Shimura varieties and modularity (3/3)

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<u>Outline</u> (1) Vanishing Conjs & results for Shimura varieties

(2) The geometry of the Hodge-Tate period morphism

(3) Complementary applications

$\rightsquigarrow$ Caraiani - Scholze

$\rightsquigarrow$ Koshikawa

(4) Applications (via local-global) compatibility

<u>Def</u> Let k = finite field of char ℓ.

<u>Setups</u> (1) Let $p \neq \ell$ prime, $L/\mathbb{Q}_p$ finite ext'n

$$\bar{\rho} : \Gamma_L = \mathrm{Gal}(\bar{L}/L) \longrightarrow GL_n(k)$$

a conti repr.

<u>Def'n</u> We say that $\bar{\rho}$ is <u>generic</u> if

- it's unramified
- the eigenvalues of $\bar{\rho}(\mathrm{Frob}_L)$ $\{\lambda_1, \dots, \lambda_n\}$ satisfy

$$\lambda_i / \lambda_j \neq (\# k_L)^{\pm 1}, \quad i \neq j.$$

(2) Let F = number field &

$$\bar{\rho} : \Gamma_F = \mathrm{Gal}(\bar{F}/F) \longrightarrow GL_n(k)$$

a conti repr.

<u>Def'n</u> We say that a prime $p \neq \ell$ is <u>decomposed generic</u> for $\bar{\rho}$ if

- p splits completely in F
- $\forall v | p$ prime of F, $\bar{\rho}|_{\Gamma_{F_v}}$ is generic.

We say that $\bar\rho$ is <u>decomposed generic</u> if
$\exists$ one (thus infinitely many) decomposed generic prime for $\bar\rho$.

<u>Rmks</u> (i) In (i), genericity guarantees that any lift of $\bar\rho$ to char 0 corresponds to irred / generic principal series repns of $GL_n(L)$.
Another way to view this:

$\bar\rho$ cannot be L-parameter of non-quasi-split G_b for $b \in B(GL_n, L)$.

(ii) Decomposed generic related to asking $\bar\rho$ to have large image.
<u>E.g.</u> if F tot real, $n=2$, $\bar\rho$ odd,
then $\mathrm{Im}(\bar\rho) \subset GL_2(k)$ non-solvable
$\Rightarrow \bar\rho$ decomposed generic.

Let $(B, *, V, \langle\cdot,\cdot\rangle)$ be a PEL datum of type A.
$\rightsquigarrow (\boxed{G}, X)$ Shimura datum
unitary similitude group
B = central simple alg w centre CM field F
$\rightsquigarrow \mathrm{Sh}_K / E$, $\quad K \subset G(\mathbb{A}^\infty)$ sufficiently small.

$$\begin{array}{l} \mathbb{T}^S(K) \curvearrowright H^*(\mathrm{Sh}_K, \mathbb{F}_\ell) \\ \ \cup \\ \ \mathfrak{m} \\ \ \downarrow \text{ by Scholze (as in S.W. Shin's talk).} \\ \bar\rho_{\mathfrak{m}} : \Gamma_F \longrightarrow GL_n(\bar{\mathbb{F}}_\ell) \end{array}$$

<u>Conj III'</u> (Koshikawa) If $\bar{\rho}_{\mathfrak{m}}$ is decomposed generic, then

(i) $H^i_c(Sh_K, \mathbb{F}_\ell)_{\mathfrak{m}} \neq 0$

$\Rightarrow i \leq d = \dim_E Sh_K$

(ii) $H^i(Sh_K, \mathbb{F}_\ell)_{\mathfrak{m}} \neq 0$

$\Rightarrow i \geq d = \dim_E Sh_K$

Poincaré duality

In particular, if either Sh_K is compact or $\mathfrak{m}$ is non-Eisenstein then $H^*_c(Sh_K, \mathbb{Z}_\ell)_{\mathfrak{m}} \simeq H^*(Sh_K, \mathbb{Z}_\ell)_{\mathfrak{m}}$ is concentrated in degree d & torsion-free.

<u>Rmks</u> (i) If we fix a prime p that splits completely in F & s.t. $K_v = GL_n(\mathcal{O}_{F,v})$, $\forall v | p$ prime of F, then can formulate version of Conj III' only using spherical Hecke algebras at $v | p$ & their systems of eigenvalues.

