1 Notes

Remark 1. *Filtered probability space is akin to saying what knowledge is allowed before time t.*

1.1 Higher Probability Theory

1.2 Continuous Semi-Martingales

$$\int_0^T f(s)da(s) := \int_{[0,T]} f(s)\mu(ds)$$
$$\int_0^T f(s)|da(s)| := \int_{[0,T]} f(s)|\mu|(ds)$$

Lemma 1. *$f : [0,T] \to \mathbb{R}$ continuous, then*

$$\int_0^T f da(s) = \lim_{n\to\infty} \sum_{i=1}^{p_n} f(t_{i-1}^n)(a(t_i^n) - a(t_{i-1}^n))$$

where the t_i^n are refinings of a partition

Proof. Follows from dominated convergence applied to the measure defining bounded variation a. Since $f_n(t) = f(t_i^n)$ for $t \in [t_{i-1}, t_i)$ converges pointwise to f and the simple sum is the integral of an elementary function ☐

Quadratic Variation

Proof of quadratic variation very long and should be looked into.

Bracket

Bracket can be thought of as an extension of the quadratic variation.

(This can be thought of as a cauchy schwarz type inequality for the processes)

Proof. Kunita-Watanabe:

Poinwise we have

$$|\langle M, N\rangle_s^t| \leq \sqrt{\langle M, M\rangle_s^t}\sqrt{\langle N, N\rangle_s^t}$$

We show this pointwise via appoximations for these+cauchy schwarz.

Then show intergral inequality using this pointwise estimate for simple functions and extend it to all functions.

☐

1.3 Continuous Martingales

1.4 Stochastic Integration

Prop 1. $\mathbb{H}^2$ *is a hilbert space*

Proof. Want to show sequence M^n cauchy then convergent.

$$\lim_{m,n\to\infty} E[(M^n_\infty - M^m_\infty)^2] = \lim_{m,n\to\infty} (M^n - M^m, M^n - M^m)_H = 0$$

So we know (M^n_∞) converging in L^2. We xtract this to get the same limit. □

We know $H \to H \cdot M$ extends to an isometry from $L^2(M)$ into $\mathbb{H}^2$. Furthermore $H \cdot M$ is unique martingale of $\mathbb{H}^2$ s.t.

$$\langle H \cdot M, N\rangle H\dot{\langle} M, N\rangle$$

This can be thought of as a version of inner product definition of derivative.

Often we say the stochastic integral "commutes" with bracket ie. for $M \in \mathbb{H}^2, H \in L^2(M)$

$$\langle H \cdot M, H \cdot M\rangle = H \cdot (H \cdot \langle M, M\rangle) = H^2 \cdot \langle M, M\rangle$$

since $\langle H \cdot M, N\rangle = H \cdot \langle M, N\rangle$

2 Lists

List

References

[1] Legal. Brownian Motion and Stochastic Calculus.

`https://drive.google.com/drive/u/1/search?q=le%20gall`