Introduction to kimberlites

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Lecture 1

§1 The v-topology

Fix prime p.

Perf $= \{$ perf'd spaces $/\mathbb{F}_p\}$ endowed w/ v-top.

Def'n A family of maps $\{X_i \xrightarrow{f_i} X\}_{i\in I}$ is a v-cover if $\forall\, U \subseteq X$, U open quasicpt,

$\exists\, J \subseteq I$ w/ $|J| < \infty$, & $\{U_j \subseteq X_j$ open, qc$\}_{j\in J}$

s.t. $\coprod_{j\in J} |U_j| \twoheadrightarrow |U|$ surjective.

Non-example Let $T \subseteq \mathbb{R}$, $T = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$, $C = \widehat{\overline{\mathbb{F}_q((t))}}$.

Let $R = C^0(T, C)$, $R^+ = C^0(T, \mathcal{O}_C)$, $|\mathrm{Spa}(R, R^+)| = T$.

$\forall\, t \in T$, $\exists$ an evaluation map

$$ev_t : R \to C, \quad f \mapsto f(t)$$

$\rightsquigarrow \quad \coprod_{t\in T} ev_t : \coprod_{t\in T} \mathrm{Spa}(C, \mathcal{O}_C) \to \mathrm{Spa}(R, R^+)$

Claim This is surjective

but NOT a v-cover.

Example Fix $X = \mathrm{Spa}(A, A^+)$ affinoid perfectoid.

Let $\varpi \in A^+$ p.u. Given $x \in X$, pick a symmetric pt

$$i_x : \mathrm{Spa}(C, C^+) \longrightarrow \mathrm{Spa}(A, A^+)$$

w/ C alg closed nonarch field

$C^+ \subseteq C$ open bounded valuation subring

s.t. if $\sigma \in \mathrm{Spa}(C, C^+)$ the most special pt.

then $i_x(\sigma) = x$.

Let $R^+ = \prod_{x \in X} C_x^+$ + ϖ-adic top. R^+ ϖ-adic complete.

$$A^+ \longrightarrow R^+, \quad \varpi \longmapsto (\varpi)_{x \in X}$$

$$R := R^+[\varpi^{-1}], \quad ev : (A, A^+) \longrightarrow (R, R^+)$$

$$a \longmapsto \{a(x)\}_{x \in X}$$

$$ev_x : (R, R^+) \longrightarrow (C_x, C_x^+)$$

$$\{r_x\}_{x \in X} \longmapsto r_x.$$

This induces a v-cover

$$\mathrm{Spa}(C_x, C_x^+) \to \mathrm{Spa}(R, R^+) \to \mathrm{Spa}(A, A^+).$$

<u>Construction</u> Consider data

$$(I, \{V_i\}_{i \in I}, \{\varpi_i \in V_i\}_{i \in I})$$

where I = index set, V_i ϖ_i-adically complete val rings w/ alg closed func field.

Let $R^+ = \prod_{i \in I} V_i$ & $\varpi = (\varpi_i)_{i \in I}$,

$R = R^+[\varpi^{-1}]$.

$\rightsquigarrow$ $\mathrm{Spa}(R, R^+)$ a basis of v-top.

<u>Def'n</u> Call affinoid perfectoids $\mathrm{Spa}(R, R^+)$ constructed in this way <u>product of points</u>.

<u>Rmks</u> (a) $\pi_0(\mathrm{Spa}(R,R^+)) = \beta I$

(b) Product of pts are a basis for the v-top.

(c) They give you strictly totally disconnected spaces.

(d) Every proétale cover splits.

<u>§2 Small v-sheaves</u>

<u>Def'n</u> Say that a v-sheaf $\mathcal{F}$ is <u>small</u> if

$\exists$ perf'd space X + surjective map of v-shv $X \to \mathcal{F}$.

<u>Def'n</u> Let X be a small v-sheaf. Define $|X|$ as

(a) set theoretically:

$$|X| := \{\text{geom pts } i: \mathrm{Spa}(C,C^+) \to X\} / \sim.$$

$i_1 \sim i_2$ if $\exists$ commutative diagram

$$\begin{tikzcd} & \mathrm{Spa}(C_1, C_1^+) \arrow[dr, "i_1"] \arrow[dd, phantom, "\circlearrowleft"] & \\ \mathrm{Spa}(C_3, C_3^+) \arrow[ur, "r_1"] \arrow[dr, "r_2"'] & & X \\ & \mathrm{Spa}(C_2, C_2^+) \arrow[ur, "i_2"'] & \end{tikzcd}$$

where r_1, r_2 are local maps of local spectral spaces.

(equivalently, r_1, r_2 surjective).

(b) Topologize it:

Find a v-cover $Y \to X$ w/ Y perf'd.

Then $R = Y \times_X Y$ is also a small v-sheaf

(can chose a cover $Z \to R$ w/ Z perf'd).

Set theoretically one shows $|X| = |Y| / |Z|$

as top spaces.

Fact This doesn't depend on Y or Z.

Construction T top space.

Let $\underline{T} : \mathrm{Perf} \longrightarrow \mathrm{Sets}$

$\underline{T}(X) = C^0(|X|, T)$.

Prop $\underline{T}$ is a v-sheaf, and:

Whenever T is compact Hausdorff

then $\underline{T}$ is a small v-sheaf.

Example If $T = \{h, \sigma\}$, $h > \sigma$,

then $\underline{T}$ is not a small v-sheaf.

