Exercise 4 (due on November 30)

Choose 4 out of 8 problems to submit.

For this exercise, we fix the coefficient field to be a finite extension E of $\mathbb{Q}_\ell$, with ring of integers $\mathcal{O}$, uniformizer ϖ, and residue field $\mathbb{F}$. The letter F is reserved to denote a number field, S a finite set of places including the ones dividing $\ell\infty$.

Problem 4.1. (Tangent space for relative deformation problem) Let T be a subset of S. For a continuous $G_{F,S}$-module M, define $\widetilde{H}^i_T(G_{F,S}, M)$ to be the cohomology of

$$\widetilde{\mathrm{R}\Gamma}_T\big(G_{F,S}, M\big) := \mathrm{Cone}\Big(\mathrm{R}\Gamma\big(G_{F,S}, M\big) \to \bigoplus_{v\in T} \mathrm{R}\Gamma\big(G_{F_v}, M\big)\Big)[-1].$$

Let $\bar\rho : G_{F,S} \to \mathrm{GL}_n(\mathbb{F})$ denote an absolutely irreducible representation and $\chi : G_{F,S} \to \mathcal{O}^\times$ a lift of $\det\bar\rho$. Assume that $\ell \nmid n$. Consider the following functor

$$\begin{aligned} \mathrm{Def}^{\square_T,\chi}_{\bar\rho} : \mathsf{CNL}_{\mathcal{O}} &\longrightarrow \mathsf{Sets} \\ A &\longmapsto \left\{ (\rho, (h_v)_{v\in T}) \;\middle|\; \begin{array}{l} \bullet\ \text{ for each } v,\ h_v \in \widehat{\mathrm{PGL}}_n(A), \\ \bullet\ \rho : G_{F,S} \to \mathrm{GL}_n(A) \text{ cont. repn. s.t.} \\ \bullet\ \rho \bmod \mathfrak{m}_A = \bar\rho \text{ and } \det\rho = \chi. \end{array} \right\} \Big/ \sim \end{aligned}$$

where $(\rho, (h_v)_{v\in T}) \sim (\rho', (h'_v)_{v\in T})$ if there exists $x \in \widehat{\mathrm{PGL}}_n(A)$ such that $\rho' = x\rho x^{-1}$ and $h'_v = h_v x^{-1}$ for every $v \in T$.

There is a natural morphism

$$\begin{aligned} \mathrm{Def}^{\square_T,\chi}_{\bar\rho}(A) &\longrightarrow \textstyle\prod_{v\in T} \mathrm{Def}^{\square,\chi_v}_{\bar\rho_v}(A) \\ (\rho, (h_v)_{v\in T}) &\longmapsto h_v \rho h_v^{-1}. \end{aligned}$$

This gives rise to a natural homomorphism

$$R^{\square_T}_{\mathrm{loc}} := \widehat{\bigotimes}_{v\in T} R^{\square,\chi_v}_{\bar\rho_v} \longrightarrow R^{\square_T,\chi}_{\bar\rho}$$

Let $\mathfrak{m}^{\square_T}_{\mathrm{loc}} := (\mathfrak{m}^{\square_v,\chi_v}_{\bar\rho_v}; v \in T)$ and let $\mathfrak{m}^{\square_T,\chi}_{\bar\rho}$ denote the maximal ideal of $R^{\square_T,\chi}_{\bar\rho}$.

(1) Show that

$$\Big(\mathfrak{m}^{\square_T,\chi}_{\bar\rho} / \big((\mathfrak{m}^{\square_T,\chi}_{\bar\rho})^2, \mathfrak{m}^{\square_T}_{\mathrm{loc}}\big)\Big)^* \;\cong\; \mathrm{Ker}\,\Big(\mathrm{Def}^{\square_T,\chi}_{\bar\rho}(\mathbb{F}[\epsilon]) \to \prod_{v\in T} \mathrm{Def}^{\square,\chi_v}_{\bar\rho_v}(\mathbb{F}[\epsilon])\Big)$$

(2) Fill in details of the proof in class that

$$\mathrm{Ker}\,\Big(\mathrm{Def}^{\square_T,\chi}_{\bar\rho}(\mathbb{F}[\epsilon]) \to \prod_{v\in T} \mathrm{Def}^{\square,\chi_v}_{\bar\rho_v}(\mathbb{F}[\epsilon])\Big) \cong \widetilde{H}^1_T(G_{F,S}, \mathrm{Ad}^0\,\bar\rho).$$

