p-adic motives

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Fix a prime p.

(I) Recollection of motives

Fix a field $k \subset \mathbb{C}$ ($k = \mathbb{Q}$ or $\mathbb{C}$).

X alg var $/k$ (sm, proj).

$\rightsquigarrow H^*(X^{an}_{\mathbb{C}}, \mathbb{Z}) \in \{\text{f.g. ab grps}\}$ Betti / singular cohom

$H^*_{dR}(X^{an}_{\mathbb{C}}) \in \{\text{fil'd } \mathbb{C}\text{-v.s.}\}$.

$H^*_{et}(X_{\bar{k}}, \mathbb{Z}/\ell\mathbb{Z}) \in \{Gal(\bar{k}/k)\text{-reps}\}$

$\circlearrowleft$ $Gal(\bar{k}/k)$

This is usual sing cohom w/ $\mathbb{Z}/\ell\mathbb{Z}$-coeff if $k = \mathbb{C}$.

Grothendieck:

$$\left\{\begin{matrix}\text{alg var}\\ /k\end{matrix}\right\}^{op} \xrightarrow{X \longmapsto M(X)} \left\{\begin{matrix}\text{motives}\\ /k\end{matrix}\right\} \begin{matrix} \xrightarrow{T_B} \{\text{f.g. ab grps}\} \\ \xrightarrow{T_{dR}} \{\text{fil'd } \mathbb{C}\text{-v.s.}\} \\ \xrightarrow{T_{et}} \{Gal(\bar{k}/k)\text{-reps}\}, \end{matrix}$$

abelian cat
"linearized".

<ins>But</ins> universal $\triangleq$ too complicated to detect.

e.g. Need to know all cycles w/ intersections in $X \times_k Y$ for input X & Y.

(which is very hard to do so.)

<u>Deligne's idea</u> Hodge theory: when $k=\mathbb{C}$,

$$\begin{Bmatrix} \text{alg vars} \\ /\mathbb{C} \end{Bmatrix}^{op} \longrightarrow \begin{Bmatrix} \mathbb{Q}\text{-mixed} \\ \text{Hodge strs} \end{Bmatrix} \begin{matrix} \xrightarrow{T_B} \{\text{f.g. ab grps}\} \\ \xrightarrow[T_{dR}]{} \{\text{fil'd } \mathbb{C}\text{-v.s.}\}. \end{matrix}$$

<u>Goal</u> Explain variations of Deligne's picture in p-adic cohom theories on p-adic spaces.

<u>(I) Cohomology of algebraic varieties</u>

C = complete alg closed p-adic field ($C = \widehat{\overline{\mathbb{Q}}_p}$)

<u>Thm</u> X/C sm proper rigid analytic space of dim d.

(1) Finiteness: Each $H^i(X, \mathbb{F}_p)$ is of fin dim & $H^i(X, \mathbb{F}_p) \neq 0$ only when $i \in [0, 2d]$.

(Scholze 2012)

(2) Poincaré duality:

$$H^i(X, \mathbb{F}_p) \cong H^{2d-i}(X, \mathbb{F}_p)^\vee, \quad \forall i.$$

(Zavyalov 2021, Mann 2022 + Gabber's talk at IHES 2015).

(3) Hodge-Tate theory:

For each $n \geq 0$, $\exists$ a filtration on $H^n(X, \mathbb{Z}_p) \otimes C$ with $gr^\bullet = \bigoplus_{i+j=n} H^i(X, \Omega_X^j)(-j)$.

(Falting 1988, Scholze 2012, Bhatt-Morrow-Scholze 2016, Conrad-Gabber)

Rmk (1) HT fil'n varies analytically in families

$\rightsquigarrow$ new HT period maps.

(2) HT fil'n is "opposite" to Hodge fil'n,

i.e. usual Hodge theory:

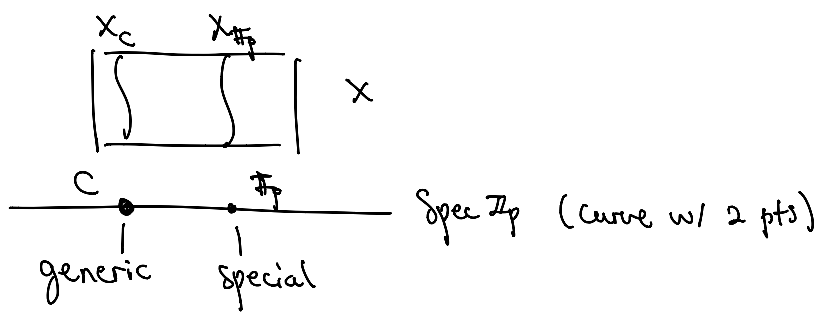
$$H^i(X, \mathbb{C}) \hookrightarrow H^i(X, \mathcal{O}_X)$$

HT theory: $\exists$ natural inj

$$H^i(X, \mathcal{O}_X) \hookrightarrow H^i(X, \mathbb{Z}_p) \otimes C$$

Variations $/ \mathbb{Z}_p$

$X / \mathbb{Z}_p$ sm proj var.



