Fall 2022 – Seminar on *Abelian Varieties* by David Mumford

**Lecture 4-8**

# ALGEBRAIC THEORY VIA VARIETIES

## 4. Definition of Abelian Varieties

Let $k$ be an algebraically closed field.

**Definition 4.1.** An **abelian variety** $X$ is a complete algebraic variety over $k$ (that is, $X$ is an integral scheme proper and of finite type over $k$) with a group law induced by the morphisms

$$m : X \times X \to X, \quad e : \mathrm{Spec}(k) \to X, \quad i : X \to X$$

such that $m$ and $i$ are both morphisms of varieties.

*Remark* 4.2.
1. As we will see later, an abelian variety is automatically projective. This is not true for abelian schemes.
2. In most of the cases, Mumford worked over an algebraically closed field. This makes the discussion much simpler in some cases. In practice, one should be aware of whether this assumption really affects the statement. For example, over a general field $k$, the correct definition of an abelian variety should be the same as the above definition except that one replaces "integral" with "geometric integral".

**Exercise 4.3.** Let $X$ be a variety over a field $k$. Show that

$$X \text{ is projective} \iff X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \overline{k} \text{ is projective.}$$

Now we give some basic properties of abelian varieties.

**Lemma 4.4.** (cf. [Har13, II, §8]) *An abelian variety $X$ is everywhere nonsingular (i.e., smooth) when $k$ is algebraically closed.*

*Proof.* It suffices to check on closed points. Since $k$ is algebraically closed, it is known that there is an open dense subset $U$ of $X$ which is nonsingular. For $x_0 \in U$ and $x \in X$, the left translation $T_{xx_0^{-1}}$ induces an isomorphism $\mathscr{O}_{X,x} \cong \mathscr{O}_{X,x_0}$. Hence $x$ is nonsingular. □

Next, we will prove that $X$ is commutative as a group.

**Lemma 4.5** (Rigidity)**.** *Let $X$ be a complete variety. Let $Y$ and $Z$ be any varieties. Assume $f : X \times Y \to Z$ is a morphism such that there exists a closed point $y_0$ of $Y$ with*

$$f(X \times \{y_0\}) = \{z_0\},$$

*a single closed point $z_0$ of $Z$. Then there exists a morphism $g : Y \to Z$ such that $f = g \circ p_2$:*

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \times Y \arrow[ur, "p_2"] \arrow[rr, "f"] & & Z \end{tikzcd}$$

*where $p_2 : X \times Y \to Y$ is the second projection to $Y$.*

*Proof.* Fix a closed point $x_0$ of $X$ and define a morphism $g : Y \to Z$ to be the composite

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$$Y \xrightarrow{\cong} \{x_0\} \times Y \hookrightarrow X \times Y \xrightarrow{f} Z.$$

It suffices to prove that $f$ and $g \circ p_2$ agree on a nonempty open subscheme of $X \times Y$. We choose an open affine neighborhood $U$ of $z_0$ in $Z$. Since $f^{-1}(U)$ is open in $X \times Y$, its complement $W = X \times Y \backslash f^{-1}(U)$ is closed in $X \times Y$.

Since $p_2$ is proper, $p_2(W)$ is closed in $Y$. By assumption, $W \cap (X \times \{y_0\}) = \emptyset$ and then $y_0 \notin p_2(W)$. We can find an open neighborhood $V$ of $y_0$ in $Y$ such that $V \cap p_2(W) = \emptyset$. Then the restriction $f|_{X \times V}$ factors through $U \subset Z$, and hence $g|_V$ factors through $U \subset Z$. For any closed point $y \in V$, $f|_{X \times \{y\}}$ is constant as $X \times \{y\}$ is proper and $U$ is affine. It shows that for any closed point $x$ of $X$,

$$f(x, y) = f(x_0, y) = (g \circ p_2)(x_0, y).$$

In other words, $f|_{X \times V}$ and $(g \circ p_2)|_{X \times V}$ agree on all closed points. Therefore, they agree on $X \times V$. This extends to

$$f = g \circ g_2 : X \times Y \to Z,$$

which completes the proof. □

**Corollary 4.6.** *If $X$ and $Y$ are abelian varieties and $f : X \to Y$ is a morphism, then $f(x) = h(x) + a$ where $h : X \to Y$ is a homomorphism and $a \in Y(k)$.*

*Proof.* Up to translations, it suffices to show that if $f(e_X) = e_Y$, then $f$ is a homomorphism. Consider the morphism $\phi : X \times X \to Y$ defined by

$$\phi(x, y) = f(xy) - f(x) - f(y) := f(xy) + i(f(x)) + i(f(y)).$$

Since $\phi(X \times \{e_X\}) = \phi(\{e_X\} \times X) = \{e_Y\}$, it follows from Lemma 4.5 that $\phi(x, x') \equiv e_Y$ on $X \times X$. Hence $f$ is always a homomorphism. □

**Corollary 4.7.** *$X$ is a commutative group.*

*Proof.* Apply the previous corollary to show the morphism attached to the group variety

$$i : X \to X, \quad x \mapsto x^{-1}$$

is a group morphism. □

**Corollary 4.8.** *Let $X$ be an abelian variety with base point $e_X$. Then on the category of complete varieties with base point, the functor*

$$S \longmapsto \mathrm{Hom}(S, X)$$

*is linear, i.e., for $S, T$ in this category, the natural map*

$$\mathrm{Hom}(S, X) \times \mathrm{Hom}(T, X) \longrightarrow \mathrm{Hom}(S \times T, X), \quad (f, g) \longmapsto h$$

*such that*

$$h(s, t) = f(s) + g(t)$$

*is a bijection.*

*Proof.* If we use $s_0$ to denote the base point then

$$h(s_0, t) = g(t), \quad h(s, t_0) = f(s), \quad \forall s \in S,\ t \in T.$$

Then the map is injective. Now given $h \in \mathrm{Hom}(S \times T, X)$, define $f : S \to X$ and $g : T \to X$ by

$$f(s) = h(s, t_0), \quad g(t) = h(s_0, t)$$

for some fixed $s_0 \in S$ and $t_0 \in T$. Then the morphism

$$k : S \times T \to X, \quad k(s, t) = h(s, t) - g(t) - f(s)$$

satisfies

$$k(S \times \{t_0\}) = k(\{s_0\} \times T) = \{e_X\} \quad \Longrightarrow \quad k(s, t) \equiv e_X$$

by Lemma 4.5. □

Now let $e_X : \mathrm{Spec}\, k \to X$ be the identity element and $\Omega^1_X = \Omega^1_{X/k}$ be the sheaf of relative differentials of $X$ over $k$. On $\mathrm{Spec}\, k$, the coherent sheaf

$$\omega_X = e_X^* \Omega^1_X$$

corresponds to the Zariski tangent space $\Omega_e$ of $X$ at $e$ (see [Har13, II, Prop 8.7]).

**Proposition 4.9.** *There is a natural isomorphism $\Omega^1_X \cong \pi^* \omega_X$ of coherent $\mathscr{O}_X$-modules.*

*Proof.* We regard the product $X \times X$ as an $X$-scheme via the second projection $p_2$. The morphism

$$\tau = (m, p_2) : X \times X \to X \times X, \quad (x, y) \mapsto (x + y, y)$$

is an automorphism of $X \times X$ over $X$, and it induces an isomorphism

$$\psi : \tau^* \Omega^1_{X \times X/X} \xrightarrow{\cong} \Omega^1_{X \times X/X}.$$

Since the following diagram is Cartesian,

$$\begin{tikzcd} X \times X \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"] & \mathrm{Spec}\, k \end{tikzcd}$$

we have $\Omega^1_{X \times X/X} \cong p_1^* \Omega^1_X$. Under this isomorphism, $\psi$ becomes

$$\psi : m^* \Omega^1_X \xrightarrow{\cong} p_1^* \Omega^1_X.$$

We pull this isomorphism back along the morphism

$$(e_X \circ \pi, \mathrm{id}_X) : X \to X \times X, \quad x \mapsto (e_X, X),$$

where $\pi : X \to \mathrm{Spec}\, k$ is the natural section. It gives rise to the desired isomorpism

$$\Omega^1_X \xrightarrow{\cong} \pi^*(e_X^* \Omega^1_X) = \pi^* \omega X.$$

□

*Remark* 4.10. The above result holds for arbitrary group scheme $\pi : G \to S$ over $S$ that is separated and of finite type.

**Proposition 4.11.** *For every $n$ that is not divisible by* $\mathrm{char}(k)$*, the endomorphism*

$$n_X : X \to X, \quad x \mapsto nx$$

*is surjective.*

*Proof.* We impose $T$ to denote the Zariski tangent space of $X$ at $e_X$.

**Claim.** The addition morphism $m : X \times X \to X$ induces the tangent map at $(e_X, e_X)$, say

$$d(m) : T_{X\times X,(e_X,e_X)} \cong T \oplus T \to T, \quad (t_1, t_2) \mapsto t_1 + t_2.$$

For this, note that the composite

$$X \xrightarrow{(\mathrm{id}_X, e_X)} X \times X \xrightarrow{m} X$$

is the identity map. One infers that $d(m)(t_1, 0) = t_1$ for all $t_1 \in T$; and similarly, $d(m)(0, t_2) = t_2$ for all $t_2 \in T$. The claim follows from the fact that $d(m)$ is additive.

Granting the claim, we are to prove the proposition. Take $K := \mathrm{Ker}(n_X)$ that sits in the left Cartesian diagram below. And consider the tangent maps on the right hand side below.

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathrm{Spec}(k) \arrow[d, "e_X"] \\ X \arrow[r, "n_X"] & X \end{tikzcd} \qquad \begin{tikzcd} T_{K,e_X} \arrow[r] \arrow[d, hook] & 0 \arrow[d] \\ T \arrow[r, "(\cdot)\times n"] & T \end{tikzcd}$$

Since $n$ is not divisible by $\mathrm{char}(k)$, we see that

$$T_{K,e_X} = 0 \quad \Longrightarrow \quad \dim \mathscr{O}_{K,e_X} = 0 \quad \Longrightarrow \quad \dim K = 0.$$

Then the dimension formula implies that $\dim(\mathrm{im}(n_X)) = \dim X$. Hence $n_X$ is surjective. □

*Remark* 4.12. We can actually show that $n_X$ is a finite flat étale morphism (recall that the finiteness is implied by quasi-finiteness and properness). When $\mathrm{char}(k) \mid n$, $n_X$ is still finite flat but no longer étale.

**Corollary 4.13.** *For all $x \in X$, $\mathscr{O}_{X,x}$ is regular and hence a UFD. So we identity the Weil divisor classes to the line bundle classes over $X$, say*

$$\mathrm{Cl}(X) \cong \mathrm{Pic}(X).$$

## 5. Cohomology and Base Change

The references for this section is [Har13, III, §12] and Conrad's lecture notes [Con00, §9].

*Setups.* Let $f : X \to Y$ be a proper morphism of noetherian schemes and $\mathscr{F}$ be a coherent sheaf of $\mathscr{O}_X$-modules. Assume that $\mathscr{F}$ is flat over $Y$, i.e., for any $x \in X$, $\mathscr{F}_x$ is flat as an $\mathscr{O}_{Y,f(x)}$-module. For any $y \in Y$, we denote

$$X_y := X \times_Y \operatorname{Spec}(k(y))$$

and $\mathscr{F}_y$ the inverse image of $\mathscr{F}$ via the morphism $X_y \to X$.

**Goal:** For any $i \geqslant 0$, we want to understand the fiber cohomology $H^i(X_y, \mathscr{F}_y)$ as a function of $y \in Y$. And the idea is to find relations between the sheaf $R^i f_* \mathscr{F}$ and the cohomology groups $H^i(X_y, \mathscr{F}_y)$.

We assume the following result.

**Theorem 5.1** (Proper base change)**.** *If $f : X \to Y$ is a proper morphism of locally noetherian schemes and $\mathscr{F}$ a coherent sheaf of $\mathscr{O}_X$-modules on $X$, then the direct image sheaves $R^p f_* \mathscr{F}$ are coherent sheaves of $\mathscr{O}_Y$-modules for all $p \geqslant 0$.*

When $f$ is projective, this follows from [Har13, III, Thm 8.8]. As for the general case, it follows from EGA III, see [GD66, III, 3.2.1].

**Theorem 5.2.** *Let $f : X \to Y$ be a proper morphism of noetherian schemes with $Y = \operatorname{Spec} A$ affine, and $\mathscr{F}$ be a coherent sheaf of $\mathscr{O}_X$-module that is flat over $Y$. Then there exists a finite complex $K^\bullet$, say*

$$0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$$

*of finitely generated projective $A$-modules and equivalences of functors*

$$H^p(X \times_Y \operatorname{Spec}(\cdot), \mathscr{F} \otimes_A (\cdot)) = H^p(K^\bullet \otimes_A (\cdot)), \quad p \geqslant 0$$

*on the category of $A$-algebras. Hence for any $B \in \mathsf{Alg}_A$,*

$$H^p(X \times_Y \operatorname{Spec} B, \mathscr{F} \otimes_A B) \cong H^p(K^\bullet \otimes_A B), \quad p \geqslant 0.$$

**Problem 5.3.** Here the sheaf $\mathscr{F} \otimes_A B$ is the inverse image sheaf of $\mathscr{F}$ under the projection $X \times_Y \operatorname{Spec} B \to X$. How to give the association $B \mapsto H^p(X \times_Y \operatorname{Spec} B, \mathscr{F} \otimes_A B)$ rise to be a functor on the category of $A$-algebras? (To remedy this, one can use Čech cohomology, but how to make it formal?)

*Remark* 5.4.
(1) Since $\mathscr{F}$ is flat over $Y = \operatorname{Spec} A$, for any affine open subset $U \subset X$, $\mathscr{F}(U)$ is flat as an $A$-module.
(2) Since $X$ is separated and noetherian, the coherent cohomology $H^*(X, \mathscr{F})$ can be computed by Čech cohomology with respect to finite affine open coverings, for any quasi-coherent sheaf $\mathscr{F}$ on $X$. The same is true for $X \times_Y \operatorname{Spec} B$.
(3) As for $H^p(K^\bullet \otimes_A B)$, it is generally not a finitely generated algebra over $A$, and the cohomology does not commute with $(\cdot) \otimes_A B$ in most cases.

*Proof of Theorem 5.2.* Let $\mathcal{U} = \{U_i\}_{i=0,\ldots,n}$ be a finite affine open covering of $X$ and $(C^\bullet(\mathcal{U}, \mathscr{F}), d^\bullet)$ be the Čech cochain complex of alternating cochains with respect to the open covering $\mathcal{U}$ and the sheaf $\mathscr{F}$. In particular,

$$C^p(\mathcal{U}, \mathscr{F}) = \bigoplus_{0 \leqslant i_0 < \cdots < i_p \leqslant n} \mathscr{F}(U_{i_0 \cdots i_p})$$

is a free $A$-module for all $p$ (being nonzero only when $0 \leqslant p \leqslant n$), and the Čech cohomology groups $H^\bullet(\mathcal{U}, \mathscr{F})$ are isomorphic to $H^\bullet(X, \mathscr{F})$.

Moreover, for any $A$-algebra $B$, $\{U_i \times_Y \operatorname{Spec} B\}_{i=0,\ldots,n}$ is an affine open covering of $X \times_Y \operatorname{Spec} B$, and $C^\bullet(\mathcal{U}, \mathscr{F}) \otimes_A B$ is the Čech cochain complex for this open covering and the sheaf $\mathscr{F} \otimes_A B$ on $X \times_Y \operatorname{Spec} B$. Therefore,

$$H^p(X \times_Y \operatorname{Spec} B, \mathscr{F} \otimes_A B) \cong H^p(C^\bullet(\mathcal{U}, \mathscr{F}) \otimes_A B), \quad p \geqslant 0,$$

and this isomorphism is functorial for $B$. □

**Lemma 5.5.** *Let $C^\bullet$ be a cochain complex of $A$-modules (but each $C^p$ may not be finitely generated over $A$) such that $H^i(C^\bullet)$ are finitely generated $A$-modules for all $i \geqslant 0$, and such that $C^\bullet$ is bounded on $[0, n]$.*[1] *Then there exists a complex $K^\bullet$ of finitely generated $A$-modules, bounded on $[0, n]$ and such that $K^p$ is free for all $1 \leqslant p \leqslant n$, and a homomorphism of cochain complexes $\phi : K^\bullet \to C^\bullet$ such that $\phi$ induces isomorphisms $H^i(K^\bullet) \to H^i(C^\bullet)$ for all $i$; namely, $\phi$ is a quasi-isomorphism.*

*Moreover, if all the $C^p$'s are $A$-flat, then $K^0$ will be $A$-flat as well.*

*Proof.* We will use descending induction on $m$ to construct the following diagram

$$\begin{tikzcd}
 & K^m \arrow[r, dashed, "d_K^m"] \arrow[d, dashed, "\phi_m"] & K^{m+1} \arrow[r, "d_K^{m+1}"] \arrow[d, "\phi_{m+1}"] & K^{m+2} \arrow[r] \arrow[d, "\phi_{m+2}"] & \cdots \\
\cdots \arrow[r] & C^m \arrow[r, "d_C^m"] & C^{m+1} \arrow[r, "d_C^{m+1}"] & C^{m+2} \arrow[r] & \cdots
\end{tikzcd}$$

with the following properties:

(1) $d_K^{p+1} \circ d_K^p = 0$ for $p \geqslant m+1$;
(2) $\phi_{p+1} \circ d_K^p = d_C^p \circ \phi_p$ for $p \geqslant m+1$;
(3) $\phi_p$ induces an isomorphism of cohomology groups $H^p(K^\bullet) \to H^p(C^\bullet)$ for $p \geqslant m+2$ and a surjective homomorphism $\operatorname{Ker}(d_K^{m+1}) \to H^{m+1}(C^\bullet)$;
(4) $K^p$ is a finite free $A$-module for $p \geqslant m+1$.

