Topics in Number Theory: 2020 Fall
Final Exam — Wenhan Dai

1. **Problem 1.**

(1) Kummer theory outputs a short exact sequence

$$1 \longrightarrow \mu_\ell \longrightarrow \mathbb{Q}_p^{\mathrm{sep},\times} \xrightarrow{x \mapsto x^\ell} \mathbb{Q}_p^{\mathrm{sep},\times} \longrightarrow 1,$$

after taking Galois cohomology it turns into

$$\mathbb{Q}_p^\times \xrightarrow{x \mapsto x^\ell} \mathbb{Q}_p^\times \longrightarrow H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \mu_\ell) \longrightarrow H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \mathbb{Q}_p^{\mathrm{sep},\times}) = 0,$$

where the last item comes to zero by Hilbert 90. Thus $H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \mu_\ell) \cong \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^\ell$.

(2) One would compute items on two sides as

$$H^1(\mathrm{Gal}_{\mathbb{F}_p}, \mu_\ell) = H^1((\mathrm{Frob}_p)^{\widehat{\mathbb{Z}}}, \mu_\ell) = \frac{\mu_\ell}{(\mathrm{Frob}_p - 1)\mu_\ell},$$

and

$$H^1(I_{\mathbb{Q}_p}, \mu_\ell)^{\mathrm{Frob}_p} \cong \mathrm{Hom}_{\mathbf{gp}}(I_{\mathbb{Q}_p}, \mu_\ell)^{\mathrm{Frob}_p} \cong \mathrm{Hom}(\mathbb{Z}_\ell(1), \mu_\ell)^{\mathrm{Frob}_p} = \mu_\ell(-1)^{\mathrm{Frob}_p}.$$

So the canonical exact sequence looks like

$$0 \longrightarrow \frac{\mu_\ell}{(\mathrm{Frob}_p - 1)\mu_\ell} \longrightarrow \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^\ell \longrightarrow \mu_\ell(-1)^{\mathrm{Frob}_p} \longrightarrow 0.$$

(3) Let $\mu_\ell(\mathbb{Q}_p) := \{x \in \mathbb{Q}_p : x^\ell = 1\}$. The previous exact sequence shows that when $\ell \neq p$, we have

$$\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^\ell \cong \mathbb{F}_\ell \oplus \mu_\ell(\mathbb{Q}_p).$$

Then the required dimension over $\mathbb{F}_\ell$ is

$$\dim H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \mu_\ell) = \begin{cases} 2, & \text{if } p \equiv 1 \bmod \ell \\ 1, & \text{otherwise.} \end{cases}$$

We then use Tate local duality with $\mathbb{F}_\ell^*(1) \cong \mu_\ell$ to deduce that

$$\begin{aligned} \dim H^2(\mathrm{Gal}_{\mathbb{Q}_p}, \mu_\ell) &= \dim H^2(\mathrm{Gal}_{\mathbb{Q}_p}, \mathbb{F}_\ell^*(1))^* = \dim H^0(\mathrm{Gal}_{\mathbb{Q}_p}, \mathbb{F}_\ell) \\ &= \dim \mathbb{F}_\ell^{\mathrm{Gal}_{\mathbb{Q}_p}} = 1. \end{aligned}$$

On the other hand,

$$\dim H^0(\mathrm{Gal}_{\mathbb{Q}_p}, \mu_\ell) = \dim \mu_\ell(\mathbb{Q}_p) = \begin{cases} 1, & \text{if } p \equiv 1 \bmod \ell \\ 0, & \text{otherwise.} \end{cases}$$

So the Euler-characteristic formula at $\ell \neq p$ is consequently verified by

$$\begin{aligned} \chi(\mu_\ell) &= \dim H^0(\mathrm{Gal}_{\mathbb{Q}_p}, \mu_\ell) - \dim H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \mu_\ell) + \dim H^2(\mathrm{Gal}_{\mathbb{Q}_p}, \mu_\ell) \\ &= \dim \mu_\ell(\mathbb{Q}_p) - \dim \mathbb{F}_\ell \oplus \mu_\ell(\mathbb{Q}_p) + 1 = 0. \end{aligned}$$

2. **Problem 2.**

(1) Using the similar argument in Problem 1, we get

$$H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \chi_{\mathrm{cycl}}) = \varprojlim_{n\in\mathbb{N}} H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \mu_{\ell^n}) \cong \varprojlim_{n\in\mathbb{N}} \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^{\ell^n}.$$

