Interview Questions on Algebra

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Set VIII: Representation Theory

1. Define "representation" of a group. Define "irreducible representation". Why can you decompose representations of finite groups into irreducible ones? Construct an invariant inner product.

   Definition. Group representations describe abstract groups in terms of vector space automorphisms; in particular, the group operation can be represented by matrix multiplication. An irreducible representation has no proper subrepresentations.

   Answer. Representations are decomposable because we are working over an algebraically closed field $\mathbb{C}$, such that every matrix in $\mathrm{GL}_n(\mathbb{C})$ turns into blocks. If we are working over K such that $\mathrm{char}(K) = p \nmid |G|$, we can use Maschke's theorem.

   Construction. Let V be a finite-dimensional $\mathbb{C}$-vector space with a representation of G on it. Taking any Hermitian inner product $\langle \cdot, \cdot \rangle$ on V and then

   $$\langle v, v' \rangle_G = \frac{1}{|G|} \sum_{g \in G} \langle g(v), g(v') \rangle$$

   is invariant under the action of G.

   Caveat. There is no modification of this over general characteristic-p-field for proving Maschke's theorem. Because a nondegenerate bilinear form gives only orthogonality but cannot guarantee the existence of set-theoretical complement.

2. State and prove Maschke's theorem. What can go wrong if you work over the real field? What can go wrong in characteristic p?

   Statement. Let V be a finite-dimensional k-vector space with a representation of G attached to. If $\mathrm{char}\, k \nmid |G|$, then each $k[G]$-submodule W of V has a complement $k[G]$-submodule, say W' such that $V = W \oplus W'$.

   Proof. The upshot is to note that $f : V \to V$ such that

   $$f(v) = \frac{1}{|G|} \sum_{g \in G} g(\mathrm{pr}(g^{-1}(v)))$$

   is a G-invariant and k-linear automorphism on V, where $\mathrm{pr} : V \to W$ is the natural projection. It satisfies $f^2 = f$, and by setting $W' = \mathrm{Ker}\, f$, we obtain $V = W \oplus W'$.

   Generalization. Under the same assumption, the group algebra $k[G]$ is semisimple.

   Answer. If k is a non-algebraically closed field such as $\mathbb{R}$, the general version must be more complicated: the group algebra $k[G]$ is a product of matrix algebras over division rings over k. The summands correspond to irreducible representations of G over k. Note that an $\mathbb{R}$-irreducible representation is possibly non-irreducible over $\mathbb{C}$.

   If $\mathrm{char}\, k = p \mid |G|$, it turns out that $|G| = 0$ and f is not constructible. In fact, some subrepresentation does not have a complement here.

3. Do you know what a group representation is? Do you know what the trace of a group representation is?

   Answer. A group representation is a group homomorphism $f : G \to \mathrm{GL}(V)$ together with a k-vector space (generally $k = \mathbb{C}$). The character of a representation is realized as the trace of image matrices of group elements.

4. State/prove/explain Schur's lemma.

   Statement. Let V be a finite-dimensional irreducible representation of G over some field k. Then $\mathrm{End}_G(V)$ is a finite-dimensional division algebra over k (i.e., its nonzero elements are all isomorphisms).

   In particular, every linear automorphism on V that is irreducible over $\mathbb{C}$, commuting with G-actions, must be a scalar.

   Proof. For any $f \in \mathrm{End}_G(V)$, the condition that f commutes with G-actions means both $\mathrm{Ker}\, f$ and $\mathrm{im}\, f$ are G-invariant. Since V is irreducible,

   $$\begin{array}{rlcl} \text{either} & \mathrm{Ker}\, f = 0, \mathrm{im}\, f = V & \implies & f \text{ is injective;} \\ \text{or} & \mathrm{Ker}\, f = V, \mathrm{im}\, f = 0 & \implies & f = 0. \end{array}$$

   If f is injective, it is naturally an isomorphism due to dimension reasons.

   Note. The endomorphism algebra $\mathrm{End}_G(V)$ is defined as the vector space of linear automorphisms on V that commute with G-actions, i.e.,

   $$\mathrm{End}_G(V) := \{f : V \to V : \rho(g) \circ f = f \circ \rho(g), \forall g \in G\}.$$

   where $\rho : G \to \mathrm{GL}(V)$ is the representation attached to V.

