Complex Analysis

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1 An Introduction

Idea 1. Proving non-analyticity. Show different limit in two diff. directions. Often useful to consider $h \in \mathbb{R}$ and $h \in i\mathbb{R}$.

Example 1. Consider $\overline{z}$

Prop 1. *Often convenient to regard holomorphic definition as*

$$f(z_0 + h) = f(z_0) + ah + o(h)$$

for some a. Ie. the function is locally complex linear.

Remark 1. The above propsition is useful for proving chain rule.

1.1 Cauchy Riemann Equations

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Remark 2. Cauchy riemann equations derived via simply differentiating f as a function of two varibles in real and comlex directions.

Question 1. Cauchy riemann conditions are necessary. Are they sufficient?

Answer 1. Yes.

Theorem 1. *Let $f : \Omega \to \mathbb{C}$ with $f = u + iv$ and satisfying cauchy riemann and u,v continuously differentiable. Then f is holomorphic.*

Proof. Fix $z_0 = x_0 + iy_0$.

$$u(x_0 + h_1, y_0 + h_2) = u(x_0, y_0) + u_x h_1 + u_y h_2 + o(h)$$

and similarly for v. Then write $f(z_0 + h) - f(z_0)$ in terms of above and massage using cauchy riemann to get form $ah + o(h)$. □

Remark 3. Determinant of jacobian is really magnitude of norm of complex derivative squared.

Example 2. $f(x,y) = \sqrt{|x||y|}$ satisfies cauchy riemann but is not holomorphic.

Theorem 2. *Radius of convergence R of power series is*

$$R = \frac{1}{limsup|a_n|^{1/n}}$$

Proof. Idea is to compare to geometric series. Set R as desired. Then just compute(since we used limsup) and see that geometric series converges and or diverges in desired cases. This is also why we have problems on boundary. □

1.2 Power Series

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Theorem 3. *If f power series than f' exists as $\sum na_nz^{n-1}$ with same R, since $n^{1/n} \to 1$.*

Proof. Fix $|z_0| < r < R$. Set $g(z) = \sum na_nz^{n-1}$. Write

$$\frac{f(z_0+h)-f(z_0)}{h} - g(z) = \frac{S_N(z_0+h)-S_N(z_0)}{h} - S_N'(z_0) + S_N'(z_0) - g(z_0) + \frac{E_N(z_0+h)-E_N(z_0)}{h}$$

The first term is small using triangle inequality and $(z_0+h)^n - z_0^h = h((z_0+h)^{n-1} + (z_0+h)^{n-2}z_0 + ...)$ which is bounded in norm by something still in R once h gets small enough. Thus this third term converges.

The second term is small since this is a convergent power series.

The first term is small by definition. And we are done.

Notice this proof simultaneously asserts existence and shows what it is. □

Proof. I had another proof by writing $f'(z_0+h) = f(z_0) + ah + o(h)$ where $a = f'(z_0)$. □

Remark 4. Power series are infinitely differentiable, since we have same disk of convergence.

Question 2. Is this how we establish holomorphic functions are infinitely differentiable? Cause they're all power series? How is this done?

Answer 2. Write holomorphic function as its taylor expansion and show they are close?

Theme 1. Anytime we prove something using algebraic facts, we have a complex argument since we haven't used the changed, rigid geometry at all.

Question 3. What function is differentiable only once? Recall weierstrass cts everytwhere diff. nowhere

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Question 4. Can we allow for a countable number of cuts? Will cutting in diff. ways countably lead to diff. integrals? Does this mess up notion of equivalency? What breaks?

1.3 Complex Integration

Theorem 4. *If f has primitive then* $\int_\gamma f(z)dz = F(\omega_2) - F(\omega_1)$

Proof. Ports from real results by construction. □

Example 3. $\int_\gamma dz/z = 2\pi i$ where γ is unit circle and hence $1/z$ must not have primitive(otherwise would be 0 on closed curve).

