An overview of prismatic theory.

§1 Introduction to prismatic theory

<u>Notations</u> $K/\mathbb{Q}_p$ fin ext'n. $\mathcal{O}_K$, ϖ, $k = \mathcal{O}_K/\varpi$.

X/K sm proj var.

$\rightsquigarrow$ · de Rham cohomology $R\Gamma_{dR}(X/K)$

· étale cohomology $R\Gamma_{et}(X, \mathbb{Q}_p)$.

If moreover X admits a smooth formal model $\mathfrak{X}/\mathrm{Spf}\,\mathcal{O}_K$, with $\mathfrak{X}_0$ special fiber, then

· crystalline cohomology $R\Gamma_{crys}(\mathfrak{X}_0/W(k))$ Witt ring $/k$.

<u>Natural Comparison thm</u>:

$$R\Gamma_{dR}(X/K) \otimes_K B_{dR} \simeq R\Gamma_{et}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR}.$$

$$R\Gamma_{crys}(\mathfrak{X}_0/W(k)) \otimes_{W(k)} B_{cris} \simeq R\Gamma_{et}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{cris}.$$

They are compatible w/ G_K, $\underset{\substack{\uparrow \\ dR}}{\mathrm{Fil}}$, $\underset{\substack{\uparrow \\ crys}}{\varphi}$.

$\rightsquigarrow$ HT comparison (by taking graded pieces)

$\forall n \geq 0$, $H^n_{et}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \simeq \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K}) \otimes_K C(-j)$

where $C := \widehat{\overline{K}}$.

<u>Question</u> Do we have a "universal" cohom theory $H^\cdot_\star$, s.t. $H^\cdot_{dR}$, $H^\cdot_{et}$, $H^\cdot_{cris}$ are all specializations of $H^\cdot_\star$?

<u>Answer</u> $H^\cdot_{\Delta}$ = prismatic cohom.

[BMS 18] A_{inf}-cohom, [BMS 19] Breuil-Kisin cohom.

[BS] Prismatic cohomology ($\Rightarrow$ BMS).

We assume $\mathfrak{X}/\mathcal{O}_K$ sm formal sch

$$\mathfrak{S} := W(k)[\![u]\!] \circlearrowleft (\varphi : u \mapsto u^p)$$

$$E(u) := \text{min poly of } \varpi \,/\, W(k).$$

$$A_{inf} := W(\mathcal{O}_C^\flat), \quad \mathcal{O}_C^\flat := \varprojlim_{\varphi} \mathcal{O}_C/p,$$

$$\varpi^\flat := (\varpi, \varpi^{1/p}, \varpi^{1/p^2}, \cdots),$$

Get $\iota : \mathfrak{S} \longrightarrow A_{inf}, \ u \longmapsto [\varpi^\flat]$.

Will see: $(\mathfrak{S}, (E))$, $(A_{inf}, (E))$ are both prisms in the sense of BS.

$(\underset{\circlearrowleft}{\phi} : A \to A$ ring homo lifting Frob: $A/p \to A/p)$.

Now, assume (A, I) is a (bounded) prism ← will explain later.

& $\mathfrak{X}/\bar{A} := A/I$ is a sm p-adic formal sch.

Thm (BS) $\exists$ a cohom theory

$$R\Gamma_{\Delta}(\mathfrak{X}/A) := R\Gamma((\mathfrak{X}/A)_{\Delta}, \mathcal{O}_{\Delta})$$

such that rel prism site, str sheaf.

(crys) (1) If $I = (p)$ then there is a ϕ-equiv quasi-isom

$$R\Gamma_{crys}(\mathfrak{X}/A) \simeq R\Gamma_{\Delta}(\mathfrak{X}/A) \hat{\otimes}^{\mathbb{L}}_{A,\phi_A} A$$

(dR) (2) $$R\Gamma_{dR}(\mathfrak{X}/\bar{A}) \simeq R\Gamma_{\Delta}(\mathfrak{X}/A) \hat{\otimes}^{\mathbb{L}}_{A,\phi_A} \bar{A}.$$

(étale) (3) If A is perfect, then $\forall n \geq 0$,

$\underset{\circlearrowleft}{\phi}$ isom $\quad R\Gamma_{et}(\mathfrak{X}_{\eta}, \mathbb{Z}/p^n\mathbb{Z}) \simeq \left((R\Gamma_{\Delta}(\mathfrak{X}/A)/p^n)[\tfrac{1}{I}]\right)^{\varphi=1}$

(4) If we have $(A, I) \to (B, J)$, then

$\exists$ canonical quasi-isom

$$R\Gamma_{\Delta}(\mathfrak{X}/A) \hat{\otimes}^{\mathbb{L}}_A B \simeq R\Gamma_{\Delta}(\mathfrak{X}_{\bar{B}}/B)$$

where $\mathfrak{X}_{\bar{B}} = \mathfrak{X} \times_{\mathrm{Spf}\,\bar{A}} \mathrm{Spf}\,\bar{B}$, $\bar{B} = B/J$.

