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Existence of Gibbs-Laguerre-Delaunay Models

Workshop Devet Skal

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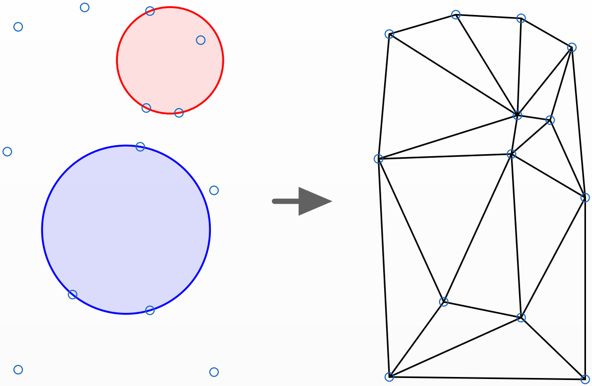
Let γ be a locally finite subset $\mathbb{R}^3$.



$$\mathcal{D}_4(\gamma) = \{\eta \subset \gamma : \mathrm{card}(\eta) = 4, \eta \text{ satisfies the empty ball property } \}$$

Point processes

$\mathcal{B}$ Borel σ-algebra on $\mathbb{R}^3$, $\mathcal{B}_0$ bounded Borel sets, $|\cdot|$ is the Lebesgue measure.

Point process: $\Phi : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathbf{N}_{lf}, \mathcal{N}_{lf})$ where

- $\mathbf{N}_{lf}$ is the set of all locally finite simple counting measures on $\mathbb{R}^3$.
- $\mathcal{N}_{lf}$ is generated by sets of the form

$$\{\nu \in \mathbf{N}_{lf} | \; \nu(\Lambda) = n\}, n \in \mathbf{N}_{lf}, \Lambda \in \mathcal{B}.$$

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Poisson point process with intensity $z > 0$ is a point process Φ such that

- $\Phi(B) \sim Pois(z|B|)$ for each $B \in \mathcal{B}_0$,
- $\Phi(B_1), \ldots, \Phi(B_n)$ are independent for each $n \in \mathbb{N}$ and $B_1, \ldots, B_n \in \mathcal{B}_0$ pairwise disjoint.

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For $\Lambda \in \mathcal{B}_0$ and Poisson point process Φ with intensity $z > 0$, denote the distribution of $\Phi_\Lambda := \Phi \cap \Lambda$ as Π^z_Λ.
We obtain the probability space $(\mathbf{N}_{lf}, \mathcal{N}_{lf}, \Pi^z_\Lambda)$.

Gibbs point process

Denote

$$\mathbf{N}_\Lambda = \{\nu \in \mathbf{N}_{lf} : \nu(\mathbb{R}^3 \setminus \Lambda) = 0\}.$$

A translation-invariant probability measure P on $(\mathbf{N}_{lf}, \mathcal{N}_{lf})$ is a Gibbs measure with activity $z > 0$ if it satisfies the Dobrushin-Lanford-Ruelle equation:

$$\int f dP = \int \frac{1}{Z_\Lambda^z(\gamma)} \int_{\mathbf{N}_\Lambda} f(\zeta \cup \gamma_{\Lambda^c}) e^{-H_\Lambda(\zeta \cup \gamma_{\Lambda^c})} \Pi_\Lambda^z(d\zeta) P(d\gamma)$$

for every $\Lambda \in \mathcal{B}_0$ and every measurable $f : \mathbf{N}_{lf} \to [0, \infty)$.
$Z_\Lambda^z(\gamma) = \int e^{H_\Lambda(\zeta \cup \gamma_{\Lambda^c})} \Pi_\Lambda^z(d\zeta)$ is the partition function
H_Λ is the energy function

$$H_\Lambda(\gamma) = \sum_{\eta \in \mathcal{E}_\Lambda(\gamma)} \varphi(\eta, \gamma),$$

such that $Z_\Lambda^z(\gamma) < \infty$.

A point process whose distribution is a Gibbs measure is called a Gibbs point process.

