Geometry of Bun_G

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§1 Introduction of Bun_G

Setups E = non-arch local field, $\pi \in \mathcal{O}_E$ uniformizer.

$\mathbb{F}_q$ = residue field $= \mathcal{O}_E/(\pi)$.

Fix $k = \overline{\mathbb{F}}_q$ alg closure.

$\rightsquigarrow \breve{E} = W_{\mathcal{O}_E}(k)[\frac{1}{\pi}]$ $/E$, G/E reductive grp.

Def'n The Bun_G is a functor

$$\mathrm{Bun}_G : \mathrm{Perf}_k^{op} \longrightarrow \text{Groupoid}$$
$$S \longmapsto \{\underbrace{G\text{-torsors}}\ \text{on } X_S\}$$

X_S = rel FF curve; sous perf'd space

i.e. structure presheaves are sheaves,

closed under étale cover, product with unit balls.

Def'n X sousperf'd space. A G-torsor on X is

a tensor-exact functor $\underbrace{\mathrm{Rep}_E G \longrightarrow \mathrm{Bun}(X)}_{\text{Tannakian formalism}}$ = cat of vb on X.

Prop As above, TFAE:

(1) The category of G-torsors on X

(2) $T \longrightarrow X$, T = adic space, s.t. étale locally on X, $T \simeq G \times X$.

$\circlearrowleft$ G-action $\circlearrowleft$

(3) $\mathcal{E} \longrightarrow X$, $\mathcal{E}$ = étale sheaf over X, étale locally $\boxed{\mathcal{E} \simeq \underline{G}}$.

$\mathcal{E} \mapsto (V \mapsto \mathcal{E} \overset{G}{\times} V = \mathcal{E} \times V / G)$.

Consider $\mathcal{O}_G \in \mathrm{Ind}\,\mathrm{Rep}_E G$. $F : \mathrm{Rep}_E G \longrightarrow \mathrm{Bun}(X)$ if $X = \mathrm{Spa}(R, R^+)$ affinoid

def'd via $F(\mathcal{O}_G) \in \mathrm{Ind}\,\mathrm{Bun}(X) \rightsquigarrow \mathrm{Alg}(\mathrm{QCoh}(\mathrm{Spec}\,R))$.

<u>Idea</u> "$\mathrm{Spec}\,F(\mathcal{O}_G)^{an}$" $= T$, relative analytification.

<u>Fact</u> Bun_G is a small v-stack.

Hopefully, we want a topology on $|\mathrm{Bun}_G|$.

<u>Thm</u> We have the classification

$$|\mathrm{Bun}_G| = \mathrm{Bun}_G(C)/\sim \;= B(G) = \text{isom classes of } \boxed{G\text{-isocrystals}}.$$

i.e. exact $\otimes$-functor $\mathrm{Rep}_E G \to \mathrm{Isoc}_E$

Now $(D, \varphi) \in \mathrm{Isoc}_E \rightsquigarrow \mathcal{E}(D) \in \mathrm{Bun}(X)$ on X_S, for any $S \in \mathrm{Perf}_k$.

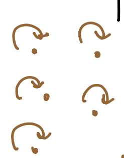
$\rightsquigarrow$ $G\text{-}\mathrm{Isoc}_E \longrightarrow \mathrm{Bun}_G(S)$, $\forall S \in \mathrm{Perf}_k$.

e.g. $\mathrm{Bun}_{GL_n}(C)/\sim$ is classified by the Newton polygons.

<u>Rmk</u> G-isocrystals can be viewed as G-torsors on $\mathrm{Spec}\,\breve{E}/\varphi_{\breve{E}} \longleftrightarrow \sigma$

Then $B(G) = G(\breve{E})/b \sim g b \sigma(g)^{-1}$

b/c $H^1(\mathrm{Spec}\,\breve{E}, G) = 0$ (Steinberg).

<u>Picture</u>  $\xleftrightarrow{\text{bijection}}$ isocrystals $\rightsquigarrow$ to study automorphisms on Bun_G.

Bun_G-stack (but not cat equiv)

<u>§2 Structure of $B(G)$</u> (Kottwitz)

$\bar{E} \supseteq E$ separable closure.

$T \subseteq B \subseteq G_{\bar{E}} \rightsquigarrow X_*(T) \circlearrowleft \Gamma = \mathrm{Gal}(\bar{E}/E)$.

