Homework problem of Stochastic Calculus

(1) Le Gall's book, Exercise 1.15
(2) 1.16
(3) 1.18
(4) Let B be a standard Brownian motion, i.e., a centered Gaussian process on $\mathbb{R}_+$ with covariance function $\Gamma(s,t) = \min(s,t)$. Fix $t > 0$, and consider the partition $0 = t_0 < t_1 \ldots < t_n = t$ and the summation depending on $p \in [0,1]$ and smooth function $f \in C_c^\infty(\mathbb{R})$

$$S_f(p) = \sum_{j=0}^{n-1} f(B_{(1-p)t_j + pt_{j+1}})(B_{t_{j+1}} - B_{t_j})$$

(i) in the case of $f(x) = x$, show $S_f(p) - S_f(0)$ converges in probability as the size of the partition goes to zero; (ii) redo (i) for general smooth f.

(5) Assuming we have a linear mapping $G : L^2([0,1]) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ satisfying (i) $G(f)$ has zero mean and preserves the norm; (ii) if $A, B \subset [0,1]$ with $A \cap B = \emptyset$, $G(A)$ is independent of $G(B)$; (iii) G is stationary in the sense that $G(A)$ has the same distribution as $G(A+x)$ for any $x \in \mathbb{R}$ and $A \subset [0,1]$ such that $A + x := \{y = x + z : z \in A\} \subset [0,1]$; (iv) for any $n \in \mathbb{Z}_+$, there exists $C_n > 0$ such that $\mathbb{E}[|G(A)/\sqrt{|A|}|^n] \le C_n$ for all $A \subset \mathbb{R}$, where $|A|$ denotes the Lebesgue measure of A.
   - prove that G is a Gaussian white noise.
   - Let $\{X_i\}$ be a Poisson point process on $[0,1]$ with Lebesgue intensity, define $\tilde{G}(f) = \sum_i f(X_i) - \int_0^1 f(x)dx$. Show that $\tilde{G}$ satisfies (i)-(iii) but not (iv).

(6) Let $\mathbb{W}$ be a Gaussian white noise on $L^2(\mathbb{R}^d)$. For any $x \in \mathbb{R}^d$ and $r > 0$, denote by $B_r(x)$ the ball centered at x with radius r. Define $W_r(x) = \frac{\mathbb{W}(B_r(x))}{|B_r(x)|}$, where $|\cdot|$ denotes the Lebesgue measure. (i) show that $\{W_r(x), x \in \mathbb{R}^d\}$ is a Gaussian process and derive its covariance function; (ii) show that the distribution of $W_r(x)$ does not converge as $r \to 0$.

(7) Denote by $\{e_n(x)\} = \{\frac{1}{\pi}\sin nx, \frac{1}{\pi}\cos nx\}$ the ONB of $L^2[0,2\pi]$. For any $\lambda = (\lambda_n) \in \ell_2$, define $V(x) = \sum_n \lambda_n e_n(x)\xi_n$ where $\{\xi_n\}$ is i.i.d. standard Gaussian. (i) show that $\{V(x) : x \in [0,2\pi]\}$ is a Gaussian process; (ii) compute the covariance function of V; (iii) show that $G(f) = \int_0^{2\pi} f(x)V(x)dx = \sum_n \lambda_n \langle f, e_n\rangle \xi_n$ is a mapping from $L^2[0,2\pi]$ to a centered Gaussian space; (iv) find the operator $K : L^2[0,2\pi] \to L^2[0,2\pi]$ such that $\mathbb{E}[G(f)G(g)] = \langle f, Kg\rangle$; (v) how do you choose a sequence of λ so that the corresponding V, which is well-defined at every $x \in [0,2\pi]$, approximates a Gaussian white noise? Simulate your result to visualize it.

