**Exercise 5** (due on December 23)

**Choose 4 out of 8 problems to submit.** (The problems are chronically ordered by the materials.) Let $\ell \geq 3$ be a prime number.

**Problem 5.1.** (Fontaine–Laffaille modules are weakly admissible) Let $M$ be a $\mathbb{Z}_\ell$-free Fontaine–Laffaille module, that is a tuple $(M, \mathrm{Fil}^\bullet M, (\Phi^i)_i)$, consisting of

- a finite free $\mathbb{Z}_\ell$-module $M$,
- equipped with a decreasing filtration $\mathrm{Fil}^\bullet M$ by saturated $\mathbb{Z}_\ell$-submodules $\mathrm{Fil}^i M$ (i.e. $M/\mathrm{Fil}^i M$ is still a free $\mathbb{Z}_\ell$-module) such that $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{\ell-1} M = 0$; and
- $\Phi^i : \mathrm{Fil}^i M \to M$ are $\mathbb{Z}_\ell$-linear maps such that $\Phi^i|_{\mathrm{Fil}^{i+1} M} = \Phi^{i+1}$, and $M = \sum_i \Phi^i(\mathrm{Fil}^i M)$.

Prove the following statements:

(1) Choose a basis of $M$ as $\{e_{0,1}, \dots, e_{0,r_0}, e_{1,1}, \dots, e_{1,r_1}, e_{2,1}, \dots, e_{\ell-2,r_{\ell-2}}\}$ so that

$$\begin{aligned}
\mathrm{Fil}^{\ell-2} M &= \mathbb{Z}_\ell e_{\ell-2,1} \oplus \cdots \oplus \mathbb{Z}_\ell e_{\ell-2,r_{\ell-2}};\\
\mathrm{Fil}^{\ell-3} M &= \mathrm{Fil}^{\ell-2} M \oplus \mathbb{Z}_\ell e_{\ell-3,1} \oplus \cdots \oplus \mathbb{Z}_\ell e_{\ell-3,r_{\ell-3}};\\
\cdots &\quad \cdots\\
\mathrm{Fil}^0 M &= \mathrm{Fil}^1 M \oplus \mathbb{Z}_\ell e_{0,1} \oplus \cdots \oplus \mathbb{Z}_\ell e_{0,r_0} = M.
\end{aligned}$$

Show that with respect to this basis, the matrix $F$ for $\Phi^0$ takes the form of

$$F = B \cdot \mathrm{Diag}\big(\underbrace{1, \dots, 1}_{r_0}, \underbrace{\ell, \dots, \ell}_{r_1}, \underbrace{\ell^2, \dots, \ell^2}_{r_2}, \dots, \underbrace{\ell^{\ell-2}, \dots, \ell^{\ell-2}}_{r_{\ell-2}}\big)$$

for a matrix $B \in \mathrm{GL}(\mathbb{Z}_\ell)$. (Note that it is not difficult to show that $B \in \mathrm{M}(\mathbb{Z}_p)$. To see that $B$ is invertible, one needs to use the condition $M = \sum_i \Phi^i(\mathrm{Fil}^i M)$.)

(2) Deduce from (1) that viewing $M \otimes \mathbb{Q}_\ell$ as a filtered $\phi$-module (with $\phi = \Phi^0$), the Newton degree of $M \otimes \mathbb{Q}_\ell$ is the same as the Hodge degree.

(3) Let $N$ be a saturated submodule of $M$ that is stable under the $\Phi^i$-actions. Then the filtration of $M$ induces a filtration on $N$. Show that $N$ admits similar basis as in $M$ so that $\Phi^0$ takes a form similar to $F$ above except that we cannot ensure that the corresponding $B$ is an invertible integral basis.

Using this, prove that $M \otimes \mathbb{Q}_\ell$ is a weakly admissible filtered $\varphi$-module.

*Solution.* (1) Write $B_i$ for the matrix for which

$$\Phi^i(e_{i,1}, \dots, e_{i,r_i}) = (e_{0,1}, \dots, e_{\ell-2,r_{\ell-2}}) \cdot B_i.$$

Then we have

$$\Phi^0(e_{i,1}, \dots, e_{i,r_i}) = (e_{0,1}, \dots, e_{\ell-2,r_{\ell-2}}) \cdot \ell^i B_i.$$

