1 Statements

Let $\mathcal{L}$ denote the class of type L random variables.

1.1 Closure Conditions

Theorem 1 (Closed Under Scaling). *Let $\lambda \in \mathbb{R}, X \in \mathcal{L}$. Then $\lambda X \in \mathcal{L}$*

Theorem 2 (Closed Under Translation). *Let $\lambda \in \mathbb{R}, X \in \mathcal{L}$. Then $X + \lambda \in \mathcal{L}$*

Theorem 3 (Closed Under Sum). *Let $X, Y \in \mathcal{L}$ independent. Then $X + Y \in \mathcal{L}$.*

Theorem 4 (Bernoulli Sums). *Let $\delta_\lambda, n \in \mathbb{Z}$ be bernoulli with parameter 1/2 taking values on 0 and n. Then $\sum \delta_{n_k} \in \mathcal{L}$ for $k \in \mathbb{N}$. Notice this corresponds to the product $\prod(1 + x^{n_k})$*

Theorem 5 (Integer Symmetrization). *Let $X \in \mathcal{L}$ with $X > 0$ integer valued. Then $\epsilon X \in \mathcal{L}$ where ϵ random sign.*

Theorem 6 (Arithmetic Symmetrization). *Let $X \in \mathcal{L}$ with $X > 0$ a uniform arithmetic progression. Then $\epsilon X \in \mathcal{L}$ where ϵ random sign.*

Theorem 7 (General Symetrization). *Let $X \in \mathcal{L}$. Suppose the powerseries representing $\psi_X(z) = \mathbb{E}e^{zX}$ is entire with strictly real coefficients in power series form. Then $\epsilon X \in \mathcal{L}$.*

Theorem 8 (Weak Convergence). *See Newman 2019 paper*

1.2 Polya's Examples

Theorem 9 (Decreasing Concave Density(173)). *Let X be a symmetric continuous random variable distributed on $[0, 1]$ density f s.t. $f', f'' < 0$. Then $X \in \mathcal{L}$.*

Theorem 10 (L1 Bounded Derivative(175)). *Let X be a symmetric continuous random variable distributed on $[0, 1]$ with density f s.t. $|f(1)| \geq \int_0^1 |f'(t)|dt$. Then $X \in \mathcal{L}$. Note in particular this works for the case f is increasing.*

Theorem 11 (Exponential Density(170)). *Let α be even integer greater than 2. Then if X a symmetric continuous random variable with density of the form e^{-t^α} then $X \in \mathcal{L}$*

Theorem 12 (Exponential Product Density(161)). *Let $1 > \alpha \geq 0, 0 < \alpha_1 \leq \alpha_2 \leq ...$ and reciprocal convergent. Then if $g(z) = e^{-\alpha z}(1 - \frac{z}{\alpha_1})(1 - \frac{z}{\alpha_2})...$ we have for symmetric X with density $e^{-t^2}g(-t^2)$ then $X \in \mathcal{L}$.*

Theorem 13 (Bessel Function(159)). *The symmetric continuous random variable X with density $\frac{2}{\pi\sqrt{1-t^2}}$ in $\mathcal{L}$.*

Theorem 14 (Large nth Coefficient(27)). *Suppose X a discrete integer valued symmetric distribution. If $p_0 + 2p_1 + ... + 2p_{n-1} < 2p_n$ then $X \in \mathcal{L}$.*

1.3 Newman's Examples

Theorem 15 (Arithmetic Sequences). *Let the sequence X above be an arbitrary arithmetic progression, ie. of the form $x_1 = d$, $x_2 = d + c$,..., $x_L = d + (L - 1)c$ for arbitrary $d \in \mathbb{R}, c > 0$. Then $S_X(z)$ has zeroes only on the imaginary axis.*

Theorem 16 (Uniform(Newman 7))**.** *Let X be random variable with density $\frac{d\mu}{dy} = 1$ if $|y| \leq A$ and 0 otherwise. $A > 0$. Then $X \in \mathcal{L}$.*

Theorem 17 (Newman (8))**.** *Density $(1-y^2)^{(d-2)/2}$ with $|y| \leq 1$ and 0 otherwise. For $d > 0$.*

Theorem 18 (Newman (9))**.** *Density $e^{-\lambda cosh(y)}$, $\lambda > 0$*

Theorem 19 (Newman (10))**.** *$e^{-ay^4-by^2}$ with $a > 0$*

1.4 Other Examples

Theorem 20 (Enestrom-Kakeya)**.** *If X integer valued symmetric with $0 \leq p_0 \leq 2p_1 \leq ... \leq 2p_n$ with $p_n > 0$ then $X \in \mathcal{L}$.*

Theorem 21 (Shifted Symmetry)**.** *Let $X \in \mathcal{L}$. Then $\exists \lambda \in \mathbb{R}$ s.t. $X - \lambda$ is symmetric.*

Theorem 22 (Renyi)**.** *Renyi paper*

1.5 Nonexamples

- The geometric sequence $(q^i)_{i=1}^L$ does not have strictly imaginary zeroes
- Binomial coefficient sequences(the first half) do not have strictly imaginary zeroes

In particular not all log-concave sequences are type L.

