Nonisometric Flows On Planar Curves via Reduced Coordinates

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Abstract

This project generalizes part of the work of Crane, Pinkall, Schroder in [**1**] on computing flows via reduced coordinates. We find nonisometric flows less stable than the isometric version, particularly on large inputs. A scaling term on our length update rule is introduced to mitigate instability and compare edge length metrics, robustness, and aesthetic.

Background

A closed C^1 curve $\gamma : [0, L] \to \mathbb{R}^2$ can be discretized by sampling a collection of points $f_1, ..., f_m \in \mathbb{R}^2$ and linearly interpolating. In the smooth setting we can perform perform curvature flow by minimizing the wilmore energy functional

$$E[\gamma] = \int_0^1 \kappa^2 dl$$

where κ is the curvature. Its discretization is given by

$$E[\gamma] = \sum_i \frac{\kappa_i^2}{(l_{i-1,i} + l_{i,i+1})/2}$$

It is natural(and easy) to minimize this in terms of reduced coordinates(lengths and angles) leading to the algorithm:

1. Evaluate lengths/curvatures
2. Calculate gradient ∇ of $E[\gamma]$
3. Project ∇ for closedness and 2π curvature
4. Solve Poisson equation to minimize discrete error
5. Update lengths/curvatures via ∇ with step h
6. Reconstruct curve

Results

We try implemented this algorithm to allow for variable edge lengths and considered three energies to minimize

$$E_w[\gamma] = \sum_i \frac{\kappa_i^2}{(l_{i-1,i} + l_{i,i+1})/2}$$

$$E_{sl} = E_w[\gamma] + \sum_i l_{i,i+1}^2 \qquad E_{sc} = \sum_i \kappa_i^2$$

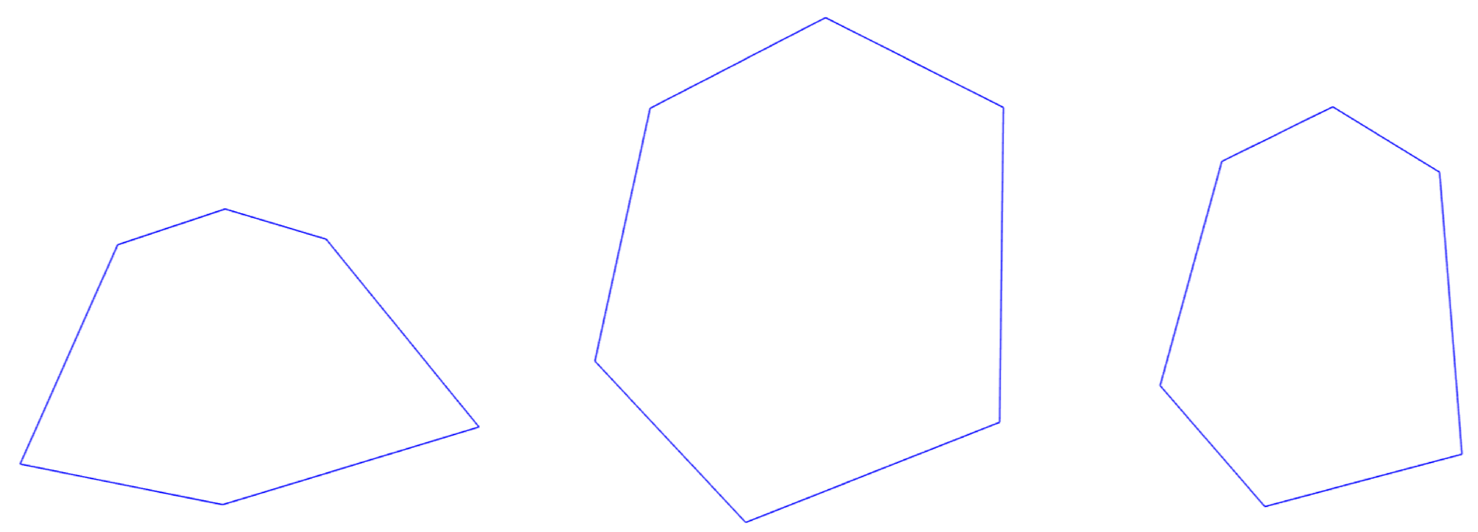

Fig 1: A wilmore flow on the leftmost initial curve produces the middle curve. Squared curvatures produces the right.

