The Ramanujan conjecture for Bianchi modular forms of weight 2

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K number field, π ↻ $GL_n(\mathbb{A}_K)$ autom rep.

There's a factorization $\pi = \bigotimes_v' \pi_v$

with π_v ↻ $GL_n(K_v)$ irred admissible.

For almost all places v of K, v is nonarch,

π_v is unramified,

⇝ can define Satake parameter

$t(\pi_v) \in GL_n(\mathbb{C})$ (def'd up to conjugacy).

<u>(Generalized) Ramanujan Conj</u>

Suppose π is cuspidal and $|\omega_\pi| = \|\cdot\|_K^r$, $r \in \mathbb{R}$.

(central char of π)

Then for almost all v, every eigenvalue α_v of $t(\pi_v)$

satisfies $|\alpha_v| = q_v^{r/n}$,

where $q_v = \# k(v)$ is the size of the residue field at v.

Another related conjecture is:

<u>Reciprocity Conj</u>

Suppose π is cuspidal and cohomological,

(i.e. $H^*(\mathfrak{gl}_n, U_\infty, \pi_\infty) \neq 0$).

Then for any prime ℓ and isom $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$,

$\exists$ a continuous, irred Gal rep

$$\rho_{\pi,\iota} : \mathrm{Gal}(\bar{K}/K) \longrightarrow GL_n(\bar{\mathbb{Q}}_\ell)$$

which is unramified at almost all places

& satisfies $\rho_{\pi,\iota}(\mathrm{Frob}_v) = \iota^{-1}\left(q_v^{\frac{n-1}{2}} t(\pi_v)\right) \in \bar{\mathbb{Q}}_\ell$

Moreover, $\rho_{\pi,\iota}$ is motivic, i.e. $\exists$ a sm proj var X/K

s.t. $\rho_{\pi,\iota}$ is a $\bar{\mathbb{Q}}_\ell[\mathrm{Gal}(\bar{K}/K)]$-subquotient of $H^*_{et}(X_{\bar{K}}, \bar{\mathbb{Q}}_\ell)$.

Rmk (1) If π is cohomological, then

reciprocity conj $\Rightarrow$ Ramanujan conj for π.

by Deligne's proof of Weil conj.

(2) Clozel formulated the reciprocity conj

for any "algebraic" π ($\Leftarrow$ "cohomological".)

Most known cases of Ramanujan:

are proved by first proving reciprocity.

e.g. (✯)

Suppose K is an imaginary CM field with tot real field K^+.

Suppose as well π is cuspidal, cohomological,

+ conjugate self-dual i.e. $\pi^c \simeq \pi^\vee$

$(\langle c \rangle = \mathrm{Gal}(K/K^+).)$

In this case one can hope to descend π ($\circlearrowright GL_n(\mathbb{A}_K)$)

an autom rep $\Pi \circlearrowright G(\mathbb{A}_{K^+})$, where

G = well-chosen unitary grp in n-variables $/K^+$.

If $\qquad G(K^+ \otimes_{\mathbb{Q}} \mathbb{R}) \simeq U(n-1,1) \times U(n)^{[K^+:\mathbb{Q}]-1}$

then you hope to construct $\rho_{\pi,\iota}$ inside $H^*_{et}(Sh_{G,\bar{K}}, \bar{\mathbb{Q}}_\ell)$

where Sh_G is the Sh var associated to G.

This strategy was first carried out by Clozel
for π s.t. π_v is square-integrable for some finite place v.
This allows you to choose G s.t. Sh_G are among the "simple Sh vars"
of Kottwitz (i.e. has no endoscopy).

We now know Ramanujan for any π satisfying $(\star)$.
We also know $\rho_{\pi,\ell}$ exists
(Clozel-Harris-Laberse, Shin, Chenevier-Harris.)

The proof of Ramanujan for all π of this type
was completed by Clozel by showing that
either $\underset{\substack{\circlearrowleft \\ U(1,n-1)\times U(n)^{d-1}}}{\rho_{\pi,\ell}}$ or $\underset{\substack{\circlearrowleft \\ U(2,n-2)\times U(n)^{d-1}}}{\Lambda^2\rho_{\pi,\ell}}$ is motivic.

We now consider a more general situation:
$(\star')$ Suppose K is imaginary CM, and π is cuspidal & cohomological.
In this case, we again know that $\rho_{\pi,\ell}$ exists.
(Harris-Lan-Taylor-Thorne, Scholze).

<u>Strategy to construct $\rho_{\pi,\ell}$</u>
Let $G^* = U(n,n)$ quasi-split unitary group $/K^+$
def'd by matrix $\begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$.
Let $P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subseteq G^*$ Siegel parabolic.

$$M = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \simeq \mathrm{Res}_{K/K^+} GL_n.$$
$$\subseteq P$$

Let $X_{G^*} = Sh_{G^*}(\mathbb{C})$ of some (Iwahori) level.

