Solutions to Exercises on Le Gall's Book: Brownian Motion, Martingales, and Stochastic Calculus

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Contents

Chapter 1

Gaussian Variables and Gaussian Processes

1.1 Exercise 1.15

Let $(X_t)_{t\in[0,1]}$ be a centered Gaussian process. We assume that the mapping $(t,w)\mapsto X_t(w)$ from $[0,1]\times\Omega$ into $\mathbb{R}$ is measurable. We denote the covariance function of X by $K(u,v)$.

1. Show that the mapping $t\mapsto X_t$ from $[0,1]$ into $L^2(\Omega)$ is continuous if and only if $K(u,v)$ is continuous on $[0,1]^2$. In what follows, we assume that this condition holds.
2. Let $h:[0,1]\to\mathbb{R}$ be a measurable function such that
$$\int_0^1 |h(t)|\sqrt{K(t,t)}dt<\infty.$$
Show that the integral, for a.e., the integral
$$\int_0^1 h(t)X_t(w)dt$$
is absolutely integral. We set $Z(w)=\int_0^1 h(t)X_t(w)dt$.
3. We now make the stronger assumption
$$\int_0^1 |h(t)|dt<\infty.$$
Show that Z is the L^2 limit of the variables
$$Z_n=\sum_{i=1}^n X_{\frac{i}{n}}\int_{\frac{i-1}{n}}^{\frac{i}{n}} h(t)dt$$
when $n\to\infty$ and infer that Z is a Gaussian random variable.
4. We assume that $K(u,v)$ is twice continuously differentiable. Show that, for every $t\in[0,1]$, the limit
$$\widetilde{X_t}=\lim_{s\to t}\frac{X_s-X_t}{s-t}$$
exists in L^2. Verify that $(\widetilde{X_t})_{t\in[0,1]}$ is a centered Gaussian process and compute its covariance function.

Proof.

1. First, we assume that $K(u,v)$ is continuous. Note that
$$||X_{t+h}-X_t||^2_{L^2(\Omega)}=\boldsymbol{E}[|X_{t+h}-X_t|^2]=K(t+h,t+h)-2K(t+h,t)+K(t,t).$$
By letting $h\downarrow 0$, we see that the mapping $t\mapsto X_t$ is continuous.
Conversely, we assume that the mapping $t\mapsto X_t$ is continuous. By using Cauchy Schwarz inequality, we get
$$\begin{aligned}&|K(u+t,v+s)-K(u,v)|\\&\le|K(u+t,v+s)-K(u,v+s)|+|K(u,v+s)-K(u,v)|\\&=\boldsymbol{E}[|(X_{u+t}-X_u)X_{v+s}|]+\boldsymbol{E}[|(X_{v+s}-X_v)X_u|]\\&=||X_{u+t}-X_u||_{L^2}||X_{v+s}||_{L^2}+||X_{v+s}-X_v||_{L^2}||X_u||_{L^2}\end{aligned}$$
Since $||X_{v+s}||_{L^2}$ is bounded for small s, we see that $K(u,v)$ is continuous.

2. It's clear that

$$\begin{aligned}
&\int_\Omega \int_0^1 |X_t(w)||h(t)|dt\boldsymbol{P}(dw) \\
&= \int_0^1 \int_\Omega |X_t(w)||h(t)|\boldsymbol{P}(dw)dt \\
&= \int_0^1 ||X_t||_{L^1}|h(t)|dt \\
&\leq \int_0^1 ||X_t||_{L^2}|h(t)|dt \\
&= \int_0^1 \sqrt{K(t,t)}|h(t)|dt < \infty
\end{aligned}$$

Thus, the integral, for a.e., the integral

$$\int_0^1 h(t)X_t(w)dt$$

is absolutely integral.

3. It suffices to show that $Z_n \to Z$ in L^2. Indeed, since $\{Z_n\}_{n\geq 1}$ are Gaussian random variables and $Z_n \to Z$ in L^2, we see that Z is a Gaussian random variable. Note that

$$Z_n(w) = \int_0^1 \sum_{i=1}^n X_{\frac{i}{n}}(w)1_{[\frac{i-1}{n},\frac{i}{n})}(t)h(t)dt.$$

Thus,

$$\begin{aligned}
&\boldsymbol{E}[|Z - Z_n|^2]^{\frac{1}{2}} \\
&= (\int_\Omega |\int_0^1 h(t)(X_t(w) - \sum_{i=1}^n X_{\frac{i}{n}}(w)1_{[\frac{i-1}{n},\frac{i}{n})}(t))dt|^2\boldsymbol{P}(dw))^{\frac{1}{2}} \\
&\leq \int_0^1 (\int_\Omega |h(t)|^2|(X_t(w) - \sum_{i=1}^n X_{\frac{i}{n}}(w)1_{[\frac{i-1}{n},\frac{i}{n})}(t))|^2\boldsymbol{P}(dw))^{\frac{1}{2}}dt \\
&= \int_0^1 |h(t)|(\int_\Omega |(X_t(w) - \sum_{i=1}^n X_{\frac{i}{n}}(w)1_{[\frac{i-1}{n},\frac{i}{n})}(t))|^2\boldsymbol{P}(dw))^{\frac{1}{2}}dt \\
&= \int_0^1 |h(t)| \times ||(X_t - \sum_{i=1}^n X_{\frac{i}{n}}1_{[\frac{i-1}{n},\frac{i}{n})}(t))||_{L^2}dt.
\end{aligned}$$

For each $t \in [0,1)$ and $n \geq 1$ such that $\frac{k-1}{n} \leq t < \frac{k}{n}$, we get

$$||(X_t - \sum_{i=1}^n X_{\frac{i}{n}}1_{[\frac{i-1}{n},\frac{i}{n})}(t))||_{L^2} = ||X_t - X_{\frac{k}{n}}||_{L^2} \leq ||X_t||_{L^2} + ||X_{\frac{k}{n}}||_{L^2} \leq 2\sup_{t\in[0,1]}\sqrt{K(t,t)} < \infty.$$

and therefore

$$|h(t)| \times ||(X_t - \sum_{i=1}^n X_{\frac{i}{n}}1_{[\frac{i-1}{n},\frac{i}{n})}(t))||_{L^2} \leq C|h(t)|$$

for each $t \in [0,1)$ and some $0 < C < \infty$.

Fix $t \in [0,1)$. Choose $\{k_n\}$ such that $\frac{k_n-1}{n} \leq t < \frac{k_n}{n}$ for each $n \geq 1$. Since $t \mapsto X_t$ is continuous, we have

$$||(X_t - \sum_{i=1}^n X_{\frac{i}{n}}1_{[\frac{i-1}{n},\frac{i}{n})}(t))||_{L^2} = ||X_t - X_{\frac{k_n}{n}}||_{L^2} \to 0 \text{ as } n \to \infty.$$

By using dominated convergence theorem, we have

$$\limsup_{n\to\infty} \boldsymbol{E}[|Z-Z_n|^2]^{\frac{1}{2}} \le \lim_{n\to\infty}\int_0^1 |h(t)| \times ||(X_t - \sum_{i=1}^n X_{\frac{i}{n}} 1_{[\frac{i-1}{n},\frac{i}{n})}(t))||_{L^2} dt = 0$$

and, hence, $Z_n \to Z$ in L^2.

4. To show that $\lim_{s\to t}\frac{X_s-X_t}{s-t}$ exists in L^2, it suffices to show that

$$||\frac{X_{t+h_1}-X_t}{h_1} - \frac{X_{t+h_2}-X_t}{h_2}||_{L^2} \to 0 \text{ as } h_1,h_2\to 0.$$

Note that

$$||\frac{X_{t+h_1}-X_t}{h_1} - \frac{X_{t+h_2}-X_t}{h_2}||_{L^2}^2 = A+B-2C,$$

where

$$A = \frac{1}{|h_1|^2}\boldsymbol{E}[(X_{t+h_1}-X_t)^2] = \frac{1}{|h_1|^2}(\boldsymbol{E}[X_{t+h_1}^2]+\boldsymbol{E}[X_t^2]-2\boldsymbol{E}[X_{t+h_1}X_t]),$$

$$B = \frac{1}{|h_2|^2}\boldsymbol{E}[(X_{t+h_2}-X_t)^2] = \frac{1}{|h_2|^2}(\boldsymbol{E}[X_{t+h_2}^2]+\boldsymbol{E}[X_t^2]-2\boldsymbol{E}[X_{t+h_2}X_t]),$$

and

$$\begin{aligned} C &= \frac{1}{|h_1|}\frac{1}{|h_2|}\boldsymbol{E}[(X_{t+h_2}-X_t)(X_{t+h_1}-X_t)] \\ &= \frac{1}{|h_2||h_1|}(\boldsymbol{E}[X_{t+h_2}X_{t+h_1}]+\boldsymbol{E}[X_t^2]-\boldsymbol{E}[X_{t+h_2}X_t]-\boldsymbol{E}[X_{t+h_1}X_t]). \end{aligned}$$

First, we show that $C \to \frac{\partial^2 K}{\partial u \partial v}(t,t)$ as $h_1,h_2\to 0$. Without loss of generality, we may suppose $h_1,h_2>0$. Set

$$g(z) = K(t+h_1,z) - K(t,z).$$

Then

$$C = \frac{1}{h_1}\frac{1}{h_2}(g(t+h_2)-g(t)).$$

Since $K \in C^2([0,1]^2)$, there exist t_1^*, t_2^* such that

$$C = \frac{1}{h_1}g'(t_2^*) = \frac{1}{h_1}(\frac{\partial K(t+h_1,t_2^*)}{\partial v} - \frac{\partial K(t,t_2^*)}{\partial v}) = \frac{\partial^2 K(t_1^*,t_2^*)}{\partial u\partial v}$$

By using the continuity of $\frac{\partial^2 K}{\partial u\partial v}$, we see that $C \to \frac{\partial^2 K}{\partial u \partial v}(t,t)$ as $h_1,h_2\to 0$.
Similarly, we have $A \to \frac{\partial^2 K}{\partial u \partial v}(t,t)$ and $B \to \frac{\partial^2 K}{\partial u \partial v}(t,t)$ as $h_1,h_2\to 0$. Therefore,

$$||\frac{X_{t+h_1}-X_t}{h_1} - \frac{X_{t+h_2}-X_t}{h_2}||_{L^2} \to 0 \text{ as } h_1,h_2\to 0$$

and, hence, $\lim_{s\to t}\frac{X_s-X_t}{s-t}$ exists in L^2. Since $\frac{X_s-X_t}{s-t}$ is a centered Gaussian random variable for all $s\neq t$, we see that $\widetilde{X_t} \equiv \lim_{s\to t}\frac{X_s-X_t}{s-t}$ is a centered Gaussian random variable. Moreover, since any linear combination $\sum_{k=1}^n c_k \frac{X_{s_k}-X_{t_k}}{s_k-t_k}$ is a centered Gaussian random, we see that $(\widetilde{X_t})_{t\in[0,1]}$ is a centered Gaussian process.
Finally, we show that

$$\widetilde{K}(t,s) = \frac{\partial^2 K}{\partial u\partial v}(t,s),$$

where $\widetilde{K}(t,s)$ is the covariance function of $(\widetilde{X_t})_{t\in[0,1]}$. By using similar argument as in (3), there exist t_h, s_h such that

$$\boldsymbol{E}[\frac{X_{t+h}-X_t}{h}\frac{X_{s+h}-X_s}{h}] = \frac{\partial^2 K}{\partial u \partial v}(t_h, s_h)$$

for each $h \neq 0$ and $t_h \to t$ and $s_h \to s$ as $h \to 0$. Since $K(u,v) \in C^2([0,1]^2)$, there exist $0 < C < \infty$ such that

$$|\boldsymbol{E}[\frac{X_{t+h}-X_t}{h}\frac{X_{s+h}-X_s}{h}]| = |\frac{\partial^2 K}{\partial u \partial v}(t_h, s_h)| \leq C$$

for all $h \neq 0$. By using dominated convergence theorem and the continuity of $\frac{\partial^2 K}{\partial u \partial v}$, we have

$$\widetilde{K}(t,s) = \boldsymbol{E}[\widetilde{X_t}\widetilde{X_s}] = \lim_{h\to 0} \boldsymbol{E}[\frac{X_{t+h}-X_t}{h}\frac{X_{s+h}-X_s}{h}] = \lim_{h\to 0}\frac{\partial^2 K}{\partial u \partial v}(t_h, s_h) = \frac{\partial^2 K}{\partial u \partial v}(t,s).$$

□

1.2 Exercise 1.16 (Kalman filtering)

Let $(\epsilon_n)_{n\geq 0}$ and $(\eta_n)_{n\geq 0}$ be two independent sequences of independent Gaussian random variables such that, for every n, ϵ_n is distributed according to $\mathcal{N}(0,\sigma^2)$ and η_n is distributed according to $\mathcal{N}(0,\delta^2)$, where $\sigma > 0$ and $\delta > 0$. We consider two other sequences $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ defined by the properties $X_0 = 0$, and , for every $n \geq 0$,

$$X_{n+1} = a_n X_n + \epsilon_{n+1} \text{ and } Y_n = cX_n + \eta_n,$$

where c and a_n are positive constants. We set

$$\hat{X}_{n/n} = \boldsymbol{E}[X_n|Y_0, ..., Y_n]$$

and

$$\hat{X}_{n+1/n} = \boldsymbol{E}[X_{n+1}|Y_0, ..., Y_n].$$

The goal of the exercise is to find a recursive formula allowing one to compute these conditional expectations.

1. Verify that $\hat{X}_{n+1/n} = a_n \hat{X}_{n/n}$, for every $n \geq 0$.
2. Show that, for every $n \geq 1$,

$$\hat{X}_{n/n} = \hat{X}_{n/n-1} + \frac{\boldsymbol{E}[X_n Z_n]}{\boldsymbol{E}[Z_n^2]} Z_n,$$

   where $Z_n = Y_n - c\hat{X}_{n/n-1}$.
3. Evaluate $\boldsymbol{E}[X_n Z_n]$ and $\boldsymbol{E}[Z_n^2]$ in terms of $P_n \equiv \boldsymbol{E}[(X_n - \hat{X}_{n/n-1})^2]$ and infer that, for every $n \geq 1$,

$$\hat{X}_{n+1/n} = a_n(\hat{X}_{n/n-1} + \frac{cP_n}{c^2 P_n + \delta^2} Z_n)$$

4. Verify that $P_1 = \sigma^2$ and that, for every $n \geq 1$, the following induction formula holds:

$$P_{n+1} = \sigma^2 + a_n^2 \frac{\delta^2 P_n}{c^2 P_n + \delta^2}.$$

Proof.

1. By observing the construction of X_n and Y_n, we see that $Y_0 = \eta_0$ and for every $n \geq 1$, X_n is a $\sigma(\epsilon_k, k = 0, ..., n)$-measurable centered Gaussian random variable and Y_n is a $\sigma(\eta_n, \epsilon_k, k = 0, ..., n)$-measurable centered Gaussian random variable. Since $\sigma(Y_0) = \sigma(\eta_0)$ and for each $n \geq 1$, $\sigma(Y_0, ..., Y_n) \subseteq \sigma(\epsilon_k, \eta_k, k = 0, ..., n)$, we have

$$\begin{aligned}
&\hat{X}_{n+1/n} = \boldsymbol{E}[X_{n+1}|Y_0, ..., Y_n] \\
&= a_n \boldsymbol{E}[X_n|Y_0, ..., Y_n] + \boldsymbol{E}[\epsilon_{n+1}|Y_0, ..., Y_n] \\
&= a_n \hat{X}_{n/n} + \boldsymbol{E}[\epsilon_{n+1}] \\
&= a_n \hat{X}_{n/n}.
\end{aligned}$$

2. Given $n \geq 1$. Set $K_n = span\{Y_0, ..., Y_n\}$. Then, for each centered Gaussian random variable $X \in L^2(\Omega, \mathcal{F}, \boldsymbol{P})$,

$$\boldsymbol{E}[X|Y_0, ..., Y_n] = p_{K_n}(X),$$

where p_{K_n} is the orthogonal projection onto K_n in the Hilbert space $L^2(\Omega, \mathcal{F}, \boldsymbol{P})$. Observe that

$$\begin{aligned}
&Z_n = Y_n - c\hat{X}_{n/n-1} \\
&= Y_n - c\boldsymbol{E}[X_n|Y_0, ..., Y_{n-1}] \\
&= Y_n + \boldsymbol{E}[\eta_n - Y_n|Y_0, ..., Y_{n-1}] \\
&= Y_n + \boldsymbol{E}[\eta_n] - \boldsymbol{E}[Y_n|Y_0, ..., Y_{n-1}] \\
&= Y_n - p_{K_{n-1}}(Y_n)
\end{aligned}$$

Set $V_n = span\{Z_n\}$. Then $K_n = span\{Y_0, ..., Y_{n-1}, Z_n\} = K_{n-1} \oplus V_n$. Thus,

$$\begin{aligned}
&\hat{X}_{n/n} = \boldsymbol{E}[X_n|Y_0, ..., Y_n] \\
&= p_{K_n}(X_n) \\
&= p_{K_{n-1}}(X_n) + p_{V_n}(X_n) \\
&= \boldsymbol{E}[X_n|Y_0, ..., Y_{n-1}] + < X_n, \frac{Z_n}{||Z_n||_{L^2(\Omega)}} >_{L^2(\Omega)} \frac{Z_n}{||Z_n||_{L^2(\Omega)}} \\
&= \hat{X}_{n/n-1} + \frac{\boldsymbol{E}[X_n Z_n]}{\boldsymbol{E}[Z_n^2]} Z_n
\end{aligned}$$

3. First, we show that

$$\boldsymbol{E}[Z_n^2] = c^2 P_n + \delta^2.$$

Note that

$$\begin{aligned}
&\boldsymbol{E}[Z_n^2] = \boldsymbol{E}[(Y_n - c\hat{X}_{n/n-1})^2] \\
&= \boldsymbol{E}[(Y_n - cX_n + cX_n - c\hat{X}_{n/n-1})^2] \\
&= \boldsymbol{E}[(\eta_n + cX_n - c\hat{X}_{n/n-1})^2] \\
&= c^2 P_n + \boldsymbol{E}[\eta_n^2] + 2c\boldsymbol{E}[\eta_n(X_n - \hat{X}_{n/n-1})] \\
&= c^2 P_n + \delta^2 + 2c\boldsymbol{E}[\eta_n(X_n - \hat{X}_{n/n-1})]
\end{aligned}$$

Since X_n is $\sigma(\epsilon_k, k = 0, ..., n)$-measurable, $\hat{X}_{n/n-1}$ is $\sigma(Y_k, k = 0, ..., n-1)$-measurable, and $\sigma(Y_k, k = 0, ..., n-1) \subseteq \sigma(\eta_k, \epsilon_k, k = 0, ..., n-1)$, we see that

$$\boldsymbol{E}[\eta_n(X_n - \hat{X}_{n/n-1})] = \boldsymbol{E}[\eta_n]\boldsymbol{E}[X_n - \hat{X}_{n/n-1}] = 0$$

and therefore

$$\boldsymbol{E}[Z_n^2] = c^2 P_n + \delta^2.$$

Next, we show that

$$\boldsymbol{E}[X_n Z_n] = cP_n.$$

Observe that

$$\begin{aligned}
&\boldsymbol{E}[\hat{X}_{n/n-1}(X_n - \hat{X}_{n/n-1})] \\
&= \boldsymbol{E}[p_{K_{n-1}}(X_n)(X_n - p_{K_{n-1}}(X_n))].
\end{aligned}$$

Since X_n is $\sigma(\epsilon_k, k = 0, ..., n)$-measurable, we have $\boldsymbol{E}[X_n \eta_n] = 0$ and therefore

$$\begin{aligned}
\boldsymbol{E}[X_n Z_n] &= \boldsymbol{E}[X_n(Y_n - c\hat{X}_{n/n-1})] \\
&= \boldsymbol{E}[X_n(Y_n - cX_n + cX_n - c\hat{X}_{n/n-1})] \\
&= \boldsymbol{E}[X_n(\eta_n + cX_n - c\hat{X}_{n/n-1})] \\
&= c\boldsymbol{E}[X_n(X_n - \hat{X}_{n/n-1})] \\
&= c\boldsymbol{E}[X_n(X_n - \hat{X}_{n/n-1})] - c\boldsymbol{E}[\hat{X}_{n/n-1}(X_n - \hat{X}_{n/n-1})] \\
&= cP_n.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
&\hat{X}_{n+1/n} = a_n \hat{X}_{n/n} \\
&= a_n(\hat{X}_{n/n-1} + \frac{\boldsymbol{E}[X_n Z_n]}{\boldsymbol{E}[Z_n^2]} Z_n) \\
&= a_n(\hat{X}_{n/n-1} + \frac{cP_n}{c^2 P_n + \delta^2} Z_n).
\end{aligned}$$

4. Note that

$$P_1 = \boldsymbol{E}[(X_1 - \boldsymbol{E}[X_1|\eta_0])^2] = \boldsymbol{E}[(\epsilon_1 - \boldsymbol{E}[\epsilon_1|\eta_0])^2] = \boldsymbol{E}[(\epsilon_1 - \boldsymbol{E}[\epsilon_1])^2] = \sigma^2$$

and

$$\begin{aligned}
&P_{n+1} = \boldsymbol{E}[(X_{n+1} - \hat{X}_{n+1/n})^2] \\
&= \boldsymbol{E}[(a_n X_n + \epsilon_{n+1} - a_n \hat{X}_{n/n})^2] \\
&= \boldsymbol{E}[(\epsilon_{n+1} - a_n(X_n - \hat{X}_{n/n}))^2] \\
&= \boldsymbol{E}[\epsilon_{n+1}^2] + a_n^2 \boldsymbol{E}[(X_n - \hat{X}_{n/n})^2] - 2a_n \boldsymbol{E}[\epsilon_{n+1}(X_n - \hat{X}_{n/n})]
\end{aligned}$$

Since X_n is $\sigma(\epsilon_k, k = 0, ..., n)$-measurable, $\hat{X}_{n/n}$ is $\sigma(Y_k, k = 0, ..., n)$-measurable, and $\sigma(Y_k, k = 0, ..., n) \subseteq \sigma(\eta_k, \epsilon_k, k = 0, ..., n)$, we see that

$$\boldsymbol{E}[\epsilon_{n+1}(X_n - \hat{X}_{n/n})] = 0$$

and therefore

$$P_{n+1} = \boldsymbol{E}[\epsilon_{n+1}^2] + a_n^2 \boldsymbol{E}[(X_n - \hat{X}_{n/n})^2] = \sigma^2 + a_n^2 \boldsymbol{E}[(X_n - \hat{X}_{n/n})^2].$$

Because Z_n and $\hat{X}_{n/n-1}$ are orthogonal and Z_n is centered Gaussian, we get $\boldsymbol{E}[Z_n\hat{X}_{n/n-1}] = 0$ and, hence,

$$\begin{aligned}
&P_{n+1} = \sigma^2 + a_n^2\boldsymbol{E}[(X_n - \hat{X}_{n/n})^2] \\
&= \sigma^2 + a_n^2\boldsymbol{E}[(X_n - \hat{X}_{n/n-1} + \hat{X}_{n/n-1} - \hat{X}_{n/n})^2] \\
&= \sigma^2 + a_n^2\boldsymbol{E}[(X_n - \hat{X}_{n/n-1} - \frac{\boldsymbol{E}[X_nZ_n]}{\boldsymbol{E}[Z_n^2]}Z_n)^2] \\
&= \sigma^2 + a_n^2(P_n + (\frac{\boldsymbol{E}[X_nZ_n]}{\boldsymbol{E}[Z_n^2]})^2\boldsymbol{E}[Z_n^2] - 2\frac{\boldsymbol{E}[X_nZ_n]}{\boldsymbol{E}[Z_n^2]}\boldsymbol{E}[Z_n(X_n - \hat{X}_{n/n-1})]) \\
&= \sigma^2 + a_n^2(P_n + \frac{\boldsymbol{E}[X_nZ_n]^2}{\boldsymbol{E}[Z_n^2]} - 2\frac{\boldsymbol{E}[X_nZ_n]}{\boldsymbol{E}[Z_n^2]}\boldsymbol{E}[Z_nX_n]) \\
&= \sigma^2 + a_n^2(P_n - \frac{\boldsymbol{E}[X_nZ_n]^2}{\boldsymbol{E}[Z_n^2]}) \\
&= \sigma^2 + a_n^2(P_n - \frac{c^2P_n^2}{c^2P_n + \delta^2}) \\
&= \sigma^2 + a_n^2\frac{\delta^2P_n}{c^2P_n + \delta^2}
\end{aligned}$$

□

1.3 Exercise 1.17

Let H be a (centered) Gaussian space and let H_1 and H_2 be linear subspaces of H. Let K be a closed linear subspace of H. We write p_K for the orthogonal projection onto K. Show that the condition

$$\forall X_1 \in H_1, \forall X_2 \in H_2, \quad \boldsymbol{E}[X_1X_2] = \boldsymbol{E}[p_K(X_1)p_K(X_2)] \tag{1}$$

implies that the σ-fields $\sigma(H_1)$ and $\sigma(H_2)$ are conditionally independent given $\sigma(K)$. (This means that, for every nonnegative $\sigma(H_1)$-measurable random variable X_1, and for every nonnegative $\sigma(H_2)$-measurable random variable X_2, one has

$$\boldsymbol{E}[X_1X_2|\sigma(K)] = \boldsymbol{E}[X_1|\sigma(K)]\boldsymbol{E}[X_2|\sigma(X_2)].) \tag{2}$$

Hint: Via monotone class arguments explained in Appendix A1, it is enough to consider the case where X_1, resp. X_2, is the indicator function of an event depending only on finitely many variables in H_1, resp. in H_2.

Proof.
To show (2), it suffices to show that

$$\begin{aligned}
&\boldsymbol{E}[1_{\{X_1^1\in\Gamma_1^1\}}...1_{\{X_{n_1}^1\in\Gamma_{n_1}^1\}} \times 1_{\{X_2^1\in\Gamma_1^2\}}...1_{\{X_{n_2}^2\in\Gamma_{n_2}^2\}} \mid \sigma(K)] \\
&= \boldsymbol{E}[1_{\{X_1^1\in\Gamma_1^1\}}...1_{\{X_{n_1}^1\in\Gamma_{n_1}^1\}}|\sigma(K)] \times \boldsymbol{E}[1_{\{X_2^1\in\Gamma_1^2\}}...1_{\{X_{n_2}^2\in\Gamma_{n_2}^2\}} \mid \sigma(K)]
\end{aligned} \tag{3}$$

for each $Z_i^s \in M_s$, $\Gamma_i^s \in \mathcal{B}_\mathbb{R}$, $m_s \in \mathbb{N}$, and $s = 1, 2$.
Let $\{Z_i^s : i = 1, 2, ..., m_s\}$ be an orthonormal basis of linear subspace space M_s of L^2 spanned by $\{X_i^s : i = 1, 2, ..., n_s\}$. Then $\{Z_1^s, Z_2^s, ..., Z_{m_s}^s\} \subseteq H_s$ are independent centered Gaussians. To show (3), it suffices to show that

$$\begin{aligned}
&\boldsymbol{E}[1_{\{Z_1^1\in\Gamma_1^1\}}...1_{\{Z_{m_1}^1\in\Gamma_{m_1}^1\}} \times 1_{\{Z_2^1\in\Gamma_1^2\}}...1_{\{Z_{m_2}^2\in\Gamma_{m_2}^2\}} \mid \sigma(K)] \\
&= \boldsymbol{E}[1_{\{Z_1^1\in\Gamma_1^1\}}...1_{\{Z_{m_1}^1\in\Gamma_{m_1}^1\}}|\sigma(K)] \times \boldsymbol{E}[1_{\{Z_2^1\in\Gamma_1^2\}}...1_{\{Z_{m_2}^2\in\Gamma_{m_2}^2\}} \mid \sigma(K)]
\end{aligned} \tag{4}$$

for each $\Gamma_i^s \in \mathcal{B}_\mathbb{R}$. Indeed, by the theorem of monotone class, we get

$$\boldsymbol{E}[1_{\{E_1\}}1_{\{E_2\}} \mid \sigma(K)] = \boldsymbol{E}[1_{\{E_1\}} \mid \sigma(K)]\boldsymbol{E}[1_{\{E_2\}} \mid \sigma(K)] \quad \forall E_s \in \sigma(M_s) \text{ and } s = 1, 2.$$

and so

$$\begin{aligned}&\boldsymbol{E}[1_{\{X_1^1\in\Gamma_1^1\}}...1_{\{X_{n_1}^1\in\Gamma_{n_1}^1\}}\times 1_{\{X_2^1\in\Gamma_1^2\}}...1_{\{X_{n_2}^2\in\Gamma_{n_2}^2\}}\mid\sigma(K)]\\ &=\boldsymbol{E}[1_{\{X_1^1\in\Gamma_1^1\}}...1_{\{X_{n_1}^1\in\Gamma_{n_1}^1\}}|\sigma(K)]\times\boldsymbol{E}[1_{\{X_2^1\in\Gamma_1^2\}}...1_{\{X_{n_2}^2\in\Gamma_{n_2}^2\}}\mid\sigma(K)]\end{aligned}$$

for each $\Gamma_i^s\in\mathcal{B}_{\mathbb{R}}$.
By independence of $\{Z_1^s,Z_2^s,...,Z_{m_s}^s\}$, we have

$$\boldsymbol{E}[(Z_i^s-p_K(Z_i^s))(Z_j^s-p_K(Z_j^s))]=0\quad\forall i\neq j,\forall s=1,2. \tag{5}$$

By (1) and Corollary 1.10, we get

$$\begin{aligned}&\boldsymbol{E}[(Z_i^1-p_K(Z_i^1))(Z_j^2-p_K(Z_j^2))]\\ &=\boldsymbol{E}[Z_i^1Z_j^2]+\boldsymbol{E}[p_K(Z_i^1)p_K(Z_j^2)]-\boldsymbol{E}[Z_i^1p_K(Z_j^2)]-\boldsymbol{E}[p_K(Z_i^1)Z_j^2]\\ &=\boldsymbol{E}[p_K(Z_i^1)p_K(Z_j^2)]+\boldsymbol{E}[p_K(Z_i^1)p_K(Z_j^2)]-\boldsymbol{E}[\boldsymbol{E}[Z_i^1|\sigma(K)]p_K(Z_j^2)]-\boldsymbol{E}[p_K(Z_i^1)\boldsymbol{E}[Z_j^2|\sigma(K)]]\\ &=\boldsymbol{E}[p_K(Z_i^1)p_K(Z_j^2)]+\boldsymbol{E}[p_K(Z_i^1)p_K(Z_j^2)]-\boldsymbol{E}[p_K(Z_i^1)p_K(Z_j^2)]-\boldsymbol{E}[p_K(Z_i^1)p_K(Z_j^2)]=0\quad\forall i,j\end{aligned}\tag{6}$$

and

$$\boldsymbol{P}(Z_i^s\in\Gamma_i^s|\sigma(K))=\frac{1}{\sigma_i^s\sqrt{2\pi}}\int_{\Gamma_i^s}\exp(-\frac{(y-p_K(Z_i^s))^2}{2(\sigma_i^s)^2})dy,$$

where $(\sigma_i^s)^2=\boldsymbol{E}[(Z_i^s-p_K(Z_i^s))^2]$. Set

$$Y_i^s=Z_i^s-p_K(Z_i^s).$$

By (5) and (6), $\{Y_i^s: s=1,2 \text{ and } i=1,2,...,m_s\}$ are independent centered Gaussians. Set

$$F(z_1^1,...,z_{m_1}^1,z_1^2,...,z_{m_2}^2)=1_{\{\Gamma_1^1\}}(z_1^1)...1_{\{\Gamma_{m_1}^1\}}(z_{m_1}^1)\times 1_{\{\Gamma_1^2\}}(z_1^2)...1_{\{\Gamma_{m_2}^2\}}(z_{m_2}^2).$$

Since $\{Y_i^s: s=1,2 \text{ and } i=1,2,...,n_s\}$ is independent of $\sigma(K)$, we get

$$\begin{aligned}&\boldsymbol{E}[1_{\{Z_1^1\in\Gamma_1^1\}}...1_{\{Z_{m_1}^1\in\Gamma_{m_1}^1\}}\times 1_{\{Z_2^1\in\Gamma_1^2\}}...1_{\{Z_{m_2}^2\in\Gamma_{m_2}^2\}}\mid\sigma(K)]\\ &=\boldsymbol{E}[F(Z_1^1,...,Z_{m_1}^1,Z_1^2,...,Z_{m_2}^2)\mid\sigma(K)]\\ &=\boldsymbol{E}[F(Y_1^1+p_K(Z_1^1),...,Y_{m_1}^1+p_K(Z_{m_1}^1),Y_1^2+p_K(Z_1^2),...,Y_{m_2}^2+p_K(Z_{m_2}^2))\mid\sigma(K)]\\ &=\int F(y_1^1+p_K(Z_1^1),...,y_{m_1}^1+p_K(Z_{m_1}^1),y_1^2+p_K(Z_1^2),...,y_{m_2}^2+p_K(Z_{m_2}^2))\\ &\boldsymbol{P}_{Y_s^1,...,Y_{m_1}^1,Y_1^2,...,Y_{m_2}^2}(dy_1^1\times...\times dy_{m_1}^1\times dy_1^2\times...\times dy_{m_2}^2)\\ &=\int F(y_1^1+p_K(Z_1^1),...,y_{m_1}^1+p_K(Z_{m_1}^1),y_1^2+p_K(Z_1^2),...,y_{m_2}^2+p_K(Z_{m_2}^2))\\ &\boldsymbol{P}_{Y_1^1}(dy_1^1)...\boldsymbol{P}_{Y_{m_1}^1}(dy_{m_1}^1)\boldsymbol{P}_{Y_1^2}(dy_1^2)...\boldsymbol{P}_{Y_{m_2}^2}(dy_{m_2}^2)\\ &=\prod_{1\le s\le 2,1\le i\le m_s}\int 1_{\{\Gamma_i^s\}}(y_i^s+p_K(Z_i^s))\boldsymbol{P}_{Y_i^s}(dy_i^s)\end{aligned}$$

□

1.4 Exercise 1.18 (Levy's construction of Brownian motion)

For each $t\in[0,1]$, we set $h_0(t)=1$, and then, for every integer $n\geq 0$ and every $k\in\{0,1,...,2^n-1\}$,

$$h_{n,k}(t)=2^{\frac{n}{2}}1_{[\frac{2k}{2^{n+1}},\frac{2k+1}{2^{n+1}})}(t)-2^{\frac{n}{2}}1_{[\frac{2k+1}{2^{n+1}},\frac{2k+2}{2^{n+1}})}(t).$$

1. Verify that the functions (**Haar system**) $H := \{h_{n,k} | n \geq 0 \text{ and } k = 0, 1, ..., 2^n - 1\} \bigcup \{h_0\}$ form an orthonormal basis of $L^2([0,1], \mathcal{B}_{[0,1]}, dt)$. (Hint: Observe that, for every fixed $n \geq 0$, any function $f : [0,1) \mapsto \mathbb{R}$ that is constant on every interval of the form $[\frac{j-1}{2^n}, \frac{j}{2^n})$, for every $1 \leq j \leq 2^n$, is a linear combination of the functions in H).

2. Suppose that $\{N_0\} \bigcup \{N_{n,k}\}$ are independent $\mathscr{N}(0,1)$ random variables. Justify the existence of the (unique) Gaussian white noise G on $[0,1]$ with intensity dt, such that $G(h_0) = N_0$ and $G(h_k^n) = N_k^n$ for every $n \geq 0$ and $0 \leq k \leq 2^n - 1$.

3. For every $t \in [0,1)$, set $B_t = G(1_{[0,t]})$. Show that

$$B_t = tN_0 + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} g_{n,k}(t) N_{n,k},$$

where the series converges in L^2 , and the functions $g_{n,k} : [0,1] \mapsto [0,\infty)$ are given by

$$g_{n,k}(t) = \int_0^t h_{n,k}(s) ds.$$

Note that the functions $g_{n,k}$ are continuous and satisfy the following property: For every fixed $n \geq 0$, the functions $g_{n,k}$, $0 \leq k \leq 2^n - 1$, have disjoint supports and are bounded above by $2^{-\frac{n}{2}}$.

4. For every integer $m \geq 0$ and every $t \in [0,1]$ set

$$B_t^m = tN_0 + \sum_{n=0}^{m-1} \sum_{k=0}^{2^n-1} g_{n,k}(t) N_{n,k}.$$

Verify that the continuous functions $t \mapsto B_t^m$ converge uniformly on $[0,1]$ as $m \to \infty$ (a.s.) (Hint: If N is $\mathscr{N}(0,1)$ distributed, prove the bound $\boldsymbol{P}(|N| \geq a) \leq \exp(-\frac{a^2}{2})$ for every $a \geq 1$, and use this estimate to bound the probability of the event $\{\sup_{0 \leq k \leq 2^n-1} |N_{n,k}| > 2^{\frac{n}{4}}\}$, for every fixed $n \geq 0$.)

5. Conclude that we can, for every $t \geq 0$, select a random variable W_t which is a.s. equal to B_t, in such a way that the mapping $t \mapsto W_t$ is continuous for every $w \in \Omega$.

Proof.

1. It's clear that H is an orthonormal system in $L^2([0,1], \mathcal{B}_{[0,1]}, dt)$. Now, we show that H is complete. Since

$$\overline{V} = L^2([0,1], \mathcal{B}_{[0,1]}, dt),$$

where $V := span(S)$, $S = \bigcup_{n=0}^{\infty} S_n$, and

$$S_n := \{f : [0,1] \mapsto \mathbb{R} : f(x) = \sum_{k=0}^{2^n-1} c_k 1_{[\frac{k}{2^n}, \frac{k+1}{2^n})}\} \quad \forall n \geq 0,$$

it suffices to show that $S \subseteq span(H)$.
Fix $f \in S_m$ such that

$$f(x) = \sum_{k=0}^{2^m-1} c_m 1_{[\frac{k}{2^m}, \frac{k+1}{2^m})}(x) \text{ for some } m \geq 0.$$

It's clear that $f \in span(H)$ if $m = 0$. Now, we assume that $m \geq 1$. To show that $f \in span(H)$, it suffices to show that there exists real numbers $\alpha_0, ..., \alpha_{2^{m-1}-1}$ such that

$$f(x) - \sum_{k=0}^{2^{m-1}-1} \alpha_k h_{m-1,k}(x) \in S_{m-1}$$

Set

$$\alpha_k = \frac{1}{2^{\frac{m+1}{2}}}(c_{2k} - c_{2k+1}) \quad \forall 0 \le k \le 2^{m-1} - 1.$$

Then

$$\begin{aligned}
& c_{2k} 1_{[\frac{2k}{2^m}, \frac{2k+1}{2^m})}(x) + c_{2k+1} 1_{[\frac{2k+1}{2^m}, \frac{2k+2}{2^m})}(x) - \alpha_k h_{m-1,k}(x) \\
&= \frac{c_{2k} + c_{2k+1}}{2} 1_{[\frac{2k}{2^m}, \frac{2k+1}{2^m})}(x) + \frac{c_{2k} + c_{2k+1}}{2} 1_{[\frac{2k+1}{2^m}, \frac{2k+2}{2^m})}(x) \\
&= \frac{c_{2k} + c_{2k+1}}{2} 1_{[\frac{k}{2^{m-1}}, \frac{k+1}{2^{m-1}})} \quad \forall 0 \le k \le 2^{m-1} - 1
\end{aligned}$$

and so $f(x) - \sum_{k=0}^{2^{m-1}-1} \alpha_k h_{m-1,k}(x) \in S_{m-1}$.

2. Let $\{N_0\} \bigcup \{N_{n,k}\}$ be independent $\mathscr{N}(0,1)$ random variables. Define

$$G(c_0 h_0 + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} c_{n,k} h_{n,k}) = c_0 N_0 + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} c_{n,k} N_{n,k}.$$

It's clear that G is a Gaussian white noise with intensity dt.

3. It's clear that

$$B_t := G(1_{[0,t]}) = tN_0 + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} g_{n,k}(t) N_{n,k},$$

where

$$g_{n,k}(t) = (1_{[0,t]}, h_{n,k})_{L^2} = \int_0^t h_{n,k}(s) ds.$$

By the definition of $h_{n,k}$, we get $g_{n,k}(t)$ is continuous, $0 \le g_{n,k}(t) \le 2^{\frac{n}{2}}$, and $supp(g_{n,k}) \subseteq [\frac{k}{2^n}, \frac{k+1}{2^n}]$ for $n \ge 0$ and $k = 0, 1, ..., 2^n - 1$.

4. Note that

$$\sum_{n=0}^{\infty} \boldsymbol{P}(\sup_{0 \le k \le 2^n - 1} |N_{n,k}| > 2^{\frac{n}{4}}) \le \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} \boldsymbol{P}(|N_{n,k}| > 2^{\frac{n}{4}}) \le \sum_{n=0}^{\infty} 2^n \exp(-2^{\frac{n}{2}-1}) < \infty.$$

By Borel Cantelli lemma, we have $\boldsymbol{P}(E) = 1$, where

$$E := \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{ \sup_{0 \le k \le 2^n - 1} |N_{n,k}| \le 2^{\frac{n}{4}} \}.$$

Fix $w \in E$. By problem 3, we get

$$\begin{aligned}
\sup_{t \in [0,1]} | \sum_{k=0}^{2^n-1} g_{n,k}(t) N_{n,k}| &\le \sup_{t \in [0,1]} \sum_{k=0}^{2^n-1} g_{n,k}(t) |N_{n,k}| = \sup_{0 \le k \le 2^n - 1} (\sup_{t \in [0,1]} g_{n,k}(t) |N_{n,k}|) \\
&\le (2^{-\frac{n}{2}} \sup_{0 \le k \le 2^n - 1} |N_{n,k}|) \le 2^{-\frac{n}{2}} \times 2^{\frac{n}{4}} = 2^{-\frac{n}{4}} \text{ for large n}
\end{aligned}$$

and so

$$\sup_{t \in [0,1]} | \sum_{n=m_1}^{m_2} \sum_{k=0}^{2^n-1} g_{n,k}(t) N_{n,k}| \le \sum_{n=m_1}^{m_2} \sup_{t \in [0,1]} | \sum_{k=0}^{2^n-1} g_{n,k}(t) N_{n,k}| \le \sum_{n=m_1}^{m_2} 2^{-\frac{n}{4}} \overset{m_1, m_2 \to \infty}{\to} 0.$$

Thus, $\sum_{n=0}^{\infty}\sum_{k=0}^{2^n-1} g_{n,k}N_{n,k}(w)$ converge uniformly on $[0,1]$ and so

$$t \in [0,1] \mapsto B_t := tN_0 + \sum_{n=0}^{\infty}\sum_{k=0}^{2^n-1} g_{n,k}(t)N_{n,k} \text{ is continuous } (a.s.).$$

Moreover, since

$$\boldsymbol{E}[(B_t - B_s)^2] = \boldsymbol{E}[G(1_{(s,t]})^2] = t - s \quad \forall 0 \le s \le t \le 1$$

and

$$\boldsymbol{E}[(B_t - B_s)B_r] = \boldsymbol{E}[G(1_{(s,t]})G(1_{[0,r]})] = 0 \quad \forall 0 \le r \le s \le t \le 1,$$

we see that $B_t - B_s \sim \mathscr{N}(0, t-s)$ and $B_t - B_s \perp\!\!\!\perp \sigma(B_r, 0 \le r \le s)$ for every $0 \le s \le t \le 1$.

5. Let $\{N_0^m : m \ge 1\}\bigcup\{N_{n,k}^m : m \ge 1, n \ge 0, 0 \le k \le 2^n - 1\}$ be independent $\mathscr{N}(0,1)$. Define Gaussian white noises

$$G^m(c_0h_0 + \sum_{n=0}^{\infty}\sum_{k=0}^{2^n-1} c_{n,k}h_{n,k}) := c_0N_0^m + \sum_{n=0}^{\infty}\sum_{k=0}^{2^n-1} c_{n,k}N_{n,k}^m \quad \forall m \ge 1$$

and

$$B_t^m := G^m(1_{[0,t]}) = tN_0^m + \sum_{n=0}^{\infty}\sum_{k=0}^{2^n-1} g_{n,k}(t)N_{n,k}^m \quad \forall m \ge 1, t \in [0,1].$$

Then $B^1, B^2, ...$ are independent. Define

$$W_t := \sum_{k=1}^{m-1} B_1^k + B_{t-\lfloor t\rfloor}^m \text{ if } m-1 \le t < m.$$

Since $(B_t^m)_{t\in[0,1]}$ is continuous for every $m \ge 1$, we see that $(W_t)_{t\ge0}$ has continuous sample path. Moreover, since

$$W_t - W_s = B_{t-\lfloor t\rfloor}^m + B_1^{m-1} + ... + B_1^{n+1} + B_1^n - B_{s-\lfloor s\rfloor}^n \sim \mathscr{N}(0, t-s) \quad \forall 0 \le s < t, n-1 \le s < n, m-1 \le t < m$$

and

$$\boldsymbol{E}[(W_t - W_s)W_r] = 0 \quad \forall 0 \le r \le s \le t,$$

we see that we see that $W_t - W_s \perp\!\!\!\perp \sigma(W_r, 0 \le r \le s)$ for every $0 \le s \le t$ and so $(W_t)_{t\ge0}$ is a Brownian motion.

$\square$

Chapter 2

Brownian Motion

2.1 Exercise 2.25 (Time inversion)

Show that the process $(W_t)_{t\geq 0}$ defined by

$$W_t = \begin{cases} tB_{\frac{1}{t}}, & \text{if } t > 0 \\ 0, & \text{if } t = 0. \end{cases}$$

is indistinguishable of a real Brownian motion started from 0.

Proof.
First, we show that $(W_t)_{t\geq 0}$ is a pre-Brownian motion. That is $(W_t)_{t\geq 0}$ is a centered Gaussian with covariance function $K(t,s) = s \wedge t$. Since $(B_t)_{t\geq 0}$ is a centered Gaussian process, we see that $(W_t)_{t\geq 0}$ is a centered Gaussian process. Let $t > 0$ and $s > 0$. Then

$$\boldsymbol{E}[W_s W_t] = \boldsymbol{E}[tsB_{\frac{1}{t}}B_{\frac{1}{s}}] = ts(\frac{1}{s} \wedge \frac{1}{t}) = t \wedge s$$

and

$$\boldsymbol{E}[W_s W_0] = 0$$

Thus, $(W_t)_{t\geq 0}$ is a pre-Brownian motion.
Next, we show that

$$\lim_{t\to\infty} W_t = \lim_{t\to\infty} \frac{B_t}{t} = 0 \text{ a.s.}$$

By considering $(B_{k+1} - B_k)_{k\geq 0}$ and using the strong law of large number, we get

$$\frac{B_n}{n} \to 0 \text{ a.s.}$$

Let $m, n \geq 0$. By using Kolmogorov's inequality, we see that

$$\boldsymbol{P}(\max_{0\leq k\leq 2^m} |B_{n+\frac{k}{2^m}} - B_n| \geq n^{\frac{2}{3}}) \leq \frac{1}{n^{\frac{4}{3}}}\boldsymbol{E}[(B_{n+1} - B_n)^2] = \frac{1}{n^{\frac{4}{3}}}.$$

By letting $m \to \infty$, we get

$$\boldsymbol{P}(\sup_{t\in[n,n+1]} |B_t - B_n| \geq n^{\frac{2}{3}}) \leq \frac{1}{n^{\frac{4}{3}}}.$$

By using Borel-Cantelli is lemma, we have a.s.

$$|\frac{B_t}{t}| \leq \frac{1}{n^{\frac{1}{3}}} + \frac{B_n}{n} \text{ for large } n \text{ and } n \leq t \leq n+1$$

and, hence,

$$\lim_{t\to\infty} \frac{B_t}{t} = 0 \text{ a.s.}$$

Therefore, W_t is continuous at $t = 0$ a.s.
Finally, we set $E = \{\lim_{t\to\infty} \frac{B_t}{t} = 0\}$ and

$$\widetilde{W_t}(w) = \begin{cases} W_t(w), & \text{if } w \in E \\ 0, & \text{otherwise} \end{cases}$$

for all $t \geq 0$. Then $(\widetilde{W_t})_{t\geq 0}$ and $(W_t)_{t\geq 0}$ are indistinguishable. Since $(\widetilde{W_t})_{t\geq 0}$ has continuous sample path, we see that $(\widetilde{W_t})_{t\geq 0}$ is the Brownian motion. Thus, $(W_t)_{t\geq 0}$ is indistinguishable of a real Brownian motion $(\widetilde{W_t})_{t\geq 0}$ started from 0. $\square$

2.2 Exercise 2.26

For each real $a \geq 0$, we set $T_a = \inf\{t \geq 0 | B_t = a\}$. Show that the process $(T_a)_{a\geq 0}$ has stationary independent increments, in the sense that, for every $0 \leq a \leq b$, the variable $T_b - T_a$ is independent of the σ-field $\sigma(T_c, 0 \leq c \leq a)$ and has the same distribution as T_{b-a}.

Proof.

1. First, we show that $T_b - T_a \overset{D}{=} T_{b-a}$ for each $0 \leq a < b$. Given $0 \leq a < b$. Set
$$\widetilde{B_t} = 1_{T_a<\infty}(B_{T_a+t} - B_{T_a}).$$
Since $T_a < \infty$ a.s., we see that $(\widetilde{B_t})_{t\geq 0}$ is a Brownian motion on probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$. Set
$$\widetilde{T_c} = \inf\{t \geq 0 | \widetilde{B_t} = c\}$$
for each $c \in \mathbb{R}$. Then we see that $\widetilde{T_{b-a}} \overset{D}{=} T_{b-a}$. Since $T_a < \infty$ a.s., we have a.s. $s \geq T_a$ if $B_s = b$. Thus, we see that a.s.
$$\begin{aligned}
\widetilde{T_{b-a}} &= \inf\{t \geq 0 | \widetilde{B_t} = b - a\} \\
&= \inf\{t + T_a | B_{T_a+t} = b \text{ and } t \geq 0\} - T_a \\
&= \inf\{s | B_s = b \text{ and } s \geq T_a\} - T_a \\
&= \inf\{s | B_s = b\} - T_a = T_b - T_a
\end{aligned}$$
and therefore
$$T_b - T_a \overset{D}{=} T_{b-a}.$$

2. Next, we show that $T_b - T_a$ is independent of the σ-field $\sigma(T_c, 0 \leq c \leq a)$. Given $0 \leq a < b$. By using strong Markov property, we see that $\widetilde{B_t}$ is independent of $\mathcal{F}_{T_a}$. Since $T_c \leq T_a$ for $0 \leq c \leq a$, we have $\mathcal{F}_{T_c} \subseteq \mathcal{F}_{T_a}$ for each $0 \leq c \leq a$. Indeed, if $A \in \mathcal{F}_{T_c}$, then
$$A \bigcap \{T_a \leq t\} = (A \bigcap \{T_c \leq t\}) \bigcap \{T_a \leq t\} \in \mathcal{F}_t.$$
Therefore
$$\{T_{c_1} \leq t_1, ..., T_{c_n} \leq t_n\} \in \mathcal{F}_{T_a}$$
for each $n \geq 1$, $0 \leq c_1 \leq .. \leq c_n \leq a$, and non-negative real number $t_1, ..., t_n$. By using monotone class theorem, we have
$$\sigma(T_c, 0 \leq c \leq a) \subseteq \mathcal{F}_{T_a}.$$
Note that $T_b - T_a = \widetilde{T_{b-a}}$ a.s. To show $T_b - T_a$ is independent of $\sigma(T_c, 0 \leq c \leq a)$, it suffices to show that $\widetilde{T_{b-a}}$ is independent of $\sigma(T_c, 0 \leq c \leq a)$. Since $\{\widetilde{T_{b-a}} \leq t\} = \{\inf_{s\in\mathbb{Q}\cap[0,t]} |\widetilde{B_s} - (b - a)| = 0\}$ and $\widetilde{B_t}$ is independent of $\mathcal{F}_{T_a}$, we see that $\widetilde{T_{b-a}}$ is independent of $\mathcal{F}_{T_a}$. Because $\sigma(T_c, 0 \leq c \leq a) \subseteq \mathcal{F}_{T_a}$, we see that $T_b - T_a$ is independent of $\sigma(T_c, 0 \leq c \leq a)$.

$\square$

2.3 Exercise 2.27 (Brownian bridge)

We set $W_t = B_t - tB_1 \quad \forall t \in [0, 1]$.

1. Show that $(W_t)_{t\in[0,1]}$ is a centered Gaussian process and give its covariance function.

2. Let $0 < t_1 < t_2 < ... < t_m < 1$. Show that the law of $(W_{t_1}, W_{t_2}, ..., W_{t_m})$ has density

$$g(x_1, x_2, ..., x_m) = \sqrt{2\pi} p_{t_1}(x_1) p_{t_2 - t_2}(x_2 - x_1) ... p_{t_m - t_{m-1}}(x_m - x_{m-1}) p_{1-t_p}(-x_m),$$

where $p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp(\frac{-x^2}{2t})$. Explain why the law of $(W_{t_1}, W_{t_2}, ..., W_{t_m})$ can be interpreted as the conditional law of $(B_{t_1}, B_{t_2}, ..., B_{t_m})$ knowing that $B_1 = 0$.

3. Verify that the two processes $(W_t)_{t\in[0,1]}$ and $(W_{1-t})_{t\in[0,1]}$ have the same distribution (similarly as in the definition of Wiener measure, this law is a probability measure on the space of all continuous functions from $[0, 1]$ into $\mathbb{R}$).

Proof.

1. Let $0 < t_1 < t_2 < ... < t_m < 1$, $Q := \sum_{i=1}^m t_i c_i$,and $R_j := \sum_{i=j}^m c_i \quad \forall 1 \le j \le m$. Then

$$\sum_{i=1}^m c_i W_{t_i} = -Q(B_1 - B_{t_m}) + (Q + R_m)(B_{t_m} - B_{t_{m-1}}) + ... + (Q + R_2)(B_{t_2} - B_{t_1}) + (Q + R_1)B_{t_1}$$

is a centered Gaussian and so $(W_t)_{t\in[0,1]}$ is a centered Gaussian process. Moreover, the its covariance function

$$\boldsymbol{E}[W_t W_s] = \boldsymbol{E}[(B_t - tB_1)(B_s - sB_1)] = t \wedge s - ts - ts + ts = t \wedge s - ts \quad \forall t, s \in [0, 1].$$

2. Let $0 = t_0 < t_1 < t_2 < ... < t_m < t_{m+1} = 1$ and $F(x_1, ..., x_m)$ be nonnegative measurable function on $\mathbb{R}^m$. Then

$$\boldsymbol{E}[F(W_{t_1}, W_{t_2}, ..., W_{t_m})] = \boldsymbol{E}[F(B_{t_1} - t_1 B_1, B_{t_2} - t_2 B_1, ..., B_{t_m} - t_m B_1)]$$
$$= \int_{\mathbb{R}^{m+1}} F(x_1 - t_1 x_{m+1}, x_2 - t_2 x_{m+1}, ..., x_m - t_m x_{m+1}) \prod_{i=1}^{m+1} p_{t_i - t_{i-1}}(x_i - x_{i-1}) dx_1 ... dx_{m+1} (x_0 = 0)$$
$$= \int_{\mathbb{R}^{m+1}} F(y_1, y_2, ..., y_m) \prod_{i=1}^{m} p_{t_i - t_{i-1}}(y_i - y_{i-1} + (t_i - t_{i-1}) y_{m+1}) p_{1-t_m}(y_{m+1} - y_m - t_m y_{m+1}) dy_1 ... dy_{m+1}$$

(Set $y_0 = 0, y_i = x_i - t_i x_{m+1}$,and $y_{m+1} = x_{m+1}$).

Note that

$$p_{t_i - t_{i-1}}(y_i - y_{i-1} + (t_i - t_{i-1}) y_{m+1}) = p_{t_i - t_{i-1}}(y_i - y_{i-1}) \exp(-y_{m+1}(y_i - y_{i-1})) \exp(-\frac{1}{2}(t_i - t_{i-1}) y_{m+1}^2)$$

for each $1 \le i \le m$ and

$$p_{1-t_m}(y_{m+1} - y_m - t_m y_{m+1}) = p_{1-t_m}(-y_m) \exp(y_m y_{m+1}) \exp(-\frac{1}{2}(1 - t_m) y_{m+1}^2).$$

Then

$$\prod_{i=1}^m p_{t_i - t_{i-1}}(y_i - y_{i-1} + (t_i - t_{i-1}) y_{m+1}) p_{1-t_m}(y_{m+1} - y_m - t_m y_{m+1}) = \prod_{i=1}^m p_{t_i - t_{i-1}}(y_i - y_{i-1}) p_{1-t_m}(-y_m) \exp(-\frac{1}{2} y_{m+1}^2)$$

and so

$$\boldsymbol{E}[F(W_{t_1}, W_{t_2}, ..., W_{t_m})]$$
$$= \int_{\mathbb{R}^{m+1}} F(y_1, y_2, ..., y_m) \prod_{i=1}^{m} p_{t_i - t_{i-1}}(y_i - y_{i-1} + (t_i - t_{i-1}) y_{m+1}) p_{1-t_m}(y_{m+1} - y_m - t_m y_{m+1}) dy_1 ... dy_{m+1}$$
$$= \int_{\mathbb{R}^{m}} F(y_1, y_2, ..., y_m) \prod_{i=1}^{m} p_{t_i - t_{i-1}}(y_i - y_{i-1}) p_{1-t_m}(-y_m) (\int_{\mathbb{R}} \exp(-\frac{1}{2} y_{m+1}^2) dy_{m+1}) dy_1 ... dy_m$$
$$= \int_{\mathbb{R}^{m}} F(y_1, y_2, ..., y_m) \prod_{i=1}^{m} p_{t_i - t_{i-1}}(y_i - y_{i-1}) p_{1-t_m}(-y_m) \sqrt{2\pi} dy_1 ... dy_m.$$

3. We have twos ways to explain why the law of Brownian bridge $(W_t)_{t\in[0,1]}$ can be interpreted as the conditional law of $(B_t)_{t\in[0,1]}$ knowing that $B_1 = 0$.

   (a) First, we show that, if $B_1(w) = 0$, then

$$\boldsymbol{E}[F(B_{t_1}, ..., B_{t_m})|B_1](w) = \int_{\mathbb{R}^m} F(x_1, ..., x_m)g(x_1, ..., x_m)dx_1...dx_m$$

   for every $0 = t_0 < t_1 < t_2 < ... < t_m < t_{m+1} = 1$ and $F(x_1, ..., x_m)$ be nonnegative measurable function on $\mathbb{R}^m$. Observe that

$$\boldsymbol{E}[F(B_{t_1}, ..., B_{t_m})|B_1] = \varphi(B_1),$$

   where $x_0 = 0$,

$$q(x_{m+1}) = \int_{\mathbb{R}^m} f_{B_{t_1},...,B_{t_m},B_1}(x_1, ..., x_m, x_{m+1})dx_1...dx_m = \int_{\mathbb{R}^m} \prod_{i=1}^{m+1} p_{t_i-t_{i-1}}(x_i - x_{i-1})dx_1...dx_m,$$

   and

$$\begin{aligned}\varphi(x_{m+1}) &= \frac{1}{q(x_{m+1})}\int_{\mathbb{R}^m} F(x_1, ..., x_m)f_{B_{t_1},...,B_{t_m},B_1}(x_1, ..., x_m, x_{m+1})dx_1...dx_m \\ &= \frac{1}{q(x_{m+1})}\int_{\mathbb{R}^m} F(x_1, ..., x_m)\prod_{i=1}^{m+1} p_{t_i-t_{i-1}}(x_i - x_{i-1})dx_1...dx_m.\end{aligned}$$

   Note that

$$q(0) = \int_{\mathbb{R}^m} \prod_{i=1}^{m} p_{t_i-t_{i-1}}(x_i - x_{i-1})p_{1-t_m}(-x_m)dx_1...dx_m = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^m} g(x_1, ..., x_m)dx_1...dx_m = \frac{1}{\sqrt{2\pi}}$$

   and

$$\begin{aligned}\varphi(0) &= \frac{1}{q(0)}\int_{\mathbb{R}^m} F(x_1, ..., x_m)\prod_{i=1}^{m+1} p_{t_i-t_{i-1}}(x_i - x_{i-1})dx_1...dx_m \\ &= \sqrt{2\pi}\int_{\mathbb{R}^m} F(x_1, ..., x_m)\prod_{i=1}^{m} p_{t_i-t_{i-1}}(x_i - x_{i-1})p_{1-t_m}(-x_m)dx_1...dx_m \\ &= \sqrt{2\pi}\int_{\mathbb{R}^m} F(x_1, ..., x_m)\frac{1}{\sqrt{2\pi}}g(x_1, ..., x_m)dx_1...dx_m \\ &= \int_{\mathbb{R}^m} F(x_1, ..., x_m)g(x_1, ..., x_m)dx_1...dx_m.\end{aligned}$$

   Thus, if $w \in \{B_1 = 0\}$, then

$$\boldsymbol{E}[F(B_{t_1}, ..., B_{t_m})|B_1](w) = \varphi(0) = \int_{\mathbb{R}^m} F(x_1, ..., x_m)g(x_1, ..., x_m)dx_1...dx_m.$$

   (b) Next, we show that

$$((B_{t_1,...,B_{t_m}})||B_1| \le \epsilon) \xrightarrow{d} (W_{t_1}, ..., W_{t_m})$$

   for every $0 < t_1 < t_2 < ... < t_m < 1$ and so the conditional law of $(B_t)_{t\in[0,1]}$ knowing that $|B_1| \le \epsilon$ converges weakly to the law of $(W_t)_{t\in[0,1]}$. Given $0 < t_1 < t_2 < ... < t_m < 1$ and $F(x_1, ..., x_m)$ be nonnegative measurable function on $\mathbb{R}^m$. Set

$$\mu_\epsilon(dx_1...dx_m) := \boldsymbol{P}((B_{t_1}, ..., B_{t_m} \in dx_1...dx_m)||B_1| \le \epsilon) \quad \forall \epsilon > 0.$$

Then

$$\begin{aligned}\int F(x_1,....,x_m)\mu_\epsilon(dx_1...dx_m) &= \boldsymbol{P}(|B_1|\le\epsilon)^{-1}\boldsymbol{E}[F(B_{t_1},...,B_{t_m})1_{\{|B_1|\le\epsilon\}}]\\ &= \boldsymbol{P}(|B_1|\le\epsilon)^{-1}\boldsymbol{E}[\boldsymbol{E}[F(B_{t_1},...,B_{t_m})|B_1]1_{\{|B_1|\le\epsilon\}}]\\ &= \boldsymbol{P}(|B_1|\le\epsilon)^{-1}\boldsymbol{E}[\varphi(B_1)1_{\{|B_1|\le\epsilon\}}]\\ &= \int_{\mathbb{R}}\varphi(x)\times(\boldsymbol{P}(|B_1|\le\epsilon)^{-1}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}1_{\{|x|\le\epsilon\}})dx.\end{aligned}$$

It's clear that $\varphi(x)$ is continuous and so

$$\int F(x_1,....,x_m)\mu_\epsilon(dx_1...dx_m)\to\varphi(0)=\int_{\mathbb{R}^m}F(x_1,...,x_m)g(x_1,...,x_m)dx_1...dx_m \text{ as } \epsilon\to 0.$$

4. Let $0=t_0<t_1<t_2<...<t_m<t_{m+1}=1$ and $F(x_1,...,x_m)$ be nonnegative measurable function on $\mathbb{R}^m$. Set $s_i=1-t_{m+1-i}$ for every $0\le i\le m+1$. Then

$$\begin{aligned}&\boldsymbol{E}[F(W_{1-t_1},...,W_{1-t_m})]=\boldsymbol{E}[F(W_{s_m},...,W_{s_1})]\\ &=\int_{\mathbb{R}^m}F(y_m,y_{m-1},...,y_1)\prod_{i=1}^m p_{s_i-s_{i-1}}(y_i-y_{i-1})p_{1-s_m}(y_m)\sqrt{2\pi}dy_1...dy_m\\ &=\int_{\mathbb{R}^m}F(x_1,...,x_m)\prod_{i=1}^m p_{s_i-s_{i-1}}(x_i-x_{i-1})p_{1-s_m}(x_m)\sqrt{2\pi}dx_1...dx_m\\ &=\int_{\mathbb{R}^m}F(x_1,...,x_m)\prod_{i=1}^m p_{t_i-t_{i-1}}(x_i-x_{i-1})p_{1-t_m}(x_m)\sqrt{2\pi}dx_1...dx_m\\ &=\boldsymbol{E}[F(W_{t_1},...,W_{t_m})]\end{aligned}$$

and so $(W_t)_{t\in[0,1]}$ and $(W_{1-t})_{t\in[0,1]}$ have the same distribution.