(HT) (5) If $X = \mathrm{Spf}\, R$, then

$$H^i(R\Gamma_{\Delta}(\mathfrak{X}/A)\otimes^{\mathbb{L}}_A \bar{A}) \simeq \Omega^i_{R/\bar{A}}\{-i\}.$$

So when $\mathfrak{X}/\mathcal{O}_K$ smooth, can choose $(A, I) = (\mathfrak{S}, (E))$.

<u>Rmk</u> What will happen if we do not fix (A, I)?

$\Rightarrow$ Absolute prismatic theory

(after Drinfeld, BS, BL).

<u>Thm</u> (Wu, BS) Let $\mathfrak{X}$ be a good p-adic formal sch with the adic generic fiber X.

Then $\exists$ equivalence of cats

$$LS^{\text{ét}}_{\mathbb{Z}_p}(X) \simeq \mathrm{Vect}^{\varphi}((\mathfrak{X})_{\Delta}, \mathcal{O}_{\Delta}[\tfrac{1}{I}]^{\wedge}_p).$$

<u>Explainations</u> Absolute prism site $(\mathfrak{X})_{\Delta} \leftsquigarrow \mathfrak{X}$

$\varphi \circlearrowright \mathcal{O}_{\Delta}[\tfrac{1}{I}]^{\wedge}_p \leftsquigarrow \mathcal{O}_{\Delta}$

$\mathcal{M} \in \mathrm{Vect}^{\varphi}((\mathfrak{X})_{\Delta}, \mathcal{O}_{\Delta}[\tfrac{1}{I}]^{\wedge}_p)$.

if $\mathcal{M}$ is sheaf of locally finite free $\mathcal{O}_{\Delta}[\tfrac{1}{I}]^{\wedge}_p$ satisfying certain condition + $\varphi_{\mathcal{M}}: \mathcal{M} \xrightarrow{\sim} \mathcal{M}$.

For example, when $\mathfrak{X} = \mathrm{Spf}\, \mathcal{O}_K$,

$$LS^{\text{ét}}_{\mathbb{Z}_p}(X) = \mathrm{Rep}_{\mathbb{Z}_p}(G_K).$$

Moreover, if $\mathbb{L}$ is a $\mathbb{Z}_p$-local system corresponding to $\mathcal{M} \in \mathrm{Vect}^{\varphi}((\mathfrak{X})_{\Delta}, \mathcal{O}_{\Delta}[\tfrac{1}{I}]^{\wedge}_p)$,

then $R\Gamma_{\text{ét}}(X, \mathbb{L}) \simeq R\Gamma((\mathfrak{X})_{\Delta}, \mathcal{M})^{\varphi=1}$.

e.g. when $\mathfrak{X} = \mathrm{Spf}\, \mathcal{O}_K$, $R\Gamma(G_K, \mathbb{L}) = R\Gamma((\mathcal{O}_K)_{\Delta}, \mathcal{M})^{\varphi=1}$.

Thm (BS, Du-Liu) $\exists$ equiv of categories

$$\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Z}_p}(G_K) \simeq \mathrm{Vect}^{\varphi}((\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}).$$

More generally, let $\mathfrak{X}/\mathcal{O}_K$ be a sm formal scheme with X generic fiber,

then $\quad \mathrm{LS}^{\mathrm{crys,et}}_{\mathbb{Z}_p}(X) \simeq \mathrm{Vect}^{\varphi}((\mathfrak{X})_{\Delta}, \mathcal{O}_{\Delta})^{\mathrm{an}}$

[DLMS, GR].

Rmk Can consider logarithmic variant of prism theory (Koshikawa, KY) s.t. the above theorem also holds true for semi-stable reps.

