Exercises in Stochastic Analysis

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Preface

This, somewhat unusual collection of problems in Measure-Theoretic Probability and Stochastic Analysis, should have been more appropriately called: "The Problems I Like". Problems borrowed from the existing literature typically have solutions different to the ones in the original sources, and for those full references and occasional historical remarks are provided. There are also original problems: Problem 3, Problem 12, Problem 23, Problem 25 (due to V. Shmarov), Problem 26 (jointly worked out with V. Kešelj), Problem 28 (jointly worked out with Y. Dolivet), Problem 38, Problem 41, Problem 51, Problem 52, Problem 53, Problem 54, Problem 59, Problem 61 (due to A. Duriez), Problem 62, and Problem 68 are original. As this collection is work in progress, and any comments from the readers will be greatly appreciated.

I look for puzzles. I look for interesting problems that I can solve. I don't care whether they're important or not, and so I'm definitely not obsessed with solving some big mystery. That's not my style.

Freeman Dyson

1 Measure Theory

Problem 1 (Давыдов Н.А. et al. (1973), Problem 275.)**.** Circles of identical size are inscribed inside equilateral triangle (Figure 1). Find the limit of the ratio of the area covered by the circles and the area of the triangle as the number of circles tends to infinity.

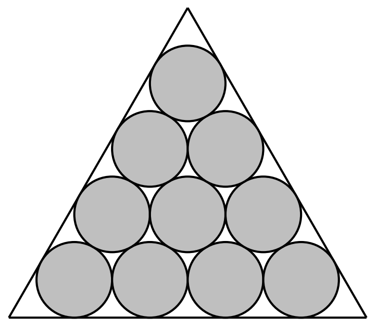
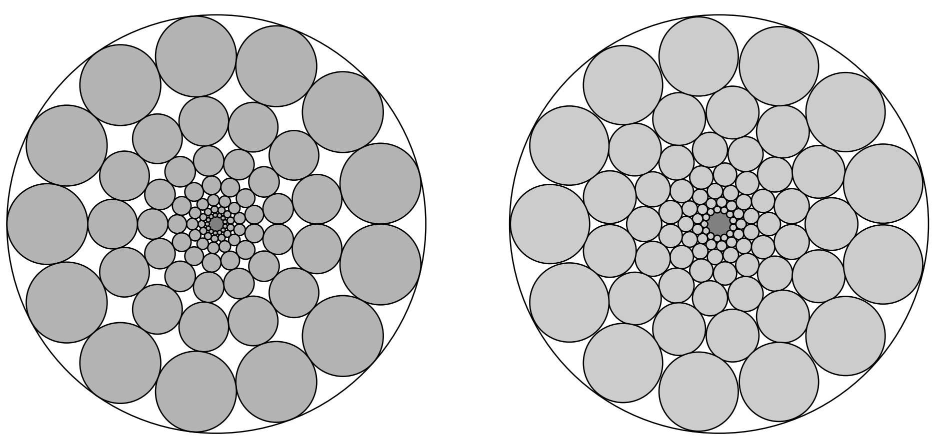


Figure 1

Solution. The limit in question equals the proportion of the area of triangle connecting the centers of three inner circles covered by those three circles, which is $\frac{\pi}{2\sqrt{3}}$.

Problem 2 (Mathematics Magazine, Problem 1919)**.** For the two pictures below, let P_n denote the proportion of the big circle covered by the small circles as function of the number of the small circles in the outer layer. Find the expressions for P_n and compute $\lim_{n\to\infty} P_n$.



Solution. Let R be the radius of the big circle, r_n be the radius of the small circles in the outer layer, and R_n be the radius of the circle encompassing all small circles except those in the outer layer. Then in both cases $P_n R^2 \pi = n r_n^2 \pi + P_n R_n^2 \pi$ $(\star)$ and $\sin \frac{\pi}{n} = \frac{r_n}{R - r_n}$. In the first case $R_n = R - 2r_n$, giving

$$P_n = \frac{n r_n}{4(R - r_n)} = \frac{n \sin \frac{\pi}{n}}{4}, \quad \lim_{n \to \infty} P_n = \frac{\pi}{4}.$$

In the second case we have $(R - r_n) \cos \frac{\pi}{n} - \sqrt{(r_n + \hat{r}_n)^2 - r_n^2} + \hat{r}_n = R_n$, where $\hat{r}_n$ is the radius of the circle in the second outer layer, $\sin \frac{\pi}{n} = \frac{\hat{r}_n}{R_n - \hat{r}_n}$. The last two relations yield R_2 to be substituted in $(\star)$, whence we get an expression for P_n and $\lim_{n \to \infty} P_n = \frac{\pi}{2\sqrt{3}}$ (which is, interestingly, the same result as in Problem 1).

□

Problem 3. Suppose we play the game where I draw a $N(0,1)$ random number A, show it to you, then draw another independent $N(0,1)$ one B, but before showing it to you ask you to guess whether B will be larger than A or not. If you are right, you win if you are wrong you lose. What is your strategy, and what will be your long term success rate if we keep playing?

Solution. Since the numbers are zero-mean, the strategy is to guess the sign of the difference $B - A$ as positive if $A < 0$, and negative otherwise. The long-term success rate will be

$$\frac{1}{2} P(B - A > 0 | A < 0) + \frac{1}{2} P(B - A \leq 0 | A \geq 0).$$

Due to the symmetry, and the fact that $P(A = 0) = 0$, the above expression simplifies to $P(B - A > 0 | A < 0)$. Plotting on the (a, b) Cartesian plane the part of the region $a < 0$ where $b - a > 0$, we see that this is one and a half quadrants out of the two comprising the full $a < 0$ region. Therefore, due to the radial symmetry of the Gaussian two-dimensional distribution with zero correlation, we conclude that the success rate is $\frac{3}{4}$.

□

Problem 4 (AMM, Problem 12150)**.** Let $X_0, X_1, \ldots, X_n$ be independent random variables, each distributed uniformly on $[0, 1]$. Find

$$E \left[\min_{1 \leq i \leq n} |X_0 - X_i| \right].$$

Solution. Let L be the expression in question. Then

$$L = \int_0^1 E \left[\min_{1 \leq i \leq n} |x - X_i| \right] dx = \int_0^1 \int_0^1 \left[P(|X_0 - x| \geq u) \right]^n du \, dx.$$

Since $P(|X_0 - x| \geq u) = \max(1-u-x, 0) + \max(x-u, 0)$, $x, u \in [0,1]$, we have

$$P(|X_0 - x| \geq u) = \begin{cases} 1-2u, & 0 \leq u < x, \\ 1-u-x, & x \leq u < 1-x, \\ 0, & 1-x \leq u \leq 1, \end{cases} \qquad x \in [0, 1/2],$$

so

$$L = 2\int_0^{1/2} \left[\int_0^x (1-2u)^n \, du + \int_x^{1-x} (1-u-x)^n \, du \right] dx = \frac{n+3}{2(n+1)(n+2)}.$$

$\square$

Problem 5. Suppose $X_1, X_2, \ldots, X_n$ are independent, identically distributed random variables. Show that

$$\frac{X_1^2}{X_1^2 + X_2^2 + \cdots + X_n^2}, \quad \frac{X_2^2}{X_1^2 + X_2^2 + \cdots + X_n^2}$$

are negatively correlated.

Solution. With $Z_i := \frac{X_i^2}{X_1^2+X_2^2+\cdots+X_n^2}$ we have $\operatorname{covar}(Z_1, \sum_{i=1}^n Z_i) = \operatorname{covar}(Z_1, 1) = 0$, so $\operatorname{var}(Z_1) + \sum_{i=2}^n \operatorname{covar}(Z_1, Z_i) = 0$. Due to symmetry $\operatorname{covar}(Z_1, Z_i) = \operatorname{covar}(Z_1, Z_2)$, $i > 2$, which thus equals $-\frac{\operatorname{var}(Z_1)}{n-1} < 0$.

$\square$

Problem 6 (Kolmogorov Probability Olympiad 2017)**.** Let X, Y be IID random variables such that $P(\{X = 0\}) = 0$. Show that

$$E\left[\frac{XY}{X^2+Y^2}\right] \geq 0.$$

Solution. Put

$$f(a) := E\left[XY \exp(-a(X^2+Y^2))\right] = E\left[X \exp(-aX^2)\right]^2 \geq 0, \quad a \geq 0,$$

and integrate over $[0, \infty)$.

$\square$

Problem 7 (Kolmogorov Probability Olympiad 2015)**.** Random vector $X = (X_1, X_2, \ldots, X_n)$ is uniformly distributed on the unit sphere in $\mathbb{R}^n$. Find $M_2 = E[X_1^2]$ and $M_4 = E[X_1^4]$.

Solution. Since X_i are identically distributed and $\sum_{i=1}^n X_i^2 = 1$, it readily follows that $M_2 = 1/n$.

For M_4, first note that by squaring $\sum_{i=1}^n X_i^2 = 1$ and taking expectation we obtain

$$nM_4 + n(n-1)V_{12} = 1, \tag{1}$$

where $V_{12} = E[X_1^2 X_2^2]$. Thus to complete the solution, we need another relation between M_4 and V_{12}. To this end, note that (X_1, X_2) and $\left(\frac{X_1+X_2}{\sqrt{2}}, \frac{X_1-X_2}{\sqrt{2}}\right)$ are identically distributed. Hence $E\left[\frac{(X_1+X_2)^2(X_1-X_2)^2}{4}\right] = E[X_1^2 X_2^2]$, which after expanding gives $M_4 = 3V_{12}$. Combined with (1) this yields $M_4 = \frac{3}{n(n+2)}$.

□

Remark 1. A common way to generate uniform distribution on the sphere is to take $X_i = \frac{Z_i}{\sqrt{Z_1^2+Z_2^2+\cdots+Z_n^2}}$, where Z_i are independent standard Normal random variables.

Problem 8 (Shiryaev (2012), Problem 2.13.4.)**.** Let Z_i, $i = 1, 2, 3$ be independent standard Gaussian random variables. Show that

$$Z := \frac{Z_1 + Z_2 Z_3}{\sqrt{1 + Z_3^2}} \sim N(0, 1).$$

Solution. The conditional distribution of Z given Z_3 is a standard Gaussian (not depending on Z_3), hence the claim.

□

Problem 9 (Torchinsky (1995), Problem VI.4.24)**.** Suppose that Ω is a set, $(\Omega, \mathscr{G})$ is a measure space, and $Z : \Omega \to \mathbb{R}$ is a given mapping. Then Z is $\mathscr{G}$ measurable iff

$$Z = \sum_{i=1}^{\infty} \lambda_i I_{A_i} \tag{2}$$

for some $\{\lambda_i\} \subset \mathbb{R}$ and $\{A_i\} \subset \mathscr{G}$. From (2) deduce the First Borel-Cantelli Lemma: if $\sum_{i=1}^{\infty} P(A_i) < \infty$, then $P(\limsup A_n) = 0$.

Solution. The "if" part follows directly from the observation that the mapping given by (2) is the pointwise limit as $n \to \infty$ of the $\mathscr{G}$ measurable mappings $\sum_{i=1}^n \lambda_i I_{A_i}$.

For the opposite direction, first suppose that Z is nonnegative. Define

$$Z_1 := I_{\{Z \geq 1\}},$$
$$S_n := \sum_{i=1}^{n} \frac{1}{i} Z_i, Z_{n+1} := I_{\{Z - \frac{1}{n+1} \geq S_n\}}, \quad n \geq 1.$$

We claim that $Z = \lim S_n = \sum_{i=1}^{\infty} \frac{1}{i} Z_i$ (note that $\lim S_n$ exists).

Using induction we first show that $S_n \leq Z$, $\forall n \in \mathbb{N}$. Clearly $S_1 = I_{\{Z \geq 1\}} \leq Z$. Suppose that the claim holds for some $m \in \mathbb{N}$, and denote $\{Z - \frac{1}{m+1} \geq S_m\}$ by A. Then on A we have $Z \geq \frac{1}{m+1} + S_m$, while, by the inductive hypothesis, on A^c we have $Z \geq S_m$. Thus $Z \geq \frac{1}{m+1} I_A + S_m = S_{m+1}$, which completes the proof by induction.

It remains to show that $\lim S_n \geq Z$. Fix an arbitrary $\omega \in \Omega$. If $Z(\omega) = \infty$, we have $Z_n(\omega) = 1$ $\forall n \in \mathbb{N}$, and the result follows. If $Z(\omega) < \infty$, suppose that on the contrary $Z(\omega) - \lim S_n > 0$, and find $k \in \mathbb{N}$ such that $\frac{1}{k} < Z(\omega) - \lim S_n$. Then, since $S_n \leq \lim S_n$, we have $Z(\omega) - \frac{1}{n+1} \geq S_n(\omega)$, $\forall n \geq k$, and so $Z_n(\omega) = 1$, $\forall n \geq k$. But then $\lim S_n = \infty$, which is a contradiction.

In the general case we write $Z = Z^+ - Z^-$ and apply the above result to both parts (since Z^+ and Z^- live on disjoint sets, there is no additional convergence issues in merging the two series).

The First Borel-Cantelli Lemma is essentially the statement that if $Z = \sum_{i=1}^{\infty} I_{A_i}$ is integrable, then $Z < \infty$ a.s.. $\square$

Problem 10 (Doob's Theorem)**.** Suppose that Ω is a set, $(F, \mathscr{F})$ is a measurable space, and $X : \Omega \to F$ is a given mapping. If a mapping $Z : \Omega \to \bar{\mathbf{R}}$ is $\sigma\{X\}$ measurable, then there exists some $\mathscr{F}$-measurable mapping $\Psi : F \to \bar{\mathbb{R}}$ such that $Z(\omega) = \Psi(X(\omega)), \quad \forall \omega \in \Omega$.

Solution. By Problem 9, we have

$$Z(\omega) = \sum_{i=1}^{\infty} \lambda_i I_{\{X \in \Gamma_i\}}(\omega),\ \forall \omega \in \Omega$$

for some $\{\lambda_i\} \subset \mathbb{R}$ and $\{\Gamma_i\} \subset \mathscr{F}$, and so

$$Z(\omega) = \sum_{i=1}^{\infty} \lambda_i I_{\Gamma_i}(X(\omega)),\ \forall \omega \in \Omega.$$

Define $\Psi : F \to \mathbb{R}$ as

$$\Psi(x) := \sum_{i=1}^{\infty} \lambda_i I_{\Gamma_i}(x),\ x \in F.$$

Since Z^+ and Z^- live on disjoint sets, for every $x \in F$ the terms of the series $\Psi(x)$ have the same sign, and so $\Psi(x)$ is well defined. Furthermore, Ψ is clearly $\mathscr{F}$ measurable and $Z(\omega) = \Psi(X(\omega)),\ \forall \omega \in \Omega$.

$\square$

Problem 11 (Torchinsky (1995), Problem VIII.3.13)**.** Let $\{X_n\}$ be a sequence of nonnegative random variables on a probability space $(\Omega, \mathscr{F}, P)$. If $\{X_n\}$ is uniformly integrable, we have

$$\limsup E[X_n] \leq E[\limsup X_n].$$

Solution. Let $\lambda > 0$ be arbitrary. By Fatou's Lemma we get

$$\liminf E[(\lambda - X_n) I_{\{X_n < \lambda\}}] \geq E[\liminf (\lambda - X_n) I_{\{X_n < \lambda\}}],$$

which implies

$$E[\limsup X_n; \{X_n < \lambda\}] \geq \limsup E[X_n; \{X_n < \lambda\}]. \tag{3}$$

Since $\{X_n\}$ is uniformly integrable for every $\epsilon > 0$ there exists $\lambda > 0$ such that

$$E[X_n] < E[X_n; \{X_n < \lambda\}] + \epsilon, \quad \forall n \in \mathbf{N},$$

Using (3) we now get

$$\begin{aligned} \limsup E[X_n] &\leq \limsup E[X_n; \{X_n < \lambda\}] + \epsilon \leq E[\limsup X_n; \{X_n < \lambda\}] + \epsilon \\ &\leq E[\limsup X_n] + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the result follows.

