# Arithmetic geometry of unitary Shimura varieties at inert places

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## Lecture 1

### § Shimura varieties

$F/\mathbb{Q}$ imag quad, $\tau : F \hookrightarrow \mathbb{C}$ fixed $\rightsquigarrow$ $\tau^c$ conjugate.

$(V, (\cdot,\cdot))$ Herm space $/F$ of dim $N \geq 2$ of sign $(N-1, 1)$ at $\infty$ $(/\mathbb{C})$.

$G = U(V)$.

$h : \mathbb{C}^\times \longrightarrow G(\mathbb{R})$ Deligne hom via $z \longmapsto \begin{pmatrix} 1_{N-1} & \\ & z^c/z \end{pmatrix}$

$$G(\mathbb{R}) = U(N-1,1) = \left\{ g \in GL_N(\mathbb{C}) \;\middle|\; {}^t\bar{g} \begin{pmatrix} 1_{N-1} & \\ & -1 \end{pmatrix} g = \begin{pmatrix} 1_{N-1} & \\ & -1 \end{pmatrix} \right\}.$$

$\rightsquigarrow$ $G(\mathbb{R})$ - conj class

$$\mathcal{D}_{N-1} \cong \left\{ z \in \mathbb{C}^{N-1} \;\middle|\; \sum_{i=1}^{N-1} |z_i|^2 < 1 \right\}.$$

$\forall K \subseteq G(\mathbb{A}^\infty)$ cpt open,

$$Sh(K)(\mathbb{C}) := G(\mathbb{Q}) \backslash (\mathcal{D}_{N-1} \times G(\mathbb{A}^\infty)/K).$$

<u>Fact</u> $Sh(K)$ quasi-proj var $/F$. $(F \overset{\tau}{\hookrightarrow} \mathbb{C}$ fixed$)$.

<u>Issue</u> • Integral model

• $Sh(K)$ is not really of PEL type (can modify).

### § Moduli interpretation & integral model

$p$ prime inert in $F$, s.t. $V$ unram at $p$,

i.e. $\exists$ $\mathbb{Z}_{p^2}$ - lattice $\Lambda_p \subseteq V \otimes \mathbb{Q}_p$ w/ $\Lambda_p^\perp = \Lambda_p$

where $\Lambda_p^{\perp} := \{x \in V \otimes \mathbb{Q}_p \mid \langle x, y\rangle \in \mathbb{Z}_{p^2},\ \forall y \in \Lambda_p\}$

$\uparrow$ Herm pairing.

Let $K_p^{\circ} := \mathrm{Stab}_{G(\mathbb{Q}_p)}(\Lambda_p)$.

$\iota_p : \bar{\mathbb{Q}}_p \simeq \mathbb{C} \rightsquigarrow \tau : F \hookrightarrow \mathbb{Q}_{p^2} \subseteq \bar{\mathbb{Q}}_p$

$\mathbb{Q}_{p^2} \supseteq \mathbb{Z}_{p^2} \leftarrow$ work over it for int model.

**Q** Extend $Sh(K^p K_p^{\circ})$ to an quasi-proj sch over $\mathbb{Z}_{p^2}$?

**Def** $\forall$ $\mathbb{Z}_{p^2}$-sch $S$, $r, s \geq 0$, define

<u>unitary ab sch $/S$ of sign $(r,s)$</u> to be a triple $(A, i, \lambda)$

- $A/S$ ab sch of dim $r+s$
- $i : \mathcal{O}_F \longrightarrow \mathrm{End}_S(A)$ s.t.

$$\det(T - i(a) \mid \mathrm{Lie}(A/S)) = (T - \tau(a))^r \cdot (T - \tau^c(a))^s$$

- $\lambda : A \longrightarrow A^\vee$ quasi-polarization

s.t. $i(a^c)^\vee \circ \lambda = \lambda \circ i(a)$, $\forall a \in \mathcal{O}_F$.

Fix an elliptic curve $A_0 / \mathbb{Z}_{p^2}$ with CM by $\tau(\mathcal{O}_F)$.

$+\ \lambda_0 : A_0 \xrightarrow{\sim} A_0^\vee$

**Exc** $E_{\mathfrak{n}} := \mathbb{C}/\tau(\mathfrak{n})$, $\forall\ \mathfrak{n} \subseteq \mathcal{O}_F$ frac ideal.

can be def'd $/ \mathbb{Z}_{p^2}$ (by CM theory).

**Def** (Rapoport–Smithling–Zhang)

Define $\mathcal{M}(K^p) : \mathrm{LocNoe}\text{-}\mathbb{Z}_{p^2}\text{-}\mathrm{Sch} \longrightarrow \mathrm{Set}$

$$S \longmapsto \{(A, \lambda, \eta^p)\}/\sim$$

- $(A,\lambda)$ is an ab sch of sgn $(N-1,1)$ / $S$
  s.t. $\lambda[p^\infty] : A[p^\infty] \xrightarrow{\sim} A^\vee[p^\infty]$.
- $\eta^p$ is a $K^p$-level str on $A$:
  $\forall \bar{s} \hookrightarrow S$ geom pt,
  $\eta^p$ is a $\pi_1(S,\bar{s})$-invariant $K^p$-orbit of isom
  $$V \otimes \mathbb{A}^{\infty,p} \xrightarrow{\sim} \mathrm{Hom}^{\lambda_0,\lambda}_{F\otimes \mathbb{A}^{\infty,p}}(\underbrace{H_1^{\text{ét}}(A_{0,\bar{s}}, \mathbb{A}^{\infty,p})}, H_1^{\text{ét}}(A_{\bar{s}}, \mathbb{A}^{\infty,p}))$$
  $$\underbrace{H_1^{\text{ét}}(A_{0,\bar{s}}, \mathbb{A}^{\infty,p})} = \hat{V}^p(A,\bar{s}) = \Big(\prod_{\ell\neq p} T_\ell(A_{0,\bar{s}})\Big)\otimes \mathbb{A}^{\infty,p}$$
  where the Hermitian on RHS is
  $$(x,y) := \lambda_{0,*}^{-1}\circ x^\vee \circ \lambda_* \circ y \in \mathrm{End}_{F\otimes\mathbb{A}^{\infty,p}}(H_1^{\text{ét}}(A_{0,\bar{s}}, \mathbb{A}^{\infty,p}))$$
- $(A,\lambda,\eta^p) \sim (A',\lambda',\eta'^p)$ if
  $\exists \varphi : A \to A'$ prime-to-$\mathcal{O}_F$ quasi-isog
  compatible w/ $\lambda, \lambda'$ & $\eta^p, \eta'^p$.

__Fact__ When $K^p$ sufficiently small,
$\underline{M}(K^p)$ is rep'ble by a quasi-proj sm $\mathbb{Z}_{p^2}$-sch of dim $N-1$
s.t. $\underline{M}(K^p)\otimes_{\mathbb{Z}_{p^2}}\mathbb{Q}_{p^2} \cong Sh(K^pK_p^\circ)\otimes_{F,\tau}\mathbb{Q}_{p^2}$.

Let $M(K^p) := \underline{M}(K^p)\otimes_{\mathbb{Z}_{p^2}}\mathbb{F}_{p^2}$, $\bar{M}(K^p) = M(K^p)\otimes_{\mathbb{F}_{p^2}}\overline{\mathbb{F}}_p$.

__Q__ What is the geometry of $M(K^p)^{ss}$ ?

__Upshot__ (1) The set $\mathrm{Irr}(M(K^p)^{ss}) = \{$geom irred components$\}$
is parametrized by a discrete Shimura set.

(2) Each irred comp of $M(K^p)^{ss}$ is
is certain unitary Deligne-Lusztig var.

§ <u>DL variety</u>

$W$ $N$-dim v.s. / $\mathbb{F}_{p^2}$.

$\{\cdot,\cdot\} : W \times W \longrightarrow \mathbb{F}_{p^2}$ s.t.

$$\{ax, y\} = a^\sigma \{x, y\} = \{x, a^\sigma y\}, \ \forall a \in \mathbb{F}_{p^2}, \ x, y \in W$$

satisfying (1) $\{x, y\} = -\{y, x\}^\sigma$.

(2) $\dim(W^\dashv) = \delta_N = \begin{cases} 1, & N \text{ even} \\ 0, & N \text{ odd} \end{cases}$

where $W^\dashv := \{x \in W \mid \{y, x\} = 0, \ \forall y \in W\}$.

<u>Exc</u> Let $\xi \in \mathbb{F}_{p^2}$, ${}^\sigma\xi = -\xi$.

Prove that $\exists$ basis of $W$ s.t. $\{\cdot,\cdot\}$ is associated to the anti-Herm matrix $\begin{cases} \xi I_N, & \text{if } N \text{ odd} \\ \xi \begin{pmatrix} I_{N-1} & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } N \text{ even}. \end{cases}$

<u>Def</u> $h \in \mathbb{Z}$. Define

$$DL(N, h) : \mathbb{F}_{p^2}\text{-Sch} \longrightarrow \text{Set}$$

$$S \longmapsto \left\{ \begin{array}{l} H \subseteq W \otimes_{\mathbb{F}_{p^2}} \mathcal{O}_S \text{ subbundle} \\ \text{s.t. } H^\dashv \subseteq H \ \& \ \text{rank } H = h \end{array} \right\}.$$

<u>Prop</u> (1) If $2h < N + \delta_N$ or $h > N$, then $DL(N, h) = \phi$.

(2) If $N + \delta_N \leq 2h \leq 2N$, then $DL(N, h)$ is rep'ble

by a geom-conn proj sm var of dim $(2h-N-\delta_N)(N-h)$.

<u>Examples</u> (1) $DL(2r+1, 2r)$ is isom to Fermat surface $\subseteq \mathbb{P}^{2r}_{\mathbb{F}_{p^2}}$
def'd by $X_0^{p+1} + \cdots + X_{2r}^{p+1} = 0$

(2) $DL(2r, h) \cong DL(2r-1, h-1)$

$$H \longmapsto H / w^{-1}.$$

<u>Def</u> (1) $\mathcal{Y}(K^p) : \ Sch/\mathbb{F}_{p^2} \longrightarrow Set$

$$T \longmapsto \{(A^*, \lambda^*, \eta^{*,p})\}/\sim$$

- $(A^*, \lambda^*)$ unitary ab sch of sgn $(N, 0)$ & $\ker(\lambda^*[p^\infty])$ of rk $p^{2\delta_N}$.
- $\eta^{*,p}$ $K^p$-level str on $A^*$

$\rightsquigarrow$ $\mathcal{Y}(K^p)$ finite étale $/\mathbb{F}_{p^2}$.

(2) $B(K^p) : \ Sch/\mathbb{F}_{p^2} \longrightarrow Set$

$$T \longmapsto \{(A, \lambda, \eta^p ; A^*, \lambda^*, \eta^{*,p} ; \alpha)\}/\sim$$

- $(A, \lambda, \eta^p) \in M(K^p)(T)$ sgn $(N-1, 1)$
- $(A^*, \lambda^*, \eta^{*,p}) \in \mathcal{Y}(K^p)(T)$ sgn $(N, 0)$
- $\alpha : A \to A^*$ $O_F$-linear quasi-$p$-isog (i.e. $\exists\, c \in \mathbb{Z}_{(p)}^\times$ s.t. $c\alpha$ is an isog).

s.t. $\ker(\alpha[p^\infty]) \subseteq A[p]$,
compatible w/ $K^p$-levels
$\alpha^\vee \circ \lambda^* \circ \alpha = p\lambda$.

<u>Rmk</u> Sgn changing is a special phenomenon / char $p$ which cannot happen / char 0.

