A survey on Beilinson–Bloch conjecture for unitary Shimura varieties via Li–Liu

CHOW GROUPS AND L-DERIVATIVES OF AUTOMORPHIC MOTIVES FOR UNITARY GROUPS

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Abstract. These notes are based on a talk by Chao Li at Columbia in February, 2021. We survey the background of the joint work by Chao Li and Yifeng Liu [LL21, LL22] on Beilinson–Bloch conjecture for unitary Shimura varieties.

Contents

1. Backgrounds

1.1. **Birch–Swinnerton-Dyer conjecture.** Let $E: y^2 = x^3 + Ax + B$ be an elliptic curve over $\mathbb{Q}$. In the sense of Birch–Swinnerton-Dyer conjecture, we define

- The *algebraic rank* of E is the rank of the finitely generated abelian group $E(\mathbb{Q})$, that is,
$$r_{\mathrm{alg}}(E) := \operatorname{rank} E(\mathbb{Q}).$$
- The *analytic rank* of E is the order of vanishing of the L-function associated to E at the central point $s = 1$, that is,
$$r_{\mathrm{an}}(E) := \mathrm{ord}_{s=1} L(E, s).$$

Conjecture 1.1 (Birch–Swinnerton-Dyer, 1960s)**.**
(1) (*Rank part*).
$$r_a n(E) = r_{\mathrm{alg}}(E).$$
(2) (*Leading coefficient*). *For* $r = r_{\mathrm{an}}(E)$,
$$\frac{L^{(r)}(E,1)}{r!} = \frac{\Omega(E)R(E)\prod_p c_p(E)\cdot|\text{Ш}(E)|}{|E(\mathbb{Q})_{\mathrm{tor}}|^2},$$
where
- $R(E) = \det(\langle P_i, P_j\rangle_{\mathrm{NT}})_{r\times r}$ *is the regulator for the Néron–Tate height pairing*
$$\langle -,-\rangle_{\mathrm{NT}} : E(\mathbb{Q}) \times E(\mathbb{Q}) \longrightarrow \mathbb{R},$$
- $\text{Ш}(E)$ *is the Tate–Shafarevich group,*
- $\Omega(E)$ *is the Néron period integral of Néron differentials* ω_E *along* $E(\mathbb{R})$, *and*
- $c_p(E)$, *called the local Tamagawa number, equals* $[E(\mathbb{Q}_p) : E^0(\mathbb{Q}_p)]$ *for an elliptic curve* E^0 *arose by some local torsion condition.*

The following remark is by Tate in *The Arithmetic of Elliptic Curves*, 1974.

> *This remarkable conjecture relates the behavior of a function L at a point where it is not at present known to be defined to the order of a group Ш which is not known to be finite!*

The BSD conjecture is still widely open in general, but much progress has been made in the rank 0 or 1 case.

Theorem 1.2 (Gross–Zagier, Kolyvagin, 1980s)**.**
$$r_{\mathrm{an}}(E) = 0 \implies r_{\mathrm{alg}}(E) = 0, \qquad r_{\mathrm{an}}(E) = 1 \implies r_{\mathrm{alg}}(E) = 1.$$

Remark 1.3. When $r = r_{\mathrm{an}}(E) \in \{0,1\}$, many cases of the formula for $L^{(r)}(E,1)$ are known.

The proof combines two inequalities:

(1) (Gross–Zagier formula, [GZ86])
$$r_{\mathrm{an}}(E) = 1 \implies r_{\mathrm{alg}}(E) \geqslant 1.$$

(2) (Kolyvagin's Euler system, [BD05])
$$r_{\mathrm{an}}(E) \in \{0,1\} \implies r_{\mathrm{alg}}(E) \leqslant r_{\mathrm{an}}(E).$$

Both steps rely on *Heegner points* on modular curves.

