**Chapter -5**

**Probability distribution**

**Definitions of Random variable**

* **A random variable** is a variable whose value is unknown so it is determined by chance, for each outcome of a procedure.
* **A probability distribution** is a description of the chance a **random variable has of taking on particular values**. It is often displayed in a graph, table, or formula.
* Random variables are often designated by capital letters and can be classified as discrete or continuous,
* **A discrete random variable:** which are variables that have specific values or it takes on a countable number of values (i.e. there are gaps between values).
* **A continuous random variable:** which are variables that can have any values within a continuous range or takes infinite number of values (i.e. there are no gaps between values).
* Random variable is a variable X whose value is determined by the outcomes of random experiment. It is classified as;
* Discrete random variable
* Continuous random variable

1. **Discrete random variable:**

* Possible values of isolated points along the number line. Random variables have their own sample space, or set of possible values. If this set is finite or countable, the random variable is said to be discrete.
* It is a random variable which can assume only a countable numbers of real values.
* If X is a discrete random variable taking a values X1,X2,**….**,Xn then P(xi)=P(X=xi),where i=1,2,3**,….,**n is called probability mass function (pmf) of random variable X
* The set of order pairs [xi = P(xi)], i=1,2,3**,….,**n gives the probability distribution of random variable X

**Discrete probability distribution:**

A probability distribution describes the possible values and their probability of occurring.

Discrete probability distribution is called probability mass function **(pmf), p(.)** and need to satisfy following conditions

**Properties of P(X=xi) where X is Discrete random variable:**

1. **0 ≤** P(X=xi) ≤ 1
2. 1

**Example:** Consider the experiment of tossing a coin three times let ‘x’ be the number of heads then write the probability distribution of the random variable x

**Solution:**

The variable ‘x’ takes the value 0,1,2,3 with probability distribution {HHH, HHT, HTH, TTH, THT, THH, HTT, TTT} then the probability distribution for ‘x’

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| X | 0 | 1 | 2 | 3 |
| P(X) | 1/8 | 3/8 | 3/8 | 1/8 |

**Example:** Suppose we record four consecutive baby births in a hospital. Let X be the difference in the number of girls and boys born. X is discrete, since it can only have the values 0, 2, or 4. (Why can’t it be 1 or 3?) Furthermore, if we write out the sample space for this procedure, we can find the probability that X equals 0, 2, or 4:

S={mmmm, mmmf, mmfm, mfmm, fmmm, mmff, mfmf, mffm, fmfm, ffmm, fmmf, mfff, fmff, ffmf, fffm, ffff}

There are 16 total cases, and each one is equally likely.

P(X = 0) = 6/16 = 0.375 (these are the cases mmff, mfmf, mffm, fmfm, ffmm, fmmf)

P(X = 2) = 8/16 = 0.5 (these are the cases mmmf, mmfm, mfmm, fmmm, mfff, fmff, ffmf,fffm)

P(X = 4) = 2/16 = 0.125 (these are the cases mmmm and ffff)

Is this a probability distribution?

* *=* 0.375+ 0.5 +0.125= 1
* 0 ≤ 0.125 < 0.375 < 0.5 ≤ 1 So it *is* a probability distribution.

Another Example: Is the following a probability distribution? If X=,Y=1,2,…,11

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| *X* | *1* | *4* | *9* | *16* | *25* | *36* | *49* | *64* | *81* | *100* | *121* |
| *P(x)* | 0.20 | 0.15 | 0.14 | 0.12 | 0.10 | 0.09 | 0.07 | 0.05 | 0.04 | 0.02 | 0.01 |

* Clearly, each probability is between 0 and 1. Now we need to see if they sum to 1.

*=* 0.20+ 0.15+ 0.14 + 0.12+ 0.10 + 0.09+ 0.07 +0.05 +0.04+ 0.02+ 0.01= 0.99 since they do not sum to 1, it is *not* a probability distribution.

