On Beilinson–Bloch conjecture for unitary Shimura varieties via Li–Liu

CHOW GROUPS AND L-DERIVATIVES OF AUTOMORPHIC MOTIVES FOR UNITARY GROUPS: A SURVEY

NOTES BY WENHAN DAI

Abstract. These notes are based on a talk by Chao Li at Columbia in 2021, and a talk by Yifeng Liu at Franco–Asian Summer School of Arithmetic Geometry at CIRM in 2022. We survey the background of the joint work by Chao Li and Yifeng Liu [LL21, LL22] on Beilinson–Bloch conjecture for unitary Shimura varieties. The first lecture by Chao Li is for an introduction. The last three lectures by Yifeng Liu aim to propose (1) an introduction to the theory of height pairings on higher dimensional algebraic varieties, which will lead to an interpretation of L-derivatives, (2) representation theory of unitary groups, and (3) the main results and the idea of the proofs.

Contents

1. Introduction

1.1. Historical background with main conjectures.

1.1.1. *Birch–Swinnerton-Dyer conjecture.* Let $E : y^2 = x^3 + Ax + B$ be an elliptic curve over $\mathbb{Q}$. In the sense of Birch–Swinnerton-Dyer conjecture, we define

- The *algebraic rank* of E is the rank of the finitely generated abelian group $E(\mathbb{Q})$, that is,
$$r_{\mathrm{alg}}(E) := \operatorname{rank} E(\mathbb{Q}).$$
- The *analytic rank* of E is the order of vanishing of the L-function associated to E at the central point $s = 1$, that is,
$$r_{\mathrm{an}}(E) := \operatorname{ord}_{s=1} L(E, s).$$

Conjecture 1.1 (Birch–Swinnerton-Dyer, 1960s)**.**

(1) (*Rank part*).
$$r_a n(E) = r_{\mathrm{alg}}(E).$$
(2) (*Leading coefficient*). *For* $r = r_{\mathrm{an}}(E)$,
$$\frac{L^{(r)}(E,1)}{r!} = \frac{\Omega(E)R(E)\prod_p c_p(E)\cdot |\text{Ш}(E)|}{|E(\mathbb{Q})_{\mathrm{tor}}|^2},$$
where
- $R(E) = \det(\langle P_i, P_j\rangle_{\mathrm{NT}})_{r\times r}$ *is the regulator for the Néron–Tate height pairing*
$$\langle\cdot,\cdot\rangle_{\mathrm{NT}} : E(\mathbb{Q}) \times E(\mathbb{Q}) \longrightarrow \mathbb{R},$$
- $\text{Ш}(E)$ *is the Tate–Shafarevich group,*
- $\Omega(E)$ *is the Néron period integral of Néron differentials* ω_E *along* $E(\mathbb{R})$*, and*
- $c_p(E)$*, called the local Tamagawa number, equals* $[E(\mathbb{Q}_p) : E^0(\mathbb{Q}_p)]$ *for an elliptic curve* E^0 *arose by some local torsion condition.*

The following remark is by Tate in *The Arithmetic of Elliptic Curves*, 1974.

> *This remarkable conjecture relates the behavior of a function L at a point where it is not at present known to be defined to the order of a group Ш which is not known to be finite!*

The BSD conjecture is still widely open in general, but much progress has been made in the rank 0 or 1 case.

Theorem 1.2 (Gross–Zagier, Kolyvagin, 1980s)**.**

$$r_{\mathrm{an}}(E)=0 \implies r_{\mathrm{alg}}(E)=0, \qquad r_{\mathrm{an}}(E)=1 \implies r_{\mathrm{alg}}(E)=1.$$

Remark 1.3. When $r=r_{\mathrm{an}}(E)\in\{0,1\}$, many cases of the formula for $L^{(r)}(E,1)$ are known.

The proof combines two inequalities:

(1) (Gross–Zagier formula, [GZ86])
$$r_{\mathrm{an}}(E)=1 \implies r_{\mathrm{alg}}(E)\geqslant 1.$$
(2) (Kolyvagin's Euler system, [BD05])
$$r_{\mathrm{an}}(E)\in\{0,1\} \implies r_{\mathrm{alg}}(E)\leqslant r_{\mathrm{an}}(E).$$

Both steps rely on *Heegner points* on modular curves.

1.1.2. *Beilinson–Bloch conjecture.* Let X be a smooth projective variety over a number field K. Denote $\mathrm{CH}^m(X)$ the Chow group of algebraic K-cycles of codimension m on X. Also denote $\mathrm{CH}^m(X)^0\subset \mathrm{CH}^m(X)$ the subgroup of geometrically cohomologically trivial cycles. Using this, we obtain the Beilinson–Bloch height pairing

$$\langle\cdot,\cdot\rangle_{\mathrm{BB}}:\mathrm{CH}^m(X)^0\times \mathrm{CH}^{\dim X+1-m}(X)^0\longrightarrow \mathbb{R}.$$

To state the conjecture, we also define $L(H^{2m-1}(X),s)$ to be the motivic L-function for $H^{2m-1}(X_{\overline{K}},\mathbb{Q}_\ell)$.

Conjecture 1.4 (Beilinson–Block, 1980s)**.**

(1) (*Rank part*).
$$\mathrm{ord}_{s=m}L(H^{2m-1}(X),s)=\operatorname{rank}\mathrm{CH}^m(X)^0.$$
(2) (*Leading coefficient*).
$$L^{(r)}(H^{2m-1}(X),m)\sim \det(\langle Z_i,Z_j'\rangle_{\mathrm{BB}})_{r\times r}.$$

Example 1.5. Let $m=1$ and $X=K$ over $E/\mathbb{Q}$. Then BB conjecture 1.4 recovers the BSD conjecture 1.1 as

$$\mathrm{CH}^1(E)^0\simeq E(\mathbb{Q}),\quad L(H^1(E),s)=L(E,s),\quad \langle\cdot,\cdot\rangle_{\mathrm{BB}}=-\langle\cdot,\cdot\rangle_{\mathrm{NT}}.$$

Remark 1.6. In general, both sides in Conjecture 1.4(1) are only conditionally defined.

- $L(H^{2m-1}(X),s)$ is not known to be analytically continued to the central point $s=m$.
- $\mathrm{CH}^m(X)^0$ is not known to be finitely generated.

Accordingly, BB conjecture is testable when X is a certain Shimura variety. Due to the works by Langlands–Kottwitz and Langlands–Rapoport, one can express the motivic L-functions of Shimura varieties $X=\mathrm{Sh}_G$ as a product of automorphic L-functions $L(s,\pi)$ on G, i.e.

$$L(H^{2m-1}(\mathrm{Sh}_G),s+m)=\prod_{\pi}L(s+1/2,\pi).$$

In the upcoming context we focus on the most interested case. For this, assume from now

(i) $2m-1=\dim X$, so that we can consider the arithmetic middle degree;
(ii) π is tempered cuspidal.

In is known that the analytic properties of $L(s,\pi)$ can be established, and hence we are able to detect those of the motivic L-function. However, $\mathrm{CH}^m(X)^0$ is not known to be finitely generated even when $X=\mathrm{Sh}_G$, but we can test if it is nontrivial.

The following is an unconditional prediction of BB conjecture, in the same spirit of Gross–Zagier.

Conjecture 1.7 (Beilinson–Bloch for Shimura varieties)**.**

$$\mathrm{ord}_{s=1/2}L(s,\pi)=1 \implies \operatorname{rank}\mathrm{CH}^m(X)^0_\pi\geqslant 1,$$

where $\mathrm{CH}^m(X)^0_\pi$ *is the* π*-isotypical component of* $\mathrm{CH}^m(X)^0$.

Remark 1.8. Conjecture 1.7 was only known for:

(1) X is a modular curve, by Gross–Zagier [GZ86];
(2) X is a Shimura curve, by S. Zhang [Zha01b, Zha01a], Kudla–Rapoport–Yang [KRY06], Yuan–Zhang–Zhang [YZZ13], Liu [Liu16, Liu19];
(3) $X = \mathrm{U}(1,1) \times \mathrm{U}(2,1)$ is a Shimura threefold and π is endoscopic, by Xue [Xue19].

Theorem 1.9 (Li–Liu, the impressionist version). *Conjecture 1.7 holds for* $\mathrm{U}(2m-1,1)$*-Shimura varieties while* π *satisfying certain local assumptions.*

1.2. **Beilinson–Bloch conjecture for** $\mathrm{U}(2m-1,1)$**-Shimura varieties.** Before setting up the unitary Shimura variety with $\mathrm{U}(2m-1,1)$, we first consider the Hermitian symmetric space for $\mathrm{U}(n-1,1)$, that is,

$$\mathbb{D}_{n-1} := \{z \in \mathbb{C}^{n-1} : |z| < 1\} \cong \frac{\mathrm{U}(n-1,1)}{\mathrm{U}(n-1) \times \mathrm{U}(1)}.$$

Moreover, we have an action on $\mathbb{D}_{n-1}$ by $\mathrm{U}(n-1,1)$. Notice that $\mathbb{D}_1$ can be regarded as a hyperbolic plane (and is hence isomorphic to the upper half complex plane $\mathbb{H}$).

1.2.1. *The unitary Shimura variety* X. Let E be a CM extension of a totally real number field F over $\mathbb{Q}$. Let $\mathbb{V}$ be a totally definite *incoherent* $\mathbb{A}_E/\mathbb{A}_F$*-hermitian space* of rank n; here $\mathbb{V}$ is incoherent if it is not the base change of a global E/F-hermitian space, or equivalently, $\prod_v \epsilon(\mathbb{V}_v) = -1$ with $\mathbb{V}_v := \mathbb{V} \otimes_{\mathbb{A}_F} F_v$. On the other hand, any place $w \mid \infty$ of F gives a nearby *coherent* E/F*-hermitian space* V such that

$$V_v \cong \mathbb{V}_v, \quad v \neq w,$$

whereas V_w has signature $(n-1,1)$.

Set $G = \mathrm{U}(\mathbb{V})$ and fix an open compact subgroup $K \subset G(\mathbb{A}_F^\infty) \cong \mathrm{U}(V)(\mathbb{A}_F^\infty)$. Then we can take X to be the unitary Shimura variety of dimension $n-1$ over its reflex field E such that for any place $w \mid \infty$ inducing the complex embedding $\iota_w : E \hookrightarrow \mathbb{C}$,

$$X(\mathbb{C}) = \mathrm{U}(V)(F)\backslash(\mathbb{D}_{n-1} \times \mathrm{U}(V)(\mathbb{A}_F^\infty)/K).$$

It turns out that X is a Shimura variety of abelian type. Its étale cohomology and L-function are computed in the forthcoming work of Kisin–Shin–Zhu [KSZ21], under the help of the endoscopic classification for unitary groups (Mok [Mok15], Kaletha–Minguez–Shin–White [KMSW14]). See also Remark 4.2.

1.2.2. *Automorphic representations* π. Resume on the setup above. Let $W = E^{2m}$ be the standard E/F-skew-hermitian space with matrix $\left(\begin{smallmatrix} 0 & 1_m \\ -1_m & 0 \end{smallmatrix}\right)$. Let $\mathrm{U}(W)$ be the quasi-split unitary group of rank $n = 2m$. Let π be the cuspidal automorphic representation of $\mathrm{U}(W)(\mathbb{A}_F)$.

Assumptions 1.10. We assume the following about π_v locally.

(1) E/F is split at all 2-adic places and $F \neq \mathbb{Q}$. Assume that $E/\mathbb{Q}$ is Galois or contains an imaginary quadratic field (for simplicity).
(2) For $v \mid \infty$, π_v is the holomorphic discrete series with Harish-Chandra parameter $\{(n-1)/2, (n-3)/2, \ldots, (-n+3)/2, (-n+1)/2\}$.
(3) For $v \nmid \infty$, π_v is tempered.
(4) For $v \nmid \infty$ ramified in E, π_v is spherical with respect to the stabilizer of $\mathcal{O}_{E_v}^{2m}$.
(5) For $v \nmid \infty$ inert in E, π_v is unramified or almost unramified. If π_v is almost unramified, then v is unramified over $\mathbb{Q}$.

Remark 1.11 (Almost unramifiedness). Saying π_v is almost unramified means that π_v has a nonzero Iwahori-fixed vector and its Satake parameter contains $\{q_v, q_v^{-1}\}$ and $2m-2$ complex numbers of norm 1. Equivalently, the theta lift of π_v to the non-quasi-split unitary group of same rank is spherical with respect to the stabilizer of an almost self-dual lattice.

1.2.3. *Main result.* The first main result of [LL21, LL22] is the verification of BB conjecture. Let S_π be the set of places v that are inert and such that π_v's are almost unramified. Then under Assumptions 1.10, the *global root number* for the (complete) standard L-function $L(s,\pi)$ equals

$$\epsilon(\pi) = (-1)^{|S_\pi|} \cdot (-1)^{m \cdot [F:\mathbb{Q}]}$$

by epsilon dichotomy (Harris–Kudla–Sweet [HKS96a], Gan–Ichino [GI16]). When $\mathrm{ord}_{s=1/2} L(s,\pi) = 1$:

- $\epsilon(\pi) = -1$,
- $\mathbb{V} = \mathbb{V}_\pi$ is the totally definite incoherent space of rank $n = 2m$ such that, for $v \nmid \infty$, we have $\epsilon(\mathbb{V}_v) = -1$ exactly for $v \in S_\pi$,
- X, the associated unitary Shimura variety, is of dimension $n-1 = 2m-1$ over E, and
- $\mathrm{CH}^m(X)^0_\pi$ is the localization of $\mathrm{CH}^m(X)^0_{\mathbb{C}}$ at the maximal ideal $\mathfrak{m}_\pi$ of the Hecke algebra associated to π.

Theorem 1.12 ([LL21, LL22])**.** *Let π be a cuspidal automorphic representation of $\mathrm{U}(W)(\mathbb{A}_F)$ satisfying Assumptions 1.10. Then the implication*

$$\mathrm{ord}_{s=1/2} L(s,\pi) = 1 \implies \mathrm{rank}\,\mathrm{CH}^m(X)^0_\pi \geqslant 1$$

holds when the level $K \subset G(\mathbb{A}_F^\infty)$ is sufficiently small.

Example 1.13 (Symmetric power L-function of elliptic curves)**.** Let A/F be a modular elliptic curve without complex multiplication such that

(i) A has bad reduction only at places v that split in E;
(ii) $\mathrm{Sym}^{2m-1} A_E$ is automorphic (Newton–Thorne, Clozel–Thorne, etc.).

Then there exists π satisfying Assumptions 1.10 such that

$$L(s+1/2,\pi) = L(\mathrm{Sym}^{2m-1} A_E, s+m).$$

As $S_\pi = \emptyset$ and $\epsilon(\pi) = (-1)^{m\cdot[F:\mathbb{Q}]}$, Theorem 1.12 applies to π when $m \cdot [F:\mathbb{Q}]$ is odd.

1.3. **Arithmetic inner product formula and arithmetic theta lifting.**

1.3.1. *Generating series of Heegner points.* Nontrivial cycles can be constructed via the method of arithmetic theta lifting by Kudla and Liu [Liu11a, Liu11b]. Here comes a baby example of Heegner points, which contributes to Gross–Zagier formula as well.

Consider the modular curve

$$X_0(N) = \Gamma_0(N)\backslash\mathbb{H} \cup \{\text{cusps}\} = \{E_1 \to E_2 : \text{cyclic } N\text{-isogeny}\}.$$

For certain imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$, we have a *Heegner divisor*

$$Z(d) := \{E_1 \to E_2 \text{ with endomorphisms by } \mathcal{O}_K\} \in \mathrm{CH}^1(X_0(N)).$$

The theory of complex multiplication asserts that $Z(d)$ is actually a divisor of $X_0(N)$ defined over K.

Let $E/\mathbb{Q}$ be an elliptic curve of conductor N who has a modular parametrization

$$\varphi_E : X_0(N) \longrightarrow E.$$

Using these, we define a *Heegner point*

$$P_K \in \varphi_E(Z(d) - \deg Z(d)\cdot\infty) \in E(K).$$

Then we are able to state the Gross–Zagier formula.

Theorem 1.14 (Gross–Zagier, [GZ86])**.** *Up to simpler nonzero factors,*

$$L'(E_K, 1) \sim \langle P_K, P_K\rangle_{\mathrm{NT}}.$$

Remark 1.15. (1) Choosing K suitably gives the implications

$$r_{\mathrm{an}}(E) = 1 \implies r_{\mathrm{alg}}(E) \geqslant 1.$$

(2) BSD formula for E_K reduces to a precise relation between P_K and $\text{Ш}(E_K)$.

To introduce Arithmetic theta liftings, we first consider the following heuristic example. Recall that $K = \mathbb{Q}(\sqrt{-d})$. Take $P_d = \mathrm{tr}_{K/\mathbb{Q}} P_K \in E(\mathbb{Q})$. It may depend on the choice of d, even when $E(\mathbb{Q}) \cong \mathbb{Z}$.

Example 1.16. Let $E = X_0^+(37) : y^2 + y = x^3 - x$. Then

- ◇ $E(\mathbb{Q}) \cong \mathbb{Z}$ with a generator $P = (0,0)$.
- ◇ E corresponds to the modular form $f \in S_2(37)$ where

$$f = q - 2q^2 - 3q^3 + 2a^4 - 2q^5 + 6q^6 - q^7 + 6q^9 + 4q^{10} - 5q^{11} - 6q^{12} - 2q^{13} + \cdots.$$

- ◇ Table of Heegner points P_d:

d	3	4	7	11	12	16	27	$\cdots$	67	$\cdots$
P_d	$(0,-1)$	$(0,-1)$	$(0,0)$	$(0,-1)$	$(0,0)$	$(1,0)$	$(-1,-1)$	$\cdots$	$(6,-15)$	$\cdots$
c_d	-1	-1	1	-1	1	2	3	$\cdots$	-6	$\cdots$

where $P_d = c_d \cdot P$.

Now the miracle is that the coefficients c_d appear as the Fourier coefficients of $\phi \in S^+_{3/2}(4 \cdot 37)$, for

$$\phi = \sum_{d \geqslant 1} c_d q^d = -q^3 - q^4 + q^7 - q^{11} + q^{12} + 2q^{16} + 3q^{27} + \cdots - 6q^{67} + \cdots,$$

which maps to f under the Shimura–Waldspurger–Kohnen correspondence

$$\theta : S^+_{3/2}(4 \cdot 37) \longrightarrow S_2(N), \quad \phi \longmapsto f.$$

1.3.2. *Arithmetic theta lifting.* The observation arising from Example 1.16 dictates that the generating series of Heegner points

$$\sum_{d \geqslant 1} P_d \cdot q^d = \sum_{d \geqslant 1} c_d P \cdot q^d = \phi \cdot P \in S^+_{3/2}(4 \cdot 37) \otimes E(\mathbb{Q})_{\mathbb{C}}$$

is a modular form valued in $E(\mathbb{Q})_{\mathbb{C}}$. More generally, we may define a generating series of Heegner divisors on $X_0(N)$,

$$Z := \sum_d Z(d) q^d \in M_{3/2}(4N) \otimes \mathrm{CH}^1(X_0(N))_{\mathbb{C}},$$

which may be viewed as an *arithmetic theta series.*

Definition 1.17. Use Z as the kernel to define *arithmetic theta lifting*

$$\Theta(\phi) := (Z, \phi)_{\mathrm{Pet}} \in \mathrm{CH}^1(X_0(N))^0_{f,\mathbb{C}} = E(\mathbb{Q})_{\mathbb{C}}.$$

Indeed, $\Theta(\phi)$ does not depend on any particular choice of d or K.

Theorem 1.18 (Gross–Kohnen–Zagier, [GKZ87])**.** *Up to simpler nonzero factors,*

$$L'(E, 1) \sim \langle \Theta(\phi), \Theta(\phi) \rangle_{\mathrm{NT}}.$$

Now let us focus on the case where X is a unitary Shimura variety as before.

Definition 1.19 (Special cycles on unitary Shimura variety)**.** Suppose $X = \mathrm{Sh}_{\mathrm{U}(V)}$.

(1) For any $y \in V$ with $(y, y) > 0$, its orthogonal complement $V_y \subset V$ has rank $n - 1$. The embedding $\mathrm{U}(V_y) \hookrightarrow \mathrm{U}(V)$ defines a Shimura subvariety of codimension1, read as
$$\mathrm{Sh}_{\mathrm{U}(V_y)} \longrightarrow X = \mathrm{Sh}_{\mathrm{U}(V)}.$$
(2) For any $x \in V(\mathbb{A}_F^\infty)$ with $(x, x) \in F_{>0}$, there exists $y \in V$ and $g \in \mathrm{U}(V)(\mathbb{A}_F^\infty)$ such that $y = gx$. Define the *special divisor*
$$Z(x) \longrightarrow X$$
to be the g-translate of $\mathrm{Sh}_{\mathrm{U}(V_y)}$.

(3) For any $\mathbf{x} = (x_1, \ldots, x_m) \in V(\mathbb{A}_F^\infty)^m$ with $T(\mathbf{x}) = ((x_i, x_j)) \in \mathrm{Herm}^\circ_m(F)^+$, define the *special cycle* (of codimension m) as
$$Z(\mathbf{x}) = Z(x_1) \cap \cdots \cap Z(x_m) \longrightarrow X.$$
(4) More generally, for a Schwartz function $\varphi \in \mathscr{S}(V(\mathbb{A}_F^\infty)^m)^K$ and $T \in \mathrm{Herm}^\circ_m(F)^+$, define the *weighted special cycle*
$$Z_\varphi(T) = \sum_{\substack{\mathbf{x} \in K \backslash V(\mathbb{A}_F^\infty)^m, \\ T(\mathbf{x}) = T}} \varphi(\mathbf{x}) Z(\mathbf{x}) \in \mathrm{CH}^m(X)_{\mathbb{C}}.$$
(5) With extra care, we can also define $Z_\varphi(T) \in \mathrm{CH}^m(X)_{\mathbb{C}}$ for any $T \in \mathrm{Herm}_m(F)_{\geqslant 0}$.

Definition 1.20. Define *Kudla's generating series of special cycles* as

$$Z_\varphi(\tau) = \sum_{T \in \mathrm{Herm}_m(E)_{\geqslant 0}} Z_\varphi(T) q^T.$$

Conjecture 1.21 (Kudla's modularity [Kud97a, Kud04])**.** *The formal generating series $Z_\varphi(\tau)$ converges absolutely and defines a modular form on $\mathrm{U}(W)$ valued in $\mathrm{CH}^m(X)_{\mathbb{C}}$.*

Remark 1.22.
(1) The analogous modularity in Betti cohomology is known by Kudla–Millson [KM90] in 1980s.
(2) Conjecture is known for $m = 1$. For general m, the modularity follows from the absolute convergence [Liu11b].
(3) The analogous conjecture for orthogonal Shimura varieties over $\mathbb{Q}$ is known by Bruinier–Westerholt-Raum [BWR15].
(4) Conjecture is known when E/F is a norm-Euclidean imaginary quadratic field, due to Xia [Xia21].

Definition 1.23. Assuming Kudla's modularity conjecture, for $\phi \in \pi$, define *arithmetic theta lifting* for Kudla's generating series of weighted special cycles as

$$\Theta_\varphi(\phi) = (Z_\varphi(\tau), \phi)_{\mathrm{Pet}} \in \mathrm{CH}^m(X)^0_\pi.$$

Theorem 1.24 ([LL21, LL22])**.** *Let π be a cuspidal automorphic representation of $\mathrm{U}(W)(\mathbb{A}_F)$ satisfying Assumptions 1.10. Assume $\epsilon(\pi) = -1$. Assume Kudla's modularity in Conjecture 1.21. Then for any $\phi \in \pi$ and $\varphi \in \mathscr{S}(V(\mathbb{A}_F^\infty)^m)$, up to simpler factors depending on ϕ and φ,*

$$L'(1/2, \pi) \sim \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi)\rangle_{\mathrm{BB}}.$$

Remark 1.25. The simpler factors can be further made explicit. For example, if

- π is unramified or almost unramified at all finite places,
- $\phi \in \pi$ is a holomorphic newform such that $(\phi, \overline{\phi})_\pi = 1$, and if
- φ is a characteristic function of self-dual or almost self-dual lattices at all finite places,

then

$$\frac{L'(1/2,\pi)}{\prod_{i=1}^{2m} L(i, \eta^i_{E/F})} C_m^{[F:\mathbb{Q}]} \prod_{v \in S_\pi} \frac{q_v^{m-1}(q_v+1)}{(q_v^{2m-1}+1)(q_v^{2m}-1)} = (-1)^m \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi)\rangle_{\mathrm{BB}},$$

where

$$C_m = 2^{m(m-1)} \pi^{m^2} \frac{\Gamma(1)\cdots\Gamma(m)}{\Gamma(m+1)\cdots\Gamma(2m)}.$$

Moreover, as an addendum,

(1) The classical Riemann hypothesis predicts that

$$L'(1/2, \pi) \geqslant 0;$$

(2) Beilinson's Hodge index conjecture predicts that

$$(-1)^m \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi)\rangle_{\mathrm{BB}} \geqslant 0.$$

The combination of (1) and (2) is compatible with our formula.

