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Hw 5

Question 1:

Let $\mathcal{M}$ be a Riemann manifold with affien connection ∇. Let $\gamma : I \to \mathcal{M}$ be a curve. Let $P_{\gamma,t_0,t} : T_{\gamma(t_0)}\mathcal{M} \to T_{\gamma(t)}\mathcal{M}$ be the mapping taking tangent vector V_0 at $\gamma(t_0)$ to $V(t)$ where V is the parallel transport of V_0 along γ.

Let X and Y be vector fields on $\mathcal{M}$. Consider curve γ as an integral curve for X. Then $\frac{d\gamma}{dt} = X|_{\gamma(t)}$.

Prop 1. *Where* ∇ *is the Riemann connection then*

$$\nabla_X Y|_{\gamma(t_0)} = \frac{d}{dt}(P^{-1}_{\gamma,t_0,t} Y|_{\gamma(t)})|_{t=t_0}$$

Proof. Set $p_t = \gamma(t)$ and $p_0 = \gamma(t_0)$. Then at $T_{p_0}\mathcal{M}$ pick an orthonormal basis $v_1, ..., v_n$. We extend these to vector fields V_1,...,V_n along γ using parallel transport. Notice via compatibility of with the metric these stay orthogonal. We can then write $Y = a^i V_i$ and thus compute

$$\nabla_X Y|_{\gamma(t_0)} = \nabla_X a^i V_i|_{p_0} = X[a^i]V_i|_{p_0} + a^i \nabla_X V_i|_{p_0} = X[a^i]v_i$$

where the second term $a^i \nabla_X V_i = 0$ since the V_i are parallel along γ.

Then in the other direction we compute

$$\frac{d}{dt}(P^{-1}_{\gamma,t_0,t} Y|_{p_t})|_{t=t_0} = \frac{d}{dt}(P^{-1}_{\gamma,t_0,t} a^i V_i|_{p_t})|_{t=t_0} = \frac{d}{dt}(a^i|_{p_t} P^{-1}_{\gamma,t_0,t} V_i|_{p_t})|_{t=t_0}$$
$$\frac{d}{dt}(a^i|_{p_t} v_i)|_{t=t_0} = X[a^i]v_i$$

where we note parallel transport is linear and thus the inverse is linear. Further $\frac{d}{dt}(a_i v_i) = X[a^i]v_i$ since $X = \frac{d\gamma}{dt}$ $\square$

Question 2:

Let $\mathcal{M}$, $\overline{\mathcal{M}}$ be as defined with : $M \to \overline{\mathcal{M}}$ an immersion. Let $g = f^*\overline{g}$ and $\nabla_X Y|_p$ as defined for vector fields X, Y on $\mathcal{M}$.

Prop 2. ∇ *as defined is the Riemannain connection on* $(\mathcal{M}, g)$.

Proof. To show ∇ is the Riemannian connection on $\mathcal{M}$ it suffices to show it is a connection, symmetric, and compatible with g. Then via uniqueness we are done.

First we show it's a connection. Note both $(df)^{-1}$ and Π_T are(locally) linear. Compute

$$\begin{aligned}\nabla_{X+Y}Z &= (df)^{-1}\Pi_T(\overline{\nabla}_{\overline{X}+\overline{Y}}df(Z)) = (df)^{-1}\Pi_T(\overline{\nabla}_{\overline{X}}df(Z) + \overline{\nabla}_{\overline{Y}}df(Z)) = \\ &= (df)^{-1}\Pi_T\overline{\nabla}_{\overline{X}}df(Z) + (df)^{-1}\Pi_T\overline{\nabla}_{\overline{Y}}df(Z) = \nabla_X Z + \nabla_Y Z\end{aligned}$$

$\nabla_X(Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$ follows similarly.

$$\begin{aligned}\nabla_X(hY) &= (df)^{-1}\Pi_T(\overline{\nabla}_{df(X)}hdf(Y)) = (df)^{-1}\Pi_T(df(X)[h]df(Y) + h\overline{\nabla}_{df(X)}df(Y)) \\ &= X[h]Y + h\nabla_X Y\end{aligned}$$

Next we show symmetry:

$$\begin{aligned}\nabla_X Y - \nabla_Y X &= (df)^{-1}\Pi_T(\overline{\nabla}_{df(X)}df(Y)) - (df)^{-1}\Pi_T(\overline{\nabla}_{df(Y)}df(X)) \\ &= (df)^{-1}\Pi_T(\overline{\nabla}_{df(X)}df(Y) - (\overline{\nabla}_{df(Y)}df(X))) = df^{-1}\Pi_T([df(X), df(Y)]) = [X, Y]\end{aligned}$$

And finally that it is compatible:

Recall that the metric is compatible with ∇ if $X[g(Y,Z)] = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$. We show

$$\begin{aligned}X(Y,Z)_g &= (df)(X)((df)(Y),(df)(Z))_{\overline{g}} \\ &= \overline{\nabla}_{(df)(X)}((df)(Y),(df)(Z))_{\overline{g}} = (\overline{\nabla}_{(df)(X)}(df)(Y), df(Z))_{\overline{g}} + (df(Y), \overline{\nabla}_{(df)(X)}(df)(Z))_{\overline{g}} \\ &= (\nabla_X Y, Z)_g + (Y, \nabla_X Z)_g\end{aligned}$$

Thus by uniqueness we have shown $\nabla_X Y$ is the affine connection $\mathcal{M}$. $\square$

Question 3:

Set $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2, y > 0\}$

with metric coefficients $g_{11} = g_{22} = \frac{1}{y^2}$ and $g_{12} = 0$.

