

Divisors on Curves and Riemann-Roch

§1 The Riemann-Roch Theorem

$X = \text{curve} / k = \bar{k}$ (proj. irreduc., 1-dim'l, nonsing).

\Rightarrow canonical sheaf $\omega_{X/k} = \Omega_{X/k}$ since $\dim X = 1$.

Def'n canonical div = div def'd by any mero $f \in \omega_{X/k}$.
 (Sorry: can div is not canonical in any sense).

$\text{Div } X \rightarrow \mathbb{Z}$, X ell curve. $\Rightarrow \text{Div } X \rightarrow \mathcal{O}(X) \rightarrow \mathbb{Z}$.

$(P) \mapsto 1, \forall P \in X(\bar{k})$ b/c any prin div has deg 0.

Shorthand $l(D) := \dim_k H^0(X, L(D))$.

Itn (Riemann-Roch)

$$\exists g = g(X) \geq 0 \text{ s.t. } \forall D \in \text{Div}(X) \text{ & } K \in \text{CanDiv}(X),$$

$$[l(D) - l(K-D)] = \deg D + 1 - g.$$

Rmk Proved by Serre duality

But in application, it has no overt relationship to cohom.

$$\text{Cor } g = l(K) = \dim_k \Gamma(X, L(K)) = \dim_k \Gamma(X, \Omega_{X/k})$$

Proof. $D=0 \Rightarrow l(D)=1$ b/c glob reg func on curve = const
 (or indeed on any proj var)

$$\Rightarrow \underbrace{l(K)}_{\text{geometric genus of } K} = g. \quad \square$$

geometric genus of K .

$k=\mathbb{C} \Rightarrow$ this matches top genus of Riemann surface of X .

Cor $\deg K = 2g - 2$.

$$\text{Proof. } D = K \Rightarrow \deg K + 1 - g = \underbrace{l(K)}_g - \underbrace{l(0)}_1$$

$$\Rightarrow \deg K = 2g - 2. \quad \square$$

Cor $\deg D > 2g - 2$ or $(\deg D = 2g - 2 \text{ & } D \neq K)$, then

$$l(D) = \deg D + 1 - g \geq g - 1.$$

$$\text{Proof. } \deg D = 2g - 2 \Rightarrow \deg(K - D) = 0$$

$$\text{If } f \in K(x), f \neq 0 \text{ s.t. } (\underbrace{f}_1 + K - D \geq 0 \leftarrow \text{i.e. } f \in L(K - D)$$

$$\Rightarrow (\underbrace{f}_1 + K - D \sim 0 \quad \deg = 0$$

$$\Rightarrow l(K - D) \neq 0 \text{ only when } K \sim D.$$

$$\deg D > 2g - 2 \Rightarrow \deg(K - D) < 0$$

$$\Rightarrow (\underbrace{f}_1 + K - D < 0 \Rightarrow l(K - D) = 0. \quad \square$$

Cor $g \geq 2$. $\forall D \in \text{Div}(X)$, $\deg D \geq 2g - 1$, the complete linear system
ass to D defines a closed imm $D \hookrightarrow \mathbb{P}^r$.

§2 The Canonical (almost) Embedding

K canonical div \Rightarrow complete lin system

$\Rightarrow K \hookrightarrow \mathbb{P}^r$ canonical (almost) emb

i.e. (almost) always a closed imm.

a few exceptions in low genus.

Lemma P any pf, $P \in \text{Div}(x)$.

$$l(D) \leq l(D+P) \leq l(D) + 1.$$

$$(\Rightarrow \deg D = \deg D + 1).$$

Proof. We have $0 \rightarrow \mathcal{L}(D) \rightarrow \mathcal{L}(D + (P)) \rightarrow \Sigma \rightarrow 0$

\mathfrak{I} = ideal sheaf defining $P \in X$.

$$\Rightarrow l(D) \leq l(D+P) .$$

$$\Gamma(x, -) \circ \rightarrow \Gamma(x, \mathcal{L}(D)) \rightarrow \Gamma(x, \mathcal{L}(D + (p))) \rightarrow \Gamma(x, \xi)$$

↑
f-limit / k

$$\Rightarrow \mathcal{Q}(\mathcal{D} + \mathcal{P}) \leq \mathcal{L}(\mathcal{D}) + 1. \quad \square$$

Prop The canonical emb is a closed imm

$\Leftrightarrow X$ not hyper-elliptic.

Proof. $D = (P) + (Q)$, $P, Q \in X(k)$. (not necessarily $P \neq Q$).

