Fall 2022 – Seminar on *Abelian Varieties* by David Mumford

Lecture 1-3

ANALYTIC THEORY OF ABELIAN VARIETIES

1. Motivation

Let C be a complex smooth projective curve of genus g. Define

$$\Omega^1 := \text{sheaf of holomorphic 1-forms on } C.$$

Then we have the following theorem.

Theorem 1.1 (Abel-Jacobi)**.** *Denote $H^0(C,\Omega^1)^*$ the complex dual of $H^0(C,\Omega^1)$ as a $\mathbb{C}$-vector space. The map*

$$\begin{aligned} H_1(C,\mathbb{Z}) &\longrightarrow H^0(C,\Omega^1)^* \\ \gamma &\longmapsto (\omega \mapsto \int_\gamma \omega) \end{aligned}$$

is injective and identifies $H_1(C,\mathbb{Z})$ with a lattice of $H^0(C,\Omega^1)^*$.

Assuming the theorem, one may define a complex torus by taking the quotient.

Definition 1.2 (Jacobian)**.** The **Jacobian variety** associated to C is

$$\mathrm{Jac}(C) := H^0(C,\Omega^1)^*/H^1(C,\mathbb{Z}).$$

It turns out that $\mathrm{Jac}(C)$ has some extra geometric structure beyond the structure of complex torus. When $g=1$, it is well known that for a fixed lattice $\Lambda \subset \mathbb{C}$, the complex torus $\mathbb{C}/\Lambda$ admits the structure of an elliptic curve. Moreover, it is projective as an algebraic variety via the morphism

$$\begin{aligned} \mathbb{C}/\Lambda &\longrightarrow \mathbb{P}^2_{\mathbb{C}} \\ z &\longmapsto [\wp_\Lambda(z), \wp'_\Lambda(z), 1]. \end{aligned}$$

Here $\wp_\Lambda(\cdot)$ denotes the Weierstrass $\wp$-function associated to Λ. To be more precise,

$$\wp_\Lambda(z) := \frac{1}{z^2} + \sum_{w\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

Through some more complicated computation, it can be verified that for each $z \in \mathbb{C}/\Lambda$, its image $[\wp_\Lambda(z), \wp'_\Lambda(z), 1]$ lies on an **elliptic curve**, which is defined to be a smooth projective algebraic curve of genus 1, on which there is a specified point O satisfying some group law[1].

Proposition 1.3. *For each $z \in \mathbb{C}/\Lambda$,*

$$\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2\wp_\Lambda(z) - g_3,$$

where the coefficients are read as

$$g_2 = 60G_4(\Lambda), \quad g_3 = 140G_6(\Lambda),$$

given by Eisenstein series

$$G_{2k}(\Lambda) := \sum_{w\in\Lambda\setminus\{0\}} w^{-2k}.$$

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[1]Possibly with at least one rational point on it, i.e., $E(\mathbb{Q}) \neq \emptyset$ for an elliptic curve E.

In general, in case when $g > 1$, there is a significant difference: for an arbitrary lattice $\Lambda \subset \mathbb{C}^g$, the quotient $\mathbb{C}^g/\Lambda$ is not automatically projective. However, we are primarily to concern about projective complex tori.

For this, we need to study (ample/very ample) line bundles on the complex tori. Also, we pay attention to the construction of abelian varieties, which can be viewed as the correct generalization of elliptic curves to higher dimensional sense, by using the language of algebraic geometry.

2. Line Bundles on a Complex Torus

Setups. Let X be a complex torus (equivalently, a compact connected complex Lie group) of dimension g. Let $V = \operatorname{Lie} X$ be the corresponding Lie algebra.[2] This naturally induces an exponential map $\exp : V \to X$ which is a surjective homomorphism of complex Lie groups. Take $U := \operatorname{Ker}(\exp)$ to be a lattice of V. (Recall: it means by saying $U \subset V$ is a lattice that U is a discrete subgroup with the maximal rank whose quotient is compact; namely, U is a free abelian group of rank $2g$ in V.)

Proposition 2.1. $(V, \exp)$ *is the universal covering of* X*. Moreover,*

$$\pi_1(X) = \pi_1(X, e) \cong U = \operatorname{Ker}(\exp).$$

Definition 2.2. The **Picard group** of an algebraic variety X over $\mathbb{C}$ is

$$\operatorname{Pic}(X) := \{\text{isomorphism classes of holomorphic line bundles on } X\}$$

In fact, this group can be computed explicitly via some sheaf cohomology

$$\boxed{\operatorname{Pic}(X) \cong H^1(X, \mathscr{O}_X^*)}$$

and this sheaf cohomology can be computed by some group cohomology. Here

$$\mathscr{O}_X := \text{the sheaf of holomorphic functions on } X,$$
$$\mathscr{O}_X^* := \text{the sheaf of invertible holomorphic functions on } X.$$

Let X be a nice topological space and G be a discrete group (i.e., equipped without any nontrivial topology) acting freely and discontinuously on X.[3] Then

$$X \circlearrowleft G \quad \rightsquigarrow \quad \begin{array}{c} X \\ \downarrow{\scriptstyle \pi} \\ X/G \end{array}$$

Let $\mathscr{F}$ be a sheaf of abelian groups on $Y = X/G$. Our goal is to compare

$$H^p(G, \Gamma(X, \pi^*\mathscr{F})) \quad \text{and} \quad H^p(Y, \mathscr{F})$$

for $p \geqslant 0$.

2.1. Group Cohomology.

Let G be as above and M be a left G-module (i.e., M is an abelian group with a left G-action). Define a **cochain complex**

$$C^\bullet = (C^p, \delta^p)_{p \geqslant 0}$$

as follows.