Existing results

D. Dereudre and F. Lavancier. Practical simulation and estimation for Gibbs Delaunay-Voronoi tessellations with geometric hardcore interaction (2011)

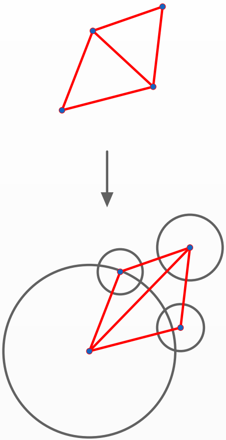
Considered Delaunay triangulation in $\mathbb{R}^2$ using the potential

$$\varphi(\eta) = \begin{cases} \infty & \text{if } l(T) \leq \epsilon, \\ \infty & \text{if } \chi(T) \geq \alpha, \\ \theta Per(T) & \text{otherwise,} \end{cases}$$

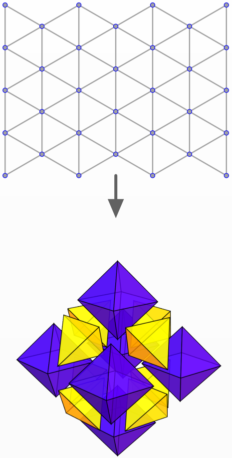
where

- $l(T)$ is the lenth of the shortest edge of the triangle T.
- $\chi(T)$ is the circumradius of T.
- $Per(T)$ is the perimeter of the triangle T.
- $\theta \in \mathbb{R}$, $\epsilon \geq 0, \alpha > 0$ such that $2\epsilon < \alpha < 1/2$.

Delaunay $\rightarrow$ Laguerre

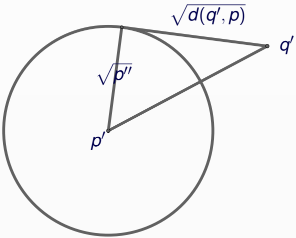


2D $\rightarrow$ 3D



- Generating points are now both locations and weights. Can be geometrically interpreted as spheres.
- $\gamma = \{p_1, \dots, p_n\} = \{(p'_1, p''_1), \dots, (p'_n, p''_n)\}$ can be thought of as marked point process.
- $\gamma \subset \mathbb{R}^3 \times S$, where $S = [0, W]$, $W > 0$.
- Distance is not Euclidean, but the power distance

$$d(q', p) = \|q' - p'\|^2 - p''.$$



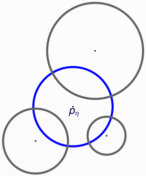
- Instead of circumscribed ball, we have the characteristic point $p_\eta = (p'_\eta, p''_\eta)$.

$$d(p'_\eta, p_i) = p''_\eta, \quad \text{for each } i = 1, \ldots, 4.$$

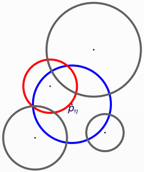
- Instead of the empty sphere property, we have regularity

η is regular if there is no other point $q \in \gamma$ such that $d(p'_\eta, q) < p''_\eta$.

$$\mathcal{LD}_4(\gamma) = \{\eta \subset \gamma : \mathrm{card}(\eta) = 4, \eta \text{ is regular}\}$$



Points of η (gray) and their characteristic point p_η (blue).



A point (red) breaking the regularity of η.

Practical results

Implementation

The partition function Z_Λ^z is unknown $\rightarrow$ Birth-Death-Move Metropolis-Hastings algorithm.
Each step has to reconstruct a new tetrahedrization.

CGAL

- Final solution: implement the MCMC algorithm and rely on Computer Graphics Algorithms Library for handling the tetrahedrization.
  Current version at https://github.com/DahnJ/Gibbs-Laguerre-Delaunay.git
- Simulations done in `C++`, numerical analysis in `Python` and `Mathematica`.

Theoretical results

Goal: To prove the existence of the Gibbs-Laguerre-Delaunay models we simulated.
Results: Proved the existence of the following two classes of models.

