Schemes

See Hartshorne II.2 / Eisenbud-Harris I.1 / EGA 1.1.

§1 Ringed and Locally Ringed Spaces

ringed space = $(X, \mathcal{O}_X)$

X : top space & $\mathcal{O}_X \in Sh_{Ring}(X)$.

Rmk Useless b/c no relation b/w top & rings.

e.g. $X = pt$. $\mathcal{O}_X : pt \mapsto R$ any ring.

More useful notion: loc. ringed space. — 0 is not a local ring.

$\forall x \in X$, $\mathcal{O}_{X,x}$ = local ring with $m_{X,x}$.

E.g. X manifold, $\mathcal{O}_X$ sheaf of $\mathbb{R}$-valued functions.

$\Rightarrow (X, \mathcal{O}_X)$ loc ringed

Check: $\forall x \in X$, $\mathcal{O}_{X,x}/\underbrace{m_{X,x}}_{= \{f : x \mapsto 0\}} \subseteq \mathbb{R}$.

- X mfd $\Rightarrow \mathcal{O}_{X,x}/m_{X,x} \neq 0 \Rightarrow m_{X,x}$ indeed max'l.
- $\forall f \in \mathcal{O}_{X,x} \setminus m_{X,x}$, $f(x) \neq 0 \in \mathbb{R}$

  $\Rightarrow \exists$ open interval $I \subseteq \mathbb{R}$ s.t. $f(x) \in I$, $0 \notin I$.

  $\Rightarrow f \in \mathcal{O}_{X,x}^{\times}$ unit

  $\Rightarrow m_{X,x}$ is the unique max'l ideal.

(Key $V = f^{-1}(I)$ open (f conti) & $0 \notin f(V) \rightsquigarrow g = f^{-1}$ s.t. $g \circ f = 1_V$.)

Rmk Similar to sm mfd / $\mathbb{C}$-mfd / abstract var.

Def Isom (of ringed spaces) = homeo & bij of sections (commutes with res of sets).

§2 The Prime Spectrum of a Ring

Goal Functor : Ring $\longrightarrow$? . $\rightsquigarrow$ affine sch.

$R \in$ Ring $\rightsquigarrow$ $\operatorname{Spec} R = \{$prime ideals of $R\}$ by Zariski.

$\mathfrak{p} \subseteq R$ prime $\Leftrightarrow R/\mathfrak{p}$ integral.

((0) is not prime.)

Caution $\phi : R \to S \rightsquigarrow \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$. But $\phi^{-1}(\mathfrak{m})$ may not be max'l.

$\mathfrak{p} \longmapsto \phi^{-1}(\mathfrak{p})$ (e.g. $\phi : \mathbb{Z} \to \mathbb{Q}$).

Check inj $R/\phi^{-1}(\mathfrak{p}) \longrightarrow S/\mathfrak{p}$ inj, $S/\mathfrak{p}$ integral $\Rightarrow R/\phi^{-1}(\mathfrak{p})$ integral.

Zariski top closed sets $= \{V(I)\}$. $V(I) = \{\mathfrak{p} \in \operatorname{Spec} R : I \subseteq \mathfrak{p}\}$

$$\Rightarrow \left.\begin{aligned} V(I) \cup V(J) &= V(I \cap J) = V(IJ) \\ \bigcap_i V(I_i) &= V\Big(\sum_i I_i\Big) \end{aligned}\right\} \Rightarrow \text{indeed a top.}$$

Distinguished open sets (top basis) : $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$.

$\Rightarrow D(f) \cap D(g) = D(fg)$ open.

Also,
$$\begin{array}{ccc} \operatorname{Spec} S & \longrightarrow & \operatorname{Spec} R \\ \cup\!\text{I} & & \cup\!\text{I} \\ D(\phi(f)) & \longrightarrow & D(f) . \end{array}$$

Lem $D(f)$ quasicompact for Zariski top.

($\Rightarrow \operatorname{Spec} R = D(1)$ quasicpt.)

Proof. If $\bigcup_i D(f_i) \supseteq D(f)$ cover $\Rightarrow \sqrt{f} \subseteq \sqrt{\text{all } f_i\text{'s}}$.

$\Rightarrow f = \sum_{j=1}^{k} r_j f_j$, $r_j \in R$. finite sum

$\Rightarrow \bigcup_{j=1}^{k} D(f_j) \supseteq D(f)$. $\square$

E.g. $k = \bar{k}$ field $\Rightarrow \operatorname{Spec} k[x] = \{$pts of $(x-a),\ a \in k\} \cup \{(0)\}$.

generic pt (i.e. $\overline{(0)}^{Zar} = \operatorname{Spec} k[x]$).

