

Projective Morphisms (I)

§1 Proj of a Graded Ring

$$S = \bigoplus_{n=0}^{\infty} S_n \text{ graded ring}, \quad S^+ = \bigoplus_{n=1}^{\infty} S_n$$

i.e. each S_n closed under addition & $S_m \cdot S_n \subseteq S_{m+n}$.

$f \in S_n$: homogeneous of deg n . Each S_n is a S_0 -mod.

$$\hookrightarrow \text{Proj } S = \{ p \subseteq S \text{ homo prime ideals}: S^+ \notin p \}$$

$\forall n, f \in S_n, S_f = \text{graded w/ negative deg}$
 $(g/f^k \text{ deg } m-kn, g \in S_m)$.

$$\hookrightarrow D(f) = \{ p \in \text{Proj } S: f \notin p \} = \text{Spec } S_{f,0}.$$

note: $D(f) \cap D(g) = D(fg)$, Proj S has sch str.

When $S = A[x_0, \dots, x_n]$. x_i homo of deg 1.

$$\hookrightarrow \text{Proj } S = \mathbb{P}_A^n.$$

Fact $S \rightarrow T$ graded rings $\leftrightarrow \text{Proj } T \rightarrow \text{Proj } S$

Say $I \subseteq S$ homo if as ab grps: $I = \bigoplus_{n=0}^{\infty} (I \cap S_n)$

$\hookrightarrow S/I$ also graded $\leftrightarrow S$

$\hookrightarrow \text{Proj } S/I \rightarrow \text{Proj } S$ closed imm

Caveat: $\text{Proj } S$ does not determine S by itself { Will come back later.
e.g. $\text{Proj } \bigoplus_{n=0}^{\infty} S_n = \text{Proj } S_0 \oplus \bigoplus_{n=2}^{\infty} S_n$

Can do the same thing towards $M = \bigoplus_{n=-\infty}^{\infty} M_n$.

§2 The Sheaf \mathcal{O}_X

M graded S -mod, i.e. $S_m \cdot M_n \subseteq M_{m+n}$.

$$\Leftrightarrow M_{\{n\}} := \bigoplus_i M_{n+i} = \bigoplus_i M_{n+i}$$

On $X = \text{Proj } S$, $\mathcal{O}_{X(n)} = \mathcal{F}\text{-coh sheaf } \widetilde{S(n)}$.

$$\Leftrightarrow \mathcal{O}_{X(n)} = \mathcal{O}_X, \quad \mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X(n)}$$

for any $\mathcal{F} \in \mathbf{Qcoh}(\mathbf{Mod}_{\mathcal{O}_X})$.

Lemma S gen'd by S_1 on an S_0 -alg i.e. $S_0[S_1] = S$

$\Rightarrow \mathcal{O}_{X(n)}$ on $\text{Proj } S$ are loc. free of rank 1.

$$\& \mathcal{O}_{X(m)} \otimes_{\mathcal{O}_X} \mathcal{O}_{X(n)} \cong \mathcal{O}_{X(m+n)}.$$

Proof. Hartshorne II.5.12. □

Note $\mathcal{F} \in \mathbf{Qcoh}(X)$, X loc ringed.

\mathcal{F} loc. free of rank 1 $\Leftrightarrow \mathcal{F}$ invertible.

b/c $\exists! \mathcal{F}^{\vee}$ st. $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^{\vee} \cong \mathcal{O}_X$ "dual sheaf"

c.f. fractional ideals of \mathcal{O}_K in number theory.

$$\Leftrightarrow (\mathcal{O}_{X(n)})^{\vee} = \mathcal{O}_{X(-n)}.$$

New interpretation:

x_i on $\text{Proj } A[x_0, \dots, x_n]$: global sections of $\mathcal{O}_{X(1)}$
(rather than \mathcal{O}_X).

Thm $S = S_0[S_1]$ f.g. alg. $\Rightarrow \forall \mathcal{F} \in \mathbf{Qcoh}(\text{Proj } S), \mathcal{F} = \tilde{M}$

\downarrow (for a canonical choice of M).

to ensure $\text{Proj } S$ "quasi-compact".

Proof. The module we want:

$$M = \Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(\text{Proj } S, \mathcal{F}(n)).$$

$$\text{where } \mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n).$$

The Rest: see Hartshorne II.5.15. \square

Caution $S = \bigoplus_{n=0}^{\infty} \Gamma(X, \mathcal{O}_X(n))$ NOT true in general!

e.g. (stupid) $S = A[x]$ w.r.t $\mathcal{O}_X(n)$ free, $\forall n$.
 $\Rightarrow \Gamma(X, \mathcal{O}_X(n)) \neq 0$ even when $n < 0$.

(less stupid) Hartshorne Ex II.5.14.