For any small v-sheaf X, $\exists$ evident map $X \to \underline{|X|}$.

Def'n A map of v-sheaves $Y \to X$ is an <u>open immersion</u> if it is rel'ly rep'ble in open immersions of perf'd spaces.

Prop If $Y \to X$ is an open immersion of v-sheaves & X is small, then Y is small and

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ \underline{|Y|} \arrow[r] & \underline{|X|} \end{tikzcd}$$

Cartisian.

& $|Y| \to |X|$ also an open immersion.

Moreover, for all open $U \subseteq |X|$, $X \times_{\underline{|X|}} \underline{U} \to X$

is an open imm w/ $|X \times_{|X|} \underline{\mathcal{U}}| = \mathcal{U}$.

Def'n A subset $S \subseteq |X|$ is <u>weakly generalizing</u> if
$\forall$ generic pts $i : \mathrm{Spa}(C, C^+) \longrightarrow X$,
if $s \in \mathrm{Spa}(C, C^+)$ is special pt & $i(s) \in S$
then $i(\mathrm{Spa}(C, C^+)) \subseteq S$.

Prop If $f : X \to Y$ is a map of small v-shvs
then $f(|X|) \subseteq |Y|$ is weakly generalizing.

Prop Let X small v-sheaf, $S \subseteq |X|$ a closed subset,
then $X \times_{|X|} \underline{S}$ is a small v-sheaf.
Moreover, $S = |X \times_{|X|} \underline{S}| \iff S$ weakly generalizing.

Example If $X = \mathrm{Spa}(C, C^+)$ and $\mathrm{rk}(C^+) > 2$,
and $s \in \mathrm{Spa}(C, C^+)$ is the special pt,
then $\{s\} \subseteq X$ is closed but $X \times_{|X|} \underline{\{s\}} = \phi$.

Def'n A map of small v-shvs $f : Y \to X$
is a <u>closed immersion</u> if $|Y| \to |X|$
is a closed immersion and

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ \underline{|Y|} \arrow[r] & \underline{|X|} \end{tikzcd}$$

Cartisian.

(note: this forces Y to be weakly generalizing.)

Summary Open imms of $X \xleftrightarrow{1-1}$ Open imms of $|X|$.
Closed imms of $X \xleftrightarrow{1-1}$ weakly generalizing closed imms of $|X|$.

§3 The functor $\diamond$

Let Y be an adic space $/\mathbb{Z}_p$.

Def'n $Y^\diamond : \mathrm{Perf} \longrightarrow \mathrm{Sets}$

$$Y^\diamond(X) := \{(X^\#, f)\}/\sim$$

where $X^\#$ = an untilt of X

$f : X^\# \to Y$ a map of adic spaces.

If (A, A^+) a Huber pair $/\mathbb{Z}_p$,
then denote $\mathrm{Spa}(A, A^+)^\diamond$ by $\mathrm{Spd}(A, A^+)$.

§4 How does $|\mathrm{Spd}(A, A^+)|$ look like?

For any adic space $Y/\mathbb{Z}_p$,
$Y^\diamond$ is a small v-sheaf!

In general, $\exists$ a cont map

$$|\mathrm{Spd}(A, A^+)| \xrightarrow{q} |\mathrm{Spa}(A, A^+)|$$

and the cardinalities of fibres $q^{-1}(x)$ is always 1 or 2.

- A analytic $\Rightarrow$ q homeomorphism.
- q homeomorphism $\Rightarrow$ $\dim(\mathrm{Spa}(A, A^+)^{\text{non-an}}) \leq 0$.

Two key failures of φ to be homeomorph

(a) $|\mathrm{Spd}(A, A^+)|$ has more pts.

(b) $|\mathrm{Spd}(A, A^+)|$ has more open subsets.

Example of failure (a)

V DVR w/ uniformizer π, $K = V[\frac{1}{\pi}]$.

Then $\mathrm{Spa}(K, V) = \{h, t\}$,

$h \rightsquigarrow |K^\times| = 1$,

$t \rightsquigarrow |k| \le 1 \ (\Leftrightarrow k \in V)$.

But $|\mathrm{Spa}(K,V)| = \{h, t, a\}$,

$h \rightsquigarrow (\mathrm{Spa}(\widehat{K((t))}, \widehat{K[[t]]}) \to \mathrm{Spa}(K,V))$

$t \rightsquigarrow (\mathrm{Spa}(\widehat{K((t))}, \widehat{V + tK[[t]]}) \to \mathrm{Spa}(K,V))$

$a \rightsquigarrow (\mathrm{Spa}(\hat{\hat{K}}_\pi, \hat{\hat{V}}_\pi) \to \mathrm{Spa}(K,V))$.

$t \in \mathrm{Spa}(K,V)$ splits into t & a

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathrm{Spa}(\widehat{K((t))}, \widehat{V + tK[[t]]}) \arrow[d] \\ \mathrm{Spa}(\hat{K}_\pi, \hat{V}_\pi) \arrow[r] & \mathrm{Spd}(K,V) \end{tikzcd}$$

Reason this is $\emptyset$:

$\pi \in \hat{K}_\pi \Rightarrow \pi$ top nilp.

but $\frac{1}{\pi} \in K((t)) \Rightarrow \frac{1}{\pi}$ power-bounded.

Example of failure (b)

$\mathrm{Spd}(\mathbb{F}_p[[t]], \mathbb{F}_p[[t]]) \longrightarrow \mathrm{Spd}(\mathbb{F}_p[t], \mathbb{F}_p[t])$

is an open immersion.