$\square$

Remark 2. A different proof and related results can be found in Darst (1980).

Remark 3. An interesting consequence of the above Problem is the following simple proof of Vitali Convergence Theorem (see e.g. Folland (1999)): assuming the statement of the theorem, let us define

$$Y_n = |X_n - X|, \quad n = 0, 1, \dots .$$

Then, according to Remark 2.6.6, the above sequence is uniformly integrable, so by Problem 11 we have

$$\limsup E[Y_n] \leq E[\limsup Y_n] = 0,$$

which in turn implies

$$\lim E[|X_n - X|] = 0.$$

Since $E[|X|] \leq E[|X_n|] + E[|X_n - X|]$, $\forall n \in \mathbf{N}$, from the above we also have $E[|X|] < \infty$, which completes the proof of the theorem.

Problem 12. Let $(X_i)_{i=1}^{\infty}$ be a sequence of uniformly integrable nonnegative random variables with $E[X_n] = 1$, $n \in \mathbb{N}$. Then $P(\limsup_{n\to\infty} X_n > 0) > 0$.

Solution. By the assumption $\lim_{K\to\infty} \sup_n E[(X_n - K)^+] = 0$, so for some $K > 0$, $\epsilon > 0$ we have $\liminf_{n\to\infty} E[X_n - K]^+ \leq \limsup_{n\to\infty} E[X_n - K]^+ < 1 - \epsilon$. In view of $1 - E[X_n \wedge K] = E[X_n - K]^+$, this gives $\limsup_{n\to\infty} E[X_n \wedge K] > \epsilon$. By Fatou lemma then $E[\limsup_{n\to\infty} X_n \wedge K] > \epsilon$, whence the claim follows.

$\square$

Problem 13 (AMM, Problem 10372(a)). Let $(X_i)_{i=1}^{\infty}$ be a sequence of nonnegative random variables such that $\sum_{n=1}^{\infty} \sqrt{M_n} < \infty$, $M_n = E[X_n]$. Show that there exists a convergent series $\sum_{n=1}^{\infty} a_n$ and a random variable N such that $P(\{X_n \leq a_n$ for $n \geq N\}) = 1$.

Solution. Set $a_n = \sqrt{c_n}$, $E_n = \{X_n > a_n\}$. Then $P(E_n) \leq a_n$, hence $\sum_{n=1}^{\infty} P(E_n) < \infty$. Therefore by the first Borel-Cantelli Lemma, $P(\limsup_{n\to\infty} E_n) = 0$, which implies the desired result.

$\square$

Problem 14 (Student Olympiad, Kiev 1983). Let $(A_n)_{n=1}^{\infty}$ be events on a probability space such that $\sum_{n=1}^{\infty} P(A_n) < \infty$. Put

$$B_n := \{\omega \,|\, \omega \in A_n \text{ for at least } m \text{ values of the index } n\}.$$

Show that B_m is an event, and that

$$P(B_m) \leq \frac{1}{m} \sum_{n=1}^{\infty} P(A_n),\ m \in \mathbb{N}.$$

Solution. Put $Z(\omega) := \sum_{n=1}^{\infty} I_{A_n}(\omega)$. Then Z is a random variable and $B_m = \{Z \geq m\}$. Futhermore, from the Markov inequality $mP(B_m) \leq E[Z] = \sum_{n=1}^{\infty} P(A_n)$.

$\square$

Problem 15 (Student Olympiad, Kiev 1983). Let ξ, η be i.i.d. random variables with $P(\xi \in [0,1]) = 1$. Then for each $\epsilon > 0$

$$P\left(|\xi - \eta| < \epsilon\right) \geq \frac{1}{1 + \left[\dfrac{1}{\epsilon}\right]}$$

This estimate is sharp.

Solution.[V. Shmarov] Put $k = \left[\frac{1}{\epsilon}\right]$,

$$A_0 = [0,\epsilon),\ A_1 = [\epsilon, 2\epsilon), ...,\ A_{k-1} = [(k-1)\epsilon, k\epsilon),\ A_k = [k\epsilon, 1].$$

Then

$$P\left(|\xi-\eta|<\epsilon\right) \geq \sum_{i=0}^{k} P(\xi,\eta \in A_i) = \sum_{i=0}^{k} P(\xi \in A_i)^2 \geq \frac{1}{k+1}\left(\sum_{i=0}^{k} P(\xi \in A_i)\right)^2 = \frac{1}{1+k},$$

where the last inequality follows by the Jensen's inequality. The equality is attained on the discreet uniform distribution on the set $\{0,\ \epsilon,\ 2\epsilon, ...,\ k\epsilon\}$.

□

Remark 4. Let F denote the common cdf of ξ and η. The probability in question can be written as $1-2\int_\epsilon^1 F(x-\epsilon)\,dF(x)$. For the special case $\epsilon = 1/2$, due to the monotonicity of the integrand, the optimal F is constant on $[1/2,1)$, with $F(1/2)$ solving $\max_{x\in[0,1]} x(1-x)$. Thus, the F achieving the optimum satisfies $F(x)=1/2$, $x \in [1/2,1)$, and is arbitrary otherwise. For any $\epsilon > 0$ analogous reasoning imples that optimal F is piecewise constant on ontervals $[i\epsilon,(i+1)\epsilon)$. Denoting by x_i, the "jumps" of F, this leads to the optimization problem

$$\max_{\sum_{i=1}^k x_i=1,\ x_i\geq 0} x_1x_2 + (x_1+x_2)x_3 + \cdots + (x_1+\cdots+x_{k-1})x_k,$$

whose solution is readily shown to be $x_i = 1/k$, $1 \leq i \leq k$.

Problem 16 (J. Karamata)**.** *Let $(a_n)_{n\geq 0}$ be a real-valued sequence such that $a_n \geq -M$, $M \geq 0$, and*

$$\lim_{t\to 1-}(1-t)\sum_{n=0}^{\infty} a_n t^n = s. \tag{4}$$

Then

$$\lim_{t\to 1-}(1-t)\sum_{n=0}^{\infty} a_n f(t^n)t^n = s\int_0^1 f(u)\,du$$

for every bounded, Riemann-integrable f.

Solution. For the case of monomial, $f(t)\equiv t^p$, $p\in\mathbb{N}$ we have

$$\begin{aligned}\lim_{t\to 1-}(1-t)\sum_{n=0}^{\infty} a_n f(t^n)t^n &= \lim_{t\to 1-}(1-t)\sum_{n=0}^{\infty} a_n t^{n(p+1)}\\ &= \lim_{t\to 1-}\frac{1-t}{1-t^{p+1}}(1-t^{p+1})\sum_{n=0}^{\infty} a_n (t^{p+1})^n = \frac{s}{p+1} = s\int_0^1 t^p\,dt.\end{aligned}$$

Therefore, the claim holds for any polynomial, and, by the density argument for any bounded, Riemann-integrable f.

□

Problem 17. *Let $f : [0,\infty) \mapsto \mathbb{R}$ be Riemann-integrable and such that $f(u)e^{\epsilon u}$ is bounded for some $\epsilon > 0$. Then*

$$\lim_{h\to 0+} h\sum_{n=1}^{\infty} f(nh) = \int_0^{\infty} f(x)\,dx.$$

Solution. Put $a = 1/\epsilon$, $g(t) \equiv \frac{f(-a\ln(t))}{t}$. According to Problem 16 we have

$$\lim_{t\to 1-}(1-t)\sum_{n=0}^{\infty} g(t^n)t^n = \int_0^1 g(x)\,dx = \int_0^{\infty} f(ax)\,dx.$$

Thus,

$$\lim_{t\to 1-}(1-t)\sum_{n=0}^{\infty} f(-na\ln(t)) = \int_0^{\infty} f(ax)\,dx,$$

whence the result follows by putting $h = -a\ln(t)$.

□

Remark 5. Problem 17 complements Problem II.30 in Pólya & Szegö (1978) by replacing the requirement of monotonicity of f with the stated boundedness condition. Other conditions can be found in Szász & Todd (1951) and references therein.

Problem 18. let S be a separable metric space and let $\mathscr{P}(S)$ and $\bar{C}(S)$ denote the sets of probability measures on $\mathscr{B}(S)$ and set of all bounded continuous functions on S respectively. A set $M \subset \bar{C}(S)$ is called *separating* for $\mathscr{P}(S)$ if whenever $P, Q \in \mathscr{P}(S)$ and

$$\int f\,dP = \int f\,dQ, \quad \forall f \in M,$$

we have $P = Q$.

Let

$$M_0 = \{f : x \in \mathbb{R} \mapsto f(x) \equiv (x-a)^+,\, a \in \mathbb{R}\}, \quad M_1 = \mathrm{span}(M_0) \cap \bar{C}(\mathbb{R}). \tag{5}$$

Show that M_1 is separating.

Solution. Note that for every $a \in \mathbb{R}$ the indicator function of the interval $(-\infty, a]$ is the pointwise limit of the uniformly bounded sequence $a_n(x) = (x-a+1/n)^+ - (x-a)^+,\ x \in \mathbb{R}$ from M_1. By the Dominated Convergence Theorem this implies that M_1 is separating.

$\square$

Problem 19 (Ethier & Kurtz (1986), Problem 3.11.7)**.** Let (S, r) be a separable metric space, let X and Y be S-valued random variables defined on a probability space $(\Omega, \mathscr{F}, P)$, and let $\mathscr{G}$ be a sub-σ-algebra of $\mathscr{F}$. Suppose that $M \subset \bar{C}(S)$ is separating and

$$E[f(X)|\mathscr{G}] = f(Y) \quad a.s.$$

for every $f \in M$. Then $X = Y$ *a.s.*

Solution. The proof below proceeds from first principles. Alternatively one may base the proof on Proposition 3.4.6 of Ethier & Kurtz (1986). Let $\sigma(M)$ denote the σ-algebra generated by M, and for every $\Gamma_1, \Gamma_2 \in \mathscr{B}(S)$ put

$$Q_1(\Gamma_1, \Gamma_2) := P(\{X \in \Gamma_1, Y \in \Gamma_2\}), \quad Q_2(\Gamma_1, \Gamma_2) := P(\{Y \in \Gamma_1, Y \in \Gamma_2\}).$$

Then for a fixed $\Gamma_2 \in \sigma(M)$ the expressions $\mu_1(\cdot) := Q_1(\cdot, \Gamma_2)$, $\mu_2(\cdot) := Q_2(\cdot, \Gamma_2)$ define measures on $\mathscr{B}(S)$, which by the initial hypothesis satisfy

$$\int f \, d\mu_1 = E[f(X) I_{\Gamma_2}(Y)] = E[f(Y) I_{\Gamma_2}(Y)] = \int f \, d\mu_2, \quad \forall f \in M.$$

Since M is separating and μ_1 and μ_2 have the same total mass, we conclude

$$Q_1(\Gamma_1, \Gamma_2) = Q_2(\Gamma_1, \Gamma_2), \quad \forall \, \Gamma_1 \in \mathscr{B}(S), \, \Gamma_2 \in \sigma(M).$$

Next, for a fixed $\Gamma_1 \in \mathscr{B}(S)$ put $\nu_1(\cdot) := Q_1(\Gamma_1, \cdot)$, $\nu_2(\cdot) := Q_2(\Gamma_1, \cdot)$, so that

$$\int f \, d\nu_1 = \int f \, d\nu_2, \quad \forall f \in M. \tag{6}$$

Furthermore, by initial hypothesis $E[f(X)] = E[f(Y)]$, $\forall f \in M$, and thus, since M is separating,

$$P(\{X \in \Gamma\}) = P(\{Y \in \Gamma\}), \quad \forall \, \Gamma \in \mathscr{B}(S), \tag{7}$$

which in turn gives $\nu_1(S) = \nu_2(S)$. Together with (6) and the fact that M is separating this implies $\nu_1 = \nu_2$, hence $Q_1 = Q_2$.

For an arbitrary $\Gamma \in \mathscr{B}(S)$ we now have

$$P(\{X \in \Gamma, \, Y \in \Gamma\}) = P(\{Y \in \Gamma\}),$$

which combined with (7) gives

$$P(\{X \in \Gamma\} \Delta \{Y \in \Gamma\}) = 0.$$

Therefore, for a countable base $(B_i)_{i\in\mathbb{N}}$ for S we can write

$$P(\{X \neq Y\}) \leq \sum_{i\in\mathbb{N}} P(\{X \in B_i\}\Delta\{Y \in B_i\}) = 0,$$

and the result follows.

□

Problem 20 (Statistica Neerlandica, Problem 130)**.** Suppose X is an integrable real-valued random variable on a probability space $(\Omega, \mathscr{F}, P)$, and let $\mathscr{G}$ be a sub-σ-algebra of $\mathscr{F}$. If X and $E[X|\mathscr{G}]$ have the same probability law, then

$$E[X|\mathscr{G}] = X a.s.. \tag{8}$$

If, in addition, $\mathscr{G}$ is P-complete, X is $\mathscr{G}$-measurable.

Solution. Let M_0, M_1 be defined in (5). Fix $f \in M_0$ and put $Y := E[X|\mathscr{G}]$. Since f is convex, Jensen's inequality implies

$$f(Y) = f(E[X|\mathscr{G}]) \leq E[f(X)|\mathscr{G}]. \tag{9}$$

On the other hand, by the uniqueness in law, $E[f(X)] = E[f(Y)]$. Thus, from (9) we have $E[f(X)|\mathscr{G}] = f(Y)$, which in turn implies

$$E[\phi(X)|\mathscr{G}] = \phi(Y), \quad \forall\phi \in M_1.$$

From Problem 19 we now get (39), hence if $\mathscr{G}$ is P-complete, we conclude that X is $\mathscr{G}$-measurable.

□

Problem 21 (Williams (1991), Exercise E9.2.)**.** Suppose X, Y are integrable real-valued random variables on $(\Omega, \mathscr{F}, P)$ such that

$$E[X|Y] = Y \ a.s. \quad \text{and} \quad E[Y|X] = X \ a.s..$$

Then $X = Y$ $a.s.$.

Solution. By Jensen's inequality for every $\phi \in M_0$ we have

$$E[\phi(Y)] = E[\phi(E[X|Y])] \leq E[E[\phi(X)|Y]] = E[\phi(X)].$$

By symmetry the opposite inequality also holds implying $E[\phi(X)] = E[\phi(Y)]$, $\forall\phi \in M_0$, hence $E[\phi(X)] = E[\phi(Y)]$, $\forall\phi \in M_1$. Since M_1 is separating, we conclude that X and Y have the same law, and the result follows from Problem 20.

$\square$

Problem 22 (Kolmogorov Probability Olympiad 2006)**.** Compute $E[X|XY]$ for independent $N(0,1)$ random variables X, Y.

Solution. $E[X|XY] = \Phi(XY)$ for a measurable mapping[1] Φ. Hence, with $\tilde{X} := -X$, $\tilde{Y} := -Y$ we get $E[\tilde{X}|\tilde{X}\tilde{Y}] = \Phi(\tilde{X}\tilde{Y}) = \Phi(XY) = E[X|XY]$. On the other hand $E[X|XY] = -E[\tilde{X}|\tilde{X}\tilde{Y}]$, whence $E[X|XY] = 0$.

$\square$

Problem 23. Suppose X, Y are nonnegative random variables, and let $m := \min(X, Y)$ and $M := \max(X, Y)$. Show that

$$E[(m + M)(E[M|m] - E[m|M])] \geq 0.$$

Solution. Since $XY = mM$ we get $E[mE[M|m]] = E[ME[m|M]]$, which in turn gives $E[XE[M|m]] \geq E[YME[m|M]]$. Flipping X and Y and adding up the two inequalities gives $E[(X + Y)(E[M|m] - E[m|M]) \geq 0$, whence the result follows as $M + m = X + Y$.

