The homotopy groups of the K(n)-local sphere

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§1 Motivation / Results

Problem: Compute $\pi_n S := \pi_{n+k} S^k$ (stabilizes for $k \gg 0$).
$\uparrow$ k-sphere

What is "S"?

- Sp = ∞-category of spectra.
  like $D(Ab)$, exact triangles.
  + $\otimes$, $\Sigma = [1]$ ← cat autom shift
  but contains top info $Top \to Sp$.
- objects of Sp are "generalized cohom theories" $E^*(x)$
- S = sphere spectrum = initial ring spectrum
  (c.f. $\mathbb{Z}$ = initial obj in Ab grps).

Localize S at a prime? (S analog of $\mathbb{Z}$).

- Division algs D in Sp = every D-mod is free.
- $D_1 \sim D_2$ if $D_1 \otimes D_2 \neq 0$.
- $\mathrm{Spec}\, Sp = \{D\} / \sim$.

<u>Thm</u> (Devinatz - Hopkins - Smith)
The fiber of $\mathrm{Spec}\, Sp \to \mathrm{Spec}\, \mathbb{Z}$ over p

is $\{K(p,n) \mid n = 1, 2, \cdots, \infty\}$.
Called Morava K-theory.

Fix p. $K(n) := K(p,n)$ the localization.
For a ring spectrum E have notion of "E-local"

$$Sp_E \subseteq Sp \quad + \quad L_E : Sp \to Sp_E \quad (E\text{-localization}).$$

<u>Thm A</u> For $n = 1, 2, \cdots$,

$$\pi_* L_{K(n)} S \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \Lambda_{\mathbb{Q}_p}[x_{-1}, x_{-3}, \cdots, x_{1-2n}]$$

exterior alg $/\mathbb{Q}_p$ on $|x_i| = i$.

"$L_{K(n)} S$ is nth graded piece of a filtration by $L_n S$".
$(\pi_n E = [S, \Sigma^{-n} E].)$

<u>§2 Morava E-theory / first reductions</u>

A comm ring spectrum E is <u>complex oriented</u> if

$$E^*(BS^1 = \mathbb{CP}^\infty) = E^*(pt)[\![t]\!], \quad |t| = 2.$$

$\otimes$-on line bundles induces a formal group law.

Fix $k = \bar{\mathbb{F}}_p$, $n \geq 1$, Γ / k formal grp height n, $W = W(k)$.
$\rightsquigarrow A_n \simeq W[\![u_1, \cdots, u_{n-1}]\!]$ (LT) deformation ring of Γ.

<u>Thm</u> (Goerss - Hopkins - Miller)
$\exists$ complex-oriented spectrum E_n
s.t. $E_n^0(pt) = A_n$, w/ universal deformation of Γ.

<u>Note</u> $\pi_* E_n = A_n[\beta^{\pm 1}]$, $|\beta| = 2$.

$E_n/(p, u_1, \cdots, u_{n-1}) = \bigoplus_i \Sigma^i K(n)$.

Let $\mathbb{G}_n = \mathrm{Aut}(\Gamma, k)$, $\mathbb{G}_n = \widehat{D^*}$, $\mathrm{End}\,\Gamma = \mathcal{O}_D$, $D/\mathbb{Q}_p$ div alg with inv $1/n$.

$$\begin{array}{c} 1 \to \mathrm{Aut}_k \Gamma \to \mathbb{G}_n \to \mathrm{Aut}\, k \to 1, \\ \| \\ \mathcal{O}_D^\times \end{array}$$

$\widehat{D}^* \xrightarrow{\mathrm{val}} \hat{\mathbb{Z}}$, $\mathbb{G}_n \circlearrowright E_n$.

<u>Thm</u> (Devinatz-Hopkins) $L_{K(n)} S \to E_n$ is Galois w/ group $\mathbb{G}_n$.