Theorem 5. *Gauss Lucas*

Proof. Exercise. Very algebraic. Probably write roots as convex combinations. Just compute by taking derivative

In fact bring out missing roots by taking quotient P'/P. Easier to work with these than the complement set of sums and products(often working with quotients easier to access new roots). □

Theorem 6. *When* $P(z) = \sum a_k a^k$ *with real coeffiecients and only real zeroes then coefficients binomial(ultra) log concave.*

Proof. One line proof via gauss-lucas. So derivatives have real roots. Reciprocal poly also has real roots. Then to get the inequality consider $z^{n-k+1}P^{(k-1)}(1/z)$ which has real roots and take $n-k-1$ derivatives resulting in two deg poly. The coefficients of this thing gives desired result by looking at the discriminant. □

Theme 2. This is common technique to show common sequences log-concave

Example 4.

- Stirling numbers of first kind
- Stirling numbers of second kind
- t_k ultra log-concave where t_k number of matchings of size k in arbitrary graph G

Remark 5. To show mulitiplicative/additive property of exponential show $e^z e^(c-z)$ constant.

Theorem 7. *(Goursout's Theorem): If* $\Omega \subseteq \mathbb{C}$ *open and* $f : \Omega \to \mathbb{C}$ *holomorphic with* $\Delta \subseteq \Omega$ *triangle then*

$$\int_{\partial\Delta} f = 0$$

Proof. Bisect Δ noting

$$\int_{\partial\Delta^{(0)}} f = \int_{\partial\Delta_1^{(1)}} f + ... + \int_{\partial\Delta_4^{(1)}} f$$

We continue recursively bisecting so that

$$|\int_{\partial\Delta^{(k)}} f| \leq 4|\int_{\Delta^{k+1}} f|$$

The diameters $d_n \to 0$ and perimeters go to 0 so $\bigcap_{n=1}^{\infty} \Delta^{(n)} = \{z_0\}$(nested compact sets with diameter going to 0).

We argue

$$4^n|\int_{\partial\Delta^{(n)}} f| \to 0$$

since f is holomorphic. Write $f(z) = f(z_0) + f'(z_0)(z - z_0) + o(z)(z - z_0)$. Clearly $f(z_0)$ and $f'(z_0)(z - z_0)$ have primitives. Thus we are only actually integrating $(z - z_0)o(z)$. Upper bounding $(z - z_0)$ by diameter and $o(z)$ by some $sup_{\Delta^{(n)}}|f(z)|$ we have the bound $p_n d_n$ sup. Recall $p_n = 1/2p_{n-1}$ and $d_n = 1/2d_{n-1}$ and so we're done. □

Prop 2. *Now since we have triangles we can also show for rectangles and by approximation most sets(tesselation).*

Theorem 8. *Let $D\mathbb{C}$ be a disc, $f : D \to \mathbb{C}$ holomorphic then f has a primitive.*

Proof. Suppose D centered at 0. Write $F(z) = \int_{\gamma_z} f(w)dw$ where γ_z is a triangular curve connecting z to 0.

Now compute with aim $F(z + h) - F(z) = hf(z) + ho(h)$

$$\int_{\gamma_{z+h}} f - \int_{\gamma_z} f = \int_{\gamma z \to z+h} f$$

by looking at paths and conidering goursout

Then we conclude by using continuity □

Remark 6. Notice this argument relies on convexity. The problem is with holes(like the punctured disk).

Theorem 9. *Cauchy's Theorem: If f holomorphic in a disk and γ closed curve in disk then $\int_\gamma f = 0$.*

Remark 7. Cauchy's theorem holds on

- rectangles
- triangles
- sectants(triangles with one side smoothed out)
- punctured half circle(set removed in neighborhood of 0)

- keyhole

Where we just repeat the same argument as above

Theme 3. Clever changes of variables are suggested by invariance of the function: for example gaussian rotationally invariant and thus should try polar coordinates

Remark 8. Rigorously showing fourier transform of gaussian: We integrate on some rectangle with width $2R$ centered and with base along $\mathbb{R}$. Call this γ. We know

$$\int_\gamma f = 0 = \int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f + \int_{\gamma_4} f$$

with

$$\int_{\gamma_1} fd = \int_{-R}^{R} e^{-\pi x^2} dx \to 1$$
$$\int_{\gamma_3} fd = -\int_{-R}^{R} f(x+iy)dx = \int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i x y} e^{\pi y^2} dx$$

and we see these will agree because the others will go to 0 as $R \to \infty$.

