HOMEWORK FOR TOPICS IN LANGLANDS PROGRAM

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Week 1

Exercise 1.1 (General properties of topological groups)**.** Let G be a topological group.

(1) All open subgroups of G are closed.
(2) A subgroup H of G is open if and only if the coset space G/H with quotient topology is discrete.
(3) If H and H' are subgroups of G such that $H' \subset H$ and H' is open in G, then H is open in G.
(4) G is Hausdorff if and only if $\{e\}$ is closed. Here e is the identity element.

Solution. (1) Let H be an open subgroup of G. Apply the set-theoretical decomposition

$$G = H \sqcup ((G-H)H).$$

Note that $G-H$ is closed and H is open, and thus $(G-H)H$ is open. This follows from the fact that the product of an open subset of G with any subset of G is open. Therefore, H is also closed.

(2) If H is open in G then every coset gH is open. It follows that every single point in G/H is open, which is the definition that G/H is discrete. The converse is proved by reversing the argument.

(3) Since H' is open, we have from (2) that G/H' is a discrete quotient. On the other hand, by assumption G/H is a quotient of G/H', which is again discrete. It again follows from (2) that H is open in G.

(4) For the "if" part, suppose $\{e\}$ is closed. Let $x, y \in G$ be such that $x \neq y$. Then $x(G-\{e\})y^{-1}$ is open, so its complement $x\{e\}y^{-1} = \{xy^{-1}\}$ is closed, where $xy^{-1} \neq e$ by assumption. Let U be an open neighborhood of 1 such that $U^2 \subset G - \{xy^{-1}\}$.[1] We check the Hausdorff property by showing that $Ux \cap Uy = \emptyset$, as Ux and Uy are open neighborhoods of x and y in G, respectively. If the intersection is non-empty, then we can find $u_1, u_2 \in U$ such that $u_1 x = u_2 y$. But then $xy^{-1} = u_1^{-1}u_2 \in \{xy^{-1}\} \cap U^2$, which is not possible.

For the "only if" part, suppose G is Hausdorff, and thus the diagonal $\Delta \coloneqq \{(x,x) \colon x \in G\}$ is closed in $G \times G$. On the other hand, given the continuous action map $\gamma \colon G \times G \to G,\ (x,y) \mapsto xy^{-1}$, we get $\gamma^{-1}(\{e\}) = \Delta \subset G \times G$, which is closed. It follows that $\{e\}$ is closed in G. $\square$

Exercise 1.2. Consider $\mathrm{GL}_n(\mathbb{C})$ equipped with the natural topology induced from that on $\mathbb{C}$. Show that there exists a neighborhood U of 1 in $\mathrm{GL}_n(\mathbb{C})$ such that no non-trivial subgroup of $\mathrm{GL}_n(\mathbb{C})$ is contained in U. (Hint: Use the exponential map.) Use this to show that any continuous homomorphism from a profinite group to $\mathrm{GL}_n(\mathbb{C})$ must have finite image.

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[1] Such U exists for the following reason. Let V be an open neighborhood of 1 in G and $m \colon G \times G \to G,\ (x,y) \mapsto xy$ be the continuous G-action on itself. Then $W \coloneqq m^{-1}(V)$ is open. We have $(1,1) \in W$ because $1^2 = 1 \in V$. By definition of the product topology on $G \times G$, there exists an open subset $\Omega \ni 1$ of G such that $\Omega \times \Omega \subset W$. We have $\Omega^2 \subset V$ by definition of W. Let $U = \Omega \cap \Omega^{-1}$. We know that Ω^{-1} is open from that Ω is open together with the fact that the inverse map $\iota \colon G \to G,\ x \mapsto x^{-1}$ is continuous. So U is open and it is symmetric by construction. We clearly have $1 \in U$ and $U^2 \subset \Omega^2 \subset V$.

Solution. For the first statement, consider $\exp\colon \mathrm{M}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ which is locally a diffeomorphism at $0 \in \mathrm{M}_n(\mathbb{C})$, i.e., there exists an open neighborhood V of $0 \in \mathrm{M}_n(\mathbb{C})$ such that $\exp|_V$ is a diffeomorphism. Take $U = \exp(V/2) \subset \mathrm{GL}_n(\mathbb{C})$, which is an open neighborhood of $1 \in \mathrm{GL}_n(\mathbb{C})$. For any $g \in U$ there is $v \in V/2$ such that $\exp(v) = g$; thus, $g^k = \exp(kv)$ with $kv \notin V/2$ for each $k \geqslant 2$. It follows that $g^k \notin U$, namely U does not contain any subgroup of $\mathrm{GL}_n(\mathbb{C})$.

For the second statement, let G be a profinite group and $f\colon G \to \mathrm{GL}_n(\mathbb{C})$ be a continuous homomorphism. Taking $U \subset \mathrm{GL}_n(\mathbb{C})$ as before, the preimage $f^{-1}(U)$ must be an open neighborhood of e. Since G is profinite, for e the identity element in G, there is a neighborhood basis consisting of compact open subgroups of G (c.f. Exercise 1.3(2)). It follows that there is an open subgroup $H \subset f^{-1}(U)$ of G. Since U contains no non-trivial subgroup of $\mathrm{GL}_n(\mathbb{C})$, we get a subgroup $f(H) \subset f(f^{-1}(U)) = U$. This leads to a contradiction unless $f(H) = \{1\}$. Again since G is profinite, it is by definition compact; this together with $f(H) = \{1\}$ force f to have finite image. $\square$

Exercise 1.3. Let X be a Hausdorff, compact, totally disconnected topological space. We show that X is homeomorphic to an inverse limit of finite sets, as follows. Let I be the set of maps $f\colon X \to \mathbb{Z}$ such that $\mathrm{im}(f)$ is finite and each $f^{-1}(n)$ is open. Informally, I is the set of ways of partitioning X into a finite disjoint union of open subsets. For $f, g \in I$, define $f \le g$ if there exists a (necessarily unique) map $\phi_{f,g}\colon \mathrm{im}(g) \to \mathrm{im}(f)$ such that $f = \phi_{f,g} \circ g$ (i.e., the partition of X corresponding to g refines that corresponding to f). Then $(\mathrm{im}(f))_{f\in I}$ is a projective system of finite sets with transition maps $\phi_{f,g}$, and we have a natural continuous map

$$\begin{aligned} \Phi\colon X &\longrightarrow \varprojlim_{f\in I} \mathrm{im}(f) \\ x &\longmapsto (f(x))_f \end{aligned}$$

Here, the right hand side is equipped with the inverse limit topology coming from the discrete topology on each $\mathrm{im}(f)$ (which is a finite set).

(1) Prove that Φ is a homeomorphism. Hint: Use the following facts (which you can also try and prove yourself):
   (a) Any bijective continuous map from a compact space to a Hausdorff space is a homeomorphism.
   (b) In a compact Hausdorff space, the connected component containing a point is the intersection of all clopen sets (i.e. sets that are simultaneously closed and open) containing that point.

(2) Use (1) to show that the topology on X has a basis consisting of compact open sets.

Solution. (1) Note that Φ is naturally continuous. Using Hint (a), since X is compact by assumption and $\varprojlim_{f\in I} \mathrm{im}(f)$ is Hausdorff, it suffices to check that Φ is bijective.

- To show Φ is surjective, let $\underline{n} = (n_f)_f \in \varprojlim_{f\in I} \mathrm{im}(f)$. We need to find a preimage of $\underline{n}$. For each $f \in I$, let $C_f = f^{-1}(n_f)$. Then C_f is clopen in X, and we have $C_g \subset C_f$ whenever $f \le g$. Note that $\Phi^{-1}(\underline{n}) = \bigcap_{f\in I} C_f$. If this were empty then by compactness of X we know that a finite sub-intersection is empty. Finding a common upper bound of the indices, we get some $f \in I$ such that $C_f = \emptyset$, a contradiction. This shows that Φ is surjective.
- To show Φ is injective, by Hint (b) above, we have for any $x \in X$ that $\{x\}$ is the intersection of all clopens containing x. If $\Phi(x) = \Phi(y)$ then for any clopen U containing x it also contains y, since the characteristic function $\mathbb{1}_U$ of U is an element of I. Taking the intersections of all such clopens we conclude $x = y$.

This completes the proof that Φ is a homeomorphism.

(2) By (1) we see X is profinite; it is the subset of $\prod_{f\in I} \mathrm{im}(f)$ consisting of $(x_f)_f$ such that $\phi_{f,g}(x_g) = x_f$ for all $f, g \in I$ with $f \le g$. We endow it with the subspace topology inherited

from the product topology on $\prod_{f\in I}\operatorname{im}(f)$, where each $\operatorname{im}(f)$ having the discrete topology. Fix a finite subset $I_0 \subset I$ and for each $f \in I_0$ we fix $X_f \subset \operatorname{im}(f)$. Let

$$U := \prod_{f\in I-I_0}\operatorname{im}(f) \times \prod_{g\in I_0} X_g.$$

Then U is clopen in $\prod_{f\in I}\operatorname{im}(f)$. Varying I_0 and $(X_g)_{g\in I_0}$ the U's form a clopen basis of the topology. Also, each such U is a closed subset of X, and thus U is Hausdorff, compact, and totally disconnected. It follows that U is profinite. Therefore, X has a basis of topology consisting of compact open profinite subsets. □

Exercise 1.4 (van Dantzig's Theorem)**.** We are to prove the following statement: Let G be a topological group which is Hausdorff, locally compact, and totally disconnected. Then G is locally profinite in the sense of the lecture, i.e., 1 has a neighborhood basis consisting of compact open subgroups of G. (Assuming Hausdorff, the converse is also true, and is easier to prove.) Prove this theorem in the following steps:

(1) Using Exercise 1.3, show that the compact open neighborhoods of 1 in G form a neighborhood basis. Let K be an arbitrary compact open neighborhood of 1. In the remaining part it suffices to construct a compact open subgroup H of G contained in K.
(2) For every $x \in K$ there is an open neighborhood V_x of 1 such that $xV_x^2 \subset K$.
(3) There is an open neighborhood V of 1 such that $KV \subset K$. In particular, $V \subset K$ and $V^2 \subset K$.
(4) We may take V to satisfy $V^{-1} = V$.
(5) Let H be the subgroup of G generated by V. Then H is an open subgroup of G, and $H \subset K$.
(6) H is compact since it is closed and contained in K.

Solution. Since G is locally compact, there is a compact open neighborhood of 1 in G, say K. Then K is a Hausdorff, compact, and totally disconnected topological space. It follows from Exercise 1.3 that K has a basis consisting of compact open sets. It suffices to construct a compact open subgroup H of G contained in K. Since the left translation $G \to G,\ g \mapsto xg$ is continuous at 1, there is an open neighborhood U_x of 1 with $xU_x \subset K$. Since the multiplication $m\colon G\times G \to G$ is continuous at $(1,1)$, there is an open neighborhood V_x of 1 such that $V_x^2 \subset U_x$ (see the footnote on page 1 for details). So we get $xV_x^2 \subset K$.

Notice that K admits an open cover $\bigcup_{x\in K} xV_x$. Then the compactness of K renders a finite subcover, say $\bigcup_{i=1}^n x_iV_{x_i} \supset K$. Let $V = V_{x_1} \cap \cdots \cap V_{x_n}$. Then

$$KV \subset \bigcup_{i=1}^n x_iV_{x_i}V \subset \bigcup_{i=1}^n x_iV_{x_i}V_{x_i} \subset \bigcup_{i=1}^n x_iU_{x_i} \subset K.$$

The inclusion $V \subset KV \subset K$ implies $V^2 \subset K$. As the inverse map $G \to G,\ g \mapsto g^{-1}$ is also continuous, we may take V to satisfy $V^{-1} = V$, i.e. $g \mapsto g^{-1}$ induces a bijection while restricting on $V \subset G$. Let H be the subgroup of G generated by V. Then H is an open subgroup of G, and $H \subset K$. By Exercise 1.1(1), H is also closed, and hence compact. This completes the proof. □

Exercise 1.5. Let G be a locally profinite group.

(1) A representation (π, V) of G is smooth if and only if the action map $G\times V \to V,\ (g, v) \mapsto \pi(g)v$ is continuous, where V is equipped with the discrete topology.
(2) Let (π, V) be a smooth representation. Then any subrepresentation or quotient representation of (π, V) is again smooth.
(3) For an arbitrary representation (π, V) of G, the subspace $V^\infty = \bigcup_{K \text{ cos of } G} V^K$ is a subrepresentation, and it is the maximal smooth subrepresentation.

Solution. (1) Since the target V is equipped with the discrete topology, each singleton $\{\pi(g)v\}$ is open. Then the given action map is continuous if and only if $\{g \in G\colon \pi(g)v = v\}$ for all fixed $v \in V$ are open in G. This is equivalent to the definition that (π, V) is smooth.

(2) By (1), if (π, V) is smooth then $G \times V \to V$, $(g, v) \mapsto \pi(g)v$ is continuous. Let W be a subrepresentation of V. Then $G \times W \to V$, $(g, w) \mapsto \pi(g)w$ is also continuous (here W is a subrepresentation implies that the image of G-action again lands in W), which implies that $\{g \in G \colon \pi(g)w = w\}$ is open in G, and thus W is smooth. For the case of quotient representation, the result is implied by the continuity of $G \times (V/W) \to V/W$.

(3) By construction, each element $v^\infty \in V^\infty$ is stabled by certain compact open subgroup K, and hence $V^\infty \subset \{v \in V \colon \mathrm{Stab}_v G \text{ is open}\}$. Since G is locally profinite, there is a neighborhood basis of $1 \in G$ consisting compact open subgroups. It follows that each $v \in V$ with open $\mathrm{Stab}_v G$ is stabilized by some open compact subgroup, and hence $V^\infty = \{v \in V \colon \mathrm{Stab}_v G \text{ is open}\}$. Note that the latter set is G-stable, and so also is V^∞. This shows that V^∞ is a subrepresentation, which is smooth by definition. Moreover, if there is $v \in V$ that is not stabilized by any compact open subgroup, i.e. $v \notin V^\infty$, then $\mathrm{Stab}_v G$ cannot be open, and hence v is not smooth. Therefore, V^∞ exactly consists of all smooth vectors in V, and hence is the maximal smooth subrepresentation. □

Exercise 1.6. Let F be a finite extension of $\mathbb{Q}_p$. Show that the natural map $\mathrm{GL}_n(\mathcal{O}_F) \to \varprojlim_i \mathrm{GL}_n(\mathcal{O}_F/\mathfrak{m}_F^i)$ is an isomorphism of topological groups. Here on the right hand side each $\mathrm{GL}_n(\mathcal{O}_F/\mathfrak{m}_F^i)$ is finite and equipped with the discrete topology.

Solution. It suffices to check the compatibility of the topologies on both $\varprojlim_i \mathrm{GL}_n(\mathcal{O}_F/\mathfrak{m}_F^i)$ and $\mathrm{GL}_n(\mathcal{O}_F)$. Since the former is a projective limit of finite discrete groups, we are to show that $\mathrm{GL}_n(\mathcal{O}_F)$ is profinite. By Exercise 1.3, it boils down to showing that $\mathrm{GL}_n(\mathcal{O}_F)$ is a Hausdorff, compact, and totally disconnected topological space. For this, identify $\mathrm{M}_n(\mathcal{O}_F)$ with $\mathcal{O}_F^{n^2}$, and consider $\mathrm{GL}_n(\mathcal{O}_F) \hookrightarrow \mathrm{M}_n(\mathcal{O}_F) \times \mathcal{O}_F \simeq \mathcal{O}_F^{n^2+1}$ via $g \mapsto (g, \det(g^{-1}))$; also, note that $\mathrm{GL}_n(\mathcal{O}_F)$ is cut out from $\mathcal{O}_F^{n^2+1}$ by polynomial equations, and is thus a closed subset with induced topology from $\mathcal{O}_F^{n^2+1}$. This proves the desired result as $\mathcal{O}_F$ itself is a Hausdorff, compact, and totally disconnected topological space. □

Week 2

Exercise 2.1. Let V be a possibly infinite-dimensional vector space over a field F.

(1) Use Zorn's lemma to show that V has a basis.
(2) For any subspace $W \subset V$, there exists a complement, i.e., a subspace $W' \subset V$ such that $V = W \oplus W'$.
(3) If $0 \to W \to V \to U \to 0$ is a short exact sequence of vector spaces, then the dual sequence $0 \to U^* \to V^* \to W^* \to 0$ is also exact.

Solution. (1) Let $(U_i)_{i\in I}$ be a totally ordered collection of linearly independent subsets in V, i.e. for each U_i and finitely many vectors $v_1, \dots, v_n \in U_i$ we have $f_1 v_1 + \cdots + f_n v_n = 0 \in V$ implies $f_1 = \cdots = f_n = 0$ in F; here the order is given by the set-theoretical inclusion, that is, $U_i \leq U_j$ if and only if $U_i \subset U_j$. From this construction, $(U_i)_{i\in I}$ forms a chain of linearly independent subsets in V, which obtains a maximal element $\bigcup_{i\in I} U_i$. It follows from Zorn's lemma that among all linearly independent subsets in V there is a set-theoretically maximal subset, whose elements form a basis of V.

(2) Let W be a subspace of V. Pick a basis $\{e_i\}_{i\in I}$ of V so that $V = \bigoplus_{i\in I} Fe_i$. Consider the set $\mathcal{J}$ of subsets J of I for which $W \cap \sum_{j\in J} Fe_j = 0$. Then $\mathcal{J} \neq \emptyset$ and $\mathcal{J}$ is inductively ordered by inclusion. If J is a maximal element of $\mathcal{J}$ then the sum $X \coloneqq W + \bigoplus_{j\in J} Fe_j$ is direct. If $X \neq V$ there is $i \in I$ with $Fe_i \not\subset X$, so the sum $X + Fe_i$ is direct, and $J \cap \{i\} \in \mathcal{J}$, contrary to hypothesis. Thus $W' = \bigoplus_{j\in J} Fe_j$ is the desired complement of W.

(3) Given an injective map $s\colon W \to V$, it induces the dual map $s^*\colon V^* \to W^*,\ \phi \mapsto \phi \circ s$. If s is surjective and $\phi \circ s = 0$, then $\phi = 0$ follows; in this case s^* is injective. Similarly, whenever $t\colon V \to U$ is injective, we have that $t^*\colon U^* \to V^*$ is surjective. Therefore, the dual functor $(\textbf{-})^*$ on vector spaces is exact. (Here we omit to check the exactness condition $\ker s^* = \operatorname{im} t^*$.) □

Exercise 2.2. Show that a countable-dimensional vector space over $\mathbb{Q}$ (the rational numbers) is countable as a set. Show that the dual of such a vector space is uncountable as a set. (Here, countable means infinite countable.)

Solution. Let V be a countable-dimensional $\mathbb{Q}$-vector space with basis $\{e_i\}_{i\in I}$. Then $\operatorname{card} I \leqslant \operatorname{card} \mathbb{Z}$. Fix an isomorphism $V = \bigoplus_{i\in I} \mathbb{Q}e_i \simeq \mathbb{Q}^I$ of vector spaces. Then by set theory, $\operatorname{card} V = \operatorname{card} \mathbb{Q}^I = \operatorname{card} I \cdot \operatorname{card} \mathbb{Q} = \operatorname{card} I \cdot \operatorname{card} \mathbb{Z} \leqslant \operatorname{card} \mathbb{Z} \cdot \operatorname{card} \mathbb{Z} = \operatorname{card} \mathbb{Z}$. It follows that V is countable as a set.

As for the dual space V^*, it suffices to show that

$$\dim_{\mathbb{Q}} V < \dim_{\mathbb{Q}} V^*.$$

Let $\{e_i\}_{i\in I}$ be a basis of V. Each element of V^* is determined by its values on $\{e_i\}_{i\in I}$. For each $i \in I$, define $\phi_i \in V^*$ by $\phi_i(e_i) = 1$ and $\phi_i(e_j) = 0$ for $j \neq i$; then $\{\phi_i\}_{i\in I}$ is a linearly independent subset of V^*. It follows that V^* is infinite-dimensional, and thus $\dim_{\mathbb{Q}} V^* \geqslant \operatorname{card} \mathbb{Q}$.

> *Lemma.* If W is a vector space over a field K, then
>
> $$\operatorname{card} W = \max(\operatorname{card} K, \dim_K W).$$
>
> *Proof of Lemma.* For a non-zero $w \in W$, the set Kw is a sunset of W with size $\operatorname{card} K$, so $\operatorname{card} K \leqslant \operatorname{card} W$. A basis of W is a subset of W with size $\dim_K W$, so $\dim_K W \leqslant \operatorname{card} W$. Next we show $\operatorname{card} W \leqslant \max(\operatorname{card} K, \dim_K W)$. Pick a basis $\{f_i\}_{i\in I}$ of W. The elements of $w \in W$ are unique finite linear combinations $\sum_{i\in I} c_i f_i$ where all but finitely many $c_i \in K$ are non-zero; so we get an embedding of W into the finite subsets of $K \times I$ by $w \mapsto \{(c_i, f_i)\colon c_i \neq 0\}$. (Note when $w = 0$ we get $\emptyset \subset K \times I$.) Since I is infinite and $K \neq \emptyset$, we have $K \times I$ is infinite, and the cardinality of the finite subsets of an infinite set equals the cardinality of the set. Thus $\operatorname{card} W \leqslant \operatorname{card}(K \times I)$. It follows that

$\operatorname{card}(W) \leqslant \max(\operatorname{card}(K), \operatorname{card}(I)) = \max(\operatorname{card} K, \dim_K W)$. Thus the equality follows.

Using the lemma, we see that $\operatorname{card} V^* = \max(\operatorname{card} \mathbb{Q}, \dim_{\mathbb{Q}} V^*) = \dim_{\mathbb{Q}} V^*$. Now to prove $\dim_{\mathbb{Q}} V < \dim_{\mathbb{Q}} V^*$, it remains to show that $\operatorname{card} V^* > \dim_{\mathbb{Q}} V$. We know that $\dim_{\mathbb{Q}} V = \operatorname{card} I$. Elements of V^* are determined by their values on that basis, and those values can be arbitrary, so as a set V^* is in bijection with $\prod_{i\in I} \mathbb{Q}$. Denote $\mathcal{P}(I)$ the power set of I, i.e. the set of all subsets of I. Then $\operatorname{card} V^* \geqslant \operatorname{card}(\prod_{i\in I} \mathbb{Q}) \geqslant \operatorname{card} \mathcal{P}(I) > \operatorname{card} I = \dim_{\mathbb{Q}} V$. This finishes the proof that V^* is uncountable. $\square$

In the following, fix a locally profinite group G, and all representations are smooth representations of G.

Exercise 2.3. Let $V = \bigoplus_{i\in I} U_i$, where each U_i is an irreducible representation. Let $W \subset V$ be a subrepresentation. Use Zorn's lemma to show that there is a maximal subset $J \subset I$ such that $W \cap \bigoplus_{j\in J} U_j = 0$. For such J, show that $V = W \oplus \bigoplus_{j\in J} U_j$.

Solution. The situation is trivial when $W = V$. If $V \neq W$ then by Exercise 2.1(2) there is a proper subspace $0 \neq W' \subsetneq V$ such that $V = W \oplus W'$; in particular, $W \cap W' = 0$. It follows that there exists $i \in I$ such that U_i is a subspace of W' with that $W \cap U_i = 0$. Again if $W \oplus U_i \neq V$ then we get $j \in I$ with $j \neq i$ and $W \cap (U_i \oplus U_j) = 0$. Repeating this process iteratively we get an increasing chain of index sets $\{i\} \subset \{i, j\} \subset \cdots \subset I$. By Zorn's lemma there is a maximal subset J in this chain with the desired property. Also, the equality $V = W \oplus \bigoplus_{j\in J} U_j$ follows from that J is maximal. $\square$

Exercise 2.4. Let V be a finitely generated representation. That is, there exist finitely many elements $v_1, \ldots, v_n$ such that V is spanned by $\bigcup_{i=1}^n Gv_i$. Prove that there exists a maximal proper subrepresentation $W \subset V$. (Proper means $W \neq V$, but we allow W to be zero.) In general, for any representation V, show that a proper subrepresentation W is maximal if and only if V/W is irreducible.

Solution. By the finitely-generated assumption, each proper subrepresentation of V does not contain the subset $\{v_1, \ldots, v_n\}$ of V. (Otherwise the G-orbit of $\{v_1, \ldots, v_n\}$ must span the whole V as it contains $Gv_1 \cup \cdots \cup Gv_n$. Let S be the set of all proper subrepresentations of V, which is non-empty since $0 \in S$. Since $n < \infty$, any totally ordered union of elements in S is still proper. Hence by Zorn's lemma there is a maximal element in S.

In general, if V is any (smooth) representation, it is clear that V/W is irreducible implies that W is maximal. Conversely, if V/W is reducible, then for non-zero element $vW \in V/W$ we have that $W \subsetneq W + Gv \subsetneq V$, which implies that W is not maximal as a subrepresentation. $\square$

Exercise 2.5. Show that a sequence of representations $0 \to U \to V \to W \to 0$ is exact if and only if $0 \to U^K \to V^K \to W^K \to 0$ is exact for every compact open subgroup $K \subset G$.

Solution. The "if" part is clear and we omit the discussion. For the "only if" part, since taking invariants $(-)^K \colon V \mapsto V^K$ is always left exact, it suffices to show that whenever $b \colon V \to W$ is surjective, so also would $b^K \colon V^K \to W^K$ be for all compact open subgroups K. By the hypothesis, given $w \in W^K \subset W$ there exists $v \in V$ such that $b(v) = w$. Since W^K is a subrepresentation of W, we have a natural projection map $p_W \colon W \to W^K$; similarly we define $p_V \colon V \to V^K$. From the commutative diagram

$$\begin{tikzcd} V \arrow[r, "b"] \arrow[d, "p_V"'] & W \arrow[d, "p_W"] \\ V^K \arrow[r, "b^K"] & W^K \end{tikzcd}$$

we have that $w = p_W(w) = p_W(b(v)) = b^K(p_V(v))$. This proves the surjectivity of b^K and thus completes the proof. $\square$

Exercise 2.6 ([BH06, p.17, Exercise (2)(3)])**.**

(1) Let (π, V) be a smooth representation of G and (σ, W) an abstract representation. Let $f \colon V \to W$ be a G-homomorphism. Show that $f(V) \subset W^\infty$, and hence $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_G(V, W^\infty)$.

(2) Let $0 \to U \to V \to W \to 0$ be an exact sequence of G-homomorphisms of abstract G-spaces. Show that the induced sequence $0 \to U^\infty \to V^\infty \to W^\infty$ is exact. Show by example that the map $V^\infty \to W^\infty$ need not be surjective.

Solution. (1) Since V is a smooth representation we have $V = V^\infty$; in particular, each $v \in V$ lies in some V^K with K a compact open subgroup of G. Fix $v \in V^K \subset V$. As f is a G-homomorphism, we have that $\sigma(g)f(v) = f(\pi(g)v)$ for all $g \in G$. Whenever $g \in K$, we get $\sigma(g)f(v) = f(\pi(g)v) = f(v)$, that is, $f(v) \in W^K$. It follows that $f(V^K) \subset W^K$, and then $f(V) = f(V^\infty) \subset W^\infty$; therefore, $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_G(V, W^\infty)$.

(2) As in Exercise 2.5, taking invariants $(\boldsymbol{-})^K$ is always left exact, and thus the induced sequence $0 \to U^\infty \to V^\infty \to W^\infty$ is exact. For the counterexample, let V, W be two abstract G-spaces such that there is a G-homomorphism $V \to W$ factoring through $V/V^\infty \to W$ and $V/V^\infty \to W$ is surjective; such a construction exists because V/V^∞ is isomorphic to a sub-representation of V when V is semi-simple and sufficiently large. On the other hand, we have $(V/V^\infty)^\infty = 0$, which cannot map onto W^∞ unless $W^\infty = 0$. $\square$

Exercise 2.7 ([BH06, p.25, Exercise])**.** Let (π, V) and (σ, W) be smooth representations of G. Let $\wp(\pi, \sigma)$ be the space of G-invariant bilinear pairings $V \times W \to \mathbb{C}$. Show that there are canonical isomorphisms

$$\mathrm{Hom}_G(\pi, \check{\sigma}) \cong \wp(\pi, \sigma) \cong \mathrm{Hom}_G(\sigma, \check{\pi}).$$

Solution. It suffices to show the first isomorphism, and the second one follows from swapping π and σ. Consider the map $\wp(\pi, \sigma) \to \mathrm{Hom}_G(\pi, \check{\sigma})$ that sends $f \colon V \times W \to \mathbb{C}$ to $\mathrm{R}_f \colon V \to W^\vee$, where $\mathrm{R}_f(v) = f(v, \boldsymbol{-})$. As a map between vector spaces, it suffices to show that the map is bijective. For the injectivity, note that $\mathrm{R}_f(v)$ is a trivial linear form on W if and only if f is degenerate. For the surjectivity, it suffices to define the inverse of R_f; given $\psi \colon V \to W^\vee$ we have $\psi(v) \colon w \mapsto \psi(v)(w) \in \mathbb{C}$, so $(v, w) \mapsto \psi(v)(w)$ is the desired inverse map. This completes the proof. $\square$

Week 3

Exercise 3.1. Show that in a profinite group, the compact open normal subgroups form a neighborhood basis of 1.

