Alex Havrilla

alumhavr

Hw 1

Presentation Problems: 4

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**Question 1:**

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This is really regular value theorem

Let $F : \mathbb{R}^n \to \mathbb{R}$ with $n \geq 2$.

**Claim:** For any regular $\alpha$, $\mathcal{M}_\alpha$ is a differentiable manifold.

Note $x_j$ denotes the first j components of x.

*Proof.* Let $x \in \mathbb{R}^n$ with $F(x) = \alpha$. Because $Df(x) \neq 0$ by assumption we know there exist coordinate $i$ with $Df(x)_i \neq 0$. Without loss of generality suppose this is the nth coordinate. The implicit function theorem tells us there exists open $U \subseteq \mathbb{R}^n$ and $g_x : \mathbb{R}^{n-1} \to \mathbb{R}$ s.t. for $y \in U$, $f(y_{n-1}, g_x(y_{n-1})) = \alpha$ with $g_x$ continuous, differentiable and $g_x(x_{n-1}) = x^n$. Define $\phi_x : U \subseteq \mathbb{R}^{n-1} \to \phi_x(U)$ via $\phi_x(y) = (y, g_x(y))$. Via sequential continuity we have this mapping is continuous under the subset topologuy. Note this is injective, so therefore bijective. Further the inverse $\phi_x^{-1} : \phi_x(U) \to U$ via $\phi_x^{-1}(y) = y_{n-1}$ a projection and hence continouous. This establishes a homeomorphism, and since x arbitrary, shows $\mathcal{M}$ a manifold.

Finally we check that $\phi^{-1} \circ \psi : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is differentiable for overlapping domains. Compute $\phi^{-1}(\psi(x)) = \phi((x, g(x))) = x$ ie. the identity. Clearly this is differentiable.

□

Note this directly generalizes via the Implict Function Theorem to the case $F : \mathbb{R}^n \to \mathbb{R}^m$ with $m < n$ and the jacobian is of rank m.

**Claim:** $\mathcal{M}_\alpha$ is orientable.

*Proof.* As computed above the composition of two overlapping maps is the identity. It has identity matrix as the jacobian and which has positiv determinant and therefore orientable.

□

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**Question 2:**

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**Claim:** $O(n)$ is a differentiable manifold when viewed as a subset of $\mathbb{R}^{n^2}$

*Proof.* We prove this using the theorem proved above in the more general case $F : \mathbb{R}^n \to \mathbb{R}^m$. Consider the function $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{\binom{n}{2}}$ defined via $f(M) = M^TM$(a function into the symmetric matrices). See that $f(M) = I \iff M^T = M^{-1}$ ie. is orthogonal. So it suffices to show $I$ is a regular value of f. To do so we show the jacobian of f has maximal rank when M orthogonal.

Examining the directional derivatives we have $Df(M)V =$

$$lim_{t\to 0}\frac{f(M+tV)-f(M)}{t} = lim\frac{M^TM+tV^TV-M^TM}{t} =$$

$$t$$

The dimension of $O(n)$ is then $n^2 - \binom{n}{2} = n(n-1)/2$ since this is the dimension of the inverse projection.

□

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**Question 3:**

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**Claim:** If $v_\gamma \in \mathcal{T}_p\mathcal{M}$ is given in the chart $(U,\phi)$, as $a^1\frac{\partial}{\partial x_1}(p) + ... + a^n\frac{\partial}{\partial x_n}(p)$ and in the chart $(V,\psi)$ as $b^1\frac{\partial}{\partial y_1}(p) + ... + b^n\frac{\partial}{\partial y_n}(p)$ satisfies

*Proof.* Recall $a^i = \frac{\partial(\phi^{-1}\circ\gamma)_i}{\partial t}|_{t=0}$ and $b^i = \frac{\partial(\psi^{-1}\circ\gamma)_i}{\partial t}|_{t=0}$. Then compute

$$\frac{\partial(\phi^{-1}\circ\gamma)}{\partial t}|_{t=0} = \frac{\partial(\phi^{-1}\circ\psi\circ\psi^{-1}\circ\gamma)}{\partial t}|_{t=0} = D\phi^{-1}\circ\psi_{\psi^{-1}(\gamma(0))}\frac{\partial\psi^{-1}\circ\gamma}{\partial t}|_{t=0}$$

