1	2	3	4	5	6	Σ
10	7	10	10	10	10	57

Alex Havrilla

Hw 3

Question 1:

Claim: For $p \geq 1$, l_p and c_0 are separable.

Proof. Note that for simplicity we show separability of sequences of real numbers. To address the complex case just consider density of complex numbers with rational coefficients in the plane(complex plane) instead of the real line. Fix $p \geq 1$ finite. We will show the set of sequences which are eventually 0 finitely many rational terms are dense in l_p. Note these sequences are in l_p as they are adding finitely many nonzero terms. If so we have separability as clearly this collection is countable.

Let $f \in l_p$. Let $\epsilon > 0$. Since $f \in l_p$ we know $\lim_{n\to\infty} f_n = 0$. Thus we can find index N s.t. $\sum_{j=N}^{\infty} |f_n|^p < \epsilon/2$. Then for the first $N-1$ terms find rationals s.t. at the ith component $|f_i - q_i|^p < \frac{\epsilon}{22^i}$ which can be done via density. Thus we find the constructed sequence $q = (q_1, q_2, ..., q_{N-1}, 0, ...)$ is s.t.

$$||f-q||_p^p = \sum_{n=1}^{\infty} |f_n - q_n|^p = \sum_{n=1}^{N-1} |f_n - q_n|^p + \sum_{n=N}^{\infty} |f_n|^p < \sum_{n=1}^{N-1} \frac{\epsilon}{22^n} + \frac{\epsilon}{2} < \epsilon$$

which establishes density since ϵ was arbitrary.

Furthermore we see that exactly the same argument works for the density of eventually 0 sequences with rational terms in c_0, which shows separability of this space. □

Claim: l_∞ is not separable

Proof. Assume for sake of contradiction l_∞ is separable. Then $\exists$ a sequence $\{f_n\}$ of $f_n \in l_\infty$ countable and dense in l_∞. Then we construct a sequence $c \in l_\infty$ s.t. $\forall f_n, ||c - f_n|| \geq 1$ via diagonalization. Set $c^n = 0$ if $|f_n^n| \geq 1$ and

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otherwise $c^n = 2$. This constructs such a sequence, showing a contradiction to density. Hence l_∞ cannot be separable.

□

Question 2:

Claim: $l_1^\star = l_\infty$ ← In fact, you need to show they're isometric isomorphism.

Proof. Let $T \in l_1^\star$. Let $e_k \in l_1$ s.t. $e_k^{(k)} = 1$ and 0 otherwise. Then every term in l_1 can be written as a limit of partial sums of linear combinations of the e_k. And since T is linear and continuous we have $T(v) = T(\sum_{k=1}^{\infty} v_k e_k) = \sum_{k=1}^{\infty} v_k T(e_k) = \sum_{k=1}^{\infty} v_k y_k$ where $y_k = T(e_k)$. Further we know $|y_k| \leq ||T||$ for all $k \in \mathbb{N}$ and hence $y = (y_1, y_2, ...) \in l_\infty$. This completes one direction.

Now if we take any $y \in l_\infty$ and $T(v) = \sum_{k=1}^{\infty} v_k y_k$ we clearly have that T is linear(since sums are linear) and furthermore T is bounded as for any $v \in l_1$ with $||v|| = 1, |T(v)| = |\sum_{k=1}^{\infty} v_k y_k| \leq \sum_{k=1}^{\infty} |v_k| ||y||_\infty < \infty$ since $v \in l_1$. This establishes the other direction.

