Robba site and Robba cohomology

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<u>Goal</u> To develop an arithmetic p-adic cohom theory for rigid-analytic varieties / local fcn field $\mathbb{F}_p((t))$.
(with overcovergent str. ⇝ dagger spaces).

§1 Motivation

p prime, F = fin ext'n of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$.

Recall: geometric realization of local Langlands & JL for $GL_n(F)$ (using rigid geometry).

<u>Def'n</u> $\mathcal{H} = \mathbb{P}_F^{n-1} - \bigcup_{\substack{Y \subseteq \mathbb{P}_F^{n-1} \\ F\text{-rational} \\ \text{hyperplane}}} Y$ Drinfeld halfspace.

<u>note</u>. $n=2$: $\mathcal{H} = \mathbb{P}^1 - \mathbb{P}^1(F)$ ⟷ analog to $\mathfrak{h}^{\pm} = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$.

· $\mathcal{H} \circlearrowleft GL_n(F)$ rigid-analytic space $/F$.

Drinfeld construction: $(\mathcal{M}_m)_m$ a tower of finite étale covers.

$$\begin{array}{c} D^\times \circlearrowright (\mathcal{M}_m)_m \\ \big\downarrow \\ \coprod_{\mathbb{Z}} \mathcal{H}_{\breve{F}} \circlearrowleft GL_n(F). \end{array}$$

where D div alg $/F$ with inv $1/n$.

For $\ell \neq p$, set

$$H_c^{n-1}(\mathcal{M}_\infty, \overline{\mathbb{Q}}_\ell) := \varinjlim_m H_{\text{ét},c}^{n-1}(\mathcal{M}_{m,\widehat{\overline{F}}}, \overline{\mathbb{Q}}_\ell).$$
$$\circlearrowright \; GL_n(F) \times D^\times \times W_F \leftarrow \text{Weil grp.}$$

<u>Thm</u> (Harris, Harris-Taylor; Hausberger), char $F = 0$.

For π supercuspidal rep of $GL_n(F)$,

$$H_c^{n-1}(\mathcal{M}_\infty, \overline{\mathbb{Q}}_\ell)[\pi] = \underbrace{JL(\pi)}_{LJL} \boxtimes \Big(\underbrace{r(\pi)}_{LLC} \otimes |\cdot|^{\frac{1-n}{2}}\Big).$$

(as rep of $D^\times \times W_F$).

Q How about p-adic étale cohom?

- char = 0, Colmez–Dospinescu–Nizioł ($n=2$, $F=\mathbb{Q}_p$)
  p-adic pro-étale cohom of Drinfeld tower realizes classical LJL + p-adic Langlands for $GL_2(\mathbb{Q}_p)$.
- char = p, p-adic étale cohom is very pathological.

Want Develop a p-adic cohom theory for rig var / $\mathbb{F}_p((t))$ s.t. cohom of $\mathcal{M}_\infty$ realizes classical LLC & LJL.

Main result (Shimizu, rough form)

To each smooth dagger sp / F ($/\mathbb{F}_p((t))$, fin),
one can functorially associate a ringed site $(X_{\mathcal{R}}, \mathcal{O}^\dagger)$
s.t. its cohom $H^*(X_{\mathcal{R}}, \mathcal{O}^\dagger)$ is a (φ, ∇)-mod / Robba ring $\mathcal{R}$.

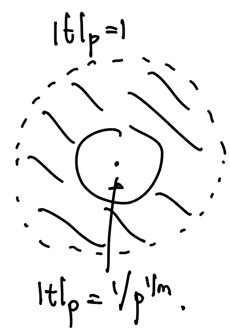
§2 Robba ring $\mathcal{R}$ & geometric ideas

From now on: $F = \mathbb{F}_p((t))$.

Q Robba ring $\mathcal{R} = \bigcup_{m>0} \Gamma(\mathcal{S}_{m,\mathbb{Q}_p}, \mathcal{O})$

$\mathcal{O}$: sheaf of rigid-analytic fcns.

where $\mathcal{S}_{m,\mathbb{Q}_p}$ = half open annulus $1/p^{1/m} \le |t|_p < 1$ over $\mathbb{Q}_p$



$\mathcal{R}$ is equipped with $\varphi \curvearrowright \mathcal{R}$ & $\nabla = \frac{d}{dt} \curvearrowright \mathcal{R}$

$\varphi(t) = t^p$. (standard (φ, ∇)-action).

Why $\mathcal{R}$?

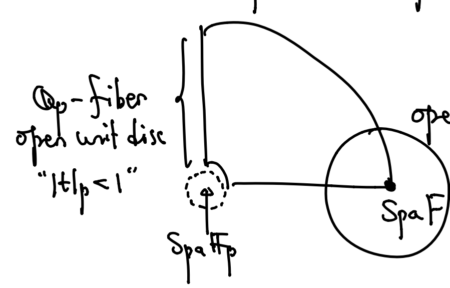
Thm (Marmora) $\exists$ that functor

$$\begin{tikzcd} D_{pst} : (\text{fin } (\varphi,\nabla)\text{-mod}/\mathcal{R}) \arrow[r] & \left(\begin{array}{c}\text{Weil-Deligne rep's of } F \\ \text{valued in } \mathbb{Q}_p^{ur}\end{array}\right) \\ (\text{sm/q-cpt rig var}/F) \arrow[u, dashed, "?"] & \end{tikzcd}$$

Rem Lazda-Pal: defined $\mathcal{R}$-valued rigid cohom
for ALGEBRAIC varieties / $F = \mathbb{F}_p((t))$,
but their construction does not seem to work in analytic setup.

