1 Lambert Calculation

Note: These are distributions of characteristics

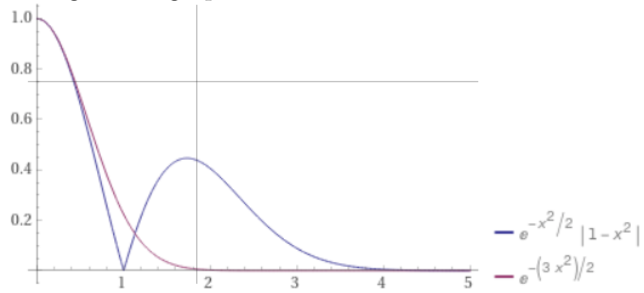
Note: Variances of these distributions should be the same.

$f(x) = e^{-3x^2/2}$(characteristic of gaussian with variance 3)

Set $g(x) = e^{-x^2/2}|1-x^2|$.(norm of characteristic of density $e^{-x^2/2}x^2$)

We seek to compute $\mu\{x > 0 : e^{-x^2/2}|1-x^2| < t\}$ for $0 < t < 1$ where $d\mu = \frac{1}{x^{p+1}}dx$ for $p > 2$

Looking at the graph:



Let F be the modified distribution of $e^{-x^2/2}$ and G be the modified distribution of the other. Set y_{lm} to be the local max of f away from 0. Then clearly for $t \geq y_{lm}$, $G(t) \geq F(t)$ since f dominates over x s.t. $g(x) > t_{lm}$. So it suffices to comopute the measure for t in the interval $(0, y_{lm})$.

We see this is the measure of two intervals: one containing 1 and one from some (x, ∞). Call these $(x_{+,0)}, x_{-,0})$ and $(x_{-,-1}, \infty)$. We must compute these in terms of the lambert W function.

For $t \in (0, y_{lm})$ write for $x > 0$

$$e^{-x^2/2}|1-x^2| = t \iff |y|e^y = \frac{e^{1/2}}{2}t$$

where $y = \frac{1-x^2}{2}$. We case if $y > 0$.

If $y \geq 0$ then $0 < x < 1$. In this case we have $ye^y = \frac{e^{1/2}}{2}t$ which we can solve with the lambert W_0 function yielding $y = W_0(\frac{e^{1/2}}{2}t)$.

In the case $y < 0$ then $x > 1$ and we have two possible solutions to $|y|e^y = \frac{e^{1/2}}{2}t$. Write $-ye^y = \frac{e^{1/2}}{2}t \implies y \iff ye^y = -\frac{e^{1/2}}{2}t$ which is solved by $y = W_0(-\frac{e^{1/2}}{2}t)$ and $y = W_{-1}(-\frac{e^{1/2}}{2}t)$.

Label

$$x_{+,0} = \sqrt{1 - 2W_0(\frac{e^{1/2}}{2}t)}$$
$$x_{-,0} = \sqrt{1 - 2W_0(-\frac{e^{1/2}}{2}t)}$$
$$x_{-,-1} = \sqrt{1 - 2W_{-1}(\frac{e^{1/2}}{2}t)}$$

Note: $W_{-1}(\frac{e^{1/2}}{2}t) \leq W_0(\frac{e^{1/2}}{2}t)$ so $x_{-,0} \leq x_{-,-1}$

Then we can compute

$$\begin{aligned}\mu\{x>0: e^{-x^2/2}|1-x^2|<t\} &= \int_{(x_{+,0},x_{-,0})}\frac{1}{x^{p+1}}dx + \int_{(x_{-,-1},\infty)}\frac{1}{x^{p+1}}dx = \\ &-\frac{1}{p}x^{-p}|_{x_{-,0},x_{+,0}} + (-\frac{1}{p}x^{-p})|_{\infty,x_{-,-1}} = \\ &-\frac{1}{p}[(1-2W_0(-\frac{e^{1/2}}{2}t))^{-p/2}-(1-2W_0(\frac{e^{1/2}}{2}t))^{-p/2}-(1-2W_{-1}(-\frac{e^{1/2}}{2}t))^{-p/2}]\end{aligned}$$