1.2. **Beilinson–Bloch conjecture.**

1.2.1. *The general statement.* Let X be a smooth projective variety over a number field K. Denote $\mathrm{Ch}^m(X)$ the Chow group of algebraic K-cycles of codimension m on X. Also denote $\mathrm{Ch}^m(X)^0 \subset \mathrm{Ch}^m(X)$ the subgroup of geometrically cohomologically trivial cycles. Using this, we obtain the Beilinson–Bloch height pairing
$$\langle -,- \rangle_{\mathrm{BB}} : \mathrm{Ch}^m(X)^0 \times \mathrm{Ch}^{\dim X+1-m}(X)^0 \longrightarrow \mathbb{R}.$$
To state the conjecture, we also define $L(H^{2m-1}(X),s)$ to be the motivic L-function for $H^{2m-1}(X_{\overline{K}},\mathbb{Q}_\ell)$.

Conjecture 1.4 (Beilinson–Block, 1980s)**.**

(1) (*Rank part*).
$$\mathrm{ord}_{s=m} L(H^{2m-1}(X),s) = \mathrm{rank}\, \mathrm{Ch}^m(X)^0.$$

(2) (*Leading coefficient*).
$$L^{(r)}(H^{2m-1}(X),m) \sim \det(\langle Z_i, Z_j' \rangle_{\mathrm{BB}})_{r\times r}.$$

Example 1.5. Let $m=1$ and $X = K$ over $E/\mathbb{Q}$. Then BB conjecture 1.4 recovers the BSD conjecture 1.1 as
$$\mathrm{Ch}^1(E)^0 \simeq E(\mathbb{Q}), \quad L(H^1(E),s) = L(E,s), \quad \langle -,- \rangle_{\mathrm{BB}} = -\langle -,- \rangle_{\mathrm{NT}}.$$

Remark 1.6. In general, both sides in Conjecture 1.4(1) are only conditionally defined.

- $L(H^{2m-1}(X),s)$ is not known to be analytically continued to the central point $s=m$.
- $\mathrm{Ch}^m(X)^0$ is not known to be finitely generated.

1.2.2. *The case of Shimura variety.* BB conjecture is testable when X is a certain Shimura variety. Due to the works by Langlands–Kottwitz and Langlands–Rapoport, one can express the motivic L-functions of Shimura varieties $X = \mathrm{Sh}_G$ as a product of automorphic L-functions $L(s,\pi)$ on G, i.e.
$$L(H^{2m-1}(\mathrm{Sh}_G), s+m) = \prod_{\pi} L(s+1/2,\pi).$$
In the upcoming context we focus on the most interested case. For this, assume from now

(i) $2m-1 = \dim X$, so that we can consider the arithmetic middle degree;
(ii) π is tempered cuspidal.

In is known that the analytic properties of $L(s,\pi)$ can be established, and hence we are able to detect those of the motivic L-function. However, $\mathrm{Ch}^m(X)^0$ is not known to be finitely generated even when $X = \mathrm{Sh}_G$, but we can test if it is nontrivial.

The following is an unconditional prediction of BB conjecture, in the same spirit of Gross–Zagier.

Conjecture 1.7 (Beilinson–Bloch for Shimura varieties)**.**
$$\mathrm{ord}_{s=1/2} L(s,\pi) = 1 \implies \mathrm{rank}\, \mathrm{Ch}^m(X)^0_\pi \geqslant 1,$$
where $\mathrm{Ch}^m(X)^0_\pi$ *is the* π*-isotypical component of* $\mathrm{Ch}^m(X)^0$.

Remark 1.8. Conjecture 1.7 was only known for:

(1) X is a modular curve, by Gross–Zagier [GZ86];
(2) X is a Shimura curve, by S. Zhang [Zha01b, Zha01a], Kudla–Rapoport–Yang [KRY06], Yuan–Zhang–Zhang [YZZ13], Liu [Liu16, Liu19];
(3) $X = \mathrm{U}(1,1) \times \mathrm{U}(2,1)$ is a Shimura threefold and π is endoscopic, by Xue [Xue19].

