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Question 1:

Let V, W be finite dimensional vector spaces. let $L_2(V \times W)$ be set of all bilinear functions from $V \times W \to \mathbb{R}$.

Prop 1. $(V \otimes W)^* \cong L_2(V \times W) \cong V^* \otimes W^*$

Proof. We nearly showed in class $(V \otimes W)^* \cong L_2(V \times W)$. We know for each $f_2 \in L_2(V \times W)$ we can find (unique) $f \in (V \otimes W)^*$ s.t. $f_2 = f \circ \psi$ where $\psi(v, w) = v \otimes w$ and uniqueness coming from $f(v \otimes w) = f_2(v, w) = g_2(v, w) = g(v \otimes w)$. This gives rise the isomorphism $\phi : L_2(V \times W) \to (V \otimes W)^*$ via $\phi(f_2) = f$. It suffices to show this is bijective and linear. To verify linearity compute $\phi(\alpha f_2 + \beta g_2)(v \otimes w) = (\alpha f_2 + \beta g_2)(v, w) = \alpha f_2(v, w) + \beta g_2(v, w) = \alpha f(v \otimes w) + \beta g(v \otimes w) = \alpha\phi(f_2)(v, w) + \beta\phi(g_2)(v, w)$ which pointwise establishes the desired equality. Clearly the map is injective since the represntation f for arbitrary f_2 is unique. Further we can establish an analogus linear injection from $(V \otimes W)^* \to L_2(V \times W)$ via $f \mapsto f_2$ and pointwise $f_2(v, w) = f(v \otimes w)$. Thus we have the dimension of the spaces is the same and our linear injections must be bijections, demontrating an isomorphism.

Now we show $L_2(V \times W) \cong V^* \otimes W^*$. Define $\phi : V^* \otimes W^* \to L_2(V \times W)$ via $\phi(f \otimes g) = fg$. Note clearly $\phi(f \otimes g)$ is multilinear for linear f,g. Linearity of ϕ is achieved by extending linearly now that we have defined ϕ for every basis element $f \otimes g$, which is well defined through the multilinearity of the tensor product.

To show the isomorphism we show the natural basis $\{v_i^* w_j^*\}$ for $V^* W^*$ gets mapped to a basis for $L_2(V, \mathbb{R})$ provided by its identification with its isomorphism with $(VW)^*$, namely $\{v_i w_j\}$ for choices of bases $\{v_i\}, \{w_j\}$ on V,W. It suffices to show $\phi(v_i^* w_j^*)(v_k, w_l) = 1$ if $k = i, w = j$ and 0 otherwise. But this is clear as $\phi(v_i^* w_j^*)(v_k, w_l) = v_i^*(v_k) w_j^*(w_l) = 1 \iff i = k, j = l$ since otherwise either $v_i^*(v_k) = 0$ or $w_j^((w_l) = 0$.

$\square$

Question 2:

Let V be a finite dimensional vector space, $Alt_k(V, \mathbb{R})$ be set of alternating multilinear maps from V^k to $\mathbb{R}$.

Prop 2. $\exists$ *natural isomorphism showing* $\Lambda^k(V^*) \cong Alt_k(V, \mathbb{R}) \cong (\Lambda^k(V))^*$

Proof. We show $Alt_k(V, \mathbb{R}) \cong (\Lambda^k(V))^*$. Define $\Phi : Alt_k(V, \mathbb{R}) \to (\Lambda^k(V))^*$ via

$$\Phi(A)(v_1 \wedge ... \wedge v_k) = A(v_1, ..., v_k).$$

We see $\Phi(A) \in (\Lambda^k(V))^*$ since $\Phi(A)$ is clearly scalar homogeneous given properties of the wedge product and for additivity it suffices to define on basis vectors and extend linearly.

Linearity of Φ is clear since $\Phi(aA + bB)(v_1 \wedge ... \wedge v_k) = (aA + bB)(v_1, ..., v_k) = aA(v_1, ..., v_k) + bB(v_1, ..., v_k)$ pointwise. Furthermore via the universal mapping property we see this corresponding $\Phi(A) \in (\Lambda^k(V))^*$ is unique for each A and hence we have injectivity. Note this uniqueness is because A completely defines $\Phi(A)$ and if $\Phi(A) = \Phi(B)$ then we may conclude $A = B$ since $A(v_1, ..., v_k) = \Phi(A)(v_1 \wedge ... \wedge v_k) = \Phi(B)(v_1 \wedge ... \wedge v_k) = B(v_1, ..., v_k)$. Since the finite dimension of $Alt_k(V, \mathbb{R})$ is the same as $(\Lambda^k(V))^*$, $\binom{n}{k}$, we may conclude surjectivity and establish the isomorphism Φ. Note, one way of seeing the dimension of $Alt_k(V, \mathbb{R})$ is $\binom{n}{k}$ is to consider the linear injection from $(\Lambda^k(V))^*$ defined via UMP, where we showed in classed dimension of this space is $\binom{n}{k}$.