(8) Let G be a Gaussian white noise on $L^2(\mathbb{R}^d)$, take $\phi \in C_c^\infty(\mathbb{R}^d)$. Define the Gaussian process $\{V(x) = G(\phi(\cdot - x))\}_{x\in\mathbb{R}^d}$. Compute the covariance function of V and show that V has a continuous modification, which we still denote by V.

(9) Consider a random process $(F(x))_{x\in\mathbb{R}^d}$ satisfying

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}[|F(x)|^p] \le C(p), \qquad \sup_{x,y\in\mathbb{R}^d} \mathbb{E}[|F(x) - F(y)|^p] \le C(p)|x-y|^{p\beta} \tag{0.1}$$

for any $p \geq 1$, where $C(p) > 0$ is some constant only depending on p. Let $w_\alpha(x) = (1+|x|)^\alpha$ be a weight, show that there exists a modification of F, which we still denote by F, such that for any $p \geq 1, \alpha > 0, \varepsilon > 0$,

$$\mathbb{E}\left[\left(\sup_{x\in\mathbb{R}^d} \frac{|F(x)|}{w_\alpha(x)}\right)^p\right] + \mathbb{E}\left[\left(\sup_{x,y\in\mathbb{R}^d, |x-y|\leq 1} \frac{|F(x)-F(y)|}{w_\alpha(x)|x-y|^{\beta-\varepsilon}}\right)^p\right] \leq C(p,\alpha,\varepsilon)$$

for some constant C depending on p, α, ε. The result shows that the condition (0.1) guarantees that F has a modification which almost surely lies in the weighted Hölder space.

(10) Exercise 2.25
(11) 2.27
(12) 2.29
(13) 2.31
(14) For any bounded function $f : \mathbb{R}^d \to \mathbb{R}$, consider the function

$$u(t,x) = \mathbb{E}[f(x+B_t)] = \int_{\mathbb{R}^d} q_t(x-y) f(y) dy$$

where $q_t(\cdot)$ is the density of B_t. Show that u solves the heat equation

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) = \frac{1}{2}(\partial_{x_1}^2 + \ldots + \partial_{x_d}^2) u(t,x)$$

for any $t > 0, x \in \mathbb{R}^d$.

(15) Let $w = \{w(i,j)\}_{i\in\mathbb{Z}_{\geq 0}, j\in\mathbb{Z}^d}$ be a family of i.i.d. random variables with standard normal distribution $N(0,1)$. Let $\{S_n\}_{n\in\mathbb{Z}_{\geq 0}}$ be a $d-$dimensional symmetric simple random walk that is independent of w, starting at the origin $S_0 = 0$. Define

$$Z_\beta(n) = \mathbb{E}_w[e^{\beta\sum_{i=0}^n w(i,S_i) - \frac{1}{2}\beta^2(n+1)}]$$

where the expectation $\mathbb{E}_w$ is only respect to S with w freezed. In other words, $Z_\beta(n)$ is a random variable with the randomness coming from w. Here $Z_\beta(n)$ refers to the partition function of a random polymer path, i.e., a symmetric simple random walk affected by the random environment, and $\beta > 0$ is the inverse temperature.

- Show that $\{Z_\beta(n)\}_{n\geq 0}$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}$ where $\mathcal{F}_n = \sigma(w(i,j) : i \leq n, j \in \mathbb{Z}^d)$.
- Show that $Z_\beta(n)$ converges a.s. as $n \to \infty$.
- Denote $Z_\beta(\infty)$ as the a.s. limit of $Z_\beta(n)$, show that there exists $\beta_0 > 0$ such that if $\beta < \beta_0$ and $d \geq 3$, then $Z_\beta(n) \to Z_\beta(\infty)$ in L^1.
- (*optional*) Show that the $Z_\beta(\infty)$ obtained in the previous step is positive almost surely.

