_s)ds,$$
$$\bar{\nu}(t, t_p, x_0) = \int_t^{t_p} \eta(s, x, \bar{y}_s)ds,$$

and

$$\partial_x \bar{\nu}(t, t_p, x_0) = \int_t^{t_p} \eta(s, x_0, \bar{y}_s)ds$$

Proof. From

$$dW^{\mathbb{Q}^{t_p}} = dW^{\mathbb{Q}} + \bar{\nu}(t, t_p)dt$$

we have that under the measure $\mathbb{Q}^{t_p}$

$$\bar{x}_{t_a} = \bar{x}_0(t_a) + \int_0^{t_a} \exp\left(-\int_s^{t_a} k_u du\right)\bar{y}_s ds - \int_0^{t_a}\exp\left(-\int_s^{t_a} k_u du\right)\bar{\nu}(s, t_p)\eta(s, \bar{x}_s, \bar{y}_s)ds$$
$$+ \int_0^{t_a} \exp\left(-\int_s^{t_a} k_u du\right)\eta(s, \bar{x}_s, \bar{y}_s)dW_s^{\mathbb{Q}^{t_p}}.$$

Now, if we apply D_s in above equality and we use the last approximation we have that

$$\mathbb{E}_s^{t_p}[D_s\bar{x}_{t_a}] \approx \eta(s, \bar{x}_s, \bar{y}_s)\exp\left(-\int_s^{t_a} k_u du\right)$$
$$\cdot \exp\left(-\int_s^{t_a}\exp\left(-\int_u^{t_a} k_u du\right)\left(\partial_x\eta(u, \bar{x}_0(t_a), \bar{y}_u)\bar{\nu}(u, t_p, \bar{x}_0(t_a)) + \eta(u, \bar{x}_0(t_a), \bar{y}_u)\right.\right.$$
$$\left.\left. \cdot \partial_x\bar{\nu}(t, t_p, \bar{x}_0(t_a))du\right)\right.$$

$\square$

A.1.2 Approximation of $\mathbb{E}_s^{\mathbb{Q}}[D_s\bar{x}_{t_a}]$

As in the previous appendix, we have

$$\bar{x}_{t_a} = \bar{x}_0(t_a) + \int_0^{t_a}\exp\left(-\int_s^{t_a} k_u du\right)\bar{y}_u du + \int_0^{t_a}\exp\left(-\int_s^{t_a} k_u du\right)\eta(u, \bar{x}_u, \bar{y}_u)dW_u^{\mathbb{Q}}$$

and therefore (see (29))

$$D_s \bar{x}_{t_a} = \exp\left(-\int_s^{t_a} k_u du\right) \eta(s, \bar{x}_s, \bar{y}_s) \bar{M}(s, t_a). \tag{30}$$

Now, if we take $\mathbb{E}_s^{\mathbb{Q}}[\cdot]$, we get

$$\mathbb{E}_s^{\mathbb{Q}}(D_s \bar{x}_{t_a}) = \exp\left(-\int_s^{t_a} k_u du\right) \eta(s, \bar{x}_s, \bar{y}_s). \tag{31}$$

Then, we obtain that

$$\mathbb{E}_s^{\mathbb{Q}}(D_s \bar{x}_{t_a}) \approx \exp\left(-\int_s^{t_a} k_u du\right) \eta(s, \bar{x}_0(t_a), \bar{y}_s). \tag{32}$$

A.1.3 Approximation of $\mathbb{E}^{01}[\bar{x}_{t_a}]$

It is easy to show that the bond dynamics under the HJM assumption is

$$\frac{dP(t,T)}{P(t,T)} = r_t dt - \nu(t,T) dW_t^{\mathbb{Q}} \tag{33}$$

where we must remember that $\nu(t,T) = \int_t^T \sigma(t,s) ds$. Therefore, if we apply Itô formula we have that

$$\frac{d01(t,t_a,t_b)}{01(t,t_a,t_b)} = r_t dt - \sigma_{01}(t,t_a,t_b) dW_t^{\mathbb{Q}} \tag{34}$$

with

$$\sigma_{01}(t,t_a,t_b) = \frac{\sum_{i=a+1}^b \delta_{i-1,i} P(t,t_i) \nu(t,t_i)}{01(t,t_a,t_b)}. \tag{35}$$

