p-adic hyperbolicity of Shimura varieties

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<u>Main thm</u> Let A_g be the moduli of ppav with torsion-free level, and A_g^* its minimal compactification.

Let $F/\mathbb{Q}_p$ be a discretely valued non-arch local field, S/F smooth rigid analytic var.

<u>Then</u> every analytic map

$$f : \Delta^\times \times S \to A_g^{an}$$

extends to an analytic map

$$f : \Delta \times S \to (A_g^*)^{an}$$

where $\Delta = \{z : |z| \le 1\}$ unit disc

& $\Delta^\times = \Delta - \{0\}$ punctured unit disc.

<u>Cor</u> If S/F is an algebraic var, then every analytic map $f : S^{an} \to A_g^{an}$ is algebraic.

Starting point: great Picard thm $/\mathbb{C}$.

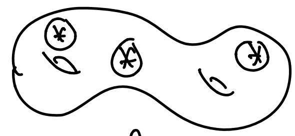
<u>Thm</u> every

$$\begin{tikzcd} \Delta^\times_{/\mathbb{C}} \arrow[r] \arrow[d, hook] & (\mathbb{P}^1 \setminus \{0,1,\infty\})^{an} \arrow[d, hook] \\ \Delta \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

extends to

$\Rightarrow$ If $S/\mathbb{C}$ algebraic var then every holo map $S^{an} \to (\mathbb{P}^1 \setminus \{0,1,\infty\})^{an}$ is algebraic.

Say $S =$ affine curve



$$\begin{tikzcd} S \arrow[r] \arrow[d, hook] & \mathbb{P}^1 \setminus \{0,1,\infty\} \arrow[d] \\ f: \bar{S} \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

GAGA $\Rightarrow$ f is algebraic.

$\Rightarrow$ Little Picard thm

Every holo map $\mathbb{C} \xrightarrow{f} \mathbb{P}^1 \setminus \{0,1,\infty\}$ is const.

(Great Picard + f & $f \circ \exp$ algebraic)

$\Leftarrow$ Uniformization thm

$\mathbb{P}^1 \setminus \{0,1,\infty\}$ is uniformized by $\mathbb{H} \simeq$ open unit disc.

Recall · A complex algebraic curve is called hyperbolic if it can be uniformized by $\mathbb{H}$.

· In higher dim, there are different notions of hyperbolicity.

Let D be a non-compact hermitian symmetric domain ($\simeq G/K$).

E.g. $D = \mathbb{H}, \mathbb{H}_g$.

Simply-conn, can be realized as an bounded domain in $\mathbb{C}^n$.

Thm Let $\Gamma \subset G$ be a sufficiently small arithmetic subgrp.

(1) (Baily-Borel)

$\exists$ canonical cptification

$$\Gamma \backslash D \hookrightarrow \Gamma \backslash D^* \xrightarrow[\text{analytic subspace}]{\text{closed}} \mathbb{P}^N .$$

(2) (Borel)

$$\begin{tikzcd} \text{every } (\Delta^\times)^a \times \Delta^b \arrow[r] \arrow[d, hook] & \Gamma\backslash D \arrow[d, hook] \\ \text{extends to } \Delta^{a+b} \arrow[r] & \Gamma\backslash D^* \end{tikzcd}$$ $\leftarrow$ given by BB.

Rmk (1) + (2) imply $\exists!$ algebraic structure on $\Gamma\backslash D$.

Recall When Γ is a congruence subgrp,
then $\Gamma\backslash D$ is a connected component of a Shimura variety
and can be defined over some number field.
(e.g. $D = \mathbb{H}_g$, $\Gamma\backslash D = A_g$.)

Rmk Same hold for Shimura varieties of abelian type $\mathcal{S}_g$ ($\hookrightarrow A_g$).

Idea of proof

Set up (1) A_g has a $\Gamma(\ell)$ level for $\ell \neq p$ odd.
(2) A_g has max parahoric level at p.
(3) $S = pt$ (for simplicity).