(16) Exercise 3.26
(17) 3.27
(18) 3.28
(19) Exercise 4.23
(20) 4.24
(21) 4.25
(22) Let X_t be a continuous and uniformly integrable martingale with $X_0 = 0$. Suppose that there exists a constant $M > 0$ such that $\mathbb{E}[|X_\infty - X_\tau| \,|\, \mathcal{F}_\tau] \leq M$ almost surely, for every stopping time τ, and define $X^* = \sup_{t\geq 0} |X_t|$.

(i) Show that for all $\lambda, \mu > 0$, we have

$$\mathbb{P}[X^* \geq \lambda + \mu] \leq \frac{M}{\mu}\mathbb{P}[X^* \geq \lambda]$$

(ii) Show that there exists $C > 0$ such that $\mathbb{P}[X^* \geq \lambda] \leq e^{2-\frac{\lambda}{C}}$ for all $\lambda > 0$.

(23) Let V be a progressively measurable process satisfying $\int_0^\infty V_s^2 ds \leq 1$ a.s. Show that for every $x \geq 0$, we have

$$\mathbb{P}\left[\sup_{t\geq 0}\int_0^t V(s)dB_s \geq x\right] \leq e^{-x^2/2}$$

(24) Using the scaling property of Brownian motion and Itô's formula to show that

$$\mathbb{E}[B_1^{2n}] = (2n-1)\cdot(2n-3)\ldots 3\cdot 1.$$

(25) The Hermite polynomial $H_n(x,t)$ is defined through the relation

$$\sum_{n=0}^{\infty} H_n(x,t)\frac{\theta^n}{n!} = e^{\theta x - \frac{1}{2}\theta^2 t}, \qquad t \geq 0, x \in \mathbb{R}, \theta \in \mathbb{R}.$$

- show that H_n satisfies the recursive relation $H_{n+1} = xH_n - ntH_{n-1}$.
- show that H_n solves the backward heat equation $\partial_t H_n + \frac{1}{2}\partial_x^2 H_n = 0$.
- show that

$$H_{n+1}(B_t,t) = \int_0^t (n+1)H_n(B_s,s)dB_s = \ldots = (n+1)!\int_0^t\int_0^{t_1}\cdots\int_0^{t_n} dB_{t_{n+1}}dB_{t_n}\ldots dB_{t_1}.$$

(26) Consider the complex Brownian motion $B(t) = B_1(t) + iB_2(t)$, where B_1, B_2 are independent one-dimensional Brownian motions, and i is the imaginary unit. Assuming $B(0) = i$, and define T to be the first time of $B(t)$ hitting the real axis. (i) Show that $T < \infty$ almost surely; (ii) find the distribution of $B(T)$ by computing $\mathbb{E}[e^{i\lambda B(T)}]$ for any $\lambda \in \mathbb{R}$.

(27) Let B_t be a standard Brownian motion. Check $X_t = e^{\frac{1}{2}t}\cos(B_t)$ is a martingale.

(28) Prove directly from the definition of Ito's integral that

$$\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds.$$

(29) Let B be a standard 1d Brownian motion with $B_0 = 0$, and $f \in C^2(\mathbb{R}), g \in C(\mathbb{R})$. Define $X_t = f(B_t)e^{-\int_0^t g(B_s)ds}$. (i) Show that X_t is a semimartingale and write down its decomposition as the sum of a continuous local martingale and a finite variation process. (ii) Show that X_t is a continuous local martingale if and only if f satisfies the differential equation $f'' = 2fg$.

(30) Let B be a standard 1d Brownian motion with $B_0 = 0$. Define $X_t = \int_0^t \mathrm{sgn}(B_s)dB_s$ where $\mathrm{sgn}(x) = 1_{x\geq 0} - 1_{x<0}$. (i) Show that X is a Brownian motion. (ii) Show that X_t is uncorrelated with B_t for any $t > 0$. (iii) Show that X_t is not independent of B_t for any $t > 0$.

(31) Let B be a 2d Brownian motion with $B_0 = (1,0)$. Assuming the fact that B never hits the origin, show that $X_t = \log|B_t|$ is a continuous local martingale but not a martingale.