Department of Probability and Mathematical Statistics

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

Daniel Jahn

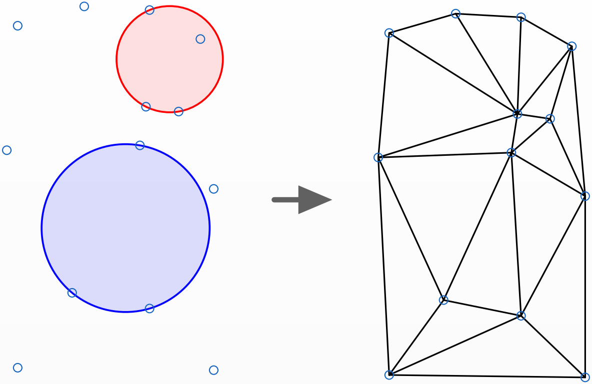
Generalized random tessellations, their properties, simulation and applications

Thesis defense

5 February 2019

Delaunay tetrahedrization

Let γ be a locally finite subset $\mathbb{R}^3$.



$$\mathcal{D}_4(\gamma) = \{\eta \subset \gamma : \mathrm{card}(\eta) = 4, \eta \text{ satisfies the empty ball property } \}$$

Point processes

$\mathcal{B}$ Borel σ-algebra on $\mathbb{R}^3$, $\mathcal{B}_0$ bounded Borel sets, $|\cdot|$ is the Lebesgue measure.

Point process: $\Phi : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathbf{N}_{lf}, \mathcal{N}_{lf})$ where

- $\mathbf{N}_{lf}$ is the set of all locally finite simple counting measures on $\mathbb{R}^3$.
- $\mathcal{N}_{lf}$ is generated by sets of the form

$$\{\nu \in \mathbf{N}_{lf} | \ \nu(\Lambda) = n\}, n \in \mathbf{N}_{lf}, \Lambda \in \mathcal{B}.$$

Point processes

$\mathcal{B}$ Borel σ-algebra on $\mathbb{R}^3$, $\mathcal{B}_0$ bounded Borel sets, $|\cdot|$ is the Lebesgue measure.

Point process: $\Phi : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathbf{N}_{lf}, \mathcal{N}_{lf})$ where

- $\mathbf{N}_{lf}$ is the set of all locally finite simple counting measures on $\mathbb{R}^3$.
- $\mathcal{N}_{lf}$ is generated by sets of the form

$$\{\nu \in \mathbf{N}_{lf} |\ \nu(\Lambda) = n\}, n \in \mathbf{N}_{lf}, \Lambda \in \mathcal{B}.$$

Poisson point process with intensity $z > 0$ is a point process Φ such that

- $\Phi(B) \sim Pois(z|B|)$ for each $B \in \mathcal{B}_0$,
- $\Phi(B_1), \dots, \Phi(B_n)$ are independent for each $n \in \mathbb{N}$ and $B_1, \dots, B_n \in \mathcal{B}_0$ pairwise disjoint.

Point processes

$\mathcal{B}$ Borel σ-algebra on $\mathbb{R}^3$, $\mathcal{B}_0$ bounded Borel sets, $|\cdot|$ is the Lebesgue measure.

Point process: $\Phi : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathbf{N}_{lf}, \mathcal{N}_{lf})$ where

- $\mathbf{N}_{lf}$ is the set of all locally finite simple counting measures on $\mathbb{R}^3$.
- $\mathcal{N}_{lf}$ is generated by sets of the form

$$\{\nu \in \mathbf{N}_{lf} |\ \nu(\Lambda) = n\}, n \in \mathbf{N}_{lf}, \Lambda \in \mathcal{B}.$$

Poisson point process with intensity $z > 0$ is a point process Φ such that

- $\Phi(B) \sim Pois(z|B|)$ for each $B \in \mathcal{B}_0$,
- $\Phi(B_1), \ldots, \Phi(B_n)$ are independent for each $n \in \mathbb{N}$ and $B_1, \ldots, B_n \in \mathcal{B}_0$ pairwise disjoint.