(a) $C^p := \operatorname{Map}(G^p, M)$;

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[2] In [Mum85], X is supposedly a group variety equipped with two morphisms $m : X \times X \to X$ and $i : X \to X$ that correspond to the multiplication and the inversion as group operations. Also, by definition there is a (closed) point $e \in X$ playing the role of the group identity. Then an equivalent definition of V is to say $V = T_e X$, the tangent space of X at some point $e \in X$.

[3] By definition, the discontinuity means that for all $x \in X$ there is some neighborhood U_x of x such that $U_x \cap \sigma(U_x) = \emptyset$ for all $\sigma \in G \backslash \{e\}$.

(b) $\delta^p : C^p \to C^{p+1}$ is defined by

$$(\delta^p f)(\sigma_0, \ldots, \sigma_p) = \sigma_0(f(\sigma_1, \ldots, \sigma_p)) + \sum_{i=0}^{p-1}(-1)^{i+1} f(\sigma_0, \ldots, \sigma_i\sigma_{i+1}, \ldots, \sigma_p)$$
$$+ (-1)^{p+1} f(\sigma_0, \ldots, \sigma_{p-1}).$$

When $p = 0$, we have $G^p = \{e\}$, and $C^0 = M$, $C^1 = \mathrm{Map}(G, M)$. Hence

$$\begin{aligned} \delta^0 : M &\longrightarrow \mathrm{Map}(G, M) \\ m &\longmapsto (g \mapsto gm - m) \end{aligned}$$

2.1.1. *Definition of Group Cohomology.* Let (C^p, δ^p) be as above. Then define

$$Z^p(G, M) = \mathrm{Ker}(\delta^p), \quad B^p(G, M) = \mathrm{im}(\delta^{p-1}).$$

There are two equivalent types of definitions for the group cohomology $H^p(G, M)$.

(1) Define $H^p(G, M) := H^p(C^\bullet) = Z^p(G, M)/B^p(G, M)$ as the cohomology of the complex.
(2) Define $H^p(G, M)$ to be the p-th right derived functor of the functor

$$\begin{aligned} \mathsf{Mod}_G &\longrightarrow \mathsf{Ab} \\ M &\longmapsto M^G = \{m \in N \mid gm = m, \ \forall g \in G\} = \mathrm{Hom}_{\mathsf{Mod}_G}(\mathbb{Z}, M). \end{aligned}$$

Example 2.3. (1) Consider the groups of 1-cocycles and 1-coboundaries:

$$Z^1(G, M) = \{f : G \to M \mid \underbrace{f(gg') = f(g) + gf(g')}_{\text{1-cocycle condition}}, \ \forall g, g' \in G\},$$
$$B^1(G, M) = \{f : G \to M \mid \exists a \in M \text{ such that } f(g) = ga - a, \ \forall g \in G\}.$$

In particular, when G acts trivially on M, we obtain $ga - a = 0$ for all $g \in G$, $a \in M$. Then $B^1(G, M)$ is trivial and hence

$$Z^1(G, M) \cong H^1(G, M) \cong \mathrm{Hom}(G, M).$$

(2) One can compute the 2-cocycle condition by definition:

$$g(f(g', g'')) - f(gg', g'') + f(g, g'g'') - f(g, g') = 0. \tag{$*$}$$

Consequently, the group of 2-cocycles is given by

$$Z^2(G, M) = \{f : G \times G \to M \mid \forall g, g', g'' \in G, \ (*) \text{ holds}\}.$$

2.1.2. *Cup Product of Group Cohomology.* Take $M, N, P \in \mathsf{Mod}_G$. Given a bilinear map

$$\varphi : M \times N \to P \quad \text{such that} \quad \varphi(gm, gn) = g\varphi(m, n),$$

we can define a **cup product** as

$$\begin{aligned} \cup : H^p(G, M) \times H^q(G, N) &\longrightarrow H^{p+q}(G, P) \\ (f, g) &\longmapsto f \cup g. \end{aligned}$$

For all $f \in H^p(G, M)$ and $g \in H^q(G, N)$,

$$(f \cup g)(\sigma_1, \ldots, \sigma_{p+q}) = f(\sigma_1, \ldots, \sigma_p) \cdot (\sigma_1, \ldots, \sigma_p) g(\sigma_{p+1}, \ldots, \sigma_{p+q}).$$

2.2. **Čech Cohomology.** Let Y be a topological space and $\mathscr{F}$ be a sheaf of abelian groups on Y. Let $\mathcal{U}=\{V_i\}_{i\in I}$ be an open covering of Y. Define the **Čech chain complex**

$$C^\bullet(\mathcal{U},\mathscr{F})=(C^p(\mathcal{U},\mathscr{F}),d^p)_{p\geqslant 0}$$

as follows.

(a) Denote $V_{i_0\cdots i_p}=V_{i_0}\cap\cdots\cap V_{i_p}$ (or more generally, $V_{i_0\cdots i_p}=V_{i_0}\times_Y\cdots\times_Y V_{i_p}$). Then

$$C^p(\mathcal{U},\mathscr{F})=\prod_{(i_0,\ldots,i_p)\in I^{p+1}}\mathscr{F}(V_{i_0\cdots i_p}).$$

(b) Define $\delta^p: C^p(\mathcal{U},\mathscr{F})\to C^{p+1}(\mathcal{U},\mathscr{F})$ as follows. For the coordinate of d^pf corresponding to $(i_0,\ldots,i_{p+1})\in I^{p+2}$,

$$(d^pf)_{i_0\cdots i_{p+1}}=\sum_{j=0}^{p+1}\mathrm{res}_j(f_{i_0\cdots\hat{i}_j\cdots i_{p+1}}).$$

Here $\mathrm{res}_j:\mathscr{F}(V_{i_0\cdots\hat{i}_j\cdots i_{p+1}})\to\mathscr{F}(V_{i_0\cdots i_{p+1}})$ is induced by the inclusion $V_{i_0\cdots i_{p+1}}\subset V_{i_0\cdots\hat{i}_j\cdots i_{p+1}}$. The **Čech cohomology** of $\mathscr{F}$ with respect to $\mathcal{U}$ is defined to be

$$\check{H}^p(\mathcal{U},\mathscr{F}):=H^p(C^\bullet(\mathcal{U},\mathscr{F}),d^\bullet).$$

Remark 2.4. Some comments about Čech cohomology.