Before introducing the proof strategy of Li–Liu, we list out a brief summary on arithmetic theta lifting, as well as the generalization from BSD conjecture to BB conjecture.

	BSD conjecture	BB conjecture
Ambient varieties	Modular curves $X_0(N)$	Unitary Shimura varieties X
Simple geometric objects	Heegner points $Z(d)$	Special cycles $Z_\varphi(T)$
Kudla's generating series	$Z = \sum_d Z(d) q^d \in \mathrm{CH}^1(X_0(N))_{\mathbb{C}}$	$Z_\varphi = \sum_T Z_\varphi(T) q^T \in \mathrm{CH}^m(X)_{\mathbb{C}}$
Arithmetic theta liftings	$\Theta(\phi) \in E(\mathbb{Q})_{\mathbb{C}}$	$\Theta_\varphi(\phi) \in \mathrm{CH}^m(X)^0_\pi$
Formulas	Gross–Zagier formula $L'(E,1) \sim \langle \Theta(\phi), \Theta(\phi)\rangle_{\mathrm{NT}}$	Arithmetic inner product formula $L'(1/2,\pi) \sim \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi)\rangle_{\mathrm{BB}}$

1.3.3. *The proof strategy (I): Doubling method.* The doubling method is introduced by Piatetski-Shapiro–Rallis [PSR86, PSR87] and Yamana [Yam14], read as

$$L(s+1/2, \pi) \sim (\phi \otimes \overline{\phi}, \mathrm{Eis}(s, g))_{\mathrm{U}(W)^2},$$

where $\mathrm{Eis}(s,g)$ is a Siegel Eisenstein series on $\mathrm{U}(W \oplus W)$.

By definition $\Theta_\varphi(\phi) = (Z_\varphi, \phi)_{\mathrm{Pet}}$ gives

$$\langle \Theta_\varphi(\phi), \Theta_\varphi(\phi)\rangle_{\mathrm{BB}} = (\phi \otimes \overline{\phi}, \langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB}})_{\mathrm{U}(W)^2}.$$

To prove $L'(1/2,\pi) \sim \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi)\rangle_{\mathrm{BB}}$, it suffices to compare

$$\mathrm{Eis}'(0,g) \stackrel{?}{=} \langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB}}.$$

This can be viewed as an *arithmetic Siegel–Weil formula*. Here the Beilinson–Bloch height pairing is a sum of local indexes

$$\langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB}} = \sum_v \langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB},v}.$$

And the nonsingular Fourier coefficient for the q^T-term decomposes as

$$\mathrm{Eis}'_T(0,g) = \sum_v \mathrm{Eis}'_{T,v}(0,g).$$

1.3.4. *The proof strategy (II): Local comparison on arithmetic Siegel–Weil formula.* For nonsingular local terms, it suffices to prove

$$\mathrm{Eis}'_{T,v}(0,g) \stackrel{?}{=} \langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB},T,v}.$$

In codimension 1 case with $m = 1$, the Gross–Zagier formula computes both sides explicitly. However, such an explicit computation is infeasible for general m.

- When $v \nmid \infty$, we use:
  - (i) the work for relating $\langle Z_\varphi, Z_\varphi\rangle_{\mathrm{BB},T,v}$ to arithmetic intersection numbers;
  - (ii) recent proof of Kudla–Rapoport conjecture due to Li–Zhang [LZ22a].
- When $v \mid \infty$, we use:
  - (i) archimedean arithmetic Siegel–Weil formula, proved by Liu [Liu11a] and Garcia–Sankaran [GS19] independently;
  - (ii) avoidance of holomorphic projections.

To finish the argument, we kill singular terms on both sides by proving the existence of special $\varphi \in \mathscr{S}(V(\mathbb{A}_F^\infty)^m)$ with *regular support* at two split places with nonvanishing local zeta integrals. Motivated by the comparison of nonsingular terms which deduced Theorem 1.24 for special φ, we can extrapolate such a proof for arbitrary φ with *multiplicity one* of doubling method in tempered case. Consequently, Theorem 1.12 is given by a same computation without Kudla's modularity, using the proof by contradiction.

Remark 1.26. We have some final remarks on Assumptions 1.10.

1. When $v \nmid \infty$, the local index $\langle \cdot,\cdot\rangle_{\mathrm{BB},v}$ is defined as an ℓ-adic linking number. It is defined on a certain subspace $\mathrm{CH}^m(X)^{\langle \ell\rangle} \subset \mathrm{CH}^m(X)^0$ (which are conjecturally equal) and its independence on ℓ is not known in general.
2. Find a Hecke operator $t \notin \mathfrak{m}_\pi$ such that $t^*Z \in \mathrm{CH}^m(X)^{\langle \ell\rangle}$, so BB height is defined. Also find another Hecke operator $s \notin \mathfrak{m}_\pi$ and BB height of s^*t^*Z can be therefore computed in terms of the *arithmetic intersection number* of a nice extension $\mathcal{Z}$ on $\mathcal{X}$. Here $\mathcal{X}$ is a regular integral model of a related unitary Shimura variety of PEL type. This step requires to prove certain vanishing of $\mathfrak{m}_\pi$-localized ℓ-adic cohomology of $\mathcal{X}$.
3. The *Kudla–Rapoport conjecture* states that
$$\mathrm{Eis}'_{T,v}(0,g) = \text{the arithmetic intersection number above.}$$
Assuming the conjecture, the ℓ-independence of $\langle Z_\varphi, Z_\varphi$
$rangle_{\mathrm{BB},T,v}$ will follow.
4. Assumptions 1.10 are required in both construction of Hecke operators and the proof of Kudla–Rapoport conjecture.
5. The condition $F \neq \mathbb{Q}$ of fields is needed to prove vanishing of $\mathfrak{m}_\pi$-localized cohomology of integral models with Drinfeld level structures at split places (with input from Mantovan [Man08], Caraiani–Scholze [CS17]).

2. The Gross–Zagier formula and height pairings

2.1. **The Gross–Zagier formula.** Consider

- a (modular) elliptic curve $E/\mathbb{Q}$ of conductor N, corresponding to a normalized new cusp form $f_E = \sum_{n\geqslant 1} a_n(E)q^n \in S_2(\Gamma_0(N))$,

- an imaginary quadratic field K (with d_K its discriminant) satisfying that every prime factor of N splits in K.

Denote by $X_0(N)$ the compactified modular curve of level $\Gamma_0(N)$ over $\mathbb{Q}$. Recall that away from cusps, $X_0(N)(\mathbb{C})$ is the set of isomorphism classes of cyclic isogenies $[E \to E']$ of complex elliptic curves of degree N.

Choose an ideal $\mathfrak{n}$ of $\mathcal{O}_K$ satisfying $\mathcal{O}_K/\mathfrak{n} = \mathbb{Z}/N\mathbb{Z}$. For a modular parametrization $\varphi : X_0(N) \to E$, we define the Heegner point

$$P_K^\varphi := \sum_{\mathfrak{a} \in \mathrm{Cl}(K)} \varphi([\mathbb{C}/\mathfrak{a} \to \mathbb{C}/\mathfrak{n}^{-1}\mathfrak{a}]) \in E(\mathbb{C}).$$

By the theory of CM elliptic curves, one sees that the above sum is independent of $\mathfrak{n}$ and the choice of representatives of $\mathrm{Cl}(K)$; moreover, P_K^φ belongs to $E(K)$.

Theorem 2.1 (Gross–Zagier, [GZ86])**.**

$$L'(1, E/K) = \frac{32\pi^2 \|f_E\|_{\mathrm{Pet}}^2}{|\mathcal{O}_K^\times|^2 \sqrt{|d_K|}} \frac{h_{\mathrm{NT}}(P_K^\varphi)}{\deg \varphi}.$$

2.2. **Néron–Tate height.** Let K be a general number field and C/K a geometrically connected smooth projective curve. We recall the definition of the Néron–Tate height on $\mathrm{Div}^0(C)$ (which is same as $Z^1(C)^0$).

Let J_C be the Jacobian variety of C over K, so that there is a canonical homomorphism $\alpha : \mathrm{Div}(C)^0 \to \mathrm{J}_C(K)$ of abelian groups. (Twice of) the theta divisor on J_C gives rise to a height function $h : \mathrm{J}_C(K) \to \mathbb{R}$.

Definition 2.2. For every $D \in \mathrm{Div}(C)^0$, we define

$$h_{\mathrm{NT}}(D) := \lim_{n\to\infty} \frac{h(\alpha(nD))}{n^2}$$

in which the limit exists.

It turns out that

(a) h_{NT} descends to a function on $\mathrm{CH}^1(C)^0 = \mathrm{Div}(C)^0/\sim_{\mathrm{rat}}$;
(b) h_{NT} is a positive definite quadratic function on $\mathrm{CH}^1(C)^0$.

In what follows, we denote by $\langle \cdot, \cdot \rangle_{\mathrm{NT}} : \mathrm{CH}^1(C)^0 \times \mathrm{CH}^1(C)^0 \to \mathbb{R}$ the associated quadratic form. The quadratic form $\langle \cdot, \cdot \rangle_{\mathrm{NT}}$ admits a decomposition into local heights over all places of K, which we review. For $? \in \{\emptyset, u\}$ where u is a place of K, put $C_? := C \otimes_K K_?$ and denote by $(\mathrm{Div}(C_?)^0 \times \mathrm{Div}(C_?)^0)^*$ the subgroup of $\mathrm{Div}(C_?)^0 \times \mathrm{Div}(C_?)^0$ consisting of pairs of degree zero divisors with disjoint support.

For every place u of K, there is a unique function (called *Néron symbol*)

$$\langle \cdot, \cdot \rangle_u : (\mathrm{Div}(C_?)^0 \times \mathrm{Div}(C_?)^0)^* \longrightarrow \mathbb{R}$$

that is bi-additive, symmetric, continuous, and satisfies

$$\langle a, b \rangle = -\sum m_x \log |f(x)|_u$$

when $a = \sum m_x x$ and $b = \mathrm{div}(f)$.

We have the identity of real-valued functions on $(\mathrm{Div}(C_?)^0 \times \mathrm{Div}(C_?)^0)^*$:

$$\langle \cdot, \cdot \rangle_{\mathrm{NT}} = -\sum_u \langle \cdot, \cdot \rangle_u.$$

Since the natural map

$$(\mathrm{Div}(C_?)^0 \times \mathrm{Div}(C_?)^0)^* \longrightarrow \mathrm{CH}^1(C)^0 \times \mathrm{CH}^1(C)^0$$

is surjective, this gives a decomposition formula for the Néron–Tate height pairing.

Definition 2.3. We review the definition of $\langle \cdot, \cdot \rangle_u$.

(1) Suppose that $u < \infty$. If C_u admits a smooth projective model $\mathcal{C}_u$ over $\mathcal{O}_{K_u}$, then

$$\langle a, b \rangle_u = \log q_u \cdot (\overline{a}, \overline{b})_{\mathcal{C}_u},$$

where $(\overline{a}, \overline{b})_{\mathcal{C}_u}$ denotes the intersection number of the Zariski closures of a and b in $\mathcal{C}_u$ and q_u denotes the residue cardinality of K_u.

More generally, C_u always admits a regular projective model $\mathcal{C}_u$ and we shall take $\overline{a}$ and $\overline{b}$ to be *flat extensions* of a and b, respectively. Here, an extension is flat if it has zero intersection number with every component of the special fiber of $\mathcal{C}_u$.

(2) Suppose that $u \mid \infty$. We have

$$\langle a, b\rangle_u = \frac{[K_u : \mathbb{R}]}{2} \sum m_x G_b(x)$$

if $a = \sum m_x x$, where G_b is a *Green function* for b, that is, a smooth function on $C_u(\mathbb{C})\backslash|b|$ such that $\mathrm{dd^c} G_b + \delta_b = 0$ as currents (also recall that $\mathrm{d^c} = (4\pi i)^{-1}(\partial - \overline{\partial})$).

2.3. **Beilinson's height pairing.** We introduce Beilinson's generalization of the Néron–Tate height to higher dimensional varieties. Suppose that X is a smooth scheme over a field K of characteristic zero.

Notation 2.4. For every prime number ℓ,

(1) We denote by $Z^m(X)^0$ the kernel of the de Rham cycle class map

$$\mathrm{cl}_{X,\mathrm{dR}} : Z^m(X) \longrightarrow H^{2m}_{\mathrm{dR}}(X/K)(m),$$

and by $\mathrm{CH}^m(X)^0$ the image of $Z^m(X)^0$ in $\mathrm{CH}^m(X)$.

(2) When K is a non-archimedean local field, we denote by $Z^m(X)^{\langle\ell\rangle}$ the kernels of the (absolute) ℓ-adic cycle class map

$$\mathrm{cl}_{X,\ell} : Z^m(X) \longrightarrow H^{2m}(X, \mathbb{Q}_\ell(m)).$$

By the comparison theorem between de Rham and ℓ-adic cohomology, we have

$$Z^m(X)^{\langle\ell\rangle} \subset Z^m(X)^0.$$

In fact, the Monodromy–Weight conjecture for X implies that when ℓ is invertible on $\mathcal{O}_{K_u}$, we have $Z^m(X)^{\langle\ell\rangle} = Z^m(X)^0$.

(3) When K is a number field, we define $Z^m(X)^{\langle\ell\rangle}$ via the following Cartesian diagram

$$\begin{array}{ccc} Z^m(X)^{\langle\ell\rangle} & \longrightarrow & \prod_{u\nmid\infty\ell} Z^m(X_u)^{\langle\ell\rangle} \\ \downarrow & & \downarrow \\ Z^m(X) & \longrightarrow & \prod_{u\nmid\infty\ell} Z^m(X_u) \end{array}$$

where the product is taken over all non-archimedean places u of K not above ℓ. We denote by $\mathrm{CH}^m(X)^{\langle\ell\rangle}$ the image of $Z^m(X)^{\langle\ell\rangle}$ in $\mathrm{CH}^m(X)$.

Now let K be a number field and consider

- a smooth projective scheme X over K of pure dimension $n-1$ (for some $n \geqslant 2$),
- a prime number ℓ such that X_u has good reduction for every place u of K above ℓ, and
- a pair of nonnegative integers (d_1, d_2) satisfying $d_1 + d_2 = n$.

For $? \in \{\emptyset, u\}$ where u is a place of K and $\#\{\langle\ell\rangle, 0\}$, denote by

$$(Z^{d_1}(X_?)^\# \times Z^{d_2}(X_?)^\#)^* \subset Z^{d_1}(X_?)^\# \times Z^{d_2}(X_?)^\#$$

consisting of pairs of cycles with disjoint support. We respectively define maps

$$\begin{aligned} \langle\cdot,\cdot\rangle_u &: (Z^{d_1}(X_u)^0 \times Z^{d_2}(X_u)^0)^* \longrightarrow \mathbb{R}, && \text{when } u \mid \infty; \\ \langle\cdot,\cdot\rangle_u &: (Z^{d_1}(X_u)^{\langle\ell\rangle} \times Z^{d_2}(X_u)^{\langle\ell\rangle})^* \longrightarrow \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell, && \text{when } u \nmid \ell\infty; \\ \langle\cdot,\cdot\rangle_u &: (Z^{d_1}(X_u)^0 \times Z^{d_2}(X_u)^0)^* \longrightarrow \mathbb{R}, && \text{when } u \mid \ell. \end{aligned}$$

Take an element (c_1, c_2) in the source of any of these maps and denote by Z_i the support of c_i so that $Z_1 \cap Z_2 = \emptyset$.

Case 1: $u \mid \infty$. We define

$$\langle c_1, c_2\rangle_u := \frac{[K_u : \mathbb{R}]}{2} \int_{X_u(\mathbb{C})} \delta_{c_1} \wedge g_{c_2} \in \mathbb{R},$$

where δ_{c_1} denotes the Dirac current of c_1 and g_{c_2} is a regular harmonic Green current for c_2, that is, a smooth (d_2-1, d_2-1)-form on $X_u(\mathbb{C})\backslash Z_2$ such that $\mathrm{dd^c} g_{c_2} + \delta_{c_2} = 0$ as currents.

Case 2: $u \nmid \infty\ell$. Let $\alpha_i \in H^{2d_i}_{Z_i}(X_u, \mathbb{Q}_\ell(d_i))$ be the refined cycle class of c_i. As α_i goes to zero in $H^{2d_i}(X_u, \mathbb{Q}_\ell(d_i))$ by definition, there exists $\gamma_i \in H^{2d_i-1}(U_i, \mathbb{Q}_\ell(d_i))$ that goes to α_i under the coboundary map

$$H^{2d_i-1}(U_i, \mathbb{Q}_\ell(d_i)) \longrightarrow H^{2d_i}_{Z_i}(X_u, \mathbb{Q}_\ell(d_i)),$$

where $U_i = X_u \backslash Z_i$. Then

$$\langle c_1, c_2 \rangle_u := \log q_u \otimes \langle c_1, c_2 \rangle'_u,$$

where $\langle c_1, c_2 \rangle'_u$ is the image of $\gamma_1 \cup \gamma_2$ under the composite map

$$\begin{aligned} H^{2n-2}(U_1 \cap U_2, \mathbb{Q}_\ell(n)) &\longrightarrow H^{2n-1}(X_u, \mathbb{Q}_\ell(n)) \\ &\longrightarrow H^1(\operatorname{Spec} K_u, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell \end{aligned}$$

in which the first arrow is the coboundary map in the Mayer–Vietoris exact sequence for the covering $X_u = U_1 \cup U_2$. Here, the identification $H^1(\operatorname{Spec} K_u, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell$ is the composition

$$H^1(\operatorname{Spec} K_u, \mathbb{Q}_\ell(1)) \longrightarrow H^2_{\operatorname{Spec} \kappa_u}(\operatorname{Spec} \mathcal{O}_{K_u}, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} H^0(\operatorname{Spec} \kappa_u, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$$

where κ_u denotes the residue field of K_u; this is *negative* to the one given by the Kummer isomorphism for Galois cohomology. It is conjectured that $\langle c_1, c_2 \rangle'_u$ belongs to $\mathbb{Q}$ and is independent of ℓ.

Case 3: $u \mid \ell$. Choose a smooth projective model $\mathcal{X}_u$ of X_u over $\mathcal{O}_{K_u}$. Then

$$\langle c_1, c_2 \rangle_u := \log q_u \cdot (\mathcal{C}_1, \mathcal{C}_2)_{\mathcal{X}_u},$$

where $\mathcal{C}_i$ denotes the Zariski closure of c_i in $\mathcal{X}_u$. Later, we will justify this definition and in particular show that it is independent of the choice of the model.

Definition 2.5. We define *Beilinson's height pairing*

$$\langle \cdot, \cdot \rangle_{\mathrm{B}} := \sum_u \langle \cdot, \cdot \rangle_u : (Z^{d_1}(X)^{\langle \ell \rangle} \times Z^{d_2}(X)^{\langle \ell \rangle})^* \longrightarrow \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell.$$

It is not hard to show that $\langle \cdot, \cdot \rangle_{\mathrm{B}}$ is symmetric and descends to a map from $\mathrm{CH}^{d_1}(X)^{\langle \ell \rangle} \times \mathrm{CH}^{d_2}(X)^{\langle \ell \rangle}$ (which we now assume). Moreover, for every correspondence $t \in \mathrm{CH}^{n-1}(X \times X)$, we have

$$\langle t^* x, y \rangle_{\mathrm{B}} = \langle x, t_* y \rangle_{\mathrm{B}}.$$

Conjecture 2.6 (Beilinson, Bloch)**.**

(1) *We have*

$$\mathrm{CH}^d(X)^{\langle \ell \rangle} = \mathrm{CH}^d(X)^0,$$

and it has a finite rank.

(2) *Beilinson's height pairing map $\langle \cdot, \cdot \rangle_{\mathrm{B}}$ takes value in $\mathbb{R}$, is independent of ℓ, and is nondegenerate.*

(3) *For every ample class L and every integer $0 \leqslant d \leqslant n/2$, the form $\langle \cdot, L^{n-2d} \cdot \rangle_{\mathrm{B}}$ is $(-1)^d$-definite on the primitive part of $\mathrm{CH}^d(X)^0$.*

(4) *For every correspondence $t \in \mathrm{CH}^{n-1}(X \times X)^0$, the form $\langle t^* \cdot, \cdot \rangle_{\mathrm{B}}$ vanishes.*

In what follows, we will often use the complex *sesquilinear* (i.e. linear in the first variable and conjugate linear in the second variable) extension of $\langle \cdot, \cdot \rangle_{\mathrm{B}}$ or $\langle \cdot, \cdot \rangle_u$.

2.3.1. *non-archimedean local index.* Now we study $\langle \cdot, \cdot \rangle'_u$ for $u \nmid \infty\ell$ when X_u admits a regular projective model $\mathcal{X}_u$ over $\mathcal{O}_{K_u}$. To ease notation, we suppress u on this and the next two pages. Denote by $Y := \mathcal{X} \otimes_{\mathcal{O}_K} \kappa$ the special fiber of $\mathcal{X}$.

Notation 2.7. For two closed subschemes $\mathcal{Z}_1$ and $\mathcal{Z}_2$ of $\mathcal{X}$ of codimension d_1 and d_2, respectively, such that $\mathcal{Z}_1 \cap \mathcal{Z}_2 \subset Y$, we have the intersection number

$$(\mathcal{Z}_1, \mathcal{Z}_2)_{\mathcal{X}} := \chi(Y, \mathcal{O}_{\mathcal{Z}_1} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{Z}_2}).$$

By sesquilinear extension, we have the intersection number $(\mathcal{C}_1, \mathcal{C}_2)_{\mathcal{X}} \in \mathbb{C}$ for every pair $(\mathcal{C}_1, \mathcal{C}_2) \in Z^{d_1}(\mathcal{X})_{\mathbb{C}} \times Z^{d_2}(\mathcal{X})_{\mathbb{C}}$ satisfying $|\mathcal{C}_1| \cap |\mathcal{C}_2| \subset Y$.

Definition 2.8. We say that an extension $\mathcal{C} \in Z^d(\mathcal{X})_{\mathbb{C}}$ of $c \in Z^d(X)^{\langle \ell \rangle}_{\mathbb{C}}$ is *ℓ-flat* if the cycle class of $\mathcal{C}$ in $H^{2d}(\mathcal{X}, \mathbb{Q}_\ell(d)) \otimes_{\mathbb{Q}} \mathbb{C}$ vanishes.

Proposition 2.9. *Given $(c_1,c_2) \in (Z^{d_1}(X)^{\langle\ell\rangle}_{\mathbb{C}} \times Z^{d_2}(X)^{\langle\ell\rangle}_{\mathbb{C}})^*$ and a pair $(\mathcal{C}_1,\mathcal{C}_2) \in Z^{d_1}(\mathcal{X})^{\langle\ell\rangle}_{\mathbb{C}} \times Z^{d_2}(\mathcal{X})^{\langle\ell\rangle}_{\mathbb{C}}$ of extensions of (c_1,c_2) in which at least one is ℓ-flat, we have*

$$\langle c_1,c_2\rangle'_u = (\mathcal{C}_1,\mathcal{C}_2)_{\mathcal{X}}.$$

In particular, when $\mathcal{X}$ is smooth over $\mathcal{O}_K$, we can take $\mathcal{C}_i$ to be the Zariski closure of c_i in $\mathcal{X}$, hence $\langle c_1,c_2\rangle'_u$ belongs to $\mathbb{C}$ and is independent of ℓ. (This justifies the definition of the local height on Page 9 when $u \mid \ell$.)

2.3.2. *Étale correspondences and flat extensions.* Keep the previous setups. We propose a method of finding flat extensions using étale correspondences, with the application to Shimura varieties and Hecke correspondences in mind. For simplicity, we assume $n = 2r$ even and $d_1 = d_2 = r$.

Definition 2.10. (1) We say that a correspondence

$$t : \mathcal{X} \xleftarrow{p} \mathcal{X}' \xrightarrow{q} \mathcal{X}$$

of $\mathcal{X}$ is *étale* if both p and q are finite étale. A *complex étale correspondence* of $\mathcal{X}$ is a complex linear combination of étale correspondence of $\mathcal{X}$.

(2) We say that a complex étale correspondence t of $\mathcal{X}$ is ℓ-tempered if $t^*(= p_! \circ q^*)$ annihilates $H^{2r}(\mathcal{X},\mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{C}$.