We are going to construct $K^m$, $d_K^m$, $\phi_m$ with the above properties. One can find finite free $A$-modules $(K')^m$ and $(K'')^m$, and surjective maps of $A$-modules:

$$(K')^m \twoheadrightarrow \operatorname{Ker}(\operatorname{Ker}(d_K^{m+1}) \to H^{m+1}(C^\bullet)),$$
$$(K'')^m \twoheadrightarrow H^m(C^\bullet).$$

Roughly speaking, the first surjection is to make $\phi_{m+1}$ into an isomorphism between cohomology groups; and the second surjection is to force $\phi_m$ to satisfy the desired property.

[1] This is not a standard notation to say that $C^p \neq 0$ implies $0 \leqslant p \leqslant n$. Indeed, using the truncation functor, one may replace $C^\bullet$ with $\tau^{\geqslant 0}\tau^{\leqslant n}C^\bullet$.

By construction, we have an inclusion $i'_m : (K')^m \to (K'')^{m+1}$ that factors through $\mathrm{Ker}(d_K^{m+1})$. Define

$$K^m := (K')^m \oplus (K'')^m, \quad d_K^m = (i'_m, 0) : K^m \to K^{m+1}.$$

Then property (1) and (4) hold for $p = m$, and $\phi_{m+1}$ induces an isomorphism $H^{m+1}(K^\bullet) \to H^{m+1}(C^\bullet)$. Since $(K'')^m$ is projective, we can lift the map $(K'')^m \to H^m(C^\bullet)$ to a map

$$\phi''_m : (K'')^m \to \mathrm{Ker}(d_C^m) \to C^m.$$

On the other hand, the composite

$$\begin{tikzcd} (K')^m \arrow[r, "i'_m"] \arrow[dr, "i'_m"'] & K^{m+1} \arrow[r, "\phi_{m+1}"] & C^{m+1} \\ & \mathrm{Ker}(d_K^{m+1}) \arrow[u, hook] \arrow[r, "\phi_{m+1}"] & \mathrm{Ker}(d_C^{m+1}) \arrow[u, hook] \end{tikzcd}$$

lies in $\mathrm{Ker}(d_C^{m+1})$ and is 0 in $H^{m+1}(C^\bullet)$. Then

$$(K')^m \xrightarrow{i'_m} \mathrm{Ker}(d_K^{m+1}) \xrightarrow{\phi_{m+1}} \mathrm{Ker}(d_C^{m+1})$$

factors through $\mathrm{im}(d_C^m)$. Since $(K')^m$ is projective, we can lift the map $(K')^m \to \mathrm{im}(d_C^m)$ to a map $\phi'_m : (K')^m \to C^m$ by the universal property. Finally we define

$$\phi_m = (\phi'_m, \phi''_m) : K^m \longrightarrow C^m.$$

It is straightforward to verify that $\phi_{m+1} \circ d_K^m = d_C^m \circ \phi_m$ and $\phi_m$ induces a surjective map

$$\mathrm{Ker}(d_K^m) = (K'')^m \longrightarrow H^m(C^\bullet).$$

This finishes the construction for $m$. Now we have the following diagram

$$\begin{tikzcd} & K^0 \arrow[r, "d_K^0"] \arrow[d, "\phi_0"] & K^1 \arrow[r, "d_K^1"] \arrow[d, "\phi_1"] & \cdots \\ 0 \arrow[r] & C^0 \arrow[r, "d_C^0"] & C^1 \arrow[r, "d_C^1"] & \cdots \end{tikzcd}$$

that satisfies (1)-(4) above. We replace $K^0$ by $K^0/(\mathrm{Ker}(d_K^0) \cap \mathrm{Ker}(\phi_0))$ and $d_K^0$, $\phi_0$ by their induced maps. Then the new diagram satisfies all the properties (1)-(4) except that $K^0$ is no longer free.

We still need to prove that $K^0$ is $A$-flat. Let $C[-1]^\bullet$ be the complex shifted by $-1$ of the cochain complex $C^\bullet$, i.e.,

$$C[-1]^p := C^{p-1}, \quad d^p_{C[-1]} := -d_C^{p-1}.$$

Consider the mapping cone of the morphism $\phi : K^\bullet \to C^\bullet$, which is defined as follows:

$$\mathrm{Cone}(\phi)^p := K^p \oplus C^{p-1} = K^p \oplus C[-1]^p,$$

together with[2]

[2]There is an alternative (and decorated) way to write the differential map as

$$d^p_{\mathrm{Cone}(\phi)} : K^p \oplus C[-1]^p \to K^{p+1} \oplus C[-1]^{p+1}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} d_K^p & 0 \\ \phi_p & d^p_{C[-1]} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\begin{aligned} d^p_{\mathrm{Cone}(\phi)} : K^p \oplus C^{p-1} &\longrightarrow K^{p+1} \oplus C^p \\ (x,y) &\longmapsto (d^p_K(x), \phi_p(x) - d^{p-1}_C(y)). \end{aligned}$$

One can easily check that $(\mathrm{Cone}(\phi)^p, d^p_{\mathrm{Cone}(\phi)})_p$ is a cochain complex. Moreover, we have an exact sequence of cochain complexes for each $p$, say

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C[-1]^p & \longrightarrow & K^p \oplus C[-1]^p & \longrightarrow & K^p & \longrightarrow & 0 \\ & & y & \longmapsto & (0,y) & & & & \\ & & & & (x,y) & \longmapsto & x & & \end{array}$$

And we have a long exact sequence of cohomology groups

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^p(C[-1]^\bullet) & \longrightarrow & H^p(\mathrm{Cone}(\phi)^\bullet) & \longrightarrow & H^p(K^\bullet) & \xrightarrow{\delta^p} & H^{p+1}(C[-1]^\bullet) & \longrightarrow \cdots \\ & \| & & & & & & \| & \\ & H^{p-1}(C^\bullet) & & & & & & H^p(C^\bullet) & \end{array}$$

Again, it is easy to verify that under the isomorphism $H^{p+1}(C[-1]^\bullet) \cong H^p(C^\bullet)$, the corresponding homomorphism $\delta^p$ is the one induced by the morphism $\phi_p^*$, which is an isomorphism as well. Hence

$$H^p(\mathrm{Cone}(\phi)^\bullet) = 0, \quad \forall p.$$

So the cochain complex

$$\mathrm{Cone}(\phi)^\bullet : \quad 0 \to K^0 = \mathrm{Cone}(\phi)^0 \to \mathrm{Cone}(\phi)^1 \to \cdots \to \mathrm{Cone}(\phi)^{n+1} = C^n \to 0$$

is exact, in which $\mathrm{Cone}(\phi)^p$ is $A$-flat for all $p \geqslant 1$. Also, $\mathrm{Cone}(\phi)^\bullet$ breaks into $n$ short exact sequences

$$0 \to \mathrm{Ker}(d^p_{\mathrm{Cone}(\phi)}) \to \mathrm{Cone}(\phi)^p \to \mathrm{Ker}(d^{p+1}_{\mathrm{Cone}(\phi)}) \to 0, \quad p = 1, \ldots, n.$$

Since $\mathrm{Ker}(d^{n+1}_{\mathrm{Cone}(\phi)}) = C^n$ is $A$-flat, so also is $\mathrm{Ker}(d^n_{\mathrm{Cone}(\phi)})$. We use descending induction and conclude that $\mathrm{Ker}(d^0_{\mathrm{Cone}(\phi)}) = K^0$ is $A$-flat. This proves the lemma. □

We apply Lemma 5.5 to the Čech cochain complex $C^\bullet = C^\bullet(\mathcal{U}, \mathscr{F})$ and obtain a cochain complex $K^\bullet$ and a cochain map $\phi : K^\bullet \to C^\bullet$ such that

(1) $K^\bullet$ is bounded on $[0, n]$;
(2) $K^0$ is finite and $A$-flat, and $K^p$ are finite free $A$-modules for $p \geqslant 1$;
(3) $\phi$ is a quasi-isomorphism, i.e., for all $p$, $\phi_p : H^p(K^\bullet) \to H^p(C^\bullet)$ is an isomorphism.

Granting these conditions, we see $K^p$ is projective as $A$-module for each $p \geqslant 0$. It remains to prove that for any $A$-algebra $B$,

$$\phi_B : H^p(K^\bullet \otimes_A B) \longrightarrow H^p(C^\bullet \otimes_A B)$$

is an isomorphism for each $p \geqslant 0$.

In fact, recall that the mapping cone $\mathrm{Cone}(\phi)^\bullet$ of $\phi$ breaks into short exact sequences

$$0 \to \mathrm{Ker}(d^p_{\mathrm{Cone}(\phi)}) \to \mathrm{Cone}(\phi)^p \to \mathrm{Ker}(d^{p+1}_{\mathrm{Cone}(\phi)}) \to 0, \quad p = 1, \ldots, n$$

and all the three terms are flat $A$-modules. Consequently, for each $p = 1, \ldots, n$,

$$0 \to \mathrm{Ker}(d^p_{\mathrm{Cone}(\phi)}) \otimes_A B \to \mathrm{Cone}(\phi)^p \otimes_A B \to \mathrm{Ker}(d^{p+1}_{\mathrm{Cone}(\phi)}) \otimes_A B \to 0$$

is also exact due to the flatness. In particular, the cochain complex $\mathrm{Cone}(\phi)^\bullet \otimes_A B$ is exact as well. On the other hand, $\mathrm{Cone}(\phi)^\bullet \otimes_A B$ is the mapping cone of $\phi_B = \phi \otimes_A B : K^\bullet \otimes_A B \to C^\bullet \otimes_A B$. So we have a long exact sequence of cohomology groups

$$\begin{tikzcd} \cdots \arrow[r] & H^p(\mathrm{Cone}(\phi)^\bullet \otimes_A B) \arrow[r] & H^p(K^\bullet \otimes_A B) \arrow[r, "\phi_B"] & H^{p+1}((C^\bullet \otimes_A B)[-1]) \arrow[r] \arrow[d, equal] & \cdots \\ & & & H^p(C^\bullet \otimes_A B) & \end{tikzcd}$$

Therefore, $\phi_B$ is an isomorphism for each $p$.

Now let $f : X \to Y$ be a proper morphism of noetherian schemes and $\mathscr{F}$ a coherent sheaf of $\mathscr{O}_X$-module on $X$ that is flat over $Y$. Recall that for $y \in Y$, we define the fiber $X_y = X \times Y \operatorname{Spec}(k(y))$ and $\mathscr{F}_y$ the inverse image of $\mathscr{F}$ on $X_y$. (Caution: $Y$ is not necessarily affine.)

**Corollary 5.6.** *Under the above notations, we have*

(1) *For every $p \geqslant 0$, the function*

$$Y \to \mathbb{Z}, \quad y \mapsto \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$$

*is upper semicontinuous on $Y$. A function $h : Y \to \mathbb{Z}$ is, by definition, upper semicontinuous, if for all $n \in \mathbb{Z}$ the set $\{y \in Y \mid h(y) \geqslant n\}$ is a closed subset of $Y$.*

(2) *The function*

$$Y \to \mathbb{Z}, \quad y \mapsto \chi(\mathscr{F}_y) = \sum_{p=0}^{\infty} (-1)^p \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$$

*is locally constant on $Y$.*

*Proof.* The question is local on $Y$ so one may assume that $Y = \operatorname{Spec} A$ is affine. We apply the pervious Theorem 5.2 to the morphism $f : X \to Y$ and the sheaf $\mathscr{F}$, and obtain a cochain complex $K^\bullet$ such that

$$H^p(X_y, \mathscr{F}_y) \cong H^p(K^\bullet \otimes_A k(y)), \quad \forall p \geqslant 0, \ y \in Y.$$

Shrinking $Y$ if necessary, we can assume that $K^p$ is free for all $p$ (the idea is to pretend $K^p$ to be the $p$th Čech complex). For $p \geqslant 0$, we define

$$W^p := \mathrm{Coker}(d_K^{p-1} : K^{p-1} \to K^p).$$

So we have an exact sequence

$$W^p \xrightarrow{d_K^p} K^{p+1} \longrightarrow W^{p+1} \longrightarrow 0.$$

Applying the functor $(\cdot) \otimes_A k(y)$, we get

$$0 \to H^p(K^\bullet \otimes_A k(y)) \to W^p \otimes_A k(y) \to K^{p+1} \otimes_A k(y) \to W^{p+1} \otimes_A k(y) \to 0.$$

This is basically because the cokernel commutes with base changes, and so we have

$$W^p \otimes_A k(y) \cong \mathrm{Coker}(d_K^{p-1} \otimes_A k(y) : K^{p-1} \otimes_A k(y) \to K^p \otimes_A k(y)).$$

Therefore,

$$\begin{aligned} \dim_{k(y)} H^p(K^\bullet \otimes_A k(y)) = \dim_{k(y)} W^p \otimes_A k(y) - \dim_{k(y)} K^{p+1} \otimes_A k(y) \\ + \dim_{k(y)} W^{p+1} \otimes_A k(y). \end{aligned}$$

Since the function

$$y \mapsto \dim_{k(y)} K^{p+1} \otimes_A k(y)$$

is (locally) constant, it suffices to prove that the function

$$y \mapsto \dim_{k(y)} W^p \otimes_A k(y)$$

is upper semicontinuous.

**Claim.** For any finitely generated $A$-module $M$, the function

$$Y \to \mathbb{Z}, \quad y \mapsto \dim_{k(y)} M \otimes_A k(y)$$

is upper semicontinuous.

The proof of the claim is leave as an exercise. Granting the claim, (2) follows by taking alternating sum of the dimension equation above. □

**Corollary 5.7.** *Under the above notations, assume further that $Y$ is reduced and connected. Then for all $p$, the following are equivalent.*

(1) *The function*

$$y \mapsto \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$$

*is constant.*

(2) *$R^p f_* \mathscr{F}$ is a locally free sheaf on $Y$, and for all $y \in Y$, the natural map*

$$R^p f_* \mathscr{F} \otimes_{\mathscr{O}_Y} k(y) \to H^p(X_y, \mathscr{F}_y)$$

*is an isomorphism.*

*If any one of* (1)(2) *hold, we also have that*

$$R^{p-1} f_* \mathscr{F} \otimes_{\mathscr{O}_Y} k(y) \cong H^{p-1}(X_y, \mathscr{F}_y)$$

*for all $y \in Y$.*

We can assume that $Y = \operatorname{Spec} A$ is affine and let $K^\bullet$ be the cochain complex in Theorem 5.2. Then (2) $\Longrightarrow$ (1) is obvious. So it boils down to prove (1) $\Longrightarrow$ (2).

**Lemma 5.8.** *Let $Y$ be a reduced affine scheme and $\mathscr{F}$ be a coherent sheaf on $Y$. If*

$$\dim_{k(y)} \mathscr{F} \otimes_{\mathscr{O}_Y} k(y) = r$$

*for all $y \in Y$ (as $k(y)$-vector spaces), then $\mathscr{F}$ is a locally free $\mathscr{O}_Y$-module of rank $r$.*

*Proof.* Let $Y = \operatorname{Spec} A$ and $\mathscr{F} = \widetilde{M}$. Fix $y \in Y$ that correspond to $\mathfrak{p} \in \operatorname{Spec} A$. We choose $x_1, \ldots, x_r \in M_\mathfrak{p}$ such that the images of $x_i$'s in $M \otimes_A k(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ form a basis of this $k(\mathfrak{p})$-vector space. By Nakayama's lemma, the $A_\mathfrak{p}$-linear homomorphism $\phi_\mathfrak{p} : A_\mathfrak{p}^r \to M_\mathfrak{p}$ determined by $x_1, \ldots, x_r$ is surjective. Then there exists $a \in A \backslash \mathfrak{p}$ such that $\phi_\mathfrak{p}$ extends to a surjective $A_a$-linear homomorphism $A_a^r \to M_a$. Replacing $A$ by $A_a$, we can assume that there exists a surjective $A$-linear map

$$\phi : A^r \longrightarrow\!\!\!\!\!\rightarrow M.$$

For any $\mathfrak{q} \in \operatorname{Spec} A$, $\phi \otimes_A k(\mathfrak{q})$ is a surjective $k(\mathfrak{q})$-linear map of $k(\mathfrak{q})$-vector spaces of dimension $r$. Then $\phi \otimes_A k(\mathfrak{q})$ is an isomorphism. Let $K = \operatorname{Ker}(\phi)$, and hence

$$K_{\mathfrak{q}} \subset (\mathfrak{q}A_{\mathfrak{q}})^r, \quad \forall \mathfrak{q} \in \operatorname{Spec} A.$$

Since $A$ is reduced, we have $K = 0$, and then $\phi$ is an isomorphism. So $M$ is free. □

**Lemma 5.9.** *Let $Y$ be a reduced noetherian affine scheme, and $\phi : \mathscr{F} \to \mathcal{G}$ be a morphism of finite and locally free $\mathscr{O}_Y$-modules. If*

$$\dim_{k(y)} \operatorname{im}(\phi \otimes_{\mathscr{O}_Y} k(y))$$

*is locally constant, then we can find a decomposition of finite and locally free $\mathscr{O}_Y$-modules*

$$\mathscr{F} = \mathscr{F}_1 \otimes \mathscr{F}_2, \quad \mathcal{G} = \mathcal{G}_1 \otimes \mathcal{G}_2$$

*such that $\phi$ factors through $\mathcal{G}_1$, $\phi|_{\mathscr{F}_1} = 0$, and $\phi : \mathscr{F}_2 \to \mathcal{G}_1$ is an isomorphism.*

*Proof.* Write $Y = \operatorname{Spec} A$ and $\mathscr{F} = \widetilde{M}$, $\mathcal{G} = \widetilde{N}$ for locally free $A$-modules $M, N$ of finite rank; $\phi : M \to N$ is an $A$-linear map. For any $\mathfrak{p} \in \operatorname{Spec} A$,

$$\dim_{k(y)} \operatorname{Coker}(\phi \otimes_A k(y)) = \dim_{k(y)} N \otimes_A k(y) - \dim_{k(y)} \operatorname{im}(\phi \otimes_A k(y))$$

is locally constant. By Lemma 5.8, $\operatorname{Coker}\phi$ is a locally free $A$-module of finite rank. Define

$$N_1 := \operatorname{Ker}(N \to \operatorname{Coker}\phi) = \operatorname{im}\phi.$$

So we have an exact sequence

$$0 \to N_1 \to N \to \operatorname{Coker}\phi \to 0.$$

We see that $N_1$ is locally free of finite rank, and there is a decomposition

$$N = N_1 \oplus N_2$$

such that $N_2 \cong \operatorname{Coker}\phi$ under the natural map $N \to \operatorname{Coker}\phi$. Also define $M_1 = \operatorname{Ker}\phi$. We have an exact sequence

$$0 \to M_1 \to M \xrightarrow{\phi} N_1 \to 0.$$

This shows that $M_1$ is locally free of finite rank. Moreover, notice that the exact sequence splits at $M$. So there is a decomposition $M = M_1 \oplus M_2$ such that $\phi|_{M_2} : M_2 \to N_1$ is an isomorphism. □

Now we are ready to prove the corollary.