Solution. From the result of Exercise 1.3, if K is an arbitrary compact open neighborhood of $1 \in G$, where G is a profinite group, then K has a basis consisting of compact open sets, and we aim to show that there is a compact open normal subgroup H of G contained in K. By the proof of van Dantzig's theorem (c.f. Exercise 1.4), there is a compact open subgroup H_0 of G contained in K. We then check that the desired H is given by

$$\bigcap_{g \in G/H_0} gH_0g^{-1}.$$

This is clearly an open normal subgroup of G, and hence is closed in G by Exercise 1.1(1); and as a closed subset of the compact H_0, it is compact as well. This completes the proof by following the guideline of Exercise 1.4. □

Exercise 3.2. Prove that the two maps specified in the proof of Frobenius Reciprocity (for Ind_G^H) are indeed inverse to each other.

Solution. Let H be a closed subgroup of G. Consider smooth representations π and σ of G and H, respectively. To prove the Frobenius reciprocity of Ind_H^G and Res_H^G, we have defined the following maps:

$$\begin{aligned} r\colon \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) &\longrightarrow \mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \sigma) \\ \Phi &\longmapsto \alpha_\sigma \circ \Phi \end{aligned}$$

together with

$$\begin{aligned} i\colon \mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \sigma) &\longrightarrow \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) \\ \Psi &\longmapsto (v \mapsto (f\colon G \to \sigma,\ g \mapsto \Phi(gv))). \end{aligned}$$

Here $\alpha_\sigma\colon \mathrm{Ind}_H^G \sigma \to \sigma$ is an H-map defined via $(f\colon G \to \sigma) \mapsto f(1)$. We first check that $r(i(\Psi)) = \Psi$. For each $v \in \pi$, its image along $r(\Psi)$ is $f\colon g \mapsto \Psi(gv)$, and thus after the pre-composition by α_σ we have $f(1) = \Psi(v) \in \sigma$, which is the same as the image of $v \in \mathrm{Res}_H^G \pi$ along $\Psi\colon \mathrm{Res}_H^G \pi \to \sigma$. We then check that $i(r(\Phi)) = \Phi$. For this we compute $i(\alpha_\sigma \circ \Phi)$ by considering $\pi \to \mathrm{Ind}_H^G \sigma \to \sigma$, $v \mapsto (g \mapsto \Phi(gv) \mapsto \Phi(v))$, which is the same as $v \mapsto \Phi(v)$. □

Exercise 3.3 ([BH06, p.19, Exercise 1–2])**.**

(1) Show that the functor c-Ind_H^G is additive and exact.
(2) Let G be a locally profinite group. Suppose H is open in G. Let $\phi\colon G \to W$ be a function, compactly supported modulo H, such that $\phi(hg) = \sigma(h)\phi(g)$ for $h \in H$ and $g \in G$. Show that $\phi \in X_{\mathrm{c}}$.

Solution. (1) Recall that for a smooth H-representation σ, $\text{c-Ind}_H^G\sigma$ consists of functions $f \in \mathrm{Ind}_H^G \sigma$ whose supports have compact projection image in $H\backslash G$. So the additivity of c-Ind_H^G follows from that of Ind_H^G, which is proved in [BH06, §2.4, Proposition]. For the exactness[2], by definition each function $f \in \text{c-Ind}_H^G\sigma$ satisfies $f(hgk) = \sigma(h)f(g)$, where $h \in H, g \in G$ and $k \in K$ for any open compact subgroup K. Thus if K is fixed then $f^K\colon G/K \to \sigma$ is determined by all $f(g)$'s whenever g runs through $H\backslash G/K$. (The double coset $H\backslash G/K$ is not necessarily finite unless $H\backslash G$ is compact.) This indeed proves that there is a natural isomorphism

[2] If we further assume H is open in G, then we have the Frobenius reciprocity $\mathrm{Hom}_G(\text{c-Ind}_H^G\sigma, \pi) \cong \mathrm{Hom}_H(\sigma, \mathrm{Res}_H^G \pi)$. In this case c-Ind_H^G has a right adjoint Res_H^G; it follows that c-Ind_H^G is right exact.

$$\begin{aligned}(\text{c-Ind}_H^G\sigma)^K &\xrightarrow{\sim} \bigoplus_{g\in H\backslash G/K} \sigma^{gKg^{-1}\cap H}\\ f &\longmapsto (f(g))_{g\in H\backslash G/K}.\end{aligned}$$

On the other hand, to show the exactness of $\text{c-Ind}_H^G(\textbf{-})$, it suffices to show for an arbitrary open compact subgroup K that $(\text{c-Ind}_H^G(\textbf{-}))^K$ is exact. Applying the result of Exercise 2.5, the functor $\sigma \mapsto \sigma^{gKg^{-1}\cap H}$ is exact as $gKg^{-1}\cap H$ is a compact open subgroup of H. So the functor sending σ to the target direct sum of the above isomorphism is exact as well. This finishes the proof that $\sigma \mapsto \text{c-Ind}_H^G\sigma$ is exact.

(2) Let (σ, W) be a smooth H-representation. By construction, X_{c} is the space of functions $f\colon G \to W$ satisfying that

(i) f is compactly supported modulo H, i.e. $\mathrm{supp}(f)$ has compact image along the projection $G \to H\backslash G$,
(ii) $f(hg) = \sigma(h)f(g)$ for $h \in H$ and $g \in G$, and
(iii) f is right G-smooth, i.e. there is a compact open subgroup K of G (depending on f) such that $f(gk) = f(g)$ for $g \in G$ and $k \in K$.

So the point is to show that the right G-smooth condition for ϕ is automatic. Here the condition (i) is equivalent to that $\mathrm{supp}(f) \subset HC$ for some compact subset $C \subset G$. The condition that H is open in G implies that H is a closed subgroup and that HC is an open subset. Since G is locally profinite, H is compact, and HC is thus an open compact subset. Again as G is locally profinite, $1 \in G$ has a neighborhood basis consisting of open compact subgroups. As a consequence, there are finitely many sufficiently small open compact subgroups $K_1, \ldots, K_n$ such that HC admits a finite cover by $\{g_iK_i\}_{i=1}^n$ for distinct $g_i \in G$. It follows that $\mathrm{supp}(f)$ admits a finite cover by $\{Hg_iK_i\}_{i=1}^n$. Note that G is totally disconnected and K is a sufficiently small neighborhood of 1. Therefore, $K \coloneqq K_1 \cap \cdots \cap K_n$ is the desired open compact subgroup such that $f(gk) = f(g)$ for all $g \in G$ and $k \in K$. □

Exercise 3.4. Let H, H_0 be closed subgroups of a locally profinite group G with $H_0 \subset H$. Show that $\mathrm{Ind}_H^G \circ \mathrm{Ind}_{H_0}^H$ is canonically isomorphic to $\mathrm{Ind}_{H_0}^G$.

Solution. We have the natural isomorphism between functors that $\mathrm{Res}_{H_0}^H \circ \mathrm{Res}_H^G \cong \mathrm{Res}_{H_0}^G$. For any smooth H_0-representation σ, we aim to show $\mathrm{Ind}_H^G \mathrm{Ind}_{H_0}^H \sigma \cong \mathrm{Ind}_{H_0}^G \sigma$. By applying Yoneda's lemma in an opposite sense, it suffices to show there is a canonical isomorphism

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \mathrm{Ind}_{H_0}^H \sigma) \cong \mathrm{Hom}_G(\pi, \mathrm{Ind}_{H_0}^G \sigma)$$

for any smooth G-representation π. Since H and H_0 are closed, we have from Frobenius reciprocity that

$$\begin{aligned}\mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \mathrm{Ind}_{H_0}^H \sigma) &\cong \mathrm{Hom}_{H_0}(\mathrm{Res}_{H_0}^H \mathrm{Res}_H^G \pi, \sigma)\\ &\cong \mathrm{Hom}_{H_0}(\mathrm{Res}_{H_0}^G \pi, \sigma).\end{aligned}$$

This proves the desired isomorphism $\mathrm{Ind}_H^G \circ \mathrm{Ind}_{H_0}^H \cong \mathrm{Ind}_{H_0}^G$ of functors. □

Exercise 3.5. Complete the proof of [BH06, §2.7, Lemma (2)].

Solution. Let G be a locally profinite group, and let $H \subset G$ be an open subgroup of finite index. If (σ, W) is a semi-simple smooth representation of H, we aim to prove that $\mathrm{Ind}_H^G \sigma$ is semi-simple as a representation of G. In class we proved it assuming H is normal. More precisely, when H is normal, we have concluded that

$$\mathrm{Res}_H^G \mathrm{Ind}_H^G \sigma = \bigoplus_{g\in G/H} \sigma^g$$

which is semi-simple. It follows that $\mathrm{Ind}_H^G \sigma$ is also semi-simple by [BH06, §2.7, Lemma (1)].

For general H, we set $H_0 \coloneqq \bigcap_{g \in G/H} gHg^{-1}$. Here the intersection is finite since $[G : H] < \infty$, so H_0 is an open normal subgroup in G (as well as in H) of finite index. Motivated by the argument above, we aim to show that $\mathrm{Res}_H^G \mathrm{Ind}_H^G \sigma$ is semi-simple; it suffices to show that $\mathrm{Res}_{H_0}^H \mathrm{Res}_H^G \mathrm{Ind}_H^G \sigma = \mathrm{Res}_{H_0}^G \mathrm{Ind}_H^G \sigma$ is semi-simple. For this, note that since σ is semi-simple and H_0 is normal in H, there exists a smooth semi-simple representation σ_0 of H_0 such that $\sigma = \mathrm{Ind}_{H_0}^H \sigma_0$. Thus we have to show $\mathrm{Res}_{H_0}^G \mathrm{Ind}_H^G \mathrm{Ind}_{H_0}^H \sigma_0 = \mathrm{Res}_{H_0}^G \mathrm{Ind}_{H_0}^G \sigma_0$ is semi-simple; but this is clear from the result that assumes the normality. □

Exercise 3.6. Let G be a locally profinite group. Prove the following statements. Every element of $C_c^\infty(G)$ is a finite linear combination of elements of the form $\mathbb{1}_{gK}$, for $g \in G$ and K compact open subgroups of G. Also, every element of $C_c^\infty(G)$ is a finite linear combination of elements of the form $\mathbb{1}_{KgK}$, for $g \in G$ and K compact open subgroups of G.

Solution. For $f \in C_c^\infty(G)$, it is locally constant and compactly supported. Since G is locally profinite, 1 has a neighborhood basis consisting of compact open subgroups of G; let K be one of these subgroups, then the local constancy of f implies that $f(1) = f(k)$ for all $k \in K$. Similarly, taking the left and right translations by $g \in G$, we get $f(kg) = f(g) = f(gk')$ for all $k, k' \in K$ for K sufficiently small. On the other hand, since $\mathrm{supp}(f)$ is compact, it admits a finite open cover of form $\{g_iU\}_{i=1}^n$, where U is an open neighborhood of 1. By shrinking U to a sufficiently small subset if necessary, we have that $KU = UK = U$. This together with the argument for Exercise 1.4 dictates that we may assume $U = K$ without loss of generality. Since f is locally constant on U, the same is true for g_iU, say $f(g_iU) \equiv a_i \in \mathbb{C}$. Then

$$f = \sum_{i=1}^n a_i \mathbb{1}_{g_iU} = \sum_{i=1}^n a_i \mathbb{1}_{g_iK}.$$

Using a similar argument with $\{g_iU\}_{i=1}^n$ replaced by $\{Ug_i\}_{i=1}^n$, we can show that f is a finite linear combination of elements of the form $\mathbb{1}_{Kg_i}$; combining these two results together deduces the desired proof for $\mathbb{1}_{KgK}$. □

Week 4

Throughout G is a locally profinite group.

Exercise 4.1. Let μ be a left Haar measure on G. Show that the functional $C_{\mathrm{c}}^{\infty}(G) \to \mathbb{C}$ sending f to

$$\int_G f(g)\delta_G(g^{-1})\mathrm{d}\mu(g)$$

is well-defined and is a right Haar measure.

Solution. We first check that the functional is a right Haar measure, i.e. the image of f and $r(x)f$ are the same, where $r(x)$ denotes the right translation by $x \in G$. For this, compute

$$\begin{aligned}\int_G f(gx)\delta_G(g^{-1})\mathrm{d}\mu(g) &= \delta_G(x)\int_G f(gx)\delta_G(x^{-1}g^{-1})\mathrm{d}\mu(g)\\ &= \delta_G(x)\int_G f(gx)\delta_G((gx)^{-1})\mathrm{d}\mu(g)\\ &= \int_G f(g)\delta_G(g^{-1})\mathrm{d}\mu(g).\end{aligned}$$

It follows that $\mathrm{d}\mu(g^{-1}) = \delta_G(g^{-1})\mathrm{d}\mu(g)$ is a right Haar measure whenever $\mathrm{d}\mu(g)$ is a left Haar measure. To show the functional is well-defined, note that by Exercise 3.6, $f \in C_{\mathrm{c}}^{\infty}(G)$ is a finite linear combination of $\mathbb{1}_{xK}$'s, where $x \in G$ and K is an open compact subgroup in G. Then the image of f is a finite linear combination of integrals of form

$$\int_G \mathbb{1}_{xK}(g)\delta_G(g^{-1})\mathrm{d}\mu(g) = \int_G \mathbb{1}_{xK}(g^{-1})\mathrm{d}\mu(g),$$

which is non-zero only when $g \in xK \cap (xK)^{-1}$. Therefore, the integral $\int_G \mathbb{1}_{xK}(g^{-1})\mathrm{d}\mu(g)$ is controlled by $\int_G \mathbb{1}_{xK}(g)\mathrm{d}\mu(g) = \mu(xK)$. So the well-definedness follows from the choice of $\mu(gK)$, which makes sense from the assumption that μ is already a left Haar measure. $\square$

Exercise 4.2. Let μ be a left Haar measure on G. Let K be a compact open subgroup. Show that $\mu(gKg^{-1}) = \delta_G(g)^{-1}\mu(K)$. Thus we have the formula

$$\delta_G(g) = [K : K \cap gKg^{-1}][gKg^{-1} : K \cap gKg^{-1}]^{-1}.$$

Solution. For the first formula, compute

$$\begin{aligned}\int_G \mathbb{1}_{gK}(xg)\mathrm{d}\mu(x) &= \delta_G(g)^{-1}\int_G \mathbb{1}_{gK}(x)\mathrm{d}\mu(x)\\ &= \delta_G(g)^{-1}\int_G \mathbb{1}_{gK}(gx)\mathrm{d}\mu(x).\end{aligned}$$

Note that the first equality above follows from definition of modulus character δ_G, and the second equality is because μ is a left Haar measure (and hence $\mathrm{d}\mu(gx) = \mathrm{d}\mu(x)$). Since we have $\mathbb{1}_{gK}(xg) = \mathbb{1}_{gKg^{-1}}(x)$ and $\mathbb{1}_{gK}(gx) = \mathbb{1}_K(x)$, we deduce

$$\mu(gKg^{-1}) = \int_G \mathbb{1}_{gKg^{-1}}(x)\mathrm{d}\mu(x) = \delta_G(g)^{-1}\int_G \mathbb{1}_K(x)\mathrm{d}\mu(x) = \delta_G(g)^{-1}\mu(K).$$

For the second formula, notice that both gKg^{-1} and $K \cap gKg^{-1}$ are compact open subgroups of G. Then

$$\delta_G(g) = \frac{\mu(K)}{\mu(gKg^{-1})} = \frac{[K : K \cap gKg^{-1}]\cdot\mu(K \cap gKg^{-1})}{[gKg^{-1} : K \cap gKg^{-1}]\cdot\mu(K \cap gKg^{-1})} = \frac{[K : K \cap gKg^{-1}]}{[gKg^{-1} : K \cap gKg^{-1}]}.$$

$\square$

Exercise 4.3. Let F be a non-archimedean local field, and let

$$G := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F) \,\middle|\, c = 0 \right\}.$$

Let

$$K := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G \,\middle|\, a, d \in \mathcal{O}_F^\times,\ b \in \mathcal{O}_F \right\}.$$

Show that K is a compact open subgroup of G. Then use the previous exercise to show that $\delta_G\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = |d/a|$, where $|\cdot|$ is the absolute value on F normalized so that a uniformizer has absolute value $|k_F|^{-1}$.

Solution. Such K is a subgroup of G because of

$$\begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ 0 & \mathcal{O}_F^\times \end{pmatrix}^{-1} \subset \begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ 0 & \mathcal{O}_F^\times \end{pmatrix}, \quad \begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ 0 & \mathcal{O}_F^\times \end{pmatrix}^2 \subset \begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ 0 & \mathcal{O}_F^\times \end{pmatrix}.$$

As for the topology, note that K is homeomorphic to $\mathcal{O}_F^\times \times \mathcal{O}_F^\times \times \mathcal{O}_F$ and G is equipped with the induced topology from $\mathrm{M}_2(F)$. We know that $\mathcal{O}_F$ is compact; to show K is compact it suffices to show $\mathcal{O}_F^\times$ is closed in $\mathcal{O}_F$, but this is implied by the definition that $\mathcal{O}_F^\times$ consists of elements $x \in \mathcal{O}_F$ with $|x| = 1$. Also, K contains the subgroup $\mathrm{R_u}(G) \cap \mathrm{GL}_2(\mathcal{O}_F)$ that is homeomorphic to $\mathcal{O}_F$, where $\mathrm{R_u}(G)$ is the unipotent subgroup of G consisting of matrices of form $\left(\begin{smallmatrix} 1 & t \\ & 1 \end{smallmatrix}\right)$ with $t \in F$; as $\mathcal{O}_F$ is the open unit ball in F, we deduce from Exercise 1.1(3) that K is open. So we have shown that K is an open compact subgroup. For $g = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$, we compute

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1} = \begin{pmatrix} x & ayd^{-1} + \cdots \\ 0 & z \end{pmatrix}.$$

It then follows from Exercise 4.2 that

$$\delta_G(g) = \frac{\mu(K)}{\mu(gKg^{-1})} = |\det(\mathrm{Ad}(g))| = |a| \cdot |d|^{-1} = |ad^{-1}|.$$

In particular, as δ_G is non-trivial while restricting to K, we conclude that G is not unimodular. $\square$

Exercise 4.4. Let $H \leqslant G$ be a closed subgroup, and let μ_H be a left Haar measure on H. Let $\theta = \delta_G|_H \cdot \delta_H^{-1} \colon H \to \mathbb{R}_{>0}$. In class we defined the G-map

$$\begin{aligned} \mathscr{A} \colon (C_c^\infty(G), \rho) &\longrightarrow (C_c^\infty(H\backslash G, \theta), \rho) \\ f &\longmapsto \tilde{f}, \end{aligned}$$

where

$$\tilde{f}(g) = \int_H \delta_G(h)^{-1} f(hg) \mathrm{d}\mu_H(h).$$

Fix a compact open subgroup K of G. Write $\ker(\mathscr{A})^K$ for the set of right K-invariant elements of $\ker(\mathscr{A})$.

(1) Show that every element of $\ker(\mathscr{A})^K$ is a linear combination of elements $f_i \in \ker(\mathscr{A})^K$ such that each f_i satisfies $\mathrm{supp}(f_i) \subset Hg_iK$ for some $g_i \in G$.
(2) Let $f \in \ker(\mathscr{A})^K$, and assume that f is supported on HgK. Show that f is a finite linear combination

$$f = \sum_{i=1}^n c_i \mathbb{1}_{h_i g K},$$

with $c_i \in \mathbb{C}$ and $h_i \in H$, satisfying

$$\sum_{i=1}^n c_i \delta_G(h_i)^{-1} = 0.$$

(3) Show that every element of $\ker(\mathscr{A})$ is killed by the right Haar measure on G. Therefore we obtain a non-zero, right G-invariant map $C_c^\infty(H\backslash G, \theta) \to \mathbb{C}$.

Solution. Let δ_G be the fixed Haar measure on H appearing in the definition of θ.

(1) Recall the conclusion from class that for a fixed $g \in G$, the image $\mathscr{A}(\mathbb{1}_{gK})$ is the unique element in $C_c^\infty(H\backslash G, \theta)$ exactly supported on HgK that is right K-invariant and has the value $\mu_H(gKg^{-1} \cap H)$ at g. On the other hand, by Exercise 3.6, $f \in \ker(\mathscr{A})^K \subset C_c^\infty(G)$ is a finite linear combination of $f_i := \mathbb{1}_{g_iK}$'s for all $i = 1, \dots, n$, and therefore $\mathscr{A}(f)$ is a finite linear combination of $\mathscr{A}(\mathbb{1}_{g_iK})$'s. Now we automatically have $\mathrm{supp}(f_i) \subset Hg_iK$ because of the prescribed conclusion, and it remains to check $f_1, \dots, f_n \in \ker(\mathscr{A})^K$. But this is implied from the linear independence of f_i's and the assumption that $\mathscr{A}(f) = 0$.

(2) By assumption each element of $\mathrm{supp}(f)$ has form hgk for some $h \in H$ and $k \in K$. And by (1) it always lies in certain compact open g_iK. It follows that there is a fixed element $g \in G$ together with $h_1, \dots, h_n \in H$ so that $g_i = h_i g$. So f is a finite linear combination

$$f = \sum_{i=1}^n c_i \mathbb{1}_{h_i gK},$$

with $c_i \in \mathbb{C}$ and $h_i \in H$. To check the desired equality, it suffices to show that if $\mathscr{A}(\mathbb{1}_{h_igK}) = \widetilde{\mathbb{1}}_{h_igK} = 0$ then $\delta_G(h_i)^{-1} = 0$; this assumption means that $\int_G \delta_G(h)^{-1}\mathbb{1}_{h_igK}(hx)\mathrm{d}\mu_H(h) = 0$ for all $x \in G$, and in particular this holds for all $x = h^{-1}h_i g$, where h runs through all elements of H. Since δ_G is a homomorphism with target $\mathbb{R}_{\geqslant 0}$, this deduces $\delta_G(h)^{-1} = 0$ for all h as desired (otherwise the integral of $\delta_G(h)^{-1}\mathbb{1}_{h_igK}(hx)$ would be positive). Therefore, we get

$$\sum_{i=1}^n c_i \delta_G(h_i)^{-1} = 0.$$

(3) We are to show that $\tilde{f}(g) = \int_H \delta_G(h)^{-1} f(hg)\mathrm{d}\mu_H(h) = 0$ implies $\int_G f(g)\mathrm{d}\mu_G(g) = 0$ for any fixed $g \in G$. By assumption, we have for any $\phi \in C_c^\infty(G)$ that

$$\begin{aligned} 0 &= \int_G \phi(g)\tilde{f}(g)\mathrm{d}\mu_G(g) \\ &= \int_G \phi(g) \int_H \delta_G(h)^{-1} f(hg)\mathrm{d}\mu_H(h)\mathrm{d}\mu_G(g) \\ &= \int_H \int_G \phi(g)\delta_G(h)^{-1} f(hg)\mathrm{d}\mu_G(g)\mathrm{d}\mu_H(h) \\ &= \int_H \int_G \phi(h^{-1}g)\delta_G(h)^{-1} f(h \cdot h^{-1}g)\mathrm{d}\mu_G(g)\mathrm{d}\mu_H(h), \end{aligned}$$

where the second-last equality is by changing the order of integrals and the last equality is by replacing g with $h^{-1}g$. Here we have used the property that μ_G is a left Haar measure and hence $\mathrm{d}\mu_G(g) = \mathrm{d}\mu_G(h^{-1}g)$. Changing the integral order again, we have that

$$\begin{aligned} 0 &= \int_G f(g) \int_H \delta_G(h)^{-1}\phi(h^{-1}g)\mathrm{d}\mu_H(h)\mathrm{d}\mu_G(g) \\ &= \int_G f(g)\tilde{\phi}(g)\mathrm{d}\mu_G(g). \end{aligned}$$

Recall from class that $\mathscr{A}$ is surjective. It follows that if C is a compact subset containing $\mathrm{supp}(f)$ then there exists $\phi \in C_c^\infty(G)$ such that $\tilde{\phi}(x)$ is a non-zero constant for all $x \in HC$ (to guarantee this we need the conclusion at the beginning of (1)). This shows that $\int_G f(g)\mathrm{d}\mu_G(g) = 0$ and we complete the proof. ∎

Week 5

Exercise 5.1. Let F be a non-archimedean local field, $G = \mathrm{GL}_2(F)$, and let B, N, T be the usual subgroups. Let $K = \mathrm{GL}_2(\mathcal{O}_F)$. Suppose the Haar measures on G, $T \cong F^\times \times F^\times$, $N \cong F$ are all induced, as explained in class, by the same Haar measure on F normalized by $\mu(\mathcal{O}_F) = C$. (See also [BH06, 7.4–7.6].) Suppose the Haar measure μ_B on B is normalized such that $\int_B f(b)\mathrm{d}\mu_B(b) = \int_T \int_N f(tn)\mathrm{d}\mu_T(t)\mathrm{d}\mu_N(n)$, and suppose the Haar measure on K is the restriction of that on G. In class, we showed that there exists a non-zero constant D such that

$$\forall f \in C_c^\infty(G), \quad \int_G f(g)\mathrm{d}\mu_G(g) = D \cdot \int_K \int_B f(bk)\mathrm{d}\mu_B(b)\mathrm{d}\mu_K(k).$$

Compute the constant D (which possibly depends on C).

Solution. As K is an open compact subgroup of G, we have $\mathbb{1}_K \in C_c^\infty(G)$. From the assumption, μ_K is restricted from μ_G, so that

$$\mu_K(K) = \mu_G(K).$$

To compute the integral with respect to $\mathrm{d}\mu_B \cdot \mathrm{d}\mu_K$, the only ambiguity lies in $H := \{(x, x) : x \in B \cap K\}$. Put the normalized Haar measure μ_H on H by restricting μ_B from B to H. Since μ_B is normalized so that $\mathrm{d}\mu_B = \mathrm{d}\mu_T \cdot \mathrm{d}\mu_N$, and both μ_T, μ_N are compatible with μ_K, we see μ_H is compatible with the restriction of μ_K. Now we have

$$\int_K \int_B \mathbb{1}_K(bk)\mathrm{d}\mu_B(b)\mathrm{d}\mu_K(k) = \mu_H(H) \cdot \mu_K(K),$$

$$\int_G \mathbb{1}_K(g)\mathrm{d}\mu_G(g) = \mu_G(K).$$

Thus it suffices to compute $\mu_H(H)$, where $H \cong B(\mathcal{O}_F) \cong \mathcal{O}_F^\times \times \mathcal{O}_F^\times \times \mathcal{O}_F$. On the other hand, we have the formula $|\det a| \cdot \mathrm{d}\mu_{F^\times}(ag) = \mathrm{d}\mu_F(g) = \mathrm{d}\mu(g)$, with $|\det a| = 1$ for $a \in \mathcal{O}_F^\times = \mathrm{GL}_1(\mathcal{O}_F)$. Combining these deduces $\mu_{F^\times} = \mu$; and then $\mu(\mathcal{O}_F^\times) = \mu(\mathcal{O}_F) = C$ because of $\mathcal{O}_F^\times = \mathcal{O}_F \cap F^\times$ and $\mathcal{O}_F = \mathcal{O}_F \cap F$. It follows that

$$D = \frac{1}{\mu_H(H)} = \frac{1}{\mu(\mathcal{O}_F^\times)^2 \cdot \mu(\mathcal{O}_F)} = \frac{1}{C^3}.$$

□

Exercise 5.2. In the previous notation, show that for given Haar measures μ_T, μ_N on T and N, the formula

$$\int_B f(b)\mathrm{d}\mu_B(b) := \int_T \int_N f(tn)\mathrm{d}\mu_N(n)\mathrm{d}\mu_T(t)$$

indeed defines a left Haar measure on B. Explain why this does not work if we replace $f(tn)$ by $f(nt)$ in the above formula.

