Alex Havrilla

alumhavr

Hw 6

Question 1:

Geodesics of a Surface of Revolution

Let ϕ as described. First we compute the metric. Recall $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \frac{\partial}{\partial x_i} \cdot \frac{\partial}{\partial x_j}$. Compute

$$\phi_u = (-f(v)sin(u), f(v)cos(u), 0)$$
$$\phi_v = (f'(v)cos(u), f'(v)sin(u), g'(v))$$

and then

$$g_{11} = \phi_u \cdot \phi_u = f(v)^2(sin^2 + cos^2) = f^2)$$
$$g_{22} = \phi_v \cdot \phi_v = f'^2(cos^2 + sin^2) + g'^2 = f'^2 + g'^2$$
$$g_{12} = \phi_u \cdot \phi_v = 0$$

ie.

$$G = \begin{bmatrix} f^2 & 0 \\ 0 & f'^2 + g'^2 \end{bmatrix}$$

and thus

$$G^{-1} = \begin{bmatrix} 1/f^2 & 0 \\ 0 & 1/(f'^2 + g'^2) \end{bmatrix}$$

Now we compute the christoffel symbols. For arbitray symbol we know $\Gamma^k_{ij} = g^{kl}/2(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l})$ so

$$
\begin{aligned}
\Gamma^1_{11} &= g^{11}/2(\frac{\partial g_{11}}{\partial u} + \frac{\partial g_{11}}{\partial u} - \frac{\partial g_{11}}{\partial u}) = 0\\
\Gamma^2_{11} &= g^{22}/2(\frac{\partial g_{12}}{\partial u} + \frac{\partial g_{12}}{\partial u} - \frac{\partial g_{11}}{\partial v}) = \frac{-ff'}{f'^2 + g'^2}\\
\Gamma^1_{12} &= g^{11}/2(\frac{\partial g_{11}}{\partial v} + \frac{\partial g_{21}}{\partial u} - \frac{\partial g_{12}}{\partial u}) = \frac{f'}{f}\\
\Gamma^2_{12} &= g^{22}/2(\frac{\partial g_{12}}{\partial v} + \frac{\partial g_{22}}{\partial u} - \frac{\partial g_{12}}{\partial v}) = 0\\
\Gamma^1_{22} &= g^{11}/2(\frac{\partial g_{21}}{\partial v} + \frac{\partial g_{21}}{\partial v} - \frac{\partial g_{22}}{\partial u}) = 0\\
\Gamma^2_{22} &= g^{22}/2(\frac{\partial g_{22}}{\partial v} + \frac{\partial g_{22}}{\partial v} - \frac{\partial g_{22}}{\partial v}) = \frac{f'f'' + g'g''}{f'^2 + g'^2}
\end{aligned}
$$

Now we compute the geodesic equations as

$$0 = \frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma^k_{ij} \frac{dx_i}{dt}\frac{dx_j}{dt}$$

so

$$0 = \frac{d^2 u}{dt^2} + 2\Gamma^1_{12}\frac{du}{dt}\frac{dv}{dt} + \Gamma^1_{11}(\frac{du}{dt})^2 + \Gamma^1_{22}(\frac{dv}{dt})^2 = \frac{d^2 v}{dt^2} + 2\frac{f'}{f}\frac{du}{dt}\frac{dv}{dt}$$

and

$$0 = \frac{d^2 v}{dt^2} + 2\Gamma^2_{12}\frac{du}{dt}\frac{dv}{dt} + \Gamma^2_{11}(\frac{du}{dt})^2 + \Gamma^2_{22}(\frac{dv}{dt})^2 = \frac{d^2 v}{dt^2} - \frac{ff'}{f'^2 + g'^2}(\frac{du}{dt})^2 + \frac{f'f'' + g'g''}{f'^2 + g'^2}(\frac{dv}{dt})^2$$

Prop 1. *Geometrically energy $|\gamma'(t)|^2$ of a geodesic is constant. Further if $\beta(t)$ is oriented angle $< \pi$ of the geodesic γ with parallel P intersecting at $\gamma(t)$ then $r\cos(\beta)$ is constant where r is the radius of the parallel P.*