Proposition 2.11. *Let t be an ℓ-tempered complex étale correspondence of $\mathcal{X}$. Then for every pair $(c_1,c_2) \in Z^r(X)_{\mathbb{C}} \times Z^r(X)_{\mathbb{C}}$ satisfying $\mathrm{Supp}(t^*c_1) \cap \mathrm{Supp}(t^*c_2) = \emptyset$, we have $(t^*c_1,t^*c_2) \in (Z^r(X)^{\langle\ell\rangle}_{\mathbb{C}} \times (Z^r(X)^{\langle\ell\rangle}_{\mathbb{C}})^*$ and*

$$\langle t^*c_1,t^*c_2\rangle'_u = (t^*\mathcal{C}_1,t^*\mathcal{C}_2)_{\mathcal{X}},$$

where $\mathcal{C}_1 \in Z^r(\mathcal{X})_{\mathbb{C}}$ is an arbitrary extension of c_i in $\mathcal{X}$ for $i = 1,2$. In particular, we have $\langle t^*c_1,t^*c_2\rangle'_u \in \mathbb{C}$.

3. Classical theta correspondence

3.1. **Beilinson–Bloch conjecture.** From now on, we take an even positive integer $n = 2r$. Let K be a number field and X a projective smooth scheme over K of (odd) dimension $n-1$. We have the L-function

$$L(s, H^{n-1}(X_{\overline{K}},\mathbb{Q}_\ell(r)))$$

for the middle degree ℓ-adic cohomology of X for every rational prime ℓ, which is conjectured to be meromorphic, independent of ℓ, and satisfy a functional equation with center $s = 0$.

The unrefined Beilinson–Bloch conjecture predicts (cf. Conjecture 1.4) that

$$\operatorname{rank} \mathrm{CH}^r(X)^0 = \operatorname{ord}_{s=0} L(s, H^{n-1}(X_{\overline{K}},\mathbb{Q}_\ell(r)))$$

holds for every ℓ. Note that when X is an elliptic curve, this recovers the (unrefined) Birch–Swinnerton-Dyer conjecture.

We have an equivariant version of the Beilinson–Bloch conjecture as follows. Suppose that X admits an action of an algebra T via étale correspondences. Then T acts on both $\mathrm{CH}^r(X)^0$ and $H^{n-1}(X_{\overline{K}},\mathbb{Q}_\ell(r))$. Let ρ be a nonzero finite-dimensional complex representation of T. Then for every ℓ and every embedding $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, we have the L-function

$$L(s, \mathrm{Hom}_{\mathrm{T}}(\rho, H^{n-1}(X_{\overline{K}},\mathbb{Q}_\ell(r))_{\mathbb{C}})).$$

Then it is expected that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{T}}(\rho, \mathrm{CH}^r(X)^0_{\mathbb{C}}) = \operatorname{ord}_{s=0} L(s, \mathrm{Hom}_{\mathrm{T}}(\rho, H^{n-1}(X_{\overline{K}},\mathbb{Q}_\ell(r))_{\mathbb{C}}))$$

holds, which can be regarded as the Beilinson–Bloch conjecture for the (conjectural Chow) motive

$$\mathrm{Hom}_{\mathrm{T}}(\rho, h^{n-1}(X)(r)_{\mathbb{C}}),$$

where $h^{n-1}(X)$ is the (conjectural Chow) motive of X of degree $n-1$.

Consequently, one can specify the equivariant Beilinson–Bloch conjecture to certain unitary Shimura varieties. From now on, we fix a subfield $E \subset \mathbb{C}$ that is a CM number field with $F \subset E$ being its maximal totally real subfield.

We say that a hermitian space V over E of rank m is *standard indefinite* if it has signature $(m-1,1)$ at the default embedding $F \subset \mathbb{R}$ and signature $(m,0)$ at other real places. For a standard indefinite hermitian space V over E of rank m, we have a system of Shimura varieties $\{X_K\}$ indexed by neat open compact subgroups $K \subset H(\mathbb{A}_F^\infty)$, where $H := \mathrm{U}(V)$, which are smooth, quasi-projective, of dimension $m-1$ over E, together with the complex uniformization:

$$X_K(\mathbb{C}) = H(F)\backslash\mathbb{P}(V_\mathbb{C})^- \times H(\mathbb{A}_F^\infty)/K,$$

where $\mathbb{P}(V_\mathbb{C})^- \subset \mathbb{P}(V_\mathbb{C})$ is the complex open domain of negative definite lines.

Conjecture 3.1 (Beilinson–Bloch for unitary Shimura varieties)**.** *Let π be a tempered cuspidal automorphic representation of $G(\mathbb{A}_F)$, where $G := G_r$ denotes the quasi-split unitary group over E/F of rank $n = 2r$. Let V be a standard indefinite hermitian space over E of rank n, with $H := \mathrm{U}(V)$. For every irreducible admissible representation $\widetilde{\pi}^\infty$ of $H(\mathbb{A}_F^\infty)$ satisfying*

(a) $\widetilde{\pi}_v^\infty \simeq \pi_v$ *for all but finitely many non-archimedean places v of F for which $H_v \simeq G_{r,v}$, and*
(b) $\mathrm{Hom}_{H(\mathbb{A}_F^\infty)}(\widetilde{\pi}^\infty, \varinjlim_K H^{n-1}_{\mathrm{dR}}(X_K/\mathbb{C})) \neq 0$,

the conclusion

$$\dim_\mathbb{C} \mathrm{Hom}_{H(\mathbb{A}_F^\infty)}(\widetilde{\pi}^\infty, \varinjlim_K \mathrm{CH}^r(X_K)^0_\mathbb{C}) = \mathrm{ord}_{s=1/2} L(s, \Pi_{\widetilde{\pi}^\infty})$$

holds. Here, $\Pi_{\widetilde{\pi}^\infty}$ is the cuspidal factor of $\mathrm{BC}(\pi)$ determined by $\widetilde{\pi}^\infty$ via Arthur's multiplicity formula; in particular, $\Pi_{\widetilde{\pi}^\infty} = \mathrm{BC}(\pi)$ if $\mathrm{BC}(\pi)$ is already cuspidal (that is, π is stable).

3.2. **Automorphic representations of unitary groups.** Recall that $n = 2r$ and we have fixed the CM subfield $E \subset \mathbb{C}$ with $F \subset E$ the maximal totally real subfield.

For a positive integer m, we equip $W_m := E^{2m}$ with the skew-hermitian form given by the matrix $\left(\begin{smallmatrix} & 1_m \\ -1_m & \end{smallmatrix}\right)$. Put $G_m := \mathrm{U}(W_m)$, the unitary group of W_m, which is a quasi-split reductive group over F. Note that the Cartan subgroup of G_m is $(\mathrm{Res}_{E/F}\,\mathrm{GL}_1)^m$. For every non-archimedean place v of F, we denote by $K_{m,v} \subset G_m(F_v)$ the stabilizer of the lattice $\mathcal{O}_{E_v}^{2m}$, which is a special maximal subgroup.

In what follows, we consider a cuspidal automorphic representation $\pi = \bigotimes_v \pi_v$ of $G(\mathbb{A}_F)$ satisfying:

(R1) If $v \mid \infty$, then π_v is a holomorphic discrete series of weights $\left(\frac{1-n}{2}, \frac{3-n}{2}, \dots, \frac{n-3}{2}, \frac{n-1}{2}\right)$.
(R2) If $v \nmid \infty$ and is nonsplit in E, then π_v is $K_{r,v}$-spherical, that is, $\pi_v^{K_{r,v}} \neq \{0\}$.
(R3) If $v \nmid \infty$, then π_v is tempered (that is, π_v is contained in a parabolic induction of a unitary discrete series representation).

In (R1), a holomorphic discrete series of weights $\left(\frac{1-n}{2}, \frac{3-n}{2}, \dots, \frac{n-3}{2}, \frac{n-1}{2}\right)$ means that it is isomorphic to a constituent of the (normalized) principal series of $(|\cdot|^{1/2}, |\cdot|^{3/2}, \dots, |\cdot|^{(n-1)/2})$. This is the minimal possible weights for a holomorphic discrete series with trivial central character. In (R2), since π_v is $K_{r,v}$-spherical, there exist unitary unramified characters $\chi_{v,1}, \dots, \chi_{v,r} : E_v^\times \to \mathbb{C}^\times$, unique up to permutation and taking inverse, such that π_v is isomorphic to the (normalized) principal series of $(\chi_{v,1}, \dots, \chi_{v,r})$.

Definition 3.2. We define the (complete) *standard L-function* $L(s,\pi) := \prod_v L(s, \pi_v)$ of π as follows:

- When $v \mid \infty$,
$$L(s, \pi_v) := \prod_{i=1}^r L(s, \arg^{2i-1}) \cdot L(s, \arg^{1-2i}),$$
where $\arg : \mathbb{C}^\times \to \mathbb{C}^\times$ is the argument character.
- When $v \nmid \infty$ and is nonsplit in E,
$$L(s, \pi_v) := \prod_{i=1}^r L(s, \chi_{v,i}) \cdot L(s, \chi_{v,i}^{-1}).$$
- When v splits in E,
$$L(s, \pi_v) := \prod_{u \mid v} L(s, \pi_u),$$
where the product is taken over (two) places of E over v, and π_u is π_v but regarded as a representation of $\mathrm{GL}_n(E_u) \simeq G(F_v)$.

As usual, $L(s,\pi)$ is absolutely convergent for $\mathrm{Re}\, s \gg 0$, has a meromorphic (holomorphic, in fact) continuation to $\mathbb{C}$ and satisfies the functional equation

$$L(1-s, \pi) = \epsilon(s, \pi) L(s, \pi),$$

in which

$$\epsilon(1/2,\pi)=(-1)^{r\cdot[F:\mathbb{Q}]}.$$

In particular, the vanishing order of $L(s,\pi)$ at the center $s=1/2$ has the same parity as $r[F:\mathbb{Q}]$.

(i) When $r[F:\mathbb{Q}]$ is even, we will study the central value $L(1/2,\pi)$ via the classical theory of theta lifting.

(ii) When $r[F:\mathbb{Q}]$ is odd, we will study the central derivative $L'(1/2,\pi)$ via the theory of arithmetic theta lifting.

Notation 3.3. We introduce some notations for future use. For a positive integer m, denote by

- $\mathrm{Herm}_m\subset\mathrm{Res}_{E/F}\mathrm{Mat}_m$ the subscheme of hermitian matrices, with the invertible part $\mathrm{Herm}^\circ_m:=\mathrm{Herm}_m\cap\mathrm{Res}_{E/F}\mathrm{GL}_m$,
- $\mathrm{Herm}_m(F)^+$ and $\mathrm{Herm}^\circ_m(F)^+$ the subsets of $\mathrm{Herm}_m(F)$ of totally semi-positive and positive definite elements, respectively.

3.3. **Weil representation.** We review the notion of Weil representation. Let $\psi:F\backslash\mathbb{A}_F\to\mathbb{C}^\times$ be the standard automorphic additive character, namely, the one satisfying $\psi_V(x)=e^{2\pi ix}$ for every $v\mid\infty$.

Definition 3.4. For every positive integer m, every place v of F and every (nondegenerate) hermitian space V_v over E_v of dimension $n=2r$, we have the representation $\omega_{m,v}$ of $G_m(F_v)\times\mathrm{U}(V_v)(F_v)$ on $\mathscr{S}(V_v^m)$ defined by the following formulae:

- For $a\in\mathrm{GL}_m(E_v)$ and $\phi\in\mathscr{S}(V_v^m)$, we have
$$\omega_{m,v}\left(\begin{pmatrix}a&\\&\overline{a}^{\mathrm{t},-1}\end{pmatrix}\right)\phi(x)=|\det a|^r_{E_v}\cdot\phi(xa).$$
Here $a\mapsto\overline{a}$ is the Galois conjugation.
- For $b\in\mathrm{Herm}_m(F_v)$ (in which Herm_m is the F-subscheme of $\mathrm{Res}_{E/F}\mathrm{Mat}_m$ given by $m\times m$ hermitian matrices b, i.e., $\overline{b}^{\mathrm{t}}=b$) and $\phi\in\mathscr{S}(V_v^m)$, we have
$$\omega_{m,v}\left(\begin{pmatrix}1_m&b\\&1_m\end{pmatrix}\right)\phi(x)=\psi_v(\mathrm{tr}\,bT(x))\cdot\phi(x),$$
where $T(x)=(x_i,x_j)_{1\leqslant i,j\leqslant m}\in\mathrm{Herm}_m(F_v)$ is the moment matrix of x.
- For $\phi\in\mathscr{S}(V_v^m)$, we have
$$\omega_{m,v}\left(\begin{pmatrix}&1_m\\-1_m&\end{pmatrix}\right)\phi(x)=\gamma^m_{V_v}\cdot\hat{\phi}(x),$$
where $\gamma_{V_v}\in\{\pm1\}$ is the Weil constant of V_v (see [KR14a, (10.3)]), and $\hat{\phi}$ is the Fourier transform of ϕ using the self-dual Haar measure on V_v^m with respect to $\psi\circ\mathrm{tr}_{E/F}$.
- For $h\in\mathrm{U}(V_v)(F_v)$ and $\phi\in\mathscr{S}(V_v^m)$, we have
$$\omega_{m,v}(h)\phi(x)=\phi(h^{-1}x).$$

3.4. **Theta lifting: when $\epsilon(1/2,\pi)=1$.** We go back to the automorphic representation π of $G(\mathbb{A}_F)$ and assume first that $r[F:\mathbb{Q}]$ is *even*. In this case, we have a hermitian space V over E of rank n, unique up to isomorphism, that is totally positive definite and split at every non-archimedean place of F. Put $H=\mathrm{U}(V)$, and in this case the adelic group $H(\mathbb{A}_F)$ for $V\otimes\mathbb{A}_E$ agrees with the $\mathbb{A}$-points of H over F. We now construct theta functions using the Weil representation $\omega_{m,v}$.

Definition 3.5. Associated to $\phi\in\mathscr{S}(V^r\otimes\mathbb{A}_F)$, define the *theta function*

$$\theta_\phi(g,h):=\sum_{x\in V^r}\omega_r(g,h)\phi(x)=\sum_{x\in V^r}\omega_r(g)\phi(h^{-1}x).$$

Then $\theta_\phi(g,h)$ is an automorphic form on $G(\mathbb{A}_F)\times H(\mathbb{A}_F)$, whose invariance under $G(F)\times H(F)$ is the consequence of the Poisson summation formula. The construction of theta function produces the *automorphic theta distribution*

$$\begin{aligned}\theta:\mathscr{S}(V^r\otimes\mathbb{A}_F)&\longrightarrow\mathscr{A}(G(\mathbb{A}_F))\otimes\mathscr{A}(H(\mathbb{A}_F))\\ \phi&\longmapsto\theta_\phi(-,-),\end{aligned}$$

a $G(\mathbb{A}_F) \times H(\mathbb{A}_F)$-equivariant distribution valued in the space of automorphic forms.

Using $\theta_\phi(g,h)$ as an integral kernel allows one to lift automorphic forms on G to automorphic forms on H (and vice versa): for an automorphic form $\varphi \in \mathscr{A}(G(\mathbb{A}_F))$, define the following.

Definition 3.6. For every $\varphi \in \pi$ (it is known that π has a unique realization in the space of cusp forms of $G(\mathbb{A}_F)$), we define the *theta lift* $\theta_\phi(\varphi)$ to $H(\mathbb{A}_F)$ by the Petersson inner product on $[G] := G(F)\backslash G(\mathbb{A}_F)$, as

$$\theta_\phi(\varphi)(h) := \langle \theta_\phi(-,h), \varphi \rangle_{G,\mathrm{Pet}} = \int_{[G]} \theta_\phi(g,h)\overline{\varphi(g)}\mathrm{d}g,$$

which is an automorphic form on $H(\mathbb{A}_F)$ (with the absolute convergence). Here $\mathrm{d}g$ is the Tamagawa measure on $G(\mathbb{A}_F)$.

We remark that, if φ is cuspidal, then this integral is absolutely convergent and defines an automorphic form $\theta_\phi(\varphi) \in \mathscr{A}(H(\mathbb{A}_F))$. Let π be a cuspidal automorphic representation of $G(\mathbb{A}_F)$, then we obtain an $G(\mathbb{A}_F) \times H(\mathbb{A}_F)$-equivariant linear map

$$\begin{aligned} \theta : \mathscr{S}(V^r \otimes \mathbb{A}_F) \otimes \pi^\vee &\longrightarrow \mathscr{A}(H(\mathbb{A}_F)) \\ (\phi, \overline{\varphi}) &\longmapsto \theta_\phi(\varphi), \end{aligned}$$

and define the *global theta lift*

$$\Theta_V(\pi) \subset \mathscr{A}(H(\mathbb{A}_F))$$

of π to be its image, and $H(\mathbb{A}_F)$-subrepresentation of $\mathscr{A}(H(\mathbb{A}_F))$. The theory of global theta correspondence provides a rather complete description of $\Theta_V(\pi)$, and we refer to Gan's article in these proceedings for more details.

3.5. **Siegel–Weil formula.** Associated to $\phi \in \mathscr{S}(V^r \otimes \mathbb{A}_F)$, consider the *theta integral*

$$I_\phi(g) := \int_{[H]} \theta_\phi(g,h)\mathrm{d}h,$$

where $\mathrm{d}h$ is the Tamagawa measure on $H(\mathbb{A}_F)$. Let α be the dimension of a maximal isotropic subspace of V. The theta integral is absolutely convergent for all ϕ if and only if the pair (V,W) satisfies *Weil's convergence condition*

- ⋄ $\alpha = 0$ (i.e., V is anisotropic), or $\alpha > 0$ and $n - \alpha > r$.

In this case $I_\phi(g)$ is an automorphic form on $G(\mathbb{A}_F)$. It can be viewed as the theta lift of the identity function on $H(\mathbb{A}_F)$, and also specializes to the weighted average of theta series within a genus class for definite hermitian forms (cf. [Li21, Example 2.2.6]). The theta integral produces a $G(\mathbb{A}_F)$-equivariant distribution

$$\begin{aligned} I : \mathscr{S}(V^r \otimes \mathbb{A}_F) &\longrightarrow \mathscr{A}(G(\mathbb{A}_F)) \\ \phi &\longmapsto I_\phi(-). \end{aligned}$$

There is another way of producing automorphic distributions like the above via Eisenstein series. For $s \in \mathbb{C}$, let

$$I_\chi(s) := \mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}(\chi |\cdot|_E^s)$$

be the degenerate principal series representation of $G(\mathbb{A}_F)$, where $\mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}$ denotes the (unnormalized) smooth parabolic induction. Associated to $\phi \in \mathscr{S}(V^r \otimes \mathbb{A}_F)$, there is a *standard Siegel–Weil section* $\Phi_\phi(g,s) \in I_\chi(s)$ defined by

$$\Phi_\phi(g,s) := \omega_m(g)\phi(0) \cdot |\det a(g)|_E^{s-s_0},$$

where

$$s_0 := \frac{n-r}{2}.$$

Here we write $g = nm(a)k$ under the Iwasawa decomposition $G(\mathbb{A}_F) = N(\mathbb{A}_F)M(\mathbb{A}_F)K$ for $K := K_r$ the standard maximal open compact subgroup of $G(\mathbb{A}_F)$, and the quantity $|\det a(g)|_E := |\det a|_E$ is well-defined. We obtain a distribution

$$\begin{aligned} \Phi(s) : \mathscr{S}(V^r \otimes \mathbb{A}_F) &\longrightarrow I_\chi(s) \\ \phi &\longmapsto \Phi_\phi(g,s), \end{aligned}$$

and the special value $s = s_0$ is the unique value such that $\Phi(s)$ is $G(\mathbb{A}_F)$-equivariant.

Definition 3.7. Define the (hermitian) *Siegel Eisenstein series*

$$E_\phi(g,s) := \sum_{\gamma\in P(F)\backslash G(F)} \Phi_\phi(\gamma g, s), \quad g\in G(\mathbb{A}_F).$$

The Siegel Eisenstein series $E_\phi(g,s)$ converges absolutely when $\mathrm{Re}\, s > n/2$. It has a meromorphic continuation to $s\in\mathbb{C}$ and satisfies a functional equation centered at $s=0$. If $E_\phi(g,s)$ is homomorphic at $s=s_0$, then its value at $s=s_0$ produces a $G(\mathbb{A}_F)$-equivariant distribution

$$\begin{aligned} E(s_0): \mathscr{S}(V^r\otimes\mathbb{A}_F) &\longrightarrow \mathscr{A}(G(\mathbb{A}_F)) \\ \phi &\longmapsto E_\phi(-,s_0). \end{aligned}$$

The Siegel–Weil formula gives a precise identity between the two distributions I and $E(s_0)$.

Theorem 3.8 (Siegel–Weil formula). *Assume that the pair (V,W) satisfies Weil's convergence condition. Then $E_\phi(g,s)$ is holomorphic at s_0 and*

$$\kappa\cdot I_\phi(g) = E_\phi(g,s_0),$$

where $\kappa = 1/2$ if $n>r$ and $\kappa=1$ otherwise.

This theorem was proved in Weil [Wei65, Theorem 5] (when $n>2r$, in which case $E_\phi(g,s)$ is also absolutely convergent at $s=s_0$), Ichino [Ich07, Theorem 1.1] (when $r<n\leqslant 2r$) and Yamana [Yam11, Theorem 2.2] (when $n\leqslant r$). If the Weil's convergence condition is not satisfied, one can still naturally define $I_\phi(g)$ via regularization and it is a long effort starting with the work of Kudla–Rallis [KR94] to generalize the Siegel–Weil formula outside the convergence range and for all reductive dual pairs of classical groups. We refer to Gan–Qiu–Takeda [GQT14] for the most general Siegel–Weil formula and a nice summary of the literature and history. The Siegel–Weil formula is an indispensable tool in the arithmetic theory of quadratic forms and hermitian forms.

3.6. **Rallis inner product formula.** Piatetski–Shapiro–Rallis [PSR86, PSR87] discovered an integral representation (the doubling method) of the standard L-function for cuspidal automorphic representations π of $G(\mathbb{A}_F)$, via integrating against a Siegel Eisenstein series on a "doubling" group. Combining with the doubling seesaw and the Siegel–Weil formula, one arrives at the *Rallis inner product formula*, which relates the Petersson inner product of theta lifts (from G to H) and a special value of the standard L-function of π.

