1 Statements

1.1 Problem ???

Theorem 1 (V ???). *If α is an even integer greater than two, $f(z) = \int_0^\infty e^{-t^\alpha} cos(zt)dt$*

1.2 Problem 173 and Relevant Theorems

Theorem 2 (V 173). *Let X be a symmetric continuous random variable distributed on $[0,1]$ density f s.t. $f', f'' < 0$. Then $X \in \mathcal{L}$.*

Theorem 3 (V 26). *Let $A_1, ..., A_n$ be non-zero real numbers and $a_1 < ... < a_n$. Then if $A_1 > 0, ..., A_{n-1} > 0$ or $A_1 > 0, ..., A_{k-1} > 0, A_{k+1} > 0, ...A_n > 0$ with $\sum A_k < 0$ then $f(x) = \frac{A_1}{x-a_1} + ... + \frac{A_n}{x-a_n}$ has only real zeroes*

Theorem 4 (III 165). *Suppose entire $F(z)$ satisfies $|F(x+iy)| < Ce^{\rho|y|}$.*
Then $\frac{d}{dz}(\frac{F(z)}{sin(\rho z)}) = -\sum_{\mathbb{Z}} \frac{\rho(-1)^n F(\frac{n\pi}{\rho})}{(\rho z - n\pi)^2}$

Theorem 5 (III 170(Precursor to 201)). *Suppose $f_1, ..., f_n, ..$ are regular in open $U \subseteq \mathbb{R}$, and convering uniformly in any closed domain inside $\mathbb{R}$. Then limit f is regular*

Theorem 6 (III 194(Rouche)). *Suppose f, ϕ regular in interior of $\mathcal{D}$, cts on closed domain,, and $|f(z)| > |\phi(z)| \forall z \in \partial\mathcal{D}$. Then $f(z) + \phi(z)$ has exactly the same number of zeores as f inside $\mathcal{D}$.*

Theorem 7 (III 201(Hurwitz Theorem main tool for controlling limit zeros)). *Suppose $f_n \to f$ pointwise with $\mathcal{Z}$ the set of all zeroes of f_n in $\mathbb{R}$. Then the zeroes of f in $\mathbb{R}$ are the limit points of $\mathcal{Z}$ in $\mathbb{R}$.*

1.3 Problem 175 and Relevant Problems

Theorem 8 (V 175). *Let $f(t)$ be real and continuously differentiable for $0 \leq t \leq 1$. If we have $|f(1)| \geq \int_0^1 |f'(t)|dt$ then the entire function $F(z) = \int_0^1 f(t)cos(zt)dt$ has only real zeroes*

Theorem 9 (V 174). *Let $\phi(t)$ be properly integrable for $0 \leq t \leq 1$. If $\int_0^1 |\phi(t)|dt \leq 1$ then entire $F(z) = sin(z) \int_0^1 \phi(t)sin(zt)dt$ has only real zeroes*

Theorem 10 (V 27). *The trignometric polynomial $f(x) = a_0 + a_1 cos(x) + ... + a_n cos(nx)$ with real coefficients has only real zeroes if $|a_0| + |a_1| + ... + |a_{n-1}| < a_n$(note this also applies to sin via differentiation and rolle's thm)*

Theorem 11 (VI 14). *A trig poly with real coefficients $g(z) = \lambda_0 + \lambda_1 cos(z) + \mu_1 sin(z) + ... + \lambda_n cos(nz) + \mu_n sin(nz)$ has exactly 2n zeroes(where shifting by 2π is not distinct)*

2 Proofs

2.1 ??? Proofs

Proof of Problem ??? Set $\alpha > 2$ even integer. We know f entire and hence has infinitely many zeroes. (why entire???).

Since α even integer $h(z) = \frac{\Gamma(z+1)\Gamma((2z=1)/\alpha)}{\Gamma(2z+1)}$ has set of poles of its numerator canceled by poles of denominator.

Compute $\int_0^\infty e^{-t^\alpha} cos(zt)dt = \sum_{k=0}^\infty \frac{(-1)^k \int_0^\infty e^{-t^\alpha} t^{2k} dt}{(2k)!} z^{2k} = \sum \frac{1}{\alpha}(-1)^k 1/(2k)! \int_0^\infty e^{-u} u^{2k+1/\alpha-1} du z^{2k} = \frac{1}{\alpha} \sum \frac{(-1)^k}{k!} (\frac{\Gamma(k+1)\Gamma(\frac{2k+1}{\alpha})}{\Gamma(2k+1)}) z^{2k}$ □

2.2 173 Proofs

Proof of Problem 173. Via integration by parts twice we write $z^2F(z) = zf(1)sin(z) - f'(0)(1 - cos(z)) + \int_0^1 f''(t)(cos(z) - cos(zt))dt$. Compute $((2m-1)\pi)^2 F((2m-1)\pi) = -2f'(0) + \int_0^1 f''(t)(-1 - cos((2m-1)\pi t) > 0$. Then compute $(2m\pi)^2 F(2m\pi) = \int_0^1 f''(t)(1 - cos(2m\pi t)) < 0$ since $f'' < 0$. Which gives infinitely many zeroes. Note $F(0) > 0$. The rational function

