Introduction to p-adic Galois representations

Jared Weinstein

<u>Tate</u> Recall $A/\mathbb{C}$ ab var of dim g,

$\rightsquigarrow A = \mathbb{C}^g/\Lambda$, $\mathrm{rk}_{\mathbb{Z}} \Lambda = 2g$.

<u>Facts</u> $H_1(A, \mathbb{Z}) = \Lambda$,

$$H^1(A, \mathbb{C}) = \mathrm{Hom}(\Lambda_{\mathbb{C}}, \mathbb{C}) \simeq H^1_{dR}(A)$$

$$(\gamma \mapsto \int_\gamma \omega) \longleftarrow \omega.$$

$$H^1_{dR}(A) = H^0(A, \Omega^1_A) \oplus H^1(A, \mathcal{O}_A)$$

Hodge decomp.

<u>Note</u> $H_1(A, \mathbb{Z}/p^n\mathbb{Z}) = \frac{1}{p^n}\Lambda/\Lambda = A[p^n]$.

$H_1^{et}(A, \mathbb{Z}_p) = \varprojlim_n A[p^n] = T_pA$ Tate mod at p.

<u>Thm</u> (Tate, good reduction)

Let $K/\mathbb{Q}_p$ finite, A/K ab var. $C = \hat{\bar{K}}$.

$$H^1_{et}(A_{\bar{K}}, \mathbb{Z}_p) \otimes C \simeq H^0(A, \Omega^1_{A/C})(-1) \oplus H^1(A_C, \mathcal{O}_{A_C})$$

$$\simeq C^g(-1) \oplus C^g$$

as C v.s. w/ semilinear G_K-actions.

(Hodge-Tate decomp.).

<u>Aside</u> (i) semilinear G_K-action:

$C \circlearrowleft G_K$. $V \in \mathrm{Vect}_{/C}$.

$\sigma : V \to V$, $\forall \sigma \in G_K$

$\sigma(\alpha v) = \sigma(\alpha)\sigma(v)$, $\alpha \in C$.

(2) Tate twist:

$$\chi_{cycl} : G_K \longrightarrow \mathbb{Z}_p^\times, \quad \sigma(\zeta_{p^n}) = \zeta_{p^n}^{\chi_{cycl}(\sigma)}.$$

$\rightsquigarrow$ $C(n) = C \cdot e$ where $e \in C$ s.t.

$$\sigma(e) = \chi_{cycl}^n(\sigma) \cdot e.$$

($/\mathbb{C}$: can imagine $e = 2\pi i$).

<u>Ex</u> $A = E/\mathbb{Q}_p$ w/ good ordinary reduction.

$$0 \to \varprojlim \hat{E}[p^n](\bar{\mathbb{Z}}_p) \to \underbrace{\varprojlim E[p^n](\bar{\mathbb{Z}}_p)}_{\substack{\| \\ T_p(E)}} \to \varprojlim \tilde{E}[p^n](\bar{\mathbb{F}}_p) \to 0$$

Here $T_pE \underset{\mathbb{Z}_p}{\otimes} C \circlearrowleft G_{\mathbb{Q}_p}$ by $\begin{pmatrix} \chi \delta^{-1} & * \\ 0 & \delta \end{pmatrix}$.

$$* \in H^1(G_{\mathbb{Q}_p}, C(1) \otimes \delta^2).$$

<u>Thm</u> Let $\eta : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ be a char, $\eta(I_{\mathbb{Q}_p})$ infinite,

then $H^i(G_{\mathbb{Q}_p}, C(\eta)) = 0$, $i = 0, 1$.

<u>pf</u> Involves study of tower K_∞ / K.

$$\begin{array}{c} \bar{K} \\ | \\ K_\infty \\ \Gamma \left(\, | \right. \\ K \end{array}$$

Assume $K_\infty = \bigcup_{n \geq 0} K_n$.

For $n \gg 0$, K_{m+n}/K_n cyclic of deg p^m + tot ramified.

e.g. $K_n = K(\mu_{p^n})$.

<u>Prop</u> $\exists\ 0 < \varepsilon < 1$ s.t. $\forall n \gg 0$, $\forall g \in \mathrm{Gal}(K_{n+1}/K_n)$, $\forall x \in \mathcal{O}_{K_{n+1}}$,

$$g(x) \equiv x \pmod{p^\varepsilon}.$$

Ex In case $K_n = K(\mu_{p^n})$,

$$\mathrm{Gal}(K_{n+1}/K_n) \simeq {}^{1+p^n\mathbb{Z}_p}/_{1+p^{n+1}\mathbb{Z}_p}$$

$$\rightsquigarrow \quad g(\zeta_{p^{n+1}}) - \zeta_{p^{n+1}} = \zeta_{p^{n+1}}(\zeta_p^a - 1) \equiv O(p^{\frac{1}{p-1}}) \leftarrow 1+p^n a.$$

Cor $\exists\, 0 < \varepsilon < 1$ s.t. $\forall n \gg 0$, $\forall x \in \mathcal{O}_{K_{n+1}}$,

$$N_{K_{n+1}/K_n}(x) \equiv x^p \pmod{p^\varepsilon}.$$

Now define some rings of char p.

$$\text{perfect} \to \tilde{E}_K^+ := \varprojlim_{\mathrm{Frob}_p} \mathcal{O}_{K_\infty}/p^\varepsilon = \{(x_0, x_1, \cdots) : x_i^p = x_{i-1}\}$$

$$\text{norm ``field''} \quad \cup$$

$$E_K^+ := \{x \in \tilde{E}_K^+ : \forall n \gg 0,\ x_n \in \mathcal{O}_{K_n}/p^\varepsilon\}.$$

Let k_n = res field of K_n,

$$k_\infty = \bigcup_n k_n = k_N \quad \text{for } N \gg 0.$$

Then $k_n \hookrightarrow \mathcal{O}_{K_n}/p^\varepsilon$ via Teichmüller lift.