Proof. Note $\gamma(t) = (u(t), v(t))$. So $\gamma'(t) = (u'(t), v'(t))$ Compute:

$$
\begin{aligned}
\frac{d}{dt}|\gamma'(t)|^2 &= \frac{d}{dt}\langle \gamma'(t), \gamma'(t)\rangle = \frac{d}{dt}(g_{11}u'^2 + g_{22}v'^2)\\
&\frac{d}{dt}(f^2u'^2 + v'^2(f'^2 + g'^2)) = 2ff'v'u'^2 + 2f^2u'u'' + 2v'v''(f'^2 + g'^2) + v'^2(2f'f''v' + 2g'g''v')
\end{aligned}
$$

which we see equals 0 when multiplying the second geodesic equation by $v'(f'^2 + g'^2)$. Hence this is constant.

Now we argue for the second geometric consideration. Compute $\beta(t)$ as $\langle \gamma'(t), p'(t) \rangle$ for parallel $p(t)$ via

$$\langle \gamma'(t), p'(t) \rangle = g_{11} u'_\gamma(t) u'_p(t) = f^2 u'_\gamma(t) u'_p(t)$$

Further we can choose p to have constant speed yielding $f^2 u'_\gamma$.

We also compute in $\mathbb{R}^3$ yielding $|p'||\gamma'|cos(\beta)$. But we know $f^2 u_\gamma$ and $|\gamma'|$ constant hence $|p'|cos(\beta)$ equal to a constant as desired. Note we know the two terms constant via the first geodesic equation.

□

Prop 2. *A geodesic of the paraboloid which is not a meridian intersects itself an infinte number of times*

Proof. Clairaut's equation tells us for a geodesic γ we know $r(t)cos(\beta(t))$ is constant for parallels p.

It suffices to show γ goes an infinite number of times around the z-axis and $r(t) \to \infty$. We consider polar coordinates. Set $\gamma(t) = (r(t), \theta(t))$. We show $\theta(t) \to \infty$ wlog. We know the geodesic cannot be tangent to a meridian and hence θ is monotone. It cannot be θ converges to a constant(otherwise would be tangent to meridian), so it diverges.

We now show $r(t) \to \infty$. Via clairaut we know $r(t)cos(\beta(t))$ constant and if $r(t)$ does not go to ∞ then it is bounded. But we know $\beta(t)$ will approach $\pi/2$ as γ approaches a meridian(asymptotically) so it must be $r(t) \to \infty$. Then it is clear γ must intersect an infinite number of times. □

Question 2:

Killing Fields

Let X be as defined via $\phi : (-\epsilon, \epsilon) \times U \to M$.

Prop 3. *Linear vector field $v : \mathbb{R}^n \to \mathbb{R}^n$ defined via $A \in \mathbb{R}^{n \times n}$ is killing $\iff$ A antisymmetric*

Proof. Suppose A anti-symmetric. So $A = -A^T$. To show A killing we need to show the flow of A for fixed t is an isometry. Recall ϕ given via

$$\phi_t(x,t) = v(\phi(x,t)) = A\phi(x,t)$$

which solves to $\phi(x,t) = e^{At}x$

We show for fixed t

$$\langle d\phi x, d\phi y\rangle = \langle e^{At}x, e^{At}y\rangle = \langle x, (e^{At})^T e^{At}\rangle = \langle x, y\rangle$$

with $(e^{At})^T = e^{A^T t} = e^{-At}$. Further clearly ϕ is a (smooth) diffeomorphism so this establishes v a killing field.

Now suppose v killing. Then the flow $\phi(x,t) = e^{At}x$ is an isometry. In particular we know for $t > 0$, and for all $x \in \mathbb{R}^n$

$$\langle x, x\rangle = \langle e^{At}x, e^{At}x\rangle = \langle x, e^{A^T t}e^{At}x\rangle$$

so it must be $e^{A^T t}e^{At} = I$ and so $A^T = -A$ as desired.

□

Prop 4. *Suppose X killing field on M. U a normal neighborhood of $p \in M$ s.t. $X(p) = 0$ uniquely in U. Then in U, X is tangent to the geodesic spheres centered at p.*

Proof. We seek to show $\langle X, V\rangle$ for all geodesic V centered at p.

