1 Notes

Remark 1. *Filtered probability space is akin to saying what knowledge is allowed before time t.*

1.1 Higher Probability Theory

Prop 1. *Gaussians converge* $\iff$ *means and variances converge.*

Remark 2. *Kolmogorov's Inequality allows us to bound sups of Sums of random variaibles in terms of variance(kind of like chebyshev).*

Theorem 1. *1.9 Let He be a centered Guassian space and* $(H_i)_{i \in I}$ *be collection of linear subspaces of H. Then the* H_i *are pairwise orthogonal in* $L^2 \iff \sigma$ *fields* $\sigma(H_i)$ *are independent.*

Proof. $\Longleftarrow$: First suppose $\sigma(H_i)$ ind. So for $X \in H_i, Y \in H_j$ we know $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

This is basically the ind iff. uncorrelated proof. □

Remark 3. *Note this is specific to gaussians and not true in general. B/c uncorrelated iff independent. Useful for making geometric arguments*

1.2 Gaussian White Noise

Gaussian white noise is isometry from space of finite variance random variables to centered gaussians. Intensity is measure of measure space

Note probability of gaussians space can depend on μ

$$E[G(f)G(g)] = \langle f, g \rangle_{L^2(E,\mathcal{E},\mu)} = \int fg d\mu$$

Prop 2. *2.3 TFAE:*

1. $(X_i)_{t \geq 0}$ *is a pre-brownian motion*

2. $(X_i)_{t \geq 0}$ *is a centered gaussian process with covariance* $K(s,t) = s \wedge t$

3. $X_0 = 0$ *a.s. and for every* $0 \leq s < t$ *rv* $X_t - X_s$ *is independent of* $\sigma(X_r, r \leq s)$ *and distributed according to* $N(0, t-s)$

4. $X_0 = 0$ *a.s. and for every choice of* $0 = t_0 < t_1 < ... < t_p$ *we have* $X_{t_i} - X_{t_{i-1}}$ *are independent and dist according to* $N(0, t_i - t_{i-1})$.

Prop 3. *First* $1 \implies 2$. *Gaussian white noise maps into gaussian space, we know* B_t *a gaussian process.*

$$\mathbb{E}[B_s B_t] = \mathbb{E}[G([0,s])G([0,t])] = \int_0^\infty dr 1_{[0,s]}1_{[0,t]} = min(s,t)$$

$2 \implies 3$. *Suppose* X_t *centered gaussian with min covariance.* X_0 *is* $N(0,0)$ *and* $X_0 = 0$ *a.s.*

Let H_s *be span of* $\{X_r, 0 \leq r \leq s\}$ *and* $\hat{H}_s$ *span of* $\{X_{s+u} - X_s, u \geq 0\}$. *We show orthogonal and therefore have desired indpendence*

Compute

$$\mathbb{E}[X_r(X_{s+u} - X_s)] = r \wedge (s+u) - r \wedge s = r - s = 0$$

Remark 4. *Subtracting an earlier value yields independence from earlier events - This is via theorem 1.9?*

Theme 1. *Show independence of objects by showing orthogonality of in a geometric space*

1.3 Pre-Brownian Motion

Prop 4. *1.* $B_t^\lambda = \frac{1}{\lambda}B_{\lambda^2 t}$ *is a prebrownian motion*

2. $B_t^{(s)} = B_{s+t} - B_s$ *is a pre-Broownian motion and ind. of* $\sigma(B_r, r \leq s)$

Proof. First 1. Clearly $B_t^\lambda = \frac{1}{\lambda}B_{\lambda^2 t}$ are centered gaussians. Suffices to show they have min covariance. Compute

$$\mathbb{E}[\frac{1}{\lambda}B_{\lambda^2 s}\frac{1}{\lambda}B_{\lambda^2 t}] = \frac{1}{\lambda^2}min(\lambda^2 s, \lambda^2 t) = min(s,t)$$

Now 2. Note by the markov property we know independence to hold. Confirmed by prop 2.3. □