$\square$

Problem 24. [V. Shmarov] Let F_1, F_2 be two distributions with the same mean. Does there necessarily exits a probability space $(\Omega, \mathscr{F}, P)$, on which we have random variable X and two sigma algebras $\mathscr{G}$, $\mathscr{H} \subset \mathscr{F}$ such that

$$E[X|\mathscr{G}] \sim F_1 \quad \text{and} \quad E[X|\mathscr{H}] \sim F_2?$$

Solution. Let m be the common mean of F_1 and F_2, and let $\Omega = [0,1] \times [0,1]$, with $\mathscr{F}$ the Borel sigma-algebra $\mathscr{B}([0,1] \times [0,1])$. Also let $\mathscr{H}$ and $\mathscr{G}$ be the sigma algebras $\Omega \times A$, $A \in \mathscr{B}([0,1])$, and $B \times \Omega$, $B \in \mathscr{B}([0,1])$, respectivelly, and put $X = F_1^{-1}(x_1 + m - 1)F_2^{-1}(x_2 + m - 1)$, $(x_1, x_2) \in \Omega$. It is straightforward to check that the above construction satisfies the required conditions.

$\square$

Problem 25. [A.A. Дороговцев] Give an example of independent random variables X and Y, and a Borel-measurable $F : \mathbb{R}^2 \to \mathbb{R}$ such that 1) X and $F(X, Y)$ are independent; 2) for every y the function $F(\cdot, y)$ is non-constant.

Solution. Let $\Omega = \{0,1\} \times \{0,1\}$, and for $\omega = (\omega_1, \omega_2) \in \Omega$ put $P(\{\omega\}) = 1/4$, $X(\omega) := \omega_1$, and $Y(\omega) := \omega_2$. Let $F(x,y) := \delta_{x=y}(x,y)$. Since X and $Z := F(X,Y)$ are binary variables, $X^n = X$, $Z^n = Z$, $n \in \mathbb{N}$, so from the fact that $E[XZ] = E[X]E[Z](= 1/4)$ it follows that $E[X^n Z^m] = E[X^n]E[Z^m]$, $n, m \in \mathbb{N}$, so X and Z are independent. Since $F(\cdot, y)$ is constant for every y, the construction is complete.

[1] C.f. Section 9.6. "Agreement with traditional usage" of Williams (1991).

$\square$

Problem 26 (Counterexample related to Hunt's lemma)**.** Suppose $(X_n)_{n=1}^{\infty}$ is a sequence of random variables such that $X := \lim_{n\to\infty} X_n$ exists a.s. and for some integrable random variable Y

$$|X_n| \le Y, \ a.s., \quad n = 1, 2, \dots. \tag{10}$$

Then *Hunt's Lemma* (e.g. Williams (1991), Exercise E14.1.) states that for any filtration $\{\mathscr{F}_n\}$, $\mathscr{F}_\infty := \sigma\big(\cup_n \mathscr{F}_n\big)$ we have

$$\lim_{n\to\infty} \mathbb{E}[X_n|\mathscr{F}_n] = \mathbb{E}[X|\mathscr{F}_\infty].$$

Show that the condition of uniform boundedness (10) cannot be replaced by uniform integrability of $(X_n)_{n=1}^{\infty}$. The counterexample constructed should have a nontrivial (i.e. strictly increasing) filtration.

Solution. Consider a circle of perimeter 1, and let A_0 an arbitrary point on the circle. Let A_1 be the point with arc $1/2$ away from A_0 in counter-clockwise direction, than move further $1/3$ to get A_2 and so on. Let σ-algebra $\mathscr{F}_n$ be defined as the σ-algebra generated by the first set of n segments $[A_{i-1}, A_i)$. Thus defined sequence of sigma algebras is clearly a filtration, $\mathscr{F}_n \subset \mathscr{F}_{n+1}$, $n \in \mathbb{N}$.

To define the random variable X_n, fix $n \in \mathbb{N}$ and observe that the interval $[A_{n-1}, A_n) \in \mathscr{F}_n$ may contain more boundary points $A_{i_1}, A_{i_2}, \dots, A_{i_m}$, $i_j < n-1$, $j = 1, \dots m$, which we denote $B_1, \dots B_m$, so that $[A_{n-1}, A_n)$ is the union of intervals $[A_{n-1}, B_1)$, $[B_1, B_2)$, $\dots$, $[B_m, A_n)$. For convenience, put $B_0 := A_{n-1}$ and $B_{m+1} := A_n$. For each interval $[B_i, B_{i+1})$, let C_i be the point on this interval such that $|B_iC_i| = |B_iB_{i+1}|/n$, where $|\cdot|$ denotes the arc length between the two points. We define the random variable X_n to be n on all intervals $[B_i, C_i)$ and 0 on the rest of the circle.

By construction, the probability of the set on which $X_n \neq 0$ is $\frac{1}{n^2}$. Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $(X_n)_{n=1}^{\infty}$ by the Borel-Cantelli lemma a.s. converges to $X \equiv 0$. Also, as

$$\mathbb{E}[X_n I_{\{X_n \ge m\}}] = \begin{cases} 0, & n < m, \\ \frac{1}{n}, & n \ge m, \end{cases} \qquad n, m = 1, 2, \dots,$$

$(X_n)_{n=1}^{\infty}$ is uniformly integrable. (Note that (10) cannot hold for any integrable Y.)

On the other hand, the conditional expectation $\mathbb{E}[X_n|\mathscr{F}_n]$ on each atomic interval $[B_i, B_{i+1})$ is 1, hence $\mathbb{E}[X_n|\mathscr{F}_n] = I_{[A_{n-1}, A_n)}$. Since $\liminf_{n\to\infty} I_{[A_{n-1}, A_n)} = 0$, $\limsup_{n\to\infty} I_{[A_{n-1}, A_n)} = 1$, the sequence $\mathbb{E}[X_n|\mathscr{F}_n]$ does not converge pointwise, while $\mathbb{E}[X|\mathscr{F}_\infty] = 0$ a.s..

$\square$

Remark 6. In Wei-an (1980) a related counterexample is given for the case where the filtration $\{\mathscr{F}_n\}$ is a constant and equal to $\mathscr{F}_\infty$.

Problem 27 (Williams (1991), Exercise E10.2.)**.** Let T be a stopping time on a filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_n, P)$ and supose that there exist $N \in \mathbb{N}$ and $\epsilon > 0$ such that

$$P(T \leq n + N | \mathscr{F}_n) > \epsilon, \quad \forall n \in \mathbb{N}.$$

Show that $E[T] < \infty$.

Solution. We have

$$E[T_{\{T>n+N\}} | \mathcal{F}_n] < 1 - \epsilon,$$

hence

$$E[T_{\{T>n+N\}} | \mathcal{F}_n] < (1 - \epsilon) I_{\{T>n\}}.$$

Putting $n = kN$ and taking expectation, and we obtain

$$P(T > kN) < (1 - \epsilon) P(T > (k+1)N),$$

giving $P(T > kN) < (1 - \epsilon)^k$. Summing over k yields $E[T/N] < \infty$. □

Problem 28. *Let ϕ be the Laplace transform of a nonnegative random variable X, $\phi(\lambda) = \mathbb{E}[\exp(-\lambda X)]$, $\lambda \geq 0$. Then for every $x, y > 0$*

$$|\phi(x) - \phi(y)| \leq \frac{1}{e} |\ln(y/x)|.$$

Solution. Since for $x \geq 0$ we have $x \exp(-x) \leq e^{-1}$, using the fact that X is nonnegative we obtain

$$\left| \frac{d\phi}{d\xi}(\xi) \right| = \mathbb{E}\left[\frac{X}{\lambda} \exp\left(-\xi \frac{X}{\lambda} \right) \right] \leq \frac{1}{e\xi}, \quad \xi > 0.$$

Thus, for every $0 < x < y$

$$|\phi(x) - \phi(y)| \leq \int_x^y \left| \frac{d\phi}{d\xi}(\xi) \right| d\xi \leq \frac{\ln(y/x)}{e}.$$

□

Problem 29 (Shenton's formula, Shenton (1954))**.** Show that

$$e^{\frac{x^2}{2}} \int_x^\infty e^{-\frac{t^2}{2}} \, dt = \int_0^\infty e^{-xt} e^{-\frac{t^2}{2}} \, dt, \quad x \in \mathbb{R}, \tag{11}$$

and conclude

$$\frac{1 - N(x)}{n(x)} < \frac{1}{x}, \quad x > 0.$$

Solution. Substituting $u = t + x$ in the second integral in (11) yields the desired equality. Thus

$$\frac{1-N(x)}{n(x)} = \int_0^\infty e^{-xt}e^{-\frac{t^2}{2}}\,dt < \int_0^\infty e^{-xt}\,dt = \frac{1}{x}, \quad x > 0.$$

□

Problem 30 (Modification of Problem X.129 in Адамов А. А. et al. (1912)). Let

$$F(x) := \left(\int_x^\infty e^{-t^2}\,dt\right)^2, \quad G(x) := \int_1^\infty \frac{e^{-x^2(t^2+1)}}{t^2+1}\,dt, \quad x \in \mathbb{R}.$$

Show that $F(x) = G(x)$, $x \in \mathbb{R}$, and use that to compute $\int_0^\infty e^{-t^2}\,dt$.

Solution. Using the Leibnitz rule we obtain

$$F'(x) = G'(x) = -2e^{-x^2}\int_x^\infty e^{-t^2}\,dt, \quad x \in \mathbb{R},$$

hence for some $C \in \mathbb{R}$

$$F(x) = G(x) + C, \quad x \in \mathbb{R}.$$

Since $\lim_{x\to\infty} F(x) = 0$ and

$$G(x) \le e^{-x^2}\int_1^\infty \frac{dt}{t^2+1} \to 0 \text{ as } x \to \infty,$$

we get $C = 0$. Thus $\int_0^\infty e^{-t^2}\,dt = \sqrt{F(0)} = \sqrt{G(0)} = \frac{\sqrt{\pi}}{2}$.

□

Problem 31 (Mitrinović (1962), Problem V.5.94). *For an integrable function f put $g(x) := f(x - 1/x)$, $x \neq 0$, $g(0) := 0$. Show that*

$$\int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} g(x)\,dx.$$

Solution. Denote by I the integral on the RHS and put $x = \cot u$. Then

$$I = \int_0^\pi \frac{f(2\cot 2u)}{\sin^2 u}\,du = \frac{1}{2}\int_0^{2\pi} \frac{f(2\cot u)}{\sin^2(u/2)}\,du.$$

Splitting the interval of integration and making the substitution $t = u - \pi$ in the second integral yields

$$I = \frac{1}{2}\int_0^\pi \frac{f(2\cot u)}{\sin^2(u/2)}\,du + \frac{1}{2}\int_0^\pi \frac{f(2\cot t)}{\cos^2(t/2)}\,dt = 2\int_0^\pi \frac{f(2\cot u)}{\sin^2 u}\,du,$$

which, after substituting $x = 2\cot u$, gives $I = \int_{\mathbb{R}} f(x)\,dx$.

□

Remark 7. This problem was also featured at 1976 and 1989 Student Olympiads in USSR. The stated reference provides a different solution.

Problem 32 (J. Liouville (Kudryavtsev et al. (2003), Problem 2.220))**.** *Show that for every continuous f we have*

$$\int_0^{\pi/2} f(\sin 2x)\cos x\,dx = \int_0^{\pi/2} f(\cos^2 x)\cos x\,dx.$$

Solution. Using

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx = \begin{cases} \frac{(m-1)!!(n-1)!!}{(m+n)!!}\frac{\pi}{2}, & n \text{ and } m \text{ are even,} \\ \frac{(m-1)!!(n-1)!!}{(m+n)!!}, & \text{else,} \end{cases}$$

we can prove the identity for the polynomials, whence the result follows by the density argument.

□

Remark 8. In Besge (1853) an alternative solution was given using the substitution $\sin 2x = \cos^2 u$. According to Lützen (2012), M. Besge was the pseudonym used by J. Liouville.

Problem 33 (Wolstenholme (1878))**.** *Show that for every continuous f we have*

$$\int_0^{\pi/2} f(\sin 2x)\cos x\,dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx.$$

Solution. Split the integral into the integrals over $[0,\pi/4]$ and $[\pi/4,\pi/2]$, and make substitutions $t = \pi/4 - x$ and $t = x - \pi/4$ respectively.

□

Problem 34 (AMM, Problem 11961)**.** *Evaluate*

$$\int_0^{\pi/2} \frac{\sin x}{1+\sqrt{\sin(2x)}}\,dx.$$

Solution. A direct application of Problem 32 yields $\pi/2 - 1$.

□

Problem 35 (Serret (1844), also Гюнтер Н.М. & Кузьмин Р.О. (2003), pg. 401)**.** *Evaluate*

$$\int_0^1 \frac{\ln(1+x)}{x^2+1}\,dx.$$

Solution. Denote by I the integral in question, and let $x = \tan\phi$. Then

$$I = \int_0^{\pi/4} \ln(\cos\phi + \sin\phi)\,d\phi - \int_0^{\pi/4} \ln(\cos\phi)\,d\phi.$$

Substituting $\varphi = \frac{\pi}{4} - \phi$ in the second integral we get

$$\int_0^{\pi/4} \ln(\cos\phi)\,d\phi = \int_0^{\pi/4} \ln\left((\cos\varphi + \sin\varphi)\frac{1}{\sqrt{2}}\right)\,d\varphi = \int_0^{\pi/4} \ln(\cos\varphi + \sin\varphi)\,d\varphi - \frac{\pi\ln 2}{8},$$

whence $I = \frac{\pi \ln 2}{8}$.

□

Problem 36 (AMM, Problem 11966)**.** *Prove that*

$$\int_0^1 \frac{x\ln(1+x)}{x^2+1}\,dx = \frac{\pi^2}{96} + \frac{(\ln 2)^2}{8}.$$

Solution. Put

$$F(a) := \int_0^1 f(x,a)\,dx, \quad f(x,a) := \frac{x\ln(1+ax)}{x^2+1}, \quad x, a \in [0,1].$$

Since f and $\frac{\partial f}{\partial a}(x,a) = \frac{x^2}{(1+ax)(x^2+1)}$ are continuous, hence also uniformly bounded on $[0,1]\times[0,1]$, F is differentiable on $[0,1]$, and, by the Leibniz rule, we have

$$F'(a) = \int_0^1 \frac{\partial f}{\partial a}(x,a)\,dx = \frac{\ln(1+a)}{a} - \int_0^1 \frac{dx}{(1+ax)(x^2+1)}. \tag{12}$$

Since

$$\int_0^1 \frac{dx}{(1+ax)(x^2+1)} = \frac{1}{a^2+1}\left[\frac{\pi}{4} - \frac{a}{2}\ln 2 + a\ln(1+a)\right],$$

integrating (12) over $[0,1]$ gives

$$F(1) = \int_0^1 \frac{\ln(1+a)}{a}\,da - \frac{\pi^2}{16} + \frac{(\ln 2)^2}{4} - F(1).$$

Since[2] $\int_0^1 \frac{\ln(1+a)}{a}\,da = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$, solving for $F(1)$ in the above equation gives the desired result.

[2]E.g. Exercise 4655 in Гюнтер Н.М. & Кузьмин Р.О. (2003).

$\square$

Problem 37 (The density of expected occupation time of the Brownian bridge has plateau, Chi et al. (2015))**.** *Show that the function*

$$F(x) := \int_0^1 \frac{1}{\sqrt{t(1-t)}} \exp\left\{-\frac{(x-t)^2}{2t(1-t)}\right\} dt$$

is constant for $x \in [0,1]$.