**Continuous random variable:**

* A random variable X is said to be continuous if it can take all possible values (integral as well as fractional) between certain limits or intervals. Continuous random variables occur when we deal with quantities that are measured on a continuous scale. **For instance**, the life length of an electric bulb, the speed of a car, weights, heights, and the like are continuous
* If X is a continuous random variable that assume a values X1,X2,X3,**….** then where i=1,2,3**,….**is called probability density function(pdf) of random variable X

**Properties of Pdf**:

1. 0 ≤ ≤ 1
2. probability density function(pdf) integral over arrange of [-∞,∞]
3. P(x1<X<x2)=
4. P(X=a)= =0

Example: suppose we have a continuous random variable’ X’ with probability density function is given by

* + - * 1. Determine the value of ‘c’
        2. Verify that f is pdf
        3. Calculate

**Solution**:

1. =1 property of pdf

= )9c=c=1/9

1. dx=1=(Then f is pdf
2. =

**Cumulative distribution function of discrete random variable**

Let X be a discrete random variable with probability mass function (pmf) then the cumulative distribution function is denoted by F(x) and it is defined by;

F(x) = P(X≤x)

=

**Example:** Tossing a coin three time, let X be getting the numbers of head, then find the CDF of x

**Solution:** The variable ‘x’ takes the value 0,1,2,3 with probability distribution

(HHH, HHT, HTH, TTH, THT, THH, HTT, TTT)

|  |  |  |
| --- | --- | --- |
| X | P(x) | F(X) |
| 0 | 1/8 | 1/8 |
| 1 | 3/8 | 4/8 |
| 2 | 3/8 | 7/8 |
| 3 | 1/8 | 1 |

**Cumulative distribution function for continuous random variable**

If X is continuous random variable with probability density function (pdf), then the Cumulative distribution function of X is F(x) which is defined as;

F(x) = =

F(X) gives the “accumulated” probability “up to x .

Properties of CDF

* 1. 0≤F(X)≤1
  2. =0
  3. F’(X)=f(x) i.e (F(X) is the anti-derivative of f(x) ).
  4. F(X) is a non –decreasing function

**Exercise: I**f the pdf of a continuous random variable x is given by

1. find the CDF of x
2. find )

**Chapter-6**

**Function of random variable**

**Equivalent set**

Let x be a random variable defined on the sample spaces S and Y be a function of X, the Y is also a random variable. Define Rx and Ry are range of spaces of X and Y respectively. Let C⊆ Ry and B⊆ Rx, then B is define as follows B={XRx,Y(x)C},then event B and C are Equivalent sets. Which means one is occurred if only if the others are occurred. For any event C⊆ Ry, P(c) defined as P(c) =P{xRx, Y(x)C}=P(B) i.e P(C)=P(B)

**Example:**

1. Suppose that Y=, then B={X: x ≥2} and C={Y:y≥4 },are equivalent sets? Why?
2. Let Y=2x+1 and Rx=(X: x>0) and Ry={Y: y>1}, suppose that event C defined as C={y≥5}. How can we define event B in order to make event C and event B are equivalent set?

**Solution:**

1. Event C and event B are equivalent set. Because in order to get the values of event C depends on the values of event B, due to this reason B and C are equivalent set

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| xi | 2 | 3 | 4 | 5 | **…….** |
| yi | 4 | 9 | 16 | 25 | **…….** |

1. First we make the table that Y=2x+1 to get C={Y:y≥5}

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| x**i** | 0 | 1 | 2 | 3 | 4 | **…….** |
| yi | 1 | 3 | 5 | 7 | 9 | **…….** |

From the table we get the values of C=(y≥5) is if x≥2. So the event B is define B=(x≥2) when x =2 is defined on the range of Rx. Therefore event C and event B are equivalent set

**Function of Discrete Random Variables**

Let us first dispose of the case when *X* is a discrete random variable, since it requires only simple point-to-point mapping. Suppose that the possible values taken by *X* can be enumerated as x1, x2, . . .. Shows that the corresponding possible values of Y may be enumerated as y1= g(x1), y2=g(x1), . . . Let the pmf of *X* be given by

The pmf of Y is simply determined as

**Example:**

1. The pmf of a random variable *X* is given as

Determine the pmf of Yi fY is related to *X* by Y =2X + 1.

