Chinese Team Selection Test for IMO, 2022

Test 3 Day 1

April 12, 2022 8:00—12:30

1. Suppose a circle Γ_2 is contained in the interior of another circle Γ_1 on the plane. Prove that there exists a point P on the plane satisfying the following condition: let l be a line not going through P; suppose l intersects Γ_1 and Γ_2 at two different points A, B and two different points C, D, respectively (where A, C, D, B lie on l in order); then $\angle APC = \angle DPB$ holds.

2. If there are positive integers α, β such that $\lfloor k_1\alpha \rfloor \neq \lfloor k_2\beta \rfloor$ holds for all positive integers k_1, k_2, where $\lfloor x \rfloor$ denotes the maximal integer not exceeding x.

Prove that there exist two positive integers m_1, m_2 such that $\dfrac{m_1}{\alpha} + \dfrac{m_2}{\beta} = 1$.

3. Fix the integer $n \geqslant 2$. Find out all n-tuples $(a_1, a_2, \ldots, a_n)$ of integers satisfying the following two conditions:

(1) a_1 is odd, $1 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$, and $M = \dfrac{1}{2^n}(a_1 - 1)a_2 \cdots a_n$ is an integer;

(2) there are M different n-tuples $(c_{i,1}, c_{i,2}, \ldots, c_{i,n})$ $(i = 1, 2, \ldots, M)$, such that for all $1 \leqslant i < j \leqslant M$, there exists $k \in \{1, 2, \ldots, n\}$ such that

$$c_{i,k} - c_{j,k} \not\equiv -1, 0, 1 \pmod{a_k}.$$

Chinese Team Selection Test for IMO, 2022

Test 3 Day 2

April 13, 2022 8:00—12:30

4. Find all positive integers k such that there are finitely many triangles in the rectangular plane coordinate system whose centers of gravity are integral points; the intersection of any two triangles is either the empty set, a common vertex, or an edge joining two common vertices; and the union of these triangles is a square with side length k. (The square's vertices may not be integral points, and the edges may not be parallel to the coordinate axes).

5. Prove that there is positive real numbers C and $\alpha > \frac{1}{2}$, such that, for any positive integer n, there is a subset A of $\{1, 2, \ldots, n\}$ with $|A| \geqslant Cn^{\alpha}$, such that the difference between any two of different numbers in A is not a perfect square.

6. (1) Prove that on the complex plane, the area for the convex hull defined by all complex roots of the equation

$$z^{20} + 63z + 22 = 0$$

is larger than π.

(2) Let n be a positive integer, and $1 \leqslant k_1 < k_2 < \cdots < k_n$ be n odd integers. Prove that for any n complex numbers $a_1, a_2, \ldots, a_n$ with sum 1 and any complex number w with length not less than 1, the equation

$$a_1 z^{k_1} + a_2 z^{k_2} + \cdots + a_n z^{k_n} = w$$

has at least one complex root with norm length not exceeding $3n|w|$.

2022 年 IMO 中国国家队选拔考试

测试四　第 1 天

2022 年 4 月 16 日　8:00—12:30

1. 在 $n \times n$ ($n \geqslant 2$) 的网格屏上, 每个单位方格初始时显示红黄蓝三种颜色之一. 每一秒钟网格屏中所有单位方格按如下方式同时变换颜色, 称为一轮变换:

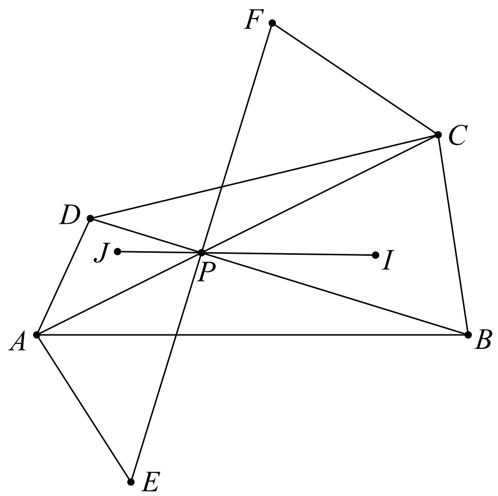
• 对当前颜色是红色的单位方格 A, 如果当前存在黄色单位方格与它有公共边, 那么下一秒钟 A 变为黄色, 否则 A 的颜色仍是红色;

• 对当前颜色是黄色的单位方格 B, 如果当前存在蓝色单位方格与它有公共边, 那么下一秒钟 B 变为蓝色, 否则 B 的颜色仍是黄色;

• 对当前颜色是蓝色的单位方格 C, 如果当前存在红色单位方格与它有公共边, 那么下一秒钟 C 变为红色, 否则 C 的颜色仍是蓝色.

证明: 如果在 $2n-2$ 轮变换后屏幕没有变成单一颜色, 那么它将永远不会变成单一颜色.

2. 在凸四边形 $ABCD$ 中, $\triangle ABC$, $\triangle ADC$ 的内心分别为 I, J. 已知 IJ, AC, BD 相交于一点 P. 过 P 且垂直于 BD 的直线与 $\angle BAD$ 的外角平分线相交于点 E, 与 $\angle BCD$ 的外角平分线相交于点 F. 证明: $PE = PF$.



3. 求所有的函数 $f: \mathbb{R} \to \mathbb{R}$ 满足: 对任意实数 x, y, 如下两个可重集相等

$$\big\{f(xf(y)+1),\ f(yf(x)-1)\big\} = \big\{xf(f(y))-1,\ yf(f(x))+1\big\}.$$

注: $\{a,b\} = \{c,d\}$ 作为可重集相等指 $a=c$, $b=d$, 或者 $a=d$, $b=c$.

2022 年 IMO 中国国家队选拔考试

测试四　第 2 天

2022 年 4 月 17 日　8:00—12:30

4. 求所有的素数 p 和正整数 a,b,c 满足

$$2^a p^b = (p+2)^c + 1.$$

5. 设 n 是正整数, $2n$ 个非负实数 $x_1, x_2, \ldots, x_{2n}$ 满足 $x_1 + x_2 + \cdots + x_{2n} = 4$. 证明: 存在非负整数 p, q 使得 $q \leqslant n-1$, 且

$$\sum_{i=1}^{q} x_{p+2i-1} \leqslant 1, \qquad \sum_{i=q+1}^{n-1} x_{p+2i} \leqslant 1.$$

注 1 : 下标按模 $2n$ 理解, 即若 $k \equiv l \pmod{2n}$, 则 $x_k = x_l$.

注 2 : 若 $q = 0$, 则第一个求和视为 0; 若 $q = n-1$, 则第二个求和视为 0.

6. 给定正整数 n, 用 D 表示 n 的所有正因子构成的集合. 设 A, B 是 D 的子集, 满足: 对任何 $a \in A$, $b \in B$, 总有 a 不整除 b 且 b 也不整除 a. 证明:

$$\sqrt{|A|} + \sqrt{|B|} \leqslant \sqrt{|D|}.$$

Solution to Test 3

1. Suppose a circle Γ_2 is contained in the interior of another circle Γ_1 on the plane. Prove that there exists a point P on the plane satisfying the following condition: let l be a line not going through P; suppose l intersects Γ_1 and Γ_2 at two different points A, B and two different points C, D, respectively (where A, C, D, B lie on l in order); then $\angle APC = \angle DPB$ holds.

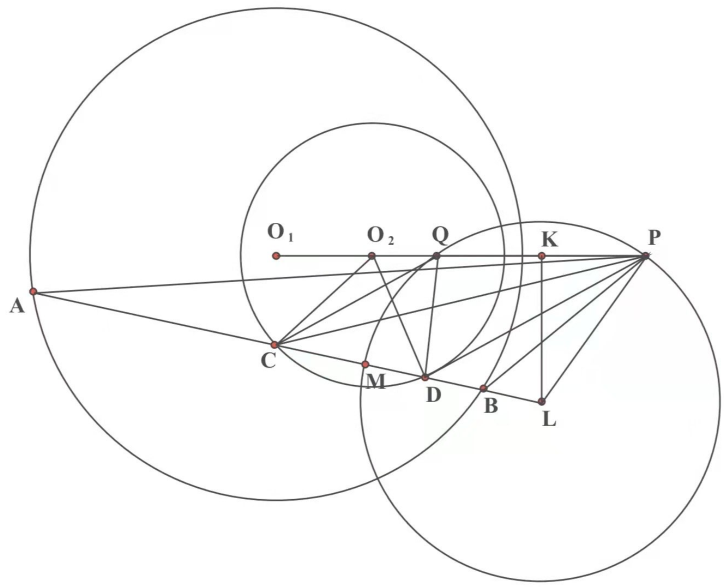
(by *Yunhao Fu*)

Proof. As shown in the diagram, let the centres of the two circles be O_1, O_2 and the radii be r_1, r_2, where $r_1 > r_2$. We first show that there are two points P, Q on the ray O_1O_2 such that $O_1P \cdot O_1Q = r_1^2$ and $O_2P \cdot O_2Q = r_2^2$.

One can choose a point K on the ray O_1O_2 such that $O_1K = \dfrac{O_1O_2^2 + r_1^2 - r_2^2}{2O_1O_2}$. From the inequality $O_1O_2 \le r_1 - r_2$, it is easy to see that $O_1K \ge r_1$. Again, one can choose two points P, Q on the line O_1O_2 such that $KP = KQ = \sqrt{O_1K^2 - r_1^2}$, and hence $O_1P \cdot O_1Q = r_1^2$. On the other hand,

$$\begin{aligned} O_2P \cdot O_2Q - O_1P \cdot O_1Q &= O_2K^2 - O_1K^2 = (O_1K - O_1O_2)^2 - O_1K^2 \\ &= O_1O_2^2 - 2O_1O_2 \cdot O_1K = r_2^2 - r_1^2, \end{aligned}$$

which deduces that $O_2P \cdot O_2Q = r_2^2$.



Note that for any given line satisfying the assumption, if the it is perpendicular to O_1O_2, the conclusion obviously holds by symmetry. If not, from $O_2C^2 = O_2Q \cdot O_2P$ we know that $\triangle O_2CQ \backsim \triangle O_2PC$, so that $\dfrac{CQ}{CP} = \dfrac{r_2}{O_2P}$. Similarly, we obtain $\dfrac{DQ}{DP} = \dfrac{r_2}{O_2P}$, and therefore $\dfrac{CQ}{CP} = \dfrac{DQ}{DP}$, which means the bisectors of $\angle CPD$ and $\angle CQD$ intersect l

at the same point, say M. Again, a similar argument shows that the bisectors of $\angle APB$ and $\angle AQB$ intersect l at the same point again, say M'.

Note that the points P, Q, M are all on an Apollonian circle with the distance ratio $\dfrac{CQ}{DQ}$ to C, D. The center of the Apollonian circle must be on l. Thus it can be nothing but the intersection of l together with the perpendicular bisector of PQ. Similarly, the points P, Q, M' are on the Apollonius circle with a distance ratio $\dfrac{AQ}{BQ}$ to A, B, whose center is the same as before. Note that K does not lie in the interior of the larger circle, so the center of this circle, denoted by L, must be outside the larger circle so that M coincides with M. Therefore,

$$\angle APC = \angle APM - \angle CPM = \angle BPM - \angle DPM = \angle BPD.$$

This completes the proof.

2. If there are positive integers α, β such that $\lfloor k_1\alpha \rfloor \neq \lfloor k_2\beta \rfloor$ holds for all positive integers k_1, k_2, where $\lfloor x \rfloor$ denotes the maximal integer not exceeding x.

Prove that there exist two positive integers m_1, m_2 such that $\dfrac{m_1}{\alpha} + \dfrac{m_2}{\beta} = 1$.

(by *Bin Wang*)

Proof. First note that $\dfrac{\beta}{\alpha}$ is an irrational number (otherwise there are positive integers k_1, k_2 such that $k_1\alpha = k_2\beta$, which leads to a contradiction).

Assume $\alpha = \dfrac{q}{p}$ is rational, then there is a positive integer k_2 such that the decimal part of $\dfrac{k_2\beta}{q}$ is less than $\dfrac{1}{q}$. Taking $k_1 = p\Big\lfloor \dfrac{k_2\beta}{q} \Big\rfloor$ to deduce that

$$k_1\alpha = q \cdot \Big\lfloor \frac{k_2\beta}{q} \Big\rfloor < k_2\beta < q\lfloor \frac{k_2\beta}{q} \rfloor + 1,$$

which contradicts to the assumption. Thus α must be irrational. Similarly, β must be irrational as well.

