Anti-cyclotomic Iwasawa theory for Rankin-Selberg products (1/3)

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Plan Lect. 1: Motivation, statement, p-adic L-function,
+ other preparation works (on both sides).
Lect. 2: Iwasawa bipartite Euler system.
Lect. 3: Finish the proof.

$F/\mathbb{Q}$ number field.

π cuspidal autom rep $\circlearrowleft GL_N(F)$.

$\rightsquigarrow L(\pi_v) \in \mathbb{C}$ (to be a family)

$\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ conti char (quasi-char).

all such χ parametrized by a $\underbrace{\text{1-dim'l}}_{\text{has only one direction}}$ ind-$\mathbb{R}$-mfld.

$\rightsquigarrow \chi = |\cdot|^s$ (can take this).

$\rightsquigarrow L((\pi \otimes |\cdot|^s)_v)$.

Define $\prod_v L((\pi \otimes |\cdot|^s)_v) =: L(s, \pi)$, $\operatorname{Re} s \gg 0$.

<u>p-adic analogue</u>

$\mathbb{C}_p$ = completed alg closure of $\mathbb{Q}_p$

$\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}_p^\times$ cont char

For simplicity, χ unr outside p.

E.g. (1) $F = \mathbb{Q}$, $\chi : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \to \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$

(2) $F/\mathbb{Q}$ imag quad, p split in F (will focus on this)

$$\chi : F^\times \backslash \mathbb{A}_F^\times \to \underset{\substack{\uparrow \\ \text{up to a finite grp}}}{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} \to \mathbb{C}_p^\times .$$

(3) $F/\mathbb{Q}$ tot real of deg d, p completely split in F

χ-space is conjectually 1-dim'l (still open).

<u>Class group for cyclotomic field</u>

p prime, $\bigcup_{c \geq 0} \mathbb{Q}(\mu_{p^c})$. $\mathbb{Q}_c := \mathbb{Q}(\mu_{p^c})$

Want $Cl(\mathbb{Q}(\mu_{p^c}))[p^k]$, $(\mathbb{Z}/p^k\mathbb{Z})[\mathrm{Gal}(\mathbb{Q}_c/\mathbb{Q})]$-module

Iwasawa's main idea: consider

$$\varprojlim_c \varprojlim_k (\mathbb{Z}/p^k\mathbb{Z})[\mathrm{Gal}(\mathbb{Q}_c/\mathbb{Q})] \cong \underbrace{\mathbb{Z}_p[\![T]\!]}_{\substack{\text{Iwasawa alg,} \\ \text{with nice structure.}}}$$

Control $Cl(\mathbb{Q}(\mu_{p^c}))[p^k]$ by $\mathbb{Z}_p[\![T]\!]$:

(Arithmetic side)

$Cl(\mathbb{Q}(\mu_{p^c}))[p^k]$

$\downarrow$

$\boxed{\mathfrak{X}}$ f.g. module over $\mathbb{Z}_p[\![T]\!]$.

(Analytic side)

$(\mathbb{Z}_p^\times)_{\text{free}} \xrightarrow{\chi} \mathbb{C}_p^\times$ parametrized by

$\cong \mathbb{Z}_p$

generic fibre of $\mathrm{Spf}\, \mathbb{Z}_p[\![T]\!]$

$\rightsquigarrow$

bounded p-adic analytification $\boxed{\mathcal{L}} \in \mathbb{Z}_p[\![T]\!]$ of L-func

<u>Iwasawa main conj</u> (char ideal of $\mathfrak{X}$) = $(\mathcal{L})$.

<u>Setup</u> $\cdot$ $\bar{\mathbb{Q}} \subseteq \mathbb{C}$ fixed.

- A_0, A_1 elliptic curves $/\mathbb{Q}$ with no CM.
- $F/\mathbb{Q}$ imag quad field.
- p odd prime split in F.
  s.t. A_0, A_1 good ordinary reduction at p

<u>Anticyclotomic</u>

Let F_∞ max abelian ext of F unram outside p,
s.t. (1) $\mathrm{Gal}(F_\infty/F)$ torsion-free
(2) complex conj acts on $\mathrm{Gal}(F_\infty/F)$ by -1.

$\Rightarrow \mathrm{Gal}(F_\infty/F) \cong \mathbb{Z}_p$.

Call F_∞ the anticyclotomic $\mathbb{Z}_p$-ext'n of F.

<u>Notations</u>
- $\Lambda := \mathbb{Z}_p[\![\mathrm{Gal}(F_\infty/F)]\!]$, $\Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.
- $F_c \subset F_\infty$ unique subfield of cond p^c.
- $T_p(A_i) = \varprojlim_k A_i[p^k](\bar{\mathbb{Q}})$, $i = 0, 1$.
  $\mathbb{Z}_p[\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})]$-mod.
- $T_0 := T_p(A_0)$, $T_1 = (\mathrm{Sym}^2 T_p(A_1))(-1)$
- $T = T_0 \otimes_{\mathbb{Z}_p} T_1$ rank 6 over $\mathbb{Z}_p$.
  (Rankin-Selberg $\leftrightarrow GL_6$ by Langlands)
- $V := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, $W := V/T$.
  (note: $V^\vee \cong V(-1)$.)

Let F' number field, v place of F'.

$$H^1_f(F', V) \subseteq H^1(F', V).$$

def'd by
$$\begin{tikzcd} H^1_f(F',V) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \prod_v H^1_f(F'_v,V) \arrow[d] \\ H^1(F',V) \arrow[r] & \prod_v H^1(F'_v,V) \end{tikzcd}$$

where the local Bloch-Kato Selmer grp

$$H^1_f(F'_v,V) = \begin{cases} 0, & v \nmid p \\ \ker(H^1(F'_v,V) \to H^1(F'_v, V \otimes_{\mathbb{Q}_p} B_{cris})), & v \mid p. \end{cases}$$

parametrizing crystalline ext'ns $\tilde{V}$

$$0 \to \underset{\circlearrowleft \atop Gal(\bar{F}_v/F_v)}{V} \to \tilde{V} \to \mathbb{Q}_p \to 0 \quad (\otimes B_{cris} + \text{good red'n}).$$

Naively,

$$\varprojlim_{F \subset F' \subset F_\infty} H^1_f(F',V) = \varprojlim_{F} \mathbb{Q}_p[Gal(F'/F)] = \mathbb{Q}_p[\![T]\!] \supset \mathbb{Z}_p[\![T]\!] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

not f.g. / $\Lambda_{\mathbb{Q}}$.

<u>Propagation of Selmer structure</u>

We have $H^1_f(F'_v,V)$. Consider

$$\begin{tikzcd} H^1_f(F'_v,T) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & H^1(F'_v,T) \arrow[d] \\ H^1_f(F'_v,V) \arrow[r, hook] \arrow[d, "\text{imag}"] & H^1(F'_v,V) \arrow[d] \\ H^1_f(F'_v,W) \arrow[r, hook] & H^1(F'_v,W) \end{tikzcd}$$

For $F' \subseteq F''$, $\quad H^1_f(F',T) \underset{\text{cores}}{\overset{\text{res}}{\rightleftarrows}} H^1_f(F'',T)$.

$$\rightsquigarrow \quad \varprojlim_{F \subset F' \subset F_\infty} H^1_f(F',T) =: \mathcal{S}(F,T) \quad \text{f.g. mod} / \Lambda.$$

& $\varinjlim_{F \subset F' \subset F_\infty} H^1_f(F', W)$ + Pontryagin dual

Define $\mathcal{X}(F, W) := \mathrm{Hom}_{\mathbb{Z}_p}\left(\varinjlim_{F \subset F' \subset F_\infty} H^1_f(F', W), \mathbb{Q}_p/\mathbb{Z}_p\right)$ f.g. mod / Λ.