Now we show $\Lambda^k(V^*) \cong Alt_k(V, \mathbb{R})$. Consider $\alpha_1 \wedge ... \wedge \alpha_k \in \Lambda^k(V^*)$. We define our isomorphism Φ pointwise s.t.

$$\Phi(\alpha_1 \wedge ... \wedge \alpha_k)(v_1, ..., v_k) = \sum_{\sigma \in S_k} sgn(\sigma) \alpha_1(v_{\sigma_1}) ... \alpha_k(v_{\sigma_k})$$

First we claim this evluates to an alternating mapping. Compute via a reindexing of the sum

$$\begin{aligned}
\Phi(\alpha_1 \wedge ... \wedge \alpha_k)(v_{\pi_1}, ..., v_{\pi_k}) &= \sum_{\sigma \in S_k} sgn(\sigma) \prod_j \alpha_j(v_{\pi_{\sigma_j}}) \\
&= \sum_{\theta \in S_k} sgn(\theta) sgn(\pi) \prod_j \alpha_j(v_{\theta_j}) = sgn(\pi) \Phi(\alpha_1 \wedge ... \wedge \alpha_k)(v_1, ..., v_k)
\end{aligned}$$

Note then we know this is alternating since if $v_i = v_j$ for some $i \neq j$ simply tranpose them and we see $-\Phi(\alpha_1 \wedge ... \wedge \alpha_k) = \Phi(\alpha_1 \wedge ... \wedge \alpha_k) \implies 0$. The multilinearity property follows clearly since the valuation is the sum of k-products of linear functions, each one present in every term.

Linearity of ϕ follows from simply defining pointwise on a basis and extending linearly, which well defined via the alternating nature of the wedge product. We show the rest of the isomorphism by showing a basis $\{\alpha_{i_1}^* \wedge ... \wedge \alpha_{i_k}^*\}$ for $\Lambda^k(V^*)$ maps to a basis for $Alt_k(V; \mathbb{R})$ induced by its isomorphism with $(\Lambda^k(V))^*$ given by $\{(\alpha_{i_1} \wedge ... \wedge \alpha_{i_k})^*\}$. As in prop 1 it suffices to show $\phi(\alpha_{i_1}^* \wedge ... \wedge \alpha_{i_k}^*)(\alpha_{j_1} \wedge ... \wedge \alpha_{j_k}) = 1$ when $i_l = j_l$ for $1 \leq l \leq k$ and 0 otherwise. But this is clear as when $i_l = j_l$ for $1 \leq l \leq k$ the sum over permutations evaluates to 1 with 1 term being 1 and the rest being 0. Otherwise the entire sum evaluates to 0 since at least one term in each of the products with be 0.

$\square$

Question 3:

Let V be a finite dimensional vector space and V^* its dual.

Prop 3. $(\alpha_1 \wedge ... \wedge \alpha_r) \bigwedge (\alpha_{r+1} \wedge ... \wedge \alpha_{r+s}) = \alpha_1 \wedge ... \wedge \alpha_{r+s}$

Proof. We know pointwise we have

$$\alpha_1 \wedge ... \wedge \alpha_{r+s}(v_1, ..., v_{r+s}) = \sum_{\beta \in S_{r+s}} sgn(\beta) \prod_j \alpha_j(v_{\beta_j})$$

so compute

$$r!s!\alpha_1 \wedge .. \wedge \alpha_r \bigwedge \alpha_{r+1} \wedge .. \wedge \alpha_{r+s}(v_1, .., v_{r+s}) = \sum_{\theta \in S_{r+s}} sgn(\theta)\alpha_1 \wedge .. \wedge \alpha_r(v_{\theta_1}, .., v_{\theta_r})\alpha_{r+1} \wedge .. \wedge \alpha_{r+s}(v_{\theta_{r+1}}, .., v_{\theta_{r+s}})$$