Consider the skew-hermitian space $W^\square = W\oplus(-W)$ of dimension $4r$ over F. Define $G^\square := \mathrm{U}(W^\square)$, a quasi-split unitary group of rank twice that of $\mathrm{U}(W)$. Associated to the parabolic subgroup $P^\square\subset G^\square$ stabilizing the maximal isotropic subspace $\{(w,-w): w\in W\}\subset W^\square$, we have a Weil representation $\omega_r^\square$ of $G^\square(\mathbb{A}_F)$ on $\mathscr{A}(V^{2r}\otimes\mathbb{A}_F)$. There is an isomorphism of $G(\mathbb{A}_F)\times G(\mathbb{A}_F)$ representations

$$\delta: \omega_r\boxtimes(\omega_r^\vee\otimes\chi)\xrightarrow{\sim}\omega_r^\square|_{G(\mathbb{A}_F)\times G(\mathbb{A}_F)}$$

such that for any $\phi_1,\phi_2\in\mathscr{S}(V^r\otimes\mathbb{A}_F)$, we have

$$\delta(\phi_1\otimes\overline{\phi}_2)(0) = \langle\phi_1,\phi_2\rangle_{\omega_r},$$

where $\langle\cdot,\cdot\rangle_{\omega_r}$ is the inner product on $\mathscr{S}(V^r\otimes\mathbb{A}_F)$. On the other hand, for any $\phi_1,\phi_2\in\mathscr{S}(V^r\otimes\mathbb{A}_F)$, we have a Siegel Eisenstein series $E_{\delta(\phi_1\otimes\overline{\phi}_2)}(g,s)$ on $G^\square$. For any $\phi_1,\phi_2\in\pi$, define the *global doubling zeta integral*

$$Z(s,\varphi_1,\varphi_2,\phi_1,\phi_2) := \int_{[G]\times[G]} \overline{\phi}_1(g_1)\phi_2(g_2)\cdot E_{\phi_1\otimes\overline{\phi}_2}((g_1,g_2),s)\chi^{-1}(\det g_2)\mathrm{d}g_1\mathrm{d}g_2.$$

It converges absolutely when $\mathrm{Re}\, s\gg 0$ and extends to a meromorphic function on $\mathbb{C}$. When $\phi_i = \bigotimes_v\phi_{i,v}$ and $\varphi_i = \bigotimes_v\varphi_{i,v}$ are factorizable, the global doubling zeta integral factorizes into a product of *local doubling zeta integrals*

$$Z(s,\varphi_1,\varphi_2,\phi_1,\phi_2) = \prod_v Z_v(s;\varphi_{1,v},\varphi_{2,v};\phi_{1,v},\phi_{2,v}),$$

where

$$Z_v(s;\varphi_{1,v},\varphi_{2,v};\phi_{1,v},\phi_{2,v}) := \int_{G(F_v)} \overline{\langle g_v\varphi_{1,v},\varphi_{2,v}\rangle}_{\pi_v}\cdot\Phi_{\phi_{1,v}\otimes\overline{\phi}_{2,v}}((g_v,1),s)\mathrm{d}g_v$$

converges absolutely when $\mathrm{Re}\, s \gg 0$ and extends to a meromorphic function on $\mathbb{C}$. When all the data are unramified at a finite place v with $\langle \varphi_{1,v}, \varphi_{2,v}\rangle = 1$, we have

$$Z_v(s; \varphi_{1,v}, \varphi_{2,v}; \phi_{1,v}, \phi_{2,v}) = \frac{L(s+1/2, \pi_v \times \pi_v)}{b_{2n,v}(s)},$$

where $L(s+1/2, \pi_v \times \pi_v)$ is the doubling L-factor (see Harris–Kudla–Sweet [HKS96b], Lapid–Rallis [LR05], Yamana [Yam14]) and agrees with standard (base change) L-factor $L(s+1/2, \mathrm{BC}(\pi_v)\otimes\chi)$ in this unramified case, and

$$b_{k,v}(s) := \prod_{i=1}^{k} L(2s+i, \eta_v^{k-i})$$

is a product of Hecke L-factors. Define the *normalized local doubling zeta integral*

$$Z_v^\natural(s; \varphi_{1,v}, \varphi_{2,v}; \phi_{1,v}, \phi_{2,v}) := \left(\frac{L(s+1/2, \pi_v \times \pi_v)}{b_{2n,v}(s)}\right)^{-1} \cdot Z_v(s; \varphi_{1,v}, \varphi_{2,v}; \phi_{1,v}, \phi_{2,v}),$$

then

$$Z_v^\natural(s; \varphi_{1,v}, \varphi_{2,v}; \phi_{1,v}, \phi_{2,v}) = 1 \quad \text{for almost all } v.$$

At $s = s_0$, the normalized local zeta integral evaluates to

$$Z_v^\natural(s_0; \varphi_{1,v}, \varphi_{2,v}; \phi_{1,v}, \phi_{2,v}) = \int_{G(F_v)} \overline{\langle g_v\varphi_{1,v}, \varphi_{2,v}\rangle}_{\pi_v} \cdot \langle g_v\phi_{1,v}, \phi_{2,v}\rangle_{\omega_{r,v}} \mathrm{d}g_v,$$

the integral of the product of matrix coefficients of π_v and $\omega_{r,v}$. Thus it produces a $G(F_v)\times G(F_v)$-equivariant distribution

$$\begin{aligned} Z_v^\natural(s_0) : \mathscr{S}(V_v^{2r}) &\longrightarrow \pi_v \boxtimes (\pi_v^\vee \otimes \chi_v) \\ \phi_{1,v}\otimes\overline{\phi}_{2,v} &\longmapsto Z_v^\natural(s_0, -, -, \phi_{1,v}, \phi_{2,v}). \end{aligned}$$

Taking product produces a $G(\mathbb{A}_F)\times G(\mathbb{A}_F)$-equivariant distribution

$$\prod_v Z_v^\natural(s_0) : \mathscr{S}(V^{2r}\otimes\mathbb{A}_F) \longrightarrow \pi\boxtimes(\pi^\vee\otimes\chi).$$

On the other hand, the Petersson inner product of theta lifts also defines a $G(\mathbb{A}_F)\times G(\mathbb{A}_F)$-equivariant distribution

$$\begin{aligned} \langle\theta,\theta\rangle : \mathscr{S}(V^{2r}\otimes\mathbb{A}_F) &\longrightarrow \pi\boxtimes(\pi^\vee\otimes\chi) \\ (\phi_1,\phi_2) &\longmapsto \langle\theta_{\phi_1}(-), \theta_{\phi_2}(-)\rangle_H. \end{aligned}$$

The Rallis inner product gives a precise identity between the two distributions $\prod_v Z_v^\natural(s_0)$ and $\langle\theta,\theta\rangle$ above.

Theorem 3.9 (Rallis inner product formula)**.** *Assume that the pair $(V, W^\square)$ satisfies Weil's convergence condition. Let π be a cuspidal automorphic representation of $G(\mathbb{A}_F)$. Then for any $\phi_i = \bigotimes_v \phi_{i,v} \in \mathscr{S}(V^r\otimes\mathbb{A}_F)$ and $\varphi_i = \bigotimes_v \varphi_{i,v}\in\pi$ with $i=1,2$,*

$$\kappa\cdot\langle\theta_{\phi_1}(\varphi_1), \theta_{\phi_2}(\varphi_2)\rangle_H = \frac{L(s_0+1/2, \pi\times\chi)}{b_{2n}(s_0)}\cdot\prod_v Z_v^\natural(s_0; \varphi_{1,v}, \varphi_{2,v}; \phi_{1,v}, \phi_{2,v}).$$

Here $s_0 = (n-2r)/2$, $\kappa = 1/2$ if $n > 2r$ and $\kappa = 1$ otherwise, are the constants in the Siegel–Weil formula (Theorem 3.8) for the pair $(V, W^\square)$.

This theorem was proved in J.-S. Li [Li92]. Rallis [Ral84] first discovered a version of this theorem in some special cases for orthogonal/symplectic dual pairs, hence its name. When Weil's convergence condition is not satisfied, one can still use the regularized Siegel–Weil formula to derive a regularized version of the Rallis inner product formula and again we refer to [GQT14] for the most general statements.

Remark 3.10. In Theorem 3.9, the two functors

$$\langle\theta_{\phi_1}(\varphi_1), \theta_{\phi_2}(\varphi_2)\rangle_H, \qquad \prod_v Z_v^\natural(-; \varphi_{1,v}, \varphi_{2,v}; \phi_{1,v}, \phi_{2,v})$$

define two elements in

$$\bigotimes_v \mathrm{Hom}_{G(F_v)\times G(F_v)}(\mathscr{S}(V_v^{2r})_{H(F_v)}, \pi_v \boxtimes \pi_v^\vee)$$

with $G = G_r$ as mentioned before. It is known that the above space has dimension 1 of which $\prod_v Z_v^\natural$ is a basis. Thus, Rallis inner product formula is nothing but the proportion of the two invariant functionals.

Proof sketch of Rallis inner product formula.

Step 1. We regard $G_r \times G_r$ as a subgroup of G_{2r} via the embedding

$$\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \longmapsto \begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & -b_2 \\ c_1 & & d_1 & \\ & -c_2 & & d_2 \end{pmatrix}.$$

Step 2. Use the Siegel–Weil formula in Theorem 3.8, say

$$\langle \theta_{\phi_1}(g_1, -), \theta_{\phi_2}(g_2, -)\rangle_H = E(0, (g_1, g_2), \phi_1 \otimes \overline{\phi}_2).$$

Here, for every $\Phi \in \mathscr{S}(V^{2r} \otimes \mathbb{A}_F)$, we have the Siegel–Eisenstein series

$$E(s, g, \Phi) := \sum_{\gamma \in P_{2r}(F)\backslash G_{2r}(F)} \omega_{2r}(\gamma g)\Phi(0) \cdot H(\gamma g)^s$$

on $G_{2r}(\mathbb{A}_F)$, where $P_{2r} \subset G_{2r}$ denotes the upper-triangle Siegel parabolic subgroup and H denotes the “height” function with respect to P_{2r}.

Step 3. Using cuspidality, we have

$$\begin{aligned}\int\int_{[G_r(F)\backslash G_r(\mathbb{A}_F)]^2} & \overline{\varphi_1(g_1)}\varphi_2(g_2)E(s, (g_1, g_2), \phi_1 \otimes \overline{\phi}_2)\mathrm{d}g_1\mathrm{d}g_2 \\ &= \frac{L(s + 1/2, \pi)}{b_n(s)} \prod_v Z_v^\natural(s; \varphi_{1,v}, \varphi_{2,v}; \phi_{1,v}, \phi_{2,v}).\end{aligned}$$

□

3.7. **Theta dichotomy in the equal rank case.** The Rallis inner product formula plays an important role in Rallis's program on the nonvanishing criterion for global theta lifts ([Ral87], cf. [GQT14, §1.2]). We recall a special case when $n = 2r$, i.e., when the two spaces V, W have equal rank. In this case, $\kappa = 1$ and as n is even we can simply choose the splitting character χ to be the trivial character. The special point $s_0 = 0$ in the Siegel–Weil formula for $(V, W^\square)$ corresponds to the center of the functional equation of the Eisenstein series, and the Rallis inner product formula relates the Petersson inner product of theta lifts and central L-values $L(1/2, \pi)$. By the Rallis inner product formula, we know that

$$\text{global theta lifting } \Theta_V(\pi) \neq 0 \iff L(1/2, \pi) \neq 0, \text{ and } \prod_v Z_v^\natural(0) \neq 0.$$

The local condition $Z_v^\natural(0) \neq 0$ turns out to be equivalent to that the local theta lift $\Theta_{V_v}(\pi_v) \neq 0$ (cf. [HKS96b, Proposition 3.1]). Moreover, at any nonsplit place v, we further have the *epsilon dichotomy* by Gan–Ichino [GI14, Theorem 11.1] (and Harris–Kudla–Sweet [HKS96b, Theorem 6.1]) pinning down exactly one of the two local hermitian spaces over F_v:

$$Z_v^\natural(0) \neq 0 \iff \epsilon(V_v) = \omega_{\pi_v} \cdot (-1) \cdot \epsilon(1/2, \pi_v, \psi_v),$$

where

- $\epsilon(V_v) = \eta_v((-1)^{n(n-1)/2}\det(V_v)) \in \{\pm 1\}$ is the local Hasse invariant,
- $\epsilon(1/2, \pi_v, \psi_v) \in \{\pm 1\}$ is the central value of the doubling epsilon factor, and
- ω_{π_v} is the central character of π_v.

For any cuspidal automorphic representation π of $G(\mathbb{A}_F)$, the theta dichotomy associates to it a *unique* collection of local hermitian spaces $\{V_v = V_v(\pi_v)\}_v$ such that $\Theta_{V_v}(\pi_v) \neq 0$ for all places v, or equivalently, a unique hermitian space $\mathbb{V} = \mathbb{V}_\pi$ of rank n over $\mathbb{A}_F$ such that $\Theta_{\mathbb{V}_v}(\pi_v) \neq 0$ for all places v, where $\mathbb{V}_v := \mathbb{V} \otimes_{\mathbb{A}} F_{0,v}$. Say that $\mathbb{V}$ is *coherent* if $\mathbb{V} \simeq V \otimes_F \mathbb{A}_F$ for some hermitian space V over F, and *incoherent* otherwise.

Define $\epsilon(\mathbb{V}) := \prod_v \epsilon(\mathbb{V}_v) \in \{\pm 1\}$. Then the Hasse principle implies that $\mathbb{V}$ is coherent if and only if $\epsilon(\mathbb{V}) = +1$. The epsilon dichotomy implies the equality of signs

$$\epsilon(\mathbb{V}_\pi) = \epsilon(1/2, \pi).$$

We have two cases:

- If $\epsilon(1/2, \pi) = +1$, then $\mathbb{V}_\pi$ is coherent. If $\mathbb{V}_\pi \simeq V \otimes_F \mathbb{A}_F$, then

  $$\text{the global theta lift } \Theta_V(\pi) \neq 0 \iff L(1/2, \pi) \neq 0.$$

  Moreover $\Theta_{V'}(\pi) = 0$ for all hermitian spaces V' of rank n over F different from V due to local reasons.
- If $\epsilon(1/2, \pi) = -1$, then $\mathbb{V}_\pi$ is incoherent. The global theta lift $\Theta_V(\pi) = 0$ for all hermitian spaces V of rank n over F due to local reasons.

In the second case there is no global theta lifting associated to the *incoherent* space $\mathbb{V} = \mathbb{V}_\pi$ and $L(1/2, \pi) = 0$ always. It is natural and interesting to study the central *derivative* $L'(1/2, \pi)$. The Birch and Swinnerton-Dyer conjecture and its generalization by Beilinson and Bloch suggests that the condition $L'(1/2, \pi) \neq 0$ should be related to the non-triviality of *algebraic cycles*. When the incoherent space $\mathbb{V}$ is totally definite, next we will canonically associate to it a unitary Shimura variety X over F and use the generating function of its special cycles to define an *arithmetic theta lift* $\Theta_V(\pi) \subset \mathrm{CH}^n(X)$. Here $\mathrm{CH}^n(X)$ is the Chow group of algebraic cycles of codimension n on X modulo rational equivalence. One of the goals of the Kudla program on arithmetic theta lifting is to establish an analogous criterion (cf. Theorem 5.16, proved in [LL21, LL22])

$$(*) \qquad \text{the arithmetic theta lift } \Theta_V(\pi) \neq 0 \overset{?}{\iff} L'(1/2, \pi) \neq 0.$$

4. Geometric theta correspondence

4.1. **Unitary Shimura varieties.** From now we assume that E/F is a CM extension of a totally real number field.

As in the previous section, V denotes a hermitian space over E of rank n and $H = \mathrm{U}(V)$. We fix an embedding $\sigma : E \hookrightarrow \mathbb{C}$ and view E (resp. F) as a subfield of $\mathbb{C}$ (resp. $\mathbb{R}$). Say V is *standard indefinite* if V has signature $(n-1, 1)$ at the real place of F induced by σ, and signature $(n, 0)$ at all other real places. When V is standard indefinite, there is a system of unitary Shimura varieties $X = \{X_K\}$ indexed by neat open compact subgroup $K \subset H(\mathbb{A}_F^\infty)$. Each X_K is a smooth quasi-projective scheme of dimension $n-1$ over $F \subset \mathbb{C}$, and is projective when V is anisotropic (e.g. when $F \neq \mathbb{Q}$, by the signature condition). It has complex uniformization

$$X_K(\mathbb{C}) = \mathrm{U}(V)(F) \backslash (\mathbb{D} \times \mathrm{U}(V)(\mathbb{A}_F^\infty)/K),$$

where $\mathbb{D}$ is the hermitian symmetric domain associated to $\mathrm{U}(V_\infty)$ given by the space of negative complex lines in $V \otimes_E \mathbb{C}$. We have isomorphisms

$$\mathbb{D} \simeq \{z \in \mathbb{C}^{n-1} : |z| < 1\} \simeq \frac{\mathrm{U}(n-1,1)}{\mathrm{U}(n-1) \times \mathrm{U}(1)}.$$

In particular, X_K can be written as a union of arithmetic quotients of complex balls.

As in the previous section, $\mathbb{V}$ denotes an incoherent hermitian space of rank n over $\mathbb{A}_E$. Say $\mathbb{V}$ is *totally definite* if $\mathbb{V}$ has signature $(n, 0)$ at all real places. If $\mathbb{V}$ is totally definite, then for any embedding $\sigma : E \hookrightarrow \mathbb{C}$, we have a unique standard indefinite hermitian space V, depending on σ, such that V_v has signature $(n-1, 1)$ at the real place of F induced by σ, and $\mathbb{V}_v \simeq V_v$ at all other places of F. By the theory of conjugation of Shimura varieties, the Shimura variety X_K associated to varying V for varying choices of σ are all conjugate, and thus can be intrinsically defined over E (without being viewed as a subfield of $\mathbb{C}$). In other word, for any totally definite incoherent hermitian space $\mathbb{V}$ over $\mathbb{A}_E$, we obtain a system of unitary Shimura varieties $X = \{X_K\}$ canonically defined over E (cf. [Zha10], [Gro20]).

From the above discussion the following dichotomy picture emerges:

- when studying the *geometric* invariants of X_K (over the algebraically closed field $\mathbb{C}$), a choice of the embedding $\sigma : E \hookrightarrow \mathbb{C}$ is involved. The *coherent* space $V(\mathbb{A}_F)$ associated to V should play a canonical role and special *values* of analytic quantities ought to appear.

- when studying the *arithmetic* invariants of X_K (over the number field E), no choice of the embedding $\sigma : E \hookrightarrow \mathbb{C}$ is involved and the *incoherent* space $\mathbb{V}$ should play a canonical role and special *derivatives* of analytic quantities ought to appear.

Remark 4.1. By the Langlands philosophy, the motivic L-function associated to the étale cohomology of X_K should be factorized into a product of automorphic L-functions for automorphic representations π of $H(\mathbb{A}_F)$. When V is standard indefinite, the L-function appearing should be the standard L-function of π. This suggests the terminology and its relevance for our goal (∗). When V has more general signature combinations, for the corresponding Shimura varieties one expects to see Langlands L-functions associated to more complicated representations of the dual group of $H = \mathrm{U}(V)$.

Remark 4.2. We remark that X_K is a Shimura variety of abelian type (rather than of PEL or Hodge type). Unlike Shimura varieties of PEL type associated to unitary similitude groups, it lacks a good moduli description in terms of abelian varieties with additional structures and thus it is technically more difficult to study. Nevertheless, its étale cohomology and L-function will be computed in terms of automorphic forms in the forthcoming work of Kisin–Shin–Zhu [KSZ21], under the help of the endoscopic classification for unitary groups due to Mok [Mok15] and Kaletha–Minguez–Shin–White [KMSW14].

4.2. **Special cycles and arithmetic theta functions.**

4.2.1. *Special cycles.* Let V be a standard indefinite hermitian space over E of rank $n = 2r$, with $H = \mathrm{U}(V)$ as before. Let $\mathbb{V}$ be the associated totally definite incoherent space and denote $\mathbb{V}^\infty := V \otimes_F \mathbb{A}_F^\infty \simeq \mathbb{V} \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty$. Take a neat open compact subgroup $K \subset H(\mathbb{A}_F^\infty)$.

We first recall the construction of Kudla's special cycle $Z(\mathbf{x})_K$ for any $\mathbf{x} \in V^m \otimes_F \mathbb{A}_F^\infty$. For any $y \in V$ with $(y, y) > 0$, i.e., with totally positive norm, its orthogonal complement $V_y \subset V$ is a standard indefinite hermitian space with rank $n-1$ over E. Let $H_y := \mathrm{U}(V_y)$, a subgroup of $H = \mathrm{U}(V)$ and X_y be the system of unitary Shimura varieties associated to H_y. We define the *special divisor* $Z(y)_K$ to be the Shimura subvariety

$$Z(y)_K := (X_y)_{K \cap H_y(\mathbb{A}^\infty)} \longrightarrow X_K.$$

More generally, we make the following definition.

Construction 4.3. For any $m \leqslant \dim X_K$ and $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{V}^{\infty, m}$,

(1) When $T(\mathbf{x}) \notin \mathrm{Herm}_m(F)^+$, we set $Z(\mathbf{x})_K = 0$.

(2) When $T(\mathbf{x}) \in \mathrm{Herm}_m^\circ(F)^+$ is positive definite, we may find elements $\mathbf{y} \in V^m$ and $h \in H(\mathbb{A}_F^\infty)$ such that $h\mathbf{x} = \mathbf{y} \in V^m \otimes_F \mathbb{A}_F^\infty$. Denote by $V_{\mathbf{y}}$ the orthogonal complement of the subspace spanned by components of $\mathbf{y}$ in V, which is standard indefinite of rank $n-m$. Put $H_{\mathbf{y}} := \mathrm{U}(V_{\mathbf{y}})$, which is naturally a subgroup of H. Define the *special divisor* $Z(\mathbf{x})_K$ to be the image cycle of the Hecke translate of a Shimura subvariety

$$Z(\mathbf{x})_K := (X_{\mathbf{y}})_{hKh^{-1} \cap H_y(\mathbb{A}_F^\infty)} \longrightarrow X_{hKh^{-1}} \xrightarrow{\cdot h} X_K,$$

where $X_{\mathbf{y}}$ denotes the system of Shimura varieties for $V_{\mathbf{y}}$. It is straightforward to check that $Z(\mathbf{x})_K$ does not depend on the choice of $\mathbf{y}$ and h. Moreover, $Z(\mathbf{x})_K$ is a well-defined element in $Z^m(X_K)$.

(3) When $T(\mathbf{x}) \in \mathrm{Herm}_m(F)^+$ is only semi-positive definite in general, define the *special cycle* of codimension n as

$$Z(\mathbf{x})_K = Z(x_1)_K \cap \cdots \cap Z(x_n)_K \longrightarrow X_K,$$

where $\cap$ denotes the fiber product over X_K, whose image cycle defines an algebraic cycle of codimension n on X_K. However, the subtlety here lies in when $(\mathbf{x}, \mathbf{x}) \in \mathrm{Herm}_m^\circ(F)^+$ but is singular, the intersection is improper, i.e., it has the wrong codimension. To botch this, let $\mathcal{L}_K$ be the Hodge line bundle on X_K, with complex uniformization

$$\mathcal{L}_K(\mathbb{C}) = H(F) \backslash (\mathcal{L} \times H(\mathbb{A}_F^\infty)/K),$$

where $\mathcal{L}$ is the tautological line bundle on $\mathbb{D} \subset \mathbb{P}(V \otimes_E \mathbb{C})$. The Hodge line bundle naturally appears when computing improper intersections: for example if $\mathbf{x} = x \in V^m$ with $(x, x) > 0$, then the excess intersection formula implies that

$$Z(x)_K \cdot Z(x)_K = Z(x)_K \cdot c_1(\mathcal{L}_K^\vee) \in \mathrm{CH}^2(X_K).$$

Here $c_1(\mathcal{L}_K^\vee) \in \mathrm{CH}^1(X_K)_\mathbb{Q}$ is the first Chern class of the dual line bundle of $\mathcal{L}_K$. This motivates us to define

$$Z(\mathbf{x})_K := Z(V_\mathbf{x})_K \cdot c_1(\mathcal{L}_K^\vee)^{m-\dim_F V_\mathbf{x}} \in \mathrm{CH}^m(X_K)_\mathbb{Q}$$

which is an element in the Chow group of correct codimension.

Therefore, we get an element of $\mathrm{CH}^m(X_K)_\mathbb{Q}$ as the special cycle in Kudla's sense (but it is not always well-defined in $Z^m(X_K)_\mathbb{Q}$).

Definition 4.4. For every K-invariant Schwartz function $\phi^\infty \in \mathscr{S}(V^m \otimes_F \mathbb{A}_F^\infty)^K = \mathscr{S}(\mathbb{V}^{\infty,m})^K$ and $T \in \mathrm{Herm}_m(F)$, define the *weighted special cycle*

$$Z_T(\phi^\infty)_K := \sum_{\substack{x \in K\backslash \mathbb{V}^{\infty,m},\\ T(\mathbf{x})=T}} \phi^\infty(\mathbf{x}) Z(\mathbf{x})_K \in \mathrm{CH}^m(X_K)_\mathbb{C}.$$

As the above summation is finite, $Z_T(\phi^\infty)_K$ is a well-defined element in $\mathrm{CH}^m(X_K)_\mathbb{C}$. For $T \in \mathrm{Herm}_m^\circ(F)^+$, the weighted special cycle $Z_T(\phi^\infty)_K$ is even a well-defined element in $Z^m(X_K)_\mathbb{C}$.

4.2.2. *Kudla's generating series.* Define *Kudla's generating function of special cycles* (a.k.a. *arithmetic theta function*, which is to be interpreted) of codimension n

$$Z_\tau(\phi^\infty)_K := \sum_{T \in \mathrm{Herm}_m(F)^+} Z_T(\phi^\infty)_K \cdot q^T,$$

as a formal generating function valued in $\mathrm{CH}^m(X_K)_\mathbb{C}$, where

$$\tau \in \mathcal{H}_n = \{\mathsf{x} + i\mathsf{y} : \mathsf{x} \in \mathrm{Herm}_m(F_\infty), \mathsf{y} \in \mathrm{Herm}_m^\circ(F_\infty)^+\}$$

lies in the the hermitian half space and

$$q^T := \prod_{v|\infty} e^{2\pi i \operatorname{tr} T\tau_v}.$$

It formally resembles the Fourier expansion of a holomorphic hermitian modular form on $\mathcal{H}_n$. In fact its modularity is the content of Kudla's modularity conjecture (see Conjecture 5.1).

