Quotient by finite group schemes

1. Action of group scheme on scheme.

G. $\quad m: G\times G \to G$

$e:$ ___

$i:$ ___

An action of ___ G on ___ X is a morphism:

$$\mu: G\times X \longrightarrow X, \quad \text{s.t.}$$

① the composite

$$X \cong \operatorname{Spec} k \times X \xrightarrow{e\times id} G\times X \xrightarrow{\mu} X$$

is id (e acts as id on X)

② the diagram is commute. $(g_1 g_2 \cdot x = g_1(g_2\cdot x))$

$$\begin{tikzcd} G\times G\times X \arrow[r, "\boxed{m\times id}"] \arrow[d, "\boxed{1\times\mu}"'] \arrow[dr, phantom, "\circlearrowright"] & G\times X \arrow[d, "\mu"] \\ G\times X \arrow[r, "\mu"] & X \end{tikzcd}$$

This is equivalent that $\underset{\triangle}{\underline{G(S)}}$ acts on $\underset{\triangle}{\underline{X(S)}}$ for every S.

For every S-valued pt $\underline{x} \in G(S)$, gives an automorphism over S.

$$\begin{tikzcd} X \times S \arrow[rr, "\mu_x"] \arrow[dr, "p_2"'] & \arrow[d, phantom, "\circlearrowleft"] & X \times \underline{\underline{S}} \arrow[dl, "p_2"] \\ & S & \end{tikzcd}$$

s.t. ① $x, y \in G(S) \Rightarrow \mu_x \circ \mu_y = \mu_{xy}$

② if $f: S_1 \to S_2$, $\underline{x: S_2 \to G}$, $\underline{x \circ f}: S_1 \to G$

$$\begin{tikzcd} X \times S_1 \arrow[r, "\mu_{x \circ f}"] \arrow[d, "(1 \times f)"] \arrow[dr, phantom, "\circlearrowleft"] & X \times S_1 \arrow[d, "(1 \times f)"'] \\ X \times S_2 \arrow[r, "\mu_x"] & X \times S_2 \end{tikzcd}$$

$\mu_x := (\mu \circ \underline{\underline{\sigma}} \circ (id_X \times x), \underline{p_2})$

σ : switch morphism : $X \times G \to G \times X$ $(x, g) \to (g, x)$

$\underline{f: X \to Y}$ is called G-invariant, if diagram

$$\begin{tikzcd} \underline{G} \times \underline{\underline{X}} \arrow[r, "\mu"] \arrow[d, "p_2"] \arrow[dr, phantom, "\circlearrowleft"] & \underline{X} \arrow[d, "\underline{f}"] \\ \underline{X} \arrow[r, "f"] & Y \end{tikzcd}$$

In particular, if we take $Y = \mathbb{A}^1_k$, f is called a G-invariant function.

An action is called free $\Leftrightarrow$ the following morphism

$$(\mu, p_2) : G \times X \longrightarrow \underline{\underline{X}} \times \underline{\underline{X}}$$

is a closed immersion.

$\underline{\underline{X}}$, F. coh sheaf on X. $\mu : G \times X \to X$

A lifting of the action μ to F.

$$\boxed{g \in G.} \quad \underline{F \otimes O_S} \longrightarrow \underline{F \otimes O_S}$$

$$\boxed{\mu_g}. \quad \underline{X \times S} \longrightarrow \underline{X \times S}.$$

① μ_g is functorial in g.

② $\boxed{\mu_{g_1 g_2} = \mu_{g_1} \circ \mu_{g_2}.}$

A lifting of $\underline{\mu}$ to $\boxed{F}$ is a isomorphism

$$\lambda : \boxed{p_2^*(F) \xrightarrow{\cong} \mu^*(F)}$$

$\downarrow$

sheaves of $G \times X$.

$$\boxed{G \times X} \underset{\mu}{\overset{p_2}{\rightrightarrows}} \underline{\underline{X}}$$

$\underline{\underline{F}}.$

sheaves on $G \times G \times X$

$$\begin{tikzcd} p_3^*(F) \arrow[rr, "(p_2, p_3)^*(\lambda)"] \arrow[dr, "(m \times id_X)^*(\lambda)"'] & \curvearrowright & \xi^*(F) \arrow[dl, "(id_G \times \mu)^*(\lambda)"] \\ & \eta^*(F) & \end{tikzcd}$$

$\triangle$

$\xi : \mu \circ (p_2, p_3)$. $\eta = \mu \circ \boxed{(m \times id_X)}$

$\boxed{G \times G \times X \to X}$

$G \times G \times X \to G \times X \to X$.

