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Hw

Question 1:

Prop 1. *$t \to X_t$ on $[0,1]$ into $L^2(\Omega)$ is cts $\iff$ K is cts on $[0,1]^2$, where $K(s,t) = COV(X_s, X_t) = \mathbb{E}(X_s X_t)$*

Proof. Suppose $t \to X_t$ is continuous. Then K is the integral of a product of continuous functions and is thus continuous. Now suppose K continuous. Then in particular $K(t,t)$ is continuous. So $t \to X_t$ via is continuous as $||X_t||_{L^2(\Omega)} = \mathbb{E}(X_t^2) = K(t,t)$ □

Prop 2. *Let $h : [0,1] \to \mathbb{R}$ be measurable s.t. $\int_0^1 |h(t)|\sqrt{K(t,t)}dt < \infty$. Then a.e.*

$$Z = \int_0^1 H(t) X_t(\omega) dt$$

is absolutely convergent.

Proof. Recall $K(t,t) = \mathbb{E}X_t^2$ Compute

$$\mathbb{E}\int_0^1 |h(t)X_t|dt \leq \int_0^1 |h(t)|\mathbb{E}|X_t|dt \leq \int_0^1 |h(t)|||X_t||_{L^2}dt < \infty$$

where we have the first inequality via tonelli(since everything nonnegative). □

Prop 3. *Suppose h integrable. Then Z is the L^2 limit of $Z_n = \sum_i^n X_{\frac{i}{n}} \int_{(i-1)/n}^{i/n} h(t)dt$. Clearly it is then gaussian as the gaussian space is closed in L^2.*

Proof. We show $Z_n \to Z$ in L^2. Recall the limit of gaussians is gaussian. Compute $\mathbb{E}[|Z - Z_n|^2] = \int(\int_0^1 h(t)(X_t - \sum X_{i/n} 1_{[(i-1)/n, i/n]}(t)dt)^2 \leq \int_0^1(\int(h(t))^2(X_t - \sum X_{i/n}1)1_{[(i-1)/n,i/n]}(t))^2 dt$ where we tonelli. This in turn yields $\int_0^1 |h(t)|(\int(X_t - \sum X_{i/n}1_{[(i-1)/n,i/n]}(t))^2 dt = \int_0^1 |h(t)|||X_t - \sum X_{i/n}1_{[(i-1)/n,i/n]}(t)||_{L^2}dt$.

We bound the L^2 difference using triangle inequality and the above prop via $2sup_{t\in[0,1]}\sqrt{K(t,t)} < \infty$. Then we can make the expectation small by making this small, and we are done, showing convergence in L^2.

□

Prop 4. *Suppose $K \in C^2$. Then $\forall t \in [0,1]$,*

$$X_t' := \lim_{s \to t} \frac{X_s - X_t}{s - t}$$

exists in $L^2(\Omega)$. Further (X_t') is a centered gaussian process.

Question 2:

Prop 5. $\hat{X}_{n+1/n} = a_n \hat{X}_{n/n}$ *for every* $n \geq 0$.

Proof.

$$\begin{aligned}\hat{X}_{n+1/n} &= E[X_{n+1}|Y_0, ..., Y_n] = E[a_n X_n + \epsilon_{n+1}|Y_0, ..., Y_n] \\ &= E[a_n X_n|Y_0, ..., Y_n] + E[\epsilon_{n+1}|Y_0, ..., Y_n] = a_n E[X_n|Y_0, ..., Y_n] = a_n \hat{X}_{n/n}\end{aligned}$$

where we note ϵ_{n+1} ind. from $Y_0, ..., Y_n$¿ □

Prop 6. $\forall n \geq 1$,

$$\hat{X}_{n/n} = \hat{X}_{n/n-1} + \frac{\mathbb{E}[X_n Z_n]}{\mathbb{E}[Z_n^2]} Z_n$$

where $Z_n := Y_n - c\hat{X}_{n/n-1}$

Proof.

$$\begin{aligned}&\hat{X}_{n/n-1} = \mathbb{E}[X_n|Y_0, ..., Y_{n-1}] \\ &Y_n = cX_n + \eta_n\end{aligned}$$

So compute

$$\hat{X}_{n/n} = \mathbb{E}[X_n|Y_0, ..., Y_n] = \Pi_n(X)$$

which we interpret as the orthogonal projection of X in $L^2(\Omega)$ onto the span of $Y_0, ..., Y_n$. Compute

$$\begin{aligned}Z_n &= Y_n - c\hat{X}_{n/n-1} = Y_n - c\mathbb{E}[X_n|Y_0, ..., Y_{n-1}] \\ &= Y_n + \mathbb{E}[\eta_n - Y_n|Y_0, ..., Y_{n-1}] = Y_n + \mathbb{E}[\eta_n] - \mathbb{E}[Y_n|Y_0, ..., Y_{n-1}] \\ &= Y_n - \Pi_{n-1}(Y_n)\end{aligned}$$

Then

$$\begin{aligned}\hat{X}_{n/n} &= \mathbb{E}[X_n|Y_0, ..., Y_n] = \Pi_n(X_n) \\ &= \Pi_{n-1}(X_n) + \Pi_{Z_n}(X_n) = \mathbb{E}[X_n|Y_0, ..., Y_{n-1}] + \langle X_n, \hat{Z_n}\rangle \hat{Z}_n \\ &= \hat{X}_{n/n-1} + \frac{\mathbb{E}[X_n Z_n]}{\mathbb{E}[Z_n^2]} Z_n\end{aligned}$$

□

Prop 7. *We compute* $\mathbb{E}[X_n Z_n]$ *and* $\mathbb{E}[Z_n^2]$ *and infer*

$$\hat{X}_{n+1/n} = a_n(\hat{X}_{n/n-1} + \frac{cP_n}{c^2 P_n + \delta^2} Z_n)$$

Proof. Compute

$$\mathbb{E}[Z_n^2] = \mathbb{E}[(Y_n - c\hat{X}_{n/n-1})^2] = \mathbb{E}[Y_n - cX_n + cX_n + c\hat{X}_{n/n-1})^2] =$$
$$\mathbb{E}[(\eta_n + cX_n - c\hat{X}_{n/n-1})^2] = c^2 P_n + \mathbb{E}[\eta_n^2] + 2c\mathbb{E}[\eta_n(X_n - \hat{X}_{n/n-1})]$$
$$= c^2 P_n + \delta^2 + 2c\mathbb{E}[\eta_n(X_n - \hat{X}_{n/n-1})]$$

we know X_n is $\sigma(\epsilon_k, 0, .., n)$ measurable so it must be the last term is 0 and so we have

$$\mathbb{E}[Z_n^2] = c^2 P_n + \delta^2$$

Now compute

$$\mathbb{E}[\hat{X}_{n/n-1}(X_n - \hat{X}_{n/n-1})] = \mathbb{E}[\Pi(X_n)(X_n - \Pi(X_n))]$$

□

Prop 8. *We show* $P_1 = \sigma^2$ *and for every* $n \geq 1$ *we have* $P_{n+1} = \sigma^2 + a_n^2 \frac{\delta^2 P_n}{c^2 P_n + \delta^2}$

Proof. Compute $P_1 = \mathbb{E}[(X_1 - \mathbb{E}[X_1|\eta_0])^2] = \mathbb{E}[(\epsilon_1 - \mathbb{E}[\epsilon_1])^2] = \sigma^2$

Further $P_{n+1} = \mathbb{E}[(X_{n+1} - \hat{X}_{n+1/n})^2] = \mathbb{E}[(a_n X_n + \epsilon_{n+1} - a_n \hat{X}_{n/n})^2] = \mathbb{E}[\epsilon_{n+1}^2] + a_n^2 \mathbb{E}[(X_n - \hat{X}_{n/n})^2] - 2a_n\mathbb{E}[\epsilon_{n+1}(X_n - \hat{X}_{n/n}]$. The last term is 0 via independence. □

Question 3:

Define

$$h_{n,k}(t) = 2^{n/2} 1_{[2k/2^{n+1}, 2k+1/2^{n+1})}(t) - 2^{n/2} 1_{[2k+1/2^{n+1}, 2k+2/2^{n+1})}$$

1.

