**CHAPTER THREE**

**3. ELEMENTARY PROBABILITY**

* 1. **Introduction**

Probability theory is the foundation upon which the logic of inference is built. It helps us to cope up with uncertainty. In general, probability is the chance of an outcome of an experiment. It is the measure of how likely an outcome is to occur.

* 1. **Deterministic and non-deterministic models**

**Deterministic models**

Deterministic modelsalways return the same result any time they are called with a specific set of input values. In deterministic models the condition under which an experiment is performed determines the outcome of the experiment.

A mathematically deterministic model is a representation y = f(x) that allows you to make predictions of y based on x.

E.g. y = 2+3X - 4. We can predict that if x = 3, then y = -25.

This type of model is "deterministic" because y is completely determined if you know x. In such models, a given *input* will always produce the *same output*.

A deterministic event always has the same outcome and is predictable 100% of the time. E.g. Distance traveled = time \* velocity

**Non-deterministic/ probability model**

Nondeterministic models may return different results each time even if the input values that they access remain the same. i.e. In nondeterministic models we cannot determine the outcome of the experiment even if we know condition under which an experiment is performed. A probability model is a representation y ~ p(y) not "*y = p(y*)". The notation "y ~ p(y)" specifically means that y is generated at random from a probability distribution whose mathematical form is p(y). This model also allows you to make "**what-if**" predictions as to the value of y, but, unlike the deterministic model, it does not allow you to say precisely what the value of y will be. A probabilistic event is an event for which the exact outcome is not predictable 100% of the time.

* 1. **Review of set theory**

A **set** is a well-defined collection of objects. Sets are usually denoted by capital letters A, B etc. Examples

1. A= describes the set consisting of positive integers 1, 2, 3 and 4.
2. A=A consisting of all real numbers between 0 and 1 inclusive or the set of all X’s where X is a real number between 0 and 1, inclusive. The individual objects making up the collection are called members or **elements of A.**

Examples: 1A and 6A

**Empty set** or **Null set: -** the set containing no members. i.e. {} or impossible event.

Example: A is the set of all real numbers X satisfying the equation. A=

**Union:** Let A and B be any two subsets of a universal set, S. Then, the union of the Sets A and B is the set of all elements in S that are ***in at least one of the sets A or B***, it is denoted by AB.

**Intersection:** Let A and B be any two subsets of universal set, S. Then the intersection of the sets A and B is the set of all elements in S that ***are in both sets A and B,*** and is denoted by AB.

**De Morgan’s rules:** for any two A and B events

Generally to know about the probability of an event, there must be know the concept of set theory about the notation of set probability.

If A and B are events then the following conditions are true.

* If A and B are **disjoint** events then AB
* If ***at least one of the events occurs*** ,then it means that **AB**
* ***Both the events are occurs***, then it means that **AB**
* If ***neither event A nor event B occurs***, then it means that
* ***Only event A occurs***, then it means that
* ***If exactly one of the events occurs***, then it means **that**
* If ***not more than one*** of the events ***A or B*** occurs, then it means that

**Exercise**:

1. Let A, B, and C be three events associated with an experiment. Express the following verbal statements in set notation.
2. At least one of the events occurs.
3. Exactly one of the events occurs.
4. Exactly two of the events occur.
5. Only A occurs.
6. Both A and B but not C occurs.
7. At least two events occur.
8. All three events occur.
9. None occurs.
10. At most one occurs.
11. At most two occurs.
    1. **Random experiments, Sample space and events**

**Experiment**: Any process of observation or measurement or any process which generates well defined outcome.

**Probability Experiment (Random Experiment):** It is an experiment that can be repeated any number of times under similar conditions and it is possible to enumerate the total number of outcomes without predicting an individual outcome.

**Example:** If a fair coin is tossed three times, it is possible to enumerate all possible eight sequences of head (H) and tail (T). But it is not possible to predict which sequence will occur at any occasion.

