TUTORIAL SESSION FOR HONORS LINEAR ALGEBRA (FALL 2024)

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This document contains notes for the tutorial sessions by W.D. attached to *Honors Linear Algebra* offered by Qiuzhen College, Tsinghua University, during the Fall 2024 semester. The tutorial lectures are held once a week (each lasting for two hours) for a total of seven weeks, covering the content before the midterm exam.

Contents

Tutorial Lecture 1

Problem 1.1. Consider the following linear maps on $\mathbb{R}^2$.

(1) Let $R_\theta(x) = R_\theta x$ with

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Prove the following assertions.

(a) R_θ is the counterclockwise rotation by angle θ about the origin.

(b) Determine when there exist $\lambda \in \mathbb{R}$ and $x \neq 0$ such that $R_\theta x = \lambda x$.

(c) Compute R_θ^n for $n \in \mathbb{N}$.

(2) Let $H_\theta(x) = H_\theta x$ with

$$H_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}, \qquad v = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad w = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}.$$

Prove the following assertions.

(a) $v^\top w = 0$, $v^\top v = w^\top w = 1$, $H_\theta v = v$, and $H_\theta w = -w$. (Interpret the geometry of H_θ.)

(b) $H_\theta^2 = I_2$ and $H_\theta = I_2 - 2ww^\top$.

(c) For any angle φ,

$$R_{-\varphi} H_\theta R_\varphi = H_{\theta-\varphi}, \qquad H_\varphi R_\theta H_\varphi = R_{-\theta},$$

and explain their geometric meaning.

(3) Let $S(x) = Sx$ with $S = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$. Show that $Sx = \lambda x$ has a nonzero solution if and only if $\lambda = 1$, list all nonzero solutions, and compute S^n for $n \in \mathbb{N}$.

Solution. (1) For part (a), the action of R_θ on the standard basis is $R_\theta e_1 = (\cos\theta, \sin\theta)^\top$ and $R_\theta e_2 = (-\sin\theta, \cos\theta)^\top$, which is the counterclockwise rotation by θ. For part (b), real eigenpairs satisfy

$$\det(R_\theta - \lambda I_2) = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = (\cos\theta - \lambda)^2 + \sin^2\theta = 0.$$

Last updated on November 10, 2024.

Hence $\lambda^2 - 2\lambda\cos\theta + 1 = 0$. Over $\mathbb{R}$ this has a real root if and only if $\sin\theta = 0$, i.e. $\cos\theta = \pm 1$, so $\theta = k\pi$ ($k \in \mathbb{Z}$). Then $R_\theta = \pm I_2$ and every nonzero vector is an eigenvector with $\lambda = \pm 1$ respectively. For part (c), using the angle-addition formulas or induction by $R_\alpha R_\beta = R_{\alpha+\beta}$,

$$R_\theta^n = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}.$$

(2) For part (a), we have $v^\top w = \cos\theta\sin\theta + \sin\theta(-\cos\theta) = 0$ and $v^\top v = w^\top w = 1$. Moreover,

$$H_\theta v = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = v,$$

$$H_\theta w = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix} = -\begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix} = -w.$$

Thus H_θ fixes the line spanned by v and flips the line spanned by w; geometrically it is the reflection about the line through angle θ. For part (b), from the reflection property, $H_\theta^2 = I_2$. Direct multiplication gives

$$I_2 - 2ww^\top = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = H_\theta.$$

For part (c), conjugating a reflection by a rotation rotates its mirror: $R_{-\varphi} H_\theta R_\varphi = H_{\theta-\varphi}$. Conjugating a rotation by any reflection reverses the angle: $H_\varphi R_\theta H_\varphi = R_{-\theta}$. Both follow either from matrix identities or from the geometric descriptions.

(3) The eigen-equation $Sx = \lambda x$ is equivalent to $(S - \lambda I_2)x = 0$ with

$$S - \lambda I_2 = \begin{bmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 1-\lambda \\ 0 & -(1-\lambda)^2 \end{bmatrix}.$$

A nonzero solution exists if and only if $(1-\lambda)^2 = 0$, i.e. $\lambda = 1$. For $\lambda = 1$, the eigenvectors are all $x = (t, 0)^\top$ with $t \neq 0$. To compute powers, note $S = I_2 + J_2$ with $J_2 = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ and $J_2^2 = 0$. Hence by the binomial theorem,

$$S^n = (I_2 + J_2)^n = I_2 + nJ_2 = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$$

for all $n \in \mathbb{N}$. $\square$

Problem 1.2. Let

$$U = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix},$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$ are invertible. Prove that U is invertible and find U^{-1}.

Solution. Guess the inverse in block upper-triangular form

$$V = \begin{bmatrix} A^{-1} & X \\ 0 & B^{-1} \end{bmatrix}.$$

Compute

$$UV = \begin{bmatrix} I_n & AX + CB^{-1} \\ 0 & I_m \end{bmatrix}.$$

The identity condition gives $AX + CB^{-1} = 0$, hence $X = -A^{-1}CB^{-1}$. Therefore

$$U^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{bmatrix}.$$

$\square$

Problem 1.3. Let $A_1 \in \mathbb{R}^{m\times m}$ and $A_2 \in \mathbb{R}^{n\times n}$. Assume there exist invertible matrices T_1, T_2 such that $T_1^{-1}A_1T_1$ and $T_2^{-1}A_2T_2$ are diagonal. Show that there is an invertible $(m+n)\times(m+n)$ matrix T such that

$$T^{-1}\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} T$$

is diagonal.

Solution. Take
$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & T_2^{-1} \end{bmatrix}.$$
Then
$$T^{-1} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} T = \begin{bmatrix} T_1^{-1} A_1 T_1 & 0 \\ 0 & T_2^{-1} A_2 T_2 \end{bmatrix},$$
which is diagonal by hypothesis. Thus such T exists. □

Problem 1.4. Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times m}$. Prove that $I_m + AB$ is invertible if and only if $I_n + BA$ is invertible.

Solution. Consider the block matrix
$$M = \begin{bmatrix} I_m & -A \\ B & I_n \end{bmatrix}.$$
Left-multiplying by $\begin{bmatrix} I_m & 0 \\ -B & I_n \end{bmatrix}$ eliminates the lower-left block and gives
$$\begin{bmatrix} I_m & 0 \\ 0 & I_n + BA \end{bmatrix}.$$
Right-multiplying M by $\begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix}$ eliminates the upper-right block and gives
$$\begin{bmatrix} I_m + AB & 0 \\ B & I_n \end{bmatrix}.$$
Elementary block operations preserve invertibility; hence
$$I_m + AB \text{ is invertible} \iff M \text{ is invertible} \iff I_n + BA \text{ is invertible.}$$
□

Alternative Solution. By symmetry it suffices to show: if $I_m + AB$ is invertible then $I_n + BA$ is invertible. Suppose $(I_n + BA)x = 0$. Set $y = Ax$. Then
$$(I_m + AB)y = A(I_n + BA)x = 0.$$
Since $I_m + AB$ is invertible, $y = 0$, so $Ax = 0$; hence
$$(I_n + BA)x = x + BAx = x = 0.$$
Thus $\ker(I_n + BA) = \{0\}$, so $I_n + BA$ is invertible. The converse follows by interchanging A and B. □

Problem 1.5. Let A, B be two row-reduced echelon matrices that are left-equivalent, i.e. $B = PA$ for some invertible P.

(1) Show that $Ax = 0$ and $Bx = 0$ have the same solution set.
(2) Write $A = [A_1 \;\; a]$ with the last column a. If the last column of A is *not* a pivot column, prove that there exists x such that
$$A \begin{bmatrix} x \\ 1 \end{bmatrix} = 0.$$
(3) If the last column of A *is* a pivot column, prove that for every x,
$$A \begin{bmatrix} x \\ 1 \end{bmatrix} \neq 0.$$
(4) Suppose $A = [A_1 \;\; a]$ and $B = [B_1 \;\; b]$ have the same first $n-1$ columns, i.e. $A_1 = B_1$, and the last column of A is not a pivot column. Prove $A = B$.
(5) Under the same hypothesis $A_1 = B_1$, but now assume the last column of A *is* a pivot column. Prove $A = B$.
(6) Use induction on the number of columns to show that two left-equivalent row-reduced echelon matrices must be equal. Conclude that the row-reduced echelon form of a matrix is unique.

Solution. Since A and B are left-equivalent, there exists an invertible P with $B = PA$.

(1) If $Ax = 0$ then $Bx = PAx = 0$. Conversely, if $Bx = 0$ then $Ax = P^{-1}Bx = 0$. Hence the solution sets coincide.

(2) Write $A = [A_1\ a]$. If the last column is not a pivot column, then in RREF the equation $A_1x = -a$ is consistent. Let x_0 be a solution. Then

$$A \begin{bmatrix} x_0 \\ 1 \end{bmatrix} = A_1x_0 + a = 0.$$

(3) If the last column of A is a pivot column, say the pivot lies in row i, then row i of A has a single 1 in the last position and zeros elsewhere. Thus for any x,

$$\left(A \begin{bmatrix} x \\ 1 \end{bmatrix}\right)_i = 1 \neq 0,$$

which proves the desired assertion.

(4) Let $A = [A_1\ a]$, $B = [B_1\ b]$ with $A_1 = B_1$. By (2) choose x with $A \begin{bmatrix} x \\ 1 \end{bmatrix} = 0$. By (1), $B \begin{bmatrix} x \\ 1 \end{bmatrix} = 0$, i.e. $B_1x + b = 0$. Hence $b = -B_1x = -A_1x = a$, so $A = B$.

(5) Keep $A_1 = B_1$. We first show the last column of B is also a pivot column. By (3), for every x, $A \begin{bmatrix} x \\ 1 \end{bmatrix} \neq 0$. Since P is invertible,

$$B \begin{bmatrix} x \\ 1 \end{bmatrix} = PA \begin{bmatrix} x \\ 1 \end{bmatrix} \neq 0 \quad \text{for all } x.$$

By (2), this forces the last column of B to be a pivot column. Now $A_1 = B_1$ are themselves in RREF; let r be the number of their nonzero rows. When the last column is a pivot column in either matrix, in RREF it must equal the vector e_{r+1}. Hence the last columns coincide and $A = B$.

(6) Induct on the number of columns. For one column, two left-equivalent RREF matrices are either both $[0; \dots; 0]$ or both $[1; 0; \dots; 0]$, hence equal. Assume the claim holds for n columns. Let $A = [A_1\ a]$ and $B = [B_1\ b]$ be left-equivalent RREF matrices with $n + 1$ columns. Since $B = PA$, we have $B_1 = PA_1$, so A_1 and B_1 are left-equivalent RREF matrices with n columns; by the induction hypothesis $A_1 = B_1$. Applying (4) and (5) according as the last column is or is not a pivot column, we get $A = B$. Finally, if R_1 and R_2 are two RREFs of the same matrix M, then both are left-equivalent to M, hence left-equivalent to each other; by the above, $R_1 = R_2$. Thus the RREF is unique. $\square$

Tutorial Lecture 2

Problem 2.1 (Fixed points of a permutation)**.** Let P be an $n \times n$ permutation matrix. If the row permutation represented by P leaves the i-th row unchanged, we call i a fixed point of P. Prove:

(1) Such i is a fixed point of P if and only if the i-th diagonal entry of P equals 1.
(2) The number of fixed points of P equals $\operatorname{trace}(P)$.
(3) For any permutation matrices P_1, P_2, the products P_1P_2 and P_2P_1 have the same number of fixed points.