We define $w_i(t) = \frac{\delta_{i-1,i} P(t,t_i)}{01(t,t_a,t_b)}$, then

$$\sigma_{01}(t,t_a,t_b) = \sum_{i=a+1}^b w_i(t) \nu(t,t_i).$$

From (35) and since $\frac{P_{ois}(t,T)}{01(t)}$ is a martingale, we have that

$$dW_t^{01} = dW_t^{\mathbb{Q}} - \sigma_{01}(t,t_a,t_b) dt.$$

Then, if we freeze the weights $w_i(t)$, we get the next approximation of (35)

$$\bar{\sigma}_{01}(t,t_a,t_b) \approx \sum_{i=a+1}^b w_i(0) \nu(t,t_i).$$

Using (6) and above approximation, we obtain that

$$\bar{x}_{t_a} \approx \bar{x}_0(t_a) + \int_0^{t_a} \exp\left(-\int_s^t k_u du\right) \bar{y}_u du + \int_0^{t_a} \exp\left(-\int_s^{t_a} k_u du\right) \eta(u, \bar{x}(u), \bar{y}_u) dW_u^{\mathbb{Q}}$$

$$= \bar{x}_0(t) + \int_0^{t_a} \exp\left(-\int_s^{t_a} k_u du\right) \bar{y}_u du + \int_0^{t_a} \exp\left(-\int_s^{t_a} k_u du\right) \eta(u, \bar{x}_u, \bar{y}_u) \bar{\sigma}_{01}(u,t_a,t_b) du$$

$$+ \int_0^{t_a} \exp\left(-\int_s^{t_a} k_u du\right) \eta(u, \bar{x}_u, \bar{y}_u) dW_u^{01}. \tag{36}$$

Therefore, we have that

$$\mathbb{E}^{01}\left[\bar{x}_t\right] \approx \bar{x}_0(t_a) + \int_0^{t_a} \exp\left(-\int_s^{t_a} k_u du\right) \bar{y}_u du$$

$$+ \int_0^{t_a} \exp\left(-\int_s^{t_a} k_u du\right) \eta(u,\bar{x}_u,\bar{y}_u)\bar{\sigma}_{01}(u,t_a,t_b,\bar{x}_0(t_a)) du \qquad (37)$$

with

$$\bar{\sigma}_{01}(u,t_a,t_b,\bar{x}_0(t_a)) = \sum_{i=a+1}^b w_i(0)\nu(t,t_i,\bar{x}_0(t_a))$$

and

$$\nu(t,t_i,\bar{x}_0(t_a)) = \int_t^{t_i} \sigma(t,u,\bar{x}_0(t_a),\bar{y}_u) du.$$

A.1.4   Approximation of $\mathbb{E}^{01}\left[D_s\bar{x}_{t_a}\right]$

Let us to remember that $\nu(t,t_i) = h(t,\bar{x}_t,\bar{y}_t)\frac{G(t,t_i)}{\beta_{t,k}(t)}$ with $\beta_{t,k}(t) = \exp\left(\int_0^t k(u)\right)$. Therefore

$$D_s\nu(t,T_i) = \partial_x\nu(t,T_i,\bar{x}_t,\bar{y}_t)D_s\bar{x}_t\frac{G(t,t_i)}{\beta_{t,k}}.$$

Then, we have that

$$D_s\sigma_{0,1}(t,t_a,t_b) = D_s\bar{x}_t\mu(t,\bar{x}_t,\bar{y}_t,t_a,t_b) \qquad (38)$$

where

$$\mu(t,\bar{x}_t,\bar{y}_t,t_a,t_b) = \sum_{i=a+1}^b w_i(t)\partial_x\nu(t,t_i,\bar{x}_t,\bar{y}_t)\frac{G(t,t_i)}{\beta_{t,k}},$$

and $\frac{\delta_{i-1,i}P(t,t_i)}{01(t,t_a,t_b)}$. From (10) and the Girsanov's theorem, we get that

$$\bar{x}_{T_a} = \bar{x}_0(t_a) + \int_0^{t_a} \exp\left(-\int_s^{t_a} k_u du\right) \bar{y}_u du$$

$$+ \int_0^{t_a} \exp\left(-\int_s^{t_a} k_u du\right) \sigma_{01}(u,t_a,t_b)\mu(u,\bar{x}_u,\bar{y}_u,t_a,t_b) du$$

$$+ \int_0^{t_a} \exp\left(-\int_s^{t_a} k_u du\right) \eta(u,\bar{x}_u,\bar{y}_u)dW_u^{\mathbb{Q}^{01}}.$$