Theorem: (A) Let G. finite group scheme, acts on $X \doteq \operatorname{Spec} A$, the orbit of ant pts in X is contained in an affine open subsets of X

Then there exist a pair (Y, π). Y scheme $\pi : X \to Y$. ___ s.t.

① As topo. sp. $(Y, \pi) = X/G$

② $\pi : X \to Y$ is G-invariant. and $\pi_*(\mathcal{O}_X)^G$ denotes the subsheaf of $\pi_*(\mathcal{O}_X)$ of G-inv. functions. Then

$$\boxed{\mathcal{O}_Y \to \pi_*(\mathcal{O}_X)^G} \quad \text{is an isomorphism.}$$

$\triangle$

The pair (Y, π) is uniquely determined up to iso.

π : $\boxed{\text{finite}}$ & surj. $Y = X/G$. π : universal.

$\forall$ G-inv. $f : X \to Z$, $\exists!$ $g : Y \to Z$, s.t.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ Y \arrow[r, dashed, "g"'] & Z \end{tikzcd}$$

(B) Furthermore, the action of G is free, $G = \operatorname{Spec} R$.

$n = \dim_k R$. Then π is a flat morphism of rank n. and the subscheme of $X \times X$: $(\mu, p_2) : \boxed{G \times X} \xrightarrow{c.i} X \times X$ is qual to $\boxed{X \times_Y X} \hookrightarrow X \times X$. Finally, if $\mathcal{F}$ is a coh. $\mathcal{O}_Y$-mod. then $\pi^* \mathcal{F}$ has a naturally defined G-action lifting on X. and $\mathcal{F} \longrightarrow \pi^*(\mathcal{F})$ give an equivalence of category of coh. $\mathcal{O}_Y$-mod and the category of coh. $\mathcal{O}_X$-mod with G-action.

Proof: (A) Assume: $X = \operatorname{Spec} A$, $G = \operatorname{Spec} R$.

$\varepsilon^* : R \to k$. $m^* : R \to R \otimes_k R$ $\mu^* : A \to R \otimes_k A$.

$Y = \operatorname{Spec} A^G$ $A^G = B = \{ a \in A \mid \mu^*(a) = 1 \otimes a \}$

$\mathrm{Nm}_A : \boxed{R \otimes_k A} \to A$ be the norm map ($R \otimes_k A$ is free of finite rank over A). Nm_A is a homegenous poly. function of degree n. ($n = \dim_k R$).

$N_A : A \to A$. $N_A(a) = \mathrm{Nm}_A(\mu^*(a))$

$$\boxed{\text{Claim. } N_A(A) \subseteq B.}$$

To prove this, we need to show for every $a \in A$.

$$\boxed{\mu^*(N_A(a)) = \underset{\phi \circ N_A}{\overset{\parallel}{1 \otimes N_A(a)}}.}$$ Define ϕ, ψ :

$\phi : A \to R \otimes_k A$ $\qquad \psi : R \otimes_k R \otimes_k A \to R \otimes R \otimes A$

$a \mapsto 1 \otimes a$ $\qquad (\xi \otimes \eta \otimes a) \to (m^*(\xi) \otimes 1) \cdot (1 \otimes \eta \otimes a)$

Note that if $f : B \to C$ homo. of k-alg. then.

$$\boxed{\begin{tikzcd} R \otimes_k B \arrow[r, "1 \otimes f"] \arrow[d, "Nm_B"'] \arrow[dr, phantom, "\circlearrowright"] & R \otimes_k C \arrow[d, "Nm_C"] \\ B \arrow[r, "f"] & C \end{tikzcd}}$$ $(Nm_C \circ (1_R \otimes f) = f \circ Nm_B)$

$\mu^* \circ N_A = \mu^* \circ Nm_A \circ \mu^*$ $\qquad (B = A,\ C = R \otimes A,\ f = \mu^*)$

$= Nm_{R \otimes A} \circ (1_R \otimes \mu^*) \circ \mu^*$

$= Nm_{R \otimes A} \circ (m^* \otimes 1_A) \circ \mu^*$

$= Nm_{R \otimes A} \circ \psi \circ (1_R \otimes \phi) \circ \mu^*$

$= Nm_{R \otimes A}$

$\mu^* \circ N_A$.

