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Hwk 7

Question 1:

Let $a_n = 3a_{n-1} - 2a_{n-2}$ with $a_0 = 0, a_1 = 1$. Consider the ogf $f(x) = \sum_{n=0} a_n x^n$. $(1 - 3x + 2x^2)f(x) = \sum_{n=0} a_n x^n - 3\sum_{n=1} a_{n-1}x^n + 2\sum_{n=2} a_{n-2}x^n = 0 + x + \sum_{n=2} a_n - 3a_{n-1} + 2a_{n-2} = x + \sum 0 = 1 \implies f(x) = \frac{x}{2x^2-3x+1} = \frac{x}{(2x-1)(x-1)} = \frac{1}{x-1} - \frac{1}{2x-1} = \frac{1}{1-2x} - \frac{1}{1-x}$.

Note that $\frac{1}{1-x} = \sum_{n=0}(-1)^n x^n$ and $\frac{1}{1-2x} = \sum_{n=0}(-1)^n 2^n x^n$. Then $f(x) = \sum_{n=0}(-1)^n 2^n x^n - \sum_{n=0}(-1)^n x^n = \sum_{n=0}(-1)^n x^n(2^n - 1) \implies a_n = 2^n - 1$

Question 2:

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i)

Let a_n be the number of sequences of 0s and 1s that do not contain consecutive 0s. Note $a_0 = 1, a_1 = 2$.

We case on the last numeral of the string of length n. If the numeral is 1 then there are a_{n-1} possibilities of the preceding $n-1$ terms. If the numeral is 0 then we know the preceeding numeral must be 1. Thus there are a_{n-2} possibilites of the first $n-2$ terms. This gives us the reccurence $a_n = a_{n-1} + a_{n-2}$.

Let $f(x) = \sum_n a_n x^n$. Consider $(1-x-x^2)f(x) = 1+2x+\sum_{n=2}(a_n - a_{n-1} - a_{n-2})x^n = 1+2x \implies f(x) = \frac{1+2x}{1-x-x^2}$. Note also that the charactersitic polynomial is $p(t) = t^2 - t - 1$ We know because this reccurence fits the form of a constant coefficient homogeneous relation the solutions must take the form $a_n = k_1(\frac{1+\sqrt{5}}{2})^n + k_2(\frac{1-\sqrt{5}}{2})^n$ where $\frac{1\pm\sqrt{5}}{2}$ are the well known roots of p. Then solving for the constanst with initial conditions, we find $a_0 = 1 = k_1 + k_2$ and $a_1 = 2 = k_1\frac{1+\sqrt{5}}{2} + k_2\frac{1-\sqrt{5}}{2} \implies 4 = k_1(1+\sqrt{5}) + k_2(1-\sqrt{5}) = k_1(1+\sqrt{5}) + (1-k_1)(1-\sqrt{5}) = k_1 + k_1\sqrt{5} + 1 - \sqrt{5} - k_1 + k_1\sqrt{5} = 1 - \sqrt{5} + 2k_1\sqrt{5} \implies \frac{3}{2\sqrt{5}} + \frac{1}{2} = k_1 = \frac{3\sqrt{5}+5}{10} \implies k_2 = 1 - \frac{3\sqrt{5}+5}{10} = \frac{5-3\sqrt{5}}{10}$ which gives us a closed form

ii)

Let b_n be defined as stated. Then let c_n be the strings counted in b_n which end with a 1. Then we know $b_n = c_n + c_{n-2} + c_{n-3}$ since for a given string, either it ends in 1, ends in 100, or ends in 1000. We also know $c_n = c_{n-3} + c_{n-4}$ either the a sequence ends in 001 or 0001.

So we solve for c_n similarly to above. Note that by assumption $c_0 = 0, c_1 = 1$. We see this is a recurrence relation of constant coefficients. Its characteristic polynomial is $p(t) = t^4 - t - 1$ which has 4 distinct roots. We know that $t^4 - t - 1 = (t^2 + at + b)(t^2 + ct + d)$ for some $a, b, c, d \in \mathbb{C}$ s.t. $a + c = 0, -a^2 + b + d = 0, ad - ba = -1$ and $bd = -1$. (Note that both polynomials therefore have distinct roots).

Because we know this has distinct roots we know the solution is of the form $c_n = k_1 r_1^n + k_2 r_2^n + k_3 r_3^n + k_4 r_4^n$ for the distinct roots r_i. This means that $b_n = k_1 r_1^n + k_2 r_2^n + k_3 r_3^n + k_4 r_4^n + k_1 r_1^n / r^2 + k_2 r_2^n / r^2 + k_3 r_3^n / r^2 + k_4 r_4^n / r^2 + k_1 r_1^n / r^3 + k_2 r_2^n / r^3 + k_3 r_3^n / r^3 + k_4 r_4^n / r^3 = d_1 r_1^n + d_2 r_2^n + d_3 r_3^n + d_4 r_4^n$ for $d_i = (1 + \frac{1}{r_i^2} + \frac{1}{r_i^3})$. Wlog order the roots $|r_1| > |r_2| > |r_3| > |r_4|$. Then $b_n^{1/p}$ is equivalent to the l_p norm on the terms. Then when we take the limit this becomes the infinity norm which is simply the largest term of the sum. In this case this is the largest root $r_1 \approx 1.2207$ which we get via numerical methods(wolfram...)

Question 3:

Let a_n denote the number of ways of decomposing a convex $n + 1$ gon into quadrilaterals via nonintersecting chords. Assume $a_0 = 0, a_1 = 1$.

We see $a_n = \sum_{k+l+m} a_k a_l a_m$ for $n \geq 3$ via the following. Consider an aribtrary edge on the perimeter. We know this is contained in some quadrilateral. Furthermore this quadrilateral may partition the n+1 gon into at most 3 disjoint convex k, l, m gons plus the quadrilateral. We know there are at most 3 resulting disjoint convex polygons since no edge can border more than two regions, all 4 edges border the quadrilateral by assumption, and we assume one the edges is on the perimeter. So the other edges may border at most one other polygon, and we have 3 other edges. Note that we know this partitioning results in polygons since the drawing of nonintersecting chords can never result in a concave polygon. Then we know that the number of way to partition the resulting 3 convex polygons is $a_m * a_k * a_l$ for some $m + k + l = n$. Then we sum over all possible $m + k + l = n$ to take into account each possible quadrilateral that contains the fixed edge.

Set $f(x) = \sum_n a_n x^n = 0x^0 + 1x + 0x^2 + \sum_{n \geq 3} \sum_{m+l+k=n} a_m a_l a_k x^n$. But we see $\sum_{n \geq 3} \sum_{m+l+k=n} a_m a_l a_k x^n = f(x)^3$ as we have no constant term(so every term in the three way product has degree at least 3). So $f(x) = x + f(x)^3 \implies f(x)^3 - f(x) + x = 0$. Then using lagrange inversion(thm 14.3) we may write $f(x) = \sum_{k=0} \binom{3k}{k} \frac{x^{2k+1}}{2k+1}$ which shows us that $a_n = \binom{3n}{n} \frac{1}{2n+1}$ for odd n and 0 otherwise