Bessel integrals and Selmer groups over ordinary eigenvariety

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Based on [LTXZZ] : BBK Conj for $GL_n \times GL_{n+1}$

[LTX] : Iwasawa main conj.

[Liu-Sun] : New constr'n of p-adic L-fct.

<u>Setup</u> F imag quad field $/\mathbb{Q}$

$\square$ "bad primes" Containing those ramified in F.

$p \notin \square$, q = p-power.

G ab grp, $H < G$ Subgrp of fin index.

$\Lambda_{G,\mathbb{Z}_q} := \varprojlim_{H<G} \mathbb{Z}_q[G/H]$.

$\Lambda_{G,\mathbb{Q}_q} := \Lambda_{G,\mathbb{Z}_q} \otimes_{\mathbb{Z}_q} \mathbb{Q}_q$.

<u>Ordinary eigenvariety</u>

$N \geq 1$. V_N herm space $/F$ of sgn $\begin{cases} (N,0) & \text{(definite)} \\ (N-1,1) & \text{(indefinite)} \end{cases}$

Split at primes off $\square$.

$G_N := U(V_N)$

Fix $P_N \subset G_{N,\mathbb{Q}_p}$ Borel subgrp.

$\downarrow$

L_N Levi quotient $\rightsquigarrow \begin{cases} (\mathbb{Q}_p^\times)^N & \text{if } p \text{ splits} \\ (\mathbb{F}_p^\times)^{\lfloor N/2 \rfloor} \times (\cdots) & \text{if } p \text{ inert} \end{cases}$

Let $I_N^\infty :=$ max cpt subgrp of $L_N(\mathbb{Q}_p)$.

$K_N^p :=$ level subgrp, hyperspecial off $\square$.

$\mathcal{K}_N$ = set of all open cpt subgrps of $G_N(\mathbb{Q}_p)$

Def (Liu-Sun) $k \to k'$ in $\mathcal{K}_N$ if $k \cap P_N(\mathbb{Q}_p) \subset k' \cap P_N(\mathbb{Q}_p)$

$(k' \subseteq k \cdot (k' \cap P_N(\mathbb{Q}_p)))$.

(e.g. $N=2$, p splits $\rightsquigarrow$ $G_N(\mathbb{Q}_p) = GL_2(\mathbb{Q}_p)$

$GL_2(\mathbb{Z}_p) \underset{a \leq 0}{\overset{a \geq 0}{\rightleftarrows}} \begin{pmatrix} p^a & \\ & 1 \end{pmatrix} GL_2(\mathbb{Z}_p) \begin{pmatrix} p^{-a} & \\ & 1 \end{pmatrix}$ in $\mathcal{K}_N$.)

$\rightsquigarrow$ $k \to H(G_N, k) = \begin{cases} H^0(Sh(G_N, K_N^p k), \mathbb{Z}_q) & , \text{ def} \\ H^{N-1}_{et}(Sh(G_N, K_N^p k)_{\bar{F}}, \mathbb{Z}_q) & , \text{ indef.} \end{cases}$

for honest unitary grp

Given $k \to k'$,

$$\begin{tikzcd} H(G_N, k') \arrow[rr] \arrow[dr, "\text{pullback}"'] & & H(G_N, k) \\ & H(G_N, k' \cap k) \arrow[ur, "\text{pushforward}"'] & \end{tikzcd}$$

(functor $H(G_N, -) : \mathcal{K}_N^{op} \longrightarrow \mathbb{Z}_q[G(\mathbb{A}^{\infty p}) /\!/ K_N^p]$-mod).

Let $I \lhd I_N^\infty$

$\rightsquigarrow$ $\mathcal{K}_{N,I} := \left\{ k \in \mathcal{K}_N \;\middle|\; \begin{array}{l} \text{image of } k \cap P_N(\mathbb{Q}_p) \\ \text{in } L_N(\mathbb{Q}_p) \text{ is } I \end{array} \right\}$

$H(G_N, I) := \varprojlim_{k \in \mathcal{K}_{N,I}^{op}} H(G_N, k)$ "ordinary submod"

Def Space of ordinary distribution

$$D_J(G_N) := \varprojlim_{J \subset I \lhd I_N^\infty} \operatorname{Hom}(H(G_N, I), \mathbb{Z}_p).$$

$J < I_N^\infty$ subgrp
not ness of fin index

Def $\quad \mathfrak{K}^{\dagger} := \left\{ \begin{array}{l} k \in \mathfrak{K} \text{ satisfying } \mathcal{U}_k := \{ p \in P_N(\mathbb{Q}_p) \mid k \to pkp^{-1} \} \\ \text{submonoid of } P_N(\mathbb{Q}_p) \text{ s.t. image of } \mathcal{U}_k \\ \text{in } L_N(\mathbb{Q}_p) \text{ generates the whole grp} \end{array} \right\}$

Fact $\quad \mathfrak{K}_N^{\dagger} \subset \mathfrak{K}_N$ cofinal

$\forall k \in \mathfrak{K}_N^{\dagger}$, e_k "Hida ordinary projector"

Lem $\quad \forall k \in \mathfrak{K}_{N,I}^{\dagger}$, $H(G_N, I) \longrightarrow H(G_N, k)$

induces an isom onto $e_k H(G_N, k)$.

