A Study of Khintchine Type Inequalities for Random Variables

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1 Known Examples of Type L Random Variables

Relatively few examples of Type L random variables are known. The majority we do have follow from results of Polya in his study of kernels producing strictly real zeroes of fourier transforms of the form:

$$\phi(z) = \int_{\mathbb{R}} K(x)cos(zx)dx$$

for some kernel $K : \mathbb{R} \to \mathbb{R}$. This can naturally be interpreted as the inverse fourier transform of a random variable X with density K. And (assuming symmetry and nice gaussanity conditions) if ϕ has strictly real zeroes then we know $\phi(iz) = \mathbb{E}e^{-zX}$ has strictly imaginary zeroes and hence X is type L.

1.1 Polya's Examples

All of these examples can be found in Polya's *Problems in Analysis* but the experience of retrieving them(and their proofs) is somewhat time consuming. Hopefully this presentation is somewhat less so. We now list the examples:

Theorem 1 (Decreasing Concave Density(173)). *Let X be a symmetric continuous random variable distributed on $[0,1]$ density f s.t. $f', f'' < 0$. Then $X \in \mathcal{L}$.*

Theorem 2 (L1 Bounded Derivative(175)). *Let X be a symmetric continuous random variable distributed on $[0,1]$ with density f s.t. $|f(1)| \geq \int_0^1 |f'(t)|dt$. Then $X \in \mathcal{L}$. Note in particular this works for the case f is increasing.*

Theorem 3 (Exponential Density(170)). *Let α be even integer greater than 2. Then if X a symmetric continuous random variable with density of the form e^{-t^α} then $X \in \mathcal{L}$*

Theorem 4 (Exponential Product Density(161)). *Let $1 > \alpha \geq 0, 0 < \alpha_1 \leq \alpha_2 \leq ...$ and reciprocal convergent. Then if $g(z) = e^{-\alpha z}(1 - \frac{z}{\alpha_1})(1 - \frac{z}{\alpha_2})...$ we have for symmetric X with density $e^{-t^2}g(-t^2)$ then $X \in \mathcal{L}$.*

Theorem 5 (Bessel Function(159)). *The symmetric continuous random variable X with density $\frac{2}{\pi\sqrt{1-t^2}}$ in $\mathcal{L}$.*

Theorem 6 (Large nth Coefficient(27)). *Suppose X a discrete integer valued symmetric distribution. If $p_0 + 2p_1 + ... + 2p_{n-1} < 2p_n$ then $X \in \mathcal{L}$.*

1.2 Newman's Examples

Newman, who initiated our study in Type L random variables, produced some examples as well.

Theorem 7 (Arithmetic Sequences)**.** *Let the sequence X above be an arbitrary arithmetic progression, ie. of the form $x_1 = d$, $x_2 = d + c$,..., $x_L = d + (L-1)c$ for arbitrary $d \in \mathbb{R}, c > 0$. Then $S_X(z)$ has zeroes only on the imaginary axis.*

Theorem 8 (Uniform(Newman 7))**.** *Let X be random variable with density $\frac{d\mu}{dy} = 1$ if $|y| \leq A$ and 0 otherwise. $A > 0$. Then $X \in \mathcal{L}$.*

Theorem 9 (Newman (8))**.** *Density $(1 - y^2)^{(d-2)/2}$ with $|y| \leq 1$ and 0 otherwise. For $d > 0$.*

Theorem 10 (Newman (9))**.** *Density $e^{-\lambda cosh(y)}$, $\lambda > 0$*

Theorem 11 (Newman (10))**.** *$e^{-ay^4 - by^2}$ with $a > 0$*

1.3 Other Examples

Theorem 12 (Enestrom-Kakeya)**.** *If X integer valued symmetric with $0 \leq p_0 \leq 2p_1 \leq ... \leq 2p_n$ with $p_n > 0$ then $X \in \mathcal{L}$.*

Theorem 13 (Absolute Value)**.** *Let $a_0, a_1, ..., a_n \in \mathbb{R}$ with $|a_0| + ..|a_{n-1}| \leq |a_n|$ then the trig polys $p_c(z) = \sum_{k=0}^{n} a_k cos(kz)$ and the sin one have only real zeroes*

Theorem 14 (Shifted Symmetry)**.** *Let $X \in \mathcal{L}$. Then $\exists \lambda \in \mathbb{R}$ s.t. $X - \lambda$ is symmetric.*

Theorem 15 (Renyi)**.** *Renyi paper*