

Triangulated and Derived Categories in Algebra and Geometry

Lecture 2

1. Functors

\mathcal{C}, \mathcal{D} - categories

Covariant

Contravariant

$$F: \mathcal{C} \rightarrow \mathcal{D}$$

$$\mathcal{C} \ni x \mapsto F(x) \in \mathcal{D}$$

$$f: x \rightarrow y \text{ in } \mathcal{C}$$

$$F(x) \rightarrow F(y)$$

$$F(y) \rightarrow F(x)$$

subject:

- $F(id_x) = id_{F(x)} \quad \forall x \in \mathcal{C}$

- respects composition

Warning Contravariant = covariant $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$.

$\forall X, Y \in \mathcal{C}$ a functor gives a map

$$\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$$

- Types - F is faithful if it's injective on Hom's
- F is full if it's surjective on Hom's
- F is essentially surjective if $\forall Y \in \mathcal{D}$
 $\exists X \in \mathcal{C}$ s.t. $Y \cong F(X)$.

- Examples
- Subcategory \hookrightarrow embedding functor.
Always faithful, full \Leftrightarrow the subcategory is full
 - Forgetful functors

$$\text{Grp} \xrightarrow{\text{For}} \text{Sets} \quad G \mapsto G \text{ as a set}$$

Since hom's of groups are maps of sets with...

- $\text{Vect}_k \xrightarrow{\oplus} \text{Vect}_k$ w - fixed vect. space
 $V \longmapsto V \otimes_k w$

- Assume that $\forall x, y \in \mathcal{C}$ their product exists.
Ex: Construct a product functor

$$\begin{aligned} \mathcal{C} \times \mathcal{C} &\longrightarrow \mathcal{C} \\ (x, y) &\longmapsto x \times y \end{aligned}$$

- homology groups are functors

$$\text{Top} \longrightarrow \text{Ab}$$

Def A presheaf on \mathcal{C} with values in \mathcal{A} is
 a functor $\mathcal{C}^{\text{op}} \rightarrow \mathcal{A}$.

Notation $\text{PSh}(\mathcal{C})$ - presheaves of sets on \mathcal{C} .

To every partially ordered set $P \rightsquigarrow$ a category:

- objects are P
- $\text{Hom}_P(x, y) = \begin{cases} \{\}, & x \leq y \\ \emptyset, & \text{otherwise.} \end{cases}$

$x \rightarrow y$ if $x \leq y$

To any topological space $X \rightsquigarrow \text{Op}(X)$ - partially ordered set of $\{U \subset X\}$, U -open. $U \subseteq W \Leftrightarrow U \subseteq W$.

Then a presheaf on X is a presheaf on $\text{Op}(X)$.

Most important examples of functors:

Given $X \in \mathcal{C}$ $h^X: \mathcal{C} \rightarrow \text{Sets}$ $Y \mapsto \text{Hom}_P(X, Y)$

$h_X: \mathcal{C}^{\text{op}} \rightarrow \text{Sets}$ $Y \mapsto \text{Hom}_P(Y, X)$

a presheaf on \mathcal{C}

2. Morphisms of functors = natural transformations

also known as
natural transformation

Def $F, G: \mathcal{A} \rightarrow \mathcal{B}$ be functors. A morphism $\eta: F \rightarrow G$ is a collection $\eta_x: F(x) \rightarrow G(x)$ s.t. $\forall f: X \rightarrow Y$

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ F(f) \downarrow & \curvearrowright & \downarrow G(f) \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array} \quad \text{commutes.}$$

Rank The identity maps $F(x) \xrightarrow{id_{F(x)}} F(x)$ form a nat'l transformation $F \rightarrow F$. Morphisms of functors compose.

We get a category $\text{Fun}(\mathcal{A}, \mathcal{B})$ of functors.

Objects: $F: \mathcal{A} \rightarrow \mathcal{B}$.

Morphisms: nat'l transformations.

Set-theoretic issues.

Can now talk about isomorphisms of functors.

Warning Isomorphism of cat's is a useless notion.

Def \mathcal{C} and \mathcal{D} are called equivalent if $\exists F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ s.t. $G \circ F \xrightarrow{\sim} \text{Id}_{\mathcal{C}}$, $F \circ G \xrightarrow{\sim} \text{Id}_{\mathcal{D}}$.

