Chapter 1 数据的表示

Bits, Bytes and Integers

Bit-level manipulation

1. Boolean Algebra

- AND: & (intersection)
- OR: | (union)
- NOT: ~ (complement)
- XOR: ^ (异或) (symmetric difference)

> ***注意***：不要和C语言中的&&,||和！混淆，后者将一切nonzero视为true，而且只返回1或0：!0x69 && 0x55 = 0x01 **同时**，C语言中的&&,||和！都有Early termination功能，即如果前面的表达式已经可以确定结果，则后面的表达式将不再计算; `p && *p` 也可以避免空指针访问

2. Shift Operation

- **left shift**: << (bitwise left shift)
  - shift bit-vector x left y positions, throw away extra bits on left, and fill 0's on right
- **right shift**: >> (bitwise right shift)，fill 0's on right
- **arithmetic right shift**: >>> replicate most significant bit on left

  > example:1010 >> 1 = 1101(1010右移一位，最高位保持不变)

- **undifined behaviour**: shift amount<0 or shift amount$\geq$word size

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Integers and Integers arithmetic

1. Unsigned and 2's Complement

- **Unsigned**：$B2T(X)=\sum_{i=0}^{w-1} x_i\cdot 2^i$
- **2's Complement**：$B2T(X)=-x_{w-1}\cdot 2^{w-1}+\sum_{i=0}^{w-2} x_i\cdot 2^i$

1. 之所以称作2的补码，是因为通过将最高位代表的数设为负,如此最高位为1则计算出负数
2. 同时注意到，当二进制数各位全为1时，最高位代表的数比其余位相加多1，即 $(111111)_{2'sComplement}=-1$
3. 如此，就可以解释为何2补码的计算方法是**取反加1加负号**了：全部取反，这样与原数相加后就变成全为1；再加上1，刚好抵消最高位-2^{w-1}，也就算出了最高位与低位的差；再加上负号，即与原始公式计算出来的结果相同了
4. 同时，对x**取负**的方法是**取反加1**

> **how to pronounce:** 2^{w-1}---- *two to the word size minus 1*

2. 原码与补码的关系：

$B2U(X) = B2T(X) + bit_{w-1} \cdot 2 \cdot 2^{w-1}$(此消彼长) $= B2T(X) + bit_{w-1} \cdot 2^w$

3. Numeric Ranges

Unsigned: $min = 0, max = 2^w - 1$ **2's Complement:** $min = -2^{w-1}, max = 2^{w-1} - 1$

- $U_{max} = 2 \cdot T_{max} + 1 : 011111 \cdot 2 = 111110$ (shift left by 1) *tips*: by default 默认情况下

4. Expression Evaluation（表达式求值）

表达式里有TU混合，就要全部转换成U：

> ***If there is a mix of unsigned and signed in single expression, signed values implicitly cast to unsigned.***

- 包括比较运算,例：

  > 0==0U -1<0 -1>0U(11111转换成unsigned)

- C语言编程可能出现的问题：

  ```
  unsigned i;
   for(i=20; i>=0; i--){
    printf("hello\n");
  }
  ```

  这样会进入死循环，因为unsigned会从0跳到Umax;

  ```
  int i;
  for(i=20; i-sizeof(int)>=0; i--){
    printf("hello\n");
   }
  ```

  同样会进入死循环，因为sizeof()返回的是unsigned类型，表达式默认切换为unsigned类型运算

5. Sign Extension

- **unsigned**: 原bit不变，高位补0；
- **2's Complement**: make k copies of sign bits: 1101(扩展k位)-->111101

  > 如果看到一个数高位是：FFF...F，肯定是负数

6. Sign Truncating

- **unsigned**: 模运算——原数mod原数但剩余位全0

> 101011（43）缩1位：01011（11）——$43 mod 32((100000)_2)$; 111011（59）缩2位：1011（11）——$59 mod 48((110000)_2)$