Also define

$$\mathcal{S}(F, V) := \mathcal{S}(F, T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \quad / \Lambda_{\mathbb{Q}}$$

$$\mathcal{X}(F, V) := \mathcal{X}(F, W) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \quad / \Lambda_{\mathbb{Q}}.$$

<u>Thm</u> $\exists!\ \mathscr{L}_p(V) \in \Lambda_{\mathbb{Q}}$ s.t. for every finite order character $\iota : \mathbb{C}_p \to \mathbb{C}$,

$$\iota\, \underbrace{\mathscr{L}_p(V)(\chi)}_{\in\, \mathbb{C}_p} = \begin{cases} C \cdot \iota\left(\dfrac{p^5}{\alpha_0^5 \alpha_1^8}\right)^{c_p(\chi)} \cdot L(0, \iota(V \otimes \chi)), & c_p(\chi) > 0 \\[2ex] C \cdot \displaystyle\prod_{\mathfrak{p} \mid p} \frac{(\chi(\mathfrak{p}) - \alpha_0^{-1}\alpha_1^{-2} p)(\chi(\mathfrak{p}) - \alpha_0^{-1})(\chi(\mathfrak{p}) - \alpha_0 \alpha_1^{-2})}{(\chi(\mathfrak{p}) - \alpha_0 \alpha_1^{2} p^{-2})(\chi(\mathfrak{p}) - \alpha_0 p^{-1})(\chi(\mathfrak{p}) - \alpha_0^{-1}\alpha_1^{2} p^{-1})} \cdot L(0, \iota(V \otimes \chi)) & c_p(\chi) = 0. \end{cases}$$

Here
- C a certain (explicit) complex number related to adjoint L-value.
- α_i is unique root of $T^2 - a_p(A_i)T + p$ in $\mathbb{Z}_p^\times$ for $i = 0, 1$.
- $c_p(\chi)$ p-conductor for χ.

<u>Main thm</u> [LTXZZ]

Suppose A_0, A_1 with no CM and are not geometrically isogeneous.
p satisfies:

(S1): $p \geq 11$

(S2): $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \longrightarrow GL_{\mathbb{F}_p}(A_0[p](\overline{\mathbb{Q}})) \times GL_{\mathbb{F}_p}(A_1[p](\overline{\mathbb{Q}}))$ $\cong GL_2(\mathbb{F}_p) \times GL_2(\mathbb{F}_p)$
has max possible image $\cong G(SL_2(\mathbb{F}_p) \times SL_2(\mathbb{F}_p))$ (det = det).

(S3): a tricky condition only excludes finitely many p's.

If $\mathcal{L}_p(V) \neq 0$, then

(1) $\mathcal{S}(F, V) = 0$

(2) $\mathcal{X}(F, V)$ is a torsion $\Lambda_{\mathbb{Q}}$-mod.

(3) $\mathcal{L}_p(V) \in \mathrm{char}(\mathcal{X}(F, V))$.

Spectrum = open disc $|T|_p \leq 1$

Def'n of char ideal : $\Lambda_{\mathbb{Q}}$ 1-dim'l Noetherian regular ring.

$$M \cong \bigoplus_i (\Lambda_{\mathbb{Q}}/f_i)^{\oplus a_i}$$

$$\rightsquigarrow \mathrm{char}(M) := \prod_i f_i^{a_i}.$$

Rmk When $\mathcal{L}_p(V) = 0$?

- For $\epsilon(V \otimes \chi) = \epsilon(V) \in \{\pm 1\}$,

Case 1 $\epsilon(V) = -1 \Rightarrow \mathcal{L}_p(V) = 0$

Consider $K(F, V) \subseteq \mathcal{S}(F, V)$ $\Lambda_{\mathbb{Q}}$-submod.

If $K(F, V) \neq 0$, then

(1) $\mathcal{S}(F, V)$ torsion-free $\Lambda_{\mathbb{Q}}$-mod of rk 1

(2) $\mathcal{X}(F, V)$ rk 1.

(3) $\mathrm{char}\left(\dfrac{\mathcal{S}(F, V)}{K(F, V)}\right)^2 \subseteq \mathrm{char}(\mathcal{X}(F, V)^{\mathrm{tor}})$.

Case 2 $\epsilon(V) = 1$, Conj $\mathcal{L}_p(V) \neq 0$

Conj $\mathrm{char}(\mathcal{X}(F, V)) = (\mathcal{L}_p(V))$.

Cor (of Iwasawa main conj)

$$\dim_{\mathbb{Q}_p} H^1_f(F, V) \leq \operatorname*{ord}_{\chi = 1} \mathcal{L}_p(\pi).$$

Cor [LTXZZ] $L(0, V) \neq 0 \Rightarrow H^1_f(F, V) = 0$.

Construction of $\mathcal{L}_p(V)$

$A_0, \quad A_1 \qquad / F$

$\rightsquigarrow \pi_0, \quad \pi_1 \qquad$ cusp autom repr of $GL_2(\mathbb{Q})$

$\rightsquigarrow \qquad Sym^2 \pi_1 \qquad$ cusp autom repr of $GL_3(\mathbb{Q})$

$\overset{BC}{\rightsquigarrow} \qquad \Pi_2 \qquad$ cusp autom repr of $GL_2(F)$ $\Big\}$ + hidden herm symm

$\qquad \Pi_3 \qquad$ cusp autom repr of $GL_3(F)$

($\Rightarrow \Pi_2, \Pi_3$ descend to unitary rep).

Let Σ^+_{min} set of rat'l primes dividing $\mathrm{cond}(F)\cdot \mathrm{cond}(A_0)\cdot \mathrm{cond}(A_1)$.

Take $q \notin \Sigma^+_{min}$.

(1) q inert in F: $\Pi_{2,q} \longleftrightarrow \begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix}$ Satake para

$\Pi_{3,q} \longleftrightarrow \begin{pmatrix} \beta & & \\ & 1 & \\ & & \beta^{-1} \end{pmatrix}$

(2) q split in F: $\Pi_{N,\mathfrak{q}_1} \simeq \Pi_{N,\mathfrak{q}_2}$ self-dual.

$q = \mathfrak{q}_1 \mathfrak{q}_2$

Let $\mathbb{T}_N$ = abstract unitary Hecke alg of rk N away from Σ^+_{min}.

$\phi_N : \mathbb{T}_N \to \mathbb{Z}_p$ Satake homomorphism determined by Π_N.

<u>Exercise</u> q inert. $T_q := [\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}] \in \mathbb{T}_2$.

Compute $\phi_2(T_q) = ?$ in terms of $a_q(A_0)$.

$\exists$ data $(V_2, V_3, \Lambda_2, \Lambda_3, K_2, K_3)$ s.t.

- V_2 2-dim herm space $/F$.
- $V_3 = (V_2)_\sharp := V_2 \oplus F\cdot e$, $(e,e) = 1$. (sublimation)

$\rightsquigarrow U(V_2) \hookrightarrow U(V_3)$

From now on, always assume $\epsilon(V)=1$.

<u>GGP conj & results</u>

(1) Local GGP:

$\exists!$ V_2 definite & $V_3=(V_2)_\#$,

s.t. $\exists$ π_2,π_3 cuspidal autom repr of $U(V_2)$ & $U(V_3)$,

s.t. $BC(\pi_2)=\Pi_2$, $BC(\pi_3)=\Pi_3$,

& $\mathrm{Hom}_{U(V_2)(\mathbb{Q}_v)}(\pi_2\otimes\pi_3,\mathbb{1})\neq 0$, $\forall$ v place of $\mathbb{Q}$.

(implied by period $\neq 0$).

(2) Global GGP:

Take Λ_2 self-dual lattice of V_2 away from Σ^+_{min}

$\Lambda_3=(\Lambda_2)_\#$.

Let K_N be of form

$$\prod_{v\in\Sigma^+_{min}} K_{N,v}\times\prod_{v\notin\Sigma^+_{min}} U(\Lambda_{N,v})\subseteq U(V_N)(\mathbb{A}^\infty)$$

open compact

Then $\exists$ $\phi\in\pi_2^{K_2}\otimes\pi_3^{K_3}$,

s.t. $$\left|\int_{U(V_2)(\mathbb{Q})\backslash U(V_2)(\mathbb{A})}\phi(\Delta(h))\,dh\right|^2=\underset{\substack{\neq\\0}}{\alpha(\phi)}\cdot L(0,V)$$

(Rmk In fact LHS = finite sum b/c $[U(V_2)]$ is discrete).

Finally be solved by

Beuzart-Plessis – Liu – Zhang – Zhu.

Namely, (2): can take level K_N s.t. spherical away from Σ^+_{min}.

Let $I_{N,p}\subseteq K_{N,p}$ standard Iwahori subgp, $N=2,3$.

$$\begin{pmatrix}1&*\\&1\end{pmatrix}\hookrightarrow\begin{pmatrix}1&*&0\\&1&0\\&&1\end{pmatrix}.$$

π_N ordinary at p.