Problem 4.2. (Relations for relative deformation problem) Continued with the previous problem and the notation therein, write J for the kernel of the map

$$R^{\square_T}_{\mathrm{loc}}[\![x_1, \dots, x_t]\!] \twoheadrightarrow R^{\square_T,\chi}_{\bar\rho}.$$

Let $\mathfrak{m}$ denote the maximal ideal $(\mathfrak{m}^{\square_T}_{\mathrm{loc}}, x_1, \dots, x_t)$ of $R^{\square_T}_{\mathrm{loc}}[\![x_1, \dots, x_t]\!]$. Show that there is a natural injective map

$$\big(J/\mathfrak{m}J\big)^* \hookrightarrow \widetilde{H}^2_T(G_{F,S}, \mathrm{Ad}^0\,\bar\rho).$$

Problem 4.3. (Equivalent description of deformation problem) Let Γ be a profinite group satisfying the finiteness condition for deformation problem. Let $\bar{\rho} : \Gamma \to \mathrm{GL}_n(\mathbb{F})$ be a residual representation and $R_{\bar{\rho}}^{\square}$ the universal deformation ring, equipped with the action of $\widehat{\mathrm{PGL}}_n$. Let I denote an ideal of $R_{\bar{\rho}}^{\square}$ that is scheme-theoretically stable under the action of $\widehat{\mathrm{PGL}}_n$. Consider the following set:

$$\mathcal{D}(I) := \left\{ (A, \rho) \;\middle|\; \begin{array}{l} A \in \mathsf{CNL}_{\mathcal{O}};\ \rho : \Gamma \to \mathrm{GL}_n(A) \text{ lifts } \bar{\rho} \text{ such that} \\ \text{the induced universal map } R_{\bar{\rho}}^{\square} \to A \text{ factors through } R_{\bar{\rho}}^{\square}/I \end{array} \right\}.$$

Prove that $\mathcal{D} = \mathcal{D}(I)$ satisfies the following list of properties:

(1) $(\mathbb{F}, \bar{\rho}) \in \mathcal{D}$.
(2) For $f : A \to B$ a morphism in $\mathsf{CNL}_{\mathcal{O}}$, $(A, \rho) \in \mathcal{D} \ \Rightarrow\ (B, f \circ \rho) \in \mathcal{D}$.
(3) If $f : A \hookrightarrow B$ is an injective morphism in $\mathsf{CNL}_{\mathcal{O}}$, then

$$(A, \rho) \in \mathcal{D} \ \Leftrightarrow\ (B, f \circ \rho) \in \mathcal{D}.$$

(4) For $A_1, A_2 \in \mathsf{CNL}_{\mathcal{O}}$ with ideals $I_1 \subset A_1$ and $I_2 \subset A_2$, if there is an isomorphism

$$f : A_1/I_1 \cong A_2/I_2 \quad \text{such that} \quad f(\rho_1 \bmod I_1) = \rho_2 \bmod I_2,$$

then $\big(A_1 \times_{A_1/I_1} A_2, \rho_1 \oplus \rho_2\big) \in \mathcal{D}$, where

$$A_1 \times_{A_1/I_1} A_2 := \{(a_1, a_2) \in A_1 \times A_2 \mid f(a_1 \bmod I_1) = a_2 \bmod I_2\}.$$

(5) If $I_1 \supset I_2 \supset \cdots$ is a nested sequence of ideals of $A \in \mathsf{CNL}_{\mathcal{O}}$ such that $\bigcap_i I_i = (0)$ and if (A, ρ) lifts $\bar{\rho}$ such that $(A/I_i, \rho \bmod I_i) \in \mathcal{D}$, then $(A, \rho) \in \mathcal{D}$.
(6) For $(A, \rho) \in \mathcal{D}$ and $a \in (1 + M_n(\mathfrak{m}_A))^{\times}$, we have $(A, a\rho a^{-1}) \in \mathcal{D}$.

