Comments on global class field theory

Let F be a number field, the correct statement for global class field theory is:

$$\mathrm{Art}_F : \mathbb{A}_F^\times \Big/ \overline{F^\times F_{\mathbb{R}}^{\times,\circ}} \xrightarrow{\cong} G_F^{\mathrm{ab}}.$$

The subtlety here is that although $F^\times$ is discrete (and hence closed) in $\mathbb{A}_F^\times$, it is no longer so in the finite ideles $\mathbb{A}_{F,f}^\times = \mathbb{A}_F^\times / F_{\mathbb{R}}^\times$.

One can describe the difference before and after taking the closures of $F^\times$ as follows. Consider the open compact subgroup $\widehat{\mathcal{O}}_F^\times = \prod_v \mathcal{O}_{F_v}^\times \subset \mathbb{A}_{F,f}^\times$. Its intersection with $F^\times$ is just the group of global units $\mathcal{O}_F^\times$. The topology induced by smaller open compact subgroups of $\widehat{\mathcal{O}}_F^\times$ on $\mathcal{O}_F^\times$ are just the usual congruence subgroup topology. So the closure of $\mathcal{O}_F^\times$ inside $\widehat{\mathcal{O}}_F^\times$ looks like

$$\mathcal{O}_F^\times = \mu(F) \times \mathbb{Z}^{r_1+r_2-1} \ \subseteq \ \mu(F) \times \widehat{\mathbb{Z}}^{r_1+r_2-1} = \overline{\mathcal{O}_F^\times}.$$

The difference $\widehat{\mathbb{Z}}/\mathbb{Z}$ is a very interesting group; it is uniquely divisible, i.e. given any $x \in \widehat{\mathbb{Z}}/\mathbb{Z}$ and any $n \in \mathbb{N}$, there exists a unique $y \in \widehat{\mathbb{Z}}/\mathbb{Z}$ such that $x = ny$. (check that!) So somehow this does not affect much of the discussion when we consider finite characters out of these groups.

Remark: When $F = \mathbb{Q}$ or an imaginary quadratic field, we do note need to take the completion, because $\mathcal{O}_F^\times$ is finite and hence $F^\times$ is discrete in $\mathbb{A}_{F,f}^\times$.