2/8 Update

February 8, 2021

Note: we use x as the variable for densities and y as the variable for characteristics.

1 Lower Bounds for Small p

I verified the density for $X_p^{(b)} = X^{(b)}/||X^{(b)}||_p$ where $X^{(b)} \sim (1 - b + bx^2)e^{-x^2/2}$ is convex. See:

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In[428]:= pth[p_, b_] = Integrate[x^p * L[x, b], {x, -Infinity, Infinity}]
```

Out[428]= $$\frac{2^{-\frac{1}{2}+\frac{1}{2}(-1+2b+p)}\left(1+(-1)^{2b+p}\right)\text{Gamma}\left[\frac{1}{2}+b+\frac{p}{2}\right]}{\sqrt{\pi}} \quad \text{if } \text{Re}[2b+p] > -1$$

```
In[429]:= Lnpb[x_, p_, b_] = L[x / pth[p, b], b] / pth[p, b]
```

Out[429]= $$\frac{2^{\frac{1}{2}(1-2b-p)}\, e^{-\frac{2^{1-2b-p}\pi x^2}{\left(1+(-1)^{2b+p}\right)^2 \text{Gamma}\left[\frac{1}{2}+b+\frac{p}{2}\right]^2}}\, \pi^b \left(\frac{2^{\frac{1}{2}+\frac{1}{2}(1-2b-p)}\, x}{\left(1+(-1)^{2b+p}\right)\text{Gamma}\left[\frac{1}{2}+b+\frac{p}{2}\right]}\right)^{2b}}{\left(1+(-1)^{2b+p}\right)\text{Gamma}\left[\frac{1}{2}+b+\frac{p}{2}\right]} \quad \text{if } \text{Re}[2b+p] > -1$$

```
(*Which is clearly convex*)
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Note the density is a gaussian times a monomial which is convex in y for arbitray b and thus the difference $f_b - f_0$ will have two zeroes.

2 Higher Order Atoms

Let

$$L_n(x) = x^{2n}e^{-x^2/2}$$

to verify this is a type L density we compute the FT and note all the zeroes are real and symmetric. So we compute

$$I_n = \int_{\mathbb{R}} x^n e^{-x^2/2} e^{-iyx} dx$$

We note via differentiation in y and DCT we have the reccurence relation $I_{n+1}(y) = iI_n'(y)$. So in particular $I_{2n+2} = -I_{2n}''$. This, in addition to the initial condition $I_0(y) = e^{-y^2/2}$, gives

$$I_{2n} = (-1)^n \frac{d}{dy} e^{-y^2/2} = (-1)^n e^{-y^2/2} H_{2n}$$

where H_{2n} denotes the 2nth hermite polynomial. It is well known these polynomials have all real zeroes(via Gauss-Lucas) and further H_{2n} is an even function(H_{2n+1} is odd). Thus we conclude L_n constitutes a valid type L density.

Remark. Perhaps there is something interesting to be done using the fact these characteristic functions are the hermite polynomials. Not sure.

Some Philosophy: I hoped to go on establishing khintchine type inequalities for these RVs and extend "atomic structure" we started with the $n = 1$ case. But the convexity technqiue we used to show the inequalities for $n = 1$ seeems to require more work. Consider a plot of the second derivative of the density:

```
In[419]:= (*Is this convex?*)
          Integrate[1 / Sqrt[a1] * L[x / Sqrt[a1], 2] * 1 / Sqrt[1 - a1] * L[(y - x) / Sqrt[1 - a1], 2], {x, -Infinity, Infinity}]

Out[419]= 1/Sqrt[2 π] e^(-y^2/2) (105 (-1 + a1)^2 a1^2 - 30 (-1 + a1) a1 (3 + 14 (-1 + a1) a1) y^2 + 3 (1 + 10 (-1 + a1) a1 (2 + 7 (-1 + a1) a1)) y^4 - 2 (-1 + a1) a1 (3 + 14 (-1 + a1) a1) y^6 + (-
            if Re[a1] ≥ Re[a1^2]

          (*Key point is that the resulting density has exponent independent of a1,
          a2(even in inhomogeneous case) since we sum to 1. Understanding why this true would be very useful.*)

In[423]:= p22[y_, a1_] = 105 (-1 + a1)^2 a1^2 - 30 (-1 + a1) a1 (3 + 14 (-1 + a1) a1) y^2 + 3 (1 + 10 (-1 + a1) a1 (2 + 7 (-1 + a1) a1)) y^4 - 2 (-1 + a1) a1 (3 + 14 (-1 + a1) a1) y^6 + (-1 + a1)^2 a1^2 y^8
          ddp22[y_, a1_] = D[D[p22[y, a1]], y], y]
```