Thm (BL) For any good p-adic formal scheme $\mathfrak{X}$, can canonically define a p-adic formal stack $\mathrm{WCart}_{\mathfrak{X}}$ ($=\mathfrak{X}^{\Delta}$)

s.t. (1) $\underbrace{\mathrm{Vect}((\mathfrak{X})_{\Delta}, \mathcal{O}_{\Delta})}_{\text{prism crystal:}} \simeq \underbrace{\mathrm{Vect}(\mathrm{WCart}_{\mathfrak{X}}, \mathcal{O}_{\mathrm{WCart}_{\mathfrak{X}}})}_{\text{cat of coh v.b.s on } \mathrm{WCart}_{\mathfrak{X}}.}$

$\mathcal{M}$ sheaf of $\mathcal{O}_{\Delta}$-mod

(2) for any prism crystal $\mathcal{M} \leftrightarrow L$ v.b.

$$R\Gamma((\mathfrak{X})_{\Delta}, \mathcal{M}) = R\Gamma(\mathrm{WCart}_{\mathfrak{X}}, L).$$

Moreover, $\mathrm{WCart} := \mathrm{WCart}_{\mathrm{Spf}\,\mathbb{Z}_p}$ then

$\forall$ bounded prism (A, I),

$\exists$ canonical map $\mathrm{Spf}\, A \xrightarrow{\rho_A} \mathrm{WCart}$

s.t. $\quad R\Gamma_{\Delta}(\mathfrak{X}_{\bar{A}}/A) \simeq \rho_A^* Rf_{\Delta,*}\mathcal{O}_{\mathrm{WCart}_{\mathfrak{X}}}$.

where $f: \mathfrak{X} \to \mathrm{Spf}\,\mathbb{Z}_p \rightsquigarrow f_{\Delta}: \mathrm{WCart}_{\mathfrak{X}} \to \mathrm{WCart}$.

§2 Overview of Hodge-Tate theory

$\exists$ canonical sheaf of invertible ideal $\mathcal{I} \subseteq \mathcal{O}_{\mathrm{WCart}}$

which defines a closed substack WCart^{HT}.

For any good $\mathfrak{X}$,

$$\mathrm{WCart}_{\mathfrak{X}}^{HT} := \mathrm{WCart}_{\mathfrak{X}} \times_{\mathrm{WCart}} \mathrm{WCart}^{HT}.$$

Morally,

$$\begin{array}{ccc} \mathrm{WCart} & \mathcal{I} & \mathrm{WCart}^{HT} \\ | & | & | \\ A & I & \bar{A} = A/I. \end{array}$$

In particular, will see:

$$\mathrm{Vect}((\mathfrak{X})_{\Delta}, \bar{\mathcal{O}}_{\Delta}) = \underset{\substack{\uparrow \\ \text{our goal to study} \\ \text{for } \mathfrak{X} \text{ sm } / \mathcal{O}_K.}}{\mathrm{Vect}}(\mathrm{WCart}_{\mathfrak{X}}^{HT}, \mathcal{O}_{\mathrm{WCart}_{\mathfrak{X}}^{HT}})$$

<u>Thm</u> (Tian) Let (A, I) be a good prism (e.g. $(A_{\inf}, (E))$ or $(\mathfrak{S}, (E))$.)

$\mathfrak{X} = \mathrm{Spf}\, R$ small affine $/\bar{A} = A/I$.

(e.g. $R = \bar{A}\langle T^{\pm 1}\rangle$.)

Then $\exists$ equiv of cats:

$$\mathrm{Vect}((\mathfrak{X}/A)_{\Delta}, \bar{\mathcal{O}}_{\Delta}) \simeq \mathrm{HIG}^{\mathrm{tril}}(R).$$

<u>Recall</u> X sm (formal) sch $/C$.

A Higgs bundle on X is a pair $(\mathcal{H}, \theta_{\mathcal{H}})$ of

- a v.b. $\mathcal{H}$ on X_{Zar},
- $\theta_{\mathcal{H}} : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_{X/C}$, $\mathcal{O}_X$-linear,

satisfying $\theta_{\mathcal{H}} \wedge \theta_{\mathcal{H}} = 0$.

$$\rightsquigarrow \mathrm{HIG}(\mathcal{H}, \theta_{\mathcal{H}}) := (\mathcal{H} \xrightarrow{\theta} \mathcal{H} \otimes \Omega \xrightarrow{\theta} \cdots \xrightarrow{\theta} \cdots)^{\theta^2 = 0}.$$

<u>Thm</u> (Tian) If $\mathcal{M} \in \mathrm{Vect}((\mathfrak{X}/A)_{\Delta}, \bar{\mathcal{O}}_{\Delta})$ via the induced Higgs bundle $(\mathcal{H}, \theta_{\mathcal{H}})$

then $R\Gamma((\mathfrak{X}/A)_{\Delta}, \mathcal{M}) \simeq R\Gamma_{\acute{e}t}(X, \mathrm{HIG}(\mathcal{H}, \theta_{\mathcal{H}}))$.

