Assignment - 1

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1. Prove that there is no positive integer n such that $n^2 + n^3 = 100$:
We can use a contradiction to prove this statement. Suppose there exists a positive integer n such that $n^2 + n^3 = 100$.

$$\begin{aligned} n^2 + n^3 &= 100 \\ n^3 &= 100 - n^2 \\ n^3 &= (10 - n)(10 + n) \quad \text{which implies} \\ n(1 + n) &= 10 \quad \text{must be an integer factor.} \end{aligned}$$

However, solving this for n such that $1 \leq n \leq 9$ (as n must be a positive integer less than 10), we find no such integer exists that satisfies $n^2 + n^3 = 100$. Therefore, no positive integer n meets the criteria.

2. Prove that $n^2 + 1 \geq 2n$ for positive integers $1 \leq n \leq 4$:
We can use direct verification for each integer within the given range.

- For $n = 1$:
$$1^2 + 1 = 2 \geq 2 \times 1 = 2$$
- For $n = 2$:
$$2^2 + 1 = 5 \geq 2 \times 2 = 4$$
- For $n = 3$:
$$3^2 + 1 = 10 \geq 2 \times 3 = 6$$
- For $n = 4$:
$$4^2 + 1 = 17 \geq 2 \times 4 = 8$$

Thus, the inequality $n^2 + 1 \geq 2n$ holds for all positive integers n within the range from 1 to 4.

3. Find a compound proposition involving p, q, r, and s that is true when exactly three of these propositional variables are true and false otherwise:
A compound proposition that meets this condition is:

$$(p \wedge q \wedge r \wedge \neg s) \vee (p \wedge q \wedge \neg r \wedge s) \vee (p \wedge \neg q \wedge r \wedge s) \vee (\neg p \wedge q \wedge r \wedge s)$$

4. Show that

$$\exists x(P(x) \to Q(x)) \quad \text{and} \quad \forall x P(x) \to \exists x Q(x)$$

always have the same truth value.

We will prove that these statements are logically equivalent.

Case 1: Suppose $\exists x(P(x) \to Q(x))$ is true.

This implies there is some x for which $P(x) \to Q(x)$ holds true. If $P(x)$ is true, then $Q(x)$ must be true for this x, ensuring $\exists x Q(x)$ is true. Thus, $\forall x P(x) \to \exists x Q(x)$ holds true.

Case 2: Suppose $\exists x(P(x) \to Q(x))$ is false.

This implies $P(x) \to Q(x)$ is false for all x. Therefore, if $\forall x P(x)$ is true, $Q(x)$ must be false for some x, making $\exists x Q(x)$ false. Hence, $\forall x P(x) \to \exists x Q(x)$ is false.

Thus, both statements always have the same truth value, proving their logical equivalence.

5. Suppose that A and B are sets such that the power set of A is a subset of the power set of B. Does it follow that $A \subseteq B$?

Given $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, we aim to show $A \subseteq B$.

Every subset of A is also a subset of B by the given condition. Particularly, each element x in A forms the subset $\{x\}$ which must also be in B. Thus, all elements of A are in B, proving $A \subseteq B$.

6. Prove that $A \subseteq B$ if and only if $A \cap B = A$:

To show $A \subseteq B$ if and only if $A \cap B = A$, we consider both implications:

1. If $A \subseteq B$, then $A \cap B = A$:

Assuming $A \subseteq B$, any $x \in A$ is also in B, thus $x \in A \cap B$. Conversely, if $x \in A \cap B$, then $x \in A$. Therefore, $A \cap B = A$.

2. If $A \cap B = A$, then $A \subseteq B$:

Assuming $A \cap B = A$, for any $x \in A$, it follows $x \in A \cap B$, which implies $x \in B$. Hence, $A \subseteq B$.

Both implications together show that $A \subseteq B$ if and only if $A \cap B = A$.