   A division algebra over k is a k-vector space and a division ring in which every nonzero element has a multiplicative inverse.

5. What can you say about characters? What are the orthogonality relations? How do you use characters to determine if a given irreducible representation is a subspace of another given representation?

   Answer. Characters are defined as traces of images of $g \in G$ under the representation $\rho : G \to \mathrm{GL}(V)$. All characters of G form a $\mathbb{C}$-vector space, which is equipped with an inner product

   $$\langle \chi, \psi \rangle_G = \sum_{g \in G} \chi(g)\psi(g)^*.$$

   Here $\psi(g)^* = \psi(g^{-1})$ is the complex conjugation. It turns out that the characters of irreducible rep-classes of G form an orthonormal basis, i.e.,

   - V is an irreducible representation with character χ if and only if $\langle \chi, \chi \rangle_G = 1$;
   - two representations are isomorphic if and only if they obtain the same character;
   - if χ, ψ are characters of two non-isomorphic representations, then $\langle \chi, \psi \rangle_G = 0$.

   If (W, ψ) and (V, χ) are given as two reps of G, then W is isomorphic to some subrep of V if and only if $\langle \psi, \chi \rangle_G \geq 1$, which means it has positive multiplicity in V.

6. What's the relation between the number of conjugacy classes in a finite group and the number of irreducible representations?

   Answer. They are the same (hence the character table is a square).

   Proof. Consider the dimension of $Z(\mathbb{C}[G])$, the space of class functions on G. Then $\dim_{\mathbb{C}} Z(\mathbb{C}[G])$ is the number of irreducible representations of G. This is because any class function is a $\mathbb{C}$-linear combination of characters for irreducible representations. On the other hand, any class function has constant values on conjugacy classes (by definition). Therefore, this dimension also equals to the number of conjugacy classes.

7. What is the character table? What field do its entries lie in?

   Answer. The rows of character table are labeled by irreducible characters (corresponding to irreducible rep-classes). The columns are labeled by conjugacy classes of G. It sorts out the values of characters at each conjugacy class. Its entries lie in $\mathbb{C}$ (or more generally, the algebraically closed base field of V).

   Note. In a character table, no two of rows or columns are completely the same. It meant to say that: the valuations at all conjugacy classes uniquely determine an irreducible character; two group elements are conjugate if and only if they have same values under all irreducible characters.

8. Why is the character table a square?

   Answer. See Question 6.

9. If $\chi(g)$ is real for every character χ, what can you say about g?

   Answer. Working over $\mathbb{C}$, this means $\chi(g) = \chi(g)^* = \chi(g^{-1})$ for any irreducible χ. From Question 7, this is equivalent to saying g and g^{-1} are conjugate.

10. What's the regular representation?

    Answer. The group algebra $\mathbb{C}[G]$ has a basis $\{e_s\}_{s\in G}$ (see Question 52 in Set VI for the definition), where e_s vanishes outside s and $e_s(s) = 1$. Then G has a left (resp. right) action

    $$\begin{aligned} G \times \mathbb{C}[G] &\longrightarrow \mathbb{C}[G] \\ (g, e_s) &\longmapsto e_{gs} \quad (\text{resp. } e_{sg^{-1}}). \end{aligned}$$

    Consider the left one, and we get $\rho : G \to \mathrm{GL}(\mathbb{C}[G])$ such that $\rho(g)(e_s) = e_{gs}$. This is the so-called regular representation. Its character $\chi_{\text{reg}}(g) = 0$ for $g \neq 1$, and $\chi_{\text{reg}}(1) = |G|$.

    Punchline. It turns out that the irreducible decomposition of all regular representations gives all G-representation classes up to isomorphisms.

11. Give two definitions of "induced representation". Why are they equivalent?

    Answer. There are two algebraic constructions for induced representations. Let H be a subgroup of G with (π, V) a given representation on it.

    (I) Say $n = [G : H]$ and $G/H = \{\overline{g_1}, \dots, \overline{g_n}\}$ are all representatives of left cosets, that is, $\overline{g_i} = g_i H$. Then

    $$\mathrm{Ind}_H^G V := \bigoplus_{i=1}^{n} \overline{g_i} V,$$

where $\overline{g_i}V = \{\overline{g_i}v, v \in V\}$ is an isomorphic copy of V.

