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AND PHYSICS
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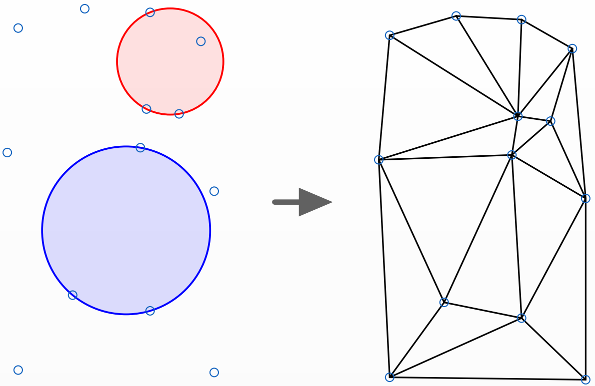
Generalized random tessellations, their properties, simulation and applications

Thesis defense

5 February 2019

Let $\mathbb{x}$ be a locally finite subset $\mathbb{R}^3$.



$$\mathcal{D}_4(\mathbb{x}) = \{\eta \subset \mathbb{x} : \eta \text{ satisfies the empty ball property } \}$$

Point processes

$\Phi : (\Omega, \mathcal{A}, P) \to (\mathbf{N}_{lf}, \mathcal{N})$ where

- $\mathbf{N}_{lf}$ is the set of all locally finite simple counting measures on $\mathbb{R}^3$.
- $\mathcal{N}$ is generated by sets of the form $\{\nu \in \mathbb{N}_{lf} | \ \nu(\Lambda) = n\}, n \in \mathbb{N}_{lf}, \Lambda \in \mathcal{B}$.

Standard Poisson point process is a point process Φ such that

- $\Phi(B) \sim Pois(|B|)$ for each $B \in \mathcal{B}_0$,
- $\Phi(B_1), \dots, \Phi(B_n)$ are independent for each $n \in \mathbb{N}$ and $B_1, \dots, B_n \in \mathcal{B}_0$ pairwise disjoint.

For $\Lambda \in \mathcal{B}_0$, denote the distribution of $\Phi \cap \Lambda$ as Π_Λ.

Consider the space $(\mathbf{N}_{lf}, \mathcal{N}_{lf}, \Pi)$.
Dobrushin-Lanford-Ruelle

Definition

Let $\mathcal{E}$ be a hypergraph structure and H an energy function on $\mathcal{E}$ such that H is non-degenerate and stable. A probability measure $P \in \mathcal{P}_\Theta$ on $(\mathbf{N}_{lf}, \mathcal{N}_{lf})$ is the *(infinite volume) Gibbs measure* with activity $z > 0$ if $P(\mathbf{N}_{cr}^{\Lambda}) = 1$ and

$$\int f dP = \int_{\mathbf{N}_{cr}^{\Lambda}} \frac{1}{Z_\Lambda^z(\gamma)} \int_{\mathbf{N}_\Lambda} f(\zeta \cup \gamma_{\Lambda^c}) e^{-H_{\Lambda,\gamma}(\zeta)} \Pi_\Lambda^z(d\zeta) P(d\gamma) \tag{1}$$

for every $\Lambda \in \mathcal{B}_0$ and every measurable $f : \mathbf{N}_{lf} \to [0, \infty)$. A point process whose distribution is a Gibbs measure is called a *(infinite volume) Gibbs point process*.

where $Z_\Delta^z(\gamma_{\Delta^c}) = \int z^{N_\Delta(\gamma)} e^{-H_\Delta(\gamma)} \Pi_\Delta(d\gamma_\Delta)$ is the normalizing constant.
Let $\mathbb{x} \in \mathbf{N}_{\mathrm{cr}}^{\Lambda}$. Then

Existing results

[Dereudre and Lavancier (2011)]
2d Delaunay

$$H(\gamma) = \sum_{T \in \mathcal{L}Del_\Lambda(\gamma)} V_1(T),$$

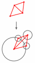
with V_1 defined as

$$V_1(T) = \begin{cases} \infty & \text{if } a(T) \leq \epsilon, \\ \infty & \text{if } R(T) \geq \alpha, \\ \theta Sur(T) & \text{otherwise,} \end{cases} \tag{2}$$

where

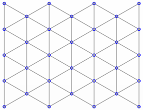
- $a(T)$ is the area of the smallest face of the tetrahedron T.
- $R(T)$ is the circumradius of T.
- $Sur(T)$ is the surface area of the tetrahedron.