For $\Lambda \in \mathcal{B}_0$ and Poisson point process Φ with intensity $z > 0$, denote the distribution of $\Phi_\Lambda := \Phi \cap \Lambda$ as Π^z_Λ.
We obtain the probability space $(\mathbf{N}_{lf}, \mathcal{N}_{lf}, \Pi^z_\Lambda)$.

Denote

$$\mathbf{N}_\Lambda = \{\nu \in \mathbf{N}_{lf} : \nu(\mathbb{R}^3 \setminus \Lambda) = 0\}.$$

A translation-invariant probability measure P on $(\mathbf{N}_{lf}, \mathcal{N}_{lf})$ is a Gibbs measure with activity $z > 0$ if it satisfies the Dobrushin-Lanford-Ruelle equation:

$$\int f dP = \int \frac{1}{Z_\Lambda^z(\gamma)} \int_{\mathbf{N}_\Lambda} f(\zeta \cup \gamma_{\Lambda^c}) e^{-H_\Lambda(\zeta \cup \gamma_{\Lambda^c})} \Pi_\Lambda^z(d\zeta) P(d\gamma)$$

for every $\Lambda \in \mathcal{B}_0$ and every measurable $f : \mathbf{N}_{lf} \to [0, \infty)$.
$Z_\Lambda^z(\gamma) = \int e^{H_\Lambda(\zeta \cup \gamma_{\Lambda^c})} \Pi_\Lambda^z(d\zeta)$ is the partition function
H_Λ is the energy function

$$H_\Lambda(\gamma) = \sum_{\eta \in \mathcal{E}_\Lambda(\gamma)} \varphi(\eta, \gamma),$$

such that $Z_\Lambda^z(\gamma) < \infty$.

A point process whose distribution is a Gibbs measure is called a Gibbs point process.

D. Dereudre and F. Lavancier. Practical simulation and estimation for Gibbs Delaunay-Voronoi tessellations with geometric hardcore interaction (2011)

Considered Delaunay triangulation in $\mathbb{R}^2$ using the potential

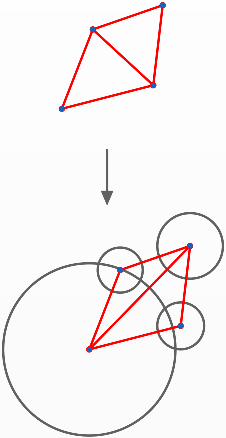
$$\varphi(\eta) = \begin{cases} \infty & \text{if } l(T) \leq \epsilon, \\ \infty & \text{if } \chi(T) \geq \alpha, \\ \theta Per(T) & \text{otherwise,} \end{cases}$$

where

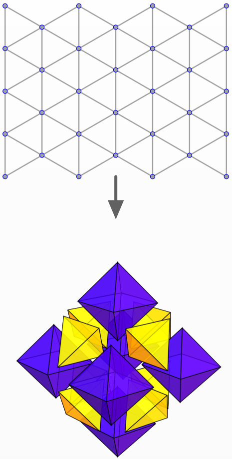
- $l(T)$ is the lenth of the shortest edge of the triangle T.
- $\chi(T)$ is the circumradius of T.
- $Per(T)$ is the perimeter of the triangle T.
- $\theta \in \mathbb{R}$, $\epsilon \geq 0, \alpha > 0$ such that $2\epsilon < \alpha < 1/2$.