(1) Čech cohomology can be computed in terms of **alternating cochains**. Say $f\in C^p(\mathcal{U},\mathscr{F})$ is alternating if
   (i) $f_{i_0\cdots i_p}=0$ when there are $r\neq s$ such that $i_r=i_s$, and
   (ii) $f_{\sigma(i_0)\cdots\sigma(i_p)}=\mathrm{sgn}(\sigma)f_{i_0\cdots i_p}$ for $\sigma\in S_{p+1}$.

   If we use $C'^p(\mathcal{U},\mathscr{F})$ to denote the subgroup of $C^p(\mathcal{U},\mathscr{F})$ consisting of alternating cochains, then

$$\check{H}^p(\mathcal{U},\mathscr{F})\cong H^p(C'^\bullet(\mathcal{U},\mathscr{F}),d^\bullet).$$

(2) Čech cohomology $\check{H}^p(\mathcal{U},\mathscr{F})$ is not always isomorphic to sheaf cohomology $H^p(Y,\mathscr{F})$. However, they are related by spectral sequences. See [Har13, III, Thm 4.5] for an example.

2.3. **A Comparison Between Sheaf Cohomologies and Group Cohomologies.** We state the main result in [Mum85, Appendix of §2]. See also [Mil08, Chap.III, Example 2.6].

Theorem 2.5 (Comparison)**.** *For any sheaf $\mathscr{F}$ on Y, there is a natural group homomorphism*

$$\phi_p: H^p(G,\Gamma(X,\pi^*\mathscr{F}))\to H^p(Y,\mathscr{F})$$

with the following properties:

(1) *If*

$$0\to\mathscr{F}'\to\mathscr{F}\to\mathscr{F}''\to 0$$

is an exact sequence of sheaves on Y, and

$$0\to\Gamma(X,\pi^*\mathscr{F}')\to\Gamma(X,\pi^*\mathscr{F})\to\Gamma(X,\pi^*\mathscr{F}'')\to 0$$

is exact, then we get a homomorphism from the cohomology sequence of $H^p(G,\cdot)$ to that of $H^p(Y,\cdot)$; i.e., the following diagram commutes:

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
H^p(G, \Gamma(X, \pi^*\mathscr{F}')) & \longrightarrow & H^p(Y, \mathscr{F}') \\
\downarrow & & \downarrow \\
H^p(G, \Gamma(X, \pi^*\mathscr{F})) & \longrightarrow & H^p(Y, \mathscr{F}) \\
\downarrow & & \downarrow \\
H^p(G, \Gamma(X, \pi^*\mathscr{F}'')) & \longrightarrow & H^p(Y, \mathscr{F}'') \\
\downarrow & & \downarrow \\
H^{p+1}(G, \Gamma(X, \pi^*\mathscr{F}')) & \longrightarrow & H^{p+1}(Y, \mathscr{F}') \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

(2) *For any $p \geqslant 0$, ϕ_p is compatible with the cup products.*[4]

(3) *If for all $i \geqslant 1$,*

$$H^i(X, \pi^*\mathscr{F}) = 0,$$

then

$$\phi_p : H^p(G, \Gamma(X, \pi^*\mathscr{F})) \to H^p(Y, \mathscr{F})$$

is an isomorphism.

Remark 2.6 (Important subtlety for beginners). In algebraic geometry, the functor

$$\begin{aligned} \mathsf{Ring} &\longrightarrow \mathsf{Sch} \\ A &\longmapsto \mathrm{Spec}(A) \end{aligned}$$

is contravariant. On the other hand, in complex geometry, the functor

$$\begin{aligned} \mathsf{Manifold}_{\mathbb{C}} &\longrightarrow \mathsf{Ring} \\ X &\longmapsto \Gamma(X, \mathscr{O}_X) \end{aligned}$$

is contravariant as well. As a consequence, if we have a left action of G on X, it induces a right action of G on $\Gamma(X, \pi^*\mathscr{F})$.

Following the convention of Mumford, we define a left action of G on $\Gamma(X, \pi^*\mathscr{F})$ by composing with the inverse. In the case of complex torus, U acts on $H = \Gamma(V, \mathscr{O}_V)$ (resp., $H^* = \Gamma(V, \mathscr{O}_V^*)$) via the formula $(uh)(z) = h(z - u)$ for $u \in U$, $h \in H$ (resp., $h \in H^*$), and $z \in V$.

Now we come to the definition of the maps ϕ_p. Choose an open covering $\mathcal{U} = \{V_i\}_{i \in I}$ of Y such that:

(1) For all $p \geqslant 0$,

$$\check{H}^p(\mathcal{U}, \mathscr{F}) \cong H^p(Y, \mathscr{F}).$$

[4]For some historical reason, Mumford meant to say by (2) that ϕ_p commutes with the cup product. However, it is difficult to define the cup product of sheaf cohomologies. Fortunately, the case that will be at work may assume $\mathscr{F}$ is a constant sheaf.

(2) Along with π, we obtain

$$\pi^{-1}(V_i) = \bigsqcup_{\sigma \in G} \sigma(U_i),$$

where $U_i \subset X$ are open such that $\pi|_{U_i} : U_i \to V_i$ are all isomorphisms. (Recall that the action of G is discontinuous, hence we take the disjoint union.)