□

2.4 Exercise 2.28 (Local maxima of Brownian paths)

Show that, a.s., the local maxima of Brownian motion are distinct: a.s., for any choice of the rational numbers $0\le p<q<r<s$, we have

$$\sup_{p\le t\le q}B_t\ne\sup_{r\le t\le s}B_t.$$

Proof.
Fixed any rational numbers $0\le p<q<r<s$. We show that

$$\boldsymbol{P}(\sup_{p\le t\le q}B_t=\sup_{r\le t\le s}B_t)=0.$$

Set

$$X=\sup_{p\le t\le q}B_t-B_r$$

and

$$Y=\sup_{r\le t\le s}B_t-B_r.$$

Since $\{B_r-B_t|p\le t\le q\}$ and $\{B_t-B_r|r\le t\le s\}$ are independent, we see that X and Y are independent

By using simple Markov property, we see that $(B_t - B_r)_{t\geq r}$ is a Brownian motion. Set $S_t = \sup_{t\geq r} B_t - B_r$. By using reflection principle, we have

$$\begin{aligned}
\boldsymbol{P}(S_t \geq a) &= \boldsymbol{P}(\sup_{t\geq r} B_t - B_r \geq a)\\
&= \boldsymbol{P}(\sup_{t\geq r} B_{t-r} \geq a)\\
&= \boldsymbol{P}(|B_{t-r}| \geq a)
\end{aligned}$$

and, hence, S_t is a continuous random variable for each $t \geq r$. Therefore,

$$\begin{aligned}
\boldsymbol{P}(\sup_{p\leq t\leq q} B_t = \sup_{r\leq t\leq s} B_t) &= \boldsymbol{P}(\sup_{p\leq t\leq q} B_t - B_r = \sup_{r\leq t\leq s} B_t - B_r)\\
&= \boldsymbol{P}(X - Y = 0)\\
&= \int_{\mathbb{R}^2} 1_{\{0\}}(x+y)\boldsymbol{P}_{(X,-Y)}(dx \times dy)\\
&= \int_{\mathbb{R}^2} 1_{\{0\}}(x+y)\boldsymbol{P}_{(X,-Y)}(dx \times dy)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} 1_{\{0\}}(x+y)\boldsymbol{P}_{-Y}(dy)\boldsymbol{P}_X(dx)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} 1_{\{-x\}}(y)\boldsymbol{P}_{-Y}(dy)\boldsymbol{P}_X(dx)\\
&= \int_{\mathbb{R}} \boldsymbol{P}(-Y = -x)\boldsymbol{P}_X(dx) = 0
\end{aligned}$$

Thus, we have

$$\boldsymbol{P}(\bigcup_{0\leq p<q<r<s \text{ are rational}} \sup_{p\leq t\leq q} B_t = \sup_{r\leq t\leq s} B_t) = 0$$

□

2.5 Exercise 2.29 (Non-differentiability)

Show that, a.s.,

$$\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{t}} = \infty \text{ and } \liminf_{t\downarrow 0} \frac{B_t}{\sqrt{t}} = -\infty,$$

and infer that, for each $s \geq 0$, the function $t \mapsto B_t$ has a.s. no right derivative at s.

Proof.

1. First, we show that a.s.,

   $$\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{t}} = \infty \text{ and } \liminf_{t\downarrow 0} \frac{B_t}{\sqrt{t}} = -\infty.$$

   Given $M > 0$. Since

   $$\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{t}} = \lim_{c\downarrow 0} \sup_{0\leq t\leq c} \frac{B_t}{\sqrt{t}} \in \mathscr{F}_{0+}$$

   and therefore

   $$\{\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{t}} \geq M\} \in \mathscr{F}_{0+}.$$

Now, by Fatou's lemma, we have

$$\begin{aligned}
&\boldsymbol{P}(\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{t}} \geq M) \\
&\geq \boldsymbol{P}(\limsup_{n\to\infty} \frac{B_{n^{-1}}}{\sqrt{n^{-1}}} \geq M) \\
&= \boldsymbol{P}(\frac{B_{n^{-1}}}{\sqrt{n^{-1}}} \geq M \text{ i.o }) \\
&= \boldsymbol{P}(\limsup_{n\to\infty}\{\frac{B_{n^{-1}}}{\sqrt{n^{-1}}} \geq M\}) \\
&\geq \limsup_{n\to\infty} \boldsymbol{P}(\frac{B_{n^{-1}}}{\sqrt{n^{-1}}} \geq M) \\
&= \int_M^\infty \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})dx > 0
\end{aligned}$$

Therefore, by zero-one law, we have a.s.

$$\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{t}} \geq M.$$

Since M is arbitrary, we get

$$\boldsymbol{P}(\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{t}} = \infty) = \lim_{n\to\infty} \boldsymbol{P}(\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{t}} \geq n) = 1.$$

Because $(-B_t)_{t\geq 0}$ is a Brownian motion, we see that

$$\boldsymbol{P}(\liminf_{t\downarrow 0} \frac{B_t}{\sqrt{t}} = -\infty) = \boldsymbol{P}(\limsup_{t\downarrow 0} \frac{-B_t}{\sqrt{t}} = \infty) = 1.$$

2. We show that, for each $s \geq 0$, the function $t \mapsto B_t$ has a.s. no right derivative at s. Given $s \geq 0$. Observe that

$$\begin{aligned}
&\boldsymbol{P}(\limsup_{t\downarrow s} \frac{B_t - B_s}{t-s} = \infty) \\
&= \boldsymbol{P}(\limsup_{t\downarrow s} \frac{B_t - B_s}{\sqrt{t-s}} \times \frac{1}{\sqrt{t-s}} = \infty) \\
&= \boldsymbol{P}(\limsup_{t\downarrow s} \frac{B_{t-s}}{\sqrt{t-s}} = \infty) = 1
\end{aligned}$$

and

$$\begin{aligned}
&\boldsymbol{P}(\liminf_{t\downarrow s} \frac{B_t - B_s}{t-s} = -\infty) \\
&= \boldsymbol{P}(\liminf_{t\downarrow s} \frac{B_t - B_s}{\sqrt{t-s}} \times \frac{1}{\sqrt{t-s}} = -\infty) \\
&= \boldsymbol{P}(\liminf_{t\downarrow s} \frac{B_{t-s}}{\sqrt{t-s}} = -\infty) = 1
\end{aligned}$$

Then the function $t \mapsto B_t$ has a.s. no right derivative at s.

$\square$

2.6 Exercise 2.30 (Zero set of Brownian motion)

Let $H = \{t \in [0,1] | B_t = 0\}$. Show that H is a.s. a compact subset of $[0,1]$ with no isolated point and zero Lebesgue measure.

Proof.
Since $(B_t)_{t\in[0,1]}$ is continuous, we see that H is closed and so H is compact. Observe that

$$\boldsymbol{E}[\lambda_{\mathbb{R}}(H)] = \int_\Omega \int_0^1 1_{\{s\in[0,1]:B_s=0\}}(t)dt\boldsymbol{P}(dw) = \int_0^1 \int_\Omega 1_{\{s\in[0,1]:B_s=0\}}(t)\boldsymbol{P}(dw)dt = \int_0^1 \boldsymbol{P}(B_t = 0)dt = 0$$

and so $\lambda_{\mathbb{R}}(H) = 0$ (a.s.).
Now, we show that H has no isolated points (a.s.). Define

$$T_q := \inf\{t \geq q : B_t = 0\} \quad \forall q \in [0,1) \bigcap \mathbb{Q}.$$

Observe that

$$\boldsymbol{P}(\sup_{0\leq s\leq \epsilon} B_{T_q+s} > 0 \text{ and } \inf_{0\leq s\leq \epsilon} B_{T_q+s} < 0 \quad \forall \epsilon \in (0, 1-q) \bigcap \mathbb{Q}, \quad \forall q \in [0,1) \bigcap \mathbb{Q}) = 1.$$

Indeed, by proposition 2.14 and the strong Markov property, we get

$$\begin{aligned}
&\boldsymbol{P}(\sup_{0\leq s\leq \epsilon} B_{T_q+s} > 0 \text{ and } \inf_{0\leq s\leq \epsilon} B_{T_q+s} < 0 \quad \forall \epsilon \in (0, 1-q) \bigcap \mathbb{Q}) \\
&= \boldsymbol{P}(\sup_{0\leq s\leq \epsilon} B_s > 0 \text{ and } \inf_{0\leq s\leq \epsilon} B_s < 0 \quad \forall \epsilon \in (0,1-q) \bigcap \mathbb{Q}) = 1 \quad \forall q \in [0,1) \bigcap \mathbb{Q}.
\end{aligned}$$

Set

$$E := \bigcap_{q\in[0,1)\bigcap\mathbb{Q}} \bigcap_{\epsilon\in(0,1-q)\bigcap\mathbb{Q}} \{\exists p \in (0,1) \bigcap \mathbb{Q} \quad T_q < T_p < T_q + \epsilon\}.$$

Then $\boldsymbol{P}(E) = 1$ and so T_q is not an isolated point for every $q \in [0,1) \bigcap \mathbb{Q}$ (a.s.). Fix $w \in E$. Let $t \in H \setminus \{T_q : q \in [0,1) \bigcap \mathbb{Q}\}$. Choose $q_n \in [0,1) \bigcap \mathbb{Q}$ such that $q_n \uparrow t$. Since $q_n < t$ and $B_t = 0$, we have

$$q_n \leq T_{q_n} \leq t \quad \forall n \geq 1$$

and so $T_{q_n} \uparrow t$. Thus, t is not an isolated. Therefore, H has no isolated points (a.s.). □

2.7 Exercise 2.31 (Time reversal)

We set $B'_t = B_1 - B_{1-t}$ for every $t \in [0,1]$. Show that the two processes $(B_t)_{t\in[0,1]}$ and $(B'_t)_{t\in[0,1]}$ have the same law (as in the definition of Wiener measure, this law is a probability measure on the space of all continuous functions from $[0,1]$ into $\mathbb{R}$).

Proof.
Let $0 = t_0 < t_1 < t_2 < ... < t_m < t_{m+1} = 1$ and $F(x_1, ..., x_m)$ be nonnegative measurable function on $\mathbb{R}^m$. Set

$s_i = 1 - t_{m+1-i}$ for every $0 \le i \le m+1$ and $p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp(-\frac{x^2}{2t})$. Then

$$\boldsymbol{E}[F(B'_{t_1}, ..., B'_{t_m})] = \boldsymbol{E}[F(B_1 - B_{s_m}, ..., B_1 - B_{s_1})]$$
$$= \int_{\mathbb{R}^{m+1}} F(x_{m+1} - x_m, x_{m+1} - x_{m-1}, ..., x_{m+1} - x_1) \prod_{i=1}^{m+1} p_{s_i - s_{i-1}}(x_i - x_{i-1}) dx_1 ... dx_{m+1} (x_0 = 0)$$
$$= \int_{\mathbb{R}^{m+1}} F(y_1, y_2, ..., y_m) \prod_{i=1}^{m+1} p_{t_{m+1-(i-1)} - t_{m+1-i}}(y_{m+1-(i-1)} - y_{m+1-i}) dy_1 ... dy_{m+1} \quad (y_i = x_{m+1} - x_{m+1-i} \quad \forall 0 \le i \le m+1)$$
$$= \int_{\mathbb{R}^{m+1}} F(y_1, y_2, ..., y_m) \prod_{i=1}^{m+1} p_{t_i - t_{i-1}}(y_i - y_{i-1}) dy_1 ... dy_{m+1}$$
$$= \int_{\mathbb{R}^m} F(y_1, y_2, ..., y_m) \prod_{i=1}^{m} p_{t_i - t_{i-1}}(y_i - y_{i-1}) \times (\int_{\mathbb{R}} p_{t_{m+1} - t_m}(y_{m+1} - y_m) dy_{m+1}) dy_1 ... dy_m$$
$$= \int_{\mathbb{R}^m} F(y_1, y_2, ..., y_m) \prod_{i=1}^{m} p_{t_i - t_{i-1}}(y_i - y_{i-1}) \times 1 dy_1 ... dy_m = \boldsymbol{E}[F(B_{t_1}, ..., B_{t_m})]$$

and so $(B_t)_{t \in [0,1]}$ and $(B'_t)_{t \in [0,1]}$ have the same distribution. □

2.8 Exercise 2.32 (Arcsine law)

Set $T := \inf\{t \ge 0 : B_t = S_1\}$.

1. Show that $T < 1$ a.s. (one may use the result of the previous exercise) and then that T is not a stopping time.
2. Verify that the three variables S_t, $S_t - B_t$ and $|B_t|$ have the same law.
3. Show that T is distributed according to the so-called arcsine law, whose density is
$$g(t) = \frac{1}{\pi \sqrt{t(1-t)}} 1_{(0,1)}(t).$$
4. Show that the results of questions 1. and 3. remain valid if T is replaced by
$$L := \sup\{t \le 1 : B_t = 0\}.$$

Proof.

1. It's clear that $\boldsymbol{P}(T \le 1) = 1$. Suppose that $\boldsymbol{P}(T = 1) > 0$. By exercise 2.31 and proposition 2.14, we get
$$\boldsymbol{P}(\inf_{0 \le s \le \epsilon} B'_s < 0 \quad \forall \epsilon \in (0,1)) = \boldsymbol{P}(\inf_{0 \le s \le \epsilon} B_s < 0 \quad \forall \epsilon \in (0,1)) = 1,$$
where $B'_t = B_1 - B_{1-t}$ for every $t \in [0,1]$. On the other hand,
$$0 < \boldsymbol{P}(T = 1) \le \boldsymbol{P}(B'_s \ge 0 \quad \forall s \in [0,1])$$
which is a contradiction. Thus, we have $\boldsymbol{P}(T < 1) = 1$.
Now, we show that T is not a stopping time by contradiction. Assume that T is a stopping time. By theorem 2.20 (strong Markov property), we see that $B^T_t = B_{T+t} - B_T$ is a Brownian motion. Since $\boldsymbol{P}(T < 1) = 1$, we get
$$\boldsymbol{P}(\sup_{0 \le s \le \epsilon} B^T_s \le 0 \text{ for some } \epsilon > 0) = 1,$$

which contradiction to (proposition 2.14)

$$\boldsymbol{P}(\sup_{0\le s\le\epsilon} B_s^T > 0 \quad \forall\epsilon>0) = 1.$$

Thus, we see that T is not a topping time.

2. Fix $t>0$. By theorem 2.21, we have $S_t \overset{d}{=} |B_t|$. Now, we show that $S_t \overset{d}{=} S_t - B_t$. By similar argument as the proof of exercise 2.31, we get $(B'_s)_{s\in[0,t]} \overset{d}{=} (B_s)_{s\in[0,t]}$, where $B'_s = B_t - B_{t-s}$ for every $s\in[0,t]$. It's clear that $(B'_s)_{s\in[0,t]} \overset{d}{=} (-B'_s)_{s\in[0,t]}$. Thus, we have

$$S_t = \sup_{0\le s\le t} B_s \overset{d}{=} \sup_{0\le s\le t} -B'_s = \sup_{0\le s\le t} B_{t-s} - B_t = \sup_{0\le s\le t} B_s - B_t = S_t - B_t.$$

3. Since

$$\boldsymbol{P}(\sup_{p_1\le s\le q_1} B_s \ne \sup_{p_2\le s\le q_2} B_s \text{ for all rational numbers } p_1<q_1<p_2<q_2) = 1,$$

we see that the global maximum of $(B_t)_{t\in[0,1]}$ is attained at a unique time (a.s.). That is,

$$\boldsymbol{P}(\exists! t\in[0,1] \quad B_t = S_1) = 1.$$

Let $r\in(0,1)$ and $Z_1, Z_2 \overset{i.i.d}{\sim} \mathcal{N}(0,1)$. Then

$$\boldsymbol{P}(T<r) = \boldsymbol{P}(\max_{0\le t\le r} B_t > \max_{r\le s\le 1} B_s) = \boldsymbol{P}(\max_{0\le t\le r} B_t - B_r > \max_{r\le s\le 1} B_s - B_r).$$

Since

$$\max_{0\le t\le r} B_t - B_r \perp\!\!\!\perp \max_{r\le s\le 1} B_s - B_r,$$

$$\max_{0\le t\le r} B_t - B_r = \max_{0\le t\le r}(B_{r-t} - B_r) \overset{d}{=} \max_{0\le t\le r} B_t = S_r \overset{d}{=} |\sqrt{r}Z_1|,$$

and

$$\max_{r\le s\le 1} B_s - B_r = \max_{r\le s\le 1}(B_s - B_r) \overset{d}{=} \max_{0\le s\le 1-r} B_s = S_{1-r} \overset{d}{=} \sqrt{1-r}|Z_2|,$$

we get

$$\boldsymbol{P}(T<r) = \boldsymbol{P}(\sqrt{r}|Z_1| > \sqrt{1-r}|Z_2|) = \boldsymbol{P}(\frac{|Z_2|^2}{|Z_1|^2+|Z_2|^2} < r)$$

and so $T \overset{d}{=} \frac{|Z_2|^2}{|Z_1|^2+|Z_2|^2}$. Since

$$\begin{aligned}
\boldsymbol{E}[f(\frac{|Z_2|^2}{|Z_1|^2+|Z_2|^2})] &= \int_{\mathbb{R}^2} f(\frac{y^2}{x^2+y^2})\frac{1}{2\pi}\exp(-\frac{x^2+y^2}{2})dxdy \\
&= 4\int_0^\infty\int_0^\infty f(\frac{y^2}{x^2+y^2})\frac{1}{2\pi}\exp(-\frac{x^2+y^2}{2})dxdy \\
&= 4\int_0^{\frac{\pi}{2}}\int_0^\infty f(\sin(\theta)^2)\frac{1}{2\pi}\exp(-\frac{x^2+y^2}{2})rdrd\theta \\
&= \frac{2}{\pi}\int_0^1 f(t)\frac{1}{2\sqrt{1-t}\sqrt{t}}dt \\
&= \int_{\mathbb{R}} \frac{1}{\pi}\frac{1}{\sqrt{t(1-t)}}1_{(0,1)}(t)dt,
\end{aligned}$$

we see that

$$g(t) = \frac{1}{\pi\sqrt{t(1-t)}}1_{(0,1)}(t)$$

is the density function of T.

4. We redefine $L(f)$ as the latest time of $f \in C([0,1])$ such that $f(t) = f(0)$. That is,
$$L(f) = \sup\{t \le 1 : f(t) = f(0)\}.$$
Then $L = L((|B_t|)_{t\in[0,1]})$. Since the global maximum of $(B_t)_{t\in[0,1]}$ is attained at a unique time (a.s.), we see that $T = L((S_t - B_t)_{t\in[0,1]})$ (a.s.). Since $S_t - B_t \overset{d}{=} |B_t|$ for every $t \ge 0$ and they have continuous sample path, we see that $(S_t - B_t)_{t\ge 0} \overset{d}{=} (|B_t|)_{t\ge 0}$ and so $L \overset{d}{=} T$. Thus, $g(t)$ is the density function of L, $L < 1$ (a.s.), and L is not a stopping time. Indeed, if L is a stopping time,
$$B'_t := B_{L+t} - B_L \overset{(a.s.)}{=} B_{L+t} \quad \forall t \ge 0$$
is a Brownian motion with 0 is an isolated point of $\{t \in [0,1] : B'_t = 0\}$ (a.s.) which contradict to Exercise 2.30.

□

2.9 Exercise 2.33 (Law of the iterated logarithm)

The goal of the exercise is to prove that
$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = 1 \ a.s.$$
We set $h(t) = \sqrt{2t\log\log t}$.

1. Show that, for every $t > 0$,
$$\boldsymbol{P}(S_t > u\sqrt{t}) \sim \frac{2}{u\sqrt{2\pi}} \exp(-\frac{u^2}{2}),$$
when $u \to \infty$.
2. Let r and c be two real numbers such that $1 < r < c^2$ and set $S_t = \sup_{s\le t} B_s$. From the behavior of the probabilities $\boldsymbol{P}(S_{r^n} > ch(r^{n-1}))$ when $n \to \infty$, infer that, a.s.,
$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log 2t}} \le 1.$$
3. Show that a.s. there are infinitely many values of n such that
$$B_{r^n} - B_{r^{n-1}} \ge \sqrt{\frac{r-1}{r}} h(r^n).$$
Conclude that the statement given at the beginning of the exercise holds.
4. What is the value of
$$\liminf_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}}?$$

Proof.

1. Given $t > 0$. By using the reflection principle, we have
$$\begin{aligned}
&\boldsymbol{P}(S_t > u\sqrt{t}) \\
&= \boldsymbol{P}(S_t > u\sqrt{t}, B_t > u\sqrt{t}) + \boldsymbol{P}(S_t > u\sqrt{t}, B_t \le u\sqrt{t}) \\
&= \boldsymbol{P}(B_t > u\sqrt{t}) + \boldsymbol{P}(B_t \ge u\sqrt{t}) \\
&= 2\boldsymbol{P}(B_t \ge u\sqrt{t}) \\
&= 2\int_{u\sqrt{t}}^{\infty} \frac{1}{\sqrt{2\pi t}} \exp(-\frac{x^2}{2t})dx \\
&= \frac{2}{\sqrt{2\pi}} \int_u^{\infty} \exp(-\frac{y^2}{2})dy
\end{aligned}$$

Note that, for $x > 0$,

$$(\frac{1}{x} - \frac{1}{x^3}) \exp(-\frac{x^2}{2}) \leq \int_x^\infty \exp(-\frac{y^2}{2}) dy \leq \frac{1}{x} \exp(-\frac{x^2}{2}).$$

Indeed, since $\exp(-\frac{z^2}{2}) \leq 1$ and

$$\int_x^\infty (1 - \frac{3}{y^4}) \exp(-\frac{y^2}{2}) dy = (\frac{1}{x} - \frac{1}{x^3}) \exp(-\frac{x^2}{2}),$$

we have

$$\int_x^\infty \exp(-\frac{y^2}{2}) dy = \int_0^\infty \exp(-\frac{(z+x)^2}{2}) dz \leq \exp(-\frac{x^2}{2}) \int_0^\infty \exp(-xz) dz = \frac{1}{x} \exp(-\frac{x^2}{2})$$

and

$$(\frac{1}{x} - \frac{1}{x^3}) \exp(-\frac{x^2}{2}) \leq \int_x^\infty \exp(-\frac{y^2}{2}) dy.$$

Thus,

$$\frac{2}{\sqrt{2\pi}} (\frac{1}{u} - \frac{1}{u^3}) \exp(-\frac{u^2}{2}) \leq \boldsymbol{P}(S_t > u\sqrt{t}) \leq \frac{2}{\sqrt{2\pi}} \frac{1}{u} \exp(-\frac{u^2}{2})$$

and therefore

$$\boldsymbol{P}(S_t > u\sqrt{t}) \sim \frac{2}{u\sqrt{2\pi}} \exp(-\frac{u^2}{2}),$$

when $u \to \infty$.

2. Given $1 < r < c^2$. By using similar argument, we have

$$\boldsymbol{P}(S_{r^n} > ch(r^{n-1})) = 2 \int_{ch(r^{n-1})}^\infty \frac{1}{\sqrt{2\pi r^n}} \exp(-\frac{x^2}{2r^n}) dx = \frac{2}{\sqrt{2\pi}} \int_{\frac{ch(r^{n-1})}{\sqrt{r^n}}} \exp(-\frac{y^2}{2}) dy.$$

Because

$$\frac{h(r^{n-1})}{\sqrt{r^n}} \to \infty \text{ as } n \to \infty$$

and

$$\int_x^\infty \exp(-\frac{y^2}{2}) dy \leq \frac{1}{x} \exp(-\frac{x^2}{2}),$$

we get

$$\lim_{n\to\infty} \boldsymbol{P}(S_{r^n} > ch(r^{n-1})) \leq \lim_{n\to\infty} \frac{2}{\sqrt{2\pi}} \frac{\sqrt{r^n}}{ch(r^{n-1})} \exp(-\frac{1}{2} \frac{c^2 h(r^{n-1})^2}{r^n}) = 0.$$

Choose $\{n_k\}$ such that

$$\sum_{k=1}^\infty \boldsymbol{P}(S_{r^{n_k}} > ch(r^{n_k - 1})) < \infty.$$

By using Borel-Cantelli lemma, we get

$$\boldsymbol{P}(\frac{S_{r^{n_k}}}{h(r^{n_k})} > c \frac{h(r^{n_k-1})}{h(r^{n_k})} \text{ i.o. }) = \boldsymbol{P}(S_{r^{n_k}} > ch(r^{n_k-1}) \text{ i.o. }) = 0.$$

Observe that

$$\lim_{k\to\infty} \frac{h(r^{n_k-1})}{h(r^{n_k})} = \frac{1}{\sqrt{r}}.$$

Then

$$\boldsymbol{P}(\limsup_{t\to\infty} \frac{S_t}{h(t)} \geq \frac{c}{\sqrt{r}}) = 0$$

and, hence,

$$\boldsymbol{P}(\limsup_{t\to\infty}\frac{B_t}{h(t)}\le\frac{c}{\sqrt{r}})\ge\boldsymbol{P}(\limsup_{t\to\infty}\frac{S_t}{h(t)}\le\frac{c}{\sqrt{r}})=1.$$

Fixed $r>1$. Choose $\{c_n\}$ such that $1<r<c_n^2$ and $c_n^2\downarrow r$. Then

$$\boldsymbol{P}(\limsup_{t\to\infty}\frac{B_t}{h(t)}\le\frac{c_n}{\sqrt{r}})=1$$

for each $n\ge 1$. By letting $n\to\infty$, we have

$$\boldsymbol{P}(\limsup_{t\to\infty}\frac{B_t}{h(t)}\le 1)=1$$

3. Given $r>1$. Set d to be the positive number such that $d=\log(r)$. By using the fact that the increments of Brownian motion are Gaussian random variables, we have

$$\begin{aligned}
&\boldsymbol{P}(B_{r^n}-B_{r^{n-1}}\ge\sqrt{\frac{r-1}{r}}h(r^n))\\
&=\boldsymbol{P}(\frac{B_{r^n}-B_{r^{n-1}}}{\sqrt{r^n-r^{n-1}}}\ge\sqrt{2\log\log r^n})\\
&=\boldsymbol{P}(\frac{B_{r^n}-B_{r^{n-1}}}{\sqrt{r^n-r^{n-1}}}\ge\sqrt{2\log dn})\\
&=\int_{\sqrt{2\log dn}}^{\infty}\frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})dx\\
&\ge\frac{1}{\sqrt{2\pi}}(\frac{1}{\sqrt{2\log dn}}-\frac{1}{(2\log dn)^{\frac{3}{2}}})\frac{1}{dn}
\end{aligned}$$

Because $\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\log n}}=\infty$ and $\sum_{n=2}^{\infty}\frac{1}{n(\log n)^{\frac{3}{2}}}<\infty$, we see that

$$\sum_{n=1}^{\infty}\boldsymbol{P}(B_{r^n}-B_{r^{n-1}}\ge\sqrt{\frac{r-1}{r}}h(r^n))=\infty.$$

Note that $\{B_{r^n}-B_{r^{n-1}}\}_{n\ge1}$ are independent. By using Borel-Cantelli lemma, we have

$$\boldsymbol{P}(B_{r^n}-B_{r^{n-1}}\ge\sqrt{\frac{r-1}{r}}h(r^n)\text{ i.o. })=1.$$

Now, we show that

$$\boldsymbol{P}(\limsup_{t\to\infty}\frac{B_t}{h(t)}=1)=1.$$

It remain to show that

$$\boldsymbol{P}(\limsup_{t\to\infty}\frac{B_t}{h(t)}\ge 1)=1.$$

Given $r>1$. Since

$$\boldsymbol{P}(B_{r^n}-B_{r^{n-1}}\ge\sqrt{\frac{r-1}{r}}h(r^n)\text{ i.o. })=1,$$

we have

$$\boldsymbol{P}(\frac{B_{r^n}}{h(r^n)}\ge\sqrt{\frac{r-1}{r}}+\sqrt{\frac{\log\log r^{n-1}}{\log\log r^n}}\sqrt{\frac{1}{r}}\frac{B_{r^{n-1}}}{h(r^{n-1})}\text{ i.o. })=1,$$

and, hence, we have a.s.

$$\limsup_{t\to\infty} \frac{B_t}{h(t)} \geq \frac{r-1}{r} + \sqrt{\frac{1}{r}} \limsup_{t\to\infty} \frac{B_t}{h(t)}.$$

Thus,

$$\boldsymbol{P}((\limsup_{t\to\infty} \frac{B_t}{h(t)})^2 \geq \frac{r-1}{r-2\sqrt{r}+1}) = 1 \text{ for each } r > 1.$$

Choose $\{r_n | r_n > 1\}$ such that $r_n \downarrow 1$. Since $\frac{r-1}{r-2\sqrt{r}+1} \to 1$ as $r \downarrow 1$, we see that

$$\boldsymbol{P}((\limsup_{t\to\infty} \frac{B_t}{h(t)})^2 \geq 1) = \lim_{n\to\infty} \boldsymbol{P}((\limsup_{t\to\infty} \frac{B_t}{h(t)})^2 \geq \frac{r_n - 1}{r_n - 2\sqrt{r_n} + 1}) = 1$$

and, hence,

$$\boldsymbol{P}(\limsup_{t\to\infty} \frac{B_t}{h(t)} \geq 1) = 1.$$

4. Since $(-B_t)_{t\geq 0}$ is a Brownian motion, we see that

$$\boldsymbol{P}(\liminf_{t\to\infty} \frac{B_t}{h(t)} = -1) = \boldsymbol{P}(\limsup_{t\to\infty} \frac{-B_t}{h(t)} = 1) = 1$$

and, hence, we have a.s.

$$\liminf_{t\to\infty} \frac{B_t}{h(t)} = -1.$$

□

Chapter 3

Filtrations and Martingales

3.1 Exercise 3.26

1. Let M be a martingale with continuous sample paths such that $M_0 = x \in \mathbb{R}_+$. We assume that $M_t \geq 0$ for each $t \geq 0$, and that $M_t \to 0$ as when $t \to \infty$, a.s. Show that, for each $y > x$,
$$\boldsymbol{P}(\sup_{t\geq 0} M_t \geq y) = \frac{x}{y}.$$

2. Give the law of
$$\sup_{t\leq T_0} B_t$$
when B is a Brownian motion started from $x > 0$ and $T_0 = \inf\{t \geq 0 | B_t = 0\}$.

3. Assume now that B is a Brownian motion started from 0, and let $\mu > o$. Using an appropriate exponential martingale, show that
$$\sup_{t\geq 0}(B_t - \mu t)$$
is exponentially distributed with parameter 2μ.

Proof.

1. Given $y > x > 0$. First, we suppose $(M_t)_{t\geq 0}$ is uniformly integrable. Then $(M_t)_{t\geq 0}$ is bounded in L^1 and, hence,
$$M_\infty = \lim_{t\to\infty} M_t = 0 \text{ a.s.}$$
Set $T = \inf\{t \geq 0 | M_t = y\}$. Then T is a stopping time. By optional stopping times, we have
$$\boldsymbol{E}[M_T] = \boldsymbol{E}[M_0] = x.$$
Observe that
$$\boldsymbol{E}[M_T] = y\boldsymbol{P}(T < \infty) + \boldsymbol{P}(T = \infty) \times 0 = y\boldsymbol{P}(T < \infty)$$
and
$$\boldsymbol{P}(T < \infty) = \boldsymbol{P}(\sup_{t\geq 0} M_t \geq y).$$
Thus, we have
$$\boldsymbol{P}(\sup_{t\geq 0} M_t \geq y) = \frac{x}{y}.$$
Next, we consider a general martingale $(M_t)_{t\geq 0}$. For each $n \geq 1$, we set
$$N_t^{(n)} = M_{t\wedge n}.$$
Then $(N_t^{(n)})_{t\geq 0}$ is an uniformly integrable martingale for each $n \geq 1$ and therefore
$$\boldsymbol{P}(\sup_{0\leq t\leq n} M_t \geq y) = \boldsymbol{P}(\sup_{t\geq 0} N_t^{(n)} \geq y) = \frac{x}{y}.$$
Letting $n \to \infty$, gives
$$\boldsymbol{P}(\sup_{t\geq 0} M_t \geq y) = \frac{x}{y}.$$

2. If $y \leq x$, it's clear that
$$\boldsymbol{P}(\sup_{t \leq T_0} B_t \geq y) = 1.$$
Now we consider $y > x$. Set
$$N_t = B_{t \wedge T_0}$$
for each $t \geq 0$. Then $(N_t)_{t \geq 0}$ is a martingale. Since $T_0 < \infty$ a.s., we get $N_t \to 0$ when $t \to \infty$. Thus,
$$\boldsymbol{P}(\sup_{t \leq T_0} B_t \geq y) = \boldsymbol{P}(\sup_{t \geq 0} N_t \geq y) = \frac{x}{y}.$$

3. Given $\mu > 0$. If $y \leq 0$, it's clear that
$$\boldsymbol{P}(\sup_{t \geq 0}(B_t - \mu t) \geq y) = 1.$$
Now, we suppose $y > 0$. Observe that
$$\begin{aligned}
&\boldsymbol{P}(\sup_{t \geq 0}(B_t - \mu t) \geq y) \\
&= \boldsymbol{P}(\sup_{t \geq 0}(B_{(\frac{1}{2\mu})^2 t} - \mu((\frac{1}{2\mu})^2 t)) \geq y) \\
&= \boldsymbol{P}(\sup_{t \geq 0}(2\mu B_{(\frac{1}{2\mu})^2 t} - \frac{1}{2}t) \geq 2\mu y) \\
&= \boldsymbol{P}(\sup_{t \geq 0}(B_t - \frac{1}{2}t) \geq 2\mu y) \\
&= \boldsymbol{P}(\sup_{t \geq 0} e^{B_t - \frac{1}{2}t} \geq e^{2\mu y})
\end{aligned}$$
Set $M_t = e^{B_t - \frac{1}{2}t}$ for each $t \geq 0$. Then $(M_t)_{t \geq 0}$ is a nonnegative martingale with continuous simple path. Since $\lim_{t \to \infty} \frac{B_t}{t} = 0$ a.s., we get
$$\lim_{t \to \infty}(B_t - \frac{1}{2}t) = \lim_{t \to \infty} t(\frac{B_t}{t} - \frac{1}{2}) = -\infty \text{ a.s.}$$
and, hence, $\lim_{t \to \infty} M_t = 0$ a.s. Because $e^{2\mu y} > 1 = M_0$, we get
$$\boldsymbol{P}(\sup_{t \geq 0}(B_t - \mu t) \geq y) = \boldsymbol{P}(\sup_{t \geq 0} M_t \geq e^{2\mu y}) = e^{-2\mu y}.$$
Therefore, we have
$$\boldsymbol{P}(\sup_{t \geq 0}(B_t - \mu t) \leq y) = \begin{cases} 1 - e^{-2\mu y}, & \text{if } y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$
and, hence, $\sup_{t \geq 0}(B_t - \mu t)$ has exponentially distributed with parameter 2μ.

□

3.2 Exercise 3.27

Let B be an $\mathscr{F}_t$-Brownian motion started from 0. Recall the notation $T_x = \inf\{t \geq 0 | B_t = x\}$, for each $x \in \mathbb{R}$. We fix two real numbers a and b with $a < 0 < b$, and we set
$$T = T_a \wedge T_b.$$

1. Show that, for every $\lambda > 0$,
$$\boldsymbol{E}[e^{-\lambda T}] = \frac{\cosh(\frac{b+a}{2}\sqrt{2\lambda})}{\cosh(\frac{b-a}{2}\sqrt{2\lambda})}.$$

2. Show similarly that, for every $\lambda > 0$,

$$\boldsymbol{E}[e^{-\lambda T}1_{\{T=T_a\}}] = \frac{\sinh(b\sqrt{2\lambda})}{\sinh((b-a)\sqrt{2\lambda})}.$$

3. Show that

$$\boldsymbol{P}(T_a < T_b) = \frac{b}{b-a}.$$

Proof.

1. Set $\alpha = \frac{b+a}{2}$ and

$$M_t = e^{\sqrt{2\lambda}(B_t-\alpha)-\lambda t} + e^{-\sqrt{2\lambda}(B_t-\alpha)-\lambda t}$$

for each $t \geq 0$.

Since

$$(U_t)_{t\geq 0} \equiv (e^{\sqrt{2\lambda}B_t - \frac{(\sqrt{2\lambda})^2}{2}t})_{t\geq 0}$$

and

$$(V_t)_{t\geq 0} \equiv (e^{-\sqrt{2\lambda}B_t - \frac{(\sqrt{2\lambda})^2}{2}t})_{t\geq 0}$$

are martingales, we see that

$$M_t = e^{-\sqrt{2\lambda}\alpha}U_t + e^{\sqrt{2\lambda}\alpha}V_t$$

is a martingale. Because

$$0 \leq U_{t\wedge T} \leq e^{\sqrt{2\lambda}b}$$

and

$$0 \leq V_{t\wedge T} \leq e^{\sqrt{2\lambda}(-a)}$$

for each $t \geq 0$, we see that $((U_{t\wedge T}))_{t\geq 0}$ and $((V_{t\wedge T}))_{t\geq 0}$ are uniformly integrable martingales and, hence, $(M_{t\wedge T})_{t\geq 0}$ is a uniformly integrable martingale. Thus, by optional stopping theorem, we get

$$\boldsymbol{E}[M_T] = \boldsymbol{E}[M_0] = 2\cosh(\sqrt{2\lambda}\frac{b+a}{2}).$$

Observe that

$$\begin{aligned}
\boldsymbol{E}[M_T] &= e^{-\sqrt{2\lambda}\frac{b-a}{2}}\boldsymbol{E}[e^{-\lambda T}1_{T_a\leq T_b}] + e^{\sqrt{2\lambda}\frac{b-a}{2}}\boldsymbol{E}[e^{-\lambda T}1_{T_a\leq T_b}] \\
&\quad + e^{\sqrt{2\lambda}\frac{b-a}{2}}\boldsymbol{E}[e^{-\lambda T}1_{T_a>T_b}] + e^{-\sqrt{2\lambda}\frac{b-a}{2}}\boldsymbol{E}[e^{-\lambda T}1_{T_a>T_b}] \\
&= \boldsymbol{E}[e^{-\lambda T}](e^{\sqrt{2\lambda}\frac{b-a}{2}} + e^{-\sqrt{2\lambda}\frac{b-a}{2}}) \\
&= \boldsymbol{E}[e^{-\lambda T}]2\cosh(\sqrt{2\lambda}\frac{b-a}{2})
\end{aligned}$$

and therefore

$$\boldsymbol{E}[e^{-\lambda T}] = \frac{\cosh(\frac{b+a}{2}\sqrt{2\lambda})}{\cosh(\frac{b-a}{2}\sqrt{2\lambda})}.$$

2. Set $\alpha = \frac{b+a}{2}$ and

$$N_t = e^{\sqrt{2\lambda}(B_t-\alpha)-\lambda t} - e^{-\sqrt{2\lambda}(B_t-\alpha)-\lambda t}$$

for each $t \geq 0$. By using similar arguments as above, we get

$$\boldsymbol{E}[N_T] = \boldsymbol{E}[N_0] = -2\sinh(\sqrt{2\lambda}\frac{a+b}{2})$$

and

$$\begin{aligned}
\boldsymbol{E}[N_T] &= e^{-\sqrt{2\lambda}\frac{b-a}{2}}\boldsymbol{E}[e^{-\lambda T}1_{T_a\le T_b}] - e^{\sqrt{2\lambda}\frac{b-a}{2}}\boldsymbol{E}[e^{-\lambda T}1_{T_a\le T_b}] \\
&+ e^{\sqrt{2\lambda}\frac{b-a}{2}}\boldsymbol{E}[e^{-\lambda T}1_{T_a> T_b}] - e^{-\sqrt{2\lambda}\frac{b-a}{2}}\boldsymbol{E}[e^{-\lambda T}1_{T_a> T_b}] \\
&= -2\sinh(\sqrt{2\lambda}\frac{b-a}{2})\boldsymbol{E}[e^{-\lambda T}1_{T_a\le T_b}] + 2\sinh(\sqrt{2\lambda}\frac{b-a}{2})\boldsymbol{E}[e^{-\lambda T}1_{T_a> T_b}]
\end{aligned}$$

Observe that

$$\begin{aligned}
&2\cosh(\sqrt{2\lambda}\frac{b+a}{2}) = \boldsymbol{E}[M_T] \\
&= 2\cosh(\sqrt{2\lambda}\frac{b-a}{2})\boldsymbol{E}[e^{-\lambda T}1_{T_a\le T_b}] + 2\cosh(\sqrt{2\lambda}\frac{b-a}{2})\boldsymbol{E}[e^{-\lambda T}1_{T_a> T_b}]
\end{aligned}$$

Thus, we have

$$\begin{cases}
\cosh(\sqrt{2\lambda}\frac{b+a}{2}) = \cosh(\sqrt{2\lambda}\frac{b-a}{2})\boldsymbol{E}[e^{-\lambda T}1_{T=T_a}] + \cosh(\sqrt{2\lambda}\frac{b-a}{2})\boldsymbol{E}[e^{-\lambda T}1_{T=T_b}] \\
-\sinh(\sqrt{2\lambda}\frac{a+b}{2}) = -\sinh(\sqrt{2\lambda}\frac{b-a}{2})\boldsymbol{E}[e^{-\lambda T}1_{T=T_a}] + \sinh(\sqrt{2\lambda}\frac{b-a}{2})\boldsymbol{E}[e^{-\lambda T}1_{T=T_b}]
\end{cases}$$

By using the formula

$$\sinh(x+y) = \sinh(x)\cosh(y) + \sinh(y)\cosh(x),$$

we get

$$\boldsymbol{E}[e^{-\lambda T}1_{\{T=T_a\}}] = \frac{\sinh(b\sqrt{2\lambda})}{\sinh((b-a)\sqrt{2\lambda})}.$$

3. By using dominated convergence theorem and the result in problem 2, we have

$$\begin{aligned}
\boldsymbol{P}(T_a < T_b) &= \boldsymbol{E}[1_{T=T_a}] \\
&= \lim_{\lambda\to 0^+}\boldsymbol{E}[e^{-\lambda T}1_{T=T_a}] \\
&= \lim_{\lambda\to 0^+}\frac{\sinh(b\sqrt{2\lambda})}{\sinh((b-a)\sqrt{2\lambda})} \\
&= \frac{b}{b-a}
\end{aligned}$$

□

3.3 Exercise 3.28

Let B be an $(\mathscr{F}_t)$-Brownian motion started from 0. Let $a > 0$ and

$$\sigma_a = \inf\{t \ge 0 \mid B_t \le t - a\}.$$

1. Show that σ_a is a stopping time and that $\sigma_a < \infty$ a.s.

2. Using an appropriate exponential martingale, show that, for every $\lambda \ge 0$,

$$\boldsymbol{E}[e^{-\lambda\sigma_a}] = e^{-a(\sqrt{1+2\lambda}-1)}.$$

The fact that this formula remains valid for $\lambda \in [-\frac{1}{2}, 0]$ can be obtained via an argument of analytic continuation.

3. Let $\mu \in \mathbb{R}$ and $M_t = e^{\mu B_t - \frac{\mu^2}{2}t}$. Show that the stopped martingale $M_{\sigma_a \wedge t}$ is closed if and only if $\mu \leq 1$.

Proof.

1. Since $\liminf_{t\to\infty} B_t = -\infty$ a.s., we see that $\liminf_{t\to\infty}(B_t - t) = -\infty$ a.s. and $\sigma_a < \infty$ a.s.

2. Given $\lambda \geq 0$. Set $\mu = 1-\sqrt{1+2\lambda}$. Then $-\frac{\mu^2}{2}+\mu = -\lambda$ and $(M_t)_{t\geq 0} \equiv (e^{\mu B_t^{\sigma_a} - \frac{\mu^2}{2}\sigma_a \wedge t})_{t\geq 0}$ is a local martingale. Moreover, since

$$-a \leq B_t^{\sigma_a} - (\sigma_a \wedge t) < \infty$$

and

$$0 \leq e^{\mu(B_t^{\sigma_a} - (\sigma_a \wedge t))} \leq e^{-\mu a}$$

for all $t \geq 0$, we see that

$$|M_t| \equiv |e^{\mu B_t^{\sigma_a} - \frac{\mu^2}{2}\sigma_a \wedge t}| = |e^{\mu B_t^{\sigma_a} - \mu(\sigma_a \wedge t)} e^{\mu(\sigma_a \wedge t) - \frac{\mu^2}{2}\sigma_a \wedge t}| \leq e^{-\mu a}$$

for all $t \geq 0$ and therefore M is an uniformly integrable martingale. By optional stopping theorem, we have

$$\boldsymbol{E}[e^{\mu\sigma_a - \mu a - \frac{\mu^2}{2}\sigma_a}] = \boldsymbol{E}[e^{\mu B_\sigma - \frac{\mu^2}{2}\sigma_a}] = 1.$$

Since

$$\mu = 1 - \sqrt{1+2\lambda}$$

and

$$-\frac{\mu^2}{2} + \mu = -\lambda,$$

we get

$$\boldsymbol{E}[e^{-\lambda\sigma_a}] = e^{\mu a} = e^{-a(\sqrt{1+2\lambda}-1)}.$$

Next, we show that the statement is true when $\lambda \in [-\frac{1}{2}, 0]$. Set $\Omega = \{z \in \mathbb{C} \mid Re(z) > -\frac{1}{2}\}$. Define $f : \Omega \mapsto \mathbb{Z}$ by

$$f(z) = \boldsymbol{E}[e^{-z\sigma_a}].$$

Note that

$$\int_0^\infty \frac{1}{s^{\frac{3}{2}}} e^{-A^2 s - \frac{B^2}{s}} ds = \frac{\sqrt{\pi}e^{-2AB}}{B}$$

for $A, B \geq 0$ and

$$\boldsymbol{P}(\sigma_a \leq t) = \int_0^t \frac{a}{\sqrt{2\pi s^3}} e^{-\frac{(a-s)^2}{2s}} ds.$$

For $z = c + id \in \Omega$, we have

$$\begin{aligned}
|\boldsymbol{E}[e^{-z\sigma_a}]| &= |\int_0^\infty e^{-zs}\frac{a}{\sqrt{2\pi s^3}} e^{-\frac{(a-s)^2}{2s}} ds| \\
&\leq \int_0^\infty e^{-cs}\frac{a}{\sqrt{2\pi s^3}} e^{-\frac{(a-s)^2}{2s}} ds \\
&= \frac{ae^a}{\sqrt{2\pi}} \int_0^\infty \frac{1}{s^{\frac{3}{2}}} e^{-\frac{a^2}{2}\frac{1}{s} - (\frac{1}{2}+c)s} ds \\
&= \frac{ae^a}{\sqrt{2\pi}} \frac{\sqrt{\pi}e^{-2\frac{a}{\sqrt{2}}\sqrt{\frac{1}{2}+c}}}{\frac{a}{\sqrt{2}}} < \infty
\end{aligned}$$

and, hence, $f(z)$ is well-defined. Let Γ be a triangle in Ω. By using Fubini's theorem, we have

$$\int_\Gamma f(z)dz = \int_\Omega \int_\Gamma e^{-z\sigma_a} dz \boldsymbol{P}(dw) = 0.$$

Thus, $f(z)$ is holomorphic in Ω. Set $g(z) = e^{-a(\sqrt{2z+1}-1)}$. Then $g(z)$ is holomorphic in Ω. Since $f(z) = g(z)$ on the positive real line, we get $g = f$ in Ω and, hence,

$$\boldsymbol{E}[e^{-\lambda\sigma_a}] = e^{\mu a} = e^{-a(\sqrt{1+2\lambda}-1)}$$

for $\lambda \in (-\frac{1}{2}, 0]$. By monotone convergence theorem, we have

$$\boldsymbol{E}[e^{\frac{1}{2}\sigma_a}] = \lim_{\lambda\downarrow -\frac{1}{2}} \boldsymbol{E}[e^{-\lambda\sigma_a}] = \lim_{\lambda\downarrow -\frac{1}{2}} e^{-a(\sqrt{1+2\lambda}-1)} = e^a$$

and, hence,

$$\boldsymbol{E}[e^{-\lambda\sigma_a}] = e^{\mu a} = e^{-a(\sqrt{1+2\lambda}-1)}$$

for $\lambda \in [-\frac{1}{2}, 0]$.

3. Note that

$$1 = \boldsymbol{E}[M_{\sigma_a}] = \boldsymbol{E}[e^{\mu(\sigma_a - a) - \frac{\mu^2}{2}\sigma_a}] = \boldsymbol{E}[e^{-(\frac{\mu^2}{2}-\mu)\sigma_a - \mu a}]$$

if and only if

$$\boldsymbol{E}[e^{-(\frac{\mu^2}{2}-\mu)\sigma_a}] = e^{\mu a}$$

Since $\frac{\mu^2}{2} - \mu \geq -\frac{1}{2}$ for $\mu \in \mathbb{R}$, we get, by the result in problem 2,

$$\boldsymbol{E}[e^{-(\frac{\mu^2}{2}-\mu)\sigma_a}] = e^{-a(\sqrt{(\mu-1)^2}-1)} = \begin{cases} e^{-a(\mu-2)}, & \text{if } \mu > 1 \\ e^{a\mu}, & \text{if } \mu \leq 1 \end{cases}$$

and, hence,

$$1 = \boldsymbol{E}[M_{\sigma_a}] \text{ if and only if } \mu \leq 1.$$

Now, we show that

$$M_{\sigma_a \wedge t} \text{ is closed if and only if } \mu \leq 1.$$

It's clear that

$$1 = \boldsymbol{E}[M_{0\wedge\sigma_a}] = \boldsymbol{E}[M_{\infty\wedge\sigma_a}] = \boldsymbol{E}[M_{\sigma_a}]$$

whenever $M_{\sigma_a\wedge t}$ is closed. It remains to show that $M_{\sigma_a\wedge t}$ is closed when $1 = \boldsymbol{E}[M_{\sigma_a}]$.
Let $t \geq 0$. By using optional stopping theorem for supermartinale(Theorem 3.25), we have

$$M_{t\wedge\sigma_a} \geq \boldsymbol{E}[M_{\sigma_a}|\mathscr{F}_{t\wedge\sigma_a}], \text{ a.s..}$$

If

$$\boldsymbol{P}(M_{t\wedge\sigma_a} > \boldsymbol{E}[M_{\sigma_a}|\mathscr{F}_{t\wedge\sigma_a}]) > 0,$$

then we have

$$1 = \boldsymbol{E}[M_{0\wedge\sigma_a}] = \boldsymbol{E}[M_{t\wedge\sigma_a}] > \boldsymbol{E}[\boldsymbol{E}[M_{\sigma_a}|\mathscr{F}_{t\wedge\sigma_a}]] = \boldsymbol{E}[M_{\sigma_a}] = 1$$

which is a contradiction. Thus, we have

$$M_{t\wedge\sigma_a} = \boldsymbol{E}[M_{\sigma_a}|\mathscr{F}_{t\wedge\sigma_a}], \text{ a.s.}$$

This shows that $M_{t\wedge\sigma_a}$ is closed.

$\square$

3.4 Exercise 3.29

Let $(Y_t)_{t\geq 0}$ be a uniformly integrable martingale with continuous sample paths, such that $Y_0 = 0$. We set $Y_\infty = \lim_{t\to\infty} Y_t$. Let $p \geq 1$ be a fixed real number. We say that Property (P) holds for the martingale Y if there exists a constant C such that, for every stopping time T, we have

$$\boldsymbol{E}[|Y_\infty - Y_T|^p|\mathscr{F}_T] \leq C$$

1. Show that Property (P) holds for Y if Y_∞ is bounded
2. Let B be an $\{\mathscr{F}_t\}$-Brownian motion started from 0. Show that Property (P) holds for the martingale $Y_t = B_{t\wedge 1}$.
3. Show that Property (P) holds for Y, with the constant C, if and only if, for any stopping time T,
$$\boldsymbol{E}[|Y_T - Y_\infty|^p] \leq C\boldsymbol{P}(T < \infty).$$
4. We assume that Property (P) holds for Y with the constant C. Let S be a stopping time and let Y^S be the stopped martingale defined by $Y_t^S = Y_{S\wedge t}$. Show that Property (P) holds for Y^S with the same constant C.
5. We assume in this question and the next one that Property (P) holds for Y with the constant $C = 1$. Let $a > 0$, and let $(R_n)_{n\geq 0}$ 0 be the sequence of stopping times defined by induction by
$$R_0 = 0 \text{ and } R_{n+1} = \inf\{t \geq R_n||Y_t - Y_{R_n}| \geq a\} \ (\inf \emptyset = \infty).$$
Show that, for every integer $n \geq 0$,
$$a^p\boldsymbol{P}(R_{n+1} < \infty) \leq \boldsymbol{P}(R_n < \infty).$$
6. Infer that, for every $x > 0$,
$$\boldsymbol{P}(\sup_{t\geq 0} Y_t > x) \leq 2^p 2^{-\frac{px}{2}}.$$

Proof.

1. Since $(Y_t)_{t\geq 0}$ is an uniformly integrable martingale,
$$Y_t = \boldsymbol{E}[y_\infty|\mathscr{F}_t]$$
for each $0 \leq t \leq \infty$. Because Y_∞ is bounded, there exists $C > 0$ such that a.s. $|Y_t| \leq C$. Since the sample path is continuous, we have a.s. $\sup_{t\geq 0}|Y_t| \leq C$ and therefore a.s. $|Y_T| \leq C$. Thus, if $p \geq 1$, then
$$\boldsymbol{E}[|Y_\infty - Y_T|^p|\mathscr{F}_T] \leq \boldsymbol{E}[(|Y_\infty| + |Y_T|)^p|\mathscr{F}_T] \leq (2C)^p$$
and therefore Property (P) holds for Y.
2. First, note that Y_t is a uniformly integrable martingale, since $Y_t = \boldsymbol{E}[Y_1|\mathscr{F}_t]$ for $t \geq 1$.
Now, we show that Property (P) holds for the martingale $Y_t = B_{t\wedge 1}$. First, we consider the case $p = 1$. Let $F \in \mathscr{F}_T$. Then
$$\boldsymbol{E}[\boldsymbol{E}[|Y_T - Y_\infty||\mathscr{F}_T]1_F] = \boldsymbol{E}[|Y_T - Y_\infty|1_F] \leq \boldsymbol{E}[|Y_\infty|1_F] + \boldsymbol{E}[|Y_T|1_F].$$
Since Y_t is a uniformly integrable martingale, $Y_T = \boldsymbol{E}[Y_\infty|\mathscr{F}_T]$ and, hence,
$$\boldsymbol{E}[|Y_T|1_F] = \boldsymbol{E}[|\boldsymbol{E}[Y_\infty|\mathscr{F}_T]|1_F] \leq \boldsymbol{E}[\boldsymbol{E}[|Y_\infty||\mathscr{F}_T]1_F] = \boldsymbol{E}[|Y_\infty|].$$
Thus,
$$\boldsymbol{E}[\boldsymbol{E}[|Y_T - Y_\infty||\mathscr{F}_T]1_F] \leq 2\boldsymbol{E}[|Y_\infty|]$$

for each $F \in \mathscr{F}_T$. Since $\boldsymbol{E}[|Y_T - Y_\infty||\mathscr{F}_T]$ is $\mathscr{F}_T$-measurable, we get

$$\boldsymbol{E}[|Y_T - Y_\infty||\mathscr{F}_T] \leq 2\boldsymbol{E}[|Y_\infty|]$$

and therefore property (P) holds for the martingale $Y_t = B_{t\wedge 1}$ when $p = 1$.

Next, we suppose $p > 1$. By Doob's inequality in L^p, we get

$$\boldsymbol{E}[\sup_{t\geq 0}|Y_t|^p] \leq \boldsymbol{E}[\sup_{0\leq t\leq 1}|B_t|^p] \leq (\frac{p}{p-1})^p\boldsymbol{E}[|B_1|^p]$$

and therefore $\sup_{t\geq 0}|Y_t|^p$ is in L^p. Then, for each $F \in \mathscr{F}_T$,

$$\begin{aligned}
\boldsymbol{E}[\boldsymbol{E}[|Y_\infty - Y_T|^p|\mathscr{F}_T]1_F] &= \boldsymbol{E}[|Y_\infty - Y_T|^p 1_F] \\
&\leq \boldsymbol{E}([|Y_\infty| + |Y_T|)^p 1_F] \\
&= \boldsymbol{E}[(2\sup_{t\geq 0}|Y_t|)^p 1_F] \\
&= 2^p\boldsymbol{E}[\sup_{t\geq 0}|Y_t|^p 1_F] \\
&\leq 2^p\boldsymbol{E}[\sup_{t\geq 0}|Y_t|^p] \\
&\leq 2^p(\frac{p}{p-1})^p\boldsymbol{E}[|B_1|^p] < \infty
\end{aligned}$$

Since $\boldsymbol{E}[|Y_\infty - Y_T|^p|\mathscr{F}_T]$ is $\mathscr{F}_T$-measurable, we get

$$\boldsymbol{E}[|Y_\infty - Y_T|^p|\mathscr{F}_T] \leq 2^p(\frac{p}{p-1})^p\boldsymbol{E}[|B_1|^p]$$

and therefore property (P) holds for the martingale $Y_t = B_{t\wedge 1}$ when $p > 1$.

3. Suppose property (P) holds for the uniformly integrable martingale $(Y_t)_{t\geq 0}$. Since $\{T < \infty\} \in \mathscr{F}_T$, we get

$$\boldsymbol{E}[|Y_\infty - Y_T|^p] = \boldsymbol{E}[|Y_\infty - Y_T|^p 1_{T<\infty}] = \boldsymbol{E}[\boldsymbol{E}[|Y_\infty - Y_T|^p|\mathscr{F}_T]1_{T<\infty}] \leq C\boldsymbol{P}(T < \infty).$$

Conversely, suppose that

$$\boldsymbol{E}[|Y_\infty - Y_T|^p] \leq C\boldsymbol{P}(T < \infty)$$

for each stopping time T. Let T be any stopping time and $F \in \mathscr{F}_T$. Then

$$\boldsymbol{E}[\boldsymbol{E}[|Y_\infty - Y_T|^p|\mathscr{F}_T]1_F] = \boldsymbol{E}[|Y_\infty - Y_T|^p 1_F] \leq C.$$

Since $\boldsymbol{E}[|Y_\infty - Y_T|^p|\mathscr{F}_T]$ is $\mathscr{F}_T$-measurable, we get

$$\boldsymbol{E}[|Y_\infty - Y_T|^p|\mathscr{F}_T] \leq C$$

and therefore property (P) holds for the martingale $(Y_t)_{t\geq 0}$

4. Let S and T be stopping times. Since $(Y_t)_{t\geq 0}$ is an uniformly integrable martingale, $(Y_t^S)_{t\geq 0}$ and $(Y_t^T)_{t\geq 0}$ are also uniformly integrable martingales. Thus, we have

$$Y_S^T = \boldsymbol{E}[Y_\infty^T|\mathscr{F}_S] = \boldsymbol{E}[Y_T|\mathscr{F}_S]$$

and therefore

$$Y_T^S = Y_{S\wedge T} = Y_S^T = \boldsymbol{E}[Y_T|\mathscr{F}_S].$$

Hence we get

$$\begin{aligned}
\boldsymbol{E}[|Y_T^S - Y_\infty^S|^p] &= \boldsymbol{E}[|\boldsymbol{E}[Y_T|\mathscr{F}_S] - Y_S|^p] \\
&= \boldsymbol{E}[|\boldsymbol{E}[Y_T|\mathscr{F}_S] - \boldsymbol{E}[Y_\infty|\mathscr{F}_S]|^p] \\
&\leq \boldsymbol{E}[|Y_T - Y_\infty|^p] \\
&\leq C\boldsymbol{P}(T < \infty).
\end{aligned}$$

and therefore property (P) holds for $(Y_t^S)_{t\geq 0}$ with the same constant C.

5. Given $a > 0$. By the definition of $\{R_n\}_{n\geq 0}$, we have $R_{n+1} \geq R_n$ for all $n \geq 0$. By considering uniformly integrable martingale $(Y_t^{R_{n+1}})_{t\geq 0}$ and using the result in problem 4, we get

$$\boldsymbol{E}[|Y_{R_{n+1}} - Y_{R_n}|^p] = \boldsymbol{E}[|Y_{R_n}^{R_{n+1}} - Y_{\infty}^{R_{n+1}}|^p] \leq \boldsymbol{P}(R_n < \infty).$$

Since $|Y_{R_{n+1}} - Y_{R_n}| \geq a$ on $\{R_{n+1} < \infty\}$, we have

$$\boldsymbol{E}[|Y_{R_{n+1}} - Y_{R_n}|^p] \geq a^p \boldsymbol{P}(R_{n+1} < \infty)$$

and, hence,

$$a^p \boldsymbol{P}(R_{n+1} < \infty) \leq \boldsymbol{P}(R_n < \infty).$$

6. Observe that if $0 < x \leq 2$, then $2^{1-\frac{x}{2}} \geq 1$ and, hence, the inequality is true. Now, we suppose $x > 2$. Set

$$R_0 = 0 \text{ and } R_{n+1} = \inf\{t \geq R_n | |Y_t - Y_{R_n}| \geq 2\}$$

for each $n \geq 0$. According the conclusion in problem 5, we get

$$\boldsymbol{P}(R_n < \infty) \leq 2^{-np}$$

for all $n \geq 1$. Let m be the smallest integer such that $2m \geq x$. Then

$$\boldsymbol{P}(\sup_{t\geq 0} Y_t > x) \leq \boldsymbol{P}(R_{m-1} < \infty) \leq 2^{-(m-1)p} \leq 2^{(-\frac{x}{2}+1)p} = 2^p 2^{-\frac{xp}{2}}.$$

$\square$

Chapter 4

Continuous Semimartingales

4.1 Exercise 4.22

Let Z be a $\mathscr{F}_0$-measurable real random variable, and let M be a continuous local martingale. Show that the process $N_t = ZM_t$ is a continuous local martingale.

Proof.
Without loss of generality, we may assume $M_0 = 0$. Set

$$T_n = \inf\{t \geq 0 || N_t| \geq n\}$$

for each $n \geq 1$. Then T_n is a stopping time for each $n \geq 1$. Clearly, $T_n \uparrow \infty$, (T_n) reduce M, and $|ZM^{T_n}| \leq n$ for all $n \geq 1$. Thus, ZM^{T_n} is bounded in L^1 for each $n \geq 1$. Now, we show that ZM^{T_n} is a martingale for each $n \geq 1$. Fix $n \geq 1$. Choose a sequence of bounded simple function $\{Z_k\}$ such that $Z_k \to Z$ and $|Z_k| \leq |Z|$ for each $k \geq 1$ and for all $w \in \Omega$. Note that,

$$|Z_k M_t^{T_n}| \leq |ZM_t^{T_n}| \leq n.$$

Fix $0 \leq s < t$. Let $\Gamma \in \mathscr{F}_s$. By Lebesgue's dominated convergence theorem, we get

$$\boldsymbol{E}[ZM_t^{T_n} 1_\Gamma] = \lim_{k\to\infty} \boldsymbol{E}[Z_k M_t^{T_n} 1_\Gamma] = \lim_{k\to\infty} \boldsymbol{E}[Z_k M_s^{T_n} 1_\Gamma] = \boldsymbol{E}[ZM_s^{T_n} 1_\Gamma].$$

Thus,

$$ZM_s^{T_n} = \boldsymbol{E}[ZM_t^{T_n} | \mathscr{F}_s]$$

for all $0 \leq s < t$ and, hence, ZM^{T_n} is a martingale. Therefore ZM is a continuous local martingale. □

4.2 Exercise 4.23

1. Let M be a martingale with continuous sample paths, such that $M_0 = 0$. We assume that $(M_t)_{t\geq 0}$ is also a Gaussian process. Show that, for every $t > 0$ and every $s > 0$, the random variable $M_{t+s} - M_t$ is independent of $\sigma(M_r, 0 \leq r \leq t)$.
2. Under the assumptions of question 1., show that there exists a continuous monotone nondecreasing function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $\langle M, M\rangle_t = f(t)$ for all $t \geq 0$.