Solution. To show the left B-invariance, take an arbitrary $b \in B$ and write $b = t'n'$ for some $t' \in T$ and $n' \in N$. Since for $T \cong F^\times \times F^\times$ and $N \cong F$, the Haar measures μ_T and μ_N are induced from $\mu_F = \mu_{F^\times}$ by Exercise 5.1, and $F^\times = \mathbb{G}_\mathrm{m}(F)$ is unimodular, we see in particular that μ_N is left N-invariant and μ_T is right T-invariant. Note that b and t commutes. So we compute

$$\begin{aligned}\int_T \int_N f(btn)\mathrm{d}\mu_N(n)\mathrm{d}\mu_T(t) &= \int_T \int_N f(tbn)\mathrm{d}\mu_N(n)\mathrm{d}\mu_T(t) \\ &= \int_T \int_N f(tt'n'n)\mathrm{d}\mu_N(n)\mathrm{d}\mu_T(t) \\ &= \int_T \int_N f(tn)\mathrm{d}\mu_N(n)\mathrm{d}\mu_T(t).\end{aligned}$$

It follows that μ_B is a left Haar measure on B. Note that n does not commute with $b \in B$ in general. Thus if we replace $f(tn)$ by $f(nt)$ then we are unable to deduce $f(tnb) = f(tbn)$. Instead, there always holds $f(tnb) = \delta_B(b)^{-1} f(tn)$ with non-trivial δ_B, because B is not unimodular by Exercise 4.3. □

Exercise 5.3. In the previous notation, prove the Iwasawa decomposition $G = BK$ by explicit matrix manipulation. Generalize it to GL_n.

Solution. We directly prove the case where $G = \mathrm{GL}_n(F)$ (as opposed to dealing with GL_2 first). It suffices to show that given $g = (g_{ij})_{i,j} \in \mathrm{GL}_n(F)$ we are able to transform it into an element of B via taking the right multiplication by matrices in $K = \mathrm{GL}_2(\mathcal{O}_F)$. The algorithm (which is an analogue of Gram–Schmidt orthonormalization over $\mathbb{C}$) is as follows:

(i) Let $|\cdot|$ be the standard normalized absolute value on F. Among $a_{n1}, \dots, a_{nn}$ there is an entry a_{nj} such that $|a_{nj}| \leqslant |a_{nk}|$ for $k \neq j$. Then swap the jth and the nth columns of g.

(ii) Apply Gauss–Jordan algorithm to eliminate the first $n-1$ columns by the nth column so that we may assume $a_{ni} = 0$ for $1 \leqslant i \leqslant n-1$.

(iii) Iterate the operation above for the submatrix in g consisting of the first $n-1$ rows and columns.

Note that each step can be realized by an element of K multiplied from the right side of g. Thus the direct product factorization $G = BK$ follows. □

Exercise 5.4. For F a non-archimedean local field with uniformizer π, use Smith normal form to prove the Cartan decomposition:

$$\mathrm{GL}_n(F) = \bigsqcup_{\substack{a_1 \geqslant \cdots \geqslant a_n \\ a_i \in \mathbb{Z}}} \mathrm{GL}_n(\mathcal{O}_F) \begin{pmatrix} \pi^{a_1} & & \\ & \ddots & \\ & & \pi^{a_n} \end{pmatrix} \mathrm{GL}_n(\mathcal{O}_F).$$

Then show that $\mathrm{GL}_n(F)$ satisfies the condition "countable at infinity", i.e. $\mathrm{GL}_n(F)/K$ is countable for any compact open subgroup K.

Solution. Since $\mathcal{O}_F$ is a PID, using the fundamental theorem of finitely generated modules over PIDs, any matrix $g \in \mathrm{GL}_n(F)$ can be turned into Smith normal form with elementary divisors $\pi^{a_1}, \dots, \pi^{a_n}$ such that π^{a_j} divides π^{a_i} for $i \leqslant j$, namely $a_1 \geqslant \cdots \geqslant a_n$ with $a_i \in \mathbb{Z}$. Similar to the argument of Exercise 5.3, this process can be done by multiplying by elementary matrices in $\mathrm{GL}_n(\mathcal{O}_F)$ from left and right, and hence we have proved the (set-theoretical) Cartan decomposition.

We then prove that for $G = \mathrm{GL}_n$, $G(F)$ is countable at infinity. Let K be any compact open subgroup. Then $K \cap G(\mathcal{O}_F)$ is compact, open, and of finite index in $G(\mathcal{O}_F)$. Therefore, the surjection $\pi\colon G(F)/(K \cap G(\mathcal{O}_F)) \twoheadrightarrow G(F)/G(\mathcal{O}_F)$ has finite fibres. Consequently, if the target of π is countable then so also is the source, and hence $G(F)/K$ is countable as well. Therefore, it suffices to show that $G(F)/G(\mathcal{O}_F)$ is countable. Indeed, by Cartan decomposition, the homogeneous space $G(\mathcal{O}_F)\backslash G(F)/G(\mathcal{O}_F)$ is countable, and each double coset contains only finitely many right cosets of form $gG(\mathcal{O}_F)$, implying that $G(F)/G(\mathcal{O}_F)$ is countable. □

Exercise 5.5 ([BH06, p.51, Exercise])**.** Let F be a non-archimedean local field. Let K be a compact subgroup of $G = \mathrm{GL}_2(F)$. Show that $gKg^{-1} \subset \mathrm{GL}_2(\mathcal{O}_F)$ for some $g \in G$. Deduce that, up to G-conjugacy, $\mathrm{GL}_2(\mathcal{O}_F)$ is the unique maximal compact subgroup of G.

Solution. Note that $\mathrm{GL}_2(\mathcal{O}_F)$ is exactly the set of $g \in G$ such that $g\mathcal{O}_F^2 = \mathcal{O}_F^2$. Because $G = \mathrm{GL}_2(F)$ acts transitively on the set of bases in F^2, it also acts transitively on the set of lattices in F^2 (defined as $\mathcal{O}_F$-submodules of finite type and maximal rank 2). We obtain an identification of $\mathrm{GL}_2(F)/\mathrm{GL}_2(\mathcal{O}_F)$ with the set of lattices in F^2. Note that the quotient topology on $\mathrm{GL}_2(F)/\mathrm{GL}_2(\mathcal{O}_F)$ is the discrete topology because $\mathrm{GL}_2(\mathcal{O}_F)$ is open in $\mathrm{GL}_2(F)$.

The statement is equivalent to the existence of a lattice $\Lambda \subset F^2$ such that $k(\Lambda) = \Lambda$ for all $k \in K$. Because K is compact, the image of $\det\colon K \to F^\times$ is compact, and so for any $k \in K$

such that $k(\Lambda) \subset \Lambda$ we actually have the equality; so it suffices to show there is a K-stable lattice Λ. Let Λ_0 be any lattice, then there is an open compact subgroup $K' \subset K$ such that Λ_0 is stable under K' (if $L_0 = \mathcal{O}_F^2$ we can take $K' = K \cap \mathrm{GL}_2(\mathcal{O}_F)$). Then we complete the proof by taking $L = \sum_{g \in K/K'} g(L_0)$. □

Alternative Solution. We refer to [Tit79, §3.2] and [Bor11, Chapter VII, Theorem 1.2]. Consider the G-action on $F \oplus F$; up to scalar there is a unique $\mathcal{O}_F$-lattice Λ_0 spanned by $(1,0)$ and $(0,1)$ whose stabilizer is exactly $\mathrm{GL}_2(\mathcal{O}_F)$. Let v_0 be the corresponding vertex of Λ_0 in the associated Bruhat–Tits tree $\mathbf{T}_G$ of G. Recall that G acts on $\mathbf{T}_G$ transitively. Thus from the prescribed observation there is a bijection between $\mathrm{GL}_2(F)/\mathrm{GL}_2(\mathcal{O}_F)$ and the set of vertices of $\mathbf{T}_G$. It follows that $gKg^{-1} \subset \mathrm{GL}_2(\mathcal{O}_F)$ if and only if gKg^{-1} stabilizes v_0 in $\mathbf{T}_G$, which is further equivalent to that K stabilizes gv_0 for some g. Again, since G acts on $\mathbf{T}_G$ transitively, it suffices to find out a lattice of $F \oplus F$ that is stabilized by K. For this, let Λ be a lattice such that $\mathfrak{p}\Lambda_0 \subset \Lambda \subset \Lambda_0$, where $\mathfrak{p}$ is the place of F above $p \in \mathbb{Q}_p$, or equivalently Λ is (up to $\mathcal{O}_F^\times$-scalars) a vertex of $\mathbf{T}_G$ in the neighborhood of Λ_0; then $K\Lambda$ is the desired lattice stabilized by K. This completes the proof. □

Exercise 5.6 (Iwahori decomposition, [BH06, (7.3.1)])**.** Show that we have $I = (I \cap N')(I \cap T)(I \cap N)$. More precisely, the product map

$$\varphi\colon (I \cap N') \times (I \cap T) \times (I \cap N) \longrightarrow I$$

is bijective, and a homeomorphism, for any ordering of the factors on the left hand side.

Solution. Denote by ϖ the uniformizer of $\mathfrak{p}_F$. To show the map is bijective, we construct the inverse map $\psi\colon I \to (I \cap N') \times (I \cap T) \times (I \cap N)$ as

$$\begin{pmatrix} 1+\varpi a & b \\ \varpi c & 1+\varpi d \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 \\ \frac{\varpi c}{1+\varpi a} & 1 \end{pmatrix} \begin{pmatrix} 1+\varpi a & 0 \\ 0 & 1+\varpi d - \frac{\varpi bc}{1+\varpi a} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{1+\varpi a} \\ 0 & 1 \end{pmatrix},$$

where $a, b, c, d \in \mathcal{O}_F$. Here the image makes sense because $1 + \mathfrak{p}_F$ is invertible in $\mathcal{O}_F$. Clearly, ψ is indeed the inverse of φ, and hence φ is bijective. By definition,

$$I = \begin{pmatrix} 1+\varpi\mathcal{O}_F & \mathcal{O}_F \\ \varpi\mathcal{O}_F & 1+\varpi\mathcal{O}_F \end{pmatrix},$$

and thus

$$I \cap N' \cong \varpi\mathcal{O}_F, \quad I \cap T \cong (1+\varpi\mathcal{O}_F)^2, \quad I \cap N \cong \mathcal{O}_F.$$

It follows that both I and $(I \cap N') \times (I \cap T) \times (I \cap N)$ are compact and Hausdorff. In this case, as a continuous bijection, φ must be a homeomorphism. Finally, note that both the set-theoretical bijection and the homeomorphism do not depend on the order of product, as there are only finitely many components on the source of φ. □

Exercise 5.7 ([BH06, p.55, Exercise])**.**

(1) Let I be the standard Iwahori subgroup; let $\mathrm{d}n', \mathrm{d}t, \mathrm{d}n$ be Haar measures on the groups $I \cap N'$, $I \cap T$, $I \cap N$, respectively. Show that the functional

$$f \longmapsto \iiint f(n'tn)\mathrm{d}n'\mathrm{d}t\mathrm{d}n, \quad f \in C_c^\infty(I),$$

is a Haar integral on I.

(2) Let $C = N'TN$. Show that C is open and dense in G, and that the product map $N' \times T \times N \to C$ is a homeomorphism.

(3) Let $\mathrm{d}g$ be a Haar measure on G. Show that there are Haar measures $\mathrm{d}n', \mathrm{d}t, \mathrm{d}n$ on N', T, N such that

$$\int_G f(g)\mathrm{d}g = \iiint f(n'tn)\delta_B(t)^{-1}\mathrm{d}n'\mathrm{d}t\mathrm{d}n, \quad f \in C_c^\infty(G).$$

Solution. (1) Since $\mathrm{d}n', \mathrm{d}t, \mathrm{d}n$ are already Haar measures, the positivity of the functional (i.e. the image of $f \geqslant 0$ is always non-negative) is clear. To check $n_0' n_0 n' t n$

(2) For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d - bc/a \end{pmatrix} \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix} \in N'TN$$

and the only condition for g is that $a \in F^\times$. Since $F^\times$ is dense in F and $N \cong F$ is open in G, we have that C is open and dense in G. To show that $\iota \colon N' \times T \times N \to C$ is a homeomorphism, note that ι is continuous in matrix entries □

Exercise 5.8. Let G be a locally profinite group. Show that for all $g \in G$ and $f_1, f_2 \in C_c^\infty(G)$, we have $(\lambda(g)f_1) * f_2 = \lambda(g)(f_1 * f_2)$. Here $\lambda(g)$ denotes the left translation of a function by g.

Solution. By definition we have $(\lambda(g)f)(x) = f(g^{-1}x)$ for all $f \in C_c^\infty(G)$. Also, from the definition of convolution product,

$$\begin{aligned} ((\lambda(g)f_1) * f_2)(x) &= \int_G (\lambda(g)f_1)(y) f_2(y^{-1}x) \mathrm{d}\mu(y) \\ &= \int_G f_1(g^{-1}y) f_2(y^{-1}x) \mathrm{d}\mu(y). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (\lambda(g)(f_1 * f_2))(x) &= \int_G f_1(y) f_2(y^{-1}g^{-1}x) \mathrm{d}\mu(y) \\ &= \int_G f_1(g^{-1}y) f_2(y^{-1}x) \mathrm{d}\mu(y). \end{aligned}$$

Here the second equality above is by replacing y by $g^{-1}y$, according to $\mathrm{d}\mu(y) = \mathrm{d}\mu(g^{-1}y)$. □

Exercise 5.9. Let G be an abelian locally profinite group. Show that any Haar measure is invariant under the automorphism $g \mapsto g^{-1}$. Then show that the Hecke algebra $\mathcal{H}(G)$ of G is commutative.

Solution. Let μ be a left Haar measure on G. Recall from Exercise 4.1 that we have $\mathrm{d}\mu(g^{-1}) = \delta_G(g^{-1})\mathrm{d}\mu(g)$. On the other hand, since G is abelian, we have $\mathrm{d}\mu(gx) = \mathrm{d}\mu(xg) = \mathrm{d}\mu(g)$ for all $x \in G$. Thus μ is also a right Haar measure. It follows that δ_G is trivial, and thus $\mathrm{d}\mu(g^{-1}) = \mathrm{d}\mu(g)$, namely μ is invariant under $g \mapsto g^{-1}$. To show that $\mathcal{H}(G)$ is commutative, consider

$$\begin{aligned} (f_1 * f_2)(x) &= \int_G f_1(y) f_2(y^{-1}x) \mathrm{d}\mu(y) \\ &= \int_G f_1(y^{-1}x) f_2(x^{-1}yx) \mathrm{d}\mu(y^{-1}x) \\ &= \int_G f_1(y^{-1}x) f_2(y) \mathrm{d}\mu(y) \\ &= (f_2 * f_1)(x). \end{aligned}$$

In the second line above, we replace y by $y^{-1}x$; in the third line above, we use the assumption that G is abelian to deduce $f_2(x^{-1}yx) = f_2(y)$ and use the previous result to deduce $\mathrm{d}\mu(y^{-1}x) = \mathrm{d}\mu(y)$. □

Exercise 5.10. Let G be a locally profinite group with a fixed a Haar measure μ. Suppose the Hecke algebra $\mathcal{H}(G)$ is commutative. Show that G is abelian.

Solution. Note that we have $\mathbb{1}_{gK} = \lambda(g)\mathbb{1}_K$ for all $g \in G$. Recall that $e_K := \mu(K)^{-1}\mathbb{1}_K$ satisfies the property $e_K * e_K = e_K$. Take $g, h \in G$ arbitrarily and compute

$$(\lambda(g)e_K) * (\lambda(h)e_K) = \lambda(g)(e_K * \lambda(h)e_K) = \lambda(g)(\lambda(h)e_K * e_K) = \lambda(gh)e_K.$$

Here the first equality is by Exercise 5.8, the second equality is because of the assumption that $\mathcal{H}(G)$ is commutative, and the last equality is again an application of Exercise 5.8. Again, since $\mathcal{H}(G)$ is commutative, we have

$$\lambda(gh)e_K = (\lambda(g)e_K) * (\lambda(h)e_K) = (\lambda(h)e_K) * (\lambda(g)e_K) = \lambda(hg)e_K.$$

It then follows that

$$\mathbb{1}_{ghK} = \mathbb{1}_{hgK}.$$

Therefore, we get $ghK = hgK$, which implies the desired result $gh = hg$ to show that G is abelian. $\square$

Week 6

Let G be a locally profinite group, and K be a compact open subgroup. Write $\mathcal{H}$ for the Hecke algebra $\mathcal{H}(G)$, and write $\mathcal{H}(G,K)$ for $e_K * \mathcal{H} * e_K$.

Exercise 6.1. For any $f \in \mathcal{H}$, show that $e_K * f$ is the average of the K-orbit of f with respect to the left translation action, and that $f * e_K$ is the average of the K-orbit of f with respect to the right translation action.

Solution. For the first assertion we need to show that

$$(e_K * f)(x) = (\lambda(e_K) \cdot f)(x)$$

for all $x \in G$ and any $f \in \mathcal{H}$. Here λ denotes the left translation. By definition of convolution product on $C_c^\infty(G)$, we have

$$(e_K * f)(x) = \int_G e_K(g) f(g^{-1}x) \mathrm{d}\mu(g).$$

On the other hand,

$$\begin{aligned}(\lambda(e_K) \cdot f)(x) &= \frac{1}{\mu(K)} \cdot \int_G \mathbb{1}_K(g) \cdot \lambda(g) f(x) \mathrm{d}\mu(g) \\ &= \frac{1}{\mu(K)} \cdot \int_G \mathbb{1}_K(g) \cdot f(g^{-1}x) \mathrm{d}\mu(g).\end{aligned}$$

So the desired equality follows from the known formula that $\mu(K) \cdot e_K = \mathbb{1}_K$. Also, the second assertion $f * e_K = \rho(e_K) \cdot f$ follows from a similar argument. $\square$

Exercise 6.2. Show that the inclusion $e_K * \mathcal{H} \to \mathcal{H}$ has a splitting. Moreover, this splitting is a map of right $\mathcal{H}$-modules if we view $e_K * \mathcal{H}$ and $\mathcal{H}$ both as right $\mathcal{H}$-modules.

Solution. Consider the map $\psi \colon \mathcal{H} \to e_K * \mathcal{H},\ f \mapsto e_K * f$. For ψ to be a splitting we need to check that the composite $e_K * \mathcal{H} \hookrightarrow \mathcal{H} \to e_K * \mathcal{H}$ is the identity; but this follows from the property that $e_K * e_K = e_K$, because an element of form $e_K * f_0$ in the source is mapped to $e_K * e_K * f_0 = e_K * f_0$ in the target.

Both $e_K * \mathcal{H}$ and $\mathcal{H}$ can be viewed as right $\mathcal{H}$-modules in the sense that $\mathcal{H}$ acts by $(-) * f$ for some $f \in \mathcal{H}$. By construction of ψ it is clearly compatible with the $\mathcal{H}$-actions from the right. $\square$

Exercise 6.3. Let W be a left unital $\mathcal{H}(G,K)$-module, and we write $*$ for the multiplication of $\mathcal{H}(G,K)$ on W. Then the $\mathcal{H}(G,K)$-module given by the K-fixed points of the smooth G-representation $\mathcal{H} \otimes_{\mathcal{H}(G,K)} W$ is isomorphic, as an $\mathcal{H}(G,K)$-module, to $(e_K * \mathcal{H}) \otimes_{\mathcal{H}(G,K)} W$. Moreover, there is a well-defined map $(e_K * \mathcal{H}) \otimes_{\mathcal{H}(G,K)} W \to W$ sending $(e_K * f) \otimes w$ to $(e_K * f * e_K) * w$.

Solution. Recall that if V is a smooth G-representation then $V^K = e_K * V$. Since W is a left unital $\mathcal{H}(G,K)$-module, it is invariant under the left action by e_K, i.e. $e_K * W = W$. It follows that

$$(\mathcal{H} \otimes_{\mathcal{H}(G,K)} W)^K = e_K * (\mathcal{H} \otimes_{\mathcal{H}(G,K)} W) = (e_K * \mathcal{H}) \otimes_{\mathcal{H}(G,K)} W.$$

Now consider the map $(e_K * \mathcal{H}) \otimes_{\mathcal{H}(G,K)} W \to W$. By Exercise 6.2 we see $\mathcal{H} \to e_K * \mathcal{H}$ via $f \mapsto e_K * f$ is surjective, and so also is the map $\mathcal{H} \otimes_{\mathcal{H}(G,K)} W \to (e_K * \mathcal{H}) \otimes_{\mathcal{H}(G,K)} W$ between $\mathcal{H}(G,K)$-modules. Moreover, this surjection admits a splitting as well, and hence we get an injection $(e_K * \mathcal{H}) \otimes_{\mathcal{H}(G,K)} W \hookrightarrow \mathcal{H} \otimes_{\mathcal{H}(G,K)} W$. On the other hand, we have the well-defined map

$$\mathcal{H} \otimes_{\mathcal{H}(G,K)} W \longrightarrow W, \quad f \otimes w \longmapsto w.$$

So it suffices to recognize $(e_K * f) \otimes w$ in terms of $(e_K * f * e_K) * w$. For this, we compute

$$\begin{aligned}(e_K * f) \otimes w &= (e_K * f) \otimes (e_K * w) \\ &= (e_K * f * e_K) \otimes w \\ &= e_K * (e_K * f * e_K) \otimes w \\ &= e_K \otimes ((e_K * f * e_K) * w).\end{aligned}$$

Here we have used the invariance property of $\mathcal{H}(G,K)$-modules with respect to $e_K * (\textbf{–}) * e_K$ for several times. $\square$

Exercise 6.4. Let F be a field. Prove the Bruhat decomposition

$$\mathrm{GL}_2(F) = B \sqcup BwB.$$

Here, as usual, B is the subgroup of upper triangular matrices, and $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Solution. We prove by elementary computation (as opposed to using the theory of Tits systems). If $b \in B \cap BwB$ then there are $b_1, b_2 \in B$ such that $b = b_1 w b_2$, and thus $w = b_1^{-1} b b_2^{-1} \in B$, which is impossible. So $B \cap BwB = \emptyset$. It suffices to show that for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(F)$, if $g \notin B$ then there are $g_1 = \left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)$ and $g_2 = \left(\begin{smallmatrix} r & s \\ 0 & t \end{smallmatrix}\right) \in B$ such that $g_1 w g_2 = g$. For this, compute

$$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} r & s \\ 0 & t \end{pmatrix} = \begin{pmatrix} yr & xt + ys \\ zr & zs \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note from the assumption that $z, r \in F^\times$. So we may take $z = 1$ and $r = c$; this makes sense because $g \notin B$ implies $c \in F^\times$. Then we can solve the matrix equation above and express x, y, s, t in terms of a, b, c, d. $\square$

Exercise 6.5. Classify and count the number of conjugacy classes in $\mathrm{GL}_3(\mathbb{F}_q)$.

Solution. Since $\mathbb{F}_q$ is not algebraically closed, we use rational canonical forms to determine the conjugacy classes. For this, we need to consider the characteristic polynomial $f(X)$ as well as elementary divisors of any $X \in \mathrm{GL}_3(\mathbb{F}_q)$.

(1) $f(X) = X^3 + aX^2 + bX + c$ is irreducible of degree 3 over $\mathbb{F}_q$. Then the arithmetic Frobenius map permutes the 3 roots of $f(X)$ in $\mathbb{F}_{q^3}$. Hence $f(X)$ is exactly determined by an element of $\mathbb{F}_{q^3} - \mathbb{F}_q$ up to Frobenius. Thus there are in total

$$\frac{1}{3}(q^3 - q)$$

conjugacy classes of $\mathrm{GL}_3(\mathbb{F}_q)$ in correspondence. In this case, the only elementary divisor is $f(X)$ itself, and the rational canonical form of X is

$$\begin{pmatrix} 0 & 0 & -c \\ 1 & 0 & -b \\ 0 & 1 & -a \end{pmatrix}.$$

(2) $f(X) = (X^2 + aX + b)(X - c)$ with $c \in \mathbb{F}_q^\times$ and $X^2 + aX + b$ an irreducible of degree 2 over $\mathbb{F}_q$. Similarly, we have $q^2 - q$ choices of $X^2 + aX + b$ to be the minimal polynomial of some elements of $\mathbb{F}_{q^2} - \mathbb{F}_q$ and $q - 1$ choices of $c \in \mathbb{F}_q^\times$. Hence there are there are in total

$$\frac{1}{2}(q^2 - q) \cdot (q - 1)$$

conjugacy classes of $\mathrm{GL}_3(\mathbb{F}_q)$ in correspondence. In this case, the elementary divisors are $X - c$ and $f(X)$, and the rational canonical form of X is

$$\begin{pmatrix} 0 & -b & 0 \\ 1 & -a & 0 \\ 0 & 0 & c \end{pmatrix}.$$

(3) $f(X)=(X-a)(X-b)(X-c)$ with $a,b,c\in\mathbb{F}_q^\times$. In this case the elementary divisors are possibly as follows.

(i) When a,b,c are mutually distinct, up to permuting a,b,c, there is essentially one case for elementary divisors, which is $f(X)$ itself. Choosing a,b,c modulo permutation, we see the number of conjugacy classes is

$$\frac{1}{6}(q-1)(q-2)(q-3).$$

(ii) When $a=b\neq c$, the elementary divisors can be $X-a,(X-a)(X-c)$ or $f(X)$ itself. In both cases there are $q-1$ choices of $a=b$ and $q-2$ choices of c. Up to permutation, the number of conjugacy classes is

$$\frac{1}{3}((q-1)+(q-1))(q-2).$$

(iii) When $a=b=c$, the elementary divisors can be either of $(X-a,X-a,X-a)$ or $(X-a,(X-a)^2)$ or $f(X)$ itself. It follows that the number of conjugacy classes is

$$3(q-1).$$

To sum up, the number of all conjugacy classes in $\mathrm{GL}_3(\mathbb{F}_q)$ is

$$\frac{1}{3}(q-1)(3q^2-q+8).$$

$\square$

Exercise 6.6. Let G be a finite group, and H,K subgroups. Let $W\in\mathsf{Rep}(K)$. State and prove Mackey's formula, which describes $\mathrm{Res}_H^G\mathrm{Ind}_K^G W$.

Solution. Let $g_1,\dots,g_r$ be a system of representatives of the double classes in $H\backslash G/K$. In other words, we have

$$G=\bigsqcup_{i=1}^r Hg_iK.$$

For every $i\in\{1,\dots,r\}$, let $K_i:=g_iKg_i^{-1}\cap H$, and let (W_i,ρ_i) be the representation of K_i on W given by $\rho_i(k)=\rho(g_i^{-1}kg_i)$ for $k\in K$.

Mackey's formula. We have an isomorphism of H-representations

$$\mathrm{Res}_H^G\mathrm{Ind}_K^G W\simeq\bigoplus_{i=1}^r\mathrm{Ind}_{K_i}^K W_i.$$

To prove it, for every $j\in\{1,\dots,r\}$, let $(x_i)_{i\in I_j}$ be a system of representatives of H/K_j. Then $\{x_ig_j\colon 1\leqslant j\leqslant r,\ i\in I_j\}$ is a system of representatives of G/K. Indeed,

$$G=\bigsqcup_{j=1}^r Hg_jK=\bigsqcup_{j=1}^r\bigsqcup_{i\in I_j}x_iK_jg_jK=\bigsqcup_{j=1}^r\bigsqcup_{i\in I_j}x_jg_j(g_j^{-1}K_jg_j)K=\bigsqcup_{j=1}^r\bigsqcup_{i\in I_j}x_jg_jK$$

because $g_j^{-1}K_jg_j\subset K$ for every j. Thus,

$$V:=\mathrm{Ind}_K^G W=\bigoplus_{j=1}^r V_j,$$

where $V_j=\bigoplus_{i\in I_j}x_ig_j\overline{F}[K]\otimes_{\overline{F}[K]}V$. Note that $V_j\subset V$ is stabilized by H. Fix $j\in\{1,\dots,r\}$. It suffices to show the following isomorphism of $\overline{F}[H]$-modules:

$$V_j\simeq\mathrm{Ind}_{K_j}^H W_j.$$

But for this, we can construct the $\overline{F}$-linear maps

$$\varphi\colon\ \mathrm{Ind}_{K_j}^H W_j\longrightarrow V_j,\quad \sum_{i\in I_j}x_i\otimes v_i\longmapsto\sum_{i\in I_j}(x_ig_j)\otimes v_i$$

and
$$\psi\colon V_i \longrightarrow \operatorname{Ind}_{K_j}^{H} W_j, \qquad \sum_{i\in H\cdot I_j} (x_i g_j)\otimes v_i \longmapsto \sum_{i\in I_j} x_i \otimes v_i.$$
It is clear that φ and ψ are inverse of each other. This completes the proof. $\square$

Exercise 6.7. Prove in detail the lemma in [BH06, §6.3].