More adelically, define

$$Z_\phi(g)_K := \sum_{T \in \mathrm{Herm}_m(F)^+} Z_T(\omega^\infty(g^\infty)\phi^\infty)_K \cdot \omega_\infty(g_\infty)\phi_\infty(T), \quad g \in G(\mathbb{A}_F)$$

as a formal sum valued in $\mathrm{CH}^m(X_K)_\mathbb{C}$. Here $\phi_\infty \in \mathscr{S}(\mathbb{V}_\infty^m)$ is the standard Gaussian function

$$\phi_\infty(\mathbf{x}) := \prod_v e^{-2\pi \operatorname{tr}(\mathbf{x},\mathbf{x})}$$

and $\omega_\infty(g_\infty)\phi_\infty(T)$ makes sense as $\omega_\infty(g_\infty)\phi_\infty$ factors through the moment map $\mathbf{x} \mapsto (\mathbf{x},\mathbf{x})$. It is the adelization of the definition of $Z_\tau(\phi^\infty)_K$ and agrees with the formal Fourier expansion of

$$Z_\phi(g)_K = \sum_{\mathbf{x} \in K\backslash \mathbb{V}^{\infty,m}} \omega(g)(\phi^\infty \otimes \phi_\infty)(\mathbf{x}) \cdot Z(\mathbf{x})_K,$$

where for $\mathbf{x} \in \mathbb{V}^{\infty,m}$ we interpret $\phi_\infty(\mathbf{x})$ as $\phi_\infty((\mathbf{x},\mathbf{x}))$ if $(\mathbf{x},\mathbf{x}) \in \mathrm{Herm}_m(F)^+$ and 0 otherwise. Moreover, $Z_\phi(g)_K$ is compatible under pullback by natural projection morphisms when varying $K \subset H(\mathbb{A}_F^\infty)$ and thus defines a formal sum

$$Z_\phi(g) := (Z_\phi(g)_K)_K$$

valued in $\mathrm{CH}^m(X)_\mathbb{C} := \varinjlim_{K \subset H(\mathbb{A}_F^\infty)} (X_K)_\mathbb{C}$.

Notice the analogy between the classical theta function

$$\theta_\phi(g,h) := \sum_{x \in V^m} \omega_m(g,h)\phi(x) = \sum_{x \in V^m} \omega_r(g)\phi(h^{-1}x)$$

and Kudla's generating function $Z_\phi(g)_K$, except two crucial modifications:

(i) the automorphic forms space $\mathscr{A}(H(\mathbb{A}_F))$ (one of the variable) is replaced by $\mathrm{CH}^m(X)$ for the system of Shimura varieties X associated to H;

(ii) the holomorphy of $Z_\phi(g)$ forces us to *fix* ϕ_∞ to be the Gaussian function, and ϕ_∞ lives on the totally definite incoherent space $\mathbb{V}$ rather than the standard indefinite space V (which matches the dichotomy philosophy as discussed in §4.1.

In this way one should view Kudla's generating function as an *arithmetic theta function*.

4.3. **Geometric modularity.** We can extract geometric invariants of an element $Z \in \mathrm{CH}^m(X_K)$ by taking its Betti cohomology class $[Z] \in H^{2m}(X_K(\mathbb{C}), \mathbb{Z})$ of the complex manifold $X_K(\mathbb{C})$. In particular, we obtain from the arithmetic theta function $Z_\phi(g)_K$ a *geometric theta function* $[Z_\phi(g)_K]$ valued in $H^{2m}(X_K(\mathbb{C}), \mathbb{C})$. Its Fourier coefficients encodes the information about the geometric intersection numbers of special cycles. The classical theorem of Kudla–Millson shows that this geometric theta function is indeed *modular*. In other words, there are many *hidden symmetry and relations* between these geometric invariants of special cycles. More precisely, denote by $\mathscr{A}_{k,\chi}(n)$ the holomorphic hermitian modular forms on $\mathcal{H}_n$ of parallel weight k and character χ, and $\mathscr{A}_{k,\chi}(G(\mathbb{A}_F)) \subset \mathscr{A}(G(\mathbb{A}_F))$ the adelization of $\mathscr{A}_{k,\chi}(n)$.

Theorem 4.5 (Geometric modularity)**.** *The formal generating function $[Z_\phi(g)_K]$ converges absolutely and defines an element in $\mathscr{A}_{m/2,\chi}(G(\mathbb{A}_F)) \otimes H^{2m}(X_K(\mathbb{C}), \mathbb{C})$.*

This theorem is proved in Kudla–Millson [KM90]. In fact [KM90] proves a much more general theorem, applicable to the generating function of special cohomology classes for locally symmetric spaces associated to any $\mathrm{U}(p,q)$ or $\mathrm{O}(p,q)$. The proof replies on the *Kudla–Millson Schwartz forms* [KM86, KM87]

$$\phi_{\mathrm{KM},v_0} \in \mathscr{S}(V_{v_0}^m) \otimes \Omega^{m,m}(\mathbb{D})$$

where v_0 is the real place of F induced by the fixed embedding $\sigma : E \hookrightarrow \mathbb{C}$, and $\Omega^{a,b}(\mathbb{D})$ is the space of smooth differential forms on $\mathbb{D}$ of type (a,b). The Schwartz form ϕ_{KM,v_0} is $H_{v_0}(\mathbb{R})$-invariant and *closed* at any $\mathbf{x} \in V_{v_0}^m$. Define

$$\widetilde{\phi}_\infty = \phi_{\mathrm{KM},v_0} \otimes \bigotimes_{v|\infty,\ v\neq v_0} \phi_v \in \mathscr{S}(V_\infty^m) \otimes \Omega^{n,n}(\mathbb{D}),$$

where $\phi_v \in \mathscr{S}(V_v^m)$ is the Gaussian function. Define

$$\widetilde{\phi}^V := \phi \otimes \widetilde{\phi}_\infty \in \mathscr{S}(V(\mathbb{A}_F)^n) \otimes \Omega^{m,m}(\mathbb{D})$$

and the *Kudla–Millson theta function*

$$\theta_{\mathrm{KM},\phi}(g,h) := \sum_{\mathbf{x} \in V^m} \omega(g)\widetilde{\phi}^V(h^{-1}\mathbf{x}), \quad g \in G(\mathbb{A}_F),\ h \in H(\mathbb{A}_F^\infty),$$

which gives a closed (m,m)-form on $X_K(\mathbb{C})$ at any $g \in G(\mathbb{A}_F)$. By the Poisson summation formula one can prove that $\theta_{\mathrm{KM},\phi}(g,h)$ defines a (nonholomorphic) automorphic form valued in closed (m,m)-forms on $X_K(\mathbb{C})$. [KM90] further proves that it represents the (holomorphic) geometric theta series $[Z_\phi(g)_K]$ in $H^{2m}(X_K(\mathbb{C}), \mathbb{C})$ (in particular, the nonholomorphic terms in $\theta_{\mathrm{KM},\phi}(g,h)$ are exact forms) and obtains the theorem.

Remark 4.6. The remarkable discovery that generating function involving intersection numbers of algebraic cycles are modular originates from the work of Hirzebruch–Zagier [HZ76] on Hilbert modular surfaces, and is one of the inspirations for Kudla's work (cf. the introduction of [KM90, Kud97a]).

4.4. **Geometric theta lifting.** Analogous to §3.4, using $[Z_\phi(g)]$ as an integral kernel allows one to lift automorphic forms on G to cohomology classes on $X_K(\mathbb{C})$.

Definition 4.7. For $\varphi \in \mathscr{A}_{r/2,\chi}(G(\mathbb{A}_F))$, define the *geometric theta lift* or *Kudla–Millson lift* $\theta_\phi^{\mathrm{KM}}(\varphi)$ to be the Petersson inner product

$$\theta_\phi^{\mathrm{KM}}(\varphi)_K := \langle [Z_\phi(g)_K, \varphi\rangle_G = \int_{[G]} [Z_\phi(g)]\overline{\varphi(g)}dg \in H^{2m}(X_K(\mathbb{C}), \mathbb{C}).$$

When varying the neat open compact subgroup $K \subset H(\mathbb{A}_F^\infty)$, it defines a class

$$\theta_\phi^{\mathrm{KM}}(\varphi) := (\theta_\phi^{\mathrm{KM}}(\varphi)_K)_K \in H^{2m}(X(\mathbb{C}), \mathbb{C}) := \varinjlim_{K \subset H(\mathbb{A}_F^\infty)} H^{2m}(X_K(\mathbb{C}), \mathbb{C}).$$

Let π be a cuspidal automorphic representation of $G(\mathbb{A}_F)$. Assume that $\pi \cap \mathscr{A}_{r/2,\chi}(G(\mathbb{A}_F)) \neq 0$, which forces that π_∞ is a holomorphic discrete series of a particular weight (cf. [Liu11a, p.852]). Then we obtain an $G(\mathbb{A}_F^\infty) \times H(\mathbb{A}_F^\infty)$-equivariant linear map

$$\begin{aligned} \theta^{\mathrm{KM}} : \mathscr{S}(\mathbb{V}^{\infty,m}) \otimes \pi^\vee &\longrightarrow H^{2m}(X(\mathbb{C}), \mathbb{C}) \\ (\phi, \overline{\varphi}) &\longmapsto \theta_\phi^{\mathrm{KM}}(\varphi). \end{aligned}$$

Definition 4.8. Define the *geometric theta lift*

$$\Theta_V^{\mathrm{KM}}(\pi) \subset H^{2m}(X(\mathbb{C}), \mathbb{C})$$

of π to be the image of the $G(\mathbb{A}_F^\infty) \times H(\mathbb{A}_F^\infty)$-equivariant linear map θ^{KM} above (here $\theta_\phi^{\mathrm{KM}}(\varphi)$ is understood to be 0 if $\varphi \notin \mathscr{A}_{r/2,\chi}(G(\mathbb{A}_F))$).

4.5. **Geometric Siegel–Weil formula.** To relate the geometric theta series to Eisenstein series, we need to extract numerical invariants from cohomology classes. To that end, assume that V is anisotropic, thus X_K is projective and we have a degree map $\deg : H^{2\dim X_K}(X_K, \mathbb{C}) \to \mathbb{C}$. For any $n \leqslant \dim X_K = m$, define the *geometric volume*

$$\begin{aligned} \mathrm{vol} : H^{2n}(X_K(\mathbb{C}), \mathbb{C}) &\longrightarrow \mathbb{C} \\ [Z] &\longmapsto \deg([Z] \cup [c_1(\mathcal{L}_K^\vee)]^{\dim X_K - n}). \end{aligned}$$

In particular when $n = \dim X_K$ we obtain the geometric volume $\mathrm{vol}([X_K])$ of the Shimura variety X_K. Define the *normalized geometric volume*

$$\begin{aligned} \mathrm{vol}^\natural : H^{2n}(X_K(\mathbb{C}), \mathbb{C}) &\longrightarrow \mathbb{C} \\ [Z] &\longmapsto \frac{\mathrm{vol}([Z])}{\mathrm{vol}([X_K])/2}. \end{aligned}$$

The Haar measure on $H(\mathbb{A}_F^\infty)$ such that K has volume $(\mathrm{vol}([X_K])/2)^{-1}$ can be viewed as an analogue of the Tamagawa measure on $H(\mathbb{A}_F)$ (cf. [LL21, Footnote 11]), hence the normalization. Then $\mathrm{vol}^\natural([Z_\phi(g)_K])$ is independent of the choice of K and can be viewed as a geometric analogue of the theta integral

$$I_\phi(g) := \int_{[H]} \theta_\phi(g, h) \mathrm{d}h,$$

and produces a $G(\mathbb{A}_F)$-equivariant distribution analogous to

$$\begin{aligned} I : \mathscr{S}(V(\mathbb{A}_F)^n) &\longrightarrow \mathscr{A}(G(\mathbb{A}_F)) \\ \phi &\longmapsto I_\phi(-). \end{aligned}$$

Such a distribution turns out to be

$$\begin{aligned} \mathrm{vol}^\natural : \mathscr{S}(\mathbb{V}^{\infty,n}) &\longrightarrow \mathscr{A}(G(\mathbb{A}_F)) \\ \phi &\longmapsto \mathrm{vol}^\natural[Z_\phi(-)]. \end{aligned}$$

On the other hand, for any $\phi \in \mathscr{S}(\mathbb{V}^{\infty,n})^K$, the Schwartz form $\widetilde{\phi}^V = \phi \otimes \widetilde{\phi}_\infty$ defined before gives an element $\phi^V \in \mathscr{S}(V(\mathbb{A}_F)^n) \otimes \Omega^{0,0}(\mathbb{D})$ such that

$$\widetilde{\phi}^V \wedge \Omega^{\dim X - n} = \phi^V \cdot \Omega^{\dim X},$$

where $\Omega \in \Omega^{1,1}(\mathbb{D})$ is the first Chern form of $\mathcal{L}^\vee$. Evaluation of ϕ^V at the base point of $\mathbb{D}$ gives a Schwartz function in $\mathscr{S}(V(\mathbb{A}_F)^n)$, which we still denote by ϕ^V by abuse of notation. Hence we obtain a coherent Siegel Eisenstein series $E_{\phi^V}(g, s)$ on $G(\mathbb{A}_F)$.

Theorem 4.9 (Geometric Siegel–Weil formula)**.** *Assume V is anisotropic. Assume that $n \leqslant \dim X_K = m - 1$. Then for any $\phi \in \mathscr{S}(\mathbb{V}^{\infty,n})$, the following identity holds*

$$\kappa \cdot \mathrm{vol}^\natural([Z_\phi(g)]) = E_{\phi^V}(g, s_0).$$

Here $s_0 = (m-n)/2$, $\kappa = 1/2$ are the constants in the Siegel–Weil formula (Theorem 3.8) for the pair (V, W).

This is [Kud04, (4,4)] (see also [Kud03, Theorem 4.23]) for orthogonal Shimura varieties. We refer to [Dun22, §2.2] for an exposition of the proof for the unitary Shimura variety X_K. The geometric Siegel–Weil formula holds more generally for non-projective X_K under Weil's convergence condition, although the geometric volume lacks a cohomological interpretation as above (see [Kud04, Theorem 4.1]).

4.6. **Geometric inner product formula.** To finish the geometric story, we introduce a geometric analogue of the Petersson inner product on $[H]$. Further assume that $2n \leqslant \dim X_K$.

Definition 4.10. Define the (normalized) *geometric inner product*

$$\begin{aligned}\langle\cdot,\cdot\rangle_{X_K(\mathbb{C})}: H^{2n}(X_K(\mathbb{C}),\mathbb{C})\times H^{2n}(X_K(\mathbb{C}),\mathbb{C}) &\longrightarrow \mathbb{C}\\ ([Z_1],[Z_2]) &\longmapsto \mathrm{vol}^\natural([Z_1]\cup\overline{[Z_2]}).\end{aligned}$$

It is again compatible with varying K and thus gives an inner product $\langle\cdot,\cdot\rangle_{X(\mathbb{C})}$ on $H^{2n}(X(\mathbb{C}),\mathbb{C})$. Combining the geometric Siegel–Weil formula and the Rallis inner product formula, we obtain the following.

Theorem 4.11 (Geometric inner product formula)**.** *Assume that V is anisotropic. Assume that $2n \leqslant \dim X_K = m-1$. Let π be a cuspidal automorphic representation of $G(\mathbb{A}_F)$ such that $\pi \cap \mathscr{A}_{m/2,\chi}(G(\mathbb{A}_F)) \neq 0$. Then for any $\phi_i = \bigotimes_v \phi_{i,v} \in \mathscr{S}(\mathbb{V}^{\infty,n})$ and $\varphi_i = \bigotimes_v \varphi_{i,v} \in \pi \cap \mathscr{S}_{m/2,\chi}(G(\mathbb{A}_F))$ with $i = 1,2$,*

$$\kappa\cdot\langle\theta^{\mathrm{KM}}_{\phi_1}(\varphi_1),\theta^{\mathrm{KM}}_{\phi_2}(\varphi_2)\rangle_{X(\mathbb{C})} = \frac{L(s_0+1/2,\pi\times\chi)}{b_{2n}(s_0)}\cdot\prod_v Z_v^\natural(s_0;\varphi_{1,v},\varphi_{2,v};\widetilde{\phi}^V_{1,v},\widetilde{\phi}^V_{2,v}).$$

Here $s_0 = (m-2n)/2$, $\kappa = 1/2$ are the constants in the Siegel–Weil formula (Theorem 3.8) for the pair $(V, W^\square)$.

Example 4.12. In the special case $2n = m-1$, each $\theta^{\mathrm{KM}}_{\phi_i}(\varphi_i)$ is the cohomology class of a middle dimensional cycle on $X(\mathbb{C})$ and the geometric inner product relates their geometric intersection number to the near central value $L(1,\pi\times\chi)$ at $s_0 = 1/2$.

Kudla–Millson's theory of geometric theta correspondence [KM90], as extended by Funke–Millson to nontrivial coefficients [FM06] and compactifications of non-compact X_K (e.g. [FM14]), have many applications to the cohomology of Shimura varieties and more general locally symmetric spaces. For example, Bergeron–Millson–Mœglin [BMM16] proved the Hodge conjecture and the Tate conjecture for the arithmetic ball quotients X_K, in cohomological degree $\leqslant (m-1)/3$ or $\geqslant 2(m-1)/3$, and geometric theta lifting is a key ingredient in the proof to generate many Hodge/Tate classes using special cycles in these degrees far away from the middle degree. Analogous to the classical theory, the geometric inner product formula and its variants (e.g. [BF10, Theorem 1.1]) are useful to prove nonvanishing results on geometric theta lifting.

5. Arithmetic theta correspondence

The modularity of classical and geometric theta functions motivates Kudla's arithmetic modularity conjecture [Kud04, Problem 1].

Conjecture 5.1 (Arithmetic modularity)**.** *The formal generating function $Z_\phi(g)_K$ converges absolutely and defines an element in $\mathscr{A}_{m/2,\chi}(G(\mathbb{A}_F))\otimes \mathrm{CH}^n(X_K)_{\mathbb{C}}$.*

The formulation in the unitary case can be found in Liu [Liu11a], who also proved the case $n=1$ and reduce the $n>1$ case to the converges. Recently Xia [Xia21] proved the desired convergence when $F=\mathbb{Q}(\sqrt{-d})$ for $d = 1,2,3,7,11$, and thus established Conjecture 5.1 in these cases.

Remark 5.2. Kudla's arithmetic modularity conjecture was originally formulated for orthogonal Shimura varieties over $\mathbb{Q}$ ([Kud97a], [Kud04, Problem 1]). In this case, Borcherds [Bor99] proved the conjecture for the divisor case $n=1$ (the special case of Heegner points on modular curves dates back to the classical work of Gross–Kohnen–Zagier [GKZ87]). Zhang [Zha09] proved the modularity for general n assuming the absolute convergence of the series. Bruinier–Westerholt-Raum [BWR15] proved the desired convergence and hence established Kudla's modularity conjecture for orthogonal Shimura varieties over $\mathbb{Q}$.

For orthogonal Shimura varieties over totally real fields, Yuan–Zhang–Zhang [YZZ13] proved the modularity for $n = 1$ (see also Bruinier [Bru12] for a different proof) and reduce the $n > 1$ case to the convergence. More recently, Bruinier–Zemel [BZ22] has extended the modularity to toroidal compactifications of orthogonal Shimura varieties when $n = 1$.

Assume Conjecture 5.1. Recall the construction of theta function produces the *automorphic theta distribution*

$$\begin{aligned} \theta : \mathscr{S}(V^r \otimes \mathbb{A}_F) &\longrightarrow \mathscr{A}(G(\mathbb{A}_F)) \otimes \mathscr{A}(H(\mathbb{A}_F)) \\ \phi &\longmapsto \theta_\phi(-,-), \end{aligned}$$

a $G(\mathbb{A}_F) \times H(\mathbb{A}_F)$-equivariant distribution valued in the space of automorphic forms. Analogous to this, we obtain an *arithmetic theta distribution*

$$\begin{aligned} Z : \mathscr{S}(\mathbb{V}^r \otimes \mathbb{A}_F^\infty) &\longrightarrow \mathscr{A}(G(\mathbb{A}_F)) \otimes \mathrm{CH}^n(X)_\mathbb{C} \\ \phi &\longmapsto Z_\phi(-). \end{aligned}$$

It is a $G(\mathbb{A}_F^\infty) \times H(\mathbb{A}_F^\infty)$-equivariant distribution, where $H(\mathbb{A}_F^\infty)$ acts on $\mathrm{CH}^n(X)_\mathbb{C}$ via the Hecke correspondences. Analogous to §4.4, using $Z_\phi(g)$ as an integral kernel allows one to lift automorphic forms on G to algebraic cycles on $X_K(\mathbb{C})$.

Definition 5.3. For $\varphi \in \mathscr{A}_{m/2,\chi}(G(\mathbb{A}_F))$, define the *arithmetic theta lift* $\Theta_\phi(\varphi)$ to be the Petersson inner product

$$\Theta_\phi(\varphi)_K := \langle Z_\phi(g)_K, \varphi\rangle_G = \int_{[G]} Z_\phi(g)\overline{\varphi(g)}\mathrm{d}g \in \mathrm{CH}^n(X_K)_\mathbb{C}.$$

When varying $K \subset H(\mathbb{A}_F^\infty)$, it defines a class

$$\Theta_\phi(\varphi) := (\Theta_\phi(\varphi)_K)_K \in \mathrm{CH}^n(X)_\mathbb{C}.$$

Let π be a cuspidal automorphic representation of $G(\mathbb{A}_F)$ and assume that $\pi \cap \mathscr{A}_{m/2,\chi}(G(\mathbb{A}_F)) \neq 0$. Then we obtain an $G(\mathbb{A}_F^\infty) \times H(\mathbb{A}_F^\infty)$-equivariant linear map

$$\begin{aligned} \Theta : \mathscr{S}(\mathbb{V}^{\infty,n}) \otimes \pi^\vee &\longrightarrow \mathrm{CH}^n(X)_\mathbb{C} \\ (\phi, \overline{\varphi}) &\longmapsto \Theta_\phi(\varphi). \end{aligned}$$

Definition 5.4. Define the *arithmetic theta lift*

$$\Theta_\mathbb{V}(\pi) \subset \mathrm{CH}^n(X)_\mathbb{C}$$

of π to be the image of the $G(\mathbb{A}_F^\infty) \times H(\mathbb{A}_F^\infty)$-equivariant linear map Θ above. (Again, here $\Theta_\phi(\varphi)$ is understood to be 0 if $\phi \neq \mathscr{A}_{m/2,\chi}(G(\mathbb{A}_F))$.)

In particular, Θ can be viewed as element

$$\Theta \in \mathrm{Hom}_{H(\mathbb{A}_F^\infty)}((\mathscr{S}(\mathbb{V}^{\infty,n}) \otimes (\pi^\infty)^\vee)_{G(\mathbb{A}_F^\infty)}, \mathrm{CH}^n(X)_\mathbb{C}).$$

Notice that $(\mathscr{S}(\mathbb{V}^{\infty,n}) \otimes (\pi^\infty)^\vee)_{G(\mathbb{A}_F^\infty)}$ is nothing but the classical theta lift $\Theta(\pi^\infty) := \bigotimes_{v\nmid\infty} \Theta_{V_v}(\pi_v)$ of π^∞, thus we may view the arithmetic theta lift as an element of the $\Theta(\pi^\infty)$-isotypic part of $\mathrm{CH}^n(X)_\mathbb{C}$,

$$\Theta \in \mathrm{Hom}_{H(\mathbb{A}_F^\infty)}(\Theta(\pi^\infty), \mathrm{CH}^n(X)_\mathbb{C}).$$

5.1. **Special cycles on integral models.** Kudla [Kud04, Problem 4] also proposed the modularity problem in the *arithmetic Chow group* $\widehat{\mathrm{CH}}^n(\mathcal{X}_K)$ of a suitable (compactified) regular integral model $\mathcal{X}_K$ (of a variant) of X_K (see [GS91, GKK07] and also [Sou92]). The problem seeks to define canonically an explicit arithmetic generating function $\hat{\mathcal{Z}}_\phi(\tau)$ valued in $\widehat{\mathrm{CH}}^n(\mathcal{X})_\mathbb{C}$ which lifts $Z_\phi(\tau)$ under the restriction map

$$\widehat{\mathrm{CH}}^n(\mathcal{X}) \longrightarrow \mathrm{CH}^n(X),$$

and such that $\hat{\mathcal{Z}}_\phi(\tau)$ is modular.

To define the integral model and special cycles on it, it is more convenient to work with a related unitary Shimura variety with an explicit moduli interpretation after [KR14b, BHK+20a, RSZ20]. Define a torus $Z^\mathbb{Q} = \{z \in \mathrm{Res}_{E/\mathbb{Q}}\,\mathbb{G}_m : \mathrm{Nm}_{E/F}(z) \in \mathbb{G}_m\}$. Fix a CM type $\Phi \subset \mathrm{Hom}(E, \overline{\mathbb{Q}})$ of E. Then associated to $\widetilde{H} := Z^\mathbb{Q} \times \mathrm{Res}_{F/\mathbb{Q}}\, H$ there is a natural Shimura datum $(\widetilde{H}, \{h_{\widetilde{H}}\})$ of PEL type [LZ22b,

§11.1]. Assume that $K_{Z^{\mathbb{Q}}} \subset Z^{\mathbb{Q}}(\mathbb{A}_F^\infty)$ is the unique maximal open compact subgroup. Then the associated Shimura variety $\mathrm{Sh}_K = \mathrm{Sh}_{K_{Z^{\mathbb{Q}}}\times K}(\widetilde{H}, \{h_{\widetilde{H}}\})$ is of dimension $n-1$ and has a canonical model over its reflex field L. Moreover, Sh_K can be identified as the product of the base change $(X_K)_L$ and a 0-dimensional Shimura variety of PEL type [LL21, Lemma 5.2].

