Lecture 1: Introduction to ghost conj and some Corollaries.

§1 Question of slopes

- Fix a prime number $p \geq 5$,
  $E/\mathbb{Q}_p$ fin ext'n (coeff field).
  $E \supseteq \mathcal{O} \twoheadrightarrow \mathcal{O}/(\varpi) = \mathbb{F}$. Assume $\sqrt{p} \in E$.
- Fix integer $N \geq 4$, $p \nmid N$.
- $S_k(p, N; \psi) :=$ space of cusp modular forms of level pN & wt k.
  with Nebentypus char $\psi : (\mathbb{Z}/p^m N\mathbb{Z})^\times \to (\mathbb{Z}/p^m\mathbb{Z})^\times \to E^\times$.
  (Want $m \geq 1$ even, if $\psi =$ trivial; m minimal.)
- Fix $\bar{r} :=$ absolute irred residue rep'n, $\bar{r} : \mathrm{Gal}_{\mathbb{Q}} \to GL_2(\mathbb{F})$

$\rightsquigarrow$ $S_k(p^m N; \psi)_{\bar{r}} =$ " localization of $S_k(p^m N; \psi)$
U_p-operator at the Hecke ideal corresp to $\bar{r}$."

i.e. $\mathbb{T} = \mathrm{Im}(\mathcal{O}[T_\ell;\ \ell \nmid pN] \to \mathrm{End}_{\mathcal{O}}(S_k))$
$\quad\ \cup$
$\mathfrak{m}_{\bar{r}} = (\varpi,\ T_\ell - \mathrm{tr}(\bar{r}(\mathrm{Frob}_\ell));\ \ell \nmid pN)$.

where $U_p(f) = \sum_{n\geq 1} a_{pn} q^n$, $a_1 = 1$, $f = \sum_{n \geq 1} a_n q^n$.

<u>Question</u> For an eigenform f of U_p (i.e. $U_p(f) = a_p \cdot f$),
what's the possible value of $\underbrace{v_p(a_p)}_{\text{slopes of } f}$?

* Why do we care this?

① Deligne: either $\psi = \mathbb{1}$ and f "p-new" $\Rightarrow a_p = \pm p^{\frac{k-2}{2}}$
o/w $|a_p|_\infty = p^{\frac{k-1}{2}}$.

($\ell \neq p$. $v_\ell(a_p) = 0$.)

③ From the point of view of (p-adic) local Langlands corresp:

$$\begin{tikzcd}
f \bmod p \arrow[r, leftrightarrow] & \bar{r} : \mathrm{Gal}_{\mathbb{Q}} \to GL_2(\bar{\mathbb{F}}) \\
f \text{ wt } k, \cdots \arrow[u, squiggly, "\text{lift}"'] \arrow[r, leftrightarrow] & r_f : \mathrm{Gal}_{\mathbb{Q}} \to GL_2(\mathcal{O}(f)) \arrow[u, squiggly, "\text{lift}"']
\end{tikzcd}$$

w/ U_p eigenval a_p s.t. $r_f|_{\mathrm{Gal}_{\mathbb{Q}_p}}$ is semistabelline

& $D_{pst}(r_{f,p}) \circlearrowleft \varphi$ has a φ-eigenval a_p.

all possible slopes of f $\hookrightarrow$ all possible φ-slopes for semistabelline lifts of $\bar{r}_p$.

More precisely, $D_{pst}(r_p^{univ}) \circlearrowleft \varphi$

(w/ def ring $R_{\bar{r}_p}^{\square,(0,k-1),\psi}$).

<u>Side goal</u> Information of φ-slopes

$\Rightarrow$ geometry of $(\mathrm{Spf}\, R_{\bar{r}_p}^{\square,(0,k-1),\psi})^{rig}$.

<u>§2 Newton polygon</u>

<u>Def'n</u> For a polynomial / power series $f(t) = 1 + c_1 t + c_2 t^2 + \cdots \in E[\![t]\!]$,

we define its <u>Newton polygon</u> to be the convex hull of $(n, v_p(c_n))$, denoted by $NP(f)$.

Then (with some growth condition)

$$\{v_p(\text{zeros of } f(t))\} = -\{\text{slopes of } NP(f)\}.$$

- Now, linear operator $T = U_p \circlearrowright V$ v.s. $/E$ (V will be $S_k^\dagger$).

$\rightsquigarrow$ characteristic power series of T

$$C(t) := \det(1 - T\cdot t\,;\, V) \quad (\text{if } \dim V = \infty, \text{ need } T \text{ to be compact.})$$

Then $\{v_p(\text{eigenvals of } T)\} = -\{v_p(\text{zeros of } C(t))\} = \{\text{slopes of } NP(C)\}$.

<u>Related tool</u> If for some basis of V, the matrix (T_{ij}) of T

satisfies $v_p(T_{ij}) \geq \lambda_i$, $\forall i, j$, $\lambda_1 \leq \lambda_2 \leq \cdots$,

$\Rightarrow C(t) = 1 + c_1 t + c_2 t^2 + \cdots$ $\rightsquigarrow \begin{pmatrix} p^{\lambda_1} & p^{\lambda_1} & \cdots \\ p^{\lambda_2} & p^{\lambda_2} & \cdots \\ & \cdots & \end{pmatrix}$.

$v_p(c_1) \geq \lambda_1$, $v_p(c_2) \geq \lambda_1 + \lambda_2$, $\cdots$.

$\Rightarrow NP(C)$ lies above the polygon w/ slopes $\lambda_1, \lambda_2, \cdots$.

"<u>Cor</u>" $c_n \approx \det(\text{upper-left } n \times n \text{ minor})$.

§3 p-adic weights

<u>Slogan</u> Even if we care about only one wt k,

it still helps to vary k p-adically.

$\Delta := (\mathbb{Z}/p\mathbb{Z})^\times$, Teichmüller char $\omega : \Delta \longrightarrow \mu_{p-1} \subseteq \mathbb{Z}_p^\times$

$$\mathbb{Z}_p^\times \simeq \Delta \times (1 + p\mathbb{Z}_p)^\times.$$

<u>p-adic wts</u> = conti char of $\mathbb{Z}_p^\times$ weight k, nebentypus $\psi : (\mathbb{Z}/p^m\mathbb{Z})^\times \longrightarrow \mathbb{C}_p^\times$

$$\rightsquigarrow \chi_{(k,\psi)} : \mathbb{Z}_p^\times \longrightarrow \mathbb{C}_p^\times$$

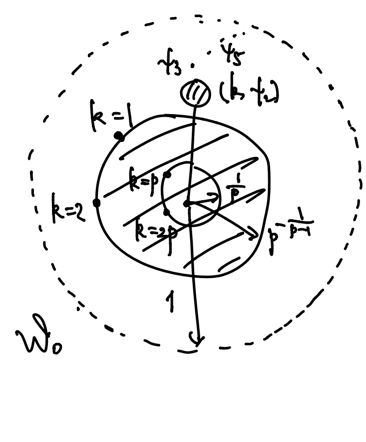
$$a \longmapsto a^k \cdot \psi(a \bmod p^m)$$

For an open disc $\mathcal{W}_0$ (w/ radius $= 1$)

$\mathcal{W} = \operatorname{Hom}_{\text{cont gp}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$

$= \underbrace{\operatorname{Hom}_{gp}(\Delta, \mathbb{C}_p^\times)}_{\substack{p-1 \text{ copies} \\ \text{like } \omega^0, \omega^1, \cdots, \omega^{p-2}}} \times \operatorname{Hom}(\underset{\circlearrowleft}{(1 + p\mathbb{Z}_p)^\times}, \mathbb{C}_p^\times)$

$\exp(p) \mapsto 1 + w$.



Each char $\chi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ corresponds to $(\chi|_\Delta, w_\chi := \chi(\exp(p)) - 1)$

E.g. $\chi = \chi_{(k,\mathbb{1})}$, $w_\chi = \chi_{(k,\mathbb{1})}(\exp(p)) - 1 = \exp(kp) - 1$.

$\rightsquigarrow v_p(w_\chi) = 1 + v_p(k) \geq 1$.

Consider $\psi_m : (\mathbb{Z}/p^m\mathbb{Z})^\times \longrightarrow \mathbb{C}_p^\times$, $\chi = \chi_{(k,\psi)}$,

$$1+p \longmapsto \zeta_{p^{m-1}}.$$

$$w_\chi = \exp(pk)\cdot \underset{\zeta_{p^{m-1}}}{\underset{\shortparallel}{\psi(\exp(p))}} - 1 = \underbrace{\zeta_{p^{m-1}} - 1}_{\text{main term}} + p(k)\zeta_{p^{m-1}} + \cdots$$

$$\Rightarrow v_p(w_\chi) = v_p(\zeta_{p^{m-1}} - 1) = \frac{1}{(p-1)p^{m-2}}.$$

<u>Major distinction</u> $v_p(w) \in (0,1)$ v.s. $v_p(w) \geq 1$

rim of the wt space (halo region) | center of weight space.

<u>Notation</u> $\omega_1 : I_{\mathbb{Q}_p} \to \mathrm{Gal}(\mathbb{Q}_p(\mu_p)/\mathbb{Q}_p) \simeq \mathbb{F}_p^\times$ first fundamental char

$\omega_2 : I_{\mathbb{Q}_p} = I_{\mathbb{Q}_{p^2}} \longrightarrow \mathrm{Gal}(\mathbb{Q}_{p^2}(\sqrt[p^2-1]{p})/\mathbb{Q}_{p^2}) \simeq \mathbb{F}_{p^2}^\times$.

<u>Started with</u> $\bar{r} : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F})$

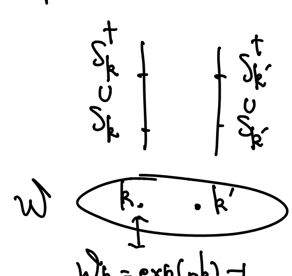
$(\det \bar{r})|_{I_{\mathbb{Q}_p}} = \omega_1^c$, $c \in \{0, 1, \cdots, p-2\}$.

Only consider $S_k(pN; \omega^{k-1-c})_{\bar{r}}$ (others are zero)

$\cap$

$S_k^\dagger(pN; \omega^{k-1-c})_{\bar{r}} := \lim_{M\to\infty} S_{k+(p-1)p^M}(pN, \omega^{k-1-c})_{\bar{r}}$.

dim $\sim$ linear in k up to $O(1)$.



<u>Theorem</u> (Coleman, Coleman-Mazur)

$$S_k(pN; \omega^{k-1-c})_{\bar{r}} = S_k^\dagger(pN; \omega^{k-1-c})_{\bar{r}}^{U_p\text{-slope} \leq k-1}.$$

$\circlearrowleft U_p$.

$\exists!$ a char power series $C_{\bar{r}}(w,t) \in \mathcal{O}[\![w,t]\!]$ (compact)

integral !

s.t. $C_{\bar{r}}(w=w_k, t) = \det(1-U_p t \,;\, S_k^{\dagger}(pN;\, \omega^{k-1-c})_{\bar{r}})$.

<u>Question of slopes</u> For any $w_\star \in \mathfrak{m}_{\mathbb{C}_p}$, what is $NP(C_{\bar{r}}(w_\star, -))$?

<u>§4 Main Theorem</u>

<u>Serre</u> $\bar{r}_p : \mathrm{Gal}_{\mathbb{Q}_p} \to GL_2(\mathbb{F})$ has two kinds:

(1) Reducible
$$\bar{r}_p = \begin{pmatrix} \omega_1^{a+1}\cdot \mathrm{unr}(\bar{\alpha}) & * \\ 0 & 1 \end{pmatrix} \otimes \omega_1^b \mathrm{unr}(\bar{\beta})$$

w/ $a, b \in \{0, \dots, p-2\}$, $\bar{\alpha}, \bar{\beta} \in \mathbb{F}^\times$.

Say $\bar{r}_p$ is generic if $1 \le a \le p-4$.

In this case, $\dim H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \omega_1^{a+1}\mathrm{unr}(\bar{\alpha})) = 1$

$\Rightarrow$ up to isom, two possibilities $\begin{cases} * = 0 \\ * \neq 0. \end{cases}$

(2) Irreducible (will not talk about)
$$\bar{r}_p = (\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{a+1}) \otimes \omega_1^b \mathrm{unr}(\bar{\beta}).$$

<u>Assume</u> $\bar{r}|_{\mathrm{Gal}_{\mathbb{Q}_p}}$ is reducible + generic, $\bar{r}$ abs irred.

<u>Fact</u> $\exists$ multiplicity $m(\bar{r}) \in \mathbb{Z}_{\ge 1}$, s.t.
$$\dim S_k(pN;\, \omega^{k-1-c})_{\bar{r}} = \frac{2k}{p-1} m(\bar{r}) + O(1).$$

<u>Theorem</u> (ghost conjecture of Bergdall-Pollack, Liu-Truong-Xiao-Zhao).

Assume $p \ge 11$, and $2 \le a \le p-5$.

(Ghost series).

There's an explicitly comb def'd $G(w,t) = 1 + \sum_{n\ge 1} g_n(w) t^n \in \mathbb{Z}_p[w][\![t]\!]$.

s.t. $\forall w_k \in \mathfrak{m}_{\mathbb{C}_p}$, $NP(G_{\bar{r}}(w_k, -)) = NP(G(w_k, -))$

(except for slope 0 part)

stretched in both x-y directions $m(\bar{r})$ times.

Rmk (1) $p=2$, $N=1$: raised by Buzzard - Calegari

... Loeffler, Lisa Clay,

... Buzzard's algorithm of slopes.

$\rightsquigarrow$ Bergdall - Pollack: more conceptual explanation of $G(w,t)$.

(2) $a=1$, $p=4$, or $p=7$? Need technical issue.

(3) irreducible case, maybe not hopeless.

<u>Working def'n of $G(w,t)$</u>

$$\bar{r}|_{I_{\mathbb{Q}_p}} = \begin{pmatrix} \omega_1^{a+1} & * \neq 0 \\ 0 & \mathbb{1} \end{pmatrix} \otimes \omega_1^b, \qquad c = 2b+a+1 \bmod p-1.$$

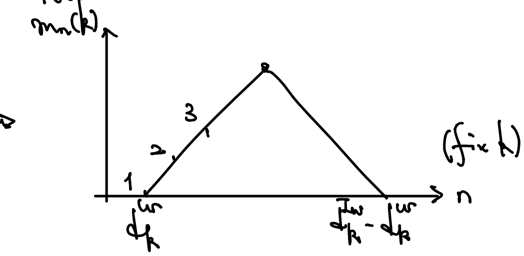
Put for $k \equiv a+2b+2 \bmod p-1$.

- $\underbrace{S_k(p^N; \mathbb{1})_{\bar{r}}}_{m(\bar{r})\cdot d_k^{Iw}} = \underbrace{S_k(N)_{\bar{r}}^{\oplus 2}}_{m(\bar{r})\cdot d_k^{ur}} \oplus S_k(pN)_{\bar{r}}^{p\text{-new}}$

- $g_n(w) = \prod\limits_{\substack{k \equiv a+2b+2 \\ \bmod p-1}} (w - \underset{\substack{\shortparallel \\ \exp(pk)-1}}{w_k})^{m_n(k)}$.

- $m_n(k) = \begin{cases} \min\{n - d_k^{ur}, d_k^{Iw} - d_k^{ur} - n\} & \text{if } d_k^{ur} < n < d_k^{Iw} - d_k^{ur}. \\ 0, & \text{o/w.} \end{cases}$

$\rightsquigarrow$



§5 Applications

All assume $\bar{r}$ irred, $\bar{r}_p$ red, $p \geq 11$, $2 \leq a \leq p-5$.

Application D (Gouvêa–Mazur Conj, 1992)

Let $n \in \mathbb{N}$. For weights $k_1, k_2 > 2n+2$

s.t. $k_1 \equiv k_2 \equiv a + 2b + 2 \mod p-1$

and $v_p(k_1 - k_2) \geq n+5$.

$\Rightarrow$ Slope seq of $S_{k_1}(pN)_{\bar{r}}$ and $S_{k_2}(pN)_{\bar{r}}$ agree up to slope n.

(originally conj'd for all $\bar{r}$, $n+5 \to n$; but $\exists$ counterexample).

- Daqing Wan showed for $n+5 \rightsquigarrow An^2 + Bn + C$.
- Combining thm with Rufei Ren.

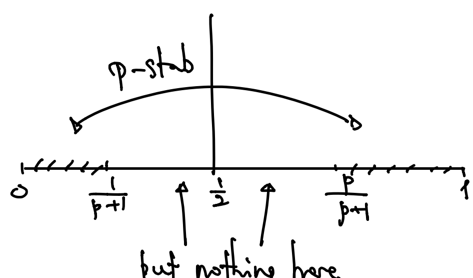
Application E (Gouvêa's slope distribution, 2001)

For each $k \equiv a + 2b + 2 \mod p-1$, write U_p-slopes on $S_k(Np)$ as

$\alpha_1(k), \cdots, \alpha_d(k) \in [0, k-1]$ (with multiplicity).

$\mu_k :=$ probability measure of $\left\{\frac{\alpha_1(k)}{k-1}, \cdots, \frac{\alpha_d(k)}{k-1}\right\}$ on $[0,1]$.

$$\Rightarrow \lim_{\substack{k \to +\infty \\ k \equiv \cdots}} \mu_k = \frac{1}{p+1}\left(\delta_{[0, \frac{1}{p+1}]} + \delta_{[\frac{p}{p+1}, 1]} + \frac{p-1}{p+1}\delta_{\frac{1}{2}}\right)$$



$$S_k(pN) = \underbrace{S_k(N)^{\oplus 2}}_{p\text{-stabilization}} \oplus \underbrace{\boxed{S_k(pN)^{p\text{-new}}}}_{U_p\text{-eigenval} = \pm p^{\frac{k-2}{2}}} \longleftarrow \dim = \frac{p-1}{p+1} \text{ of total dim.}$$

Lecture 2: Local ghost conj and overview of the proof

§1 Geometry of spectral curve / Eigencurve

Recall $C_{\bar{r}}(w,t) \in \mathcal{O}[\![w,t]\!]$ char power series of U_p.

Define $\mathrm{Spc}(\bar{r}) :=$ zero locus of $C_{\bar{r}}(w,t)$ in $\mathcal{W} \times \mathbb{G}_m^{\mathrm{rig}}$.

Cor $\mathrm{Spc}(\bar{r}) \approx$ zero locus of $G(w,t)$ w/ multi $m(\bar{r})$.

Always assume $\bar{r}$ abs irred.
$\bar{r}_p$ reducible, $2 \leq a \leq p-5$, $p \geq 11$

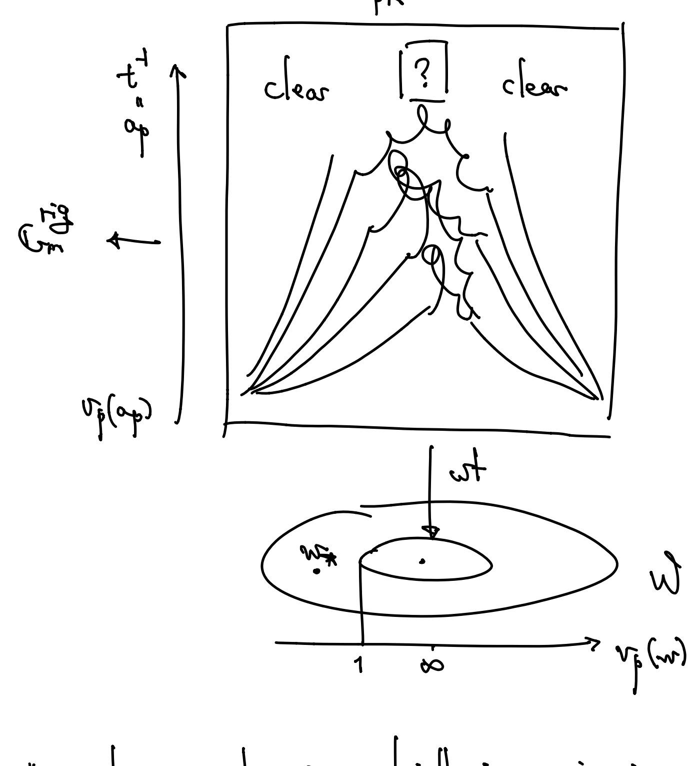
$\mathcal{W}^{(0,1)} = \{ w \in \mathcal{W} \mid v_p(w) \in (0,1) \}$

$\mathrm{Spc}(\bar{r})^{(0,1)} := \mathrm{wt}^{-1}(\mathcal{W}^{(0,1)})$.

Key When $w_\star \in \mathcal{W}^{(0,1)}$

$$v_p(g_n(w_\star)) = v_p\Big(\prod_{k} (\underbrace{w_\star - w_k}_{v_p = v_p(w_\star)})^{m_n(k)} \Big) = v_p(w_\star) \cdot \deg g_n.$$

(with k ranging over ss pk)



Fact $1 \leq a \leq p-4$, $\deg g_n - \deg g_{n-1}$ strictly increasing in n.

Application F (Refined spectral halo conj of Coleman, 1998)

$$\mathrm{Spc}(\bar{r})^{(0,1)} = Y_1 \sqcup Y_2 \sqcup Y_3 \sqcup \cdots$$

s.t. (1) for each point $(w_\star, a_p) \in Y_n$, $v_p(a_p) = (\deg g_n - \deg g_{n-1}) \cdot v_p(w_\star)$

possible issue with ordinary part.

(2) $wt : Y_n \longrightarrow W^{(0,1)}$ is finite & flat of deg $m(\bar{r})$.

Remark Weaker version but no constraint on $\bar{r}$: by Liu-Wan-Xiao 2017

HMF(p splits) Ren-Zhao 2022

modular symbol Diao-Yao 2023.

Q: Analogy b/n Hida family v.s. spectral halo?

Application C Write $\mathrm{Spc}(\bar{r}) = \mathrm{Spc}(\bar{r})^{\mathrm{ord}} \sqcup \mathrm{Spc}(\bar{r})^{\mathrm{nord}}$.

Thm $\mathrm{Spc}(\bar{r})^{\mathrm{nord}}$ has finitely many irred components.

(asked by Coleman-Mazur 1998.)

- $\lambda \in (0,1)$: A "converging" power series $F(w,t) = 1 + \sum_{n\geq 1} f_n(w) t^n \in E\langle \frac{w}{p^\lambda} \rangle [\![t]\!]$.

$\forall w_\star \in \mathfrak{m}_{\mathbb{C}_p}$, $v_p(w_\star) \geq \lambda$, $NP(F(w_\star, -)) = NP(G(w_\star, -))$.

stretched m times.

Thm If $F(w,t)$ is a power series of ghost type w/ mult m

and $F(w,t) = F_1(w,t) \cdot F_2(w,t)$

$\Rightarrow$ both $F_i(w,t)$ are of ghost type, and $m = m_1 + m_2$.

(Rigidity of ghost type power series).

§2. Local ghost conjecture

Key point 1 Don't use modular forms.

Use Betti's realization ← better integral structure.

Key point 2 (Emerton) Complete cohomology.

Assume $\bar{r}$ abs irred.

Define $\widetilde{H}_{1,\bar{r}} := \varprojlim_m H_1(X(K^p(1+p^m M_2(\mathbb{Z}_p)))(\mathbb{C}), \mathcal{O})_{\bar{r}}^{\text{cplx}=1}$

completed cohom

right $GL_2(\mathbb{Q}_p)$-action

projective $GL_2(\mathbb{Z}_p)$-mod.

Upshot This $\widetilde{H}_{1,\bar{r}}$ contains all p-adic info we need.

Note For $K_p = \mathrm{Iw}_p$ or $GL_2(\mathbb{Z}_p)$,

$$H^1(X(K^pK_p), \mathrm{Sym}^{k-2}(\mathcal{O}^{\oplus 2}))_{\bar{r}}^{\text{cplx}=1} = \mathrm{Hom}^{\text{cont}}_{\mathcal{O}[K_p]}(\widetilde{H}_{1,\bar{r}}, \mathrm{Sym}^{k-2}\mathcal{O}^{\oplus 2}).$$

as Hecke mods

$$\cong \quad S_k(K^pK_p)_{\bar{r}} \curvearrowleft U_p \qquad U_p$$

Q: (Emerton) What's the minimum setup for ghost to work?

Def'n Let $\bar{\rho}: I_{\mathbb{Q}_p} \to GL_2(\mathbb{F})$ be the rep'n $\begin{pmatrix} \omega_1^{a+b+1} & * \neq 0 \\ 0 & \omega_1^b \end{pmatrix}$, $1 \leq a \leq p-1$.

$$K_p = GL_2(\mathbb{Z}_p) \supseteq \mathrm{Iw}_p = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \supseteq \mathrm{Iw}_{p,1} = \begin{pmatrix} 1+p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}$$

$$\cup\!\!\mid \quad B^{op}(\mathbb{Z}_p) = \begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$$

$$\cup\!\!\mid \quad \begin{pmatrix} \Delta & \\ & \mathbb{Z}_p^\times \end{pmatrix} \cong \begin{pmatrix} \Delta & \\ & \Delta \end{pmatrix}.$$

An $\mathcal{O}[\![K_p]\!]$-projective augmented module primitive type $\bar{\rho}$ is

$\widetilde{H}$ = proj envelop of $\mathrm{Sym}^a \mathbb{F}^{\oplus 2} \otimes \det^b$ as a right $\mathcal{O}[\![K_p]\!]$-mod s.t. the right action extends to a cont. $GL_2(\mathbb{Q}_p)/p^{\mathbb{Z}}$ - action

& (technical assumption)

$$\widetilde{H} \simeq \widetilde{H}_0 \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[\![(1+p\mathbb{Z}_p)^\times]\!] \circlearrowleft (1+p\mathbb{Z}_p)^\times$$
$$\circlearrowleft$$
$$GL_2(\mathbb{Q}_p)/p^{\mathbb{Z}}(1+p\mathbb{Z}_p)^\times.$$

Rmk $\bar{\rho}$ is on Galois side $\longleftrightarrow$ corresponding Serre wt $\mathrm{Sym}^a \otimes \det^b$.

<u>Weight discs</u> $\longleftrightarrow$ characters of $\begin{pmatrix} \Delta & \\ & \Delta \end{pmatrix}$: $\omega^? \times \omega^?$ $(\omega : \Delta \to \mathbb{Z}_p^\times)$.

"relevant character" $\varepsilon = \omega^{-s+b} \times \omega^{a+s+b}$ for some $s \in \{0, 1, \cdots, p-2\}$.