Prop 3. *The christoffel symbols of the Riemannian connection are* $\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = 0$ *and* $\Gamma^2_{11} = \frac{1}{y}, \Gamma^1_{12} = \Gamma^2_{22} = \frac{-1}{y}$

Proof. It suffices to compute the christoffel symbols.

First note we have for $y > 0$ ie. the half plance:

$$G^{-1} = \begin{bmatrix} y^2 & 0 \\ 0 & y^2 \end{bmatrix}$$

Recall we know for arbitrary symbol

$$\Gamma_{ij}^k g^{kl}/2(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l})$$

Compute

$$\Gamma_{11}^1 = \frac{g^{11}}{2}(\frac{\partial g_{11}}{\partial x} + \frac{\partial g_{11}}{\partial x} - \frac{\partial g_{11}}{\partial x}) + \frac{g^{12}}{2}(\frac{\partial g_{12}}{\partial x} + \frac{\partial g_{12}}{\partial x} - \frac{\partial g_{11}}{\partial x}) = 0$$

since $\frac{\partial \cdot}{\partial x} = 0$ and $g^{12} = 0$.

A similar computation holds for $\Gamma_{12}^2, \Gamma_{22}^1$

Now compute

$$\Gamma_{12}^1 = \frac{g^{11}}{2}(\frac{\partial g_{11}}{\partial y} + \frac{\partial g_2}{\partial x} - \frac{\partial g_{12}}{\partial x}) + \frac{g^{12}}{2}(\frac{\partial g_{12}}{\partial y} + \frac{\partial g_{22}}{\partial x} - \frac{\partial g_{12}}{\partial y}) = \frac{y^2}{2}(-\frac{2}{y^3}) = -\frac{1}{y}$$

Computation follows similarly for Γ_{22}^2 and Γ_{11}^2.

$\square$

Let $V_0 = (0,1)^T$ tangent vector at $(0,1) \in \mathbb{R}^2_+$. Let $V(t)$ be the parallel transport of V_0 along curve $x = t, y = 1$.

Prop 4. *V(t) makes angle t with direction of y-axis(measured clockwise)*

Proof. Set $v(t) = (a(t), b(t))$

We know

$$\frac{\partial a}{\partial t} + \Gamma_{12}^1 b = 0$$
$$\frac{\partial b}{\partial t} + \Gamma_{11}^2 a = 0$$

We have $a = cos\theta(t), b = sin\theta(t)$ and long the curve we have $y = 1$, we obtain from the equations above that $\frac{d\theta}{dt} = -1$. With $v(0) = v_0$ then $\theta(t) = \pi/2 - t$. $\square$

Question 4:

Let $L : T_p(\mathcal{M}) \to T_p(\mathcal{M})$ via $L(Y) = \nabla_Y X|_p$ which is linear. Then write $DIV(X) = trace(L)$ since L is a linear operator on finite dimensional vector spaces.

Prop 5. $DIV(X) = div(X) = \frac{di_X(\omega_g)}{\omega_g}$

Proof. $DIV(X)$ is defined as the trace of L where L takes $Y \to \nabla_Y X$.

We follow do Carmo's plan.

First we establish we can write at a point $p \in \mathcal{M}$

$$DIVX = \sum_i E_i(f_i)(p)$$

where $X = f^i E_i$ with $\{E_i\}$ an orthonormal basis s.t. at p.

Note for each E_i, $a_i^j E = \nabla_{E_i} X = \nabla_{E_i} f^i E_i = E^i[f] E_i$

$$trace(L) = \sum_i a_i^i$$

But note exactly $E_i(f_i)(p) = a_i^i$.

Now pick 1-forms ω_i s.t. $\omega = \omega_i(E_j) = \delta_{i,j}$. Since $\mathcal{M}$ is oriented this is a volume form via an argument similar to the direction orientability $\implies$ existance of volume form. In particular we know $dx_1 \wedge ... \wedge dx_n$ can be well defined globally as a volume form, which is equivalent to $f\omega_1 \wedge ... \wedge \omega_n$ for some f which we can take to be positive by negating appropriate E_i and ω_i.

Set $\theta_i = \omega_1 \wedge ... \wedge \hat{\omega_i} \wedge ... \wedge \omega_n$. We compute

$$i_X(\omega)(v_2, ..., v_n) = \omega(X, v_2, ..., v_n) = \omega(f^i E_i, v_2, ..., v_n) = \sum_i \omega(f_i E_i, v_2, ..., v_n)$$
$$= \sum_i \omega(f_i E_i, v_2, ..., v_n) = \sum_i (-1)^{i-1} \omega_i(f_i E_i) \theta_i(v_2, ..., v_n)$$

since if we have $\omega_j(f_i E_i)$ for some $j \neq i$ then we have 0 in a neighborhood. Note $\omega_i(f_i E_i) = f_i$ at a point. So

$$\sum_i (-1)^{i-1} \omega_i(f_i E_i) \theta_i(v_2, ..., v_n) = \sum_i (-1)^{i+1} f_i \theta_i(v_2, ..., v_n)$$

which gives $i_X(\omega) = \sum_i (-1)^{i+1} f_i \theta_i$

Then write

$$d(i(X)\omega) = \sum_i (-1)^{i+1} df_i \wedge \theta_i + \sum_i (-1)^{i+1} f_i \wedge d\theta_i = \sum_i E_i(f_i)\omega + \sum_i (-1)^{i+1} f_i \wedge d\theta_i$$

$\square$

Note $d\theta_i = 0$ since

$$d\omega_k(E_i, E_j) = E_i\omega_k(E_j) - E_j\omega(E_i) - \omega_k([E_i, E_j]) = \omega_k(\nabla_{E_i}E_j - \nabla_{E_j}E_i)$$

So really

$$d(i(X)\omega)(p) = \sum_i E_i(f_i)(p)\omega = divX(p)\omega$$