Hartshorne 2.3.4: $\operatorname{Spec} k[x,y]$ (See it!)

§3 Presheaf of Rings

<u>Upshot</u> $\mathfrak{p}$ prime, $f \in \mathfrak{p} \Leftrightarrow f^n \in \mathfrak{p}$.

($X = \operatorname{Spec} R$ does not distinguish $f \in R$ & $f^n \in R$.)

<u>Lem</u> $f, g \in R$. $D(f) \subseteq D(g) \Leftrightarrow f = hg$ $(h \in R)$.

Proof. · $D(f) = D(f^n) \Rightarrow f^n = tg$ implies $D(f) = D(f^n) \subseteq D(g)$.

· $D(f) \subseteq D(g) \Rightarrow V(g) \subseteq V(f) \Rightarrow \sqrt{(f)} \subseteq \sqrt{(g)} = \bigcap_{(g) \subseteq \mathfrak{p}} \mathfrak{p}$

$\Rightarrow f \in \sqrt{(g)} \Rightarrow f = tg^n = hg$ $(h = tg^{n-1})$. $\square$

<u>Def</u> $S \subseteq R$ multi subset.

S is <u>saturated</u> if $\forall x \in R$ $(x = u \cdot s,\ s \in S$ & $u \in R^\times \Rightarrow x \in S)$.

<u>Fact</u> $\forall S$, $\exists!\ \tilde{S} \supseteq S$ saturated.

<u>Cor</u> $f, g \in R$, $D(f) = D(g) \Leftrightarrow \tilde{S}_f = \tilde{S}_g$, where $S_f = \{1, f, f^2, \dots\}$.

Now $S \subseteq R \rightsquigarrow S^{-1}R \in \mathrm{Alg}_R$ localization at S.

<u>Note</u> · Univ property $\Rightarrow \tilde{S}^{-1}R \cong S^{-1}R$.

· $\forall f \in R$, $R_f := S_f^{-1}R \cong \tilde{S}_f^{-1}R \cong R[x]/(xf-1)$ "$= R[\frac{1}{f}]$".

Let $\mathcal{D} = \{D(f) : f \in R\}$, $X = \operatorname{Spec} R$.

$\rightsquigarrow$ presheaf $\mathcal{O}_X : D(f) \longmapsto R_f$ well-def'd by Cor.

· $D(g) \subseteq D(f) \Rightarrow R_f \subseteq R_g$, $\exists$ canonical $R_f \to R_g$.

Less canonically: $g = hf^n$, $\mathcal{O}_X(D(f)) = R[x]/(xf-1)$.

$\mathcal{O}_X(D(g)) = R[y]/(yg-1)$.

$$\rightsquigarrow \quad R[x]/(xf-1) \longrightarrow R[y]/(yg-1)$$

$$\underset{f^{-1}}{\overset{x}{\shortparallel}} \longmapsto \underset{f^{-1}}{\overset{f^{n-1}\cdot hy}{\shortparallel}} \qquad \left(\tfrac{1}{f} \mapsto \tfrac{1}{f}\right).$$

§4 The Fundamental Theorem of Affine Schemes

<u>Thm 1</u> $X = \operatorname{Spec} R$. $\mathcal{O}_X$ on $\mathcal{D}$ satisfies sheaf axiom of coverings of dist opens

($\Rightarrow \mathcal{O}_X$ extends uniquely to Sh_{Ring} on $\operatorname{Spec} R$.)

A stronger ver:

<u>Thm 2</u> $M \in \mathrm{Mod}_R$. Presheaf $\widetilde{M} : D(f) \longmapsto M \otimes_R R_f$.

$\Rightarrow \widetilde{M}$ satisfies sheaf axiom of coverings of dist opens.

<u>Proof</u> $X = \bigcup D(f_i)$. Need: $M \xrightarrow{\psi} \prod (M \otimes_R R_{f_i})$ inj.

Suppose $m \in \ker\psi \Rightarrow \operatorname{Ann} m \subseteq R$ s.t. $\operatorname{Ann} m \nsubseteq \mathfrak{p}$, $\forall \mathfrak{p} \in \operatorname{Spec} R$.

(or else $\exists \mathfrak{p} \in D(f_i)$ s.t. $\psi(m) \in M \otimes_R R_{f_i} \setminus \{0\}$.)