Given a Huber pair (A, A^+), let

$$\mathrm{Spo}(A, A^+) := \left\{ (|\cdot|_h, |\cdot|_a) \in \mathrm{Spa}(A, A^+)^2 \;\middle|\; \begin{array}{l} |\cdot|_a \text{ vertical generalization of } |\cdot|_h, \\ |\cdot|_a \text{ is of rk } 1 \text{ or trivial} \end{array} \right\}$$

└ olivine (a kind of rock).

Given $f, g \in A$,

$$U_{f \le g \ne 0} := \{ x \in \mathrm{Spo}(A, A^+) \mid |f(x)|_h \le |g(x)|_h \ne 0 \}$$

$$N_{f \ll g \ne 0} := \{ x \in \mathrm{Spo}(A, A^+) \mid \underbrace{|f(x)|_a < |g(x)|_a}_{\Updownarrow \; f/g \text{ is top nilp}} \ne 0 \}$$

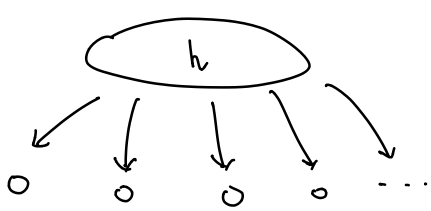
<u>Theorem</u> For all Huber pairs $(A, A^+) / \mathbb{Z}_p$,

$\exists$ a cont bijective map

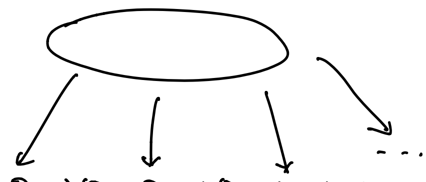
$$|\mathrm{Spd}(A, A^+)| \longrightarrow \mathrm{Spo}(A, A^+).$$

It is a homeomorph when (A, A^+) is top'ly of fin rank over a ring of def'n.

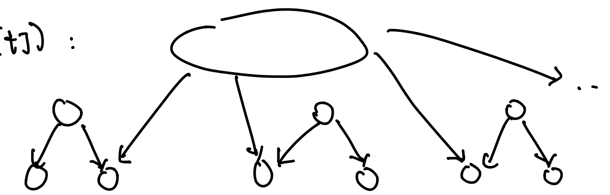
<u>Pictures</u> $\mathrm{Spec}\, \overline{\mathbb{F}}_p[t]$:



$\mathrm{Spa}(\overline{\mathbb{F}}_p[t], \overline{\mathbb{F}}_p[t])$:



$\mathrm{Spd}(\overline{\mathbb{F}}_p[t], \overline{\mathbb{F}}_p[t])$:



<u>Lecture 2</u>

Recall that

$$\mathrm{Spo}(A, A^+) := \left\{ (|\cdot|_h, |\cdot|_a) \in \mathrm{Spa}(A, A^+)^2 \;\middle|\; \begin{array}{l} |\cdot|_a \text{ a vertical generalization of } |\cdot|_h, \\ |\cdot|_a \text{ is of rk 1 or trivial} \end{array} \right\}$$

Most cases $\mathrm{Spo}(A, A^+) = |\mathrm{Spd}(A, A^+)|$.

Basis of top: $U_{f \leq g \neq 0}$ – classical

$N_{f \ll g \neq 0}$ – locus where f/g is top nilp.

$\exists$ projection map

$$\begin{aligned} \mathrm{Spo}(A, A^+) &\longrightarrow \mathrm{Spa}(A, A^+) \\ (|\cdot|_h, |\cdot|_a) &\longmapsto |\cdot|_h . \end{aligned}$$

<u>Example</u> Let V a DVR w/ discrete top (e.g. $V = \mathbb{F}_p[\![t]\!]$)

$\pi \in V$ a uniformizer, $K = V[\frac{1}{\pi}]$, $k = V/\pi$.

- $\mathrm{Spec}\, V = \{ g > s \}$

  $g \rightsquigarrow (0)$, $s \rightsquigarrow \langle \pi \rangle$

  $$g \longrightarrow s$$

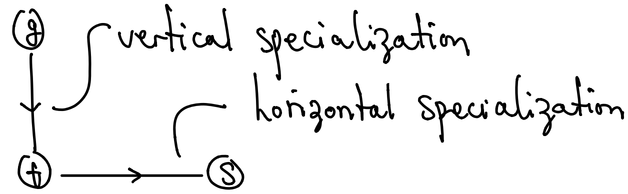
  horizontal specialization

- $\mathrm{Spa}(V, V) = \{ g > t > s \}$

$g \rightsquigarrow |k^\times|_g = 1,\ |0|_g = 0,\ \mathrm{Supp}(g) = (0).$

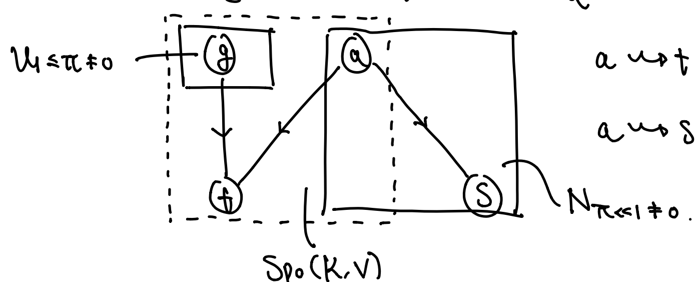
$t \rightsquigarrow \pi^n u,\ u \in V^\times.\ |\pi^n u|_t = e^n$ for $e \in (0,1)$

