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Hw 7

Question 1:

Prop 1. *Show* $x : \mathbb{R}^2 \to \mathbb{R}^4$ *given by*

$$x(\theta, \phi) = \frac{1}{\sqrt{2}}(cos(\theta), sin(\theta), cos(\phi), sin(\phi)), (\theta, \phi) \in \mathbb{R}^2$$

is an immersion

Proof. Clearly the image of x lies in S^3. To show an immersion we just show the differential is injective. Compute

$$dx(\theta, \phi) = \frac{1}{\sqrt{2}} \begin{bmatrix} -sin(\theta) & 0 \\ cos(\theta) & 0 \\ 0 & -sin(\phi) \\ 0 & cos(\phi) \end{bmatrix}$$

Which is clearly injective for all θ, ϕ. Note we've showed previously this a torus. So to conclude we show this has sectional curvature 0.

Recall we define

$$K(x, y) = \frac{\langle R(x, y)x, y\rangle}{|x \wedge y|^2}$$

We take the area formula to be 1 and compute for basis tangent vectors $\frac{d}{d\theta}, \frac{d}{d\phi}$ using Gauss theorem

$$\begin{aligned} K(\frac{d}{d\theta}, \frac{d}{d\phi}) &= \langle B(\frac{d}{d\theta}, \frac{d}{d\phi}), B(\frac{d}{d\theta}, \frac{d}{d\phi})\rangle - |B(\frac{d}{d\theta}, \frac{d}{d\phi})|^2 \\ &= \langle \overline{\nabla}_{\overline{\frac{d}{d\theta}}} \overline{\frac{d}{d\theta}} - \nabla_{d/d\theta} \frac{d}{d\theta}, \overline{\nabla}_{\overline{\frac{d}{d\phi}}} \overline{\frac{d}{d\phi}} - \nabla_{d/d\phi} \frac{d}{d\phi}\rangle - |\overline{\nabla}_{\overline{d/d\theta}} \overline{\frac{d}{d\phi}} - \nabla_{d/d\theta} \frac{d}{d\phi}|^2 \\ &= \langle \overline{\nabla}_{\overline{d/d\theta}} \overline{\frac{d}{d\theta}}, \overline{\nabla}_{d/d\phi} \overline{\frac{d}{d\phi}}\rangle - |\overline{\nabla} \overline{\frac{d}{d\phi}}|^2 \end{aligned}$$

Further note $\nabla_{d/d\theta} \frac{d}{d\theta} = \nabla_{d/d\phi} \frac{d}{d\phi} = 0$ and $\nabla_{d/d\theta} \frac{d}{d\phi} = 0$. Further extending $d/d\theta$ we find $\overline{\nabla}_{\overline{d/d\theta}} \overline{\frac{d}{d\theta}} = 1/2(-x_1, -x_2, 0, 0)$ and similarly for $d/d\phi$.

We then have $\langle \overline{\nabla}_{\overline{d/d\theta}} \overline{\frac{d}{d\theta}}, \overline{\nabla}_{d/d\phi} \overline{\frac{d}{d\phi}}\rangle - |\overline{\nabla} \overline{\frac{d}{d\phi}}|^2 = \langle -(x_1, -x_2, 0, 0), (0, 0, -x_3, -x_4)\rangle - 0 = 0$ to conclude the proof

$\square$

Question 2:

Prop 2. *Prove that the sectional curvature of the Riemannian manifold $M = S^2 \times S^2 = M_1 \times M_2$ with the product metric, where S^2 is the unit sphere in $\mathbb{R}^3$, is non-negative. Find a totally geodesic, flat torus, T^2, embedded in $S^2 \times S^2$.*

Proof. We know our product metric defined as $\langle (X_1, X_2), (Y_1, Y_2) \langle_M = \langle X_1, Y_1 \rangle_{M_1} + \langle X_2, Y_2 \rangle_{M_2}$. We know M_1, M_2 have constant sectional curvature. Let $X_1, .., X_m$ basis of open set in M and $Y_1, ..., Y_n$ basis of open set in N. Note we have $\nabla_{X_i} Y_j = \nabla_{Y_j} X_i = 0$ in where ∇ is the corresponding riemannian connection to the product metric. Then we see for u, v, w which are written as a sum of both X_i and Y_i, the curvature tensor $R(u, v)w$ is 0 since we have $R(u, v)w = \nabla_u \nabla_v w -_v \nabla_u w + \nabla_{[u,v]} w$. Then since S^2 has nonnegative curvature, $S^2 \times S^2$ has nonnegative curvature, since it is otherwise 0.

Now we consider $S^1 \times S^1$ embedded in $S^2 \times S^2$ flat and totally geodesic. The total geodesic part is clear since locally at each point S^1 is a geodesic in S^2. Further it's also flat via an argument to the above. Thus the product is totally geodesic follows from problem 4 in chapter 6. □

Question 3:

Let x be the immersion defined in Question 1.

Prop 3. *The vectors $e_1 = (-sin(\theta), cos(\theta), 0, 0)$ and $e_2 = (0, 0, -sin(\phi), cos(\phi))$ form an orthonormal basis of the tangent space, and that the vectors $n_1 = \frac{1}{\sqrt{2}}(cos(\theta), sin(\theta), cos(\phi), sin(\phi))$ with $n_2 = \frac{1}{\sqrt{2}}(-cos(\theta), -sin(\theta), cos(\phi), sin(\phi))$ form an orthonormal basis of the normal space*

Proof. First note under the metric inherited from $\mathbb{R}^4$. e_1 and e_2 are orthnormal. Further note $e_1 = \frac{d}{d\theta}$ and $e_2 = \frac{d}{d\phi}$ and hence span the tangent space.

