A Study of Khintchine Type Inequalities for Random Variables

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1 Introduction

2 Ultra Sub-Gaussanity and Strong Log Concavity

Here we discuss the notions of Ultra Sub-Gaussanity and Strong Log Concavity of a Random Variable. First recall

Definition 1. A sequence $(a_i)_{i=0}^{\infty}$ of non-negative real numbers is called *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$.

Then we can define the notion of *Ultra Sub-Gaussanity* for $\mathbb{R}^n$ valued random vectors.

Definition 2. $\mathbb{R}^n$ valued X is *Ultra sub-Gaussian* if $X = 0$ or X is rotation invariant, has finite moments, and has gaussian log-concave even moments ie. $a_i = \mathbb{E}||X||^{2i}/\mathbb{E}||G||^{2i}$ are log-concave.

Given X Ultra Sub-Gaussian we can extract a khintchine type inequality of the following form.

Theorem 3. *Let n,d positive integers and $p > q \geq 2$ even integers. If $X_1, ..., X_n$ independent $\mathbb{R}^n$ valued random vectors are ultra sub-Gaussian then*

$$(\mathbb{E}|S|^p)^{1/p} \leq \frac{(\mathbb{E}|G|^p)^{1/p}}{(\mathbb{E}|G|^q)^{1/q}}\mathbb{E}(|S|^q)^{1/q}$$

where $S = \sum_i X_i$

The proof of which rests crucially on the clsoure of **USG** under sums:

Lemma 4. *If $X, Y \in$ **USG** are independent random vectors then $X + Y \in$ **USG**.*

3 A Discrete Generalization of Random Signs

We recall that a random sign ϵ takes values on $\{-1, 1\}$ with uniform probability. We now consider the generalization to X $\{-L, , .., 0, ..., L\}$ with some mass $\mathbb{P}(X = 0) = \rho_0$ and otherwise uniformly distributed on the $\{-L, ..., -1\} \cup \{1, ..., L\}$. We have the following results.

Theorem 5. *Let $\rho_0 \in [0, 1]$ and let L be a positive integer. Let $X_1, X_2, \ldots$ be i.i.d. copies of a random variable X with $\mathbb{P}(X = 0) = \rho_0$ and $\mathbb{P}(X = -j) = \mathbb{P}(X = j) = \frac{1-\rho_0}{2L}$, $j = 1, \ldots, L$. Then X is ultra sub-Gaussian if and only if $\rho_0 = 1$, or*

$$\rho_0 \leq 1 - \frac{2}{5}\frac{3L^2 + 3L - 1}{(L + 1)(2L + 1)}. \tag{1}$$

If this holds, then, consequently, for positive even integers $q > p \geq 2$, every $n \geq 1$ and reals $a_1, \ldots, a_n$, we have

$$\left(\mathbb{E}\left|\sum_{i=1}^n a_i X_i\right|^q\right)^{1/q} \leq C_{p,q} \left(\mathbb{E}\left|\sum_{i=1}^n a_i X_i\right|^p\right)^{1/p} \tag{2}$$

with $C_{p,q} = \frac{[1\cdot 3\cdot \ldots \cdot (q-1)]^{1/q}}{[1\cdot 3\cdot \ldots \cdot (p-1)]^{1/p}}$ which is sharp.

Here we first need to recall the classical notions of majorisation and Schur-convexity. Given two nonnegative sequences $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$, we say that $(b_i)_{i=1}^n$ *majorises* $(a_i)_{i=1}^n$, denoted $(a_i) \prec (b_i)$ if

$$\sum_{i=1}^n a_i = \sum_{i=1}^n b_i \qquad \text{and} \qquad \sum_{i=1}^k a_i^* = \sum_{i=1}^k b_i^* \ \text{ for all } \ k = 1, \ldots, n,$$

where $(a_i^*)_{i=1}^n$ and $(b_i^*)_{i=1}^n$ are nonincreasing permutations of $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ respectively. For example, $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) \prec (a_1, a_2, \ldots, a_n) \prec (1, 0, \ldots, 0)$ for every nonnegative sequence (a_i) with $\sum_{i=1}^n a_i = 1$. A function $\Psi\colon [0,\infty)^n \to \mathbb{R}$ which is symmetric (with respect to permuting the coordinates) is said to be *Schur-convex* if $\Psi(a) \leq \Psi(b)$ whenever $a \prec b$ and *Schur-concave* if $\Psi(a) \geq \Psi(b)$ whenever $a \prec b$. For instance, a function of the form $\Psi(a) = \sum_{i=1}^n \psi(a_i)$ with $\psi\colon [0, +\infty) \to \mathbb{R}$ being convex is Schur-convex.