Thm (BMS 2016) $\dim_{\mathbb{F}_p} H^i(X, \mathbb{F}_p) \leq \dim_{\mathbb{F}_p} H^i_{dR}(X_{\mathbb{F}_p})$.

Application: Kisin - Farb - Wolfson

(essential dim via prismatic cohom).

Thm (Drinfeld 2020, Bhatt-Lurie 2022)

$\exists$ a natural "Sen operator" $\Theta \circlearrowright R\Gamma_{dR}(X_{\mathbb{F}_p})$

s.t. $gr_i^{conj}(\Theta) = -i \in \mathbb{F}_p$.

$\Rightarrow$ If $\dim X < p$, then Hodge-to-dR spec seq degenerates.

(Deligne - Illusie).

Rmk Think of Θ as "$\frac{d}{dp}$".

Architecture of F-gauges

To any p-adic formal sch S, can attach a cat $\mathrm{FGauge}(S)$,

So $\left\{\begin{array}{c}\text{sm proj} \\ \text{vars } / S\end{array}\right\} \longrightarrow \mathrm{FGauge}(S)$

$$\mathrm{FGauge}(S) \to \left\{\begin{array}{c}\mathbb{Z}_p\text{-local systems} \\ \text{on generic fibre } S[\tfrac{1}{p}]\end{array}\right\}$$

$$\mathrm{FGauge}(S) \to \left\{\begin{array}{c}F\text{-crystals on} \\ \text{special fibre}\end{array}\right\}$$

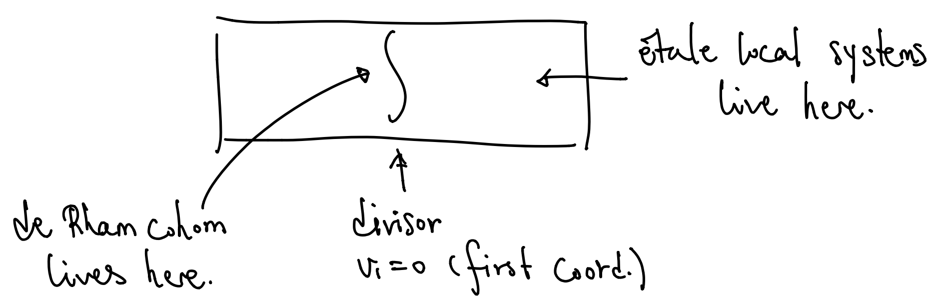
$$\mathrm{FGauge}(S) \xrightarrow{\text{"}T_{dR}\text{"}} \left\{\begin{array}{c}\text{fil'd "bdls" on } S \\ + \text{flat connection}\end{array}\right\}$$

In fact, $\exists$ a stack S^{Syn} s.t.

$$\mathrm{FGauge}(S) = \mathrm{Perf}(S^{Syn}).$$

Relevance to thms

For $S = \mathrm{Spf}\,\mathbb{Z}_p$, picture of $S^{Syn} \otimes \mathbb{F}_p$:



Commutative algebra

Thm (Bhatt 2020) Set $R = \mathbb{Z}[x_1, \cdots, x_n]$.

R^+ = integral closure of R in $\overline{\mathrm{Frac}(R)}$.

$\Rightarrow$ R^+ is p-adically Cohen-Macauley.

note $\Leftrightarrow$ $p = x_0, x_1, \cdots, x_n$ is a regular seq on R^+

$\Leftrightarrow$ $\forall i$, x_i is a nonzero divisor on $R^+/(x_0, \cdots, x_{i-1})$.

$\Leftrightarrow$ R^+/p is flat over R/p.

$\Leftrightarrow$ $H^{<n+1}_{(x_1,\dots,x_n)}(R^+) = 0$ (as local cohom).

<u>Rmk</u> (1) $n+1 = 2$ case can be done elementarily.

(2) Thm $\Rightarrow$ "Direct summand Conj" (André 2016)

<u>Thm</u> If $S \subset R^+$ is a f.g. R-alg, then $R \to S$ splits as R-mods.

(3) Global version of the thm ("Kodaira vanishing up to finite covers") has applications in birat'l geom (e.g. MMP in $\dim \leq 3$ / $\mathbb{Z}_p$ [BMPSTWW] [TY]).