For short $R = \mathcal{O}_C\langle T^{\pm 1}\rangle \rightsquigarrow \hat{R}_\infty := \mathcal{O}_C\langle T^{\pm 1/p^\infty}\rangle$

perfectoid ring.

$\Gamma := \mathrm{Gal}(\hat{R}_\infty[\frac{1}{p}]/R[\frac{1}{p}]) \simeq \mathbb{Z}_p \cdot \gamma$.

$\gamma(T^{1/p^n}) = \zeta_{p^n} \cdot T^{1/p^n}$.

$\hat{R}_\infty \rightsquigarrow (A_{inf}(\hat{R}_\infty), (E))$

$A_{inf}\langle X^{\pm 1/p^\infty}\rangle$, where $X := (T, T^{1/p}, T^{1/p^2}, \dots)$

$\circlearrowleft \Gamma$. $\quad \gamma(X) = [\varepsilon] X$, $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots)$

$\rightsquigarrow V := M(A_{inf}(\hat{R}_\infty), (E)) \circlearrowleft \Gamma$.

Can check If M corresponds to $\mathcal{H} \xrightarrow{\Theta_{\mathcal{H}}} \mathcal{H}\, d\log T$,

then $V = \mathcal{H} \otimes_R \hat{R}_\infty$, $\gamma = \exp(-(\zeta_p - 1)\cdot \Theta_{\mathcal{H}})$.

Cor When $R = \mathcal{O}_C\langle T^{\pm 1}\rangle$ (for example), $\exists$ functor

$$\mathrm{HIG}^{tril}(R) \longrightarrow \mathrm{Rep}_\Gamma(\hat{R}_\infty).$$

not fully faithful (but can remedy this by inverting p.)

$\rightsquigarrow$ p-adic nonabelian Hodge theory.

Recall (Non-abelian Hodge theory)

$X/\mathbb{C}$ proj sm var.

Then (naive version) $\exists$ equiv of cats

$$\mathrm{Rep}_{\pi_1^{top}(X(\mathbb{C}))}(\mathbb{C}) \simeq \{\text{stable Higgs bundle}/X \text{ with vanishing Chern classes}\}$$

$$R\Gamma_B(X, L) = R\Gamma(X, \mathrm{HIG}(\mathcal{H}, \Theta_{\mathcal{H}})).$$

In particular, when $L = \mathbb{C} \Rightarrow$ Hodge decomp

$$H^n_B(X, \mathbb{C}) = \bigoplus_{i+j=n} H^i(X, \Omega^j_X)$$

In p-adic world: When X/K sm proj var

<u>Thm</u> (LZ) For any $\mathbb{Z}_p$-local system $\mathbb{L}$ on $X_{\acute{e}t}$,

$\exists$ Higgs bundle $(\mathcal{H}, \theta_{\mathcal{H}})$ on $X_{C,\acute{e}t}$,

s.t. $R\Gamma_{\acute{e}t}(X_C, \mathbb{L}) \otimes_{\mathbb{Z}_p} C \underset{G_K\text{-equiv}}{\simeq} R\Gamma(X_C, HIG(\mathcal{H}, \theta_{\mathcal{H}}))$

($\Rightarrow$ HT decomp when $\mathbb{L} = \mathbb{Z}_p$.)

<u>Rmk</u> (1) $\mathbb{L} \rightsquigarrow (\mathcal{H}, \theta_{\mathcal{H}})$ if LZ thm is not fully faithful.

(2) Unlike the classical case, $K \neq \bar{K}$.

<u>Analogue</u> p-adic non-abelian Hodge theory

$\updownarrow$

classical non-abelian HT

for X sm (rigid) var $/ C$.

<u>Def</u> (Generalized rep) X/C sm rigid var.

A generalized rep of X is a vector bundle on $X_{pro\acute{e}t}$.

<u>Prop</u> $\exists$ fully faithful $\mathrm{Rep}_{\pi^{\acute{e}t}(X)}(C) \longrightarrow \mathrm{Vect}(X, \hat{\mathcal{O}}_X)$.