Let q be a point in the geodesic sphere of radius $a > 0$with $exp_p v = q$ for some $v \in T_pM$. γ is the geodesic $t \to exp_p(tv)$. We know $q \to \phi(q,s)$ is an isometry for fixed s, the image of γ under ϕ is a geodesic, since the curve stays autoparallel. Denote this images as $\gamma_s(t)$. Since $\gamma_s(0) = \phi(p,s) = 0$ since $X(p) = 0$ and the curves have same speed we know $\gamma_s(t) = exp_p(tv(s))$ for $v(s) \in T_pM$. So $\gamma_s(1)$ is on the geodesic sphere for arbitrary s. We have $\gamma_s(1) = \phi(q,s)$ and $X(q) = \frac{d}{ds}|_{s=0}\phi(q,s)$ so $X(q)$ tangent to the geodesic sphere. □

Prop 5. *Let X a differentiable vector field on M and $f : M \to N$ an isometry. Y VF on N via $Y(f(p)) = df_p(X(p))$ for $p \in M$. Then Y killing field $\iff$ X killing field.*

Proof. Via invertability both directions are the same so it suffices to consider the forward direction. Suppose Y is a killing field. Let $V \subseteq N$ open subset. Then want to show for flow $\phi(t,p)$ of X an isometry.

Consider flow on N via $\psi_t(q) = f(\phi(t, f^{-1}(q)))$ as a flow for Y. We know $f(\phi(t,\cdot)) = \psi_t(f(\cdot))$ and ψ an isometry. Note clearly ψ is a diffeomorphism(composition of diffeomorphisms). Compute for $v, w \in T_pM$,

$$\langle d\phi_t v, d\phi_t w\rangle_M = \langle df d\phi_t v, df d\phi_t w\rangle_N = \langle d\psi_t \circ fv, d\psi_t \circ fw\rangle_N = \langle dfv, dfw\rangle_N = \langle v, w\rangle_M$$

showing ϕ an isometry □

Prop 6. *X is killing* $\iff$ $\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0$ *for* $Y, Z \in X(M)$

Proof. For the forwards direction by continuity it suffices to prove for $X(q) \neq 0$. We pick a submanifold orthogonal to $X(q)$ of dimension n-1 and pick coordinates $X_i = \frac{d}{dx_i}$ for the tangent bundle. Then for U neighborhood of q and V n-1 dim neighborhood, for small enough ϵ we know $V \times (-\epsilon, \epsilon) \subseteq U$ (in coordinates).

Compute

$$\langle \nabla_{X_j} X, X_i\rangle + \langle \nabla_{X_i} X, X_j\rangle = X\langle X_i, X_j\rangle - \langle [X, X_i], X_j\rangle - \langle [X, X_j], X_i\rangle = \frac{\partial}{\partial t}\langle X_i, X_j\rangle = 0$$

where we get the last equality because X is a killing field.

For the backwards direction assume $\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0$ for $Y, Z \in X(M)$. Let $\phi_t(x)$ be the flow at time t. Compute at point p:

$$\frac{d}{dt}\langle d\phi_t Y, d\phi_t Z\rangle = \langle \nabla_Y X, d\phi_t Z\rangle + \langle d\phi_t Y, \nabla_Z X\rangle = 0$$

so in particular $\langle d\phi_t Y, d\phi_t Z\rangle = \langle Y, Z\rangle$ where we check the value at $t = 0$. □

Prop 7. *Let X a nonzero killing field. Then ∃ a system of coordinates* $(x_1, ..., x_n)$ *in neighborhood of q s.t.* g_{ij} *of the metric does not depend on* x_n*.*

Proof. Recall because X is nonzero we can find a coordinate s.t. locally x is $\frac{\partial}{\partial x_n}$. But then it must be the metric independent of x_n vai the killing equation since when we differentiate an expression of the metric with respect to x_n we get 0. □

Question 3:

More on Killing Fields

Prop 8. *Let* $A_X : X(M) \to X(M)$ *be as defined. Then* $\langle A_X(Z), X\rangle\rangle(p) = 0$ *where p is a critical point of* $f(q) = \langle X, X\rangle_q$ *for arbitrary Z*

Proof. This constructs what we want.

