Projective Morphisms (I)

Wenhan Dai

§1 Proj of a Graded Ring

$S = \bigoplus_{n=0}^{\infty} S_n$ graded ring, $S^+ = \bigoplus_{n=1}^{\infty} S_n$

i.e. each S_n closed under addition & $S_m \cdot S_n \subseteq S_{m+n}$.

$f \in S_n$: homogeneous of deg n. Each S_n is a S_0-mod.

$\rightsquigarrow \operatorname{Proj} S = \{ \mathfrak{p} \subseteq S$ homo prime ideals: $S^+ \nsubseteq \mathfrak{p} \}$

$\forall n$, $f \in S_n$, S_f = graded w/ negative deg

(g/f^k deg $m-kn$, $g \in S_m$).

$\rightsquigarrow D(f) = \{ \mathfrak{p} \in \operatorname{Proj} S : f \notin \mathfrak{p} \} = \operatorname{Spec} S_{f,0}$.

note: $D(f) \cap D(g) = D(fg)$, $\operatorname{Proj} S$ has sch str.

When $S = A[x_0, \dots, x_n]$. x_i homo of deg 1.

$\rightsquigarrow \operatorname{Proj} S = \mathbb{P}^n_A$.

<u>Fact</u> $S \rightarrow T$ graded rings $\mapsto \operatorname{Proj} T \rightarrow \operatorname{Proj} S$

Say $I \subseteq S$ homo if as <u>ab grps</u>: $I = \bigoplus_{n=0}^{\infty} (I \cap S_n)$

$\rightsquigarrow S/I$ also graded $\twoheadleftarrow S$

$\rightsquigarrow \operatorname{Proj} S/I \longrightarrow \operatorname{Proj} S$ closed imm

<u>Caveat</u> $\operatorname{Proj} S$ does not determine S by itself $\Big\}$ Will come back later.

e.g. $\operatorname{Proj} \bigoplus_{n=0}^{\infty} S_n = \operatorname{Proj} S_0 \oplus \bigoplus_{n=2}^{\infty} S_n$

Can do the same thing towards $M = \bigoplus_{n=-\infty}^{\infty} M_n$.

§2 The Sheaf $\mathcal{O}(1)$

M graded S-mod, i.e. $S_m \cdot M_n \subseteq M_{m+n}$.

$\rightsquigarrow M(n) := \bigoplus_i M(n)_i = \bigoplus_i M_{n+i}$

On $X = \operatorname{Proj} S$, $\mathcal{O}_X(n) =$ q-coh sheaf $\widetilde{S(n)}$.

$\rightsquigarrow \mathcal{O}_X(0) = \mathcal{O}_X$, $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$

for any $\mathcal{F} \in \operatorname{Qcoh}(\operatorname{Mod}_{\mathcal{O}_X})$.

<u>Lemma</u> S gen'd by S_1 as an S_0-alg i.e. $S_0[S_1] = S$

$\Rightarrow$ $\mathcal{O}_X(n)$ on $\operatorname{Proj} S$ are loc. free of rank 1.

& $\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \cong \mathcal{O}_X(m+n)$.

Proof. Hartshorne II.5.12. $\square$

<u>Note</u> $\mathcal{F} \in \operatorname{Qcoh}(X)$, X loc ringed.

$\mathcal{F}$ loc. free of rank 1 $\Leftrightarrow$ $\mathcal{F}$ invertible.

b/c $\exists! \mathcal{F}^\vee$ s.t. $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^\vee \cong \mathcal{O}_X$ "dual sheaf"

c.f. fractional ideals of $\mathcal{O}_K$ in number theory.

$\rightsquigarrow (\mathcal{O}_X(n))^\vee = \mathcal{O}_X(-n)$.

New interpretation:

x_i on $\operatorname{Proj} A[x_0, \cdots, x_n]$: global sections of $\mathcal{O}_X(1)$

(rather than $\mathcal{O}_X$).

<u>Thm</u> $S = S_0[S_1]$ $\underbrace{\text{f.g. alg.}}$ $\Rightarrow \forall \mathcal{F} \in \operatorname{Qcoh}(\operatorname{Proj} S)$, $\mathcal{F} = \widetilde{M}$

$\Downarrow$ (for a canonical choice of M).

to ensure $\operatorname{Proj} S$ "quasi-compact".

Proof. The module we want:

$$M = \Gamma_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(\operatorname{Proj} S, \mathcal{F}(n)).$$

where $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

The Rest: see Hartshorne II.5.15. $\square$

<u>Caution</u> $S = \bigoplus_{n=0}^{\infty} \Gamma(X, \mathcal{O}_X(n))$ NOT true in general!

e.g. (stupid) $S = A[x] \rightsquigarrow \mathcal{O}_X(n)$ free, $\forall n$.

$\Rightarrow \Gamma(X, \mathcal{O}_X(n)) \neq 0$ even when $n<0$.

(less stupid) Hartshorne Ex II.5.14.

However:

<u>Lemma</u> $n \geq 1$. $S = A[x_0, \cdots, x_n]$ with usual grading

$\Rightarrow S = \bigoplus_{n=0}^{\infty} \Gamma(X, \mathcal{O}_X(n))$.