Prop 9. *The* $h_{n,k}$ *form an orthonormal basis on* $L^2([0,1])$

Proof. We clearly have the $h_{n,k}$ orthonormal on $L^2([0,1])$. It suffices to show the system complete.

We know the dyadic decomposition $1_{[k/2^n, k+1/2^n)}$ are dense in $L^2([0,1])$. So we just show these are contained in the span of the $h_{n,k}$.

□

2.

Prop 10. *Let* $\{N_0\} \cup \{N_{n,k}\}$ *are independent* $N(0,1)$. *Then* $\exists!$ *gaussian whit enoise* G *on* $[0,1]$ *with intensity dt s.t.* $G(h_0) = N_0$ *and* $G(h_k^n) = N_k^n$.

Proof. Uniqueness given existence is clear.

Define

$$G(c_0 h_0 + \sum_{h=0}^{\infty} \sum_{k=0}^{2^n-1} c_{n,k} h_{n,k}) = c_0 N_0 + \sum_{h=0}^{\infty} \sum_{k=0}^{2^n-1} c_{n,k} N_{n,k}$$

Via is an isometry with intensity dt since these are sums of gaussians on one hand and sums of orthonormal coordinates on the other. □

Prop 11. *For* $t \in [0,1)$ *we write* $B_t = G(1_{[0,t]})$. *Then*

$$B_t = tN_0 + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} g_{n,k}(t) N_{n,k}$$

with convergence in L^2 *and* $g_{n,k}(t) = \int_0^t h_{n,k}(s)ds$

Proof. This follows directly from the linearity of inner product and Gaussian white noise. □

Prop 12. *For every integer* $m \geq 0$ *and* $t \in [0,1]$ *we have*

$$B_t^m = tN_0 + \sum_{n=0}^{m-1} \sum_{k=0}^{2^n-1} g_{n,k}(t) N_{n,k}$$

Then $t \to B_t^m$ *converges uniformly on* $[0,1]$.

Proof. We want to use Borel Cantelli. To do so compute $\sum P(sup_{0 \leq k \leq 2^n-1} |N_{n,k} > 2^{n/4}) \leq \sum^{\infty} \sum^{2^n-1} P(|N_{n,k}| > 2^{n/4}) \leq \sum 2^n e^{-2^{n/2-1}} < \infty$. So borel cantelli tells us

$$P(\bigcup \bigcap \{sup_{0 \leq k \leq 2^n-1} |N_{n,k}| \leq 2^{n/4}\})$$

Now compute $sup_{t \in [0,1]} |\sum_{n=m_1}^{m_2} \sum_{k=0}^{2^n-1} g_{n,k}(t) N_{n,k}| \leq \sum_{m_1}^{m_2} 2^{-n/4} \to 0$. This gives uniform convergence on $[0,1]$ and yields continuity of the map $t \to B_t$. We can then compute $\mathbb{E}[(B_t - B_s)^2] = \mathbb{E}[G(1_{(s,t]})^2] = t - s$ and $\mathbb{E}[(B_t - B_s)B_r] = 0$. □

Prop 13. *We know we can find* W_t *equal to* B_t *almost surely which has cts sample path*

Proof. We define gaussian gaussian white noises $G^m(c_0 h_0 + \sum \sum c_{n,k} h_{n,k}) = c_0 N_0^m + \sum \sum c_{n,k} N^{n,km}$ as above and define B_t^m correspondingly. Then define $W_t = \sum_{k=1}^{m-1} B_1^k + B_{t-floor(t)}^m$. This has the desired continouous sample path. □

Question 4:

Prop 14. *Let B standard brownain motion. Let $f \in C_c^\infty$ with $p \in [0,1]$ and partition up to time t*

$$S_f(p) = \sum_{j=0}^{n-1} f(B_{(1-p)t_j + pt_{j+1}})(B_{t_{j+1}} - B_{t_j})$$

For smooth f we have $S_f(p) - S_f(0)$ converges in probability as the size of the partition goes to zero.

Proof. We claim the expression converges in probabiliy to pt.

First we consider $f(x) = x$. In this case we have

$S(p) = \sum_{j=0}^{n-1}(B_{(1-p)t_j+pt_{j+1}})(B_{t_{j+1}} - B_{t_j})$ and then

$$\begin{aligned} S_f(p) - S_f(0) &= \sum_{j=0}^{n-1}(B_{(1-p)t_j+pt_{j+1}})(B_{t_{j+1}} - B_{t_j}) - \sum_{j=0}^{n-1}(B_{t_j}(B_{t_{j+1}} - B_{t_j}) \\ &= \sum_{j=0}^{n-1}(B_{(1-p)t_j+pt_{j+1}} - B_{t_j})(B_{t_{j+1}} - B_{t_j}) \end{aligned}$$

Note by proposition 2.16 we know for $p = 1$ we get convergence to the variance of B_t ie. t in L^2 and therefore in probability. The first term also does, yielding t as an upper bound. But as $p \to 0$ clearly via via continuity of sample paths we get 0. Write $\sum_{j=0}^{n-1}(B_{(1-p)t_j+pt_{j+1}} - B_{t_j})(B_{t_{j+1}} - B_{t_j}) = \sum_{j=0}^{n-1}(G(1_{[t_j,pt_{j+1})}))(G(1_{[t_j,t_{j+1})]})) \to tp$

The same analysis applies to the general case to compute $f(t)p$ as the limit.

∎

Question 5:

Prop 15. *Let $G : L^2([0,1]) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ linear s.t.*

1. *$G(f)$ has zero mean and preserves norm*
2. *If $A, B \subseteq [0,1]$ with $A \cap B = \emptyset$, $G(A)$ ind. from $G(B)$*
3. *G is stationary in the sense $G(A)$ has same dist as $G(A + x)$ when $A + x \subseteq [0,1]$ with $A \subseteq [0,1]$.*
4. *For any $n \in \mathbb{Z}_+$ we find $C_n > 0$ s.t. $\mathbb{E}[G(A)/\sqrt{|A|}^n] \leq C_n$ for $A \subseteq \mathbb{R}$.*

Then G is a gaussian white noise.

Proof. Linearity and norm preservation prove G is isometry. It suffices to show its image is a gaussian space. In particular suffices to show $G(1_A)$ is a gaussian as then via linearity and a density argument we have the image gaussian.