$s \rightsquigarrow |\pi V|_s = 0,\ |k^\times|_s = 1,\ \mathrm{Supp}(s) = \langle \pi \rangle.$



- $\mathrm{Spa}(V,V) = \{ (|\cdot|_g, |\cdot|_g),\ (|\cdot|_t, |\cdot|_g),\ (|\cdot|_t, |\cdot|_t),\ (|\cdot|_s, |\cdot|_s) \}$

$\qquad\qquad\quad \overset{\shortparallel}{g} \qquad\qquad \overset{\shortparallel}{t} \qquad\qquad \overset{\shortparallel}{a} \qquad\qquad \overset{\shortparallel}{s}$



$a \rightsquigarrow t$: mero specialization

$a \rightsquigarrow s$: formal specialization

<u>Thm A</u> (SW Berkeley notes, Gleason)

Let X, Y adic spaces $/\mathbb{Z}_p$.

If Y is perfect & non-analytic

then $\mathrm{Hom}(Y^\diamond, X^\diamond) = \mathrm{Hom}(Y, X)$.

In particular, $\diamond$ is fully faithful

on perfect non-analytic adic spaces.

<u>Definition</u> Let (A, A^+) be a Huber pair $/\mathbb{Z}_p$.

Say $S \subseteq \mathrm{Spa}(A, A^+)$ is <u>schematic</u> if $S = \bigcup_{i \in I} T_i$

where $T_i = f_i(\mathrm{Spa}(B,B)) \subseteq \mathrm{Spa}(A, A^+)$

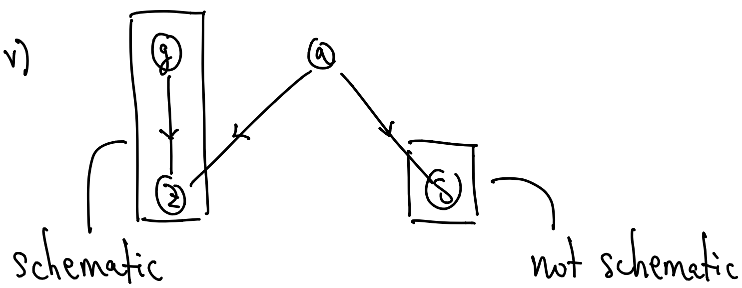
$f_i : (A, A^+) \to (B, B)$ map of Huber pairs

with B equipped w/ disc top.

Lem B Suppose $Z \subseteq \mathrm{Spo}(B, B^+)$ is schematic and closed. Then $\exists$ an open ideal $I \subseteq B$ s.t.

$$x \in Z \iff \mathrm{Supp}(x) \supseteq I.$$

Example $\mathrm{Spo}(V, V)$



§5 Analytifications

$$\begin{tikzcd}
\{\text{adic spaces}/\mathbb{Z}_p\} \arrow[r, "\lozenge"] & \{v\text{-sheaves}\} \\
\{\text{schemes}/\mathbb{Z}_p\} \arrow[u, dashed] \arrow[ur]
\end{tikzcd}$$

$? \in \{\diamond, \dagger, \lozenge\}$

$$X^{?\,\mathrm{pre}}(\mathrm{Spa}(R, R^+)) = \{(\underbrace{R^\sharp}_{\text{an untilt}}, \alpha)\}/\sim$$

$$\text{with } \alpha \in \begin{cases} X(\mathrm{Spec}\, R^\sharp), & ? = \lozenge \\ X(\mathrm{Spec}\, R^{\circ\sharp}), & ? = \dagger \\ X(\mathrm{Spec}\, R^{+\sharp}), & ? = \diamond \end{cases}$$

$X^? =$ sheafification of $X^{?\,\mathrm{pre}}$.

Example $X = \mathbb{A}^1_{\mathbb{Z}_p}$. Then

$$X^\diamond = (\mathbb{B}^1_{\mathbb{Z}_p})^\lozenge = \mathrm{Spd}(\mathbb{Z}_p\langle T\rangle)$$

$$X^\lozenge = (\mathbb{A}^1_{\mathbb{Z}_p})^\lozenge = (\mathbb{P}^1_{\mathbb{Z}_p} \setminus \mathrm{Spd}\,\mathbb{Z}_p)^\lozenge, \quad \mathrm{Spd}\,\mathbb{Z}_p = \text{pt at } \infty.$$

$$X^{\dagger} = (\overline{\mathbb{B}^1_{\mathbb{Z}_p}})^{\diamondsuit}$$

└ canonical compactification

<u>Example</u> $X = \mathbb{A}^1_{\mathbb{F}_p}$. Then

$$X^{\diamond} = \mathrm{Spd}(\mathbb{F}_p[t], \mathbb{F}_p[t]) = \mathrm{Spd}(\mathbb{F}_p[t]^{\mathrm{perf}}, \mathbb{F}_p[t]^{\mathrm{perf}}).$$

$X^{\dagger}$ — does not come from adic space

$X^{\diamondsuit} = \mathrm{Spd}(\mathbb{F}_p[t], \mathbb{F}_p)$

(with $X^{\diamond} \cong X^{\dagger} \cong X^{\diamondsuit}$ written vertically)

<u>Rmks</u> (a) If X is of fin type $/\mathbb{Z}_p$, then $X^{\diamond} \subseteq X^{\dagger}$ & $X^{\dagger} \subseteq X^{\diamondsuit}$

$X^{\diamond} \cong X^{\diamondsuit}$ open imms, closed imm.