**Solution:** the corresponding values of *Y* are: g(-1) =2(-1)+ 1= 1; *g*(0) =1; *g*(1)= 3; and

*g*(2) =5. Hence, the pmf of *Y* is given by

1. For the same *X* as given in Example.1, determine the pmf of Y if Y=2X2 +1.

**Solution:** in this case, the corresponding values of *Y* are: g(-1) =2(-1)2+1= 3; *g*(0)=1;

*g*(1)= 3; and *g*(2)= 9, resulting in

P(y)=

1. **A** random variable X has the following probability mass function

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| X | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| P(x) | k | 3k | 5k | 7k | 9k | 11k | 13k |

Find

1. the value of K
2. evaluate P(x<4),P(x>5)and P(3<x≤6)
3. what is the smallest values of X for which P(X≤x)>

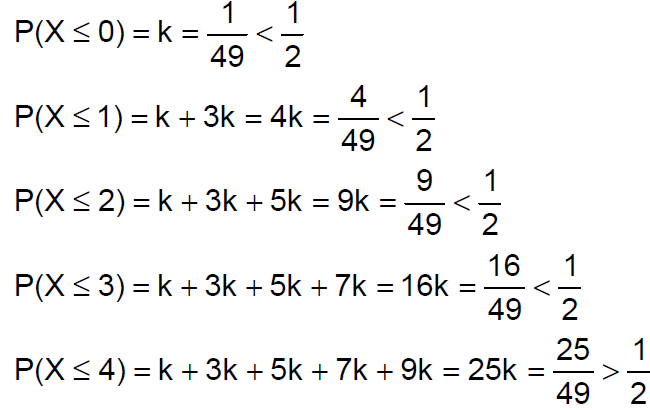
**Solution:**

1. Since P(X=x) is the probability mass function
2. P(x<4)=p(x=0)+ p(x=1)+ p(x=2)+ p(x=3)= k+3k+5k+7k=16k

**P(x≥5) =** p(x=5)+ p(x=6)= 11k+13k=24k

P(3<x≤6) = p(x=4)+p(x=5)+ p(x=6)= 9k+11k+13k=33k

1. The minimum value of x may be determined by trial and error method.

****

Therefore the smallest value of x for which P(X ≤ x) > is 4

**Function of Continuous Random Variables**

Suppose that X is a continuous random variable with pdf f(x) and Y is a function of X, Y=f(x) random variable. Then we can obtain pdf of Y,g(y) from pdf of X, f(x).

The general procedure to find pdf of Y from pdf of X

Step 1: obtain CDF of Y, G(y)

G(y)=P(Y),by finding the event C in the range space of X

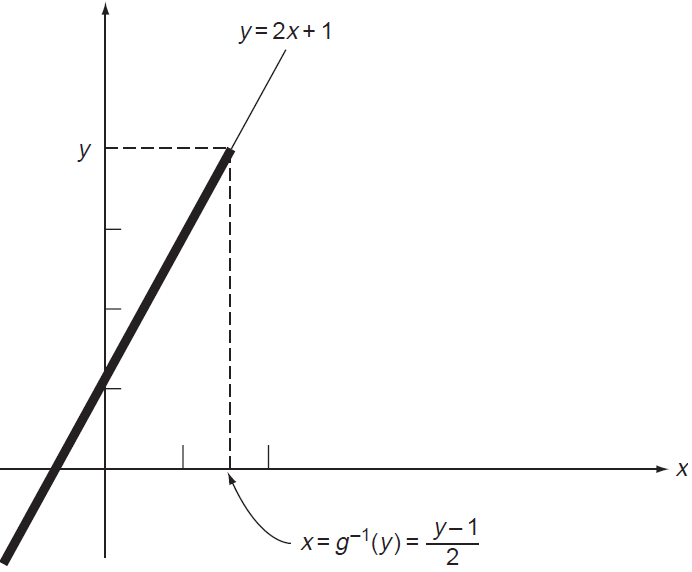
Step 2: differentiate G(y) with respect to y to obtain g(y)

