Hilbert Polynomials and Flatness

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See Hartshorne III.9 again.

§1 Hilbert Polynomials

k field (not necessarily $k = \bar{k}$). $j : X \to \mathbb{P}^r_k$ closed imm ($r \geq 1$).

Write $\mathcal{O}_X(1) = j^{-1}\mathcal{O}_{\mathbb{P}^r_k}(1)$. $\mathcal{F} \in \mathrm{QCoh}(X)$ f.g.

Def'n Euler char of $\mathcal{F}$ is

$$\chi(X, \mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i \underbrace{\dim_k H^i(X, \mathcal{F})}_{\text{finite } (\leq r)}.$$

(by Serre's finiteness thm).

Lemma $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ short exact

$\rightsquigarrow$ $\chi(\mathcal{G}) = \chi(\mathcal{F}) + \chi(\mathcal{H})$.

(use the $H^i(\cdot)$ - long exact sequence to prove).

Cor $0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$ exact of f.g. Qcohs.

$\rightsquigarrow$ $\sum_{i=1}^{n} (-1)^i \chi(X, \mathcal{F}_i) = 0$.

Thm $\exists P(z) \in \mathbb{Q}[z]$ poly s.t. $\underline{\chi(X, \mathcal{F}(n)) = P(n)}$, $\forall n \in \mathbb{Z}$

Morover, $\deg P = \deg \chi(X, \mathcal{F}) = \dim X$.

Proof. (Dévessage)

Replacing $\mathcal{F}$ by $j_*\mathcal{F}$ ($j : X \to \mathbb{P}^r_k$) $\rightsquigarrow$ reduce to $X = \mathbb{P}^r_k$.

b/c $H^i(X, \mathcal{F}) \cong H^i(\mathbb{P}^r_k, j_*\mathcal{F})$ under closed imm.

Also, may assume $k = \bar{k}$

b/c changing k doesn't change any dim

(e.g. by looking at Čech cohom,
this is a special case of <u>flat base change thm</u>)

Induction on $\dim(\operatorname{supp}\mathcal{F})$.

① $\operatorname{supp}\mathcal{F} = \phi$ i.e. $\mathcal{F} \equiv 0 \Rightarrow P(z) = 0$ works.

② $0 \to \mathcal{G} \to \mathcal{F}(-1) \xrightarrow{\times x_r} \mathcal{F} \to \mathcal{H} \to 0$ exact

$\dim(\operatorname{supp}\mathcal{G}) \le \dim(\operatorname{supp}\mathcal{F}) \ge \dim(\operatorname{supp}\mathcal{H})$

$\Rightarrow H \cap \operatorname{supp}\mathcal{F} = \phi$, $H: x_r = 0$.

Ind hyp: $\chi(\mathbb{P}^r_k, \mathcal{F}(n)) - \chi(\mathbb{P}^r_k, \mathcal{F}(n-1)) = p(n)$

$\deg p \le \dim(\operatorname{supp}\mathcal{F}) - 1$

elementary alg. $\Rightarrow \chi(\mathbb{P}^r_k, \mathcal{F}(n)) = q(n)$, $\deg q \le \dim(\operatorname{supp}\mathcal{F})$. □

<u>Def'n</u> $P(n) =$ Hilb poly of $\mathcal{F}$.
When $\mathcal{F} = \mathcal{O}_X$: $P(n) =$ Hilb poly of X.

<u>Note</u> By Serre vanishing ($H^i(X, \mathcal{O}_X(n)) = 0$, $n \gg 0$, $i > 0$)

$P(n) = \dim_k H^0(X, \mathcal{F})$, $n \gg 0$.

(original def'n of Hilb poly).

E.g. (1) For $\mathbb{P}^r_k$, $P(n) = \binom{n+r}{n} = \chi(\mathbb{P}^r_k, \mathcal{O}(n))$.

(2) For $\operatorname{Spec} k[x]/(x^2) \hookrightarrow \mathbb{P}^1_k$, $P(n) = 2$.

(For $X =$ two reduced pts, $P(n) = 2$).

can be written as a flat lim of pairs of distinct pts

<u>§2 Flatness and Hilbert Polynomials</u>

Numerical Criterion for Flatness:

Thm T $\underbrace{\text{integral}}_{\text{can be replaced by conn't red.}}$, $\underbrace{\text{loc noe}}_{\text{essential}}$ sch. $X \subseteq \mathbb{P}^r_T$ closed subsch.

$\mathcal{F} \in \mathrm{Coh}(X)$.

$\kappa(t) = \mathcal{O}_{T,t}/\mathfrak{m}_{T,t}$ res field at t.

$\forall t \in T$, $P_t \in \mathbb{Q}[z]$ Hilb poly of $j_t^{-1}\mathcal{F}$, $j_t : X_t \longrightarrow \mathbb{P}^r_{\kappa(t)}$.

<u>Then</u> $\mathcal{F}$ flat relative to $X \to T$

$\Leftrightarrow$ $P_t \equiv$ const in t.

<u>In particular</u> X flat over T $\Leftrightarrow$ Hilb poly of X_t is const in t.

Proof. (c.f. Hartshorne Thm III.9.9, or EGA III §7.9).

Note: (1) Can reduce to $X = \mathbb{P}^r_T$ by $\mathcal{F} \rightsquigarrow i_*\mathcal{F}$, $i : X \subseteq \mathbb{P}^n_T$.

(2) it suffices to consider $T = \mathrm{Spec}\, A$,

A local int noe ring.