1.4 Brownian Motion

Remark 5. *Brownian motion constructed constructed from pre-brownian as a modification via Kolmogorov's lemma(a technical point involving holder continuity).*

Strong Markov Property of Brownian Motion

1.5 Optional Stopping

Remark 6. *Optional Stopping for martingales tells us we don't expect to change from what we already know.*

$X_S = \mathbb{E}[X_T|\mathcal{F}_S]$

Remark 7. *Stopping early preserves martingales and uniform integrability.*

1.6 Continuous Semi-Martingales

$$\int_0^T f(s)da(s) := \int_{[0,T]} f(s)\mu(ds)$$
$$\int_0^T f(s)|da(s)| := \int_{[0,T]} f(s)|\mu|(ds)$$

Lemma 1. *$f : [0,T] \to \mathbb{R}$ continuous, then*

$$\int_0^T f da(s) = \lim_{n\to\infty} \sum_{i=1}^{p_n} f(t_{i-1}^n)(a(t_i^n) - a(t_{i-1}^n))$$

where the t_i^n are refinings of a partition

Proof. Follows from dominated convergence applied to the measure defining bounded variation a. Since $f_n(t) = f(t_i^n)$ for $t \in [t_{i-1}, t_i)$ converges pointwise to f and the simple sum is the integral of an elementary function □

Quadratic Variation

Proof of quadratic variation very long and should be looked into.

Bracket

Bracket can be thought of as an extension of the quadratic variation.

(This can be thought of as a cauchy schwarz type inequality for the processes)

Proof. Kunita-Watanabe:

Poinwise we have

$$|\langle M, N\rangle_s^t| \leq \sqrt{\langle M, M\rangle_s^t}\sqrt{\langle N, N\rangle_s^t}$$

We show this pointwise via appoximations for these+cauchy schwarz.

Then show intergral inequality using this pointwise estimate for simple functions and extend it to all functions.

□

1.7 Martingales

Prop 5. *If X is a martingale with right-continuous sample paths. Then TFAE:*

i) X is closed

ii) (X_t) is uniformly integrable

iii) X_t converges a.s. and in L^1 as $t \to \infty$

Proof. SHoudl learn this □

Prop 6. *If X_t is a martingale then*

i) If $X_t \in L^1$ then $Z_t = X_t - \mathbb{E}[X_t]$ martingale

ii) If $X_t \in L^2$ then $Y_t = Z_t^2 - \mathbb{E}(Z_t)^2$

ie. subtracting off mean and variance

iii) If we have for some r, $\mathbb{E}[e^{rX_t} < \infty$ for all t then $W_t = \frac{e^{rX_t}}{\mathbb{E}[e^{rX_t}]}$ is a martingale

1.8 Continuous Martingales

Theorem 2. *Let $M_0 \in L^2$ a CLM. TFAE:*

i) M is a martingale and $M_t \in L^2$ for arbitrary t

ii) $E[\langle M, M\rangle_t] < \infty$

1.9 Stochastic Integration

Prop 7. *$\mathbb{H}^2$ is a hilbert space*

Proof. Want to show sequence M^n cauchy then convergent.

$$\lim_{m,n\to\infty} E[(M_\infty^n - M_\infty^m)^2] = \lim_{m,n\to\infty} (M^n - M^m, M^n - M^m)_H = 0$$

So we know (M_∞^n) converging in L^2. We xtract this to get the same limit. □

We know $H \to H \cdot M$ extends to an isometry from $L^2(M)$ into $\mathbb{H}^2$. Furthermore $H \cdot M$ is unique martingale of $\mathbb{H}^2$ s.t.

$$\langle H \cdot M, N\rangle H \dot{\langle} M, N\rangle$$

This can be thought of as a version of inner product definition of derivative.

Often we say the stochastic integral "commutes" with bracket ie. for $M \in \mathbb{H}^2, H \in L^2(M)$

$$\langle H \cdot M, H \cdot M \rangle = H \cdot (H \cdot \langle M, M \rangle) = H^2 \cdot \langle M, M \rangle$$

since $\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle$

?Why do we have the second equality?