$$= \sum_{\theta \in S_{r+s}} sgn(\theta) \sum_{\pi \in S_{\theta([r])}} sgn(\pi) \prod_{j \leq r} \alpha_j(v_{\pi_{\theta_j}}) \sum_{\sigma \in S_{\theta([r+s] \setminus [r])}} sgn(\sigma) \prod_{j \leq s} \alpha_{r+j}(v_{\sigma_{\theta_{r+j}}})$$

$$= \sum_{\theta \in S_{r+s}} \sum_{\pi \in S_{\theta([r])}} \sum_{\sigma \in S_{\theta([r+s] \setminus [r])}} sgn(\theta)sgn(\pi)sgn(\sigma) \prod_{j \leq r} \alpha_j(v_{\pi_{\theta_j}}) \prod_{j \leq s} \alpha_{r+j}(v_{\sigma_{\theta_{r+j}}})$$

Fix $\beta \in S_{r+s}$ and consider the term $sgn(\beta) \prod_j \alpha_j(v_{\beta_j})$. The triple sum has $r!s!(r+s)!$ terms and we claim $r!s!$ of these terms are equal to $sgn(\beta) \prod_j \alpha_j(v_{\beta_j})$ for each $\beta \in S_{r+s}$.

Suppose $sgn(\beta) \prod_j \alpha_j(v_{\beta_j}) = sgn(\theta)sgn(\pi)sgn(\sigma) \prod_{j \leq r} \alpha_j(v_{\pi_{\theta_j}}) \prod_{j \leq s} \alpha_{r+j}(v_{\pi_{\sigma_{r+j}}})$ for some $\theta \in S_{r+s}, \pi \in S_{\theta([r])}, \sigma \in S_{\theta([r+s] \setminus [r])}$. Then it must be $\beta = \pi \circ \sigma \circ \theta$, where we extend π, σ to S_{r+s} via the identity. So in particular $sgn(\beta) = sgn(\pi)sgn(\sigma)sgn(\theta)$. Further for arbitrary θ we can only have $\beta = \pi \circ \sigma \circ \theta$ for some π, σ if $\beta([r]) = \theta([r])$ ie. the image of the first r numbers are permutations of each other(in which case we can then find satisfying σ, π). The number of permutations on S_{r+s} satisfying this for β is $r!s!$(first we order the image of [r] then we order the other half of the partition). Note further the permutations σ, π satisfying $\beta = \pi \circ \sigma \circ \theta$ for suitable θ are unqiue. Hence this shows the claim. We may thus conclude

$$\sum_{\theta \in S_{r+s}} \sum_{\pi \in S_{\theta([r])}} \sum_{\sigma \in S_{\theta([r+s] \setminus [r])}} sgn(\theta)sgn(\pi)sgn(\sigma) \prod_{j \leq r} \alpha_j(v_{\pi_{\theta_j}}) \prod_{j \leq s} \alpha_{r+j}(v_{\sigma_{\theta_{r+j}}}) = \sum_{\beta \in S_{r+s}} r!s!sgn(\beta) \prod_j \alpha_j(v_{\beta_j})$$

which shows the proposition.

$\square$

Question 4:

Prop 4. *Pull-back of a $(0,s)$ tensor. Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a differentiable mapping and S a $(0,s)$ tensor on $\mathcal{N}$. This satisfies*

1. $\Phi^*(S_1 \otimes S_2) = \Phi^*(S_1) \otimes \Phi^*(S_2)$
2. $\Phi^*(\omega_1 \wedge \omega_2) = \Phi^*(\omega_1) \wedge \Phi^*(\omega_2)$
3. $\Phi^*(d\omega) = d\phi^*(\omega)$

Proof. First we show 1.. Let S_1, S_2 be tensors forms. Compute for vectors $u_1, ..., u_r, v_1, ..., v_r$ at an arbitrary point p:

$$\begin{aligned}\Phi^*(S_1 \otimes S_2)|_p(u_1, ..., u_r, v_1, ..., v_r) &= S_1 \otimes S_2|_{\Phi(p)}(d\Phi u_1, ..., d\Phi v_r)\\ &= S_1|_{\Phi(p)}(d\Phi u_1, ..., d\Phi u_r) S_2|_{\Phi(p)}(d\Phi v_1, ..., v_s)\end{aligned}$$

where we evaluate the tensor $S_1 \otimes S_2$ using the isomorphism constructed in problem 1.

