$1 \leq p \leq 2$ For Type L with enough Gausannity

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1 Results

Let $X^{(b)}$ be a random variable with density given by $\frac{1}{2\pi}(1-b+bx^2)e^{-x^2/2}$. Set $X_p^{(b)} = X^{(b)}/||X^{(p)}||_p$. We rewrite with where S is a sum of iid $X_p^{(b_i)}$

$$\mathbb{E}|X_p^{(b)} + S|^p = \frac{\mathbb{E}||X^{(b)}|_p + S|^p + \mathbb{E}||X_p^{(b)}| - S|^p}{2}$$

Then using technique of interlacing densities and the convexity of

$$h(x) = |x^{1/p} + 1|^p + |x^{1/p} - 1|^p$$

via Lemma 12 from

`https://www.math.cmu.edu/~ttkocz/mypapers/mathematics/khintchBpn.pdf`

we show for $b > 0$

$$\mathbb{E}h(X_p^{(b)}) \geq \mathbb{E}h(X_p^{(0)})$$

which gives us a khintchine type lower bound since $X^{(0)}$ is gaussian. So it suffices to show the difference of densities $f_p^{(b)}(x) - f_p^{(0)}(x)$ has at most two zeroes. Note we already know there are at least 2 via both being probability distributions and agreeing on pth moments. We compute

$$f_p^{(b)}(x) = \frac{1}{\sqrt{2}} e^{-1/2(\pi^{-1/2/p}\sqrt{2}} x((1+bp)\Gamma(1+p/2))^{1/p})^{2/p} \pi^{-1/2-1/2p} (2^{p/2}(1+bp)\Gamma(1+p/2))^{1/p} \quad program@epst$$
$$(1 - b + b(\pi^{-1/2/p}\sqrt{2}x(1+bp)\Gamma(1+p/2)^{1/p})^{2/p})$$

ie. setting $||X^{(b)}||_p = C_p^{(b)} = (\frac{\sqrt{2^p}}{\sqrt{\pi}}(1+bp)\Gamma(1+p/2))^{1/p}$ which is clearly increasing in b we have

$$f_p^{(b)}(x) = C_p^{(b)}(1 - b + b(C_p^{(b)}x)^2)e^{-(C_p^{(b)}x)^2/2}$$

and in particular

$$f_p^{(0)}(x) = C_p^{(0)}e^{-(C_p^{(0)}x)^2/2}$$

The taking logs we know $f_p^{(b)} = f_p^{(0)}$ when

$$\begin{aligned}&log(C_p^{(b)}) + log(1 - b + b(C_p^{(b)}x)^2) - (C_p^{(b)}x)^2/2 = log(C_p^{(0)}) - (C_p^{(0)}x)^2/2 \iff \\ &log(1 - b + b(C_p^{(b)}x)^2) = (C_p^{(b)2} - C_p^{(0)2})x^2/2 + A_p^{(b)}\end{aligned}$$

where we clearly get two intersections when the affine term has a positive slope. Since the pth moment increasing in b, we have this.

Note this also establishes comparisons between $1 > b_1, b_2 > 0$ since pth moment increasing in b.

2 Continuations

1. Comparing arbitrary b_1, b_2 instead $b, 0$? - This seems to be resolved
2. $p \in (0, 1)$?