Solution. (1) If i is fixed, then $Pe_i = e_i$, hence $e_i^\top Pe_i = p_{ii} = 1$. Conversely, if $p_{ii} = 1$, then in row i all other entries are 0 (permutation matrix property). For any x,

$$(Px)_i = \sum_{j=1}^{n} p_{ij}x_j = p_{ii}x_i = x_i,$$

so left-multiplication by P leaves the i-th component and hence the i-th row unchanged. Thus i is a fixed point.

(2) Each diagonal entry of a permutation matrix is either 0 or 1, and it equals 1 exactly for fixed points by (1). Therefore the number of fixed points is $\sum_{i=1}^{n} p_{ii} = \operatorname{trace}(P)$.

(3) Both P_1P_2 and P_2P_1 are permutation matrices, so their numbers of fixed points equal their traces. Using cyclicity of trace,

$$\#\operatorname{Fix}(P_1P_2) = \operatorname{trace}(P_1P_2) = \operatorname{trace}(P_2P_1) = \#\operatorname{Fix}(P_2P_1).$$

□

Problem 2.2 (Affine maps)**.** On $\mathbb{R}^n$ consider maps of the form $f(x) = Ax + b$; these are called *affine* maps.

(1) Prove
$$\begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} f(x) \\ 1 \end{bmatrix}.$$
(2) For an affine map $f(x) = Ax + b$ set $M_f := \left[\begin{smallmatrix} A & b \\ 0 & 1 \end{smallmatrix}\right]$. Show that for any affine f, g, we have $M_f M_g = M_{f \circ g}$.
(3) Show that if f is invertible, then M_f is invertible and $M_{f^{-1}} = (M_f)^{-1}$.

Solution. (1) Block multiplication gives

$$\begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Ax + b \\ 1 \end{bmatrix} = \begin{bmatrix} f(x) \\ 1 \end{bmatrix}.$$

(2) Let $g(x) = Bx + b'$. Then

$$(f \circ g)(x) = A(Bx + b') + b = ABx + (Ab' + b),$$

hence

$$M_{f \circ g} = \begin{bmatrix} AB & Ab' + b \\ 0 & 1 \end{bmatrix}.$$

On the other hand,

$$M_f M_g = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} B & b' \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} AB & Ab' + b \\ 0 & 1 \end{bmatrix} = M_{f \circ g}.$$

(3) If f is invertible then necessarily A is invertible, and

$$f^{-1}(y) = A^{-1}(y - b) = A^{-1}y - A^{-1}b,$$

so

$$M_{f^{-1}} = \begin{bmatrix} A^{-1} & -A^{-1}b \\ 0 & 1 \end{bmatrix}.$$

Directly,

$$(M_f)^{-1} = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}b \\ 0 & 1 \end{bmatrix} = M_{f^{-1}},$$

which also shows M_f is invertible. $\square$

Problem 2.3. Work over $\mathbb{C}$. Any complex matrix can be written uniquely as $A+\mathrm{i}B$ with real matrices A, B; any complex vector as $v+\mathrm{i}w$ with real vectors v, w. Write $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ for real and imaginary parts.

(1) Express $\mathrm{Re}\big((A+\mathrm{i}B)(v+\mathrm{i}w)\big)$ and $\mathrm{Im}\big((A+\mathrm{i}B)(v+\mathrm{i}w)\big)$ in terms of A, B, v, w.
(2) For given real A, B, find a real matrix X such that for all real v, w,

$$\begin{bmatrix} \mathrm{Re}\left((A+\mathrm{i}B)(v+\mathrm{i}w)\right) \\ \mathrm{Im}\left((A+\mathrm{i}B)(v+\mathrm{i}w)\right) \end{bmatrix} = X \begin{bmatrix} v \\ w \end{bmatrix}.$$

(3) Define $f : \mathbb{C} \to \mathrm{M}_2(\mathbb{R})$ by

$$f(a+\mathrm{i}b) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Prove $f\big((a+\mathrm{i}b)(c+\mathrm{i}d)\big) = f(a+\mathrm{i}b)\, f(c+\mathrm{i}d)$.

Solution. (1) Notice that

$$(A+\mathrm{i}B)(v+\mathrm{i}w) = (Av - Bw) + \mathrm{i}(Aw + Bv).$$

Hence

$$\mathrm{Re}\big((A+\mathrm{i}B)(v+\mathrm{i}w)\big) = Av - Bw, \qquad \mathrm{Im}\big((A+\mathrm{i}B)(v+\mathrm{i}w)\big) = Aw + Bv.$$

(2) From part (1),

$$\begin{bmatrix} \mathrm{Re} \\ \mathrm{Im} \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}.$$

Thus the required real matrix is

$$X = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}.$$

(3) Since

$$(a+\mathrm{i}b)(c+\mathrm{i}d) = (ac-bd) + \mathrm{i}(bc+ad),$$

we have

$$f\big((a+\mathrm{i}b)(c+\mathrm{i}d)\big) = \begin{bmatrix} ac-bd & -(bc+ad) \\ bc+ad & ac-bd \end{bmatrix}.$$

On the other hand,

$$f(a+\mathrm{i}b)f(c+\mathrm{i}d) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac-bd & -(bc+ad) \\ bc+ad & ac-bd \end{bmatrix}.$$

Hence $f\big((a+\mathrm{i}b)(c+\mathrm{i}d)\big) = f(a+\mathrm{i}b)f(c+\mathrm{i}d)$. $\square$

Problem 2.4 (Interleaved block-diagonal matrix)**.** For 2×2 matrices $A = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{smallmatrix}\right]$, define

$$A \triangle B = \begin{bmatrix} a_{11} & 0 & a_{12} & 0 \\ 0 & b_{11} & 0 & b_{12} \\ a_{21} & 0 & a_{22} & 0 \\ 0 & b_{21} & 0 & b_{22} \end{bmatrix}.$$

Prove:

(1) $(A_1 \triangle B_1)(A_2 \triangle B_2) = (A_1A_2) \triangle (B_1B_2)$.
(2) $A \triangle B$ is invertible if and only if both A and B are invertible; in that case $(A \triangle B)^{-1} = A^{-1} \triangle B^{-1}$.
(3) Find X such that for all 2×2 matrices A, B,

$$X(A \triangle B)X^{-1} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Solution. Let

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then for all A, B,

$$A \triangle B = P \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} P.$$

(1) Using $P^2 = I_4$, we compute

$$\begin{aligned} (A_1 \triangle B_1)(A_2 \triangle B_2) &= P \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix} PP \begin{bmatrix} A_2 & 0 \\ 0 & B_2 \end{bmatrix} P \\ &= P \begin{bmatrix} A_1 A_2 & 0 \\ 0 & B_1 B_2 \end{bmatrix} P \\ &= (A_1 A_2) \triangle (B_1 B_2). \end{aligned}$$

(2) $A \triangle B$ is similar to $\operatorname{diag}(A, B)$ via P, so it is invertible if and only if A and B are invertible. Moreover,

$$(A \triangle B)^{-1} = \big(P \operatorname{diag}(A, B) P\big)^{-1} = P \operatorname{diag}(A^{-1}, B^{-1})\, P = A^{-1} \triangle B^{-1}.$$

(3) Take $X = P$. Then

$$X (A \triangle B) X^{-1} = P \big(P \operatorname{diag}(A, B) P\big) P = \operatorname{diag}(A, B).$$

□

Problem 2.5. Let $A \in \mathbb{R}^{n \times n}$ be written in block form

$$A = \begin{bmatrix} a_{11} & w^\top \\ v & B \end{bmatrix}, \qquad v \neq 0,$$

and suppose A admits an LU factorization $A = LU$ (without pivoting).

1. Is the leading entry a_{11} necessarily nonzero?
2. If we perform forward row operations on A to zero out v, how many additions and multiplications are needed?
3. Now assume A is symmetric. What is the relation between v and w? What property does B have?
4. Still assuming A is symmetric: after using row operations to make $v = 0$, apply the corresponding column operations to make $w^\top = 0$. Prove that the resulting $(n-1) \times (n-1)$ block becomes symmetric. How many additions and multiplications are needed in this step?

Conclude that the arithmetic cost of LU for symmetric matrices can be reduced by about one half.

Solution. (1) Having an LU factorization without pivoting is equivalent to all leading principal minors being nonsingular. The first such minor equals a_{11}, hence $a_{11} \neq 0$.

(2) The elimination step updates

$$B \longleftarrow B - \frac{1}{a_{11}} v w^\top.$$

We count the desired operations. Compute w / a_{11} in $(n-1)$ multiplications; form the outer product $v (w / a_{11})^\top$ in $(n-1)^2$ multiplications; subtract from B using $(n-1)^2$ additions. In total, for this step, it requires $(n-1)^2$ addition operations and $n(n-1)$ multiplication operations.

(3) If A is symmetric, then $w = v$ and $B = B^\top$.

(4) Use the elementary matrices

$$L_1 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{a_{11}} v & I_{n-1} \end{bmatrix}, \qquad U_1 = \begin{bmatrix} 1 & -\frac{1}{a_{11}} w^\top \\ 0 & I_{n-1} \end{bmatrix}.$$

Then

$$L_1 A U_1 = \begin{bmatrix} a_{11} & 0 \\ 0 & B - \frac{1}{a_{11}} v w^\top \end{bmatrix}.$$

For symmetric A we have $w = v$, so the trailing block becomes $B - \frac{1}{a_{11}} v v^\top$, which is symmetric. Exploiting symmetry, only the upper (or lower) triangular part needs updating: there are $\frac{(n-1)n}{2}$ subtractions. For the multiplications, first scale v by $1/a_{11}$ in $(n-1)$ multiplications, then form the upper-triangular products in $\frac{(n-1)n}{2}$ multiplications. Thus this step uses

$$\text{additions} = \frac{n(n-1)}{2}, \qquad \text{multiplications} = \frac{n(n-1)}{2} + (n-1),$$

for a total of $n^2 - 1$ arithmetic operations. Summing over stages $i = 1, \dots, n-1$ gives

$$\sum_{i=1}^{n-1} (i^2 - 1) = \frac{(n-1)n(n+1)}{3}$$

operations for the symmetric case, i.e., about half of the general case. $\square$

Problem 2.6. Let $T \in \mathbb{R}^{n \times n}$ be the tridiagonal matrix

$$T = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

Show, using elementary transformations, that $T = LU$ with L lower triangular and U upper triangular. Then compute T^{-1}.

Solution. Perform forward elimination by adding each row to the next one successively: $R_2 \leftarrow R_2 + R_1$, $R_3 \leftarrow R_3 + R_2$, $\dots$. The accumulated left multiplier equals the unit lower bidiagonal

$$L = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}.$$

After these operations the matrix becomes the unit upper bidiagonal

$$U = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & \ddots & \\ & & & \ddots & -1 \\ & & & & 1 \end{bmatrix}.$$

Hence $T = LU$. Since $T = LU$, we have $T^{-1} = U^{-1} L^{-1}$. The inverses of the bidiagonal factors are

$$L^{-1} = \left[\mathbf{1}_{i \geqslant j} \right]_{i,j=1}^{n}, \qquad U^{-1} = \left[\mathbf{1}_{i \leqslant j} \right]_{i,j=1}^{n},$$

i.e., L^{-1} is lower triangular with all 1's on and below the diagonal, and U^{-1} is upper triangular with all 1's on and above the diagonal. Therefore

$$(T^{-1})_{ij} = \sum_{k=1}^{n} \mathbf{1}_{i \leqslant k} \, \mathbf{1}_{k \geqslant j} = \#\{k \colon k \geqslant \max(i,j)\} = n + 1 - \max\{i, j\}.$$

Equivalently,

$$T^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ & 1 & \cdots & 1 \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

This completes the factorization and the explicit inverse. $\square$

Tutorial Lecture 3

Problem 3.1. Let $a_1, \ldots, a_n \in \mathbb{R}^m$. Fix indices $1 \leqslant i_1 < \cdots < i_s \leqslant m$ and, from each vector, delete the $i_1, \ldots, i_s$ components to obtain $a_1', \ldots, a_n' \in \mathbb{R}^{m-s}$. Prove:

(1) If $a_1, \ldots, a_n$ are linearly dependent, then $a_1', \ldots, a_n'$ are linearly dependent.
(2) If $a_1', \ldots, a_n'$ are linearly independent, then $a_1, \ldots, a_n$ are linearly independent.