□

Claim:

$\exists$ linear functional T onl_∞ s.t. for all $a \in l_\infty$ with real components,

$$\liminf_{n\to\infty} a_n \leq T(a) \leq \limsup_{n\to\infty} a_n$$

Proof. We show the result in the $l_\infty(\mathbb{R})$ first and note we can lift to the complex case by using hahn banach on real and imaginary terms component wise. Set $\rho : l_\infty \to \mathbb{R}$ via $\rho(x) = ||x||_\infty$. Note that $V = \{a \in l_\infty : a_n \text{ converges}\}$ is a subspace of l_∞. We check the conditions:

$0 = (0, ...) \in V$. If $r \in \mathbb{R}, a \in V$ then $\lim_{n\to\infty} ra_n = rlim_{n\to\infty} a_n = rc$ so $ra \in V$. Finally if $a, b \in V$ then $a + b \in V$ since the sequence $a_n + b_n$ has limit $c + d$:

Define $T : V \to \mathbb{R}$ to be $T(a) = \lim_{n\to\infty} a_n$ which we know is linear on this subspace. Further we have $T(a) \leq ||a||_\infty$. Hence we may apply Hahn-Banach to lift to $T' : \mathbb{R} \to \mathbb{R}$ continuously. ← this extension may not satisfy the requirement.

To extend to the complex case simply consider $T''(a+ib) = T'(a) + iT'(b)$ ∎

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Claim: The dual of $l_\infty^\star \neq l_1$

Proof. Consider the $T \in l_\infty^\star$ we constructed above. Assume for sake of contradiction $T(v) = \sum_{k=1}^\infty b_k v_k$ for some $b = (b_1, ...) \in l_1$. Consider again $e_k = (0, ..., 1, 0, ...)$ which is 1 in the kth index and 0 otherwise. Then

$$0 = lim_{n\to\infty} e_k^{(n)} = \sum_{j=1}^\infty e_k^{(n)} b_j = b_k$$

which is clearly a contradiction as this holds for arbitrary k. ∎

Question 3:

Let X be a banach space and (ϵ_n) be a sequence of positive real numbers converging to zero. Suppose (f_n) is a sequence in the dual X^* having the property that for all $x \in X$ with $||x|| < 1$ we have $C(x) > 0$ s.t. $|f_n(x)| \leq \epsilon_n ||f_n|| + C(x)$

Claim: (f_n) is bounded

Proof. Set $g_n = \frac{f_n}{1+\epsilon_n ||f_n||}$. Then for fixed x we have

$$|g_n(x)| = \frac{|f_n(x)|}{1 + \epsilon_n ||f_n||} \leq \frac{\epsilon_n ||f||_n + C(x)}{1 + \epsilon_n ||f_n||} \leq 1 + C(x)$$

then via uniform boundedness principle we know $\sup_{n\in\mathbb{N}} ||g_n|| = C < \infty$. But

$$||g_n|| = \frac{||f_n||}{1 + \epsilon_n ||f_n||} \leq C \implies ||f_n|| \leq \frac{C}{1 - \epsilon_n} C$$

Then if we pick N s.t. for $n \geq N$, $\epsilon_n < 1/2$ we know $\forall n, ||f_n|| \leq max\{||f_1||, ..., ||f_N||, 2C\} = D$ showing the desired result

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Question 4:

Let X be a banach space and $T : X \to X^*$ linear s.t. when $f = T(x)$ then $f(x) \geq 0$.

Claim: T is bounded.

Proof. In order to show boundedness we show continuity by showing it's graph is closed via the closed graph theorem. Let $\Gamma \subseteq X \times X^*$ be the graph of T.

Since T is linear it suffices to consider limit points of the form $(0, z)$ s.t. $\{(x_n, T(x_n))\} \to (0, z)$. In particular we know $T(0) = 0$ so we seek to show $z = 0$. Fix $y \in X$. We know

$$(T(x_n - y))(x_n - y) \geq 0 \iff T(x_n)(x_n) - T(x_n)(y) - T(y)(x_n) + T(y)(y) \geq 0$$