Mean 0 comes from 1. To show $G(1_A)$, $A \subseteq [0,1]$ gaussian we consider its characteristic $\mathbb{E}e^{itG(1_A)}$, which is shown to be the characteristic of a gaussian. □

Question 6:

Prop 16. *Let $\mathbb{W}$ be a gaussian white enoise on $L^2(\mathbb{R}^d)$. Define $W_r(x) = \frac{\mathbb{W}(B_r(x))}{|B_r(x)|}$. Then $\{W_r(x), x \in \mathbb{R}^d\}$ is a gaussian process. ITs distribution does not converge as $r \to 0$*

Proof. First we show we have a gaussiain process for fixed r. Clearly for each $x \in \mathbb{R}$ we have $W_r(x)$ is a gaussian random variable since $\mathbb{W}$ is a gaussian white noise. Then we consdier a linear combination $\sum_i c_i W_{x_i}$ which is also gaussian since if a ball is nonintersecting then it's independenct and if we have an intersection it can be rewritten as the sum of limits of balls which are disjoint(and thus independent gaussians). Let $\alpha = |B_r(x)|$. We compute its covariance

$$\mathbb{E}[W_r(x)W_y(x)] = \frac{1}{\alpha^2}\int_{\mathbb{R}^d} 1_{B_r(x)}1_{B_r(y)}dm = \frac{m(B_r(x) \cap B_r(y))}{\alpha^2}$$

Now we argue $W_r(x)$ does not converge in distribution as $r \to 0$ for fixed x. Note if it did converge it must converge to a gaussian. But if this were true the variances would converge. Note the variance is $m(B_r(x) \cap B_r(x))/\alpha^2 = \alpha/\alpha^2 = \alpha = 1/|B_r(x)|$. Clearly these blow up as $r \to 0$ so this cannot be the case. □

Question 7:

Denote by $\{e_n(x)\} = \{1/\pi sin(nx), 1/\pi cos(nx)\}$ ONB of $L^2[0, 2\pi]$. For any $\lambda = (\lambda_n) \in l_2$. Define $V(x) = \sum_n \lambda_n e_n(x)\xi_n$ with $\{\xi_n\}$ is iid standard gaussian.

Prop 17. *We know $\{V(x) : x \in [0, 2\pi]\}$ is a gaussian process*

Proof. Clearly for fixed x, $V(x)$ is a centered gaussian. Note linear combinations of these are simply linear combinatinos of the $\{\xi_n\}$ iid basis standard gaussians, which is itself a gaussian. We compute the covariance:

$$\mathbb{E}[V(x)V(y)] = \mathbb{E}[\sum_n \lambda_j e_j(x)\xi_j \sum_n \lambda_i e_i(x)\xi_i] = \mathbb{E}[\sum_n \lambda_j^2 e_j^2 \xi_j^2] = \sum_n \lambda_j^2 e_j^2(x)$$

□

Prop 18. *We have* $G(f) = \int_0^{2\pi} f(x)V(x)dx = \sum_n \lambda_n \langle f, e_n \rangle \xi_n$ *is a mapping from* $L^2[0, 2\pi]$ *to a centered gaussian space*

Proof. Clearly the outputs are centered gaussians □

Prop 19. *Find the operator* $K : L^2[0, 2\pi] \to L^2[0, 2\pi]$ *s.t.* $\mathbb{E}[G(f)G(g)] = \langle f, Kg \rangle$

Proof. Compute:

$$\mathbb{E}[G(f)G(g))] = \mathbb{E}[\sum_n \lambda_i \langle f, e_i \rangle \xi_i \sum_n \lambda_j \langle g, e_j \rangle \xi_j] = \sum_n \lambda_i^2 \langle f, e_i \rangle \langle g, e_i \rangle$$
$$= \langle f, \sum_i \lambda_i^2 \langle g, e_i \rangle e_i \rangle$$

which gives K. □

Prop 20. *We choose a sequence of* λ *so that for corresponding* V *approximates a gaussian white nosie.*

Proof. We simply need to approximate an isometry ie. want $\langle f, g \rangle = \langle f, \sum_i \lambda_i^2 \langle g, e_i \rangle e_i$. So we choose eventually 0 sequences of 1 since the e_i form an ONB. □

Question 8:

Prop 21. *Let* G *be a gaussian white noise on* $L^2(\mathbb{R}^d)$ *and let* $\phi \in C_c^\infty(\mathbb{R}^d)$*. Set* $\{V(x) = G(\phi(\cdot - x))\}_{x \in \mathbb{R}^d}$*. We compute the covariance and show* V *has a continuous modification*

Proof. Without loss of generality suppose $d = 1$.

Let K be the compact support of ϕ and M its measure. Let m be the maximum value of ϕ.

First computing the covariance:

$$\mathbb{E}[V(x)V(y)] = \mathbb{E}[G(\phi(\cdot - x))G(\phi(\cdot - y))] = \int_{\mathbb{R}^d} \phi(z - x)\phi(z - y)dm$$
$$= \int_{\mathbb{R}^d} \phi(z)\phi(z + x - y)dm$$

ie. the covariance is a measure of how much the compact support and shifted compact support overlap.

We seek to use Kolmogorov's lemma. Write

$$\mathbb{E}[|X_s - X_t|^2] = \int_{\mathbb{R}^d} (\phi(z - s) - \phi(z - t))^2 dm \leq 2mM|s - t|$$

where we get the last inequality by considering the geometry in 1d and looking at the overlap. Thus we may apply Kolmgorov. □

Question 9:

Prop 22. *Consider a random process $(F(x))_{x\in\mathbb{R}^d}$ satisfying*

$$\sup_{x\in\mathbb{R}^d}\mathbb{E}[|F(x)|^p]\leq C(p),\ \sup_{x,y\in\mathbb{R}^d}\mathbb{E}[|F(x)-F(y)|^p]\leq C(p)|x-y|^{p\beta}$$

for $p\geq 1$ where where $C(p)>0$ is some constant only depending on p. Let $w_\alpha(x)=(1+|x|)^\alpha$ be a weight, then there exists a modification of F, which we still denote by F, s.t. for any $p\geq 1,\alpha>0,\epsilon>0$ we have

$$\mathbb{E}[(\sup_{x\in\mathbb{R}^d}\frac{|F(x)|}{w_\alpha(x)})^p]+\mathbb{E}[(\sup_{x,y\in\mathbb{R}^d,|x-y|\leq 1}\frac{|F(x)-F(y)|}{w_\alpha(x)|x-y|^{\beta-\epsilon}})^p]\leq C(p,\alpha,\epsilon)$$

for some constant C depending on p,α,ϵ.

Proof. It suffices to consider $d=1$. Otherwise we simply biject and reindex.