## § Basic correspondence

$$\begin{tikzcd} & B(K^p) \arrow[dl, "\pi"'] \arrow[dr, "\iota"] & \\ S(K^p) & & M(K^p) \end{tikzcd}$$

**Thm** (Wedhorn–Vollaard, LTXZZ)

(1) $\pi$ is proj sm of rel dim

$$\left\lceil \frac{N-1}{2} \right\rceil = \begin{cases} r, & N = 2r \\ r-1, & N = 2r+1. \end{cases}$$

(2) $\mathrm{Im}(\iota) = M(K^p)^{ss}$

(3) $\forall s \in S(K^p)(\overline{\mathbb{F}}_p)$, $\quad B_s := \pi^{-1}(s)$,

$\iota|_{B_s}$ is a locally closed imm

& $\qquad B_s \cong DL(N, \lfloor \frac{N+1}{2} \rfloor)$.

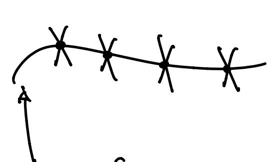
$$(A, \lambda, \eta^p; A^*, \lambda^*, \eta^{*p}; \alpha) \longmapsto H := (\check{\alpha}_{*,\tau})^{-1}(\omega_{A^\vee/S,\tau})$$
$$\subseteq H_1^{dR}(A^*/S)_\tau = W_s \otimes_{\mathbb{F}_{p^2}} \mathcal{O}_S$$

- $W_s := \left(H_1^{dR}(A^*/\overline{\mathbb{F}}_p)_\tau\right)^{F=V}$,
- $\check{\alpha} : A^* \longrightarrow A$ s.t. $\check{\alpha} \circ \alpha = p \cdot \mathrm{id}_A$

$\rightsquigarrow \check{\alpha}_{*,\tau} : H_1^{dR}(A^*/S)_\tau \longrightarrow H_1^{dR}(A/S)_\tau$
$$\cup\!|\;①$$
$$\omega_{A^\vee/S,\tau}.$$

**Illustration** $N = 3, 4$:

$$\overline{M}(K^p)^{ss} = \bigcup_{S(K^p)(\overline{\mathbb{F}}_p)} (\underbrace{\text{Fermat surfaces in } \mathbb{P}^3}_{= DL(3,2)})$$



$(p^3+1)$ supersingular locus $\updownarrow$ $DL(3,2)(\mathbb{F}_{p^2})$

Fermat surface

· every special pt is contained in
$p+1$ (resp. $p^2+1$) irred components for $N=3$ (resp. $N=4$).

## Lecture 2

Recall

$$\begin{tikzcd} & B(K^p) \arrow[dl, "\pi"'] \arrow[dr, "\iota"] & \\ \mathcal{S}(K^p) & & M(K^p) \end{tikzcd} \quad /\mathbb{F}_{p^2}.$$

$\pi$ sm proj w/ fibres DL var.

$\iota$ locally closed embedding.

Let $V^*$ herm space $/F$ of sgn $(N,0)$

$i : V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \xrightarrow{\sim} V^* \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ fixed isometry.

$\Lambda^*$ $\mathbb{Z}_{p^2}$-lattice of $V^* \otimes_{\mathbb{Q}} \mathbb{Q}_p$ satisfying

$$p\Lambda_p^* \subset (\Lambda_p^*)^\vee \quad \& \quad (\Lambda_p^*)^\vee / p\Lambda_p^* \text{ of length } s_N = \begin{cases} 0, & N \text{ odd} \\ 1, & N \text{ even}. \end{cases}$$

$K_p^* = \mathrm{Stab}_{U(V^*)(\mathbb{Q}_p)}(\Lambda_p^*)$

Define Shimura set

$$Sh(iK^pK_p^*) := U(V^*)(\mathbb{Q}) \backslash U(V^*)(\mathbb{A}^\infty) / iK^pK_p^* \quad \text{as a set.}$$

<u>Claim</u> (uniformization) $\mathcal{S}(K^p)(\overline{\mathbb{F}}_p) \xrightarrow{\sim} Sh(iK^pK_p^*)$, $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$-inv.

Fix $s^* = (A^*, \lambda^*, \eta^{p*}) \in \mathcal{S}(K^p)(\overline{\mathbb{F}}_p)$

$\rightsquigarrow V_{s^*} = \mathrm{Hom}_{O_F}(A_0 \otimes \overline{\mathbb{F}}_p, A^*)_{\mathbb{Q}}$ equipped w/ $(\cdot,\cdot)_{s^*}$

$$(x,y)_{s^*} = p^{-1}\lambda_0^{-1} \circ y^\vee \circ \lambda^* \circ x \in F.$$

**Exc** $V_{S^*}$ is isometric to $V^*$.

Moreover, $\Lambda_{S^*,p} := \mathrm{Hom}_{\underset{\mathbb{Z}_{p^2}}{\overset{\parallel}{O_F}}}(A_0[p^\infty], A^*[p^\infty])$

satisfies same conditions as $\Lambda_p^*$.

$$V_{S^*} \otimes_{\mathbb{Q}} \mathbb{A}^{p,\infty} = \mathrm{Hom}_{F\otimes_{\mathbb{Q}}\mathbb{A}^{p,\infty}}(H_1^{et}(A_0 \otimes \bar{\mathbb{F}}_p, \mathbb{A}^{p,\infty}), H_1^{et}(A^*, \mathbb{A}^{p,\infty})).$$
$$\uparrow \eta^{*,p}$$
$$V \otimes_{\mathbb{Q}} \mathbb{A}^{p,\infty}.$$

<u>Construction</u> Given isometry $\eta_{rat} : V_{S^*} \longrightarrow V^*$, want $g^p g_p \in U(V^*)(\mathbb{A}^\infty)$.

$$\begin{tikzcd} V^* \otimes_{\mathbb{Q}} \mathbb{A}^{p,\infty} \arrow[r, "i^{-1}"] \arrow[drr] & V \otimes_{\mathbb{Q}} \mathbb{A}^{p,\infty} \arrow[r, "\eta^{*,p}"] & V_{S^*} \otimes_{\mathbb{Q}} \mathbb{A}^{p,\infty} \arrow[d, "\eta_{rat}"] \\ & & V^* \otimes_{\mathbb{Q}} \mathbb{A}^{p,\infty} \end{tikzcd}$$

$g^p := \eta_{rat} \circ \eta^{*,p} \circ i^{-1}$

each map is an isometry of Herm spaces.

$\rightsquigarrow g^p \in U(V^*)(\mathbb{A}^{p,\infty}) / iK^p$.

Also, $g_p \in U(V^*)(\mathbb{Q}_p) / K_p^*$ s.t. $g_p \Lambda_p^* = \Lambda_{S^*,p}$

$\rightsquigarrow g^p \cdot g_p \in U(V^*)(\mathbb{A}^\infty) / iK^p K_p^*$

well-def'd whenever $\eta_{rat}$ is chosen.

(But $\eta_{rat}$ is not unique.)

<u>note</u> Two $\eta_{rat}$'s are differed by an $V^*$-isometry.

<u>Corrected constr'n</u> image of $g^p \cdot g_p$ along

$$U(V^*)(\mathbb{A}^\infty) / iK^p K_p^* \longrightarrow Sh(iK^p K_p^*)$$

$\Rightarrow$ image indep of $\eta_{rat}$ ($\Rightarrow$ depending only on $S^*$).

Back to basic corr:

$$\begin{tikzcd} & B(K^p) \arrow[dl, "\pi"'] \arrow[dr, "\imath"] & \\ \mathcal{S}(K^p) & & M(K^p) \end{tikzcd}$$

$\mathcal{S}(K^p) = Sh(iK^pK_p^*)$ viewed as a discrete sch / $\mathbb{F}_{p^2}$

And

$$\begin{tikzcd} & \underline{M}(K^p) \text{ generic fibre} & \\ M(K^p) \arrow[ur] & & \underline{M}(K^p) \otimes_{\mathbb{Z}_{p^2}} \mathbb{C} \overset{(V,\Lambda_p)}{=\!=} Sh(K^pK_p^\circ) \arrow[ul] \end{tikzcd}$$

<u>Note</u> All diagrams are equivariant w.r.t. away-from-p Hecke $\mathbb{Z}[K^p \backslash U(V)(\mathbb{A}^{p,\infty})/K^p]$.

* <u>$N = 2r+1$ odd</u>

$\dim M(K^p) = N-1 = 2r$,

$\dim B(K^p) = \frac{N-1}{2} = r$.

Want to study $H^*_?(\bar{M}(K^p), L)$, $? \in \{\phi, c\}$

$L$: coeff field, fin ext'n / $\mathbb{F}_\ell$ or $\mathbb{Q}_\ell$, $\ell \neq p$.

Consider $H_c^{2r}(\bar{M}(K^p), L(r)) \circlearrowleft \Gamma := Gal(\bar{\mathbb{F}}_p/\mathbb{F}_{p^2})$

<u>Tate conj</u> $H_c^{2r}(\bar{M}(K^p), L(r))^{\Gamma\text{-fin}}$ generated by alg cycles of codim $r$.

Let $\Sigma :=$ finite set of places not containing $p$,

$\forall v \notin \Sigma \cup \{p\}$, $K_v^p$ hyperspecial maximal.

$\mathbb{T}^\Sigma := \mathbb{Z}[K^{p,\Sigma}K_p^\circ \backslash U(V)(\mathbb{A}^{\infty,\Sigma})/K^{p,\Sigma}K_p^\circ]$.

hyperspecial max $\Rightarrow \mathbb{T}^{\Sigma}$ Comm ring.

Consider $\mathbb{T}^{\Sigma} \xrightarrow{\phi} L$ homo of rings.

$\cup\!\mathsf{I}$

$\mathfrak{m}_\phi := \ker \phi$.

$$\begin{tikzcd} H^0(\mathcal{S}(K^p), L) \arrow[r, "\pi^*"] \arrow[d, "="] & H^0(B(K^p), L) \arrow[r, "\iota_!"] & H_c^{2r}(M(K^p), L(r)) \arrow[d, "BC"] \\ L[Sh(:K^p K_\phi^*)] \arrow[rr, "Tate"] & & H_c^{2r}(\bar{M}(K^p), L(r))^{\Gamma} \end{tikzcd}$$

Recall: JL corr – Cohom from 0-dim to 2r-dim.

**Thm** (Xiao – Zhu) $L/\mathbb{Q}_\ell$ fin ext'n, $\phi$ automorphic.

Satake parameter $\phi$ at $p$ contains 1 exactly once.

$( (\alpha_1, \cdots, \alpha_r, 1, \alpha_1^{-1,c}, \cdots, \alpha_r^{-1,c})$

in $U(V)(\mathbb{Q}_p)$ of rk $2r+1$.

Then $\text{Tate}[\mathfrak{m}_\phi \cap \mathbb{T}^{\Sigma,p}]$ is injective.

* $N = 2r$ even

$\dim M(K^p) = 2r-1$

$\dim B(K^p) = r-1$.

Want $H^1(\mathbb{F}_{p^2}, H_c^{2r-1}(\bar{M}(K^p), L(r)))$.

$L/\mathbb{F}_\ell$ fin ext'n. Hoschchild – Serre spectral seq:

$$H^p(\mathbb{F}_{p^2}, H_c^q(\bar{M}(K^p), L(r))) \Rightarrow H^{2r}(M(K^p), L(r)),$$

$$p+q = 2r$$

Let $H^0(\mathcal{S}(K^p), L)^{\diamond} := \ker(H^0(\mathcal{S}(K^p), L) \to H^{2r}(\bar{M}(K^p), L(r)))$

$\rightsquigarrow H^0(\mathcal{S}(K^p), L)^{\diamond} \xrightarrow{\alpha} H^1(\mathbb{F}_{p^2}, H_c^{2r-1}(M(K^p), L(r)))$.

Now let $N=2$.

Conj $\quad \ell \nmid p(p^2-1)$.