Solution. Put $t = \cos^2 u$. Then

$$F(x) = 2\int_0^{\frac{\pi}{2}} \exp\left\{-\frac{(\cos^2 u - x)^2}{2\sin^2 u \cos^2 u}\right\} du = \int_0^{\pi} \exp\left\{-\frac{(\cos u - 2x + 1)^2}{2\sin^2 u}\right\} du,$$

hence

$$F(x) = F(1-x),\ x \in [0,1] \quad \text{and} \quad F(1/2) = F(1) = F(0). \tag{13}$$

On the other hand, from the definition of the Hermite polynomials $\exp(xt - t^2/2) = \sum_{n=0}^{\infty} He_n(x)\frac{t^n}{n!}$ we obtain (taking $z^n \exp(-1/z^2) = 0$ for $z = \pm\infty$, $n \geq 0$)

$$\exp\left\{-\frac{(\cos u - x)^2}{2\sin^2 u}\right\} = \sum_{n=0}^{\infty} e^{-\frac{\cot^2 u}{2}} He_n(\cot u)\left(\frac{x}{\sin u}\right)^n \frac{1}{n!}, \quad x, u \in \mathbb{R}.$$

Integrating both sides gives

$$F\left(\frac{x+1}{2}\right) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!},\ |x| < 1, \quad a_n := \int_0^{\pi} e^{-\frac{\cot^2 u}{2}} He_n(\cot u)\frac{1}{\sin^n u}\, du, \tag{14}$$

where the interchanging the order of integration and summation on for $|x| < 1$ is justified using inequalities $\exp(-x^2/4)|He_n(x)| \leq \sqrt{n!}$, $\exp(-x^2/4)|x^n| \leq (2n)^{\frac{n}{2}} e^{-\frac{n}{2}} \sim 2^n\sqrt{n!}$, $x \in \mathbb{R}$. Since $He_{2k+1}(-x) = -He_{2k+1}(x)$, $k = 0, 1, \ldots$, it follows that the odd terms in (14) are zero. Next, with $t = \cot u$ we have

$$a_{2k} = \int_{-\infty}^{\infty} (t^2+1)^{k-1} e^{-\frac{t^2}{2}} He_{2k}(t)\, dt, \quad k = 0, 1, 2, \ldots$$

Since for $k > 0$ the polynomial $(t^2+1)^{k-1}$ belongs to the linear span of $He_0, He_1, \cdots, He_{2k-2}$, from the orthogonality of the Hermite polynomials it follows that $a_{2k} = 0$ for $k > 0$. Therefore F is constant on $(0,1)$, which together with (13) gives the desired result.

$\square$

Remark 9. We have shown that $F(x) = \int_{-\infty}^{\infty}(t^2+1)^{-1}e^{-\frac{t^2}{2}}He_0(t)\,dt$ for $x \in [0,1]$, which can be written as

$$\frac{F(x)}{\sqrt{2\pi}} = \mathbb{E}\left[\frac{1}{1+\xi^2}\right], \quad \xi \sim N(0,1),\ x \in [0,1].$$

This is the form given in Proposition 1 of Chi et al. (2015).

Problem 38 (A generalization of the "Problem of the Nile", SIAM Review, Problem 89-6). Suppose that P_θ, $\theta \in [0,1)$ is a family of probability measures on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ such that (i) P_0 is a product measure, (ii) P_θ is absolutely continuous wrt the Lebesgue measure for every $\theta \in [0,1)$, (iii) $\lim_{\theta\to 1-} P_\theta\big([a,b)\times[a,b)\big) = P_0\big([a,b)\times\mathbb{R}\big)$. Then if for some bounded $A \in \mathcal{B}(\mathbb{R}^2)$ the value of $P_\theta(A)$ does not depend on $\theta \in [0,1)$, the set A is of Lebesgue measure zero.

Solution. Fix a rectangle $[a,b)\times[c,d) \in \mathcal{B}(\mathbb{R}^2)$ and let $[a_i,b_i)\times[a_i,b_i)$, $i \in \mathbb{N}$, be a collection of pairwise disjoint squares, such that $[a,b)\times[c,d) = \cup_i[a_i,b_i)\times[a_i,b_i)$. In that case, from (i) we have $P_0\big(\cup_i[a_i,b_i)\times[a_i,b_i)\big) = \sum_i P_0\big([a_i,b_i)\times\mathbb{R}\big)^2$, which by (iii) and the fact that $P_\theta\big((\cup_i[a_i,b_i)\times[a_i,b_i)\big)$ is constant for $\theta \in [0,1)$, implies $\sum_i P_0\big([a_i,b_i)\times\mathbb{R}\big)^2 = \sum_i P_0\big([a_i,b_i)\times\mathbb{R}\big)$. So each $P_0\big([a_i,b_i)\times\mathbb{R}\big)$ is either zero or one, the former, being precluded by the boundedness of the intervals considered. Thus $P_0\big([a_i,b_i)\times\mathbb{R}\big) = 0$, $1 \le i \le n$, so the result holds for all rectangles in $\mathcal{B}(\mathbb{R}^2)$. Finally, the collection of sets $\{A \in \mathcal{B}(\mathbb{R}^2) : A \text{ is infinite or } A \text{ is a rectangle}\}$ is algebra such that the the smallest monotone class containing it, $\{A \in \mathcal{B}(\mathbb{R}^2) : A \text{ is infinite or the claim holds for } A\}$, is the whole $\mathcal{B}(\mathbb{R}^2)$, giving the desired result.

□

Remark 10. In the source above, the concrete form of P_θ was given as the two-dimensional Gaussian density

$$P_\theta(A) := \frac{1}{2\pi\sqrt{1-\theta^2}}\int_A \exp\left(\frac{2\theta xy - x^2 - y^2}{2(1-\theta^2)}\right)dxdy, \quad A \in \mathcal{B}(\mathbb{R}^2),$$

where the problem was attributed to Fisher (1936).

Problem 39 (AMM, Problem 11522). *Let E be the set of all real 4-tuples (a,b,c,d) such that if $x, y \in \mathbb{R}$, then $(ax+by)^2 + (cx+dy)^2 \le x^2+y^2$. Find the volume of E in $\mathbb{R}^4$.*

Solution. Introduce the polar coordinates

$$\begin{aligned} x &= \rho\cos\phi, \quad y = \rho\sin\phi \\ a &= \rho_1\cos\phi_1, \quad b = \rho_1\sin\phi_1 \\ c &= \rho_2\cos\phi_2, \quad d = \rho_2\sin\phi_2, \end{aligned}$$

where the angles are taken in $[0, 2\pi)$, $\rho, \rho_1, \rho_2 \geq 0$. Then we have

$$E = \{(\rho_1, \rho_2, \phi_1, \phi_2) \,:\, \Phi(\rho_1, \rho_2, \phi, \phi_1, \phi_2) \leq 1, \quad \forall \phi \in [0, 2\pi)\}$$

where

$$\Phi(\rho_1, \rho_2, \phi, \phi_1, \phi_2) := \rho_1^2(\cos\phi\cos\phi_1 + \sin\phi\sin\phi_1)^2 + \rho_2^2(\cos\phi\cos\phi_2 + \sin\phi\sin\phi_2)^2$$

After elementary transformations we see that $(\rho_1, \rho_2, \phi_1, \phi_2) \in E$ iff for every $\phi \in [0, 2\pi)$

$$\sqrt{(\rho_1^2\cos(2\phi_1) + \rho_2^2\cos(2\phi_2))^2 + (\rho_1^2\sin(2\phi_1) + \rho_2^2\sin(2\phi_2))^2}\,\sin(2\phi + \theta) + \rho_1^2 + \rho_2^2 \leq 2,$$

where

$$\sin\theta = \frac{\rho_1^2\cos(2\phi_1) + \rho_2^2\cos(2\phi_2)}{\sqrt{(\rho_1^2\cos(2\phi_1) + \rho_2^2\cos(2\phi_2))^2 + (\rho_1^2\sin(2\phi_1) + \rho_2^2\sin(2\phi_2))^2}},$$

hence $(\rho_1, \rho_2, \phi_1, \phi_2) \in E$ iff

$$\sqrt{(\rho_1^2\cos(2\phi_1) + \rho_2^2\cos(2\phi_2))^2 + (\rho_1^2\sin(2\phi_1) + \rho_2^2\sin(2\phi_2))^2} + \rho_1^2 + \rho_2^2 \leq 2,$$

so

$$E = \left\{(\rho_1, \rho_2, \phi_1, \phi_2) \,:\, \sqrt{\rho_1^4 + \rho_2^4 + 2\rho_1^2\rho_2^2\cos(2(\phi_1 - \phi_2))} + \rho_1^2 + \rho_2^2 \leq 2,\ \phi_1, \phi_2 \in [0, 2\pi)\right\}$$

The integral in question can now be written as

$$I = \int_0^{2\pi}\int_0^{2\pi} \Psi(\phi_1 - \phi_2)\, d\phi_1 d\phi_2,$$

where

$$\Psi(z) := \int\int_{\left\{(\rho_1,\rho_2)\,:\,\sqrt{\rho_1^4+\rho_2^4+2\rho_1^2\rho_2^2\cos(2z)}+\rho_1^2+\rho_2^2\leq 2\right\}} \rho_1\rho_2\, d\rho_1 d\rho_2.$$

Let now $u = \phi_1 + \phi_2$, $v = \phi_1 - \phi_2$. Then the square $[0, 2\pi]^2$ maps into the square with vertices at $(0, 0)$, $(2\pi, \pm 2\pi)$, and $(4\pi, 0)$. The Jacobian of the mapping is $\frac{1}{2}$, so since Ψ

is even we have

$$
\begin{aligned}
I &= 2\left(\int_0^{2\pi}\int_0^u \Psi(v)\frac{1}{2}\,dvdu + \int_{2\pi}^{4\pi}\int_0^{4\pi-u}\Psi(v)\frac{1}{2}\,dvdu\right)\\
&= \int_0^{2\pi}\int_0^u \Psi(v)\,dvdu + \int_0^{2\pi}\int_0^{2\pi-u}\Psi(v)\,dvdu\\
&= \int_0^{2\pi}\int_0^u \Psi(v)\,dvdu + \int_0^{2\pi}\int_u^{2\pi}\Psi(2\pi-v)\,dvdu\\
&= \int_0^{2\pi}\int_0^u \Psi(v)\,dvdu + \int_0^{2\pi}\int_u^{2\pi}\Psi(v)\,dvdu \qquad (15)\\
&= \int_0^{2\pi}\int_0^{2\pi}\Psi(v)\,dvdu = 2\pi\int_0^{2\pi}\Psi(v)\,dv,
\end{aligned}
$$

where in (15) we used $\Psi(2\pi - v) = \Psi(v)$.

Put $x = \rho_1^2$, $y = \rho_2^2$. Then we have

$$
I = \frac{\pi}{2}\int_0^{2\pi}\int\int_{\hat{E}_v} dxdydv,
$$

where $\hat{E}_v := \{(x,y) \,:\, x,y \geq 0,\ \sqrt{x^2+y^2+2\cos(2v)xy} + x + y \leq 2\}$. Since for $0 \leq y \leq x$ we have $2x = |x-y| + x + y \leq \sqrt{x^2+y^2+2\cos(2v)xy} + x + y \leq 2$, we conclude that $x, y \in [0,1]$, and then rewrite $\hat{E}_v$ as

$$
\hat{E}_v = \left\{(x,y) \,:\, 0 \leq x \leq 1,\ 0 \leq y \leq \frac{2(1-x)}{2-(1-\cos(2v))x}\right\}.
$$

We now have

$$
I = \pi\int_0^1\int_0^{2\pi}\frac{1-x}{2-(1-\cos(2v))x}\,dxdv = 2\pi\int_0^1\int_0^{\pi}\frac{1-x}{2-(1-\cos(2v))x}\,dxdv.
$$

It is elementary to check that

$$
\int_0^{\pi}\frac{d\phi}{\alpha\cos\phi + 1} = \frac{\pi}{\sqrt{1-\alpha^2}}, \quad |\alpha| < 1,
$$

so we finally obtain

$$
I = \pi^2\int_0^1\sqrt{1-x}\,dx = \frac{2}{3}\pi^2.
$$

□

2 Stochastic Analysis

Problem 40. Let $\{X_t,\ t \in [0,\infty)\}$ be a stochastic process on a measurable space $(\Omega, \mathscr{F})$ with state space a separable metric space E, and let for some $t \in [0,\infty)$ Z be an $\mathscr{F}^X_{t_+}$-measurable random variable taking values in a Polish metric space (S,r). Then there exists an $\mathscr{E}^{[0,\infty)}$-measurable function $\Psi : E^{[0,\infty)} \to S$ such that for every $s > t$ we have $Z = \Psi(X(\cdot \wedge s))$.

Solution. Since Z is $\mathscr{F}^X_{t+\frac{1}{n}}$ measurable for every $n \in \mathbb{N}$, we have that there exists a sequence of $\mathscr{E}^{\mathbb{N}}$-measurable functions $\Phi_n : E^{\mathbb{N}} \to S$ so that

$$Z = \Phi_n(X_{t_1^n}, X_{t_2^n}, \ldots)$$

for some sequence $(t_i^n)_{i=1}^\infty,\ t_i^n \in [0, t+\frac{1}{n}]$, $\forall i \in \mathbb{N}$.

Let us define $\Psi_n : E^{[0,\infty)} \to S$ as $\Psi_n := \Phi_n \circ \pi_{\mathbf{t^n}}$, where, as usual, $\pi_{\mathbf{t^n}} : E^{[0,\infty)} \to E^{\mathbb{N}}$ is the projection mapping $e \to (e(t_1^n), e(t_2^n), \ldots)$. From the definition of Ψ_n we immediately have that Ψ_n is $\mathscr{E}^{[0,\infty)}$ measurable and

$$Z = \Psi_n(X) = \Psi_n(X(\cdot \wedge t + \frac{1}{n})).$$

Since all Ψ_n's are $\mathscr{E}^{[0,\infty)}$ measurable, the set $A := \{e \in E^{[0,\infty)} \ :\ \lim \Psi_n(e) \text{ exists}\}$ belongs to $\mathscr{E}^{[0,\infty)}$[3], and thus some fixed $a \in S$ the function

$$\Psi(e) := \begin{cases} \lim \Psi_n(e) & \text{for } e \in A, \\ a & \text{otherwise} \end{cases}$$

is $\mathscr{E}^{[0,\infty)}$ measurable. Also, for every $s > t$ and every $n \in \mathbb{N}$ so that $\frac{1}{n} < s - t$ we have $Z = \Psi_n(X(\cdot \wedge s))$, which in turn implies that $Z = \Psi(X(\cdot \wedge s))$.

$\square$

Problem 41. Let $\{\mathscr{F}_t\}$ be a filtration in the measurable space $(\Omega, \mathscr{F})$, and let T and S be $\{\mathscr{F}_t\}$-stopping times. Then

$$\mathscr{F}_{S\vee T} = \pi\{\mathscr{F}_S, \mathscr{F}_T\}. \tag{16}$$

Solution. We first show that

$$\mathscr{F}_{S\vee T} = \{A \cup B \ :\ A \in \mathscr{F}_T,\ B \in \mathscr{F}_S\}, \tag{17}$$

and then complete the proof by showing that the right-hand sides of the above relation and (16) are equal.

[3] Using the fact that S is complete, it easily follows that $A = \cap_{k\in\mathbb{N}} \cup_{n\in\mathbb{N}} \cap_{m\in\mathbb{N}} \{e \in E^{[0,\infty)} : r(\Psi_n(e), \Psi_{n+m}(e)) < \frac{1}{k}\}$.

Let $C \in \{A \cup B \,:\, A \in \mathscr{F}_S,\ B \in \mathscr{F}_T\}$ be arbitrary. Then $C = A \cup B$ for some $A \in \mathscr{F}_S$ and $B \in \mathscr{F}_T$. Since

$$\mathscr{F}_S,\ \mathscr{F}_T \subset \mathscr{F}_{S\vee T}, \tag{18}$$

we have that $A, B \in \mathscr{F}_{S\vee T}$, which in turn implies that $C = A \cup B \in \mathscr{F}_{S\vee T}$. Thus

$$\{A \cup B \,:\, A \in \mathscr{F}_T,\ B \in \mathscr{F}_S\} \subset \mathscr{F}_{S\vee T}.$$

For the opposite set inclusion, we proceed as follows. Let $C \in \mathscr{F}_{S\vee T}$ be arbitrary. Since we clearly have

$$C = C \cap \{S \geq T\} \cup C \cap \{T \geq S\},$$

in order to establish the result it suffices to show that $C \cap \{S \geq T\} \in \mathscr{F}_S$ and $C \cap \{T \geq S\} \in \mathscr{F}_T$. Since we have that $\{S \geq T\} \in \mathscr{F}_{S\vee T}$, by the choice of C we have that $C \cap \{S \geq T\} \in \mathscr{F}_{S\vee T}$. Hence, by definition we have

$$C \cap \{S \geq T\} \cap \{S \vee T \leq t\} \in \mathscr{F}_t,\ \forall t \in [0, \infty],$$

which clearly implies that

$$C \cap \{S \geq T\} \cap \{S \leq t\} \in \mathscr{F}_t,\ \forall t \in [0, \infty],$$

and consequently $C \cap \{S \geq T\} \in \mathscr{F}_S$. By the analogous derivation we get that $C \cap \{T \geq S\} \in \mathscr{F}_T$, which in view of the previous remark establishes (17).

