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Hw 7

Question 1:

Take $\mathcal{H} = L^2_{per}$ and let $Bf(x) = \int_{-\pi/2}^{x} f(x)dx$. By arzela ascoli this is a compact operator. Then consider the sequence $A_n f(x) = 1_{(-1/n,1/n)} f(x)$(with $A_n f(0) = 0$). It is linear and bounded(will not increase L^2 norm) and clearly converging pointwise(and thus strongly) to 0. However it is not converging to 0 in norm as each of the operator norms of A_n are at least 1 if we consider functions whose support is inside $(-1/n, 1/n)$. Further $||BA_n||$ are also at least 1 if we consider exponential functions with support inside $(-1/n, 1/n)$. Hence BA_n do not converge in norm to 0.

Question 2:

a)

Let $U : \mathbb{R} \to \mathcal{L}(X)$ strongly differentiable at $t \in \mathbb{R}$.

Claim: $x \to Ax$ is linear and continuous

Proof. Fix $t \in \mathbb{R}$. Then Clearly $Ax = \lim_{h\to 0} \frac{U(t+h)x - U(t)x}{h}$ is linear given the linearity of limits and $U(t+h), U(t)$. We show it is bounded.

Compute

$$|| \lim_{h\to 0} \frac{U(t+h)x - U(t)}{h} || \leq || \frac{U(t+h) - U(t)}{h} || || |x|| \leq Mx$$

where $M = \sup_{h>0} || \frac{U(t+h)-U(t)}{h} ||$ which we are guaranteed is finite via the uniform boundedness principle(since the limit exists for each fixed x). This shows boundedness and hence continuity.

$\square$

b)

Define $U : \mathbb{R} \to \mathcal{L}(l_1)$ via $U(t)x = (e^{int} x_n)_{n\in\mathbb{N}}$. Then U is strongly continuous but not strongly differentiable.

Proof. First we show strong continuity. Fix $x \in X$. Then we show $\lim_{t\to 0} x_t = \lim_{t\to 0} U(t)x = x$ in norm on X. Let $\epsilon > 0$. Compute

$$||x_t - x||_{l^1} = \sum_{k=1}^{\infty} |e^{int}x_k - x_k|$$

Pick N s.t. the tail after N is less than $\epsilon/4$. Because the map $t_n \to e^{int}$ is continuous for fixed n we may find t s.t. $|e^{int} - 1| < \epsilon/(2|x_n|N)$. Set $T = min(t_1, ..., t_N)$. Then for $t \geq T$ we have

$$\sum_{k=1}^{N} |e^{int}x_k - x_k| + \sum_{k=N+1}^{\infty} |e^{int}x_k - x_k| \leq \sum_{k=1}^{N} |e^{int}x_k - x_k| + 2\epsilon/4 =$$

$$\sum_{k=1}^{N} |e^{int} - 1||x_k| + 2\epsilon/4 < sum_{k=1}^{N} \epsilon/(2|x_k|N)|x_k| + \epsilon/2 = \epsilon$$

which shows strong continuity since ϵ and x arbitrary.

However it is not strongly differentiable as when we look at the limit for arbitrary $x \in l^1$:

$$\lim_{h\to 0} \frac{U(t+h)x - U(t)x}{h} = \lim_{h\to 0}\left(\frac{e^{inh}e^{int}x - e^{int}x}{h}\right)_n = \lim_{h\to 0}\left(\frac{(e^{inh} - 1)}{h}e^{int}x\right)_n$$

Compute

$$\lim_{h\to 0} \frac{e^{inh} - 1}{h} = \lim_{h\to 0} ine^{inh}$$

which will blow up as n gets large, ie. the limit may not be l^1.

□

Question 3:

Let $\mathcal{H}$ Hilbert and $A \in \mathcal{L}(\mathcal{H})$. Suppose λ is in the essential spectrum of A.

Claim: If $B \in \mathcal{L}(\mathcal{H})$ compact then λ in the essential spectrum of $A + B$.

Proof. Because λ in the essential specturm of A we know $\exists\{u_n\} \subseteq H$ s.t. $||(\lambda I - A)u_n||$ and $\{u_n\}$ orthonormal. Then $\{u_n\}$ and $\{Bu_n\}$ therefore has convergent subsequence $\{Bu_{n_k}\}$ since B compact and $\{u_n\}$ bounded. Then consider

$$\lim_{n\to\infty} ||(\lambda I - A - B)u_{n_k}|| \leq \lim_{n\to\infty} ||(\lambda I - A)u_{n_k}|| + \lim_{n\to\infty} ||Bu_{n_k}||$$

However we know the first term goes to 0 by assumption as it is a subsequence of $\{u_n\}$. It suffices to show $Bu_{n_k} \to 0$(in norm). But this is implied by $Bu_n \to 0$(in norm) since B is a compact operator and $u_n \to 0$ weakly. Hence we are done.

