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Hw 4

Question 1:

Suppose (u_n) is a sequence in hilbert H weakly converging to $u \in H$ with $||u_n|| \to ||u||$. Then $u_n \to u$ in norm metric on H.

Proof. Compute

$$||v - v_n||^2 = (v - v_n, v - v_n) = (v, v) + (v_n, v_n) - (v, v_n) - (v_n, v)$$

via weak convergence and riesz representation we know $(v, v_n) \to (v, v) = ||v||^2$ and similarly for (v_n, v). Then given convergence in norm we also have $(v_n, v_n) \to (v, v)$ which shows $||v - v_n||^2$ can be made arbitrarily small, completing the proof.

□

Question 2:

Recall that $S \subseteq X$ banach is weakly bounded if for each $\lambda \in X^*$ is bouned on S. S is strongly bounded if $\sup_{x \in S} ||x|| < \infty$. Then S is strongly bounded $\iff$ it is weakly bouned.

Proof. This follows from the uniform boundedness principle. To see this let our banach space be $X^\star$ and our set of linear functionals T the evaluation mappings of $\lambda \in X^*$ on $x \in S$. Uniform boundedness principle tells us that $\sup_{x \in S} ||x||_{op} < \infty \iff$ for each $\lambda \in X^{ast}$, $\sup_{x \in S} |x(\lambda)| < \infty$. But $||x||_{op} = ||x||$. And $x(\lambda) = \lambda(x)$. So we are done.

□

Question 3:

Can complete proof by arguing with if a sequence converges on dense subset in X^*(extracted using projections and riesz representation) and it is bounded(which we know since subset is bounded) then we get convergence everywhere. Sequence extracted via pointwise argument where we simply keep adding more open sets and restricing those which are already there.

Suppose S is a bounded subset of l^2. If $x \in \overline{S}^w$ then there is a sequence in S weakly converging to x.

Proof. Case on $x \in \overline{S}^w$. Clearly if $x \in S$ then we are done so consider $x \in (S')^w$. We know projections of the nth coordinate are linear so we may look at intersections of pullbacks of open sets around $\pi_n(x)$. For the nth coordinate let $U_n^k = \{y : |\pi_n(y) - \pi_n(x)| < 1/k\}$. Since x is a limit point of S we know we can find an approximating sequence in every finite intersection of the U_n^k. Then we may intersect the the sets $\{U_n^k\}$, $n \in \mathbb{N}$, $k \in \mathbb{N}$ to form a nested sequence of neighborhoods hoods and extract a sequence of sequences converging pointwise to x. (We get pointwise convergence since the number of inputs to an element of l^2 is countable).

Note that pointwise convergence gives us convergence on a dense subse of $(l^2)^*$ namely rational scalings of the projections operators(we see this via Riesz representation). So then coupled with boundedness via our lemma we have convergence on every $\lambda \in (l^2)^*$ giving us the result.

□

Question 4:

Let $\{x_n\}$ be a sequence in banach X weakly converging to $x \in X$ and $\{l_n\}$ a sequence in X^* converging weak-$*$ to $l \in X^*$.

a)

First note it is not necessarily the case $l_n(x_n) \to l(x)$ as $n \to \infty$. Consider the counterexample with $X = l^2$, $x_n = e_n$, and $l_n = \pi_n$, the projection onto the nth coordinate. We have that $x_n \to 0$ weakly and $l_n \to \not{l}$ weak-$*$. But

$l_n(x_n) = 1$ for all n with $l(x) = 0(0) = 0$.

b)

If either (x_n) or (l_n) converges strongly then we have $l_n(x_n) \to l(x)$.

Proof. First suppose $l_n \to l$ strongly. Then compute

$$|l(x)-l_n(x_n)| \leq |(l-l_n)(x)|+|l_n(x-x_n)| \leq |(l-l_n)(x)|+|l(x-x_n)|+|(l-l_n)(x-x_n)|$$

The first term can be made arbitrarily small via an operator norm bound and sending $l_n \to l$. The second term is made small by recalling $l(x_n) \to l(x)$ as $n \to \infty$ via weak convergence. The last term follows similarly to the first by noting weak convergence implies a bound in norm on x_n(and hence $x-x_n$) and then sending $l_n \to l$ sufficiently in norm. This establishes $l_n(x_n) \to l(x)$.