Assume that $K = \prod_{v\nmid\infty} K_v \subset H(\mathbb{A}_F^\infty)$ and $K_v \subset H(F_v)$ is given by

- the stabilizer of a self-dual or almost self-dual lattice $\Lambda_v \subset V_v$ if v is inert in E,
- the stabilizer of a self-dual lattice $\Lambda_v \subset V_v$ if v is ramified in E,
- a principal congruence subgroup of $H_v(F_v) \simeq \mathrm{GL}_n(F_v)$ if v is split in F.

Let $\mathcal{V}_{\mathrm{ram}}$ be the set of finite places v of F such that v is unramified in E (resp. v is inert in E and Λ_v is almost self-dual). Further assume that all places of F above $\mathcal{V}_{\mathrm{ram}} \cup \mathcal{V}_{\mathrm{asd}}$ are unramified over E. Then we obtain a global regular integral model $\mathcal{X}_K$ of Sh_K over $\mathcal{O}_E$ after Rapoport–Smithling–Zhang [RSZ20] (see [LZ22b, §14.1–14.2]) for the construction and more precise technical assumptions), which is semistable at all places of F above $\mathcal{V}_{\mathrm{ram}} \cup \mathcal{V}_{\mathrm{asd}}$ and smooth everywhere else. When K_G is the stabilizer of a global self-dual lattice, the regular integral model $\mathcal{X}_K$ recovers that in [BHK+20a] if $F = \mathbb{Q}$. Let $\phi \in \mathscr{S}(\mathbb{V}^{\infty,n})^K$ be a factorizable Schwartz function such that $\phi_v = \mathbf{1}_{(\Lambda_v)^n}$ at all v nonsplit in E. Let $T \in \mathrm{Herm}_n(F)$ be nonsingular. Associated to (T, ϕ) we have an arithmetic special cycle $\mathcal{Z}(T, \phi)_K$ over $\mathcal{X}_K$ [LZ22b, §14.3].

5.2. **Modularity in arithmetic Chow groups.** The integral model $\mathcal{X}_K$ and $\mathcal{Z}(T,\phi)_K$ are constructed as the moduli spaces of certain abelian varieties with additional structures. To describe them more precisely, in this subsection we consider the special case $F = \mathbb{Q}$ (so $L = E$ is an imaginary quadratic field), $n = 1$, and there is a global self-dual lattice Λ such that $\Lambda_v = \Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_{F,v}$ and $\phi_v = \mathbf{1}_{\Lambda_v}$ at all finite places v. In this special case, the special cycles are indexed by $T \in \mathrm{Herm}_n(\mathcal{O}_F)^+ = \mathbb{Z}_{\geqslant 0}$. Assume that E/F is unramified at 2 for simplicity.

Definition 5.5. Define an integral $\mathcal{X}_K$ of Sh_K over $\mathcal{O}_E$ as follows. For an $\mathcal{O}_E$-scheme S, we consider $\mathcal{X}_K(S)$ to be the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \mathcal{F}_A)$, where

(1) A_0 (resp. A) is an abelian scheme over S.
(2) ι_0 (resp. ι) is an action of $\mathcal{O}_E$ on A_0 (resp. A).
(3) λ_0 (resp. λ) is a polarization of A_0 (resp. A).
(4) $\mathcal{F}_A \subset \mathrm{Lie}_S A$ is an $\mathcal{O}_E$-stable $\mathcal{O}_S$-module local direct summand.

We require that

(1) $\mathcal{O}_E$ acts on the $\mathcal{O}_S$-module $\mathrm{Lie}_S A_0$ via the structure morphism $\mathcal{O}_E \hookrightarrow \mathcal{O}_S$. This is the *Kottwitz condition* of signature $(1,0)$:
$$\det(\iota_0(a) \mid \mathrm{Lie}_S A_0) = T - a \in \mathcal{O}_S[T]$$
for all $a \in \mathcal{O}_E$.
(2) $\mathcal{F}_A$ satisfies the *Krämer condition*: $\mathcal{O}_E$ acts on $\mathcal{F}_A$ via the structure morphism and acts on the line bundle $\mathrm{Lie}_S(A/\mathcal{F}_A)$ via the conjugate of the structure morphism. This implies (and in characteristic 0 is equivalent to) the Kottwitz condition of signature $(m-1,1)$:
$$\det(\iota(a) \mid \mathrm{Lie}_S A) = (T-a)^{m-1}(T-\overline{a}) \in \mathcal{O}_S[T]$$
for all $a \in \mathcal{O}_E$.
(3) The Rosati involution on $\mathrm{End}_S A_0$ (resp. $\mathrm{End}_S A$) induces the conjugation on $\mathcal{O}_E$ via ι_0 (resp. ι).
(4) At every geometric point s of S, there is an isomorphism of hermitian $\mathcal{O}_{F,\ell}$-modules
$$\mathrm{Hom}_{\mathcal{O}_E}(T_\ell A_{0,s}, T_\ell A_s) \simeq \mathrm{Hom}_{\mathcal{O}_E}(\Lambda_0, \Lambda) \otimes \mathbb{Z}_\ell$$
for any prime ℓ different from the residue characteristic of s. Here Λ_0 is a fixed self-dual hermitian lattice of rank 1 over $\mathcal{O}_E$. Notice that $\mathrm{Hom}_{\mathcal{O}_E}(\Lambda_0, \Lambda)$ has a natural hermitian module structure given by $(x, y) := y^\vee \circ x \in \mathrm{End}_{\mathcal{O}_E}(\Lambda_0) \subset E$ and similarly for the left-hand side.

Then the functor $S \mapsto \mathcal{X}_K(S)$ is represented by a Deligne–Mumford stack $\mathcal{X}_K$ regular over $\mathrm{Spec}\, \mathcal{O}_E$.

The extra data $(A_0, \iota_0, \lambda_0)$ of a CM elliptic curve allows us to consider a motivic version of the lattice Λ. For $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \mathcal{F}_A) \in \mathcal{X}_K(S)$, define the *special homomorphisms* to be
$$\Lambda(A_0, A) := \mathrm{Hom}_{\mathcal{O}_E}(A_0, A),$$

equipped with a natural hermitian form $(x, y) \in \mathcal{O}_E$ given by

$$(A_0 \xrightarrow{x} A \xrightarrow{\lambda} A^\vee \xrightarrow{y^\vee} A_0^\vee \xrightarrow{\lambda_0^{-1}} A_0) \in \mathrm{End}_{\mathcal{O}_E}(A_0) = \iota_0(\mathcal{O}_E) \simeq \mathcal{O}_E.$$

When $T > 0$, define the *special divisor* $\mathcal{Z}(T, \phi)_K$ by requiring an additional special homomorphism of norm T. More precisely, the functor $S \mapsto \{(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \mathcal{F}_A, x)\}$, where

(1) $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \mathcal{F}_A) \in \mathcal{X}_K(S)$,
(2) $x \in \Lambda(A_0, A)$ such that $T(x) = (x, x) = T$,

is represented by a Deligne–Mumford stack $\mathcal{Z}(T, \phi)_K$, which is finite and unramified over $\mathcal{X}_K$. It extends to a compactified special divisor $\mathcal{Z}^*(T, \phi)_K$ on the canonical toroidal compactification $\mathcal{X}_K^*$ by taking the Zariski closure. [BHK+20a] further defines a *total special divisor* $\mathcal{Z}^{\mathrm{tot}}(T, \phi)_K$ by adding an explicit boundary divisor to $\mathcal{Z}^*(T, \phi)_K$ [BHK+20a, (1.1.3)]. Using regularized theta lifts of harmonic Maass forms, $\mathcal{Z}^*(T, \phi)_K$ is equipped with an *automorphic Green function* with log-log singularities along the boundary [BHK+20a, §7.2], hence it defines an element in

$$\widehat{\mathcal{Z}}^{\mathrm{tot}}(T, \phi)_K \in \widehat{\mathrm{CH}}^1(\mathcal{X}_K^*).$$

When $T = 0$, define

$$\widehat{\mathcal{Z}}^{\mathrm{tot}}(0, \phi)_K = \widehat{\mathcal{L}}_K^\vee + (\mathrm{Exc}, -\log|\mathrm{disc}(F)|) \in \widehat{\mathrm{CH}}^1(\mathcal{X}_K^*)$$

where $\widehat{\mathcal{L}}_K^\vee$ is the metrized dual Hodge line bundle over $\mathcal{X}_K^*$, Exc is an effective vertical divisor supported above $\mathcal{V}_{\mathrm{ram}}$ equipped with the constant Green function $-\log|\mathrm{disc}(F)|$. Define the *generating function in arithmetic Chow groups*

$$\widehat{\mathcal{Z}}^{\mathrm{tot}}(\tau, \phi)_K := \sum_{T \geqslant 0} \widehat{\mathcal{Z}}^{\mathrm{tot}}(T, \phi)_K \cdot q^T,$$

as a formal generating function valued in $\widehat{\mathrm{CH}}^1(\mathcal{X}_K^*)$, where $\tau \in \mathcal{H}_1$ lies in the usual upper half plane.

Theorem 5.6 (Modularity in arithmetic Chow groups: the divisor case)**.** *The formal generating function $\widehat{\mathcal{Z}}^{\mathrm{tot}}(\tau, \phi)_K$ converges absolutely and defines an elliptic modular form valued in $\widehat{\mathrm{CH}}^1(\mathcal{X}_K^*)$ of weight m, level $|\mathrm{disc}\, F|$ and character η^m.*

This is proved in Bruinier–Howard–Kudla–Rapoport–Yang [BHK+20a, Theorem B]. Analogous to §5.1, Theorem 5.6 allows us to construct arithmetic theta lifts valued in $\widehat{\mathrm{CH}}^1(\mathcal{X}_K^*)$. As applications, [BHK+20b, Theorem A,B] proves formulas relating the arithmetic intersection of these arithmetic theta lifts and small/big CM points to the central derivative of certain convolution L-functions of two elliptic modular forms, generalizing the Gross–Zagier formula [GZ86].

Remark 5.7. One can also use *Kudla's Green function* [BHK+20a, (7.4.1)] in place of the automorphic Green function to define an arithmetic divisor $\widehat{\mathcal{Z}}^{\mathrm{tot}}(\mathsf{y}, T, \phi)_K \in \widehat{\mathrm{CH}}^1(\mathcal{X}_K^*)$ depending on a parameter $\mathsf{y} = \mathrm{Im}(\tau) \in \mathbb{R}_{>0}$ (here T is also allowed to be < 0, in which case the divisor is supported at the archimedean fiber). Then the generating function

$$\sum_{T \in \mathbb{Z}} \widehat{\mathcal{Z}}^{\mathrm{tot}}(\mathsf{y}, T, \phi)_K \cdot q^T$$

becomes a nonholomorphic modular form [BHK+20a, Theorem 7.4.1]. This is a consequence of Theorem 5.6 and the modularity of the difference of the two generating function due to Ehlen–Sankaran [ES18].

Remark 5.8. The proof of Theorem 5.6 uses the arithmetic theory of Borcherds products, which requires the assumption $F = \mathbb{Q}$. For $F \neq \mathbb{Q}$, a version of Theorem 5.6 is proved in Qiu [Qiu22] by a different method and formulation. A version of Theorem 5.6 is proved in Howard–Madapusi Pera [HP20] for (open) orthogonal Shimura varieties over $\mathbb{Q}$.

Remark 5.9. The generating functions of arithmetic divisors have also found many applications outside the Kudla program. To name some recent arithmetic applications:

(1) Theorem 5.6 is used in Zhang's proof of the arithmetic fundamental lemma over $\mathbb{Q}_p$ in [Zha21]. Variants over general totally real fields also play a key role for the arithmetic fundamental lemma over p-adic fields in Mihatsch–Zhang [MZ] and the arithmetic transfer conjecture in Z. Zhang [Zha], in the framework of the arithmetic Gan–Gross–Prasad conjectures for unitary groups. We refer to Zhang's article in these proceedings for more details.
(2) The arithmetic modularity in [HP20] is used in Shankar–Shanker–Tang–Tayou [SSTT19] on the Picard rank jumps of K3 surfaces over number fields.
(3) The proof of the averaged Colmez conjecture in Andreatta–Goren–Howard–Madapusi Pera [AGHP18] relies on relating arithmetic intersection of special divisors on orthogonal Shimura varieties and big CM points to central derivatives of certain L-functions.

Remark 5.10. The modularity problem in arithmetic Chow groups [Kud04, Problem 4] remains open in higher codimension $n > 1$. When $n > 1$, even when $T > 0$ the special cycle $\mathcal{Z}(T, \phi)$ in general has the wrong codimension due to improper intersection in positive characteristics, and the consideration of *derived intersection* is necessary to obtain the correct class $\widehat{\mathcal{Z}}(T, \phi)$ in arithmetic Chow groups. It is also subtle to find the correction terms at places of bad reduction and at boundary (both issues already appear when $n = 1$) and to find the correct construction of Green currents to ensure modularity. The forthcoming works of Howard–Madapusi Pera and Madapusi Pera address some of these issues when $n > 1$.

5.3. **Arithmetic Siegel–Weil formula.** If the arithmetic theta function $\widehat{\mathcal{Z}}(\tau, \phi) \in \widehat{\mathrm{CH}}^n(\mathcal{X}_K)$ can be constructed, then we may apply the *arithmetic volume*

$$\begin{aligned} \widehat{\mathrm{vol}} : \widehat{\mathrm{CH}}^n(\mathcal{X}_K) &\longrightarrow \mathbb{C} \\ \widehat{\mathcal{Z}} &\longmapsto \widehat{\deg}(\widehat{\mathcal{Z}} \cdot (c_1(\widehat{\mathcal{L}}_K^\vee))^{\dim \mathcal{X}_K - n}) \end{aligned}$$

and try to relate $\widehat{\mathrm{vol}}(\widehat{\mathcal{Z}}(\tau, \phi))$ to the special derivatives of Siegel Eisenstein series. However, as discussed in Remark 5.10 the definition of $\widehat{\mathcal{Z}}(\tau, \phi)$ is rather subtle when $n > 1$. Moreover, the special derivatives are nonholomorphic function (also including terms indexed by $T \notin \mathrm{Herm}_n(F)^+$, cf. Remark 5.7).

In this subsection we assume that $m = n$, so $s_0 = 0$ and $\kappa = 1$ in the Siegel–Weil formula for the pair (V, W). In this special case, the arithmetic volume is simply the arithmetic degree and we can define the nonsingular terms in the generating function in a more explicit way. Even for $T > 0$ terms, the relation to Siegel Eisenstein series is more complicated due to contribution at places of bad reduction, a phenomenon first discovered by Kudla–Rapoport [KR00] via explicit computation in the context of Shimura curves uniformized by the Drinfeld p-adic half plane.

For nonzero $t_1, \ldots, t_n \in E$ and $\phi_1, \ldots, \phi_n \in \mathscr{S}(\mathbb{V}_F^\infty)^K$ such that $\phi_v = \mathbf{1}_{\Lambda_v}$ at all v nonsplit in E, we have a natural decomposition [KR14b, (11.2)]

$$\mathcal{Z}(t_1, \phi_1)_K \cap \cdots \cap \mathcal{Z}(t_n, \phi_n)_K = \bigsqcup_{T \in \mathrm{Herm}_n(F)} \mathcal{Z}(T, \phi)_K,$$

here $\cap$ denotes taking fiber product over $\mathcal{X}_K$, the indexes T have diagonal entries $t_1, \ldots, t_n$, and $\phi = \bigotimes_{i=1}^n \phi_i$.

Definition 5.11. For $v \nmid \infty$ and ν a place of L (which is the reflex field of Sh_K) over v, define the *local arithmetic intersection number*

$$\mathrm{Int}_{T,\nu}(\phi) := \chi(\mathcal{Z}(T, \phi)_K, \mathcal{O}_{\mathcal{Z}(t_1,\phi_1)_K} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_n,\phi_n)_K}) \cdot \log q_\nu,$$

where

- q_ν denotes the size of the residue field k_ν of L_ν,
- $\mathcal{O}_{\mathcal{Z}(t_i,\phi_i)_K}$ denotes the structure sheaf of the special divisor $(\mathcal{Z}(t_i, \phi_i)_K)_{\mathcal{O}_{K_\nu}}$,
- $\otimes^{\mathbb{L}}$ denotes the derived tensor product of coherent sheaves on $(\mathcal{X}_K)_{\mathcal{O}_{K_\nu}}$, and
- χ denotes the Euler–Poincaré characteristic (an alternating sum of lengths of $\mathcal{O}_{K_\nu}$-modules).

Notice that the derived tensor product $\mathcal{O}_{\mathcal{Z}(t_1,\phi_1)_K} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_n,\phi_n)_K}$ has the structure of a complex of $\mathcal{O}_{\mathcal{Z}(t_1,\phi_1)_K \cap \cdots \cap \mathcal{Z}(t_n,\phi_n)_K}$-modules, hence has a natural decomposition by support according to the decomposition above. Define

$$\mathrm{Int}_{T,v}(\phi) := \frac{1}{[K : F]} \cdot \sum_{\nu | v} \mathrm{Int}_{T,\nu}(\phi).$$

Using the star product of *Kudla's Green functions*, we can also define its local arithmetic intersection number $\mathrm{Int}_{T,v}(\mathsf{y},\phi)$ at infinite places [LZ22b, §15.3], which depends on a parameter $\mathsf{y} \in \mathrm{Herm}_n^{\circ}(F_\infty)^+$. Combining all the local arithmetic numbers together, define the *global arithmetic intersection number*, or the (normalized) *arithmetic degree* of the special cycle $\mathcal{Z}(T,\phi)_K$

$$\widehat{\deg}_T(\mathsf{y},\phi) := \frac{1}{\mathrm{vol}([\mathrm{Sh}_K])/2}\left(\sum_{v\nmid\infty}\mathrm{Int}_{T,v}(\phi)+\sum_{v\mid\infty}\mathrm{Int}_{T,v}(\mathsf{y},\phi)\right).$$

We form the *generating function of arithmetic degrees*

$$\widehat{\deg}(\tau,\phi) := \sum_{\substack{T\in\mathrm{Herm}_n(F),\\ \det T\neq 0}} \widehat{\deg}_T(\mathbf{y},\phi)q^T.$$

On the other hand, associated to

$$\phi^{\mathbb{V}} := \phi\otimes\phi_\infty \in \mathscr{S}(\mathbb{V}^n),$$

where ϕ_∞ is the Gaussian function, we obtain a classical *incoherent Eisenstein series* $E(\tau,s,\phi^{\mathbb{V}})$. The central value $E(\tau,0,\phi^{\mathbb{V}})=0$ by the incoherence. We thus consider its *central derivative*

$$\mathrm{Eis}'(\tau,\phi) := \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E(\tau,s,\phi^{\mathbb{V}}).$$

To match the arithmetic degree, we need to modify $\mathrm{Eis}'(\tau,\phi)$ by central values of coherent Eisenstein series at places of bad reduction. For $v\in\mathcal{V}_{\mathrm{ram}}\cup\mathcal{V}_{\mathrm{asd}}$, let ${}^v\mathbb{V}$ be the *coherent* hermitian space over $\mathbb{A}_F^\infty$ nearby $\mathbb{V}$ at v, namely $({}^v\mathbb{V})_w\simeq\mathbb{V}_w$ exactly for all places $w\neq v$. For any vertex lattice $\Lambda_{t,v}\subset({}^v\mathbb{V})_v$ of type t, the Schwartz function $\phi^v\otimes\mathbf{1}_{(\Lambda_{t,v})^n}\otimes\phi_\infty\in\mathscr{S}(({}^v\mathbb{V})^n)$ gives a classical *coherent Eisenstein series* $E(\tau,s,\phi^v\otimes\mathbf{1}_{(\Lambda_{t,v})^n}\otimes\phi_\infty)$. Define the (normalized) *central values*

$${}^v\mathrm{Eis}_t(\tau,\phi) := \frac{\mathrm{vol}(K_{G,v})}{\mathrm{vol}(K_{\Lambda_{t,v}})}\cdot E(\tau,0,\phi^v\otimes\mathbf{1}_{(\Lambda_{t,v})^n}\otimes\phi_\infty).$$

Define the *modified central derivative*

$$\partial\mathrm{Eis}(\tau,\phi) := \mathrm{Eis}'(\tau,\phi)+(-1)^n\sum_{v\in\mathcal{V}_{\mathrm{ram}}\cup\mathcal{V}_{\mathrm{asd}}}{}^v\mathrm{Eis}(\tau,\phi).$$

Here ${}^v\mathrm{Eis}(\tau,\phi)$ is an explicit $\mathbb{Q}$-linear combination of ${}^v\mathrm{Eis}_t(\tau,\phi)$ for certain t's as defined in [HLSY22]. It has a decomposition into Fourier coefficients

$$\partial\mathrm{Eis}(\tau,\phi) = \sum_{T\in\mathrm{Herm}_n(F)}\partial\mathrm{Eis}_T(\tau,\phi).$$

Now we can state the arithmetic Siegel–Weil formula, which is an identity between the arithmetic degrees and the modified central derivative of the incoherent Eisenstein series.

Theorem 5.12 (Arithmetic Siegel–Weil formula: nonsingular terms)**.** *Assume that E/F is split at all places above 2. Let $\phi\in\mathscr{S}(\mathbb{V}^{\infty,n})^K$ be a factorizable Schwartz function such that $\phi_v=\mathbf{1}_{(\Lambda_v)^n}$ at all v nonsplit in E. Let $T\in\mathrm{Herm}_n(F)$ be nonsingular. Then*

$$\widehat{\deg}_T(\mathsf{y},\phi)\cdot q^T = (-1)^n\cdot\partial\mathrm{Eis}_T(\tau,\phi).$$

In particular, $\widehat{\deg}(\tau,\phi)$ is a nonholomorphic hermitian modular form on $\mathcal{H}_n$.

The proof of this theorem boils down to a local arithmetic Siegel–Weil formula computing $\mathrm{Int}_{T,v}(\phi)$ at each place v nonsplit in E:

(1) At $v\mid\infty$, this is the archimedean arithmetic Siegel–Weil formula proved by Liu [Liu11a] and Garcia–Sankaran [GS19] independently.
(2) At $v\nmid\infty$ inert in E such that Λ_v is self-dual, this is the content of the *Kudla–Rapoport conjecture* [KR14b, Conjecture 11.10], proved by Li–Zhang [LZ22b]. We refer to [Li21, §5] for an exposition. An analogous theorem is also proved for orthogonal Shimura varieties over $\mathbb{Q}$ at a place of good reduction [LZ22c].
(3) At $v\nmid\infty$ inert in E such that Λ_v is almost self-dual, this is a variant of the Kudla–Rapoport conjecture formulated and proved by Li–Zhang [LZ22b].

(4) At $v \nmid \infty$ ramified in E such that Λ_v is self-dual, this is the Kudla–Rapoport conjecture for Krämer models formulated by He–Shi–Yang [HSY21] and proved by He–Li–Shi–Yang [HLSY22].

Remark 5.13. The precise formulation of the singular part of the arithmetic Siegel–Weil [Kud04, Problem 6] remains an open problem. As a special case, the constant term of the arithmetic Siegel–Weil formula should roughly relate the arithmetic volume of $\mathcal{X}_K$ to logarithmic derivatives of Dirichlet L-functions. Such an explicit arithmetic volume formula was proved by Bruinier–Howard [BH21], though a precise comparison with the constant term of $\partial\mathrm{Eis}(\tau,\phi)$ is yet to be formulated and established.

Remark 5.14. In contrast to the classical and geometric Siegel–Weil formula, here the choice of K is fixed at all nonsplit places v in order to construct a regular integral model $\mathcal{X}_K$, which prevents us from formulating a full adelic version of the arithmetic Siegel–Weil formula. Nevertheless, the flexibility at split places (due to the regular integral models with Drinfeld level structure) allow us to choose ϕ to be nonsingular at split places to kill all singular terms on both sides. This extra flexibility is crucial for applications such as the arithmetic inner product formula, via making use of the multiplicity one result of the doubling method and bypassing the need of proving the singular part of the arithmetic Siegel–Weil formula.

5.4. **Arithmetic inner product formula.** In this subsection we come back to the equal rank situation consider in §3.7 and assume that $m = 2n$, so we can take χ to be the trivial character, and $s_0 = 0$, $\kappa = 1$ in the Siegel–Weil formula for the pair $(V, W^\square)$. Assume $\epsilon(\pi) = -1$ so $\mathbb{V} = \mathbb{V}_\pi$ is incoherent. In this case we may use the Beilinson–Bloch height pairing to define an arithmetic inner product between arithmetic theta lifts.