A pair of positive integers (a, b) is called *distinguished* if $0 < b\beta - a\alpha < 1$. From the assumption, there is a unique positive integer t such that $a\alpha < t < b\beta$. Denoting $u = t - a\alpha$, $v = b\beta - t$, we say a distinguished pair (a, b) corresponds to the *middle number* t together with a *difference pair* (u, v). We then prove the following result.

Lemma. Assume two distinguished pairs (a_1, b_1), (a_2, b_2) are in correspondence with difference pairs (u_1, v_1), (u_2, v_2), respectively. Then $\dfrac{u_1}{v_1} = \dfrac{u_2}{v_2}$.

Proof of Lemma. Suppose the middle numbers of these distinguished pairs are t_1, t_2. Assume for the sake of contradiction that $\dfrac{u_1}{v_1} > \dfrac{u_2}{v_2}$. We also take $\epsilon = \dfrac{u_1v_2 - u_2v_1}{2} > 0$.

Since $\dfrac{\beta}{\alpha}$ is an irrational number, there are positive integers a_0, b_0 such that $0 < a_0\alpha - b_0\beta < \epsilon$. On the other hand, the assumption implies that there is another integer t_0 satisfying $b_0\beta < t_0 < a_0\alpha$. Denote $u_0 = a_0\alpha - t_0$ and $v_0 = t_0 - b_0\beta$. If $\dfrac{u_1}{u_0} - \dfrac{v_1}{v_0} > 1$, then one can take $L = \Big\lfloor \dfrac{v_1}{v_0} \Big\rfloor + 1$ such that $\dfrac{u_1}{u_0} > L > \dfrac{v_1}{v_0}$, i.e.,

$$u_1 - Lu_0 = (t_1 + Lt_0) - (a_1 + La_0)\alpha > 0,$$
$$v_1 - Lv_0 = (t_1 + Lt_0) - (b_1 + Lb_0)\beta < 0.$$

Let $k_1 = a_1 + La_0$, $k_2 = b_1 + Lb_0$. And then $\lfloor k_1\alpha \rfloor = \lfloor k_2\beta \rfloor = t_1 + Lt_0 - 1$, which leads to a contradiction. This further dictates that $\dfrac{u_1}{u_0} - \dfrac{v_1}{v_0} \leqslant 1$. From a similar approach, we obtain $\dfrac{u_2}{u_0} - \dfrac{v_2}{v_0} \geqslant -1$. Therefore,

$$u_1 - \frac{u_0}{v_0}v_1 \leqslant u_0,\ \ u_2 - \frac{u_0}{v_0}v_2 \geqslant -u_0,\ \ \Rightarrow\ \ u_1v_2 - u_2v_1 \leqslant u_0(v_1 + v_2) < 2\epsilon.$$

However, this is not compatible with the definition of ϵ. So the lemma is proved.

Now we see all distinguished pairs share the same ratio $\frac{u}{v} = \frac{t - a\alpha}{b\beta - t}$, where (a, b) is any one of all distinguished pairs. Let's denote $\lambda = \frac{v}{u+v} \in (0, 1)$, and we obtain the equality $\lambda \cdot a\alpha + (1 - \lambda) \cdot b\beta = t \in \mathbb{Z}$. In the present context, the linear combinations of distinguished pairs with integer coefficients are called *nice*. For each nice pair (c, d), it is not hard to see

$$\lambda \cdot c\alpha + (1 - \lambda) \cdot d\beta = c \cdot \lambda\alpha + d \cdot (1 - \lambda)\beta \in \mathbb{Z}.$$

We then choose a nice pair (a, b) and denote that $\delta = b\beta - a\alpha \in (0, 1)$. We claim for all $M > 0$ that any pair (c, d) of integers satisfying $0 < d\beta - c\alpha < M$ and $c > \frac{a}{\delta}M$ is nice. To see this, it suffices to take $L = \left\lfloor \frac{d\beta - c\alpha}{\delta} \right\rfloor < \frac{M}{\delta} < \frac{c}{a}$, such that $(d - Lb)\beta - (c - La)\alpha = (d\beta - c\alpha) - L\delta \in (0, \delta) \subset (0, 1)$. If so, the pair $(c - La, d - Lb)$ must be distinguished as well as the pair $(c, d) = (c - La, d - Lb) + L \cdot (a, b)$ must be nice.

Now we let $M = \alpha + 2\beta$. Note that $c_0 > \frac{a}{\delta}M + 1$ and $d_0 = \left\lceil \frac{c_0\alpha}{\beta} \right\rceil$ make the condition $0 < d_0\beta - c_0\alpha < \beta$ valid. By our argument above, (c_0, d_0), $(c_0, d_0 + 1)$, and $(c_0 - 1, d_0)$ are nice pairs whose linear combinations $(0, 1)$ and $(1, 0)$ are nice as well. Therefore, $\lambda\alpha$ and $(1 - \lambda)\beta$ are (positive) integers. Via taking $m_2 = \lambda\alpha, m_1 = (1 - \lambda)\beta$ such that $\frac{m_2}{\alpha} + \frac{m_1}{\beta} = \lambda + (1 - \lambda) = 1$, or equivalently $m_1\alpha + m_2\beta = \alpha\beta$, we finish the proof.

3. Fix the integer $n \geqslant 2$. Find out all n-tuples $(a_1, a_2, \ldots, a_n)$ of integers satisfying the following two conditions:

(1) a_1 is odd, $1 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$, and $M = \dfrac{1}{2^n}(a_1 - 1)a_2 \cdots a_n$ is an integer;

(2) there are M different n-tuples $(c_{i,1}, c_{i,2}, \ldots, c_{i,n})$ $(i = 1, 2, \ldots, M)$, such that for all $1 \leqslant i < j \leqslant M$, there exists $k \in \{1, 2, \ldots, n\}$ such that

$$c_{i,k} - c_{j,k} \not\equiv -1, 0, 1 \pmod{a_k}.$$

(by *Jian Ding* and *Liang Xiao*)

Solution. The n-tuples in need are to satisfy the following condition:

$$\text{if there are exactly } r \text{ odd numbers in } a_2, \ldots, a_n, \text{ then } 2^r \mid a_1 - 1. \qquad (*)$$

Let's verify the necessity of $(*)$ at first. For this, we drop the condition $a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_1$ and assume that $a_1, \ldots, a_r$ are odd and $a_{r+1}, \ldots, a_n$ are even.

Suppose that the M tuples satisfying the requirement (2) has been retrieved. For each $s \in \mathbb{Z}$, denote $B_s = \big\{i | 1 \leqslant i \leqslant M, c_{i,n} \equiv s \pmod{a_n}\big\}$. Then

$$|B_1| + |B_2| + \cdots + |B_{a_n}| = M.$$

Consequently, there exists s such that $|B_s| + |B_{s+1}| \geq \frac{M}{a_n/2}$. This means that we can choose at least $\frac{M}{a_n/2}$ tuples such that the differences between their n-th coordinate $c_{i,n}$ are all congruent to 0 or ± 1 modulo a_n.

By running the same process for all the tuples iteratively, i.e., to consider the $n-1$-th coordinate modulo a_{n-1}, the $n-2$-th coordinate modulo a_{n-2}, etc., it eventually renders that there are at least $\dfrac{M}{\frac{a_n}{2} \cdots \frac{a_2}{2}} = \dfrac{a_1 - 1}{2}$ tuples such that for each two of them, the difference between their k-th coordinates $c_{i,k}$'s are congruent to 0 or ± 1 modulo a_k, where $2 \leqslant k \leqslant n$. However, the given conditions imply that the mutual difference between the first coordinate of these tuples can not be $0, \pm 1 \pmod{a_1}$, for which it remains to consider at most $\frac{a_1-1}{2}$ tuples. Hence all equalities hold in the argument above. Namely, for each t we have exactly $\dfrac{M}{\frac{a_n}{2} \cdots \frac{a_t}{2}}$ different tuples. In particular, taking $t = r + 1$ leads to

$$\frac{M}{\frac{a_n}{2} \cdots \frac{a_{r+1}}{2}} = \frac{1}{2^r}(a_1 - 1)a_2 \cdots a_r \in \mathbb{Z}.$$

It renders that $2^r | a_1 - 1$.

In the following context we construct the tuples under the condition $2^r | a_1 - 1$. For convenience, let's first reduce the condition $a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_1$ to $a_1 = \min\{a_1, \ldots, a_n\}$. Whenever there is any even number in $a_2, \ldots, a_n$, we say a_n is even without loss of generality. Apply the induction on n. Suppose all tuples are found for given $a_1, \ldots, a_{n-1}$, say $(c_{i,1}, c_{i,2}, \ldots, c_{i,n-1})$ $(1 \leq i \leq M' = \frac{1}{2^{n-1}}(a_1 - 1)a_2 \cdots a_n = \frac{2}{a_n}M)$. Then it is sufficient to take $(c_{i,1}, c_{i,2}, \ldots, c_{i,n-1}, 2c)$ $(1 \leq i \leq M', 1 \leq c \leq \frac{a_n}{2})$.

Now it boils down to prove for the case where all $a_2, a_3, \ldots, a_n$ are odd. Let's denote $a_1 = 2^n t + 1$ and consider the case where $a_1 = a_2 = \cdots = a_n$. Now $M = ta_1^{n-1}$. Define the function $f(x_1, x_2, \ldots, x_{n-1}) = \sum\limits_{i=1}^{n-1} 2^i x_i$ and take the following $M = ta_1^{n-1}$ tuples:

$$\big(x_1, x_2, \ldots, x_{n-1}, f(x_1, \ldots, x_{n-1}) + 2^n s\big),$$

where $x_1, \ldots, x_{n-1} \in \{1, 2, \ldots, a_1\}$, $s \in \{1, \ldots, t\}$.

The following shows that these tuples satisfy the conditions as desired. Consider the above tuple together with another one: $\big(x'_1, x'_2, \ldots, x'_{n-1}, f(x'_1, \ldots, x'_{n-1}) + 2^n s'\big)$. If for any k, the difference between their k-th coordinate $\equiv 0, \pm 1 \pmod{a_1}$, then

$$x_i - x'_i \equiv 0, \pm 1 \pmod{a_1}, \quad i = 1, \ldots, n-1; \text{ and}$$

$$\sum_{i=1}^{n-1} 2^{i-1}(x_i - x'_i) + 2^n(s - s') \equiv 0, \pm 1 \pmod{a_1}. \tag{$*$}$$

The first equality implies that $\sum\limits_{i=1}^{i-1} 2^i(x_i - x'_i)$ can be only congruent to $2^{n-1} - 1, 2^{n-2} - 1, \ldots, 1 - 2^{n-1} \pmod{a_1}$. On the other hand, $s - s' \in \{1 - t, 2 - t, \ldots, t - 1\}$. So s is forced to be equal to s' by $(*)$, and $\sum\limits_{i=1}^{n-1} 2^{i-1}(x_i - x'_i) = 0 \pmod{a_1}$. From the uniqueness of binary expression, we get $x_i = x'_i \pmod{a_1}$, which is to say these two tuples are the same. This finishes the verification that the tuples we have constructed satisfies the conditions.

As for the general case where $a_1, \ldots, a_n$ are all odd numbers, we claim if $a_2 > a_1$, the argument can be reduced to the case where $a_1, a_2 - 2, a_3, \ldots, a_n$. Hence by induction, it suffices to consider the case where all a_i are equal. However, this has been discussed as before.

Assume now there exist $M' = \frac{1}{2^n}(a_1 - 1)(a_2 - 2)a_3 \cdots a_n$ tuples as desired when $a'_1 = a_1, a'_2 = a_2 - 2, a'_3 = a_3, \ldots, a'_n = a_n$. We may assume the k-th coordinates of all these tuples lie in the set $\{0, 1, \ldots, a'_k - 1\}$.