Proof.

1. Observe that

   $$\boldsymbol{E}[M_{s+t} M_t] = \boldsymbol{E}[M_t^2]$$

   for all $s > 0$ and $t > 0$. Since

   $$\boldsymbol{E}[(M_{t+s} - M_t)M_r] = \boldsymbol{E}[M_r^2] - \boldsymbol{E}[M_r^2] = 0$$

   for all $0 \leq r \leq t$, we get $span\{M_{t+s} - M_t\}$ and $span\{M_r | 0 \leq r \leq t\}$ are orthogonal. It followings form Theorem 1.9 that $M_{t+s} - M_t$ is independent of $\sigma(M_r, 0 \leq r \leq t)$.
2. Observe that if B is Brownian motion, B is both continuous martingale and a Gaussian process. Moreover, we have

   $$\langle B, B\rangle_t = t = \boldsymbol{E}[B_t^2].$$

   Therefore we consider the function

   $$f(t) = \boldsymbol{E}[M_t^2].$$

Now, we set $\mathscr{F}_t = \sigma(M_r|0 \le r \le t)$ for all $t \ge 0$. First, we show that $f(t)$ is a continuous monotone nondecreasing function. Let $0 \le s < t$. Since

$$M_s^2 = \boldsymbol{E}[M_t|\mathscr{F}_s]^2 \le \boldsymbol{E}[M_t^2|\mathscr{F}_s],$$

we have

$$f(s) = \boldsymbol{E}[M_s^2] \le \boldsymbol{E}[M_t^2] = f(t)$$

and, hence, $f(t)$ is monotone nondecreasing function. Let $T > 0$ and $\{t_n\} \bigcup \{t\} \subseteq [0, T]$ such that $t_n \to t$. By using Doob's maximal ieuqiality in L^2, we have

$$\boldsymbol{E}[\sup_{0 \le s \le T} |M_s|^2] \le 4\boldsymbol{E}[|M_T|^2] < \infty.$$

By using dominated convergence theorem, we get

$$\lim_{n\to\infty} f(t_n) = \lim_{n\to\infty} \boldsymbol{E}[M_{t_n}^2] = \boldsymbol{E}[M_t^2] = f(t)$$

and, hence, $f(t)$ is continuous.

Next, we show that $\langle M, M\rangle_t = f(t)$ for all $t \ge 0$. Set $\mathscr{N}$ to be the class of all $(\sigma(M_t|t \ge 0), \boldsymbol{P})$-negligible sets. That is,

$$\mathscr{N} := \{A : \exists A' \in \sigma(M_t|t \ge 0) \quad A \subseteq A' \text{ and } \boldsymbol{P}(A') = 0\}.$$

Define

$$\mathscr{G}_t := \sigma(M_s|s \le t) \vee \sigma(\mathscr{N}) \quad t \ge 0$$

and

$$\mathscr{G}_\infty := \sigma(M_t|t \ge 0) \vee \sigma(\mathscr{N}) \quad t \ge 0.$$

Then $(\mathscr{G}_t)_{t\in[0,\infty]}$ is a complete filtration, $\mathscr{G}_t \subseteq \mathscr{F}_t$ for every $0 \le t \le \infty$, $M_{t+s} - M_t \perp\!\!\!\perp \mathscr{G}_t$ for every $t, s > 0$, and $(M_t)_{t\ge 0}$ is a $(\mathscr{G}_t)_{t\in[0,\infty]}$-martingale.
To show that $\langle M, M\rangle_t = f(t)$ for every $t \ge 0$, it suffices to show that $M_t^2 - f(t)$ is a $(\mathscr{G}_t)_{t\in[0,\infty]}$- continuous local martingale. Indeed, since

$$\sum_{i=1}^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})^2 \xrightarrow{P} \langle M, M\rangle_t,$$

we see that finite variation process $(\langle M, M\rangle_t)_{t\ge 0}$ does not depend on the filtration of $(M_t)_{t\ge 0}$.
Now, we show that $M_t^2 - f(t)$ is a $(\mathscr{G}_t)_{t\in[0,\infty]}$-martingale. Let $0 \le s < t$. Observe that

$$\boldsymbol{E}[(M_t - M_s)^2|\mathscr{G}_s] = \boldsymbol{E}[M_t^2 - M_s^2|\mathscr{G}_s]$$

Since $M_t - M_s$ is independent of $\mathscr{G}_s$, we have

$$\boldsymbol{E}[(M_t - M_s)^2|\mathscr{G}_s] = \boldsymbol{E}[(M_t - M_s)^2] = \boldsymbol{E}[M_t^2 - M_s^2].$$

Thus, if $0 \le s < t$, we get

$$\boldsymbol{E}[M_t^2|\mathscr{G}_s] - \boldsymbol{E}[M_t^2] = \boldsymbol{E}[M_t^2 - M_s^2|\mathscr{F}_s] + M_s^2 - \boldsymbol{E}[M_t^2] = \boldsymbol{E}[M_t^2 - M_s^2] + M_s^2 - \boldsymbol{E}[M_t^2] = M_s^2 - \boldsymbol{E}[M_s^2]$$

and therefore $M_t^2 - f(t)$ is a $(\mathscr{G}_t)_{t\in[0,\infty]}$-martingale.

$\square$

4.3 Exercise 4.24

Let M be a continuous local martingale with $M_0 = 0$.

1. For every integer $n \geq 1$, we set $T_n = \inf\{t \geq 0 | |M_t| = n\}$. Show that, a.s.
$$\{\lim_{t\to\infty} M_t \text{ exists and finite }\} = \bigcup_{n\geq 1}\{T_n = \infty\} \subseteq \{\langle M, M\rangle_\infty < \infty\}.$$

2. We set
$$S_n = \inf\{t \geq 0 | \langle M, M\rangle_t = n\}$$
for each $n \geq 1$. Show that, a.s.,
$$\{\langle M, M\rangle_\infty < \infty\} = \bigcup_{n\geq 1}\{S_n = \infty\} \subseteq \{\lim_{t\to\infty} M_t \text{ exists and finite }\}$$
and conclude that
$$\{\lim_{t\to\infty} M_t \text{ exists and is finite }\} = \{\langle M, M\rangle_\infty < \infty\} \text{ , a.s.}$$

Proof.

1. Since M has continuous sample paths, we see that
$$T_n = \inf\{t \geq 0 | |M_t| \geq n\}$$
and $(T_n)_{n\geq 1}$ reduces M and, hence, M^{T_n} is a uniformly integrable martingale for each $n \geq 1$. Thus, for each $n \geq 1$,
$$M^{T_n}_\infty \text{ exists a.s.}$$
Since $|M^{T_n}| \leq n$ for each $n \geq 1$, M^{T_n} is bounded in L^2 and, hence, $\boldsymbol{E}[\langle M^{T_n}, M^{T_n}\rangle_\infty] < \infty$. Thus, for each $n \geq 1$,
$$\langle M, M\rangle_{T_n} < \infty \text{ a.s.}$$
Set
$$E = \bigcup_{n\geq 1}\{M^{T_n}_\infty \text{ exists and } \langle M, M\rangle_{T_n} < \infty\}.$$
Then $\boldsymbol{P}(E) = 1$. To complete the proof, it suffices to show that the statement is true for each $w \in E$. Let
$$w \in \{\lim_{t\to\infty} M_t \text{ exists and finite }\} \bigcap E.$$
Since $M(w)$ has continuous sample path and $M_\infty(w) < \infty$, there exists $K > 0$ such that $|M_t(w)| \leq K$ for all $t \geq 0$ and, hence, $T_m(w) = \infty$ for each $m > K$. Thus, $w \in E \bigcap (\bigcup_{n\geq 1}\{T_n = \infty\})$. Conversely, let $w \in E$ and $T_m(w) = \infty$ for some $m \geq 1$. Then
$$M_\infty(w) = M^{T_m}_\infty(w) \text{ exists}$$
and
$$|M_t(w)| = |M^{T_m}_t(w)| < m \text{ for all } 0 \leq t \leq \infty.$$
Thus, $w \in \{M_\infty \text{ exists and } M_\infty < \infty\} \bigcap E$. Moreover, since $w \in E$, we have
$$\langle M, M\rangle_\infty(w) = \langle M, M\rangle_{T_m}(w) < \infty$$
Thus, we get
$$E \bigcap \{\lim_{t\to\infty} M_t \text{ exists and finite }\} = E \bigcap (\bigcup_{n\geq 1}\{T_n = \infty\}) \subseteq E \bigcap \{\langle M, M\rangle_\infty < \infty\}$$
and therefore a.s.
$$\{\lim_{t\to\infty} M_t \text{ exists and finite }\} = \bigcup_{n\geq 1}\{T_n = \infty\} \subseteq \{\langle M, M\rangle_\infty < \infty\}.$$

2. Since $\langle M, M \rangle$ is an increasing process, it's clear that

$$\{\langle M, M\rangle_\infty < \infty\} = \bigcup_{n\geq 1}\{S_n = \infty\}.$$

Let $n \geq 1$. Then

$$\langle M^{S_n}, M^{S_n}\rangle_t = \langle M, M\rangle_{S_n \wedge t} \leq n$$

for all $t \geq 0$ and, hence, $\boldsymbol{E}[\langle M^{S_n}, M^{S_n}\rangle_\infty] \leq n$. Thus, we see that M^{S_n} is a L^2 bounded martingale and, hence, $\lim_{t\to\infty} M_t^{S_n}$ exists and finite (a.s.). Set

$$F = \bigcup_{n\geq 1}\{\lim_{t\to\infty} M_t^{S_n} \text{ exists and is finite }\}.$$

Then $\boldsymbol{P}(F) = 1$. Fix $w \in F \bigcap(\bigcup_{n\geq 1}\{S_n = \infty\})$. Then $S_m(w) = \infty$ for some $m \geq 1$ and, hence,

$$\lim_{t\to\infty} M_t(w) = \lim_{t\to\infty} M_t^{S_m}(w)$$

exists and is finite. Thus, a.s.,

$$\{\langle M, M\rangle_\infty < \infty\} = \bigcup_{n\geq 1}\{S_n = \infty\} \subseteq \{\lim_{t\to\infty} M_t \text{ exists and is finite }\}.$$

Combining the result with the above, we get

$$\{\lim_{t\to\infty} M_t \text{ exists and finite }\} = \{\langle M, M\rangle_\infty < \infty\} \text{ , a.s.}$$

□

4.4 Exercise 4.25

For every integer $n \geq 1$, let $M^n = (M_t^n)_{t\geq 0}$ 0 be a continuous local martingale with $M_0^n = 0$. We assume that

$$\lim_{n\to\infty} \langle M^n, M^n\rangle_\infty = 0 \text{ in probability.}$$

1. Let $\epsilon > 0$, and, for every $n \geq 1$, let

$$T_\epsilon^n = \inf\{t \geq 0 | \langle M^n, M^n\rangle_t \geq \epsilon\}.$$

Justify the fact that T_ϵ^n is a stopping time, then prove that the stopped continuous local martingale

$$M_t^{n,\epsilon} = M_{t\wedge T_\epsilon^n}^n, \ \forall t \geq 0$$

is a true martingale bounded in L^2.

2. Show that

$$\boldsymbol{E}[\sup_{0\leq t} |M_t^{n,\epsilon}|^2] \leq 4\epsilon.$$

3. Writing, for every $a > 0$,

$$\boldsymbol{P}(\sup_{t\geq 0} |M_t^n| \geq a) \leq \boldsymbol{P}(\sup_{t\geq 0} |M_t^{n\epsilon}| \geq a) + \boldsymbol{P}(T_\epsilon^n < \infty),$$

show that

$$\lim_{n\to\infty} (\sup_{t\geq 0} |M_t^n|) = 0$$

in probability.

Proof.

1. Since $\langle M^n, M^n \rangle$ has continuous sample paths, it follows form proposition 3.9 (iii) that
$$T^n_\epsilon = \inf\{t \geq 0 || \langle M^n, M^n \rangle_t| \in [\epsilon, \infty)\}$$
is a stopping time. Hence $M^{n,\epsilon} = (M^n)^{T^n_\epsilon}$ is a continuous local martingale with
$$\langle M^{n,\epsilon}, M^{n,\epsilon} \rangle_\infty \leq \epsilon.$$
Thus, $M^{n,\epsilon}$ is a L^2 bounded martingale.

2. Since $(M^{n,\epsilon}_t)_{t\geq 0}$ is a martingale bounded in L^2, we see that
$$\boldsymbol{E}[(M^{n,\epsilon}_\infty)^2] = \boldsymbol{E}[\langle M^{n,\epsilon}, M^{n,\epsilon} \rangle_\infty] \leq \epsilon.$$
By Doob's maximal inequality, we get
$$\boldsymbol{E}[\sup_{0\leq s\leq t} |M^{n,\epsilon}_s|^2] \leq 4\boldsymbol{E}[|M^{n,\epsilon}_t|^2]$$
for each $t > 0$. Since $M^{n,\epsilon}$ is a martingale, we see that
$$\boldsymbol{E}[(M^{n,\epsilon}_s)^2] \leq \boldsymbol{E}[(M^{n,\epsilon}_t)^2]$$
for each $s \leq t$. Thus,
$$\boldsymbol{E}[\sup_{0\leq s\leq t} |M^{n,\epsilon}_s|^2] \leq 4\boldsymbol{E}[|M^{n,\epsilon}_t|^2] \leq 4\boldsymbol{E}[|M^{n,\epsilon}_\infty|^2] \leq 4\epsilon.$$
By the Monotone convergence theorem, we have
$$\boldsymbol{E}[\sup_{s\geq 0} |M^{n,\epsilon}_s|^2] \leq 4\epsilon.$$

3. Given $a > 0$ and $\epsilon > 0$. It's clear that
$$\begin{aligned}\boldsymbol{P}(\sup_{t\geq 0} |M^n_t| \geq a) &\leq \boldsymbol{P}(\sup_{t\geq 0} |M^n_t| \geq a, T^n_\epsilon = \infty) + \boldsymbol{P}(T^n_\epsilon < \infty)\\ &= \boldsymbol{P}(\sup_{t\geq 0} |M^{n,\epsilon}_t| \geq a, T^n_\epsilon = \infty) + \boldsymbol{P}(T^n_\epsilon < \infty)\\ &\leq \boldsymbol{P}(\sup_{t\geq 0} |M^{n,\epsilon}_t| \geq a) + \boldsymbol{P}(T^n_\epsilon < \infty).\end{aligned}$$
Note that
$$\boldsymbol{P}(\sup_{t\geq 0} |M^{n,\epsilon}_t| \geq a) \leq \frac{1}{a^2}\boldsymbol{E}[\sup_{0\leq t} |M^{n,\epsilon}_t|^2] \leq \frac{4\epsilon}{a^2}$$
and
$$\boldsymbol{P}(T^n_\epsilon < \infty) = \boldsymbol{P}(\langle M^n, M^n \rangle_\infty \geq \epsilon).$$
Thus,
$$\boldsymbol{P}(\sup_{t\geq 0} |M^n_t| \geq a) \leq \frac{4\epsilon}{a^2} + \boldsymbol{P}(\langle M^n, M^n \rangle_\infty \geq \epsilon).$$
By letting $n \to \infty$ and then $\epsilon \downarrow 0$, we get
$$\lim_{n\to\infty} \boldsymbol{P}(\sup_{t\geq 0} |M^n_t| \geq a) = 0.$$
Since a is arbitrary, we have
$$\lim_{n\to\infty} \sup_{t\geq 0} |M^n_t = 0 \text{ in probability.}$$

$\square$

4.5 Exercise 4.26

1. Let A be an increasing process (adapted, with continuous sample paths and such that $A_0 = 0$) such that $A_\infty < \infty$ a.s., and let Z be an integrable random variable. We assume that, for every stopping time T,

$$\boldsymbol{E}[A_\infty - A_T] \leq \boldsymbol{E}[Z1_{\{T<\infty\}}].$$

   Show, by introducing an appropriate stopping time, that, for every $\lambda > 0$,

$$\boldsymbol{E}[(A_\infty - \lambda)1_{\{A_\infty>\lambda\}}] \leq \boldsymbol{E}[Z1_{\{A_\infty>\lambda\}}].$$

2. Let $f : \mathbb{R}_+ \mapsto \mathbb{R}$ be a continuously differentiable monotone increasing function such that $f(0) = 0$ and set $F(x) = \int_0^x f(t)dt$ for each $x \geq 0$. Show that, under the assumptions of question 1., one has

$$\boldsymbol{E}[F(A_\infty)] \leq \boldsymbol{E}[Zf(A_\infty)].$$

3. Let M be a (true) martingale with continuous sample paths and bounded in L^2 such that $M_0 = 0$, and let M_∞ be the almost sure limit of M_t as $t \to \infty$. Show that the assumptions of question 1 hold when $A_t = \langle M, M\rangle_t$ and $Z = M_\infty^2$. Infer that, for every real $q \geq 1$,

$$\boldsymbol{E}[(\langle M, M\rangle_\infty)^{q+1}] \leq (q + 1)\boldsymbol{E}[(\langle M, M\rangle_\infty)^q M_\infty^2].$$

4. Let $p \geq 2$ be a real number such that $\boldsymbol{E}[(\langle M, M\rangle_\infty)^p] < \infty$. Show that

$$\boldsymbol{E}[(\langle M, M\rangle_\infty)^p] \leq p^p\boldsymbol{E}[|M_\infty|^{2p}].$$

5. Let N be a continuous local martingale such that $N_0 = 0$, and let T be a stopping time such that the stopped martingale N^T is uniformly integrable. Show that, for every real $p \geq 2$,

$$\boldsymbol{E}[(\langle N, N\rangle_T)^p] \leq p^p\boldsymbol{E}[|N_T|^{2p}].$$

6. Give an example showing that this result may fail if N^T is not uniformly integrable.

Proof.

1. Set $T = \inf\{t \geq 0|A_t > \lambda\}$. Then $\{T < \infty\} = \{A_\infty > \lambda\}$ and therefore

$$\begin{aligned}\boldsymbol{E}[Z1_{\{A_\infty>\lambda\}}] &= \boldsymbol{E}[Z1_{\{T<\infty\}}] \geq \boldsymbol{E}[A_\infty - A_T]\\ &= \boldsymbol{E}[(A_\infty - A_T)1_{\{T<\infty\}}]\\ &= \boldsymbol{E}[(A_\infty - \lambda)1_{\{T<\infty\}}]\\ &= \boldsymbol{E}[(A_\infty - \lambda)1_{\{A_\infty>\lambda\}}].\end{aligned}$$

2. Note that

$$F(x) = xf(x) - \int_0^x \lambda f'(\lambda)d\lambda$$

   and $f'(\lambda) \geq 0$ for all $x, \lambda \geq 0$. Since

$$\{1_{\{A_\infty>\lambda\}} = 1\} = \{(w, \lambda) \in \Omega \times \mathbb{R}_+|A_\infty > \lambda\} = \bigcup_{q\in\mathbb{Q}_+} (\{A_\infty > q\} \bigcap [0, q]) \in \mathscr{F} \otimes \mathcal{B}_{\mathbb{R}_+}$$

   for all $\lambda \in \mathbb{R}_+$, we see that $1_{\{A_\infty>\lambda\}}(w, \lambda)f'(\lambda)$ is $\mathscr{F} \otimes \mathcal{B}_{\mathbb{R}_+}$-measurable and, hence,

$$\boldsymbol{E}[\int_0^\infty 1_{\{A_\infty>\lambda\}}f'(\lambda)d\lambda] = \boldsymbol{E}[\int_0^{A_\infty} f'(\lambda)d\lambda]$$

is well-defined. Then

$$
\begin{aligned}
&\boldsymbol{E}[F(A_\infty)] \\
&= \boldsymbol{E}[A_\infty f(A_\infty)] - \boldsymbol{E}[\int_0^{A_\infty} \lambda f'(\lambda) d\lambda] \\
&= \boldsymbol{E}[A_\infty \int_0^\infty 1_{\{A_\infty > \lambda\}} f'(\lambda) d\lambda] - \boldsymbol{E}[\int_0^\infty 1_{\{A_\infty > \lambda\}} \lambda f'(\lambda) d\lambda] \\
&= \int_0^\infty \boldsymbol{E}[A_\infty 1_{\{A_\infty > \lambda\}}] f'(\lambda) d\lambda - \int_0^\infty \boldsymbol{E}[\lambda 1_{\{A_\infty > \lambda\}}] f'(\lambda) d\lambda \\
&\leq \int_0^\infty \boldsymbol{E}[Z 1_{\{A_\infty > \lambda\}}] f'(\lambda) d\lambda
\end{aligned}
$$

By using Fubini's theorem, we get

$$\int_0^\infty \boldsymbol{E}[Z 1_{\{A_\infty > \lambda\}}] f'(\lambda) d\lambda = \boldsymbol{E}[Z \int_0^\infty 1_{\{A_\infty > \lambda\}} f'(\lambda) d\lambda] = \boldsymbol{E}[Z f(A_\infty)]$$

and, hence,

$$\boldsymbol{E}[F(A_\infty)] \leq \boldsymbol{E}[Z f(A_\infty)].$$

3. First, we show that the assumptions of question 1. hold when $A_t = \langle M, M\rangle_t$ and $Z = M_\infty^2$. Let T be any stopping time. Since M is L^2- bounded martingale, we see that $M^2 - \langle M, M\rangle$ is an uniformly integrable martingale and, hence,

$$\boldsymbol{E}[M_T^2 - \langle M, M\rangle_T] = \boldsymbol{E}[M_\infty^2 - \langle M, M\rangle_\infty].$$

Thus,

$$
\begin{aligned}
\boldsymbol{E}[\langle M, M\rangle_\infty - \langle M, M\rangle_T] &= \boldsymbol{E}[M_\infty^2 - M_T^2] \\
&= \boldsymbol{E}[(M_\infty^2 - M_T^2) 1_{\{T<\infty\}}] \\
&\leq \boldsymbol{E}[M_\infty^2 1_{\{T<\infty\}}]
\end{aligned}
$$

and therefore

$$\boldsymbol{E}[A_\infty - A_T] \leq \boldsymbol{E}[Z 1_{\{T<\infty\}}].$$

Next, by taking $F(x) = x^{q+1}$ in problem 2, we have

$$\boldsymbol{E}[(\langle M, M\rangle_\infty)^{q+1}] \leq (q+1) \boldsymbol{E}[(\langle M, M\rangle_\infty)^q M_\infty^2].$$

4. Given $p \geq 2$. Set $q = \frac{p}{p-1}$. Then $\frac{1}{p} + \frac{1}{q} = 1$. By Holder's inequality, we get

$$
\begin{aligned}
\boldsymbol{E}[(\langle M, M\rangle_\infty)^p] &\leq p \boldsymbol{E}[(\langle M, M\rangle_\infty)^{p-1} M_\infty^2] \\
&\leq p \boldsymbol{E}[(\langle M, M\rangle_\infty)^{q(p-1)}]^{\frac{1}{q}} \boldsymbol{E}[|M_\infty|^{2p}]^{\frac{1}{p}} \\
&= p \boldsymbol{E}[(\langle M, M\rangle_\infty)^p]^{\frac{1}{q}} \boldsymbol{E}[|M_\infty|^{2p}]^{\frac{1}{p}}.
\end{aligned}
$$

By assumption, we have $\boldsymbol{E}[(\langle M, M, \rangle_\infty)^p] < \infty$ and, hence,

$$\boldsymbol{E}[(\langle M, M, \rangle_\infty)^p]^{q-1} \leq p^q \boldsymbol{E}[|M_\infty|^{2p}]^{\frac{q}{p}}.$$

That is,

$$\boldsymbol{E}[(\langle M, M, \rangle_\infty)^p] \leq p^{\frac{q}{q-1}} \boldsymbol{E}[|M_\infty|^{2p}]^{\frac{q}{(q-1)p}} = p^p \boldsymbol{E}[|M_\infty|^{2p}].$$

5. Given $p \geq 2$. If $\boldsymbol{E}[|N_T|^{2p}] = \infty$, then there is nothing to prove. Now, we suppose $\boldsymbol{E}[|N_T|^{2p}] < \infty$. Observe that N^T is a L^{2p}- bounded martingale. Indeed, since N^T is uniformly integrable martingale, one has

$$N_{T\wedge t} = \boldsymbol{E}[N_T|\mathscr{F}_t]$$

for all $t \geq 0$ and, hence,

$$\boldsymbol{E}[|N_{T\wedge t}|^{2p}] \leq \boldsymbol{E}[|N_T|^{2p}] < \infty$$

for all $t \geq 0$. Thus we see that N^T is a L^{2p}- bounded martingale, which implies that N^T is a L^2- bounded martingale. Set

$$\tau_n = \{t \geq 0 | \langle N^T, N^T\rangle_t \geq n\}$$

for each $n \geq 1$. Since N^T is uniformly integrable martingale, we have

$$N_{T\wedge\tau_n} = \boldsymbol{E}[N_T|\mathscr{F}_{T\wedge\tau_n}]$$

for each $n \geq 1$ and, hence,

$$\boldsymbol{E}[|N_{T\wedge\tau_n}|^{2p}] \leq \boldsymbol{E}[|N_T|^{2p}]$$

for each $n \geq 1$. Note that $N^{T\wedge\tau_n} = (N^T)^{\tau_n}$ is a L^2–martingale with continuous sample paths and

$$\boldsymbol{E}[\langle N^{T\wedge\tau_n}, N^{T\wedge\tau_n}\rangle_\infty^p] \leq n^p.$$

By using the result in problem 4, we get

$$\boldsymbol{E}[(\langle N, N\rangle_{T\wedge\tau_n})^p] = \boldsymbol{E}[(\langle N^{T\wedge\tau_n}, N^{T\wedge\tau_n}\rangle_\infty)^p] \leq p^p \boldsymbol{E}[|N_{T\wedge\tau_n}|^{2p}]$$

for each $n \geq 1$. By using monotone convergence theorem, we have

$$\boldsymbol{E}[(\langle N, N\rangle_T)^p] = \lim_{n\to\infty} \boldsymbol{E}[(\langle N, N\rangle_{T\wedge\tau_n})^p] \leq \limsup_{n\to\infty} p^p \boldsymbol{E}[|N_{T\wedge\tau_n}|^{2p}] \leq p^p \boldsymbol{E}[|N_T|^{2p}].$$

6. Let $a \neq 0$, $p \geq 1$, and B is a Brownian motion starting from 0. Then B is a marintgale and $\langle B, B\rangle_t = t$. Set $T = \inf\{t \geq 0 | B_t = a\}$. Note that $T < \infty$ (a.s.) and

$$\boldsymbol{E}[|B_T|^{2p}] = |a|^{2p} < \infty.$$

By using the result in Chapter 2(Corollary 2.22), we see that $\boldsymbol{E}[T] = \infty$ and, hence, $\boldsymbol{E}[T^p] = \infty$. Thus,

$$\infty = \boldsymbol{E}[T^p] = \boldsymbol{E}[(\langle B, B\rangle_T)^p] > p^p|a|^{2p} = p^p \boldsymbol{E}[|B_T|^{2p}]$$

and, hence, the inequality fails.
Finally, B^T isn't uniformly integrable. Indeed, if B^T is uniformly integrable, then

$$0 = \boldsymbol{E}[B_0^T] = \boldsymbol{E}[B_\infty^T] = \boldsymbol{E}[B_T] = a \neq 0$$

which is a contradiction.

□

4.6 Exercise 4.27

Let $(X_t)_{t\geq 0}$ be an adapted process with continuous sample paths and taking nonnegative values. Let $(A_t)_{t\geq 0}$ be an increasing process (adapted, with continuous sample paths and such that $A_0 = 0$). We consider the following condition:

(D) For every bounded stopping time T, we have $\boldsymbol{E}[X_T] \leq \boldsymbol{E}[A_T]$.

1. Show that, if M is a square integrable martingale with continuous sample paths and $M_0 = 0$, the condition (D) holds for $X_t = M_t^2$ and $A_t = \langle M, M\rangle_t$.

2. Show that the conclusion of the previous question still holds if one only assumes that M is a continuous local martingale with $M_0 = 0$.

3. We set $X_t^* = \sup_{s \le t} X_s$. Show that, under the condition (D), we have, for every bounded stopping time S and every $c > 0$,
$$\boldsymbol{P}(X_S^* \ge c) \le \frac{1}{c}\boldsymbol{E}[A_S].$$

4. Infer that, still under the condition (D), one has, for every (finite or not) stopping time S,
$$\boldsymbol{P}(X_S^* > c) \le \frac{1}{c}\boldsymbol{E}[A_S].$$
(when S takes the value ∞, we of course define $X_\infty^* = \sup_{s \ge 0} X_s$)

5. Let $c > 0$ and $d > 0$, and $S = \inf\{t \ge 0 | A_t \ge d\}$. Let T be a stopping time. Noting that
$$\{X_T^* > c\} \subseteq \{X_{T\wedge S}^* > c\} \bigcup \{A_T \ge d\}.$$
Show that, under the condition (D), one has
$$\boldsymbol{P}(X_T^* > c) \le \frac{1}{c}\boldsymbol{E}[A_T \wedge d] + \boldsymbol{P}(A_T \ge d).$$

6. Use questions (2) and (5) to verify that, if $M^{(n)}$ is a sequence of continuous local martingales and T is a stopping time such that $\langle M^{(n)}, M^{(n)}\rangle_T$ converges in probability to 0 as $n \to \infty$, then,
$$\lim_{n\to\infty} (\sup_{s \le T} |M_s^{(n)}|) = 0, \text{ in probability.}$$

Proof.

1. Let T be a bounded stopping time. Since M is a L^2-bounded martingale, we see that $M^2 - \langle M, M\rangle$ is uniformly integrable and, hence,
$$\boldsymbol{E}[M_T^2 - \langle M, M\rangle_T] = \boldsymbol{E}[M_0^2 - \langle M, M\rangle_0] = 0.$$
Thus,
$$\boldsymbol{E}[X_T] = \boldsymbol{E}[M_T^2] = \boldsymbol{E}[\langle M, M\rangle_T] = \boldsymbol{E}[A_T].$$

2. Let T be a bounded stopping time. Set
$$\tau_n = \inf\{t \ge 0 | |M_t| \ge n\}$$
for each $n \ge 1$. Then $\tau_n \to \infty$ as $n \to \infty$, (τ_n) reduce M,and M^{τ_n} is a bounded martingale for each $n \ge 1$. By (1), we have
$$\boldsymbol{E}[M_{T\wedge\tau_n}^2] \le \boldsymbol{E}[\langle M, M\rangle_{\tau\wedge T}]$$
for each $n \ge 1$. By Fatou's lemma and monotone convergence theorem, we get
$$\boldsymbol{E}[(M_T)^2] \le \liminf_{n\to\infty} \boldsymbol{E}[(M_{\tau_n\wedge T})^2] = \lim_{n\to\infty} \boldsymbol{E}[\langle M, M\rangle_{\tau_n\wedge T}] = \boldsymbol{E}[\langle M, M\rangle_T].$$

3. Given a bounded stopping time S and $c > 0$. Set $R = \inf\{t \ge 0 | X_t \ge c\}$ and $T = S \wedge R$. According to the assumption, we have
$$\boldsymbol{E}[X_T] \le \boldsymbol{E}[A_T] \le \boldsymbol{E}[A_S].$$
Note that
$$\{T = R\} = \{R \le S\} = \{X_S^* \ge c\}.$$

Since X is continuous and S is bounded, we see that

$$X_R = c \text{ on } \{T = R\}$$

and, hence,

$$\boldsymbol{E}[X_T 1_{\{T=R\}}] = c\boldsymbol{P}(T = R) = c\boldsymbol{P}(X_S^* \geq c).$$

Therefore

$$\boldsymbol{P}(X_S^* \geq c) = \frac{1}{c}\boldsymbol{E}[X_T 1_{\{T=R\}}] \leq \frac{1}{c}\boldsymbol{E}[X_T] \leq \frac{1}{c}\boldsymbol{E}[A_S].$$

4. Given a stopping time S (finite or not) and $c > 0$. Set $S_n = S \wedge n$. Then $S_n \uparrow S$ and S_n is a bounded stopping time for each $n \geq 1$. By using the result in problem 3, we get

$$\boldsymbol{P}(X_{S_n}^* > c) \leq \frac{1}{c}\boldsymbol{E}[A_{S_n}].$$

By using monotone convergence theorem, we get

$$\boldsymbol{E}[A_S] = \lim_{n\to\infty} \boldsymbol{E}[A_{S_n}].$$

Note that

$$\{X_{S_n}^* > c\} \subseteq \{X_{S_{n+1}}^* > c\}$$

for each $n \geq 1$ and

$$\bigcup_{n\geq 1} \{X_{S_n}^* > c\} = \{X_S^* > c\}.$$

Thus

$$\boldsymbol{P}(X_S^* > c) = \lim_{n\to\infty} \boldsymbol{P}(X_{S_n}^* > c) \leq \frac{1}{c} \lim_{n\to\infty} \boldsymbol{E}[A_{S_n}] = \frac{1}{c}\boldsymbol{E}[A_S].$$

5. Note that

$$\begin{aligned}
&\{X_T^* > c\} \subseteq \{A_T < d, X_T^* > c\} \bigcup \{A_T \geq d\} \\
&\subseteq \{T \leq S, X_{T\wedge S}^* > c\} \bigcup \{A_T \geq d\} \\
&\subseteq \{X_{T\wedge S}^* > c\} \bigcup \{A_T \geq d\}.
\end{aligned}$$

and, hence,

$$\boldsymbol{P}(X_T^* > c) \leq \boldsymbol{P}(X_{S\wedge T}^* > c) + \boldsymbol{P}(A_T \geq d).$$

Since $A_{S\wedge T} = A_T \wedge d$, by using the result in problem 4, we get

$$\boldsymbol{P}(X_{S\wedge T}^* > c) \leq \frac{1}{c}\boldsymbol{E}[A_{T\wedge S}] = \frac{1}{c}\boldsymbol{E}[A_T \wedge d].$$

and, so,

$$\boldsymbol{P}(X_T^* > c) \leq \frac{1}{c}\boldsymbol{E}[A_T \wedge d] + \boldsymbol{P}(A_T \geq d).$$

6. Given $\epsilon > 0$. Let $d > 0$. Set $X^{(n)} = (M^{(n)})^2$ and $A^{(n)} = \langle M^{(n)}, M^{(n)} \rangle$. Then $A_T^{(n)} \to 0$ in probability. By using the result in problem 5, we get

$$\boldsymbol{P}(\sup_{0\leq s\leq T} |M_s^{(n)}|^2 > \epsilon) \leq \frac{1}{\epsilon}\boldsymbol{E}[A_T^{(n)} \wedge d] + \boldsymbol{P}(A_T^{(n)} \geq d) \leq \frac{d}{\epsilon} + \boldsymbol{P}(A_T^{(n)} \geq d).$$

By letting $n \to \infty$ and $d \downarrow 0$, we have

$$\lim_{n\to\infty} \boldsymbol{P}(\sup_{0\leq s\leq T} |M_s^{(n)}| > \sqrt{\epsilon}) = \lim_{n\to\infty} \boldsymbol{P}(\sup_{0\leq s\leq T} |M_s^{(n)}|^2 > \epsilon) = 0$$

and therefore

$$\lim_{n\to\infty} (\sup_{s\leq T} |M_s^{(n)}|) = 0, \text{ in probability.}$$

$\square$

Chapter 5

Stochastic Integration

5.1 Exercise 5.25

Let B be an $(\mathscr{F}_t)$-Brownian motion with $B_0 = 0$, and let H be an adapted process with continuous sample paths. Show that $\frac{1}{B_t}\int_0^t H_s dB_s$ converges in probability when $t \to 0$ and determine the limit.

Proof.
To determine the limit of $\frac{1}{B_t}\int_0^t H_s dB_s$, consider the special case

$$H_s(w) = \sum_{i=0}^{p-1} H_{(i)}(w) 1_{(t_i, t_{i+1}]}(s),$$

where $H_{(i)}$ be $\mathscr{F}_{t_i}$-measurable and $0 < t < t_1$. We see that

$$\frac{1}{B_t}\int_0^t H_s dB_s = \frac{1}{B_t}(\sum_{i=0}^{p-1} H_{(i)}(B_{t_{i+1}\wedge t} - B_{t_i \wedge t})) = \frac{1}{B_t} H_{(0)} B_t = H_{(0)}.$$

From the above observation, we will show that

$$\frac{1}{B_t}\int_0^t H_s dB_s \xrightarrow{p} H_0$$

and we may suppose that $H_0 = 0$.
First, we consider the case that H is bounded. By Cauchy–Schwarz's inequality and Jensen's inequality, we get

$$\begin{aligned}
\boldsymbol{E}[|\frac{1}{B_t}\int_0^t H_s dB_s|^{\frac{1}{4}}] &\le \boldsymbol{E}[|B_t|^{-\frac{1}{2}}]^{\frac{1}{2}} \boldsymbol{E}[(|\int_0^t H_s dB_s|^2)^{\frac{1}{4}}]^{\frac{1}{2}} \\
&\le \boldsymbol{E}[|B_t|^{-\frac{1}{2}}]^{\frac{1}{2}} \boldsymbol{E}[|\int_0^t H_s dB_s|^2]^{\frac{1}{8}} \\
&= \boldsymbol{E}[|B_t|^{-\frac{1}{2}}]^{\frac{1}{2}} \boldsymbol{E}[\int_0^t H_s^2 ds]^{\frac{1}{8}} \\
&\le \boldsymbol{E}[|B_t|^{-\frac{1}{2}}]^{\frac{1}{2}} \boldsymbol{E}[\sup_{0\le s\le t} H_s^2 \times t]^{\frac{1}{8}} \\
&\le \boldsymbol{E}[|B_t|^{-\frac{1}{2}}]^{\frac{1}{2}} \boldsymbol{E}[\sup_{0\le s\le t} H_s^2]^{\frac{1}{8}} t^{\frac{1}{8}}.
\end{aligned}$$

Note that

$$\begin{aligned}
\boldsymbol{E}[|B_t|^{-\frac{1}{2}}]^{\frac{1}{2}} &= (2\int_0^\infty \frac{1}{\sqrt{x}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx)^{\frac{1}{2}} \\
&= (2\int_0^\infty \frac{1}{\sqrt{y}} \frac{1}{(2t)^{\frac{1}{4}}} \frac{1}{\sqrt{\pi}} e^{-y^2} dy)^{\frac{1}{2}} \\
&= c \times t^{-\frac{1}{8}},
\end{aligned}$$

where $0 < c = (\frac{2}{2^{\frac{1}{4}}\sqrt{\pi}}\int_0^\infty \frac{1}{\sqrt{y}} e^{-y^2} dy)^{\frac{1}{2}} < \infty$. By Lebesgue dominated convergence theorem, we shows that

$$\boldsymbol{E}[\sup_{0\le s\le t} H_s^2]^{\frac{1}{8}} \to 0 \text{ as } t \to 0^+$$

and therefore

$$\begin{aligned}\boldsymbol{P}(|\frac{1}{B_t}\int_0^t H_s dB_s| \geq \epsilon) &\leq \frac{1}{\epsilon^{\frac{1}{4}}}\boldsymbol{E}[|\frac{1}{B_t}\int_0^t H_s dB_s|^{\frac{1}{4}}] \\ &\leq \frac{1}{\epsilon^{\frac{1}{4}}}\boldsymbol{E}[|B_t|^{-\frac{1}{2}}]^{\frac{1}{2}}\boldsymbol{E}[\sup_{0\leq s\leq t} H_s^2]^{\frac{1}{8}} t^{\frac{1}{8}} \\ &\leq \frac{1}{\epsilon^{\frac{1}{4}}} c \times t^{-\frac{1}{8}}\boldsymbol{E}[\sup_{0\leq s\leq t} H_s^2]^{\frac{1}{8}} t^{\frac{1}{8}} \\ &= \frac{1}{\epsilon^{\frac{1}{4}}} c\boldsymbol{E}[\sup_{0\leq s\leq t} H_s^2]^{\frac{1}{8}} \to 0 \text{ as } t \to 0^+.\end{aligned}$$

Next, we prove the statement for unbounded case. Set

$$H_s^{(R)}(w) = \begin{cases} H_s(w) & \text{if } |H_s(w)| < R \\ R, & \text{if } H_s(w) \geq R \\ -R, & \text{if } H_s(w) \leq -R. \end{cases}$$

Then $H_s^{(R)}(w)$ is an adapted process with continuous sample paths. Now, we show that, for $0 < a < 1$, a.s.

$$\int_0^a H_s dB_s = \int_0^a H_s^{(R)} dB_s \text{ in } \{\sup_{0\leq s\leq 1} |H_s| < R\}.$$

That is,

$$\boldsymbol{P}(\int_0^a H_s dB_s = \int_0^a H_s^{(R)} dB_s, \sup_{0\leq s\leq 1} |H_s| < R) = 1.$$

Given $0 < a < 1$. Note that, if $0 = t_0 < ... < t_p$ and $w \in \{\sup_{0\leq s\leq 1} |H_s| < R\}$, then

$$\sum_{i=0}^{p-1} H_{(i)}(w)(B_{t_{i+1}\wedge a}(w) - B_{t_i\wedge a}(w)) = \sum_{i=0}^{p-1} H_{(i)}^{(R)}(w)(B_{t_{i+1}\wedge a}(w) - B_{t_{i-1}\wedge a}(w)).$$

Choose $0 = t_0^n < ... < t_{p_n}^n = a$ of subdivisions of $[0, a]$ whose mesh tends to 0. By using Proposition 5.9, we have

$$A_n \equiv \sum_{i=0}^{p_n-1} H_{t_i^n}(B_{t_{i+1}^n\wedge a} - B_{t_i^n\wedge a}) \to \int_0^a H_s dB_s \text{ in probability}$$

and

$$B_n \equiv \sum_{i=0}^{p_n-1} H_{t_i^n}^{(R)}(B_{t_{i+1}^n\wedge a} - B_{t_i^n\wedge a}) \to \int_0^a H_s^{(R)} dB_s \text{ in probability..}$$

Choose some subsequences A_{n_k} and B_{n_k} such that a.s.

$$A_{n_k} \to \int_0^a H_s dB_s$$

and

$$B_{n_k} \to \int_0^a H_s^{(R)} dB_s.$$

Since $A_{n_k} = B_{n_k}$ in $\{\sup_{0\leq s\leq 1} |H_s| < R\}$, we see that a.s.

$$\int_0^a H_s dB_s = \int_0^a H_s^{(R)} dB_s \text{ in } \{\sup_{0\leq s\leq 1} |H_s| < R\}.$$

Given $\epsilon > 0$. Let $R > 0$ and $0 < t < 1$. Then

$$\begin{aligned}\boldsymbol{P}(|\frac{1}{B_t}\int_0^t H_s dB_s| \geq \epsilon) &\leq \boldsymbol{P}(\sup_{0\leq s\leq 1}|H_s| < R, |\frac{1}{B_t}\int_0^t H_s dB_s| \geq \epsilon) + \boldsymbol{P}(\sup_{0\leq s\leq 1}|H_s| \geq R)\\ &= \boldsymbol{P}(\sup_{0\leq s\leq 1}|H_s| < R, |\frac{1}{B_t}\int_0^t H_s^{(R)} dB_s| \geq \epsilon) + \boldsymbol{P}(\sup_{0\leq s\leq 1}|H_s| \geq R)\\ &\leq \boldsymbol{P}(|\frac{1}{B_t}\int_0^t H_s^{(R)} dB_s| \geq \epsilon) + \boldsymbol{P}(\sup_{0\leq s\leq 1}|H_s| \geq R).\end{aligned}$$

By using the result in first case, we get

$$\lim_{t\to 0^+}\boldsymbol{P}(|\frac{1}{B_t}\int_0^t H_s^{(R)} dB_s| \geq \epsilon) = 0.$$

Because H is continuous and $H_0 = 0$, we see that

$$\boldsymbol{P}(\sup_{0\leq s\leq 1}|H_s| \geq R) \to 0 \text{ as } R \to \infty.$$

By letting $t \to 0^+$ and then $R \to \infty$, we get

$$\boldsymbol{P}(|\frac{1}{B_t}\int_0^t H_s dB_s| \geq \epsilon) \to 0 \text{ as } t \to 0^+.$$

□

5.2 Exercise 5.26

1. Let B be a one-dimensional $(\mathscr{F}_t)$-Brownian motion with $B_0 = 0$. Let f be a twice continuously differentiable function on $\mathbb{R}$, and let g be a continuous function on $\mathbb{R}$. Verify that the process

$$X_t = f(B_t)e^{-\int_0^t g(B_s)ds}$$

is a semimartingale, and give its decomposition as the sum of a continuous local martingale and a finite variation process.

2. Prove that X is a continuous local martingale if and only if the function f satisfies the differential equation

$$f'' = 2gf.$$

3. From now on, we suppose in addition that g is nonnegative and vanishes outside a compact subinterval of $(0, \infty)$. Justify the existence and uniqueness of a solution f_1 of the equation $f'' = 2fg$ such that $f_1(0) = 1$ and $f_1'(0) = 0$. Let $a > 0$ and $T_a = \inf\{t \geq 0 \mid B_t = a\}$. Prove that

$$\boldsymbol{E}[e^{-\int_0^{T_a} g(B_s)ds}] = \frac{1}{f_1(a)}.$$

Proof.

1. Set $F(x, y) = f(x)e^{-y}$. Then $F \in C^2(\mathbb{R}^2)$. Note that $(\int_0^t g(B_s)ds)_{t\geq 0}$ is a finite variation process. By using Itô's formula, we get

$$\begin{aligned}X_t &= F(B_t, \int_0^t g(B_s)ds)\\ &= f(0) + \int_0^t f'(B_s)e^{-\int_0^s g(B_r)dr}dB_s + \int_0^t -f(B_s)e^{-\int_0^s g(B_r)dr}g(B_s)ds + \frac{1}{2}\int_0^t f''(B_s)e^{-\int_0^s g(B_r)dr}ds.\end{aligned}$$

Since

$$f(0) + \int_0^t f'(B_s) e^{-\int_0^s g(B_r)dr} dB_s$$

is a continuous local martingale and

$$\int_0^t -f(B_s) e^{-\int_0^s g(B_r)dr} g(B_s) ds + \frac{1}{2} \int_0^t f''(B_s) e^{-\int_0^s g(B_r)dr} ds$$

is a finite variation process, we see that

$$X_t = f(B_t) e^{-\int_0^t g(B_s)ds}$$

is a simimartingale.

2. Note that X is a continuous local martingale if and only if

$$\int_0^t e^{-\int_0^s g(B_r)dr} (f''(B_s) - 2f(B_s)g(B_s)) ds = 0, \forall t \geq 0 \text{ a.s.}$$

It's clear that X is a continuous local martingale whenever $f'' = 2fg$. Now, we show that $f'' = 2fg$ when

$$\int_0^t e^{-\int_0^s g(B_r)dr} (f''(B_s) - 2f(B_s)g(B_s)) ds = 0, \forall t \geq 0 \text{ a.s.}$$

We prove it by contradiction. Without loss of generality, we assume that there exists $a \in \mathbb{R}$ and $\delta > 0$ such that

$$f''(x) - 2f(x)g(x) > 0 \text{ on } B(a, \delta).$$

Choose $t_a > a + \delta$. Set $T = \inf\{t \geq 0 \mid B_t = a\}$. Then

$$\boldsymbol{P}(\int_0^t e^{-\int_0^s g(B_r)dr} (f''(B_s) - 2f(B_s)g(B_s)) ds \neq 0 \text{ for some } t \in (0, t_a)) \geq \boldsymbol{P}(T < t_a) > 0$$

which is a contradiction.

3. We show that existence and uniqueness of the problem:

$$\begin{cases} f''(x) = 2g(x)f(x), & \forall x \in \mathbb{R} \\ f \in C^2(\mathbb{R}) \\ f(0) = 1 \text{ and } f'(0) = 0. \end{cases}$$

(a) Choose $[\alpha, \beta] \subseteq (0, \infty)$ such that $g(x) = 0$ for every $x \notin [\alpha, \beta]$. Observe that if f is a solution of the problem, then $f''(x) = 0$ for every $x \leq \alpha$ and so

$$f(x) = 1 \quad \forall x \leq \alpha.$$

(b) Let $f(x)$ be a solution of the problem. By continuity, we see that $f(\alpha) = 1$ and $f'(\alpha) = 0$. By [[2], Theorem 4.1.1], there exists a unique solution $F \in C^2([\alpha, \beta])$ such that

$$\begin{cases} F''(x) = 2g(x)F(x), & \forall x \in [\alpha, \beta] \\ F(\alpha) = 1 \text{ and } F'(\alpha) = 0. \end{cases}$$

(c) Since $g(x) = 0$ for every $x \geq \beta$, we see that $f''(x) = 0$ for every $x \geq \beta$ and so

$$f(x) = F'(\beta)x + F(\beta) - F'(\beta)\beta \quad \forall x \geq \beta.$$

Thus, we define

$$f_1(x) = \begin{cases} 1, & \text{if } -\infty < x \le \alpha \\ F(x), & \text{if } \alpha \le x \le \beta \\ F'(\beta)x + F(\beta) - F'(\beta)\beta, & \text{if } \beta \le x < \infty. \end{cases}$$

and so f_1 is a solution of the problem. Moreover, by the construction as mentioned above, f_1 is the unique solution of the problem.

4. Now, we show that

$$\boldsymbol{E}[\exp(-\int_0^{T_a} g(B_s)ds)] = \frac{1}{f_1(a)}.$$

Fix $a > 0$. Define $T_a := \inf\{t \ge 0 : B_t = a\}$. Let $c > 0$. Then

$$M_t^c := X_{t \wedge T_a \wedge c} \quad \forall t \ge 0$$

is a continuous local martingale. It's clear that $\sup_{x \le a} |f_1'(x)| \le M < \infty$ for some $M > 0$. Thus,

$$\boldsymbol{E}[\langle M^c, M^c \rangle_\infty] = \boldsymbol{E}[\int_0^{c \wedge T_a} f_1'(B_s)^2 \exp(-2\int_0^s g(B_u)du)ds] \le M^2 c < \infty$$

and so M^c is a L^2-bounded martingale. Therefore, we have

$$\boldsymbol{E}[f_1(B_{c\wedge T_a})\exp(-\int_0^{c\wedge T_a} g(B_s)ds)] = \boldsymbol{E}[M_\infty^c] = \boldsymbol{E}[M_0^c] = f_1(0) = 1.$$

Note that $\sup_{x \le a} |f(x)| < \infty$ and $\boldsymbol{P}(T_a < \infty) = 1$. By dominated convergence theorem, we get

$$\boldsymbol{E}[f_1(a)\exp(-\int_0^{T_a} g(B_s)ds)] = \lim_{c\to\infty} \boldsymbol{E}[f_1(B_{c\wedge T_a})\exp(-\int_0^{c\wedge T_a} g(B_s)ds)] = 1$$

and so

$$\boldsymbol{E}[\exp(-\int_0^{T_a} g(B_s)ds)] = \frac{1}{f_1(a)}.$$

□

5.3 Exercise 5.27 (Stochastic calculus with the supremum)

1. Let $m : \mathbb{R}_+ \mapsto \mathbb{R}$ be a continuous function such that $m(0) = 0$, and let $s : \mathbb{R}_+ \mapsto \mathbb{R}$ be the monotone increasing function defined by

$$s(t) = \sup_{0 \le r \le t} m(r).$$

Show that, for every bounded Borel function h on $\mathbb{R}$ and every $t > 0$,

$$\int_0^t (s(r) - m(r))h(r)ds(r) = 0.$$

2. Let M be a continuous local martingale such that $M_0 = 0$, and for every $t \ge 0$, let

$$S_t = \sup_{0 \le r \le t} M_t.$$

Let $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}$ be a twice continuously differentiable function. Justify the equality

$$\varphi(S_t) = \varphi(0) + \int_0^t \varphi'(S_s)dS_s.$$

3. Show that
$$(S_t - M_t)\varphi(S_t) = \Phi(S_t) - \int_0^t \varphi(S_s)dM_s$$
where $\Phi(x) = \int_0^x \varphi(y)dy$ for each $x \in \mathbb{R}$.

4. Infer that, for every $\lambda > 0$,
$$e^{-\lambda S_t} + \lambda(S_t - M_t)e^{-\lambda S_t}$$
is a continuous local martingale.

5. Let $a > 0$ and $T = \inf\{t \geq 0 \mid S_t - M_t = a\}$. We assume that a.s. $\langle M, M\rangle_\infty = \infty$. Show that $T < \infty$ a.s. and S_T is exponentially distributed with parameter $\frac{1}{a}$.

Proof.

1. Given $t > 0$ and a bounded Borel function h on $\mathbb{R}$. Observe that $s(r)$ is a nonnegative continuous function. Then
$$E \equiv \{r \in [0,t] \mid s(r) - m(r) > 0\}$$
is an open subset in $[0,t]$ and, hence, there exists a sequence of disjoint intervals $\{I_n\}_{n\geq 1}$ in $[0,t]$ (these intervals may be open or half open) such that
$$E = \bigcup_{n\geq 1} I_n.$$
Moreover, s is a constant in I_n for each $n \geq 1$. Indeed, if $r_0 \in I_n = (a_n, b_n)$ (I_n may be half open interval, but the argument remain the same) for some $n \geq 1$, there exists $\delta > 0$ such that
$$m(r) < s(r_0) \text{ in } B(r_0, \delta)$$
and, hence, s is a constant in $B(r_0,\delta)$. By using the connectedness of I_n, we see that s is a constant in I_n. Thus
$$\int_{I_n} (s(r) - m(r))h(r)ds(r) = 0$$
for each $n \geq 1$ and, hence,
$$\begin{aligned}\int_0^t (s(r) - m(r))h(r)ds(r) &= \int_E (s(r) - m(r))h(r)ds(r) + \int_{[0,t]\setminus E} (s(r) - m(r))h(r)ds(r)\\ &= \sum_{n=1}^{\infty} \int_{I_n} (s(r) - m(r))h(r)ds(r) + 0 = 0\end{aligned}$$

2. Since S is an increasing process, we see that S is a finite variation process and, hence, $\langle S, S\rangle = 0$. By Itô's formula, we get
$$\varphi(S_t) = \varphi(0) + \int_0^t \varphi'(S_s)dS_s + \frac{1}{2}\int_0^t \varphi''(S_s)d\langle S,S\rangle_s = \varphi(0) + \int_0^t \varphi'(S_s)dS_s.$$

3. Set
$$F(x,y) = (y-x)\varphi(y) - \Phi(y).$$
Then $F \in C^2(\mathbb{R}^2)$, $\frac{\partial F}{\partial y}(x,y) = (y-x)\varphi'(y)$, and $\frac{\partial^2 F}{\partial x^2}(x,y) = 0$. By Itô's formula, we get
$$\begin{aligned}&(S_t - M_t)\varphi(S_t) - \Phi(S_t) = F(M_t, S_t)\\ &= F(0,0) + \int_0^t \frac{\partial F}{\partial x}(M_s,S_s)dM_s + \int_0^t \frac{\partial F}{\partial y}(M_s,S_s)dS_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(M_s,S_s)d\langle M,M\rangle_s\\ &= -\int_0^t \varphi(S_s)dM_s + \int_0^t (S_s - M_s)\varphi'(S_s)dS_s.\end{aligned}$$

Fix $w \in \Omega$. Note that $s \in [0,t] \mapsto \varphi'(S_s(w))$ is continuous and, hence $\varphi'(S_s(w))$ is bounded in $[0,t]$. It followings for, problem 1 that

$$(\int_0^t (S_s - M_s)\varphi'(S_s)dS_s)(w) = 0$$

and therefore

$$(S_t - M_t)\varphi(S_t) = \Phi(S_t) - \int_0^t \varphi(S_s)dM_s.$$

4. Given $\lambda > 0$. Set $\varphi(x) = \lambda e^{-\lambda x}$. Then $\Phi(x) = 1 - e^{-\lambda x}$. Fix $t \geq 0$. By using the result in problem 4, we get

$$e^{-\lambda S_t} + \lambda(S_t - M_t)e^{-\lambda S_t} = 1 - \int_0^t \lambda e^{-\lambda S_s}dM_s.$$

Because $\int_0^t \lambda e^{-\lambda S_s}dM_s$ is a continuous local martingale, so is

$$e^{-\lambda S_t} + \lambda(S_t - M_t)e^{-\lambda S_t}.$$

5. Fix $a > 0$. By Theorem 5.13, we see that there exists a Brownian motion $(\beta_s)_{s\geq 0}$ such that

$$M_t = \beta_{\langle M,M\rangle_t}, \forall t \geq 0, \text{ a.s.}$$

By Proposition 2.14, we have a.s. $\liminf_{t\to\infty} \beta_t = -\infty$. Because $\langle M,M\rangle_\infty = \infty$ a.s., we have a.s.

$$\liminf_{t\to\infty} M_t = -\infty.$$

Since S is nonnegative, we have a.s. $T = \inf\{t \geq 0 \mid S_t - M_t = a\} < \infty$. Now, we show that S_T is exponentially distributed with parameter $\frac{1}{a}$. For this, it suffices to show that

$$\boldsymbol{E}[e^{-\lambda S_T}] = \frac{1}{1 + \lambda \times a}$$

for each $\lambda \geq 0$. Let $\lambda > 0$. By using the result in problem 4, we see that

$$e^{-\lambda S_t} + \lambda(S_t - M_t)e^{-\lambda S_t}$$

is a continuous local martingale and, hence, there exists a sequence of stopping times $\{\sigma_n\}_{n\geq 1}$ such that $\sigma_n \uparrow \infty$ and

$$e^{-\lambda S_{t\wedge T_n}} + \lambda(S_{t\wedge T_n} - M_{t\wedge T_n})e^{-\lambda S_{t\wedge T_n}}$$

is an uniformly integrable martingale where $T_n \equiv \sigma_n \wedge T$ and $n \geq 1$. Then $T_n \uparrow T$ and

$$\boldsymbol{E}[e^{-\lambda S_{T_n}}] + \lambda \boldsymbol{E}[(S_{T_n} - M_{T_n})e^{-\lambda S_{T_n}}] = \boldsymbol{E}[e^{-\lambda S_{0\wedge T_n}}] + \lambda \boldsymbol{E}[(S_{0\wedge T_n} - M_{0\wedge T_n})e^{-\lambda S_{0\wedge T_n}}] = 1$$

for each $n \geq 1$. Note that

$$0 \leq S_{T_n} - M_{T_n} \leq a$$

for all $n \geq 1$. By using Lebesgue dominated convergence theorem, we see that

$$\begin{aligned}
1 &= \lim_{n\to\infty} \boldsymbol{E}[e^{-\lambda S_{T_n}}] + \lim_{n\to\infty} \lambda \boldsymbol{E}[(S_{T_n} - M_{T_n})e^{-\lambda S_{T_n}}] \\
&= \boldsymbol{E}[e^{-\lambda S_T}] + \lambda \boldsymbol{E}[(S_T - M_T)e^{-\lambda S_T}] \\
&= \boldsymbol{E}[e^{-\lambda S_T}](1 + \lambda \times a).
\end{aligned}$$

and, hence,

$$\boldsymbol{E}[e^{-\lambda S_T}] = \frac{1}{1 + \lambda \times a}.$$

□

5.4 Exercise 5.28

Let B be an $(\mathscr{F}_t)$-Brownian motion started from 1. We fix $\epsilon \in (0,1)$ and set $T_\epsilon = \{t \geq 0 \mid B_t = \epsilon\}$. We also let $\lambda > 0$ and $\alpha \in \mathbb{R} \setminus \{0\}$.

1. Show that $Z_t = (B_{t\wedge T_\epsilon})^\alpha$ is a semimartingale and give its canonical decomposition as the sum of a continuous local martingale and a finite variation process.
2. Show that the process
$$Z_t = (B_{t\wedge T_\epsilon})^\alpha e^{-\lambda \int_0^{t\wedge T_\epsilon} \frac{1}{B_s^2} ds}$$
is a continuous local martingale if α and λ satisfy a polynomial equation to be determined.
3. Compute
$$\boldsymbol{E}[e^{-\lambda \int_0^{T_\epsilon} \frac{1}{B_s^2} ds}].$$

Proof.

1. Observe that
$$T_\epsilon < \infty \text{ a.s.}$$
and
$$B_{t\wedge T_\epsilon} \geq \epsilon \ \forall t \geq 0 \text{ a.s.}$$
Define $F : \mathbb{R}^+ \mapsto \mathbb{R}$ by $F(x) = x^\alpha$. By Itô's formula, we have
$$(B_{t\wedge T_\epsilon})^\alpha = 1 + \alpha \int_0^t (B_{s\wedge T_\epsilon})^{\alpha-1} dB_s + \frac{\alpha(\alpha-1)}{2} \int_0^t (B_{s\wedge T_\epsilon})^{\alpha-2} ds \text{ a.s.}$$
for all $t \geq 0$.
2. Define $F : \mathbb{R}^+ \mapsto \mathbb{R}$ by $F(x) = \ln(x)$. By Itô's formula, we have
$$\ln(B_{t\wedge T_\epsilon})^\alpha = \alpha \ln(B_{t\wedge T_\epsilon}) = \alpha \int_0^{t\wedge T_\epsilon} \frac{1}{B_s} dB_s - \frac{\alpha}{2} \int_0^{t\wedge T_\epsilon} \frac{1}{B_s^2} ds.$$
and, hence,
$$\begin{aligned} Z_t = (B_{t\wedge T_\epsilon})^\alpha e^{-\lambda \int_0^{t\wedge T_\epsilon} \frac{1}{B_s^2} ds} &= e^{\ln(B_{t\wedge T_\epsilon})^\alpha} e^{-\lambda \int_0^{t\wedge T_\epsilon} \frac{1}{B_s^2} ds} \\ &= e^{\alpha \int_0^{t\wedge T_\epsilon} \frac{1}{B_s} dB_s - \frac{\alpha}{2} \int_0^{t\wedge T_\epsilon} \frac{1}{B_s^2} ds - \lambda \int_0^{t\wedge T_\epsilon} \frac{1}{B_s^2} ds} \end{aligned}$$
is a continuous loacl martingal whenever $\frac{\alpha^2}{2} = \frac{\alpha}{2} + \lambda$ (i.e. $\alpha = \frac{1\pm\sqrt{1+8\lambda}}{2}$).
3. Let $\lambda > 0$. Set $\alpha = \frac{1-\sqrt{1+8\lambda}}{2}$ be a negative real number. Choose stopping times $(T_n)_{n\geq 1}$ such that $T_n \to \infty$ and Z^{T_n} is an uniformly integrable martingale for $n \geq 1$. Then
$$1 = \boldsymbol{E}[Z_0^{T_n}] = \boldsymbol{E}[Z_{T_\epsilon}^{T_n}] = \boldsymbol{E}[(B_{T_n\wedge T_\epsilon})^\alpha e^{-\lambda \int_0^{T_n\wedge T_\epsilon} \frac{1}{B_s^2} ds}]$$
for all $n \geq 1$. Observe that
$$0 \leq (B_{T_n\wedge T_\epsilon})^\alpha e^{-\lambda \int_0^{T_n\wedge T_\epsilon} \frac{1}{B_s^2} ds} \leq (B_{T_n\wedge T_\epsilon})^\alpha \leq \epsilon^\alpha \text{ a.s.}$$
for all $n \geq 1$. By using the Lebesgue dominated convergence theorem, we have
$$1 = \lim_{n\to\infty} \boldsymbol{E}[(B_{T_n\wedge T_\epsilon})^\alpha e^{-\lambda \int_0^{T_n\wedge T_\epsilon} \frac{1}{B_s^2} ds}] = \boldsymbol{E}[\epsilon^\alpha e^{-\lambda \int_0^{T_\epsilon} \frac{1}{B_s^2} ds}]$$
and therefore
$$\boldsymbol{E}[e^{-\lambda \int_0^{T_\epsilon} \frac{1}{B_s^2} ds}] = \frac{1}{\epsilon^\alpha}.$$

□

5.5 Exercise 5.29

Let $(X_t)_{t\geq 0}$ be a semimartingale. We assume that there exists an $(\mathscr{F}_t)$-Brownian motion $(B_t)_{t\geq 0}$ started from 0 and a continuous function $b : \mathbb{R} \mapsto \mathbb{R}$, such that

$$X_t = B_t + \int_0^t b(X_s)ds. \tag{7}$$

1. Let $F : \mathbb{R} \mapsto \mathbb{R}$ be a twice continuously differentiable function on $\mathbb{R}$. Show that, for $F(X_t)$ to be a continuous local martingale, it suffices that F satisfies a second-order differential equation to be determined.