A hyperedge potential ϕ is unary for the hypergraph structure $\mathcal{E}$ if there exists a measurable function $\hat{\varphi} : \mathbf{N}_{lf} \to \mathbb{R} \cup \{+\infty\}$ such that

$$\varphi(\eta, \gamma) = \hat{\varphi}(\eta) \text{ for } \eta \in \mathcal{E}(\gamma).$$

Bounded interaction: For $\eta \in \mathcal{LD}_4(\gamma)$ define the potential φ_S as a unary potential such that

$$\varphi_S(\eta, \gamma) \leq K_0 + K_1 \chi(\eta)^\beta$$

for some $K_0, K_1 \geq 0, \beta > 0$
Hard-core interaction: For $\eta \in \mathcal{LD}_4(\gamma)$ define the potential φ_{HC} as a unary potential such that

$$\sup_{\eta : d_0 \leq \chi(\eta) \leq d_1} \varphi_{HC}(\eta, \gamma) < \infty \text{ and } \varphi_{HC}(\eta, \gamma) = \infty \text{ if } \chi(\eta) > \alpha$$

for some $0 \leq d_0 < d_1 \leq \alpha$.

Tessellations as hypergraphs

Based on *Dereudre, Drouilhet, Georgii: Existence of gibbsian point processes with geometry-dependent interactions (2012)*.

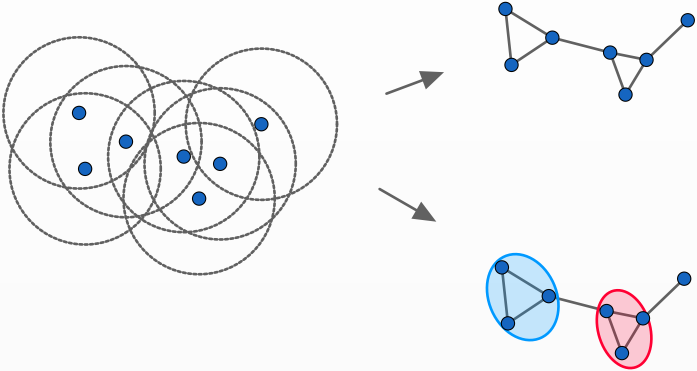
Definition. Hypergraph structure

A *hypergraph structure* is a measurable subset $\mathcal{E}$ of $(\mathbf{N}_f \times \mathbf{N}_{lf}, \mathcal{N}_f \otimes \mathcal{N}_{lf})$ such that $\eta \subset \gamma$ for all $(\eta, \gamma) \in \mathcal{E}$. We call η a *hyperedge* of γ and write $\eta \in \mathcal{E}(\gamma)$, where $\mathcal{E}(\gamma) = \{\eta : (\eta, \gamma) \in \mathcal{E}\}$. For a given $\gamma \in \mathbf{N}_{lf}$, the pair $(\gamma, \mathcal{E}(\gamma))$ is called a *hypergraph*.

A hyperedge potential is a measurable function $\varphi : \mathcal{E} \to \mathbb{R} \cup \{+\infty\}$. We define $\varphi = 0$ on $\mathcal{E}^c$.

Many-body interaction

$$\mathrm{LC}_r = \{(\eta,\gamma) : \eta \subset \gamma, \mathrm{diam}(\eta) \le r, \gamma \in \mathbb{N}_{lf}\}$$

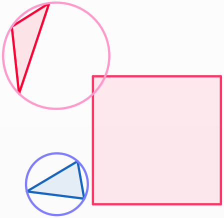


Let $\Lambda \in \mathcal{B}_0$. Define the set

$$\mathcal{E}_\Lambda(\gamma) := \{\eta \in \mathcal{E}(\gamma) : \varphi(\eta, \zeta \cup \gamma_{\Lambda^c}) \neq \varphi(\eta, \gamma) \text{ for some } \zeta \in \mathbf{N}_\Lambda\}.$$

For $\eta \in \mathcal{E}(\gamma)$ such that $\varphi(\eta, \gamma) \neq 0$ we have the following implication:

$$\eta \notin \mathcal{E}(\zeta \cup \gamma_{\Lambda^c}) \text{ for some } \zeta \in \mathbf{N}_\Lambda \Rightarrow \eta \in \mathcal{E}_\Lambda(\gamma).$$



Finite horizon, range confinement, range condition.