However, $p \not\equiv 1 \bmod \ell$ implies that $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^{\ell^n} \cong \mathbb{Z}/\ell^n\mathbb{Z}$, which blots out the torsion of $H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \chi_{\mathrm{cycl}})$. Considering $\chi_{\mathrm{cycl}} : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathbb{Z}_\ell^\times$ as a $\mathbb{F}_\ell[\mathrm{Gal}_{\mathbb{Q}_p}]$-module, the result in part (3) of Problem 1 does become

$$\dim_{\mathbb{F}_\ell} H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \chi_{\mathrm{cycl}}) = \mathrm{rank}_{\mathbb{Z}_\ell} H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \chi_{\mathrm{cycl}}) = \begin{cases} 2, & \text{if } p \equiv 1 \bmod \ell \\ 1, & \text{otherwise.} \end{cases}$$

Thus $H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \chi_{\mathrm{cycl}}) \cong \mathbb{Z}_\ell$ is free of rank one over $\mathbb{Z}_\ell$.

(2) Fix a geometric Frobenius Frob_p and a tame generator τ in $G_{\mathbb{Q}_p}/P_{\mathbb{Q}_p} \cong \widehat{\mathbb{Z}}^{(p)}(1) \rtimes \widehat{\mathbb{Z}}$, which is the quotient of the Galois group by the wild inertia subgroup. The relation $\mathrm{Frob}_p^{-1}\,\tau\,\mathrm{Frob}_p = \tau^p$ says that the subspace of fixed vectors of $E(\tau)$ is invariant under the action of $E(\mathrm{Frob}_p)$. $E|_{P_{\mathbb{Q}_p}}$ is trivial (under some twist if necessary), and $E|_{I_{\mathbb{Q}_p}}$, which is stretched out from $\chi_{\mathrm{cycl}}|_{I_{\mathbb{Q}_p}/P_{\mathbb{Q}_p}}$, must be an extension of the trivial representation by the trivial representation.

(3) Let (r, N) be the Weil–Deligne representation attached to $E \otimes \mathbb{Q}_\ell$. From (2) the $G_{\mathbb{Q}_p}/P_{\mathbb{Q}_p}$-action of $E \otimes \mathbb{Q}_\ell$ is given by

$$(E \otimes \mathbb{Q}_\ell)(\mathrm{Frob}_p) = \begin{pmatrix} 1 & \\ & p \end{pmatrix}, \quad (E \otimes \mathbb{Q}_\ell)(\tau) = \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}.$$

Thus for m sufficiently divisible, by Grothendieck monodromy theorem

$$N = \frac{1}{m} \log \tau^m = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}.$$

Suppose $a \in \mathbb{Z}$ and $x \in I_{\mathbb{Q}_p}$, and by definition $r : \mathrm{Gal}_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_2(\mathbb{C})$ satisfies

$$r(\mathrm{Frob}_p^a\, x) = (E \otimes \mathbb{Q}_\ell)(\mathrm{Frob}_p^a\, x) \cdot \exp(-t_{\zeta,\ell}(x)N).$$

Note that $\exp(-t_{\zeta,\ell}(\tau)N) = 1 - N$. Taking $(a, x) = (0, \mathrm{Frob}_p)$ and $(a, x) = (0, \tau)$ respectively, we finally get

$$r(\mathrm{Frob}_p) = \begin{pmatrix} 1 & \\ & p \end{pmatrix}, \quad r(\tau) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}.$$

3. **Problem 3.**

Compute the universal representation $\rho^{\text{univ}} : \mathrm{Gal}_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_2(R_{\bar\rho}^{\square,\chi})$ as follows:

$$\rho^{\text{univ}}(\mathrm{Frob}_p) = \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix} \begin{pmatrix} 1+Z & \\ & p+Z \end{pmatrix} \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix}^{-1}$$

