From Geometric Representation Theory to the Langlands Program

AN INTERVIEW WITH ICM2022 INVITED PRESENTER – ALUMNUS XINWEN ZHU

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The quadrennial International Congress of Mathematicians (ICM) is a global academic conference on mathematics organised by the International Mathematical Union (IMU). It aims to promote high-level academic exchange, with world-renowned mathematical prizes such as the Fields Medal being awarded at the opening ceremony. ICM status is a high academic accolade and an important indication that a mathematician's work has been recognised and noticed by the international academic community.

The 29th ICM will be held in July 2022 and six Peking University mathematics faculty members: Weinan E, Xiaohua Zhu, Zhifei Zhang, Bin Dong, Yi Liu and Jian Ding have been invited to be presenters, including a one-hour presentation by academician Weinan E. Another seven PKU alumni will give 45-minute presentations: Chi Li, Gang Liu, Lu Wang, Guozhen Wang, Zhouli Xu, Xin Zhou and Xinwen Zhu.

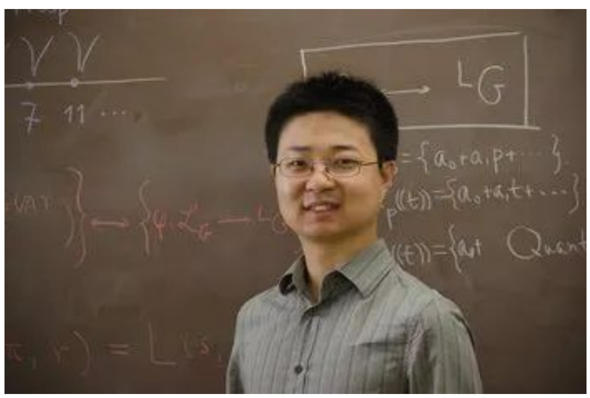

FIGURE 1. Xinwen Zhu

Xinwen Zhu received his undergraduate degree from the School of Mathematical Sciences, Peking University in 2004, and his Ph.D. in mathematics from the University of California, Berkeley, U.S.A. He was an Assistant Professor at Harvard University and Northwestern University from 2009 to 2014, and has been an Associate Professor in the Department of Mathematics at the California Institute of Technology since 2014 and a Professor since 2016. Xinwen Zhu focuses on geometric representation theory, in particular on geometric Langlands programs. He has studied the geometric and topological properties of flag manifolds of toric groups and has achieved important results in applying the theory of geometric Langlands' program to the field of arithmetic algebraic geometry. He was awarded the 2013-2014 American Mathematical Society Centennial Fellowship and the 2020 Scientific Breakthrough Award – New Horizons in Mathematics.

Interviewer Ruiqi Bai, Zekun Chen, and Jiedong Jiang.
Reporter Wenhan Dai and Xiangqian Yang.

Jiang: Congratulations on your invitation to give a 45-minute presentation at ICM 2022. What are you planning to talk about?

Zhu: I haven't thought about exactly what I'm going to talk about yet. There is still some progress in the work, and I don't know how far it will have progressed by then. But the general direction should be the geometric Langlands and the classical Langlands correspondence, and some applications in arithmetic geometry.

Jiang: Your PhD research was closer to representation theory. How did you go from representation theory, especially geometric Langlands, to arithmetic algebraic geometry, especially classical Langlands?

Zhu: Yes, I basically studied geometric representation theory as part of my PhD, and geometric Langlands was part of it. The change in research direction didn't happen all of a sudden. During my post-doc, I found that some of the methods in my first PhD paper *Affine Demazure modules and T-fixed point subschemes in the affine Grassmannian* could be used to solve The problem posed by some arithmetic geometers, specifically a conjecture of Pappas-Rapoport. They were motivated by the study of Shimura varieties, but the problem itself can be separated from the Shimura varieties and is more of an algebraic geometry problem. It just so happens that I found that some of the techniques in the article I wrote could be used to solve this conjecture, and by the way I learned a bit about what a shimura variety is. So the best time to learn a concept or a theory is when you need to use it in your research, and if you learn it right away, you'll see that it's not that advanced. After solving the conjecture, the algebraic geometry of the Shimura variety, called the local model, has been greatly advanced.