<u>Thm</u> (1) $\nu : B(G) \longrightarrow (X_*(T)^+_{\mathbb{Q}})^{\Gamma}$ Newton point

$\kappa : B(G) \longrightarrow \pi_1(G)_\Gamma$, $\pi_1(G) = \underset{\circlearrowleft\,\Gamma}{X_*(T)}$ / coinvariant lattice (e.g. $\pi_1(GL_n) \cong \mathbb{Z}$).

are maps that are functorial in G.

(2) Moreover, $\exists$ an injection
$$B(G) \xhookrightarrow[\text{injection}]{(\nu,\kappa)} (X_*(T)^+_{\mathbb{Q}})^\Gamma \times \pi_1(G)_\Gamma .$$

(3) ν is determined by
$$B(GL_n) \longrightarrow X_*(T)^+_{\mathbb{Q}} \qquad \text{Newton polygons.}$$

e.g. $G = GL_2$: $D(\text{slope} = \frac{1}{2}) \longmapsto (\frac{1}{2}, \frac{1}{2})$ for isocrystals

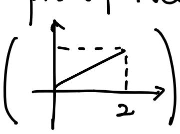
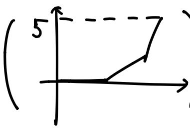
(4) κ is determined by choosing
$$B(G) \simeq X_*(T)_\Gamma \cong \pi_1(T)_\Gamma .$$

(5) Compatibility:
$$\begin{tikzcd} & (X_*(T)^+_{\mathbb{Q}})^\Gamma \arrow[dr] \arrow[d, phantom, "\circlearrowright"] & \\ B(G) \arrow[ur, "\nu"] \arrow[r, "\kappa"'] & \pi_1(G)^\Gamma \arrow[r] & \pi_1(G)^\Gamma_{\mathbb{Q}} \end{tikzcd}$$

replace Γ by finite quotient averaging by $\frac{1}{|\Gamma|}\sum_{g\in\Gamma} g$.

If $\pi_1(G)_\Gamma$ is torsion-free, can determine κ by ν.

E.g. $G = GL_n$, κ of pts of Newton polygon are

$\kappa\Big($  $\Big) = 1$, $\quad \kappa\Big($  $\Big) = 5$.

note $(X_*(T)^+_{\mathbb{Q}})^\Gamma$ is ordered

$\rightsquigarrow$ order on $B(G)$: $b < b' \iff \nu_b < \nu_{b'}$, $\kappa(b) = \kappa(b')$.

Thm $|\mathrm{Bun}_G| \simeq B(G)$ homeomorphism

$\iff$ $b < b'$ $\iff$ $\mathcal{E}_b, \mathcal{E}_{b'} \in |\mathrm{Bun}_G|$, $\exists\ \mathcal{E}_{b'} \longrightarrow \mathcal{E}_b$ specialization

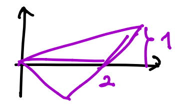
Define the basic locus
$$B(G)_{bas} := \{\text{min'l } b \text{ in } B(G)\} \quad (\longleftrightarrow \mathcal{E}_b \text{ semistable } G\text{-bundles})$$
$$\Rightarrow \quad \pi_0(\mathrm{Bun}(G)) \simeq B(G)_{bas} \xrightarrow[\kappa]{\sim} \pi_1(G)_\Gamma .$$

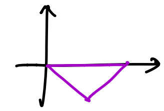
Example $G = GL_n$, $\pi_1(G) \simeq \mathbb{Z}$, $\pi_0(|\mathrm{Bun}_G|) \xrightarrow{\kappa} \mathbb{Z}$.

$2\rho = (1,-1)$

GL_2
- deg 2: $\cdot\ \mathcal{O}(1)\oplus\mathcal{O}(1) \rightsquigarrow \cdots$
- deg 1: $\cdot\ \mathcal{O}(\tfrac{1}{2}) \rightsquigarrow \cdot\ \underset{\dim 1}{\overset{(0,-1)}{\boxed{\mathcal{O}\oplus\mathcal{O}(1)}}} \rightsquigarrow \cdot\ \underset{\dim 3}{\mathcal{O}(-1)\oplus\mathcal{O}(2)}$
- deg 0: $\cdot\ \mathcal{O}\oplus\mathcal{O} \rightsquigarrow \cdot\ \underset{\dim 2}{\mathcal{O}(-1)\oplus\mathcal{O}(1)} \rightsquigarrow \cdot\ \underset{\dim 4}{\mathcal{O}(-2)\oplus\mathcal{O}(2)}$
- deg -1: $\cdot\ \mathcal{O}(-\tfrac{1}{2}) \rightsquigarrow \cdots$