(b) $X^{\diamond} \xrightarrow{s} X^{\dagger} \xrightarrow{r} X^{\diamondsuit}$

(c) If X is proper $/\mathbb{Z}_p$ then $X^{\diamond} = X^{\dagger} = X^{\diamondsuit}$

(d) Scholze constructs a fully faithful

$$c^* : \mathrm{D\acute{e}t}(X, \mathbb{F}_\ell) \longrightarrow \mathrm{D\acute{e}t}(X^{\diamondsuit}, \mathbb{F}_\ell).$$

Let $b^* := r^* c^*$, $a^* := s^* r^* c^*$

which are also fully faithful.

<u>§6 The schematic v-topology</u>

<u>Def'n</u> A map of qcqs schs $X \to Y$ is a <u>v-cover</u> if for any map $\mathrm{Spec}\, V \to Y$ w/ V valuation ring, $\exists$ an ext'n of val rings $W \supseteq V$, + a comm diagram

$$\begin{array}{ccc} \mathrm{Spec}\, W & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, V & \longrightarrow & Y. \end{array}$$

Prop Let $f : X \to Y$ be a map of qcqs perfect schs $/ \mathbb{F}_p$.

TFAE: (a) $f : X \to Y$ is a (schematic) v-cover.

(b) $f^\diamond : X^\diamond \to Y^\diamond$ is a v-cover.

(c) $f^\dagger : X^\dagger \to Y^\dagger$ is a v-cover.

Warning Even if $f : X \to Y$ is a schematic v-cover, $f^\diamond : X^\diamond \to Y^\diamond$ might not be a v-cover.

Prop If f is of fin presentation & is a (sch'tic) v-cover, then $f^\diamond$ is a v-cover.

§7 The reduction functor

$$\begin{tikzcd} \{\text{perfect schemes} / \mathbb{F}_p\} \arrow[r, "\diamond"] \arrow[d] & \{\text{small v-stacks}\} \\ \left\{\begin{matrix}\text{small scheme-}\\ \text{theoretic v-stacks}\end{matrix}\right\} \arrow[ur, "\diamond"'] & \end{tikzcd}$$

There is a right adjoint

$$\mathrm{red} : \{\text{small v-stacks}\} \longrightarrow \left\{\begin{matrix}\text{small scheme-}\\ \text{theoretic v-stacks}\end{matrix}\right\}$$

More concretely, if $\mathcal{F}$ is a small v-sheaf

$$\begin{aligned} \mathcal{F}^{\mathrm{red}} : \mathrm{Perf\,Sch} &\longrightarrow \mathrm{Sets} \\ X &\longmapsto \mathrm{Hom}(X^\diamond, \mathcal{F}) \end{aligned}$$

Example Thm A implies

(a) $(\mathrm{Spd}\,\mathbb{Q}_p)^{red} = \phi$

(b) $(\mathrm{Spd}\,\mathbb{Z}_p)^{red} = \mathrm{Spd}\,\mathbb{F}_p$

(c) $(X^{\diamond})^{red} = X$ for $X \in \mathrm{PerfSch}$.

(d) $\mathrm{Spd}\,(B,B)^{red} = (\mathrm{Spec}\, B/I)^{perf}$

whenever B has I-adic top.

By adjunction, have a map $(\mathcal{F}^{red})^{\diamond} \longrightarrow \mathcal{F}$.

Abbreviation: $\mathcal{F}^{red} := (\mathcal{F}^{red})^{\diamond}$.

<u>Definition</u> A map of v-sheaves $\mathcal{F} \to \mathcal{G}$ is <u>formally adic</u> if

$$\begin{tikzcd} \mathcal{F}^{red} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{G}^{red} \arrow[d] \\ \mathcal{F} \arrow[r] & \mathcal{G} \end{tikzcd}$$

Cartisian.

<u>Rmk</u> Let X scheme & $\mathcal{F} \to X^{\diamond}$ a formally adic closed imm.

Lem B $\Rightarrow$ $\mathcal{F} = Y^{\diamond}$ for $Y \hookrightarrow X$ Zariski closed subset.

<u>§8 Specializing v-sheaves</u>

Big picture of objects:

$\{\text{formal schemes}\}^{\diamond} \subseteq \{\text{spatial kimberlites}\}$

$\subseteq \{\text{kimberlites}\}$

$\subseteq \{\text{pre-kimberlites}\}$

$\subseteq \{\text{specializing v-sheaves}\}$

$\subseteq \{\text{small v-sheaves}\}$

Def'n A v-sheaf $\mathcal{F}$ is <u>formally separated</u> if $\Delta : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ is a formally adic closed immersion.

Def'n A v-sheaf $\mathcal{F}$ is <u>v-locally formal</u> if $\exists$ index set I
+ a family (R_i, R_i^+) affinoid perfectoid
and a v-cover $\coprod_{i \in I} \mathrm{Spd}(R_i, R_i^+) \to \mathcal{F}$.

Def'n A v-sheaf $\mathcal{F}$ is <u>specializing</u> if it is v-locally formal and formally separated.

Prop Let $\mathcal{F}$ be a specializing v-sheaf.
There is a conti map $\mathrm{Sp} : |\mathcal{F}| \to |\mathcal{F}^{red}|$
This is functorial in $\mathcal{F}$.