Examples • $\text{Sets}^f \cong$ full subcategory formed by
 $\emptyset, \{1\}, \{1, 2\}, \dots$

- $\text{Vect}_k \cong$ category with objects All ,
 $\text{Map}(n, m) = \text{Mat}_{n \times m}(k)$.
- Affine schemes $^{op} \cong$ Comm rings

Observation The rule $\mathcal{C} \ni X \mapsto h_X \in \text{Fun}(\mathcal{C}^{op}, \text{Sets}) = \text{PSh}(\mathcal{C})$
is a functor!

Exc Check if.

What can one say about this functor?

Most of the times not essentially surjective.

Def If $F \in \text{Psh}(\mathcal{C})$ and $F \cong h_x$ for some $x \in \mathcal{C}$, then F is called representable.

But $\mathcal{C} \rightarrow \text{Psh}(\mathcal{C})$ is fully faithful. Follows immediately from the following.

Prop (Yoneda Lemma)

$\forall x \in \mathcal{C}, \forall F: \mathcal{C}^{\text{op}} \rightarrow \text{Sets}$ there is a natural bijection

$$\text{Hom}_{\text{Psh}(\mathcal{C})}(h_x, F) \xleftrightarrow{1:1} F(x).$$

More precisely, there is an isomorphism of functors

$$\text{Hom}_{\text{Psh}(\mathcal{C})}(h_-, -), -(-): \mathcal{C}^{\text{op}} \times \text{Psh}(\mathcal{C}) \rightarrow \text{Sets}.$$

In diagrams:

$$V \xrightarrow{\eta} F \rightarrow G$$

$$V \xrightarrow{f} X \rightarrow Y$$

$$\begin{array}{ccc} \text{Hom}(h_x, F) & \xrightarrow{\sim} & F(x) \\ \downarrow & \curvearrowleft & \downarrow \eta_x \\ \text{Hom}(h_x, G) & \xrightarrow{\sim} & G(x) \end{array}$$

$$\begin{array}{ccc} \text{Hom}(h_x, F) & \xrightarrow{\sim} & F(x) \\ \uparrow & \curvearrowright & \uparrow F(f) \\ \text{Hom}(h_y, F) & \xrightarrow{\sim} & F(y) \end{array}$$

Proof Need a map

$$\begin{array}{ccc} \text{Hom}_{\text{PSh}(e)}(h_x, F) & \longrightarrow & F(x) \\ \Downarrow & & \Downarrow \\ \eta_x & \longmapsto & \eta_x(\text{id}_x) \end{array}$$

$$\eta_y : \text{Hom}_e(Y, X) \rightarrow F(y)$$

$$\begin{array}{ccc} \eta_x : \text{Hom}_e(X, X) & \longrightarrow & F(x) \\ \Downarrow & & \Downarrow \\ \text{id}_x & \longmapsto & \eta_x(\text{id}_x) \end{array}$$

Exc Construct a map in the other direction and show that they are inverse to each other. 15

Cor \mathcal{C} embeds fully faithfully as a subcategory in $\text{PSh}(\mathcal{C})$.

Def $F \in \text{PSh}(\mathcal{C})$ is representable if $F \cong h_x$ for some $x \in \mathcal{C}$.

3. Limits & colimits

Def A diagram indexed by a category I in \mathcal{C} is just a functor $F: I \rightarrow \mathcal{C}$. index category

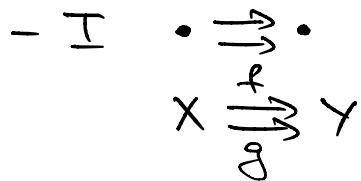
Examples - Take $I = \cdots$

Diagram = pair of objects $X, Y \in \mathcal{C}$.