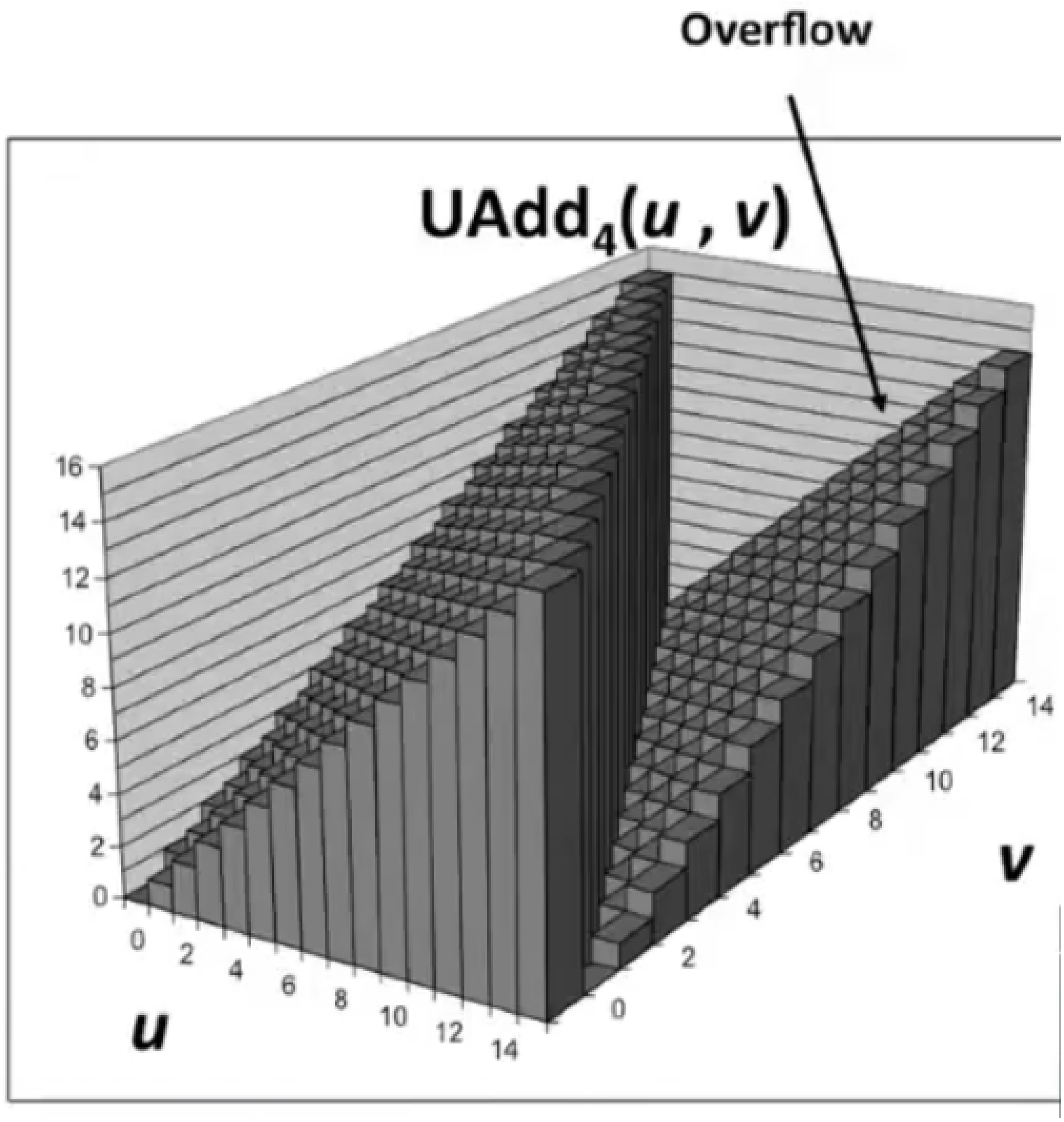
- **2's Complement**:

