On the cohomology of Shimura varieties with torsion coefficients

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<u>Setting</u> F CM field, G quasi-split unitary grp $/F^+$ in $2n$ variables.

$K \subset G(\mathbb{A}_{F,f})$ sufficiently small compact open subgrp.

$\rightsquigarrow$ $S_K / \mathbb{Q}$ Shimura var of dim d.

Want to understand

$$H^*(S_K(\mathbb{C}), \mathbb{F}_\ell) \circlearrowleft \mathbb{T} = \bigotimes'_{v \notin S} \mathcal{H}_v$$

where S = finite set of bad places.

<u>Thm 1</u> (Scholze, 13)

Given $\mathfrak{m} \subset \mathbb{T}$ a max ideal in support of $H^*(S_K(\mathbb{C}), \mathbb{F}_\ell)$,

$\exists$ conti semisimple Gal rep

$$\bar{\rho}_{\mathfrak{m}} : \mathrm{Gal}(\bar{F}/F) \longrightarrow GL_{2n}(\bar{\mathbb{F}}_\ell)$$

characterized by:

- at $v \notin S$, $\bar{\rho}_{\mathfrak{m}}|_{G_{F_v}}$ unram & char poly of $\bar{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ is given in terms of Hecke eigenvals at v.

<u>Thm 2</u> (Caraini-Scholze '19, Koshikawa '21', Hamann-Lee '23).

If $\bar{\rho}_{\mathfrak{m}}$ is sufficiently generic at an auxilliary prime p,

then $H^i(S_K(\mathbb{C}), \mathbb{F}_\ell)_{\mathfrak{m}} \neq 0 \Rightarrow i \geq d$

& $H^i_c(S_K(\mathbb{C}), \mathbb{F}_\ell)_{\mathfrak{m}} \neq 0 \Rightarrow i \leq d$.

In particular, if $\mathfrak{m}$ is non-Eisenstein (i.e. $\bar{\rho}_{\mathfrak{m}}$ is abs irred)

$$H^i(S_K(\mathbb{C}), \mathbb{F}_\ell)_{\mathfrak{m}} \neq 0 \Rightarrow i = d.$$

* "<u>Sufficiently generic</u>" means

(1) In setting of [CS] & [Koshikawa],

- $\exists\, p$ which splits completely in F,
- $\forall\, v | p$ prime of F,

  $\bar{\rho}_{\mathfrak{m}}|_{G_{F_v}}$ is unramified
- if $\{\alpha_1, \cdots, \alpha_{2n}\}$ eigenvals of $\bar{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$

  then $\alpha_i/\alpha_j \neq p^{\pm 1} \pmod{p}$ for $i \neq j$.

(2) In setting of [Hamann-Lee],

- $\exists\, p$ unramified in F, every $\bar{v}|p$ in F^+ splits in F.
- $\forall\, v|p$ prime of F,

  $\bar{\rho}_{\mathfrak{m}}|_{G_{F_v}}$ is unramified
- if $\{\alpha_1, \cdots, \alpha_{2n}\}$ eigenvals of $\bar{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$

  then $\alpha_i/\alpha_j \in \{1, q_v^{\pm 1}\}$, $i \neq j$.

Can reformulate Thm 2 by localizing at $\mathfrak{m}_p \subset \mathcal{H}_p = \bigotimes_{v|p} \mathcal{H}_v$.

<u>Similar results for other Shimura varieties</u>:

- Boyer (Harris-Taylor Sh vars)
- Caraiani-Tamiozzo (Hilbert)
- Santos '23 (PEL type A).
- Hamann-Lee (e.g. GSp_4):

robust strategy, assuming compatibility of Fargues-Scholze,
w/ classical local Langlands.
(conditional).

Using work of Arthur: Can pass to other grps.

<u>Note</u> All above use geometry of HT period map.

<u>Application</u>

$P = M \ltimes U \subset G$ Siegel parabolic

$M = \mathrm{Res}_{F/F^+} GL_n$

S_K^{BS} = Borel-Serre cpt'n of S_K.

∂S_K^{BS} contains a torus bundle over the loc sym space $X^M_{K_M}$.

