Dynamic Models with R

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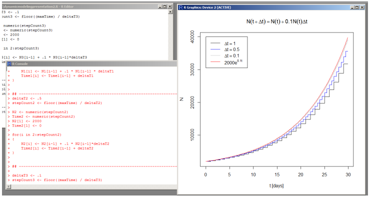

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2. We know the **initial state** of a system at a given time (e.g. leaf area ten days after emergence)
3. We know a **rule** or **formula**, how to determine the rate from the state (e. g. the increase of leaves is related to the actual size of leaves). The formula might depend additionally on *parameters* (e.g. specific leaf area) or *input variables* (e.g. radiation).

The day after tomorrow

From a given state we can calculate the state of the next timestep by calculating the rate and add it to the state. In most cases, we can not calculate the state for a future time directly. We have to calculate it step by step.

If we want to know the state of *the day after tomorrow*, we have first to calculate from the *today*'s state the state of *tomorrow*, and from the tomorrow's state we calculate the state of *the day after tomorrow*.

$$
\begin{aligned}
S_1 &= a && \text{start value} \\
&\ldots \\
\Delta S_{i-1} &= \ldots && \text{rate calculation formula} \\
S_i &= S_{i-1} + \Delta S_{i-1} && \text{calculate new state} \\
&\ldots
\end{aligned}
$$

NB: This calculation scheme (algorithm) is known as the **Euler method**.

Unlimited growth

Let's model a population of rabbits, that live in paradise: enough space, enough food, no foxes, no hunters and plenty of time for doing pleasant things. Their prosperity and growth is not limited by any conditions.

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3. The number of rabbit babies (increase rate) depends on the number of actual rabbits (state). We assume, that the ratio between yearly increase rate and the population size is constant: $\frac{\Delta N}{N} = 0.1$, giving us the **formula** $\Delta N_i = 0.1 N_i$ for all i.

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- (Question: What about **inputs** and **parameters** in this formula?)

Unlimited growth - calculation

How many rabbits will be there in the fourth year N_4?

With *R* we can calculate the number:

```
N1 <- 2000

deltaN1 <- 0.1 * N1
N2 <- N1 + deltaN1

deltaN2 <- 0.1 * N2
N3 <- N2 + deltaN2

deltaN3 <- 0.1 * N3
N4 <- N3 + deltaN3

N4
```

```
# [1] 2662
```

A first model - unlimited growth

- And how many 100 years later?

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- And how many 100 years later?
- Before starting to type 100 times the formulas above, try to do it in a more efficient way.

A first model - unlimited growth

- And how many 100 years later?
- Before starting to type 100 times the formulas above, try to do it in a more efficient way.
- Use the for loop!

Unlimited growth - with for loop

We create vectors with 100 elements for the state and rates and perform the calculation with the help of `for`.

```
EndTime <- 100
N <- numeric(EndTime)
deltaN <- numeric(EndTime)
N[1] <- 2000
for(n in 2:EndTime)
{
    deltaN[n-1] <- 0.1 * N[n-1]
    N[n] <- N[n-1] + deltaN[n-1]
}
N[EndTime]
```

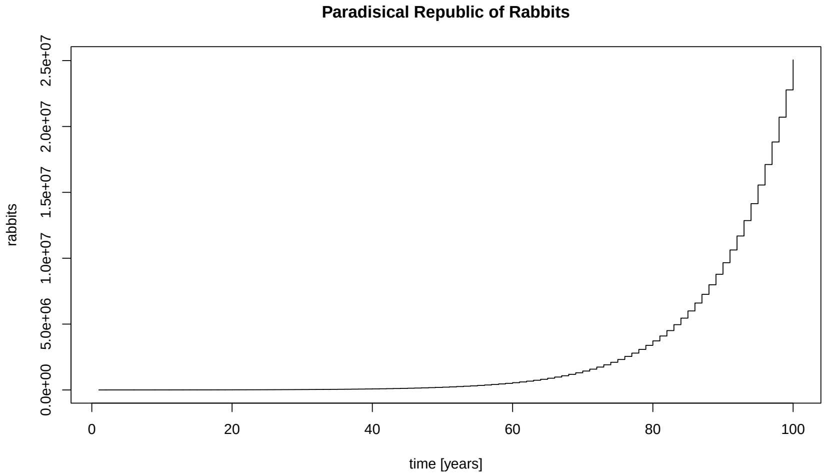
```
# [1] 25055659
```

Wow! - more than 25 millions of rabbits!