Conversely, let $\mathcal{D}$ be a subset of pairs (A, ρ) such that $A \in \mathsf{CNL}_{\mathcal{O}}$ and $\rho : \Gamma \to \mathrm{GL}_n(A)$ lifts $\bar{\rho}$, we define an ideal $I(\mathcal{D})$ of $R_{\bar{\rho}}^{\square}$ as follows: consider

$$J := \big\{\text{ideals } I \subset R_{\bar{\rho}}^{\square} \;\big|\; \big(R_{\bar{\rho}}^{\square}, \rho^{\mathrm{univ}}/I\big) \in \mathcal{D}\big\}.$$

Show that

(a) J contains a unique minimal ideal, denoted by $I(\mathcal{D})$.
(b) $(A, \rho) \in \mathcal{D}$ if and only if the map $R_{\bar{\rho}}^{\square} \to A$ induced by ρ factors through the quotient $R_{\bar{\rho}}^{\square}/I(\mathcal{D})$.
(c) Show that $\mathcal{D}(I(\mathcal{D})) = \mathcal{D}$ and $I(\mathcal{D}(I)) = I$.

Optional question: Show that when I is a radical ideal. Then, to get $\mathcal{D}(I)$, it is enough to require that I is point-wise stable under the action of $\widehat{\mathrm{PGL}}_n(R_{\bar{\rho}}^{\square})$.

Problem 4.4. (Example of local deformation ring) Consider the *trivial* representation $\bar{\rho} : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F}_\ell)$. Assume moreover that $p \equiv 1 \bmod \ell$. We hope to analysis the generic fiber of the deformation ring $R_{\bar{\rho}})^{\square}$. For each point $x \in R_{\bar{\rho}}^{\square}[\frac{1}{\ell}](E)$, let ρ_x denote the corresponding representation, and let (r, N) denote the Weil–Deligne representation associated to ρ_x.

(1) Using the fact that $\bar{\rho} = \mathrm{triv}$, prove that ρ_x is trivial on the wild inertia subgroup. Therefore, ρ_x is determined by the image of two elements: geometric Frobenius ϕ and a generator τ of tame group.

Recall that $N = \log(\tau)$ is always nilpotent. So we separate the two cases (up to conjugation): $N = 0$ and $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

(2) When $N = 0$, $\rho_x(\tau)$ has finite index. In particular, it is given by a semisimple element. Prove that in this case, ρ_x takes form of (up to conjugation)

$$\rho_x(\tau) = \begin{pmatrix} \zeta_{p-1}^a & 0 \\ 0 & \zeta_{p-1}^b \end{pmatrix}, \qquad \rho_x(\phi) = \begin{pmatrix} \alpha & * \\ 0 & \beta \end{pmatrix}$$

where $*$ is zero unless $a = b$ and $\alpha = \beta$. Show that the subspace $\operatorname{Spec} R_{\bar{\rho}}^{\square}[\frac{1}{\ell}]$ defined by $N = 0$ is the union of *smooth* irreducible components labeled by the pair (a, b).

(3) When $N \neq 0$, show that the two Frobenius eigenvalues have ratio equal to p, and in this case, commuting with the action of Frobenius, (in the Weil representation r_x), $r_x(\tau) = \begin{pmatrix} \zeta_{p-1}^a & 0 \\ 0 & \zeta_{p-1}^a \end{pmatrix}$. Show that the subspace $\operatorname{Spec} R_{\bar{\rho}}^{\square}[\frac{1}{\ell}]$ cut out by the condition: Frobenius eigenvalues have ratio equal to p, or equivalently $p\mathrm{tr}(\rho(\phi))^2 = (1+p)^2 \det(\rho(\phi))$, is a union of smooth irreducible components, labeled by the number a.

Problem 4.5. (Tilting process) Let K be a complete discrete valuation field of mixed characteristic and perfect residue field k. Let $\mathbb{C}_p$ denote the p-adic completion of K^{alg}, and $\mathcal{O}_{\mathbb{C}_p}$ its ring of integers.

Show that there is a bijection

$$\varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/(p) \;\cong\; \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}.$$

Indeed, given a sequence $(x_0, x_1, \dots)$ from the first inverse limit, we define $(x^{(0)}, x^{(1)}, \dots)$ in the second sequence as follows:

$$x^{(n)} := \lim_{m \to \infty} (\tilde{x}_{m+n})^{p^m},$$

where $\tilde{x}_{m+n}$ is a lift of $x_{m+n} \in \mathcal{O}_{\mathbb{C}_p}/(p)$ in $\mathcal{O}_{\mathbb{C}_p}$. Show that this definition is well-defined and it does give the needed bijection above. (Hint: prove first that if $a \equiv b \bmod p^m\mathcal{O}_{\mathbb{C}_p}$ then $a^p \equiv b^p \bmod p^{m+1}\mathcal{O}_{\mathbb{C}_p}$. This implies that the limit above converges.)

