A Study of Khintchine Type Inequalities for Random Variables

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1 Introduction

2 Known Examples of Type L Random Variables

Relatively few examples of Type L random variables are known. The majority we do have follow from results of Polya in his study of kernels producing strictly real zeroes of fourier transforms of the form:

$$\phi(z) = \int_{\mathbb{R}} K(x)cos(zx)dx$$

for some kernel $K : \mathbb{R} \to \mathbb{R}$. This can naturally be interpreted as the inverse fourier transform of a random variable X with density K. And (assuming symmetry and nice gaussanity conditions) if ϕ has strictly real zeroes then we know $\phi(iz) = \mathbb{E}e^{-zX}$ has strictly imaginary zeroes and hence X is type L.

2.1 Polya's Examples

All of these examples can be found in Polya's *Problems in Analysis* but the experience of retrieving them(and their proofs) is somewhat time consuming. Hopefully this presentation is somewhat less so. We attach proofs of these examples in an appendix.

Theorem 1 (Decreasing Concave Density(173)). *Let X be a symmetric continuous random variable distributed on $[0, 1]$ density f s.t. $f', f'' < 0$. Then $X \in \mathcal{L}$.*

Theorem 2 (L1 Bounded Derivative(175)). *Let X be a symmetric continuous random variable distributed on $[0, 1]$ with density f s.t. $|f(1)| \geq \int_0^1 |f'(t)|dt$. Then $X \in \mathcal{L}$. Note in particular this works for the case f is increasing.*

Theorem 3 (Exponential Density(170)). *Let α be even integer greater than 2. Then if X a symmetric continuous random variable with density of the form e^{-t^α} then $X \in \mathcal{L}$*

Theorem 4 (Exponential Product Density(161)). *Let $1 > \alpha \geq 0, 0 < \alpha_1 \leq \alpha_2 \leq ...$ and reciprocal convergent. Then if $g(z) = e^{-\alpha z}(1 - \frac{z}{\alpha_1})(1 - \frac{z}{\alpha_2})...$ we have for symmetric X with density $e^{-t^2}g(-t^2)$ then $X \in \mathcal{L}$.*

Theorem 5 (Bessel Function(159)). *The symmetric continuous random variable X with density $\frac{2}{\pi\sqrt{1-t^2}}$ in $\mathcal{L}$.*

Theorem 6 (Large nth Coefficient(27)). *Suppose X a discrete integer valued symmetric distribution. If $p_0 + 2p_1 + ... + 2p_{n-1} < 2p_n$ then $X \in \mathcal{L}$.*

2.2 Newman's Examples

Newman, who initiated our study in Type L random variables, produced some examples as well.

Theorem 7 (Arithmetic Sequences). *Let the sequence X above be an arbitrary arithmetic progression, ie. of the form $x_1 = d$, $x_2 = d + c$,..., $x_L = d + (L-1)c$ for arbitrary $d \in \mathbb{R}, c > 0$. Then $S_X(z)$ has zeroes only on the imaginary axis.*

Theorem 8 (Uniform(Newman 7)). *Let X be random variable with density $\frac{d\mu}{dy} = 1$ if $|y| \leq A$ and 0 otherwise. $A > 0$. Then $X \in \mathcal{L}$.*

Theorem 9 (Newman (8)). *Density $(1-y^2)^{(d-2)/2}$ with $|y| \leq 1$ and 0 otherwise. For $d > 0$.*

Theorem 10 (Newman (9)). *Density $e^{-\lambda cosh(y)}$, $\lambda > 0$*

Theorem 11 (Newman (10)). *$e^{-ay^4-by^2}$ with $a > 0$*

2.3 Other Examples

Theorem 12 (Enestrom-Kakeya). *If X integer valued symmetric with $0 \leq p_0 \leq 2p_1 \leq ... \leq 2p_n$ with $p_n > 0$ then $X \in \mathcal{L}$.*

Theorem 13 (Absolute Value). *Let $a_0, a_1, ..., a_n \in \mathbb{R}$ with $|a_0| + ..|a_{n-1}| \leq |a_n|$ then the trig polys $p_c(z) = \sum_{k=0}^{n} a_k cos(kz)$ and the sin one have only real zeroes*