*Proof of Corollary 5.7.* Applying Theorem 5.2 to $f : X \to Y$ and $\mathscr{F}$, we attain a cochain complex $K^\bullet$ such that for each $p \geqslant 0$,

$$H^p(X_y, \mathscr{F}_y) = H^p(K^\bullet \otimes_A k(y)).$$

Therefore,

$$\begin{aligned} &\dim_{k(y)} H^p(X_y, \mathscr{F}_y) \\ =&\dim_{k(y)} \operatorname{Ker}(d_K^p \otimes_A k(y)) - \dim_{k(y)} \operatorname{im}(d_K^{p-1} \otimes_A k(y)) \\ =&\dim_{k(y)} K^p \otimes_A k(y) - \dim_{k(y)} \operatorname{im}(d_K^p \otimes_A k(y)) - \dim_{k(y)} \operatorname{im}(d_K^{p-1} \otimes_A k(y)) \end{aligned}$$

is constant. Hence

$$\underbrace{\dim_{k(y)} \mathrm{im}(d_K^p \otimes_A k(y))}_{=\phi_1(y)} - \underbrace{\dim_{k(y)} \mathrm{im}(d_K^{p-1} \otimes_A k(y))}_{=\phi_2(y)}$$

is locally constant. Shrinking $Y$ if necessary, we can assume that $\phi_1(y) + \phi_2(y) = C$ (constant) on $Y$. Since $\phi_1(y)$ and $\phi_2(y)$ are lower semicontinuous, there is a natural stratification on $Y$, read as

$$\begin{aligned} Y &= \bigsqcup_{n=0}^{c} \{y \in Y \mid \phi_1(y) = n,\ \phi_2(y) = c - n\} \\ &= \bigsqcup_{n=0}^{c} \{y \in Y \mid \phi_1(y) \leqslant n,\ \phi_2(y) \leqslant c - n\}. \end{aligned}$$

Since $Y$ is connected, $\phi_1$ and $\phi_2$ are constant on $Y$. Now we can apply Lemma 5.9 to $d_K^p : K^p \to K^{p+1}$ and $d_K^{p-1} : K^{p-1} \to \mathrm{Ker}(d_K^p)$, to see there is a decomposition of locally free $A$-modules of finite rank:

$$\begin{array}{ccccccccc} & & Z^{p-1} \oplus (K')^{p-1} & & B^p \oplus H^p \oplus (K')^p & & B^{p+1} \oplus (K')^{p+1} & & \\ & & \| & & \| & & \| & & \\ \cdots & \longrightarrow & K^{p-1} & \xrightarrow{d_K^{p-1}} & K^p & \xrightarrow{d_K^p} & K^{p+1} & \longrightarrow & \cdots \end{array}$$

such that

$$\begin{aligned} Z^{p-1} &= \mathrm{Ker}(d_K^{p-1}), \qquad & d_K^{p-1} &: (K')^{p-1} \xrightarrow{\cong} B^p = \mathrm{im}(d_K^{p-1}); \\ B^p \oplus H^p &= \mathrm{Ker}(d_K^p), \qquad & d_K^p &: (K')^p \xrightarrow{\cong} B^{p+1} = \mathrm{im}(d_K^p). \end{aligned}$$

Therefore, for any $A$-algebra $B$,

$$H^p(K^\bullet \otimes_A B) \cong H^p \otimes_A B \cong H^p(K^\bullet) \otimes_A B.$$

Since $R^p f_* \mathscr{F}$ corresponds to the $A$-module

$$H^p(X, \mathscr{F}) \cong H^p(K^\bullet) \cong H^p,$$

we have that $R^p f_* \mathscr{F}$ is a locally free $A$-module of finite rank, and

$$(R^p f_* \mathscr{F}) \otimes_A B \cong H^p \otimes_A B \cong H^p(K^\bullet \otimes_A B) \cong H^p(X_y, \mathscr{F}_y).$$

This proves (2). Moreover, in this case,

$$\begin{aligned} (R^{p-1} f_* \mathscr{F}) \otimes_A k(y) &\cong H^{p-1}(X, \mathscr{F}) \otimes_A k(y) \\ &\cong \mathrm{Ker}(d_K^{p-1}) \otimes_A k(y) / \mathrm{im}(d_K^{p-1}) \otimes_A k(y) \\ &\cong Z^{p-1} \otimes_A k(y) / \mathrm{im}(d_K^{p-1} \otimes_A k(y)) \\ &\cong H^{p-1}(K^\bullet \otimes_A k(y)). \end{aligned}$$

Therefore,

$$(R^{p-1} f_* \mathscr{F}) \otimes_A k(y) \cong H^{p-1}(X_y, \mathscr{F}_y)$$

for all $y \in Y$. □

**Corollary 5.10.** *Under the above notations ($Y$ may not be reduced or connected), assume that $H^p(X_y, \mathscr{F}_y) = 0$ for some $p$ and all $y \in Y$. Then the rational map*

$$R^{p-1} f_* \mathscr{F} \otimes_{\mathscr{O}_Y} k(y) \xrightarrow{\cong} H^{p-1}(X_y, \mathscr{F}_y)$$

*is an isomorphism for all $y \in Y$.*

*Proof.* Let $K^\bullet$ be the cochain complex by Theorem 5.2. Fix $y \in Y$ such that

$$H^p(X_y, \mathscr{F}_y) \cong H^p(K^\bullet \otimes_A k(y)) = 0.$$

Then the sequence

$$K^{p-1} \otimes_A k(y) \xrightarrow{d_K^{p-1} \otimes_A k(y)} K^p \otimes_A k(y) \xrightarrow{d_K^p \otimes_A k(y)} K^{p+1} \otimes_A k(y)$$

is exact. We can decompose the $k(y)$-vector space $K^p \otimes_A k(y)$ as $\overline{W}_1 \oplus \overline{W}_2$ such that

$$\overline{W}_1 = \mathrm{im}(d_K^{p-1} \otimes_A k(y))$$

and $d_K^p \otimes_A k(y)|_{\overline{W}_2}$ is injective. Let $\{\overline{x}_1, \ldots, \overline{x}_r\}$ be a basis of $\overline{W}_1$ and $\{\overline{y}_1, \ldots, \overline{y}_s\}$ be a basis of $\overline{W}_2$. For $i = 1, \ldots, s$, denote

$$\overline{z}_i = d_K^p \otimes_A k(y)(\overline{y}_i) \in K^{p+1} \otimes_A k(y),$$

and extend $\{\overline{z}_1, \ldots, \overline{z}_s\}$ to a basis $\{\overline{z}_1, \ldots, \overline{z}_n\}$ of $K^{p+1} \otimes_A k(y)$. We choose lifting $x_i \in \mathrm{im}(d_K^{p-1})$ of $\overline{x}_i$ for $i = 1, \ldots, r$, $y_i \in K^p$ of $\overline{y}_j$ for $j = 1, \ldots, s$, and $z_i \in K^{p+1}$ of $\overline{z}_l$ for $l = 1, \ldots, s$. Shrinking $A$ by a localization $A_a$ at $a$ such that $a(y) \neq 0$, one may assume that $\{x_1, \ldots, x_r, y_1, \ldots, y_r\}$ is a basis of $K^p$, and $\{z_1, \ldots, z_n\}$ is a basis of $K^{p+1}$. Let $W_1$, $W_2$ be the free modules generated by $x_1, \ldots, x_r$ and $y_1, \ldots, y_s$, respectively. Then $K^p = W_1 \oplus W_2$, where $W_1 \subset \mathrm{im}(d_K^{p-1})$ and $d_K^p|_{W_2}$ is injective. Hence $W_1 = \mathrm{im}(d_K^{p-1})$. As $W_1$ is free, it is projective. So there is a decomposition $K^{p-1} = W_1 \oplus \mathrm{Ker}(d_K^{p-1})$. Now we have two exact sequences

$$K^{p-2} \xrightarrow{d_K^{p-2}} \mathrm{Ker}(d_K^{p-1}) \longrightarrow H^{p-1}(K^\bullet) \cong H^{p-1}(X, \mathscr{F}) \longrightarrow 0,$$

and

$$\begin{array}{ccccccc} K^{p-2} \otimes_A k(y) & \xrightarrow{d_K^{p-2} \otimes_A k(y)} & \mathrm{Ker}(d_K^{p-1} \otimes_A k(y)) & \longrightarrow & H^{p-1}(K^\bullet \otimes_A k(y)) & \longrightarrow & 0. \\ & & \| & & \| & & \\ & & \mathrm{Ker}(d_K^{p-1}) \otimes_A k(y) & & H^{p-1}(X_y, \mathscr{F}_y) & & \\ & & (\text{by } K^{p-1} = W_1 \oplus \mathrm{Ker}(d_K^{p-1})) & & & & \end{array}$$

Since the cokernel is stable under base changes, we have an isomorphism

$$\begin{array}{ccc} \boxed{H^{p-1}(X, \mathscr{F})} \otimes_A k(y) & \xrightarrow{\cong} & H^{p-1}(X_y, \mathscr{F}_y). \\ \| & & \\ R^{p-1} f_* \mathscr{F} & & \end{array}$$

This completes the proof. □

**Corollary 5.11.** *If $R^k f_* \mathscr{F} = 0$ for $k \geqslant k_0$, then*

$$H^k(X_y, \mathscr{F}_y) = 0, \quad \forall y \in Y, \ k \geqslant k_0.$$

**Corollary 5.12** (Flat base change)**.** *If $B$ is a flat $A$-algebra, then*

$$H^p(X \times_Y \mathrm{Spec}\, B, \mathscr{F} \otimes_A B) \cong H^p(X, \mathscr{F}) \otimes_A B.$$

**Corollary 5.13** (Seesaw's theorem)**.** *Let $X$ be a complete*[3] *variety and $T$ be any variety. Choose a line bundle $\mathscr{L} \in \mathrm{Pic}(X \times T)$. Then the set*

$$T_1 := \{t \in T \mid \mathscr{L}|_{X\times\{t\}} \text{ is trivial on } X \times \{t\}\}$$

*is closed in $T$, and $\mathscr{L}|_{X\times T_1} \cong p_2^*\mathscr{M}$ for some $\mathscr{M} \in \mathrm{Pic}(T_1)$, where $p_2 : X \times T_1 \to T_1$ is the second projection.*

**Lemma 5.14.** *A line bundle (i.e., an invertible sheaf) $\mathscr{M}$ on a complete variety $X$ is trivial if and only if*

$$\dim H^0(X, \mathscr{M}) > 0, \quad \dim H^0(X, \mathscr{M}^{-1}) > 0.$$

*Proof.* Exercise. □

*Proof of Seesaw's Theorem.* It follows from Lemma 5.14 that

$$\begin{aligned} T_1 &= \{t \in T \mid \mathscr{L}|_{X\times\{t\}} \text{ is trivial on } X \times \{t\}\} \\ &= \left\{ t \in T \;\middle|\; \begin{array}{c} \dim_{k(t)} H^0((X\times T)\times_T \mathrm{Spec}(k(t)), \mathscr{L}\otimes_{\mathscr{O}_T} k(t)) > 0, \text{ and} \\ \dim_{k(t)} H^0((X\times T)\times_T \mathrm{Spec}(k(t)), \mathscr{L}^{-1}\otimes_{\mathscr{O}_T} k(t)) > 0 \end{array} \right\}. \end{aligned}$$

By the semicontinuity theorem (Corollary 5.6), $T_1$ is closed in $T$. We regard $T_1$ as a reduced closed subscheme of $T$, and $p_2 : X \times T_1 \to T_1$ is a proper morphism of noetherian schemes. Denote for simplicity that $\mathscr{L}_1 = \mathscr{L}|_{X\times T_1}$. By definition of $T_1$, for any $t \in T_1$,

$$\dim_{k(t)} H^0((X\times T_1)\times_{T_1} \mathrm{Spec}(k(t)), \mathscr{L}_1\otimes_{\mathscr{O}_{T_1}} k(t)) > 0$$

By Corollary 5.7, $\mathscr{M} := p_{2,*}\mathscr{L}_1$ is an invertible sheaf on $T_1$ and the natural map

$$p_{2,*}\mathscr{L}_1 \otimes_{\mathscr{O}_{T_1}} k(t) \longrightarrow H^0(X\times\{t\}, \mathscr{L}_1|_{X\times\{t\}})$$

is an isomorphism for any $t \in T_1$.

We prove that the natural morphism $p_2^*\mathscr{M} \to \mathscr{L}_1$ is an isomorphism. In fact, for any $t \in T_1$, the sheaf $p_2^*\mathscr{M}|_{X\times\{t\}}$ is the inverse image of $\mathscr{M}$ under

$$X \times \{t\} \hookrightarrow X \times T_2 \xrightarrow{p_2} T_2.$$

It is the trivial invertible sheaf on $X \times \{t\}$ and is the pullback of the $k(t)$-vector space $p_{2,*}\mathscr{L}_1 \otimes_{\mathscr{O}_{T_1}} k(t)$ under $X \times \{t\} \to \{t\} = \mathrm{Spec}(k(t))$. On the other hand, $\mathscr{L}_1|_{X\times\{t_1\}}$ is also trivial and the restriction of $p_2^*\mathscr{M} \to \mathscr{L}_1$ on $X \times \{t\}$ corresponds to the morphism

$$p_{2,*}\mathscr{L}_1 \otimes_{\mathscr{O}_{T_1}} k(t) \longrightarrow H^0(X\times\{t\}, \mathscr{L}_1|_{X\times\{t\}})$$

of global sections. Therefore, the restriction of $p_2^*\mathscr{M} \to \mathscr{L}_1$ on $X \times \{t\}$ is an isomorphism for each $t \in T_1$. This is enough to show that $p_2^*\mathscr{M} \to \mathscr{L}_1$ is itself an isomorphism. □

*Remark* 5.15. We can assume that $T$ is a (reduced) scheme of finite type over an algebraically closed field $k$.

[3] Can be replaced with properness.

## 6. The Theorem of the Cube (I)

All varieties live on an algebraically closed field $k$.

### 6.1. **Statement and the Primary Ingredients.**

**Theorem 6.1** (The theorem of the cube)**.** *Let $X$, $Y$ be complete normal varieties, and $Z$ be any normal variety. Take $x_0$, $y_0$, and $z_0$ as base (closed) points on $X$, $Y$, and $Z$, respectively. If $\mathscr{L}$ is any line bundle on $X \times Y \times Z$ where restrictions to $\{x_0\} \times Y \times Z$, $X \times \{y_0\} \times Z$, and $X \times Y \times \{z_0\}$ are all trivial, then $\mathscr{L}$ is trivial.*

This section is primarily dedicated to the proof of this theorem. We begin with introducing two lemmas.

**Lemma 6.2** (Arcwise connectedness of complete varieties)**.** *Let $X$ be a complete variety and $x_0$, $x_1$ be two closed points of $X$. Then there exists an irreducible curve $C$ on $X$ containing $x_0$ and $x_1$.*

*Proof.* Use induction on $\dim_k X$. May assume $\dim X > 1$, and by Chow's lemma[4] $X$ can be taken as a projective variety. Now we can find a birational morphism $f : X' \to X$ with $X'$ projective, satisfying $\dim f^{-1}(x_i) \geqslant 1$ for $i = 0, 1$. For example, we can take $X'$ to be the blow-up of $X$ along the closed subscheme $\{x_0, x_1\}$.