Putting these together, we have

$$\begin{aligned}
&\Phi^0(e_{0,1}, \dots, e_{\ell-2,r_{\ell-2}}) = (e_{0,1}, \dots, e_{\ell-2,r_{\ell-2}})\big(B_0, \ell B_1, \dots, \ell^{\ell-2} B_{\ell-2}\big)\\
&= (e_{0,1}, \dots, e_{\ell-2,r_{\ell-2}})\big(B_0, B_1, \dots, B_{\ell-2}\big) \cdot \mathrm{Diag}\big(\underbrace{1, \dots, 1}_{r_0}, \underbrace{\ell, \dots, \ell}_{r_1}, \underbrace{\ell^2, \dots, \ell^2}_{r_2}, \dots, \underbrace{\ell^{\ell-2}, \dots, \ell^{\ell-2}}_{r_{\ell-2}}\big)
\end{aligned}$$

Write $B = (B_0, \dots, B_{\ell-2})$. Since $\Phi^i|_{\mathrm{Fil}^{i+1} M} = \ell \Phi^{i+1}$, we have for $i < j$, $\Phi^i(\mathrm{Fil}^j M) \subseteq \Phi^j(\mathrm{Fil}^j M)$. Thus

$$M = \sum \Phi^i(\mathrm{Fil}^i M) = \sum \Phi^i(\langle e_{i,1}, \dots, e_{i,r_i} \rangle) = \langle (e_{1,1}, \dots, e_{\ell-2,r_{\ell-2}}) B \rangle$$

It follows from this that $B \in \mathrm{GL}(\mathbb{Z}_\ell)$.

(2) $t_H(M\otimes\mathbb{Q}_\ell)=\sum ir_i$ and

$$t_N(M\otimes\mathbb{Q}_\ell)=v_\ell(\det B\cdot \mathrm{Diag}(1,\dots,\ell^{\ell-2})=v_\ell(\ell^{\sum ir_i})=\sum ir_i.$$

So $t_H(M\otimes\mathbb{Q}_\ell)=t_N(M\otimes\mathbb{Q}_\ell)$.

(3) Since $N/\mathrm{Fil}^iN=N/(\mathrm{Fil}^iM\cap N)$ is a submodule of $M/\mathrm{Fil}^iM$, $N/\mathrm{Fil}^iN$ is a free $\mathbb{Z}_\ell$-module. Hence we may provide $N$ with a basis $\{e'_{0,1},\dots,e'_{0,r'_0},\dots,e'_{\ell-2,r'_{\ell-2}}\}$ with the similar property as in (1), except that the corresponding $B'$ may not be invertible. Thus

$$t_N(N\otimes\mathbb{Q}_\ell)=\sum_i ir'_i+v_\ell(\det B')\geq\sum ir'_i=t_H(N\otimes\mathbb{Q}_\ell).$$

So $M$ is weakly admissible. □

**Problem 5.2.** (Deformation of Fontaine–Laffaille modules) Let $\overline{M}$ denote the following Fontaine–Laffaille module: $\overline{M}=\mathbb{F}_\ell e_1\oplus\mathbb{F}_\ell e_2$, with $\mathrm{Fil}^0\overline{M}=\overline{M}\supset\mathrm{Fil}^1\overline{M}=\mathbb{F}_\ell e_1\supset\mathrm{Fil}^2\overline{M}=0$; $\Phi^1(e_1)=e_2$ and $\Phi^0(e_2)=e_1$.

Consider the following deformation functor:

$$\begin{aligned}\mathrm{Def}_{\overline{M}}:\mathsf{CNL}_{\mathbb{Z}_\ell}&\longrightarrow \mathsf{Sets}\\ A&\longmapsto\big\{\text{Fontaine–Laffaille module } M \text{ such that } M\otimes_A\mathbb{F}_\ell\cong\overline{M}\big\}/\sim\end{aligned}$$

(Note that this an unframed deformation.) Show that $\mathrm{Def}_{\overline{M}}$ is represented by a two-variable formal power series over $\mathbb{Z}_\ell$. (Had we fixed the determinant, it would be only one-variable.)

*Solution.* For any $M\in\mathrm{Def}_{\overline{M}}(A)$, we may assume that $M=Ae_1\oplus Ae_2$, $\mathrm{Fil}^1M=Ae_1$, and $\Phi^1(e_1)=\alpha e_1+\beta e_2$. Then $\beta\equiv 1 \bmod \varpi$. In particular, $\beta$ is a unit in $A$.