Now suppose $x_n \to x$ strongly. Compute

$$|l_n(x_n) - l(x)| \leq |l_n(x_n) - l_n(x)| + |l(x_n - x)| + |l_n(x) - l(x_n)|$$

We know that the uniform boundedness principle tells us $||l_n||$ are uniformly bounded(since each evaluation of the collection on a point is ptwise convergent and hence bounded in norm). Then we may make the first term small by taking an operator norm bound and using strong convergence of $x_n \to x$(plus a uniform bound on l_n norm). The second term is made small via strong convergence of $x_n \to x$. The last term is made small as $|\cdot|$ is continous so as we make n large $l_n(x)$ approaches $l(x)$, $l(x_n)$ approaches $l(x)$, and the norm becomes small.

This finishes both cases.

□

Question 5:

Let X be an infinte dimensional banach space with weak topology. Then the weak closure of the unit sphere $S = \{x \in X : ||x|| = 1\}$ is the unit ball $B = \{x \in X : ||x|| \leq 1\}$.

Proof. First note that we may write

$$B = \bigcap_{||l||=1} \{x \in X : |l(x)| \leq 1\}$$

Clearly we have left inclusion. Right inclusion comes from noting for each $x \in X$ there is an l_x s.t. $l_x(x) = ||x||$. Further we have that each set in the intersection is weakly closed since these are pullbacks of closed complex sets and each $l \in X^*$ is continuous. Thus B being the arbitrary intersection of weakly closed sets is weakly closed. This shows $\overline{S} \subseteq B$. We now show the other inclusion.

Let $x \in B$. Let U be a neighborhood of x. Without loss of generality we may assume it is basic, ie. $U = \bigcap_{i \in F} \{y : |l_i(y) - l_i(x)| < \epsilon_i\}$ for finite index set F.

Pick $z \in X$ s.t. for all $i \in F$, $l_i(z) = 0$. Note this is possible since the dimension of X is infinite.

Then via the continuity of $|| \cdot ||$ we may pick $c \in \mathbb{C}$ so that $||x + cy|| = 1$. Then we have $x + cy \in U$ since for each $l_i, l_i(x + cy) = l_i(x)$. Further $x + cy \in S$. Since U was arbitrary and x was arbitrary, this concludes the proof.

$\square$

Question 6:

Recall that a weakly convergent sequence is bounded in norm. Call this bound M. Then this is exactly what we need to show uniform convergence as we argue in the case that a sequence is cauchy(we have uniform pointwise bounds and equicontinuity estimates). This allows us to apply AA extracting uniformly conv. subsequence. (Note that uniform norms and H1 norms bound each other). Then if a sequence doesn't converge, there must be a suqsequence not getting close enough. Show that this subsquence in turn has a uniformly converging subsequence, a contradiction. Then convergence in norm shows convergence uniformly.(Actually can just do this in uniform norm topology).

Let $(f_k) \subseteq H^1_{per}$ converge weakly to some $f \in H^1_{per}$. Then f_k converge uniformly.

Proof. Recall that since the f_k converge weakly they are bounded in norm(H_1 norm). Let M be such a bound. Then using sobolev inequality variants we may establish equicontinuity and uniform boundedness. Equicontinuity is gained since $|f_k(x) - f_k(y)| \leq |x-y|^{1/2}||\frac{df_k}{dx}||_{L^2_{per}} \leq |x-y|^{1/2}M$ independent of k. A uniform pointwise bound(independent of k and x) is given via $|f_k(x)| \leq \sqrt{2\pi}||u||_{H^1} \leq \sqrt{2\pi}M$. Note that these conditions also hold for any subsequence.

Suppose f_k does not converge uniformly. Then there exists a sequence of f_{k_n} s.t. $|\sup_{x\in X}|f_{k_n}(x) - f(x)| > \epsilon$ for some ϵ. But we may then extract a subequence using Arzela-Ascoli converging uniformly to f from f_{n_k}, a contradiction. Hence $f_k \to f$ uniformly.

□