2. Give the solution of this differential equation which is such that $F(0) = 0$ and $F'(0) = 1$. In what follows, F stands for this particular solution, which can be written in the form

$$F(x) = \int_0^x e^{-2\beta(y)}dy,$$

   with a function β that will be determined in terms of b.

3. In this question only, we assume that b is integrable, i.e $\int_{\mathbb{R}} |b(x)|dx < \infty$.
   (a) Show that the continuous local martingale $M_t = F(X_t)$ is a martingale.
   (b) Show that $\langle M, M\rangle_\infty = \infty$ a.s.
   (c) Infer that

$$\limsup_{t\to\infty} X_t = +\infty, \liminf_{t\to\infty} X_t = -\infty, \text{ a.s.}$$

4. We come back to the general case. Let $c < 0$ and $d > 0$, and

$$T_c = \inf\{t \geq 0 \mid X_t \leq c\}, T_d = \inf\{t \geq 0 \mid X_t \geq d\}.$$

   Show that, on the event $\{T_c \wedge T_d\}$, the random variables $|B_{n+1} - B_n|$ for $n \geq 0$, are bounded above by a (deterministic) constant which does not depend on n. Infer that

$$\boldsymbol{P}(T_c \wedge T_d = \infty) = 0.$$

5. Compute $\boldsymbol{P}(T_c < T_d)$ in terms of $F(c)$ and $F(d)$.

6. We assume that b vanishes on $(-\infty, 0]$ and that there exists a constant $\alpha > \frac{1}{2}$ such that $b(x) \geq \frac{\alpha}{x}$ for all $x \geq 1$. Show that, for every $\epsilon > 0$, one can choose $c < 0$ such that

$$\boldsymbol{P}(T_n < T_c, \forall n \geq 1) \geq 1 - \epsilon.$$

   Infer that $X_t \to \infty$ as $t \to \infty$ a.s.

7. Suppose now $b(x) = \frac{1}{2x}$ for all $x \geq 1$. Show that

$$\liminf_{t\to\infty} X_t = -\infty, \text{ a.s.}$$

Proof.

1. By Itô's formula, we get

$$F(X_t) = \int_0^t F'(X_s)dB_s + \int_0^t F'(X_s)b(X_s)ds + \frac{1}{2}\int_0^t F''(X_s)ds.$$

Thus,

$$F(X_t) = \int_0^t F'(X_s)dB_s \ \forall t \geq 0 \text{ a.s.} \tag{8}$$

is a continuous local martingale whenever

$$\frac{1}{2}F''(x) + F'(x)b(x) = 0 \text{ for all } x \in \mathbb{R}.$$

2. By integrating both sides of the equation, we get

$$F'(x) = e^{\int_0^x -2b(t)dt} \tag{9}$$

and, hence,

$$F(x) = \int_0^x e^{\int_0^y -2b(t)dt}dy \tag{10}$$

3. (a) Since $b \in L^1(\mathbb{R})$, there exists $0 < l < L < \infty$ such that

$$l \leq e^{\int_0^x -2b(t)dt} \leq L \tag{11}$$

for all $x \in \mathbb{R}$. By the formula (1), we get

$$l \leq F'(X_s)(w) \leq L \tag{12}$$

for all $s \geq 0$ and $w \in \Omega$ and, hence, $(F'(X_t))_{t\geq 0} \in L^2(B^a)$ for all $a > 0$. Thus $(\int_0^{t\wedge a} F'(X_s)dB_s)_{t\geq 0}$ is a L^2-bounded martingale for $a > 0$ and therefore $(\int_0^t F'(X_s)dB_s)_{t\geq 0}$ is a martingale. By (32), we see that $M_t = F(X_t)$ is a martingale.

(b) By (32) and (12)

$$\langle M, M\rangle_t = \int_0^t F'(X_s)^2 ds \geq l^2 \times t \ \forall t \geq 0 \text{ a.s.}$$

and, hence, $\langle M, M\rangle_\infty = \infty$ a.s.

(c) Since

$$M_t = \beta_{\langle M,M\rangle_t} \ \forall t \geq 0 \text{ a.s.}$$

for some Brownian motion β and $\langle M, M\rangle_\infty = \infty$ a.s., we see that

$$\limsup_{t\to\infty} M_t = +\infty, \liminf_{t\to\infty} M_t = -\infty, \text{ a.s.}$$

By (9), (10), and (11), we see that F is nondecreasing and

$$F(\pm\infty) \equiv \lim_{x\to\pm\infty} F(x) = \pm\infty.$$

Since $M_t = F(X_t)$, we have

$$\limsup_{t\to\infty} X_t = +\infty, \liminf_{t\to\infty} X_t = -\infty, \text{ a.s.}$$

4. Given $c < 0$ and $d > 0$. Let $w \in \{T_c \wedge T_d = \infty\}$. Then $c < X_t(w) < d$ for all $t \geq 0$. By (7), we get

$$\begin{aligned}|B_n - B_{n-1}| = |X_n - X_{n-1} - \int_{n-1}^n b(X_s)ds| &\leq |X_n| + |X_{n-1}| + \int_{n-1}^n |b(X_s)|ds \\ &\leq 2 \times (d \vee (-c)) + \sup_{t\in[c,d]} |b(t)| \equiv R < \infty.\end{aligned}$$

for all $n \geq 1$. Thus, we see that

$$\{T_c \wedge T_d = \infty\} \subseteq \{|B_n - B_{n-1}| \leq R, \forall n \geq 1\}.$$

Because $\{B_n - B_{n-1} \mid n \geq 1\}$ are independent and

$$0 < \boldsymbol{P}(|B_n - B_{n-1}| \leq R) \equiv c < 1$$

for all $n \geq 1$, we see that

$$\boldsymbol{P}(|B_n - B_{n-1}| \leq R, \forall n \geq 1) = \lim_{m \to \infty} \boldsymbol{P}(|B_n - B_{n-1}| \leq R, \forall 1 \leq n \leq m) \qquad = \lim_{m \to \infty} c^m = 0$$

and, hence,

$$\boldsymbol{P}(T_c \wedge T_d = \infty) = 0. \tag{13}$$

5. Set $T = T_c \wedge T_d$. Because $\boldsymbol{P}(T < \infty) = 1$ and M is a continuous local martingale, we get

$$|M_t^T| = |F(X_t^T)| \leq \sup_{x \in [c,d]} |F(x)| < \infty, \forall t \geq 0, \ a.s.$$

and, hence, M^T is an uniformly integrable martingale. Thus,

$$0 = \boldsymbol{E}[M_0^T] = \boldsymbol{E}[M_\infty^T] = \boldsymbol{E}[M_T] = \boldsymbol{E}[1_{T_c < T_d} M_{T_c}] + \boldsymbol{E}[1_{T_d \leq T_c} M_{T_d}] = F(c)\boldsymbol{P}(T_c < T_d) + F(d)\boldsymbol{P}(T_d \leq T_c)$$

and, hence,

$$\boldsymbol{P}(T_c < T_d) = \frac{F(d)}{F(d) - F(c)}, \ \boldsymbol{P}(T_d \leq T_c) = \frac{-F(c)}{F(d) - F(c)}. \tag{14}$$

6. Observe that, for each $x \geq 1$ and $z < 0$,

$$\begin{aligned}
F(x) &= \int_0^x e^{-2\int_0^y b(t)dt} dy \\
&= \int_0^1 e^{-2\int_0^y b(t)dt} dy + e^{-2\int_0^1 b(t)dt} \int_1^x e^{-2\int_1^y b(t)dt} dy \\
&\leq \int_0^1 e^{-2\int_0^y b(t)dt} dy + e^{-2\int_0^1 b(t)dt} \int_1^x e^{-2\int_1^y \frac{\alpha}{t}dt} dy \\
&= \int_0^1 e^{-2\int_0^y b(t)dt} dy + e^{-2\int_0^1 b(t)dt} \int_1^x \frac{1}{y^{2\alpha}} dy
\end{aligned}$$

and

$$F(z) = -\int_z^0 e^{\int_y^0 2b(t)dt} dy = -\int_z^0 1 dy = z.$$

This implies that

$$0 < F(\infty) < \infty \text{ and } F(-\infty) = -\infty. \tag{15}$$

Given $\epsilon > 0$. By (15), there exists $c < 0$ such that $\frac{F(\infty)}{F(\infty) - F(c)} < \epsilon$. Since $T_n \geq T_{n-1}$, we see that

$$\boldsymbol{P}(T_n < T_c, \forall n \geq 1) = \lim_{n \to \infty} \boldsymbol{P}(T_n < T_c) = 1 - \frac{F(\infty)}{F(\infty) - F(c)} \geq 1 - \epsilon.$$

For $k \geq 1$, there exists $c_k < 0$ such that

$$\boldsymbol{P}(T_n \geq T_{c_k} \text{ for some } n \geq 1) \leq 2^{-k}.$$

By Borel Cantelli's lemma, we see that $\boldsymbol{P}(E^c)=0$, where

$$E^c = \{\{T_n \geq T_{c_k} \text{ for some } n\geq 1\} \text{ i.o k}\}.$$

For $k\geq 1$, since $F(c_k)\leq M_{t\wedge T_{c_k}} = F(X_{t\wedge T_{c_k}}) \leq F(\infty)<\infty$, we see that $M^{T_{c_k}}$ is an uniformly integrable martingale and, hence, $\lim_{t\to\infty} M_t^{T_{c_k}}$ exists (a.s.). Set

$$G = \bigcap_{k\geq 1}\{\lim_{t\to\infty} M_t^{T_{c_k}} \text{ exists }\}.$$

Then $\boldsymbol{P}(G\bigcap E)=1$. Let $w\in E\bigcap G$. Then $T_n(w)<T_{c_k}(w)$ for some $k\geq 1$ and all $n\geq 1$. Since $T_n(w)\uparrow\infty$, we see that $T_{c_k}(w)=\infty$, and, hence, $\lim_{t\to\infty} M_t(w) = \lim_{t\to\infty} M_t^{T_{c_k}}(w)$ exist. Because

$$\lim_{t\to\infty} M_t(w) = \lim_{n\to\infty} M_{T_n}(w) = \lim_{n\to\infty} F(n) = F(\infty),$$

we get $\lim_{t\to\infty} X_t(w)=\infty$. Therefore $\lim_{t\to\infty} X_t=\infty$ (a.s.).

7. Let $x>1$. We see that
$$F(x) = \int_0^1 e^{-2\int_0^y b(t)dt}dy + e^{-2\int_0^1 b(t)dt}\int_1^x \frac{1}{y}dy$$
and, hence, $F(\infty)=\infty$. Choose $\{c_k\}\subseteq\mathbb{R}_-$ such that $c_k\to-\infty$. For $k\geq 1$, by (14), there exists $d_k>0$ such that
$$\boldsymbol{P}(T_{c_k}\geq T_{d_k})\leq 2^{-k}.$$
By Borel Cantelli's lemma, we see that $\boldsymbol{P}(\Gamma^c)=0$, where
$$\Gamma^c = \{\{T_{c_k}\geq T_{d_k}\} \text{ i.o. k}\}.$$
Let $w\in\Gamma$. There exists $K\geq 1$ such that $T_{c_k}(w)<T_{d_k}(w)$ for all $k\geq K$ and, hence, $T_{c_k}(w)<\infty$ for all $k\geq K$. Thus,
$$\lim_{k\to\infty} X_{T_{c_k}}(w) = \lim_{k\to\infty} c_k = -\infty.$$
Therefore $\liminf_{t\to\infty} X_t = -\infty$ (a.s.).

□

5.6 Exercise 5.30 (Lévy Area)

Let $(X_t, Y_y)_{t\geq 0}$ be a two-dimensional $(\mathscr{F}_t)$-Brownian motion started from 0. We set, for every $t\geq 0$:

$$\mathscr{A}_t = \int_0^t X_s dY_s - \int_0^t Y_s dX_s \text{ (Lévy area)}$$

1. Compute $\langle \mathscr{A},\mathscr{A}\rangle_t$ and infer that $(\mathscr{A}_t)_{t\geq 0}$ is a square-integrable (true) martingale.
2. Let $\lambda>0$. Justify the equality
$$\boldsymbol{E}[e^{i\lambda\mathscr{A}_t}] = \boldsymbol{E}[\cos(\lambda\mathscr{A}_t)].$$
3. Let $f\in C^3(\mathbb{R}_+)$. Give the canonical decomposition of the semimartingales
$$Z_t = \cos(\lambda\mathscr{A}_t), W_t = -\frac{f'(t)}{2}(X_t^2+Y_t^2)+f(t).$$
Verify that $\langle Z, W\rangle_t = 0$.

4. Show that, for the process $Z_t e^{W_t}$ to be a continuous local martingale, it suffices that f solves the differential equation

$$f''(t) = f'(t)^2 - \lambda^2.$$

5. Let $r > 0$. Verify that the function

$$f(t) = -\ln(\cosh(\lambda(r-t)))$$

solves the differential equation of question 4. and derive the formula

$$\boldsymbol{E}[e^{i\lambda \mathscr{A}_r}] = \frac{1}{\cosh(\lambda r)}.$$

Proof.

1. By Fubini's theorem, we get

$$\begin{aligned}\boldsymbol{E}[\langle \mathscr{A}, \mathscr{A}\rangle_t] &= \boldsymbol{E}[\int_0^t X_s^2 ds] + \boldsymbol{E}[\int_0^t Y_s^2 ds] \\ &= \int_0^t \boldsymbol{E}[X_s^2]ds + \int_0^t \boldsymbol{E}[Y_s^2]ds \\ &= \int_0^t sds + \int_0^t sds = t^2\end{aligned}$$

for all $t \geq 0$. By Theorem 4.13, we see that $\mathscr{A}$ is a true martingale and $\mathscr{A}_t \in L^2$ for all $t \geq 0$.

2. Fix $\lambda > 0$ and $t > 0$. Let $0 = t_0^n < t_1^n < ... < t_{p_n}^n = t$ be a sequence of subdivisions of $[0,t]$ whose mesh tends to 0. By Proposition 5.9, we have

$$\sum_{i=0}^{p_n-1} X_{t_i^n}(Y_{t_{i+1}^n} - Y_{t_{i-1}^n}) - \sum_{i=0}^{p_n-1} Y_{t_i^n}(X_{t_{i+1}^n} - X_{t_{i-1}^n}) \xrightarrow{p} \int_0^t X_s dY_s - \int_0^t Y_s dX_s = \mathscr{A}_t$$

and

$$\sum_{i=0}^{p_n-1} Y_{t_i^n}(X_{t_{i+1}^n} - X_{t_{i-1}^n}) - \sum_{i=0}^{p_n-1} X_{t_i^n}(Y_{t_{i+1}^n} - Y_{t_{i-1}^n}) \xrightarrow{p} \int_0^t Y_s dX_s - \int_0^t X_s dY_s = -\mathscr{A}_t.$$

Let

$$p(x) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{t_1(t_2-t_1)...(t_p-t_{p-1})}} e^{-\sum_{k=0}^{p_n-1}\frac{(x_{i+1}-x_i)^2}{2(t_{i+1}-t_i)}}.$$

Since $(X_t, Y_y)_{t\geq 0}$ is two-dimensional Brownian motion, we get

$$\begin{aligned}&\boldsymbol{E}[e^{i\xi(\sum_{i=0}^{p_n-1} X_{t_i^n}(Y_{t_{i+1}^n}-Y_{t_{i-1}^n}) - \sum_{i=0}^{p_n-1} Y_{t_i^n}(X_{t_{i+1}^n}-X_{t_{i-1}^n}))}] \\ &= \int_{\mathbb{R}^p}\int_{\mathbb{R}^p} e^{i\xi(\sum_{k=0}^{p_n-1} x_i(y_{i+1}-y_i) - \sum_{k=0}^{p_n-1} y_i(x_{i+1}-x_i))} p(x)p(y)dxdy \\ &= \boldsymbol{E}[e^{i\xi(\sum_{i=0}^{p_n-1} Y_{t_i^n}(X_{t_{i+1}^n}-X_{t_{i-1}^n}) - \sum_{i=0}^{p_n-1} X_{t_i^n}(Y_{t_{i+1}^n}-Y_{t_{i-1}^n}))}]\end{aligned}$$

for all $n \geq 1$ and $\xi \in \mathbb{R}$. By Lévy's continuity theorem, we see that

$$\boldsymbol{E}[e^{i\xi\mathscr{A}_t}] = \boldsymbol{E}[e^{i\xi(-\mathscr{A}_t)}]$$

for all $\xi \in \mathbb{R}$ and, hence $\mathscr{A}_t \stackrel{D}{=} -\mathscr{A}_t$ Therefore

$$\boldsymbol{E}[\cos(\lambda\mathscr{A}_t)] + i\boldsymbol{E}[\sin(\lambda\mathscr{A}_t)] = \boldsymbol{E}[\cos(\lambda\mathscr{A}_t)] - i\boldsymbol{E}[\sin(\lambda\mathscr{A}_t)]$$

and, hence $\boldsymbol{E}[\sin(\lambda\mathscr{A}_t)] = 0.$

3. By Itô's formula, we get

$$
\begin{aligned}
Z_t &= 1-\lambda\int_0^t \sin(\lambda\mathscr{A}_s)d\mathscr{A}_s - \frac{1}{2}\lambda^2\int_0^t \cos(\lambda\mathscr{A}_s)d\langle\mathscr{A},\mathscr{A}\rangle_s \\
&= 1-\lambda\int_0^t \sin(\lambda\mathscr{A}_s)d\mathscr{A}_s - \frac{1}{2}\lambda^2\int_0^t \cos(\lambda\mathscr{A}_s)(X_s^2+Y_s^2)ds \\
&= 1-\lambda\int_0^t \sin(\lambda\mathscr{A}_s)d\mathscr{A}_s - \frac{1}{2}\lambda^2\int_0^t Z_s(X_s^2+Y_s^2)ds.
\end{aligned}
$$

Also we have

$$
\begin{aligned}
&f'(t)(X_t^2+Y_t^2) \\
&= \int_0^t f''(s)(X_s^2+Y_s^2)ds + \int_0^t f'(s)2X_s dX_s + \int_0^t f'(s)2Y_s dY_s + \frac{1}{2}\int_0^t f'(s)\times 2ds + \frac{1}{2}\int_0^t f'(s)\times 2ds \\
&= \int_0^t f''(s)(X_s^2+Y_s^2)ds + \int_0^t f'(s)2X_s dX_s + \int_0^t f'(s)2Y_s dY_s + 2(f(t)-f(0))
\end{aligned}
$$

and, hence,

$$
W_t = \frac{-1}{2}f'(t)(X_t^2+Y_t^2)+f(t) = f(0)-\int_0^t f'(s)X_s dX_s - \int_0^t f'(s)Y_s dY_s - \frac{1}{2}\int_0^t f''(s)(X_s^2+Y_s^2)ds.
$$

Therefore

$$
\begin{aligned}
\langle W, Z\rangle_t &= X_t f'(t)\lambda\sin(\lambda\mathscr{A}_t)\langle X,\mathscr{A}\rangle_t + Y_t f'(t)\lambda\sin(\lambda\mathscr{A}_t)\langle Y,\mathscr{A}\rangle_t \\
&= X_t f'(t)\lambda\sin(\lambda\mathscr{A}_t)\times(-Y_t t) + Y_t f'(t)\lambda\sin(\lambda\mathscr{A}_t)(X_t t) = 0
\end{aligned}
$$

4. By Itô's formula, we get

$$
Z_t e^{W_t} = \int_0^t e^{W_s}dZ_s + \int_0^t Z_s e^{W_s}dW_s + \frac{1}{2}\int_0^t Z_s e^{W_s}d\langle W, W\rangle_s.
$$

Note that

$$
dZ_s = -\lambda\sin(\lambda\mathscr{A}_s)d\mathscr{A}_s - \frac{1}{2}\lambda^2 Z_s(X_s^2+Y_s^2)ds,
$$

$$
dW_s = f'(s)X_s dX_s - f'(s)Y_s dY_s - \frac{1}{2}f''(s)(X_s^2+Y_s^2)ds,
$$

and

$$
d\langle W, W\rangle_s = (X_s^2 f'(s)^2 + Y_s^2 f'(s)^2)ds.
$$

Thus, $Z_t e^{W_t}$ is a continuous local martingale when

$$
f''(t) = f'(t)^2-\lambda^2.
$$

5. Fix $r>0$ and $\lambda>0$. It's clear that $f(t) = -\ln(\cosh(\lambda(r-t)))\in C^3(\mathbb{R}_+)$ and satisfy

$$
f''(t) = f'(t)^2-\lambda^2.
$$

Thus $(Z_t e^{W_t})_{t\geq 0}$ is a continuous local martingale. Choose $(T_n)_{n\geq 1}$ such that $(Z_t^{T_n}e^{W_t^{T_n}})_{t\geq 0}$ is an uniformly integrable martingale for $n\geq 1$ and $T_n\uparrow\infty$. Then

$$
\boldsymbol{E}[\cos(\lambda\mathscr{A}_{T_n\wedge r})e^{-\frac{1}{2}f'(T_n\wedge r)(X_{T_n\wedge r}^2+Y_{T_n\wedge r}^2)+f(T_n\wedge r)}] = \boldsymbol{E}[Z_r^{T_n}e^{W_r^{T_n}}] = \boldsymbol{E}[Z_0^{T_n}e^{W_0^{T_n}}] = \frac{1}{\cosh(\lambda r)}.
$$

Because $r - T_n \wedge r \geq 0$ for all $n \geq 1$, we see that

$$f'(T_n \wedge r) = \frac{\sinh(\lambda(r - T_n \wedge r))}{\cosh(\lambda(r - T_n \wedge r))}\lambda \geq 0$$

and, hence,

$$0 \leq e^{-\frac{1}{2}f'(T_n\wedge r)(X^2_{T_n\wedge r}+Y^2_{T_n\wedge r})} \leq 1$$

for all $n \geq 1$. Since $\cosh(\lambda(r - T_n \wedge r)) \geq 1$ for all $n \geq 1$, we get

$$f(T_n \wedge r) = -\ln(\cosh(\lambda(r - T_n \wedge r))) \leq 0$$

and, hence

$$0 \leq e^{f(T_n\wedge r)} \leq 1.$$

By Lebesgue dominated convergence theorem, we see that

$$\begin{aligned}\frac{1}{\cosh(\lambda r)} &= \lim_{n\to\infty} \boldsymbol{E}[\cos(\lambda \mathscr{A}_{T_n\wedge r})e^{-\frac{1}{2}f'(T_n\wedge r)(X^2_{T_n\wedge r}+Y^2_{T_n\wedge r})+f(T_n\wedge r)}] \\ &= \boldsymbol{E}[\cos(\lambda\mathscr{A}_r)e^{-\frac{1}{2}f'(r)(X_r^2+Y_r^2)+f(r)}]\end{aligned}$$

Since $f'(r) = \frac{\sinh(\lambda(r-t))}{\cosh(\lambda(r-t))}|_{t=r} = 0 = f(r)$, we have

$$\boldsymbol{E}[\cos(\lambda\mathscr{A}_r)e^{-\frac{1}{2}f'(r)(X_r^2+Y_r^2)+f(r)}] = \boldsymbol{E}[\cos(\lambda\mathscr{A}_r)].$$

By the result in problem 2,

$$\boldsymbol{E}[e^{i\lambda\mathscr{A}_r}] = \boldsymbol{E}[\cos(\lambda\mathscr{A}_r)] = \frac{1}{\cosh(\lambda r)}.$$

□

5.7 Exercise 5.31 (Squared Bessel processes)

Let B be an $(\mathscr{F}_t)_{t\geq 0}$-Brownian motion started from 0, and let X be a continuous semimartingale. We assume that X takes values in $\mathbb{R}_+$, and is such that, for every $t \geq 0$,

$$X_t = x + 2\int_0^t \sqrt{X_s}dB_s + \alpha t$$

where x and α are nonnegative real numbers.

1. Let $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a continuous function, and let φ be a twice continuously differentiable function on $\mathbb{R}_+$, taking strictly positive values, which solves the differential equation

$$\varphi'' = 2f\varphi$$

and satisfies $\varphi(0) = 1$ and $\varphi'(1) = 0$. Observe that the function φ must then be decreasing over the interval $[0, 1]$. We set

$$u(t) = \frac{\varphi'(t)}{2\varphi(t)}$$

for every $t \geq 0$. Verify that we have, for every $t \geq 0$,

$$u'(t) + 2u(t)^2 = f(t),$$

then show that, for every $t \geq 0$,

$$u(t)X_t - \int_0^t f(s)X_s ds = u(0)x + \int_0^t u(s)dX_s - 2\int_0^t u(s)^2 X_s ds.$$

We set

$$Y_t = u(t)X_t - \int_0^t f(s)X_s ds.$$

2. Show that, for every $t \geq 0$,

$$\varphi(t)^{-\frac{\alpha}{2}} e^{Y_t} = \mathscr{E}(N)_t$$

where $\mathscr{E}(N)_t = \exp(N_t - \frac{1}{2}\langle N, N\rangle_t)$ denotes the exponential martingale associated with the continuous local martingale

$$N_t = u(0)x + 2\int_0^t u(s)\sqrt{X_s}dB_s.$$

3. Infer from the previous question that

$$\boldsymbol{E}[\exp(-\int_0^1 f(s)X_s ds)] = \varphi(1)^{\frac{\alpha}{2}} \exp(\frac{x}{2}\varphi'(0)).$$

4. Let $\lambda > 0$. Show that

$$\boldsymbol{E}[\exp(-\lambda\int_0^1 X_s ds)] = (\cosh(\sqrt{2\lambda}))^{-\frac{\alpha}{2}} \exp(-\frac{x}{2}\sqrt{2\lambda}\tanh(\sqrt{2\lambda})).$$

5. Show that, if $\beta = (\beta_t)_{t\geq 0}$ is a real Brownian motion started from y, one has, for every $\lambda > 0$,

$$\boldsymbol{E}[\exp(-\lambda\int_0^1 \beta_s^2 ds)] = (\cosh(\sqrt{2\lambda}))^{-\frac{1}{2}} \exp(-\frac{y^2}{2}\sqrt{2\lambda}\tanh(\sqrt{2\lambda})).$$

Proof.

1. Since $f \geq 0$ and $\varphi > 0$, we see that $\varphi'' = 2f\varphi \geq 0$. Because $\varphi'(1) = 0$ and φ' is nondecreasing, one has $\varphi' \leq 0$ in $[0,1]$ and, hence, φ is decreasing over the interval $[0,1]$. Note that

$$u'(t) + 2u(t)^2 = \frac{\varphi''(t)2\varphi(t) - 2\varphi(t)^2}{4\varphi(t)^2} + 2\frac{\varphi'(t)^2}{4\varphi(t)^2} = \frac{\varphi''(t)}{2\varphi(t)} = f(t).$$

By Itô's formula, we get

$$\begin{aligned} u(t)X_t &= u(0)x + \int_0^t u'(s)X_s ds + \int_0^t u(s)dX_s \\ &= u(0)x + \int_0^t f(s)X_s ds - 2\int_0^t u(s)^2 X_s ds + \int_0^t u(s)dX_s. \end{aligned}$$

and, hence,

$$u(t)X_t - \int_0^t f(s)X_s ds = u(0)x + \int_0^t u(s)dX_s - 2\int_0^t u(s)^2 X_s ds.$$

2. Note that

$$\begin{aligned} Y_t &= u(0)x + \int_0^t u(s)dX_s - 2\int_0^t u(s)^2 X_s ds \\ &= u(0)x + \int_0^t u(s)\sqrt{X_s}dB_s + \alpha\int_0^t u(s)ds - 2\int_0^t u(s)^2 X_s ds \\ &= u(0)x + \int_0^t u(s)\sqrt{X_s}dB_s - 2\int_0^t u(s)^2 X_s ds + \alpha\int_0^t \frac{\varphi'(s)}{2\varphi(s)}ds \\ &= u(0)x + \int_0^t u(s)\sqrt{X_s}dB_s - 2\int_0^t u(s)^2 X_s ds + \frac{\alpha}{2}\ln(\varphi(t)). \end{aligned}$$

Then we have

$$\begin{aligned}
\mathscr{E}(N)_t &= \exp(N_t - \langle N, N\rangle_t)\\
&= \exp(u(0)x + 2\int_0^t u(s)\sqrt{X_s}dB_s - 2\int_0^t u(s)^2 X_s ds)\\
&= \exp(u(0)x + 2\int_0^t u(s)\sqrt{X_s}dB_s - 2\int_0^t u(s)^2 X_s ds + \frac{\alpha}{2}\ln(\varphi(t)))\varphi(t)^{-\frac{\alpha}{2}}\\
&= \exp(Y_t)\varphi(t)^{-\frac{\alpha}{2}}.
\end{aligned}$$

3. Choose m such that $\ln(\varphi(t)) \geq m$ for all $t \in [0,1]$. Fix $t \in [0,1]$. Because $\varphi' \leq 0$ in $[0,1]$ (problem 1), we see that $u \leq 0$ in $[0,1]$. Because $f \geq 0$ in $[0,1]$ and $X_t, \alpha \geq 0$, we see that

$$\mathscr{E}(N)_t = \exp(Y_t)\varphi(t)^{-\frac{\alpha}{2}} = \exp(u(t)X_t - \int_0^t f(s)X_s ds - \frac{\alpha}{2}\ln(\varphi(t))) \leq \exp(-\frac{\alpha}{2}m) < \infty.$$

and, hence, $\mathscr{E}(N)_{t\wedge 1}$ is a uniformly integrable martingale. Because $u(1) = \varphi'(1) = 0$ and $\varphi(0) = 1$, we have

$$\begin{aligned}
\varphi(1)^{-\frac{\alpha}{2}}\boldsymbol{E}[\exp(-\int_0^1 f(s)X_s ds)] &= \varphi(1)^{-\frac{\alpha}{2}}\boldsymbol{E}[\exp(u(1)X_1 - \int_0^1 f(s)X_s ds)] = \boldsymbol{E}[\varphi(1)^{-\frac{\alpha}{2}}\exp Y_1]\\
&= \boldsymbol{E}[\mathscr{E}(N)_1] = \boldsymbol{E}[\mathscr{E}(N)_0] = \boldsymbol{E}[\exp(N_0)] = \exp(u(0)x)\\
&= \exp(x\frac{\varphi'(0)}{2\varphi(0)}) = \exp(\frac{x\varphi'(0)}{2})
\end{aligned}$$

and, so

$$\boldsymbol{E}[\exp(-\int_0^1 f(s)X_s ds)] = \varphi(1)^{\frac{\alpha}{2}}\exp(\frac{x}{2}\varphi'(0)).$$

4. Set $f = \lambda$. Then we have $\varphi''(t) - 2\lambda\varphi(t) = 0$ and, hence, $\varphi(t) = c_1\exp(\sqrt{2\lambda}t) + c_2\exp(-\sqrt{2\lambda}t)$. Combining with initial conditions, we get

$$\varphi(t) = \frac{\exp(-\sqrt{2\lambda})}{\exp(\sqrt{2\lambda}) + \exp(-\sqrt{2\lambda})}\exp(\sqrt{2\lambda}t) + \frac{\exp(\sqrt{2\lambda})}{\exp(\sqrt{2\lambda}) + \exp(-\sqrt{2\lambda})}\exp(-\sqrt{2\lambda}t).$$

Thus,

$$\varphi(1) = \frac{2}{\exp(\sqrt{2\lambda}) + \exp(-\sqrt{2\lambda})} = \frac{1}{\cosh(\sqrt{2\lambda})}$$

and

$$\varphi'(0) = \sqrt{2\lambda}\frac{-\exp(\sqrt{2\lambda}) + \exp(-\sqrt{2\lambda})}{\exp(\sqrt{2\lambda}) + \exp(-\sqrt{2\lambda})} = -\sqrt{2\lambda}\tanh(\sqrt{2\lambda}).$$

By problem 3, we get

$$\boldsymbol{E}[\exp(-\lambda\int_0^1 X_s ds)] = (\cosh(\sqrt{2\lambda}))^{-\frac{\alpha}{2}}\exp(-\frac{x}{2}\sqrt{2\lambda}\tanh(\sqrt{2\lambda})).$$

5. Suppose β is a $(\mathscr{F}_t)_{t\geq 0}$-real Brownian motion. By Itô's formula, we get

$$\beta_t^2 = y^2 + 2\int_0^t \beta_s d\beta_s + t$$

Set $B_t = \int_0^t sgn(\beta_s)d\beta_s$. Then $(B_t)_{t\geq 0}$ is a process $\langle B, B\rangle_t = t$, we see that B is a $(\mathscr{F}_t)_{t\geq 0}$-real Brownian motion and

$$\beta_t^2 = y^2 + 2\int_0^t |\beta_s|dB_s + t.$$

Thus, by problem 4, we get

$$\boldsymbol{E}[\exp(-\lambda \int_0^1 \beta_s^2 ds)] = (\cosh(\sqrt{2\lambda}))^{-\frac{1}{2}} \exp(-\frac{y^2}{2}\sqrt{2\lambda}\tanh(\sqrt{2\lambda})).$$

$\square$

5.8 Exercise 5.32 (Tanaka's formula and local time)

Let B be an $(\mathscr{F}_t)_{t\geq 0}$-Brownian motion started from 0. For every $\epsilon > 0$, we define a function $g_\epsilon : \mathbb{R} \mapsto \mathbb{R}$ by setting $g_\epsilon(x) = \sqrt{\epsilon^2 + x^2}$.

1. Show that
$$g_\epsilon(B_t) = g_\epsilon(0) + M_t^\epsilon + A_t^\epsilon$$
where M^ϵ is a square integrable continuous martingale that will be identified in the form of a stochastic integral, and A^ϵ is an increasing process.

2. We set $sgn(x) = 1_{\{x>0\}} - 1_{\{x<0\}}$ for all $x \in \mathbb{R}$. Show that, for every $t \geq 0$,
$$M_t^\epsilon \to \int_0^t sgn(B_s) dB_s \text{ in } L^2 \text{ as } \epsilon \to 0.$$
Infer that there exists an increasing process L such that, for every $t \geq 0$,
$$|B_t| = \int_0^t sgn(B_s) dB_s + L_t.$$

3. Observing that $A_t^\epsilon \to L_t$ as $\epsilon \to 0$ (It seems that the author want us to prove
$$A_t^\epsilon \to L_t \text{ as } \epsilon \to 0 \ \forall t \geq 0 \text{ (a.s.)},$$
but this statement is to strong to prove. You can prove the following problems without this statement). Show that, for every $\delta > 0$, for every choice of $0 < u < v$, the condition ($|B_t| \geq \delta$ for every $t \in [u, v]$) a.s. implies that $L_u = L_v$. Infer that the function $t \mapsto L_t$ is a.s. constant on every connected component of the open set $\{t \geq 0 \mid B_t \neq 0\}$.

4. We set $\beta_t = \int_0^t sgn(B_s) dB_s$ for all $t \geq 0$. Show that $(\beta_t)_{t\geq 0}$ is a $(\mathscr{F}_t)_{t\geq 0}$ Brownian motion started from 0.

5. Show that $L_t = \sup_{s\leq t}(-\beta_s)$ (a.s.). (In order to derive the bound $L_t \leq \sup_{s\leq t}(-\beta_s)$, one may consider the last zero of B before time t, and use question 3.) Give the law of L_t.

6. For every $\epsilon > 0$, we define two sequences of stopping times $(S_n^\epsilon)_{n\geq 1}$ and $(T_n^\epsilon)_{n\geq 1}$, by setting
$$S_1^\epsilon = 0,\ T_1^\epsilon = \inf\{t \geq S_1^\epsilon \mid |B_t| = \epsilon\}$$
and then, by induction,
$$S_{n+1}^\epsilon = \inf\{t \geq T_n^\epsilon \mid |B_t| = 0\},\ T_{n+1}^\epsilon = \inf\{t \geq S_{n+1}^\epsilon \mid |B_t| = \epsilon\}.$$
For every $t \geq 0$, we set
$$N_t^\epsilon = \sup\{n \geq 1 \mid T_n^\epsilon \leq t\},$$
where $\sup \emptyset = 0$. Show that
$$\epsilon N_t^\epsilon \xrightarrow{L^2} L_t \text{ as } \epsilon \to 0.$$
(One may observe that
$$L_t + \int_0^t \sum_{n=1}^\infty 1_{[S_n^\epsilon, T_n^\epsilon]}(s) sgn(B_s) dB_s = \epsilon N_t^\epsilon + r_t^\epsilon \text{ (a.s.)},$$
where the "remainder" r_t^ϵ satisfies $|r_t^\epsilon| \leq \epsilon$.)

7. Show that $\frac{N_t^1}{\sqrt{t}}$ converges in law as $t \to \infty$ to $|U|$, where U is $\mathcal{N}(0,1)$-distributed.

Proof.

1. By Itô's formula, we get
$$g_\epsilon(B_t) = g_\epsilon(0) + \int_0^t \frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} dB_s + \frac{1}{2}\int_0^t \frac{\epsilon^2}{(\epsilon^2 + B_s^2)^{\frac{3}{2}}} ds.$$
It's clear that
$$A_t^\epsilon \equiv \frac{1}{2}\int_0^t \frac{\epsilon^2}{(\epsilon^2 + B_s^2)^{\frac{3}{2}}} ds \tag{16}$$
is an increasing process. For $t \geq 0$,
$$\boldsymbol{E}[\langle \int_0^t \frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} dB_s, \int_0^t \frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} dB_s \rangle_t] = \boldsymbol{E}[\int_0^t \frac{B_s^2}{\epsilon^2 + B_s^2} ds] \leq t.$$
By theorem 4.13, we see that
$$M_t^\epsilon \equiv \int_0^t \frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} dB_s \tag{17}$$
is a sequare integrable continuous martingale.

2. Fix $t > 0$. Then
$$\frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} \to \frac{B_s}{|B_s|} = sgn(B_s) \text{ as } \epsilon \to 0 \ \forall s \in [0,t] \text{ (a.s.)},$$
where $\frac{B_s}{|B_s|} = 0$ when $B_s = 0$.
By Proposition 5.8, we see that
$$\int_0^t \frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} dB_s \xrightarrow{P} \int_0^t sgn(B_s) dB_s \text{ as } \epsilon \to 0.$$
Recall that

Lieb's theorem [1, Theorem 6.2.3].
Let $(E, \mathcal{B}, \mu)$ be a measure space, $p \in [1, \infty)$, and $\{f_n\} \bigcup \{f\} \subseteq L^p(\mu; \mathbb{R})$. If $\sup_{n \geq 1} ||f_n||_{L^p(\mu;\mathbb{R})} < \infty$ and $f_n \to f$ in μ-measure, then
$$||f_n - f||_{L^p(\mu;\mathbb{R})} \to 0 \textit{ whenever } ||f_n||_{L^p(\mu;\mathbb{R})} \to ||f||_{L^p(\mu;\mathbb{R})}.$$
Since
$$|| \int_0^t \frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} dB_s ||_{L^2}^2 = \boldsymbol{E}[(\int_0^t \frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} dB_s)^2] = \boldsymbol{E}[\int_0^t \frac{B_s^2}{\epsilon^2 + B_s^2} ds] \leq t$$
for all $\epsilon > 0$ and
$$\lim_{\epsilon \to 0} || \int_0^t \frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} dB_s ||_{L^2}^2 = t = \boldsymbol{E}[(\int_0^t sgn(B_s) dB_s)^2] = || \int_0^t sgn(B_s) dB_s ||_{L^2}^2,$$
we get
$$M_t^\epsilon = \int_0^t \frac{B_s}{\sqrt{\epsilon^2 + B_s^2}} dB_s \to \int_0^t sgn(B_s) dB_s \text{ in } L^2 \text{ as } \epsilon \to 0.$$

Let us now construct the corresponding increasing process $(L_t)_{t\geq 0}$. We just define

$$L_t = |B_s| - \int_0^t sgn(B_s)dB_s. \tag{18}$$

It remains to show that $(L_t)_{t\geq 0}$ is an increasing process. Fix $t > 0$. By Lieb's theorem, we see that

$$g_\epsilon(B_t) = \sqrt{\epsilon^2 + |B_s|^2} \xrightarrow{L^2} |B_t| \text{ as } \epsilon \to 0$$

and therefore

$$A_t^\epsilon = g_\epsilon(B_t) - g_\epsilon(0) - M_t^\epsilon \xrightarrow{L^2} |B_t| - \int_0^t sgn(B_s)dB_s = L_t.$$

Since $(A_t^\epsilon)_{t\geq 0}$ is an increasing process for all $\epsilon > 0$, we see that $(L_t)_{t\geq 0}$ is an increasing process.

3. First we show that the condition ($|B_t| \geq \delta$ for every $t \in [u,v]$) a.s. implies that $L_u = L_v$. Fix $\delta > 0$ and $0 < u < v$. Since $A_i^\epsilon \xrightarrow{L^2} L_i$ for $i = u, v$, there exists $\{\epsilon_k\}$ such that $\epsilon_k \downarrow 0$ and $A_i^{\epsilon_k} \xrightarrow{a.s.} L_i$ for $i = u, v$. Let

$$w \in \{\lim_{k\to\infty} A_u^{\epsilon_k} = L_u\} \bigcap \{\lim_{k\to\infty} A_v^{\epsilon_k} = L_v\} \bigcap \{|B_t| \geq \delta \text{ for all } t \in [u,v]\}.$$

Then

$$\frac{\epsilon_k^2}{(\epsilon_k^2 + B_s^2(w))^{\frac{3}{2}}} \leq \frac{1}{\delta^3}$$

for $s \in [u,v]$ and $k \geq 1$. By Lebesgue's dominated convergence theorem, we get

$$L_v(w) - L_u(w) = \lim_{k\to\infty} \frac{1}{2} \int_u^v \frac{\epsilon_k^2}{(\epsilon_k^2 + B_s^2(w))^{\frac{3}{2}}} ds = 0.$$

Thus, the condition ($|B_t| \geq \delta$ for every $t \in [u,v]$) a.s. implies that $L_u = L_v$.
Next, we show that the function $t \mapsto L_t$ is a.s. constant on every connected component of the open set $\{t \geq 0 \mid B_t \neq 0\}$. Set

$$Z_{\delta,u,v}^c = \{(|B_t| \geq \delta \text{ for every } t \in [u,v]) \text{ implies that } L_u = L_v\}$$

for all positive rational numbers δ and $u < v$. Then

$$Z \equiv \bigcup_{\delta,u,v} Z_{\delta,u,v} \tag{19}$$

is a zero set. Let $w \in Z^c$. Let (a,b) be a connected component of $\{t \geq 0 \mid B_t(w) \neq 0\}$. For any two rational numbers u and v such that $a < u < v < b$, there exists positive rational number δ such that $|B_t(w)| \geq \delta$ for all $t \in [u,v]$ and therefore $L_u(w) = L_v(w)$. Since $t \in (a,b) \mapsto L_t(w)$ is increasing, we see that $t \in (a,b) \mapsto L_t(w)$ is a constant. Hence $t \mapsto L_t$ is a.s. constant on every connected component of the open set $\{t \geq 0 \mid B_t \neq 0\}$.

4. It's clear that $(\beta_t)_{t\geq 0}$ is a $(\mathscr{F}_t)_{t\geq 0}$-continuous local martingale with $\langle \beta, \beta \rangle_t = t$ for all $t \geq 0$. Thus, $(\beta_t)_{t\geq 0}$ is a $(\mathscr{F}_t)_{t\geq 0}$ Brownian motion started from 0.

5. Fix $t_0 > 0$. Since $|B_t| = \beta_t + L_t$ $\forall t \geq 0$ (a.s.), we have $\sup_{s\leq t_0}(-\beta_s) \leq \sup_{s\leq t_0} L_s = L_{t_0}$ (a.s.). We show that

$$\sup_{s\leq t_0}(-\beta_s) \geq L_{t_0} \text{ (a.s.)}.$$

Let $w \in Z^c \bigcap \{|B_t| = \beta_t + L_t \ \forall t \geq 0\}$, where Z is defined in (19). Set $r = \sup\{0 \leq s \leq t_0 \mid B_s(w) = 0\}$. Then $B_r(w) = 0$ and

$$L_{t_0}(w) = -\beta_t(w) \leq \sup_{s\leq t_0}(-\beta_s)(w) \text{ whenever } B_{t_0}(w) = 0.$$

Since $t \in \mathbb{R}_+ \mapsto L_t(w) \in C(\mathbb{R}_+)$ is constant on every connected component of $\{t \geq 0 \mid B_t(w) \neq 0\}$, we have

$$L_t(w) = L_r(w) = -\beta_r(w) \leq \sup_{s \leq t}(-\beta_s)(w) \text{ whenever } B_t(w) \neq 0.$$

Thus

$$\sup_{s \leq t_0}(-\beta_s) \geq L_{t_0} \text{ (a.s.)}$$

and therefore

$$\sup_{s \leq t_0}(-\beta_s) = L_{t_0} \text{ (a.s.)}. \tag{20}$$

To find the law of L_t, we define stopping times

$$\Gamma_a = \inf\{t \geq 0 \mid -\beta_t = a\} \tag{21}$$

for $a \in \mathbb{R}$. By the result of problem 4 and Corollary 2.22, we get

$$\boldsymbol{P}(L_t \leq a) = \boldsymbol{P}(\sup_{s \leq t}(-\beta_s) \leq a) = \boldsymbol{P}(\Gamma_a \geq t) = \int_t^\infty \frac{a}{\sqrt{2\pi s^3}} \exp(-\frac{a^2}{2s}) ds.$$

6. Fix $t > 0$ and $\epsilon > 0$. Note that N_t^ϵ is the number of upcrossing from 0 to $\pm\epsilon$ by $(B_s)_{s \in [0,t]}$. First, we show that

$$L_t + \int_0^t \sum_{n=1}^\infty 1_{[S_n^\epsilon, T_n^\epsilon]}(s) sgn(B_s) dB_s = \epsilon N_t^\epsilon + r_t^\epsilon \text{ (a.s.)},$$

where $|r_t^\epsilon| \leq \epsilon$. By (18) and Proposition 5.8, we get

$$\begin{aligned} L_t + \int_0^t \sum_{n=1}^\infty 1_{[S_n^\epsilon, T_n^\epsilon]}(s) sgn(B_s) dB_s &= |B_t| - \int_0^t \sum_{n=1}^\infty 1_{(T_n^\epsilon, S_{n+1}^\epsilon)}(s) sgn(B_s) dB_s \\ &= |B_t| - \sum_{n=1}^\infty \int_0^t 1_{(T_n^\epsilon, S_{n+1}^\epsilon)}(s) sgn(B_s) dB_s \end{aligned}$$

outside a zero set N. Let $w \in N^c$. We consider the following cases:

(a) Suppose that $0 = S_1^\epsilon(w) < T_1^\epsilon(w) < S_2^\epsilon(w) ... < T_{m-1}^\epsilon(w) < S_m^\epsilon(w) < t < T_m^\epsilon(w)$ for some $m \geq 1$. Then $|B_t(w)| \leq \epsilon$, $N_t^\epsilon = m-1$, and $sgn(B_s)(w) = sgn(B_{T_k^\epsilon})(w)$ for $s \in [T_k^\epsilon(w), S_{k+1}^\epsilon(w))$ for each $k = 1, ..., m-1$. If we set $r_t^\epsilon(w) = |B_t(w)|$, then we have

$$\begin{aligned} &|B_t(w)| - (\sum_{k=1}^\infty \int_0^t 1_{(T_k^\epsilon, S_{k+1}^\epsilon)}(s) sgn(B_s) dB_s)(w) \\ &= r_t^\epsilon(w) - (\sum_{k=1}^{m-1} sgn(B_{T_k^\epsilon}) \int_0^t 1_{(T_k^\epsilon, S_{k+1}^\epsilon)}(s) dB_s)(w) \\ &= r_t^\epsilon(w) - \sum_{k=1}^{m-1} sgn(B_{T_k^\epsilon})(w)(B_{S_{k+1}^\epsilon}(w) - B_{T_k^\epsilon}(w)) \\ &= r_t^\epsilon(w) - \sum_{k=1}^{m-1} sgn(B_{T_k^\epsilon})(w)(0 - sgn(B_{T_k^\epsilon})(w) \times \epsilon) \\ &= r_t^\epsilon(w) + (m-1)\epsilon \\ &= r_t^\epsilon(w) + N_t^\epsilon(w)\epsilon. \end{aligned}$$

(b) Suppose that $0 = S_1^\epsilon(w) < T_1^\epsilon(w) < S_2^\epsilon(w) ... < T_{m-1}^\epsilon(w) < S_m^\epsilon(w) < T_m^\epsilon(w) \leq t < S_{m+1}^\epsilon(w)$ for some $m \geq 1$. Similar, we get $N_t^\epsilon = m$, and $sgn(B_s)(w) = sgn(B_{T_k^\epsilon})(w)$ for $s \in [T_k^\epsilon(w), S_{k+1}^\epsilon(w))$ for each $k = 1, ..., m+1$. If we set $r_t^\epsilon(w) = \epsilon$, then we have

$$\begin{aligned}
&|B_t(w)| - (\sum_{k=1}^{\infty} \int_0^t 1_{(T_k^\epsilon, S_{k+1}^\epsilon)}(s) sgn(B_s) dB_s)(w) \\
&= |B_t(w)| - (\sum_{k=1}^{m} sgn(B_{T_k^\epsilon}) \int_0^t 1_{(T_k^\epsilon, S_{k+1}^\epsilon)}(s) dB_s)(w) - sgn(B_t) \int_0^t 1_{(T_m^\epsilon, t)}(s) dB_s)(w) \\
&= |B_t(w)| - \sum_{k=1}^{m} sgn(B_{T_k^\epsilon})(w)(B_{S_{k+1}^\epsilon}(w) - B_{T_k^\epsilon}(w)) - sgn(B_t)(w)(B_t(w) - B_{T_m^\epsilon}(w)) \\
&= |B_t(w)| - \sum_{k=1}^{m} sgn(B_{T_k^\epsilon})(w)(0 - sgn(B_{T_k^\epsilon})(w) \times \epsilon) - sgn(B_t)(w)(B_t(w) - sgn(B_t)(w) \times \epsilon) \\
&= \epsilon + m\epsilon \\
&= r_t^\epsilon(w) + N_t^\epsilon(w)\epsilon.
\end{aligned}$$

Thus we have, a.s.,

$$L_t + \int_0^t \sum_{n=1}^{\infty} 1_{[S_n^\epsilon, T_n^\epsilon]}(s) sgn(B_s) dB_s = \epsilon N_t^\epsilon + r_t^\epsilon,$$

where $|r_t^\epsilon| \leq \epsilon$.
Next, we show that

$$\epsilon N_t^\epsilon \xrightarrow{L^2} L_t \text{ as } \epsilon \to 0.$$

Fix $t \geq 0$. Note that

$$\sum_{k=1}^{\infty} 1_{[S_n^\epsilon(w), T_n^\epsilon(w)]}(s) \leq 1_{\{|B_s| \leq \epsilon\}}(w) \text{ for all } 0 \leq s \leq t \text{ and } w \in \Omega. \tag{22}$$

and so

$$\begin{aligned}
||\epsilon N_t^\epsilon - L_t||_{L^2} &\leq || \int_0^t \sum_{n=1}^{\infty} 1_{[S_n^\epsilon, T_n^\epsilon]}(s) sgn(B_s) dB_s||_{L^2} + ||r_t^\epsilon||_{L^2} \\
&= \boldsymbol{E}[\int_0^t \sum_{n=1}^{\infty} 1_{[S_n^\epsilon, T_n^\epsilon]}(s) ds] + ||r_t^\epsilon||_{L^2} \\
&= \int_0^t \boldsymbol{E}[\sum_{n=1}^{\infty} 1_{[S_n^\epsilon, T_n^\epsilon]}(s)] ds + ||r_t^\epsilon||_{L^2} \\
&\leq \int_0^t \boldsymbol{E}[1_{\{|B_s| \leq \epsilon\}}(w)] ds + ||r_t^\epsilon||_{L^2} \\
&= \int_0^t \boldsymbol{P}(|B_s| \leq \epsilon) ds + ||r_t^\epsilon||_{L^2} \overset{\epsilon \to 0}{\to} \int_0^t \boldsymbol{P}(|B_s| = 0) ds = 0.
\end{aligned}$$

7. First we show that $\frac{L_t}{\sqrt{t}} \overset{d}{=} |U|$ for all $t > 0$. Define stopping times Γ_a as (33). Fix $t_0 > 0$. By (20) and Corollary 2.22, we get

$$\boldsymbol{P}(\frac{L_{t_0}}{\sqrt{t_0}} \leq a) = \boldsymbol{P}(\sup_{s \leq t_0}(-\beta_s) \leq a \times \sqrt{t_0}) = \boldsymbol{P}(\Gamma_{a\sqrt{t_0}} \geq t_0) = \int_{t_0}^{\infty} \frac{\sqrt{t_0}a}{\sqrt{2\pi t^3}} \exp(-\frac{t_0 a^2}{2t}) dt.$$

Set $x = \frac{\sqrt{t_0}a}{\sqrt{t}}$. Then $dx = \frac{1}{2}\frac{\sqrt{t_0}a}{t^{\frac{3}{2}}}dt$ and

$$\int_{t_0}^{\infty} \frac{\sqrt{t_0}a}{\sqrt{2\pi t^3}} \exp(-\frac{t_0 a^2}{2t})dt = \int_0^a \frac{2}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})dx = \boldsymbol{P}(|U| \leq a).$$

Recall that if $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} 0$, then $X_n + Y_n \xrightarrow{d} X$. To show that $\frac{N_t^1}{\sqrt{t}} \xrightarrow{d} |U|$, it suffices to show that, as $t \to \infty$,

$$\frac{1}{\sqrt{t}}(N_t^1 - L_t) = \frac{1}{\sqrt{t}}(\int_0^t \sum_{n=1}^{\infty} 1_{[S_n^1, T_n^1]}(s) sgn(B_s) dB_s - r_t^1) \xrightarrow{L^2} 0.$$

Note that

$$||\frac{1}{\sqrt{t}}(\int_0^t \sum_{n=1}^{\infty} 1_{[S_n^1, T_n^1]}(s) sgn(B_s) dB_s - r_t^1)||_{L^2} \leq ||\frac{1}{\sqrt{t}}\int_0^t \sum_{n=1}^{\infty} 1_{[S_n^1, T_n^1]}(s) sgn(B_s) dB_s||_{L^2} + ||\frac{1}{\sqrt{t}} r_t^1||_{L^2}$$

and

$$||\frac{1}{\sqrt{t}} r_t^1||_{L^2} \leq \frac{1}{\sqrt{t}}.$$

It suffices to show that

$$\frac{1}{\sqrt{t}}\int_0^t \sum_{n=1}^{\infty} 1_{[S_n^1, T_n^1]}(s) sgn(B_s) dB_s \xrightarrow{L^2} 0 \text{ as } t \to \infty.$$

By (32), we get

$$\begin{aligned}
&||\frac{1}{\sqrt{t}}\int_0^t \sum_{n=1}^{\infty} 1_{[S_n^1, T_n^1]}(s) sgn(B_s) dB_s||_{L^2}^2 \\
&= \boldsymbol{E}[\frac{1}{t}\int_0^t \sum_{n=1}^{\infty} 1_{[S_n^1, T_n^1]}(s) sgn(B_s) ds] \leq \boldsymbol{E}[\frac{1}{t}\int_0^t 1_{\{|B_s| \leq 1\}} ds] \\
&= \frac{1}{t}\int_0^t \boldsymbol{P}(|B_s| \leq 1) ds = \frac{1}{t}\int_0^t \boldsymbol{P}(|B_1| \leq \frac{1}{\sqrt{s}}) ds \\
&= \frac{2}{t}\int_0^t \int_0^{\frac{1}{\sqrt{s}}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx ds \\
&= \frac{2}{t}(\int_0^{\frac{1}{\sqrt{t}}} \int_0^t \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) ds dx + \int_{\frac{1}{\sqrt{t}}}^{\infty} \int_0^{\frac{1}{x^2}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) ds dx) \\
&= \frac{2}{t}(\int_0^{\frac{1}{\sqrt{t}}} \frac{t}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx + \int_{\frac{1}{\sqrt{t}}}^{\infty} \frac{1}{x^2} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx) \\
&\leq \frac{2}{t}(\int_0^{\frac{1}{\sqrt{t}}} \frac{t}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx + \int_{\frac{1}{\sqrt{t}}}^{\infty} \frac{1}{x^2} \frac{1}{\sqrt{2\pi}} dx) \\
&= \frac{2}{t}(\int_0^{\frac{1}{\sqrt{t}}} \frac{t}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx + \frac{1}{\sqrt{2\pi}} \sqrt{t}) \\
&= \int_0^{\frac{1}{\sqrt{t}}} \frac{2}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) dx + \frac{2}{\sqrt{2\pi}} \frac{1}{\sqrt{t}} \overset{t \to \infty}{\to} 0.
\end{aligned}$$

$\square$

5.9 Exercise 5.33 (Study of multidimensional Brownian motion)

Let $B_t = (B_1^N, ..., B_t^N)$ be an N-dimensional $(\mathscr{F}_t)$-Brownian motion started from $x = (x_1, ..., x_N)$. We suppose that $N \geq 2$.

1. Verify that $|B_t|^2$ is a continuous semimartingale, and that the martingale part of $|B_t|^2$ is a true martingale.
2. We set
$$\beta_t = \sum_i^N \int_0^t \frac{B_s^i}{|B_s|} dB_s^i$$
with the convention that $\frac{B_s^i}{|B_s|} = 0$ if $|B_s| = 0$. Justify the definition of the stochastic integrals appearing in the definition of β_t, then show that the process $(\beta_t)_{t\geq 0}$ is an $(\mathscr{F}_t)$-Brownian motion started from 0.
3. Show that
$$|B_t|^2 = |x|^2 + 2\int_0^t |B_s| d\beta_s + Nt.$$
4. From now on, we assume that $x \neq 0$. Let $\epsilon \in (0, |x|)$ and $T_\epsilon = \inf\{t \geq 0 \mid |B_t| \leq \epsilon\}$. Define $f : (0, \infty) \mapsto \mathbb{R}$ by
$$f(a) = \begin{cases} \log(a), & \text{if } N = 2 \\ a^{2-N}, & \text{if } N \geq 3 \end{cases}$$
Verify that $f(|B_{t\wedge T_\epsilon}|)$ is a continuous local martingale.
5. Let $R > |x|$ and set $S_R = \inf\{t \geq 0 \mid |B_t| \geq R\}$. Show that
$$\boldsymbol{P}(T_\epsilon < S_R) = \frac{f(R) - f(|x|)}{f(R) - f(\epsilon)}.$$
Observing that $\boldsymbol{P}(T_\epsilon < S_R) \to 0$ as $\epsilon \to 0$, show that $B_t \neq 0$ for all $t \geq 0$, a.s.
6. Show that, a.s., for every $t \geq 0$,
$$|B_t| = |x| + \beta_t + \frac{N-1}{2}\int_0^t \frac{ds}{|B_s|}.$$
7. We assume that $N \geq 3$. Show that $\lim_{t\to\infty} |B_t| = \infty$ (a.s.) (Hint: Observe that $|B_t|^{2-N}$ is a nonnegative supermartingale.)
8. We assume $N = 3$. Using the form of the Gaussian density, verify that the collection of random variables $(|B_t|^{-1})_{t\geq 0}$ is bounded in L^2. Show that $(|B_t|^{-1})_{t\geq 0}$ is a continuous local martingale but is not a (true) martingale.

Proof.

1. By Itô's formula and Doob's inequality in L^2, we get
$$|B_t|^2 = |x|^2 + \sum_{i=1}^N \int_0^t 2B_s^i dB_s^i + Nt$$
and
$$\boldsymbol{E}[\langle \int_0^t 2B_s^i dB_s^i, \int_0^t 2B_s^i dB_s^i \rangle] = 4\boldsymbol{E}[\int_0^t (B_s^i)^2 ds] \leq 4t\boldsymbol{E}[\sup_{0\leq s\leq t} (B_s^i)^2] \leq 4t2^2\boldsymbol{E}[(B_t^i)^2] \leq 16t(t + x_i^2)$$
for $1 \leq i \leq N$. Thus, $(\int_0^t 2B_s^i dB_s^i)_{t\geq 0}$ is a true $(\mathscr{F}_t)$-martingale for $1 \leq i \leq N$.

2. Since $(\frac{B^i}{|B|})^2 \le 1$, we see that $\frac{B^i}{|B|} \in L^2_{loc}(B^i)$ and, hence, $\int_0^t \frac{B^i_s}{|B_s|}dB^i_s$ is well-defined continuous local martingale. Thus, $(\beta_t)_{t\ge 0}$ is a $(\mathscr{F}_t)$-continuous local martingale. Because

$$\langle \beta, \beta \rangle_t = \sum_{i=1}^N \int_0^t \frac{(B^i_s)^2}{|B_s|^2} ds = t,$$

we see that $(\beta_t)_{t\ge 0}$ is an $(\mathscr{F}_t)$-Brownian motion started from 0.