(II) Any representation (π, V) of H is regarded as a $K[H]$-module over the group algebra. Then

$$\mathrm{Ind}_H^G \pi := K[G] \otimes_{K[H]} V$$

is naturally a $K[G]$-module.

They are equivalent because when (π, V) is given, $\mathrm{Ind}_H^G \pi$ and $\mathrm{Ind}_H^G V$ can be mutually determined.

Note. The induced representation has a universal property: there is an H-equivariant map $f : V \to \mathrm{Ind}_H^G V$ such that for any H-equivariant $\psi : V \to \hat{V}$, there is a G-equivariant $\varphi : \mathrm{Ind}_H^G V \to \hat{V}$ such that $\psi = \varphi \circ f$.

12. If you have a representation of H, a subgroup of a group G, how can you induce a representation of G?

    Answer. See Question 11 for the definition of induced representations.

13. If you have an irreducible representation of a subgroup, is the induced representation of the whole group still irreducible?

    Answer. Typically not. But the restricted representation is always irreducible if the primitive one is.

    Theorem. (Mackey's Irreducibility Criterion) $\mathrm{Ind}_H^G V$ is irreducible if and only if:

    - V is irreducible;
    - for any $g \in G - H$, V and V^g has no common irreducible component restricting to $H \cap H^g$. Here V^g is the conjugate representation of $H^g = gHg^{-1}$.

    Counterexample. Inducing a restriction of some irreducible representation (which is still irreducible) outputs some copies of the original representation.

14. What can you say about the kernel of an irreducible representation? How about kernels of direct sums of irreducibles? What kind of functor is induction? Left or right exact?

    Answer. The kernel of $\pi : G \to \mathrm{GL}(V)$ must be a normal subgroup of G if π is irreducible. Consider the algebra homomorphism

$$\mathbb{C}[G] \longrightarrow \bigoplus_{\pi \text{ irred}} \mathrm{End}(V)$$

    where the direct sum goes through all irreducible representations. This is morally a one-to-one correspondence and the kernel must be trivial. Namely, the kernel of some direct sum of different irreducible representations is trivial.

    The induction functor, say

$$\begin{aligned} \mathrm{Ind}_H^G : \mathsf{Rep}(H) &\longrightarrow \mathsf{Rep}(G) \\ V &\longmapsto \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V, \end{aligned}$$

    is essentially a tensor product (as a left-adjoint of restriction functor, which is exact). Hence Ind_H^G is always right-exact.

15. What is Frobenius reciprocity?

    Answer. For any group G, there is an inner product $\langle\cdot,\cdot\rangle$ on the vector space of class functions $G\to\mathbb{C}$. Let H be a subgroup and $\varphi: G\to C$, $\psi: H\to C$ be two class functions. Then
    $$\langle \operatorname{Ind}_H^G \psi,\varphi\rangle_G = \langle\psi,\operatorname{Res}_H^G\varphi\rangle_H.$$
    In other words, the functors Ind_H^G and Res_H^G are Hermitian adjoint.

16. Given a normal subgroup H of a finite group G, we lift all the representations of G/H to representations of G. Show that the intersection of the kernels of all these representations is precisely H. What can you say when H is the commutator subgroup of G?

    Proof. Since all irreducible representations of G/H constitute the regular representation of G/H, it suffices to consider $\operatorname{Ind}_{G/H}^G \operatorname{reg}$. But its kernel is exactly H.

    Answer. When $H=[G,G]$ is the commutator subgroup, G/H is abelian and every irreducible representation of it is 1-dimensional. The liftings of these characters to G are decomposed into $\rho_H\otimes\chi_{G/H}$ and H equals the intersection of all $\operatorname{Ker}\rho_H$.

17. If you have two linear representations π_1 and π_2 of a finite group G such that $\pi_1(g)$ is conjugate to $\pi_2(g)$ for every g in G, is it true that the two representations are isomorphic?

    Answer. Yes. This condition guarantees $\pi_1(g)$ and $\pi_2(g)$ have the same trace, hence $\chi_1(g)=\chi_2(g)$ for all $g\in G$ at the level of characters. This is equivalent to saying π_1,π_2 are isomorphic.

18. What's special about using $\mathbb{C}$ in the definition of group algebra? Is it possible to work over other fields? What goes wrong if the characteristic of the field divides the order of the group?