(3) For all i, j, there is at most one $\sigma_{ij} \in G$ such that $U_i \cap \sigma_{ij} U_j \neq \emptyset$.

Note that in (3), σ_{ij} exists if and only if $V_i \cap V_j \neq \emptyset$. Also, $\sigma_{ij}^{-1} = \sigma_{ji}$. If $V_i \cap V_j \cap V_k \neq \emptyset$, then $\sigma_{ik} = \sigma_{ij}\sigma_{jk}$. Now we are ready to construct the map ϕ_p. For $i \in I$, define α_i to be the composite

$$\begin{tikzcd} \Gamma(X, \pi^* \mathscr{F}) \arrow[r, "\text{res}"] \arrow[rr, bend left, "\alpha"] & \Gamma(U_i, \pi^* \mathscr{F}) \arrow[r, "\cong"] & \Gamma(V_i, \mathscr{F}). \end{tikzcd}$$

Define ϕ_p as the group homomorphism

$$\phi_p : C^p(G, \Gamma(X, \pi^* \mathscr{F})) \to C^p(\mathcal{U}, \mathscr{F}),$$

for which

$$(\phi_p f)_{i_0 \cdots i_p} = \text{res} \circ \alpha_{i_0}(f(\sigma_{i_0 i_1}, \sigma_{i_1 i_2}, \ldots, \sigma_{i_{p-1} i_p}))$$

for all $(i_0, \ldots, i_p) \in I^{p+1}$. Note that res is basically the restriction map

$$\Gamma(V_{i_0}, \mathscr{F}) \to \Gamma(V_{i_0 \cdots i_p}, \mathscr{F}).$$

Exercise 2.7. Check that ϕ_p induces a morphism of cohomology groups.

2.4. **Geometric Description of Holomorphic Line Bundles.** We concern about the case of complex torus $X = V/U$ say. Here X is a connected compact complex Lie group, $V = \text{Lie}\, X$, and U is a fixed lattice in V. There is a natural projection

$$\pi = \exp : V \to X = V/U.$$

Denote $H^* := \Gamma(V, \pi^* \mathscr{O}_X^*) = \Gamma(V, \mathscr{O}_V^*)$. Theorem 2.5 (3) dictates that

$$H^p(U, \Gamma(V, \pi^* \mathscr{O}_V^*)) \xrightarrow[\cong]{\phi_p} H^p(V, \mathscr{O}_V^*).$$

At the level of line bundles (when $p = 1$), here comes another construction of the isomorphism

$$H^1(X, \mathscr{O}_X^*) \xrightarrow{\cong} H^1(U, H^*).$$

Theorem 2.8. *For $p \geqslant 1$, we have*

$$H^p(\mathbb{C}^g, \mathscr{O}) \cong H^p(V, \mathscr{O}_V) = 0$$

by viewing $\mathbb{C}^g = \mathbb{C}^{\dim X} \approx V$ as an algebraic variety over $\mathbb{C}$. In particular,

$$H^p(\mathbb{C}^g, \mathscr{O}^*) \cong H^p(V, \mathscr{O}_V^*) \cong \text{Pic}(V) = 0$$

for $p \geqslant 1$.

Let $\mathscr{L}$ be a (holomorphic) line bundle on X. By the theorem, $\pi^* \mathscr{L} \in \text{Pic}(V)$ is trivial on V. So we can choose and fix an isomorphism

$$\chi : \pi^* \mathscr{L} \xrightarrow{\cong} \mathbb{C} \times V.$$

Conversely, to define a line bundle $\mathscr{L}$ on X, it suffices to define an action of U on $\mathbb{C}\times V$ that covers the translation action of U on V. Recall that H^* is the group of nowhere vanishing holomorphic functions on V. We define the action of U on H^* by

$$(uf)(z) = f(z+u).$$

The U-action on $\mathbb{C}\times V$ is of the following form:

$$u(\alpha, z) = (e_u(z)\cdot\alpha, z+u), \quad \alpha\in\mathbb{C},\ z\in V,\ u\in U,\ e_u\in H^*.$$

Exercise 2.9. Check that the association

$$\varphi : U\to H^*, \quad e\mapsto e_u$$

is indeed a 1-cocycle, i.e., an element in $Z^1(U, H^*)$.

If we modify the trivialization χ by a function $f\in H^*$, each of e_u will be replaced by

$$e'_u(z) = e_u(z)f(z+u)f(z)^{-1}.$$

This is because

$$\begin{tikzcd}
(\alpha, z) \arrow[r, mapsto] \arrow[d, mapsto, "u"] & (\alpha f(z), z) \arrow[d, mapsto, "u"] \\
(e_u(z)\alpha, z+u) \arrow[r, mapsto] & (e_u(z)\alpha f(z+u), z+u)
\end{tikzcd}$$

in which the right vertical arrow denotes the action under the new trivialization. So we have a well-defined map

$$H^1(X, \mathscr{O}_X^*)\to H^1(U, H^*),$$

which is compatible with the isomorphism established before. The upshot here is that if $u\mapsto e_u$ is a 1-cocycle under the action $(uf)(z) = f(z+u)$, then $u\mapsto f_u := e_{-u}$ is also a 1-cocycle under the action $(uh)(z) = h(z-u)$.