Solution. (1) Assume $x_1, \ldots, x_n$ are not all zero and $\sum_{k=1}^n x_k a_k = 0$. Writing $a_i = \begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{mi} \end{bmatrix}^\top$, we have

$$\sum_{k=1}^n x_k a_k = \begin{bmatrix} x_1 a_{11} + x_2 a_{12} + \cdots + x_n a_{1n} \\ x_1 a_{21} + x_2 a_{22} + \cdots + x_n a_{2n} \\ \vdots \\ x_1 a_{m1} + x_2 a_{m2} + \cdots + x_n a_{mn} \end{bmatrix} = \mathbf{0}.$$

Deleting the $i_1, \ldots, i_s$ components of this zero vector yields $\sum_{k=1}^n x_k a_k' = 0$, so $a_1', \ldots, a_n'$ are linearly dependent.

Here comes a comment. Indeed, one can find a matrix A such that $Aa_k = a_k'$ for all k. Let A be the $(m-s) \times m$ matrix obtained from I_m by removing rows $i_1, \ldots, i_s$. Then $Aa_k = a_k'$ for all k, so $\sum x_k a_k' = A \sum x_k a_k = 0$.

(2) This is the contrapositive of (1): if $a_1, \ldots, a_n$ were dependent, then (1) would force $a_1', \ldots, a_n'$ to be dependent as well. Hence independence of the a_k' implies independence of the a_k. □

Problem 3.2. Let $k_1, \ldots, k_n \in \mathbb{R} \setminus \{0\}$ satisfy

$$\frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n} + 1 \neq 0.$$

Define vectors in $\mathbb{R}^n$ by

$$a_1 = \begin{bmatrix} 1 + k_1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 \\ 1 + k_2 \\ \vdots \\ 1 \end{bmatrix}, \quad \ldots, \quad a_n = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 + k_n \end{bmatrix}.$$

Find the rank of $\{a_1, \ldots, a_n\}$.

Solution. Let $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$. Then

$$A = \operatorname{diag}(k_1, \ldots, k_n) + \mathbf{1}\mathbf{1}^\top,$$

where $\mathbf{1} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^\top$. Set $D = \operatorname{diag}(k_1, \ldots, k_n)$ and $v = \mathbf{1}$. By Sherman–Morrison theorem, $D + vv^\top$ is invertible if and only if

$$1 + v^\top D^{-1} v \neq 0.$$

Here $v^\top D^{-1} v = \sum_{i=1}^n \frac{1}{k_i}$, so the given hypothesis $1 + \sum_{i=1}^n \frac{1}{k_i} \neq 0$ implies A is invertible. Hence the columns $a_1, \ldots, a_n$ are linearly independent and

$$\operatorname{rank}\{a_1, \ldots, a_n\} = n.$$

□

Problem 3.3 (Steinitz exchange lemma)**.** Let $S = \{a_1, \ldots, a_r\}$ be linearly independent and suppose every a_i is a linear combination of $T = \{b_1, \ldots, b_t\}$. Prove:

(1) $r \leqslant t$.
(2) One can replace r vectors in T by $a_1, \ldots, a_r$ to obtain a new t-tuple that is linearly equivalent to T (each spans the other).

Solution. Since each a_i is in $\operatorname{span} T$, there exist scalars c_{ij} such that

$$[a_1 \ a_2 \ \cdots \ a_r] = [b_1 \ b_2 \ \cdots \ b_t]\, C, \qquad C = (c_{ij}) \in \mathrm{M}_{t \times r}.$$

(1) If $r > t$ then the homogeneous system $Cx = 0$ has a nonzero solution x_0. Hence $[a_1 \ \cdots \ a_r]x_0 = [b_1 \ \cdots \ b_t]Cx_0 = 0$, so $\{a_1, \ldots, a_r\}$ is dependent, a contradiction. Thus $r \leqslant t$.

(2) We argue by induction on r. First consider the base case $r = 1$. Since $a_1 = \sum_{j=1}^t x_j b_j$ with some $x_j \neq 0$, assume $x_1 \neq 0$ and replace b_1 by a_1 to form $T' = \{a_1, b_2, \dots, b_t\}$. Clearly $T \subset \operatorname{span} T'$. Conversely,

$$b_1 = x_1^{-1}\left(a_1 - \sum_{j=2}^t x_j b_j\right) \in \operatorname{span} T,$$

so $\operatorname{span} T' = \operatorname{span} T$; hence T' and T are linearly equivalent.

Then we deal with the inductive step. Assume the claim holds for r and let $S = \{a_1, \dots, a_{r+1}\}$. By the inductive hypothesis we may choose indices $1 \leqslant j_1 < \cdots < j_r \leqslant t$ and replace $b_{j_1}, \dots, b_{j_r}$ by $a_1, \dots, a_r$ to obtain

$$T' = \{a_1, \dots, a_r, b_{r+1}, \dots, b_t\}$$

that is linearly equivalent to T. Because $a_{r+1} \in \operatorname{span} T = \operatorname{span} T'$, there exist scalars $y_1, \dots, y_t$ such that

$$a_{r+1} = y_1 a_1 + \cdots + y_r a_r + y_{r+1} b_{r+1} + \cdots + y_t b_t.$$

Not all of $y_{r+1}, \dots, y_t$ can be zero; otherwise $y_1 a_1 + \cdots + y_r a_r - a_{r+1} = 0$ would contradict the independence of S. Pick $k \in \{r+1, \dots, t\}$ with $y_k \neq 0$ and replace b_k by a_{r+1}. Exactly as in the base case, this yields a new t-tuple

$$T'' = \{a_1, \dots, a_r, a_{r+1},\ b_{r+1}, \dots, \widehat{b}_k, \dots, b_t\}$$

that is linearly equivalent to T', hence to T. This completes the induction. ☐

Problem 3.4. Let A have n rows and B have m rows.

(1) For the block diagonal matrix $C = \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$, prove that

$$\operatorname{rank}(C) = \operatorname{rank}(A) + \operatorname{rank}(B).$$

(2) For the block upper-triangular matrix $C = \left[\begin{smallmatrix} A & X \\ 0 & B \end{smallmatrix}\right]$, prove that

$$\operatorname{rank}(C) \geqslant \operatorname{rank}(A) + \operatorname{rank}(B).$$

Deduce that if A, B are invertible, then C is invertible.

Solution. (1) Perform row operations on the first n rows to reduce A to its row-reduced echelon form R_A while leaving the last m rows unchanged; then perform row operations on the last m rows to reduce B to R_B while leaving the first n rows unchanged:

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} R_A & 0 \\ 0 & B \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} R_A & 0 \\ 0 & R_B \end{bmatrix}.$$

Finally swap zero rows of R_A with nonzero rows of R_B so that all nonzero rows are stacked on top:

$$\begin{bmatrix} R_A & 0 \\ 0 & R_B \end{bmatrix} \rightsquigarrow \begin{bmatrix} R_A' & 0 \\ 0 & R_B' \\ 0 & 0 \end{bmatrix},$$

where R_A', R_B' consist of the nonzero rows of R_A, R_B. The number of nonzero rows equals $\operatorname{rank}(A) + \operatorname{rank}(B)$, so

$$\operatorname{rank}(C) = \operatorname{rank}(A) + \operatorname{rank}(B).$$

(2) Apply the same two-stage row reductions to A and B inside C. Row operations on the first n rows transform C into

$$\begin{bmatrix} R_A' & C_1 \\ 0 & B \end{bmatrix},$$

where $\begin{bmatrix} C_1 & C_2 \end{bmatrix}^\top$ is the result of applying those operations to $\begin{bmatrix} X & B \end{bmatrix}^\top$. Then reduce the last m rows to get

$$\begin{bmatrix} R_A' & C_1 \\ 0 & R_B' \end{bmatrix}.$$

This matrix is in row-echelon form above the zero block, hence its number of nonzero rows is at least $\operatorname{rank}(A) + \operatorname{rank}(B)$. Therefore

$$\operatorname{rank}(C) \geqslant \operatorname{rank}(A) + \operatorname{rank}(B).$$

If A and B are invertible, then $\operatorname{rank}(A) = n$, $\operatorname{rank}(B) = m$, so $\operatorname{rank}(C) \geqslant n+m$; since $\operatorname{rank}(C) \leqslant n+m$, we have $\operatorname{rank}(C) = n + m$, hence C is invertible. □

Problem 3.5 (Rank of skew-symmetric matrices)**.** Let $A \in \mathrm{M}_{m\times m}(\mathbb{R})$ be skew-symmetric ($A^\top = -A$).

(1) Prove that $\operatorname{rank}(A) \neq 1$.

(2) Delete the first row and first column of A to obtain $B \in \mathrm{M}_{(m-1)\times(m-1)}(\mathbb{R})$. Show that B is skew-symmetric and that

$$\operatorname{rank}(B) \in \{\operatorname{rank}(A),\ \operatorname{rank}(A) - 2\}.$$

Hint. Write $A = \left[\begin{smallmatrix} 0 & -v^\top \\ v & B \end{smallmatrix}\right]$ and split into the cases $v \in \mathcal{R}(B)$ or $v \notin \mathcal{R}(B)$.

(3) Deduce that the rank of a skew-symmetric matrix is always even; in particular, an odd-dimensional skew-symmetric matrix is never invertible.

Solution. (1) If $\operatorname{rank}(A) = 1$, then $A = vw^\top$ for some nonzero v, w. Since $A^\top = -A$, we have $wv^\top = -vw^\top$, so $\mathcal{R}(A^\top) = \operatorname{span}(w) = \mathcal{R}(A) = \operatorname{span}(v)$. Hence $w = cv$ for some c, and $A = cvv^\top$. But then $A^\top = cvv^\top = -A = -cvv^\top$, forcing $c = 0$, a contradiction.

(2) Writing $A = \left[\begin{smallmatrix} 0 & -v^\top \\ v & B \end{smallmatrix}\right]$, we see $B^\top = -B$, so B is skew-symmetric. If $v \in \mathcal{R}(B)$, choose x with $Bx = v$ and observe the block factorizations

$$\begin{bmatrix} 1 & -x^\top \\ 0 & I_{m-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x & I_{m-1} \end{bmatrix} = \begin{bmatrix} 0 & -v^\top \\ v & B \end{bmatrix} = A.$$

Since the two outer factors are invertible, $\operatorname{rank}(A) = \operatorname{rank}(B)$. If $v \notin \mathcal{R}(B)$, consider the augmented matrix $[\,v \ \ B\,]$. Elementary row/column operations show

$$\operatorname{rank}(B) \leqslant \operatorname{rank}\big[\,v \ \ B\,\big] \leqslant \operatorname{rank}(A) \leqslant \operatorname{rank}\big[\,v \ \ B\,\big] + 1.$$

The upper bound is strict in this case (otherwise $v \in \mathcal{R}(B)$ would follow), hence $\operatorname{rank}(A) = \operatorname{rank}\big[\,v \ \ B\,\big] + 1 = \operatorname{rank}(B) + 2$. Equivalently, $\operatorname{rank}(B) = \operatorname{rank}(A) - 2$.