Then, taking D_s, we have that

$$D_s\bar{x}_{t_a} = \exp\left(-\int_s^{t_a} k_u du\right) \eta(s,\bar{x}_0,\bar{y}_{t_a})\bar{M}^{01}(s,t_a)$$

where

$$\bar{M}^{01}(s,t_a) = \exp\left(-\int_s^{t_a}\left(\frac{(\partial_x\beta(u,t_a,\bar{x}_u,\bar{y}_u))^2}{2} + \partial_x(\beta(u,t_a,\bar{x}_u,\bar{y}_u)\mu(u,\bar{x}_u,\bar{y}_u,t_a,t_b)) du\right)\right)$$

$$\exp\left(\int_s^{t_a} \partial_x\beta(u,t_a,\bar{x}_0,\bar{y}_{t_a})dW_u^{\mathbb{Q}^{01}}\right)$$

and $\beta(u,t_a,x,y) = \exp\left(-\int_u^{t_a} k_u du\right)\eta(u,x,y)$.
Then,

$$\mathbb{E}_s^{01}\left[D_s\bar{x}_{t_a}\right] = \beta(s,t_a,\bar{x}_s,\bar{y}_s)\mathbb{E}_s^{01}\left(\exp\left(-\int_s^{t_a} \partial_x(\beta(u,t_a,\bar{x}_u,\bar{y}_u)\mu(u,\bar{x}_u,\bar{y}_u,t_a,t_b)) du\right)\right)$$

$$\approx \beta(s,t_a,\bar{x}_s,\bar{y}_s)\exp\left(-\int_s^{t_a} \partial_x(\beta(u,t_a,\bar{x}_u,\bar{y}_u)\mu(u,\bar{x}_u,\bar{y}_u,t_a,t_b))|_{\bar{x}_u=\bar{x}_0(t_a)} du\right) \qquad (39)$$

B.2 Proofs

B.2.1 Proof Theorem 4.1

Observe that $L_E(t, t_1, t_2)$ is a martingale under the forward measure $\mathbb{Q}^{t_2}$. Let us define the future rate as:

$$\hat{L}_E(t, t_0, t_1, t_2) = \mathbb{E}_t^{\mathbb{Q}}[L_E(t_0, t_1, t_2)] \tag{40}$$

where $\mathbb{Q}$ is the measure associated to the numeraire $B_t = \exp\left(\int_0^t r_{ois,s} ds\right)$ with $r_{ois,t}$ the risk free short rate. Using (12) and (40), then the convexity adjustment definition is

$$CA(t, t_0, t_1, t_2) = \hat{L}_E(t, t_0, t_1, t_2) - \mathbb{E}_t^{\mathbb{Q}^{t_2}}[L_E(t_0, t_1, t_2)].$$

From (4) and since $f_{ois}(t, T)$ is a $\mathbb{Q}^T$ martingale, we have that

$$dW^{\mathbb{Q}^{t_2}} = dW^{\mathbb{Q}} + \nu(t, t_2) dt. \tag{41}$$

Applying (11) with $f(x_t) = L_E(t, t_0, t_1, t_2)$, $\mathbb{Q}_1 = \mathbb{Q}$, $\mathbb{Q}_1 = \mathbb{Q}^{t_2}$ and $\lambda_t = \nu(t, t_2)$, we get that

$$CA(t, t_0, t_1, t_2) = \mathbb{E}^{\mathbb{Q}^{t_2}}\left[\int_0^{t_0} \mathbb{E}_s^{\mathbb{Q}}\left[D_s L_E(t_0, t_1, t_2)\right] \nu(s, t_2) ds\right] \tag{42}$$

where $\nu(t, T)$ has been defined in (4). Calculating the Malliavin derivative of $L_E(t_0, t_1, t_2)$ we have that

$$D_s L_E(t_0, t_1, t_2) = \frac{H(t_0, t_1)}{\delta_{t_1, t_2} H(t_0, t_2)} D_s \left(\frac{P_{ois}(t_0, t_1)}{P_{ois}(t_0, t_2)}\right).$$

Now from the zero-coupon representation formula (7), we get that

$$D_s \left(\frac{P_{ois}(t_0, t_1)}{P_{ois}(t_0, t_2)}\right) = \frac{\partial_x P_{ois}(t_0, t_1) P_{ois}(t_0, t_2) - \partial_x P_{ois}(t_0, t_2) P_{ois}(t_0, t_1)}{P_{ois}^2(t_0, t_2)} D_s x_{t_0}.$$