$\phi$ corresponds to an abs irred (conj self-dual)

Galois rep $\rho_\phi : \mathrm{Gal}(\bar{F}/F) \longrightarrow GL_N(\ell)$.

If the Satake param of $\phi$ at $p$ contains $\{p, p^{-1}\}$ at most once,

then $\alpha_{m_\phi^p}$ is surjective.

where $m_\phi^p = m_\phi \cap \mathbb{T}^{\Sigma, p}$.

note multiplicity of $\{p, p^{-1}\}$ = dim of target of $\alpha_{m_\phi^p}$.

Thm Assume further

- $\ell \nmid p \prod_{k=1}^{N} (1-\underbrace{(-p)^k}_{})$

  (up to p-power) is the card of finite unitary grp.

- $\ell \geq 2(N+1)$, $\ell \notin \Sigma$
- $\phi$ cohom generic
- $\rho_\phi|_{\mathrm{Gal}(\bar{F}/F(\zeta_\ell))}$ is absolutely irred.
- $\exists$ prime $p'$ inert in $F$, $\ell \nmid p'(p'^2-1)$,

  $K_{p'}^p$ = stabilizer of an almost self-dual lattice

  s.t. $\rho_\phi$ is unram at $p'$, whose

  "Satake param" contains $\{p', p'^{-1}\}$ exactly once.

- $\forall v \in \Sigma \setminus \{p'\}$ every lifting of $\rho_\phi|_{D_v}$

  is minimally ramified

Then conj holds.

# Lecture 3

## § Ribet's level raising

$N \geq 1$, $S_2(N)$ wt 2, level $\Gamma_0(N)$ cusp forms.

$\ell$ prime, $\bar{\mathbb{Q}}_\ell \simeq \mathbb{C}$ fixed.

Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_2(N)$ eigenform $(a_1 = 1)$.

<u>Eichler-Shimura</u> $\exists$ Gal rep $\rho_{f,\ell} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\bar{\mathbb{Q}}_\ell)$

s.t. $\rho_{f,\ell}$ unram outside $N\ell$,

& $\forall\, p \nmid N\ell$, $\det(T - \rho_{f,\ell}(\mathrm{Frob}_p)) = T^2 - a_p T + p$.

$\rightsquigarrow$ $\bar{\rho}_{f,\ell} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\bar{\mathbb{F}}_\ell)$

well-def'd up to semisimplification.

<u>Thm 1</u> (Ribet) Assume $\bar{\rho}_{f,\ell}$ irred. Then

$\exists$ eigenform $g \in S_2(Np)$ new at $p$

s.t. $\bar{\rho}_{f,\ell} = \bar{\rho}_{g,\ell}$ iff $a_p \equiv \pm(p+1) \bmod \mathfrak{m}_{\bar{\mathbb{Z}}_\ell}$.

Here "$g$ new at $p$" means $g$ does not lie in the image

of $\delta_1 + \delta_p : S_2(N)^{\oplus 2} \longrightarrow S_2(Np)$

$(f_1, f_2) \longmapsto f_1(z) + f_2(pz)$.

<u>Exc</u> (Baby Ihara) Prove that $\delta_1 + \delta_p$ is injective.

<u>Rmk</u> Since $g$ is new at $p$,

local-global compatibility for $\rho_{g,\ell}$

$\Rightarrow$ $\rho_{g,\ell}$ tamely ramified at $p$

w/ a nontriv monodromy operator.

$\Rightarrow$ $\forall$ $\mathrm{Frob}_p \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \subseteq \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ Frob lift,

$\rho_{g,\ell}(\mathrm{Frob}_p) \underset{\mathrm{conj}}{\sim} \pm \begin{pmatrix} p & \\ & 1 \end{pmatrix}$.

## § Geometric formulation

$Y_0(N) = \Gamma_0(N)\backslash \mathfrak{h}$ affine sm curve / $\mathbb{Z}[\frac{1}{N}]$.

└ coarse moduli space of $(E, C_N)$

- $E$ elliptic curve
- $C_N \subseteq E[N]$ cyclic subgrp of order $N$.

$\rightsquigarrow$ $Y_0(N)_{\mathbb{F}_p} = \underbrace{Y_0(N)^{\mathrm{ord}}_{\mathbb{F}_p}}_{\text{open}} \amalg \underbrace{Y_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}}_{\text{closed}}$ as var / $\mathbb{F}_p$.

Prop (Deuring - Serre)

Let $B_{p,\infty}$ be the unique quaternion alg / $\mathbb{Q}$.

ramified exactly at $p, \infty$.

Fix $(B_{p,\infty} \otimes \mathbb{A}^{\infty,p})^\times \cong GL_2(\mathbb{A}^{\infty,p})$.

Put $K^p := \left\{ g \in GL_2(\hat{\mathbb{Z}}^{(p)}) \mid g \equiv \begin{pmatrix} * & * \\ & * \end{pmatrix} \bmod N \right\} \subseteq GL_2(\mathbb{A}^{p,\infty})$.

Then

$$Y_0(N)^{\mathrm{ss}}_{\mathbb{F}_p}(\bar{\mathbb{F}}_p) \cong Sh(B^\times_{p,\infty}, K^p \mathcal{O}^\times_{B_p})$$
$$= B^\times_{p,\infty} \backslash (B_{p,\infty} \otimes \mathbb{A}^\infty)^\times / K^p \mathcal{O}^\times_{B_p}$$

equivariant under the prime-to-p Hecke corr,

where $\mathcal{O}_{B_p} \leq (B_{p,\infty} \otimes \mathbb{Q}_p)^\times$ unique max order.

(This is the simplest case of basic corr.)

<u>Notation</u> $\mathbb{T}^{Np} := \mathbb{Z}[T_q \mid q \nmid Np]$.

$$f \rightsquigarrow \bar{\phi}_f : \mathbb{T}^{Np} \longrightarrow \bar{\mathbb{F}}_\ell, \quad T_q \longmapsto \bar{a}_q.$$

$$\mathbb{T}^{Np} \supseteq \mathfrak{m}_f := \ker(\bar{\phi}_f).$$

$$Y_0(N)^{\mathrm{ord}}_{\bar{\mathbb{F}}_p} \hookrightarrow Y_0(N)_{\bar{\mathbb{F}}_p} \hookleftarrow Y_0(N)^{\mathrm{ss}}_{\bar{\mathbb{F}}_p}$$

By formelism of étale cohom:

$$\rightsquigarrow \quad 0 \to H^1(Y_0(N)_{\bar{\mathbb{F}}_p}, \mathbb{F}_\ell) \to H^1(Y_0(N)^{\mathrm{ord}}_{\bar{\mathbb{F}}_p}, \mathbb{F}_\ell) \to H^2_{Y_0(N)^{\mathrm{ss}}_{\bar{\mathbb{F}}_p}}(Y_0(N)_{\bar{\mathbb{F}}_p}, \mathbb{F}_\ell) \to 0$$

$$\circlearrowleft \ \mathbb{T}^{Np} \times \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \qquad\qquad H^2_{Y_0(N)^{\mathrm{ss}}_{\bar{\mathbb{F}}_p}}(Y_0(N)_{\bar{\mathbb{F}}_p}, \mathbb{F}_\ell) \cong H^0(Y_0(N)^{\mathrm{ss}}_{\bar{\mathbb{F}}_p}, \mathbb{F}_\ell)(-1)$$

$$\rightsquigarrow \quad \Phi_{\mathfrak{m}_f} : H^0(Y_0(N)^{\mathrm{ss}}_{\bar{\mathbb{F}}_p}, \mathbb{F}_\ell)^{\mathrm{Gal}_{\mathbb{F}_{p^2}}}_{\mathfrak{m}_f} \longrightarrow H^1(\mathbb{F}_{p^2}, H^1(Y_0(N)_{\bar{\mathbb{F}}_p}, \mathbb{F}_\ell(1))_{\mathfrak{m}_f}).$$

Abel-Jacobi map for $r=1$.

<u>Thm 2</u> (Ribet when $\ell \nmid p^2-1$).

Assume $\ell \nmid p^2-1$ & $\bar{\rho}_{f,\ell}$ absolute irred.

Then $\Phi_{\mathfrak{m}_f}$ is surjective.

<u>Thm 2 $\Rightarrow$ Thm 1</u> $\quad H^1(Y_0(N)_{\bar{\mathbb{F}}_p}, \mathbb{F}_\ell(1))_{\mathfrak{m}_f} \twoheadrightarrow H^1(Y_0(N)_{\bar{\mathbb{F}}_p}, \mathbb{F}_\ell(1))/\mathfrak{m}_f \simeq \bar{\rho}_{f,\ell}^{\oplus r}$

$\rightsquigarrow$ Pass to $H^1(\mathbb{F}_{p^2}, \bar{\rho}_{f,\ell}) = T_{\mathfrak{m}_f}/(\bar{\rho}_{f,\ell}(\mathrm{Frob}_{p^2})-1)$

So $\quad H^1(\mathbb{F}_{p^2}, \bar{\rho}_{f,\ell}) \neq 0 \iff \bar{\rho}_{f,\ell}(\mathrm{Frob}_p) \sim \pm\begin{pmatrix} p & \\ & 1\end{pmatrix}$

$$\iff a_p \equiv \pm(p+1) \pmod{\mathfrak{m}_{\mathbb{T}_\ell}}.$$

If this is the case then $\Phi_{\mathfrak{m}_f}$ surj

$$\Rightarrow \quad H^0(Y_0(N)^{\mathrm{ss}}_{\bar{\mathbb{F}}_p}, \mathbb{F}_\ell)_{\mathfrak{m}_f} \neq 0$$

$$\quad\quad \mathbb{F}_\ell[\mathrm{Sh}(B^\times_{p,\infty}, K^p\mathcal{O}^\times_{B_p})]_{\mathfrak{m}_f}.$$

$\Rightarrow$ Ribet's level raising by JL transfer.

## § Ihara's lemma

$p \nmid N$.

$$\begin{array}{ccccccc} (E, C_N, C_p) & & & Y_0(Np) & & & (E, C_N, C_p) \\ \downarrow & & \swarrow^{\pi_1} & & \searrow^{\pi_2} & & \downarrow \\ (E, C_N) & & Y_0(N) & & Y_0(N) & & (E/C_p,\ C_N + C_p/C_p) \end{array} \quad / \mathbb{Z}[\tfrac{1}{N}].$$

**Thm** (Ihara) Assume $\bar{\rho}_{f,\ell}$ also irred. $\ell \nmid p(p^2-1)$.

Then $H^1(Y_0(N)_{\bar{\mathbb{Q}}}, \mathbb{F}_\ell)_{\mathfrak{m}_f}^{\oplus 2} \xrightarrow{\pi_1^* + \pi_2^*} H^1(Y_0(Np)_{\bar{\mathbb{Q}}}, \mathbb{F}_\ell)_{\mathfrak{m}_f}$ injective.

Or equivalently,

$H^1_c(Y_0(N)_{\bar{\mathbb{Q}}}, \mathbb{F}_\ell)_{\mathfrak{m}_f} \xrightarrow{(\pi_{1,!}, \pi_{2,!})} H^1_c(Y_0(N)_{\bar{\mathbb{Q}}}, \mathbb{F}_\ell)_{\mathfrak{m}_f}^{\oplus 2}$ surjective.

<u>Historical rmk</u> 1973 Ihara: Group theory.

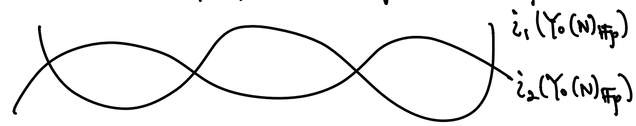
1994 Diamond-Taylor: p-adic comparison.

2021 Manning-Shotton: Galois deformation.

(valid for $\ell = p$ & curve / tot real fld).

