A-parameters and eigensheaves

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(Joint with A. Bertoloni-Meli)

F nonarch local field, res field $\mathbb{F}_q$.

G/F q-split reductive.

$\ell \neq p$. Choose $\sqrt{q} \in \bar{\mathbb{Q}}_\ell$ + Whittaker datum.

<u>Conj</u> (Fargues) $\phi : W_F \longrightarrow {}^L G / \bar{\mathbb{Q}}_\ell$ discrete L-param.

Then $\exists$ $\mathcal{F}_\phi \in D_{lis}(\underbrace{\mathrm{Bun}_G}_{\text{moduli of } G\text{-bdl on FF curve}}, \bar{\mathbb{Q}}_\ell)$

satisfying

(1) Eigen property: $\mathcal{F}_\phi$ is an Hecke eigensheaf.

i.e. $\forall$ $V \in \mathrm{Rep}(\hat{G})$,

$$T_V : D_{lis}(\mathrm{Bun}_G, \bar{\mathbb{Q}}_\ell) \longrightarrow D_{lis}(\mathrm{Bun}_G, \bar{\mathbb{Q}}_\ell)^{W_F}$$

$$T_V(\mathcal{F}_\phi) = \underset{\substack{\circlearrowleft \\ W_F\text{-equiv.}}}{(V \circ \phi)} \otimes \mathcal{F}_\phi$$

(2) $\mathcal{F}_\phi$ perverse (c.f. Caraiani-Scholze)

Recall $\underbrace{\mathrm{Bun}_G^b}_{\text{smooth}} \subset \mathrm{Bun}_G$, $b \in B(G)$ See Hansen's Beijing notes.

$b \in B(G)$ basic $\Rightarrow$ i_b open imm

$$\mathrm{Bun}_G^b \cong * / \underline{G_b(F)},$$

$$D_{lis}(\mathrm{Bun}_G^b, \bar{\mathbb{Q}}_\ell) \cong D(\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(G_b(F))).$$

(3) $\mathcal{F}_\phi|_{\mathrm{Bun}_G^b} = \bigoplus_{\underbrace{\pi \in \Pi_\phi(G_b)}_{\text{L-packet}}} \pi^{\oplus \dim \underbrace{\langle \pi, - \rangle}_{\text{rep of } S_\phi = \mathrm{Cent}\,\phi \text{ corresp to } \pi}}$

(4) local-global compatibility :

$\exists$ relation w/ $\pi_{HT} : \mathrm{Igs} \to \mathrm{Bun}_G$

$\rightsquigarrow R\pi_{HT,*}\overline{\mathbb{Q}}_\ell \rightsquigarrow$ vast of eigensheaves.

Conj (FS, Categorical LLC)

$$D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell) \overset{\exists !}{\cong} \mathrm{Ind\,Coh}^{b,qc}(\underbrace{Z^1(W_F, \hat{G})}_{\text{moduli stack of L-param.}} / \hat{G})$$

$i_{1!}\, c\mathrm{Ind}_N^G \psi \longmapsto \mathcal{O}$ str sheaf.

(N, ψ) fixed Whit datum

$T_V \longleftrightarrow \underline{V} \otimes (-)$.

$\rightsquigarrow \exists$ Hecke eigensheaf on $Z^1(W_F, \hat{G}) / \hat{G}$.

These are easy to describe !

E.g. ϕ supercusp + G ss. Then

$$BS_\phi \hookrightarrow Z^1(W_F, \hat{G}) / \hat{G} \quad \text{clopen \& regular}$$

$\rightsquigarrow$ eigensheaf = pushforward of regular rep.

<u>Recently</u> It was observed that Fargues's conj / this constr above should work for a wider class of L-param

$\cup$

"tempered" for $\mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$.

(in progress: Hamann + Hansen-Koshikawa).

This works for general A-parameters.

<u>Conj</u> (Bertoloni-Meli - Koshikawa)

$$\psi : W_F \times SL_2^D \times SL_2^A \longrightarrow {}^LG \quad \text{generalized A-param.}$$

Then $\exists\ \mathcal{F}_\psi \in D_{lis}(Bun_G, \bar{\mathbb{Q}}_\ell)$ satisfying

(1) Sheared eigenproperty: (also, c.f. BZSV).

$$T_V(\mathcal{F}_\psi) = \bigoplus_i V_i \otimes \mathcal{F}_\psi[-i].$$

$$V \circ \psi|_{\mathbb{G}_m^A \subseteq SL_2^A} = \bigoplus_i \underbrace{V_i}_{\text{wt } i \text{ rep of } \mathbb{G}_m^A}$$

(2) $\mathcal{F}_\psi$ is perverse.

(3) G ss, b basic $\rightsquigarrow G_b$ pure inner form.

Consider p-adic Adams-Barbasch-Vogan packet Π_ψ

(Vogan, Cunningham-Fiori-Moussaoui-Mracek-Xu).

Then $\mathcal{F}_\psi|_{Bun_G^b} = \bigoplus_{\pi \in \Pi_\psi(G_b)} \pi^{\oplus \dim \kappa_\pi, -s}$.

(4) Local-global compatibility.

<u>Thm</u> (BM-K) $\exists\ \mathcal{F}_\psi$ on $Z^1(W_F, \hat{G})/\hat{G}$ satisfying (1) (2) (3),

in progress.

<u>Rmk</u> Certain indcoh sheaves $\overset{\text{Conj}}{\longleftrightarrow}$ parabolic inductions of universal twist of sc. — projective generators.

<u>Q</u> Which (ind-) coh shvs would corresp to $i_{b!}\pi \in Rep(G_b(F))$? (b basic)

Have some progress (preprint w/ T. Leake).

Rmk In Conj (3), relation with ABV:

work of Ben-Zvi – Chen – Helm – Nadler (in the unipotent case).

Example Case of $G = GL_2$ or PGL_2, $\hat{G} = SL_2$.

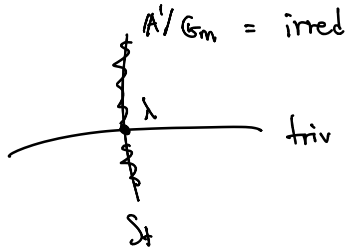
ϕ_{St} : L-param of Steinberg $W_F \times SL_2^D \longrightarrow \hat{G}$.

$\psi_{\mathbb{1}}$: A-param of triv rep $W_F \times SL_2^D \times SL_2^A \xrightarrow{pr_3} \hat{G}$.

Then $\phi_{St} \rightsquigarrow \lambda$ the same semi-simplification.
$\psi_{\mathbb{1}} \rightsquigarrow$

$\mathcal{F}_{\psi_{\mathbb{1}}}$ on Bun_G should be $\bar{\mathbb{Q}}_\ell$ on Bun_G.

$\mathbb{A}^1/\mathbb{G}_m$ = irred comp of $Z^1(W_F, \hat{G}) /\!/ \hat{G}$



$$\downarrow$$

$$\hat{G} /\!/ \hat{G} \ni \lambda$$

$$Z^1(W_F, \hat{G}) /\!/ \hat{G} \Big|^{\wedge}_{\mathbb{A}^1/\mathbb{G}_m} \xleftrightarrow[\text{duality}]{\text{Koszul}} T^*(\mathbb{A}^1/\mathbb{G}_m)$$

Cotangent stack.