E.g. When $X = \mathrm{Spa}(C\langle T^{\pm 1}\rangle)$, $\hat{R}_\infty = \mathcal{O}_C\langle T^{\pm 1/p^\infty}\rangle$.

then $\mathrm{Vect}(X, \hat{\mathcal{O}}_X) = \mathrm{Rep}_\Gamma(\hat{R}_\infty[\frac{1}{p}])$.

So Tian $\Rightarrow$ $HIG^{tril}(R[\frac{1}{p}]) \longrightarrow \mathrm{Vect}(\mathrm{Spf}(R[\frac{1}{p}], \mathcal{O}))$.

Recall **Thm** (BL, GMW, AHLB)

$\exists$ fully faithful functor

$$\mathrm{Vect}((\mathcal{O}_K)_{\Delta}, \bar{\mathcal{O}}_{\Delta}[\tfrac{1}{p}]) \longrightarrow \mathrm{Rep}_{G_K}(C).$$

More precisely, $\exists$ equiv

$$\mathrm{Vect}((\mathcal{O}_K)_{\Delta}, \bar{\mathcal{O}}_{\Delta}) \simeq \mathrm{End}^{nHT}_{\mathcal{O}_K} = \{(M, \phi_M)\}$$

- M finite free $\mathcal{O}_K$-mod,
- $\phi \in \mathrm{End}_{\mathcal{O}_K}(M)$ + certain conditions.

And for any HT crystal $\mathbb{L}$ w/ induced pair (M, ϕ_M)

we have $V = M \otimes_K C$

& ϕ_M is the *Sen operator* of V (up to a scalar).

Recall (Sen theory) $K_{cyc} := K(\zeta_{p^\infty})$, $\chi : \mathrm{Gal}(K_{cyc}/K) \hookrightarrow \mathbb{Z}_p^\times$.

By (almost) descent

$$\mathrm{Rep}_{G_K}(C) \simeq \mathrm{Rep}_{\Gamma}(\hat{K}_{cyc})$$
$$W \otimes C \longleftarrow\!\shortmid\; W.$$

Sen For any $W \in \mathrm{Rep}_{\Gamma}(\hat{K}_{cyc})$,

define $W_0 := \{w \in W : \Gamma \cdot w \text{ is a finite set}\}$.

(e.g. $W = \hat{K}_{cycl}$, $W_0 = K_{cycl}$).

$\Rightarrow$ W_0 is fin-dim'l / K_{cycl} + Γ-action.

Thm $W \longmapsto W_0$ induces an equiv

$$\mathrm{Rep}_{\Gamma}(\hat{K}_{cycl}) \longrightarrow \mathrm{Rep}_{\Gamma}(K_{cyc}).$$

$\forall\, V \in \mathrm{Rep}_{G_K}(C)$, $V = W \otimes_{\hat{K}_{cyc}} C$, $W \rightsquigarrow W_0$.

Can check $V = W_0 \otimes_{K_{cyc}} C$.

Sen operator ϕ_V is scalar ext'n of $\log\gamma$ on W_0.

Rmk $\forall w \in W_0$, $\Gamma \cdot w$ finite

$\rightsquigarrow \log = \lim_{n\to\infty} \frac{\gamma^{p^n}-1}{p^n}$ is well-def'd.

Facts (1) V is uniquely determined by ϕ_V

(2) $R\Gamma(G_K, V) = [V \xrightarrow{\phi_V} V]$.

Cor For any HT crystal $\mathbb{L}$ w/ induced pair (M, ϕ_M).

$$R\Gamma((\mathcal{O}_K)_\Delta, \mathbb{L}) = [\underset{\substack{\circlearrowleft \\ G_K\text{-rep.}}}{M} \xrightarrow{\phi_M} M] \simeq R\Gamma(G_K, V)$$

Also, $V \in \mathrm{Rep}_{G_K}(C)$ lies in the essential image

$\Longleftrightarrow$ the eigenval's of ϕ_V is contained in $\mathbb{Z} + m_{\bar{K}} \cdot a$ for some $a \in K$.

Rmk For a V with ϕ_V, say V is HT if ϕ_V ss & eigenval$(\phi_V) \subset \mathbb{Z}$.

say V is almost HT if eigen$(\phi_V) \subset \mathbb{Z}$.

Plan
- Review of [BS, §2-3]
- Work of Tian & GMW.
- Quick review of WCart in [BL, §3].
  Study HT stack for $\mathcal{O}_K$ [AHLB 1]
- Relationship to p-adic NAHT. [AHLB 2]