We then define the $M = \frac{1}{2^n}(a_1 - 1)a_2 a_3 \cdots a_n$ tuples as follows. One can first choose the M' tuples that are already defined. As for $x_2 = a_2 - 2$, we choose $(x_1, a_2 - 2, x_3, \ldots, x_n)$ as a desired tuple if and only if there is a selected tuple of the form $(x_1, a_2 - 4, x_3, \ldots, x_n)$; again, for $x_2 = a_2 - 1$, we choose $(x_1, a_2 - 1, x_3, \ldots, x_n)$ if and only if there is a selected tuple of the form $(x_1, a_2 - 3, x_3, \ldots, x_n)$. It's easy to check that these tuples satisfy our requirement. Also, via the proof for $2^r | a_1 - 1$ on induction, the condition for equality implies the following conclusion. Among the M' tuples we have selected before, the number of tuples whose second coordinate is either $a_2 - 1$ or $a_2 - 2$ is $M' \cdot \frac{2}{a_2 - 2}$. Hence we can check the number of new tuples is $M' + M' \cdot \frac{2}{a_2 - 2} = M$.

4. Find all positive integers k such that there are finitely many triangles in the rectangular plane coordinate system whose centers of gravity are integral points; the intersection of any two triangles is either the empty set, a common vertex, or an edge joining two common vertices; and the union of these triangles is a square with side length k. (The square's vertices may not be integral points, and the edges may not be parallel to the coordinate axes).

(by *Zhenhua Qu*)

Solution. The desired positive integers are given by all k's divisible by 3.

We assume $k = 3t$ whenever $3 \mid k$. Consider the square with vertices $(0, 0)$, $(3t, 3t)$, $(3t, 0)$, and $(0, 3t)$. Divide it into t^2 different smaller squares with the same side length 3 along the lines $x = 3i$ $(i = 1, \ldots, t)$ and $y = 3j$ $(j = 1, \ldots, t)$. After this, each square is then divided into 2 isosceles right triangles along the diagonal, and the centre of gravity of each triangle is an integral point. This process is called a *triangulation.*

On the other hand, suppose that a square with side k has a triangulation such that the center of gravity of each small triangle is an integral point. Let V be the set of all vertices of the triangulation. Define a binary relation $A \sim_0 B$ in V if two triangles exist in the triangulation of the form $\triangle ACD$, $\triangle BCD$. The equivalence relation $\sim$ on V is generated by $\sim_0$, i.e. $A \sim B$ if and only if there exists $A_1, \ldots, A_r$ such that $A \sim_0 A_1 \sim_0 \cdots \sim_0 A_r \sim_0 B$. Let the horizontal and vertical coordinates of any point P be x_P, y_P, respectively. We have:

(i) If $A \sim B$, then $3 \mid x_A - x_B$, $3 \mid y_A - y_B$. This is due to the following reason. By transfer property, one may set $A \sim_0 B$ without loss of generality, i.e., there are two triangles in the triangulation of the form $\triangle ACD$, $\triangle BCD$, whose centers of gravity are both integral points. Therefore, $3 \mid x_A + x_C + x_D$, $3 \mid x_B + x_C + x_D$, and so $3 \mid x_A - x_B$; and similarly $3 \mid y_A - y_B$.

(ii) The set V has at most 3 equivalence classes with respect to $\sim$. The reason is as follows. After fixing a triangle T_0, for each point A in V, there is always a sequence of triangles $T_0, \ldots, T_r$ such that T_{i-1} and T_i shares a same side for all $i = 1, \ldots, r$. Moreover, A is a vertex of T_r. By definition of $\sim$, all three vertices of T_{i-1} are respectively equivalent to three vertices of T_i. Hence by induction, A is equivalent to one of the vertices of T_0.

By (ii) together with the pigeon principle, two of the four vertices of a square must be equivalent. Then from (i), we know that $3 \mid k^2$ or $3 \mid 2k^2$, which gives us $3 \mid k$ as expected.

5. Prove that there is positive real numbers C and $\alpha > \frac{1}{2}$, such that, for any positive integer n, there is a subset A of $\{1, 2, \ldots, n\}$ with $|A| \geqslant Cn^{\alpha}$, such that the difference between any two of different numbers in A is not a perfect square.

(by *Hongbing Yu*)

Proof. For $n \geq 25$, let's consider $5^{2t} \leq n < 5^{2t+2}$, where $t \in \mathbb{N}^*$. Take the set

$$A = \{(\alpha_{2t}, \ldots, \alpha_1)_5 \mid \alpha_{2i} \in \{0, 1, 2, 3, 4\} \text{ and } \alpha_{2i-1} \in \{1, 3\} \text{ for } i = 1, \ldots, t\}.$$

where $(\alpha_{2t}, \ldots, \alpha_1)_5$ denotes the number $m = 5^{2t-1}\alpha_{2t} + \cdots + 5\alpha_2 + \alpha_1$ in quinary. It's obvious that $A \subset \{1, 2, \ldots, n\}$.

For $u_1, u_2 \in A$, $u_1 = (a_{2t}, \ldots, a_1)$, $u_2 = (b_{2t}, \ldots, b_1)$, we may assume $u_1 > u_2$ and consider $u_1 - u_2$. Now let s be the minimal index such that $a_s \neq b_s$. Namely, we have $a_1 = b_1, \ldots, a_{s-1} = b_{s-1}, a_s \neq b_s$. Note that in this case, $u_1 - u_2 = (a_{2t} - b_{2t})5^{2t-1} + \cdots + (a_s - b_s)5^{s-1}$.

If $2 \mid s$, then $5^{s-1} \| (u_1 - u_2)$ i.e., $a_s - b_s \neq 0$ and $-4 \leq a_s - b_s \leq 4$. Hence $u_1 - u_2$ cannot be a perfect square.

If $2 \nmid s$, then $\frac{u_1 - u_2}{5^{s-1}} = (a_{2t} - b_{2t})5^{2t-1} + \cdots + (a_s - b_s) \in \mathbb{Z}$. Now assume that $u_1 - u_2$ is a perfect square. Since $2 \mid s - 1$, we see $\frac{u_1 - u_2}{5^{s-1}}$ is a perfect square as well. On the other hand, the condition $2 \mid s - 1$ and $a_s \neq b_s$ render that $\{a_s, b_s\} = \{1, 3\}$ and $\frac{u_1 - u_2}{5^{s-1}} \equiv 2, 3 \pmod 5$, which can never be a perfect square.

Thus for $u_1, u_2 \in A$ such that $u_1 > u_2$, the number $u_1 - u_2$ is not a perfect square. Hence A satisfies the requirement. Also, note that $|A| = 10^t$, by considering $\alpha = \log_{25} 10 > 1/2$, we obtain

$$n^{\alpha} < 5^{(2t+2)\log_{25} 10} = 10^{t+1} = 10|A|.$$

So it suffices to take $C = \frac{1}{24}$, $\alpha = \log_{25} 10 \in (0, 1)$. As for $n \leq 24$, let's take $A = \{1\}$ and then $|A| \geqslant \frac{1}{24}n \geqslant \frac{1}{24}n^{\alpha}$.

To sum up, for all $n \in \mathbb{N}^*$, one can find some subset A such that $|A| \geqslant Cn^{\alpha}$.

Remark. One may also write a similar proof in a hexadecimal sense, i.e., beginning with

$$A = \{(\alpha_t, \ldots, \alpha_1)_{16} \mid \alpha_i \in \{2, 5, 7, 13, 15\} \text{ for } i = 1, \ldots, t\}.$$

6. (1) Prove that on the complex plane, the area for the convex hull defined by all complex roots of the equation

$$z^{20} + 63z + 22 = 0$$

is larger than π.

(2) Let n be a positive integer, and $1 \leqslant k_1 < k_2 < \cdots < k_n$ be n odd integers. Prove that for any n complex numbers $a_1, a_2, \ldots, a_n$ with sum 1 and any complex number w with length not less than 1, the equation

$$a_1 z^{k_1} + a_2 z^{k_2} + \cdots + a_n z^{k_n} = w$$

has at least one complex root with modulus length not exceeding $3n|w|$.

(by *Yijuan Yao*)

Proof: (1) We first prove the following lemma.

Lemma: [Gauss-Lucas Theorem][1] All zeros of the polynomial $f'(z)$ lie in the closure formed by zeros of $f(z)$.

Proof of Lemma: By the fundamental theorem of algebra, we know that $f(z)$ admits a decomposition, read as

$$f(z) = A(z - z_1)^{\alpha_1}(z - z_2)^{\alpha_2} \cdots (z - z_n)^{\alpha_n}.$$

Thus, z_1, z_2, $\ldots$, z_n are zeros of $f'(z)$ with respective multiplicities $\alpha_1 - 1$, $\alpha_2 - 1$, $\ldots$, $\alpha_n - 1$. Say Z is any other zero of $f'(z)$, then

$$\frac{f'(Z)}{f(Z)} = \frac{\alpha_1}{Z - z_1} + \frac{\alpha_2}{Z - z_2} + \cdots + \frac{\alpha_n}{Z - z_n} = 0.$$

That is,

$$\frac{\alpha_1}{|Z - z_1|^2}(\overline{Z} - \bar{z}_1) + \frac{\alpha_2}{|Z - z_2|^2}(\overline{Z} - \bar{z}_2) + \cdots + \frac{\alpha_n}{|Z - z_n|^2}(\overline{Z} - \bar{z}_n) = 0.$$

After taking complex conjugations, the equality becomes

$$Z = \frac{\frac{\alpha_1}{|Z-z_1|^2} z_1 + \frac{\alpha_2}{|Z-z_2|^2} z_2 + \cdots + \frac{\alpha_n}{|Z-z_n|^2} z_n}{\frac{\alpha_1}{|Z-z_1|^2} + \frac{\alpha_2}{|Z-z_2|^2} + \cdots + \frac{\alpha_n}{|Z-z_n|^2}}.$$

Hence Z is a convex linear combination of z_1, z_2, $\ldots$, z_n. This proves the lemma.

Back to the original problem. By the lemma, we see the given convex hull contains the convex hull defined by all zeros of the polynomial equation

$$20z^{19} - 63 = 0.$$

It turns out to be an inscribed 19-sided regular polygon of a circle with radius $\left(\dfrac{63}{20}\right)^{1/19}$.

[1] This proposition was first used implicitly by Gauss in 1836, and the first proof was given by the French mathematician Félix Lucas in 1874. Here we essentially adopt Lucas's idea, as described by the Hungarian mathematician Lipót Fejér in his thesis *Sur la racine de moindre module d'une équation algébrique (C. R. A. S. 145 (1907), pp. 459-461)* in 1907.

The inscribed circle of this polygon has a radius

$$\begin{aligned}\left(\frac{63}{20}\right)^{1/19}\cos\frac{\pi}{19} &> \left(\frac{63}{20}\right)^{1/19}\left(1-\frac{1}{2}\left(\frac{\pi}{19}\right)^2\right)\\ &> \left(\frac{63}{20}\right)^{1/19}\left(1-\frac{1}{2}\left(\frac{1}{6}\right)^2\right)=\frac{71}{72}\left(\frac{63}{20}\right)^{1/19}.\end{aligned}$$

On the other hand, since

$$\left(\frac{72}{71}\right)^{19}=\left(1+\frac{1}{71}\right)^{19}<1+\frac{19}{71}+18\cdot C_{19}^2\frac{1}{72^2}<3<\frac{63}{20},$$

the radius of the inscribed circle is larger than 1. This proves (1).

(2) We are to show the following.

(A) Assume $a_1, a_2, \ldots, a_n$ and w satisfy the assumption. Then the minimal length for roots of

$$a_1z^{k_1}+a_2z^{k_2}+\cdots+a_nz^{k_n}=w \tag{$*$}$$

is less than or equal to $3mn$. Equivalently, it is to say:

(B) Assume $a_1, a_2, \ldots, a_n$ and w satisfy the assumption. Then the maximal length for roots of

$$wz^{k_n}-a_1z^{k_n-k_1}-a_2z^{k_n-k_2}-\cdots-a_{n-1}z^{k_n-k_{n-1}}-a_n=0, \tag{$**$}$$

denoted by M_0, is not less than $1/(3mn)$.