(3) Define A_1 to be A with its first row and column removed, A_2 to be A_1 with its first row and column removed, and so on, until $A_m = [0]$. By part (2), each removal changes the rank by either 0 or -2. Thus $\operatorname{rank}(A), \operatorname{rank}(A_1), \ldots, \operatorname{rank}(A_m) = 0$ all have the same parity; hence $\operatorname{rank}(A)$ is even. If m is odd, then $\operatorname{rank}(A) \leqslant m - 1 < m$, so A cannot be invertible. □

Problem 3.6. (1) Find vectors u, v such that

$$uv^\top = \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix}.$$

(2) Let $A \in \mathbb{R}^{m\times n}$ have rank $r > 0$. Let C be the $m \times r$ matrix formed by the pivot columns of A in order, and let R be the $r \times n$ matrix formed by the nonzero rows of the row-reduced echelon form of A in order. Determine the sizes of C and R, and prove

$$A = CR.$$

(3) Deduce that every $A \in \mathbb{R}^{m\times n}$ of rank $r > 0$ factors as $A = CR$ with $C \in \mathbb{R}^{m\times r}$ column-full-rank, $R \in \mathbb{R}^{r\times n}$ row-full-rank, and $\operatorname{rank}(C) = \operatorname{rank}(R) = r$.

(4) Conclude that every linear map $f\colon \mathbb{R}^n \to \mathbb{R}^m$ admits a factorization $f = g \circ h$ where h is surjective and g is injective.

Solution. (1) Take

$$u = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \qquad v = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

More generally, for any $c \in \mathbb{R}^\times$, $u = c\begin{bmatrix} 3 & 1 & 4 \end{bmatrix}^\top$ and $v = c^{-1}\begin{bmatrix} 1 & 2 & 2 \end{bmatrix}^\top$ also work.

(2) $\operatorname{rank}(A) = r$ implies A has r pivot columns, hence C has size $m \times r$. Its RREF has r nonzero rows, hence R has size $r \times n$. Write $A = [a_1 \ \cdots \ a_n]$ and let $C = [c_1 \ \cdots \ c_r]$ be the pivot columns.

Each a_j is a linear combination of the c_i with coefficients given by the RREF, so $a_j = C\,r_j$ (where r_j is the j-th column of R). Therefore $A = CR$.

A constructive proof proceeds by induction on the number of columns. Write $A = [A'\ a]$ and assume $A' = C'R'$ with the stated properties. If a is a pivot column, set

$$C = \begin{bmatrix} C' & a \end{bmatrix}, \qquad R = \begin{bmatrix} R' \\ e_k^\top \end{bmatrix}$$

for a suitable unit row $e_k^\top$ that places the new pivot; then $CR = A$. If a is not a pivot column, there exists $x \in \mathbb{R}^r$ with $a = C'x$, so $A = [C' \ \ a] \begin{bmatrix} R' & x \end{bmatrix} = CR$ with $C = C'$ and $R = [R'\ x]$. In both cases C has r independent columns and R has r independent rows.

(3) The statement is exactly the conclusion of part (2): C consists of the pivot columns (hence column-full-rank r) and R of the nonzero RREF rows (hence row-full-rank r), with $A = CR$.

(4) Let A be the matrix of f. By part (3), write $A = CR$ with $C \in \mathbb{R}^{m\times r}$ column-full-rank and $R \in \mathbb{R}^{r\times n}$ row-full-rank. Define linear maps $h\colon \mathbb{R}^n \to \mathbb{R}^r$ by $h(x) = Rx$ and $g\colon \mathbb{R}^r \to \mathbb{R}^m$ by $g(y) = Cy$. Since R has rank r, h is surjective; since C has rank r, g is injective. Moreover $g \circ h(x) = C(Rx) = Ax = f(x)$. □

Problem 3.7 (Fredholm alternative in finite dimension)**.** Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Show that the linear system

$$Ax = b$$

has a solution if and only if the stacked system

$$\begin{bmatrix} A^\top \\ b^\top \end{bmatrix} y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

has no solution. (In the first system the unknown is x, in the second it is y.)

Solution. By the rank (Rouché–Capelli) criterion, a system $My = c$ is inconsistent if and only if

$$\operatorname{rank}\bigl([\,M\ c\,]\bigr) = \operatorname{rank}(M) + 1.$$

Take $M = \begin{bmatrix} A^\top \\ b^\top \end{bmatrix}$ and $c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then

$$[\,M\ c\,] = \begin{bmatrix} A^\top & 0 \\ b^\top & 1 \end{bmatrix} \qquad \text{and} \qquad \operatorname{rank}(M) = \operatorname{rank} \begin{bmatrix} A^\top \\ b^\top \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A & b \end{bmatrix}.$$

Elementary row/column operations (or transposition) give

$$\operatorname{rank} \begin{bmatrix} A^\top & 0 \\ b^\top & 1 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} = \operatorname{rank}(A) + 1.$$

Hence

$$\begin{bmatrix} A^\top \\ b^\top \end{bmatrix} y = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ is inconsistent } \iff \operatorname{rank}(A) = \operatorname{rank} \begin{bmatrix} A & b \end{bmatrix}.$$

By the Rouché–Capelli theorem again, the latter is equivalent to $Ax = b$ being solvable. □

Tutorial Lecture 4

Problem 4.1. Let $A \in \mathbb{R}^{n\times n}$. Prove:

(1) $A^2 = A$ if and only if $\operatorname{rank}(A) + \operatorname{rank}(I_n - A) = n$.
(2) $A^2 = I_n$ if and only if $\operatorname{rank}(I_n + A) + \operatorname{rank}(I_n - A) = n$.

Solution. (1) Set $V = \mathcal{R}(A)$ and $W = \mathcal{R}(I - A)$. For any x,

$$x = Ax + (I - A)x \in V + W \quad \Rightarrow \quad \mathbb{R}^n = V + W,$$

so

$$n = \dim(V + W) = \operatorname{rank}(A) + \operatorname{rank}(I - A) - \dim(V \cap W).$$

If $A^2 = A$ and $x \in V \cap W$, then $x = Av = (I - A)w$ for some v, w, hence

$$(I - A)x = (I - A)Av = 0, \qquad Ax = A(I - A)w = 0,$$

so $x = (I - A)x + Ax = 0$. Thus $V \cap W = \{0\}$ and the stated rank identity holds.

If $\operatorname{rank}(A) + \operatorname{rank}(I - A) = n$, then

$$\dim \mathcal{N}(A) + \dim \mathcal{N}(I - A) = n,$$

and $\mathcal{N}(A) \cap \mathcal{N}(I - A) = \{0\}$. Hence $\mathbb{R}^n = \mathcal{N}(A) \oplus \mathcal{N}(I - A)$. Write any $x = x_1 + x_2$ with $x_1 \in \mathcal{N}(A)$, $x_2 \in \mathcal{N}(I - A)$. Then

$$(A^2 - A)x = (A^2 - A)x_1 + (A^2 - A)x_2 = (I - A)Ax_1 + A(I - A)x_2 = 0,$$

so $A^2 = A$.

(2) Let $V = \mathcal{R}(I + A)$ and $W = \mathcal{R}(I - A)$. For any x,

$$x = \tfrac{1}{2}(I + A)x + \tfrac{1}{2}(I - A)x \in V + W,$$

hence $\mathbb{R}^n = V + W$ and

$$n = \operatorname{rank}(I + A) + \operatorname{rank}(I - A) - \dim(V \cap W).$$

If $A^2 = I$, and $x = (I + A)v = (I - A)w$, then

$$(I - A)x = (I - A)(I + A)v = 0, \qquad (I + A)x = (I + A)(I - A)w = 0,$$

so $x = \frac{1}{2}[(I + A)x + (I - A)x] = 0$. Thus $V \cap W = \{0\}$ and the rank sum equals n.

If $\operatorname{rank}(I + A) + \operatorname{rank}(I - A) = n$, then

$$\dim \mathcal{N}(I + A) + \dim \mathcal{N}(I - A) = n, \qquad \mathcal{N}(I + A) \cap \mathcal{N}(I - A) = \{0\}.$$

Hence $\mathbb{R}^n = \mathcal{N}(I + A) \oplus \mathcal{N}(I - A)$. For $x = x_1 + x_2$ with $x_1 \in \mathcal{N}(I + A)$ and $x_2 \in \mathcal{N}(I - A)$,

$$(I - A^2)x = (I - A)(I + A)x_1 + (I + A)(I - A)x_2 = 0,$$

so $A^2 = I_n$. □

Problem 4.2. (1) Find three vectors in $\mathbb{R}^2$ whose pairwise inner products are negative.
(2) Find four vectors in $\mathbb{R}^3$ whose pairwise inner products are negative.
(3) What is the maximal size of a set in $\mathbb{R}^n$ whose pairwise inner products are all negative?

Solution. (1) In $\mathbb{R}^2$ take

$$a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \quad a_3 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

Then $a_i \cdot a_j < 0$ for $i \neq j$.

(2) In $\mathbb{R}^3$ take

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{100} \end{bmatrix}, \quad a_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ \frac{1}{100} \end{bmatrix}, \quad a_3 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ \frac{1}{100} \end{bmatrix}, \quad a_4 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix},$$

and again $a_i \cdot a_j < 0$ for all $i \neq j$.

(3) The maximum is $n+1$. We first show that $n+1$ is an upper bound. Assume, for contradiction, that $n+2$ vectors $a_1, \ldots, a_{n+2} \in \mathbb{R}^n$ satisfy $a_i \cdot a_j < 0$ for all $i \neq j$. They are linearly dependent, so choose scalars $(x_1, \ldots, x_{n+2}) \not\equiv 0$ with $\sum_{i=1}^{n+2} x_i a_i = 0$. Both positive and negative x_i must occur. Without loss of generality, $x_1, \ldots, x_r > 0$ and $x_{r+1}, \ldots, x_{n+2} < 0$, and

$$\sum_{i=1}^{r} x_i a_i = -\sum_{j=r+1}^{n+2} x_j a_j.$$

Taking the inner product of both sides with $\sum_{i=1}^{r} x_i a_i$ gives

$$0 \leqslant \Big\|\sum_{i=1}^{r} x_i a_i\Big\|^2 = -\sum_{i=1}^{r}\sum_{j=r+1}^{n+2} x_i x_j \,(a_i \cdot a_j) \;<\; 0,$$

because $x_i x_j < 0$ and $a_i \cdot a_j < 0$. This contradiction shows that at most $n+1$ such vectors can exist.

Now it remains to check that the maximum can reach $n+1$. We already constructed examples for $n = 2$ and $n = 3$. Assume there exist $n+1$ vectors $a_1, \ldots, a_{n+1} \in \mathbb{R}^n$ with $a_i \cdot a_j < 0$ for $i \neq j$. Embed into $\mathbb{R}^{n+1}$ by

$$b_i = \begin{bmatrix} a_i \\ \varepsilon \end{bmatrix} \quad (1 \leqslant i \leqslant n+1), \qquad b_{n+2} = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

where $\varepsilon > 0$ is chosen small enough so that $\varepsilon^2 < \min_{i \neq j}(-a_i \cdot a_j)$. Then $b_i \cdot b_{n+2} = -\varepsilon < 0$, and for $i \neq j$,

$$b_i \cdot b_j = a_i \cdot a_j + \varepsilon^2 < 0.$$

Thus $\mathbb{R}^{n+1}$ contains $n+2$ vectors with pairwise negative inner products. By induction, $\mathbb{R}^n$ admits $n+1$ such vectors. □

Problem 4.3 (Higher-dimensional Pythagoras)**.** Let $a, b, c \in \mathbb{R}^n$.