Problem 4.6. (θ-map for $\mathbb{A}_{\mathrm{inf}}$) Continued with the setup of the previous problem. Write $\mathbb{A}_{\mathrm{inf}} := W(\mathcal{O}_{\mathbb{C}_p})$ and consider the following map $\theta : \mathbb{A}_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_p}$: each element of $\mathbb{A}_{\mathrm{inf}}$ can be written as $\sum_{n \geq 0} p^n [a_n]$, where $[a_n]$ is the Teichmüller lift of $a_n = (a_n^{(0)}, a_n^{(1)}, \dots)$; we define

$$\theta\Big(\sum_{n \geq 0} p^n [a_n]\Big) \;=\; \sum_{n \geq 0} p^n a_n^{(0)}.$$

Show that θ is a homomorphism.

Problem 4.7. (More general Selmer duality) We start with local situation. For K a nonarchimedean local field, and let M a continuous $\mathcal{O}[G_K]$-module. Let

$$\mathcal{D} := \big[D^0 \to D^1 \to \dots\big] \quad \text{and} \quad \mathcal{D}^* := \big[(D^*)^0 \to (D^*)^1 \to \cdots\big]$$

be two complexes with morphisms $\mathcal{D} \to \mathrm{R}\Gamma(G_K, M)$ and $\mathcal{D}^* \to \mathrm{R}\Gamma(G_K, M^*(1))$ that induce *injective* maps on all cohomology groups. If under the natural cup product

$$\mathcal{D} \otimes \mathcal{D}^* \to \mathrm{R}\Gamma(G_K, M) \times \mathrm{R}\Gamma(G_K, M^*(1)) \xrightarrow{\cup} \mathrm{R}\Gamma(G_K, E/\mathcal{O}(1)) \to E/\mathcal{O}[-2],$$

we have

$$\begin{array}{ccccc} H^i(\mathcal{D}) & & H^{2-i}(\mathcal{D}^*) & & \\ \bigcap & & \bigcap & & \\ H^i(G_K, M) & \times & H^{2-i}(G_K, M^*(1)) & \xrightarrow{\cup} & H^2(G_K, E/\mathcal{O}(1)) \cong E/\mathcal{O}, \end{array}$$

$H^i(\mathcal{D})$ and $H^{2-i}(\mathcal{D}^*)$ are exact annihilators of each other, we say that $\mathcal{D}$ and $\mathcal{D}^*$ are *dual local conditions* for Galois cohomology.

A trivial example of dual local condition is $\mathcal{D} = 0$ and $\mathcal{D}^* = R\Gamma(G_K, M^*(1))$.

(1) Assume that ℓ does not divide the residual characteristic of K. Consider the continuous cochain complex $\mathrm{R}\Gamma(G_K, M)$:

$$C^0(G_K, M) \to C^1(G_K, M) \to C^2(G_K, M) \to \cdots$$

Set

$$\mathcal{D}^K(G_K, M) := \big[C^0(G_K, M) \to Z^1(G_K, M)\big]$$

$$\mathcal{D}_K(G_K, M^*(1)) := \big[C^0(G_K, M^*(1)) \to B^1(G_K, M^*(1))\big]$$

Show that $\mathcal{D}^K(G_K, M)$ and $\mathcal{D}_K(G_K, M^*(1))$ are dual local conditions.

(2) Continued with the previous setup. Let $Z^1_{\mathrm{ur}}(G_K, M)$ denote the preimage of $H^1_{\mathrm{ur}}(G_K, M) \subseteq H^1(G_K, M)$ under the natural quotient $Z^1(G_K, M) \twoheadrightarrow H^1(G_K, M)$. Consider

$$\mathcal{D}_{\mathrm{ur}}(G_K, M) := \big[C^0(G_K, M) \to Z^1_{\mathrm{ur}}(G_K, M)\big]$$

which admits a natural morphism $\mathcal{D}_{\mathrm{ur}}(G_K, M) \to \mathrm{R}\Gamma(G_K, M)$.

