Localy symmetric spaces and torsion classes

Ana Caraini

(1) Quadratic reciprocity

$$x^2 - q \equiv 0 \pmod p \quad (p, q \text{ primes})$$

Fact # sol'ns depends on p mod q.

(2) Eichler's reciprocity law for

$$y^2 + y \equiv x^3 - x^2 \pmod p$$

Say this has N_p sol'ns.

Q How does N_p vary?

Error $N_p - p$ can be described by rec law.

$$q \prod_{i=1}^{\infty} (1-q^n)^2 (1-q^{11n})^2 = \sum_{n=1}^{\infty} a_n q^n$$

$a_p = p - N_p$.

More sophisticated ver:

modularity of ell curves.

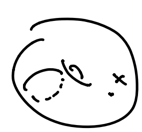
i.e. Gal rep'ns $\rho_E \simeq \rho_f : \underset{\substack{\| \\ G_{\mathbb{Q}}}}{\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})} \to GL_2(\mathbb{Q}_p)$.

Let $E/\mathbb{Q}$ ell curve.

e.g. $y^2 + y = x^3 - x^2$

$\rightsquigarrow E(\mathbb{C}) \simeq \mathbb{C}/\Lambda$, Λ lattice

has a grp str



& $E[p^n] \simeq (\mathbb{Z}/p^n\mathbb{Z})^2$ (fix this isom).

Then construct $\rho_E : G_{\mathbb{Q}} \longrightarrow GL(\varprojlim_n E[p^n])$

$$\begin{array}{c} \wr\wr \\ GL_2(\varprojlim_n (\mathbb{Z}/p^n\mathbb{Z})) = GL_2(\mathbb{Z}_p). \end{array}$$

<u>Rmk</u> Étale cohom gives a $G_{\mathbb{Q}}$-action on

$$H^i(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p$$

when $X/\mathbb{Q}$ alg var.

<u>Back to q-expansion</u>

f mod form (i.e holo fct on $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ + many symmetries)

$$q = e^{2\pi i z} \Rightarrow f(z) = \sum_{n=1}^{\infty} a_n q^n.$$

<u>Geometry</u> $\mathcal{H} \hookrightarrow \mathbb{P}^1(\mathbb{C})$, $SL_2(\mathbb{R}) \circlearrowright \mathcal{H}$, $GL_2(\mathbb{C}) \circlearrowright \mathbb{P}^1(\mathbb{C})$

Möbius transforms

<u>Fact</u> $\mathcal{H} \simeq SL_2(\mathbb{R}) / SO_2(\mathbb{R})$

$\mathcal{H}$: + natural cplx str

$SO_2(\mathbb{R})$: max cpt subgrp.

<u>Level</u> $\Gamma := \Gamma_0(11) = \left\{ \gamma \in SL_2(\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod 11 \right\}$

$\cap$

$SL_2(\mathbb{Z}) \subset SL_2(\mathbb{R}) \circlearrowright \mathcal{H}$

Congruence subgrp

<u>So</u> symmetries of f w.r.t. $\Gamma \subset SL_2(\mathbb{R})$

$\rightsquigarrow$ f descends to holo diff on $\Gamma \backslash \mathcal{H} \simeq Y_\Gamma(\mathbb{C})$

where Y_Γ = curve $/ \mathbb{Q}$.

grp isom + cplx str

<u>Upshot</u> $f \rightsquigarrow \omega_f$ = differential on Y_Γ

$\rightsquigarrow f$ contributes to $H^1_{et}(Y_\Gamma, \mathbb{Q}_p)$

<u>Reciprocity law</u> $\rho_E \simeq \rho_f$

(1) Fermat's last thm : consequence $E \rightsquigarrow f$

(2) Ramanujan conj for f: $|a_\ell| \le 2\sqrt{\ell}$,

where $\forall\, \ell \neq p, 11$, $a_\ell = \operatorname{tr} \rho_f(\mathrm{Frob}_\ell)$.

$$\mathrm{Gal}(\bar{\mathbb{F}}_\ell / \mathbb{F}_\ell) \ni \mathrm{Frob}_\ell : x \mapsto x^\ell$$

$\Rightarrow$ $a_\ell = \operatorname{tr} \rho_E(\mathrm{Frob}_\ell) = \ell - N_\ell$

(Lefschetz counting-pt formula.