(1) For the triangle spanned by a and b, prove that its area squared is

$$\frac{1}{4}\big(\|a\|^2\|b\|^2 - (a^\top b)^2\big).$$

(2) If a, b, c are pairwise orthogonal and form a tetrahedron, show that the area squared of the oblique face equals the sum of the area squares of the other three right-triangle faces.

Solution. (1) If θ is the angle between a and b, then the area is $S = \frac{1}{2}\|a\|\,\|b\| \sin\theta$ and $\cos\theta = \frac{a^\top b}{\|a\|\,\|b\|}$. Hence

$$S^2 = \frac{1}{4}\|a\|^2\|b\|^2(1 - \cos^2\theta) = \frac{1}{4}\big(\|a\|^2\|b\|^2 - (a^\top b)^2\big).$$

(2) Let $S_{a,b}, S_{a,c}, S_{b,c}$ be the areas of the right triangles on the mutually perpendicular edges, and let S be the area of the oblique face whose edge vectors are $c - a$ and $b - a$. Then

$$S_{a,b}^2 = \frac{\|a\|^2\|b\|^2}{4}, \qquad S_{a,c}^2 = \frac{\|a\|^2\|c\|^2}{4}, \qquad S_{b,c}^2 = \frac{\|b\|^2\|c\|^2}{4}.$$

By part (1),

$$S^2 = \frac{1}{4}\Big(\|b-a\|^2\|c-a\|^2 - \big((b-a)^\top(c-a)\big)^2\Big).$$

Since a, b, c are pairwise orthogonal,

$$\|b-a\|^2 = \|b\|^2 + \|a\|^2, \quad \|c-a\|^2 = \|c\|^2 + \|a\|^2, \quad (b-a)^\top(c-a) = \|a\|^2.$$

Therefore

$$S^2 = \frac{1}{4}\big(\|a\|^2\|b\|^2 + \|a\|^2\|c\|^2 + \|b\|^2\|c\|^2\big) = S_{a,b}^2 + S_{a,c}^2 + S_{b,c}^2.$$

This completes the proof. □

Problem 4.4 (Riesz representation in $\mathbb{R}^n$)**.** Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be linear. Prove that there exists $b \in \mathbb{R}^n$ such that $f(a) = b^\top a$ for all $a \in \mathbb{R}^n$.

Solution. Let $e_1, \dots, e_n$ be the standard basis, set $b_i = f(e_i)$, and $b = (b_1, \dots, b_n)^\top$. For any $a = (a_1, \dots, a_n)^\top = \sum_{i=1}^n a_i e_i$,

$$f(a) = \sum_{i=1}^n a_i f(e_i) = \sum_{i=1}^n a_i b_i = b^\top a.$$

This completes the proof. □

Problem 4.5 (Hadamard matrix)**.** An $n \times n$ matrix A with entries ± 1 is called an nth-order *Hadamard matrix* if $A^\top A = nI_n$. Equivalently, $A/\sqrt{n}$ is an orthogonal matrix whose entries have equal absolute value. It is known that a Hadamard matrix can only have order $1, 2$, or $4k$ ($k \in \mathbb{N}$). Whether a Hadamard matrix exists for every order $4k$ is the (open) *Hadamard conjecture*.

(1) List all 1×1 and all 2×2 Hadamard matrices.
(2) Show that no 3×3 Hadamard matrix exists.
(3) Exhibit a 4×4 Hadamard matrix.
(4) Prove that if A is an $n \times n$ Hadamard matrix, then

$$\begin{bmatrix} A & A \\ A & -A \end{bmatrix}$$

is a $2n \times 2n$ Hadamard matrix. Conclude that Hadamard matrices exist for all orders 2^m.

Solution. (1) The Hadamard matrices of order 1 are exactly $[1]$ and $[-1]$. Up to row/column permutations and sign flips, every 2×2 Hadamard matrix is the Sylvester matrix $H_2 = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$. Explicitly, the eight 2×2 Hadamard matrices are

$$\pm\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \pm\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \pm\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \pm\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

(2) If A is 3×3 with entries ± 1, then every entry of $A^\top A$ is a sum of three elements of $\{\pm 1\}$, hence an odd integer. Therefore no entry of $A^\top A$ is 0, so it cannot equal $3I_3$. Contradiction. Thus no 3×3 Hadamard matrix exists.

(3) A 4×4 example (Sylvester construction):

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \qquad H_4^\top H_4 = 4I_4.$$

(4) If $A^\top A = nI_n$, then

$$\begin{bmatrix} A & A \\ A & -A \end{bmatrix}^\top \begin{bmatrix} A & A \\ A & -A \end{bmatrix} = \begin{bmatrix} A^\top & A^\top \\ A^\top & -A^\top \end{bmatrix} \begin{bmatrix} A & A \\ A & -A \end{bmatrix} = \begin{bmatrix} 2A^\top A & 0 \\ 0 & 2A^\top A \end{bmatrix} = 2n\, I_{2n}.$$

Hence $\left[\begin{smallmatrix} A & A \\ A & -A \end{smallmatrix}\right]$ is Hadamard of order $2n$. Starting from the order-1 and order-2 cases and iterating this doubling, Hadamard matrices exist for all orders 2^m. □

Problem 4.6 (Gram matrix)**.** Let $a_1, a_2, \dots, a_m \in \mathbb{R}^n$. Define the Gram matrix

$$G(a_1, \dots, a_m) := \begin{bmatrix} a_1^\top a_1 & a_1^\top a_2 & \cdots & a_1^\top a_m \\ a_2^\top a_1 & a_2^\top a_2 & \cdots & a_2^\top a_m \\ \vdots & \vdots & \ddots & \vdots \\ a_m^\top a_1 & a_m^\top a_2 & \cdots & a_m^\top a_m \end{bmatrix}.$$

Prove:

(1) $a_1, \dots, a_m$ form an orthonormal set if and only if $G(a_1, \dots, a_m) = I_m$.
(2) $G = G(a_1, \dots, a_m)$ is an $m \times m$ symmetric matrix and $x^\top G x \geqslant 0$ for all $x \in \mathbb{R}^m$.
(3) $a_1, \dots, a_m$ are linearly independent if and only if G is invertible, equivalently $x^\top G x > 0$ for all nonzero $x \in \mathbb{R}^m$.

Solution. (1) The set $a_1, \dots, a_m$ is orthonormal if and only if

$$a_i^\top a_j = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases}$$

which is equivalent to $G(a_1, \dots, a_m) = I_m$.

(2) Let $A = \begin{bmatrix} a_1 & \cdots & a_m \end{bmatrix} \in \mathbb{R}^{n \times m}$. Then $G(a_1, \dots, a_m) = A^\top A$, hence G is symmetric and

$$x^\top G x = x^\top A^\top A x = (Ax)^\top (Ax) = \|Ax\|_2^2 \geqslant 0 \quad \forall\, x \in \mathbb{R}^m.$$

(3) The vectors $a_1, \dots, a_m$ are linearly independent if and only if $Ax = 0$ has only the trivial solution. This holds if and only if $\|Ax\|_2^2 > 0$ for all $x \neq 0$, i.e. $x^\top G x > 0$ for all $x \neq 0$, which is equivalent to G being positive definite and thus invertible. Conversely, if G is invertible, then $x^\top G x = 0$ implies $x = 0$, hence $Ax = 0$ has only the trivial solution and the columns are linearly independent. □

Tutorial Lecture 5

Problem 5.1. Compute $\det A$ for
$$A = \left[\, 1 + x_i y_j \,\right]_{i,j=1}^{n}.$$

Solution. We claim that
$$\det(A) = \begin{cases} 1 + x_1 y_1, & n = 1, \\ (x_1 - x_2)(y_1 - y_2), & n = 2, \\ 0, & n \geqslant 3. \end{cases}$$

For $n = 1$ the claim is immediate. For $n = 2$,
$$\det(A) = (1 + x_1 y_1)(1 + x_2 y_2) - (1 + x_1 y_2)(1 + x_2 y_1) = (x_1 - x_2)(y_1 - y_2).$$
For $n \geqslant 3$, write with $\mathbf{1} = (1, \dots, 1)^\top$, $x = (x_1, \dots, x_n)^\top$, $y = (y_1, \dots, y_n)^\top$:
$$A = \mathbf{1}\mathbf{1}^\top + x\, y^\top.$$
Hence $\operatorname{rank}(A) \leqslant \operatorname{rank}(\mathbf{1}\mathbf{1}^\top) + \operatorname{rank}(x\, y^\top) \leqslant 1 + 1 = 2$, so A is not full rank when $n \geqslant 3$, and $\det(A) = 0$. □

Problem 5.2. Compute
$$\det \begin{bmatrix} a_{11} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & \cdots & a_{2,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & 0 & \cdots & 0 \end{bmatrix}.$$

Solution. We claim that
$$\det = (-1)^{\lfloor n/2 \rfloor}\, a_{1n}\, a_{2,n-1} \cdots a_{n1} = (-1)^{\frac{n(n-1)}{2}}\, a_{1n}\, a_{2,n-1} \cdots a_{n1}.$$

Swap column 1 with n, column 2 with $n-1$, …, column i with $n+1-i$. After $\lfloor n/2 \rfloor$ such swaps the matrix becomes upper triangular with diagonal entries $a_{1n}, a_{2,n-1}, \dots, a_{n1}$, hence
$$\det = (-1)^{\lfloor n/2 \rfloor}\, a_{1n}\, a_{2,n-1} \cdots a_{n1}.$$
Alternatively, move the last column step by step to the first position, which uses $(n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2}$ swaps, giving the same sign and product. □

Problem 5.3 (Prove or give a counterexample)**.** Decide whether each statement is true; prove it or provide a counterexample.

(1) $\det(AB - BA)$ must be zero for all square matrices A, B of the same size.
(2) $\det(A)$ equals the determinant of the row-reduced echelon form of A.
(3) If A is an $n \times n$ skew-symmetric matrix and n is odd, then $\det(A) = 0$.
(4) If A is an $n \times n$ skew-symmetric matrix and n is even, then $\det(A) = 0$.
(5) If $|\det(A)| > 1$, then as $n \to \infty$ some entry of A^n has absolute value $\to \infty$.
(6) If $|\det(A)| < 1$, then as $n \to \infty$ all entries of A^n tend to 0.

Solution. (1) **False.** Take
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}.$$
Then
$$AB - BA = \begin{bmatrix} 2 & 7 \\ -7 & -2 \end{bmatrix} \quad \text{and} \quad \det(AB - BA) = (-4) + 49 = 45 \neq 0.$$

(2) **False.** With
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \operatorname{rref}(A) = I_2, \quad \det(A) = -1 \neq \det(I_2) = 1.$$

(3) **True.** For skew-symmetric A, $A^\top = -A$. Hence
$$\det(A) = \det(A^\top) = \det(-A) = (-1)^n \det(A).$$

If n is odd, then $\det(A) = -\det(A)$, so $\det(A) = 0$.

(4) **False.** For

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \det(A) = 1.$$

(5) **True.** Let A be $m \times m$. The expansion of $\det(A^n)$ is a sum of $m!$ products of m entries of A^n. If all entries of A^n were bounded by M, then

$$|\det(A^n)| \leqslant m!\, M^m \quad \text{(bounded)}.$$

But $|\det(A^n)| = |\det(A)|^n \to \infty$ when $|\det(A)| > 1$, a contradiction. Hence some entry of A^n must be unbounded.

(6) **False.** Consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \implies A^n = \begin{bmatrix} 1 & 0 \\ 0 & 2^{-n} \end{bmatrix}.$$

Not all entries tend to 0 since the $(1,1)$-entry is 1 for all n. □

Problem 5.4 (Find the mistake)**.** Below are typical incorrect arguments. Locate the error and give the correct statement or a counterexample.