Kolmogorov's lemma immediately tells us we can find a modification satisfying a pointwise bound(but not necessarily uniform as is desired). Because we have $sup_{x\in\mathbb{R}^d}\mathbb{E}[|F(x)|^p]\leq C(p)$ we know our random variables in L^p and uniformly bounded. Because we have this integrability, we know our random variables cannot be too large most of the time. In particular we will have $\mathbb{E}[(sup_{x\in\mathbb{R}^d}\frac{|F(x)|}{w_\alpha(x)})^p]$ uniformly bounded dependent on α and p since otherwise we would not have this uniform integrability. In particular if this quantity was not finite we know with nonzero probability. we can find a process $F(x)$ which is infinite for some input. Further since we are weighting with w_α which blows up nonintegrably we know for large x we must find correspondingly larger $|F(x)|$ as $|x|\to\infty$. We can repeat this argument for every input which has inifinte supremum. Then using this plus the second condition allows us to construct an $F(x)$ which does not have finite L^p norm, as a we finite a sequence of random variables which get large on a nonnegligable subset, a contradiction.

We can argue similarly for the other term to get the overall bound. □

Question 10:

Prop 23. *The process $(W_t)_{t\geq 0}$ via $W_0=0$ and $W_t=tB_{1/t}$ is a real brownian motion started from 0. Then argue $lim_{t\to\infty}B_t/t=0$ a.s.*

Proof. First we establish W a pre-Brownian motion and then conclude indistinduishable from brownian motion up to a modification.

Clearly the process is centered as each B_t centered. Then we compute the covariance:

$$\mathbb{E}[W_sW_t]=\mathbb{E}[stB_{1/s}B_{1/t}]=stmin(1/s,1/t)=min(s,t)$$

Then up to a modification W_t is a brownian motion(has continuous sample paths).

We know $\frac{B_n}{n} \to 0$ via summing $B_{n+1} - B_n$ and SLLN. So we have the desired result in the discrete case.

To extend to the continuos case we apply the maximal inequality(Clearly brownian motion supermartingale) to see

$$nP(sup_{0 \le s \le n}|B_n - B_s| > n) \le E[|B_n - B_0|] = E[|B_n|]$$

Then applying borel-cantelli we see $\frac{B_s}{s} \to 0$ almost surely. Since this occurs almost everywhere redfine W_s to appropriately on a set of measure 0. This completes the proof.

□

Question 11:

Set $W_t = B_t - tB_1$ for $t \in [0,1]$

Prop 24. *$(W_t)_{t\in[0,1]}$ is a centered gaussian process*

Proof. To show W_t is a gaussian process we show an arbitrary linear combination $\sum_i c_i W_{t_i}$ is normally distributed. But we have

$$\sum_i c_i W_{t_i} = -S(B_1 - B_{t_m}) + (S + c_m)(B_{t_m} - B_{t_{m-1}}) + ... + (S + \sum_{i=1}^m c_i)B_{t_1}$$

where S is the dot product of time ts and coefficient cs. This is a sum of independent gaussians and is hence gaussian.

We compute covariance:

$$\mathbb{E}]W_s W_t] = \mathbb{E}(B_s - sB_1)(B_t - tB_1) = t \wedge s - ts - ts + ts = t \wedge s - ts$$

□

Prop 25. *Let $0 < t_1 < t_2 < ... < t_p < 1$. Then the law of $(W_{t_1}), ..., W_{t_p})$ has density*

$$g(x_1, ..., x_p) = \sqrt{2\pi} p_{t_1}(x_1) p_{t_2 - t_1}(x_2 - x_1) ... p_{1-t_p}(-x_p)$$

Then the law of $(W_{t_1}, ..., W_{t_m})$ can be interpreted as conditional law on $(B_{t_1}, .., B_{t_m})$ when $B_1 = 0$.

Proof. We set a partitioning t_i of $[0,1]$ and let F be a measurbale function. Compute

$$\begin{aligned}
\mathbb{E}[F(W_{t_1}, ..., W_{t_m})] &= \int_{\mathbb{R}^{m+1}} F(x_1 - t_1 x_{m+1}, ..., x_m - t_m x_{m+1}) \prod p_{t_i - t_{i-1}}(x_i - x_{i-1}) dx_1 .... dx_{m+1} \\
&= \int_{\mathbb{R}^{m+1}} F(y_1, ..., y_m) \prod p_{t_i - t_{i-1}}(y_i - y_{i-1} + (t_i - t_{i-1} y_{m+1})) p_{1-t_m}(y_{m+1} - y_m - t_m y_{m+1}) dy_1 ... \\
&= \int_{\mathbb{R}^m} F(y_1, ..., y_m) \prod p_{t_i - t_{i-1}}(y_i - y_{i-1}) p_{1-t_m}(-y_m) \sqrt{2\pi} dy_1 ... dy_m
\end{aligned}$$

$\square$

Question 12:

Prop 26. *$limsup_{t \to 0} \frac{B_t}{\sqrt{t}} = \infty, liminf_{t \to 0} \frac{B_t}{\sqrt{t}} = -\infty$*

Proof. Let $M > 0$. Compute

$$P(limsup_{t \to 0} \frac{B_t}{\sqrt{t}} \geq M) \geq P(limsup_{n \to \infty} \frac{B_{1/n}}{\sqrt{1/n}} \geq M) \geq limsup_{n \to \infty} P(\frac{B_{1/n}}{\sqrt{1/n}} \geq M)$$
$$= \int_M^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx > 0$$

via Fatou.

The 0-1 Law tells us $\mathcal{F}_{0+}$ is trivial, hence it must be, $P(limsup_{t \to 0} \frac{B_t}{\sqrt{t}} \geq M) = 1$. Note $limsup_{t \to 0} \frac{B_t}{\sqrt{t}} \geq M \in \mathcal{F}_{0+}$ clearly since it is in each element of the intersection. Taking the negation gives a similar result for the liminf. $\square$

Prop 27. *The above implies we have no right derivative for $t \to B_t$.*

Proof. Compute

$$P(limsup_{t \to s} \frac{B_t - B_s}{t - s} = \infty) = P(limsup_{t \to s} \frac{B_{t-s}}{\sqrt{t-s}} = \infty) = 1$$

by rescaling by the variance. A similar computation shows the liminf goes to $-\infty$, establishing the lack of a right derivative. $\square$

Question 13:

Prop 28. *Time Reversal Set $B_t' = B_1 - B_{1-t}$ for $t \in [0, 1]$. This has the same law as B_t.*

Proof. The idea is to compute the expectation under arbitrary (measurable) function F of a collection of B_i' and show it agrees with the expectation of B_i.

Set $\{t_i\}$ as a partitoining of 1. Let $F : \mathbb{R}^m \to \mathbb{R}$. We compute

$$\mathbb{E}[F(B_{t_1}', ..., B_{t_m}')] = \mathbb{E}[F(B_1 - B_{1-t_1}, ..., B_1 - B_{1-t_m})] = \int_{\mathbb{R}^{m+1}} F(x_{m+1} - x_m, ..., x_{m+1} - x_1) \prod_{i=1}^{m+1} p_{t_i - t_{i-1}}(x_i - x_{i-1}$$

Via independence This simplifies to with a change of variables

$$\int_{\mathbb{R}^{m+1}} F(y_1, ..., y_m) \prod p_{t_i - t_{i-1}}(y_i - y_{i-1}) dy_1 ... dy_{m+1}$$

Now note we can factor out $\int_{\mathbb{R}} p_{t_{m+1} - t_m}(y_{m+1} - y_m) dy_{m+1} = 1$ giving the desired result which is $\mathbb{E}[F(B_{t_1}, ..., B_{t_m})]$. Since F arbitrary, this completes the proof. $\square$

Question 14:

Prop 29. *For any bounded function $f : \mathbb{R}^d \to \mathbb{R}$, consider*

$$u(t,x) = \mathbb{E}[f(x+B_t)] = \int_{\mathbb{R}^d} q_t(x-y)f(y)dy$$

Then u solves the heat equation.

