Sheaves

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§1 Presheaves

Fix $\mathcal{C}$ cat, X top space, $\underline{X}$ = cat of opens of X.

Define a presheaf $\mathcal{F}: \underline{X} \longrightarrow \mathcal{C}$ by

objs: $\forall U \in \underline{X}$, $\mathcal{F}(U) \in \mathcal{C}$

Mors: $\forall V \subseteq U$, $\mathrm{Res}_{U,V} = \mathrm{Res}_{U,V}(\mathcal{F}) : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ s.t.

(i) $\forall U \in \underline{X}$, $\mathrm{Res}_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$.

(ii) $\forall W \subseteq V \subseteq U$, $\mathrm{Res}_{V,W} \circ \mathrm{Res}_{U,V} = \mathrm{Res}_{U,W} \in \mathrm{Hom}_{\underline{X}}(U,W)$.

<u>Ambiguity</u>. What is $\mathcal{F}(\phi)$? (This does not matter)

EGA avoids this by omitting the "presheaves".

$\mathcal{F}(U) :=$ section of $\mathcal{F}$ on U = $\underbrace{\Gamma(U,\mathcal{F})}_{\text{contravariant functor } \Gamma(U,\cdot)}$.

§2 Sheaves

E.g. Presheaf $\mathcal{F}: \underline{X} \longrightarrow$ Sets

$U \longmapsto \mathrm{Hom}_{\underline{Top}}(U, Y)$ by fixing $Y \in$ Top.

Special Feature: a conti. func. can be specified locally

<u>Motivation</u> $\left\{ \begin{array}{c} \mathcal{F}(V_i),\ i \in I \\ \text{agrees on intersections} \end{array} \right\} \xleftrightarrow{\approx} \left\{ \mathcal{F}(U),\ U \subseteq \bigcup V_i \right\}$

Axiom for sheaves:

(i) $s_1, s_2 \in \mathcal{F}(U)$ s.t. $s_1|_{V_i} = s_2|_{V_i}$, $\forall i \Rightarrow s_1 = s_2$. (when $\mathcal{C} = \underline{Ab}$, just check for $s_2 = 0$). } uniqueness.

(ii) $s_i \in \mathcal{F}(V_i)$ s.t. $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, $\forall i \neq j \Rightarrow \exists s \in \mathcal{F}(U)$ s.t. $s|_{V_i} = s_i$.

E.g. for sheaves:

① On manifolds: $\mathcal{F}(U) = \{\text{conti func } f: U \to Y\}$, $Y \in \mathcal{T}op$.

  $\rightsquigarrow$ $Y = C$ discrete $\Rightarrow$ $\mathcal{F}$ = loc. const sheaf.

② On differentiable mfld: $\mathcal{F}(U) = \{\text{diff func on } U\}$.

③ On cplx mflds: $\mathcal{F}(U) = \{\text{holo func on } U\}$.

④ On $X \in Var_{alg}(\bar{k})$. $\mathcal{F}$ = reg funcs = $\mathcal{O}_X$ or $\mathcal{F} = \Omega_{X/\bar{k}}$.

$\rightsquigarrow$ locally ringed spaces.

(E-Hypothesis) $\exists$ forgetful functor $\mathcal{C} \to \underline{Sets}$ reflecting small limits & colimits.

§3 Defining Sheaves on a Basis

$\mathcal{B}$ = basis of X, i.e. $\forall U \in X$, $U = \cup V_i$, $V_i \in \mathcal{B}$.

<u>Vakil's Definition</u>: $\mathcal{B}$ nice $\Leftrightarrow$ $\forall U_1, U_2 \in \mathcal{B}$, $U_1 \cap U_2 \in \mathcal{B}$.

$\rightsquigarrow$ Basic Lemma:

$$\begin{tikzcd} \underline{\mathcal{B}} \arrow[r, "\mathcal{F}_B"] \arrow[d, hook] & \mathcal{C} \\ \underline{X} \arrow[ur, dashed, "\mathcal{F}"'] & \end{tikzcd}$$

& $\mathcal{F}_B \to \mathcal{F}_B'$ also extends to $\mathcal{F} \to \mathcal{F}'$.

<u>Philosophy</u> "extending sections $s_i \rightsquigarrow s$" $\rightsquigarrow$ "extending sheaves $\mathcal{F}_i \rightsquigarrow \mathcal{F}$".

$\Rightarrow$ we can glue sheaves.