By the lemma in (1), the maximal length for roots of $(**)$ cannot be less than that for roots of

$$wk_nz^{k_n-1}-a_1(k_n-k_1)z^{k_n-k_1-1}-\cdots-a_{n-1}(k_n-k_{n-1})z^{k_n-k_{n-1}-1}=0. \tag{1}$$

That is, the maximal length, say M_1, for roots of

$$wk_nz^{k_{n-1}}-a_1(k_n-k_1)z^{k_{n-1}-k_1}-a_2(k_{n-1}-k_2)z^{k_{n-1}-k_2}-\cdots-a_{n-1}(k_n-k_{n-1})=0. \tag{1'}$$

Beginning with $(1')$, we repeat the same process for $n-s$ times. It reveals that the maximal norm length for the roots decreases as the number of steps increases. Therefore,

$$M_0 \geqslant M_1 \geqslant \cdots \geqslant M_{n-s},$$

where M_{n-s} is the maximal length for the roots of the equation

$$\begin{aligned}wk_nk_{n-1}\cdots k_{s+1}z^{k_s}-a_1(k_n-k_1)(k_{n-1}-k_1)\cdots(k_{s+1}-k_1)z^{k_s-k_1}\\ -\cdots-a_s(k_n-k_s)\cdots(k_{s+1}-k_s)=0. \quad ((n-s)')\end{aligned}$$

And Vieta's theorem further tells us that

$$M_0 \geqslant M_1 \geqslant \cdots \geqslant M_{n-s} \geqslant \left[\frac{(k_n - k_s)\cdots(k_{s+1}-k_s)}{k_n k_{n-1}\cdots k_{s+1}} \cdot \left|\frac{a_s}{w}\right|\right]^{1/k_s}.$$

Hence it suffices to show the existence of $s \in \{1, 2, \ldots, n\}$ that satisfies

$$\left[\frac{k_n k_{n-1}\cdots k_{s+1}}{(k_n - k_s)(k_{n-1}-k_s)\cdots(k_{s+1}-k_s)}\right]^{1/k_s} \cdot \left[\left|\frac{w}{a_s}\right|\right]^{1/k_s} \leqslant 3mn.$$

Now we note that

$$\frac{k_p}{k_p - k_s} = 1 + \frac{k_s}{k_p - k_s} \leqslant \left(1 + \frac{1}{k_p - k_s}\right)^{k_s}.$$

So it boils down to showing there exists some s such that

$$\left[\prod_{p=s+1}^{n}\left(1 + \frac{1}{k_p - k_s}\right)\right] \cdot \left(\frac{m}{|a_s|}\right)^{1/k_s} \leqslant 3mn$$

holds.[2] We also make the following observations.

- Since all k_j's are odd numbers, $k_p - k_s \geqslant 2(p-s)$. Therefore,
$$1 + \frac{1}{k_p - k_s} \leqslant 1 + \frac{1}{2(p-s)} < \frac{p-s+1}{p-s};$$
- From the relation $\sum_{j=1}^{n} a_j = 1$, we see $\sum_{j=1}^{n} |a_j| \geqslant 1$. Thus there must be s satisfying $|a_s| \geqslant \dfrac{1}{2^s}$, or equivalently, $\dfrac{1}{|a_s|} \leqslant 2^s$.

We consider the following two cases.

(1) If there is s, such that $k_s \geqslant 3$ (where s is possibly equal to 1) and $|a_s| \geqslant \dfrac{1}{2^s}$, then (notice that $s \leqslant 2s - 1 \leqslant k_s$)

$$\begin{aligned}\left[\prod_{p=s+1}^{n}\left(1 + \frac{1}{k_p - k_s}\right)\right] \cdot \left(\frac{m}{|a_s|}\right)^{1/k_s} &< \left[\prod_{p=s+1}^{n}\frac{p-s+1}{p-s}\right] \cdot m^{1/k_s} \cdot 2^{s/k_s} \\ &< n \cdot \sqrt[3]{m} \cdot 2 < 3mn.\end{aligned}$$

(2) Otherwise, there must be $k_1 = 1$ as well as $|a_1| \geqslant \frac{1}{2}$. Hence

$$\begin{aligned}\left[\prod_{p=2}^{n}\left(1 + \frac{1}{k_p - k_1}\right)\right] \cdot \frac{m}{|a_s|} &\leqslant \left[\prod_{p=2}^{n}\frac{2(p-1)+1}{2(p-1)}\right] \cdot m \cdot 2 \\ &< \sqrt{\prod_{k=1}^{2n-2}\frac{k+1}{k}} \cdot 63 \cdot 2 = \sqrt{2n-1} \cdot m \cdot 2 < 3mn.\end{aligned}$$

This finishes the proof.

[2] The above argument refers to a paper by the Hungarian mathematician M. Fekete. See *Analoga zu den Sätze von Rolle und Bolzano fÿr komplexe Polynome und Potenzreihen mit Lüchen* (*Jahr. der deutschen Math. Verieiningung, 32(1924), pp. 299-306).*

2022 年 IMO 中国国家队选拔考试测试四解答

1. 在 $n \times n$ $(n \geqslant 2)$ 的网格屏上, 每个单位方格初始时显示红黄蓝三种颜色之一. 每一秒钟网格屏中所有单位方格按如下方式同时变换颜色, 称为一轮变换:

• 对当前颜色是红色的单位方格 A, 如果当前存在黄色单位方格与它有公共边, 那么下一秒钟 A 变为黄色, 否则 A 的颜色仍是红色;

• 对当前颜色是黄色的单位方格 B, 如果当前存在蓝色单位方格与它有公共边, 那么下一秒钟 B 变为蓝色, 否则 B 的颜色仍是黄色;

• 对当前颜色是蓝色的单位方格 C, 如果当前存在红色单位方格与它有公共边, 那么下一秒钟 C 变为红色, 否则 C 的颜色仍是蓝色.

证明: 如果在 $2n-2$ 轮变换后屏幕没有变成单一颜色, 那么它将永远不会变成单一颜色.

(冷福生 供题)

证法一: 用 0, 1, 2 分别标记红黄蓝三种颜色. 对两个方格 u, v (或两个颜色), 定义

$$w(u,v)=\begin{cases}-1, & u \text{ 是红 (黄、蓝) 色而 } v \text{ 是黄 (蓝、红) 色;}\\ 1, & u \text{ 是黄 (蓝、红) 色而 } v \text{ 是红 (蓝、黄) 色;}\\ 0, & u,v \text{ 颜色相同.}\end{cases}$$

即 $w(u,v)$ 是 $\{-1,0,1\}$ 中与 $u-v$ 模 3 同余的数.

以单位方格为顶点，具有公共边的方格之间连边，构造简单图 G. 对每个单位方格 v，记 $v^{(t)}$ 为第 t 秒 v 的颜色. 对于 G 的每一个 (有向) 圈 $\alpha = v_1 \cdots v_k v_1$, 定义

$$w_t(\alpha)=\sum_{j=1}^{k} w(v_j^{(t)}, v_{j+1}^{(t)})$$

这里顶点下标 mod k 考虑. 我们证明, 在题目所述变换下, $w_t(\alpha)$ 不依赖于 t. 只需证明

$$\sum_{j=1}^{k}\left(w(v_j^{(t+1)}, v_{j+1}^{(t+1)}) - w(v_j^{(t)}, v_{j+1}^{(t)})\right)=0.$$

为此, 只需证明对每一个 j,

$$w(v_j^{(t+1)}, v_{j+1}^{(t+1)}) - w(v_j^{(t)}, v_{j+1}^{(t)}) = w(v_j^{(t+1)}, v_j^{(t)}) - w(v_{j+1}^{(t+1)}, v_{j+1}^{(t)}). \qquad (*)$$

首先注意到 $(*)$ 两边自动模 3 同余, 再者根据颜色改变规则, $(*)$ 右边每项都属于 $\{0,1\}$. 所以 $(*)$ 右边属于 $\{-1,0,1\}$. 只需证明 $(*)$ 左边不为 ± 2. 假如 $(*)$ 左边为 2, 则 $w(v_j^{(t+1)}, v_{j+1}^{(t+1)}) = 1$ 且 $w(v_j^{(t)}, v_{j+1}^{(t)}) = -1$. 不妨 $v_j^{(t)}$ 为红, 则 $v_{j+1}^{(t)}$ 为黄. 根据颜色变化规则得 $v_j^{(t+1)}$ 为黄, 故 $v_{j+1}^{(t+1)}$ 为红. 但红色格子 $v_{j+1}^{(t)}$ 不允许在下一秒变成黄色, 矛盾. 同理可证 $(*)$ 左边不为 -2, 故 $(*)$ 成立. 所以, $w_t(\alpha)$ 不依赖于 t.

显然若某个 $w_0(\alpha)\neq 0$, α 中顶点颜色永远不单一, 因此要想屏幕最终变为单一颜色, 图 G 的每个圈 α 均需满足 $w_0(\alpha)=0$. 这意味着我们可以对每个方格 v 赋值 $h_0(v)\in\mathbb{Z}$, 使得 G 中连接任意给定顶点 u,v 的任何 (有向) 路径 $\rho=uv_1\cdots v_kv$, 均满足和式

$$w_0(u,v_1)+\left(\sum_{j=1}^{k-1}w_0(v_j,v_{j+1})\right)+w_0(v_k,v)=h_0(u)-h_0(v).$$

假设 u 是使得赋值 h_0 最大的单位方格. 由赋值的最大性, u 在下一秒不变色. 由于 $w_t(\alpha)$ 不依赖于 t, 故 $w_1(\alpha)=0$ 对每个圈 α 成立. 故我们可以对每个方格 v 类似赋值 $h_1(v)$, 并满足条件 $h_1(u)=h_0(u)$. 下说明 $h_1(u)$ 仍是函数 h_1 的最大值点. 假设不是, 即存在任一点 v 使得 $h_1(v)>h_1(u)$. 选取路线 $uv_1\cdots v_kv$, 对每条边使用 $(*)$ 并相加得到

$$(h_1(u)-h_1(v))-(h_0(u)-h_0(v))=w(u^{(1)},u^{(0)})-w(v^{(1)},v^{(0)})=-w(v^{(1)},v^{(0)}).$$

由此推出必然 $h_0(u)=h_0(v)$ 且 $v^{(1)}\neq v^{(0)}$, 即 v 点也是 h_0 的最大点. 但由赋值 h_0 的定义知, v 点的颜色不改变, 矛盾!

此外, 注意到与 u 点相邻的点 v, $h_0(v)$ 只能为 $h_0(u)$ 或者 $h_0(u)-1$. 不论哪种情况, 第 1 秒后其颜色与点 u 相同.

归纳使用上述结论知赋值 h_0 的极大值点 u 一直不变色且可以对任意时间 t 给出类似赋值 h_t 使得 $h_t(u)=h_0(u)$. 现在对任意方格 v, 若存在路径 $\rho=uv_1\cdots v_kv$, 则最多 $k+1$ 秒后, v 将变成 u 的颜色. 由于屏幕上任何两个单位方格的距离 (连接两者的最短路径长度) 不超过 $2n-1$, 因此, 若屏幕最终变为单一颜色, 最后用时不超过 $2n-2$ 秒. 证毕.

证法二: 先证明断言: 如果最终屏幕变成单一颜色, 不妨设为蓝色, 那么必存在某一方格始终为蓝色.

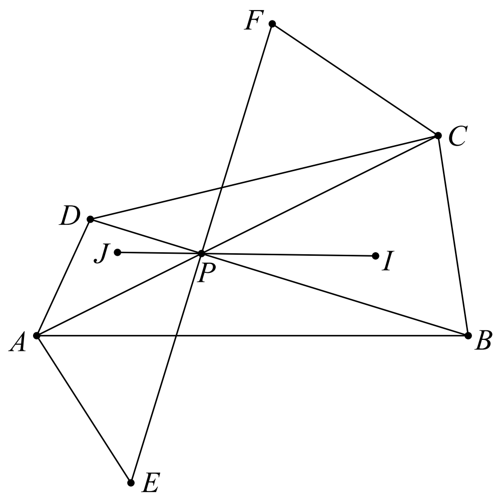
反证之, 假设最终方格全为蓝色, 但每个方格均改变过颜色. 我们构造有向图 G 如下: 以所有方格为顶点, 连接有向边 $A\to B$, 如果方格 A 和 B 相邻且存在时刻 t_0 使得: 方格 A 在 t_0-1 时不为蓝色, 而在 $t\geqslant t_0$ 时恒为蓝色; 方格 B 在 t_0-1 时为蓝色 (换言之, 方格 A 最后一次变色是由方格 B 导致的). 由于我们假定每个方格均改变过颜色, 故图 G 中每点的出度都 $\geqslant 1$, 进而图 G 中必存在一个圈 $A_1\to A_2\to\cdots\to A_k\to A_1$. 设 A_i 在 t_i 轮变换后最终变蓝, 即 A_i 在 t_i-1 时不是蓝色, 在 $t\geqslant t_i$ 时恒为蓝色. 不妨设 t_1 是 $t_1,\ldots,t_k$ 中最大者. 由 $A_1\to A_2$ 知, 在 t_1-1 时 A_1 为黄色; 由 $A_k\to A_1$ 知, 在 t_k-1 时 A_1 为蓝色, 因此一定有 $t_k<t_1$. 然而蓝色方格必须经由红色才能变成黄色, 即存在 $t_k<t'<t_1$, 使得 A_1 在 $t'-1$ 时为红色. 但是由于 $t'-1\geqslant t_k$, A_k 在 $t'-1$ 时是蓝色, 于是 A_1 迫使 A_k 在 t' 时变为红色. 这同 t_k 的定义, 即 A_k 从 t_k 时刻开始恒为蓝色, 相矛盾.

