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Best Khintchine type inequalities for sums of independent, rotationally invariant random vectors

H. König*, S. Kwapień†

Abstract

Let $X_i : (\Omega, P) \to \mathbb{R}^n$ be an i.i.d. sequence of rotationally invariant random vectors in $\mathbb{R}^n$. If $\|X_1\|^2$ is dominated (in the sense defined below) by $\|Z\|^2$ for a rotationally invariant normal random vector Z in $\mathbb{R}^n$, then for each $k \in \mathbb{N}$ and $(\alpha_i) \subseteq \mathbb{R}$

$$\left(\mathbb{E}\,\|\sum_{i=1}^{k} \alpha_i X_i\|^p\right)^{1/p} \;\le\; \text{(resp. } \ge\text{)}\;\; (\mathbb{E}\,\|Z\|^p)^{1/p} \left(\sum_{i=1}^{k} |\alpha_i|^2\right)^{1/2}$$

for $p \ge 3$ or $p, n \ge 2$ (resp. for $1 \le p \le 2$, $n \ge 3$). The constant $(\mathbb{E}\,\|Z\|^p)^{1/p}$ is the best possible. The result applies, in particular, for variables uniformly distributed on the sphere S^{n-1} or the ball B_n. In the case of the sphere, the best constant is

$$(\mathbb{E}\,\|Z\|^p)^{1/p} = \sqrt{\frac{2}{n}} \left(\Gamma\left(\frac{p+n}{2}\right) / \Gamma\left(\frac{n}{2}\right)\right)^{1/p}.$$

With this constant, the Khintchine type inequality in this case also holds for $1 \le p \le 2$, $n = 2$.

1 Introduction and main results

The Khintchine inequality is important in the theory of Banach spaces when studying unconditional convergence of series or p-summing operators in Banach spaces. It is also much used for series in Banach spaces of type p or cotype q. The Rademacher variables used in the real case have to be replaced by the Steinhaus variables in the complex case which are distributed like e^{it} on $S^1 \subseteq \mathbb{C} = \mathbb{R}^2$, $t \in [0, 2\pi)$. We determine the optimal

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constants in the analogue of the more general Khintchine inequality where the variables map equidistributedly onto $S^{n-1} \subseteq \mathbb{R}^n$ when $n \geq 2$ and, more generally, for rotationally invariant random vectors if $p, n \geq 2$ or $1 \leq p \leq 2$, $n \geq 3$.

Let (Ω, P) always denote a probability space. Let $X : (\Omega, P) \to \mathbb{R}^n$ be a **rotationally invariant** random vector, i.e.

$$P(X \in A) = P(X \in R^{-1}(A))$$

for all Borel sets $A \subseteq \mathbb{R}^n$ and all orthogonal transformations $R \in O(n)$. If X takes its values in the euclidean sphere $S^{n-1} \subseteq \mathbb{R}^n$, X is **uniformly distributed** on S^{n-1}. Another example are normal random vectors.

To compare moments of sums of independent, rotationally invariant random vectors, we use a method based on the concept of domination of random variables. An extensive study of the concept is found in the book [MaO], see also the paper [FHJSZ]. However, we only need the definition and some basic facts which we provide.

If $\xi, \eta : (\Omega, P) \to \mathbb{R}$ are real random variables in $L_1(\Omega, P)$, we say that ξ **dominates** η, denoted by $\eta \prec \xi$, if

$$\mathbb{E}\,\eta = \mathbb{E}\,\xi \text{ and } \int_s^\infty P(\eta > t)dt \leq \int_s^\infty P(\xi > t)dt$$

for all $s \in \mathbb{R}$.

Let $\|\cdot\|$ denote the euclidean norm on $\mathbb{R}^n$. Our basic result is

Proposition 1 *Let $(X_i), (Y_i)$ be two sequences of independent, rotationally invariant random vectors in $\mathbb{R}^n$ such that $\|X_i\|^2 \prec \|Y_i\|^2$ for $i = 1, \ldots, k$ Then*

$$\mathbb{E}\,\|\sum_{i=1}^k X_i\|^p \leq \quad (resp. \geq) \quad \mathbb{E}\,\|\sum_{i=1}^k Y_i\|^p$$

for $p \geq 3$ or $p, n \geq 2$ (resp. for $1 \leq p \leq 2$ and $n \geq 3$).

If one of the sequences is normal, the p-th moment is easily calculated. The consequence then is

Theorem 2 *Let (X_i) be an i.i.d. sequence of rotationally invariant random vectors in $\mathbb{R}^n$. Let Z be a rotationally invariant, normal random vector in $\mathbb{R}^n$ such that $\|X_1\|^2 \prec \|Z\|^2$. Then for all $k \in \mathbb{N}$ and $(\alpha_i) \subseteq \mathbb{R}$,*

$$\left(\mathbb{E}\,\|\sum_{i=1}^k \alpha_i X_i\|^p\right)^{1/p} \leq \ (resp. \geq) \ (\mathbb{E}\,\|Z\|^p)^{1/p} \left(\sum_{i=1}^k |\alpha_i|^2\right)^{1/2}$$

for $p \geq 3$ or $p, n \geq 2$ (resp. for $1 \leq p \leq 2$ and $n \geq 3$). The constant $(\mathbb{E}\,\|Z\|^p)^{1/p}$ is the best possible.

Obviously, Theorem 2 applies to an i.i.d. sequence X_i of uniformly distributed random vectors in S^{n-1}. The resulting inequalities in this case will be also shown to hold if $1 \leq p \leq 2$ and $n = 2$, by a different method. We find

Theorem 3 *Let $1 \leq p < \infty$ and $n \geq 2$. If (X_i) is an i.i.d. sequence of uniformly distributed random vectors in the sphere S^{n-1} of $\mathbb{R}^n$, we have for all $k \in \mathbb{N}$, $(\alpha_i)_{i=1}^k \subseteq \mathbb{R}$ that*

$$a_p(n) \left(\sum_{i=1}^{k} |\alpha_i|^2 \right)^{1/2} \leq \left(\mathbb{E} \, \| \sum_{i=1}^{k} \alpha_i X_i \|^p \right)^{1/p} \leq b_p(n) \left(\sum_{i=1}^{k} |\alpha_i|^2 \right)^{1/2} \tag{1.1}$$

where

$$a_p(n) := \min \left(1, \sqrt{\frac{2}{n}} \left(\Gamma \left(\frac{p+n}{2} \right) / \Gamma \left(\frac{n}{2} \right) \right)^{1/p} \right)$$

$$b_p(n) := \max \left(1, \sqrt{\frac{2}{n}} \left(\Gamma \left(\frac{p+n}{2} \right) / \Gamma \left(\frac{n}{2} \right) \right)^{1/p} \right)$$

The constants $a_p(n)$ and $b_p(n)$ are the best possible in the range of indices p, n given.

The case of the classical Rademacher variables $X_i = r_i$ ($n = 1$, $S^0 = \{+1, -1\}$) was treated by U. Haagerup [H], the case of $p = 1$ before by Szarek [Sz]. In that case, the formula for $a_p = a_p(1)$ changes to $a_p = 2^{1/2-1/p}$ for $p \leq p_0 \approx 1.847$. The case of the complex Steinhaus variables $X_i = S_i$ ($n = 2$) has been proved for $p = 1$ by J. Sawa [S]. He also announced the formula for $p \geq p_0 \approx .475617$ but never published a proof. Here p_0 is the unique root of the equation

$$2^{p/2}\, \Gamma\left((p+1)/2\right) = \sqrt{\pi}\, \Gamma\left((p+2)/2\right)^2.$$

The above theorem is also valid with the same formulas for $a_p(n)$ and $b_p(n)$ if $0 < p < 1$ and $n \geq 3$, although we do not prove this here. For $n = 2$ and $0 < p < 1$, Haagerup conjectured that there is $p_0 < 1$ such that the formula for $a_p(2)$ and $0 < p < p_0$ has to be changed, see [P]. It is likely that p_0 is the above value and that for $0 < p < p_0$

$$a_p(2) = \sqrt{2} \left(\Gamma \left(\frac{p+1}{2} \right) \Big/ \left(\Gamma \left(\frac{p}{2} + 1 \right) \sqrt{\pi} \right) \right)^{1/p}.$$

Equality in the left inequality would be attained for $k = 2$ and $|\alpha_1| = |\alpha_2|$. We do not consider this case $0 < p < 1$, however. Theorem 2 has been proved for $p > 2$, recently and independently, also by A. Baernstein and R. Culverhouse [BC].

There is a close relation between these variables and uniformly distributed variables on the ball $B_n(r) := \{x \in \mathbb{R}^n \mid \|x\| \leq r\} \subseteq \mathbb{R}^n$ of radius $r > 0$ in $\mathbb{R}^n$.

Proposition 4 *Let $0 < p < \infty$ and $n \in \mathbb{N}$. Let (X_i) [resp. (U_i)] be an i.i.d. sequence of uniformly distributed random vectors in the ball $B_n(r)$ of $\mathbb{R}^n$ [resp. in the sphere S^{n+1} of $\mathbb{R}^{n+2}$]. Let $k \in \mathbb{N}$, $(\alpha_i)_{i=1}^k \subseteq \mathbb{R}$. Then*

$$\mathbb{E} \, \| \sum_{i=1}^k \alpha_i X_i \|^p = r^p \, \frac{n}{n+p} \, \mathbb{E} \, \| \sum_{i=1}^k \alpha_i U_i \|^p$$

This clearly yields a similar inequality as (1.1) for the variables X_i on balls with optimal constants r times $A_p(n) = (n/(n+p))^{1/p} a_p(n+2)$, $B_p(n) = (n/(n+p))^{1/p} b_p(n+2)$, which in most cases can be also proved by a direct application of Theorem 2 to these variables X_i.

Theorem 5 *Let $1 \leq p < \infty$ and $n \geq 1$. Let (X_i) be an i.i.d. sequence of uniformly distributed random vectors on the ball $B_n(r)$ of radius $r > 0$ in $\mathbb{R}^n$. Then for all $k \in \mathbb{N}$ and $(\alpha_i)_{i=1}^k \subseteq \mathbb{R}$,*

$$r \, A_p(n) \left(\sum_{i=1}^n |\alpha_i|^2 \right)^{1/2} \leq \left(\mathbb{E} \, \| \sum_{i=1}^k \alpha_i X_i \|^p \right)^{1/p} \leq r \, B_p(n) \left(\sum_{i=1}^k |\alpha_i|^2 \right)^{1/2}$$

where

$$A_p(n) := \min \left((n/(n+p))^{1/p}, \; (2/(n+2))^{1/2} \left(\Gamma\left(\frac{n+p}{2}\right) / \, \Gamma\left(\frac{n}{2}\right) \right)^{1/p} \right)$$

$$B_p(n) := \max \left((n/(n+p))^{1/p}, \; (2/(n+2))^{1/2} \left(\Gamma\left(\frac{n+p}{2}\right) / \, \Gamma\left(\frac{n}{2}\right) \right)^{1/p} \right)$$

are the optimal constants.

Theorem 5 for $n = 1$ is due to Latala-Oleszkiewicz [LO], for $n \geq 2$ the result was also shown by Culverhouse [C].

2 Comparison theorems for sums of rotationally invariant random vectors

We need the following known facts on the domination of real random variables with finite first moments [Ma0].

Lemma 6 *Let $\xi, \eta : (\Omega, P) \to \mathbb{R}$ be real random variables in $L_1(\Omega, P)$.*

(a) The following are equivalent:

(1) ξ dominates η, $\eta \prec \xi$.

(2) For each convex function $k : \mathbb{R} \to \mathbb{R}_+$ such that $k(\xi)$ is integrable

$$\mathbb{E}\, k(\eta) \leq \mathbb{E}\, k(\xi) \quad .$$

(b) If η is a constant random variable equal to $\mathbb{E}\,\xi$, then $\eta \prec \xi$.

(c) If ξ and η have densities g_ξ, g_η which have the same mean and are such that for some $a < b$ the difference $g_\xi - g_\eta$ is nonnegative on $(-\infty, a)$ and (b, ∞) and nonpositive on (a, b), then $\eta \prec \xi$.

For the convenience of the reader, we provide the simple

Proof.

(a) The extreme rays of the cone $\mathcal{C}$ of nonnegative convex functions $k : \mathbb{R} \to \mathbb{R}_+$ are given by the nonnegative constants and the functions $k_s, l_s : \mathbb{R} \to \mathbb{R}_+$ for $s \in \mathbb{R}$ where $k_s(t) := \max(0, t-s)$, $l_s(t) := \max(s-t, 0) = k_s(t) - s + t$. The equivalence of (1) and (2) thus follows from the observation that

$$\int\limits_s^\infty P(\xi > t)dt = \mathbb{E}\, k_s(\xi)$$

holds for each real random variable $\xi : (\Omega, P) \to \mathbb{R}$ in L_1.

(b) This is a consequence of Jensen's inequality.

(c) The function $v : \mathbb{R} \to \mathbb{R}$ given by

$$\begin{aligned} v(s) \quad &:= \int\limits_s^\infty P(\xi > t)dt - \int\limits_s^\infty P(\eta > t)dt \\ &= \int\limits_s^\infty \left(\int\limits_t^\infty (g_\xi(u) - g_\eta(u))du \right) dt \end{aligned}$$

is convex on $(-\infty, a)$ and (b, ∞) and concave on (a, b). Further $\lim\limits_{s \to \pm\infty} v(s) = 0$ since $\mathbb{E}\,\xi = \mathbb{E}\,\eta$. Hence v has to be nonnegative on $\mathbb{R}$ and thus $\eta \prec \xi$.

$\square$

In the sequel, $U : (\Omega', P') \to S^{n-1}$ will always be a random vector uniformly distributed on the unit sphere of $\mathbb{R}^n$. The main technical result needed to prove Proposition 1 is

Proposition 7 *Let $1 \le p < \infty$, $n \in \mathbb{N}$ and $a \in \mathbb{R}^n$. Then the function $h^a_{n,p} : \mathbb{R}_+ \to \mathbb{R}$ defined by*

$$h^a_{n,p}(t) := \mathbb{E}\,\|a + \sqrt{t}\,U\|^p$$

is convex for $(p \ge 3, n \ge 1)$ or $(p \ge 2, n \ge 2)$ and it is concave for $(1 \le p \le 2, n \ge 3)$.

We postpone the proof of Proposition 7 and first show that most of the results of section 1 follow for these cases of (p, n) from Lemma 6 and Proposition 7.

Proof of Proposition 1.