Similarly:

$$\begin{aligned}\Phi^* S_1 \otimes \Phi^* S_2|_p(u_1, ..., u_r, v_1, ..., v_s) &= \Phi^* S_1|_{\Phi(p)}(u_1, ..., u_r)\Phi^* S_2|_{\Phi(p)}(v_1, ..., v_s)\\ & S_1|_p(d\Phi u_1, ..., d\Phi u_r) S_2|_p(d\Phi v_1, ..., d\Phi v_s)\end{aligned}$$

which demonstrates the equality pointwise.

Item 2. follows similarly

Lastly we show 3. Note it suffices to show the result for basis vectors $dx_{i_1} \wedge ... \wedge dx_{i_k}$ since the pullback and derivative will distribute over sums. First we examine 0-forms at a point p:

$$\begin{aligned}\Phi^*(df)[v] &= df[d\Phi v] = d\Phi v[f] = v[f \circ \Phi]\\ d\Phi^*(f)[v] &= v[\Phi^* f] = v[f \circ \Phi]\end{aligned}$$

Now we consider arbitrary basis k-form:

$$\begin{aligned}\Phi^*(d(f dx_{i_1} \wedge ... \wedge dx_{i_k})) &= \Phi^*(df \wedge dx_{i_1} ... \wedge dx_{i_k}) = \Phi^*(df) \wedge \Phi^*(dx_{i_1}) \wedge .. \wedge \Phi^*(dx_{i_k})\\ &= d\Phi^*(f) \wedge d\Phi^*(x_{i_1}) \wedge .. \wedge d\Phi^*(x_{i_k}) = d(\Phi^*(f) d\Phi^*(x_{i_1}) \wedge .. \wedge d\Phi^*(x_{i_k}))\\ &= d\Phi^*(f dx_{i_1} \wedge ... \wedge dx_{i_k})\end{aligned}$$

which finishes the proof

□

Question 5:

Prop 5. *Let ω a 1-form on S^2. Suppose for any $\phi \in SO(3)$, $\phi^*\omega = \omega$. Then $\omega = 0$.*

Proof. Fix point $p \in S^2$ and compute for arbitrary vector v at p:

$$\omega|_p[v] = \phi^*\omega|_p[v] = \omega|_{\phi(p)}[d\phi v] = \omega|_{\phi(p)}[\phi v] = 0$$

for the correct choice of rotation ϕ. No vector is rotation invariant under every rotation, and the differential of the rotation is rotation, so this should always be possible. □

Question 6:

Prop 6. *Given an vector field X, we have*

1. *If α and β are forms then $L_X(\alpha \wedge \beta) = L_X\alpha \wedge \beta + \alpha \wedge L_X\beta$*
2. *If ω is a form then $L_X(d\omega) = dL_X(\omega)$*
3. *If Y is a vector field and ω a form then $L_X(i_Y\omega) - i_Y(L_X\omega) = i_{[X,Y]}\omega$*

Proof. First we show 1. Let α, β be forms and compute

$$\begin{aligned}
L_X(\alpha \wedge \beta)|_p = \frac{d}{dt}|_{t=0}\Phi_t^*(\alpha \wedge \beta)|_p = \lim_{t\to 0} \frac{\Phi_t^*(\alpha \wedge \beta)|_p - \alpha \wedge \beta|_p}{t} = \lim_{t\to 0} \frac{\Phi_t^*(\alpha)|_p \wedge \Phi_t^*(\beta)|_p - \alpha \wedge \beta|_p}{t} \\
= \lim_{t\to 0} \frac{\Phi_t^*(\alpha)|_p \wedge \Phi_t^*(\beta)|_p - \Phi_t^*(\alpha)|_p \wedge \beta|_p}{t} + \lim_{t\to 0} \frac{\Phi_t^*(\alpha)|_p \wedge \beta|_p - \alpha \wedge \beta|_p}{t} \\
= \lim_{t\to 0} \Phi_t^*(\alpha)|_p \wedge \frac{\Phi_t^*(\beta)|_p - \beta|_p}{t} + \lim_{t\to 0} \frac{\Phi_t^*(\alpha)|_p - \alpha|_p}{t} \wedge \beta|_p = \alpha|_p \wedge L_X\beta|_p + L_X\alpha|_p \wedge \beta|_p
\end{aligned}$$

which establishes the desired equality pointwise.