Remark 9. This is an example of contour integration: Extracting integral we're interested in by pulling it out as a piece of a contour. In general allows us to compute one integral as another along a different path

Prop 3. *Computing:*

$$\int_0^\infty \frac{1-cos(x)}{x^2} dx = \frac{\pi}{2}$$

via computing contour for $f(z) = \frac{1-e^{iz}}{z^2}$. We integrate over contour:

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In the limit we'll get what we want. Note integral is 0 to start. So suffices to compute another integral along different path

$$\int_{\gamma_\epsilon} \frac{1-e^{iz}}{z^2}dz = \int_{\gamma_\epsilon} \frac{-iz+g(z)}{z^2}dz = \int_{\gamma_\epsilon} \frac{-i}{z}dz = -\pi$$

where $g(z)$ is remainder of power series which is holomorphic. Value of the integral comes from the singularity

Theme 4. Note we used linearity of the integral + the powerseries to make integration easier, and recover a holomorphic function(which is often made nonholomorphcic by singularities 1/z)

Theorem 10. *Suppose f is holomorphic in an open set that contains the closure of a disc D. If C denotes the boundary circle of this disc with the positive orientation, then*

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta$$

for any point in closure of disc D.

Proof. Fix $z \in D$ and consider with keyhole $\Gamma_{\delta,\epsilon}$, omitting z.

Clearly $\int_{\Gamma_{\delta,\epsilon}} F(\zeta)d\zeta = 0$ since holomorphic away from singularity. We let $\delta \to 0$ and use continuity of F to see the limit the integrals of corridors cancel.

$F(\zeta) = \frac{f(\zeta)-f(z)}{\zeta-z} + \frac{f(z)}{\zeta-z}$ this give us the result in the limit.

The key is to use holomorphicity to rewrite expression and then factor out an $f(z)$.

□

Prop 4. *If f is holomorphic in an open set Ω, the f has infinitely many complex derivatives in Ω. Moreover if $C \subseteq \Omega$ is a circle whose interior is also contained in Ω then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}$$

Prop 5. *Cauchy's Inequalities: If f is holomorphic in an open set that contains the closure of disc D centered at z_0 and of radius R, then*

$$|f^{(n)}(z_0)| \leq \frac{n!}{R^n}||f||_C$$

Theorem 11. *Rigidity: If $f : \Omega \to \mathbb{C}$ holomorphic, and has z_n which converges in Ω to w, with all $f(z_n) = 0$. Then f = 0*

Proof. First we show f 0 near w.

Use power series expansion. AFSOC not 0 near w. Then we have an $a_n \neq 0$. Take minimal one. Compute $f(z) = a_m(z-w)^m(1+g(z))$ where g holomorphic. But as we take a point close enough, we know $f(z_n)$ should be 0 but the product above cannot be. So done in a set

Let $U = int\{z : f(z) = 0\}$. We know U nonempty. U open, but U also closed because f continuous. Furthermore we can find a neighborhood around z in U(by first part). So it must be U everything since U connected. ☐

Theorem 12. *Morera's Theorem: If cts f vanishes on triangles then it's holomorphic.*

Proof. We know we can find a primitive via the proof of cauchy's theorem. Further it's holomorphic. Then f is holomorphic. ☐

Prop 6. *Uniqueness of analytic continuation. If two holomorphic f,g equal on sequence of points then equal everywhere.*

Proof. True by regularity(look at difference). ☐

Question 5. When is there a continuation?

Answer 3. There should only be a problem at boundary(can think of compact functions going to 0).