3. Note that

$$B^i_t = \frac{B^i_t}{|B_t|}|B_t|,$$

where $\frac{B^i_t}{|B_t|}$ is defined in problem 2, and

$$d\beta_t = \sum_{i=1}^N \frac{B^i_t}{|B_t|} dB^i_t.$$

Then

$$|B_t|^2 = |x|^2 + \sum_{i=1}^N \int_0^t 2B^i_s dB^i_s + Nt = |x|^2 + 2\int_0^t |B_s| d\beta_s + Nt.$$

4. Define $F : \mathbb{R}^N \setminus \{0\} \mapsto \mathbb{R}$ by $F(x) = f(|x|)$. Then we have

$$\frac{\partial F}{\partial x_i}(x) = \begin{cases} \frac{(2-N)x_i}{|x|^N}, & \text{if } N \ge 3 \\ \frac{x_i}{|x|^2}, & \text{if } N = 2 \end{cases}$$

and

$$\frac{\partial^2 F}{\partial x_i^2}(x) = \begin{cases} \frac{N-2}{|x|^N}(1 - \frac{Nx_i^2}{|x|^2}), & \text{if } N \ge 3 \\ 1 - \frac{2x_i^2}{|x|^2}, & \text{if } N = 2. \end{cases}$$

Note that $|B_{t\wedge T_\epsilon}(w)| \ge \epsilon$ for all $t \ge 0$ and $w \in \Omega$. By Itô's formula, we get

$$\begin{aligned} f(|B_{t\wedge T_\epsilon}|) &= F(B_{t\wedge T_\epsilon}) \\ &= f(|x|) + \sum_{i=1}^N \int_0^t \frac{\partial F}{\partial x_i}(B_{s\wedge T_\epsilon}) dB^i_s + \frac{1}{2}\sum_{i=1}^N \int_0^t \frac{\partial^2 F}{\partial x_i^2}(B_{s\wedge T_\epsilon}) ds \\ &= \begin{cases} f(|x|) + \sum_{i=1}^N \int_0^t \frac{(2-N)B^i_{s\wedge T_\epsilon}}{|B_{s\wedge T_\epsilon}|^N} dB^i_s + \frac{1}{2}\sum_{i=1}^N \int_0^t \frac{N-2}{|B_{s\wedge T_\epsilon}|^N}(1 - \frac{N(B^i_{s\wedge T_\epsilon})^2}{|B_{s\wedge T_\epsilon}|^2}) ds, & \text{if } N \ge 3 \\ f(|x|) + \sum_{i=1}^N \int_0^t \frac{B^i_{s\wedge T_\epsilon}}{|B_{s\wedge T_\epsilon}|^2} dB^i_s + \frac{1}{2}\sum_{i=1}^N \int_0^t (1 - \frac{2(B^i_{s\wedge T_\epsilon})^2}{|B_{s\wedge T_\epsilon}|^2}) ds, & \text{if } N = 2 \end{cases} \\ &= \begin{cases} f(|x|) + \sum_{i=1}^N \int_0^t \frac{(2-N)B^i_{s\wedge T_\epsilon}}{|B_{s\wedge T_\epsilon}|^N} dB^i_s, & \text{if } N \ge 3 \\ f(|x|) + \sum_{i=1}^N \int_0^t \frac{B^i_{s\wedge T_\epsilon}}{|B_{s\wedge T_\epsilon}|^2} dB^i_s, & \text{if } N = 2 \end{cases} \end{aligned}$$

and, hence, $f(|B_{t\wedge T_\epsilon}|)$ is a continuous local martingale.

5. Set $T = T_\epsilon \wedge S_R$. Then $|f(|B^T_t|)| \le M$ for some $M > 0$ and all $t \ge 0$ (a.s.). Since $f(|B_{t\wedge T_\epsilon}|)$ is a continuous local martingale, we see that $f(|B^T_t|)$ is a bounded continuous local martingale and, hence, $f(|B^T_t|)$ is an uniformly bounded martingale. Then we have

$$f(|x|) = \boldsymbol{E}[f(|B^T_0|)] = \boldsymbol{E}[f(|B_T|)] = f(\epsilon)\boldsymbol{P}(T_\epsilon < S_R) + f(R)\boldsymbol{P}(T_\epsilon \ge S_R).$$

Since $\boldsymbol{P}(T_\epsilon < S_R) + \boldsymbol{P}(T_\epsilon \ge S_R) = 1$,we get

$$\boldsymbol{P}(T_\epsilon < S_R) = \frac{f(R) - f(|x|)}{f(R) - f(\epsilon)}.$$

Because $f(\epsilon) \to \pm\infty$(depending on N) as $\epsilon \to 0$, we see that $\boldsymbol{P}(T_\epsilon < S_R) \to 0$ as $\epsilon \to 0$. Next we show that $B_t \neq 0$ for all $t \geq 0$ (a.s.). Choose a sequence of positive real number $\{\epsilon_n\}$ such that $\epsilon_n \downarrow 0$ and

$$\sum_{n=1}^{\infty} \boldsymbol{P}(T_{\epsilon_n} < S_n) < \infty.$$

By Borel Cantelli's lemma, we get $\boldsymbol{P}(Z) = 0$, where $Z = \limsup_{n\to\infty}\{T_{\epsilon_n} < S_n\}$. Then $B_t \neq 0$ for all $t \geq 0$ in Z^c. Indeed, if $w \in Z^c$ and $B_t(w) = 0$ for some $t > 0$, then $T_{\epsilon_n}(w) < t$ for all $n \geq 1$ and, hence, $S_n(w) < t$ for some $m \geq 1$ and all $n \geq m$. Since $\{S_n(w)\}$ is nondecreasing, we see that $\lim_{n\to\infty} S_n(w)$ exists, $s \equiv \lim_{n\to\infty} S_n(w) \leq t$ and, hence, $B_s(w) = \infty$ which is a contradiction. Thus, $B_t \neq 0$ for all $t \geq 0$, a.s.

6. Define $F : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}_+$ by $F(x) = |x|$. Then $F \in C^\infty(\mathbb{R}^N \setminus \{0\})$, $\frac{\partial F}{\partial x_i}(x) = \frac{x_i}{|x|}$, and $\frac{\partial^2 F}{\partial x_i^2}(x) = \frac{|x|^2 - x_i^2}{|x|^3}$. Since $B_t \in \mathbb{R}^N \setminus \{0\}$ for all $t \geq 0$ (a.s.), we get

$$|B_t| = F(B_t) = |x| + \sum_{i=1}^{N} \int_0^t \frac{B_s^i}{|B_s|} dB_s^i + \frac{1}{2}\sum_{i=1}^{N} \int_0^t \frac{|B_s|^2 - (B_s^i)^2}{|B_s|^3} ds = |x| + \beta_t + \frac{N-1}{2}\int_0^t \frac{ds}{|B_s|}$$

7. Define $F : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}_+$ by $F(x) = |x|^{2-N}$. Then $F \in C^\infty(\mathbb{R}^N \setminus \{0\})$. Since $B_t \in \mathbb{R}^N \setminus \{0\}$ for all $t \geq 0$ (a.s.), we get (see the proof of problem 4)

$$|B_t|^{2-N} = |x|^{2-N} + \sum_{i=1}^{N} \int_0^t \frac{(2-N)B_s^i}{|B_s|^N} dB_s^i.$$

Then $|B_t|^{2-N}$ is a non-negative continuous local martingale and, hence, $|B_t|^{2-N}$ is a non-negative supermartingale. Thus,

$$\boldsymbol{E}[|B_t|^{2-N}] \leq \boldsymbol{E}[|B_0|^{2-N}] = |x|^{2-N}$$

for all $t \geq 0$. By Theorem 3.19, $|B_\infty|^{2-N}$ exists (a.s.) and, hence, $\lim_{t\to\infty} |B_t|$ exists (a.s.). Since $\limsup_{t\to\infty} B_t^1 = \infty$ (a.s.), we see that $\lim_{t\to\infty} |B_t| = \infty$ (a.s.).

8. First, we show that $(|B_t|^{-1})_{t\geq 0}$ is bounded in L^2. Set $\delta = \frac{|x|}{2} > 0$. Then

$$\boldsymbol{E}[|B_t|^{-2}] = \int_{\mathbb{R}^3} \frac{1}{|y|^2(2\pi t)^{\frac{3}{2}}} \exp(\frac{-|y-x|^2}{2t})dy = \int_{|y|<\delta} + \int_{|y|\geq\delta}.$$

Since

$$\int_{|y|\geq\delta} \frac{1}{|y|^2(2\pi t)^{\frac{3}{2}}} \exp(\frac{-|y-x|^2}{2t})dy \leq \frac{1}{\delta^2}\int_{\mathbb{R}^3} \frac{1}{(2\pi t)^{\frac{3}{2}}} \exp(\frac{-|y-x|^2}{2t})dy \leq \frac{1}{\delta^2}$$

for all $t > 0$, it suffices to show that

$$\int_{|y|<\delta} \frac{1}{|y|^2(2\pi t)^{\frac{3}{2}}} \exp(\frac{-|y-x|^2}{2t})dy$$

is bounded in $t > 0$. Note that, if $|y| < \delta = \frac{|x|}{2}$, then $|y-x| \geq |x| - |y| \geq \frac{|x|}{2}$. Then we see that

$$\int_{|y|<\delta} \frac{1}{|y|^2(2\pi t)^{\frac{3}{2}}} \exp(\frac{-|y-x|^2}{2t})dy \leq \frac{1}{(2\pi t)^{\frac{3}{2}}}\exp(\frac{-|x|^2}{8t})\int_{|y|<\delta}\frac{1}{|y|^2}dy = \frac{1}{(2\pi t)^{\frac{3}{2}}}\exp(\frac{-|x|^2}{8t})w_3,$$

where w_3 is the area of unit sphere in $\mathbb{R}^3$. Define $\varphi : (0,\infty) \to \mathbb{R}_+$ by

$$\varphi(t) = \frac{1}{(2\pi t)^{\frac{3}{2}}}\exp(\frac{-|x|^2}{8t}).$$

Then $\varphi \in C_0((0,\infty))$ and $\lim_{t\downarrow 0}\varphi(t)=0$. There exists $M>0$ such that $\sup_{t>0}|\varphi(t)|\le M<\infty$. Thus,

$$\sup_{t>0}\int_{|y|<\delta}\frac{1}{|y|^2(2\pi t)^{\frac{3}{2}}}\exp(\frac{-|y-x|^2}{2t})dy \le Mw_3$$

and therefore $(|B_t|^{-1})_{t\ge 0}$ is bounded in L^2. Now we show that $(|B_t|^{-1})_{t\ge 0}$ is a continuous local martingale but is not a true martingale. Assume that $(|B_t|^{-1})_{t\ge 0}$ is a true martingale. Then $(|B_t|^{-1})_{t\ge 0}$ is a L^2-bounded martingale. Recall that $\lim_{t\to\infty}|B_t|=\infty$ (a.s.). Together with Theorem 4.13, we get

$$0=\boldsymbol{E}[|B_\infty|^{-2}]=\boldsymbol{E}[|B_0|^{-2}]+\boldsymbol{E}[\langle |B|^{-1},|B|^{-1}\rangle_\infty]$$

which is a contradiction. Thus $(|B_t|^{-1})_{t\ge 0}$ is a continuous local martingale (see the proof of problem 7) but is not a true martingale.

$\square$

Chapter 6

General Theory of Markov Processes

6.1 Exercise 6.23 (Reflected Brownian motion)

We consider a probability space equipped with a filtration $(\mathscr{F}_t)_{t\in[0,\infty]}$. Let $a \geq 0$ and let $B = (B_t)_{t\geq 0}$ be an $(\mathscr{F}_t)$-Brownian motion such that $B_0 = a$. For every $t > 0$ and every $z \in \mathbb{R}$, we set

$$p_t(z) = \frac{1}{\sqrt{2\pi t}} \exp(-\frac{z^2}{2t}).$$

1. We set $X_t = |B_t|$ for every $t \geq 0$. Verify that, for every $s \geq 0$ and $t \geq 0$, for every bounded measurable function $f : \mathbb{R}_+ \mapsto \mathbb{R}$,
$$\boldsymbol{E}[f(X_{s+t}) \mid \mathscr{F}_s] = Q_t f(X_s),$$
where $Q_0 f = f$ and, for every $t > 0$, for every $x \geq 0$,
$$Q_t f(x) = \int_0^\infty (p_t(y-x) + p_t(y+x)) f(y) dy.$$

2. infer that $(Q_t)_{t\geq 0}$ is a transition semigroup, then that $(X_t)_{t\geq 0}$ is a Markov process with values in $E = \mathbb{R}_+$, with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$, with semigroup $(Q_t)_{t\geq 0}$.

3. Verify that $(Q_t)_{t\geq 0}$ is a Feller semigroup. We denote its generator by L.

4. Let f be a twice continuously differentiable function on $\mathbb{R}_+$, such that f and f'' belong to $C_0(\mathbb{R}_+)$. Show that, if $f'(0) = 0$, f belongs to the domain of L, and $Lf = \frac{1}{2} f''$. (Hint: One may observe that the function $g : \mathbb{R} \mapsto \mathbb{R}$ defined by $g(y) = f(|y|)$ is then twice continuously differentiable on $\mathbb{R}$.) Show that, conversely, if $f(0) \neq 0$, f does not belong to the domain of L.

Proof.

1. Set Q_t^B to be the semigroup of real Brownian motion (i.e. $Q_t^B(x, dy) = p_t(y-x)dy$). Given a bounded measurable function $f : \mathbb{R}_+ \mapsto \mathbb{R}$. Define $g : \mathbb{R} \mapsto \mathbb{R}$ by $g(y) = f(|y|)$. By definition of Markov process,
$$\begin{aligned}
\boldsymbol{E}[f(X_{s+t}) \mid \mathscr{F}_s] &= \boldsymbol{E}[g(B_{s+t}) \mid \mathscr{F}_s] = Q_t^B g(B_s) \\
&= \int_{-\infty}^\infty f(|y|) \frac{1}{\sqrt{2\pi t}} \exp(-\frac{(y-B_s)^2}{2t}) dy \\
&= \int_0^\infty f(|y|) \frac{1}{\sqrt{2\pi t}} \exp(-\frac{(y-B_s)^2}{2t}) dy + \int_{-\infty}^0 f(|y|) \frac{1}{\sqrt{2\pi t}} \exp(-\frac{(y-B_s)^2}{2t}) dy \\
&= \int_0^\infty f(|y|) \frac{1}{\sqrt{2\pi t}} \exp(-\frac{(y-B_s)^2}{2t}) dy + \int_0^\infty f(|y|) \frac{1}{\sqrt{2\pi t}} \exp(-\frac{(y+B_s)^2}{2t}) dy \\
&= \int_0^\infty f(y) \frac{1}{\sqrt{2\pi t}} \exp(-\frac{(y-B_s)^2}{2t}) dy + \int_0^\infty f(y) \frac{1}{\sqrt{2\pi t}} \exp(-\frac{(y+B_s)^2}{2t}) dy \\
&= Q_t f(X_s).
\end{aligned}$$

2. It's clear that
$$(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \mapsto Q_t(x, A) = \int_0^\infty (\frac{1}{\sqrt{2\pi t}} \exp(-\frac{(y-x)^2}{2t}) + \frac{1}{\sqrt{2\pi t}} \exp(-\frac{(y+x)^2}{2t})) 1_A(y) dy$$

is a measurable function. Thus, it suffices to show that $(Q_t)_{t\geq 0}$ satisfy Chapman-Kolmogorov's identity. Let f be a bounded measuable function on $\mathbb{R}_+$. Define $g:\mathbb{R}\mapsto\mathbb{R}$ by $g(y)=f(|y|)$. By using similar argument as the proof of problem 1, we have

$$Q_tf(|x|)=Q_t^Bg(x)\qquad\forall x\in\mathbb{R}. \tag{23}$$

and therefore

$$\begin{aligned}
Q_{t+s}f(x)=Q_{t+s}^Bg(x)=Q_t^BQ_s^Bg(x)&=\int_{\mathbb{R}}Q_s^Bg(y)\frac{1}{\sqrt{2\pi t}}\exp(-\frac{(y-x)^2}{2t})dy\\
&=\int_{\mathbb{R}_+}Q_s^Bg(y)\frac{1}{\sqrt{2\pi t}}\exp(-\frac{(y-x)^2}{2t})dy+\int_{\mathbb{R}_-}Q_s^Bg(y)\frac{1}{\sqrt{2\pi t}}\exp(-\frac{(y-x)^2}{2t})dy\\
&=\int_{\mathbb{R}_+}Q_s^Bg(y)\frac{1}{\sqrt{2\pi t}}\exp(-\frac{(y-x)^2}{2t})dy+\int_{\mathbb{R}_+}Q_s^Bg(-y)\frac{1}{\sqrt{2\pi t}}\exp(-\frac{(y+x)^2}{2t})dy\\
&=\int_{\mathbb{R}_+}Q_sf(y)\frac{1}{\sqrt{2\pi t}}\exp(-\frac{(y-x)^2}{2t})dy+\int_{\mathbb{R}_+}Q_sf(y)\frac{1}{\sqrt{2\pi t}}\exp(-\frac{(y+x)^2}{2t})dy\\
&=Q_tQ_sf(x)\qquad\forall x\in\mathbb{R}_+.
\end{aligned}$$

3. Given $f\in C_0(\mathbb{R}_+)$. Then $g(x)\equiv f(|x|)\in C_0(\mathbb{R})$. Since $(Q_t^B)_{t\geq 0}$ is Feller semigroup, we see that $Q_tf(x)=Q_t^Bg(x)\in C_0(\mathbb{R}_+)$ and

$$\sup_{x\in\mathbb{R}_+}|Q_tf(x)-f(x)|\leq\sup_{x\in\mathbb{R}}|Q_t^Bg(x)-g(x)|\overset{t\to 0}{\to}0.$$

Therefore $(Q_t)_{t\geq 0}$ is a Feller semigroup.

4. Let f be a twice continuously differentiable function on $\mathbb{R}_+$, such that f and f'' belong to $C_0(\mathbb{R}_+)$. Define $g:\mathbb{R}\mapsto\mathbb{R}$ by $g(y)=f(|y|)$. Observe that

$$\lim_{t\to 0^+}\frac{g(x)-g(0)}{x}=\lim_{t\to 0^+}\frac{f(x)-f(0)}{x}=f'(0).$$

and

$$\lim_{t\to 0^-}\frac{g(x)-g(0)}{x}=\lim_{t\to 0^-}\frac{f(-x)-f(0)}{x}=-f'(0).$$

Since $f'(0)=0$, $g'(0)$ exists and therefore

$$g'(y)=f'(|y|)sgn(y)$$

and

$$g''(y)=f''(|y|),$$

where $sgn(y)=1_{\{y>0\}}-1_{\{y<0\}}$. Thus g is a twice continuously differentiable function on $\mathbb{R}$, such that g and g'' belong to $C_0(\mathbb{R})$. Let L^B be the generator of $(Q_t^B)_{t\geq 0}$. Then $L^Bh=\frac{1}{2}h''$ (see the example after Corollary 6.13). By (32), we have

$$Lf(x)=\lim_{t\to 0}\frac{Q_tf(x)-f(x)}{t}=\lim_{t\to 0}\frac{Q_t^Bg(x)-g(x)}{t}=\frac{1}{2}g''(x)=\frac{1}{2}f''(x)\qquad\forall x\in\mathbb{R}_+$$

and therefore $Lf=\frac{1}{2}f''$. Conversely, assume that there exists $f\in C_0(\mathbb{R}_+)\bigcap D(L)$ such that $f'(0)\neq 0$. Then $g'(0)$ doesn't exist and $\lim_{t\to 0}\frac{Q_tf(x)-f(x)}{t}$ exists for all $\forall x\in\mathbb{R}_+$. By (32), we see that

$$\lim_{t\to 0}\frac{Q_t^Bg(x)-g(x)}{t}=\lim_{t\to 0}\frac{Q_tf(x)-f(x)}{t}=L_tf(x)\qquad\forall x\geq 0,$$

$$\lim_{t\to 0}\frac{Q_t^Bg(x)-g(x)}{t}=\lim_{t\to 0}\frac{Q_tf(-x)-f(-x)}{t}=L_tf(-x)\qquad\forall x<0,$$

and therefore $L_t^B g(x) = L_t f(|x|)$ for all $x \in \mathbb{R}$. Since $L_t f \in C_0(\mathbb{R}_+)$, we see that $L^B g \in C_0(\mathbb{R})$ and, hence, $g \in D(L^B) = \{h \in C^2(\mathbb{R}) \mid h \text{ and } h'' \in C_0(\mathbb{R})\}$ (see the example after Corollary 6.13) which is a contradiction. Thus, we see that

$$D(L) = \{h \in C^2(\mathbb{R}_+) \mid h, h'' \in C_0(\mathbb{R}_+) \text{ and } h'(0) = 0\}.$$

and $Lf = \frac{1}{2} f''$.

□

6.2 Exercise 6.24

Let $(Q_t)_{t\geq 0}$ be a transition semigroup on a measurable space E. Let π be a measurable mapping from E onto another measurable space F. We assume that, for any measurable subset A of F, for every $x, y \in E$ such that $\pi(x) = \pi(y)$, we have

$$Q_t(x, \pi^{-1}(A)) = Q_t(y, \pi^{-1}(A)) \qquad \forall t > 0. \tag{24}$$

We then set, for every $z \in F$ and every measurable subset A of F, for every $t > 0$,

$$Q'_t(z, A) = Q_t(x, \pi^{-1}(A)) \tag{25}$$

where x is an arbitrary point of E such that $\pi(x) = z$. We also set $Q'_0(z, A) = 1_A(z)$. We assume that the mapping $(t, z) \mapsto Q'_t(z, A)$ is measurable on $\mathbb{R}_+ \times F$, for every fixed A.

1. Verify that $(Q'_t)_{t\geq 0}$ forms a transition semigroup on F.
2. Let $(X_t)_{t\geq 0}$ be a Markov process in E with transition semigroup $(Q_t)_{t\geq 0}$ with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$. Set $Y_t = \pi(X_t)$ for every $t \geq 0$. Verify that $(Y_t)_{t\geq 0}$ is a Markov process in F with transition semigroup $(Q'_t)_{t\geq 0}$ with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$.
3. Let $(B_t)_{t\geq 0}$ be a d-dimensional Brownian motion, and set $R_t = B_t$ for every $t \geq 0$. Verify that $(R_t)_{t\geq 0}$ is a Markov process and give a formula for its transition semigroup (the case d = 1 was treated via a different approach in Exercise 6.23).

Proof.

1. To show that $(Q'_t)_{t\geq 0}$ forms a transition semigroup on F, it remain to show that $(Q'_t)_{t\geq 0}$ satisfies Chapman–Kolmogorov identity. Since

$$\int_F 1_A(y) Q'_t(\pi(x), dy) = \int_E 1_A(\pi(y)) Q_t(x, dy),$$

we get

$$(Q'_t f)(\pi(x)) = Q_t g(x), \tag{26}$$

where f is a bounded measurable function on F, $g = f \circ \pi$, and $x \in E$. Given $z \in F$. Since π is surjective, there exists $x \in E$ such that $z = \pi(x)$. By (26) and (25), we get

$$\begin{aligned} Q'_{t+s} f(z) = Q_{t+s} g(x) = Q_t Q_s g(x) &= \int_E Q_s g(y) Q_t(x, dy) \\ &= \int_E Q'_s f(\pi(y)) Q_t(x, dy) = \int_F Q'_s f(w) Q_t(\pi(x), dw) \\ &= Q'_t Q'_s f(\pi(x)) = Q'_t Q'_s f(z). \end{aligned}$$

2. It's clear that $(Y_t)_{t\geq 0}$ is an adapted process. It remain to show that has $(Y_t)_{t\geq 0}$ Markov property. Let f be a bounded measurable function on F and $g = f \circ \pi$. By (26), we get

$$\boldsymbol{E}[f(Y_{t+s}) \mid \mathscr{F}_s] = \boldsymbol{E}[g(X_{t+s}) \mid \mathscr{F}_s] = Q_t g(X_s) = Q'_t f(\pi(X_s)) = Q'_t f(Y_s).$$

3. The case $d = 1$ was solved in Exercise 6.23. Now we assume that $d \geq 2$. Recall that

$$Q_t f(x) = \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\pi t^d}} \exp(-\frac{|w-x|^2}{2t}) f(w) dw.$$

for all bounded measurable function f on $\mathbb{R}^d$. Define $\pi(x) = |x|$ and $Q'_t(z, A)$ as (25) for $z \in \mathbb{R}_+$ and $A \in \mathcal{B}_{\mathbb{R}_+}$. First we show that $(Q_t)_{t \geq 0}$ satisfies condition (24). Let $A \in \mathcal{B}_{\mathbb{R}_+}$ and $B = \pi^{-1}(A)$. Then

$$OB \equiv \{Ox \mid x \in B\} = B$$

for all orthogonal matrix O. Given $x, y \in \mathbb{R}^d$ such that $\pi(x) = \pi(y)$. Choose an orthogonal matrix O such that $x = Oy$. Then

$$\begin{aligned}
Q_t(x, \pi^{-1}(A)) = Q_t(x, B) &= \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\pi t^d}} \exp(-\frac{|w-x|^2}{2t}) 1_B(w) dw \\
&= \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\pi t^d}} \exp(-\frac{|Ou-Oy|^2}{2t}) 1_B(Ou) du \qquad (w = Ou) \\
&= \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\pi t^d}} \exp(-\frac{|u-y|^2}{2t}) 1_{O^{-1}B}(u) du \\
&= \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\pi t^d}} \exp(-\frac{|u-y|^2}{2t}) 1_B(u) du \\
&= Q_t(y, B) = Q_t(y, \pi^{-1}(A))
\end{aligned}$$

Next we show that the mapping $(t, z) \mapsto Q'_t(z, A)$ is measurable on $\mathbb{R}_+ \times \mathbb{R}_+$ for all $A \in \mathcal{B}_{\mathbb{R}_+}$. Given a bounded measurable function f on $\mathbb{R}_+$ and $z \in \mathbb{R}_+$. Set $x = (z, 0, ..., 0)$ and $g = f \circ \pi$. By (26), we have

$$Q'_t f(z) = Q_t g(x) = \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\pi t^d}} \exp(-\frac{1}{2t}((w_1 - z)^2 + \sum_{k=2}^{d} w_k^2)) f(|w|) dw. \tag{27}$$

This shows that the mapping $(t, z) \mapsto Q'_t(z, A)$ is measurable on $\mathbb{R}_+ \times \mathbb{R}_+$ for all $A \in \mathcal{B}_{\mathbb{R}_+}$. By problem 2, we see that $(R_t)_{t \geq 0}$ is a Markov process with semigroup (27).

□

In the remaining exercises, we use the following notation. (E, d) is a locally compact metric space, which is countable at infinity, and $(Q_t)_{t \geq 0}$ is a Feller semigroup on E. We consider an E-valued process $(X_t)_{t \geq 0}$ with càdlàg sample paths, and a collection $(\boldsymbol{P}_x)_{x \in E}$ of probability measures on E, such that, under $\boldsymbol{P}_x$, $(X_t)_{t \geq 0}$ is a Markov process with semigroup $(Q_t)_{t \geq 0}$ with respect to the filtration $(\mathscr{F}_t)_{t \geq 0}$ and $\boldsymbol{P}_x(X_0 = x) = 1$. We write L for the generator of the semigroup $(Q_t)_{t \geq 0}$, $D(L)$ for the domain of L and R_λ for the λ-resolvent, for every $\lambda > 0$.

6.3 Exercise 6.25 (Scale Function)

In this exercise, we assume that $E = \mathbb{R}_+$ and that the sample paths of X are continuous. For every $x \in \mathbb{R}_+$, we set

$$T_x \equiv \inf\{t \geq 0 \mid X_t = x\}$$

and

$$\varphi(x) \equiv \boldsymbol{P}_x(T_0 < \infty).$$

1. Show that, if $0 \leq x \leq y$,

$$\varphi(y) = \varphi(x) \boldsymbol{P}_y(T_x < \infty).$$

2. We assume that $\varphi(x) < 1$ and $\boldsymbol{P}_x(\sup_{t \geq 0} X_t = \infty) = 1$, for every $x > 0$. Show that, if $0 < x \leq y$,

$$\boldsymbol{P}_x(T_0 < T_y) = \frac{\varphi(x) - \varphi(y)}{1 - \varphi(y)}.$$

Proof.

1. By strong Markov property, we have

$$\boldsymbol{P}_y(T_0 < \infty) = \boldsymbol{P}_y(T_0 < \infty, T_x < \infty) = \boldsymbol{E}_y[1_{\{T_x<\infty\}}1_{\{T_0<\infty\}}] = \boldsymbol{E}_y[1_{\{T_x<\infty\}}\boldsymbol{E}_{X_{T_x}}[1_{\{T_0<\infty\}}]].$$

Since $(X_t)_{t\geq 0}$ has continuous sample path, we get $X_{T_x} = x$ on $\{T_x < \infty\}$ and therefore

$$\boldsymbol{P}_y(T_0 < \infty) = \boldsymbol{E}_y[1_{\{T_x<\infty\}}\boldsymbol{E}_{X_{T_x}}[1_{\{T_0<\infty\}}]] = \boldsymbol{P}_y(T_x < \infty)\boldsymbol{P}_x(T_0 < \infty) = \varphi(x)\boldsymbol{P}_y(T_x < \infty).$$

2. Because $\boldsymbol{P}_x(T_y < \infty) = 1$, we get

$$\boldsymbol{P}_x(T_0 < \infty) = \boldsymbol{P}_x(T_0 < T_y) + \boldsymbol{P}_x(T_0 < \infty, T_y < T_0).$$

By strong Markov property, we have

$$\boldsymbol{E}_x[1_{\{T_y<T_0\}}1_{\{T_0<\infty\}}] = \boldsymbol{E}_x[1_{\{T_y<T_0\}}\boldsymbol{E}_{X_{T_y}}[1_{\{T_0<\infty\}}]].$$

Since $(X_t)_{t\geq 0}$ has continuous sample path, we get $X_{T_y} = y$ (a.s.) and therefore

$$\boldsymbol{E}_x[1_{\{T_y<T_0\}}1_{\{T_0<\infty\}}] = \boldsymbol{P}_x(T_y < T_0)\boldsymbol{P}_y(T_0 < \infty).$$

Hecen

$$\varphi(x) = \boldsymbol{P}_x(T_0 < \infty) = \boldsymbol{P}_x(T_0 < T_y) + \boldsymbol{P}_x(T_y < T_0)\boldsymbol{P}_y(T_0 < \infty) = \boldsymbol{P}_x(T_0 < T_y) + \boldsymbol{P}_x(T_y < T_0)\varphi(y).$$

Since

$$1 = \boldsymbol{P}_x(T_0 < T_y) + \boldsymbol{P}_x(T_y < T_0)$$

and

$$\varphi(x) < 1 \qquad \forall x > 0,$$

we have

$$\boldsymbol{P}_x(T_0 < T_y) = \frac{\varphi(x) - \varphi(y)}{1 - \varphi(y)}.$$

□

6.4 Exercise 6.26 (Feynman–Kac Formula)

Let v be a nonnegative function in $C_0(E)$. For every $x \in E$ and every $t \geq 0$, we set, for every $\varphi \in B(E)$,

$$Q_t^*\varphi(x) \equiv \boldsymbol{E}_x[\varphi(X_t)\exp(-\int_0^t v(X_s)ds)].$$

1. Show that, for every $\varphi \in B(E)$, and $s, t \geq 0$, $Q_{s+t}^*\varphi = Q_t^*(Q_s^*\varphi)$.

2. After observing that

$$1 - \exp(-\int_0^t v(X_s)ds) = \int_0^t v(X_s)\exp(-\int_s^t v(X_u)du)ds,$$

show that, for every $\varphi \in B(E)$,

$$Q_t\varphi - Q_t^*\varphi = \int_0^t Q_s(vQ_{t-s}^*\varphi)ds. \tag{28}$$

3. Assume that $\varphi \in D(L)$. Show that

$$\frac{d}{dt}Q_t^*\varphi|_{t=0} = L\varphi - v\varphi.$$

Proof.

1. Fix $s, t \geq 0$. Define $\Phi^{(s)}(f) = \varphi(f(s)) \exp(-\int_0^s v(f(u))du)$. By simple Markov property, we get

$$\begin{aligned}
Q_t^*(Q_s^*\varphi)(x) &= \boldsymbol{E}_x[\boldsymbol{E}_{X_t}[\varphi(X_s)\exp(-\int_0^s v(X_u)du)]\exp(-\int_0^t v(X_u)du)] \\
&= \boldsymbol{E}_x[\mathbb{E}_{X_t}[\Phi^{(s)}]\exp(-\int_0^t v(X_u)du)] \\
&= \boldsymbol{E}_x[\boldsymbol{E}_x[\Phi^{(s)}((X_{t+r})_{r\geq 0}) : \mathscr{F}_t]\exp(-\int_0^t v(X_u)du)] \\
&= \boldsymbol{E}_x[\Phi^{(s)}((X_{t+r})_{r\geq 0})\exp(-\int_0^t v(X_u)du)] \\
&= \boldsymbol{E}_x[\varphi(X_{s+t})\exp(-\int_0^s v(X_{u+t})du)\exp(-\int_0^t v(X_u)du)] \\
&= \boldsymbol{E}_x[\varphi(X_{s+t})\exp(-\int_t^{t+s} v(X_u)du)\exp(-\int_0^t v(X_u)du)] = Q_{s+t}^*\varphi(x)
\end{aligned}$$

2. Observe that

$$\frac{d}{ds}\exp(-\int_s^t v(X_u)du) = v(X_s)\exp(-\int_s^t v(X_u)du).$$

   Then we have

$$1 - \exp(-\int_0^t v(X_s)ds) = \int_0^t v(X_s)\exp(-\int_s^t v(X_u)du)ds.$$

   By Fubini's theorem and simple Markov property, we get

$$\begin{aligned}
Q_t\varphi(x) - Q_t^*\varphi(x) &= \boldsymbol{E}_x[\varphi(X_t)] - \boldsymbol{E}_x[\varphi(X_t)\exp(-\int_0^t v(X_s)ds)] \\
&= \boldsymbol{E}_x[\varphi(X_t)(1-\exp(-\int_0^t v(X_s)ds))] \\
&= \boldsymbol{E}_x[\varphi(X_t) \times \int_0^t v(X_s)\exp(-\int_s^t v(X_u)du)ds] \\
&= \int_0^t \boldsymbol{E}_x[\varphi(X_t) \times v(X_s)\exp(-\int_s^t v(X_u)du)]ds \\
&= \int_0^t \boldsymbol{E}_x[v(X_s) \times \varphi(X_t)\exp(-\int_0^{t-s} v(X_{u+s})du)]ds \\
&= \int_0^t \boldsymbol{E}_x[v(X_s)\Phi^{(t-s)}((X_{s+r})_{r\geq 0})]ds \\
&= \int_0^t \boldsymbol{E}_x[v(X_s)\boldsymbol{E}_x[\Phi^{(t-s)}((X_{s+r})_{r\geq 0}) : \mathscr{F}_s]ds \\
&= \int_0^t \boldsymbol{E}_x[v(X_s)\mathbb{E}_{X_s}[\Phi^{(t-s)}]ds \\
&= \int_0^t \boldsymbol{E}_x[v(X_s)\boldsymbol{E}_{X_s}[\varphi(X_{t-s})\exp(-\int_0^{t-s} v(X_u)du)]ds \\
&= \int_0^t \boldsymbol{E}_x[v(X_s)Q_{t-s}^*\varphi(X_s)]ds \\
&= \int_0^t Q_s(vQ_{t-s}^*\varphi)(x)ds
\end{aligned}$$

3. Note that
$$Q_t\varphi(x) = \varphi(x) + \int_0^t Q_s(L\varphi)(x)ds$$
and $Q_0^*\varphi(x) = \varphi(x)$. By differentiating (32), we have
$$\frac{d}{dt}Q_t^*\varphi(x)|_{t=0} = L\varphi(x) - v(x)\varphi(x).$$

□

6.5 Exercise 6.27 (Quasi left-continuity)

Throughout the exercise we fix the starting point $x \in E$. For every $t > 0$, we write $X_{t-}(w)$ for the left-limit of the sample path $s \mapsto X_s(w)$ at t.
Let $(T_n)_{n\geq 1}$ be a strictly increasing sequence of stopping times, and $T = \lim_{n\to\infty} T_n$. We assume that there exists a constant $C < \infty$ such that $T \leq C$. The goal of the exercise is to verify that $X_T = X_{T-}$, $\boldsymbol{P}_x$-a.s.

1. Let $f \in D(L)$ and $h = Lf$. Show that, for every $n \geq 1$,
$$\boldsymbol{E}_x[f(X_T) \mid \mathscr{F}_{T_n}] = f(X_{T_n}) + \boldsymbol{E}_x[\int_{T_n}^T h(X_s)ds \mid \mathscr{F}_{T_n}].$$
2. We recall from the theory of discrete time martingales that
$$\boldsymbol{E}_x[f(X_T) \mid \mathscr{F}_{T_n}] \overset{a.s., L^1}{\to} \boldsymbol{E}_x[f(X_T) \mid \widetilde{\mathscr{F}_T}],$$
where
$$\widetilde{\mathscr{F}_T} = \bigvee_{n=1}^{\infty} \mathscr{F}_{T_n}.$$
Infer from question (1) that
$$\boldsymbol{E}[f(X_T) \mid \widetilde{\mathscr{F}_T}] = f(X_{T-}).$$
3. Show that the conclusion of question (2) remains valid if we only assume that $f \in C_0(E)$, and infer that, for every choice of $f, g \in C_0(E)$,
$$\boldsymbol{E}_x[f(X_T)g(X_{T-})] = \boldsymbol{E}_x[f(X_{T-})g(X_{T-})].$$
Conclude that $X_{T_-} = X_T$, $\boldsymbol{P}_x$-a.s.

Proof.

1. By Theorem 6.14, we see that $(f(X_t) - \int_0^t h(X_s)ds)_{t\geq 0}$ is a martingale with respect to $(\mathscr{F}_t)_{t\geq 0}$. By Corollary 3.23, we have
$$\boldsymbol{E}_x[f(X_T) - \int_0^T h(X_s)ds \mid \mathscr{F}_{T_n}] = f(X_{T_n}) - \int_0^{T_n} h(X_s)ds$$
and so
$$\boldsymbol{E}_x[f(X_T) \mid \mathscr{F}_{T_n}] = f(X_{T_n}) + \boldsymbol{E}_x[\int_{T_n}^T h(X_s)ds \mid \mathscr{F}_{T_n}].$$
2. Note that
$$\boldsymbol{E}_x[f(X_T) \mid \widetilde{\mathscr{F}_T}] \leq ||f||_u < \infty,$$
where $||f||_u = \sup_{x\in E} |f(x)|$. Then the discrete time martingale
$$(\boldsymbol{E}_x[f(X_T) \mid \mathscr{F}_{T_n}])_{n\geq 0} = (\boldsymbol{E}_x[\boldsymbol{E}_x[f(X_T) \mid \widetilde{\mathscr{F}_T}] \mid \mathscr{F}_{T_n}])_{n\geq 0}$$

is closed and, hence,

$$f(X_{T_n}) + \boldsymbol{E}_x[\int_{T_n}^{T} h(X_s)ds \mid \mathscr{F}_{T_n}] = \boldsymbol{E}_x[f(X_T) \mid \mathscr{F}_{T_n}] \overset{a.s.,L^1}{\to} \boldsymbol{E}_x[f(X_T) \mid \widetilde{\mathscr{F}_T}].$$

Note that $\lim_{n\to\infty} X_{T_n} = X_{T-}$, $\boldsymbol{P}_x$-a.s. and $||h||_u < \infty$. By Lebesgue's dominated convergence theorem, we get

$$\begin{aligned}
&||f(X_{T-}) - f(X_{T_n}) - \boldsymbol{E}_x[\int_{T_n}^{T} h(X_s)ds \mid \mathscr{F}_{T_n}]||_{L^1} \\
&\le ||f(X_{T-}) - f(X_{T_n})||_{L^1} + ||\boldsymbol{E}_x[\int_{T_n}^{T} h(X_s)ds \mid \mathscr{F}_{T_n}]||_{L^1} \\
&\le \boldsymbol{E}_x[|f(X_{T-}) - f(X_{T_n})|] + \boldsymbol{E}_x[\int_{T_n}^{T} |h(X_s)|ds] \\
&\le \boldsymbol{E}_x[|f(X_{T-}) - f(X_{T_n})|] + ||h||_u \boldsymbol{E}_x[T - T_n] \overset{n\to\infty}{\to} 0
\end{aligned}$$

and therefore $\boldsymbol{E}[f(X_T) \mid \widetilde{\mathscr{F}_T}] = f(X_{T-})$, $\boldsymbol{P}_x$-a.s.

3. First, we show that

$$\boldsymbol{E}[f(X_T) \mid \widetilde{\mathscr{F}_T}] = f(X_{T-}) \qquad \forall f \in C_0(E).$$

By proposition 6.8 and proposition 6.12, we see that

$$D(L) = \mathscr{R} \equiv \{R_\lambda f \mid f \in C_0(E)\}$$

is dense in $C_0(E)$. Given $f \in C_0(E)$ and $\epsilon > 0$. Choose $g \in D(L)$ such that $||f - g||_u < \epsilon$. Then

$$\boldsymbol{E}[g(X_T) \mid \widetilde{\mathscr{F}_T}] = g(X_{T-})$$

and, hence,

$$\begin{aligned}
&\boldsymbol{E}_x[|\boldsymbol{E}[f(X_T) \mid \widetilde{\mathscr{F}_T}] - f(X_{T-})|] \\
&\le \boldsymbol{E}_x[|\boldsymbol{E}[f(X_T) \mid \widetilde{\mathscr{F}_T}] - \boldsymbol{E}[g(X_T) \mid \widetilde{\mathscr{F}_T}]|] + \boldsymbol{E}_x[|g(X_{T-}) - f(X_{T-})|] \\
&\le \boldsymbol{E}_x[|g(X_T) - f(X_T)|] + \boldsymbol{E}_x[|g(X_{T-}) - f(X_{T-})|] \\
&\le 2||f - g||_u \le 2\epsilon.
\end{aligned}$$

By letting $\epsilon \to 0$, we get

$$\boldsymbol{E}[f(X_T) \mid \widetilde{\mathscr{F}_T}] = f(X_{T-}).$$

Next, we show that $X_{T_-} = X_T$. Let $f, g \in C_0(E)$. Then $g(X_{T-})$ is $\widetilde{\mathscr{F}_T}$-measurable and, hence,

$$\boldsymbol{E}_x[f(X_T)g(X_{T-})] = \boldsymbol{E}_x[\boldsymbol{E}_x[f(X_T) \mid \widetilde{\mathscr{F}_T}]g(X_{T-})] = \boldsymbol{E}_x[f(X_{T-})g(X_{T-})].$$

Thus, we have

$$\boldsymbol{E}_x[f(X_T)g(X_{T-})] = \boldsymbol{E}_x[f(X_{T-})g(X_{T-})] \qquad \forall f, g \in C_0(E).$$

Hence

$$\boldsymbol{E}_x[f(X_T)g(X_{T-})] = \boldsymbol{E}_x[f(X_{T-})g(X_{T-})] \qquad \forall f, g \in B(E)$$

and therefore

$$\boldsymbol{E}_x[h(X_T, X_{T-})] = \boldsymbol{E}_x[h(X_{T-}, X_{T-})] \qquad \forall h \in B(E \times E).$$

For $\epsilon > 0$, if we set $h(x, y) = 1_{d(x,y)>\epsilon}(x, y)$, then

$$\boldsymbol{P}_x(d(X_T, X_{T-}) > \epsilon) = \boldsymbol{E}_x[h(X_T, X_{T-})] = \boldsymbol{E}_x[h(X_{T-}, X_{T-})] = 0.$$

Therefore $X_{T_-} = X_T$, $\boldsymbol{P}_x$-a.s.

$\square$

6.6 Exercise 6.28 (Killing operation)

In this exercise, we assume that X has continuous sample paths. Let A be a compact subset of E and

$$T_A = \inf\{t \geq 0 \mid X_t \in A\}.$$

1. We set, for every $t \geq 0$ and every bounded measurable function φ on E,

$$Q_t^* \varphi(x) = \boldsymbol{E}_x[\varphi(X_t) 1_{\{t < T_A\}}], \qquad \forall x \in E.$$

   Verify that $Q_{t+s}^* \varphi = Q_t^*(Q_s^* \varphi)$, for every $s, t > 0$.

2. We set $\overline{E} = (E \setminus A) \bigcup \{\Delta\}$, where Δ is a point added to $E \setminus A$ as an isolated point. For every bounded measurable function φ on $\overline{E}$ and every $t \geq 0$, we set

$$\overline{Q}_t \varphi(x) = \begin{cases} \boldsymbol{E}_x[\varphi(X_t) 1_{\{t < T_A\}}] + \boldsymbol{P}_x(T_A \leq t) \varphi(\Delta), & \text{if } x \in E \setminus A \\ \varphi(\Delta), & \text{if } x = \Delta. \end{cases}$$

   Verify that $(\overline{Q}_t)_{t \geq 0}$ is a transition semigroup on $\overline{E}$. (The proof of the measurability of the mapping $(t, x) \mapsto \overline{Q}_t \varphi(x)$ will be omitted.)

3. Show that, under the probability measure $\boldsymbol{P}_x$, the process $\overline{X}$ defined by

$$\overline{X}_t = \begin{cases} X_t, & \text{if } t < T_A \\ \Delta, & \text{if } t \geq T_A. \end{cases}$$

   is a Markov process with semigroup $(\overline{Q}_t)_{t \geq 0}$, with respect to the canonical filtration of X.

4. We take it for granted that the semigroup $(\overline{Q}_t)_{t \geq 0}$ is Feller, and we denote its generator by $\overline{L}$. Let $f \in D(L)$ such that f and Lf vanish on an open set containing A. Write $\overline{f}$ for the restriction of f to $E \setminus A$, and consider $\overline{f}$ as a function on $\overline{E}$ by setting $\overline{f}(\Delta) = 0$. Show that $\overline{f} \in D(\overline{L})$ and $\overline{L f}(x) = Lf(x)$ for every $x \in E \setminus A$.

Proof.

1. By the simple Markov property, we have

$$\begin{aligned} Q_t^*(Q_s^* \varphi)(x) &= \boldsymbol{E}_x[Q_s^* \varphi(X_t) 1_{\{t < T_A\}}] \\ &= \boldsymbol{E}_x[\boldsymbol{E}_{X_t}[\varphi(X_s) 1_{\{s < T_A\}}] 1_{\{t < T_A\}}] \\ &= \boldsymbol{E}_x[\boldsymbol{E}_x[\varphi(X_{s+t}) 1_{\{s < \inf\{r \geq 0 \mid X_{r+t} \in A\}\}} \mid \mathscr{F}_t] 1_{\{t < T_A\}}] \\ &= \boldsymbol{E}_x[\varphi(X_{s+t}) 1_{\{s < \inf\{r \geq 0 \mid X_{r+t} \in A\}\}} 1_{\{t < T_A\}}] \\ &= \boldsymbol{E}_x[\varphi(X_{s+t}) 1_{\{t+s < T_A\}}] = Q_{t+s}^* \varphi(x) \end{aligned}$$

2. First, we show that $x \in \overline{E} \mapsto \overline{Q}_t \varphi(x)$ is measurable for every bounded measurable function φ on $\overline{E}$ and every $t \geq 0$. Observe that

$$\{x \in \overline{E} \mid \overline{Q}_t \varphi(x) \in \Gamma\} = (\{\overline{Q}_t \varphi \in \Gamma\} \bigcap (E \setminus A)) \bigcup \begin{cases} \{\Delta\}, & \text{if } \varphi(\Delta) \in \Gamma \\ \emptyset, & \text{otherwise.} \end{cases}$$

   Define $\widetilde{\varphi} : E \mapsto \mathbb{R}$ by

$$\widetilde{\varphi}(x) = \begin{cases} \varphi(x), & \text{if } x \in E \setminus A \\ 0, & \text{if } x \in A. \end{cases}$$

Then $\widetilde{\varphi}$ is a bounded measurable function on E and, hence,

$$x \in E \mapsto \boldsymbol{E}_x[\widetilde{\varphi}(X_t)1_{\{t<T_A\}}]$$

is measurabale on E. Note that

$$\widetilde{\varphi}(X_t) = \varphi(X_t) \text{ in } \{t < T_A\}.$$

Then we see that

$$x \in E \setminus A \mapsto \boldsymbol{E}_x[\widetilde{\varphi}(X_t)1_{\{t<T_A\}}] = \boldsymbol{E}_x[\varphi(X_t)1_{\{t<T_A\}}]$$

is measurable on $E \setminus A$. Similarly, we see that

$$x \in E \setminus A \mapsto \boldsymbol{P}_x(T_A \le t)$$

is measurable on $E \setminus A$. Thus,

$$x \in E \setminus A \mapsto \boldsymbol{E}_x[\varphi(X_t)1_{\{t<T_A\}}] + \boldsymbol{P}_x(T_A \le t)\varphi(\Delta) = \overline{Q}_t\varphi(x)$$

is measurable on $E \setminus A$ and, hence,

$$\{x \in \overline{E} \mid \overline{Q}_t\varphi(x) \in \Gamma\} = (\{\overline{Q}_t\varphi \in \Gamma\} \bigcap (E \setminus A)) \bigcup \begin{cases} \{\Delta\}, & \text{if } \varphi(\Delta) \in \Gamma \\ \emptyset, & \text{otherwise.} \end{cases}$$

is a meausbale set on $E \setminus A$.
Next, we show that $\overline{Q}_t\overline{Q}_s\varphi = \overline{Q}_{t+s}\varphi$ for all bounded meausable funciotn φ on $\overline{E}$. It's clear that

$$\overline{Q}_t\overline{Q}_s\varphi(\Delta) = \overline{Q}_s\varphi(\Delta) = \varphi(\Delta) = \overline{Q}_{t+s}\varphi(\Delta).$$

Now, we suppose $x \in E \setminus A$. By the simple Markov property, we get

$$\begin{aligned}
&\overline{Q}_t\overline{Q}_s\varphi(x) \\
&= \boldsymbol{E}_x[\overline{Q}_s\varphi(X_t)1_{\{t<T_A\}}] + \boldsymbol{P}_x(T_A \le t)\overline{Q}_s\varphi(\Delta) \\
&= \boldsymbol{E}_x[\overline{Q}_s\varphi(X_t)1_{\{t<T_A\}}] + \boldsymbol{P}_x(T_A \le t)\varphi(\Delta) \\
&= \boldsymbol{E}_x[(\boldsymbol{E}_{X_t}[\varphi(X_s)1_{\{s<T_A\}}] + \boldsymbol{P}_{X_t}(T_A \le s)\varphi(\Delta))1_{\{t<T_A\}}] + \boldsymbol{P}_x(T_A \le t)\varphi(\Delta) \\
&= \boldsymbol{E}_x[\boldsymbol{E}_{X_t}[\varphi(X_s)1_{\{s<T_A\}}]1_{\{t<T_A\}}] + \boldsymbol{E}_x[\boldsymbol{P}_{X_t}(T_A \le s)\varphi(\Delta)1_{\{t<T_A\}}] + \boldsymbol{P}_x(T_A \le t)\varphi(\Delta) \\
&= \boldsymbol{E}_x[\varphi(X_{s+t})1_{\{s<\inf\{r\ge 0 \mid X_{r+t}\in A\}\}}1_{\{t<T_A\}}] + \boldsymbol{E}_x[1_{\{\inf\{r\ge 0\mid X_{r+t}\in A\}\le s\}}\varphi(\Delta)1_{\{t<T_A\}}] + \boldsymbol{P}_x(T_A \le t)\varphi(\Delta) \\
&= \boldsymbol{E}_x[\varphi(X_{s+t})1_{\{t+s<T_A\}}] + \varphi(\Delta)\boldsymbol{E}_x[(1_{\{\inf\{r\ge 0\mid X_{r+t}\in A\}\le s\}}1_{\{t<T_A\}} + 1_{\{T_A\le t\}})] \\
&= \boldsymbol{E}_x[\varphi(X_{s+t})1_{\{t+s<T_A\}}] + \boldsymbol{P}_x(T_A \le s+t)\varphi(\Delta) = \overline{Q}_{s+t}(x).
\end{aligned}$$

3. For $t \ge 0$ and a measurable set Γ of $\overline{E}$ such that $\Delta \notin \Gamma$,

$$\{\overline{X}_t \in \Gamma\} = \{X_t \in \Gamma\} \bigcap \{t < T_A\} \in \mathscr{F}_t$$

and, hence, $(\overline{X}_t)_{t\ge0}$ is a $(\mathscr{F}_t)_{t\ge0}$-adapted process. Now, we show that $(\overline{X}_t)_{t\ge0}$ is a $(\mathscr{F}_t)_{t\ge0}$-Markov process on $\overline{E}$. Let $\varphi \in B(\overline{E})$. Note that

$$\varphi(\overline{X}_t) = \begin{cases} \varphi(X_t), & \text{if } t < T_A \\ \varphi(\Delta), & \text{if } t \ge T_A. \end{cases}$$

By the simple Markov property, we get

$$
\begin{aligned}
&\boldsymbol{E}_x[\varphi(\overline{X}_{t+s}) \mid \mathscr{F}_s] \\
&= \boldsymbol{E}_x[\varphi(\overline{X}_{t+s})1_{\{t+s<T_A\}} \mid \mathscr{F}_s] + \boldsymbol{E}_x[\varphi(\overline{X}_{t+s})1_{\{t+s\geq T_A\}} \mid \mathscr{F}_s] \\
&= \boldsymbol{E}_x[\varphi(X_{t+s})1_{\{t+s<T_A\}} \mid \mathscr{F}_s] + \boldsymbol{E}_x[\varphi(\Delta)1_{\{t+s\geq T_A\}} \mid \mathscr{F}_s] \\
&= \boldsymbol{E}_x[\varphi(X_{t+s})1_{\{s<T_A\}}1_{\{t<\inf\{r\geq 0|X_{s+r}\in A\}\}} \mid \mathscr{F}_s] + \boldsymbol{E}_x[\varphi(\Delta)(1_{\{s<T_A\}}1_{\{t\geq\inf\{r\geq 0|X_{s+r}\in A\}\}} + 1_{\{s\geq T_A\}}) \mid \mathscr{F}_s] \\
&= 1_{\{s<T_A\}}\boldsymbol{E}_{X_s}[\varphi(X_t)1_{\{t<T_A\}}] + \varphi(\Delta)1_{\{s<T_A\}}\boldsymbol{P}_{X_s}(t\geq T_A) + \varphi(\Delta)1_{\{s\geq T_A\}} \\
&= 1_{\{s<T_A\}}(\boldsymbol{E}_{\overline{X}_s}[\varphi(X_t)1_{\{t<T_A\}}] + \varphi(\Delta)\boldsymbol{P}_{\overline{X}_s}(t\geq T_A)) + \varphi(\overline{X}_s)1_{\{s\geq T_A\}} \\
&= \overline{Q}_t\varphi(\overline{X}_s).
\end{aligned}
$$

4. Let us show that

$$\overline{L}\,\overline{f}(x) = \begin{cases} Lf(x), & \text{if } x \in E\setminus A \\ 0, & \text{if } x = \Delta. \end{cases}$$

Since Δ is an isolated point of $E\setminus A$ and $f, Lf \in C_0(E)$, we see that $\overline{f}, \overline{Lf} \in C_0(\overline{E})$. By thoerem 6.14, it suffices to show that $(\overline{f}(\overline{X}_t) - \int_0^t \overline{Lf}(\overline{X}_s)ds)_{t\geq 0}$ is a $(\mathscr{F}_t)_{t\geq 0}$-martingale under $\boldsymbol{P}_x$ for all $x \in \overline{E}$. If $x = \Delta$, then

$$\overline{X}_t = \Delta \qquad \forall t \geq 0 \quad \boldsymbol{P}_x\text{-a.s.}$$

and so

$$\overline{f}(\overline{X}_t) = \overline{Lf}(\overline{X}_t) = 0 \qquad \forall t\geq 0 \quad \boldsymbol{P}_x\text{-a.s.}$$

Thus $(\overline{f}(\overline{X}_t) - \int_0^t \overline{Lf}(\overline{X}_s)ds)_{t\geq 0}$ is a zero process. Now, we suppose $x \in E\setminus A$. Since f and Lf vanish on an open set containing A, we see that

$$f(X_{t\wedge T_A}) = Lf(X_{t\wedge T_A}) = 0 \qquad \forall t \geq T_A.$$

Thus, we have

$$\overline{f}(\overline{X}_t) = f(X_{t\wedge T_A}) \qquad \forall t\geq 0$$

and

$$\int_0^t \overline{Lf}(\overline{X}_s)ds = \int_0^t Lf(X_{s\wedge T_A})ds = \int_0^{t\wedge T_A} Lf(X_s)ds \qquad \forall t \geq 0.$$

Since $(f(X_t) - \int_0^t Lf(X_s)ds)_{t\geq 0}$ is a $(\mathscr{F}_t)_{t\geq 0}$-martingale under $\boldsymbol{P}_x$, we get

$$(\overline{f}(\overline{X}_t) - \int_0^t \overline{Lf}(\overline{X}_s)ds)_{t\geq 0} = (f(X_{t\wedge T_A}) - \int_0^{t\wedge T_A} Lf(X_s)ds)_{t\geq 0}$$

is a $(\mathscr{F}_t)_{t\geq 0}$-martingale under $\boldsymbol{P}_x$. Thus $\overline{f} \in D(\overline{L})$ and

$$\overline{L}\,\overline{f}(x) = \overline{Lf}(x) = \begin{cases} Lf(x), & \text{if } x \in E\setminus A \\ 0, & \text{if } x = \Delta. \end{cases}$$

□

6.7 Exercise 6.29 (Dynkin's formula)

1. Let $g \in C_0(E)$ and $x \in E$, and let T be a stopping time. Justify the equality

$$\boldsymbol{E}_x[1_{\{T<\infty\}}e^{-\lambda T}\int_0^\infty e^{-\lambda t}g(X_{T+t})dt] = \boldsymbol{E}_x[1_{\{T<\infty\}}e^{-\lambda T}R_\lambda g(X_T)] \tag{29}$$

2. Infer that
$$R_\lambda g(x) = \boldsymbol{E}_x[\int_0^T e^{-\lambda t} g(X_t)dt] + \boldsymbol{E}_x[1_{\{T<\infty\}} e^{-\lambda T} R_\lambda g(X_T)]. \tag{30}$$

3. Show that, if $f \in D(L)$,
$$f(x) = \boldsymbol{E}_x[\int_0^T e^{-\lambda t}(\lambda f - Lf)(X_t)dt] + \boldsymbol{E}_x[1_{\{T<\infty\}} e^{-\lambda T} f(X_T)].$$

4. Assuming that $\boldsymbol{E}_x[T] < \infty$, infer from the previous question that
$$\boldsymbol{E}_x[\int_0^T Lf(X_t)dt] = \boldsymbol{E}_x[f(X_T)] - f(x). \qquad (Dynkin's formula) \tag{31}$$
How could this formula have been established more directly?

5. For every $\epsilon > 0$, we set $T_{\epsilon,x} = \inf\{t \geq 0 \mid d(x, X_t) > \epsilon\}$. Assume that $\boldsymbol{E}_x[T_{\epsilon,x}] < \infty$, for every sufficiently small ϵ. Show that (still under the assumption $f \in D(L)$) one has
$$Lf(x) = \lim_{\epsilon \downarrow 0} \frac{\boldsymbol{E}_x[f(X_{T_{\epsilon,x}})] - f(x)}{\boldsymbol{E}_x[T_{\epsilon,x}]}.$$

6. Show that the assumption $\boldsymbol{E}_x[T_{\epsilon,x}] < \infty$ for every sufficiently small ϵ holds if the point x is not absorbing, that is, if there exists a $t > 0$ such that $Q_t(x, \{x\}) < 1$. (Hint: Observe that there exists a nonnegative function $h \in C_0(E)$ which vanishes on a ball centered at x and is such that $Q_t h(x) > 0$. Infer that one can choose $\alpha > 0$ and $\eta \in (0, 1)$ such that $\boldsymbol{P}_x(T_{\alpha,x} > nt) \leq (1 - \eta)^n$ for every integer $n \geq 1$.)

Proof.