Note $\quad D_J(G_N)$ is a module over $\underbrace{\mathbb{Z}_q[G_N(\mathbb{A}^{\infty p}) /\!/ K_N^p]}_{\shortparallel} \otimes \Lambda_{I_N^{\infty}/J, \mathbb{Z}_q}$

$\mathbb{Z}_q[G_N(\mathbb{Q}_\square) /\!/ K_{N,\square}^p] \otimes \underbrace{\mathbb{T}_N^{\square,p}}_{\text{spherical Hecke alg, commutative}}$

Def $\quad E_J(G_N)$ as the $\Lambda_{I_N^{\infty}/J, \mathbb{Z}_q}$-subring

of $\mathrm{End}_\Lambda(D_J(G_N))$ generated by the action of $\mathbb{T}_N^{\square,p}$.

$\rightsquigarrow \quad \mathcal{D}_J(G_N) := D_J(G_N) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$,

$\mathcal{E}_J(G_N) := E_J(G_N) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

$\rightsquigarrow$ Then Spec $\mathcal{E}_J(G_N)$ ordinary eigenvariety.

Let $\mathfrak{m}_N$ max ideal of $\mathbb{T}_N^{\square,p}$ w/ res field $\subseteq \mathbb{F}_q$

which is "coh gen"

$\iff H^i(Sh(\cdots)_{\bar{F}}, \mathbb{F}_q)_{\mathfrak{m}_N} = 0$ if $i \neq N-1$

(not a strong condition, c.f. C-S).

<u>Prop</u> (1) $\mathcal{D}_J(G_N)_{\mathfrak{m}_N}$ is a fin free mod / $\mathcal{E}_J(G_N)_{\mathfrak{m}_N}$

<u>Aside</u> A closed pt χ of $\operatorname{Spec} \Lambda_{I_N^\infty/J, \mathbb{Q}_q}$ is <u>classical</u> if
$\chi : I_N^\infty/J \longrightarrow \overline{\mathbb{Q}}_q$ s.t. $\chi = \chi_1 \cdot \chi_2$,

- χ_1 = alg char of dominant wt ξ_{χ_1}
- χ_2 = smooth char.

(2) $\forall$ classical pt χ of $\operatorname{Spec} \mathcal{E}_J(G_N)_{\mathfrak{m}_N}$,

$$\mathcal{D}_J(G_N)_{\mathfrak{m}_N}|_\chi \simeq H^0(Sh(G_N, -), \mathcal{L}_{\xi_\chi})[\phi_\chi]^{\mathrm{ord}}$$

with $\phi_\chi : \mathbb{T}_N^{D,p} \to \overline{\mathbb{Q}}_q$.

(3) $\operatorname{Spec} \mathcal{E}_J(G_N)_{\mathfrak{m}_N} \longrightarrow \operatorname{Spec} \Lambda_{I_N^\infty/J, \mathbb{Z}_q}$ is finite dominant.
Its generic fibre is étale at all classical pts.

This is a new constr'n of eigenvar.

<u>Bessel period</u>

Let $G = G_n \times G_{n+1}$, $V = (V_n, V_{n+1})$
$\quad \cup \qquad\qquad\qquad V_{n+1} = V_n \oplus \mathbb{1}$.
$H = \Delta G_n$

Fix $K^p \leq G(\mathbb{A}^{\infty p})$, $P = P_n \times P_{n+1} \subseteq G_{\mathbb{Q}_p}$ Borel
$\qquad\qquad\downarrow$
$\qquad L = L_n \times L_{n+1}$

- $\mathfrak{K} := \{\text{open cpt of } G(\mathbb{Q}_p)\}$.
$\quad\cup$
- $k \rightsquigarrow H(G, k) \rightsquigarrow H(G, I)$ for $I < I^\infty$

· For subgrp $J < I^\infty$, $\mathcal{D}_J(G)$, $E_J(G)$

· $\mathfrak{m}_n$, $\mathfrak{m}_{n+1}$, $\mathfrak{m} = (\mathfrak{m}_n, \mathfrak{m}_{n+1})$

<u>Def</u> For any $\mathbb{Z}[G(\mathbb{Q}_\square) /\!/ K^p_\square]$-mod M,

denote by M^H the max submod of M

on which the Hecke operators from $\mathbb{Z}[H(G_\square) /\!/ (K^p_\square \cap H(G_\square))]$

acts by the deg map.

