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Hw 2

Question 1:

Set $X : \mathcal{M} \to \mathcal{TM}$ a vector field, continuous.

Claim: The following are equivalent:

i) X is a smooth vector field

ii) $\forall p \in \mathcal{M}$ ther exists (U, ϕ) containing p s.t. on $\Omega = \phi(U)$ we have $X = \sum_i^d a^i \frac{\partial}{\partial x_i}$

for smooth $a^i : \Omega \to \mathbb{R}$

iii) For every $f \in C^\infty(\mathcal{M}), X[f]$ is smooth.

Proof. Note $X(p) = (p, \sum a^i(p) \frac{\partial}{\partial x_i})$. Further for chart $\psi : U \to \Omega \subseteq \mathcal{TM}$ we know $\psi^{-1}((p, v)) = (\psi_{\mathcal{M}}^{-1}(p), \psi_{\mathcal{T}_p\mathcal{M}}^{-1}(v))$ where $\psi_{\mathcal{M}}^{-1}$ is some chart on $\mathcal{M}$. Thus it suffices to show the second coordinate smooth when composed(in order to show smoothness of X).

We prove $i) \iff ii)$ and $i) \iff iii)$.

First we show $i) \iff ii)$. Suppose the a^i are smooth. Then X is a sum of smooth function and is thus smooth(since the $\frac{\partial}{\partial x_i}$ are smoothly transitioning). Suppose X smooth. Note in particular then $\psi^{-1}(\sum a^i \partial/\partial x_i \circ \phi$ smooth for charts ψ^{-1} and ϕ. Breaking ψ^{-1} into components ψ_j^{-1} shows each component smooth and hence each a^i smooth.

Now we do $i) \iff iii)$. If X is smooth then we know each a^i is smooth and hence $X[f] = \sum a^i \frac{\partial f}{\partial x_i}$ is a sum of product of smooth functions. Note $\partial f/\partial x_i$ smooth since f smooth. However if each $X[f]$ smooth we know $X[x_i] = a^i$ are smooth and so by $i) \iff ii)$ we have X smooth.

□

Question 2:

Let $V \in \mathcal{X}(\mathcal{M})$ and $p \in \mathcal{M}$. Assume $V(p) \neq 0$.

Claim: There exists a coordinate chart (U, ϕ) containing p s.t. on $\Omega = \phi(U)$ $V = \frac{\partial}{\partial x_1}$

Proof. We can choose some chart (ϕ, U) s.t. $X(p) = \frac{\partial}{\partial x_1}$ at the point via a change of basis(which is possible since $X(p) \neq 0$ and we have a change of basis formula).

We have $\phi : U \subseteq \mathbb{R}^n \to M$ with $\Omega = \phi(U)$ without loss f generality centered at the origin($\phi(0) = p$). We find a chart and open set W satisfying $X = \frac{\partial}{\partial x_1}$. To do this we use flow defined via X as $\Phi_t(p) = \gamma(t)$ where γ solves $\gamma'(t) = X(\gamma(t))$. Set $W = \{p \in \Omega : \phi^{-1}(p)_1 = 0\}$. Then considera map f given by $x \to \phi_{x^1}(0, x^2, ..., x^n)$. The idea is to flow out from this hyperplane along the vector field(which is $\frac{\partial}{\partial x_1}$ at the point p). However geometrically we only ever flow along $\frac{\partial}{\partial x_1}$ since we start at p with $f(0) = p$. So within the image $\widehat{\Omega}$ we have $X = \frac{\partial}{\partial x_1}$. So we must simply argue this is a valid parameterization. Bijectivity is clear via definition of the flow. To argue smoothness we use inverse function theorem at 0. Recall that the derivative of the solution to our flow is $X(\gamma(t))$ where we know $X = \frac{\partial}{\partial x_1}$ at 0. Thus we may apply inverse function theorem showing smoothness and inverse smoothness.

□

Question 3:

Let $(G, \cdot)$ be a group, $\mathcal{M}$ a manifold and $\phi : G \times \mathcal{M} \to \mathcal{M}$ a properly discontinuous action.

Claim: $\mathcal{M} \backslash G$ is orientable $\iff$ $\exists$ an orientation of $\mathcal{M}$ that is preserved by all $\phi_g : \mathcal{M} \to \mathcal{M}$ for all $g \in G$.

