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Gibbs-Delaunay Tessellations

Simulation and estimation

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Section 1

Point processes

Poisson point process

We're on $(\mathbb{R}^d, \mathcal{B})$.
Denote $\mathcal{B}_0$ the set of bounded Borel sets.

Definition. Poisson point process

Let μ be a locally finite non-atomic measure on $\mathbb{R}^d$. A point process Φ satisfying

- $\Phi(B) \sim \mathit{Pois}(\mu(B))$ for each $B \in \mathcal{B}_0$,
- $\Phi(B_1), \ldots, \Phi(B_n)$ for each $n \in \mathbb{N}$ and $B_1, \ldots, B_n \in \mathcal{B}_0$ pairwise disjoint.

is called a Poisson point process with the intensity measure μ.

If $\mu = z\lambda^d$ we call the process homogenous and z the intensity.
For $\Lambda \in \mathcal{B}_0$, denote the distribution of $\Theta \cap \Lambda$ as π^z_Λ.
For the case $z = 1$, use π_Λ.

Poisson point process as a reference measure

$\Phi : (\Omega, \mathcal{A}, P) \to (\mathcal{F}_{lf}, \mathscr{F})$ where

- $\mathcal{F}_{lf} = \{\gamma \subset \mathbb{R}^d | \ \gamma \cap \Lambda \text{ is finite for all } \Lambda \in \mathcal{B}_0\}$ and
- $\mathscr{F}$ is generated by sets of the form $\{\gamma \in \mathcal{F}_{lf} | \ N_\Lambda(\gamma) = n\}, n \in \mathbb{N}, \Lambda \in \mathcal{B}$, where $N_\Lambda(\gamma) = \text{Card}(\gamma \cap \Lambda)$.

We can view π_Λ as a reference measure on $(\mathcal{F}_{lf}, \mathscr{F}, \pi_\Lambda)$.
Then we can define new point processes through defining their density w.r.t. π_Λ.

Poisson point process with intensity z:

$$\pi_\Lambda^z(d\gamma) \propto z^{N_\Lambda(\gamma)} \pi_\Lambda(d\gamma).$$

Add a new term to obtain the finite volume Gibbs point process:

$$z^{N_\Lambda(\gamma)} e^{-H(\gamma)} \pi_\Lambda(d\gamma).$$

Take $\Lambda \in \mathcal{B}_0$.

Definition. Finite volume Gibbs point process

The finite-volume Gibbs point process (fGPP) is a point process defined by its density with respect to π_Λ:

$$f(\gamma) = \frac{1}{C_\Lambda^z} z^{N_\Lambda(\gamma)} e^{-H(\gamma)} \qquad \gamma \in \mathcal{F}_{lf},$$

where

- $z > 0$,
- $H : \mathcal{F}_{lf} \mapsto \mathbb{R} \cup \{+\infty\}$ is a measurable function called the energy function,
- $C_\Lambda^z = \int z^{N_\Lambda} e^{-H} d\pi_\Lambda$ is the normalizing constant.

Examples and usefulness of (f)GPP

- Physical motivation
- Other examples of energy functions
- Allows working explicitly with geometrical structures such as random tessellations

Local energy and GNZ equations

For $\gamma \in \mathcal{F}_{lf}$ and $x \in \mathbb{R}^d$, define the local energy of x in γ by

$$h(x,\gamma) = H(\gamma \cup \{x\}) - H(\gamma).$$

Proposition (Georgii, Nguyen, Zessin). GZN equations

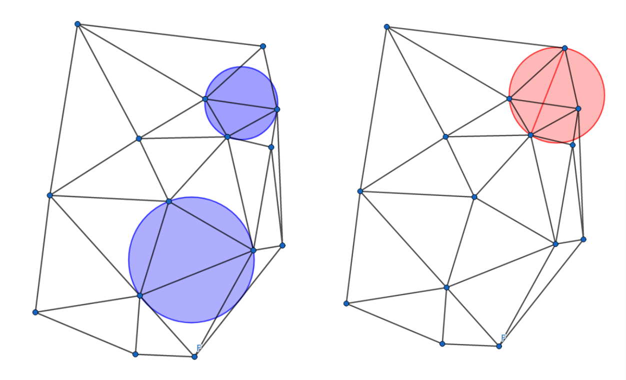
For any positive measurable function $f : \mathbb{R}^d \times \mathcal{F}_{lf} \to \mathbb{R}$,

