Dynamic Models - Showcase

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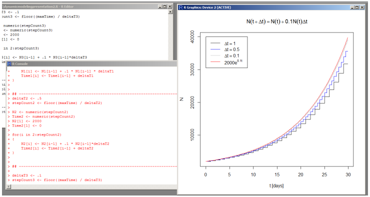
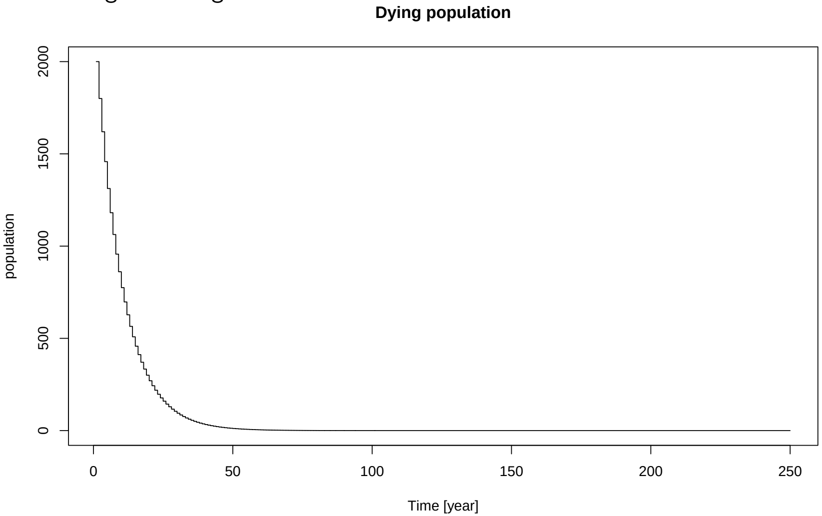
January 18, 2022



Foxes in paradise

Foxes have to eat rabbits to survive. But in paradise this is a *no go*.
Therefore their population decreases each year by a constant rate: $\frac{\Delta N}{N} = -.1$.
Calculating in R we get



Real world rabbits

... meet and ~~eat~~ are eaten by foxes. Let `R` denote the number of rabbits and `F` the number of foxes, then we get the formulas:

$$\begin{aligned} \Delta R_n &= \quad .1R_n - .2R_nF_n \\ \Delta F_n &= -.1F_n + .15R_nF_n \end{aligned}$$

If we enter these formulas in our R loop (and do all the initialisation), we can calculate the number of rabbits and foxes over time.

Foxes and rabbits

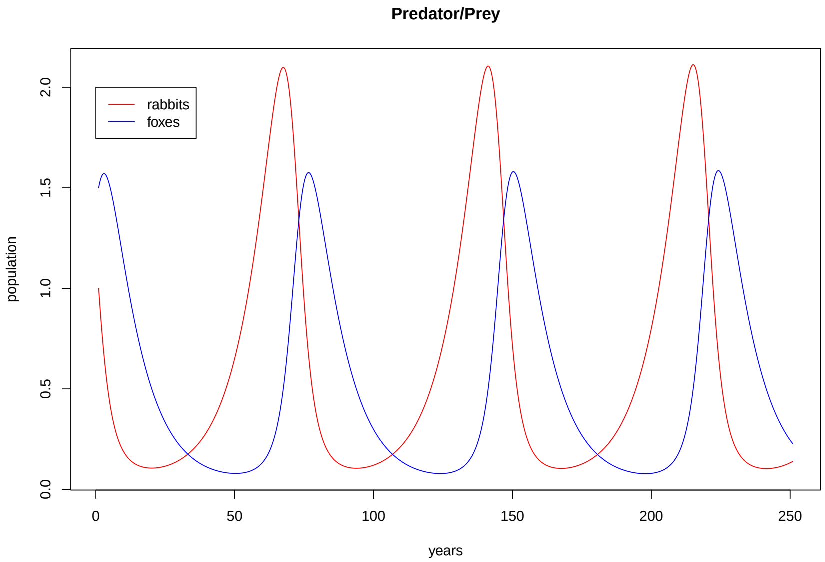
```r
EndTime <- 250
steps <- 100
A <- numeric(EndTime*steps)
B <- numeric(EndTime*steps)
deltaA <- numeric(EndTime)
deltaB <- numeric(EndTime)

A[1] <- 1.0
B[1] <- 1.5

for(i in 2:(steps*EndTime))
{
  deltaA[i-1] <-   .1 * A[i-1] - .20 *  A[i-1]*B[i-1]
  deltaB[i-1] <-  -.1 * B[i-1] + .15 *  A[i-1]*B[i-1]

  A[i] <- A[i-1] + deltaA[i-1]*1/steps
  B[i] <- B[i-1] + deltaB[i-1]*1/steps
}
```

The plot


Predator/Prey
rabbits
foxes
population
years
0.0
0.5
1.0
1.5
2.0
0
50
100
150
200
250

Not part of the exercises - just for your information:

Fibonacci numbers as a dynamic model?