The condition $\mathrm{Frob}_p^{-1}\,\tau\,\mathrm{Frob}_p = \tau^p$ forces τ to be the form:

$$\rho^{\text{univ}}(\tau) = \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix} \begin{pmatrix} 1+A & B \\ & 1+D \end{pmatrix} \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix}^{-1}$$

with

$$\begin{pmatrix} 1+Z & \\ & p+Z \end{pmatrix}^{-1} \begin{pmatrix} 1+A & B \\ & 1+D \end{pmatrix} \begin{pmatrix} 1+Z & \\ & p+Z \end{pmatrix} = \begin{pmatrix} 1+A & B \\ & 1+D \end{pmatrix}^p,$$

which leads to $(1+Z)^{-1}(1+A)(1+Z) = (1+A)^p$ or equivalently $(1+A)^{p-1} = 1$. However, $p \not\equiv 1 \bmod \ell$ and then $A = 0$. The same argument shows that $D = 0$. Finally we obtain $(p+Z)(1+Z)^{-1}B = pB$, so $BZ = 0$. Hence

$$\begin{aligned} \rho^{\text{univ}} : \mathrm{Gal}_{\mathbb{Q}_p} &\longrightarrow \mathrm{GL}_2(R_{\bar\rho}^{\square,\chi}) \\ \mathrm{Frob}_p &\longmapsto \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix} \begin{pmatrix} 1+Z & \\ & p+Z \end{pmatrix} \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix}^{-1} \\ \tau &\longmapsto \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix} \begin{pmatrix} 1 & B \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix}^{-1} \end{aligned}$$

and $R_{\bar\rho}^{\square,\chi} \cong \mathcal{O}[\![X, Y, Z, B]\!]/(BZ)$.

The natural surjections, which are $R_{\bar\rho}^{\square,\chi} \to R_{\bar\rho}^{\square,\chi,\text{ur}}$ by taking $B = 0$, and $R_{\bar\rho}^{\square,\chi} \to R_{\bar\rho}^{\square,\chi,\text{St}}$ by taking $Z = 0$, induces two formally smooth irreducible components:

$$\mathrm{Spec}\, R_{\bar\rho}^{\square,\chi} = \mathrm{Spec}\, R_{\bar\rho}^{\square,\chi,\text{ur}} \cup \mathrm{Spec}\, R_{\bar\rho}^{\square,\chi,\text{St}}.$$

Therefore, ρ^{univ} goes to be unframed over the unramified subspace because of $\tau = 1$, and $\mathrm{tr}(\rho(\mathrm{Frob}_p)) = p + 1$ over the Steinberg subspace.

4. **Problem 4.**

The group $H^0(\mathrm{Gal}_{\mathbb{Q}_p}, \mathrm{Ad}^\circ \bar{\rho}(1))$ is said to be nontrivial if $\bar{\rho}$ is the sum of characters $\mathbf{1} \oplus \bar{\chi}_{\mathrm{cycl}}^{-1}$. Moreover, taking advantage of the computations in Problem 1, in this case

$$\dim H^2(\mathrm{Gal}_{\mathbb{Q}_p}, \mathrm{Ad}^\circ \bar{\rho}) = \dim H^0(\mathrm{Gal}_{\mathbb{Q}_p}, \mathrm{Ad}^\circ \bar{\rho}(1)) = 1$$

by Tate local duality. We next consider the exact sequence as follows

$$0 \longrightarrow H^0(\mathrm{Gal}_{\mathbb{Q}_p}, \mathrm{Ad}^\circ \bar{\rho}) \longrightarrow \mathrm{Ad}^\circ \bar{\rho} \longrightarrow Z^1(\mathrm{Gal}_{\mathbb{Q}_p}, \mathrm{Ad}^\circ \bar{\rho}) \longrightarrow H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \mathrm{Ad}^\circ \bar{\rho}) \longrightarrow 0$$

in which $\dim \mathrm{Ad}^\circ \bar{\rho} = 2^2 - 1 = 3$. Consequently, by the Euler-characteristic formula,

$$\begin{aligned} \dim Z^1(\mathrm{Gal}_{\mathbb{Q}_p}, \mathrm{Ad}^\circ \bar{\rho}) &= \dim H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \mathrm{Ad}^\circ \bar{\rho}) + \dim \mathrm{Ad}^\circ \bar{\rho} - \dim H^0(\mathrm{Gal}_{\mathbb{Q}_p}, \mathrm{Ad}^\circ \bar{\rho}) \\ &= \dim H^2(\mathrm{Gal}_{\mathbb{Q}_p}, \mathrm{Ad}^\circ \bar{\rho}) + \dim \mathrm{Ad}^\circ \bar{\rho} \\ &= 4. \end{aligned}$$

Then because of $Z^1(\mathrm{Gal}_{\mathbb{Q}_p}, \mathrm{Ad}^\circ \bar{\rho}) \cong \mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}_{\bar{\rho}}^{\square,\chi}/((\mathfrak{m}_{\bar{\rho}}^{\square,\chi})^2, \varpi), \mathbb{F})$, the tangent space of $\mathrm{Spec}\, R_{\bar{\rho}}^{\square,\chi}$ has dimension 4. From another point of view, it is essentially deduced by the smoothness of $R_{\bar{\rho}}^{\square,\chi}$ in the previous problem. On the other hand, the minimal cardinality of the set of relations should be $\dim H^2(\mathrm{Gal}_{\mathbb{Q}_p}, \mathrm{Ad}^\circ \bar{\rho}) = 1$.