Proposition 2.10. *For any complex manifold Y, we have an exact sequence*

$$0\longrightarrow\mathbb{Z}\longrightarrow\mathscr{O}_Y\xrightarrow{\exp(2\pi i(\cdot))}\mathscr{O}_Y^*\longrightarrow 0.$$

Applying the proposition to X, it induces a long exact sequence of cohomology:

$$\begin{tikzcd}
 & & \mathrm{Pic}(X) \arrow[d, equal] & \\
\cdots \arrow[r] & H^1(X,\mathbb{Z}) \arrow[r] & \boxed{H^1(X,\mathscr{O}_X)} \arrow[r] & H^1(X,\mathscr{O}_X^*) \arrow[dll, out=0, in=180] \\
 & H^2(X,\mathbb{Z}) \arrow[r] & H^2(X,\mathscr{O}_X) \arrow[r] & H^2(X,\mathscr{O}_X^*) \arrow[r] & \cdots
\end{tikzcd}$$

On the other hand, one shall notice that the following is an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(V,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^0(V,\pi^*\mathscr{O}_X) \arrow[r, "\exp(2\pi i(\cdot))"] \arrow[d, equal] & H^0(V,\pi^*\mathscr{O}_X^*) \arrow[r] \arrow[d, equal] & 0. \\
 & \mathbb{Z} & H & H^* &
\end{tikzcd}$$

For the middle and the right terms above, use an open covering of V to see the equality to H and H^*, respectively. This sequence terminates on the right side because

$$H^i(V,\mathbb{Z}) = 0, \quad \forall i > 0.$$

Then by Theorem 2.5 (3) again, we have isomorphisms

$$H^p(U,\mathbb{Z}) \xrightarrow{\cong} H^p(X,\mathbb{Z})$$

for all $p \geqslant 0$. Here on the left side, we make the abelian group U acts trivially on $\mathbb{Z}$; on the right side, we regard $\mathbb{Z}$ as a constant sheaf on X. Moreover, the following diagram commutes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & H^1(U,H) \arrow[d, "\exp(2\pi i(\cdot))"'] & H^1(X,\mathscr{O}_X) \arrow[d] & \\
 & H^1(U,H^*) \arrow[r, "\cong"] \arrow[dd, "\delta"'] & \boxed{H^1(X,\mathscr{O}_X^*)} \arrow[dd] & \cong \mathrm{Pic}(X) \ni [\mathscr{L}] \arrow[dd, mapsto] \\
 \wedge^2 \mathrm{Hom}_{\mathbb{Z}}(U,\mathbb{Z}) \arrow[d, equal] & & & \\
 \mathrm{Hom}_{\mathbb{Z}}(\wedge^2 U,\mathbb{Z}) \cong & \boxed{H^2(U,\mathbb{Z})} \arrow[r, "\cong"] \arrow[d] & H^2(X,\mathbb{Z}) \arrow[d] & \ni \ c_1(\mathscr{L}) \\
 & \vdots & \vdots &
\end{tikzcd}$$

Here $c_1(\mathscr{L})$ denotes the first Chern class of $\mathscr{L}$.

Lemma 2.11. *The map*

$$\begin{aligned} A : Z^2(U,\mathbb{Z}) &\longrightarrow \mathrm{Hom}(\wedge^2 U,\mathbb{Z}) \\ F &\longmapsto AF(u_1,u_2) := F(u_1,u_2) - F(u_2,u_1) \end{aligned}$$

induces an isomorphism

$$A : H^2(U,\mathbb{Z}) \longrightarrow \mathrm{Hom}(\wedge^2 U,\mathbb{Z}) \xrightarrow{\cong} \wedge^2 \mathrm{Hom}(U,\mathbb{Z}).$$

Moreover, we have the commutative diagram

$$\begin{tikzcd}
H^1(U,\mathbb{Z}) \times H^1(U,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^2(U,\mathbb{Z}) = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[d, "\cong"', "A"] \\
\mathrm{Hom}(U,\mathbb{Z}) \times \mathrm{Hom}(U,\mathbb{Z}) \arrow[r, "\wedge"] & \wedge^2 \mathrm{Hom}(U,\mathbb{Z})
\end{tikzcd}$$

Remark 2.12. In fact we have isomorphism of graded rings

$$H^*(U,\mathbb{Z}) \xrightarrow{\cong} H^*(X,\mathbb{Z}).$$

Given $[\mathscr{L}] \in H^1(X,\mathscr{O}_X^*)$ that corresponds to the element $[\{e_u\}]$ in $H^1(U,H^*)$, let $E \in \mathrm{Hom}(\wedge^2 U,\mathbb{Z})$ be the alternating form obtained by

$$H^1(U,H^*) \xrightarrow{\delta} H^2(U,\mathbb{Z}) \xrightarrow{\cong} \mathrm{Hom}(\wedge^2 U,\mathbb{Z})$$

along the isomorphism given by Lemma 2.11.

Lemma 2.13. *If we $\mathbb{R}$-linearly extend E to a map $E : V \times V \to \mathbb{R}$, then this extended E satisfies the identity*

$$E(ix,iy) = E(x,y), \quad \forall x,y \in V.$$

Proof. The proof requires some Hodge theory that is explained in the following two commutative diagrams.

(1) Regarding $\mathbb{C}$ as a constant sheaf over X, we have by Theorem 2.5 that $H^1(U,\mathbb{C}) \cong H^1(X,\mathbb{C})$. Also, by the Hodge decomposition, $H^1(X,\mathbb{C}) \cong H^{1,0}(X) \oplus H^{0,1}(X)$. Therefore,

$$\begin{tikzcd} H^1(U,\mathbb{C}) \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) \arrow[r, "\cong"] & \Omega \oplus \overline{\Omega} \arrow[d] \\ H^1(X,\mathbb{C}) \arrow[rr, "\text{Hodge Decomposition}", "\cong"'] & & H^{1,0}(X) \oplus H^{0,1}(X). \end{tikzcd}$$

(2) Again, the Hodge decomposition for $H^2(X,\mathbb{C})$ leads to a natural projection $H^2(X,\mathbb{C}) \to H^{0,2}(X)$.

