CHAPTER 4:

Conditional probability and Independency

Conditional Events: If the occurrence of one event has an effect on the next occurrence of the other event then the two events are conditional or dependent events. Conditional Event is a dependent event that occurs only if another event (on which it depends) has occurred.

Conditional probability of an event

The conditional probability of an event A given that B has already occurred, denoted by P(A|B). Since A is known to have occurred, it becomes the new sample pace replacing the original sample space.

From this we are led to the definition

$P(A|B)=\frac{p(A\cap B}{P(B)}$, P (B) $\neq 0$ or P (A$\cap$ B) = P (A|B).P(B)

Remark:

1. $P(A^{-}|B)=1-P(A|B)$
2. $P(B^{-}|A)=1-P(B|A)$
3. For three events then
4. Generalization of multiplication theorem, for events we have

Example:

1. For a student enrolling at freshman at certain university the probability is 0.25 that he/she will get scholarship and 0.75 that he/she will graduate. If the probability is 0.2 that he/she will get scholarship and will also graduate. What is the probability that a student who get a scholarship graduate?

Solution: Let A= the event that a student will get a scholarship

B= the event that a student will graduate

$$given \quad p(A)=0.25, \qquad p(B)=0.75, \qquad p(A\cap B)=0.20$$

$$\mathrm{Re}\,quired \quad p(B/A)$$

$$p(B/A)=\frac{p(A\cap B)}{p(A)}=\frac{0.20}{0.25}=0.80$$

2. A lot consists of 20 defective and 80 non-defective items from which two items are chosen without replacement. Events A & B are defined as A = {the first item chosen is defective}, B = {the second item chosen is defective}

a. What is the probability that both items are defective?
b. What is the probability that the second item is defective?

3. In the above example 2, if we choose 3 items after other without replacement, what is the probability that all items are defective?

The law of total probability

Law of Total Probability: The "Law of Total Probability" (also known as the "Method of Conditioning") allows one to compute the probability of an event B by conditioning on cases, according to a partition of the sample space.

Definition (Partition):- We say that events represent a partition of sample space S

A_1	A_2	A_3	…An

Partition is the collection of non-overlapping, non-empty subset of a sample space whose union is the sample space itself. So the property of partition is;

I.

II.

III.

Proof: from the partition sample space let B any event in sample space **S**

A_1	A_2	A_3	**…** An

From the above diagram =

Bayes' theorem

Bayes' theorem (also known as Bayes' rule or Bayes' law) is a result in probability theory that relates conditional probabilities. If A and B denote two events, P(A|B) denotes the conditional probability of A occurring, given that B occurs. The two conditional probabilities P(A|B) and P(B|A) are in general different. Bayes' theorem gives a relation between P(A|B) and P(B|A).

An important application of Bayes' theorem is that it gives a rule how to update or revise the strengths of evidence-based beliefs in light of new **evidence a posteriori**.

Bayes' theorem relates the conditional and marginal probabilities of stochastic events A and B:

Each term in Bayes' theorem has a conventional name:

- P(A) is the **prior** probability or marginal probability of A. It is **"prior"** in the sense that it does not take into account any information about B.
- $P(A_i|B)$ is the conditional probability of A, given B. It is also called the **posterior** probability because it is derived from or depends upon the **specified** value of B.
- P(B|A) is the conditional probability of B given A.
- P(B) is the prior or marginal probability of B, and acts as a normalizing constant
  - A prior probability is an initial probability value originally obtained before any additional information is obtained.
  - A posterior probability is a probability value that has been revised by using additional information that is later obtained

Let be a partition of the sample space S and let B be the event associated with S. Applying the definition of conditional probability, we have

Proof:- but $P(A\cap B)= P(B\cap A)=P(A)P(B|A)$

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Example 1: An aircraft emergency locator transmitter (ELT) is a device designed to transmit a signal in the case of a crash. The Altigauge Manufacturing Company makes 80% of the ELTs, the Bryant Company makes 15% of them, and the Chartair Company makes the other 5%. The ELTs made by Altigauge have a 4% rate of defects, the Bryant ELTs have a 6% rate of defects, and the Chartair ELTs have a 9% rate of defects (which helps to explain why Chartair has the lowest market share).

A. If an ELT is randomly selected from the general population of all ELTs, find the probability that it was made by the Altigauge Manufacturing Company.
B. If a randomly selected ELT is then tested and is found to be defective, find the probability that it was made by the Altigauge Manufacturing Company.

Solution: We use the following notation: Let:-

A = ELT manufactured by Altigauge

B = ELT manufactured by Bryant

C = ELT manufactured by Chartai

D= ELT is defective

= ELT is not defective (or it is good)

A. If an ELT is randomly selected from the general population of all ELTs, the probability that it was made by Altigauge is 0.8 (because Altigauge manufactures 80% of them).
B. If we now have the additional information that the ELT was tested and was found to be defective, we want to revise the probability from part (a) so that the new information can be used. We want to find the value of P(A|D), which is the probability that the ELT was made by the Altigauge company given that it is defective. Based on the given information, we know these probabilities:

P(A) = 0.80 because Altigauge makes 80% of the ELTs

P(B) = 0.15 because Bryant makes 15% of the ELTs

P(C) = 0.05 because Chartair makes 5% of the ELTs

P(D|A) = 0.04 because 4% of the Altigauge ELTs are defective

P(D|B) = 0.06 because 6% of the Bryant ELTs are defective

P(D|C) = 0.09 because 9% of the Chartair ELTs are defective

Here is Bayes' theorem extended to include three events corresponding to the selection of ELTs from the three manufacturers (A, B, C)

P(A|D)= = ==0.703

Example 2: Three urns contain colored balls:

Urn	Red	White	blue
1	3	4	1
2	1	2	3
3	4	3	2

One urn is chosen at random and a ball is withdrawn.