$\Rightarrow 1\cdot m = m = 0 \Rightarrow \psi$ inj.

Check gluing property:

X quasicpt $\Rightarrow D(f_1) \cup \cdots \cup D(f_n) \supseteq X$.

Suppose $D(f) \subseteq \bigcup D(f_i) \rightsquigarrow m_i/f_i^{h_i} \in M \otimes_R R_{f_i}$

s.t. $m_i/f_i^{h_i}$ & $m_j/f_j^{h_j}$ have same image in $R_{f_i f_j}$.

only finitely many $f_i \Rightarrow$ can take $h \gg 0$ s.t.

$$(f_i f_j)^{g_{ij}}(f_i^h m_j - f_j^h m_i) = 0.$$

Rechoose $m_i \rightsquigarrow$ can force $g_{ij} = 0$ $\forall i,j$. $\Rightarrow f_i^h m_j = f_j^h m_i$.

And $D(f_i^h) = D(f_i) \Rightarrow D(f_i^h)$ cover $X \Rightarrow (1) = (f_1^h, \cdots, f_n^h)$.

Pick $g_1, \cdots, g_n \in R$ s.t. $g_1 f_1^h + \cdots + g_n f_n^h = 1$

Put $m = g_1 m_1 + \cdots + g_n m_n$.

$\Rightarrow \quad f_i^h m = \sum_j f_i^h g_j m_j = \sum_j f_j^h g_j m_i = m_i$.

$\Rightarrow \quad m \in M$ restricts to m_i / f_i^h. $\square$

§5 Schemes

$\forall \mathfrak{p} \in \operatorname{Spec} R$, $R_\mathfrak{p} \cong \mathcal{O}_{X,\mathfrak{p}}$ canonically.

$\rightsquigarrow$ $\operatorname{Spec} R$ locally ringed space.

Def Affine scheme $= (X \cong \operatorname{Spec} R)$ with $\underset{\substack{\uparrow \\ \text{by Thm 2.}}}{\Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) = R}$.

Scheme = each pt has a nbhd as aff sch.

Warning $X = \operatorname{Spec} R$ aff $\Rightarrow$ each $D(f) = \operatorname{Spec} R_f$ aff sch (exercise.)

$\{D(f)\}$ form a Zar top basis

$\Rightarrow \exists\, U$ Zar open s.t. $(U, \mathcal{O}_X|_U)$ aff but $\underline{\underline{U \text{ not dist.}}}$

§6 Schemes by Gluing

$X_1 \supseteq U_1$, $X_2 \supseteq U_2$ open with $U_1 \cong U_2 \rightsquigarrow X = X_1 \coprod_{U_{12}} X_2$ by gluing.

Gluing data $(X_i)_{i \in I}$, $U_{ij} \subseteq X_i$ s.t. $U_{ii} = X_i$.

$\phi_{ij} : U_{ij} \longrightarrow U_{ji}$ isom, $\phi_{ii} = \mathrm{id}_{X_i}$.

Suppose $\forall i, j, k$, $\phi_{ij}|_{U_{ij} \cap U_{ik}} = \phi_{ij}|_{U_{ji} \cap U_{jk}}$

& cocycle condition $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_{ij} \cap U_{ik}$.

$(\Rightarrow \phi_{ji} = \phi_{ij}^{-1}.)$

§7 Examples of Gluing

Gluing $\rightsquigarrow$ both good & evil.

<u>The good</u> $X_i = \operatorname{Spec} R[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]$

Define $U_{ij} = D(x_j/x_i) \subseteq X_i$.

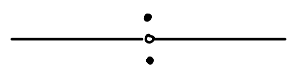
$\rightsquigarrow$ $U_{ij} \cong U_{ji}$ via $x_k/x_i \longmapsto (x_k/x_j)(\overset{\neq 0}{x_j/x_i})$.

Easy to check cocycle condition $\rightsquigarrow$ get a sch $\mathbb{P}^n_R$.

(Alternatively: Proj(graded rings) = $\mathbb{P}^n$.)

<u>The evil</u> $k = \bar{k}$ field. $X_1 = X_2 = \operatorname{Spec} k[x]$.

$U_1 = U_2 = (\operatorname{Spec} k[x]) \setminus \{0\} = D(x)$.

$\Rightarrow$ line with double pt =  not Hausdorff

not required for Sch.

$\rightsquigarrow$ notion of separatedness.