5. **Problem 5.**

(1) As $\pi_*\mathcal{O}_X$ is a finite locally free sheaf over Y there exists a canonical trace for π which is an $\mathcal{O}_Y$-linear map $\pi_*\mathcal{O}_X \to \mathcal{O}_Y$, sending a local section $f \in \Gamma(Y, \pi_*\mathcal{O}_X)$ to the trace of multiplication by f on $\pi_*\mathcal{O}_X$. Simultaneously, for any given section $s \in \Gamma(Y, \mathcal{L})$ we even construct another map $\mathcal{O}_Y \to \mathcal{O}_Y \otimes_{\mathcal{O}_Y} \mathcal{L}$ also by multiplication by s. The composition of these two maps induces that

$$H^0(Y, \pi_*\mathcal{O}_X) \longrightarrow H^0(Y, \mathcal{L}).$$

Both this and the same argument over X essentially help to lift the underlying trace map $H^0(X, \mathcal{O}_X) \to H^0(Y, \pi_*\mathcal{O}_X)$ to the needed natural trace map

$$\mathrm{Tr} : H^0(X, \pi^*\mathcal{L}) \longrightarrow H^0(Y, \mathcal{L}).$$

(2) Using p-stabilization process, we compute that U_p acts on $S_2(\Gamma_1(N); \mathbb{F}_\ell)^{\oplus 2}_{\mathfrak{m}_{\bar\rho}}$ as matrix in the following:

$$(U_p)^{\oplus 2}_{\mathfrak{m}_{\bar\rho}} = \begin{pmatrix} T_p & p \\ -p(\langle p\rangle^*)^{-1} & 0 \end{pmatrix} : S_2(\Gamma_1(N); \mathbb{F}_\ell)^{\oplus 2}_{\mathfrak{m}_{\bar\rho}} \longrightarrow S_2(\Gamma_1(N); \mathbb{F}_\ell)^{\oplus 2}_{\mathfrak{m}_{\bar\rho}}$$

By newform theory, the stability of the old subspace and the new subspace under Hecke operators deduces that the following diagram commutes:

$$\begin{tikzcd}
S_2(\Gamma_1(N); \mathbb{F}_\ell)^{\oplus 2}_{\mathfrak{m}_{\bar\rho}} \arrow[r] \arrow[d, "(U_p)^{\oplus 2}_{\mathfrak{m}_{\bar\rho}}"'] & S_2(\Gamma_1(N) \cap \Gamma_0(p); \mathbb{F}_\ell)_{\mathfrak{m}_{\bar\rho}} \arrow[r] \arrow[d, "U_p"'] & S_2(\Gamma_1(N); \mathbb{F}_\ell)^{\oplus 2}_{\mathfrak{m}_{\bar\rho}} \arrow[d, "(U_p)^{\oplus 2}_{\mathfrak{m}_{\bar\rho}}"'] \\
S_2(\Gamma_1(N); \mathbb{F}_\ell)^{\oplus 2}_{\mathfrak{m}_{\bar\rho}} \arrow[r] & S_2(\Gamma_1(N) \cap \Gamma_0(p); \mathbb{F}_\ell)_{\mathfrak{m}_{\bar\rho}} \arrow[r] & S_2(\Gamma_1(N); \mathbb{F}_\ell)^{\oplus 2}_{\mathfrak{m}_{\bar\rho}}
\end{tikzcd}$$

in which two horizontal maps are given by $(\pi_{1,*} \oplus \pi_{2,*}) \circ (\pi_1^*, \pi_2^*)$. After choosing the basis $f(z), f(pz)$ in p-stabilization process, the composition

$$(\pi_{1,*} \oplus \pi_{2,*}) \circ (\pi_1^*, \pi_2^*) \circ (U_p)^{\oplus 2}_{\mathfrak{m}_{\bar\rho}} = (U_p)^{\oplus 2}_{\mathfrak{m}_{\bar\rho}} \circ (\pi_{1,*} \oplus \pi_{2,*}) \circ (\pi_1^*, \pi_2^*)$$