Let X and Y be two rotationally invariant random vectors in $\mathbb{R}^n$ such that $\|X\|^2 \prec \|Y\|^2$. If the uniformly distributed random vector U is chosen to be independent of X and Y, then X has the same distribution as $\|X\|U$ and Y has the same distribution as $\|Y\|U$. Indicating the variables of integration for X (or Y) by $\omega \in \Omega$ and for U by $\omega' \in \Omega'$, one thus has for any $a \in \mathbb{R}^n$

$$\begin{aligned} \mathbb{E}_\omega\,\|a + X\|^p &= \mathbb{E}_{\omega,\omega'}\,\|a + \|X\|U\|^p = \mathbb{E}_\omega\,h^a_{n,p}(\|X\|^2), \\ \mathbb{E}_\omega\,\|a + Y\|^p &= \mathbb{E}_{\omega,\omega'}\,\|a + \|Y\|U\|^p = \mathbb{E}_\omega\,h^a_{n,p}(\|Y\|^2). \end{aligned}$$

Hence Proposition 7 and Lemma 6 (a) yields

$$\mathbb{E}_\omega\,\|a + X\|^p \le \quad [\mathit{resp.} \ge] \quad \mathbb{E}_\omega\,\|a + Y\|^p \tag{2.1}$$

if $(p \ge 3, n \ge 1)$ or $(p \ge 2, n \ge 2)$ [resp. $(1 \le p \le 2, n \ge 3)$]. An easy induction argument based on (2.1) implies Proposition 1. □

Proof of Theorem 2.

Let Z_i be a sequence of independent random vectors in $\mathbb{R}^n$, each of them distributed as Z. Then $\|\alpha_i X_i\|^2 \prec \|\alpha_i Z_i\|^2$ and hence by Proposition 1

$$\begin{aligned} \left(\mathbb{E}\,\|\sum_{i=1}^k \alpha_i X_i\|^p\right)^{1/p} &\le (\mathit{resp.} \ge) \left(\mathbb{E}\,\|\sum_{i=1}^k \alpha_i Z_i\|^p\right)^{1/p} \\ &= (\mathbb{E}\,\|Z\|^p)^{1/p} \left(\sum_{i=1}^k |\alpha_i|^2\right)^{1/2}. \end{aligned}$$

To get that the constant $(\mathbb{E}\,\|Z\|^p)^{1/p}$ is the best possible, it suffices to put $\alpha_i = 1/\sqrt{k}$, $i = 1, \ldots, k$ and apply the Central Limit Theorem with $k \to \infty$. □

Proof of Theorem 3.

Except for the case of $n = 2$, $1 \le p \le 2$, this is a direct application of Theorem 2 to the case where $X_i = U_i$ is an i.i.d. sequence uniformly distributed on the sphere S^{n-1} of $\mathbb{R}^n$. In this case $\mathbb{E}\,\|X_i\|^2 = 1$ and so the normal random vector Z is chosen with $\mathbb{E}\,\|Z\|^2 = 1$. Since $\mathbb{E}\,\|X_1\|^2 = 1 = \mathbb{E}\,\|Z\|^2$ is constant, Lemma 6 (b) yields that $\|X_1\|^2 \prec \|Z\|^2$. The best constant from Theorem 2 is

$$(\mathbb{E}\,\|Z\|^p)^{1/p} = \sqrt{\frac{2}{n}} \left(\Gamma\left(\frac{p+n}{2}\right) / \,\Gamma\left(\frac{n}{2}\right)\right)^{1/p}.$$

This, together with the monotonicity of the L_p-norms, yields the estimate with the constants $a_p(n)$ and $b_p(n)$ in Theorem 3, except for the case of $1 \le p \le 2$, $n = 2$ which will be considered later using a different method. □

Proposition 4 will be proved at the end of this chapter. Combined with Theorem 3, it immediately gives a proof of Theorem 5. Here we give another argument based also on Theorem 2 (in the index cases considered there).

Proof of Theorem 5.
We apply Theorem 2 in the case when (X_i) is an i.i.d. sequence of independent random vectors, each of them uniformly distributed in the ball $B_n(r)$ of radius $r > 0$ in $\mathbb{R}^n$. In this case

$$P(\|X_1\|^2 > t) = \int_t^{r^2} g_{\|X_1\|^2}(u)du = 1 - t^{n/2}/r^n, \ 0 \le t \le r^2$$

with density $g_{\|X_1\|^2}(u) = \frac{n}{2} r^{-n}\, u^{n/2-1}\, \chi_{(0,r^2)}(u)$. Thus

$$\mathbb{E}\,\|X_1\|^2 = \int_0^{r^2} u\, g_{\|X_1\|^2}(u)du = \frac{n}{n+2}\, r^2.$$

Choosing a rotationally invariant normal vector Z with $\mathbb{E}\,\|Z\|^2 = \frac{n}{n+2}\, r^2$, the density of $\|Z\|^2$ is

$$g_{\|Z\|^2}(u) = \beta^{n/2}/\Gamma\left(\frac{n}{2}\right) u^{n/2-1}\, e^{-\beta u}\, \chi_{(0,\infty)}(u)$$

where $\beta = (n+2)/(2r^2)$. The difference $g_{\|Z\|^2} - g_{\|X_1\|^2}$ satisfies the assumption of Lemma 6 (c) since

$$g_{\|Z\|^2}(u) - g_{\|X_1\|^2}(u) = \frac{n}{2}\, u^{n/2-1} r^{-n} \left(c_n\, e^{-(n+2)u/(2r^2)}\, \chi_{(0,\infty)}(u) - \chi_{(o,r^2)}(u)\right),$$

with $c_n := \left(\frac{n+2}{2}\right)^{n/2} / \,\Gamma\left(\frac{n}{2} + 1\right) > 1$ and $c_n e^{-(n+2)/2} < 1$ so that there is a unique $0 < u = a < r^2$ with $c_n\, e^{-(n+2)u/(2r^2)} = 1$. Then $g_{\|Z\|^2} - g_{\|X_1\|^2}$ is ≥ 0 in $(-\infty, a) \cup (r^2, \infty)$ and < 0 in (a, r^2). Hence by Lemma 6 (c) we have $\|X_1\|^2 \prec \|Z\|^2$. The inequality constants $A_p(n)$ and $B_p(n)$ we get in this case are the ones from Theorem 3 multiplied by $(\mathbb{E}\,\|X_1\|^2)^{1/2} = (n/(n+2))^{1/2}$. This yields the value of $A_p(n)$ if $1 \le p \le 2$ and

$n \geq 3$ and the value of $B_p(n)$ if $2 \leq p < \infty$ and $n \geq 2$. The case $n = 1$ was considered in [LO], as mentioned already.

The remaining index cases follow from Theorem 3 after Proposition 4 has been shown. $\square$

Remarks.

(i) If $p \geq 3$, under the assumption of Proposition 1

$$\mathbb{E}\,\|\sum_{i=1}^{n} L_i X_i\|^p \leq \mathbb{E}\,\|\sum_{i=1}^{n} L_i Y_i\|^p$$

for each sequence of linear operators (L_i) in $\mathbb{R}^n$. The similar statement is not true for other values p.

As shown in the beginning of the proof of Proposition 7, for $p \geq 3$ and any vectors $x, z \in \mathbb{R}^n$ the function

$$f(t) := \mathbb{E}\,||x + \sqrt{t}\,\varepsilon\, z||^p$$

is convex in $t > 0$ if ε denotes the Rademacher random variable. Therefore for any $x, z \in \mathbb{R}^n$

$$\mathbb{E}\,||x + ||X||\varepsilon z||^p \leq \mathbb{E}\,||x + ||Y||\varepsilon z||^p,$$

if X and Y are rotationally invariant random vectors in $\mathbb{R}^n$ independent of ε and if $||X||^2$ is dominated by $||Y||^2$. If U is uniformly distributed on S^{n-1} and independent of X, Y and ε, then for any linear operator L on $\mathbb{R}^n$ the random vector $L(X)$ is equidistributed with $||X||\,L(X)\varepsilon$ and $L(Y)$ is equidistributed with $||Y||\;L(Y)\varepsilon$. Thus by Fubini's theorem and the last inequality

$$\mathbb{E}\,||x + L(X)||^p \leq \mathbb{E}\,||x + L(Y)||^p$$

for any $x \in \mathbb{R}^n$ and any linear operator L on $\mathbb{R}^n$. An easy induction (as in Proposition 1) then proves the claim.

(ii) If $(\alpha_i)_{i=1}^n$ and $(\beta_i)_{i=1}^n$ are two sequences of numbers then we put $(\alpha_i)_{i=1}^n \prec (\beta_i)_{i=1}^n$ if for each convex function k it is $\sum\limits_{i=1}^{n} k(\alpha_i) \leq \sum\limits_{i=1}^{n} k(\beta_i)$. If $(\alpha_i^2)_{i=1}^2 \prec (\beta_i^2)_{i=1}^2$ then $\alpha_1^2 + \alpha_2^2 = \beta_1^2 + \beta_2^2$ and $|\beta_1\beta_2| \leq |\alpha_1\alpha_2|$. Therefore if U_1, U_2 are independent vectors, uniformly distributed on S^{n-1}, then $\|\beta_1 U_1 + \beta_2 U_2\|^2 = \beta_1^2 + \beta_2^2 + 2\beta_1\beta_2 < U_1, U_2 > \prec \alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2 < U_1, U_2 > = \|\alpha_1 U_1 + \alpha_2 U_2\|^2$. Hence by Proposition 1 and an easy induction we can prove that the sequence (U_i) has the Schur domination property, i.e. if $(\alpha_i^2)_{i=1}^n \prec (\beta_i^2)_{i=1}^n$ then $\mathbb{E}\|\sum\limits_{i=1}^{n} \beta_i U_i\|^p \leq$ $(resp. \geq)$ $\mathbb{E}\|\sum\limits_{i=1}^{n} \alpha_i U_i\|^p$ for p, n as in Theorem 2.

Proof of Proposition 7.

(i) At first we consider the case of $p \geq 3$. Let ε be random variable distributed by $P(\varepsilon = \pm 1) = 1/2$. It is known that the function $h(t) = \mathbb{E}\,|\alpha + \sqrt{t}\,\beta\,\varepsilon|^p$ is convex (if $p \geq 3$) for all $\alpha, \beta \in \mathbb{R}$, see [FHJSZ]. Since any euclidean space is isometric to a subspace of L_p, the same is true if α, β are vectors a, b in a euclidean space H and $|\cdot|$ is replaced by $\|\cdot\|$. Choosing U independent of ε, U has the same distribution as $U\varepsilon$ and therefore the function

$$h^a_{n,p}(t) = \mathbb{E}\,\|a + \sqrt{t}\,U\|^p = \mathbb{E}\,\|a + \sqrt{t}\,U\varepsilon\|^p$$

is convex as an average of convex functions.

(ii) We now consider the case of $1 \leq p \leq 3$. The function $h^a_{n,p}$ only depends on the norm of a, $\|a\|$. We may assume without loss of generality that $\|a\| = 1$. For $n = 2$ and $\|a\| = 1$, let σ be the uniform distribution on the circle $T = \{z \in \mathbb{C} \,\big|\, |z| = 1\}$. Then

$$\begin{aligned} h^a_{2,p}(t) = h_{2,p}(t) &= \int_T |1 + \sqrt{t}\,z|^p \sigma(dz) \\ &= \int_T (1 + \sqrt{t}\,z)^{p/2}\,(1 + \sqrt{t}\,\overline{z})^{p/2} \sigma(dz) \end{aligned}$$

which by Taylor expansion gives for $0 < t < 1$

$$h_{2,p}(t) = \int_T \sum_{k,l=0}^{\infty} \binom{p/2}{k} \binom{p/2}{l} \sqrt{t}^{k+l} z^k\,\overline{z}^{\,l}\,\sigma(dz) = \sum_{k=0}^{\infty} \binom{p/2}{k}^2 t^k.$$

This function is clearly convex on $(0,1)$ for all p. For $t > 1$ similarly

$$\begin{aligned} h_{2,p}(t) &= t^{p/2} \sum_{k=0}^{\infty} \binom{p/2}{k}^2 t^{-k} \\ &= \left[t^{p/2} + \left(\frac{p}{2}\right)^2 t^{p/2-1} \right] + \left[\sum_{k=2}^{\infty} \binom{p/2}{k}^2 t^{p/2-k} \right]. \end{aligned}$$

For $2 \leq p \leq 3$ both functions in the brackets $[\ldots]$ are convex on $(1, \infty)$. Since $h_{2,p}$ is continuously differentiable on $\mathbb{R}_+$, we get that it is convex on $\mathbb{R}_+$ for $2 \leq p \leq 3$ and $n = 2$.

(iii) For general $n \geq 3$ and $\|a\| = 1$,

$$\|a + \sqrt{t}\,U\| = (1 + t + 2\sqrt{t}\ < a, U >)^{1/2}.$$

The density of the random variable $< a, U >$ is equal to

$$c_n (1 - u^2)^{(n-3)/2}\,\chi_{(-1,1)}(u),\ c_n = 2^{2-n}\,\Gamma(n-1)/\,\Gamma\left(\frac{n-1}{2}\right)^2.$$

Hence

$$\begin{aligned} h^a_{n,p}(t) &= h_{n,p}(t) = \mathbb{E}\,\|a+\sqrt{t}\,U\|^p \\ &= c_n \int\limits_{-1}^{1} (1+t+2u\sqrt{t})^{p/2}\,(1-u^2)^{(n-3)/2}du \end{aligned} \tag{2.2}$$

For $n=3$ this gives

$$h_{3,p}(t) = \frac{1}{2(p+2)}\left((1+\sqrt{t})^{p+2} - |1-\sqrt{t}|^{p+2}\right)\,/\,\sqrt{t}.$$

Again, by Taylor expansion for $0<t<1$ we find

$$h_{3,p}(t) = \frac{1}{p+2}\sum_{k=0}^{\infty}\binom{p+2}{2k+1}\;t^k.$$

If $2\le p\le 3$, then $\binom{p+2}{2k+1}\ge 0$ for all $k\in\mathbb{N}_0$ and hence $h_{3,p}$ is convex on $(0,1)$. If $1\le p\le 2$, then $\binom{p+2}{2k+1}\le 0$ for all $k\ge 2$ and $h_{3,p}$ is concave on $(0,1)$. For $t>1$, we find similarly that

$$h_{3,p}(t) = \frac{1}{p+2}\sum_{k=0}^{\infty}\binom{p+2}{2k+1}\;t^{p/2-k}.$$

For $2\le p\le 3$, the sum of the first two terms and each term with $k\ge 2$ define convex functions on $(1,\infty)$; hence $h_{3,p}$ is convex on $(1,\infty)$. For $1\le p\le 2$, the sum of the first three terms and each of the terms with $k\ge 3$ is concave on $(1,\infty)$; thus $h_{3,p}$ is concave on $(1,\infty)$. Since $h_{3,p}$ is continuously differentiable, we obtain that $h_{3,p}$ is convex on $\mathbb{R}_+$ for $2\le p\le 3$ and concave on $\mathbb{R}_+$ for $1\le p\le 2$.

