Special divisors in toroidal compactifications

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§1 Borcherds higher dim GKZ

Let Λ an even lattice of sgn $(2,n)$.

Consider the (complex) Shimura var

$$X = \Gamma \backslash O(2,n) / O(2)\cdot O(n)$$

where $\Gamma \subseteq \mathrm{Aut}(\Lambda)$ arith subgrp

- $\forall m \in \mathbb{Q}_{<0}$, $\gamma \in \Lambda^\vee/\Lambda$ set

$$C_{m,\gamma} := \bigcup_{\substack{v \in \Lambda^\vee \\ [v,v] = -2m \\ [v] = \gamma \in \Lambda^\vee/\Lambda}} \Gamma \backslash v^\perp \rightsquigarrow [C_{m,\gamma}] \in CH^1(X, \mathbb{Q})$$

- $[C_{0,0}] = c_1(\omega) \in CH^1(X, \mathbb{Q})$

inverse Hodge bdl.

Thm I (Borcherds)

$$\Phi(q) = \sum_{m,\gamma} [C_{m,\gamma}]\, q^m \cdot e_\gamma \in CH^1(X) \otimes_{\mathbb{Q}} \mathbb{Q}[[q]] \otimes \mathbb{Q}[\Lambda^\vee/\Lambda]$$

is a vector-valued mod form

valued in $CH^1(X)_{\mathbb{Q}}$

w.r.t. Weil rep $Mp_2(\mathbb{Z}) \otimes \mathbb{Q}[\Lambda^\vee/\Lambda]$

of weight $\frac{n+2}{2}$.

Cor If $[B] \in H_2(X)$, then

$$\underbrace{\Phi(q)}_{\substack{\cap \\ CH^1(X)_{\mathbb{Q}}}} \cdot [B] \in \mathrm{Mod\,Form}\left(\tfrac{n}{2}+1, \mathbb{Q}[\Lambda^\vee/\Lambda]\right)$$

e.g. if $\begin{matrix}\mathcal{X} \\ \downarrow \\ B\end{matrix}$ sm family of K3 surfaces over proper curve.

Q What if $\begin{matrix}\mathcal{X} \\ \downarrow \\ B\end{matrix}$ has singular fibres?

§2 Compactification of X

$$X \subseteq X^{BB} = X \amalg \coprod_{\{J\}/\Gamma} (\text{mod curves}) \amalg \coprod_{\{I\}/\Gamma} \text{pts}$$

J = isotropic plane in $\Lambda_{\mathbb{Q}}$.

I = isotropic line in $\Lambda_{\mathbb{Q}}$.

$\varepsilon \uparrow$

X^Σ = toroidal cpt'n

$$= X \amalg \coprod_{\{J\}/\Gamma} \Delta_J \amalg \coprod_{\{I\}/\Gamma} \coprod_{c} \Delta_{I,c}.$$

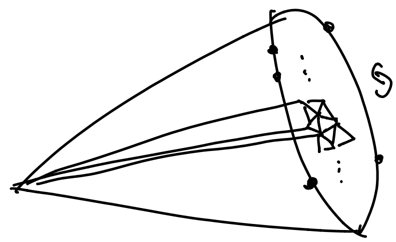
Σ is a choice of <u>admissible fan</u>:

$\forall I$ isotropic line, $I^\perp/I$ is a hyperbolic lattice of sgn $(1, n-1)$

Σ = rat'l polyhedral cone decomp'n on $\overline{\mathcal{C}}^+ \subseteq (I^\perp/I) \otimes \mathbb{R}$
(rat'l closure)

s.t. ① $\mathrm{Stab}_\Gamma(I)$ preserves it

② $\mathrm{Stab}_\Gamma(I) \curvearrowright$ Cones w/ finite orbits.



c = ray through some vertex of Σ.

⇝ $\Delta_{I,c}$.

We form the naive generating series by taking Zar closure of $C_{m,\gamma}$ in X^Σ (as $\bar{C}_{m,\gamma}$):

$$\Phi^{?}(q) = \sum_{m,\gamma} [\bar{C}_{m,\gamma}]\, q^m e_\gamma$$

<u>Thm II</u> (EGT) $\quad X^{\mathrm{II}} := X \sqcup \coprod_{\{J\}/\Gamma} \Delta_J$ partial cpt'n.

⇝ $\Phi^{\mathrm{II}}(q) := \sum_{m,\gamma} [\bar{C}_{m,\gamma}]\, q^m \cdot e_\gamma \in CH^1(X^{\mathrm{II}}) \otimes \mathbb{Q}[\Lambda^\vee/\Lambda][[q]]$.

↑ Zar closure in X^{II}.

is a vec-valued quasi-mod form valued in $CH^1(X)_{\mathbb{Q}}$

w.r.t. Weil rep $Mp_2(\mathbb{Z}) \circlearrowright \mathbb{Q}[\Lambda^\vee/\Lambda]$ of wt $\frac{n+2}{2}$.

f quasi-mod means

$f = \sum f_j \cdot E_2(q)^j$ w/ f_j modular form.