Note that by definition the first term $D\phi^{-1}\psi_{\psi^{-1}(\gamma(0))}$. If we look at the ith component of the expression we find this to be $a^i$. Yet the ith component of $\frac{\partial\psi^{-1}\circ\gamma}{\partial t}|_{t=0}$ is $b^i$. So the change of basis is given by the matrix $D\phi^{-1}\psi_{\psi^{-1}(\gamma(0))}$. □

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**Question 4:**

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***I could present this one***

**Claim:** Suppose $\mathcal{M}$ is a connected manifold and $f : \mathcal{M} \to \mathbb{R}$ s.t. $df = 0$ everywhere on $\mathcal{M}$. Then $f$ is constant function

*Proof.* If $df = 0$ we have $\forall p \in \mathcal{M}$ $df|_p = 0$ ie. $\forall v_\gamma \in \mathcal{T}_p\mathcal{M}$ $df|_p[v] = 0$ or $v_\gamma[f] = \frac{\partial f\circ\gamma}{\partial t}|_{t=0} = 0$ for arbitrary curve $\gamma$.

Now consider arbitrary $p \in \mathcal{M}$. Let $\phi : U \to M$ be a chart with $p \in \phi(U)$. Without loss of generality take $U$ to be convex. Select $q \in \phi(U)$. Consider the line connecting $\phi^{-1}(p)$ to $\phi^{-1}(q)$. Then taking the image of this under $\phi$ defines a curve $\gamma$ with $\gamma(0) = p$ and $\gamma(1) = q$(note that if we have any issues with differentiability we can extend the curve by some small amount $\epsilon$). We know $\frac{\partial f \circ \gamma}{\partial t}|_{t=0} = 0$ but the parameterization of this curve is arbitrary, so we may simply reparameterize s.t. all along the curve we see $\frac{\partial f \circ \gamma}{\partial t} = 0$. This tells us $f$ constant along the curve, and in particular $f(p) = f(q)$. But q was arbitrary so we see for all $x, y \in \phi(U)$, $f(x) = f(p) = f(y)$ so f is constant on $\phi(U)$.

Since this is true for arbitary p we see f is constant in a neighborhood of every point $p \in M$. And then since M is connected, f must be constant on M, since otherwise we could find two open sets U and V disjoint but covering U by taking unions of neighborhoods of points with constant $c_1$ and unions of neighborhoods of poitns with constant $c_2$.

$\square$

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**Question 5:**

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Try inverse function theorem when looking at inverse

**Claim:** $f : P^2 \to \mathbb{R}^4$ defined by $f([x], [y], [z]) = (x^2 - y^2, xy, xz, yz)$. Then $F$ is an embedding

*Proof.* Recall $f : P^2 \to \mathbb{R}^4$ is an embedding if F is a homeomorphism between its domain and image and $df_p$ is injective for all $p \in P^2$(F is an immersion).

Take $p \in P^2$ and compute $dF_p(v_\gamma) = v_\gamma[f] = \sum a^i \frac{\partial}{\partial x^i}[f] = \sum a^i df_p[\frac{\partial}{\partial x_i}]$. So to compute the injectivity of $df_p$ it suffices to study the standard tangent basis vectors at p.

Set $p = ([x], [y], [z])$ and compute $\frac{\partial}{\partial x^i}[f] = \frac{d}{dx^i} f \circ \phi|_{\phi^{-1}(p)} = \frac{d}{dx^i} f(p)$ which is just the $x^i$ partial derivative of f. So we compute the jacobian

$$\begin{bmatrix} 2x & -2y & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{bmatrix}$$

For $p \in P^2$ we see the column vectors are linearly independent. This shows injectivity and hence an immersion.

Now we show f a homeomorphism. Note if $(x^2 - y^2, xy, xz, yz) = (a^2 - b^2, ab, ac, bc)$ then we know $z^2 = c^2$ everything nonzero. This quickly yields injectivity in the nonzero case. If two components are 0 then injectivity is clear. If only one component is 0 then this is a contradiction. This establishes a bijection.

Continuity is clear from sequential convergence. Continuity of the inverse comes from the inverse value theorem,

□