Solution. Let π be an irreducible representation of $G = \mathrm{GL}_2(F)$, where F is a non-archimedean local field. We are to prove that π is contained in $\operatorname{Ind}_B^G \chi$ for some character $\chi\colon T \to \mathbb{C}^\times$ if and only if $\operatorname{Res}_N^G \pi$ contains the trivial representation of N. Here the character χ of T can be viewed as a representation of B for the following reason.

Note that given the canonical projection $B \to B/N$ and the isomorphism $B/N \cong T$, the data of the character $T \to \mathbb{C}^\times$ is equivalent to the data of the character $B \to B/N \xrightarrow{\sim} T \to \mathbb{C}^\times$ that is trivial on N. Therefore, $\operatorname{Res}_N^G \pi$ contains the trivial representation of N if and only if $\operatorname{Res}_B^G \pi$ contains an irreducible representation σ of B such that σ contains a trivial representation of N. Further, this is equivalent to $\sigma \cong \operatorname{Ind}_T^B \chi$ for some character $\chi\colon T \to \mathbb{C}^\times$. By Frobenius reciprocity, we have that
$$\operatorname{Hom}_G(\pi, \operatorname{Ind}_B^G \sigma) \cong \operatorname{Hom}_B(\operatorname{Res}_B^G \pi, \sigma) \cong \operatorname{Hom}_B(\operatorname{Res}_B^G \pi, \operatorname{Ind}_T^B \chi).$$
Therefore, the multiplicity of $\operatorname{Ind}_T^B \chi$ in $\operatorname{Res}_B^G \pi$ is the same as that of $\operatorname{Ind}_B^G \sigma$ in π, which proves the desired equivalence. $\square$

Week 7

Exercise 7.1. Let k be a field, and let l/k be a Galois quadratic extension, with Galois group $\{1, \sigma\}$. Fix a k-linear isomorphism $l \cong k^2$. Then the left multiplication action of $l^\times$ on l defines an injective group homomorphism $i\colon l^\times \to \mathrm{GL}_2(k)$. Denote the image by E. Let N be the normalizer of E in $\mathrm{GL}_2(k)$.

(1) Show that there exists g in N such that the automorphism $\mathrm{Int}(g)\colon E \to E$ is the same as σ on $E \cong l^\times$.
(2) Show that for any $g \in N$, the automorphism $\mathrm{Int}(g)\colon E \to E$ must either be the identity, or σ as above. Conclude that the centralizer of E in $\mathrm{GL}_2(k)$ is of index 2 in N.
(3) Show that the centralizer of E in $\mathrm{GL}_2(k)$ is E.
(4) Let $g \in E - Z$, where Z denotes the center of $\mathrm{GL}_2(k)$. Show that the centralizer of g in $\mathrm{GL}_2(k)$ is E. Then show that if $h \in \mathrm{GL}_2(k)$ is such that $hgh^{-1} \in E$, then h normalizes E.

Solution. (1) Consider $l \cong k^2$ as an l-module via $l \times l \to l,\ (a, b) \mapsto \sigma(a)b$. Also consider l as the standard l-module via $l \times l \to l,\ (a, b) \mapsto ab$. Then $\sigma\colon l \to l$ is an isomorphism of l-modules because it is a bijective homomorphism in the sense that $\sigma(ab) = \sigma(a)\sigma(b)$. Further, it descends to an automorphism $\sigma\colon l^\times \to l^\times$ of $l^\times$-modules; through the isomorphism $l^\times \cong E$ induced by i, we get an automorphism $E \to E$ of E-modules induced from σ. Using the compatibility with E-action, it must be inner, written as $\mathrm{Int}(g)\colon E \to E$.

(2) Notice that $i\colon l^\times \to \mathrm{GL}_2(k)$ maps $k^\times$ to $k^\times \cdot \mathrm{id} \subset \mathrm{GL}_2(k)$, and $k^\times \cdot \mathrm{id}$ is clearly invariant under $\mathrm{Int}(g)$. On the other hand, via the isomorphism $E \cong l^\times$ of groups, each $\mathrm{Int}(g)$ gives rise to an automorphism of $l^\times$. Thus it must keep $k^\times$ invariant, and it is thus an element of $\mathrm{Aut}_k(l^\times)$. Moreover, such an action extends thereof to an element in $\mathrm{Gal}(l/k)$ uniquely because l is a field, so the description in need follows. As each element $n \in N$ normalizes E, or namely defines an automorphism $\mathrm{Int}(n)\colon E \to E$, the set $\mathrm{Inn}(E)$ is in bijection with $\{1, \sigma\}$, meaning that $N/Z_G(E)$ for $G = \mathrm{GL}_2(k)$ contains a unique non-trivial element, and therefore $Z_G(E)$ is of index 2 in N.

(3) As E is isomorphic to the abelian group $l^\times$, it is clear that E centralizes itself, or equivalently E is a subgroup of $Z_G(E)$; it suffices to show the converse set-theoretical inclusion. Using part (2), any $g \in Z_G(E)$ can be regarded as $1 \in \mathrm{Gal}(l/k)$, and hence defines $\mathrm{id}\colon l^\times \to l^\times$ up to isomorphism, meaning that it is an automorphism of $l^\times$ admitting a trivialization, which can only be scalar multiplication of $l^\times$. Thus the data of g-action is equivalent to the data of an abstract map $E \times E \xrightarrow{\sim} l^\times \times E \to E$. So we conclude that $E = Z_G(E)$.

(4) Denote by $Z_G(g)$ the centralizer of g. Then from (3) we have $E \subset Z_G(g)$ and the subgroup $\langle g \rangle$ generated by g is normal in E, since $g \notin Z$ and any $x \in E$ commutes with g. Then

$$Z_G(g) = E = Z_G(E),$$

where the first equality follows from the definition and the second equality is by part (3). Now it remains to show such h normalizes E. For this, we only need that $h^{-1}xh \in E = Z_G(g)$ for all $x \in E$. Note that by considering G-conjugacy action on itself, we can identify $Z_G(g)$ with $\mathrm{Stab}_G(g)$. So the desired inclusion is implied by $h^{-1}xh \cdot g = g \cdot h^{-1}xh$; but this is equivalent to $hgh^{-1} \cdot x = x \cdot hgh^{-1}$, i.e. $x \in Z_G(hgh^{-1})$ for all $x \in E$. Hence we finish the proof, because this is implied by part (3) that $E = Z_G(E) \subset Z_G(hgh^{-1})$. □

Exercise 7.2. Keep the same notation as in [BH06, p.48] and in addition let χ_1 be the character of $\mathrm{Ind}_{ZN}^G \theta_\psi$, and let χ_2 be the character of $\mathrm{Ind}_E^G \theta$.

(1) Verify the character formula (6.4.1).
(2) Verify that

$$\langle \chi_1, \chi_1 \rangle = q, \quad \langle \chi_1, \chi_2 \rangle = q - 1.$$

Solution. (1) The desired formula (6.4.1) computes $\operatorname{tr}\pi_\theta = \chi_1 - \chi_2$. By construction and character formulae for inductions we have

$$\chi_1(g) = \sum_{\substack{h\in G/ZN \\ h^{-1}gh\in ZN}} \theta_\psi(h^{-1}gh), \qquad \chi_2(g) = \sum_{\substack{h\in G/E \\ h^{-1}gh\in E}} \theta(h^{-1}gh).$$

Here $\theta_\psi\colon zu \mapsto \theta(z)\psi(u)$ for $z\in Z$ and $u \in N$ is a character of ZN.

(i) Suppose $g = z \in Z$. Then $h^{-1}zh = z \in ZN \cap E$, and hence

$$\begin{aligned}\operatorname{tr}\pi_\theta(g) &= |G/ZN|\cdot\theta_\psi(z) - |G/E|\cdot\theta(z)\\ &= (q^2-1)\cdot\theta(z) - (q^2-q)\cdot\theta(z)\\ &= (q-1)\cdot\theta(z).\end{aligned}$$

Here we have used the observations $|Z| = |\mathbb{F}_q^\times| = q-1$, $|N| = |\mathbb{F}_q| = q$, and $|E| = |\mathbb{F}_{q^2}^\times| = q^2-1$, as well as the known fact that $|G| = |\mathrm{GL}_2(\mathbb{F}_q)| = (q^2-1)(q^2-q)$.

(ii) Suppose $g = zu$ for $z \in Z$ and $u \in N\backslash\{1\}$. We first claim that g is not conjugate to E. Indeed, if this is not true, then $h^{-1}uh \in E$ for some $h \in G$ with $\det(h^{-1}uh) = \det(u) = 1$; so the image of $h^{-1}uh$ in $l^\times$ is an element with norm 1. On the other hand, as l/k is a quadratic extension, there are at most 2 candidates for the image of $h^{-1}uh$ in $l^\times$, but the condition $\operatorname{tr}(h^{-1}uh) = \operatorname{tr}(u) > 0$ forces $h^{-1}uh$ to be trivial in $l^\times$. This contradicts to $u \in N\backslash\{1\}$ and proves the claim. Consequently, we have

$$\chi_2(zu) = 0,$$

so it remains to compute $\chi_1(zu)$. For this, notice that $\theta_\psi(h^{-1}zuh) = \theta_\psi(z\cdot h^{-1}uh) = \theta(z)\psi(h^{-1}uh) = \theta(z)\psi(u)$. So we have

$$\operatorname{tr}\pi_\theta(g) = \chi_1(zu) = \theta(z)\sum_{u'\sim u}\psi(u'),$$

where the sum runs through elements $u' \in N\backslash\{1\}$ that are G-conjugate to u. Since $N\backslash\{1\}$ is in bijection with $F^\times$, if we fix an element $u = \left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right) \in N\backslash\{1\}$ then any other $u' \in N\backslash\{1\}$ is of form $\left(\begin{smallmatrix}1 & ax\\ 0 & 1\end{smallmatrix}\right)$ for some $a \in F^\times$; according to the relation $huh^{-1} = u'$ with $h = \left(\begin{smallmatrix}a & 0\\ 0 & 1\end{smallmatrix}\right) \in G$, any two elements of $N\backslash\{1\}$ are G-conjugate. On the other hand, $\psi\colon N \to \mathbb{C}^\times$ gives rise to a class function

$$\Psi\colon N \longrightarrow \mathbb{C}^\times, \quad u \longmapsto \sum_{u'\sim u}\psi(u').$$

Because both θ and $\operatorname{tr}\pi_\theta$ are group homomorphisms, so also is Ψ. Combining the two observations above, the image of Ψ consists of two values that form a subgroup of $\mathbb{C}^\times$, namely $\Psi(N) = \{\pm 1\}$. It follows that $\Psi(N\backslash\{1\}) = \{-1\}$, and we deduce

$$\operatorname{tr}\pi_\theta(g) = -\theta(z).$$

(iii) Suppose $g = y \in E\backslash Z$. For a similar reason as in proving the claim in (ii), we see g is not conjugate to ZN, and thus $\chi_1(g) = 0$. By Exercise 7.1 all the automorphisms of E of form $\operatorname{Int}(g)\colon E \to E$ with $g \in N_G(E)$ are exactly elements of $\operatorname{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q) = \{1, \sigma\}$. It follows that the orbit of θ under $N_G(E)$ is exactly $\{\theta, \theta\circ\sigma\} = \{\theta, \theta^q\}$. Therefore,

$$\operatorname{tr}\pi_\theta(g) = -\chi_2(y) = -(\theta(y) + \theta^q(y)).$$

So far we have verified the desired formula about $\operatorname{tr}\pi_\theta$.

(2) We begin with verifying $\langle \chi_1, \chi_1 \rangle = q$. We have from part (1) that

$$\begin{aligned} |G| \cdot \langle \chi_1, \chi_1 \rangle &= \sum_{g \in G} \chi_1(g)\overline{\chi_1(g)} \\ &= \sum_{g \in Z} |(q^2-1)\theta(g)|^2 + \sum_{\substack{g \sim uz \\ uz \in NZ\backslash Z}} |-\theta(z)|^2 \\ &= (q^2-1)^2 \cdot |Z| + \sum_{\substack{g \sim uz \\ uz \in NZ\backslash Z}} 1. \end{aligned}$$

Here in the last equality we have used $|\theta(g)|^2 = \theta(g) \cdot \overline{\theta(g)} = \theta(g) \cdot \theta(g)^{-1} = 1$. As for the second sum, modulo the center, the number of ways that uz is G-conjugate to some $g \in G$ is the same as that for u conjugate to g. Now the only ambiguity lies in the choice of $u \in N$. Thus we have

$$\sum_{\substack{g \sim uz \\ uz \in NZ\backslash Z}} 1 = \frac{|G/Z|}{|N\backslash\{1\}| \cdot |N|} \sum_{uz \in NZ\backslash Z} 1 = (q+1) \cdot |NZ\backslash Z|.$$

Therefore,

$$\langle \chi_1, \chi_1 \rangle = \frac{1}{|G|}((q^2-1)^2 \cdot |Z| + (q+1) \cdot |NZ\backslash Z|) = q.$$

We then verify that $\langle \chi_1, \chi_2 \rangle = q-1$. Again by the formulae of part (1),

$$\begin{aligned} |G| \cdot \langle \chi_1, \chi_2 \rangle &= \sum_{g \in G} \chi_1(g)\overline{\chi_2(g)} \\ &= \sum_{g \in Z} (q^2-1)(q^2-q)|\theta(g)|^2 + \sum_{\substack{g \sim g' \\ g' \in (NZ \cap E)\backslash Z}} \chi_1(g)\overline{\chi_2(g)} \\ &= (q^2-1)(q^2-q) \cdot |Z|. \end{aligned}$$

Here the second sum in the second line vanishes simply because $(NZ \cap E)\backslash Z = \emptyset$. So we get

$$\langle \chi_1, \chi_2 \rangle = \frac{|Z|}{|G|} \cdot (q^2-1)(q^2-q) = q-1.$$

□

Exercise 7.3 ([BH06, p.48, Exercise])**.** Let ψ be a non-trivial character of N, and consider the representation

$$\mathcal{F} = \operatorname{Ind}_N^G \psi.$$

Let σ be an irreducible representation of G. Show that:

(1) If $\sigma = \phi \circ \det$, for a character ϕ of $k^\times$, then σ does not occur in $\mathcal{F}$.
(2) Otherwise, σ occurs in $\mathcal{F}$ with multiplicity one.

Solution. (1) Note that for unipotent radical N we always have $\det(n) = 1 \in k^\times$ for all $n \in N$. It follows that $\sigma(N) = \phi(\det(N))$ is trivial. If σ occurs in $\mathcal{F}$ then there is $f \in \sigma$ such that

$$f(ng) = \psi(n) \cdot f(g)$$

for all $n \in N$ and $g \in G$. Since ψ is non-trivial on N, we may assume $\psi(n)$ is non-trivial for a specified n and $g = n' \in N$; then the equality above breaks. This leads to a contradiction.

(2) It suffices to show the space of G-maps $\operatorname{Hom}_G(\mathcal{F}, \sigma)$ is exactly 1-dimensional. By Frobenius reciprocity we need to look at $\operatorname{Hom}_G(\operatorname{Ind}_N^G \psi, \sigma) = \operatorname{Hom}_N(\psi, \sigma|_N)$. Here $\sigma|_N$ is necessarily identified with a character because of the isomorphism $N \cong F$. As ψ is another non-trivial character of N, we see $\dim \operatorname{Hom}_N(\psi, \sigma|_N) = 1$ as desired. □

Exercise 7.4 ([BH06, pp.62–63, Exercises 1,2])**.**

(1) Let (π, V) be an irreducible smooth representation of G with a non-trivial $\pi(N)$-fixed vector. Show that $\pi = \phi \circ \det$, for some character ϕ of $F^\times$.
(2) Let (π, V) be an irreducible smooth representation of G such that $\dim V$ is finite. Show that V has a non-zero $\pi(N)$-fixed vector. Deduce that $\dim V = 1$ and π is of the form $\phi \circ \det$, for some character ϕ of $F^\times$.

Solution. (1) Since (π, V) is irreducible, we have either $V(N) = 0$ or $V(N) = V$. But the existence of non-trivial $v \in V$ such that $v = \pi(n)v$ for all $n \in N$ forces $V(N)$ to be 0. It follows that $V = V_N$ as vector spaces, and in particular $V_N \neq 0$, which means π is non-cuspidal. So π is contained in $\operatorname{Ind}_B^G \chi$ for some character $\chi \colon T \to \mathbb{C}^\times$ at the level of B-representations, and the Jordan–Hölder theory implies that V is 1-dimensional from $V(N) = 0$. To show that $\pi = \phi \circ \det$, note that $\ker \pi$ contains the commutator subgroup $\mathrm{SL}_2(F)$ of G because of $\dim(\pi, V) = 1$, where $\phi \colon F^\times \to \mathbb{C}^\times$ is a group homomorphism. Since $\det \colon G = \mathrm{GL}_2(F) \to F^\times$ is surjective and open, such ϕ is a character. This finishes the proof.

(2) If $V(N) \neq 0$ then $V(N) = V$ and $V_N = 0$, by the irreducibility, and therefore $V_\vartheta \neq 0$ for all non-trivial characters $\vartheta \colon N \to \mathbb{C}^\times$; it implies that $\dim V$ is infinite, which contradicts to the assumption. Thus $V(N) = 0$ and then N acts trivially on V, so that V has a non-zero $\pi(N)$-fixed vector. Again by $V_N \neq 0$ and the Jordan–Hölder theory of non-cuspidal representation, we have $\dim V = 1$ and $\pi = \phi \circ \det$ for some character $\phi \colon F^\times \to \mathbb{C}^\times$. $\square$

Week 8

In the following F denotes a non-archimedean local field, and $\mathfrak{m}$ denotes the maximal ideal of $\mathcal{O}_F$. All characters $F \to \mathbb{C}^\times$ are assumed to be smooth.

Exercise 8.1. Show that if $\theta, \theta' \colon F \to \mathbb{C}^\times$ are two non-trivial characters, then there exists $a \in F^\times$ such that $\theta'(x) = \theta(ax)$ for all $x \in F$.

Solution. It suffices to show that if we fix a non-trivial character θ of F then the function $a\theta \colon F \to \mathbb{C}^\times,\ x \mapsto \theta(ax)$ ranges over the characters of F as a ranges over F. If this is the case then $a \in F^\times$ follows from the non-triviality of θ'. As θ is fixed, we are to show that there is a group isomorphism

$$\iota \colon F \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont}}(F, \mathbb{C}^\times), \quad a \longmapsto a\theta.$$

It is clear that ι is an injective group homomorphism. So it remains to show the surjectivity.

Recall that for non-trivial $\psi \in \mathrm{Hom}_{\mathrm{cont}}(F, \mathbb{C}^\times)$ the level of ψ is defined as the least integer d such that $\mathfrak{m}^d \subset \ker \psi$; if we fix d then the subset of $\mathrm{Hom}_{\mathrm{cont}}(F, \mathbb{C}^\times)$ of level at most d is the subgroup consisting of those ψ such that $\psi|_{\mathfrak{m}^d} = \mathbb{1}$.

Let ϖ be a chosen uniformizer of F and $u \in \mathcal{O}_F^\times$. Let the levels of θ and θ' be l and l' respectively. Then the character $u\varpi^{l-l'}\theta$ has level l' and so agrees with θ' on $\mathfrak{m}^{l'}$; moreover, the characters $u\varpi^{l-l'}\theta$ and $u'\varpi^{l-l'}\theta$ with $u, u' \in \mathcal{O}_F^\times$ agree on $\mathfrak{m}^{l'-1}$ if and only if $u \equiv u' \bmod \mathfrak{m}$. The group $\mathfrak{m}^{l'-1}$ has $q-1$ non-trivial characters which are trivial on $\mathfrak{m}^{l'}$. (Here q denotes the residual cardinality of ϖ.) As u ranges over $\mathcal{O}_F^\times/(1+\mathfrak{m})$, the $q-1$ characters $u\varpi^{l-l'}\theta|_{\mathfrak{m}^{l'-1}}$ are distinct, non-trivial, but trivial on $\mathfrak{m}^{l'}$. Therefore one of them, say $u_1\varpi^{l-l'}\theta|_{\mathfrak{m}^{l'-1}}$, equals $\theta'|_{\mathfrak{m}^{l'-1}}$. Iterating this procedure, we find a sequence of elements $u_n \in \mathcal{O}_F^\times$ such that $u_n\varpi^{l-l'}\theta$ agrees with θ' on $\mathfrak{m}^{l'-n}$ and $u_{n+1} \equiv u_n \bmod \mathfrak{m}^n$. Then the Cauchy sequence $\{u_n\}$ converges to some $u \in \mathcal{O}_F^\times$ and we have $\theta' = u\varpi^{l-l'}\theta = \iota(u\varpi^{l-l'})$. This completes the proof that ι is an isomorphism. $\square$

Exercise 8.2. Let $\theta \colon F \to \mathbb{C}^\times$ be a non-trivial character. Let $j \geqslant 1$ be a positive integer, and let $a \in F^\times$ be such that $a \notin 1 + \mathfrak{m}^j$. Show that there exists $x \in F$ such that the map $1 + \mathfrak{m}^j \to \mathbb{C},\ u \mapsto \theta(aux) - \theta(x)$ is a non-zero constant.

Solution. Referring to the proof of Exercise 8.1, let $k \geqslant 1$ be the level of θ, so that $\theta|_{\mathfrak{m}^k}$ is trivial whilst $\theta|_{\mathfrak{m}^{k-1}}$ is non-trivial. Again by Exercise 8.1 we may assume $k = j$ by replacing θ with $\varpi^{k-j}\theta$ if necessary. Then there is some $y \in \mathfrak{m}^{j-1}$ such that $\theta(y) \in \mathbb{C}\backslash\{0\}$. Fix such y and let $x = y/a \in F$, so we get

$$\theta(aux) = \theta(uy) = \theta(y + (u-1)y) = \theta(y) \cdot \theta|_{\mathfrak{m}^j}((u-1)y).$$

As $\theta|_{\mathfrak{m}^j}$ is trivial by hypothesis, we have $\theta(aux)$ equal to the constant $\theta(y)$. Also, as x is independent of u, we see $\theta(aux) - \theta(x) \equiv \theta(y) - \theta(x) \in \mathbb{C}$ is a constant. To show this is non-zero under the assumption $a \notin 1 + \mathfrak{m}^j$, it suffices to show $\theta(y) = \theta(x)$ implies $a \in 1 + \mathfrak{m}^j$. But for this, we notice that $\theta(ax) = \theta(x) \cdot \theta((a-1)x) = \theta(x)$ implies $a - 1 \in \mathfrak{m}^j$, which is as desired. $\square$

Exercise 8.3. Let $(V_i)_{i \in I}$ be a family of smooth representations of $N \cong F$. Show that there are natural isomorphisms

$$\bigoplus_{i \in I} V_i(N) \cong \left(\bigoplus_{i \in I} V_i\right)(N), \qquad \bigoplus_{i \in I} V_{i,N} \cong \left(\bigoplus_{i \in I} V_i\right)_N.$$

Solution. It suffices to show the first isomorphism, and the second one can be deduced from the same argument with replacing V_i by their contragredients V_i^*. Note that N acts on the direct sum (π, V) for $V = \bigoplus_{i \in I} V_i$ diagonally, i.e. $\pi(n)((v_i)_{i \in I}) = (\pi_i(n)v_i)_{i \in I}$. Then $\pi(n)((v_i)_{i \in I}) -$

$(v_i)_{i\in I} = (\pi_i(n)v_i)_{i\in I} - (v_i)_{i\in I} = (\pi_i(n)v_i - v_i)_{i\in I}$, and thus by definition,

$$\begin{aligned} V(N) &= \operatorname{span}\{(\pi_i(n)v_i - v_i)_{i\in I} \colon n \in N,\ v_i \in V_i \text{ for all } i \in I\} \\ &\cong \bigoplus_{i\in I} \operatorname{span}\{\pi_i(n)v_i - v_i \colon n \in N,\ v_i \in V_i\}. \end{aligned}$$

This proves the desired result. □

Exercise 8.4. Let G be a locally profinite group. A smooth representation V of G is said to be of *finite length*, if there exists a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ where each V_i is a subrepresentation, and V_{i+1}/V_i is non-zero irreducible. The integer n is called the length of V, and the isomorphism classes of the irreducible representations V_{i+1}/V_i are called the Jordan–Hölder factors (or *composition factors*) of V (counting multiplicities).

(1) Show that both n and the set of Jordan–Hölder factors (counting multiplicities) depend only on V, and are independent of the choice of the filtration.
(2) Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be a short exact sequence of smooth representations. Show that V_2 is of finite length if and only if V_1 and V_3 are of finite length.
(3) Let B be a closed subgroup of G. Let V be a smooth representation of G such that it is of finite length as a B-representation. Show that it is of finite length as a G-representation, and that the length is at most equal to the length as a B-representation.

Solution. (1) We implicitly suppose V is of finite length n. It suffices to show that any two Jordan–Hölder filtrations are equivalent, in the sense that for $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ and $0 = V_0' \subset V_1' \subset \cdots \subset V_m' = V$ we have $m = n$ and $V_{i+1}/V_i \cong V'_{\sigma(i)+1}/V'_{\sigma(i)}$ for all $0 \leqslant i \leqslant m-1$ with some $\sigma \in S_n$. We will prove that if the statement is true for any subrepresentation of V then it is true for V. If V is irreducible then the statement is trivial; otherwise we only need to consider the case where $V_{n-1} \neq V'_{m-1}$, which implies $V_{n-1} + V'_{m-1} = V$, and hence

$$V/V_{n-1} \cong V'_{m-1}/(V'_{m-1} \cap V_{n-1}), \quad V/V'_{m-1} \cong V_{n-1}/(V'_{m-1} \cap V_{n-1}).$$

By assumption both V/V_{n-1} and V/V'_{m-1} are irreducible, so that

$$\begin{gathered} 0 = W_0 \subset W_1 \subset \cdots \subset W_{r-1} = V'_{m-1} \cap V_{n-1} \subset V_{n-1} \subset V_n = V \\ 0 = W_0 \subset W_1 \subset \cdots \subset W_{r-1} = V'_{m-1} \cap V_{n-1} \subset V'_{m-1} \subset V_m = V \end{gathered}$$

are two Jordan–Hölder filtrations of V; they are clearly equivalent and the desired result follows from the inductive assumption on $V'_{m-1} \cap V_{n-1}$.

(2) Suppose V_2 is of finite length of length k with Jordan–Hölder filtration $0 = V_{2,0} \subset \cdots \subset V_{2,k} = V_2$. Then for each $0 \leqslant i \leqslant k-1$, since $V_1 \subset V_2$, the quotient $(V_1 \cap V_{2,i+1})/(V_1 \cap V_{2,i})$ is a subrepresentation of the non-zero irreducible representation $V_{2,i+1}/V_{2,i}$, so it is either 0 or $V_{2,i+1}/V_{2,i}$. It follows that after deleting some steps to get rid of zero quotients, the sequence

$$0 = V_1 \cap V_{2,0} \subset \cdots \subset V_1 \cap V_{2,k} = V_1$$

is a Jordan–Hölder filtration for V_1 with length at most k. As for V_3, the proof is similar by first considering the images of all $V_{2,i}$ in V_3 and then deleting some indices.