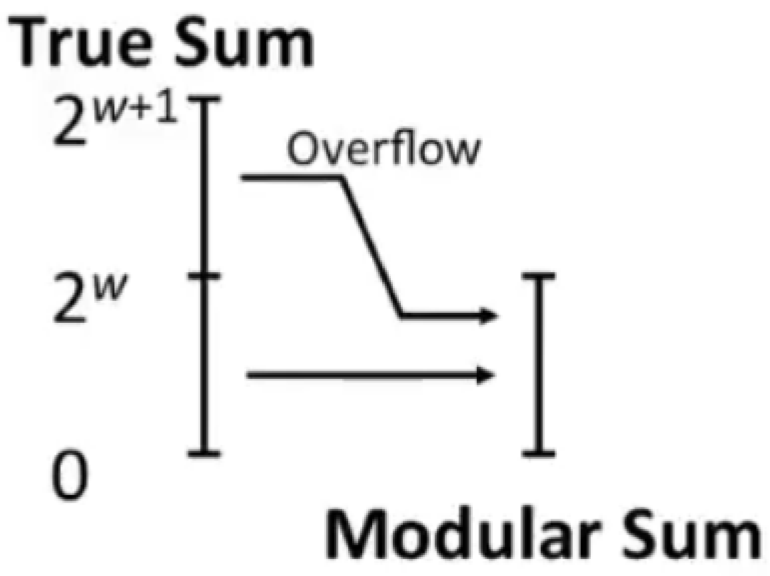
7. Addition

unsigned: Ignores carry output $s = UAdd_w(u, v) = (u + v), mod, 2^w$

> $1101 + 0101 = 10010 \xrightarrow{cast} 0010 = 2 = (5 + 13), mod, 16$

上面的现象称为溢出（overflow）：

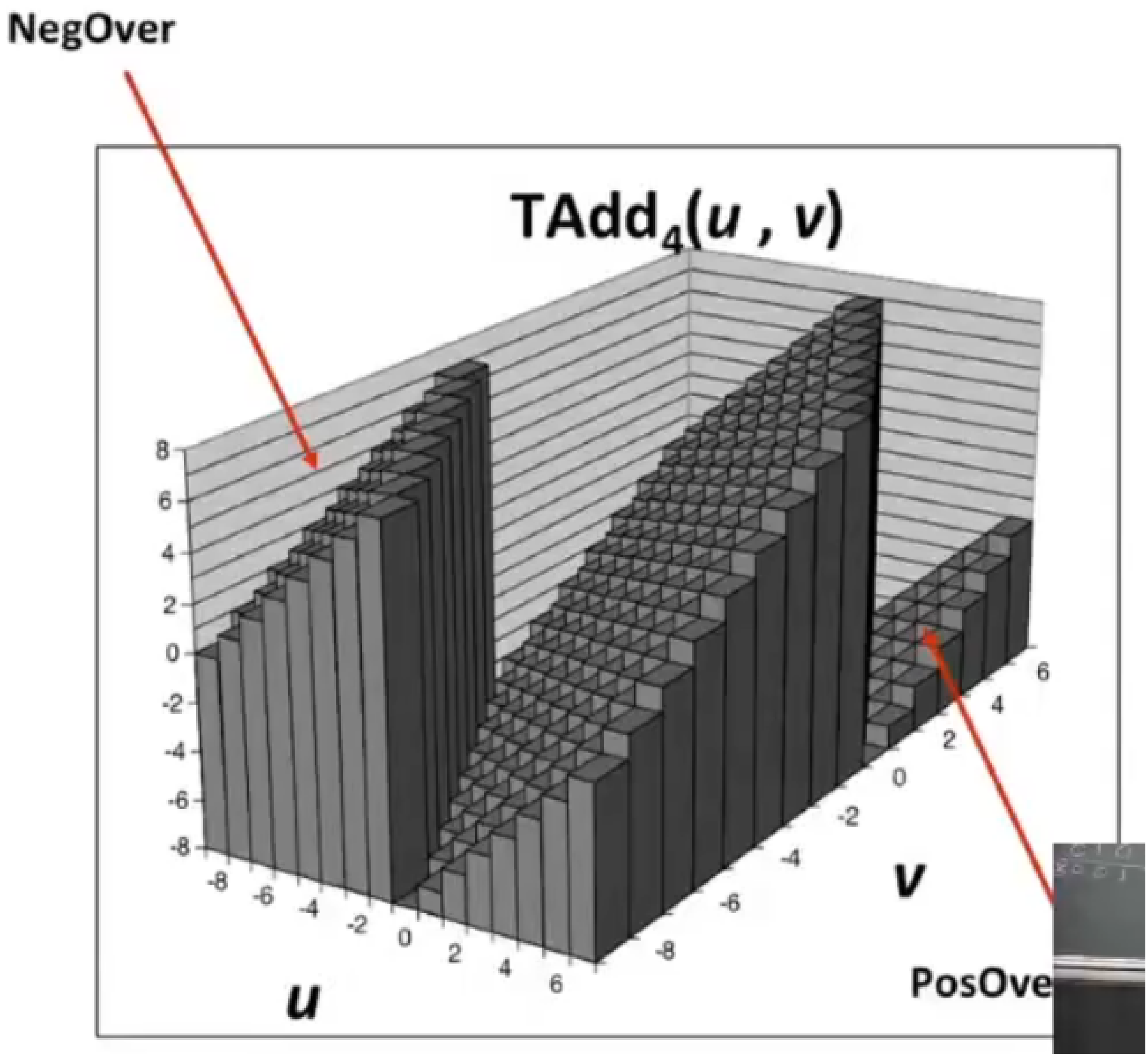


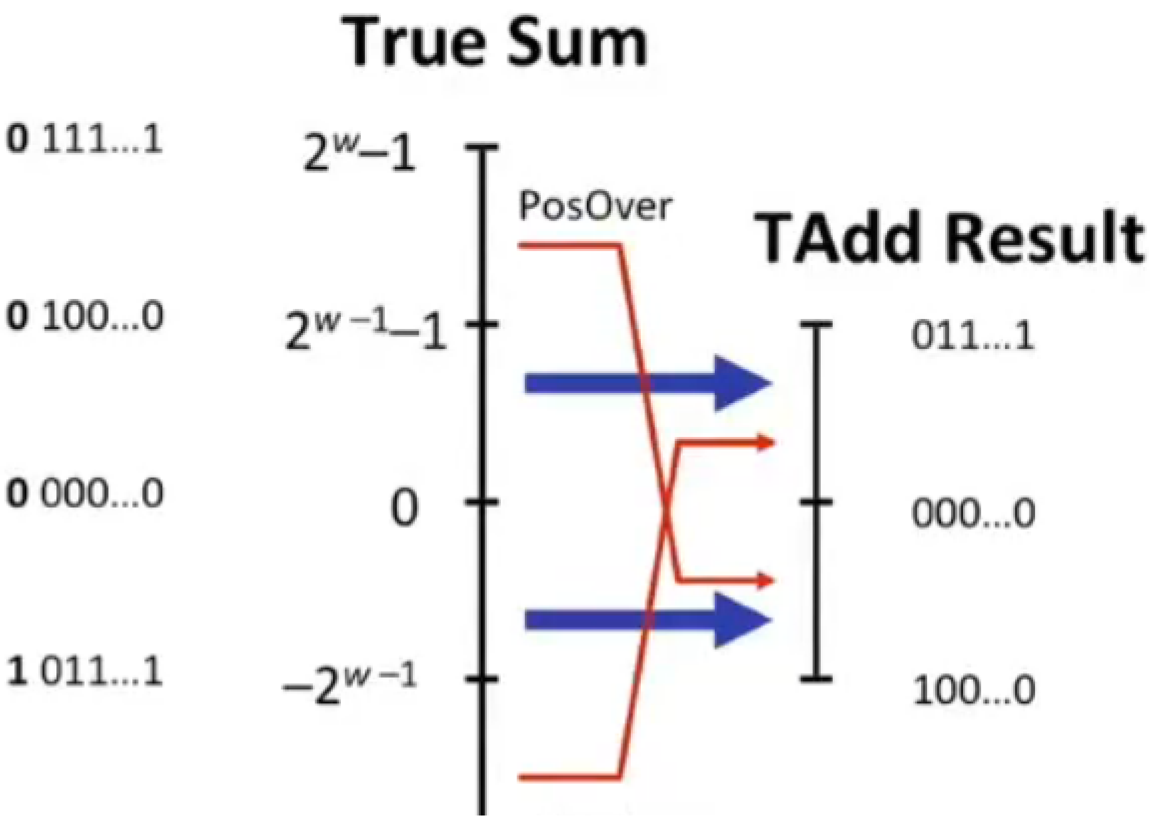


2's Complement

- 补码加法与原码加法类似，也可以通过补码加法进行减法运算（如 $5-3=5+(-3)=0101+1011=1110=-2$ ）
- 同样，补码也会有溢出，但是分为正溢出和负溢出,如：

> negative overflow： $1101+1010=0111(-3+-6=7)$ positive overflow： $0111+0101=1100111(7+5=-4)$


NegOver
TAdd4(u , v)
8
6
4
2
0
-2
-4
-6
-8
-8 -6 -4 -2 0 2 4 6
u
-8 -6 -4 -2 0 2 4 6
v
PosOve


True Sum
0 111…1
0 100…0
0 000…0
1 011…1
2^w–1
2^(w–1)–1
0
–2^(w–1)
PosOver
TAdd Result
011…1
000…0
100…0

1 000...0 -2^w $\perp$ NegOver

8. Multiplication

unsigned: $UMul_w(u,v) = (u \cdot v), mod, 2^w$

2's Complement:

- truncating 后最高位决定了最终结果是否为负数，因此会出现正数相乘得负数的情况，如:

> $5 \times 5 = 00011001 = 1001 = -7$

- 但是，将补码转换为原码后做乘法，最终的结果仍正确，如:

> $1101(-3/13) \times 1110(-2/14) = ...0110 = 6(-3 \times -2)/182(13 \times 14)$
> $1101(-3/13) \times 0010(2) = 00011010(26) = 1010(-3 \times 2)$

9. Power-of-2 Operations with Shifts

1. **Power-of-2 Multiply with Shifts** $u << k = u * 2^k$ for both unsigned and 2's Complement
2. **Power-of-2 Divide with Shifts unsigned**: $u >> k = u/2^k$, using logical shift **2's Complement**: arithmetic shift

