1 Polya Schur Notes

1.1 Multiplier Sequences

Definition 1. *$T = \{\gamma_k\}_{k=0}^{\infty} \subseteq \mathbb{R}$ is a multiplier sequence if, when arbitrary polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$ ahs only real zeroes, $T[p(z)] = \sum_{k=0}^{n} \gamma_k a_k z^k$ has only real zeroes. This extends to arbitrary $f \in \mathcal{LP}$ via $T[f(z)] = \sum_{k=0}^{\infty} \gamma_k a_k z^k$*

Power series analog of universal factors. Whereas universal factors correspond to integral representations(as we will see below).

Classification:

Theorem 2. *T is a multiplier sequence $\iff$*

1. *The function*

$$\phi(z) = T[e^z] = \sum \gamma_k \frac{z^k}{k!} \in \mathcal{LP}^+$$

2. *Jensen Polynomials*

$$T[(1+z)^n] = \sum_{k=0}^{n} \binom{n}{k} \gamma_k z^k \in \mathcal{LP}^+$$

Also we have the connection via Laguerre

Theorem 3. *If $\phi(z) \in \mathcal{LP}$ and ϕ has no zeroes in $(0, n)$ then ϕ evaluated at the integers $[n]$ acts as a multiplier sequence on polynomials of degree up to n.*

1.2 Universal Factors

Definition 4. *Universal Factors*

Let $K(t) \in O(e^{-|t|^b})$, $b > 2$. Then $\phi(t)$ is a universal factor of class r if for K with $\int_{-r}^{r} K(t) e^{izt} dt \in \mathcal{LP}$ then

$$\int_{-r}^{r} \phi(t) K(t) e^{izt} dt \in \mathcal{LP}$$

is entire with only real zeroes

Theorem 5. *Let $\phi(z) = \sum_{k=0}^{\infty} \gamma_k z^k$ with $\phi, f \in \mathcal{LP}$. Then the differential operator $\phi(D)$ when acting on f has real zeroes.*

So differentiation via functions in the $\mathcal{LP}$ of functions in $\mathcal{LP}$ is closed.

Prop 6. *Let $0 < r \leq \infty$. If $\phi(iz) \in \mathcal{LP}$ then $\phi(t)$ is a universal factor of class r.*

Does this go the other way? Probably not. Is there a restriction allowing us to go other way?

Theorem 7. *If $K(t) = e^{-\alpha_1 t^2} K_0(t)$, $\alpha_1 > 0$ and $K_0(t)$ is s.t.*

$$K_0(z) = c e^{\alpha_0 z^2 + bz} z^m \prod_{k=1}^{\infty} (1 + z/z_k) e^{-z/z_k}$$

for $b, c, iz_k \in \mathbb{R}$ with $\alpha_1 > \alpha_0$ then $f(z) = \int_{-\infty}^{\infty} K(t) e^{izt} dt \in \mathcal{LP}$

We think of K_0 as ϕ_X for some $x \in \mathcal{L}$. Then if we regard this as a density for some rv Y, $y \in \mathcal{L}$. What is this operation called?

Theorem 8. *If real analytic $\phi(t)$ is universal factor, then $\phi(iz) \in \mathcal{LP}$.*

So universal factors, ie. functions that preserve real zeroes via pointwise product with the kernel, yield type $\mathcal{L}$ characteristics(assuming positivity). Analytically universal factors are exactly those functions in $\mathcal{LP}$.

Definition 9. *Mellin Transform*

If $K(t) : [0, \infty) \to \mathbb{R}$ integrable on $[0, \infty)$ then

$$H(z) = \int_0^\infty K(t) t^{z-1} dt$$

is the Mellin transform

Connects log of a random varible with with the random variable(scaling the density by an exponential). Sometimes easier to show property of Mellin transform instead of fourier transform to show some $f \in \mathcal{LP}$.

The proofs of the above characterizations heavily lie on differential operators. This is how we can view the pointwise product.

1.3 Polya-Schur Theory

Reading of [2].

Key questions in polya-schur theory:

Let $U \subseteq \mathbb{C}$ with $Z(U)$ the set of all complex polynomials whose zeroes lie in U. Set $Z_n(U)$ to be the subset of such polynomials of degree at most n.

- What is the set of all linear transformations $T : Z(U) \to Z(U) \cup \{0\}$
- What is the set of all linear transformations $T : Z_n(U) \to Z(U) \cup \{0\}$

So what linear operators preserve the zeroes of such polynomials? [2] solves this problem for U a line, circle, closed half plane, closed disk, complement of open disk.

Note in particular this addresses problems of what linear operators preserve the zeroes of functions $\mathcal{LP}$, via approximation by Jensen polynomials: Apply operator to Jensen polynomials and converge zeroes via hurwitz.

Theorem 10. *(Polya-Schur Theorem)*

Let $\lambda : \mathbb{N} \to \mathbb{R}$ be a sequence of real numbers and $T : \mathbb{R}[z] \to \mathbb{R}[z]$ be the corresponding diagonal linear operator given by $T(z^n) = \lambda(n) z^n$. Define $\Phi(z)$ as

$$\Phi(z) = \sum_{k=0}^{\infty} \frac{\lambda(k)}{k!} z^k$$

Then the following are equivalent:

i) λ is a multiplier sequence

ii) Φ defines an entire function which is the limit, uniformly on compact sets, of polynomials with only real zeros of the same sign

iii) Either $\Phi(z)$ or $\Phi(-z)$ is entire of the form $Cz^n e^{az} \prod_{k=1}^{\infty}(1 + \alpha_k z)$ with $a, \alpha_k \geq 0$ and $\sum \alpha_k < \infty$

iv) For all non-negative integers n th polynomials $T[(z+1)^n]$ is hyperolic with all zeroes same sign

This is the classical (partial) characterization of Polya-Schur. Note the multiplier sequences correspond to diagonal matrices on the monomial basis. The problem was solved more generally in the following form:

Set $\mathcal{H}_1(\mathbb{C})$ to be the set of stable polynomials in n varibles.

Theorem 11. *Let $n \in \mathbb{N}$ and $T : \mathbb{R}_n[z] \to \mathbb{R}[z]$ be linear operator. Then T preserves hyperbolicity $\iff$ either*

1) T has range of dimension at most two and is of the form $T(f) = \alpha(f)P + \beta(f)Q$ for $\alpha, \beta \in \mathbb{R}_n[z] \to \mathbb{R}$ are linear functionals and $P, Q \in \mathcal{H}_1(\mathbb{R})$ have interlacing zeroes

2) $T[(z+w)^n] \in \mathcal{H}_2(\mathbb{R})$

3) $T[(z-w)^n] \in \mathcal{H}_2(\mathbb{R})$

References

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