Alex Havrilla 4/3

1. Let X, Y be Banach spaces. $\forall\, k \in \mathbb{N}$, $T_k : X \to Y$ bdd. linear s.t. $\|T_k\| \leq M$. $X_0 = \{x \in X : \{T_k(x)\} \text{ Cauchy in } Y\}$.

<u>Claim:</u> X_0 is a closed subspace of X.

[Note immediately that X_0 is a v.s. since the T_k linear and sums of Cauchy sequences are Cauchy. It remains to show X_0 closed. Suppose y is a limit point of X_0. We will show $\{T_k(y)\}$ Cauchy in Y.

Compute

$$\|T_n(y) - T_m(y)\| \leq \|T_n(y) - T_n(x_k) + T_n(x_k) - T_m(x_k) + T_m(x_k) -$$

$$\leq \|T_n(y - x_k)\| + \|T_m(y - x_k)\| + \|T_n(x_k) - T_m(x_k)\|$$

$$\leq M\|y - x_k\| + M\|y - x_k\| + \cancel{\|(T_n - T_m)(x_k)\|}$$

$$\leq M \ldots \quad \ldots + \cancel{\ldots}\ \|T_n - T$$

~~The first term~~ Let $\varepsilon > 0$. The first term can be made small by letting k large enough as $x_k \to y$. Same for the 2nd term. Make these both $< \varepsilon/3$. We know $\{T_m(x_k)\}$ is Cauchy so we make this small by making m, n large enough. Makes $\|T_n(x_k) - T_m(x_k)\| < \varepsilon/3$. Since ε arbitrary we know $\{T_n(y)\}$ Cauchy and hence $y \in X_0$, completing the proof.

(2) Let X, Y banach. $T: X \to Y$ linear and $\exists\, c \geq 0$ w/

$$\|Tx\|_Y \geq c\|x\|_X \quad \forall\, x \in X.$$

Claim: If $Im(T)$ is closed then T bdd.

Suppose $Im(T)$ is closed. We know it is a VS, but then it is also banach as it is a closed subspace of banach Y.

Closed graph theorem?

Recall T is cts iff T is bdd.

$\|Tx\| \geq C\|x\| \;\forall x \Rightarrow \|T\| \geq C \ldots$

Note that T must be injective since we cannot map $x \neq 0$ to 0 due to our inequality. Then it is bijective on its range. Then we know T^{-1} exists.

Hence by a mapping theorem T is cts and hence bdd.

*Its graph must be closed since it is a bijection ~~[illegible]~~ and range is closed.

(3.) Claim: Each neighborhood of 0 in weak* top. on X^* has nontrivial subspace.

Let U be a neighborhood of $0 \in X^*$.

Recall that wlog U can be written as the infinite product of X, finitely of many whose terms are nontrivial open sets in X. Call these terms $U_{x_1}, \dots, U_{x_n}$. Then we construct a subspace around these. Set $S = \{\lambda \in X : \lambda(x_j) = 0,\ x_j \in \{x_1, \dots, x_n\}\}$.

Clearly this constitutes a vectorspace as if $\lambda, \gamma \in S$, then $(\lambda + \gamma)(x_j) = 0 = \lambda(x_j) + \gamma(x_j)$ and similarly for scalar mult. Further $S \subseteq U$ since $\forall\, \lambda \in S$, $\lambda(x) \in U_x$ where U_x is the corresponding open set for $x \in X$ in the product U.

Note that all the U_x must be neighborhoods of $0 \in X$.

Since U arbitrary this concludes the proof.

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4. Let g cts 2π-periodic. Claim: $f \mapsto (g, f)_{L^2}$ is weakly seq. cts on H^1_{per}.

WTS if f_k converges weakly to f in H_1 then $(g, f_k) \to (g, f)_{L^2}$. Recall that if $f_k \to f$ weakly then $\forall\, h \in H^1$, $(h, f_k) \to (h, f)$, via Riesz representation.

~~[illegible]~~

$$(g, f_k) = \int_0^{2\pi} g(x) f_k(x)\, dx.$$
$$(g, f) = \int_0^{2\pi} g(x) f(x)\, dx$$

* Note that if $h_k \to h$ weakly then ~~[illegible]~~ h_k is norm bounded. If $f_k \to f$ pointwise could use DCT to push through limit since H_1 norm bounds L_∞ norm. But we do know $f_k(x) \to f(x)$ via projections (since projections linear transformations in H^{1*}). Hence via DCT we may conclude

$$\lim_{k \to \infty} \int_0^{2\pi} g(x) f_k(x)\, dx = \int_0^{2\pi} g(x) f(x)\, dx$$

where our dominating function is $2\pi M$ where M bounds the H_1 norm of f_k.

(Note cty ~~of~~ g is used to ensure dominator integrable).

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(5.) Claim: $u(x) = x$ mod 2π has no weak deriv $v \in L^2_{per}$

If u had a weak derivative $\forall f \in C^\infty_{2\pi}$, $(u, f) = -(v, f')$

$\to \int_0^{2\pi} x f(x) dx = -\int_0^{2\pi} v(x) f'(x) dx$

Recall if $v \in H^1$ then u has a weak derivative.

Recall the complex exponentials $\left\{\frac{1}{\sqrt{2\pi}} e^{ikx}\right\}_{k \in \mathbb{Z}}$ form an

ON basis.

Compute $c_k = (e_k, u) = \int_0^{2\pi} \frac{1}{\sqrt{2\pi}} x e^{ikx} dx =$

$-\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \frac{1}{ik} e^{ikx} dx = -\frac{1}{ik\sqrt{2\pi}} \int_0^{2\pi} e^{ikx} dx.$

But these sequence is then not in $\ell^2(\mathbb{Z})$ and hence x cannot then be in H^1_{per} and $\therefore$ does not have a weak derivative

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