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Hw 6

Question 1:

Let X be an infinite diensional banach space with $T : X \to X$ compact bounded linear map.

Claim: T is not bijective

Proof. Assume for sake of contradiction T is bijective. Then we know T^{-1} is linear and bounded via the inverse function theorem. This implies $T^{-1}T = I$ is compact which cannot be the case(as for example then every weakly converging sequence would be stronlgy converging). Hence we have a contradiction.

□

Question 2:

Let X be a reflexive banach space and $T : X \to X$ be a bounded and linear. Suppose for every weakly converging sequence $\{x_n\}$, $\{Tx_n\}$ converges in norm.

Claim: T is compact

Proof. Let $\{x_n\}$ be a bounded sequence in X. We aim to extract a weakly converging subsequence. Then by assumption its image is strongly converging, showing compactness of T.

Note that since $\{x_n\} \exists M \geq 0$ s.t. arbitrary $||x_n|| \leq M$. Then wlog let $M = 1$ via a scaling argument. So $\{x_n\} \subseteq B_(0, 1)$. But we know the unit ball in the weak topology is sequentially compact in reflexive banach spaces. Hence we may extract a weakly convergent subsequence from $\{x_n\}$, finishing the proof.

□

Question 3:

Let M, N be subspaces of banach space X s.t. $M \oplus N = X$ with $M \cap N = \{0\}$. Let P be the projection on M.

Claim: P is bounded $\iff$ both M and N are closed.

Proof. $\implies$: Suppose P bounded. Then it is continuous. Let x be a limit point of M. Then $\exists x_n \to x$ in X, $\{x_n\} \subseteq M$. Compute

$$x = \lim_{n\to\infty} x_n = lim_{n\to\infty} P(x_n) = P(\lim_{n\to\infty} x_n) = P(x)$$

so $x \in M$. Similarly let x be a limit point of N with x_n approaching and compute

$$0 = \lim_{n\to\infty} P(x_n) = P(x) \implies x \in N$$

So both M and N closed.

$\impliedby$: Now suppose M, N closed. By the closed graph theorem it suffices to show the graph of P, $\Gamma \subseteq X \times M$ is closed.

Let $p = (x, y)$ a limit point of Γ. Then we have sequence $\{p_k\} \subseteq \Gamma, p_k = (x_k, P(x_k))$ converging in the product norm(sum of norms of components). So in particular we know $P(x_k) \to y$. But we know $\{P(x_k)\} \subseteq M$ is closed, so its limit must be in M. Hence $y \in M$ which shows the graph closed and then P bounded.

$\square$

Question 4:

Let $f \in L^2_{per}$ and define $Tf \in L^2_{per}$ via

$$Tf(x) = \int_{-\pi}^{\pi} (x + y) f(y) dy$$

Claim: T is compact and self-adjoint.

Proof. First we show self-adjointness. Compute

$$(f, Tg) = \int_{-\pi}^{\pi} \overline{f}(x)Tg(x)dx = \int_{-\pi}^{\pi} \overline{f}(x) \int_{-\pi}^{\pi} (x+y)g(y)dydx =$$

via fubini

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \overline{f}(x)(x+y)g(y)dxdy = \int_{-\pi}^{\pi} g(y) \int_{-\pi}^{\pi} (x+y)\overline{f}(x)dxdy =$$

$$\int_{-\pi}^{\pi} \overline{Tf}(x)g(x)dx = (Tf, g) \implies T^* = T$$

We use Arezela Ascoli to show convergence compactness. Suppose the sequence $\{f_k\}$ bounded by $M \geq 0$ in L^2_{per} norm. Then we have a uniform bound on $\{Tf_k\}$ via:

$$||Tf_k||^2 = \int_{-\pi}^{\pi} (Tf_k(x))^2 dx = \int_{-\pi}^{\pi} (\int_{-\pi}^{\pi} (x+y)f_k(y)dy)^2 dx \leq$$

via cauchy schwarz

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (x+y)^2 dy \int_{-\pi}^{\pi} f_k(y)^2 dydx = ||f_k||^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (x+y)^2 dxdy \leq C_M$$

where C is a constant depending on M. This establishes uniform boundedness. Now we show equicontinuity. Let $\epsilon > 0$. Then

$$|Tf(x) - Tf(y)| = \int_{-\pi}^{\pi} (x+z)f(z) - (y+z)f(z)dz =$$

$$\int_{-\pi}^{\pi} f(z)(x-y)dz \leq ||f||2\pi(x-y) \leq 2\pi M(x-y) < \epsilon$$

by making $||x-y||$ small enough. Note that we know $||T||_{op} < \infty$ via a computation identical to one for f_k except we do not bound $||f||$ by M(since clearly $(x+y)^2 \in L^2_{per}$). Then arezela ascoli gives us a uniformly converging subsquence, which in tuerm gives us a subsequence converging in norm.