Theorem 6. *Let L be a positive integer. Let $X_1, X_2, \ldots$ be i.i.d. copies of a random variable X with $\mathbb{P}(X = -j) = \mathbb{P}(X = j) = \frac{1}{2L}$, $j = 1, \ldots, L$. For every $n \geq 1$, reals $a_1, \ldots, a_n$ and $p \geq 3$, we have*

$$\left(\mathbb{E}\left|\sum_{i=1}^n a_i X_i\right|^p\right)^{1/p} \leq C_p \left(\mathbb{E}\left|\sum_{i=1}^n a_i X_i\right|^2\right)^{1/2} \tag{3}$$

with $C_p = \sqrt{2}\left(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\right)^{1/p}$ which is sharp.

Theorem 7. *Let $\rho_0 \in [0, \frac{1}{2}]$. Let $X_1, X_2, \ldots$ be i.i.d. copies of a random variable X with $\mathbb{P}(X = 0) = \rho_0$ and $\mathbb{P}(X = -1) = \mathbb{P}(X = 1) = \frac{1-\rho_0}{2}$. Let $p \geq 3$. For every $n \geq 1$ and reals $a_1, \ldots, a_n, b_1, \ldots, b_n$ such that $(a_i^2)_{i=1}^n \prec (b_i^2)_{i=1}^n$, we have*

$$\mathbb{E}\left|\sum_{i=1}^n a_i X_i\right|^p \geq \mathbb{E}\left|\sum_{i=1}^n b_i X_i\right|^p. \tag{4}$$

Corollary 8. *Under the assumptions of Theorem 7 for every $n \geq 1$ and reals $a_1, \ldots, a_n$, we have*

$$\left(\mathbb{E}\left|\sum_{i=1}^n a_i X_i\right|^p\right)^{1/p} \leq C_p \left(\mathbb{E}\left|\sum_{i=1}^n a_i X_i\right|^2\right)^{1/2} \tag{5}$$

with $C_p = \sqrt{2}\left(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\right)^{1/p}$ which is sharp.

Theorem 9. *Let $\rho_0 \in [\frac{1}{2}, 1]$ and let L be a positive integer. Let $X_1, X_2, \ldots$ be i.i.d. copies of a random variable X with $\mathbb{P}(X = 0) = \rho_0$ and $\mathbb{P}(X = -j) = \mathbb{P}(X = j) = \frac{1-\rho_0}{2L}$, $j = 1, \ldots, L$. For every $n \geq 1$ and reals $a_1, \ldots, a_n$, we have*

$$\mathbb{E}\left|\sum_{i=1}^n a_i X_i\right| \geq c_1 \left(\mathbb{E}\left|\sum_{i=1}^n a_i X_i\right|^2\right)^{1/2} \tag{6}$$

with $c_1 = \frac{\mathbb{E}|X|}{\sqrt{\mathbb{E}|X|^2}} = \sqrt{\frac{3(1-\rho_0)L(L+1)}{2(2L+1)}}$ which is sharp.

4 Known Examples of Type L Random Variables

Relatively few examples of Type L random variables are known. The majority we do have follow from results of Polya in his study of kernels producing strictly real zeroes of fourier transforms of the form:

$$\phi(z) = \int_{\mathbb{R}} K(x) cos(zx) dx$$

for some kernel $K : \mathbb{R} \to \mathbb{R}$. This can naturally be interpreted as the inverse fourier transform of a random variable X with density K. And (assuming symmetry and nice gaussanity conditions) if ϕ has strictly real zeroes then we know $\phi(iz) = \mathbb{E}e^{-zX}$ has strictly imaginary zeroes and hence X is type L.

