Semistable local systems

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(Joint with Du, Liu, Shimizu)

Introduction

- K complete DVF of mixed char $(0,p)$.
  w/ perfect residue field k.
- $\pi \in \mathcal{O}_K$ uniformizer.
- X proper $\underbrace{\text{Semistable}}$ p-adic formal sch $/\mathcal{O}_K$.
  étale locally,
  étale over $\mathcal{O}_K\langle T_1, \cdots, T_m, T_{m+1}^{\pm 1}, \cdots, T_d^{\pm 1}\rangle / (T_1 \cdots T_m - \pi)$
- log str M_X: subsheaf assoc to $\mathcal{O}_{X,\text{ét}} \cap \mathcal{O}_{X,\text{ét}}[\frac{1}{p}]^\times \hookrightarrow \mathcal{O}_{X,\text{ét}}$.

<u>Cst - Conj</u> (Fontaine - Jansen) Have rat'l isom

$$H^i_{\text{ét}}(X_{\bar{\eta}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{st} \simeq H^i_{\text{log-cris}}(X_k / W(k)) \otimes_{W(k)} B_{st}$$

+ compatible with action of $G_K = \mathrm{Gal}(\bar{K}/K)$,
Frob, monodromy, etc.

Proved by: Tsuji, Faltings, Nizioł,
Bhatt, Beilinson, Colmez - Nizioł,
Česnavičius - Koshikawa, Koshikawa - Yao.

$\rightsquigarrow$ abstract notion of semistable G_K-rep (Fontaine) via B_{st}.

<u>Q</u> More geometric & integral formulation of semistable rep'ns ?

<u>Thm</u> (Bhatt - Scholze, crys case; Du - Liu)

$$\left\{ \begin{array}{c} \mathbb{Z}_p\text{-lattices of semistable} \\ \text{rep'ns of } G_K \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{log - prismatic F-crystals} \\ \text{on } (\mathcal{O}_K, M_{\mathcal{O}_K})_{\Delta} \end{array} \right\}.$$

Q Comparison of log prismatic cohom with log-cris / étale cohom ?

A Bhatt - Scholze (sm, crys), Koshikawa - Yao.

<u>Main results</u> · Semistable loc sys on X via log prism F-Crystals.

· Comparison b/w log prism cohom & log crys cohom for X w/ coeff's given by F-crystals.

Semistable local system

Let $X_\eta := X^{rig}$, $X_1 := X \times_{\mathbb{Z}_p} \mathbb{Z}_p/p$.

$\mathbb{L}$: $\mathbb{Z}_p$-loc sys on $X_{\eta, \text{proét}}$.

<u>Def</u> (Faltings) $\mathbb{L}$ is <u>semistable</u> if $\exists$ F-isoc $\mathcal{E}$ on $(X_1, M_{X_1})_{CRIS}$

+ isom $\mathcal{E}(\mathbb{B}_{cris}) \xrightarrow{\sim} \mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{B}_{cris}$.

of sheaves of $\mathbb{B}_{cris}$-mods on $X_{\eta, \text{proét}}$ compatible w/ Frob.

<u>Rmk</u> Can show:

· above isom is compatible w/ fil'n for unique filtered F-isoc str on $\mathcal{E}$ (using rigidity of de Rham, c.f. Liu - Zhu).

· $\mathbb{L}$ being semistable is indep on sst model of X_η.

· $(X, M_X)_\Delta$ bdd log prisms $(A, I, M_{\text{Spf} A})$ w/ strict map $(\text{Spf } A/I, M_{\text{Spf } A/I}) \to (X, M_X)$.

Strict flat top

<u>E.g.</u> $X = \text{Spf } \mathcal{O}_K \langle T_1, \cdots, T_m, T_{m+1}^{\pm 1}, \cdots, T_d^{\pm 1} \rangle / (T_1 \cdots T_m - \pi)$

Breuil-Kisin log prism: $\mathfrak{S} = W(k)\langle T_1, \cdots, T_m, T_{m+1}^{\pm 1}, \cdots, T_d^{\pm 1}\rangle [[u]] / (T_1 \cdots T_m - t)$

$\rightsquigarrow \ (\mathfrak{S}, (E(u)), N^m)^a \in (X, M_X)_{\Delta}$

$N: p_i \longmapsto T_i \quad (\delta(T_i) = 0)$.