To complete the proof, define

$$\pi^c\{\mathscr{F}_S, \mathscr{F}_T\} \triangleq \{A \subset \Omega \,:\, A^c \in \pi\{\mathscr{F}_S, \mathscr{F}_T\}\}.$$

We first observe that from (18) and the definition of a π-class, it follows that $\pi\{\mathscr{F}_S, \mathscr{F}_T\} \subset \mathscr{F}_{S\vee T}$, and consequently

$$\pi^c\{\mathscr{F}_S, \mathscr{F}_T\} \subset \mathscr{F}_{S\vee T}. \tag{19}$$

To establish the opposite set inclusion, fix some arbitrary $C \in \mathscr{F}_{S\vee T}$. Then, according to (17), for some $A \in \mathscr{F}_S$ and $B \in \mathscr{F}_T$ we have $C = A \cup B$. Since $A^c \in \mathscr{F}_S$ and $B^c \in \mathscr{F}_T$, we have that $A^c \cap B^c \in \pi(\mathscr{F}_S, \mathscr{F}_T)$, and consequently that $C = (A^c \cap B^c)^c \in \pi^c\{\mathscr{F}_S, \mathscr{F}_T\}$. Therefore, we have $\mathscr{F}_{S\vee T} \subset \pi^c\{\mathscr{F}_S, \mathscr{F}_T\}$, which together with (19) implies that

$$\mathscr{F}_{S\vee T} = \pi^c\{\mathscr{F}_S, \mathscr{F}_T\}. \tag{20}$$

Since $\pi^c\{\mathscr{F}_S, \mathscr{F}_T\}$ is a σ-algebra, we clearly have

$$\pi^c\{\mathscr{F}_S, \mathscr{F}_T\} = \{A \subset \Omega \,:\, A^c \in \pi^c\{\mathscr{F}_S, \mathscr{F}_T\}\},$$

which together with the definition of $\pi^c\{\mathscr{F}_S, \mathscr{F}_T\}$ implies that[4]

$$\pi\{\mathscr{F}_S, \mathscr{F}_T\} = \pi^c\{\mathscr{F}_S, \mathscr{F}_T\}.$$

Hence, from (20) we have that

$$\mathscr{F}_{S\vee T} = \pi\{\mathscr{F}_S, \mathscr{F}_T\}.$$

$\square$

Remark 11. Since from (16) and (18) we have

$$\mathscr{F}_{S\vee T} = \sigma\{\mathscr{F}_S, \mathscr{F}_T\},$$

it follows that for arbitrary $\{\mathscr{F}_t\}$-stopping times S and T we have

$$\sigma\{\mathscr{F}_S, \mathscr{F}_T\} = \pi\{\mathscr{F}_S, \mathscr{F}_T\}.$$

Problem 42. Let $\{\mathscr{F}_n\}$ be a filtration in the measurable space $(\Omega, \mathscr{F})$, and let T be an $\{\mathscr{F}_n\}$-stopping time. Define

$$\mathscr{F}_T^* \triangleq \left\{ \bigcup_{1\leq n\leq\infty} A_n \, : \, A_n \in \mathscr{F}_n \cap \{T = n\}, \ n = 0, 1, \ldots, \infty \right\}.$$

Then we have $\mathscr{F}_T^* = \mathscr{F}_T$.

Solution. Let $A \in \mathscr{F}_T^*$ be arbitrary. Then since $\{T = n\} \in \mathscr{F}_n, \ n = 0, 1, \ldots, \infty$, we have that $A \cap \{T = n\} \in \mathscr{F}_n, \ n = 0, 1, \ldots, \infty$, which by definition implies that $A \in \mathscr{F}_T$. Hence we have $\mathscr{F}_T^* \subset \mathscr{F}_T$.

For the opposite set inclusion, fix some arbitrary $A \in \mathscr{F}_T$ and put

$$A_n \triangleq A \cap \{T = n\}, \ n = 0, 1, \ldots, \infty.$$

Then we have that $A = \cup_{1\leq n\leq\infty} A_n$, and (since $A \in \mathscr{F}_T$) $A_n \in \mathscr{F}_n \cap \{T = n\}, \ n = 0, 1, \ldots, \infty$. Hence we have $\mathscr{F}_T \subset \mathscr{F}_T^*$, which establishes the result.

$\square$

Problem 43 (Revuz & Yor (1999), Exercise IV.1.35)**.** If $X \in M^c(\{\mathscr{F}_t\}, P)$ and it is a Gaussian process, then its quadratic variation is deterministic, i.e. there is a function f on $\mathbf{R}_+$ such that $[X](t) = f(t)$ a.s.

[4]Let $\mathscr{B}, \mathscr{C}$ be two arbitrary collections of subsets of Ω such that $\{A \subset \Omega : A^c \in \mathscr{B}\} = \{A \subset \Omega : A^c \in \mathscr{C}\}$. We claim that $\mathscr{B} = \mathscr{C}$. Let $X \in \mathscr{B}$ be arbitrary. Then $X^c \in \{A \subset \Omega : A^c \in \mathscr{C}\}$, which implies that $X \in \mathscr{C}$. Hence, $\mathscr{B} \subset \mathscr{C}$. Since the opposite set inclusion follows by the analogous argument, we have that the two collections of sets are equal.

Solution. Let us denote the variance of X_t by $f(t)$. Since $X \in M^c(\{\mathscr{F}_t\}, P)$ it has uncorrelated increments. Since X is also Gaussian so are its (zero-mean) increments, and thus we have that X is a process with independent increments. Let us define

$$\tilde{\mathscr{F}}_t := \sigma\{\mathscr{F}_t^X, \mathscr{Z}^P[\mathscr{F}]\}.$$

Then we have a.s.

$$E[(X_t - X_s)^2 | \tilde{\mathscr{F}}_t] = E[(X_t - X_s)^2].$$

Clearly $X \in M_2^c(\{\tilde{\mathscr{F}}_t\}, P)$, so we have a.s.

$$\begin{aligned} E[X_t^2 - X_s^2 | \tilde{\mathscr{F}}_t] &= E[(X_t - X_s)^2 | \tilde{\mathscr{F}}_t] = E[(X_t - X_s)^2] \\ &= E[X_t^2 - X_s^2] = f(t) - f(s),\ 0 \le t < s < \infty. \end{aligned}$$

Thus the process $\{(X_t^2 - f(t), \tilde{\mathscr{F}}_t); t \in [0, \infty)\}$ is an L_2-martingale.

Furthermore, for each $a \in \mathbf{R}_+$ the submartingale $X^2(t \wedge a)$ converges in L_1 to $X^2(a)$, which according to the Scheffé's Lemma (see Williams (1991)) implies continuity of f on $\mathbf{R}_+$. Thus $\{(X_t^2 - f(t), \tilde{\mathscr{F}}_t); t \in [0, \infty)\}$ is a continuous L_2-martingale, and

$$[X](t) = f(t) - f(0)\ a.s.$$

$\square$

Problem 44. Let X be a stochastic process and T a stopping time of $\{\mathscr{F}_t^X\}$. Suppose that for some $\omega_1, \omega_2 \in \Omega$ we have $X_t(\omega_1) = X_t(\omega_2)$ for all $t \in [0, T(\omega_1)] \cap [0, \infty)$. Then $T(\omega_1) = T(\omega_2)$.

Solution. By the definition of a stopping time we have that

$$\{T(\omega) = T(\omega_1)\} \in \mathscr{F}_{T(\omega_1)}, \tag{21}$$

which, implies that there exist a sequence $\{t_i\},\ t_i \in [0, T(\omega_1)] \cap [0, \infty),\ i = 1, 2, \ldots$ and a $\mathscr{B}(\mathbf{R}^\infty)$-measurable mapping $\Psi : \mathbf{R}^\infty \to \bar{\mathbf{R}}$, such that

$$I_{\{T(\omega) = T(\omega_1)\}} = \Psi(X_{t_1}, X_{t_2}, \ldots).$$

Using the initial assumption we now have

$$\begin{aligned} 1 = I_{\{T(\omega) = T(\omega_1)\}}(\omega_1) &= \Psi(X_{t_1}(\omega_1), X_{t_2}(\omega_1), \ldots) = \\ &= \Psi(X_{t_1}(\omega_2), X_{t_2}(\omega_2), \ldots) = I_{\{T(\omega) = T(\omega_1)\}}(\omega_2), \end{aligned}$$

and the result follows.

$\square$

Remark 12. It is easy to see that the discrete-time analog (obtained by replacing "t" with "n") of Problem 44 holds. To establish the result we follow the same idea : the relation (21) implies the existence of a $\mathscr{B}(\mathbf{R}^{T(\omega_1)})$-measurable mapping $\Psi : \mathbf{R}^{T(\omega_1)} \to \bar{\mathbf{R}}$ such that we have

$$I_{\{T(\omega)=T(\omega_1)\}} = \Psi(X_1, X_2, \ldots, X_{T(\omega_1)}).$$

Similarly as above, the result follows from the equality

$$\begin{aligned} 1 = I_{\{T(\omega)=T(\omega_1)\}}(\omega_1) &= \Psi(X_1(\omega_1), X_2(\omega_1), \ldots, X_{T(\omega_1)}(\omega_1)) = \\ &= \Psi(X_1(\omega_2), X_2(\omega_2), \ldots, X_{T(\omega_1)}(\omega_2)) = I_{\{T(\omega)=T(\omega_1)\}}(\omega_2). \end{aligned}$$

Problem 45 (Galmarino's test (discrete-time))**.** Let X be a discrete-parameter stochastic process, and let T be a stopping time of $\{\mathscr{F}_t^X\}$. If Z is an $\mathscr{F}_T$-measurable random variable, then there exists a $\mathscr{B}(\mathbf{R}^\infty)$-measurable mapping $\Psi : \mathbf{R}^\infty \to \bar{\mathbf{R}}$, such that

$$Z = \Psi(X_{1\wedge T}, X_{2\wedge T}, \ldots).$$

Solution. It suffices to establish the result for the special case $Z = I_A,\ A \in \mathscr{F}_T$, the general result then being an immediate consequence the monotone class theorem and Doob's theorem.

Let $A \in \mathscr{F}_T$ be arbitrary. Then by the definition of a pre-σ-algebra, we have

$$A \cap \{T = n\} \in \mathscr{F}_n^X,\ \forall n = 1, 2, \ldots,$$

which by Doob's Theorem implies existence of a sequence of functions $\{\Phi_n\}$, $\Phi_n : \mathbf{R}^n \to \mathbf{R}$, such that for each $n \in \mathbf{N}$ Φ_n is $\mathscr{B}(\mathbf{R}^n)$-measurable and we have

$$I_A I_{\{T=n\}} = \Phi_n(X_1, X_2, \ldots, X_n).$$

Hence, we have

$$\begin{aligned} I_A I_{\{T \le n\}} &= \textstyle\sum_{i=1}^n \Phi_i(X_1, X_2, \ldots, X_i) = \sum_{i=1}^n \Phi_i(X_1, X_2, \ldots, X_i) I_{\{T=i\}} \\ &= \textstyle\sum_{i=1}^n \Phi_i(X_{1\wedge T}, X_{2\wedge T}, \ldots, X_{i\wedge T}) I_{\{T=i\}}. \end{aligned} \tag{22}$$

We now observe the following. Since for each $n = 1, 2, \ldots, \infty$ we have $\{T = n\} \in \mathscr{F}_n^X$, by Exercise 1.1 we have that there exists a set $\Gamma_n \in \mathscr{B}(\mathbf{R}^n)$ such that $\{T = n\} = \{(X_1, X_2, \ldots, X_n) \in \Gamma_n\}$; this in turn implies that $\{T = n\} \subset \{(X_{1\wedge T}, X_{2\wedge T}, \ldots, X_{n\wedge T}) \in \Gamma_n\}$. The opposite set inclusion holds as well: let us suppose that there exists an $\omega_1 \in \{T \neq n\} \cap \{(X_{1\wedge T}, X_{2\wedge T}, \ldots, X_{n\wedge T}) \in \Gamma_n\}$. Since on $\{T = n\}$ we have $X_i = X_{i\wedge T},\ i = 1, 2, \ldots, n$, from the definition of Γ_n it follows that there exists an $\omega_2 \in \{T = n\}$ such that

$$(X_{1\wedge T}(\omega_1), X_{2\wedge T}(\omega_1), \ldots, X_{n\wedge T}(\omega_1)) = (X_{1\wedge T}(\omega_2), X_{2\wedge T}(\omega_2), \ldots, X_{n\wedge T}(\omega_2)),$$

which is by Remark 12 a contradiction. Therefore, from (22), according to Doob's Theorem it follows that there exists a sequence $\{\Psi_n\}_{n=1}^{n=\infty}$ of functions $\Psi_n : \mathbf{R}^n \to \mathbf{R}$ such that each Ψ_n is $\mathscr{B}(\mathbf{R}^n)$ measurable and we have

$$I_A I_{\{T\leq n\}} = \sum_{i=1}^{n} \Psi_i(X_{1\wedge T}, X_{2\wedge T}, \ldots, X_{i\wedge T}), \quad \forall n \in \mathbf{N}$$

and

$$I_A I_{\{T=\infty\}} = \Psi_\infty(X_{1\wedge T}, X_{2\wedge T}, \ldots). \tag{23}$$

Hence, since for each $n \in \mathbf{N}$ the function $I_A I_{\{T\leq n\}}$ is $\sigma\{\mathbf{X}^T\}$ measurable, the limit $I_A I_{\{T\leq\infty\}}$ is $\sigma\{\mathbf{X}^T\}$ measurable as well. Together with (23), this implies that I_A is $\sigma\{\mathbf{X}^T\}$ measurable, and the result follows by another application of Doob's Theorem.

□

Problem 46. Let $\{X_t,\ t \in [0,\infty)\}$ be a process ona a measurable space $(\Omega, \mathscr{F})$, and let T be an $\{\mathscr{F}_t^X\}$-stopping time. Then T is also an $\{\mathscr{F}_t^{X^T}\}$-stopping time.

Solution. Let $t \in [0,\infty)$ be arbitrary. To establish the result, we show that $\{T > t\} \in \mathscr{F}_t^{X^T}$.

Then there exist a sequence $\{t_i\}_{i=1}^{\infty},\ t_i \in [0,t]$ and a set $\Gamma \in \mathscr{B}(R^\infty)$ such that

$$\{T > t\} = \{(X_{t1}, X_{t2}, \ldots) \in \Gamma\}. \tag{24}$$

Since on the set $\{T > t\}$ we have $X_{t_i\wedge T} = X_{t_i}$, from (24) we have

$$\{T > t\} \subset \{(X_{t_1\wedge T}, X_{t_2\wedge T}, \ldots) \in \Gamma\}. \tag{25}$$

We now show that the opposite set inclusion holds as well. Suppose, on the contrary, that there exists $\omega_1 \in \{T \leq t\}$ such that $(X_{t_1\wedge T}(\omega_1), X_{t_2\wedge T}(\omega_1), \ldots) \in \Gamma$. Then, by the definition of Γ, there exists $\omega_2 \in \{T > t\}$ such that

$$(X_{t_1\wedge T}(\omega_1), X_{t_2\wedge T}(\omega_1), \ldots) = (X_{t_1\wedge T}(\omega_2), X_{t_2\wedge T}(\omega_2), \ldots).$$

But, since $T(\omega_1) \neq T(\omega_2)$, according to Problem 44, this is a contradiction. Hence, we have set equality in (25), and thus T is an $\{\mathscr{F}_t^{X^T}\}$-stopping time.