(iv) If $n>3$, equality (2.2) may be integrated by parts, giving

$$\begin{aligned} h_{n,p}(t) &= c_n(n-3)\int\limits_0^1\left(\int\limits_{-u}^{u}(1+t+2v\sqrt{t})^{p/2}dv\right)\,u(1-u^2)^{(n-5)/2}du \\ &= c_n\frac{n-3}{p+2}\int\limits_0^1\left[(1+t+2u\sqrt{t})^{p/2+1}-(1+t-2u\sqrt{t})^{p/2+1}\right]/\sqrt{t}\,u(1-u^2)^{(n-5)/2}du \end{aligned}$$

Therefore, to prove that $h_{n,p}$ is convex [resp. concave] it suffices to prove that for each $0<u\le 1$ the function f_u,

$$f_u(t) := \left[(1+t+2u\sqrt{t})^{p/2+1}-(1+t-2u\sqrt{t})^{p/2+1}\right]/\sqrt{t}$$

is convex [resp. concave] in $t\in\mathbb{R}_+$ if $2\le p\le 3$ [resp. $1\le p\le 2$].

To prove this, we introduce the following notations: let $a:=p/2+1$, for $u\in[-1,1]$ let $w_u(t):=1+t+2u\sqrt{t}$ and $g_u(t):=w_u(t)^a/\sqrt{t}$. Then $f_u=g_u-g_{-u}$ for $u\in[0,1]$.

We will show that

$$f_u''(t) \left\{ \begin{array}{ll} \geq 0 & for\ a \geq 2, t > 0 \\ \leq 0 & for\ 3/2 \leq a \leq 2, t > 0 \end{array} \right\} \tag{2.3}$$

This will prove the convexity [concavity] claim about f_u in the range of p's given. Note that for $u = 1$, $f_1 = 2(p+2)h_{3,p}$ which was shown to be convex [resp. concave] in part (iii).

Also, for $u = 0$, $f_0 = 0$. For any $u \in [-1, 1]$,

$$\begin{aligned} g_u''(t) &= w_u(t)^{a-2}/(4t^{5/2})\{4a(a-1)(tw_u'(t))^2 \\ &\quad +4a(t^2 w_u''(t))w_u(t) - 4a(tw_u'(t))w_u(t) + 3w_u(t)^2\} \end{aligned} \tag{2.4}$$

We substitute $x = (1+t)/(2|u|\sqrt{t})$, $y = \sqrt{t}/(2|u|)$. Then $x = y + \frac{1}{4u^2}\,\frac{1}{y}$, $x \geq y + \frac{1}{4y} \geq 1$ and $t = y/(x-y)$ as well as $|u| = 1/(2\sqrt{y(x-y)}\,)$. Further, for $u \in [0, 1]$

$$\begin{aligned} w_{\pm u}(t) &= 1 + t \pm 2u\sqrt{t} = 2\sqrt{t}\,u(x \pm 1) \\ tw_{\pm u}'(t) &= t \pm u\sqrt{t} = 2\sqrt{t}\,u(y \pm 1/2) \\ t^2 w_{\pm u}''(t) &= \mp\sqrt{t}\,u/2. \end{aligned}$$

Inserting these new variables into formula (2.4), we obtain for any $u \in (0, 1]$

$$g_{\pm u}''(t) = (2u\sqrt{t}\,)^a (x \pm 1)^{a-2}\ Q_\pm(x, y)/(4t^{5/2}),$$

where

$$Q_\pm(x, y) = 4a(a-1)(y \pm 1/2)^2 - 4a(y \pm 3/4)(x \pm 1) + 3(x \pm 1)^2.$$

Let $D := \{(x, y) \in \mathbb{R}^2 \,|\, x \geq 1, y > 0, x \geq y + 1/(4y)\}$. Since $f_u'' = g_u'' - g_{-u}''$, the inequality $f_u'' \geq 0$ (resp. ≤ 0) is equivalent to

$$(x+1)^{a-2} Q_+(x, y) \geq\ (resp.\ \leq)\ (x-1)^{a-2} Q_-(x, y) \tag{2.5}$$

for all $(x, y) \in D$. Fixing $x \geq 1$ and differentiating $Q_-(x, y)$ in y, one finds that the minimum in y is attained when $\overline{y} = \frac{1}{2}\left(1 + \frac{x-1}{a-1}\right)$. But then

$$Q_-(x, \overline{y}) = (x-1)\left(\frac{2a-3}{a-1}\,(x-1) + a\right) \geq 0$$

since $a \geq 3/2$. Hence $Q_-(x, y) \geq 0$ for all $(x, y) \in D$, and $Q_-(x, y) > 0$ if $x > 1$. Therefore (2.5) can be rewritten as

$$\left(\frac{x-1}{x+1}\right)^{a-2} \leq\ (resp.\ \geq)\ \frac{Q_+(x, y)}{Q_-(x, y)} \tag{2.6}$$

for $(x, y) \in D$. Again we fix $x > 1$ and differentiate $Q_+(x,y)/Q_-(x,y)$ with respect to y to find after some calculation

$$\frac{\partial}{\partial y}\left(\frac{Q_+(x,y)}{Q_-(x,y)}\right) = 8a(a-1)(a-2)(-4ay^2 + 6xy + a - 3)/Q_-^2. \tag{2.7}$$

Recall that $p \geq 2$ $[p \leq 2]$ corresponded to $a \geq 2$ $[a \leq 2]$. Hence, for fixed $x > 1$, the derivative in (2.7) is positive [negative] if and only if $y \in (y_-, y_+)$ where

$$y_\pm = \frac{1}{2}\left(\frac{3x}{2a} \pm \sqrt{\left(\frac{3x}{2a}\right)^2 + \frac{a-3}{a}}\right).$$

But $(x, y) \in D$ if and only if $y \in [\overline{y}_-, \overline{y}_+]$ where

$$\overline{y}_\pm = \frac{1}{2}\left(x \pm \sqrt{x^2 - 1}\right).$$

Simple estimates show that $y_- \leq \overline{y}_-$ (with equality only if $a = 3/2$). Hence the ratio $Q_+(x,y)/Q_-(x,y)$ for fixed x attains its minimum (resp. maximum) on the interval $[\overline{y}_-, \overline{y}_+]$ at the boundary of this interval. Then (x, y) is on the boundary of D. Hence we proved that (2.6) holds for all $(x, y) \in D$ if it holds for all (x, y) on the boundary of D. The boundary of D, however, corresponds to the case of $u = 1$ when $f_1 = 2(p+2)h_{3,p}$. In (iii) we just showed that $f_1'' = 2(p+2)h_{3,p}''$ is ≥ 0 for $2 \leq p \leq 3$ $[\leq 0$ for $1 \leq p \leq 2]$, so (2.5) holds on the boundary of D and thus in all of D. This ends the proof of Proposition 7. $\square$

We end this section with a proof of Proposition 4. The proof we present here is due to Rafal Latala. The author's previous proof was obtained as an immediate consequence of Proposition 10 below and the remark thereafter. To show Proposition 4, we need

Lemma 8 *Let $X_j : (\Omega, P) \to \mathbb{R}^n$ be independent, rotationally invariant random vectors in $\mathbb{R}^n$ and X_{j1} be the first coordinate function of X_j, $j = 1, \ldots, k$. let $0 < p < \infty$ and $\alpha = (\alpha_j)_{j=1}^k \subseteq \mathbb{R}$. Then*

$$\mathbb{E}\left|\sum_{j=1}^k \alpha_j X_{j1}\right|^p = e_p(n)\mathbb{E}\,||\sum_{j=1}^k \alpha_j X_j||^p. \tag{2.8}$$

where $e_p(n) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{p+n}{2}\right)}$.

For $n = 2, p = 1$ (with $e_1(2) = 2/\pi$), this can be found in J. Sawa [S].

Proof. Let $U : (\Omega', P') \to S^{n-1}$ be a random vector uniformly distributed vector on S^{n-1} and independent of all X_j's. By rotational invariance of U, for any $x \in \mathbb{R}^n$,

$$\mathbb{E}\,|\langle x, U\rangle|^p = c_p(n)\ ||x||^p$$

where

$$\begin{aligned} c_p(n) &= \mathbb{E}\,|U_1|^p \\ &= \frac{\int_0^\pi (\cos t)^p (\sin t)^{n-2} dt}{\int_0^\pi (\sin t)^{n-2} dt} = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{p+n}{2}\right)}. \end{aligned}$$

Again by rotational invariance, we have for all $y \in S^{n-1}$

$$\begin{aligned} \mathbb{E}\left|\sum_{j=1}^n \alpha_j X_{j1}\right|^p &= \mathbb{E}\,|\langle y, \sum_{j=1}^k \alpha_j X_j\rangle|^p \\ &= \mathbb{E}_{p\times p}|\langle U, \sum_{j=1}^k \alpha_j X_j\rangle|^p \\ &= c_p(n)\ \mathbb{E}\,||\sum_{j=1}^n \alpha_j X_j||^p. \end{aligned}$$

□

Proof of Proposition 4.
Under the assumptions of Proposition 4, the first coordinate functions $r^{-1}X_{j1}$ and U_{j1} have the same distribution, with density $\Gamma(\frac{n+2}{2})/(\sqrt{\pi}\Gamma(\frac{n+1}{2}))\ (1-y^2)^{\frac{n-1}{2}}\chi_{[-1,1]}(y)$, for $j = 1, \ldots, k$. Hence

$$r^{-p}\ \mathbb{E}\left|\sum_{j=1}^k \alpha_j X_{j1}\right|^p = \mathbb{E}\left|\sum_{j=1}^k \alpha_j U_{j1}\right|^p,$$

and thus, applying (2.8) of Lemma 8 to X_j and U_j,

$$r^{-p}c_p(n)\ \mathbb{E}\,||\sum_{j=1}^k \alpha_j X_j||^p = c_p(n+2)\ \mathbb{E}\,||\sum_{j=1}^k \alpha_j U_j||^p$$

which implies Proposition 4 since $c_p(n+2)/c_p(n) = n/(n+p)$. □

We note that, for $n \geq 3$, the complete proof of Theorem 3 was given already. Since $n+2 \geq 3$ for all $n \geq 1$, Theorem 5 follows from Theorem 3 and Proposition 4 in all cases of $1 \leq p < \infty$, $n \geq 1$, with constants $A_p(n) = (n/(n+p))^{1/p} a_p(n+2)$ and $B_p(n) = (n/n+p))^{1/p} b_p(n+2)$. This also holds for $0 < p < 1$ since Theorem 3 is true for $0 < p < 1$ and $n \geq 3$, though we do not prove it.

3 Moments for sums of rotationally invariant random vectors and Bessel functions

The convexity argument used to prove Theorem 3 does not work in the case of the Steinhaus variables, $n = 2$, when $1 \leq p \leq 2$. The case $p = 1$ was done by Sawa [S]. For

$1 < p < 2$, we use a generalization of Haagerup's method to derive the best Khintchine constants when $n = 1$ [H]. This will also yield the proof of Proposition 4. We start by deriving formulas for the expectations $\mathbb{E} \| \sum_{j=1}^{k} \alpha_j X_j \|^p$, using

Lemma 9 *Let $< \cdot, \cdot >$ denote the standard scalar product on $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. Then for $0 < p < 2$:* $\quad \|x\|^p = c_{p,n} \int\limits_{\mathbb{R}^n} \frac{1 - \cos < x, y >}{\|y\|^{n+p}} \, dy$
where $c_{p,n} := \left(\sin \frac{\pi p}{2}\right) 2^p \, \Gamma\left(\frac{p}{2} + 1\right) \Gamma\left(\frac{p+n}{2}\right) / \pi^{n/2+1}$.

Proof. The Lemma is well-known to probabilists. For $0 < p < 2$ and $\alpha \in \mathbb{R}$

$$\int\limits_0^\infty \frac{1 - \cos(\alpha s)}{s^{p+1}} \, ds = |\alpha|^p \int\limits_0^\infty \frac{1 - \cos u}{u^{p+1}} \, du = |\alpha|^p \, \pi / \left(2 \sin \frac{\pi p}{2} \, \Gamma(p+1)\right),$$

cf. Haagerup [H]. In the following integral, we may assume by rotation invariance that $x = re_1, \; e_1 = (1, 0, \ldots, 0)$. Introducing polar coordinates on $\mathbb{R}^n$ we find

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} \frac{1 - \cos < x, y >}{\|y\|^{n+p}} &= \int\limits_{\mathbb{R}^n} \frac{1 - \cos(ry_1)}{\|y\|^{n+p}} \, dy \\
&= |S^{n-2}| \int\limits_0^\pi \left(\int\limits_0^\infty \frac{1 - \cos(rs \cos \varphi_1)}{s^{n+p}} \, s^{n-1} \, ds \right) (\sin \varphi_1)^{n-2} \, d\varphi_1 \\
&= |S^{n-2}| \, \pi / \left(2 \sin \frac{\pi p}{2} \, \Gamma(p+1)\right) \cdot \int\limits_0^\pi r^p \, |\cos \varphi_1|^p \, (\sin \varphi_1)^{n-2} \, d\varphi_1 \\
&= c_{p,n}^{-1} \, \|x\|^p.
\end{aligned}$$

$\square$

Remark. For larger p, integration by parts yields modifications of this formula, e.g.

$$\|x\|^p = -c_{p,n} \int\limits_{\mathbb{R}^n} \frac{\cos < x, y > -1 + 1/2 < x, y >^2}{\|y\|^{n+p}} \, dy, \; 2 < p < 4.$$

Proposition 10 *Let $X_j : (\Omega, P) \to S^{n-1}$ be i.i.d. random vectors uniformly distributed in the sphere of $\mathbb{R}^n$. Let $\alpha = (\alpha_j)_{j=1}^k \subseteq \mathbb{R}$. Then for $0 < p < 2$*

$$\mathbb{E}\,\|\sum_{j=1}^{k}\alpha_j X_j\|^p = d_{p,n}\int\limits_0^\infty \frac{1-\prod\limits_{j=1}^{k} j_{n/2-1}(\alpha_j r)}{r^{p+1}}\,dr$$

where

$$d_{p,n} := \sin\frac{\pi p}{2}\,2^{p+1}\,\Gamma\left(\frac{p}{2}+1\right)\,\Gamma\left(\frac{p+n}{2}\right)\Big/\left(\pi\Gamma\left(\frac{n}{2}\right)\right)$$

and

$$j_{n/2-1}(r) := 2^{n/2-1}\,\Gamma\left(\frac{n}{2}\right)\,J_{n/2-1}(r)/r^{n/2-1},\ \ r>0.$$

is defined in terms of the standard Bessel functions.