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From the Fibonacci formula

$$t_{n+1} = t_n + t_{n-1}$$

we get

$$\Delta t_n = t_{n+1} - t_n = t_{n-1}$$

Difference to our dynamic models:

- Daily increment depends on the state of the day before

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Difference to our dynamic models:

- Daily increment depends on the state of the day before
- We need initial values for first two days

In the rabbit model, the ratio between rate and state is constant $\frac{\Delta N}{N} = c$.

Let's look at the ratio for fibonacci numbers:

$$\frac{\Delta t_n}{t_n} = \frac{t_{n-1}}{t_n}$$

We calculate this ratio by R:

```r
t[1:(m-1)]/t[2:m]
```

```
#  [1] 1.0000000 0.5000000 0.6666667 0.6000000 0.6250000 0.6153846 0.6190476
#  [8] 0.6176471 0.6181818 0.6179775 0.6180556 0.6180258 0.6180371 0.6180328
# [15] 0.6180344 0.6180338 0.6180341 0.6180340 0.6180340 0.6180340 0.6180340
# [22] 0.6180340 0.6180340 0.6180340 0.6180340 0.6180340 0.6180340 0.6180340
# [29] 0.6180340 0.6180340 0.6180340 0.6180340 0.6180340 0.6180340 0.6180340
# [36] 0.6180340 0.6180340 0.6180340 0.6180340 0.6180340 0.6180340 0.6180340
# [43] 0.6180340 0.6180340 0.6180340 0.6180340 0.6180340 0.6180340 0.6180340
```

Fibonacci numbers approximated by dynamic model

For the fibonacci numbers above stage 18, the ratio becomes nearly constant 0.618034.

For higher stages, we can calculate the fibonacci numbers as a dynamic model:

$$t_{19} = 4181$$

$$\Delta t_n \approx 0.618034 t_n \quad \text{for} \quad n \geq 19$$

By the way: italian mathematician Fibonacci used his numbers to calculate rabbit population.

Conclusion: for small numbers / stages, Fibonacci numbers are different from our dynamic models. For higher stages / numbers, the series become similar.

Fibonacci numbers as a system of difference equations

To overcome the problem that the rate of Fibonacci numbers depend on the previous day's value, we do a trick by declaring a new state variable, that represents the previous day's value. We define $s_i := t_{i-1}$ and rewrite the formula as $t_{i+1} = t_i + t_{i-1} = t_i + s_i$.

Then we have

$$\Delta t_i = t_{i+1} - t_i = t_i + s_i - t_i = s_i$$

and

$$\Delta s_i = s_{i+1} - s_i = t_i - s_i$$

with $t_1 = 1$ and $s_1 = t_0 = 0$. The differences depend now formally only on the state of the day, not on the day before, so that we can use the Euler method.

Calculating as a dynamic system

```
EndTime <- 50
s <- numeric(EndTime)
t <- numeric(EndTime)
deltas <- numeric(EndTime)
deltat <- numeric(EndTime)

s[1] <- 0
t[1] <- 1

for(n in 2:EndTime)
{
  deltas[n-1] <- t[n-1] - s[n-1]
  deltat[n-1] <- s[n-1]

  s[n] <- s[n-1] + deltas[n-1]
  t[n] <- t[n-1] + deltat[n-1]

}
```

Show the result

```
t
```

```
#  [1]          1           1          2          3          5          8
#  [7]         13          21         34         55         89        144
# [13]        233         377        610        987       1597       2584
# [19]       4181        6765      10946      17711      28657      46368
# [25]      75025      121393     196418     317811     514229     832040
# [31]    1346269     2178309    3524578    5702887    9227465   14930352
# [37]   24157817    39088169   63245986  102334155  165580141  267914296
# [43]  433494437   701408733 1134903170 1836311903 2971215073 4807526976
# [49] 7778742049 12586269025
```