$\Rightarrow$ "Sheaf of sheaves is a sheaf".

i.e. $\forall i \in I$, $\mathcal{F}_i : \underline{U_i} \to \mathcal{C}$ satisfying axioms, $X \subseteq \cup U_i$

$\Rightarrow$ $\mathcal{F}: \underline{X} \to \mathcal{C}$ s.t. $\mathcal{F}|_{U_i} = \mathcal{F}_i$.

$\boxed{\exists! \text{ gluing}}$

§4 Stalks

Look in the nbhd of a point.

Direct system = contravariant functor $(P) \xrightarrow{F} \mathcal{C}$

directed set.

[Say $S, T \in P$, $x \in F(S)$, $y \in F(T)$. $x \sim y$ if

$$\begin{tikzcd} S \arrow[dr, "f"'] & & T \arrow[dl, "g"] \\ & U & \end{tikzcd}$$

s.t. $Ff(x) = Fg(y) \in F(U)$.

$\rightsquigarrow \varinjlim F = \bigsqcup_{S \in P} F(S)/\sim$]

e.g. R integral, $\operatorname{Frac} R = \varinjlim_{f \in R\setminus\{0\}} R[x]/(xf-1)$.

Here $P = R\setminus\{0\}$ (ordered under divisibility)

$R[x]/(xf-1) \mapsto R[x]/(xfg-1) \rightsquigarrow x \mapsto xg$.

Def'n $\mathcal{F}_x = \varinjlim \mathcal{F}$, $\mathcal{F} : P_x \to \mathcal{C}$. $\mathcal{F}_x$ = stalk at x = $\bigsqcup$ germs

$f, g \in \Gamma(U, \mathcal{F})$ defines the same germ at $x \in U$ if $f(V) = g(V)$

Caution In the case of (cplx) manifolds:

germs contain more info than values.

Also define stalk at $Z \subseteq X$ (any subset) by $\mathcal{F}_Z = \varinjlim_{U \supseteq Z} \mathcal{F}(U)$.

§5 Stalks and Morphisms

Prop $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$. $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$.

① ϕ_x inj./bij., $\forall x \iff \phi(U)$ inj./bij. $\forall U$

② ϕ_x surj. $\forall x \Rightarrow \phi(U)$ surj. $\forall U$.

E.g. $X = \mathbb{C}\setminus\{0\}$. $\mathcal{F}_1$ = sheaf of holo on X

$\mathcal{F}_2$ = sheaf of holo (nowhere vanishing)

$\rightsquigarrow \phi : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$

$(f : U \to \mathbb{C}) \mapsto (\exp \circ f : U \to \mathbb{C})$

<u>Then</u> (i) ϕ is surj b/c $\log(\exp\circ f)$ always exists <u>locally</u>

(ii) but not <u>globally</u> : $z \in \Gamma(X, \mathcal{F}_2)$, $z \notin \operatorname{im} \phi(X)$.

§6 Sheafification

$\mathcal{F} : \underline{X} \longrightarrow \mathcal{C}$ presheaf. Define $\mathcal{F}^+ : \underline{X} \to \mathcal{C}$

$$\mathcal{F}^+(U) = \left\{ s = \prod_{x \in U} s_x,\ s_x \in \mathcal{F}_x \;\middle|\; \begin{array}{l} \forall x \in U,\ \exists x \in V \subseteq U \ \& \\ t \in \mathcal{F}(V) \text{ s.t. } s_y = t_y,\ \forall y \in V \end{array} \right\}$$

$\rightsquigarrow \mathcal{F}^+$ sheaf, with $\mathcal{F}^+_x = \mathcal{F}_x$.

<u>Note</u> $(\cdot)^+$ & forgetful functor are adjoint.

§7 Direct and Inverse Image

$f : X \longrightarrow Y$ conti. $\mathcal{F} \in Sh(X)$, $\mathcal{G} \in Sh(Y)$.

$\rightsquigarrow (f_* \mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ defines $f_* \mathcal{F} \in Sh(Y)$.

And presheaf $(f^{-1}_{-})\mathcal{G} \in Sh(X)$: $(f^{-1}_{-}\mathcal{G})(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$, $U \in \underline{X}$.

$\rightsquigarrow (f^{-1}_{-})^+ = f^{-1}$ 'inverse image' functor

<u>Prop</u> f_*, f^{-1} are adjoint, i.e.

$$\mathrm{Hom}_{Sh(Y)}(f_* \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{Sh(X)}(\mathcal{F}, f^{-1}\mathcal{G})$$

Define the <u>restriction</u> of $\mathcal{F}$: $Z \subseteq X$ arbitrary,

$$\mathcal{F}|_Z = i^{-1}\mathcal{F}, \quad i : Z \hookrightarrow X.$$