    Answer. Because $\mathbb{C}$ is an algebraically closed field of characteristic 0 (not dividing $|G|$). The most significant advantage is there is a Hermitian inner product of characters over $\mathbb{C}$.

    If we're using an algebraically closed field $k\neq\mathbb{C}$, all proofs with respect to inner products should be modified. If we're working over non-algebraically closed fields, e.g. the normality of characters in $Z(k[G])$ fails (but the orthogonality still holds).

    Consider the case where $\operatorname{char}(k)\mid |G|$, this leads to the failure of Maschke's theorem hence the group algebra $k[G]$ is possibly not semisimple.

19. Suppose you have a finite p-group, and you have a representation of this group on a finite-dimensional vector space over a finite field of characteristic p. What can you say about it?

    Answer. In this case, any irreducible representation should be trivial. Therefore, any representation is copies of trivial ones.

    Proof. Let V be the irreducible finite-dimensional representation space of the given p-group over $\mathbb{F}_p$. Consider the action of G on $V^\times = V-\{0\}$. By the orbit-stabilizer formula, for any $v\in V^\times$, the size for the orbit of v must be a power of p. On the other hand, the sum of these orbit should be $p^{\dim V}-1$ since V is defined over $\mathbb{F}_p$.

But a sum of powers of p cannot equal to $p^{\dim V} - 1$ unless $\dim V = 0$. Hence V is trivial.

20. Let (π, V) be a faithful finite-dimensional representation of G. Show that, given any irreducible representation of G, the n-th tensor power of $\mathrm{GL}(V)$ will contain it for some large enough n.

    Proof. This statement is known as the **tensor power trick**. It follows from the fact that any irreducible representation of G is a constituent of the regular representation of G. The regular representation can be realized as a subrepresentation of $\mathrm{GL}(V)^{\otimes |G|}$, where $|G|$ denotes the order of G. Thus, any irreducible representation of G can be realized as a subrepresentation of $\mathrm{GL}(V)^{\otimes n}$ for some large enough n.

21. What are the irreducible representations of finite abelian groups?

    Answer. In an abelian group G, each element $g \in G$ forms a single conjugacy class. Hence there are $|G|$ irreducible representation classes up to isomorphisms. By $|G| = m_1^2 + \cdots m_{|G|}^2 < \infty$, each multiplicity $m_i = 1$, which shows each irreducible representation of a finite abelian group is 1-dimensional. In fact, G has a faithful irreducible representation if and only if G is cyclic.

22. What are the group characters of the multiplicative group of a finite field?

    Answer. The group $\mathbb{F}_q^\times$ is cyclic of order $q - 1 = p^r - 1$ for some prime p. Then each $x \in \mathbb{Z}/(q-1)\mathbb{Z}$ forms a single conjugacy class. There are $q - 1$ of them, say $C_0, \ldots, C_{q-2}$, and hence there are irreducible characters $\chi_0, \ldots, \chi_{q-2}$ such that

    $$\chi_i(C_j) = \zeta_{q-1}^{ij}.$$

    Note. This construction gives a canonical way to construct characters of cyclic groups.

23. Are there two nonisomorphic groups with the same representations?

    Answer. Yes. Two groups whose all irreducible representations are the same means equivalently that they share the same character table. For example, Q_8 and D_4 do this. See Question 26 for the table.

24. If you have a $\mathbb{Z}/5\mathbb{Z}$ action on a complex vector space, what does this action look like? What about an S_3 action? A dihedral group of any order?

    Answer. It can be realized as a rotation on some regular pentagon in $\mathbb{C}^r$ for $r \geq 2$. Also, S_3 acts on an equilateral triangle and D_n acts on an regular n-gon with rotations and reflexions.

25. What are the representations of S_3? How do they restrict to S_2?

    Answer. There are 3 conjugacy classes in S_3, and hence there are 3 irreducible representations. The character tables for S_3 and its restriction to S_2 are in the following.

