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Hw 6

Task 1:

Claim: The algorithm described is a 3 approximation.

Proof. Consider n boxes with widths w_i and heights h_i. Enumerate the boxes via decreasing height. Suppose our algorithm produces k levels. Let the indices $i_1, ..., i_k$ denote the indices of the leftmost boxes on each level(for example $i_1 = 1$ and in general i_j is the index of the first box on the jth level).

Set $A = \sum_{i=1}^{n} h_i w_i$ ie. the total area of the boxes. Note that $A \leq OPT$ since the best height we can achieve the the area divided by the width of the container(this way we "waste no space"). But since the container width is exactly 1 this is A.

Now break the area into a sum over levels and lower bound

$$A = \sum_{i=1}^{n} h_i w_i = \sum_{i=i_1}^{i_2-1} h_i w_i + ... + \sum_{i=i_k}^{n} h_i w_i \geq h_{i_2} \sum_{i=i_1}^{i_2-1} w_i + ... + h_{i_k} \sum_{i=i_{k-1}}^{i_k} w_i$$

where we lower bound the height of each block on the jth level by the height of the block i_{j+1}(note then we lower bound the last sum by 0). Adding $h_1 + h_{i_2}(1 - \sum_{i=i_1}^{i_2-1} w_i) + ... + h_{i_k}(1 - \sum_{i=i_{k-1}}^{i_k} w_i)$ yields the inequality

$$h_1 + h_2 + ... + h_k \leq A + h_1 + h_{i_2}(1 - \sum_{i=i_1}^{i_2-1} w_i) + ... + h_{i_k}(1 - \sum_{i=i_{k-1}}^{i_k} w_i)$$

But $SOL = h_1 + h_2 + ... + h_k$. And clearly $h_1 \leq OPT$. So if we can show $h_{i_2}(1 - \sum_{i=i_1}^{i_2-1} w_i) + ... + h_{i_k}(1 - \sum_{i=i_{k-1}}^{i_k} w_i) \leq OPT$ then we have $SOL \leq 3OPT$ ie. the 3 approximation.

Note that $1 - \sum_{i=i_j}^{i_{j+1}-1} w_i \leq w_{i_{j+1}}$ since if this were not the case then we could move the block i_{j+1} onto the jth level(there would be enough width). This gives the bound

$$h_{i_2}(1-\sum_{i=i_1}^{i_2-1} w_i)+...+h_{i_k}(1-\sum_{i=i_{k-1}}^{i_k} w_i) \leq h_{i_2}w_{i_2}+...+h_{i_k}w_{i_k} \leq A \leq OPT$$

as desired where we know $h_{i_2}w_{i_2}+...+h_{i_k}w_{i_k} \leq A$ since these are terms in the area

□

Task 2:

Claim: Ham-Cycle is NP-Complete.

Proof. Note that clearly Ham-Cycle is NP(we can check if a cycle is hamiltonian in poly time). So to show NP completeness it suffices to show a reduction from Dir-Ham-Cycle. Consider the following.

Let graph $G = (V, E)$ with n vertices. Replace each vertex $v \in V$ with the triple of vertices v_{in}, v_{dum}, v_{out} where all the in-edges of v are now undirected edges incident to v_{in} and all the outedges are now incident to v_{out} with v only adjacent to v_{in} and v_{out}. Ie. we have a vertex for indedges and a vertex for outedges in between which there is a dummy vertex.

Note that this conversion runs in polynomial time as the number of edges increases by $2n$ and the number of vertices by $2n$.

Then to check if there is a hamiltonian cycle in G we ask if there is a hamiltonian cycle in G'. We now argue correctness:

Suppose we have a hamiltonian cycle in the undirected graph G. Call this $v_1, ..., v_n$. Then to produce a hamiltonian path in G' consider $v_{in}^1, v_{dum}^1, v_{out}^1 v_{in}^2, v_{dum}^2, v_{out}^2 ... v_{in}^n, v_{dum}^n, v_{out}^n$ Observe that this constructs a hamiltonian cycle. In particular we know v_{out}^i is adjacent to v_{in}^{i+1} since we have the forward edge in the hamiltonian cycle $v^i v^{i+1}$. Also note this is a cycle.

Now suppose we have a hamiltonian cycle in G'. Note that this must consist of blocks of $v_{in}^i v_{dum}^i v_{out}^i$ since the only vertices v_{dum} is adjacent to are v_{in} and v_{out}. So if we have the path $v_{in}^1 v_{dum}^1 v_{out}^1 ... v_{in}^n v_{dum}^n v_{out}^n$ then consider

$v_1 ... v_n$ as hamiltonian cycle on G. Because we have the adjacency $v^i_{out} v^{i+1}_{in}$ we know there exist forward edges from v_i to v_{i+1}. Further we know this is a cycle as $v^1_{in} v^1_{dum} v^1_{out} ... v^n_{in} v^n_{dum} v^n_{out}$ a cycle.

