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1) We have X VS over $\mathbb{F}$ Banach w/ $\|\cdot\|_1$, $\|\cdot\|_2$ w/ $M_2 > 0$ s.t.

$\|v\|_2 \leq M_1 \|v\|_1 \quad \forall v \in X$.

$\|\cdot\|_2$ is bdd. by $\|\cdot\|_1$.

Let $J : X_{\|\cdot\|_1} \to X_{\|\cdot\|_2}$ via the natural injection.

Clearly b/c X is a VS, J is linear. Further by the inequality it is bdd. Further it is bijective. So by the inverse mapping thm we know the inverse injection J^{-1} is linear and bdd. Hence $\forall x \in X_2$, $\|J^{-1}x\| = \|x\|_1 \leq \underbrace{\|J^{-1}\|}_{M_2} \|x\|_2$ ✓

As desired.

a) Note: WTS $\langle Tv_k, Tv_k\rangle \to \langle Tv, Tv\rangle \iff$
$\langle v_k, T^*T v_k\rangle \to \langle v, T^*Tv\rangle$.

Compute $\langle v_k, T^*Tv_k\rangle = \langle v_k, T^*T(v_k - v + v)\rangle$
$= \langle v_k, T^*T(v_k - v)\rangle + \langle v_k, T^*Tv\rangle$

Note $\langle v_k, T^*Tv\rangle \to \langle v, T^*Tv\rangle$ since $v_k \to v$ weakly.

Further $\langle v_k, T^*T(v_k - v)\rangle \to 0$ since $T^*Tv_k \to T^*Tv$ in norm and $\{\|v_k\|\}$ bdd. ✓

This proves the claim

b) Note that if we can show $\|Tv_k - Tv\| \to 0$ then we know T compact since our H is reflexive and we proved a homework this condition implies T compact.

So we show $\|Tv_k - Tv\| \to 0$.

But this is clear as T bounded and the v_k converge weakly to v and also converge w/ norm to v.

(we also proved this a hws.

3a) ~~[illegible]~~ We will use Arzela-Ascoli to show compactness. Thus we must show $\{T_\varepsilon f_n\}$ uniformly bdd. and equicontinuous for some bounded set $\{f_n\} \subseteq L^2_{per}$. Once this is done we extract a unif. conv. subsequence which is conv. in L^2_{per}. ✓ Set M the bound on $\|f_n\|$.

The uniform bound comes from Young's inequality.

Compute $\|T_\varepsilon f\|_{L^2} = \sqrt{\int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} K_\varepsilon(x-y) f(y) dy \right|^2 dx}$

Min. Integral inequality (Proved via Tonelli and a change of variables) $\rightarrow$ $\leq \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |K_\varepsilon(x-y)| dy \sqrt{\int |f(y)|^2 dy}\, dx$

$\leq \|f\|_{L^2} \leq M.$

We show Equicontinuity similarly: Let $\varepsilon > 0$

$|T_\varepsilon f_n(x) - T_\varepsilon f_n(z)| = \left| \int_{-\pi}^{\pi} K(x-y) f(y) dy - \int_{-\pi}^{\pi} K(z-y) f(y) dy \right| \leq$

$\leq \int_{-\pi}^{\pi} |K(x-y) - K(z-y)| \, |f(y)| dy = \int_{-\pi}^{\pi} |K(y)| \, |f(x-y) - f(z-y)| dy$

$\leq |x-z| M$ Since Lipschitz ~~[illegible]~~ c.ts.

So we have uniform bound, equicontinuity, and so apply AA.

b) $(e_k, T_\varepsilon f) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{+ikx} K(x-y) f(y)\, dy\, dx$

$= \int_{-\pi}^{\pi} f(y)\, e^{+iky} \int_{-\pi}^{\pi} K(x-y)\, e^{ik(x-y)}\, dx\, dy$

$= c_k^\varepsilon f_k$

c) Compute $(\lambda I - T_\varepsilon) f = 0$. (This suffices since T_ε compact).