Prop 1 Let $f : \Delta^\times \to A_g^{an}$ $/F$.
Then the abelian rank of $x^* A^{univ}$ is constant for all classical pts $x \in \Delta^\times$.

$$\begin{tikzcd} \text{punctured disc } \otimes \arrow[r] \arrow[d] & A_g \arrow[d] \\ \odot \arrow[r] & A_g^* \end{tikzcd}$$ $\rightsquigarrow \mathcal{A}_g^*$ (integral model)

Note Good reduction: $\{0\} \to A_g \subset A_g^*$

Bad reduction: $\{0\} \to A_g^* - A_g$ (in some boundary strata).

Prop 2 Let $f: \Delta^\times \to \hat{A}_g^{rig}$ be a analytic map $/F$.

Then f extends to $f: \Delta \to \hat{A}_g^{rig}$.

Prop 3 Let $f: \Delta^\times \to \hat{A}_g^{rig}$ as above. Then

$\exists$ a lifting

$$\begin{tikzcd} & RZ_{\bar{x}}^{rig} \arrow[d] \\ \Delta \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & \hat{A}_g \end{tikzcd} \qquad \left(\begin{tikzcd} RZ_{\bar{x}}^{rig} \arrow[r] & T\backslash RZ_{\bar{x}}^{rig} \\ \hat{A}_g \arrow[ur] & \end{tikzcd} \right)$$

where $RZ_{\bar{x}}$ is the Rapoport-Zink space.

$$R \in Nilp_{\breve{\mathbb{Z}}_p} \longrightarrow \left\{ (A, \lambda, \iota) \;\middle|\; \begin{array}{l} (A,\lambda) \text{ p.a.v. } / R \\ \iota: (A,\lambda)_{(R/p)} \xrightarrow{\sim} (A_i, \underset{\substack{\uparrow \\ \text{Some given one}}}{\lambda_i}) \otimes_{\bar{\mathbb{F}}_p} R/p \end{array} \right\}$$

representable by a formal scheme $/ \breve{\mathbb{Z}}_p$.

Thm 4 Every $f: \Delta^\times \to RZ_{\bar{x}}^{rig}$ extends to

$f: \Delta \to RZ_{\bar{x}}^{rig}$.

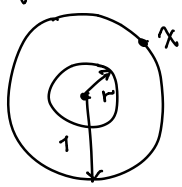
~~Idea of proof of thm 4~~

$$\left\{ \begin{array}{l} \mathbb{Z}_p\text{-étale de Rham} \\ \text{local system on } \Delta^\times \end{array} \right\} \xrightarrow[\text{Liu-Zhu}]{\text{Diao-Lan-}} \left\{ \begin{array}{l} \text{v.b. with log connection on } \Delta \\ + \text{ a descending filn satisfying} \\ \quad \text{Griffith transversality} \\ + \text{ residue has exponent in } [0,1) \cap \mathbb{Q} \end{array} \right\}$$

Note

$$\begin{tikzcd} \Delta^\times \arrow[r] \arrow[d, hook] & RZ_x^{rig} \arrow[d, "\text{“HT”}"] \\ \Delta \arrow[r] \arrow[ur, dashed] & \mathcal{F}\ell \end{tikzcd}$$

<u>Idea of proof of Prop 1</u>



$$\Delta^{\times} = \bigcup \mathbb{T}_{[r,1]}$$

$$\begin{tikzcd}
\pi_1'(\mathbb{T}_{[1,1]}) \arrow[r, "\sim"] \arrow[d] & \pi_1'(\mathbb{G}_m) \arrow[d] & \pi_1'(x) \arrow[l] \arrow[d] \\
GL(\Gamma_{\bar{x}}) \arrow[r] & \pi_1'(\mathbb{G}_m, 0) & \pi_1'(0) \arrow[l]
\end{tikzcd}$$