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Hw 2

1 Confusions

For curve γ, $\frac{d\gamma}{dt}$ is a Vector Field. Gives the vector corresponding to "direction of curve". At a point p

$$\frac{d\gamma}{dt}|_p[f] = \frac{d}{dt}(f \circ \gamma)|_{t=0}$$

Why can T_pM be thought of as a copy of $\mathbb{R}^n$?

Meridians and parallels on surface S like longitudes(lines with constant width) and latitudes(geodesics with constant height) on sphere.

Geodesic sphere: Seems to be points of distance r away from point p determined by radiating geodesics of length r from p. Denotes $S_r(p)$.

?How does differential of compositions work? I think trivially: $df(dg) = df \circ g$ since

$$df \circ g[v] = v[f \circ g] = dgv[f] = dfdgv$$

Note here I'm not applying all the way: should really be applying resulting vector to functions $M \to \mathbb{R}$.

?What is the relationship between differential and Affine Connection/Covariant Derivative?

Defining 3 tensors like R' via

$$g(R'(X, Y, W), Z) = \langle X, W\rangle\langle Y, Z\rangle - \langle Y, W\rangle\langle X, Z\rangle$$

ie. an implicit definition. Much like how we can define directional derivatives.

A lot of the formulas for curvatures don't seem to be typechecking for me?

2 Content

Claim: For $X, Y \in \mathbb{X}(\mathcal{M})$ for some manifold $\mathcal{M}$, we have

$$[X, Y] = L_X Y$$

(Note this means lie derivative produces another vector field)

Proof. Let Φ_t be the flow of X.

Recall this is defined as $\Phi : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ via $\Phi_t(p) = \gamma(t)$ where γ solves ODE $\gamma'(t) = X(\gamma(t)), \gamma(0) = p$.(A collection of paths over time flowing along vector field X for some initial condition). (So each vector field produces a flow).

$\forall g \in \mathcal{D}$, $X|_p[g] = \frac{d\Phi_t(p)}{dt}|_{t=0}[g] = \frac{d}{dt}_{t=0} g(\phi_t(p))$ which is true by definition of the flow

Let ψ_s be the flow of Y. For $f \in \mathcal{D}$ set $H(t,s) = f(\Phi_{-t}(\psi_s(\Phi_t(p))))$(flow forward t along X, then s along Y, then back -t along X).

Then $\frac{\partial H}{\partial s}_{(t,0)} = Y|_{\phi_t(p)}[f \circ \phi_{-t}]$ since symbolically this is the same as two lines above(making some substitutions).

Taking a derivative in t yields $\frac{\partial^2 H}{\partial t \partial s}|_{(0,0)} = \frac{d}{dt}|_{t=0} Y|_{\phi_t(p)}[f \circ \phi_{-t}]$

But we know $L_X Y|_p[f] = \frac{d}{dt}|_{t=0} d\phi_{-t}(Y|_{\phi_t(p)})[f]$.

Recall the lie derivative is defined as

$$L_X Y|_p = lim_{t \to 0} \frac{d\phi_{-t} Y|_{\phi_t(p)} - Y|_p}{t} = \frac{d}{dt}|_{t=0} d\phi_{-t}(Y_{\phi_t(p)})$$

where we measure the change in Y at a point p against flows forward along X. Ie. the change in Y against X

And $\frac{d}{dt}|_{t=0} d\phi_{-t}(Y|_{\phi_t(p)})[f] = \frac{d}{dt}|_{t=0} Y|_{\phi_t(p)}[f \circ \phi_{-t}]$. So lie derivative is cross term of second derivative of H.

Then define $K(r, s, t)$ to show cross terms of second derivative of H also equal to lie bracket.

□

Prop 1. ***Levi-Civita*** *Let $(\mathcal{M}, g)$ be a Riemanian manifold. Then $\exists$ a unique affine connection ∇ which is compatible with g and symmetric. Such connection is called Riemann connection.*

Proof. Suppose ∇ compatible and symmetric connection. Then

$$\begin{aligned} Xg(Y,Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Yg(Z,X) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ -Zg(X,Y) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \end{aligned}$$

which comes from compatibility, since X acting on function yields s.t. corollary to compatibility.(We may say this pointwise since at any ponit we have an integral curve γ for X with $\gamma(0) = p$).

