Tate Cycles on Unitary Shimura Varieties mod p

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<u>Phylosophy</u> X even dim Sh var over $\mathbb{F}_q$.

supersingular locus (or more generally, basic locus)

$X^{ss} \rightsquigarrow$ some obvious generic Tate cycles in middle dim.

<u>Notation</u> F real quardratic, E/F CM

p inert in F, split in E

$\tau_1, \tau_2 : E \hookrightarrow \overline{\mathbb{Q}}_p \cong \mathbb{C}$ inducing v

$(D, *)/E$: division alg of dim n^2, with positive involution

s.t. $D \otimes \overline{\mathbb{Q}}_p = M_n(E_v) \times M_n(E_{\bar v})$.

$$\begin{array}{ccc} \{\tau_1,\tau_2\} & & \{\bar\tau_1,\bar\tau_2\} \\ | & & | \\ v & & \bar v \\ & \diagdown \quad \diagup & \\ & p & \\ & | & \\ & p & \end{array} \qquad \begin{array}{c} \overline{\mathbb{Q}}_p \simeq \mathbb{C} \\ | \quad \diagup \\ E \\ | \\ F \\ | \\ \mathbb{Q} \end{array}$$

$V = D$ left D-mod

$\psi : V \times V \longrightarrow \mathbb{Q}$

$(x, y) \longmapsto \mathrm{Tr}_{D/\mathbb{Q}}(x \delta y^*)$, for $\delta \in D^\times$, $\delta^* = -\delta$.

$\rightsquigarrow G = GU(V, \psi)$ unitary similitute grp associated to Herm space V.

<u>Assumption</u> (1) $G(\mathbb{R}) \cong G(U(1, n-1) \times U(n-1, 1))$ $\quad n \geq 2$

$\uparrow$ $\mathbb{R}$-valued points of this alg grp.

(2) $G(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times GL_n(E_v)$

Fix $\mathcal{O}_D \subseteq D$ order, stable under $(*)$ and $\mathcal{O}_{D(p)}$ is maximal and self-dual under ψ.

$K^p \subseteq G(\mathbb{A}_f)$: suff small open compact subgroup

$\mathbb{Z}_{p^2}$: deg 2 unram ext'n of $\mathbb{Z}_p$.

$Sh_{1,n-1}/\mathbb{Z}_{p^2}$: {loc noeth $\mathbb{Z}_{p^2}$-schemes} $\to$ Sets

$$S \longmapsto (A/S, \iota, \lambda, \eta)/\cong$$

- A/S: abelian scheme of dim $2n^2$
- ι: embedding $\mathcal{O}_D \hookrightarrow \mathrm{End}(A)$ s.t. $\forall b \in \mathcal{O}_E$

$$\det(T - \iota(b) \mid \mathrm{Lie}(A)) = (T - \tau_1(b))^n \cdot (T - \bar{\tau}_1(b))^{n(n-1)} (T - \tau_2(b))^{n(n-1)} \cdot (T - \bar{\tau}_2(b))^n$$

- $\lambda: A \to A^\vee$ prime-to-p polarization
- η: K^p-level str.

Kottwitz: $Sh_{1,n-1}/\mathbb{Z}_{p^2}$ is rep'd by a proj. smooth scheme over $\mathbb{Z}_{p^2}$ of dim $2(n-1)$.

$Sh_{1,n-1,\mathbb{F}_p} := Sh_{1,n-1}/\mathbb{Z}_{p^2} \otimes \mathbb{F}_{p^2}$ special fiber of it

$\ell \neq p$: $H^{2(n-1)}_{\text{ét}}(Sh_{1,n-1,\bar{\mathbb{F}}_p}, \bar{\mathbb{Q}}_\ell)\underbrace{(n-1)}_{\text{twisted by } n-1} \circlearrowleft \mathcal{H}_K \times \mathrm{Gal}_{\mathbb{F}_{p^2}}$

$\mathcal{H}_K = \bar{\mathbb{Q}}_\ell[K \backslash G(\mathbb{A}_f)/K]$

$K = K^p K_p$, $K_p = \mathbb{Z}_{p^2} \times GL_n(\mathcal{O}_{E_v})$

$$H^{2(n-1)}_{\text{ét}}(Sh_{1,n-1,\bar{\mathbb{F}}_p}, \bar{\mathbb{Q}}_\ell)(n-1) = \bigoplus_{\pi_f \in \mathrm{Irr}(G(\mathbb{A}_f))} \underbrace{\pi_f^K \otimes R_\ell(\pi_f)}$$

$\mathrm{Irr}(G(\mathbb{A}_f))$: irred. adm rep'ns

$\pi_f^K \otimes R_\ell(\pi_f)$: π_f-component of the étale coh. $=: H^{2(n-1)}_{\text{ét}}(Sh_{1,n-1,\bar{\mathbb{F}}_p}, \bar{\mathbb{Q}}_\ell)(n-1)_{\pi_f}$.