**Outcome:** The result of a single trial of a random experiment

**Sample Space(S)**: Set of all possible outcomes of a probability experiment.

Example: Sample space of a trial conducted by three tossing of a coin is

S= {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

Sample space can be

* **Countable (finite or infinite)**

1. *S*={0, 1, 2, 3, …}
2. *S*={ … -3, -2, -1, 0, 1, 2, 3, …}

* **Uncountable**

1. *S*=(0, ∞)
2. *S*=[5,6]
3. *S*=(–∞, ∞)
4. *S*=[0, 1]\*(0, ∞) = {(*x*, *y*): 0≤ *x*≤ 1; *y*>0

**Event (Sample Point)**: It is a subset of sample space. It is a statement about one or more outcomes of a random experiment. It is denoted by capital letter A, B, C - - -.

**For example**, in the event, that there are exactly two heads in three tossing of a coin, it would consist of three points HTH, HHT and THH.

**Equally Likely Events**: Events which have the same chance of occurring.

**Complement of an Event:** the complement of an event A means non- occurrence of A and is denoted by, contains those points of the sample space which don’t belong to A.

**Elementary (simple) Event**: an event having only a single element or sample point.

**Mutually Exclusive (Disjoint) Events**: Two events which cannot happen at the same time.

**Independent Events**: Two events are said to be independent if the occurrence of one does not affect the probability of the other occurring.

**Dependent Events**: Two events are dependent if the first event affects the outcome or occurrence of the second event in a way the probability is changed.

* 1. **Counting Techniques**

In order to determine a probability of certain event, first it is mandatory to know the number of possibility that this event going to occur as well as the number of possible outcome of an experiment that results this event. If we perform an experiment repeatedly to solve such type of problems we are going to use different mathematical technique which is known to be counting techniques.

That is in order to judge what is *probable,* we have to know what is *possible****.* So, we have to know**

* **The number of elements of an event**
* **The number of elements of the sample space**
* In order to determine the number of outcomes, one can use several rules of counting
* The addition rule
* The multiplication rule
* The permutation rule
* The combination rule

1. **Addition Rule:**

Suppose that for an experiment the procedure is designated by 1, can be performed ways, the procedure is designated by 2, can be performed ways … the procedure is designated by K can be performed ways and each procedure/stps ***cannot be performed together,*** then the total number of possibility of the experiment can be performed (

i.e. ,

* To list the outcomes of the sequence of events, **tree diagram** is used.

**Example:**

1. Supposes we are planning a trip to some place. If there are 3 bus routes & 2 train routes that we can take, then there are 3+2=5 different routes that we can take.
2. A student goes to the nearest restaurant to have a breakfast. He can take tea, coffee, or milk with bread, cake and sandwich. How many possibilities does he have?

**Solutions:**

* He can take ***tea with*** *Bread, Cake or Sandwich.* ***i.e. has 3 possibilities.***
* He can take ***Coffee with*** *Bread, Cake or Sandwich.* ***i.e. has 3 possibilities.***
* He can take ***Milk with*** *Bread, Cake or Sandwich.* ***i.e. has 3 possibilities.***

So, there are nine Possibilities

1. **Multiplication rule**

If an operation consists of **k steps** and the 1st step can be done in n1 ways, the 2nd step can be done in n2 ways (regardless of how the 1st step was performed), the kth step can be done in nk ways, (regardless of how the preceding steps were performed), then the entire operation can be performed in **n1**· **n2**·**…** · **nk** ways.

**Example:**

1. How many different ways can seven true or false questions be answered?

**Solution**: if there are 7 different true or false questions to be answered, we can answer the first question in n1 = 2 ways, the second question in n2 = 2 ways, … , and the last (7th) question can answered in n7 =2 ways. Therefore, we can answer in n1\*n2\* … \* n7= 2\*2\*2\*2\*2\*2\*2 = 128 ways.