gets a unique expression given by matrix. Whereas we check explicitly that

$$\begin{aligned}
&\begin{pmatrix} T_p & p \\ -p(\langle p\rangle^*)^{-1} & 0 \end{pmatrix} \begin{pmatrix} p+1 & T_p \\ T_p \circ (\langle p\rangle^*)^{-1} & p+1 \end{pmatrix} \\
=& \begin{pmatrix} (p+1)T_p - p(\langle p\rangle^*)^{-1} T_p & p(p+1) \\ T_p \circ (\langle p\rangle^*)^{-1} \circ T_p - p(p+1)(\langle p\rangle^*)^{-1} & pT_p \circ (\langle p\rangle^*)^{-1} \end{pmatrix} \\
=& \begin{pmatrix} p+1 & T_p \\ T_p \circ (\langle p\rangle^*)^{-1} & p+1 \end{pmatrix} \begin{pmatrix} T_p & p \\ -p(\langle p\rangle^*)^{-1} & 0 \end{pmatrix},
\end{aligned}$$

and therefore

$$(\pi_{1,*} \oplus \pi_{2,*}) \circ (\pi_1^*, \pi_2^*) = \begin{pmatrix} p+1 & T_p \\ T_p \circ (\langle p\rangle^*)^{-1} & p+1 \end{pmatrix}.$$

6. **Problem 6.**

As for the eigenvalues 1 and p of $\bar{\rho}(\mathrm{Frob}_p)$, let α_0 and β_0 be lifts of them to $\mathcal{O}$, respectively. The universal representation $\rho^{\mathrm{univ}} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(R_{\bar{\rho}}^{\chi}) = \mathrm{GL}(W)$ reduces to $\bar{\rho}$ modulo $\mathfrak{m}_{R_{\bar{\rho}}^{\chi}}$, where W is a free $R_{\bar{\rho}}^{\chi}$-module of rank 2 with a continuous action of $G_{\mathbb{Q},S}$. Suppose that the characteristic polynomial of $\rho^{\mathrm{univ}}(\mathrm{Frob}_p)$ is

$$\mathrm{char}(\rho^{\mathrm{univ}}(\mathrm{Frob}_p)) = (x - \alpha_0 - a)(x - \beta_0 - b)$$

for some $a, b \in \mathfrak{m}_{R_{\bar{\rho}}^{\chi}}$. By Hensel's Lemma it does have roots in $R_{\bar{\rho}}^{\chi}$ reducing to 1 and p. Using Cayley–Hamilton Theorem, there is a decomposition

$$W = (\rho^{\mathrm{univ}}(\mathrm{Frob}_p) - \alpha_0 - a)W \oplus (\rho^{\mathrm{univ}}(\mathrm{Frob}_p) - \beta_0 - b)W.$$

It is also crucial that $\alpha_0 + a, \beta_0 + b, \alpha_0 - \beta_0 + a - b$ are all invertible in $R_{\bar{\rho}}^{\chi}$. If $\bar{u}, \bar{v}$ is a basis of eigenvectors of $\rho^{\mathrm{univ}}(\mathrm{Frob}_p)$ in $V \otimes \mathbb{F}$ and u, v is a basis of V lifting $\bar{u}, \bar{v}$, then there are unique $X, Y \in \mathfrak{m}_{R_{\bar{\rho}}^{\chi}}$ such that

$$u + Xv, \quad v + Yu$$

are eigenvectors of $\rho^{\mathrm{univ}}(\mathrm{Frob}_p)$. Moreover, scalaring u, v to ku, kv for $k \in 1 + \mathfrak{m}_{R_{\bar{\rho}}^{\chi}}$ does not change our X and Y.