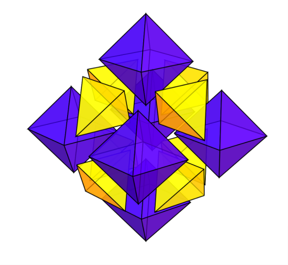
Our work



Left Part

Delaunay -¿ Laguerre-Delaunay
2d -¿ 3d



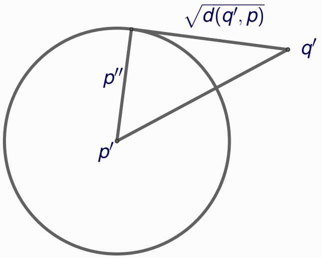


Right Part

Laguerre-Delaunay

- Generators are not points, but spheres.
- $\gamma = \{p_1, \ldots, p_n\} = \{(p'_1, p''_1), \ldots, (p'_n, p''_n)\}$ can be thought of as marked point process.
- Metric is not Euclidean, but power distance.

$$d(q', p) = \|q' - p'\|^2 - p''^2$$



Theoretical

Based on [Dereudre et al. 2012]
Understand them as hypergraphs
Provides very general conditions for existence BUT - not Laguerre, not MPP
First step - rephrase it to MPP

Range condition

Best would be finite range, but we don't have that.

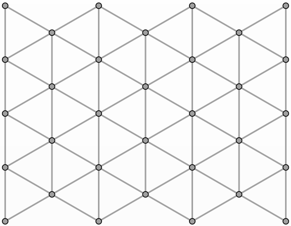
Definition

A set $\Delta \in \mathcal{B}_0$ is a *finite horizon* for the pair $(\eta, \mathbb{x}) \in \mathcal{E}$ and the hyperedge potential φ if for all $\tilde{\mathbb{x}} \in \mathbf{N}_{lf}, \tilde{\mathbb{x}} = \mathbb{x}$ on $\Delta \times S$

$$(\eta, \tilde{\mathbb{x}}) \in \mathcal{E} \text{ and } \varphi(\eta, \tilde{\mathbb{x}}) = \varphi(\eta, \mathbb{x}).$$

The pair $(\mathcal{E}, \varphi)$ satisfies the *finite-horizon property* if each $(\eta, \mathbb{x}) \in \mathcal{E}$ has a finite horizon.

- *Range condition*. There exist constants $\ell_R, n_R \in \mathbb{N}$ and $\delta_R < \infty$ such that for all $(\eta, \mathbb{x}) \in \mathcal{E}$ there exists a finite horizon Δ satisfying: For every $x, y \in \Delta$ there exist ℓ open balls $B_1, \dots, B_\ell$ (with $\ell \leq \ell_R$) such that
  - the set $\cup_{i=1}^{\ell} \bar{B}_i$ is connected and contains x and y, and
  - for each i, either $\operatorname{diam} B_i \leq \delta_R$ or $\#(\mathbb{x} \cap (B_i \times S)) \leq n_R$.



3D much more difficult
Need to recalculate tessellation each step
Decided to write own, more general in C++
Failed, went to CGAL
MCMC MH
Additional analysis done in Python and Mathematica

CGAL

https://github.com/DahnJ/General-Increment-Decrement.git
https://github.com/DahnJ/Gibbs-Laguerre-Delaunay.git

Practical

Some results, e.g. role of theta (+reduction to PPP)

Summary and references, I guess?

