Laumon sheaf and the mod p Langlands program for GL_2 of a finite extension of $\mathbb{Q}_p$

Laurent Fargues

Let $E/\mathbb{Q}_p$, $d = [E:\mathbb{Q}_p] < \infty$.

$\rho \in \mathrm{Irr}_{\overline{\mathbb{F}}_q}(\mathrm{Gal}(\overline{E}/E))$

<u>Main formula</u> $\pi(\rho)\Big|_{\begin{pmatrix} E^\times & E \\ & 1\end{pmatrix}} = \mathcal{F}\left(j_! \, RH(\mathrm{Sym}_d \mathcal{F}_\rho)\right)$

- $\pi(\rho)$ = image of ρ under p-adic local Langlands
- $\mathcal{F}_\rho$ = an étale local sys
- $\mathrm{Sym}_d \mathcal{F}_\rho$ = Symmetrization
- RH = Riemann-Hilbert corresp
- $j_!(\cdots)$ = ext'n by zero, as a solid qcoh sheaf.
- $\mathcal{F}$ = Fourier transform (of "Mukai type").

Philosophy in MSRI 2014: $(\underset{\text{space}}{p}, \underset{\text{coeffs}}{\ell})$, $p \neq \ell$ $\rightsquigarrow$ Fargues-Scholze. local Langlands

Scholze's liquid work: (∞, ∞)

<u>Today</u> (p, p) w/ $\ell = p$

p-adic local Langlands.

<u>Key word</u> $\boxed{\text{Holonomy}}$.

Warm up Recall Laumon's work:

X/k sm proj curve

$\mathcal{E}/X$ irred $\bar{\mathbb{Q}}_\ell$-local sys.

For $d \geq 1$ s.t. $d!$ invertible in k,

$$\pi_d : X^d \longrightarrow X^d / \underset{\text{symm grp}}{S_d} =: \mathrm{Div}_X^d$$
$$(x_1, \cdots, x_d) \longmapsto \sum_{i=1}^d [x_i].$$

Def Symmetrization $\mathrm{Sym}_d \mathcal{E} := [\pi_{d*} \mathcal{E}^{\boxtimes d}]^{S_d} \in \mathrm{Perv}(\mathrm{Div}_X^d, \bar{\mathbb{Q}}_\ell)$

Fact $\mathrm{Sym}_d \mathcal{E} = j_{!*} \mathcal{F}_d$ for some $\mathcal{F}_d$:

$U_d \subset X^d$ open
$= \{(x_1, \cdots, x_d) \mid \forall i \neq j,\ x_i \neq x_j\}$

$$\rightsquigarrow S_d\text{-torsor} \left(\begin{tikzcd} U_d \arrow[r, hook] \arrow[d] & X^d \\ \pi_d(U_d) \arrow[r, hook, "j"', "\text{open}"] & \mathrm{Div}_X^d \end{tikzcd} \right.$$

& $\mathcal{F}_d$ = local system
Corresp to $\mathcal{E}^{\boxtimes d}|_{U_d} \circlearrowleft S_d$.

Let $\mathcal{E}$ = vec bdl on X.

Def $\mathrm{Sym}_d \mathcal{E} := [\pi_{d*} \mathcal{E}^{\boxtimes d}]^{S_d}$ coh sheaf on Div_X^d.

Lem $\mathrm{Sym}_d \mathcal{E}$ is a vec bdl as well.

Stratification by multiplicities of Div_X^d:

$\mathrm{Div}_X^{d,\underline{\lambda}}$, $\underline{\lambda}$ = partition of d.

If $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ w/ $\sum_i \lambda_i = d$,

$$S_{\underline{\lambda}} := \{ \sigma \in S_n \mid \lambda_{\sigma(i)} = \lambda_i \} \quad \text{Stabilizer}$$

$\rightsquigarrow$

$$\begin{tikzcd} U_n \arrow[r, hook] \arrow[d, "S_{\underline{\lambda}}\text{-torsor}"] & X^n \ni (x_1, \dots, x_n) \arrow[dl, mapsto] \\ \mathrm{Div}_X^{d,\underline{\lambda}} \ni \sum_i \lambda_i [x_i] & \end{tikzcd}$$

Prop For $i^{\underline{\lambda}} : \mathrm{Div}_X^{d,\underline{\lambda}} \hookrightarrow \mathrm{Div}_X^d$ closed,

$$(i^{\underline{\lambda}})^* \mathrm{Sym}_d \mathcal{E} = \text{vec bdl assoc to the } S_{\underline{\lambda}}\text{-equiv vec bdl}$$

$$\tau_{\lambda_1} \mathcal{E} \boxtimes \cdots \boxtimes \tau_{\lambda_n} \mathcal{E}$$

$\forall k \in \mathbb{Z}_{\geq 0}$, $\tau_k(-) = \underbrace{\text{twisted}}_{\text{locally on } X} \text{ form of } (-)^{\otimes k}$

Compatibility with RH

Recall $X / \mathbb{F}_q$ sch.

$$\text{Katz} : \left\{ \begin{matrix} \mathbb{F}_q\text{-étale local} \\ \text{systems on } X \end{matrix} \right\} \xrightarrow{\sim} \left\{ (\mathcal{E}, \Phi) \;\middle|\; \begin{matrix} \mathcal{E} = \text{vec bdl} \\ \Phi : \mathrm{Frob}_q^* \mathcal{E} \xrightarrow{\sim} \mathcal{E} \end{matrix} \right\}$$

$$\mathcal{F} \longmapsto (\mathcal{F} \otimes_{\mathbb{F}_q} \mathcal{O}, \mathrm{Id} \otimes \mathrm{Frob}_q)$$

By Emerton-Kisin & Bhatt-Lurie:

$$\text{RH} : \left\{ \begin{matrix} \text{étale schs of} \\ \mathbb{F}_q\text{-vec spaces} / X \end{matrix} \right\} \hookrightarrow \left\{ (\mathcal{E}, \varphi) \;\middle|\; \begin{matrix} \mathcal{E} \text{ quasi-coh} \\ \varphi : \mathcal{E} \xrightarrow{\sim} \mathcal{E} \\ \mathrm{Frob}_q\text{-linear} \end{matrix} \right\}$$

Then $\text{RH}(\mathcal{F}) = \text{Katz}(\mathcal{F})^{1/p^\infty}$

perfection ($\rightsquigarrow$ holonomy condition).