Our work

Delaunay → Laguerre

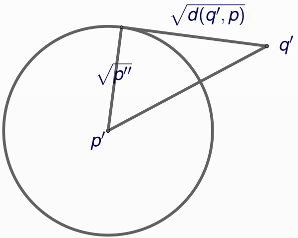


2D → 3D



- Generating points are now both locations and weights. Can be geometrically interpreted as spheres.
- $\gamma = \{p_1, \ldots, p_n\} = \{(p'_1, p''_1), \ldots, (p'_n, p''_n)\}$ can be thought of as marked point process.
- $\gamma \subset \mathbb{R}^3 \times S$, where $S = [0, W]$, $W > 0$.
- Distance is not Euclidean, but the power distance

$$d(q', p) = \|q' - p'\|^2 - p''.$$



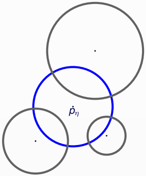
- Instead of circumscribed ball, we have the characteristic point $p_\eta = (p'_\eta, p''_\eta)$.

$$d(p'_\eta, p_i) = p''_\eta, \quad \text{for each } i = 1, \dots, 4.$$

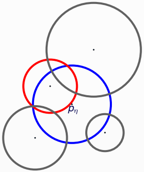
- Instead of the empty sphere property, we have regularity

η is regular if there is no other point $q \in \gamma$ such that $d(p'_\eta, q) < p''_\eta$.

$$\mathcal{LD}_4(\gamma) = \{\eta \subset \gamma : \mathrm{card}(\eta) = 4, \eta \text{ is regular}\}$$



Points of η (gray) and their characteristic point p_η (blue).



A point (red) breaking the regularity of η.

Theoretical results

Theoretical contribution: Existence of Gibbs-measures for a class of potentials for the Laguerre-Delaunay tetrahedrization in $\mathbb{R}^3$.

- Based on *Dereudre, Drouilhet, Georgii: Existence of gibbsian point processes with geometry-dependent interactions (2012)*.
- Instead of treating edges, faces, etc. individually, it treats the structure of a the tessellation as a hypergraph.
- General existence results in $\mathbb{R}^d$.

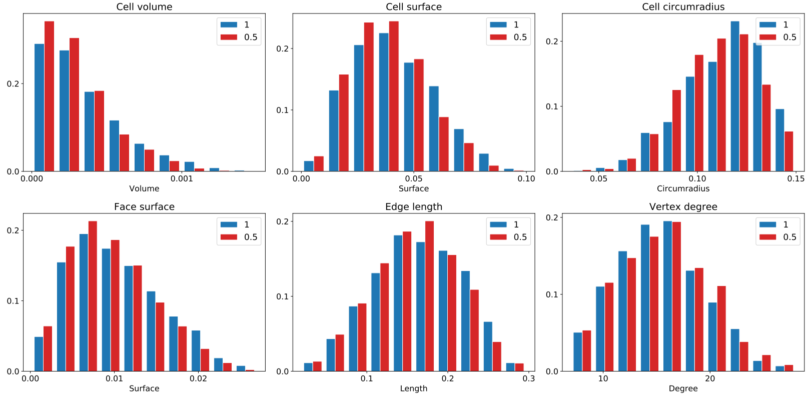
Limitations

- Details only in $\mathbb{R}^2$.
- Does not the treat marked case - only Delaunay, not Laguerre.

Practical results
Implementation

The partition function Z_Λ^z is unknown $\rightarrow$ Birth-Death-Move Metropolis-Hastings algorithm.
Each step has to reconstruct a new tetrahedrization.

- Initial plan: to build a more general tetrahedronization algorithm.
  Discontinued, available at https://github.com/DahnJ/General-Increment-Decrement.git
- Final solution: implement the MCMC algorithm and rely on Computer Graphics Algorithms Library for handling the tetrahedrization.
  Current version at https://github.com/DahnJ/Gibbs-Laguerre-Delaunay.git
- Simulations done in `C++`, numerical analysis in `Python` and `Mathematica`.



Comparison of the distribution of facet statistics for one realization of a model with maximum circumradius $\alpha = 0.15$, $z = 500$, and $\theta = 0.5, 1$.