Therefore

$$D_s L_E(t_0, t_1, t_2) = \frac{H(t_0, t_1)}{\delta_{t_1, t_2} H(t_0, t_2)} \frac{\partial_x P_{ois}(t_0, t_1) P_{ois}(t_0, t_2) - \partial_x P_{ois}(t_0, t_2) P_{ois}(t_0, t_1)}{P_{ois}^2(t_0, t_2)} D_s x_{t_0}. \tag{43}$$

If we use (29) with $T_a = t_0$ and $\beta(t, t_0, x, y) = \exp\left(-\int_s^{T_a} k_u du\right) \eta(u, x, y)$, we have that

$$D_s L_E(t_0, t_1, t_2) \approx \frac{H(t_0, t_1)}{\delta_{t_1, t_2} H(t_0, t_2)} \frac{\partial_x P_{ois}(t_0, t_1) P_{ois}(t_0, t_2) - \partial_x P_{ois}(t_0, t_2) P_{ois}(t_0, t_1)}{P_{ois}^2(t_0, t_2)} \beta(t, t_0, \bar{x}_s, \bar{y}_s) \bar{M}(s, t_0)$$

$$\approx \frac{P_E(0, t_1)}{\delta_{t_1, t_2} P_E(0, t_2)} \left(G(t_0, t_2) \frac{P_{ois}(0, t_2)}{P_{ois}(0, t_0)} - G(t_0, t_1) \frac{P_{ois}(0, t_1)}{P_{ois}(0, t_0)}\right) \beta(t, t_0, \bar{x}_s, \bar{y}_s) \bar{M}(s, t_0).$$

Therefore

$$\mathbb{E}_s[D_s L_E(t_0, t_1, t_2)] = \frac{P_E(0, t_1)}{\delta_{t_1, t_2} P_E(0, t_2)} (G(t_0, t_2) - G(t_0, t_1)) \beta(t, t_0, x_0, \bar{y}_s). \tag{44}$$

Then from (42) and (44) we find the approximation for the convexity adjustment for futures.

B.2.2 Proof Theorem 4.3

To compute $\mathbb{E}^{\mathbb{Q}}[R(t_0, t_1)]$ we will use (11), with $\mathbb{Q}_2 = \mathbb{Q}$, $\mathbb{Q}_1 = \mathbb{Q}^{t_1}$ and $x_{t_1} = R(t_0, t_1)$. If we apply D_s on $I(t_0, t_1)$ we obtain that

$$D_s I(t_0, t_1) = \int_{\max(s, t_0)}^{t_1} D_s x_u du.$$

Now, if $t < t_0$, then from (1) and (32), we have that

$$I(t_0, t_1) = \mathbb{E}_t^{\mathbb{Q}}\left[I(t_0, t_1)\right] + \int_t^{t_1} \int_{\max(s, t_0)}^{t_1} \mathbb{E}_s^{\mathbb{Q}}\left[\beta(s, u, x_s, y_s)\bar{M}(s, u)\right] du dW_s^{\mathbb{Q}}$$

$$\approx \mathbb{E}_t^{\mathbb{Q}}\left[I(t_0, t_1)\right] + \int_t^{t_1} \int_{\max(s, t_0)}^{t_1} \beta(s, u, \bar{x}_0(t_1), \bar{y}_s) du dW_s^{\mathbb{Q}}$$

$$= \mathbb{E}_t^{\mathbb{Q}}\left[I(t_0, t_1)\right] + \int_t^{t_1} g(s)h(s, \bar{x}_0(t_1), \bar{y}_s) \int_{\max(s, t_0)}^{t_1} \exp\left(-\int_s^u k_{s'} ds'\right) du dW_s^{\mathbb{Q}}. \quad (45)$$

Then, using the previous approximation, we get that

$$1 + \delta_{t_0, t_1} \mathbb{E}_t^{\mathbb{Q}}\left[R(t_0, t_1)\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\exp(I(t_0, t_1))\right]$$

$$\approx \exp\left(\mathbb{E}_t^{\mathbb{Q}}\left[I(t_0, t_1)\right]\right) \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_t^{t_1} \Gamma(s, t_0, t_1) dW_s^{\mathbb{Q}}\right)\right]$$

where

$$\Gamma(s, t_0, t_1) = g(s)h(s, x_0, y_0) \int_{\max(s, t_0)}^{t_1} \exp\left(-\int_s^u k_{s'} ds'\right) du.$$

Therefore, we have that

$$1 + \delta_{t_0, t_1} \mathbb{E}^{\mathbb{Q}}\left[R(t_0, t_1)\right] \approx \exp\left(\mathbb{E}^{\mathbb{Q}}\left[I(t_0, t_1)\right]\right) \exp\left(-\frac{1}{2}\int_t^{t_1} \Gamma^2(s, t_0, t_1) ds\right). \quad (46)$$

Then, we obtain (16).