Step 3: determine the values of y in the range space of Y for with g(y) ≥ 0

**Note**g(y**) =** f[]||

A more frequently encountered case arises when *X* is continuous with known PDF, *F*(*x*), or pdf, *f* x(*x*). To carry out the mapping steps as outlined at the beginning of this section, care must be exercised in choosing appropriate corresponding regions in range spaces *R*xand *R*y, this mapping being governed by the transformation Y= *g*(x). Thus, the degree of complexity in determining the probability distribution of *Y* is a function of complexity in the transformation g(x). Let us start by considering a simple relationship Y=g(x)=2x+1, The transformation y=g(x) is presented graphically in Figure below Consider the pdf of Y,*F*(y); it is defined by: F**y**(y) =P(Y≤y)

The region defined by *Y y*in the range space *RY* covers the heavier portion of the transformation curve, as shown in Figure below, which, in the range space *RX*, corresponds to the region g(X ) ≤y, or X≤ g**-1**(y), where



**Figure:** Transformation defined by Equation Y=g(x)=2x+1is the inverse function of *g*(*x*), or the solution for *x* in Equation Y=g(x)=2x+1 in terms of *y*. Hence,

This gives the relationship between the PDF of *X* and that of Y, our desired result.

The relationship between the pdfs of Xand Yare obtained by differentiating both sides of the above Equation with respect to y. We have:

**Theorem:** Let *X* be a continuous random variable and Y *=*g(x) where g(x) is continuous in *X* and strictly monotone. Then

Where|u| denotes the absolute value of u.

**Example**:

1. Let *X* has the following *p.d.f.*, Then find the *p.d.f.* of *Y* = .

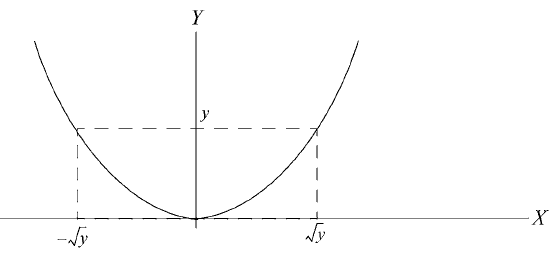
**Solution:**

1. Let *X* have a p.d.f. *f*(*x*) and *Y* = ,find𝑝.𝑑.𝑓.of Y.

**Solution:**

**Example:** Consider the quadratic function *Y* = *X*2. The plot of Yagainst Xis shown in

Figure below where we see that for one value of *Y* there are two values of *X*, namelyand. Thus, the CDF of *Y* is given by



**Figure:** Plot of Y = X2

Let U== thus, and

If *f*(*x*) is an even function, then f(x) = *f*(−*x*) and *F*(−*x*) = 1 − *F*(*x*). Thus, we have

**Example:** Find the PDF of the random variable *Y* = , where *X* is the standard normal random variable.

**Solution:** Since the PDF of *X* is given by *f*(*x*) =which is an even function, we have that and

Therefore, if we let *u* =, then

**Exercise:**

1. The random variable *X* has the following PDF

If we define Y = 2X +3, what is the PDF of Y?