Thm (Du-Liu-Moon-Shimizu) Have nat'l equiv

$$\left\{ \begin{array}{c} \text{semist } \mathbb{Z}_p\text{-loc sys} \\ \text{on } X_\eta \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{analytic prism F-crys} \\ \text{on } (X, M_X)_{\Delta} \end{array} \right\}.$$

For $(A, I, M_{\mathrm{Spf}\, A}) \in (X, M_X)_{\Delta}$,

$$\mathrm{Vect}^{\varphi}(A, I, M_{\mathrm{Spf}\, A}) := \left\{ (\mathcal{E}_{\Delta}, \varphi_{\mathcal{E}_{\Delta}}) : \begin{array}{l} \mathcal{E}_{\Delta} \in VB(\mathrm{Spec}\, A) \\ \varphi_{\mathcal{E}_{\Delta}} : \varphi_{\Delta}^* \mathcal{E}_{\Delta}[\tfrac{1}{I}] \xrightarrow{\sim} \mathcal{E}_{\Delta}[\tfrac{1}{I}] \end{array} \right\}$$

$$\mathrm{Vect}^{\varphi, an}(A, I, M_{\mathrm{Spf}\, A}) := \left\{ (\mathcal{E}_{\Delta}, \varphi_{\mathcal{E}_{\Delta}}) : \begin{array}{l} \mathcal{E}_{\Delta} \in VB(\mathrm{Spec}\, A \setminus V(p, I)) \\ \varphi_{\mathcal{E}_{\Delta}} : \varphi_{\Delta}^* \mathcal{E}_{\Delta}[\tfrac{1}{I}] \xrightarrow{\sim} \mathcal{E}_{\Delta}[\tfrac{1}{I}] \end{array} \right\}$$

$\rightsquigarrow$ (anal) prism F-crys $:= \lim\limits_{\substack{(A, I, M_{\mathrm{Spf}\, A}) \\ \in (X, M_X)_{\Delta}}} \mathrm{Vect}^{\varphi, (an)}(A, I, M_{\mathrm{Spf}\, A})$.

<u>One direction</u> Given anal prism F-crys $\mathcal{E}_{\Delta}$, can construct its étale realization & crys realization $\mathcal{E}_{cris} \in (X, M_X)_{CRIS}$

And show they are associated.

<u>For converse</u> Purity!

Let $\{\eta_1, \cdots, \eta_m\}$ generic fibres of irred comps of $|X|$.

Thm (DLMS) Purity for anal prism F-crys / semist loc sys holds.

E.g. R = above, $X = \mathrm{Spf}\, R$

$\rightsquigarrow \ \widehat{\mathcal{O}}_{X, \eta_i} = \widehat{R_{T_i}}, \quad i = 1, \cdots, m \quad$ CDVR, π = unif.

Then $\mathbb{Z}_p$-loc sys $\mathbb{L}$ is semist on $X \iff \mathbb{L}|_{\eta_i}$ semist, $\forall i$.

Rmk · Purity $\Rightarrow$ $\mathbb{L}$ being ss is indep of ss model of X_η.

· Guo-Yang: used purity + rigidity of de Rham (Liu-Zhu)
& p-adic loc monodromy for inperfet res field
(Morita, Ohkuki).

To show: $\mathbb{L}$ ss $\Leftrightarrow$ its red'n to any cl pt of X_η is ss.

<u>Log-prism - Cryst Comparison</u>

$f: (X, M_X) \longrightarrow (Y, M_Y)$

Assume X proper "Semist" p-adic formal sch / Y.

Y qcqs sm p-adic formal sch / $\mathcal{O}_K$.

$(\mathcal{E}_\Delta, \varphi_{\mathcal{E}_\Delta})$: prism F-crys on $(X, M_X)_\Delta$.

cryst real'n $\rightsquigarrow$ $(\mathcal{E}_{cris}, \varphi_{\mathcal{E}_{cris}})$: F-isoc on $(X, M_X)_{CRIS}$.

Let $(A, (p), M_A) \in (Y, M_Y)_\Delta$ + some conditions.

$J \subset A$ p-completion, PD-ideal, $p \in J$.

$$A/p \longrightarrow A/J \xrightarrow{\ \tilde\varphi\ } A/p$$

with composite $A/p \xrightarrow{\varphi} A/p$.

$\rightsquigarrow$ $X_{A/J} = X \times_Y A/J$, $X^{(1)}_{A/J} := X_{A/J} \times_{A/J, \tilde\varphi} A/p$

Thm (DLMS) (1) $(Rf_{cris*}\mathcal{E}_{cris}, Rf_{cris*}\varphi_{\mathcal{E}_{cris}})$ is F-crys
on (Y, M_Y) in derived sense.

(2) $Rf_{\Delta,*}(\mathcal{E}_\Delta)$ is a perfect complex of $\mathcal{O}_\Delta$-crys on $(Y, M_Y)_\Delta$

(3) Have nat'l Frob-equiv isom of perfect $A[\frac{1}{p}]$-complex:

$$R\Gamma\left(\left((X^{(1)}_{A/J}, M^{(1)}_{A/J})/(\varphi_* A, \varphi_* M_A)\right)_\Delta, \mathcal{E}^{(1)}_\Delta\right)[\tfrac{1}{p}]$$
$$\simeq \varphi_* Rf_{cris,*}\mathcal{E}_{cris}(A, A/J, M_A)[\tfrac{1}{p}]$$

<u>Rmk</u>

- For (2), Tian: sm case.
- For (3), Koshikawa: const coeff, case for $X/\mathcal{O}_K$ log-sm of Cartier type.