Theorem 1.9 (Li–Liu, the impressionist version)**.** *Conjecture 1.7 holds for* $\mathrm{U}(2m-1,1)$*-Shimura varieties while* π *satisfying certain local assumptions.*

2. Beilinson–Bloch conjecture for U$(2m-1,1)$-Shimura varieties

2.1. Setups. Before setting up the unitary Shimura variety with U$(2m-1,1)$, we first consider the Hermitian symmetric space for U$(n-1,1)$, that is,

$$\mathbb{D}_{n-1} := \{z \in \mathbb{C}^{n-1} : |z| < 1\} \cong \frac{\mathrm{U}(n-1,1)}{\mathrm{U}(n-1) \times \mathrm{U}(1)}.$$

Moreover, we have an action on $\mathbb{D}_{n-1}$ by U$(n-1,1)$. Notice that $\mathbb{D}_1$ can be regarded as a hyperbolic plane (and is hence isomorphic to the upper half complex plane $\mathbb{H}$).

2.1.1. *The unitary Shimura variety X.* Let E be a CM extension of a totally real number field F over $\mathbb{Q}$. Let $\mathbb{V}$ be a totally definite *incoherent $\mathbb{A}_E/\mathbb{A}_F$-hermitian space* of rank n; here $\mathbb{V}$ is incoherent if it is not the base change of a global E/F-hermitian space, or equivalently, $\prod_v \varepsilon(\mathbb{V}_v) = -1$ with $\mathbb{V}_v := \mathbb{V} \otimes_{\mathbb{A}_F} F_v$. On the other hand, any place $w \mid \infty$ of F gives a nearby *coherent E/F-hermitian space V* such that

$$V_v \cong \mathbb{V}_v, \quad v \neq w,$$

whereas V_w has signature $(n-1,1)$.

Set $G = \mathrm{U}(\mathbb{V})$ and fix an open compact subgroup $K \subset G(\mathbb{A}_F^\infty) \cong U(V)(\mathbb{A}_F^\infty)$. Then we can take X to be the unitary Shimura variety of dimension $n-1$ over its reflex field E such that for any place $w \mid \infty$ inducing the complex embedding $\iota_w : E \hookrightarrow \mathbb{C}$,

$$X(\mathbb{C}) = \mathrm{U}(V)(F)\backslash(\mathbb{D}_{n-1} \times \mathrm{U}(V)(\mathbb{A}_F^\infty)/K).$$

It turns out that X is a Shimura variety of abelian type. Its étale cohomology and L-function are computed in the forthcoming work of Kisin–Shin–Zhu [KSZ], under the help of the endoscopic classification for unitary groups (Mok [Mok15], Kaletha–Minguez–Shin–White [KMSW14]).

2.1.2. *Automorphic representations π.* Resume on the setup above. Let $W = E^{2m}$ be the standard E/F-skew-hermitian space with matrix $\begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix}$. Let U$(W)$ be the quasi-split unitary group of rank $n = 2m$. Let π be the cuspidal automorphic representation of U$(W)(\mathbb{A}_F)$.

Assumptions 2.1. We assume the following about π_v locally.

1. E/F is split at all 2-adic places and $F \neq \mathbb{Q}$. Assume that $E/\mathbb{Q}$ is Galois or contains an imaginary quadratic field (for simplicity).
2. For $v \mid \infty$, π_v is the holomorphic discrete series with Harish-Chandra parameter $\{(n-1)/2, (n-3)/2, \ldots, (-n+3)/2, (-n+1)/2\}$.
3. For $v \nmid \infty$, π_v is tempered.
4. For $v \nmid \infty$ ramified in E, π_v is spherical with respect to the stabilizer of $\mathcal{O}_{E_v}^{2m}$.
5. For $v \nmid \infty$ inert in E, π_v is unramified or almost unramified. If π_v is almost unramified, then v is unramified over $\mathbb{Q}$.