1. By Fubini's theorem and the strong Markov properpty, we get
$$\begin{aligned}
\boldsymbol{E}_x[1_{\{T<\infty\}} e^{-\lambda T} \int_0^\infty e^{-\lambda t} g(X_{T+t})dt] &= \int_0^\infty \boldsymbol{E}_x[1_{\{T<\infty\}} e^{-\lambda T} e^{-\lambda t} g(X_{T+t})]dt \\
&= \int_0^\infty \boldsymbol{E}_x[1_{\{T<\infty\}} e^{-\lambda T} e^{-\lambda t} \boldsymbol{E}_x[g(X_{T+t}) \mid \mathscr{F}_T]]dt \\
&= \int_0^\infty \boldsymbol{E}_x[1_{\{T<\infty\}} e^{-\lambda T} e^{-\lambda t} \boldsymbol{E}_{X_T}[g(X_t)]]dt \\
&= \int_0^\infty \boldsymbol{E}_x[1_{\{T<\infty\}} e^{-\lambda T} e^{-\lambda t} Q_t g(X_T)]dt \\
&= \boldsymbol{E}_x[1_{\{T<\infty\}} e^{-\lambda T} \int_0^\infty e^{-\lambda t} Q_t g(X_T)dt] \\
&= \boldsymbol{E}_x[1_{\{T<\infty\}} e^{-\lambda T} R_\lambda g(X_T)].
\end{aligned}$$

2. By (29), we get
$$\begin{aligned}
&\boldsymbol{E}_x[\int_0^T e^{-\lambda t} g(X_t)dt] + \boldsymbol{E}_x[1_{\{T<\infty\}} e^{-\lambda T} R_\lambda g(X_T)] \\
&= \boldsymbol{E}_x[\int_0^T e^{-\lambda t} g(X_t)dt] + \boldsymbol{E}_x[1_{\{T<\infty\}} e^{-\lambda T} \int_0^\infty e^{-\lambda t} g(X_{T+t})dt] \\
&= \boldsymbol{E}_x[\int_0^T e^{-\lambda t} g(X_t)dt] + \boldsymbol{E}_x[1_{\{T<\infty\}} \int_T^\infty e^{-\lambda t} g(X_t)dt] \\
&= \boldsymbol{E}_x[\int_0^\infty e^{-\lambda t} g(X_t)dt] = \int_0^\infty e^{-\lambda t} \boldsymbol{E}_x[g(X_t)]dt = \int_0^\infty e^{-\lambda t} Q_t g(x)dt = R_\lambda g(x).
\end{aligned}$$

3. Fix $f \in D(L)$. By proposition 6.12, there exists $g \in C_0(E)$ such that $f = R_\lambda g \in D(L)$ and $(\lambda - L)f = g$. By (30), we get

$$f(x) = \boldsymbol{E}_x[\int_0^T e^{-\lambda t}(\lambda f - Lf)(X_t)dt] + \boldsymbol{E}_x[1_{\{T<\infty\}}e^{-\lambda T}f(X_T)].$$

4. Note that $f, L(f)$ are bounded and $\boldsymbol{E}_x[T] < \infty$. By Lebesgue's dominated convergence theorem, we get

$$\begin{aligned}
&\lim_{\lambda\to 0} \boldsymbol{E}_x[\int_0^T e^{-\lambda t}(\lambda f - Lf)(X_t)dt] \\
&= \lim_{\lambda\to 0} \boldsymbol{E}_x[1_{\{T<\infty\}} \int_0^T e^{-\lambda t}(\lambda f - Lf)(X_t)dt] \\
&= \boldsymbol{E}_x[1_{\{T<\infty\}} \lim_{\lambda\to 0} \int_0^T e^{-\lambda t}(\lambda f - Lf)(X_t)dt] \\
&= \boldsymbol{E}_x[1_{\{T<\infty\}} \int_0^T \lim_{\lambda\to 0} e^{-\lambda t}(\lambda f - Lf)(X_t)dt] \\
&= -\boldsymbol{E}_x[\int_0^T Lf(X_t)dt]
\end{aligned}$$

and therefore

$$f(x) = \lim_{\lambda\to 0} \boldsymbol{E}_x[\int_0^T e^{-\lambda t}(\lambda f - Lf)(X_t)dt] + \lim_{\lambda\to 0} \boldsymbol{E}_x[1_{\{T<\infty\}}e^{-\lambda T}f(X_T)] = -\boldsymbol{E}_x[\int_0^T Lf(X_t)dt] + \boldsymbol{E}_x[f(X_T)].$$

Next, we prove (31) directly. By theorem 6.14, we see that $(M_t)_{t\geq 0} \equiv (f(X_t) - \int_0^t Lf(X_s)ds)_{t\geq 0}$ is a $(\mathscr{F}_t)_{t\geq 0}$-martingale. Let $K > 0$. Then $(M_{t\wedge K})_{t\geq 0}$ is a uniformly integrable martingale. By optional stopping theorem, we have

$$\boldsymbol{E}_x[f(X_{T\wedge K}) - \int_0^{T\wedge K} Lf(X_s)ds] = f(x).$$

Since $\boldsymbol{E}_x[T] < \infty$, we see that

$$\lim_{K\to\infty} f(X_{T\wedge K}) = f(X_T) \qquad \boldsymbol{P}_x\text{-a.s.}$$

By Lebesgue's dominated convergence theorem, we get

$$f(x) = \boldsymbol{E}_x[f(X_T)] - \boldsymbol{E}_x[\int_0^T Lf(X_s)ds].$$

5. Fix $f \in D(L)$. Given $\eta > 0$. Since Lf is continuous at x, there exists $\delta > 0$ such that $|Lf(y) - Lf(x)| < \eta$ whenever $d(y, x) < \delta$. For sufficiently small ϵ such that $\boldsymbol{E}_x[T_{\epsilon,x}] < \infty$ and $\epsilon < \delta$, we have

$$|Lf(X_t) - Lf(x)| < \eta \qquad \forall 0 \leq t \leq T_{\epsilon,x}, \boldsymbol{P}_x\text{-a.s.}$$

and therefore

$$\begin{aligned}
&|\frac{\boldsymbol{E}_x[\int_0^{T_{\epsilon,x}} Lf(X_t)dt]}{\boldsymbol{E}_x[T_{\epsilon,x}]} - Lf(x)| \\
&= |\frac{\boldsymbol{E}_x[\int_0^{T_{\epsilon,x}} Lf(X_t) - Lf(x)dt]}{\boldsymbol{E}_x[T_{\epsilon,x}]}| \\
&= \frac{\boldsymbol{E}_x[\int_0^{T_{\epsilon,x}} |Lf(X_t) - Lf(x)|dt]}{\boldsymbol{E}_x[T_{\epsilon,x}]} \\
&< \frac{\boldsymbol{E}_x[T_{\epsilon,x}]}{\boldsymbol{E}_x[T_{\epsilon,x}]}\eta = \eta
\end{aligned}$$

By (31), we get

$$\lim_{\epsilon\downarrow 0}\frac{\boldsymbol{E}_x[f(X_{T_{\epsilon,x}})]-f(x)}{\boldsymbol{E}_x[T_{\epsilon,x}]}=\lim_{\epsilon\downarrow 0}\frac{\boldsymbol{E}_x[\int_0^{T_{\epsilon,x}}Lf(X_t)dt]}{\boldsymbol{E}_x[T_{\epsilon,x}]}=Lf(x).$$

6. Since $Q_t(x,\{x\})<1$, there exists $r>0$ such that $Q_t(x,\overline{B(x,r)})<1$. Then $E\setminus\overline{B(x,r)}$ is an open set and $Q_t(x,E\setminus\overline{B(x,r)})>0$. Choose $z\in E\setminus\overline{B(x,r)}$. Then there exists $R>0$ such that $Q_t(x,(E\setminus\overline{B(x,r)})\bigcap B(z,R))>0$. Set $G=(E\setminus\overline{B(x,r)})\bigcap B(z,R)$. Then G is an bounded open set and $Q_t1_G(x)=Q_t(x,G)>0$. Set

$$f_k(y)=(\frac{d(y,E\setminus G)}{1+d(y,E\setminus G)})^{\frac{1}{k}}\qquad\forall k\geq 1.$$

Then

$$0\leq f_k(y)\uparrow 1_G(y)\qquad\forall y\in E$$

and $f_k\in C_0(E)$ for all $k\geq 1$. Since $(Q_t)_{t\geq 0}$ is Feller,

$$Q_tf_k\in C_0(E)\qquad\forall k\geq 1$$

and

$$Q_tf_k(x)\overset{k\to\infty}{\to}Q_t(x,G).$$

Choose large k such that $Q_tf_k(x)>0$ and set $h=f_k$. Then $0<Q_th(x)\leq 1$ and, hence, there exists $0<\alpha<r$ and $0<\eta<1$ such that

$$Q_t(y,G)\geq Q_th(y)>\eta>0\qquad\forall y\in B(x,\alpha).$$

Thus,

$$Q_t(y,E\setminus G)\leq(1-\eta)\qquad\forall y\in B(x,\alpha).$$

For $n\geq 1$, by the simple Markov property, we get

$$\begin{aligned}
&\boldsymbol{P}_x(T_{\alpha,x}>nt)\\
&\leq\boldsymbol{E}_x[1_{\{X_t\in B(x,\alpha)\}}...1_{\{X_{(n-1)t}\in B(x,\alpha)\}}1_{\{X_{nt}\in B(x,\alpha)\}}]\\
&=\boldsymbol{E}_x[1_{\{X_t\in B(x,\alpha)\}}...1_{\{X_{(n-1)t}\in B(x,\alpha)\}}\boldsymbol{E}_{X_{(n-1)t}}[1_{X_t\in B(x,\alpha)}]]\\
&=\boldsymbol{E}_x[1_{\{X_t\in B(x,\alpha)\}}...1_{\{X_{(n-1)t}\in B(x,\alpha)\}}Q_t(X_{(n-1)t},B(x,\alpha))]\\
&\leq\boldsymbol{E}_x[1_{\{X_t\in B(x,\alpha)\}}...1_{\{X_{(n-1)t}\in B(x,\alpha)\}}Q_t(X_{(n-1)t},E\setminus G)]\\
&\leq\boldsymbol{E}_x[1_{\{X_t\in B(x,\alpha)\}}...1_{\{X_{(n-1)t}\in B(x,\alpha)\}}](1-\eta)\\
&...\\
&\leq(1-\eta)^n.
\end{aligned}$$

Therefore

$$\boldsymbol{E}_x[T_{\epsilon,x}]\leq\boldsymbol{E}_x[T_{\alpha,x}]=\sum_{n=1}^{\infty}\int_{(n-1)t}^{nt}\boldsymbol{P}_x(T_{\alpha,x}>t)dt\leq\sum_{n=1}^{\infty}(1-\eta)^n<\infty$$

for all $\epsilon<\alpha$.

$\square$

Chapter 7

Brownian Motion and Partial Differential Equations

7.1 Exercise 7.24

Let $B(0,1)$ be the open ball of $\mathbb{R}^d$ ($d \geq 2$), and $B(0,1)^* \equiv B(0,1) \setminus \{0\}$. Let g be the continuous function defined on $\partial B(0,1)^*$ by

$$g(x) = \begin{cases} 0, & \text{if } |x| = 1 \\ 1, & \text{if } x = 0. \end{cases}$$

Prove that the Dirichlet problem in $B(0,1)^*$ with boundary condition g has no solution.

Proof.
We prove this by contradiction. Assume that there exists a $u \in C^2(B(0,1)^*) \bigcap C(\overline{B(0,1)})$ such that

$$\begin{cases} \Delta u(x) = 0, & \text{if } x \in B(0,1)^* \\ \lim_{y \in B(0,1)^* \to x \in \partial B(0,1)^*} u(y) = g(x), & \text{if } x \in \partial B(0,1)^*. \end{cases}$$

By proposition 7.7, we see that

$$u(x) = \boldsymbol{E}_x[g(B_T)] \qquad \forall x \in B(0,1)^*,$$

where $T = U_0 \wedge U_1$ and $U_a = \inf\{t \geq 0 \mid |B_t| = a\}$. By proposition 7.16, we see that

$$\boldsymbol{P}_x(U_0 < U_1) = \lim_{\epsilon \downarrow 0} \boldsymbol{P}_x(U_\epsilon < U_1) = \begin{cases} \lim_{\epsilon \downarrow 0} \frac{0 - \log(|x|)}{0 - \log(\epsilon)}, & \text{if } d = 2 \\ \lim_{\epsilon \downarrow 0} \frac{1 - |x|^{2-d}}{1 - \epsilon^{2-d}}, & \text{if } d \geq 3 \end{cases} = 0$$

and, hence,

$$u(x) = \boldsymbol{E}_x[g(B_T)] = \boldsymbol{E}_x[g(B_{U_1}) 1_{\{U_1 < U_0\}}] = 0 \qquad \forall x \in B(0,1)^*$$

which contradict to

$$\lim_{y \in B(0,1)^* \to 0} u(y) = 0 \neq 1 = g(0).$$

□

7.2 Exercise 7.25 (Polar sets)

Throughout this exercise, we consider a nonempty compact subset K of $\mathbb{R}^d$ ($d \geq 2$). We set $T_K = \inf\{t \geq 0 \mid T_t \in K\}$. We say that K is polar if there exists an $x \in K^c$ such that $\boldsymbol{P}_x(T_K < \infty) = 0$.

1. Using the strong Markov property as in the proof of Proposition 7.7 (ii), prove that the function $x \mapsto \boldsymbol{P}_x(T_K < \infty)$ is harmonic on every connected component of K^c.

2. From now on until question 4., we assume that K is polar. Prove that K^c is connected, and that the property $\boldsymbol{P}_x(T_K < \infty) = 0$ holds for every $x \in K^c$. Hint: Observe that $\{x \in K^c \mid \boldsymbol{P}_x(T_K < \infty) = 0\}$ is both open and closed.

3. Let D be a bounded domain containing K, and $D' = D \setminus K$. Prove that any bounded harmonic function h on D' can be extended to a harmonic function on D. Does this remain true if the word "bounded" is replaced by "positive"?

4. Define
$$g(x) = \begin{cases} 0, & \text{if } x \in \partial D \\ 1, & \text{if } x \in \partial D' \setminus \partial D. \end{cases}$$
Prove that the Dirichlet problem in D' with boundary condition g has no solution. (Note that this generalizes the result of Exercise 7.24.)

5. If $\alpha \in (0, d]$, we say that the compact set K has zero α-dimensional Hausdorff measure if, for every $\epsilon > 0$, we can find an integer $N_\epsilon \geq 1$ and N_ϵ open balls $B(c_k, r_k)$, $k = 1, 2, ..., N_\epsilon$, such that
$$K \subseteq \bigcup_{k=1}^{N_\epsilon} B(c_k, r_k) \text{ and } \sum_{k=1}^{N_\epsilon} r_k^\alpha \leq \epsilon.$$
Prove that if $d \geq 3$ and K has zero $d-2$-dimensional Hausdorff measure then K is polar.

Proof.
We define $T_A = \inf\{t \geq 0 \mid B_t \in A\}$ for all closed subset A of $\mathbb{R}^d$.

1. Define $\varphi : K^c \mapsto \mathbb{R}$ by $\varphi(x) = \boldsymbol{P}_x(T_K < \infty)$. To show that φ is harmonic on every connected component of K^c, it suffices to show that φ satisfies the mean value property for every $x \in K^c$. Fix $x \in K^c$. Let $r > 0$ such that $B(x, r) \subseteq K^c$. Set $T_{x,r} = \inf\{t \geq 0 \mid |B_t - x| = r\}$. Then
$$T_{x,r} < T_K, \quad T_{x,r} < \infty \qquad \boldsymbol{P}_x\text{-a.s.}$$
By the strong Markov property, we get
$$\varphi(x) = \boldsymbol{E}_x[1_{\{T_K<\infty\}}] = \boldsymbol{E}_x[\boldsymbol{E}_{B_{T_{x,r}}}[1_{\{T_K<\infty\}}]] = \boldsymbol{E}_x[\varphi(B_{T_{x,r}})].$$
Since the distribution of $B_{T_{x,r}}$ under $\boldsymbol{P}_x$ is the uniform probability measure $\sigma_{x,r}$ on the $\partial B(x, r)$, we have
$$\varphi(x) = \boldsymbol{E}_x[\varphi(B_{T_{x,r}})] = \int_{\partial B(x,r)} \varphi(y)\sigma_{x,r}(dy).$$

2. First, we show that K^c is connected. We prove this by contradiction. Assume that $K^c = \bigcup_{n=1}^m G_n$, where G_n is a connected component of K^c and $2 \leq m \leq \infty$. Then
$$\bigcup_{n=1}^m \partial G_n \subseteq K.$$
For $x \in G_i$, choose $y \in G_j$, where $i \neq j$, and $r > 0$ such that $\overline{B(y, r)} \subseteq G_j$. By proposition 7.16, we get
$$\boldsymbol{P}_x(T_K < \infty) \geq \boldsymbol{P}_x(T_{\partial G_i} < \infty) \geq \boldsymbol{P}_x(T_{\overline{B(y,r)}} < \infty) > 0.$$
Thus, we get
$$\boldsymbol{P}_x(T_K < \infty) > 0 \qquad \forall x \in K^c$$
which contradict to K is polar.
Next, we show that
$$\boldsymbol{P}_x(T_K < \infty) = 0 \qquad \forall x \in K^c.$$
Since K^c is connected, it suffices to show that
$$\Gamma \equiv \{x \in K^c \mid \boldsymbol{P}_x(T_K < \infty) = 0\}$$
is both open and closed in K^c. Indeed, since K is polar, we see that Γ is nonempty and, hence, $\Gamma = K^c$. By problem 1., we see that $\varphi(z) = \boldsymbol{P}_z(T_K < \infty)$ is continuous in K^c and so
$$\Gamma = \varphi^{-1}(\{0\})$$

is closed in K^c. Now, we show that Γ is open in K^c. Fix $x \in \Gamma$. We choose $r > 0$ such that $B(x,r) \subseteq K^c$. Assume that there exists $y \in B(x,r)$ such that $\boldsymbol{P}_y(T_K < \infty) > \eta$ for some $\eta > 0$. Since $\varphi(z) = \boldsymbol{P}_z(T_K < \infty)$ is continuous in K^c, there exists exists $r' > 0$ such that $\overline{B(y,r')} \subseteq B(x,r)$ and

$$\boldsymbol{P}_z(T_K < \infty) > \frac{\eta}{2} \qquad \forall z \in \overline{B(y,r')}.$$

By the strong Markov property, we get

$$\boldsymbol{P}_x(T_K < \infty) \geq \boldsymbol{P}_x(T_{\overline{B(y,r')}} < T_K < \infty) = \boldsymbol{E}_x[\boldsymbol{E}_{B_{T_{\overline{B(y,r)}}}}[1_{\{T_K<\infty\}}]] \geq \frac{\eta}{2} > 0$$

which is a contradiction. Thus, $B(x,r) \subseteq \Gamma$ and therefore Γ is open in K^c.

3. (a) Choose a sequence of bounded domains $\{\Gamma_n\}$ such that

$$K \subseteq \Gamma_n, \quad \overline{\Gamma_n} \subseteq \Gamma_{n+1} \qquad \forall n \geq 1, \text{ and } \overline{\Gamma_n} \uparrow D.$$

Define $u : D \mapsto \mathbb{R}$ by

$$u(x) = \lim_{n\to\infty} \boldsymbol{E}_x[h(B_{T_{\partial\Gamma_n}})].$$

Now we show that u satisfy

$$\begin{cases} \Delta u(x) = 0, & \text{if } x \in D \\ u(x) = h(x), & \text{if } x \in D'. \end{cases}$$

First, we show that $u = h$ in D' and u is well-defined.

i. Fix $x \in D'$. Choose large n such that $x \in \Gamma_n$. Since $x \in K^c$ and K is polar, we get $T_K = \infty$ $\boldsymbol{P}_x$-(a.s.) and so

$$B_{T_{\partial\Gamma_n}\wedge t} \in D' \qquad \forall t \geq 0 \quad \boldsymbol{P}_x\text{-(a.s.)}.$$

By Itô's formula, we have

$$h(B_{t\wedge T_{\partial\Gamma_n}}) = h(x) + \int_0^{t\wedge T_{\partial\Gamma_n}} \triangledown h(B_s) \cdot dB_s \qquad \forall t \geq 0 \quad \boldsymbol{P}_x\text{-(a.s.)}$$

and therefore $(h(B_{t\wedge T_{\partial\Gamma_n}}))_{t\geq 0}$ is a continuous local martingale. Since h is bounded in D', $(h(B_{t\wedge T_{\partial\Gamma_n}}))_{t\geq 0}$ is a uniformly integrable martingale and, hence,

$$h(x) = \boldsymbol{E}_x[h(B_{T_{\partial\Gamma_n}})].$$

Therefore, if $x \in \Gamma_m$ for some $m \geq 1$, then

$$E_x[h(B_{T_{\partial\Gamma_n}})] = h(x) \qquad \forall n \geq m. \tag{32}$$

Moreover,

$$u(x) = \lim_{n\to\infty} \boldsymbol{E}_x[h(B_{T_{\partial\Gamma_n}})] = h(x).$$

ii. Fix $x \in K$. We show that

$$\boldsymbol{E}_x[h(B_{T_{\partial\Gamma_n}})] = \boldsymbol{E}_x[h(B_{T_{\partial\Gamma_m}})] \qquad \forall n > m \geq 1. \tag{33}$$

Fix $n > m$. Then $\Gamma_m \subseteq \Gamma_n$. By the strong Markov property, we get

$$\boldsymbol{E}_x[h(B_{T_{\partial\Gamma_n}})] = \boldsymbol{E}_x[\boldsymbol{E}_{B_{T_{\partial\Gamma_m}}}[h(B_{T_{\partial\Gamma_n}})]].$$

By (32), we have

$$\boldsymbol{E}_{B_{T_{\partial\Gamma_m}}}[h(B_{T_{\partial\Gamma_n}})] = h(B_{T_{\partial\Gamma_m}}) \quad \boldsymbol{P}_x\text{-(a.s.)}$$

and so

$$\boldsymbol{E}_x[h(B_{T_{\partial\Gamma_n}})] = \boldsymbol{E}_x[h(B_{T_{\partial\Gamma_m}})].$$

Moveover,

$$\lim_{n\to\infty} \boldsymbol{E}_x[h(B_{T_{\partial\Gamma_n}})] = \boldsymbol{E}_x[h(B_{T_1})]$$

and, hence, u is well-defined.

Next, we show that u is harmonic on D. It suffices to show that u satisfies the mean value property. Fix $x \in D$ and $r > 0$ such that $\overline{B(x,r)} \subseteq D$. Choose $n \geq 1$ such that $\overline{B(x,r)} \subseteq \Gamma_n$. Set $T_{x,r} = \inf\{t \geq 0 \mid |B_t - x| = r\}$. By (32) and (33), we have

$$\boldsymbol{E}_z[h(B_{T_{\partial\Gamma_n}})] = u(z) \qquad \forall z \in \Gamma_n.$$

By the strong Markov property, we get

$$u(x) = \boldsymbol{E}_x[h(B_{T_{\partial\Gamma_n}})] = \boldsymbol{E}_x[\boldsymbol{E}_{B_{T_{x,r}}}[h(B_{T_{\partial\Gamma_n}})]] = \boldsymbol{E}_x[u(B_{T_{x,r}})].$$

Since the distribution of $B_{T_{x,r}}$ under $\boldsymbol{P}_x$ is the uniform probability measure $\sigma_{x,r}$ on the $\partial B(x,r)$, we have

$$u(x) = \int_{\partial B(x,r)} u(y)\sigma_{x,r}(dy).$$

Therefore u is a harmonic function on D such that $u(x) = h(x)$ for all $x \in D'$.

(b) Now we show that boundedness is necessary for this statement. Set $K = \{0\}$. By proposition 7.16, K is a polar. Choose $D = B(0,r)$ for some $0 < r < 1$. Then $D' = B(0,r) \setminus \{0\}$. Define Φ to be the fundamental solution of Laplace equation. That is,

$$\Phi(x) = \begin{cases} \frac{-1}{2\pi}\log(|x|), & \text{if } d = 2 \\ \frac{1}{n(n-2)w_n}\frac{1}{|x|^{d-2}}, & \text{if } d \geq 3. \end{cases}$$

Then Φ is a unbounded, positive harmonic function on D' and Φ can't be extended to a harmonic function on D.

4. We prove this by contradiction. Assume that there exists a $u \in C^2(D') \bigcap C(\overline{D'})$ such that

$$\begin{cases} \Delta u(x) = 0, & \text{if } x \in D' \\ \lim_{y\in D' \to x \in \partial D'} u(y) = g(x), & \text{if } x \in \partial D'. \end{cases}$$

By proposition 7.7, we see that

$$u(x) = \boldsymbol{E}_x[g(B_T)] \qquad \forall x \in D',$$

where $T = T_{\partial D} \wedge T_{\partial D' \setminus \partial D}$. Note that

$$T_{\partial D' \setminus \partial D} = T_K \quad \boldsymbol{P}_x\text{-a.s.} \qquad \forall x \in D'.$$

Fix $x \in D'$. Since $T_K = \infty$ $\boldsymbol{P}_x$-(a.s.), we see that $T = T_{\partial D}$ $\boldsymbol{P}_x$-(a.s.) and, hence,

$$u(x) = \boldsymbol{E}_x[g(B_T)] = \boldsymbol{E}_x[g(B_{T_{\partial D}})] = 0.$$

Thus, we see that

$$u(x) = 0 \qquad \forall x \in D'$$

which contradict to

$$\lim_{x \in D' \to y \in \partial D' \setminus \partial D} u(x) = 0 \neq 1 = g(y) \qquad \forall y \in \partial D' \setminus \partial D.$$

5. To show that K is polar, we show that $\boldsymbol{P}_x(T_K < \infty) = 0$ for all $x \in K^c$. Fix $x \in K^c$. Then

$$h_{x,K} \equiv \inf\{|x - z| \mid z \in K\} > 0.$$

Given $\epsilon > 0$. There exists $N_\epsilon \geq 1$ and N_ϵ open balls $B(c_k, r_k)$, $k = 1, 2, ..., N_\epsilon$, such that

$$K \subseteq \bigcup_{k=1}^{N_\epsilon} B(c_k, r_k) \text{ and } \sum_{k=1}^{N_\epsilon} r_k^{d-2} \leq \epsilon.$$

Without loss of generality, we assume that

$$B(c_k, r_k) \bigcap K \neq \emptyset \qquad \forall k = 1, 2, ..., N_\epsilon.$$

Choose $\widetilde{c}_k \in B(c_k, r_k) \bigcap K$ and set $\widetilde{r}_k = 2r_k$ for all $k = 1, 2, ..., N_\epsilon$. Then

$$K \subseteq \bigcup_{k=1}^{N_\epsilon} B(\widetilde{c}_k, \widetilde{r}_k) \text{ and } \sum_{k=1}^{N_\epsilon} \widetilde{r}_k^{d-2} \leq 2^{d-2}\epsilon.$$

Set $T_k = \inf\{t \geq 0 \mid |B_t - \widetilde{c}_k| = \widetilde{r}_k\}$ for all $k = 1, 2, ..., N_\epsilon$. Then

$$\boldsymbol{P}_x(T_K < \infty) \leq \boldsymbol{P}_x(\wedge_{k=1}^{N_\epsilon} T_k < \infty) \leq \sum_{k=1}^{N_\epsilon} \boldsymbol{P}_x(T_k < \infty).$$

By proposition 7.16, we get

$$\boldsymbol{P}_x(T_k < \infty) = (\frac{\widetilde{r}_k}{|x - \widetilde{c}_k|})^{d-2} \qquad \forall k = 1, 2, ..., N_\epsilon$$

and, hence,

$$\boldsymbol{P}_x(T_K < \infty) \leq \sum_{k=1}^{N_\epsilon} (\frac{\widetilde{r}_k}{|x - \widetilde{c}_k|})^{d-2} \leq \sum_{k=1}^{N_\epsilon} (\frac{\widetilde{r}_k}{h_{x,K}})^{d-2} < \frac{2^{d-2}}{h_{x,K}^{d-2}}\epsilon.$$

By letting $\epsilon \downarrow 0$, we have $\boldsymbol{P}_x(T_K < \infty) = 0$.

$\square$

7.3 Exercise 7.26

In this exercise, $d \geq 3$. Let K be a compact subset of the open unit ball of $\mathbb{R}^d$, and $T_K = \inf\{t \geq 0 : B_t \in K\}$. We assume that $D := \mathbb{R}^d \setminus K$ is connected. We also consider a function g defined and continuous on K. The goal of the exercise is to determine all functions $u : \overline{D} \mapsto \mathbb{R}$ that satisfy:
(P) u is bounded and continuous on $\overline{D}$, harmonic on D, and $u(y) = g(y)$ if $y \in \partial D$.
(This is the Dirichlet problem in D, but in contrast with Sect. 7.3 above, D is unbounded here.) We fix an increasing sequence $\{R_n\}_{n\geq 1}$ of reals, with $R_1 \geq 1$ and $R_n \uparrow \infty$ as $n \to \infty$. For every $n \geq 1$, we set $T_n = \inf\{t \geq 0 : |B_t| \geq R_n\}$.

1. Suppose that u satisfies (P). Prove that, for every $n \geq 1$ and every $x \in D$ such that $|x| < R_n$,

$$u(x) = \boldsymbol{E}_x[g(B_{T_K})1_{\{T_K \leq T_n\}}] + \boldsymbol{E}_x[u(B_{T_n})1_{\{T_n \leq T_K\}}].$$

2. Show that, by replacing the sequence $\{R_n\}$ with a subsequence if necessary, we may assume that there exists a constant $\alpha \in \mathbb{R}$ such that, for every $x \in D$,

$$\lim_{n\to\infty} \boldsymbol{E}_x[u(B_{T_n})] = \alpha,$$

and that we then have

$$\lim_{|x|\to\infty} u(x) = \alpha.$$

3. Show that, for every $x \in D$,
$$u(x) = \boldsymbol{E}_x[g(B_{T_K})1_{\{T_K<\infty\}}] + \alpha \boldsymbol{P}_x(T_K = \infty).$$
4. Assume that D satisfies the exterior cone condition at every $y \in \partial D$ (this is defined in the same way as when D is bounded). Show that, for any choice of $\alpha \in \mathbb{R}$ the formula of question 3. gives a solution of the problem (P).

Proof.
We define $T_A := \inf\{t \geq 0 : B_t \in A\}$ for all closed subset A of $\mathbb{R}^d$.

1. Fix $n \geq 1$. Set continuous function
$$f(x) = \begin{cases} u(x), & \text{if } y \in \partial B(0, R_n) \\ g(x), & \text{if } y \in \partial K, \end{cases}$$
By using proposition 7.7 on the bounded domain $B(0, R_n) \setminus K$, we get
$$u(x) = \boldsymbol{E}_x[g(B_{T_K})1_{\{T_K \leq T_n\}}] + \boldsymbol{E}_x[u(B_{T_n})1_{\{T_n \leq T_K\}}] \quad \forall x \in D \bigcap B(0, R_n).$$
2. Denote $M := \sup_{z \in \overline{D}} |u(z)|$.
   (a) We show that there exists $1 \leq n_1 < n_2 < n_3 < ...$ such that $\lim_{k\to\infty} \boldsymbol{E}_x[u(B_{T_{n_k}})]$ converges uniformly on every compact subset $K \subseteq \mathbb{R}^d$ for every $x \in \mathbb{R}^d$. Denote
$$f_n(x) := \boldsymbol{E}_x[u(B_{T_n})] \quad \forall x \in B(0, R_n), \quad n \geq 1.$$
   By the strong Markov property, we get f_n is harmonic on $B(0, R_n)$ for every $n \geq 1$.
   First, we show that $\{f_n\}$ is equicontinuous on $\overline{B(p,r)}$ for every $p \in \mathbb{Q}^d$ and $r \in \mathbb{Q}_+$. Fix $p \in \mathbb{Q}^d$ and $r \in \mathbb{Q}_+$. Choose $N \geq 1$ such that $B(p, r) \subseteq B(0, R_N)$ and $\eta := d(B(p, r), \partial B(0, R_N)) > 0$. By local estimates for harmonic function, there exists $C_1 > 0$ such that
$$|Df_n(x)| \leq \frac{C_1}{(\eta/2)^{d+1}} ||f_n||_{L^1(B(x,\eta/2))} \leq \frac{C_1 M}{\eta/2} \quad \forall x \in B(p, r + \eta/2), \quad n \geq N.$$
   Fix $\epsilon > 0$. Let $x, y \in \overline{B(p,r)}$ such that $|x - y| < \frac{\eta}{2C_1 M}\epsilon$. Then
$$|f_n(x) - f_n(y)| \leq \sup_{z \in B(p, r+\eta/2)} |Df_n(z)||x - y| < \epsilon \quad \forall n \geq N.$$
   Moreover, by Arzelà–Ascoli theorem, there exists a subsequence $N \leq n_1 < n_2 < n_3 < ...$ such that $f_{n_k}(x)$ converges uniformly on $\overline{B(p,r)}$.
   Next, by a standard diagonalization procedure, there exists $1 \leq n_1 < n_2 < n_3 < ...$ such that $f_{n_k}(x)$ converges uniformly on $\overline{B(p_i, r_i)}$ for each $i \geq 1$, where $Q^d = \{p_i\}_{i \geq 1}$ and $Q_+ = \{r_i\}_{i\geq 1}$, and so, $\lim_{k\to\infty} f_{n_k}(x)$ uniformly on every compact subset K of $\mathbb{R}^d$.
   (b) We show that there exists $\alpha \in \mathbb{R}$ such that
$$\lim_{k\to\infty} \boldsymbol{E}_x[u(B_{T_{n_k}})] = \alpha \quad \forall x \in D.$$
   Set
$$f(x) := \lim_{k\to\infty} f_{n_k}(x) \quad \forall x \in \mathbb{R}^d.$$
   By the strong Markov property, we get
$$\int f(y)\sigma_{x,r}(dy) = \lim_{k\to\infty} \int \boldsymbol{E}_y[u(B_{T_{n_k}})]\sigma_{x,r}(dy) = \lim_{k\to\infty} \boldsymbol{E}_x[u(B_{T_{n_k}})] = f(x)$$
   and so f is a bounded, harmonic function. By Liouville's theorem, we see that $f = \alpha$ for some $\alpha \in \mathbb{R}$.

(c) We show that $\lim_{|x|\to\infty} u(x) = \alpha$. Fix $\epsilon > 0$. Choose $R > 0$ such that $\frac{1}{R^{d-2}} < \epsilon$. Let $|x| \geq R$. Choose large $j \geq 1$ such that $|x| \leq R_{n_j}$,

$$|\boldsymbol{E}_x[u(B_{T_{n_j}})] - \alpha| < \epsilon,$$

and

$$\frac{R_{n_j}^{2-d} - |x|^{2-d}}{R_{n_j}^{2-d} - 1} \leq |x|^{2-d} + \epsilon.$$

Set $B := \overline{B(0,1)}$. Then

$$\boldsymbol{P}_x(T_B < T_{n_j}) = \frac{R_{n_j}^{2-d} - |x|^{2-d}}{R_{n_j}^{2-d} - 1} \leq |x|^{2-d} + \epsilon \leq R^{2-d} + \epsilon < 2\epsilon$$

and so

$$\begin{aligned}|u(x) - \alpha| &= |\boldsymbol{E}_x[g(B_{T_K})1_{\{T_K \leq T_{n_j}\}}] - \boldsymbol{E}_x[u(B_{T_{n_j}})1_{\{T_j > T_K\}}] + \boldsymbol{E}_x[u(B_{T_{n_j}})] - \alpha| \\ &\leq M\boldsymbol{P}_x(T_{n_j} > T_K) + M\boldsymbol{P}_x(T_{n_j} > T_K) + \epsilon \leq (4M+1)\epsilon.\end{aligned}$$

3. Since $\lim_{t\to\infty} |B_t| = \infty$ and $u(x) \overset{|x|\to\infty}{\to} \alpha$, we get $T_{n_k} < \infty$ for every $k \geq 1$ (a.s.) and so

$$\boldsymbol{E}_x[u(B_{T_{n_k}})1_{\{T_{n_k} \leq T_K\}}] = \boldsymbol{E}_x[u(B_{T_{n_k}})1_{\{T_{n_k} \leq T_K < \infty\}}] + \boldsymbol{E}_x[u(B_{T_{n_k}})1_{\{T_{n_k} < \infty\} \bigcap \{T_K = \infty\}}] \overset{k\to\infty}{\to} 0 + \alpha\boldsymbol{P}_x(T_K = \infty).$$

By problem 1 and problem 2, we have

$$u(x) = \lim_{k\to\infty} \boldsymbol{E}_x[g(B_{T_K})1_{\{T_K \leq T_n\}}] + \lim_{k\to\infty} \boldsymbol{E}_x[u(B_{T_n})1_{\{T_n \leq T_K\}}] = \boldsymbol{E}_x[g(B_{T_K})1_{\{T_K < \infty\}}] + \alpha\boldsymbol{P}_x(T_K = \infty).$$

4. It suffices to show that $\lim_{x\in D\to y} u(x) = g(y)$ for every $y \in \partial D$. Denote $M := \sup_{z\in K} |g(z)|$. Fix $\epsilon > 0$ and $y \in \partial D$. Choose $\delta > 0$ such that

$$|g(z) - g(y)| < \epsilon \quad \forall z \in K \bigcap B(y,\delta).$$

Choose $\eta > 0$ such that

$$\boldsymbol{P}_0(\sup_{t\leq\eta} |B_t| \geq \frac{\delta}{2}) < \epsilon.$$

Observe that

$$\lim_{x\in D\to y} \boldsymbol{P}_x(T_K > \eta) = 0$$

(This proof is the same as the proof of lemma 7.9) and so there exists $\delta' > 0$ such that

$$\boldsymbol{P}_x(T_K > \eta) < \epsilon \quad \forall x \in D \bigcap B(y,\delta').$$

Let $x \in D \bigcap B(y, \delta' \wedge \frac{\delta}{2})$. Then

$$\boldsymbol{P}_x(\sup_{t\leq\eta} |B_t - x| \geq \frac{\delta}{2}) = \boldsymbol{P}_0(\sup_{t\leq\eta} |B_t| \geq \frac{\delta}{2}) < \epsilon$$

and so

$$\begin{aligned}&|u(x) - g(y)| \\ &\leq \boldsymbol{E}_x[|g(B_{T_K}) - g(y)|1_{\{T_K \leq \eta\}}] + \boldsymbol{E}_x[|g(B_{T_K}) - g(y)|1_{\{\eta < T_K < \infty\}}] + (g(y)+\alpha)\boldsymbol{P}_x(T_K = \infty) \\ &\leq \boldsymbol{E}_x[|g(B_{T_K}) - g(y)|1_{\{T_K \leq \eta\}}1_{\{\sup_{t\leq\eta} |B_t - x| < \frac{\delta}{2}\}}] + 2M\boldsymbol{P}_x(\sup_{t\leq\eta} |B_t - x| \geq \frac{\delta}{2}) + \\ &\boldsymbol{E}_x[|g(B_{T_K}) - g(y)|1_{\{\eta < T_K < \infty\}}] + (g(y)+\alpha)\boldsymbol{P}_x(T_K = \infty) \\ &\leq \epsilon + 2M\epsilon + 2M\boldsymbol{P}_x(\eta < T_K < \infty) + (g(y)+\alpha)\boldsymbol{P}(T_K = \infty) \\ &\leq \epsilon + 2M\epsilon + (3M+\alpha)\boldsymbol{P}_x(T_K > \eta) < \epsilon + 2M\epsilon + (3M+\alpha)\epsilon.\end{aligned}$$

□

7.4 Exercise 7.27

Let $f : \mathbb{C} \mapsto \mathbb{C}$ be a nonconstant holomorphic function. Use planar Brownian motion to prove that the set $\{f(x) : z \in \mathbb{C}\}$ is dense in $\mathbb{C}$. (Much more is true, since Picard's little theorem asserts that the complement of $\{f(x) : z \in \mathbb{C}\}$ in $\mathbb{C}$ contains at most one point: This can also be proved using Brownian motion, but the argument is more involved)

Proof.
We prove this by contradiction. Assume that there exists $z \in \mathbb{C}$ and $r > 0$ such that $\overline{B(z,r)} \subseteq G^c$, where $G = \{f(z) : z \in \mathbb{C}\}$. For any filtration $(\mathscr{G}_t)_{t\geq 0}$ and $(\mathscr{G}_t)_{t\geq 0}$-adapted process $(A_t)_{t\geq 0}$ on $\mathbb{C}$, we define a stopping time

$$T_F^A = \inf\{t \geq 0 : A_t \in F\}$$

for closed subset F of $\mathbb{C}$. Let $(B_t)_{t\geq 0}$ be a complex Brownian motion that starts from 0 under the probability measure $\boldsymbol{P}_0$. Since $\overline{B(z,r)} \subseteq G^c$, we get

$$\boldsymbol{P}_0(T^{f(B)}_{\overline{B(z,r)}} < \infty) = 0.$$

By Theorem 7.18, there exists a complex Brownian motion Γ that starts from $f(0)$ under $\boldsymbol{P}_0$, such that

$$f(B_t) = \Gamma_{C_t} \qquad \forall t \geq 0 \quad \boldsymbol{P}_0\text{-(a.s.)},$$

where

$$C_t = \int_0^t |f'(B_s)|^2 ds \qquad \forall t \geq 0.$$

By Proposition 7.16, we see that

$$\boldsymbol{P}_0(T^{\Gamma}_{\overline{B(z,r)}} < \infty) = 1.$$

Since $(C_t)_{t\geq 0}$ is a continuous increasing process and $C_\infty = \infty$ $\boldsymbol{P}_0$-(a.s.), we have

$$\boldsymbol{P}_0(T^{f(B)}_{\overline{B(z,r)}} < \infty) = \boldsymbol{P}_0(T^{\Gamma_C}_{\overline{B(z,r)}} < \infty) = 1$$

which is a contradiction. □

7.5 Exercise 7.28 (Feynman–Kac formula for Brownian motion)

This is a continuation of Exercise 6.26 in Chap. 6. With the notation of this exercise, we assume that $E = \mathbb{R}^d$ and $X_t = B_t$. Let v be a nonnegative function in $C_0(\mathbb{R}^d)$, and assume that v is continuously differentiable with bounded first derivatives. As in Exercise 6.26, set, for every $\varphi \in B(\mathbb{R}^d)$,

$$Q_t^*\varphi(x) = \boldsymbol{E}_x[\varphi(X_t)e^{-\int_0^t v(X_s)ds}].$$

1. Using the formula derived in question 2. of Exercise 6.26, prove that, for every $t > 0$, and every $\varphi \in C_0(\mathbb{R}^d)$, the function $Q_t^*\varphi$ is twice continuously differentiable on $\mathbb{R}^d$, and that $Q_t^*\varphi$ and its partial derivatives up to order 2 belong to $C_0(\mathbb{R}^d)$. Conclude that $Q_t^*\varphi \in D(L)$.
2. Let $\varphi \in C_0(\mathbb{R}^d)$ and set $u_t(x) = Q_s^*\varphi(x)$ for every $t > 0$ and $x \in \mathbb{R}^d$. Using question 3. of Exercise 6.26, prove that, for every $x \in \mathbb{R}^d$, the function $t \mapsto u_t(x)$ is continuously differentiable on $(0, \infty)$, and

$$\frac{\partial}{\partial t}u_t = \frac{1}{2}\Delta u_t - vu_t.$$

Proof.

1. For $f : \mathbb{R}^d \mapsto \mathbb{R}$, we set $||f|| = \sup_{x\in\mathbb{R}^d} |f(x)|$. Observe that we have the following facts:

(a) Fix $\varphi \in B(\mathbb{R}^d)$ and $t \geq 0$. By the definition of $Q_t^*\varphi$, we get

$$||Q_t^*\varphi|| \leq ||\varphi||.$$

(b) Fix $\varphi \in C_0(\mathbb{R}^d)$ and $t \geq 0$. By question 2. of Exercise 6.26, we get

$$Q_t^*\varphi(x) = Q_t\varphi(x) - \int_0^t Q_s(vQ_{t-s}^*\varphi)(x)ds \qquad \forall x \in \mathbb{R}^d,$$

where $\{Q_t\}$ is the semigroup of $(B_t)_{t\geq 0}$.

(c) Fix $f \in C_0(\mathbb{R}^d)$ and $t \geq 0$. Since $Q_tf(x) = f * k_s(x)$, where

$$k(x) := (2\pi)^{-\frac{d}{2}}e^{-\frac{|x|^2}{2}} \text{ and } k_s(x) := (s)^{-\frac{d}{2}}k(\frac{x}{\sqrt{s}}),$$

we see that $Q_tf \in C^\infty(\mathbb{R}^d)$, and that Q_tf and all its partial derivatives belong to $C_0(\mathbb{R}^d)$. Moreover, if $t > 0$, then

$$||D_jQ_tf|| \leq \frac{1}{\sqrt{t}}||D_jk||_{L^1(\mathbb{R}^d)}||f||. \tag{34}$$

Indeed, since

$$D_jQ_tf(x) = D_j(f * k_t)(x) = \int_{\mathbb{R}^d}(2\pi t)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{2t}}(-\frac{x-y}{t})f(y)dy = \frac{-1}{\sqrt{t}}(((D_jk)_t) * f)(x),$$

we have

$$||D_jQ_tf(x)|| \leq \frac{1}{\sqrt{t}}||((D_jk)_t) * f|| \leq \frac{1}{\sqrt{t}}||D_jk||_{L^1(\mathbb{R}^d)}||f||.$$

(d) Let $s > 0$. Then

$$D_ik_s(x) = \frac{1}{\sqrt{s}}(D_ik)_s(x) \qquad \forall x \in \mathbb{R}^d.$$

(e) Let $\varphi \in C_0(\mathbb{R}^d)$. Then

$$||Q_r^*\varphi|| \leq ||\varphi||$$

for all $r \geq 0$. We will show that $x \in \mathbb{R}^d \mapsto Q_r^*\varphi(x)$ is continuous for all $r \geq 0$. Therefore $vQ_r^*\varphi \in C_0(\mathbb{R}^d)$,

$$Q_s(vQ_r^*\varphi)(x) = ((vQ_r^*\varphi) * k_s)(x) \in C^\infty(\mathbb{R}^d),$$

and that $Q_s(vQ_r^*\varphi)(x)$ and all its derivatives belong to $C_0(\mathbb{R}^d)$ for all $r, s \geq 0$. Moreover,

$$\int_0^t Q_s(vQ_{t-s}^*\varphi)(x)ds = \int_0^t ((vQ_{t-s}^*\varphi) * k_s)(x)ds \qquad \forall x \in \mathbb{R}^d.$$

(f) Note that

$$\{h \in C^2(\mathbb{R}^d) \mid h \text{ and } \Delta h \in C_0(\mathbb{R}^d)\} \subseteq D(L),$$

where L is the generator of B and $D(L)$ is the domain of L.

Fix $\varphi \in C_0(\mathbb{R}^d)$. To prove problem 1, it suffices to show that $x \in \mathbb{R}^d \mapsto \int_0^t Q_s(vQ_{t-s}^*\varphi)(x)ds$ is twice continuously differentiable, and that $x \in \mathbb{R}^d \mapsto \int_0^t Q_s(vQ_{t-s}^*\varphi)(x)ds$ and its partial derivatives up to order 2 belong to $C_0(\mathbb{R}^d)$.

(a) We show that $x \in \mathbb{R}^d \mapsto \int_0^t Q_s(vQ^*_{t-s}\varphi)(x)ds$ belong to $C_0(\mathbb{R}^d)$. It suffices to show that $x \in \mathbb{R}^d \mapsto Q^*_r\varphi(x)$ is continuous for all $r \geq 0$. Indeed, since

$$Q_s(vQ^*_{t-s}\varphi) \in C_0(\mathbb{R}^d) \qquad \forall s \in [0,t]$$

and

$$||Q_s(vQ^*_{t-s}\varphi)|| \leq ||v||||\varphi|| \qquad \forall s \in [0,t],$$

we get

$$\lim_{x\to a}\int_0^t Q_s(vQ^*_{t-s}\varphi)(x)ds = \int_0^t \lim_{x\to a} Q_s(vQ^*_{t-s}\varphi)(x)ds = \begin{cases} \int_0^t Q_s(vQ^*_{t-s}\varphi)(a)ds, & \text{if } a \neq \infty \\ 0, & \text{otherwise} \end{cases}$$

and, hence, $x \in \mathbb{R}^d \mapsto \int_0^t Q_s(vQ^*_{t-s}\varphi)(x)ds$ belong to $C_0(\mathbb{R}^d)$.
Now we show that $x \in \mathbb{R}^d \mapsto Q^*_r\varphi(x)$ is continuous for all $r \geq 0$. Fix $r \geq 0$. Observe that

$$\boldsymbol{E}_x[\varphi(X_r)e^{-\frac{r}{n}\sum_{i=1}^n v(X_{\frac{ir}{n}})}] \overset{n\to\infty}{\to} Q^*_r\varphi(x) := \boldsymbol{E}_x[\varphi(X_r)e^{-\int_0^r v(X_s)ds}] \text{ uniformly on } \mathbb{R}^d.$$

Indeed, since

$$\boldsymbol{E}_x[\varphi(X_r)e^{-\frac{r}{n}\sum_{i=1}^n v(X_{\frac{ir}{n}})}] = \boldsymbol{E}_0[\varphi(X_r+x)e^{-\frac{r}{n}\sum_{i=1}^n v(X_{\frac{ir}{n}}+x)}] \qquad \forall n \geq 1,$$
$$\boldsymbol{E}_x[\varphi(X_r)e^{-\int_0^r v(X_s)ds}] = \boldsymbol{E}_0[\varphi(X_r+x)e^{-\int_0^r v(X_s+x)ds}] \qquad \forall n \geq 1,$$

and

$$\frac{r}{n}\sum_{i=1}^n v(X_{\frac{ir}{n}}+x) \overset{n\to\infty}{\to} \int_0^r v(X_s+x)ds \text{ uniformly on } \mathbb{R}^d \qquad \boldsymbol{P}_0\text{-(a.s.)},$$

we get

$$\begin{aligned}\lim_{n\to\infty}\boldsymbol{E}_x[\varphi(X_r)e^{-\frac{r}{n}\sum_{i=1}^n v(X_{\frac{ir}{n}})}] &= \lim_{n\to\infty}\boldsymbol{E}_0[\varphi(X_r+x)e^{-\frac{r}{n}\sum_{i=1}^n v(X_{\frac{ir}{n}}+x)}] \\ &= \boldsymbol{E}_0[\varphi(X_r+x)e^{-\int_0^r v(X_s+x)ds}] \\ &= \boldsymbol{E}_x[\varphi(X_r)e^{-\int_0^r v(X_s)ds}] \text{ uniformly on } \mathbb{R}^d.\end{aligned}$$

By Lebesgue's dominated convergence theorem, we get

$$x \in \mathbb{R}^d \mapsto \boldsymbol{E}_0[\varphi(X_r+x)e^{-\frac{r}{n}\sum_{i=1}^n v(X_{\frac{ir}{n}}+x)}] = \boldsymbol{E}_x[\varphi(X_r)e^{-\frac{r}{n}\sum_{i=1}^n v(X_{\frac{ir}{n}})}]$$

is continuous for all $n \geq 1$ and so

$$x \in \mathbb{R}^d \mapsto \boldsymbol{E}_x[\varphi(X_r)e^{-\int_0^r v(X_s)ds}] = Q^*_r\varphi(x)$$

is continuous.

(b) We show that

$$D_i\int_0^t Q_s(vQ^*_{t-s}\varphi)(x)ds = D_i\int_0^t ((vQ^*_{t-s}\varphi)*k_s)(x)ds = \int_0^t ((vQ^*_{t-s}\varphi)*(D_ik_s))(x)ds$$

for all $x \in \mathbb{R}^d$ and

$$x \in \mathbb{R}^d \mapsto D_i\int_0^t Q_s(vQ^*_{t-s}\varphi)(x)ds$$

belong to $C_0(\mathbb{R}^d)$ for all $i = 1, 2, ..., d$. Since $vQ^*_{t-s}\varphi$ is bounded, we have

$$D_i((vQ^*_{t-s}\varphi)*k_s)(x) = ((vQ^*_{t-s}\varphi)*(D_ik_s))(x) \qquad \forall x \in \mathbb{R}^d.$$

Note that, if $s \in [0, t]$, then

$$\begin{aligned} ||(vQ^*_{t-s}\varphi) * (D_i k_s)|| &\leq ||vQ^*_{t-s}\varphi|| \times ||D_i k_s||_{L^1(\mathbb{R}^d)} \\ &\leq ||v|| ||\varphi|| \times \frac{1}{\sqrt{s}} ||(D_i k)_s||_{L^1(\mathbb{R}^d)} \\ &\leq ||v|| ||\varphi|| \times \frac{1}{\sqrt{s}} ||D_i k||_{L^1(\mathbb{R}^d)} \in L^1([0, t]). \end{aligned}$$

By mean value theorem and Lebesgue's dominated convergence theorem, we have

$$D_i \int_0^t ((vQ^*_{t-s}\varphi) * k_s)(x) ds = \int_0^t D_i((vQ^*_{t-s}\varphi) * k_s)(x) ds = \int_0^t ((vQ^*_{t-s}\varphi) * (D_i k_s))(x) ds$$

for all $x \in \mathbb{R}^d$. Given $a \in \mathbb{R}^d \bigcup \{\infty\}$. By Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} \lim_{x \to a} D_i \int_0^t ((vQ^*_{t-s}\varphi) * k_s)(x) ds &= \lim_{x \to a} \int_0^t ((vQ^*_{t-s}\varphi) * (D_i k_s))(x) ds \\ &= \int_0^t \lim_{x \to a} ((vQ^*_{t-s}\varphi) * (D_i k_s))(x) ds \\ &= \int_0^t \lim_{x \to a} D_i((vQ^*_{t-s}\varphi) * (k_s))(x) ds \\ &= \int_0^t \lim_{x \to a} D_i(Q_s(vQ^*_{t-s}\varphi))(x) ds. \end{aligned}$$

Since $D_i Q_s(vQ^*_{t-s}\varphi) \in C_0(\mathbb{R}^d)$, we see that

$$\begin{aligned} \int_0^t \lim_{x \to a} D_i(Q_s(vQ^*_{t-s}\varphi))(x) ds &= \begin{cases} \int_0^t D_i(Q_s(vQ^*_{t-s}\varphi))(a) ds, & \text{if } a \neq \infty \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} D_i \int_0^t (Q_s(vQ^*_{t-s}\varphi))(a) ds, & \text{if } a \neq \infty \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

and so

$$x \in \mathbb{R}^d \mapsto D_i \int_0^t Q_s(vQ^*_{t-s}\varphi)(x) ds$$

belong to $C_0(\mathbb{R}^d)$.

(c) We show that

$$D_{j,i} \int_0^t Q_s(vQ^*_{t-s}\varphi)(x) ds = D_{j,i} \int_0^t ((vQ^*_{t-s}\varphi) * k_s)(x) ds = \int_0^t ((D_j(vQ^*_{t-s}\varphi)) * (D_i k_s))(x) ds$$

for all $x \in \mathbb{R}^d$ and

$$x \in \mathbb{R}^d \mapsto D_{j,i} \int_0^t Q_s(vQ^*_{t-s}\varphi)(x) ds$$

belong to $C_0(\mathbb{R}^d)$ for all $i, j = 1, 2, ..., d$. Since we have shown that

$$D_j Q^*_r \varphi(x) = D_j Q_r \varphi(x) - D_j \int_0^r Q_s(vQ^*_{r-s}\varphi)(x) ds$$

and

$$D_j Q_r \varphi(x), D_j \int_0^r Q_s(vQ^*_{r-s}\varphi)(x) ds \in C_0(\mathbb{R}^d)$$

for all $r \geq 0$ and $j = 1, 2, ..., d$, we see that

$$vQ_r^*\varphi \in C^1(\mathbb{R}^d) \text{ and } D_j(vQ_r^*\varphi) \in C_0(\mathbb{R}^d).$$

Thus $\int_0^t ((D_j(vQ_{t-s}^*\varphi)) * (D_ik_s))(x)ds$ is well-defined.
Fix $0 < s < t$. First, we show that

$$D_{j,i}Q_s(vQ_{t-s}^*\varphi)(x) = D_j((vQ_{t-s}^*) * (D_ik_s))(x) = ((D_j(vQ_{t-s}^*\varphi)) * (D_ik_s))(x)$$

for all $x \in \mathbb{R}^d$. Note that $D_ik_s \in L^1(\mathbb{R}^s)$ and

$$\begin{aligned}
||D_j(vQ_{t-s}^*\varphi)|| &= ||(D_jv)Q_{t-s}^*\varphi + vD_jQ_{t-s}^*\varphi|| \\
&= ||(D_jv)Q_{t-s}^*\varphi + vD_jQ_{t-s}\varphi - vD_j\int_0^{t-s} Q_u(vQ_{t-s-u}^*\varphi)du|| \\
&= ||(D_jv)Q_{t-s}^*\varphi + vD_jQ_{t-s}\varphi - v\int_0^{t-s} D_jQ_u(vQ_{t-s-u}^*\varphi)du|| \\
&= ||(D_jv)Q_{t-s}^*\varphi + vD_jQ_{t-s}\varphi - v\int_0^{t-s} D_j(vQ_{t-s-u}^*\varphi) * (k_u)du|| \\
&= ||(D_jv)Q_{t-s}^*\varphi + vD_jQ_{t-s}\varphi - v\int_0^{t-s} (vQ_{t-s-u}^*\varphi) * (D_jk_u)du|| \\
&\leq ||D_jv||||\varphi|| + ||v||||D_jQ_{t-s}\varphi|| + \int_0^t ||(vQ_{t-s-u}^*\varphi) * (D_jk_u)||du \\
&\leq ||D_jv||||\varphi|| + ||v||||D_jQ_{t-s}\varphi|| + \int_0^t ||(vQ_{t-s-u}^*\varphi)||||(D_jk_u)||_{L^1(\mathbb{R}^d)}du \\
&\leq ||D_jv||||\varphi|| + ||v||||D_jQ_{t-s}\varphi|| + \int_0^t ||v||||\varphi||\frac{1}{\sqrt{u}}||D_jk||_{L^1(\mathbb{R}^d)}du.
\end{aligned}$$

By (34), we get

$$||D_j(vQ_{t-s}^*\varphi)|| \leq C(1 + \frac{1}{\sqrt{t-s}}),$$

where C is a constant independent of s and j (We may set $C = \max_{1\leq i\leq d} C_i$ and so C is independent of i). Fix $x \in \mathbb{R}^d$. By mean value theorem, we get

$$|D_ik_s(y)(\frac{(vQ_{t-s}^*\varphi)(x-y+he_j) - (vQ_{t-s}^*\varphi)(x-y+he_j)}{h})| \leq C(1 + \frac{1}{\sqrt{t-s}})|D_ik_s(y)| \in L^1(\mathbb{R}^d).$$

By Lebesgue's convergence theorem, we have

$$D_{j,i}Q_s(vQ_{t-s}^*\varphi)(x) = D_j((vQ_{t-s}^*) * (D_ik_s))(x) = ((D_j(vQ_{t-s}^*\varphi)) * (D_ik_s))(x).$$

Next, we show that

$$D_{j,i}\int_0^t Q_s(vQ_{t-s}^*\varphi)(x)ds = D_{j,i}\int_0^t ((vQ_{t-s}^*\varphi) * k_s)(x)ds = \int_0^t ((D_j(vQ_{t-s}^*\varphi)) * (D_ik_s))(x)ds$$

for all $x \in \mathbb{R}^d$. Note that we already have

$$D_i\int_0^t Q_s(vQ_{t-s}^*\varphi)(x)ds = \int_0^t ((vQ_{t-s}^*\varphi) * (D_ik_s))(x)ds.$$

It suffices to show that

$$D_j\int_0^t ((D_j(vQ_{t-s}^*\varphi)) * (D_ik_s))(x)ds = \int_0^t ((D_j(vQ_{t-s}^*\varphi)) * (D_ik_s))(x)ds.$$

Fix $x \in \mathbb{R}^d$. If $0 < s < t$, then

$$\begin{aligned}
&|\frac{((vQ^*_{t-s}\varphi) * (D_i k_s))(x + he_j) - ((vQ^*_{t-s}\varphi) * (D_i k_s))(x)}{h}| \\
&\leq ||(D_j(vQ^*_{t-s}\varphi)) * (D_i k_s)|| \\
&\leq ||D_j(vQ^*_{t-s}\varphi)|| ||D_i k_s||_{L^1(\mathbb{R}^d)} \\
&\leq C(1 + \frac{1}{\sqrt{t-s}})\frac{1}{\sqrt{s}}||(D_i k)_s||_{L^1(\mathbb{R}^d)} \\
&= C(1 + \frac{1}{\sqrt{t-s}})\frac{1}{\sqrt{s}}||D_i k||_{L^1(\mathbb{R}^d)} \in L^1((0,t)).
\end{aligned}$$

By Lebesgue's dominated convergence theorem, we have

$$D_j D_i \int_0^t Q_s(vQ^*_{t-s}\varphi)(x)ds = D_j \int_0^t ((vQ^*_{t-s}\varphi) * (D_i k_s))(x)ds = \int_0^t ((D_j(vQ^*_{t-s}\varphi)) * (D_i k_s))(x)ds.$$

Given $a \in \mathbb{R}^d \bigcup\{\infty\}$. Note that

$$\begin{aligned}
D_{j,i} \int_0^t Q_s(vQ^*_{t-s}\varphi)(x)ds &= \int_0^t ((D_j(vQ^*_{t-s}\varphi)) * (D_i k_s))(x)ds \\
&= \int_0^t D_{j,i}((vQ^*_{t-s}\varphi)) * (k_s)(x)ds \\
&= \int_0^t D_{j,i} Q_s(vQ^*_{t-s}\varphi)(x)ds
\end{aligned}$$

and

$$D_{j,i} Q_s(vQ^*_{t-s}\varphi) \in C_0(\mathbb{R}^d) \qquad \forall s \in (0,t).$$

By Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{x \to a} D_{j,i} \int_0^t Q_s(vQ^*_{t-s}\varphi)(x)ds \\
&= \int_0^t \lim_{x \to a} D_{j,i} Q_s(vQ^*_{t-s}\varphi)(x)ds \\
&= \begin{cases} \int_0^t D_{j,i} Q_s(vQ^*_{t-s}\varphi)(a)ds, & \text{if } a \neq \infty \\ 0, & \text{otherwise.} \end{cases} \\
&= \begin{cases} D_{j,i} \int_0^t Q_s(vQ^*_{t-s}\varphi)(a)ds, & \text{if } a \neq \infty \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

2. Since $u_t(x) = Q_t\varphi(x) - \int_0^t Q_s(vQ^*_{t-s}\varphi)(x)ds$, we show that

$$\frac{\partial}{\partial t}(Q_t\varphi - \int_0^t Q_s(vQ^*_{t-s}\varphi)ds) = \frac{1}{2}\Delta u_t - vu_t$$

and

$$t \in [0, \infty) \mapsto \frac{1}{2}\Delta u_t(x) - v(x)u_t(x)$$

is continuous for all $x \in \mathbb{R}^d$. Note that

$$u_t(x) = Q_t\varphi - \int_0^t Q_s(vQ^*_{t-s}\varphi)ds = Q_t\varphi - \int_0^t Q_{t-s}(vQ^*_s\varphi)ds.$$

By Theorem 7.1 and Leibniz integral rule, we get

$$\begin{aligned}\frac{\partial}{\partial t}u_t(x) &= \frac{\partial}{\partial t}Q_t\varphi(x) - v(t)Q_t^*\varphi(x) - \int_0^t \frac{\partial}{\partial t}Q_{t-s}(vQ_s^*\varphi)ds. \\ &= \frac{1}{2}\Delta Q_t\varphi(x) - v(t)Q_t^*\varphi(x) - \int_0^t \frac{1}{2}\Delta Q_{t-s}(vQ_s^*\varphi)ds.\end{aligned}$$

Since we have shown that

$$D_{i,j}\int_0^t Q_{t-s}(vQ_s^*\varphi)ds = D_{i,j}\int_0^t Q_s(vQ_{t-s}^*\varphi)ds = \int_0^t D_{i,j}Q_s(vQ_{t-s}^*\varphi)ds = \int_0^t D_{i,j}Q_{t-s}(vQ_s^*\varphi)ds,$$

we get

$$\frac{\partial}{\partial t}u_t(x) = \frac{1}{2}\Delta(Q_t\varphi(x) - \int_0^t Q_{t-s}(vQ_s^*\varphi)(x)ds) - vQ_t^*\varphi(x) = \frac{1}{2}\Delta u_t(x) - v(x)u_t(x).$$

Now we show that

$$t \in [0,\infty) \mapsto \frac{1}{2}\Delta u_t(x) - v(x)u_t(x)$$

is continuous for all $x \in \mathbb{R}^d$. Fix $x \in \mathbb{R}^d$. By Lebesgus's dominated convergence theorem, we see that

$$t \in [0,\infty) \mapsto u_t(x) = Q_t^*(x) = \boldsymbol{E}_x[\varphi(X_t)e^{-\int_0^t v(X_s)ds}]$$

is continuous. It remain to show that $t \in [0,\infty) \mapsto \Delta u_t(x)$ is continuous. Let $h > 0$. Because

$$D_{i,i}u_t(x) = \int_0^t ((D_j(vQ_{t-s}^*\varphi)) * (D_ik_s))(x)ds \qquad \forall t \geq 0,$$

we get

$$\begin{aligned}&|D_{i,i}u_{t+h}(x) - D_{i,i}u_t(x)| \\ &\leq |\int_0^{t+h}((D_j(vQ_{t+h-s}^*\varphi)) * (D_ik_s))(x)ds - \int_0^t((D_j(vQ_{t+h-s}^*\varphi)) * (D_ik_s))(x)ds| \\ &+ |\int_0^t((D_j(vQ_{t+h-s}^*\varphi)) * (D_ik_s))(x)ds - \int_0^t((D_j(vQ_{t-s}^*\varphi)) * (D_ik_s))(x)ds|. \\ &\leq \int_t^{t+h}||(D_j(vQ_{t+h-s}^*\varphi)) * (D_ik_s)||ds + \int_0^t|((D_j(vQ_{t+h-s}^*\varphi)) - (D_j(vQ_{t-s}^*\varphi))) * (D_ik_s))(x)|ds \\ &= \alpha + \beta.\end{aligned}$$

Note that

$$\begin{aligned}\alpha &\leq \int_t^{t+h}||D_j(vQ_{t+h-s}^*\varphi)||||D_ik_s||_{L^1(\mathbb{R}^d)}ds \\ &\leq \int_t^{t+h}C(1+\frac{1}{\sqrt{t+h-s}})\frac{1}{\sqrt{s}}||D_ik||_{L^1(\mathbb{R}^d)}ds \overset{h\to 0}{\to} 0.\end{aligned}$$

Now we show that $\beta \overset{h\to 0}{\to} 0$. Fix $0 < s < t$. First, we show that

$$|((D_j(vQ_{t+h-s}^*\varphi)) - (D_j(vQ_{t-s}^*\varphi))) * (D_ik_s))(x)| \overset{h\to 0}{\to} 0$$

for all $x \in \mathbb{R}^d$. Note that

$$\begin{aligned}&|((D_j(vQ_{t+h-s}^*\varphi))(x-y) - (D_j(vQ_{t-s}^*\varphi))(x-y)) \times (D_ik_s))(y)| \\ &\leq (||D_j(vQ_{t+h-s}^*\varphi)|| + ||D_j(vQ_{t-s}^*\varphi)||)|(D_ik_s))(y)| \\ &\leq (C(1+\frac{1}{t+h-s}) + C(1+\frac{1}{t-s}))|(D_ik_s))(y)| \\ &\leq 2C(1+\frac{1}{t-s})|(D_ik_s))(y)| \in L^1(\mathbb{R}^d).\end{aligned}$$

By Lebesgue convergence theorem, we have

$$|((D_j(vQ^*_{t+h-s}\varphi)) - (D_j(vQ^*_{t-s}\varphi))) * (D_i k_s))(x)| \stackrel{h\to 0}{\to} 0.$$

Next, we show that $\beta \stackrel{h\to 0}{\to} 0$. Note that

$$\begin{aligned}
&||((D_j(vQ^*_{t+h-s}\varphi)) - (D_j(vQ^*_{t-s}\varphi))) * (D_i k_s))|| \\
&\leq ||((D_j(vQ^*_{t+h-s}\varphi)) - (D_j(vQ^*_{t-s}\varphi)))|| \times ||(D_i k_s))||_{L^1(\mathbb{R}^d)} \\
&\leq (||((D_j(vQ^*_{t+h-s}\varphi))|| + ||(D_j(vQ^*_{t-s}\varphi)))||) \times ||(D_i k_s))||_{L^1(\mathbb{R}^d)} \\
&\leq (C(1 + \frac{1}{\sqrt{t+h-s}}) + C(1 + \frac{1}{\sqrt{t-s}})) \times \frac{1}{\sqrt{s}}||D_i k||_{L^1(\mathbb{R}^d)} \\
&\leq 2C(1 + \frac{1}{\sqrt{t-s}}) \times \frac{1}{\sqrt{s}}||D_i k||_{L^1(\mathbb{R}^d)} \in L^1((0,t)).
\end{aligned}$$

By Lebesgue's convergence theorem, we have $\beta \stackrel{h\to 0}{\to} 0$ and so $t \in [0,\infty) \mapsto \Delta u_t(x)$ is right continuous. By using similar way, we get $t \in [0,\infty) \mapsto \Delta u_t(x)$ is left continuous and, hence, $t \in [0,\infty) \mapsto \Delta u_t(x)$ is continuous which complete the proof.

