Type L Easy Regime

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We seek to show $(1,1,0,0,...)$ a minimizer and $(\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}},...)$ a maximizer for the expression

$$\mathbb{E}|\sqrt{a_1}X_1+\sqrt{a_2}X_2+W|^p$$

where $X_i \sim \frac{1}{\sqrt{2\pi}}x^2e^{-x^2/2} = L(x)$

Suppose $a_1 > a_2$. Set $X_\epsilon = \sqrt{a_1-\epsilon}X_1+\sqrt{a_2+\epsilon}X_2$. Wlog we may suppose $a_1+a_2=1$. Then we show

$$\begin{aligned}&\mathbb{E}_W\mathbb{E}_X[|X_\epsilon+W|^p-|X_0+W|^p] = \\ &\mathbb{E}_W\int_0^\infty[|\sqrt{x}+W|^p+|-\sqrt{x}+W|^p+\alpha x+\beta](f_\epsilon(\sqrt{x})-f_0(\sqrt{x}))\frac{1}{2\sqrt{x}}dx \geq 0\end{aligned}$$

where α,β depend on the densities f_ϵ. By known arguments it suffices to show $f_\epsilon(\sqrt{x})-f_0(\sqrt{x})$ has exactly two zeroes(note it must have at least two since X_ϵ have fixed variances and both f are probability distributions).

Fix $a_1 > a_2 \in \mathbb{R}_+$. Compute for arbitrary a_1, a_2 the density

$$\begin{aligned}f_0(y) = \int_{-\infty}^\infty L(x/\sqrt{a_1})L((y-x)/\sqrt{a_2})dx &= \frac{e^{-y^2/2}}{\sqrt{2\pi}}\frac{3a_1a_2+(a_1^2-4a_1a_3+a_2^2)y^2+a_1a_2y^4}{\sqrt{1/a_1a_2}} \\ &= \frac{e^{-y^2/2}}{\sqrt{2\pi}}((a_1a_2)^{3/2}(y^4-2y^2+3)+\sqrt{a_1a_2}(a_1-a_2)^2y^2)\end{aligned}$$

Then considering $g(y) = f_\epsilon(\sqrt{y})-f_0(\sqrt{y})$ we can only be 0 when the polynomial expression $(b_1b_2)^{3/2}(y^4-2y^2+3)+\sqrt{b_1b_2}(b_1-b_2)^2y^2-(a_1a_2)^{3/2}(y^4-2y^2+3)+\sqrt{a_1a_2}(a_1-a_2)^2y^2$ is 0 where $b_1 = a_1-\epsilon$, and similarly for b_2. Then it suffices to show this expression convex. But trivially by taking two derivatives in y we see

$$g''(y) = 2(b_1 b_2)^{3/2} - 2(a_1 a_2)^{3/2} \geq 0$$

since $b_1 b_2 = (a_1 - \epsilon)(a_2 + \epsilon) \geq a_1 a_2$ when $a_1 > a_2$.

More generally for $b \in [0, 1]$ we have if $L_b(x) = (1 - b + bx^2)\frac{e^{-x^2/2}}{\sqrt{2\pi}}$ then

$$f_{a,b,0}(x) = e^{y^2/2}(1 - b + 3a_1 a_2 b^2 + b(1 - 6a_1 a_2 b)y^2 + a_1 a_2 b^2 y^4)$$

(via mathematica)

so exactly the same analysis carries over since we have the highest order term $a_1 a_2 b^2 y^4$.