(1) For an invertible 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\det(A^{-1}) = \det\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) = \frac{ad-bc}{ad-bc} = 1.$$

This looks suspicious. Where is the mistake?

(2) For the block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, one computes

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = AD - BC.$$

This treats blocks like scalars and is invalid. If A is invertible, what is the correct formula?

(3) For the orthogonal projection P onto $\operatorname{col}(A)$ one writes

$$P = A(A^\top A)^{-1}A^\top, \quad \det(P) = \frac{\det(A)\det(A^\top)}{\det(A^\top A)} = 1,$$

yet a projection is often non-invertible. What went wrong?

(4) If $AB = -BA$, then

$$\det(A)\det(B) = -\det(B)\det(A) \implies 2\det(A)\det(B) = 0,$$

so one of A, B must be non-invertible. Is this correct? If not, give a counterexample.

(5) For $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, do the simultaneous row changes

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} a+sc & b+sd \\ c+ta & d+tb \end{bmatrix}.$$

For which s, t is the determinant preserved? (This is *not* an elementary row operation.)

Solution. (1) The error is using $\det(\lambda M) = \lambda \det(M)$. For 2×2, $\det(\lambda M) = \lambda^2 \det(M)$. Hence

$$\det(A^{-1}) = \frac{1}{(ad-bc)^2}\det\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(ad-bc)^2}(ad-bc) = \frac{1}{ad-bc} = \frac{1}{\det(A)}.$$

(2) The identity $AD - BC$ is meaningless for noncommuting blocks. If A is invertible, block Gaussian elimination gives

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix},$$

so

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A)\ \det(D - CA^{-1}B).$$

Similarly, if D is invertible then $\det = \det(D)\det(A - BD^{-1}C)$.

(3) A is typically rectangular, so $\det(A)$ is undefined; the whole determinant computation is invalid. Indeed, P is a projection with eigenvalues 0 or 1, hence $\det(P)$ is 0 unless the projection is the identity.

(4) For $n \times n$ matrices,

$$\det(-BA) = (-1)^n \det(BA) = (-1)^n \det(B)\det(A).$$

Only when n is odd does this force $\det(A)\det(B) = 0$. Counterexample for $n = 2$:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

both invertible and

$$AB = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -BA.$$

(5) Direct computation:

$$\det \begin{bmatrix} a + sc & b + sd \\ c + ta & d + tb \end{bmatrix} = (a + sc)(d + tb) - (b + sd)(c + ta) = (1 - st)(ad - bc).$$

Thus the determinant is preserved if and only if $st = 0$. □

Problem 5.5 (Determinantal rank equals rank)**.** Let A be an $m \times n$ matrix. For any choice of k rows and k columns of A, the determinant of the resulting $k \times k$ submatrix is called a kth-order minor of A. Define

$$\operatorname{rank}_{\det}(A) := \max\{\, k \mid A \text{ has a nonzero } k \times k \text{ minor} \,\}.$$

Prove that $\operatorname{rank}_{\det}(A) = \operatorname{rank}(A)$.

Proof. Write $r = \operatorname{rank}(A)$ and $d = \operatorname{rank}_{\det}(A)$. Let $A = [a_1\, a_2\, \cdots\, a_n]$ with column vectors a_i.

We first show that $r \leqslant d$. Choose a maximal linearly independent set of columns $a_{i_1}, \dots, a_{i_r}$, and let $B = [a_{i_1}\ \cdots\ a_{i_r}]$. Then $\operatorname{rank}(B) = \operatorname{rank}(A) = r$. Select a maximal linearly independent set of rows of B; it contains r rows, say with indices $j_1 < \cdots < j_r$. Let C be the $r \times r$ submatrix of B formed by these rows. Then $\operatorname{rank}(C) = r$, hence C is invertible and $\det(C) \neq 0$. Since C is also the submatrix of A using rows $j_1, \dots, j_r$ and columns $i_1, \dots, i_r$, A has a nonzero $r \times r$ minor, so $d \geqslant r$.

Now it suffices to show that $d \leqslant r$. Take rows $j_1, \dots, j_d$ and columns $i_1, \dots, i_d$ of A forming a $d \times d$ submatrix C with $\det(C) \neq 0$. Then $\operatorname{rank}(C) = d$. Let B be the submatrix of A consisting of rows $j_1, \dots, j_d$ and all columns. Since C is a submatrix of B, $\operatorname{rank}(B) \geqslant \operatorname{rank}(C) = d$; thus $\operatorname{rank}(A) \geqslant \operatorname{rank}(B) \geqslant d$, i.e., $r \geqslant d$.

Combining both steps gives $\operatorname{rank}_{\det}(A) = \operatorname{rank}(A)$. □

Problem 5.6 (Hadamard inequality)**.** Let $T = \begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix}$ be an $n \times n$ real matrix with column vectors t_i.

(1) Using the QR decomposition, prove

$$|\det(T)| \leqslant \|t_1\|\,\|t_2\| \cdots \|t_n\|.$$

(2) Show that equality holds when T is a Hadamard matrix (see the definition in Problem 4.5).

Solution. (1) If T is singular, then $\det(T) = 0$ and the inequality is trivial. Assume T is nonsingular and write the QR decomposition $T = QR$ with Q orthogonal and R upper triangular with positive diagonal. Then $|\det(T)| = |\det(Q)\det(R)| = |\det(R)| = \prod_{i=1}^n |r_{ii}|$. Let v_i be the i-th column of R. Since r_{ii} is the i-th entry of v_i,

$$|r_{ii}| \leqslant \|v_i\|_2.$$

Moreover $t_i = Qv_i$, hence $\|t_i\|_2 = \|v_i\|_2$ because Q is orthogonal. Therefore

$$|\det(T)| = \prod_{i=1}^n |r_{ii}| \leqslant \prod_{i=1}^n \|v_i\| = \prod_{i=1}^n \|t_i\|.$$

(2) Let $A = [\, a_1 \;\; \cdots \;\; a_n \,]$ be an $n \times n$ Hadamard matrix, so $A^\top A = nI_n$. Then $a_i^\top a_i = n$ for all i, i.e. $\|a_i\| = \sqrt{n}$. Also

$$\det(A)^2 = \det(A^\top A) = \det(nI_n) = n^n,$$

so $|\det(A)| = n^{n/2} = \prod_{i=1}^n \|a_i\|$, which achieves equality in the inequality of part (1). $\square$

Tutorial Lecture 6

Problem 6.1. Let A be an $n \times n$ symmetric matrix with an $LDL^\top$ factorization $A = LDL^\top$, where $D = \operatorname{diag}(d_1, \dots, d_n)$. Let A_i denote the i-th leading principal submatrix of A. Prove

$$d_i = \frac{\det(A_i)}{\det(A_{i-1})} \qquad (1 \leqslant i \leqslant n),$$

where $\det(A_i)$ is the i-th leading principal minor of A.

Proof. Fix $1 \leqslant i \leqslant n$. Partition

$$L = \begin{bmatrix} L_{1i} & 0 \\ L_{2i} & L_{3i} \end{bmatrix}, \qquad D = \operatorname{diag}(D_{1i}, D_{2i}),$$

where $L_{1i} \in \mathbb{R}^{i \times i}$ and $L_{3i} \in \mathbb{R}^{(n-i)\times(n-i)}$ are unit lower triangular, and $D_{1i} = \operatorname{diag}(d_1, \dots, d_i)$, $D_{2i} = \operatorname{diag}(d_{i+1}, \dots, d_n)$. Then

$$A = \begin{bmatrix} L_{1i} & 0 \\ L_{2i} & L_{3i} \end{bmatrix} \begin{bmatrix} D_{1i} & 0 \\ 0 & D_{2i} \end{bmatrix} \begin{bmatrix} L_{1i}^\top & L_{2i}^\top \\ 0 & L_{3i}^\top \end{bmatrix}.$$

By block multiplication, the $i \times i$ leading principal block of A is

$$A_i = L_{1i} D_{1i} L_{1i}^\top.$$

Since L_{1i} is unit lower triangular, $\det(L_{1i}) = \det(L_{1i}^\top) = 1$, hence

$$\det(A_i) = \det(D_{1i}) = d_1 \cdots d_i.$$

Similarly, $\det(A_{i-1}) = d_1 \cdots d_{i-1}$, so

$$d_i = \frac{\det(A_i)}{\det(A_{i-1})}.$$

□

Problem 6.2 (Determinants in multivariable calculus)**.** For a multivariable function $f(x_1, \dots, x_n)$, the derivative with respect to x_i (keeping all other variables constant) is the partial derivative $\frac{\partial f}{\partial x_i}$. For example, if $f(x, y) = x^2 y$, then $\frac{\partial f}{\partial x} = 2xy$ and $\frac{\partial f}{\partial y} = x^2$.

In $\mathbb{R}^2$ consider Cartesian coordinates (x, y) and polar coordinates (r, θ), where $r \geqslant 0$ is the distance to the origin and $\theta \in [0, 2\pi)$ is the counterclockwise angle from the positive x-axis. They are related by

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Compute the determinants of

$$J_1 = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{bmatrix}, \qquad J_2 = \begin{bmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{bmatrix},$$

expressing the answers as functions of r, θ. What is the relationship between J_1 and J_2?

Solution. From $x = r\cos\theta$ and $y = r\sin\theta$,

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta,$$

so

$$J_1 = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}, \qquad \det(J_1) = r(\cos^2\theta + \sin^2\theta) = r.$$

Since $r = (x^2 + y^2)^{1/2}$ and $\tan\theta = y/x$, we get

$$\frac{\partial r}{\partial x} = \frac{x}{(x^2+y^2)^{1/2}} = \cos\theta, \qquad \frac{\partial r}{\partial y} = \frac{y}{(x^2+y^2)^{1/2}} = \sin\theta,$$

and using $\theta = \arctan(y/x)$,

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r}, \qquad \frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}.$$

Hence

$$J_2 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\dfrac{\sin\theta}{r} & \dfrac{\cos\theta}{r} \end{bmatrix}, \qquad \det(J_2) = \frac{1}{r}.$$

A direct multiplication shows $J_1 J_2 = I_2$, so $J_2 = J_1^{-1}$. Equivalently, by the chain rule the entries satisfy

$$\begin{cases} \cos\theta \dfrac{\partial r}{\partial x} - r\sin\theta \dfrac{\partial \theta}{\partial x} = 1, \\ \cos\theta \dfrac{\partial r}{\partial y} - r\sin\theta \dfrac{\partial \theta}{\partial y} = 0, \\ \sin\theta \dfrac{\partial r}{\partial x} + r\cos\theta \dfrac{\partial \theta}{\partial x} = 0, \\ \sin\theta \dfrac{\partial r}{\partial y} + r\cos\theta \dfrac{\partial \theta}{\partial y} = 1, \end{cases}$$

which compactly encode $J_1 J_2 = I_2$. □

Problem 6.3. Let $f(a, b, c, d) = \ln(ad - bc)$.

(1) Compute the partial derivatives $\frac{\partial f}{\partial a}$, $\frac{\partial f}{\partial b}$, $\frac{\partial f}{\partial c}$, $\frac{\partial f}{\partial d}$.

(2) Prove

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{\partial f}{\partial a} & \dfrac{\partial f}{\partial b} \\ \dfrac{\partial f}{\partial c} & \dfrac{\partial f}{\partial d} \end{bmatrix}^{\top}.$$

(3) Is there an analogous statement for 3×3 matrices?