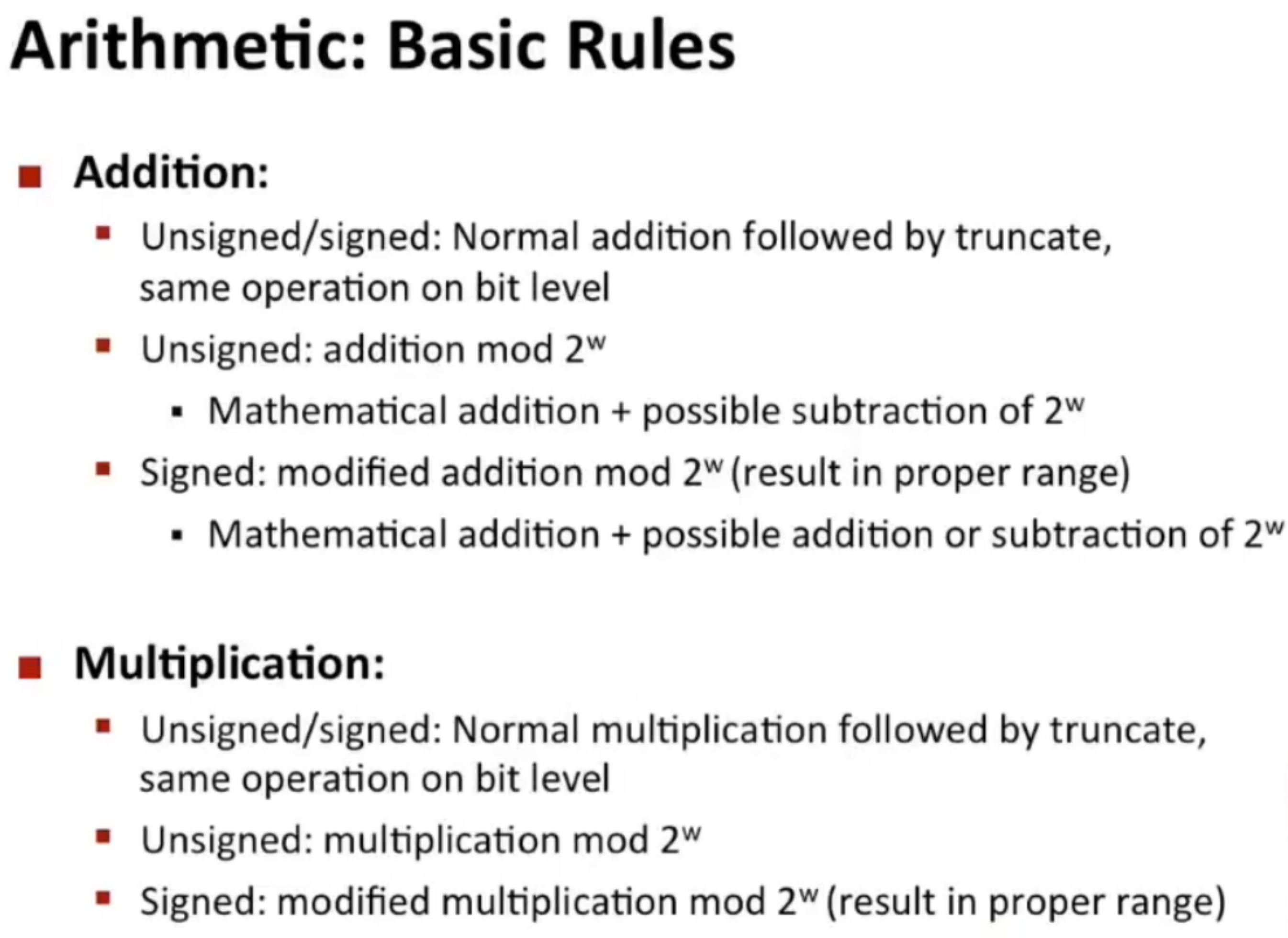
   > $-6/2 = 1010 >> 2_(arithmetic) = 1101(-3)$

