Tate classes & endoscopy for GSp_4

Naomi Sweeting

Let g = wt 2 cusp Hecke eigenform, new of level $\Gamma_0(N)$.

Look at $\underbrace{H^1_{et}(X_0(N)_{\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_\ell)}_{\circlearrowright \; \mathbb{T}_N \times G_{\mathbb{Q}}}[g] = \rho_g$ assoc 2-dim'l Gal rep to g.

__Also__ Image in dR cohom of $\mathrm{Span}\{\omega_g, \bar{\omega}_g\} \subseteq H^1_{dR}(X_0(N), \mathbb{C})$

But ρ_g also appears in the cohom of (many) other Sh vars

e.g. $Sh_K(GSp_4) = \{$2-dim'l ppav + level str$\}$.

"Siegel 3-fold", $K \subseteq GSp_4(\mathbb{A}_f)$.

$$\underset{\mathcal{H}_K \times G_{\mathbb{Q}}}{\circlearrowright} \quad H^3_{et}(Sh_K(GSp_4)_{\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_\ell) = \bigoplus_{\pi_f \otimes \pi_\infty} \pi_f^K \otimes \rho_{\pi_f}$$

$\pi_f \otimes \pi_\infty$: autom rep of GSp_4

Recall: $\mathcal{A}(G) = \{$nice fcts $G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}\}$

Autom rep: irred constituents of $\mathcal{A}(G) \circlearrowleft G(\mathbb{A})$.

__Note__ When $\neq 0$, ρ_{π_f} is typically 4-dim'l + irred

but for some "special" π_f's,

$$\rho_{\pi_f} = \rho_g(-1)$$

$$\Rightarrow H^4_{et}(Sh_K(GSp_4) \times X_0(N), \bar{\mathbb{Q}}_\ell(2))^{G_{\mathbb{Q}}} \neq 0.$$

Q Can you find alg cycles generating these Tate classes?

__Convention__ Drop all level strs

(e.g. work on infinity / small level str).

modular curve $= Sh(GL_2)$.

<u>Candidate cycle</u> $\quad Sh(GSp_4) \times Sh(GL_2) = \left\{ (A, E) \,\middle|\, \begin{array}{l} A \text{ ppa surface,} \\ E \text{ ell curve} \end{array} \right\}$

$$\begin{array}{ccc} \cup\!\mathsf{I} & & \cup\!\mathsf{I} \\ Sh(H) & = & \{ (E_1 \times E_2, E_1) \} \end{array}$$

$$H = GL_2 \times_{\mathbb{G}_m} GL_2 \underset{(\iota, p_1)}{\hookrightarrow} GSp_4 \times GL_2$$

ι given by $\left(\begin{matrix} \times & & \times & \\ & \times & & \times \\ \times & & \times & \\ & \times & & \times \end{matrix} \right) \Big\} GSp_4.$

GL_2 $\quad GL_2$

<u>Thm A</u> Suppose
- π is a cusp autom rep of GL_2 corresp to a mod form of wt 2
- Π is an autom rep of GSp_4 s.t. $H^4_{et}(Sh(GSp_4) \times Sh(GL_2), \bar{\mathbb{Q}}_\ell(2))^{G_{\mathbb{Q}}}[\Pi_f^\vee \times \pi_f] \neq 0$.

Then $[Sh(H)]_{\Pi_f^\vee \times \pi_f} \neq 0 \iff \Pi$ is "globally $\underbrace{\text{generic}}_{\text{autom condition}}$".

<u>Rmks</u> (1) Higher wts + totally real fields

(2) Francesco Lemma: "$\Leftarrow$" part.

<u>Thm B</u> In any case, $H^4_{et}(Sh(GSp_4) \times Sh(GL_2), \bar{\mathbb{Q}}_\ell(2))^{G_{\mathbb{Q}}}[\Pi_f^\vee \times \pi_f]$

is spanned by $\underbrace{\text{Hodge classes}}$,

i.e. $H^4(Sh(GSp_4) \times Sh(GL_2), \mathbb{Q}) \cap H^{2,2}_{dR}(Sh(GSp_4) \times Sh(GL_2), \mathbb{C})$

(weaker than Tate conj)

<u>Relation to Tate</u>

Let X sm var $/\mathbb{Q}$

Conj essentially surj.

$$\begin{tikzcd}
 & H^4_{et}(X, \bar{\mathbb{Q}}_\ell(2))^{G_\mathbb{Q}} \arrow[r, hook] \arrow[d, phantom, "\circlearrowright"] & H^4_{et}(X, \bar{\mathbb{Q}}_\ell(2)) \\
CH^2(X) \arrow[ur] \arrow[dr] & & \\
 & H^4(X, \mathbb{Q}) \cap H^{2,2}_{dR}(X, \mathbb{C}) \arrow[r, hook] & H^4_{dR}(X, \mathbb{C}) \arrow[uu, "\wr", "\text{comparison}"']
\end{tikzcd}$$

<u>Which π makes contribution?</u>

<u>Θ corresp</u> If H, G form a "dual red pair"

e.g. $GSO_{2m} \times GSp_{2n}$

have a θ-lift $\theta : \mathcal{A}(H) \longrightarrow \mathcal{A}(G)$

<u>Properties</u> (1) θ respects cusp rep'ns

$\rightsquigarrow \Theta : \{(\text{cusp})\text{ ARs of } H\} \longrightarrow \{(\text{cusp})\text{ ARs of } G\}$

(2) Local-global compatibility:

$\Theta(\pi) \neq 0 \Rightarrow \Theta(\pi) = \widetilde{\bigotimes}' \Theta_v(\pi_v)$.