S_3	1	(123)	(12)
χ_0	1	1	1
χ_1	1	1	-1
χ_2	2	-1	0

S_2	1	(12)
$\mathrm{Res}_{S_2}^{S_3} \chi_0$	1	1
$\mathrm{Res}_{S_2}^{S_3} \chi_1$	1	-1

Example. Here comes an explicit construction of the irreducible representation π_2 for irreducible character χ_2. Note that $S_3 \cong \mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, so

$$S_3 = \langle x, y : x^3 = y^2 = 1, yx = x^{-1}y \rangle, \quad x = (123), \quad y = (12).$$

One may take

$$X = \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

such that the representation $\pi_2 : S_3 \to \mathrm{GL}_2(\mathbb{C})$ sends $x \mapsto X$, $y \mapsto Y$. One can check that $\chi_2(x) = \zeta_3 + \zeta_3^{-1} = -1$, and $\chi_2(y) = 0$. Since χ_1, χ_2 restrict to $S_2 = \{1, y\}$, this π_2 has only trivial restriction to S_2.

26. Tell me about the representations of D_4. Write down the character table. What is the 2-dimensional representation? How can it be interpreted geometrically?

Notes. In a character table, if all but one characters are already obtained, there are 3 different approaches to compute the last one:

- using the equality $\chi_{\text{reg}} = \sum m_i \chi_i$;
- using the column orthogonality relation, i.e., for conjugacy classes $C_0, \dots, C_n$,

$$\chi_0(C_i)\chi_0(C_j)^* + \cdots + \chi_n(C_i)\chi_n(C_j)^* = 0, \quad i \neq j;$$
$$\chi_0(C_i)\chi_0(C_i)^* + \cdots + \chi_n(C_i)\chi_n(C_i)^* = g/|C_i|.$$

- computing it as an induced representation from some subgroup:

$$(\mathrm{Ind}_H^G \chi)(x) = \sum_{g_i \in G/H} \chi(g_i x g_i^{-1})$$

  for $x \in G$ and all representatives g_i of G/H.

Solution. By definition,

$$D_4 = \langle x, y : x^4 = y^2 = 1, yxy^{-1} = x^{-1} \rangle.$$

We first compute $D_4^{\text{ab}} = D_4/[D_4, D_4]$: since $[x, y] = xyx^{-1}y^{-1} = x^2$, and if $x^2 = 1$ then $yxy^{-1} = x^{-1} = x$ (i.e., x, y mutually commutes), we get $[D_4, D_4] = \langle x^2 \rangle$; now $|D_4^{\text{ab}}| = 4$, which is the number of 1-dimensional irreducible representations. On the other hand, D_4-action has a 2-dimensional geometric realization, so there is one 2-dimensional irreducible representation. Note that all 5 conjugacy classes in D_4 are given by

$$\{1\}, \{x, x^3\}, \{y, x^2y\}, \{x^2\}, \{xy, x^3y\}.$$

In the following character table, $\chi_0, \dots, \chi_3$ permutes ± 1 over x and y, and there are 3 different ways to determine χ_4: using the χ_{reg}, using the column orthogonality, and computing it as an induce representation from the subgroup $\langle x \rangle$.

D_4	1	x	y	x^2	xy
χ_0	1	1	1	1	1
χ_1	1	-1	1	1	-1
χ_2	1	1	-1	1	-1
χ_3	1	-1	-1	1	1
χ_4	2	0	0	-2	0

Interpretation. Since D_4 has only one 2-dimensional irreducible representation, this can be viewed as the action of D_4 on a rectangle in $\mathbb{C}^2$. Hence $D_4 \to \mathrm{GL}_2(\mathbb{C})$ is constructed. Here r is represented by the rotation by $90°$, and s is represented by the reflection whose axis doesn't go through any vertices.

Remark. Any dihedral group D_n generally admits this interpretation on $\mathbb{C}^2$, hence the 2-dimensional irreducible representation always exists for D_n (and the number of them is morally $n/2 - 1$). See Question 27 below.

27. How would you work out the orders of the irreducible representations of the dihedral group D_n? Why is the the sum of squares of dimensions equal to the order of the group?

Answer. Say $D_n = \langle r, s : r^n = s^2 = 1, srs^{-1} = r^{-1}\rangle$. Then any element in D_n can be written in the form r^k or sr^k for $0 \le k \le n-1$.