Then substituting expression via symmetry of the expression $\nabla_X Z - \nabla_Z X = [X, Z]$ and the like and cancelling terms when adding the above three lines yields

$$g(\nabla_X Y, Z) = \frac{1}{2}(Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g(X,[Y,Z]) + g(Y,[X,Z]) - g(Z,[X,Y]))$$

so then we define $\nabla_X Y$ to be the vector s.t. the relation holds(note this is how inner product leads to definition of derivative) $\square$

3 Curvature

Recall Curvature $R : X(M) \times X(M) \to X(M)$ is defined as

$$R(X,Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$
$$(X,Y,Z,W) := g(R(X,Y)Z,W)$$

Curvature tensor example of covariant tensor. Differential forms example of contravariant tensor(measuring the dual?)

4 Jacobi Fields

Jacobi field is vector field along geodisic γ which corresponds to a perturbation of γ iin the space of geodesic.

Also thought of as tangent vectors in space of geodesics.

We study the following:

$$(dexp_p)_{tv}(tw) = \frac{df}{ds}(t,0)$$

along $\gamma(t) = exp_p(tv)$

5 Killing Fields

Killing fields flows preserve orientation of vectors at points. Flow smoothly(diffeomorphism assumption).

6 Jacobi Fields

Remark 1. *A jacobi field is a vector field along a geodesic γ which corresponds to a perturbation of γ in the space of geodesics.*

Also can be thought of as tanget vectors in space of geodesics(along a geodesic).

Theme 1. *Formalizes connection between curvature and geodesics*

Def 1. $(dexp_p)_{tv}(tw) = \frac{df}{ds}(t,0)$ *along* $\gamma(t) = exp_p(tv)$ *where* $f(t,s) = exp_p tv(s)$

Remark 2. $(dexp_p)_{tv}$ *map from tangent space of tangent space to tangent space of tangent space*

Def 2. $0 = \frac{D^2J}{dt^2} + R(\gamma'(t), J(t))\gamma'(t)$

Remark 3. *Curvatue indicates spread of geodesics*

Prop 2. *Jacobi field is determined by initial conditions* $J(0), \frac{DJ}{dt}(0)$.

Proof. Let $e_1(t), ..., e_n(t)$ parallel orthonormal fields along γ. Write

$$J(t) = \sum_i f_i(t)e_i(t)$$

with $a_{ij} = g(R(\gamma'(t), e_i(t))\gamma'(t), e_j(t))$

Then $\frac{D^2J}{dt^2} = \sum_i f_i''(t)e_i(t)$ and $R(\gamma', J)\gamma' = \sum_j g(R(\gamma', J)\gamma', e_j)e_j = \sum_{i,j} f_i a_{i,j} e_j$

since the e_i parallel.

Jacobi eqution equivalent to $f_j''(t) + \sum_i a_{ij}(t)f_i(t) = 0$ which is a linear second order system.

Thsu given initial conditions we have $2n$ linearly separable independent jacobi fields along γ. □

Theme 2. *A lot of these things connected to DEs on the manifold(I guess point of differentials).*

Prop 3. *Let M have constant sectional curvature K. $\gamma : [0,l] \to M$ be a geodesic arclength parameterized. Let J jacobi field normal to γ'.*

Then $R(\gamma', J)\gamma' = KJ$ ie. curvature of γ', J along γ' is just KJ ie. multiple of J. J describes turning away of γ. In some sense an eigenvalue of the curvature.

Proof. For all T along γ we have

$$g(R(\gamma', J)\gamma', T) = K(g(\gamma', \gamma')g(J, T) - g(\gamma', T)g(J, \gamma'))$$

why??? - I think becaue we have 0 sectional curvature so we have this form for the curvature.