Another way of construction is in [Mum85]. Choose a rational function $h$ on $X$ with indeterminacies at $x_1$ and $x_2$. Let $X'$ be the graph[5] of $h$. Then $X'$ is projective and the first projection $X \times \mathbb{P}^1 \to X$ induces a birational morphism $f : X' \to X$. If $\dim f^{-1}(x_i) = 0$ for $i = 0, 1$, by dimension theory, there are open neighborhoods $V_i$ of $x_i$ in $X$ such that

$$g = f|_{f^{-1}(V_i)} : f^{-1}(V_i) \longrightarrow V_i$$

is quasi-finite. As $g$ is proper, by Zariski Main Theorem, we infer that $g$ is finite. Suppose $V_i$ is normal without loss of generality (otherwise one can replace $X$ by its normalization), and $g$ is birational, we see $g$ must be an isomorphism. Therefore, $h$ is well-defined at $x_i$, which is a contradiction. Therefore,

$$\dim f^{-1}(x_i) \geqslant 1, \quad i = 0, 1.$$

Now we choose a projective embedding $X' \hookrightarrow \mathbb{P}^N$ for some $N$. By Bertini's theorem, there is a hyperplane of $\mathbb{P}^N$, say $H$, that does not contain $X'$, such that $Y' := H \cap X'$ is irreducible. Again, since $\dim f^{-1}(x_i) \geqslant 1$, we see $H \cap f^{-1}(x_i) \neq \emptyset$.

Let $Y = f(Y') \subset X$ be with the reduced irreducible closed subscheme structure that contains $x_0, x_1$ and $\dim Y = \dim X - 1$. By induction, we can find an irreducible curve $C \subset Y$ containing $x_0, x_1$. So we are done. $\square$

[4]The topological completeness is interpreted as the properness in an algebraic sense. And Chow's lemma is a machine to turn the conditions for projective varieties into those for simply proper varieties. More precisely, if $X$ itself is not projective but complete, then there is a projective variety $X'$ and a birational morphism $X' \to X$. This is moreover surjective as a map.

[5]Let $U$ be the maximal open subvariety of $X$ on which $h$ is well-defined. Then the graph of $h$ is defined to be the image of the morphism $(i, h) : U \to X \times \mathbb{P}^1$, where $i : U \hookrightarrow X$ is the open immersion.

**Lemma 6.3.** *Let $X$ be a smooth projective curve with a fixed line bundle $\mathscr{L}$ on it. For any divisor $D$ on $X$ with*

$$h^0(D) := \dim H^0(X, \mathscr{L}_X(D)) > 0,$$

*we have*

$$h^0(D-P) = h^0(D) - 1$$

*for all but finitely many closed points $P$ on $X$.*

*Proof.* In fact, we have an exact sequence of sheaves on $X$:

$$0 \to \mathscr{L}(D-P) \to \mathscr{L}(D) \to k(P) \to 0.$$

This induces the left-exact cohomological sequence

$$0 \to H^0(X, \mathscr{L}_X(D-P)) \to H^0(X, \mathscr{L}_X(D)) \xrightarrow{\varphi} k(P).$$

Pick a nonzero section $f \in H^0(X, \mathscr{L}_X(D))$, the following set is finite:

$$\#\{P \in X(k) \mid f_P \in \mathfrak{m}_P \mathscr{L}_X(D)_P\} < \infty.$$

For those $P$ landing outside this set, $\varphi : H^0(X, \mathscr{L}_X(D)) \to k(P)$ is surjective (so that the second sequence is right-exact). Hence $h^0(D-P) = h^0(D) - 1$. □

6.2. **Proof of the Theorem of Cube.**

*Proof of Theorem 6.1.* We begin with some reductions.

(1) **First reduction:** *by symmetry, it suffices to show that for any closed point $(x,z) \in X \times Z$, the invertible sheaf $\mathscr{L}|_{\{x\}\times Y\times\{z\}}$ is trivial.*

 In fact, notice that $X \times Z$ is Jacobson and hence closed points are dense in $X \times Z$. By Seesaw's theorem (Corollary 5.13), $\mathscr{L}|_{\{x\}\times Y\times\{z\}}$ is trivial for any point $(x,z)$ of $X \times Z$ and hence there is a line bundle $\mathscr{M}$ on $X \times Z$ such that $\mathscr{L} \cong \pi^*\mathscr{M}$ along the projection $\pi : X \times Y \times Z \to X \times Z$. Since $\mathscr{L}|_{X\times\{y_0\}\times Z}$ is trivial, we see $\mathscr{M}$ is trivial. This deduces the triviality of $\mathscr{L}$ itself.

(2) **Second reduction:** *it suffices to prove the theorem under the assumption that $X$ is a smooth projective curve and $Y, Z$ are normal varieties.*

 By Step (1), we need to show that $\mathscr{L}|_{\{x\}\times Y\times\{z\}}$ is trivial for any closed point $(x,y) \in X \times Z$. By Lemma 6.2 we can find an irreducible curve $C$ of $X$ that contains $x_0$ and $x$. Let $\pi : C' \to C$ be the normalization of $C$. By assumption, $C'$ is a smooth projective curve. Pick a closed point $x' \in \pi^{-1}(x)$. We also denote $\pi : C' \times Y \times Z \to C \times Y \times Z$. So that

$$(\pi^*\mathscr{L})|_{\{x'\}\times Y\times\{z\}} \cong \mathscr{L}|_{\{x\}\times Y\times\{z\}}.$$

 So we can assume that $X$ is a smooth projective curve. Consider the normalizations of $Y$ and $Z$. By a similar argument as above, they are assumed to be normal.

(3) **Third reduction:** *it boils down to find a nonempty open subset $Z'$ of $Z$ such that $\mathscr{L}|_{X\times Y\times Z'}$ is trivial.*

 If so, $\mathscr{L}|_{X\times Y\times\{z\}}$ is trivial for any point $z \in Z'$. Since $Z'$ is dense in $Z$, we see that $\mathscr{L}|_{X\times Y\times\{z\}}$ is trivial for all $z \in Z$ by Seesaw's theorem (Corollary 5.13). Therefore, $\mathscr{L}|_{\{x\}\times Y\times\{z\}}$ is trivial for any closed point $(x,z) \in X \times Z$.

Now we are ready to prove the theorem of cubes. Let $\Omega_X^1$ be the sheaf of differentials of $X/k$, and $g := \dim H^0(X, \Omega_X^1)$ be the genus of $X$. We can find $g$ closed points $p_1, \ldots, p_g$ such that

$$H^0(X, \Omega_X^1 \otimes \mathscr{L}_X(-D)) = 0,$$

where $D = \sum_{i=1}^g P_i$. This follows from Lemma 6.3. For such a divisor $D$, we define

$$\mathscr{L}' = \mathscr{L} \otimes p_1^* \mathscr{L}_X(D),$$

where $p_1 : X \times Y \times Z \to X$ is the first projection. For any point $y \in Y$, we have

$$\mathscr{L}'|_{X \times \{y\} \times \{z_0\}} \cong \mathscr{L}_X(D),$$

and

$$\dim H^1(X, \mathscr{L}_X(D)) = \dim H^0(X, \Omega_X^1 \otimes \mathscr{L}_X(-D)) = 0$$

by Serre duality. If one uses $F$ to denote the closed subset

$$\{(y,z) \in Y \times Z \mid \dim H^1(X, \mathscr{L}'_{(y,z)}) \geqslant 1\} \subset Y \times Z,$$

where $\mathscr{L}'_{(y,z)} = \mathscr{L}'|_{X \times \{y\} \times \{z\}}$, we see that $F \cap (Y \times \{z_0\}) = \emptyset$. Since $Y \times Z \to Z$ is proper, we can find an open subset $Z'$ of $Z$ such that $F \cap (Y \times Z') = \emptyset$. By Step (3) above, it suffices to prove that $\mathscr{L}|_{X \times Y \times Z'}$ is trivial. Replacing $Z$ by $Z'$, we can assume that for all points $(y,z) \in Y \times Z$, $F \cap (Y \times Z) = \emptyset$, i.e., $H^1(X, \mathscr{L}'_{(y,z)}) = 0$. Consequently, by Corollary 5.6 in the previous lecture,

$$\begin{aligned}\dim H^0(X, \mathscr{L}'_{(y,z)}) &= \chi(\mathscr{L}'_{(y,z)}) \overset{(5.6)}{=} \chi(\mathscr{L}'_{(y_0,z_0)}) = \chi(\mathscr{L}_X(D)) \\ &= \deg D + 1 - g = 1.\end{aligned}$$

Consider the natural projection $p_{23} : X \times Y \times Z \to Y \times Z$. By Corollary 5.7, we see that $p_{23,*}\mathscr{L}'$ is an invertible sheaf on $Y \times Z$ and

$$p_{23,*}\mathscr{L}' \otimes k(y,z) \longrightarrow H^0(X, \mathscr{L}'_{(y,z)})$$

is an isomorphism for all points $(y,z)$ of $Y \times Z$.

Denote $\mathscr{M} = p_{23,*}\mathscr{L}'$.[6] We define an effective Cartier divisor $\widetilde{D}$ on $X \times Y \times Z$ that corresponds to the invertible sheaf $\mathscr{L}' \in \mathrm{Pic}(X \times Y \times Z)$ as follows: for any open subset $U$ of $Y \times Z$ such that $\mathscr{M}|_U$ is trivial, we choose a generating section $\sigma_U \in \Gamma(U, \mathscr{M})$. Since

$$\Gamma(U, \mathscr{M}) \cong \Gamma(U, p_{23,*}\mathscr{L}') \cong \Gamma(X \times U, \mathscr{L}'),$$

we obtain a nonzero section $f_U \in \Gamma(X \times U, \mathscr{L}')$. Let $\widetilde{D}_U$ be the effective Weil divisor on $X \times U$ associated to $f_U$ (i.e., the divisor of zeros of $f_U$, see [Har13, II, §7]). Note that two different generating sections of $\mathscr{M}|_U$ in $\Gamma(U, \mathscr{M})$ are differed by an element in $\Gamma(U, \mathscr{O}_U^*)$.

It follows that the collections $\{U, \widetilde{D}_U\}_U$ (where $U$ runs through open subsets of $Y \times Z$) defines an effective Weil divisor $\widetilde{D}$ (by abuse of notation) that correspond to $\mathscr{L}'$ under the natural isomorphism

$$\mathrm{CaCl}(X) \cong \mathrm{Pic}(X)$$

(see [Har13, II, §6]).

[6] A priori we have this, whereas the natural pushforward map $p_{23}^*\mathscr{M} \to \mathscr{L}'$ is NOT an isomorphism in general. In fact, compared with the proof of Seesaw's theorem (Corollary 5.13), we need to assume that $\mathscr{L}'|_{X \times \{y\} \times \{z\}}$ is trivial for all points $(y,z) \in Y \times Z$. But this is not the case in our discussion. As a consequence, $\mathscr{L}'|_{X \times U}$ is NOT trivial in general.

A key property for $\widetilde{D}$ is in the following. For any closed point $(y,z) \in Y \times Z$, $\widetilde{D}|_{X\times\{y\}\times\{z\}}$ is the effective Cartier divisor associated to a nonzero section of $H^0(X, \mathscr{L}'_{(y,z)})$. The condition $\dim H^0(X, \mathscr{L}'_{(y,z)}) = 1$ implies that the linear system for $\mathscr{L}'_{(y,z)}$ consists of a single effective divisor, i.e., there is a unique effective divisor $E$ with $\mathscr{L}'_{(y,z)} = \mathscr{L}_X(E)$. This fact is implicitly but crucially used in the argument below.

Fix a closed point $P$ of $X$ such that $P \neq P_i$ for $i = 1, \ldots, g$. Let $S$ be the support of $\widetilde{D}|_{\{P\}\times Y\times Z}$ which is a closed subset of $\{P\} \times Y \times Z$, and all the irreducible components of $S$ have codimension one in $\{P\} \times Y \times Z$.[7]

Since $\widetilde{D}|_{X\times\{y\}\times\{z_0\}} \cong \mathscr{L}_X(D)$, we have

$$S \cap \{P\} \times Y \times \{z_0\} = \emptyset.$$

Hence the image of $S$ under $Y \times Z \to Z$ is a proper closed subset of $Z$. Thus,

$$S = \bigcup (\{P\} \times Y \times T_j)$$

where $T_j \subset Z$ are closed irreducible subvarieties of codimension 1. However, $S \cap (\{P\} \times \{y_0\} \times Z) = \emptyset$ for $P \neq P_i$, $i = 1, \ldots, g$. Denote $D'$ the Weil divisor on $X \times Y \times Z$ associated to $\widetilde{D}$. It follows that

$$D' = \sum_{i=1}^{g} n_i \{P_i\} \times Y \times Z.$$

Restricting to $X \times \{y_0\} \times \{z_0\}$, we see each $n_i = 1$. Then

$$\mathscr{L}'_{(y,z)} \cong \mathscr{L}_X(D), \quad \forall (y,z) \in Y \times Z.$$

Consequently, $\mathscr{L}|_{X\times\{y\}\times\{z\}}$ is trivial for all $(y,z) \in Y \times Z$. Finally, we apply Step (1) to $Y \times Z$ to see that $\mathscr{L}$ is trivial on $X \times Y \times Z$. This finishes the proof. $\square$

### 6.3. **Consequences of the Main Theorem.**

**Corollary 6.4.** *Let $X$, $Y$, and $Z$ be the same as in Theorem 6.1. Then any line bundle on $X \times Y \times Z$ is isomorphic to*

$$\pi_1^*\mathscr{L}_1 \otimes \pi_2^*\mathscr{L}_2 \otimes \pi_3^*\mathscr{L}_3,$$

*where $\pi_1 : X \times Y \times Z \to Y \times Z$, $\pi_2 : X \times Y \times Z \to X \times Z$, $\pi_3 : X \times Y \times Z \to X \times Y$ are natural projections; $\mathscr{L}_1 \in \mathrm{Pic}(Y \times Z)$, $\mathscr{L}_2 \in \mathrm{Pic}(X \times Z)$, and $\mathscr{L}_3 \in \mathrm{Pic}(X \times Y)$.*

*Proof.* Using the same method as in the proof of Theorem 6.1. Define

$$\sigma_1 : Y \times Z \to X \times Y \times Z, \quad (y,z) \mapsto (x_0, y, z),$$

and also $\sigma_2$, $\sigma_3$ in respective similar ways. And

$$\pi_X : X \times Y \times Z \to X, \quad \pi_Y : X \times Y \times Z \to Y, \quad \pi_Z : X \times Y \times Z \to Z.$$

Again, define

$$\sigma_X : X \to X \times Y \times Z, \quad x \mapsto (x, y_0, z_0)$$

and also $\sigma_Y$, $\sigma_Z$ in respective similar ways. Define

$$\mathscr{L}_? = \sigma_?^*\mathscr{L}, \quad ? = 1, 2, 3, X, Y, Z,$$

[7] In fact, this is Krull's Hauptidealsatz in a general case.

and
$$\begin{aligned}\mathscr{M} = \mathscr{L} \otimes \pi_1^*\mathscr{L}_1^{-1} \otimes \pi_2^*\mathscr{L}_2^{-1} \otimes \pi_3^*\mathscr{L}_3^{-1} \otimes \pi_X^*\mathscr{L}_X^{-1} \otimes \pi_Y^*\mathscr{L}_Y^{-1} \otimes \pi_Z^*\mathscr{L}_Z^{-1} \\ \in \operatorname{Pic}(X \times Y \times Z).\end{aligned}$$
It is straightforward to verify that $\sigma_i^*\mathscr{M}$ is trivial for $i = 1, 2, 3$. By theorem of cube (Theorem 6.1), $\mathscr{M}$ is trivial and hence
$$\mathscr{L} \cong \pi_1^*\mathscr{L}_1 \otimes \pi_2^*\mathscr{L}_2 \otimes \pi_3^*\mathscr{L}_3 \otimes \pi_X^*\mathscr{L}_X^{-1} \otimes \pi_Y^*\mathscr{L}_Y^{-1} \otimes \pi_Z^*\mathscr{L}_Z^{-1},$$
which is of the form in the corollary. □

**Corollary 6.5.** *Let $X$ be any variety and $Y$ be an abelian variety. Let $f, g, h : X \to Y$ be morphisms. Then for each $\mathscr{L} \in \operatorname{Pic}(Y)$, we have*
$$(f+g+h)^*\mathscr{L} \cong (f+g)^*\mathscr{L} \otimes (f+h)^*\mathscr{L} \otimes (g+h)^*\mathscr{L} \otimes f^*\mathscr{L}^{-1} \otimes g^*\mathscr{L}^{-1} \otimes h^*\mathscr{L}^{-1}.$$

*Proof.* Let $p_i : Y \times Y \times Y \to Y$ be the projection to the $i$th factor for $i = 1, 2, 3$. Also, denote $m_{ij} = p_i + p_j$, and $m = p_1 + p_2 + p_3 : Y \times Y \times Y \to Y$. Define
$$\begin{aligned}\mathscr{M} = m^*\mathscr{L} \otimes m_{12}^*\mathscr{L}^{-1} \otimes m_{23}^*\mathscr{L}^{-1} \otimes m_{13}^*\mathscr{L}^{-1} \otimes p_1^*\mathscr{L} \otimes p_2^*\mathscr{L} \otimes p_3^*\mathscr{L} \\ \in \operatorname{Pic}(Y \times Y \times Y).\end{aligned}$$
One can verify that $\mathscr{M}|_{\{e_Y\} \times Y \times Y}$, $\mathscr{M}|_{Y \times \{e_Y\} \times Y}$, and $\mathscr{M}|_{Y \times Y \times \{e_Y\}}$ are all trivial. By Theorem 6.1, $\mathscr{M}$ itself is trivial. We pull $\mathscr{M}$ back along $(f, g, h) : X \to Y \times Y \times Y$ and get the desired isomorphism. □

**Corollary 6.6.** *If $X$ is an abelian variety and $n \in \mathbb{Z}$, then for all $\mathscr{L} \in \operatorname{Pic}(X)$,*
$$n_X^*\mathscr{L} \cong \mathscr{L}^{\frac{n^2+n}{2}} \otimes (-1)_X^*\mathscr{L}^{\frac{n^2-n}{2}}.$$

*Proof.* In Corollary 6.5, take $f = (n+1)_X$, $g = 1_X$, and $h = (-1)_X$ to deduce
$$(n+1)_X^*\mathscr{L} \otimes n_X^*\mathscr{L}^{-2} \otimes (n-1)_X^*\mathscr{L} \cong \mathscr{L} \otimes (-1)_X^*\mathscr{L},$$
and hence
$$n_X^*\mathscr{L} \otimes (n-1)_X^*\mathscr{L}^{-1} \cong \mathscr{L}^n \otimes ((-1)_X^*\mathscr{L})^{n-1}.$$
We can infer from it that
$$n_X^*\mathscr{L} \cong \mathscr{L}^{\frac{n^2+n}{2}} \otimes ((-1)_X^*\mathscr{L})^{\frac{n^2-n}{2}}.$$
□

**Corollary 6.7** (Theorem of square)**.** *For any $\mathscr{L} \in \operatorname{Pic}(X)$ and closed points $x, y \in X$, where $X$ is an abelian variety, we have*
$$T_{x+y}^*\mathscr{L} \otimes \mathscr{L} \cong T_x^*\mathscr{L} \otimes T_y^*\mathscr{L}.$$
*Here $T_x : X \to X$ is the translation by $x$. In other words, the map*
$$\phi_{\mathscr{L}} : X(k) \to \operatorname{Pic}(X), \quad x \mapsto T_x^*\mathscr{L} \otimes \mathscr{L}^{-1}$$
*is a group homomorphism.*

*Proof.* In Corollary 6.5, we take $X = Y$ and
$$f : X \to k \xrightarrow{x} X, \quad g : X \to k \xrightarrow{y} X, \quad h = \mathrm{id}_X : X \to X$$
to complete the proof. □

*Remark* 6.8. The map $\phi_{\mathscr{L}} : X(k) \to \operatorname{Pic}(X)$ defined above has the following properties:

(1) $\phi_{\mathscr{L}_1 \otimes \mathscr{L}_2} = \phi_{\mathscr{L}_1} +_{\operatorname{Pic}(X)} \phi_{\mathscr{L}_2}$;
(2) $\phi_{T_x^* \mathscr{L}} = \phi_{\mathscr{L}}$.