Fact X^{II} is sm & quasi-proj
but not proper.

$$-12 \cdot \frac{\partial}{\partial E_2} \Phi^{\mathrm{II}}(q) = \sum_{\{J\}/\Gamma} \Delta_J \otimes \Theta_{J^\perp/J}(q^{-1})$$

definite lattice

classical Θ-func.

Thm II (EGT)

$$\Phi^{\Sigma}(q) \in CH^1(X^{\Sigma}) \otimes \mathbb{Q}[\Lambda^\vee/\Lambda] \otimes \underbrace{\mathrm{MMMod}}\left(\tfrac{n+2}{2}\right).$$

mixed mock-modular forms

Def A mixed mock modular form is $\sum f_i g_i$

f_i : mod form

g_i : weak modular, i.e. the holo part of a weak harmonic Maass form.

Operator of Zagier-Shallow ?:

$$\xi_{3/2} \cdot \Phi^{\Sigma}(q) = \sum_{\{I\}/\Gamma} \sum_{c} A_{I,c} \otimes p\left(\Theta_{c^\perp}(q^{-1}) \otimes \Theta_{\mathbb{Z}c}(q)\right)$$

$$(\xi_k : H_k \longrightarrow M_{2-k}).$$

Thm II$'$ $\Phi^{\Sigma}(q) = \sum_{m,\gamma} [\bar{C}_{m,\gamma}] \cdot q^m e_\gamma.$

$$\Rightarrow \Phi^{\Sigma}(q) = \frac{1}{12} \sum_{J} \Delta_J \otimes E_2(q) \otimes \Theta_{J^\perp/J}(q)$$

$$- \frac{1}{2} \sum_I \sum_c \Delta_{I,c} \otimes \rho(\Theta_{c^\perp}(q) \times F_N^+)$$

$$\in \mathrm{Mod}\left(\tfrac{n}{2}+1, \mathbb{Q}[\Lambda^\vee/\Lambda]\right)$$

F_N^+ is a mock mod form

& s.t. $\xi_{3/2}(F_N) = \Theta_{\mathbb{Z}c}(q) \quad (c^2 = 2N)$.

Pf sketch $n \geq 3$ or $n=2$ & Γ is irred,

Margulis' super rigidity $\Rightarrow H^1(X, \mathbb{Q}) = 0$, $H^1(X^\Sigma, \mathbb{Q}) = 0$.

$\Rightarrow \mathrm{Pic}^0(X^\Sigma) = 0$.

$$c_1 : CH^1(X^\Sigma) \xrightarrow{\sim} H^{1,1}(X^\Sigma, \mathbb{Z}).$$

By Poincaré duality:

it suffices to $\Phi^\Sigma(q) \cdot \mu$, $\mu \in H_2(X^\Sigma)$.

Lem $\forall \mu \in H_2(X^\Sigma)$, μ splits relative to the boundary

i.e. $\mu = \mu_0 + \mu_1$ w/ $\mu_0 \in H_2(X)$,

$$\mu_1 \in \bigoplus_J H_2(\Delta_J) \otimes \bigoplus_{I,c} H_2(\Delta_{I,c})$$

$\varepsilon_* : X^{\mathrm{II}} \longrightarrow X^{BB}$ (blow-down map)

actually the $H_2(\Delta_J) \in \ker \varepsilon_*$ components.

Exercise in spectral sequences

Use a moment map for an infinite type toric variety.

$$\Rightarrow \quad \Phi^{\Sigma}(q)\cdot \mu = \underbrace{\Phi^{\Sigma}(q)\cdot\mu_0}_{\text{modular form by Borcherds}} + \underbrace{\Phi^{\Sigma}(q)\cdot\mu_1}_{\text{compute explicitly}}.$$

§3 Geometry of the boundary

For each isotropic plane $J \subseteq \Lambda_{\mathbb{C}}$,

$$\begin{array}{ccc} \text{open part } \overset{\circ}{\Delta}_J & \cong & \mathcal{E}^{univ} \\ \downarrow \pi & \swarrow & \\ \text{modular curve} & & \end{array}$$

family of sm AVs.

The special divisors $C_{m,\gamma}$ meet $\overset{\circ}{\Delta}_J$ transversally in a codim 1 sub-AV.

$j : E^{n-2} \hookrightarrow X^{\mathrm{II}}$

$$\rightsquigarrow \quad \Phi^{\mathrm{II}}(q)\cdot\mu_1 = \sum_{x\in J^{\perp}/J} f(x)\cdot q^{\boxed{\ }}$$

depending on x.

linear algebra on E^{n-2} (power of EC)

$f(x)$ is quadratic on x.

$\mu_1 \in H_2(E^{n-2})$

$$q\frac{d}{dq}\Theta = E_2\cdot\Theta + (\text{modular form}).$$

For each I isotropic line, c a ray in fan Σ,

$A_{I,c}$ = toric variety def'd by $(I^\perp/I)_{\mathbb{R}}/\mathbb{R}\cdot c$

$\rightsquigarrow$ $H_2(A_{I,c})$ gen'd by $\mathbb{P}^1$'s in the toric boundary.

$$\Phi^\Sigma(q)\cdot \varphi_1 = \sum_{x\in I^\perp/I} g(x)\cdot q^{(x,x)/2}$$

w/ $g(x)$ piecewise linear