- $I = \bullet \rightarrow \bullet \rightsquigarrow X \rightarrow Y$



- I - opposite category to N^{op} as a partially ordered set
- I - opposite category to N
 - $\dots \rightarrow X_i \xrightarrow{f_i} \dots \rightarrow X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$
 - condition



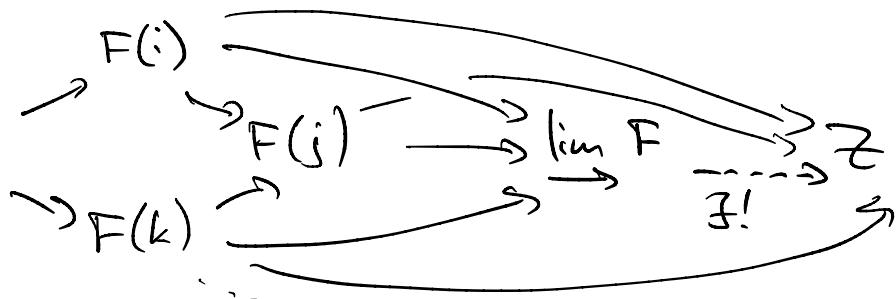
Def A limit of $F: I \rightarrow \mathcal{C}$ is an object $\lim_{\leftarrow} F \in \mathcal{C}$ and a collection of $p_i: \lim_{\leftarrow} F \rightarrow F(i)$ for all $i \in I$.
 Satisfying: $\forall \varphi: i \rightarrow j$ in I

$$\begin{array}{ccc} \lim_{\leftarrow} F & \xrightarrow{\varphi_i} & F(i) \\ & \downarrow F(\varphi) & \\ & \xrightarrow{\varphi_j} & F(j) \end{array}$$

And it's universal: $\forall X, \{f_i: X \rightarrow F(i)\}$ s.t.

$$\forall i \xrightarrow{\varphi} j \quad X \begin{array}{c} \xrightarrow{f_i} F(i) \\ \downarrow G \quad \downarrow F(\varphi) \\ \xrightarrow{f_j} F(j) \end{array} \quad \exists! X \xrightarrow{\eta} \varprojlim F \text{ s.t. } f_i = \varphi_i \circ \eta.$$

A colimit:



Example - $I = \bullet \cdot \rightsquigarrow$ limit = product
colimit = coproduct

I - discrete \rightsquigarrow products indexed by I

- $I = \bullet \rightrightarrows \bullet \rightsquigarrow$ limit = equalizer
 colimit = coequalizer

- In algebra: I - category of $\mathbb{Z}_{\geq 0}$,
 $n \rightarrow m \Leftrightarrow n|m$.

$$\begin{array}{ccc} I & \xrightarrow{\quad} & \text{Ab} \\ \downarrow & & \downarrow \\ n & \longmapsto & \mathbb{Z}/n\mathbb{Z} \end{array} \quad \begin{array}{ccc} & & \mathbb{Z}/n\mathbb{Z} \hookrightarrow \mathbb{Z}/m\mathbb{Z} \\ & & \downarrow \\ i & \longmapsto & \frac{m}{n}i \end{array}$$

Exc $\varinjlim \mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Q}/\mathbb{Z} \simeq \mu \leftarrow$ complex roots
 of unity

- $\varprojlim \left(\dots \rightarrow \mathbb{Z}/p^3 \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p \right) = \mathbb{Z}_p$ p -adic numbers

Prop In Sets all limits and colimits exist.

Pf Via products and equalizers. \square

Prop Assume all products & equalizers exist in \mathcal{C} .
Then all limits exist in \mathcal{C} .

Pf Consider $F: I \rightarrow \mathcal{C}$. Construct two morphisms

$$\prod_{i \in I} F(i) \xrightarrow{\psi} \prod_{i:j \rightarrow j} F(j)$$

In order to construct $\rightarrow \prod$ enough (same thing)
as to construct morphisms into its elements.

$\forall f: i \rightarrow j$ need $\prod_{i \in I} F(i) \xrightarrow{i \in I} F(j)$.

$$\psi_f: \prod_{i \in I} F(i) \xrightarrow{P_j} F(j) \xrightarrow{id} F(j)$$

$$\psi_f: \prod_{i \in I} F(i) \xrightarrow{P_i} F(i) \xrightarrow{F(f)} F(j)$$

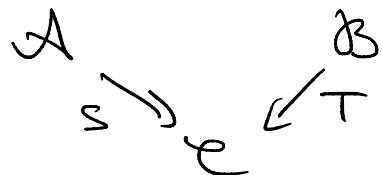
Finish the proof: from the equalizer $\rightsquigarrow \varprojlim F$. \square

Exc State and prove a similar statement for colimits.

Cor Sets has all products & equalizers \Rightarrow limits.
 — n — coproduct & coequalizers \Rightarrow colimits.