(ii) Previous results towards Conj III' of

Lan - Suh $\Big\}$ integral p-adic Hodge theory

Emerton - Gee

Shin, — supercuspidal cond.

Boyer — Harris-Taylor case, Conj III.

<u>Thm</u> (Caraiani - Scholze, Koshikawa)

If Sh_K is compact, or $B = F$, $V = F^{2m}$ & G quasi-split, then the conjecture is true.

Rmk (i) Both approaches use geometry of Hodge-Tate morphism.

(ii) M. Santos is working on full Conj III

(where PEL type A, p splits completely in F).

via Caraiani-Scholze semi-perversity

+ Koshikawa approach.

§2 The geometry of the Hodge-Tate period morphism

$$
\begin{tikzcd}
 & \mathcal{S}^{tor}_{K^p} \arrow[d] \arrow[ddr, bend left, "\pi^{tor}_{HT}"] & \\
\mathcal{S}_{K^p} \arrow[ur] \arrow[r] \arrow[dr, "\pi_{HT}"'] & \mathcal{S}^{min}_{K^p} \arrow[d, "\pi^{min}_{HT}"] & \\
 & \mathcal{F}\ell_{G,\mu} & \\
 & \mathcal{F}\ell \arrow[u, equal] &
\end{tikzcd}
$$

toroidal cpt

minimal (BB) cpt

For $? \in \{\emptyset, \text{tor}, \text{min}\}$, as diamonds,

$$\mathcal{S}^{?}_{K^p} = \varprojlim_{K_p} \mathcal{S}^{?}_{K^pK_p} .$$

(1) $\exists$ Newton stratification

$$\mathcal{F}\ell = \coprod_{b \in B(G,\mu)} \boxed{\mathcal{F}\ell^b} \leftarrow \text{loc closed strata}$$

(Viehmann) $\overline{\mathcal{F}\ell^b} = \coprod_{b' \geq b} \mathcal{F}\ell^{b'} \leftarrow$ accounting Bruhat order.

· $\mathcal{F}\ell^{ord} = \mathcal{F}\ell(\mathbb{Q}_p)$, $\mathcal{F}\ell^{basic}$ open.

$$
\begin{tikzcd}
\mathcal{F}\ell \simeq \mathrm{Gr}^{B^+_{dR}}_{G,\mu} \arrow[r] \arrow[dr] & \mathrm{Gr}^{B^+_{dR}}_{G} \arrow[d] \\
 & \mathrm{Bun}_G
\end{tikzcd}
$$

Newton stratification is

- pulled back from Bun_G.
- compatible under π_{HT} on rk 1 pts w/ Newton stratification on $\mathcal{S}_{K^p}$.

(2) Igusa varieties

$\bar{S}_{K^pK_p} / \bar{\mathbb{F}}_p$, K_p hyperspecial.

$b \in B(G,\mu) \rightsquigarrow$ can find p-div gp w/ G-structure $\mathbb{X}_b / \bar{\mathbb{F}}_p$ s.t.

- isocrystal w/ G-str is b
- compatible w/ μ.

Oort central leaf

$$\mathscr{C}_{\mathbb{X}_b} = \left\{ x \in \bar{S}^b_{K^pK_p} \;\middle|\; \begin{array}{l} \exists \text{ isom } A[p^\infty] \times_{\bar{S}^b_{K^pK_p}} \kappa(\bar{x}) \simeq \mathbb{X}_b \times_{\bar{\mathbb{F}}_p} \kappa(\bar{x}) \\ \text{compatible w/ G-structures} \end{array} \right\}$$

$$\begin{array}{l} \mathrm{Ig}^b / \bar{\mathbb{F}}_p \text{ perfect scheme} \\ \big\downarrow \text{ profinite} \\ \mathscr{C}_{\mathbb{X}_b} \end{array}$$

universal object which trivializes $A[p^\infty]|_{\mathscr{C}_{\mathbb{X}_b}}$.

$\exists$ isom $\quad A[p^\infty] \times_{\mathscr{C}_{\mathbb{X}_b}} \mathrm{Ig}^b \simeq \mathbb{X}_b \times_{\bar{\mathbb{F}}_p} \mathrm{Ig}^b$

compatible w/ G-str

$\mathrm{Ig}^{b,\text{pre-perf}} / \mathrm{Spa}(\breve{\mathbb{Q}}_p, \breve{\mathbb{Z}}_p)$.