Let $\quad Sh(V_N, K_N) = U(V_n)(\mathbb{Q}) \backslash U(V_N)(\mathbb{A}^{\infty}) / K_N \quad$ (Shimura set).

Take $\quad P_N : \mathbb{Z}_p[Sh(V_N, K_N)] \longrightarrow \mathbb{Z}_p[Sh(V_N, K_N^p I_{N,p})]$

ord proj assoc to π_N.

For $c>0$, $\quad g_2^{(c)} = \begin{pmatrix} p^{2c} & \\ & p^c \end{pmatrix}$, $g_3^{(c)} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p^{2c} & & \\ & p^c & \\ & & 1 \end{pmatrix}$

Take $\quad I_{3,p}^{(c)} := g_3^{(c)} \cdot I_{3,p} (g_3^{(c)})^{-1}$.

$\quad I_{2,p}^{(c)} := \underbrace{g_2^{(c)} \cdot I_{2,p} (g_2^{(c)})^{-1}}_{\text{does not contained in } I_{3,p}^{(c)}} \cap I_{3,p}^{(c)}$ $\rightsquigarrow$ shrink level grp.

$K_N^{(c)} := K_N^p \cdot I_{N,p}^{(c)}$. right translation

$P_N^{(c)} := R_{(g_N^{(c)})^{-1}} \circ P_N : \mathbb{Z}_p[Sh(V_N, K_N)] \longrightarrow \mathbb{Z}_p[Sh(V_N, K_N^{(c)})]$

$P^{(c)} := P_2^{(c)} \times P_3^{(c)}$.

<u>Lem</u> For $\mathbb{G} := \mathrm{Ker}(\mathrm{Res}\, \mathbb{G}_{m,F} \xrightarrow{Nm} \mathbb{G}_{m,\mathbb{Q}})$. Then

$$\begin{tikzcd} Sh(V_2, K_2^{(c)}) \arrow[d, "\det"] \arrow[dd, bend right] \\ \mathbb{G}(\mathbb{Q}) \backslash \mathbb{G}(\mathbb{A}^{\infty}) / \det K_2^{(c)} \arrow[d] \\ \mathrm{Gal}(F_c/F) \end{tikzcd}$$

(factors through det)

$\leftarrow$ of interest!

$\forall \sigma \in \mathrm{Gal}(F_c/F)$, set $Sh(V_2, K_2^{(c)})_\sigma$ to be the fiber over σ.

$\forall \phi \in \mathbb{Z}_p[Sh(V_2, K_2)][\ker \phi_2] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[Sh(V_3, K_3)][\ker \phi_3]$, define

$$\lambda(\phi)_c := (\alpha_0 \alpha_1^\sigma)^{-c} \sum_{\sigma \in \mathrm{Gal}(F_c/F)} \Big(\sum_{h \in Sh(V_2, K_2^{(c)})_\sigma} (P^{(c)} \phi)(\Delta(h)) \Big) [\sigma]$$

$$\in \mathbb{Z}_p[\mathrm{Gal}(F_c/F)]$$

Thm (Liu)

(1) $(\lambda(\phi)_c)_c$ is compatible under corestriction,

i.e. $\exists!\ \lambda(\phi) \in \Lambda$ s.t. $\lambda(\phi)_c$ is the image of $\lambda(\phi)$ in $\mathbb{Z}_p[Gal(F_c/F)]$

(2) $\exists\, \phi$ s.t. $(\lambda(\phi)^2) = (\mathcal{L}_p(V))$ as ideals of $\Lambda_{\mathbb{Q}}$.

(a priori $\lambda(\phi)^2 \sim \mathcal{L}_p(V)$ up to local terms).

$\exists\, \epsilon \in \Lambda^\times$. $c \in \mathbb{Q}_p^\times$. $c \in \lambda(p)^2$.

Anti-cyclotomic Iwasawa theory for Rankin-Selberg products (2/3)

Yifeng Liu

<u>Last time</u> A_0, A_1 ell curves $/F$.

$T = T_p(A_0) \otimes \mathrm{Sym}^2 T_p(A_1)(-1)$, $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. $W = V/T$.

$F_\infty = \bigcup_{c \geq 0} F_c$, anticycl $\mathbb{Z}_p$-ext of F.

$\Sigma^+_{\min}$ bad primes.

$\phi_N : \mathbb{T}_N \to \mathbb{Z}_p$ $(N = 2, 3)$ given by Π_2, Π_3.

$\mathcal{L}_p(V) \in \Lambda_{\mathbb{Q}} = \mathbb{Z}_p[\![\mathrm{Gal}(F_\infty/F)]\!] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

$\rightsquigarrow$ $\mathfrak{X}(F_\infty, V)$, $\mathcal{Y}(F_\infty, V)$.

<u>Note</u> $\mathcal{L}_p(V) \in \mathbb{Z}_p^\times \Rightarrow H^1_f(F, V) = 0$

b/c char ideal $\ni$ unit $\Rightarrow$ original mod $= 0$.

Also, $H^1_f(F, V) = 0 \Leftarrow H^1_f(F, V)[p] = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad H^1_f(F, W[p])$

(
$$\begin{tikzcd} & T \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[d] \\ W[p^k] \arrow[r, phantom, "\cong"] & T/p^kT \arrow[r] & W \end{tikzcd}$$
Propogation of Selmer str to torsion mods.
)

Have Tate pairing

$$\langle \cdot, \cdot \rangle_w : H^1(F_w, W[p]) \times H^1(F_w, T/p^kT) \longrightarrow \mathbb{F}_p.$$

$$H^1(F, W[p]) \times H^1(F, T/p^kT) \longrightarrow 0$$

$$\rightsquigarrow \langle \cdot, \cdot \rangle_w : H^1_f(F_w, W[p]) \times H^1_f(F_w, T/p^kT) \longrightarrow 0$$

Need $s \neq 0$, $s \in H^1_f(F, W[p])$.

- $\ell \gg 0$ prime, inert in F, $\mathrm{loc}_\ell(s) \neq 0$.
- $c \in H^1(F, T/pT)$, s.t. $\mathrm{loc}_w(c) \in H^1_f(F_w, T/pT)$, $w \nmid \ell$,

  with $\langle \mathrm{loc}_\ell(c), \mathrm{loc}_\ell(s) \rangle_\ell \neq 0$.

Ideal situation: $\dim H^1_{\mathrm{unr}}(F_\ell, W[p]) = 1$.

<u>Recall initial datum</u>

Fix $\vartheta := (V_2, V_3, \Lambda_2, \Lambda_3, K_2, K_3)$ as before.

Fix set $\square$ of rat'l primes away from $\Sigma^+_{\min} \cup \{p\}$,

s.t. the image of

$$\underbrace{\mathbb{T}_N \otimes \mathbb{Z}_p}_{f.g.} \longrightarrow \mathrm{End}_{\mathbb{Z}_p}(\mathbb{Z}_p[\mathrm{Sh}(V_N, K_N)]).$$

is generated by the finite product $\bigotimes_{v \in \square} \mathbb{T}_{N,v}$.

(So the level raising on K_N dismisses primes outside $\square$ later).

<u>Def</u> $k \geq 1$ integer, let

$\mathcal{L}_k :=$ set of rat'l primes ℓ satisfying

(c1) $\ell \notin \Sigma^+_{\min} \cup \square$,

(c2) $p \nmid \ell(\ell^2 - 1)$,

(c3) ℓ inert in F,

(c4) $a_\ell(A_0)^2 \equiv (\ell+1)^2 \mod p^k$,

Fourier coeff of mod form for A_0

& $\ell^{-2} a_\ell(A_1)^4 - 4\ell^{-1} a_\ell(A_1)^2 + 2 \mod p \notin \{2, \ell^2 + \ell^{-2}\}$.

(c5) $\ell^{-2} a_\ell(A_1)^4 - 4\ell^{-1} a_\ell(A_1)^2 + 2 \mod p \neq -(\ell + \ell^{-1})$.

Explanation of congruence conditions

$T^2 - a_\ell(A_i)T + \ell$ has two roots $\beta_i, \ell\beta_i^{-1}$.