Taking the limit as $n \to \infty$ we see $T(x_n)(y) \to z(y)$, since convergence in operator norm implies pointwise convergence, $T(y)(x_n) \to 0$, and $T(x_n)(x_n) \to z(0) = 0$. Thus we know in the limit we have

$$T(y)(y) - z(y) \geq 0 \iff (z - T(y))(0 - y) \geq 0$$

Write then the function for

$$f(t) = (z - T(ty))(0 - ty) \geq 0 \iff T(ty)(ty) \geq z(ty)$$

Then when $t > 0$ we know

$$T(ty)(y) \geq z(y) \implies tT(y)(y) \geq z(y)$$

but we may send t to 0 to see $0 \geq z(y)$(since $T(y)(y) \geq 0$). But since y is arbitrary we may take its negation to see

$$(z + T(ty))(0 + ty) \geq 0 \implies tT(y)(y) \geq -z(y)$$

so $tT(y)(y) \geq |z(y)| \implies |z(y)| = 0 \implies z(y) = 0$ for arbitrary y as we send $t \to 0$. Thus $z = 0$, completing the proof.

This gives what we want.

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□

Question 5:

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a)

Claim: $(u, v) = \sum_{k\in\mathbb{Z}} \overline{\hat{u}(k)}\hat{v}(k)$ for $u, v \in L^2_{per}$

Proof. Let $u, v \in L^2_{per}$. We know $u = \sum_{k\in\mathbb{Z}} \hat{u}(k)e_k$ and $v = \sum_{k\in\mathbb{Z}} \hat{v}(k)e_k$. So then

$$(u, v) = (\sum_{k\in\mathbb{Z}} \hat{u}(k)e_k, \sum_{k\in\mathbb{Z}} \hat{v}(k)e_k) = \sum_{j\in Z} \hat{v}(j)(\sum_{k\in\mathbb{Z}} \hat{u}(k)e_k, e_j) =$$

$$\sum_{j\in Z}\sum_{k\in\mathbb{Z}} \hat{v}(j)\overline{\hat{u}(k)}(e_k, e_j)$$

via linearity in the second coordinate and sesquilinearity in the first coordinate.

But we know the $\{e_k\}$ form an orthonormal basis, so this sum reduces to

$$\sum_{j\in\mathbb{Z}} \overline{\hat{u}(j)}\hat{v}(j)(e_j, e_j) = \sum_{j\in\mathbb{Z}} \overline{\hat{u}(j)}\hat{v}(j)$$

as desired

Note the interchange of limits and inner products is justified via continuity and linearity.

$\square$

b)

Claim: We show the isoperimetric inequality:

Proof. We know $P = \int_{-\pi}^{\pi} \sqrt{u'(x)^2 + v'(x)^2}dx = 2\pi c$. Further $(f', f') = \int_{-\pi}^{\pi} |f'(x)|^2 dx = 2\pi c^2$ since $(u')^2 + (v')^2 = c^2$. So we have the relation via Parseval:

$$\frac{P^2}{2\pi} = (f', f') = \int_{-\pi}^{\pi} |f'(x)|^2 dx = \int_{-\pi}^{\pi} u'(x)^2 + v'(x)^2 dx = (u', u') + (v', v') = \sum_{k\in\mathbb{Z}} |k\pi|^2(|\hat{u}(k)|^2 + |\hat{v}(k)|^2)$$

Further we compute $(f', f) = (u' + iv', f + iv) = (u', u) + (v', v) + i(u', v) - i(v', u) = (u', u) + (v', v) - 2iA$. Compute

$$(f', f) = -\sum_{k\in\mathbb{Z}} ik\pi|\hat{f}(k)|^2$$

so algebra gives

$$A = \frac{1}{2}|\sum_{k\in\mathbb{Z}} k\pi(|\hat{f}(k))|^2 - |\hat{u}(k)|^2 - |\hat{v}(k)|^2|$$

and we conclude the argument by noting

$$\hat{f}(k) = \hat{u}(k) + i\hat{v}(k)$$

so triangle inequality and $(a + b)^2 \leq 2a^2 + 2b^2$ gives

$$A = \frac{1}{2}|\sum_{k\in\mathbb{Z}} |k\pi|(|\hat{f}(k))|^2 - |\hat{u}(k)|^2 - |\hat{v}(k)|^2| \leq$$

$$\frac{1}{2}\sum_{k\in\mathbb{Z}} |k\pi|(|\hat{u}(k)|^2 + |\hat{v}(k)|^2) \leq \sum_{k\in\mathbb{Z}} |k\pi|^2(|\hat{u}(k)|^2 + |\hat{v}(k)|^2) = \frac{P^2}{2\pi}$$

finishing the proof

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$\square$

Question 6:

