Alex Havrilla

Hw 5

Question 1:

Let $X = l^1$ and consider $T(x)$ defined on its action on each componenent via $T(x)_i = x_i/2^i$. We claim the image is dense but not closed.

Note that this clearly defines a linear operation $T : l^1 \to l^1$. It is also bounded(as it in fact decreases the norm of its input). Its image is not closed as the l^1 sequence $a = (1, 1/2, 1/4, ...)$ is not in the range but the sequence $[1]^n$ ie. the sequence of n ones and then 0 is s.t. $T([1]^n) \to a$ in l^1. Further the image is dense as for any $a \in l^1$, we can find $b \in im(T)$ ϵ close by finding index N s.t. $\sum_{i=N}^{\infty} |a_i| < \epsilon$ then considering the $b = (a_1, 2a_2, 4a_3, ...)$ where then $T(b)$ is ϵ close to a.

Thus T is a linear bounded operator with dense but unclosed image.

Question 2:

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a)

Suppose $\{A_\alpha\}_{\alpha\in I}$, $\{B_\alpha\}_{\alpha\in I}$ as in the problem statement. Then we show $A_\alpha B_\alpha \to AB$

Proof. To show weak convergence we need to show convergence under every $l \in H^*$ or equivalently $\forall x \in H, \forall z \in H$

$$(z, A_\alpha B_\alpha x) \to (z, ABx)$$

Compute

$$||(z, A_\alpha B_\alpha x - ABx)|| \leq ||(z, A_\alpha B_\alpha x - A_\alpha Bx) + (z, A_\alpha Bx - AB_\alpha x) + (z, AB_\alpha x - ABx)|| \leq$$

$$||(z, A_\alpha B_\alpha x - A_\alpha Bx)|| + ||(z, A_\alpha Bx - AB_\alpha x)|| + ||(z, AB_\alpha x - ABx)||$$

We now argue each term can be made arbitrarily small.

The first term

$$||(z, A_\alpha B_\alpha x - A_\alpha Bx)|| = ||(A_\alpha z, B_\alpha x - Bx)|| \leq M||z||||B_\alpha x - Bx||$$

where we use the fact that strong convergence gives a bound in norm. Then we make this term small by letting α large enough as $B_\alpha \to B$ strongly.

The second term

$$||(z, A_\alpha Bx - AB_\alpha x)|| = ||(A^*_\alpha z, Bx) - (z, AB_\alpha x)|| \to ||(z, ABx - ABx) = 0$$

since via continuity of the norm inner product we can send a the limit inside and recall $AB_\alpha x \to ABx$ via strong convegence and $A^*_\alpha z \to A^* z$.

Finally the last term

$$||(z, AB_\alpha x - ABx)||$$

is made small via continuity of the inner product and strong convergence as $AB_\alpha x \to ABx$.

$\square$

b)

Consider the hilbert space L^2_{per} and define the bounded linear operators $T_n, S_n : H \to H$

$$T_n f(x) = f(x)e^{-inx}$$
$$S_n f(x) = f(x)e^{inx}$$

We proved $T_n \to 0$ weakly(and a simiarl argument shows $S_n \to 0$ weakly). Yet for arbitrary $f \in L^2_{per}$,

$$f = S_n T_n f \implies S_n T_n = I \forall n$$

hence the product does not converge weakly to $ST = 0 * 0 = 0$.

Question 3:

Let X be a banach space and $P \in \mathcal{L}(X)$ be a projection. Assume P has finite rank.

Claim: $rank(P) = rank(P')$

Proof. Let n be the dimension of im(P) and $v_1, ..., v_n$ a basis for $im(P)$. Define $\lambda_i \in X^*$ via $\lambda_i(v_i) = 1$ and 0 for vectors not a rescaling of v_i. We claim $\lambda_1, ..., \lambda_n$ are a basis for $im(P')$.

Let $\lambda \in im(P')$. We know that $\lambda = P'(\mu) = \mu(P(\cdot))$ for $\mu \in X^*$. Thus we know it suffices to consider the action of μ on elements of $im(P)$. Then on $im(P)$ $\mu = \sum_{i=1}^{n} \mu(b_i)\lambda_i$ as for $x = \sum_{i=1}^{n} c_i b_i \in im(P)$,

$$\mu(x) = \sum_{i=1}^{n} c_i \mu(b_i)$$

$$\sum_{i=1}^{n} \mu(b_i)\lambda_i(x) = \sum_{i=1}^{n} c_i \mu(b_i)$$

Thus we may span all continuous linear functions over $im(P)$ with n basis linear functionals. Since we know the $im(P')$ is precisely these linear functionals(extended to be 0 on the kernel of P and the value of Px for $x \notin Im(P)$, where we extend our basis functions in the same way).

$\square$

Question 4:

Claim: If $P, Q \in \mathcal{L}(X)$ are both finite rank projection with $||P - Q|| < 1$ then $rank(P) = rank(Q)$.

Proof. We prove the contrapositive. Suppose $m = rank(P) \neq rank(Q) = n$. Wlog suppose $m > n$. Suppose we can find $v \in im(P)$ s.t. $Q(v)$. Suppose it has unit norm. Then

$$||(P-Q)v|| = ||Pv|| = ||v|| = 1$$

Thus $||P-Q|| \geq 1$. It remains to justify we can find such a v.

Assume for sake of contradiction $ker(Q) \cap im(P) = \emptyset$. Then we know $Q : im(P) \to im(Q)$ is injective. Yet this is a contradiction since we cannot have a linear injection from a higher dimensional space to a lower dimensional one.

□

Question 5:

Proof. Note that for a bounded operator $T \in X^*$ we know T is bounded from below $\iff$ it is injective and has a closed graph. We see this since if T is bounded from below then it must be injective and its graph must be closed(via continuity).

Further if T closed and injective then $T : X \to im(X)$ is a continouous bijection from a banach space to banach space and then by inverse mapping theorem the inverse is also bdd. We may thus compute

$$||x|| = ||T^{-1}Tx| \leq ||T^{-1}||||Tx|| \implies ||Tx|| \geq \frac{||x||}{||T^{-1}||}$$

as desired. This characterizes the set of $(\lambda I - A)$ we are studying. Therefore we know there some m_λ s.t.

$$||\lambda x - Ax|| \geq m_\lambda ||x||$$

then via continuity of the norm we may adjust λ s.t. for any $z \in B(\lambda, r)$,

$$||zx - Ax|| \geq m_\lambda/2||x||$$

This ensures we are still in the desired class of functions and shows the set open as we put a ball around λ.

□