**Definition 6.9.** Let $X$ be an abelian variety as above. For $\mathscr{L} \in \operatorname{Pic}(X)$, define

$$K(\mathscr{L}) := \operatorname{Ker}(\phi_{\mathscr{L}}) = \{x \in X(k) \mid T_x^* \mathscr{L} \cong \mathscr{L}\}.$$

**Proposition 6.10.** *$K(\mathscr{L})$ is a Zariski closed subset of $X$ (here we view $X$ as an algebraic variety over $k$).*

*Proof.* Consider the line bundle $\mathscr{M} = m^* \mathscr{L} \otimes p_2^* \mathscr{L}^{-1}$ on $X \times X$, where $m$ is the addition map and $p_2$ is the second projection. By Seesaw's theorem 5.13,

$$F := \{x \in X \mid \mathscr{M}|_{\{x\} \times X} \text{ is trivial}\}$$

is a closed subset of $X$. When $x \in X(k)$, namely, $x$ is a closed point, we have

$$\mathscr{M}|_{\{x\} \times X} \cong T_x^* \mathscr{L} \otimes \mathscr{L}^{-1},$$

then $K(\mathscr{L})$ is the set of closed points of $F$. Therefore, $K(\mathscr{L})$ is Zariski closed in the algebraic variety $X$. □

6.4. **Some Further Applications.** The following theorem is the first explicit application of theorem of cube and its corollaries.

**Theorem 6.11.** *Let $D$ be an effective divisor on an abelian variety $X$, and $\mathscr{L} \cong \mathscr{L}_X(D)$ be the associated invertible sheaf. Then the following are equivalent:*

(1) *The (complete) linear system $|2D|$ has no base point and defines a finite morphism $X \to \mathbb{P}^N$ with $N = \dim H^0(X, \mathscr{L}(2D)) - 1$.*
(2) *$\mathscr{L}$ is ample on $X$.*
(3) *$K(\mathscr{L})$ is finite.*
(4) *The subgroup $H = \{x \in X(k) \mid T_x^*(D) = D\}$ of $X(k)$ is finite. Here $T_x^*(D) = D$ is an equality of divisors rather than divisor classes.*

*Proof.* We first prove (1) ⇒ (2) ⇒ (3) ⇒ (4). After this, we setup a lemma in order to prove (4) ⇒ (1).

(1) ⇒ (2) This follows from the fact that under a finite morphism of complete varieties, the inverse image of an ample line bundle is again ample (cf. [Har13, III, Exer 5.7]).

(2) ⇒ (3) Suppose that $K(\mathscr{L})$ is a positive dimensional $k$-scheme[8] and let $Y$ be the connected component of $e_X$ in $K(\mathscr{L})$. Then $Y$ is an abelian variety of positive dimension. Since $Y \hookrightarrow X$ is a closed immersion, $\mathscr{L}_Y := \mathscr{L}|_X$ is ample on $Y$. Since $\mathscr{L}_Y$ is stable under the translation $T_y$ for any $y \in Y(k)$, we have $T_y^* \mathscr{L}_y \cong \mathscr{L}_y$ (now we view $T_y$ as the translation by $y$ on $Y$ rather than on $X$). Hence the line bundle

$$\mathscr{M} = m^* \mathscr{L}_Y \otimes p_1^* \mathscr{L}_Y^{-1} \otimes p_2^* \mathscr{L}_Y^{-1}$$

[8] See [Har13, II, Prop 2.6] for the functor $t : \mathsf{Var}_k \to \mathsf{Sch}_k$ from the category of varieties over $k$ to schemes over $k$.

on $Y \times Y$ such that $\mathscr{M}|_{\{y\}\times Y}$ and $\mathscr{M}|_{Y\times\{y\}}$ are trivial for all $y \in Y(k)$. Here $m$ is addition and $p_i$ is the $i$th projection. By Seesaw's theorem (Corollary 5.13), $\mathscr{M}$ is trivial. We pull back $\mathscr{M}$ along the morphism $(1_Y, (-1)_Y) : Y \to Y \times Y$ and see that $\mathscr{L}_Y \otimes (-1)_Y^* \mathscr{L}_Y$ is trivial on $Y$. Since $\mathscr{L}_Y$ is ample and $(-1)_Y$ is an automorphism of $Y$, $(-1)_Y^*\mathscr{L}_Y$ is ample. Then $\mathscr{L}_Y \otimes (-1)_Y^*\mathscr{L}_Y$ is ample, i.e., $\mathscr{O}_Y$ itself is ample on $Y$. But this is impossible when $\dim Y > 0$. Then $\dim K(\mathscr{L}) = 0$ and $K(\mathscr{L})$ is finite.

(3) $\Rightarrow$ (4) This is obvious as $H \subset K(\mathscr{L})$.

(4) $\Rightarrow$ (1) By the theorem of the square (Corollary 6.7), the (complete) linear system $|2D|$ contains the divisor $T_x^*(D) + T_{-x}^*(D)$ for all $x \in X(k)$. For any $P \in X(k)$, we can find $x \in X(k)$ such that $P \pm x \notin \mathrm{Supp}(D)$ if and only if $P \notin \mathrm{Supp}(T_x^*(D) + T_{-x}^*(D))$. It follows that $|2D|$ is base-point-free and any basis of $H^0(X, \mathscr{L}_X(2D))$ gives a morphism $\phi : X \to \mathbb{P}^N$. Since $\phi$ is proper, it follows from Zariski Main Theorem that to prove $\phi$ is finite, it suffices to prove that $\phi$ is quasi-finite. Suppose it is not the case for the sake of contradiction. Then we can find an irreducible curve $C$ on $X$ such that $\phi(C)$ is a single closed point of $\mathbb{P}^N$. It follows that for any (Weil) divisor $D'$ in $|2D|$, we have either $C \cap D' = \emptyset$ or $C \subset D'$.[9]

We now introduce the following lemma.

**Lemma 6.12.** *If $C$ is an irreducible curve and $E$ is a prime divisor on $X$ such that $C \cap E = \emptyset$, then $E$ is invariant under translations defined by $x_1 - x_2$ for all $x_1, x_2 \in C(k)$.*

*Proof of Lemma.* We use $\mathscr{L}$ to denote the invertible sheaf $\mathscr{L}_X(E)$ associated to $E$. Since $E \cap C = \emptyset$, $\mathscr{L}|_C$ is trivial and hence $\deg(\mathscr{L}|_C) = 0$. Let $\mathscr{M}$ be the pullback of $\mathscr{L}$ along the morphism $X \times C \hookrightarrow X \times X \xrightarrow{m} X$. We infer that $\mathscr{M}|_{\{x\}\times C} \cong (T_x^*\mathscr{L})|_C$.

For any invertible sheaf $\mathscr{N}$ on $X \times C$, since $p_1 : X \times C \to X$ is proper and flat, by Corollary 5.6 (2), we see that the function $x \mapsto \chi(\mathscr{N}|_{\{x\}\times C})$ is constant, i.e., $\chi(\mathscr{N}|_{\{x\}\times C})$ is independent of $x \in X(k)$. Replacing $\mathscr{N}$ by $\mathscr{N}^n$ for all $n \in \mathbb{Z}_{>0}$, we get $x \longmapsto \chi(\mathscr{N}^n|_{\{x\}\times C})$ is independent of $x \in X(k)$ as a function in $n$. However, $C$ is a curve and it is well-known that

$$n \longmapsto \chi(\mathscr{N}^n|_{\{x\}\times C})$$

is a linear function on $n$ with the linear coefficient $\deg(\mathscr{N}|_{\{x\}\times C})$. Therefore, the function

$$x \mapsto \deg(\mathscr{N}|_{\{x\}\times C})$$

is constant. To summarize, we have $\deg(T_x^*\mathscr{L})|_C = 0$ for all $x \in X(k)$. Then $E$ and $T_x(C)$ cannot intersect at only finitely many (but not empty) closed points. Thus, either $E \cap T_x(C) = \emptyset$ or $T_x(C) \subset E$. Fix $y \in E(k)$ and $x_1, x_2 \in C(k)$. Since $y \in T_{y-x_2}(C) \cap E$, we get $T_{y-x_2}(C) \subset E$. Therefore, $y - x_2 + x_1 \in E$. This proves the lemma. ☐

*Resume on.* Now we are to finish the proof of (4) $\Rightarrow$ (1).

(4) $\Rightarrow$ (1) Fix $P \in C(k)$. By our previous discussion, there exists $x \in X(k)$ such that $P \notin \mathrm{Supp}(T_x^*(D) + T_{-x}^*(D))$. Then $C \cap \mathrm{Supp}(T_x^*(D) + T_{-x}^*(D)) = \emptyset$. Let $C' = T_x(C)$ and then $C' \cap D = \emptyset$. If $D = \sum_i n_i D_i$ with $n_i > 0$ and $D_i$ prime divisors, then

[9] In fact, $D'$ corresponds to a nonzero section in $\Gamma(X, \mathscr{L}_X(2D))$ and hence $D'$ is the preimage of a hyperplane under $\phi$.

$C' \cap D_i = \emptyset$ follows. By the lemma, $D_i$ must be stable under translations by $x_1 - x_2$ for all $x_1, x_2 \in C(k)$. This contradicts with the condition that $H$ is finite. Hence $\phi$ is finite.

This completes the proof of the theorem. □

**Corollary 6.13.** *An abelian variety $X$ is projective.*

*Proof.* It suffice to find an effective Weil divisor $D$ on $X$ such that $H = \{x \in X(k) \mid T_x^*(D) = D\}$ is finite. We first prove the following lemma.

**Lemma 6.14.** *Let $X$ be a noetherian, separable, normal, and integral scheme; let $U \subset X$ be a nonempty affine open subset and $U \neq X$. Then every irreducible component of $X\backslash U$ is of codimension 1 in $X$.*

*Proof of Lemma.* By the noetherian condition, $X\backslash U$ has only finitely many irreducible components. Let $\xi$ be a generic point of $X\backslash U$. We can find an affine open neighborhood $V$ of $\xi$ in $X$ such that $\xi$ is the only generic point of $X\backslash U$ in $V$. It suffices to prove that $\dim \mathscr{O}_{X,\xi} = 1$.

Suppose not, then $V\backslash(U \cap V)$ has codimension $\geqslant 2$ in $V$. Recall the following result: if $A$ is an normal integral domain of dimension $\geqslant 1$, then we have the following equality in $\mathrm{Frac}(A)$:

$$A = \bigcap_{\substack{\mathfrak{p} \in \mathrm{Spec}(A), \\ \mathrm{ht}(\mathfrak{p})=1}} A_{\mathfrak{p}}.$$

As a result, the restriction map $\Gamma(V, \mathscr{O}_X) \to \Gamma(V \cap U, \mathscr{O}_X)$ is an isomorphism (here we use the fact that $U \cap V$ is affine as $X$ is separated). Therefore, that $V \cap U \to V$ is an isomorphism and $V\backslash V \cap U = \emptyset$ are both valid, which is a contradiction. Then $\dim \mathscr{O}_{X,\xi} = 1$. □

*Resume on.* For the abelian variety $X$, we choose an affine open neighborhood $U$ of $e_X$ and the above lemma implies that the irreducible component $D_1, \ldots, D_n$ of $X\backslash U$ are all of codimension one. So $D = \sum_{i=1}^n D_i$ is a Weil divisor on $X$. Also,

$$H = \{x \in X(k) \mid T_x^*(D) = D\}$$

is a closed subgroup of $X(k)$.[10] In particular, $H$ is proper.

On the other hand, for each $x \in H$, $U$ is stable under $T_x$. As $e_x \in U$, we have $x \in U$ and then $H \subset U$. However, $H$ itself is proper and $U$ is affine. This forces $H$ to be finite. □

Here comes the second application of the result.

**Proposition 6.15.** *An abelian variety is a divisible group, and for each $n \geqslant 1$, $X_n = \mathrm{Ker}(n_X : X \to X)$ is finite over $k$.*

[10] Alternatively, use the description

$$H = \{x \in X(k) \mid T_x^*(D) \subset D\} = \bigcap_{d \in D(k)} (d - D).$$

*Proof.* By dimension theory, it suffices to prove that $X_n$ is finite. We view $X_n$ as a (reduced) closed subscheme of $X$. Let $\mathscr{L} \in \mathrm{Pic}(X)$ be an ample line bundle.[11] Clearly $(n_X^*\mathscr{L})|_{X_n}$ is trivial. On the other hand,

$$n_X^*\mathscr{L} \cong \mathscr{L}^{\frac{n(n+1)}{2}} \otimes ((-1)_X^*\mathscr{L})^{\frac{n(n-1)}{2}}.$$

Since $(-1)_X^*\mathscr{L}$ is also ample, the pullback $n_X^*\mathscr{L}$ is ample. In particular, so also is $(n_X^*\mathscr{L})|_{X_n}$ and hence $X_n$ is finite. □

It is known tat $n_X : X \to X$ is a finite surjective homomorphism; in particular, $n_X$ is dominated. Also, $n_X$ induces a field embedding $n_X^* : k(X) \to k(X)$. Denote $\deg(n_X)$ the degree of this field extension, which is called the *degree of* $n_X$. One can similarly define the separable degree and the inseparable degree.

By intersection theory, we have

$$(n_X^*D_1, \dots, n_X^*D_g) = \deg(n_X)(D_1, \dots, D_g), \quad g = \dim X$$

for arbitrary Cartier divisors $D_1, \dots, D_g$ on $X$. We take $D$ to be an ample symmetric divisor on $X$, i.e., $(-1)_X^*D = D$.[12] Consequently, $n_X^*D \sim n^2D$ as a linear equivalence. So $\deg(n_X) = n^{2g}$.

When $p \mid n$ we have seen before that the induced map on tangent spaces by $n_X$, say $dn_X : T_{X,e_X} \to T_{X,e_X}$ is 0. Recall that $\omega_X = e_X^*\Omega_X^1$ can be identical with the cotangent space of $X$ at $e_X$ and $\pi^*\omega_X \cong \Omega_X^1$, where $\pi : X \to k$ is the structure map. So the canonical map $n_X^* : \Omega_X^1 \to \Omega_X^1$ is the zero map, and so also is $n_X^*\Omega^1_{k(X)/k} \to \Omega^1_{k(X)/k}$ under the field extension $k \to k(X) \xrightarrow{n_X^*} k(X)$. In particular, the composition of $n_X^*$ with the canonical derivation $d$ is zero:

$$\begin{tikzcd} k(X) \arrow[r, "n_X^*"] \arrow[rr, bend left, "0"] & k(X) \arrow[r, "d"] & \Omega^1_{k(X)/k}. \end{tikzcd}$$

Therefore,

$$n_X^*(k(X) \subset \mathrm{Ker}(d) = k(X)^p \subset k(X),$$

for which the proof of the fact $\mathrm{Ker}(d) = k(X)^p$ is leave as an exercise.