For $i=0$, $\mathrm{Sat}(\Pi_{2,\ell}) = \{(\ell^{-1/2}\beta_0)^2, (\ell^{1/2}\beta_1^{-1})^2\} = \{\ell^{-1}\beta_0^2, \ell\beta_0^{-2}\}$

$i=1$, $\mathrm{Sat}(\Pi_{3,\ell}) = \{\ell^{-2}\beta_1^4, 1, \ell^2\beta_1^{-4}\}$. $\left(\begin{matrix}\beta_0^2 \in \{1, \ell^2\} \\ \Downarrow \\ \beta_0 \in \{\pm 1, \pm\ell\}\end{matrix}\right)$

Have $\mathbb{I} \le T/p^kT$ triv rep semisimple.

$\rightsquigarrow$ $H^1_{ur}(F_\ell, T/p^kT) \cong \mathbb{Z}/p^k\mathbb{Z}$, non-canonical.

(c4) $\Rightarrow$ $\mathrm{Sat}(\Pi_{2,\ell}) = \{\ell^{-1}, \ell\}$,

$\mathrm{Sat}(\Pi_{3,\ell})$ contains exactly one ℓ & ℓ^{-1} mod p.

(c5) $\Rightarrow$ $\mathrm{Sat}(\Pi_{3,\ell})$ mod p $\neq \{-\ell, 1, -\ell^{-1}\}$.

<u>Remark</u> (1) $\mathcal{L}_1 \supseteq \mathcal{L}_2 \supseteq \cdots$

(2) $\forall k$, $\mathcal{L}_k$ is infinite

by $(S_1) + (S_2)$ ($p \geq 11$. Galois image as big as possible)

(3) $\bigcap_k \mathcal{L}_k = \phi$.

Def Let $\mathcal{N}_k$ be the set of multi-free product of elements in $\mathcal{L}_k$

$\rightsquigarrow$ similarly for $\mathcal{N}_k^{odd/even}$.

Then $\bigcap_k \mathcal{N}_k = \{1\}$.

$\forall n \in \mathcal{N}_1$, fix $\mathcal{V}^n = (V_2^n, V_3^n, \Lambda_2^n, \Lambda_3^n, K_2^n, K_3^n)$ & (j_2^n, j_3^n) in which

(i) V_2^n has sign $(2,0)$, $(1,1)$ if $n \in \mathcal{N}_1^{even}$, $\mathcal{N}_1^{odd}$.

(ii) For $v < \infty$, $(V_2^n)_v \simeq (V_2)_v$ iff $v \nmid n$.

Also, $V_3^n = (V_2^n)_\sharp$.

(iii) Λ_2^n lattice of $V_2^n \otimes \mathbb{A}^{\infty, \Sigma^+_{min}}$ that is self-dual / almost self-dual at $v \nmid n$ / $v \mid n$.

$(\Lambda_2^n)_v^\vee \cong (\Lambda_2^n)_v$ $\qquad$ $(\Lambda_2^n)_v^\vee / (\Lambda_2^n)_v$ has length one.

Also, $\Lambda_3^n = (\Lambda_2^n)_\#$.

(iv) $\breve{j}_2^n : V_2 \otimes \mathbb{A}^{\infty,n} \xrightarrow{\sim} V_2^n \otimes \mathbb{A}^{\infty,n}$ s.t. $\breve{j}_2^n(\Lambda_2) = (\Lambda_2^n)_v$, $v \nmid n$.

$\breve{j}_3^n = (\breve{j}_2^n)_\#$.

(v) $K_N^n = (\breve{j}_N^n K_N)_{\Sigma^+_{min}} \times \mathrm{Stab}(\Lambda_N^n)$, $\quad N = 2, 3$.

Also fix $(\breve{j}_2^a, \breve{j}_3^a)$ for $a = a(n, n\ell)$ in which

$$\breve{j}_2^a : V_2^n \otimes \mathbb{A}^{\infty,\ell} \xrightarrow{\sim} V_2^{n,\ell} \otimes \mathbb{A}^{\infty,\ell}$$

s.t. $(\breve{j}_2^a)^{n\ell} \circ \breve{j}_2^n = \breve{j}_2^{n\ell}$.

Denote $\mathfrak{m}_N^{n,k} := \ker(\mathbb{T}_N^n \to \mathbb{T}_N \xrightarrow{\phi_N} \mathbb{Z}_p \to \mathbb{Z}_p/p^k)$, $\quad N = 2, 3$.

$\mathrm{Sh}_N^n := \mathrm{Sh}(V_N^n, K_N^n)$

(Sh var or Sh set up to parity of N.)

Def $\forall n \in \mathcal{N}_k$, define <u>congruence module</u> (mod p^k) at n to be

$$\mathcal{C}^{n,k} := \begin{cases} \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[\mathrm{Sh}_2^n]/\mathfrak{m}_2^{n,k} \otimes \mathbb{Z}_p[\mathrm{Sh}_3^n]/\mathfrak{m}_3^{n,k}, \mathbb{Z}/p^k), & n \in \mathcal{N}_k^{even}, \\ \mathrm{Hom}_{\mathbb{Z}_p[\mathrm{Gal}_F]}(H^3((\mathrm{Sh}_2^n \times_F \mathrm{Sh}_3^n)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(2))/(\mathfrak{m}_2^{n,k}, \mathfrak{m}_3^{n,k}), T/p^k T), & n \in \mathcal{N}_k^{odd}. \end{cases}$$

In the next two talks:

$\forall a = a(n, n\ell)$, construct an isom

$$\rho^{a,k} : \mathcal{C}^{n,k} \xrightarrow{\sim} \mathcal{C}^{n\ell,k} \quad \text{of } \mathbb{Z}/p^k\text{-mods}$$

called <u>reciprocity law</u>.

(First reciprocity law when $n \in \mathcal{N}_k^{even}$.)

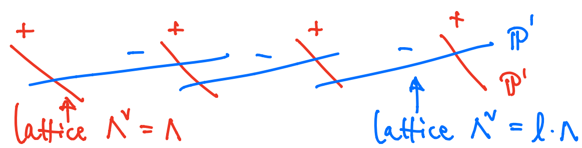
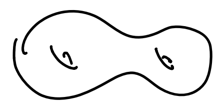
Need to relate $\mathcal{C}^{n,k}$ with

$$H^1_{sing}(F_\ell, H^3((\mathrm{Sh}_2^n \times_F \mathrm{Sh}_3^n)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(2))/(\mathfrak{m}_2^k, \mathfrak{m}_3^k)) \quad (:= H^1/H^1_{unr}).$$

$$(C4) \Rightarrow \quad H^1_{sing}(F_\ell, H^3((Sh^n_2 \times_F Sh^n_3)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(2))/(m_2^k, m_3^k))$$
$$\cong H^1_{sing}(F_\ell, H^1(Sh^n_{2,\bar{\mathbb{Q}}}, \mathbb{Z}_p(1))/m_2^k) \otimes H^1_{sing}(F_\ell, H^1(Sh^n_{3,\bar{\mathbb{Q}}}, \mathbb{Z}_p(1))/m_3^k)$$

<u>Note</u> $Sh^{n\ell}_N$ has a canonical integral model $\mathcal{M}_N$ over $\mathbb{Z}_{\ell^2} (= \mathcal{O}_{F_\ell})$ that is strictly semistable.

$M_N := \mathcal{M}_N \otimes_{\mathbb{Z}_{\ell^2}} \mathbb{F}_{\ell^2} \rightsquigarrow M_N^+, M_N^-$.

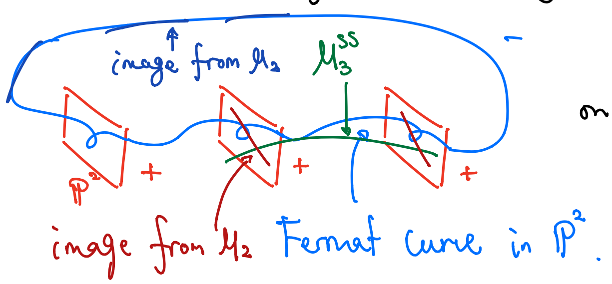
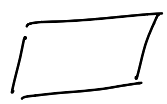
- $N=2$,  on 

intersection determined by BT building.

