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Hw 2

1 Confusions

For curve γ, $\frac{d\gamma}{dt}$ is a Vector Field. Gives the vector corresponding to "direction of curve". At a point p

$$\frac{d\gamma}{dt}|_p[f] = \frac{d}{dt}(f \circ \gamma)|_{t=0}$$

2 Proofs

Claim: For $X, Y \in \mathbb{X}(\mathcal{M})$ for some manifold $\mathcal{M}$, we have

$$[X, Y] = L_X Y$$

(Note this means lie derivative produces another vector field)

Proof. Let Φ_t be the flow of X.

Recall this is defined as $\Phi : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ via $\Phi_t(p) = \gamma(t)$ where γ solves ODE $\gamma'(t) = X(\gamma(t)), \gamma(0) = p$.(A collection of paths over time flowing along vector field X for some initial condition). (So each vector field produces a flow).

$\forall g \in \mathcal{D}$, $X|_p[g] = \frac{d\Phi_t(p)}{dt}|_{t=0}[g] = \frac{d}{dt}_{t=0} g(\phi_t(p))$ which is true by definition of the flow

Let ψ_s be the flow of Y. For $f \in \mathcal{D}$ set $H(t, s) = f(\Phi_{-t}(\psi_s(\Phi_t(p))))$(flow forward t along X, then s along Y, then back -t along X).

Then $\frac{\partial H}{\partial s}_{(t,0)} = Y|_{\phi_t(p)}[f \circ \phi_{-t}]$ since symbolically this is the same as two lines above(making some substitutions).

Taking a derivative in t yields $\frac{\partial^2 H}{\partial t \partial s}|_{(0,0)} = \frac{d}{dt}|_{t=0} Y|_{\phi_t(p)}[f \circ \phi_{-t}]$

But we know $L_X Y|_p[f] = \frac{d}{dt}|_{t=0} d\phi_{-t}(Y|_{\phi_t(p)})[f]$.

Recall the lie derivative is defined as

$$L_X Y|_p = lim_{t \to 0} \frac{d\phi_{-t} Y|_{\phi_t(p)} - Y|_p}{t} = \frac{d}{dt}|_{t=0} d\phi_{-t}(Y_{\phi_t(p)})$$

where we measure the change in Y at a point p against flows forward along X. Ie. the change in Y against X

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And $\frac{d}{dt}|_{t=0} d\phi_{-t}(Y|_{\phi_t(p)})[f] = \frac{d}{dt}|_{t=0} Y|_{\phi_t(p)}[f \circ \phi_{-t}]$. So lie derivative is cross term of second derivative of H.

Then define $K(r, s, t)$ to show cross terms of second derivative of H also equal to lie bracket.

$\square$