We may replace $e_2$ by $\alpha e_1+\beta e_2$ so that $\Phi^1(e_1)=e_2$. Now, let $\Phi^0(e_2)=\gamma e_1+\delta e_2$. By compatibility, we must have $\gamma\equiv 1 \bmod \varpi$ and $\delta\equiv 0\bmod\varpi$. The corresponding Fontaine–Laffaille module is uniquely determined by $\gamma,\delta\in A$. So the deformation ring is represented by $\mathbb{Z}_\ell[\![X,Y]\!]$. □

**Problem 5.3.** (Some computation from Weil's conjecture) This is partially copied from Hartshorne GTM 52's Appendix C. Let $X$ denote a projective smooth variety of dimension $n$ over a finite field $\mathbb{F}_q$. Let $|X|$ denote the set of closed points of $X$; and for $x\in|X|$, write $\deg(x)=[k_x:\mathbb{F}_q]$ for the degree at $x$. Define the zeta function of $X$ to be

$$\zeta_X(t):=\prod_{x\in|X|}\frac{1}{1-t^{\deg(x)}}\in\mathbb{Q}[\![t]\!].$$

Here $t$ is a proxy of $q^{-s}$ in the usual Riemann zeta function; but in function field, luckily, all residue fields at closed points are extensions of the *same* finite field $\mathbb{F}_q$.

(1) Write $N_r:=\#X(\mathbb{F}_{q^r})$. Show that $\zeta_X(t)=\exp\Big(\sum_{r=1}^{\infty}N_r\dfrac{t^r}{r}\Big)$.
(2) Compute $\zeta_X(t)$ for $X=\mathbb{P}^n$ over $\mathbb{F}_q$.
(3) Consider the geometric $q$-Frobenius $\phi_q$ on each of the cohomology group $H^i_{\mathrm{et}}(X_{\overline{\mathbb{F}}_q},\mathbb{Q}_\ell)$. The Lefschetz trace formula says that

$$\#X(\mathbb{F}_{q^r})=\sum_{i=0}^{2n}(-1)^i\mathrm{Tr}\big(\phi_q^r;\ H^i_{\mathrm{et}}(X_{\overline{\mathbb{F}}_q},\mathbb{Q}_\ell)\big)$$

Show that plugging Lefschetz trace formula into the definition of $\zeta_X(t)$ gives

$$\zeta_X(t) = \frac{\prod_{i \text{ odd}} \det\left(1-\phi_q t;\ H^i_{\text{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)\right)}{\prod_{i \text{ even}} \det\left(1-\phi_q t;\ H^i_{\text{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)\right)}.$$

(In particular, $\zeta_X(t) \in \mathbb{Q}(t)$.

(4) (Optional) The Poincaré duality for étale cohomology says that there is a $\phi_q$-equivariant isomorphism

$$H^i_{\text{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \cong H^{2n-i}_{\text{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)^*(-\dim X).$$

Show that this implies the following functional equation:

$$\zeta_X\left(\frac{1}{q^n t}\right) = \pm q^{nE/2} t^E \zeta_X(t),$$

where $E$ is the Euler characteristic of $X$, namely, $E = \sum_i (-1)^i \dim H^i_{\text{et}}\left(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell\right)$.

*Solution.* (1) By taking formal integrals and formal derivatives, $\zeta_X(t) = \exp\left(\sum_{r=1}^{\infty} N_r \frac{t^r}{r}\right)$ is equivalent to

$$t \cdot \frac{d}{dt} \log \zeta_X(t) = \sum_{r=1}^{\infty} N_r t^r.$$

Note that $N_r = \#X(\mathbb{F}_{q^r}) = \sum_{\substack{x \in |X| \\ \deg(x) | r}} \deg(x)$.

$$\begin{aligned}
t \cdot \frac{d}{dt} \log \zeta_X(t) &= t \cdot \sum_{x \in |X|} \frac{d}{dt} \log\left(\left(1-t^{\deg(x)}\right)^{-1}\right) \\
&= \sum_{x \in |X|} \deg(x) \cdot \frac{t^{\deg(x)}}{1-t^{\deg(x)}} = \sum_{x \in |X|} \deg(x) \cdot \sum_{k \geq 1} t^{k \cdot \deg(x)} \\
&= \sum_{x \in |X|} \sum_{r \geq 1} \sum_{\deg(x) | r} \deg(x) t^r = \sum_{r \geq 1} N_r t^r.
\end{aligned}$$

(1) is proved.

