

Technical Lemmas and extended proofs

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The proof of Lemma 2.1:

Proof. Recalling the definition of the BESS power law, for $B \in \mathcal{B}(S)$

$$\Pi_{n+1}(A) = \mathbb{P}(B_{n+1} \in A). \quad (1)$$

We may express any measurable set as a random variable using indicator functions:

$$\Pi_{n+1} = \mathbb{P}\{B_{n+1} \in A\} = \mathbb{E} [\mathbf{1}_{\{B_{n+1} \in A\}}] \quad (2)$$

By the Law of Total Probability (Tower Rule), we may write:

$$\mathbb{E} [\mathbf{1}_{\{B_{n+1} \in A\}}] = \mathbb{E} [\mathbb{E} [\mathbf{1}_{\{B_{n+1} \in A\}} | \mathcal{F}_n]] \quad (3)$$

Once more by the Tower Property, we may express conditional expectations in terms of the sigma algebra given by the battery. The inner term can be manipulated

$$\mathbb{E} [\mathbb{E} [\mathbf{1}_{\{B_{n+1} \in A\}} | \mathcal{F}_n] | \sigma(B_n)] = \mathbb{E} [\mathbf{1}_{\{B_{n+1} \in A\}} | \sigma(B_n)] . \quad (4)$$

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By the Markov Property, this inner expectation is given by the regular conditional probability

$$\mathbb{E} [\mathbf{1}_{\{B_{n+1} \in A\}} | \sigma(B_n)] = \mathbb{E} [\mathbf{1}_{\{B_{n+1} \in A\}} | B_n = s] = \kappa(s, A). \quad (5)$$

Hence, we substitute back into Eq.(3), with the kernel as a random variable

$$\Pi_{n+1}(A) = \mathbb{E} [\kappa(B_n, A)] \quad (6)$$

Since $\kappa(s, A)$ is measurable and bounded by definition, then

$$\mathbb{E} [\kappa(s, A)] = \int_S \kappa(s, A) \Pi_n(ds). \quad (7)$$

If we assume $S \subseteq \mathbb{R}$, and we choose $A = (-\infty, b]$, then

$$G_{n+1}(b) = \Pi_{n+1}((-\infty, b]) = \int_S \kappa(s, (-\infty, b]) \Pi_n(ds), \quad (8)$$

so that the CDF evolution equation is given by

$$G_{n+1}(b) = \int_S F_\kappa(b|s) \Pi_n(ds). \quad (9)$$

□

The proof of remark 2.2:

Proof. Let $S \subseteq \mathbb{R}$ be a measurable state space with Borel sigma-algebra $\mathcal{B}(S)$. We assume that the probability measures Π_n and $\kappa(s, \cdot)$ have no singular continuous part and that the atoms lie in a fixed countable set, $C \subset S$. We had defined the reference measure:

$$\rho := \lambda_{S \setminus C} + \sum_{c \in C} \delta_c. \quad (10)$$

The Lebesgue Decomposition Theorem on \mathbb{R} states that, for any finite measure μ on $S \subseteq \mathbb{R}$, there exist unique measures $\mu_{a.c.} \ll \lambda$ and $\mu_s \perp \lambda$ such that $\mu = \mu_{a.c.} + \mu_s$. Furthermore, μ_s splits uniquely into an atomic part, $\mu_{a.t.}$ and a singular-continuous part, $\mu_{s.c.}$. We assume that $\mu_{s.c.} = 0$ for all relevant measures $\mu \in \{\Pi_n, \kappa(s, \cdot)\}$. Hence, for each μ ,

$$\mu = \mu_{a.c.} + \mu_{a.t.}, \quad \mu_{a.c.} \ll \lambda, \quad \mu_{a.t.} = \sum_{c \in C} \mu(\{c\})\delta_c. \quad (11)$$

The decomposition then becomes,

$$\mu(A) = \int_{A \cap (S \setminus C)} f_\mu d\lambda + \sum_{c \in C} \mu(\{c\})\delta_c(A), \quad (12)$$

where f_μ is given by the RN derivative:

$$f_\mu = \frac{d\mu_{a.c.}}{d\lambda}. \quad (13)$$

Given any $A \in \mathcal{B}(S)$ with $\rho(A) = 0$. Then, by the definition of $\rho(A) = \lambda(A \cap (S \setminus C)) + \sum_{c \in C} \delta_c(A) = 0$,

$$\lambda(A \cap (S \setminus C)) = 0, \quad (14)$$

$$\delta_c(A) = 0 \text{ for every } c \in C. \quad (15)$$

Introducing such a set into the general decomposed measure μ :

$$\mu(A) = \int_{A \cap (S \setminus C)} f_\mu d\lambda + \sum_{c \in C} \mu(\{c\})\delta_c(A) = 0 \quad (16)$$

since $\lambda(A \cap (S \setminus C)) = 0$ and $\delta_c(A) = 0 \forall c$. Since this holds for every A with $\rho(A) = 0$, then $\mu \ll \rho$. Since $\mu \in \{\Pi_n, \kappa(s, \cdot)\}$ was arbitrary, $\Pi_n \ll \rho$ and $\kappa(s, \cdot) \ll \rho$, \square