2.4 Our Examples

Theorem 14 (Rapidly Decreasing Polynomial). *Suppose X has density $e^{-x^2/2}x^2$. Then $X \in \mathcal{L}$.*

Theorem 15 (General Symmetrization). *Suppose $X - c$ is type L for some $c \in \mathcal{L}$. Then $\epsilon X \in \mathcal{L}$.*

3 Type L Example Proofs

3.1 Problem ???

Theorem 16 (V ???). *If α is an even integer greater than two, $f(z) = \int_0^\infty e^{-t^\alpha} cos(zt)dt$*

3.2 Problem 173 and Relevant Theorems

Theorem 17 (V 173). *Let X be a symmetric continuous random variable distributed on $[0, 1]$ density f s.t. $f', f'' < 0$. Then $X \in \mathcal{L}$.*

Theorem 18 (V 26). *Let $A_1, ..., A_n$ be non-zero real numbers and $a_1 < ... < a_n$. Then if $A_1 > 0, ..., A_{n-1} > 0$ or $A_1 > 0, ..., A_{k-1} > 0, A_{k+1} > 0, ...A_n > 0$ with $\sum A_k < 0$ then $f(x) = \frac{A_1}{x-a_1} + ... + \frac{A_n}{x-a_n}$ has only real zeroes*

Theorem 19 (III 165). *Suppose entire $F(z)$ satisfies $|F(x+iy)| < Ce^{\rho|y|}$.*

Then $\frac{d}{dz}(\frac{F(z)}{sin(\rho z)}) = -\sum_{\mathbb{Z}} \frac{\rho(-1)^n F(\frac{n\pi}{\rho})}{(\rho z - n\pi)^2}$

Theorem 20 (III 170(Precursor to 201)). *Suppose $f_1, ..., f_n, ..$ are regular in open $U \subseteq \mathbb{R}$, and convering uniformly in any closed domain inside $\mathbb{R}$. Then limit f is regular*

Theorem 21 (III 194(Rouche)). *Suppose f, ϕ regular in interior of $\mathcal{D}$, cts on closed domain,, and $|f(z)| > |\phi(z)| \forall z \in \partial\mathcal{D}$. Then $f(z) + \phi(z)$ has exactly the same number of zeores as f inside $\mathcal{D}$.*

Theorem 22 (III 201(Hurwitz Theorem main tool for controlling limit zeros)). *Suppose $f_n \to f$ pointwise with $\mathcal{Z}$ the set of all zeroes of f_n in $\mathbb{R}$. Then the zeroes of f in $\mathbb{R}$ are the limit points of $\mathcal{Z}$ in $\mathbb{R}$.*

3.3 Problem 175 and Relevant Problems

Theorem 23 (V 175). *Let $f(t)$ be real and continuously differentiable for $0 \leq t \leq 1$. If we have $|f(1)| \geq \int_0^1 |f'(t)|dt$ then the entire function $F(z) = \int_0^1 f(t)cos(zt)dt$ has only real zeroes*

Theorem 24 (V 174). *Let $\phi(t)$ be properly integrable for $0 \leq t \leq 1$. If $\int_0^1 |\phi(t)|dt \leq 1$ then entire $F(z) = sin(z)\int_0^1 \phi(t)sin(zt)dt$ has only real zeroes*

Theorem 25 (V 27). *The trignometric polynomial $f(x) = a_0 + a_1 cos(x) + ... + a_n cos(nx)$ with real coefficients has only real zeroes if $|a_0| + |a_1| + ... + |a_{n-1}| < a_n$(note this also applies to sin via differentiation and rolle's thm)*

Theorem 26 (VI 14). *A trig poly with real coefficients $g(z) = \lambda_0 + \lambda_1 cos(z) + \mu_1 sin(z) + ... + \lambda_n cos(nz) + \mu_n sin(nz)$ has exactly 2n zeroes(where shifting by 2π is not distinct)*