In order to get an approximation of $\mathbb{E}^{\mathbb{Q}}\left[R_{avg}(t_0, t_1)\right]$ with base $\mathbb{E}^{\mathbb{Q}}\left[R(t_0, t_1)\right]$, we must note that

$$\mathbb{E}_t^{\mathbb{Q}}\left[R_{avg}(t_0, t_1)\right] = \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\log\left(1 + \delta_{t_0, t_1} R(t_0, t_1)\right)\right]}{\delta_{t_0, t_1}}.$$

Then from (46), we get (17).

B.2.3 Proof Theorem 4.7

$L_E(t, t_1, t_2)$ is martingale under the measure $\mathbb{Q}^{t_2}$, therefore the expected value of (19) is taken with respect to the wrong martingale. To calculate the convexity adjustment, we use the Clark-Ocone to get a representation for $L_E(t_1, t_1, t_2)$, i.e

$$L_E(t_1, t_1, t_2) = \mathbb{E}^{t_2}\left[L_E(t_1, t_1, t_2)\right] + \int_0^{t_1} \mathbb{E}^{t_2}\left[D_s L_E(t_1, t_1, t_2)\right] dW_s^{\mathbb{Q}^{t_2}}. \quad (47)$$

Under the HJM dynamics, we have the relation

$$dW_s^{\mathbb{Q}^{t_2}} = dW_s^{\mathbb{Q}^{t_1}} + \left(\nu(s, t_2) - \nu(s, t_1)\right) ds.$$

Taking $\mathbb{E}^{t_1}(\cdot)$, we get that

$$\mathbb{E}^{t_1}\left[L_E(t_1, t_1, t_2)\right] = L_E(0, t_1, t_2) + \mathbb{E}^{t_1}\left[\int_0^{t_1} \mathbb{E}_s^{t_2}\left[D_s L_E(t_1, t_1, t_2)\right] dW_s^{t_2}\right]$$

$$= L_E(0, t_1, t_2) + \mathbb{E}^{t_1}\left[\int_0^{t_1} \mathbb{E}_s^{t_2}\left[D_s L_E(t_1, t_1, t_2)\right] \left(\nu(s, t_2) - \nu(s, t_1)\right) ds\right].$$

Now from (28) we have that

$$
\begin{aligned}
D_s L(t_1, t_1, t_2) &= \frac{G(t_1, t_2)}{\delta_{t_1, t_2} P_E(t_1, t_2)} D_s x_{t_1} \\
&\approx \frac{G(t_1, t_2)}{\delta_{t_1, t_2} P_E(0, t_1, t_2)} \beta(s, t_1, \bar{x}_0(t_1), \bar{y}_s) \overline{DM}(s, t_1).
\end{aligned}
$$

Using the above approximation, we get

$$
CA(t_0, t_1) \approx \frac{G(t_1, t_2)}{\delta_{t_1, t_2} P_E(0, t_1, t_2)} \beta(s, t_1, \bar{x}_0(t_1), \bar{y}_s) \int_0^{t_1} \overline{DM}(s, t_1) \Big(\bar{\nu}(s, t_2, \bar{x}_0(t_1)) - \bar{\nu}(s, t_1, \bar{x}_0(t_1)) \Big) ds
$$

with

$$
\bar{\nu}(t, t_p, \bar{x}_0(t_a)) = \int_t^{t_p} \eta(s, \bar{x}_0(t_a), \bar{y}_s) ds.
$$

B.2.4   Proof Theorem 4.10

Assume we have a cash flow in $t_a < t_p < t_b$ with value $S_{a,b}(t_a)$. $S_{a,b}(t_a)$ is a martingale under the measure $\mathbb{Q}^{01}$, but not under the measure $\mathbb{Q}^{t_p}$. Therefore, we take into consideration the effect to compute the expected value of $S_{a,b}(t_a)$ in a measure that is not its natural measure. Then, the convexity adjustment for a CMS is

$$
CA_{CMS}(t_p) = \mathbb{E}^{t_p} \left[S_{a,b}(t_a) \right] - S_{a,b}(0). \tag{48}
$$