$$A \xrightarrow{N_A} A \xrightarrow{\mu^*} R \otimes_k A$$

$$\underline{A} \xrightarrow{\mu^*} \boxed{R \otimes_k A \xrightarrow{N_m} A' \xrightarrow{\mu^*} \underline{R \otimes A}}$$

$$A \xrightarrow{\mu^*} \boxed{\underline{R \otimes A} \xrightarrow[\Delta]{1_R \otimes \mu^*} \underline{R \otimes R \otimes A} \xrightarrow{N_{m_{R \otimes A}}} R \otimes A}$$

$$a \longmapsto 1 \otimes a.$$

$$\underline{A} \xrightarrow{\mu} \boxed{\underline{R \otimes A} \xrightarrow[\Delta]{m^* \otimes 1_A} \underline{R \otimes R \otimes A}} \xrightarrow{N_{m_{R \otimes A}}} \underline{R \otimes A}$$

$R \otimes \to \underline{R} \otimes R \otimes A$

$$A \xrightarrow{\mu^*} \boxed{R \otimes A \xrightarrow{1_R \otimes \phi} R \otimes R \otimes A} \boxed{\xrightarrow{\cancel{\psi}} R \otimes R \otimes A \xrightarrow{N_m} R \otimes A}$$

$$\boxed{N_{m_{R \otimes A}} \circ \psi = N_{m_{R \otimes A}}}$$

$$A \xrightarrow{\mu^*} R\otimes A \xrightarrow[\Delta]{1_R\otimes\phi} R\otimes R\otimes A \xrightarrow[G]{N_m} R\otimes A.$$

$$A \xrightarrow{\mu^*} R\otimes A \xrightarrow{N_{m_A}} A \xrightarrow{\phi} R\otimes A.$$

$$\underbrace{\underline{A} \xrightarrow[\Delta]{N_A} A \xrightarrow[\Delta]{\phi} R\otimes A.}_{1\otimes a}$$

$$\boxed{\mu^*}\circ N_A = \boxed{\phi}\circ N_A$$

$N_A(A) \subseteq B.$

$$\boxed{N_{m_{R\otimes A}} \circ \psi = N_m{}_{R\otimes A}.}$$

consider $R\otimes R\otimes A$ as an $R\otimes A$-alg.

$$\boxed{\begin{aligned} R\otimes_k A &\to R\otimes R\otimes A. \\ \eta\otimes a &\mapsto 1\otimes\eta\otimes a. \end{aligned}}$$

$\boxed{\psi : \text{ is an } \boxed{\text{automorphism}} \text{ of } \boxed{R\otimes_k A}\text{-alg.}}$

$N_A(A) \subseteq B$. ✓.

$G \curvearrowright X$. $\qquad G \curvearrowright \boxed{X \times A'} \xrightarrow{} A[T]$ (G acts trivially on A')

Similarly we can define : $N_{A[T]} : A[T] \to A[T]$ $\quad B[T]$.

$\forall\, a \in A$, $\quad \chi_a(T) = N_{A[T]}(T-a)$

$\chi_a(T) = T^n + s_1 T^{n-1} \cdots + s_n$ is G-inv

$s_1 - s_n \in B$. $\chi_a(T)$ is the characteristic poly

of endomorphism of $\boxed{R \otimes A}$ defined by $\boxed{\mu^*(a)}$

free A-mod.

$$N_{A[T]}(T-a) = \boxed{N_{A[T]}(T-\mu^*(a))}$$

$\varepsilon \otimes 1 : \ R \otimes A \to A$ is surj. $\boxed{\boxed{(\varepsilon \otimes 1)}(\mu^*(a)) = \boxed{a}}$

$\boxed{\mu^*(a) - a}$ defines zero map on $A \cong \big/ (\varepsilon\otimes 1)(R\otimes A)$

$\boxed{\mu^*(a) - 1 \otimes a}$ $\qquad R \otimes A \big/ (\qquad)$

$\boxed{\mu^*(a) - a}$; ~ on quotient A of $R \otimes A$ via

$\det(\mu^*(a) - a) = \boxed{\chi_a(a) = 0}$

$\mu^*(a) - 1 \otimes a$ in $k \otimes A$

$a^n + s_1 a^{n-1} \cdots + s_n = 0$

$a \in A$.

$s_1 - s_n \in B$

$\Rightarrow$ a is integral over B. $\Rightarrow$ A integral over B.

$$\mathcal{O}_Y \xrightarrow{\cong} \pi_*(\mathcal{O}_X)^G$$

π is G-inv.