(I) The case $2 \mid n$. Corresponding ± 1 with r and s alternatively, we get 4 irreducible linear characters:

	r^k	sr^k
ψ_1	1	1
ψ_2	1	-1
ψ_3	$(-1)^k$	$(-1)^k$
ψ_4	$(-1)^k$	$(-1)^{k+1}$

Consider 2-dimensional representations. One can define $\rho_j : D_n \to \mathrm{GL}_2(\mathbb{C})$ by

$$\rho_j(r^k) = \begin{pmatrix} \zeta_n^{jk} & 0 \\ 0 & \zeta_n^{-jk} \end{pmatrix}, \quad \rho_j(sr^k) = \begin{pmatrix} 0 & \zeta_n^{-jk} \\ \zeta_n^{jk} & 0 \end{pmatrix}$$

that correspond to character χ_j. Note that $\chi_0 = \psi_1 + \psi_2$ and $\chi_{n/2} = \psi_3 + \psi_4$ (here $\rho_j = \rho_{n-j}$, we may assume $0 < j < n/2$). Also,

$$\chi_j(r^k) = \zeta_n^{jk} + \zeta_n^{-jk} = 2\cos(\frac{2\pi}{n} jk), \quad \chi_j(sr^k) = 0.$$

Hence there are 4 irreducible 2-dimensional representations and $n/2 - 1$ irreducible 2-dimensional representations.

(II) The case $2 \nmid n$. There are only 2 irreducible linear characters: ψ_1, ψ_2 as before. The remaining constructions are the same. It turns out to have $(n-1)/2$ irreducible 2-dimensional representations.

Check: Since he sum of squares of dimensions equal to the order of the group, there are no more irreducible representations. This fact holds because

$$|G| = \chi_{\mathrm{reg}}(1) = \sum m_i \chi_i(1) = \sum m_i^2.$$

28. What about representations of A_4? Give a nontrivial one. What else is there? How many irreducible representations do we have? What are their degrees? Write the character table of A_4.

Solution. Symmetries of a tetrahedron form a nontrivial representation of A_4. Since $V \lhd A_4$, there is a 2-dimensional (non-irreducible) representation whose kernel is $V \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:

Cosets of V in A_4	Representation Images
$\{(1),(12)(34),(13)(24),(14)(23)\}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$
$\{(123),(243),(142),(134)\}$	$\begin{pmatrix}0&-1\\1&-1\end{pmatrix}$
$\{(132),(234),(124),(143)\}$	$\begin{pmatrix}-1&1\\-1&0\end{pmatrix}$

Note that A_4 admits a doubly transitive action on $\{1,2,3,4\}$ such that the standard representation gives an irreducible character of degree 3, say χ_3. In the following, χ_1 and χ_2 are linear and they can be determined by orthogonality relations.

A_4	(1)	(12)	(123)	(132)
χ_0	1	1	1	1
χ_1	1	1	ω	ω^2
χ_2	1	1	ω^2	ω
χ_3	3	-1	0	0

29. Write the character table for S_4.

Recipe. While computing characters of G that acts double-transitively on some set X, an irreducible character can be attained by taking the number of fixed points (of each representatives) and subtracting 1. Also, there is a nontrivial sign-character by taking -1 at all transitions on S_n. Using the following lemma, we can get some $\chi\theta$ with $\deg\chi\theta > 1$ from χ for free: "linear $\times$ irreducible = irreducible".

Lemma. If χ is a character and θ is a linear character, then $\chi\theta$ is a character, and is irreducible if and only if χ is.

Solution. Say S_4 has 5 conjugacy classes:

$$\begin{aligned}
&C_0 = \{(1)\},\\
&C_1 = \{(12),(13),(14),(23),(24),(34)\},\\
&C_2 = \{(123),(132),(213),(231),(312),(321)\},\\
&C_3 = \{(1234),(1243),(1324),(1342),(1423),(1432)\},\\
&C_4 = \{(12)(34),(13)(24),(14)(23)\}.
\end{aligned}$$

In the character table, χ_1 is given by alternating signs on A_4. And χ_3 is given by "$\#\{\text{fixed points}\} - 1$". Then we get irreducible $\chi_4 = \chi_1\chi_3$ for free. Finally, from the regular character, $\deg\chi_2 = 2$ and $2\chi_2 = \chi_{\text{reg}} - \chi_0 - \chi_1 - 3\chi_3 - 3\chi_4$.