Proof. Since the variables X_j are equidistributed on S^{n-1}, they are symmetric. Hence for any $y \in \mathbb{R}^n$

$$\mathbb{E}\,\cos\Big\langle\sum_{j=1}^{k}\alpha_j X_j, y\Big\rangle = \mathbb{E}\exp\left(i\Big\langle\sum_{j=1}^{k}\alpha_j X_j, y\Big\rangle\right).$$

Using this, Lemma 9, Fubini's theorem and the independence of the variables $\alpha_j X_j$, we find

$$\begin{aligned}
\mathbb{E}\,\|\sum_{j=1}^{k}\alpha_j X_j\|^p &= c_{p,n}\int\limits_{\mathbb{R}^n}\frac{1-\mathbb{E}\,\exp\left(i\langle\sum\limits_{j=1}^{k}\alpha_j X_j, y\rangle\right)}{\|y\|^{n+p}}\,dy \\
&= c_{p,n}\int\limits_{\mathbb{R}^n}\frac{1-\prod\limits_{j=1}^{k}\mathbb{E}\,\exp\left(i\,\alpha_j < X_j, y >\right)}{\|y\|^{n+p}}\,dy \\
&= c_{p,n}\int\limits_{\mathbb{R}^n}\frac{1-\prod\limits_{j=1}^{k}\mathbb{E}\,\cos\left(\alpha_j < X_j, y >\right)}{\|y\|^{n+p}}\,dy.
\end{aligned}$$

To calculate the expectation $\mathbb{E}\,\cos(\alpha_j < X_j, y >)$, we may assume that $y = r\,e_1,\ r = \|y\|$, since the X_j's are equidistributed on S^{n-1}. Using polar coordinates, we get

$$\begin{aligned}
&\mathbb{E}\,\cos(\alpha_j < X_j, y >) = \mathbb{E}\,\cos(\alpha_j r X_{j1}) \\
&= \int\limits_0^\pi \cos(\alpha_j r\cos\varphi_1)(\sin\varphi_1)^{n-2}d\varphi_1\Big/\int\limits_0^\pi(\sin\varphi_1)^{n-2}d\varphi_1 \\
&= 2^{n/2-1}\,\Gamma\left(\frac{n}{2}\right)\,J_{n/2-1}(\alpha_j r)/(\alpha_j r)^{n/2-1} =: j_{n/2-1}(\alpha_j r),
\end{aligned}$$

cf. Gradshtein-Ryzhik [GR]. This yields Proposition 10 with $d_{p,n} = c_{p,n}\,|S^{n-1}|$. $\square$

Remark. Let $X_j : (\Omega, P) \to B_n$ be i.i.d. random vectors uniformly distributed in the ball B_n of radius 1 in $\mathbb{R}^n$. Let $(\alpha_j)_{j=1}^k \subseteq \mathbb{R}$. Then for $0 < p < 2$

$$\mathbb{E}\Big\|\sum_{j=1}^k \alpha_j X_j\Big\|^p = d_{p,n} \int_0^\infty \frac{1 - \prod_{j=1}^k j_{n/2}(\alpha_j r)}{r^{p+1}}\, dr$$

The previous proof applies except that now

$$\begin{aligned}
\mathbb{E}\cos(\alpha_j < X_{j,}y >) &= \mathbb{E}\cos(\alpha_j r X_{j1}) \\
&= \frac{\int_0^1 \left(\int_0^\pi \cos(\alpha_j r s\cos\varphi_1)(\sin\varphi_1)^{n-2}\, d\varphi_1\right) s^{n-1}\, ds}{\int_0^1 \left(\int_0^\pi (\sin\varphi_1)^{n-2}\, d\varphi_1\right) s^{n-1}\, ds} \\
&= n\; 2^{n/2-1}\,\Gamma\left(\frac{n}{2}\right) \int_0^1 \left(J_{n/2-1}(\alpha_j r s)/(\alpha_j r s)^{n/2-1}\right) s^{n-1}\, ds \\
&= 2^{n/2}\,\Gamma\left(\frac{n}{2}+1\right)(\alpha_j r)^{-n} \int_0^{\alpha_j r} J_{n/2-1}(u)\, u^{n/2}\, du,
\end{aligned}$$

substituting $u = \alpha_j r s$. Since $(J_{n/2}(u)\, u^{n/2})' = J_{n/2-1}(u)\, u^{n/2}$, the last expression equals

$$= 2^{n/2}\,\Gamma\left(\frac{n}{2}+1\right)\; J_{n/2}(\alpha_j r)/(\alpha_j r)^{n/2} = j_{n/2}(\alpha_j r).$$

Hence the only difference is that $j_{n/2-1}$ is replaced by $j_{n/2}$.

The proof of Theorem 3 in the missing case $n = 2,\ 1 \le p \le 2$ will be given by extending Haagerup's argument for the best Khintchine inequality constants ($n = 1$, [H]). It is fairly technical, relying on properties of the Bessel function J_0, and based on

Proposition 11 *Let $X_j : (\Omega, P) \to S^1$ be i.i.d. Steinhaus variables, i.e. random vectors equidistributed on the circle $S^1 \subseteq \mathbb{C} = \mathbb{R}^2$. Let $0 < p < 2$, $(\alpha_j)_{j=1}^m \subseteq \mathbb{R}$ with $\sum_{j=1}^k \alpha_j^2 = 1$ and $s_0 = \min_{1\le j\le k} \alpha_j^{-2}$. Clearly $s_0 \ge 1$. Assume that*

$$G(p, s) := \int_0^\infty \frac{(\exp(-u^2/4))^s - |J_0(u)|^s}{u^{p+1}} du \ge 0 \tag{3.1}$$

holds for all $s \geq s_0$ and the p given. Then

$$\left(\Gamma\left(\frac{p}{2}+1\right)\right)^{1/p} = a_p(2) \leq \mathbb{E}\left\|\sum_{j=1}^{k} \alpha_j X_j\right\|^p \leq 1,$$

i.e. the claim of Theorem 3 holds in this case.

Proof. Let $s_j > 0,\ b_j \geq 0$ with $\sum\limits_{j=1}^{k} b_j = 1$. We use the generalized arithmetic-geometric mean inequality

$$s_1^{b_1} \dots s_k^{b_k} \leq \sum_{j=1}^{k} s_j b_j$$

for $b_j = \alpha_j^2$ and $s_j := |J_0(\alpha_j r)|^{\alpha_j^{-2}}$ to conclude

$$\prod_{j=1}^{k} J_0(\alpha_j r) \leq \prod_{j=1}^{k} |J_0(\alpha_j r)| \leq \sum_{j=1}^{k} \alpha_j^2\, |J_0(\alpha_j r)|^{\alpha_j^{-2}}. \tag{3.2}$$

Note that $j_0 = J_0$. Thus for $0 < p < 2$, by Proposition 10 and (3.2),

$$\begin{aligned} \mathbb{E}\left\|\sum_{j=1}^{k} \alpha_j X_j\right\|^p &\geq d_{p,2} \int_0^\infty \frac{1 - \sum\limits_{j=1}^{k} \alpha_j^2\, |J_0(\alpha_j r)|^{\alpha_j^{-2}}}{r^{p+1}}\, dr \\ &= \sum_{j=1}^{k} \alpha_j^2\, F_p(\alpha_j^{-2}) \end{aligned} \tag{3.3}$$

where we have put

$$F_p(s) := d_{p,2}\ s^{-p/2} \int_0^\infty \frac{1 - |J_0(u)|^s}{u^{p+1}}\ du$$

and substituted $s := \alpha_j^{-2},\ u = r/\sqrt{s}$. Note that for even integers $k \in \mathbb{N}$

$$F_p(k) = \mathbb{E}\,\|\sum_{j=1}^{k} \frac{1}{\sqrt{k}}\ X_j\|^p$$

and hence by the Central Limit Theorem

$$\begin{aligned} F_p(\infty) &= \lim_{k\to\infty} F_p(k) = \mathbb{E}\|Z\|^p \\ &= d_{p,2}\, s^{-p/2} \int_0^\infty \frac{1 - \exp(-u^2 s/4)}{u^{p+1}}\, du = a_p(2)^p \end{aligned} \tag{3.4}$$

Near zero, $J_0(u) \approx \exp(-u^2/4)$. Hence by assumption, for $s \geq s_0$,

$$\begin{aligned} F_p(s) - F_p(\infty) &= d_{p,2}\ s^{-p/2} \int_0^\infty \frac{\exp(-u^2 s/4) - |J_0(u)|^s}{u^{p+1}}\, du \\ &= G(p,s) \geq 0 \end{aligned}$$

and hence (3.3) and (3.4) imply

$$\begin{aligned} \mathbb{E}\, \| \sum_{j=1}^k \alpha_j X_j \|^p &\geq \sum_{j=1}^k \alpha_j^2\, F_p(\alpha_j^{-2}) \\ &\geq \sum_{j=1}^k \alpha_j^2\, F_p(\infty) = a_p(2)^p. \end{aligned}$$

□

In chapter 4, we will prove (3.1) provided that $p+s \geq 2.5$, i.e. $\max_j |\alpha_j| \leq (2.5-p)^{-1/p}$. If p and s are both close to 1, (3.1) is actually false. If $\max_j |\alpha_j| \geq a_p(2)$,

$$\mathbb{E}\, \| \sum_{j=1}^k \alpha_j X_j \|^p \geq \max_j |\alpha_j|^p \geq a_p(2)^p$$

is trivial. This leaves a gap for p close to 1 when

$$(2.5-p)^{-1/p} < \max_j |\alpha_j| < a_p(2) = \Gamma(p/2+1)^{1/p}. \tag{3.5}$$

This case will be treated by an ad-hoc argument in chapter 5.

For the estimates in chapters 4 and 5, we need some facts about the Bessel function J_0.

Proposition 12

$$|J_0(t)| \leq \exp(-t^2/4 - t^4/64) \qquad \textit{for } 0 < t < 2.59,$$

$$|J_0(t)| \leq \exp(-t^2/4 - t^4/64 - t^6/576) \quad \textit{for } 0 < t < 2.41.$$

Proof. By Watson [W], the Bessel functions satisfy for $\alpha \geq 0$

$$\begin{aligned} &J_{\alpha-1}(t) + J_{\alpha+1}(t) = 2\alpha\, J_\alpha(t)/t \\ &\Big(J_{\alpha-1}(t)/t^{\alpha-1}\Big)' = -J_\alpha(t)/t^{\alpha-1} \end{aligned} \tag{3.6}$$

Let $j(\alpha)$ denote the first positive zero of J_α. Then j_α is increasing in α, $j(0) \simeq 2.4048$, and we find for $0 < t < j(2) \simeq 5.13$

$$\begin{aligned} J_0(t) &< J_0(t) + J_2(t) = 2\, J_1(t)/t \\ &< 2/t(J_1(t) + J_3(t)) = 2/t \cdot 4\, J_2(t)/t = 8\, J_2(t)/t^2 \ . \end{aligned} \tag{3.7}$$

Let $\varphi(t) := \exp(t^2/4 + t^4/64) J_0(t)$. Then by (3.6) and (3.7)

$$\begin{aligned} \varphi'(t) &= \varphi(t)\{-J_1(t) + (t/2 + t^3/16) J_0(t)\} \\ &= \varphi(t)\{-t/2\, J_2(t) + t^3/16\, J_0(t)\} < 0. \end{aligned}$$

Hence φ is decreasing for $0 \le t \le j(2)$ with $\varphi(0) = 1$. Thus

$$0 \le J_0(t) \le \exp(-t^2/4 - t^4/64) \quad \text{for } \ 0 \le t \le j(0) \simeq 2.4.$$

For $j(0) < t < 2.59$, $J_0(t)$ is negative, $|J_0(t)|$ increasing and $|J_0(t)| \exp(t^2/4{+}t^4/64) \le 1$ for $t = 2.59$.

The second inequality in Proposition 12 is proved similarly. □

Remark. A modification of the proof shows that for $n \in \mathbb{N}$, $n \ge 3$, $t_0(n) := \frac{n}{2} + 2$ if $n \ge 4$, $t_0(3) := 3.38$ we have for all $0 \le t \le t_0(n)$

$$|j_{n/2-1}(t)| \le \exp\left(-t^2/(2n) - t^4/(4n^2(n+2))\right).$$

Lemma 13 *For* $0 \le t \le \pi$

$$(i) \qquad \sin t \le 4/\pi\, t(1 - t/\pi) =: f_1(t)$$

$$(ii) \qquad \sin t \le t(1 - t/\pi) + \frac{4}{\pi}\left(\frac{4}{\pi} - 1\right) t^2(1 - t/\pi)^2 =: f_2(t).$$

Proof. Since $\sin(t) = t \prod_{k=-\infty, k\neq 0}^{+\infty} \left(1 - \dfrac{t}{\pi k}\right)$, we get

$$\frac{\sin(t)}{t(1-t/\pi)} = \prod_{k=1}^{+\infty}\left(1 + \frac{t}{\pi k}\right)\left(1 - \frac{t}{\pi(k+1)}\right) = \prod_{k=1}^{+\infty}\left(1 + \frac{1}{k(k+1)}\, t/\pi(1 - t/\pi)\right) := F\left(t\left(1 - \frac{t}{\pi}\right)\right).$$

Obviously the function $F(u)$ is a convex function of $u \in \mathbb{R}^+$. Hence $F(u) \le F(0) + \frac{4}{\pi}\left(F\left(\frac{\pi}{4}\right) - F(0)\right) u = 1 + \frac{4}{\pi}\left(\frac{4}{\pi} - 1\right) u \le \frac{4}{\pi}$ for $0 \le u \le \frac{\pi}{4}$ which corresponds to $0 \le t \le \pi$. This proves (ii) and (i). □

Lemma 14 *Let* $1 \le p, s \le 2$. *Then*

$$\int_{1.8}^{\infty} \frac{|J_0(t)|^s}{t^{p+1}}\, dt \le 1.4506\ (.2638)^s\ (.2934)^p.$$

Proof. Let $\alpha := 1.8$. We will reduce this by interpolation to the 4 cases of $p, s \in \{1, 2\}$.