Show that $\mathcal{D}_{\mathrm{ur}}(G_K, M)$ and $\mathcal{D}_{\mathrm{ur}}(G_K, M^*(1))$ are dual local conditions.

(3) Let ϕ_K denote a geometric Frobenius element of G_K. Let $\mathcal{D}'_{\mathrm{ur}}(G_K, M)$ denote the complex

$$\mathcal{D}'_{\mathrm{ur}}(G_K, M) := \big[M^{I_K} \xrightarrow{\phi_K - 1} M^{I_K}\big]$$

Construct a natural quasi-isomorphism $\mathcal{D}'_{\mathrm{ur}}(G_K, M) \xrightarrow{\sim} \mathcal{D}_{\mathrm{ur}}(G_K, M)$, i.e. there are two ways to represent this unramified local conditions.

<u>Caveat:</u> The unramified condition is not exact in M, i.e. for $0 \to M \to M' \to M'' \to 0$ a short exact sequence of continuous $\mathcal{O}[G_K]$-modules, $\mathcal{D}_{\mathrm{ur}}(G_K, M) \to \mathcal{D}_{\mathrm{ur}}(G_K, M') \to \mathcal{D}_{\mathrm{ur}}(G_K, M'') \to$ is NOT a distinguished triangle in general, because taking I_K-invariants is not exact.

(4) Now we switch to the global setup. Let F be a number field and let S be a finite set of places including those dividing $\ell\infty$. Let S_∞ denote the archimedean places of F. Let M be a continuous $\mathcal{O}[G_{F,S}]$-module. Assume that $\ell \geq 3$ to avoid archimedean troubles.

For each $v \in S \backslash S_\infty$, suppose that we are given a dual pair of local conditions $\mathcal{D}_v$ and $\mathcal{D}^*_v$ for $\mathrm{R}\Gamma(G_{F_v}, M)$ and $\mathrm{R}\Gamma(G_{F_v}, M^*(1))$. We define

$$\mathrm{R}\Gamma_{\mathcal{D}}(G_{F,S}, M) := \mathrm{Cone}\Big[\mathrm{R}\Gamma(G_{F,S}, M) \oplus \bigoplus_{v \in S\backslash S_\infty} \mathcal{D}_v \longrightarrow \bigoplus_{v \in S\backslash S_\infty} \mathrm{R}\Gamma(G_{F_v}, M)\Big][-1],$$

and $\mathrm{R}\Gamma_{\mathcal{D}^*}(G_{F,S}, M^*(1))$ similarly. We write $\widetilde{H}^i_{\mathcal{D}}(G_{F,S}, M)$ for the cohomology of $\mathrm{R}\Gamma_{\mathcal{D}}(G_{F,S}, M)$. Deduce from the global duality that there is a natural isomorphism

$$\widetilde{H}^i_{\mathcal{D}}(G_{F,S}, M)^* \cong \widetilde{H}^{3-i}_{\mathcal{D}^*}(G_{F,S}, M^*(1)).$$

<u>Remark:</u> In general, we do not need the injectivities on the cohomology groups of $\mathcal{D} \to \mathrm{R}\Gamma(G_K, M)$ and $\mathcal{D}^* \to \mathrm{R}\Gamma(G_K, M^*(1))$ to define dual local conditions. Instead, we need a certain derived version of duality.

Problem 4.8. (Another interpretation of the conditions for Taylor–Wiles primes) Assume that $\ell \geq 3$. (This problem requires some input from the previous problem.)

Let M be a finite dimensional $\mathbb{F}$-vector spaces with continuous $G_{F,S}$-actions. Let Q be a finite set of places disjoint from S (in particular, each place in Q is relatively prime to ℓ).