**Fact.** $k(X)/k(X)^p$ is a purely inseparable extension of degree $p^g$. (We use the fact that $k$ is algebraically closed and $\mathrm{trdeg}_k\, k(X) = g$.)[13]

**Proposition 6.16.** *Keep the notations as above. We obtain the following.*

(1) $\deg(n_X) = n^{2g}$.
(2) $n_X$ *is separable if and only if* $p \nmid n$. *In fact,* $n_X$ *is separable if and only if it is étale as a morphism.*
(3) *If* $p \nmid n$, *then* $X_n(k) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$.
(4) *There exists* $0 \leqslant i \leqslant g$ *such that for each* $m \geqslant 1$, $X_{p^m}(k) \cong (\mathbb{Z}/p^m\mathbb{Z})^i$.

[11] Indeed, one may be able to prove that $n_X : X \to X$ is flat (cf. [Har13, III, Exer 10.9]).

[12] This is possible because one may choose an ample divisor $D'$ on $X$ and then let $D = D' + (-1)X^*D'$.

[13] On separating the transcendental basis: let $K/k$ be a finitely generated field extension and $k$ is perfect; then there exists a transcendental basis $x_1, \dots, x_m$ of $K/k$ such that $K/k(x_1, \dots, x_m)$ is an algebraic separable extension.

## 7. Dividing Varieties by Finite Groups

**Definition 7.1** (Étale morphism)**.** Let $f : X \to Y$ be a morphism of algebraic varieties over an algebraically closed field $k$. Then $f$ is called **étale** if

(1) $f$ is flat;
(2) $f$ is unramified, i.e., for each closed point $x \in X$, let $y = f(x) \in Y$ and $\mathfrak{m}_x$ (resp. $\mathfrak{m}_y$) be the maximal ideal of $\mathscr{O}_{X,x}$ (resp. $\mathscr{O}_{Y,y}$), then $f^*(\mathfrak{m}_y)\mathscr{O}_{X,x} = \mathfrak{m}_x$ for $f^* : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$. (In general, we also need to assume that $k(y) \to k(x)$ is separable.

Or equivalently (cf. [Har13, III, Exercise 10.4]),

(2') for any $x \in X$, let $y = f(x) \in Y$ and let $\widehat{\mathscr{O}}_{X,x}$ (resp. $\widehat{\mathscr{O}}_{Y,y}$) be the completion of $\mathscr{O}_{X,x}$ (resp. $\mathscr{O}_{Y,y}$). Then $f^*$ insudes an isomorphism $\widehat{f}^* : \widehat{\mathscr{O}}_{Y,y} \to \widehat{\mathscr{O}}_{X,x}$.

### 7.1. The Quotient along an Étale Morphism.

**Theorem 7.2.** *Let $X$ be an algebraic variety and $G$ be a finite group of automorphisms of $X$. Suppose that for any $x \in X$, the orbit $G_x$ of $x$ is contained in an affine open subset of $X$. Then there is a pair $(Y, \pi)$ where $Y$ is a variety and $\pi : X \to Y$ is a morphism with the following conditions.*

(1) *As a topological space, $(Y, \pi)$ is the quotient of $X$ under the $G$-action.*
(2) *Denote $\pi_*(\mathscr{O}_X)^G$ the subsheaf of $G$-invariants of $\pi_*\mathscr{O}_X$ for the action of $G$ on $\pi_*\mathscr{O}_X$ deduced from* (1)*, the natural homomorphism $\mathscr{O}_Y \to \pi_*(\mathscr{O}_X)G$ is an isomorphism.*

*Moreover, the pair is uniquely determined (up to isomorphisms) by* (1) *and* (2)*. The morphism $\pi$ is finite, surjective, and separable. If $X$ is affine then so also is $Y$. If further $G$ acts freely on $X$, i.e., $gx \neq x$ for all $x \in X$ for any $g \in G \backslash \{e\}$, $\pi$ shall be étale.*

*Remark* 7.3. (1) We essentially use the language of varieties instead of schemes in this lecture.
(2) $G$ acts on $X$ from the left and acts on $\mathscr{O}_X$ (the sheaf of regular functions on $X$) via the formula
$$(g(f))(x) = f(g^{-1}x), \quad \forall f \in \mathscr{O}_X(U),\ x \in U,\ g \in G,$$
where $U \subset X$ is an open subset. This is also viewed as a left action.
(3) When $X$ is quasi-projective, any finitely many points of $X$ is contained in an affine open subset of $X$, and hence the assumption in the theorem is satisfied. To be more explicit, by definition, we have
$$X \overset{j}{\hookrightarrow} \overline{X} \overset{f}{\hookrightarrow} \mathbb{P}^N,$$
where $j$ is an open immersion and $f$ is a closed immersion. We know that, by the prime avoidance, if $I$ is an ideal and $\mathfrak{p}_1, \ldots, p_r$ are prime ideals such that $I \not\subset \mathfrak{p}_i$ for each $i$, then $I \not\subset \bigcup \mathfrak{p}_i$. Thus there is a homogeneous element $f \in k[x_0, \ldots, x_N]$ such that $\overline{X} \backslash X \subset V_+(f)$ and $x_i \notin V_+(f)$ for all $i$; $D_+(f) \cap X = D_+(f) \cap \overline{X}$ is affine and does not contain the points $x_1, \ldots, x_r$.

*Proof of Theorem 7.2.* Note that (1) determines the topology on $Y$ and (2) determines the structure sheaf on $Y$, so the uniqueness follows. We are to show that if one takes $Y = X/G$

as a topological space[14] then the pair $(Y, (\pi_*\mathscr{O}_X)^G)$ is an algebraic variety. First we reduce to the affine case. For any closed point $x$ of $X$ and an affine open neighborhood $U$ of $x$ in $X$, the intersection $\bigcap_{g\in U} gU$ is an affine open neighborhood of $x$, and is $G$-stable. So we can find an affine open $G$-stable subset $U$ containing $x$. This renders that $pi^{-1}(\pi(U)) = U$ where $\pi : X \to Y$ is the quotient map and $\pi(U)$ is open in $Y$.

So it is harmless to assume that $X = \operatorname{Spec} A$ affine. Since $A$ is a finitely generated $k$-algebra, we let $\{x_1, \dots, x_n\}$ be a set of generators of $A$ over $k$. Let $v = |G|$ with $G = \{g_1, \dots, g_v\}$. For each $f \in A$ and $1 \leqslant k \leqslant v$ we use $\sigma_k(f)$ to denote the elementary symmetric function of degree $k$ in $\{g_1(f), \dots, g_v(f)\}$. Let $B'$ be the $k$-subalgebra of $A$ generated by $\{\sigma_k(x_i) \mid i = 1, \dots, n\ k = 1, \dots, v\}$ and $B = A^G$. Then we have $B' \subset B \subset A$. For each $1 \leqslant i \leqslant n$, the $x_i$ satisfies the monic equation over $B'$, say

$$x^v - \sigma_1(x_i)x^{v-1} + \cdots + (-1)^v\sigma_v(x_i) = 0.$$

Hence $x_i$ is integral over $B'$ and then $A$ is integral over $B'$. Again, since $A$ is a finitely generated $k$-algebra, it is finite over $B'$. As $B'$ is noetherian, $B$ is finite over $B'$ and so also is $A$ over $B$. In particular, $B$ is a finitely generated $k$-algebra.

Let $Y = \operatorname{Spec} B$ and let $\pi : X \to Y$ be the morphism corresponding to the inclusion $B \hookrightarrow A$. Then $Y$ is an algebraic variety and $\pi$ is finite surjective. Let $K$ (resp. $L$) be the quotient field of $B$ (resp. $A$). The $G$-action on $A$ extends to $L$ in the obvious way. Clearly, we have $K \subset L^G$. On the other hand, if $a/b \in L^G$, one can verify that

$$a \cdot \prod_{g\in G\backslash\{e\}} g(b) \in B, \quad \prod_{g\in G} g(b) \in B.$$

Thus $a/b \in K$, and then actually $K = L^G$; so $L/K$ is a Galois extension. This shows that $\pi$ is separable.

Since $\pi$ is finite, $\pi_*\mathscr{O}_X$ is a coherent sheaf on $Y$. Note that the $G$-invariant part is

$$(\pi_*\mathscr{O}_X)^G = \operatorname{Ker}(\pi_*\mathscr{O}_X \overset{(g_1,\dots,g_v)}{\longrightarrow} \prod_{i=1}^{v} \pi_*\mathscr{O}_X),$$

which is coherent on $Y$. Since the natural morphism $\mathscr{O}_Y \to (\pi_*\mathscr{O}_X)^G$ induces an isomorphism of global sections, and $Y$ is itself affine, we see

$$\mathscr{O}_Y \cong (\pi_*\mathscr{O}_X)^G.$$

Now we check both the set-theoretical and the topological properties. Let $x_1, x_2$ be two closed points of $X$ such that $Gx_1 \cap Gx_2 = \emptyset$. By the Chinese remainder theorem, there is $f \in A$ such that $f(gx_1) = 1$ and $f(gx_2) = 0$ for each $g \in G$. Let

$$\phi = \prod_{g\in G} g(f) \in A^G = B,$$

and $\phi(\pi(x_1)) = 1$, $\phi(\pi(x_2)) = 0$. Then $\pi(x_1) \neq \pi(x_2)$. Thus, the equality $Y = X\backslash G$ holds set-theoretically. Again, note that $\pi : X \to Y$ is continuous and finite, and hence a closed map; we see that $Y \approx X/G$ as topological spaces.

We still need to check that when $G$ acts on $X$ freely, the morphism $\pi$ is étale. Fix a closed point $x \in X$ and let $y = \pi(x)$. Let $\mathfrak{m} \in \operatorname{Spec} A$ (resp. $\mathfrak{n} \in \operatorname{Spec} B$) be the maximal

[14]Strictly, it should be written as $Y = G\backslash X$.

ideal that corresponds to the point $x$ (resp. $y$). Let $\widehat{A}$ and $\widehat{B}$ be the $\mathfrak{n}$-adic completions of $A$ and $B$, respectively. Then

$$\widehat{B} = \widehat{\mathscr{O}}_{Y,y}, \quad \widehat{A} \cong \widehat{B} \otimes_B A$$

as $A$ is finite over $B$. Also note that elements in the form $g\mathfrak{m}$ for $g \in G$ are exactly all the prime ideals of $A$ lying over $\mathfrak{n} \in \operatorname{Spec} B$. (In fact, these $g\mathfrak{m}$'s are all distinct as the $G$-action is free.) Using the Chinese remainder theorem, we have

$$\widehat{B} \otimes_B A \cong \widehat{A} \xrightarrow{\cong} \prod_{g \in G} \widehat{\mathscr{O}}_{X,gx}.$$

Since $\mathfrak{n}$ is stable under $G$, the $G$-action on $A$ induces a $G$-action on $\widehat{A}$. Under the isomorphism $\widehat{B} \otimes_B A \cong \widehat{A}$, this action is given by $g(\hat{b} \otimes a) = \hat{b} \otimes g(a)$. The fact that $B = A^G$ can be expressed as the following exact sequence of $B$-modules:

$$\begin{array}{ccccccc} 0 & \longrightarrow & B & \longrightarrow & A & \longrightarrow & \prod_{g \in G} A \\ & & & & a & \longmapsto & (ga - a)_{g \in G}. \end{array}$$

Since $\widehat{B}$ is flat over $B$, we have the exactness of

$$0 \longrightarrow \widehat{B} \longrightarrow \widehat{B} \otimes_B A \longrightarrow \prod_{g \in G} \widehat{B} \otimes_B A.$$

Therefore, $\widehat{B} = \widehat{A}^G$. On the other hand, for any $g \in G$, the action of $g$ on $A$ induces an isomorphism

$$\widehat{\mathscr{O}}_{X,x} \xrightarrow{\cong} \widehat{\mathscr{O}}_{X,gx}.$$

So we can identify the products of them over $g \in G$:

$$\prod_{g \in G} \widehat{\mathscr{O}}_{X,x} \xrightarrow{\cong} \prod_{g \in G} \widehat{\mathscr{O}}_{X,gx}.$$

Again, the freeness condition is essential to infer that

$$\left( \prod_{g \in G} \widehat{\mathscr{O}}_{X,x} \right)^G = \widehat{\mathscr{O}}_{X,x};$$

note that the right hand side is viewed as the diagonal elements in the product. Thus, we finally attain the isomorphism

$$\widehat{\mathscr{O}}_{Y,y} = \widehat{B} \xrightarrow{\cong} \widehat{\mathscr{O}}_{X,x}.$$

□

**Notation 7.4.** We call the pair $(Y, \pi)$ the quotient of $X$ by $G$ and it is denoted by $X/G$ (again, this is strictly $G \backslash X$).

7.2. **Coherent Sheaves under Group Actions.** Now the goal is to study and understand how (coherent) sheaves behave under the group actions.

Let $G, X$ and $(Y, \pi)$ be as before and $\mathscr{F}$ be a coherent sheaf on $Y$. Fix a $g \in G$. From the following commutative diagram

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "\pi"] & & X \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

we obtain an $\mathscr{O}_X$-linear isomorphism[15]

$$\phi_g : g^*(\pi^*\mathscr{F}) \xrightarrow{\cong} \pi^*\mathscr{F}.$$

The isomorphism $\{\phi_g \mid g \in G\}$ satisfy the cocycle condition[16]

$$\phi_{gh} = \phi_h \circ h^*(\phi_g) : (gh)^*(\pi^*\mathscr{F}) \to \pi^*\mathscr{F}.$$

**Definition 7.5.** Let $\mathfrak{g}$ be a coherent sheaf on $X$. Then $\mathfrak{g}$ is called a **coherent $G$-sheaf on $X$** if for each $g \in G$, we have an isomorphism of $\mathscr{O}_X$-modules $\phi_g : g^*\mathfrak{g} \to \mathfrak{g}$ satisfying the above cocycle conditions.

*Remark* 7.6. To understand this sheaf, let us see what happens in the affine case. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and $\mathscr{F}$ a sheaf of $\mathscr{O}_Y$-module that corresponds to a $B$-module $M$. Then

$$\pi^*\mathscr{F} = (M \otimes_B A)^{\sim}.$$

For $g \in G$, we use $g^* : A \to A$ to denote the action of $g$ on $\Gamma(X, \mathscr{O}_X)$. Then we have a map

$$g^* : M \otimes_B A \to M \otimes_B A, \quad m \otimes a \mapsto m \otimes g^*(a).$$

Unfortunately, this is NOT $A$-linear. More precisely, we have

$$g^*((m \otimes a)b) = g^*(m \otimes a) \cdot g^*(b).$$

Then $g^*$ induces an $A$-linear isomorphism

$$\begin{aligned} \phi_g : (M \otimes_B A) \otimes_{A, g^*} A &\longrightarrow M \otimes_B A \\ (m \otimes a) \otimes b &\longmapsto g * (m \otimes a) \cdot b. \end{aligned}$$

Note that the left hand side gives the $A$-module structure.

**Definition 7.7.** For a finitely generated $A$-module $N$, we say that $N$ is a **$(G, A)$-module** if we have an additive map $\psi_g : N \to N$ and $\psi_g(an) = g^*(a)\psi_g(n)$ for all $g \in G$, $a \in A$, and $n \in N$.

The condition in the definition makes $\psi_g$ to be $B$-linear. And it satisfies the cocycle condition of $\phi_g$, read as $\psi_{hg} = \psi_h \circ \psi_g$.

---

[15]This can be equivalently translated to $\phi_g : (g^{-1})^*(\pi^*\mathscr{F}) \to \pi^*\mathscr{F}$ if one prefers to consider the left action. In practice, to make it into a left action, $g$ should be replaced by $g^{-1}$. Possibly it is better to understand this notation via vector bundles.

[16]Similarly, if we use the left action, this becomes $\phi_{gh} = \phi_h \circ (g^{-1})^*(\phi_h) : ((gh)^{-1})^*(\pi^*\mathscr{F}) \to \pi^*\mathscr{F}$.

**Proposition 7.8.** *Let $G$ acts freely on $X$ and $Y = X/G$. Then the functor $\mathscr{F} \mapsto \pi^*\mathscr{F}$ is an equivalence between the category of coherent $\mathscr{O}_Y$-modules and that of coherent $G$-sheaves on $X$, whose inverse in given by $\mathfrak{g} \mapsto \pi_*\mathfrak{g}^G$. The locally free sheaves correspond to the locally free sheaves of the same rank.*

*Proof.* We can reduce to the affine case. Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. We need to show that the functors

$$\begin{aligned} \mathsf{Mod}_B &\longleftrightarrow \mathsf{Mod}_{(G,A)} \\ M &\longmapsto M \otimes_B A \\ N^G &\longleftarrow N \end{aligned}$$

are truly inverses of each other. In other words, we are to prove that, for each $B$-module $M$,

$$S(M) : M \to (M \otimes_B A)^G, \quad m \mapsto m \otimes I,$$

and for each $(G, A)$-module $N$,

$$T(N) : N^G \otimes_B A \to N, \quad n \otimes a \mapsto an$$

are isomorphisms. Since the composite

$$M \otimes_B A \xrightarrow{S(M)\otimes_B A} (M \otimes_B A)^G \otimes_B A \xrightarrow{T(M\otimes_B A)} M \otimes_B A$$

is the identity map and $B \to A$ is faithfully flat, $T(M \otimes_B A)$ is an isomorphism and hence $S(M)$ is an isomorphism. So it suffices to show that $T(N)$ is an isomorphism for all $(G, A)$-module $N$. Regard $T(N) : N^G \otimes_B A \to N$ as a homomorphism of $B$-modules. Then $T(N)$ is an isomorphism if and only if for all maximal ideal $\mathfrak{n}$ of $B$, the localization $T(N)_{\mathfrak{n}} : (N^G \otimes_B A) \otimes_B \widehat{B}_n \to N \otimes_B \widehat{B}$ is an isomorphism.