This equals $Kg(J, T)$ as terms cancel via unit norm and parallelism. □

Example 1. *Set* $M = \mathbb{R}^2$ *with* $\gamma(t) = (t, 0)$. $w(t) = (0, 1)$. *Then* $J(t) = (0, t)$.

Showing existence of jacobi field satisfying general form(with 0 sectional curvature).

Theme 3. *Typically focus on jacobi fields normal to curve.*

Prop 4. *Can taylor expand* $|J(t)|^2$. *Involves curvature*

$|J(t)|^2 = t^2 - (1/3)g(R(v,w)v,w)t^4 + ...$

Notice involves sectional curvature $K(v,w) = g(R(v,w)v,w)$

Proof. Know $J(0) = 0$ with $J'(0) = w$. So

$$\begin{aligned}
&g(J,J)(0) = 0 \\
&(g(J,J))'(0) = 2g(J,J')(0) = 0 \text{ via a product rule likely} \\
&(g(J,J))''(0) = 2g(J',J')(0) + 2g(J'',J)(0) = 2 \text{ since second term cancels(parallelism?)}
\end{aligned}$$

Last expression also probably comes from DE

Via the DE also get third order term 0 and fourth order term $8g(J', J''')(0)$

We have

$$J'''(0) = -\frac{D}{dt}|_{t=0}(R(\gamma'(t), J(t))\gamma'(t))$$

via DE. Substituting this into expression gives us what we want.

□

Remark 4. ***Geometric Interpretation:***

Geodesics $t \to exp_p(tv(s))$ *spread from* $t \to exp_p(tv)$ *at a rate which differs from that of flat space by* $\approx -1/6K(p,\sigma)t^3$.

6.1 Conjugate Points

Remark 5. *Conjugate point sinks for tangent geodesics along* γ

Example 2. *Consdier* $\mathbb{S}^2$. *Antipodes conjugate along* $(cos(t), sin(t), 0)$ *for* $t \in [0, \pi]$ *with* $J(t) = (0, 0, sin(t))$

Example 3. *Cut locus of* $p \in S^n$ *is* $\{-p\}$

Prop 5. *A point* $q = \gamma(t_0)$ *is conjugate to* p *along* $\gamma \iff v_0 := t_0\gamma'(0)$ *is a critical point of* exp_p. *also mult* $= dim(ker((dexp_p)_{v_0}))$

Proof. We say q conjugate $\iff$ we can find a J s.t. $J(0) = J(t_0) = 0$. We set $v = \gamma'(0)$ and $w = J'(0)$.

We know $J(t) = (dexp_p)_{tv}(tw)$. So $0 = ((dexp_p)_0)(0) = (dexp_p)_{t_0 v}(t_0 w)$ □

Prop 6. $g(J, \gamma') = g(J'(0), \gamma'(0))t + g(J(0), \gamma'(0))$

Proof. Compute

$g(J', \gamma')' = g(J'', \gamma')$ Because $\gamma'' parallel J'?$= -g(R$(\gamma', J)\gamma', \gamma') = 0$ because we know curvature is constant multip

7 Complete Manifolds

Remark 6. *Covering map covers each open set with image of disjoint open sets on which f is a diffeomorphism(f locally a diffeomorphism on open sets)*

Remark 7. *The manifold is extendible if it can be isometrically emebedded in larger space(open).*

Example 4. *Torus geodesically complete, $\mathbb{R}^n$ geodesically complete, $B(0,1)$ not complete since geodesic runs out of manifold.*

Prop 7. *If M is extendible then it is not geodesically complete.*

Proof. Suppose M extendible. Then $M \subseteq M'$. Since $M' \backslash M$ is not open(since M' connected) we can find $p \in \partial M = \overline{M} \backslash M$. Let U be normal neighborhood of p. Let $q \in U \cap M$. Then we can find geodesic γ from p to q. But this is not in M.

Ie. just look at boundary and find normal neighborhood. □

Theorem 1. *Hopf-Rinow*

Proof. Need to show $i) \implies *$ and $i) \iff ii) \iff iii) \iff iv)$ □

8 Confusions

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n't have a very good understanding of how to convert bewteen derivatives?