(a) We start when $s = 2$ and $p \in \{1, 2\}$ and claim that

$$\int_{\alpha}^{\infty} \frac{J_0(t)^2}{t^2}\, dt \le .029096, \quad \int_{\alpha}^{\infty} \frac{J_0(t)^2}{t^3}\, dt \le .008685 \tag{3.8}$$

By Sawa's paper [S], cf. also Watson [W], for $0 \le t \le \alpha$

$$1 - J_0(t)^2 = -\sum_{i=1}^{\infty} \frac{(-1)^i}{i!^2} \binom{2i}{i} \left(\frac{t}{2}\right)^{2i} \le -\sum_{i=1}^{N} \frac{(-1)^i}{i!^2} \binom{2i}{i} \left(\frac{t}{2}\right)^{2i} \tag{3.9}$$

provided that N is odd ≥ 3. Hence for $1 \le p < 2$

$$\begin{aligned} I(p) := \int_{\alpha}^{\infty} \frac{J_0(t)^2}{t^{p+1}}\, dt &= \int_0^{\alpha} \frac{1 - J_0(t)^2}{t^{p+1}}\, dt - \int_0^{\infty} \frac{1 - J_0(t)^2}{t^{p+1}}\, dt + \int_{\alpha}^{\infty} \frac{dt}{t^{p+1}} \\ &= \int_0^{\alpha} \frac{1 - J_0(t)^2}{t^{p+1}}\, dt - \frac{2}{p} \int_0^{\infty} \frac{J_0(t) J_1(t)}{t^p}\, dt + \frac{1}{p} \frac{1}{\alpha^p} \end{aligned} \tag{3.10}$$

By Gradshtein-Ryzhik [GR],

$$\frac{2}{p} \int_0^{\infty} \frac{J_0(t) J_1(t)}{t^p}\, dt = \frac{1}{2^p} \frac{\Gamma(p)\,\Gamma(1 - p/2)}{\Gamma(p/2 + 1)^3} = \frac{1}{2^{p-1}} \frac{\Gamma(p)\,\Gamma(2 - p/2)}{\Gamma(p/2 + 1)^3 (2 - p)} \tag{3.11}$$

For $p = 1$, (3.9)-(3.11) imply for $N = 5$ the first estimate in (3.8),

$$I(1) \le -\sum_{i=1}^{5} \frac{(-1)^{i+1}}{i!^4} \frac{(2i)!}{2^{2i}} \frac{\alpha^{2i-1}}{2i - 1} - \frac{4}{\pi} + \frac{5}{9} \le .029096.$$

For the second estimate, we choose $N = 7$ and $1 < p < 2$ with the aim of taking the limit as $p \to 2$:

$$\begin{aligned} I(p) \le & - \sum_{i=2}^{7} \frac{(-1)^i}{i!^4} \frac{(2i)!}{2^{2i}} \frac{\alpha^{2i-p}}{2i-p} \\ & + \left\{ \frac{\alpha^{2-p}}{2(2-p)} - \frac{1}{2^{p-1}} \frac{\Gamma(p)\,\Gamma(2-p/2)}{\Gamma(p/2+1)^3(2-p)} \right\} + \frac{1}{p} \frac{1}{\alpha^p} . \end{aligned} \tag{3.12}$$

Using l'Hospital's rule, we find that the limit of the expression in the brackets $\{\ldots\}$ as $p \to 2$ is equal to $\dfrac{\gamma + \ln\alpha - \ln 2}{2} - \dfrac{1}{4}$.

This, together with (3.12) gives the numerical result in (3.8),

$$I(2) = \lim_{p \to 2} I(p) \le .00868445.$$

(b) We now take $s = 1$ and $p \in \{1, 2\}$ and claim that

$$\int_\alpha^\infty \frac{|J_0(t)|}{t^2}\, dt \le .112243, \quad \int_\alpha^\infty \frac{|J_0(t)|}{t^3}\, dt \le .032928. \tag{3.13}$$

Let $\nu_1 = 2.4048\ldots$ denote the first positive zero of J_0. For $t \in [0, \nu_1]$,

$$0 \le J_0(t) \le \sum_{j=0}^{4} (-1)^j\, t^{2j} \,/\, (j!^2\, 2^{2j}) =: p_8(t) \tag{3.14}$$

from the Taylor series of J_0. For $t \ge \nu_1$, we will use the approximation

$$|J_0(t) - \sqrt{\frac{2}{\pi t}} \left(\cos\left(t - \frac{\pi}{4}\right) + \frac{\sin(t - \pi/4)}{8t} \right) | \le \sqrt{\frac{2}{\pi}} \left(\frac{9}{128} \frac{1}{t^{5/2}} + \frac{75}{1024} \frac{1}{t^{7/2}} \right) \tag{3.15}$$

which can be found in Watson [W], 7.3 or Magnus-Oberhettinger [MO], p. 22. For $t \in [2.41, 5/4\pi]$

$$|\cos(t - \pi/4) + \sin(t - \pi/4)/(8t)| = |\cos(t - \pi/4)| - |\sin(t - \pi/4)/(8t)|. \tag{3.16}$$

Using (3.14)-(3.16) and the fact that $\nu_1 \geq 3\pi/4$, we find

$$\begin{aligned}
\mathcal{J}(p) &:= \int_{\alpha}^{\infty} \frac{|J_0(t)|}{t^{p+1}}\, dt \\
&\leq \int_{\alpha}^{\nu_1} \frac{p_8(t)}{t^{p+1}}\, dt + \sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty} \int_{(m+3/4)\pi}^{(m+7/4)\pi} \frac{|\cos(t-\pi/4)|}{t^{p+3/2}}\, dt \\
&+ \sqrt{\frac{2}{\pi}} \left(\int_{\nu_1}^{2.41} - \int_{2.41}^{5\pi/4} + \int_{5\pi/4}^{9\pi/4} \right) \frac{|\sin(t-\pi/4)|}{8\, t^{p+5/2}}\, dt + \sqrt{\frac{2}{\pi}} \int_{9\pi/4}^{\infty} \frac{dt}{8\, t^{p+5/2}} \\
&+ \sqrt{\frac{2}{\pi}} \int_{\nu_1}^{\infty} \Big(\frac{9}{128} \frac{1}{t^{p+7/2}} + \frac{75}{1024} \frac{1}{t^{p+9/2}} \Big)\, dt.
\end{aligned}$$

Denote the first integral by I_1, the term $\sqrt{\frac{2}{\pi}} \int_{\nu_1}^{2.41} (\ldots)$ by R_1 and the last two integrals by R_2 and R_3. By Lemma 13, with f_1 and f_2 as given there,

$$\begin{aligned}
\int_{(m+3/4)\pi}^{(m+7/4)\pi} \frac{|\cos(t-\pi/4)|}{t^{p+3/2}}\, dt &= \int_0^{\pi} \frac{\sin u}{(u+(m+3/4)\pi)^{p+3/2}}\, du \\
&\leq \int_0^{\pi} \frac{f_2(u)}{(u+(m+3/4)\pi)^{p+3/2}}\, du.
\end{aligned}$$

For $m \geq 2$, it suffices to estimate $\sin u \leq f_1(u)$. Treating the integrals involving $\sin(t-\pi/4)$ similarly gives for $m_0 > 2$ to be chosen

$$\mathcal{J}(p) \leq I_1 + \left(I_2 + \sum_{m=2}^{m_0} K(m) + R_4(m_0) \right) + (R_1 - I_3 + I_4) + R_2 + R_3, \tag{3.17}$$

where

$$\begin{aligned}
I_2 &= \sqrt{\frac{2}{\pi}} \sum_{m=0}^{1} \int_0^{\pi} \frac{f_2(u)}{(u+(m+3/4)\pi)^{p+3/2}}\, du, \\
K(m) &= \sqrt{\frac{2}{\pi}} \int_0^{\pi} \frac{f_1(u)}{(u+(m+3/4)\pi)^{p+3/2}}\, du,
\end{aligned}$$

$$
\begin{aligned}
R_4(m_0) &= \sqrt{\frac{2}{\pi}} \int\limits_{(m_0+7/4)\pi}^{\infty} du \,/\, u^{p+3/2}, \\
-I_3 &= -\sqrt{\frac{2}{\pi}} \int\limits_{2.41-3\pi/4}^{\sqrt{2}} \frac{1-u^2/2}{8(u+3\pi/4)^{p+5/2}}\, du, \\
I_4 &= \sqrt{\frac{2}{\pi}} \int\limits_{0}^{\pi} \frac{f_1(u)}{8(u+5\pi/4)^{p+5/2}}\, du.
\end{aligned}
$$

For I_3, we also used $\cos u \geq \max(0, 1-u^2/2)$ for $u \in [0, \pi/2]$. Since p_8, f_1, f_2 are polynomials, all integrals I_l, $l = 1, \ldots, 4$ and $K(m)$ as well as R_2 and R_3 can be calculated explicitely; R_1 is quickly estimated. The values of $K(m)$ and I_4 are obtained from the formula

$$
\int \frac{f_1(u)}{(u+k\pi)^{5/2+l}}\, du = \alpha_l \, \frac{1}{\pi^2 (u+k\pi)^{3/2+l}} \left[(2\pi^2 k + (3+2l)\pi)(4k+1-2l) + \beta_l u^2 \right].
$$

This is needed for $l = 0, 1, 2$ and $k = (m+3/4)\pi,\ 5/4\pi$. Here $\alpha_0 = -8/3,\ \alpha_1 = 8/15,\ \alpha_2 = 8/105,\ \beta_0 = 3,\ \beta_1 = 15,\ \beta_2 = 35$.

For $p = 1$, we choose $m_0 = 30$ and for $p = 2$, $m_0 = 5$. Integration then yields the numerical upper bounds

	$I_1 \leq$	$I_2 \leq$	$\sum\limits_{m=2}^{m_0} K(m) \leq$	$-I_3 \leq$	$I_4 \leq$
p=1	.0256286	.0729538	.0134986	-.0023029	.0006130
p=2	.0129997	.0191242	.0008500	-.0006870	.0001203

	$R_1 \leq$	$R_2 \leq$	$R_3 \leq$	$R_4(m_0) \leq$
p=1	.0000243	.0003004	.0009936	.0005334
p=2	.0000102	.0000304	.0003256	.0001542

Adding these values for $p = 1$ and $p = 2$, respectively, and using (3.17), gives the estimates for $\mathcal{J}(1)$ and $\mathcal{J}(2)$ stated in (3.13).

(c) Take $1 \leq p,\ s \leq 2$. The function

$$
\int\limits_{\alpha}^{\infty} \left(\frac{|J_0(t)|}{.26376} \right)^{s} \left(\frac{1}{.29337\, t} \right)^{p} \frac{dt}{t}
$$

is convex in (p,s) for $1 \leq p,s \leq 2$. Therefore its maximum is attained when $p,s \in \{1,2\}$ which by (a) and (b) is ≤ 1.4506. This proves Lemma 14. $\square$

Remark. Let $I_- := \{t \in \mathbb{R} \mid J_0(t) < 0\} \subseteq [2.4, \infty)$. A variant of the method in (b) also yields that

$$\int_{I_-} |J_0(t)|/t^2 \, dt < 7/100 \tag{3.18}$$

The essential contribution comes from the part of I_2 resulting from $m = 0$ (about $6/100$) and $K(2)$ in the estimate corresponding to (3.17).

4 Positivity of integrals involving exponential and Bessel functions

The aim of this chapter is to prove that the positivity assumption (3.1) in Proposition 11 is satisfied if $p + s \geq 2.5$.

Proposition 15 *Let $1 \leq p \leq 2$, $s \geq 1$ and $p + s \geq 2.5$. Then*

$$G(p,s) := \int_0^\infty \frac{(\exp(-u^2/4))^s - |J_0(u)|^s}{u^{p+1}} \, du \geq 0. \tag{4.1}$$

Remark. The integrand is positive for $u \leq j(0)$. For $u > j(0)$ it is negative for the most part. The estimate is delicate since the order of magnitude of $G(p,s)$ is only about 10^{-2} for small p,s.

We start with a simple lemma.

Lemma 16 *Let $q, x > 0$. Then*

(a) $\displaystyle\int_x^\infty t^{-q} e^{-t} \, dt \geq (x+1)^{-q} e^{-x}$

(b) $(x+1)^{-q} \geq \dfrac{3}{25}(7 - 2q)\, x^{-q}$ *for* $x \geq 8/5$, $q \geq 3/2$.

Proof. (a) By Jensen's inequality

$$\int_x^\infty t^{-q}\, e^{x-t}\, dt \geq \left(\int_x^\infty t\, e^{x-t}\, dt \right)^{-q} = (x+1)^{-q}.$$

(b) The function $\psi(x) = (x/(x+1))^q$ is decreasing since $q > 0$. Hence for $x \geq 8/5$, $(x+1)^{-q} \geq (8/13)^q\, x^{-q}$. But, with $f(q) := (8/13)^q$, f is convex with $f(3/2) \geq 12/25$, $f'(3/2) \geq -6/25$. Thus for $q \geq 3/2$

$$f(q) \geq f(3/2) + f'(3/2)(q - 3/2) \geq 3/25\,(7 - 2q).$$

□

Proof of Proposition 15.

We consider two cases (i) $s \geq 2$ and (ii) $1 \leq s < 2$, $p + s \geq 2.5$.