## § From Ihara lem to Ribet's Thm 2

Deligne-Rapoport: $Y_0(Np)_{\mathbb{F}_p} = Y_0(N)_{\mathbb{F}_p} \cup Y_0(N)_{\mathbb{F}_p}$.




$i_1 : Y_0(N)_{\mathbb{F}_p} \longrightarrow Y_0(Np)_{\mathbb{F}_p}$

$(E, C_N) \longmapsto (E, C_N, \ker(\mathrm{Fr}_E))$

$\mathrm{Fr}_E : E \longrightarrow E^{(p)}$.

$i_2 : Y_0(N)_{\mathbb{F}_p} \longrightarrow Y_0(Np)_{\mathbb{F}_p}$ Verschiebung

$(E, C_N) \longmapsto (E^{(p)}, C_N^{(p)}, \ker(V_E))$

Recall: $E \xrightarrow{Fr_E} E^{(p)} \xrightarrow{V_E} E$, with the composite $E \xrightarrow{\times p} E$.

Have $\pi_1 \circ i_1 = id$, $\pi_2 \circ i_1 = Fr_{Y_0(N)_{\mathbb{F}_p}}$

$\pi_1 \circ i_2 = Fr_{Y_0(N)_{\mathbb{F}_p}}$, $\pi_2 \circ i_2 = id$.

$$\begin{tikzcd} H^1(Y_0(N)_{\overline{\mathbb{F}}_p})_{m_f}^{\oplus 2} \arrow[r, "\pi_1^* + \pi_2^*"] \arrow[dr, "\theta"'] & H^1(Y_0(Np)_{\overline{\mathbb{F}}_p})_{m_f} \arrow[d, "(i_1^*, i_2^*)"] \\ & H^1(Y_0(Np)_{\overline{\mathbb{F}}_p})_{m_f}^{\oplus 2}. \end{tikzcd}$$

given by $\theta = \begin{pmatrix} 1 & Fr_p \\ Fr_p & 1 \end{pmatrix}$

$\rightsquigarrow$ $\ker \theta \cong H^1(Y_0(N)_{\overline{\mathbb{F}}_p})^{Fr_p^2 = 1}$

$\quad \downarrow \Phi_{m_f}^*$ $\qquad \Rightarrow$ desired injectivity.

$\ker(i_1^*, i_2^*) = H^0(Y_0(N)^{ss}_{\overline{\mathbb{F}}_p})_{m_f}$.

<u>Upshot</u> $\Phi_{m_f}^*$ is dual to $\Phi_{m_f}$!

## § <u>Another formulation</u>

$K_p := GL_2(\mathbb{Z}_p) \supseteq Iw_p = \{ g \equiv \begin{pmatrix} * & * \\ & * \end{pmatrix} \bmod p \}$.

Let $Y_0(N) = Sh(GL_2, K^p K_p^0)$,

$Y_0(Np) = Sh(GL_2, K^p Iw_p)$.

$K_p^1 := \ker(GL_2(\mathbb{Z}_p) \longrightarrow GL_2(\mathbb{F}_p))$

$Y(N,p)_{\mathbb{Q}} = Sh(GL_2, K^p K_p^1) \longrightarrow Y_0(N)_{\mathbb{Q}}$

$GL_2(\mathbb{F}_p)$ - torsor.

$\forall$ $GL_2(\mathbb{F}_p)$-rep $\sigma \rightsquigarrow \mathcal{L}(\sigma)$ : $\mathbb{F}_\ell$-loc system on $Y_0(N)$.

So $\quad \mathrm{Ind}_{B(\mathbb{F}_p)}^{GL_2(\mathbb{F}_p)}(\mathbb{F}_\ell) \rightsquigarrow \pi_{1,*}(\mathbb{F}_\ell)$.

& $\quad \ell \nmid p(p^2-1) \Rightarrow \mathrm{Ind}_{B(\mathbb{F}_p)}^{GL_2(\mathbb{F}_p)}(\mathbb{F}_\ell) = \underset{1}{\mathbb{F}_\ell} \oplus \underset{p}{St_\ell}$

& $\quad \pi_{1,*} = \mathbb{F}_\ell \oplus \mathcal{L}(St_\ell)$.

**Prop** (Reformulated Ihara)

Assume $\bar\rho_{f,\ell}$ is abs irred, $\ell \nmid p(p^2-1)$.

Then the map

$$\begin{tikzcd}
 & H^1(Y_0(N)_{\bar{\mathbb{Q}}}, \pi_{1,*}\mathbb{F}_\ell) \arrow[d, phantom, "\parallel"] \\
H^1(Y_0(N)_{\bar{\mathbb{Q}}}, \mathcal{L}(St_\ell))_{m_f} \arrow[r, hook] \arrow[dr] & H^1(Y_0(Np)_{\bar{\mathbb{Q}}}, \mathbb{F}_\ell)_{m_f} \arrow[d, "\pi_{2,!}"] \\
 & H^1(Y_0(N)_{\bar{\mathbb{Q}}}, \mathbb{F}_\ell)_{m_f}
\end{tikzcd}$$

is surjective.

## Lecture 4

**Recall** Need $\alpha_{m_f}$ surjective.

This is implied by Ihara lem.

(analogue in unitary).

Assume $N = 2r$ even.

$$\mathrm{Ind}_{B(\mathbb{F}_p)}^{GL_2(\mathbb{F}_p)}(\mathbb{F}_\ell) = \mathbb{1} \oplus \underset{St_\ell}{\underset{\parallel}{\Omega}}, \quad \ell \nmid p(p^2-1).$$

Let $K_p^\circ$ = Stabilizer of $\Lambda_p^\circ$.
$\quad\ \cup$
$\quad K_p^1$ Siegel parahoric.

Take $L \underset{\text{Lagrangian}}{\subseteq} \Lambda_p^\circ \otimes \mathbb{F}_p \leftarrow$ 2r-dim herm space / $\mathbb{F}_{p^2}$.

$K_p^1 = \{ g \in K_p^0 \mid g \ (\mathrm{mod}\ p) \text{ preserving } L \}$.

$\Rightarrow \quad (K_p^0 : K_p^1) = (1+p)(1+p^3)\cdots(1+p^{2r-1})$.

Assume $\ell \nmid p \prod_{k=1}^{N} (1-(-p)^k)$.

$L / \mathbb{F}_\ell$ finite ext.

$\rightsquigarrow \quad L[K_p^1 \backslash K_p^0 / K_p^1]$ finite Hecke alg., comm,

generated by the char fct of

$$Q_1 = \{ \gamma \in K_p^0 : \gamma L \cap L \text{ corank } 1 \text{ in } L \}.$$

$\rightsquigarrow \quad L[K_p^1 \backslash K_p^0 / K_p^1] \circlearrowright L[K_p^1 \backslash K_p^0] \circlearrowleft L[K_p^0]$.

<u>Thm</u> $\exists$ unique decomposition

$$L[K_p^1 \backslash K_p^0] = \bigoplus_{k=0}^{r} \Omega_{N,L}^k$$

of $L[K_p^1 \backslash K_p^0 / K_p^1] \otimes_L L[K_p^0]$-bimods

s.t. $Q_1$ acts on $\Omega_{N,L}^k$ by

$$\ell_{N,p}^k := \frac{1}{p^2-1}\left(-(-p)^{N+1-k} - (-p)^k - p + 1\right).$$

Moreover,

(1) $\Omega_{N,L}^k$ is an irred $L[K_p^0]$-mod.

(2) $\Omega_{N,L}^0$ = triv rep

(3) $\Omega_{N,L}^1$ has dim $\dfrac{p(p^{2r-1}+1)}{p+1}$.

(So $\dim = p$ if $r=1 \longleftrightarrow \mathrm{St}_\ell$).

For deep reasons, only need to consider $\Omega_{N,L}^1$ & $\Omega_{N,L}^0$.

<u>Rmk</u> $\mathrm{Iso}(\Lambda_p^0 \otimes \mathbb{F}_p)$ sch $/ \mathbb{F}_{p^2}$, $\dim = 2r-2$,

$K_p^0 \circlearrowright$ parametrizing isotropic lines in $\Lambda_p^0 \otimes \mathbb{F}_p$.

& $H^{2r-2}(\mathrm{Iso}(\Lambda_p^\circ \otimes \mathbb{F}_p)_{\bar{\mathbb{F}}_p}, L) \simeq \Omega^0_{N,L} \oplus \Omega^1_{N,L}$.

$\circlearrowleft K_p^\circ \to \mathcal{U}(\Lambda_p^\circ \otimes \mathbb{F}_p)$.

$f : \mathrm{Sh}(K^p K_p^1) \xrightarrow{\text{forgetful}} \mathrm{Sh}(K^p K_p^\circ)$

finite étale of deg $[K_p^\circ : K_p^1]$.

"Atkin-Lehner involution"

$$\begin{tikzcd} \mathrm{Sh}(K^p K_p^1) \arrow[rr, "i^\eta"] \arrow[dr] & & \mathrm{Sh}(K^p K_p^1) \arrow[dl] \\ & \mathrm{Sh}(K^p K_p^\circ) & \end{tikzcd}.$$

Cor $f_*^\eta L = \bigoplus_{k=0}^{r} \Omega^{\eta,k}_{N,L}$ corresponding decomp.

Denote $\Omega^\eta_{N,L} := \Omega^{1,\eta}_{N,L}$.

$\rightsquigarrow$ Tate-Thompson local system.

Define

$$\begin{aligned} \nabla^\eta_{\mathfrak{m}_\phi^\Sigma} : H^{2r-1}(\mathrm{Sh}(K^p K_p^\circ)_{\bar{\mathbb{F}}}, \Omega^\eta_{N,L}) &\longrightarrow H^{2r-1}(\mathrm{Sh}(K^p K_p^\circ)_{\bar{\mathbb{F}}}, f_*^\eta L) \\ = H^{2r-1}(\mathrm{Sh}(K^p K_p^1)_{\bar{\mathbb{F}}}, L) &\xrightarrow{f_* \circ i_*} H^{2r-1}(\mathrm{Sh}(K^p K_p^\circ)_{\bar{\mathbb{F}}}, L) \end{aligned}$$

Conj (Ihara lemma)

$\ell \nmid p \prod_{k=1}^{N} (1-(-p)^k)$,

$\phi : \mathbb{T}^\Sigma \to L$ corresponding to an abs irred Gal repr

s.t. its Satake param contains $\{p, p^{-1}\}$ at most once.

Then $\nabla^\eta_{\mathfrak{m}_\phi^\Sigma}$ is surj.

**Thm** $\forall \phi : \mathbb{T}^{\Sigma} \to L$,

$$\theta^{?}_{m_\phi^p} \text{ surj} \Rightarrow \alpha_{m_\phi^p} \text{ surj}.$$

From now on, $K^p$ will be omitted.

$$\begin{tikzcd} & & & \underline{M} & \\ \mathcal{Y} & B \arrow[l] \arrow[r] & M \arrow[ur] & & M^? = Sh(K^pK_p^\circ) \arrow[ul] \end{tikzcd}$$

**Def** $\underline{P} / \mathbb{Z}_{p^2}$ parametrizing

$$(A^\Delta, \lambda^\Delta, \eta^{p\Delta}; A^\nabla, \lambda^\nabla, \eta^{p\nabla}; \psi^\Delta)$$

in which

- $(A^\Delta, \lambda^\Delta, \eta^{p\Delta}), (A^\nabla, \lambda^\nabla, \eta^{p\nabla}) \in \underline{M}$
- $\psi^\Delta : A^\Delta \to A^\nabla$ $O_F$-linear quasi-p-isog

  s.t. $p \cdot \lambda^\Delta = \psi^{\Delta\vee} \circ \lambda^\nabla \circ \psi^\Delta$

  & commutes w/ level str.

Also, $\psi^\Delta \rightsquigarrow \psi^\nabla : A^\nabla \to A^\Delta$ s.t. $\psi^\nabla \psi^\Delta = \psi^\Delta \psi^\nabla = p$.