$\mathcal{O}_Y \subseteq \pi_*(\mathcal{O}_X)^G$

$$\mathcal{O}_Y \hookrightarrow \boxed{\pi_*(\mathcal{O}_X)^G}$$

$\boxed{\pi_*(\mathcal{O}_X)^G}$ is a coh. $\mathcal{O}_Y$-mod. and is the kernel of

$$\pi_*(\mathcal{O}_X)^G \xrightarrow{\lambda} \pi_*(\mathcal{O}_X) \otimes_k R$$

$$f \longmapsto \underbrace{\mu^*(f) - f \otimes 1}_{\lambda}$$

$\ker \lambda = \pi_*(\mathcal{O}_X)^G$.

$\pi_*(\cancel{\mathcal{O}}A) = A^G$

$\shortparallel$

B.

$\ker \lambda = \mathcal{O}_Y = B = A^G$.

(B) $\boxed{\text{lifting}}$

$$\boxed{\begin{array}{ccc} G\times X & \hookrightarrow & X\times X \\ \wr\vert & & \\ X\times_Y X & & \end{array}}$$

$\pi : X \to Y$

$\mathcal{F}$ is $\mathcal{O}_Y$-mod $\qquad \pi^*(\mathcal{F})$ is $\mathcal{O}_X$-mod.

$$\boxed{\lambda : p_2^* \circ \pi^*(\mathcal{F}) \xrightarrow{\sim} \mu^* \circ \pi^*(\mathcal{F})} \qquad G\times X$$

Lemma: $X \xrightarrow{f} Y \xrightarrow{g} Z$ morphism $\qquad \mathcal{F}$: $\mathcal{O}_Z$-mod.

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ A & \leftarrow & B & \leftarrow & C \end{array} \qquad M \otimes_C A = (M\otimes_C B)\otimes_B A.$$

$\lambda_{f,g}(\mathcal{F}) : (g\circ f)^* \mathcal{F} \xrightarrow{\sim} f^* \circ g^*(\mathcal{F})$ s.t.

if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T$. following diagram commute:

$$\begin{array}{ccc} f^* g^* h^*(\mathcal{F}) & \xrightarrow{f^*\circ(\lambda_{g,h})} & f^*(h\circ g)^* \mathcal{F} \\ \downarrow \lambda_{f,g}\circ h^* & \circlearrowright & \downarrow \lambda_{f,h\circ g} \\ (g\circ f)^* h^*(\mathcal{F}) & \xrightarrow{\quad} & (h\circ g\circ f)^* \mathcal{F}. \end{array}$$

Return to our case

$$G\times X \underset{p_2}{\overset{\mu}{\rightrightarrows}} X \xrightarrow{\pi} Y$$

$$\mathcal{F}$$

π is G-inv $\Leftrightarrow$ $\boxed{\pi \circ \mu} = \boxed{\pi \circ p_2} = f.$

$$\mu^*(\pi^* \mathcal{F}) \underset{\triangle}{\cong} f^* \mathcal{F} \underset{\Diamond}{\cong} p_2^*(\pi^* \mathcal{F})$$

$$\lambda : p_2^*(\pi^* \mathcal{F}) \underset{\triangle}{\cong} \mu^*(\pi^* \mathcal{F})$$

gives a lifting of action of G on $\pi^* \mathcal{F}$

$(\mu, p_2) : \boxed{G \times X} \to X \times X$

$\qquad\qquad G \times X \cong X \times_Y X$

$X \xrightarrow{\pi} Y$ is G-inv.

$$\boxed{G \times X} \xrightarrow{\exists !} \boxed{X \times_Y X} \hookrightarrow X \times_k X$$

$$\begin{tikzcd}
G \times X \arrow[dr, dashed] \arrow[drr, bend left, "\triangle"] \arrow[ddr, bend right, "\triangle"'] & & \\
& X \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
& \boxed{X} \arrow[r] & Y
\end{tikzcd}$$

$$G \times X \xrightarrow{\text{cl. im.}} X \times_Y X$$

$$\boxed{\begin{array}{rcl} & a_1 \otimes a_2 \longmapsto & \underline{\mu^*(a_1)} \cdot (1 \otimes a_2) \\ \lambda : & A \otimes_B A \longrightarrow & R \otimes_k A \end{array}}$$

λ is surj. need to prove λ is inj.