Therefore, by taking $\alpha = \alpha_0 + a \equiv \alpha_0 \equiv 1 \bmod \mathfrak{m}_{R_{\bar{\rho}}^{\chi}}$ and $\beta = \beta_0 + b \equiv \beta_0 \equiv p \bmod \mathfrak{m}_{R_{\bar{\rho}}^{\chi}}$ we obtain that

$$\rho^{\mathrm{univ}}(\mathrm{Frob}_p) = \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix} \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix}^{-1},$$

in which $v_+ = u + Xv$ and $v_- = Yu + v$. Next, we care about equations

$$\tau_p(v_-) = 0, \quad \tau_p(v_+) = Vv_-.$$

Using the similar argument in Problem 3 with the condition $\mathrm{Frob}_p^{-1} \tau_p \mathrm{Frob}_p = \tau_p^p$, we see that under the universal representation,

$$\rho^{\mathrm{univ}}(\tau_p) = \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix} \begin{pmatrix} 1+U & \\ V & 1+W \end{pmatrix} \begin{pmatrix} 1 & Y \\ X & 1 \end{pmatrix}^{-1},$$

and this gives the needed $V \in \mathfrak{m}_{R_{\bar{\rho}}^{\chi}}$.

The scheme where the universal deformation is unramified is the subspace cut by the condition $V = 0$, for both framed and unframed deformation rings. The natural restriction depicted in Problem 3 gives closed subscheme $\mathrm{Spec}\, R_{\bar{\rho}}^{\square_S,\chi}/(V) \cong \mathrm{Spec}\, R_{\bar{\rho}}^{\square_S,\chi,\mathrm{ur}} \hookrightarrow \mathrm{Spec}\, R_{\bar{\rho}}^{\square_S,\chi}$. As for unframed deformation rings, taking advantage from the image of $\rho^{\mathrm{univ}} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(R_{\bar{\rho}}^{\chi})$, we obtain

$$\mathrm{Spec}\, R_{\bar{\rho}}^{\chi,\mathrm{ur}} \cong \mathrm{Spec}\, R_{\bar{\rho}}^{\chi}/(V).$$

Thus $\mathrm{Spec}\, R_{\bar{\rho}}^{\chi,\mathrm{ur}} \hookrightarrow \mathrm{Spec}\, R_{\bar{\rho}}^{\chi}$ is a closed subscheme and the left part of the diagram is commutative Cartesian. On the other hand, under the local framed universal representation

$$\rho_p^{\mathrm{univ}} : \mathrm{Gal}_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_2(R_{\bar{\rho}_p}^{\square,\chi}),$$

the unramified local deformation ring is as follows:

$$R_{\bar{\rho}_p}^{\square,\chi,\mathrm{ur}} \cong R_{\bar{\rho}_p}^{\square,\chi}/(\rho_p^{\mathrm{univ}}(g) - 1; \forall g \in I_{\mathbb{Q}_p}).$$

View it in terms of schemes, $\mathrm{Spec}\, R_{\bar{\rho}_p}^{\square,\chi,\mathrm{ur}} \hookrightarrow \mathrm{Spec}\, R_{\bar{\rho}_p}^{\square,\chi}$ is also a closed subscheme. Then the right part is commutative Cartesian.

7. **Problem 7.**

First denote $J_\infty = \mathbb{Z}_\ell[\![z_1,\dots,z_r,w_1,\dots,w_{3\#S}]\!]$ and run the patching argument with using the notation defined in class. Let

$$S_m := S_2(\Gamma_1(N)\cap\Gamma_1^*(Q_m);\mathbb{Z}_\ell)^{\vee,\square_S}_{\mathfrak{m}_{\bar\rho}} \cong S_2(\Gamma_1(N)\cap\Gamma_1^*(Q_m);\mathbb{Z}_\ell)^{\vee}_{\mathfrak{m}_{\bar\rho}}[\![w_1,\dots,w_{3\#S}]\!].$$

For open idea $\mathfrak{a}\subset J_\infty$, we know that over S_m is free over $\mathbb{Z}_\ell[\Delta_{Q_m}]$ and $R^{\mathcal{D}}_{\mathrm{loc}}[\![y_1,\dots,y_r]\!]$ acts on $U_{\mathfrak{F}}(S_m/\mathfrak{a}S_m)$ in a $\mathbb{Z}_\ell[\Delta_{Q_m}]/\mathfrak{a}$-manner, where $\mathfrak{F}$ is a non-principal ultrafilter on $\mathbb{Z}_{>0}$. Then take inverse limit over $\mathfrak{a}$, we get

$$\mathcal{M}_\infty(\Gamma_0(N)) := \varprojlim_{\mathfrak{a}\subset J_\infty} U_{\mathfrak{F}}(S_m/\mathfrak{a}S_m),$$

and the similar way draws up $\mathcal{M}_\infty(\Gamma_0(Np))$.