Conversely, suppose both V_1 and V_3 are of finite length. Let $0 = V_{1,0} \subset \cdots \subset V_{1,m} = V_1$ and $0 = V_{3,0} \subset \cdots \subset V_{3,n} = V_3$ be Jordan–Hölder filtrations of V_1 and V_3, respectively. Note that each subrepresentation $V_{3,i}$ of V_3 corresponds to a unique subrepresentation $\tilde{V}_{3,i}$ of V_2 containing V_1; here we can take $\tilde{V}_{3,i}$ to be the inverse image of $V_{3,i}$ in V_2. Then

$$0 = V_{1,0} \subset \cdots \subset V_{1,m} = V_1 = \tilde{V}_{3,0} \subset \cdots \subset \tilde{V}_{3,n} = V_2$$

is a Jordan–Hölder filtration of V_2 of finite length $m+n$.

(3) Let $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ be the Jordan–Hölder filtration of B-representations. Since V is a priori a G-representation we have $V^G = V$. On the other hand, provided the

Iwasawa decomposition $G = BK$, we see $V_i^G = V_i^K$ because each V_i is a B-representation. Thus we have an induced filtration

$$0 = V_0^K \subset V_1^K \subset \cdots \subset V_n^K = V$$

of G-representations. By Exercise 2.5, it follows from the exactness of $(-)^K$ that $V_{i+1}^K/V_i^K = (V_{i+1}/V_i)^K$ for all $0 \leqslant i \leqslant n-1$. Consequently, similar to part (2), it turns out that either $(V_{i+1}/V_i)^K = 0$ or $(V_{i+1}/V_i)^K = V_{i+1}/V_i$. Therefore, by deleting the zero Jordan–Hölder factors in G-filtration above we get the desired Jordan–Hölder filtration with length at most n. $\square$

Week 9

Exercise 9.1. Let F be a non-archimedean local field, with valuation v. Let $\phi\colon F^\times \to \mathbb{C}^\times$ be a homomorphism. Assume one of the following two conditions holds:

(i) There exists $n \in \mathbb{Z}$ such that ϕ is constant on $\{x \in F^\times : v(x) \geqslant n\}$.
(ii) There exists $n \in \mathbb{Z}$ such that ϕ is constant on $\{x \in F^\times : v(x) \leqslant n\}$.

Show that ϕ is trivial.

Solution. Choose ϖ to be a uniformizer of $\mathcal{O}_F$. We claim that $\phi(\varpi) = 1$. Indeed, in either of cases (i) or (ii), there is $m \in \mathbb{Z}$ with $|m| \gg 0$ such that $\phi(\varpi^k) \equiv C \in \mathbb{C}^\times$ whenever $|k| \geqslant |m|$. We may assume $k \geqslant m \geqslant 0$ and then $\phi(\varpi^{k+1}) = C = \phi(\varpi^k)$, which implies the claim, because ϕ is a homomorphism of multiplicative groups.

For general $x \in F^\times$, again by assumption there is $m \in \mathbb{Z}$ with $|m| \gg 0$ such that $v(\varpi^m x)$ satisfies either (i) or (ii). It follows that $C = \phi(\varpi^m x) = \phi(\varpi)^m \phi(x) = \phi(x)$. This proves that ϕ is a constant on the whole $F^\times$, and hence equal to $\phi(\varpi) = 1$, meaning that ϕ is trivial. $\square$

Exercise 9.2. Let G be a locally profinite group, and K a compact open subgroup. Fix a left Haar measure on G, and equip $G \times G$ with the product Haar measure. Show that there is a natural algebra isomorphism

$$\mathcal{H}(G \times G, K \times K) \cong \mathcal{H}(G, K) \otimes_{\mathbb{C}} \mathcal{H}(G, K).$$

Solution. Note that $K \times K$ is a compact open subgroup of $G \times G$. Consider the natural map

$$\Phi\colon \mathcal{H}(G, K) \otimes_{\mathbb{C}} \mathcal{H}(G, K) \longrightarrow \mathcal{H}(G \times G, K \times K), \quad f_1 \otimes f_2 \longmapsto f_1 f_2,$$

where $\Phi(f_1 \otimes f_2)(g_1, g_2) = (f_1 f_2)(g_1, g_2) = f_1(g_1) f_2(g_2)$. Notice that the compact support condition is automatic. To check the image of Φ is bi-$(K \times K)$-invariant, compute for any $(k_1, k_2), (k_1', k_2') \in K \times K$ that

$$\Phi(f_1 \otimes f_2)((k_1, k_2)(g_1, g_2)(k_1', k_2')) = f_1(k_1 g_1 k_1') f_2(k_2 g_2 k_2') = f_1(g_1) f_2(g_2),$$

because f_1 and f_2 are bi-K-invariant. To check Φ is a homomorphism, we compute

$$\begin{aligned}
\Phi((f_1 \otimes f_2) * (f_1' \otimes f_2'))(g_1, g_2) &= \int_{G \times G} f_1(x) f_2(y) f_1'(x^{-1} g_1) f_2'(y^{-1} g_2) \mathrm{d}(x, y) \\
&= \int_G f_1(x) f_1'(x^{-1} g_1) \mathrm{d}x \cdot \int_G f_2(y) f_2'(y^{-1} g_2) \mathrm{d}y \\
&= (f_1 * f_1')(g_1) \cdot (f_2 * f_2')(g_2),
\end{aligned}$$

which matches the convolution $*$ in $\mathcal{H}(G \times G, K \times K)$. The second equality above uses the condition that $G \times G$ is equipped with the normalized product Haar measure. Now it remains to check the bijectivity. By Exercise 3.6, every function in $\mathcal{H}(G \times G, K \times K)$ can be written as finite linear combination of functions of form

$$\mathbb{1}_{(g, g')(K \times K)} = \mathbb{1}_{gK} \otimes \mathbb{1}_{g'K} \in \mathcal{H}(G, K) \otimes_{\mathbb{C}} \mathcal{H}(G, K),$$

and it follows that Φ is surjective; also, the injectivity is clear because $f_1(g_1) f_2(g_2) = 0$ for all $g_1, g_2 \in G$ implies $f_1 = 0$ or $f_2 = 0$, and thus $f_1 \otimes f_2 = 0$. $\square$

Exercise 9.3. Prove [BH06, (9.5.2)].

Solution. Suppose $\chi = \chi_1 \otimes \chi_2$ is a character of T and ϕ is a character of $F^\times$. We need to prove that

$$\operatorname{Ind}_B^G(\phi \cdot \chi) \cong \phi \operatorname{Ind}_B^G \chi.$$

For this, inflate $\phi \cdot \chi \coloneqq \phi\chi_1 \otimes \phi\chi_2$ to a B-representation that is trivial on N, and then

$$\phi \cdot \chi = (\phi \circ \det|_B) \otimes \chi.$$

To complete the proof, check for each $g \in G$ that

$$\begin{aligned}(\phi \operatorname{Ind}_B^G \chi)(g) &= \phi(\det g) \cdot (\operatorname{Ind}_B^G \chi)(g) \\ &= ((\phi \circ \det) \otimes (\operatorname{Ind}_B^G \chi))(g) \\ &= (\operatorname{Ind}_B^G((\phi \circ \det)|_B \otimes \chi))(g) \\ &= (\operatorname{Ind}_B^G((\phi \circ \det|_B) \otimes \chi))(g) \\ &= (\operatorname{Ind}_B^G(\phi \cdot \chi))(g).\end{aligned}$$

The third equality above is due to the projection formula (or the associativity of tensor product of $\mathbb{C}[B]$-modules). ☐

Exercise 9.4. The *Burnside's theorem for matrix algebras* is stated as follows:

> *Theorem.* Let C be an algebraically closed field, and V a finite-dimensional C-vector space. Let A be a unital subalgebra of $\operatorname{End}_C(V)$ (i.e., a subalgebra containing 1) such that V is a simple A-module. Then $A = \operatorname{End}_C(V)$.

Show that the finite-dimensionality assumption is necessary by constructing a counter-example when this assumption is dropped.

Solution. Let $C = \mathbb{C}$ and $V = \mathbb{C}[x]$ as an infinite-dimensional vector space over $\mathbb{C}$. Note that the derivative $D\colon p(x) \mapsto p'(x)$ is an element of $\operatorname{End}_{\mathbb{C}}(V)$. Define A as the subalgebra of $\operatorname{End}_{\mathbb{C}}(V)$ generated by D over $\mathbb{C}$. We check the desired properties on A and V as follows.

(i) It is clear that A is unital. Also, if $p(x) \in V$ is such that $D(p(x)) = p(x)$, then $p(x) = 0$. It follows that there is no non-trivial A-invariant subspace in V, and thus V is a simple A-module.

(ii) Note that the shift operator $T\colon p(x) \mapsto xp(x)$ is also an element of $\operatorname{End}_{\mathbb{C}}(V)$ but is not contained in A. So $A \neq \operatorname{End}_{\mathbb{C}}(V)$.

☐

Week 10

No class in this week.

Week 11

Exercise 11.1. Let (π, V) be a smooth representation of a locally profinite group G.

(1) Show that $\langle \check{v}, v\rangle = \langle \check{\pi}(g)\check{v}, \pi(g)v\rangle$ for all $g \in G$ and $v \in \pi,\ \check{v} \in \check{\pi}$.
(2) Let K be a compact open subgroup of G. Show that for $v \in \pi^K$ and $\check{v} \in \check{\pi}^K$, the function $\gamma\colon G \to \mathbb{C},\ g \mapsto \langle \check{v}, \pi(g)v\rangle$ is left and right K-invariant.
(3) For arbitrary $\check{v} \in \check{\pi}$ and $v \in \pi$, show that $\langle \check{v}, \pi(e_K)v\rangle = \langle \check{\pi}(e_K)\check{v}, v\rangle$.

Solution. (1) By definition of contragredient we have $\check{\pi} = (\pi^*)^\infty\colon G \to \mathrm{Aut}_{\mathbb{C}}(\check{V})$. For $v^* \in \pi^*$, recall that $\langle \pi^*(g)v^*, v\rangle = \langle v^*, \pi(g^{-1})v\rangle$ for all $g \in G$, which still holds by taking the smooth subrepresentation; so we deduce the equality

$$\langle \check{\pi}(g)\check{v}, v\rangle = \langle \check{v}, \pi(g^{-1})v\rangle$$

for all $g \in G$ with $v \in \pi$ and $\check{v} \in \check{\pi}$. It follows that

$$\langle \check{\pi}(g)\check{v}, \pi(g)v\rangle = \langle \check{v}, \pi(g^{-1})\pi(g)v\rangle = \langle \check{v}, v\rangle.$$

(2) The image of γ under K-action on γ is $g \mapsto \langle \check{v}, \pi(k_1)\pi(g)\pi(k_2)v\rangle$ with $k_1, k_2 \in K$. Since $v \in \pi^K$ and $\check{v} \in \check{\pi}^K$, we have

$$\langle \check{v}, \pi(k_1)\pi(g)\pi(k_2)v\rangle = \langle \check{v}, \pi(k_1)\pi(g)v\rangle = \langle \check{\pi}(k_1^{-1})\check{v}, \pi(g)v\rangle = \langle \check{v}, \pi(g)v\rangle.$$

and hence γ is bi-K-invariant. Here the first and third equalities respectively follow from $v \in \pi^K$ and $\check{v} \in \check{\pi}^K$; the second equality is because of the formula in (1).

(3) By the preparation work in (1), we get

$$\langle \check{v}, \pi(e_K)v\rangle = \langle \check{\pi}(e_K^{-1})\check{v}, v\rangle = \langle \check{\pi}(e_K)\check{v}, v\rangle.$$

□

Exercise 11.2. Let F be a non-archimedean local field, and we write $\mathcal{O}$ for $\mathcal{O}_F$, and $\mathfrak{p}$ for the maximal ideal. Let n be a positive integer. Let $K_n = 1 + \mathrm{M}_2(\mathfrak{p}^n)$, and $T_n = T \cap K_n$, $N_n = N \cap K_n$, $N'_n = N' \cap K_n$. Here T is the subgroup of $\mathrm{GL}_2(F)$ consisting of diagonal matrices, and N (resp. N') is the subgroup of $\mathrm{GL}_2(F)$ consisting of upper triangular (resp. lower triangular) matrices with eigenvalues 1.

(1) Show that the map $N_n \times T_n \times N'_n \to K_n,\ (x,t,y) \mapsto xty$ is a homeomorphism. Previously, we showed the similar statement with K_n replaced by the Iwahori subgroup $I = \left(\begin{smallmatrix} \mathcal{O}^\times & \mathcal{O} \\ \mathfrak{p} & \mathcal{O}^\times \end{smallmatrix}\right)$.
(2) Let $t = \left(\begin{smallmatrix} \pi & \\ & 1 \end{smallmatrix}\right)$ where $\pi \in F$ is a uniformizer. Let a be an integer. Show that $t^{-a}K_nt^a = N_{n-a}T_nN'_{n+a}$.
(3) Let (π, V) be a smooth representation of $\mathrm{GL}_2(F)$, and let $v \in V^{K_n}$. Let a be a non-negative integer. Show that the $t^{-a}K_nt^a$-average of v is a non-zero constant times the N_{n-a}-average of v.

Solution. (1) Since K_n is compact and $N_n \times T_n \times N'_n$ is Hausdorff as spaces, to show they are homeomorphism it suffices to show that $N_n \times T_n \times N'_n \to K$ is bijective. Note that $K_n = N_nT_nN'_n$ as a group, so the map $N_n \times T_n \times N'_n \to N_nT_nN'_n,\ (x,t,y) \mapsto xty$ is clearly surjective. For the injectivity, we have $N_n \cap T_n = T_n \cap N'_n = N_n \cap N'_n = \{1\}$, and hence $xty = 1$ implies $x = t = y = 1$. To conclude, we get a homeomorphism $N_n \times T_n \times N'_n \simeq K_n$.

(2) By (1) we get an identity $K_n = N_nT_nN'_n$ between topological groups for $n \in \mathbb{N}$. Then we notice that

$$t^{-a}K_nt^a = (t^{-a}N_nt^a)(t^{-a}T_nt^a)(t^{-a}N'_nt^a).$$

So it reduces to the following computation. We have

$$t^{-a}N_nt^a = \begin{pmatrix} \pi^{-a} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \mathfrak{p}^n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \pi^a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mathfrak{p}^{n-a} \\ 0 & 1 \end{pmatrix} = N_{n-a},$$
$$t^{-a}N'_nt^a = \begin{pmatrix} \pi^{-a} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \mathfrak{p}^n & 1 \end{pmatrix}\begin{pmatrix} \pi^a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mathfrak{p}^{n-a} & 1 \end{pmatrix} = N'_{n-a}.$$

Also, it is immediate that

$$t^{-a}T_nt^a = T_n$$

since all the matrices are diagonal. So we conclude that

$$t^{-a}K_nt^a = N_{n-a}T_nN'_{n+a}.$$

(3) From (2) there is a non-zero constant $C_0 \neq 0$ such that the Haar measure on $t^{-a}K_nt^a = N_{n-a}T_nN'_{n+a}$ is normalized as

$$\mathrm{d}\mu_{t^{-a}K_nt^a}(g) = \mathrm{d}\mu_{N_{n-a}T_nN'_{n+a}}(xty) = C_0 \cdot \mathrm{d}\mu_{N_{n-a}}(x)\mathrm{d}\mu_{T_n}(t)\mathrm{d}\mu_{N'_{n+a}}(y).$$

On the other hand, as both T_n and N'_{n+a} are subgroups of K_n, we see $\pi(t)\pi(y)v = v$ for all $t \in T_n$ and $y \in N'_{n+a}$ because of $v \in V^{K_n}$. Thus, we compute the K_n-average of v as

$$\begin{aligned}\int_{t^{-a}K_nt^a} \pi(g)v\mathrm{d}g &= C_0 \cdot \int_{N_{n-a}T_nN'_{n+a}} \pi(xty)v\mathrm{d}x\mathrm{d}t\mathrm{d}y \\ &= C_0 \cdot \int_{N_{n-a}} \pi(x)v\mathrm{d}x \cdot \int_{T_n} \pi(t)v\mathrm{d}t \int_{N'_{n-a}} \pi(y)v\mathrm{d}y \\ &= C \cdot \int_{N_{n-a}} \pi(x)v\mathrm{d}x,\end{aligned}$$

where C is the product of C_0 with T_n-average and N'_{n-a}-average of v, which is non-zero. $\square$

Exercise 11.3. Let G be a locally profinite group. Let K be a closed subgroup of G. Let (ρ, W) be an irreducible smooth representation of K.

(1) Let $\phi\colon W \to \text{c-Ind}_K^G W$ be a K-map. Fix $g \in G$. Show that the map $f\colon W \to W,\ w \mapsto \phi(w)(g)$ is an element of $\mathrm{Hom}_{g^{-1}Kg\cap K}(\rho, \rho^g)$. Here ρ^g is the representation of $g^{-1}Kg$ given by

$$g^{-1}Kg \xrightarrow{\mathrm{Int}(g)} K \xrightarrow{\rho} \mathrm{Aut}_{\mathbb{C}}(W).$$

(2) Assume that K is compact open, and let $\alpha\colon W \to \text{c-Ind}_K^G W$ be the canonical K-map sending w to the unique $f \in \text{c-Ind}_K^G W$ that is supported on K and satisfies $f(1) = w$. Show that $\text{c-Ind}_K^G W$ is spanned by the G-translates of the image of α.

Solution. (1) Since ϕ is K-equivariant, we have by definition that, for each $k \in K$,

$$\phi(\rho(k)w)(x) = ({}^k\phi(w))(x) = \phi(w)(xk), \quad \forall x \in G.$$

It follows that f is K-equivariant. Also, by definition of compact induction, we have

$$\rho^g(g^{-1}kg)\phi(w)(g) = \rho(k)\phi(w)(g) = \phi(w)(kg).$$

This means $f(W)$ admits the interior action of $g^{-1}Kg$ and hence f can be regarded as a K-map $f\colon \rho \to \rho^g$. Moreover, in the above argument, replace K with gKg^{-1}, we see such f can also be $g^{-1}kg$-equivariant for those $k \in K$ such that $g^{-1}kg \in K$. To conclude, we see f is compatible with the action of $g^{-1}Kg \cap K$, and hence an element of $\mathrm{Hom}_{g^{-1}Kg\cap K}(\rho, \rho^g)$.

(2) From the construction of α we see that $\alpha(w)(k) = \rho(k)w$ for all $k \in K$ and $w \in W$, and $\mathrm{supp}(\alpha(w)) = K$. Using this, notice that $\mathrm{im}(\alpha) = \{f \in \text{c-Ind}_K^G W\colon \mathrm{supp}(f) = K\}$. Take an

arbitrary $f \in \text{c-Ind}_K^G W$, there are finitely many $g_1, \dots, g_m \in G$ such that $\operatorname{supp}(f) = \bigcup_{i=1}^m g_i K$, and hence we may write

$$f = \sum_{i=1}^m f_i, \quad f_i = f|_{g_i K} \in \text{c-Ind}_K^G W.$$

For our purpose we aim to show there exists $g \in G$ such that $\lambda_g f \in \operatorname{im}(\alpha)$. It suffices to check this for each f_i in replace of f. But by construction we have $\operatorname{supp}(f_i) = g_i K$, and then the left translation $\lambda_{g_i^{-1}} f_i \colon x \mapsto f_i(g_i x)$ is supported on K; consequently, we have $f_i \in \lambda_{g_i} \operatorname{im}(\alpha)$, so

$$\text{c-Ind}_K^G W = \sum_{g \in G} \lambda_g \operatorname{im}(\alpha).$$

□

Exercise 11.4. Let G be a locally profinite group. Let Z be a closed subgroup of G such that it is central in G and G/Z is compact. Let (π, V) be a smooth representation of G such that Z acts on V by a smooth character. Show that (π, V) is semi-simple.

Solution. Suppose Z acts by the central character $\omega \colon Z \to \mathbb{C}^\times$ so that for all $v \in V$, we have $\pi(z)v = \omega(z)v$. Fix a vector $v \in V$, then v is fixed by some open compact subgroup K of G by smoothness assumption, i.e. $v \in V^K$; it implies $\omega(z)v \in V^K$ as well. Since G/Z is compact, it is clear that G/KZ is finite and equipped with discrete topology, so KZ is an open subgroup of G of finite index. Thus we can apply the following result from [BH06, §2.7, Lemma].

> *Lemma.* For any open and finite index subgroup H of G, the semi-simplicity of π is equivalent to the semi-simplicity of $\operatorname{Res}_H^G \pi$.

Using this, it suffices to show that π is semi-simple as a KZ-representation. Note that in our case, the space spanned by

$$\pi(kz)v = \pi(k)\omega(z)v = \omega(z)v$$

with varying $k \in K$ and $z \in Z$ is one-dimensional and equal to an irreducible G-space. It follows that $\operatorname{Res}_{KZ}^G \pi$ is semi-simple, and so also is π as a G-representation. □

Exercise 11.5. Let G be a locally profinite group with center Z. Let K be a closed subgroup of G. Let (ρ_1, W_1) and (ρ_2, W_2) be irreducible smooth representations of K. Show that for $g \in G$, the property $\operatorname{Hom}_{g^{-1}Kg \cap K}(\rho_1^g, \rho_2) \neq 0$ depends only on the image of g in $K \backslash G / KZ$. (In other words, for any $g' \in KgKZ$, either g and g' both have this property, or they both do not have this property.)

Solution. To simplify the notation, we denote by $\mathtt{P}_g$ the property $\operatorname{Hom}_{g^{-1}Kg \cap K}(\rho_1^g, \rho_2) \neq 0$ for $g \in G$. Fix $g \in G$ and consider $g' = k_1 g k_2 z \in KgKZ$ for some $k_1, k_2 \in K$ and $z \in Z$. It suffices to show that $\mathtt{P}_g$ implies $\mathtt{P}_{g'}$, and the converse direction can be verified by a similar argument. Since $\mathtt{P}_g$ is true by assumption, there is a non-zero map $f \colon W_1 \to W_2$ such that

$$f \circ \rho_1^g(y) = \rho_2(y) \circ f, \qquad \forall y \in g^{-1}Kg \cap K.$$

To prove $\mathtt{P}_{g'}$, we need a non-zero map $f^\flat_{g,g'} \colon W_1 \to W_2$ in terms of g, g' and f, such that

$$f^\flat_{g,g'} \circ \rho_1^{g'}(x) = \rho_2(x) \circ f^\flat_{g,g'}, \qquad \forall x \in {g'}^{-1}Kg' \cap K.$$

For this, define that

$$f^\flat_{g,g'} := \rho_2(k_2^{-1}) \circ f \circ \rho_1(k_1^{-1}),$$

and it then remains to verify the desired intertwining property, i.e. $({g'}^{-1}Kg \cap K)$-equivariance. By definition, if $x \in g^{-1}kg$ we have $\rho_1^g(x) = \rho_1(gxg^{-1})$, and the similar holds for g'. Thus, given $x \in {g'}^{-1}Kg' \cap K$, there is some $k \in K$ so that

$$x = {g'}^{-1}kg' = z^{-1}k_2^{-1}g^{-1}k_1^{-1}kk_1gk_2z = k_2^{-1}g^{-1}k_1^{-1}kk_1gk_2.$$

From this, we see $k_2xk_2^{-1} \in g^{-1}Kg$, and hence $k_2xk_2^{-1} \in g^{-1}Kg \cap K$ as $x \in K$. Thus, we can apply the intertwining property of f to get

$$\rho_2(k_2xk_2^{-1}) \circ f = f \circ \rho_1^g(k_2xk_2^{-1}) = f \circ \rho_1(gk_2xk_2^{-1}g^{-1}).$$

Finally, to check the desired intertwining property of $f^\flat_{g,g'}$, we compute for $x \in g'^{-1}Kg \cap K$ that

$$\begin{aligned} f^\flat_{g,g'} \circ \rho_1^{g'}(x) &= \rho_2(k_2^{-1}) \circ f \circ \rho_1(k_1^{-1}) \circ \rho_1(k_1gk_1kk_2^{-1}g^{-1}k_1^{-1}) \\ &= \rho_2(k_2^{-1}) \circ (f \circ \rho_1(gk_1kk_2^{-1}g^{-1})) \circ \rho_1(k_1^{-1}) \\ &= \rho_2(k_2^{-1}) \circ (\rho_2(k_2xk_2^{-1}) \circ f) \circ \rho_1(k_1^{-1}) \\ &= \rho_2(x) \circ (\rho_2(k_2^{-1}) \circ f \circ \rho_1(k_1^{-1})) \\ &= \rho_2(x) \circ f^\flat_{g,g'}. \end{aligned}$$

It means that $f^\flat_{g,g'}$ is an element of $\mathrm{Hom}_{g'^{-1}Kg' \cap K}(\rho_1^{g'}, \rho_2)$, and it is non-zero if and only if f is non-zero, because both ρ_1 and ρ_2 are non-zero irreducible representations of K. This completes the proof. $\square$

Week 12

In the following F is a local non-archimedean field, with ring of integers $\mathcal{O}$, uniformizer π, and maximal ideal $\mathfrak{p} = \pi\mathcal{O}$ in $\mathcal{O}$.

Exercise 12.1. Let $\mathcal{L} = \{L_i\}_{i\in\mathbb{Z}}$ be a collection of lattices in F^2 such that $L_{i+1} \subsetneq L_i$ for all i. Show that there exists $e \in \mathbb{Z}$ such that $L_{i+e} = \pi L_i$ for all $i \in \mathbb{Z}$ if and only if $\mathcal{L}$ is stable under multiplication by $F^\times$. (These are two equivalent definitions of a lattice chain.)

Solution. Suppose $\mathcal{L}$ is stable under multiplication by $F^\times$. In particular, for each $L_i \in \mathcal{L}$ and $\pi \in F^\times$, we have $\pi L_i \in \mathcal{L}$. Since $\pi L_i \subset L_i$ always holds, for each $i \in \mathbb{Z}$ there exists some $0 \leqslant e(i) < \infty$ such that $\pi L_i = L_{i+e(i)}$. Note that the assumption $L_{i+1} \subsetneq L_i$ for all $i \in \mathbb{Z}$ forces such $e(i)$ to be uniquely determined by i. Then the following map of ordered sets

$$\iota\colon (\mathbb{Z}, \leqslant) \longrightarrow (\mathbb{Z}, \leqslant), \quad i \longmapsto i + e(i)$$

is a bijection. Thus $e(i)$ must be a constant $e \in \mathbb{Z}$. This proves that $L_{i+e} = \pi L_i$ for all $L_i \in \mathcal{L}$.

Conversely, given $e \in \mathbb{Z}_{\geqslant 0}$ such that $L_{i+e} = \pi L_i$ for all $i \in \mathbb{Z}$, then for each $a \in F^\times$ there exists $u \in \mathcal{O}^\times$ and $n \in \mathbb{Z}$ such that $a = u\pi^n$. Note that each $L_i \in \mathcal{L}$ is an $\mathcal{O}$-module so that $L_i = uL_i$. Thus it suffices to show π^n stabilizes $\mathcal{L}$. But this is clear because the assumption implies that π stabilizes $\mathcal{L}$. To conclude, we get $\{aL_i\}_{i\in\mathbb{Z}} \subset \mathcal{L}$ for all $a \in F^\times$. ☐

Exercise 12.2. Let $\mathfrak{A}_\mathcal{L}$ be the chain order associated with a lattice chain $\mathcal{L}$. Show that $\mathfrak{A}_\mathcal{L}$ is indeed an order in $A = \mathrm{M}_2(F)$, i.e., a subring which is also an $\mathcal{O}$-lattice.

Solution. By definition we have $\mathfrak{A}_\mathcal{L} = \{x \in A\colon xL_i \subset L_i \text{ for all } i \in \mathbb{Z}\}$. It is clear that $\mathfrak{A}_\mathcal{L}$ is closed under matrix addition and multiplication, and hence a subring of A. By classification of chain orders in A (c.f. [BH06, (12.1.2)]), there exists some $g \in \mathrm{GL}_2(F)$ such that

$$g\mathfrak{A}_\mathcal{L}g^{-1} \in \{\mathfrak{M}, \mathfrak{I}\} = \left\{ \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}, \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix} \right\}.$$

Since the G-translation of an $\mathcal{O}$-lattice is still an $\mathcal{O}$-lattice (see Exercise 12.7(4)), and both $\mathfrak{M}$ and $\mathfrak{I}$ are clearly $\mathcal{O}$-lattices, we have proved that $\mathfrak{A}_\mathcal{L}$ is also an $\mathcal{O}$-lattice. ☐

Exercise 12.3. Let $\mathfrak{A}_\mathcal{L}$ be as above. Show that a lattice L in F^2 is a member of $\mathcal{L}$ if and only if it is stabilized by $\mathfrak{A}_\mathcal{L}$.