4.1 Polya's Examples

All of these examples can be found in Polya's *Problems in Analysis* but the experience of retrieving them(and their proofs) is somewhat time consuming. Hopefully this presentation is somewhat less so. We attach proofs of these examples in an appendix.

Theorem 10 (Decreasing Concave Density(173)). *Let X be a symmetric continuous random variable distributed on $[0,1]$ density f s.t. $f', f'' < 0$. Then $X \in \mathcal{L}$.*

Theorem 11 (L1 Bounded Derivative(175)). *Let X be a symmetric continuous random variable distributed on $[0,1]$ with density f s.t. $|f(1)| \geq \int_0^1 |f'(t)| dt$. Then $X \in \mathcal{L}$. Note in particular this works for the case f is increasing.*

Theorem 12 (Exponential Density(170)). *Let α be even integer greater than 2. Then if X a symmetric continuous random variable with density of the form e^{-t^α} then $X \in \mathcal{L}$*

Theorem 13 (Exponential Product Density(161)). *Let $1 > \alpha \geq 0, 0 < \alpha_1 \leq \alpha_2 \leq \ldots$ and reciprocal convergent. Then if $g(z) = e^{-\alpha z}(1 - \frac{z}{\alpha_1})(1 - \frac{z}{\alpha_2})\ldots$ we have for symmetric X with density $e^{-t^2} g(-t^2)$ then $X \in \mathcal{L}$.*

Theorem 14 (Bessel Function(159)). *The symmetric continuous random variable X with density $\frac{2}{\pi\sqrt{1-t^2}}$ in $\mathcal{L}$.*

Theorem 15 (Large nth Coefficient(27)). *Suppose X a discrete integer valued symmetric distribution. If $p_0 + 2p_1 + \ldots + 2p_{n-1} < 2p_n$ then $X \in \mathcal{L}$.*

4.2 Newman's Examples

Newman, who initiated our study in Type L random variables, produced some examples as well.

Perhaps one of the most basic examples of type L random variables are arithmetic progressions and uniform random variables.

Theorem 16 (Arithmetic Sequences). *Let the sequence X above be an arbitrary arithmetic progression, ie. of the form $x_1 = d$, $x_2 = d + c, \ldots, x_L = d + (L-1)c$ for arbitrary $d \in \mathbb{R}, c > 0$. Then $S_X(z)$ has zeroes only on the imaginary axis.*

Theorem 17 (Uniform(Newman 7)). *Let X be random variable with density $\frac{d\mu}{dy} = 1$ if $|y| \leq A$ and 0 otherwise. $A > 0$. Then $X \in \mathcal{L}$.*

We also know marginals of uniformly random vectors on spheres are types L.

Theorem 18 (Newman (8)). *Density $(1-y^2)^{(d-2)/2}$ with $|y| \leq 1$ and 0 otherwise. For $d > 0$.*

Theorem 19 (Newman (9)). *Density $e^{-\lambda cosh(y)}$, $\lambda > 0$*

(Some physics field theory context)

Theorem 20 (Newman (10)). *$e^{-ay^4-by^2}$ with $a > 0$*

4.3 Other Examples

Theorem 21 (Enestrom-Kakeya). *If X integer valued symmetric with $0 \leq p_0 \leq 2p_1 \leq ... \leq 2p_n$ with $p_n > 0$ then $X \in \mathcal{L}$.*

Theorem 22 (Absolute Value). *Let $a_0, a_1, ..., a_n \in \mathbb{R}$ with $|a_0| + ..|a_{n-1}| \leq |a_n|$ then the trig polys $p_c(z) = \sum_{k=0}^n a_k cos(kz)$ and the sin one have only real zeroes*

4.4 Our Examples

Theorem 23 (Rapidly Decreasing Polynomial). *Suppose X has density $e^{-x^2/2}x^2$. Then $X \in \mathcal{L}$.*