Alternative definition through comma categories

$\mathcal{A}, \mathcal{B}, \mathcal{C}$ — categories, $S: \mathcal{A} \rightarrow \mathcal{C} \leftarrow T: \mathcal{B}$



Def The comma category $(S \downarrow T)$ is the category whose objects are triples $A \in \mathcal{A}, B \in \mathcal{B}, f: S(A) \rightarrow T(B)$. Morphisms are $(\varphi, \chi): A \times B \rightarrow A' \times B'$ (in $\mathcal{A} \times \mathcal{B}$) such that

$$\begin{array}{ccc} S(A) & \xrightarrow{f} & T(B) \\ S(\varphi) \downarrow & & \downarrow T(\chi) \\ S(A') & \xrightarrow{f'} & T(B') \end{array}$$

Example Take $\mathcal{A} = \mathcal{A}$, $\mathcal{C} = \mathcal{A}$, $\mathcal{B} = \text{id}\&f$. $S = \text{Id}$, $T = A \in \mathcal{C}$.
 Get the slice category \mathcal{A}/A :
 objects are $A' \rightarrow A$, morphisms are

$$\begin{array}{ccc} A' & \xrightarrow{\quad S \quad} & A'' \\ & \downarrow & \downarrow \\ & A & \end{array}.$$

Given an object $X \in \mathcal{C}$, define the constant diagram
 $X : I \rightarrow \mathcal{C}$, sends all objects to X , all morphisms
 to the identity morphism.

Exc Check that you get a functor $\text{Const} : \mathcal{C} \rightarrow \text{Fun}(I, \mathcal{C})$.
 Consider the comma category of

$$\begin{array}{ccc} \mathcal{C} & & \text{id}\&f \\ & \searrow \text{Const} & \swarrow F \\ & \text{Fun}(I, \mathcal{C}) & \end{array}.$$

Exc Check that $\varprojlim F$ is the same as a terminal object in this comma category.

Exc Do the same for colimits.

Alternative definition via representable functors

For every $X \in \mathcal{C}$ consider the composition

$$\begin{array}{ccccc} I & \xrightarrow{F} & \mathcal{C} & \xrightarrow{h^X} & \text{Sets} \\ & \Downarrow & \Downarrow & & \Downarrow \\ i & \longrightarrow & F(i) & \longrightarrow & \text{Hom}(X, F(i)) \end{array}$$

Since all limits exist in Sets, we can compute (pick one)

$$\varprojlim (h^X \circ F) = \varprojlim \text{Hom}(X, F(i)).$$

Ex This limit, $\varprojlim (h^X \circ F)$ produces a functor
 $F^? : \mathcal{C}^{\text{op}} \rightarrow \text{Sets}$.

Ex: $\varprojlim F$ exists in \mathcal{C} $\Leftrightarrow F \in \text{PSh}(\mathcal{C})$ is representable.

4. Adjoint functors

Def $F: \mathcal{A} \rightleftarrows \mathcal{B}: G$ are called adjoint (F is left adjoint to G , G - right adjoint to F). If there exists a natural bijection of sets

$$\text{Hom}_{\mathcal{B}}(F(x), Y) \xrightarrow{\sim} \text{Hom}_{\mathcal{A}}(x, G(Y))$$

for all $x \in \mathcal{A}, Y \in \mathcal{B}$.

Both sides are functors $\mathcal{A}^{\text{op}} \times \mathcal{B} \rightarrow \text{Sets}$.
As functors must be isomorphic.

Examples 1) $V \in \text{Vect}^f-k$

$$-\otimes_k V: \text{Vect}^f-k \rightleftarrows \text{Vect}^f-k: \text{Hom}_k(V, -),$$

2) free objects

Free is left adjoint to For:

For: $\text{Grp} \rightarrow \text{Sets}$

$$\text{Hom}_{\text{Grp}}(\text{Free}(x), G) = \text{Maps}_{\text{Sets}}(x, G).$$

Same for Comm, Ab, ...

Notation $F \dashv G$.

Problem 1

Show that $F: A \rightarrow B$ has a right adjoint
($G: B \rightarrow A$ + an isom of bifunctors)
if and only if the functor

$$\text{Hom}_B(F(-), Y): A^{\text{op}} \rightarrow \text{Sets}$$

is representable $\forall Y \in B$.

Want a proof with all the details.