Define $\underline{f} : \underline{P} \to \underline{M}$ remembering $(A^\Delta, \lambda^\Delta, \eta^{p\Delta})$

$\underline{i} : \underline{P} \to \underline{P}$ via

$$(A^\Delta, \lambda^\Delta, \eta^{p\Delta}; A^\nabla, \lambda^\nabla, \eta^{p\nabla}; \psi^\Delta) \mapsto (A^\nabla, \lambda^\nabla, \eta^{p\nabla}; A^\Delta, \lambda^\Delta, p\eta^{p\Delta}; \psi^\nabla)$$

$$\begin{tikzcd} P \arrow[rr, "i"] \arrow[dr, "f"'] & & P \arrow[dl, "f"] & \underline{P} \arrow[rr, "\underline{i}"] \arrow[dr, "\underline{f}"'] & & \underline{P} \arrow[dl, "\underline{f}"] & Sh(K^pK_p^1) \arrow[rr, "i^?"] \arrow[dr, "f^?"'] & & Sh(K^pK_p^1) \arrow[dl, "f^?"] \\ & M & & & \underline{M} & & & Sh(K^pK_p^\circ) & \end{tikzcd}$$

$\otimes_{\mathbb{Z}_{p^2}} \mathbb{F}_{p^2}$ (from the second diagram to the first), $\otimes_{\mathbb{Z}_{p^2}} \mathbb{C}$ (from the second diagram to the third)

$\kappa \supset \mathbb{F}_{p^2}$ alg closed, $(A, \lambda, \eta^p) \in M(\kappa)$

$$\rightsquigarrow \quad \underset{\substack{\circlearrowright \\ \mathbb{F}_{p^2}}}{H_1^{cris}(A/\mathcal{K})} = \underbrace{H_1^{dR}(A/\mathcal{K})^{\leftarrow}}_{\substack{\text{isotypic part} \\ \text{for } \mathbb{F}_{p^2} \subset \mathcal{K}}} \oplus \underbrace{H_1^{dR}(A/\mathcal{K})^{\rightarrow}}_{\substack{\text{isotypic part} \\ \text{for } \mathbb{F}_{p^2} \xrightarrow{c} \mathbb{F}_{p^2} \subset \mathcal{K}.}}$$

<u>Hodge fil'n</u> $\quad 0 \longrightarrow \omega_{A^\vee} \longrightarrow H_1^{dR}(A/\mathcal{K}) \longrightarrow \mathrm{Lie}_A \longrightarrow 0$

$\dim \omega^{\leftarrow}_{A^\vee} = 1.$

**Def** $P^\Delta$ reduced closed locus on which $\psi^\nabla_*(\omega^{\leftarrow}_{A^{\nabla,\vee}}) = 0$

$P^\nabla$ reduced closed locus on which $\psi^\Delta_*(\omega^{\leftarrow}_{A^{\Delta,\vee}}) = 0$.

$P^\dagger := P^\Delta \cap P^\nabla$.

**Prop** We have

(1) $\underline{P}$ quasi-proj strictly semistable / $\mathbb{Z}_{p^2}$ of rel dim $N-1$.

$P = P^\Delta \cap P^\nabla$.

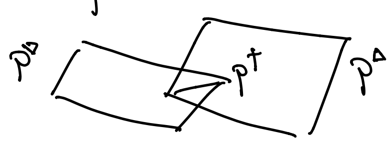
( <u>note</u> strictly semistable:

Zariski locally, $\underline{P}$ sm / $\mathbb{Z}_{p^2}[T_1, \cdots, T_s]/(T_1 \cdots T_s - p)$. )

(2) $\underline{f}$ is proper.

(3) $P^\Delta, P^\nabla, P^\dagger$ are sm of dim $N-1, N-1, N-2$, resp.

<u>Picture</u>




nearly smooth (sm = sm / $\mathbb{Z}_{p^2}$).

(4) $f^\Delta$ étale away from $P^\dagger$.

(5) $f^\Delta : P^\Delta \longrightarrow M$, $f^\nabla : P^\nabla \longrightarrow M$ small

(blowing of fibre is not too wild).

* <u>Ekedahl-Oort stratification on $M$</u>:

unitary Dieudonné space assoc to $x=(A,\lambda,\eta^p)\in M(\kappa)$

is $\mathcal{D}_x = H_1^{dR}(A/\kappa) \curvearrowleft \mathbb{F}_{p^2}, F, V$ w/ $FV=VF=0$.

$$= \mathcal{D}_x^{\leftarrow} \oplus \mathcal{D}_x^{\rightarrow}$$

$F, V$

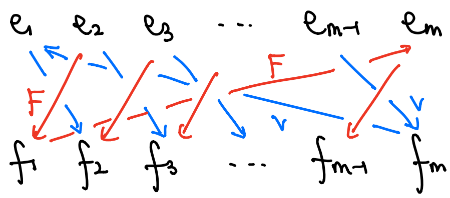
$\langle\cdot,\cdot\rangle : \mathcal{D}_x\times\mathcal{D}_x \longrightarrow \kappa$ non-deg

s.t. $\langle Fx,y\rangle = \langle x,Vy\rangle^p$, $\mathcal{D}_x^{\leftarrow}, \mathcal{D}_x^{\rightarrow}$ tot isom.

$\mathcal{S}$ supersingular unitary Dieudonné space

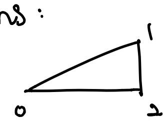
(assoc w/ ss elliptic curve).

$m\geq 1$, $\mathcal{B}(m) \longleftrightarrow$

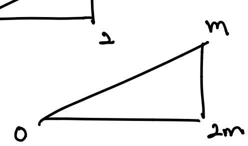



<u>Fact</u> About Newton polygons:

- $\mathcal{S}$ has slope $\frac{1}{2}$:

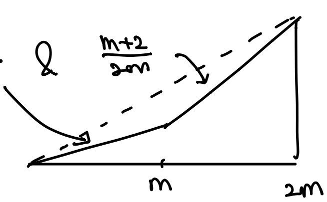



- $\mathcal{B}(m)$ has slope $\frac{1}{2}$:




if $m$ odd

or slopes $\frac{m-2}{2m}$ & $\frac{m+2}{2m}$




if $m$ even

<u>Prop</u> (Bültel-Wedhorn, two building blocks of $\mathcal{D}_x$).

$\forall x\in \bar{M}(\kappa)$, $\exists!\ 1\leq m_x\leq N$

s.t. $\mathcal{D}_x \simeq \mathcal{B}(m_x)\oplus \mathcal{S}^{\oplus N-m_x}$.

Let $M^{(m)}$ locally closed locus on which $m_x = m$.

$M^{[m]}$ Zariski closure of $M^{(m)}$.

$M^{]m[} := \underline{M} \setminus M^{[m]}$.

$M^{b} := 2(B) \subseteq M$ basic locus.

**Prop** (EO strata)

(1) $\forall\, 1 \le m \le N$,

$$M^{[m]} = \begin{cases} \bigcup\limits_{\substack{m' \ge m \\ m' \text{ even}}} M^{(m')} \cap \bigcup\limits_{m' \text{ odd}} M^{(m')}, & m \text{ even} \\ \bigcup\limits_{\substack{m' \le m \\ m' \text{ odd}}} M^{(m')}, & m \text{ odd} \end{cases}$$

(2) $M^{b} = \bigcup\limits_{m' \text{ odd}} M^{(m')}$

(3) $M^{(m)}$ pure dim $\begin{cases} \frac{m-1}{2}, & m \text{ odd} \\ N - \frac{m}{2}, & m \text{ even.} \end{cases}$

| $m$ | 2 | 4 | 6 | $\cdots$ | $2r$ | | $2r-1$ | $2r-3$ | $\cdots$ | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\dim M^{(m)}$ | $N-1$ | $N-2$ | $N-3$ | | $N-r$ | $>$ | $r-1$ | $r-2$ | $\cdots$ | 0 |
| $M^{(m)}$ | ordinary | | | | | | supersingular | | | superspecial. |

Denote $\underline{M}^{?}$, $? \in \{\,]m[,\ [m],\ (m),\ b\,\}$.

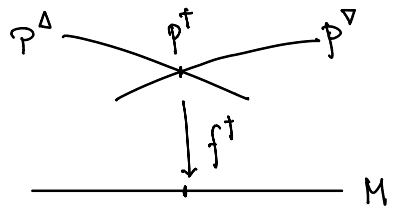
$\underline{P}^{?} = M^{?} \times_{M^{+}} P$

$\underline{f}^{?} = \underline{f}|_{\underline{P}^{?}}$, $f^{?} = f|_{P^{?}}$.

**Prop** (continued)

(6) $\underline{f}^{]2r-3[}$ is finite flat

(7) $x \in M^{b}$, $f^{+}(x)_{red}$ is DL var of dim $\frac{N-m_x-1}{2}$.




(8) image($f^\dagger$) disjoint with $M^{(2)}$.

$f^\dagger : P^\dagger \longrightarrow M \setminus M^{(2)}$ is semi-small

w/ rel strata $M^{(4)}, M^{(6)}$.

<u>Convention</u> $M^{(m)} = \phi$ if it does not make sense.

## <u>Lecture 5</u>

<u>Recall</u> $P = P^\Delta \cup P^\nabla$, $f^\Delta, f^\nabla$ small,

$f \downarrow$ $\quad f^\dagger = f^\nabla \cap f^\Delta : P^\dagger = P^\Delta \cap P^\nabla \to M \setminus M^{(2)}$ semi-small

$M$ $\quad M^{(2)}$ : ordinary locus.

### <u>§ Perverse sheaves</u>

$X/k$ alg var, pure dim $d$. $\ell \neq \operatorname{char} k$.

$L = \mathbb{Z}/\ell^n\mathbb{Z}$, $\mathbb{Q}_\ell$ $\rightsquigarrow D^b_c(X, L)$.

$\mathbb{D}_X : D^b_c(X, L) \longrightarrow D^b_c(X, L)$ Verdier duality.

Define ${}^pD^{\leq 0}(X, L) := \{ F \in D^b_c(X, L) \mid \dim \operatorname{supp} \mathcal{H}^{-i}(F) \leq i, \ \forall i \}$

${}^pD^{\geq 0}(X, L) := \{ F \in D^b_c(X, L) \mid \mathbb{D}_X(F) \in {}^pD^{\leq 0}(X, L) \}$.

$\rightsquigarrow \operatorname{Perv}(X, L) := \{ K \in D^b_c(X, L) \mid K[d] \in {}^pD^{\leq 0} \cap {}^pD^{\geq 0} \}$.

<u>Fact</u> $\operatorname{Perv}(X, L)$ ab cat.

<u>Example</u> $X$ smooth. $\forall$ loc system $\mathcal{L} \in \mathrm{Perv}(X, L)$,

$j : U \hookrightarrow X$ open immersion

- $j_! : D_c^b(U, L) \longrightarrow D_c^b(X, L)$ right t-exact

  $\Rightarrow j_! : {}^pD^{\leq 0}(U, L) \longrightarrow {}^pD^{\leq 0}(X, L)$

- $Rj_* : D_c^b(U, L) \longrightarrow D_c^b(X, L)$ left t-exact

  $\Rightarrow Rj_* : {}^pD^{\geq 0}(U, L) \longrightarrow {}^pD^{\geq 0}(X, L)$.

- $\forall \mathcal{F} \in \mathrm{Perv}(U, L)$, define

  $$j_{!*}\mathcal{F} := \mathrm{Im}(j_!\mathcal{F} \to Rj_*\mathcal{F}) \in \mathrm{Perv}(X, L).$$

Assume $U$ sm, $\mathcal{L}$ loc sys on $U$.

then $IC(X, \mathcal{L}) = j_{!*}\mathcal{L}$.