Zhu: Further on it was a collaboration with Liang Xiao. It just so happened that he had previously done some Tate conjectures and some geometric results with Yichao Tian on quaternion Shimura varieties, something that had some resemblance to some moduli spaces over function fields inside geometric Langlands, like shtuka and such. It just so happened that around that time V. Lafforgue made a breakthrough that advanced Langlands on functional domains considerably. Then I realized that the geometric Langlands approach could be used to study all these Tate conjectures on the Shimura varieties, the Jacquet-Langlands correspondence and so on, a whole set of things that could be done. In order to apply it to Shimura varieties, it was necessary to develop something of mixed characteristic. So I wrote *Affine Grassmannians and the geometric Satake in mixed characteristic*, which is equivalent to building some basic tools so that one can actually apply some of the results of geometric Langlands to arithmetic geometry, or methods to arithmetic geometry. We then systematically constructed this Jacquet-Langlands correspondence between different groups, as well as the Tate conjecture for the generic case on the Shimura variety mod p. This was then applied to the paper *Beilinson-Bloch-Kato conjecture for Rankin-Selberg motives.*

Zhu: Later I myself understood the deeper phenomena behind this work, hence a paper from last August, "Coherent sheaves on the stack of Langlands parameters". This article systematically made a framework for proposing many of the conjectures inside classical Langlands. Of course, the paper does not prove any big theorems itself, just some basic constructions, such as the construction of Galois representation moduli spaces and Langlands parameter moduli spaces. But mainly it presents

FIGURE 2. Xinwen Zhu (the left) with Liang Xiao at AIM seminar in 2019

some conjectures that I think unify many of the problems and phenomena in those previous arithmetic Langlands, for example it is very closely related to Taylor-Wiles' modularity theorem, so now I am considering with M. Emerton and T. Gee whether the conjectures I describe can be used to prove more problems like modularity lifting.

Chen: Why study the local model of the Shimura cluster? Is it to turn it into a perfectoid? Is there a purely global way to do it?

Zhu: This was first developed to study some of the geometry of Shimura varieties, in particular the nearby cycle, and thus to calculate the cohomology of some Shimura varieties. The concept of the nearby cycle was first introduced from topology, but later algebraic geometry abstracted it into a language of sheaf theory. What the sheaf looks like is related to the singularity of the varieties, so the local model is mainly used to study the nearby cycle by studying the singularity. This theory has been around since the 1990s and has nothing to do with perfectoid space, which is something that has been around since 2010.

Zhu: Of course, people started to use more prototypical methods in their research. The geometry of the Shimura varieties was very complicated, and people converted it into a more purely commutative algebra or linear algebra problem to study it, so what could be done was very limited at the beginning, and it might have to be discussed on a case-by-case basis. Later on, as I mentioned earlier, because I do geometric representation theory, I found that I could introduce some more representational-theoretic or more algebro-geometric methods to study these singularities of Shimura varieties directly and systematically.

Zhu: What is interesting about this story is that the conjecture about "what the nearby cycles on the Shimura variety should look like" was first made by Kottwitz in the 1990s. This conjecture was difficult at the time, but it happened to inspire some developments in geometric Langlands, in particular Gaitsgory's 2001 paper using nearby cycles to construct some of the elements in the Bernstein centre. So the whole process was first driven by Kottwitz's conjecture, and then by some powerful technical developments in geometric Langlands, which eventually led to a systematic calculation of all the nearby cycles. Then I took these geometric Langlands techniques, such as my earlier work on the Pappas-Rapoport conjecture, and applied them back to the computation of the nearby cycles, so that I could compute the very general case of Shimura varieties with the so-called parahoric level.