Unlimited growth - plot

Let's see, how their number developed over time:

```
plot(N, type="s", main="Paradisical Republic of Rabbits",
  xlab="time [years]", ylab="rabbits")
```



Restricted growth

Let's introduce a sort of birth control: when the number reaches one million, the increase rate is limited to 100000 rabbit babies per year.

$$\Delta N_i = 0.1 N_i \quad \text{if} \quad N_i \leq 1000000$$

$$\Delta N_i = 100000 \quad \text{if} \quad N_i > 1000000$$

The two conditional formulas can be written as

$$\Delta N_i = \begin{cases} 0.1 N_i & \text{if} \quad N_i \leq 1000000 \\ 100000 & \text{else} \end{cases}$$

Implement this model in *R*. (You have to change only the rate calculation within the for loop, making use of `if(...){...}else{...}`.)

Translating a conditional formula to `if ... else`

Mathematical notation

$$X = \begin{cases} formula1 & \text{if} \quad condition \\ formula2 & \text{else} \end{cases}$$

R Code

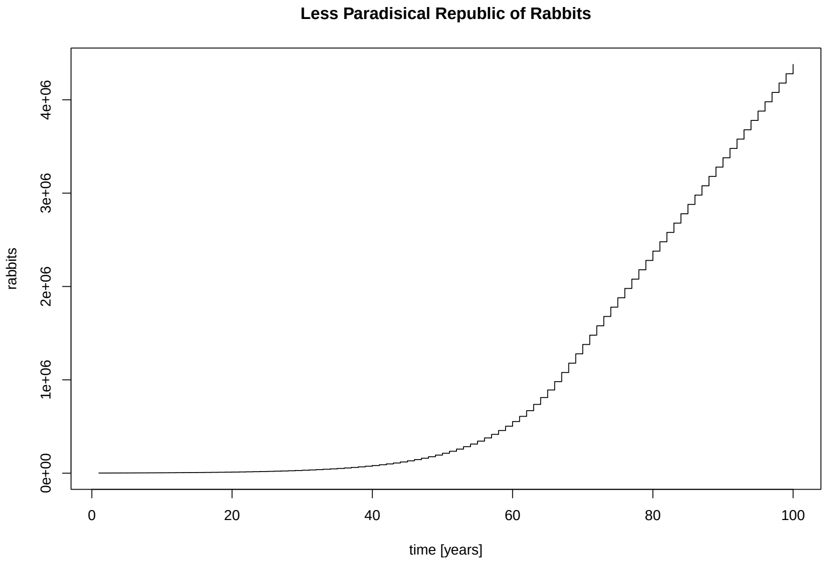
```
if(condition)
{
  X <- formula1
}
else {
  X <- formula2
}
```

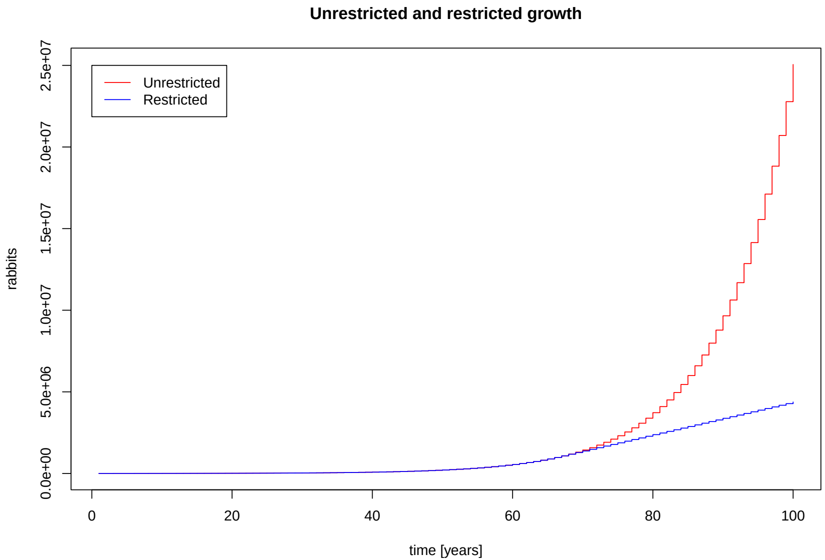
Restricted growth - calculation

```
EndTime <- 100
 NR <- numeric(EndTime)
 deltaNR <- numeric(EndTime)
 NR[1] <- 2000
 for(n in 2:EndTime)
 {
     if(NR[n-1] <= 1000000)
     {
       deltaNR[n-1] <- 0.1 * NR[n-1]
     }
     else
     {
       deltaNR[n-1] <- 100000
     }
     NR[n] <- NR[n-1] + deltaNR[n-1]
 }
 NR[EndTime]
```

```
# [1] 4378816
```

Restricted growth - plot




Unrestricted and restricted growth
Unrestricted
Restricted
rabbits
0.0e+00
5.0e+06
1.0e+07
1.5e+07
2.0e+07
2.5e+07
0
20
40
60
80
100
time [years]