References

1. *D. Dereudre and F. Lavancier. Practical simulation and estimation for Gibbs Delaunay-Voronoi tessellations with geometric hardcore interaction. Computational Statistics and Data Analysis, 55(1):498-519, 2011.*
2. *D. Dereudre, R. Drouilhet, and H.O. Georgii. Existence of gibbsian point processes with geometry-dependent interactions. Probability Theory and Related Fields, 153(3):643-670, 2012*
3. *Fropuff. The vertex configuration of a tetrahedral-octahedral honeycomb., 2006. URL https://en.wikipedia.org/wiki/File:TetraOctaHoneycomb-VertexConfig.svg*

(1) (Reinforced) general position

Definition. General position

Let $\gamma \in \mathbf{N}_{lf}$. We say γ is in **general position** if

$$\eta \subset \gamma, 2 \leq \mathrm{card}(\eta) \leq 4 \Rightarrow \eta' \text{ is affinely independent in } \mathbb{R}^3.$$

Denote $\mathbf{N}_{gp} \subset \mathbf{N}_{lf}$ the set of all locally finite configurations in general position.

We call points $\{x_0', x_1', \ldots, x_k'\} \subset \mathbb{R}^3, k \in \mathbb{N}$ *cospherical* if there exists a sphere $S \subset \mathbb{R}^3$ such that $\{x_0', \ldots, x_k'\} \subset S$. In this text, a sphere will always refer to the boundary of a ball, never to the interior.

Definition. Reinforced general position

Let $\gamma \in \mathbf{N}_{gp}$. We say γ is in **reinforced general position** if

$$\eta \subset \gamma, \mathrm{card}(\eta) = 4 \Rightarrow \eta' \text{ is not cospherical.}$$

Denote $\mathbf{N}_{rgp}$ the set of all locally finite configurations in reinforced general position.

(2) Set $\mathcal{E}_\Lambda$

$$\mathbf{N}_\Lambda = \{\nu \in \mathbf{N}_{lf} : \nu((\mathbb{R}^3 \setminus \Lambda) \times S) = 0\}$$

Definition.

Let $\Lambda \in \mathcal{B}_0$. Define the set

$$\mathcal{E}_\Lambda(\gamma) := \{\eta \in \mathcal{E}(\gamma) : \varphi(\eta, \zeta \cup \gamma_{\Lambda^c}) \neq \varphi(\eta, \gamma) \text{ for some } \zeta \in \mathbf{N}_\Lambda\}.$$

Recall that we have defined $\varphi = 0$ on $\mathcal{E}^c$. This means that for $\eta \in \mathcal{E}(\gamma)$ such that $\varphi(\eta, \gamma) \neq 0$ we have

$$\eta \notin \mathcal{E}(\zeta \cup \gamma_{\Lambda^c}) \text{ for some } \zeta \in \mathbf{N}_\Lambda \Rightarrow \eta \in \mathcal{E}_\Lambda(\gamma).$$

$$\mathbf{N}_\Lambda = \{\nu \in \mathbf{N}_{lf} : \nu((\mathbb{R}^3 \setminus \Lambda) \times S) = 0\}$$

Definition.

Let $\Lambda \in \mathcal{B}_0$. Define the set

$$\mathcal{E}_\Lambda(\gamma) := \{\eta \in \mathcal{E}(\gamma) : \varphi(\eta, \zeta \cup \gamma_{\Lambda^c}) \neq \varphi(\eta, \gamma) \text{ for some } \zeta \in \mathbf{N}_\Lambda\}.$$

Recall that we have defined $\varphi = 0$ on $\mathcal{E}^c$. This means that for $\eta \in \mathcal{E}(\gamma)$ such that $\varphi(\eta, \gamma) \neq 0$ we have the following implication:

$$\eta \notin \mathcal{E}(\zeta \cup \gamma_{\Lambda^c}) \text{ for some } \zeta \in \mathbf{N}_\Lambda \Rightarrow \eta \in \mathcal{E}_\Lambda(\gamma).$$