Pseudo-periodic configurations

$M \in \mathbb{R}^{3\times 3}$ ... an invertible 3×3 matrix, column vectors (M_1, M_2, M_3). For each $k \in \mathbb{Z}^3$ define the cell

$$C(k) = \{Mx \in \mathbb{R}^3 : x - k \in [-1/2, 1/2)^3\}.$$

Let $\Gamma \in \mathcal{N}_C$ be non-empty. Then we define the *pseudo-periodic* configurations $\bar{\Gamma}$ as

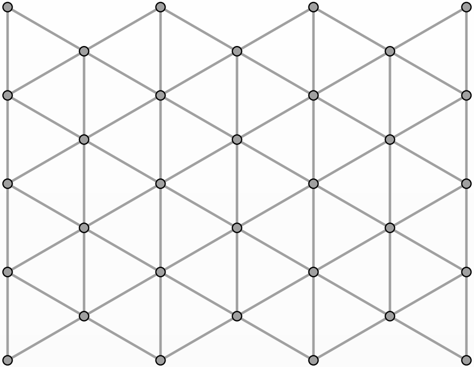
$$\bar{\Gamma} = \{\gamma \in \mathbf{N}_{lf} : \vartheta_{Mk}(\gamma_{C(k)}) \in \Gamma \text{ for all } k \in \mathbb{Z}^3\},$$

Fix some $A \subset C \times S$ and define

$$\Gamma^b = \{\zeta \in \mathbf{N}_C : \zeta = \{p\}, p \in B(0, b)\},$$

Let M be such that $|M_i| = a > 0$ for $i = 1, 2, 3$ and $\angle(M_i, M_j) = \pi/3$ for $i \neq j$.

In 2D: Equilateral triangles



In 3D: Regular tetrahedra?

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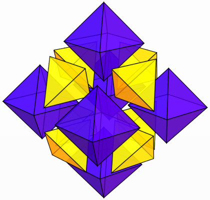
Aristotle

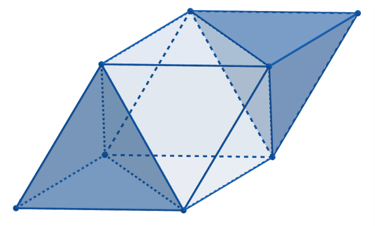
In 3D: ~~Regular tetrahedra~~

Tetrahedrons do not tessellate.

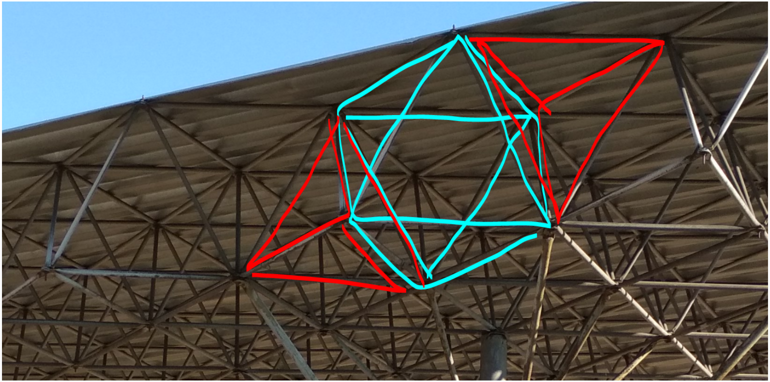
In 3D: Tetrahedral-octahedral honeycomb

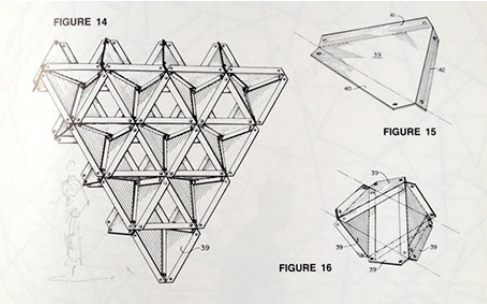


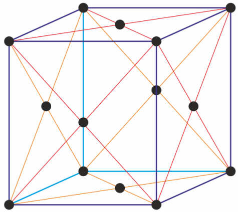
Tetrahedral-octahedral honeycomb in an exploded view.