<u>Sketch of constr of Sp</u>

Given $x \in |\mathcal{F}|$, want to construct $\mathrm{sp}(x) \in |\mathcal{F}^{red}|$.
x is rep'd by a geom pt

$$\begin{tikzcd} \mathrm{Spa}(C_x', C_x') \arrow[r, two heads] \arrow[d] \arrow[drr, bend left] & \mathrm{Spa}(C_x, C_x^+) \arrow[r] & \mathcal{F} \\ \mathrm{Spd}(C_x'^{+}, C_x'^{+}) \arrow[rr] & & \coprod_{i \in I} \mathrm{Spd}(R_i, R_i^+) \arrow[u] \end{tikzcd}$$

Get a map
$$(\mathrm{Spd}(C_x'^{+}, C_x'^{+}) \longrightarrow \mathcal{F})^{red}$$
$$\rightsquigarrow \quad \mathrm{Spec}(C_x'^{+}/C_x^{\circ\circ}) \longrightarrow \mathcal{F}^{red}.$$

If k_x = res field of $C_x'^{+}$, then

$$\mathrm{Spa}\, k_x \longrightarrow \mathrm{Spec}(C_x'^{+}/C_x'^{\circ\circ}) \longrightarrow \mathfrak{F}^{red}$$

defines a schematic pt.

This is $sp(x) \in |\mathfrak{F}^{red}|$.

Rmk Formal separatedness is used to show $sp(x)$ does not depend on choices.

<u>Lecture 3</u>

specializing v-sheaf:
- v-localy formal
- formally separated

Given a specializing v-sheaf $\mathfrak{F}$

$\rightsquigarrow$ $sp : |\mathfrak{F}| \longrightarrow |\mathfrak{F}^{red}|$

functorial on specializing v-sheaves.

§1 Prekimberlites

Def'n A specializing v-sheaf $\mathfrak{F}$ is a <u>prekimberlite</u> if

(a) $\mathfrak{F}^{red}$ is a perfect scheme.

(b) The adjunction map

$$(\mathfrak{F}^{red})^{\diamond} =: \mathfrak{F}^{Red} \longrightarrow \mathfrak{F} \quad \text{is a closed imm.}$$

Def'n Let $\mathfrak{F}^{an} := \mathfrak{F} \setminus \mathfrak{F}^{Red}$ whenever $\mathfrak{F}$ is a prekimberlites.

called analytic locus of $\mathfrak{F}$.

<u>Construction</u> Let X perfect sch / $\mathbb{F}_p$.

For $X^{\diamond/\circ}$: Perf $\longrightarrow$ Sets,

$$X^{(\diamond/\circ)\mathrm{pre}}(R,R^+) = \mathrm{Hom}(\mathrm{Spec}(R^+/R^{\circ\circ}), X).$$

Let $X^{\diamond/\circ}$ be the sheafification.

<u>Thm</u> (Heuer) If $X = \mathrm{Spec}\,A$ is affine then $X^{(\diamond/\circ)\mathrm{pre}} = X^{\diamond/\circ}$ (i.e. $X^{(\diamond/\circ)\mathrm{pre}}$ is a v-sheaf).

<u>Notation</u> When $\mathfrak{F}$ is prekimberlite, let $\mathfrak{F}^H := (\mathfrak{F}^{\mathrm{red}})^{\diamond/\circ}$.

<u>Aside</u> $X^{\diamond\mathrm{pre}}(R,R^+) = \mathrm{Hom}(\mathrm{Spec}\,R^+, X)$

$X^{(\diamond/\circ)\mathrm{pre}}(R,R^+) = \mathrm{Hom}(\mathrm{Spec}\,R^+/R^{\circ\circ}, X)$

e.g. $(\mathbb{A}^1_{\mathbb{F}_p})^{\diamond} = \mathbb{B}^1_{\mathbb{F}_p}$ – closed unit ball

$(\mathbb{A}^1_{\mathbb{F}_p})^{\diamond/\circ} \approx \mathbb{B}^1_{\mathbb{F}_p} / \mathbb{B}^{<1}_{\mathbb{F}_p}$.

roughly — open unit ball

v-sheaf theoretic specialization map:

If $\mathfrak{F}$ is a kimberlite, $\exists$ a map of v-sheaves

$$\mathrm{sp} : \mathfrak{F} \longrightarrow \mathfrak{F}^H \text{ functorial in } \mathfrak{F}.$$

<u>Sketch</u> If $\alpha \in \mathfrak{F}(R,R^+)$,

$$\begin{tikzcd} \mathrm{Spa}(S,S^+) \arrow[r, two heads] \arrow[dr, hook] & \mathrm{Spa}(R,R^+) \arrow[r] & \mathfrak{F} \\ & \mathrm{Spd}(S,S^+) \arrow[ur, "\alpha'"'] & \end{tikzcd}$$

$$\underbrace{(\alpha')^{red} : \operatorname{Spec}(S^+/S^{\circ\circ})}_{(\mathcal{F}^{red})^{\diamond/\circ}(S,S^+)} \longrightarrow \mathcal{F}^{red} \rightsquigarrow Sp(\alpha').$$

Def'n Let X be a kimberlite,

$S \to X^{red}$ a map of perfect schs.