<u>Relation to F-gauges</u>

$\mathrm{Perf}(R^{\mathrm{Syn}})$ perfect complexes

$$\begin{tikzcd}
\mathrm{Perf}(R^{\mathrm{Syn}}) \arrow[d, phantom, "\shortparallel"] & \\
\mathrm{FGauge}(R) \arrow[r] \arrow[d, "\text{pullback}"] & \left\{\begin{array}{c}\mathbb{Z}_p\text{-local systems} \\ \text{on } R[\frac{1}{p}]\end{array}\right\} \arrow[d, phantom, "\cap"] \\
\mathrm{QCoh}(X^{\Delta,\mathrm{perf}})^{\phi_* = 1} & \left\{\begin{array}{c}\text{constructible } \mathbb{Z}_p\text{-sheaves} \\ \text{on } R[\frac{1}{p}]\end{array}\right\} \arrow[l, "\mathrm{RH}"']
\end{tikzcd}$$

(Bhatt-Lurie)

<u>Fact</u> $\exists$ open substack carrying lifted Frob:

$$X^{\Delta} \subset X^{\mathrm{Syn}}$$
$$\circlearrowleft \phi$$

Application: Gal rep'ns

$K = \mathbb{Q}_p$ (fin ext'ns should be ok).

Q What is a "$\mathbb{Z}_p$-local system" on $\mathrm{Spf}\ \mathcal{O}_K$?
(need to figure out a correct def'n).

Obs $\mathbb{Z}_\ell$-local systems (à la Grothendieck) are good for $\ell \neq p$
but not for $\ell = p$.

Proposed ans $\mathbb{Z}_p$-local system / $\mathrm{Spf}\ \mathcal{O}_K$
$$\text{"} = \text{"}\ \mathrm{FGauge}(\mathcal{O}_K).$$

* Evidence for this to make sense

(i) With $\mathbb{Q}_p$-coeffs:

(Fontaine) $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_K) \subseteq \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \ni \rho : G_K \xrightarrow{\text{Cont}} GL_n(\mathbb{Q}_p)$

proposed correct def'n of $\mathbb{Q}_p$-loc sys on $\mathrm{Spf}\ \mathcal{O}_K$.

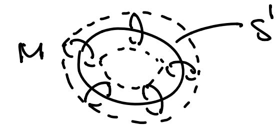
(loc sys on generic fibre)

gen pt — $\mathrm{Spf}\ \mathcal{O}_K$

Thm (Bhatt-Lurie)
$$\mathrm{FGauge}(\mathcal{O}_K)[\tfrac{1}{p}] = \mathrm{Perf}(\mathcal{O}_K^{\mathrm{Syn}})[\tfrac{1}{p}] \simeq D^b(\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_K)).$$

Rmk Thm $\Rightarrow$ integral ver of $H^*_f(G_K, -)$.

(2) Primes / knots analogy

Number theory	Topology
$\operatorname{Spec} \mathbb{F}_p$	Knot $S^1 \subset$ 3-mfld
$\operatorname{Spec} \mathcal{O}_K$	Solid nbhd M of this knot S^1 
$\operatorname{Spec} K$	boundary ∂M
$\operatorname{Rep}_{\mathbb{F}_p} G_K$	$\mathbb{F}_p$-local systems on ∂M.
local Tate duality	Poincaré duality

Topology (1) Poincaré duality $\Rightarrow$ $\underline{\operatorname{Loc}}_{\mathbb{F}_p}(\partial M)$ has a symplectic str

(2) $\underline{\operatorname{Loc}}_{\mathbb{F}_p}(M) \longrightarrow \underline{\operatorname{Loc}}_{\mathbb{F}_p}(\partial M)$ is a Lagrangian

Thm (Bhatt - Lurie) $\forall$ $E \in \operatorname{FGauge}(\mathcal{O}_K)$,

$\exists$ a natural exact triangle on Gal rep'ns

$$R\Gamma(\mathcal{O}_K^{syn}, E) \xrightarrow{\eta_E} R\Gamma(G_K, T(E))$$
$$\text{IS local Tate duality}$$
$$R\Gamma(G_K, T(E)^*(1))[2]^\vee \xrightarrow{\eta_{E^*}} R\Gamma(\mathcal{O}_K^{syn}, E^*(1))[2]^\vee$$

where $T(E)$ = étale realization of E.

Algebraic K-theory

Topology X space, E generalized cohom theory.

Thm $\exists$ a fil'n on $E^*(X)$ with gr^* controlled by $H^*(X)$.

Thm (CMM, BMS)

Say R is a p-adically complete ring.

Then $\exists$ a fil'n of $K_{\acute{e}t}(R, \mathbb{Z}_p)$ w/ assoc graded

(a slight modification of K-thy)

given by (prismatic) cohom of F-gauges / R.

Two applications

(1) Odd vanishing: $\pi_{odd} K(-, \mathbb{Z}_p)$ vanish

quasi-syntomic locally on qsyn rings

e.g. $\pi_{odd} K(\overline{\mathbb{Z}}_p / p^n) = 0$, $\forall n$.

(2) (Amazingly)

Even vanishing (AKN, 2022)

$\pi_{2k} K(\mathbb{Z}/p^n) = 0$, $\forall\ k \gg 0$.

Rmk Lurie conjectured that the entire picture above extends to all gen cohom theories.