Get $k_\infty \hookrightarrow E_K^+$.

e.g. Cyclotomic case $K_n = \mathbb{Q}_p(\mu_{p^n})$:

E_K^+ contains $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \cdots) \in E_K^+$

$x^p \quad x^p$

$$T = \varepsilon - 1 = (0, \zeta_p - 1, \zeta_{p^2} - 1, \cdots)$$

uniformizers.

$\rightsquigarrow$ T is a top nilp unit.

Easy to check: $\mathbb{F}_p[\![T]\!] \xrightarrow{\sim} E_K^+$.

<u>Prop</u> If $n \gg 0$, $\pi_n \in \mathcal{O}_{K_n}$ is a uniformizer,

then $\exists\, \pi_{n+1} \in \mathcal{O}_{K_{n+1}}$ s.t. $\pi_{n+1}^p \equiv \pi_n \mod p^{\varepsilon}$.

pf Let $\varpi_{n+1} \in \mathcal{O}_{K_{n+1}}$ be a unif.

Then $\underbrace{N(\varpi_{n+1})}_{\text{unif of } K_n} \equiv \sum_{i=1}^{\infty} [a_i] \varpi_n$, $a_i \in k_n$.

Want to find: $\pi_{n+1} = \sum_i [b_i] \varpi_{n+1}^i$

so that $\pi_{n+1}^p \equiv \pi_n \pmod{p^{\varepsilon}}$.

$$\Rightarrow \quad \sum [b_i^p] \underset{\substack{\| \\ N(\varpi_{n+1}^i)}}{\varpi_{n+1}^{pi}} \equiv \pi_n \pmod{p^{\varepsilon}}$$

Solve for b_i's. $\square$

Let $\bar{\pi} = (\cdots, \pi_n, \pi_{n+1}, \cdots) \in E_K^+$

unifs.

<u>Thm</u> (a) $E_K^+ \xrightarrow{\sim} k_\infty[\![x]\!]$ (let $E_K = k_\infty(\!(x)\!)$).

$\bar{\pi} \longleftarrow\!\shortmid\; x$

(b) $\tilde{E}_K^+ \xrightarrow{\sim} k_\infty[\![x^{1/p^\infty}]\!]$.

Have a map $\theta : W(\tilde{E}_K^+) \longrightarrow \mathcal{O}_{\hat{K}_\infty}$

$[x] \longmapsto \lim_{n\to\infty} \tilde{x}_n^{p^n}$

where $x = (x_0, x_1, \cdots)$, $x_n \in \mathcal{O}_{k_\infty}/p^{\varepsilon}$.

$$\tilde{x}_0, \tilde{x}_1, \cdots \in \mathcal{O}_{K_\infty}.$$

& $\ker \theta = (\omega)$.

e.g. In cycl case: $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \cdots) \in \tilde{E}_K^+ \subseteq \tilde{E}_K$

$$\theta([\varepsilon]) = 1, \quad \theta([\varepsilon^{1/p}]) = \zeta_p \quad \rightsquigarrow \quad \omega = \frac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1}.$$

If M/K_∞ a fin ext'n,

$$M = \bigcup_n M_n, \quad M_n = M_0 K_n, \quad n \gg 0.$$

then E_M / E_K fin sep ext'n.

$$E_K \cong k_\infty((x))$$

<u>Thm</u> (Fontaine-Wintenberger) $\exists$ equiv

$$\begin{tikzcd}
\left\{ \begin{array}{c} \text{fin ext'n} \\ \text{of } K_\infty \end{array} \right\} \arrow[r, "\sim", "M \mapsto E_M"'] \arrow[d] & \left\{ \begin{array}{c} \text{fin sep ext'ns} \\ \text{of } E_K \end{array} \right\} \arrow[d, "\wr"] \\
\left\{ \begin{array}{c} \text{fin ext'ns} \\ \text{of } \hat{K}_\infty \end{array} \right\} \arrow[r, "\sim"] & \left\{ \begin{array}{c} \text{fin sep ext'ns} \\ \text{of } \tilde{E}_K \end{array} \right\} \\
W(\mathcal{O}_L) \otimes_{W(\tilde{E}_K^+), \theta} \hat{K}_\infty & L \arrow[l, mapsto]
\end{tikzcd}$$

$$W(\mathcal{O}_L) \otimes_{W(\tilde{E}_K^+), \theta} \hat{K}_\infty = W(\mathcal{O}_L)[\tfrac{1}{p}] / \omega.$$

<u>Cor</u> $G_{K_\infty} \simeq G_{E_K}$.

<u>Rmk</u> Facts on $E_K \simeq k_\infty((x))$:

In cycl case $\Gamma \simeq \mathbb{Z}_p^\times$

$\gamma_a \longmapsfrom a$

$\gamma_a(x) = (1+x)^a - 1.$

$$\begin{array}{c} \overline{K} \\ | \\ K_\infty \\ \Gamma \left(\; | \right. \\ K \end{array}$$