Technical Lemmas and extended proofs

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The proof of Lemma 2.1:

Proof. Recalling the defition of the BESS power law, for $B \in \mathcal{B}(S)$

$$\Pi_{n+1}(A) = \mathbb{P}(B_{n+1} \in A). \tag{1}$$

We may express any measurable set as a random variable using indicator functions:

$$\Pi_{n+1} = \mathbb{P}\{B_{n+1} \in A\} = \mathbb{E}\left[\mathbf{1}_{\{B_{n+1} \in A\}}\right] \tag{2}$$

By the Law of Total Probability (Tower Rule), we may write:

$$\mathbb{E}\left[\mathbf{1}_{\{B_{n+1} \in A\}}\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{B_{n+1} \in A\}} | \mathcal{F}_n\right]\right] \tag{3}$$

Once more by the Tower Property, we may express conditional expectations in terms of the sigma algebra given by the battery. The inner term can be manipulated

$$\mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{B_{n+1} \in A\}} | \mathcal{F}_n\right] | \sigma(B_n)\right] = \mathbb{E}\left[\mathbf{1}_{\{B_{n+1} \in A\}} | \sigma(B_n)\right]. \tag{4}$$

By the Markov Property, this inner expectation is given by the regular conditional probability

$$\mathbb{E}\left[\mathbf{1}_{\{B_{n+1}\in A\}}|\sigma(B_n)\right] = \mathbb{E}\left[\mathbf{1}_{\{B_{n+1}\in A\}}|B_n = s\right] = \kappa(s, A). \tag{5}$$

Hence, we substitute back into Eq.(3), with the kernel as a random variable

$$\Pi_{n+1}(A) = \mathbb{E}\left[\kappa(B_n, A)\right] \tag{6}$$

Since $\kappa(s, A)$ is measurable and bounded by definition, then

$$\mathbb{E}\left[\kappa(s, A)\right] = \int_S \kappa(s, A)\Pi_n(ds). \tag{7}$$

If we assume $S \subseteq \mathbb{R}$, and we choose $A = (-\infty, b]$, then

$$G_{n+1}(b) = \Pi_{n+1}((-\infty, b]) = \int_S \kappa(s, (-\infty, b])\Pi_n(ds), \tag{8}$$

so that the CDF evolution equation is given by

$$G_{n+1}(b) = \int_S F_\kappa(b|s)\Pi_n(ds). \tag{9}$$

□

The proof of remark 2.2:

Proof. Let $S \subseteq \mathbb{R}$ be a measurable state space with Borel sigma-algebra $\mathcal{B}(S)$. We assume that the probability measures Π_n and $\kappa(s, \cdot)$ have no singular continuous part and that the atoms lie in a fixed countable set, $C \subset S$. We had defined the reference measure:

$$\rho := \lambda_{S\setminus C} + \sum_{c\in C} \delta_c. \tag{10}$$

The Lebesgue Decomposition Theorem on $\mathbb{R}$ states that, for any finite measure μ on $S \subseteq \mathbb{R}$, there exist unique measures $\mu_{a.c.} \ll \lambda$ and $\mu_s \perp \lambda$ such that $\mu = \mu_{a.c.} + \mu_s$. Furthermore, μ_s splits uniquely into an atomic part, $\mu_{a.t.}$

and a singular-continuous part, $\mu_{s.c.}$. We assume that $\mu_{s.c.} = 0$ for all relevant measures $\mu \in \{\Pi_n, \kappa(s, \cdot)\}$. Hence, for each μ,

$$\mu = \mu_{a.c.} + \mu_{a.t.}, \quad \mu_{a.c.} \ll \lambda, \; \mu_{a.t.} = \sum_{c \in C} \mu(\{c\})\delta_c. \tag{11}$$

The decomposition then becomes,

$$\mu(A) = \int_{A \cap (S \setminus C)} f_\mu d\lambda + \sum_{c \in C} \mu(\{c\})\delta_c(A), \tag{12}$$

where f_μ is given by the RN derivative:

$$f_\mu = \frac{d\mu_{a.c.}}{d\lambda}. \tag{13}$$

Given any $A \in \mathcal{B}(S)$ with $\rho(A) = 0$. Then, by the definition of $\rho(A) = \lambda(A \cap (S \setminus C)) + \sum_{c \in C} \delta_c(A) = 0$,

$$\lambda(A \cap (S \setminus C)) = 0, \tag{14}$$

$$\delta_c(A) = 0 \text{ for every } c \in C. \tag{15}$$

Introducing such a set into the general decomposed measure μ:

$$\mu(A) = \int_{A \cap (S \setminus C)} f_\mu d\lambda + \sum_{c \in C} \mu(\{c\})\delta_c(A) = 0 \tag{16}$$

since $\lambda(A \cap (S \setminus C)) = 0$ and $\delta_c(A) = 0 \; \forall c$. Since this holds for every A with $\rho(A) = 0$, then $\mu \ll \rho$. Since $\mu \in \{\Pi_n, \kappa(s, \cdot)\}$ was arbitrary, $\Pi_n \ll \rho$ and $\kappa(s, \cdot) \ll \rho$, $\square$