Remark 2.2 (Almost unramifiedness). Saying π_v is almost unramified means that π_v has a nonzero Iwahori-fixed vector and its Satake parameter contains $\{q_v, q_v^{-1}\}$ and $2m-2$ complex numbers of norm 1. Equivalently, the theta lift of π_v to the non-quasi-silt unitary group of same rank is spherical with respect to the stabilizer of an almost self-dual lattice.

2.2. Main result. The first main result of [LL21, LL22] is the verification of BB conjecture. Let S_π be the set of places v that are inert and such that π_v's are almost unramified. Then under Assumptions 2.1, the *global root number* for the (complete) standard L-function $L(s,\pi)$ equals

$$\varepsilon(\pi) = (-1)^{|S_\pi|} \cdot (-1)^{m \cdot [F:\mathbb{Q}]}$$

by epsilon dichotomy (Harris–Kudla–Sweet [HKS96], Gan–Ichino [GI16]). When $\mathrm{ord}_{s=1/2} L(s,\pi) = 1$:

- $\varepsilon(\pi) = -1$,
- $\mathbb{V} = \mathbb{V}_\pi$ is the totally definite incoherent space of rank $n = 2m$ such that, for $v \nmid \infty$, we have $\varepsilon(\mathbb{V}_v) = -1$ exactly for $v \in S_\pi$,
- X, the associated unitary Shimura variety, is of dimension $n-1 = 2m-1$ over E, and
- $\mathrm{Ch}^m(X)^0_\pi$ is the localization of $\mathrm{Ch}^m(X)^0_{\mathbb{C}}$ at the maximal ideal $\mathfrak{m}_\pi$ of the Hecke algebra associated to π.

Theorem 2.3 ([LL21, LL22])**.** *Let π be a cuspidal automorphic representation of $\mathrm{U}(W)(\mathbb{A}_F)$ satisfying Assumptions 2.1. Then the implication*

$$\mathrm{ord}_{s=1/2}L(s,\pi)=1 \implies \mathrm{rank}\,\mathrm{Ch}^m(X)^0_\pi \geqslant 1$$

holds when the level $K\subset G(\mathbb{A}_F^\infty)$ is sufficiently small.

Example 2.4 (Symmetric power L-function of elliptic curves)**.** Let A/F be a modular elliptic curve without complex multiplication such that

(i) A has bad reduction only at places v that split in E;
(ii) $\mathrm{Sym}^{2m-1} A_E$ is automorphic (Newton–Thorne, Clozel–Thorne, etc.).

Then there exists π satisfying Assumptions 2.1 such that

$$L(s+1/2,\pi)=L(\mathrm{Sym}^{2m-1}A_E, s+m).$$

As $S_\pi=\emptyset$ and $\varepsilon(\pi)=(-1)^{m\cdot[F:\mathbb{Q}]}$, Theorem 2.3 applies to π when $m\cdot[F:\mathbb{Q}]$ is odd.

3. Arithmetic inner product formula and arithmetic theta lifting

3.1. **Generating series of Heegner points.** Nontrivial cycles can be constructed via the method of arithmetic theta lifting by Kudla and Liu [Liu11a, Liu11b]. Here comes a baby example of Heegner points, which contributes to Gross–Zagier formula as well.

Consider the modular curve

$$X_0(N)=\Gamma_0(N)\backslash\mathbb{H}\cup\{\text{cusps}\}=\{E_1\to E_2:\text{cyclic } N\text{-isogeny}\}.$$

For certain imaginary quadratic field $K=\mathbb{Q}(\sqrt{-d})$, we have a *Heegner divisor*

$$Z(d):=\{E_1\to E_2 \text{ with endomorphisms by } \mathcal{O}_K\}\in \mathrm{Ch}^1(X_0(N)).$$

The theory of complex multiplication asserts that $Z(d)$ is actually a divisor of $X_0(N)$ defined over K.

