Correspondences of Shimura varieties via the geometric Satake

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Starting pt $X_0(N)^{ss}_{\bar{\mathbb{F}}_p} = D^\times \backslash (D_{p,\infty} \otimes \mathbb{A}_f)^\times / K_N \hookrightarrow X_0(N)_{\bar{\mathbb{F}}_p}$.

- $F = \mathbb{F}_q(C)$, G/F (split) red grp, $V \in \operatorname{Rep} \hat{G}^n$.

$\rightsquigarrow \mathrm{Sht}_V / C^n$

$\rightsquigarrow \{H^*(\mathrm{Sht}_V)\} \overset{\text{V. Lafforgue}}{\rightsquigarrow}$ Langlands corresp.

Key Consider family of cohom of shtukas.

Shimura datum (G, X).

- G red grp $/\mathbb{Q}$.
- $X = \{h : \mathbb{C}^\times \to G_{\mathbb{R}} + \text{certain conditions}\} = G_{\mathbb{R}} / K_\infty$.

$\rightsquigarrow h_{\mathbb{C}} : (\mathbb{C} \otimes \mathbb{C})^\times \simeq \mathbb{C}^\times \times \mathbb{C}^\times \to G_{\mathbb{C}}$

Fact $\mu_h(z) = h_{\mathbb{C}}(z, 1) : \mathbb{C}^\times \to G_{\mathbb{C}}$.

$\updownarrow$

$V = V_{\mu_h} \in \operatorname{Rep} \hat{G}$.

Fix $K \subset G(\mathbb{A}_f)$ open cpt subgrp.

$\rightsquigarrow \mathrm{Sh}_V = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K$ over $E = E(G, X)$

$\llcorner V$ determined by X.

$\rightsquigarrow H^*_c(\mathrm{Sh}_V \otimes \bar{E}, \underset{\substack{| \\ (\text{e.g. } \Lambda = \mathbb{F}_\ell, \mathbb{Q}_\ell, \mathbb{Z}_\ell)}}{\Lambda}) \circlearrowleft \mathrm{Gal}(\bar{E}/E) \times \underset{\substack{\| \\ C_c(K \backslash G(\mathbb{A}_f) / K)}}{\mathcal{H}_K}$.

Define bad stack :

$$\mathcal{L}oc = \mathrm{Hom}(\Gamma_E, {}^L G(\Lambda)) / \hat{G}.$$

Then $\mathrm{Rep}\,\hat{G} \rightsquigarrow$ vect bdls on $\mathcal{L}oc$

via $V \mapsto \tilde{V}$.

<u>Expectation</u> $\exists$ a (complex of) quasi-coh sheaf $\mathcal{M}$ on $\mathcal{L}oc$

w/ an action of $\mathcal{H}_K$

$$\text{s.t.} \quad R\Gamma_c(Sh_W \otimes \bar{E}, \Lambda) = R\Gamma(\mathcal{L}oc, \mathcal{M} \otimes \tilde{V})$$

$$\circlearrowleft \qquad\qquad\qquad \circlearrowleft \quad \circlearrowleft$$

$$\mathcal{H}_K \times \mathrm{Gal}(\bar{E}/E) \quad \Leftarrow \quad \mathcal{H}_K \quad \mathrm{Gal}(\bar{E}/E)$$

<u>note</u> $\mathcal{H}_K$ depends only on $G(\mathbb{A}_f)$.

Corollary of expect :

Let (G, X), (G', X') s.t. $G(\mathbb{A}_f) = G'(\mathbb{A}_f)$

$$\mathrm{Hom}(\tilde{V}, \tilde{V}') \longrightarrow \mathrm{Hom}_{\mathcal{H}_K}(R\Gamma_c(Sh_W), R\Gamma_c(Sh_{W'}))$$

Assume $G/\mathbb{Z}_p$ red grp.