In the following discussion we write $\widehat{B} = \widehat{B}_n$, $\widehat{A} = A \otimes_B \widehat{B}$, $\widehat{N} = N \otimes_B \widehat{B}$ for simplicity. As we have seen before,

$$\prod_{g\in G} \widehat{\mathscr{O}}_{X,gx} \cong \widehat{A} \cong \prod_{g\in G} \widehat{B}$$

and the $G$-action on $\widehat{A}$ is simply a permutation of the product factors.

Since $N$ is a $(G, A)$-module, $\widehat{N} = N \otimes_A (A \otimes_B \widehat{B})$ is a $(G, \widehat{A})$-module. Under the isomorphism $\widehat{A} \cong \prod_{g\in G} \widehat{B}$, we see that $\widehat{N} \cong \prod_{g\in G} \widehat{N}_1$ for some $\widehat{B}$-module $\widehat{N}_1$ and $G$ acts on $\widehat{N}$ via the permutation of factors. As

$$N^G = \operatorname{Ker}(N \xrightarrow{\psi_g - 1} \prod_{g\in G} N),$$

and $B \to \widehat{B}$ is flat, we have $N^G \otimes_B \widehat{B} \cong (N \otimes_B \widehat{B})^G$. Under the above notations, the morphism $\widehat{T(N)}$ becomes

$$\begin{tikzcd} (N \otimes_B A)^G \otimes_B \widehat{B} \arrow[r, "\cong"] \arrow[d, "\widehat{T(N)}"'] & (N \otimes_B \widehat{B})^G \otimes_{\widehat{B}} \widehat{A} \arrow[r, "\cong"] & \widehat{N}^G \otimes_{\widehat{B}} \widehat{A} \arrow[d] \\ \widehat{N} \arrow[rr, "="] & & \widehat{N} \end{tikzcd}$$

Here the right vertical map is clearly an isomorphism, hence $\widehat{T(N)}$ is an isomorphism. This completes the proof. $\square$

In the following discussion, we assume that $X$ is complete and $G$ acts freely on $X$. Denote

$$\widehat{G} := \mathrm{Hom}_{\mathsf{Grp}}(G, k^*).$$

**Proposition 7.9.** *For all $\alpha \in \widehat{G}$, define*

$$\mathscr{L}_\alpha = \{a \in \pi_* \mathscr{O}_X \mid g(a) = \alpha(g) \cdot a,\ \forall g \in G\}.$$

*Then $\mathscr{L}_\alpha$ is an invertible sheaf on $Y$ and the multiplication in $\pi_*\mathscr{O}_X$ induces an isomorphism*[17]

$$\mathscr{L}_\alpha \otimes \mathscr{L}_\beta \xrightarrow{\sim} \mathscr{L}_{\alpha+\beta}.$$

*The association $\alpha \mapsto \mathscr{L}_\alpha$ defines an isomorphism*

$$\widehat{G} \xrightarrow{\sim} \mathrm{Ker}(\mathrm{Pic}\, Y \to \mathrm{Pic}\, X).$$

*Proof.* It follows from the previous Proposition 7.8 that

$$\mathrm{Ker}(\mathrm{Pic}\, Y \xrightarrow{\pi^*} \mathrm{Pic}\, X) \longleftrightarrow \{\text{coherent } G-\text{sheaf structure on } \mathscr{O}_X\}.$$

Given a $G$-action on the coherent sheaf $\mathscr{O}_X$,[18] for any $g \in G$ and $\Gamma(X, \mathscr{O}_X) \cong k$ say,

$$g : \Gamma(X, \mathscr{O}_X) \to \Gamma(X, \mathscr{O}_X)$$

is determined by $g(1) \in k^*$. We define $\alpha(g) := g(1)^{-1}$. Then $\alpha : G \to k^*$ is a group homomorphism. Conversely, given $\alpha \in G \to k^*$, we define an action of $G$ on $\mathscr{O}_X$ via $g(f) = \alpha^{-1}(g) \cdot f \circ g^{-1}$. Then $g(af) = g(a) \cdot g(f)$ for all $g \in G$, $f \in \mathscr{O}_X$ ($\mathscr{O}_X$ as a coherent sheaf), and $a \in \mathscr{O}_X$ ($\mathscr{O}_X$ as the structure sheaf) such that $g(a) = a \circ g^{-1}$. This makes $\mathscr{O}_X$ a coherent $G$-sheaf.

In this way we establish an isomorphism

$$\mathrm{Ker}(\mathrm{Pic}\, Y \xrightarrow{\pi^*} \mathrm{Pic}\, X) \xrightarrow{\sim} \widehat{G}.$$

Fix $\alpha \in \widehat{G}$, we use $\sigma$ to denote the $G$-action on $\mathscr{O}_X$ corresponding to $\alpha$. This comes from

$$\sigma(g)(f) = \alpha^{-1}(g) \cdot g(f)$$

where $g(f) = f \circ g^{-1}$ is the action of $G$ on the structure sheaf $\mathscr{O}_X$. Then

$$\mathscr{L}_\alpha = (\pi_*\mathscr{O}_X)^\sigma = \{a \in \pi_*\mathscr{O}_X \mid g(a) = \alpha(g) \cdot a\}.$$

For two $\alpha, \beta \in \widehat{G}$, we have a natural map $\mathscr{L}_\alpha \otimes \mathscr{L}_\beta \to \mathscr{L}_{\alpha+\beta}$. For $U \subset Y$ an open subset such that $\mathscr{L}_\alpha|_U$ is trivial. We can find a generating section $f \in \Gamma(U, \mathscr{L}_\alpha) \subset \Gamma(\pi^{-1}(U), \mathscr{O}_X)$, which vanishes nowhere on $\pi^{-1}(U)$. Therefore, $f^{-1}$ is a well-defined nowhere-vanishing section on $\Gamma(\pi^{-1}(U), \mathscr{O}_X)$, and for any $g \in \Gamma(U, \mathscr{L}_{\alpha+\beta})$, $f^{-1}g \in \Gamma(U, \mathscr{L}_\beta)$. Thus, the map is an isomorphism: $\mathscr{L}_\alpha \otimes \mathscr{L}_\beta \cong \mathscr{L}_{\alpha+\beta}$. $\square$

[17] We use Mumford's notation. However, it is better to replace $\mathscr{L}_{\alpha+\beta}$ by $\mathscr{L}_{\alpha\beta}$.

[18] Caution: in case this coincides with the $G$-action on the structure sheaf $\mathscr{O}_X$ but in general they differ from each other.

*Remark* 7.10. If $G$ is commutative and $\mathrm{char}(k) \nmid |G|$, we have the decomposition

$$\pi_* \mathscr{O}_X \cong \bigoplus_{\alpha \in \widehat{G}} \mathscr{L}_\alpha.$$

**Theorem 7.11.** *Let $X$ be an abelian variety. Then there is a one-to-one correspondence between the following two sets of objects:*

1. *finite subgroups $K \subset X$;*[19]
2. *separable isogenies, i.e., finite separable (surjective) homomorphisms $f : X \to Y$, where two isogenies $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ are considered equal if there is an isomorphism $h : Y_1 \to Y_2$ such that $f_2 = h_2 \circ f_1$.*

*Explicitly, these maps are given by $K \mapsto (\pi : X \to X/K)$ and $(f : X \to Y) \mapsto K = \mathrm{Ker}(f)$.*

*Sketchy Proof.* Given a finite subgroup $K \subset X(k)$, $K$ acts on $X$ via translation, and this is a free action. Let $(X/K, \pi)$ be the quotient. The multiplication map $m : X \times X \to X$ induces a morphism $m'$ as

$$\begin{tikzcd}
X \times X \arrow[r, "m"] \arrow[d, "\pi \times \pi"'] & X \arrow[d, "\pi"] \\
X/K \times X/K \arrow[r, "m'"] \arrow[d, equal] & X/K \\
(X \times X)/(K \times K) &
\end{tikzcd}$$

Moreover, $m'$ makes $X/K$ an abelian variety and $\pi : X \to X/K$ is a separable isogeny. Conversely, given a separable isogeny $f : X \to Y$ with $K = \mathrm{Ker}(f)$, the condition that $f$ is separable implies $\#K(k) = \deg(f)$. Let $g : X \to X/K$ be the natural morphism. Then $f$ factors through $g$, i.e.,

$$\begin{tikzcd}
& X \arrow[dl, "g"'] \arrow[dr, "f"] & \\
X/K \arrow[rr, dashed, "h"] & & Y
\end{tikzcd}$$

where $h$ is bijective on points. Note that when $f$ is separable, so also is $h$. So that

$$\deg(h) \cdot \deg(g) = \deg(f).$$

However, on the other hand, $\deg(g) = \deg(f) = \#K(k)$. This forces $\deg(h)$ to be 1, namely, $h$ is birational. Finally, via the Zariski Main Theorem, $h$ is an isomorphism (cf. [Har13, III, Cor 11.4]). $\square$

**Corollary 7.12.** *A separable isogeny $f : X \to Y$ is an étale morphism.*

**Corollary 7.13.** *Let $f : X \to Y$ be an isogeny of order prime to $p = \mathrm{char}(k)$. Then the kernel of $f$ and the kernel of $f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ are dual as finite abelian groups.*

For this, by the previous Theorem 7.11, $f : X \to Y$ can be identified with $f : X \to X/K$ with $\#K$ prime to $\mathrm{char}(k)$. Then one can apply Proposition 7.9 to the morphism $f : X \to X/K$.

[19]Strictly, finite subgroups $K \subset X(k)$.

## 8. The Dual Abelian Variety in Characteristic 0

**Goal.** This section is to construct dual abelian variety over $k$ with $\mathrm{char}(k) = 0$. Being out of tune, the characteristic assumption will be used at the end of our discussion.

Recall that in the theorem of the square (Corollary 6.7) we have defined the map

$$\phi_{\mathscr{L}} : X(k) \to \mathrm{Pic}(X), \quad x \mapsto T_x^*\mathscr{L} \otimes \mathscr{L}^{-1}.$$

**Definition 8.1.** The **principal Picard group** is defined as the subgroup

$$\mathrm{Pic}^0(X) := \{\mathscr{L} \in \mathrm{Pic}(X) \mid \phi_{\mathscr{L}} \equiv 0\}.$$

in $\mathrm{Pic}(X)$.

One can check that the map $\phi_{\mathscr{L}}$ takes values in $\mathrm{Pic}^0(X)$. Moreover, we get an exact sequence of abelian groups

$$\begin{array}{ccccccc} & & & & & & \text{finitely generated free } \mathbb{Z}\text{-module} \\ 0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \boxed{\mathrm{Hom}(X(k), \mathrm{Pic}^0(X))} \\ & & & & \mathscr{L} & \longmapsto & \phi_{\mathscr{L}}. \end{array}$$

In a natural sense one may want to endow $\mathrm{Pic}^0(X)$ with a structure of abelian varieties; after this, it will be shown that $\mathrm{Pic}^0(X)$ is isomorphic to another abelian variety that is called **the dual of** $X$ and denoted by $\widehat{X}$.

### 8.1. General Observations on $\mathrm{Pic}^0(X)$.

(1) *By definition, $\mathscr{L} \in \mathrm{Pic}^0(X)$ if and only if $T_x^*\mathscr{L} \cong \mathscr{L}$ for all $x \in X(k)$. This is also equivalent to say on $X \times X$ that*

$$m^*\mathscr{L} \cong p_1^*\mathscr{L} \otimes p_2^*\mathscr{L}.$$

*Here $m : X \times X \to X$ is the group operation.*

*Proof.* By Seesaw (Corollary 5.13), the sheaf

$$\mathscr{M} := m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^*\mathscr{L}^{-1}$$

is trivial on $X \times X$. Equivalently, for all $x \in X(k)$, $\mathscr{M}|_{X \times \{x\}} \cong T_x^*\mathscr{L} \otimes \mathscr{L}^{-1}$ is trivial and $\mathscr{M}|_{\{e_X\} \times X}$ is trivial (the latter is always true). Therefore, it is to say $\mathscr{L} \in \mathrm{Pic}^0(X)$. □

(2) *If $\mathscr{L} \in \mathrm{Pic}^0(X)$, then for all schemes $S$ and all morphisms $f, g : S \to X$, we have*

$$(f + g)^*\mathscr{L} \cong f^*\mathscr{L} \otimes g^*\mathscr{L}.$$

*Proof.* Pull back the isomorphism $m^*\mathscr{L} \cong p_1^*\mathscr{L} \otimes p_2^*\mathscr{L}$ along the morphism $(f, g) : S \to X \times X$. □

(3) *If $\mathscr{L} \in \mathrm{Pic}^0(X)$ then $n_X^*\mathscr{L} \cong \mathscr{L}^{\otimes n}$.*

*Proof.* Use (2) and the induction on $n$. □

(4) *For any $\mathscr{L} \in \mathrm{Pic}(X)$, we have*

$$n_X^*\mathscr{L} \cong \mathscr{L}^{n^2} \otimes \mathscr{L}_1$$

*with $\mathscr{L}_1 \in \mathrm{Pic}^0(X)$.*

*Proof.* Recall that

$$n_X^*\mathscr{L} \cong \mathscr{L}^{\frac{n^2+n}{2}} \otimes ((-1)_X^*\mathscr{L})^{\frac{n^2-n}{2}} \cong \mathscr{L}^{n^2} \otimes (\mathscr{L} \otimes (-1)_X^*\mathscr{L}^{-1})^{-\frac{n^2-n}{2}}.$$

So it suffices to show that $\mathscr{L} \otimes (-1)_X^*\mathscr{L}^{-1} \in \operatorname{Pic}^0(X)$. For any $x \in X(k)$,

$$\begin{aligned} & T_x^*(\mathscr{L} \otimes (-1)_X^*\mathscr{L}^{-1}) \otimes (\mathscr{L} \otimes (-1)_X^*\mathscr{L}^{-1})^{-1} \\ \cong\ & T_x^*\mathscr{L} \otimes (-1)_X^*T_{-x}^*\mathscr{L}^{-1} \otimes \mathscr{L}^{-1} \otimes (-1)_X^*\mathscr{L} \\ \cong\ & T_x^*\mathscr{L} \otimes \mathscr{L}^{-1} \otimes (-1)_X^*(T_x^*\mathscr{L}^{-1} \otimes \mathscr{L}) \\ \cong\ & T_x^*\mathscr{L} \otimes \mathscr{L}^{-1} \otimes (T_x^*\mathscr{L}^{-1} \otimes \mathscr{L})^{-1} \qquad \text{by (3)} \\ \cong\ & T_x^*\mathscr{L} \otimes T_{-x}^*\mathscr{L} \otimes \mathscr{L}^{-2}. \end{aligned}$$

Hence it is trivial by the theorem of the square (Corollary 6.7). □

(5) *If $\mathscr{L} \in \operatorname{Pic}(X)$ has finite order, then $\mathscr{L} \in \operatorname{Pic}^0(X)$.*

*Proof.* Let $n$ be the order of $\mathscr{L}$, then by definition we have $\phi_{\mathscr{L}^n}(x) \equiv 0$ for each $x \in X(k)$. But also

$$\phi_{\mathscr{L}^n}(x) = \underbrace{\phi_{\mathscr{L}}(x) + \cdots + \phi_{\mathscr{L}}(x)}_{n \text{ terms}} = \phi_{\mathscr{L}}(nx),$$

and $X(k)$ is a divisible group. This forces $\phi_{\mathscr{L}}(x) \equiv 0$ and $\mathscr{L} \in \operatorname{Pic}^0(X)$. □

(6) *For any variety $S$ over $k$ and any line bundle $\mathscr{L}$ on $X \times S$, if we denote $\mathscr{L}_s := \mathscr{L}|_{X\times\{s\}}$ for $s \in S(k)$, then for all $s_0, s_1 \in S(k)$,*

$$\mathscr{L}_{s_1} \otimes \mathscr{L}_{s_0}^{-1} \in \operatorname{Pic}^0(X).$$

*Proof.* Since $S$ is irreducible, the question is local on $S$. So we can assume that $\mathscr{L}|_{\{e_X\}\times S}$ is trivial. Fix $s_0 \in S(k)$. Replacing $\mathscr{L}$ by $\mathscr{L} \otimes p_1^*\mathscr{L}_{s_0}^{-1}$, we can assume that $\mathscr{L}_{s_0}$ is trivial. We need to show that $\mathscr{L}_{s_1} \in \operatorname{Pic}^0(X)$ for each $s_1 \in S(k)$. By (1), it further boils down to show that

$$m^*\mathscr{L}_{s_1} \otimes p_1^*\mathscr{L}_{s_1}^{-1} \otimes p_2^*\mathscr{L}_{s_1}^{-1}$$

is trivial on $X \times X$. We view this as a family of line bundles on $X \times X \times S$. More precisely, define

$$\mathscr{M} := m^*\mathscr{L} \otimes p_{13}^*\mathscr{L}^{-1} \otimes p_{23}^*\mathscr{L}^{-1},$$

where $p_{13}$, $p_{23}$ are the natural projections and

$$m : X \times X \times S \to X \times S, \quad (x, y, s) \mapsto (x +_X y, s).$$

Then for each $s \in S(k)$,

$$\mathscr{M}|_{X\times X\times\{s\}} \cong m^*\mathscr{L}_s \otimes p_1^*\mathscr{L}_s^{-1} \otimes p_2^*\mathscr{L}_s^{-1}.$$

In particular, $\mathscr{M}|_{X\times X\times\{s_0\}}$ is trivial.