S_4	(1)	(12)	(123)	(1234)	(12)(34)
χ_0	1	1	1	1	1
χ_1	1	-1	1	-1	1
χ_2	2	0	-1	0	2
χ_3	3	1	0	-1	-1
χ_4	3	-1	0	1	-1

30. Start constructing the character table for S_5.

Sketchy Solution. Say S_5 has 7 different conjugacy classes:

Representatives	(1)	(12)	(12)(34)	(123)	(12)(345)	(12345)	(1234)
Size of Classes	1	10	15	20	20	24	30
Size of Centralizers	120	12	8	6	6	5	4

In the following character table, χ_1 is the sign character and χ_2 is the standard character, i.e., the natural representation on $\{1, 2, 3, 4, 5\}$ subtracting χ_0. From these we get $\chi_3 = \chi_2\chi_1$ by "linear $\times$ irreducible = irreducible". From orthogonality on rows and columns (see Question 26), we can completely determine χ_4, $\chi_5 = \chi_4\chi_1$, and χ_6 (given by χ_{reg}).

	1	(12)	(12)(34)	(123)	(123)(45)	(12345)	(1234)
χ_0	1	1	1	1	1	1	1
χ_1	1	-1	1	1	-1	1	-1
χ_2	4	2	0	1	-1	-1	0
χ_3	4	-2	0	1	1	-1	0
χ_4	5	-1	1	-1	-1	0	1
χ_5	5	1	1	-1	1	0	-1
χ_6	6	0	-2	0	0	1	0

31. How many irreducible representations does S_n have? What classical function in mathematics does this number relate to?

    Answer. See Question 18 in Set I for conjugacy classes in S_n. The number of irreducible representations is the number of partitions $\lambda = (\lambda_1, \ldots, \lambda_m)$ for n, where $\lambda_i \geq \lambda_{i+1}$ and $\lambda_1 + \cdots + \lambda_m = n$. This is given by the partition function, which doesn't have an explicit expression. Young diagrams are useful in computing this function.

32. Discuss representations of $\mathbb{Z}$, the infinite cyclic group. What is the group algebra of $\mathbb{Z}$? What is the connection with modules over PIDs? When is a representation of $\mathbb{Z}$ completely reducible? Why not always? Which are the indecomposable modules?

    Answer. Every infinite cyclic group is isomorphic to $\mathbb{Z}$. Each representation $\rho : \mathbb{Z} \to \mathrm{GL}(V) = \mathrm{GL}_1(\mathbb{C})$ is determined by $\rho(1) \in \mathbb{C}$. So each such ρ is characterized by a complex number $\rho(1) = z$. The group algebra

$$\mathbb{C}[\mathbb{Z}] = \left\{ \sum_{n\in\mathbb{Z}} na_n \in \mathbb{C} : a_i \in \mathbb{C},\ a_i = 0 \text{ for all but finitely many } i \in \mathbb{Z} \right\}.$$

33. State Artin's theorem and Brauer's theorem.

    Artin's Theorem. Let k be a field of characteristic 0 (but not necessarily algebraically closed) and G be any group. Let X be a set of subgroups in G. Consider the map

$$\mathrm{Ind}_X = \bigoplus_{H\in X} \mathrm{Ind}_H^G : \mathsf{Rep}(H) \to \mathsf{Rep}(G)$$

    between representations over k. Then the following are equivalent:

    (1) Up to conjugation, G can be set-theoretically covered by subgroups in X, i.e.

$$G = \bigcup_{H\in X} \bigcup_{g\in G} gHg^{-1}.$$

(2) The map $\mathrm{Ind}_X \otimes_{\mathbb{Z}} \mathbb{Q}$ is surjective. That is, for any $\rho \in \mathsf{Rep}(G)$, there exist an integer $d \geqslant 1$ and $\rho_H \in \mathsf{Rep}(H)$ for each $H \in X$, such that

$$d \cdot \rho = \sum_{H \in X} \mathrm{Ind}_H^G \rho_H.$$

Conventions. Let p be a prime number. A finite group H is called *p-elementary* if $H = C \times P$, where C is a cyclic group with prime-to-p order and P is a p-group. Denote $X(p)$ the set of all p-elementary subgroups of G.

Brauer's Theorem. Suppose further k is algebraically closed of characteristic 0 and G is a finite group. If X is the union of all the $X(p)$ for p prime, then Ind_X is surjective.

34. What is a Lie group? What is the Lie algebra? How do you get from a Lie algebra to a Lie group? The Jacobi identity?

    Definition. A Lie group is a topological group G (such that the operation and the inverse function are smooth) that is a smooth manifold as well.

