

## Hilbert Polynomials and Flatness

See Hartshorne III.9 again.

### §1 Hilbert Polynomials

$k$  field (not necessarily  $k = \bar{k}$ ).  $j: X \rightarrow \mathbb{P}_k^r$  closed imm ( $r \geq 1$ ).

Write  $\mathcal{O}_{X(i)} = j^* \mathcal{O}_{\mathbb{P}_k^r}(i)$ .  $\mathcal{F} \in \mathbf{Qcoh}(X)$  f.g.

Def'n Euler char of  $\mathcal{F}$  is

$$\chi(X, \mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i \lim_{\leftarrow} H^i(X, \mathcal{F}).$$

finite ( $\leq r$ )

(by Serre's finiteness thm).

Lemma  $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$  short exact

$$\Rightarrow \chi(\mathcal{G}) = \chi(\mathcal{F}) + \chi(\mathcal{H}).$$

(use the  $H^i(\cdot)$ -long exact sequence to prove).

Cor  $0 \rightarrow \mathcal{F}_1 \rightarrow \dots \rightarrow \mathcal{F}_n \rightarrow 0$  exact of f.g.  $\mathbf{Qcoh}$ s.

$$\Rightarrow \sum_{i=1}^n (-1)^i \chi(X, \mathcal{F}_i) = 0.$$

Thm  $\exists P(z) \in \mathbf{Q}[z]$  poly s.t.  $\underline{\chi(X, \mathcal{F}_n)} = P(n)$ ,  $\forall n \in \mathbb{Z}$

Moreover,  $\deg P = \deg \chi(X, \mathcal{F}) = \dim X$ .

Proof. (Dévissage)

Replacing  $\mathcal{F}$  by  $j_* \mathcal{F}$  ( $j: X \rightarrow \mathbb{P}_k^r$ ) we reduce to  $X = \mathbb{P}_k^r$ .

b/c  $H^i(X, \mathcal{F}) \cong H^i(\mathbb{P}_k^r, j_* \mathcal{F})$  under closed imm.

Also, may assume  $k = \bar{k}$

b/c changing  $k$  doesn't change any dim

(e.g. by looking at Čech cohom,  
this is a special case of flat base change thm)

Induction on  $\dim(\text{supp } F)$ .

①  $\text{supp } F = \emptyset$  i.e.  $F = 0 \Rightarrow P(n) = 0$  works.

②  $0 \rightarrow g \rightarrow F(-1) \xrightarrow{\times X_r} F \rightarrow H \rightarrow 0$  exact  
 $\dim(\text{supp } g) = \dim(\text{supp } F) \geq \dim(\text{supp } H)$   
 $\Rightarrow H \cap \text{supp } F = \emptyset, H: X_r = 0.$

Ind hyp:  $\chi(\mathbb{P}_k^r, F(m)) - \chi(\mathbb{P}_k^r, F(n-r)) = p(n)$

$$\deg p = \dim(\text{supp } F) - 1$$

elementary alg.  $\rightarrow \chi(\mathbb{P}_k^r, F(n)) = q(n), \deg q = \dim(\text{supp } F).$

□

Def:  $P(n) = \text{Hilb poly of } F.$

When  $F = \mathcal{O}_X$ :  $P(n) = \text{Hilb poly of } X.$

Note By Serre vanishing ( $H^i(X, \mathcal{O}_X(n)) = 0, n \gg 0, i > 0$ )

$$P(n) = \dim_k H^0(X, F), \quad n \gg 0.$$

(original def'n of Hilb poly).

E.g. (1) For  $\mathbb{P}_k^r$ ,  $P(n) = \binom{n+r}{r} = \chi(\mathbb{P}_k^r, \mathcal{O}(n)).$

(2) For  $\text{Spec } k[X]/(X^2) \hookrightarrow \mathbb{P}_k^1$ ,  $P(n) = 2$ .

(For  $X = \text{two reduced pts}, P(n) = 2$ ).

can be written as a flat lim of pairs of distinct pts

## §2 Flatness and Hilbert Polynomials

Numerical Criterion for Flatness:

Thm  $T$  integral, loc noe sch.  $X \subseteq \mathbb{P}_T^r$  closed subsch.  
 can be replaced essential by conn + red.  $\mathcal{F} \in \text{Coh}(X)$ .  
 $K(t) = Q_{T,t}/m_{T,t}$  res field at  $t$ .

$\forall t \in T, P_t \in \mathbb{Q}[z]$  Hilb poly of  $j_t^*\mathcal{F}$ ,  $j_t : X_t \rightarrow \mathbb{P}_{k(t)}^r$ .

Then  $\mathcal{F}$  flat relative to  $X \rightarrow T$

$\Leftrightarrow P_t = \text{const in } t$ .

In particular  $X$  flat over  $T \Leftrightarrow$  Hilb poly of  $X_t$  is const in  $t$ .

Proof. (c.f. Hartshorne Thm III.9.9, or EGA III §7.9).

Note: (1) can reduce to  $X = \mathbb{P}_T^r$  by  $\mathcal{F} \rightarrow i^*\mathcal{F}$ ,  $i : X \subseteq \mathbb{P}_T^r$ .

(2) it suffices to consider  $T = \text{Spec } A$ ,

A local int noe ring.

Step 1 To show that  $\mathcal{F}$  flat over  $T$

$\Leftrightarrow H^0(X, \mathcal{F}(m))$  fin free over  $A$ ,  $m \gg 0$ .

$(\Rightarrow)$   $\mathcal{F}$  flat over  $T \Rightarrow$  so are  $\check{C}^i(A, \mathcal{F}(m))$ .

$\frac{1}{f/c}$  open imms are flat.