1. The digits 0, 1, 2, 3, and 4 are to be used in 4 digit identification card. How many different cards are possible if
   * 1. Repetitions are permitted.
     2. Repetitions are not permitted.

**Solutions** a)

|  |  |  |  |
| --- | --- | --- | --- |
| 1st digit | 2nd digit | 3rd digit | 4th digit |
| 5 | 5 | 5 | 5 |

There are four steps

1. Selecting the 1st digit, this can be made in 5 ways.

2. Selecting the 2nd digit, this can be made in 5 ways.

3. Selecting the 3rd digit, this can be made in 5 ways.

4. Selecting the 4th digit, this can be made in 5 ways.

**So there are 5\*5\*5\*5= 625 *different cards are possible***

*b)*

|  |  |  |  |
| --- | --- | --- | --- |
| 1st digit | 2nd digit | 3rd digit | 4th digit |
| 5 | 4 | 3 | 2 |

There are four steps

1. Selecting the 1st digit, this can be made in 5 ways.

2. Selecting the 2nd digit, this can be made in 4 ways.

3. Selecting the 3rd digit, this can be made in 3 ways.

4. Selecting the 4th digit, this can be made in 2 ways.

So there are 5\*4\*3\*2 =120 *different cards are possible*

1. **Permutation**

An arrangement of objects with ***attention given to order of arrangement*** is called permutation. The number of permutation of n different objects taken r at a time is obtained by:



**Permutation Rule:**

a) The number of permutations of n objects taken all together is n!

i.e. n!= n\*(n-1)\*(n-2)\*…\*3\*2\*1 = 

Note: By definition 0! = 1

b) The arrangement of n distinct objects in a specific order using r objects at a time is called the permutation of n objects taken r objects at a time. It is written as nPr and the formula is



c) The number of distinct permutation of n objects in which n1 are alike, n2 are alike, ...,nk are alike is

****for

**Example:**

1. Find number of permutations of the letters in the word **‘‘statistics’’**.
2. Suppose we have a letters A, B, C, D

a) How many permutations are there taking all the four?

b) How many permutations are there two letters at a time?

1. How many different permutations can be made from the letters in the word  
   “CORRECTION”?

**Solution:**

1. There are 3s’s, 3t’s, 1a’s, 2i’s and 1c’s. i.e. ,,

Therefore = 50,400.

1. a) *Here n there are four distinct object*

*There are permutations*

*b)* 

3. Here n=10 of which 2 are C, 2 are O, 2 are R, 1 E, 1 T, 1 I AND 1 N

1. **Combination**

A **selection** of objects considered ***without regard to order*** in which they occur is called Combination. The number of combination of n different objects taking r of them at a time is , for.

**Example:**

1. Given the letters A, B, C, and D list the permutation and combination for selecting two letters.

**Solution:**

**Permutation Combination**

AB BA CA DA AB BC

AC BC CB DB AC BD

AD BD CD DC AD DC

Note that in permutation AB is different from BA but in combination AB is the same as BA.

1. In a club containing 7 members a committee of 3 people is to be formed. In how many ways can the committee be formed?

**Solution**: 7C3 =🡺 = 35

1. Among 15 clocks there are two defectives .In how many ways can an inspector chose three of the clocks for inspection so that:
2. There is no restriction.
3. None of the defective clock is included.
4. Only one of the defective clocks is included.
5. Two of the defective clock is included.

**Solution**: **a)**n=15 of which 2 are defective and 13 are non-defective.

r = 3, If there is no restriction select three clocks from 15 clocks and this can be done in:

b) This is equivalent to zero defective and three non-defectives, which can be done in:

c) This is equivalent to one defective and two non-defectives, which can be done in:

d) This is equivalent to two defective and one non defective, which can be done in:

**Exercise:** Out of 5 Mathematician and 7 Statistician a committee consisting of 2 Mathematician and 3 Statistician is to be formed. In how many ways this can be done if

1. There is no restriction
2. One particular Statistician should be included
3. Two particular Mathematicians cannot be included on the committee.
   1. **Definitions of probability**

**Definition:** Probability is a numerical measure of the chance or likelihood that a particular event will occur & it lies in the range from 0-1, inclusive. Probability is a building block of inferential statistics.