2 Proofs

Proof of Theorem **??**. Do after Newman and Polya. □

Proof of Theorem 1. Suppose $X \in \mathcal{L}$. Then $\mathbb{E}e^{z\lambda X} = 0 \implies \lambda z \in i\mathbb{R} \implies z \in i\mathbb{R}$. □

Proof of Theorem 2. Suppose $X \in \mathcal{L}$. Then $0 = \mathbb{E}e^{z(X+c)} = e^{cz}\mathbb{E}e^{zX} \iff 0 = \mathbb{E}e^{zX}$. □

Proof of Theorem 3. See Newman's [3]. □

Proof of Theorem 4. This is a sum of type L random variables. □

Proof of Theorem 5. Let $X \in \mathcal{L}$ with $X > 0$. Then $\psi_X(iz) = \mathbb{E}e^{iz\epsilon X} = \sum_{k=1}^{n} \frac{1}{2}p_k(e^{ix_kz} + e^{-ix_kz}) = \sum_{k=1}^{n} p_k cos(x_kz)$. Theorem 2.18 from [5] tells us if $P_c(z) = p_1z^{x_1} + ... + p_nz^{x_n}$ has zeroes only on the unit circle, then $\psi_X(iz)$ as only real zeroes, ie. $\psi_X(z)$ has only imaginary zeroes. But with a change of variables P_c is exactly $\mathbb{E}e^{zX}$ which has strictly imaginary zeroes, implying P_c has zeroes strictly on the unit circle. □

Proof of Theorem 15. Write

$$S_X(z) = \sum_{n=1}^{L} e^(x_nz) = 0 \iff e^{x_1z}(\sum_{n=1}^{L} e^{(x_n-1x_1)z}) = 0 \iff \sum_{n=1}^{L} e^{(x_n-x_1)z} = 0$$

since e^{x_1z} has no zeroes. So wlog we may assume $x_1 = 0$, since the translation still results in an arithmetic sequence. Then we sum

$$\sum_{n=1}^{L} e^{x_1z} = \sum_{n=1}^{L} e^{(n-1)cz} = \sum_{n=0}^{L-1} (e^{cz})^n = \frac{e^{Lcz}-1}{e^{cz}-1}$$

So

$$S_X(z) = 0 \implies \frac{e^{Lcz} - 1}{e^{cz} - 1} = 0 \implies e^{Lcz} = 1 \implies z = ib$$

for some $b \in \mathbb{R}$. In fact we must have $Lcz = 2\pi n \implies z = \frac{2\pi n}{Lc}$ for some $n \in \mathbb{Z}$ $\square$

Proof of Theorem 7. Let $X \in \mathcal{L}$. Set $\psi_X(z) = \mathbb{E}e^{zX}$. Then because of real coefficients $\psi_X(z) = 0 \implies \psi_X(\overline{z}) = 0 \implies \psi_X(-z) = 0$ since $z \in i\mathbb{R}$. Further $\psi_X(-z) = 0 \implies \psi_X(\overline{-z}) = 0 \implies \psi_X(z) = 0$. So $h(z) = \mathbb{E}e^{z\epsilon X} = \frac{1}{2}\mathbb{E}e^{zX} + \frac{1}{2}\mathbb{E}e^{-zX} = \frac{1}{2}\psi_X(z) + \frac{1}{2}\psi(-z)$. Then applying Rouche's theorem, the zeroes of the h are completely determined by $\psi, \rho\psi$ where ρ is the reflection operator. Always either $|\psi| \leq |\rho\psi|$ or $|\rho\psi| \leq |\psi|$ and so the number of zeroes in any region completely determined by the dominator(which have identical zeroes). And since h as at least all the same zeroes as $\psi, \rho\psi$, this is uniquely determining(as there cannot be any extra in any region).

Alternatively $\mathbb{E}e^{z\epsilon X} = \mathbb{E}e^{z\epsilon(X-c)}e^{z\epsilon c} = \frac{1}{2}e^{zc}\mathbb{E}e^{z(X-c)} + \frac{1}{2}e^{-zc}\mathbb{E}e^{-z(X-c)} = \frac{1}{2}\mathbb{E}e^{z(X-c)}(e^{zc} + ^{-zc})$ where we use symmetry. Note the last term only has roots on the imaginary axis. $\square$

Proof of Theorem 16. Wlog suppose $A = 1$. Then $f' = 0$ on $[-1, 1]$ and hence by 10 type L. $\square$

Proof of Theorem 17. Follows from a generalization of Iliya. $\square$

Proof of Theorem 18. $\square$

Proof of Theorem 20. Suppose X symmetric, integer valued with probability distribution $0 \leq p_0 \leq 2p_1 \leq ... \leq 2p_n$. Then $\psi_X(iz) = p_0 + \sum 2p_k cos(kz)$. The Enestrom-Kakaya Theorem(2.17 from [5]) tells us all the zeroes of the polynomial $p_0 + 2p_1 x + ... + 2p_n x^n$ has all zeroes in the closed unit disk. So then ψ_X has all zeroes on the imaginary axis via the argument from 5. $\square$

References

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[2] Keel. L, Bhattarcharyya. S, A New Proof of the Jury Test. https://ieeexplore.ieee.org/document/703305

[3] Newman, An Extension of Khintchines Inequality.

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[5] Hallum P. Zeroes of Entire Functions Represented By Fourier Transforms

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