Define moduli of local shtukas :

modification at $0 \in D$.

$$Sht^{loc}_{/\mathbb{F}_p}(R) = \left\{ \mathcal{E}_0 \dashrightarrow \mathcal{E}_1 \simeq {}^\tau\mathcal{E}_0 \right\}$$

two G-torsors on $D \hat{\times} \mathrm{Spec}\, R = \mathrm{Spec}\, W(R)$

any perfect ring

$D = \mathrm{Spec}\, \mathbb{Z}_p$

Bhatt - Scholze $\Downarrow$ $\mathrm{Perv}(Sht^{loc})$

This is a local analog of Drinfeld's shtuka.

Have
$$\begin{array}{c} Sht^{loc}_{/\bar{\mathbb{F}}_p}(R) \\ \pi \downarrow \\ Hk^{loc}(R) = \{ \mathcal{E}_0 \dashrightarrow \mathcal{E}_1 \} = L^+G \backslash LG / L^+G . \end{array} \qquad (L^+G \simeq X_*(T),\ T \subset G).$$

Will define "Hecke corresp." between $Sht^{loc}_{\mu_1|\mu_2}$:

$$Sht^{loc}_{\mu_1|\mu_2} = \left\{ \begin{array}{ccc} \mathcal{E}_0 & \overset{\mu_1}{\dashrightarrow} & \mathcal{E}_1 = {}^{\tau}\mathcal{E}_0 \\ \beta \downarrow & & \downarrow \tau(\beta) \\ \mathcal{E}_0' & \overset{\mu_2}{\dashrightarrow} & \mathcal{E}_1' = {}^{\tau}\mathcal{E}_1 \end{array} \right\}$$

$$\begin{tikzcd} & Sht^{loc}_{\mu_1|\mu_2} \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\ Sht^{loc}_{\mu_1} & & Sht^{loc}_{\mu_2} \end{tikzcd}$$

$\rightsquigarrow$ $Perv^{Corr}(Sht^{loc})$ by pull-push.

- obj: perv sheaves on Sht^{loc}
- morph: $Hom(\mathcal{F}_1, \mathcal{F}_2) := Hom(\overleftarrow{h}^* \mathcal{F}_1, \overrightarrow{h}^! \mathcal{F}_2)$.

<u>Thm</u> $\exists$ a functor S s.t.

$$(\sigma_p = Frob_p) \quad \begin{tikzcd} Coh_{Fr}(\hat{G}\sigma_p/\hat{G}) \arrow[r, "S"] & Perv^{Corr}(Sht^{loc}) . \\ Coh(\mathbb{B}\hat{G}) = Rep(\hat{G}) \arrow[u] \arrow[r, "Sat", "\sim"'] \arrow[ur, phantom, "\circlearrowright"] & Perv(Hk^{loc}) . \arrow[u, "\pi^*"'] \end{tikzcd}$$

<u>Rmk</u> Have an integral ver (e.g. $\Lambda = \mathbb{Z}_p$).
Expect a version of Iwahori level of Sh_K.
Expect $Perv(Hk^{loc})$ to have a motivic nature
(b/c it is a Satake cat).

Given (G, X), in many cases:

$$\begin{tikzcd} Sh_{V/\overline{\mathbb{F}}_p} = \{(A, \lambda, \iota, \eta)\}/\sim \arrow[d] \\ Sht^{loc}_{\mu} = \{(A[p^\infty], \lambda, \iota, \eta)\}/\sim \end{tikzcd}$$

- (A, λ) polarized ab var
- (ι, η) additional strs.

$(\mu = \mu_h, \ V = V_\mu)$.

<u>Obs</u> Can pull from local theory to global theory.

$(G_1, X_1), \ (G_2, X_2)$ s.t. $G_1 \otimes \mathbb{A}_f = G_2 \otimes \mathbb{A}_f$ pure inner form.

p unram prime

$\Rightarrow G_1 \otimes \mathbb{Z}_p = G_2 \otimes \mathbb{Z}_p$.

<u>Then</u> local-global diagram:

$$\begin{tikzcd} & Sh_{V_1|V_2/\overline{\mathbb{F}}_p} \arrow[dl] \arrow[dr] \arrow[dd] & \\ Sh_{V_1/\overline{\mathbb{F}}_p} \arrow[dd] & & Sh_{V_2/\overline{\mathbb{F}}_p} \arrow[dd] \ \circlearrowleft \ \mathcal{H}_{K^p} \\ & Sht^{loc}_{\mu_1|\mu_2} \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\ Sht^{loc}_{\mu_1} & & Sht^{loc}_{\mu_2} \end{tikzcd}$$

<u>Rmk</u> Over $\mathbb{Q}$: generic fiber has no way to make an isog b/w two ab vars w/ different Mumford-Tate grps.

