Sheaf Cohomology

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§1 Having Enough Injectives

<u>Lemma</u> AbGrp has enough injs. (by elementary arguments).

Rmk Grothendieck's criterion: so does Mod_R.

<u>Categories of sheaves have enough injs</u>:

X loc ringed, $\mathcal{C}$ ab cat., $\mathcal{D} = \{\mathcal{F} \in Sh(X) : \mathcal{F} : X \to \mathcal{C}\} = Sh_{\mathcal{C}}(X)$.

$\mathcal{D}$: ab cat, again.

(Need $\mathcal{D}$ to have enough injs)

Method: constructing a large class of inj objs by using <u>skyscraper</u> sheaves

$$i_x(G) = i_* G, \quad i : \{x\} \hookrightarrow X.$$

$$\text{section: } (i_x(G))(U) = \begin{cases} G, & x \in U \\ 0, & \text{otherwise} \end{cases}$$

$\rightsquigarrow$ $\mathrm{Hom}_{Sh_{\mathcal{C}}(X)}(\mathcal{F}, i_x G) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}_x, G)$ by adjointness, where $\mathcal{F}_x = i_x^{-1}\mathcal{F}$.

$\rightsquigarrow$ mono $\mathcal{F}_x \to G_x$, $\mathcal{F} \hookrightarrow \prod_{x \in X} i_x(G_x)$.

$\forall U \subseteq X$ open,

$$\mathcal{F}(U) \longrightarrow \Big(\prod_{x \in X} i_x(G_x)\Big)(U) = \prod_{x \in U} G_x = \prod_{x \in U} \mathcal{F}_x$$

$$s \longmapsto (s_x)_{x \in U} \quad \text{germs.}$$

<u>Prop</u> $Sh_{\mathrm{Mod}_{\mathcal{O}_X}}(X)$ has enough injs (X ringed space).

Caution: X loc. ringed $\not\Rightarrow$ $\mathrm{Qcoh}(\mathrm{Mod}_{\mathcal{O}_X})$ has enough injs
but true for X affine b/c Mod_R does.

§2 Grothendieck's Criterion

Thm $\mathcal{C}$ ab cat s.t.

(a) $\mathcal{C}$ admits arbitrary (small) direct sums

(b) $X \to Y$, I totally ordered, $Y_i \hookrightarrow Y$ increasing family (i.e. $Y_i \to Y_j \to Y$, $\forall i \le j$ in I). Then
$$(\sum Y_i) \cap X = (\sum (Y_i \cap X)).$$
in other words, forming direct lim of Y_i commutes with taking the fibred product $(\cdot) \times_Y X$.

weak condition

(c) $\exists\, U \in \mathcal{C}$ s.t. $X \to Y$ not epi
$$\Rightarrow \mathrm{Hom}(U, X) = \mathrm{Hom}(U, Y) \text{ not epi}$$
i.e. $\exists\, U \to Y$ not through X.

Also, the class of isom classes of monos in U is small.
(C admits a forgetful additive fun to $\underline{\mathrm{Ab}}$ implies this)

Then $\mathcal{C}$ has enough injs.

Mod_R satisfies (a)(b) easily. To check (c):
$$U = \bigoplus_{V \in \underline{X}} j_{V*}(\mathbb{Z}_V), \quad \mathbb{Z}_V = \text{const sheaf over } V \text{ w/ values in } \mathbb{Z}.$$

Upshot
$$\begin{aligned}\mathrm{Hom}(\bigoplus_V j_{V*}(\mathbb{Z}_V), \mathcal{G}) &= \bigoplus_V \mathrm{Hom}(j_{V*}(\mathbb{Z}_V), \mathcal{G}) \\ &= \bigoplus_V \mathrm{Hom}(\mathbb{Z}_V, \mathcal{G}|_V) \quad \text{adj.} \\ &= \bigoplus_V \Gamma(V, \mathcal{G}) \quad \text{by def.}\end{aligned}$$

§3 Sheaf Cohomology for Top & Ringed Spaces

Define $H^i : Sh_{\mathcal{C}}(X) \longrightarrow \mathcal{C}$ as

$$R^i\Gamma(X,-) = H^i(X,-).$$

effaceable δ-functor as well.