<u>D-H spectral sequence</u>

$$H^s_{\mathrm{cont}}(\mathbb{G}_n, \pi_t E_n) \Rightarrow \pi_{t-s} L_{K(n)} S.$$

<u>Thm B</u> The inclusion $W \hookrightarrow A_n$ induces an isom

$$H^*_{\mathrm{cont}}(\mathbb{G}_n, W)[\tfrac{1}{p}] \xrightarrow{\sim} H^*_{\mathrm{cont}}(\mathbb{G}_n, A_n)[\tfrac{1}{p}].$$

$$(B) \Rightarrow (A): \text{D-H} + H^*_{\mathrm{cont}}(\mathbb{G}_n, W)[\tfrac{1}{p}] \simeq H^*_{\mathrm{cont}}(\mathcal{O}_D^*, \mathbb{Q}_p) \overset{\text{Lazard}}{\simeq} \Lambda_{\mathbb{Q}_p}(x_1, \cdots, x_{2n-1})$$

<u>Prop</u> $W \hookrightarrow A_n$ admits a $\mathbb{G}_n$-equiv splitting

$$\begin{array}{ccc} A_n = & W & \oplus\ A_n^c. \\ & \circlearrowleft & \circlearrowleft \\ & \mathbb{G}_n & \mathbb{G}_n \end{array}$$

$\rightsquigarrow$

<u>Thm B</u> $H^*_{\mathrm{cont}}(\mathbb{G}_n, A_n^c)[\tfrac{1}{p}] = 0$.

§3 The two towers

Let $K = W[\frac{1}{p}]$, $LT = \mathrm{Spf}\, A_n$,

LT_K = rigid generic fibre = $(\mathrm{Spa}\, A_n)_\eta$, open ball $/K$.

(with $\mathrm{Spa}\, W = \{\eta, s\}$).

Thm (Faltings, Fargues, Scholze–Weinstein)

$\exists$ diagram

$$\begin{tikzcd} & \mathcal{X} \arrow[dl, "GL_n(\mathbb{Z}_p)"'] \arrow[dr, "\mathbb{G}_n"] & \\ \mathbb{G}_n \circlearrowright LT_K & & \mathcal{H} = \mathbb{P}^{n-1}_{\mathbb{Q}_p} \setminus \bigcup_{\substack{L \subseteq \mathbb{P}^{n-1}_{\mathbb{Q}_p} \\ \mathbb{Q}_p\text{-rat'l}}} L \end{tikzcd}$$

pro-étale torsors

$\underbrace{\circlearrowleft GL_n(\mathbb{Z}_p)}_{\text{Drinfeld side}}$

For an adic space $\mathcal{X}$, have site $X_{\text{proét}}$.

$\mathcal{U} \in X_{\text{proét}}$ is $\mathcal{U} = \varprojlim \mathcal{U}_i$, $\mathcal{U}_i \in X_{\text{ét}}$.

$X_{\text{proét}}$ has structure sheaf

$$\widehat{\mathcal{O}}^+(\mathcal{U}) = \left(\varinjlim \mathcal{O}^+(\mathcal{U}_i)\right)^\wedge_{(p)}.$$

Thm on towers implies

$$R\Gamma_{\text{cont}}(\mathbb{G}_n, R\Gamma(LT_{K\,\text{proét}}, \widehat{\mathcal{O}}^+)) \simeq R\Gamma(GL_n(\mathbb{Z}_p), R\Gamma(\mathcal{H}_{\text{proét}}, \widehat{\mathcal{O}}^+)).$$

Rmk Need condensed math.

Thm C (1) $R\Gamma(LT_{K,\text{proét}}, \widehat{\mathcal{O}}^+)[\frac{1}{p}] \underset{\mathbb{G}_n\text{-equiv}}{\simeq} A_n \otimes_W K[\varepsilon]$, $|\varepsilon| = 1$ & $\varepsilon^2 = 0$.

(2) $R\Gamma(\mathcal{H}_{\text{proét}}, \widehat{\mathcal{O}}^+)[\frac{1}{p}] \underset{GL_n(\mathbb{Z}_p)\text{-equiv}}{\simeq} \mathbb{Q}_p[\varepsilon]$.

$(C) \Rightarrow (B)$: $\left(R\Gamma_{cont}(\mathbb{G}_n, W) \oplus R\Gamma_{cont}(\mathbb{G}_n, A_n^c)\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p[\varepsilon]$

$\simeq R\Gamma_{cont}(GL_n(\mathbb{Z}_p), \mathbb{Q}_p)[\varepsilon]$.