   但是，

   > $-3/2 = 1101 >> 2_(arithmetic) = 1110(-2)$

   此时并没有向0舍入，而是向负无穷舍入，因此需要加一个bias（偏移量）：

   > $(-3+1)/2 = 1110 >> 2_(arithmetic) = 1111(-1)$



10. Counting Down with Unsigned

11. examples

Floating Point

1. Representatin

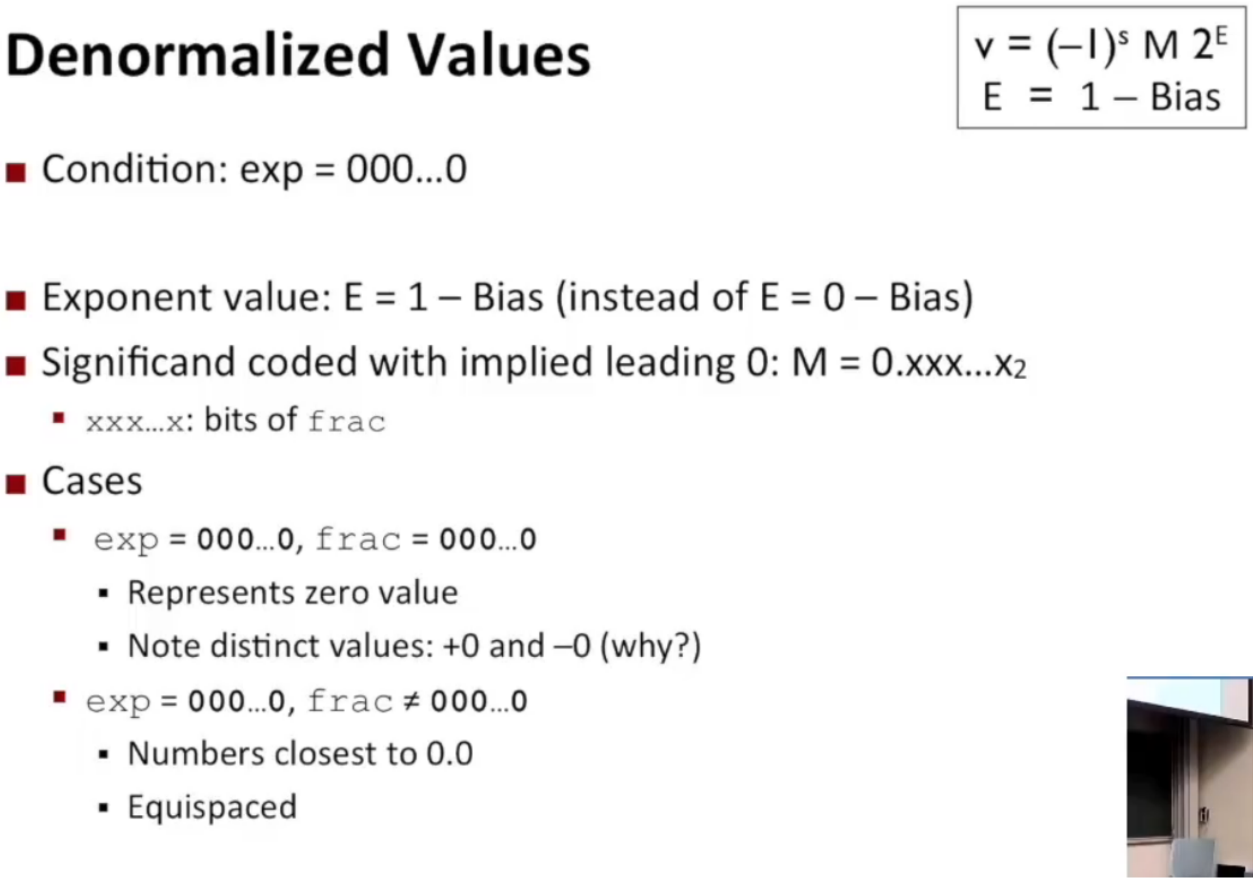
1. **Numerical Form**: $(-1)^S \cdot M \cdot 2^E$
   - S: sign bit (0 for positive, 1 for negative)
   - M: mantissa, siginificand that is normally a fraction value in range $[1.0, 2.0)$
   - E: exponent, weighing value by power of 2
2. **Encoding:**
   - MSB **s** is sign bit **S**
   - **exp** field encodes **E** (but is not equal to E)
   - **frac** field encodes **M** (but is not equal to M)

> **single** precision: 1 sign bit, 8 exponent bits, 23 fraction bits **double** precision: 1 sign bit, 11 exponent bits, 52 fraction bits **extended** precision: 1 sign bit, 15 exponent bits, 63 fraction bits

3. **Normalized Representation:**
   - $exp \neq 000...0 \ \ and \ \ exp \neq 111...1$
   - $E = exp - bias$
   - $bias = 2^{k-1} - 1$, where k is the number of exponent bits
     - Single precision: 127 $(exp: 1...254, E: -126...127)$
     - Double precision: 1023 $(exp: 1...2046, E: -1022...1023)$
   - M=1.xxxxx...x, where **x** are bits of frac field(因为默认整数部分是1所以将其省略，还能多表示一个小数位)
     - Minimun when frac=000...0($M = 1.0$)
     - Maximum when frac=111...1($M = 2.0 - \varepsilon$)