(3) Rapoport-Zink spaces

$\mathcal{M}^b / \mathrm{Spf}\, \breve{\mathbb{Z}}_p$ formal sch w/ moduli theoretic description

$R = \breve{\mathbb{Z}}_p$-alg on which p nilpotent

$$R \longrightarrow \mathfrak{M}^b(R) = \left\{ (\mathscr{G}, \rho) : \begin{array}{l} \mathscr{G} \text{ p-div gp w/ G-str } / R \\ \rho : \mathbb{X}_b \times_{\bar{\mathbb{F}}_p} R/p \longrightarrow \mathscr{G} \times_R R/p \text{ quasi-isog} \end{array} \right\}$$

Set $\mathcal{M}^b = (\mathfrak{M}^b)^{ad}_\eta$

adic space assoc to $\mathfrak{M}^b$.

$\rightsquigarrow \mathcal{M}^b_\infty \xrightarrow{\pi_{HT}} \mathscr{F}\ell^b$ pre-perfectoid over $/ \mathbb{Q}_p(\zeta_{p^\infty})$

(4) Product formula

$\exists$ cartesian diagram of diamonds $(\forall b \in B(G, \mu))$

$$\begin{tikzcd} \mathcal{M}^b_\infty \times_{\mathrm{Spa}(\breve{\mathbb{Q}}_p, \breve{\mathbb{Z}}_p)} Ig^{b, \text{pre-perf}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{M}^b_\infty \arrow[d, "\pi^b_{HT}"] \\ \underbrace{(\mathscr{S}^\circ_{K^p})}^b \arrow[r, "\pi_{HT}"] & \mathscr{F}\ell^b \end{tikzcd}$$

"good reduction locus" pro-étale torsor for $\tilde{G}_b$.

- infinite-level version of Mantovian product formula
- Cohomological consequence

$$R\Gamma_c(Sh_{K^p}, \bar{\mathbb{F}}_\ell)$$

has a "filtration" by

$$R\Gamma(Ig^b, \bar{\mathbb{F}}_\ell)^{op} \otimes^{L}_{C_c(G_b(\mathbb{Q}_p))} R\Gamma_c(\mathcal{M}_\infty, \bar{\mathbb{F}}_\ell(d_b))[2d_b]$$

$\circlearrowleft$ $G_b(\mathbb{Q}_p)$

(i) $R\Gamma(Ig^b, \bar{\mathbb{F}}_\ell)^{op}_{\mathfrak{m}} = 0$ by Caraiani-Scholze

$\Leftarrow$ $\mathfrak{m}$ generic + G_b not quasi-split.

Shin: computed $[H(Ig^b, \bar{\mathbb{Q}}_\ell)_{\mathfrak{m}}]$

(ii) $R\Gamma_c(\mathcal{M}_\infty, \bar{\mathbb{F}}_\ell(d_b))_{\mathfrak{m}}[2d_b] = 0$ by Kashikawa.

$L / \mathbb{Q}_p$ fin ext'n, (G, b, μ) local Shimura datum

s.t. $G = \prod_{i \in I} GL_{n_i} \ / L$

$$K = \prod_{i \in I} GL_{n_i}(\mathcal{O}_L) \subset G(L).$$

$\rightsquigarrow \mathcal{M}_K$ RZ space / local Shimura variety.

Thm Assume $\ell \neq p$. If $\mathfrak{m}' \subset \mathcal{H}_K \curvearrowright H^i_c(\mathcal{M}_K, \mathbb{Z}_\ell)$,

s.t. $\rho_{\mathfrak{m}'}$ is generic & G_b non-quasi-split,

then $\forall i$, $H^i_c(\mathcal{M}_K, \mathbb{Z}_\ell)_{\mathfrak{m}} = 0$.

Pf idea $\quad H^i_c(\mathcal{M}_K, \overline{\mathbb{F}}_\ell)$ with actions $\mathcal{H}_K \curvearrowright$ and $\curvearrowleft G_b(L)$

- π sm irrep / $\overline{\mathbb{F}}_\ell$ of $G_b(L)$

  $\rightsquigarrow \varphi_\pi$ not generic whenever $G_b(L)$ not quasi-split.

  L-parameter constr'd by Fargue-Scholze.

- π irr subquotient of $H^i_c(\mathcal{M}_K, \overline{\mathbb{F}}_\ell)_{\mathfrak{m}}$

  $\rightsquigarrow \varphi_\pi = \rho_{\mathfrak{m}}$

  Fargues-Scholze excursion operators.