Geom idea Let's consider $\mathrm{Spa}(\mathbb{Z}_p[[t]], \mathbb{Z}_p[[t]]) - \mathrm{Spa}(\mathbb{F}_p)$
with (p,t)-adic top.

This looks like



$\mathbb{Q}_p$-fiber $= \mathcal{Y}_{m,\mathbb{Q}_p}$
$\mathbb{F}_p$-fiber $=$ Spa F.

$\mathbb{F}_p$-fiber = single pt Spa F.

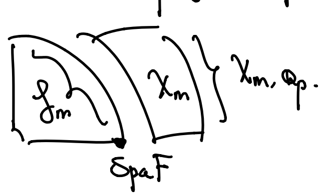
$\mathcal{Y}_m = \mathrm{Spa}(R_m, R_m^\circ)$, $R_m^\circ = \mathbb{Z}_p[[t]]\langle p/t^m\rangle$, $R_m = R_m^\circ[1/t]$.

normalization: $|p/t^m|_p \le 1 \iff \frac{1}{p} \le |t|_p^m$, $|t|_p < 1$.

Upshot (1) Spa F admits open nbd basis $(\mathcal{Y}_m)_m$.

(2) $\mathcal{R} = \bigcup_m \Gamma(\underset{\substack{\uparrow \\ \mathbb{Q}_p\text{-fiber of } \mathcal{Y}_m}}{\mathcal{Y}_{m,\mathbb{Q}_p}}, \mathcal{O})$

Naively



§3 Def & properties of Robba site $(X_{\mathcal{R}}, \mathcal{O}^{+})$

Fix sm dagger sp $X / F = \mathbb{F}_p((t))$.

<u>Def</u> Robba site $X_{\mathcal{R}}$ is def'd as :

<u>obj</u> $((\mathrm{Spa}^{\dagger} B_m)_m, \imath)$ where B_m dagger alg / R_m.

s.t. $B_{m+1} = B_m \otimes^{\dagger}_{R_m} R_{m+1}$.

& $\imath : \mathrm{Spa}^{\dagger} \bar{B} \longrightarrow X$ mor of dagger spaces / F. $\bar{B} := B_m/(p)$.

Also : natural notion of morphisms + coverings $\rightsquigarrow$ $X_{\mathcal{R}}$ site

+ sheaf of $\mathcal{R}$-algebras $\mathcal{O}^{+}((\mathrm{Spa}^{\dagger} B_m)_m, \imath) = \bigcup_m \Gamma((\mathrm{Spa}^{\dagger} B_m)_{\mathbb{Q}_p}, \mathcal{O}^{+})$.

$\rightsquigarrow$ Get a ringed site $(X_{\mathcal{R}}, \mathcal{O}^{+})$, $F\langle T_1, \cdots, T_n\rangle^{\dagger} \subset F\langle T_1, \cdots, T_n\rangle$.

<u>Thm</u> (1) Robba cohom $H^{*}(X_{\mathcal{R}}, \mathcal{O}^{+})$ is a (φ, ∇)-mod / $\mathcal{R}$.

(2) Assume X/F admits a compatible family of lifts $\mathcal{X}_m / R_m$.

Then $H^{*}(X_{\mathcal{R}}, \mathcal{O}^{+}) \simeq \varinjlim_m H^{*}_{\mathrm{odR}}(\mathcal{X}_{m,\mathbb{Q}_p} / \mathcal{S}_{m,\mathbb{Q}_p})$.

where H_{odR} = relative overconvergent de Rham cohom.

(3) If $\dim X = 1$ & φ-cpt, then $H^{*}(X_{\mathcal{R}}, \mathcal{O}^{+})$ is finite free / $\mathcal{R}$.

$\rightsquigarrow$ $D_{\mathrm{pst}}(H^{*}(X_{\mathcal{R}}, \mathcal{O}^{+}))$ WD-rep'n of F.

§4 Proof ideas

(1) $\varphi \curvearrowright H^{*}(X_{\mathcal{R}}, \mathcal{O}^{+})$ comes from functoriality w.r.t. X/F,

∇ : $X_{\mathcal{R}}^{\mathrm{sm}} \subset X_{\mathcal{R}}$ consisting of "smooth" objects.

$\rightsquigarrow$ $H^{*}(X_{\mathcal{R}}, \mathcal{O}^{+}) = H^{*}(X_{\mathcal{R}}^{\mathrm{sm}}, \mathcal{O}^{+})$

$\uparrow$ imitate Katz-Oda's constr of Gauss-Manin conn.

$$0 \longrightarrow \Omega^{\dagger, \bullet - 1}_{X_{\mathcal{R}}^{\mathrm{sm}}/\mathcal{R}} \otimes dt \longrightarrow \Omega^{\dagger, \bullet}_{X_{\mathcal{R}}^{\mathrm{sm}}/\mathbb{Q}_p} \longrightarrow \Omega^{\dagger, \bullet}_{X_{\mathcal{R}}^{\mathrm{sm}}/\mathcal{R}} \longrightarrow 0 .$$

(3) <u>Fact</u> $\dim X = 1$, $X \subseteq C$ open, C = (alg) sm proj curve

s.t. $Y = C - X = \coprod(\text{open disk}/F)$.

$\rightsquigarrow$ $\underbrace{H^n(C_R, X_R)} \to H^n(C_R) \to H^n(X_R) \to H^{n+1}(C_R, X_R) \to H^{n+1}(C_R)$.

Gysin isom to reduce to H(open disk).

C liftable $\Rightarrow$ finite by (2).