Let $\widehat{X}_{/S}$ & $X^{\circ}_{/S}$ be the v-sheaves fitting in the following Cartesian diagram

$$\begin{tikzcd} \widehat{X}_{/S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "Sp"] \\ S^{\diamond/\circ} \arrow[r] & X^{H} \end{tikzcd} \qquad \begin{tikzcd} X^{\circ}_{/S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X^{an} \arrow[d, "Sp"] \\ S^{\diamond/\circ} \arrow[r] & X^{H} \end{tikzcd}.$$

Example If $S \to X$ is a locally closed imm, then

$$\widehat{X}_{/S}(R,R^+) = \left\{ \alpha \in X(R,R^+) \text{ s.t. } |Spa(R,R^+)| \xrightarrow{\alpha} |X| \xrightarrow{Sp} |X^{red}| \text{ factors through } S \right\}.$$

When $f : S \to X^{red}$ is a locally closed imm,

Call $\widehat{X}_{/S}$ a formal nbhd around S.

Call $X^{\circ}_{/S}$ a tubular nbhd around S.

Example If $X = (\mathbb{B}^1_{\mathbb{Z}_p})^{\diamond} = \operatorname{Spd} \mathbb{Z}_p\langle T\rangle$,

then $X^{red} = \mathbb{A}^{1,perf}_{\mathbb{F}_p}$, $X^{an} = (\mathbb{B}^1_{\mathbb{Q}_p})^{\diamond}$.

For $0 : \operatorname{Spec} \mathbb{F}_p \to X^{red}$,

$\widehat{X}_{/0} = \operatorname{Spd}(\underbrace{\mathbb{Z}_p[\![t]\!]}_{(p,t)\text{-adic}})$. $X^{\circ}_{/0} = (\mathbb{B}^{<1}_{\mathbb{Q}_p})^{\diamond}$

Berthelot's tube.

<u>Example</u> If $Y = \mathrm{Spec}\,A$, A perf ring $/\mathbb{F}_p$,
& if $X^{red} = Y$, $X = Y^{\diamond}$,
$S \hookrightarrow Y$ Zariski closed given by ideal $I \subseteq A$ f.g.
& $I = (i_1, \dots, i_n)$.

Then $\hat{X}_{/S} = \bigcap_{j=1}^{n} N_{i_j \ll 1 \neq 0} \subseteq X^{\diamond}$
representable by $\mathrm{Spd}(B, B)$,
(B = I-adic completion of A).

<u>Prop</u> Let $S \to Y$ be a map of perf schs $/\mathbb{F}_p$. Then
(a) If S is an open imm (resp. is étale)
then $S^{\diamond/\circ} \longrightarrow Y^{\diamond/\circ}$ is formally adic
& is an open imm (resp. is étale)
(b) If $S \to Y$ is a constructible closed imm
then $S^{\diamond/\circ} \longrightarrow Y^{\diamond/\circ}$ is an open imm.
(c) If $Y = X^{red}$ for X a prekimberlite,
& $S \to Y$ as in (a) or (b),
then $\hat{X}_{/S}$ is a prekimberlite.

<u>Def'n</u> Let X be a prekimberlite.
Let $(X)_{\text{for-ét}}$ be the cat of maps $f : Y \to X$
where Y prekimberlite & f satisfies
(i) f formally adic
(ii) f étale
(iii) f quasicompact.

Thm Let X be a prekimberlite.

$$\mathrm{red} : (X)_{\text{for-ét}} \xrightarrow{\sim} (X^{\mathrm{red}})_{\text{ét, qc, sep}} \text{ is an equiv.}$$

The inverse functor takes

$$\{U \to X^{\mathrm{red}}\} \text{ to } \{\hat{X}_{/U} \to X\}.$$

Cor If $X = \mathrm{Spd}(B, B)$ with B I-adically complete (formal sch), and $Y \in (X)_{\text{for-ét}}$ is a prekimberlite, then Y is rep'ble by a formal sch.

Rmk If $X = \mathrm{Spd}(B, B)$ is a formal sch, B I-adic & $I \subseteq J$ defining $S \hookrightarrow X^{\mathrm{red}} = \mathrm{Spec}(B/I)^{\mathrm{perf}}$

then $\hat{X}_{/S} = \mathrm{Spd}(\widehat{B}_{/J}, \widehat{B}_{/J})$.

Rmk Thm above leads to a constr'n of nearby cycle functors for prekimberlites.

§10 Kimberlites

Def'n Let $\mathcal{F}$ be a v-sheaf, say it is <u>quasicompact</u> if

$\forall$ families $\{X_i \to \mathcal{F}\}_{i \in I}$ of v-surjections

$\exists\, J \subseteq I$, $|J| < \infty$ s.t. $\{X_j \to \mathcal{F}\}_{j \in J}$ is still v-surjective.

Say $\mathcal{F}$ is <u>quasiseparated</u> if

$\forall$ pair of maps $X_1 = \mathrm{Spa}(R_1, R_1^+) \to \mathcal{F} \leftarrow \mathrm{Spa}(R_2, R_2^+) = X_2$,

X_1, X_2 affinoid perf'd,

always have $X_1 \times_{\mathcal{F}} X_2$ quasicompact.

Warning The final obj {v-sheaves} $* = \mathrm{Spa}(\mathbb{F}_p)$ is NOT quasisep.

$\{\mathcal{F} \text{ quasisep}\} \neq \{\Delta : \mathcal{F} \to \mathcal{F} \times \mathcal{F} \text{ quasicpt map}\}$.