Solution. The "only if" part is clear by definition of $\mathfrak{A}_\mathcal{L}$. As for the "if" part, suppose $x \in A$ stabilizes L together with all $L_i \in \mathcal{L}$. Up to G-conjugacy, we may assume $\mathfrak{A}_\mathcal{L} \in \{\mathfrak{M}, \mathfrak{I}\}$. In both cases that $\mathfrak{A}_\mathcal{L} = \mathfrak{M}$ or $\mathfrak{I}$, we have idempotent elements $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathfrak{A}_\mathcal{L}$, and that

$$L = e_1L \oplus e_2L.$$

This is because each vector $v \in L$ has the expression $v = e_1v + e_2v$. It follows that there are $a, b \in \mathbb{Z}$ such that $e_1L = \mathfrak{p}^a$ and $e_2L = \mathfrak{p}^b$, and then $L = \mathfrak{p}^a \oplus \mathfrak{p}^b$. Also, in both cases we have

$$\begin{pmatrix} 1 & \mathcal{O} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathfrak{p}^a \\ \mathfrak{p}^b \end{pmatrix} = \begin{pmatrix} \mathfrak{p}^a + \mathcal{O}\mathfrak{p}^b \\ \mathfrak{p}^b \end{pmatrix} \subset \begin{pmatrix} \mathfrak{p}^a \\ \mathfrak{p}^b \end{pmatrix},$$

which forces $\mathfrak{p}^a + \mathfrak{p}^b \subset \mathfrak{p}^a$, namely $b \geqslant a$. We then split the argument into two cases as follows.

(i) Whenever $\mathfrak{A}_\mathcal{L} = \mathfrak{M}$, by a symmetric argument, we have

$$\begin{pmatrix} 1 & 0 \\ \mathcal{O} & 1 \end{pmatrix} \begin{pmatrix} \mathfrak{p}^a \\ \mathfrak{p}^b \end{pmatrix} = \begin{pmatrix} \mathfrak{p}^a \\ \mathcal{O}\mathfrak{p}^a + \mathfrak{p}^b \end{pmatrix} \subset \begin{pmatrix} \mathfrak{p}^a \\ \mathfrak{p}^b \end{pmatrix},$$

which deduces $a \geqslant b$, and hence $L = \mathfrak{p}^a \oplus \mathfrak{p}^a$. In this case, $e_\mathcal{L} = 1$ and

$$\mathcal{L} = \{\pi^i\mathcal{O} \oplus \pi^i\mathcal{O}\}_{i\in\mathbb{Z}} = \{\mathfrak{p}^i \oplus \mathfrak{p}^i\}_{i\in\mathbb{Z}},$$

so we have proved that $L \in \mathcal{L}$.

(ii) Whenever $\mathfrak{A}_{\mathcal{L}} = \mathfrak{I}$, the stability condition implies
$$\begin{pmatrix} 1 & 0 \\ \mathfrak{p} & 1 \end{pmatrix} \begin{pmatrix} \mathfrak{p}^a \\ \mathfrak{p}^b \end{pmatrix} = \begin{pmatrix} \mathfrak{p}^a \\ \mathfrak{p}^{a+1} + \mathfrak{p}^b \end{pmatrix} \subset \begin{pmatrix} \mathfrak{p}^a \\ \mathfrak{p}^b \end{pmatrix}.$$
This gives $\mathfrak{p}^{a+1} + \mathfrak{p}^b \subset \mathfrak{p}^b$, and hence $a + 1 \geqslant b$. On the other hand we have $b \geqslant a$, so $b = a$ or $a + 1$. In this case, $e_{\mathcal{L}} = 2$ and
$$\mathcal{L} = \{\pi^i(\mathfrak{p} \oplus \mathfrak{p}), \pi^i(\mathcal{O} \oplus \mathfrak{p})\}_{i\in\mathbb{Z}},$$
so we have proved that $L = \mathfrak{p}^a \oplus \mathfrak{p}^b \in \mathcal{L}$.

This completes the proof. □

Exercise 12.4. Give an example of an order in A which is not a chain order.

Solution. Fix $r, s \in \mathcal{O}$ and consider the order
$$B = \begin{pmatrix} r\mathcal{O} & \mathcal{O} \\ \mathcal{O} & s\mathcal{O} \end{pmatrix} \subset A.$$
It is clear that B is a subring of A and we have the equality $B \otimes_{\mathcal{O}} F = A$ between $\mathcal{O}$-modules. So B is an order in A. If B is a chain order then there exists $g \in \mathrm{GL}_2(F)$ such that $gBg^{-1} \in \{\mathfrak{M}, \mathfrak{I}\}$. However, in both $\mathfrak{M}$ and $\mathfrak{I}$ the trace of any matrix can be arbitrarily chosen in $\mathcal{O}$, whereas gBg^{-1} only have matrices of trace divisible by $r + s$. This shows B is not a chain order. □

Exercise 12.5. Let $\mathfrak{A}$ be a chain order.

(1) Show that each $U_{\mathfrak{A}}^n$ with $n \in \mathbb{Z}_{\geqslant 0}$ is a compact open subgroup of $G = \mathrm{GL}_2(F)$, and normal in $U_{\mathfrak{A}}^0$. Moreover, show that $\{U_{\mathfrak{A}}^n\}_n$ is a neighborhood basis of 1 in G.
(2) Show that for $2m \geqslant n > m \geqslant 1$, there is a group isomorphism
$$\vartheta \colon \mathfrak{P}_{\mathfrak{A}}^m / \mathfrak{P}_{\mathfrak{A}}^n \xrightarrow{\sim} U_{\mathfrak{A}}^m / U_{\mathfrak{A}}^n, \quad x \longmapsto 1 + x,$$
where $\mathfrak{P}_{\mathfrak{A}}$ is the radical of $\mathfrak{A}$.

Solution. (1) Since $\mathfrak{A}$ is a chain order, we have $\mathfrak{A} = \mathfrak{A}_{\mathcal{L}}$ for some $\mathcal{L} = \{L_i\}_{i\in\mathbb{Z}}$. By definition,
$$U_{\mathfrak{A}}^n = 1 + \mathfrak{P}_{\mathfrak{A}}^n = \{1 + x^n \colon x \in \mathfrak{A} \text{ such that } xL_i \subset L_{i+1} \text{ for all } i \in \mathbb{Z}\}.$$
Notice that each $g \in U_{\mathfrak{A}}^n$ can be written as $g = 1 + y$ for some $y \in \mathfrak{A}$ such that $yL_i \subset L_{i+n}$. It follows that $f(y) \in \mathfrak{P}_{\mathfrak{A}}^n$ for all $f(y) \in \mathbb{Z}[\![y]\!]/\mathbb{Z}$. In particular, for $f(y) = \sum_{i=1}^{\infty} (-1)^i y^i$, we get $g^{-1} = 1 + f(y) \in U_{\mathfrak{A}}^n$. So $U_{\mathfrak{A}}^n$ is a subgroup of G. To show that $U_{\mathfrak{A}}^n$ is compact open, it suffices to show $\mathfrak{P}_{\mathfrak{A}}$ is compact open. For this, note that $\mathfrak{P}_{\mathfrak{A}}$ is a subring in $\mathfrak{A}$; Exercise 12.2 dictates that $\mathfrak{A}$ is an order in A, so $\mathfrak{P}_{\mathfrak{A}}$ is an $\mathcal{O}$-lattice in A as well. Thus, $\mathfrak{P}_{\mathfrak{A}}$ is an $\mathcal{O}$-lattice in finite-dimensional vector space $A \simeq F^4$, and hence an open compact subgroup in A. Since the topology of $G = \mathrm{GL}_2(F)$ and that of A are both induced from non-archimedean topology of F, we see $\mathfrak{P}_{\mathfrak{A}}$ is open compact in G.

We have proved that $U_{\mathfrak{A}}^n$ is an open compact subgroup of G, and then we check it is normal in $U_{\mathfrak{A}}^0$. For each $x \in U_{\mathfrak{A}}^n$ and $y \in U_{\mathfrak{A}}^0$, we have for all $i \in \mathbb{Z}$ that $xL_i \subset L_{i+n}$ and $yL_i \subset L_{i+1}$. The latter condition implies $y^{-1}L_{i+1} \subset L_i$. Thus $(y^{-1}xy)L_j \subset y^{-1}xL_{j+1} \subset y^{-1}L_{j+1+n} \subset L_{j+n}$ for any $j \in \mathbb{Z}$, and hence $y^{-1}xy \in U_{\mathfrak{A}}^n$, implying the normality of $U_{\mathfrak{A}}^n$ in $U_{\mathfrak{A}}^0$. To show that $\{U_{\mathfrak{A}}^n\}_n$ form a neighborhood basis of 1 in G, we apply the following fact based on Zorn's lemma.

> *Filter Criterion.* Let $\mathcal{B}$ be a collection of open subsets in a topological space X containing a special point $x \in X$. If for any $U_1, U_2 \in \mathcal{B}$ there exists $V \in \mathcal{B}$ such that $x \in V \subset U_1 \cap U_2$, then $\mathcal{B}$ is an open neighborhood basis of x in X.

Provided this, we see from definition that each $U_{\mathfrak{A}}^n$ contains $1 \in G$; also, if $m \geqslant n$ then $\mathfrak{P}_{\mathfrak{A}}^m \subset \mathfrak{P}_{\mathfrak{A}}^n$, and hence $U_{\mathfrak{A}}^m \subset U_{\mathfrak{A}}^n$. So the condition of filter criterion holds because $U_{\mathfrak{A}}^{\max(i,j)} \subset U_{\mathfrak{A}}^i \cap U_{\mathfrak{A}}^j$ for all $i, j \in \mathbb{Z}_{\geqslant 0}$. To conclude, $\{U_{\mathfrak{A}}^n\}_n$ is a compact open neighborhood basis of 1 in G.

(2) To check ϑ is a group homomorphism, note that the source of ϑ is an additive group and the target is a multiplicative group. For each $x, y \in \mathfrak{P}_{\mathfrak{A}}^m$, we have in $U_{\mathfrak{A}}^m$ that

$$\begin{aligned}\vartheta(x+y)^{-1} \cdot \vartheta(x)\vartheta(y) &= (1+x+y)^{-1}(1+x)(1+y) \\ &= 1 + (1+x+y)^{-1}xy.\end{aligned}$$

So we need to show $(1+x+y)^{-1}xy \in \mathfrak{P}_{\mathfrak{A}}^n$. Since $2m \geqslant n > m \geqslant 1$, we see $xy \in \mathfrak{P}_{\mathfrak{A}}^{2m}$ implies $xy \in \mathfrak{P}_{\mathfrak{A}}^n$. On the other hand, if $x, y \in \mathfrak{P}_{\mathfrak{A}}^m/\mathfrak{P}_{\mathfrak{A}}^n$ then $1+x+y \in U_{\mathfrak{A}}^n$ as well, which proves that ϑ is a group homomorphism. Moreover, if $\vartheta(x) = 1+x = 1$ then $x = 0$, so ϑ is injective; for each $y \in U_{\mathfrak{A}}^m/U_{\mathfrak{A}}^n$ we have $y = \vartheta(y-1)$, so ϑ is surjective. The above operations make sense in the ring $A = \mathrm{M}_2(F)$, so ϑ is an isomorphism of groups. □

Exercise 12.6 ([BH06, p.89, Exercises])**.** Define

$$\mathcal{K}_{\mathfrak{A}} = \{g \in G\colon g\mathfrak{A}g^{-1} = \mathfrak{A}\}.$$

(1) Let $g \in G$; show that $g \in \mathcal{K}_{\mathfrak{A}}$ if and only if $g\mathfrak{A} = \mathfrak{P}^m$ for some $m \in \mathbb{Z}$.
(2) Show that $\mathcal{K}_{\mathfrak{A}}$ is the normalizer $N_G(U_{\mathfrak{A}})$ of $U_{\mathfrak{A}}$ in G.
(3) Show that $\mathcal{K}_{\mathfrak{A}}$ is the G-normalizer of $U_{\mathfrak{A}}^m$ for any $m \geqslant 0$.

Solution. (1) We first introduce the following result at work.

> *Lemma.* If $\mathfrak{A} = \mathfrak{A}_{\mathcal{L}}$, then
>
> $$\mathcal{K}_{\mathfrak{A}} = \mathrm{Aut}_{\mathcal{O}}(\mathcal{L}) := \{g \in G\colon gL_i \in \mathcal{L} \text{ for all } i \in \mathbb{Z}\}.$$
>
> *Proof of Lemma.* It suffices to show the set-theoretical equality; we begin with the following observation. Let $g \in G$ be satisfying that $gL_i \in \mathcal{L}$ for all $i \in \mathbb{Z}$. Then there exists $f(g,i) \in \mathbb{Z}$ such that $gL_i = L_{f(g,i)}$, which gives rise to a map
>
> $$f\colon G \times \mathbb{Z} \longrightarrow \mathbb{Z}, \quad (g,i) \longmapsto f(g,i).$$
>
> Fix such $g \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{L})$ in the following. Since g is an $\mathcal{O}$-automorphism on $\mathcal{L}$, it preserves the partially ordered set $(\mathcal{L}, \subsetneq)$, and so also $(\mathbb{Z}, \leqslant)$; in other words, $i \leqslant j$ implies $f(g,i) \leqslant f(g,j)$, namely f is monotonely increasing in its second variable. Again as $g \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{L})$, we see $f(g, -)$ is a bijection $\mathbb{Z} \to \mathbb{Z}$. It follows that there exists a constant $m_g \in \mathbb{Z}$ (depending on g) such that $f(g,i) = i + m_g$ for all $i \in \mathbb{Z}$. In summary, we have
>
> $$gL_i = L_{i+m_g}, \quad \forall i \in \mathbb{Z}.$$
>
> Now we need to prove that $g\mathfrak{A}g^{-1} = \mathfrak{A}$ whenever $g \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{L})$. Using the observation above, for each $x \in \mathfrak{A}$, we have $(gxg^{-1})L_i = gxL_{i-m_g} \subset gL_{i-m_g} = L_i$ by definition of $\mathfrak{A}$. Then $gxg^{-1} \in \mathfrak{A}$, and hence $g\mathfrak{A}g^{-1} \subset \mathfrak{A}$. Conversely, note that $g \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{L})$ implies $g^{-1} \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{L})$, so the same argument applies (with $m_{g^{-1}} = -m_g$) to see $g^{-1}\mathfrak{A}g \subset \mathfrak{A}$, which implies $g\mathfrak{A}g^{-1} = \mathfrak{A}$. This gives the first inclusion $\mathrm{Aut}_{\mathcal{O}}(\mathcal{L}) \subset \mathcal{K}_{\mathfrak{A}}$ and it remains to prove the converse. Suppose g is such that $g\mathfrak{A}g^{-1} = \mathfrak{A}$, and also $g^{-1}\mathfrak{A}g = \mathfrak{A}$ holds. Then for each $i \in \mathbb{Z}$, if $xL_i \subset L_i$ then $g^{-1}xgL_i \subset L_i$ for all $x \in \mathfrak{A}$, and hence $x(gL_i) \subset gL_i$ for all $x \in \mathfrak{A}$. By Exercise 12.3, this then deduces $gL_i \in \mathcal{L}$ because $\mathfrak{A} = \mathfrak{A}_{\mathcal{L}}$, and thus $g \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{L})$ as desired. This proves the lemma.

We tackle with part (1). Suppose $g \in \mathcal{K}_{\mathfrak{A}}$, then there exists $m_g \in \mathbb{Z}$ as in the proof of lemma, and $gL_i = L_{i+m_g}$ for each $i \in \mathbb{Z}$ because $\mathcal{K}_{\mathfrak{A}} = \mathrm{Aut}_{\mathcal{O}}(\mathcal{L})$. Then for all $x \in \mathfrak{A}$, we have $(gx)L_i \subset gL_i = L_{i+m_g}$, and hence $gx \in \mathfrak{P}^{m_g}$ with $m_g \in \mathbb{Z}$. This proves $g\mathfrak{A} \subset \mathfrak{P}^{m_g}$. On the other hand, if $y \in \mathfrak{P}^{m_g}$ then $yL_i \subset L_{i+m_g}$ for all $i \in \mathbb{Z}$, and further $(g^{-1}y)L_i \subset g^{-1}L_{i+m_g} = L_i$; this shows $g^{-1}y \in \mathfrak{A}$, so $g\mathfrak{A} \subset \mathfrak{P}^{m_g}$ since y lies in $g\mathfrak{A}$.

Now we have proved the "only if" part and are remained to the "if" part. Suppose $g \in G$ satisfies $g\mathfrak{A} = \mathfrak{P}^m$ for some $m \in \mathbb{Z}$; a priori we cannot assign $m = m_g$ at this point. We need to show $gL_i \in \mathcal{L}$ for each $i \in \mathbb{Z}$. But by Exercise 12.3 this is equivalent to showing that gL_i is stabilized by $\mathfrak{A} = \mathfrak{A}_{\mathcal{L}}$, i.e. $xgL_i \subset gL_i$ for all $x \in \mathfrak{A}$. Note that there exists $p \in \mathfrak{P}^m$ such that $pL_i = L_{i+m}$[3]. From the assumption $g\mathfrak{A} = \mathfrak{P}^m$, there exists $\alpha \in \mathfrak{A}$ such that $p = g\alpha$. Then $g^{-1}L_{i+m} \subset \alpha L_i \subset L_i$ for all $i \in \mathbb{Z}$, and it follows that $g^{-1}xgL_i \subset g^{-1}L_{i+m} \subset L_i$, which is the desired relation to deduce $g \in \mathcal{K}_{\mathfrak{A}}$.

(2) Recall the definition that $U_{\mathfrak{A}} = \mathfrak{A}^\times$, so $N_G(U_{\mathfrak{A}}) = \{g \in G\colon g\mathfrak{A}^\times g^{-1} = \mathfrak{A}^\times\}$. It is clear that $\mathcal{K}_{\mathfrak{A}} \subset N_G(U_{\mathfrak{A}})$, because $x \in \mathfrak{A}^\times$ if and only if $gxg^{-1} \in \mathfrak{A}^\times$. For the converse inclusion, we need to show $g\mathfrak{A}^\times g^{-1} = \mathfrak{A}^\times$ implies $g\mathfrak{A}g^{-1} = \mathfrak{A}$. For this purpose, we only need to show that $\mathfrak{A}$ is contained in the $\mathcal{O}$-algebra generated by $\mathfrak{A}^\times$. Observe that if $x \in \mathfrak{A}^\times$ then $x^{-1} \in \mathfrak{A}^\times$, and by definition for all $i \in \mathbb{Z}$ we have $xL_i \subset L_i$ and $x^{-1}L_i \subset L_i$, so $L_i \subset xL_i \subset L_i$. It follows that $xL_i = L_i$ for all $i \in \mathbb{Z}$, and consequently $x \in \mathcal{O}^\times$. From this we see $\mathcal{O}^\times$ is a multiplicative subgroup of $U_{\mathfrak{A}}$. Consequently, $\mathcal{O}[\![\mathcal{O}^\times \mathbb{1}_2]\!] \subset \mathcal{O}[\![\mathfrak{A}^\times]\!]$ as $\mathcal{O}$-subalgebras of $\mathrm{M}_2(\mathcal{O})$. So it suffices to show $\mathfrak{A} \subset \mathcal{O}[\![\mathcal{O}^\times \mathbb{1}_2]\!]$ as subalgebras of $\mathrm{M}_2(\mathcal{O})$, where $\mathbb{1}_2$ is the identity matrix. Since the desired condition is invariant up to G-conjugation, we may replace $\mathfrak{A}$ with either $\mathfrak{M} = \left(\begin{smallmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{smallmatrix}\right)$ or $\mathfrak{I} = \left(\begin{smallmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{smallmatrix}\right)$. Then the inclusion clearly holds in both cases[4].

(3) Recall from proof of part (1) that there exists $\Pi \in G$ such that $m_\Pi = 1$, where the map $G \to \mathbb{Z},\ g \mapsto m_g$ is defined in the proof of lemma. It follows that for each $m \in \mathbb{Z}$, there exists $g \in G$ such that $m = m_g$. Again by part (1), for $\mathfrak{P}$ the radical of $\mathfrak{A}$, the condition $g\mathfrak{A}g^{-1} = \mathfrak{A}$ is equivalent to $g\mathfrak{A} = \mathfrak{P}^{m_g}$. So $g \in \mathcal{K}_{\mathfrak{A}}$ if and only if $g\mathfrak{A}g^{-1} = \mathfrak{A} = \mathfrak{P}^m g^{-1} = g^{-1}\mathfrak{P}^m$, namely $g\mathfrak{P}^m g^{-1} = \mathfrak{P}^m$ for some $m \in \mathbb{Z}$, and it further makes sense to take $m = m_g$ for the prescribed reason. This proves that $\mathcal{K}_{\mathfrak{A}} = \{g \in G\colon g\mathfrak{P}^{m_g}g^{-1} = \mathfrak{P}^{m_g}\} = \{g \in G\colon gU_{\mathfrak{A}}^{m_g}g^{-1} = U_{\mathfrak{A}}^{m_g}\}$; here the second equality is because of $U_{\mathfrak{A}}^m = 1 + \mathfrak{P}^m$ for $m \geqslant 0$. Therefore, $\mathcal{K}_{\mathfrak{A}}$ is the G-normalizer of $U_{\mathfrak{A}}^m$ for any $m \geqslant 0$. □

Exercise 12.7. Fix a non-trivial character $\psi\colon F \to \mathbb{C}^\times$. Then for every lattice P in $A = \mathrm{M}_2(F)$, we have defined $P^* = \{x \in A\colon \psi_A(xy) = 1 \text{ for all } y \in P\}$.

(1) Show that P^* is also a lattice, and we have $(P^*)^* = P$.
(2) Let Q be another lattice in A. Show that $P \cap Q$ and $P + Q$ are both lattices in A.
(3) Show that $(P+Q)^* = P^* \cap Q^*$ and $(P \cap Q)^* = P^* + Q^*$.
(4) Let $g \in \mathrm{GL}_2(F)$. Show that gP and Pg are lattices in A, and $(gP)^* = P^*g^{-1}$. Here $gP = \{gx\colon x \in P\}$, where gx is the matrix product.

Solution. (1) Recall that for $x \in A$, we define $\psi_A(x) \coloneqq \psi(\mathrm{tr}(x))$; it follows that for $x_1, x_2 \in P^*$, we have $\psi((x_1+x_2)y) = \psi(x_1y)\psi(x_2y) = 1$ for all $y \in P$ and hence $x_1 + x_2 \in P^*$. Also, if $x \in P^*$ and $a \in \mathcal{O}$, then for all $y \in P$ we have $\psi_A((ax)y) = \psi(a \cdot \mathrm{tr}(xy)) = \psi_A(xy)^a = 1$; here the second last equality makes sense because $a \in \mathcal{O}$ and ψ_A is $\mathcal{O}$-linear. Finally, P^* is clearly discrete because $\psi_A(x) = 1$ leads to discrete $\mathcal{O}$-module structure. So P^* is also a lattice.

We then check $(P^*)^* = P$. Note that $\psi_A(xy) = \psi_A(yx)$ for all $x, y \in A$. For any $y \in P$ and $x \in P^*$, by definition we have $\psi_A(xy) = 1$, and then $y \in (P^*)^*$. This proves $P \subset (P^*)^*$. For the converse inclusion, assume $x \in (P^*)^*$; this means $\psi_A(xy) = 1$ for all $y \in P^*$. Suppose $x \notin P$ then there must be some $y_0 \in P^*$ such that $\psi_A(y_0x) \neq 1$, which is a contradiction, and thus $x \in P$. This gives $(P^*)^* \subset P$.

(2) By assumption $P \cap Q$ is clearly a discrete and compactly generated $\mathcal{O}$-module in A, and hence a lattice. As for $P + Q$, it is closed under addition and $\mathcal{O}$-multiplication, equipped with the discrete topology inherited from P and Q. So $P + Q$ is also a lattice in A.

(3) For $x \in (P+Q)^*$ with $p \in P$ and $q \in Q$, we have $\psi_A(x(p+q)) = \psi_A(xp)\psi_A(xq) = 1$. Since this holds for arbitrary p, q, we see $\psi_A(xp) = \psi_A(xq) = 1$ must hold individually, and hence

[3]To see the existence of such p, recall that up to G-conjugacy we can identify $\mathfrak{A}$ with either $\mathfrak{M}$ or $\mathfrak{I}$. In both cases we have $\mathfrak{P}^m = \Pi^m\mathfrak{A}$ for some $\Pi \in \mathrm{GL}_2(\mathcal{O})$. In particular, we have $\Pi L_i = L_{i+1}$ and in this case we can take $p = \Pi^m$. Note that this is the only step of the proof that relies on the special feature of $G = \mathrm{GL}_2(F)$.

[4]Indeed, for $\mathfrak{A} \in \{\mathfrak{M}, \mathfrak{I}\}$ we have $\mathfrak{A} = \mathcal{O}[\![\mathcal{O}^\times \mathbb{1}_2]\!] = \mathcal{O}[\![\mathfrak{A}^\times]\!]$ unless when $\mathfrak{A} = \mathfrak{I}$ and $\mathcal{O}/\mathfrak{p} = \mathbb{F}_2$.

$x \in P^* \cap Q^*$. This proves $(P+Q)^* \subset P^* \cap Q^*$. The converse inclusion $P^* \cap Q^* \subset (P+Q)^*$ is clear because $\psi_A = \psi_A(xq) = 1$ always implies $\psi_A(x(p+q)) = 1$.

Now we prove the second equality. By part (1) it suffices to show $(P^* + Q^*)^* = P \cap Q$. But for this, using the first equality of (3), we get $(P^* + Q^*)^* = (P^*)^* \cap (Q^*)^* = P \cap Q$, where we used part (1) again. This proves $(P \cap Q)^* = P^* + Q^*$.

(4) Note that gP is a translation of P, and hence they have the same discrete topology. Also, for each $a \in \mathcal{O}$, $a(gP) = g(aP)$ holds and then gP is an $\mathcal{O}$-lattice. The same argument works for Pg. For each $x \in (gP)^*$ we have $\psi_A(xgp) = 1$ for all $p \in P$, and hence $xg \in P^*$, which means $x \in P^* g^{-1}$; this proves $(gP)^* \subset P^* g^{-1}$. The converse inclusion also follows from $\psi_A(x(gp)) = \psi_A((xg)p)$. Then we have done. □

Exercise 12.8. In the setting of [BH06, §12.8, Proposition], the following are equivalent:

(1) The coset $a + \mathfrak{P}^{1-n}$ contains a nilpotent element of $\mathfrak{A}$.
(2) There is an integer $r \geqslant 1$ such that $a^r \in \mathfrak{P}^{1-rn}$.

Show that each of the two conditions does not change if we replace $(\mathfrak{A}, n, a)$ by $(\mathfrak{A}, n - e_{\mathfrak{A}}, \pi a)$; that is, condition (1) is satisfied by $(\mathfrak{A}, n, a)$ if and only if it is satisfied by $(\mathfrak{A}, n - e_{\mathfrak{A}}, \pi a)$, and similarly for condition (2).

Solution. In the following we do not a priori admit the proposition that (1) and (2) are equivalent. So we need to check both conditions respectively.

(1) Suppose condition (1) holds for $(\mathfrak{A}, n, a)$ then there exists $t \in G$ such that $a+t$ is nilpotent in $a + \mathfrak{P}^{1-n}$; for this we need to require that $tL_i \subset L_{i+1-n}$ for all $i \in \mathbb{Z}$. Recall from periodicity that $\pi L_{i-e_{\mathfrak{A}}} = L_i$ for all $i \in \mathbb{Z}$. So we equivalently have $t(\pi L_{i-e_{\mathfrak{A}}}) \subset L_{i+1-n}$ for all $i \in \mathbb{Z}$, and hence $(\pi t) L_i \subset L_{i+(1-n+e_{\mathfrak{A}})}$. It is clear that this means $\pi t \in \mathfrak{P}^{1-n+e_{\mathfrak{A}}}$. Also, $a+t$ is nilpotent if and only if $\pi a + \pi t$ is nilpotent. So the condition (1) for $(\mathfrak{A}, n + e_{\mathfrak{A}}, \pi a)$ follows. The converse verification is by simply reverse the argument above.