On the other hand, since $\mathscr{L}|_{\{e_X\}\times S}$ is trivial, we have $\mathscr{M}|_{\{e_X\}\times X\times S}$ and $\mathscr{M}|_{X\times\{e_X\}\times S}$ both being trivial. By the theorem of the cube (Theorem 6.1), $\mathscr{M}$ is trivial. Thus, $\mathscr{M}|_{X\times X\times\{s_1\}} \cong m^*\mathscr{L}_{s_1} \otimes p_1^*\mathscr{L}_{s_1}^{-1} \otimes p_2^*\mathscr{L}_{s_1}^{-1}$ is trivial and $\mathscr{L}_{s_1} \in \operatorname{Pic}^0(X)$. □

(7) *If $\mathscr{L} \in \operatorname{Pic}^0(X)$ and $\mathscr{L}$ is not trivial, then $H^i(X, \mathscr{L}) = 0$ for all $i \geqslant 0$.*[20]

[20] We will see later that

$$\dim_k H^i(X, \mathscr{O}_X) = \binom{g}{i}, \quad g = \dim_k X.$$

*Proof.* Suppose $H^0(X,\mathscr{L}) \neq 0$, then we can find an effective Weil divisor $D$ such that $\mathscr{L} \cong \mathscr{O}_X(D)$. As $\mathscr{L} \in \mathrm{Pic}^0(X)$, by (3) it turns out that $(-1)_X^*\mathscr{L} \cong \mathscr{L}^{-1}$. Since $(-1)_X^*\mathscr{L} \cong \mathscr{O}_X((-1)_X^*D)$, we get

$$\mathscr{O}_X \cong \mathscr{L} \otimes \mathscr{L}^{-1} \cong \mathscr{O}_X(D + (-1)_X^*D).$$

On the other hand, recall that

$$H^i(X,\mathscr{O}_X) = \begin{cases} k, & i = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the condition $H^0(X,\mathscr{O}_X) = k$ implies $D + (-1)_X^*D = 0$. (This is an equality of divisors rather than divisor classes.) Since $D$ is effective, we have $D = 0$ and then $\mathscr{L} = \mathscr{O}_X(D) = \mathscr{O}_X$. This contradicts to the assumption that $\mathscr{L}$ is not trivial. Then $H^0(X,\mathscr{L}) = 0$.

Assume the claim is not true and there exists $k > 0$ such that $H^k(X,\mathscr{L}) \neq 0$. We may choose $k$ to be the smallest index. Then the morphisms

$$\begin{tikzcd} X \arrow[r, "s_1"] \arrow[rr, bend left, "\mathrm{id}_X"] & X \times X \arrow[r, "m"] & X \\ x \arrow[r, mapsto] & (x, e_X) & \end{tikzcd}$$

which induces morphisms of cohomology groups:

$$\begin{tikzcd} H^k(X,\mathscr{L}) & H^k(X,\times X, m^*\mathscr{L}) \arrow[l, "s_1^*"'] & H^k(X,\mathscr{L}) \arrow[l, "m^*"'] \arrow[ll, bend right, "\mathrm{id}"'] \end{tikzcd}$$

Since $\mathscr{L} \in \mathrm{Pic}^0(X)$, by (1), $m^*\mathscr{L} \cong p_1^*\mathscr{L} \otimes p_2^*\mathscr{L}$. Applying the Künneth formula to $X \times X$, we obtain

$$H^k(X \times X, m^*\mathscr{L}) \cong H^k(X \times X, p_1^*\mathscr{L} \otimes p_2^*\mathscr{L}) \cong \bigoplus_{i+j=k} H^i(X,\mathscr{L}) \otimes H^j(X,\mathscr{L}).$$

Since $H^i(X,\mathscr{L}) = 0$ for all $0 \leqslant i < k$, $H^k(X \times X, m^*\mathscr{L}) = 0$ and therefore $H^k(X,\mathscr{L}) = 0$. This leads to a contradiction. $\square$

8.2. **The Key Theorem.**

**Theorem 8.2.** *Let $\mathscr{L}$ be an ample line bundle on $X$, and $\mathscr{M} \in \mathrm{Pic}^0(X)$. Then there exists $x \in X(k)$ such that*

$$\mathscr{M} \cong T_x^*\mathscr{L} \otimes \mathscr{L}^{-1}.$$

*In other words, the map*

$$\phi_{\mathscr{L}} : X(k) \longrightarrow \mathrm{Pic}^0(X)$$

*is surjective.*

*Proof.* We consider the following line bundle on $X \times X$:

$$K = m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^*(\mathscr{L}^{-1} \otimes \mathscr{M}^{-1}).$$

It enjoys the property that

$$K|_{\{x\}\times X} \cong T_x^*\mathscr{L} \otimes \mathscr{L}^{-1} \otimes \mathscr{M}^{-1}, \quad K|_{X\times\{x\}} \cong T_x^*\mathscr{L} \otimes \mathscr{L}^{-1}.$$

For the two projections $p_1, p_2 : X \times X \to X$, we have the Leray spectral sequences

$$E_2^{p,q} := H^p(X, R^q p_{1,*} K) \Rightarrow H^{p+q}(X \times X, K), \tag{1}$$

and

$$E_2^{p,q} := H^p(X, R^q p_{2,*} K) \Rightarrow H^{p+q}(X \times X, K). \tag{2}$$

May assume that the statement in the theorem does not hold, i.e., $T_x^*\mathscr{L} \otimes \mathscr{L}^{-1} \ncong \mathscr{M}$ for all $x \in X(k)$. Then $K|_{\{x\}\times X}$ is a nonzero element in $\mathrm{Pic}^0(X)$. By (7), considering an arbitrary fiber $\{x\} \times X$ of $p_1$,

$$H^q(\{x\} \times X, K|_{\{x\}\times X}) = 0, \quad q \geqslant 0.$$

By Corollary 5.7 (2), $R^q p_{1,*} K = 0$ while restricting to $X \backslash K(\mathscr{L})$. Therefore $H^n(X \times X, K) = 0$ by spectral sequence (1) for all $n \geqslant 0$.

On the other hand, $K|_{X\times\{x\}} \cong T_x^*\mathscr{L} \otimes \mathscr{L}^{-1}$ is a nonzero element in $\mathrm{Pic}^0(X)$ if and only if $x \notin K(\mathscr{L})$, which is a finite closed subgroup of $X(k)$. For $x \in X(k)\backslash K(\mathscr{L})$, by (7) again, we have

$$H^q(X \times \{x\}, K|_{X\times\{x\}}) = 0, \quad q \geqslant 0.$$

Similarly, $R^q p_{2,*} K = 0$ while restricting to $X \backslash K(\mathscr{L})$. Hence

$$\mathrm{Supp}(R^q p_{2,*} K) \subset K(\mathscr{L}).$$

For a coherent sheaf $\mathscr{F}$ on $X$ with $\mathrm{Supp}(\mathscr{F}) \subset K(\mathscr{L})$, we have

$$H^p(X, \mathscr{F}) = 0, \quad p \geqslant 1.$$ [21]

By the spectral sequence (2), we have for all $n$ that

$$H^0(X, R^n p_{2,*} K) = \bigoplus_{x \in K(\mathscr{L})} (R^n p_{2,*} K)_x \cong H^n(X \times X, K).$$

But $H^n(X \times X, K) = 0$ for all $n$ and hence $(R^n p_{2,*} K)_x = 0$. This shows that $R^n p_{2,*} K$ vanishes for all $n$. By Corollary 5.11 before, it turns out that

$$H^n(X \times \{x\}, K_{X\times\{x\}}) = 0, \quad n \geqslant 0.$$

In particular, we take $n = 0$ and $x = e_X$ to get

$$H^0(X, \mathscr{O}_X) = 0.$$

But this is a contradiction. □

[21] There are several ways to see this. For example, if we use $U$ to denote the open subvariety $X \backslash K(\mathscr{L})$, then $\mathscr{F}|_U = 0$ and so for all $p \geqslant 0$, $H^p_{K(\mathscr{L})}(X, \mathscr{F}) \cong H^0(X, \mathscr{F})$. Then use excision to reduce to the case for $X$ affine. Then $H^p_{K(\mathscr{L})}(X, \mathscr{F}) \cong H^0(X, \mathscr{F}) = 0$ for all $p \geqslant 1$.

8.3. **Characterizing the Dual Abelian Variety.** The key theorem implies that as an abstract group, $\mathrm{Pic}^0(X)$ is isomorphic to $X(k)/K(\mathscr{L})$, and we can endow $\mathrm{Pic}^0(X)$ with an abelian variety structure, and is called the dual abelian variety of $X$, denoted by $\widehat{X}$. But this definition depends on the choice of $\mathscr{L}$. How to characterize $\widehat{X}$? The following answer can be interpreted as the definition of the dual abelian variety in characteristic 0.

**Theorem 8.3.** *The* ***dual abelian variety*** $\widehat{X}$ *of* $X$ *is an abelian variety* $\widehat{X}$ *with an isomorphism of groups* $\widehat{X}(k) \cong \mathrm{Pic}^0(X)$ *satisfying the following conditions.*

(1) *There is a line bundle $P$ on $X \times \widehat{X}$ called the Poincaré bundle, such that for any $\alpha \in \widehat{X}(k)$, the line bundle $P|_{X\times\{\alpha\}}$ represents the line bundle in $\mathrm{Pic}^0(X)$ given by $\alpha$ under the above isomorphism $\widehat{X}(k) \cong \mathrm{Pic}^0(X)$; moreover, $P|_{\{e_X\}\times X}$ is trivial.*[22]
(2) *For every normal variety $S$ and a line bundle $K$ on $X \times S$, suppose that*
   (i) *$K_s := K|_{X\times\{s\}} \in \mathrm{Pic}^0(X)$ for one $s \in S$ – and hence for all by (6);*
   (ii) *$K|_{\{e_X\}\times S}$ is trivial.*

   *Then the map of sets $f : S \to \widehat{X}$ satisfying $K_s \cong P_{f(s)}$ is a morphism of varieties and $K \cong (\mathrm{id}_X \times f)^*P$.*

*Proof.* The above two properties imply that $(\widehat{X}, P)$, if it exists, is unique up to canonical isomorphism. Fix an ample line bundle on $X$. We define $\widehat{X}$ to be the quotient $X/K(\mathscr{L})$ and let $\pi : X \to \widehat{X}$ be the natural morphism. We apply property (2) to the line bundle $K = m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^*\mathscr{L}^{-1}$ on $X \times X$, we see that the Poincaré bundle $P$ on $X \times \widehat{X}$ satisfies the property that $(\mathrm{id}_X \times \pi)^*P \cong K$ on $X \times X$. Hence it suffices to define an $\{e_X\} \times K(\mathscr{L})$-action on $K$ hat is compatible with its action on $X \times X$. For simplicity, we use $T_{0,a}$ to denote the translation

$$T_{0,a} : X \times X \to X \times X, \quad (x, y) \mapsto (x, y + a).$$

By a direct computation we obtain

$$\begin{aligned} T_{0,a}^*K &\cong T_{0,a}^*m^*\mathscr{L} \otimes T_{0,a}^*p_1^*\mathscr{L}^{-1} \otimes T_{0,a}^*p_2^*\mathscr{L}^{-1} \\ &\cong m^*T_a^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^*T_a^*\mathscr{L}^{-1} \\ &\cong m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^*\mathscr{L}^{-1} = K. \end{aligned}$$

This induces an isomorphism $\phi_a : T_{0,a}^*K \xrightarrow{\sim} K$, once the isomorphism $T_a^*\mathscr{L} \cong \mathscr{L}$ is fixed. But we still need to choose $\phi_a$'s carefully to make the qualities $\phi_{a+b} = \phi_a \circ T_{0,a}^*(\phi_b)$ hold for all $a, b \in K(\mathscr{L})$.

Since $X$ is complete, if $\mathscr{L}_1, \mathscr{L}_2$ are two line bundles on $X \times X$ and $\phi, \psi : \mathscr{L}_1 \to \mathscr{L}_2$ are two isomorphisms, then $\phi$ and $\psi$ are differed by a scalar in $k^\times$. To remedy this, we must kill the ambiguity of this scalar. Consider the restriction $K|_{\{e_X\}\times X}$. If we denote $\mathscr{L}^{-1}(0)$ for $e_X^*(\mathscr{L}^{-1})$ on $\mathrm{Spec}\, k$, then $K|_{\{e_X\}\times X} \cong p_1^*(\mathscr{L}^{-1}(0))$.

Fix such an isomorphism, we get a canonical isomorphism

$$\psi_a : T_a^*(K|_{\{e_X\}\times X}) \to K|_{\{e_X\}\times X}.$$

We require that $\phi_a|_{\{e_X\}\times X} = \psi_a$ for all $a \in K(\mathscr{L})$. Then we get a well-defined $\{e_X\} \times X$-action on $K$ and hence obtain a line bundle $P$ on $X \times \widehat{X}$ with $(\mathrm{id}_X \times \pi)^*P \cong K$.

[22]Note that by Seesaw's theorem, these properties characterize $P$.

Let us verify the pair $(X \times \widehat{X}, P)$ satisfies (1) and (2). For $\alpha \in \widehat{X}$, $\alpha = \pi(x)$ for some $x \in X(k)$. Then

$$P_\alpha = P|_{X\times\{\alpha\}} \cong \pi^*(P)|_{X\times\{x\}} \cong T_x^*\mathscr{L} \otimes \mathscr{L}^{-1} = \phi_{\mathscr{L}}(x) \in \operatorname{Pic}^0(X).$$

Since $K|_{\{e_X\}\times X} \cong p_1^*(\mathscr{L}^{-1}(0))$, by our construction above, $P|_{\{e_X\}\times X}$ is trivial. This proves (1). As for (2), we consider the line bundle $E = p_{12}^*(K) \otimes p_{13}^*(P^{-1})$ on $X \times S \times \widehat{X}$. Thus,

$$E|_{X\times\{s,\alpha\}} \cong K|_{X\times\{s\}} \times P^{-1}|_{X\times\{\alpha\}}.$$

By Seesaw, the set

$$\Gamma = \{(s,\alpha) \mid E|_{X\times\{\alpha,s\}} \text{ is trivial}\}$$

is Zariski closed in $S\times\widehat{X}$. We see $\Gamma(k)$ is the set-theoretic graph of the map $f : S(k) \to \widehat{X}(k)$ such that $K_s \cong P_{f(s)}$. Then the first projection $\Gamma \to S$ is bijective on the closed points.

Now we use the assumption that $\operatorname{char}(k) = 0$, which infers that $\Gamma \to S$ is birational. Since

$$\Gamma \to \widehat{X} \times S \to S$$

is proper, quasi-finite, and birational, and $S$ is normal, we see $\Gamma \to S$ is an isomorphism. $\Gamma$ induces a morphism of algebraic varieties $f : S \to \widehat{X}$ and by Seesaw again, $(\mathrm{id}_X \times f)^*(P) \cong K$ as desired. □

*Remark* 8.4. (1) For any $\mathscr{L} \in \operatorname{Pic}(X)$ the map $\phi_{\mathscr{L}} : X(k) \to \widehat{X}(k)$ is a morphism of varieties. To see this, apply property (2) in Theorem 8.3 to the line bundle

$$K = m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^*\mathscr{L}^{-1}$$

on $X \times X$.

(2) Let $f : X \to Y$ be a homomorphism of abelian varieties. The map $f^* : \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$ maps $\operatorname{Pic}^0(Y)$ to $\operatorname{Pic}^0(X)$ (check this), and induces a map

$$\widehat{f}(k) : \widehat{Y}(k) \to \widehat{X}(k).$$

I claim that this is a morphism. In fact, if we use $Q$ to denote the Poincaré bundle on $Y \times \widehat{Y}$ and let $Q' = (f \times \mathrm{id}_{\widehat{Y}})^*Q \in \operatorname{Pic}(X \times \widehat{Y})$, then

$$Q'|_{X\times\{\widehat{y}\}} \cong f^*(Q|_{Y\times\{\widehat{y}\}}) \in \operatorname{Pic}^0(X)$$

for all $\widehat{y} \in \widehat{Y}(k)$, and $Q'|_{\{e_X\}\times\widehat{Y}} \cong Q|_{\{e_Y\}\times\widehat{Y}}$ is trivial. Then there is a morphism $\widehat{f} : \widehat{Y} \to \widehat{X}$ such that $Q' \cong (\mathrm{id}_X \times \widehat{f})^*P$, where $P$ is the Poincaré bundle on $X \times \widehat{X}$. The morphism of $\widehat{f}$ on $k$-points is just the $\widehat{f}(k)$ defined above.

(3) If $f : X \to Y$ is an isogeny, so is $\widehat{f} : \widehat{Y} \to \widehat{X}$; and there exists a canonical duality of abelian groups between $\operatorname{Ker}(f)$ and $\operatorname{Ker}(\widehat{f})$. In fact, by Proposition 7.9 and Corollary 7.13, we have a canonical duality between $\operatorname{Ker}(f)$ and $\operatorname{Ker}(f^* : \operatorname{Pic}^0(Y) \to \operatorname{Pic}^0(X))$. Since $\operatorname{Ker}(f^* : \operatorname{Pic}(Y) \to \operatorname{Pic}(X))$ is finite, any $\mathscr{L}$ in $\operatorname{Ker}(f^*)$ is of finite order. Thus, $\mathscr{L} \in \operatorname{Pic}^0(Y)$ and $\operatorname{Ker}(f^*) = (\operatorname{Ker}(\widehat{f}))(k)$. By the dimension argument, we see $\widehat{f}$ is also an isogeny.

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School of Mathematical Sciences, Peking University, 100871, Beijing, China
*Email address*: daiwenhan@pku.edu.cn