Solution. (1) Using $f = \ln(ad - bc)$,

$$\frac{\partial f}{\partial a} = \frac{d}{ad - bc}, \qquad \frac{\partial f}{\partial b} = -\frac{c}{ad - bc}, \qquad \frac{\partial f}{\partial c} = -\frac{b}{ad - bc}, \qquad \frac{\partial f}{\partial d} = \frac{a}{ad - bc}.$$

(2) Substitute these into the right-hand side and multiply:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \dfrac{\partial f}{\partial a} & \dfrac{\partial f}{\partial b} \\ \dfrac{\partial f}{\partial c} & \dfrac{\partial f}{\partial d} \end{bmatrix}^{\top} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = I_2,$$

hence

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial a} & \dfrac{\partial f}{\partial b} \\ \dfrac{\partial f}{\partial c} & \dfrac{\partial f}{\partial d} \end{bmatrix}^{\top}.$$

(3) Yes. In general, for an $n \times n$ matrix $A = [x_{ij}]$ and

$$f(x_{ij}) = \ln\det(A),$$

the matrix of partial derivatives satisfies

$$\left[\frac{\partial f}{\partial x_{ji}}\right] = (A^{-1}) \quad \text{equivalently} \quad \left[\frac{\partial f}{\partial x_{ij}}\right] = (A^{-1})^{\top}.$$

Thus $A\,[\partial f/\partial x_{ji}] = [\partial f/\partial x_{ji}]\,A = I_n$. For 3×3 (and any n), the identity follows from the expansion of $\det(A)$ and the Laplace cofactor formula, or from the matrix differential $\mathrm{d}\ln\det A = \operatorname{tr}(A^{-1}\mathrm{d}A)$. □

Problem 6.4. Compute the determinant

$$D_n(\lambda; a_1, \dots, a_n) = \begin{vmatrix} \lambda & & & & a_n \\ -1 & \lambda & & & a_{n-1} \\ & \ddots & \ddots & & \vdots \\ & & -1 & \lambda & a_2 \\ & & & -1 & \lambda + a_1 \end{vmatrix}.$$

Solution. For $n = 1, 2, 3$ one checks

$$D_1 = \lambda + a_1, \quad D_2 = \lambda^2 + a_1\lambda + a_2, \quad D_3 = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3.$$

We claim for all $n \geqslant 1$,

$$(*) \qquad D_n = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n.$$

Assume $(*)$ holds for size n. For size $n+1$, expand the determinant along the first row:

$$D_{n+1} = \lambda\, D_n + (-1)^{1+(n+1)}\, a_{n+1} \begin{vmatrix} -1 & \lambda & & \\ & \ddots & \ddots & \\ & & -1 & \lambda \\ & & & -1 \end{vmatrix}.$$

The last minor is upper triangular with diagonal entries all -1, hence its determinant equals $(-1)^n$. Therefore

$$D_{n+1} = \lambda D_n + (-1)^{n+2} a_{n+1} (-1)^n = \lambda D_n + a_{n+1}.$$

Using the induction hypothesis,

$$D_{n+1} = \lambda\big(\lambda^n + a_1\lambda^{n-1} + \cdots + a_n\big) + a_{n+1} = \lambda^{n+1} + a_1\lambda^n + \cdots + a_n\lambda + a_{n+1}.$$

Thus $(*)$ holds for $n+1$, and by induction for all n. □

Problem 6.5. Compute

$$f_n(\lambda) := \begin{vmatrix} \lambda & 1 & & & \\ n & \lambda & 2 & & \\ & \ddots & \ddots & \ddots & \\ & & 2 & \lambda & n \\ & & & 1 & \lambda \end{vmatrix}.$$

Solution. Define $f_n(\lambda)$ as above. For $1 \leqslant i \leqslant n-1$, replace row i by the sum of rows $i, i+1, \ldots, n$. This does not change the determinant, and yields a first row whose entries are all $\lambda + n$:

$$\begin{vmatrix} \lambda + n & \lambda + n & \cdots & \lambda + n \\ n & \lambda + n - 1 & \cdots & \lambda + n \\ \vdots & \vdots & \ddots & \vdots \\ 2 & \lambda + 1 & \cdots & \lambda + n \\ 1 & \lambda & \cdots & \lambda + n \end{vmatrix}.$$

For $2 \leqslant j \leqslant n$, replace column j by column j minus column $j-1$. Again the determinant is unchanged, and we get

$$\begin{vmatrix} \lambda + n & 0 & \cdots & 0 \\ n & \lambda - 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & & & \lambda - 1 \\ 1 & & & \end{vmatrix}.$$

Expanding along the first row gives the recurrence

$$f_n(\lambda) = (\lambda + n)\, f_{n-1}(\lambda - 1).$$

In the base cases of induction, we check that

$$f_1(\lambda) = \lambda, \quad f_2(\lambda) = (\lambda + 2)(\lambda - 1).$$

Then, by induction,

$$f_n(\lambda) = \left(\prod_{k=0}^{n-2} (\lambda + n - 2k)\right)\big(\lambda - (n-1)\big).$$

Equivalently,

$$f_n(\lambda) = \begin{cases} \prod_{i=1}^{m}(\lambda + 2i) \prod_{i=1}^{m-1}(\lambda - 2i)\,(\lambda - (2m-1)), & n = 2m, \\ \prod_{i=0}^{m}(\lambda + (2i+1)) \prod_{i=1}^{m}(\lambda - 2i), & n = 2m+1. \end{cases}$$

$\square$

Tutorial Lecture 7

Problem 7.1. Given

$$A_n = \begin{bmatrix} 2 & 1 & & & \\ -1 & 2 & & & \\ & \ddots & \ddots & & \\ & & -1 & 2 & \\ & & & -1 & 2 \end{bmatrix}_{n\times n}, \qquad B_n = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{n\times n}.$$

(1) Use Laplace expansion to derive a recurrence in n for $\det(B_n)$, and compute $\det(B_n)$.
(2) Use the relation between $\det(A_n)$ and $\det(B_n)$ to compute $\det(A_n)$.

Solution. (1) For $n = 1, 2$ we have $\det(B_1) = 1$, $\det(B_2) = 1$. For $n \geqslant 3$, expand $\det(B_n)$ along the last column to get

$$\det(B_n) = -\det(B_{n-2}) + 2\det(B_{n-1}).$$

Equivalently,

$$\det(B_n) - \det(B_{n-1}) = \det(B_{n-1}) - \det(B_{n-2}),$$

so $\{\det(B_n)\}$ is an arithmetic progression. With the initial values,

$$\det(B_n) = 1 \quad \text{for all } n \geqslant 1.$$

(Alternatively, add row 1 to row 2 and then expand along column 1 to obtain $\det(B_n) = \det(B_{n-1})$, which again yields $\det(B_n) = 1$.)

(2) The first rows satisfy

$$(2,\ 1,\ 0, \dots, 0) = (1,\ -1,\ 0, \dots, 0) + (1,\ 0,\ 0, \dots, 0).$$

By linearity of the determinant in the first row,

$$\det(A_n) = \det(B_n) + \det\begin{bmatrix} 1 & 0 & \cdots & 0 \\ * & & & \\ \vdots & & A_{n-1} & \\ * & & & \end{bmatrix} = \det(B_n) + \det(A_{n-1}) = 1 + \det(A_{n-1}).$$

Since $\det(A_1) = 2$, the recurrence gives

$$\det(A_n) = n + 1 \quad (n \geqslant 1).$$

$\square$

Problem 7.2 (Fibonacci sequences in determinants)**.** A matrix is called upper (resp. lower) Hessenberg if it differs from an upper (resp. lower) triangular matrix by allowing one more nonzero subdiagonal (resp. superdiagonal). For example,

$$H_4 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

is an upper Hessenberg matrix.

(1) Let H_n be the $n \times n$ upper Hessenberg matrix whose diagonal entries are 2 and all other nonzero entries are 1. Prove

$$\det(H_{n+2}) = \det(H_{n+1}) + \det(H_n),$$

so these determinants form a Fibonacci sequence.

(2) Let S_n be the $n \times n$ tridiagonal matrix with main diagonal 3 and both off-diagonals 1, e.g.

$$S_4 = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Find a recurrence for $S_n \coloneqq \det(S_n)$ and relate it to the Fibonacci numbers.

(3) In the full cofactor expansion of the determinant of an $n \times n$ tridiagonal matrix, at most t_n terms can be nonzero. Find a recurrence for t_n.

Solution. (1) Use linearity in the first row of H_{n+2}. Rewriting the first row $[\,2, 1, 1, \dots, 1\,]$ as $[\,1, 1, 1, \dots, 1\,]+[\,1, 0, 0, \dots, 0\,]$, we get

$$\det(H_{n+2}) = \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ 0 & 1 & \ddots & 1 \\ \vdots & \vdots & \ddots & 2 \end{bmatrix} + \det \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 2 & \cdots & 1 \\ 0 & 1 & \ddots & 1 \\ \vdots & \vdots & \ddots & 2 \end{bmatrix}.$$

Expanding the second determinant along the first row gives $\det(H_{n+1})$. For the first determinant, replace the second row by (second row)$-$(first row); this creates a 1 in entry $(2, 1)$ and zeros in the rest of row 2 except a 1 on the diagonal, after which a cofactor expansion reduces it to $\det(H_n)$. Hence $\det(H_{n+2}) = \det(H_{n+1}) + \det(H_n)$. Since $\det(H_1) = 2$ and $\det(H_2) = 3$, these determinants form a Fibonacci sequence.

(2) Expanding $\det(S_{n+2})$ along the first row yields

$$S_{n+2} = 3S_{n+1} - S_n, \qquad S_1 = 3,\ S_2 = 8,\ S_3 = 21.$$

Let $\{F_k\}$ be the Fibonacci numbers with $F_0 = F_1 = 1$ and $F_{k+2} = F_{k+1} + F_k$. The even-index subsequence satisfies $F_{k+4} = 3F_{k+2} - F_k$ (indeed, $F_{k+4} = F_{k+3} + F_{k+2} = (F_{k+2} + F_{k+1}) + F_{k+2} = 3F_{k+2} - F_k$). Therefore $S_n = F_{2n+2}$, which matches $S_1 = F_4 = 3$, $S_2 = F_6 = 8$, $S_3 = F_8 = 21$.

(3) Let t_n be the maximum number of nonzero terms in the full cofactor expansion of an $n \times n$ tridiagonal determinant. Clearly $t_1 = 1$, $t_2 = 2$, $t_3 = 3$. Expanding along the first row, the only possibly nonzero cofactors arise from the first two columns and lead to tridiagonal submatrices of sizes $n - 1$ and $n - 2$. Hence

$$t_{n+2} = t_{n+1} + t_n \qquad (n \geqslant 1).$$

This is the Fibonacci recursion relation (in closed form). □

Problem 7.3. Let $A = [a_{ij}]$ be an $n \times n$ matrix that is (row) strictly diagonally dominant and whose diagonal entries are all positive. Prove $\det(A) > 0$.

Solution. We argue by induction on n. The case $n = 1$ is trivial. Assume the claim holds for all sizes $\leqslant n$ and consider an $(n + 1) \times (n + 1)$ strictly diagonally dominant matrix $A = [a_{ij}]$ with $a_{11} > 0$.

Perform elementary column operations

$$C_j \longleftarrow C_j - \frac{a_{1j}}{a_{11}}\, C_1 \qquad (j \geqslant 2),$$

which do not change the determinant. The resulting matrix has the block form

$$\begin{bmatrix} a_{11} & 0 \\ * & B \end{bmatrix}, \qquad B = [b_{ij}]_{2\leqslant i,j\leqslant n+1}, \quad b_{ij} = a_{ij} - \frac{a_{i1}a_{1j}}{a_{11}}.$$

Hence $\det(A) = a_{11} \det(B)$, so it suffices to show B is strictly diagonally dominant with positive diagonal.