A. What is the probability that a white ball is drawn?

B. Suppose a red ball is drawn. What is the probability that it came from urn 2?

Solution:

Let be the event that the urn is selected, i= 1, 2, 3. Let S be the sample space of this experiment –selecting a urn and drawing a ball. Then, form a partition of S. Moreover, since the urn is selected at random, it must be

A. Let Wbe the event that a white ball is drawn and the conditional probability is defined as

   Then =

B. Let B be the event that the ball withdrawn is red. The probability that the chosen red ball is from urn 2 is By Bayes' theorem

By using the information from the table:

So we obtain

Exercise: 1. Box I contains 3 red and 2 blue marbles while Box II contains 2 red and 8 blue marbles. A fair coin is tossed. If the coin turns up heads a marble is chosen from Box I; if it turns up tails a marble is chosen from Box II.

A. Find the probability that a red marble is chosen
B. What is the probability that Box I was chosen given that a red marble is known to have been chosen?

2. Of the travelers arriving at a small airport, 60 % fly on major airlines, 30 % fly on privately owned planes, and the remainder fly on commercially owned planes not belonging to a major airline. Of those travelling on major airlines, 50 % are travelling for business reasons, whereas 60 % of those arriving on private planes and 90 % of those arriving on other commercially owned planes are travelling for business reasons. Suppose that we randomly select one person arriving at this airport. What is the probability that the person

A. Is travelling on business?
B. Is travelling for business on a privately owned plane?
C. Arrived on privately owned plane, given that the person is travelling for business reasons?
D. Is travelling on business, given that the person is flying on a commercially owned plane?

Probability of Independent Events

The probability of B occurring is not affected by the occurrence or nonoccurrence of A, then we say that A and B are independent events i.e. P (B/A) = P (B). This is equivalent to $P(A \cap B) = P(A).P(B)$

Let A and B be events with $P(B) \neq 0$. Then A and B are independent if and only If $P(A \mid B) = P(A)$.

Proof: So first suppose that A and B are independent. Remember that this means that

P(AB) = P(A)P(B). Then

P(A | B)=== P(A)

Collary: If A and B are independent, then A andare independent

We are given that P(AB) = P(A) P(B), and asked to prove that P(A) = P(A) ·P()we know that P() = 1−P(B). The events AB and A $B^c$are disjoint (since no outcome can be both in B and), and their union is A (since every event in A is either in B or in); we have that P(A) = P(AB)+P(A).

P(A) = P(A)− P(AB)

= P(A)−P(A) P(B) (since A and B are independent)

= P(A)[1−P(B)]

= P(A) P(),

Collary: If A and B are independent, so and are independent.

Collary Let events A, B, C are mutually independent. Then A and BC are independent, and A and BC are independent

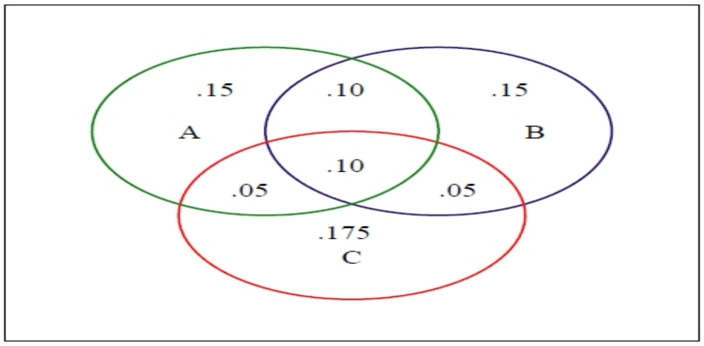
Remarks: If A_1, A_2, A_3 are to be independent then they must be pair wise independent,

$P(A_j \cap A_k) = P(A_j).P(A_k)$ $j \neq k$ Where j, k=1, 2, 3 and we must also have

$P(A_1 \cap A_2 \cap A_3) = P(A_1).P(A_2).P(A_3)$

Exercise:

1. A ball is drawn at random from a box containing 6 red balls, 4 white balls and 5 blue balls. Find the probability that they are drawn in the order red, white and blue if each ball is?
   a. Replacement
   b. Not replacement
2. In a factory machine A_1, A_2, A_3 manufacturing 25%, 35% and 40% of the total output respectively. Out of their product 5%, 4% and 2% are respectively defective. An item is drawn at random from the product is found to be defective.
   A. What is the probability that defective item is produce by all machine?
   B. What is the probability that this item is produce by machine A_1?
3. Suppose the probabilities of three events, *A*, *B* and *C* are as depicted in the following diagram:



a. Are the three events pair wise independent?

b. Are the three events independent?

c. What is P(AB)?

d. What is P($A\ B|C$)?

e. What is P($C\ |A\ B$)?

f. What is P($C\ |A\ B$)?

4. An industry A has three machines A_1, A_2, A_3 which produced the same item. It is known that 30%, 30% and 40% of the total output respectively. It is known that 2%, 3% and 3% are defective respectively. All the items are put in to one stockpile and then one item is chosen at random. Then find the probability that;
   A. This item is defective
   B. Given that the item selected in (as defective) then what is the probability that this item was produced by machine A_3?

5. **A** laboratory blood test is **95** percent effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 1 percent of the healthy persons tested.(That is, if a healthy person is tested, then, with probability 0.01, the test result will imply he has the disease.) If **0.5** percent of the population actually has the disease, what is the probability a person has the disease given that his test result is positive?