Further we have n_1, n_2 orthonormal under the inerited metric. They are also orthogonal to e_1, e_2. We know the normal space is 2-dimensional and thus this constitutes a basis. □

Prop 4. *We know $\langle S_{n_k}(e_i), e_j \rangle = -\langle \overline{\nabla}_{e_i} n_k, e_j \rangle = \langle \overline{\nabla}_{e_i} e_j, n_k \rangle$ where $\overline{\nabla}$ is covariant derivative of $\mathbb{R}^4$ and $i, j, k = 1, 2$. Then we claim the matrices of S_{n_1} and S_{n_2} with respet to the basis $\{e_1, e_2\}$ are*

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Proof. Recall the shape operator can be written as

$$S_\eta(x) = -(\overline{\nabla}_X \overline{\eta})^T$$

for $\eta \in T_pM^\perp$ and $x \in T_pM$. Then we compute

$$\begin{aligned}\langle S_{n_1}(e_1), e_1\rangle &= \langle \overline{\nabla}_{e_1} e_1, n_1\rangle = \langle \overline{\nabla}_{\sqrt{2}\frac{d}{d\theta}} \sqrt{2}\frac{d}{d\theta}, n_1\rangle \\ &= \langle(-cos(\theta), -sin(\theta), 0, 0), \frac{1}{\sqrt{2}}(cos(\theta), sin(\theta), cos(\phi), sin(\phi))\rangle \\ &= -1\end{aligned}$$

where we recall $e_1 = \sqrt{2}\frac{d}{d\theta}$. The rest of the computations follow similarly.

□

Prop 5. *x is a minimal immersion*

Proof. We know minimality is equivalent to having 0 mean curvature. Clearly the trace of S_{n_2} is 0, ie the sum of the eigenvalues. Further we know n_2 is orthogonal to T_pT^2 since n_1 orthogonal to S^3, establishing 0 mean curvature since S_{n_2} is the corresponding shape operator and thus minimality.

□

Question 4:

Let $f : \overline{M}^{n+1} \to \mathbb{R}$ be a differentiable function. Let *Hessian*, $Hess(f)$ of f at $p \in \overline{M}$ as the linear operator $Hess(f)Y = \overline{\nabla}_Y grad(f)$ for $Y \in T_p\overline{M}$. Let a be a regular value of f and $M^n \subseteq \overline{M}^{n+1}$ be the hypersurface in $\overline{M}$ defined by $M = \{p \in \overline{M}; f(p) = a\}$.

Prop 6. *The laplacian $\overline{\Delta} f$ is given by $\overline{\Delta} f = trace(Hess(f))$*

Proof. For $p \in \overline{M}$ we let E_i be an orthonormal basis of $T_p\overline{M}$. Then compute

$$\overline{\Delta} f = div_{\overline{M}} \overline{\nabla} f = \sum_i \langle \overline{\nabla}_{E_i} \overline{\nabla} f, E_i\rangle = \sum_i \langle Hess(f)E_i, E_i\rangle = trace(Hess(f))$$

□

Prop 7. *If $X, Y \in X(\overline{M})$ then $\langle Hess(f)Y, X\rangle = \langle Y, Hess(f)X\rangle$. Then $Hess(f)$ is self-adjoint and determines symmetric bilinear form via $Hess(f)(X, Y) = \langle Hess(f)X, Y\rangle$.*

Proof. Compute

$$\begin{aligned}\langle Hess(f)Y, X\rangle &= \langle \overline{\nabla}_Y \overline{\nabla} f, X\rangle = Y\langle \overline{\nabla} f, X\rangle - \langle \overline{\nabla} f, \overline{\nabla}_Y X\rangle \\ &= YX[f] - (\overline{\nabla}_Y X)[f] = XY[f] - (\overline{\nabla}_Y X)[f] = \langle Y, Hess(f)X\rangle\end{aligned}$$

This shows $Hess(f)$ self adjoint and we can then defines the symmetric bilinear form given above, where symmetry comes from self-adjointness. $\square$

Prop 8. *The mean curvature H of $M \subseteq \overline{M}$ is given by*

$$nH = -div(\frac{grad(f)}{|grad(f)|})$$

Proof. We again fix orthonormal vector fields E_i and compute for $\eta = \overline{\nabla} f$ which we wlog assume to have norm 1,

$$\begin{aligned} nH = trace(S_\eta) &= \sum_i \langle S_\eta(E_i), E_i \rangle = \sum_i \langle \overline{\nabla}_\eta \eta, \eta \rangle - \langle \overline{\nabla}_{E_i} \eta, E_i \rangle \\ &= -\sum_i \langle \overline{\nabla}_{E_i} \eta, E_i \rangle = -div_{\overline{M}} \eta = -div(\overline{\nabla} f) \end{aligned}$$

$\square$

Prop 9. *We know every embedded hypersurface $M^n \subseteq \overline{M}^{n+1}$ is locally the inverse image of a regular value. Then the mean curvature H of such a hypersurface is given by $H = -\frac{1}{n} div(N)$ where N is local extension of unit normal vector field on M^n.*

Proof. We can use implicit function theorem then calculate for the corresponding f,

$$\langle \overline{\nabla} f, \frac{d}{dx_i} \rangle = \frac{d}{dx_i}[f] = 0$$

for $i \in [n]$ and then we can conclude $H = (-1/n)div(\overline{\nabla} f) = (-1/n)div(N)$ for some extension N. $\square$