(2) Unwinding the definition, for the stratum $(\mathfrak{A}, n, a)$, condition (2) (i.e. $a^r \in \mathfrak{P}^{1-rn}$ for some $r \geqslant 1$) means $L_i \subset a^r L_{i+rn-1}$ for all $i \in \mathbb{Z}$. For the same r in the following, condition (2) for the stratum $(\mathfrak{A}, n - e_{\mathfrak{A}}, \pi a)$ (i.e. $(\pi a)^r \in \mathfrak{P}^{1-r(n-e_{\mathfrak{A}})}$) means $L_i \subset (\pi a)^r L_{i+r(n-e_{\mathfrak{A}})-1}$ for all $i \in \mathbb{Z}$. To show the two conditions above are equivalent, we compute

$$(\pi a)^r L_{i+r(n-e_{\mathfrak{A}})-1} = a^r (\pi^r L_{(i+rn-1)-re_{\mathfrak{A}}}) = a^r L_{i+rn-1}.$$

Here in the last equality, we recursively used the property $\pi L_j = L_{j+e_{\mathfrak{A}}}$ for all $j \in \mathbb{Z}$ to deduce $\pi^r L_{i-re_{\mathfrak{A}}} = L_i$. □

Exercise 12.9. Let $\mathfrak{I} = \left(\begin{smallmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{smallmatrix}\right)$ be the standard period-2 chain order. Let $\mathfrak{P}$ be its radical.

(1) Show that
$$\mathfrak{P} = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix} \mathfrak{I} = \mathfrak{I} \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix} = \begin{pmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix}$$
using the definition $\mathfrak{P} = \{x \in \mathfrak{I} \colon xL_i \subset L_{i+1} \text{ for all } i \in \mathbb{Z}\}$. Here $\{L_i\}_{i \in \mathbb{Z}}$ is the lattice chain corresponding to $\mathfrak{I}$.
(2) Let $a \in \mathfrak{I}$ be such that some positive power of a lies in $\mathfrak{P}$. Show that $a + \mathfrak{P}$ contains a nilpotent element.
(3) Let $a \in \mathfrak{P}$ be such that $a^r \in \mathfrak{P}^{1+r}$ for some positive integer r. Show that $a + \mathfrak{P}^2$ contains a nilpotent element.

Solution. (1) The equalities $\left(\begin{smallmatrix} 0 & 1 \\ \pi & 0 \end{smallmatrix}\right)\mathfrak{I} = \mathfrak{I}\left(\begin{smallmatrix} 0 & 1 \\ \pi & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right)$ are immediate from direct computation. So it suffices to check $\mathfrak{P} = \left(\begin{smallmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right)$. Let $g \in \mathfrak{P}$ with the condition $gL_i \subset L_{i+1}$ for all $i \in \mathbb{Z}$. Since $\mathcal{L} = \{L_i\}_{i \in \mathbb{Z}}$ is a period-2 chain order, for each fixed $i \in \mathbb{Z}$, this condition can be written as

$$\begin{aligned} &\text{either} && gL_i = g(\mathfrak{p}^{n-1} \oplus \mathfrak{p}^n) \subset L_{i+1} = \mathfrak{p}^n \oplus \mathfrak{p}^n, \\ &\text{or} && gL_i = g(\mathfrak{p}^n \oplus \mathfrak{p}^n) \subset L_{i+1} = \mathfrak{p}^n \oplus \mathfrak{p}^{n+1}, \end{aligned}$$

for some $n \in \mathbb{Z}$. If we write $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then it satisfies the two relations above simultaneously for i varying, i.e., for all $n \in \mathbb{Z}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \mathfrak{p}^{n-1} \\ \mathfrak{p}^{n} \end{pmatrix} \subset \begin{pmatrix} \mathfrak{p}^{n} \\ \mathfrak{p}^{n} \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \mathfrak{p}^{n} \\ \mathfrak{p}^{n} \end{pmatrix} \subset \begin{pmatrix} \mathfrak{p}^{n} \\ \mathfrak{p}^{n+1} \end{pmatrix}.$$

The first inclusion implies $a, c \in \mathfrak{p}$ and $b, d \in \mathcal{O}$; the second inclusion together with $c \in \mathfrak{p}$ deduce $a, b \in \mathcal{O}$ and $d \in \mathfrak{p}$. Combining these, we have $g \in \left(\begin{smallmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right)$. Conversely, following the computation above it is clear that any element $x \in \left(\begin{smallmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right)$ satisfies $xL_i \subset L_{i+1}$ for all $i \in \mathbb{Z}$, and hence lies in $\mathfrak{P}$. This proves the desired expression of $\mathfrak{P}$.

(2) Write $a = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_4 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & a_2 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ a_3 & 0 \end{smallmatrix}\right) \in \left(\begin{smallmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{smallmatrix}\right)$ and suppose that $a^r \in \mathfrak{P}$ for some $r \geqslant 1$. Note that the latter two matrices are nilpotent in $\mathrm{GL}_2(\mathcal{O})$. Using part (1), the assumption means

$$a^r = \begin{pmatrix} a_1^r & * \\ * & a_4^r \end{pmatrix} + b \in \begin{pmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix} = \mathfrak{P},$$

where the non-diagonal entries can be computed in terms of a_2 and a_3, and $b \in \mathrm{M}_2(\mathcal{O})$ has diagonal entries with degrees in a_1, a_4 no more than $r-1$. It now follows that $a_1^r, a_4^r \in \mathfrak{p}$, and hence $a_1, a_4 \in \mathfrak{p}$. Therefore, we see

$$a \in \begin{pmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix} = \mathfrak{P},$$

so $-a \in \mathfrak{P}$ as well; then $a + \mathfrak{P}$ contains a nilpotent element because it contains $0 = a + (-a)$.

(3) Based on the proof of part (2), we directly copy the argument of [BH06, §12.8, Proposition]. Note that if $a = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right) \in \mathfrak{P}$, then, by replacing a with its $U_{\mathfrak{I}}$-conjugate if necessary, we have

$$a + \mathfrak{P}^2 = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix} + \mathfrak{P}^2.$$

By a similar computation, the condition $a^r \in \mathfrak{P}^{1+r}$ for some $r \geqslant 1$ implies $a_1 a_4 \in \mathfrak{p}$. This can be done through the matrix computation on condition $a^r L_i \subset L_{i+1+r}$ for all $i \in \mathbb{Z}$ as in part (1), with $L_i \in \{\mathfrak{p}^{n-1} \oplus \mathfrak{p}^n, \mathfrak{p}^n \oplus \mathfrak{p}^n\}$. In this case, as $a_3 \in \mathfrak{p}$, there exists $\jmath \in \mathfrak{P}^2$ such that

$$a + \jmath = a + \begin{pmatrix} -a_1 & 0 \\ -a_3 & -a_4 \end{pmatrix} = \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix}, \quad a_2 \in \mathcal{O}.$$

This is clearly a nilpotent element so we are done. □

Exercise 12.10. Classify the lattice chains in F^3. More specifically, for each $e \geqslant 1$, classify the $\mathrm{GL}_3(F)$-orbits of lattice chains in F^3 with period e.

Solution. Let $\mathcal{L} = \{L_i\}_{i \in \mathbb{Z}}$ be a lattice chain in F^3 with period $e \geqslant 1$, so we have $L_{i+e} = \pi L_i$ for all $i \in \mathbb{Z}$. Consider the chain $L_i \supsetneq L_{i+1} \supsetneq \cdots \supsetneq L_{i+e}$, in which each subquotient of lattices is viewed as an $\mathcal{O}$-submodule of $L_i/L_{i+e} = L_i/\pi L_i$. Applying $(-) \otimes_{\mathcal{O}} k$ where k is the residue field of $\mathfrak{p}$, we can identify each subquotient L_{i+m}/L_{i+m+n} with $0 \leqslant m < m+n \leqslant e$ with some k-subspace in k^3 of dimension either of $1, 2, 3$. In particular, we have $\dim_k(L_i/L_{i+e}) \otimes_{\mathcal{O}} k \leqslant 3$ but $\dim_k(L_{i+j}/L_{i+j+1}) \otimes_{\mathcal{O}} k \geqslant 1$ for all $j \in \{0, \ldots, e-1\}$. This forces $e \leqslant 3$.

We then split the case into cases with $e = 1, 2, 3$ respectively. Before we start, fix $i \in \mathbb{Z}$ and note that the lattice $L_i \subset F^3$ is in the G-orbit of the standard lattice $\mathcal{O}^3$, i.e. there exists $g \in G = \mathrm{GL}_3(F)$ such that $L_i = g\mathcal{O}^3 \subset F^3$. Thus, up to G-translate, we may assume $L_i = \mathcal{O}^3$ without loss of generality.

Case I. Suppose $e = 1$. Then $L_{i+1} = \pi L_i = \pi \mathcal{O}^3$. which further implies that $L_j = \pi^{i-j} L_i = \pi^{i-j}\mathcal{O}^3$ for each $j \in \mathbb{Z}$. Therefore, in this case the lattice chain is read as

$$\mathcal{L} = \{L_j\}_{j \in \mathbb{Z}} = \{\pi^{i-j}\mathcal{O}^3\}_{j \in \mathbb{Z}} = \{\mathfrak{p}^n \oplus \mathfrak{p}^n \oplus \mathfrak{p}^n\}_{n \in \mathbb{Z}}.$$

Case II. Suppose $e = 2$. Then $L_{i+2} = \pi L_i = \pi\mathcal{O}^3$ and then $\pi\mathcal{O}^3 = L_{i+2} \subsetneq L_{i+1} \subsetneq L_i = \mathcal{O}^3$. For the prescribed reason L_i/L_{i+2} corresponds to a 2-dimensional k-subspace of k^3. This forces L_i/L_{i+1} to correspond to a subspace of k^3 of dimension 1 or 2, and L_{i+1}/L_{i+2} then corresponds to the direct sum complement of L_i/L_{i+1}. To conclude, in this case the chain lattice can be either

$$\mathcal{L} = \{\cdots \subsetneq (\mathfrak{p}^n)^3 \subsetneq \mathfrak{p}^{n-1} \oplus (\mathfrak{p}^n)^2 \subsetneq (\mathfrak{p}^{n-1})^3 \subsetneq \cdots\}$$

or

$$\mathcal{L} = \{\cdots \subsetneq (\mathfrak{p}^n)^3 \subsetneq (\mathfrak{p}^{n-1})^2 \oplus \mathfrak{p}^n \subsetneq (\mathfrak{p}^{n-1})^3 \subsetneq \cdots\}.$$

Case III. Suppose $e = 3$. Then $L_{i+3} = \pi L_i = \pi\mathcal{O}^3$. Similar to Case II above, for $l \in \{0, 1, 2\}$, L_{i+l}/L_{i+l+1} must correspond to a 1-dimensional subspace of k^3, and the direct sum of these three subspaces equals to k^3. So in this case, the chain lattice must look like

$$\mathcal{L} = \{\cdots \subsetneq (\mathfrak{p}^n)^3 \subsetneq \mathfrak{p}^{n-1} \oplus (\mathfrak{p}^n)^2 \subsetneq (\mathfrak{p}^{n-1})^2 \oplus \mathfrak{p}^n \subsetneq (\mathfrak{p}^{n-1})^3 \subsetneq \cdots\}.$$

□

Week 13

Exercise 13.1. Let π be an irreducible representation of $G = \mathrm{GL}_2(F)$ with $\ell(\pi) = 0$. Show that π contains a stratum of the form $\left(\mathfrak{I}, 1, \left(\begin{smallmatrix} 0 & 0 \\ a_0 & 0 \end{smallmatrix}\right)\right)$, where $a_0 \in \mathcal{O}_F$.

Solution. By definition, $\ell(\pi) = 0$ means π contains a trivial character on $U^1_{\mathfrak{M}}$. Observe that

$$U^2_{\mathfrak{I}} \subset U^1_{\mathfrak{M}} \subset U^1_{\mathfrak{I}},$$

so π contains a character χ on $U^1_{\mathfrak{I}}$ that is trivial on $U^1_{\mathfrak{M}}$, and hence χ is trivial on $U^2_{\mathfrak{I}}$. In other words, π contains a trivial character on $U^2_{\mathfrak{I}}$. This further means π contains some stratum $(\mathfrak{I}, 1, \alpha)$ with $\alpha \in U^1_{\mathfrak{I}}/U^2_{\mathfrak{I}}$. In order to determine α, note that

$$U^1_{\mathfrak{I}}/U^2_{\mathfrak{I}} = \frac{1+\mathfrak{P}_{\mathfrak{I}}}{1+\mathfrak{P}^2_{\mathfrak{I}}} = \begin{pmatrix} 1+\mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix} \Big/ \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{pmatrix} \simeq \begin{pmatrix} 0 & \boldsymbol{k} \\ \boldsymbol{k} & 0 \end{pmatrix},$$

where $\boldsymbol{k} = \mathcal{O}/\mathfrak{p}$; the last isomorphism above is between a multiplicative group and an additive group. It follows that the image of α in $\mathrm{M}_2(\boldsymbol{k})$ must be of form $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ with $b, c \in \boldsymbol{k}$. On the other hand, if we fix $\psi\colon F \to \mathbb{C}^\times$ such that $\psi(0) = 1$, then as π contains $\psi_\alpha|_{U^2_{\mathfrak{I}}} = \mathbb{1}$, we have that $\psi(\mathrm{tr}(\alpha(x-1))) = 1 \in \mathbb{C}^\times$ for all $x \in U^2_{\mathfrak{I}}$, or equivalently the trace of $\alpha(x-1)$ modulo $\mathfrak{P}^2_{\mathfrak{I}} = \left(\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} \end{smallmatrix}\right)$ must be zero. For this, we identify $b, c \in \boldsymbol{k}$ with their liftings to $\mathcal{O}$ and compute

$$\alpha(x-1) \in a\mathfrak{P}^2_{\mathfrak{I}} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\begin{pmatrix} \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} \end{pmatrix} = \begin{pmatrix} b\mathfrak{p}^2 & 0 \\ 0 & c\mathfrak{p} \end{pmatrix} \equiv \begin{pmatrix} b\mathfrak{p}^2 & 0 \\ 0 & 0 \end{pmatrix} \bmod \mathfrak{P}^2_{\mathfrak{I}}.$$

This forces $b = 0$, and therefore, there exists $a_0 \in \mathcal{O}$ that is congruent with a lift of c modulo $\mathfrak{I}$, such that $\alpha = \left(\begin{smallmatrix} 0 & 0 \\ a_0 & 0 \end{smallmatrix}\right) \in \mathrm{M}_2(\mathcal{O})$. So we have proved that π contains the stratum $(\mathfrak{I}, 1, \alpha)$. ☐

Exercise 13.2. Consider a stratum of the form $(\mathfrak{I}, -1, \alpha)$. Check that it is fundamental if and only if $\alpha\mathfrak{I} = \mathfrak{P}_{\mathfrak{I}}$.

Solution. Recall from definition of stratum that $(\mathfrak{I}, -1, \alpha)$ is fundamental if and only if $\alpha + \mathfrak{P}^2_{\mathfrak{I}}$ contains no nilpotent element of A; we aim to prove this is equivalent to $\alpha\mathfrak{I} \neq \mathfrak{P}_{\mathfrak{I}}$. Also by definition of stratum, it is automatic that $\alpha \in U^1_{\mathfrak{I}}/U^2_{\mathfrak{I}}$, so α can be viewed as the lift of an element of the additive group $\left(\begin{smallmatrix} 0 & \boldsymbol{k} \\ \boldsymbol{k} & 0 \end{smallmatrix}\right)$, using the argument in the proof of Exercise 13.1; without loss of generality, we may assume $\alpha = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ with $b, c \in \mathcal{O}$, and see

$$\alpha\mathfrak{I} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix} = \begin{pmatrix} b\mathfrak{p} & b\mathcal{O} \\ c\mathcal{O} & c\mathcal{O} \end{pmatrix} \neq \begin{pmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix} = \mathfrak{P}_{\mathfrak{I}}$$

unless $b = 1$ and $c = \pi$ up to scalar by $\mathcal{O}^\times$. Further, since $\mathfrak{P}_{\mathfrak{I}}$ is closed under addition, modulo $\mathfrak{P}_{\mathfrak{I}}$ we may assume $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ c & 0 \end{smallmatrix}\right)$; this makes sense because we only concern about the condition $\alpha\mathfrak{I} = \mathfrak{P}_{\mathfrak{I}}$. Therefore, we only need to consider $\alpha \in A = \mathrm{M}_2(\mathcal{O})$ of form $\left(\begin{smallmatrix} 0 & 1 \\ c & 0 \end{smallmatrix}\right)$, where c is a lift of some element in $\mathcal{O}/\mathfrak{p}$. On the other hand, reducing everything from $\mathrm{M}_2(\mathcal{O})$ to $\mathrm{M}_2(\boldsymbol{k})$, we observe that

$$\alpha\mathfrak{I} \otimes_{\mathcal{O}} \boldsymbol{k} = \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{k} & \boldsymbol{k} \\ 0 & \boldsymbol{k} \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{k} \\ c\boldsymbol{k} & c\boldsymbol{k} \end{pmatrix} \neq \begin{pmatrix} 0 & \boldsymbol{k} \\ 0 & 0 \end{pmatrix} = \mathfrak{P}_{\mathfrak{I}} \otimes_{\mathcal{O}} \boldsymbol{k}$$

unless the image of c is $0 \in \boldsymbol{k}$, or equivalently π divides c in $\mathcal{O}$. Again, as c is lifted from $\mathcal{O}/\mathfrak{p}$, we see $c \neq 0$ if and only if $c = \pi$ under the prescribed situation.

To proceed on, suppose $\alpha + \mathfrak{P}^2_{\mathfrak{I}}$ contains no nilpotent element of A. In particular, $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ c & 0 \end{smallmatrix}\right)$ is not nilpotent, and hence $c \neq 0$; but this is equivalent to $c = \pi$, implying that $\alpha\mathfrak{I} = \mathfrak{P}_{\mathfrak{I}}$. Conversely, suppose there is a nilpotent element in $\alpha + \mathfrak{P}^2_{\mathfrak{I}}$, then $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ c & 0 \end{smallmatrix}\right) \in A$ is nilpotent modulo $\mathfrak{P}^2_{\mathfrak{I}}$, and thus $c = 0$; if this is true then $\alpha\mathfrak{I} \neq \mathfrak{P}_{\mathfrak{I}}$. This proves the desired equivalence. ☐

Exercise 13.3. Let $(\mathfrak{A}, n, \alpha)$ be a stratum with $e_{\mathfrak{A}} = 1$. Define $\tilde{f}_\alpha(t)$ as in [BH06, §13.2].

(1) Show that $\tilde{f}_\alpha(t)$ depends only on the G-conjugacy class of $(\mathfrak{A}, n, \alpha)$.
(2) Show that $(\mathfrak{A}, n, \alpha)$ is fundamental if and only if $\tilde{f}_\alpha(t) \neq t^2$.

Solution. Recall from definition that for $\alpha = \pi^{-n}\alpha_0$ with $\alpha_0 \in \mathfrak{A}$, the polynomial $\tilde{f}_\alpha(t) \in \boldsymbol{k}[t]$ is defined as the characteristic polynomial of image of $\alpha_0 \in \mathfrak{A}$ in $\mathfrak{A}/\mathfrak{P}_{\mathfrak{A}} \cong \mathrm{M}_2(\boldsymbol{k})$.

(1) The conjugation by $g \in G$ maps the stratum $(\mathfrak{A}, n, \alpha)$ to $(g\mathfrak{A}g^{-1}, n, g\alpha g^{-1})$. If we write $\alpha = \pi^{-n}\alpha_0$ with $\alpha_0 \in \mathfrak{A}$ then we have $g\alpha g^{-1} = \pi^{-n}(g\alpha_0 g^{-1})$ with $g\alpha_0 g^{-1} \in g\mathfrak{A}g^{-1}$. Thus, $\tilde{f}_{g\alpha g^{-1}}(t)$ is the characteristic polynomial of image of $g\alpha_0 g^{-1}$ in $\mathfrak{A}/\mathfrak{P}_{\mathfrak{A}}$, which is the same as that of α_0. This proves $\tilde{f}_{g\alpha g^{-1}}(t) = \tilde{f}_\alpha(t)$.

(2) By part (1) we can replace $(\mathfrak{A}, n, \alpha)$ by its G-conjugation $(\mathfrak{M}, n, g\alpha g^{-1})$, where $g \in G$ is such that $\mathfrak{M} = g\mathfrak{A}g^{-1}$. By [BH06, (12.8.1)], this stratum is non-fundamental if and only if $g\alpha g^{-1} = \pi^{-n}\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Therefore, if $(\mathfrak{A}, n, \alpha)$ is non-fundamental then

$$\tilde{f}_\alpha(t) = \tilde{f}_{g\alpha g^{-1}}(t) = t^2.$$

Conversely, recall from Exercise 13.1 that we can view α as a lift of an element in $\left(\begin{smallmatrix} 0 & \boldsymbol{k} \\ \boldsymbol{k} & 0 \end{smallmatrix}\right)$ with $\boldsymbol{k} = \mathcal{O}/\mathfrak{p}$. If $\tilde{f}_\alpha(t) = t^2$ then α must be nilpotent in $\mathfrak{A}$ up to G-conjugacy, so there exists $g \in G$ such that

$$g\mathfrak{A}g^{-1} = \mathfrak{M}, \quad g\alpha g^{-1} = \pi^{-n}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This is equivalent to $(\mathfrak{A}, n, \alpha)$ being fundamental. $\square$

Exercise 13.4 ([BH06, p.98, Exercise])**.** Let E/F be a quadratic field extension, let $\alpha \in E^\times$, and write $n = -v_E(\alpha)$. Show that α is minimal over F if and only if $\alpha + \mathfrak{p}_E^{1-n} \cap F = \emptyset$.

Solution. Choose a uniformizer π of F. Recall from definition that α minimal means the subalgebra $E = F[\alpha]$ of $\mathrm{M}_2(\mathcal{O}_F)$ is a field and

- If E/F is totally ramified, then $n = -v_E(x)$ is odd.
- If E/F is unramified, then the coset $\pi^n\alpha + \mathfrak{p}_E$ generates the field extension $\boldsymbol{k}_E/\boldsymbol{k}$.

For the "if" part, assume $\alpha + \mathfrak{p}_E^{1-n} \cap F \neq \emptyset$. In this case, if E/F is totally ramified, then there exists $x \in \mathcal{O}_E$ such that $\alpha + \pi_E^{1-n}x \in F$. In particular,

$$v_E(\alpha + \pi_E^{1-n}x) = \min(v_E(\alpha), v_E(\pi_E^{1-n}x)) = \min(-n, 1 - n + v_E(x)) \in \mathbb{Z}$$

must be even because $\pi_E^2 = \pi$. On the other hand, we have $v_E(x) \geqslant 0$ so this valuation must equal to $-n$, which proves that n is even and contradicts to the definition of minimality. So it suffices to consider the case where E/F is unramified, for which we can identify π_E with π. By assumption, there exists $x \in \mathcal{O}_E$ such that $\alpha + \pi^{1-n}x \in F$, or equivalently $\pi^n\alpha + \pi x \in F$. Notice that $v_E(\pi^n\alpha) = v_F(\pi^n\alpha) = 0$ and $\pi^n\alpha \in \mathcal{O}_E$. We then assume that $E = F[\alpha]$ as well as α is minimal to deduce the contradiction. Under this assumption, $\mathcal{O}_E = \mathcal{O}_F[\pi^n\alpha]$ by [BH06, §13.4, Lemma], and $\alpha \notin F$; the latter further implies $\pi x \notin F$ and hence $x \notin F$. Since α is minimal, $\boldsymbol{k}_E/\boldsymbol{k}$ is generated by $\pi^n\alpha + \mathfrak{p}_E$, then $x \in \mathcal{O}_E$ but $x \notin \mathcal{O}_F$, namely $x \in \mathcal{O}_F[\pi^n\alpha]$ but $x \notin \mathcal{O}_F$. So there exists $s, t \in \mathcal{O}_E$ such that $x = s + t(\pi^n\alpha)$ because $\pi^n\alpha \in \mathcal{O}_E$ must be a root of a quadratic polynomial over $\mathcal{O}_F$. This forces

$$\pi^n\alpha + \pi x = s\pi + (t+1)(\pi^n\alpha) = s\pi + \alpha \cdot (t+1)\pi^n \in \mathcal{O}_F + \alpha\mathcal{O}_F,$$

and it follows that $(\mathcal{O}_F + \alpha\mathcal{O}_F) \cap F \neq \emptyset$. This is a contradiction to $\alpha \notin F$. So we have proved that α cannot be minimal.

For the "only if" part, assume $\alpha + \mathfrak{p}_E^{1-n} \cap F = \emptyset$ and we need to show the minimality of α. By assumption we have $\alpha \notin F$, and hence $E = F[\alpha]$ holds. When E/F is totally ramified, by assumption $\pi_E^{1-n}x + \alpha \notin F$ for all $x \in \mathcal{O}_E$. Notice that $\alpha \notin F$ implies $\pi_E^{1-n}x \notin F$. In this case, if $n = -v_E(\alpha)$ is even, then $\pi_E x \notin F$ for all $x \in \mathcal{O}_E$, which fails to be true; for a counter-example, one can take $x = \pi_E \in \mathcal{O}_E$ so that $\pi_E x = \pi_E^2 = \pi \in \mathcal{O}_F$. This proves that n is odd. Now it remains to consider the case when E/F is unramified with $\pi_E = \pi$. To show the extension $\boldsymbol{k}_E/\boldsymbol{k}$ of degree 2 is generated by $\pi^n\alpha + \mathfrak{p}_E$, it suffices to show $\pi^n\alpha \in \mathcal{O}_E - \mathcal{O}_F$. For this we only need $\pi^n\alpha \notin \mathcal{O}_F$. But for all $x \in \mathcal{O}_E$, note that $\pi^{1-n}x + \alpha \notin F$ if and only if $\pi x + \pi^n\alpha \notin F$. On the other hand, whenever $\pi^n\alpha \in \mathcal{O}_F$ we have $\pi x \notin F$ and hence $x \notin F$ for all $x \in \mathcal{O}_E$, which is again impossible. This completes the proof. $\square$

Exercise 13.5. Prove [BH06, §14.2, Proposition (2)(3)]. Here the level of a character of $F^\times$ is defined in [BH06, §1.8], and the terminology *essentially scalar fundamental stratum* is defined in [BH06, §13.2].

Solution. Let $\chi = \chi_1 \otimes \chi_2$ be a character of T and set $\Sigma = \mathrm{Ind}_B^G \chi$. Let n_i be the level of χ_i. We need to prove the following:

(2) If $n_1 = n_2 = n \neq 0$ and $\chi_1\chi_2^{-1}|_{U_F^n}$ is the trivial character, then Σ contains an essentially scalar fundamental stratum.

(3) If $n_1 = n_2 = 0$, then Σ contains the trivial character of $U_{\mathfrak{J}}^1$.

In the following, fix a character $\psi\colon F \to \mathbb{C}^\times$.

To prove (2), for $i = 1, 2$ there exists $a_i \in \mathfrak{p}^{-n}$ such that $\chi_i(1+x) = \psi(a_i x)$ for all $x \in \mathfrak{p}^n$. By definition of level, n_i is the least integer such that

$$\psi(a_i x) = \chi_i(1+x) = 1, \quad \forall x \in \mathfrak{p}^{n_i+1}.$$

Since $n_1 = n_2 = n \neq 0$, we see $v_F(a_1) = v_F(a_2) = -n$. Moreover, since $\chi_1\chi_2^{-1}|_{U_F^n}$ is trivial, we see $\chi_1(1+x) = \chi_2(1+x)$ for all $x \in \mathfrak{p}^n$, or equivalently $\psi(a_1x) = \psi(a_2x)$ for all $x \in \mathfrak{p}^n$. This implies $a_1 \equiv a_2 \bmod \mathfrak{p}^{1-n}$. In particular, we have $\pi^n a_1 \equiv \pi^n a_2 \bmod \mathfrak{p}$, namely $\pi^n a_1$ and $\pi^n a_2$ have the same image in $\boldsymbol{k} = \mathcal{O}/\mathfrak{p}$. Thus,

$$(\mathfrak{M}, n, a) = \left(\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}, n, \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\right)$$

must be an essentially scalar stratum, because $\tilde{f}_a(t)$ has doubled root in $\boldsymbol{k}^\times$.