Proof. Recall the density $q_t(x) = \frac{1}{\sqrt{2\pi t}^d} e^{-|x|^2/2t}$. Immediatley we should recognize this as the fundamental solution to the heat equation in $\mathbb{R}^d$. So from PDE theory we know the convolution with an initial condition $f : \mathbb{R}^d \to \mathbb{R}$ is a solution to the heat equation.

Probabilistically we compute $u_{x_i x_i}(t,x) = \mathbb{E}[f''(x+B_t)]$ via DCT. Compute

$$u_t(t,x) = \frac{d}{dt}\int_{\mathbb{R}^d} q_t(x-y)f(y)dy = \int_{\mathbb{R}^d} \frac{d}{dt} q_t(x-y)f(y)dy$$

where we pass the limit inside via DCT. We compute the differential w.r.t t of q_t as $-(2/d\sqrt{2\pi}^d)\frac{e^{-|x|^2/2t}}{\sqrt{2\pi}^d\sqrt{t}^{d+1}} + \frac{1}{\sqrt{2\pi t}^d} e^{-|x|^2/2t}(\frac{|x|^2}{2t^2})$

We similarly two derivatives of x:

$$u_{xx}(t,x) = \frac{d^2}{dx^2}\int_{\mathbb{R}^d} q_t(x-y)f(y)dy = \int_{\mathbb{R}^d} \frac{d^2}{dx^2} q_t(x-y)f(y)dy$$

We compute $\frac{d}{dx}q_t(x) = frac1\sqrt{2\pi t}^d e^{-|x|^2/2t}(-\frac{-\sum x_i}{t})$. Differentiating again we see we agree with the time derivative(except for a factor of (1/2)) showing the desired result.

Note we justify DCT via the boundedness of f and by picking a small ball around the origing, keeping the kernel away from 0. □

Question 15:

Let w be as defined and S_n a d-dimensional symmetric simple random walk ind. of d. Set $Z_\beta(n)$ as defined.

Prop 30. $\{Z_\beta(n)\}$ *is a martingale with respect to* $\mathcal{F}_n = \sigma(w(i,j) : i \le n, j \in \mathbb{Z}^d\}$

Proof. Note the process is adapted by design of the filtration(since we take the expectation over the random walk) and we see each term in L^1 since we can compute it explicity(as the product of expectations of exponential gaussians).

We compute

$$\mathbb{E}[Z_\beta(n+k)|\mathcal{F}_n] = \mathbb{E}[e^{\beta\sum^{n+k} w(i,S_i)-(1/2)\beta^2(n+k+1)}|\mathcal{F}_n] = \mathbb{E}[e^{\beta\sum_{n+1}^{n+k} w(i,S_i)-(1/2)\beta^2 k}|\mathcal{F}_n]e^{\beta\sum^{n} w(i,S_i)-(1/2)\beta^2(n+1)}$$

so we argue remainder of expectation is 1.

Write $\mathbb{E}[e^{\beta\sum_{n+1}^{n+k} w(i,S_i)-1/2)\beta^2 k}] = \mathbb{E}[e^{\beta\sum_{n+1}^{n+k} w(i,S_i)}]e^{(-1/2)\beta^2 k} = 1$ since the expectation of the gaussian random variables is $e^{(1/2)\beta^2}$ where the variance is β^2. This shows the random variable a martingale. ☐

Prop 31. *$Z_\beta(n)$ converges a.s. as $n \to \infty$.*

Proof. $Z_\beta(n)$ converges almost surely to 1. To see this we compute the expectation of one term, which is $e^{(1/2)\beta^2} * e^{-(1/2)\beta^2}$. Then as $n \to \infty$, since we have integrability we apply the SLLN the log and see this goes to 0. Hence pointwise the exponential of this goes to 1. ☐

Question 16:

Prop 32. *Let M be a martingale with continuous sample paths s.t. $M_0 = x > 0$. Suppose $M_t \geq 0$ for $t \geq 0$ and $M_t \to 0$ when $t \to \infty$ a.s. Then for $y > x$ we have*

$$P(sup_{t\geq 0}M_t \geq y) = \frac{x}{y}$$

Proof. The idea is to first show the uniformly integrable case and then extend to the general case via a stopping time argument.

In the uniformly integrable case we know M bounded in L1 and $M_\infty = 0$. We choose the stopping time $T = inf\{t \geq 0 | M_t = y\}$ ie. the first time at which we get to y. Via optional stopping we know $\mathbb{E}[M_T] = \mathbb{E}[M_0] = x$. Then $\mathbb{E}[M_T] = yP(T < \infty)$ since $M_t \to 0$. We compute

$$P(T < \infty) = P(sup_{t\geq 0}M_t \geq y)$$

which establishes what is desired.

In the general case for $n \geq 1$ we write $N_t^{(n)} = M_{t\wedge n}$. Thus for fixed n we know $\{N_t^{(n)}\}$ uniformly integrable and we have the result. Letting $n \to \infty$ finished the proof. ☐

Prop 33. *We compute the law of $sup_{t\leq T_0}B_t$ started from $x > 0$ and $T_0 = inf\{t \geq 0 : B_t = 0\}$*

Proof. Clearly for $y \leq x$ we know $P(sup_{t\leq T_0}B_t \geq y) = 1$. For $y > x$, write $N_t = B_{t\wedge T_0}$. We know this is a martingale since B_t martingale. Further we know $N_t \to 0$ since $T_0 < \infty$. Then we apply the previous propsition to finish, getting $\frac{x}{y}$. ☐

Prop 34. *Suppose B is a brownia motion started at 0, and $\mu > 0$. Then $sup_{t\geq 0}(B_t - \mu t)$ is exponentially distributed with parameter 2μ*

Proof. The idea is to massage the expression into one computable with martingale technology using the scaling property of brownian motion.

For nonpositive y clearly we have probability 1. Fix $y > 0$. Note from a direct computation we know $e^{B_t-(1/2)t}$ is martingale. Also recall if B_t brownian motion then $\frac{1/\lambda}{B}{}_{\lambda^2 t}$ brownian motion. So massage:

$$\begin{aligned}P(sup_{t\geq 0}(B_t - \mu t) \geq y) &= P(sup_{t\geq 0}(B_{t/4\mu^2} - 1/(4\mu)t) \geq y)\\&= P(sup_{t\geq 0}(2\mu B_{t/4\mu^2} - 1/2t) \geq 2\mu y)\\&= P(sup_{t\geq 0}(B_t - 1/2t) \geq 2\mu y) = P(sup_{t\geq 0}e^{(B_t-1/2t)} \geq e^{2\mu y})\end{aligned}$$

To conclude we argue $P(sup_{t\geq 0}e^{(B_t-1/2t)} \geq e^{2\mu y}) = e^{-2\mu y}$ which establishes the distribution. We get this since $e^{2\mu y} > 1 = e^{B_0-0}$ and $e^{B_t-1/2t} \to 0$ since $B_t/t \to 0$.