(1) For each $v \in Q$, consider the unramified local condition $\mathcal{D}_{\mathrm{ur}}(G_{F_v}, M)$. Collectively, let $\mathcal{D}_{Q\text{-ur}}$ denote the local condition which is trivial at places in S and $\mathcal{D}_{\mathrm{ur}}(G_{F_v}, M)$ at each $v \in Q$. Show that we have an isomorphism

$$H^i(G_{F,S}, M) \cong \widetilde{H}^i_{\mathcal{D}_{Q\text{-ur}}}(G_{F,S\cup Q}, M).$$

Hint: Here is one way to prove this. In fact, we prove a statement that is more general than this. Let $\mathcal{D}_S$ and $\mathcal{D}_S^*$ denote a dual pair of local conditions for $\mathrm{R}\Gamma(G_{F,S}, M)$ and $\mathrm{R}\Gamma(G_{F,S}, M^*(1))$. Then we may extend these dual pair to a dual pair of local conditions $\mathcal{D}_S \oplus \mathcal{D}_{Q\text{-ur}}$ and $\mathcal{D}_S^* \oplus \mathcal{D}^*_{Q\text{-ur}}$ for $\mathrm{R}\Gamma(G_{F,S\cup Q}, M)$ and $\mathrm{R}\Gamma(G_{F,S\cup Q}, M^*(1))$, by taking the unramified local conditions at places at Q. Then, we have natural isomorphisms

$$\widetilde{H}^i_{\mathcal{D}_S \oplus \mathcal{D}_{Q\text{-ur}}}(G_{F,S}, M) \cong \widetilde{H}^i_{\mathcal{D}_S^* \oplus \mathcal{D}^*_{Q\text{-ur}}}(G_{F,S\cup Q}, M). \tag{4.8.1}$$

The reason for this generalization is that one can prove (4.8.2) and (4.8.1) relatively directly for $\widetilde{H}^0$ and $\widetilde{H}^1$. Then one can invoke the duality from the previous for $\widetilde{H}^2$ and $\widetilde{H}^3$. For the empty local condition, one needs to consider full local condition as its dual.

(2) Now assume that $M^{G_{F,S}} = 0$. Let $\mathcal{D}_{Q\text{-full}}$ denote the full local condition at Q, that is $\bigoplus_{v\in Q} R\Gamma(G_{F_v}, M)$. Let $\mathcal{D}_{S\text{-}\emptyset}$ denote the empty local condition at S. Using the natural isomorphism (4.8.1), we deduce an (injective) natural map (which is equivalent to (4.8.2))

$$\widetilde{H}^1_{\mathcal{D}_{S\text{-}\emptyset}}(G_{F,S}, M) \overset{(4.8.1)}{\cong} \widetilde{H}^1_{\mathcal{D}_{S\text{-}\emptyset} \oplus \mathcal{D}_{Q\text{-ur}}}(G_{F,S\cup Q}, M) \longrightarrow \widetilde{H}^1_{\mathcal{D}_{S\text{-}\emptyset} \oplus \mathcal{D}_{Q\text{-full}}}(G_{F,S\cup Q}, M).$$

Show that this is surjective (and thus an isomorphism) if and only if the natural map

$$H^1(G_{F,S}, M^*(1)) \to \bigoplus_{v\in Q} H^1(G_{F_v}, M^*(1))$$

is surjective.

Hint: In fact, one can prove that there is a long exact sequence

$$\widetilde{H}^1_{\mathcal{D}_{S\text{-}\emptyset} \oplus \mathcal{D}_{Q\text{-ur}}}(G_{F,S\cup Q}, M) \longrightarrow \widetilde{H}^1_{\mathcal{D}_{S\text{-}\emptyset} \oplus \mathcal{D}_{Q\text{-full}}}(G_{F,S\cup Q}, M) \to \bigoplus_{v\in Q} H^1(I_v, M)^{\mathrm{Gal}_{k_v}} \to \widetilde{H}^2_{\mathcal{D}_{S\text{-}\emptyset} \oplus \mathcal{D}_{Q\text{-ur}}}(G_{F,S\cup Q}, M)$$

So the surjectivity of the first map is equivalent to the injectivity of the second map. Dualizing the second map gives what we want.

(3) Show that the injectivity/surjectivity of (4.8.2) is equivalent to the injectivity/surjectivity of

$$\widetilde{H}^1_S(G_{F,S}, M) \to \widetilde{H}^1_S(G_{F,S\cup Q}, M). \tag{4.8.2}$$

Applying this problem to the case when $M = \mathrm{Ad}^\circ \bar{\rho}$ for an absolutely irreducible representation of $G_{F,S}$, we give an alternative proof of a step in the search of Taylor–Wiles primes, without relying on numerical computations (and under less additional conditions).