Neumann Series requirements for convergence: Extended derivation of the relation $||K|| \leq M_X(\theta)$

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Neumann series generalizes geometric series to operators in the form

$$[I - K]^{-1} = \sum_{n=0}^{\infty} K^n \tag{1}$$

Convergence of the series is guaranteed if and only if the operator lives in a banach space and if its norm is bounded to 1 as $||K|| < 1$. As the markov kernel follows the form of a Laplace distribution (SL) the (ADD A LOGICAL EXPLANATION OF THE PROPOSED WEIGHTED SPACE). Assuming the operator K lives in this space, we define the norm as

$$||g(\tilde{b})||_\theta = \sup_{\tilde{b} \geq 0} \; e^{\theta \tilde{b}} |g(\tilde{b})| \tag{2}$$

From this definition, then, the following is true:

$$|g(b)| \leq e^{-\theta b} ||g(b)||_\theta \tag{3}$$

We can define the norm of the applied operator as

$$||(Kg)(b)|| = e^{\theta b} |(Kg)(b)| = e^{\theta b} \left| \int_0^\infty f_x(b - s) g(s) ds \right| \tag{4}$$

From the *Integral Inequality-abolute value* we may express the norm of the operator as follows

$$||(Kg)(b)||_\theta = e^{\theta b} \left| \int_0^\infty f_X(b - s) g(s) ds \right| \; \leq \; e^{\theta b} \int_0^\infty |f_X(b - s)| \cdot |g(b)| ds \tag{5}$$

Invoking equation (3) we can express the norm of the operator as

$$||(Kg)(b)||_\theta \ \leq \ e^{\theta b} \int_0^\infty |f_X(b-s)| \cdot \left[e^{-\theta s} ||g(s)|| \right] ds \tag{6}$$

Using the next change of variables $x = b - s$ we then have

$$||(Kg)(b)|| \leq ||g(s)||_\theta \int_{-\infty}^{b} f_X(x) e^{\theta x} dx \tag{7}$$

As the markov kernel is a density function, must be positive, so we know there will not be a change of sign as we extend the integration domain $f_X(x)\ e^{\theta x} \geq 0$ and therefore, the integral will increase. Then, the following is true.

$$||(Kg)(b)||_\theta \leq ||g(s)||_\theta \int_{-\infty}^{b} f_X(x) e^{\theta x} dx \ \leq \ ||g(s)||_\theta \int_{-\infty}^{\infty} f_X(x) e^{\theta x} dx \tag{8}$$

Is easy to see that, at extending the integration domain, the integral becomes the moment generating function of the random variable X

$$\begin{gathered} ||Kg||_\theta \leq ||g||_\theta \ M_X(\theta) \\ ||K|| \leq M_X(\theta) \end{gathered} \tag{9}$$

As mentioned in section 2.9, the distribution function of the effective increment $X = -(Y + a)$ is expressed as a shifted Laplace distribution $f_X(x) = \beta/2 e^{-|\beta x|}$ where $\mathbb{E}[X] = -a$. Therefore, the specific form of the MGF is

$$||K||_\theta \leq \frac{e^{-\theta a}}{1 - \theta^2/\beta^2} \ |\theta| < \beta \tag{10}$$

In accordance with the mathematical developement presented in section 2.7, using normalized variables, equation 10 simplifies to

$$||K||_\theta \leq \frac{e^{-\theta \tilde{a}}}{1 - \theta^2} \ |\theta| < 1 \tag{11}$$

This relation itself does not proof that the *MGF* is bounded as requirement (2). It range is bounded as $[m, \infty)$ where $0 < m < 1$ given negative drift. Therefore, there must exist a neighborhood with the form

$$e^{\tilde{a}\theta} < 1 - \theta^2 \tag{12}$$

inside its θ-domain where equirement (2) $||K|| < 1$ is satisfied. Which guarantees the series converges within this domain.

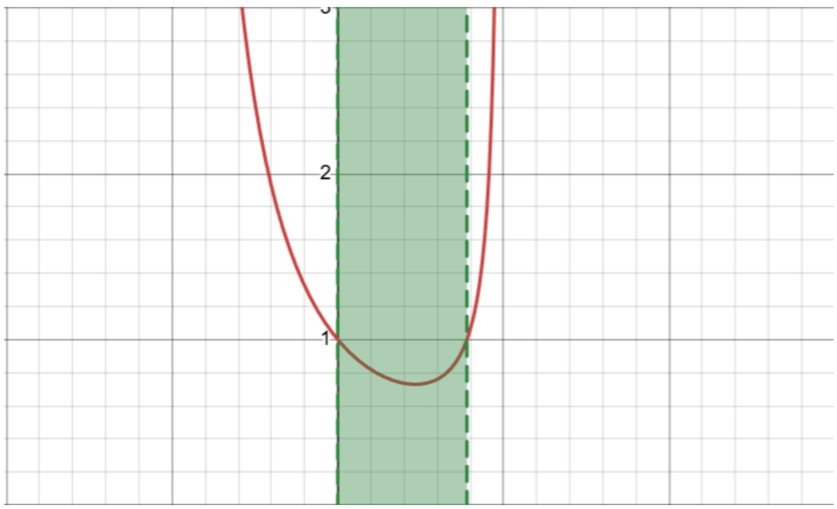

Figure 1: $M_X(\theta)$ with negative drift (red) showing the neighborhood for which the series converges