Consider the monotone strictly increasing sequence of nonnegative numbers $X = (x_i)_{i=1}^L$, $L \in \mathbb{N}$. We consdier the exponential sum defined by $S_X(z) = \sum_{n=1}^L e^{x_n z}$ and consider its zeroes.

1 On Concavity of Geometric Sequence

This argument needs some small fixing(I misplace some terms in the computation) but the core idea should still work: show concavity of $F(t) = log(t \sum e^(kt))$ by looking at level sets of second derivative of $log(\sum e^(kt)$ and bounding decay via integral approximation. Lots of annoying algebra. But clearly exponentially decaying.

2 On Imaginary Zeroes of Arithmetic Progressions and Symmetries

Theorem 1. *Let the sequence X above be an arbitrary arithmetic progression, ie. of the form $x_1 = d$, $x_2 = d + c$,..., $x_L = d + (L-1)c$ for arbitrary $d \in \mathbb{R}, c > 0$. Then $S_X(z)$ has zeroes only on the imaginary axis.*

Proof. Write

$$S_X(z) = \sum_{n=1}^{L} e^(x_n z) = 0 \iff e^{x_1 z}(\sum_{n=1}^{L} e^{(x_n - 1x_1)z}) = 0 \iff \sum_{n=1}^{L} e^{(x_n - x_1)z} = 0$$

since $e^{x_1 z}$ has no zeroes. So wlog we may assume $x_1 = 0$, since the translation still results in an arithmetic sequence. Then we sum

$$\sum_{n=1}^{L} e^{x_1 z} = \sum_{n=1}^{L} e^{(n-1)cz} = \sum_{n=0}^{L-1} (e^{cz})^n = \frac{e^{Lcz} - 1}{e^{cz} - 1}$$

So

$$S_X(z) = 0 \implies \frac{e^{Lcz} - 1}{e^{cz} - 1} = 0 \implies e^{Lcz} = 1 \implies z = ib$$

for some $b \in \mathbb{R}$. In fact we must have $Lcz = 2\pi n \implies z = \frac{2\pi n}{Lc}$ for some $n \in \mathbb{Z}$ ☐

Using Rouche's Theorem we can extend this result to the symmetric case:

Theorem 2. *Consider the sequence $Y = -X \cup X + c$ where X is arithmetic as defined above and $c \in \mathbb{R}$ is an arbitrary translation. Then $S_Y(z)$ has zeroes only on the imaginary axis.*

Proof. Via shifting it suffices to consider $c = 0$. Then we argue wlog for $z \in \mathbb{C}$ s.t. $Re(z) > 0$, $|S_X(z)| \geq |S_{-X}(z)|$. Since we already know S_X has no zeroes in an arbitrarily small ball around $z \in \mathbb{C}, Re(z) > 0$, this completes the proof via Rouche.

Simply compute

$$S_X(z) = \frac{e^{Lcz}-1}{e^{cz}-1}, S_{-X}(z) = \frac{e^{-Lcz}-1}{e^{-cz}-1}$$
$$|S_{-X}| \leq |S_X| \iff |\frac{e^{-Lcz}-1}{e^{-cz}-1}| \leq |\frac{e^{Lcz}-1}{e^{cz}-1}| \iff |\frac{e^{-Lcz}}{e^{-cz}}||\frac{e^{Lcz}-1}{e^{cz}-1}| \leq |\frac{e^{Lcz}-1}{e^{cz}-1}|$$
$$\iff 1 \leq |\frac{e^{Lcz}}{e^{cz}}| = e^{Re(z)(L-1)c}$$

□

3 On Imaginary Zeroes of Complement Sequences

Let the sequence X defined above be integer valued. We prove

Theorem 3. $S_X(z) = 0 \implies Re(z) = 0 \iff \forall n \in [L], x_i \in X \iff x_L - x_n \in X$ *or X is a translation of such a sequence.*

Proof. Write

$$S_X(z) = \sum_{n=1}^{L} e^(x_n z) = 0 \iff e^{x_1 z}(\sum_{n=1}^{L} e^{(x_n - 1x_1)z}) = 0 \iff \sum_{n=1}^{L} e^{(x_n - x_1)z} = 0$$

since $e^{x_1 z}$ has no zeroes. So wlog we may assume $x_1 = 0$. Then make the change of variables $z = log(z')$ where we note complex log is surjective since $z = log(z') \iff e^z = z'$. Also $z = ib \iff log(z') = ib \iff z' = e^(ib) \implies z' \in \mathbb{S}^1$. Compute

$$S_X(log(z')) = 1 + \sum_{n=2}^{L} e^{log((z')^{x_n})} = 1 + \sum_{n=2}^{L} (z')^{x_n} = P_X(z')$$

which is a polynomial in terms of z'. Thus $S_X(z)$ has an imaginary root $\iff P_X(z')$ has a root on the complex unit circle. So S_X has strictly imaginary axis roots iff P_X has strictly unit modulus roots. It is known that the only polynomials which have strictly unit modulus roots are the palindromic polynomials($a_i = a_{n-i}$) and the anti-palindromic polynomials($a_i = -a_{n-i}$)[1]. It s clear our polynomial P cannot be anti-palindromic, so to have all roots on $\mathbb{S}^1$ it must be palindromic.

The rest does not work since palindromic polynomials need not have all unit modulus roots.

□

Remark 4. This result includes a farily wide class of sequences, including integer arithmetic progressions and their symmetries. Call this class the set of complement sequences $\mathcal{C}$. So in particular this implies the random variable defined as $P(X = k) = 1/2L$ for $k \in \{-L, ..., -1\} \cup \{1, ..., L\}$ is type L(and hence Ultra-Sub Gaussian). This also fully characterizes discrete integer valued uniform type L random variables.

4 On The Zeroes of Other Types of Sequences

Don't yet have a full characterization but we know through testing

- The geometric sequence $(q^i)_{i=1}^L$ does not have strictly imaginary zeroes
- Binomial coefficient sequences(the first half) do not have strictly imaginary zeroes

so this breaks for the full log-concave class.

5 Other Ideas

I came across a theorem from Control Theory? called the Jury Test[2] which tells us when a polynomial has zeroes outside of the unit disc by looking at operations on the coefficients. I don't think this is useful anymore since we now have a complete characterization via these palindromic polynomials, but I'm not sure. It's proved using Rouche's theorem so it may be useful.

I think we should be able to extend this proof invovling palindromic polynomials to a wider class via some shifting arguments, but I haven't thought about how very thoroughly yet.

My notation changed several times over the course of writing this out, sorry about that.

References

[1] Markovsky. I, Shodhan. R, Palindromic Polynomials, Time-Reversible Systems, and Conserved Quantities. https://eprints.soton.ac.uk/266592/1/Med08.pdf

[2] Keel. L, Bhattarcharyya. S, A New Proof of the Jury Test. https://ieeexplore.ieee.org/document/703305