(2) For $X = \mathbb{P}^n$, $\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \cdots \sqcup \mathbb{A}^1 \sqcup \{\text{pt}\}$. Then

$$N_r = \#\mathbb{P}^n(\mathbb{F}_{q^r}) = 1 + q^r + q^{2r} + \cdots + q^{nr}.$$

So we have

$$\log \zeta_{\mathbb{P}^n}(t) = \sum_{r \geq 1} \left(1 + q^r + q^{2r} + \cdots + q^{nr}\right) \cdot \frac{t^r}{r} = \log \frac{1}{1-t} + \log \frac{1}{1-qt} + \cdots + \log \frac{1}{1-q^n t}.$$

Thus

$$\zeta_{\mathbb{P}^n}(t) = \frac{1}{(1-t)(1-qt)\cdots(1-q^n t)}.$$

(3) By Lefschetz trace formula,

$$\log \zeta_X(t) = \sum_{i=0}^{2n} \sum_{r \geq 1} (-1)^i \text{Tr}\left(\phi_q^r; H^i_{\text{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)\right)$$

Suppose that the eigenvalues of $\phi_q$ acting on $H^i_{\mathrm{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ are $\alpha_{i,1}, \dots, \alpha_{i,d_i} \in \overline{\mathbb{Q}}_\ell$ (which in fact belong to $\overline{\mathbb{Q}}$), then

$$\sum_{r\geq 1} \mathrm{Tr}\Big(\phi_q^r; H^i_{\mathrm{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)\Big)\frac{t^r}{r} = \sum_{r\geq 1}\big(\alpha_{i,1}^r + \cdots + \alpha_{i,d_i}^r\big)\frac{t^r}{r}$$
$$= \log\frac{1}{1-\alpha_{i,1}t} + \cdots \frac{1}{1-\alpha_{i,d_i}t} = \log\frac{1}{\det\big(1-\phi_q t;\ H^i_{\mathrm{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)\big)}.$$

Combining this with the expression of $\log \zeta_X(t)$, we deduce immediately that

$$\zeta_X(t) = \frac{\prod_{i\text{ odd}} \det\Big(1-\phi_q t; H^i_{\mathrm{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)\Big)}{\prod_{i\text{ even}} \det\Big(1-\phi_q t; H^i_{\mathrm{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)\Big)}.$$

In particular, $\zeta_X(t)$ is the power series expansion of a rational function of $t$.

(4) By Poincaré duality, $H^i_{\mathrm{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \cong H^{2n-i}_{\mathrm{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)(-\dim X)$. So if $\alpha_{i,1}, \dots, \alpha_{i,d_i}$ denotes the $\varphi_q$-eigenvalues on $H^i_{\mathrm{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$, then $q^n/\alpha_{i,1}, \dots, q^n/\alpha_{i,d_i}$ are the $\varphi_q$-eigenvalues on $H^{2n-i}_{\mathrm{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$. This implies that

$$\begin{aligned}
\zeta_X(t)^2 &= \prod_{i=1}^{2n}\Big(\det\big(1-\phi_q t;\ H^i_{\mathrm{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)\big)\cdot \det\big(1-\phi_q t;\ H^{2n-i}_{\mathrm{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)\big)\Big)^{(-1)^{i+1}}\\
&= \prod_{i=0}^{2n}\Big(\prod_{j=1}^{d_i}(1-\alpha_{i,j}t)\Big(1-\frac{q^n}{\alpha_{i,j}}t\Big)\Big)^{(-1)^{i+1}}\\
&= \prod_{i=0}^{2n}\Big(\prod_{j=1}^{d_i}\Big(\frac{1}{\alpha_{i,j}t}-1\Big)\alpha_{i,j}t\cdot\Big(\frac{\alpha_{i,j}}{q^n t}-1\Big)\frac{q^n}{\alpha_{i,j}}t\Big)^{(-1)^{i+1}}\\
&= \prod_{i=0}^{2n}\Big(\prod_{j=1}^{d_i}\Big(1-\frac{q^n}{\alpha_{i,j}}\frac{1}{q^n t}\Big)\Big(1-\alpha_{i,j}\frac{1}{q^n t}\Big)\cdot q^n t^2\Big)^{(-1)^{i+1}}\\
&= q^{nE}t^{2E}\zeta_X(1/q^n t)^2.
\end{aligned}$$

Statement (4) follows. □

**Problem 5.4.** (Traces of differential forms) Consider the simplest case $h : \mathbb{C} \to \mathbb{C}$ with $h(z) = w^n$. Show that the trace map of differential forms is well-defined.

*Solution.* The only ramification point of $h$ is at 0; so the trace map is well defined for the restriction of $h$ to $\mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$. It suffices to show that $\mathrm{Tr}_h$ extends holomophically from $\mathbb{C}\backslash\{0\}$ to $\mathbb{C}$.