We next consider the dimension of local deformation ring (with the crystalline part at $v=p$ included), which is

$$\dim R^{\square,\mathcal{D}_v}_{\bar\rho_v} = \begin{cases} 3, & v\neq \ell\infty \\ 4, & v=\ell \\ 2, & v=\infty. \end{cases}$$

So $R^{\mathcal{D}}_{\mathrm{loc}}$ is a power series ring in $3(\#S-2)+4+2=3\#S$ elements, and then

$$\mathrm{Krull}\dim R^{\mathcal{D}}_{\mathrm{loc}} = \mathrm{Krull}\dim \widehat{\bigotimes}_{v\in S} R^{\square,\mathcal{D}_v}_{\bar\rho_v} = 3\#S+1.$$

For checking the projective dimension of $\mathcal{M}_\infty(\Gamma_0(N))$, notice that

$$\mathrm{depth}_{J_\infty}(\mathcal{M}_\infty(\Gamma_0(N))) = 3\#S+1+r.$$

Yet applying Auslander-Buchbaum Theorem, we obtain

$$\mathrm{depth}_{J_\infty}(\mathcal{M}_\infty(\Gamma_0(N))) + \mathrm{pdim}_{J_\infty}(\mathcal{M}_\infty(\Gamma_0(N))) = \mathrm{depth}_{J_\infty}(J_\infty) = 3\#S+1+r,$$

and this forces $\mathrm{pdim}_{J_\infty}(\mathcal{M}_\infty(\Gamma_0(N)))$ to be zero, namely $\mathcal{M}_\infty(\Gamma_0(N))$ is already projective over J_∞, hence free because J_∞ is a local ring. Moreover, the same argument would be valid for $\mathcal{M}_\infty(\Gamma_0(Np))$.

8. **Problem 8. (The Latter-Half Part)**

   To get the matrix representation of the composition, we need to check the kernel of $(\pi_1^*, \pi_2^*)^\vee \circ (\pi_{1,*} \oplus \pi_{2,*})^\vee$. We make acknowledgement from the newform theory that

   $$\mathrm{Tr}(\rho^{\mathrm{univ}}(\mathrm{Frob}_p)) \equiv p+1 \bmod \mathfrak{m}_{\bar{\rho}}.$$

   Since $T_p^\vee$ is given by multiplication by $\mathrm{Tr}(\rho^{\mathrm{univ}}(\mathrm{Frob}_p))$, after correctly choosing a basis with scalaring -1 on $\mathcal{M}_\infty(\Gamma_0(N))^{\oplus 2}$, we construct the following

   $$\{(x,y) \in \mathcal{M}_\infty(\Gamma_0(N))^{\oplus 2} \mid T_p^\vee x = -(p+1)y, T_p^\vee y = -(p+1)x\}$$

   to be $\ker(\pi_1^*, \pi_2^*)^\vee \circ (\pi_{1,*} \oplus \pi_{2,*})^\vee$. In other words, this is exactly

   $$\begin{pmatrix} p+1 & T_p^\vee \\ T_p^\vee & p+1 \end{pmatrix} : \mathcal{M}_\infty(\Gamma_0(N))^{\oplus 2} \longrightarrow \mathcal{M}_\infty(\Gamma_0(N))^{\oplus 2}.$$

9. **Problem 9.**

(1) Begin with the natural composition

$$J_\infty := \mathbb{Z}_\ell[\![z_1, \ldots, z_r, w_1, \ldots, w_{3\#S}]\!] \longrightarrow R^{\mathcal{D}}_{\mathrm{loc}}[\![y_1, \ldots, y_r]\!] \longrightarrow R^{\mathcal{D},\mathrm{ur}}_{\mathrm{loc}}[\![y_1, \ldots, y_r]\!].$$

Applying the argument in Problem 7, by Auslander-Buchbaum Theorem,

$$\begin{aligned} & \mathrm{depth}_{R^{\mathcal{D},\mathrm{ur}}_{\mathrm{loc}}[\![y_1,\ldots,y_r]\!]}(\mathcal{M}_\infty(\Gamma_0(N))) + \mathrm{pdim}_{R^{\mathcal{D},\mathrm{ur}}_{\mathrm{loc}}[\![y_1,\ldots,y_r]\!]}(\mathcal{M}_\infty(\Gamma_0(N))) \\ = & \mathrm{depth}_{R^{\mathcal{D},\mathrm{ur}}_{\mathrm{loc}}[\![y_1,\ldots,y_r]\!]}(R^{\mathcal{D},\mathrm{ur}}_{\mathrm{loc}}[\![y_1, \ldots, y_r]\!]) \\ = & \mathrm{depth}_{J_\infty}(\mathcal{M}_\infty(\Gamma_0(N))) \\ \le & \mathrm{depth}_{R^{\mathcal{D},\mathrm{ur}}_{\mathrm{loc}}[\![y_1,\ldots,y_r]\!]}(\mathcal{M}_\infty(\Gamma_0(N))) \end{aligned}$$

