Hasse-Weil zeta functions of orthogonal Shimura varieties

Yihang Zhu

§ Motivation

X proper sm $/\mathbb{Q}$, p prime of good reduction,

$$\rightsquigarrow \quad \zeta_p(X,s) := \exp\Big(\sum_{n=1}^{\infty} \# X(\mathbb{F}_{p^n}) \cdot p^{-ns}/n\Big)$$

$$= \prod_{i=1}^{\infty} \det(1-\mathrm{Frob}_p T \mid H^i_{\text{ét}})^{(-1)^{i+1}}_{T=p^{-s}}.$$

Conjectured: $\zeta(X,s) = \prod_p \zeta_p(X,s)$ (a.a. p) should have good analytic properties.

<u>Thm</u> (Eichler–Shimura)

$X = X_0(N)/\mathbb{Q}$. Then

$$\zeta(X,s) = \zeta(s)\,\zeta(s-1)\cdot \prod_{i=1}^{g} (\text{Mellin transforms of } f_i)$$

with (f_i) = basis of $S_2(\Gamma_0(N))$.

Generalization: To find Hasse-Weil ζ for Shimura varieties

$\updownarrow$

Automorphic L-functions.

Langlands's strategy:

ζ of Shimura vars $\longleftrightarrow$ traces of Hecke operators & Frob on Cohom of Sh vars.

$\updownarrow$ precise conjs today

Spectral info $\xleftrightarrow{\text{Arthur-Selberg trace formula}}$ Group theory info

<u>Rmk</u> Grp-theoretical info: geometry of red gps,
conj classes inside the gps,
integral test fcns on its orbits, etc.

§ Results

Fix a Shimura datum (G, X).

$\rightsquigarrow$ $E/\mathbb{Q}$ reflex field, $\forall K \subset G(\mathbb{A}_f)$ cpt open, small enough

$\rightsquigarrow$ Sh_K over E (q-proj & sm).

w/ $Sh_K(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K$.

<u>E.g.</u> (1) $G = GL_2$, $X = \mathcal{H}^{\pm}$ $\rightsquigarrow$ modular curves

(2) V = quadratic space / $\mathbb{Q}$ of sgn $(n, 2)$

$G = SO(V)$, $X = \{$negative def'te oriented planes in $V_{\mathbb{R}}\}$.

$\rightsquigarrow$ orthogonal S.V. ($E = \mathbb{Q}$, $\dim_E = n$).

Assume $E = \mathbb{Q}$. Fix p: hyperspecial for K.

i.e. $K = K^p K_p$, $K^p \subset G(\mathbb{A}_f^p)$, $K_p \subset G(\mathbb{Q}_p)$ hyperspecial.

<u>Thm</u> (Zhu) The following conj is true for orthogonal Shimura vars.

<u>Conj</u> (Kottwitz, 1990)

$$\mathrm{Tr}(f^{p,\infty} \times \Phi^j \mid IH^*) = \sum_H \iota(G, H) \cdot ST^H(f_H).$$

where · $f^{p,\infty} \in \mathcal{H}(G(\mathbb{A}_f^p), K^p)$ Hecke operator

· Φ geom Frob at p, $j \in \mathbb{N}$.

· IH^* intersection cohom of $\overline{Sh}_K$.

with $\overline{Sh}_K$ = Baily-Borel (minimal) compactification of Sh_K.

(Implicit in conj: $Gal \circlearrowright IH^*$ should be unram at p.)

Rmk Sh_K is a proj & normal / E

But it can be quite singular.

- H endoscopic data of G
- ST^H geometric side of stable trace formula for H
- f_H prescribed function in $H(\mathbb{A})$.

Rmks (1) Also have:

Conj 2 (Kottwitz)

$$\mathrm{Tr}(f^{p,\infty} \times \Phi^j \mid H^*_c(Sh_K \times \operatorname{Spec} \bar{\mathbb{Q}})) = \sum_H \iota(G,H)\, ST^H_e(f_H)$$

{ both sides are easier to compute

ST^H_e: elliptic part of ST^H

Conj 2 is proved by Kottwitz for PEL type S.V.s

by Kisin-Shin-Zhu for Hodge type S.V.s

& some abelian type S.V.s.

Problem ST^H_e does not reveal spectral info well.

(2) Conj is proved for GSp, GU(p,q) by Morel (2010).

(3) Conj + Arthur's Conjectures

(about parametrizing & multiplicities of autom reps)

$\Rightarrow$ a full description of IH^*

$\Rightarrow$ automorphy of $\zeta(IH^*)$.

(4) Analogue of Conj w/o Frob is due to

$$\text{Arthur} \xlongequal[\text{(known)}]{\text{Zucker's conj}} \text{Goresky-Kottwitz}$$

+ Arthur's stabilization of invariant trace formula

§ Idea of proof

By Calculating both sides.

By work of Morel $\Rightarrow$ $Tr(IH^*) = \sum_{M \subset G} Tr_M$.

$\uparrow$ Levi subgps /$\mathbb{Q}$ up to $G_{\mathbb{Q}}$-conj

where $Tr_G = Tr(H_c^*(Sh_K \times Spec\,\bar{\mathbb{Q}}))$.

Also, $ST^H(f_H) = \sum_{M' \subset H} ST^H_{M'}(f_H)$ w/ $ST^H_H = ST^H_e$

By conj 2 in the case proved by KSZ:

$$Tr_G = \sum_H \iota(G,H)\, ST^H_H(f_H).$$

$\leadsto$ Remains to show

$$\sum_{M \subsetneq G} Tr_M = \sum_H \iota(G,H) \sum_{M' \subsetneq H} ST^H_{M'}(f_H).$$

In fact,

Can classify some of pairs (H, M') in terms of $M \subset G$.

$\leadsto$ Need to show:

- $Tr_M = \sum_{(H,M') \sim M} \iota(G,H)\, ST^H_{M'}(f_H)$, $\forall M \subsetneq G$.
- $0 = \sum_{\substack{\text{the rest} \\ (H,M') \not\sim M}} \iota(G,H)\, ST^H_{M'}(f_H)$.

Contributions from $\mathbb{R}$:

LHS: truncated grp cohom of unipotent radicals of parabolic subgps of G.

RHS: Harish-Chandra characters of descrete series reps of $H(\mathbb{R})$, valued on non-compact maximal tori of $H(\mathbb{R})$.

$\leadsto$ Need to find an identity b/w linear combinations of them (signs matter!)

Contributions from p:

LHS: count pts on boundary strata (modular curves & pts).

RHS: e.g. $M \subset G$ Levi, $M_{\mathbb{R}} = GL_2 \times SO(n-2,0)$

Two kinds of contributions from p to RHS

(i) Counting pts on $Sh(GL_2)$

(ii) Counting pts on $Sh(SO(n-2,0))$ do not exist.

$\rightsquigarrow$ To show good terms remain & bad terms cancel (after intersection w/ signs from arch calculation).

- To get correct signs at ∞, need to compute the sign b/w different normalizations of transfer functors for $(G_{\mathbb{R}}, H_{\mathbb{R}})$.