D. Dereudre and F. Lavancier. Practical simulation and estimation for Gibbs Delaunay-Voronoi tessellations with geometric hardcore interaction. Computational Statistics and Data Analysis, 55(1):498-519, 2011.

D. Dereudre, R. Drouilhet, and H.O. Georgii. Existence of gibbsian point processes with geometry-dependent interactions. Probability Theory and Related Fields, 153(3):643-670, 2012

Fropuff. The vertex configuration of a tetrahedral-octahedral honeycomb., 2006. URL https://en.wikipedia.org/wiki/File:TetraOctaHoneycomb-VertexConfig.svg

(1) (Reinforced) general position

Definition

Let $\mathbb{x} \in \mathbf{N}_{lf}$. We say $\mathbb{x}$ is in **general position** if

$$\eta \subset \mathbb{x}, 2 \leq \mathrm{card}(\eta) \leq 4 \Rightarrow \eta' \text{ is affinely independent in } \mathbb{R}^3.$$

Denote $\mathbf{N}_{gp} \subset \mathbf{N}_{lf}$ the set of all locally finite configurations in general position.

We call points $\{x_0', x_1', \ldots, x_k'\} \subset \mathbb{R}^3, k \in \mathbb{N}$ *cospherical* if there exists a sphere $S \subset \mathbb{R}^3$ such that $\{x_0', \ldots, x_k'\} \subset S$. In this text, a sphere will always refer to the boundary of a ball, never to the interior.

Definition

Let $\mathbb{x} \in \mathbf{N}_{gp}$. We say $\mathbb{x}$ is in **reinforced general position** if

$$\eta \subset \mathbb{x}, \mathrm{card}(\eta) = 4 \Rightarrow \eta' \text{ is not cospherical.}$$

Denote $\mathbf{N}_{rgp}$ the set of all locally finite configurations in reinforced general position.

(2) Set $\mathcal{E}_\Lambda$

$$\mathbf{N}_\Lambda = \{\nu \in \mathbf{N}_{lf} : \nu((\mathbb{R}^3 \setminus \Lambda) \times S) = 0\}$$

Definition

Let $\Lambda \in \mathcal{B}_0$. Define the set

$$\mathcal{E}_\Lambda(\mathbb{x}) := \{\eta \in \mathcal{E}(\mathbb{x}) : \varphi(\eta, \zeta \cup \mathbb{x}_{\Lambda^c}) \neq \varphi(\eta, \mathbb{x}) \text{ for some } \zeta \in \mathbf{N}_\Lambda\}.$$

Recall that we have defined $\varphi = 0$ on $\mathcal{E}^c$. This means that for $\eta \in \mathcal{E}(\mathbb{x})$ such that $\varphi(\eta, \mathbb{x}) \neq 0$ we have

$$\eta \notin \mathcal{E}(\zeta \cup \mathbb{x}_{\Lambda^c}) \text{ for some } \zeta \in \mathbf{N}_\Lambda \Rightarrow \eta \in \mathcal{E}_\Lambda(\mathbb{x}).$$

The condition in the definition can also be equivalently stated as

$$\text{There is no point } q \in \mathbb{x} \text{ such that } d(p'_\eta, q) < p''_\eta. \tag{3}$$

[$\mathcal{E}_\Lambda(\mathbb{x})$ for $\mathcal{D}$ and $\mathcal{LD}$] For $\mathcal{D}$, we have that

$$\eta \in \mathcal{D}_\Lambda(\mathbb{x}) \iff B(\eta) \cap \Lambda \neq \emptyset.$$

For $\mathcal{LD}$, using the characterization (3), we obtain

$$\eta \in \mathcal{LD}_\Lambda(\mathbb{x}) \iff d(p'_\eta; \Lambda) < \sqrt{p''_\eta + W},$$

where $d(p'_\eta; \Lambda) = \inf\{\|p'_\eta - x\| : x \in \Lambda\}$ is the distance of p'_η from Λ.