$R_\ell(\pi_f)$: some Galois module.

Put $H^{2(n-1)}_{\text{ét}}(Sh_{1,n-1,\bar{\mathbb{F}}_p}, \bar{\mathbb{Q}}_\ell)(n-1)_{\pi_f}^{G_{\mathbb{F}_{p^2}}\text{-finite}} := \bigcup_{\mathbb{F}_q/\mathbb{F}_{p^2} \text{ finite}} H_{\text{ét}}^{2(n-1)}(\cdots)^{G_{\mathbb{F}_q}}$

"Galois-finite part".

somehow a geom. version of Tate cycles

<u>Conj</u> (Tate$[\pi_f]$) : $\forall \pi_f \in \mathrm{Irr}(G(\mathbb{A}_f))$,

$H^{2(n-1)}_{\acute{e}t}(Sh_{1,n-1,\bar{\mathbb{F}}_p}, \bar{\mathbb{Q}}_\ell)(n-1)^{G_{\mathbb{F}_{p^2}}\text{-finite}}_{\pi_f}$ is generated by

coh classes of alg. cycles on $Sh_{1,n-1,\bar{\mathbb{F}}_p}$

special fiber

<u>Assumptions on π_f</u> (Technical. To garantee some automorphic multiplicity to be 1).

(1) $\pi_f^K \neq 0$, $R_\ell(\pi_f) \neq 0$.

K-invariant space

(*) (2) π_f is the finite part of an automorphic representation of $G(\mathbb{A}_f)$ which admits a cuspidal base change to $\mathbb{A}_E^\times \times GL_n(\mathbb{A}_E)$.

<u>TheoremA</u> (Helm-Tian-Xiao)

Let π_p be the p-component of π_f.

Assume that the Satake parameters of π_p are distinct modulo roots of unity.

Then Tate$[\pi_f]$ is true,

$H^{2(n-1)}_{\acute{e}t}(Sh_{1,n-1,\bar{\mathbb{F}}_p}, \bar{\mathbb{Q}}_\ell)(n-1)^{G_{\mathbb{F}_p}\text{-finite}}_{\pi_f}$

is generated by the (irred. comps of) $Sh^{ss}_{1,n-1,\bar{\mathbb{F}}_p}$.

Write $\pi_p = \pi_{p,0} \otimes \pi_v$ as rep of $G(\mathbb{Q}_p) = \mathbb{Q}_p^\times \times GL_n(E_v)$

$a_v^{(i)}$ be the eigenvalue of

$$T_v^{(i)} = GL_n(\mathcal{O}_{E_v}) \begin{pmatrix} \overbrace{p \, \ddots}^{i} & & & \\ & p & & \\ & & \underbrace{1 \, \ddots \, 1}_{n-i} & \end{pmatrix} GL_n(\mathcal{O}_{E_v}) \quad \text{on } \pi_v^{GL_n(\mathcal{O}_{E_v})}$$

Denote by $\alpha_{\pi_w,1}, \dots, \alpha_{\pi_w,n}$ the roots of

$$X^n + \sum_{i=1}^{n} (-1)^i a_w^{(i)} p^{i(i-1)} X^{n-i}. \qquad (**)$$

Assumption on π_p about Satake para $\Leftrightarrow$ $\alpha_{\pi_w,i}/\alpha_{\pi_w,j} \notin \{\text{roots of } 1\}$.

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autom. multi involved

<u>Kottwitz</u> $[R_\ell(\pi_f)] = \#\ker^1(\mathbb{Q}, G) \cdot m_G(\pi_f) \, [\Lambda^{n-1}\rho_{\pi_w} \otimes \rho_{\pi_w} \otimes \chi_{p,0}^{-1}(\tfrac{n(n-1)}{2})]$.

A very explicit description of Galois module $R_\ell(\pi_f)$ in Grothendieck group of finite dim'l rep'ns of some finite Galois group.