However:

Lemma $n \geq 1$. $S = A[x_0, \dots, x_n]$ with usual grading
 $\Rightarrow S = \bigoplus_{n=0}^{\infty} \Gamma(X, \mathcal{O}_X(n)).$

Proof. see Hartshorne II.5.13. \square

§3 Closed Subschemes of Projective Spaces

Prop $\forall n \geq 1$, any $X \xrightarrow{i} \mathbb{P}_A^n$ closed imm is def'd by
some homogeneous ideal $I \subseteq A[x_0, \dots, x_n]$.

Proof. I ideal sheaf defining i .
 $\Rightarrow \Gamma_*(I) = \text{ideal of } \Gamma_*(\mathcal{O}_X).$ { canonical way
 $\overset{''}{I}$ to find I .
 $S = A[x_0, \dots, x_n]$ } \square

In general, $I \hookrightarrow X \subseteq \mathbb{P}_A^n \hookrightarrow \overline{I} = \text{saturation of } I.$
 $\underbrace{\quad}_{\text{as above}}$

$$\overline{I} = \{ f \in A[x_0, \dots, x_n] \text{ s.t. } x_0^i f, \dots, x_n^i f \in I \}$$

"Projective Nullstellensatz":

$I \rightarrow X$ multi-to-one

$\bar{I} \leftrightarrow X$ (saturated) one-to-one.

Cor $n \geq 1$. $I \subseteq S = A[x_0, \dots, x_n]$ homo ideal. TFAE:

(a) $X \subseteq \mathbb{P}_A^n$ def'd by I , $X = \emptyset$.

(b) $\bar{I} = S^\dagger$,

(c) $S_n \subseteq I$, $n \gg 0$.

Proof. (a) \Leftrightarrow (b) known. (c) \Rightarrow (b) clear.

(b) \Rightarrow (c): $\forall f \in \{x_0, \dots, x_n\}$, $x_i^i \mid f, \dots, x_n^i \mid f \in \bar{I} = S^\dagger$,

$\Rightarrow \exists x_0^i, \dots, x_n^i \in I$ for some i . (true for $i \gg 0$).

$\Rightarrow S_{(n+1)(i-1)+1} \subseteq I$

Pigeonhole principle } b/c \exists one of x_0^i, \dots, x_n^i dividing
each monomial of $\deg(n+1)(i-1)+1$. \square

§4 Projective Implies Proper

Goal Complete the proof that $f: \mathbb{P}_\mathbb{Z}^n \rightarrow \text{Spec } \mathbb{Z}$ is proper.

Recall The missing step:

to show that f is wiv. closed

i.e. $\forall X \in \text{Sch}, \mathbb{P}_X^n \xrightarrow{f} X$ closed.

\Leftarrow Do this in $X = \text{Spec } A$ affine.

Proof. Let $Z \subseteq \mathbb{P}_X^n \rightarrow X$, $z \in X \setminus \text{im } Z$. Put $k = K(z)$.

We must exhibit $x \in U \subseteq X$ s.t. $Z \cap \mathbb{P}_U^n = \emptyset$.

Let $I = \bigoplus_{n=0}^{\infty} I_n$ saturated $\subseteq S = A[x_0, \dots, x_n]$ defining Z .

$\Rightarrow I \otimes_A k$ defines $\phi \in \text{Proj } k[x_0, \dots, x_n]$.

(but may not be saturated).

$$\exists m \text{ s.t. } I_m \otimes_A k = S_m \otimes_A k \text{ & } (S_m/I_m) \otimes_A k = 0.$$

Now S_n/I_n f.g. A -mod, by Nakayama:

$$\Rightarrow (S_n/I_n) \otimes_A A_p = 0, \forall p \in A \text{ defining } \mathfrak{z}.$$

Again, since S_n/I_n f.g.

$$\Rightarrow (S_n/I_n) \otimes_A A_g = 0, g \in A \setminus p.$$

$$\Rightarrow \mathfrak{z} \in D(g), D(g) \cap \text{im } \mathfrak{z} = \emptyset. \quad \square$$

§5 What is a Projective Morphism?

Hartshorne / Eisenbud - Harris:

$f: Y \rightarrow X$ proj if $f: Y \rightarrow \mathbb{P}_X^n \rightarrow X$ for some n .

This def'n:

① evidently stable under base changes

(!) ② not local on the base.

Better to say "globally proj", and say

f loc proj if $\forall x \in X, \exists x \in U \subseteq X$ open

s.t. $f: Y_{x \times U} \rightarrow U$ globally proj.

i.e. $Y_{x \times U} \rightarrow \mathbb{P}_U^r \rightarrow U$.

Morally Globally proj = loc proj

if X is "not too large".

e.g. X globally quasi-proj / affine sch.

$$\begin{pmatrix} f: Y \rightarrow X & Y \rightarrow \mathfrak{Z} \text{ open imm} \\ \downarrow & \swarrow \\ \mathfrak{Z} & \mathfrak{Z} \rightarrow X \text{ glob proj} \end{pmatrix}$$

EGA: $\xleftarrow{\quad}$? $\xrightarrow{\quad}$
loc. proj proj glob proj
(more on this later)

Warning Eisenbud-Harris claim that

loc proj = proj (X)
but counterexamples are rather pathological.