(i) If $s \geq 2$, we let $t_0 := \sqrt{6.4}$, $c := t_0^2/4 = 8/5$, $u_0 := cs$,

$$\varphi_l(u) := u^{l-p/2}\, e^{-u} \text{ for } l \in \mathbb{Z},$$

$$\mathcal{J} := \int_{t_0}^\infty |J_0(t)|^s / t^{p+1}\, dt.$$

Integration by parts gives

$$\int_0^{u_0} \varphi_l(u)\, du = -\varphi_l(u_0) + \left(l - \frac{p}{2} \right) \int_0^{u_0} \varphi_{l-1}(u)\, du, \;\; l = 3, 2 \tag{4.2}$$

Using Proposition 12, the substitution $u = t^2 s/4$, the inequality $1 - e^{-x} \geq x - x^2/2$ for $x = u^2/(4s) > 0$ and (4.2), we find

$$\begin{aligned} G(p,s) \;\; &\geq \int_0^{t_0} \exp\left(-t^2 s/4\right) \left(1 - \exp\left(-t^4 s/64\right)\right) \Big/ t^{p+1}\, dt \\ &\quad + \int_{t_0}^\infty \exp\left(-t^2 s/4\right) \Big/ t^{p+1}\, dt - \mathcal{J} \\ &= s^{p/2}/2^{p+1} \int_0^{u_0} e^{-u} \left(1 - \exp\left(-u^2/(4s)\right)\right) \Big/ u^{p/2+1}\, du + F - \mathcal{J} \end{aligned}$$

$$
\begin{aligned}
&\geq \quad s^{p/2-1}/2^{p+3} \int\limits_0^{u_0} \left[\varphi_1(u) - \varphi_3(u)/(8s)\right]\, du + F - \mathcal{J} \\
&= \quad s^{p/2-1}/2^{p+3} \left\{ \left(\Gamma\left(2 - p/2\right) - \int\limits_{u_0}^{\infty} \varphi_1(u)\, du \right) (1 - H) + K \right\} + F - \mathcal{J}
\end{aligned}
\tag{4.3}
$$

where we have put

$$
F = \frac{s^{p/2}}{2^{p+1}} \int\limits_{u_0}^{\infty} \varphi_{-1}(u)\, du, \quad H = \frac{(3-p/2)(2-p/2)}{8s}, \quad K = \frac{\varphi_3(u_0) + (3-p/2)\varphi_2(u_0)}{8s}.
$$

Integration by parts and Lemma 16 yield

$$
\begin{aligned}
-\int\limits_{u_0}^{\infty} \varphi_1(u)\, du \quad &= \quad -(\varphi_1(u_0) + (1-p/2)\varphi_0(u_0)) + \frac{p}{2}\left(1 - \frac{p}{2}\right) \int\limits_{u_0}^{\infty} \varphi_{-1}(u)\, du \\
&\geq \quad -(\varphi_1(u) + (1-p/2)\varphi_0(u_0)) + \frac{p}{2}\left(1 - \frac{p}{2}\right) \frac{3}{25}\,(5-p)\,\varphi_{-1}(u).
\end{aligned}
$$

Using this in (4.3), where $H < 1$ holds, a rearrangement of terms implies the estimate

$$
G(p,s) \geq s^{p/2-1}/2^{p+3} \cdot \Gamma(2-p/2)(1-H) + \psi(p,s)/[8(2\sqrt{c})^p (e^c)^s] - \mathcal{J}, \tag{4.4}
$$

where

$$
\psi(p,s) := As + B + C/s + D/s^2 + E/s^3,
$$

$$
\begin{aligned}
&A := \frac{64}{125},\ B := -\frac{4}{25}(p+4),\ C := (p^2 - 6p + 34)/20 \geq C' := 13/10, \\
&D := \left[(3-p/2)(2-p/2) + 3(5-p)\right](1-p/2)/8 \\
&E := -3/320\ (3-p/2)(2-p/2)(1-p/2)p/2(5-p).
\end{aligned}
$$

Since $|E| \leq D,\ D/s^2 + E/s^3 > 0$ and hence

$$
\psi(p,s) \geq As + B + C'/s =: \overline{\psi}(p,s)
$$

$\overline{\psi}(p,\cdot)$ is increasing in $s \in [2,\infty)$. Therefore

$$\psi(p,s) \geq \overline{\psi}(p,2) = \frac{521}{500} - \frac{4}{25}\,p > 0. \tag{4.5}$$

For the integral $\mathcal{J}$, we know by (3.8) with $\sup\limits_{t\geq\sqrt{6.4}} |J_0(t)| \leq \dfrac{1}{2.4}$

$$\mathcal{J} \;=\; \int\limits_{\sqrt{6.4}}^{\infty} \frac{|J_0(t)|^s}{t^{p+1}}\,dt \leq \frac{1}{2.4^{s-2}}\,\frac{1}{2.5^{p-1}} \int\limits_{2.5}^{\infty} \frac{J_0(t)^2}{t^2}\,dt \leq \frac{.0291}{2.4^{s-2}\cdot 2.5^{p-1}}\ .$$

Using this, (4.5) and $\Gamma(2-p/2)(1-H) \geq \dfrac{49}{128}\sqrt{\pi}$ in (4.4), we find

$$\begin{aligned} G(p,s) &\geq 49/128\sqrt{\pi}/(8\sqrt{s}\,2^p) - .0291/(2.4^{s-2}\cdot 2.5^{p-1}) \\ &\geq .0299\,\sqrt{2/s}\,/2^{p-1} - .0291/(2.4^{s-2}\cdot 2.5^{p-1}) > 0\ . \end{aligned}$$

(ii) Now let $1 \leq s \leq 2$ with $p+s \geq 2.5$. In this case, put $t_0 := 1.8$, $c := t_0^2/4 = 81/100$, $u_0 := cs$ and define φ_l and $\mathcal{J}$ as in (i) with these values of t_0, c and u_0. We modify the estimate of (i) by using the second estimate for J_0 in Proposition 12. Letting F as before, we have

$$\begin{aligned} G(p,s) \;&\geq\; \int\limits_0^{t_0} \exp\left(-t^2 s/4\right)\left(1-\exp\left(-t^2 s/64 + t^6 s/576\right)\right)\Big/\, t^{p+1}\,dt \\ &\quad + \int\limits_{t_0}^{\infty} \exp\left(-t^2 s/4\right)\,/\,t^{p+1}\,dt - \mathcal{J} \\ &= s^{p/2}/2^{p+1} \int\limits_0^{u_0} e^{-u}\left(1-\exp\left(-u^2/(4s) - u^3/(9s^2)\right)\right)\Big/\, u^{p/2+1}\,du + F - \mathcal{J} \\ &\geq s^{p/2-1}/2^{p+1} \int\limits_0^{u_0} \left(\varphi_1(u)/4 + \varphi_2(u)/(9s) - 289/5000\,\varphi_3(u)/s^2\right)\,du + F - \mathcal{J}\ . \end{aligned}$$

Here $1-e^{-x} \geq x - x^2/2$ was used for $x = u^2/(4s) + u^3/(9s^2)$. Integration by parts yields

$$\begin{aligned} -\int\limits_0^{u_0} \varphi_3(u)\,du \;&\geq\; \varphi_3(u_0) - \frac{5}{2}\int\limits_0^{u_0} \varphi_3(u)\,du \\ &= \varphi_3(u_0) - \frac{5}{2}\left[\varphi_2(u_0) - (2-p/2)\int\limits_0^{u_0} \varphi_1(u)\,du\right]. \end{aligned}$$

Hence with $M := \int\limits_0^{u_0} \varphi_1(u)\, du$ and $g_1(s) := \dfrac{289}{5000}\, c^3 s + \dfrac{601}{18000}\, c^2$

$$G(p,s) \geq s^{p/2}/2^{p+1} \left(g_1(s)\, \varphi_0(cs) + \left[\frac{1}{4s} - \frac{601}{9000}\ \frac{1-p/4}{s^2} \right] M \right) + F - \mathcal{J}. \tag{4.6}$$

We estimate F and M by linear and quadratic approximation near $p = 1,\ s = 1$ which is proved below in Lemma 17,

$$\begin{aligned} F &\geq \frac{1}{2 \cdot 1.8^p\, (e^c)^s}\, g_2(p,s), \quad M \geq g_3(p,s), \qquad (4.7)\\ g_2(p,s) &:= .54347 - .28805(s-1) - .11778(p-1), \\ g_3(p,s) &= .30586 + .3243(s-1) + .13538(p-1) - .06315(s-1)^2 + .0604(p-1)^2. \end{aligned}$$

Since $1 \leq p, s \leq 2$, we can use the estimate for $\mathcal{J}$ given in Lemma 14. This, (4.6) and (4.7) imply

$$\begin{aligned} G(p,s) \ \geq\ & \frac{1}{2 \cdot 1.8^p\, (e^c)^s}\ (g_1(s) + g_2(p,s)) + \frac{s^{p/2-1}}{2^{p+3}} \left(1 - \frac{601}{2250}\ \frac{1-p/4}{s} \right) g_3(p,s) \\ & - 1.4506\ (.2638)^s\ (.2934)^p. \end{aligned}$$

To show that the right side is positive, we multiply by $(.2638)^{-s}\ (.2934)^{-p}$. Inserting the formulas for g_1, g_2, g_3 and $c = .81$ shows that $G(p,s) \geq 0$ provided that

$$F(p,s) := F_1(p,s) + F_2(p,s) > 1.4506 \tag{4.8}$$

where

$$\begin{aligned} F_1(p,s) &:= 1.8935^p \cdot 1.6863^s\, (.4856 - .1287\, s - .0589\, p) \\ F_2(p,s) &:= 1.7041^p \cdot 3.7907^s / s^{1-p/2}\, (1 - .2672\, (1-p/4)/s) \\ & \qquad \cdot (-.01958 + .05632\, s + .00182\, p - .0079\, s^2 + .0075\, p^2). \end{aligned}$$

Let $A, B, C > 0$ and $x \in (0, \infty)$. It is easy to check that

$$A^x(B - Cx) \text{ is increasing (decreasing) for } x < (>)\, B/C - 1/\ln A. \tag{4.9}$$

For fixed $1 \le s \le 2$, $F_1(\cdot, s)$ is increasing in $1 \le p \le 2$: in (4.9) take $A = 1.8935$, $B' := .4856 - .1287s \ge B := .2282$, $C := .0589$, then $B/C - 1/\ln A > 2$. For fixed $1 \le s \le 2$, $F_2(\cdot, s)$ is increasing in $1 \le p \le 2$, too, since all terms involving p are increasing in p. We claim that

$$\begin{aligned} F(1,s) &\ge 1.4506 \qquad \text{for all } 1.5 \le s \le 2 & (4.10) \\ F(2.5 - s, s) &\ge 1.4506 \qquad \text{for all } 1.5 \le s \le 2 & (4.11) \end{aligned}$$

This is sufficient to verify (4.8) since by (4.10), (4.11) and the monotonicity properties of F_1, F_2 just mentioned for $p + s > 2.5$

$$\begin{aligned} &- \text{ for } s \ge 1.5: \quad F(p,s) \ge F(1,s) \ge 1.4506 \\ &- \text{ for } s \le 1.5: \quad F(p,s) \ge F(2.5 - s, s) \ge 1.4506\ . \end{aligned}$$

The function $F_1(1, \cdot)$ is concave in $1.5 \le s \le 2$, thus

$$\begin{aligned} F_1(1,s) &\ge F_1(1, 1.5) + \frac{F(1,2) - F(1,1.5)}{2 - 1.5}\,(s - 1.5) \\ &\ge 1.1405 - .1145\, s\ . \end{aligned}$$

In $F_2(1, \cdot)$, the brackets are increasing functions in $s \in [1.5, 2]$, thus putting $s = 1.5$ in $F_2(1, \cdot)$ except for the term in front of the brackets,

$$F_2(1,s) \ge .083337 \cdot 3.7907^s/\sqrt{s} =: .0833 \cdot R(x)\ .$$

The function R convex in $s \in [1,2]$ with derivative ≥ 6, hence for $s \ge 1.5$

$$\begin{aligned} F_2(1,s) &\ge .083337(R(1.5) + R'(1.5)(s - 1.5)) \\ &\ge .5s - .25, \\ F(1,s) &= F_1(1,s) + F_2(1,s) \ge .8905 + .3855s \ge 1.4687, \end{aligned}$$

proving (4.10). To check (4.11) for $1 \le s \le 1.5$, let

$$\begin{aligned} h_1(s) &:= F_1(2.5-s,s) = .8905^s\,(1.6692 - .3444\,s) \\ h_2(s) &:= F_2(2.5-s,s) = \frac{2.2244^s}{s^{s/2+3/4}}\,(-.01210 + .10619\,s + .06029\,s^2 - .00142\,s^3). \end{aligned}$$

We claim that $h := h_1 + h_2$ is decreasing in $[1, 1.5]$, so that

$$F(2.5-s,s) = h(s) \ge h(1.5) \ge 1.4706,$$

i.e. (4.11) holds. To prove that $h' < 0$ in $[1, 1.5]$, we will show

$$h_1' < -2/5,\ h_2' < 2/5 \text{ in } [1, 1.5] \tag{4.12}$$

The function h_1 is decreasing and convex in $[1, 1.5]$, with

$$\begin{aligned} h_1'(s) &= .8905^s\,(-.5380 + .0399\,s) < 0 \\ h_1''(s) &= .8905^s\,(.1023 - .0046\,s) > 0. \end{aligned}$$

Hence $h_1'(s) \le h_1'(1.5) = -.4017 < -2/5$. As for h_2,

$$h_2'(s) = (2.2244^s/s^{s/2})\ s^{-7/4}\,q(s),$$

where $q(s)$ has the form

$$q(s) = \sum_{j=0}^{4} (a_j + b_j \ln s)\,s^j.$$

The calculation shows q to be positive, increasing and concave in $[1, 1.5]$, which implies

$$q(s) \le q(1) + q'(1)(s-1) = -.0501 + .2037\,s =: r(s).$$

But $s^{-7/4}r(s)$ is decreasing in [1,1.5]: $s^{-7/4}(Bs - A)$ is decreasing provided that $s \geq 7A/3B = .57$ for $B = .203,\ A = -.0496$. Thus $s^{-7/2}q(s) \leq r(1) = .1536$.

Moreover, $k(s) := 2.2244^s/s^{s/2}$ is increasing since

$$k'(s) = k(s)\,(.2995 - (\ln s)/2) \geq 0,\ s \geq 1.82.$$

Thus $k(s) \leq k(1.5) \leq 2.4477$ and

$$h_2'(s) \leq k(1.5)\, r(1) \leq .376 < 2/5.$$

Therefore (4.12) and (4.11) hold.