FIGURE 3. June 2017 Xinwen Zhu returned to Peking University to attend and present at an academic conference at BICMR

Zhu: In fact the whole picture of the discipline is similar. The earliest Langlands program was proposed by Langlands in the 1960s, then in the 1980s Drinfeld and Laumon proposed geometric Langlands, and of course the impetus at the beginning was all classical Langlands, i.e. to formulate and solve some similar problems in the case of function fields of Riemann surfaces. But then it slowly started to develop some new methods and tools independently, and roughly in the two or three decades from the 1980s until 2010, it became more independent and more connected to algebraic geometry and even mathematical physics, and more and more distant from number theory or from classical Langlands, so much so that when I was a student I felt as if they were two different disciplines. That's why you asked in the interview: I started out in representation theory, how did I get involved with number theory and all this stuff? The discipline started out with classical Langlands, but it slowly developed on its own, it took in other disciplines, like mathematical physics and so on, and developed a lot of tools, and eventually it became clear that these things could come back and feed into the original classical Langlands. I don't think it's an exaggeration to say that more and more mathematicians who are doing classical Langlands now are also learning the methods that come out of geometric representation theory.

Zhu: In a more general sense, it seems that mathematics and physics developed in the same way. In the early days, mathematics and physics influenced each other and developed together; by the middle of the 20th century, they became more and more distant and completely different; and later, by the end of the 20th century, mathematics and physics were basically connected again. So the development of the disciplines is quite amazing, "when they are divided, they are to be united, and when they are united, they are to be divided".

Bai: You mentioned the "general" Shimura variety, so how general is it? For example, is it of Abelian type or Hodge type?

Zhu: This depends on what kind of problem is being done. For example, the work I did with Liang Xiao was that Shimura varieties of general Abelian types can be made to construct Jacquet-Langlands cohomological correspondence, and also to prove the Tate conjecture. But for example, my work with Ruochuan Liu is not just for the Abelian, we proved for all Shimura varieties that all those p-adic local systems above are de Rham in the sense of p-adic Hodge theory. Yes, it is not only Abelian, that is, it is also provable for Shimura varieties without any moduli problem. So the development of p-adic Hodge theory in the last decade or so has still made it possible to do some things that could not be done before, and to ask questions that might not even have been asked before. I should say that arithmetic algebraic geometry has developed quite rapidly in the last decade or so, and of course Peter Scholze has played a big role in this.

Chen: We noticed that you have a lot of articles on arXiv with multiple collaborators, such as the one posted in August (2021) with five authors. So how did you start collaborating? We understood that you would have your own skills in certain areas, and then when someone else was working on a problem, they could come to you and use that part of your skills, was that a model?

Zhu: About half of the articles now are produced through collaboration. The one from August 2021 was a collaboration between Yifeng Liu, Yichao Tian, Liang Xiao, Wei Zhang and myself. This article is actually an appendix to the previous article, and we have separated it out again. Because the previous article was too long, the reviewer suggested that this be separated out because it is a more independent part.

Zhu: There are various modes of collaboration. Nowadays, mathematics is probably more and more collaborative, because it's becoming more and more refined, more and more extensive, requiring more and more knowledge, and everyone has their own specialization, and sometimes you have to put all kinds of different things together to make it. But actually it doesn't necessarily mean that if you have to solve a really difficult problem, it's probably better to work on your own. And of course it depends on what the problem is.

Chen: Is it that papers on number theory are generally on the long side? How do you go about determining if an essay has the right length?

Zhu: I don't think there is such a thing as an appropriate or inappropriate length of a paper, that is, if you write all the reasoning for a problem, it is natural for it to be as long as it should be. But it is true that papers in areas like number theory and arithmetic-algebraic geometry are quite a bit longer than those in other directions.