□

Problem 47. Let $\{X_n,\ n = 0, 1, \ldots\}$ be a process on a measurable space $(\Omega, \mathscr{F})$, and let T be an $\{\mathscr{F}_n^{X^T}\}$-stopping time. Then T is also an $\{\mathscr{F}_n^X\}$-stopping time.

Solution. We use induction to show that for each $n = 0, 1, \ldots$, we have $\{T = n\} \in \mathscr{F}_n^X$.

Since we have $\{T = 0\} \in \sigma\{X_0\}$, the claim holds for $n = 0$. Suppose that the claim holds for some $n > 0$. Since

$$I_{\{T=n+1\}} = \Phi(X_{1\wedge T}, X_{2\wedge T}, \ldots, X_{n+1\wedge T}). \tag{26}$$

Since for each integer $0 \leq i \leq n+1$ we have

$$X_{i\wedge T} = X_i I_{\{T>i\}} + \sum_{j=1}^{n} X_j I_{\{T=j\}},$$

by the inductive hypothesis we have that $X_{i\wedge T}$ is $\mathscr{F}_i^X$ measurable for each integer $0 \leq i \leq n+1$, so by (26) we have that $I_{\{T=n+1\}}$ is $\mathscr{F}_{n+1}^X$ measurable, and the claim holds for $n+1$.

From the above we have that $\{T < \infty\}$ is $\mathscr{F}^X$-measurable, so we have $\{T = \infty\} \in \mathscr{F}^X$.

$\square$

Problem 48 (Vrkoč et al. (1978), Exercise 1)**.** Let

$$Y_0 = \frac{1}{\sqrt{2\pi}} W_{2\pi},\ Y_n(t) = \frac{1}{\sqrt{\pi}} \int_0^t \cos ns\, dW_s,\ X_n(t) = \frac{1}{\sqrt{\pi}} \int_0^t \sin ns\, dW_s,\ n = 1, 2, \ldots.$$

Then $Y_0, Y_n(2\pi), X_m(2\pi),\ n \in \mathbf{N} \cup \{0\},\ m \in \mathbf{N}$ are mutually independent random variables having Normal distribution $N(0, 1)$.

Solution. Let $u, v \in \mathbf{R}$; $n, m \in \mathbf{N}$ be arbitrary. By Itô's formula we have

$$\begin{aligned}
&\exp[\imath(uX_m(t) + vY_n(t))] = \\
&\quad 1 + \imath u \int_0^t \exp[\imath(uX_m(s) + vY_n(s))] dX_n(s) + \imath v \int_0^t \exp[\imath(uX_m(s) + vY_n(s))] dY_n(s) \\
&\quad -\frac{1}{2} uv \int_0^t \exp[\imath(uX_m(s) + vY_n(s))] d[X_m, Y_n] - \frac{1}{2} u^2 \int_0^t \exp[\imath(uX_m(s) + vY_n(s))] d[X_m] \\
&\quad -\frac{1}{2} v^2 \int_0^t \exp[\imath(uX_m(s) + vY_n(s))] d[Y_n],\ \forall t \in [0, \infty).
\end{aligned}$$

We observe that the second and third term on the right-hand side are continuous martingales; thus, taking expectations on both sides and denoting $\exp[\imath(uX_m(t) +$

$vY_n(t))]$ by $f(t)$, we get

$$\begin{aligned} E[f(t)] &= 1 - \frac{1}{2\pi} uvE[\int_0^t f(s)\sin ms \cos ns \, ds] - \frac{1}{2\pi} u^2 E[\int_0^t f(s)\sin^2 ms \, ds] \\ &- \frac{1}{2\pi} v^2 E[\int_0^t f(s)\cos^2 ns \, ds]. \end{aligned}$$

By Fubini's Theorem we now have

$$\begin{aligned} E[f(t)] &= 1 - \frac{1}{2\pi} uv \int_0^t E[f(s)]\sin ms \cos ns \, ds - \frac{1}{2\pi} u^2 \int_0^t E[f(s)]\sin^2 ms \, ds \\ &- \frac{1}{2\pi} v^2 \int_0^t E[f(s)]\cos^2 ns \, ds, \end{aligned}$$

so we have that $E[f(t)]$ is the solution of the differential equation

$$\frac{dE[f(t)]}{dt} = -\frac{1}{2\pi} E[f(t)](uv \sin mt \cos nt + u^2 \sin^2 mt + v^2 \cos^2 nt), \; E[f(0)] = 1.$$

Thus we have

$$E[f(t)] = \exp[-\frac{1}{2\pi}\int_0^t (uv \sin ms \cos ns + u^2 \sin^2 ms + v^2 \cos ns) ds],$$

which implies

$$E[f(2\pi)] = \exp[-\frac{1}{2\pi}\int_0^{2\pi} u^2 \sin^2 ms + v^2 \cos ns \, ds].$$

From the above we immediately have

$$E[\exp[\imath(uX_m + vY_n)]] = E[\exp[\imath uX_m]]E[\exp[\imath vY_n]],$$

and (after evaluating the integrals)

$$E[\exp[\imath uX_n]] = E[\exp[\imath uY_n]] = \exp[-\frac{1}{2}u^2].$$

The argument for the case $m = 0$ is identical.

$\square$

Problem 49 (Kailath-Segal identity, Revuz & Yor (1999), Exercise IV.3.30)**.** Let $X \in M^c_{loc}(\{\mathscr{F}_t\}, P)$ such that $X_0 = 0$, and define the iterated stochastic integrals of X by

$$I_0 = 1, \;\; I_n = (I_{n-1} \bullet X), \; n \in \mathbf{N}.$$

Then we have

$$nI_n = I_{n-1}X - I_{n-2}[X], \; n \geq 2.$$

Solution. For n=2 the claim directly follows from the integration-by-parts formula. Suppose that the claim holds for $n-1, n \geq 3$:

$$(n-1)I_{n-1} = I_{n-2}X - I_{n-3}[X]. \tag{27}$$

Applying integration by parts we get

$$\begin{aligned} I_{n-1}X &= \\ &= (I_{n-1} \bullet X) + (X \bullet I_{n-1}) + [I_{n-1}, X] \\ &= I_n + (I_{n-2}X \bullet X) + [I_{n-1}, X]. \end{aligned}$$

By the inductive hypothesis (27) we now have

$$I_{n-1}X = I_n + (I_{n-3}[X] \bullet X) + [I_{n-1}, X]. \tag{28}$$

Since

$$[I_{n-1}, X] = [(I_{n-2} \bullet X), X] = (I_{n-2} \bullet [X])$$

and

$$(I_{n-3}[X] \bullet X) = ([X] \bullet I_{n-2}),$$

from (28) it follows that

$$nI_n = I_{n-1}X - (I_{n-2} \bullet [X]) - ([X] \bullet I_{n-2}),$$

which, after another application of the inegration-by-parts formula, gives

$$nI_n = I_{n-1}X - I_{n-2}[X],$$

and the proof by induction is complete.

$\square$

Problem 50 (Chung & Williams (1990), Exercise 5.5.6)**.** Let $\{(X(t), \mathscr{F}_t),\ t \in [0,\infty)\}$ and $\{(Y(t), \mathscr{F}_t)\ t \in [0,\infty)\}$ be continuous local martingales on $(\Omega, \mathscr{F}, P)$, and write

$$\mathscr{F}_X = \sigma\{X(t),\ t \in [0,\infty)\}, \quad \mathscr{F}_Y = \sigma\{Y(t),\ t \in [0,\infty)\}.$$

If the σ-algebras $\mathscr{F}_X$ and $\mathscr{F}_Y$ are independent, and $\mathscr{F}_0$ includes all P-null events in $\mathscr{F}$, then a.s.:

$$[X,Y](t) = 0,\ \forall t \in [0,\infty).$$

Solution. Let us first establish the result for the case of continuous bounded martingales. Fix some arbitrary $X, Y \in M^c(\{\mathscr{F}_t\}, P)$ for which $\mathscr{F}^X$ and $\mathscr{F}^Y$ are independent and whose magnitudes are bounded by some $C \in (0,\infty)$. With $\tau_k^n =$

$k2^{-n}$, $\forall k, n = 0, 1, \ldots$ let us define $\forall n = 0, 1, \ldots,$ $\forall t \in [0, \infty)$ (for the sake of brevity we suppress the dependence on ω)

$$\begin{aligned} A_n(t) &= \sum_{0 \le k < \infty} [X(t \wedge \tau^n_{k+1}) - X(t \wedge \tau^n_k)][Y(t \wedge \tau^n_{k+1}) - Y(t \wedge \tau^n_k)], \\ \Delta X^n_k(t) &= X(t \wedge \tau^n_{k+1}) - X(t \wedge \tau^n_k),\ \Delta Y^n_k = Y(t \wedge \tau^n_{k+1}) - Y(t \wedge \tau^n_k), \\ \Delta [X]^n_k(t) &= [X](t \wedge \tau^n_{k+1}) - [X](t \wedge \tau^n_k),\ \Delta [Y]^n_k(t) = [Y](t \wedge \tau^n_{k+1}) - [Y](t \wedge \tau^n_k). \end{aligned}$$

So $A_n(t) = \sum_{i=0}^{\infty} \Delta X^n_i(t) \Delta Y^n_i(t)$, and $\forall t \in [0, \infty)$ we have

$$\begin{aligned} &E[A^2(t)] = \\ &\quad = E[\sum_{i=0}^{\infty} (\Delta X^n_i(t) \Delta Y^n_i(t))^2 + 2 \sum_{i \ne j} \Delta X^n_i(t) \Delta X^n_j(t) \Delta Y^n_i(t) \Delta Y^n_j(t)] \\ &\quad = \sum_{i=0}^{\infty} E[(\Delta X^n_i(t) \Delta Y^n_i(t))^2] + 2 \sum_{i \ne j} E[\Delta X^n_i(t) \Delta X^n_j(t) \Delta Y^n_i(t) \Delta Y^n_j(t)]. \end{aligned}$$

Using the independence we now have

$$\begin{aligned} &E[A^2(t)] = \\ &\quad = \sum_{i=0}^{\infty} E[(\Delta X^n_i(t))^2] E[(\Delta Y^n_i(t))^2] + 2 \sum_{i \ne j} E[\Delta X^n_i(t) \Delta X^n_j(t)] E[\Delta Y^n_i(t) \Delta Y^n_j(t)]. \end{aligned}$$

Furthermore, since we have

$$E[(\Delta X^n_i(t))^2] = E[\Delta [X]^n_i(t)], \quad E[\Delta X^n_i(t) \Delta X^n_j(t)] = 0,\ i \ne j,$$

from the above, using the Cauchy-Schwarz inequality and Monotone Convergence theorem, we get

$$\begin{aligned} &E[A^2(t)] = \qquad (29) \\ &\quad = \sum_{i=0}^{\infty} E[\Delta [X]^n_i(t)] E[\Delta [Y]^n_i(t)] \le \sqrt{\sum_{i=0}^{\infty} (E[\Delta [X]^n_i(t)])^2} \sqrt{\sum_{i=0}^{\infty} (E[\Delta [Y]^n_i(t)])^2} \\ &\quad \le \sqrt{\sum_{i=0}^{\infty} E[(\Delta [X]^n_i(t))^2]} \sqrt{\sum_{i=0}^{\infty} E[(\Delta [Y]^n_i(t))^2]} = \sqrt{E[\sum_{i=0}^{\infty} (\Delta [X]^n_i(t))^2]} \sqrt{E[\sum_{i=0}^{\infty} (\Delta [Y]^n_i(t))^2]}. \end{aligned}$$

But

$$E[\sum_{i=0}^{\infty} (\Delta [X]^n_i(t))^2] \le E[[X](t) \sup_k \Delta [X]^n_k(t)],$$

and

$$[X](t) \sup_k \Delta[X]^n_k(t) \le [X]^2(t),$$

so in view of uniform continuity of $[X](\cdot)$ on $[0,t]$ and Dominated Convergence Theorem (note that $E[[X]^2(t)] < \infty$), we have $\lim_{n\to\infty} E[[X](t)\sup_k \Delta[X]^n_k(t)] = 0$. Since the analogous derivations obviously hold for Y, by (30) we have that $\{A_n(t)\}$ converges in probability to zero for all $t \in [0,\infty)$, which (since $\mathscr{F}_0$ contains all P-null events in $\mathscr{F}$) implies that a.s.:

$$[X,Y](t) = 0, \;\; \forall t \in [0,\infty).$$

Let's now consider the general case. Suppose that X and Y are continuous local martingales for which $\mathscr{F}^X$ and $\mathscr{F}^Y$ are independent. Then, taking in Example 3.5.5 (b) the filtration to be $\{\mathscr{F}^X_t\}$ and Γ to be $(-\infty,-k] \cup [k,\infty)$, we get that

$$S_k := \inf\{t \in [0,\infty) \;:\; |X_t(\omega)| \ge k\}$$

defines an $\{\mathscr{F}^X_t\}$-stopping time for each $k \in \mathbf{N}$. Moreover, it follows that $\{S_k,\;\, k = 1,2,\ldots\}$ is a localizing sequence for $\{(X_t,\mathscr{F}_t); t \in [0,\infty)\}$ for which $X^{S_k} \in M^c_2(\{\mathscr{F}_t\},P)$ for each $k \in \mathbf{N}$. Repeating the above procedure for Y, we come up with the localizing sequence $\{R_k,\;\; k = 1,2,\ldots\}$ for Y with the analogous properties.

In view of the fact that $\{S_k\}$ is an $\{\mathscr{F}^X_t\}$-stopping time for each $k \in \mathbf{N}$ we have that $X^{S_k}(t)$ is $\mathscr{F}^X$ measurable for each $t \in [0,\infty)$ and each $k \in \mathbf{N}$. Since the analogous observation obviously holds for Y, we have that $\mathscr{F}^{X^{S_k}}$ and $\mathscr{F}^{Y^{S_k}}$ are independent for each $k \in \mathbf{N}$. Thus, since $X^{S_k}, Y^{R_k} \in M^c(\{\mathscr{F}_t\},P)$ and are bounded, according to the first part of the proof, we have that $X^{S_k}Y^{R_k} \in M^c(\{\mathscr{F}_t\},P)$. This by Corollary 3.5.20 implies that $X^{S_k\wedge R_k}Y^{S_k\wedge R_k} \in M^c(\{\mathscr{F}_t\},P)$ for each $k \in \mathbf{N}$ as well, so in view of the uniqueness of co-quadratic variation process we have a.s.:

$$[X,Y](t) = 0, \;\; \forall t \in [0,\infty).$$

$\square$

Problem 51. Let $X_t^{(i)}$, $i = 1, 2$ be square-integrable martingales such that

$$E\left[\left(X_t^{(1)}\right)^2\right] \le E\left[\left(X_t^{(2)}\right)^2\right], \quad t \ge 0. \tag{30}$$

Then

$$E\left[\left(\int_0^t X_u^{(1)}\,du\right)^2\right] \le E\left[\left(\int_0^t X_u^{(2)}\,du\right)^2\right], \quad t \ge 0.$$

Solution. Put $\bar{X}_t^{(i)} := \int_0^t X_u^{(i)}\,du$, $i = 1, 2$. Repeated application of Itô's formula gives

$$\left(\bar{X}_t^{(i)}\right)^2 = 2\int_0^t X_u^{(i)}\bar{X}_u^{(i)}\,du = 2\int_0^t \int_0^u \left(X_z^{(i)}\right)^2 dz\,du + 2\int_0^t \int_0^u \bar{X}_z^{(i)}\,dX_z^{(i)}\,du, \; t \ge 0, \; i = 1, 2.$$

In view of (30), the result now follows by taking expectation and using Fubini's theorem together with the fact that $X_t^{(i)}$, $i = 1, 2$ are martingales.