□

Question 17:

TODO

Let B be a brownian motion at 0. Set $T_x = inf\{t \geq 0 : B_t = x\}$. Fix $a < 0 < b$ and write $T = T_a \wedge T_b$.

Prop 35. *For every $\lambda > 0$ we have*

$$\mathbb{E}[e^{-\lambda T}] = \frac{cosh(\frac{b+a}{2}\sqrt{2\lambda})}{cosh(\frac{b-a}{2}\sqrt{2\lambda})}$$

Proof. Write $M_t = e^{\sqrt{2\lambda}(B_t-(b+a)/2)-\lambda t} + e^{-\sqrt{2\lambda}(B_t-(b+a)/2)-\lambda t}$. Note $C_t = e^{\sqrt{2\lambda}B_t - \frac{2\lambda}{2}t}$ and $D_t = e^{-\sqrt{2\lambda}B_t - \frac{2\lambda}{2}t}$ are martingales. Then the sum scaled sum M_t is a martingale which is uniformly integrable(since the terms uniformly integrable). By optional stopping we have $\mathbb{E}[M_T] = \mathbb{E}[M_0] = 2cosh(\sqrt{2\lambda}(b+a)/2)$. Furter we can compute $\mathbb{E}[M_T] = e^{-\sqrt{2\lambda}(b-a)/2}\mathbb{E}[e^{-\lambda T}1_{T_a\leq T_b}] + e^{\sqrt{2\lambda}(b-a)/2}\mathbb{E}[e^{-\lambda T}1_{T_a\leq T_b}] + e^{\sqrt{2\lambda}(b-a)/2}\mathbb{E}[e^{-\lambda T}1_{T_a>T_b}] + e^{-\sqrt{2\lambda}(b-a)/2}\mathbb{E}[e^{-\lambda T}1_{T_a>T_b}]$. In turn this equals $\mathbb{E}[e^{-\lambda T}](e^{\sqrt{2\lambda}(b-a)/2} + e^{-\sqrt{2\lambda}}(b-a)/2) = \mathbb{E}[e^{-\lambda T}]2cosh(\sqrt{2\lambda}(b-a)/2)$ which gives the desired result □

Prop 36. *For $\lambda > 0$*

$$\mathbb{E}[e^{-\lambda T}1_{T=T_0}] = \frac{sinh(b\sqrt{2\pi})}{sinh((b-a)\sqrt{2\lambda})}$$

Proof. We can compute $\mathbb{E}[e^{\sqrt{2\lambda}(B_t-(b+a)/2)} - e^{-\sqrt{2\lambda}(B_t-(b+a)/2)-\lambda t}] = -2sinh(\sqrt{2\lambda}(a+b)/2)$. We also compute it as $-2sinh(\sqrt{2\lambda}(b-a)/2)\mathbb{E}[e^{-\lambda T}1_{T_a \le T_b}] + 2sinh(\sqrt{2\lambda}(b-a)/2)\mathbb{E}[e^{-\lambda T}1_{T_a > T_b}]$.

Using the *sinh* additivity formula yields the result. ☐

Prop 37. *We compute $P(T_a < T_b)$*

Proof. We simply use DCT on part 2. To see $P(T_a < T_b) = \mathbb{E}[1_{T=T_a}] = lim_{\lambda \to 0}\mathbb{E}[e^{-\lambda T}1_{T=T_a}] = \frac{b}{b-a}$ when comparing exponentials. ☐

Question 18:

Let B_t be a brownian motion starting at 0. Let $a > 0$ with $\sigma_a = inf\{t \ge 0 : B_t \le t - a\}$

Prop 38. *$\sigma_a < \infty$ is a stopping time*

Proof. Clearly σ_a respects the sigma field since it is a function of B_t. Further the probability $B_t - t$ is greater than $-a$ always is 0. More formally $liminf B_t - t = -\infty$. ☐

Prop 39. *For every $\lambda \ge 0$ we have*

$$\mathbb{E}[e^{-\lambda\sigma_a}] = e^{-a(\sqrt{1+2\lambda}-1)}$$

Proof. We know that for $\mu = 1 - \sqrt{1+2\lambda}$ we have $e^{\mu B_t^{\sigma_a} - \mu^2/2\sigma_a \wedge t}$ is a (local) martingale. ☐

Prop 40. *Let $\mu \in \mathbb{R}$ and set $M_t = e^{\mu B_t - \frac{\mu^2}{2}t}$. Then the stopped martingale $M_{\sigma_a \wedge t}$ is closed $\iff$ $\mu \le 1$.*

Proof. Clearly $1 = \mathbb{E}[M_{\sigma_a}] = \mathbb{E}[e^{\mu(\sigma_a - a) - \mu^2/2\sigma_a}] = \mathbb{E}[e^{-(\mu^2/2-\mu)\sigma_a - \mu a}] \iff \mathbb{E}[e^{-(\mu^2/2-\mu)\sigma_a}] = e^{\mu a}$. By the second proposition we know $\mathbb{E}[e^{-\mu^2/2-\mu)\sigma_a}] = e^{-a(\mu-2)}$ when $\mu > 1$ and $e^{a\mu}$ otherwise.

We know $1 = \mathbb{E}[M_{0\wedge\sigma_a}] = \mathbb{E}[M_{\infty\wedge\sigma_a}] = \mathbb{E}[M_{\sigma_a}]$ when we have closedness. We further know in the other direction $M_{t\wedge\sigma_a} \ge \mathbb{E}[M_{\sigma_a}|\mathcal{F}_{t\wedge\sigma_a}]$ and then if probability the RV exceeds its expectation is nonzero we will have $1 = \mathbb{E}[M_{0\wedge\sigma_a}] = \mathbb{E}[M_{t\wedge\sigma_a}] > \mathbb{E}[M_{\sigma_a}] = 1$, a contradiciton. This concludes the proof ☐

Question 19:

Prop 41. *Let M be a martingale with cts sample path with $M_0 = 0$. Further M a gaussian process. Then for every $t \ge 0$ and $s > 0$ we have $M_{t+s} - M_t$ is independent o $\sigma(M_r, 0 \le\le t)$.*

Proof. Compute $\mathbb{E}[M_{s+t}M_t] = \mathbb{E}[M_{s+t}|M_t]\mathbb{E}[M_t] = \mathbb{E}[M_t^2]$. We then also have $\mathbb{E}[(M_{t+s} - M_t)M_r] = 0$ for $0 \leq r \leq t$. Then we must have independence which follows from orthogonality of these gaussian random variables. □

Prop 42. *We show using 1.* $\exists$ *a continuous monotone nondecreasing* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *s.t.* $\langle M, M\rangle_t = f(t)$

Proof. We use the function $f(t) = \mathbb{E}[M_t^2]$ inspired by the brownian motion case($\langle B, B\rangle = t = \mathbb{E}[B_t^2]$).

Via Jensen's inequality and the martingale property this is nondecreasing. To show continuity consider Doob's inequality which says $\mathbb{E}[sup_{0\leq s\leq T}|M_s|^2] \leq 4\mathbb{E}[|M_T|^2| < \infty$ for arbitrary $T > 0$. Then dominated convergence justifies $lim_{n\to\infty}f(t_n) = lim_{n\to\infty}\mathbb{E}[M_{t_n}^2] = E[M_t^2] = f(t)$

We establish $f(t) = \langle M, M\rangle_t$. □

Question 20:

Let M be a CLM with $M_0 = 0$.