Let $\mathrm{CH}^n(X_K)^0 \subset \mathrm{CH}^n(X_K)$ be the subgroup of cohomologically trivial cycles. Since $\dim X_K = 2n-1$ we have a (conditional) symmetric bilinear height pairing

$$\langle\cdot,\cdot\rangle_{\mathrm{BB}} : \mathrm{CH}^n(X_K)^0 \times \mathrm{CH}^n(X_K)^0 \longrightarrow \mathbb{R},$$

constructed by Beilinson [Bei87] and Bloch [Blo84]. It generalizes the Néron–Tate height pairing when $n = 1$. When $n > 1$, the Beilinson–Bloch height pairing is only defined assuming certain conjectures on algebraic cycles on X_K (see [Bei87, Conjectures 2.2.1 and 2.2.3]). This important technical issue is addressed in [LL21, LL22] so that the left-hand side of the arithmetic inner product formula in Theorem 5.16 can be defined unconditionally, but we will intentionally ignore it for the purpose of this article. Then we naturally obtain an inner product on $\mathrm{CH}^n(X_K)^0_{\mathbb{C}}$ and define the (normalized) *arithmetic inner product*

$$\begin{aligned}\langle\cdot,\cdot\rangle_{X_K} : \mathrm{CH}^n(X_K)^0_{\mathbb{C}} \times \mathrm{CH}^n(X_K)^0_{\mathbb{C}} &\longrightarrow \mathbb{C}\\ (Z_1, Z_2) &\longmapsto \frac{\langle Z_1, Z_2\rangle_{\mathrm{BB}}}{\mathrm{vol}([X_K])/2},\end{aligned}$$

which also gives a well-defined inner product $\langle\cdot,\cdot\rangle_X$ on $\mathrm{CH}^n(X)^0_{\mathbb{C}}$.

Assumptions 5.15. We impose the following (mild) local assumptions on E/F and π.

(1) E/F is split at all 2-adic places and $F \neq \mathbb{Q}$. If $v \nmid \infty$ is ramified in E, then v is unramified over $\mathbb{Q}$. Assume that $E/\mathbb{Q}$ is Galois or contains an imaginary quadratic field (for simplicity).
(2) For $v \mid \infty$, π_v is the holomorphic discrete series with Harish-Chandra parameter $\{(m-1)/2, (m-3)/2, \ldots, (-m+3)/2, (-m+1)/2\}$.
(3) For $v \nmid \infty$, π_v is tempered.
(4) For $v \nmid \infty$ ramified in E, π_v is spherical with respect to the stabilizer of $\mathcal{O}_{E_v}^{2n}$.
(5) For $v \nmid \infty$ inert in E, π_v is unramified or almost unramified [Liu21b] with respect to the stabilizer of $\mathcal{O}_{F_v}^{2n}$. If π_v is almost unramified, then v is unramified over $\mathbb{Q}$.

Under Assumptions 5.15, the arithmetic theta lift $\Theta_\phi(\varphi)$ is in fact cohomologically trivial and thus $\Theta_{\mathbb{V}}(\pi) \subset \mathrm{CH}^n(X)^0_{\mathbb{C}}$ (see [LL21, Proposition 6.10]) and we can apply the arithmetic inner product.

Theorem 5.16 (Arithmetic inner product formula, proved by Li–Liu)**.** *Let π be a cuspidal automorphic representation of $G(\mathbb{A}_F)$ satisfying Assumptions 5.15. Assume that $\epsilon(\pi) = -1$. Assume that Kudla's arithmetic modularity Conjecture 5.1 holds. Then for any $\phi_i = \bigotimes_v \phi_{i,v} \in \mathscr{S}(\mathbb{V}^{\infty,n})$ and*

$\varphi_i = \bigotimes_v \varphi_{i,v} \in \pi \cap \mathscr{A}_n(G(\mathbb{A}_F))$ *with* $i = 1, 2$,

$$\langle \Theta_{\phi_1}(\varphi_1), \Theta_{\phi_2}(\varphi_2)\rangle_X = \frac{L'(1/2,\pi)}{b_{2n}(0)} \cdot \prod_v Z_v^{\natural}(0; \varphi_{1,v}, \varphi_{2,v}; \phi_{1,v}^{\mathbb{V}}, \phi_{2,v}^{\mathbb{V}}).$$

In particular,

$$L'(1/2,\pi) \neq 0 \implies \Theta_{\mathbb{V}}(\pi) \neq 0,$$

and the converse also holds if $\langle\cdot,\cdot\rangle_X$ *is nondegenerate.*

This is proved by Li–Liu in [LL21, LL22]. The conjectural arithmetic inner product formula was formulated (in the orthogonal case) by Kudla [Kud97b] using the Gillet–Soulé height and in more generality by Liu [Liu11a] using the Beilinson–Bloch height. This theorem verifies (under local assumptions) the conjecture formulated by Liu (who also completely proved the case $n = 1$ in [Liu11b]).

Remark 5.17. The formula can further be made explicit by computing the local doubling zeta integrals. For example, if

- π is unramified or almost unramified at all finite places,
- $\varphi \in \pi$ is a holomorphic newform such that $(\varphi, \overline{\varphi})_\pi = 1$, and
- ϕ_v is the characteristic function of self-dual or almost self-dual lattices at all finite places v,

then we have

$$\langle \Theta_\phi(\varphi), \Theta_\phi(\varphi)\rangle_X = (-1)^n \cdot \frac{L'(1/2,\pi)}{b_{2n}(0)} \cdot C_n^{[F:\mathbb{Q}]} \cdot \prod_{v \in S_\pi} \frac{q_v^{n-1}(q_v+1)}{(q_v^{2n-1}+1)(q_v^{2n}-1)},$$

where $C_n = 2^{-2n}\pi^{n^2}\frac{\Gamma(1)\cdots\Gamma(n)}{\Gamma(n+1)\cdots\Gamma(2n)}$ is an archimedean doubling zeta integral computed by Eischen–Liu [EL20] and $S_\pi = \{v \text{ inert} : \pi_v \text{ almost unramified}\}$.

Notice that the Grand Riemann Hypothesis predicts that $L'(1/2,\pi) \geqslant 0$, while Beilinson's Hodge index conjecture [Bei87, Conjecture 3.5] predicts that $(-1)^n\langle \Theta_\phi(\varphi), \Theta_\phi(\varphi)\rangle_X \geqslant 0$. It is a good reality check that these two (big) conjectures are compatible with the formula above.

The arithmetic inner product formula can be viewed as a higher dimensional generalization of the Gross–agier formula [GZ86] and has applications to the Beilinson–Bloch conjecture for higher dimensional Shimura varieties. Without assuming Kudla's modularity conjecture, we cannot define $\Theta_\phi(\varphi)$ but we may still obtain unconditional nonvanishing results [LL21, LL22] on Chow groups as predicted by the Beilinson–Bloch conjecture (using a proof by contradiction argument). See the argument below Theorem 5.27 for more details.

Theorem 5.18 (Application to the Beilinson–Bloch conjecture)**.** *Let* π *be a cuspidal automorphic representation of* $G(\mathbb{A}_F)$ *satisfying Assumptions 5.15. Let* $\mathrm{CH}^n(X)^0_{\mathfrak{m}_\pi}$ *be the localization of* $\mathrm{CH}^n(X)^0_{\mathbb{C}}$ *at the maximal ideal* $\mathfrak{m}_\pi$ *of the spherical Hecke algebra of* $H(\mathbb{A}_F^\infty)$ *(away from all ramification) associated to* π*. Then the implication*

$$\mathrm{ord}_{s=1/2} L(s,\pi) = 1 \implies \operatorname{rank} \mathrm{CH}^n(X)^0_{\mathfrak{m}_\pi} \geqslant 1$$

holds when the level subgroup $K \subset H(\mathbb{A}_F^\infty)$ *is sufficiently small.*

Remark 5.19. Disegni–Liu [DL22] proved a p-adic version of the arithmetic inner product formula, relating the central derivative of the cyclotomic p-adic L-function $L_p(\pi)$ to the p-adic height pairing [Nek93] of arithmetic theta lifts. As an application, they prove implications of the form

$$\begin{aligned} &\text{central order of vanishing of } L_p(\pi) \text{ is } 1\\ \implies &\text{Bloch–Kato Selmer group } H^1_f(E, \rho_\pi(n)) \text{ has rank } \geqslant 1, \end{aligned}$$

where ρ_π is the Galois representation associated to π. This verifies part of the p-adic Bloch–Kato conjecture.

Remark 5.20. Xue [Xue19] used the arithmetic inner product formula in the case $n = 1$ and the Gan–Gross–Prasad conjecture for $\mathrm{U}(2) \times \mathrm{U}(2)$ to prove endoscopic cases of the arithmetic Gan–Gross–Prasad conjecture for $\mathrm{U}(2) \times \mathrm{U}(3)$. In general, one also expects a similar relation between the arithmetic inner product formula for $\mathrm{U}(m)$ and endoscopic cases of arithmetic Gan–Gross–Prasad conjecture for $\mathrm{U}(m) \times \mathrm{U}(m+1)$.

Remark 5.21. Throughout we have assumed the skew-hermitian space W has even dimension $2n$. When the skew-hermitian space W has odd dimension, Liu [Liu21a] has defined mixed arithmetic theta lifting and used it to formulate a conjectural arithmetic inner product formula.

5.5. **Arithmetic theta lifting: when $\epsilon(1/2,\pi)=-1$.** Now we state another quantitative version of Theorem 5.16, based on the argument in Remark 5.17, by giving a height formula for arithmetic theta liftings.

5.5.1. *Setups and assumptions.* As before, we state $n=2r$ together with the CM extension E/F in $\mathbb{C}$ of the totally real field F over $\mathbb{Q}$. We equip $W_r := E^n$ with the skew-hermitian form given by the matrix $\begin{pmatrix} & 1_r \\ -1_r & \end{pmatrix}$. Put $G_r := \mathrm{U}(W_r)$, the unitary group of W_r, which is a quasi-split reductive group over F. For every non-archimedean place v of F, we denote by $K_{r,v} \subset G_r(F_v)$ the stabilizer of the lattice $\mathcal{O}_{E_v}^n$, which is a special maximal subgroup. Denote $\mathscr{A}_r(G_m(\mathbb{A}_F))$ the space of automorphic forms on $G_m(\mathbb{A}_F)$ of "parallel weight r" (for which we fix a choice of χ). We have the Siegel–Fourier expansion

$$\varphi \sim \sum_{T\in \mathrm{Herm}_m(F)} \varphi_T \cdot q^T,$$

where

$$\varphi_T := \int_{\mathrm{Herm}_m(F)\backslash \mathrm{Herm}_m(\mathbb{A}_F)} \varphi\left(\begin{pmatrix} 1_m & b \\ & 1_m \end{pmatrix}\right) \cdot \psi(\mathrm{tr}\, bT)^{-1} \mathrm{d}b.$$

Then Kudla's modularity Conjecture 5.1 can be precisely read as follows.

Conjecture 5.22 (Kudla's modularity hypothesis)**.** *For $\phi^\infty \in \mathscr{S}(\mathbb{V}^{\infty,m})^K = \mathscr{S}(V^m \otimes_F \mathbb{A}_F^\infty)^K$, there exists a (necessarily unique) holomorphic element*

$$\mathcal{Z}(\phi^\infty)_K \in \mathscr{A}_r(G_m(\mathbb{A}_F)) \otimes \mathrm{CH}^m(X_K)$$

such that for every $g^\infty \in G_m(\mathbb{A}_F^\infty)$, the Siegel–Fourier expansion of $g^\infty \mathcal{Z}(\phi^\infty)_K$ coincides with

$$\sum_{T\in \mathrm{Herm}_m(F)} Z_T(\omega_r^\infty(g^\infty)\phi^\infty)_K \cdot q^T.$$

This conjecture is formally known (that is, ignore the issue of convergence), and is only rigorously known when $m=1$.

We also consider a cuspidal automorphic representation $\pi = \bigotimes_v \pi_v$ of $G_r(\mathbb{A}_F)$ satisfying

- (R1) If $v \mid \infty$, then π_v is a holomorphic discrete series of weights $\{(n-1)/2, (n-3)/2, \dots, (3-n)/2, (1-n)/2\}$.
- (R2) If $v \nmid \infty$ and is nonsplit in E, then π_v is $K_{r,v}$-spherical, that is, $\pi_v^{K_{r,v}} \neq \{0\}$.
- (R3) If $v \nmid \infty$, then π_v is tempered, i.e., π_v is contained in a parabolic induction of a unitary discrete series representation.

We have studied in §3.4 the case where $r[F:\mathbb{Q}]$ is even (and hence the root number $\epsilon(1/2,\pi)=1$) via the classical theta lifting. To step in the case where the root number $\epsilon(1/2,\pi)=-1$, we assume now $r[F:\mathbb{Q}]$ is *odd* and apply the theory of arithmetic theta liftings. Under this assumption, there is a standard indefinite hermitian space V over E of rank n, unique up to isomorphism, that is totally positive definite and split at every non-archimedean place of F. Put $H := \mathrm{U}(V)$. Recall that V is *standard indefinite* if it has signature $(n-1,1)$ at the default embedding $F \hookrightarrow \mathbb{R}$ and signature $(n,0)$ at other real places. For a standard indefinite hermitian space V over E of rank n, we have a system of Shimura varieties $\{X_K\}$ indexed by neat open compact subgroups $K \subset H(\mathbb{A}_F^\infty)$, which are smooth, quasi-projective, of dimension $n-1$ over E, together with the complex uniformization:

$$X_K(\mathbb{C}) = H(F)\backslash \mathbb{P}(V_\mathbb{C})^- \times H(\mathbb{A}_F^\infty)/K,$$

where $\mathbb{P}(V_\mathbb{C})^- \subset \mathbb{P}(V_\mathbb{C})$ is the complex open domain of negative definite lines.

5.5.2. *Evidence toward Beilinson–Bloch conjecture.* Denote by Σ^{spl} the set of places of F that are split in E, and $\Sigma_\pi \subset \Sigma^{\mathrm{spl}}$ the (finite) subset at which π is ramified.

Since V is split at every non-archimedean place of F, we may fix an $\mathcal{O}_E$-lattice Λ of V satisfying

$$\{x \in V \mid \mathrm{Tr}_{E/\mathbb{Q}}(x,\Lambda)_V \in \mathbb{Z}\} = \Lambda.$$

Denote by $K_0 \subset H(\mathbb{A}_F^\infty)$ the stabilizer of $\widehat{\Lambda}$.

For every finite set Σ of non-archimedean places of F containing Σ_π, we have the *Satake homomorphism*

$$\chi_\pi^\Sigma := \mathbb{T}^\Sigma \otimes \mathbb{C} \longrightarrow \mathbb{C}$$

which is the eigen-character of the action of $\mathbb{T}^\Sigma$ on $\pi^{K_r^\Sigma}$. Put $\mathfrak{m}_\pi^\Sigma := \ker \chi_\pi^\Sigma$.

We propose the following assumption that limits our later theorems.

Assumption 5.23. The field E properly contains an imaginary quadratic subfield E_0 in which 2 splits and satisfying $(d_{E_0}, d_F) = 1$.

Theorem 5.24 ([LL21, LL22])**.** *Assume Assumption 5.23. If* $\mathrm{ord}_{s=1/2} L(s,\pi) = 1$*, then for every finite set* Σ *of non-archimedean places of* F *containing* Σ_π *and satisfying* $|\Sigma \cap \Sigma^{\mathrm{spl}}| \geqslant 2$*, we have*

$$\varinjlim_{K=K_\Sigma K_0^\Sigma} (\mathrm{CH}^r(X_K)^0_{\mathbb{C}})_{\mathfrak{m}_\pi^\Sigma} \neq \{0\}.$$

Remark 5.25. If we apply the Beilinson–Bloch conjecture for unitary Shimura varieties to our particular V and $\tilde{\pi}^\infty = \mathrm{Hom}_{G_r(\mathbb{A}_F^\infty)}(\omega_r^\infty \otimes \pi^\infty, 1)$, then

$$\dim \varinjlim_{K=K_\Sigma K_0^\Sigma} \mathrm{CH}^r(X_K)^0_{\mathbb{C}}[(\tilde{\pi}^\infty)^K] = 1.$$

Indeed, the theory of local theta lifting tells us that when

$$\mathrm{ord}_{s=1/2} L(s,\pi) = \mathrm{ord}_{s=1/2} L(s, \mathrm{BC}(\pi)) = 1,$$

$\Pi_{\tilde{\pi}^\infty}$ is exactly the isobaric factor of $\mathrm{BC}(\pi)$ such that $\mathrm{ord}_{s=1/2} L(s, \Pi_{\tilde{\pi}^\infty}) = 1$.

Thus, Theorem 5.24 aligns with and provide evidence toward the Beilinson–Bloch conjecture (cf. Conjecture 1.4) for higher dimensional unitary Shimura varieties. In fact, the theorem provided by Li–Liu is even stronger. First, we only need a weaker version of Assumption 5.23; in particular, E does not have to contain an imaginary quadratic field. Second, we allow a finite set Σ' of primes of F in E at which π can be almost unramified, which means "slightly ramified"; in response, we have $\epsilon(1/2, \pi) = (-1)^{r[F:\mathbb{Q}]+|\Sigma'|}$.

Example 5.26. We give examples of π satisfying (R1)–(R3) under Assumption 5.23 and the additional assumption that $F/\mathbb{Q}$ is *solvable*.

Let A be an elliptic curve over $\mathbb{Q}$ without complex multiplication such that every prime factor of its conductor d_A splits in E_0. By the modularity of A and the very recent breakthrough on the automorphy of symmetric powers of holomorphic modular forms obtained by Newton–Thorne [NT21], there exists a unique cuspidal automorphic representation $\Pi(\mathrm{Sym}^{n-1} A)$ of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ satisfying

- the base change of $\Pi(\mathrm{Sym}^{n-1} A)_\infty$ to $\mathrm{GL}_n(\mathbb{C})$ is the principal series of
$$(\arg^{1-n}, \arg^{3-n}, \dots, \arg^{n-3}, \arg^{n-1});$$
- for every prime $p \nmid d_A$, $\Pi(\mathrm{Sym}^{n-1} A)_p$ is unramified with the Satake polynomial
$$\prod_{j=0}^{n-1} (T - \alpha_{p,1}^j \alpha_{p,2}^{n-1-j}) \in \mathbb{Q}[T],$$
where $\alpha_{p,1}$ and $\alpha_{p,2}$ are the two roots of the polynomial $T^2 - a_p(A)T + p$.

Let $\Pi(\mathrm{Sym}^{n-1} A_E)$ be the (solvable) base change of $\Pi(\mathrm{Sym}^{n-1} A)$ to E, which is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. The representation $\Pi(\mathrm{Sym}^{n-1} A_E)$ satisfies

$$\Pi(\mathrm{Sym}^{n-1} A_E)^\vee \simeq \Pi(\mathrm{Sym}^{n-1} A_E) \simeq \Pi(\mathrm{Sym}^{n-1} A_E)^{\mathsf{c}}$$

and that $L(s, \Pi(\mathrm{Sym}^{n-1} A_E), \mathrm{As}^+)$ is regular at $s = 1$.

By the endoscopic classification for quasi-split unitary groups, there exists a cuspidal automorphic representation $\pi(\mathrm{Sym}^{n-1} A_E)$ of $G_r(\mathbb{A}_F)$ satisfying (R1)–(R3) and that for every $v \nmid \infty d_A$, the base change of $\pi(\mathrm{Sym}^{n-1} A_E)_v$ to $\mathrm{GL}_n(E_v)$ is isomorphic to $\Pi(\mathrm{Sym}^{n-1} A_E)_v$.

Now we are ready to state the arithmetic theta lifting. We first point out that the construction of arithmetic theta lifting relies on Kudla's modularity conjecture (see Conjecture 5.1 and Conjecture 5.22), which is unknown when $r > 1$. Thus, our height formula will be conditional (on the construction of the object). From now on, we assume Assumption 5.23 and the modularity hypothesis 5.22 for V with

$m = r$, that is, for every $\phi^\infty \in \mathscr{S}(\mathbb{V}^{\infty,m})^K$, there exists a (necessarily unique) holomorphic element $\mathcal{Z}(\phi^\infty)_K \in \mathscr{A}_r(G_r) \otimes \mathrm{CH}^r(X_K)$ such that for every $g^\infty \in G_r(\mathbb{A}_F^\infty)$, the Siegel–Fourier expansion of $g^\infty \mathcal{Z}(\phi^\infty)_K$ coincides with

$$\sum_{T \in \mathrm{Herm}_r(F)} Z_T(\omega_r^\infty(g^\infty)\phi^\infty)_K \cdot q^T.$$

Recalling Definition 5.3 and 5.4, for every $\phi^\infty \in \mathscr{S}(\mathbb{V}^{\infty,r})^K$ and $\varphi \in \pi$ that has parallel weight r (the lowest weight), we define the *arithmetic theta lift* to be

$$\Theta_{\phi^\infty}(\varphi)_K := \int_{G_r(F)\backslash G_r(\mathbb{A}_F)} \overline{\varphi(g)}\mathcal{Z}(\phi^\infty)_K(g)\mathrm{d}g \in \mathrm{CH}^r(X_K)_\mathbb{C}.$$

In fact, we show that $\Theta_{\phi^\infty}(\varphi)_K$ belongs to $\mathrm{CH}^r(X_K)_\mathbb{C}^{\langle \ell \rangle}$ for every sufficiently large prime ℓ.

Theorem 5.27 (Generalized arithmetic inner product formula, proved by Li–Liu)**.** *For $\phi_i^\infty = \bigotimes_v \phi_{i,v}^\infty \in \mathscr{S}(\mathbb{V}^{\infty,r})^K$ and $\varphi_i = \bigotimes_v \varphi_{i,v} \in \pi$ with $i = 1, 2$ that have parallel weight r, we have*

$$\mathrm{vol}(L) \cdot \langle \Theta_{\phi_1^\infty}(\varphi_1)_K, \Theta_{\phi_2^\infty}(\varphi_2)_K \rangle_{X_K} = \frac{L'(1/2,\pi)}{b_n(0)} \cdot C_r^{[F:\mathbb{Q}]} \cdot \prod_{v \nmid \infty} Z_v^\natural(\varphi_{1,v}^\infty, \varphi_{2,v}^\infty; \phi_{1,v}^\infty, \phi_{2,v}^\infty).$$

Here,

- $\mathrm{vol}(K)$ *is the normalized volume of K such that the degree of the Hodge line bundle on X_K equals* $2\mathrm{vol}(L)^{-1}$*;*
- *in the decomposition* $\mathrm{d}g = \prod_v \mathrm{d}g_v$ *of the Haar measure on $G_r(\mathbb{A}_F)$, we take* $\mathrm{d}g_v$ *to be the "standard" measure when $v \mid \infty$;*
- *in the decomposition $\langle \cdot, \cdot \rangle_\pi = \prod_v \langle \cdot, \cdot \rangle_{\pi_v}$ of the inner product, we assume $\langle \varphi_{1,v}, \varphi_{2,v} \rangle = 1$ when $v \mid \infty$;*
- $C_r \in (-1)^r \mathbb{R}_{>0}$ *is an explicit constant depending only on r (which was computed by Eischen–Liu* [EL20]*).*

Corollary 5.28. *The arithmetic inner product*

$$\langle \Theta_{\phi_1^\infty}(\varphi_1)_K, \Theta_{\phi_2^\infty}(\varphi_2)_K \rangle_{X_K}$$

belongs to $\mathbb{C}$ and is independent of the choice of ℓ.

Remark 5.29. If we take $\phi_1^\infty = \phi_2^\infty$ and $\varphi_1 = \varphi_2$ in the theorem, the signs on both sides are compatible under the Beilinson–Bloch conjecture and the generalized Riemann Hypothesis (see Remark 5.17).

5.5.3. *Height pairing between special cycles.* Fix a sufficiently large prime ℓ. We compare the height pairing between $Z_{T_1}(\phi_1^\infty)_K$ and $Z_{T_2}(\phi_2^\infty)_K$ and the derivative

$$\sum_{T = \begin{pmatrix} T_1 & * \\ * & T_2 \end{pmatrix} \in \mathrm{Herm}_{2r}(F)} \mathcal{E}'(0, \phi_1^\infty \otimes \overline{\phi}_2^\infty)_T,$$

where $\mathcal{E}'(0, \phi_1^\infty \otimes \overline{\phi}_2^\infty)$ denotes the Siegel–Eisenstein series $E(s, g, \Phi_\infty^0 \otimes (\phi_1^\infty \otimes \overline{\phi}_2^\infty))$ on $G_{2r}(\mathbb{A}_F)$, in which Φ_∞^0 is the standard hermitian Gaussian function on $(E^n \otimes_\mathbb{Q} \mathbb{R})^{2r}$.

Such study was originally proposed in the Kudla program. However, the height pairing we adopt is Beilinson's height (whose definition does not require a global integral model), instead of the Gillet–Soulé arithmetic intersection pairing in the Kudla program.

Even before the start of our comparison, there are three major difficulties:

(D1) For $i = 1, 2$, $Z_{T_i}(\phi_i^\infty)_K$ has no reason to belong to $\mathrm{CH}^r(X_K)_\mathbb{C}^{\langle \ell \rangle}$.