回到原题: 假设最终方格全为蓝色, 证明 $2n-2$ 轮变换后全为蓝色.

根据断言, 存在一个方格 A 始终为蓝色. 定义两方格的距离为水平距离与竖直距离之和, 则两方格的最远距离为 $2n-2$. 若 B 与 A 距离为 1, 则 B 始终不能为红色, 这意

味着: 若 B 初始为蓝色, 则 B 始终为蓝色; 若 B 初始为黄色, 则 B 在一轮变换后固定为蓝色. 现在对从某点 B 到点 A 的距离 k 从 1 到 $2n-2$ 归纳证明 B 在 k 轮变换后固定为蓝色. 假设此结论已经对 $k-1$ 证明, 若 B 与 A 距离为 k, 取与 B 相邻的点 B' 与 A 距离为 $k-1$, 由归纳知 B' 在 $k-1$ 轮变换后固定为蓝色, 那么同样的论证可以说明 B 在 k 轮变换后固定为蓝色. 此归纳证明推出所有方格在 $2n-2$ 轮变换后全变为蓝色. 证毕.

2. 在凸四边形 $ABCD$ 中, $\triangle ABC$, $\triangle ADC$ 的内心分别为 I, J. 已知 IJ, AC, BD 相交于一点 P. 过 P 且垂直于 BD 的直线与 $\angle BAD$ 的外角平分线相交于点 E, 与 $\angle BCD$ 的外角平分线相交于点 F. 证明: $PE = PF$.

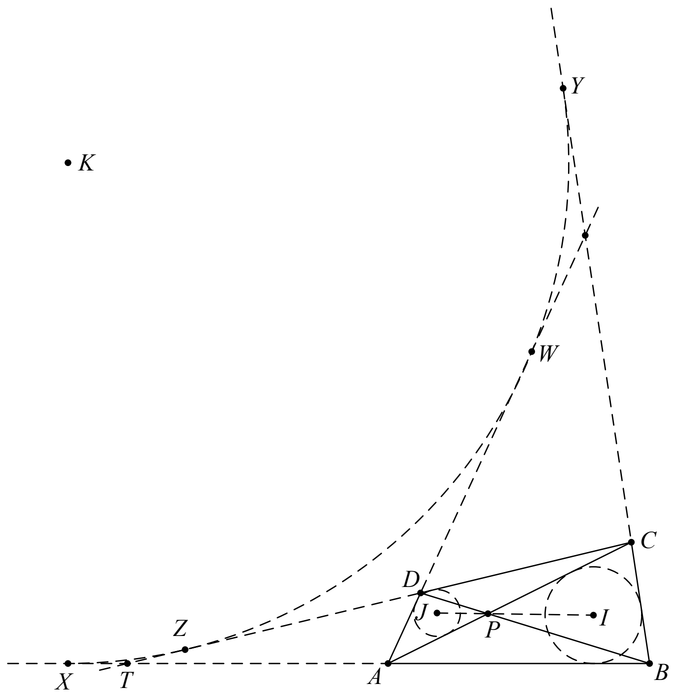


(林天齐 供题)

证法一: 若 $AB//CD$ 且 AD/BC, 则 $ABCD$ 为平行四边形. 此时, P 为 AC 中点且 $AE//CF$, 故 $PE = PF$.

以下假设 AB 与 CD 不平行. 先证明 $AB + AD = CB + CD$.

如图所示, 不妨设 BA, CD 的延长线相交于点 T. 作 $\triangle TBC$ 的 B-旁切圆 $\odot K$, 它们分别与直线 AB, BC, CD 相切于点 X, Y, Z.



因为 $\odot I, \odot J$ 的内位似中心在直线 IJ 上, 而 AC 是它们的一条内公切线, 故结合已知条件可知, $\odot I, \odot J$ 的内位似中心是 P.

熟知 $\odot I$ 与 $\odot J$ 的内位似中心 P, $\odot I$ 与 $\odot K$ 的外位似中心 B, $\odot J$ 与 $\odot K$ 的内位似中心, 三点共线, 故 $\odot J$ 与 $\odot K$ 的内位似中心在直线 BP 上. 又 CD 是 $\odot J$ 与 $\odot K$ 的一条内公切线, 故 $\odot J$ 与 $\odot K$ 的内位似中心是 D, 进而 AD 也与 $\odot K$ 相切, 记切点为 W.

由切线长定理可得

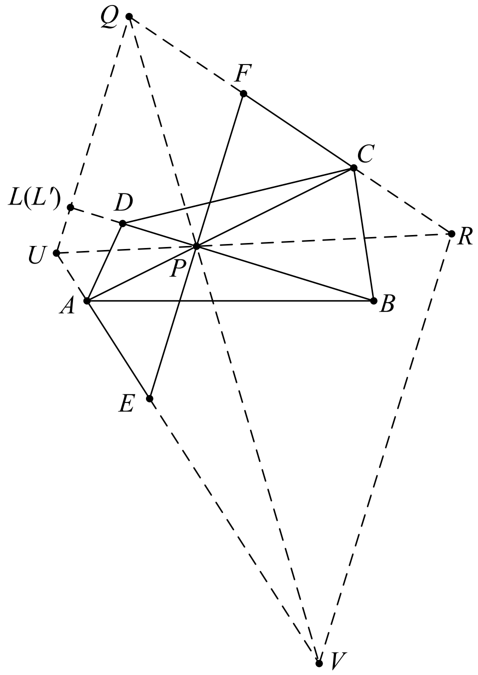
$$\begin{aligned} AB + AD &= (BX - AX) + (AW - DW) = BX - DW \\ &= BY - DZ = (CB + CY) - (CZ - CD) = CB + CD. \end{aligned}$$

即

$$AB + AD = CB + CD. \tag{1}$$

设 $\triangle ABD$ 的 B-旁切圆, D-旁切圆分别是 $\odot U$, $\odot V$; $\triangle BCD$ 的 B-旁切圆, D-旁切圆分别是 $\odot Q, \odot R$.

下面证明 UR, VQ 相交与点 P. 考虑 $\odot K, \odot U, \odot R$, 熟知 $\odot K$ 与 $\odot U$ 的外位似中心 A, $\odot K$ 与 $\odot R$ 的内位似中心 C, $\odot U$ 与 $\odot R$ 的内位似中心, 三点共线. 而 BD 是 $\odot U$ 与 $\odot R$ 的内公切线, 故 $\odot U$ 与 $\odot R$ 的内位似中心是 P. 进而 UR 过点 P. 同理 VQ 也过点 P.



再证明 $UQ \perp BD, VR \perp BD$. 设 $\odot U, \odot Q$ 分别与 BD 的延长线相切于点 L, L', 则

$$BL = \frac{1}{2}(AB + AD + BD),\ BL' = \frac{1}{2}(CB + CD + BD).$$

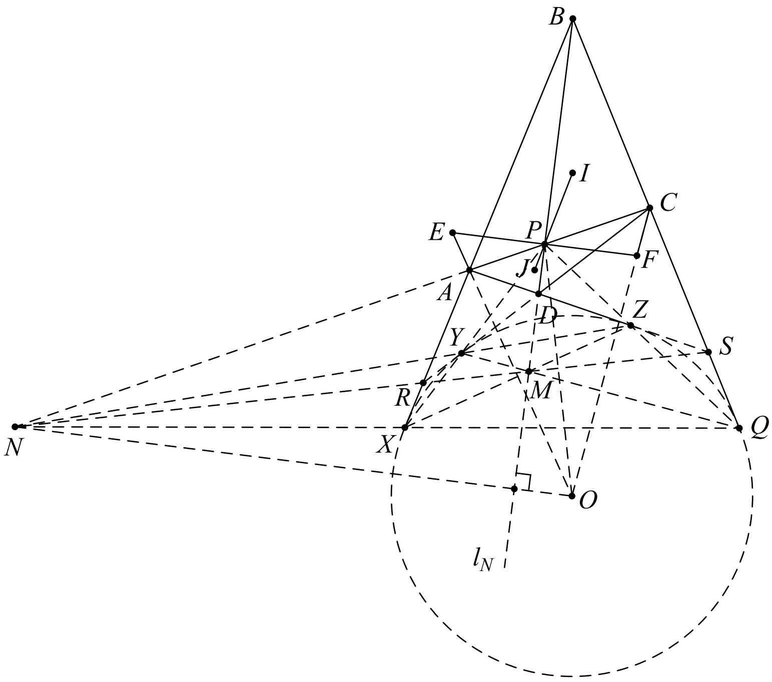
结合(1)式可知 $BL = BL'$, 故 L, L' 重合. 因此 $UQ \perp BD$. 同理 $VR \perp BD$.

又 $EF \perp BD$, 故 $UQ // EF // VR$. 于是

$$\frac{PE}{UQ} = \frac{VP}{VQ} = \frac{RF}{RQ} = \frac{PF}{UQ},$$

因此 $PE=PF$. 证毕.

证法二: (根据陈梓青同学的解答整理)



若 $AB//CD$ 且 $AD//BC$, 同证法一易证. 以下不妨假设 AB 与 CD 不平行. 同证法一得到四边形 $ABCD$ 有旁切圆 $\odot O$, 与四边所在直线 AB, BC, CD, DA 分别相切于点 X, Q, Y, Z. 设 $AD\cap BC=S$, $AB\cap CD=R$.

由圆外切四边形在旁切圆下的牛顿定理知: AC, BD, XY, ZQ 四线交于点 P; AC, RS, YZ, XQ 四线交于点 N; BD, RS, XZ, QY 四线交于点 M.

此时考虑圆内接四边形 $XYZQ$ 可知, 点 M, P 皆位于点 N 关于 $\odot O$ 的极线 ℓ_N 上, 即 $MP=\ell_N$. 故 $ON\perp PM$, 即 $ON\perp BD$. 在完全四边形 $BARDSC$ 中, $A, C; P, N$ 成调和点列, 故 $ON, OP; OA, OC$ 成调和线束. 因为 $EF\perp BD$, 故 $EF//NO$, 从而 P 为 EF 中点. 证毕.

3. 求所有的函数 $f:\mathbb{R}\to\mathbb{R}$ 满足: 对任意实数 x,y, 如下两个可重集相等

$$\big\{f(xf(y)+1),\ f(yf(x)-1)\big\}=\big\{xf(f(y))-1,\ yf(f(x))+1\big\}. \qquad (*)$$

注: $\{a,b\}=\{c,d\}$ 作为可重集相等指 $a=c$, $b=d$, 或者 $a=d$, $b=c$.

(肖梁 供题)

解答: 所求的所有函数为 $f(x)=x$ 和 $f(x)=-x$. 易验证这两个函数满足(*).

以下所有集合理解为可重集. 在(*)中取 $x=y=0$ 得 $\{f(1),f(-1)\}=\{1,-1\}$. 先考虑 $f(1)=1$ 的情形. 往证 $f(x)=x$ $(x\in\mathbb{R})$.

第 1 步: 证明 $f(n)=n$ $(n\in\mathbb{Z})$.

首先说明 $f(0)=0$. 在(*)中取 $x=0$ 得 $\big\{f(1),\ f(yf(0)-1)\big\}=\big\{-1,\ yf(f(0))+1\big\}$. 若 $f(0)\neq 0$, 在上式取 $y=\dfrac{2}{f(0)}$, 左边为 $\{1,\ 1\}$ 不可能包含右边的 -1.

下面归纳证明 $f(n)=n$ $(n\in\mathbb{N})$. $n=1$ 已知. 假设 $f(m)=m$ 对 $m\leqslant n$ 成立. 在(*)中取 $x=1$, $y=n$ 得

$$\big\{f(n+1),\ f(n-1)\big\}=\big\{n-1,n+1\big\}$$

由此推出 $f(n+1)=n+1$, 完成归纳证明. 用类似的方法取 $x=1$, $y=-n$ 可以归纳证明 $f(-n)=-n$ 对 $n\in\mathbb{N}$ 成立. 这完成第 1 步.

