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We use $\vdash$ to mean equality up to a shift(multiplication by z^α).

1.1 Quadratic Forms

We consider whether $\sum_{i,j} a_{ij}\epsilon_i\epsilon_j = \langle A\epsilon, \epsilon\rangle$ is type L for vector of independently distributed random signs ϵ and matrix A.

Compute $\sum_{i,j} a_{ij}\epsilon_i\epsilon_j = \sum_{1\le i,j\le n} a_{ij}\epsilon_i\epsilon_j + \sum_{i=1}^n \epsilon_{n+1}\epsilon_i(a_{ij}+a_{ji}) = \sum_{1\le i,j\le n} a_{ij}\epsilon_i\epsilon_j + \sum_{i=1}^n \epsilon_{n+1}\epsilon_i(b_i)$.

Then $\mathbb{E}z^{\langle A\epsilon,\epsilon\rangle} = \mathbb{E}z^{\sum_{1\le i,j\le n} a_{ij}\epsilon_i\epsilon_j + \sum_{i=1}^n \epsilon_{n+1}\epsilon_i(b_i)} =$
$\frac{1}{2}\mathbb{E}_{\epsilon_{[n]}} z^{\sum_{1\le i,j\le n} a_{ij}\epsilon_i\epsilon_j + \sum_{i=1}^n \epsilon_i(b_i)} + \frac{1}{2}\mathbb{E}_{\epsilon_{[n]}} z^{\sum_{1\le i,j\le n} a_{ij}\epsilon_i\epsilon_j - \sum_{i=1}^n \epsilon_i(b_i)} =$
$\mathbb{E}z^{\sum_{1\le i,j\le n} a_{ij}\epsilon_i\epsilon_j + \sum_{i=1}^n \epsilon_i(b_i)}$ via symmetry. Since we are only interested in the zeroes we may factor out constant terms $a_{i,i}\epsilon_i$ to examine $\mathbb{E}z^{\sum_{1\le i<j\le n} c_{ij}\epsilon_i\epsilon_j + \sum_{i=1}^n \epsilon_i(b_i)}$ where $c_{ij} = a_{ij} + a_{ji}$. Wlog we may suppose all coefficients > 0.

Consider the case $n = 2$. We have $\mathbb{E}z^{a\epsilon_1\epsilon_2 + b\epsilon_1 + c\epsilon_2} \propto z^{a+b+c} + z^{-a+b-c} + z^{-a-b+c} + z^{a-b-c}$. After a shift this becomes $z^{2a+2b+2c} + z^{2a} + z^{2b} + z^{2c}$. In the integer case, the polynomial must be palindromic. If $a = b = c$ then we have $z^{6a} + 3z^{2a} \vdash z^{4a} + 3$ which is not palindromic. Suppose wlog $a = b$. Then $z^{2a+2b+2c} + z^{2a} + z^{2b} + z^{2c} = z^{4a+2c} + 2z^{2a} + z^{2c}$. Then to be palindromic it must be that $2a = \frac{1}{2}(4a+2c)$ implying $c = 0$. Note if this is the case the polynomial is unirooted. Now suppose wlog $a < b < c$. We have $z^{2a+2b+2c} + z^{2a} + z^{2b} + z^{2c} \vdash z^{2b+2c} + z^{2(c-a)} + z^{2(b-a)} + 1$. For palindromicity we need $2b+2c = 2c+2b-4a \implies a = 0$. So again we require the smallsest term to be 0. So $z^{2(b+c)} + z^{2b} + z^{2c} + 1$. Note this does have all roots on unit circle then.

In summary: All three terms cannot be equal. If two are equal the other must be zero. If none are equal the smallest must be 0. Note that if any of the terms are 0, the result is trivially type l(as it is either a sum of two type L random variables or a symmerization of one).

Further note in general the diagonal of A, $(a_{ii})_{i=1}^n$ is irrelevant due to shifting. So in particular Positive semi definiteness is not required. Since $tr(A) = \sum \lambda_i$ where the trace is arbitrary.

