

# Neumann Series requirements for convergence: Extended derivation of the relation $\|K\| \leq M_X(\theta)$

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Neumann series generalizes geometric series to operators in the form

$$[I - K]^{-1} = \sum_{n=0}^{\infty} K^n \quad (1)$$

Convergence of the series is guaranteed if and only if the operator lives in a banach space and if its norm is bounded to 1 as  $\|K\| < 1$ . As the markov kernel follows the form of a Laplace distribution (SL) the (ADD A LOGICAL EXPLANATION OF THE PROPOSED WEIGHTED SPACE). Assuming the operator  $K$  lives in this space, we define the norm as

$$\|g(\tilde{b})\|_{\theta} = \sup_{\tilde{b} \geq 0} e^{\theta\tilde{b}} |g(\tilde{b})| \quad (2)$$

From this definition, then, the following is true:

$$|g(b)| \leq e^{-\theta b} \|g(b)\|_{\theta} \quad (3)$$

We can define the norm of the applied operator as

$$\|(Kg)(b)\| = e^{\theta b} |(Kg)(b)| = e^{\theta b} \left| \int_0^{\infty} f_x(b-s) g(s) ds \right| \quad (4)$$

From the *Integral Inequality-absolute value* we may express the norm of the operator as follows

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$$\|(Kg)(b)\|_\theta = e^{\theta b} \left| \int_0^\infty f_X(b-s)g(s)ds \right| \leq e^{\theta b} \int_0^\infty |f_X(b-s)| \cdot |g(s)| ds \quad (5)$$

Invoking equation (3) we can express the norm of the operator as

$$\|(Kg)(b)\|_\theta \leq e^{\theta b} \int_0^\infty |f_X(b-s)| \cdot [e^{-\theta s} \|g(s)\|] ds \quad (6)$$

Using the next change of variables  $x = b - s$  we then have

$$\|(Kg)(b)\| \leq \|g(s)\|_\theta \int_{-\infty}^b f_X(x)e^{\theta x} dx \quad (7)$$

As the markov kernel is a density function, must be positive, so we know there will not be a change of sign as we extend the integration domain  $f_X(x) e^{\theta x} \geq 0$  and therefore, the integral will increase. Then, the following is true.

$$\|(Kg)(b)\|_\theta \leq \|g(s)\|_\theta \int_{-\infty}^b f_X(x)e^{\theta x} dx \leq \|g(s)\|_\theta \int_{-\infty}^\infty f_X(x)e^{\theta x} dx \quad (8)$$

Is easy to see that, at extending the integration domain, the integral becomes the moment generating function of the random variable  $X$

$$\begin{aligned} \|Kg\|_\theta &\leq \|g\|_\theta M_X(\theta) \\ \|K\| &\leq M_X(\theta) \end{aligned} \quad (9)$$

As mentioned in section 2.9, the distribution function of the effective increment  $X = -(Y + a)$  is expressed as a shifted Laplace distribution  $f_X(x) = \beta/2e^{-|\beta x|}$  where  $\mathbb{E}[X] = -a$ . Therefore, the specific form of the MGF is

$$\|K\|_\theta \leq \frac{e^{-\theta a}}{1 - \theta^2/\beta^2} |\theta| < \beta \quad (10)$$

In accordance with the mathematical developement presented in section 2.7, using normalized variables, equation 10 simplifies to

$$\|K\|_\theta \leq \frac{e^{-\theta \tilde{a}}}{1 - \theta^2} |\theta| < 1 \quad (11)$$

This relation itself does not proof that the *MGF* is bounded as requirement (2). Its range is bounded as  $[m, \infty)$  where  $0 < m < 1$  given negative drift. Therefore, there must exist a neighborhood with the form

$$e^{\tilde{a}\theta} < 1 - \theta^2 \quad (12)$$

inside its  $\theta$ -domain where requirement (2)  $\|K\| < 1$  is satisfied. Which guarantees the series converges within this domain.

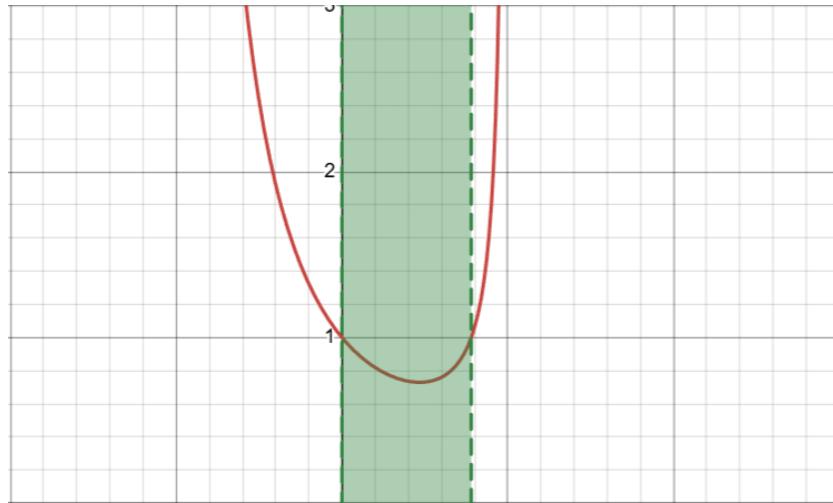


Figure 1:  $M_X(\theta)$  with negative drift (red) showing the neighborhood for which the series converges