

1 Polya Schur Notes

1.1 Multiplier Sequences

Definition 1. $T = \{\gamma_k\}_{k=0}^\infty \subseteq \mathbb{R}$ is a multiplier sequence if, when arbitrary polynomial $p(z) = \sum_{k=0}^n a_k z^k$ has only real zeroes, $T[p(z)] = \sum_{k=0}^n \gamma_k a_k z^k$ has only real zeroes. This extends to arbitrary $f \in \mathcal{LP}$ via $T[f(z)] = \sum_{k=0}^\infty \gamma_k a_k z^k$

Power series analog of universal factors. Whereas universal factors correspond to integral representations(as we will see below).

Classification:

Theorem 2. T is a multiplier sequence \iff

1. The function

$$\phi(z) = T[e^z] = \sum \gamma_k \frac{z^k}{k!} \in \mathcal{LP}^+$$

2. Jensen Polynomials

$$T[(1+z)^n] = \sum_{k=0}^n \binom{n}{k} \gamma_k z^k \in \mathcal{LP}^+$$

Also we have the connection via Laguerre

Theorem 3. If $\phi(z) \in \mathcal{LP}$ and ϕ has no zeroes in $(0, n)$ then ϕ evaluated at the integers $[n]$ acts as a multiplier sequence on polynomials of degree up to n .

1.2 Universal Factors

Definition 4. Universal Factors

Let $K(t) \in O(e^{-|t|^b})$, $b > 2$. Then $\phi(t)$ is a universal factor of class r if for K with $\int_{-r}^r K(t) e^{izt} dt \in \mathcal{LP}$ then

$$\int_{-r}^r \phi(t) K(t) e^{izt} dt \in \mathcal{LP}$$

is entire with only real zeroes

Theorem 5. Let $\phi(z) = \sum_{k=0}^\infty \gamma_k z^k$ with $\phi, f \in \mathcal{LP}$. Then the differential operator $\phi(D)$ when acting on f has real zeroes.

So differentiation via functions in the \mathcal{LP} of functions in \mathcal{LP} is closed.

Prop 6. Let $0 < r \leq \infty$. If $\phi(iz) \in \mathcal{LP}$ then $\phi(t)$ is a universal factor of class r .

Does this go the other way? Probably not. Is there a restriction allowing us to go other way?

Theorem 7. If $K(t) = e^{-\alpha_1 t^2} K_0(t)$, $\alpha_1 > 0$ and $K_0(t)$ is s.t.

$$K_0(z) = ce^{\alpha_0 z^2 + bz} z^m \prod_{k=1}^\infty (1 + z/z_k) e^{-z/z_k}$$

for $b, c, iz_k \in \mathbb{R}$ with $\alpha_1 > \alpha_0$ then $f(z) = \int_{-\infty}^\infty K(t) e^{izt} dt \in \mathcal{LP}$

We think of K_0 as ϕ_X for some $x \in \mathcal{L}$. Then if we regard this as a density for some rv Y , $y \in \mathcal{L}$. What is this operation called?

Theorem 8. *If real analytic $\phi(t)$ is universal factor, then $\phi(iz) \in \mathcal{LP}$.*

So universal factors, ie. functions that preserve real zeroes via pointwise product with the kernel, yield type \mathcal{L} characteristics (assuming positivity). Analytically universal factors are exactly those functions in \mathcal{LP} .

Definition 9. *Mellin Transform*

If $K(t) : [0, \infty) \rightarrow \mathbb{R}$ integrable on $[0, \infty)$ then

$$H(z) = \int_0^\infty K(t)t^{z-1}dt$$

is the Mellin transform

Connects log of a random variable with the random variable (scaling the density by an exponential). Sometimes easier to show property of Mellin transform instead of Fourier transform to show some $f \in \mathcal{LP}$.

The proofs of the above characterizations heavily lie on differential operators. This is how we can view the pointwise product.

1.3 Polya-Schur Theory

Reading of [2].

Key questions in Polya-Schur theory:

Let $U \subseteq \mathbb{C}$ with $Z(U)$ the set of all complex polynomials whose zeroes lie in U . Set $Z_n(U)$ to be the subset of such polynomials of degree at most n .

- What is the set of all linear transformations $T : Z(U) \rightarrow Z(U) \cup \{0\}$
- What is the set of all linear transformations $T : Z_n(U) \rightarrow Z(U) \cup \{0\}$

So what linear operators preserve the zeroes of such polynomials? [2] solves this problem for U a line, circle, closed half plane, closed disk, complement of open disk.

Note in particular this addresses problems of what linear operators preserve the zeroes of functions \mathcal{LP} , via approximation by Jensen polynomials: Apply operator to Jensen polynomials and converge zeroes via Hurwitz.

Theorem 10. *(Polya-Schur Theorem)*

Let $\lambda : \mathbb{N} \rightarrow \mathbb{R}$ be a sequence of real numbers and $T : \mathbb{R}[z] \rightarrow \mathbb{R}[z]$ be the corresponding diagonal linear operator given by $T(z^n) = \lambda(n)z^n$. Define $\Phi(z)$ as

$$\Phi(z) = \sum_{k=0}^{\infty} \frac{\lambda(k)}{k!} z^k$$

Then the following are equivalent:

- i) λ is a multiplier sequence
- ii) Φ defines an entire function which is the limit, uniformly on compact sets, of polynomials with only real zeros of the same sign
- iii) Either $\Phi(z)$ or $\Phi(-z)$ is entire of the form $Cz^n e^{az} \prod_{k=1}^{\infty} (1 + \alpha_k z)$ with $a, \alpha_k \geq 0$ and $\sum \alpha_k < \infty$
- iv) For all non-negative integers n the polynomials $T[(z+1)^n]$ is hyperbolic with all zeroes same sign

This is the classical (partial) characterization of Polya-Schur. Note the multiplier sequences correspond to diagonal matrices on the monomial basis. The problem was solved more generally in the following form:

Set $\mathcal{H}_1(\mathbb{C})$ to be the set of stable polynomials in n variables.

Theorem 11. *Let $n \in \mathbb{N}$ and $T : \mathbb{R}_n[z] \rightarrow \mathbb{R}[z]$ be linear operator. Then T preserves hyperbolicity \iff either*

1) T has range of dimension at most two and is of the form $T(f) = \alpha(f)P + \beta(f)Q$ for $\alpha, \beta \in \mathbb{R}_n[z] \rightarrow \mathbb{R}$ are linear functionals and $P, Q \in \mathcal{H}_1(\mathbb{R})$ have interlacing zeroes

2) $T[(z + w)^n] \in \mathcal{H}_2(\mathbb{R})$

3) $T[(z - w)^n] \in \mathcal{H}_2(\mathbb{R})$

References

- [1] Hallum P. Zeroes of Entire Functions Represented by Fourier Transforms.
- [2] Borcea J, Branden P. Polya-schur Master Theorems for Circular Domains and Their Boundaries
- [3] Markovsky. I, Shodhan. R, Palindromic Polynomials, Time-Reversible Systems, and Conserved Quantities. <https://eprints.soton.ac.uk/266592/1/Med08.pdf>
- [4] Keel. L, Bhattacharyya. S, A New Proof of the Jury Test. <https://ieeexplore.ieee.org/document/703305>