Let $D$ denote a small disk around $0 \in \mathbb{C}$. Let $\omega = f(z)dz \in H^0(D, \Omega^1)$ be a holomorphic differential form. Take any small enough disc $U \subset D\backslash\{0\}$ so that its preimage under $h$ splits as $h^{-1}(U) = U_1 \sqcup \cdots \sqcup U_n$. The inverse map $U \to U_i$ is given by $z \mapsto \zeta_n^i z^{1/n}$ (for a fixed choice of $n$th root of $z$ on $U$). Then by our definition,

$$\mathrm{Tr}_h(\omega)|_U = \sum_{i=1}^n (h|_{U_i}^{-1})^*(\omega_{U_i}) = \sum_{i=1}^n f(\zeta_n^i z^{\frac{1}{n}})d(\zeta_n^i z^{\frac{1}{n}}) = \frac{1}{n}\sum_{i=1}^n f(\zeta_n^i z^{\frac{1}{n}})\zeta_n^i z^{\frac{1-n}{n}}dz.$$

So if $f(z)$ has Taylor expansion $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ near $z = 0$, using the equality $\sum_{i=1}^n \zeta_n^{ir} = \begin{cases} n & \text{if } n|r \\ 0 & \text{if } n\nmid r \end{cases}$, we conclude that

$$\mathrm{Tr}_h(\omega)|_U = \sum_{m=0}^{\infty} a_{(m+1)n-1} z^m dz.$$

It can be extended to 0. $\square$

**Problem 5.5.** (Compatibility of Abel–Jacobi map with respect to functoriality maps) For $X$ a projective smooth curve over $\mathbb{C}$, let $\mathrm{Jac}_X := H^0(X, \Omega_X^1)^\vee / H_1(X^{\mathrm{an}}, \mathbb{Z})$ denote the associated Jacobian, and let $\mathrm{Pic}^0(X)$ denote the Picard group of $X$, parametrizing line bundles of degree 0. Let $h : X \to Y$ be a finite (flat) morphism of projective smooth curves over $\mathbb{C}$. Show that the following diagrams of functorial maps commute:

$$\begin{tikzcd} \mathrm{Pic}^0(X) \arrow[r, "\mathcal{O}_X(D)\mapsto\mathcal{O}_Y(h(D))"] \arrow[d, "\mathrm{AJ}"', "\cong"] & \mathrm{Pic}^0(Y) \arrow[d, "\mathrm{AJ}"', "\cong"] \\ \mathrm{Jac}_X \arrow[r, "(h^*)^\vee"] & \mathrm{Jac}_Y \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Pic}^0(Y) \arrow[r, "\mathcal{L}\mapsto h^*\mathcal{L}"] \arrow[d, "\mathrm{AJ}"', "\cong"] & \mathrm{Pic}^0(X) \arrow[d, "\mathrm{AJ}"', "\cong"] \\ \mathrm{Jac}_Y \arrow[r, "(\mathrm{Tr}_h)^\vee"] & \mathrm{Jac}_X. \end{tikzcd}$$

*Solution.* Recall that the Abel–Jacobi map is given by

$$\begin{aligned} \mathrm{Pic}^0(X) &\longrightarrow \mathrm{Jac}_X \\ \sum n_x[x] &\longmapsto \sum n_x \int_{x_0}^{x} \end{aligned}$$

(This is independent of the choice of $x_0$ because $\sum n_x = 0$.)

For the left diagram, it is the following obvious commutative diagram

$$\begin{tikzcd} \sum n_x[x] \arrow[r, mapsto] \arrow[d, mapsto] & \sum n_x[h(x)] \arrow[d, mapsto] \\ \sum n_x \int_{x_0}^{x} \arrow[r, mapsto] & \sum n_x \int_{h(x_0)}^{h(x)}. \end{tikzcd}$$

For the right diagram, we start with $D = \sum n_y[y] \in \mathrm{Pic}^0(Y)$; in fact, as $h$ is only ramified at finitely points; we may modify $D$ by a principal divisor so that for all $n_y \neq 0$, $h : X \to Y$ is étale at $y$. Now, we want to understand

$$\begin{tikzcd} \sum n_y[y] \arrow[r, mapsto] \arrow[d, mapsto] & \sum n_y \sum_{h(x)=y}[x] \arrow[d, mapsto] \\ \sum n_y \int_{y_0}^{y} \arrow[r, mapsto] & ?? \end{tikzcd}$$

Let $d$ denote $\deg h$. We may choose $y_0$ so that $h : X \to Y$ is étale at $y_0$ and let $f^{-1}(y_0) = x_0^{(1)}, \dots, x_0^{(d)}$.