Here $\rho_{\pi_w} : G_{\mathbb{F}_{p^2}} \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$

s.t. $\rho_{\pi_w}(\mathrm{Frob}_{p^2})$ has char poly $(**)$

& $\chi_{p,0} : G_{\mathbb{F}_{p^2}} \longrightarrow \overline{\mathbb{Q}}_\ell^{\times}$

$\mathrm{Frob}_{p^2} \longmapsto \pi_{p,0}(p^2) \in \{\text{roots of } 1\}$.

the eigenvalues of Frob_{p^2} on $\Lambda^{n-1}\rho_{\pi_w} \otimes \rho_{\pi_w}$

are $\left(\prod_{1 \le i_1 \le \dots \le i_{n-1} \le n} \alpha_{\pi_w, i_j}\right) \cdot \alpha_{\pi_w,k}$

which is of the form $p^{n(n-1)}\zeta$,

only when $\{i_1, \dots, i_{n-1}\} \cup \{k\} = \{1, \dots, n\}$.

$\Rightarrow$ $\dim R_\ell(\pi_f)^{G_{\mathbb{F}_{p^2}}\text{-finite}} \le n \cdot \#\ker^1(\mathbb{Q}, G) \cdot m_G(\pi_f)$

<u>Construction of Cycles</u>

<u>Fact</u> $\exists$ unique unitary similitude group G' for E/F s.t.

- $G'(\mathbb{A}_f) \cong G(\mathbb{A}_f)$
- $(G')^{\mathrm{der}}$ has signature $(0,n) \times (n,0)$ at ∞.

$\rightsquigarrow$ $Sh_{0,n}/\mathbb{F}_{p^2}$ 0-dim Shimura var associated to G'

Consider $i = 1, \dots, n$

$$Y_i : \begin{pmatrix} \text{loc. noetherian} \\ \mathbb{F}_{p^2}\text{-schemes} \end{pmatrix} \longrightarrow \text{Sets}$$

$$S \longmapsto (A/S, \iota, \lambda, \eta ; B/S, \iota', \lambda', \eta' ; \phi)$$

- $(A/S, \iota, \lambda, \eta) \in Sh_{1,n-1}(S)$ bigger
- $(B/S, \iota', \lambda', \eta') \in Sh_{0,n}(S)$ smaller
- $\phi : B \to A$ isogeny, compatible with $\mathcal{O}_D$, K^p-level strs

& s.t. (1) $\ker \phi \subseteq B[p]$

(2) $\phi^\vee \circ \lambda \circ \phi = p \cdot \lambda'$

(3) $\operatorname{coker}(\phi_{*,1} : \mathcal{H}_1^{dR}(B/S)_{\tau_1} \to \mathcal{H}_1^{dR}(A/S)_{\tau_1})$ covariant

$$\mathcal{O}_E \otimes_{\mathbb{F}_p} \mathcal{O}_S = \bigoplus_{\tau_i : \mathcal{O}_E \to \mathbb{F}_{p^2}} \mathcal{O}_S$$

is a loc. direct factor of $\mathcal{H}_1^{dR}(A/S)$ of rank $(i-1) \cdot n$.

$\operatorname{coker}(\phi_{*,2} : \mathcal{H}_1^{dR}(B/S)_{\tau_2} \to \mathcal{H}_1^{dR}(A/S)_{\tau_2})$ is $\cdots$ of rank $i \cdot n$.

Prop (1) Each Y_i is rep'd by a proj. smooth $\mathbb{F}_{p^2}$-var of dim $n-1$

(2)
$$\begin{tikzcd} & Y_i \arrow[dl, "pr_1^{(i)}"'] \arrow[dr, "pr_2^{(i)}"] & & Y_{i,z} \arrow[d, mapsto] \\ Sh_{1,n-1} & & Sh_{0,n} & z \end{tikzcd}$$

$\forall z \in Sh_{0,n}(\overline{\mathbb{F}}_p)$, $(pr_2^{(i)})^{-1}|_{Y_{i,z}}$ is a closed immersion (for K^p suff. small) ($Y_{i,z}$: fiber).

and $Y_{i,z}$ is isomorphic to

$$Z_i^{(n)} \subseteq Gr(n, i) \times Gr(n, i-1)$$ Frobenius twist

$$" \{ (H_1, H_2) \mid H_2 \subseteq H_1^{(p)} \ \& \ H_1 \subseteq H_2^{(p)} \}$$

(3) $\bigcup_{i=1}^{n} pr_1^{(i)}(Y_i)$ is the supersingular locus of some big Shimura variety.