$$\begin{tikzcd} & H^2(U,\mathbb{Z}) \arrow[d, equal] & \\ H^1(X,\mathscr{O}_X^*) \arrow[r, "c_1"] & H^2(X,\mathbb{Z}) \arrow[r] \arrow[d, hook] & H^2(X,\mathscr{O}_X) \arrow[d, equal] \\ & H^2(X,\mathbb{C}) \arrow[r, two heads, "\text{Hodge}"] \arrow[d, "\cong"] & H^{0,2}(X) \arrow[d, "\cong"] \\ & \wedge^2\Omega \oplus (\Omega \times \overline{\Omega}) \oplus \wedge^2\overline{\Omega} \arrow[r] & \wedge^2\overline{\Omega}. \end{tikzcd}$$

Then the image of E in $H^2(X,\mathbb{C})$ lies in the mixed part $\Omega \times \overline{\Omega}$, i.e., it is a Hodge class, and the lemma follows. □

Upshot. In the proof of Lemma 2.13, the idea is to pretend E to be the first Chern class of some line bundle representative. Then chase along the diagram to find out how it can be realized as a Hodge class.

Lemma 2.14. *Let V be a complex vector space. We have a bijective correspondence*

$$\begin{array}{ccc} \{\text{Hermitian forms } H \text{ on } V\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{real skew-symmetric form } E \text{ on } V \text{ such that} \\ E(iu,iv) = E(u,v) \text{ for all } u,v \in V \end{array} \right\} \\ H & \longmapsto & \mathrm{Im}\, H \\ H(x,y) = E(x,y) + iE(x,y) & \longleftarrow\!\shortmid & E \end{array}$$

Definition 2.15 (Néron-Severi group)**.** Define the Néron-Severi group of $X = V/U$ to be

$$\mathrm{NS}(X) := \{H \text{ Hermitian form on } V \mid \mathrm{Im}\, H|_{U\times U} \subset \mathbb{Z}\}.$$

Loosely, it consists of the Hermitian forms that take integral imaginary part on the lattice. Seriously,

$$\mathrm{NS}(X) := \mathrm{im}(H^1(X,\mathscr{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z})).$$

Also, $\mathrm{NS}(X)$ has a natural abelian group structure under addition.

Lemma 2.16. *Fix $H \in \mathrm{NS}(X)$ and let $E = \mathrm{Im}\, H$.*

(1) *There exists (but not necessarily unique) a map*

$$\alpha : U \longrightarrow \mathbb{C}_1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

such that

$$\alpha(u_1+u_2)=e^{i\pi E(u_1,u_2)}\alpha(u_1)\alpha(u_2),\quad \forall u_1,u_2\in U.$$

(2) *For such an α as in (1), define*

$$e_u(z):=\alpha(u)e^{\pi H(z,u)+\frac{1}{2}\pi H(u,u)},\quad u\in U,\ z\in V.$$

Then $u\mapsto e_u$ defines an element in $H^1(U,H^*)$ and its Chern class of the associated line bundle is $E\in H^2(U,\mathbb{Z})\cong H^2(X,\mathbb{Z})$.

Now we define

$$P(X):=\{(H,\alpha)\mid H\in \mathrm{NS}(X),\ \alpha \text{ be as in (1)}\}.$$

Then $P(X)$ has an abelian group structure as well: say

$$(H_1,\alpha_1)(H_2,\alpha_2)=(H_1+H_2,\alpha_1\alpha_2).$$

And therefore,

$$\mathscr{L}(H_1,\alpha_1)\otimes\mathscr{L}(H_2,\alpha_2)=\mathscr{L}(H_1+H_2,\alpha_1\alpha_2).$$

We infer that there is an exact sequence of abelian groups

$$\begin{tikzcd}
& \alpha \arrow[r, mapsto] & (0,\alpha) & & \\
0 \arrow[r] & \mathrm{Hom}_{\mathsf{Grp}}(U,\mathbb{C}_1) \arrow[r] \arrow[d, dashed, "\lambda"] & P(X) \arrow[r] \arrow[d, "(2)"] & \mathrm{NS}(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] & H^1(X,\mathscr{O}_X^*) \arrow[r] \arrow[d, equal] & \mathrm{NS}(X) \arrow[r] & 0 \\
& & \mathrm{Pic}(X) & &
\end{tikzcd}$$

where $\mathrm{Pic}^0(X):=\mathrm{Ker}(H^1(X,\mathscr{O}_X^*)\xrightarrow{c_1}H^2(X,\mathbb{Z}))$. And the middle vertical map above is given by Lemma 2.16 (2).

Proposition 2.17. *The map $P(X)\to\mathrm{Pic}(X)$ induces an isomorphism*

$$\lambda:\mathrm{Hom}_{\mathsf{Grp}}(U,\mathbb{C}_1)\xrightarrow{\cong}\mathrm{Pic}^0(X).$$

Proof. First we prove that λ is injective. For $\alpha\in\mathrm{Hom}_{\mathsf{Grp}}(U,\mathbb{C}_1)$, if $\lambda(\alpha)=1$, the 1-cocycle

$$\alpha\in\mathrm{Hom}_{\mathsf{Grp}}(U,\mathbb{C}_1)\subset\mathrm{Hom}_{\mathsf{Grp}}(U,\mathbb{C}^*)=H^1(U,\mathbb{C}^*)$$

will become trivial in $H^1(U,H^*)\cong H^1(X,\mathscr{O}_X^*)$, which renders that we can find $g(z)\in H^*$ such that