(D2) When $T_i \notin \mathrm{Herm}_r^\circ(F)$, $Z_{T_i}(\phi_i^\infty)_K$ is only well-defined as a Chow cycle, which is very obscure in terms of height pairing.

(D3) When $T_i \in \mathrm{Herm}_r^\circ(F)$ for $i = 1, 2$ so that $Z_{T_1}(\phi_1^\infty)_K$ and $Z_{T_2}(\phi_2^\infty)_K$ are both well-defined in $Z^r(X_K)_\mathbb{C}$, they may share support, which prohibits us to apply the decomposition formula directly.

By the multiplicity one property of local theta lifting, it suffices to consider K of the form $K = (\prod_{v \in \Sigma} K_v) \times K_0^\Sigma$ for some finite set $\Sigma_\pi \subset \Sigma \subset \Sigma^{\mathrm{spl}}$ of cardinality at least 2.

To solve (D1), we use Hecke operators. We show that there exists an element $t \in \mathbb{T} \backslash \mathfrak{m}$ (depending on K) such that $t^* Z_{T_i}(\phi_i^\infty)_K$ belongs to $Z^r(X_K)_\mathbb{C}^{\langle \ell \rangle}$ for $i = 1, 2$, every $T_i \in \mathrm{Herm}_r^\circ(F)^+$, and every

$\phi_i^\infty \in \mathscr{S}(\mathbb{V}^{\infty,r})^K$. Considering $t^*Z_{T_i}(\phi_i^\infty)_K$ will not lose information toward the arithmetic inner product formula since

$$\int \overline{\varphi}_i \cdot t^*\mathcal{Z}(\phi_i^\infty)_K = \int \overline{\varphi}_i \cdot \mathcal{Z}(t\phi_i^\infty)_K = \chi(t) \int \overline{\varphi}_i \cdot \mathcal{Z}(\phi_i^\infty)_K.$$

To solve (D2) and (D3), we use the same trick. For $v \nmid \infty$, we say that $(\phi_{1,v}^\infty, \phi_{2,v}^\infty)$ is a *regular pair* if the support of $\phi_{1,v}^\infty \otimes \overline{\phi}_{2,v}^\infty$ is contained in the subset $\{x \in V_v^{2r} \mid T(x) \in \mathrm{Herm}_{2r}^\circ(F_v)\}$.

Lemma 5.30. (1) *Suppose that $(\phi_{1,v}^\infty, \phi_{2,v}^\infty)$ is a regular pair for some $v \in \Sigma$, then*
- *$Z_{T_i}(\phi_i^\infty)_K = 0$ if $T_i \notin \mathrm{Herm}_r^\circ(F)^+$ for $i = 1, 2$;*
- *$Z_{T_1}(\phi_1^\infty)_K$ and $Z_{T_2}(\phi_2^\infty)_K$ have disjoint support for every pair $(T_1, T_2) \in \mathrm{Herm}_r^\circ(F)^+ \times \mathrm{Herm}_r^\circ(F)^+$.*

(2) *For every $v \in \Sigma$, there exists a regular pair $(\phi_{1,v}^\infty, \phi_{2,v}^\infty)$ such that the functional*

$$(\varphi_{1,v}, \varphi_{2,v}) \in \pi_v^\vee \times \pi_v \longmapsto Z_v^\natural(\varphi_{1,v}, \varphi_{2,v}; \phi_{1,v}^\infty, \phi_{2,v}^\infty)$$

is nontrivial.

5.5.4. *Derivative of Eisenstein series.* The above discussion provides us with a decomposition

$$\langle t^*Z_{T_1}(\phi_1^\infty)_K, t^*Z_{T_2}(\phi_2^\infty)_K\rangle_{X_K} = \sum_u \langle t^*Z_{T_1}(\phi_1^\infty)_K, t^*Z_{T_2}(\phi_2^\infty)_K\rangle_u$$

over all places u of E for every pair $(T_1, T_2) \in \mathrm{Herm}_r^\circ(F)^+ \times \mathrm{Herm}_r^\circ(F)^+$ and every pair $(\phi_1^\infty, \phi_2^\infty)$ that is regular at some place in Σ.

Now we provide a similar decomposition for Eisenstein series $\mathcal{E}'(0, t\phi_1^\infty \otimes \overline{t\phi_2}^\infty)$. We say that an element $T \in \mathrm{Herm}_{2r}(F)$ is *nearby to V* if it satisfies:

(1) T belongs to $\mathrm{Herm}_{2r}^\circ(F)$;
(2) if we denote by V_T the hermitian space over E defined by T, then there exists a unique place v_T of F such that V_T and V are isomorphic exactly away from $\{F \subset \mathbb{R}\} \triangle \{v_T\} = (\{F \subset \mathbb{R}\} \cup \{v_T\}) - (\{F \subset \mathbb{R}\} \cap \{v_T\})$.

Lemma 5.31. *Suppose that $(\phi_1^\infty, \phi_2^\infty)$ is regular at at least two places in Σ. Then*

$$\mathcal{E}'(0, t\phi_1^\infty \otimes \overline{t\phi_2}^\infty) = \sum_v \sum_{\substack{T \in \mathrm{Herm}_{2r}(F) \\ \text{nearby to } V, \\ v_T = v}} W_T'(0, \Phi_v) W_T(0, \Phi^\vee).$$

Here, $\Phi := \Phi_\infty^0 \otimes (t\phi_1^\infty \otimes \overline{t\phi_2}^\infty) \in \mathscr{S}(\mathbb{V}^{2r})$ and W_T denotes the T-th Siegel–Whittaker function.

5.5.5. *Comparison of local terms.* We will maintain the setups before.

Proposition 5.32. *For every place u of E of degree 1 over F, there exists an element $s_u \in \mathbb{T} \backslash \mathfrak{m}$, such that*

$$\langle s_u^* t^* Z_{T_1}(\phi_1^\infty)_K, s_u^* t^* Z_{T_2}(\phi_2^\infty)_K\rangle_u = 0$$

for every pair $(T_1, T_2) \in \mathrm{Herm}_r^\circ(F)^+ \times \mathrm{Herm}_r^\circ(F)^+$ and every pair $(\phi_1^\infty, \phi_2^\infty)$ that is regular at at least two places in Σ. Moreover, one can take $s_u = 1$ if u is not above Σ.

Proposition 5.33. *For every non-archimedean place u of E of degree 2 over F, we have*

$$\mathrm{vol}(L) \cdot \langle t^*Z_{T_1}(\phi_1^\infty)_K, t^*Z_{T_2}(\phi_2^\infty)_K\rangle_u = \sum_{\substack{T = \begin{pmatrix} T_1 & * \\ * & T_2 \end{pmatrix} \in \mathrm{Herm}_{2r}(F) \\ \text{nearby to } V}} W_T'(0, \Phi_v) W_T(0, \Phi^\vee)$$

for every pair $(T_1, T_2) \in \mathrm{Herm}_r^\circ(F)^+ \times \mathrm{Herm}_r^\circ(F)^+$ and every pair $(\phi_1^\infty, \phi_2^\infty)$ that is regular at at least two places in Σ.

We also have a comparison theorem in [LL22] for terms indexed by $u \mid \infty$, whose form is rather technical, which we omit here. We now explain the idea for Proposition 5.32 of comparison. Take a place u of E of degree 1 over F, with $v \in \Sigma^{\mathrm{spl}}$ the underlying place.

For every integer $m \geqslant 0$, let $K_{m,v} \subset K_{0,v}$ be the unique subgroup such that $K_{0,v}/K_{m,v} \simeq \mathrm{GL}_n(\mathcal{O}_F/v^m)$. Without loss of generality, we may assume $K_v = K_{m,v}$ for some $m \geqslant 0$.

- When $m = 0$, we denote by $\mathcal{X}_K$ the canonical (projective) smooth model of $X_{K,u}$ over $\mathcal{O}_{E_u}$.

- When $m > 0$, we denote by $\mathcal{X}_K$ the normalization of the above smooth model in $X_{K,u}$.

Proposition 5.34. *For every $m \geqslant 0$, we have*

$$H^{2r}(\mathcal{X}_K, \mathbb{Q}_\ell(r))_\mathfrak{m} = 0.$$

Let $\mathcal{Z}_{T_i}(\phi_i^\infty)_K$ be the Zariski closure of $Z_{T_i}(\phi_i^\infty)_K$ in $\mathcal{X}_K$. Then $t^*\mathcal{Z}_{T_i}(\phi_i^\infty)_K$ may not be ℓ-flat in general. However, the above proposition enables us to apply a further Hecke operator $s_u \in \mathbb{T}\backslash\mathfrak{m}$ such that $s_u^*t^*\mathcal{Z}_{T_i}(\phi_i^\infty)_K$ is ℓ-flat. Now under the condition that $(\phi_1^\infty, \phi_2^\infty)$ is regular at at least two places in Σ. It is easy to show that $s_u^*t^*\mathcal{Z}_{T_1}(\phi_1^\infty)_K$ and $s_u^*t^*\mathcal{Z}_{T_2}(\phi_2^\infty)_K$ have disjoint supports even in $\mathcal{X}_K$, which implies the proposition.

The second proposition of comparison will boil down to a similar comparison for the intersection number of special cycles on a certain Raporport–Zink space, known as the *Kudla–Rapoport (type) conjecture*, whose content and proof require another short course.

References

[AGHP18] F. Andreatta, E.Z. Goren, B. Howard, and K. Madapusi Pera. Faltings heights of abelian varieties with complex multiplication. *Annals of Mathematics (2)*, 187(2):391–531, 2018.

[BD05] M. Bertolini and H. Darmon. Iwasawa's main conjecture for elliptic curves over anticyclotomic $\mathbb{Z}_p$-extensions. *Ann. Math. (2)*, 162(1):1–64, 2005.

[Bei87] A.A. Beilinson. Height pairing between algebraic cycles. In *K-Theory, Arithmetic and Geometry (Moscow, 1984–1986)*, volume 1289 of *Lecture Notes in Mathematics*, pages 1–25. Springer, Berlin, 1987.

[BF10] J.H. Bruinier and J. Funke. On the injectivity of the Kudla–Millson lift and surjectivity of the Borcherds lift. In *Moonshine: The First Quarter Century and Beyond*, volume 372 of *London Mathematical Society Lecture Note Series*, pages 12–39. Cambridge University Press, Cambridge, 2010.

[BH21] J.H. Bruinier and B. Howard. Arithmetic volumes of unitary Shimura varieties. 2021. arXiv:2105.11274.

[BHK+20a] J.H. Bruinier, B. Howard, S. Kudla, M. Rapoport, and T. Yang. Modularity of generating series of divisors on unitary Shimura varieties. *Astérisque*, (421, Diviseurs arithmétiques sur les variétés orthogonales et unitaires de Shimura):7–125, 2020.

[BHK+20b] J.H. Bruinier, B. Howard, S. Kudla, M. Rapoport, and T. Yang. Modularity of generating series of divisors on unitary Shimura varieties II: Arithmetic applications. *Astérisque*, (421, Diviseurs arithmétiques sur les variétés orthogonales et unitaires de Shimura):127–186, 2020.

[Blo84] S. Bloch. Height pairings for algebraic cycles. In *Proceedings of the Luminy Conference on Algebraic K-theory (Luminy, 1983)*, volume 34, pages 119–145. 1984.

[BMM16] N. Bergeron, J. Millson, and C. Mœglin. The Hodge conjecture and arithmetic quotients of complex balls. *Acta Mathematica*, 216(1):1–125, 2016.

[Bor99] R.E. Borcherds. The Gross–Kohnen–Zagier theorem in higher dimensions. *Duke Mathematical Journal*, 97(2):219–233, 1999.

[Bru12] J.H. Bruinier. Regularized theta lifts for orthogonal groups over totally real fields. *Journal für die Reine und Angewandte Mathematik*, 672:177–222, 2012.

[BWR15] J.H. Bruinier and M. Westerholt-Raum. Kudla's modularity conjecture and formal Fourier–Jacobi series. *Forum Math. Pi*, 3:e7, 30, 2015.

[BZ22] J.H. Bruinier and S. Zemel. Special cycles on Toroidal compactifications of orthogonal Shimura varieties. *Mathematische Annalen*, 384:1–63, 2022.

[CS17] A. Caraiani and P. Scholze. On the generic part of the cohomology of compact unitary Shimura varieties. *Ann. of Math.*, 186(3):649–766, 2017.

[DL22] D. Disegni and Y. Liu. A p-adic arithmetic inner product formula. 2022. arXiv:2204.09239.

[Dun22] N.C. Dung. Geometric pullback formula for unitary Shimura varieties, 2022.

[EL20] E. Eischen and Z. Liu. Archimedean zeta integrals for unitary groups. 2020. arXiv:2006.04302.

[ES18] S. Ehlen and S. Sankaran. On two arithmetic theta lifts. *Compositio Mathematica*, 154(10):2090–2149, 2018.

[FM06] J. Funke and J. Millson. Cycles with local coefficients for orthogonal groups and vector-valued Siegel modular forms. *American Journal of Mathematics*, 128(4):899–948, 2006.

[FM14] J. Funke and J. Millson. The geometric theta correspondence for Hilbert modular surfaces. *Duke Mathematical Journal*, 163(1):65–116, 2014.

[GI14] W.T. Gan and A. Ichino. Formal degrees and local theta correspondence. *Inventiones Mathematicae*, 195(3):509–672, 2014.

[GI16] W.T. Gan and A. Ichino. The Gross–Prasad conjecture and local theta correspondence. *Inventiones Mathematicae*, 206(3):705–799, 2016.

[GKK07] J.I. Burgos Gil, J. Kramer, and U. Kühn. Cohomological arithmetic Chow rings. *Journal of the Institute of Mathematics of Jussieu*, 6(1):1–172, 2007.

[GKZ87] B.H. Gross, W. Kohnen, and D.B. Zagier. Heegner points and derivatives of L-series, II. *Math. Ann.*, 278(1-4):497–562, 1987.

[GQT14] W.T. Gan, Y. Qiu, and S. Takeda. The regularized Siegel–Weil formula (the second term identity) and the rallis inner product formula. *Invent. Math.*, 198(3):739–831, 2014.

[Gro20] B.H. Gross. Incoherent definite spaces and Shimura varieties. 2020. arXiv:2005.05188.

[GS91] H. Gillet and C. Soulé. Arithmetic intersection theory. *Institut des Hautes Études Scientifiques. Publications Mathématiques*, (72):93–174, 1991. Published in 1990.

[GS19] L.E. Garcia and S. Sankaran. Green forms and the arithmetic Siegel–Weil formula. *Invent. Math.*, 215(3):863–975, 2019.

[GZ86] B.H. Gross and D. Zagier. Heegner points and derivatives of L-series. *Invent. Math.*, 84(2):225–320, 1986.

[HKS96a] M. Harris, S. Kudla, and W.J. Sweet. Theta dichotomy for unitary groups. *Journal of the American Mathematical Society*, 9(4):941–1004, 1996.

[HKS96b] M. Harris, S. Kudla, and W.J. Sweet. Theta dichotomy for unitary groups. *Journal of the American Mathematical Society*, 9(4):941–1004, 1996.

[HLSY22] Q. He, C. Li, Y. Shi, and T. Yang. A proof of the Kudla–Rapoport conjecture for Krämer models. 2022. Preprint, to appear in Inventiones Mathematicae.

[HP20] B. Howard and K. Madapusi Pera. Arithmetic of Borcherds products. *Astérisque*, (421, Diviseurs arithmétiques sur les variétés orthogonales et unitaires de Shimura):187–297, 2020.

[HSY21] Q. He, Y. Shi, and T. Yang. Kudla–Rapoport conjecture at ramified primes. 2021. URL.

[HZ76] Friedrich Hirzebruch and Don Zagier. Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus. *Inventiones Mathematicae*, 36:57–113, 1976.

[Ich07] A. Ichino. On the Siegel–Weil formula for unitary groups. *Math. Z.*, 255(4):721–729, 2007.

[KM86] S. Kudla and J. Millson. The theta correspondence and harmonic forms, I. *Mathematische Annalen*, 274(3):353–378, 1986.

[KM87] S. Kudla and J. Millson. The theta correspondence and harmonic forms, II. *Mathematische Annalen*, 277(2):267–314, 1987.

[KM90] S. Kudla and J. Millson. Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables. *Inst. Hautes Études Sci. Publ. Math.*, 71:121–172, 1990.

[KMSW14] T. Kaletha, A. Minguez, S.W. Shin, and P.-J. White. Endoscopic classification of representations: Inner forms of unitary groups. 2014. arXiv:1409.3731.

[KR94] S. Kudla and S. Rallis. A regularized Siegel–Weil formula: the first term identity. *Ann. of Math. (2)*, 140(1):1–80, 1994.

[KR00] S. Kudla and M. Rapoport. Height pairings on Shimura curves and p-adic uniformization. *Inventiones Mathematicae*, 142(1):153–223, 2000.

[KR14a] S. Kudla and M. Rapoport. Special cycles on unitary Shimura varieties II: Global theory. *J. Reine Angew. Math.*, 697:91–157, 2014.

[KR14b] S. Kudla and M. Rapoport. Special cycles on unitary Shimura varieties II: Global theory. *Journal für die Reine und Angewandte Mathematik*, 697:91–157, 2014.

[KRY06] S. Kudla, M. Rapoport, and T. Yang. Modular forms and special cycles on Shimura curves. 161:x+373, 2006.

[KSZ21] M. Kisin, S.W. Shin, and Y. Zhu. The stable trace formula for Shimura varieties of abelian type. 2021. arXiv:2110.05381.

[Kud97a] S. Kudla. Algebraic cycles on Shimura varieties of orthogonal type. *Duke Math. J.*, 86(1):39–78, 1997.

[Kud97b] S. Kudla. Central derivatives of Eisenstein series and height pairings. *Annals of Mathematics (2)*, 146(3):545–646, 1997.

[Kud03] S. Kudla. Integrals of Borcherds forms. *Compositio Mathematica*, 137(3):293–349, 2003.

[Kud04] S. Kudla. Special cycles and derivatives of Eisenstein series. In *Heegner points and Rankin L-series*, volume 49 of *Math. Sci. Res. Inst. Publ.*, pages 243–270. Cambridge Univ. Press, Cambridge, 2004.

[Li92] J.-S. Li. Nonvanishing theorems for the cohomology of certain arithmetic quotients. *Journal für die Reine und Angewandte Mathematik*, 428:177–217, 1992.

[Li21] C. Li. From sum of two squares to arithmetic Siegel–Weil formulas. 2021. arXiv:2110.07457.

[Liu11a] Y. Liu. Arithmetic theta lifting and L-derivatives for unitary groups, I. *Algebra Number Theory*, 5(7):849–921, 2011.

[Liu11b] Y. Liu. Arithmetic theta lifting and L-derivatives for unitary groups, II. *Algebra Number Theory*, 5(7):923–1000, 2011.

[Liu16] Y. Liu. Hirzebruch–Zagier cycles and twisted triple product Selmer groups. *Invent. Math.*, 205(3):693–780, 2016.

[Liu19] Y. Liu. Bounding cubic–triple product Selmer groups of elliptic curves. *J. Eur. Math. Soc. (JEMS)*, 21(5):1411–1508, 2019.

[Liu21a] Y. Liu. Mixed arithmetic theta lifting for unitary groups. In W. Müller, S.W. Shin, and N. Templier, editors, *Relative Trace Formulas*, pages 329–350, Cham, 2021. Springer International Publishing.

[Liu21b] Y. Liu. Theta correspondence for almost unramified representations of unitary groups. 2021. arXiv:2105.14223.

[LL21] C. Li and Y. Liu. Chow groups and l-derivatives of automorphic motives for unitary groups. *Annals of Mathematics*, 194(3):817–901, 2021.

[LL22] C. Li and Y. Liu. Chow groups and L-derivatives of automorphic motives for unitary groups, II. *Forum of Mathematics, Pi*, 10:Paper No. e5, 2022.

[LR05] E.M. Lapid and S. Rallis. On the local factors of representations of classical groups. In James W. Cogdell, James B. Keating, and Yannick Rivaud, editors, *Automorphic Representations, L-functions and Applications: Progress and Prospects*, volume 11 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 309–359. de Gruyter, Berlin, 2005.

[LZ22a] C. Li and W. Zhang. Kudla–rapoport cycles and derivatives of local densities. *J. Amer. Math. Soc.*, 35(3):705–797, 2022.

[LZ22b] C. Li and W. Zhang. Kudla–Rapoport cycles and derivatives of local densities. *Journal of the American Mathematical Society*, 35(3):705–797, 2022.

[LZ22c] C. Li and W. Zhang. On the arithmetic Siegel–Weil formula for GSpin Shimura varieties. *Inventiones Mathematicae*, 228(3):1353–1460, 2022.

[Man08] Elena Mantovan. A compactification of Igusa varieties. *Math. Ann.*, 340(2):265–292, 2008.

[Mok15] C. P. Mok. Endoscopic classification of representations of quasi-split unitary groups. *Memoirs of the American Mathematical Society*, 235(1108):vi+248, 2015.

[MZ] A. Mihatsch and W. Zhang. On the Arithmetic Fundamental Lemma conjecture over a general p-adic field. *to appear JEMS*.

[Nek93] J. Nekovář. On p-adic height pairings. In *Séminaire de Théorie des Nombres, Paris, 1990–91*, volume 108 of *Progress in Mathematics*, pages 127–202. Birkhäuser Boston, Boston, MA, 1993.

[NT21] J. Newton and J.A. Thorne. Symmetric power functoriality for holomorphic modular forms, II. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 134:117–152, 2021.

[PSR86] I. Piatetski-Shapiro and S. Rallis. ε factor of representations of classical groups. *Proc. Nat. Acad. Sci. U.S.A.*, 83(13):4589–4593, 1986.

[PSR87] I. Piatetski-Shapiro and S. Rallis. L-functions for the classical groups. In *Automorphic Forms, Representations, and L-functions*, pages 1–52. Springer Berlin Heidelberg, Berlin, Heidelberg, 1987.

[Qiu22] C. Qiu. Arithmetic modularity of special divisors and arithmetic mixed Siegel–Weil formula (with an appendix by y. xu). 2022. arXiv:2204.13457.

[Ral84] S. Rallis. Injectivity properties of liftings associated to Weil representations. *Compositio Mathematica*, 52(2):139–169, 1984.

[Ral87] S. Rallis. *L-functions and the Oscillator Representation*, volume 1245 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.

[RSZ20] M. Rapoport, B. Smithling, and W. Zhang. Arithmetic diagonal cycles on unitary Shimura varieties. *Compositio Mathematica*, 156(9):1745–1824, 2020.

[Sou92] Christophe Soulé. *Lectures on Arakelov Geometry*, volume 33 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1992. With the collaboration of D. Abramovich, J.-F. Burnol and J. Kramer.

[SSTT19] A.N. Shankar, A. Shankar, Y. Tang, and S. Tayou. Exceptional jumps of Picard ranks of reductions of K3 surfaces over number fields. 2019. arXiv:1909.07473.

[Wei65] A. Weil. Sur la formule de siegel dans la théorie des groupes classiques. *Acta Math.*, 113:1–87, 1965.

[Xia21] J. Xia. Some cases of Kudla's modularity conjecture for unitary Shimura varieties. 2021. arXiv:2101.06304.

[Xue19] H. Xue. Arithmetic theta lifts and the arithmetic Gan–Gross–Prasad conjecture for unitary groups. *Duke Mathematical Journal*, 168(1):127–185, 2019.

[Yam11] S. Yamana. On the Siegel–Weil formula: the case of singular forms. *Compos. Math.*, 147(4):1003–1021, 2011.

[Yam14] S. Yamana. L-functions and theta correspondence for classical groups. *Invent. Math.*, 196(3):651–732, 2014.

[YZZ13] X. Yuan, S.-W. Zhang, and W. Zhang. *The Gross–Zagier formula on Shimura curves*, volume 184 of *Annals of Mathematics Studies*. Princeton University Press, 2013.

[Zha] Z. Zhang. Maximal parahoric arithmetic transfers, resolutions and modularity. arXiv:2112.11994.

[Zha01a] S. Zhang. Gross–Zagier formula for GL(2). *Asian J. Math.*, 5(2):183–290, 2001.

[Zha01b] S. Zhang. Heights of Heegner points on Shimura curves. *Ann. of Math. (2)*, 153(1):27–147, 2001.

[Zha09] W. Zhang. Modularity of generating functions of special cycles on Shimura varieties, 2009.

[Zha10] S. Zhang. Linear forms, algebraic cycles, and derivatives of L-series. 2010. preprint.

[Zha21] W. Zhang. Weil representation and Arithmetic Fundamental Lemma. *Ann. of Math. (2)*, 193(3):863–978, 2021.

School of Mathematical Sciences, Peking University, 100871, Beijing, China
Email address: daiwenhan@pku.edu.cn