For each $y$ such that $n_y \neq 0$, we may choose a path $\gamma_y : y_0 \rightsquigarrow y$ which only runs through unramified locus of $Y$; then $\gamma_y$ lifts precisely to $d$ paths connecting $x_0^{(1)} \rightsquigarrow x_y^{(1)}$, ..., $x_0^{(d)} \rightsquigarrow$

$x_y^{(d)}$, where $f^{-1}(y) = \{x_y^{(1)}, \ldots, x_y^{(d)}\}$. Then

$$\int_{y_0}^{y} \mathrm{Tr}_h(\omega) = \sum_{i=1}^{d} \int_{x_0^{(i)}}^{x_y^{(i)}} \omega$$

By definition, we have

$$(Tr_h)^\vee \circ \mathrm{AJ}\big(\sum n_y[y]\big) = \sum n_y \sum_{i=1}^{d} \int_{x_0^{(i)}}^{x_y^{(i)}} = \sum n_y \sum_{x \in f^{-1}(y)} \int_{x_0}^{x} = \mathrm{AJ} \circ h^*\big(\sum n_y[y]\big).$$

The second equality is zero because $\sum n_y \cdot \int_{x_0}^{x_0^{(i)}} = 0$ (as $\sum n_y = 0$). $\square$

**Problem 5.6.** ($L$-function for modular forms and its functional equations) Let $f = \sum_{n \geq 1} a_n q^n$ denote a cuspidal modular forms of weight $k$, level $\Gamma_0(N)$. Suppose that $f$ is normalized so that $a_1 = 1$, and that $f$ is an eigenform for all Hecke operators.

(1) Prove that

$$\int_0^\infty f(iy) y^s \frac{dy}{y} = (2\pi)^{-s}\Gamma(s) L(f, s)$$

where $L(f, s) = \sum_{n \geq 1} a_n / n^s$ is the $L$-function associated to $f$.

(2) Suppose for simplicity that $f$ has level $\mathrm{SL}_2(\mathbb{Z})$. Let $L_\infty(f, s) := (2\pi)^{-s}\Gamma(s)$ denote the "$L$-factor at infinity", and write $\Lambda(f, s) := L(f, s) L_\infty(f, s)$ for the "complete $L$-function". Show that

$$\Lambda(f, s) = (-1)^{k/2} \Lambda(f, k - s).$$

(Hint: breaks up the integral above as $\int_0^1$ and $\int_1^\infty$, and use the relation $f\big(\frac{az+b}{cz+d}\big) = (cz + d)^k f(z)$ for $\big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big) = \big(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\big)$.)

*Solution.* (1) When $f = \sum_{n \leq 1} a_n q^n$ is a cuspidal form, the following integral is absolutely convergent when $\mathrm{Re}\, s \gg 0$:

$$\int_0^\infty f(iy) y^s \frac{dy}{y} = \int_0^\infty \sum_{n \geq 1} a_n e^{-2\pi n y} y^s \frac{dy}{y} = \sum_{n \geq 1} a_n \int_0^\infty e^{-2\pi n y} y^s \frac{dy}{y}$$
$$\overset{w = 2\pi n y}{=} \sum_{n \geq 1} \frac{a_n}{(2\pi n)^s} \int_0^\infty e^{-w} w^s \frac{dw}{w} = (2\pi)^{-s}\Gamma(s) \sum_{n \geq 1} \frac{a_n}{n^s} = (2\pi)^{-s}\Gamma(s) L(f, s).$$

(2) The relation $f(\frac{az+b}{cz+d}) = (cz + d)^k f(z)$ when $\big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big) = \big(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\big)$ says that

$$f\left(-\frac{1}{z}\right) = (-z)^k f(z).$$

When $z = iy$, $-\frac{1}{z} = i/y$. So we have when $\operatorname{Re} s \gg 0$,

$$\begin{aligned}
\Lambda(f,s) &= \int_0^\infty f(iy)y^s \frac{dy}{y} = \int_0^1 f(iy)y^s\frac{dy}{y} + \int_1^\infty f(iy)y^s\frac{dy}{y} \\
&= (-i)^k \int_0^1 f\Big(\frac{i}{y}\Big)y^{s-k}\frac{dy}{y} + \int_1^\infty f(iy)y^s\frac{dy}{y} \\
&\overset{w=1/y}{=} (-1)^{k/2}\int_1^\infty f(iw)w^{k-s}\frac{dw}{w} + \int_1^\infty f(iy)y^s\frac{dy}{y} \\
&= \int_1^\infty f(it)\big((-1)^{k/2}t^{k-s} + t^{s-1}\big)\frac{dt}{t}
\end{aligned}$$