A lot of these results also extend to the CLM case.

Also the Semimartingale case

1.10 Wiener Integral

Remark 8. *Wiener integral of h coincides with stochastic integral $(h \cdot B)_t$ which makes sense when viewing h as a determinsitc progressive process*

Proof. True for simple functions and then true for arbitrary functions via density □

Remark 9. *Wiener integral integrates functions against RVs, Stochastic integral integrates processes against random variables*

Theorem 3. *5.1.3 Properties*

Proof. WTS $(H \cdot X)_t = \sum_{i=0}^{p-1} H_{(i)}(X_{t_{i+1}} - X_{t_i \wedge t})$

Enough to consider $X = M$ is a CLM in H^2. Set $T_n = inf\{t \geq 0 : |H_t| \geq\} = inf\{t_i : |H_{(i)}| \geq n\}$

This is a stopping time. $H_s 1_{[0,T_n]}(s) = \sum_{i=0}^{p-1} H_{(i)}^n 1_{(t_i,t_{i+1}]}(s)$.

$(H \cdot M)_{t \wedge T_n} = (H 1_{[0,T_n]} \cdot M)_t = \sum_{i=0}^{p-1} H_{(i)}^n (M_{t_{i+1} \wedge t} - M_{t_i \wedge t})$ □

1.11 Stochastic Integral Convergence Theorems

Theorem 4. *Dominated Convergence Theorem*

$$\int_0^t H_s^n dX_s \to \int_0^t H_s dX_s$$

Proof. $\int_0^t H_s^n dV_s \to \int_0^t H_s dV_s$ we have pointwise via dominated convergence theorem. So we show convergence in probability

Set stopping time $T_p := inf\{r \in [0,t] : \int_0^r K_s^2 d\langle M, M \rangle_s \geq p\} \wedge t$

$E[(\int_0^{T_p} H_s^n dM_s - \int_0^{T_p} H_s dM_s)^2] \leq E[\int_0^{T_p} (H_s^n - H_s)^2 d\langle M, M \rangle_s]$

goes to 0 by DCT. Note $\int_0^{T_p} K_s^2 d\langle M, M \rangle_s \leq p$. $P(T_P = t)$ tends to 1 $p \to \infty$, and the desired result follows. □

Theme 2. *Creating stopping times to analyze finite behavior.*

1.12 Itos Formula

Theorem 5. *Itos Formula*

Let $X^1, ..., X^p$ CSM. F twice cts diff. Then for every $t \geq 0$.

$$F(X_t^1, ..., X_t^p) = F(X_0^1, ..., X_0^p) + \sum_{i=1}^{p} \int_0^t \frac{dF}{dX^i}(X_s^1, ..., X_s^p) dX_s^i + (1/2) \sum_{i,j=1}^{p} \int_0^t \frac{d^2F}{dx^i dx^j}(X_s^1, ..., X_s^p) d\langle X^i, X^j \rangle_s$$

Proof. First we address the $p = 1$. Set $t > 0$. Increasing partition $0 = t_0^n < ... < t_{p_n}^n = t$. For every n write

We apply Taylor-Lagrange:

$$F(X_{t_{i+1}^n}) - F(X_{t_i^n}) = F'(X_{t_i^n})(X_{t_{i+1}^n} - X_{t_i^n}) + (1/2) f_{n,i}(X_{t_{i+1}^n} - X_{t_i^n})^2$$

□

Remark 10. *Remarks on itos formula.*

Prop 8. *In one dimension CLM with t as quadratic variation must be a brownian Motion(BM).*

Remark 11. *Often easier to estimate moments of $\sqrt{\langle M, M \rangle_t}$ than M hence why Burkholder-Davis-Gundy Useful*

2 Girsanov's Theorem

Need to study this

2.1 Applications of Girsanov

3 SDEs

Example 1. *Ornstein-Uhlenbeck Process*

Example 2. *Geometric Brownian Motion: SDE. $\sigma > 0$, $X_0 = 1$*

$$dX_t = rX_t dt + \sigma X_t dB_t$$

Apply Ito to $Y_t = log(X_t)$.