$\mathcal{M}_2$

$\downarrow$

$\mathcal{M}_3$

- $N=3$,  on 

Take $\Lambda^{n,+}_{N,\ell} := \Lambda^n_{N,\ell}$. supersingular loci in $M_N^\pm$.

set parametrizing $\mathbb{P}^1$'s of $M_N^{\pm,ss}$ $= B_N^\pm$

$$\begin{tikzcd} & B_N^\pm \arrow[dl, "\pi_N^\pm"', "(j_N^{-q})"] \arrow[dr, "\iota_N^\pm"] & \\ Sh(V_N^n, K_N^{n,\pm}) & & M_N^\pm . \end{tikzcd}$$

Choose $\Lambda^{n,-}_{N,\ell}$ lattice of $V_N^n \otimes \mathbb{Z}_\ell$, s.t.

- $\ell \Lambda^{n,-}_{N,\ell}$ is contained in $(\Lambda^{n,-}_{N,\ell})^\vee$ of length $N-2$.
- $\Lambda^n_{2,\ell} \subseteq \Lambda^{n,-}_{2,\ell} \subseteq \ell^{-1} \Lambda^n_{2,\ell}$
- $(\Lambda^{n,-}_{2,\ell})_\# \subseteq \Lambda^{n,-}_{3,\ell} \subseteq \ell^{-1} \cdot ((\Lambda^{n,-}_{2,\ell})_\#)^\vee$.

Compute H^1_{sing} by $E_1^{p,q}$ $\quad (M^n_N := M^+_N \cap M^-_N,\ \bar{M} := M \otimes_{\mathbb{F}_{\ell^2}} \bar{\mathbb{F}}_{\ell^2})$,

$$
\begin{array}{llll}
q=2 & H^0(\bar{M}^n_2, \mathbb{Z}_p) \xrightarrow{d_1^{-1,2}} & H^2(M^+_2, \mathbb{Z}_p(1)) \oplus H^2(M^-_2, \mathbb{Z}_p(1)) \longrightarrow & 0 \\
q=1 & 0 \longrightarrow & 0 \longrightarrow & 0 \\
q=0 & 0 \longrightarrow & H^0(\bar{M}^+_2, \mathbb{Z}_p(1)) \oplus H^0(\bar{M}^-_2, \mathbb{Z}_p(1)) \xrightarrow{d_1^{0,0}} & H^0(\bar{M}^n_2, \mathbb{Z}_p(1)) \\
 & p=-1 & p=0 & p=1
\end{array}
$$

μ monodromy

It degenerates at $E_2^{p,q}$.

Then $\quad H^1_{sing}(F_\ell, V) = (V_{I_{F_\ell}})^{\phi_{F_\ell}}$

$$
\begin{tikzcd}
0 \arrow[r] & E_2^{1,0} \arrow[r] & H^1(Sh_2^{n\ell}) \arrow[r] & E_2^{-1,2} \arrow[r] \arrow[ll, bend left, "\mu"] & 0 .
\end{tikzcd}
$$

$$\Rightarrow \quad H^1(Sh_2^{n\ell}) = E_2^{1,0}(-1)/\mu E_2^{-1,2}.$$

Define $\nabla_2 : E_1^{0,2} \longrightarrow \mathbb{Z}_p[Sh_2^n] \qquad K_2^n \qquad$ as follows:

$$
\begin{array}{ccc}
E_1^{0,2} & & \\
\| & & \\
H^2(\bar{M}^+_2, \mathbb{Z}_p(1)) & \xrightarrow{(\pi_2^+)_!} & \mathbb{Z}_p[Sh(V_2^n, K_2^{n,+})] \\
\oplus & & \\
H^2(\bar{M}^-_2, \mathbb{Z}_p(1)) & \xrightarrow{(\pi_2^-)_!} & \mathbb{Z}_p[Sh(V_2^n, K_2^{n,-})]
\end{array}
$$

$$\mathbb{Z}_p[Sh(V_2^n, K_2^{n,+})] \underset{T^{+-}}{\overset{T^{-+}}{\rightleftarrows}} \mathbb{Z}_p[Sh(V_2^n, K_2^{n,-})] \quad \rightsquigarrow \quad 1_{K_2^{n+} K_2^{n-}}.$$

$$\nabla_2 := T^{+,-} \circ (T^{-,+}(\pi_2^+)_! + (\ell+1)(\pi_2^-)_!)$$
$$E_1^{0,2} \longrightarrow \mathbb{Z}_p[Sh_2^n].$$

<u>Thm</u> ∇_2 induces an isom
$$(E_2^{1,0}(-1)/\mu E_2^{-1,2})/m_2^{n\ell,k} \xrightarrow{\sim} \mathbb{Z}_p[Sh_2^n]/m_2^{n\ell,k}.$$

$$H^{0,2}(\bar{M}^+_2, \mathbb{Z}_p(1)) = \mathbb{Z}_p[Sh(V_2^n, K_2^{n,+})]$$
$$H^{0,2}(\bar{M}^-_2, \mathbb{Z}_p(1)) = \mathbb{Z}_p[Sh(V_2^n, K_2^{n,-})]$$

Compute
$$\begin{pmatrix} \ell+1 & -T^{+-} \\ -T^{-+} & \ell+1 \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^- \end{pmatrix} = \begin{pmatrix} (\ell+1)\phi^+ - T^{+-}\phi^- \\ -T^{-+}\phi^+ + (\ell+1)\phi^- \end{pmatrix}$$

$$\rightsquigarrow \quad T^{+-}\left(T^{-+}\left((\ell+1)\phi^+ - T^{+-}\phi^-\right) + (\ell+1)\left(-T^{-+}\phi^+ + (\ell+1)\phi^-\right)\right)$$
$$= T^{+-}\left(-T^{-+}T^{+-}\phi^- + (\ell+1)^2\phi^-\right)$$
$$= -\left((T^{+-}\circ T^{-+}) - (\ell+1)^2\right)(T^{+-}\phi^-) = 0 \quad \text{after mod } m_2^{n\ell,k}.$$

Before we obtain (exercise) $\phi_2(T^{+-}\circ T^{-+}) = a_\ell(A_0)^2$.

Back to weight spectral seq for $N=3$:

$$E_2^{p,q}: \quad \begin{array}{lcccccc} q=3 & H^1(\bar{M}_3^n, \mathbb{Z}_p) & \longrightarrow & H^3(\bar{M}_3^+, \mathbb{Z}_p(1)) & \oplus & H^3(\bar{M}_3^-, \mathbb{Z}_p(1)) & \longrightarrow 0 \\ q=2 & H^0(\bar{M}_3^n, \mathbb{Z}_p) & \longrightarrow & H^2 & \oplus & H^2 & \longrightarrow H^2 \\ q=1 & 0 & \longrightarrow & H^1 & \oplus & H^1 & \longrightarrow H^1. \end{array}$$

<u>Lem</u> $\quad H^1(\bar{M}_3^n, \mathbb{Z}_p)_{m_3^{n\ell,1}} \overset{(C5)}{=} 0.$

(hidden neat rep theory).

Define $\nabla_3 : E_1^{0,2} \longrightarrow \mathbb{Z}_p[Sh_3^n]$.

<u>Thm</u> ∇_3 induces an isom

$$(E_2^{0,2}/m_3^{n\ell,k})_{\phi_{F_\ell}} \xrightarrow{\sim} \mathbb{Z}_p[Sh_3^n]/m_3^{n\ell,k}.$$

(used (C4)).

<u>Thm</u> (First reciprocity law)

Put $\nabla := \nabla_2 \boxtimes \nabla_3$.