$\square$

Claim: $\sigma_p(T) = \{0, -2\pi^2, 2\pi^2\}$

Proof. First note $0 \in \sigma_p(T)$ since any odd function is sent to 0. Now suppose $\lambda f = Tf$ with $\lambda \neq 0$. Then we know

$$f(x) = \frac{Tf(x)}{\lambda} = \frac{\int_{-\pi}^{\pi} f(y)}{\lambda} x + \int_{-\pi}^{\pi} yf(y)$$

ie. f is a line. Set $m = \frac{\int_{-\pi}^{\pi} f(y)}{\lambda}$ and $b = \int_{-\pi}^{\pi} yf(y)dy/\lambda$ to compute

$$\lambda m = \int_{-\pi}^{\pi} f(y) = \int_{-\pi}^{\pi} mx + b = 2\pi b$$

and

$$\lambda b = \int_{-\pi}^{\pi} mx^2 + bxdx = 2m\pi^3$$

so

$$\lambda b = 2\pi^3 (2\pi b/\lambda)\pi^3 \implies \lambda^2 = 4\pi^4 \implies \lambda = \pm 2\pi^2$$

□

Claim: $\rho(T) = \mathbb{C} \backslash \sigma_p(T)$

Proof. Suppose $\lambda I - T$ is injective. We will show it is surjective and hence bijective.

Let $g \in L^2_{per}$. Set $g = (\lambda I - T)f \implies \lambda f = g + Tf = g + \int f(y)dyx + \int yf(y)dy$. Wlog suppose $\lambda = 1$(the computation is the same regardless). So if we can find an appropriate f then clearly f is in L^2_{per} and thus our operator will be surjective. Let $C = \int f(y)dy$ and $D = \int yf(y)dy$

Compute

$$C = \int f(y)dy = \int g(x) + Cx + Ddx = \int g(x)dx + 2\pi D$$

and

$$D = \int xf(x)dx = \int xg(x) + x^2C + xDdx = \int xg(x)dx + C\int x^2dx$$

where we use symmetry and oddness to conclude some integrals are 0. This defines f in terms of g, showing surjectivity. Hence T_λ is bijective.

□

Question 5:

For 2π periodic u, f on $\mathbb{R}$ we have the differential equation

$$\lambda u - u' = f$$

Claim: Whenever $\lambda \notin i\mathbb{Z}$ there is an operator on L^2_{per} s.t. when f smooth then $u = T_\lambda f$ is a solution.

Proof. Applying the ft gives

$$\lambda \hat{u}(m) - (im)\hat{u} = \hat{f} \implies \hat{u} = \frac{\hat{f}}{\lambda - im}$$

where we may divide in the case $\lambda \notin i\mathbb{Z}$. So we know u is given by $\mathcal{F}^-(\frac{\hat{f}(m)}{\lambda - im})$ where $\mathcal{F}^-$ is the inverse fourier transform. Furthermore we compute for $f \in L^2_{per}$,

$$||T_\lambda f|| = ||\mathcal{F}^-(\frac{\hat{f}(m)}{\lambda - im})|| = ||\frac{\hat{f}(m)}{\lambda - im}||$$

Note that unless $|\lambda - im| < 1$ for some $m \in \mathbb{Z}$, $||\frac{\hat{f}(m)}{\lambda - im}|| \leq ||\hat{f}||$. Note in the worst case $||\frac{\hat{f}(m)}{\lambda - im}|| \leq \frac{1}{|\lambda - im|}||\hat{f}||$ where we attain equality via $1_m = (0, ..., 1, 0, ..)$ ie a 1 in the mth term. Hence $||T_\lambda|| = \frac{1}{min|\lambda - im|}$.

□

We compute $K_\lambda(x, y) = e^{\lambda(x-y)}$

Proof. Then K_λ is bounded since

$$||K_\lambda f|| = \int_{-\pi}^{\pi} |e^{\lambda x} \int_{-\infty}^{\infty} e^{-\lambda y} f(y)dy|dx \leq \int_{-\pi}^{\pi} |e^{\lambda x}||f||||e^{-\lambda x}||dx \leq$$

$$||e^{\lambda}||||e^{-\lambda}||||f||$$

which gives boundedness. Furthermore have $T_\lambda = K_\lambda$ since this integral is computing the inverse fourier transform.

□

Claim: For any bounded set $S \subseteq L^2_{per}$, $K_\lambda S$ is an equicontinuous family of functions.

Proof. Note that if $K_\lambda S$ is also uniformly bounded since S is bounded and K_λ a bounded operator on L^2_{per} so Arzela Ascoli tells us we can extract a uniformly converging subsequence of an arbitrary sequence in S if we have $K_\lambda S$ equicontinuous. This will show the operator compact.

Let $\epsilon > 0$. Then compute

$$|K_\lambda(x) - K_\lambda(y)| = |\int_{-\infty}^{\infty} [k_\lambda(x,z) - k_\lambda(y,z)] f(z)dz| \leq ||f|| \int_{-\infty}^{\infty} |k_\lambda(x,z) - k_\lambda(y,z)|dz \leq M 2\pi C_{k_\lambda} |x-y|$$

where M is a bound on $||f||$ and we use the continuity of k_λ to bound its integral. Then making $|x-y|$ small enough meets ϵ independent of f. This shows equicontinuity.

□

Proof. Finally we compute the point spectrum. Suppose $K_\lambda f = af$ for some $a \in \mathbb{C}$. Then

$$af = \int e^{\lambda(x-y)} f(y)dy = e^{\lambda x} \int e^{-\lambda y} f(y)dy$$

and $\sigma_p(K_\lambda) = \mathbb{C}$ since the solutions are not unique.

□