> ***Example***: `float F = 15213.0` $15213.0_{10} = 11101101101101_2 = 1.1101101101101 \times 2^{13}$ **Significand**: $M = 1101101101101_2$ $frac = 11011011011010000000000_2$ **Exponent:** $E = 13$ $bias = 127$ $exp = 140 = 10001100_2$ **Sign:** $S = 0$ **Result:**$\underbrace{0}*{s},\underbrace{10001100}*{exp},\underbrace{11011011011010000000000}_{frac}$

4. **Denormalized values:**



> you can't express 0 in normalized form for there's always a 1 before .xxx...x

5. **Special Values:**
   - $exp = 111...1, frac = 000...0$
     - represents ∞(infinity)
     - 当operation overflow时得到
     - 有$+\infty$和$-\infty$

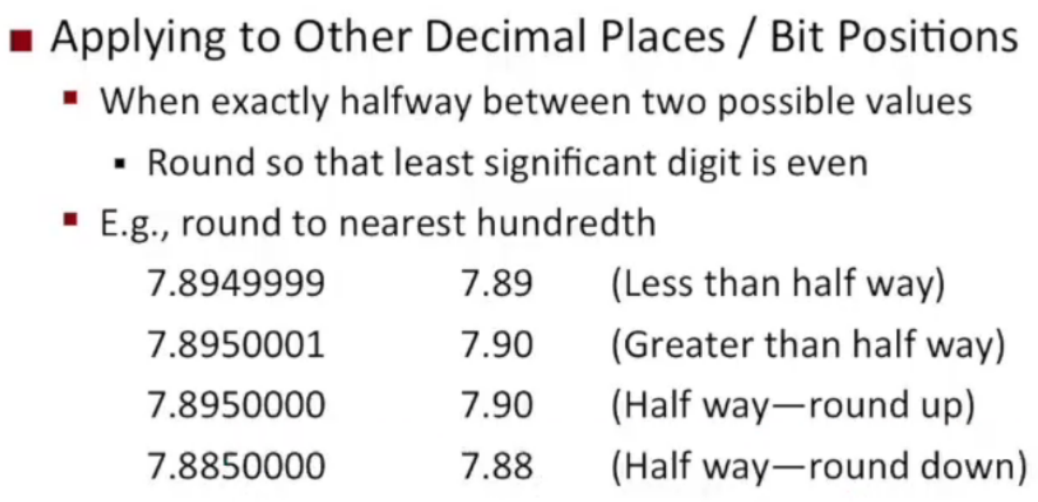
        - 溢出后进行任何操作仍是∞，不会回到正常范围，**which is different from 2's Complement case**
        - E.g., $1.0/0.0 = -1.0/-0.0 = +\infty, \quad 1.0/-0.0 = -\infty$
    - $exp = 111...1, frac \neq 000...0$
        - represents NaN(Not-a-Number)
        - get when operation is undefined(e.g., take the square root of -1)

2. Operations

1. **Basic idea:**
    - first compute exact result
    - make it fit into desired precision
        - possibly overflow if exponent too large
        - possible **round** to fit into frac
2. **Rounding Rules:**
    - towards zero
    - round down($-\infty$)
    - round up($+\infty$)
    - round-to-nearest-even(default)
    - **IEEE: 小于最近偶数的一半，就向下舍入，大于则向上舍入，正好一半(halfway)就向最近偶数舍入**

    > e.g., $1.40 \to 1, , 1.5 \to 2, , 2.5 \to 2, , -1.5 \to -2$

        - 数字均匀分布，因此向上或向下舍入的概率相等
        - **怎样找到最近偶数：若舍入位为奇就加上half**
        - 举例：十进制下百分位精度舍入