<u>Bessel period</u> (Def) $\lambda_J \in \mathcal{D}_J(G)^H$

(Indef) $\kappa_J \in H^1_f(F, \mathcal{D}_J(G))^H$

<u>Characteristic divisor</u>

Let X normal noetherian sch,

$\mathcal{F}$ torsion coh sheaf on X.

$$\mathrm{char}_X(\mathcal{F}) := \sum_{\substack{z \text{ codim } 1 \\ \text{pt of } X}} \mathrm{length}_{\mathcal{O}_{X,z}}(\mathcal{F}_z) \in \mathrm{Div}(X).$$

characteristic divisor

$\exists$ unique Γ_F repr $R_1(G_N)_{\mathfrak{m}_N}$ on $E_1(G_N)_{\mathfrak{m}_N}$ of rk N.

R_J, E_J for J trivial

$\rightsquigarrow$ $$\underbrace{R_J(G_N)_{\mathfrak{m}}}_{} := (R_1(G_N)_{\mathfrak{m}_n} \otimes R_1(G_N)_{\mathfrak{m}_{n+1}}) \otimes_{E_1(G)_{\mathfrak{m}}} E_J(G)_{\mathfrak{m}}(-)$$

fin free mod / $E_J(G_N)_{\mathfrak{m}}$ of rk $n(n+1)$.

$$\mathcal{X}_J(G)_{\mathfrak{m}} := H^1_f(R_J(G)^*_{\mathfrak{m}}(1))^* \otimes_{\mathbb{Z}_q} \mathbb{Q}_q.$$

Conj (Iwasawa main conj)

$\mathcal{E}$ be an irred comp of the normal locus of $\operatorname{Spec} \mathcal{E}_J(G)_m$

(Def) If $\lambda_J \neq 0$ on $\mathcal{E}$, then

(1) $H^1_f(F, R_J(G)_m) = 0$

(2) $\mathcal{X}_J(G)_m$ is torsion over $\mathcal{E}$

(3) $2\,\mathrm{char}_{\mathcal{E}}(\mathcal{D}_J(G)^H/\lambda_J) = \mathrm{char}_{\mathcal{E}}(\mathcal{X}_J(G)_m)$

(Indef) If $\kappa_J \neq 0$ on $\mathcal{E}$, then

(1) $H^1_f(F, R_J(G)_m)$ torsion-free of rk 1 on $\mathcal{E}$

(2) $\mathcal{X}_J(G)_m$ rank 1 on $\mathcal{E}$

(3) $2\,\mathrm{char}_{\mathcal{E}}(H^1_f(F, \mathcal{D}_J(G))^H/\kappa_J) = \mathrm{char}_{\mathcal{E}}(\mathcal{X}_J(G)_m^{tor})$.

Thm Assuming $p > n$, I^∞/J torsion-free

+ lots of conditions on m,

Then we know (1) (2) + part of (3).

(Def) For $I \triangleleft I^\infty$, $\mathcal{K}_I \supseteq \mathcal{K}_I^H$ cofinal

$$\mathcal{K}_I^H = \{ k \in \mathcal{K}_I \mid k = (k \cap H(\mathbb{Q}_p)) \cdot (k \cap P(\mathbb{Q}_p)) \}.$$

$k \in \mathcal{K}_I^H \rightsquigarrow Sh(H, K^p k) \xrightarrow{\Delta} Sh(G, K^p k)$

$\lambda_k := \Delta_! \mathbb{1}_{Sh(H, K^p k)}$.

Fact This is indep of k, $\lambda_I \rightsquigarrow \lambda_J$.

Let x classical pt of $\operatorname{Spec} \mathcal{E}_J(G)_m$

$\chi \Rightarrow$ autom repr $\pi_{n,\chi}$, $\pi_{n+1,\chi}$ of $GL_n(\mathbb{A}_F)$ or $GL_{n+1}(\mathbb{A}_F)$.

Input local conditions for GGP:

<u>GGP</u> (A) $\lambda_K|_\chi \neq 0$

(B) $L(\frac{1}{2}, \pi_{n,\chi} \times \pi_{n+1,\chi}) \neq 0$

& $\exists\ \pi_\square$ repr of $G(\mathbb{Q}_\square)$ with base change $\pi_{n,\square} \boxtimes \pi_{n+1,\square}$

s.t. $(\pi_\square^{K_\square^p})^H \neq 0$.

<u>(A) $\Rightarrow$ (B)</u> ok

<u>(B) $\Rightarrow$ (A)</u> If $(\xi_{n,\chi}, \xi_{n+1,\chi})$ interlacing

+ $\pi_{n,\chi}$, $\pi_{n+1,\chi}$ potentially cran.

[BPLZZ] + [Liu-Sun] Then (B) $\Rightarrow$ (A).