(1) $\mathbb{Z}_p[\mathrm{Gal}(\bar{\mathbb{Q}}/F)]$-mod isom

$$Q_3 := H^3(Sh_2^{n\ell} \underset{F}{\times} Sh_3^{n\ell})_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(2))/(m_2^{n\ell,k}, m_3^{n\ell,k}) \cong \bigoplus_{k'\le k} T/p^{k'}T$$

(2) Have isom $\quad H^1_{sing}(F_\ell, H^3(Sh_2^{n\ell} \times_F Sh_3^{n\ell})_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(2)) / (m_2^{n\ell,k}, m_3^{n\ell,k}))$

$$\big\downarrow \nabla$$

$$\mathbb{Z}_p[Sh_2^n]/m_2^{n\ell,k} \otimes \mathbb{Z}_p[Sh_3^n]/m_3^{n\ell,k}.$$

(3) $\left(\begin{array}{l} \underline{\text{Recap}} \quad Sh_2^{n\ell} \to Sh_3^{n\ell} \rightsquigarrow \Delta(Sh_2^{n\ell}) \in CH^2(Sh_2^{n\ell} \times Sh_3^{n\ell}) \\ \text{with} \quad CH^2(Sh_2^{n\ell} \times Sh_3^{n\ell}) \xrightarrow{loc_\ell} H^1(F_\ell, Q_3) \longrightarrow H^1_{sing}(F_\ell, Q_3). \end{array}\right)$

Have

$$\nabla(\partial_\ell \, loc_\ell(\Delta(Sh_2^{n\ell})))(\phi) = (\ell+1)^3 (a_\ell(A_1)^4 - 4\ell a_\ell(A_1)^2) \cdot \sum_{Sh_2^n} \phi(\Delta(h_2))$$

$$\forall \phi \in \mathcal{C}^{n,k} = (\mathbb{Z}/p^k)[Sh_2^n][m_2^{n\ell,k}] \otimes (\mathbb{Z}/p^k)[Sh_3^n][m_3^{n\ell,k}].$$

By (1), $\mathcal{C}^{n\ell,k} \xrightarrow{\sim} \mathrm{Hom}(H^1_{sing}(F_\ell, Q_3), \mathbb{Z}/p^k)$.

By first reciprocity law, this is

$$[(\ell+1)^3 (a_\ell(A_1)^4 - 4\ell a_\ell(A_1)^2)]^{-1} \cdot \nabla^\vee.$$

Anti-cyclotomic Iwasawa theory for Rankin-Selberg products (3/3)

Yifeng Liu

<u>Recall</u>

$$\mathbb{1}_{\Delta Sh_2^n} := \begin{cases} \text{char func of } \Delta Sh_2^n \text{ in} \\ \quad \mathbb{Z}_p[Sh_2^n]/m_2^{n,k} \otimes \mathbb{Z}_p[Sh_3^n]/m_3^{n,k}, \quad n \in \mathcal{N}_k^{even} \\ \text{AJ image of } \Delta Sh_2^n \text{ in} \\ \quad H_f^1(F, H^3((Sh_2^n \times_F Sh_3^n)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(2))/(m_2^{n,k}, m_3^{n,k})), \quad n \in \mathcal{N}_k^{odd}. \end{cases}$$

When $n \in \mathcal{N}_k^{even}$, have 1st reciprocity law (bad red'n at ℓ)

$$\begin{array}{ccccc} \phi & \mathcal{C}^{n,k} & \xrightarrow[\text{fixed}]{\rho^{n,k}\ \sim} & \mathcal{C}^{n\ell,k} & \phi \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ \langle \phi, \mathbb{1}_{\Delta Sh_2^n} \rangle & \mathbb{Z}/p^k & \xrightarrow{=} & H^1_{sing}(F_\ell, T/p^kT) & \phi_* \operatorname{loc}_\ell \mathbb{1}_{\Delta Sh_2^{n\ell}} \end{array}$$

<u>Refinement</u> $\ell \in \mathcal{L}_k$,

$$0 \to \underset{\mathbb{Z}/p^k}{\underset{\cong}{H^1_{unr}(F_\ell, T/p^kT)}} \to \underset{(\mathbb{Z}/p^k)^{\oplus 2}}{\underset{\cong}{H^1(F_\ell, T/p^kT)}} \to \underset{\mathbb{Z}/p^k}{\underset{\cong}{H^1_{sing}(F_\ell, T/p^kT)}} \to 0$$

The quotient map has a canonical splitting:

$\exists!$ decomposition $T/p^kT = \underset{\mathbb{Z}/p}{\underset{\cong}{L}} \oplus \underset{\underbrace{\mathbb{Z}/p^k}_{\text{coinv} = 0}(1)}{\underset{\cong}{L'}} \oplus T'$,

s.t.

So $H^1_{unr}(F_\ell, T/p^kT) = $ image of $H^1(F_\ell, L) \to H^1(F_\ell, T/p^kT)$

$\underbrace{H^1_{ord}(F_\ell, T/p^kT)}_{\text{this is an ad hoc def'n}} :=$ image of $H^1(F_\ell, L') \to H^1(F_\ell, T/p^kT)$.

depending on the decomp of T/p^kT

Lem $\mathrm{loc}_\ell \mathbb{1}_{\Delta Sh_2^{n\ell}} \in H^1_{\mathrm{ord}}(F_\ell, T/p^kT)$, $\phi_* \mathbb{1}_{\Delta Sh_2^{n\ell}} \in H^1(F, T/p^kT)$.

$\Downarrow$

Can replace H^1_{sing} in 1st reciprocity law by H^1_{ord} (strengthen).

$\uparrow$ canonical lift of H^1_{sing} in H^1

<u>Second reciprocity law</u> (good red'n)

Take $n \in \mathcal{N}_k^{\mathrm{odd}}$, $a = a(n, n\ell)$,

$$\begin{tikzcd} \phi \arrow[d, mapsto] & \mathcal{E}^{n,k} \arrow[r, "p^{a,k}", "\sim"'] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{E}^{n\ell,k} \arrow[d] & \phi \arrow[d, mapsto] \\ \phi_*(\mathrm{loc}_\ell \mathbb{1}_{\Delta Sh_2^{n\ell}}) & H^1_{\mathrm{unr}}(F_\ell, T/p^kT) \arrow[r, "="] & \mathbb{Z}/p^k & \langle \phi, \mathbb{1}_{\Delta Sh_2^{n\ell}} \rangle \end{tikzcd}$$

Here $p^{a,k}$ surj: Ribet's level lowering

$p^{a,k}$ inj: need input (S3).

<u>Note</u> $\langle \phi, \mathbb{1}_{\Delta Sh_2^n} \rangle = \sum_{h \in Sh_2^n} \phi(\Delta h)$

$\updownarrow$ GGP

(mod p^k) special L-values.

For $n \in \mathcal{N}_k^{\mathrm{even}}$, construct a natural map

$$\lambda^{n,k} : \mathcal{E}^{n,k} \longrightarrow \Lambda/p^k\Lambda$$

s.t. $(\lambda^{1,k} : \mathcal{E}^{1,k} \to \Lambda/p^k\Lambda) = \lambda \bmod p^k$,

where $\lambda = \varprojlim_k \mathcal{E}^{n,k} \longrightarrow \varprojlim_k \Lambda/p^k\Lambda = \Lambda$

For $n \in \mathcal{N}_k^{\mathrm{odd}}$, construct a natural map $\kappa^{n,k}$ as follows:

Lem (Shapiro) $\varinjlim_{F\subset F'\subset F_\infty} H^1(F', T/p^kT) = H^1(F, T\otimes \Lambda/p^k\Lambda)$.

$$\mathbb{Q} \,-\, F \overset{\mathbb{Z}_p}{\text{ ---- }} F_\infty, \quad \mathrm{Frob}_\ell^2 = 1 \text{ in } \mathbb{Z}_p \rtimes \mathfrak{g}_2$$

with $\mathbb{Z}_p \rtimes \mathfrak{g}_2$ over $\mathbb{Q} - F_\infty$;

ℓ inert tot split

$\Rightarrow \mathrm{Gal}_{F_\ell} \curvearrowright \Lambda/p^k\Lambda$ trivially

$\Rightarrow H^1(F_\ell, T\otimes\Lambda/p^k\Lambda) = H^1(F_\ell, T)\otimes_{\mathbb{Z}_p}\Lambda$

$\rightsquigarrow \kappa^{n,k} : \mathcal{C}^{n,k} \longrightarrow H^1(F, T\otimes\Lambda/p^k\Lambda)$

s.t. $\forall \ell | n$, $\mathrm{loc}_\ell\circ\kappa^{n,k}$ has image in $H^1_{\mathrm{ord}}(F_\ell, T\otimes\Lambda/p^k\Lambda)$.