□

7.6 Exercise 7.29

In this exercise $d = 2$ and $\mathbb{R}^2$ is identified with the complex plane $\mathbb{C}$. Let $\alpha \in (0, 2\pi)$, and consider the open cone

$$\mathscr{C}_\alpha = \{re^{i\theta} : r > 0, \theta \in (-\alpha, \alpha)\}.$$

Set $T := \inf\{t \geq 0 : B_t \notin \mathscr{C}_\alpha\}$.

1. Show that the law of $\log|B_T|$ under $\boldsymbol{P}_1$ is the law of $\beta_{\inf\{t\geq 0:|\gamma_t|=\alpha\}}$, where β and γ are two independent linear Brownian motions started from 0.
2. Verify that, for every $\lambda \in \mathbb{R}$,
$$\boldsymbol{E}_1[e^{i\lambda \log|B_T|}] = \frac{1}{\cosh(\alpha\lambda)}.$$

Proof.

1. By the skew-product representation (Theorem 7.19), there exist two independent linear Brownian motions β and γ that start from 0 under $\boldsymbol{P}_1$ such that
$$B_t = e^{\beta_{H_t} + i\gamma_{H_t}} \qquad \forall t \geq 0 \quad \boldsymbol{P}_1\text{-(a.s.)},$$
where $H_t = \int_0^t \frac{1}{|B_s|^2} ds$. Set $S := \inf\{t \geq 0 : |\gamma_t| = \alpha\}$. Since $(H_t)_{t\geq 0}$ is a continuous increasing process and $H_\infty = \infty$ $\boldsymbol{P}_1$-(a.s.), we have
$$H_T = H_{\inf\{t\geq 0:|\gamma_{H_t}|=\alpha\}} = \inf\{t \geq 0 : |\gamma_t| = \alpha\} = S$$
and so $\log|B_T| = \beta_{H_T} = \beta_S = \beta_{\inf\{t\geq 0:|\gamma_t|=\alpha\}}$ $\boldsymbol{P}_1$-(a.s.).
2. Note that $\cosh(x)$ is an even function. By taking complex conjugate in both side of the identity, we may assume that $\lambda \geq 0$. By problem 1., we get
$$\boldsymbol{E}_1[e^{i\lambda \log|B_T|}] = \boldsymbol{E}_1[e^{i\lambda\beta_S}] = \boldsymbol{E}_1[\boldsymbol{E}_1[e^{i\lambda\beta_S} \mid \sigma(\gamma_t, t \geq 0)]].$$

Recall that, if $X \sim \mathcal{N}(\mu, \sigma)$, then the characteristic function of X is

$$\boldsymbol{E}[e^{i\xi X}] = e^{i\mu\xi - \frac{\sigma^2}{2}\xi^2}.$$

Since β and γ are independent, we get

$$\boldsymbol{E}_1[\boldsymbol{E}_1[e^{i\lambda\beta_S} \mid \sigma(\gamma_t, t \geq 0)]] = \boldsymbol{E}_1[\int_{\mathbb{R}} e^{i\lambda y} \frac{1}{\sqrt{2\pi S}} e^{-\frac{y^2}{2S}} dy] = \boldsymbol{E}_1[e^{-\frac{S}{2}\lambda^2}].$$

Since $(e^{\lambda\gamma_{t\wedge S} - \frac{\lambda^2}{2}(t\wedge S)})_{t\geq 0}$ is an uniformly integrable martingale, we see that

$$\boldsymbol{E}_1[e^{\lambda\gamma_S - \frac{\lambda^2}{2}S}] = 1.$$

and so

$$e^{\lambda\alpha}\boldsymbol{E}_1[e^{-\frac{\lambda^2}{2}S}1_{\{\gamma_S=\alpha\}}] + e^{-\lambda\alpha}\boldsymbol{E}_1[e^{-\frac{\lambda^2}{2}S}1_{\{\gamma_S=-\alpha\}}] = 1.$$

By symmetry ($-\gamma$ is a Brownian motion), we have

$$\boldsymbol{E}_1[e^{-\frac{\lambda^2}{2}S}1_{\{\gamma_S=\alpha\}}] = \boldsymbol{E}_1[e^{-\frac{\lambda^2}{2}S}1_{\{\gamma_S=-\alpha\}}] = \frac{1}{2}\boldsymbol{E}_1[e^{-\frac{\lambda^2}{2}S}]$$

and, hence,

$$\boldsymbol{E}_1[e^{-\frac{\lambda^2}{2}S}] = \frac{1}{\cosh(\alpha\lambda)}.$$

□

Chapter 8

Stochastic Differential Equations

8.1 Exercise 8.9 (Time change method)

We consider the stochastic differential equation

$$E(\sigma, 0): \qquad dX_t = \sigma(X_t)dB_t$$

where the function $\sigma : \mathbb{R} \mapsto \mathbb{R}$ is continuous and there exist constants $\epsilon > 0$ and M such that $\epsilon \leq \sigma \leq M$.

1. In this question and the next one, we assume that X solves $E(\sigma, 0)$ with $X_0 = x$, for every $t \geq 0$,

$$A_t = \int_0^t \sigma(X_s)^2 ds, \quad \tau_t = \inf\{s \geq 0 \mid A_s > t\}.$$

   Justify the equalities

$$\tau_t = \int_0^t \frac{1}{\sigma(X_{\tau_r})^2} dr, \quad A_t = \inf\{s \geq 0 \mid \int_0^s \frac{1}{\sigma(X_{\tau_r})^2} dr > t\}.$$

2. Show that there exists a real Brownian motion $\beta = (\beta_t)_{t \geq 0}$ started from x such that, a.s. for every $t \geq 0$,

$$X_t = \beta_{\inf\{s \geq 0 \mid \int_0^s \sigma(\beta_r)^{-2} dr > t\}}.$$

3. Show that weak existence and weak uniqueness hold for $E(\sigma, 0)$. (Hint: For the existence part, observe that, if X is defined from a Brownian motion β by the formula of question 2., X is (in an appropriate filtration) a continuous local martingale with quadratic variation $\langle X, X \rangle_t = \int_0^t \sigma(X_r)^2 dr$.

Proof.
For the sake of simplicity, sometimes we denote A_t and τ_t as $A(t)$ and $\tau(t)$, respectively.

1. Since $\sigma \in C(\mathbb{R})$ and $A'(t) = \sigma(X_t)^2 \geq \epsilon^2 > 0$, we see that $A(t)$ is strickly increasing and so $A(t)$ is injective. Because $A(\tau(t)) = t$ for all $t \geq 0$, we see that $\tau(t) = A^{-1}(t)$ and, hence, $\tau(t) \in C^1(R)$. By setting $s = \tau(r)$, we get $r = A(s)$, $dr = A'(s)ds$, and so

$$\int_0^t \frac{1}{\sigma(X_{\tau(r)})^2} dr = \int_0^t A'(\tau(r))^{-1} dr = \int_0^{\tau(t)} A'(s)^{-1} A'(s) ds = \tau(t).$$

   Moreover,

$$A(t) = \inf\{s \geq 0 \mid s > A(t)\} = \inf\{s \geq 0 \mid \tau(s) > t\} = \inf\{s \geq 0 \mid \int_0^s \frac{1}{\sigma(X_{\tau(r)})^2} dr > t\}.$$

2. Note that $X_t = X_0 + \int_0^t \sigma(X_s) dB_s$ is a continuous local maringale and

$$\langle X, X \rangle_t = \int_0^t \sigma(X_s)^2 ds = A(t) \qquad \forall t \geq 0.$$

   Since $\sigma \geq \epsilon > 0$, we see that $\langle X, X \rangle_\infty = \infty$ and, hence, there exists a Brownian motion $\beta = (\beta_t)_{t \geq 0}$ such that

$$X_t = \beta_{\langle X, X \rangle_t} = \beta_{A(t)} \qquad \forall t \geq 0 \text{ (a.s.)}.$$

   By problem 1., we get $X_{\tau(r)} = \beta_r$ and

$$X_t = \beta_{A(t)} = \beta_{\inf\{s \geq 0 \mid \int_0^s \frac{1}{\sigma(X_{\tau_r})^2} dr > t\}} = \beta_{\inf\{s \geq 0 \mid \int_0^s \sigma(\beta_r)^{-2} dr > t\}}.$$

3. (a) We prove that weak existence hold for $E(\sigma,0)$. Fix $x\in\mathbb{R}$. We show that there exists a solution $(X,B),(\Omega,\mathscr{F},(\mathscr{C}_t)_{t\ge0},\boldsymbol{P})$ of $E_x(\sigma,0)$. Let $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\ge0},\boldsymbol{P})$ be a filtered probability space ($(\mathscr{F}_t)_{t\ge0}$ is complete) and $(\beta_t)_{t\ge0}$ is a $(\mathscr{F}_t)_{t\ge0}$-Brownian motion such that $\beta_0=x$. Define

$$\tau(t):=\int_0^t\sigma(\beta_r)^{-2}dr \text{ and } A(t):=\inf\{s\ge0\mid\tau(s)>t\}.$$

As the proof in probelm 1., we have $\tau(A(t))=t$ for all $t\ge0$ and $A(t),\tau(t)\in C^1(\mathbb{R})$. Moreover, since $A'(\tau(t))=\tau'(t)^{-1}=\sigma(\beta_t)^2$, we see that

$$A(t)=\int_0^t\sigma(\beta_r)^2dr.$$

Set

$$X_t:=\beta_{A(t)} \text{ and } \mathscr{C}_t:=\mathscr{F}_{A_t}.$$

Then X is continuous. Because $(\mathscr{F}_t)_{t\ge0}$ is complete, we see that $(\mathscr{C}_t)_{t\ge0}$ is complete. Since $A_t<\infty$ (a.s.) and A_t is a $(\mathscr{F}_t)_{t\ge0}$-stopping time for all $t\ge0$, we see that X_t is $\mathscr{C}_t$-measuable for all $t\ge0$. Define

$$Y_t:=\int_0^t\sigma(\beta_s)^{-1}d\beta_s,\quad B_t:=Y_{A_t}.$$

Then $B_0=0$ and B_t is $\mathscr{C}_t$-measurable for all $t\ge0$. Now, we show that $(B_t)_{t\ge0}$ is a $(\mathscr{C}_t)_{t\ge0}$-Brownian motion such that $B_0=0$. It suffices to show that $(B_t)_{t\ge0}$ is a $(\mathscr{C}_t)_{t\ge0}$-martingale and $\langle B,B\rangle_t=t$ for all $t\ge0$. Fix $s\le r<t$. Since Y is a $(\mathscr{F}_t)_{t\ge0}$-continuous local martingale, Y^{A_t} is a $(\mathscr{F}_t)_{t\ge0}$-continuous local martingale. Moreover, since

$$\langle Y^{A_t},Y^{A_t}\rangle_\infty=\int_0^{A_t}\sigma(X_r)^{-2}dr\le\delta^2A_t\le\delta^{-2}M^2t<\infty,$$

we see that Y^{A_t} is a uniform integrable $(\mathscr{F}_t)_{t\ge0}$-martingale. By optional stopping theorem, we get

$$\boldsymbol{E}[B_r\mid\mathscr{C}_s]=\boldsymbol{E}[Y^{A_t}_{A_r}\mid\mathscr{F}_{A_s}]=Y^{A_t}_{A_s}=Y_{A_s}=B_s$$

and so $(B_t)_{t\ge0}$ is a $(\mathscr{C}_t)_{t\ge0}$-martingale. Moreover, since $\langle Y,Y\rangle_t=\tau(t)$, we get

$$\langle B,B\rangle_t=\langle Y,Y\rangle_{A_t}=\tau(A(t))=t\qquad\forall t\ge0$$

and, hence, $(B_t)_{t\ge0}$ is a $(\mathscr{C}_t)_{t\ge0}$-Brownian motion. Observe that

$$\int_0^t\sigma(\beta_{A_s})dY_{A_s}=\int_0^{A_t}\sigma(\beta_s)dY_s.$$

Indeed, since

$$\sum_{i=0}^{n-1}\sigma(\beta_{A_{\frac{it}{n}}})(Y_{A_{\frac{(i+1)t}{n}}}-Y_{A_{\frac{it}{n}}})\xrightarrow{P}\int_0^t\sigma(\beta_{A_s})dY_{A_s}\text{ as }n\to\infty,$$

there exists $\{n_k\}$ such that

$$\sum_{i=0}^{n_k-1}\sigma(\beta_{A_{\frac{it}{n_k}}})(Y_{A_{\frac{(i+1)t}{n_k}}}-Y_{A_{\frac{it}{n_k}}})\overset{(a.s.)}{\to}\int_0^t\sigma(\beta_{A_s})dY_{A_s}\text{ as }n\to\infty.$$

Because

$$\sum_{i=0}^{n_k-1}\sigma(\beta_{A_{\frac{it}{n_k}}})(Y_{A_{\frac{(i+1)t}{n_k}}}-Y_{A_{\frac{it}{n_k}}})\overset{(a.s.)}{\to}\int_0^{A_t}\sigma(\beta_s)dY_s\text{ as }n\to\infty,$$

we have

$$\int_0^t \sigma(\beta_{A_s})dY_{A_s} = \int_0^{A_t} \sigma(\beta_s)dY_s \text{ (a.s.)}$$

and so

$$\int_0^t \sigma(X_s)dB_s = \int_0^t \sigma(\beta_{Y_{A_s}})dY_{A_s} = \int_0^{A_t} \sigma(\beta_s)dY_s = \int_0^{A_t} \sigma(\beta_s)\sigma(\beta_s)^{-1}d\beta_s = \beta_{A_t} - \beta_0 = X_t - x.$$

Therefore $(X,B),(\Omega,\mathscr{F},(\mathscr{C}_t)_{t\geq 0},\boldsymbol{P})$ is a soltion of $E_x(\sigma,0)$.

(b) We prove that weak uniqueness hold for $E(\sigma,0)$. Fix $x \in \mathbb{R}$. Let $(X,B),(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq 0},\boldsymbol{P})$ be a soltion of $E_x(\sigma,0)$. By problem 2., there exists a Borwnian motion $(\beta_t)_{t\geq 0}$ such that

$$X_t = \beta_{\inf\{s\geq 0 | \int_0^s \sigma(\beta_r)^{-2}dr > t\}} \text{ (a.s.)} \quad \forall t \geq 0.$$

Define $\Phi_t : C(\mathbb{R}_+,\mathbb{R}) \mapsto \mathbb{R}$ by

$$\Phi_t(b) := b(\inf\{s \geq 0 \mid \int_0^s \sigma(b(r))^{-2}dr > t\}).$$

Let $f_i : \mathbb{R} \mapsto \mathbb{R}$ be bounded measuable functions for $i = 1,2,...,m$ and $0 \leq t_1 < t_2 < ... < t_m$. Then

$$\begin{aligned}\boldsymbol{E}[f_1(X_{t_1})f_2(X_{t_2})...f_m(X_{t_m})] &= \boldsymbol{E}[f_1(\Phi_{t_1}(\beta))f_2(\Phi_{t_2}(\beta))...f_m(\Phi_{t_m}(\beta))] \\ &= \int f_1(\Phi_{t_1}(w))f_2(\Phi_{t_2}(w))...f_m(\Phi_{t_m}(w))W(dw),\end{aligned}$$

where $W(dw)$ is the Wiener measure on $C(\mathbb{R}_+,\mathbb{R})$. Thus, weak uniqueness hold for $E_x(\sigma,0)$.

□

8.2 Exercise 8.10

We consider the stochastic differential equation

$$E(\sigma,b): \quad dX_t = \sigma(X_t)dB_t + b(X_t)dt$$

where the function $\sigma, b : \mathbb{R} \mapsto \mathbb{R}$ are bounded and continuous, and such that $\int_{\mathbb{R}} |b(x)|dx < \infty$ and $\sigma \geq \epsilon$ for some $\epsilon > 0$.

1. Let X be a solution of $E(\sigma,b)$. Show that there exists a monotone increasing function $F : \mathbb{R} \mapsto \mathbb{R}$, which is also twice continuously differentiable, such that $F(X_t)$. Give an explicit formula for F in terms of σ and b.
2. Show that the process $Y_t = F(X_t)$ solves a stochastic differential equation of the form $dY_t = \sigma'(Y_t)dB_t$, with a function σ' to be determined.
3. Using the result of the preceding exercise, show that weak existence and weak uniqueness hold for $E(\sigma,b)$. Show that pathwise uniqueness also holds if σ is Lipschitz.

Proof.
For the sake of simplicity, we define $||f||_u := \sup_{x\in\mathbb{R}} |f(x)|$ and $||f||_{L^1(\mathbb{R})} := \int_{\mathbb{R}} |f(x)|dx$.

1. Suppose $F \in C^2(\mathbb{R})$. By Itô's formula, we get

$$\begin{aligned}F(X_t) &= F(X_0) + \int_0^t F'(Xs)dX_s + \frac{1}{2}\int_0^t F''(X_s)d\langle X,X\rangle_s \\ &= F(X_0) + \int_0^t F'(Xs)\sigma(X_s)dB_s + \int_0^t F'(X_s)b(X_s)ds + \frac{1}{2}\int_0^t F''(X_s)\sigma(X_s)^2ds.\end{aligned}$$

Define $F : \mathbb{R} \mapsto \mathbb{R}$ by

$$F(x) := \int_0^x e^{-\int_0^s \frac{2b(r)}{\sigma(r)^2} dr} ds.$$

Note that

$$F'(x) = e^{-\int_0^x \frac{2b(r)}{\sigma(r)^2} dr}, F''(x) = -e^{-\int_0^x \frac{2b(r)}{\sigma(r)^2} dr} \frac{2b(x)}{\sigma(x)^2},$$

and

$$2F'(x)b(x) + F''(x)\sigma(x)^2 = 0.$$

Then F is a monotone increasing, twice continuously differentiable function and

$$F(X_t) = F(X_0) + \int_0^t F'(X_s)\sigma(X_s)dB_s$$

is a continuous local martingale. Since

$$\boldsymbol{E}[\langle F(X), F(X)\rangle_t] = \boldsymbol{E}[\int_0^t F'(X_s)^2\sigma(X_s)^2 ds] \leq t \times ||(F')^2||_u ||\sigma^2||_u \leq t \times e^{\frac{4}{\epsilon^2}\int_{\mathbb{R}} |b(r)|dr} ||\sigma^2||_u < \infty,$$

we see that $(F(X_t))_{t\geq 0}$ is a martingale.

2. Since $F'(x) > 0$ for all $x \in \mathbb{R}$, F is strictly increasing and so F^{-1} exist. Observe that

$$e^{-\int_0^s \frac{2b(r)}{\sigma(r)^2} dr} \geq e^{-|\int_0^s \frac{2b(r)}{\sigma(r)^2} dr|} \geq e^{-\frac{2}{\epsilon^2}||b||_{L^1(\mathbb{R})}} > 0.$$

Then

$$\lim_{x\to\pm\infty} F(x) = \lim_{x\to\pm\infty} \int_0^x e^{-\int_0^s \frac{2b(r)}{\sigma(r)^2} dr} ds = \pm\infty$$

and so the domain of F^{-1} is $\mathbb{R}$. Moreover, since $F \in C^2(\mathbb{R})$, we see that $F^{-1} \in C^2(\mathbb{R})$. Set

$$H(x) := F'(x)\sigma(x) \text{ and } \sigma'(y) := H(F^{-1}(y)).$$

Then

$$E'(\sigma') : \quad dY_t = H(X_t)dB_t = H(F^{-1}(Y_t))dB_t = \sigma'(Y_t)dB_t.$$

3. First, we show that weak existence and weak uniqueness hold for $E'(\sigma')$. By Exercise 8.9, it suffices to show that $\sigma' : \mathbb{R} \mapsto \mathbb{R}$ is a continuous function and the exist $\epsilon, M > 0$ such that $\delta \leq \sigma'(y) \leq M$ for all $y \in \mathbb{R}$. Since F^{-1} and H are continuous,

$$H(x) = e^{-\int_0^x \frac{2b(s)}{\sigma(s)^2} ds}\sigma(x) \geq e^{-|\int_0^x \frac{2b(s)}{\sigma(s)^2} ds|}\sigma(x) \geq e^{-\frac{2}{\epsilon^2}||b||_{L^1(\mathbb{R})}}\epsilon := \delta > 0 \qquad \forall x \in \mathbb{R},$$

and

$$H(x) = e^{-\int_0^x \frac{2b(s)}{\sigma(s)^2} ds}\sigma(x) \leq e^{|\int_0^x \frac{2b(s)}{\sigma(s)^2} ds|}\sigma(x) \leq e^{\frac{2}{\epsilon^2}||b||_{L^1(\mathbb{R})}}||\sigma||_u := M < \infty \qquad \forall x \in \mathbb{R},$$

we see that $\sigma'(y) = H(F^{-1}(y))$ is continuous and $\delta \leq \sigma'(x) \leq M$ for all $x \in \mathbb{R}$. Thus, weak existence and weak uniqueness hold for $E'(\sigma')$.
Now, we show that weak existence hold for $E(\sigma, b)$. Fix $x \in \mathbb{R}$. Set $y = F(x)$. There exists a solution $(Y, B), (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \boldsymbol{P})$ of $E'_y(\sigma')$. Define

$$X_t := F^{-1}(Y_t).$$

By Itô's formula, we get

$$X_t = x + \int_0^t \frac{dF^{-1}}{dy}(Y_s)dY_s + \frac{1}{2}\int_0^t \frac{d^2F^{-1}}{dy^2}(Y_s)d\langle Y, Y\rangle_s.$$

By $F^{-1}(F(x)) = x$, we get

$$\frac{dF^{-1}}{dy}(F(x))\frac{dF}{dx}(x) = 1 \text{ and } \frac{d^2F^{-1}}{dy^2}(F(x))(\frac{dF}{dx}(x))^2 + \frac{dF^{-1}}{dy}(F(x))\frac{d^2F}{dx^2}(x) = 0.$$

Thus,

$$\frac{dF^{-1}}{dy}(Y_s) = \frac{dF^{-1}}{dy}(F(X_s)) = (\frac{dF}{dx}(X_s))^{-1} = e^{\int_0^{X_s} \frac{2b(r)}{\sigma(r)^2}dr}$$

and

$$\begin{aligned}\frac{d^2F^{-1}}{dy^2}(Y_s) &= \frac{d^2F^{-1}}{dy^2}(F(X_s)) = (-\frac{dF^{-1}}{dy}(F(X_s))\frac{d^2F}{dx^2}(X_s)) \times (\frac{dF}{dx}(X_s))^{-2} \\ &= (-e^{\int_0^{X_s} \frac{2b(r)}{\sigma(r)^2}dr} \times -e^{-\int_0^{X_s} \frac{2b(r)}{\sigma(r)^2}dr}(\frac{2b(X_s)}{\sigma(X_s)^2})) \times e^{2\int_0^{X_s} \frac{2b(r)}{\sigma(r)^2}dr} \\ &= \frac{2b(X_s)}{\sigma(X_s)^2}e^{2\int_0^{X_s} \frac{2b(r)}{\sigma(r)^2}dr}.\end{aligned}$$

By

$$dY_t = \sigma'(Y_t)dB_t = H(F^{-1}(Y_t))dB_t = H(X_t)dB_t = e^{-\int_0^{X_t} \frac{2b(r)}{\sigma(r)^2}dr}\sigma(X_t)dB_t,$$

we get

$$\begin{aligned}X_t &= x + \int_0^t \frac{dF^{-1}}{dy}(Y_s)dY_s + \frac{1}{2}\int_0^t \frac{d^2F^{-1}}{dy^2}(Y_s)d\langle Y, Y\rangle_s \\ &= x + \int_0^t e^{\int_0^{X_s} \frac{2b(r)}{\sigma(r)^2}dr}e^{-\int_0^{X_s} \frac{2b(r)}{\sigma(r)^2}dr}\sigma(X_s)dB_s + \frac{1}{2}\int_0^t \frac{2b(X_s)}{\sigma(X_s)^2}e^{2\int_0^{X_s} \frac{2b(r)}{\sigma(r)^2}dr}e^{-2\int_0^{X_s} \frac{2b(r)}{\sigma(r)^2}dr}\sigma(X_s)^2ds \\ &= x + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds\end{aligned}$$

and so $(X, B), (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \boldsymbol{P})$ is a solution of $E_x(\sigma, b)$.

Now, we show that weak uniqueness hold for $E(\sigma, b)$. Fix $x \in \mathbb{R}$ and $y = F(x)$. Let $(X, B), (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \boldsymbol{P})$ and $(X', B'), (\Omega', \mathscr{F}', (\mathscr{F}'_t)_{t\geq 0}, \boldsymbol{P}')$ be solutions of $E_x(\sigma, b)$. By problem 2., we see that $(Y_t)_{t\geq 0} := (F(X_t))_{t\geq 0}$ and $(Y'_t)_{t\geq 0} := (F(X'_t))_{t\geq 0}$ are solutions of $E'_y(\sigma')$. Since weak uniqueness hold for $E'(\sigma')$ and F is injective, we get

$$\begin{aligned}\boldsymbol{E}[1_{X_{t_1}\in\Gamma_1}...1_{X_{t_k}\in\Gamma_k}] &= \boldsymbol{E}[1_{Y_{t_1}\in F(\Gamma_1)}...1_{Y_{t_k}\in F(\Gamma_k)}] \\ &= \boldsymbol{E}'[1_{Y'_{t_1}\in F(\Gamma_1)}...1_{Y'_{t_k}\in F(\Gamma_k)}] \\ &= \boldsymbol{E}'[1_{X'_{t_1}\in\Gamma_1}...1_{X'_{t_k}\in\Gamma_k}]\end{aligned}$$

and, hence, weak uniqueness hold for $E(\sigma, b)$.

Finally, we show that pathwise uniqueness hold for $E(\sigma, b)$ whenever σ is Lipshitz. To show this, it suffices to show that σ' is Lipshitz. Indeed, by Theorem 8.3 and σ' is Lipshitz, we see that pathwise uniquness hold for $E'(\sigma')$. Let X and X' are solutions of $E(\sigma, b)$ under $(\Omega, \mathscr{F}, (\mathscr{F})_{t\geq 0}, \boldsymbol{P})$ and $(\mathscr{F})_{t\geq 0}$-Brownian motion $(B_t)_{t\geq 0}$ started from 0 such that $\boldsymbol{P}(X_0 = X'_0) = 1$. By problem 2., we get $(Y_t)_{t\geq 0} := (F(X_t))_{t\geq 0}$ and $(Y'_t)_{t\geq 0} := (F(X'_t))_{t\geq 0}$ are solutions of $E'(\sigma')$ such that $\boldsymbol{P}(Y_0 = Y'_0) = 1$ and so

$$F(X_t) = Y_t = Y'_t = F(X'_t) \qquad \forall t \geq 0 \quad \boldsymbol{P}\text{-(a.s.)}.$$

Since F is injective, we get

$$X_t = X'_t \qquad \forall t \geq 0 \quad \boldsymbol{P}\text{-(a.s.)}.$$

Now, we show that $\sigma'(y) := H(F^{-1}(y))$ is Lipshitz whenever σ is Lipshitz. Choose $C > 0$ such that

$$|\sigma(x_1) - \sigma(x_2)| \leq C|x_1 - x_2|.$$

Fix real numbers y_1 and y_2. Set $x_i = F^{-1}(y_i)$ for $i = 1, 2$. Note that

$$||F'||_u \le e^{\frac{2}{\epsilon^2}||b||_{L^1(\mathbb{R})}} < \infty.$$

and

$$||F''||_u \le \frac{2||b||_u}{\epsilon^2} e^{\frac{2}{\epsilon^2}||b||_{L^1(\mathbb{R})}} < \infty.$$

By mean value theorem, we get

$$\begin{aligned}
|\sigma'(y_1) - \sigma'(y_2)| = |H(x_1) - H(x_2)| &= |F'(x_1)\sigma(x_1) - F'(x_2)\sigma(x_2)| \\
&\le |F'(x_1)\sigma(x_1) - F'(x_1)\sigma(x_2)| + |F'(x_1)\sigma(x_2) - F'(x_2)\sigma(x_2)| \\
&\le ||F'||_u C|x_1 - x_2| + ||\sigma||_u||F''||_u|x_1 - x_2| := C'|x_1 - x_2|,
\end{aligned}$$

where $C' := (||F'||_u C) \vee (||\sigma||_u||F''||_u)$. Because

$$|\frac{dF^{-1}}{dy}(y)| = |F'(F^{-1}(y))| \le ||(F')^{-1}||_u = \sup_{x\in\mathbb{R}} e^{\int_0^x \frac{2b(r)}{\sigma(r)^2}dr} \le e^{\frac{2}{\epsilon^2}||b||_{L^1(\mathbb{R})}} < \infty,$$

we get

$$|x_2 - x_1| = |F^{-1}(y_2) - F^{-1}(y_1)|^{-1} \le ||\frac{dF^{-1}}{dy}||_u|y_2 - y_1|$$

and so

$$|\sigma'(y_1) - \sigma'(y_2)| \le C|y_1 - y_2|,$$

where $C := ||\frac{dF^{-1}}{dy}||_u C'$.

$\square$

8.3 Exercise 8.11

We suppose that, for every $x \in \mathbb{R}_+$, one can construct on the same filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F})_{t\ge0}, \boldsymbol{P})$ a process X^x taking nonnegative values, which solves the stochastic differential equation

$$\begin{cases} dX_t = \sqrt{2X_t}dB_t \\ X_0 = x. \end{cases}$$

and that the processes X^x are Markov processes with values in $\mathbb{R}_+$, with the same semigroup $(Q_t)_{t\ge0}$, with respect to the filtration $(\mathscr{F}_t)_{t\ge0}$ (This is, of course, close to Theorem 8.6, which however cannot be applied directly because the function $\sqrt{2x}$ is not Lipschitz.)

1. We fix $x \in \mathbb{R}_+$, and real $T > 0$. We set, for every $t \in [0, T]$

$$M_t = e^{-\frac{\lambda X_t^x}{1+\lambda(T-t)}}.$$

Show that the process $(M_{t\wedge T})$ is a martingale.

2. Show that $(Q_t)_{t\ge0}$ is the semigroup of Feller's branching diffusion (see the end of Chap. 6).

Proof.
Note that $\lambda \ge 0$.

1. Fix $T > 0$. By Itô's formula, we get

$$\begin{aligned}
M_t &= e^{\frac{-\lambda X_t^x}{1+\lambda(T-t)}} \\
&= e^{\frac{-\lambda x}{1+\lambda(T)}} + \int_0^t \frac{-\lambda}{1+\lambda(T-s)} e^{\frac{-\lambda X_s^x}{1+\lambda(T-s)}} dX_s^x + \int_0^t \frac{-\lambda^2 X_s^x}{(1+\lambda(T-s))^2} e^{\frac{-\lambda X_s^x}{1+\lambda(T-s)}} ds \\
&+ \frac{1}{2}\int_0^t \frac{\lambda^2}{(1+\lambda(T-s))^2} e^{\frac{-\lambda X_t^x}{1+\lambda(T-t)}} d\langle X^x, X^x\rangle_s \\
&= e^{\frac{-\lambda x}{1+\lambda(T)}} + \int_0^t \frac{-\lambda}{1+\lambda(T-s)} e^{\frac{-\lambda X_s^x}{1+\lambda(T-s)}} \sqrt{2X_s^x} dB_s + \int_0^t \frac{-\lambda^2 X_s^x}{(1+\lambda(T-s))^2} e^{\frac{-\lambda X_s^x}{1+\lambda(T-s)}} ds \\
&+ \frac{1}{2}\int_0^t \frac{\lambda^2}{(1+\lambda(T-s))^2} e^{\frac{-\lambda X_t^x}{1+\lambda(T-t)}} (2X_s^x) ds \\
&= e^{\frac{-\lambda x}{1+\lambda(T)}} + \int_0^t \frac{-\lambda}{1+\lambda(T-s)} e^{\frac{-\lambda X_s^x}{1+\lambda(T-s)}} \sqrt{2X_s^x} dB_s
\end{aligned}$$

is a continuous local martingale. Since $x \le e^x$ for all $x \ge 0$, we have

$$\begin{aligned}
\boldsymbol{E}[\langle M, M\rangle_T] = \boldsymbol{E}[\int_0^T \frac{\lambda^2 2X_s^x}{(1+\lambda(T-s))^2} e^{\frac{-2\lambda X_s^x}{1+\lambda(T-s)}} ds] &\le \boldsymbol{E}[\int_0^T \frac{\lambda}{1+\lambda(T-s)} ds] \\
= \int_0^T \frac{\lambda}{1+\lambda(T-s)} ds < \infty
\end{aligned}$$

and so $(M_{t\wedge T})_{t\ge 0}$ is an uniformly integrable martingale.

2. Fix $T > 0$. By optional stopping theorem and problem 1., we get

$$e^{\frac{-\lambda x}{1+\lambda T}} = \boldsymbol{E}[M_{0\wedge T}] = \boldsymbol{E}[M_{\infty\wedge T}] = \boldsymbol{E}[e^{-\lambda X_T^x}] = \int e^{-\lambda y} Q_T(x, dy).$$

Thus, we have

$$\int e^{-\lambda y} Q_t(x, dy) = e^{-x\psi_t(\lambda)},$$

where $\psi_t(\lambda) := \frac{\lambda}{1+\lambda t}$ and $t > 0$. By the last example in chapter 6., we see that $(Q_t)_{t\ge 0}$ is the semigroup of Feller's branching diffusion.

□

8.4 Exercise 8.12

We consider two sequences $(\sigma_n)_{n\ge 1}$ and $(b_n)_{n\ge 1}$ of real functions defined on $\mathbb{R}$. We assume that:

1. There exists a constant $C > 0$ such that $|\sigma_n(x)| \vee |b_n(x)| \le C$ for every $n \ge 1$ and $x \in \mathbb{R}$.
2. There exists a constant $K > 0$ such that, for every $n \ge 1$ and $x, y \in \mathbb{R}$,

$$|\sigma_n(x) - \sigma_n(y)| \vee |b_n(x) - b_n(y)| \le K|x - y|.$$

Let B be an $(\mathscr{F}_t)_{t\ge 0}$-Brownian motion and, for every $n \ge 1$, let X^n be the unique adapted process satisfying

$$X_t^n = \int_0^t \sigma_n(X_s^n) dB_s + \int_0^t b_n(X_s^n) ds.$$

1. Let $T > 0$. Show that there exists a constant $A > 0$ such that, for every real $M > 0$ and for every $n \ge 1$,

$$\boldsymbol{P}(\sup_{t\le T} |X_t^n| \ge M) \le \frac{A}{M^2}.$$

2. We assume that the sequences $\{\sigma_n\}$ and $\{b_n\}$ converge uniformly on every compact subset of $\mathbb{R}$ to limiting functions denoted by σ and b respectively. Justify the existence of an adapted process $X = (X_t)_{t\geq 0}$ with continuous sample paths, such that

$$X_t = \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds,$$

then show that there exists a constant A' such that, for every real $M > 0$, for every $t \in [0, T]$ and $n \geq 1$,

$$\begin{aligned}\boldsymbol{E}[\sup_{s\leq t}|X^n_s - X_s|^2] &\leq 4(4+T)K^2\int_0^t \boldsymbol{E}[|X^n_s - X_s|^2]ds + \frac{A'}{M^2}\\ &\quad + 4T(4\sup_{|x|\leq M}|\sigma_n(x)-\sigma(x)|^2 + T\sup_{|x|\leq M}|b_n(x)-b(x)|^2).\end{aligned}$$

3. Infer from the preceding question that

$$\lim_{n\to\infty}\boldsymbol{E}[\sup_{s\leq T}|X^n_s - X_s|^2] = 0.$$

Proof.

1. Fix $T > 0$ and $M > 0$. By Burkholder–Davis–Gundy inequalities (Theorem 5.16), we get

$$\begin{aligned}\boldsymbol{P}(\sup_{t\leq T}|X^n_t| \geq M) &\leq \frac{1}{M^2}\boldsymbol{E}[\sup_{t\leq T}|X^n_t|^2] \leq \frac{C_2}{M^2}\boldsymbol{E}[\langle X^n, X^n\rangle_T]\\ &= \frac{C_2}{M^2}\boldsymbol{E}[\int_0^T \sigma_n(X^n_s)^2 ds] \leq \frac{C_2 T C^2}{M^2} := \frac{A}{M^2},\end{aligned}$$

where $A = A(T) := C_2 T C^2$.

2. Since $\sigma_n \to \sigma$ and $b_n \to b$ uniformly on every compact subset of $\mathbb{R}$, we get

$$|\sigma(x)-\sigma(y)| \vee |b(x)-b(y)| \leq K|x-y| \quad \forall x, y \in \mathbb{R},$$

and

$$|\sigma(x)| \vee |b(x)| \leq C \quad \forall x \in \mathbb{R}.$$

By Theorem 8.5, there exists an adapted process $X = (X_t)_{t\geq 0}$ with continuous sample paths, such that

$$X_t = \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds \quad \forall t \geq 0 \quad \boldsymbol{P}\text{-(a.s.)}.$$

By similar argument, we have

$$\boldsymbol{P}(\sup_{t\leq T}|X_t| \geq M) \leq \frac{A(T)}{M^2} \quad \forall T > 0 \text{ and } M > 0.$$

Fix $T > 0$, $t \in [0, T]$, and $M > 0$. Now, we show that

$$\begin{aligned}\boldsymbol{E}[\sup_{s\leq t}|X^n_s - X_s|^2] &\leq 2\times 4^2K^2(4+T)\int_0^t \boldsymbol{E}[|X^n_s - X_s|^2]ds + \frac{(4+T)T4^3C^2 2A(T)}{M^2}\\ &\quad + 4T(4^2\sup_{|x|\leq M}|\sigma_n(x)-\sigma(x)|^2 + 4T\sup_{|x|\leq M}|b_n(x)-b(x)|^2)\end{aligned}$$

for all $n \geq 1$. (Note that this upper bound is larger then the upper bound in problem 2. However, this doesn't affect of the proof of problem 3.) Let $n \geq 1$. Then

$$\boldsymbol{E}[\sup_{s\leq t}|X^n_s - X_s|^2] \leq 4\boldsymbol{E}[\sup_{s\leq t}|\int_0^s \sigma_n(X^n_r) - \sigma(X_r)dB_r|^2] + 4\boldsymbol{E}[\sup_{s\leq t}|\int_0^s b_n(X^n_r) - b(X_r)dr|^2].$$

Since $|\sigma_n(x)| \vee |\sigma(x)| \leq C$ for all $x \in \mathbb{R}$, we see that $(\int_0^s \sigma_n(X^n_r) - \sigma(X_r)dB_r)_{s\geq 0}$ is a martingale. By Doob's inequality in L^2 and Hölder's inequality, we have

$$\begin{aligned}
&4\boldsymbol{E}[\sup_{s\leq t}|\int_0^s \sigma_n(X^n_r) - \sigma(X_r)dB_r|^2] + 4\boldsymbol{E}[\sup_{s\leq t}|\int_0^s b_n(X^n_r) - b(X_r)dr|^2] \\
&\leq 4\times 4\boldsymbol{E}[|\int_0^t \sigma_n(X^n_s) - \sigma(X_s)dB_s|^2] + 4T\boldsymbol{E}[\int_0^t |b_n(X^n_s) - b(X_s)|^2 ds] \\
&\leq 4\times 4\boldsymbol{E}[\int_0^t |\sigma_n(X^n_s) - \sigma(X_s)|^2 ds] + 4T\boldsymbol{E}[\int_0^t |b_n(X^n_s) - b(X_s)|^2 ds] \\
&\leq 4\times 4\boldsymbol{E}[\int_0^t |\sigma_n(X^n_s) - \sigma(X_s)|^2 ds 1_{\{\sup_{s\leq T}|X^n_s|\geq M\}\bigcup\{\sup_{s\leq T}|X_s|\geq M\}}] \\
&+ 4\times 4\boldsymbol{E}[\int_0^t |\sigma_n(X^n_s) - \sigma(X_s)|^2 ds 1_{\{\sup_{s\leq T}|X^n_s|\leq M\}\bigcap\{\sup_{s\leq T}|X_s|\leq M\}}] \\
&+ 4\times T\boldsymbol{E}[\int_0^t |b_n(X^n_s) - b(X_s)|^2 ds 1_{\{\sup_{s\leq T}|X^n_s|\geq M\}\bigcup\{\sup_{s\leq T}|X_s|\geq M\}}] \\
&+ 4\times T\boldsymbol{E}[\int_0^t |b_n(X^n_s) - b(X_s)|^2 ds 1_{\{\sup_{s\leq T}|X^n_s|\leq M\}\bigcap\{\sup_{s\leq T}|X_s|\leq M\}}] \\
&\leq 4\times 4\boldsymbol{E}[\int_0^t 4|\sigma_n(X^n_s) - \sigma_n(X_s)|^2 ds 1_{\{\sup_{s\leq T}|X^n_s|\geq M\}\bigcup\{\sup_{s\leq T}|X_s|\geq M\}}] \\
&+ 4\times 4\boldsymbol{E}[\int_0^t 4|\sigma_n(X_s) - \sigma(X_s)|^2 ds 1_{\{\sup_{s\leq T}|X^n_s|\geq M\}\bigcup\{\sup_{s\leq T}|X_s|\geq M\}}] \\
&+ 4\times 4\boldsymbol{E}[\int_0^t 4|\sigma_n(X^n_s) - \sigma_n(X_s)|^2 ds 1_{\{\sup_{s\leq T}|X^n_s|\leq M\}\bigcap\{\sup_{s\leq T}|X_s|\leq M\}}] \\
&+ 4\times 4\boldsymbol{E}[\int_0^t 4|\sigma_n(X_s) - \sigma(X_s)|^2 ds 1_{\{\sup_{s\leq T}|X^n_s|\leq M\}\bigcap\{\sup_{s\leq T}|X_s|\leq M\}}] \\
&+ 4\times T\boldsymbol{E}[\int_0^t 4|b_n(X^n_s) - b_n(X_s)|^2 ds 1_{\{\sup_{s\leq T}|X^n_s|\geq M\}\bigcup\{\sup_{s\leq T}|X_s|\geq M\}}] \\
&+ 4\times T\boldsymbol{E}[\int_0^t 4|b_n(X_s) - b(X_s)|^2 ds 1_{\{\sup_{s\leq T}|X^n_s|\geq M\}\bigcup\{\sup_{s\leq T}|X_s|\geq M\}}] \\
&+ 4\times T\boldsymbol{E}[\int_0^t 4|b_n(X^n_s) - b_n(X_s)|^2 ds 1_{\{\sup_{s\leq T}|X^n_s|\leq M\}\bigcap\{\sup_{s\leq T}|X_s|\leq M\}}] \\
&+ 4\times T\boldsymbol{E}[\int_0^t 4|b_n(X_s) - b(X_s)|^2 ds 1_{\{\sup_{s\leq T}|X^n_s|\leq M\}\bigcap\{\sup_{s\leq T}|X_s|\leq M\}}]
\end{aligned}$$

$$
\begin{aligned}
&\leq 4^2\boldsymbol{E}[\int_0^t 4K^2|X_s^n-X_s|^2ds] + 4^3(T4C^2\boldsymbol{P}(\{\sup_{s\leq T}|X_s^n|\geq M\}\bigcup\{\sup_{s\leq T}|X_s|\geq M\}))\\
&+4^2\boldsymbol{E}[\int_0^t 4K^2|X_s^n-X_s|^2ds] + 4^3T\sup_{|x|\leq M}|\sigma_n(x)-\sigma(x)|^2\\
&+4T\boldsymbol{E}[\int_0^t 4K^2|X_s^n-X_s|^2ds] + 4^2T(T4C^2\boldsymbol{P}(\{\sup_{s\leq T}|X_s^n|\geq M\}\bigcup\{\sup_{s\leq T}|X_s|\geq M\}))\\
&+4T\boldsymbol{E}[\int_0^t 4K^2|X_s^n-X_s|^2ds] + 4^2T\times T\sup_{|x|\leq M}|b_n(x)-b(x)|^2\\
&= 2\times 4^2K^2(4+T)\int_0^t \boldsymbol{E}[|X_s^n-X_s|^2]ds + (4+T)T4^3C^2\boldsymbol{P}(\{\sup_{s\leq T}|X_s^n|\geq M\}\bigcup\{\sup_{s\leq T}|X_s|\geq M\})\\
&+4T(4^2\sup_{|x|\leq M}|\sigma_n(x)-\sigma(x)|^2+4T\sup_{|x|\leq M}|b_n(x)-b(x)|^2)\\
&= 2\times 4^2K^2(4+T)\int_0^t \boldsymbol{E}[|X_s^n-X_s|^2]ds + (4+T)T4^3C^2(\boldsymbol{P}(\sup_{s\leq T}|X_s^n|\geq M)+\boldsymbol{P}(\sup_{s\leq T}|X_s|\geq M))\\
&+4T(4^2\sup_{|x|\leq M}|\sigma_n(x)-\sigma(x)|^2+4T\sup_{|x|\leq M}|b_n(x)-b(x)|^2)\\
&= 2\times 4^2K^2(4+T)\int_0^t \boldsymbol{E}[|X_s^n-X_s|^2]ds + (4+T)T4^3C^2(2\frac{A(T)}{M^2})\\
&+4T(4^2\sup_{|x|\leq M}|\sigma_n(x)-\sigma(x)|^2+4T\sup_{|x|\leq M}|b_n(x)-b(x)|^2).
\end{aligned}
$$

3. Fix $M, T > 0$ and $n \geq 1$. By problem 2., we get

$$
\begin{aligned}
\boldsymbol{E}[\sup_{s\leq t}|X_s^n-X_s|^2] &\leq 2\times 4^2K^2(4+T)\int_0^t \boldsymbol{E}[|X_s^n-X_s|^2]ds + (4+T)T4^3C^2(2\frac{A(T)}{M^2})\\
&+4T(4^2\sup_{|x|\leq M}|\sigma_n(x)-\sigma(x)|^2+4T\sup_{|x|\leq M}|b_n(x)-b(x)|^2)\\
&\leq 2\times 4^2K^2(4+T)\int_0^t \boldsymbol{E}[\sup_{r\leq s}|X_r^n-X_r|^2]ds + (4+T)T4^3C^2(2\frac{A(T)}{M^2})\\
&+4T(4^2\sup_{|x|\leq M}|\sigma_n(x)-\sigma(x)|^2+4T\sup_{|x|\leq M}|b_n(x)-b(x)|^2)
\end{aligned}
$$

for all $t \in [0, T]$. Define $g : [0, T] \mapsto \mathbb{R}_+$ by

$$
g(t) := \boldsymbol{E}[\sup_{s\leq t}|X_s^n - X_s|^2].
$$

Set positive real numbers

$$
a := (4+T)T4^3C^2(2\frac{A(T)}{M^2}) + 4T(4^2\sup_{|x|\leq M}|\sigma_n(x)-\sigma(x)|^2+4T\sup_{|x|\leq M}|b_n(x)-b(x)|^2)
$$

and

$$
b := 2\times 4^2K^2(4+T).
$$

Then we have

$$
g(t) \leq b\int_0^t g(s)ds + a \quad \forall t \in [0, T].
$$

By Burkholder–Davis–Gundy inequalities (Theorem 5.16) and Hölder's inequality, we get

$$\begin{aligned}
|g(t)| &= \boldsymbol{E}[\sup_{s\le t}|X^n_s - X_s|^2] \\
&\le 4\boldsymbol{E}[\sup_{s\le t}|\int_0^s \sigma_n(X^n_r) - \sigma(X_r)dBr|^2] + 4\boldsymbol{E}[\sup_{s\le t}|\int_0^s b_n(X^n_r) - b(X_r)dr|^2] \\
&\le 4C_2\boldsymbol{E}[\int_0^t |\sigma_n(X^n_s) - \sigma(X_s)|^2 ds] + 4t\boldsymbol{E}[|\int_0^t |b_n(X^n_s) - b(X_s)|^2 ds] \\
&\le 4C_2(4C^2T) + 4T(4C^2T) < \infty
\end{aligned}$$

and so g is bounded. By Gronwall's lemma (Lemma 8.4), we have

$$\begin{aligned}
&\boldsymbol{E}[\sup_{s\le T}|X^n_s - X_s|^2] = g(T) \le a \times e^{bT} \\
&\le ((4+T)T4^3C^2(2\frac{A(T)}{M^2}) + 4T(4^2\sup_{|x|\le M}|\sigma_n(x) - \sigma(x)|^2 + 4T\sup_{|x|\le M}|b_n(x) - b(x)|^2)) \\
&\times \exp(2\times 4^2K^2(4+T)\times T)
\end{aligned}$$

and so

$$\limsup_{n\to\infty}\boldsymbol{E}[\sup_{s\le T}|X^n_s - X_s|^2] \le (4+T)T4^3C^2(2\frac{A(T)}{M^2})\exp(2\times 4^2K^2(4+T)\times T).$$

By letting $M\to\infty$, we get

$$\lim_{n\to\infty}\boldsymbol{E}[\sup_{s\le T}|X^n_s - X_s|^2] = 0.$$

□

8.5 Exercise 8.13

Let $\beta = (\beta_t)_{t\ge 0}$ be an $(\mathscr{F}_t)_{t\ge 0}$-Brownian motion started from 0. We fix two real parameters α and r, with $\alpha > \frac{1}{2}$ and $r > 0$. For every integer $n \ge 1$ and every $x \in \mathbb{R}$, we set

$$f_n(x) = \frac{1}{|x|} \wedge n.$$

1. Let $n \ge 1$. Justify the existence of unique semimartingale Z^n that solves the equation

$$Z^n_t = r + \beta_t + \alpha\int_0^t f_n(Z^n_s)ds.$$

2. We set $S_n := \inf\{t \ge 0 \mid Z^n_t \le \frac{1}{n}\}$. After observing that, for $t \le S_{n+1}\wedge S_n$,

$$Z^{n+1}_t - Z^n_t = \alpha\int_0^t \frac{1}{Z^{n+1}_s} - \frac{1}{Z^n_s}ds,$$

show that $Z^{n+1}_t = Z^n_t$ for every $t \in [0, S_{n+1}\wedge S_n]$ (a.s.). Infer that $S_{n+1} \ge S_n$.

3. Let g be a twice continuously differentiable function on $\mathbb{R}$. Show that the process

$$g(Z^n_t) - g(r) - \int_0^t (\alpha g'(Z^n_s)f_n(Z^n_s) + \frac{1}{2}g''(Z^n_s))ds$$

is a continuous local martingale.

4. We set $h(x) = x^{1-2\alpha}$ for every $x > 0$. Show that, for every integer $n \geq 1$, $h(Z^n_{t\wedge S_n})$ is a bounded martingale. Infer that, for every $t' \geq 0$, $\boldsymbol{P}(S_n \leq t') \to 0$ as $n \to \infty$, and consequently $S_n \to \infty$ as $n \to \infty$ $\boldsymbol{P}$-(a.s.).
5. Infer from questions 2. and 4. that there exists a unique positive semimartingale Z such that, for every $t \geq 0$,
$$Z_t = r + \beta_t + \alpha \int_0^t \frac{ds}{Z_s}.$$
6. Let $d \geq 3$ and let B be a d-dimensional Brownian motion started from $y \in \mathbb{R}^d \setminus \{0\}$. Show that $Y_t = |B_t|$ satisfies the stochastic equation in question 5. (with an appropriate choice of β) with $r = |y|$ and $\alpha = \frac{d-1}{2}$. One may use the results of Exercise 5.33.

Proof.

1. To prove the existence of unique of soltion of
$$E^n_r: \quad dZ^n_t = d\beta_t + \alpha f_n(Z^n_t)dt$$
it suffices to show that f_n is Lipschitz. Observe that, if $|x|, |y| \geq \frac{1}{n}$, and if $|v| < \frac{1}{n} \leq |u|$, then
$$|f_n(x) - f_n(y)| = |\frac{1}{|x|} - \frac{1}{|y|}| = |\frac{|x| - |y|}{|x||y|}| \leq n^2|x-y|$$
and
$$|f_n(v) - f_n(u)| = n - \frac{1}{|u|} = \frac{|u| - |\pm\frac{1}{n}|}{\frac{1}{n}|u|} \leq n^2(|u + \frac{1}{n}| \wedge |u - \frac{1}{n}|) \leq n^2|u-v|.$$
Hence f_n is Lipschitz. By Theorem 8.5.(iii), there exists a unique solution of E^n_r.
2. Obsreve that, if $0 \leq t \leq S_{n+1} \wedge S_n$, then
$$Z^k_t = r + \beta_t + \alpha \int_0^t \frac{1}{Z^k_s} ds \quad \forall k = n, n+1$$
and
$$Z^{n+1}_t - Z^n_t = \alpha \int_0^t \frac{1}{Z^{n+1}_s} - \frac{1}{Z^n_s} ds.$$
Then $Z^n_t \geq \frac{1}{n} > 0$ and $Z^{n+1}_t \geq \frac{1}{n+1} > 0$ for every $0 \leq t \leq S_n \wedge S_{n+1}$. Fix $0 \leq t \leq S_n \wedge S_{n+1}$. Note that $\frac{1}{a} \leq \frac{1}{b}$ whenever $0 < b \leq a$. Suppose $Z^{n+1}_s \geq Z^n_s$ for all $s \in [0,t]$. Then
$$0 \leq Z^{n+1}_s - Z^n_s = \alpha \int_0^s \frac{1}{Z^{n+1}_r} - \frac{1}{Z^n_r} dr \leq 0$$
and so $Z^{n+1}_s = Z^n_s$ for all $s \in [0,t]$. Similarly, if $Z^{n+1}_s \leq Z^n_s$ for all $s \in [0,t]$, then $Z^{n+1}_s = Z^n_s$ for all $s \in [0,t]$. Thus, we get
$$Z^{n+1}_t = Z^n_t \qquad \forall t \in [0, S_n \wedge S_{n+1}] \quad \boldsymbol{P}\text{-(a.s.)}.$$
Now, we show that $S_{n+1} \geq S_n$ for every $n \geq 1$ by contradiction. Fix $n \geq 1$. Aussme that $\boldsymbol{P}(S_{n+1} < S_n) > 0$. Then
$$\boldsymbol{P}(S_{n+1} < S_n, Z^{n+1}_t = Z^n_t \quad \forall t \in [0, S_n \wedge S_{n+1}]) > 0.$$
Fix $w \in \{S_{n+1} < S_n\} \bigcap \{Z^{n+1}_t = Z^n_t \quad \forall t \in [0, S_n \wedge S_{n+1}]\}$. Set $\lambda = S_{n+1}(w)$. Since $Z^{n+1}_t(w) = Z^n_t(w)$ for all $0 \leq t \leq S_n(w) \wedge S_{n+1}(w) = S_{n+1}(w) = \lambda$, we get
$$Z^n_\lambda(w) = Z^{n+1}_\lambda(w) = \frac{1}{n+1} < \frac{1}{n}$$
and so $S_{n+1}(w) = \lambda \geq S_n(w)$ which contradict to $S_{n+1}(w) < S_n(w)$. Therefore, we have
$$S_{n+1} \geq S_n \quad \forall n \geq 1 \quad \boldsymbol{P}\text{-(a.s.)}.$$

3. By Itô's formula, we get

$$
\begin{aligned}
g(Z^n_t) &= g(r) + \int_0^t g'(Z^n_s)dZ^n_s + \frac{1}{2}\int_0^t g''(Z^n_s)d\langle Z^n, Z^n\rangle_s \\
&= g(r) + \int_0^t g'(Z^n_s)d\beta_s + \int_0^t g'(Z^n_s)\alpha f_n(Z^n_s)ds + \frac{1}{2}\int_0^t g''(Z^n_s)ds
\end{aligned}
$$

and so

$$
g(Z^n_t) - g(r) - \int_0^t (\alpha g'(Z^n_s)f_n(Z^n_s) + \frac{1}{2}g''(Z^n_s))ds = \int_0^t g'(Z^n_s)d\beta_s
$$

is a continuous local martingale.

4. Fix large $n \geq 1$ such that $n > \frac{1}{r}$. Then $S_n > 0$. Since $Z^n_{t\wedge S_n} \geq \frac{1}{n}$ for every $t \geq 0$, we have $f_n(Z^n_{t\wedge S_n}) = \frac{1}{Z^n_{t\wedge S_n}}$ for every $t \geq 0$ and so

$$
\int_0^t 1(s)_{\{s\leq S_n\}}dZ^n_s = \int_0^t 1(s)_{\{s\leq S_n\}}d\beta_s + \alpha\int_0^t \frac{1}{Z^n_{s\wedge S_n}}1(s)_{\{s\leq S_n\}}ds.
$$

By Itô's formula, we get

$$
\begin{aligned}
M_t &:= h(Z^n_{t\wedge S_n}) \\
&= r^{1-2\alpha} + \int_0^t (1-2\alpha)(Z^n_{s\wedge S_n})^{-2\alpha}1(s)_{\{s\leq S_n\}}dZ^n_s \\
&+ \frac{(-2\alpha)(1-2\alpha)}{2}\int_0^t (Z^n_{s\wedge S_n})^{-2\alpha-1}1(s)_{\{s\leq S_n\}}d\langle Z^n, Z^n\rangle_s \\
&= r^{1-2\alpha} + \int_0^t (1-2\alpha)(Z^n_{s\wedge S_n})^{-2\alpha}1(s)_{\{s\leq S_n\}}d\beta_s + \int_0^t (1-2\alpha)(Z^n_{s\wedge S_n})^{-2\alpha}\alpha\frac{1}{Z^n_{s\wedge S_n}}1(s)_{\{s\leq S_n\}}ds \\
&+ \frac{(-2\alpha)(1-2\alpha)}{2}\int_0^t (Z^n_{s\wedge S_n})^{-2\alpha-1}1(s)_{\{s\leq S_n\}}ds \\
&= r^{1-2\alpha} + \int_0^t (1-2\alpha)(Z^n_{s\wedge S_n})^{-2\alpha}1(s)_{\{s\leq S_n\}}d\beta_s
\end{aligned}
$$

is a continuous local martingale. Moreover, since

$$
\boldsymbol{E}[\langle M, M\rangle_t] = \boldsymbol{E}[(1-2\alpha)^2\int_0^t (Z^n_{s\wedge S_n})^{-4\alpha}1(s)_{\{s\leq S_n\}}ds] \leq (1-2\alpha)^2 \times t \times n^{4\alpha} < \infty
$$

for every $t \geq 0$, we see that $(h(Z^n_{t\wedge S_n}))_{t\geq 0} = (M_t)_{t\geq 0}$ is a martingale. Because

$$
0 < M_t = h(S^n_{t\wedge S_n}) = (Z^n_{t\wedge S_n})^{1-2\alpha} \leq n^{2\alpha-1} < \infty
$$

for every $t \geq 0$, we get $(h(Z^n_{t\wedge S_n}))_{t\geq 0} = (M_t)_{t\geq 0}$ is a bounded martingale.
Now, we show that $\lim_{n\to\infty}\boldsymbol{P}(S_n \leq t') = 0$ for every $t' \geq 0$. Fix $t' \geq 0$. Choose large $n \geq 1$ such that $n > \frac{1}{r}$. Since $(h(Z^n_{t\wedge S_n}))_{t\geq 0}$ is a bounded martingale and h is positive, we get

$$
\begin{aligned}
r^{1-2\alpha} = h(r) &= \boldsymbol{E}[h(Z^n_{0\wedge S_n})] = \boldsymbol{E}[h(Z^n_{t'\wedge S_n})] \\
&= \boldsymbol{P}(S_n \leq t')n^{2\alpha-1} + \boldsymbol{E}[h(Z^n_{t'\wedge S_n})1_{t'<S_n}] \\
&\geq \boldsymbol{P}(S_n \leq t')n^{2\alpha-1}
\end{aligned}
$$

and, hence,

$$
\boldsymbol{P}(S_n \leq t') \leq (\frac{1}{nr})^{2\alpha-1} \to 0 \text{ as } n \to \infty.
$$

Moreover, since $S_{n+1} \geq S_n$ for every $n \geq 1$, $S := \lim_{n\to\infty} S_n$ exist and so

$$\boldsymbol{P}(S \leq t) = \lim_{n\to\infty} \boldsymbol{P}(S_n \leq t) = 0$$

for every $t \geq 0$. Thus,

$$\lim_{n\to\infty} S_n = S = \infty \quad \boldsymbol{P}\text{-(a.s.)}.$$

5. (a) We show that there exists a positive semimartingale Z such that, for every $t \geq 0$,

$$Z_t = r + \beta_t + \alpha \int_0^t \frac{ds}{Z_s}.$$

By problem 2., we have

$$Z_t^{n+1} = Z_t^n \quad \forall t \in [0, S_n] \text{ and } n \geq 1 \text{ outside a zero set } N.$$

For the sake of simplicity, we redefine N as

$$N \bigcup \Big(\bigcap_{n\geq 1} \{Z_t^n = r + \beta_t + \alpha \int_0^t f_n(Z_s^n) ds \quad \forall t \geq 0\} \Big)^c.$$

Define

$$Z_t(w) = \begin{cases} Z_t^n(w), & \text{if } w \notin N \text{ and } t \leq S_n(w) \\ 0, & \text{otherwise.} \end{cases}$$

Then Z is a positive, adapted, continuous process. Fix $w \notin N$ and $t \geq 0$. Choose large $n \geq 1$ such that $S_n(w) \geq t$. Then

$$\begin{aligned} Z_t(w) = Z_t^n(w) &= r + \beta_t(w) + \int_0^t f_n(Z_s^n(w)) ds \\ &= r + \beta_t(w) + \int_0^t \frac{1}{Z_s^n(w)} ds \\ &= r + \beta_t(w) + \int_0^t \frac{1}{Z_s(w)} ds. \end{aligned}$$

Thus, Z is a positive semimartingale such that

$$Z_t = r + \beta_t + \alpha \int_0^t \frac{ds}{Z_s} \quad \forall t \geq 0 \quad \boldsymbol{P}\text{-(a.s.)}.$$

(b) Let Z and Z' are postive semimartingales such that

$$Z_t = r + \beta_t + \alpha \int_0^t \frac{ds}{Z_s} \quad \forall t \geq 0 \quad \boldsymbol{P}\text{-(a.s.)}$$

and

$$Z'_t = r + \beta_t + \alpha \int_0^t \frac{ds}{Z'_s} \quad \forall t \geq 0 \quad \boldsymbol{P}\text{-(a.s.)}$$

under filered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \boldsymbol{P})$and Brownian motion β started from 0. Note that $\frac{1}{a} \leq \frac{1}{b}$ whenever $0 < b \leq a$. Fix $w \in \Omega$. Observe that, if there exists real number $T > 0$ such that

$$Z_t \geq Z'_t \quad \forall t \in [0, T],$$

then

$$Z_t = r + \beta_t + \alpha \int_0^t \frac{1}{Z_s} ds \le r + \beta_t + \alpha \int_0^t \frac{1}{Z'_s} ds = Z'_t$$

for all $t \in [0,T]$ and so $Z_t = Z'_t$ for all $t \in [0,T]$. Similarly, if there exists real number $T > 0$ such that

$$Z_t \le Z'_t \quad \forall t \in [0,T],$$

then $Z_t = Z'_t$ for all $t \in [0,T]$. This shows that

$$Z_t = Z'_t \quad \forall t \ge 0 \quad \boldsymbol{P}\text{-(a.s.)}.$$

6. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\ge 0}, \boldsymbol{P})$ be filered probability space and B be d-dimensional Brownian motion started from $y \in \mathbb{R}^d \setminus \{0\}$. By Exercise 5.33, we get

$$|B_t| = |y| + \beta_t + \frac{d-1}{2} \int_0^t \frac{ds}{|B_s|},$$

where

$$\beta_t = \sum_{i=1}^d \int_0^t \frac{B_s^i}{|B_s|} dB_s^i$$

is a $(\mathscr{F}_t)_{t\ge 0}$ 1-dimensional Brownian motion started from 0. Thus, $(|B|, \beta), (\Omega, \mathscr{F}, (\mathscr{F})_{t\ge 0}, \boldsymbol{P})$ is a solution of the stochastic equation in question

$$Z_t = |y| + \beta_t + \frac{d-1}{2} \int_0^t \frac{ds}{Z_s}.$$

□

8.6 Exercise 8.14 (Yamada–Watanabe uniqueness criterion)

The goal of the exercise is to get pathwise uniqueness for the one-dimensional stochastic differential equation

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt$$

when the functions σ and b satisfy the conditions

$$|\sigma(x) - \sigma(y)| \le K\sqrt{|x-y|}, \quad |b(x) - b(y)| \le K|x-y|,$$

for every $x, y \in \mathbb{R}$, with a constant $K < \infty$.