Note that the end expression is absolutely convergent for *all* $s \in \mathbb{C}$ (giving the complex analytic continuation of $\Lambda(f,s)$). Moreover, changing $s$ into $k-s$ will only change the expression by a sign $(-1)^k$. So we deduce that

$$\Lambda(f,s) = (-1)^{k/2}\Lambda(f,k-s).$$

□

**Problem 5.7.** (Geometric realization of Hecke operator $T_p$) Recall that $Y_0(N)$ parameterizes elliptic curves together with a cyclic subgroup $C_N$ or order $N$. Write $\mathcal{E}$ for the universal elliptic curve, and $\pi : \mathcal{E} \to Y_0(N)$ for the structure map and $e : Y_0(N) \to \mathcal{E}$ the zero section. Set $\omega_{E/Y_0(N)} := e^*\Omega^1_{E/Y_0(N)}$.

To a modular form $f \in S_k(\Gamma_0(N))$ and a prime $p \nmid N$, one can associate a section

$$f(z) \otimes d\tau^{\otimes k} \in H^0(Y_0(N), \omega).$$

We define the Hecke operator $T_p$ on $S_k(\Gamma_0(N))$ using the following diagram:

$$\begin{tikzcd}
& Y_0(pN) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
Y_0(N) & & Y_0(N)
\end{tikzcd}
\qquad
\begin{tikzcd}
& (E, C_N, C_p) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
(E, C_N) & & (E/C_p, C_N + C_p/C_p).
\end{tikzcd}$$

Here, we may alternatively view $Y_0(pN)$ as the moduli space of isogenies of elliptic curves $E \to E/C_p$ of degree $p$ together with a cyclic subgroup $C_N$ of $E$ of order $N$. Let $\varphi : \pi_1^*\mathcal{E} \to \pi_2^*\mathcal{E}$ denote the universal isogeny on $Y_0(pN)$. Define

$$T_p : H^0(Y_0(N), \omega_2^{\otimes k}) \xrightarrow{\pi_2^*} H^0(Y_0(Np), \omega_2^{\otimes k}) \xrightarrow{\varphi^*} H^0(Y_0(Np), \omega_1^{\otimes k}) \xrightarrow{\frac{1}{p}\mathrm{Tr}} H^0(Y_0(N), \omega_1^{\otimes k})$$

Here we write $\omega_1$ and $\omega_2$ to distinguish sheaves on the left and right $Y_0(N)$. Show that this agrees with the usual definition of $T_p$.

*Solution.* We want to express $T_p(f)(z) \otimes d\tau^{\otimes k}$ (on the $\pi_1$-side) in terms of the form $f(z) \otimes d\tau^{\oplus k}$ (on the $\pi_2$-side). Under the map $\pi_1$, the preimage of the elliptic curve $(\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}z), \langle \frac{1}{N}\rangle)$ is given by

(1) $\big(\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}z), \langle \frac{1}{N}\rangle, \langle \frac{z+i}{p}\rangle\big)$ for $i = 0, \dots, p-1$,
(2) $\big(\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}z), \langle \frac{1}{N}\rangle, \langle \frac{1}{p}\rangle\big)$.

In case (1), the image of $\big(\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}z), \langle\frac{1}{N}\rangle, \langle\frac{z+i}{p}\rangle\big)$ under the map $\pi_2$ is given by quotienting out this subgroup of order $p$, namely we get $\big(\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\frac{z+i}{p}), \langle\frac{1}{N}\rangle\big)$. So the contribution to $T_p(f)(z)\otimes d\tau^{\oplus k}$ is given by $\frac{1}{p}f(\frac{z+i}{p})\otimes d\tau^{\otimes k}$. If $f$ has $q$-expansion $\sum_{n\geq 1}a_nq^n$, summing over $i$, we get

$$\frac{1}{p}\sum_{i=0}^{p-1}f\big(\frac{z+i}{p}\big)\otimes d\tau^{\otimes k}=\frac{1}{p}\sum_{i=0}^{p-1}\sum_{n\geq 1}a_n\zeta_p^{ni}q^n\otimes d\tau^{\otimes k}=\sum_{n\geq 1}a_{pn}q^n\otimes d\tau^{\otimes k}.$$