**Definition:** Let E be an experiment. Let S be a sample space associated with E. With each event A in S we associate a real number designated by P (A) and called the probability of A.

**Approaches to measuring Probability**

There are four different conceptual approaches to study probability theory. These are

* The classical approach
* The frequencies approach
* The axiomatic approach
* The subjective approach
  + - 1. **Classical approach**:

**Definition:** If there are n equally likely outcomes of an experiment, and out of the n outcomes event A occur only k times the probability of the event A is denoted by P (A) is defined as

P(A) = = =

**Note:** Classical approach of measuring probability fails to answer for the following conditions:

* If total number of outcomes is infinite or if it is not possible to enumerate all elements of the sample space.
* If each outcome is not equally likely.

**Example:** Compute

1. The probability of having two boys & one girl is a three child family using the classical method, assuming boys & girls are equally likely.
2. Using (a) compute the probability of having three boys in a three-child family.
3. Using (a) compute the probability of having three girls in a three –child family.
4. Using (a) compute the probability of having two girls & one boy in three child family.

**Solution**

The sample space S or the experiment is

S= {BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}

So n(S)=8

1. For the event A= ''two boys & a girl'' = {BBG,BGB,GBB} , we have n(A)=3,Since the outcome are equally likely , the probability of A is P(A)= n(A)/n(S)=3/8 =0.375
2. Compute the probability of having three boys in a three-child family. For the event B= ''three boys'' = {BBB} , we have n(B)=1,Since the outcome are equally likely , the probability of B is P(B)= n(B)/n(S)=1/8 = 0.125.
3. Compute the probability of having three girls in a three –child family. For the event C= ''three girls'' = {GGG} , we have n(C)=1,Since the outcome are equally likely , the probability of C is P(C)= n(C)/n(S)=1/8 = 0.125
4. Compute the probability of having two girls & one boy in three child family.

For the event D= ''two girls & one boy'' = {BGG, GBG,GGB}, we have n(A)=3,Since the outcome are equally likely, the probability of D is P(D)= n(D)/n(S)=3/8 =0.375.

**Exercise:** A fair die is tossed once. What is the probability of getting

* 1. Number 4?
  2. An odd number?
  3. An even number?
  4. Number 8?

1. **The Frequentist Approach (Empirical Probability)**

This approach to probability is based on relative frequencies.

**Definition:** Suppose we do again and again a certain experiment n times and let A be an event of the experiment and let k be the number of times that event A occurs. Therefore the probability of the event A happening in the long run is given by:

In other words given a frequency distribution, the probability of an event (A) being

in a given class is P(A) =

**Example:** The national center for health statistics reported that of every 539 deaths in recent years, 24 resulted that from automobile accident, 182 from cancer, and 353 from other disease. What is the probability that particular death is due to an automobile accident?

**Solution**

P (automobile) = death due to automobile /total death =24/539 = 0.445

The probability that particular death is due to an automobile accident is 0.445.

1. **The axiomatic approach**

Let E be a random experiment and S be a sample space associated with E. With each event A a real number called the probability of A satisfies the following properties called axioms of probability or postulates of probability.

1. 0*1*

2. P(S) =1, S is the sure/certain event.

3. If A1 and A2 are mutually exclusive events, the probability that one or the other occur equals the sum of the two probabilities. i. e. P (A1A2) =P(A1)+P(A2)

Similarly P(A1A2 . . . An) = P(A1)+P(A2) +. . . +P(An) =

4. P (A') =1-P (A)

5. P (ø) =0, ø is the impossible event.

1. **Subjective Approach**

It is always based on some prior body of knowledge. Hence subjective measures of uncertainty are always conditional on this prior knowledge. The subjective approach accepts unreservedly that different people (even experts) may have vastly different beliefs about the uncertainty of the same event.