$$\int \sum_{x\in\gamma} f(x,\gamma \setminus \{x\}) P_\Lambda^z d(\gamma) = z \int \int_\Lambda f(x,\gamma) e^{-h(x,\gamma)} dx P_\Lambda^z(d\gamma).$$

Section 2

Triangulations

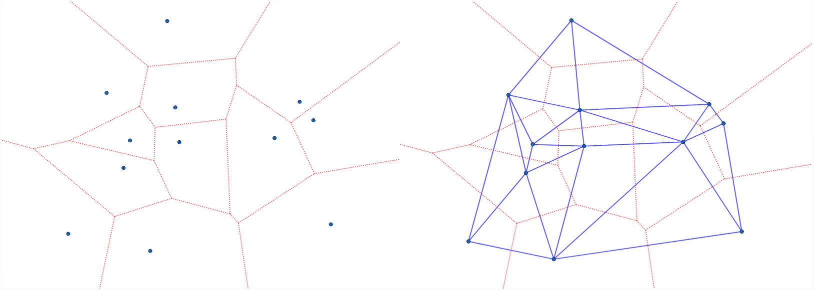
A $d+1$-tuplet $\{x_1, \ldots, x_{d+1}\} \subset \gamma$ has the empty sphere property if the open circumscribed ball $\mathcal{B}(T)$ does not contain any points from γ.



Additional assumption on γ (No cospherical points): no $d+2$ points $x_1, \ldots, x_{d+2}$ are cospherical, i.e. there is no point $x \in \mathbb{R}^d$ such that $d(x, x_1) = \cdots = d(x, x_2)$.

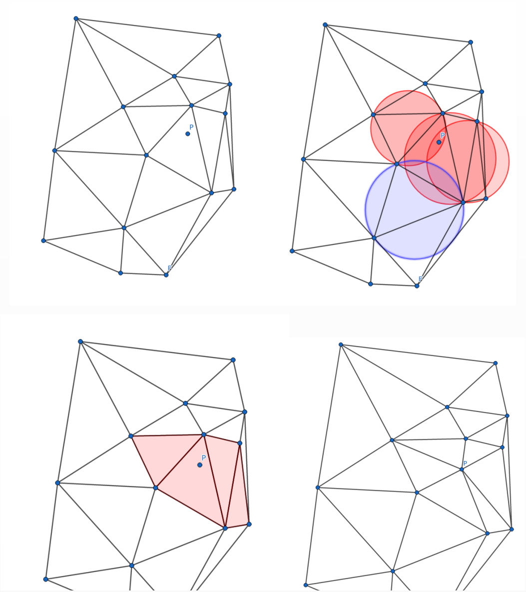
For $x \in \gamma$, the Voronoi cell of x in γ is

$$C(x,\gamma) = \{z \in \mathbb{R}^d : \ \|x - z\| \leq \|y - z\| \ \forall y \in \gamma\}.$$



Then the Delaunay tessellation can be defined as

$$Del(\gamma) = \{\{x, y\} \subset \gamma : \ C(x,\gamma) \cap C(y,\gamma) \neq \emptyset\}.$$

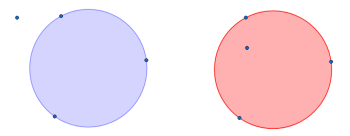


In $2D$

$$INCIRCLE(P_1, P_2, P_3, P_4) = \begin{vmatrix} x_1 & y_1 & w_1 & 1 \\ x_2 & y_2 & w_2 & 1 \\ x_3 & y_3 & w_3 & 1 \\ x_4 & y_4 & w_4 & 1 \end{vmatrix}$$

where $w_i = x_i^2 + y_i^2, i = 1, \ldots, 4$ and

$$ORIENTATION(P_1, P_2, P_3) = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} > 0$$