Solve for X_t by getting Y_t.

Question 1. *Long term behavior of X_t?*

Answer 1. *Depends on relationship between* r, σ^2

Example 3. $dX_t = \sigma X_t dB_t$ *with* $x_0 = 1$

Remark 12. $dX_t = f(X_t)dB_t$ *equivalent to* $X_t = 1 + \int_0^t f(X_s)dB_s$ *with initial condition* $X_0 = 0$

Long term behavior? Because positive martingale converges a.s(Even if f nonlinear).

Example 4. *Brownian motin on a circle. B standard BM. Set* $Y_t = e^{iB_t}$.

Apply Ito to differentiate:

$$dY_1(t) = dcos(B_t) = -sin(B_t)dB_t - (1/2)cos(B_t)dt$$
$$dY_2(t) = dsin(B_t) = cos(B_t)dB_t - (1/2)sin(B_t)dt$$

Sets up matrix equation

Example 5. $dX_t = rdt + \sigma X_t dB_t$ *with* $r, \sigma > 0$

rdt is drift term and $\sigma X_t dB_t$ *is volatility term. To solve start by guessing.*

For $r = 0$ *solution takes form* $X_t = X_0 e^{\sigma B_t - (1/2)\sigma^2 t}$ *via geometric brownian motion.*

So try $Y_t = X_t e^{-\sigma B_t + (1/2)\sigma^2 t} = X_t Z_t$.

Compute

$dZ_t = Z_t(-\sigma dB_t + (1/2)\sigma^2 dt) + (1/2)Z_t\sigma^2 dt$ *these terms we're adding are the Ito correction terms*

$dY_t = X_t dZ_t + Z_t dX_t + d\langle X, Z\rangle_t$ *via Ito*

Solve Y_t *explicitly and verify is solves SDE.*

Question 2. *Long time behavior?*

Blow up, because of drift. Otherwise would go to 0, as in BM case.

Theme 3. *To differentiate processes we must add correction terms via Ito's formula.*

Why? Look at Ito proof

3.1 Forward Kolmogorov Equation

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

Suppose X_t has density q_t.

Question 3. *Does* q_t *solve certain PDE?*

Remark 13. *If random variable solves SDE, does density solve PDE?*

Theme 4. *Extracting information: computing in two ways.*

Theme 5. *If integrating two different functions against all test functions give same results, then two functions must be the same(point of test function).*

Extracting local information from global info

4 Lists

4.1 Stochastic Integral Properties

1. When a the martingale square integrable, $\mathbb{E}[(\int_0^\infty H_s dM_s)^2] = \mathbb{E}\int_0^\infty H_s^2 d\langle M, M\rangle_s$

5 Questions

Question 4. *Equivalences to gaussian white noise*

Answer 2. *All we seem to have is prop 1.13* **??** *which shows existence of gaussian white noise*

Question 5. *Ways of deducing a random variable is gaussian?*

Answer 3. *Rotational invariance?*

Prop 9. *RV ind. components and rotationally invariant* $\iff$ *components are gaussian*

Unique characteristic.

Question 6. *How do deal with sups?*

Answer 4. *Maximal inequality, doob's martingale inequality.*

Question 7. *How to argue two processes have same dist?*

Answer 5. *Show expectation same under any function. Why does this work?*

Question 8. *When does a limit of martingales exist? (Over time, ie.* M_∞*)*

Answer 6. *Cts sample paths is a good start. I think* M_∞ *a limit in* L^2*.*

Question 9. *How to compute differentials in stochastic calculus?*

Answer 7. *Seems like it satisfies some kind of chain rule. See the wikipedia:*

`https://en.wikipedia.org/wiki/It%C3%B4%27s_lemma`

References

[1] Legal. Brownian Motion and Stochastic Calculus.

`https://drive.google.com/drive/u/1/search?q=le%20gall`