<u>Thm</u> $(G_1, X_1), (G_2, X_2)$ satisfying certain conditions.

Then $\mathrm{Hom}_{\hat{G}_0/\hat{G}}(\tilde{V}_1, \tilde{V}_2) \xrightarrow{\text{"}S_*\text{"}} \mathrm{Hom}_{\mathrm{Perv}^{\mathrm{Corr}}(Sht^{loc})}(\pi^* \mathrm{Sat}(V_1), \pi^* \mathrm{Sat}(V_2))$

$\to \mathrm{Corr}_{\mathcal{H}_{K^p}}(\overline{\mathbb{Q}}_\ell \, Sh_{V_1/\overline{\mathbb{F}}_p}\langle \cdots \rangle, \overline{\mathbb{Q}}_\ell \, Sh_{V_2/\overline{\mathbb{F}}_p}\langle \cdots \rangle)$

$\to \mathrm{Hom}_{\mathcal{H}_{K^p}}(H^*_c(Sh_{V_1}), H^*_c(Sh_{V_2}))$

Compatible with compositions for $(G_1, X_1), (G_2, X_2), (G_3, X_3)$.

<u>Case (I)</u> If $(G_1, X_1) = (G_2, X_2)$, $V_1 = V_2$,

$$\begin{tikzcd} \mathrm{End}(\tilde{V}) \arrow[r, phantom, "\curvearrowright"] & H_c^*(\mathrm{Sh}_V) \arrow[d, phantom, "\circlearrowleft"] \\ J = \Gamma(\hat{G}\sigma/\hat{G}, \mathcal{O}) \arrow[u] \arrow[r, "\mathrm{Sat}^{cl}", "\sim"'] & \mathcal{H}_{K_p} \end{tikzcd}$$

<u>note</u> $\mathrm{End}(\hat{V}) \ni \gamma_{taut}$

$$\begin{tikzcd} \mathrm{End}(\hat{V}) \\ J[\gamma_{taut}]/\text{Cayley-Hamilton}. \arrow[u, "\leftarrow \text{isom if } p \text{ is split}"'] \end{tikzcd}$$

<u>Conj</u> The diagram is commutative.

<u>Prop</u> The conj is true when $G_{\mathbb{R}}$ compact mod center.

<u>Case (II)</u> $(G_1)_{\mathbb{R}}$ is cpt mod center.

$$\Rightarrow \quad H_c^*(\mathrm{Sh}_{V_1}) = C(G_1(\mathbb{Q})\backslash G_1(\mathbb{A}_f)/K)$$

$\circlearrowleft$ J by usual Hecke corr.

$$\rightsquigarrow \quad \bigotimes_J \mathrm{Hom}(\tilde{V}_1, \tilde{V}_2) \longrightarrow H_c^{mid}(\mathrm{Sh}_{V_2}). \qquad (*)$$

<u>Thm</u> (i) $(*)$ is given by cycle class map for middle-dim cycles in $\mathrm{Sh}_{V_2, \overline{\mathbb{F}}_p}$.

(ii) Let π_f be an $\mathcal{H}_K$-mod. Then the π_f-isotypic part of $(*)$ is injective if the Satake param of $\pi_{f,p}$ is generic w.r.t. V_2.

(iii) For certain Shimura varieties,

the π_f-isotypic part of $(*)$ is surjective

to the space of Tate classes

if the Satake param of $\pi_{f,p}$ is "strongly generic".

The intersection matrix of the cycles are encoded in

$$\mathrm{Hom}(\tilde{V}_1, \tilde{V}_2) \otimes_J \mathrm{Hom}(\tilde{V}_2, \tilde{V}_1) \longrightarrow \mathrm{Hom}(\tilde{V}_1, \tilde{V}_1) = J.$$

Rmk Do not know how to calculate entries of matrix

(but only know about det).