# Gelfand pairs and gamma factors mod $\ell$

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<u>Question</u> When is $\mathrm{Ind}_H^G \rho$ multiplicity-free?

$\rho$ irred rep of $H$, $H \leq G$ finite.

Over $\mathbb{C}$, $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(\pi, \mathrm{Ind}_H^G \rho) \overset{?}{\leq} 1$, $\forall$ irrep $\pi$ of $G$.

($\mathrm{soc}(\mathrm{Ind}_H^G \rho)$ or $\mathrm{cosoc}(\mathrm{Ind}_H^G \rho)$ multi-free).

<u>Def</u> $(G, H, \rho)$ <u>multi-free triple</u>, if

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(\pi, \mathrm{Ind}_H^G \rho) \leq 1, \quad \forall \pi \in \mathrm{Irrep}_G.$$

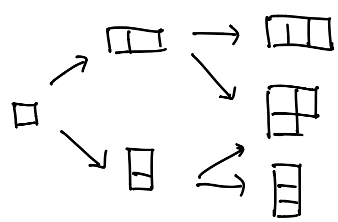
$(G, H)$ <u>Gelfand pair</u> if $(G, H, \mathrm{triv}_H)$ multi-free triple.

e.g. $G = S_n$, $H = S_k$, $\mathbb{C}$.

Irrep $S_n \longleftrightarrow S^\lambda$ Specht mod. $\lambda$ partition of $n$.

$\mathrm{Ind}_H^G \longleftrightarrow$ Adding blocks to $\lambda$.

For $\mathrm{Ind}_{S_1}^{S_3}(\mathrm{triv}_{S_1})$:



$\rightsquigarrow \mathrm{Ind}_{S_1}^{S_3}(\mathrm{triv}_{S_1}) = S^{(3)} \oplus (S^{(2,1)})^{\oplus 2} \oplus S^{(1,1,1)}$

<u>Fact</u> $\mathrm{Ind}_{S_k}^{S_n}(S^\lambda)$ multi-free $\Leftrightarrow n = k+1$.

## A criterion over $\mathbb{C}$

Consider the Hecke algebra

$$\mathcal{H}(G,H,\rho) := \left\{ \begin{array}{l} \text{cont } \Delta : G \to \mathrm{End}_{\mathbb{C}}(\rho) \\ \text{s.t. } \Delta(h_2 g h_1) = \rho(h_2)\,\Delta(g)\,\rho(h_1) \end{array} \right\}$$
$$= \mathrm{End}(\mathrm{Ind}_H^G \rho).$$

**Thm** (Gelfand 1950, Gelfand-Graev 1962)

Over $\mathbb{C}$, when $G$ & $H$ finite grps,

$$\mathcal{H}(G,H,\mathrm{triv}_H) \text{ commutative}$$
$$\Leftrightarrow (G,H) \text{ a Gelfand pair.}$$

**Idea** Maschke's thm $\Rightarrow \mathrm{Ind}_H^G \rho = \bigoplus_i d_i [\pi_i]$ $/\mathbb{C}$.

Schur's lem $\Rightarrow \mathrm{End}(\mathrm{Ind}_H^G \rho) = \bigoplus_i \mathrm{Mat}_{d_i}(\mathbb{C})$

Commutativity $\Rightarrow d_i \leq 1$, $\forall i$.

**Usual application** (Gelfand's trick)

$\exists\, c : G \to G$ anti-involution s.t. $f(c(g)) = f(g)$ $\forall f \in \mathcal{H}(G,H,\rho)$.

& s.t. $Hc(g)H = HgH$.

$\Rightarrow \mathcal{H}(G,H,\rho)$ Commutative.

e.g. (Uniqueness of Whittaker models)

$G = GL_n(\mathbb{F}_q)$, $H = \mathcal{U}_n = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \right\} \subseteq G$.

Fix $\psi : \mathbb{F}_q \to \mathbb{C}^\times$, $\psi_{\mathcal{U}_n}((u_{ij})) = \psi(u_{12}) + \psi(u_{23}) + \cdots + \psi(u_{n-1,n})$.

Then: $(G, H, \psi_{\mathcal{U}_n})$ is a multi-free triple.

(pf by Gelfand's trick).

e.g. (Prasad thesis)

$k$ local field, $D_k$ division alg / $k$.

$$(D_k^* \times D_k^* \times D_k^*,\ \Delta D_k^*,\ \mathrm{triv}_{\Delta D_k^*}) \quad / \mathbb{C}$$

multi-free tripe

$\Rightarrow$ $\exists!$ trilinear form $\pi_1 \otimes \pi_2 \otimes \pi_3 \to \mathbb{C}$

$\rightsquigarrow$ Harris-Kudla nonvanishing $L(\pi_1 \otimes \pi_2 \otimes \pi_3, \frac{1}{2})$.

or GGP.

<u>Thm</u> (Zhang, 2023)

Over $\overline{\mathbb{F}}_\ell$, when $G, H$ finite grps.

$\mathcal{H}(G, H, \rho)$ commutative $\Rightarrow$ $(G, H)$ Gelfand pair.