Remark 10. Miracles of holomorphicity: Closedness, regularity, unique extension(rigidity).

Theorem 13. *Morera's Theorem: Converse to cauchy's theorem.*

Suppose f cts on $f : D \to \mathbb{C}$ s.t. for all triangles we have 0 then f holomorphic.

Proof. From cauchy we know we have F holo on D with $F' = f$.(via this triangle property). Since F holomorphic then f holomorphic. ☐

Prop 7. *Uniform limits of holo. functions are holo*

Proof. If we can show the limit vanishes on triangles then done by Morera. Note we know unif. limits of cts functions are cts. But this is clear just from DCT. ☐

Remark 11. Recall this does not hold in real case: think weierstrass's theorem(polynomials can uniformly approximate any cts function).

Theorem 14. *Derivative of limit is limit of derivatives of holo. function*

Proof. We go for a bound on the derivative in terms of f: $||F'||_{\infty,\Omega_\delta} \leq \frac{1}{\delta}||F||_{\infty,\Omega_\Omega}$. This is enough when we look at diferences of the derivatives(take out limit)

We go by cauchy's formula:

$$|F'(z)| = |\frac{1}{2\pi i}\int_{\partial D_\delta(z)} \frac{F(w)}{(w-z)^2}dw|$$

and apply triangle inequality ☐

Remark 12. True for higher order derivatives. Amazing that knowing holomorphicity gives control of limit of all derivatives.

Theme 5. Constructing holomorphic functions: look at series of holomorphic functions(assuming unif. convergence on compact sets)

Also often useful to define as integrals of holomorphic functions of a parameter.

Prop 8. *Let $F : \Omega \times [0,1] \to \mathbb{C}$. be s.t.*

i) $z \to F(z,s)$ is holo. for all s

ii) F cts on $\Omega \times [0,1]$. Then

$f(z) = \int_0^1 F(z,s)ds$

Proof. We argue the integral is the uniform limit of the riemanns sums.

Write $f_n(z) = \frac{1}{n}\sum^n F(z,k/n)$. Uniform bound comes from continuity: cause we're on a comapct set ☐

Theorem 15. *Reflection Principle: Let $\Omega \subseteq \mathbb{C}$ sym. about $\mathbb{R}$. Let Ω^+ be part above real line. Let I be real part.*

Symmetry Princple: Let $f^{+/-} : \Omega^{+/-} \to \mathbb{C}$ holo, both extend cts to I(maybe not holomorphcally). And extensions agree. Then gluing everyting together we get f a holomorphic function.

Proof. Simply need to check boundary. After showing continuity use morera's theorem(converse to gousard).

Hardest case is triangle over interval. Then we simply subdivide and integrate each function along restricted sub-divisions. We use continuous extension by shifting triangles down by ϵ. ☐

Remark 13. Real line is irrelevant, can be any curve. Also note holomorphicity is extended by continuity.

Theorem 16. *Schwarz Refletion Principle: Let $f^+ : \Omega^+ \to \mathbb{C}$ holo, f^+ extends cts to I, $f|_I \in \mathbb{R}$ and reflecting over it.*

Proof. Define f^- and argue using reflection. Set $f^-(z) = \overline{f(\overline{z})}$. Simply need to check holomorphicity(since agreement on real line).

To show holo. we use power series. Note bar distributes over multiplication and we're done. ☐

Theorem 17. *Runge's Theorem: Let $K \subseteq \mathbb{C}$ with f holo. in neighborhood of K. Then f can be approximuated uniformly by rational functions with singularities in K^C.*

If K^C connected then the rational functions can be polynomials.

Proof. Use cauchy's formula plus riemann sums.

Connectedness(in K^C) is used to push away singularities. □

Example 5. $e^{1/z}$ has essential singularity at 0(blows up faster than any rational function at 0).

Remark 14. We can locally write $f(z) = (z - z_0)^n g(z)$ for a zero z_0 of f for g nonvanishing locally.