This ends the proof of Proposition 15. $\square$

It remains to prove the estimates for the integrals F and M given in (4.7) and used there. We state this as

Lemma 17 *Let* $F := \frac{s^{p/2}}{2^{p+1}} \int\limits_{u_0}^{\infty} \varphi_{-1}(u)\,du,\ M := \int\limits_0^{u_0} \varphi_1(u)\,du$ *where* $\varphi_l(u) = u^{l-p/2}e^{-u}$ *and* $u_0 = cs,\ c = .81,\ 1 \leq s \leq 2,\ 1 \leq p \leq 2$. *Then*

$$\begin{aligned} F &\geq \frac{1}{2 \cdot 1.8^p (e^c)^s}\, g_2(p,s),\ M \geq g_3(p,s), \\ g_2(p,s) &:= .54347 - .28805\,(s-1) - .11778\,(p-1) \\ g_3(p,s) &:= .30586 + .3243\,(s-1) + .13538\,(p-1) - .06315\,(s-1)^2 + .0604\,(p-1^2). \end{aligned}$$

Proof. (i) Substitution of $t = u/u_0 - 1$ yields

$$F = \frac{s^{p/2}}{2^{p+1}}\, \varphi_0(u_0)\ F(p,u_0) = \frac{1}{2 \cdot 1.8^p (e^c)^s}\ F(p,u_0),$$

$$F(p,u_0) = \int\limits_0^{\infty} (1+t)^{-(p/2+1)}\, e^{-u_0 t}\, dt.$$

We have to prove that $F(p,u_0)=F(p,cs)\geq g_2(p,s)$. One has

$$\begin{aligned}\frac{\partial F}{\partial p}(p,u_0) &= \int_0^\infty \Big(-\frac{1}{2}\ln(1+t)\Big)\,(1+t)^{-(p/2+1)}\,e^{-u_0 t}\,dt<0\\ \frac{\partial F}{\partial u_0}(p,u_0) &= \int_0^\infty (-t)\,(1+t)^{-(p/2+1)}\,e^{-u_0 t}\,dt<0.\end{aligned}$$

All second derivatives are positive, one has to replace the factors $-1/2\ln(1+t)$, $(-t)$ above for $\dfrac{\partial^2 F}{\partial p\partial u_0}$, $\dfrac{\partial^2 F}{\partial p^2}$, $\dfrac{\partial^2 F}{\partial u_0^2}$ by the positive functions $t\ln(1+t)$, $\ln(1+t)^2$, t^2, respectively. Thus for $u=cs$

$$F(p,cs)\geq F(1,cs)+\frac{\partial F}{\partial p}\,(1,c)(p-1)+\frac{\partial F}{\partial u_0}\,(1,c)\,c\,(s-1)$$

We show that

$$F(1,c)\geq .54347,\ \frac{\partial F}{\partial p}\,(1,c)\geq -.11778,\ \frac{\partial F}{\partial u_0}\,(1,c)\cdot c\geq .28805; \tag{4.13}$$

then $F(p,u_0)\geq g_2(p,s)$. But

$$\begin{aligned}F(1,c) &= \int_0^\infty (1+t)^{-3/2}\,e^{-ct}\,dt=c^{1/2}\,e^c\int_c^\infty u^{-3/2}\,e^{-u}\,du\\ &= c^{1/2}\,e^c\left(2\,c^{-1/2}\,e^{-c}-2\int_c^\infty u^{-1/2}\,e^{-u}\,du\right)\\ &= 2-2\,c^{1/2}\,e^c\,\sqrt{\pi}\ \text{erfc}\ (\sqrt{c}),\ \text{erfc}\ (x):=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-v^2}\,dv\\ &\geq .543471\end{aligned}$$

using the known values of error function. Similarly,

$$\int_0^\infty (1+t)^{-1/2}\, e^{-ct}\, dt = c^{-1/2}\, e^c \sqrt{\pi}\ \mathrm{erfc}\,(\sqrt{c}) \le .899091$$

$$\frac{\partial F}{\partial u_0}\,(1,c) = -\int_0^\infty t\,(1+t)^{-3/2}\, e^{-ct}\, dt$$

$$= -\int_0^\infty \left((1+t)^{-1/2} - (1+t)^{-3/2}\right) e^{-ct}\, dt \ge -.355620$$

so that $c\,\frac{\partial F}{\partial u_0}\,(1,c) \ge -.28805$. In the case of the logarithmic integral

$$\frac{\partial F}{\partial p}\,(1,c) = -\int_0^\infty \ln(1+t)(1+t)^{-3/2}\, e^{-ct}\, dt$$

there is no easy formula in terms of the error function. Since $\ln(1+t)(1+t)^{-3/2}$ is decreasing,

$$\frac{1}{2}\int_5^\infty \ln(1+t)(1+t)^{-3/2}\, e^{-ct}\, dt \le \frac{\ln 6}{2\cdot 6^{3/2}}\ \frac{e^{5c}}{c} \le .001312.$$

For $a > 0$ and $t \ge a$

$$\begin{aligned}
\exp(-ct) &\le \exp(-ca)\left(1 - c(t-a) + \frac{c^2}{2}\,(t-a)^2\right) =: \psi_a(t) \\
\frac{\partial F}{\partial p}\,(1,c) &\ge -\frac{1}{2}\int_0^5 \ln(1+t)(1+t)^{-3/2}\, e^{-ct}\, dt - .001312 \\
&\ge -\frac{1}{2}\sum_{j=1}^{10}\ \int_{(j-1)/2}^{j/2} \ln(1+t)(1+t)^{-3/2}\, \psi_{(j-1)/2}(t)\, dt - .001312.
\end{aligned}$$

The latter integrals can be calculated exactly, since ψ_a is a polynomial of degree 2 in t. This yields the numerical estimate

$$\frac{\partial F}{\partial p}\,(1,c) \ge -.116461 - .001312 \ge -.11778.$$

Hence (4.13) is shown, and the estimate for F follows.

(ii) Let

$$M = \int_0^{u_0} u^{1-p/2}\, e^{-u}\, du =: G(p, u_0).$$

Then, fixing u_0,

$$\begin{aligned} \frac{\partial G}{\partial p}(p, u_0) &= -\frac{1}{2} \int_0^{u_0} (\ln u)\ u^{1-p/2}\, e^{-u}\, du, \\ \frac{\partial^2 G}{\partial p^2}(p, u_0) &= \frac{1}{4} \int_0^{u_0} (\ln u)^2\ u^{1-p/2}\, e^{-u}\, du \ge 0, \end{aligned}$$

i.e. $\dfrac{\partial G}{\partial p}$ is increasing in p,

$$\frac{\partial G}{\partial p}(p, u_0) \ge \frac{\partial G}{\partial p}(1, u_0) = -\frac{1}{2} \int_0^{u_0} (\ln u)\ \sqrt{u}\, e^{-u}\, du.$$

Let $\psi_k(x) := e^{-1} \sum_{j=0}^{k} \dfrac{(-1)^j}{j!}\,(x-1)^j$ for $k \in \mathbb{N}$. Then $e^{-x} - \psi_4(x)$ is positive (negative) for $x < 1$ $(x > 1)$ and

$$\frac{\partial G}{\partial p}(p, u_0) \;\; = \;\; -\frac{1}{2} \int_0^{u_0} (\ln \frac{1}{u})\ \sqrt{u}\ \psi_4(u)\, du =: h(u_0).$$

Recall that $u_0 = cs$, $0.81 \le cs \le 1.62$ for $1 \le s \le 2$. Clearly, the last integral is increasing in $u_0 \in [.81, 1]$ and decreasing in $u_0 \in [1, 1.62]$. Since ψ_4 is a fourth order polynomial, the values can be calculated exactly, giving

$$\begin{aligned} h\,(.81) &\ge .1551, \quad h(1.62) \ge .135389, \\ \frac{\partial G}{\partial p}(p, u) &\ge \frac{\partial G}{\partial p}(1, u_0) \ge h(1.62) \ge .13538, \;\; 1 \le s \le 2 \end{aligned} \tag{4.14}$$

Further,

$$G(1,c) = -\sqrt{c}\, e^{-c} + \sqrt{\pi}/2 \ \mathrm{erf}\ (\sqrt{c}) \geq .30586.$$

Using that $1 \leq p, s \leq 2$ and $c < 1$, we also get

$$\begin{aligned} \frac{\partial^2 G}{\partial p^2}(p,u) &\geq \frac{1}{4}\int_0^c (\ln u)^2\, u^{1-p/2}\, e^{-u}\, du \\ &\geq \frac{1}{4}\int_0^c (\ln u)^2 \sqrt{u}\, e^{-u}\, du \\ &\geq \frac{1}{4}\int_0^c (\ln u)^2 \sqrt{u}\, \psi_3(u)\, du \geq .120814; \end{aligned}$$

the latter integral again can be evaluated exactly. This and (4.14) implies for all $.81 \leq u_0 = cs \leq 1.62$

$$G(p,u_0) \geq G(1,u_0) + .13538\,(p-1) + .060407\,(p-1)^2. \tag{4.15}$$

A lower estimate for $G(1,u_0)$ follows from

$$\begin{aligned} \frac{\partial G}{\partial u_0}(1,u_0) &= \sqrt{u_0}\, e^{-u_0}, \ \frac{\partial G}{\partial u_0}(1,c) = \sqrt{c}\, e^{-c} \geq .400372 \\ \frac{\partial^2 G}{\partial u_0^2}(1,u_0) &= \left(\frac{1}{2\sqrt{u_0}} - \sqrt{u_0}\right) e^{-u_0} \geq -.192474 : \\ G(1,u_0) &\geq G(1,c) + .400372 \cdot c(s-1) - .096237\, c^2 (s-1)^2 \\ &\geq .30586 + .3243(s-1) - .06315(s-1)^2. \end{aligned}$$

This, together with (4.15), gives $M \geq q_3(p,s)$. □

Remark. Using the same method, one can show that the integral corresponding to (3.1) in Proposition 11 for $n \geq 3$ and implying Theorem 3 in the same way,

$$G(p,s,n) := \int_0^\infty \frac{(\exp(-u^2/(2n)))^s - |j_{n/2-1}(u)|^s}{u^{p+1}}\, du,$$

is positive also for $0 < p < 1$, $s \geq 1$ provided that $n \geq 4$ and for $p + s \geq 1.6$ if $n = 3$. For $n = 3$ and p close to 0 and s close to 1, one can modify the approach given in the next chapter 5, yielding Theorem 3 with the same formulas for $a_p(n)$ and $b_p(n)$ also for $0 < p < 1$ if $n \geq 3$. For $n = 2$, this is not true as mentioned earlier; the formula for $a_p(2)$ definitely has to be changed if $p \leq p_0 \approx .4756$.

5 The case of one dominating coefficient

To complete the proof of Theorem 3 for $n = 2$, we still have to consider the case of (3.5). We state the required estimate as

Proposition 18 *Let $\alpha = (\alpha_j)_{j=1}^k \subseteq \mathbb{R}$ with $\sum\limits_{j=1}^k \alpha_j^2 = 1$ and $|\alpha_1| \geq \ldots \geq |\alpha_k|$, $X_j : (\Omega, P) \to S^1$ be i.i.d. uniformly distributed vectors on S^1. Assume that $(2.5 - p)^{-1/2} < |\alpha_1| < a_p(2) := \Gamma(p/2 + 1)^{1/p}$ where $1 \leq p < 2$. Then*

$$\left(\mathbb{E} \, \| \sum_{j=1}^k \alpha_j \, X_j \|^p \right)^{1/p} \geq a_p(2).$$

Proof. Our proof is again based on Proposition 10. Take α as in Proposition 18, and let $s := \alpha_1^{-2}$. Then

$$\Gamma(p/2 + 1)^{-2/p} < s < 2.5 - p.$$

(a) We claim that

$$\begin{aligned} \{(p, s) \,|\, 1 \leq p \leq 2, \; \Gamma\,(p/2 + 1)^{-2/p} < s < 2.5 - p\} \\ \subseteq \{(p, s) \,|\, 1 \leq p \leq 1.35, \; 1.45 - p/4 < s < 2.5 - p\}. \end{aligned} \tag{5.1}$$

Let $f_1(p) := 2.5 - p$, $f_2(p) := \Gamma\,(p/2 + 1)^{-2/p}$ for $p \in [1, 2]$. We claim that $f_1 - f_2$ is strictly decreasing. Further, $(f_1 - f_2)(1) = \dfrac{3}{2} - \dfrac{4}{\pi} > 0$ and $(f_1 - f_2)(1.35) \simeq -.011 < 0$. Thus $f_1(p) = f_2(p)$ for exactly one value $p = p_0 < 1.35$ ($p_0 \approx 1.334$). One has with $\Psi := (\ln \Gamma)'$

$$f_2'(p) = f_2(p)\ (2\ln\Gamma\,(p/2+1)/p^2 - \Psi(p/2+1)/p).$$

Since $1 \le s \le 2.5 - p,\ 1 \le p \le 1.5$. Then

$$\begin{aligned} \sqrt{\pi}/2 = \Gamma\,(3/2) &\le \Gamma(p/2+1), \\ |\ln\Gamma\,(p/2+1)| &\le \ln(2/\sqrt{\pi}) < 1/8, \\ 0 \le \Psi(p/2+1) &\le \Psi(7/4) < 1/4; \end{aligned}$$

for the properties of Γ and Ψ cf. Artin [A] or Abramowitz-Stegun [AS]. Hence

$$f_2'(p) < 0,\ |f_2'(p)| \le \frac{4}{\pi}\left(2\cdot\frac{1}{8}+\frac{1}{4}\right) = \frac{2}{\pi},\ (f_1 - f_2)'(p) \le \frac{2}{\pi} - 1 < 0,$$

i.e. $f_1 - f_2$ is decreasing.
Consider $f_3(p) := (1.45 - p/4)^p$ and $f_4(p) := \Gamma\,(p/2+1)^{-2}$ with $p \in [1, 1.35]$. Since

$$\begin{aligned} f_3'(p) &= f_3(p)\,(\ln(1.45 - p/4) - p/(5.8-p)) \\ &\le f_3(p)\,((.45 - p/4) - p/(5.8-p)) < 0, \end{aligned}$$

f_3 is decreasing. Also, f_4 is decreasing so that for $p \in [1, 1.35]$

$$f_3(p) \le f_3(1) = 1.2 < 1.223 \simeq f_4(1.35) \le f_4(p).$$

Thus $1.45 - p/4 \le \Gamma\,(p/2+1)^{-2/p}$, which implies (5.1).