In particular: $H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F}) = \mathcal{F}(X)$.

Let $(X, \mathcal{O}_X)$ ringed space, can define $H^i : Sh(Mod_{\mathcal{O}_X}) \to \mathcal{C}$.

Remember we can compute H^i by resolutions.

Say $\mathcal{F} \in Sh(X)$ is flasque (flabby) if

$\forall\, U \subseteq V \subseteq X$ opens. $\mathcal{F}(V) \twoheadrightarrow \mathcal{F}(U)$

e.g. X irred. $\Rightarrow$ $\underline{C}_X$ const sheaf are flasque.

<u>Reminder</u> $\forall C \in \mathcal{C}$, $\underline{C}_X = (U \mapsto C)^+$.

However: $X = \mathbb{R}$ w/ usual top

<u>Lemma</u> $(X, \mathcal{O}_X)$ ringed, any inj $\mathcal{O}_X$-mod is flasque.

In particular, any inj sheaf of ab grps on X is flasque.

$\mathcal{O}_X = \underline{\mathbb{Z}}_X$

(c.f. Hartshorne Lem III.2.4)

Proof. $\mathcal{I}$ inj. $\mathcal{O}_X$-mod. $\forall\, U \in \underline{X}$, $\mathcal{O}_U$ = ext by 0 of $\mathcal{O}_X|_U$ to X.

$$\text{i.e. } \mathcal{O}_U = \left(V \longmapsto \begin{cases} \mathcal{O}_X(V), & V \subseteq U \\ 0, & \text{otherwise} \end{cases} \right)^+$$

$\rightsquigarrow \mathcal{O}_{U,x} = \mathcal{O}_{X,x}$, $x \in U$; $\mathcal{O}_{U,x} = 0$, $x \notin U$.

(differs from $i_*\mathcal{O}_U$, $i : U \to X$, $i_*\mathcal{O}_U(W) \neq 0$ when $W \cap V \neq \phi$.)

Take $V \subseteq U \rightsquigarrow \mathcal{O}_V \xrightarrow{\text{mono}} \mathcal{O}_U$ / $Sh(Mod_{\mathcal{O}_X})$.

$$\mathcal{I} \text{ inj.} \Rightarrow \begin{array}{ccc} \mathrm{Hom}(\mathbb{Q}_U, \mathcal{I}) & \twoheadrightarrow & \mathrm{Hom}(\mathbb{O}_V, \mathcal{I}) \\ \| & & \| \\ \mathcal{I}(U) & \longrightarrow & \mathcal{I}(V) \end{array} \Rightarrow \mathcal{I} \text{ flasque.} \quad \square$$

Prop $\mathcal{F}$ flasque $: X \to \mathrm{AbGrp}. \Rightarrow H^i(X, \mathcal{F}) = 0, \ \forall \boxed{i \geq 0}$.

Proof. AbGrp has enough injs

$$\Rightarrow \quad 0 \to \mathcal{F} \to \underset{\substack{\uparrow \\ \text{inj.}}}{\mathcal{I}} \to \mathcal{G} = \underset{\substack{\uparrow \\ \text{coker.}}}{\mathcal{I}/\mathcal{F}} \to 0$$

$\mathcal{F}$ flasque $\Rightarrow 0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{I}) \to H^0(X, \mathcal{G}) \to 0$ exact

$\Rightarrow H^1(X, \mathcal{F}) = 0$

$\mathcal{I}$ acyclic (since H^i is effaceable)

$\Rightarrow H^i(X, \mathcal{I}) = 0 \ (i > 0)$

$\Big\} \Rightarrow H^i(X, \mathcal{F}) \cong H^{i-1}(X, \mathcal{G}) \ (i > 1)$

Also, $\mathcal{G}$ is flasque ($\mathcal{F}, \mathcal{I}$ flasque)

$\Rightarrow H^i(X, \mathcal{F}) \cong H^{i-1}(X, \mathcal{G}) = 0$ by induction. $\square$

Watch: <u>Dimension Shifting</u> Recipe!

§4 Sheaf Cohomology & Top Cohomology

Pretend that $H^i_{sing}(X, \mathbb{Z}) \cong H^i(X, \underline{\mathbb{Z}}_X)$ (and skip this sec) if you're not familiar with H^*_{sing}.