<u>Step 1</u> To show that $\mathcal{F}$ flat over T

$\Leftrightarrow$ $H^0(X, \mathcal{F}(m))$ fin free over A, $m \gg 0$.

($\Rightarrow$) $\mathcal{F}$ flat over T $\Rightarrow$ so are $\check{C}^i(\mathcal{U}, \mathcal{F}(m))$,

b/c open imms are flat.

$\overset{\Gamma(X,-)}{\rightsquigarrow}$ $0 \to H^0(X, \mathcal{F}(m)) \to \check{C}^0(\mathcal{U}, \mathcal{F}(m)) \to \cdots \to \check{C}^r(\mathcal{U}, \mathcal{F}(m)) \to 0$

are all flat except possibly for H^0-term.

$\Rightarrow$ $H^0(X, \mathcal{F}(m))$ flat as well.

By Serre's finiteness, $H^0(X, \mathcal{F}(m))$ f.g. over A, $\boxed{m \gg 0}$

$\Rightarrow$ $H^0(X, \mathcal{F}(m))$ free.

($\Leftarrow$) Pick $m_0 \gg 0$ s.t. $H^0(X, \mathcal{F}(m))$ fin free $/A$, $\forall m \geq m_0$.

$\rightsquigarrow$ can recover $\mathcal{F}$ as $\tilde{M}$ for

$$M = \bigoplus_{m \geq m_0} H^0(X, \mathcal{F}(m)) \quad (\text{flat})$$

$\Rightarrow \mathcal{F} = \widetilde{M}$ flat.

Step 2 To show $H^0(X, \mathcal{F}(m))$ fin. free, $m \gg 0 \iff P_t = $ const.

Claim: it follows by checking

$$H^0(X_t, \mathcal{F}_t(m)) = H^0(X, \mathcal{F}(m)) \otimes_A \kappa(t).$$

for $m \gg 0$. even I don't prove this uniformly in t.

Namely, $H^0(X, \mathcal{F}(m))$ fin free $/A$, $\forall t$, $m \gg 0$,

$\rightsquigarrow P_t = P_\eta$, $\eta \in T$ generic pt. (also the converse).

May reduce to $t \in T$ closed pt

by replacing A by $\mathcal{O}_{T,t}$.

A noe $\rightsquigarrow$ short exact seq

$$A^{\oplus n} \to A \to \kappa(t) \to 0 \quad \text{of } A\text{-Mods}.$$

$(\cdot) \otimes \mathcal{F} \rightsquigarrow H^0(X, \mathcal{F}(m)^{\oplus n}) \to H^0(X, \mathcal{F}(m)) \to H^0(X, \mathcal{F}_t(m)) \to 0, \quad m \gg 0$

can be pulled out ... not X_t (!) $\square$

§3 Hilbert Schemes

$\exists$ univ family of closed schs of $\mathbb{P}^r$ with a fixed Hilb poly.

Thm (Grothendieck) Fix k field, $r \in \mathbb{Z}$. $P(z) \in \mathbb{Q}[z]$.

$\Rightarrow \exists H \in \mathrm{Sch}_k^{noe}$ & $X \subseteq \mathbb{P}^r_H$ closed subsch,

flat with Hilb poly P.

s.t. $\forall T \in \mathrm{Sch}_k^{noe}$, $Y \subseteq \mathbb{P}^r_T$ closed subsch which is

flat with Hilb poly P,

$\exists!$ $T \to H$ s.t. $Y = X \times_H T$ as closed subschs

of $\mathbb{P}^r_T \cong \mathbb{P}^r_H \times_H T$.

e.g. One can show that the closed subsch

$H \subseteq \mathbb{P}^r_k$ is a plane of dim $= d$
$\Leftrightarrow$ Hilb poly of $H = P(n) = \binom{n+d-1}{n}$ $\Bigg\}$ parameter sch $= \underset{\text{dim } d \text{ planes in } \mathbb{P}^r_k.}{\underset{\uparrow}{Gr(d, r)}}$

§4 Hilbert Polynomials, Degree, and Dimension

Lemma $P(z) =$ Hilb poly of closed subsch $X \subseteq \mathbb{P}^n_k$.

(a) $\deg P = \dim X$

(b) Put $d = \dim X$. $\forall P \subseteq \mathbb{P}^n_k$ d-dim'l plane s.t. $\dim(X \cap P) = 0$,

$\Rightarrow$ length $(X \cap P) = d! \cdot \underset{\text{leading coeff of } P(z).}{\underset{\uparrow}{a_d}}$

Proof. May assume $k = \bar{k}$.

① For P generic d-dim'l plane, $\dim(X \cap P) = 0$.
i.e. chosen outside some closed subsch of $Gr(d, n)$.

$\Leftarrow$ $\forall H$ generic hyperplane, $\dim(X \cap H) < \dim X$.

② Put $\mathcal{F} = j_* \mathcal{O}_X$, $j : X \to \mathbb{P}^n_k$ closed imm.

$\forall H$ hyper s.t. $\dim(X \cap H) < \dim X$, we have

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \underset{\text{direct im of } \mathcal{O}_{X \cap H}.}{\underset{\uparrow}{\mathcal{G}}} \to 0$$

Hilbert polys : $P(z) \leftrightarrow \chi(\mathbb{P}^n_k, \mathcal{F}(m))$

$P(z-1) \leftrightarrow \chi(\mathbb{P}^n_k, \mathcal{F}(-1)(m))$

$\rightsquigarrow$ $P(z) - P(z-1) \leftrightarrow \chi(\mathbb{P}^n_k, \mathcal{G}(m))$.

$\Rightarrow$ both claims follow. $\square$