$X_P = P(K^+) \backslash P(\mathbb{A}_{K^+}) / U_P$.

$X_M = M(K^+) \backslash M(\mathbb{A}_{K^+}) / U_M$. (Bianchi manifold when $G = GL_2$)

$\rightsquigarrow$ $\exists$ a diagram of symm spaces

$$X_{G^+} \subseteq X_{G^+}^{BS} \supseteq \partial X_{G^*}^{BS} \supseteq X_P \longrightarrow X_M.$$

BS = Borel-Serre cpt'n.

- Can show $\pi^\infty \hookrightarrow H^*(X_M, \mathbb{C})$
- Using the long exact seq of

$$X_{G^*} \hookrightarrow X_{G^*}^{BS} \hookleftarrow \partial X_{G^*}^{BS}$$

can show $\mathrm{Ind}_P^G \pi^\infty$ appears as a subquot of $H^*(X_{G^*}, \mathbb{C})$

<u>Trial</u> Can also look at $H^*_{\acute{e}t}(Sh_{G^*, \bar{K}}, \bar{\mathbb{Q}}_\ell)[\mathrm{Ind}_P^G \pi^\infty]$

(recall: Eichler-Shimura).

<u>but</u> this will almost never contain $\rho_{\pi,\iota}$.

<u>Scholze</u> Construct a $2n$-dim'l pseudo-rep

$$t_{G^*} : \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathbb{T}_{G^*} \subseteq \mathrm{End}_{\bar{\mathbb{Q}}_\ell}(H^*(Sh_{G^*, \bar{K}}, \bar{\mathbb{Q}}_\ell)).$$

Here $\mathbb{T}_{G^*}$ = algebra gen'd by unramified Hecke operators.

t_{G^*} satisfies for almost all v,

- t_{G^*} is unram at v
- $t_{G^*}(\mathrm{Frob}_v) = T_v$.

The apperance of $\mathrm{Ind}_P^G \pi^\infty$ in $H^*(X_{G^*}, \mathbb{C})$ gives a homomorphism

$\mathbb{T}_{G^*} \longrightarrow \bar{\mathbb{Q}}_\ell$ assoc to the Hecke eigenvalues of $\mathrm{Ind}_P^G\, \iota^{-1}\pi^\infty$.

The assoc pseudo-rep

$$t_{\pi^\infty} : \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathbb{T}_{G^*} \longrightarrow \bar{\mathbb{Q}}_\ell$$

is the one assoc to $\rho_{\pi,\iota} \oplus (\rho^c_{\pi,\iota})^\vee \cdot \varepsilon^{1-n}$.

Two most important ingredients in constructing t_{G^*}:

(1) Scholze's theory of perfectoid Shimura varieties.

(2) Classification of autom rep'ns of G^* (Base change).

<u>Theorem</u> (Allen, Calegari, Caraiani, Gee, Helm, Le Hung, Newton, Scholze, Taylor, Thorne).

Suppose π satisfies ($\star'$) for $n=2$.

(i.e. $\pi \curvearrowright GL_2(\mathbb{A}_K)$ cuspidal & cohomological.)

Then π satisfies Ramanujan.

<u>Note</u> We don't know that any tensor power of $\rho_{\pi,\iota}$ is motivic!

<u>Sketch of argument</u>

We show for any $m \geq 1$, $\mathrm{Sym}^m \rho_{\pi,\iota}$ is <u>potentially autom</u>

WLOG π is not of dihedral type,

$\forall m \geq 1$, $\exists$ CM number field K_m/K & $\Pi_m \curvearrowright GL_{m+1}(\mathbb{A}_{K_m})$ satisfying ($\star'$)

and $\rho_{\Pi_m,\iota} \simeq \mathrm{Sym}^m \rho_{\pi,\iota}|_{\mathrm{Gal}(\bar{K}/K_m)}$.

Jacquet-Shalika (classification of unitary generic reps)

$\Rightarrow$ if w is an unram place of Π_m

& β_w = an eigenval of $t(\Pi_w)$,

then $$q_w^{-1/2} < |\beta_w| < q_w^{1/2}.$$

If v is a place below w & α_v = an eigenval of $t(\pi_v)$,

then take $\beta_w = \alpha_v^{m(\log_{q_v} q_w)}$

$$\Rightarrow \quad q_v^{-1/2m} < |\alpha_v| < q_v^{1/2m}.$$

Now let $m \to \infty$.

To prove potential automorphy,

need to prove automorphy lifting thm.

This is made possible by

- Calegari-Geraghty (beyond the Taylor-Wiles method)
- Caraiani-Scholze

(Vanishing thms for cohom of Sh vars)

$\Rightarrow$ local-global compatibility for $H^*(X_M, \bar{\mathbb{Z}}_\ell)$.