1. Preliminary question. Let Z be a semimartingale such that $\langle Z, Z\rangle_t = \int_0^t h_s ds$, where $0 \le h_s \le C|Z_s|$, with a constant $C < \infty$. Show that, for every $t \ge 0$,

$$\lim_{n\to\infty} n\boldsymbol{E}[\int_0^t 1_{\{0<|Z_s|\le \frac{1}{n}\}} d\langle Z, Z\rangle_s] = 0.$$

(Hint: Observe that, $\boldsymbol{E}[\int_0^t |Z_s|^{-1} 1_{\{0<|Z_s|\le 1\}} d\langle Z, Z\rangle_s] \le Ct < \infty$.)

2. Fir every $n \ge 1$, let φ_n be the function defined on $\mathbb{R}$ by

$$\varphi_n(x) = \begin{cases} 0, & \text{if } |x| \ge \frac{1}{n} \\ 2n(1-nx), & \text{if } 0 \le x \le \frac{1}{n} \\ 2n(1+nx), & \text{if } \frac{-1}{n} \le x \le 0. \end{cases}$$

Also write F_n for the unique twice continuously differentiable function on $\mathbb{R}$ such that $F_n(0) = F'_n(0) = 0$ and $F''_n = \varphi_n$. Note that, for every $x \in \mathbb{R}$, one has $F_n(x) \to |x|$ and $F'_n(x) \to sgn(x) := 1_{\{x>0\}} - 1_{\{x<0\}}$ when

$n \to \infty$.

Let X and X' be two solutions of $E(\sigma, b)$ on the same filtered probability space and with the same Brownian motion B. Infer from question 1. that

$$\lim_{n\to\infty} \boldsymbol{E}[\int_0^t \varphi_n(X_s - X'_s) d\langle X - X', X - X'\rangle_s] = 0.$$

3. Let T be a stopping time such that the semimartingale $X_{t\wedge T} - X'_{t\wedge T}$ is bounded. By applying Itô's formula to $F_n(X_{t\wedge T} - X'_{t\wedge T})$, show that

$$\boldsymbol{E}[|X_{t\wedge T} - X'_{t\wedge T}|] = \boldsymbol{E}[|X_0 - X'_0|] + \boldsymbol{E}[\int_0^{t\wedge T} (b(X_s) - b(X'_s)) sgn(X_s - X'_s) ds].$$

4. Using Gronwall's lemma, show that, if $X_0 = X'_0$, one has $X_t = X'_t$ for every $t \geq 0$ (a.s.).

Proof.

1. Note that

$$\begin{aligned}
\boldsymbol{E}[\int_0^t |Z_s|^{-1} 1_{\{0<|Z_s|\leq 1\}} d\langle Z, Z\rangle_s] &= \boldsymbol{E}[\int_0^t |Z_s|^{-1} 1_{\{0<|Z_s|\leq 1\}} h_s ds] \\
&= \boldsymbol{E}[\int_0^t |Z_s|^{-1} 1_{\{0<|Z_s|\leq 1\}} 1_{\{h_s>0\}} h_s ds] \\
&\leq \boldsymbol{E}[\int_0^t \frac{C}{h_s} 1_{\{0<|Z_s|\leq 1\}} 1_{\{h_s>0\}} h_s ds] \\
&\leq Ct
\end{aligned}$$

and

$$\int_0^t n 1_{\{0<|Z_s|\leq \frac{1}{n}\}} d\langle Z, Z\rangle_s \leq \int_0^t |Z_s|^{-1} 1_{\{0<|Z_s|\leq 1\}} d\langle Z, Z\rangle_s \quad \forall n \geq 1.$$

By Lebesgue's dominated convergence theorem, we get

$$\begin{aligned}
\lim_{n\to\infty} \boldsymbol{E}[\int_0^t n 1_{\{0<|Z_s|\leq \frac{1}{n}\}} d\langle Z, Z\rangle_s] &= \boldsymbol{E}[\lim_{n\to\infty} \int_0^t n 1_{\{0<|Z_s|\leq \frac{1}{n}\}} d\langle Z, Z\rangle_s] \\
&= \boldsymbol{E}[\lim_{n\to\infty} \int_0^t n 1_{\{0<|Z_s|\leq \frac{1}{n}\}} h_s ds] \\
&\leq \boldsymbol{E}[\lim_{n\to\infty} \int_0^t n 1_{\{0<|Z_s|\leq \frac{1}{n}\}} C|Z_s| ds] \\
&\leq \boldsymbol{E}[\lim_{n\to\infty} \int_0^t n 1_{\{0<|Z_s|\leq \frac{1}{n}\}} C\frac{1}{n} ds] \\
&= \boldsymbol{E}[\lim_{n\to\infty} \int_0^t 1_{\{0<|Z_s|\leq \frac{1}{n}\}} C ds] \\
&= \boldsymbol{E}[\int_0^t \lim_{n\to\infty} 1_{\{0<|Z_s|\leq \frac{1}{n}\}} C ds] = 0
\end{aligned}$$

2. Since $\varphi_n \in C(\mathbb{R})$, we get $F_n \in C^2(\mathbb{R})$. Note that

$$F'_n(x) = \int_0^x \varphi_n(t) dt = \begin{cases} (2nx - n^2x) 1_{[0,\frac{1}{n})}(x) + 1_{[\frac{1}{n},\infty)}(x), & \text{if } x \geq 0 \\ (2nx + n^2x) 1_{(-\frac{1}{n},0]}(x) - 1_{(-\infty,-\frac{1}{n}]}(x), & \text{if } x \leq 0 \end{cases}$$

and

$$F_n(x) = \int_0^x F_n'(t)dt = \begin{cases} (x - \frac{1}{n})1_{[\frac{1}{n},\infty)}(x) + (n(x \wedge \frac{1}{n})^2 - \frac{n^2}{3}(x \wedge \frac{1}{n})^3), & \text{if } x \geq 0 \\ -(x + \frac{1}{n})1_{(-\infty,-\frac{1}{n}]}(x) + (n(x \vee \frac{-1}{n})^2 + \frac{n^2}{3}(x \vee \frac{-1}{n})^3), & \text{if } x \leq 0. \end{cases}$$

Then $F_n'(x) \to sgn(x)$ and $F_n(x) \to |x|$ as $n \to \infty$. Indeed, if $x > 0$ and $y < 0$, choose large $N \geq 1$ such that $\frac{1}{N} \leq x$ and $-\frac{1}{N} \geq y$, we have

$$F_n(x) = x - \frac{1}{n} + (n\frac{1}{n^2} - \frac{n^2}{3}\frac{1}{n^3}) = x - \frac{1}{3n} \quad \forall n \geq N,$$

$$F_n(y) = -y - \frac{1}{n} + (n\frac{1}{n^2} - \frac{n^2}{3}\frac{1}{n^3}) = -y - \frac{1}{3n} \quad \forall n \geq N$$

and so $F_n(x) \to x$ and $F_n(y) \to -y$ as $n \to \infty$.
Let X and X' be two solutions of $E(\sigma, b)$ on the same filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \boldsymbol{P})$ and with the same Brownian motion $(B_t)_{t\geq 0}$. Then

$$X_t = X_0 + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds$$

and

$$X_t' = X_0' + \int_0^t \sigma(X_s')dB_s + \int_0^t b(X_s')ds$$

for all $t \geq 0$. Set $Z_t := X_t - X_t'$ and $h_t := (\sigma(X_t) - \sigma(X_t'))^2$ for all $t \geq 0$. Then

$$\langle Z, Z\rangle_t = \int_0^t h_s ds$$

and

$$0 \leq h_t \leq K^2|X_t - X_t'| = K^2|Z_t|$$

for all $t \geq 0$. By problem 1., we get

$$\begin{aligned} &\lim_{n\to\infty} \boldsymbol{E}[\int_0^t \varphi_n(X_s - X_s')d\langle X - X', X - X'\rangle_s] \\ &= \lim_{n\to\infty} \boldsymbol{E}[\int_0^t \varphi_n(X_s - X_s')1_{0<|X_s - X_s'|\leq\frac{1}{n}}(s)d\langle X - X', X - X'\rangle_s] \\ &\leq \lim_{n\to\infty} \boldsymbol{E}[\int_0^t (2n + 2n^2|Z_s|)1_{0<|Z_s|\leq\frac{1}{n}}(s)d\langle Z, Z\rangle_s] \\ &\leq \lim_{n\to\infty} 2n\boldsymbol{E}[\int_0^t 1_{0<|Z_s|\leq\frac{1}{n}}(s)d\langle Z, Z\rangle_s] + \lim_{n\to\infty} \boldsymbol{E}[\int_0^t 2n^2 \times \frac{1}{n}1_{0<|Z_s|\leq\frac{1}{n}}(s)d\langle Z, Z\rangle_s] = 0. \end{aligned}$$

3. Fix $M > 0$. Define $T_M := \inf\{t \geq 0 \mid |X_t| + |X_t'| \geq M\}$. For the sake of simplicity, we denote T as T_M. Then $(X_{t\wedge T} - X_{t\wedge T}')_{t\geq 0}$ is a bounded martingale. Fix $t \geq 0$. By Itô's formula, we get

$$\begin{aligned} F_n(X_{t\wedge T} - X_{t\wedge T}') = &F_n(X_0 - X_0') \\ &+ \int_0^{t\wedge T} F_n'(X_s - X_s')(\sigma(X_s) - \sigma(X_s'))dB_s (:= Y_t) \\ &+ \int_0^{t\wedge T} F_n'(X_s - X_s')(b(X_s) - b(X_s'))ds \\ &+ \frac{1}{2}\int_0^{t\wedge T} \varphi_n(X_s - X_s')d\langle X - X', X - X'\rangle_s. \end{aligned}$$

Since

$$\begin{aligned}
\boldsymbol{E}[\langle Y,Y\rangle_t] &= \boldsymbol{E}[\int_0^{t\wedge T} |F_n'(X_s-X_s')|^2|\sigma(X_s)-\sigma(X_s')|^2 ds] \\
&\le \boldsymbol{E}[\int_0^{t\wedge T} 1\times K^2|X_s-X_s'|ds] \quad (|F_n'(x)|\le 1) \\
&\le K^2 2Mt<\infty \quad \forall t\ge 0,
\end{aligned}$$

we see that Y is a martingale and so

$$\begin{aligned}
\boldsymbol{E}[F_n(X_{t\wedge T}-X'_{t\wedge T})] &= \boldsymbol{E}[F_n(X_0-X_0')] \\
&\quad + \boldsymbol{E}[\int_0^{t\wedge T} F_n'(X_s-X_s')(b(X_s)-b(X_s'))ds] \\
&\quad + \boldsymbol{E}[\frac{1}{2}\int_0^{t\wedge T} \varphi_n(X_s-X_s')d\langle X-X',X-X'\rangle_s].
\end{aligned}$$

Note that $|X_{s\wedge T}|\vee|X'_{s\wedge T}|\le M$, $\sup_{|x|\le M}|b(x)|<\infty$, and $F_n(x)$ are uniformly bounded over $[-2M,2M]$. By Lebesgue's domainated theorem, we get

$$\begin{aligned}
\boldsymbol{E}[|X_{t\wedge T}-X'_{t\wedge T}|] &= \lim_{n\to\infty}\boldsymbol{E}[F_n(X_{t\wedge T}-X'_{t\wedge T})] \\
&= \lim_{n\to\infty}\boldsymbol{E}[F_n(X_0-X_0')] \\
&+ \lim_{n\to\infty}\boldsymbol{E}[\int_0^{t\wedge T} F_n'(X_s-X_s')(b(X_s)-b(X_s'))ds] \\
&+ \lim_{n\to\infty}\boldsymbol{E}[\frac{1}{2}\int_0^{t\wedge T} \varphi_n(X_s-X_s')d\langle X-X',X-X'\rangle_s] \\
&= \boldsymbol{E}[|X_0-X_0'|] + \boldsymbol{E}[\int_0^{t\wedge T} sgn(X_s-X_s')(b(X_s)-b(X_s'))ds] \\
&+ \lim_{n\to\infty}\boldsymbol{E}[\frac{1}{2}\int_0^{t\wedge T} \varphi_n(X_s-X_s')d\langle X-X',X-X'\rangle_s].
\end{aligned}$$

By problem 2., we get

$$\lim_{n\to\infty}\boldsymbol{E}[\frac{1}{2}\int_0^{t\wedge T} \varphi_n(X_s-X_s')d\langle X-X',X-X'\rangle_s] \le \lim_{n\to\infty}\boldsymbol{E}[\frac{1}{2}\int_0^{t} \varphi_n(X_s-X_s')d\langle X-X',X-X'\rangle_s]=0$$

and so

$$\boldsymbol{E}[|X_{t\wedge T}-X'_{t\wedge T}|] = \boldsymbol{E}[|X_0-X_0'|] + \boldsymbol{E}[\int_0^{t\wedge T} sgn(X_s-X_s')(b(X_s)-b(X_s'))ds].$$

4. Fix $t_0\ge 0$, $t_0\le L$, and $M>0$. Define $g:[0,L]\mapsto\mathbb{R}_+$ by

$$g(t):=\boldsymbol{E}[|X_{t\wedge T_M}-X'_{t\wedge T_M}|].$$

Then $0\le g(t)\le 2M$. By problem 3., we get

$$\begin{aligned}
g(t) &\le |\boldsymbol{E}[\int_0^{t\wedge T_M} sgn(X_s-X_s')(b(X_s)-b(X_s'))ds]| \\
&\le \boldsymbol{E}[\int_0^{t} |sgn(X_{s\wedge T_M}-X'_{s\wedge T_M})(b(X_{s\wedge T_M})-b(X'_{s\wedge T_M}))|ds] \\
&\le \boldsymbol{E}[\int_0^{t} K^2|X_{s\wedge T_M}-X'_{s\wedge T_M}|ds] = K^2\int_0^t g(s)ds.
\end{aligned}$$

By Gronwall's lemma, we get $g = 0$ and so

$$\boldsymbol{E}[|X_{t_0 \wedge T_M} - X'_{t_0 \wedge T_M}|] = 0.$$

Bt letting $M \to \infty$, we get $\boldsymbol{E}[|X_{t_0} - X'_{t_0}|] = 0$ and, hence, $X_{t_0} = X'_{t_0}$ (a.s.). Since X and X' have continuous sample path, we get

$$X_t = X'_t \quad \forall t \geq 0 \quad \boldsymbol{P}\text{-(a.s.)}.$$

□

Chapter 9

Local Times

9.1 Exercise 9.16

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a monotone increasing function, and assume that f is a difference of convex functions. Let X be a semimartingale and consider the semimartingale $Y_t = f(X_t)$. Prove that, for every $a \in \mathbb{R}$,

$$L_t^a(Y) = f'_+(a)L_t^a(X) \text{ and } L_t^{a-}(Y) = f'_-(a)L_t^{a-}(X).$$

In particular, if X is a Brownian motion, the local times of $f(X)$ are continuous in the space variable if and only if f is continuously differentiable.

Remark.
Note that $(L^a(X), a \in \mathbb{R})$ is the càdlàg modification of local time of X. The formula

$$L_t^a(Y) = f'_+(a)L_t^a(X)$$

doesn't hold for all increasing function $f = \varphi_1 - \varphi_2$, where φ_i is a convex function on $\mathbb{R}$. For example, if $\varphi_1(x) = 2e^x$ and $\varphi_2(x) = e^x$, and if X is a continuous semimartingale such that $\boldsymbol{P}(L_t^a(X) \neq 0) > 0$ for some $a < 0$ and $t > 0$, then $f(x) = e^x$ and so

$$L_t^a(Y) = L_t^a(f(X)) = 0 \neq e^a L_t^a(X) = f'(a)L_t^a(X)$$

on $\{L_t^a(X) \neq 0\}$.
To avoid this problem, we restatement Exercise 9.16 as following: Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a strictly increasing function such that $f = \varphi_1 - \varphi_2$, where φ_i is a convex function on $\mathbb{R}$. Let X be a semimartingale and consider the semimartingale $Y_t = f(X_t)$. Prove that, a.s.

$$L_t^{f(a)}(Y) = f'_+(a)L_t^a(X) \text{ and } L_t^{f(a)-}(Y) = f'_-(a)L_t^{a-}(X) \quad \forall a \in \mathbb{R}, t \geq 0$$

In particular, if X is a Brownian motion and $(u, v) \subseteq R(f) := \{a \in \mathbb{R} \mid f(a)\}$, we have, a.s. $a \in (u, v) \mapsto L^a(Y)$ is continuous if and only if $a \in (u, v) \mapsto f(a)$ is continuously differentiable.

Proof.

1. Since $f = \varphi_1 - \varphi_2$, we see that f is continuous and f'_+ is right continuous. We show that, a.s.

   $$L_t^{f(a)}(Y) = f'_+(a)L_t^a(X) \quad \forall t > 0, a \in \mathbb{R}.$$

   To show this, it suffices to show that $\boldsymbol{P}(L_t^{f(a)}(Y) = f'_+(a)L_t^a(X)) = 1$ for all $t \geq 0$ and $a \in \mathbb{R}$. Indeed, since $a \in \mathbb{R} \mapsto f'_+(a)L_t^a(X)$ is right continuous for $t \geq 0$ and

   $$E_a := \{L_t^{f(a)}(Y) = f'_+(a)L_t^a(X) \quad \forall t \geq 0\} = \bigcap_{s \in \mathbb{Q}_+} E_{a,s} \quad \forall a \in \mathbb{R},$$

   where

   $$E_{a,s} := \{L_s^{f(a)}(Y) = f'_+(a)L_s^a(X)\} \quad \forall a \in \mathbb{R}, s > 0,$$

   we see that

   $$\boldsymbol{P}(L_t^{f(a)}(Y) = f'_+(a)L_t^a(X) \quad \forall a \in \mathbb{R}, t \geq 0) = \boldsymbol{P}(\bigcap_{q \in \mathbb{Q}} E_q) = 1.$$

   Fix $a \in \mathbb{R}$ and $t > 0$. Now, we show that $\boldsymbol{P}(L_t^{f(a)}(Y) = f'_+(a)L_t^a(X)) = 1$. By generalized Itô formula, we see that

   $$d\langle Y, Y\rangle_s = f'_-(X_s)^2 d\langle X, X\rangle_s.$$

By Proposition 9.9 and Corollary 9.7, we have, a.s.

$$\begin{aligned}
L_t^{f(a)}(Y) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t 1_{\{f(a)\le f(X_s)\le f(a)+\epsilon\}} f'_-(X_s)^2 d\langle X,X\rangle_s \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\mathbb{R}} 1_{\{f(a)\le f(b)\le f(a)+\epsilon\}} f'_-(b)^2 L_t^b(X)db \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\mathbb{R}} 1_{\{f(a)\le f(b)\le f(a)+\epsilon\}} f'_+(b)^2 L_t^b(X)db.
\end{aligned}$$

We show that, a.s.

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\mathbb{R}} 1_{\{f(a)\le f(b)\le f(a)+\epsilon\}} f'_+(b)^2 L_t^b(X)db = f'_+(a)L_t^a(X).$$

Fix w. Given $\eta > 0$. Choose $h > 0$ such that

$$|f'_+(a)L_t^a(X) - f'_+(b)L_t^b(X)| < \eta$$

whenever $a \le b < a+h$. Note that f is a continuous strictly increasing function. For $\epsilon > 0$, define

$$a_\epsilon := \inf\{b \in \mathbb{R} \mid f(b) = f(a)+\epsilon\}.$$

Choose $j > 0$ such that $a < a_\epsilon < a+h$ for every $0 < \epsilon < j$. Let $0 < \epsilon < j$. Then $-\infty < a < a_\epsilon < \infty$, $f(a_\epsilon) = f(a)+\epsilon$,

$$|f'_+(a)L_t^a(X) - f'_+(b)L_t^b(X)| < \eta \quad \forall b \in [a, a_\epsilon],$$
$$\{b \in \mathbb{R} \mid f(a) \le f(b) \le f(a)+\epsilon\} = [a, a_\epsilon],$$

and so

$$\frac{1}{\epsilon}\int 1_{\{f(a)\le f(b)\le f(a)+\epsilon\}} f'_+(b)db = \frac{1}{\epsilon}\int_a^{a_\epsilon} f'_+(b)db = \frac{f(a_\epsilon)-f(a)}{\epsilon} = 1.$$

Thus,

$$\begin{aligned}
&|\frac{1}{\epsilon}\int_{\mathbb{R}} 1_{\{a\le f(b)\le a+\epsilon\}} f'_+(b)^2 L_t^b(X)db - f'_+(a)L_t^a(X)| \\
&= |\frac{1}{\epsilon}\int_a^{a_\epsilon} f'_+(b)^2 L_t^b(X)db - \frac{1}{\epsilon}\int_a^{a_\epsilon} f'_+(b)f'_+(a)L_t^a(X)db| \\
&\le \frac{1}{\epsilon}\int_a^{a_\epsilon} f'_+(b)|f'_+(b)L_t^b(X) - f'_+(a)L_t^a(X)|db \\
&< \eta\frac{1}{\epsilon}\int_a^{a_\epsilon} f'_+(b)db = \eta\frac{1}{\epsilon}(f(a_\epsilon)-f(a)) = \eta\frac{1}{\epsilon}\epsilon = \eta.
\end{aligned}$$

Therefore, we have, a.s.

$$L_t^{f(a)}(Y) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\mathbb{R}} 1_{\{f(a)\le f(b)\le f(a)+\epsilon\}} f'_+(b)^2 L_t^b(X)db = f'_+(a)L_t^a(X).$$

2. We show that, a.s.

$$L_t^{f(a)-}(Y) = f'_-(a)L_t^{a-}(X) \quad \forall t > 0, a \in \mathbb{R}.$$

To show this, it suffices to show that $\lim_{b\uparrow a} f'_+(b) = f'_-(a)$ for every $a \in \mathbb{R}$. Indeed, if $w \in E$, where $E = \{L_t^{f(a)}(Y) = f'_+(a)L_t^a(X) \quad \forall a \in \mathbb{R}, t \ge 0\}$, then

$$L_t^{f(a)-}(Y) = \lim_{b\uparrow a} L_t^{f(b)}(Y) = \lim_{b\uparrow a} f'_+(b)L_t^b(X) = f'_-(a)L_t^{a-}(X) \quad \forall a \in \mathbb{R}, t \ge 0.$$

Fix $a \in \mathbb{R}$. Now, we show that $\lim_{b\uparrow a} f'_+(b) = f'_-(a)$. Since $f = \varphi_1 - \varphi_2$, it suffices to show that $\lim_{b\uparrow a} \varphi'_{i,+}(b) = \varphi'_{i,-}(a)$ for $i = 1, 2$. We denote φ_i as φ. It's clear that

$$\varphi'_+(b) \leq \varphi'_-(a) \quad \forall b < a.$$

Given $\eta > 0$. There exists $c < a$ such that

$$\varphi'_-(a) - \eta \leq \frac{\varphi(a) - \varphi(c)}{a - c}.$$

By continuity, there exists $c < d < a$ such that

$$\frac{\varphi(a) - \varphi(c)}{a - c} - \eta \leq \frac{\varphi(d) - \varphi(c)}{d - c}$$

and so

$$\varphi'_-(a) - 2\eta \leq \frac{\varphi(d) - \varphi(c)}{d - c} \leq \varphi'_+(b) \quad \forall d < b < a.$$

Thus, we get

$$\varphi'_-(a) - 2\eta \leq \varphi'_+(b) \leq \varphi'_-(a) \quad \forall d < b < a$$

and, hence, $\lim_{b\uparrow a} f'_+(b) = f'_-(a)$.

3. Assume that X is a Brownian motion and $(u, v) \subseteq R(f)$. Then $a \mapsto L^a(X)$ is continuous and so, a.s.

$$L^a_t(X) = L^{a-}_t(X) \quad \forall a \in \mathbb{R}, t \geq 0.$$

Note that, a.s.

$$a \in (u, v) \mapsto L^a(Y) \text{ is continuous if and only if } L^{a-}_t(Y) = L^a_t(Y) \quad \forall a \in (u, v), t \geq 0.$$

Thus, if f is continuously differentiable, then we have, a.s.

$$L^a_t(Y) = f'(f^{-1}(a))L^{f^{-1}(a)}_t(X) = f'(f^{-1}(a))L^{f^{-1}(a)-}_t(X) = L^{a-}_t(Y) \quad \forall a \in (u, v), t \geq 0.$$

Now, we suppose $a \in (u, v) \mapsto L^a(Y)$ is continuous. Note that $-\infty = \liminf_{t\to\infty} X_t$ and $\limsup_{t\to\infty} X_t = \infty$. By Theorem 9.12, we get, a.s.

$$\forall a \in \mathbb{R} \quad \exists t_a > 0 \quad \forall t > t_a \quad L^a_t(X) > 0$$

(t_a also depend on w). Fix $\alpha \in (u, v)$. Choose w and $t > 0$ such that $L^\alpha_t(X) > 0$, $L^{f(a)}_t(Y) = f'_+(a)L^a_t(X)$ and, $L^{f(a)-}_t(Y) = f'_-(a)L^{a-}_t(X)$ for all $a \in \mathbb{R}$. Thus,

$$f'_+(\alpha)L^\alpha_t(X) = L^{f(\alpha)}_t(Y) = L^{f(\alpha)-}_t(Y) = f'_-(\alpha)L^{\alpha-}_t(X) = f'_-(\alpha)L^\alpha_t(X)$$

and so $f'_+(\alpha) = f'_-(\alpha)$. Therefore f is differentiable at α. Moreover, since $(a, s) \mapsto L^a_s(X)$ is continuous, there exists $\delta > 0$ such that

$$L^a_s(X) > 0 \quad \forall (a, s) \in (\alpha - \delta, \alpha + \delta) \times (t - \delta, t + \delta)$$

and so $a \in (\alpha - \delta, \alpha + \delta) \mapsto f'(a) = \frac{L^{f(a)}_t(Y)}{L^a_t(X)}$ is continuous.

$\square$

9.2 Exercise 9.17

Let M be a continuous local martingale such that $\langle M, M, \rangle = \infty$ (a.s.) and let B be the Brownian motion associated with M via the Dambis–Dubins–Schwarz theorem (Theorem 5.13). Prove that, a.s. for every $a \geq 0$ and $t \geq 0$,

$$L_t^a(M) = L^a_{\langle M,M \rangle_t}(B).$$

Proof.
Note that $(L^a(X), a \in \mathbb{R})$ is the càdlàg modification of local time of continuous semimartingale X. Set

$$E_{a,t} := \{L_t^a(M) = L^a_{\langle M,M \rangle_t}(B)\} \quad \forall t > 0, a \in \mathbb{R}.$$

Then it suffices to show that $\boldsymbol{P}(E_{a,t}) = 1$ for all $t > 0$ and $a \in \mathbb{R}$. Indeed, since

$$E_a := \{L_t^a(M) = L^a_{\langle M,M \rangle_t}(B) \quad \forall t \geq 0\} = \bigcap_{q \in \mathbb{Q}_+} E_{a,q} \quad \forall a \in \mathbb{R}$$

and

$$E := \{L_t^a(M) = L^a_{\langle M,M \rangle_t}(B) \quad \forall t \geq 0\ , a \in \mathbb{R}\} = \bigcap_{a \in \mathbb{Q}} E_a,$$

we see that $\boldsymbol{P}(E) = 1$. Fix $t > 0$ and $a \in \mathbb{R}$. Now, we show that $\boldsymbol{P}(E_{a,t}) = 1$. Note that $M_s = B_{\langle M,M \rangle_s} \quad \forall s \geq 0$ (a.s.). By Tanaka's formula, we get, a.s.

$$|M_t - a| = |M_0 - a| + \int_0^t sgn(M_s - a) dM_s + L_t^a(M)$$

and

$$|M_t - a| = |B_{\langle M,M \rangle_t} - a| = |M_0 - a| + \int_0^{\langle M,M \rangle_t} sgn(B_s - a) dB_s + L^a_{\langle M,M \rangle_t}(B).$$

By Proposition 5.9, there exists $\{n_k\}$ such that, a.s.

$$\begin{aligned}\int_0^t sgn(M_s - a) dM_s &= \lim_{k \to \infty} \sum_{i=0}^{n_k - 1} sgn(M_{\frac{it}{n_k}} - a)(M_{\frac{t(i+1)}{n_k}} - M_{\frac{it}{n_k}}) \\ &= \lim_{k \to \infty} \sum_{i=0}^{n_k - 1} sgn(B_{\langle M,M \rangle_{\frac{it}{n_k}}} - a)(B_{\langle M,M \rangle_{\frac{(i+1)t}{n_k}}} - B_{\langle M,M \rangle_{\frac{it}{n_k}}}).\end{aligned}$$

Since $s \in \mathbb{R}_+ \mapsto \langle M, M \rangle_s$ is increasing continuous function, we have, a.s.

$$\lim_{k \to \infty} \sum_{i=0}^{n_k - 1} sgn(B_{\langle M,M \rangle_{\frac{it}{n_k}}} - a)(B_{\langle M,M \rangle_{\frac{(i+1)t}{n_k}}} - B_{\langle M,M \rangle_{\frac{it}{n_k}}}) = \int_0^{\langle M,M \rangle_t} sgn(B_s - a) dB_s$$

and so

$$\int_0^t sgn(M_s - a) dM_s = \int_0^{\langle M,M \rangle_t} sgn(B_s - a) dB_s.$$

Thus, we have, a.s.

$$L_t^a(M) = L^a_{\langle M,M \rangle_t}(B).$$

$\square$

9.3 Exercise 9.18

Let X be a continuous semimartingale, and assume that X can be written in the form

$$X_t = X_0 + \int_0^t \sigma(w,s)dB_s + \int_0^t b(w,s)ds,$$

where B is a Brownian motion and σ and b are progressive and locally bounded. Assume that $\sigma(w,s) \neq 0$ for Lebesgue a.e. $s \geq 0$ a.s. Show that the local times $L_t^a(X)$ are jointly continuous in the pair (a,t).

Proof.
By the proof of theorem 9.4, it suffices to show that

$$\int_0^t 1_{\{X_s=a\}}(s)b(w,s)ds = 0 \quad \forall t \geq 0, a \in \mathbb{R} \quad (a.s.)$$

and so we show that $1_{\{X_s=a\}} = 0$ for almost every $s \geq 0$ and for every $a \in \mathbb{R}$ (a.s.). By density of occupation time formula (Corollary 9.7), we have

$$\int_0^t \varphi(X_s)\sigma(w,s)^2 ds = \int_{\mathbb{R}} \varphi(a) L_t^a(X) da$$

for all nonnegative measurable function $\varphi : \mathbb{R} \mapsto \mathbb{R}_+$ and $t \geq 0$ (a.s.) and so

$$\int_0^t 1_{\{X_s=a\}}\sigma(w,s)^2 ds = 0 \quad \forall t \geq 0, a \in \mathbb{R} \quad (a.s.).$$

Since $\sigma(w,s) \neq 0$ for almost every $s \geq 0$ (a.s.), we get $1_{\{X_s=a\}} = 0$ for almost every $s \geq 0$ and for every $a \in \mathbb{R}$ (a.s.). □

9.4 Exercise 9.19

Let X be a continuous semimartingale. Show that the property

$$supp(d_s L_s^a(X)) \subseteq \{s \geq 0 \mid X_s = a\}$$

holds simultaneously for all $a \in \mathbb{R}$, outside a single set of probability zero.

Proof.
Note that $(L^a(X), a \in \mathbb{R})$ is the càdlàg modification of local time of X. Set

$$E_a := \{w \in \Omega \mid supp(d_s L_s^a(X)) \subseteq \{s \geq 0 \mid X_s = a\}\} \quad \forall a \in \mathbb{R}$$

and

$$E = \bigcap_{q \in \mathbb{Q}} E_q.$$

By Proposition 9.3, $\boldsymbol{P}(E) = 1$ and so it suffices to show that

$$supp(d_s L_s^a(X)) \subseteq \{s \geq 0 \mid X_s = a\} \quad \forall a \in \mathbb{R} \text{ on E}.$$

Fix $w \in E$. Assume that there exists $b \in \mathbb{R}$ and $0 \leq s < t$ such that $L_s^b(X)(w) < L_t^b(X)(w)$ and $X_r(w) \neq b$ for all $s \leq r \leq t$. Suppose that $b < \min_{s \leq r \leq t} X_r(w)$. Choose $\epsilon > 0$ such that

$$L_s^b(X)(w) + \epsilon < L_t^b(X)(w) - \epsilon.$$

Since $a \mapsto L^a(X)(w)$ is right continuous, there exists $q \in \mathbb{Q}$ such that $b < q < \min_{s \leq r \leq t} X_r$ and

$$|L_s^q(X)(w) - L_s^b(X)(w)| \vee |L_t^q(X)(w) - L_t^q(X)(w)| < \epsilon.$$

Thus, we get $X_r(w) \neq q$ for all $s \leq r \leq t$ and $L_s^q(X)(w) < L_t^q(X)(w)$ which is a contradiction. By similar argument, we see that $b > \max_{s \leq r \leq t} X_r(w)$ is a contradiction and so

$$supp(d_s L_s^a(X)(w)) \subseteq \{s \geq 0 \mid X_s(w) = a\} \quad \forall a \in \mathbb{R}.$$

□

9.5 Exercise 9.20

Let B be a Brownian motion started from 0. Show that a.s. there exists an $a \in \mathbb{R}$ such that the inclusion $supp(d_sL_s^a(X)) \subseteq \{s \geq 0 \mid X_s = a\}$ is not an equality. (Hint: Consider the maximal value of B over $[0,1]$.)

Proof.
We denote B as X. Note that $(L^a(B), a \in \mathbb{R})$ is the càdlàg modification of local time of B. First, we show that, a.s.

$$\max_{0\leq t\leq 1} B_t > B_1.$$

Note that

$$\boldsymbol{P}(B_1 \geq \max_{0\leq t\leq 1} B_s) = \boldsymbol{P}(\min_{0\leq t\leq 1} B_1 - B_t \geq 0) = \boldsymbol{P}(\min_{0\leq t\leq 1} B_1 - B_{1-t} \geq 0).$$

Define

$$B'_t = B_1 - B_{1-t} \quad \forall t \in [0,1].$$

By Exercise 2.31, we see that $(B'_t)_{[0,1]}$ and $(B_t)_{[0,1]}$ have the same law and so

$$\boldsymbol{P}(\min_{0\leq t\leq 1} B_1 - B_{1-t} \geq 0) = \boldsymbol{P}(\min_{0\leq t\leq 1} B_t \geq 0).$$

By Proposition 2.14, we get

$$\boldsymbol{P}(\max_{0\leq t\leq 1} B_t > B_1) = 1 - \boldsymbol{P}(B_1 \geq \max_{0\leq t\leq 1} B_s) = 1 - \boldsymbol{P}(\min_{0\leq t\leq 1} B_t \geq 0) = 1.$$

Next, we show that a.s. there exists an $a \in \mathbb{R}$ such that the inclusion

$$supp(d_sL_s^a(X)) \subseteq \{s \geq 0 \mid X_s = a\}$$

is not an equality. Fix

$$w \in \{\max_{0\leq t\leq 1} B_t > B_1\} \bigcap \{L_t^a(B) = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_0^t 1_{\{a\leq B_s\leq a+\epsilon\}} ds \quad \forall a \in \mathbb{R}, t > 0\}.$$

Choose $a = \max_{0\leq t\leq 1} B_s$. Since $\max_{0\leq t\leq 1} B_t > B_1$, there exists $t \in (0,1)$ such that $B_t = a$. Let $b > a$. Then

$$L_1^b(B) = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_0^1 1_{\{b\leq B_s\leq b+\epsilon\}} ds = 0.$$

By right continuity, we get

$$L_1^a(B) = \lim_{b\downarrow a} L_1^b(B) = 0$$

and so

$$t \in \{s \geq 0 \mid B_s = a\} \bigcap (supp(d_sL_s^a(B)))^c.$$

□

9.6 Exercise 9.21

Let B be a Brownian motion started from 0. Note that

$$\int_0^\infty 1_{\{B_s>0\}} ds = \infty$$

a.s. and set, for every $t \geq 0$,

$$A_t = \int_0^t 1_{\{B_s>0\}} ds, \quad \sigma_t = \inf\{s \geq 0 \mid A_s > t\}.$$

1. Verify that the process
$$\gamma_t = \int_0^{\sigma_t} 1_{\{B_s>0\}} dB_s$$
is a Brownian motion in an appropriate filtration.

2. Show that the process $\Lambda_t = L^0_{\sigma_t}(B)$ has nondecreasing and continuous sample paths, and that the support of the measure $d_s\Lambda_s$ is contained in $\{s \geq 0 \mid B_{\sigma_s} = 0\}$.

3. Show that the process $(B_{\sigma_t})_{t\geq 0}$ has the same distribution as $(|B_t|)_{t\geq 0}$.

Proof.

1. Since $\limsup_{t\to\infty} B_s = \infty$, we see that $\int_0^\infty 1_{\{B_s>0\}} ds = \infty$ (a.s.) and so
$$\sigma_t < \infty \quad \forall t \geq 0 \quad (a.s.).$$
Note that γ_t is $\mathscr{F}_{\sigma_t}$-measurable for every $t \geq 0$ and $(\sigma_t)_{t\geq 0}$ is nondecreasing. It's clear that $t \mapsto \sigma_t$ is right continuous and so $(\gamma_t)_{t\geq 0}$ has a right continuous sample path. Observe that
$$B_s \leq 0 \quad \forall s \in (\sigma_{t-}, \sigma_t), \quad \forall t > 0 \quad (a.s.).$$
Then
$$\lim_{t\uparrow u} \gamma_t = \lim_{t\uparrow u} \int_0^{\sigma_t} 1_{\{B_s>0\}} dB_s = \int_0^{\sigma_{u-}} 1_{\{B_s>0\}} dB_s = \int_0^{\sigma_u} 1_{\{B_s>0\}} dB_s = \gamma_u \quad \forall u > 0 \quad (a.s.)$$
and so $(\gamma_t)_{t\geq 0}$ has a continuous sample path.
Now, we show that $(\gamma_t)_{t\geq 0}$ is a $(\mathscr{F}_{\sigma_t})_{t\geq 0}$-martingale. Fix $s_1 < s_2$. Since
$$\boldsymbol{E}[\langle \int_0^{\cdot\wedge\sigma_{s_2}} 1_{\{B_s>0\}} dB_s, \int_0^{\cdot\wedge\sigma_{s_2}} 1_{\{B_s>0\}} dB_s \rangle_\infty] \leq \boldsymbol{E}[\int_0^{\sigma_{s_2}} 1_{\{B_s>0\}} ds] = \boldsymbol{E}[A_{\sigma_{s_2}}] = s_2,$$
we get $(\int_0^{t\wedge\sigma_{s_2}} 1_{\{B_s>0\}} dB_s)_{t\geq 0}$ is a L^2-bounded $(\mathscr{F}_t)_{t\geq 0}$-martingale and so $(\int_0^{t\wedge\sigma_{s_2}} 1_{\{B_s>0\}} dB_s)_{t\geq 0}$ is an uniformly integrable $(\mathscr{F}_t)_{t\geq 0}$-martingale. By optional stopping theorem, we get
$$\boldsymbol{E}[\int_0^{\sigma_{s_2}} 1_{\{B_s>0\}} dB_s \mid \mathscr{F}_{\sigma_{s_1}}] = \int_0^{\sigma_{s_1}} 1_{\{B_s>0\}} dB_s$$
and so $(\int_0^{t\wedge\sigma_{s_2}} 1_{\{B_s>0\}} dB_s)_{t\geq 0}$ is a $(\mathscr{F}_t)_{t\geq 0}$-martingale. Moreover, since
$$\langle \gamma, \gamma \rangle_\infty = \int_0^\infty 1_{\{B_s>0\}} ds = \infty \text{ and } \langle \gamma, \gamma \rangle_t = t \quad \forall t \geq 0,$$
we see that $(\gamma_t)_{t\geq 0}$ is a $(\mathscr{F}_{\sigma_t})_{t\geq 0}$-Brownian motion.

2. It's clear that $(\Lambda_t)_{t\geq 0} = (L^0_{\sigma_t}(B))_{t\geq 0}$ has nondecreasing and right continuous sample paths. Note that
$$B^+_{\sigma_t} = \int_0^{\sigma_t} 1_{\{B_s>0\}} dB_s + \frac{1}{2} L^0_{\sigma_t}(B) = \gamma_t + \frac{1}{2} L^0_{\sigma_t}(B) \quad \forall t \geq 0 \quad (a.s.).$$
Recall that
$$B_s \leq 0 \quad \forall s \in (\sigma_{t-}, \sigma_t), \quad \forall t > 0 \quad (a.s.).$$
Observe that if $\sigma_{t-} < \sigma_t$, then $\lim_{u\uparrow t} B^+_u = B^+_{\sigma_{t-}} = 0 = B^+_{\sigma_t}$ and so $(L^0_{\sigma_t}(B))_{t\geq 0}$ has a continuous sample path.
Now, we show that $supp(d_s\Lambda_s) \subseteq \{s \geq 0 \mid B_{\sigma_s} = 0\}$. Recall that
$$supp(d_s L^0_s(B)) = \{s \geq 0 \mid B_s = 0\} \quad (a.s.).$$

Fix $w \in \{supp(d_sL^0_s(B)) = \{s \geq 0 \mid B_s = 0\}\}$. Let $t \in supp(d_s\Lambda_s)$. If $\sigma_{t-} < \sigma_t$, it's clear that $B_{\sigma_t} = 0$. Now, we assume that $(\sigma_t)_{t\geq 0}$ is continuous at t. Let $\alpha < \sigma_t < \beta$. Then there exists $u < t < v$ such that $(\sigma_u, \sigma_v) \subseteq (\alpha, \beta)$,

$$L^0_\alpha(B) \leq L^0_{\sigma_u}(B) < L^0_{\sigma_v}(B) \leq L^0_\beta(B),$$

and so $\sigma_t \in supp(d_sL^0_s(B)) = \{s \geq 0 \mid B_s = 0\}$.

3. Observe that $B_{\sigma_t} \geq 0 \quad \forall t \geq 0 \quad (a.s.)$ and so $B_{\sigma_t} = B^+_{\sigma_t} \quad \forall t \geq 0 \quad (a.s.)$. Then

$$B_{\sigma_t} = B^+_{\sigma_t} = \gamma_t + \frac{1}{2}L^0_{\sigma_t}(B) \quad \forall t \geq 0 \quad (a.s.).$$

By Skorokhod's Lemma (Appendices), we see that

$$\sup_{s\leq t}(-\gamma_s) = \frac{1}{2}L^0_{\sigma_t}(B) \quad \forall t \geq 0 \quad (a.s.).$$

By Theorem 9.14, we get

$$B_{\sigma_t} = \sup_{s\leq t}(-\gamma_s) + \gamma_t = \sup_{s\leq t}(-\gamma_s) - (-\gamma_t) \stackrel{d}{=} |-\gamma_{\sigma_t}| \stackrel{d}{=} |B_t| \quad \forall t \geq 0$$

and so

$$(B_{\sigma_t})_{t\geq 0} \stackrel{d}{=} (|B_t|)_{t\geq 0}.$$

□

9.7 Exercise 9.22

9.8 Exercise 9.23

Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a real integrable function ($\int_{\mathbb{R}} |g(x)|dx < \infty$). Let B be a Brownian motion started from 0, and set

$$A_t = \int_0^t g(B_s)ds.$$

1. Justify the fact that the integral defining A_t makes sense, and verify that, for every $c > 0$ and every $u \geq 0$, A_{c^2u} has the same distribution as
$$c^2 \int_0^u g(cB_s)ds.$$

2. Prove that
$$\frac{A_t}{\sqrt{t}} \xrightarrow{d} (\int_{\mathbb{R}} g(x)dx)|N| \text{ as } t \to \infty,$$
where N is $\mathcal{N}(0,1)$.

Proof.

1. Let $t > 0$. Then

$$\boldsymbol{E}[\int_0^t |g(B_s)|ds] = \int_{\mathbb{R}}\int_0^t \frac{1}{\sqrt{2\pi s}} \exp(-\frac{x^2}{2s})ds|g(x)|dx \leq \int_{\mathbb{R}}\int_0^t \frac{1}{\sqrt{2\pi s}} \times 1ds|g(x)|dx$$
$$= \sqrt{\frac{2t}{\pi}} \int_{\mathbb{R}} |g(x)|dx < \infty$$

and so $\int_0^t |g(B_s)|ds < \infty$ (a.s.). Since

$$\int_0^t |g(B_s)|ds < \infty \quad \forall t \in \mathbb{Q}_+ \quad (a.s.),$$

we see that

$$\int_0^t |g(B_s)|ds < \infty \quad \forall t \in \mathbb{R} \quad (a.s.)$$

and so $(A_t)_{t\geq 0}$ is well-defined. Moreove, by changing of variable, we get

$$A_{c^2u} = \int_0^{c^2u} g(B_s)ds = c^2 \int_0^u g(B_{c^2s})ds = c^2 \int_0^u g(c\frac{1}{c}B_{c^2s})ds \overset{d}{=} c^2 \int_0^u g(cB_s)ds.$$

2. By Density of occupation time formula, we get

$$\frac{A_u}{\sqrt{u}} = \int_{\mathbb{R}} g(a)\frac{1}{\sqrt{u}}L_u^a(B)da \quad (a.s.)$$

for every $u > 0$. First, we show that

$$(\frac{1}{\sqrt{u}}L_u^a(B))_{a\in\mathbb{R}} \overset{d}{=} (L_1^{\frac{a}{\sqrt{u}}}(B))_{a\in\mathbb{R}} \quad \forall u > 0.$$

Fix $u > 0$ and $a \in \mathbb{R}$. Define Brownian motion $\widetilde{B}$ by $\widetilde{B}_t = \frac{1}{\sqrt{u}}B_{tu}$. By Tanaka's formula, we get

$$|\widetilde{B}_1 - \frac{a}{\sqrt{u}}| = |\frac{a}{\sqrt{u}}| + \frac{1}{\sqrt{u}}\int_0^u sgn(B_s - a)dB_s + \frac{1}{\sqrt{u}}L_u^a(B) \quad (a.s.).$$

Choose increasing sequence $\{n_k\}_{k\geq 1}$ such that (1),(2) hold (a.s.):

$$\begin{aligned}
\frac{1}{\sqrt{u}}\int_0^u sgn(B_s - a)dB_s &\overset{(1)}{=} \frac{1}{\sqrt{u}}\lim_{k\to\infty}\sum_{i=0}^{n_k-1} sgn(B_{\frac{i}{n_k}u} - a)(B_{\frac{i+1}{n_k}u} - B_{\frac{i}{n_k}u}) \\
&= \lim_{k\to\infty}\sum_{i=0}^{n_k-1} sgn(\widetilde{B}_{\frac{i}{n_k}} - \frac{a}{\sqrt{u}})(\widetilde{B}_{\frac{i+1}{n_k}} - \widetilde{B}_{\frac{i}{n_k}}) \\
&\overset{(2)}{=} \int_0^1 sgn(\widetilde{B}_s - a)d\widetilde{B}_s.
\end{aligned}$$

Thus,

$$|\widetilde{B}_1 - \frac{a}{\sqrt{u}}| = |\frac{a}{\sqrt{u}}| + \int_0^1 sgn(\widetilde{B}_s - a)d\widetilde{B}_s + \frac{1}{\sqrt{u}}L_u^a(B) \quad (a.s.)$$

and so $\frac{1}{\sqrt{u}}L_u^a(B) = L_1^{\frac{a}{\sqrt{u}}}(\widetilde{B})$ (a.s.). By right continuity, we get

$$\frac{1}{\sqrt{u}}L_u^a(B) = L_1^{\frac{a}{\sqrt{u}}}(\widetilde{B}) \quad \forall a \in \mathbb{R} \quad (a.s.)$$

and so

$$(\frac{1}{\sqrt{u}}L_u^a(B))_{a\in\mathbb{R}} \overset{d}{=} (L_1^{\frac{a}{\sqrt{u}}}(B))_{a\in\mathbb{R}} \quad \forall u > 0.$$

Next, we show that

$$\frac{A_u}{\sqrt{u}} \overset{d}{\to} (\int_{\mathbb{R}} g(x)dx)|N| \text{ as } u \to \infty.$$

Note that
$$\boldsymbol{E}[\exp(i\xi\frac{A_u}{\sqrt{u}})] = \boldsymbol{E}[\exp(i\xi\int_{\mathbb{R}} g(a)\frac{1}{\sqrt{u}}L_u^a(B)da)] = \boldsymbol{E}[\exp(i\xi\int_{\mathbb{R}} g(a)L_1^{\frac{a}{\sqrt{u}}}(B)da)].$$
Since
$$L_1^a(B) = 0 \quad \forall a \notin [\min_{0\le s\le 1} B_s, \max_{0\le s\le 1} B_s] \quad (a.s.),$$
we get
$$|L_1^a(B)| \le M \text{ for some } M = M(w) < \infty \quad (a.s.)$$
and so
$$|L_1^{\frac{a}{\sqrt{u}}}(B)| \le M(w) < \infty \quad \forall a \in \mathbb{R}, u \in \mathbb{R}_+ \quad (a.s.).$$
By dominated convergence theorem and right continuity, we get
$$\begin{aligned}\lim_{u\to\infty}\boldsymbol{E}[\exp(i\xi\frac{A_u}{\sqrt{u}})] &= \lim_{u\to\infty}\boldsymbol{E}[\exp(i\xi\int_{\mathbb{R}} g(a)L_1^{\frac{a}{\sqrt{u}}}(B)da)] = \boldsymbol{E}[\exp(i\xi\lim_{u\to\infty}\int_{\mathbb{R}} g(a)L_1^{\frac{a}{\sqrt{u}}}(B)da)] \\ &= \boldsymbol{E}[\exp(i\xi\int_{\mathbb{R}} g(a)L_1^0(B)da)].\end{aligned}$$
By Theorem 9.14 and Theorem 2.21, we have
$$L_1^0(B) \overset{d}{=} \sup_{0\le s\le 1} B_s \overset{d}{=} |B_1|$$
and so
$$\lim_{u\to\infty}\boldsymbol{E}[\exp(i\xi\frac{A_u}{\sqrt{u}})] = \boldsymbol{E}[\exp(i\xi\int_{\mathbb{R}} g(a)L_1^0(B)da)] = \boldsymbol{E}[\exp(i\xi\int_{\mathbb{R}} g(a)da|B_1|)].$$

□

9.9 Exercise 9.24

Let σ and b be two locally bounded measurable functions on $\mathbb{R}_+ \times \mathbb{R}$, and consider the stochastic differential equation
$$E(\sigma, b): \quad dX_t = \sigma(t, X_t)dB_t + b(t, X_t)dt.$$
Let X and X' be two solutions of $E(\sigma, b)$ on the same filtered probability space and with the same Brownian motion B.

1. Suppose that $L_t^0(X - X') = 0$ for every $t \ge 0$. Show that both $X \vee X'$ and $X \wedge X'$ are solutions of $E(\sigma, b)$. (Hint: Write $X_t \vee X_t' = X_t + (X_t' - X_t)^+$, and use Tanaka's formula.)
2. Suppose that $\sigma(t, x) = 1$ for all t,a. Show that the assumption in question 1. holds automatically. Suppose in addition that weak uniqueness holds for $E(\sigma, b)$. Show that, if $X_0 = X_0' = x \in \mathbb{R}$, the two processes X and X' are indistinguishable.

Proof.

1. Note that
$$X_t \vee X_t' = X_t + (X_t' - X_t)^+.$$
By Tanaka's formula, we get
$$(X_t' - X_t)^+ = (X_0' - X_0)^+ + \int_0^t 1_{\{X_s' > X_s\}}(\sigma(s, X_s') - \sigma(s, X_s))dB_s + \int_0^t 1_{\{X_s' > X_s\}}(b(s, X_s') - b(s, X_s))ds$$

for all $t \geq 0$ (a.s.). Since

$$\sigma(s, (X'_s \vee X_s)) = 1_{\{X'_s > X_s\}}\sigma(s, X'_s) + 1_{\{X_s \geq X'_s\}}\sigma(s, X_s)$$

and

$$b(s, (X'_s \vee X_s)) = 1_{\{X'_s > X_s\}}b(s, X'_s) + 1_{\{X_s \geq X'_s\}}b(s, X_s),$$

we get

$$\begin{aligned}
(X'_t \vee X_t) &= X_t + (X'_t - X_t)^+ \\
&= X_0 + \int_0^t \sigma(s, X_s)dB_s + \int_0^t b(s, X_s)ds \\
&+ (X'_0 - X_0)^+ + \int_0^t 1_{\{X'_s > X_s\}}(\sigma(s, X'_s) - \sigma(s, X_s))dB_s + \int_0^t 1_{\{X'_s > X_s\}}(b(s, X'_s) - b(s, X_s))ds \\
&= (X'_0 \vee X_0) + \int_0^t \sigma(s, (X'_s \vee X_s))dB_s + \int_0^t b(s, (X'_s \vee X_s))ds
\end{aligned}$$

for all $t \geq 0$ (a.s.) and so $X \vee X'$ is a soltion of $E(\sigma, b)$. Note that

$$(X_t \wedge X'_t) = X_t - (X_t - X'_t)^+.$$

By similar argument, we see that $X \wedge X'$ is a soltion of $E(\sigma, b)$.

2. Suppose $\sigma(t, x) = 1$ for all t, x. Then

$$X_t - X'_t = X_0 - X'_0 + \int_0^t (b(s, X_s) - b(s, X_s))ds$$

for all $t \geq 0$ (a.s.) and so $L^0_t(X - X') = 0$ for all $t \geq 0$ (a.s.). Suppose in addition that weak uniqueness holds for $E(\sigma, b)$ and $X_0 = X'_0 = x \in \mathbb{R}$. By question 1, $X \vee X'$ and $X \wedge X'$ are solution of $E(\sigma, b)$ and so $X \vee X' \overset{d}{=} X \wedge X'$. It's clear that

$$X_t \vee X'_t = X_t \wedge X'_t \quad (a.s.)$$

for all $t \geq 0$. Indeed, if $P(X_t \vee X'_t > X_t \wedge X'_t) > 0$, then $\boldsymbol{E}[X_t \wedge X'_t] < \boldsymbol{E}[X_t \vee X'_t]$ which contradict to $X_t \vee X'_t \overset{d}{=} X_t \wedge X'_t$. Thus, we have $X_p = X'_p$ for all $p \in \mathbb{Q}_+$ (a.s.) and so

$$X_t = \lim_{p \in \mathbb{Q}_+ \to t} X_p = \lim_{p \in \mathbb{Q}_+ \to t} X'_p = X'_t$$

for all $t \geq 0$ (a.s.). Therefore X and X' are indistinguishable.

□

9.10 Exercise 9.25 (Another look at the Yamada–Watanabe criterion)

Let ρ be a nondecreasing function from $[0, \infty)$ into $[0, \infty)$ such that, for every $\epsilon > 0$,

$$\int_0^\epsilon \frac{du}{\rho(u)} = \infty.$$

Consider then the one-dimensional stochastic differential equation

$$E(\sigma, b): \qquad dX_t = \sigma(X_t)dB_t + b(X_t)dt$$

where one assumes that the functions σ and b satisfy the conditions

$$(\sigma(x) - \sigma(y))^2 \leq \rho(|x - y|), \quad |b(x) - b(y)| \leq K|x - y|,$$

for every $x, y \in \mathbb{R}$, with a constant $K < \infty$. Our goal is use local times to give a short proof of pathwise uniqueness for $E(\sigma, b)$ (this is slightly stronger than the result of Exercise 8.14).

1. Let Y be a continuous semimartingale such that, for every $t > 0$,

$$\int_0^t \frac{d\langle Y, Y\rangle_s}{\rho(|Y_s|)} < \infty \quad (a.s.).$$

Prove that $L_t^0(Y) = 0$ for every $t \geq 0$ (a.s.).

2. Let X and X_0 be two solutions of $E(\sigma, b)$ on the same filtered probability space and with the same Brownian motion B. By applying question 1. to $Y = X - X'$, prove that $L_t^0(X - X')$ for every $t \geq 0$ (a.s.) and therefore,

$$|X_t - X_t'| = |X_0 - X_0'| + \int_0^t (\sigma(X_s) - \sigma(X_s'))sgn(X_s - X_s')dB_s + \int_0^t (b(X_s) - b(X_s'))sgn(X_s - X_s')ds.$$

3. Using Gromwall's lemma, prove that if $X_0 = X_0'$, then $X_t = X_t'$ for every $t \geq 0$ (a.s.).

Proof.

1. Since $L_t^a(Y) \overset{a\downarrow 0}{\to} L_t^0(Y) \quad \forall t \geq 0$ (a.s.), there exists $C = C(w) > 0$ and $\epsilon = \epsilon(w) > 0$ such that

$$L_t^a(Y) \geq CL_t^0(Y) \quad \forall 0 < a < \epsilon \quad \forall t \geq 0 \quad (a.s.).$$

By Density of occupation time formula (Corollary 9.7), we have

$$\infty > \int_0^t \frac{d\langle Y, Y\rangle_s}{\rho(|Y_s|)} = \int_{\mathbb{R}} \frac{1}{\rho(|a|)} L_t^a(Y)da \geq CL_t^0(Y) \int_0^\epsilon \frac{1}{\rho(a)}da \quad \forall t \geq 0 \quad (a.s.).$$

Since $\int_0^\epsilon \frac{du}{\rho(u)} = \infty$ for all $\epsilon > 0$, we get $L_t^0(Y) = 0$ for all $t \geq 0$ (a.s.).

2. Set $Y = X - X'$. Then

$$Y_t = X_0 - X_0' + \int_0^t (\sigma(X_s) - \sigma(X_s'))dB_s + \int_0^t (b(X_s) - b(X_s'))ds$$

and so

$$d\langle Y, Y\rangle_t = (\sigma(X_t) - \sigma(X_t'))^2 dt.$$

Thus,

$$\int_0^t \frac{d\langle Y, Y\rangle_s}{\rho(|Y_s|)} = \int_0^t \frac{(\sigma(X_s) - \sigma(X_s'))^2}{\rho(|X_s - X_s'|)}ds \leq \int_0^t \frac{\rho(|X_s - X_s'|)}{\rho(|X_s - X_s'|)}ds = t < \infty \quad \forall t \geq 0 \quad (a.s.).$$

By question 1., we get $L_t^0(X - X') = 0$ for every $t \geq 0$ (a.s.). By Tanaka's formula, we have

$$|X_t - X_t'| = |X_0 - X_0'| + \int_0^t (\sigma(X_s) - \sigma(X_s'))sgn(X_s - X_s')dB_s + \int_0^t (b(X_s) - b(X_s'))sgn(X_s - X_s')ds$$

for every $t \geq 0$ (a.s.).

3. By continuity, it suffices to show that $X_t = X_t'$ (a.s.) for every $t \geq 0$. Fix $t_0 > 0$ and choose $L > t_0$. Define

$$T_M = \inf\{s \geq 0 \mid |X_s| \geq M \text{ or } |X_s'| \geq M\} \quad \forall M > 0.$$

Fix $M > 0$. Since

$$\begin{aligned}
&\boldsymbol{E}[\langle \int_0^\cdot (\sigma(X_s) - \sigma(X_s'))sgn(X_s - X_s')1_{[0,T_M]}dB_s, \int_0^\cdot (\sigma(X_s) - \sigma(X_s'))sgn(X_s - X_s')1_{[0,T_M]}dB_s\rangle_t] \\
&= \boldsymbol{E}[\int_0^t (\sigma(X_s) - \sigma(X_s'))^2 1_{[0,T_M]}ds] \leq \boldsymbol{E}[\int_0^t \rho(|X_s - X_s'|)1_{[0,T_M]}ds] \leq \rho(2M)t < \infty \quad \forall t > 0,
\end{aligned}$$

we see that $(\int_0^t(\sigma(X_s)-\sigma(X'_s))sgn(X_s-X'_s)1_{[0,T_M]}dB_s)_{t\geq 0}$ is a martingale. Thus

$$0 \leq g(t) \equiv \boldsymbol{E}[|X_t - X'_t|1_{[0,T_M]}(t)] \leq 2M$$

and

$$g(t) = \boldsymbol{E}[|X_t - X'_t|1_{[0,T_M]}(t)] = \boldsymbol{E}[\int_0^t (b(X_s)-b(X'_s))sgn(X_s-X'_s)1_{[0,T_M]}ds] \leq 2K\int_0^t g(s)ds$$

for every $t \in [0,L]$. By Gromwall's lemma, we get $g(t)=0$ in $[0,L]$ and so $\boldsymbol{E}[|X_{t_0\wedge T_M} - X'_{t_0\wedge T_M}|]=0$. By letting $M\uparrow\infty$, we have $\boldsymbol{E}[|X_{t_0}-X'_{t_0}|]=0$ and so $X_{t_0}=X'_{t_0}$.

□

Chapter 10

Appendices

10.1 Skorokhod's Lemma

Let y be a real-valued continuous function on $[0, \infty)$ such that $y(0) \geq 0$. There exists a unique pair (z, a) of functions on $[0, \infty)$ such that

1. $z(t) = y(t) + a(t)$,
2. $z(t)$ is nonnegative,
3. $a(t)$ is increasing, continuous, vanishing at zero and $supp(da_s) \subseteq \{s \geq 0 : z(s) = 0\}$.

Moreover, the function $a(t)$ is given by

$$a(t) = \sup_{s \leq t}(-y(s) \vee 0).$$

Proof.
It's clear that $(y - a, a)$ satisfies all properties above, where $a(t) = sup_{s \leq t}(-y(s) \vee 0)$, and so, it suffices to prove the uniqueness of the pair (z, a). Suppose that (z, a) and $(\overline{z}, \overline{a})$ satisfy all properties above. Then

$$z(t) - \overline{z}(t) = a(t) - \overline{a}(t) \quad \forall t \geq 0$$

and so

$$0 \leq (a(t) - \overline{a}(t))^2 = 2\int_0^t z(s) - \overline{z}(s) d(a - \overline{a})(s) \quad \forall t \geq 0.$$

Since

$$\int_0^t z_s da(s) = \int_0^t \overline{z}(s) d\overline{a}(s) = 0 \quad \forall t \geq 0,$$

we see that

$$2\int_0^t z(s) - \overline{z}(s) d(a - \overline{a})(s) = -2(\int_0^t z(s) d\overline{a}(s) + \int_0^t \overline{z} da(s)) \leq 0 \quad \forall t \geq 0$$

and so $z(t) = \overline{z}(t)$ for every $t \geq 0$.

□

References

[1] Daniel W. Stroock, Essentials of Integration Theory for Analysis.

[2] Dennis G. Zill, Warren S. Wright, Differential Equations with Boundary-Value Problems, Eighth Edition.