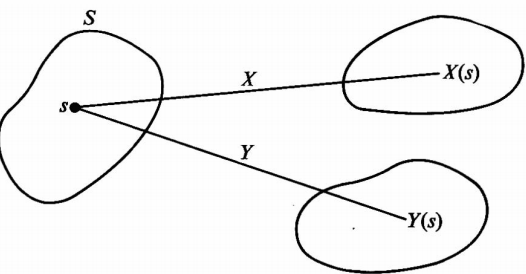
1. Let X be a random variable with pdff , let Y=3x+1 then find the pdf of Y, g(y)?
2. Let X be a random variable with pdff , let Y= then find the pdf of Y,g(y)?
3. Let X be a random variable with pdff , let Y= +2 then find the pdf of Y,g(y)?

Chapter-7

Two-dimensional random variable

What is Two-dimensional random variable?

**Definition:** Let E be an experiment and S a sample space associated with E. Let X=X(s) and Y= Y(s) be two functions each assigning a real number to each outcomes sS.We call (X, Y) a two-dimensional random variable (sometimes called a random vector).



Example:

1. recording the amount of precipitate (p) and volume of gas (q) for a given locality, (p,q)
2. Observing the rainfall (R) and temperature (T) of certain town (R, T)

**Note**: If the possible values (x, y) are finite countable, then (x, y) is called two-dimensional discrete random variable.

If (x, y) assumes all values in a specified region R in the xy-plane then (x,y) is called two-dimensional continuous random variable.

**Joint probability distribution**

If X and Y are two random variable the probability distribution for their simultaneous occurrences can be represented by a function f(x,y), for any pair values (X,Y) within the range of the random variable X and Y. This function is known as joint probability distribution (X, Y).

**Definition: 1**. Let (x, y) is a two-dimensional discrete random variable with each possible outcome (Xi, Yi) we associate a number (Xi, Yi) representing P(X=Xi, Y=Yi) and satisfying the following conditions.

The function P is joint probability mass function.

The set of triples, i=j=1, 2,3,…, is the joint probability distribution of .

**Definition: 2**. Let (X, Y) be a continuous random variable. If it assuming all values in some region R of the Euclidean plane. Let (X, Y) be two-dimensional continuous random variable then the joint probability density function f is a function satisfying the following conditions:

Example:

1. Two production lines 1 and 2 have a capacity of producing 5 and 3 items per day respectively, assume the numbers of items produced by each line is a random variable. Let be a two-dimensional random variable yielding the number of items produced by line 1 and line 2 respectively.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Y|X | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 0 | 0.01 | 0.03 | 0.05 | 0.07 | 0.09 |
| 1 | 0.01 | 0.02 | 0.04 | 0.05 | 0.06 | 0.08 |
| 2 | 0.01 | 0.03 | 0.05 | 0.05 | 0.05 | 0.06 |
| 3 | 0.01 | 0.02 | 0.04 | 0.06 | 0.06 | 0.05 |

1. Show that is a legitimate probability function of .
2. What is the probability that both lines produce the same numbers of items?
3. What is the probability that more items are produce by line 2?
4. Let be a two-dimensional discrete random variable with
5. Show that is probability pmf?
6. Find
7. Suppose that is a two-dimensional random variable with joint pdf is given by
8. Show that
9. Find
10. Given
11. Find the value of K
12. Find
13. Find

**Solution**

1. A.
2. =0.01+0.03+…+0.06+0.05

1B.

1, C.

=

1. A.

=

=

=

=

2B.

* Consider

=

=

⇒

=

=

1. A
2. ⇒

3B.

**Definition**: Let (X,Y) be a two-dimensional random variable. The cumulative distribution function (cdf) F of the two-dimensional random variable (X, Y) is defined by

Properties of

**Example**: Suppose that the two-dimensional continuous random variable (X, Y) has jointpdf given by

1. Determine the joint CDF of (x,y)
2. Obtaine F(1,2) and F(1,1.5)
3. Obtaine

**Solution**

=

Therefore

=

=

=

***Example:*** If the joint probability density of and is given by,

Find the joint cumulative distribution function of these two random variables.