<u>This afternoon</u> (Lec 3)

" $S_{\widetilde{H}}^{(\varepsilon), p\text{-adic}} := \mathrm{Hom}_{\mathrm{Iw}_p}(\widetilde{H}, \mathrm{Ind}_{B^{op}(\mathbb{Z}_p)}^{\mathrm{Iw}_p} \chi_{univ}^{(\varepsilon)})$

$\circlearrowleft$

U_p $\quad \infty$-dim'l Banach mod/ $\mathcal{O}[\![w]\!]$.

Space of abstract p-adic form.

Here $\chi_{univ}^{(\varepsilon)} : \begin{pmatrix} \Delta & \\ & \mathbb{Z}_p^\times \end{pmatrix} \longrightarrow \mathcal{O}[\![w]\!]^\times$

$$(\bar{\alpha}, \delta) \longmapsto \varepsilon(\bar{\alpha}, \bar{\delta}) \cdot (1+w)^{\log(\delta/\omega(\bar{\delta}))/p}$$

$$(1, \exp(p)) \longmapsto 1+w$$

Put $C_{\widetilde{H}}^{(\varepsilon)}(w,t) := \det(\mathbb{1} - U_p t ; S_{\widetilde{H}}^{(\varepsilon), p\text{-adic}})$.

Similarly, we have ghost series $\quad k \equiv k_\varepsilon := a + 2b + 2s + 2 \mod p-1$

$d_k^{\mathrm{Iw}} \rightsquigarrow S_{\widetilde{H}, k}^{\mathrm{Iw}}(\omega^s) := \mathrm{Hom}_{\mathrm{Iw}_p}(\widetilde{H}, \mathrm{Sym}^{k-2}\mathcal{O}^{\oplus 2} \otimes \omega^s \otimes \det)$ $\qquad \left(U_p(\varphi)(x) = \sum_{j=0}^{p-1} \varphi\left(x \begin{pmatrix} p & \\ pj & 1 \end{pmatrix}^{-1}\right) \right)$

$i_1 \quad i_2 \quad \mathrm{pr}_1 \quad \mathrm{pr}_2$

$d_k^{ur} \rightsquigarrow S_{\widetilde{H}, k}^{ur}(\omega^s) := \mathrm{Hom}_{K_p}(\widetilde{H}, \mathrm{Sym}^{k-2}\mathcal{O}^{\oplus 2} \otimes \omega^s \otimes \det)$

<u>Define</u> $G_{\bar{\rho}}^{(\varepsilon)}(w,t) = 1 + \sum_{n \geq 1} g_n(w) t^n \in \mathbb{Z}_p[w][\![t]\!]$ with $g_n(w) = \prod_{\substack{k \equiv a+2b+2 \\ \mod p-1}} (w - w_k)^{m_n(k)}$.

where $m_n(k) = \begin{cases} \min\{n - d_k^{ur}, d_k^{\mathrm{Iw}} - d_k^{ur} - n\} & \text{if } d_k^{ur} < n < d_k^{\mathrm{Iw}} - d_k^{ur}. \\ 0, & o/w. \end{cases}$

$$\mathrm{rk}_{\mathcal{O}}\left(\mathrm{Hom}_{K_p}\left(\tilde{H}, \mathrm{Sym}^{k-2}\mathcal{O}^{\oplus 2}\otimes\omega^s\otimes\det\right)\right.$$
$$= \dim_{\mathbb{F}}\left(\mathrm{Hom}_{K_p}\left(\mathrm{Proj}_{\mathbb{F}[GL_2(\mathbb{Z}_p)]}(\mathrm{Sym}^a\otimes\det^b), \mathrm{Sym}^{k-2}\otimes\det^s\right)\right)$$
$$= \#\mathrm{JH}_{\mathrm{Sym}^a\otimes\det^b}\left(\mathrm{Sym}^{k-2}\otimes\det^b\right).$$

<u>Local ghost theorem</u>

If $2\le a\le p-5$, $p\ge 11$, $\forall w_\star\in\mathfrak{m}_{\mathbb{C}_p}$,

$$NP\left(C_{\tilde{H}}^{(\varepsilon)}(w_\star,-)\right) = NP\left(G_{\bar{\rho}}^{(\varepsilon)}(w_\star,-)\right).$$

<u>Amazing fact</u> $\bar{\rho}$ has a comparison $\bar{\rho}' = \begin{pmatrix}\omega^{a+b+1} & 0\\ *\neq 0 & \omega^b\end{pmatrix}$

$\hookrightarrow$ Serre wt $\mathrm{Sym}^{p-3-a}\otimes\det^{a+b+1}$.

$G_{\bar{\rho}}^{(\varepsilon)}(w,t) = G_{\bar{\rho}'}^{(\varepsilon)}(w,t)$ except for the ordinary part

$\quad\; \| \qquad\qquad\quad \|$

$\; G^{(\varepsilon)} \qquad\quad G'^{(\varepsilon)}$

for some ε : $G^{(\varepsilon)} = 1 + tG'^{(\varepsilon)}$

for some other ε : $G'^{(\varepsilon)} = 1 + tG^{(\varepsilon)}$.

This reflects on the Galois side (when slope >0)

$\bar{r}_p$ lifts to irred rep'n V/E of $\mathrm{Gal}_{\mathbb{Q}_p}$.

But V has a lattice whose reduction is $\bar{r}_p$

$\Longleftrightarrow$ V has a lattice whose reduction is $\bar{r}_p'$. (Ribet's lemma).

For a fixed $\bar{\rho}$, $\exists!\ \varepsilon = \omega^b\times\omega^{a+b}$, $G_{\bar{\rho}}^{(\varepsilon)}(w,t)$ has slope 0 part.

<u>§3 Local ghost $\Rightarrow$ B-P ghost</u>

<u>Idea</u> Local ghost $\Rightarrow$ slopes on trianguline deform spaces $\Rightarrow$ B-P ghost

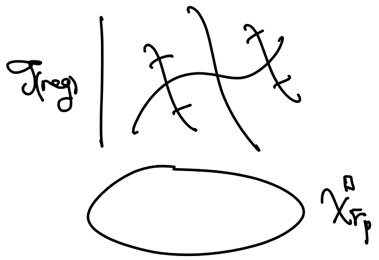
applying local ghost to Paškūnas module — remove the reducible nonsplit constraint

$\mathcal{T}$:= rigid space of 2 conti chars $\delta_1,\delta_2:\mathbb{Q}_p^\times\to\mathbb{C}_p^\times$.

$\cup\!|$

$\mathcal{T}_{reg} = \{(\delta_1,\delta_2)\in\mathcal{T} \mid \frac{\delta_2}{\delta_1}\neq x^n,\ x^n\chi_{cycl}(x) \text{ for some } n\in\mathbb{Z}_{\geq 0}\}$

Fix $\bar{r}_p \rightsquigarrow R^{\square}_{\bar{r}_p}$ = framed deformation ring $\mathcal{X}^{\square}_{\bar{r}_p} := (\mathrm{Spf}\, R^{\square}_{\bar{r}_p})^{rig}$



$U^{\square,tri}_{\bar{r}_p} = \{(x,\delta_1,\delta_2)\in\mathcal{X}^{\square}_{\bar{r}_p}\times\mathcal{T}_{reg} \mid$

$0\to\mathcal{R}(\delta_1)\to\mathbb{D}_{rig}(V_x)\to\mathcal{R}(\delta_2)\to 0\}$

$\mathcal{X}^{\square,tri}_{\bar{r}_p}$ = Zariski closure of $U^{\square,tri}_{\bar{r}_p}$

<u>Theorem</u> Same assumption:

For $(x,\delta_1,\delta_2)\in\mathcal{X}^{\square,tri}_{\bar{r}_p}$, put $w_k := \delta_1\delta_2^{-1}\chi_{cycl}^{-1}(\exp(p))-1$

$\varepsilon = \delta_2|_{\Delta}\times\delta_1|_{\Delta}\cdot\omega^{-1}.$

Then if $v_p(\delta_1(p)) = -v_p(\delta_2(p)) > 0$, then

$v_p(\delta_1(p))$ is a slope in $NP(G^{(\varepsilon)}_{\bar{p}}(w_k,-))$.

Conversely, all slopes in $NP(G^{(\varepsilon)}_{\bar{p}}(w_k,-))$ appears this way.

<u>Consequences</u>

<u>Application A</u> (Breuil-Buzzard-Emerton, ~2005)

(1) If r_p is a crystalline rep'n of HT wts $\{0,k-1\}$ lifting $\bar{r}_p$,

then $v_p(\varphi$-eigenval on $\mathbb{D}_{cris}(\bar{r}_p)) \in \begin{cases}\mathbb{Z}, & \text{if } a \text{ even}\\ \frac{1}{2}\mathbb{Z}, & \text{if } a \text{ odd.}\end{cases}$

(2) If r_p is a crystalline rep'n with wild char of conductor $p^m \geq p^2$.

φ-slope $\in \frac{1}{p^{m-2}(p-1)}\mathbb{Z}$.

<u>Rmk</u> Known when $v_p(-)$ small e.g. $\leq p$ by Ghate, Buzzard-Gee, Rozensztan, Berger, ...

or wt $\leq 3p$ by Breuil, Mézard (?)

(didn't have assumption on $\bar{r}_p$).

Application B (Gouvêa, $\lfloor\frac{k-1}{p+1}\rfloor$-conj.)

In above (1), less or φ-slope $\le \lfloor\frac{k-1}{p+1}\rfloor$.

Rmk Proved by Berger-Li-Zhu for $\lfloor\frac{k-1}{p+1}\rfloor$, Bergdall-Levin for $\lfloor\frac{k-1}{p}\rfloor$.

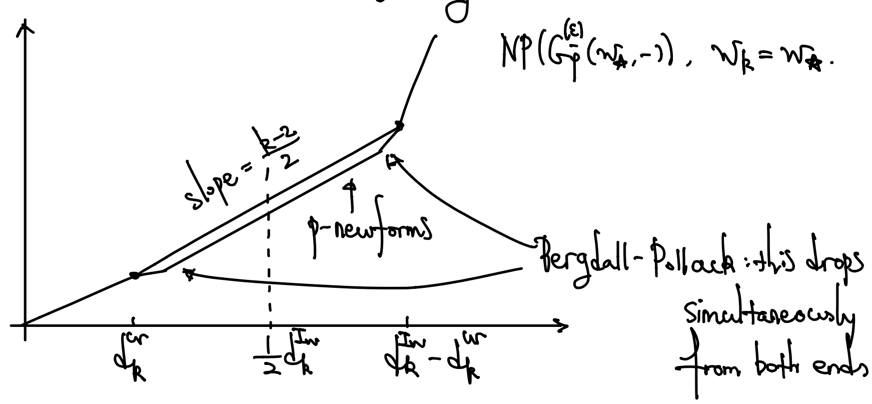
§4 Sketch of local ghost

Step 0 Criterion for vertices of $NP(G_{\bar{p}}^{(\varepsilon)}(w,-))$.

Thm $(n, g_n(w_\star))$ is <u>not</u> a vertex iff $\exists\, k \equiv a+2b+2s+2 \mod p-1$

s.t. $v_p(w_\star - w_k) \ge \cdots$ (some explicit number.)

"too close to a Steinberg wt".



Step 1 Have a matrix for U_p (power bases).

$$C_{\bar{H}}^{(\varepsilon)}(w,t) = \sum c_n(w)t^n.$$

Hope $c_n(w) \approx g_n(w)$.

Important Enough to estimate $A_k^{(n)} \cdot s + g_n(w)h_n(w)$.

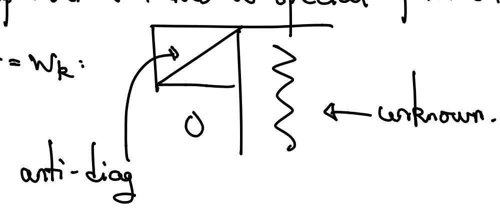
Lagrange interpolation

$$c_n(w) = \sum_k g_{n,\hat{k}}(w)\cdot\left(A_{k,0}^{(n)} + A_{k,1}^{(n)}(w-w_k)+\cdots\right)$$

Will prove this for all $n\times n$ minors by induction on size.

Step 2 **Key** U_p-matrix takes a special form at $w=w_k$.

At $w=w_k$:



Lecture 3: Abstract p-adic forms & corank thm

<u>Notations</u> $\tilde{H}$ primitive $\mathcal{O}[\![K_p]\!]$-proj augmented module of type $\bar{\rho}$.

- $\mathcal{E}: \Delta^2 \to \mathcal{O}^\times$, $\mathcal{E}(\bar{\alpha}, \bar{x}) = \bar{\alpha}^{a+2b}$, $\bar{\alpha} \in \Delta$.

- $\mathcal{O}[\![w]\!]^{(\varepsilon)} = \mathcal{O}[\![\Delta \times \mathbb{Z}_p^\times]\!] \otimes_{\mathcal{O}[\Delta^2], \varepsilon} \mathcal{O} \simeq \mathcal{O}[\![w]\!]$

$$w = [\![(1, \exp(p))]\!] - 1.$$

- $$\chi_{univ}^{(\varepsilon)} : B^{op}(\mathbb{Z}_p) = \begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ \mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \longrightarrow (\mathcal{O}[\![w]\!]^{(\varepsilon)})^\times$$
$$\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \longmapsto [(\bar{\alpha}, \delta)] \otimes 1 \overset{\shortparallel}{} \mathcal{E}(\bar{\alpha}, \bar{\delta})(1+w)^{\log(\delta/\omega(\bar{\delta}))/p}$$

- $$\mathrm{Ind}_{B^{op}(\mathbb{Z}_p)}^{\mathrm{Iw}_p}(\chi_{univ}^{(\varepsilon)}) := \left\{ \text{cont } f : \mathrm{Iw}_p \to \mathcal{O}[\![w]\!]^{(\varepsilon)} \;\middle|\; f(gb) = \chi_{univ}^{(\varepsilon)}(b) f(g) \right\}$$
$$f\big|_{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}(g) = f\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} g\right)$$

right action convolution.

$$\rightsquigarrow \mathrm{Ind}_{B^{op}(\mathbb{Z}_p)}^{\mathrm{Iw}_p}(\chi_{univ}^{(\varepsilon)}) \simeq C^{\cdot}(\mathbb{Z}_p, \mathcal{O}[\![w]\!]^{(\varepsilon)}).$$
$$f \longmapsto h(z) = f\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right).$$

- The action extends to $M_1 = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathbb{Z}_p) : p \mid \gamma,\ p \nmid \delta,\ \det \neq 0 \right\}$.

$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = p^t d$, for some $d \in \mathbb{Z}_p^\times$.

$$h\big|_{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}} = [(\bar{d}/\bar{\delta}, \gamma z + \delta)]\, h\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right)$$
$$= \mathcal{E}(\bar{d}/\bar{\delta}, \bar{\delta})(1+w)^{\log((\gamma z + \delta)/\omega(\bar{\delta}))/p}\, h\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right).$$

- $$S_{\tilde{H}}^{(\varepsilon)} = \mathrm{Hom}_{\mathrm{Iw}_p}(\tilde{H}, \mathrm{Ind}_{B^{op}(\mathbb{Z}_p)}^{\mathrm{Iw}_p}(\chi_{univ}^{(\varepsilon)}))$$
$$\simeq \mathrm{Hom}_{\mathrm{Iw}_p}(\tilde{H}, C^{\cdot}(\mathbb{Z}_p, \mathcal{O}[\![w]\!]^{(\varepsilon)}))$$ families of p-adic forms.

$$\mathrm{Iw}_p \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} \mathrm{Iw}_p = \coprod v_j \mathrm{Iw}_p, \quad \text{e.g. } v_j = \begin{pmatrix} p & 0 \\ pj & 1 \end{pmatrix}.$$

Similarly, families of overconvergent forms

- $$S_{\tilde{H}}^{\dagger, (\varepsilon)} = \mathrm{Hom}_{\mathrm{Iw}_p}(\tilde{H}, \mathcal{A}^{(\varepsilon), s^{\frac{1}{p}}}\langle z \rangle),$$

$\circlearrowleft U_p$ $\circlearrowleft M_1$

where $\Lambda^{\le\frac{1}{p}} = \mathcal{O}\langle w/p\rangle$, $\Lambda^{\ge\frac{1}{p}} = \mathcal{O}[\![w]\!]\langle p/w\rangle$, $\Lambda^{\frac{1}{p}} = \mathcal{O}\langle w/p, p/w\rangle$,

and $\Lambda^{(\varepsilon),?} = \mathcal{O}[\![w]\!]^{(\varepsilon)} \otimes_{\mathcal{O}[\![w]\!]} \Lambda^{?}$.

(U_p is compact on $S_{\widetilde{H}}^{\dagger,(\varepsilon)}$.)

- $\psi : \Delta^2 \longrightarrow \mathcal{O}^\times$, $k \ge 2$,

$$S^{\dagger}_{\widetilde{H},k}(\psi) := \mathrm{Hom}_{\mathrm{Iw}_p}(\widetilde{H}, \mathcal{O}\langle z\rangle \otimes \psi).$$

For $f \in \mathcal{O}\langle z\rangle$, $f|_{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}} := (\gamma z+\delta)^{k} f\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)$.

$$S^{\mathrm{Iw}}_{\widetilde{H},k}(\psi) := \mathrm{Hom}_{\mathrm{Iw}_p}(\widetilde{H}, \mathcal{O}[z]^{k-2} \otimes \psi).$$

- $(\psi, k) \rightsquigarrow \varepsilon = \varepsilon(\psi, k) = \psi(1 \times w^{k})$

It turns out $S^{\mathrm{Iw}}_{\widetilde{H},k}(\psi) \subset S^{\dagger}_{\widetilde{H},k}(\psi) \subset S^{\dagger,(\varepsilon)}_{\widetilde{H}} \otimes_{w \mapsto w_k} \mathcal{O}$.

- $\eta : \Delta \longrightarrow \mathcal{O}^\times \rightsquigarrow S^{ur}_k(\eta) := \mathrm{Hom}_{K_p}(\widetilde{H}, \mathcal{O}[z]^{\le k-2} \otimes \eta\circ\det)$

$$\circlearrowleft \quad M_2 = \left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in M_2(\mathbb{Z}_p) : \det \ne 0\right\}.$$

s.t. $(\eta\circ\det)\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \eta(d)$, $\qquad p^r d = \det\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$, $d \in \mathbb{Z}_p^\times$.

<u>Char power series of U_p</u>

$U_p \circlearrowright S^{\dagger,(\varepsilon)}_{\widetilde{H}}$ compact $\rightsquigarrow$ only defined over $\Lambda^{(\varepsilon),\le\frac{1}{p}}$.

$\rightsquigarrow$ sublattice $C^0(\mathbb{Z}_p, \Lambda^{(\varepsilon),\ge\frac{1}{p}})^{\mathrm{mod}}$

"modified".

$$\widehat{\bigoplus} w^{[n/p]}\binom{p}{n} \Lambda^{(\varepsilon),\ge\frac{1}{p}} \subseteq C^0(\mathbb{Z}_p, \Lambda^{(\varepsilon),\ge\frac{1}{p}}).$$

resp. $p^{[n/p]}\binom{z}{n}$ with "$\le\frac{1}{p}$".

<u>note</u> p/w is a unit in $\Lambda^{\frac{1}{p}} \Rightarrow C^0(\mathbb{Z}_p, \Lambda^{(\varepsilon),\ge\frac{1}{p}})^{\mathrm{mod}}$ & $C^0(\mathbb{Z}_p, \Lambda^{(\varepsilon),\le\frac{1}{p}})^{\mathrm{mod}}$

coincide on $\Lambda^{(\varepsilon),\frac{1}{p}}$.

They glue to a Banach subsheaf of $C(\mathbb{Z}_p, \Lambda^{(p)})$, $\Lambda^{(p)} = \mathcal{O}[\![w]\!]^{(\varepsilon)}$

$$w^{[\frac{n}{p}]}\binom{z}{n}\Big|_{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in M_1} = \sum_{m \ge 0} P_{m,n}\, w^{[m/p]}\binom{z}{m}.$$

Fact If $p \mid \alpha$, $p \mid \gamma$, $p \nmid \delta$), then $P_{m,n} \in w^{m\lceil \frac{n}{p} \rceil} \Lambda^{\geq \frac{1}{p}}$.

Similar for $p^{\lceil n/p \rceil} \binom{z}{n}$ $\Rightarrow$ U_p is compact on this Banach subsheaf!

$\Downarrow$

Can define $C_{\tilde{H}}^{(\varepsilon)}(w,t)$ as an element of $\mathcal{O}[[w,t]]$.

(Stronger than Coleman-Mazur.)

From now on, assume $b=0$ ($\Rightarrow \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ acts trivially on $\tilde{H}$.)

$$\bar{T} \cdot \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \in \mathcal{O}[[\mathrm{Iw}_p]].$$

$$\rightsquigarrow \tilde{H} \simeq \bigoplus_{i=1}^{r} e_i \mathcal{O} \otimes_{\chi_i, \mathcal{O}[\bar{T}]} \mathcal{O}[[\mathrm{Iw}_p]], \quad r = \mathrm{rank}_{\underset{\uparrow}{\mathrm{Iw}_p}} \tilde{H}.$$

noetherian local ring

Since $\tilde{H}$ is primitive,

$$\mathrm{Proj}_{a,0} \simeq \bigoplus_{i=1}^{2} e_i \mathcal{O} \otimes_{\chi_i} \mathbb{F}(\bar{B}).$$

Fact $\chi_1 = 1 \times \omega^a$, $\chi_2 = \omega^a \times 1$. (Possibly, will exchange χ_1 & χ_2.)

Lemma We may replace e_2 by $e_2' = e_1 \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ ($\Rightarrow e_1 = e_2' \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$).

Proof If e_1 has $\bar{T}$-char $1 \times \omega^a$, then $\bar{e_2'}$ has $\bar{T}$-char $\omega^a \times 1$. □

$\rightsquigarrow$ From now on, assume $e_2 = e_1 \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$.

$$S_{\tilde{H}}^{\dagger,(\varepsilon)} = \mathrm{Hom}_{\mathrm{Iw}_p}(\tilde{H}, \Lambda^{(\varepsilon), \leq 1/p}\langle z \rangle \otimes \varepsilon), \quad \varepsilon = \omega^{-s_\varepsilon} \times \omega^{a+s_\varepsilon}, \quad s_\varepsilon \in \{0, \dots, p-2\},$$

$$\simeq e_1^*(\Lambda^{(\varepsilon), \leq 1/p}_{\langle z \rangle} \otimes \varepsilon)^{\bar{T} = 1 \times \omega^a} \oplus e_2^*(\Lambda^{(\varepsilon), \leq 1/p}_{\langle z \rangle} \otimes \varepsilon)^{\bar{T} = \omega^a \times 1}.$$

$\equiv a + s_\varepsilon \bmod p-1$

Prop $\{e_1^* \cdot z^{s_\varepsilon + (p-1)i}\}_{i \geq 0} \cup \{e_2^* \cdot z^{\{a+s_\varepsilon\} + (p-1)i}\}_{i \geq 0}$

is a basis of $S_{\tilde{H}}^{\dagger,(\varepsilon)}$ as a Banach module over $\Lambda^{\leq 1/p}$.

or of $S_{\tilde{H},k}^{\dagger}(\varepsilon(1 \times \omega^{2k}))$ as a Banach $\mathcal{O}$-mod.

Moreover, terms whose (powers in z) $\leq k-2$ form an $\mathcal{O}$-basis of $S_k^{\mathrm{Iw}}(\varepsilon(1 \times \omega^{2k}))$.

Proof · z^j has $\bar{T}$-char $\omega^j \times \omega^{-j}$

· $(j - s_\varepsilon,\ j + a + s_\varepsilon) \equiv (a,a)$ or $(a,0)$ $\Big\}$ (mod $p-1$). □

$\Rightarrow$ $j \equiv s_\varepsilon$ or $j \equiv \{a + s_\varepsilon\}$

$B^{(\varepsilon)}$ or $B_k^{(\varepsilon)} \circlearrowleft U_p$

Remark $U_{(k)}^{\dagger,(\varepsilon)}$ = matrix of U_p w.r.t. $B_{(k)}^{(\varepsilon)}$ $\rightsquigarrow$ $C_{\mathcal{H}}^{(\varepsilon)}(w,t) = \mathrm{char}(1 - U_p \cdot ?)$

Prop (Theta map) $\overset{\varepsilon'(\psi,\,2-k)}{\overset{\|}{\varepsilon'}} = \varepsilon(\omega^{k-1} \times \omega^{1-k})$, $\psi = \varepsilon(1 \times \omega^{2-k})$, $\psi' = \varepsilon'(1 \times \omega^{k})$.

(1) $\exists$ short exact sequence

$$0 \longrightarrow \overset{U_p \circlearrowleft}{S_k^{Iw}(\psi)} \longrightarrow \overset{U_p \circlearrowleft}{S_k^{\dagger}(\psi)} \xrightarrow{\left(\frac{d}{dz}\right)^{k-1} \circ (\cdot)} \overset{p^{k-1}U_p \circlearrowleft}{S_{2-k}^{\dagger}(\psi)} \quad \text{(not surj)}$$

$$\left(\left(\tfrac{d}{dz}\right)^{k-1} \circ \varphi\right)(x) := \left(\tfrac{d}{dz}\right)^{k-1}(\varphi(x)).$$

equivariant w/ U_p-action

$p^{k-1}U_p$-equivariant here

(2) $C^{(\varepsilon)}(w_k, t) = C^{(\varepsilon)}(w_{2-k}, p^{k-1}t) = \mathrm{char}(U_p, S_k^{Iw}(\psi))$.

(3) All finite U_p-slopes that are $< k-1$ belong to $S_k^{Iw}(\psi)$.

Proof (1) By direct computation.

(2) · $\mathrm{Ker}\left(\left(\frac{d}{dz}\right)^{k-1} \circ (-)\right)$ is spanned by $B_k^{(\varepsilon)}$

$\rightsquigarrow$ $\left(\frac{d}{dz}\right)^{k-1}(e_1^* \cdot z^j) = \underbrace{j \cdot (j-1) \cdots (j-k+2)}_{\text{"integrating is not integral"}}\, e_1^* \cdot z^{j-k}$.

· $U_k^{\dagger,(\varepsilon)} = \begin{pmatrix} U_k^{Iw} & 0 \\ 0 & p^{k-1} D^{-1} U_{2-k}^{\dagger,(\varepsilon)} D \end{pmatrix}$

D = diagonal matrix given by the coefficients $j(j-1)\cdots(j-k+2)$.