第 2 步: 证明 f 是双射.

f 是满射因为在(*)中取 $y=1$ 右边有一项 $xf(f(1))-1=x-1$ 可以取得所有实数. 根据左边的形式, 它一定在 f 的像集内.

再证 $f(y_0)=0$ 推出 $y_0=0$. 反设 $y_0\neq 0$, 在(*)中取 $y=y_0$ 得

$$\big\{f(1),\ f(y_0f(x)-1)\big\}=\big\{-1,\ y_0f(f(x))+1\big\}.$$

必然有 $1=f(1)=y_0f(f(x))+1$. 所以 $f(f(x))=0$ 对所有 x 成立. 这与 $f(1)=1$ 矛盾.

下证 f 是单射. 若 $y_1\neq y_2$ 满足 $f(y_1)=f(y_2)\neq 0$. 在(*)中分别取 $y=y_1$ 和 $y=y_2$ 得

$$\begin{aligned}\big\{f(xf(y_1)+1),\ f(y_1f(x)-1)\big\} &= \big\{xf(f(y_1))-1,\ y_1f(f(x))+1\big\},\\ \big\{f(xf(y_2)+1),\ f(y_2f(x)-1)\big\} &= \big\{xf(f(y_2))-1,\ y_2f(f(x))+1\big\}.\end{aligned}$$

注意到上面两式中左右两边的第一个元素都对应相等.

若某个 $x_0\in\mathbb{R}$ 使得 $f(x_0f(y_1)+1)\neq x_0f(f(y_1))-1$, 则

$$y_1f(f(x_0))+1=f(xf(y_1)+1)=f(xf(y_2)+1)=y_2f(f(x_0))+1$$

由此得 $y_1f(f(x_0))=y_2f(f(x_0))$, 得 $f(f(x_0))=0$ 推出 $x_0=0$.

所以若 $x\neq 0$, $f(xf(y_1)+1)=xf(f(y_1))-1$. 由此得到 $f(x)=ax+b$ $(a,b\in\mathbb{R},\ x\neq 1)$ 是线性函数. 但 $f(n)=n$ $(n\in\mathbb{Z})$. 得到 $a=1$, $b=0$. 故 $f(x)=x$ 亦是单射.

第 3 步: 证明对任意 $n\in\mathbb{Z}$, $y\in\mathbb{R}$, 有

$$f\Big(f\Big(\frac{n}{f(y)}\Big)\Big)=\frac{n}{y}. \tag{1}$$

只需考虑 $n\neq 0$ 且 $y\neq 0$ 的情形. 在(*)中取 $x=\dfrac{n}{f(y)}$ 得

$$\Big\{n+1,\ f\Big(yf\Big(\frac{n}{f(y)}\Big)-1\Big)\Big\}=\Big\{n\frac{f(f(y))}{f(y)}-1, yf\Big(f\Big(\frac{n}{f(y)}\Big)\Big)+1\Big\}. \tag{2}$$

同样在(*)中用 y 代替 x, $\dfrac{n}{f(y)}$ 代替 y 得

$$\Big\{n-1,\ f\Big(yf\Big(\frac{n}{f(y)}\Big)+1\Big)\Big\}=\Big\{n\frac{f(f(y))}{f(y)}+1, yf\Big(f\Big(\frac{n}{f(y)}\Big)\Big)-1\Big\}. \tag{3}$$

若某个 $y_0\in\mathbb{R}\backslash\{0\}$ 使得 $n\neq y_0f\Big(f\Big(\dfrac{n}{f(y_0)}\Big)\Big)$, 则由(2)和(3)分别得

$$n+1=n\frac{f(f(y_0))}{f(y_0)}-1, \qquad n-1=n\frac{f(f(y_0))}{f(y_0)}+1.$$

取两式的差得到 $2=-2$ 矛盾.

第 4 步: 对 $\alpha\in\mathbb{Q}$ 和 $y\in\mathbb{R}$, 证明 $f(\alpha y)=\alpha f(y)$.

只需讨论 $\alpha\neq 0$ 且 $y\neq 0$ 的情况. 在(1)中以 $f\Big(\dfrac{m}{f(y)}\Big)$ $(m\in\mathbb{Z}\backslash\{0\})$ 代替 y 得到

$$f\Big(f\Big(\frac{n}{f\big(f\big(\frac{m}{f(y)}\big)\big)}\Big)\Big)=\frac{n}{f\big(\frac{m}{f(y)}\big)}.$$

再带入(1)到左边括号内的分母中

$$f\Big(f\Big(\frac{n}{m}y\Big)\Big)=\frac{n}{f\big(\frac{m}{f(y)}\big)}$$

再在上式中 $\dfrac{t}{f(y)}$ $(t\in\mathbb{Z}\backslash\{0\})$ 代替 y 得到

$$\frac{nt/m}{y}=f\Big(f\Big(\frac{n}{m}\frac{t}{f(y)}\Big)\Big)=\frac{n}{f\big(\frac{m}{f(\frac{t}{f(y)})}\big)}$$

整理得 $f\Bigg(\dfrac{m}{f(\frac{t}{f(y)})}\Bigg)=\dfrac{m}{t}y$. 对两边取 f 得到

$$\frac{m}{t}f(y)=f\Big(f\Big(\frac{m}{f(\frac{t}{f(y)})}\Big)\Big)=f\Big(\frac{m}{t}y\Big).$$

完成第 4 步的证明.

第 5 步: 证明 $f(y+a)=f(y)+a$ 对 $y\in\mathbb{R}$ 和 $a\in\mathbb{Q}$ 成立.

在(*)中取 $x=\dfrac{1}{r}\in\mathbb{Q}\backslash\{0\}$ 得

$$\Big\{f\Big(\frac{1}{r}f(y)+1\Big),\ f\Big(\frac{1}{r}y-1\Big)\Big\}=\Big\{\frac{1}{r}f(f(y))-1,\ \frac{1}{r}y+1\Big\}$$

利用第 4 步并两边乘以 r 得

$$\big\{f(f(y)+r),\ f(y-r)\big\}=\big\{f(f(y))-r,\ y+r\big\}. \tag{4}$$

我们往证 $f(y-r)=f(f(y))-r$. 假若有某个 $y_0\in\mathbb{R}, r_0\in\mathbb{R}\backslash\{0\}$ 使得 $f(y_0-r_0)=y_0+r_0$. 在(4)中取 $y=y_0-r_0, r=-2r_0$ 得

$$\big\{f(f(y_0-r_0)-2r_0),\ f(y_0-r_0+2r_0)\big\}=\big\{f(f(y_0-r_0))+2r_0,\ y_0-3r_0\big\}.$$

整理得

$$\big\{y_0+r_0,\ f(y_0+r_0)\big\}=\big\{f(y_0+r_0)+2r_0,\ y_0-3r_0\big\}.$$

因为左边的 $y_0+r_0\neq y_0-3r_0$, 故 $y_0+r_0=f(y_0+r_0)+2r_0$, 即 $f(y_0+r_0)=y_0-r_0$. 但这样左右两边变成 $\big\{y_0+r_0,\ y_0-r_0\big\}=\big\{y_0+r_0,\ y_0-3r_0\big\}$. 它们显然不等, 矛盾.

所以我们得到 $f(y-r)=f(f(y))-r$ 对所有 $y\in\mathbb{R}, r\in\mathbb{Q}\backslash\{0\}$ 成立. 取两个不同的 r 相减易得 $f(y+a)=f(y)+a$ 对所有 $y\in\mathbb{R}, a\in\mathbb{Q}$ 成立.

<u>第 6 步</u>: 证明 $f(y)=y$ $(y\in\mathbb{R})$

结合第 5 步和(4)得到

$$\big\{f(f(y))+r,\ f(y)-r\big\}=\big\{f(f(y))-r,\ y+r\big\}$$

两边集合元素分别求和得到 $f(f(y))+f(y)=f(f(y))+y$, 即 $f(y)=y$.

现在考虑 $f(1)=-1$ 的情形. 往证 $f(x)=-x$ $(x\in\mathbb{R})$

<u>第 1 步</u>: 证明 $f(n)=-n$ $(n\in\mathbb{Z})$.

在(*)中取 $y=1$ 和分别取 $x=1,-1$ 得

$$\big\{f(0),\ f(-2)\big\}=\big\{0,\ 2\big\},\quad \big\{f(2),\ f(0)\big\}=\big\{-2,\ 0\big\}.$$

由此得 $h(0)=0, h(2)=2, h(-2)=-2$.

下面对 $|n|$ 进行归纳法证明 $f(n)=-n$. $|n|=1,2$ 已证. 假设已经证明 $|n|\leqslant n_0$ $(n_0\geqslant 2)$ 时已证. 在(*)中取 $y=1$ 和分别取 $x=n_0,-n_0$ 得

$$\begin{aligned}\big\{f(-n_0+1),\ f(f(n_0)-1)\big\} &= \big\{n_0-1,\ f(f(n_0))+1\big\},\\ \big\{f(n_0+1),\ f(f(-n_0)-1)\big\} &= \big\{-n_0-1,\ f(f(-n_0))+1\big\}\end{aligned}$$

由此可导出 $f(n_0+1)=-n_0-1, f(-n_0-1)=n_0+1$. 这完成第 1 步的归纳证明.

<u>第 2 步</u>: 证明 $f(x)=-x$ 对所有 $x\in\mathbb{R}$ 成立.

首先和 $f(1)=1$ 的情形一样可以证明 f 为满射.

对非零整数 n 和 $z\in\mathbb{R}\backslash\{0\}$ 在(*)中分别取 $x=\dfrac{n}{f(z)}$, $y=z$, 以及 $x=z$, $y=\dfrac{n}{f(z)}$ 得到

$$\begin{aligned}\Big\{-n-1,\ f\Big(zf\Big(\frac{n}{f(z)}\Big)-1\Big)\Big\}&=\Big\{n\frac{f(f(z))}{f(z)}-1,zf\Big(f\Big(\frac{n}{f(z)}\Big)\Big)+1\Big\},\\ \Big\{-n+1,\ f\Big(zf\Big(\frac{n}{f(z)}\Big)+1\Big)\Big\}&=\Big\{n\frac{f(f(z))}{f(z)}+1,zf\Big(f\Big(\frac{n}{f(z)}\Big)\Big)-1\Big\}.\end{aligned}\tag{5}$$

若对某个 $z=z_0\in\mathbb{R}\backslash\{0\}$, $f(f(z_0))\neq -f(z_0)$, 由(5)中两式得

$$-n-1=zf\Big(f\Big(\frac{n}{f(z)}\Big)\Big)+1,\qquad -n+1=zf\Big(f\Big(\frac{n}{f(z)}\Big)\Big)-1.$$

两式相减得 $2=-2$. 矛盾.

所以 $f(f(z_0))=-f(z_0)$. 但 f 是满射. 所以 $f(x)=-x$ 对 $x\in\mathbb{R}\backslash\{0\}$ 成立. 而 $f(0)=0$ 已证.

综合两种情况得, 所求的函数为 $f(x)=x$ $(x\in\mathbb{R})$ 和 $f(x)=-x$ $(x\in\mathbb{R})$.

4. 求所有的素数 p 和正整数 a,b,c 满足

$$2^a p^b = (p+2)^c + 1.$$

(瞿振华 供题)

解答: 显然 p 是奇数, $p \geq 3$. 若 $c=1$, 则

$$p+3 = 2^a p^b \geq 2p \geq p+3,$$

只能 $p=3$, $a=b=1$, 得一组解 $(p,a,b,c)=(3,1,1,1)$. 以下假设 $c \geq 2$.

情形 1: c 是奇数. 设 q 是 c 的一个素因子, 由于 $(p+2)^q+1 \mid (p+2)^c+1$, 故

$$(p+2)^q + 1 = 2^\alpha p^\beta. \tag{1}$$

显然 $\alpha > 0$. 注意到 $(p+2)^q+1 = (p+3)A$, 其中

$$A = (p+2)^{q-1} - (p+2)^{q-2} + \cdots + 1 > (p+2)^{q-1} - (p+2)^{q-2} = (p+2)^{q-2}(p+1) > p^{q-1},$$

且 A 是奇数, 故 A 是 p 的方幂, $A \geq p^q$, $\beta \geq q$. (1) 两边模 p 得 $2^q \equiv -1 \pmod{p}$, 故 2 模 p 的阶为 2 或 $2q$.