Let $E/\mathbb{Q}$ be an elliptic curve of conductor N who has a modular parametrization

$$\varphi_E: X_0(N)\longrightarrow E.$$

Using these, we define a *Heegner point*

$$P_K\in\varphi_E(Z(d)-\deg Z(d)\cdot\infty)\in E(K).$$

Then we are able to state the Gross–Zagier formula.

Theorem 3.1 (Gross–Zagier, [GZ86])**.** *Up to simpler nonzero factors,*

$$L'(E_K,1)\sim\langle P_K,P_K\rangle_{\mathrm{NT}}.$$

Remark 3.2. (1) Choosing K suitably gives the implications

$$r_{\mathrm{an}}(E)=1\implies r_{\mathrm{alg}}(E)\geqslant 1.$$

(2) BSD formula for E_K reduces to a precise relation between P_K and $\text{Ш}(E_K)$.

To introduce Arithmetic theta liftings, we first consider the following heuristic example. Recall that $K=\mathbb{Q}(\sqrt{-d})$. Take $P_d=\mathrm{tr}_{K/\mathbb{Q}}P_K\in E(\mathbb{Q})$. It may depend on the choice of d, even when $E(\mathbb{Q})\cong\mathbb{Z}$.

Example 3.3. Let $E=X_0^+(37): y^2+y=x^3-x$. Then

- $E(\mathbb{Q})\cong\mathbb{Z}$ with a generator $P=(0,0)$.
- E corresponds to the modular form $f\in S_2(37)$ where

$$f=q-2q^2-3q^3+2a^4-2q^5+6q^6-q^7+6q^9+4q^{10}-5q^{11}-6q^{12}-2q^{13}+\cdots.$$

- Table of Heegner points P_d:

d	3	4	7	11	12	16	27	$\cdots$	67	$\cdots$
P_d	$(0,-1)$	$(0,-1)$	$(0,0)$	$(0,-1)$	$(0,0)$	$(1,0)$	$(-1,-1)$	$\cdots$	$(6,-15)$	$\cdots$
c_d	-1	-1	1	-1	1	2	3	$\cdots$	-6	$\cdots$

where $P_d=c_d\cdot P$.

Now the miracle is that the coefficients c_d appear as the Fourier coefficients of $\phi \in S^+_{3/2}(4 \cdot 37)$, for

$$\phi = \sum_{d \geqslant 1} c_d q^d = -q^3 - q^4 + q^7 - q^{11} + q^{12} + 2q^{16} + 3q^{27} + \cdots - 6q^{67} + \cdots,$$

which maps to f under the Shimura–Waldspurger–Kohnen correspondence

$$\theta : S^+_{3/2}(4 \cdot 37) \longrightarrow S_2(N), \quad \phi \longmapsto f.$$

3.2. **Arithmetic theta lifting.** The observation arising from Example 3.3 dictates that the generating series of Heegner points

$$\sum_{d \geqslant 1} P_d \cdot q^d = \sum_{d \geqslant 1} c_d P \cdot q^d = \phi \cdot P \in S^+_{3/2}(4 \cdot 37) \otimes E(\mathbb{Q})_{\mathbb{C}}$$

is a modular form valued in $E(\mathbb{Q})_{\mathbb{C}}$. More generally, we may define a generating series of Heegner divisors on $X_0(N)$,

$$Z := \sum_d Z(d) q^d \in M_{3/2}(4N) \otimes \mathrm{Ch}^1(X_0(N))_{\mathbb{C}},$$

which may be viewed as an *arithmetic theta series.*

Definition 3.4. Use Z as the kernel to define *arithmetic theta lifting*

$$\Theta(\phi) := (Z, \phi)_{\mathrm{Pet}} \in \mathrm{Ch}^1(X_0(N))^0_{f,\mathbb{C}} = E(\mathbb{Q})_{\mathbb{C}}.$$

Indeed, $\Theta(\phi)$ does not depend on any particular choice of d or K.