$\Rightarrow$ (2).

(3) U_p-slopes of $S_{2-k}^{\dagger}(\psi)$ is nonnegative

$\Rightarrow$ if U_p-slope of f is $< k-1$, then $\left(\frac{d}{dz}\right)^{k-1}(f) = 0$.

Prop (Atkin-Lehner involution). $\psi = \psi_1 \times \psi_2$, $\psi^s = \psi_2 \times \psi_1$ chars of Δ^2.

$$\varepsilon'' = \varepsilon \cdot \psi^s \cdot \psi^{-1}.$$

(1) $\exists$ natural morphism

$$AL_{(k,\psi)} : S_k^{Iw}(\psi) \longrightarrow S_k^{Iw}(\psi^s)$$
$$\varphi \longmapsto \left(x \mapsto \varphi\left(x \begin{pmatrix} 0 & p^{-1} \\ 1 & 0 \end{pmatrix}\right)\right)\Big|_{\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}}.$$

(2) $\forall j \geq 0$, $l = 1, \dots, d_k^{Iw}(\psi^s)$, $i = 1, 2$

$$AL_{(k,\psi)}(e_i^* z^j) = p^{k-2-j} \cdot e_{3-i}^* \cdot z^{k-2-j}$$

$$\left(\Rightarrow \quad AL_{(k,\psi^s)} \circ AL_{(k,\psi)} = p^{k-2}. \right)$$

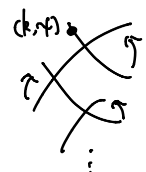
(3) When $\psi_1 \neq \psi_2$ ($\Leftrightarrow k \not\equiv k_\varepsilon \bmod p-1$), we have

$$U_p \circ AL_{(k,\psi)} \circ U_p = p^{k-1} \cdot AL_{(k,\psi)}.$$

In this case, can pair U_p-slopes on $S_k^{Iw}(\psi)$ & $S_k^{Iw}(\psi^s)$ s.t. each pair adds up to $k-1$.

Rmk If $\psi_1 \neq \psi_2$, then (k,ψ), (k,ψ^s) are often on different wt disks.

Some Ping-Pong type phenomenon: $(k,\psi) \rightsquigarrow (k',\psi') \rightsquigarrow (k,\psi)$.



From now on, assume $k \equiv k_\varepsilon$.

Then Atkin-Lehner $\rightsquigarrow S_k^{Iw}(\tilde{\varepsilon}_1) \longrightarrow S_k^{Iw}(\tilde{\varepsilon}_2)$ $\quad (\varepsilon = \varepsilon_1 \times \varepsilon_2)$.

Fix $\eta : \Delta \to \mathcal{O}^\times$, $\tilde{\eta} := \eta \times \eta : \Delta^2 \to \mathcal{O}^\times$.

Def'n

$$\begin{tikzcd} S_k^{ur}(\varepsilon_1) \arrow[r, bend left=40, "i_1"] \arrow[r, bend left=15, "i_2"] & S_k^{Iw}(\tilde{\varepsilon}_1) \arrow[l, bend left=15, "pr_1"] \arrow[l, bend left=40, "pr_2"] \end{tikzcd}.$$

$i_1(\varphi) = \varphi$, $\quad i_2(\varphi) = AL(\varphi)$

$$pr_1(\varphi)(x) = \sum_{j=0,\dots,p-1,*} \varphi(x u_j)\big|_{u_j^{-1}}$$

$$pr_2(\varphi)(x) = \sum_{j=0,\dots,p-1,*} \varphi\left(x \begin{pmatrix} 0 & p^{-1} \\ 1 & 0 \end{pmatrix} u_j\right)\Big|_{u_j^{-1}\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}}.$$
$$= pr_1(AL(\varphi))(x).$$

Prop $U_p(\varphi) = i_2(pr_1(\varphi)) - AL(\varphi)$, $\varphi \in S_k^{Iw}(\tilde{\varepsilon})$

$$= AL(pr_1(\varphi)) - AL(\varphi).$$

Put U_k^{Iw} = matrix of U_p w.r.t. $B_k^{(\varepsilon)}$

L_k^{cl} = matrix of AL w.r.t. $B_k^{(\varepsilon)}$.

Prop (1) L_k^{cl} is anti-diag with entries $p^{\deg e_1^{(\varepsilon)}}, p^{\deg e_2^{(\varepsilon)}}, \dots$

(where $e_1^{(\varepsilon)}, e_2^{(\varepsilon)}, \dots$ is the power basis indexed by exponents of z)

from upper right to lower left.

(2) $\underset{\substack{\uparrow \\ \text{size } (d_k^{Iw} \times d_k^{Iw})}}{U_k^{Iw}} = -L_k^{cl} + \square$, where $\square$ has $\mathrm{rk} \le d_k^{ur}(\varepsilon_1)$.

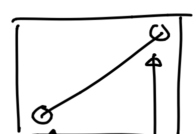
Proof (2) $\square = U_k^{Iw} + L_k^{cl}$ is the matrix corresponds to $\varphi \mapsto L_2(pr_1(\varphi))$. $\square$

Consider $d_k^{new}(\varepsilon_1) = d_k^{Iw}(\varepsilon_1) - 2d_k^{ur}(\varepsilon_1)$.

Cor The multiplicities of $\pm p^{(k-2)/2}$ as eigenvals of U_p on $S_k^{Iw}(\tilde{\varepsilon}_1)$

are at least $\frac{1}{2} d_k^{new}(\varepsilon_1)$ each.

Proof $U_k^{Iw} \pm p^{\frac{k-2}{2}} I = -(\underbrace{L_k^{cl} \pm p^{\frac{k-2}{2}} I}_{\text{rank } \frac{1}{2} d_k^{Iw}}) + \underset{\substack{\uparrow \\ \text{rank} \le d_k^{ur}}}{\square}$



L_k^{cl} has eigenvals $\pm p^{\frac{k-2}{2}}$ each with multiplicity $\frac{1}{2} d_k^{Iw}$. two entries for eigenvals.

$\Rightarrow$ rank $U_k^{Iw} \pm p^{\frac{k-2}{2}} I \le d_k^{ur} + \frac{1}{2} d_k^{Iw}$

$\Rightarrow$ corank $U_k^{Iw} \pm p^{\frac{k-2}{2}} I \ge \frac{1}{2} d_k^{Iw} - d_k^{ur} = \frac{1}{2} d_k^{new}$. $\square$

Cor (Weak corank thm)

Write $U^{\dagger,(\varepsilon)}(\underline{n}) \in M_n(\mathcal{O}\langle \frac{w}{p} \rangle)$ for the upper-left $n \times n$-submatrix of $U^{\dagger,(\varepsilon)}$.

Then $p^{-\deg g_n^{(\varepsilon)}} g_n^{(\varepsilon)}(w) \mid \det(U^{\dagger,(\varepsilon)}(\underline{n})) \in \mathcal{O}\langle \frac{w}{p} \rangle$.

Proof Need to show: $\forall\, k \equiv k_\varepsilon \bmod p-1$ s.t. $m_n(k) > 0$,

$$\left(\frac{w}{p} - \frac{w_k}{p}\right)^{m_n(k)} \Big|\ \det\left(U^\dagger(\underline{n})\right).$$

(note: coeffs $\in \mathcal{O}\langle \frac{w}{p}\rangle \rightsquigarrow$ need to devide each ghost factor by p).

$\Leftarrow$ (eval $U^\dagger(\underline{n})$ at $w = w_k$) $= U_k^\dagger(\underline{n})$ has corank $\geq m_n(k)$.

Indeed, for $L_k^{cl}(\underline{n}) :=$ upper left $n\times n$-submat of L_k^{cl}.

Prop (1)(2) above $\Rightarrow$ $\operatorname{rank}(U_k^\dagger(\underline{n})) \leq d_k^{ur} + \operatorname{rank} L_k^{cl}(\underline{n})$

$$= \begin{cases} d_k^{ur}, & n \leq \frac{1}{2} d_k^{Iw}, \\ d_k^{ur} + 2\left(n - \frac{1}{2} d_k^{Iw}\right), & n \geq \frac{1}{2} d_k^{Iw}. \end{cases}$$

$$\Rightarrow \operatorname{corank} U_k^\dagger(\underline{n}) \geq \begin{cases} n - d_k^{ur} & \text{if } n \leq \frac{1}{2} d_k^{Iw}. \\ d_k^{Iw} - d_k^{ur} - n & \text{if } n \geq \frac{1}{2} d_k^{Iw}. \end{cases}$$

$\Rightarrow$ $\operatorname{corank} U_k^\dagger(\underline{n}) \geq m_n(k)$. $\square$

Lecture 4: Basic properties of ghost series

Fix $p \geq 5$. $E/\mathbb{Q}_p$ finite ext'n, $\varpi \in \mathcal{O} \subseteq E \rightsquigarrow \mathbb{F} = \mathcal{O}/(\varpi)$.

Fix a residual rep'n $\bar{\rho}: \begin{pmatrix} \omega_1^{a+1} & * \neq 0 \\ 0 & 1 \end{pmatrix} \cdot I_{\mathbb{Q}_p} \longrightarrow GL_2(\mathbb{F})$ with $1 \leq a \leq p-4$.

Let $\widetilde{H}$ be a primitive $\mathcal{O}[\![K_p]\!]$-proj augmented mod of type $\bar{\rho}$ on which $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ acts trivially.

$\rightsquigarrow \widetilde{H}$ $\mathcal{O}[\![K_p]\!] = \mathcal{O}[\![GL_2(\mathbb{Z}_p)]\!]$-mod where $GL_2(\mathbb{Z}_p)$-action extends to $GL_2(\mathbb{Q}_p)$.

Fix a character $\varepsilon = \varepsilon_1 \times \varepsilon_2 = \omega^{-s_\varepsilon} \times \omega^{a+s_\varepsilon} : \Delta^2 \rightarrow \mathcal{O}^\times$

$\Delta \subseteq \mathbb{Z}_p^\times$ torsion subgrp, $\omega: \Delta \rightarrow \mathcal{O}^\times$ Teichmuller char.

Relevant to $\bar{\rho}$: $s_\varepsilon \in \{0, \dots, p-2\}$.

for $k \geq 2$, $\psi: \Delta^2 \rightarrow \mathcal{O}^\times$

$\rightsquigarrow$ we define $S_k^{Iw}(\psi) = \mathrm{Hom}_{Iw_p}(\widetilde{H}, \mathcal{O}[z]^{\leq k-2} \otimes \psi) \leftrightarrow S_k(\Gamma \cap \Gamma_0(p), \psi)$.

and $d_k^{Iw}(\psi) = \mathrm{rank}_{\mathcal{O}} S_k^{Iw}(\psi)$.

$\rightsquigarrow \mathcal{O}[z]^{\leq k-2}$ subspace of $\mathcal{O}\langle z\rangle$ consisting of polynomials of deg $\leq k-2$.

$$f\Big|\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(z) = (\gamma z + \delta)^{k-2} f\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right), \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in K_p, \ f \in \mathcal{O}[z]^{\leq k-2}.$$

For $\eta: \Delta \rightarrow \mathcal{O}^\times$ we define

$S_k^{ur}(\eta) = \mathrm{Hom}_{K_p}(\widetilde{H}, \mathcal{O}[z]^{\leq k-2} \otimes \eta \circ \det) \longleftrightarrow S_k(\Gamma, \eta)$, $\Gamma = \Gamma_0(N)$, $(N,p)=1$.

$d_k^{ur}(\eta) = \mathrm{rank}_{\mathcal{O}} S_k^{ur}(\eta)$.

<u>Goal</u> To give formulas to compute $d_k^{Iw}(\psi)$ & $d_k^{ur}(\eta)$ in terms of k, ψ, and η (with a, p).

For $\eta: \Delta \rightarrow \mathcal{O}^\times$ we set $\tilde{\eta}: \Delta^2 \rightarrow \mathcal{O}^\times$, $\tilde{\eta} = \eta \times \eta$

if $S_k^{Iw}(\psi)$ belongs to $\mathcal{W}^{(\varepsilon)}$

$\Leftrightarrow (1\times\omega^{k-2})\cdot\psi = \varepsilon \Rightarrow \psi = \varepsilon(1\times\omega^{2-k})$.

<u>Computation</u> $d_k^{Iw}(\psi) = d_k^{Iw}(\varepsilon(1\times\omega^{2-k}))$.

Recall $\tilde{H}$ is a free $\mathcal{O}[[\mathrm{Iw}_{p,1}]]$-mod of rk 2.

We can choose a basis $\{e_1, e_2\}$

s.t. Δ^2 acts on e_1 (resp. e_2) via the character $1\times\omega^a$ (resp. $\omega^a\times 1$).

$\Delta^2 \subseteq \mathrm{Iw}_p$, $\mathrm{Iw}_{p,1} = \begin{pmatrix} 1+p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix} \subseteq K_p$.

<u>Rmk</u> Can further assume $e_1\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} = e_2$,

$\mathcal{O}[z]^{\leq k-2}\otimes\psi$ has a basis $\{1, z, \cdots, z^{k-2}\}$

Δ^2 acts on z^i via the character $\omega^{i-s_\varepsilon}\times\omega^{a+s_\varepsilon-i}$

$\Rightarrow$ the $\mathcal{O}$-module $S_k^{Iw}(\psi)$ has an $\mathcal{O}$-basis

$\{e_1^*\cdot z^i \mid i\equiv s_\varepsilon \bmod p-1,\ 0\leq i\leq k-2\}\cup\{e_2^*\cdot z^j \mid j\equiv a+s_\varepsilon \bmod p-1,\ 0\leq j\leq k-2\}$.

$\rightsquigarrow (e_1^*\cdot z^i : e_1\longmapsto z^i,\ e_2\longmapsto 0)\in \mathrm{Hom}_{\mathrm{Iw}_p}(\tilde{H}, \mathcal{O}[z]^{\leq k-2}\otimes\psi)$.

<u>Prop 4.1</u> We have

$$d_k^{Iw}(\psi) = d_k^{Iw}(\varepsilon\cdot(1\times\omega^{2-k})) = \left\lfloor\frac{k-2-s_\varepsilon}{p-1}\right\rfloor + \left\lfloor\frac{k-2-\{a+s_\varepsilon\}}{p-1}\right\rfloor + 2.$$

$\forall m\in\mathbb{Z}$, $\{m\}$ is the unique integer $\in\{0,\cdots,p-2\}$ s.t. $m\equiv\{m\} \bmod p-1$.

In particular, $d_{k+(p-1)}^{Iw}(\varepsilon(1\times\omega^{2-k})) - d_k^{Iw}(\varepsilon(1\times\omega^{2-k})) = 2$.

$d_k^{Iw}(\varepsilon(1\times\omega^{2-k})) = \frac{2k}{p-1} + O(1)$.

- When $\psi = \varepsilon(1\times\omega^{2-k}) = \varepsilon_1\times\varepsilon_2\omega^{2-k} = \varepsilon_1\times\varepsilon_1 = \tilde{\varepsilon}_1$,

  we may have $S_k^{ur}(\varepsilon_1)\subseteq S_k^{Iw}(\psi)$.

- Recall we define $k_\varepsilon = 2+\{a+2s_\varepsilon\}$.

  we consider $k\equiv k_\varepsilon \bmod (p-1) \Rightarrow \psi = \varepsilon\cdot(1\times\omega^{2-k}) = \tilde{\varepsilon}_1$

The number $d_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1)$ ($d_k^{\mathrm{ur}}(\varepsilon_1)$) is used in the def'n of ghost series.

- $k = k_\varepsilon + (p-1)k_0$

$\rightsquigarrow$ Define $\delta_\varepsilon = \left\lfloor \dfrac{s_\varepsilon + \{a + s_\varepsilon\}}{p-1} \right\rfloor = \begin{cases} 0, & s_\varepsilon + \{a + s_\varepsilon\} < p-1 \\ 1, & s_\varepsilon + \{a + s_\varepsilon\} \geq p-1. \end{cases}$

<u>Cor</u> For the above k,

$$d_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1) = 2k + 2 - 2\delta_\varepsilon.$$

In particular, $d_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1)$, $d_k^{\mathrm{new}} = d_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1) - 2d_k^{\mathrm{ur}}(\varepsilon_1)$ are even.

($k \mapsto k + (p-1)$, $d_k^{\mathrm{Iw}}(\tilde{\varepsilon}) \longrightarrow d_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1) + 2$)

Computation of $d_k^{\mathrm{ur}}(\tilde{\varepsilon}_1)$:

$k \equiv k_\varepsilon \mod p-1$, $k_\varepsilon = 2 + \{a + 2s_\varepsilon\}$

For two integers $a \geq 0$, b,

use $\sigma_{a,b} :=$ right rep of $GL_2(\mathbb{F}_p)$, say $\mathrm{Sym}^a \mathbb{F}^{\oplus 2} \otimes \det^b$.

When $0 \leq a \leq p-1$, $\sigma_{a,b}$ irred, we let

$\mathrm{Proj}_{a,b} =$ proj envelop of $\sigma_{a,b}$ in $\mathrm{Rep}_{\mathbb{F}[GL_2(\mathbb{F}_p)]}$.

$\tilde{H} = \mathcal{O}[\![K_p]\!]$-module,

$\rightsquigarrow$ $\tilde{H}/(\varpi, \underbrace{I_{1+pM_2(\mathbb{Z}_p)}}_{K_1}) \cong \mathrm{Proj}_{a,0}$ as $\mathbb{F}[GL_2(\mathbb{F}_p)]$-mod.

For $k \equiv k_\varepsilon \mod p-1$,

$$\begin{aligned} d_k^{\mathrm{ur}}(\varepsilon_1) &= \mathrm{rank}_{\mathcal{O}} \mathrm{Hom}_{K_p}(\tilde{H}, \mathcal{O}[z]^{\leq k-2} \otimes \varepsilon_1 \circ \det) \\ &= \mathrm{rank}_{\mathbb{F}} (\mathrm{Proj}_{a,0}, \sigma_{k-2} \otimes \det^{s_\varepsilon}) \\ &= \mathrm{rank}_{\mathbb{F}} (\mathrm{Proj}_{a,s_\varepsilon}, \sigma_{k-2}) = \mathrm{Multi}_{\sigma_{a,s_\varepsilon}}(\sigma_{k-2,0}). \end{aligned}$$

We define $t_1 < t_2$ as follows:

(i) When $a + s_\varepsilon < p-1$, set $t_1 = s_\varepsilon + \delta_\varepsilon$, $t_2 = a + s_\varepsilon + \delta_\varepsilon + 2$,

(ii) When $a+\delta_\varepsilon \geq p-1$, set $t_1 = \{a+\delta_\varepsilon\} + \delta_\varepsilon + 1$, $t_2 = \delta_\varepsilon + \delta_\xi + 1$.

<u>Prop</u> $k = k_\varepsilon + (p-1)k_\bullet$. We have

$$d_k^{ur}(\tilde{\varepsilon}_1) = \left\lfloor \frac{k_\bullet - t_1}{p+1} \right\rfloor + \left\lfloor \frac{k_\bullet - t_2}{p+1} \right\rfloor + 2. \quad (k \mapsto d_k^{ur}).$$

In particular, we have

$$d^{ur}_{k+(p-1)(p+1)}(\tilde{\varepsilon}_1) - d^{ur}_k(\tilde{\varepsilon}_1) = 2$$

and $d_k^{ur}(\varepsilon_1) = \frac{2k}{p^2-1} + O(1)$, $d_k^{new}(\varepsilon_1) = \frac{2k}{p+1} + O(1)$.

§ Application of dimension formulas

(1) Refined spectral halo of eigencurves.

Recall $k \in \mathbb{Z}$, $w_k = \exp((k-2)p) - 1 \in \mathfrak{m}_{\mathbb{C}_p}$.

For $k \equiv k_\varepsilon \bmod p-1$,

we define $\{m_n(k)\}_{n\geq 1}$ as a seq of integers:

$$\underbrace{0, \dots, 0}_{d_k^{ur}(\varepsilon_1)}, 1, 2, 3, \dots, \tfrac{1}{2}d_k^{new}, \tfrac{1}{2}d_k^{new} - 1, \dots, 3, 2, 1, 0, \dots$$

Take $g_n(w) = \prod_{\substack{k \equiv k_\varepsilon \\ k \geq 2}} (w - w_k)^{m_n(k)} \in \mathbb{Z}_p[w]$.

Define $G^{(\varepsilon)}(w,t) = G(w,t) = 1 + \sum_{n \geq 1} g_n(w) \cdot t^n \in \mathcal{O}[\![w,t]\!]$.

<u>Goal</u> Compute $NP(G^{(\varepsilon)}(w_\star, -))$ when $w_\star \in \mathfrak{m}_{\mathbb{C}_p}$ lies in the halo range

i.e. $v_p(w_\star) \in (0,1)$.

For such $w_\star$ we have $v_p(w_\star - w_k) = v_p(w_\star)$

$\Rightarrow v_p(g_n(w_\star)) = \deg(g_n(w)) \cdot v_p(w_\star)$.

We define $\{d_n\}_{n \geq 1} = \{d \equiv \delta_\varepsilon \text{ or } a + \delta_\varepsilon \bmod p-1,\ d \geq 0\}$.

$\lambda_n = d_n - \lfloor \frac{d_n}{p} \rfloor \longleftarrow$ Hodge bound.

We define the Hodge polygon whose nth slope $= \lambda_n v_p(w_\star)$, $n \geq 1$.

[LWX] $\Rightarrow$ $NP(C^{(\varepsilon)}(w_\star, -))$ lies on or above this ghost Hodge polygon.

$\uparrow$

essentially gives an estimation of the matrix for the operator

$$(-)\Big|\begin{pmatrix} pa & b \\ pc & d \end{pmatrix} \text{ on } C(\mathbb{Z}_p, \mathcal{O}[[w]]\langle \tfrac{p}{w} \rangle) \text{ w.r.t. } \left\{ \binom{z}{n} \,\Big|\, n \geq 0 \right\}.$$

<u>Prop</u> If $a + s_\varepsilon < p-1$ for $n \geq 0$,

$$\deg g_{n+1}(w) - \deg g_n(w) - \lambda_{n+1} = \begin{cases} 1, & n - 2s_\varepsilon \equiv 1, 3, \cdots, 2a+1 \mod 2p \\ -1, & n - 2s_\varepsilon \equiv 2, 4, \cdots, 2a+2 \mod 2p \\ 0, & \text{otherwise.} \end{cases}$$

If $a + s_\varepsilon \geq p-1$,

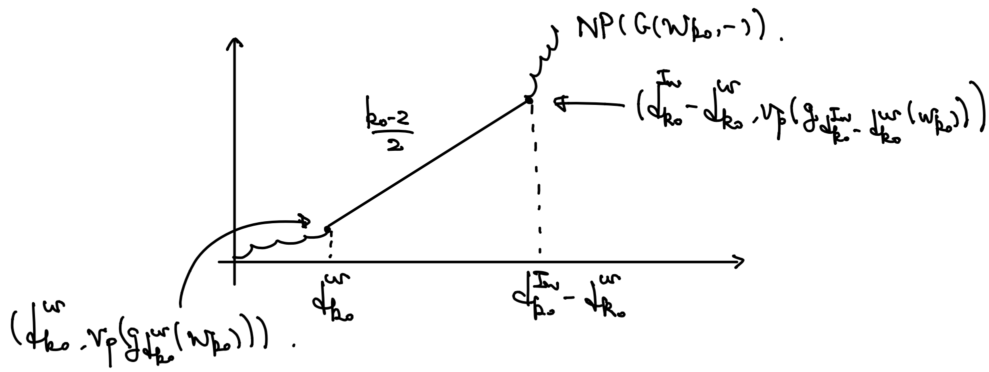
$$\deg g_{n+1}(w) - \deg g_n(w) - \lambda_{n+1} = \begin{cases} 1, & n - 2s_\varepsilon \equiv 2, 4, \cdots, 2a+2 \mod 2p \\ -1, & n - 2s_\varepsilon \equiv 3, 5, \cdots, 2a+3 \mod 2p. \\ 0, & \text{otherwise.} \end{cases}$$

<u>Rmk</u> If we set $a_n = \deg g_n(w) - \deg g_{n-1}(w)$, $n \geq 1$,

then (1) $\{a_n\}_{n \geq 1}$ is strictly increasing

(2) $a_{n+2p} - a_n = (p-1)^2$, $n \geq 1$.

$\Rightarrow \{a_n\}_{n\geq 1}$ is a disjoint union of $2p$ arithmetic progressions.



__Application__ (Ghost duality)

For $k \equiv k_\varepsilon \bmod p-1$, for each $l = 0, \dots, \frac{1}{2} d_{k_0}^{\mathrm{new}} - 1$,

we have

$$v_p\left(g_{d_{k_0}^{\mathrm{Iw}} - d_{k_0}^{\mathrm{ur}} - l, \hat{k}_0}(w_{k_0})\right) - v_p\left(g_{d_{k_0}^{\mathrm{ur}} + l, \hat{k}_0}(w_{k_0})\right) = \frac{k_0 - 2}{2}\left(d_{k_0}^{\mathrm{new}} - 2l\right).$$

Here $\displaystyle g_{n, \hat{k}_0}(w) := \prod_{\substack{k \equiv k_\varepsilon (p-1) \\ k \neq k_0}} (w - w_k)^{m_n(k)}.$

$$\left(g_n(w) = \prod_{\substack{k \equiv k_\varepsilon (p-1) \\ k \geq 2}} (w - w_k)^{m_n(k)}. \right)$$

When $l = 0$, the slope of the line segment connecting $(d_{k_0}^{\mathrm{ur}}, v_p(g_{d_{k_0}^{\mathrm{ur}}}(w_{k_0})))$ and $(d_{k_0}^{\mathrm{Iw}} - d_{k_0}^{\mathrm{ur}}, v_p(g_{d_{k_0}^{\mathrm{Iw}} - d_{k_0}^{\mathrm{ur}}}(w_{k_0})))$ is $\frac{k_0 - 2}{2}$.

__Question__ What is the meaning of ghost duality for $l \geq 1$?

Lecture 5: Further properties of ghost NP.

<u>Question</u> Fix $\varepsilon: \Delta^2 \to \mathcal{O}^\times$, a positive int n, and $w_\star \in m_{\mathbb{C}_p}$, $v_p(w_\star) \geq 1$.
How to determine when $(n, v_p(g_n^{(\varepsilon)}(w_\star)))$ is a vertex of $NP(G^{(\varepsilon)}(w_\star, -))$ or not?