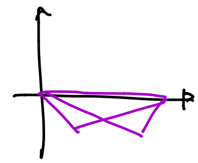
$\mathrm{Aut}(\mathcal{O}\oplus\mathcal{O}(1)) = \begin{cases} E^\times, & \dim \mathcal{BC}(\mathcal{O}(1)) = 1 \end{cases}$





$2\rho = (2,0,-2)$

GL_3
- deg 0: $\mathcal{O}\oplus\mathcal{O} \rightsquigarrow \cdot\ \overset{(\frac{1}{2},\frac{1}{2},-1)}{\mathcal{O}(-\tfrac{1}{2})\oplus\mathcal{O}(1)}$ dim3 $\rightsquigarrow \cdot\ \mathcal{O}(-1)\oplus\mathcal{O}\oplus\mathcal{O}(1)$; $\mathcal{O}\oplus\mathcal{O} \rightsquigarrow \cdot\ \mathcal{O}(-1)\oplus\mathcal{O}(\tfrac{1}{2}) \rightsquigarrow \cdot\ \mathcal{O}(-1)\oplus\mathcal{O}\oplus\mathcal{O}(1)$
- deg -1: $\cdot\ \mathcal{O}(-\tfrac{1}{3}) \rightsquigarrow \cdot\ \mathcal{O}(-\tfrac{1}{2})\oplus\mathcal{O} \rightsquigarrow \cdots$



Over $S = \mathrm{Spa}(C, C^+)$, $\dim(\mathcal{BC}(\mathcal{O}(\tfrac{r}{s}))) = r > 0$.

Thm $\mathrm{Bun}_G^{ss} = \coprod_{b\in B(G)_{bas}} \mathrm{Bun}_G^b$, where $\mathrm{Bun}_G^b = [*/G_b(E)]$.

$\updownarrow$

$b \in B(G)_{bas}$

For $b \in B(G)$, $b \in G(\breve{E})$, and

$$G_b(R) := \{ g \in G(R \otimes_E \breve{E}) \mid g = b\sigma(g)b^{-1} \}.$$

$\rightsquigarrow$
- G_b conn reductive grp when $b \in B(G)_{bas}$
- G_b inner form of G.
- G quasi-split $\Rightarrow$ b nonbasic, G_b inner form of some Levi subgrp.
- $\forall G$ red, $\exists!\ b \in B(G)_{bas}$ s.t. G_b quasi-split

Have $G\text{-Isoc} = \coprod_{b\in B(G)} [*/G_b(E)] \longrightarrow \mathrm{Bun}_G(S)$

$b \longmapsto \mathcal{E}_b$

If $b \in B(G)\setminus B(G)_{bas}$, then $\mathrm{Bun}_G^b \simeq [*/\tilde{G}_b]$,

$$\tilde{G}_b = \underbrace{\boxed{\tilde{G}_b^{>0}}}_{\text{unipotent grp diamond.}} \rtimes G_b(E)$$

successive ext'n of positive BC space with $\dim \tilde{G}_b = \langle 2\rho, \nu_b \rangle$.

Prop (1) $\underline{G} \in \mathrm{Bun}_G$ trivial G-torsors.

(2) $\forall S \in \mathrm{Perf}_k$, $\mathrm{Aut}(\mathcal{G})(S) = G(X_S) = G(\mathcal{O}_{X_S}(X_S)) = G(\underline{E}(S)) = \underline{G(E)}(S)$.

(3) For $\mathcal{G} \in [*/G]$, $\mathrm{Aut}(\mathcal{G}) = G$,

e.g. $\mathrm{Aut}(\mathcal{O}(\tfrac{1}{2}) \oplus \mathcal{O}(1)) = \begin{pmatrix} D^\times & \mathrm{Hom}(\mathcal{O}(\frac{1}{2}), \mathcal{O}(1)) \\ 0 & \underline{E}^\times \end{pmatrix}$

<u>Rmk</u> Diamonds $= \left\{ \begin{array}{l} \text{proétale sheaves } X/\mathrm{Perf}_k \text{ s.t.} \\ \exists \text{ perf'd } S \twoheadrightarrow X \text{ quasi-pro-étale} \end{array} \right\}$

Small v-sheaves $= \left\{ \begin{array}{l} \text{v-sheaves } X/\mathrm{Perf}_k \text{ s.t.} \\ \exists \text{ perf'd } S \twoheadrightarrow X \text{ v-cover} \end{array} \right\}$

reduce the basis for perf'd to strictly tot discom perf'd space S.