INCIRCLE > 0 INCIRCLE < 0

In $3D$

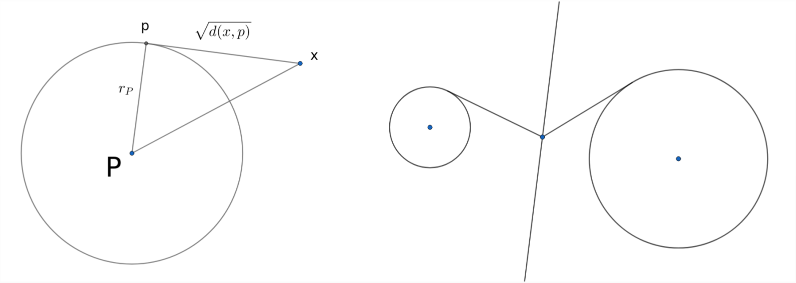
$$INCIRCLE(P_1, P_2, P_3, P_4, P_5) = \begin{vmatrix} x_1 & y_1 & z_1 & w_1 & 1 \\ x_2 & y_2 & z_2 & w_2 & 1 \\ x_3 & y_3 & z_3 & w_3 & 1 \\ x_4 & y_4 & z_4 & w_4 & 1 \\ x_5 & y_5 & z_5 & w_5 & 1 \end{vmatrix}$$

where $w_i = x_i^2 + y_i^2 + z_i^2, i = 1, \dots, 5$
if the following condition is satisfied

$$ORIENTATION(P_1, P_2, P_3, P_4) = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} > 0$$

Generators are not points, but spheres.
Metric is not Euclidean, but power distance.

$$d(x, P) = d(x, p)^2 - r_P^2$$



$$INCIRCLE(P_1, P_2, P_3, P_4, P_5) = \begin{vmatrix} x_1 & y_1 & z_1 & w_1 & 1 \\ x_2 & y_2 & z_2 & w_2 & 1 \\ x_3 & y_3 & z_3 & w_3 & 1 \\ x_4 & y_4 & z_4 & w_4 & 1 \\ x_5 & y_5 & z_5 & w_5 & 1 \end{vmatrix}$$

where $w_i = x_i^2 + y_i^2 + z_i^2 - r_i^2, i = 1, \ldots, 5$
if the following condition is satisfied

$$ORIENTATION(P_1, P_2, P_3, P_4) = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} > 0$$

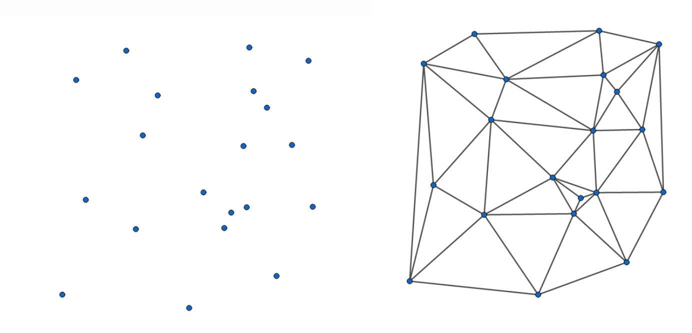
Why? Because both are regular triangulations - convex hulls of lifted sets of points.

Interlude: CGAL

Section 3

Random triangulations

For Poisson point process Φ, define Poisson-Delaunay triangulation as $Del(\Phi)$.



Geometric aspects of the triangulation can be used to define H.
In general, the energy can have the form

$$H(\gamma) = \sum_{T \in Del(\gamma)} V_1(T) + \sum_{\{T,T'\} \subset Del(\gamma)} V_2(T, T')$$

to take interaction into account. V_1 and V_2 can be any function from d-dimension simplices to $\mathbb{R} \cup \{+\infty\}$.

Add example(s)?