Prop 43. *For every* $n \geq 1$ *let* $T_n = inf\{t \geq 0 : |M_t| = n\}$. *Then a.s.* $\{lim_{t\to\infty}M_t \text{ exists and finite}\} = \bigcup_{n\geq 1}\{T_n = \infty\} \subseteq \{\langle M, M\rangle_\infty < \infty\}$

Proof. Recall M has continuous sample paths so $T_n = inf\{t \geq 0||M_t| \geq n\}$ reduces M and then M^{T_n} is uniformly integrable. Thus we have $M_\infty^{T_n}$. We also know $M, M\rangle_{T_n} < \infty$. Let $A = \bigcup_{n\geq 1}\{M_\infty^{T_n} existsand \langle M, M\rangle_{T_n} < \infty\}$. We know $P(A) = 1$.

For fixed sample w in A s.t. M_t limit exists we know $|M_t(w)| \leq M$ uniformly bounded. Then $T_m(w) = \infty$ for $m > M$. Therefore $w \in A \cap \bigcup\{T_n = \infty\}$. Now suppose $T_m(w) = \infty$ for some m. Then it must be $M_\infty(w) = M_\infty^{T_m}(w)$ and is less than m. Also $\langle M, M\rangle_\infty(w) = \langle M, M\rangle_{T_m}(w) < \infty$. □

Prop 44. *Set* $S_n = inf\{t \geq 0 : \langle M, M\rangle_t = n\}$. *Then a.s.* $\{\langle M, M\rangle_\infty < \infty\} = \bigcup_{n\geq 1}\{S_n = \infty\} \subseteq \{lim_{n\to\infty}M_t \text{ exists and finite}\}$

Proof. We have since $\langle M, M\rangle$ increasing $\{\langle M, M\rangle_\infty < \infty\} \bigcup_{n\geq 1}\{S_n = \infty\}$. Fixing $n \geq 1$ we write $\langle M^{S_n}, M^{S_n}\rangle_t = \langle M, M\rangle_{S_n\wedge t} \leq n$. So $\mathbb{E}[\langle M^{S_n}, M^{S_n}\rangle \leq n$. So we have L^2 boundedness and thus convergence a.s. Sending to the limit gives the desired result.

Since we have double inclusion this gives set equality. □

Question 21:

For $n \geq 1$ let $M^n = (M_t^n)$ be a CLM 0 at $t = 0$. Suppose $lim_{n\to\infty}\langle M^n, M^n\rangle_\infty$ in probability

Prop 45. *Let $\epsilon > 0$ and for all $n \geq 1$ let $T_\epsilon^n = inf\{t \geq 0 : \langle M^n, M^n\rangle_t \geq \epsilon\}$. Then T_ϵ^n is a stopping time and $M_t^{n,\epsilon} = M_{t\wedge T_\epsilon^n}^n$ is a martingale bounded in L^2*

Proof. Because $\langle M^n, M^n\rangle$ has continuos sample paths we know T_ϵ^n is a stopping time. Then the stopping $(M^n)^{T_\epsilon^n}$ is a CLM s.t. $\langle M^{n,\epsilon}, M^{n,\epsilon}\rangle \leq \epsilon$. Thus it must be $M^{n,\epsilon}$ bounded in L^2. □

Prop 46. *We have $\mathbb{E}[sup|M_t^{n,\epsilon}|^2] \leq 4\epsilon$*

Proof. We showed $M_t^{n,\epsilon}$ bounded in L^2. So we know $\mathbb{E}[M_\infty^{n,\epsilon 2}] = \mathbb{E}[\langle M^{n,\epsilon}, M^{n,\epsilon}\rangle_\infty] \leq \epsilon$.

Next since we want to bound the sup we apply Doob's maximal inequality yielding $\mathbb{E}[sup_{0\leq s\leq t}|M_s^{n,\epsilon}|^2 \leq 4\mathbb{E}[|M_t^{n,\epsilon}|^2]$. Because we have a martingale we also know $\mathbb{E}[(M_s^{n,\epsilon})^2] \leq \mathbb{E}[(M_t^{n,\epsilon})^2]$. Then applying our bound on the limit yield which upper bounds the sup yields the result for fixed t. Since it holds for all t it holds and then limit, giving the desired inequality. □

Prop 47. *For $a > 0$ write $P(sup_{t\geq 0}|M_t^n| \geq a) \leq P(sup_{t\geq 0}|M_t^{n\epsilon}| \geq a) + P(T_\epsilon^n < \infty)$.*

Then $lim_{n\to\infty}(sup_{t\geq 0}|M_t^n|) = 0$ in probability

Proof. We can compute

$$P(sup_{t\geq 0}|M_t^n| \geq a) \leq P(sup_{t\geq 0}|M_t^n| \geq a, T_\epsilon^n = \infty) + P(T_\epsilon^n < \infty) \leq P(sup_{t\geq 0}|M_t^{n,\epsilon}| \geq a) + P(T_\epsilon^n < \infty)$$

The first term in the upper bound can be bound using chebyshev and applying the previous proposition. The second term goes to 0 as $n \to \infty$ and then as $\epsilon \to 0$. Hence we get convergence to 0 in probability. □

Question 22:

Let X_t be a continuous and uniformly integrable martingale with $X_0 = 0$. Suppose we can find $M > 0$ s.t. $\mathbb{E}[|X_\infty - X_\tau||\mathcal{F}_\tau] \leq M$ for every stopping time τ. Define X^* as the sup. Then

Prop 48. *For all $\lambda, \mu > 0$ we have*

$$P(X^* \geq \lambda + \mu) \leq M/\mu P(X^* \geq \lambda)$$

Question 23:

Prop 49. *Let V be a progressively measurable process s.t. $\int_0^\infty V_S^2 ds \leq 1$ a.s. Then for $x \geq 0$ we have $P(sup_{t\geq 0} \int_0^t V(s)dB_s \geq x) \leq e^{-x^2/2}$*

Proof. □

Question 24:

Prop 50. $\mathbb{E}[B_1^{2n}] = (2n-1)...(3)(1)$

Proof. Note B_1^{2n} is just a normal gaussian. We prove this via induction. Clearly the base case holds. We proceed with integration by parts via Ito.

$$B_1^{2n} = B_1 B_1^{2n-1} = B_0^{2n} + \int_0^1 B_s dB_s^{2n-1} + \int_0^1 B_s^{2n-1} dB_s + \langle B, B^{2n-1} \rangle_1$$

Note $dB_s^{2n-1} = (2n-1)(2n-2)B_s^{2n-3}ds + (2n-1)B_s^{2n-2}dB_s$. When we take the expectation we see the last two terms disappear(via symmetry) and hence are left with $\mathbb{E}[B_1^{2n}] = \mathbb{E}\int_0^1 B_s((2n-1)(2n-2)B_s^{2n-3}ds + (2n-1)B_s^{2n-2}dB_s) = \mathbb{E}\int_0^1 (2n-1)(2n-2)B_s^{2n-2}ds = (2n-1)\mathbb{E}[B_1^{2n-2}]$ which completes the induction and the proof. □

Question 25:

We define the Hermite Polynomial as

$$H_n(x,t) = \sum_{n=0}^{\infty} H_n(x,t)\frac{\theta^n}{n!} = e^{\theta x - \frac{1}{2}\theta^2 t} = F(x,t,\theta)$$

for $t \geq 0, x, \theta \in \mathbb{R}$.