Iwasawa reciprocity laws

1st $n\in\mathcal{N}_k^{\mathrm{even}}$.

$$\begin{tikzcd} \mathcal{C}^{n,k} \arrow[r, "p^{a,k}", "\sim"'] \arrow[d, "\lambda^{n,k}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{C}^{n\ell,k} \arrow[d, "\mathrm{loc}_\ell\circ\kappa^{n\ell,k}"] \\ \Lambda/p^k\Lambda \arrow[r, "=", "\text{(up to }\Lambda^\times)"'] & H^1_{\mathrm{ord}}(F_\ell, T\otimes\Lambda/p^k\Lambda) \end{tikzcd}$$

2nd $n\in\mathcal{N}_k^{\mathrm{even}}$, $a = a(n, n\ell)$,

$$\begin{tikzcd} \mathcal{C}^{n,k} \arrow[r, "p^{a,k}", "\sim"'] \arrow[d, "\mathrm{loc}_\ell\circ\kappa^{n,k}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{C}^{n\ell,k} \arrow[d, "\lambda^{n,k}"] \\ H^1_{\mathrm{unr}}(F_\ell, T\otimes\Lambda/p^k\Lambda) \arrow[r, "="] & \Lambda/p^k\Lambda \end{tikzcd}$$

Rmk $\lambda^{n,k}$ for L-func, $\kappa^{n,k}$ for cohom class.

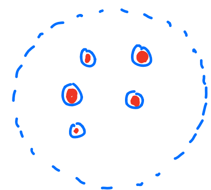
Def'n An <u>Iwasawa bipartite Euler system</u> is a sort of data

(i) $\{\gamma^n, j^n, \mathcal{C}^{n,k} \mid n\in\mathcal{N}_k\}$,

(ii) $\{ j^{a}, p^{a,k} \mid a = a(n, n\ell) \}$,

(iii) $\{ \lambda^{n,k} \mid n \in \mathcal{N}_k^{\mathrm{even}} \}$, $\{ \kappa^{n,k} \mid n \in \mathcal{N}_k^{\mathrm{odd}} \}$,

satisfying Iwasawa reciprocity laws.



Recall our goal $\quad \mathcal{L}_p(V) \in \mathrm{char}_{\Lambda_{\mathbb{Q}}}(\mathcal{X}(F_\infty, V))$

Specialization of Selmer group

V ordinary crystalline at p.

$\exists$ unique $\mathrm{Fil}_p^{j} T$ of $\mathbb{Z}_p[\mathrm{Gal}_{\mathbb{Q}_p}]$-module

s.t. (1) $\forall j \in \mathbb{Z}$, $\mathrm{Gr}_p^{j} T := \mathrm{Fil}_p^{j} T / \mathrm{Fil}_p^{j-1} T$ is a free $\mathbb{Z}_p$-mod on which inertia group acts by $\chi_{\mathrm{cycl}}^{-j}$.

(2) $\mathrm{Fil}_p^{j} T = 0$ if $j < -2$, $\mathrm{Fil}_p^{j} T = T$ if $j > 1$.

Lem1 For $\forall$ finite ext'n $K/\mathbb{Q}_p$,

$$H^1(K, \mathrm{Fil}_p^{-1} V) \to H^1(K, V) \to H^1(K, V \otimes B_{\mathrm{cris}})$$

is exact. i.e.

$$H^1_f(K, V) = \mathrm{Ker}(H^1(K, V) \to H^1(K, V/\mathrm{Fil}_p^{-1} V)).$$

$\rightsquigarrow$ can forget about B_{cris} and define Bloch-Kato in this way.

Lem2 For every regular integral Λ-ring Θ,

Θ-mod $H^1(F, T \otimes_{\mathbb{Z}_p} \Theta)$ is torsion-free.

pf. $\forall f \in \Theta$, $\underbrace{H^0(F, T \otimes \Theta/f)}_{\text{constituent ``}T/pT \otimes \chi\text{''}} \to H^1(F, T \otimes \Theta) \xrightarrow{f} H^1(F, T \otimes \Theta)$

Set $\mathcal{Y}(F_\infty, T) := \varprojlim_{F \subset F' \subset F_\infty} H^1_f(F', T) \subset \varprojlim_{F \subset F' \subset F_\infty} H^1(F', T) \underset{\substack{\uparrow \\ \text{Shapiro}}}{=} H^1(F, T \otimes \Lambda)$.

Lem 2 $\Rightarrow$ $H^1(F, T \otimes \Lambda)$ is Λ-torsion free.

$\Rightarrow$ $\mathcal{Y}(F_\infty, T)$ is Λ-torsion free.

$\mathcal{Y}(F_\infty, V)$ is $\Lambda_{\mathbb{Q}}$-torsion free.

<u>Lem 3</u> $\forall\, s \neq 0 \in H^1_f(F, T/p^k T)$ & $k \geq 1$,

$\exists$ infinitely many $\ell \in \mathcal{L}_k$ s.t. $\mathrm{loc}_\ell(s) \neq 0$.

<u>Notation</u> M $\mathbb{Z}_p[\mathrm{Gal}(\bar{\mathbb{Q}}/F)]$-mod. Θ Λ-ring.

Put $M_\Theta := M \otimes_{\mathbb{Z}_p} \Theta$ as $\mathbb{Z}_p[\mathrm{Gal}(\bar{\mathbb{Q}}/F)]$-mod.

<u>Def</u> Θ p-torsion free Λ-ring, w place of F.

$$H^1_f(F_w, V_\Theta) := \begin{cases} 0, & w \mid \infty, \\ \mathrm{Ker}(H^1(F_w, V_\Theta) \to H^1(I_w, V_\Theta)), & w \nmid \infty p, \\ \mathrm{Ker}(H^1(F_w, V_\Theta) \to H^1(F_w, (V/\mathrm{Fil}^{-1}_p V)_\Theta)), & w \mid p. \end{cases}$$

Let $H^1_f(F_w, T_\Theta)$ be the Selmer propagation of $H^1_f(F_w, V_\Theta)$.

Define $W^\Theta := \mathrm{Hom}_{\mathbb{Z}_p}(T_\Theta, \mathbb{Q}_p/\mathbb{Z}_p(1))$ = Cartier dual of T_Θ.

($W^\Theta = W_\Theta \iff \Theta$ finite $\mathbb{Z}_p$-mod).

Pontryagin dual pair

$$\langle \cdot, \cdot \rangle_w : H^1(F_w, T_\Theta) \times H^1(F_w, W^\Theta) \xrightarrow{\sim} \mathbb{Q}_p/\mathbb{Z}_p$$

$\rightsquigarrow$ $H^1_f(F_w, W^\Theta)$ def'd as the annihilator of $H^1_f(F_w, T_\Theta)$ under $\langle \cdot, \cdot \rangle_w$

e.g. of Θ (1) $\Theta = \Lambda$

(2) $F \subset F' \subset F_\infty$, $\Theta = \mathbb{Z}_p[\mathrm{Gal}(F'/F)]$

(key case) (3) $\forall$ closed pt x of $\mathrm{Spec}\,\Lambda_{\mathbb{Q}}$,

$$\begin{array}{ccccccc} & & \mathrm{Ker} & & \Lambda_{\mathbb{Q}} & \longrightarrow & \mathbb{Q}_x \\ & & \| & & \cup & & \cup \\ 0 & \to & \mathfrak{P}_x & \longrightarrow & \Lambda & \twoheadrightarrow & \Theta_x \end{array} \quad \text{(image)}$$

Let $\mathbb{I}_x$ be the integral closure of Θ_x in $\mathbb{Q}_x$.

<u>Remark</u> (1) In Eg (2) above two def'ns coincide.

(2) (Functoriality) $\Theta \to \Theta'$

$\rightsquigarrow H^1_f(F_v, T_\Theta) \to H^1_f(F_v, T_{\Theta'})$.

<u>Thm</u> Denote by $\mathcal{Y} \subseteq \mathrm{Spec}\,\Lambda_{\mathbb{Q}}$ minimal Zariski closed subset away from which $\Lambda_{\mathbb{Q}}$-mod $H^1(\mathbb{Q}_p, (V/\mathrm{Fil}^{-1}_F V)_\Lambda)$ is locally free.

(1) $\forall$ closed pt x of $(\mathrm{Spec}\,\Lambda_{\mathbb{Q}}) \setminus \mathcal{Y}$ the natural map

$$H^1_f(F, T_\Lambda) / \mathfrak{P}_x H^1_f(F, T_\Lambda) \to H^1_f(F, T_{\mathbb{I}_x})$$

is injective.

(2) $\forall$ closed pt x of $\mathrm{Spec}\,\Lambda_{\mathbb{Q}}$, both ker & coker of

$$H^1_f(F, W^{\mathbb{I}_x}) \longrightarrow H^1_f(F, W^\wedge)[\mathfrak{P}_x]$$

are finite, with order bounded by a const depending on $\#(\mathbb{I}_x/\Theta_x)$.