Now we show 2. First consider a 0-form f. Then

$$L_X(df)(Y) = \frac{d}{dt}|_{t=0}\Phi_t^*(df) = \frac{d}{dt}|_{t=0}d\Phi_t^*(f) = d\frac{d}{dt}|_{t=0}\Phi_t^*(f) = dL_xf$$

where we justify the third equality pointwise.

Note the result for 0-forms is sufficient to show Cartan's theorem. Then we may use Cartan to conclude for arbitrary ω:

$$d((d \circ i_X + i_X \circ d)(\omega)) = d(d \circ i_X\omega + i_X \circ d\omega) = di_X \circ d\omega = (d \circ i_X + i_X \circ d)(d\omega) = L_X(d\omega)$$

Finally 3. First we show the result is true for 1-forms gdf(it is trivially true for 0-forms). We have

$$\begin{aligned} &i_{[X,Y]}gdf = df([X,Y]) = [X,Y](f) \\ &= L_X(i_Y df) = L_X(df(Y)) = X[df(Y)] = X[Y[f]] \\ &= i_Y(L_X df) = i_Y(dL_X f) = i_Y(X[f]) = Y[X[f]] \end{aligned}$$

so we may conclue $i_{[X,Y]} = L_X(i_Y df) - i_Y(L_X df)$ since $[X,Y] = XY - YX$. This extends to abitrary 1-forms gdf since

$$\begin{aligned} &i_{[X,Y]}gdf = i_{[X,Y]}g \wedge df + g \wedge i_{[X,Y]}df = gi_{[X,Y]}df \\ &L_X(i_Y gdf) = L_X(fi_Y dg) = (L_X f)i_Y dg + fL_X i_Y dg \\ &i_Y(L_X fdg) = L_X fi_Y dg + fi_Y L_X dg \end{aligned}$$

and the cross terms cancel.

We can extend this to arbitrary k-forms via induction. Note via linearity of the lie derivative and interior derivative over sums it suffices to consider a form which can be written $f\alpha \wedge \beta$. We seek to show $L_X(i_Y f\alpha \wedge \beta) - i_Y(L_X f\alpha \wedge \beta) = i_{[X,Y]}f\alpha \wedge \beta$. Compute:

$$L_X(i_Y(\alpha \wedge \beta)) = L_X(\alpha \wedge i_Y\beta + (-1)^k i_Y\alpha \wedge \beta) = L_X\alpha + i_Y\beta + \alpha \wedge L_X i_Y\beta + (-1)^k L_X i_Y\alpha \wedge \beta + (-1)^k i_Y\alpha \wedge \beta$$
$$i_Y(L_X\alpha \wedge \beta) = L_X\alpha \wedge i_Y\beta + (-1)^k i_Y L_X\alpha \wedge \beta + \alpha \wedge i_Y L_x\beta + (-1)^k i_y\alpha \wedge L_X\beta$$

Then the difference is

$$\begin{aligned} &\alpha \wedge L_X i_Y\beta - \alpha \wedge i_Y L_X\beta + (-1)^{k+1}[L_X i_Y\alpha \wedge \beta - i_Y L_X\alpha \wedge \beta] \\ &= \alpha \wedge (L_X i_Y\beta - i_Y L_X\beta) + (-1)^{k+1}(L_X i_Y\alpha - i_Y L_X\alpha) \wedge \beta) \\ &= \alpha \wedge i_{[X,Y]}\beta + (-1)^{k-1}(i_{[X,Y]}\alpha \wedge \beta) \\ &= i_{[X,Y]}\alpha \wedge \beta \end{aligned}$$

Then since every higher order form can be written as this wedge, we are done.

□

Question 7:

Prop 7. *Suppose $\mathcal{M}$ a compact manifold and (U, ϕ) coordinate chart s.t. U bounded. If ω is a 1-form supported in $\phi(U)$ with $d\omega = 0$ then $\omega = df$ for some f.*

Proof. Using Cartan's formula we see for arbitrary vector field X

$$L_X\omega = (d \circ i_X + i_X \circ d)\omega = d \circ i_X(\omega) = d(\omega[X])$$

So if we can produce vector field X s.t. $L_X\omega = \omega$ we would have the desired result.

We could also try integrating over interior areas A with smooth boundary γ in U(since ω subordianted by U in compact $\mathcal{M}$) to see via stokes to see

$$0 = \int_A d\omega = \int_{\partial A} \omega = \int_\gamma \omega$$

which suggests ω conservative and therefore can be written as df for some f. $\square$