<u>Idea</u> Replace $\rho$ with $\mathrm{soc}\,\rho$ or $\mathrm{cosoc}\,\rho = \rho/\mathrm{rad}\,\rho$.

Hope that $\mathrm{End}(\rho/\mathrm{rad}\,\rho) \cong \mathrm{End}\,\rho/\mathrm{rad}(\mathrm{End}\,\rho)$.

$\Big\downarrow$ Gelfand's argument

multi-freeness of cosoc.

Can write two uniqueness results in char $p$.

- Whitaker: $\psi : \mathbb{F}_q \to \overline{\mathbb{F}}_\ell$, $\ell \nmid q$.

using $c : g \mapsto \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} {}^t g \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$.

## Relative projectivity / injectivity

**Def** $M$ $R$-mod,

$$\begin{tikzcd} K \arrow[r, hook] \arrow[d] & N \arrow[dl, dashed] \\ M & \end{tikzcd} \qquad \begin{tikzcd} & N \arrow[d] \\ M \arrow[ur, dashed] \arrow[r] & K \end{tikzcd}$$

$M$ injective $\qquad$ $M$ projective

**Lem** (1) $M$ is self-projective & $\operatorname{rad} M$ is superfluous + injective

$$\left( \begin{array}{l} N \text{ superfluous} \Leftrightarrow (N + L = K \Rightarrow L = K) \\ N \text{ essential} \Leftrightarrow (N \cap K = 0 \Rightarrow K = 0) \end{array} \right)$$

$\Rightarrow \operatorname{End}(\rho / \operatorname{rad} \rho) \cong \operatorname{End} \rho / \operatorname{End}(\operatorname{rad} \rho)$.

(2) $\operatorname{soc} \rho$ essential $\Rightarrow \operatorname{End}(\operatorname{soc} \rho) \cong \operatorname{End} \rho / \operatorname{End}(\operatorname{rad} \rho)$.

**Thm** $F = \bar{F}$, $R$ $F$-alg, $M$ f.g. $R$-mod.

Suppose $\operatorname{End}_R M$ commutative. Then

(1) $M$ self-proj $\Rightarrow \dim_R \operatorname{Hom}(M, N) \leq 1$, $\forall N \in \operatorname{Mod}_R$ simple

using $\operatorname{rad} M$ superfluous

(2) $M$ self-inj $\Rightarrow \dim_R \operatorname{Hom}(N, M) \leq 1$, $\forall N \in \operatorname{Mod}_R$ simple.

using $\operatorname{soc} M$ essential.

Gelfand's case $R = F[G]$, $M = \operatorname{Ind}_H^G \rho$.

**Question** What about p-adic grps?

**Question** Does the Rankin-Selberg $\gamma$-factor

$\gamma(\pi \times \pi', s, \psi)$ determine $\pi$?

<u>Another motivation</u>

$K/\mathbb{Q}_p$ finite. LLC for $GL_n(k)/\mathbb{C}$ states that

$$\left\{ \begin{array}{c} \text{irr adm supercusp} \\ \text{reps of } GL_n(k)/\mathbb{C} \end{array} \right\} \xleftrightarrow[\text{rec}_n]{1\text{-}1} \left\{ \begin{array}{c} \text{irr semisimple} \\ \text{WD-reps } G_K/\mathbb{C} \end{array} \right\}$$

characterized by

(1) $n=1$, local CFT

(2) twists by $\lambda$, det, etc.

(3) $\underbrace{L(\pi \times \pi', s)}_{\gamma\text{-factors}} = \underbrace{L(\text{rec}(\pi) \times \text{rec}(\pi'), s)}_{\gamma\text{-factors}}$

<u>Thm</u> (Henniart 1993)

$\pi_1, \pi_2$ supercusp s.t.

$$\gamma(\pi_1 \times \pi', s) = \gamma(\pi_2 \times \pi', s),$$

$\forall$ adm irr $\pi'$ of $GL_r$, $1 \le r \le n-1$

$\Rightarrow \quad \pi_1 \cong \pi_2$.

Again, LLC for $GL_n(k) / \overline{\mathbb{F}}_\ell$ (Vignéras, 1994)

can be characterized by compatibility w/ LLC $/\mathbb{C}$

$$\begin{tikzcd} \square \arrow[r, "\text{rec}_{\mathbb{C}}"] \arrow[d] & \square \\ \square \arrow[r, "\text{rec}_{\overline{\mathbb{F}}_\ell}"] & \square \arrow[u] \end{tikzcd}$$

Same thm / $\overline{\mathbb{F}}_\ell$. characterized by $\gamma$ mod $\ell$.

<u>Thm</u> (BGGGIMZ, 2023)

$\ell \neq p$. $\pi_1, \pi_2$ cusp irrep of $GL_2(\mathbb{F}_q)$, $q = p^f$.

$$\tilde{\gamma}(\pi_1 \times \pi', \psi) = \tilde{\gamma}(\pi_2 \times \pi', \psi)$$

$\forall \pi'$ irred generic rep of $GL_{n-1}$.

$\Rightarrow$ $\pi_1 \cong \pi_2$.