Numerical solution to the integral equation

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A common numerical method for solving integral equations is the Nyström method. In this method, the continuous problem is broken into discrete intervals and the integral is replaced by a weighted sum. The kernel follows the familiar Toeplitz structure.

A discrete Toeplitz matrix is an $N \times N$ matrix defined by

$$A_{ij} = c_{i-j}, \tag{1}$$

for constants $c_{1-N}, \ldots, c_{N-1}$ and a total number of elements N. This structure gives the Toeplitz matrix its characteristic diagonal form. The discrete convolution operation can be written as a matrix–vector multiplication, where one of the inputs is converted into a Toeplitz matrix. Whenever there exists a bi-infinite Toeplitz matrix, for which the index set of the constants is infinite in nature (for example, the natural numbers), the matrix induces a linear operator. This linear operator is known as a Toeplitz operator.

In our case, the kernel only depends on the difference $b - s$. Consequently, K is a Toeplitz operator on the half–line. This becomes apparent when the problem is discretized for a numerical solution and the operator is approximated in matrix form. Define the discrete grid

$$b_i = ih, \qquad h = \frac{B_{\max}}{N}, \qquad i = 0, \ldots, N.$$

Next, compute the trapezoidal weights for the quadrature,

$$\mathbf{w} = \left(\frac{1}{2}, 1, \ldots, 1, \frac{1}{2}\right). \tag{2}$$

This turns every integral into a weighted sum, namely

$$\int_0^{B_{\max}} (\cdot)\,\mathrm{d}s = h\sum_j w_j(\cdot)_j, \tag{3}$$

where B_max is the upper limit that should numerically be representative of infinity. The integral operator, $\mathbf{K}$, can be constructed by use of this trapezoid quadrature; the Toeplitz structure given by $b - s$ should be recovered by defining a difference grid. The difference grid can be calculated by performing substractions between values in b_i (the battery value discrete grid):

$$D_{ij} = (i - j)h \tag{4}$$

where we recall that i and j range from 0 to N. Then the $\mathbf{K}$ operator can be represented by:

$$(\mathbf{K}g) \approx (\mathbf{K}^N g) = h\sum_{j=0}^{N} f((i-j)h)w_j g_j, \tag{5}$$

where f is the given form of the PDF, f_X. One may define the vector form of $f_X(b)$ by evaluating it using the discrete grid:

$$\vec{f} = f_X(b_i). \tag{6}$$

Then the linear system defined earlier can be solved using a numerical resolvent:

$$\begin{aligned} (\mathbf{I} - \mathbf{K}^N)\vec{u} &= \vec{f} \\ \vec{u} &\approx (\mathbf{I} - \mathbf{K}^N)^{-1}\vec{f}. \end{aligned} \tag{7}$$

The value of p_0 should be determined using the same quadrature and the normalization condition. Discretely, the integral in equation (59) can be represented using the quadrature.

$$\Omega^{(N)} = h\sum_j w_j u_j, \tag{8}$$

where u_j are the elements of $\vec{u}$. Then, the point mass is given by

$$p_= \approx \frac{1}{1 + \Omega^{(N)}} \tag{9}$$

An alternative method stems from from the analytical convolution equation and also insolves using the quadrature approximation of the kernel, $\mathbf{K}^N$. The **Piccard iteration method** performs the following operation.

$$\vec{u}^{(k+1)} = \vec{f} + \mathbf{K}^N \vec{u}^{(k)} \tag{10}$$

where $k = 1, 2..., K$ and K is the total number of iterations. Therefore, the Piccard method is implementing the approximation:

$$\vec{u} \approx \sum_{n=0}^{K} \vec{f}^{*(n+1)} = \vec{f} + \vec{f} * \vec{f} + \vec{f} * \vec{f} * \vec{f} + ... \tag{11}$$