Thm X loc. contractible top space.

$\Rightarrow H^i(X, \mathbb{Z}_X) \cong H^i_{sing}(X, \mathbb{Z})$ canonically.

(loc. contractible = each pt has a basis of contractible nbhds).

§5 Čech Cohomology

X top space. $\mathcal{U} = \{U_i\}$ open cover of X $(i \in I)$

(i.e. $x \in X$ appears in only finitely many U_i's).

$J \subseteq I$ finite $\rightsquigarrow$ $U_J = \bigcap_{i \in I} U_i$, $U_\phi = X$.

$\mathcal{F} \in Sh_{Ab}(X)$. Define ~~Čech complex~~ of $\mathcal{F}$ w.r.t. $\{U_i\}$:

$\forall j \geq 0$, $\check{C}^j(\mathcal{U}, \mathcal{F}) = \prod_{J} \Gamma(U_J, \mathcal{F})$

over all $\underline{(j+1)}$-element subset of I

$d^j : \check{C}^j(\mathcal{U}, \mathcal{F}) \to \check{C}^{j+1}(\mathcal{U}, \mathcal{F})$

$\alpha = (\alpha_J) \longmapsto (d^j(\alpha)_J)$

where $d^j(\alpha)_J = \sum_{k=0}^{j+1} (-1)^k \mathrm{Res}_{U_{J - \{i_k\}}, J} (\alpha_{J - \{i_k\}})$

$J = \{i_0 \leq \cdots \leq i_{j+1}\}$ ← $j+2$ elements.

E.g. $\mathcal{U} = \{U_1, U_2\}$:

$$0 \to \Gamma(U_1, \mathcal{F}) \oplus \Gamma(U_2, \mathcal{F}) \xrightarrow{d^0} \Gamma(U_1 \cap U_2, \mathcal{F}) \to 0.$$

__Lemma__ $0 \to \mathcal{F} \to \check{C}^0(\mathcal{U}, \mathcal{F}) \to \check{C}^1(\mathcal{U}, \mathcal{F}) \to \cdots$ exact.

(c.f. Hartshorne Lemma III.4.2)

$\rightsquigarrow$ Write $\check{H}^i(\mathcal{U}, \mathcal{F}) = h^i(\check{C}^\cdot(\mathcal{U}, \mathcal{F}))$.

__NOT__ a δ-functor if $\mathcal{U}$ is fixed.

(See Hartshorne Caution 4.0.2)

- If $\mathcal{U} = \{X\}$, then $\Gamma(X, -)$ is not exact.

__Lemma__ $\mathcal{F}$ flasque $\Rightarrow$ $\check{H}^i(\mathcal{U}, \mathcal{F}) = 0$, $\forall i > 0$

Refinement limit: $\check{H}^i(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^i(\mathcal{U}, \mathcal{F})$.

__Thm__ X paracpt (i.e. Hausdorff + every $\mathcal{U}$ refines to a loc. finite subcovering).

$\Rightarrow \check{H}^{\cdot}(X, \mathcal{F})$ effaceable $\Rightarrow$ universal-δ

$\Rightarrow \check{H}^{\cdot}(X, \mathcal{F}) \cong H^{i}(X, \mathcal{F})$

$\rightsquigarrow$ $\forall \mathcal{U}$ particular covering, $\check{H}^{\cdot}(\mathcal{U}, \mathcal{F}) \to H^{\cdot}(X, \mathcal{F})$.

__Thm__ (Leray) $\mathcal{U}$ good for $\mathcal{F}$ (i.e. $\forall J \subseteq I$, $\mathcal{F}|_{U_J}$ acyclic)

$\Rightarrow \check{H}^{i}(\mathcal{U}, \mathcal{F}) \cong H^{i}(X, \mathcal{F})$, $\forall i \geq 0$

namely: $\check{C}^{i}(\mathcal{U}, \mathcal{F})$ computes the sheaf cohom.

__Rmk__ Analogues: (1) contractible open $\longleftrightarrow$ affine

(2) good covers $\longleftrightarrow$ quasi-coherent sheaves