Specification of the GLD model

In the model we used, the energy function is of the form

$$H(\gamma) = \sum_{T \in Del_{\Lambda}(\gamma)} V_1(T),$$

with V_1 defined as

$$V_1(T) = \begin{cases} \infty & \text{if } a(T) \leq \epsilon, \\ \infty & \text{if } R(T) \geq \alpha, \\ \theta Sur(T) & \text{otherwise,} \end{cases} \tag{1}$$

where

- $a(T)$ is the area of the smallest face of the tetrahedron T.
- $R(T)$ is the circumradius of T.
- $Sur(T)$ is the surface area of the tetrahedron.

Section 4

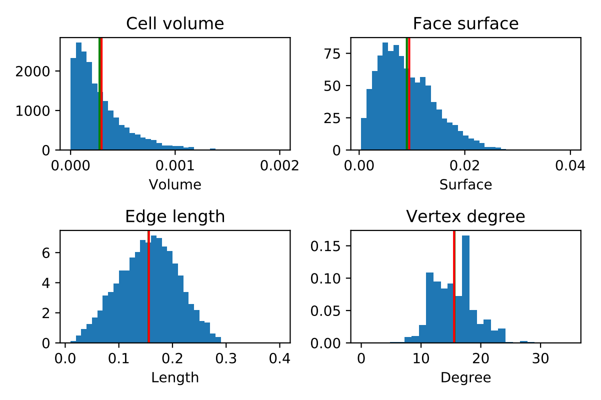
Simulation

Comparison with Poisson-Delaunay

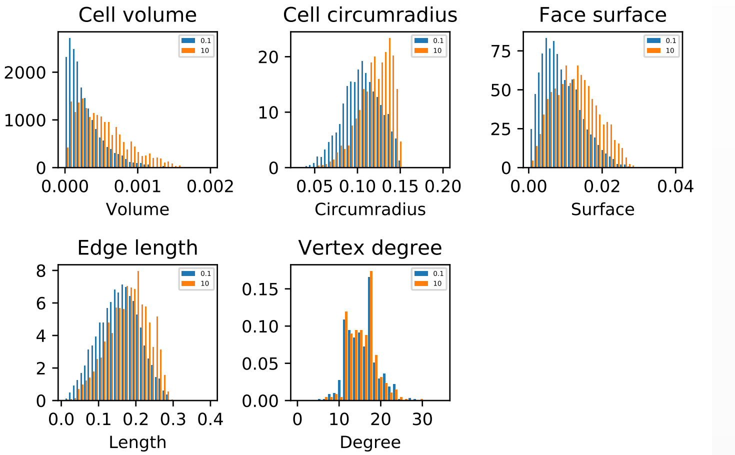
$$\pi_\Lambda^z \propto z^{N_\Lambda} \pi_\Lambda$$

$$P_\Lambda^{,z} \propto z^{N_\Lambda} e^{-\theta H} \pi_\Lambda$$

$\theta = 0 \Rightarrow$ GPP becomes PPP with intensity z.

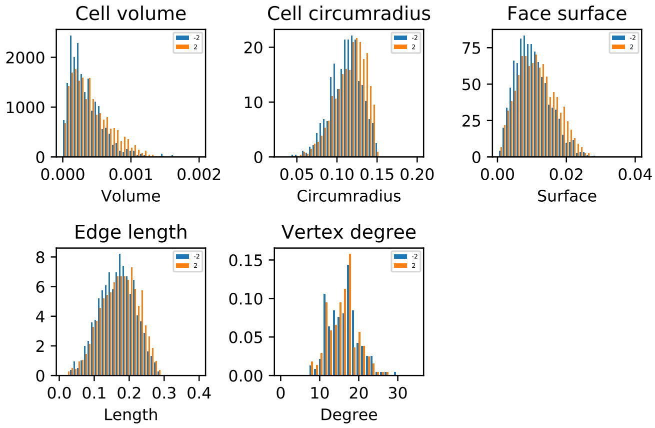


The model prefers configurations with lower energy.
θ multiplies the total surface area of all cells, thus with higher θ, the cells are forced to become large.