<u>Later</u> To prove $NP(G(w_\star, -)) = NP(C(w_\star, -))$,
it suffices to prove two claims:

<u>Claim 1</u> $\forall n \geq 1$, $(n, v_p(C_n(w_\star)))$ lies on or above $NP(G(w_\star, -))$.

<u>Claim 2</u> If $(n, v_p(g_n(w_\star)))$ is a vertex of $NP(G^{(\varepsilon)}(w_\star, -))$
then $v_p(g_n(w_\star)) = v_p(C_n(w_\star))$.

If $w_\star = w_k$ for some $k \equiv k_\varepsilon \bmod p-1$, and
n lies in the Steinberg range of w_k, i.e. $n \in \left(\frac{1}{2}d_k^{\mathrm{Iw}} - \frac{1}{2}d_k^{\mathrm{ur}}, d_k^{\mathrm{Iw}} - d_k^{\mathrm{ur}}\right)$
then $(n, v_p(g_n(w_\star)))$ is not a vertex.

<u>Intuition</u> (1) If $w_\star$ is "very closed" to w_k and n lies in some subinterval of $(d_k^{\mathrm{ur}}, d_k^{\mathrm{Iw}} - d_k^{\mathrm{ur}})$, then
$$NP(G(w_\star, -)) \approx NP(G(w_k, -))$$
and $(n, v_p(g_n(w_\star)))$ will not be a vertex.

(2) Consider $n = d_k^{\mathrm{ur}} + 1$.
Assume $w_\star$ is "closed" to w_k, so that
$$v_p(w_\star - w_k) > v_p(w_k' - w_k),$$
where w_k' is any zero of $g_l(w)$, $l = d_k^{\mathrm{ur}}, d_k^{\mathrm{ur}}+1, d_k^{\mathrm{Iw}} - d_k^{\mathrm{ur}}$.
$\Rightarrow$ $v_p(w_\star - w_k') = v_p(w_k' - w_k)$ other than w_k.

Under this assumption,

we have $v_p(g_l(w_\star)) = v_p(g_l(w_k))$ for $l = d_k^{ur}, d_k^{Iw} - d_k^{ur}$.

$$P_1 = (d_k^{ur}, v_p(g_{d_k^{ur}}(w_\star))), \quad P_2 = (d_k^{Iw} - d_k^{ur}, v_p(g_{d_k^{Iw} - d_k^{ur}}(w_\star))).$$

Ghost duality $\Rightarrow$ the slope of $\overline{P_1P_2}$ is $\frac{k-2}{2}$.

Consider the pt $P(d_k^{ur}+1, v_p(g_{d_k^{ur}+1}(w_\star)))$

$$v_p(g_{d_k^{ur}+1}(w_\star)) = v_p(w_\star - w_k) + v_p(g_{d_k^{ur}+1, \hat{k}}(w_\star))$$

$$= v_p(w_\star - w_k) + v_p(g_{d_k^{ur}+1, \hat{k}}(w_k)).$$

- P lies on or above $\overline{P_1P_2}$

$$\Longleftrightarrow \text{ slope of } \overline{P_1P_2} \geq \frac{k-2}{2}$$

$$\Longleftrightarrow v_p(w_\star - w_k) + v_p(g_{d_k^{ur}+1, \hat{k}}(w_k)) \geq \frac{k-2}{2}$$

$$\Longleftrightarrow v_p(w_\star - w_k) \geq v_p(g_{d_k^{ur}}(w_k)) - v_p(g_{d_k^{ur}+1, \hat{k}}(w_k)) + \frac{k-2}{2}. \qquad (*)$$

- Introduce $\Delta'_{k,l} = v_p(g_{\frac{1}{2}d_k^{Iw}+l, \hat{k}}(w_k)) - \frac{k-2}{2} \cdot l$

$$l = -\tfrac{1}{2}d_k^{new}, -\tfrac{1}{2}d_k^{new}+1, \cdots, 0, 1, \cdots, \tfrac{1}{2}d_k^{new}.$$

(Ghost duality $\Longleftrightarrow \Delta'_{k,l} = \Delta'_{k,-l}$).

$$(*) \Longleftrightarrow v_p(w_\star - w_k) \geq \Delta'_{k,-\frac{1}{2}d_k^{new}} - \Delta'_{k,-\frac{1}{2}d_k^{new}+1}$$

$$= \Delta'_{k,\frac{1}{2}d_k^{new}} - \Delta'_{k,\frac{1}{2}d_k^{new}-1}.$$

- Consider the pt $P(d_k^{ur}+2, v_p(g_{d_k^{ur}+1}(w_\star)))$.

$$(*) \xRightarrow{\text{Ghost duality}} P'(d_k^{Iw} - d_k^{ur} - 1, v_p(g_{d_k^{Iw} - d_k^{ur} - 1}(w_\star)))$$

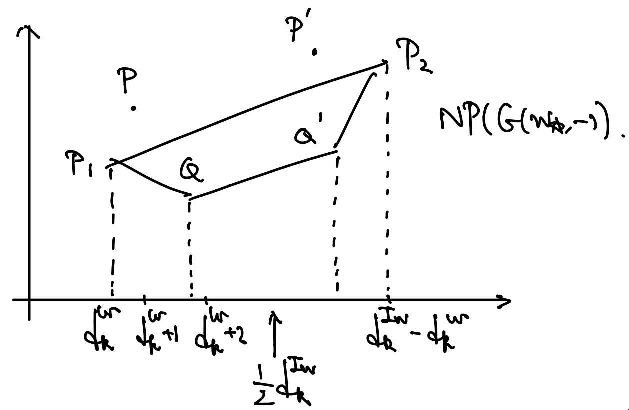
lies on or above $\overline{P_1P_2}$.

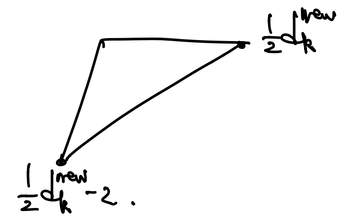
- Consider the pt $Q(d_k^{ur}+2, v_p(g_{d_k^{ur}+2}(w_\star)))$.

A similar computation

$$\rightsquigarrow Q \text{ lies below } \overline{P_1P_2} \Longleftrightarrow v_p(w_\star - w_k) < \tfrac{1}{2}(\Delta'_{k,\frac{1}{2}d_k^{new}} - \Delta'_{k,\frac{1}{2}d_k^{new}-2})$$

$$P \text{ lies below } \overline{P_1P_2} \Longleftrightarrow v_p(w_\star - w_k) < \Delta'_{k,\frac{1}{2}d_k^{new}} - \Delta'_{k,\frac{1}{2}d_k^{new}-1}.$$





If $\frac{1}{2}\left(\Delta'_{k,\frac{1}{2}d_k^{new}} - \Delta'_{k,\frac{1}{2}d_k^{new}-2}\right) > \Delta'_{k,\frac{1}{2}d_k^{new}} - \Delta'_{k,\frac{1}{2}d_k^{new}-1}$

$\iff 2\Delta'_{k,\frac{1}{2}d_k^{new}-1} > \Delta'_{k,\frac{1}{2}d_k^{new}} + \Delta'_{k,\frac{1}{2}d_k^{new}-2}$.

$(l, \Delta'_{k,l})$ for $l = \frac{1}{2}d_k^{new}-2, \dots, \frac{1}{2}d_k^{new}$ are not lower vertices.

<u>Def'n</u> Let Δ_k = lower convex hull of the pts $(l, \Delta'_{k,l})$, $|l| \le \frac{1}{2}d_k^{new}$.
For such l, $(l, \Delta_{k,l})$ is the corresponding pt on Δ_k.

<u>Def'n</u> (Near Steinberg range)
Fix ε and $w_\star \in \mathfrak{m}_{\mathbb{C}_p}$.
For $k \equiv k_\varepsilon (p-1)$ we define $L_{w_\star,k}$ to be the largest int (if any) in $\{1, \dots, \frac{1}{2}d_k^{new}\}$ s.t. $v_p(w_\star - w_k) \ge \Delta_{k,L_{w_\star,k}} - \Delta_{k,L_{w_\star,k}-1}$.
$(L_{w_\star,k}, \Delta_{k,L_{w_\star,k}})$ must be a vertex of Δ_k.
$(\Delta_{k,L_{w_\star,k}} = \Delta'_{k,L_{w_\star,k}})$

<u>Rmk</u> (1) The intuition of $nS_{w_\star,k}$ is that
$w_\star$ is close to w_k s.t. $NP(G(w_\star,-)) \approx NP(G(w_k,-))$
on the interval $\overline{nS_{w_\star,k}}$ where $NP(G(w_\star,-))$ has a long straight line on $\overline{nS_{w_\star,k}}$.

(2) In the def'n of $(w_\star, n)$ being near-Steinberg

the wt w_k may not be unique!

(3) In our previous discussion we assume

$$v_p(w_\star - w_k) > v_p(w_\star - w_{k'}) \text{ for all other ghost zeroes.}$$

<u>Some results on $\Delta'_{k,l}$ or $\Delta_{k,l}$</u> $(l, \Delta'_{k,l})$

<u>Lemma</u> $\Delta'_{k,l+1} - 2\Delta'_{k,l} + \Delta'_{k,l-1} \geq l - 2v_p(l)$, $\forall\, l \geq 1$.

$\Rightarrow \Delta_{k,l} = \Delta'_{k,l}$ for $1 \leq l \leq 2p$, $l \neq p$.

<u>Lemma</u> $\Delta'_{k,l} - \Delta'_{k,l-1} \geq \frac{1}{2}\min\{a+2, p-1-a\} + \frac{1}{2}(p-1)(l-1)$

$\geq \frac{3}{2} + \frac{p-1}{2}(l-1)$.

(This inequality is very sharp.)

<u>Theorem</u> Fix $w_\star$.

(1) The set of near Steinberg ranges $nS_{w_\star,k}$ for all k is nested,

i.e. for any two such open intervals,

either they are disjoint

or one is contained in the other.

(2) The X-coordinates of vertices of $NP(G(w_\star, -))$

are exactly those integers which do not lie in any $nS_{w_\star,k}$

i.e. $\forall n \geq 1$, $(n, w_\star)$ is near-Steinberg

$\Leftrightarrow$ $(n, v_p(g_n(w_\star)))$ is NOT a vertex of $NP(G(w_\star, -))$.

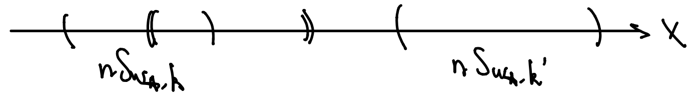
<u>Rmk</u> (1) If $nS_{w_\star,k_1} \cap nS_{w_\star,k_2} \neq \phi$

and we assume $L_{w_\star,k_1} \geq L_{w_\star,k_2}$,

then $L_{w_\star,k_1} \geq p^{1+\frac{p-1}{2}(L_{w_\star,k_2}-1)} - L_{w_\star,k_2} \gg L_{w_\star,k_2}$.

(2) Consider a pair $(w_\star, n)$.

Fix $w_\star$, those n s.t. $(n, v_p(g_n(w_\star)))$ is a vertex looking like



Consider another view. Fix n.

Q What is the set of $w_\star$

s.t. $(n, v_p(g_n(w_\star)))$ is not a vertex of $NP(G(w_\star,-))$

$\Updownarrow$ s.t. $(w_\star, n)$ is near Steinberg.

<u>Answer</u>. First $\exists!\ k = k(n)$ s.t. $k \equiv k_\varepsilon \bmod p-1$.

$(n = \frac{1}{2} d_k^{\mathrm{Iw}})$

then $n \in S_{w_\star,k} \iff v_p(w_\star - w_k) \geqslant \Delta_{k,1} - \Delta_{k,0} \approx \frac{a+2}{2}$ or $\frac{p-1-a}{2}$

$\approx \frac{p+1}{4}$.

These $w_\star$ corr to a disc at w_k of radius $\frac{p+1}{4}$.

- Next consider: $k' = k \pm (p-1) \Rightarrow \frac{1}{2} d_{k'}^{\mathrm{Iw}} = n \pm 1$.

$n \in nS_{w_\star,k'} \iff v_p(w_\star - w_{k'}) \geqslant \Delta_{k,2} - \Delta_{k,1} \approx \frac{p+1}{2}$.

These corr to two discs centered at k's with radius $p^{-\frac{p+1}{2}}$.

Lecture 6: Integrality of slopes of ghost series at classical weights

Goal Compute $NP(G^{(\varepsilon)}(w_k, -))$ for $k \equiv k_\varepsilon \bmod p-1$.

Notation Suppose we have a list of pts $P_i = (i, A_i)$, $A_i \in \mathbb{R}$, $i = m, m+1, \cdots, n$.

We shift them down relative to a linear function $y = ax + b$ $(a, b \in \mathbb{R})$ by transforming them into $Q_i = (i, A_i - ai - b)$, $i = m, m+1, \cdots, n$.

Facts (1) $\forall i \neq j$ slope of $\overline{Q_i Q_j}$ = slope of $\overline{P_i P_j} - a$.

(2) if the convex hull of P_i's is a straight line then the same is true for Q_i's.

(3) Fix i_0. If P_{i_0} is a vertex of the convex hull of P_i's, then Q_{i_0} is a vertex of the convex hull of Q_i's.

- $\Delta'_{k,l} = v_p\left(g_{\frac{1}{2}d_k^{\mathrm{Iw}} + l, \hat{k}}(w_k)\right) - \frac{k-2}{2} \cdot l$
- Δ_k = Convex hull of $(l, \Delta'_{k,l})$ with $|l| \le \frac{1}{2} d_k^{\mathrm{new}}$ $\longleftrightarrow$ Convex hull of $(n, v_p(g_n(w_\star)))$, $n \in \overline{nS_{w_\star,k}} \subseteq \left[\frac{1}{2}d_k^{\mathrm{Iw}} - \frac{1}{2}d_k^{\mathrm{new}}, \frac{1}{2}d_k^{\mathrm{Iw}} + \frac{1}{2}d_k^{\mathrm{new}}\right]$.

Here $nS_{w_\star,k} = \left(\frac{1}{2}d_k^{\mathrm{Iw}} - L_{w_\star,k}, \frac{1}{2}d_k^{\mathrm{Iw}} + L_{w_\star,k}\right)$.

$L_{w_\star,k}$ is the largest integer in $\{1, \cdots, \frac{1}{2}d_k^{\mathrm{new}}\}$ s.t. $v_p(w_\star - w_k) \ge \Delta_{k, L_{w_\star,k}} - \Delta_{k, L_{w_\star,k}-1}$.

Goal Show that the convex hull of the pts $(n, v_p(g_n(w_\star)))$ for $n \in \overline{nS_{w_\star,k}}$ is a straight line and we compute its slope.

Note $v_p(g_n(w_\star)) = v_p(g_{n,\hat{k}}(w_\star)) + m_n(k) \cdot v_p(w_\star - w_k)$, $g_n(w) = \prod (w - w_{k'})^{m_n(k')}$.

Lemma The convex hull of the pts

$$(n, v_p(g_{n,\tilde{k}}(w_k)) + m_n(k)\cdot v_p(w_* - w_k)), \quad n \in \overline{nS_{w_*,k}}$$

is a straight line of slope $\frac{k-2}{2}$.

Proof Write $n = \frac{1}{2}d_k^{Iw} + l \Rightarrow m_n(k) = \frac{1}{2}d_k^{new} - |l|$

$\rightsquigarrow$ the above pts can be written as

$$P_l = \left(\tfrac{1}{2}d_k^{Iw} + l,\ \Delta'_{k,l} + \tfrac{k-2}{2}l + \left(\tfrac{1}{2}d_k^{new} - |l|\right)\cdot v_p(w_* - w_k)\right)$$

for $l \in [-L, L]$, $L = L_{w_*,k}$ for simplicity.

We shift P_l's relative to the linear function

$$y = \frac{k-2}{2}\left(x - \tfrac{1}{2}d_k^{Iw}\right) + \Delta_{k,L} - \left(\tfrac{1}{2}d_k^{Iw} - L\right)v_p(w_* - w_k)$$

and we get

$$Q_l = \left(\tfrac{1}{2}d_k^{Iw} + l,\ \Delta'_{k,l} - \Delta_{k,L} + (L - |l|)\cdot v_p(w_* - w_k)\right).$$

By def'n $nS_{w_*,k} \Rightarrow v_p(w_* - w_k) \geq \Delta_{k,L} - \Delta_{k,L-1}$

$$\geq \Delta_{k,L-1} - \Delta_{k,L-2} \geq \cdots \geq \Delta_{k,|l|+1} - \Delta_{k,|l|}.$$

$$(L - |l|)\,v_p(w_* - w_k) \geq \Delta_{k,L} - \Delta_{k,|l|} \geq \Delta_{k,L} - \Delta'_{k,l}.$$

$\Rightarrow$ y coordinate of $Q_l \geq 0$,

"=" holds when $l = \pm L$.

$\Rightarrow$ convex hull of Q_l's is a straight line of slope 0. $\square$

Lemma Let $k = k_\varepsilon + (p-1)k_0$. $l = 1, \dots, \frac{1}{2}d_k^{new}$.

Let $k' = k_\varepsilon + (p-1)k'_0$ be another wt

s.t. one of the following conditions holds:

(1) either $d_{k'}^{ur}$ or $d_{k'}^{Iw} - d_{k'}^{ur} \in \left(\frac{1}{2}d_k^{Iw} - l,\ \frac{1}{2}d_k^{Iw} + l\right)$

(2) $\frac{1}{2}d_{k'}^{Iw} \in \left[\frac{1}{2}d_k^{Iw} - l,\ \frac{1}{2}d_k^{Iw} + l\right]$.

Then $\Delta_{k,l} - \Delta'_{k,l-1} - v_p(w_k - w_{k'}) \geq \frac{1}{2}(2l-1)$

$$\left(\Delta'_{k,l} - \Delta'_{k,l-1} \geq \frac{3}{2} + \frac{p-1}{2}(l-1). \right)$$

Prop Let $\mathrm{nS}_{w_\star,k} = \left(\frac{1}{2} d_k^{\mathrm{Iw}} - L_{w_\star,k},\ \frac{1}{2} d_k^{\mathrm{Iw}} + L_{w_\star,k}\right)$ be a near-Steinberg range.

(1) $\forall k' = k_\varepsilon + (p-1)k'_\bullet$. $v_p(w_{k'} - w_k) \geq \Delta_{k,L_{w_\star,k}} - \Delta_{k,L_{w_\star,k}-1}$

then $\frac{1}{2} d_{k'}^{\mathrm{Iw}} \notin \overline{\mathrm{nS}_{w_\star,k}}$. $d_{k'}^{ur}$, $d_{k'}^{\mathrm{Iw}} - d_{k'}^{ur} \notin \mathrm{nS}_{w_\star,k}$.

(2) For k' in (1), $m_n(k')$ is linear in n when $n \in \overline{\mathrm{nS}_{w_\star,k}}$.

(3) The following two lists of pts

$$P_n = (n, v_p(g_{n,\hat{k}}(w_\star))), \quad Q_n = (n, v_p(g_{n,\hat{k}}(w_k))), \quad n \in \overline{\mathrm{nS}_{w_\star,k}}$$

are shifts of each other relative to a linear function

with slope in $\mathbb{Z} + \mathbb{Z}\alpha$, $\alpha = \max\{v_p(w_\star - w_{k'}) : w_{k'}$ is a zero of $g_n(w)$, $n \in \mathrm{nS}_{w_\star,k}\}$.

(4) More generally, let $\underline{k} = \{k_1, \cdots, k_r\}$, $k_i \equiv k_\varepsilon \bmod p-1$.

Suppose $\exists$ an interval $[n_-, n_+]$ s.t. $\forall k' \notin \underline{k}$

with $v_p(w_{k'} - w_k) \geq v_p(w_\star - w_k)$

then the ghost multiplicity $m_n(k')$ is linear in n for $n \in [n_-, n_+]$.

Then the lists of pts $P_n = (n, A_n, v_p(g_{n,\hat{\underline{k}}}(w_\star)))$

$$Q_n = (n, A_n, v_p(g_{n,\hat{\underline{k}}}(w_k))) \qquad (A_n \in \mathbb{R})$$

are shifts of each other by a linear function w/ slope

in $\mathbb{Z} + \mathbb{Z}\beta$. $\beta = \max\{v_p(w_\star - w_{k'}) \mid w_{k'}$ is a zero of $g_{n,\hat{\underline{k}}}(w)$, $n \in [n_-, n_+]\}$.

Proof (1) is an immediate conseq'ce of the previous lemma.

(2) follows from the def of $\{m_n(k')\}_{n\geq 1}$:

$$\underbrace{0, \cdots,}_{d_{k'}^{ur}} \Big| \underbrace{0,\ 1,\ 2,\ \cdots,\ \tfrac{1}{2} d_{k'}^{\mathrm{new}}}_{\longleftarrow \text{ linear } \longrightarrow} \Big| \underbrace{\tfrac{1}{2} d_{k'}^{\mathrm{new}} - 1, \cdots, 2, 1, 0}_{\longleftarrow \text{ linear } \longrightarrow} \Big| \underbrace{0, \cdots}_{\longleftarrow \text{ linear } \longrightarrow} \Big|$$

(3) $v_p(g_{n,\hat{k}}(w_*)) - v_p(g_{n,\hat{k}}(w_k))$

$$= \sum_{\substack{k' \equiv k \,(p-1) \\ k' \neq k}} m_n(k') \left(\boxed{v_p(w_* - w_{k'}) - v_p(w_k - w_{k'})} \right). \qquad \boxed{\in \mathbb{Z} + \mathbb{Z}\alpha}$$

If $v_p(w_{k'} - w_k) < \Delta_{k,L} - \Delta_{k,L-1}$ $(\leq v_p(w_* - w_k))$.

$\Rightarrow v_p(w_* - w_{k'}) = v_p(w_k - w_{k'})$.

If $v_p(w_{k'} - w_k) \geq \Delta_{k,L} - \Delta_{k,L-1}$ by (2).

$m_n(k')$ is linear for $n \in \overline{nS_{w_*,k}}$. $\quad (L = L_{w_*,k})$ $\square$

Rmk The previous lemma gives

$$\Delta_{k,l} - \Delta'_{k,l-1} - v_p(w_{k'} - w_k) \geq \tfrac{1}{2}(2l-1)$$

if we have $w_{k'}$ very close to w_k,

then the exclusions in (1) most holds.

Cor Fix $\overline{nS_{w_*,k}}$ as before. $w_* \neq w_k$.

(1) Assume that w_* is not a zero of $g_n(w)$ for any (or some) $n \in \overline{nS_{w_*,k}}$.

Then the lower convex hull of pts $(n, v_p(g_n(w_*)))$ for $n \in \overline{nS_{w_*,k}}$

is a straight line.

Moreover, the slope of this straight line belongs to $\frac{a}{2} + \mathbb{Z} + \mathbb{Z}\alpha$. $(*)$

(2) Assume that $w_* = w_{k_1}$ is a zero of $g_n(w)$ for $\underline{\text{any}}$ (or some) $n \in \overline{nS_{w_*,k}}$.

Then the lower hull of pts $(n, v_p(g_{n,\hat{k}}(w_{k_1})))$, $n \in \overline{nS_{w_*,k}}$

is a straight line w/ slope $\in \frac{a}{2} + \mathbb{Z}$. b/c $m_n(k_1)$ linear in n.

Proof (1) For $n \in \overline{nS_{w_*,k}}$.

$$(n, v_p(g_n(w_*))) \xleftrightarrow[\text{with slope} \in \mathbb{Z} + \mathbb{Z}\alpha]{\text{shift rel to a lin func}} (n, v_p(g_{n,\hat{k}}(w_k) + m_n(k) v_p(w_* - w_k)).$$

$\| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow$

$(n, v_p(g_{n,\hat{k}}(w_*)) + m_n(k) v_p(w_* - w_k))$ $\qquad$ lower convex hull = stright line of slope $\frac{k-2}{2}$.

note $k \equiv k_\varepsilon \mod p-1 \equiv 2 + \{a + 2s_\varepsilon\} \equiv 2 + a + 2s_\varepsilon \mod p-1$.

$\frac{k-2}{2} \in \frac{a}{2} + \mathbb{Z}$.

Cor: For any $k \equiv k_\varepsilon \mod p-1$,

(1) The slopes of $NP(G^{(\varepsilon)}(w_k, -))$ with multi 1 are all integers

(2) Other slopes of $NP(G^{(\varepsilon)}(w_k, -))$ always have even multiplicities and belong to $\frac{a}{2} + \mathbb{Z}$.

For (1): $v_p(g_n(w_k)) \in \mathbb{Z}$.

For (2): we need to compute the slope of lower convex hull of $NP(G^{(\varepsilon)}(w_k, -))$ on a max'l $nS_{w_k, k'}$.

This follows from the previous corollary (1).

Prop Fix $k_0 \equiv k_\varepsilon \mod p-1$. $l \in \{1, \ldots, \frac{1}{2} d_{k_0}^{new} - 1\}$. TFAE:

(1) $(l, \Delta'_{k_0, l})$ is not a vertex of Δ_{k_0}.

(2) $(w_{k_0}, \frac{1}{2} d_{k_0}^{Iw} + l, k_1)$ is near-Steinberg for some $k_1 > k_0$

i.e. $\frac{1}{2} d_{k_0}^{Iw} + l \in nS_{w_{k_0}, k_1}$.

(3) $(w_{k_0}, \frac{1}{2} d_{k_0}^{Iw} - l, k_2)$ is near-Steinberg for some $k_2 < k_0$.

Lecture 7: Irreducible components of eigencurves

<u>Goal</u> Ghost $\Rightarrow$ finiteness of irred comps of eigencurves.

<u>Main Thm</u> $\widetilde{H}$: $\mathcal{O}[\![K_p]\!]$-proj arith mods of type $\bar{r}_p$ ($\bar{\rho}$)
with multiplicity $m(\widetilde{H})$

ε: relevant char

$C_{\widetilde{H}}^{(\varepsilon)}(w,t)$: char power series of U_p on
abstract p-adic forms assoc to $\widetilde{H}$.

$G_{\bar{\rho}}^{(\varepsilon)}(w,t)$: ghost series (depends only on $\bar{\rho}$). $\quad$ ($\bar{r}_p|_{I_{\mathbb{Q}_p}} \simeq \bar{\rho}^{ss}$).