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a)

Claim: $u(x) = |sin(x/2)|$ is in H^1_{per} and is a weak solution of $-u''(x) + u(x) = f$ where f is a linear combination of dirac delta and $v \to \int_0^{2\pi} \overline{g(x)} v(x)dx$

Proof. We have $||u||_H =$

$$\int_0^{2\pi} sin(x/2)^2 + 1/4cos(x/2)^2 dx \leq 8 < \infty$$

so $u \in H$. Further we show it is a weak solution. Fix smooth test function v and write

$$\int_0^{2\pi} u'(x)v'(x) + u(x)v(x)dx = -v(0) + \int_0^{2\pi} g(x)v(x) \iff$$

$$\int_0^{2\pi} u'(x)v'(x)+v(x)(u(x)-g(x))dx = -v(0) = \int_0^{2\pi} \frac{1}{2}cos(1/2x)v'(x)v(x)(sin(1/2x)-g(x))dx$$

We now choose $g(x) = 5/4sin(1/2x)$ to make the product rule work:

$$\frac{1}{2}\int_0^{2\pi} (v(x)\frac{1}{2}cos(1/2x))'dx = \frac{1}{2}v(x)\frac{1}{2}cos(1/2x)|_0^{2\pi} = -v(0)$$

as desired. This shows u a weak solution

$\square$

b)

The function $w(x) = \sqrt{u(x)}$ does not belong to H^1_{per}. In particular w has no weak derivative in L^2_{per}.

Proof. Assume for sake of contradiction there exists a weak derivative ie. a map of the form $T(f) = -\int_0^{2\pi} w(x)f'(x)dx$ for all $f \in C^\infty_{per}$. We will show contradiction.

First we seek to approximate an integration by parts. Define $1_{\epsilon^c} = 1_{(-\pi,-\epsilon)\cup(\epsilon,\pi)}$. Then we can find $f_\epsilon \in L_2$ s.t. $\int_{-\pi}^{\pi} w' 1_{\epsilon^c} f dx = ||w' 1_{\epsilon^c}||^2$. Clearly as $\epsilon \to 0$ we know $||w' 1_{\epsilon^c}||^2$ blows up as we have a $1/x$ like term in the denominator. Further we may assume f 2π periodic and with L^2 norm 1. We may even assume it is even as we compute the integral from 0 to π and pick f, then reflect across the origin. We may approximate f arbitrarily with a C_{per}^{∞} function, labeled g_ϵ with precision to be picked later. In effect

$$|\int_{-\pi}^{\pi} w' f_\epsilon 1_{\epsilon^c} - \int_{-\pi}^{\pi} w' g_\epsilon 1_{\epsilon^c}|$$

can be made arbitrarily small. But now we may justify an integration by parts, computing(the boundary terms are small)

$$\int_{-\pi}^{\pi} w(x) g_\epsilon'(x) 1_{\epsilon^c} dx \approx \int_{-\pi}^{\pi} w'(x) g_\epsilon(x) 1_{\epsilon^c} dx$$

But we know this term becomes arbitrarily large so it simply remains to justify $\int_{-\pi}^{\pi} w(x) g_\epsilon'(x) 1_{\epsilon^c} dx$ is close to $\int_{-\pi}^{\pi} w(x) g_\epsilon'(x) dx$. In particular we must compute

$$\int_{-\epsilon}^{\epsilon} w(x) g_\epsilon'(x) dx$$

Note however that we may pick f without loss of generality to be even. Therefore with some low degree of error g is even and therefore g' is odd. Thus the above integral is the(approximately) the product of an even function with an odd function. We can pick g to make this term arbitrarily small by improving its approximation of f.

Thus we know as f improves its approximation of f, the integral $T(g_\epsilon)$ blows up, as desired.

$\square$

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