Proof. see Hartshorne II.5.13. $\square$

<u>§3 Closed Subschemes of Projective Spaces</u>

<u>Prop</u> $\forall n \geq 1$, any $X \overset{i}{\hookrightarrow} \mathbb{P}^n_A$ closed imm is def'd by some homogeneous ideal $I \subseteq A[x_0, \cdots, x_n]$.

Proof. $\mathcal{I}$ ideal sheaf defining i.

$$\left.\begin{array}{l} \Rightarrow \underset{\substack{\parallel \\ I}}{\Gamma_*(\mathcal{I})} = \text{ideal of } \underset{\substack{\parallel \\ S = A[x_0, \cdots, x_n]}}{\Gamma_*(\mathcal{O}_X)}. \end{array}\right\} \text{canonical way to find } I.$$

$\square$

In general, $I \rightsquigarrow X \subseteq \mathbb{P}^n_A \underbrace{\rightsquigarrow \mathcal{I} \rightsquigarrow}_{\text{as above}} \bar{I} =$ saturation of I.

$\bar{I} = \{ f \in A[x_0, \cdots, x_n] \text{ s.t. } x_0^i f, \cdots, x_n^i f \in I \}$

"Projective Nullstellensatz":

$I \longrightarrow X$ multi-to-one

$\bar{I} \longleftrightarrow X$ (saturated) one-to-one.

Cor $n \geq 1$. $I \subseteq S = A[x_0, \dots, x_n]$ homo ideal. TFAE:

(a) $X \subseteq \mathbb{P}^n_A$ def'd by I, $X = \phi$.

(b) $\bar{I} = S^+$.

(c) $S_n \subseteq I$, $n \gg 0$.

Proof. (a) $\Leftrightarrow$ (b) known. (c) $\Rightarrow$ (b) clear.

(b) $\Rightarrow$ (c): $\forall f \in \{x_0, \dots, x_n\}$, $x_0^i f, \dots, x_n^i f \in \bar{I} = S^+$.

$\Rightarrow \exists\, x_0^i, \dots, x_n^i \in I$ for some i. (true for $i \gg 0$).

$\Rightarrow S_{(n+1)(i-1)+1} \subseteq I$

pigeonhole principle $\Big\}$ b/c $\exists$ one of $x_0^i, \dots, x_n^i$ dividing each monomial of deg $(n+1)(i-1)+1$. $\square$

§4 Projective Implies Proper

Goal Complete the proof that $f : \mathbb{P}^n_{\mathbb{Z}} \longrightarrow \operatorname{Spec} \mathbb{Z}$ is proper.

Recall The missing step:

to show that f is univ. closed

i.e. $\forall X \in Sch$, $\mathbb{P}^n_X \longrightarrow X$ closed.

$\Leftarrow$ Do this in $X = \operatorname{Spec} A$ affine.

Proof. Let $Z \subseteq \mathbb{P}^n_X \longrightarrow X$. $z \in X \setminus \operatorname{im} Z$. Put $k = \kappa(z)$.

We must exhibit $x \in U \subseteq X$ s.t. $Z \cap \mathbb{P}^n_U = \phi$.

Let $I = \bigoplus_{n=0}^{\infty} I_n$ saturated $\subseteq S = A[x_0, \dots, x_n]$ defining Z.

$\Rightarrow I \otimes_A k$ defines $\phi \subseteq \operatorname{Proj} k[x_0, \dots, x_n]$.

(but may not be saturated).

$\exists m$ s.t. $I_m \otimes_A k = S_m \otimes_A k$ & $(S_m/I_m) \otimes_A k = 0$.

Now S_n/I_n f.g. A-mod, by Nakayama:

$\Rightarrow (S_n/I_n) \otimes_A A_\mathfrak{p} = 0$, $\forall \mathfrak{p} \subseteq A$ defining z.

Again, since S_n/I_n f.g.

$\Rightarrow (S_n/I_n) \otimes_A A_g = 0$, $g \in A \setminus \mathfrak{p}$.

$\Rightarrow z \in D(g)$, $D(g) \cap \operatorname{im} Z = \phi$. $\square$

§5 What is a Projective Morphism?

Hartshorne / Eisenbud-Harris:

$f: Y \to X$ proj if $f: Y \to \mathbb{P}^n_X \to X$ for some n.

This def'n:

① evidently stable under base changes

(!) ② not local on the base.

Better to say "globally proj", and say

f loc proj if $\forall x \in X$, $\exists x \in U \subseteq X$ open

s.t. $f: Y \times_X U \to U$ globally proj.

i.e. $Y \times_X U \to \mathbb{P}^n_U \to U$.

Morally Globally proj = loc proj

if X is "not too large".

e.g. X globally quasi-proj / affine sch.

$$\left(\begin{tikzcd} f: Y \arrow[r] \arrow[d] & X \\ Z \arrow[ur] & \end{tikzcd} \quad \begin{array}{l} Y \to Z \text{ open imm} \\ Z \to X \text{ glob proj} \end{array}\right)$$

EGA: loc. proj —— ? proj —— glob proj

(more on this later)

<u>Warning</u> Eisenbud-Harris claim that

$$\text{loc proj} = \text{proj} \qquad (X)$$

but counterexamples are rather pathological.