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MASTER THESIS

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Generalized Random Tessellations

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Contents

Introduction

1. Geometric preliminaries

Are graphs geometric? I mean, geometric graphs are geometric. But graphs in general? Are potentials part of this?

Before diving into the mathematics of Gibbs-Laguerre-Delaunay tetrihedrization models, we must first lay out the fundamentals of their geometric and combinatorial structure. The key geometric component is the empty sphere property [...] which determines the edge structure, which is in turn analyzed in terms of hypergraphs.

$\mathcal{F}$ or $\mathcal{N}$

Let $\mathcal{F}_{lf}$ be the set of locally finite sets on $\mathbb{R}^3$, and $\mathcal{F}_f \subset \mathcal{F}_{lf}$ the set of all finite sets on $\mathbb{R}^3$. An elements of F_{lf} will be usually denoted $\mathbb{x}$ and called a *configuration* and its subset η. If $|\eta| = 4$, as will be the case for the majority of this text, then η will be called *tetrahedron*.

1.1 Tetrahedrizations

The aim of this section is to introduce the geometric concepts necessary for the definition of the hypergraph structures in the following section. Definitions might be postponed. This text is concerned with two types of tetrihedrizations.

We introduce the notion of (reinforced) general position. This requirement will be later relaxed.

Definition 1. Let $\mathbb{x} \in \mathcal{F}_{lf}$. We say $\mathbb{x}$ is in **general position** if

$$\eta \subset \mathbb{x}, 2 \leq |\eta| \leq 3 \Rightarrow \eta \text{ is affinely independent.}$$

Denote $\mathcal{F}_{gp} \subset \mathcal{F}_{lf}$ the set of all locally finite configurations in general position.

Commment on measurability of the set of locally finite sets in general position. This comes from cite[Zessin2008] and the $\mathcal{F}$ $\mathcal{M}$ equivalence?

It's sufficient to check only subsets with $d+1$ points

Definition 2. Let $\mathbb{x} \in \mathcal{F}_{gp}$. We say $\mathbb{x}$ is in **reinforced general position** if

$$\eta \subset \mathbb{x}, 3 \leq |\eta| \leq 4 \Rightarrow \eta \text{ is non-circular.}$$

Denote $\mathcal{F}_{rgp}$ the set of all locally finite configurations in reinforced general position.

Define cocircular in general

Again, only need to check $d+2$

1.1.1 Delaunay tetrihedrization

Definition 3. Let $\eta \in \mathcal{F}_{gp}$, $|\eta| = 4$ be a tetrahedron. The open ball $B(\eta)$ such that $\eta \subset \partial B(\eta)$ is called a *circumball*. The boundary $\partial B(\eta)$ is called a *circumsphere*.

Note that the circumball is uniquely defined by η.

Definition 4. Let $\mathbb{x} \in \mathcal{F}_{lf}$ and $\eta \subset \mathbb{x}$. We say that $(\eta, \mathbb{x})$ satisfies the *empty sphere property* if $B(\eta) \cap \mathbb{x} = \emptyset$.

1.1.2 Laguerre tetrihedrization

1.2 Hypergraph structures

Both Delaunay and Laguerre tetrihedrizations can be seen as graphs where two points $p, q \in \mathbb{x}$ are joined if they are part of the same tetrahedron. For the purposes of this text, a more natural structure will be the hypergraph.

satisfying ESP or sth

Definition 5. A *hypergraph structure* is a measurable subset $\mathcal{E}$ of $(F_f \times N, \mathcal{F}_f \otimes \mathcal{F})$ such that $\eta \subset \mathbb{x}$ for all $(\eta, \mathbb{x}) \in \mathcal{E}$. We call η a *hyperedge* of $\mathbb{x}$ and write $\eta \in \mathcal{E}(\mathbb{x})$, where $\mathcal{E}(\mathbb{x}) = \{\eta : (\eta, \mathbb{x}) \in \mathcal{E}\}$. For a given $\mathbb{x} \in \mathcal{F}_{lf}$, the pair $(\mathbb{x}, \mathcal{E}(\mathbb{x}))$ is called a hypergraph.

A hypergraph is thus a generalization of a graph in the sense that edges are now allowed to "join" any number of points. A hypergraph structure can be thought of as a rule that turns a configuration $\mathbb{x}$ into a hypergraph $(\mathbb{x}, \mathcal{E}(\mathbb{x}))$.

2. Stochastic geometry

2.1 Gibbs point process

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A.1 Section

List of Abbreviations