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2.1 Multiplier Sequences

Definition 1. $T = \{\gamma_k\}_{k=0}^\infty \subseteq \mathbb{R}$ is a multiplier sequence if, when arbitrary polynomial $p(z) = \sum_{k=0}^n a_k z^k$ ahs only real zeroes, $T[p(z)] = \sum_{k=0}^n \gamma_k a_k z^k$ has only real zeroes. This extends to arbitrary $f \in \mathcal{LP}$ via $T[f(z)] = \sum_{k=0}^\infty \gamma_k a_k z^k$

Discrte analog of Universal Factors.

Classification:

Theorem 2. *T is a multiplier sequence $\iff$*

1. *The function*

$$\phi(z) = T[e^z] = \sum \gamma_k \frac{z^k}{k!} \in \mathcal{LP}^+$$

2. *Jensen Polynomials*

$$T[(1+z)^n] = \sum_{k=0}^n \binom{n}{k}\gamma_k z^k \in \mathcal{LP}^+$$

Also we have the connection via Laguerre

Theorem 3. *If $\phi(z) \in \mathcal{LP}$ and ϕ has no zeroes in $(0, n)$ then ϕ evaluated at the integers $[n]$ acts as a multiplier sequence on polynomials of degree up to n.*

Multiplier sequences correspond to a subset of $\mathcal{LP}$ whereas universal factors correspond to the entire class(as we see below).

2.2 Universal Factors

Definition 4. *Universal Factors*

Let $K(t) \in O(e^{-|t|^b})$, $b > 2$. Then $\phi(t)$ is a universal factor of class r if for K with $\int_{-r}^{r} K(t)e^{izt}dt \in \mathcal{LP}$ then

$$\int_{-r}^{r} \phi(t)K(t)e^{izt}dt \in \mathcal{LP}$$

is entire with only real zeroes

Theorem 5. *Let $\phi(z) = \sum_{k=0}^{\infty} \gamma_k z^k$ with $\phi, f \in \mathcal{LP}$. Then the differential operator $\phi(D)$ when acting on f has real zeroes.*

So differentiation via functions in the $\mathcal{LP}$ of functions in $\mathcal{LP}$ is closed.

Prop 6. *Let $0 < r \leq \infty$. If $\phi(iz) \in \mathcal{LP}$ then $\phi(t)$ is a universal factor of class r.*

Does this go the other way? Probably not. Is there a restriction allowing us to go other way?

Theorem 7. *If $K(t) = e^{-\alpha_1 t^2} K_0(t)$, $\alpha_1 > 0$ and $K_0(t)$ is s.t.*

$$K_0(z) = ce^{\alpha_0 z^2 + bz} z^m \prod_{k=1}^{\infty} (1 + z/z_k)e^{-z/z_k}$$

for $b, c, iz_k \in \mathbb{R}$ with $\alpha_1 > \alpha_0$ then $f(z) = \int_{-\infty}^{\infty} K(t)e^{izt}dt \in \mathcal{LP}$

We think of K_0 as ϕ_X for some $x \in \mathcal{L}$. Then if we regard this as a density for some rv Y, $y \in \mathcal{L}$. What is this operation called?

Theorem 8. *If real analytic $\phi(t)$ is universal factor, then $\phi(iz) \in \mathcal{LP}$.*

So universal factors, ie. functions that preserve real zeroes via pointwise product with the kernel, yield type $\mathcal{L}$ characteristics(assuming positivity). Analytically universal factors are exactly those functions in $\mathcal{LP}$.

Definition 9. *Mellin Transform*

If $K(t) : [0, \infty) \to \mathbb{R}$ integrable on $[0, \infty)$ then

$$H(z) = \int_{0}^{\infty} K(t)t^{z-1}dt$$

is the Mellin transform

Connects log of a random varible with with the random variable(scaling the density by an exponential). Sometimes easier to show property of Mellin transform instead of fourier transform to show some $f \in \mathcal{LP}$.

The proofs of the above characterizations heavily lie on differential operators. This is how we can view the pointwise product.

2.3 Polya-Schur Theory

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