(3) "Seesaw identity"

<u>Slogan</u> Certain periods are easier to compute for theta lifts.

(4) "Siegel-Weil": $\theta(\mathbb{1}) =$ an Eis series.

<u>For us</u> $H = GSO_4$, $G = GSp_4$

$GSO_4 = GL_2 \times GL_2 / \mathbb{G}_m$

$\rightsquigarrow \{\text{ARs of } H\} = \left\{ \begin{array}{l} \text{pairs } (\pi_1, \pi_2) \text{ of ARs of } GL_2 \\ \text{w/ same central char} \end{array} \right\}$

$\rightsquigarrow \Theta(\pi_1 \boxtimes \pi_2)$ AR of GSp_4

For B a quat alg, $GSO_4^B = B^\times \times B^\times / \mathbb{G}_m$

inner form corresp. to $B^\times$ in GL_2.

If π_1, π_2 have JL transfers π_1^B, π_2^B to $B^\times$, Can lift $\Theta_B(\pi_1^B \boxtimes \pi_2^B)$.

(2) $\Rightarrow$ this is nearly equiv to $\Theta(\pi_1 \boxtimes \pi_2)$

<u>Fact</u> $\{\Theta_B(\pi_1^B \boxtimes \pi_2^B)\}$ is an L-packet on GSp_4 !

"endoscopic Yoshida lift L-packet".

The rep'ns $\Theta_B(\pi_1^B \boxtimes \pi_2^B)$ are cohomological (w/ triv coeffs)

$\Leftrightarrow$ π_1, π_2 have wts 4 & 2, resp.

<u>Fact</u> (Weissauer) $\Pi = \Theta_B(\pi_1^B \boxtimes \pi_2^B)$,

$$H^3_{et}(Sh(GSp_4), \bar{\mathbb{Q}}_\ell)[\Pi_f] = \begin{cases} \rho_{\pi_2}(-1), & B \otimes \mathbb{R} \text{ split} \\ \rho_{\pi_1}, & \text{else} \end{cases}$$

In particular, if $B \otimes \mathbb{R}$ split,

$$H^4_{et}(Sh(GSp_4) \times Sh(GL_2), \bar{\mathbb{Q}}_\ell(2))^{G_{\mathbb{Q}}}[\Pi_f^\vee \times \pi_{2,f}] \neq 0.$$

Reformulations:

<u>Thm A</u> Let $\Pi = \Theta_B(\pi_1^B \boxtimes \pi_2^B)$

Then $[Sh(H)]_{\Pi_f^\vee \times \pi_{2,f}} (\neq 0) \in H^4_{et}(Sh(GSp_4) \times Sh(GL_2), \bar{\mathbb{Q}}_\ell(2))^{G_{\mathbb{Q}}}[\Pi_f^\vee \times \pi_{2,f}]$

$\Leftrightarrow$ $B = Mat_2$.

<u>Pf</u> Look at $\int_{[H]} \alpha(h_1, h_2)\, \beta(h_2)\, d(h_1, h_2)$

$$H = GL_2 \times_{G_m} GL_2 \xhookrightarrow{(\iota, p_1)} GSp_4 \times GL_2 \ni (\alpha, \beta), \quad \alpha \in \Pi,\ \beta \in \pi_2^\vee.$$

STS: This is 0, $\forall \alpha \in \Pi$ & $\beta \in \pi_2^\vee$ unless $B = M_2$.

<u>Aside</u> GGP pair: $GSpin_4 \subset GSpin_5$,

$SO_4 \subset SO_5$.

(3) Seesaw pair:

$$\begin{array}{ccc} G_2 & & G_1 \\ | & \times & | \\ H_1 & & H_2 \\ f_1 & & f_2 \end{array} \quad \rightsquigarrow \quad \int_{[H_1]} \theta(f_2)\, f_1 = \int_{[H_2]} \theta(f_1)\, f_2$$

Apply to our case:

$$\begin{array}{cccc} GSp_4^B \times_{\mathbb{G}_m} GSp_4^B & & GSp_4 & \alpha \in \pi \\ | & \times & | & \\ GSp_4^B & & GL_2 \times_{\mathbb{G}_m} GL_2 & \beta \boxtimes \mathbb{1} \\ \pi_1^B \boxtimes \pi_2^B & & & \beta \in \pi_2^\vee \end{array}$$

(4) $\theta(\mathbb{1})$ = an Eis series on GSO_4^B, $= 0$ if $B \neq M_2$

$$GSO_4 = GL_2 \times GL_2 / \mathbb{G}_m$$