若 2 模 p 的阶为 2, 则 $p=3$. 此时(1)为 $5^q+1 = 2^\alpha 3^\beta$. 由于 $5^q+1 \equiv 2 \pmod{4}$, 故 $\alpha=1$. 又由升幂定理, $v_3(5^q+1) = v_3(5+1) + v_3(q) \leq 2$, 故 $\beta \leq 2$. 检验 $\beta = 1,2$ 可知均不合要求.

若 2 模 p 的阶为 $2q$, 则有 $2q \mid p-1$, 从而 $q \leq \frac{p-1}{2} < \frac{p}{2}$. 在(1)两边除以 p^q, 并利用 $(1+\frac{1}{x})^x < e$, $x \geq 1$, 有

$$2^\alpha p^{\beta-q} = \left(1+\frac{2}{p}\right)^q + p^{-q} < \left(1+\frac{2}{p}\right)^{\frac{p}{2}} + p^{-q} < e + 3^{-3} < 3,$$

故 $\beta = q$, $\alpha = 1$. 再由 $2 \cdot p^q = (p+2)^q + 1 = (p+3)A$, 可得 $A = p^q$, $p+3=2$, 矛盾.

情形 2: c 是偶数, 设 $2^d \| c$, $d \geq 1$. 则 $(p+2)^{2^d}+1 \mid (p+2)^c+1$, 故

$$(p+2)^{2^d} + 1 = 2^\alpha p^\beta.$$

由于 $(p+2)^{2^d}+1 \equiv 2 \pmod{4}$, 故 $\alpha = 1$.

$$(p+2)^{2^d} + 1 = 2 \cdot p^\beta. \tag{2}$$

(2)两边模 p 可得 $2^{2^d} \equiv -1 \pmod{p}$, 故 2 模 p 的阶为 2^{d+1}, 于是 $2^d < \frac{p}{2}$. 由于

$$p^{\beta+1} > 2 \cdot p^\beta = (p+2)^{2^d} + 1 > p^{2^d},$$

故 $\beta \geq 2^d$. (2)两边除以 p^{2^d}, 有

$$2 \cdot p^{\beta - 2^d} = \left(1+\frac{2}{p}\right)^{2^d} + p^{-2^d} < \left(1+\frac{2}{p}\right)^{\frac{p}{2}} + p^{-2^d} < e + 3^{-2} < 3,$$

故 $\beta = 2^d$.

若 $d \geq 2$, 则由 $2^{d+1} \mid p-1$ 可得 $p \equiv 1 \pmod 8$. 由(2)有 $p^{2^d} - 1 = (p+2)^{2^d} - p^{2^d}$. 分析两边的 2 因子个数,

$$v_2(p^{2^d} - 1) = v_2(p^2 - 1) + d - 1 \geq d + 3,$$

而

$$v_2((p+2)^{2^d} - p^{2^d}) = v_2((p+2)^2 - p^2) + d - 1 = v_2(2) + v_2(2p+2) + d - 1 = d + 2,$$

矛盾. 故 $d = 1$, $2p^2 = (p+2)^2 + 1$, 得 $p = 5$. 再回到原方程, $c = 2k$, k 是奇数, 而 $a = 1$, 我们有

$$2 \cdot 5^b = 7^{2k} + 1.$$

由升幂定理, $b = v_5(7^{2k} + 1) = v_5(7^2 + 1) + v_5(k) \leq 2 + \dfrac{k}{5}$. 若 $k \geq 3$, 则

$$2 \cdot 5^b \leq 2 \cdot 5^{2+\frac{k}{5}} = (7^2 + 1) \cdot 5^{\frac{k}{5}} < 7^{2k} + 1,$$

矛盾. 故 $k = 1$, 从而 $c = 2$, $b = 2$, 又得到一组解 $(p, a, b, c) = (5, 1, 2, 2)$.

综上所述, 共有两组满足条件的解 (p, a, b, c), 是 $(3, 1, 1, 1)$ 和 $(5, 1, 2, 2)$.

5. 设 n 是正整数, $2n$ 个非负实数 $x_1, x_2, \ldots, x_{2n}$ 满足 $x_1 + x_2 + \cdots + x_{2n} = 4$. 证明: 存在非负整数 p, q 使得 $q \leqslant n-1$, 且

$$\sum_{i=1}^{q} x_{p+2i-1} \leqslant 1, \qquad \sum_{i=q+1}^{n-1} x_{p+2i} \leqslant 1.$$

注 1 : 下标按模 $2n$ 理解, 即若 $k \equiv l \pmod{2n}$, 则 $x_k = x_l$.

注 2 : 若 $q = 0$, 则第一个求和视为 0; 若 $q = n-1$, 则第二个求和视为 0.

(命题组 供题)

证法一: 按奇偶位置分组, 记 $A = x_1 + x_3 + \cdots + x_{2n-1}, B = x_2 + x_4 + \cdots + x_{2n}$, 并定义奇/偶组的部分和序列: $A(0) = B(0) = 0$,

$$\begin{cases} A(2k+1) = A(2k) + x_{2k+1} \\ B(2k+1) = B(2k) \end{cases}, \quad \begin{cases} A(2k+2) = A(2k+1) \\ B(2k+2) = B(2k+1) + x_{2k+2} \end{cases}, \quad k = 0, 1, 2, \ldots$$

角标模 $2n$ 理解, 部分和周期增长: $A(k+2n) = A(k) + A, B(k+2n) = B(k) + B$. 我们再连续化 (变成分段线性), 即对非负实数 t, 定义

$$A(t) = A(\lfloor t \rfloor) + (t - \lfloor t \rfloor)[A(\lfloor t \rfloor + 1) - A(\lfloor t \rfloor)], \;\; B(t) = B(\lfloor t \rfloor) + (t - \lfloor t \rfloor)[B(\lfloor t \rfloor + 1) - B(\lfloor t \rfloor)].$$

这样 $A(\cdot), B(\cdot)$ 均为 $\mathbb{R}_{\geqslant 0}$ 上的连续不减函数, 并仍然周期增长.

由于 $A + B = 4$ 知存在正整数 L 满足 $\lfloor \frac{L}{A} \rfloor + \lfloor \frac{L}{B} \rfloor \geqslant L$. 对 $l = 0, 1, 2, \ldots, L$, 由 $A(\cdot)$ 的连续性知可以选取适当的 $t_l \in \mathbb{R}_{\geqslant 0}$ 满足 $A(t_l) = l$, (我们选取 $t_0 = 0$). 由于 $L \geqslant A\lfloor \frac{L}{A} \rfloor = A(2n \cdot \lfloor \frac{L}{A} \rfloor)$, 我们可以要求 $t_L \geqslant 2n \cdot \lfloor \frac{L}{A} \rfloor$. 这样

$$B(t_L) - B(t_0) = B(t_L) \geqslant B\Big(2n \cdot \Big\lfloor \frac{L}{A} \Big\rfloor\Big) \geqslant B\Big\lfloor \frac{L}{A} \Big\rfloor \geqslant B\Big(L - \Big\lfloor \frac{L}{B} \Big\rfloor\Big) \geqslant (B-1) \cdot L.$$

因此存在某个 $l = 0, 1, \ldots, L-1$ 满足 $B(t_{l+1}) - B(t_l) \geqslant (B-1)$, 即 $B(t_l + 2n) - B(t_{l+1}) \leqslant 1$. 取非负整数 c, d 满足 $2c \leqslant t_l \leqslant 2c+2, \;\; 2d-1 \leqslant t_{l+1} \leqslant 2d+1$. 我们有:

$$A(t_1) \leqslant A(2c+1) = A(2c+2), \;\; A(t_{l+1}) \geqslant A(2d) = A(2d-1)$$
$$B(t_1) \geqslant B(2c+1) = B(2c), \;\; B(t_{l+1}) \leqslant B(2d) = B(2d+1)$$

这样

$$\begin{aligned} x_{2c+3} + x_{2c+5} + \cdots + x_{2d-1} = & \quad A(2d) - A(2c+1) \leqslant & A(t_{l+1}) - A(t_l) = & \;\; 1 \\ x_{2d+2} + x_{2d+4} + \cdots + x_{2c+2n} = & \quad B(2c+1+2n) - B(2d) \leqslant & B(t_l + 2n) - B(t_{l+1}) \leqslant & \;\; 1 \end{aligned}$$

即 $p = 2c+2, q = d-c-1$ 满足题目要求. (若 $q < 0$, 则重置 $q = 0$; 若 $q \geqslant n$, 则重置 $q = n-1$.)

证毕.

证法二: 设 $x_1 + x_3 + \cdots + x_{2n-1} = A, \;\; x_2 + x_4 + \cdots + x_{2n} = B$.

若 A, B 中有一个小于等于 1, 结论显然. 若 $A>1, B>1$, 对 $0 \leq k \leq n-1$, 设 $m(k) \in \{1,2,\cdots,n-1\}$ 满足

$$\sum_{i=0}^{m(k)} x_{2k+2i+1} > 1 \tag{1}$$

$$\sum_{i=0}^{m(k)-1} x_{2k+2i+1} \leq 1. \tag{2}$$

注意到, 若 $x_{2k+2m(k)+2}+x_{2k+2m(k)+4}+\cdots+x_{2k+2n-2} \leq 1$, 则此式与(2)即构成符合题意的情形 $(p=2k, q=m(k))$.

考虑

$$x_{2k+2m(k)+2}+x_{2k+2m(k)+4}+\cdots+x_{2k+2n-2} > 1. \tag{3}$$

现在, 以 $0,1,2,\cdots,n-1$ 为顶点, 从 k 向 $k+m(k)+1$ 引一条边 (模 n 意义下). 易知此图中有圈, 不妨设 $k_1 \to k_2 \to \cdots \to k_t \to k_1$ 是一个最短圈.

若 $t=1$, 则 $\sum\limits_{i=0}^{n-2} x_{2k_1+2i+1} \leq 1$, 令 $p=2k_1$, $q=n-1$ 即可.

若 $t>1$, 设

$$\left\{\frac{k_2-k_1}{n}\right\}+\left\{\frac{k_3-k_2}{n}\right\}+\cdots+\left\{\frac{k_t-k_{t-1}}{n}\right\}+\left\{\frac{k_1-k_t}{n}\right\}=s.$$

即当 $k=k_1,k_2,\ldots,k_t$ 时, (1)的加数共 $s\cdot n$ 个, 且由引边的条件知 $x_1,x_3,\ldots,x_{2n-1}$ 中的每一项都被加了 s 次. 另一方面, 当 $k=k_1,k_2,\ldots,k_t$ 时, (3)的加数共 $(t-s)\cdot n$ 个, 且由引边的条件知 $x_2,x_4,\ldots,x_{2n}$ 中的每一项都被加了 $t-s$ 次, 这是因为 x_{2j} 在(3)被加当且仅当 x_{2j+1} 没在(1)中被加.

现将 $k=k_1,k_2,\ldots,k_t$ 时的(1)加起来得 $s\cdot A>t$. 同理, 将 $k=k_1,k_2,\cdots,k_t$ 时的(3)加起来得 $(t-s)\cdot B>t$. 此时, $A+B>t\left(\frac{1}{s}+\frac{1}{t-s}\right)\geq 4$, 矛盾.

证毕.

证法三: 设 $x_1+x_3+\cdots+x_{2n-1}=A,\ \ x_2+x_4+\cdots+x_{2n}=B$. 同样, 不妨设 $A>1, B>1$. 我们证明存在 k 使得

$$x_{2k} \geq x_{2k+1}\cdot\frac{B}{A}$$

$$x_{2k}+x_{2k+2} \geq (x_{2k+1}+x_{2k+3})\cdot\frac{B}{A}$$

$$\cdots$$

$$x_{2k}+x_{2k+2}+\cdots+x_{2k+2n-2} \geq (x_{2k+1}+x_{2k+3}+x_{2k+2n-1})\cdot\frac{B}{A}$$

令 $a_i = x_{2i} - x_{2i+1} \cdot \dfrac{B}{A}$, 则 $a_0 + a_1 + \cdots + a_{n-1} = 0$. 由熟知结论, 存在 k 使得

$$\begin{aligned} a_k &\geq 0 \\ a_k + a_{k+1} &\geq 0 \\ &\cdots \\ a_k + a_{k+1} + \cdots + a_{k+n-1} &\geq 0 \end{aligned}$$

即得.