Prop 51. *H_n satisfies $H_{n+1} = xH_n - ntH_{n-1}$*

Proof. Note:

$$e^{\theta x - 1/2\theta^2 t} = \sum_{n\geq 0} \frac{(\theta x - 1/2\theta^2 t)^n}{n!}$$

Compute

$$F_\theta(x,t,\theta) = (x-\theta t)F(x, heta) = x\sum_{n=0}^{\infty} H_n(x,t)\frac{\theta^n}{n!} - \theta t \sum_{n=0}^{\infty} H_n(x,t)\frac{\theta^n}{n!}$$
$$F_\theta(x,t,\theta) = \sum_{n=1}^{\infty} H_n(x,t)\frac{\theta^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} H_{n+1}(x,t)\frac{\theta^n}{(n)!}$$

Collecting terms give the desired result since is true for arbitrary θ. □

Prop 52. *H_n solves $\partial_t H_n + \frac{1}{2}\partial_x^2 H_n = 0$*

Proof. Compute $F_{xx}(x,t,\theta) = \theta^2 e^{\theta x - (1/2)\theta^2 t}$ and $F_t(x,t,\theta) = -\frac{1}{2}\theta^2 e^{\theta x-(1/2)\theta^2 t}$. By dct we also know $F_{xx}(x,t,\theta) = \sum_{n=0}^{\infty} H_{n,xx}(x,t)\frac{\theta^n}{n!}$ and $F_t(x,t,\theta) = \sum_{n=0}^{\infty} H_{n,t}(x,t)\frac{\theta^n}{n!}$. Then we know $F_t(x,t,\theta) + (1/2)F_{xx}(x,t,\theta) = 0$ and since this holds for all θ we know this holds for each H_n. □

Prop 53. *We have*

$$H_{n+1}(B_t,t) = \int_0^t (n+1)H_n(B_s,s)dB_s = (n+1)! \int_0^t \int_0^{t_1} \dots \int_0^{t_n} dB_{t_{n+1}} \dots dB_{t_1}$$

Proof. This follows from the previous propositions and Ito's formula.

$$H_{n+1}(B_t,t) = H_{n+1}(B_0,0) + \int_0^t \frac{dH_{n+1}}{dx}(B_s,s)dB_s + \int_0^t (\frac{dH_{n+1}}{dt} + \frac{1}{2}\frac{d^2H_{n+1}}{dx^2})(B_s,s)ds$$

But we know the second term is 0 by the previous proposition. And further by the first proposition and an inductive argument we hget $\frac{dH_{n+1}}{dx} = H_n$ as desired. Applying the result iteratively yields the rest of the equalities. □

Question 26:

Consider the complex Brownian motion $B(t) = B_1(t) + iB_2(t)$ where B_1, B_2 ind. one-dimenstional brownian motions. Suppose $B(0) = i$ and set T to be first time $B(t)$ hits real line.

Prop 54. *$T < \infty$ almost surely*

Prop 55. *We know $B(t)$ hits the real line when $B_2(t) = 0$. But we know the probability $B_2(t) \neq 0$ is 0 since it is a brownian motion, in particular via proposition 2.14.*

Prop 56. *We compute the distribution of $B(T)$ by computing $\mathbb{E}[e^{i\lambda B(T)}]$ for $\lambda \in \mathbb{R}$.*

Proof. □

Question 27:

Prop 57. *If B_t is a brownian motion then $X_t = e^{(1/2)t}cos(B_t)$ is a martingale*

Prop 58. *We apply Ito's formula to show this a martingale. Write $F(t,x)$*

Ito tells us $X_t = F(t,B_t) = F(0,B_0) + \int_0^t \frac{df}{dx}(s,B_s)dB_s + \int_0^t(\frac{dF}{dx} + (1/2)\frac{d^2F}{dx^2})(t,B_s)dt$

Further note via a simple computation $\frac{dF}{dx} + (1/2)\frac{d^2F}{dx^2} = 0$ so $X_t = F(0,B_0) + \int_0^t \frac{df}{dx}(s,B_s)dB_s$ which shows it a martingale.

Question 28:

Prop 59. *Via Ito integration we have*

$$\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds$$

Proof. Fix a sequence of partitions $\{t_{n_i}\}$ from 0 to with $B_t^n = B_{t_j} = B_J, t \in [t_j, t_{j+1}]$. We set $\mu_n = sup|t_{n_i} - t_{n_{i+1}}| \to 0$ as $n \to \infty$. Note $F(x) = x^3/3 = \int_0^x f(s)ds$. Then $F(B_{k+1}) = F(B_k) + (B_k)^2(B_{k+1} - B_k) + B_k(B_{k+1} - B_k)^2 + ...$ via a taylor series argument.

Then we get $f(B_k)(B_{k+1} - B_k) = F(B_{k+1}) - F(B_k) - B_k(B_{k+1} - B_k)^2 - ....$ Summing the expression over the partition and refining sends the LHS to our desired ito integral ie. $\int_0^t B_s^2 dB_s$. We compute the right hand side.

Note the first two terms telescope, yielding $F(B_t) = \frac{B_t^3}{3}$. So we want to show the rest of the terms yield $\int_0^t B_s ds$. It suffices to argue $B_k((B_{k+1} - B_k)^2 - (t_{k+1} - t_k)) \to 0$ when summed as then we add and subtract $t_{k+1} - t_k)$ and see $B_k(t_{k+1} - t_k) \to \int_0^t B_s ds$ when summed.

To show this is 0 we compute the variance of the sum which we see goes to 0. A similar argument works for the higher order terms. Write $\sum_{k=0}^n Var(B_k)(t_{k+1} - t_k)^2 Var(B_{k+1} - B_k)^2/(t_{k+1} - t_k) - 1) = \sum_{k=0}^n t_k(t_{k+1} - t_k)^2 Var(B_{k+1} - B_k)^2/(t_{k+1} - t_k) - 1) \to 0$ as the partition gets finer

□

Question 29:

Let B be a standard 1d Brownian motion with $B_0 = 0$ and $f \in C^2(\mathbb{R})$ and $g \in C(\mathbb{R})$. Let $X_t = f(B_t)e^{-\int_0^t g(B_s)ds}$

Prop 60. *X_t is a semi-martingale*

Question 30:

Let X_t be as defined.

Prop 61. *X_t is a brownian motion*

Proof. We know X is a CLM(since B_t is martingale) with $X_0 = 0$. Further we can compute its quadratic variation as

$$\langle X, X \rangle_t = \int_0^t sgn(B_t)^2 ds = t$$

which shows it a brownian motion. □

Prop 62. *X_t is uncorrelated with B_t.*

Proof. We simply apply Ito's formula and compute. □

Question 31:

Prop 63. *Let B be a 2d brownian motion with $B_0 = (1, 0)$. Suppose B never hits the origin, then $X_t = log|B_t|$ is a CLM but not a martingale.*