$$\frac{g(z+u)}{g(z)}=\alpha(u),\quad \forall z\in X.$$

Since $|\alpha(u)|=1$ for all $u\in U$, we see $|g(z+u)|=|g(z)|$. As $V/U=X$ is compact, $g(z)$ should be bounded on V. Then $g(z)$ is a constant function, and then $\alpha\equiv 1$. Therefore, λ is injective. It suffices to prove that it is surjective as well. Let us consider the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^1(X,\mathbb{Z}) \arrow[r] & H^1(X,\mathscr{O}_X) \arrow[r, "\exp(2\pi i(\cdot))"] & H^1(X,\mathscr{O}_X^*) \arrow[r, "c_1"] \arrow[d, equal] & H^2(X,\mathbb{Z}) \\
 & & H^1(X,\mathbb{C}) \arrow[u, two heads, "\text{Hodge Decomp.}", "p"'] \arrow[r, "\exp(2\pi i(\cdot))"] \arrow[d, equal, "\text{Theorem 2.5}"'] & H^1(X,\mathscr{O}_X^*) \arrow[d, equal, "\text{by construction}"] & \\
 & & H^1(U,\mathbb{C}) \arrow[r, "\exp(2\pi i(\cdot))"] & H^1(U,H^*) &
\end{tikzcd}$$

By the Hodge decomposition, $p : H^1(X,\mathbb{C}) \to H^1(X,\mathscr{O}_X)$ is surjective. This gives rise to

$$\mathrm{im}(H^1(X,\mathbb{C}) \to H^1(X,\mathscr{O}_X^*)) = \mathrm{im}(H^1(X,\mathscr{O}_X) \to H^1(X,\mathscr{O}_X^*)) = \mathrm{Pic}^0(X).$$

It follows that any line bundle $\mathscr{L}$ corresponds to a 1-cocycle $e_u(z) = e^{2\pi i f(u)}$ for some $f \in \mathrm{Hom}_{\mathsf{Grp}}(U,\mathbb{C})$. We extend $\mathbb{R}$-linearly that

$$\mathrm{Im}\, f : U \to \mathbb{R} \quad \rightsquigarrow \quad \mathrm{Im}\, f : V \to \mathbb{R}.$$

Also define

$$\begin{aligned} l : V &\longrightarrow \mathbb{C} \\ v &\longmapsto \mathrm{Im}(f(iv)) + i\,\mathrm{Im}(f(v)) \end{aligned}$$

together with

$$e'_u(z) := e_u(z) \cdot e^{2\pi i(l(z)-l(u+z))} = e^{2\pi i(f(u)+l(z)-l(u+z))}.$$

Notice that e'_u equals to e_u in $H^1(U,H^*)$ and

$$f(u) + l(z) - l(u+z) = \mathrm{Re}(f(u)) - \mathrm{Im}(f(iu)) \in \mathbb{R}$$

does not depend on z. Hence $e'_u(z) \in H^1(U,\mathbb{C}_1)$, which completes the proof. $\square$

3. Algebraizability of Tori

Goal. We have previously mentioned that the complex torus $\mathbb{C}/\Lambda$ can be realized as a projective algebraic curve. Or more generally, we are to realize the complex Lie group $X = V/U$ as a projective algebraic variety over $\mathbb{C}$, via the existence of a positive definite Hermitian forms.

3.1. **On Projective Morphisms.** The prototypical reference of this is [Har13, II, §7].

(1) To give a projective morphism $\varphi : X \to \mathbb{P}^n_k$ is equivalent to giving an invertible sheaf $\mathscr{L} = \varphi^*(\mathscr{O}(1))$ and sections $s_i = \varphi^*(x_i)$ $(i = 0, \ldots, n)$ that generate $\mathscr{L}$.

(2) Under the notations in (1), φ is a closed immersion if and only if for the subspace $V = \mathrm{Span}\{s_i\} \subset \Gamma(X, \mathscr{L})$ the following two conditions hold:
- elements of V separate points, i.e., for all $P, Q \in X$, there is $s \in V$ such that $s \in \mathfrak{m}_P \mathscr{L}_P$, $s \notin \mathfrak{m}_Q \mathscr{L}_Q$, or vice versa;
- elements of V separate tangent vectors, i.e., for each $P \in X$, the set $\{s \in V \mid s_P \in \mathfrak{m}_P \mathscr{L}_P\}$ spans the k-vector space $\mathfrak{m}_P \mathscr{L}_P / \mathfrak{m}_P^2 \mathscr{L}_P$.

Remark 3.1. (1) Sections of $\Gamma(X, \mathscr{L})$ cannot simultaneously vanish (resp., non-vanish) on some subvariety of X.

(2) φ induces injective maps on tangent spaces at all closed points.

3.2. **Global Sections of a Line Bundle.** Therefore, it is important to study the global sections of a line bundle $\mathscr{L} \in \mathrm{Pic}(X)$. We now explain why it is necessary to assume that H is positive definite and non-degenerate.

(a) Suppose that $\mathscr{L} = \mathscr{L}(H, \alpha)$, where H is a Hermitian form on V such that $E = \operatorname{Im} H$ being integral on $U \times U$ and $\alpha : U \to \mathbb{C}_1$ satisfying
$$\alpha(u_1 + u_2) = e^{i\pi E(u_1, u_2)} \alpha(u_1)\alpha(u_2).$$
It turns out that
$$\begin{aligned} &\Gamma(X, \mathscr{L}(H, \alpha)) \\ &= \{\text{sections of } \mathbb{C} \times V \to V \text{ that are invariant under the action of } U\} \\ &= \{\theta : V \to \mathbb{C} \text{ theta function} \mid \theta(z + u) = e_u(z)\theta(z),\ \forall z \in V,\ u \in U\}. \end{aligned}$$

(b) Suppose H is degenerate. Let
$$N := \{x \in V \mid H(x, y) = 0,\ \forall y \in V\} = \{x \in V \mid E(x, y) = 0,\ \forall y \in V\}.$$
Then $N \subset V$ is a complex subspace and $N \cap U$ is a lattice in N as $E|_{U \times U} \subset \mathbb{Z}$. For a theta function $\theta \in \Gamma(X, \mathscr{L}(H, \alpha))$ such that $\theta(z + u) = \alpha(u)\theta(z)$ for all $u \in N \cap U$,
$$\theta(z + z') = \theta(z), \quad \forall z \in V,\ z' \in N.$$
Consider the complex subtorus $X' = N/N \cap U$ of X. Elements in $\Gamma(X, \mathscr{L}(H, \alpha))$ cannot separate points in X'. Therefore, $\mathscr{L}(H, \alpha)$ can never ever be ample.