***Solution:***

***Example:*** Find the joint probability density function of the two random variables and whose joint distribution function is given by,

Also use the joint probability density to determine

**Solution*:***Since partial differentiation yields,for and 0 elsewhere, we find that the joint probability density and is given by,

Thus, integration yields,

Marginal and Conditional Probability Distributions

**Definition*:*** If and are discrete two-dimensional random variables and is the valueof their joint probability distribution at:

* for each within the range of : is the marginal probability mass function of .
* for each within the range of : isthe marginal probability mass function of *.*

**Definition*:*** If and are continuous two-dimensional random variables and is the value of their joint probability density at:

* for each : is the marginal probability density function of .
* for each : isthe marginal probability density function of *.*

NB

**Example**:

1. let (x,y) be the joint probability function given by

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Y|X | 0 | 1 | 2 |  |
| 0 | 0.25 | 0.15 | 0.1 | 0.50 |
| 1 | 0.1 | 0.08 | 0.1 | 0.28 |
| 2 | 0.05 | 0.07 | 0.1 | 0.22 |
|  | 0.40 | 0.30 | 0.30 | 1 |

Find the marginal distribution function of X and Y

**Solution:**

1. The marginal distribution function of X is

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| X= | 0 | 1 | 2 | Total |
|  | 0.40 | 0.30 | 0.30 | 1 |

1. The marginal distribution function of Y are

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Y= | 0 | 1 | 2 | Total |
|  | 0.50 | 0.28 | 0.22 | 1 |

1. The joint PMF of two random variables *X* and *Y* is given by

P*X,Y*(*x*, *y*) =

Wherek is a constant:

1. What is the value of K?
2. Find the marginal PMFs of *X* and *Y*.

Solution:

1. To evaluate *k*, we remember that

*==1*

*Thus==1*

*=*

*=*

1. Themarginal pmf’s are

*=*

*=*

*= =*

*=*

1. Let the joint pdf of (x,y) is given by

Find the marginal distribution function of X and Y

**Solution:**

1. The marginal distribution density function of X is

=

Therefore so that is, X is uniformly distributed over [0, 1]

1. The marginal distribution density function of X is

=

Therefore so that is, y is uniformly distributed over [0, 1]

1. Given the joint probability density,

Find the marginal densities of and .

**Solution*:*** Performing the necessary integrations, we get

Likewise,

**Exercise**: Let the joint pdf of (x,y) is given by

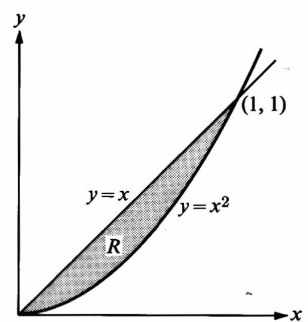
Find the marginal distribution function of X and Y

**Definition**: We say that the two-dimensional continuous random variable is uniformly distributed over a region R in the Euclidean plane if

Because of the requirement the above implies that the constant equals. We are assuming that R is a region with finite, nonzero area. i.e

**Note**: This definition represents the two-dimensional analog to the one-dimensional uniformly distributed random variable.

**Example**: Suppose that the two-dimensional random variable (X, Y) is uniformly distributed over the region R., Rx={(x, y): 0<y<x<1,0}



1. Find the value of C
2. Find the marginal pdf of X and Y

Solution:

1. we find that

Therefore the pdf is given by

1. In the following equations we find the marginal pdf's of X and Y
2. The marginal distribution density function of X is
3. The marginal distribution density function of y is

Conditional Probability Distributions

**Definition**: Let (X, Y) be a continuous two-dimensional random variable with joint pmf. Let and be the marginal pmf‘s of X and Y, respectively. Then the conditional pmf of X given that Y = y is defined by>0 and the conditional pmf of Y given that X= is defined by>0

**Definition**: Let (X, Y) be a continuous two-dimensional random variable with joint pdf. Let and be the marginal pdf‘s of X and Y, respectively. Then the conditional pdf of X given that Y = y is defined by >0 and the conditional pdf of Y given that X = x is defined by >0

**Generally:** If X and Y have a joint distribution with joint density or probability functionthen the marginal distribution of X has a probability function or density function denoted which is equal to in the continuous case andin the discrete case.The density function for the marginal distribution of Y is found in a similar way; is equal to either in the continuous case and in the discrete case.