This shows both directions establishing a valid redution, and hence showing Ham-Cycle NP Complete.

□

Task 3:

Claim: The algorithm described produces a 6 addtive spanner with high probability.

Proof. a) First we argue there are $O(n^{4/3} log^3(n))$.

Step 1 includes $O(n^{4/3}) = O(n * n^{1/3})$ edges. We get this bound since there are at most n light vertices(vertices with $n^{1/3}$ degree or less). And each of them has at most $n^{1/3}$ associated edges.

Step 2 includes $O(n)$ edges since we include at most one edge for each $v \in V$.

Step 4 includes $O(n^{4/3} log^3(n))$ edges for each pair $u \in A$, $v \in B^i$, we include at most 2^i edges. Then since the number the number of pairs between A and B^i is $O(n^{4/3} 2^{-i} log(n)^2)$, this is a total of $O(n^{4/3} log(n))^2$ per B^i. We get $O(n^{4/3} log^3(n))$ from $log(n)$ B^i.

Now we argue for an arbitrary pair (u, v), if the shortest path P in G has k heavy edges, then the neighboring set $S(P)$ of the path has cardinality at $\Omega(kn^{1/3})$. Note that if there are k heavy edges then there are at least $k + 1$ heavy vertices(vertices with degree at least $n^{1/3}$). Note that since these are in a path we can find a set of at least $k/2$ no of which are non adjacent. This gives us at least $\frac{1}{2} kn^{1/3}$ distinct vertices in $S(P)$ since if any of these nonadjacent vertices share a neighbor, this shrinks the shortest path.

Now fix arbitrary $a, b \in G$. Fix P to be an arbitrary shortest path. Suppose the number of heavy edges in P is in the range $[2^{i-1}, 2^i)$. Then the probability $B^i \cap S(P) = \emptyset$ is $1/n^4$. We compute this via

$$P(B^i \cap S(P) = \emptyset) \leq (1 - \frac{1/2 2^{i-1} n^{1/3}}{n})^{c 2^{-i} n^{2/3} log(n)} \leq e^{\frac{-1/2 2^{i-1} n^{1/3} c 2^{-i} n^{2/3} log(n)}{n}} \leq$$

$$e^{-1/4 c log(n)} = n^{-1/4c} = \frac{1}{n^4}$$

where we choose $c = 16$.

Finally we argue the construction is 6 additive. Let u, v be arbitrary vertices in V. Let P be an arbitrary shortest path in G between them. Suppose this shortest path has k heavy edges for $k \in [2^{i-1}, 2^i)$ for some $i \in log(n)$(note this range is sufficient as we cannot have more than n-1 edges). Now consider the same path in G'. If there are no heavy edges missing then we are done. Otherwise let $h_1 = (a, \cdot)$ be the first missing heavy edge and $h_2 = (\cdot, b)$ be the last missing heavy edge(where the order is determined by a traversal from u to v).

Suppose $\exists$ a neighbor to P w_i s.t. $w_i \in B_i$. Further suppose there exist edges between a and b and nodes a_1, a_2 in A. Then we have paths P_1 and P_2 from a_1 and a_2 to w_i, respectively. Say w_i is adjacent to c in P. We have the bounds

$$|P_1| \leq dist(a, c) + 2, |P_2| \leq dist(c, b) + 2$$

since we have a path from w_i to a_1, a_2 going through the path P and node a, b which has at most $O(2^i)$ heavy edges(and we know we fill in the shortest such path). This constructs a new path from u to v via

$P' = u - a - a_1 - w_1 - a_2 - b - v$. We bound

$$|P'| = |u-a| + |a-a_1| + |P_1| + |P_2| + |a_2 - b| + |b - v| \leq |u-a| + |a-c| + |c-b| + |b-v| + 4 + 2 = |P| + 6$$

as desired. (Note that u-a denotes the path from u to a on P and $a_1 - w_1$ denotes P_1). This shows 6 additivity under the assumptions.

Now we show the assumptions occur with high probabilty. The probability there does not exist a neighbor w_i in B_i is less than n^4. The probability wlog a is not adjacent to A is bounded by

$$(1 - \frac{n^{2/3} log(n)}{n})^{n^{1/3}} \leq e^{-log(n)} = 1/n$$

Thus via a union bound the probability that one of these things does not occur is bouned by $\frac{1}{n^4} + \frac{1}{n} + \frac{1}{n} \implies$ the probability of 6 additivity is $1 - \frac{1}{O(n)}$ as desired.

$\square$