4 Proofs

4.1 173 Proofs

Proof of Problem 173. Via integration by parts twice we write $z^2F(z) = zf(1)sin(z) - f'(0)(1 - cos(z)) + \int_0^1 f''(t)(cos(z) - cos(zt))dt$. Compute $((2m-1)\pi)^2 F((2m-1)\pi) = -2f'(0) + \int_0^1 f''(t)(-1 - cos((2m-1)\pi t) > 0$. Then compute $(2m\pi)^2 F(2m\pi) = \int_0^1 f''(t)(1 - cos(2m\pi t)) < 0$ since $f'' < 0$. Which gives infinitely many zeroes. Note $F(0) > 0$. The rational function

$f_n(z) = (-1)^n \frac{F(-n\pi)}{z+n\pi} + ... + \frac{F(-2\pi)}{z+2\pi} - \frac{F(-\pi)}{z+\pi} + \frac{F(0)}{z} - \frac{F(\pi)}{z-\pi} + ... + (-1)^n \frac{F(n\pi)}{z-n\pi}$ can have via **26** either all real zeroes of 2n-2 real zeroes and 2 imaginary. Further it converges to $\frac{F(z)}{sin(z)}$ by integrating the result of **165**. So as we take the limit, the nonreal zeroes $\frac{F(z)}{sin(z)}$ as via **201** they are the limit points of approaching f_n zeroes. In the case we have two nonreal zeroes, it must be the case they are strictly imaginary, as $F(z) = F(-z) = F(\overline{z}) = 0$. But $F(ix) = \int_0^1 f(t)\frac{e^{xt}+e^{-xt}}{2}dt > 0$ ☐

Proof of Problem 26. Idea is to count zeroes intervals (a_i, a_{i+1}) via changes of sign. Write $f(x) = P(x)/Q(x)$ for $Q = (x - a_1)...(x - a_n)$ and P a sum of n-1 deg polys. Via ϵ approximations $f(a_1 + \epsilon) > 0$ and $f(a_2 - \epsilon) < 0$ which continues alternating. Note that poles blow up as we get very close, dominating sign. Then regarding polynomial in numerator this is a full accounting.

(Can we still use IVT despite singularities? We look at intervals in between which do have continuity.) ☐

Proof of Problem 165. Somewhere in German Hurwitz-Courant ☐

Proof of Problem 170(Precursor to 201). Use Cauchy integral theorem on closed cts curve $L \subseteq \mathbb{R}$. Write $f_n(z) = \frac{1}{2\pi i}\int_L \frac{f_n(\xi)}{\xi - z}d\xi$ via cauchy integral formula. We know uniformly on L $f_n(\xi)/(\xi - z) \to f(\xi)/(\xi - z)$ and further f cts(as uniform limit of continuous functions). So $f_n(z) \to \frac{1}{2\pi i}\int_L \frac{f(\xi)}{\xi - z}d\xi$(why?). And the last function is regular(why?). ☐

Proof of problem 194(Rouche)! We the stronger symmetric form of Rouche: Let $C : [0, 1] \to \mathbb{C}$ be simple closed curve whose image is boundary of ∂K. If f, g holomorphic on K with $|f(z) - g(z)| < |f(z)| + |g(z)|$ on ∂K then they have the same number of zeroes.

Via the argument principle the number of zeroes of f in K is $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{f\circ C} \frac{dz}{z} = Ind_{f\circ C}(0)$ ie. the winding number of closed curve $f \circ C$. https://en.wikipedia.org/wiki/Rouch

☐

Proof of Problem 201(Hurwitz Limit Theorem!) We use Rouche. Note $|f(z)| > |f_n(z) - f(z)|$ on boundary of D when n large(since no zeroes on boundary). Then apply rouche. So f has same number of zeroes as close f_n. Thus same zeroes(as we must have at least those approaching, and no more besides via rouche). ☐

4.2 175 Proofs

Proof of Problem 175. Write $\frac{z}{f(1)}\int_0^1 f(t)cos(zt)dt = sin(z) - \int_0^1 \frac{f'(t)}{f(1)}sin(zt)dt$ with integration by parts where the RHS has all real zeroes via **174** ☐

