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Hw 5

Question 1(1.15):

Le Gall, 1.15

Prop 1. *$t \to X_t$ on $[0,1]$ into $L^2(\Omega)$ is cts $\iff$ K is cts on $[0,1]^2$, where $K(s,t) = COV(X_s, X_t) = \mathbb{E}(X_s X_t)$*

Proof. Suppose $t \to X_t$ is continuous. Then K is the integral of a product of continuous functions and is thus continuous. Now suppose K continuous. Then in particular $K(t,t)$ is continuous. So $t \to X_t$ via is continuous as $||X_t||_{L^2(\Omega)} = \mathbb{E}(X_t^2) = K(t,t)$ □

Prop 2. *Let $h : [0,1] \to \mathbb{R}$ be measurable s.t. $\int_0^1 |h(t)|\sqrt{K(t,t)}dt < \infty$. Then a.e.*

$$Z = \int_0^1 H(t) X_t(\omega) dt$$

is absolutely convergent.

Proof. Recall $K(t,t) = \mathbb{E}X_t^2$ Compute

$$\mathbb{E}\int_0^1 |h(t)X_t| dt \leq \int_0^1 |h(t)|\mathbb{E}|X_t| dt \leq \int_0^1 |h(t)|||X_t||_{L^2} dt < \infty$$

where we have the first inequality via tonelli(since everything nonnegative). □

Prop 3. *Suppose h integrable. Then Z is the L^2 limit of $Z_n = \sum_i^n X_{\frac{i}{n}} \int_{(i-1)/n}^{i/n} h(t)dt$. Clearly it is then gaussian as the gaussian space is closed in L^2.*

Proof. Write

$$\sum_i^n X_{\frac{i}{n}} \int_{(i-1)/n}^{i/n} h(t)dt = \int_0^1 h(t) \sum_i 1_{(i-1/n, i/n)} X_{i/n} dt$$

so we can see pointwise $Z_n \to Z$ as $n \to \infty$ for fixed ω. By uniqueness of limits it STS the Z_n cauchy in L^2.

NTF

□

Prop 4. *Suppose $K \in C^2$. Then $\forall t \in [0,1]$,*

$$X_t' := \lim_{s \to t} \frac{X_s - X_t}{s - t}$$

exists in $L^2(\Omega)$. Further (X_t') is a centered gaussian process.

Proof. NTF □

Question 2(1.16: Kalman Filtering):

Prop 5. *$\hat{X}_{n+1/n} = a_n \hat{X}_{n/n}$ for every $n \geq 0$.*

Proof.

$$\begin{aligned}\hat{X}_{n+1/n} &= E[X_{n+1}|Y_0, ..., Y_n] = E[a_n X_n + \epsilon_{n+1}|Y_0, ..., Y_n] \\ &= E[a_n X_n|Y_0, ..., Y_n] + E[\epsilon_{n+1}|Y_0, ..., Y_n] = a_n E[X_n|Y_0, ..., Y_n] = a_n \hat{X}_{n/n}\end{aligned}$$

where we note ϵ_{n+1} ind. from $Y_0, ..., Y_n$¿ □

Prop 6. *$\forall n \geq 1$,*

$$\hat{X}_{n/n} = \hat{X}_{n/n-1} + \frac{\mathbb{E}[X_n Z_n]}{\mathbb{E}[Z_n^2]} Z_n$$

where $Z_n := Y_n - c\hat{X}_{n/n-1}$

Proof.

$$\begin{aligned}&\hat{X}_{n/n-1} = \mathbb{E}[X_n|Y_0, ..., Y_{n-1}] \\ &Y_n = cX_n + \eta_n\end{aligned}$$

So compute

$$\hat{X}_{n/n} = \mathbb{E}[X_n|Y_0, ..., Y_n] = \Pi_n(X)$$

which we interpret as the orthogonal projection of X in $L^2(\Omega)$ onto the span of $Y_0, ..., Y_n$. Compute

$$\begin{aligned}Z_n &= Y_n - c\hat{X}_{n/n-1} = Y_n - c\mathbb{E}[X_n|Y_0, ..., Y_{n-1}] \\ &= Y_n + \mathbb{E}[\eta_n - Y_n|Y_0, ..., Y_{n-1}] = Y_n + \mathbb{E}[\eta_n] - \mathbb{E}[Y_n|Y_0, ..., Y_{n-1}] \\ &= Y_n - \Pi_{n-1}(Y_n)\end{aligned}$$

Then

$$\begin{aligned}\hat{X}_{n/n} &= \mathbb{E}[X_n|Y_0,...,Y_n] = \Pi_n(X_n)\\ &= \Pi_{n-1}(X_n) + \Pi_{Z_n}(X_n) = \mathbb{E}[X_n|Y_0,...,Y_{n-1}] + \langle X_n, \hat{Z}_n\rangle \hat{Z}_n\\ &= \hat{X}_{n/n-1} + \frac{\mathbb{E}[X_n Z_n]}{\mathbb{E}[Z_n^2]} Z_n\end{aligned}$$

$\square$

Question 3(1.18: Levy's Construction of Brownian Motion):