Theorem 3.5 (Gross–Kohnen–Zagier, [GKZ87])**.** *Up to simpler nonzero factors,*

$$L'(E, 1) \sim \langle \Theta(\phi), \Theta(\phi) \rangle_{\mathrm{NT}}.$$

Now let us focus on the case where X is a unitary Shimura variety as before.

Definition 3.6 (Special cycles on unitary Shimura variety)**.** Suppose $X = \mathrm{Sh}_{\mathrm{U}(V)}$.

- For any $y \in V$ with $(y, y) > 0$, its orthogonal complement $V_y \subset V$ has rank $n - 1$. The embedding $\mathrm{U}(V_y) \hookrightarrow \mathrm{U}(V)$ defines a Shimura subvariety of codimension1, read as
$$\mathrm{Sh}_{\mathrm{U}(V_y)} \longrightarrow X = \mathrm{Sh}_{\mathrm{U}(V)}.$$
- For any $x \in V(\mathbb{A}_F^\infty)$ with $(x, x) \in F_{>0}$, there exists $y \in V$ and $g \in \mathrm{U}(V)(\mathbb{A}_F^\infty)$ such that $y = gx$. Define the *special divisor*
$$Z(x) \longrightarrow X$$
to be the g-translate of $\mathrm{Sh}_{\mathrm{U}(V_y)}$.
- For any $\mathbf{x} = (x_1, \ldots, x_m) \in V(\mathbb{A}_F^\infty)^m$ with $T(\mathbf{x}) = ((x_i, x_j)) \in \mathrm{Herm}_m(F)_{>0}$, define the *special cycle* (of codimension m) as
$$Z(\mathbf{x}) = Z(x_1) \cap \cdots \cap Z(x_m) \longrightarrow X.$$
- More generally, for a Schwartz function $\varphi \in \mathscr{S}(V(\mathbb{A}_F^\infty)^m)^K$ and $T \in \mathrm{Herm}_m(F)_{>0}$, define the *weighted special cycle*
$$Z_\varphi(T) = \sum_{\substack{\mathbf{x} \in K \backslash V(\mathbb{A}_F^\infty)^m, \\ T(\mathbf{x}) = T}} \varphi(\mathbf{x}) Z(\mathbf{x}) \in \mathrm{Ch}^m(X)_{\mathbb{C}}.$$
- With extra care, we can also define $Z_\varphi(T) \in \mathrm{Ch}^m(X)_{\mathbb{C}}$ for any $T \in \mathrm{Herm}_m(F)_{\geqslant 0}$.

Definition 3.7. Define *Kudla's generating series of special cycles* as

$$Z_\varphi(\tau) = \sum_{T \in \mathrm{Herm}_m(E)_{\geqslant 0}} Z_\varphi(T) q^T.$$

Conjecture 3.8 (Kudla's modularity [Kud97, Kud04])**.** *The formal generating series $Z_\varphi(\tau)$ converges absolutely and defines a modular form on $\mathrm{U}(W)$ valued in $\mathrm{Ch}^m(X)_{\mathbb{C}}$.*

Remark 3.9.
(1) The analogous modularity in Betti cohomology is known by Kudla–Millson [KM90] in 1980s.
(2) Conjecture is known for $m = 1$. For general m, the modularity follows from the absolute convergence [Liu11b].

(3) The analogous conjecture for orthogonal Shimura varieties over $\mathbb{Q}$ is known by Bruinier–Westerholt-Raum [BWR15].
(4) Conjecture is known when E/F is a norm-Euclidean imaginary quadratic field, due to Xia [Xia21].