Theorem 24 (General Symmetrization). *Suppose $X - c$ is type L for some $c \in \mathcal{L}$. Then $\epsilon X \in \mathcal{L}$.*

5 Type L Example Proofs

5.1 Problem ???

Theorem 25 (V ???). *If α is an even integer greater than two, $f(z) = \int_0^\infty e^{-t^\alpha} cos(zt)dt$*

5.2 Problem 173 and Relevant Theorems

Theorem 26 (V 173). *Let X be a symmetric continuous random variable distributed on $[0, 1]$ density f s.t. $f', f'' < 0$. Then $X \in \mathcal{L}$.*

Theorem 27 (V 26). *Let $A_1, ..., A_n$ be non-zero real numbers and $a_1 < ... < a_n$. Then if $A_1 > 0, ..., A_{n-1} > 0$ or $A_1 > 0, ..., A_{k-1} > 0, A_{k+1} > 0, ...A_n > 0$ with $\sum A_k < 0$ then $f(x) = \frac{A_1}{x-a_1} + ... + \frac{A_n}{x-a_n}$ has only real zeroes*

Theorem 28 (III 165). *Suppose entire $F(z)$ satisfies $|F(x+iy)| < Ce^{\rho|y|}$.
Then $\frac{d}{dz}(\frac{F(z)}{sin(\rho z)}) = -\sum_{\mathbb{Z}} \frac{\rho(-1)^n F(\frac{n\pi}{\rho})}{(\rho z - n\pi)^2}$*

Theorem 29 (III 170(Precursor to 201)). *Suppose $f_1, ..., f_n, ..$ are regular in open $U \subseteq \mathbb{R}$, and convering uniformly in any closed domain inside $\mathbb{R}$. Then limit f is regular*

Theorem 30 (III 194(Rouche)). *Suppose f, ϕ regular in interior of $\mathcal{D}$, cts on closed domain,, and $|f(z)| > |\phi(z)| \forall z \in \partial\mathcal{D}$. Then $f(z) + \phi(z)$ has exactly the same number of zeores as f inside $\mathcal{D}$.*

Theorem 31 (III 201(Hurwitz Theorem main tool for controlling limit zeros))**.** *Suppose $f_n \to f$ pointwise with $\mathcal{Z}$ the set of all zeroes of f_n in $\mathbb{R}$. Then the zeroes of f in $\mathbb{R}$ are the limit points of $\mathcal{Z}$ in $\mathbb{R}$.*

5.3 Problem 175 and Relevant Problems

Theorem 32 (V 175)**.** *Let $f(t)$ be real and continuously differentiable for $0 \leq t \leq 1$. If we have $|f(1)| \geq \int_0^1 |f'(t)|dt$ then the entire function $F(z) = \int_0^1 f(t)cos(zt)dt$ has only real zeroes*

Theorem 33 (V 174)**.** *Let $\phi(t)$ be properly integrable for $0 \leq t \leq 1$. If $\int_0^1 |\phi(t)|dt \leq 1$ then entire $F(z) = sin(z)\int_0^1 \phi(t)sin(zt)dt$ has only real zeroes*

Theorem 34 (V 27)**.** *The trignometric polynomial $f(x) = a_0 + a_1cos(x) + ... + a_ncos(nx)$ with real coefficients has only real zeroes if $|a_0| + |a_1| + ... + |a_{n-1}| < a_n$(note this also applies to sin via differentiation and rolle's thm)*

Theorem 35 (VI 14)**.** *A trig poly with real coefficients $g(z) = \lambda_0 + \lambda_1cos(z) + \mu_1sin(z) + ... + \lambda_ncos(nz) + \mu_nsin(nz)$ has exactly 2n zeroes(where shifting by 2π is not distinct)*

6 Proofs

6.1 173 Proofs

Proof of Problem 173. Via integration by parts twice we write $z^2F(z) = zf(1)sin(z) - f'(0)(1 - cos(z)) + \int_0^1 f''(t)(cos(z) - cos(zt))dt$. Compute $((2m-1)\pi)^2F((2m-1)\pi) = -2f'(0) + \int_0^1 f''(t)(-1 - cos((2m-1)\pi t) > 0$. Then compute $(2m\pi)^2F(2m\pi) = \int_0^1 f''(t)(1 - cos(2m\pi t)) < 0$ since $f'' < 0$. Which gives infinitely many zeroes. Note $F(0) > 0$. The rational function