    A Lie algebra is a vector space $\mathfrak{g}$ equipped with a bilinear map $[\cdot,\cdot]$, say Lie bracket, satisfying $[x,x] = 0$ and the Jacobi identity: $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

    Answer. Given $\mathfrak{g}$, embed it as a subalgebra of $\mathfrak{gl}_n$ and then exponentiate to get G as a subgroup of GL_n.

35. Define a unitary representation. What is the Peter–Weyl theorem?

    Definition. A representation (π, V) of G is unitary if V is a complex Hilbert space and $\pi(g)$ is a unitary operator on V for all $g \in G$.

    Statement. (Peter–Weyl) Let π be a unitary representation of a compact group G on a complex Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ splits into an orthogonal direct sum of irreducible finite-dimensional unitary representations of G.

    Corollary. Every compact Lie group has a faithful finite-dimensional representation and is therefore isomorphic to a closed subgroup of $\mathrm{GL}_n(\mathbb{C})$ for some n.

36. What is the adjoint representation of a Lie algebra? What is the commutator of two vector fields on a manifold?

    Answer. The adjoint representation of a Lie algebra is a way of representing the elements of the Lie algebra as linear transformations of itself. This is done by associating each element x of the Lie algebra with a linear transformation ad_x defined by $\mathrm{ad}_x(y) = [x, y]$, where $[x, y]$ denotes the Lie bracket of x and y.

    The commutator of two vector fields on a manifold is another vector field defined by its action on functions. If X and Y are two vector fields on a manifold, then their commutator is given by $[X, Y]f = X(Yf) - Y(Xf)$ for any smooth function f on the manifold.

37. Talk about the representation theory of compact Lie groups. How do you know you have a finite-dimensional representation?

Answer. Representation theory of a compact Lie group G is concerned with studying continuous group homomorphisms $G \to \mathrm{GL}(V)$ with $\dim V < \infty$. One important result is the Peter–Weyl theorem (Question 35). Consequently, if we have a continuous homomorphism from $\rho : G \to \mathrm{GL}(W)$ with $\dim W = \infty$, then ρ cannot be irreducible.

38. How do you prove that any finite-dimensional representation of a compact Lie group is equivalent to a unitary one?

    Proof. For a compact Lie group G, we are able to stick an arbitrary inner product $\langle \cdot, \cdot \rangle$ on the given finite-dimensional representation space V. Then the average inner product in Hermitian. To be more precise, for the compact group,

    $$\langle x, y \rangle' = \int_G \langle g(x), g(y) \rangle dg.$$

    This endows a unitary structure on G.

39. Do you know a Lie group that has no faithful finite-dimensional representations?

    Philosophy. Since every group has a trivial finite-dimensional unitary representation, non-faithful representations (but not necessarily finite-dimensional) are very easy to construct. There have morally been some finite-dimensional ones: the nontrivial covers for $\mathrm{SL}_2(\mathbb{R})$.

    Answer. Any finite-dimensional Lie algebra $\mathfrak{g}$ can be realized as the algebra of a simply connected universal cover Lie group $\widetilde{G}$. Recall that

    $$\mathrm{Hom}(G, H) \longrightarrow \mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$$

    is always injective for connected Lie groups G, H, and is bijective if G is simply connected. Then take $G = \mathrm{SL}_2(\mathbb{R})$ and $H = \mathrm{GL}_n(\mathbb{R})$. Now G is a semisimple Lie group with simple Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$. The finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{R})$ are completely reducible, and the irreducible representations of $\mathfrak{sl}_2(\mathbb{R})$ are precisely the symmetric powers $\mathrm{Sym}^n(\mathbb{R}^2)$ of the defining representation. It turns out that every irreducible representation of any nontrivial cover of G factors through the covering map to G. It follows that nontrivial covers of $G = \mathrm{SL}_2(\mathbb{R})$ (which exist) have no faithful finite-dimensional representations.

40. What do you know about representations of SO(2)? SO(3)?

    Comment. The answer should be complicated and we choose to omit. For representation theory of compact Lie groups, it would be worthwhile to completely memorize everything about $\mathrm{SU}(n)$, $\mathrm{SO}(2n)$, $\mathrm{Sp}(n)$, and $\mathrm{SO}(2n+1)$ (i.e. their Lie algebras, maximal tori, Cartan subalgebras, simple roots, positive roots, roots, Weyl chambers, Weyl groups, Dynkin diagrams, etc.).