Proof. First suppose $\mathcal{M}M \backslash G$ is orientable. For arbitrary charts on $\mathcal{M} \backslash G$ we can write $\pi \circ \phi$ for charts on $\mathcal{M}$ since locally the projection operator will be bijective since the action properly discontinuous. So then we know for overlapping charts $|D(\pi \circ \psi)^{-1} \circ (\pi \circ \phi)| > 0$ given the correct orientation. Further we know for fixed ϕ, ψ, $(\pi \circ \psi)^{-1} \circ (\pi \circ \phi) = \psi^{-1} \circ f_g \circ \phi$ for some diffeomorphism $f_g : \mathcal{M} \to \mathcal{M}$ again via proper discontinuity. So we have for some $g \in G$, $|D\psi^{-1} f_g \phi| > 0$. Note also for arbitrary well-defined (overlapping) composition $\psi^{-1} \circ f_g \circ \phi$ we know this corresponds to a composition of charts on $\mathcal{M} \backslash G$ via $(\pi \circ \psi)^{-1} \circ (\pi \circ \phi)$ where $(\pi \circ \psi)^{-1}$ is defined via f_g with π pulling back to element which is in the image of f_g(well-defined via proper discontinuity). Thus since we can do this for arbitrary composition, we know $|D\psi^{-1} \circ f_g \circ \phi| > 0$ and we are done.

In the other direction suppose we have an orientation of $\mathcal{M}$ which is preserved by all $f_g : \mathcal{M} \to \mathcal{M}, g \in G$. Then we know for arbitrary $g \in G$ and ϕ, ψ s.t. $f_g \circ \phi$ overaps with ψ we have $|D\psi^{-1} \circ f_g \circ \phi| > 0$ without loss of generality. But as described above this is exactly the condition showing orientation of $\mathcal{M} \backslash G$. So we take the atlas $\{\pi \circ \phi : \phi \in A_{\mathcal{M}}\}$ which produces an orientation on $\mathcal{M} \backslash G$.

□

Claim: $\mathbb{P}^2$ is non-orientable, $\mathbb{P}^3$ is orientable

Proof. First note $\mathbb{P}^3$ orientable because we can find an atlas giving $\mathbb{S}^3$ orientation invariant under multiplication by -1. Namely the one given via sterographic projection. Then for arbitrary ϕ, ψ^{-1} we know $|D\psi^{-1}(-\phi)| = (-1)^{n+1}|D\psi^{-1} \circ \phi| > 0$ since $n = 3$. So by the above proof we know $\mathbb{P}^3$ orientable. Yet we see for the same reason that $|D\psi^{-1}(-\phi)| = (-1)^3|D\psi^{-1} \circ \phi| < 0$ for $\mathbb{S}^2$ which flips the orientation and thus $\mathbb{P}^2$ is not orientable

□

Question 4:

Set $\mathcal{M} = SL(n)$. Then $sl(n) = T_I SL(n)$.

Claim: $[[A, B]] = AB - BA$ for $A, Y \in sl(n)$.

Proof. Compute $[[A, B]] = [X_A, Y_B]|_I = (X_A Y_B - Y_B X_A)|_I$ where X_A, Y_B are the left invariant vector fields corresponding to A, B and I is the identity matrix which is the identitiy element of $SL(n)$. We write $X_A(M) = dL_M(A)$ and evaluated on a function is $dL_M(A)[f] = A[f \circ L_M] = A[fM] = \langle \nabla fM, A\rangle = \langle (\nabla fM), MA\rangle$ which justifies $dL_M(A) = M^T A$(which still has trace 0). So

$$(X_A Y_B - Y_B X_A)|_I = X_A Y_B|_I - Y_B X_A|_I =$$

$$X_A(IB) - Y_B(IA) = X_A(B) - Y_B(A) = AB - BA$$

as desired

□

Question 5:

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i)

First we compute the exponential map $exp : sl(n) \to SL(n)$.

We know $\gamma'(t) = X(\gamma(t)) = dL_{\gamma(t)}(X_e)$ but we have $dL_{\gamma(t)}(X_e) = X_e\gamma(t)$ from problem 4 which suggests $\gamma(t) = e^{X_e t}$ via ODE theory. So then $exp(X_e) = \gamma(1) = e^{X_e} = \sum_{k=1}^{\infty} \frac{X_e^k}{k!}$

ii)

Claim: For every lie group G there exists an open neighborhood U of 0 in g s.t. $\Omega = exp(U)$ is an open neighborhood of e and furthermore $exp : U \to \Omega$ is a diffeomorphism.

Proof. Chain rule tells us $exp(tX) = \gamma(t) \implies exp(0) = \gamma(0) = e$. We use implicit function theorem to show a diffeomorphism since the differential of an exponential map is nonzero at 0(invertible).

□