and therefore $\mathrm{pdim}_{R^{\mathcal{D},\mathrm{ur}}_{\mathrm{loc}}[\![y_1,\ldots,y_r]\!]}(\mathcal{M}_\infty(\Gamma_0(N))) = 0$. So $\mathcal{M}_\infty(\Gamma_0(N))$ defines a free module over $R^{\mathcal{D},\mathrm{ur}}_{\mathrm{loc}}[\![y_1, \ldots, y_r]\!]$.

(2) Suppose that $\ker(\pi_1^*, \pi_2^*)^\vee$ is supported on $R^{\square_S,\mathcal{D},\mathrm{St}}_{\bar\rho}$. The main theorem of patching argument says that $\mathrm{Supp}_{R^{\mathcal{D},\mathrm{St}}_{\mathrm{loc}}[\![y_1,\ldots,y_r]\!]}(\ker(\pi_1^*, \pi_2^*)^\vee)$ is a union of irreducible components of $\mathrm{Spec}\, R^{\mathcal{D},\mathrm{St}}_{\mathrm{loc}}[\![y_1, \ldots, y_r]\!]$, which covers the entire

$$\mathrm{Spec}\, R^{\square_S,\mathcal{D},\mathrm{St}}_{\bar\rho} = \mathrm{Spec}\, R^{\mathcal{D},\mathrm{St}}_{\mathrm{loc}}[\![y_1, \ldots, y_r]\!]/(f_1, \ldots, f_r).$$

Since the irreducible components of $\mathrm{Spec}\, R^{\square_S,\mathcal{D},\mathrm{St}}_{\bar\rho}$ are in bijection with the irreducible component of $\mathrm{Spec}\, R^{\mathcal{D},\mathrm{St}}_{\mathrm{loc}}[\![y_1, \ldots, y_r]\!]$, this implies that

$$\mathrm{Supp}_{R^{\mathcal{D},\mathrm{St}}_{\mathrm{loc}}[\![y_1,\ldots,y_r]\!]} \ker(\pi_1^*, \pi_2^*)^\vee = \mathrm{Spec}\, R^{\mathcal{D},\mathrm{St}}_{\mathrm{loc}}[\![y_1, \ldots, y_r]\!].$$

The similar argument as that in (1) with switching the unramified component into the Steinberg component shows that $\ker(\pi_1^*, \pi_2^*)^\vee$ defines a finite free module over $R^{\mathcal{D},\mathrm{St}}_{\mathrm{loc}}[\![y_1, \ldots, y_r]\!]$.

(3) Let f be a Hecke eigenform in $S_2(\Gamma_0(N))$ after tensoring with $\mathbb{F}_\ell$. By rescaling f if necessary, we may assume the image $\bar f$ of f in $S_2(\Gamma_0(N))_{\mathfrak{m}_{\bar\rho}}$ is nonzero. Then the Hecke operator acts on $\bar f$ by multiplication by the scalar. Hence we can restate the congruence of $\mathrm{Tr}(\bar\rho(\mathrm{Frob}_p))$ in the form of

$$T_p^\vee \equiv p + 1 \bmod \mathfrak{m}_{\bar\rho},$$

and therefore our map, which is given by the previous problem, deduces that

$$\det \begin{pmatrix} p+1 & T_p^\vee \\ T_p^\vee & p+1 \end{pmatrix} = 0.$$

As a result, the map

$$(\pi_1^*, \pi_2^*)^\vee \circ (\pi_{1,*} \oplus \pi_{2,*})^\vee : S_2(\Gamma_0(N))^{\vee,\oplus 2}_{\mathfrak{m}_{\bar\rho}} \longrightarrow S_2(\Gamma_0(N))^{\vee,\oplus 2}_{\mathfrak{m}_{\bar\rho}}$$

is not injective. Now under the assumption that $\ker(\pi_1^*, \pi_2^*)^\vee = 0$, the former map cannot be injective. This gives a contradiction.