<u>Then</u> for any $w_\star \in \mathfrak{m}_{\mathbb{C}_p}$, $NP(G_{\widetilde{H}}^{(\varepsilon)}(w_\star,-))$ is the same as
$NP(G_{\bar{\rho}}^{(\varepsilon)}(w_\star,-))$ stretched in both x-, y-directions by $m(\widetilde{H})$
except the slope-zero part.

- has length $m'(\widetilde{H})$ when $\bar{r}_p$ is split, $\varepsilon = \omega^b \times \omega^{a+b}$.
- has length $m''(\widetilde{H})$ when $\bar{r}_p$ is non-split. $\varepsilon = \omega^{a+b+1} \times \omega^{b-1}$.

(<u>Note</u> $\bar{r}_p$ split $\rightsquigarrow$ $\widetilde{H} \simeq (\mathrm{Proj}_{\mathcal{O}[\![K_p]\!]}\sigma_{a,b})^{\oplus m'(\widetilde{H})} \oplus (\mathrm{Proj}_{\mathcal{O}[\![K_p]\!]}\sigma_{b,a})^{\oplus m''(\widetilde{H})}$.)

<u>Def'n</u> Fix $\lambda \in (0,1)$. Take $\mathcal{W}_{\geq\lambda} := \mathrm{Sp}\, E\langle w/p^\lambda\rangle$.

(1) A Fredholm series of $\mathcal{W}_{\geq\lambda}$ is $F(w,t) \in E\langle w/p^\lambda\rangle[\![t]\!]$.
s.t. $F(w,0)=1$ and $F(w,t)$ converges on $\underbrace{\mathcal{W}_{\geq\lambda}\times(\mathbb{A}^1)^{\mathrm{rig}}}$.
$\quad\quad \cup\!\!\mid$
$\mathcal{Z}(F)$ = zero locus of F.

(2) A Fredholm series $F(w,t)$ is called of ghost type $(\bar{r}_p, \varepsilon)$
if $\forall w_\star \in \mathcal{W}_{\geq\lambda}(\mathbb{C}_p)$, $NP(F(w_\star,-))$ is the same as $NP(G_{\bar{\rho},\mathrm{nord}}^{(\varepsilon)}(w_\star,-))$
stretched in both x-, y-directions by $m(F) \in \mathbb{N}$.
multiplicity

Lemma $C^{(\varepsilon)}_{\tilde H} = C^{(\varepsilon)}_{\tilde H,\mathrm{ord}} \cdot C^{(\varepsilon)}_{\tilde H,\mathrm{nord}}$,

where $C^{(\varepsilon)}_{\tilde H,\mathrm{nord}}$ is of ghost type $(\bar r_p, \varepsilon)$ w/ multi $m(\tilde H)$.

Proof Main thm + Weierstrass preparation

$\Rightarrow C^{(\varepsilon)}_{\tilde H}(w,t) = C^{(\varepsilon)}_{\tilde H,\mathrm{ord}} \cdot C^{(\varepsilon)}_{\tilde H,\mathrm{nord}}$,

- $C^{(\varepsilon)}_{\tilde H,\mathrm{nord}}(w,t)$ is of ghost type $\tilde m$, with multiplicity $m(\tilde H)$
- $C^{(\varepsilon)}_{\tilde H,\mathrm{ord}}$ is a polynomial of deg $m'(\tilde H)$ or $m''(\tilde H)$.

Key Technical lemma

$\check{\mathcal O}$:= completion of max unram ext'n of $\mathcal O$, $\check E$:= Frac $\check{\mathcal O}$.

$r \in \mathbb{Q}_{>0}$, $D(w_\star, r) = \{w \in \mathcal{W}_{\geq\lambda}(\mathbb{C}_p) : v_p(w - w_\star) \geq r\}$

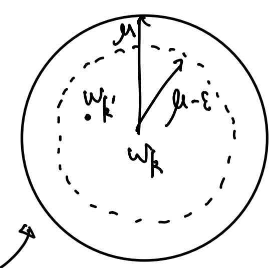
$\rightsquigarrow \eta_{w_\star, r}$ assoc Gauss pt.

Slope derivatives: $\mu \in (\lambda, \infty) \cap \mathbb{Z}$.

$$\rightsquigarrow V^{+}_{w_\star,\mu}(f) := \lim_{\varepsilon \to 0^+}\left(-\frac{\ln|f(\eta'_{w_\star,\mu-\varepsilon})| - \ln|f(\eta_{w_\star,\mu})|}{(\ln p)\cdot\varepsilon}\right)$$

$$V^{\bar\alpha}_{w_\star,\mu}(f) := \lim_{\varepsilon \to 0^+}\left(-\frac{\ln|f(\eta_{w_\star+\alpha p^\mu,\, \mu+\varepsilon})| - \ln|f(\eta_{w_\star+\alpha p^\mu,\, \mu})|}{(\ln p)\cdot\varepsilon}\right)$$

depends only on $\bar\alpha$.



Then $V^{+}_{w_\star,\mu}(f) + \sum_{\bar\alpha \in \bar{\mathbb{F}}} \underbrace{V^{\bar\alpha}_{w_\star,\mu}(f)}_{=\,0 \text{ for almost all } \bar\alpha} = 0$

E.g. $g_n(w)$ "ghost polynomial". $g_n(w) = \prod (w - w_k)^{m_n(k)}$.

$$\Rightarrow V^{+}_{w_\star,\mu}(g_n) = \sum_{v_p(w_{k'} - w_k) > \mu} m_n(k') - \frac{|w - w_{k'}|_{\eta_{w_k,\mu-\varepsilon}} - |w - w_{k'}|_{\eta_{w_k,\mu}}}{\ln p \cdot \varepsilon} = m_n(k!)$$

Thm $F(w,t)$ Fredholm series of ghost type with multi $m(F)$.

& If Fredholm series $H(w,t) \mid F(w,t)$, then H is of ghost type with multi $m(H) \leq m(F)$.

Cor $Z(C^{(\varepsilon)}_{\tilde{H},\mathrm{nord}})$ has only $\underbrace{\text{fin many}}_{\leq m(\tilde{H})}$ irred components.

Proof of Thm $F(w,t) = H(w,t)\cdot H'(w,t)$,

$w_* \in \mathcal{W}_{\geq\lambda}(\mathbb{C}_p)$ s.t. $(n, v_p(g_n(w_*)))$ is a vertex of ghost NP.

Form an open subspace of $\mathcal{W}_{\geq\lambda}$:

$$\underset{\substack{\uparrow \\ \text{Connected}}}{\mathrm{Vtx}_{n,\geq\lambda}} := \mathcal{W}_{\geq\lambda} \setminus \bigcup_{\substack{k\equiv k_\varepsilon (p-1) \\ n\in(d_k^{ur},\, d_k^{Iw}-d_k^{ur})}} D\left(w_k,\ \Delta_{k,|\frac{1}{2}d_k^{Iw}-n|+1} - \Delta_{k,|\frac{1}{2}d_k^{Iw}-n|}\right)$$

$\rightsquigarrow$ assoc Berkovich space $\mathrm{Vtx}^{\mathrm{Berk}}_{n,\geq\lambda}$.

Step I The total multiplicity of n smallest slopes of ghost NP in H is constructed for $w_* \in \mathrm{Vtx}^{\mathrm{Berk}}_{n,\geq\lambda}$, denoted by $m(H,n)$ ($\overset{?}{=} m(H)\cdot n$).

Step II It is known $\exists!\, k\equiv k_\varepsilon$ s.t. $\frac{1}{2}d_k^{Iw} = n-1$.

Claim $\forall\, \varepsilon\in(0,\frac{1}{2})$, $\forall\, \alpha\in\mathcal{O}_{\mathbb{C}_p}$,

(1) $\eta_{w_k,\ \Delta_{k,1}-\Delta_{k,0}-\varepsilon}$ belongs to $\mathrm{Vtx}^{\mathrm{Berk}}_{n,\geq\lambda}$, $\mathrm{Vtx}^{\mathrm{Berk}}_{n-1,\geq\lambda}$

(2) $\eta_{w_k+\alpha p^{\Delta_{k,1}-\Delta_{k,0}},\ \Delta_{k,1}-\Delta_{k,0}+\varepsilon}$ belongs to $\mathrm{Vtx}^{\mathrm{Berk}}_{n,\geq\lambda}$, $\mathrm{Vtx}^{\mathrm{Berk}}_{n-2,\geq\lambda}$, but not $\mathrm{Vtx}^{\mathrm{Berk}}_{n-1,\geq\lambda}$.

Step III Granting step I, II, we conclude the proof.

$m(H) := m(H,1)$. Will prove inductively that $m(H,n) = n\cdot m(H)$.

- $n=1$: ok.
- Assume the statement holds for smaller n.

Take k as in Step II. $H(w,t) = \sum_{m\geq 0} h_m(w)\cdot t^m$.

Step II (1) $\Rightarrow$ $|h_{m(H,n)}(\eta_{w_k,\Delta_{k,1}-\Delta_{k,0}-\varepsilon})|$

$$= |g_{n-1}^{m(H)}(\eta_{w_k,\Delta_{k,1}-\Delta_{k,0}-\varepsilon})\cdot (g_n/g_{n-1})^{m(H,n)-m(H,n-1)}(\eta_{w_k,\Delta_{k,1}-\Delta_{k,0}-\varepsilon})|.$$

By continuity this holds for $\varepsilon=0$ as well.

$$\Rightarrow V^+_{w_k,\Delta_{k,1}-\Delta_{k,0}}(h_{m(H,n)}) = V^+_{w_k,\Delta_{k,1}-\Delta_{k,0}}\left(g_{n-1}^{m(H)}\cdot\left(g_n/g_{n-1}\right)^{m(H,n)-m(H,n-1)}\right)$$

Step II (2) $\Rightarrow \eta w_k + \alpha p^{\Delta_{k,1}-\Delta_{k,0}}, \Delta_{k,1}-\Delta_{k,0}+\varepsilon$

ghost NP at these pts is a straight line from $n-2$ to n

$$\Rightarrow V^{\bar{\alpha}}_{w_k, \Delta_{k,1}-\Delta_{k,0}}(h_{m(H,n)}) = V^{\bar{\alpha}}_{w_k, \Delta_{k,1}-\Delta_{k,0}}\left(g_{n-2}^{m(H)} \cdot (g_n/g_{n-2})^{\frac{m(H,n)-m(H,n-2)}{2}}\right).$$

Sum up

$$0 = V^{+}_{w_k, \Delta_{k,1}-\Delta_{k,0}}\Big(\underbrace{g_{n-1}^{m(H)} \cdot (g_n/g_{n-1})^{m(H,n)-m(H,n-1)}}_{A}\Big)$$

$$+ V^{\bar{\alpha}}_{w_k, \Delta_{k,1}-\Delta_{k,0}}\Big(\underbrace{g_{n-2}^{m(H)} \cdot (g_n/g_{n-2})^{\frac{m(H,n)-m(H,n-2)}{2}}}_{B}\Big).$$

$$= V^{+}_{w_k, \Delta_{k,1}-\Delta_{k,0}}(A/B).$$

where $A/B = \left(\frac{g_n \cdot g_{n-2}}{g_{n-1}^2}\right)^{\frac{m(H,n)-m(H,n-1)-m(H)}{2}}$

(Use $m(H,n-1) = m(H,n-2) + m(H)$.)

To show that $m(H,n) - m(H,n-1) - m(H) = 0$,

it suffices to show $V^{+}_{w_k, \Delta_{k,1}-\Delta_{k,0}}\left(\frac{g_n g_{n-2}}{g_{n-1}^2}\right) \neq 0$

$$\overset{\shortparallel}{V^+(g_n) + V^+(g_{n-2}) - 2V^+(g_{n-1})} \overset{\text{claim}}{=\!=} -2.$$

Pf of the claim

For $i \in \{n-2, n-1, n\}$,

$$V^{+}_{w_k, \Delta_{k,1}-\Delta_{k,0}}(g_i) = \sum_{\substack{v_p(w_{k'}-w_k) > \Delta_{k,1}-\Delta_{k,0} \\ k' \equiv k_\varepsilon}} m_i(k')$$

$m_{n-2}(k') + m_n(k') - 2m_{n-1}(k')$

$m_i(k')$ is linear except for $i = d_{k'}^{ur}, \frac{1}{2}d_{k'}^{Iw}, d_{k'}^{Iw} - d_{k'}^{ur}$.

Recall $\frac{1}{2}d_k^{Iw} = n-1 \Rightarrow$ If $v_p(w_{k'} - w_k) > \Delta_{k,1} - \Delta_{k,0}$

then $\{n-2, n-1, n\} \subseteq nS_{w_{k'}, k}$.

On the other hand, if $k' \neq k$ then $d_{k'}^{ur}, \frac{1}{2}d_{k'}^{Iw}, d_{k'}^{Iw} - d_{k'}^{ur} \notin nS_{w_{k'}, k}$

$\Rightarrow m_i(k')$ is linear for $i \in \{n-2, n-1, n\}$ except $k' = k$.

$m_{n-2}(k) + m_n(k) - 2m_{n-1}(k) = -2.$

Lecture 8: On Paškūnas modules

§1 (φ,Γ)-modules and p-adic LLC for $GL_2(\mathbb{Q}_p)$

$\mathcal{O}_\mathcal{E} := \mathcal{O}[\![T]\!][\frac{1}{T}]^\wedge_p$, $\mathcal{O}_\mathcal{E}/(p) = \mathbb{F}((T))$

$\varphi, T \underset{\mathcal{O}\text{-linear}}{\curvearrowright} \mathcal{O}_\mathcal{E}, \mathbb{F}((T))$, $\varphi(T) = (1+T)^p - 1$

$\gamma \in \Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \xrightarrow[\chi]{\sim} \mathbb{Z}_p^\times$, $\gamma(T) = (1+T)^{\chi(\gamma)} - 1$.

<u>Def'n</u> $? = \mathcal{O}_\mathcal{E}, \mathbb{F}((T))$ with $\mathcal{E} = \mathcal{O}_\mathcal{E}[\frac{1}{p}]$.

A (φ,Γ)-mod over $?$ is a finite free $?$-mod M equipped with commuting semilinear actions of φ.

- For $? = \mathcal{O}_\mathcal{E}, \mathbb{F}((T))$, M is called étale if $\underset{M \otimes_{?,\varphi} ?}{\overset{\|}{\varphi^* M}} \simeq M$.

- For $? = \mathcal{E}$, M is called étale if it is the base change of an étale (φ,Γ)-mod $/\mathcal{O}_\mathcal{E}$.

<u>Thm.</u> (Fontaine) $\exists$ rank-preserving equiv of cats

$$\left\{\begin{matrix}\text{étale } (\varphi,\Gamma)\text{-mods} \\ \text{over } \mathcal{E} \text{ or } \mathcal{O}_\mathcal{E} \text{ or } \mathbb{F}((T))\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} G_{\mathbb{Q}_p}\text{-rep'ns over} \\ E \text{ or } \mathcal{O} \text{ or } \mathbb{F}\end{matrix}\right\}$$

$$D \longmapsto V(D).$$

<u>Colmez's functor</u>

$P^+ := \begin{pmatrix} \mathbb{Z}_p - \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \curvearrowright \mathbb{F}\text{-v.s. } M.$

$\rightsquigarrow$ a structure of (φ,Γ)-mods over $\mathbb{F}((T))$

- $\mathbb{F}[\![\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}]\!] \longrightarrow \mathbb{F}[\![T]\!]$, $\begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & 1 \end{pmatrix} \simeq \Gamma$,

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1 \longmapsto T$ $\qquad \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \rightsquigarrow \varphi$.

$\rightsquigarrow \varphi, \Gamma \curvearrowright M / \mathbb{F}[\![T]\!]$.

<u>Def'n</u> π sm adm finite length rep'n of $GL_2(\mathbb{Q}_p)$ over $\mathbb{F}$.

$D(\pi) := \mathbb{F}((T)) \hat{\otimes}_{\mathbb{F}[\![T]\!]} \pi^\vee$, $P^+ \curvearrowright \pi^\vee = \mathrm{Hom}_{\mathbb{F}}(\pi, \mathbb{F})$.

$D(\pi)$ is an étale (φ, Γ)-mod over $\mathbb{F}((T))$

$\rightsquigarrow$ $V(\pi) := V(D(\pi))(1)$ Gal rep'n assoc to π.

$\uparrow$ twist by ω

<u>Thm</u> For any $\bar{\rho} : G_{\mathbb{Q}_p} \longrightarrow GL_2(\mathbb{F})$, $\exists !$ sm adm fin-length rep'n $\pi(\bar{\rho})$ of $GL_2(\mathbb{Q}_p)$ over $\mathbb{F}$

s.t. $\cdot$ $V(\pi(\bar{\rho})) \simeq \bar{\rho}$

$\cdot$ $\pi(\bar{\rho})$ has central char $\det(\bar{\rho}) \cdot \omega$

$\cdot$ $\pi(\bar{\rho})$ has no fin-dim'l $GL_2(\mathbb{Q}_p)$-subrep.

(normalization: $\mathrm{rec}(p) =$ geom Frob.)

<u>Rmk</u> $\cdot$ $\pi(\bar{\rho})$ is supersingular $\Leftrightarrow$ $\bar{\rho}$ is irred.

$\cdot$ $\bar{\rho}^{ss} \simeq \chi_1 \oplus \chi_2 \rightsquigarrow \pi(\bar{\rho})^{ss} \simeq \mathrm{Ind}_B^G(\chi_2 \otimes \chi_1 \omega^{-1})^{ss} \oplus \mathrm{Ind}_B^G(\chi_1 \otimes \chi_2 \omega^{-1})^{ss}$.

<u>Generic condition</u> $\chi_1/\chi_2 \neq \omega^{\pm 1}$ (mod p LLC)

<u>Def'n</u> π = unitary adm residually finite length E-Banach space rep'n of $GL_2(\mathbb{Q}_p)$ with a central char.

$\pi^\circ = \{v \in \pi \mid |v| \le 1\} \rightsquigarrow V(\pi^\circ) := \varprojlim_n V(\pi^\circ / \varpi^n \pi^\circ)$

$V(\pi) := V(\pi^\circ)[\frac{1}{p}]$

<u>Thm</u> For any $\rho : G_{\mathbb{Q}_p} \longrightarrow GL_2(E)$, $\exists !$ unitary adm residually fin-length E-Banach space rep'n $\pi(\rho)$ of $GL_2(\mathbb{Q}_p)$

s.t. $\cdot\ V(\pi(\rho)) \simeq \rho$

$\cdot\ \pi(\rho)^{\circ}/\varpi \simeq \kappa(\bar{\rho})$,

$\cdot\ \pi(\rho)$ has central char $\det\rho \cdot \chi$.

<u>Rmk</u> $\pi(\rho)$ is ss $\Longleftrightarrow$ ρ irred.

<u>§2 Deformation theory & Paškūnas modules</u>

<u>Galois side</u> $\bar{\rho}: G_{\mathbb{Q}_p} \longrightarrow GL_2(\mathbb{F})$ s.t. $\mathrm{End}_{G_{\mathbb{Q}_p}}(\bar{\rho}) = \mathbb{F}$.

$\overset{\text{Mazur}}{\rightsquigarrow}$ universal deformation $(R_{\bar{\rho}}, V_{univ})$,

$$R_{\bar{\rho}} \simeq \mathcal{O}[\![x_1, \cdots, x_5]\!]$$

$$\begin{tikzcd} \zeta: G_{\mathbb{Q}_p} \arrow[rr] \arrow[dr] & & \mathbb{F}^{\times} \\ & G_{\mathbb{Q}_p}^{ab} \arrow[ur] & \end{tikzcd}$$

$\rightsquigarrow R_{\bar{\rho}}^{\zeta}$ parametrizing deformations of $\bar{\rho}$ with det ζ

$R_{\bar{\rho}}^{\zeta} \simeq \mathcal{O}[\![x_1, x_2, x_3]\!]$.

$(R_{\bar{\rho}}^{\square}, V_{univ}^{\square})$: universal framed deformations

$$R_{\bar{\rho}}^{\square} \simeq R_{\bar{\rho}}^{\zeta} \underset{\substack{\uparrow \\ \text{non-canonical.}}}{\hat{\otimes}_{\mathcal{O}}} \mathcal{O}[\![u, v, z_1, z_2, z_3]\!]$$

<u>$GL_2(\mathbb{Q}_p)$-side</u>

<u>Kisin</u>: Colmez functor extends to the level of deformations

$\mathfrak{C}$: Cat of profinite $\mathcal{O}$-mods M equipped w/ cont right $GL_2(\mathbb{Q}_p)$-actions

s.t. $\cdot$ $GL_2(\mathbb{Z}_p)$-action extends to $\mathcal{O}[\![GL_2(\mathbb{Z}_p)]\!]$-action

$\cdot$ for any $v \in M^{\vee} = \mathrm{Hom}(M, E/\mathcal{O})$, $\mathcal{O}[GL_2(\mathbb{Q}_p)]$-submod generated by v is of finite length.

$\mathfrak{C}_{\zeta}$:= subcat of $\mathfrak{C}$ consisting of objs w/ central char ζ.

$\widetilde{P}_{\zeta}$:= universal deformation of $\kappa(\bar{\rho})^{\vee}$ in $\mathfrak{C}_{\zeta}$.

Kisin's observation $\Rightarrow$ $R_{\bar{\rho}}^{\zeta}$ naturally acts on $\widetilde{P}_{\zeta}$.

<u>Thm</u> (Colmez, Paškūnas)

(1) $\tilde{P}_\xi$ flat over $R_{\bar\rho}^{\xi}$ and $\tilde{P}_\xi \otimes_{R^{\xi}_{\bar\rho}} \mathbb{F} \simeq \kappa(\bar\rho)^\vee$

(2) $\mathrm{End}_{GL_2(\mathbb{Q}_p)}(\tilde{P}_\xi) \simeq R^{\xi}_{\bar\rho}$. $V(\tilde{P}_\xi) \simeq V_{univ}$ as $R^{\xi}_{\bar\rho}[G_{\mathbb{Q}_p}]$-rep'ns.

For any $x : \mathrm{Spec}\, R^{\xi}_{\bar\rho}[\frac{1}{p}] \longrightarrow \bar{\mathbb{Q}}_p$,

$$V(\tilde{P}_\xi \otimes_{R^{\xi}_{\bar\rho}, x} \bar{\mathbb{Q}}_p) \simeq V_{univ,x}.$$

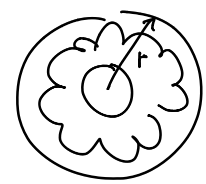
(3) $\tilde{P}_\xi$ is the proj envelope of $\kappa(\bar\rho)^\vee$ in $\mathcal{C}_\xi$,

$\bar\rho = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ non-split. $\kappa(\bar\rho) = (\pi_1 - \pi_2)$.

In this case, $\tilde{P}_\xi$ is also the proj envelope of $\pi_1^\vee$.

<u>§3 Trianguline deformation space</u>

Robba ring $\mathcal{R} = \{ f(T) = \sum_{i \in \mathbb{Z}} a_i T^i \mid a_i \in E,\ f$ convergent on $1 > |T| \geq r \}$.



$\rightsquigarrow \varphi, \Gamma \curvearrowright \mathcal{R}$ (φ,Γ)-mod over $\mathcal{R}$.

$\mathcal{E}^\dagger = \{ f \in \mathcal{R} \mid \{|a_i|\}$ is bounded$\}$.

$\mathcal{R} \supset \underset{\varphi,\Gamma}{\circlearrowleft} \subset \mathcal{E}$

<u>Thm</u> (Cherbonnier-Colmez, Kedlaya)

$\exists$ rank-preserving equiv of cats

$\{$étale (φ,Γ)-mods$/\mathcal{R}\} \longleftrightarrow \{G_{\mathbb{Q}_p}$-rep'ns over $E\}$.

$D^\dagger_{rig}(V) \longleftarrow\!\shortmid V$

<u>Def'n</u> A rank d (φ,Γ)-mod D over $\mathcal{R}$ is called <u>trianguline</u> if $\exists$ a filtration

$$0 = \mathrm{Fil}^0 D \subset \mathrm{Fil}^1 D \subset \cdots \subset \mathrm{Fil}^d D = D$$

of (φ,Γ)-submods s.t. $\mathrm{Fil}^{i+1}/\mathrm{Fil}^i$ is a rank 1 (φ,Γ)-mod $\mathcal{R}(\delta_i)$.

A trianguline (φ,Γ)-mod $\rightsquigarrow (\delta_1, \delta_2, \dots, \delta_d)$.

Thm (Kisin) f finite slope overconvergent p-adic modular forms.
Then $D^{+}_{rig}(\rho_f)$ is trianguline (ρ_f is a trianguline rep'n).

Thm V = 2-dim'l rep'n of $G_{\mathbb{Q}_p}$. Then
V trianguline $\Longleftrightarrow$ V crystabelian
i.e. V becomes crystalline
after an abelian ext'n of $\mathbb{Q}_p$.

Def D rank 2 trianguline (φ,Γ)-mods $/\mathcal{R}$.
$\rightsquigarrow 0 \to \mathcal{R}(\delta_1) \to D \to \mathcal{R}(\delta_2) \to 0$.
Say D is étale if $\cdot\ v_p(\delta_1(p)) + v_p(\delta_2(p)) = 0$,
$\nexists$ rank 1 (φ,Γ)-submod of D has negative slope.
In particular, $v_p(\delta_1(p)) \geq 0$.

§4 Trianguline deformation space à la BHS

T = rigid analytic space parametrizing cont chars of $(\mathbb{Q}_p^\times)^2 \to E^\times$
$= (\mathbb{G}_m^{rig})^2 \times (\mathrm{Spf}\ \mathcal{O}[\![(\mathbb{Z}_p^\times)^2]\!])^{rig}$.
$T_{reg} = \{(\delta_1,\delta_2) \in T \mid (\delta_1/\delta_2)^{\pm 1} \neq x^n \cdot \chi,\ n \geq 0\}$ $\quad (\dim H^1(\delta_1/\delta_2) = 2.)$
generic condition.
$\mathcal{X}_{\bar\rho}^{\square} = (\mathrm{Spf}\ R_{\bar\rho}^{\square})^{rig}$, $\mathcal{X}_{\bar\rho}^{\square}$ is of dim 8 $/E$.

Def $U_{\bar\rho,reg}^{\square,tri}$ = set of pts $(x,\delta_1,\delta_2) \in \mathcal{X}_{\bar\rho}^{\square} \times T_{reg}$
s.t. $0 \to \mathcal{R}(\delta_1) \to D^{+}_{rig}(V_x) \to \mathcal{R}(\delta_2) \to 0$.