For $i \geqslant 2$,

$$b_{ii} = a_{ii} - \frac{a_{i1}a_{1i}}{a_{11}} = \frac{a_{ii}a_{11} - a_{i1}a_{1i}}{a_{11}} > \frac{a_{ii}a_{11} - |a_{i1}||a_{1i}|}{a_{11}} > 0,$$

since $a_{ii}, a_{11} > 0$ and $|a_{i1}| < a_{ii}$, $|a_{1i}| < a_{11}$. Moreover, using the triangle inequality,

$$\sum_{\substack{2\leqslant j\leqslant n+1 \\ j\neq i}} |b_{ij}| \leqslant \sum_{\substack{2\leqslant j\leqslant n+1 \\ j\neq i}} |a_{ij}| + \frac{|a_{i1}|}{a_{11}} \sum_{\substack{2\leqslant j\leqslant n+1 \\ j\neq i}} |a_{1j}|.$$

Strict diagonal dominance of rows i and 1 gives

$$\sum_{\substack{2\leqslant j\leqslant n+1 \\ j\neq i}} |a_{ij}| < a_{ii} - |a_{i1}|, \qquad \sum_{\substack{2\leqslant j\leqslant n+1 \\ j\neq i}} |a_{1j}| < a_{11} - |a_{1i}|.$$

Therefore

$$\sum_{\substack{2\leqslant j\leqslant n+1\\ j\neq i}} |b_{ij}| < a_{ii} - |a_{i1}| + \frac{|a_{i1}|}{a_{11}}\left(a_{11} - |a_{1i}|\right) = a_{ii} - \frac{|a_{i1}||a_{1i}|}{a_{11}} \leqslant a_{ii} - \frac{a_{i1}a_{1i}}{a_{11}} = b_{ii}.$$

Hence B is strictly diagonally dominant with positive diagonal. By the induction hypothesis, $\det(B) > 0$, and thus

$$\det(A) = a_{11}\det(B) > 0.$$

This completes the proof. □

Alternative Solution. Let $f(t) = \det(tI_n + A)$ for $t \geqslant 0$. Then f is a degree-n polynomial with leading coefficient 1, so f is continuous and $f(t) \to +\infty$ as $t \to +\infty$. For every $t \geqslant 0$, the matrix $tI_n + A$ is still strictly diagonally dominant with positive diagonal, hence $\det(tI_n + A) \neq 0$. Thus f has no zeros on $[0, +\infty)$. Since f is continuous and $f(t) \to +\infty$, we have $f(t) > 0$ for all $t \geqslant 0$, in particular

$$\det(A) = f(0) > 0.$$

This also gives the desired positivity. □

Problem 7.4. Let A, B be $n \times n$ matrices with $AB = BA$.

(1) Show that
$$\left|\det\begin{bmatrix} A & B\\ -B & A\end{bmatrix}\right| = \left|\det(A^2+B^2)\right|.$$

(2) Further, show that
$$\det\begin{bmatrix} A & B\\ -B & A\end{bmatrix} = \det(A^2+B^2).$$

Solution. (1) For any A, B,

$$\begin{bmatrix} A & B\\ -B & A\end{bmatrix}\begin{bmatrix} A & -B\\ B & A\end{bmatrix} = \begin{bmatrix} A^2+B^2 & -AB+BA\\ -BA+AB & A^2+B^2\end{bmatrix}.$$

If $AB = BA$, this equals $\operatorname{diag}(A^2+B^2,\ A^2+B^2)$. Taking determinants,

$$\det(A^2+B^2)^2 = \det\begin{bmatrix} A & B\\ -B & A\end{bmatrix}\det\begin{bmatrix} A & -B\\ B & A\end{bmatrix}.$$

Let $J = \left[\begin{smallmatrix} 0 & I\\ I & 0\end{smallmatrix}\right]$. Then $\left[\begin{smallmatrix} A & -B\\ B & A\end{smallmatrix}\right] = J\left[\begin{smallmatrix} A & B\\ -B & A\end{smallmatrix}\right]J$, hence the two determinants are equal. Therefore

$$\left|\det\begin{bmatrix} A & B\\ -B & A\end{bmatrix}\right| = \left|\det(A^2+B^2)\right|.$$

(2) First assume A is invertible. Using the Schur complement,

$$\det\begin{bmatrix} A & B\\ -B & A\end{bmatrix} = \det(A)\det\big(A-(-B)A^{-1}B\big) = \det(A)\det\big(A+BA^{-1}B\big).$$

Since $AB = BA$, B commutes with A^{-1}, so $A + BA^{-1}B = A + A^{-1}B^2 = A^{-1}(A^2+B^2)$. Hence

$$\det\begin{bmatrix} A & B\\ -B & A\end{bmatrix} = \det(A)\det(A^{-1})\det(A^2+B^2) = \det(A^2+B^2).$$

If A is not invertible, set $A_t = A + tI_n$ and define

$$F(t) = \det\begin{bmatrix} A_t & B\\ -B & A_t\end{bmatrix}, \qquad G(t) = \det(A_t^2+B^2).$$

Both are polynomials in t of degree $2n$ with leading coefficient 1. For all large t, A_t is invertible, so by the previous step $F(t) = G(t)$ for infinitely many t. Hence $F \equiv G$ as polynomials, and at $t = 0$,

$$\det\begin{bmatrix} A & B\\ -B & A\end{bmatrix} = \det(A^2+B^2).$$

□

Problem 7.5 (Sylvester equation)**.** Let $A_1 \in \mathrm{M}_m(\mathbb{R})$, $A_2 \in \mathrm{M}_n(\mathbb{R})$ be upper triangular, and $B \in \mathrm{M}_{m\times n}(\mathbb{R})$. Assume A_1 and A_2 have no common eigenvalues. Show that the matrix equation

$$A_1X - XA_2 = B$$

has a unique solution $X \in \mathrm{M}_{m\times n}(\mathbb{R})$.

Proof. We first show the uniqueness. Write $A_2 = [a_{ij}]$ (upper triangular), so its eigenvalues are $a_{11}, \dots, a_{nn}$, none of which is an eigenvalue of A_1 by hypothesis. Let $X = [x_1 \ \cdots \ x_n]$ with $x_j \in \mathbb{R}^m$. From $A_1X = XA_2$ we obtain the column relations

$$A_1x_1 = a_{11}x_1, \quad A_1x_2 = a_{12}x_1 + a_{22}x_2, \ \dots, \ A_1x_n = a_{1n}x_1 + \cdots + a_{nn}x_n.$$

Since a_{11} is not an eigenvalue of A_1, $A_1 - a_{11}I_m$ is invertible, hence $x_1 = 0$. Inductively, using that a_{kk} is not an eigenvalue of A_1, we get $x_k = 0$ for all k, so the homogeneous equation $A_1X - XA_2 = 0$ has only the zero solution. Therefore the solution to $A_1X - XA_2 = B$ is unique if it exists.

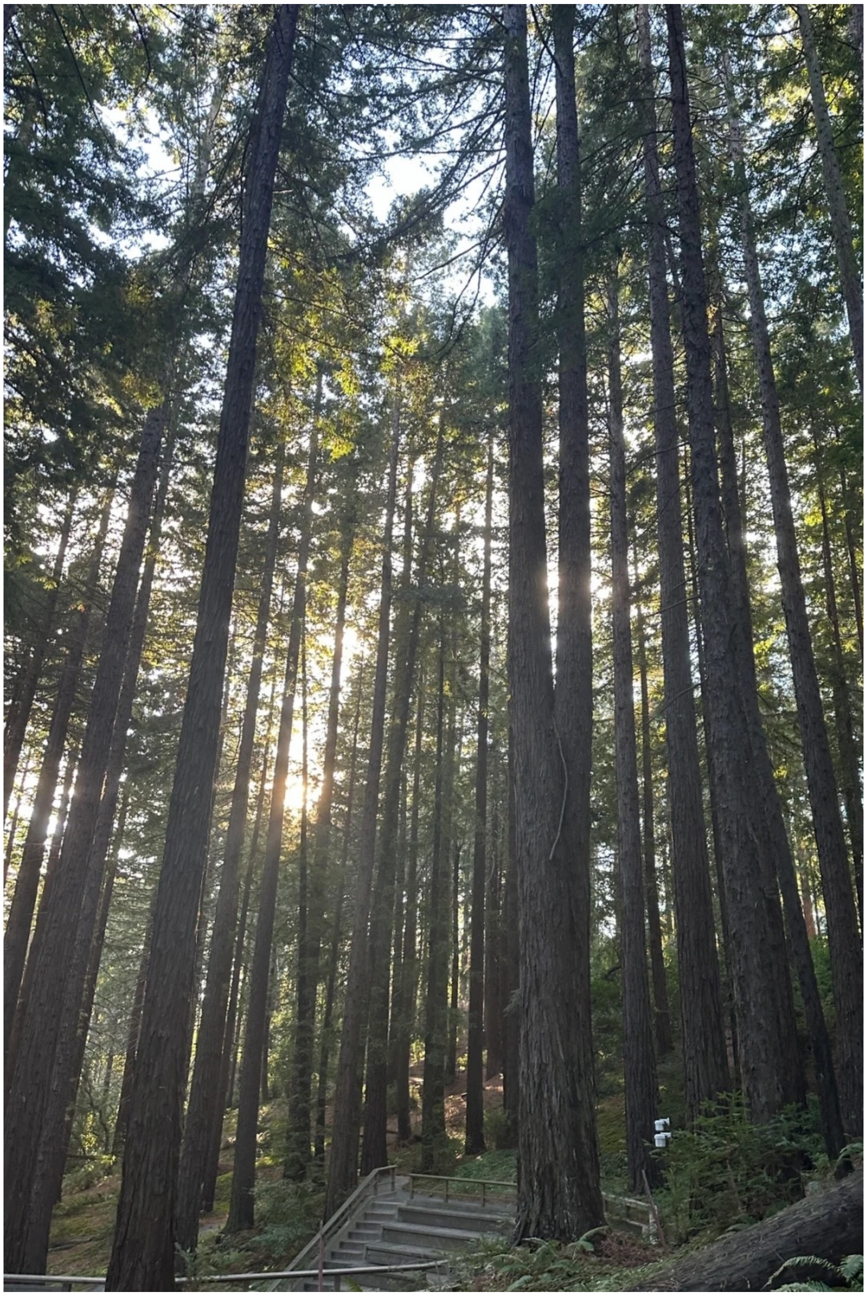
Now it remains to show the existence. Identify $M_{m\times n}(\mathbb{R})$ with $\mathbb{R}^{mn}$ via

$$\varphi\colon \mathrm{M}_{m\times n}(\mathbb{R}) \longrightarrow \mathbb{R}^{mn}, \qquad X = [x_1 \ \cdots \ x_n] \longmapsto \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Define the linear map

$$T\colon \mathrm{M}_{m\times n}(\mathbb{R}) \to \mathrm{M}_{m\times n}(\mathbb{R}), \qquad T(X) = A_1X - XA_2,$$

and $f = \varphi \circ T \circ \varphi^{-1}$ on $\mathbb{R}^{mn}$. From the uniqueness step, $\ker T = \{0\}$, hence T is injective. Since domain and codomain have the same finite dimension, T is bijective. Thus for every B there exists X with $T(X) = B$, i.e. $A_1X - XA_2 = B$. $\square$



PHOTOGRAPH — DECEMBER 17, 2023; AT UC BOTANICAL GARDEN AT BERKELEY, CA. *The Mather Redwood Grove includes majestic coast redwoods surrounding a classic amphitheater, creating a magnificent natural cathedral that evokes a fantasy forest.*

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