(c) When H is non-degenerate but not positive definite, one can show that $\Gamma(X, \mathscr{L}(H, \alpha)) = 0$ and hence $\mathscr{L}(H, \alpha)$ cannot be ample.

Therefore, in the upcoming context, we **always assume** H **is positive definite** to make all of the constructions to be reasonable.

Proposition 3.2. *When H is positive definite, we have*

$$\dim H^0(X, \mathscr{L}(H,\alpha)) = \sqrt{\det(E)}.$$

Sketch of Proof. By the construction, U is a lattice in V of rank $2g$. We choose a sublattice U' of U or rank g such that $E|_{U'\times U'} \equiv 0$, and if $W = U' \otimes_{\mathbb{Z}} \mathbb{R}$ then $W \cap U = U'$ (this is to ensure that U is the maximal sublattice with respect to the previous condition).

Define a map

$$\begin{aligned} \beta : U &\longrightarrow \widehat{U} := \mathrm{Hom}_{\mathbb{Z}}(U, \mathbb{Z}) \\ u &\longmapsto (u' \mapsto E(u, u')). \end{aligned}$$

By some complex analysis calculation, one can prove that

$$\dim H^0(X, \mathscr{L}(H,\alpha)) = \#\,\mathrm{Coker}(U \xrightarrow{\beta} \widehat{U} \to \widehat{U}').$$

Then the proposition follows from the following commutative diagram:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & U' \arrow[r] \arrow[d, "\alpha"] & U \arrow[r] \arrow[d, "\beta"] & U/U' \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & \widehat{U/U'} \arrow[r] & \widehat{U} \arrow[r] & \widehat{U}' \arrow[r] & 0 \end{tikzcd}$$

and the fact that

$$\#\,\mathrm{Coker}\,\beta = (\det E)^2, \quad \#\,\mathrm{Coker}\,\alpha = \#\,\mathrm{Coker}\,\beta.$$

□

3.3. **Dual Complex Tori.**

Definition 3.3. The **$\mathbb{C}$-vector space of $\mathbb{C}$-antilinear maps** is defined to be

$$\overline{V} := \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C}).$$

We have a canonical $\mathbb{R}$-linear isomorphism

$$\begin{aligned} \mathrm{Hom}_{\overline{\mathbb{C}}}(V,\mathbb{C}) &\xrightarrow{\ \cong\ } \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}) \\ l &\longmapsto \mathrm{Im}\, l \\ l(v) = -k(iv) + ik(v) &\longleftarrow\!\!\shortmid\ k. \end{aligned}$$

This leads to a non-degenerate bilinear map

$$\begin{aligned} \langle \cdot,\cdot \rangle : \overline{V} \times V &\longrightarrow \mathbb{R} \\ (l, v) &\longmapsto \mathrm{Im}\, l(v). \end{aligned}$$

Define

$$\widehat{U} = \{ l \in \overline{V} \mid \langle l, U \rangle \subset \mathbb{Z} \},$$

which is a lattice of $\overline{V}$. The complex torus

$$\widehat{X} := \overline{V}/\widehat{U}$$

is called the dual complex torus of X.

Proposition 3.4. *We list out the following basic facts about $\widehat{X}$.*

(1) *The map*

$$\begin{aligned} \overline{V} &\longrightarrow \mathrm{Hom}_{\mathsf{Grp}}(U, \mathbb{C}_1) = \mathrm{Pic}^0(X) \\ l &\longmapsto e^{2\pi i \langle l, \cdot \rangle} \end{aligned}$$

induces an isomorphism $\widehat{X} \cong \mathrm{Pic}^0(X)$.

(2) *For every $\mathscr{L} = \mathscr{L}(H, \alpha) \in \mathrm{Pic}(X)$, the map*

$$\begin{aligned} V &\longrightarrow \overline{V} \\ v &\longmapsto H(v, \cdot) \end{aligned}$$

induces a homomorphism of complex torus $\phi_{\mathscr{L}} : X \to \widehat{X}$.

(3) *Moreover, as in (2),*

$$\phi_{\mathscr{L}} \text{ is surjective as an isogeny} \quad \Longleftrightarrow \quad H \text{ is non-degenerate.}$$

And in this case,

$$\deg(\phi_{\mathscr{L}}) = \det \mathrm{Im}(H).$$

When H is positive definite, we see that

$$\deg(\phi_{\mathscr{L}}) = (\dim H^0(X, \mathscr{L}))^2.$$

Theorem 3.5 (Lefschetz)**.** *Let X be a complex torus and $\mathscr{L} = \mathscr{L}(H, \alpha)$ be a line bundle on X. Then the following are equivalent:*

(1) *H is positive definite;*
(2) *$\mathscr{L}$ is ample;*
(3) *$\mathscr{L}^{\otimes n}$ is very ample for all $n \geqslant 3$.*

We will give an algebraic proof of the results above.

Corollary 3.6. *If $\mathscr{L} = \mathscr{L}(H, \alpha)$ is ample and $\det E = 1$, i.e., $\mathscr{L}$ gives a principal polarization of X, then X admits a closed immersion into some projective space:*

$$X \hookrightarrow \mathbb{P}^{3^g - 1}, \quad g = \dim_{\mathbb{C}} X.$$

In the future, this corollary is the key to construct the moduli space of abelian varieties.

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