We claim that $(\mathfrak{M}, n, a)$ is a fundamental stratum. Indeed, it suffices to check that a modulo $\mathfrak{P}_{\mathfrak{M}}^{1-n}$ is not nilpotent in $\mathrm{M}_2(F)$, for which we only need to consider $\pi^n a$ modulo $\mathfrak{P}_{\mathfrak{M}} = \left(\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right)$. Then the claim is clear because $v_F(\pi^n a_i) = 0$, meaning that $\pi^n a_i \in \mathcal{O} - \mathfrak{p}$.

Now it remains to verify that $\Sigma = \mathrm{Ind}_B^G \chi$ contains the character

$$\psi_a\colon U_{\mathfrak{M}}^n = 1 + \mathfrak{P}_{\mathfrak{M}}^n \longrightarrow \mathbb{C}^\times, \quad 1 + x \longmapsto \psi(\mathrm{tr}(ax)).$$

For this, let $f \in \Sigma$ be supported on $BU_{\mathfrak{M}}^n = BN'_n$, where $N'_n = \left(\begin{smallmatrix} 1 & 0 \\ \mathfrak{p}^n & 1 \end{smallmatrix}\right)$ as in Exercise 11.2. Then for any $u \in U_{\mathfrak{M}}^n$, we can write $u = bn'$ for some $b \in B$ and $n' \in N'_n$. Note that

$$a(n'-1) \in \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \mathfrak{p}^n & 0 \end{pmatrix} \subset \begin{pmatrix} 0 & 0 \\ a_2\mathfrak{p}^n & 0 \end{pmatrix}$$

and in particular it has trace zero. Thus, for our purpose we may assume f is fixed by N'_n, so $u = bn'$ acts on f by $\chi(b)$. By definition of smooth induction, we get

$$b \cdot f = \chi(b)f = \psi(\mathrm{tr}(a(b-1)))f = \psi_a(b)f,$$

where the second equality is because of the construction $\chi = \chi_1 \otimes \chi_2$ and $\chi_i(1+x) = \psi(a_i x)$. So Σ contain ψ_a when restricted to $U_{\mathfrak{M}}^n$. This gives the conclusion that Σ contains an essentially scalar fundamental stratum $(\mathfrak{M}, n, a)$.

To prove (3), we only need to modify the argument for part (2). Note that when $n_1 = n_2 = 0$ we have $\chi_1(1+x) = \chi_2(1+x) = 1$ for all $x \in \mathfrak{p}$. So for any $u \in U_{\mathfrak{J}}^1 = \left(\begin{smallmatrix} 1+\mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & 1+\mathfrak{p} \end{smallmatrix}\right)$, with the same assumption as above, we have $\mathrm{tr}(a(u-1)) = 0$, and hence $\psi_a(u)$ is trivial. So Σ contains the trivial character of $U_{\mathfrak{J}}^1$. $\square$

Week 14

Exercise 14.1. Let F be a local field. Let $n \geqslant 1$, and let $\chi\colon U_F^n = 1 + \mathfrak{p}^n \to \mathbb{C}^\times$ be a smooth character. Show that χ can be extended to a smooth character $F^\times \to \mathbb{C}^\times$.

Solution. Choose π to be a uniformizer of $\mathcal{O}_F$. Since $F^\times \simeq \pi^{\mathbb{Z}} \times \mathcal{O}_F^\times = \pi^{\mathbb{Z}} \times U_F^0$, if χ extends to a smooth character of U_F^0, it then further extends to a smooth character of $F^\times$ by defining $\chi(\pi) = 1$. Since χ is smooth and U_F^m's form a neighborhood basis of $1 \in \mathcal{O}_F^\times$, there exists $m \gg 0$ such that χ is constant on each coset of $\mathcal{O}_F^\times / U_F^m$. In the case where $m < n$ there is nothing to prove because χ is a constant modulo U_F^n. As for the case where $m \geqslant n$, we can extend χ to $\widetilde{\chi}\colon \mathcal{O}_F^\times \to \mathbb{C}^\times$ by defining $\widetilde{\chi}(tU_F^n) = 1$ for each $t \in \mathcal{O}_F^\times / U_F^n$. This clearly admits the group homomorphism property on χ and is smooth (for which we only need the locally constant property). Thus $\widetilde{\chi}$ further extends to $F^\times$ by assigning $\widetilde{\chi}(\pi) = 1$. □

In the following exercises, let G be a unimodular locally profinite group. Fix a Haar measure μ, and let $\mathcal{H}$ be the Hecke algebra of G.

Exercise 14.2. Let $K \subset G$ be a compact open subgroup. Let (ϕ, W) be a smooth irreducible representation of K (which is necessarily of finite dimension). Define a function $e_\phi\colon G \to \mathbb{C}$ by

$$e_\phi(g) := \begin{cases} \mu(K)^{-1} \dim W \cdot \operatorname{tr}(\phi(g)^{-1}), & g \in K, \\ 0, & g \notin K. \end{cases}$$

(1) Show that $e_\phi \in \mathcal{H}$.
(2) Show that for each smooth representation (π, V) of G, the operator $\pi(e_\phi)\colon V \to V$ is the projection to the ϕ-isotypic component V^ϕ[5].
(3) Show that e_ϕ is idempotent, i.e., $e_\phi * e_\phi = e_\phi$.
(4) Show that $\mathcal{H}_\phi := e_\phi * \mathcal{H} * e_\phi$ is a subalgebra of $\mathcal{H}$ with multiplicative identity e_ϕ.

Solution. (1) By definition e_ϕ is supported on the open compact subset K of G. Since (ϕ, W) is a smooth representation of G and K is an open compact subgroup, ϕ is locally constant on K. Thus $\mu(K)^{-1} \dim W \cdot \operatorname{tr}(\phi(g)^{-1})$ is a locally constant function in $g \in K$. It follows that $e_\phi\colon G \to \mathbb{C}$ is an element of $\mathcal{H}$.

(2) We copy the argument in [BH06, §4.4] here. Let K' be the kernel of ϕ, then K' is an open compact subgroup of K. Also, e_ϕ is constant on cosets gK' and $K'g$, so $e_{K'} * e_\phi = e_\phi * e_{K'} = e_\phi$. Thus, e_ϕ lies in the subalgebra $e_{K'} * \mathcal{H} * e_{K'}$ of $\mathcal{H}$. However, $e_{K'} \mapsto 1$ induces an algebra isomorphism $e_{K'} * \mathcal{H} * e_{K'} \to \mathbb{C}[K/K']$. This isomorphism takes e_ϕ to the idempotent of the group algebra corresponding to the irreducible representation ϕ of the finite group K/K'. It follows that $\pi(e_\phi)V = V^\phi$, where V^ϕ is a module over the subalgebra $e_\phi * \mathcal{H} * e_\phi$.

(3) It suffices to show for any smooth representation (π, V) of G that $\pi(e_\phi)$ is idempotent, because we always have $\pi(e_\phi * e_\phi) = \pi(e_\phi)^2$ (this is given by the general formula $\pi(f * g) = \pi(f)\pi(g)$ in $\operatorname{End}_{\mathbb{C}}(V)$ for $f, g \in \mathcal{H}$). But it is clear from part (2) that $\pi(e_\phi)^2 = \pi(e_\phi)$, namely $\pi(e_\phi)$ is idempotent.

(4) To check that $\mathcal{H}_\phi$ is a subalgebra of $\mathcal{H}$, we only need for $f, g \in \mathcal{H}$ that $(e_\phi * f * e_\phi) * (e_\phi * g * e_\phi) = e_\phi * (f * e_\phi * g) * e_\phi$ with $f * e_\phi * g \in \mathcal{H}$; here we have used the result of (3). Also, by part (3) and the associativity of convolution, for each $e_\phi * f * e_\phi \in \mathcal{H}_\phi$ with $f \in \mathcal{H}$, we have $e_\phi * (e_\phi * f * e_\phi) = (e_\phi * e_\phi) * f * e_\phi = e_\phi * f * e_\phi$, and similarly for e_ϕ acting on the right. So e_ϕ is the identity of $\mathcal{H}_\phi$. □

Exercise 14.3. Assume that G is countable at infinity. Prove the following theorem in steps.

[5] Recall that the K-representation V is semi-simple, so we have the isotypic decomposition $V = \bigoplus_\rho V^\rho$, where ρ runs through isomorphism classes of irreducible smooth representations of K, and V^ρ is the sum of all K-subrepresentations of V which are isomorphic to ρ.

Theorem. For every non-zero $f \in \mathcal{H}$, there exists an irreducible smooth representation (π, V) of G such that the operator $\pi(f)\colon V \to V$ is non-zero.

(1) Let R be a unital $\mathbb{C}$-algebra of countable $\mathbb{C}$-dimension. (We do not assume that R is commutative.) Let $r \in R$ be a non-nilpotent element. Show that there exists $\lambda \in \mathbb{C}^\times$ such that $r - \lambda \in R$ is not invertible.
(2) Show that there exists a non-zero simple left unital R-module M such that $rM \neq 0$. Here unital means that $1 \in R$ acts as the identity.
(3) Show that for every compact open subgroup K of G, the algebra $\mathcal{H}_K = e_K * \mathcal{H} * e_K$ is of countable dimension over $\mathbb{C}$.
(4) Define $f^* \in \mathcal{H}$ by $f^*(g) \coloneqq \overline{f(g^{-1})}$. Let $h = f * f^* \in \mathcal{H}$. Show that $h^* = h$, and $h(1) > 0$.
(5) Show that h is not nilpotent.
(6) Show that there exists a compact open subgroup $K \subset G$ such that $h \in \mathcal{H}_K$ and there exists a simple $\mathcal{H}_K$-module on which h is non-zero.
(7) Show that there exists an irreducible smooth representation π of G such that $\pi(f) \neq 0$.

Solution. (1) Suppose R is a unital $\mathbb{C}$-algebra of countable $\mathbb{C}$-dimension. Then there exists a homomorphism $\rho\colon R \to D$ of $\mathbb{C}$-algebras, where D is a matrix algebra with identity element $1 \in D$; note that there is a natural determinant map $\det\colon D \to \mathbb{C}$. Given r non-nilpotent, $\rho(r)$ is non-nilpotent neither, and hence there exists $\lambda \in \mathbb{C}^\times$ such that $\det(\rho(r) - t \cdot 1) \in \mathbb{C}[\![t]\!]$ vanishes at λ. It follows that $\det(\rho(r-\lambda) - t \cdot 1)$ vanishes at 0, meaning that $\rho(r-\lambda)$ is not invertible in D. This proves that $r - \lambda$ is not invertible in R.

(2) Keep the notations in part (1). Let $I \subsetneq R$ be a maximal ideal of R containing $r - \lambda$. Then $M = R/I$ is simple left R-module. Moreover, it is unital because I does not contain $R^\times$. If $rM = 0$ then $r \in I$, which further implies $\lambda \in I$ as $r - \lambda \in I$. However, this cannot be true while assuming $\lambda \in \mathbb{C}^\times \subset R^\times$. So we have $rM \neq 0$.

(3) Since G is countable at infinity, G/K is countable for any open subgroup K. For each element $f \in \mathcal{H}_K$ that is stabilized by K, its G-orbit has size at most G/K, which is countable. Thus, $\mathcal{H}_K$ is generated by at most countably many elements as a $\mathbb{C}$-algebra, and hence has a countable dimension over $\mathbb{C}$.

(4) We first show that $h^* = h$, i.e. $(f * f^*)^*(x) = (f * f^*)(x)$ for all $x \in G$. For this, it suffices to show that $\overline{(f * f^*)(x^{-1})} = (f * f^*)(x)$, which can be verified through

$$\begin{aligned}
\overline{\int_G f(g) f^*(g^{-1}x^{-1}) \mathrm{d}\mu(g)} &= \int_G \overline{f(g)} \cdot \overline{f^*(g^{-1}x^{-1})} \mathrm{d}\mu(g) \\
&= \int_G f^*(g^{-1}) f(xg) \mathrm{d}\mu(g) \\
&= \int_G f(g) f^*(g^{-1}x) \mathrm{d}\mu(g).
\end{aligned}$$

Here the last equality is by replacing g with $x^{-1}g$ in the integral.

As for the assertion $h(1) = (f * f^*)(1) > 0$, we compute

$$(f * f^*)(1) = \int_G f(g) f^*(g^{-1}) \mathrm{d}\mu(g) = \int_G f(g)\overline{f(g)} \mathrm{d}\mu(g) = \int_G |f(g)|^2 \mathrm{d}\mu(g).$$

Since f is assumed to be non-zero, we have $|f(g)|^2 > 0$ for some $g \in G$, and hence $h(1) > 0$ follows.

(5) Consider $h^{*2} \coloneqq h * h \in \mathcal{H}$. By part (4), we have $h * h = h * h^*$ where h is a nonzero element of $\mathcal{H}$. So we can apply the same argument before with f replaced by h, to deduce that $(h * h^*)^* = h * h^*$ and $(h * h^*)(1) > 0$. Thus, for each $n \geqslant 1$, the self-convolution of h on itself in 2^n times, denoted by $h^{*(2^n)}$, must be non-zero. This shows that h is not nilpotent.

(6) Since G is assumed to be locally profinite, $1 \in G$ has a neighborhood basis $\{K_i\}_{i\in I}$ consisting of compact open subgroups of G. It follows that $\mathcal{H} = \bigcup_{i\in I} \mathcal{H}_{K_i}$, and hence there is some $K = K_i$ such that $h \in \mathcal{H}_K$. By parts (3) and (5), we know $h \in \mathcal{H}_K$ is non-nilpotent and $\mathcal{H}_K$ is a unital $\mathbb{C}$-algebra of countable dimension. So part (2) implies there exists a non-zero simple (left) unital $\mathcal{H}_K$-module W on which $hW \neq 0$, namely h is non-zero on W.

(7) Continuing with part (6), the simple $\mathcal{H}_K$-module W satisfies $\mathcal{H}_K W \subset W$, and hence $\mathcal{H}_K W$ is either 0 or W; but $hW \neq 0$ leads to $\mathcal{H}_K W = W$. Thus, W can be regarded as an irreducible smooth G-representation (π, W), such that $\pi(h) \neq 0$. From the construction above, it follows that $\pi(f * f^*) = \pi(f)\pi(f^*) \neq 0$, and in particular $\pi(f) \neq 0$. □

Exercise 14.4. Assume that G is countable at infinity and such that every irreducible smooth representation of G is admissible (e.g. $G = \mathrm{GL}_2(F)$). Let $e \in \mathcal{H}$ be a non-zero idempotent.

(1) Show that for each smooth representation (π, V) of G, we have a canonical decomposition $V = \ker(\pi(e)) \oplus \mathrm{im}(\pi(e))$. Moreover, $\mathrm{im}(\pi(e))$ is finite-dimensional if (π, V) is admissible.

(2) Suppose $f_1, f_2 \in e * \mathcal{H} * e$ satisfies $f_1 * f_2 = e$. Show that $f_2 * f_1 = e$.

Solution. (1) Since e is non-zero idempotent, $\pi(e)$ is idempotent as well. So we have the canonical decomposition

$$V = (1 - \pi(e))V \oplus \pi(e)V = \ker(\pi(e)) \oplus \mathrm{im}(\pi(e)).$$

Further, as G is countable at infinity, by the proof of Exercise 14.3(6), there is an open compact subgroup K of G such that $e \in \mathcal{H}_K$, and hence $e = e_{K'}$ for some open compact subgroup K' of K. So we have $\mathrm{im}(\pi(e)) = \pi(e)V = \pi(e_{K'})V = V^{K'}$. By assumption (π, V) is admissible, and hence $\dim V^{K'} < \infty$, which is as desired.

(2) We have $\pi(e) = \pi(f_1 * f_2) = \pi(f_1)\pi(f_2)$. On the other hand, $\pi(e)V = \mathrm{im}(\pi(e))$ must be finite-dimensional by part (1). As $\pi(e)$ is non-zero idempotent, both $\pi(f_1)$ and $\pi(f_2)$ can be regarded as linear operators on $\pi(e)V$, so $\pi(e) = \pi(f_1)\pi(f_2)$ implies $\pi(e) = \pi(f_2)\pi(f_1) = \pi(f_2 * f_1)$. By assumption on G, such (π, V) can be replaced by any other irreducible smooth representation, and hence $f_2 * f_1 = e$ holds. □

Week 15

Exercise 15.1. Let $I = U_{\mathfrak{J}}$ be the Iwahori subgroup of $G = \mathrm{GL}_2(F)$, and let ϕ be a character of I trivial on $U_{\mathfrak{J}}^1$.

(1) Let $f \in \mathcal{H}_\phi$ be a function supported on ItI, and let $h \in \mathcal{H}_\phi$ be a function supported on $It^{-1}I$. Assume that $f(1) \neq 0$ and $h(1) \neq 0$. Compute $(f * h)(1)$ in terms of $f(1)$ and $h(1)$.

(2) Let (π, V) be a smooth representation of G. Let $w \in V$ be a vector satisfying $\pi(y)w = \phi(y)w$ for all $y \in N_1' T^0 N_1$. (Here $N_j := \left(\begin{smallmatrix} 1 & \mathfrak{p}^j \\ 0 & 1 \end{smallmatrix}\right)$ and $N_j' := \left(\begin{smallmatrix} 1 & 0 \\ \mathfrak{p}^j & 1 \end{smallmatrix}\right)$.) Show that w is fixed by N_1, and that $\pi(e_\phi)w$ is equal to a non-zero scalar times $\sum_{g \in N_0/N_1} \pi(g)w$.

Solution. (1) By definition the function $f*h$ has support contained in $ItIt^{-1}I = IN_0'I$. Suppose $z \in N_0'$ lies in the support of $f * h$. Then by [BH06, §11.2, Lemma], z intertwines ϕ. However, recall that $U_{\mathfrak{M}} = \mathrm{GL}_2(\mathcal{O})$ with $\mathfrak{M} = \mathrm{M}_2(\mathcal{O})$, so we clearly have $z \in U_{\mathfrak{M}}$, and the image $\bar{z}$ of z in $\mathrm{GL}_2(\boldsymbol{k})$ therefore intertwines the character $\bar{\phi}$ of the Borel group of upper triangular matrices in $\mathrm{GL}_2(\boldsymbol{k})$. However, the character $\bar{\phi}$ induces irreducibly to $\mathrm{GL}_2(\boldsymbol{k})$ by [BH06, §6.3, Proposition], so $\bar{z} = 1$ and $z \in N_1'$. Thus, the support of $f * h$ is contained in I. Recall that for each $g \in G$ together with $F \in \mathcal{H}$, we have

$$\int_G F(x)\mathbb{1}_{IgI}(x)\mathrm{d}\mu(x) = \mu(IgI)F(g).$$

In particular, letting $F(x) = f(x)h(x^{-1})$ and $g = 1$, it thus follows that

$$(f * h)(1) = \int_G f(x)h(x^{-1})\mathbb{1}_I(x)\mathrm{d}\mu(x) = \mu(I)f(1)h(1).$$

(2) For any $n_1 \in N_1$, it is clear that $n_1 \in N_1'T^0N_1$, and hence $\pi(n_1)w = \phi(n_1)w$. Since ϕ is trivial on $U_{\mathfrak{J}}^1 = 1 + \mathfrak{P}_{\mathfrak{J}} = \left(\begin{smallmatrix} 1+\mathfrak{p} & \mathcal{O}_F \\ \mathfrak{p} & 1+\mathfrak{p} \end{smallmatrix}\right)$, it is in particular trivial on $N_1 \subset U_{\mathfrak{J}}^1$, and thus $\pi(n_1) = 1$. So w is fixed by N_1. To compute $\pi(e_\phi)w$ in terms of a sum running through $g \in N_0/N_1$, recall from Exercise 14.1 that for the character ϕ on I,

$$e_\phi(g) = \mu(I)^{-1} \dim\phi \cdot \mathrm{tr}(\phi(g)^{-1}) = \mu(I)^{-1}\phi(g)^{-1}$$

for any $g \in G$. Thus,

$$\begin{aligned}\pi(e_\phi)w &= \int_G e_\phi(g)\pi(g)w\mathrm{d}\mu(g) \\ &= \mu(I)^{-1}\int_I \phi(g)^{-1}\pi(g)w\mathrm{d}\mu(g) \\ &= \mu(I)^{-1}\mu(N_1)\sum_{x \in I/N_1}\phi(x)^{-1}\pi(x)w.\end{aligned}$$

Here to deduce the last equality, note that w is fixed by N_1 and so also is e_ϕ (as ϕ is trivial on N_1). We then notice

$$N_1'T^0N_0 = \begin{pmatrix} 1 & 0 \\ \mathfrak{p} & 1 \end{pmatrix}\begin{pmatrix} \mathcal{O}_F^\times & 0 \\ 0 & \mathcal{O}_F^\times \end{pmatrix}\begin{pmatrix} 1 & \mathcal{O}_F \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ \mathfrak{p} & \mathcal{O}_F^\times \end{pmatrix} = I.$$

So in the last row of formula of $\pi(e_\phi)w$ above, we may write $x = n_1't^0g$ for some $n_1' \in N_1'$, $t^0 \in T^0$, and $g \in N_0/N_1$. Again as ϕ is trivial on $U_{\mathfrak{J}}^1$, it is trivial on both N_1' and N_0; thus, we have $\phi(x)^{-1} = \phi(n_1't^0g)^{-1} = \phi(t^0)^{-1}$. It now remains to compute the sum

$$\sum_{x \in I/N_1}\phi(x)^{-1}\pi(x)w = \sum_{n_1't^0 \in N_1'T^0}\phi(t^0)^{-1}\pi(n_1't^0)\sum_{g \in N_0/N_1}\pi(g)w.$$

By assumption of w, it is clear that $N_1'T^0N_1$ acts on $\sum_{g \in N_0/N_1}\pi(g)w$ via ϕ, so the sum above becomes

$$\sum_{n_1't_0 \in N_1'T^0}\phi((t^0)^{-1}n_1't^0)\sum_{g \in N_0/N_1}\pi(g)w$$

where $(t^0)^{-1}n_1't^0 \in N_0$. Therefore, $\pi(e_\phi)w$ is a non-zero scalar times $\sum_{g\in N_0/N_1}\pi(g)w$. $\square$

Exercise 15.2. Let I be the Iwahori subgroup of G, and let $K = \mathrm{GL}_2(\mathcal{O}_F) \subset G$. Let k denote the residue field of F.

(1) Show that reduction modulo $\mathfrak{p}$ induces surjective homomorphisms $K \to \mathrm{GL}_2(k)$ and $I \to B_k$, where B_k is the subgroup of $\mathrm{GL}_2(k)$ consisting of upper triangular matrices.
(2) Use the Bruhat decomposition $\mathrm{GL}_2(k) = B_k \cup B_k w B_k$, where $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, to show that
$$K = I \cup IwI.$$
(3) Use the Iwasawa decomposition $G = BK$ to show that
$$G = BI \cup BwI.$$
(4) Show that the Steinberg representation $\mathrm{St}_G = (\pi, V)$ satisfies $\dim V^I = 1$ and $V^K = 0$.
(5) Show that every double coset in $I\backslash G/I$ has a representative of the form either $\left(\begin{smallmatrix} \pi^a & 0 \\ 0 & \pi^b \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & \pi^a \\ \pi^b & 0 \end{smallmatrix}\right)$ for some $a, b \in \mathbb{Z}$.

Solution. (1) The construction of homomorphism $K \to \mathrm{GL}_2(k)$ is clear. As for the Iwahori subgroup, we have $I = \left(\begin{smallmatrix} U_F & \mathcal{O}_F \\ \mathfrak{p} & U_F \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} k^\times & k \\ 0 & k^\times \end{smallmatrix}\right) \bmod \mathfrak{p}$, and hence the map $I \to B_k$ follows. This is also a group homomorphism.

(2) We only need to show the set-theoretical equality. If $K \neq I \cup IwI$, then along the maps in (1), modulo $\mathfrak{p}$ we have $\mathrm{GL}_2(k) \neq B_k \cup B_k w B_k$. But this contradicts to the Bruhat decomposition, so $K = I \cup IwI$.

(3) The Iwasawa decomposition together with part (2) shows
$$G = BK = B(I \cup IwI) = BI \cup BIwI = BI \cup BwI.$$
Here the last equality is deduced as follows. It is automatic that $B \subset BI$ and hence $BwI \subset BIwI$; but in G both BwI and $BIwI$ have only trivial intersection with BI, so we must have $BIwI = BwI$.

(4) By definition, St_G sits in the exact sequence
$$0 \longrightarrow \mathbb{C} \longrightarrow \mathrm{Ind}_B^G \delta_B^{-1/2} \longrightarrow \mathrm{St}_G \longrightarrow 0.$$
By part (3) we have $G = BI \cup BwI$, and then
$$\dim(\mathrm{Ind}_B^G(\delta^{-1/2}))^I = 2,$$
because it has the same property when viewed as a B-representation. As for the subrepresentation $\mathbb{C}$, we have $\dim \mathbb{C} = \dim \mathbb{C}^I = 1$. It follows that $\dim \mathrm{St}_G^I = 1$. Again, if we view $\mathrm{Ind}_B^G \delta_B^{-1/2}$ as a B-representation, then $G = BK$ acts on it with $\mathrm{St}_G^K = 0$.

(5) By Cartan decomposition (Exercise 5.4), we have
$$G = \mathrm{GL}_2(F) = \bigsqcup_{a \geqslant b} K \begin{pmatrix} \pi^a & 0 \\ 0 & \pi^b \end{pmatrix} K.$$
So there are representatives $\left(\begin{smallmatrix} \pi^a & 0 \\ 0 & \pi^b \end{smallmatrix}\right)$ for $K\backslash G/K$. Modulo the center of G, it suffices to consider the cosets $Kt^{-a}K$ for $a \geqslant 0$, where $t = \left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right)$. If $a = 0$ then the coset is K itself, and $K = I \cup IwI$ is already known in part (2). Now we assume $a \geqslant 1$. Again, since $K = I \cup IwI$, each $Kt^{-a}K$ decomposes into four double cosets:
$$It^{-a}I, \quad It^{-a}IwI, \quad IwIt^{-a}I, \quad IwIt^{-a}IwI.$$
It is clear that they respectively contain double cosets of form
$$It^{-a}I, \quad It^{-a}wI, \quad Iwt^{-a}I, \quad Iwt^{-a}wI,$$
and all of $t^{-a}, t^{-a}w, wt^{-a}, wt^{-a}$ are of the form either $\left(\begin{smallmatrix} \pi^a & 0 \\ 0 & \pi^b \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & \pi^a \\ \pi^b & 0 \end{smallmatrix}\right)$ as desired.

To show these four are all double cosets in $Kt^{-a}K$, choose a Haar measure μ such that $\mu(I) = 1$, and it suffices to show that

$$\mu(Kt^{-a}K) = \mu(It^{-a}I) + \mu(It^{-a}wI) + \mu(Iwt^{-a}I) + \mu(Iwt^{-a}wI).$$

Note that $\mu(K) = q + 1$ and then

$$\mu(Kt^{-a}K) = (q + 1)^2 q^{a-1}.$$

On the other hand, to compute $\mu(It^{-a}I)$, we have $t^{-a}It^a = N_{-a}T^0N'_{a+1}$, and hence $I \cap t^{-a}It^a = N_0T^0N'_{a+1}$. This group has index q^a in I, so

$$\mu(It^{-a}I) = [It^{-a}I : I] \cdot \mu(I) = [It^{-a}It^a : It^a] = [I : I \cap t^{-a}It^a] = q^a.$$

By replacing t^{-a} above with either of $t^{-a}w$, wt^{-a}, or $wt^{-a}w$ and applying the similar argument, we also deduce

$$\mu(It^{-a}wI) = q^{a-1}, \quad \mu(Iwt^{-a}I) = q^{a+1}, \quad \mu(Iwt^{-a}wI) = q^a.$$

So the measures of the four double cosets sum up to $\mu(Kt^{-a}K)$ as desired. Finally, the only remaining ambiguity lies in $It^{-a}I \cap Iwt^{-a}wI$. If this is non-empty then $It^{-a}I = Iwt^{-a}wI$, which means there are $x, y \in I$ such that $xt^{-a} = wt^{-a}wy$; but this turns out to be impossible by testing the matrix entries. $\square$

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