The case (2) is a bit more tricky. the image of $\big(\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}z), \langle\frac{1}{N}\rangle, \langle\frac{1}{p}\rangle\big)$ under the map $\pi_2$ is given by quotienting out this subgroup of order $p$, namely we get $\big(\mathbb{C}/(\frac{1}{p}\mathbb{Z}\oplus\mathbb{Z}z), \langle\frac{1}{N}\rangle\big)$. But this is not of the standard form; so we need to make an identification:

$$\begin{array}{ccc}\mathbb{C}/(\frac{1}{p}\mathbb{Z}\oplus\mathbb{Z}z) & \xrightarrow{\text{mult.}p} & \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}pz)\\ \tau & \longmapsto & p\tau\end{array}$$

Pulling back the form $f(pz)d\tau^{\otimes k}$ to the elliptic curve, we get $p^kf(pz)\otimes d\tau^{\oplus k}$. So the contribution to $T_p(f)(z)\otimes d\tau^{\oplus k}$ is given by $\frac{1}{p}\cdot p^kf(pz)\otimes d\tau^{\otimes k}$.

Summing up the contribution from case (1) and case (2), we obtain the usual expression for $T_p$, namely

$$T_p(f)(z)\otimes d\tau^{\otimes k}=\sum_{n\geq 1}a_{pn}q^n\otimes d\tau^{\otimes k}+p^{k-1}\sum_{n\geq 1}a_nq^{pn}\otimes d\tau^{\otimes k}.$$

$\square$

**Problem 5.8.** (Monodromy of moduli space of elliptic curves at a cusp) Fix $N\geq 5$ and consider the quotient $Y_1(N):=\Gamma_1(N)\backslash\mathcal{H}$. This space parametrizes elliptic curves $E$ over $\mathbb{C}$ together with a point $P\in E$ of exact order $N$. Let $\pi:\mathcal{E}\to Y_1(N)$ be the universal elliptic curve; its fiber over $z\in Y_1(N)$ is denoted by $\mathcal{E}_z:=\pi^{-1}(z)$. We study the pushforward of the sheaf $R^1\pi_*\underline{\mathbb{Z}}$, which is locally free sheaf of rank 2 over $Y_1(N)$; such that the stalk at the point $z$ is precisely $H^1(E_z,\mathbb{Z})$. Consider the loop $\gamma:[0,1]\to Y_1(N)$ given by $\gamma(a)=a+10i$. Looping around $\gamma$ defines a homomorphism $H^1(E_{10i},\mathbb{Z})\to H^1(E_{10i},\mathbb{Z})$, called the monodromy operator at $i\infty$. What is this operator in terms of 2-by-2 matrix?

*Solution.* It is more intuitive to consider the homology groups instead, i.e. moving $H_1(E_z,\mathbb{Z})$ as $z$ varies. At each $z\in\mathcal{H}$, the corresponding elliptic curve $\mathcal{E}_z$ is $\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}z$. A natural basis of $H_1(\mathcal{E}_z,\mathbb{Z})$ is given by the loops

$$\begin{aligned}\ell_1:[0,1]\to\mathcal{E}_z;\qquad & \ell_1(x)=x\in\mathcal{E}_z=\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}z);\\ \ell_2:[0,1]\to\mathcal{E}_z;\qquad & \ell_2(x)=x\cdot z\in\mathcal{E}_z=\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}z).\end{aligned}$$

As the point $z$ moves from $10i$ to $1+10i$ on the modular curve, the natural basis $\{\ell_{1@10i},\ell_{2@10i}\}$ continuously moved continuously to $\{\ell_{1@1+10i},\ell_{2@1+10i}\}$. Yet, in the moduli problem of $\Gamma_1(N)\backslash\mathcal{H}$, we will need to identify $\mathcal{E}_{10i}$ with $\mathcal{E}_{1+10i}$ by

$$\begin{array}{ccc}\mathcal{E}_{10i}=\mathbb{C}/(\mathbb{Z}\oplus(1+10i)\mathbb{Z}) & \xrightarrow{\cong} & \mathcal{E}_{1+10i}/(\mathbb{Z}\oplus 10i\mathbb{Z})\\ \tau & \longmapsto & \tau.\end{array}$$

Under this isomorphism, $\ell_{1@1+10i}$ is sent to $\ell_{1@10i}$, yet $\ell_{2@1+10i}$ is sent to the path that is homotopic to $\ell_{1@10i}+\ell_{2@10i}$. Thus, looping around $\gamma$, the monodromy operator acts on

$H_1(E_{10i}, \mathbb{Z})$ by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. So the induced monodromy operator on $H^1(E_z, \mathbb{Z})$ is, by transpose, $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. $\square$