(Can see non-alg objs here)

<u>Cor</u> (of Thm 2) Have a $\mathbb{T}$-equivariant diagram

$$\begin{tikzcd} \underbrace{H^d(S_K(\mathbb{C}), \mathbb{Z}_\ell)_{\mathfrak{m}}[\tfrac{1}{\ell}]}_{\substack{\text{well-understood} \\ \text{via essentially self-dual case.}}} & H^d(S_K(\mathbb{C}), \mathbb{Z}_\ell)_{\mathfrak{m}} \arrow[l, hook'] \arrow[r, two heads] & H^d(\partial S_K^{BS}, \mathbb{Z}_\ell)_{\mathfrak{m}} \\ & (d = \text{middle dim}) & H^*(X^M_{K_M}, \mathbb{Z}/\ell^n)_{\mathfrak{m}'} \arrow[u, hook] \end{tikzcd}$$

<u>Method of proof</u>

- Study the geometry of π_{HT}.
  base change from $\mathbb{Q}$ to $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$,
  work w/ adic spaces.

$$\mathcal{S}_{K^p} \xrightarrow{\pi_{HT}} \mathcal{F}\ell_{G,\mu} = (G/P_\mu)^{ad}_{\mathbb{C}_p}, \quad P_\mu = \text{Siegel parabolic}$$

$$\downarrow BL$$

$$\mathrm{Bun}_{G_{\mathbb{Q}_p}} = \coprod_{b \in B(G)} \underbrace{\mathrm{Bun}^b_{G_{\mathbb{Q}_p}}}_{\simeq [*/\tilde{G}_b]}, \quad \tilde{G}_b = G_b(\mathbb{Q}_p) \ltimes \cdots$$

where G_b = inner form of Levi subgrp of G.

$\rightsquigarrow \mathcal{F}\ell_{G,\mu} = \coprod_{b \in B(G,\mu)} \mathcal{F}\ell^b_{G,\mu}$

Thm (Zhang '23)

$$\begin{tikzcd} \mathcal{S}_{K^p} \arrow[r, "\pi_{HT}"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{F}\ell_{G,\mu} \arrow[d, "BL"] \\ \mathrm{Igs}_{K^p} \arrow[r, "\bar{\pi}_{HT}"] & \mathrm{Bun}_{G_{\mathbb{Q}_p}} \end{tikzcd}$$

Igusa stack (small v-stack).

- If $x : \mathrm{Spa}(C, C^+) \longrightarrow \mathcal{F}\ell^b_{G,\mu}$ $(b \in B(G,\mu))$
  then $\mathrm{Ig}^b \sim \pi^{-1}_{HT}(x)$.

<u>Want to study</u> $(R\pi_{HT,*}\mathbb{F}_\ell)_{\mathfrak{m}^p}$ on $\mathcal{F}\ell_{G,\mu}$
or at $\mathfrak{m}_p \subset \mathcal{H}_p$.

This descends to $R\bar{\pi}_{HT,*}\mathbb{F}_\ell$ on $\mathrm{Bun}_{G_{\mathbb{Q}_p}}$.

ULA sheaves on Bun_G

Let $\mathcal{F} = R\bar{\pi}_{HT,*}\mathbb{F}_\ell$ + pretend S_K compact.
perverse for certain t-str on $\mathrm{Bun}_{G_{\mathbb{Q}_p}}$.

Look at $\mathcal{F}_{\varphi_{\mathfrak{m}_p}}$ = localization at Langland param using spectral action of FS.

$\rightsquigarrow$ Hamann-Lee : $\mathcal{F}_{\varphi_{\mathfrak{m}_p}} \simeq \bigoplus_{b \in B(G,\mu)_{un}} H^{d_b}(\mathrm{Ig}^b, \mathbb{F}_\ell)_{\varphi_{\mathfrak{m}_p}}$.

(assuming $\varphi_{\mathfrak{m}^p}$ generic).

Hamann: Recover $R\Gamma(\mathcal{S}_K, \mathbb{F}_\ell)_{\mathfrak{m}^p}$ by applying T_μ to $\mathcal{F}_{\varphi_{\mathfrak{m}^p}}$.

T_μ is perverse t-exact.

If $\mathfrak{m}^p \subset \mathbb{T}^p$ is non-Eisenstein & generic at p,

then can show · $\mathcal{F}_{\mathfrak{m}^p}$ is perverse.

· $\mathcal{F}_{\mathfrak{m}^p, \varphi_{\mathfrak{m}^p}} \simeq \mathcal{F}_{\mathfrak{m}^p}$ where $\varphi_{\mathfrak{m}^p} = ((\bar{\rho}_{\mathfrak{m}}|_{G_{F_v}})^{ss})_{v|p}$.

Hamann: computed a Hecke eigensheaf $\mathcal{G}_\varphi$.