Trianguline deformation space
$\mathcal{X}_{\bar\rho}^{\square,tri} :=$ Zariski closure of $U_{\bar\rho,reg}^{\square,tri}$ in $\mathcal{X}_{\bar\rho}^{\square} \times T$.

Thm
- $U^{\square,\mathrm{tri}}_{\bar{\rho},\mathrm{reg}}$ is the set of closed pts of a Zariski open and dense subspace $\mathcal{U}^{\square,\mathrm{tri}}_{\bar{\rho},\mathrm{reg}}$ of $\mathcal{X}^{\square,\mathrm{tri}}_{\bar{\rho}}$.
- $\mathcal{X}^{\square,\mathrm{tri}}_{\bar{\rho}}$ is equidim'l of dim 7.

Lecture 9: Bootstrapping argument

"Automorphic side"		"Galois side"
local ghost thm	$\xrightarrow{\text{Paškūnas module, Emerton's Jacquet functor}}$	slopes on trianguline deformation space
"global" ghost thm of Bergdall–Pollack	$\xleftarrow{\text{"bootstrapping"}}$	

<u>Setup</u> $p \geq 11$, $2 \leq a \leq p-5$. (will assume $b=0$ for simplicity.)

$F/\mathbb{Q}_p \supset \mathcal{O} \twoheadrightarrow \mathcal{O}/(\varpi) = \mathbb{F}$.

$$\omega_1 : \mathrm{Gal}_{\mathbb{Q}_p} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_p(\mu_p)/\mathbb{Q}_p) \simeq \mathbb{F}_p^\times$$

$$\mathrm{unr}(\bar\alpha) : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathrm{Gal}_{\mathbb{F}_p} \to \mathbb{F}^\times$$
$$\mathrm{geomFr}_p \mapsto \bar\alpha.$$

$$\bar r_p = \begin{pmatrix} \mathrm{unr}(\bar\alpha_1)\cdot\omega_1^{a+1} & * \\ 0 & \mathrm{unr}(\bar\alpha_2) \end{pmatrix} : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F}).$$

Always denote $\bar\rho : I_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$
$$\bar\rho = \begin{pmatrix} \omega_1^{a+1} & * \neq 0 \\ 0 & 1 \end{pmatrix}$$
comes from an ext'n of rep'ns of $\mathrm{Gal}_{\mathbb{Q}_p}$.

$$\bar\rho^{ss} := \omega_1^{a+1} \oplus 1.$$

<u>Have explained</u>: $\mathcal{X}^{\square,\mathrm{tri}}_{\bar r_p}$:= trianguline deform space

$\ni (x, \delta_1, \delta_2)$ s.t. $0 \to \mathcal{R}(\delta_1) \to \mathcal{D}_{\mathrm{rig}}(V_x) \to \mathcal{R}(\delta_2) \to 0$.

$$\mathcal{X}^{\square,\mathrm{tri}}_{\bar r_p} \longrightarrow \mathcal{W}^{(\varepsilon)} \qquad \Delta = \mathbb{F}_p^\times \xhookrightarrow{\omega} \mathbb{Z}_p^\times.$$

$(x, \delta_1, \delta_2) \longmapsto \varepsilon = \delta_2|_\Delta \times \delta_1|_\Delta \cdot \omega^{-1}$ (relevant to $\bar\rho$).

(<u>Normalizations</u> $w_\star = (\delta_1\delta_2^{-1}\chi_{\mathrm{cycl}}^{-1})(\exp(p)) - 1$.

$w_k = \exp(pk) - 1$.)

Theorem Suppose $(x, \delta_1, \delta_2) \in \mathcal{X}^{\square, tri}_{\bar{r}_p}$.

(1) If $v_p(\delta_1(p)) = -v_p(\delta_2(p)) > 0$

$\Rightarrow v_p(\delta_1(p))$ is a slope of

$$NP(G_{\bar{p}}^{(\varepsilon)}(w_\star, -)), \quad w_\star = (\delta_1 \delta_2^{-1} \chi_{cycl}^{-1})(\exp(p)) - 1.$$

(2) If $v_p(\delta_1(p)) = 0$ then $\varepsilon = \begin{cases} 1 \times \omega^a \\ \omega^{a+1} \times \omega^{-1} \text{ \& } \bar{r}_p \text{ is split.} \end{cases}$

(3) If $v_p(\delta_1(p)) = \frac{k}{2} - 1$ and $w_\star = w_k$ for an integer k,

then $\delta_1(p) = p^{k-2}\delta_2(p)$.

} standard (φ, Γ)-mod theory.

* Conversely, given any slope in $NP(G_{\bar{p}}^{(\varepsilon)}(w_\star, -))$, $\exists (x, \delta_1, \delta_2)$ as above

Proof Only in the case when $\bar{r}_p$ is nonsplit

$$\bar{r}_p = \underbrace{unr(\bar{\alpha}_1) \cdot \omega_1^{a+1}}_{} \text{ --- } \underbrace{unr(\bar{\alpha}_2)}_{}$$

char χ_1 of $\mathbb{Q}_p^\times$ $\qquad \chi_2 : \mathbb{Q}_p^\times \xrightarrow{v_p(-)} \mathbb{Z} \longrightarrow \mathbb{F}^\times$

$$1 \longmapsto \bar{\alpha}_2 .$$

$$\bar{\pi}_1 = \operatorname{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)}(\bar{\chi}_2 \otimes \bar{\chi}_1 \cdot \bar{\chi}_{cycl}^{-1}), \quad \bar{\pi}_2 = \operatorname{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)}(\bar{\chi}_1 \otimes \bar{\chi}_2 \cdot \bar{\chi}_{cycl}^{-1})$$

$$\bar{\pi}(\mathbb{Q}_p) = \bar{\pi}_1 - \bar{\pi}_2 .$$

Fix a central char $\zeta : \mathbb{Q}_p^\times \longrightarrow \mathcal{O}^\times$ s.t. $\zeta|_\Delta \equiv \omega^a \bmod \varpi$.

Main subject In $\mathcal{C}_\zeta$, $\widetilde{P}_\zeta \longrightarrow\!\!\!\!\rightarrow \pi_1^\vee$

$R_{\bar{r}_p}^{\zeta} \circlearrowright$ " proj envelope of $\pi_1^\vee$.

Put $\widetilde{P}^\square := \widetilde{P}_\zeta \boxtimes \mathbb{1}_{tw}$

$R^\square_{\bar{r}_p} \circlearrowright \quad GL_2(\mathbb{Q}_p) \circlearrowright \quad$ central twist

a char $\mathbb{Q}_p^\times \longrightarrow \mathcal{O}[[u, v]]$

$$p \longmapsto 1+u$$
$$\exp(p) \longmapsto 1+v$$

Key (Hu–Paškūnas) $\exists x \in \mathfrak{m}_{R^\square_{\bar{r}_p}} \setminus \mathfrak{m}^2$, s.t. as an $\underbrace{\mathcal{O}[[u, x, z_1, z_2, z_3]]}_{=: S^\square}$, $[[GL_2(\mathbb{Z}_p)]]$-mod,

$\widetilde{P}^\square$ is the proj envelope of $\mathrm{Sym}^a(\mathbb{F}^{\oplus 2})$.

$$\begin{tikzcd} R^\square_{\bar r_p} \curvearrowright \tilde P^\square \arrow[d, dash] \\ S^\square \end{tikzcd}$$

<u>Rmk</u> for any evaluation $s^*: S \longrightarrow \mathcal{O}'$, $u \mapsto u_0$, $x \mapsto x_0$,

$\mathcal{O}'/\mathcal{O}$ fin ext'n,

$s^*\tilde P^\square := \boxed{\tilde P^\square \otimes_{S, s^*} \mathcal{O}'}$ is a primitive K_p-proj augmented module of type $\bar\rho$.

can apply local ghost thm to this

- Put $\Pi^\square := \mathrm{Hom}^{\mathrm{cont}}(\tilde P^\square, E)$.

<u>Define</u> $\mathcal{M}^\square := \mathrm{swap}^*\left(\left(J_B(\Pi^\square)^{S^\square\text{-an}}\right)'_b\right)$

$\mathcal{M}^\square$ over $\mathcal{T}$; J_B: Emerton's Jacquet functor

$\mathcal{T} = \mathrm{Hom}((\mathbb{Q}_p^\times)^2, \mathbb{C}_p^\times)$, $\quad \mathrm{swap}: \mathcal{T} \longrightarrow \mathcal{T}$

$(\delta_1, \delta_2) \longmapsto (\delta_2, \delta_1)$

$\mathcal{E}ig(\rho^\square) = \mathrm{Supp}(\mathcal{M}^\square)$ over $R^\square_{\bar r_p} \times \mathcal{T} \times \mathcal{Y}^\square$, $\mathcal{Y}^\square = (\mathrm{Spf}\, S^\square)^{\mathrm{rig}}$.

<u>Key input</u> (Breuil-Ding, Breuil-Hellman-Schraen)

$\mathcal{E}ig(\rho^\square)^{(\mathrm{red})} \cong X^{\square,\mathrm{tri}}_{\bar r_p}$ ("global trianguline + density of classical pts")

Slopes here are governed by ghost series

$$\begin{tikzcd} \boxed{\mathrm{Spc}(s^*\tilde P^\square)} \arrow[r, hook] \arrow[dd] \arrow[ddrr, phantom, "\square"] & \mathrm{Spc}(\tilde P^\square) \arrow[dr] & & \\ & & \mathcal{Y}^\square \times \mathcal{T} \arrow[r, "\delta_2(p)"] \arrow[d] & \mathbb{G}_m^{\mathrm{rig}} \\ \mathcal{W} \arrow[rr, "s"] & & \mathcal{Y}^\square \times \mathcal{W} & \end{tikzcd}$$

§ Bootstrapping

* <u>Local-global compatibility @ p</u> $\qquad K_p = GL_2(\mathbb{Z}_p)$.

Let $\tilde H$ be a K_p-proj augmented mod

(i.e. fin proj right $\mathcal{O}[\![K_p]\!]$-mod whose K_p-action extends to a $GL_2(\mathbb{Q}_p)$-action.)

s.t. $\forall \bar\alpha_1 \in \Delta$, $\begin{pmatrix} \bar\alpha_1 & \\ & \bar\alpha \end{pmatrix}$ acts on $\tilde H$ by $\bar\alpha^a$.

Fix $\varepsilon : \Delta^2 \to \mathbb{F}^\times$ relevant, i.e. $\Delta(\bar\alpha, \bar\alpha) = \bar\alpha^a$,

$\varepsilon = \omega^{-s} \times \omega^{a+s}$ for $s \in \{0, \dots, p-2\}$.

For $k = k_\varepsilon := a + 2s + 2 \mod p-1$ (and $k \geq 2$).

$T_p, S_p \curvearrowright S_k^{\mathrm{cris}}(\omega^{-s}) := \mathrm{Hom}_{\mathcal{O}[\![K_p]\!]}(\tilde H, \mathcal{O}[z]^{\deg \leq k-2} \otimes \omega^{-s} \circ \det)$

$S_p(\varphi)(x) := \varphi\left(x \begin{pmatrix} p^{-1} & \\ & p^{-1} \end{pmatrix}\right)$.

The eigenvalues of T_p, S_p are expected to correspond to sth on Gal side:

$R_{\bar r_p}^{\square, 1-k, \omega^{-s}}$:= crystalline framed deform space of $\bar r_p$ of HT wts $(1-k, 0)$ and $\mathrm{Gal}_{\mathbb{Q}_p}$ acts on $\mathbb{D}_{\mathrm{pcrys}}(-)$ by ω^{-s}.

$$\rightsquigarrow \begin{array}{l} \mathbb{D}_{\mathrm{pcrys}}(\mathcal{V}_{1-k}) \circlearrowleft \text{ crystalline Frob } \phi \\ \quad \Big| \\ \mathcal{X}_{\bar r_p}^{\square, 1-k, \omega^{-s}} \end{array}$$

loc free of rk 2

(six authors) $\Rightarrow \exists$ elements $s_p, t_p \in R_{\bar r_p}^{\square, 1-k, \omega^{-s}}[\frac{1}{p}]$.

s.t. $\det(\phi) = p^{k-1} \cdot s_p^{-1}$, $\mathrm{tr}(\phi) = s_p^{-1} t_p$.

Def'n An $\mathcal{O}[\![K_p]\!]$-proj arith modular of type $\bar r_p$ is an $\mathcal{O}[\![K_p]\!]$-proj augmented mod $\tilde H$ equipped with a cont. left action of $R_{\bar r_p}^{\square}$.

s.t. ① left $R_{\bar r_p}^{\square}$-action and right $GL_2(\mathbb{Q}_p)$-action commute

② $\tilde H$ as a right $\mathcal{O}[\![K_p]\!]$-mod is isom to

* $\mathrm{Proj}_{\mathcal{O}[\![K_p]\!]}(\mathrm{Sym}^a)^{\oplus m(\tilde H)}$ if $\bar r_p$ is non-split (Serre wt $= \mathrm{Sym}^a \mathbb{F}^2$).
* $\mathrm{Proj}(\mathrm{Sym}^a)^{\oplus m'(\tilde H)} \oplus \mathrm{Proj}(\mathrm{Sym}^{p-3-a} \otimes \det^{a+1})^{\oplus m''(\tilde H)}$ if $\bar r_p$ is split (Serre wts Sym^a, $\mathrm{Sym}^{p-3-a} \otimes \det^{a+1}$).

. $m(\tilde H) = m'(\tilde H) + m''(\tilde H)$

③ $\forall \varepsilon = \omega^{-s} \times \omega^{a+s}$ relevant, $k \equiv a+2s+2 \mod p-1$, $k \geq 2$

$R_{\bar{r}_p}^{\square}$ - action on $S_k^{\mathrm{ur}}(\omega^{-s})$ factors through $R_{\bar{r}_p}^{\square, 1-k, \omega}$.

$$\begin{array}{ccc} t_p, s_p \in R_{\bar{r}_p}^{\square, 1-k, \omega^{-s}} & \twoheadleftarrow & R_{\bar{r}_p}^{\square} \\ \circlearrowright & & \circlearrowright \\ & S_k^{\mathrm{ur}}(\omega^{-s}) := \mathrm{Hom}_{\mathcal{O}[\![K]\!]}(\tilde{H}, \mathcal{O}[z]^{\deg \leq k-2} \otimes \omega^{-s} \circ \det), & \\ & \circlearrowleft S_p, T_p & \end{array}$$

<u>Example</u> "Essentially" for any $G/\mathbb{Q}$ s.t. $G_{\mathbb{Q}_p} \approx GL_2(\mathbb{Q}_p) \times H$

(e.g. $G = \mathrm{Res}_{F/\mathbb{Q}} GL_2$, F totally real, $F / \mathbb{Q}$, $\mathfrak{p}_1 \cdots \cdots$ over p, $F_{\mathfrak{p}_1} = \mathbb{Q}_p$.)

$$R_{\bar{r}_p} \circlearrowright \tilde{H} := \varprojlim_N H_{\mathrm{mid}}(Sh_G(K^p \cdot K_H \cdot (1+p^N M_2(\mathbb{Z}_p))(\mathbb{C})^{\mathrm{an}}, \mathbb{Z}_p)_{\mathfrak{m}_{\bar{r}}}$$

* for some $\bar{r}$ irred + "large image" (to apply Caraiani-Scholze.)

<u>Or</u>, a patched version of this!

$$\begin{array}{ccc} R_{\bar{r}_p}^{\square} \curvearrowright \tilde{H}_\infty & & \tilde{H}_\infty \otimes_{J_\infty, \underline{z}^*} \mathcal{O}' \circlearrowleft R_{\bar{r}_p}^{\square}. \\ \Big| & & \Big| \\ J_\infty = \mathcal{O}[\![z_1, \cdots, z_g]\!] & \xrightarrow{\underline{z}^*} & \mathcal{O}' \end{array}$$

<u>Rmk 1</u> What if $\bar{r}$ is not irred?

(Some combinatorics to be done, see Diao-Yao's recent work)

<u>Rmk 2</u> Why Paškūnas mod but not patched module of six author?

<u>Thm</u> Let $\tilde{H}$ be an $\mathcal{O}[\![K_p]\!]$-proj arith mod of type $\bar{r}_p$ and multiplicity $m(\tilde{H})$.

Let $C_{\tilde{H}}^{(\varepsilon)}(w,t) :=$ char power series of $U_p \curvearrowright S_{\tilde{H}}^{p\text{-adic},(\varepsilon)}$.

Then for any $w_\star \in \mathfrak{m}_{\mathbb{C}_p}$, $NP(C_{\tilde{H}}^{(\varepsilon)}(w_\star, -)) = NP(G_{\bar{p}}^{(\varepsilon)}(w_\star, -))$

stretched in both x-, y-directions $m(\tilde{H})$ times.

(except for the ordinary part.)

<u>Proof</u>

$$\text{Length of ord part} = \begin{cases} m(\tilde{H}) & \text{if } \bar{r}_p \text{ non-split and } \varepsilon = 1 \times \omega^a \\ m'(\tilde{H}) & \text{if } \bar{r}_p \text{ split and } \varepsilon = 1 \times \omega^a \\ m''(\tilde{H}) & \text{if } \bar{r}_p \text{ split and } \varepsilon = \omega^{a+1} \times \omega^{-1}. \\ 0 & \text{o/w.} \end{cases}$$

$$\begin{tikzcd}
\mathcal{X}^{\square,\mathrm{tri}}_{\bar{r}_p} \arrow[d] & \underset{\text{at least the reduced part}}{\overset{\text{density of classical pts}}{\supseteq}} & \mathcal{E}ig^{(\varepsilon)}(\tilde{H}) \arrow[r, "\text{“finite”}"] \arrow[d] & \mathrm{Spc}^{(\varepsilon)}(\tilde{H}) \arrow[d] \arrow[loop above, "R^{\square}_{\bar{r}_p}"] \arrow[drr, dashed, "\mathrm{wt}"] & = \text{zero locus of } C^{(\varepsilon)}(w,t). & \\
\mathcal{X}^{\square}_{\bar{r}_p} \times \mathcal{T} & \longleftrightarrow & \mathcal{X}^{\square}_{\bar{r}_p} \times \mathcal{W}^{(\varepsilon)} \times \mathbb{G}_m^{\mathrm{rig}} \arrow[ll, bend left] \arrow[r] & \mathcal{W}^{(\varepsilon)} \times \mathbb{G}_m^{\mathrm{rig}} \arrow[rr] & & \mathcal{W}^{(\varepsilon)}
\end{tikzcd}$$

need to choose this normalization here.

$\Rightarrow$ Pointwise, slopes on $\mathrm{Spc}^{(\varepsilon)}(\tilde{H})$ are the slopes of ghost series

but not sure about multiplicity yet.

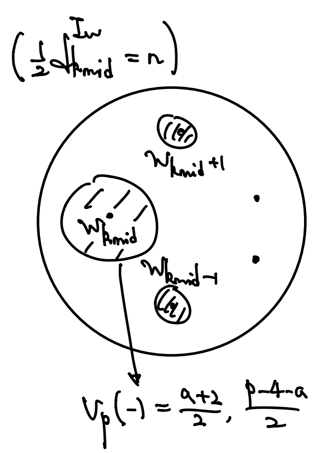
Fix ε relevant. $\forall n \in \mathbb{N}$, define

$$\mathrm{Vtx}_n := \{ w_\star \in \mathfrak{m}_{\mathbb{C}_p} : (n, v_p(g_n(w_\star))) \text{ is a vertex of } \mathrm{NP}(G_{\bar{\rho}}, (w_\star, -)) \}$$

$$= \mathcal{W}^{(\varepsilon)} \setminus \bigcup_k \{ w_\star \in \mathfrak{m}_{\mathbb{C}_p} \mid v_p(w_\star - w_k) \geq \Delta_{k, |\frac{1}{2} d_k^{\mathrm{Iw}} - n| + 1} - \Delta_{k, |\frac{1}{2} d_k^{\mathrm{Iw}} - n|} \}$$

quasi-Stein, irred.

$$= \bigcup_{\delta \to 0^+} \underbrace{\mathrm{Vtx}_n^{\delta}}_{} = \left\{ w_\star \in \mathfrak{m}_{\mathbb{C}_p} \;\middle|\; \begin{array}{l} v_p(w_\star) \geq \delta, \\ v_p(w_\star - w_k) \leq \cdots - \delta, \ \forall k \end{array} \right\}$$



<u>Upshot</u> at each pt $w_\star \in \mathrm{Vtx}_n^{\delta}$,

the left slope at $x = n$ of $\mathrm{NP}(G_{\bar{\rho}})$

$\leq$ (the right slope at $x = n$ of $\mathrm{NP}(G_{\bar{\rho}})$) $- \epsilon(\delta)$.

<u>Upshot</u>

$$\mathrm{Spc}(\tilde{H})_n^{\delta} := \left\{ (w_{\ast}, a_p) \in \mathrm{Spc}(\tilde{H}) \;\middle|\; \begin{array}{l} w_{\ast} \in \mathcal{V}tx_n^{\delta} \\ -v_p(a_p) \le \text{left slope at } x = n \text{ of } NP(G) \end{array} \right\}$$

strict relative to $\mathcal{V}tx_n$ $\quad \downarrow \quad \|$

$$\mathrm{Spc}(\tilde{H})_n^{\delta,+} := \left\{ (w_{\ast}, a_p) \in \mathrm{Spc}(\tilde{H}) \;\middle|\; \begin{array}{l} w_{\ast} \in \mathcal{V}tx_n^{\delta} \\ -v_p(a_p) \le \text{right slope at } x = n \text{ of } NP(G) + \epsilon(\delta) \end{array} \right\}$$

$$\xRightarrow{\text{Kiehl's argument}} \quad \left.\begin{array}{l} wt_{\ast}(\mathrm{Spc}(\tilde{H})_n^{\delta}) = \text{finite over } \mathcal{V}tx_n^{\delta} \\ \qquad \text{flat by construction} \end{array}\right\} \Rightarrow \text{Constant degree}.$$

<u>Technical lemma</u> (next lecture)

$\forall k,\ n = d_k^{\mathrm{Iw}}(\varepsilon\cdot(1\times\omega^{2-k}))$ (usually essential Iw_p-level)

$\Rightarrow (n, v_p(g_n(w_k)))$ is a vertex for $NP(G)$.

By dim formula, done!

Lecture 10: Proof of local ghost conjecture (I)

Fix $\bar{\rho} = \begin{pmatrix} \omega_1^{a+1} & * \neq 0 \\ 0 & 1 \end{pmatrix} : I_{\mathbb{Q}_p} \longrightarrow GL_2(\mathbb{F})$,

$\varpi \in \mathcal{O}, \mathbb{F}$, $\mathcal{O} \subseteq E/\mathbb{Q}_p$

$\tilde{H}$ = primitive $\mathcal{O}[\![K_p]\!]$-proj augmented module of type $\bar{\rho}$

$\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ acts trivially on $\tilde{H}$. $(1 \le a \le p-4.)$

$\varepsilon = \omega^{-s_\varepsilon} \times \omega^{a+s_\varepsilon} : \Delta^\times \longrightarrow \mathcal{O}^\times$ a relevant char

$(s_\varepsilon \in \{0, \cdots, p-2\})$.

$\circlearrowleft S_{p\text{-adic}}^{(\varepsilon)} = \mathrm{Hom}_{\mathrm{Iw}_p}(\tilde{H}, C^0(\mathbb{Z}_p, \mathcal{O}[\![w]\!]^{(\varepsilon)}))$

$U_p \circlearrowright S^{\dagger, (\varepsilon)} := \mathrm{Hom}_{\mathrm{Iw}_p}(\tilde{H}, \mathcal{O}\langle \frac{w}{p} \rangle^{(\varepsilon)} \langle z \rangle)$.

$C^{(\varepsilon)}(w,t) = \sum_{n \ge 0} c_n^{(\varepsilon)}(w) t^n \in \mathcal{O}[\![w,t]\!]$.

Ghost series $G^{(\varepsilon)}(w,t) = \sum_{n \ge 0} g_n^{(\varepsilon)}(w) \cdot t^n \in \mathbb{Z}_p[w][\![t]\!] \subseteq \mathcal{O}[\![w,t]\!]$.

Our goal in the last 3 lectures: to prove

Thm When $p \ge 11$, $2 \le a \le p-5$, we have

$$NP(C^{(\varepsilon)}(w_*, -)) = NP(G^{(\varepsilon)}(w_*, -)), \quad w_* \in \mathfrak{m}_{\mathbb{C}_p}.$$

Warm up **Prop 4.1** Fix ε.

If (1) $g_1^{(\varepsilon)}(w) \in \mathcal{O}[\![w]\!]^\times \iff \varepsilon = 1 \times \omega^a$

(2) For $k \ge 2$, let $d_{\varepsilon,k} = d_k^{\mathrm{Iw}}(\varepsilon \cdot (1 \times \omega^{2-k}))$.

Then $(d_{\varepsilon,k}, v_p(c_{d_{\varepsilon,k}}^{(\varepsilon)}(w_k)))$ (resp. $(d_{\varepsilon,k}, v_p(g_{d_{\varepsilon,k}}^{(\varepsilon)}(w_k)))$)

is a vertex of $NP(C^{(\varepsilon)}(w_k, -))$ (resp. $NP(G^{(\varepsilon)}(w_k, -))$).

Proof (1) We write $\varepsilon = \omega^{-s_\varepsilon} \times \omega^{a+s_\varepsilon} : \Delta^2 \longrightarrow \mathcal{O}^\times$

Assume first $s_\varepsilon = 0$. We consider $k = 2 + s_\varepsilon + \{a + s_\varepsilon\}$.

By dim formula, $d_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1) = 2$, $d_k^{\mathrm{ur}}(\varepsilon_1) = 0$.

$\Rightarrow$ U_p-slopes on $S_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1)$ is $\frac{k-2}{2} = \frac{s_\varepsilon + \{a + s_\varepsilon\}}{2} > 0$.

$\Rightarrow$ $v_p(C_1^{(\varepsilon)}(w_k)) > 0$

$\Rightarrow$ $C_1^{(\varepsilon)}(w)$ is not a unit.

Now we assume $s_\varepsilon = 0$, $\varepsilon = 1 \times \omega^a$, $\varepsilon' = \omega^a \times 1$. $\psi = \omega^a \times \omega^a$.