<u>Example</u> $X$ curve. $U \subseteq X$ open sm.

$$\begin{array}{c} \mathcal{L} \\ | \\ U \overset{j}{\hookrightarrow} X \end{array} \quad \text{gives } IC(X, \mathcal{L}) = j_*\mathcal{L}.$$

$\forall \bar{x} \in X$ geom pt, $IC(X, \mathcal{L})_{\bar{x}} = \mathcal{L}_{\bar{\eta}}^{I_x}$,

$\bar{\eta} \in \mathrm{Spec}(\hat{\mathcal{O}}_{X,x})$ geom generic pt.

<u>§ (Semi-)small maps</u>

$f : Y \to X$ proper surjective.

Assume $\exists$ a stratification of $f$ s.t.

$X = \coprod_{i \in I} X_i$,
- $X_i \subset X$ locally closed
- $\bar{X}_i$ disjoint union of strata
- $f^{-1}(X_i) \to X_i$ étale w/ locally trivial fibration.

**Def** (1) $f$ is <u>semismall</u>, if $\dim(Y \times_X Y) \le \dim Y$

i.e. $2 \dim f^{-1}(X_i) - \dim(X_i) \le \dim Y$.

Those $X_i$'s with equality are called relevant strata.

(2) $f$ is <u>small</u> if $f$ is semismall

& $\forall X_i$ non-generic strata,

$$2 \dim f^{-1}(X_i) - \dim(X_i) < \dim Y.$$

($X_i$ non-generic strata $\Leftrightarrow$ $\bar{X}_i$ is not an irred comp of $X$.)

<u>Rmk</u> $f$ semismall $\Rightarrow$ $f$ generically finite & $\dim X = \dim Y$.

<u>Prop</u> Assume $f: Y \to X$ semismall, $Y$ sm of dim $d$.

(1) $Rf_* \underline{L} \in \mathrm{Perv}(X, L)$.

(2) If $f$ small, then $Rf_* \underline{L} = IC(X, \mathcal{L}_U)$

$U$ some open sm s.t. $\mathcal{L}_U := f_* \underline{L}|_U$ is a loc sys on $U$.

<u>Back to Shimura vars</u>

$f^\eta: Sh(K^p K'_p) \longrightarrow Sh(K^p K^\circ_p)$, $\quad \Omega^\eta_{N,L} \subset f^\eta_* L$ direct summand.

Siegel parabolic $K'_p \subset K^\circ_p$.

<u>§ Hecke correspondence on $\underline{P}$</u>

<u>Def</u> $\underline{N}: Sch/\mathbb{Z}_{p^2} \longrightarrow Sets$

$S \longmapsto (A, \lambda, \eta^p, H)$

- $(A, \lambda, \eta^p) \in \underline{M}(S)$
- $H \subseteq A[p]$ $O_F$-stable subgrp,

loc free of fin type of rk $p^{2(r-1)}$,

s.t. • $H$ is isotropic under $\lambda$-Weil pairing on $A[p]$

$$A[p] \times A[p] \longrightarrow \mu_p,$$

• $\pi : A \to A' := A/H$, $\lambda' : A' \to A'^{\vee}$

s.t. $p\lambda = \pi^{\vee} \circ \lambda' \circ \pi$

where $(A', \lambda')$ unitary ab sch / $\mathbb{Q}_F$ of sgn $(N-1, 1)$.

<u>Def</u> $\underline{R} : Sch/\mathbb{Z}_{p^2} \longrightarrow Set$

$$S \longmapsto (\underbrace{A^{\Delta}, \lambda^{\Delta}, \eta^{p\Delta}; A^{\nabla}, \lambda^{\nabla}, \eta^{p\nabla}; \psi^{\Delta}}_{\in \underline{P}(S)}; H)$$

• $(A^{\Delta}, \lambda^{\Delta}, \eta^{p\Delta}; H) \in \underline{N}(S)$

• $H \subseteq \ker(\psi^{\Delta})[p]$.

<u>Def</u> $\underline{Q} := \underline{R} \times_{\underline{N}} \underline{B}$

$g_< := g_0 \circ h_<$ , $g_> := g_0 \circ h_>$

$$\begin{tikzcd} & \underline{Q} \arrow[dl, "h_<"'] \arrow[dr, "h_>"] \arrow[d, phantom, "\Diamond"] & & & & & \underline{Q} \arrow[dl, "g_<"'] \arrow[dr, "g_>"] \arrow[d, phantom, "\Diamond"] & \\ \underline{R} \arrow[dr, "g_1"'] & {} & \underline{B} \arrow[dl, "g_1"'] \arrow[dr, "g_0"] & & \rightsquigarrow & \underline{P} \arrow[dr, "f"'] & {} & \underline{P} \arrow[dl, "f"] \\ & \underline{N} \arrow[dr, "f"'] & & \underline{P} \arrow[dl, "f"] & & & \underline{M} & \\ & & \underline{M} & & & & & \end{tikzcd}$$

$\underline{Q} := \{(A, A^{\triangleleft}, A^{\triangleright}, \psi^{\triangleleft}, \psi^{\triangleright}; H)\}$

$\psi^{\triangleleft} : A \longrightarrow A^{\triangleleft}$, $\psi^{\triangleright} : A \longrightarrow A^{\triangleright}$.

$H \subseteq \ker(\psi^{\triangleleft})[p] \cap \ker(\psi^{\triangleright})[p]$.

<u>Construction</u> Given diagram w/ $c_2$ finite flat:

$$\begin{tikzcd} & C \arrow[dl, "c_1"'] \arrow[d] \arrow[dr, "c_2"] & \\ X_1 \arrow[r, "f_1"'] & S & X_2 \arrow[l, "f_2"] \end{tikzcd}$$

$$Rf_{1*}L \xrightarrow{adj} Rf_{1*}Rc_{1*}(c_1^*L) \xrightarrow{\sim} Rf_{2*}Rc_{2*}L \xrightarrow{tr\, c_2} Rf_{2*}L.$$

Applying to $Q^?$, we get

$$q^? : f_*^? L \longrightarrow f_*^? L.$$

$\forall\, 0 \le k \le N+1$, $\Omega^?_{N,L} := \Omega^{?,1}_{N,L}$, assuming $\ell \nmid p \cdot \prod_{k=1}^{N}(1-(-p)^k)$.

$$\Omega^{?,k}_{N,L} := \ker\left(q^? - \left(\ell^k_{N,p} + \frac{p^{2r}-1}{p^2-1}\right) \middle| f_*^? L\right)$$

$$\rightsquigarrow \quad f_*^? L = \bigoplus_{k=0}^{r} \Omega^{?,k}_{N,L}.$$

<u>§ Nearby cycles</u>

$$\bar{P} \xrightarrow{\bar{i}} P \otimes_{\mathbb{Z}_{p^2}} \bar{\mathbb{Z}}_p \xleftarrow{\bar{j}} P^{\bar{\eta}}$$

<u>Def</u> $R\Psi_p(L) := \bar{i}^* R\bar{j}_*(L)$

$\circlearrowleft$ $I_{\mathbb{Q}_{p^2}} \twoheadrightarrow I^{tame}_{\mathbb{Q}_{p^2}} \cong \prod_{q \ne p} \mathbb{Z}_q(1) \ni \xi$ top generator

$$\Rightarrow (\xi - 1)^2 R\Psi_p(L) = 0.$$

Gabber: $R\Psi_p(L) \in \mathrm{Perv}(\bar{P}, L) \subseteq D^b_c(\bar{P}, L)$.

Monodromy filtration:

$$0 \subseteq \underset{\substack{\| \\ \mathrm{Im}(\xi-1)}}{F_{-1}R\Psi_p(L)} \subseteq \underset{\substack{\| \\ \ker(\xi-1)}}{F_0 R\Psi_p(L)} \subseteq F_1 R\Psi_p(L) = R\Psi_p(L)$$

$gr^F_{-1} \cong p^\dagger_* L[-1]$ $\qquad p^? : P^? \hookrightarrow P.$

$gr^F_0 \cong \bar{p}^\Delta_* L \oplus \bar{p}^\nabla_* L$

$gr^F_1 \cong p^\dagger_* L(-1)[-1].$

$f : \underline{P} \longrightarrow \underline{M}$ proper.

$$Rf_* R\psi_p(L) \cong R\psi_p(f_*^{?} L).$$

$\rightsquigarrow$ $f^{\circ}, f^{\Delta}, f^{*}$ semi-small.

$$\Rightarrow \quad Rf_* F_i R\psi_p(L) \in \mathrm{Perv}(\bar{M}, L)$$

$$\quad \| \ F_i R\psi_p(f_*^{?} L)$$

This is monodromy fil'n.

$$F_i R\psi_p(\Omega_{N,L}^{?,k}) \subseteq F_i R\psi_p(f_*^{?} L) = Rf_*(F_i R\psi_p(L)).$$

$$gr_i^F R\psi_p(f_*^{?} L) = \begin{cases} Rf_*^{\dagger} L[-1], & i=-1 \\ Rf_*^{\Delta} L \oplus Rf_*^{\nabla} L, & i=0 \\ Rf^{\dagger} L(-1)[-1], & i=1 \end{cases}$$

## § Reduction of Ihara's lemma

Atkin-Lehner inv $\ i : \underline{P} \longrightarrow \underline{P}$

$$(A^{\Delta}, A^{\nabla}, \psi^{\Delta}) \longmapsto (A^{\nabla}, A^{\Delta}, \psi^{\nabla}). \quad \psi^{\nabla} \circ \psi^{\Delta} = p.$$

$$\begin{tikzcd}
H^{2r-1}(Sh(K^p K_p^{\circ}), \Omega_{N,L}^{?}(r)) \arrow[r, hook] \arrow[d, "="'] \arrow[ddr, "\vartheta^{?}"] & H^{2r-1}(Sh(K^p K_p'), L(r)) \arrow[d, "f_! \circ i_!"] \\
H^{2r-1}(\bar{M}, R\psi(\Omega_{N,L}^{?})(r)) \arrow[d] & H^{2r-1}(Sh(K^p K_p^{\circ}), L(r)) \arrow[d, "="] \\
H^{2r-1}(\bar{M}, F_{\geq 0} R\psi(\Omega_{N,L}^{?}(r))) \arrow[r, "\xi"] & H^{2r-1}(\bar{M}, \underbrace{R\psi L(r)}_{\| \ L(r)})
\end{tikzcd}$$

quotient of $F_{-1}$. $\qquad$ $L(r)$ by smoothness.

<u>Ihara</u> $\vartheta_{\phi}^{?}$ is surjective. (Can extend surj to AJ map).

$$\begin{tikzcd}
H^{2r-1}(\bar{M}, gr_0^F R\psi(\Omega^?_{N,L}(r))) \arrow[r, "\tau"] \arrow[dr, "\theta"'] & H^{2r-1}(\bar{M}, \underbrace{F_{\geq 0} R\psi(\Omega^?_{N,L}(r))}_{\shortparallel\ R\psi/F_{-1}.}) \arrow[d, "\xi"] \\
& H^{2r-1}(\bar{M}, L(r))
\end{tikzcd}$$

$\rightsquigarrow \quad \beta : \operatorname{coker} \tau \longrightarrow \operatorname{coker} \theta.$

$$1 \longrightarrow gr_0^F \longrightarrow F_{\geq 0} \longrightarrow \underset{\substack{\shortparallel \\ \ker\left(H^{2r-1}(\underset{\substack{\shortparallel \\ {}^{\Omega}E_2^{-1,2r}}}{gr_{-1}^F}) \xrightarrow{d_1^{-1,2r}} H^{2r}(gr_0^F)\right).}}{gr_1^F} \longrightarrow \operatorname{coker} \tau$$

where ${}^{\Omega}E_1^{a,b} := H^{a+b}(\bar{M}, gr^F_{-a}(R\psi\, \Omega^?_{N,L})) \Rightarrow H^{a+b}(\bar{M}, R\psi\, \Omega^?_{N,L})$

<u>Lemma</u> $\vartheta^?_\phi$ surjective $\Rightarrow$ $\beta_\phi$ surjective.