To check $l(k-d) = l(k)-2 = g-2$. (P)+(Q).

$$\text{Riemann-Roch} \Rightarrow l(K-D) = l(D) - \deg D - 1 + g$$

$$= l(D) + g - 3$$

$\Leftrightarrow \exists$ embedding $X \hookrightarrow \mathbb{P}^1 \Leftrightarrow l(D) = 1, \forall D$ of $\deg 2$.

but a failure defines 2-to-1 map to \mathbb{P}^1

when X hyperelliptic.

Strictly also to check for D of deg 1.

But $\text{Im}(D) > 0 \Rightarrow \exists f \in k(x)$ w/ single pole

$\Leftrightarrow X \rightarrow \mathbb{P}^1$ deg 1. i.e. $X \cong \mathbb{P}^1$.

Birk canonical emb and variants of it } key tools to study
 (e.g. higher mult of k) } moduli spaces M_g
 of curves with given g .

$\Rightarrow M_g$ reps $\text{Fun}(\text{Sch}, \text{C}_\infty^k)$

↑
family of curves of genus g

it becomes replete in DM

↑
cat of Deligne-Mumford stacks.
ext'n of Sch.

(the same as orbifolds extends manifolds).

§3 The Riemann-Hurwitz Formula

$f: X \rightarrow Y$ fin sep of curves

(i.e. induced field ext'n $k(x)/k(y)$ sep).

Def'n The ramification div of f :

$$R = \sum_{p \in X(k)} \underset{\substack{\uparrow \\ \text{mod of K\"ahler diff.}}}{\text{length}}(\Omega_{X/Y})_p(P).$$

Prop $K_X \sim f^*K_Y + R$

Proof. (c.f. Hartshorne Prop IV.2.3) Note that

$$\circ \rightarrow \underbrace{f^*\Omega_{Y/k}}_{\substack{\text{check at } \eta \\ (\phi/k \text{ fin})/k(y) \text{ sep}}} \rightarrow \Omega_{X/k} \rightarrow \Omega_{X/Y} \rightarrow \circ \quad \text{exact.}$$

by properties of K\"ahler diff.

$(\phi/k \text{ fin})/k(x) \text{ sep}$

$$(1) \otimes \Omega_{X/k}^\vee \rightsquigarrow \circ \rightarrow f^*\Omega_{Y/k} \otimes \Omega_{X/k}^\vee \rightarrow \mathcal{O}_X \rightarrow \underbrace{\Omega_{X/Y}^\vee}_{\substack{\text{(fin)}}} \otimes \Omega_{X/k}^\vee \rightarrow \circ.$$

$(\phi/k \text{ fin})$ sup on fin many pts
 $(\Rightarrow \Omega_{X/Y}^\vee \cong \Omega_{X/Y} \otimes \Omega_{X/k}^\vee)$

$$\Rightarrow (f^*\Omega_{Y/k}) \otimes \Omega_{X/k}^\vee \cong \mathcal{O}_X/\Omega_{X/Y}. \\ \Leftrightarrow f^*K_Y - K_X \sim -R \quad \square$$

Riemann-Roch \Rightarrow Riemann-Hurwitz formula

Prop $2g(X) - 2 = (\deg f)(2g(Y) - 2) + \deg R$
 $(K_X \sim f^*K_Y + R)$
where $\deg f = [k(X): k(Y)]$.

Moreover, contribution of $P \in X(k)$ can be simply computed.

i.e. $Q = f(P)$, $t \in k(Y)$ s.t. $M_{Y,Q} = (t)$.

$\Rightarrow f^*(t)$ generates $M_{X,P}^e$, $e \geq 0$

call $e = e_P$ ramif index of P .

$\Rightarrow \text{length } (\Omega_{X/Y})_P \geq e_P - 1$.

"=" $\Leftrightarrow f$ tamely ram, i.e. $\text{char } k \nmid e_P$.

(Recall In alg number theory, L/k fin ext'n
of complete DVR. $k = \mathcal{O}_L/\pi_L$.
 L/k tamely ram $\Leftrightarrow \text{char } k \nmid [L:k]$)

When $k = \mathbb{C}$: R-H formula has a top meaning.

Euler char X additive,

- f unram $\Rightarrow \deg R = 0 \Rightarrow \chi(X) = \deg f \cdot \chi(Y)$.
- Otherwise $\Rightarrow \chi(X) = \deg f \cdot \chi(Y) - \left(\sum_P (e_P - 1) \right)$
those P s.t. $e_P > 1$
 $(\chi(p) = 1)$.