Example: Abebe’s belief about the chances of Ethiopia Buna club winning the FA Cup this year may be very different from Daniel's. Abebe, using only his knowledge of the current team and past achievements may rate the chances at 30%. Daniel, on the other hand, may rate the chances as 10% based on some inside knowledge he has about key players having to be sold in the next two months.

**3.7. Derived theorems of probability**

**Rule l**: let A be an event and A' be the complement of A with respect to a given sample space of an experiment, then P(A')=1-P(A)

**Proof**: let S be a sample space S=AUA' and, A and A' are mutually exclusive

A∩A' = ø

P(S) = P (AUA') = P (A') + P (A) and P(S) = 1

1= P (A') + P (A) => P (A') = 1-P (A)

**Rule 2**: let A and B are events of a sample space S, then

P (A' ∩ B) = P (B) - P (A ∩ B)

**Proof**: B =S ∩ B = (AUA') ∩ B = (A∩ B) U (A'∩ B)

If A∩B ø , then P(B) =P (A∩ B) +P (A' ∩ B)

P (A' ∩ B) = P(B) – P(A ∩ B).

**Rule 3**: Suppose A and B are two events of a sample space, then

P(AUB) = P(A) + P(B) – P(A ∩ B)

Proof:

(AUB) = AU(A' ∩ B), A and A' ∩ B are disjoint sets

P(AU B) = p(A) + p(A' ∩ B) . . . .\*

But we have already proved that P (A’ n B) = P (B) – P (A ∩ B)

Put this in equation \*

P(A U B) = P(A) + P (B) – P (A ∩ B)

**Example:** A fair die is thrown twice. Calculate the probability that the sum of spots on the face of the die that turn up is divisible by 2 or 3.

**Solution**

S={(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,1),(5,2),(5,3),(5,4),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

This sample space has 6\*6 =36 elements let A be the event that the sum of the spots on the die is divisible by 2 and B be the event that the sum of the spots on the die is divisible by three, then

A = {(1,1), (1,3), (1,5), (2,2), (2,4), (2,6), (3,1), (3,3), (3,5), (4,2), (4,4), (4,6), (5,1), (5,3), (5,5), (6,2), (6,4), (6,6)}

B = {(1,2), (1,5), (2,1), (2,4), (3,3), (3,6), (4,2), (4,5), (5,1), (5,4), (6,3), (6,6)}

A∩B = {(1, 5), (2,4), (3,3), (4,2), (5,1), (6,6)}

P (A or B) = P (A U B) = P (A) +P (B) – P (A∩B) = 18/36 + 12/36 -6/36 = 24/36 = 2/3

**Exercise:**

1. If two dice are thrown, what is the probability that the sum is
2. Greater than 8?
3. Neither 7 nor 11?
4. An urn contains 8 white balls and 2 green balls. A sample of three balls is selected at random. What is the probability that the sample contains at least one green ball?
5. A box contains 12 light bulbs of which 5 are defective. All the bulbs look alike and have equal probability of being chosen. Three bulbs are picked up at random. What is the probability that at least 2 are defective?
6. A problem in Mathematics is given to three students, whose chances of solving it are respectively. What is the probability that problem will be solved?
7. The probabilities of A, B and C solving a problem are respectively. If all the three try to solve the problem simultaneously, find the probability that exactly one of them will solve it.
8. A husband and wife appear in an interview for two vacancies in the same department. The probability of husband's selection that of wife's selection. What is the probability that
   * 1. Only one of them will be selected?
     2. Both of them will be selected?
     3. None of them will be selected?
     4. At least one of them will be selected?