Example:

1. The joint PMF of two random variables *X* and *Y* is given by
2. What is the conditional PMF of *Y* given that*X*?
3. What is the conditional PMF of *X* given that *Y*?

**Solution:**we know that the marginal PMFs are given by

*x=1, 2* and

, *y=1, 2*

Thus, the conditional PMFs are given by

1. the conditional PMF of Ygiven thatXis
2. the conditional PMF of Xgiven that Y is
3. Suppose that the two-dimensional continuous random variable (X, Y) has joint pdf given by

Determine the conditional pdf of Xgiven that Y and the conditional pdf of Ygiven that X

**Solution:** To determine the conditional PDFs, we first evaluate the marginal pdf’s, which are given by

Hence,

**Exercise**: verify that is pdf

1. Two random variables Xand Yhave the following joint PDF:

Determine the conditional pdf of Xgiven thatY and the conditional pdf of Ygiven thatX.

**Solution:** To determine the conditional PDFs, we first evaluate the marginal pdf’s, which are given by *=*

==

=

Let u = x, which means that *du = dx*; and let *dv = e****−x(y+1)****dx*, which means that

. Integrating by parts we obtain

*=0−*

*=*

Thus, the conditional PDFs are given by:

**Independent random variables**

**Definition:** Let denote a continuous bivariate random variable with joint pdf and marginal pdfs and. Then X and Y are called independent random variables if, for every x ∈ X and y ∈ Y i.e. and Let denote a discrete bivariate random variable with joint pmf and marginal pmf and. Then X and Y are called independent random variables if, for every x ∈ X and y ∈ Y i.e..

**Example:** ,

then X and Y are not independent

, then X and Y are independent

1. Suppose (X, Y) are discrete random variables with probability function given by

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Y x | -1 | 0 | 1 | *Px(x)* |
| -1 | 1/8 | 1/8 | 1/8 |  |
| 0 | 1/8 | 0 | 1/8 |  |
| 1 | 1/8 | 1/8 | 1/8 |  |
| *PY(y)* |  |  |  |  |

1. find the marginal pmf of X and Y
2. Are X and Y independent?
3. Given the joint probability density,
4. Find the marginal densities of X and Y.
5. Are X and Y independent?

***Solution:***

1. Performing the necessary integrations, we get

Likewise,

1. X and Y are not independent because

**Function of two dimensional random variables**

Let *X* and *Y* be two random variables with a given joint PDF Assume that *U* and *V* are two functions of *X* and *Y*; that is, *U* = and *V* = . Sometimes it is necessary to obtain the joint PDF of *U* and *W*, in terms of the pdf’s of *X* and *Y*.

It can be shown that is given by:

Where are real solutions of the equations U= and

V=; and J(x, y) is called the Jacobian of the transformation

**Example:**Let U = = X + Y and V= = X − Y. Find

**Solution** The unique solution to the equations U = x + y and V= x − y is x =and y = . Thus, there is only one set of solutions. Since

= =-2

We obtain

**Exercise**:

1. Find if U = + and V=
2. Let X and Y have the joint probability density function

Let. Find the probability that (X, Y) falls into A

1. Let X and Y have the joint probability function
2. Find the conditional probability function of given that .
3. Find the conditional probability function of , given that .
4. Let X and Y have the joint probability density function
5. Find the marginal probability density functions.
6. ?
7. Calculate