Def'n A v-sheaf is spatial if it is qcqs

& $|\mathcal{F}|$ has a basis of open nbhds $U \subseteq |\mathcal{F}|$

s.t. $\mathcal{F} \times_{|\mathcal{F}|} U$ is also quasicpt.

Prop If $\mathcal{F}$ is a spatial v-sheaf,

then $|\mathcal{F}|$ is a spatial top space.

Non-example Prekimberlites are never spatial v-shvs.

Def'n (Alternative) A spatial diamond $\mathcal{F}$ is a spatial v-sheaf

admitting a quasi-proet surjection

from a perf'd space.

Example $\diamond : \left\{ \begin{array}{c} \text{qcqs analytic} \\ \text{adic spaces } / \mathbb{Z}_p \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{spatial diamonds} \\ / \operatorname{Spd} \mathbb{Z}_p \end{array} \right\}$

Definition A prekimberlite X is called valuative if

$\mathrm{Sp} : X \to X^H$ is partially proper.

i.e.
$$\begin{tikzcd} \mathrm{Spa}(R, R^\circ) \arrow[r] \arrow[d] & X \arrow[d, "\mathrm{Sp}"] \\ \mathrm{Spa}(R, R^+) \arrow[r] \arrow[ur, dashed, "\exists!"] & X^H \end{tikzcd}$$

Example If $X = \mathrm{Spd}(B, B)$,
then X is valuative.

$B \to R^\circ \rightsquigarrow \mathrm{Spa}(R, R^+) \to \mathrm{Spd}(B, B)$

Get $(B/I)^{\mathrm{perf}} \to R^\circ/R^{\circ\circ}$,
and ask that this map factors through
$(B/I)^{\mathrm{perf}} \to R^+/R^{\circ\circ} \subseteq R^\circ/R^{\circ\circ}$.

Can show that $B \to R^\circ$ factors through R^+ as well.

Prop If X valuative prekimberlite,
$S \to X^{\mathrm{red}}$ locally closed imm / étale
or compositions of such,
then $\widehat{X}_{/S}$ is a valuative prekimberlite.

Def'n A prekimberlite $\mathcal{F}$ is called a kimberlite if
(a) X is valuative
(b) X^{an} is a locally spatial diamond,
(c) $\mathrm{Sp} : |X^{\mathrm{an}}| \to |X^{\mathrm{red}}|$ is quasicpt.

Thm If X is a kimberlite, then
the specialization map $\mathrm{Sp} : |X^{\mathrm{an}}| \to |X^{\mathrm{red}}|$ is spectral and closed.

Question (a) Are kimberlites stable under products?
(b) If X is a kimberlite and $S \hookrightarrow X$ closed,
is $\widehat{X}_{/S}$ also a kimberlite?

§11 Spatial Kimberlite

<u>Def'n</u> Let X be a kimberlite s.t. X^{red} is qcqs.

Say X is <u>spatial</u> if there is a v-cover

$$f : \mathrm{Spd}(B, B) \longrightarrow X$$

s.t. (a) B is I-adic for f.g. $I \subseteq B$ (even allow $I = 0$),

(b) f formally adic

(c) f quasicpt

(d) $f^{an} : \mathrm{Spd}(B,B)^{an} \longrightarrow X^{an}$ quasi-pro-étale.

Let X a kimberlite,

& $f : S \longrightarrow X^{red}$ qcqs map of schemes.

If f is étale or a constructible closed imm,

then $\hat{X}/S$ is also a spatial kimberlite.

<u>Thm</u> The cat of spatial kimberlites

is stable under fibre product

& contains $\mathrm{Spd}(B, B)$ for B I-adic, $I \subseteq B$ f.g. ideal.

(in particular, $X^{\diamond}$ for X a sch is a spatial kbl).

<u>Example</u> $X = \mathrm{Spd}\, \mathbb{Z}_p$, $X^{an} = \mathrm{Spd}\, \mathbb{Q}_p = Y$.

If we take $Y \times Y$, this is not qc.

But if we take $X \times X$, this is also a spatial kimberlite

qc — $(X \times X)^{an}$ is a spatial diamond.

$\cup$

not qc — $Y \times Y$.

__Def'n__ A prekimberlite X is a __locally spatial Kimberlite__
if for all affine $\mathcal{U} \subseteq X^{red}$, $\mathfrak{X}_{/\mathcal{U}}$ is a spatial kimberlite.

__Prop__ Let $f: X \to Y$ be a map of locally spatial Kimberlites.
Then f is representable in locally spatial diamonds.
Moreover, f quasicpt $\iff$ f formally adic
& f^{red} is quasicpt

* __List of kimberlites__

(a) $X^{\diamond}$ for X a formal sch
(b) Beilinson-Drinfeld Grassmannian over $(\mathrm{Spd}\,\mathbb{Z}_p)^n$.
(c) Non-negative Banach-Colmez spaces
(d) $\widetilde{G}_b = \mathrm{Aut}(\mathcal{E}_b)$ for $\mathcal{E}_b$ vec bdl on FF curve
(e) Moduli spaces of shtukas

$$\mathrm{Sht}_{\mathcal{G}}^{\le\mu}(b) \longrightarrow \mathrm{Spd}\,\mathbb{Z}_p$$

at parahoric level $\mathcal{G}$.

__Note__ Its special fibre $(\mathrm{Sht}_{\mathcal{G}}^{\le\mu}(b))^{red} = X_{\mathcal{G}}^{\le\mu}(b)$ ADLV
which is a locally spatial Kimberlite.