Geometry of $Z_i^{(n)}$:

$Z_1^{(n)} = Z_n^{(n)} \cong \mathbb{P}^{n-1}$

- $Z_2^{(3)} \subseteq \underset{(x_1, x_2, x_3)}{\mathbb{P}^2} \times \underset{(y_1, y_2, y_3)}{\mathbb{P}^2}$ defined by $\begin{cases} x_1^p y_1 + x_2^p y_2 + x_3^p y_3 = 0 \\ x_1 y_1^p + x_2 y_2^p + x_3 y_3^p = 0. \end{cases}$

Theorem B. Assume $\alpha_{\pi_v, i} \neq \alpha_{\pi_v, j}$, $\forall i \neq j$.

Then the map

$$\mathcal{J}\mathcal{L}_{\pi_f} : H^0(Sh_{0,n}, \overline{\mathbb{F}}_p)^{\oplus n} \xrightarrow{\bigoplus_{i=1}^{n} pr_2^{(i)*}} \bigoplus_{i=1}^{n} H^0(Y_i, \overline{\mathbb{F}}_p) \xrightarrow{\sum_{i=1}^{n} pr_{1,!}^{(i)}} \underbrace{H^{2(n-1)}(Sh_{1,n-1}, \overline{\mathbb{F}}_p)(n-1)_{\pi_f}}_{\text{Tate cycles}}$$

Gysin's map / Cycle class map

everthing is Hecke-equivariant under $\mathcal{H}_K$

$\Rightarrow \mathcal{J}\mathcal{L}_{\pi_f}$ is injective.

Rmk This is the main theorem, which means the contribution of those cycles to the Tate classes is of the max'l dimension.

- Thm B + $\underbrace{(m_G(\pi_f) = m_{G'}(\pi_f))}_{\text{Key Fact, due to White et. al.}} \Rightarrow$ Thm A

Pf of Thm B $M_{\pi_f} : H^0(Sh_{0,n}, \overline{\mathbb{F}}_p)^{\oplus n}_{\pi_f} \xrightarrow{\mathcal{J}\mathcal{L}_{\pi_f}} H^{2(n-1)}(Sh_{1,n-1})(n-1)_{\pi_f}$

$$\xrightarrow{\oplus pr_1^{(i),*}} \bigoplus_{i=1}^{n} H^{2(n-1)}(Y_i)(n-1) \xrightarrow[\mathrm{Tr}\, pr_2^{(i)}]{\cong} H^0(Sh_{0,n}, \overline{\mathbb{F}}_p)^{\oplus n}$$

We can compute the matrix of $M_{\pi f}$ explicitly:

e.g. $n=2$:

$$M_{\pi f} = \begin{pmatrix} -2p & a_v^{(1)} \\ S_v^{-1} a_v^{(1)} & -2p \end{pmatrix} \qquad S_v = a_v^{(n)}$$

$$a_v^{(i)} \longleftrightarrow \begin{pmatrix} \underbrace{p \; \cdots \; p}_{i} & & & \\ & & \underbrace{1 \; \cdots \; 1}_{n-i} & \end{pmatrix}$$ eigenvalue associated to the Hecke operator

$$\det(M_{\pi f}) = 4p^2 - (a_v^{(1)})^2 S_v^{-1} = -p^2 \frac{(\alpha_{\pi_v,1} - \alpha_{\pi_v,2})^2}{\alpha_{\pi_v,1} \cdot \alpha_{\pi_v,2}}.$$

e.g. $n=3$:

$$M_{\pi f} = \begin{pmatrix} 3p^2 & -2p a_v^{(1)} & a_v^{(2)} \\ -2p a_v^{(2)} S_v^{-1} & 3p^4 + a_v^{(1)} a_v^{(2)} S_v^{-1} & -2p a_v^{(1)} \\ a_v^{(1)} S_v^{-1} & -2p a_v^{(2)} S_v^{-1} & 3p^2 \end{pmatrix}$$

$$\det(M_{\pi f}) = \pm p^2 \cdot \frac{\prod_{i \neq j} (\alpha_{\pi_v,i} - \alpha_{\pi_v,j})^2}{\left(\prod_{i=1}^{n} \alpha_{\pi_v,i}\right)^{n-1}}.$$

$\rightsquigarrow$ $Y_{i,z} \cap Y_{j,z'} = \emptyset \Longleftrightarrow z' \in$ Hecke orbit of z

$\uparrow$ by some calculation on Dieudonné modules