(b) From now on, we may assume that

$$1 \le p \le 1.35,\ 1.45 - p/4 < s = |\alpha_1|^{-2} < 2.5 - p. \tag{5.2}$$

Let $y_1 = 0,\ y_j = \alpha_j/\alpha_1$ for $j = 2, \ldots, m$. Then $\|y\|_2^2 = \alpha_1^{-2} - 1 = s - 1 \le 1/2$. Let $d_p = \left(\sin\frac{\pi p}{2}\right) 2^{p+1}/\pi\ \Gamma(p/2+1)^2$. By Proposition 10, with $u = \alpha_1 t$,

$$\begin{aligned}
\mathbb{E}\,\|\sum_{j=1}^{k}\alpha_j\,X_j\|^p &= d_p\int_0^\infty \frac{1-J_0(\alpha_1 t)\,\prod_{j=2}^{k}J_0(\alpha_j t)}{t^{p+1}}\,dt \\
&= |\alpha_1|^p\,d_p\int_0^\infty \frac{1-J_0(u)\,\prod_{j=2}^{k}J_0(\alpha_j u)}{u^{p+1}}\,du \\
&= s^{-p/2}\left(d_p\int_0^\infty \frac{1-J_0(u)}{u^{p+1}}\,du+\int_0^\infty \frac{J_0(u)}{u^{p+1}}\left(1-\prod_{j=2}^{k}J_0(y_j u)\right)du\right).
\end{aligned}$$

Since $\mathbb{E}\,\|X_1\|^p = 1$, $d_p\int_0^\infty \frac{1-J_0(u)}{u^{p+1}}\,du = 1$. The point of the argument is that the second term yields a positive contribution which is sufficiently large. The factor $\left(1-\prod_{j=2}^{k}J_0(y_j u)\right)$ is positive. By Proposition 12

$$0\le J_0(v)\le \exp\left(-v^2/4\right),\ 0\le v\le j(0)\simeq 2.4048.$$

Let $a := 2^{5/4}$, $a<j(0)$ and $I_- := \{v\in\mathbb{R}_+ \mid J_0(v)<0\}$. Using (3.18) (after the proof of Lemma 14) and $1-e^{-z}\ge z-z^2/2$, we can conclude that

$$\begin{aligned}
\mathbb{E}\,\|\sum_{j=1}^{k}\alpha_j\,X_j\|^p &\ge s^{-p/2}\left(1+d_p\int_0^a \frac{J_0(u)}{u^{p+1}}\left(1-\exp\left(-\frac{\|y\|_2^2}{4}\,u^2\right)\right)-d_p\int_{I_-}\frac{|J_0(u)|}{u^{p+1}}\,du\right) \\
&\ge s^{-p/2}\left(1+d_p\int_0^a \frac{J_0(u)}{u^{p+1}}\left(\frac{s-1}{4}\,u^2-\left(\frac{s-1}{4}\,u^2\right)^2/2\right)du-d_p\,\frac{7}{100}\,\frac{1}{2.4^{p-1}}\right) \\
&= s^{-p/2}\left(1+d_p\,\frac{s-1}{4}\int_0^a J_0(u)\left\{u^{1-p}-\frac{s-1}{8}\,u^{3-p}\right\}du-d_p\,\frac{7}{100}\,\frac{1}{2.4^{p-1}}\right).
\end{aligned}$$

The bracket $\{\ldots\}$ is positive, and by the alternating Taylor series expansion of J_0,

$$J_0(u)\ge\sum_{j=0}^{3}(-1)^j\,\frac{u^{2j}}{2^{2j}(j!)^2},\ 0\le u\le a.$$

Using this, integration yields

$$\mathbb{E}\,\|\sum_{j=1}^{k} \alpha_j\, X_j\|^p \quad \geq \quad s^{-p/2}\left(1+\delta_p\left[\frac{s-1}{4}\tilde{\varphi}\,(p,s)-\frac{7}{100}\right]\right) \tag{5.3}$$

where

$$\delta_p := d_p/a^p = \frac{4}{\pi}\sin\left(\frac{\pi p}{2}\right)\;\Gamma\,(p/2+1)^2/2^{(p-1)/4},$$

$$\tilde{\varphi}\,(p,s) := \frac{a}{2-p}-\frac{s+1}{8}\;\frac{a^3}{4-p}+\frac{2s-1}{64}\;\frac{a^5}{6-p}-\frac{9s-7}{4608}\;\frac{a^7}{8-p}.$$

Let $\varphi(p,s) := \frac{1}{4}(2-p)\,\tilde{\varphi}\,(p,s)$. Then

$$\frac{\partial^2\varphi}{\partial p^2}\,(p,s) = g_0(p)+sg_1(p),\;\; g_j(p) = \sum_{k=2}^{4} h_{jk}/(2k-p)^3,\;\; j=0,1,$$

$$h_{02} = 2^{3/4},\;\; h_{03} = 2^{5/4},\;\; h_{04} = -7/(3\cdot 2^{1/4}),\;\; h_{12} = 2^{3/4},\;\; h_{13} = -2^{9/4},\;\; h_{14} = 3/2^{1/4}.$$

It is easily checked that $g_0(p) > 0$, $g_1(p) > 0$ and thus $\frac{\partial^2\varphi}{\partial p^2} > 0$. By Taylor series expansion,

$$\begin{aligned}\varphi(p,s) \;&\geq\; \varphi(1,s)+\frac{\partial\varphi}{\partial p}\,(1,s)(p-1)\\ &\geq\; h_0(s)+h_1(s)(p-1)\end{aligned} \tag{5.4}$$

where

$$h_0(s) = .41835 - .05126\,s,\;\; h_1(s) = .12098 + .02403\,s.$$

Letting

$$\lambda(p,s) := (s-1)(h_0(s)+h_1(s)(p-1)) - .07\,(2-p),$$

inequalities (5.3) and (5.4) yield

$$\mathbb{E}\,\|\sum_{j=1}^{k} \alpha_j\, X_j\|^p \geq s^{-p/2}(1+\delta_p\,\lambda(p,s)/(2-p)) \tag{5.5}$$

Calculation shows $\dfrac{\partial \lambda}{\partial s}(p,s) \geq 0$ for (p,s) with (5.2). Thus λ is increasing in s and $s \geq 1.45 - p/4$ implies

$$\lambda(p,s) \geq \lambda(p, 1.45 - p/4) \geq \sum_{j=0}^{3} k_j (p-1)^j,$$

where $k_0 = .00136$, $k_1 = .01331$, $k_2 = -.04187$, $k_3 = .00150$. This third order polynomial is positive if $p \leq 1.4$. Hence $\lambda(p,s) \geq 0$.

To estimate δ_p from below, we show that δ_p is concave as a function of p in [1,1.35]; then with $p_0 = 1.35$

$$\begin{aligned} \delta_p \geq \beta_p \quad &:= \quad \delta_1 + ((\delta_{p_0} - \delta_1)/(p_0 - 1))(p-1) \\ &= \quad 1.4708 - .4708\, p. \end{aligned} \tag{5.6}$$

The concavity of δ_p follows from

$$\begin{aligned} \delta_p'' = -\frac{1}{4}\,\delta_p \Big([\pi^2 - (\ln 2/2 - 2\Psi(p/2+1))^2 - 2\Psi'(p/2+1)] \\ + \operatorname{ctg}\left(\frac{\pi p}{2}\right) [4\Psi(p/2+1) - \ln 2] \Big) \end{aligned}$$

where, again, $\Psi = (\ln \Gamma)'$. By Abramowitz-Stegun [AS], Ψ is increasing and Ψ' decreasing (in the range considered), leading to

$$.04 \leq \Psi\,(p/2+1) \leq .2, \;\; .8 \leq \Psi'(p/2+1) \leq .94.$$

This and the formula for δ_p'' implies $\delta_p'' < 0$. Hence (5.6) holds and

$$\mathbb{E}\, \| \sum_{j=1}^{k} \alpha_j\, X_j \|^p \geq s^{-p/2} (1 + \beta_p\, \lambda(p,s)/(2-p)). \tag{5.7}$$

Since λ is a polynomial of degree 2 in s,

$$\lambda(p,s) = \gamma_0(p) + \gamma_1(p)(s-1) + \gamma_2(p)(s-1)^2,$$

with

$$\gamma_0(p) = -\frac{7}{100}(2-p),\ \gamma_1(p) = .22207 + .14502\,p,\ \gamma_2(p) = -.07530 + .02404\,p.$$

Thus, letting

$$A(p) := 1 - \frac{7}{100}\,\beta_p,\ B(p) := \frac{\beta_p \gamma_1(p)}{2-p},\ C(p) := \frac{\beta_p \gamma_2(p)}{2-p},$$

(5.7) yields

$$\mathbb{E}\,\|\sum_{j=1}^{k} \alpha_j\, X_j\|^p \geq s^{-p/2}(A(p) + B(p)(s-1) + C(p)(s-1)^2). \tag{5.8}$$

We claim that the right side of (5.8) is decreasing in $s \in [1, 1.5]$: $s^{-p/2}(s-1)^2$ is increasing and $C(p) < 0$. The other term,

$$F(s) := s^{-p/2}(A(p) + B(p)(s-1))$$

is also decreasing:

$$F'(s) = \frac{s\,B(p)(2-p) + (B(p) - A(p))p}{2s^{p/2-1}} \leq 0$$

is equivalent to $s \leq D(p) := \dfrac{p}{2-p}\ \dfrac{A(p) - B(p)}{B(p)}$. But the latter is satisfied since $\min\{D(p)\,|\,p \in [1, 1.35]\} > 1.5$: This means that

$$E(p) := (A(p) - B(p))\,p - 1.5\,B(p)(2-p) \geq 0.$$

But $E(p) = .01227 + .28599(p-1) - .77323(p-1)^2 - .06709(p-1)^3$ is concave with $E(1.35) > E(1) > 0$, hence $E > 0$ in [1,1.35]. Thus F and the right side of (5.8) is decreasing in $s \in [1, 1.5]$, thus larger than the value at $s = 1.5$

$$\mathbb{E}\,\|\sum_{j=1}^{k} \alpha_j\, X_j\|^p \geq H(p)/1.5^{p/2},\ H(p) := A(p) + B(p)/2 + C(p)/4. \tag{5.9}$$

To show that this is bigger than $a_p(2)^p = \Gamma\,(p/2+1)$, we estimate

$$r(p) := 1.5^{p/2}\, \Gamma\,(p/2+1) \le R(p) := 1.08541 + .29534(p-1). \tag{5.10}$$

Inequality (5.10) holds since r is convex,

$$r''(p) = \frac{1}{4} r(p)\{[(\ln(3/2) + \Psi(p/2+1)]^2 + \Psi'(p/2+1)\} \ge 0,$$

with $\Psi = (\ln\Gamma)'$, $\Psi' > 0$, thus with $p_0 = 1.35$

$$r(p) \le r(1) + ((r(p_0) - r(1))/(p_0-1))(p-1) = R(p).$$

By (5.9) and (5.10)

$$\mathbb{E}\,\|\sum_{j=1}^{k} \alpha_j\, X_j\|^p - \Gamma(p/2+1) \ge 1.5^{-p/2}\,\{H(p) - R(p)\}.$$

Inserting the known values for β_p, $\gamma_1(p)$, $\gamma_2(p)$, thus $A(p)$, $B(p)$, $C(p)$, $H(p)$ and $R(p)$, one finds

$$H(p) - R(p) \ge k(p)/(2-p),\ k(p) = .01532 - .10883(p-1) + .22541(p-1)^2.$$

In [1,1.35], k has a minimum of 0.0022 at $\overline{p} \simeq 1.236$ and hence is > 0. This proves that

$$\mathbb{E}\,\|\sum_{j=1}^{k} \alpha_j\, X_j\|^p > \Gamma(p/2+1) = a_p(2)^p.$$

This proves Proposition 18. □

This ends the proof of Theorem 3 for $n=2$.

6 Some consequences

The following modification of the Khintchine inequality is a corollary to Theorem 3 and Lemma 8.

Proposition 19 *Let $X_j : (\Omega, P) \to S^{n-1} \subseteq \mathbb{R}^n$ be i.i.d. uniformly distributed vectors on S^{n-1}. Let $n \geq 2,\ 1 \leq p < \infty,\ k \in \mathbb{N}$ and $y_j \in \mathbb{R}^n$ for $j = 1, \ldots, k$. Then*

$$\alpha_p(n) \left(\sum_{j=1}^{k} \|y_j\|_{l_2^n}^2 \right)^{1/2} \leq \left(\mathbb{E} \Big| \sum_{j=1}^{k} < y_j, X_j >_{\mathbb{R}^n} \Big|^p \right)^{1/p} \leq \beta_p(n) \left(\sum_{j=1}^{k} \|y_j\|_{l_2^n}^2 \right)^{1/2},$$

where

$$\alpha_p(n) \ := \ \left\{ \begin{array}{cc} \sqrt{2/n}\ \left(\Gamma\left(\frac{p+1}{2}\right)/\sqrt{\pi}\right)^{1/p} & 1 \leq p \leq 2 \\ \left(\Gamma(n/2)/\Gamma\left(\frac{p+n}{2}\right)\right)^{1/p} \left(\Gamma\left(\frac{p+1}{2}\right)/\sqrt{\pi}\right)^{1/p} & 2 \leq p < \infty \end{array} \right\}$$

$$\beta_p(n) \ := \ \left\{ \begin{array}{cc} \left(\Gamma(n/2)/\Gamma\left(\frac{p+n}{2}\right)\right)^{1/p} \left(\Gamma\left(\frac{p+1}{2}\right)/\sqrt{\pi}\right)^{1/p} & 1 \leq p < 2 \\ \sqrt{2/n}\ \left(\Gamma\left(\frac{p+1}{2}\right)/\sqrt{\pi}\right)^{1/p} & 2 \leq p < \infty \end{array} \right\}$$

are the best possible constants.

Proof. Since the variables X_j are uniformly distributed on S^{n-1} and independent, we may rotate all y_j's into $\|y_j\|_{l_2^n} \cdot e_1$. Using Lemma 8 as well, we find

$$\begin{aligned} \mathbb{E} \Big| \sum_{j=1}^{k} < y_j, X_j > \Big|^p &= \mathbb{E} \Big| \sum_{j=1}^{k} < \|y_j\|_{l_2^n} \cdot e_1,\ X_j > \Big|^p \\ &= \mathbb{E} \Big| \sum_{j=1}^{k} \|y_j\|_{l_2^n}\ X_{j_1} \Big|^p \\ &= e_p(n)\ \mathbb{E} \Big\| \sum_{j=1}^{m} \|y_j\|_{l_2^n}\ X_j \Big\|_{l_2^n}^p . \end{aligned}$$

Theorem 3 then yields Proposition 19 with

$$\alpha_p(n) = a_p(n)\, e_p(n), \quad \beta_p(n) = b_p(n)\, e_p(n).$$

□

Remarks.

(1) Theorem 5 yields a similar analogue for functions with values on balls B_n.

(2) The variables $X_j : (\Omega, P) \to S^{n-1}$ have been used in [KST] to define isometric imbeddings $i : l_1^m(l_2^n) \to L_\infty(\Omega, P)$ by

$$\sum_{j=1}^m y_j\, e_j \mapsto \sum_{j=1}^m < y_j, X_j >;\ y_j \in l_2^n,\ e_j = (1, 0, \ldots, 0) \in l_1^m.$$

"Projecting" back onto $l_1^m(l_2^n)$ by $P : L_\infty(\Omega, P) \to l_1^m(l_2^n)$,

$$f \mapsto n \left(\int_\Omega f(w) X_j(w) dP(w) \right)_{j=1}^m,$$

one can calculate the projection constant λ and the 1-summing norm of $l_1^m(l_2^n)$ as

$$\lambda\,(l_1^m(l_2^n)) = n\,\mathbb{E}\Big|\sum_{j=1}^m X_{j1}\Big|,\ \pi_1\,(l_1^m(l_2^n)) = m/\mathbb{E}\Big|\sum_{j=1}^m X_{j1}\Big|.$$

With the formulas in Lemma 8 and Proposition 10, this is expressed in term of Bessel functions. One finds that $m \to \infty$

$$\lambda\,(l_1^m(l_2^n)) \simeq \sqrt{\frac{2}{\pi}}\,\sqrt{nm},\ \pi_1\,(l_1^m(l_2^n)) \simeq \sqrt{\frac{\pi}{2}}\,\sqrt{nm}.$$

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