<u>Def'n</u> Std perf'd = qcqs & every étale cover splits

($\Rightarrow$ pro-étale $\underline{G(E)}$-torsors on S is trivial)

Proof of the main thm.

<u>Proof</u> $T \xrightarrow{G(E)} S$, $K \subseteq G(E)$ open compact, $\underline{K} \backslash T \to S$ étale cover.

It reduces to consider $\underline{K}$-torsor $T = \varprojlim_{U} U \backslash T$, $U \backslash T \to S$ étale

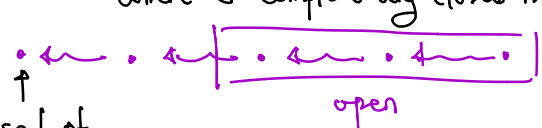
open subgrp of K.

Let S be std perf'd.

$\rightsquigarrow |S| \twoheadrightarrow \pi_0(S)$ (profinite grp)

every fiber is of the form $\mathrm{Spa}(C, C^+)$

where C = complete alg closed non-arch field.



total ordered

$\exists!$ closed pt.

Say $\mathrm{Def}(x)$ is glued from

To prove $|\mathrm{Bun}_G| \simeq B(G)$, we need

$\nu : |\mathrm{Bun}_G| \longrightarrow (X_*(T)^+_{\mathbb{Q}})^{\Gamma}$ semi-conti,

to reduce to vbs.

<u>Want</u> $\kappa : |\mathrm{Bun}_G| \longrightarrow \pi_1(G)_\Gamma$ locally constant.

If G is induced torus, $\pi_1(G)_\Gamma$ will be torsion free and determined by ν.

Need $\ker(G' \twoheadrightarrow G)$ central torus

So
$$\begin{tikzcd} \mathrm{Bun}_{G'} \arrow[r] \arrow[d, "\kappa"'] \arrow[dr] & \mathrm{Bun}_G \arrow[d, "\kappa"] \\ \pi_1(G')_\Gamma \arrow[r, two heads] & \boxed{\pi_1(G)_\Gamma} \end{tikzcd}$$
$\rightsquigarrow |\mathrm{Bun}_{G'}| \twoheadrightarrow |\mathrm{Bun}_G|$ with quotient top.

discrete top.

$\Rightarrow$ can reduce G to G'.

<u>Idea</u> (1) $\mathrm{Gr}_G \longrightarrow \mathrm{Bun}_G$

B^+_{dR} - affine Grassmanian.

(2) $G' \twoheadrightarrow G$ as above. Then
$$\begin{tikzcd} \mathrm{Gr}_{G'} \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\Downarrow"] & \mathrm{Gr}_G \arrow[d, two heads] \\ \mathrm{Bun}_{G'} \arrow[r, two heads] & \mathrm{Bun}_G \end{tikzcd}$$

To continue the proof, need the thm:

Let $\mathrm{Bun}_G^1 \subseteq \mathrm{Bun}_G$ substack parametrizing geom fiberwise trivial G-bundles.

<u>Thm</u> ([FS], III.2.4) $\mathrm{Bun}_G^1 \subset \mathrm{Bun}_G$ open, and

$$[*/\underline{G(E)}] \xrightarrow{\sim} \mathrm{Bun}_G^1.$$

<u>Proof</u> Need $S \in \mathrm{Perf}_k$ qcqs, std perf'd, $(S \longrightarrow \mathrm{Bun}_G) \longleftrightarrow \mathcal{E}$

- Subset of $|S|$ s.t. $\mathcal{E}$ trivial is actually open in $|S|$.

<u>Fact</u> The locus where the Newton pt is identically zero. is an open subset of $|S|$

$\rightsquigarrow$ finite free $C^0(|S|, E)$-module.

Fact $\varinjlim_{U \ni S} C^0(|S|, E)$ Henselian.