$f_n(z) = (-1)^n\frac{F(-n\pi)}{z+n\pi} + ... + \frac{F(-2\pi)}{z+2\pi} - \frac{F(-\pi)}{z+\pi} + \frac{F(0)}{z} - \frac{F(\pi)}{z-\pi} + ... + (-1)^n\frac{F(n\pi)}{z-n\pi}$ can have via **26** either all real zeroes of 2n-2 real zeroes and 2 imaginary. Further it converges to $\frac{F(z)}{sin(z)}$ by integrating the result of **165**. So as we take the limit, the nonreal zeroes $\frac{F(z)}{sin(z)}$ as via **201** they are the limit points of approaching f_n zeroes. In the case we have two nonreal zeroes, it must be the case they are strictly imaginary, as $F(z) = F(-z) = F(\overline{z}) = 0$. But $F(ix) = \int_0^1 f(t)\frac{e^{xt}+e^{-xt}}{2}dt > 0$ □

Proof of Problem 26. Idea is to count zeroes intervals (a_i, a_{i+1}) via changes of sign. Write $f(x) = P(x)/Q(x)$ for $Q = (x - a_1)...(x - a_n)$ and P a sum of n-1 deg polys. Via ϵ approximations $f(a_1 + \epsilon) > 0$ and $f(a_2 - \epsilon) < 0$ which continues alternating. Note that poles blow up as we get very close, dominating sign. Then regarding polynomial in numerator this is a full accounting.

(Can we still use IVT despite singularities? We look at intervals in between which do have continuity.) □

Proof of Problem 165. Somewhere in German Hurwitz-Courant □

Proof of Problem 170(Precursor to 201). Use Cauchy integral theorem on closed cts curve $L \subseteq \mathbb{R}$. Write $f_n(z) = \frac{1}{2\pi i}\int_L \frac{f_n(\xi)}{\xi - z}d\xi$ via cauchy integral formula. We know uniformly on L $f_n(\xi)/(\xi - z) \to f(\xi)/(\xi - z)$ and further f cts(as uniform limit of continuous functions). So $f_n(z) \to \frac{1}{2\pi i}\int_L \frac{f(\xi)}{\xi - z}d\xi$(why?). And the last function is regular(why?). □

Proof of problem 194(Rouche)! We the stronger symmetric form of Rouche: Let $C : [0,1] \to \mathbb{C}$ be simple closed curve whose image is boundary of ∂K. If f, g holomorphic on K with $|f(z) - g(z)| < |f(z)| + |g(z)|$ on ∂K then they have the same number of zeroes.

Via the argument principle the number of zeroes of f in K is $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{f\circ C}\frac{dz}{z} = Ind_{f\circ C}(0)$ ie. the winding number of closed curve $f \circ C$. https://en.wikipedia.org/wiki/Rouch ☐

Proof of Problem 201(Hurwitz Limit Theorem!) We use Rouche. Note $|f(z)| > |f_n(z) - f(z)|$ on boundary of D when n large(since no zeroes on boundary). Then apply rouche. So f has same number of zeroes as close f_n. Thus same zeroes(as we must have at least those approaching, and no more besides via rouche). ☐

6.2 175 Proofs

Proof of Problem 175. Write $\frac{z}{f(1)}\int_0^1 f(t)cos(zt)dt = sin(z) - \int_0^1 \frac{f'(t)}{f(1)}sin(zt)dt$ with integration by parts where the RHS has all real zeroes via **174** ☐