Definition 3.10. Assuming Kudla's modularity conjecture, for $\phi \in \pi$, define *arithmetic theta lifting* for Kudla's generating series of weighted special cycles as

$$\Theta_\varphi(\phi) = (Z_\varphi(\tau), \phi)_{\mathrm{Pet}} \in \mathrm{Ch}^m(X)^0_\pi.$$

Theorem 3.11 ([LL21, LL22])**.** *Let π be a cuspidal automorphic representation of $\mathrm{U}(W)(\mathbb{A}_F)$ satisfying Assumptions 2.1. Assume $\varepsilon(\pi) = -1$. Assume Kudla's modularity in Conjecture 3.8. Then for any $\phi \in \pi$ and $\varphi \in \mathscr{S}(V(\mathbb{A}_F^\infty)^m)$, up to simpler factors depending on ϕ and φ,*

$$L'(1/2, \pi) \sim \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi)\rangle_{\mathrm{BB}}.$$

Remark 3.12. The simpler factors can be further made explicit. For example, if

- ◦ π is unramified or almost unramified at all finite places,
- ◦ $\phi \in \pi$ is a holomorphic newform such that $(\phi, \overline{\phi})_\pi = 1$, and if
- ◦ φ is a characteristic function of self-dual or almost self-dual lattices at all finite places,

then

$$\frac{L'(1/2,\pi)}{\prod_{i=1}^{2m} L(i, \eta^i_{E/F})} C_m^{[F:\mathbb{Q}]} \prod_{v \in S_\pi} \frac{q_v^{m-1}(q_v+1)}{(q_v^{2m-1}+1)(q_v^{2m}-1)} = (-1)^m \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi)\rangle_{\mathrm{BB}},$$

where

$$C_m = 2^{m(m-1)} \pi^{m^2} \frac{\Gamma(1)\cdots\Gamma(m)}{\Gamma(m+1)\cdots\Gamma(2m)}.$$

Moreover, as an addendum,

(1) The classical Riemann hypothesis predicts that
$$L'(1/2, \pi) \geqslant 0;$$
(2) Beilinson's Hodge index conjecture predicts that
$$(-1)^m \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi)\rangle_{\mathrm{BB}} \geqslant 0.$$

The combination of (1) and (2) is compatible with our formula.

Before introducing the proof strategy of Li–Liu, we list out a brief summary on arithmetic theta lifting, as well as the generalization from BSD conjecture to BB conjecture.

	BSD conjecture	BB conjecture
Ambient varieties	Modular curves $X_0(N)$	Unitary Shimura varieties X
Simple geometric objects	Heegner points $Z(d)$	Special cycles $Z_\varphi(T)$
Kudla's generating series	$Z = \sum_d Z(d) q^d \in \mathrm{Ch}^1(X_0(N))_\mathbb{C}$	$Z_\varphi = \sum_T Z_\varphi(T) q^T \in \mathrm{Ch}^m(X)_\mathbb{C}$
Arithmetic theta liftings	$\Theta(\phi) \in E(\mathbb{Q})_\mathbb{C}$	$\Theta_\varphi(\phi) \in \mathrm{Ch}^m(X)^0_\pi$
Formulas	Gross–Zagier formula $L'(E,1) \sim \langle \Theta(\phi), \Theta(\phi)\rangle_{\mathrm{NT}}$	Arithmetic inner product formula $L'(1/2,\pi) \sim \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi)\rangle_{\mathrm{BB}}$

3.3. **The proof strategy.**

3.3.1. *Doubling method.* The doubling method is introduced by Piatetski-Shapiro–Rallis [PSR86, PSR87] and Yamana [Yam14], read as

$$L(s + 1/2, \pi) \sim (\phi \otimes \overline{\phi}, \mathrm{Eis}(s, g))_{\mathrm{U}(W)^2},$$

where $\mathrm{Eis}(s, g)$ is a Siegel Eisenstein series on $\mathrm{U}(W \oplus W)$.