$\Rightarrow$ it is G-torsor iff it is trivial at some pt of S

($\Rightarrow$ $\underbrace{\text{iff it is trivial in a nbhd}}_{\text{the open condition.}}$)

For $\mathrm{Bun}_G^1 \simeq [*/\underline{G(E)}]$, need $* \longrightarrow \mathrm{Bun}_G^1$ (v-cover)

$\forall \mathcal{E} \in \mathrm{Bun}_G^1(S)$, to find cover of S s.t. $\mathcal{E}$ trivial.

Need $* \twoheadrightarrow \mathrm{Bun}_G^b$ for any $b \in B(G)$. (To be finished). $\square$

§3 Non-semistable points

$b \in B(G) \longleftrightarrow \mathcal{E}_b \in \mathrm{Bun}_G$, $\widetilde{G}_b = \mathrm{Aut}(\mathcal{E}_b)$

with HN fil'n for any $\rho \in \mathrm{Rep}_E G$.

$\rightsquigarrow \widetilde{G}_b^{\geq \lambda} \subseteq \widetilde{G}_b$, with $\widetilde{G}_b^{>\lambda} = \bigcup_{\lambda' > \lambda} \widetilde{G}_b^{\geq \lambda'}$.

$\rightsquigarrow \underline{G_b(E)} \hookrightarrow \widetilde{G}_b = \underline{G_b(E)} \ltimes \widetilde{G}_b^{>0}$.

Prop $\exists$ natural isom $\widetilde{G}_b^{\geq\lambda}/\widetilde{G}_b^{>\lambda} \xrightarrow{\sim} \mathcal{BC}(\underbrace{(\mathrm{ad}\,\mathcal{E}_b)}^{\geq\lambda}/(\mathrm{ad}\,\mathcal{E}_b)^{>\lambda})$

where $\mathrm{ad}: G \longrightarrow GL(\mathrm{Lie}(G)) = GL(\mathfrak{g})$.

$(-\lambda)$-isoclinic part of $(\mathrm{Lie}(G) \otimes_E \breve{E}, \mathrm{Ad}(b)\sigma)$.

Proof Take $S = \mathrm{Spa}(R, R^+)$, X_R^{alg}, with G-bundle $\mathcal{E}_b / X_R^{alg}$.

H / X_R^{alg} inner twisting of $G \times X_R^{alg}$ by $\mathcal{E}_b$ $\rightsquigarrow (H^{\geq\lambda})_{\lambda \geq 0}$

- $H^{\geq 0}/H^{>0} \simeq G_b \times_E X_R^{alg}$
- $\lambda > 0$, $H^{\geq\lambda}/H^{>\lambda} = (\mathrm{ad}\,\mathcal{E}_b)^{\geq\lambda}/(\mathrm{ad}\,\mathcal{E}_b)^{>\lambda}$
- $\widetilde{G}_b^{\geq\lambda}(S) = H^{\geq\lambda}(X_R^{alg})$.

$H^{\geq\lambda}(X_R^{alg}) / H^{>\lambda}(X_R^{alg}) = (H^{\geq\lambda}/H^{>\lambda})(X_R^{alg})$ as $H^1_{\text{ét}}(X_R^{alg}, H^{>\mu}) = 0$.

$$\dim \widehat{G}_b = \sum_{\lambda} \dim \mathcal{BC}\left((\mathrm{ad}\, \xi_b)^{\geq \lambda} / (\mathrm{ad}\, \xi_b)^{> \lambda} \right)$$

$$= \sum_{\lambda} \lambda \cdot \dim (\mathrm{ad}\, \xi_b)^{\geq \lambda} / (\mathrm{ad}\, \xi_b)^{> \lambda}$$

$$= \sum_{\alpha \text{ positive}} \langle \alpha, \nu_b \rangle \quad \Leftarrow \quad \nu_b \in (X_*^+(T)_{\mathbb{Q}})^{\Gamma}. \quad G \xrightarrow{\mathrm{ad}} GL(\mathrm{Lie}\, G)$$

$$= \langle 2\rho, \nu_b \rangle \qquad \square \qquad \nu_b \longmapsto \begin{pmatrix} t^{\langle \alpha, \nu_b \rangle} & & \\ & 1 & \\ & & t^{\langle -\alpha, \nu_b \rangle} \end{pmatrix}.$$

Rmk $\mathcal{BC}(\xi(\alpha, \nu_b))$ will appear in $\widetilde{G}_b^{>0}$.

Can calculate $\langle \alpha, \nu_b \rangle$ for all $\alpha \in \Phi^+$.