不妨设 $k = 0$ (否则所有角标减 $2k$), 设 $m \in \{0, 1, \ldots, n-1\}$ 使

$$\sum_{i=0}^{m} x_{2i+1} > 1, \quad \sum_{i=0}^{m-1} x_{2i+1} \leq 1.$$

我们取 $p = 0, q = m$, 只需证

$$x_{2m+2} + x_{2m+4} + \cdots + x_{2n-2} \leq 1.$$

事实上,

$$\begin{aligned} &x_{2m+2} + x_{2m+4} + \cdots + x_{2n-2} \\ &= B - (x_0 + x_2 + \cdots + x_{2m}) \\ &\leq B - (x_1 + x_3 + \cdots + x_{2m+1}) \cdot \frac{B}{A} \\ &< B - \frac{B}{A} = 1 + \frac{AB - A - B}{A} \\ &\leq 1. \quad (AB \leq 4 = A + B) \end{aligned}$$

证毕.

6. 给定正整数 n, 用 D 表示 n 的所有正因子构成的集合. 设 A, B 是 D 的子集, 满足: 对任何 $a \in A$, $b \in B$, 总有 a 不整除 b 且 b 也不整除 a. 证明:

$$\sqrt{|A|}+\sqrt{|B|} \leqslant \sqrt{|D|}.$$

(艾颖华 供题)

证法一: 将 D 分解为如下四个子集的不交并 $D = X \cup Y \cup Z \cup W$, 其中

$$X=\{x\in D: \exists a|x, \exists b|x\}, \quad Y=\{x\in D: \exists a|x, \nexists b|x\},$$
$$Z=\{x\in D: \nexists a|x, \exists b|x\}, \quad W=\{x\in D: \nexists a|x, \nexists b|x\}.$$

注意到, 题目条件表明 $A\subseteq Y, B\subseteq Z$, 我们只需证明更强的结论: 对 D 的任何两个非空子集 A, B, 总有 $\sqrt{|Y|}+\sqrt{|Z|}\leq\sqrt{|D|}$. 此不等式等价于

$$|Y|+|Z|+2\sqrt{|Y|\cdot|Z|}\leq |D| = |X|+|Y|+|Z|+|W| \iff 2\sqrt{|Y|\cdot|Z|}\leq |X|+|W|,$$

此不等式可由不等式 $|Y|\cdot|Z|\leq|X|\cdot|W|$ 推出, 后者又可以改写为

$$(|X|+|Y|)(|X|+|Z|) = |X|(|X|+|Y|+|Z|)+|Y|\cdot|Z| \leq |X|(|X|+|Y|+|Z|)+|X|\cdot|W| = |X|\cdot|D|.$$

令 $U=X\cup Y, V=X\cup Z$, 则上述不等式为 $|U|\cdot|V|\leq|U\cap V|\cdot|D|$. 注意到, $U=\{x\in D: \exists a|x\}$ 满足: 如果 $x\in U$ 且 $x|x'$, 则 $x'\in U$, 称 D 的这种子集为向上封闭的; 类似的 $V=\{x\in D:\exists b|x\}$ 也是 D 的向上封闭的子集.

下面证明: 对于 D 的非空的向上封闭的子集 U, V, 有 $|U|\cdot|V|\leq|U\cap V|\cdot|D|$.

设 n 的素因子分解为 $n=p_1^{\alpha_1}\cdots p_k^{\alpha_k}$, 我们对 k 进行归纳. 将 p_k, α_k 分别记为 p, α, 记 $n=p^\alpha n'$. 定义 $D_k=\{x\in D|v_p(x)=k\}$, $U_k=U\cap D_k$, $V_k=V\cap D_k$. 对每个 $k=0,1,\ldots,\alpha-1$, 对任何 $x\in U_k$, 由 U 向上封闭可知 $px\in U_{k+1}$, 这表明 $|U_k|\leq|U_{k+1}|$, 即 $\{|U_k|\}_k$ 是递增的; 类似的, $\{|V_k|\}_k$ 也是递增的. 注意到 $\frac{1}{p^k}U_k, \frac{1}{p^k}V_k$ 是 $\frac{1}{p^k}D_k=D(n')$ 的向上封闭子集, 由归纳假设可得

$$|(\frac{1}{p^k}U_k)\cap(\frac{1}{p^k}V_k)|\geq\frac{1}{|D(n')|}\cdot|(\frac{1}{p^k}U_k)|\cdot|(\frac{1}{p^k}V_k)|,$$

即有 $|U_k\cap V_k|\geq\dfrac{1+\alpha}{|D|}|U_k|\cdot|V_k|$. 结合排序不等式可得

$$\begin{aligned}
|U\cap V| &= \sum_{k=0}^{\alpha}|U_k\cap V_k| \geq \frac{1+\alpha}{|D|}\sum_{k=0}^{\alpha}|U_k|\cdot|V_k| \\
&\geq \frac{1+\alpha}{|D|}\cdot\frac{1}{1+\alpha}\left(\sum_{k=0}^{\alpha}|U_k|\right)\cdot\left(\sum_{k=0}^{\alpha}|V_k|\right) \\
&= \frac{1}{|D|}|U|\cdot|V|.
\end{aligned}$$

证毕.

证法二: (根据张志成同学的解答整理)

引理: 设 t 为正整数, $M_1,\dots,M_t$ 是有限集 P 的 t 个两两不等的子集, 则集合 $\{M_i \setminus M_j | 1 \leqslant i,j \leqslant t\}$ 至少有 t 个元素.

引理的证明: 对 $|M_1 \cup \cdots \cup M_t| = k$ 归纳. 当 $k=0$ 时, 结论显然. 下设 $k \geqslant 1$ 且小于 k 时引理成立. 任取 $a \in M_1 \cup \cdots \cup M_t$, 不妨设 a 属于 $M_1,\dots,M_s$, 不属于 $M_{s+1},\dots,M_t$, $1 \leqslant s \leqslant t$. 若 $s=t$, 则用 $M_i \setminus \{a\}$ 代替 M_i, 由归纳假设知结论成立. 下设 $s<t$, 定义四个集族

$$S_1 = \{M_i, 1 \leqslant i \leqslant s | \exists s+1 \leqslant j \leqslant t, M_i \setminus \{a\} = M_j\}, \quad S_2 = \{M_1,\dots,M_s\} \setminus S_1,$$
$$S_3 = \{M \setminus \{a\} | M \in S_1\}, \quad S_4 = \{M_{s+1},\dots,M_t\} \setminus S_3,$$

则 S_1,S_2,S_3,S_4 构成 $\{M_1,\dots,M_t\}$ 的一个划分. 再记 $S_5 = \{M \setminus \{a\} | M \in S_2\}$, 知 $|S_1| = |S_3|$, $|S_2| = |S_5|$.

(i) 对 S_3 应用归纳假设知, $\{M \setminus N | M,N \in S_3\}$ 至少有 $|S_3|$ 个元素. 从而

$$\{(M \cup \{a\}) \setminus N | M,N \in S_3\} = \{M \setminus N | M \in S_1, N \in S_3\}$$

至少有 $|S_3|$ 个元素.

(ii) 对 $S_3 \cup S_4 \cup S_5$ 应用归纳假设知, $\{M \setminus N | M,N \in S_3 \cup S_4 \cup S_5\}$ 至少有 $|S_3| + |S_4| + |S_5|$ 个元素. 对 $M,N \in S_3 \cup S_4 \cup S_5$ 作如下讨论:

- 若 M,N 均不属于 S_5, 则 $M \setminus N$ 形如 $M_i \setminus M_j$, 其中 $M_i,M_j \in S_3 \cup S_4$.
- 若 M,N 均属于 S_5, 则 $M \setminus N = (M \cup \{a\}) \setminus (N \cup \{a\})$ 形如 $M_i \setminus M_j$, 其中 $M_i,M_j \in S_2$.
- 若 $M \notin S_5, N \in S_5$, 则 $M \setminus N = M \setminus (N \cup \{a\})$ 形如 $M_i \setminus M_j$, 其中 $M_i \in S_3 \cup S_4$, $M_j \in S_2$.
- 若 $M \in S_5, N \notin S_5$, 且 $M \setminus N$ 不能表示为前三种, 则 $(M \setminus N) \cup \{a\} = (M \cup \{a\}) \setminus N$ 形如 $M_i \setminus M_j$, 其中 $M_i \in S_2$, $M_j \in S_3 \cup S_4$, 且它与之前的 $(M' \cup \{a\}) \setminus N'$, $M',N' \in S_3$ 不重复（否则, $M \setminus N = M' \setminus N'$, 同 $M \setminus N$ 不能表示为前三种相矛盾）.

综合 (i) 和 (ii) 知, $\{M_i \setminus M_j | 1 \leqslant i,j \leqslant t\}$ 的元素个数至少为

$$|S_3| + |S_3| + |S_4| + |S_5| = |S_1| + |S_3| + |S_4| + |S_5| = t.$$

引理得证.

回到原题. 设 $G = \{\gcd(a,b) | a \in A, b \in B\}$, $L = \{\operatorname{lcm}(a,b) | a \in A, b \in B\}$. 由题设条件知, A,B,G,L 两两不交且 $A \cup B \cup G \cup L \subseteq D$. 故只需证:

$$\sqrt{|A|} + \sqrt{|B|} \leqslant \sqrt{|A| + |B| + |G| + |L|},$$

即: $2\sqrt{|A||B|} \leqslant |G|+|L|$, 进而只需证: $|A|\cdot|B| \leqslant |G|\cdot|L|$.

我们定义两个映射

$$\rho_1 : A\times B \to D\times D, \quad (a,b)\mapsto (\gcd(a,b), \operatorname{lcm}(a,b)),$$
$$\rho_2 : G\times L \to D\times D, \quad (x,y)\mapsto (\gcd(x,y), \operatorname{lcm}(x,y)),$$

并试图证明

$$(*) \qquad \text{对于每个 } (g,l)\in D\times D, |\rho_1^{-1}(g,l)| \leqslant |\rho_2^{-1}(g,l)|.$$

这会推出 $|A\times B| \leqslant |G\times L|$, 从而完成证明.

现在证明 $(*)$. 若 $|\rho_1^{-1}(g,l)|=0$, 则结论显然. 下设 $|\rho_1^{-1}(g,l)|=t>0$, 并设

$$\rho_1^{-1}(g,l) = \{(gm_i, gn_i)|i=1,\ldots,t\},$$

其中 $gm_i\in A$, $gn_i\in B$, $m_in_i = l/g$, 且 m_i 和 n_i 互素. 记

$$P=\{p^\alpha|p\text{是素数}, \alpha\geqslant 1, p^\alpha||l/g\}, \quad P_i=\{p^\alpha|p\text{是素数}, \alpha\geqslant 1, p^\alpha||m_i\},$$

则 $P_i\subseteq P$, 且 $m_i = \prod\limits_{p^\alpha\in P_i} p^\alpha$, $n_i = \prod\limits_{p^\alpha\in P\setminus P_i} p^\alpha$. 于是对任意 $1\leqslant i,j\leqslant t$,

$$\gcd(m_i,n_j) = \prod_{p^\alpha\in P_i\setminus P_j} p^\alpha, \quad \operatorname{lcm}(m_j,n_i) = \prod_{p^\alpha\in P_j\cup(P\setminus P_i)} p^\alpha,$$

其中 $P_j\cup(P\setminus P_i)$ 恰为 $P_i\setminus P_j$ 在 P 中的补集, 因此 $\gcd(m_i,n_j)$ 和 $\operatorname{lcm}(m_j,n_i)$ 互素且乘积为 l/g. 记

$$x_{ij} := g\cdot\gcd(m_i,n_j) = \gcd(gm_i,gn_j), \quad y_{ij} := g\cdot\operatorname{lcm}(m_j,n_i) = \operatorname{lcm}(gm_j,gn_i),$$

则 $x_{ij}\in G$, $y_{ij}\in L$, 且 $\rho_2(x_{ij},y_{ij}) = (g,l)$. 根据引理, 由于 $P_1,\ldots,P_t$ 是 P 的 t 个两两不等的子集, 故集合 $\{P_i\setminus P_j|1\leqslant i,j\leqslant t\}$ 至少有 t 个元素, 从而 $\{(x_{ij},y_{ij})|1\leqslant i,j\leqslant t\}$ 至少有 t 个元素, 即知 $|\rho_2^{-1}(g,l)|\geqslant t$. 证毕.