$T(X, -) \circ \rightarrow H^0(X, \mathcal{F}(m)) \rightarrow \check{C}^0(A, \mathcal{F}(m)) \rightarrow \dots \rightarrow \check{C}^r(A, \mathcal{F}(m)) \rightarrow 0$

are all flat except possibly for  $H^0$ -term.

$\Rightarrow H^0(X, \mathcal{F}(m))$  flat as well.

By Serre's finiteness,  $H^0(X, \mathcal{F}(m))$  f.g. over  $A$ ,  $\boxed{m \gg 0}$

$\Rightarrow H^0(X, \mathcal{F}(m))$  free.

$(\Leftarrow)$  Pick  $m_0 \gg 0$  s.t.  $H^0(X, \mathcal{F}(m))$  fin free /  $A$ ,  $\forall m \geq m_0$ .

$\Rightarrow$  we can recover  $\mathcal{F}$  on  $\tilde{M}$  for

$$M = \bigoplus_{m \geq m_0} H^0(X, \mathcal{F}(m)) \text{ (flat)}$$

$\Rightarrow \mathcal{G} = \tilde{M}$  flat.

Step 2 To show  $H^0(X, \mathcal{F}(m))$  fin. free,  $m \gg 0 \Leftrightarrow P_+ = \text{const.}$

Claim: it follows by checking

$$H^0(X+, \mathcal{F}_{+(m)}) = H^0(X, \mathcal{F}_{(m)}) \otimes_A K(t).$$

for  $m \gg 0$ .  $\leftarrow$   
even I don't prove this uniformly in  $t$ .

Namely,  $H^0(X, \mathcal{F}(m))$  for free  $\mathbb{A}$ ,  $\mathbb{H}^t$ ,  $m \gg 0$ ,

$\Rightarrow P_t = P_\eta$ ,  $\eta \in T$  generic pt. (also the converse).

May reduce to  $t \in T$  closed pt

try replacing A by  $G\Gamma t$ .  
 A has no short exact seq

$$A^{\oplus n} \rightarrow A \rightarrow \eta_{\mathcal{C}(+) \rightarrow 0} \quad \text{of } A\text{-Mod}_\bullet.$$

$$\hookrightarrow H^0(X, \mathcal{F}^{(m)^{\otimes n}}) \rightarrow H^0(X, \mathcal{F}_{(m)}) \rightarrow H^0(X, \mathcal{F}_{\geq t(m)}) \rightarrow 0, \quad m \gg 0$$

can be pulled out

not  $\nexists t$  (!)

四

## §3 Hilbert Schemen

$\exists$  univ family of closed schs of  $\mathbb{P}^r$  with a fixed Hilb poly.

Hm (Grothendieck) Fix  $k$  field,  $r \in \mathbb{Z}$ .  $P(\frac{1}{x}) \in \mathbb{Q}[\frac{1}{x}]$ .

$\Rightarrow \exists H \in \text{Schr}_k^{\text{ne}} \quad \mathcal{Q} \subset X \subseteq P_H$  closed subsch.

flat with Hilb poly P.

s.t.  $\forall T \in \text{Schr}^{\text{noe}}$ ,  $y \in P_T$  closed subspace which is

flat with Hilb poly P.

(3!)  $T \rightarrow H$  s.t.  $\mathcal{Y} = X \times_H T$  as closed subschs

$${}^0\phi \quad \tilde{P}_T^r \cong \tilde{P}_{H \times_H T}^r.$$

e.g. One can show that the closed subsch

$$\begin{aligned} H \subseteq \mathbb{P}_k^n & \text{ is a plane of } \dim = d \\ \Leftrightarrow \text{Hilb poly of } H = P(n) = \binom{n+d-1}{n} & \left\{ \begin{array}{l} \text{parameter sch} \\ = \text{Gr}(d, n) \\ \dim d \text{ planes in } \mathbb{P}_k^n. \end{array} \right. \end{aligned}$$

### §4 Hilbert Polynomials, Degree, and Dimension

Lemma  $P(z) = \text{Hilb poly of closed subsch } X \subseteq \mathbb{P}_k^n.$

$$(a) \deg P = \dim X$$

(b) Put  $d = \dim X$ .  $\forall P \subseteq \mathbb{P}_k^n$   $d$ -dim'l plane s.t.  $\dim(X \cap P) = 0$ ,

$$\Rightarrow \text{length}(X \cap P) = d! \cdot (a_d)$$

↑ leading coeff of  $P(z)$ .

Proof. May assume  $k = \bar{k}$ .

① For  $P$  generic  $d$ -dim'l plane,  $\dim(X \cap P) = 0$ .

↑ i.e. chosen outside some closed subsch of  $\text{Gr}(d, n)$ .

$\Leftarrow \forall H$  generic hyperplane,  $\dim(X \cap H) < \dim X$ .

② Put  $\mathcal{F} = j_* \mathcal{O}_X$ ,  $j: X \rightarrow \mathbb{P}_k^n$  closed imm.

$\forall H$  hyper s.t.  $\dim(X \cap H) < \dim X$ , we have

$$0 \rightarrow \mathcal{F}(-) \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$$

↑ Direct im of  $\mathcal{O}_{X \cap H}$ .

Hilbert polys :  $P(z) \leftrightarrow \chi(\mathbb{P}_k^n, \mathcal{F}_m)$

$$P(z-1) \leftrightarrow \chi(\mathbb{P}_k^n, \mathcal{F}_{(-1)m})$$

$$\therefore P(z) - P(z-1) \leftrightarrow \chi(\mathbb{P}_k^n, \mathcal{G}_m).$$

$\Rightarrow$  both claims follows.  $\square$

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