$\square$

Problem 52 (Bond volatility at maturity is zero)**.** Considered a collection of adapted, bounded processes r_t, $\sigma(t, T)$, $T \in [0, \infty]$ on a filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$, together with a continuous curve $P(0, T)$, $T \in [0, \infty]$. Let for every $T > 0$

$$dP(t, T) = P(t, T)\left(r_t\,dt + \sigma(t, T)\,dW_t\right), \quad 0 \le t \le T. \tag{31}$$

Show that if $P(T, T) = 1$ for every $T \in [0, \infty)$, then $\sigma(T, T) = 0$ for every $T \in [0, \infty)$.

Solution. For every fixed $T > 0$ we have

$$P(t, T) = P(0, T)B_t \exp\left(-\frac{1}{2}\int_0^t \sigma^2(u, T)\,du + \int_0^t \sigma(u, T)\,dW_u\right), \quad 0 \le t \le T,$$

where $B_t := \exp\left(\int_0^t r_u\,du\right)$. Since $P(T, T) = 1$, we have for $t \in [0, T]$

$$\frac{1}{B_t} = P(0, t)\exp\left(-\frac{1}{2}\int_0^t \sigma^2(u, t)\,du + \int_0^t \sigma(u, t)\,dW_u\right),$$

so

$$\frac{d}{dt}\log(B_t) = \cdots du - \sigma(t, t)\,dW_t,$$

whence the result follows since $\log(B_t)$ is a process of bounded variation.

$\square$

Remark 13. The equation (31) defines a Price Based Formulation of the HJM model of interest rates presented in Carverhill (1995).

Problem 53. For $A < B$, an arbitrary partition $A = t_0 < t_1 < \cdots < t_N = B$, and the standard Wiener process W_t, compute

$$E\left[\left(\sum_{i=0}^{N-1} W_{t_i} - \frac{1}{t_{i+1} - t_i}\int_{t_i}^{t_{i+1}} W_u\,du\right)^2\right].$$

Solution. Straightforward calculation using independence of the Wiener process increments yields $\frac{B-A}{3}$ (so, in particular, the result does not depend on the underlying partition).

□

Problem 54 (Lucic (2020): A variation on the theme of Carr & Pelts (2015))**.** Let

$$S_T = \frac{\omega(Z + \tau(T))}{\omega(Z)}, \quad T \geq 0, \tag{32}$$

where $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function with $\tau(0) = 0$, and Z is a random variable with a unimodal pdf $\omega(\cdot)$. Show that $C(K,T) := \mathbb{E}[(S_T - K)^+]$, $T, K \geq 0$, is a non-decreasing function of T for every $K \geq 0$, with $\lim_{T\to\infty} C(K,T) = 1$, $K \geq 0$.

Solution. We have

$$C(K,T) = \int_{\mathbb{R}} \left[\frac{\omega(z + \tau(T))}{\omega(z)} - K\right]^+ \omega(z)\,dz = \int_{\mathbb{R}} [\omega(z + \tau(T)) - K\omega(z)]^+\,dz,\ T \geq 0. \tag{33}$$

Fix $K \geq 1$, $T_0 \geq 0$, and let z^* denote the global maximum of $\omega(\cdot)$. Since for $z \geq z^*$ we have $\omega(z + \tau(T_0)) - K\omega(z) \leq (1-K)\omega(z) \leq 0$, from (33)

$$C(K,T_0) = \int_{-\infty}^{z^*} [\omega(z + \tau(T_0))) - K\omega(z)]^+\,dz.$$

For $T_1 > T_0$, $\Delta := \tau(T_1) - \tau(T_0) \geq 0$ we have

$$\begin{aligned} C(K,T_1) &\geq \int_{-\infty}^{z^*+\Delta} [\omega(z + \tau(T_1)) - K\omega(z)]^+\,dz \\ &= \int_{-\infty}^{z^*} [\omega(z + \tau(T_0)) - K\omega(z - \Delta)]^+\,dz \geq C(K,T_0), \end{aligned}$$

where the last inequality follows from the fact that $\omega(\cdot)$ is increasing on $(-\infty, z^*]$.

For $0 < K < 1$, note that $\tilde{\omega}(x) := \omega(-x)$, $x \in \mathbb{R}$ is a unimodal distribution, so applying the first part of the proof we conclude that

$$T \mapsto \int_{\mathbb{R}} \left[\tilde{\omega}(z+\tau(T)) - \frac{1}{K}\tilde{\omega}(z)\right]^+ d = \frac{1}{K}\int_{\mathbb{R}} [K\omega(z) - \omega(z+\tau(T))]^+ \, dz = 1 - \frac{1}{K} + \frac{C(K,T)}{K}$$

is increasing. The limiting behavior follows by the Lebesgue dominated convergence theorem.

□

Remark 14. *Carr & Pelts (2015) (see also Tehranchi (2020), and Antonov et al. (2019) for a probabilistic interpretation) postulate $\omega(\cdot)$ of the form*

$$\omega(z) = e^{-h(z)}, \quad z \in \mathbb{R},$$

for some convex $h(\cdot)$, with $\lim_{z\to\pm\infty} h'(z) = \pm\infty$. This in nontrivial cases necessarily implies a unique zero of $h'(\cdot)$, hence unimodality of $\omega(\cdot)$.

Problem 55. *[Basket option formula, Dhaene & Goovaerts (1996)]* Suppose X and Y are nonnegative integrable random variables with cumulative density function $F(x, y)$. Show that for $K \geq 0$,

$$E[(X+Y-K)^+] = E[X+Y] - K + \int_0^K F(u, K-u)\, du.$$

Solution.

Write

$$E[(X+Y-K)^+] = E[X+Y] - K + E[(K-X-Y)^+]. \tag{34}$$

For $x, y \geq 0$ we have

$$[K-x-y]^+ = \int_0^K I_{\{x+y\leq u\leq K\}}\, du = \int_0^K I_{\{u\geq x\}} I_{\{u\leq K-y\}}\, du,$$

which after integrating wrt the CDF of (X, Y) gives

$$E[(K-X-Y)^+] = \int_0^K F(u, K-u)\, du. \tag{35}$$

Combining (34) and (35) gives the desired result.

□

Remark 15. Since $(x, y) \mapsto C(x, y) \equiv [x+y-1]^+$ is a copula, $(x, y) \mapsto B_K(x, y) \equiv [x+y-K]^+$ is is a quasimonotone (hence measure-inducing) map with $\frac{\partial^2 B_K(x,y)}{\partial x \partial y} = \delta_{\{x+y=K\}}$, we can interpret the statement of Exercise 55 as the integration-by-parts formula

$$\int_{\mathbb{R}^+\times\mathbb{R}^+} B_K(x, y)\, F(dx, dy) = \text{boundary terms} + \int_{\mathbb{R}^+\times\mathbb{R}^+} F(x, y)\, B_K(dx, dy).$$

Integration-by-parts for higher dimensions was developed in Young (1917) (see also Hobson (1957)), and was used in a similar context for generalizing Hoeffding's lemma in Brendan (2009).

Remark 16. Related to Exercise 55 is Theorem 2.1 of Quesada-Molina (1992), which implies the following: for nonnegative integrable random variables X and Y with cumulative density function $F(x, y)$, random variable $\tilde{Y}$ independent of X, having the same distribution as Y we have

$$E[(X+Y-K)^+] = E[(X+\tilde{Y}-K)^+] + \int_0^K F(x, K-x)\, dx - \int_0^K F_1(x)\, dx \int_0^K F_2(x)\, dx, K \geq 0,$$

where $F_1(x)$ and $F_2(x)$ are CDFs of X and Y respectively.

Remark 17. From Exercise 15 it follows that the cdf of $X + Y$ can be written as

$$F_{X+Y}(K) = \int_0^K F_x(K-u, u)\, dx = \int_0^K F_y(u, K-u)\, du.$$

Problem 56. *[Spread option formula]* Suppose X and Y are nonnegative integrable random variables with cumulative density function $F(x, y)$. Show that for $K \geq 0$,

$$E[(X - Y - K)^+] = \int_0^\infty F_2(u) - \tilde{F}(K+u, u)\, du,$$

where $F_2(x)$ is the CDF of Y and $\tilde{F}(x, y) := F(x-, y)$.

Solution. We have

$$(X - Y - K)^+ = \int_0^\infty I_{\{u \geq Y+K\}} I_{\{u \leq X\}}\, du = \int_0^\infty \left(I_{\{Y \leq u-K\}} - I_{\{Y \leq u-K\}} I_{\{X<u\}}\right) du,$$

which after integrating wrt the CDF of (X, Y), gives

$$E[(X - Y - K)^+] = \int_0^\infty F_2(u-K) - \tilde{F}(u, u-K)\, du = \int_0^\infty F_2(u) - \tilde{F}(K+u, u)\, du.$$

$\square$

Remark 18. The spread option formula of Exercise 56 appears in Andersen & Piterbarg (2010) (see also Berrahoui (2004)).

Problem 57. Suppose X is a nonnegative, integrable random variable with strictly increasing, continuous cdf. Show that

$$\mathrm{VaR}_\alpha(X) = \operatorname*{arg\,min}_{K\in\mathbb{R}} \left\{ K + \frac{1}{1-\alpha} E(X-K)^+ \right\}.$$

Solution. Let F be the cdf of X. Then

$$E(X-K)^+ = E[X] - K + \int_0^K F(u)\,du,$$

so to establish the claim we need to show that

$$\mathrm{VaR}_\alpha(X) = \operatorname*{arg\,min}_{K\in\mathbb{R}} \left\{ -\alpha K + \int_0^K F(u)\,du \right\}.$$

Since the mapping

$$\phi(K) : K \mapsto -\alpha K + \int_0^K F(u)\,du$$

is differentiable, with strictly increasing, continuous derivative

$$\phi'(K) = -\alpha + F(K),$$

it reaches its unique minimum at $F^{-1}(\alpha)$.

$\square$

Remark 19. Lemma 3.2 of Embrechts & Wang (2015) using different methods shows that for an arbitrary integrable random variable

$$\mathrm{VaR}_\alpha(X) \in \operatorname*{arg\,min}_{K\in\mathbb{R}} \left\{ K + \frac{1}{1-\alpha} E(X-K)^+ \right\}.$$

Problem 58 (Value-at-Risk(VaR) inequality)**.** Suppose X and Y are IID random variables. Then

$$\mathrm{VaR}_\alpha(X+Y) \geq \mathrm{VaR}_{\sqrt{\alpha}}(X), \quad \alpha \in (0,1).$$

Solution. This follows by integrating the inequality

$$I_{x+y\leq K} \leq I_{x\leq K} I_{y\leq K}, \quad x,y,K \in \mathbb{R}$$

wrt the cdf of (X,Y).

$\square$

Problem 59 (Unique characterization of VaR via loss contributions)**.** Let L_1 and L_2 be integrable random variables with strictly positive probability density function $f(l_1, l_2)$. Suppose that for some differentiable $R : \mathbb{R}^2 \to \mathbb{R}$ such that $R(0,0) = 0$ we have

$$\frac{\partial R}{\partial x_i}(x_1, x_2) = \mathbb{E}\big[L_i | x_1 L_1 + x_2 L_2 = R(x_1, x_2)\big], \; x_1, x_2 \in \mathbb{R}, \; i = 1, 2. \tag{36}$$

Then there exists $\alpha \in (0,1)$ such that $R(x_1, x_2) = \mathrm{VaR}_\alpha(x_1 L_1 + x_2 L_2)$ for every $x_1, x_2 \in \mathbb{R}$.

Solution. Fix $x_1, x_2 \neq 0$. Since

$$\mathbb{E}\big[L_1 | x_1 L_1 + x_2 L_2 = R(x)\big] = \frac{\int_{\mathbb{R}} l_1 f\big(l_1, (R(x) - x_1 l_1)/x_2\big)\, dl_1}{\int_{\mathbb{R}} f\big(l_1, \big(R(x) - x_1 l_1\big)/x_2\big)\, dl_1},$$

from (36) we have

$$\int_{\mathbb{R}} \left(\frac{\partial R}{\partial x_1}(x) - l_1 \right) f\big(l_1, (R(x) - x_1 l_1)/x_2\big)\, dl_1 = 0. \tag{37}$$

On the other hand,

$$\mathbb{P}(x_1 L_1 + x_2 L_2 \leq R(x)) = \int_{\mathbb{R}} \int_{-\infty}^{(R(x) - x_1 l_1)/x_2} f(l_1, l_2)\, dl_2\, dl_1,$$

so differentiating the both sides (using the integrability of L_1 to justify differentiation under the integral sign, e.g. (Billingsley 1995)[Theorem 16.8., Problem 16.5.]), from (37) we get

$$\frac{\partial}{\partial x_1} \mathbb{P}(x_1 L_1 + x_2 L_2 \leq R(x)) = 0, \; x_1, x_2 \neq 0, \tag{38a}$$

and analogously

$$\frac{\partial}{\partial x_2} \mathbb{P}(x_1 L_1 + x_2 L_2 \leq R(x)) = 0, \; x_1, x_2 \neq 0. \tag{38b}$$

Using the continuity of R, from (38) it now follows that there exists $\alpha \in [0,1]$ such that

$$\mathbb{P}(x_1 L_1 + x_2 L_2 \leq R(x)) = \alpha, \; x_1, x_2 \in \mathbb{R}. \tag{39}$$

Also, since for $K \in \mathbb{R}$

$$\frac{\partial}{\partial K} \mathbb{P}(x_1 L_1 + x_2 L_2 \leq K) = \frac{1}{x_2} \int_{\mathbb{R}} f\big(l_1, (K - x_1 l_1)/x_2\big)\, dl_1, \; x_2 \neq 0,$$

with the analogous relationship holding for x_2, from the strict positivity of f we conclude that for $(x_1, x_2) \neq 0$ the cdf of $x_1 L_1 + x_2 L_2$ is strictly increasing. Therefore, the claim follows from (39), the definition of VaR_α, and the strict monotonicity of the cdf of $x_1 L_1 + x_2 L_2$.

□

Remark 20. If one drops the condition of strict positivity of f, the same argument yields existence of $\alpha \in [0, 1]$ such that $R(x) = \mathrm{VaR}_\alpha(x_1 L_1 + x_2 L_2)$ for every $x = (x_1, x_2) \in \mathbb{R}^2$ such that $R(x)$ is not the point of constancy of the cdf of $x_1 L_1 + x_2 L_2$.

Remark 21. The equality in the opposite direction, i.e. showing that the partial derivatives of VaR satisfy equality given in the statement of the exercise, was established in Gourieroux et al. (2000). In Qian (2006) it was used to give a financial interpretation of to the concept of risk contribution.