# <u>Lecture 6</u>

<u>Preview</u>

$$\begin{tikzcd}
\operatorname{coker} \tau \arrow[r, "\beta"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowright"] & \operatorname{coker} \theta \arrow[d, "\text{claim}"] \\
H^0(B, L)^{\diamond} \arrow[r, "\alpha"] & H^1(\mathbb{F}_{p^2}, H^{2r-1}(\bar{M}, L(r))).
\end{tikzcd}$$

Have diagram

$$\begin{tikzcd}
& \underline{Q} \arrow[dl, "h_<"'] \arrow[dr, "h_>"] \arrow[drr, bend left, "g_>"] \arrow[dd, phantom, "\lozenge"] & & \\
\underline{R} \arrow[dr, "g_1"'] & & \underline{B} \arrow[dl, "g_1"'] \arrow[dr, "g_0"] & \\
& \underline{N} \arrow[dr, "f"'] & & \underline{P} \arrow[dl, "f"] \\
& & \underline{M} &
\end{tikzcd}$$

Eo strata $M = \bigcup_{1 \leq m \leq 2r} M^{(m)}$

Write $P^? = P \times_M M^?$ & similarly $Q^?$, $N^?$, $R^?$.

**Lem** $\forall$ even $1 \le m \le 2r$,

$\mathfrak{g}^{(m)}_>, \mathfrak{g}^{(m)}_<, \mathfrak{g}^{(m)}_0, \mathfrak{g}^{(m)}_1$ are all finite

$$\underline{Q}^\eta = \frac{p^N-1}{p^2-1} Q_0 + Q_1 \rightsquigarrow \mathfrak{L}^{\eta,k}_{N,\underline{L}} := \ker\left(\mathfrak{g}^\eta - \left(\ell^k_{N,p} + \frac{p^N-1}{p^2-1}\right)\right)$$

**Prop** $\mathfrak{g}_0^{]2r-1[}, \mathfrak{g}_1^{]2r-1[}, \mathfrak{g}_>^{]2r-1[}, \mathfrak{g}_<^{]2r-1[}$ are all finite flat of degs $\frac{p^{2r}-1}{p^2-1}, p+1, \frac{p^{2r}-1}{p-1}, \frac{p^{2r}-1}{p-1}$.

**Pf** $\mathfrak{g}_0^{]2r-1[}$ finite flat by miraculous flatness + lem.

$\mathfrak{g}_1^{]2r-1[}$ same as $\mathfrak{g}_0^{]2r-1[}$.

$h_<^{]2r-1[}, h_>^{]2r-1[}$ base change of $\mathfrak{g}_1^{]2r-1[}$

$\Rightarrow \mathfrak{g}_>^{]2r-1[}, \mathfrak{g}_<^{]2r-1[}$ finite flat. $\square$

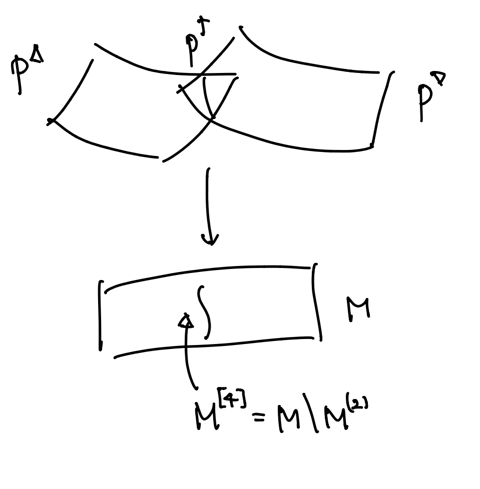
Assume $N = 2r \geq 4$.

Study $R\Psi(Rf_* L)|_{M^{(m)}}$, $m$ even, $0 \le m \le 2r$.

**Rmk** In fact, only need $M^{(2)}$ & $M^{(4)}$

b/c relevant strata: $f^\Delta, f^\nabla, f^\dagger$

$M^{(2)} \quad M^{(4)}, M^{(1)}$.




Can detect:

$p^{(2)} = p^{(2|0)} \amalg p^{(2|1)}$

$p^{(2|0)} \subset p^\Delta \setminus p^\dagger \quad p^{(2|1)} \subset p^\nabla \setminus p^\dagger$

$p^{(4)} = p^{(4|0)} \amalg p^{(4|1)} \amalg p^{(4|2)}$

$p^{(4|0)} \subset p^\Delta \setminus p^\dagger \quad p^{(4|1)} \subset p^\dagger \quad p^{(4|2)} \subset p^\nabla \setminus p^\dagger$

(set-theoretical inclusion).

In general, $P^{(m)} = \coprod_{l=0}^{m/2} P^{(m|l)}$, $P^{(m|0)} \subset P^{\Delta} \setminus P^{\dagger}$, $P^{(m|\frac{m}{2})} \subset P^{\nabla} \setminus P^{\dagger}$.

Now we have

$$\begin{tikzcd}
 & Q^{(m)} \arrow[dl, "g_<^{(m)}"'] \arrow[dr, "g_>^{(m)}\ \text{finite flat}"] & \\
P^{(m)} \arrow[dr] & & P^{(m)} \arrow[dl] \\
 & M^{(m)} &
\end{tikzcd}$$

$$\varphi^{(m)} : Rf_*^{(m)} L \longrightarrow Rf_*^{(m)} L = \bigoplus_{l=0}^{m/2} Rf_*^{(m|l)} L \quad \text{convolution map}$$

$$\rightsquigarrow \quad \Omega^{(m),k}_{N,L} := \ker\left(\varphi^{(m)} - \left(l^k_{N,p} + \frac{p^N-1}{p^2-1}\right) : Rf_*^{(m)} L \to Rf_*^{(m)} L\right).$$

Know that $F_0 R\Psi L = L$ on $\bar{P}$.

Lem Let $0 \le m \le 2r$ even. Then

$$F_0 R\Psi(\Omega^{\eta,k}_{N,L})|_{M^{(m)}} \simeq \Omega^{(m),k}_{N,L}.$$

Def Let $0 \le l_<, l_> \le \frac{m}{2}$.

$$Q^{(m|l_<,l_>)} := g_<^{-1} P^{(m|l_<)} \cap g_>^{-1} P^{(m|l_>)}.$$

Then

$$\begin{tikzcd}
 & Q^{(m|l_<,l_>)} \arrow[dl] \arrow[dr, "g_>^{(m|l_<,l_>)}\ \text{finite flat}"] & \\
P^{(m|l_<)} \arrow[dr] & & P^{(m|l_>)} \arrow[dl, "f^{(m|l_>)}"] \\
 & M^{(m)} &
\end{tikzcd}$$

Here $\varphi^{(m|l_<,l_>)} : Rf_*^{(m|l_<)} L \longrightarrow Rf_*^{(m|l_>)} L$

$$\varphi^{(m)} = \begin{pmatrix} \varphi^{(m|0,0)} & \varphi^{(m|1,0)} & \cdots & \varphi^{(m|\frac{m}{2},0)} \\ \vdots & & \ddots & \vdots \\ \varphi^{(m|0,\frac{m}{2})} & & \cdots & \varphi^{(m|\frac{m}{2},\frac{m}{2})} \end{pmatrix}.$$

Lem $Q^{(m|l_<,l_>)} = \emptyset$ if $|l_< - l_>| > 1$ ($\Rightarrow \varphi^{(m|l_<,l_>)} = 0$).

It turns out that can reduce study of $g^{(m)}$ to study of $g^{(m|0,0)}$.

<u>Idea</u> Given $x \in M^{(m)}(\kappa)$,

$$\rightsquigarrow \quad \mathcal{D}_x \cong \underset{\substack{\uparrow \\ \text{slopes} \neq \frac{1}{2}}}{\mathcal{B}(m)} \oplus \underset{\substack{\uparrow \\ \text{slope} = \frac{1}{2}}}{\mathcal{S}}{}^{\oplus N-m}. \qquad y \in \mathcal{P}^{(m|0)} \text{ over } x.$$

$\mathcal{C} \subseteq \mathcal{D}_x$ Lagrangian subspace, $F, V$-stable.

$\Rightarrow \quad \mathcal{C} \cap \mathcal{B}(m)$ Lagrangian, fixed.

On $M^{(m)}$: $\quad 0 \subseteq A[p]^{>\frac{1}{2}} \subseteq A[p]^{\geq \frac{1}{2}} \subseteq A[p]$,

$A[p]^{\frac{1}{2}} := A[p]^{\geq\frac{1}{2}} / A[p]^{>\frac{1}{2}}$ fin flat grp sch $/ M^{(m)}$. (slope $\frac{1}{2}$)

$\rightsquigarrow \; P^{((m))} / M^{(m)}$, parametrizing $\mathbb{F}_{p^2}$-linear Lagrangian subgrps of $A[p]^{1/2}$.

<u>Obs</u> If $m = 2r = N$, $P^{((2r))} = M^{(2r)}$.

Similarly, $N^{((m))}$, $R^{((m))}$, $Q^{((m))}$.

<u>Construction</u>

$$\begin{tikzcd}
 & Q^{((m))} \arrow[dl, "g_{\leftarrow}^{((m))}"'] \arrow[dr, "g_{\rightarrow}^{((m))}"] & \\
P^{((m))} \arrow[dr, "f^{((m))}"'] & & P^{((m))} \arrow[dl, "f^{((m))}"] \\
 & M^{(m)} &
\end{tikzcd}
\quad \text{finite étale diagram.}$$

$\Delta P^{((m))} \subseteq P^{((m))} \times_{M^{(m)}} P^{((m))}$ open closed

$\rightsquigarrow \quad {}^{\natural}Q^{((m))} := Q^{((m))} \times_{P^{((m))} \times_{M^{(m)}} P^{((m))}} \left( P^{((m))} \times_{M^{(m)}} P^{((m))} \setminus \Delta P^{((m))} \right)$.

Get

$$\begin{tikzcd}
 & {}^{\natural}Q^{((m))} \arrow[dl, "{}^{\natural}g_{\leftarrow}^{((m))}"'] \arrow[dr, "{}^{\natural}g_{\rightarrow}^{((m))}"] & \\
P^{((m))} \arrow[dr, "f^{((m))}"'] & & P^{((m))} \arrow[dl, "f^{((m))}"] \\
 & M^{(m)} &
\end{tikzcd}
\quad \text{finite étale diagram } (\Rightarrow \text{no derived info}).$$

$\rightsquigarrow \quad {}^{\natural}g^{((m))} : f_*^{((m))} L \longrightarrow f_*^{((m))} L \quad$ $\natural$-convolution.