$\lambda f(x) = \int_{-\pi}^{\pi} K(x-y) f(y)\, dy \Rightarrow 0 = \int_{-\pi}^{\pi} K(x-y) [f(y) - \lambda f(x)]\, dx$

since $\int_{-\pi}^{\pi} K = 1$

or otherwise $\lambda f = K_\varepsilon * f \Rightarrow \lambda \hat{f} = \hat{K}_\varepsilon \hat{f}$

$\forall k$ $\Rightarrow \lambda f_k = f_k c_k^\varepsilon$

So if $f_k \neq 0$, we know $\lambda = C_k^\varepsilon$, $\forall k$ and $f_k \neq 0$ (Note if $f_k = 0\ \forall k$ then $f = 0$).

So ~~the~~ spectrum λ given by the C_k^ε.

3d) Claim: $\forall f \in L^2_{per.}$ $\|T_\varepsilon f - f\|_{L^2} \to 0$

~~We show $T_\varepsilon f \to f$ ... with ... the ...~~

~~As ... ...~~

Compute

$$\|T_\varepsilon f - f\|_{L^2} = \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} K_\varepsilon(x-y)\left[f(y) - f(x)\right] dy \right|^2 dx$$

$$\leq \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} K_\varepsilon(y) \, |f(x-y) - f(x)| \, dy \right)^2 dx$$

$$= \int_{-\pi}^{\pi} \left(\int_{-\varepsilon}^{\varepsilon} K_\varepsilon(y) \, |f(x-y) - f(x)| \, dy + \int_{B(0,\varepsilon)^c} K_\varepsilon(y) \, |f(x-y) - f(x)| \, dy \right)^2$$

Note that we can make the second integral arbitrarily small by making $\varepsilon \to 0$ and noting $M_q \to 0$.

So it suffices to make $\int_{-\varepsilon}^{\varepsilon} K_\varepsilon(x-y)\left[f(y) - f(x)\right] dy$ small.

But trig polys are cts so we can make this term small via the continuity of translation $\int [f(x) - f(y-x)] \approx 0$ for small perturbation.

4. X, Y Banach spaces w/ $T \in \mathcal{B}(X, Y)$ a bdd. lin.

WTS $Ker(T')$ is seq. closed weak*.

Let $\ell \in Y^*$ be a weak* limit point of $Ker(T')$.

$\exists$ then $\{\ell_n\} \subseteq X^*$ w/ $\ell_n(y) \to \ell(y)$ $\forall\, y \in Y$ (i.e. conv. pointwise).

We know $\forall n$, $T'\ell_n = 0$. We must show $T'\ell = 0$.

Fix $x \in X$. $T'\ell(x) = \ell(T(x)) = \lim_{n \to \infty} \ell_n(T(x))$

$= \lim_{n \to \infty} T' \ell_n(x) = 0$

by continuity of T' — since $T'\ell_n = 0$

This shows $T'\ell = 0$ since x arb. Hence $\ell \in Ker(T')$ and it is $\therefore$ closed.

(5.)

a) we aim to show the graph of the derivative operator is closed. This amounts to showing if $\{f_k\} \subseteq C'$ w/ $f_k \to f$ then also $\frac{df_k}{dx} \to \frac{df}{dx}$.

I.e. the uniform limit of continuously differentiable functions is differentiable.

But this is clear thanks to uniform convergence. We know for $x \in X$,

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \lim_{n\to\infty} \frac{f_n(x+h) - f_n(x)}{h}$$

$$\underset{\text{b/c uniform conv.}}{=} \lim_{n\to\infty} \lim_{h\to 0} \frac{f_n(x+h) - f_n(x)}{h} = \lim_{n\to\infty} f_n'(x).$$

So we define f' pointwise this way. Further we know $f_n' \to f'$ uniformly, and the uniform limit of continuous functions is cts, so $f' \in X$. This shows closedness.

(since $f_n \to f$ uniformly).

b) We deduce a function which is a limit of C^2 functions but ~~which is not differentiable~~ or whose derivative is not ~~continuous~~ C^1. (And hence D not closed).

Consider arbitrary $f \in C^1$ but not in C^2. Then $Af \in C^0$ ~~which~~ but $Af \notin C^1$.

Since C^2 dense (Really C^∞ dense), we can find an approximating sequence $\{f_k\} \subseteq C^2$ to f.

Via an argument similar to 2S we know $Df_k \to \frac{df}{dx}$ pointwise. ~~But~~ Hence f is a limit point of C^2 $\Gamma(D)$.

(and uniformly).

But by construction $f \notin C^2$, so it cannot be $(f, df) \in \Gamma(D)$

$\therefore$ D not closed.