Compute:

$$\langle A_X(Z), X\rangle(p) = \langle \nabla_Z X, X\rangle(p)$$

Recall the Killing equation:

$$\langle \nabla_Y X, Z \rangle + \langle Y, \nabla_Z X \rangle = 0$$

Note then $\langle A_X(Z), X \rangle + \langle Z, A_X(X) \rangle = 0$. And then $\langle Z, A_X(X) \rangle = 0$ since p is a critical point and $\nabla_X X|_p = 0$.

□

Prop 9. $\langle A_X(Z), A_X(Z) \rangle(p) = \frac{1}{2} Z_p(Z\langle X, X \rangle) + \langle R(X, Z)X, Z \rangle$

Proof. We finish using results from Question 2. Let $S = 1/2ZZ\langle X, X \rangle - \langle R(X, Z)X, Z \rangle$. The killing equation from above tells us $\langle \nabla_Z X, X \rangle + \langle \nabla_X X, Z \rangle = 0$

Then

$$\langle \nabla_{[X,Z]} X, Z \rangle - \langle \nabla_X X, \nabla_Z Z \rangle - \langle \nabla_X \nabla_Z, Z \rangle$$

We conclude

$$-\langle \nabla_Z X, \nabla_X Z \rangle + \rangle \nabla_Z X, \nabla_Z X \rangle + \langle \nabla_Z X, \nabla_X Z \rangle - \langle \nabla_X X, \nabla_Z Z \rangle = \langle \nabla_Z X, \nabla_Z X \rangle$$

since $[X, Y] = \nabla_X Y - \nabla_Y X$ and $\nabla_X X|_p = 0$ □

Question 4:

More on Killing Fields

Prop 10. *Let M be a compact Riemannian manifold of even dimension with positive sectional curvature. Then every kiling field X on M has a singularity.*

Proof. Set $f : M \to R$ via $f(q) = \langle X, X \rangle(q)$ and $p \in M$ a minimum. Suppose $X(p) \neq 0$. We set $A : T_pM \to T_pM$ via $A(y) = A_X Y = \nabla_Y X$ where Y extends $y \in T_pM$.

Set $E \subseteq T_pM$ orthogonal to $X(p)$. We claim $A : E \to E$ is an antisymmetric isomorphism. Then it must be $dimE = dimM - 1$ is even given the antisymmetry, a contradiction since the manifold is even degree. $X(p) = 0$.

We now argue A antisymmetric isomorphism. First we argue the isomorphism. Suffices to show bijectivity since the mapping clearly linear. Notice via the above problem equation i, $A(Z)$ stays orthogonal to X. Note if $A_X Y = \nabla_Y X = 0$ then it must be $Y = 0$ by part 2 of the above since otherwise it must have positive norm given by equation ii, which shows injectivity since the mapping linear. Surjectivity then follows since the codomain is the domain. Antisymmetry comes from the first equation i) in above by transposing. □

Question 5:

Schur's Theorem

Prop 11. *Let M^n a connected Riemannian manifold with $n \geq 3$. Suppose M is isotropic, that is, for each $p \in M$, sectional curvature does not depend on $\sigma \subseteq T_pM$. Then M has constant sectional curvature.*

Proof. We define the 4-tensor

$$R'(W, Z, X, Y) = \langle W, X\rangle\langle Z, Y\rangle - \langle Z, X\rangle\langle W, Y\rangle$$

Because sectional curvature K is constant w.r.t σT_pM we know $R = KR'$ from our lemma. So for $U \in X(M)$, $\nabla_U R = (UK)R'$ via application.

The second Bianchi identity tells us

$$\nabla R(W, Z, X, Y, U) + \nabla R(W, Z, Y, U, X) + (W, Z, U, X, Y) = 0$$

Choose Y and Z s.t. at p $\langle X, Y\rangle = \langle Y, Z\rangle = \langle Z, X\rangle = 0$ with $\langle Z, Z\rangle = 1$. Then setting $U = Z$ we have

$$\langle (XK)Y - (YK)X, W\rangle = 0$$

for arbitrary W via the second bianchi identity and $\nabla_U R = (UK)R'$. Then since X, Y linearly independent at p conclude $XK = 0$ for all $X \in T_pM$. Hence K must be constant.

$\square$