Problem 60. Let S_t be a geometric Brownian motion

$$S_t = \exp\left(-\frac{1}{2}\sigma^2 t + \sigma W_t\right), \quad t \geq 0, \tag{40}$$

and for some $0 \leq s \leq t$ put

$$X_u = \begin{cases} 1 & u \in [0, s], \\ S_u / S_s & u \in (s, t] \end{cases}.$$

Show that for deterministic $a_t \geq 0$

$$E\left[\max_{u \in [0,t]} (a_u X_u)\right] \leq E\left[\max_{u \in [0,t]} (a_u S_u)\right]. \tag{41}$$

Solution.[K. Zhao] Let

$$\hat{S}_u = \begin{cases} X_u e^{-\frac{1}{2}\sigma^2 u + \sigma Z_u} & u \in [0, s], \\ X_u e^{-\frac{1}{2}\sigma^2 s + \sigma Z_s} & u \in (s, t] \end{cases},$$

where Z is a Wiener process independent of W. Then $\hat{S}$ and S have the same distribution, so with $u^* := \arg\max_{u \in [0,t]} (a_u X_u)$ for (41) we have

$$\begin{aligned}
\text{RHS} &\geq E\left[a_{u^*} X_{u^*} e^{-\frac{1}{2}\sigma^2 u^* + \sigma Z_{u^*}} I_{u^* \in [0,s]}\right] + E\left[a_{u^*} X_{u^*} e^{-\frac{1}{2}\sigma^2 s + \sigma Z_s} I_{u^* \in (s,t]}\right] \\
&= E\left[a_{u^*} I_{u^* \in [0,s]} E\left[e^{-\frac{1}{2}\sigma^2 u^* + \sigma Z_{u^*}} \middle| u^*\right]\right] + E\left[a_{u^*} X_{u^*} I_{u^* \in (s,t]}\right] E\left[e^{-\frac{1}{2}\sigma^2 s + \sigma Z_s}\right] \\
&= E\left[a_{u^*} I_{u^* \in [0,s]}\right] + E\left[a_{u^*} X_{u^*} I_{u^* \in (s,t]}\right] \\
&= \text{LHS}.
\end{aligned}$$

$\square$

Problem 61 (Duriez' inequality)**.** Let S_t be a geometric Brownian motion

$$S_t = \exp\left(-\frac{1}{2}\sigma^2 t + \sigma W_t\right), \quad t \geq 0, \tag{42}$$

and put $X_{s,t} := \max_{u\in[s,t]} S_u$, $\hat{X}_{s,t} := \max_{u\in[s,t]} a_u S_u$, $0 \leq s \leq t$, where $a_t \geq 0$ are deterministic. Show that

$$E\left[\frac{\hat{X}_{0,t}}{X_{0,s}}\middle| \mathcal{F}_s\right] \leq E[\hat{X}_{0,t}] \quad a.s., \quad 0 \leq s \leq t.$$

Solution. We have

$$\begin{aligned} E\left[\frac{\hat{X}_{0,t}}{X_{0,s}}\middle| \mathcal{F}_s\right] &= E\left[\frac{\max(\hat{X}_{0,s}, \hat{X}_{s,t})}{X_{0,s}}\middle| \mathcal{F}_s\right] \\ &\leq E\left[\max\left(\max_{u\in[0,s]} a_u, \frac{\hat{X}_{s,t}}{S_s}\right)\middle| \mathcal{F}_s\right] \\ &= E\left[\max\left(\max_{u\in[0,s]} a_u, \max_{u\in[s,t]} a_u S_{u-s}\right)\right], \end{aligned}$$

whence the proof follows from Problem 60.

$\square$

Problem 62. Let N_t be a continuous-time taking values in $\{a, b\}$, $a, b \in \mathbb{R}$. Show that $N_t = N_0$ a.s.If N_t is corlor, it is constant a.s..

Solution. If $a = b$ the claim is trivial. Suppose $a \neq b$ and put $\hat{N}_t := (N_t - a)/(b - a)$, so that $\hat{N}_t \in \{0, 1\}$. Then, since $\hat{N}_s^2 = \hat{N}_s$ for every $s \geq 0$, we have

$$E[(\hat{N}_t - \hat{N}_0)^2] = E[\hat{N}_t] - 2E[\hat{N}_t\hat{N}_0] + E[\hat{N}_0] = E[\hat{N}_t] - 2E[\hat{N}_0] + E[\hat{N}_0] = 0, \quad t \geq 0.$$

Therefore $N_t = N_0$ a.s. If N_t is corlor, by the right-continuity we further conclude that it is constant a.s..

$\square$

Problem 63 (Feller)**.** Consider the Feller diffusion

$$dv_t = \lambda(\bar{v} - v_t)\, dt + \eta\sqrt{v_t}\, dW_t, \quad v_0, \lambda, \bar{v}, \eta > 0.$$

Show that the origin is unattainable for v_t if the Feller condition $2\lambda\bar{v} > \eta^2$ holds.

Solution. We put $x_t = -\ln(v_t)$ and show that x_t does not explode. To this end, note that with

$$Af(x) := \left[\left(\frac{1}{2}\eta^2 - \lambda\bar{v}\right)e^x + \lambda\right]\partial f(x) + \frac{1}{2}\eta^2 e^x \partial^2 f(x), \quad f \in C^\infty(\mathbb{R})$$

the process

$$f(x_t) - \int_0^t Af(x_u)\,du, \quad t \geq 0$$

is a martingale for every $f \in C^\infty(\mathbb{R})$ with $\|f\|, \|Af\| < \infty$.

Fix a $g \in C^\infty(\mathbb{R})$ such that[5] $g(x) = 1$ for $x \leq 1$, $g(x) = 0$ for $x \geq 2$, and g is strictly decreasing on $[1,2]$. Let $g_n(x) := g(x/n)$, $n \in \mathbb{N}$. Then

$$nAg_n(x) = \left[\left(\frac{1}{2}\eta^2 - \lambda\bar{v}\right)\partial g(x/n) + \frac{1}{2n}\eta^2\partial^2 g(x/n)\right]e^x + \lambda\partial g(x/n),$$

whence, since $\eta^2/2 - \lambda\bar{v} < 0$, there exists[6] n_0 such that $n\inf_x Ag_n(x) > -\lambda\|\partial g\|$ for every $n > n_0$. Therefore, since $\|g_n\|, \|Ag_n\| < \infty$ for every n, and g_n is a bounded sequence converging pointwise to 1 on $\mathbb{R}$ with $\inf_{n>n_0}\inf_x Ag_n(x) > -\infty$, the result follows by Proposition 4.3.9 of Ethier & Kurtz (1986).

$\square$

Problem 64. Suppose X_t, $t \geq 0$ is a corlol supermartingale taking values in $[0,\infty]$, such that $X_0 \neq \infty$ a.s.. Let T be the first contact time with ∞,

$$T := \inf\{t : X_t = \infty \text{ or } X_{t-} = \infty\} \quad (\inf\{\emptyset\} = \infty).$$

Then $T = \infty$ a.s..

Solution. For $\alpha > 0$ let

$$f_\alpha(x) := \begin{cases} 1 - e^{-\alpha x}, & x \in [0,\infty), \\ 1, & x = \infty. \end{cases}$$

Fix $\alpha > 0$ and note that in view of the optional stopping theorem and Jensen's inequality the stopped process $f_\alpha(X_t^T)$, $t \geq 0$ is a corlol, bounded supermartingale. Therefore, $2E[f_\alpha(X_0)] \geq E[f_\alpha(X_t^T) + f_\alpha(X_{t-}^T)]$ for every $t > 0$, whence by taking $\alpha \to 0+$, $t \to \infty$ we obtain $0 \geq P(T < \infty)$.

[5]E.g. the classical *molifier* can be used: $g(x) := e^{\frac{(x-1)^2}{(x-1)^2-1}}$ for $x \in [1,2]$.

[6]If $C := \min_{x\in[1,2]}(Ag(x) - \lambda\partial g(x)) \geq 0$ the claim is trivial, so suppose $C < 0$, and let $x^* := \arg\min_{x\in[1,2]}(Ag(x) - \lambda\partial g(x))$. Then n_0 can be taken as $\left\lceil \frac{2\eta^2\partial^2 g(x^*)}{(2\lambda\bar{v}-\eta^2)\partial g(x^*)} \right\rceil$ (note that $\partial^2 g(x^*) < 0$, due to strict monotonicity of g, implies $\partial g(x^*) < 0$).

$\square$

Problem 65 (Feller's test for explosion)**.** Suppose $A \subset C(\bar{\mathbb{R}}) \times B(\bar{\mathbb{R}})$ is such that there exists $\lambda \geq 0$, $C \in \mathbb{R}$ and $f \in \mathscr{D}(A)$ with $f(\infty) = \infty$, $f(x) \geq 0$, $Af(x) \leq \lambda f(x)$ for every $x \in [0, \infty)$. Let X_t be a $\bar{\mathbb{R}}$-valued, corlol process with $X_0 = 0$, and put $T_n := \inf\{t : X_t \geq n \;\text{ or }\; X_{t-} \geq n\}$, $n \in \mathbb{N}$. Suppose that X_t is a solution of the stopped martingale problem for (A, T_n) for every $n \in \mathbb{N}$. Then X_t is a.s. $\mathbb{R} \cup \{-\infty\}$-valued.

Solution. Fix an arbitrary $n \in \mathbb{N}$. Let $T_0 := \inf\{t : X_t \leq 0 \;\text{ or }\; X_{t-} \leq 0\}$, $Y_t := X(T_0 \wedge T_n \wedge t)$, $t \geq 0$. It is routine to verify (see, e.g. Proposition 4.3.1 of Ethier & Kurtz (1986)) that the process

$$e^{-\lambda t} f(Y_t) + \int_0^t e^{-\lambda u}(\lambda f(Y_u) - Af(Y_u))\, du, \quad t \geq 0$$

is a corlol martingale. Since $Af(x) \leq \lambda f(x)$, $f(x) \geq 0$, $\forall x \in [0, \infty)$, the process $e^{-\lambda t} f(Y_t)$ is a nonnegative (corlol) supermartingale, and the claim follows from Problem 64.

$\square$

Problem 66 (Martingale lemma of Kurtz & Ocone (1988))**.** Let $\{(U_t, \mathscr{F}_t),\ t \in [0, \infty)\}$ and $\{(V_t, \mathscr{F}_t)\ t \in [0, \infty)\}$ be two continuous adapted processes on $(\Omega, \mathscr{F}, P)$ such that

$$N_t := U_t - \int_0^t V_s\, ds, \quad t \geq 0$$

is an $\{\mathscr{F}_t\}$-martingale. If $\{\mathscr{G}_t\}$ is a filtration with $\mathscr{G}_t \subset \mathscr{F}_t$, $t \geq 0$, show that the process

$$M_t := E[U_t|\mathscr{G}_t] - \int_0^t E[V_s|\mathscr{G}_s]\, ds, \quad t \geq 0$$

is a $\{\mathscr{G}_t\}$-martingale.

Solution. Using repeadately the tower propery of conditional expectation we obtain

$$E[M_t - M_s|\mathscr{G}_s] = E[N_t - N_s|\mathscr{G}_s] = EE[[N_t - N_s|\mathscr{F}_s]|\mathscr{G}_s] = 0, \quad 0 \leq s \leq t.$$

$\square$

Problem 67 (Kurtz & Ocone (1988))**.** Let $\{(U_t, \mathscr{F}_t),\ t \in [0, \infty)\}$ and $\{(V_t, \mathscr{F}_t)\ t \in [0, \infty)\}$ be two continuous adapted processes on $(\Omega, \mathscr{F}, P)$. Show that if

$$f(U_t) - \int_0^t V_s f'(U_s)\, ds, \quad t \geq 0$$

is an $\{\mathscr{F}_t\}$-martingale for each $f \in \hat{C}^{(1)}(\mathbb{R})$, then a.s.

$$U_t = U_0 + \int_0^t V_s\, ds, \quad t \geq 0.$$

Solution. Put $T_n := \inf\{t \geq 0 : |U_t| > n\}$. Then for every $f \in \hat{C}^{(1)}(\mathbb{R})$

$$M_t^{(f)} := f(U_t^{T_n}) - \int_0^t V_s I_{\{s \leq T_n\}} f'(U_s^{T_n})\, ds, \quad t \geq 0 \tag{43}$$

is an $\{\mathscr{F}_t\}$-martingale, hence by the density argument on the compact $[-n, n]$ the expression in (43) is also martingale for every $f \in C^{(2)}(\mathbb{R})$. Therefore, by Proposition VII.2.4 of Revuz & Yor (1999) it follows that $[M^{(f)}]_t \equiv 0$, so

$$U_t^{T_n} = U_0 + \int_0^{t \wedge T_n} V_s\, ds, \quad t \geq 0,$$

whence the result follows by taking $n \to \infty$.

□

Problem 68. Let E be a metric space, $A \subset B(E) \times B(E)$, and suppose that X_t is a solution of the martingale problem for A with sample paths in $D_E[0, \infty)$. Show that, if $\Gamma \in \mathscr{B}(E)$ is such that $(I_\Gamma, 0)$ belongs to the bp-closure of $A \cap (\bar{C}(E) \times B(E))$, then $P(\{X \in D_E[0, \infty)\} \Delta \{X_0 \in \Gamma\}) = 0$.

Solution. Fix an $\{\mathscr{F}_t^X\}$-stopping time T with $P\{T < \infty\} = 1$, and put $N_t \equiv I_\Gamma(X_t)$. By the Optional Sampling Theorem for every $(f, g) \in A \cap (\bar{C}(E) \times B(E))$

$$f(X_t^T) - \int_0^{t \wedge T} g(X_s)\, ds, \quad t \geq 0$$

is an $\{\mathscr{F}_t^X\}$-martingale. Since $(I_\Gamma, 0)$ belongs to the bp-closure of $A \cap (\bar{C}(E) \times B(E))$, N_t^T is an $\{\mathscr{F}_t^X\}$-martingale, so by Problem 62 $N_t^T = N_0$ a.s., whence by letting $t \to \infty$ we obtain $N_T = N_0$ a.s.. Therefore, by the uniqueness of the Optional Projection (see, e.g. Remark 2.4.3 of Ethier & Kurtz (1986)), the bounded $\{\mathscr{F}_t^X\}$-optional process N_t is indistinguishable from the constant process equal to N_0 for all $t \geq 0$, hence the result.

□

Remark 22. Problem 68 provides an alternative proof of Proposition 4.3.10 of Ethier & Kurtz (1986).

Problem 69 (Markovian Projection)**.** Consider a real-valued process X wich on some filtered probability space satisfies

$$dX_t = b_t\, dt + \sigma_t\, dW_t$$

for some adapted, real-valued and uniformly bounded processes b and σ . Put

$$\hat{b}(t,x) = E[b_t|X_t = x], \quad \hat{\sigma}(t,x) = \sqrt{E[\sigma_t^2|X_t = x]}, \quad t \in [0,\infty),\ x \in \mathbb{R}.$$

Then there exists a weak solution $\hat{X}$ of

$$d\hat{X}_t = \hat{b}(t,\hat{X}_t)\, dt + \hat{\sigma}(t,\hat{X}_t)\, dW_t \tag{44}$$

which has the same one-dimensional distributions as X.

Solution. By Ito formula, for any $f \in \bar{C}^2(\mathbb{R})$, $t \geq 0$ we have

$$E[f(X_t)] = E[f(X_0)] + \int_0^t E\left[b_u + \sigma_u^2/2\right] du = E[f(X_0)] + \int_0^t E\left[\hat{b}(u,X_u) + \hat{\sigma}^2(u,X_u)/2\right] du.$$

Define $\mu : [0,\infty) \mapsto \mathcal{P}(\mathbb{R})$ via $\mu_t f \equiv \int_{\mathbb{R}} f(x)\mu_t(dx) = E[f(X_t)]$, $f \in \bar{C}(\mathbb{R})$, so that we have

$$\mu_t f = \mu_0 f + \int_0^t \mu_u(\mathcal{L}_u f)\, du, \quad t \geq 0, \tag{45}$$

where

$$\mathcal{L}_t f = \hat{b}(t,x)\frac{\partial f}{\partial x} + \frac{1}{2}\hat{\sigma}^2(t,x)\frac{\partial^2 f}{\partial x^2}, \quad t \in [0,\infty),\ x \in \mathbb{R}.$$

By Theorem 1.2. of Kurtz (2011) there exists a weak solution of (44) with $E[f(\hat{X}_t)] = \mu_t f$, $t \geq 0$, i.e. with the same one-dimensional distributions as X.

□

Remark 23. This result first appeared in Gyöngy (1986) under the condition of bounded b and bounded, non-degenerate σ. An alternative proof, requiring $\int_0^t |b_u| + \sigma_u^2\, du < \infty$, $t \geq 0$, was given in Brunick & Shreve (2013). Further relaxing of the boundedness conditions is provided in Corollary 4.3. of Kurtz & Stockbridge (1998).

Remark 24. According to the classical Stroock-Varadhan theory, the uniqueness of solution for (44) is equivalent to the uniqueness for the martingale problem for $\mathcal{L}_t$, which in turn is equivalent to the uniqueness for the measure-valued equation (45) (see, e.g. Corollary 1.3. of Kurtz (2011)). This holds, for instance, under the assumption of the continuity of σ and the bounds

$$|\hat{b}(t,x)| + |\hat{\sigma}(t,x)| \leq K(1 + |x|), \quad t \in [0,\infty),\ x \in \mathbb{R}.$$

(See Condition 4.25 (U") of of Bain & Crisan (2009) which deals with time-homogeneous case; the time-nonhomogeneous extension following by Theorem 5.2. of Bhatt & Karandikar (1993).)

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