Proof of Problem 174. Wlog suppose $\int_0^1 |\phi(t)|dt < 1$. Otherwise just scale by a multiplicative factor. Then for large $n \in \mathbb{N}$, $\frac{1}{n}|\phi(\frac{1}{n})| + ... + \frac{1}{n}|\phi(\frac{n-1}{n})| < 1$. So by **27** $sin(\frac{nz}{n}) - \frac{1}{n}\phi(\frac{1}{n})sin(\frac{z}{n}) - \frac{1}{n}\phi(\frac{2}{n})sin(\frac{2z}{n}) - ... - \frac{1}{n}\phi(\frac{n-1}{n})sin(\frac{n-1}{n}z)$ has no complex zeroes ☐

Proof of Problem 27. We count the changes of sign. In particular $f(0) > 0, f(\pi/n) < 0, ..., f(\frac{2n\pi}{n}) > 0$ where the largest term alternates sign. Hence we have $2n$ real zeroes on $[0, 2\pi]$. Further via definition of complex sine and substiution of $x = e^{iz}$ this is the full number of zeroes we can have(since we have a polynomial of degree n). This is **14** ☐

Proof of Problem 14. Use complex definitions of sine and cosine and make substutution $z = e^{i\theta}$. ☐

4.3 Newman Proofs

Proof of Theorem 7. Write

$$S_X(z) = \sum_{n=1}^{L} e^(x_n z) = 0 \iff e^{x_1 z}(\sum_{n=1}^{L} e^{(x_n - 1x_1)z}) = 0 \iff \sum_{n=1}^{L} e^{(x_n - x_1)z} = 0$$

since $e^{x_1 z}$ has no zeroes. So wlog we may assume $x_1 = 0$, since the translation still results in an arithmetic sequence. Then we sum

$$\sum_{n=1}^{L} e^{x_1 z} = \sum_{n=1}^{L} e^{(n-1)cz} = \sum_{n=0}^{L-1} (e^{cz})^n = \frac{e^{Lcz} - 1}{e^{cz} - 1}$$

So

$$S_X(z) = 0 \implies \frac{e^{Lcz}-1}{e^{cz}-1} = 0 \implies e^{Lcz} = 1 \implies z = ib$$

for some $b \in \mathbb{R}$. In fact we must have $Lcz = 2\pi n \implies z = \frac{2\pi n}{Lc}$ for some $n \in \mathbb{Z}$ $\square$

Proof of Theorem **??**. Alternatively $\mathbb{E}e^{z\epsilon X} = \mathbb{E}e^{z\epsilon(X-c)}e^{z\epsilon c} = \frac{1}{2}e^{zc}\mathbb{E}e^{z(X-c)} + \frac{1}{2}e^{-zc}\mathbb{E}e^{-z(X-c)} = \frac{1}{2}\mathbb{E}e^{z(X-c)}(e^{zc} + ^{-zc})$ where we use symmetry. Note the last term only has roots on the imaginary axis. $\square$

Proof of Theorem 8. Wlog suppose $A = 1$. Then $f' = 0$ on $[-1, 1]$ and hence by 2 type L. $\square$

Proof of Theorem 9. Follows from a generalization of Iliya. $\square$

Proof of Theorem 10. Currently unknown. $\square$

Proof of Theorem 11. Proof by Newman. Comes from field theories. $\square$

Proof of Theorem 12. Suppose X symmetric, integer valued with probability distribution $0 \leq p_0 \leq 2p_1 \leq ... \leq 2p_n$. Then $\psi_X(iz) = p_0 + \sum 2p_k cos(kz)$. The Enestrom-Kakaya Theorem(2.17 from [5]) tells us all the zeroes of the polynomial $p_0 + 2p_1x + ... + 2p_nx^n$ has all zeroes in the closed unit disk. So then ψ_X has all zeroes on the imaginary axis via the argument from **??**. $\square$

Proof of Theorem 14. Take the inverse fourtier transform. $\square$

Proof of Theorem **??**. Alternatively $\mathbb{E}e^{z\epsilon X} = \mathbb{E}e^{z\epsilon(X-c)}e^{z\epsilon c} = \frac{1}{2}e^{zc}\mathbb{E}e^{z(X-c)} + \frac{1}{2}e^{-zc}\mathbb{E}e^{-z(X-c)} = \frac{1}{2}\mathbb{E}e^{z(X-c)}(e^{zc} + ^{-zc})$ where we use symmetry. Note the last term only has roots on the imaginary axis. $\square$

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