By dimention, for $k = 1 + p - a$ on $\mathcal{W}^{(\varepsilon')}$,

we have $d_k^{\mathrm{Iw}}(\psi) = 2$ and $d_k^{\mathrm{ur}}(\omega^a) = 0$.

$\Rightarrow$ U_p-slopes on $S_k^{\mathrm{Iw}}(\psi)$ is $\frac{p+1-a-2}{2} > 1$.

$\Rightarrow$ $v_p(C_1^{(\varepsilon')}(w_k)) \geq \frac{p+1-a-2}{2} > 1$.

$C_1^{(\varepsilon')}(w) \in \mathcal{O}[\![w]\!] \Rightarrow v_p(C_1^{(\varepsilon')}(w_2)) \geq 1$.

By dim formula, $d_2^{\mathrm{Iw}}(\varepsilon) = d_2^{\mathrm{Iw}}(\varepsilon') = 1$.

By classicality the U_p-slope on $S_2^{\mathrm{Iw}}(\varepsilon')$ is 1.

By Atkin-Lehner $S_2^{\mathrm{Iw}}(\varepsilon) \longleftrightarrow S_2^{\mathrm{Iw}}(\varepsilon')$

the U_p-slope on $S_2^{\mathrm{Iw}}(\varepsilon)$ is $0 \Rightarrow v_p(C_1^{(\varepsilon)}(w_2)) = 0$.

$\Rightarrow C_1^{(\varepsilon)}(w) \in \mathcal{O}[\![w]\!]^\times$.

The 1st slope of $NP(C^{(\varepsilon)}(\underset{\substack{\uparrow \\ w_\star \in \mathfrak{m}_{\mathbb{C}_p}}}{w_\star}, -))$ is $0 \Leftrightarrow \varepsilon = 1 \times \omega^a$.

(2) $\psi = \varepsilon(1 \times \omega^{2-k}) = \omega^{-s_\varepsilon} \times \omega^{a+s_\varepsilon+2-k}$

$\psi^s = \omega^{a+s_\varepsilon+2-k} \times \omega^{-s_\varepsilon}$

$\psi' = \psi \cdot (\omega^{k-1} \times \omega^{k-1}) = \omega^{-s_\varepsilon+k-1} \times \omega^{a+s_\varepsilon+1}$.

By Atkin-Lehner:

$$AL_{(k,\psi)} : S_k^{\mathrm{Iw}}(\psi) \longrightarrow S_k^{\mathrm{Iw}}(\psi^s).$$

The $(d_{\varepsilon,k})$th slope of $S_k^{Iw}(\psi)$ is $\leq k-1$.

and equality holds $\Longleftrightarrow a+s_\varepsilon+2-k\equiv 0 \mod p-1$ ①

By Theta map:

$$0\to S_k^{Iw}(\psi)\to S_k^{\dagger}(\psi)\xrightarrow{\theta} S_{2-k}^{\dagger}(\psi')$$

the $(d_{\varepsilon,k}+1)$st slope of $S_k^{\dagger}(\psi)\geq k-1$

and equality holds $\Longleftrightarrow -s_\varepsilon+k-1\equiv 0 \mod p-1$ ②

But ① & ② cannot simultaneously happen

(by assumption on a). $\square$

Def (Lagrangian interpolation formula)

Let $f(w)\in\mathcal{O}\langle\frac{w}{p}\rangle$ (later: $f(w)=C_n^{(\varepsilon)}(w)\in\mathcal{O}[\![w]\!]$).

and $g(w)=(w-x_1)^{m_1}\cdots(w-x_s)^{m_s}\in\mathbb{Z}_p[w]$ (later: $g(w)=g_n(w)$.)

$x_i\in p\mathbb{Z}_p$, $m_1,\dots,m_s\in\mathbb{Z}_{\geq 0}$.

Then we write $f(w)$ uniquely as

$$f(w)=\sum_{i=1}^{s}\Big(\underbrace{A_i(w)}_{E[w]^{<m_i}}\frac{g(w)}{(w-x_i)^{m_i}}\Big)+\underbrace{h(w)}_{E\langle\frac{w}{p}\rangle}\cdot g(w)$$

Lagrange interpolation of $f(w)$ along $g(w)$.

s.t. $f(w)\equiv A_j(w)\frac{g(w)}{(x-x_j)^{m_j}} \mod (w-x_j)^{m_j}$ in $E[\![w-x_j]\!]$, $\forall j=1,\dots,s$.

Fix n & ε. Consider the Lagrange interpolation of $C_n^{(\varepsilon)}(w)\in\mathcal{O}[\![w]\!]$

along $g_n^{(\varepsilon)}(w)\in\mathbb{Z}_p[w]$.

$$(\bigstar)\quad C_n^{(\varepsilon)}(w)=\sum_{\substack{k\equiv k_\varepsilon(p-1)\\ m_n(k)\neq 0}}\Big(A_k^{(n,\varepsilon)}(w)\cdot g_{n,\hat{k}}(w)\Big)+h_n^{(\varepsilon)}(w)\,g_n^{(\varepsilon)}(w)$$

$$A_k^{(n,\varepsilon)}(w)=\sum_{i=0}^{m_n(k)-1}A_{k,i}^{(n,\varepsilon)}(w-w_k)^i\in E[w].$$

Prop 4.4 The local ghost conj is true if the following holds:

$\forall n, \varepsilon$ and every ghost zero w_k of $g_n^{(\varepsilon)}(w)$,

we have $v_p(A_{k,i}^{(n,\varepsilon)}) \geq \Delta^{(\varepsilon)}_{k,\frac{1}{2}d_k^{new}-i} - \Delta^{(\varepsilon)\prime}_{k,\frac{1}{2}d_k^{new}-m_n(k)}$.

for all $i = 0, 1, \cdots, m_n(k)-1$.

Recall $\Delta^{(\varepsilon)\prime}_{k,l} = v_p(g^{(\varepsilon)}_{\frac{1}{2}d_k^{Iw}+l,\hat{k}}(w_k)) - \frac{k-2}{2}\cdot l$.

$|l| \leq \frac{1}{2}d_k^{new}$, $\underline{\Delta}_k^{(\varepsilon)}$ lower convex hull of $(l, \Delta'_{k,l})$.

Proof It suffices to prove $\forall w_* \in \mathfrak{m}_{\mathbb{C}_p}$,

Claim 1 The pt $(n, v_p(C_n^{(\varepsilon)}(w_*)))$ lies on or above $NP(G^{(\varepsilon)}(w_*,-))$.

Claim 2 If $(n, v_p(g_n^{(\varepsilon)}(w_*)))$ is a vertex of $NP(G^{(\varepsilon)}(w_*,-))$

then $v_p(g_n^{(\varepsilon)}(w_*)) = v_p(C_n^{(\varepsilon)}(w_*))$.

Lemma $A = A_{k,i}^{(n,\varepsilon)}$. The pt $(n, v_p(A(w-w_*)^i g_{n,\hat{k}}(w_*)))$

lies on or above $NP(G^{(\varepsilon)}(w_*,-))$ and it lies strictly on this NP

if $(n, v_p(g_n^{(\varepsilon)}(w_*)))$ is a vertex of $NP(G^{(\varepsilon)}(w_*,-))$.

Pf of Claim 1 We know $A_k^{(n,\varepsilon)} \in \mathcal{O}[w] \Rightarrow h_n^{(\varepsilon)}(w) \in \mathcal{O}[\![w]\!]$.

Pf of Claim 2 Can show $h_n^{(\varepsilon)}(w) \in \mathcal{O}[\![w]\!]^\times$.

We take a wt $k \equiv k_\varepsilon \bmod p-1$ s.t. $d_k^{Iw}(\varepsilon(1\times\omega^{2-k})) = n$

$S_{\varepsilon''} = \{k-2-a-S_\varepsilon\}$.

Then the pt $(n, v_p(g_n^{(\varepsilon)}(w_k)))$ (resp. $(n, v_p(g_n^{(\varepsilon'')}(w_k)))$)

is a vertex of $NP(G^{(\varepsilon)}(w_*,-))$ (resp. $NP(G^{(\varepsilon'')}(w_*,-))$).

Lem $\Rightarrow$ $v_p(C_n^{(\varepsilon)}(w_k)) \geq v_p(g_n^{(\varepsilon)}(w_k))$, equality holds iff $h_n^{(\varepsilon)}(w_k) \in \mathcal{O}^\times$

$\Downarrow$

and similar result for ε''. $h_n(w) \in \mathcal{O}[\![w]\!]^\times$.

$$v_p(C_n^{(\varepsilon)}(w_k)) + v_p(C_n^{(\varepsilon)}(w_k)) = (k-1)n$$
$$= v_p(g_n^{(\varepsilon)}(w_k)) + v_p(g_n^{(\varepsilon'')}(w_k)).$$

Rmk (1) intuition of Prop 4.4.

When w_* is not close to any ghost zero w_k of $g_n(w)$.

$\Rightarrow$ $(n, v_p(g_n^{(\varepsilon)}(w_*)))$ is a vertex of $NP(G^{(\varepsilon)}(w_*, -))$.

Because $v_p(A_{k,i}^{(n,\varepsilon)})$ is big $\Rightarrow v_p(C_n^{(\varepsilon)}(w_*)) = v_p(g_n^{(\varepsilon)}(w_*))$.

When w_* is close to some w_k

$\Rightarrow v_p(C_n^{(\varepsilon)}(w_*))$ is large.

(2) In $(*)$, let $w = w_k$ for some ghost zero w_k of $g_n(w)$.

$\Rightarrow A_{k,0}^{(n,\varepsilon)} = C_n^{(\varepsilon)}(w_k) / g_{n,\hat{k}}^{(\varepsilon)}(w_k)$

$\rightsquigarrow$ the equality in Prop 4.4 becomes $\quad (n = \frac{1}{2} d_k^{Iw})$.

$$v_p(C_n^{(\varepsilon)}(w_k)) \geq v_p(\underbrace{g_{d_k^{ur}}(w_k)}_{\text{sum of } d_k^{ur} \ U_p\text{-slopes}}) + (n - d_k^{ur}) \cdot \frac{k-2}{2}.$$

$S^{\dagger,(\varepsilon)} = \mathrm{Hom}_{I_{\mathrm{up}}}(\widetilde{H}, \mathcal{O}\langle \frac{w}{p} \rangle^{(\varepsilon)} \langle z \rangle)$.

We have a power basis $\quad \{e_1^{(\varepsilon)}, e_2^{(\varepsilon)}, \cdots\}$

$\mathcal{B}^{(\varepsilon)} = \{ e_1^* \cdot z^i \cdot e_2^* \cdot z^j : i \equiv s_\varepsilon \bmod p-1,\ j \equiv a + s_\varepsilon \bmod p-1 \}$

$U^\dagger = U^{\dagger,(\varepsilon)} \in M_\infty(\mathcal{O}\langle \frac{w}{p} \rangle)$ matrix of the U_p-operator on $S^\dagger$ w.r.t. the power basis $\mathcal{B}^{(\varepsilon)}$.

Take $\underline{\zeta} = \{\zeta_1 < \zeta_2 < \cdots < \zeta_n\}$, $\underline{\xi} = \{\xi_1 < \cdots < \xi_n\}$.

(1) $U^\dagger(\underline{\zeta}, \underline{\xi})$ = $n \times n$ submatrix of $U^\dagger$ with row indices in $\underline{\zeta}$ & column indices in $\underline{\xi}$.

(2) $\deg(\underline{\zeta}) = \sum_{i=1}^n \deg e_{\zeta_i}$. $\det(U^\dagger(\underline{\zeta}, \underline{\xi})) \in \mathcal{O}\langle \frac{w}{p} \rangle$.

Fix ε and n. Consider the Lagrange interpolation of $\det(U^\dagger(\underline{\zeta}, \underline{\xi}))$ along $g_n^{(\varepsilon)}(w) \in \mathbb{Z}_p[w]$.

$$\det(U^{\dagger}(\underline{\zeta}, \underline{\xi})) = \sum_{\substack{k = k_\varepsilon(p-1) \\ m_n(k) \neq 0}} \left(A_k^{(\underline{\zeta} \times \underline{\xi})}(w) \cdot g_{n,\hat{k}}(w) \right) + h_{\underline{\zeta} \times \underline{\xi}}(w) \cdot g_n(w)$$

$$= \sum_{i=0}^{m_n(k)-1} A_{k,i}^{(\underline{\zeta} \times \underline{\xi})} (w - w_k)^i \in E[w].$$

<u>Thm 5.2</u> Assume $p \geq 11$ and $2 \leq a \leq p-5$.

For $\forall \underline{\zeta}, \underline{\xi}$ of size n, w_k zero of $g_n(w)$,

we have

$$v_p(A_{k,i}^{(\underline{\zeta} \times \underline{\xi})}) \geq \Delta_{k, \frac{1}{2} d_k^{\mathrm{new}} - i} - \Delta'_{k, \frac{1}{2} d_k^{\mathrm{new}} - m_n(k)} + \frac{1}{2}(\deg \underline{\zeta} - \deg \underline{\xi}).$$

Lecture 11: Proof of ghost conjecture (II) — cofactor expansion

Setup $p>11$, $2\le a\le p-5$, $E\ni\mathcal{O}\twoheadrightarrow\mathcal{O}/(\varpi)=\mathbb{F}$.

$$\bar\rho=\begin{pmatrix}\omega_1^{a+1} & * \\ 0 & 1\end{pmatrix}: I_{\mathbb{Q}_p}\longrightarrow GL_2(\mathbb{F})$$

- $K_p=GL_2(\mathbb{Z}_p)\supseteq \mathrm{Iw}_p=\begin{pmatrix}\mathbb{Z}_p^\times & \mathbb{Z}_p\\ p\mathbb{Z}_p & \mathbb{Z}_p^\times\end{pmatrix}\overset{\omega}{\hookleftarrow}\Delta=\begin{pmatrix}\mathbb{F}_p^\times & \\ & \mathbb{F}_p^\times\end{pmatrix}\supseteq\begin{pmatrix}1+p\mathbb{Z}_p & \mathbb{Z}_p\\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p\end{pmatrix}=\mathrm{Iw}_{p,1}$.

$\widetilde{H}$ = primitive K_p-augmented module of type $\bar\rho$:

i.e. $\widetilde{H}=\mathrm{Proj}_{\mathcal{O}[\![K_p]\!]}^{\mathrm{right}}(\mathrm{Sym}^a\mathbb{F}^{\oplus 2})\circlearrowleft$ extends the K_p-action to $GL_2(\mathbb{Q}_p)/p^{\mathbb{Z}}$.

+ centralizer, and $\cdots$

- As $\mathcal{O}[\![\mathrm{Iw}_p]\!]$-mods, $\widetilde{H}=e_1\mathcal{O}\otimes_{1\times\omega^a,\mathcal{O}[\Delta]}\mathcal{O}[\![\mathrm{Iw}_p]\!]\oplus e_2\mathcal{O}\otimes_{\omega^a\times 1,\mathcal{O}[\Delta]}\mathcal{O}[\![\mathrm{Iw}_p]\!]$.

Here, we choose basis e_1,e_2, s.t. $e_1\begin{pmatrix}0&1\\p&0\end{pmatrix}=e_2$.

- $\varepsilon:\Delta=(\mathbb{F}_p^\times)^2\longrightarrow\mathcal{O}^\times$. $\varepsilon=\omega^{-s_\varepsilon}\times\omega^{a+s_\varepsilon}$.

$$S_{\widetilde{H}}^{(\varepsilon),p\text{-adic}}:=\mathrm{Hom}_{\mathrm{Iw}_p}\Big(\widetilde{H},\underbrace{\mathrm{Ind}_{B^{\mathrm{op}}(\mathbb{Z}_p)}^{GL_2(\mathbb{Z}_p)}\chi_{\mathrm{univ}}^{(\varepsilon)}}_{=\,\mathcal{C}(\mathbb{Z}_p,\mathcal{O}[\![w]\!])}\Big)\circlearrowleft U_p$$

$\chi_{\mathrm{univ}}^{(\varepsilon)}:\mathbb{Z}_p^\times\times\mathbb{Z}_p^\times\longrightarrow\mathcal{O}[\![w]\!]^\times$

$(\alpha,\omega(d))\longmapsto\varepsilon(\bar\alpha,\bar d)$

$(1,\exp(p))\longmapsto 1+w$

(ε parametrizes wt disc)

$$=e_1^*\mathcal{C}(\mathbb{Z}_p,\mathcal{O}[\![w]\!])^{\deg_z\equiv s_\varepsilon \bmod p-1}\oplus e_2^*\underbrace{\mathcal{C}(\mathbb{Z}_p,\mathcal{O}[\![w]\!])^{\deg_z\equiv a+s_\varepsilon \bmod p-1}}_{=\,\widehat{\bigoplus}_{n\ge0}\mathcal{O}[\![w]\!]\binom{z}{n}}.$$

"power basis": $e_1^*z^{s_\varepsilon},\ e_1^*z^{s_\varepsilon+p-1},\ e_1^*z^{s_\varepsilon+2(p-1)},\ \cdots$

$e_2^*z^{\{a+s_\varepsilon\}},\ e_2^*z^{\{a+s_\varepsilon\}+(p-1)},\ \cdots$ degree of basis (on z).

$\{A\}\equiv A \bmod p-1$, with $\{A\}\in\{0,\cdots,p-2\}$.

Rename these by $e_1^{(\varepsilon)},e_2^{(\varepsilon)},\cdots$, ordered by deg

$\rightsquigarrow U_p^\dagger:=U_p$-action on this basis.

Define $C_{\bar{H}}^{(\varepsilon)}(w,t) := \det(I - U^{\dagger} t) = \sum C_n^{(\varepsilon)}(w)\, t^n \in \mathbb{Q}[[w,t]]$.

* Ghost series $G_{\bar{p}}^{(\varepsilon)}(w,t) = \sum g_n^{(\varepsilon)}(w)\, t^n$

where $g_n^{(\varepsilon)}(w) = \prod (w - w_k)^{m_n^{(\varepsilon)}(k)}$

$$m_n^{(\varepsilon)}(k) = \begin{cases} \min\{n - d_k^{ur},\ d_k^{Iw} - d_k^{ur} - n\} & \text{if } d_k^{ur} \le n \le d_k^{Iw} - d_k^{ur} \\ 0, & \text{o/w.} \end{cases}$$

Thm (Local ghost) For any $w_\star \in \mathfrak{m}_{\mathbb{C}_p}$

$$NP(C_{\bar{H}}^{(\varepsilon)}(w_\star, -)) = NP(G_{\bar{p}}^{(\varepsilon)}(w_\star, -)) \qquad (\text{Omit } (\varepsilon) \text{ from this notation.})$$

Step I: Lagrange interpolation

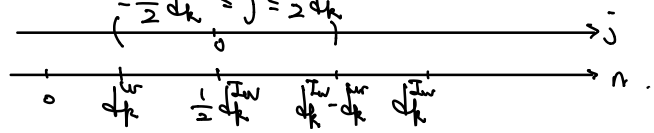
Write
$$C_n(w) = \sum_{k \equiv a+2s+2 \bmod p-1} \underbrace{g_{n,\hat{k}}(w)}_{=\, g_n(w)/(w-w_k)^{m_n(k)}} \left(A_{k,0}^{(n)} + A_{k,1}^{(n)}(w-w_k) + \cdots + A_{k,m_n(k)-1}^{(n)}(w-w_k)^{m_n(k)-1}\right) + h_n(w)\, g_n(w)$$

Last time To prove local ghost conj. it suffices to prove

$$v_p(A_{k,i}^{(n)}) \ge \Delta_{k,\frac{1}{2}d_k^{new}-i} - \Delta'_{k,\frac{1}{2}d_k^{new}-m_n(k)} \quad \text{for all } i = 0, 1, \dots, m_n(k)-1.$$

Here $\Delta'_{k,j} = v_p\big(g_{\frac{1}{2}d_k^{Iw}+j,\hat{k}}(w_k)\big) - \frac{k-2}{2} j$, $\Delta'_{k,j} := \Delta'_{k,-j}$.

$$-\tfrac{1}{2} d_k^{new} \le j \le \tfrac{1}{2} d_k^{new}$$



$\{(j, \Delta_{k,j})\}$ is the convex hull of $(j, \Delta'_{k,j})$.

We prove a stronger statement.

$$\underline{\xi} = (\xi_1 < \cdots < \xi_n), \quad \underline{\zeta} = (\zeta_1 < \cdots < \zeta_n).$$

Apply the same Lagrange interpolation to $\det U^{\dagger}(\underline{\xi} \times \underline{\zeta})$. Same as conjugating by $\begin{pmatrix} p^{\frac{1}{2}\deg e_1} & & \\ & \ddots & \\ & & p^{\frac{1}{2}\deg e_{d_k^{Iw}}} \end{pmatrix}$.

$\rightsquigarrow A_{k,i}^{(\underline{\xi},\underline{\zeta})} \in E$.

Need to show

$$(\bigstar) \quad v_p(A_{k,i}^{(\underline{\xi},\underline{\zeta})}) \ge \Delta_{k,\frac{1}{2}d_k^{new}-i} - \Delta'_{k,\frac{1}{2}d_k^{new}-m_n(k)} + \underbrace{\tfrac{1}{2}\big(\overset{\text{total deg of } e_{\xi_i}\text{'s}}{\deg(\underline{\xi})} - \deg(\underline{\zeta})\big)}$$

by induction on size n.

Step II Tells about how to understand this.

Step III Cofactor expansion

Key Input When $k \equiv a+2s+2 \mod p-1$.

$$T_p \circlearrowright \; \begin{tikzcd} S_k^{ur}(\omega^s) \arrow[r, shift left=3, bend left, "i_1"] \arrow[r, shift left=1, "i_2"] & S_k^{I_w}(\omega^s \times \omega^s) \arrow[l, shift left=1, "pr_1"] \arrow[l, shift left=3, bend left, "pr_2"] \end{tikzcd} \; \begin{matrix} \circlearrowleft U_p \\ \circlearrowleft AL \end{matrix}$$

$$e_1 \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} = e_2$$

We have $U_p = \underbrace{i_1 \circ T_p \circ pr_2}_{\text{rank} \le d_k^{ur}} - \underset{\uparrow \atop \text{easy}}{AL}$ $\qquad (AL(e_i) = p^{\deg e_i} \cdot e)$.

anti-diagonal,

Contributions $= p^{k-2}$

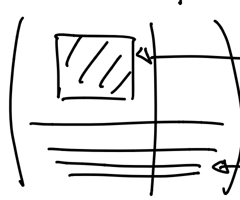
$$U^\dagger|_{w=w_k} = \left(\begin{array}{c|c} \text{rank} \le d_k^{ur} & 0 \\ \hline 0 & 0 \end{array}\right) - \left(\begin{array}{c|c} \begin{matrix} 0 & & p^{\deg e_1} \\ & \cdot^{\cdot^{\cdot}} & \\ p^{\deg e_{d_k^{I_w}}} & & 0 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right)$$

$(\text{i}\deg: k-2 \approx \deg e_{d_k^{I_w}})$

Naive bound: ith row lies in $p^{\deg e_i} \mathcal{O}\langle \frac{w}{p} \rangle$.

Write

$$\underset{\underset{M_{\infty\times\infty}(\mathcal{O}\langle \frac{w}{p}\rangle)}{\in}}{U^\dagger} = \left(\begin{array}{c|c} \begin{matrix} & & -p^{\deg e_1} \\ & -p^{\deg e_2} & \\ \cdot^{\cdot^{\cdot}} & & \end{matrix} & U^\dagger|_{w=w_k} \\ \hline 0 & U^\dagger|_{w=w_k} \end{array}\right) + T_k$$

Here $T_k =$ 

Can do elementary row operation so that at least $d_k^{I_w} - d_k^{ur}$ rows are div by $w - w_k$

all div by $w - w_k$.

Precise version $U^\dagger(\underline{\xi} \times \underline{\xi}) = L_k(\underline{\xi} \times \underline{\xi}) + T_k(\underline{\xi} \times \underline{\xi})$.

For simplicity, $\underline{\xi}, \underline{\xi} \subseteq \{1, \dots, d_k^{I_w}\}$, $d_k^{I_w} \le n \le \frac{1}{2} d_k^{I_w}$.

$$r_{\underline{\xi} \times \underline{\xi}} := \#\{\eta \in \underline{\xi} \mid \eta^{op} \in \underline{\xi}\}, \quad \text{rank } L_k = r_{\underline{\xi} \times \underline{\xi}}.$$

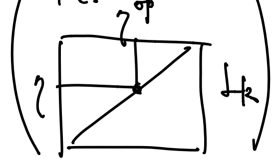
$$\text{Corank } U^\dagger(\underline{\xi}, \underline{\xi})|_{w=w_k} \ge \underbrace{n - d_k^{ur}}_{m_n(k)} - r_{\underline{\xi} \times \underline{\xi}}.$$

If $r_{\underline{\xi} \times \underline{\xi}} = 0$, i.e. $L_k(\underline{\xi} \times \underline{\xi}) = 0$ so that $\det U^\dagger(\underline{\xi} \times \underline{\xi})$ is divisible by $(w - w_k)^{m_n(k)}$

$\Rightarrow$ All $A_{k,i}^{(\underline{\xi} \times \underline{\xi})} = 0$. (nothing to prove.)

If $r_{\underline{\xi}\times\underline{\xi}} = 1$, in this case, $\det U^{\dagger}(\underline{\xi}\times\underline{\xi})$ is div by $(w-w_k)^{m_n(k)-1}$

i.e.



$\Rightarrow$ So suffices to look at $A^{(\underline{\xi},\underline{\xi})}_{k,m_n(k)-1}$.

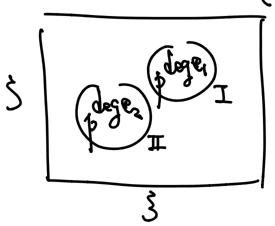
Write $U^{\dagger}(\underline{\xi}\times\underline{\xi}) = T_k(\underline{\xi}\times\underline{\xi}) + L_k(\underline{\xi}\times\underline{\xi})$.

$$\Rightarrow \det U^{\dagger}(\underline{\xi}\times\underline{\xi}) = \underbrace{\det T_k(\underline{\xi}\times\underline{\xi})}_{\text{div by } (w-w_k)^{m_n(k)}} \pm p^{\deg e_\eta}\cdot \underbrace{\det U^{\dagger}(\underline{\xi}\setminus\eta,\ \underline{\xi}\setminus\eta^{op})}_{\text{use "}A^{(\underline{\xi}\setminus\eta,\ \underline{\xi}\setminus\eta^{op})}_{k,m_n(k)-1}\text{"}}.$$

$m_{n-1}(k)$.