A single cell of a tetrahedral-octahedral honeycomb.

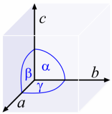






CRYSTAL LATTICE

face-centered cubic



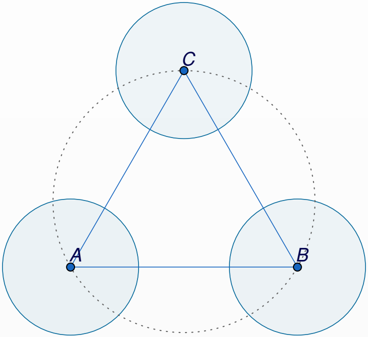
$$a = b = c$$

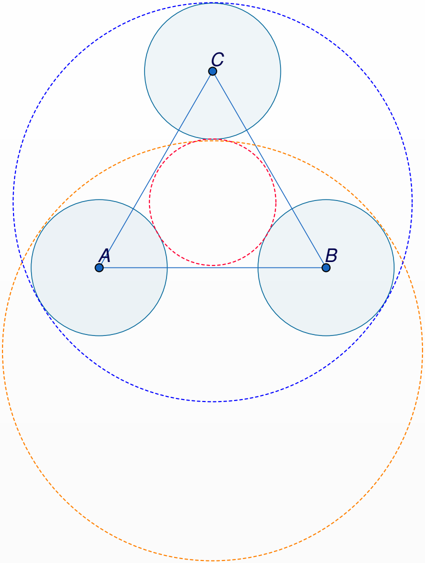
$$\alpha = \beta = \gamma = 90°$$

Bounding the diameter

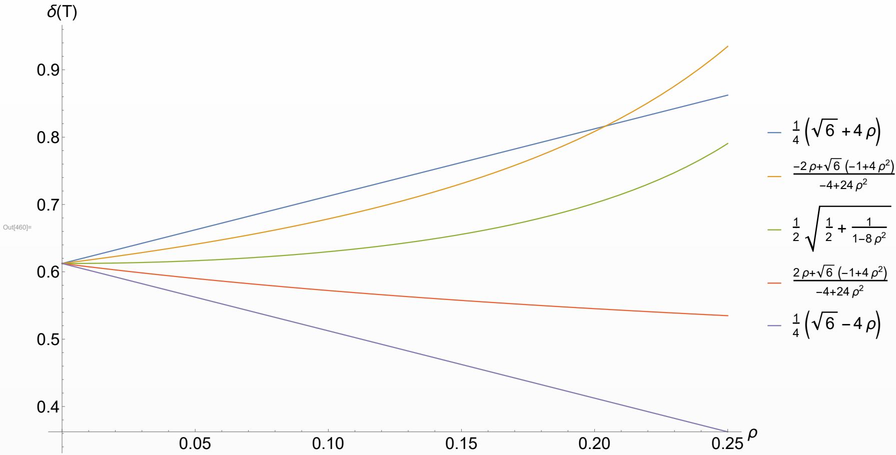
$$\varphi_S(\eta, \gamma) \leq K_0 + K_1 \chi(\eta)^{\beta}$$

Need to bound the potential on the pseudo-periodic configurations $\tilde{\Gamma}$.




C
A
B

Apollonius spheres



$\delta(T)$ - circumdiameter of the tetrahedron
ρ - radius of the sphere in which the points can move

References

1. *D. Dereudre and F. Lavancier. Practical simulation and estimation for Gibbs Delaunay-Voronoi tessellations with geometric hardcore interaction. Computational Statistics and Data Analysis, 55(1):498-519, 2011.*
2. *D. Dereudre, R. Drouilhet, and H.O. Georgii. Existence of gibbsian point processes with geometry-dependent interactions. Probability Theory and Related Fields, 153(3):643-670, 2012*
3. *Fropuff. The vertex configuration of a tetrahedral-octahedral honeycomb., 2006. URL https://en.wikipedia.org/wiki/File:TetraOctaHoneycomb-VertexConfig.svg*