On small inputs the flows behave as expected without modification. A wilmore flow on nonisometric inputs degenerates as edge lengths blow up to reduce energy. However it is useful for reducing edge length variance, producing a "smoother" quality than squared curvatures(and squared lengths). In the above figure edge lengths are roughly doubled by wilmore but variance is notably reduced.

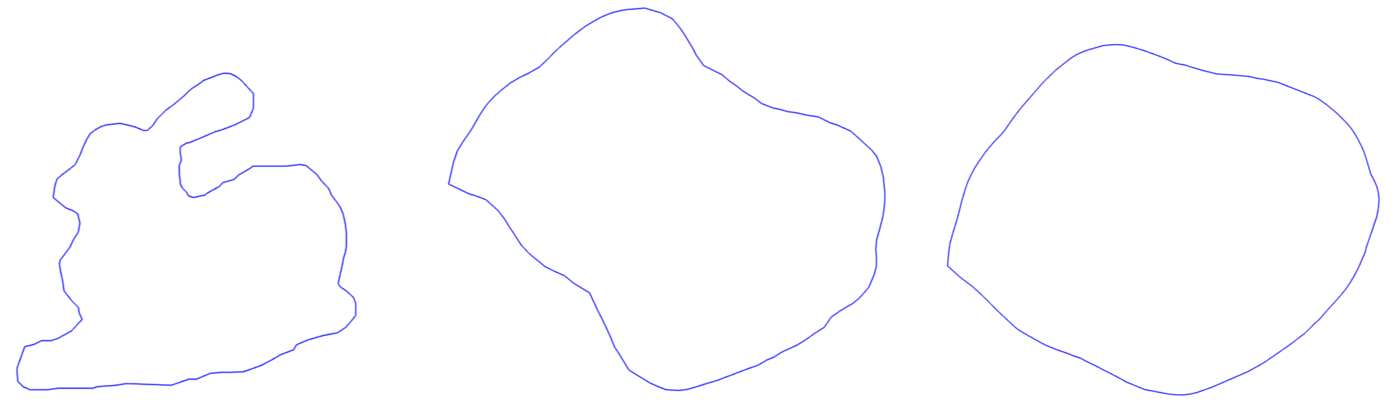

Fig 2: Devolution of a bunny. Wilmore in the middle and squared curvatures on the right.

Results cont.

Unfortunately the nonisometric flows do not scale well to large inputs(> 1000 vertices). The wilmore flow and squared lengths are unable to take large time steps without degenerating. To address this we introduce a paramter $g > 0$ modifying the edge length step size relative to curvature step size. Empirically we found $g \approx .1$ works well when $h \approx .001$. The squared curvatures flow fairs much better due to independence from edge length. We take are able to take step size $h < 0.5$ while retaining stability, leading to a faster convergence.
All three flows tend to develop cusps after reaching global minima, likely due to specifics of the reconstruction procedure.
Try it here!

Future Work

We plan to address the formation of sharp cusps. Further we would like to continue experimenting with various types of energies and descent methods. This also naturally generalizes to flows on space curves as well as surfaces.

References

[1] Keenan Crane, Ulrich Pinkall, and Peter Schröder. 2013. Robust fairing via conformal curvature flow. ACM Trans. Graph. 32, 4, Article 61 (July 2013)
[2] Alexander I. Bobenko and Peter Schröder. 2005. Discrete Willmore flow. In Proceedings of the third Eurographics symposium on Geometry processing (SGP '05) Article 101.