If $r_{\underline{\xi}\times\underline{\xi}} = 2$, say $\eta_1, \eta_2 \in \underline{\xi}$, $\eta_1^{op}, \eta_2^{op} \in \underline{\xi}$:

Write $T_k(\underline{\xi}\times\underline{\xi}, \eta_i) = U^{\dagger}(\underline{\xi}\times\underline{\xi}) + p^{\deg e_{\eta_i}}$ at (η_i, η_i^{op})-entry



$$\begin{cases} \det U^{\dagger}(\underline{\xi}\times\underline{\xi}) \text{ div by } (w-w_k)^{m_n(k)-2} \rightsquigarrow \text{need to consider } A^{(\underline{\xi},\underline{\xi})}_{k,m_n(k)-1},\ A^{(\underline{\xi},\underline{\xi})}_{k,m_n(k)-2}. \\ \det T_k(\underline{\xi}\times\underline{\xi}, \eta_i) \text{ div by } (w-w_k)^{m_n(k)-1} \\ \det T_k(\underline{\xi}\times\underline{\xi}) \text{ div by } (w-w_k)^{m_n(k)}. \end{cases}$$

items \ choices	neither	only $p^{\deg e_{\eta_1}}$	only $p^{\deg e_{\eta_2}}$	both
$\det U^{\dagger}(\underline{\xi}\times\underline{\xi})$	✓	✓	✓	✓
$\det T_k(\underline{\xi}\times\underline{\xi}, \eta_1)$	✓		✓	
$\det T_k(\underline{\xi}\times\underline{\xi}, \eta_2)$	✓	✓		
$\det T_k$	✓			
$p^{\deg e_{\eta_1}}\cdot \det U^{\dagger}(\underline{\xi}\setminus\eta_1, \underline{\xi}\setminus\eta_1)$		✓		✓
$p^{\deg e_{\eta_2}}\cdot \det U^{\dagger}(\underline{\xi}\setminus\eta_2, \underline{\xi}\setminus\eta_2)$			✓	✓
$p^{\deg e_{\eta_1}+\deg e_{\eta_2}} \det U^{\dagger}(\underline{\xi}\setminus\{\eta_1,\eta_2\}, \underline{\xi}\setminus\{\eta_1,\eta_2\})$				✓

$$\det U^{\dagger}(\underline{\xi}\times\underline{\xi}) = \underbrace{\det(T_k(\underline{\xi}\times\underline{\xi}, \eta_1))}_{\text{div by } (w-w_k)^{m_n(k)-1}} + p^{\deg e_{\eta_1}} \det U^{\dagger}(\underline{\xi}\setminus\eta_1,\ \underline{\xi}\setminus\eta_1^{op})$$

$\rightsquigarrow A^{(\underline{\xi}\times\underline{\xi})}_{k,m_n(k)-2}(w-w_k)^{m_n(k)-2}$.

* note also do a cofactor expansion of $\det U^{\dagger}(\underline{\xi}\setminus\eta_1,\ \underline{\xi}\setminus\eta_1^{op})$

$$\det U^{\dagger}(\underline{\xi}\backslash\eta_1, \underline{\xi}\backslash\eta_1^{op}) = \underbrace{\det T_k(\underline{\xi}\backslash\eta_1, \underline{\xi}\backslash\eta_1^{op})}_{\text{div by } (w-w_k)^{m_n(k)-1}} + p^{\deg e_2} \det U^{\dagger}(\underline{\xi}\backslash\eta, \underline{\xi}\backslash\eta^{op})$$

$$\underbrace{\det U^{\dagger}(\underline{\xi}\times\underline{\xi})}_{\text{div by } (w-w_k)^{m_n(k)-2}} = \underbrace{\det T_k}_{\text{div by } (w-w_k)^{m_n(k)}} + \sum_{i=1}^{2} p^{\deg e_i} \det U^{\dagger}(\underline{\xi}\backslash\eta_i, \underline{\xi}\backslash\eta_i^{op}) - p^{\deg e_1 + \deg e_2} \det U^{\dagger}(\underline{\xi}\backslash\eta, \underline{\xi}\backslash\eta^{op})$$

<u>Recall</u> $\det U^{\dagger}(\underline{\xi}\times\underline{\xi}) = \sum_{k'} g_{n,\hat{k}}(w)\left(A^{(\underline{\xi}\times\underline{\xi})}_{k',0} + \cdots\right) + h^{(\underline{\xi}\times\underline{\xi})}(w)\, g_n(w)$.

Put $$p^{\frac{1}{2}(\deg(\underline{\xi}) - \deg(\underline{\eta}))} \cdot \frac{\det U^{\dagger}(\underline{\xi}\times\underline{\xi})}{g_{n,\hat{k}}(w)/g_{n,\hat{k}}(w_k)} = \sum B^{(\underline{\xi}\times\underline{\xi})}_{k,i}(w-w_k)^i \quad \text{in } E[\![w-w_k]\!].$$

<u>NTS</u>: $v_p(B^{(\underline{\xi}\times\underline{\xi})}_{k,i}) \geq \Delta_{k,\frac{1}{2}d_k^{new}-i} - \frac{k-2}{2}(d_k^{Iw} - n)$, $i = 0, \dots, m_n(k)-1$.

<u>More generally</u>

$$p^{\frac{1}{2}(\deg\underline{\xi} - \deg\underline{\xi})} \cdot \frac{\sum_{\substack{\text{minors of } L_k \\ \text{of size } \ell}} \det(\text{minor}^{\ell\times\ell}) \cdot \det(U^{\dagger}(\text{compl}))}{g_{n-\ell,\hat{k}}(w)/g_{n-\ell,\hat{k}}(w_k)}$$

$$=: \sum B^{(\underline{\xi}\times\underline{\xi};\ell)}_{k,i}(w-w_k)^i.$$

By induction hypothesis (Step II)

$$v_p(B^{(\underline{\xi}\times\underline{\xi};\ell)}_{k,i}) \geq \Delta_{k,\frac{1}{2}d_k^{new}-i} - \frac{k-2}{2}\left(\frac{1}{2}d_k^{Iw} - n\right)$$

when $i \geq m_{n-\ell}(k)$ and $i \leq m_n(k)-1$, $\ell \neq 0$.

When $m = m_n(k)$,

$$\det U^{\dagger}(\underline{\xi}\times\underline{\xi}) = \det(T_k(\underline{\xi}\times\underline{\xi}, \eta_1)) + p^{\deg e_1} \det U^{\dagger}(\underline{\xi}\backslash\eta_1, \underline{\xi}\backslash\eta_1^{op})$$

$$\Rightarrow (g_{n,\hat{k}}(w)/g_{n,\hat{k}}(w_k)) \cdot B^{(\underline{\xi},\underline{\xi})}_{k,m-2}(w-w_k)^{m-2} = \frac{1}{2}\cdot(g_{n-1,\hat{k}}(w)/g_{n-1,\hat{k}}(w_k)) \cdot B^{(\underline{\xi},\underline{\xi};1)}_{k,m-2}(w-w_k)^{m-2}$$

$$= (g_{n-2,\hat{k}}(w)/g_{n-2,\hat{k}}(w_k)) \cdot B^{(\underline{\xi},\underline{\xi};2)}_{k,m-2}(w-w_k)^{m-2}.$$

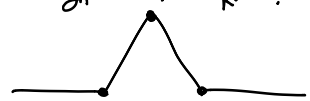
$$\overset{\text{mod } w-w_k}{\Longrightarrow} \quad B^{(\underline{\xi},\underline{\xi})}_{k,m-2} = \frac{1}{2}B^{(\underline{\xi},\underline{\xi};1)}_{k,m-2} = B^{(\underline{\xi},\underline{\xi};2)}_{k,m-2}.$$

$$B_{k,m-2} + B_{k,m-1}(w-w_k) \equiv \overbrace{\boxed{\frac{g_{n-1,\hat{k}}(w)/g_{n-1,\hat{k}}(w_k)}{g_{n,\hat{k}}(w)/g_{n,\hat{k}}(w_k)}}}^{\eta(w)} \cdot \left(B^{(1)}_{k,m-2} + B^{(1)}_{k,m-1}(w-w_k)\right)$$

$$+ \underbrace{\boxed{\frac{g_{n-2,\hat{k}}(w)/g_{n-2,\hat{k}}(w_k)}{g_{n,\hat{k}}(w)/g_{n,\hat{k}}(w_k)}}}_{\hat{\eta}(w)} \cdot \left(B^{(2)}_{k,m-2} + B^{(2)}_{k,m-1}(w-w_k)\right) \mod (w-w_k)^2$$

<u>Key observation</u> $(g_{n-1}/g_n)^2 \approx g_{n-2}/g_n$

$g_n = \prod (w - w_k)^{m_n(k)}$.



$$\eta(w) = 1 + \eta_1 \cdot (w - w_k) + \cdots$$

Compare coeff of $w - w_k$

$$\Rightarrow B_{k,m-1} \approx \underbrace{B^{(1)}_{k,m-1} - B^{(1)}_{k,m-1}}_{=0} + \underbrace{B^{(1)}_{k,m-2} \cdot \eta_1 - 2\eta_1 \cdot B^{(2)}_{k,m-2}}_{=0} .$$

Lecture 12: Completing the proof of local ghost conjecture

Fix $\underline{\zeta} = \{\zeta_1 < \cdots < \zeta_n\}$, $\underline{\xi} = \{\xi_1 < \cdots < \xi_n\}$.

$$\det U^{\dagger}(\underline{\zeta}\times\underline{\xi}) = \sum_{m_n(k)\neq 0} \underbrace{A_k^{(\underline{\zeta}\times\underline{\xi})}(w)}\, g_{n,\hat{k}}(w) + h_{\underline{\zeta}\times\underline{\xi}}(w)\, g_n(w).$$

$$A_k^{(\underline{\zeta}\times\underline{\xi})}(w) = \sum_{i=0}^{m_n(k)-1} A_{k,i}^{(\underline{\zeta}\times\underline{\xi})}(w-w_k)^i \in E[w], \quad h_{\underline{\zeta}\times\underline{\xi}} \in E\langle \tfrac{w}{p}\rangle.$$

Goal Prop 5.4 Assume that ghost zero w_k of $g_n(w)$,

we have $v_p(A_{k,i}^{(\underline{\zeta}\times\underline{\xi})}) \geq \Delta_{k,\frac{1}{2}d_k^{new}-i} - \Delta'_{k,\frac{1}{2}d_k^{new}-m_n(k)} + \frac{1}{2}(\deg(\underline{\zeta}) - \deg(\underline{\xi}))$ ①

for $i = 0, \dots, m_n(k)-1$.

Then (1) $h_{\underline{\zeta}\times\underline{\xi}}(w) \in p^{\frac{1}{2}(\deg\underline{\zeta} - \deg\underline{\xi})}$

(2) for every ghost zero w_{k_0} of $g_n(w)$, if we expand

$$\det(U^{\dagger}(\underline{\zeta}\times\underline{\xi}))/g_{n,\hat{k}_0}(w) = \sum_{i\geq 0} a_{k_0,i}^{(\underline{\zeta}\times\underline{\xi})}(w-w_{k_0})^i$$

then $v_p(a_{k_0,i}^{(\underline{\zeta}\times\underline{\xi})}) \geq \frac{1}{2}(\deg\underline{\zeta} - \deg\underline{\xi}) + \frac{1}{2}(d_{k_0}^{new}-i)^2 - (\frac{1}{2}d_{k_0}^{new} - m_n(k_0))^2$

$+ \Delta_{k_0,\frac{1}{2}d_k^{new}-i} - \Delta'_{k_0,\frac{1}{2}d_k^{new}-i}$

for $i = m_n(k_0), \dots, \frac{1}{2}d_{k_0}^{new}$.

Let $p \geq 11$, $2 \leq a \leq p-5$.

Can show (1) $\Rightarrow$ (2) by a direct computation.

In the rest of the lecture, we will focus on (1).

We have $\Delta_{k,l'} - \Delta'_{k,l} \geq l' - l$, $\forall\, l' > l \geq 0$.

$\overset{①}{\Longrightarrow}$ $v_p(A_{k,i}^{(\underline{\zeta}\times\underline{\xi})}) \geq m_n(k) - i + \frac{1}{2}(\deg\underline{\zeta} - \deg\underline{\xi})$

$\Rightarrow$ $A_k^{(\underline{\zeta}\times\underline{\xi})}(w)\cdot g_{n,\hat{k}}(w) \in p^{\frac{1}{2}(\deg\underline{\zeta} - \deg\underline{\xi}) + \deg g_n}$.

To show $h_{\underline{\zeta}\times\underline{\xi}}(w) \in p^{\frac{1}{2}(\deg\underline{\zeta} - \deg\underline{\xi})}\mathcal{O}\langle\frac{w}{p}\rangle$,

it suffices to show $\det U^{\dagger}(\underline{\zeta}\times\underline{\xi}) \in p^{\frac{1}{2}(\deg\underline{\zeta} - \deg\underline{\xi}) + \deg g_n(w)}\mathcal{O}\langle\frac{w}{p}\rangle$.

Modified Mahler basis

Recall that on $\mathcal{C}^0(\mathbb{Z}_p; \mathcal{O}[\![w]\!]^{(\varepsilon)})$, we have a right action of

$$M_1 = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathbb{Z}_p) \,\middle|\, \alpha\delta - \beta\gamma \neq 0,\ p \nmid \delta \right\}$$

$$\rightsquigarrow\ h\Big|_{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}(z) = \varepsilon(\bar{\alpha}/\bar{\delta}, \bar{\delta})\,(1+w)^{\frac{1}{p}\log\frac{\gamma z+\delta}{\omega(\bar{\delta})}} \cdot h\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)$$

$$(\alpha\delta - \beta\gamma = p^m \cdot d,\ d \in \mathbb{Z}_p^\times).$$

[LWX] Let $P = (P_{m,n})_{m,n\geq 0} \in M_\infty(\mathcal{O}[\![w]\!])$ be the matrix of the operator $\cdot\Big|_{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}$ on $\mathcal{C}^0(\mathbb{Z}_p, \mathcal{O}[\![w]\!]^{(\varepsilon)})$ w.r.t. Mahler basis $\left\{\binom{z}{n} : n \geq 0\right\}$.

<u>Then</u> $P_{m,n} \in (p,w)^{\max\{m-n,0\}}\mathcal{O}[\![w]\!] \subseteq p^{\max\{m-n,0\}}\mathcal{O}\langle\frac{w}{p}\rangle$ with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_1$

and $\in (p,w)^{\max\{m-\lfloor\frac{n}{p}\rfloor,0\}}\mathcal{O}[\![w]\!] \subseteq p^{\max\{m-\lfloor\frac{n}{p}\rfloor,0\}}\mathcal{O}\langle\frac{w}{p}\rangle$ with

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}.$$

In our application, the elements $\binom{z}{n}$ is not eigenvectors under the action of $\bar{T} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}$.

On $\mathrm{Hom}_{\mathrm{Iw}_p}(\tilde{H}, \mathcal{C}^0(\mathbb{Z}_p, \mathcal{O}[\![w]\!]^{(\varepsilon)}))$,

Consider $f(z) = f_1(z) = \frac{1}{p}(z^p - z) \quad \leftrightarrow \binom{z}{p}$

$$f_{i+1}(z) = f \circ f_i(z) = \frac{1}{p}(f_i(z)^p - f_i(z)), \quad i \geq 1.$$

$\forall n \geq 0$, write $n = n_0 + pn_1 + \cdots$ with $n_i \in \{0, \ldots, p-1\}$

and we define

$$\mathbb{m}_n(z) = z^{n_0} f_1(z)^{n_1} f_2(z)^{n_2} \cdots \in \mathcal{C}^0(\mathbb{Z}_p, \mathcal{O}[\![w]\!]^{(\varepsilon)}).$$

<u>Lemma</u> (1) $\forall n \geq 0$, deg of each monomial term in $\mathbb{m}_n(z)$ is convergent to n mod $p-1$.

In particular, $\bar{T}$ acts on $\mathbb{m}_n(z)$ via the character $\omega^n \times \omega^{-n}$.

(2) $\{\mathbb{m}_n(z) : n \geq 0\}$ is an orthonormal basis of $\mathcal{C}^0(\mathbb{Z}_p; \mathcal{O}[\![w]\!]^{(\varepsilon)})$ and is called the modified Mahler basis.

(3) If $P=(P_{m,n})_{m,n\geq 0}$ denote the matrix of $\cdot|_{\left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right)}$ on $C^0(\mathbb{Z}_p,\mathcal{O}[[w]]^{(\varepsilon)})$
w.r.t. the modified Mahler basis,
then we have the same estimation as before.

It admits an orthonormal basis $\mathbb{C}=\mathbb{C}^{(\varepsilon)}$

$$= \{e_1^*\cdot m_i(z),\ e_2^*\cdot m_j(z) : i\equiv s_\varepsilon \bmod p-1,\ j\equiv a+s_\varepsilon \bmod p-1\}$$

$$\varepsilon=\omega^{-s_\varepsilon}\times\omega^{a+s_\varepsilon},\ s_\varepsilon\in\{0,\cdots,p-2\}.$$

Define $\deg(e_i^*\cdot m_j(z))=\deg(m_j(z))$.
Write $\mathbb{C}^{(\varepsilon)}=\{f_1^{(\varepsilon)}, f_2^{(\varepsilon)},\cdots\}$ with increasing degrees.
If $e_n^{(\varepsilon)}=e_i^*\cdot z^j$, then $f_n^{(\varepsilon)}=e_i^*\cdot m_j(z)$ for $i=1,2$.
Define two matrices of the U_p-operator:

$U^{\dagger,(\varepsilon)} := (U^{\dagger,(\varepsilon)}_{e_m,e_n})_{m,n\geq 0}$ for $U_p : S^\dagger\to S^\dagger$ w.r.t. $\mathbb{B}^{(\varepsilon)}$ (power basis)

$U_{\mathbb{C}}=U^{(\varepsilon)}_{\mathbb{C}}=(U^{(\varepsilon)}_{\mathbb{C},f_m,f_n})_{m,n\geq 0}$ for $U_p : S^{(\varepsilon)}_{p\text{-adic}}\to S^{(\varepsilon)}_{p\text{-adic}}$
w.r.t. $\mathbb{C}^{(\varepsilon)}\to$ modified Mahler basis.

<u>Prop 3.18</u> We have $U^{(\varepsilon)}_{\mathbb{C},f_m,f_n}\in p^{\deg f_m^{(\varepsilon)}-\lfloor\frac{\deg f_n^{(\varepsilon)}}{p}\rfloor}\cdot\mathcal{O}\langle\frac{w}{p}\rangle\in\mathcal{O}[[w]]$.

<u>Rmk</u> On $E[z]^{\deg\leq k-2}$ we have two basis

$\{1,\cdots,z^{k-2}\}$ (corank thm)

$\{m_0(z),\cdots,m_{k-2}(z)\}$ (halo bound).

Let $Y=(Y_{m,n})_{m,n\geq 0}\in M_\infty(\mathbb{Q}_p)$ be the change of basis matrix b/w $\{m_n(z)\}$ & $\{z^n\}$.

$\rightsquigarrow\quad m_n(z)=\sum_{m\geq 0}Y_{m,n}z^m.$

Define $Y=(Y_{e_m,f_n})_{m,n\geq 0}$ be the change of basis matrix from $\mathbb{C}$ to $\mathbb{B}$.
Then $Y_{e_m,f_n}=Y_{\deg e_m,\deg f_n}$.

<u>Lemma</u> $Y \in M_\infty(\mathbb{Q}_p)$ is upper-triangular with diagonal entries $Y_{n,n} \in (n!)^{-1} \cdot \mathbb{Z}_p^\times$.

$Y_{m,n} = 0$ unless $m \equiv n \mod p-1$.

Moreover, for $m < n$,

$$v_p(Y_{m,n}) \geq -v_p(m!) + \left\lfloor \frac{m}{p} \right\rfloor - \left\lfloor \frac{n}{p} \right\rfloor - \left\lfloor \frac{m-n}{p^2-p} \right\rfloor.$$

$$v_p((Y^{-1})_{m,n}) \geq v_p(n!) + \left\lfloor \frac{m}{p} \right\rfloor - \left\lfloor \frac{n}{p} \right\rfloor - \left\lfloor \frac{m-n}{p^2-p} \right\rfloor.$$

$\rightsquigarrow$ $\mathcal{U}^\dagger = Y \cdot \mathcal{U}_C \cdot Y^{-1}$.

<u>Notation</u> For $m, n \geq 0$, we write $m = m_0 + pm_1 + \cdots$

and $n = n_0 + pn_1 + \cdots$ with $m_i, n_i \in \{0, \dots, p-1\}$.

We define $D(m,n) = \#\{i : n_{i+1} > m_i\}$.

<u>Example</u> $n = (p-1) + (p-1)p + \cdots + (p-1)p^k$, $k \geq 1$.

$m = n+1 = p^{k+1} \Rightarrow D(m,n) = k$.

<u>Prop</u> Let $P = (P_{m,n})_{m,n \geq 0}$ be the matrix of $\cdot \Big|_{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}$,

$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \begin{pmatrix} p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}^{\det \neq 0}$ on $C^0(\mathbb{Z}_p, \mathcal{O}[\![w]\!]^{(\varepsilon)})$

w.r.t. $\{m_n(z) : n \geq 0\}$.

Then $P_{m,n} \in p^{D(m,n) + m - \lfloor \frac{n}{p} \rfloor} \cdot \mathcal{O}\langle \frac{w}{p} \rangle$.

(<u>note</u> $v_p\left(\binom{m}{m - \lfloor \frac{n}{p} \rfloor}\right) \geq D(m,n)$.)

<u>Notation</u> Take $\underline{\eta} = \{\eta_1 < \cdots < \eta_n\}$, $\underline{\lambda} = \{\lambda_1 < \cdots < \lambda_n\}$.

For each λ_i, write $\deg e_{\lambda_i} = \lambda_{i,0} + p \cdot \lambda_{i,1} + \cdots$.

$\forall j \geq 0$, define $D^{(\varepsilon)}_{=0}(\underline{\lambda}, j) = \#\{i \mid \lambda_{i,j} = 0\}$.

We define $D^{(\varepsilon)}_{=0}(\underline{\eta}, j+1)$ similarly.

$\rightsquigarrow$ Also define

$$D(\underline{\lambda}, \underline{\eta}) = \sum_{j \geq 0} \left(\max\{D_{=0}(\underline{\lambda}, j) - D_{=0}(\underline{\eta}, j+1), 0\} \right)$$

tuple version of $D(\lambda, \eta) = \#\{i : \lambda_i < \eta_{i+1}\}$.

Cor 7.27 $v_p(\det(U_C^{(\varepsilon)}(\underline{\lambda}\times\underline{\eta}))) \geq D(\underline{\lambda},\underline{\eta}) + \underbrace{\sum_{i=1}^{n}\left(\deg e_{\lambda_i} - \lfloor\frac{\deg e_{\eta_i}}{p}\rfloor\right)}_{\text{LWX halo bound}}.$

Lemma Let $\underline{n} = \{1,\dots,n\}$. Then

(1) $D_{=0}(\underline{n},j) \leq D_{=0}(\underline{n},j+1)$, $\forall j \geq 0$.

(2) Write $\deg e_n = n_0 + pn_1 + \cdots$. If $n_{j+1} = p-1$ or $n_j = n_{j+1} = 0$ then $D_{=0}(\underline{n},j) = D_{=0}(\underline{n},j+1)$.

In particular, $D(\underline{n},\underline{n}) = 0$.

Let $\widetilde{D}_{=0}(\underline{n},j) = \{m \mid m \leq \deg e_n,\ m_j = 0,\ m \equiv s_\varepsilon \text{ or } a+s_\varepsilon \bmod p-1\}$.

$\Rightarrow D_{=0}(\underline{n},j) = \#\widetilde{D}_{=0}(\underline{n},j)$,

$\widetilde{D}_{=0}(\underline{n},j) \longrightarrow \widetilde{D}_{=0}(\underline{n},j+1)$,

$m = \sum_{i\geq 0} p^i m_i \xrightarrow[\text{digits}]{\text{switching } j\text{th } \& \ (j+1)\text{st}} m' = m_0 + pm_1 + \cdots + p^{j-1}m_{j-1} + p^{j+1}m_{j+1} + \cdots$.

From $(*)$, to prove $v_p(U^\dagger(\underline{\xi}\times\underline{\zeta})) \geq \cdots$

it suffices to show

$$v_p(\det(U_C(\underline{\lambda}\times\underline{\eta}))) \geq \deg g_n(w) + \frac{1}{2}(\deg\underline{\lambda} - \deg\underline{\eta}) + \sum_{i=1}^{n} v_p(\deg e_{\lambda_i}!) - v_p(\deg e_{\eta_i}!) \qquad (**)$$

Consider the special case

$\underline{\lambda} = \{1,2,\dots,n-1,n+1\}$, $\underline{\eta} = \underline{n}$.

$\delta = \deg g_n(w) - \sum_{i=1}^{n}\left(\deg e_i - \lfloor\frac{\deg e_i}{p}\rfloor\right) \in \{0,1\}$

& $\delta = 1$ only when $\deg e_{i+1} - \deg e_i = p-1-a$.

Take $r = \max\{v_p(i) : \deg e_n + 1 \leq i \leq \deg e_{n+1}\}$.

In case $(**)$ becomes

$$v_p(\det(U_C(\underline{\lambda}\times\underline{\eta}))) \geq \deg g_n(w) + \frac{1}{2}(\deg e_{n+1} - \deg e_n) + r.$$

By the refined halo bound,

$$\mathrm{LHS} \geq D(\underline{\lambda}, \underline{n}) + \sum_{i=1}^{n} \left(\deg e_{\lambda_i} - \left\lfloor \frac{\deg e_i}{p} \right\rfloor \right).$$

$\rightsquigarrow$ Enough to show

$$D(\underline{\lambda}, \underline{n}) + \frac{1}{2}(\underbrace{\deg e_{n+1} - \deg e_n}_{\in \{a,\, p-1-a\}}) \geq \delta + r.$$

We use $2 \leq a \leq p-5$

$$\Rightarrow \frac{1}{2}(\deg e_{n+1} - \deg e_n) \geq \delta + 1.$$

Key: $D(\underline{\lambda}, \underline{n}) \geq r - 1$.