(3) Characterization of sets $\mathcal{D}_\Lambda$ and $\mathcal{LD}_\Lambda$

Convention: $\varphi(\eta, \gamma) = 0$ if $\eta \notin \mathcal{D}(\gamma)$.
For $\mathcal{D}$, we have that

$$\eta \in \mathcal{D}_\Lambda(\gamma) \iff B(\eta) \cap \Lambda \neq \emptyset.$$

For $\mathcal{LD}$, we obtain

$$\eta \in \mathcal{LD}_\Lambda(\gamma) \iff d(p'_\eta; \Lambda) < \sqrt{p''_\eta + W},$$

where $d(p'_\eta; \Lambda) = \inf\{\|p'_\eta - x\| : x \in \Lambda\}$ is the distance of p'_η from Λ.

(3) Characterization of sets $\mathcal{D}_\Lambda$ and $\mathcal{LD}_\Lambda$

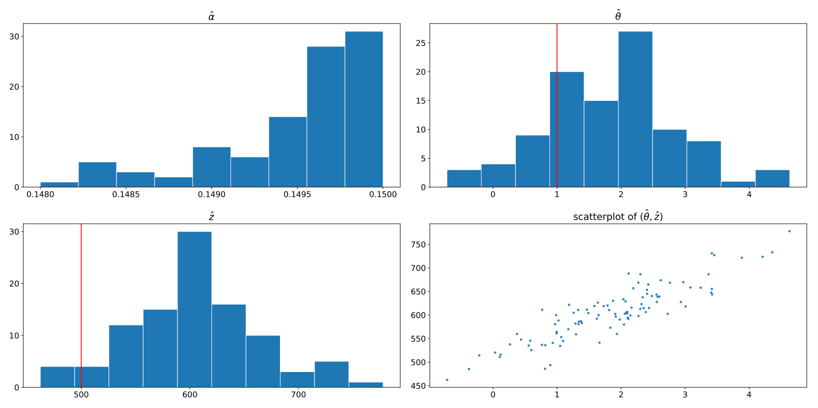
Convention: $\varphi(\eta, \gamma) = 0$ if $\eta \notin \mathcal{D}(\gamma)$.
Assume $\varphi(\eta, \gamma) \neq 0$ For $\mathcal{D}$, we have that

$$\eta \in \mathcal{D}_\Lambda(\gamma) \iff B(\eta) \cap \Lambda \neq \emptyset.$$

For $\mathcal{LD}$, we obtain

$$\eta \in \mathcal{LD}_\Lambda(\gamma) \iff d(p'_\eta; \Lambda) < \sqrt{p''_\eta + W},$$

where $d(p'_\eta; \Lambda) = \inf\{\|p'_\eta - x\| : x \in \Lambda\}$ is the distance of p'_η from Λ.



Estimation results for the model D+ with parameters $\theta = 1, \alpha = 0.15, z = 500, W = 0.01$ for 100 realizations. Average number of removable points: 516.

Range condition

Simplest condition is finite range, not satisfied by our models.

A set $\Delta \in \mathcal{B}_0$ is a finite horizon for the pair (η, γ) and the potential φ if for all $\tilde{\gamma} \in \mathbf{N}_{lf}, \tilde{\gamma} = \gamma$ on $\Delta \times S$

$$(\eta, \tilde{\gamma}) \in \mathcal{E} \text{ and } \varphi(\eta, \tilde{\gamma}) = \varphi(\eta, \gamma).$$

- Range condition. There exist constants $\ell_R, n_R \in \mathbb{N}$ and $\chi_R < \infty$ such that for all $(\eta, \gamma) \in \mathcal{E}$ there exists a finite horizon Δ satisfying: For every $x, y \in \Delta$ there exist ℓ open balls $B_1, \ldots, B_\ell$ (with $\ell \leq \ell_R$) such that
  - the set $\cup_{i=1}^{\ell} \bar{B}_i$ is connected and contains x and y, and
  - for each i, either $\text{diam} B_i \leq \chi_R$ or $\gamma(B_i \times S) \leq n_R$.

Apollonius problem