$\square$

Question 4:

Claim: Let X,Y,Z three nested banach spaces. Let $I : X \to Y$ a compact identity and $J : Y \to Z$ a continuous identity embedding. Then $\forall \epsilon > 0 \exists C_\epsilon \geq 0$ s.t.

$$||u||_Y \leq \epsilon ||u||_X + C_\epsilon ||u||_Z, \forall u \in X$$

Proof. Assume for sake of contradiction $\exists \epsilon > 0$ s.t. $\forall C \geq 0$ $\exists u \in X$ with

$$||u||_Y > \epsilon ||u||_X + C||u||_Z$$

Consider the sequence $C_n = n$ which generates $\{u_n\} \subseteq X$ satisfying the above for $C = C_n$.

First suppose $\{u_n\}$ is bounded in norm. Because I compact we have a convergent subsequence $\{u_{n_k}\}$ in Y with $u_{n_k} \to u$ for some u. Further by continuity we also have $u_{n_k} \to u$ in Z. So as we send $n \to \infty$ we must have the inequality

$$||u||_Y > C||u||_Z$$

for all $C \geq 0$ which is clearly not possible. So it cannot be that $\{u_n\}$ bounded.

Consider $J^{-1} : Im(J) \subseteq Z \to Y$. We know $J : Y \to Im(J)$ is a bdd. bijection and hence J^{-1} is bdd. In particular this tells us for all $y \in Y, z_y = J(y)$,

$$||y||_Y = ||J^{-1}z_y||_Y \leq ||J^{-1}||_{op}||z_y||_Z = ||J^{-1}||_{op}||y||_Z$$

So then for $C > ||J^{-1}||_{op}$ we cannot have

$$||u||_Y > \epsilon||u||_X + C||u||_Z$$

for any u. A contradiction.

Thus the statement holds.

□

We now use it to show $\forall \epsilon > 0 \exists C_\epsilon \geq 0$ s.t.

$$\max_{x\in[0,1]} |u(x)| \leq \epsilon \max_{x\in[0,1]} |u'(x)| + C_\epsilon \int_0^1 |u(y)|dy, \forall u \in C^1([0,1])$$

Proof. Take $X = C^1([0,1]), Y = C^0([0,1])$ and $Z = L^1([0,1])$. Clearly $X \subseteq Y \subseteq Z$. With $||u||_X = \max_{x\in[0,1]} |u(x)| + \max_{x\in[0,1]} |u'(x)|$, $||u||_Y = \max_{x\in[0,1]} |u(x)|$ and $||u||_Z = \int_0^1 |u(x)|dx$. Then to show the statement is suffices to show the embedding $I : X \to Y$ compact and $J : Y \to Z$ continuous.

Note that $I : X \to Y$ compact by arzela ascoli as if $\{u_n\}$ bounded in X then it is bounded in Y and furthermore equicontinuous as we have a bound on its derivative.

$J : Y \to Z$ is bounded(and hence continuous) via $\int_0^1 |u(y)|dy \leq \max_{x\in[0,1]} |u(x)| = ||u||_Y$

$\square$

Question 5:

Let A be a bounded self-adjiont operator on Hilbert $\mathcal{H}$ and set

$$M = \sup\{|(u, Au)| : u \in \mathcal{H}, ||u|| = 1\}$$

Claim: $M = ||A||$

Proof. First note for unit $u \in \mathcal{H}$

$$|(u, Au)| \leq ||Au||||u|| \leq ||A||||u||^2 = ||A||$$

so $M \leq ||A||$. We now show $||A|| \leq M$.

Note that if $|(v, Au)| \leq M$ for arbitrary $||v|| = ||u|| = 1$ then take for arbitrary u with $||u|| = 1$

$$||Au|| = \frac{(Au, Au)}{||Au||} = |(\frac{Au}{||Au||}, Au)| \leq M \implies ||A|| \leq M$$

so it suffices to show the hypothesis. Compute

$$|(v, Au)| = \sqrt{(}Re(v, Au)^2 + Im(v, Au)^2) \leq |Re(v, Au)| + |Im(v, Au)| =$$

$$|\frac{1}{4}[(v{+}u, A(v{+}u)) - (v{-}u, A(v{-}u))]| + |\frac{1}{4}[(v{+}u, iA(v{+}u)) - (v{-}u, iA(v{-}u))]| \leq$$

$$\frac{1}{4}[M||v+u||^2 + M||v-u||^2 + M||v+u||^2 + M||v-u||^2] = M(||u||^2 + ||v||^2)/2$$

as desired

$\square$