Proof of Problem 174. Wlog suppose $\int_0^1 |\phi(t)|dt < 1$. Otherwise just scale by a multiplicative factor. Then for large $n \in \mathbb{N}$, $\frac{1}{n}|\phi(\frac{1}{n})| + ... + \frac{1}{n}|\phi(\frac{n-1}{n})| < 1$. So by **27** $sin(\frac{nz}{n}) - \frac{1}{n}\phi(\frac{1}{n})sin(\frac{z}{n}) - \frac{1}{n}\phi(\frac{2}{n})sin(\frac{2z}{n}) - ... - \frac{1}{n}\phi(\frac{n-1}{n})sin(\frac{n-1}{n}z)$ has no complex zeroes ☐

Proof of Problem 27. We count the changes of sign. In particular $f(0) > 0, f(\pi/n) < 0, ..., f(\frac{2n\pi}{n}) > 0$ where the largest term alternates sign. Hence we have $2n$ real zeroes on $[0, 2\pi]$. Further via definition of complex sine and substiution of $x = e^{iz}$ this is the full number of zeroes we can have(since we have a polynomial of degree n). This is **14** ☐

Proof of Problem 14. Use complex definitions of sine and cosine and make substutution $z = e^{i\theta}$. ☐

6.3 Newman Proofs

Proof of Theorem 16. Write

$$S_X(z) = \sum_{n=1}^{L} e^(x_n z) = 0 \iff e^{x_1 z}(\sum_{n=1}^{L} e^{(x_n - 1x_1)z}) = 0 \iff \sum_{n=1}^{L} e^{(x_n - x_1)z} = 0$$

since $e^{x_1 z}$ has no zeroes. So wlog we may assume $x_1 = 0$, since the translation still results in an arithmetic sequence. Then we sum

$$\sum_{n=1}^{L} e^{x_1 z} = \sum_{n=1}^{L} e^{(n-1)cz} = \sum_{n=0}^{L-1} (e^{cz})^n = \frac{e^{Lcz} - 1}{e^{cz} - 1}$$

So

$$S_X(z) = 0 \implies \frac{e^{Lcz} - 1}{e^{cz} - 1} = 0 \implies e^{Lcz} = 1 \implies z = ib$$

for some $b \in \mathbb{R}$. In fact we must have $Lcz = 2\pi n \implies z = \frac{2\pi n}{Lc}$ for some $n \in \mathbb{Z}$ ☐

Proof of Theorem **??**. Alternatively $\mathbb{E}e^{z\epsilon X} = \mathbb{E}e^{z\epsilon(X-c)}e^{z\epsilon c} = \frac{1}{2}e^{zc}\mathbb{E}e^{z(X-c)} + \frac{1}{2}e^{-zc}\mathbb{E}e^{-z(X-c)} = \frac{1}{2}\mathbb{E}e^{z(X-c)}(e^{zc} +^{-zc})$ where we use symmetry. Note the last term only has roots on the imaginary axis. □

Proof of Theorem 17. Wlog suppose $A = 1$. Then $f' = 0$ on $[-1, 1]$ and hence by 11 type L. □

Proof of Theorem 18. Follows from a generalization of Iliya. □

Proof of Theorem 19. Currently unknown. □

Proof of Theorem 20. Proof by Newman. Comes from field theories. □

Proof of Theorem 21. Suppose X symmetric, integer valued with probability distribution $0 \leq p_0 \leq 2p_1 \leq ... \leq 2p_n$. Then $\psi_X(iz) = p_0 + \sum 2p_k cos(kz)$. The Enestrom-Kakaya Theorem(2.17 from [7]) tells us all the zeroes of the polynomial $p_0 + 2p_1 x + ... + 2p_n x^n$ has all zeroes in the closed unit disk. So then ψ_X has all zeroes on the imaginary axis via the argument from **??**. □

Proof of Theorem 23. Take the inverse fourtier transform. □

Proof of Theorem **??**. Alternatively $\mathbb{E}e^{z\epsilon X} = \mathbb{E}e^{z\epsilon(X-c)}e^{z\epsilon c} = \frac{1}{2}e^{zc}\mathbb{E}e^{z(X-c)} + \frac{1}{2}e^{-zc}\mathbb{E}e^{-z(X-c)} = \frac{1}{2}\mathbb{E}e^{z(X-c)}(e^{zc} +^{-zc})$ where we use symmetry. Note the last term only has roots on the imaginary axis. □

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