After some changes of measure, we can see that

$$
\begin{aligned}
\mathbb{E}^{t_p} \left(S_{a,b}(t_a) \right) &= \frac{1}{M(0, t_p)} \mathbb{E}^{01} \left[S_{a,b}(t_a) M(t_a, t_p) \right] \\
&= \frac{1}{M(0, t_p)} \mathbb{E}^{01} \left[S_{a,b}(t_a) \mathbb{E}^{01} \left[M(t_a, t_p) | S_{a,b}(t_a) \right] \right]
\end{aligned}
$$

with

$$
M(t, t_p) = \frac{P_{ois}(t, t_p)}{01(t, t_a, t_p)}.
$$

Then, we can approximate it by

$$
\mathbb{E}^{t_p} \left(S_{a,b}(t_a) \right) \approx \frac{1}{M(0, t_p)} \mathbb{E}^{01} \left[S_{a,b}^{ois}(t_a) \mathbb{E}^{01} \left[M(t_a, t_p) | S_{a,b}(t_a) \right] \right] \tag{49}
$$

$$
+ \frac{\sum_{i=1}^{n_E} \delta_{t_{i-1}^E, t_i^E} \alpha(0, t_{i-1}^E, t_i^E) P_{ois}(0, t_i^E)}{01(0, t_a, t_b)}. \tag{50}
$$

From the previous expression, we must note that under the assumption that there are not stochastic basis, we must compute the convexity adjustment for the OIS swap rate. But a complicated point is calculating the expected value

$$
\mathbb{E}^{01} \left[M(t_a, t_p) | S_{a,b}(t_a) \right]. \tag{51}
$$

To reduce this complexity, it is a common practice to assume that $M(t_a, t_p)$ is a function of the swap rate $S_{a,b}(t_a)$, i.e $M(t_a, t_p) = f(S_{a,b}(t_a))$. Under this assumption (51) is trivial to calculate it. The function $f(\cdot)$ is known as the mapping function. There is a vast literature about how to choose it (see Andersen and Piterbarg (2010b) or Hagan et al. (2020)).

In this case, we will choose $\bar{x}_0(t_a)$ such that $S_{a,b}(t_a, \bar{x}_0(t_a), \bar{y}_{t_a}) = S_{a,b}(0)$. Now, if we apply Clark-Ocone formula to $M(t_a, t_p)$ we get that

$$M(t_a, t_p) = M(0, t_p) + \int_0^{t_a} \mathbb{E}_s^{01}\Big[D_s M(t_a, t_p)\Big] dW_s^{01}. \tag{52}$$

Then, if we substitute the last expressions in (49), we obtain that

$$
\begin{aligned}
\mathbb{E}^{t_p}\left(S_{a,b}(t_a)\right) =& S_{a,b}^{ois}(0) + \frac{1}{M(0,t_p)}\mathbb{E}^{01}\left[S_{a,b}^{ois}(t_a)\int_0^{t_a}\mathbb{E}_s^{0,1}\Big[D_s x_{t_a}\partial_x M(t_a, t_p)\Big] dW_s^{0,1}\right] \\
=& S_{a,b}^{ois}(0) + \frac{1}{M(0,t_p)}\mathbb{E}^{01}\left[\int_0^{t_a} D_s x_{t_a}\partial_x S_{a,b}(t_a)\mathbb{E}_s^{0,1}\Big[D_s x_{t_a}\partial_x M(t_a, t_p)\Big] ds\right]. \tag{53}
\end{aligned}
$$

Now from (39) and choosing $\bar{x}_0(t_a)$ such that $S_{a,b}(t_a, \bar{x}_0(t_a), \bar{y}_{t_a}) = S_{a,b}(0)$, we have that

$$
\begin{aligned}
\mathbb{E}^{t_p}\left[S_{a,b}(t_a)\right] \approx & S_{a,b}^{ois}(0) + \frac{\partial_x S_{a,b}(t_a, \bar{x}_0(t_a), \bar{y}_0)\partial_x M(t_a, t_p, \bar{x}_0(t_a), \bar{y}_0)}{M(0, t_p)} \\
& \cdot \int_0^{t_a} \beta^2(s, t_a, \bar{x}_0(t_a), \bar{y}_s)\exp\left(-\int_s^{t_a}\partial_x(\beta(u, t_a, \bar{x}_u, \bar{y}_u)\mu(u, \bar{x}_u, \bar{y}_u, t_a, t_b))\big|_{\bar{x}_u=\bar{x}_0(t_a)}du\right) ds
\end{aligned} \tag{54}
$$

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