    > 第三行7.895中舍入位为9是奇数，因此加上half=0.01(hundredth)/2=0.005，得到7.90

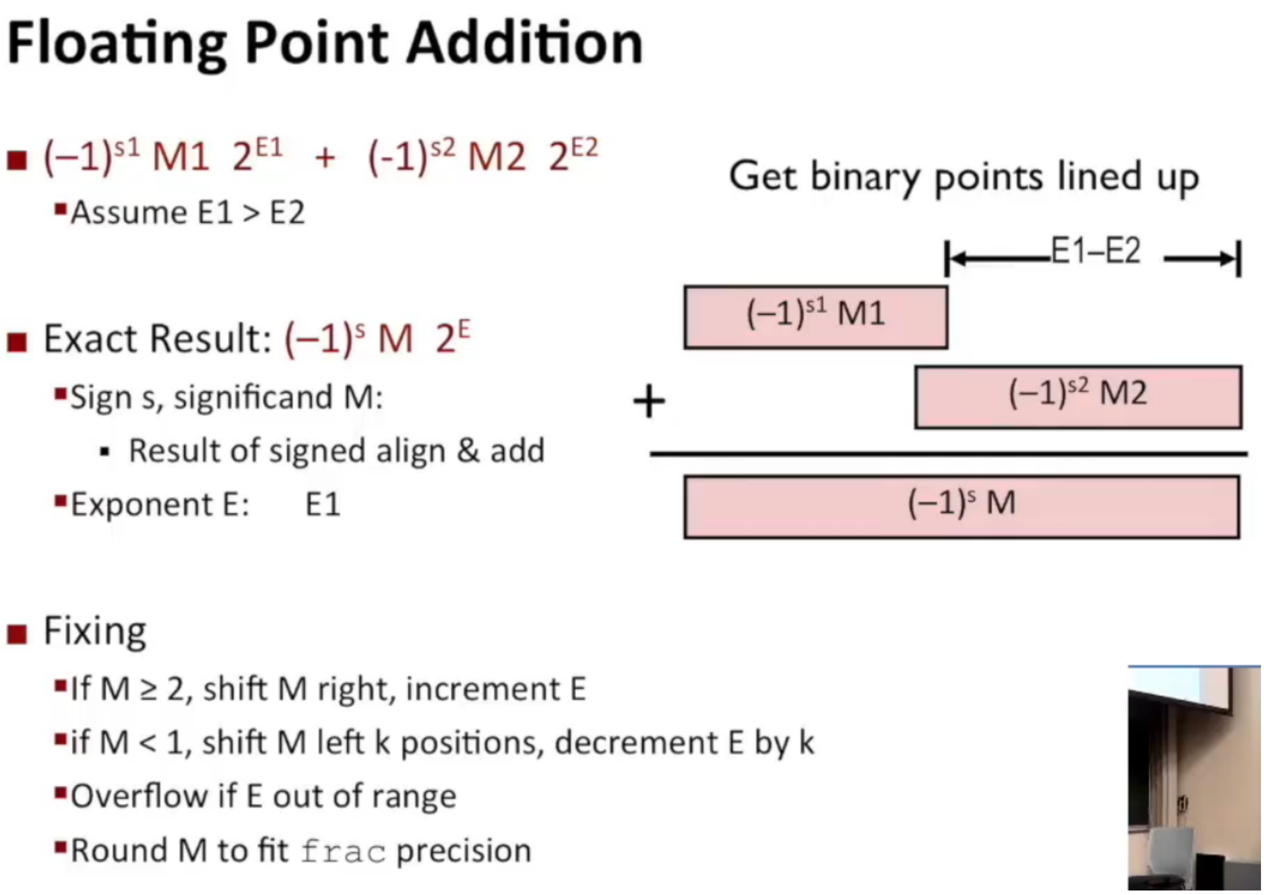
3. **Rounding Binary Numbers:**
    - "Even" when least significant bit is 0
    - "Halfway" when bits to right of rounding position = 100...0

- Easier to get nearest even：舍入位为0罢了，为1就再加1

  - **Examples**
    - Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

4. **Multiplication:** $(-1)^{s1}M_1,2^{E_1} \times (-1)^{s2}M_2,2^{E_2}$:
   - Exact result: $(-1)^s M, 2^E$
     - s: s1 ^ s2
     - M: $M_1 \times M_2$
     - E: $E_1 + E_2$
   - Fixing:
     - If $M \geq 2$, shift M right and increment E(M得是1点几)
     - If E out of range, overflow
     - Round M to fit frac presicion
5. **Addition:** 根据E对齐，然后进行计算和fixing：



6. **Mathematical Properties of FP Add:**

- 加法和乘法均满足交换律，不满足结合律(commutative but not associative)
  - Overflow and inexactness of rounding

> e.g., $(3.14+1e10)-1e10=0, 3.14+(1e10-1e10)=3.14$, $(1e10\cdot 1e10)\cdot 1e-10=inf, 1e10\cdot(1e10\cdot 1e-10)=1e10$

- 除infinity和NaN外都有加法逆元
- 除infinity和NaN外满足单调性(monotonicity)：a\geq b\Rightarrow a+c\geq b+c, \quad a\geq b,&, c\geq 0 \Rightarrow a\cdot c\geq b\cdot c
- 不满足乘法分配律
  - possibility of overflow and inexactness of rounding

> e.g., $1e20\cdot(1e20-1e20)=0.0,, 1e20\cdot 1e20-1e20\cdot 1e20=NaN$

3. Floating Point in C