Then (1) $\Rightarrow$ suffices to investigate $\mathcal{S}(F_\infty, T)$.

(2) $\Rightarrow$ if $H^1_f(F, W^\wedge) = \varinjlim_{F \subset F' \subset F_\infty} H^1_f(F', W)$

then $\mathfrak{X}(F_\infty, W) = H^1_f(F, W^\wedge)^\vee$.

Now let x closed pt of $\mathrm{Spec}\,\Lambda_{\mathbb{Q}}$.

$\lambda_x^{n,k}$, $\kappa_x^{n,k}$ specializations of $\lambda^{n,k}$, $\kappa^{n,k}$ at x.

Need $H^1(F, T\otimes\Lambda/p^k\Lambda)$

<u>Def</u> $\forall n\in\mathcal{N}_k$, consider modified Selmer group

$$H^1_{f(n)}(F_w, T\otimes\Lambda/p^k\Lambda) := \begin{cases} 0, & w\mid\infty \\ H^1_{unr}(F_w, T\otimes\Lambda/p^k\Lambda), & w\nmid\infty pn, \\ H^1(F_w, T\otimes\Lambda/p^k\Lambda), & w\mid p, \\ H^1_{ord}(F_w, T\otimes\Lambda/p^k\Lambda), & w\mid n. \end{cases}$$

<u>Prop</u> $\exists\, k_{min}\geq 0$, s.t. $\forall k$, $\forall n\in\mathcal{N}_k^{odd}$,

$$p^{k_{min}}\cdot\kappa^{n,k}\in H^1_{f(n)}(F_w, T\otimes\Lambda/p^k\Lambda).$$

From now on: pretend $k_{min}=0$ (ok for ultimate goal).

<u>Prop</u> We have

(1) $H^1_f(F, T_{\mathbb{Z}_x})=0$ if $\exists\,\phi\in\mathcal{C}^{1,\infty}=\varprojlim_k \mathcal{C}^{1,k}$ s.t. $\lambda_x(\phi)\neq 0$.

(2) $\lambda_x(\phi)^2$ belongs to Fitting $H^1_f(F, W^{\mathbb{Z}_x})$ for every $\phi\in\mathcal{C}^{1,\infty}$.

<u>Prop $\Rightarrow$ Main thm</u>

(1) $\mathcal{L}_p(V)\neq 0 \Rightarrow \exists\,\phi\in\mathcal{C}^{1,\infty}$ s.t. $\lambda(\phi)\neq 0$.

Can choose $x\in\mathrm{Spec}\,\Lambda_{\mathbb{Q}}\setminus(\mathcal{Y}\cup Z(\lambda(\phi)=0))$

$\Rightarrow \mathfrak{F}(F_\infty, T)=0$ b/c $H^1_f(F, T_{\mathbb{Z}_x})=0$ at fiber & tor-free.

(control Thm (1) + Prop (1) $\Rightarrow$ Main Thm (1)).

(2) Prop (2) $\Rightarrow \mathcal{L}_p(V)\in\mathrm{char}(\mathfrak{X}(F_\infty, V))$.

<u>Pf of Prop</u>

Let O = valuation ring of $\mathbb{I}_x$, π uniformizer of O.

$\mathrm{val}(\pi) = 1$.

For (2) ((2) $\Rightarrow$ (1)), $\quad \forall k \geq 1,\ \forall \phi \in \mathcal{C}^{1,\infty}$

$$\lambda(\phi)^2 \in \mathrm{Fitt}(H^1_f(F, W \otimes O/p^k O)).$$

To prove a stronger version of (2):

- $\forall\, n \in \mathcal{N}_k^{even}$, fix $k \gg 0$,

  $\underline{\wp(n)}$ $\quad \forall \phi \in \mathcal{C}^{n,2k},\quad \lambda^{n,k}(\phi)^2 \in \mathrm{Fitt}(H^1_{f(n)}(F, W \otimes O/p^k O))$.

  (Then (2) $\Leftrightarrow \wp(1)$.)

<u>Goal</u> To prove $\wp(n)$ for a fixed $k \gg 0$.

$\forall \phi \in \mathcal{C}^{n,2k},\quad t_\phi := \mathrm{val}(\lambda^{n,2k}(\phi))$

$\forall\, n \in \mathcal{N}_{2k}^{even},\quad t_n := \min\{t_\phi \mid \phi \in \mathcal{C}^{n,2k}\}$

$$t := \min\{t_n \mid n \in \mathcal{N}_k^{even}\}.$$

Prove $\wp(n)$ by increasing induction on $t_n - t$ ($\in \mathbb{Z}_{\geq 0}$).

Let $a = a(n, n\ell)$, $n \in \mathcal{N}_{2k}^{even}$, $\quad \kappa^{n\ell,2k}(p^{a,2k}(\phi)) \quad \phi \in \mathcal{C}^{n,2k}$

$$\in H^1_{f(n)}(F, T \otimes \Lambda/p^{2k}\Lambda)$$

Then $\quad \mathrm{loc}_\ell(\kappa^{n\ell,2k}(p^{a,2k}(\phi))) = \lambda^{n,2k}(\phi)$.

Write $\quad t_\phi^\ell := \mathrm{val}(\kappa^{n\ell,2k}(p^{a,2k}(\phi))) \leq t_\phi$.

<u>Lem</u> $\exists\, \kappa^{n\ell}(\phi) \in H^1_{f(n)}(F, T \otimes \Lambda/p^k\Lambda)$ s.t.

$$\pi^{t_\phi^\ell} \cdot \kappa^{n\ell}(\phi) = \kappa^{n\ell,2k}(p^{a,2k}(\phi)) \pmod{p^k\Lambda}.$$

Have $\mathrm{val}(\kappa^{n\ell}(\phi)) = 0$, $\quad \mathrm{val}(\mathrm{loc}_\ell\, \kappa^{n\ell}(\phi)) = t_\phi - t_\phi^\ell$.

(A) Now suppose $t_n - t = 0$

$$\Rightarrow \quad H^1_{f(n)}(F, W \otimes \mathcal{O}/p^k\mathcal{O}) = 0 .$$

It suffices to show $H^1_{f(n)}(F, W \otimes \mathcal{O}/p^k\mathcal{O})[\pi] \overset{?}{=} 0$.

$$\cong \quad H^1_{f(n)}(F, W \otimes \mathcal{O}/\pi) \overset{?}{=} 0 .$$

Take $s \neq 0 \in H^1_{f(n)}(F, W \otimes \mathcal{O}/\pi)$.

$\rightsquigarrow$ $\exists\, \ell \in \mathcal{L}_{2k} \setminus n$ s.t. $\mathrm{loc}_\ell(s) \neq 0$

$$\kappa^{n\ell}(\phi) \in H^1_{f(n\ell)}(F, T \otimes \mathcal{O}/\pi) .$$

$\uparrow$ Choose ϕ s.t. $t_\phi = t_n$.

But $\mathrm{val}(\mathrm{loc}_\ell\, \kappa^{n\ell}(\phi)) = t_\phi - t_\phi^\ell$

$\rightsquigarrow$ it remains to show $t_\phi^\ell = t_\phi$.

If $t_\phi^\ell < t_\phi$, Lem 3 $\Rightarrow$ $\exists\, \ell' \in \mathcal{L}_{2k} \setminus n\ell$ s.t. $\mathrm{loc}_{\ell'}(\kappa^{n\ell}(\phi)) \neq 0$.

By 2nd reciprocity law,

$$\pi^{t_\phi^\ell} \cdot \mathrm{loc}_{\ell'}(\kappa^{n\ell}(\phi)) = \lambda^{n\ell\ell'}(\phi') \quad \text{for some } \phi' \in \mathcal{C}^{n\ell\ell', 2k} .$$

$$\Rightarrow \quad \mathrm{val}(\lambda^{n\ell\ell'}(\phi')) = t_\phi^\ell < t_\phi = t_n .$$

$$\Rightarrow \quad t_{n\ell\ell'} < t_n . \qquad \text{Contradiction.}$$

(B) Induction $t_n - t \rightsquigarrow t_n - t + 1$:

more sophistecated but with a similar flavour. $\square$