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ListAnimate[Table[Plot[{ddp22[y, a1 / 10]}, {y, 0, 1}], {a1, 1, 10}]]
(*So I seems convexity just breaks for higher atoms.*)
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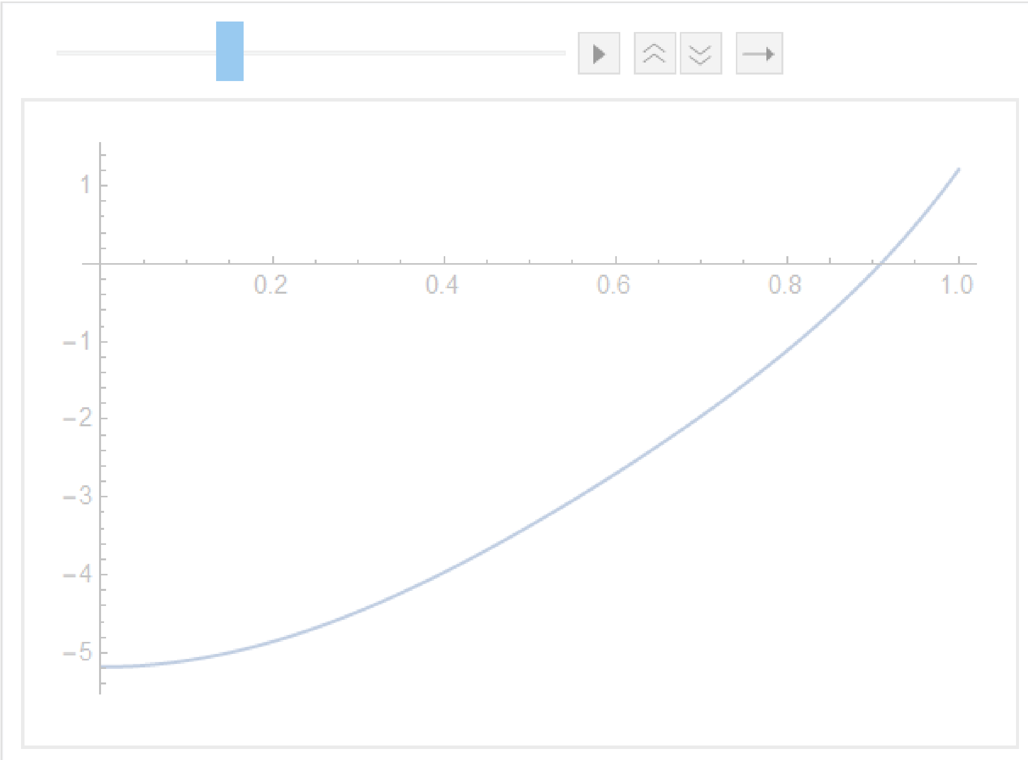
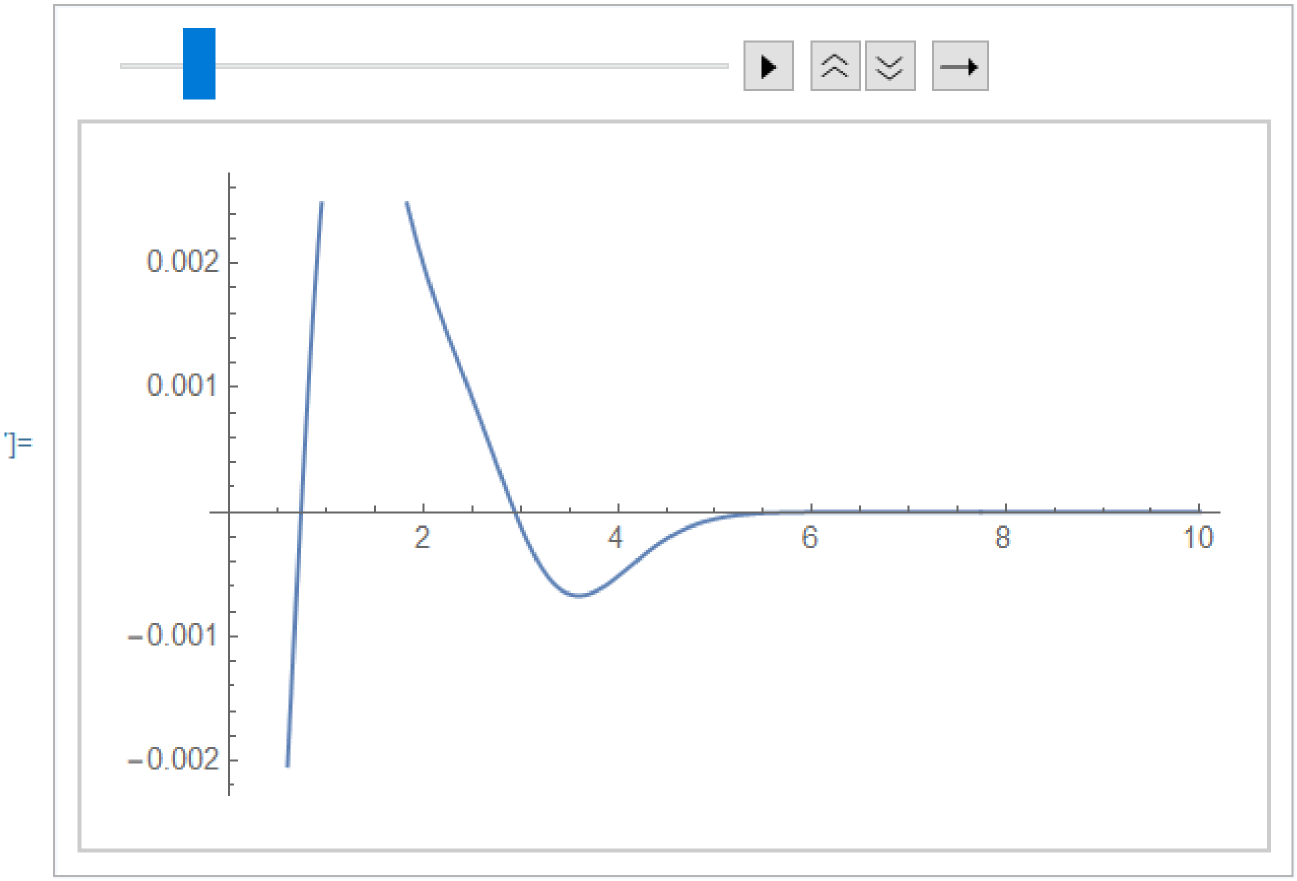
:427]=



Ie. the function is not convex. Yet there do seem to be 2 zeroes when we look at an ϵ difference:



So we could still hope to recover khintchine inequalities here using the technique. But some more work is required.

Question. A key point in all these computations is that the dependence on a_1, a_2 disappears. After putting some work into doing the computation

$$\int_{\mathbb{R}} x^{2n} e^{-x^2/2a_1^2} x^{2m} e^{(y-x)^2/2a_2^2} dx$$

I am still not sure why. Understanding would help greatly in this general case(mathematica fails here).