View as A-mod

Now since λ is surj, $\boxed{R \otimes_k A}$ is generated

by $\underline{\mu^*(A)}$ $\quad (1 \otimes a_2 \cdot \text{scalar})$

$\exists \{a_i\}_{1 \le i \le n}$ s.t. $\mu^*(a_i)$ from a basis of $R \otimes_k A$ ($/A$)

Claim: For $a, \lambda_1, \lambda_2 \cdots \lambda_n \in A$.

$$\boxed{\begin{array}{c} [\mu^*(a) = \sum_{i=1}^n (1 \otimes \lambda_i) \mu^*(a_i)] \\ \Leftrightarrow [a = \sum_{i=1}^n \lambda_i a_i, \quad \lambda_1, \lambda_2 \cdots \lambda_n \in B] \end{array}}$$

$\Leftarrow$ is trivial. $\quad \mu^*(a) = \sum \mu^*(\lambda_i a_i)$

$$= \sum (1 \otimes \lambda_i) \mu^*(a_i)$$

$\Rightarrow$ $\quad (1 \otimes_k \mu^*)(\mu^* a) = (m^* \otimes 1_A)(\mu^*(a))$ $\quad \boxed{a = \sum \lambda_i a_i}$ $\quad \boxed{\lambda_i \in A}$

$$\sum_{i=1}^n (1 \otimes \mu^* \lambda_i)(1 \otimes \mu^*) \boxed{(\mu^* a_i)}$$

$$= \sum_{i=1}^n (1 \otimes 1 \otimes \lambda_i) \cdot (m^* \otimes 1)(\mu^* a_i)$$

$$= \sum (1\otimes 1\otimes \lambda_i)(1\otimes \mu^*)\boxed{(\mu^* a_i)}$$

$\mu^* a_i$ form a basis $\Rightarrow$ $1\otimes \mu^* \lambda_i = 1\otimes 1\otimes \lambda_i$

$\Rightarrow$ $\mu^* \lambda_i = 1\otimes \lambda_i$ $\Rightarrow$ $\lambda_i \in B$.

$\Rightarrow$ A is a free B-mod with basis $a_1 - a_n$.

$$\lambda : \quad A\otimes_B A \longrightarrow k\otimes_k A$$
$$a_i \otimes 1 \longrightarrow \boxed{\mu^*(a_i)}$$

$\sum \mu^* a_i = 0$ $\Rightarrow$ $a_i = 0$

$\ker \lambda = 0$ $\Rightarrow$ $\begin{cases} \lambda \text{ is injective.} \\ \lambda \text{ is surj.} \end{cases}$

$\Rightarrow$ λ : is an isomorphism

Corollary : X itself is a group scheme, G normal f.g.s

(G is normal if $\forall S$. $G(S)$ is a normal subgp of $X(S)$)

$\pi : X \to \boxed{X/G}$ is an epimorphism, $G = \ker f$

(epi : $f: \underline{X} \to \underline{Y}$ morphism of group scheme, surj.

$\underline{f^*: \mathcal{O}_Y \to f_*\mathcal{O}_X \text{ is inj.}}$)

Conversely: $f: \underline{X} \to \underline{Y}$ epi let $G = \ker f$

$\Rightarrow$ $Y \cong X/G$.

$$\begin{tikzcd}
\boxed{X \times X} \arrow[r, "m"] \arrow[d, "(\pi\times\pi)"'] & X \arrow[d, "\pi"] \\
X/G \times X/G \arrow[r, dashed, "m'"] & \boxed{X/G}
\end{tikzcd}$$

$\boxed{G \times G}$

$m \circ \pi$

$\pi \circ m$

Coro 2. $\boxed{Y = X/G}$ $\mathcal{G}$: coh sheaf on X.

acted by G

natural iso : $\boxed{\pi^*(\pi_* \mathcal{G}) \cong \mathcal{G} \otimes_k R}$

$\underline{G \overset{\text{free}}{\curvearrowright} X}$. $\underline{G \curvearrowright F \text{ finite scheme}}$

$\Downarrow$

$G \curvearrowright X \times F$

$$\mathcal{U} = (\underline{X} \times F)/G.$$

$$V = X/G.$$

$\underline{\mathcal{U} \xrightarrow{\pi} V}$ is called the fibration with fibre F

associated to the principal G-bundle.

Theorem: $\pi : \mathcal{U} \to V$. for coh. $\mathcal{F}$ on V. we have

$$\boxed{\chi(\pi^*(\mathcal{F})) = (\deg \pi) \cdot \chi(\mathcal{F})}$$

<u>degree of π</u>. is defined as $\boxed{\pi_*(\mathcal{O}_\mathcal{U})}$ <u>rank</u> over $\boxed{\underline{\mathcal{O}_V}}$