$f_n(z) = (-1)^n \frac{F(-n\pi)}{z+n\pi} + ... + \frac{F(-2\pi)}{z+2\pi} - \frac{F(-\pi)}{z+\pi} + \frac{F(0)}{z} - \frac{F(\pi)}{z-\pi} + ... + (-1)^n \frac{F(n\pi)}{z-n\pi}$ can have via **26** either all real zeroes of 2n-2 real zeroes and 2 imaginary. Further it converges to $\frac{F(z)}{sin(z)}$ by integrating the result of **165**. So as we take the limit, the nonreal zeroes $\frac{F(z)}{sin(z)}$ as via **201** they are the limit points of approaching f_n zeroes. In the case we have two nonreal zeroes, it must be the case they are strictly imaginary, as $F(z) = F(-z) = F(\overline{z}) = 0$. But $F(ix) = \int_0^1 f(t) \frac{e^{xt} + e^{-xt}}{2} dt > 0$ □

Proof of Problem 26. Idea is to count zeroes intervals (a_i, a_{i+1}) via changes of sign. Write $f(x) = P(x)/Q(x)$ for $Q = (x - a_1)...(x - a_n)$ and P a sum of n-1 deg polys. Via ϵ approximations $f(a_1 + \epsilon) > 0$ and $f(a_2 - \epsilon) < 0$ which continues alternating. Note that poles blow up as we get very close, dominating sign. Then regarding polynomial in numerator this is a full accounting.

(Can we still use IVT despite singularities? We look at intervals in between which do have continuity.) □

Proof of Problem 165. Somewhere in German Hurwitz-Courant □

Proof of Problem 170(Precursor to 201). Use Cauchy integral theorem on closed cts curve $L \subseteq \mathbb{R}$. Write $f_n(z) = \frac{1}{2\pi i} \int_L \frac{f_n(\xi)}{\xi - z} d\xi$ via cauchy integral formula. We know uniformly on L $f_n(\xi)/(\xi - z) \to f(\xi)/(\xi - z)$ and further f cts(as uniform limit of continuous functions). So $f_n(z) \to \frac{1}{2\pi i} \int_L \frac{f(\xi)}{\xi - z} d\xi$(why?). And the last function is regular(why?). □

Proof of problem 194(Rouche)! We the stronger symmetric form of Rouche: Let $C : [0,1] \to \mathbb{C}$ be simple closed curve whose image is boundary of ∂K. If f, g holomorphic on K with $|f(z) - g(z)| < |f(z)| + |g(z)|$ on ∂K then they have the same number of zeroes.

Via the argument principle the number of zeroes of f in K is $\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{f \circ C} \frac{dz}{z} = Ind_{f \circ C}(0)$ ie. the winding number of closed curve $f \circ C$. https://en.wikipedia.org/wiki/Rouch □

Proof of Problem 201(Hurwitz Limit Theorem!) We use Rouche. Note $|f(z)| > |f_n(z) - f(z)|$ on boundary of D when n large(since no zeroes on boundary). Then apply rouche. So f has same number of zeroes as close f_n. Thus same zeroes(as we must have at least those approaching, and no more besides via rouche). □

2.3 175 Proofs

Proof of Problem 175. Write $\frac{z}{f(1)} \int_0^1 f(t)cos(zt)dt = sin(z) - \int_0^1 \frac{f'(t)}{f(1)} sin(zt)dt$ with integration by parts where the RHS has all real zeroes via **174** □

Proof of Problem 174. Wlog suppose $\int_0^1 |\phi(t)|dt < 1$. Otherwise just scale by a multiplicative factor. Then for large $n \in \mathbb{N}$, $\frac{1}{n}|\phi(\frac{1}{n})|+...+\frac{1}{n}|\phi(\frac{n-1}{n})| < 1$. So by **27** $sin(\frac{nz}{n}) - \frac{1}{n}\phi(\frac{1}{n})sin(\frac{z}{n}) - \frac{1}{n}\phi(\frac{2}{n})sin(\frac{2z}{n}) - ... - \frac{1}{n}\phi(\frac{n-1}{n})sin(\frac{n-1}{n}z)$ has no complex zeroes □

Proof of Problem 27. We count the changes of sign. In particular $f(0) > 0, f(\pi/n) < 0, ..., f(\frac{2n\pi}{n}) > 0$ where the largest term alternates sign. Hence we have $2n$ real zeroes on $[0, 2\pi]$. Further via definition of complex sine and substiution of $x = e^{iz}$ this is the full number of zeroes we can have(since we have a polynomial of degree n). This is **14** □

Proof of Problem 14. Use complex definitions of sine and cosine and make substutution $z = e^{i\theta}$. □

References

[1] Polya G. Szego G. Problems and Theorems in Analysis.