# BASIC NUMBER THEORY: LECTURE 12

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This lecture explains the solution to the mid-term exam and uses the rest of the lesson to teach the following.

## Genus field revisited

Recall the whole picture of genus field and Hilbert class field of any imaginary quadratic field $K$ as follows.

$$\begin{tikzcd}
L \arrow[d, dash, "C(\mathcal{O}_K)^2"'] \arrow[dd, dash, bend left, "C(\mathcal{O}_K)"] & \text{Hilbert class field} \\
M \arrow[d, dash, "\{\pm1\}^{\mu-1}\simeq C(\mathcal{O}_K)/C(\mathcal{O}_K)^2"'] \arrow[dd, dash, bend left, "\{\pm1\}^{\mu}"] & \text{genus field} \\
K \arrow[d, dash] & \text{imaginary quadratic field} \\
\mathbb{Q} &
\end{tikzcd}$$

Let $p_1, \dots, p_r$ be all odd prime divisors of $d_K$. Then we obtain

$$M = K(\sqrt{p_1^*}, \dots, \sqrt{p_r^*}) = \begin{cases} \mathbb{Q}(\sqrt{p_1^*}, \dots, \sqrt{p_r^*}) & d_K \equiv 1 \bmod 4, \\ \mathbb{Q}(\sqrt{-1}, \sqrt{p_1^*}, \dots, \sqrt{p_r^*}) & \frac{d_K}{4} \equiv 1 \bmod 4, \\ \mathbb{Q}(\sqrt{-2}, \sqrt{p_1^*}, \dots, \sqrt{p_r^*}) & \frac{d_K}{4} \equiv 2 \bmod 8, \\ \mathbb{Q}(\sqrt{2}, \sqrt{p_1^*}, \dots, \sqrt{p_r^*}) & \frac{d_K}{4} \equiv 6 \bmod 8. \end{cases}$$

Also, we have defined the Artin map

$$\left(\frac{M/K}{\cdot}\right) : I_K \to \mathrm{Gal}(M/K) \to \prod_{i=1}^{r} \mathrm{Gal}(K_i/K) \xrightarrow{\sim} \{\pm 1\}^r,$$

where $\Phi_K : \mathrm{Gal}(M/K) \to \{\pm 1\}^r$ and $K_i = K(\sqrt{p_i^*})$. For each fractional ideal $\mathfrak{a} \subseteq \mathcal{O}_K$,

$$\Phi_K(\mathfrak{a}) = \left( \left(\frac{N(\mathfrak{a})}{p_1}\right), \dots, \left(\frac{N(\mathfrak{a})}{p_r}\right) \right).$$

**Claim:** Denote $P$ the group of principal genus in $C(\mathcal{O}_K)$. Then the following diagram commutes:

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$$\begin{tikzcd}
C(d_K)/C(d_K)^2 \arrow[r] \arrow[d, "\sim"'] & C(\mathcal{O}_K)/P \arrow[r] & \{\pm 1\}^{\mu} \arrow[d] \\
C(\mathcal{O}_K)/C(\mathcal{O}_K)^2 \arrow[rr, "(\chi_1,\dots,\chi_r)", "\sim"'] & & \{\pm 1\}^r.
\end{tikzcd}$$

$\boxed{\text{To be continued in Lecture 13.}}$

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