By definition $\Theta_\varphi(\phi) = (Z_\varphi, \phi)_{\mathrm{Pet}}$ gives

$$\langle \Theta_\varphi(\phi), \Theta_\varphi(\phi)\rangle_{\mathrm{BB}} = (\phi \otimes \overline{\phi}, \langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB}})_{U(W)^2}.$$

To prove $L'(1/2, \pi) \sim \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi)\rangle_{\mathrm{BB}}$, it suffices to compare

$$\mathrm{Eis}'(0, g) \overset{?}{=} \langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB}}.$$

This can be viewed as an *arithmetic Siegel–Weil formula*. Here the Beilinson–Bloch height pairing is a sum of local indexes

$$\langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB}} = \sum_v \langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB},v}.$$

And the nonsingular Fourier coefficient for the q^T-term decomposes as

$$\mathrm{Eis}'_T(0,g) = \sum_v \mathrm{Eis}'_{T,v}(0,g).$$

3.3.2. *Local comparison on arithmetic Siegel–Weil formula.* For nonsingular local terms, it suffices to prove

$$\mathrm{Eis}'_{T,v}(0,g) \overset{?}{=} \langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB},T,v}.$$

In codimension 1 case with $m=1$, the Gross–Zagier formula computes both sides explicitly. However, such an explicit computation is infeasible for general m.

- When $v \nmid \infty$, we use:
  - (i) the work for relating $\langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB},T,v}$ to arithmetic intersection numbers;
  - (ii) recent proof of Kudla–Rapoport conjecture due to Li–Zhang [LZ22].
- When $v \mid \infty$, we use:
  - (i) archimedean arithmetic Siegel–Weil formula, proved by Liu [Liu11a] and Garcia–Sankaran [GS19] independently;
  - (ii) avoidance of holomorphic projections.

To finish the argument, we kill singular terms on both sides by proving the existence of special $\varphi \in \mathscr{S}(V(\mathbb{A}_F^\infty)^m)$ with *regular support* at two split places with nonvanishing local zeta integrals. Motivated by the comparison of nonsingular terms which deduced Theorem 3.11 for special φ, we can extrapolate such a proof for arbitrary φ with *multiplicity one* of doubling method in tempered case. Consequently, Theorem 2.3 is given by a same computation without Kudla's modularity, using the proof by contradiction.

Remark 3.13. We have some final remarks on Assumptions 2.1.

(1) When $v \nmid \infty$, the local index $\langle -,-\rangle_{\mathrm{BB},v}$ is defined as an ℓ-adic linking number. It is defined on a certain subspace $\mathrm{Ch}^m(X)^{\langle \ell\rangle} \subset \mathrm{Ch}^m(X)^0$ (which are conjecturally equal) and its independence on ℓ is not known in general.

(2) Find a Hecke operator $t \notin \mathfrak{m}_\pi$ such that $t^*Z \in \mathrm{Ch}^m(X)^{\langle \ell\rangle}$, so BB height is defined. Also find another Hecke operator $s \notin \mathfrak{m}_\pi$ and BB height of s^*t^*Z can be therefore computed in terms of the *arithmetic intersection number* of a nice extension $\mathcal{Z}$ on $\mathcal{X}$. Here $\mathcal{X}$ is a regular integral model of a related unitary Shimura variety of PEL type. This step requires to prove certain vanishing of $\mathfrak{m}_\pi$-localized ℓ-adic cohomology of $\mathcal{X}$.

(3) The *Kudla–Rapoport conjecture* states that

$$\mathrm{Eis}'_{T,v}(0,g) = \text{the arithmetic intersection number above.}$$

Assuming the conjecture, the ℓ-independence of $\langle Z_\varphi, Z_\varphi$ $rangle_{\mathrm{BB},T,v}$ will follow.

(4) Assumptions 2.1 are required in both construction of Hecke operators and the proof of Kudla–Rapoport conjecture.

(5) The condition $F \neq \mathbb{Q}$ of fields is needed to prove vanishing of $\mathfrak{m}_\pi$-localized cohomology of integral models with Drinfeld level structures at split places (with input from Mantovan [Man08], Caraiani–Scholze [CS17]).

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