Prop X curve. $\mathcal{F} = \overline{\mathbb{F}}_q$-étale loc sys on X.

Then $RH(\mathrm{Sym}_d \mathcal{F}) = \varinjlim_{\varphi} \mathrm{Sym}_d \mathcal{E}$

with $(\mathcal{E}, \varphi) = \mathrm{Katz}(\mathcal{F})$.

<u>The real thing</u>

$E/\mathbb{Q}_p$ of deg d.

$$\begin{array}{l} \mathbb{B}^{\varphi=\pi} = \text{absolute BC space} \cong \mathrm{Spd}(\overline{\mathbb{F}}_q[\![T^{1/p^\infty}]\!]) \\ \downarrow \\ * = \mathrm{Spd}(\overline{\mathbb{F}}_q) \end{array}$$

$\mathrm{Rep}_{\overline{\mathbb{F}}_q}(\Gamma_E) \simeq$ étale loc sys on $\underbrace{(\mathbb{B}^{\varphi=\pi} \setminus \{0\})/\underline{E}^\times}_{= \mathrm{Div}^1}$ (really a curve).

$$RH: \mathrm{Rep}_{\overline{\mathbb{F}}_q}(\Gamma_E) \xrightarrow{\sim} \{\text{VBs on } \mathrm{Div}^1\}.$$

$$\rho \longmapsto \underbrace{\mathcal{F}_\rho}_{\text{Lubin-Tate } (\varphi,\Gamma)\text{-mod assoc to } \rho} \otimes \mathcal{O}$$

$\mathcal{E}_\rho := \mathcal{F}_\rho \otimes \mathcal{O}$ seen as an $\underline{E}^\times$-equiv VB on $\mathbb{B}^{\varphi=\pi} \setminus \{0\}$.

Consider $\pi_d : (\mathbb{B}^{\varphi=\pi} \setminus \{0\})^d \longrightarrow \mathbb{B}^{\varphi=\pi^d} \setminus \{0\}$

$$(x_1, \dots, x_d) \longmapsto x_1 \cdots x_d.$$

Thm (Fargues) $\Delta_d = \{(\mu_1, \dots, \mu_d) \in (\underline{E}^\times)^d \mid \prod_{i=1}^d \mu_i = 1\}$.

Then π_d is quasi-proét surj, inducing

$$(\mathbb{B}^{\varphi=\pi}\setminus\{0\})^d / \Delta_d \rtimes S_d \xrightarrow{\sim} \mathbb{B}^{\varphi=\pi^d}\setminus\{0\}.$$

proet quotient.

Def $S_d\mathcal{F}_\rho := (\pi_{d*}\mathcal{F}_\rho^{\boxtimes d})^{\Delta_d \rtimes S_d}$ an étale sheaf.

$S_d\mathcal{E}_\rho := (\pi_{d*}\mathcal{E}_\rho^{\boxtimes d})^{\Delta_d \rtimes S_d}$ v-sheaf of $\mathcal{O}$-mods.

Note $\mathbb{B}^{\varphi=\pi^d}\setminus\{0\} = \mathrm{Spa}(\overline{\mathbb{F}}_q[[x_1^{1/p^\infty},\dots,x_d^{1/p^\infty}]]_\Gamma)\setminus V(x_1,\dots,x_d)$

$=$ qc perfectoid space.

Have deg d polynomial functor

$$S_d : \mathrm{Rep}_{\overline{\mathbb{F}}_q}(\Gamma_E) \longrightarrow \left\{ \begin{array}{l} \text{overconv étale sheaves of } \overline{\mathbb{F}}_q\text{-v.s.} \\ \text{on } \underbrace{\mathrm{Spa}(\overline{\mathbb{F}}_q[[x_1,\dots,x_d]])\setminus V(x_1,\dots,x_d)}_{\text{noetherian adic space.}} \end{array} \right\}$$

Conj (Holonomicity) $S_d\mathcal{E}_\rho =$ Completion of the projection of an $\mathcal{O}$-mod that is a perfect complex.

Rmk Ok if replace E with $E=\mathbb{F}_q((\pi))$ (equal char case).

Thm Holonomy conj $\Rightarrow$ $S_d\mathcal{E}_\rho$ generated by the global sections.

Point $\overline{\mathbb{F}}_q[[E]] = \mathcal{O}(\mathbb{B}^{\varphi=\pi^d})$

$\circlearrowleft$
$E^\times$

global sections of $\mathcal{E}_\rho$ defines a functor

$$\mathrm{Rep}_{\overline{\mathbb{F}}_q}(\Gamma_E) \longrightarrow \mathrm{Rep}_{\overline{\mathbb{F}}_q}\begin{pmatrix} E^\times & E \\ 0 & 1 \end{pmatrix}.$$

expected to be the mod p Langlands for $GL_2(E)$.