This works for $E : y^2 + y = x^3 - x^2 \pmod{\ell}$ $/ \mathbb{F}_\ell$

Generally: $X / \mathbb{F}_\ell$ sm proj <u>curve</u>

$H^1_{et}(X, \mathbb{Q}_p) \circlearrowleft \mathrm{Frob}_\ell$ $\rightsquigarrow$ eigenvals of Frob_ℓ have abs value $\ell^{1/2}$.

<u>Shimura var</u>

Let G conn red grp $/ \mathbb{Q}$

$\rightsquigarrow X = G(\mathbb{R}) / K_\infty$ symm domain for G.

e.g. $G = SL_2$ $/ \mathbb{Q}(i)$ $\rightsquigarrow X \simeq \mathcal{H}^3$

$\Gamma < G(\mathbb{Q})$ congruence subgrp.

$\rightsquigarrow X_\Gamma := \Gamma \backslash X$ locally symm spaces.

Fact If $G = GSp_{2g}$ or unitary grp,

$\rightsquigarrow X_\Gamma$ has alg str (= Shimura var).

Thm 1 (Harris-Taylor, Clozel, Shin, Caraiani)

Let π be a cuspidal autom rep'n of $GL_n / F = F^+ \cdot K$ CM field

s.t. (1) π regular algebraic

(2) π is self-dual.

Then π satisfies the Ramanujan-Petersson conj

at all finite places $\wp \in \mathcal{O}_F$, $\wp | p$.

i.e. (roughly) π appears in $H^*(X_\Gamma, \mathbb{C})$.

is fixed $\mathbb{Q}_p$ $\forall p$.

Key of pf π is locally generic + Weil conj.

N.B. Conditions on π = generalizing symm of mod forms

(c.f. ω_f appears in $H^*(Y_\Gamma, \mathbb{C})$.)

Rmk generic = "far away from triv rep'n".

Q How about torsion coeff $H^*(X_\Gamma, \mathbb{F}_\ell)$?

Thm 2 (Caraiani-Scholze)

Let X_Γ be a compact unitary Shimura var.

Let $\psi \in H^*(X_\Gamma, \mathbb{F}_\ell)$ be a system of Hecke eigenvals.

If ψ is sufficiently generic

$\Rightarrow \psi$ occurs in middle deg only.

Idea: Input analogy of geom:

Hodge:

$$\begin{tikzcd} & SL_2(\mathbb{R}) \circlearrowright \mathcal{H} \arrow[dl] \arrow[dr, hook] & \\ Y_\Gamma(\mathbb{C}) & & \mathbb{P}^1(\mathbb{C}) \\ \Gamma \backslash \mathcal{H} \arrow[u, equal] & & GL_2(\mathbb{C}) \arrow[u, phantom, "\circlearrowright"] \end{tikzcd}$$

- $\mathcal{H} \hookrightarrow \mathbb{P}^1(\mathbb{C})$
  Hodge decomp. $\mapsto$ Hodge fil'n
- pt $\in \mathcal{H}$ trivializing H_*^{dR} of $E \in Y_\Gamma(\mathbb{C})$ ell curve.

Records info of diff of mod forms.

Hodge-Tate:

$$\begin{tikzcd} & \mathcal{Y}_{\Gamma(p^\infty)} \arrow[dl] \arrow[dr, "\pi_{HT}"] & \\ \mathcal{Y}_\Gamma & & \mathbb{P}^1(\mathbb{C}_p) \end{tikzcd}$$

$\mathcal{Y}_{\Gamma(p^\infty)}$ — perfectoid mod curve

- very complicated
- pt $E \in \mathcal{Y}_{\Gamma(p^\infty)}$ ell curve w/ trivialized H^1

$\mathcal{Y}_\Gamma$ — "p-adic analytic space" = Huber's adic space