Zhu: It is still related to the characteristics of the discipline, but not to the level of the discipline. In some disciplines an essay is a single idea, a single technique to prove a problem, in a few pages. For example, Grothendieck wrote thousands of pages on the foundations of algebraic geometry, and in recent years, Jacob Lurie, an American mathematician, has written thousands and thousands of pages on the direction of so-called derived algebraic geometry. It is characteristic that often a problem requires the development of a whole set of proof tools, so it is particularly long. In contrast there are disciplines that are less instrumental and less systematic and perhaps more skillful. But I am not saying that number theory or algebraic geometry is difficult, because sometimes it is difficult to come up with a technique 'out of nothing' without a particularly good system. On the contrary, because there is a large system of number theory or algebraic geometry, there are certainly many techniques in between, but more often than not you know that there is a general direction or some guiding principles, so you can go in that direction, and it is always right, but it is not clear how much further you can go.

Bai: You are now teaching at Caltech, what did you find attractive about the school? Or what do you think is good about Caltech compared to other schools and why did you choose it? What do you think of the environment since you came here? For example, the research atmosphere, or the campus environment.

Zhu: Caltech is in Los Angeles, and Southern California is actually a good place for Chinese people to live in the US. Caltech is different from other universities in that it is very

FIGURE 4. Zhu Xinwen returns to Peking University in 2019 for *the Young Mathematicians Forum*

small, with only about 900 undergraduates, about the same size as the mathematics faculty at Peking University, and it is a very traditional, small and sophisticated university. But Caltech is a very strong academic institution in the United States. In terms of the number of achievements, although the absolute number is small (compared to those of large comprehensive universities), if we look at the number of Nobel Prizes produced in proportion to the number of students and faculty, it is probably the highest in the United States. I think Caltech has about 2,000 students plus graduate students and 300 faculty members, so it's a small, public school with a lot of resources.

Zhu: But of course one problem is that if the school is small, you need to be very motivated to do your own research. For example, there may be 70 to 80 PhDs in mathematics at Peking University in a year, but at Caltech, there are only 5 to 7 PhDs in a year. If you don't have strong self-discipline or determination, you may feel lonely.

Zhu: One of the advantages of this side of the university is that it is quieter, and if you have a strong motivation, you can do your academic work in a practical way, which is the advantage of a small but sophisticated university. Anyway, it depends on your personal habits: some people like a lively place with all sorts of presentations from morning to night, while some people prefer a quieter place where they don't have to listen to so many presentations or discussion classes. Personally, I find that I can listen to one or two presentations a week, because if there is too much information, it would be very difficult for me to process.

Bai: The final question. Over this century, more and more Chinese mathematicians have been invited to give presentations. During your undergraduate studies, Beijing hosted the International Congress of Mathematicians in 2002; now you have been invited to be a speaker at the International Congress of Mathematicians in 2022. As a witness and participant in the development of Chinese mathematics, what role do you think Chinese mathematicians will play on the international stage in the future?

Zhu: The development of Chinese mathematics over the years has definitely become better and better in every way. I think the first point is that when we were studying, our horizons were still very narrow, for example, geometric representation theory was not even known until I was a graduate student. Now you have much better access to information in China than we did then, and there are now many scholars who can come back to give lectures, seminars and lectures at any time, so the starting point is definitely getting higher and higher. I think the number of mathematicians and their average level are much higher than 20 years ago, and we can say that a group of international mathematics stalwarts has grown up. I believe that people like you will certainly get better and better in the future and will occupy an increasingly important position.

Zhu: But I feel that we are still a bit short of seminal figures. Of course, I think some of our mathematicians are leading the development of their respective fields, but there is still a gap between leading a field and really pioneering a field. Judging from the last two decades, we have a good thickness now, but there are not yet any great mathematicians like Shishiang Chern who have pioneered a field. Geniuses are usually difficult to cultivate, but with such a large population base in China, there must be some, but the main thing is to discover them and provide them with a good environment. The main thing is to discover them and provide them with a good environment. For the time being, I hope that your next generation will be able to produce someone who will create a big scene.

Figure 5. Spring Landscape at Peking University

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