The model prefers configurations with lower energy.

- $\theta > 0$. The sum needs to be minimized $\Rightarrow$ fewer larger tetrahedra.
- $\theta < 0$. The sum needs to be maximized $\Rightarrow$ many smaller tetrahedra.



Section 5

Estimation

Two-step procedure

Thanks to the fact that the hardcore parameter ϵ satisfies

$$\text{if } \epsilon > \epsilon' \text{ then } \forall \Lambda,\ E_\Lambda^{\epsilon,\alpha,\theta}(\gamma_\Lambda, \gamma_{\Lambda^c}) < \infty \Rightarrow E_\Lambda^{\epsilon',\alpha,\theta}(\gamma_\Lambda, \gamma_{\Lambda^c}) < \infty,$$

and the hardcore parameter α satisfies

$$\text{if } \alpha < \alpha' \text{ then } \forall \Lambda,\ E_\Lambda^{\epsilon,\alpha,\theta}(\gamma_\Lambda, \gamma_{\Lambda^c}) < \infty \Rightarrow E_\Lambda^{\epsilon,\alpha',\theta}(\gamma_\Lambda, \gamma_{\Lambda^c}) < \infty,$$

their consistent estimators are:

$$\hat{\epsilon} = \inf\{\epsilon > 0, E_\Lambda(\gamma_\Lambda, \gamma_\Lambda^c) < \infty\},$$

$$\hat{\alpha} = \sup\{\alpha > 0, E_\Lambda(\gamma_\Lambda, \gamma_\Lambda^c) < \infty\}.$$

In practice, the parameters are estimated as

$$\hat{\epsilon} = \min\{a(T), T \in Del_\Lambda(\gamma)\},$$

$$\hat{\alpha} = \max\{r(T), T \in Del_\Lambda(\gamma)\}.$$

The estimate $\hat{\beta} = (\hat{\epsilon}, \hat{\alpha})$ is then used in the pseudo-likelihood function in the second estimation step.

Maximum pseudolikelihood

$\mathbb{R}^{\beta}(\gamma)$ is the set of removable points.

$$PLL_{\Lambda_n}(\gamma, z, \beta, \theta) = \int_{\Lambda'_n} z \exp(-h^{\beta,\theta}(x,\gamma))dx + \sum_{x \in \mathcal{R}^{\beta}(\gamma) \cap \Lambda_n} (h^{\beta}, \theta(x, \gamma \backslash \{x\}) - \ln($$

where Λ'_n is the set of all addable points in Λ_n and $h^{,\theta}(x, \gamma \setminus \{x\})$ is local energy of x in γ defined for every $x \in \mathcal{R}^{\beta}(\gamma)$ by:

$$h^{\beta,\theta}(x, \gamma \setminus \{x\}) = H_{\Lambda}^{\beta,\theta}(\gamma_{\Lambda}, \gamma_{\Lambda^c}) - H_{\Lambda}^{\beta,\theta}(\gamma_{\Lambda} \setminus \{x\}, \gamma_{\Lambda^c}).$$

The estimates $\hat{\theta}$ and $\hat{z}$ are obtained through minimizing the PLL_{Λ_n} function.

$$(\hat{z}, \hat{\theta}) = \text{argmin}_{z,\theta} PLL_{\Lambda_n}(\gamma, z, \hat{\beta}, \theta).$$

We obtain the estimate of θ by substituting the expression for $\hat{z}$ into the equation for θ. This eventually leads to the equation

$$\int_{\Lambda'_n} \exp\left(-\theta h(x)\right)(h(x) - c)dx = 0.$$

After $\hat{\theta}$ is estimated, we then obtain the estimate $\hat{z}$ with $\hat{\theta}$ instead of θ. All integrals are estimated by MC-integration.

Estimation results

are not great so far.

Possible future directions

- Variational estimate
- Energy with explicit interaction
- Periodic outside configuration
- ...