**Def** $\Omega^{((m)),k}_{N,L} := \mathrm{Ker}\left({}^{H}g^{((m))} - \ell^{k}_{N-m,p} : f^{((m))}_{*} L \longrightarrow f^{((m))}_{*} L\right).$

**Lem** $\displaystyle\bigoplus_{k=0}^{r-\frac{m}{2}} \Omega^{((m)),k}_{N,L} \xrightarrow{\ \sim\ } f^{((m))}_{*} L$

$$\begin{tikzcd}
P^{(m|\ell)} \arrow[rr, "\pi^{(m|\ell)}"] \arrow[dr] & & P^{((m))} \arrow[dl] \\
& M^{(m)} &
\end{tikzcd}$$

$\pi^{(m|\ell)}$ flat universal homomorphism.

$$f^{((m))}_{*} L = Rf^{((m))}_{*} L \xrightarrow{\ \sim\ } Rf^{((m))}_{*} R\pi^{(m|\ell)}_{*} L = Rf^{(m|\ell)}_{*} L.$$

Here $f^{((m))}_{*} L \longleftrightarrow \begin{pmatrix} Rf^{(m|0)}_{*} L \\ \vdots \\ Rf^{(m|\frac{m}{2})}_{*} L \end{pmatrix}.$

Need to compute $g^{(m)} = (g^{(m|\ell_<,\ell_>)}) \in \mathrm{End}\left((f^{((m))}_{*} L)^{\oplus \frac{m}{2}+1}\right).$

**Thm**

$$g^{(m)} = \begin{array}{c|c}
 & \begin{array}{ccccc} 0 & \quad & \ell & \ell+1 & \cdots \end{array} \\ \hline
\begin{array}{c} 0 \\ \\ \\ \ell \\ \ell+1 \\ \vdots \end{array} &
\begin{pmatrix}
{}^{H}g^{((m))} + \frac{p^N-1}{p^2-1} & \ddots & & 0 & 0 \\
 & \ddots & & & \\
 & & \ddots & & \\
 & & & p^{2\ell}\cdot {}^{H}g^{((m))} + \frac{p^{2\ell-1}}{p+1} + \frac{p^N-1}{p^2-1} & p\cdot\frac{p^N-p^{N-m+2\ell}}{p^2-1} \\
 & & & \frac{p^{2\ell+2}-1}{p^2-1} & \ddots \\
 & 0 & & & \ddots
\end{pmatrix}
\end{array}$$

## Lecture 7

### § Computation of nearby cycles

$M = \bigcup_{m=1}^{N} M^{(m)}$, $\quad P^{(m)} := f^{-1}(M^{(m)}) = \coprod_{\ell=0}^{m/2} P^{(m|\ell)}$, $\quad 2 \le m \le N$ even.

$P^{((m))} \longrightarrow M^{(m)}$ finite étale,

$\forall\ 0 \le \ell \le \frac{m}{2}$, $\pi^{(m|\ell)} : P^{(m|\ell)} \longrightarrow P^{(m)}$ finite universal homeomorph, flat, & is isom for $\ell = 0$.

Let $m = 2$.

$$\begin{tikzcd}
P^{\nabla} \setminus P^{\dagger} & & P^{\Delta} \setminus P^{\dagger} \\
P^{(2|1)} \arrow[u, phantom, "\cup" description] \arrow[rr, "j^{(2)} = \pi^{(2|0),-1} \circ \pi^{(2|1)}"] \arrow[dr, "\pi^{(2|1)}"'] & & P^{(2|0)} \arrow[u, phantom, "\cup" description] \arrow[dl, "\pi^{(2|0)}", "\sim"'] \\
& P^{(2)} &
\end{tikzcd}$$

<u>Lem</u> $j^{(2)}$ extends uniquely to $j : P^{\nabla} \to P^{\Delta}$.

(1) $j$ purely inseparable of deg $p^{2r-1}$

(2) $f^{\Delta} \circ j = f^{\nabla} \Rightarrow Rf^{\Delta}_{*} L \xrightarrow[\sim]{ad_j} Rf^{\Delta}_{*}(Rj_{*} j^{*} L) = Rf^{\nabla}_{*} L$ isom.

(3)
$$\begin{tikzcd}
P^{\Delta} \arrow[r, "i", "\sim"'] \arrow[d, "f^{\Delta}"'] & P^{\nabla} \arrow[r, "j"] \arrow[d, phantom, "\circlearrowright"] & P^{\Delta} \arrow[r, "i", "\sim"'] & P^{\nabla} \arrow[d, "f^{\nabla}"] \\
M \arrow[rrr, "\phi_M := p^2\text{-Frob of } M"] & & & M
\end{tikzcd}$$

<u>Notation</u> $\forall\, b \in L$, let
$$j_b : Rf^{\Delta}_{*} L \xrightarrow{(id,\, b \circ ad_j)} Rf^{\Delta}_{*} L \oplus Rf^{\nabla}_{*} L$$
be the graph of $b \circ j$.

<u>Goal</u> $gr^F_i R\psi(\Omega^{?}_{N,L}) = ?$

Known: $i = 0$: $gr^F_0 R\psi(\Omega^{?}_{N,L}) \subseteq gr^F_0 R\psi(f^{?}_{*} L) = Rf^{\Delta}_{*} L \oplus Rf^{\nabla}_{*} L$.

$f^{\Delta}, f^{\nabla}$ small $\Rightarrow gr^F_0 R\psi(\Omega^{?}_{N,L}) = IC(\bar{M}, \Omega^{(2)}_{N,L})$

where $\Omega^{(2)}_{N,L}$ = loc sys on $\bar{M}^{(2)}$.

<u>Prop</u> $\mathrm{gr}^F_0 R\Psi(\Omega^?_{N,L}) = j_{(-p)^{-1}} IC(\bar{M}, \Omega^{((2))}_{N,L})$.

<u>sketch pf</u> It suffices to prove

$$\mathrm{gr}^F_0 R\Psi(\Omega^?_{N,L})|_{\bar{M}^{(2)}} = \dot{j}_{(-p)^{-1}} \Omega^{((2))}_{N,L}$$

as a sub-loc sys of

$$\begin{aligned}&(f^{\Delta}_{*} L \oplus f^{\nabla}_{*} L)|_{\bar{M}^{(2)}} \\ &= f^{(2|0)}_{*} L \oplus \underbrace{f^{(2|1)}_{*}}_{= f^{(2|0)}_{*} \hat{j}_{*}} L \\ &= f^{(2|0)}_{*} L \oplus f^{(2|0)}_{*} \hat{j}_{*} L.\end{aligned}$$

Have corr

$$\begin{tikzcd} & Q^{(2)} = Q^{(2|0,0)} \sqcup Q^{(2|1,0)} \sqcup Q^{(2|0,1)} \sqcup Q^{(2|1,1)} \arrow[dl] \arrow[dr] & \\ P^{(2)} \arrow[dr] & & P^{(2)} = P^{(2|0)} \sqcup P^{(2|1)} \arrow[dl] \\ & M^{(2)} & \end{tikzcd}$$

represented by matrix

$$q^{(2)} = \begin{pmatrix} {}^{\hbar}q^{((2))} + \frac{p^{2r}-1}{p^2-1} & p^{2r-1} \\ 1 & p^2\,{}^{\hbar}q^{((2))} + (p-1) + \frac{p^{2r}-1}{p^2-1} \end{pmatrix}.$$

Need to compute

$$\det\left(q^{(2)} - l^1_{N,p} - \frac{p^{2r}-1}{p^2-1}\right) = p^2\left({}^{\hbar}q^{((2))} - \underbrace{l^1_{N-2,p}}_{\updownarrow\ \Omega^{((2))}}\right)\left({}^{\hbar}q^{((2))} - l^{-1}_{N-2,p}\right)$$

Have that

$$\mathrm{Ker}\left(q^{(2)} - l^1_{N,p} - \frac{p^{2r}-1}{p^2-1}\right) = \begin{pmatrix} 1 \\ (-p)^{-1} \end{pmatrix}$$

when ${}^{\hbar}q^{((2))} = l^1_{N-2,p}$. □

$$\mathrm{gr}^F_{-1} R\Psi(\Omega^?_{N,L}) \subseteq \mathrm{gr}^F_{-1} R\Psi(f^?_{*} L) = Rf^{\dagger}_{*} L[-1].$$

$$f^{\dagger} : P^{\dagger} \longrightarrow M^{[4]}$$

dim $\quad 2r-2 \qquad 2r-2$

Semismall w/ relevant strata $M^{(4)}, M^{(1)} \leftarrow$ 0-dim'l

Let $P^{\langle 1\rangle}$ := reduced subsch of $P^{(1)} := P \times_M M^{(1)}$.

<u>Lem</u> $P^{\langle 1\rangle}$ is sm of dim $r-1$ w/ geom connected fibres

$$f^{\langle 1\rangle} : P^{\langle 1\rangle} \longrightarrow M^{(1)}.$$

$\rightsquigarrow$ $f^{\langle 1\rangle !} L = L_{P^{\langle 1\rangle}}(r-1)[2r-2]$

$\rightsquigarrow$ $\gamma : Rf^{\dagger}_{*} L \to Rf^{\dagger}_{*} L_{P^{\langle 1\rangle}} = Rf^{\langle 1\rangle}_{*} L \xrightarrow{\mathrm{Tr}_{f^{\langle 1\rangle}}} L_{M^{\langle 1\rangle}}(1-r)[2-2r]$.

<u>Lem</u> The above morph fits into:

$$0 \longrightarrow IC(M^{[4]}_{-}, (Rf^{\dagger}_{*} L)|_{M^{(4)}}) \longrightarrow Rf^{\dagger}_{*} L \xrightarrow{\gamma} L_{M^{\langle 1\rangle}}(1-r)[2-2r] \longrightarrow 0$$

in $\mathrm{Perv}(M^{[4]}_{-}, L)$.

<u>Rmk</u>

$$\begin{tikzcd} Rf^{\dagger}_{*} L \arrow[r, "\gamma"] & L_{M^{\langle 1\rangle}}(1-r)[2-2r] \\ L_{M^{(1)}}(1-r)[2-2r] \arrow[u] \arrow[ur, dashed] & \end{tikzcd}$$

self-intersection number of $P^{\langle 1\rangle}$ in $P^{\dagger}$.

<u>Prop 1</u>

$$\begin{tikzcd} gr^{F}_{-1} R\Psi(\Omega^{?}_{N,L}) \arrow[r, hook] \arrow[dr, "\simeq"'] & gr^{F}_{-1} R\Psi(f^{?}_{*} L) = Rf^{\dagger}_{*} L[-1] \arrow[d, "\gamma"] \\ & L_{M^{(1)}}(1-r)[2-2r] \end{tikzcd}$$

This is an isom.

<u>Lem 1</u> $gr^{F}_{-1} R\Psi(\Omega^{?}_{N,L}) \cap IC(M^{[4]}_{-}(Rf^{\dagger}_{*} L)|_{M^{(4)}}) = 0$.

Pf Suffices to check this on $M^{(4)}$ that

$$gr^{F}_{-1} R\Psi(\Omega^{?}_{N,L})|_{M^{(4)}} = 0.$$

But one obtains

$$0 \to gr^F_{-1} R\Psi(\Omega^{\eta}_{N,L})|_{M^{(4)}} \to F_0 R\Psi(\Omega^{\eta}_{N,L})|_{M^{(4)}} \to gr^F_0 R\Psi(\Omega^{\eta}_{N,L})|_{M^{(4)}} \to 0 .$$

$\rightsquigarrow$ reduces to show:

(A) The proj of $f^{(4)}_* L = f^{(4|0)}_* L \oplus f^{(4|1)}_* L \oplus f^{(4|2)}_* L$ to $f^{(4|0)}_* L$

$$\left( P^{(4)} = \coprod_{l=0}^{2} P^{(4|l)} \right)$$

induces an isom

$$\Omega^{((4))}_{N,L} \cong F_0 R\Psi(\Omega^{\eta}_{N,L}).$$

(B) $gr^F_0 R\Psi(\Omega^{\eta}_{N,L})|_{M^{(4)}} \cong \Omega^{((4))}_{N,L}$.

$$( Rf_* F_0 R\Psi_p(L) = F_0 R\Psi_p(f^{\eta}_* L) = Rf_*(L_p) .$$

<u>Sketch of (A)</u>

$$\mathfrak{q}^{(4)} = \begin{pmatrix} \mathfrak{q}^{(4|0,0)} & \mathfrak{q}^{(4|0,1)} & \mathfrak{q}^{(4|0,2)} \\ \mathfrak{q}^{(4|1,0)} & \mathfrak{q}^{(4|1,1)} & \mathfrak{q}^{(4|1,2)} \\ \mathfrak{q}^{(4|2,0)} & \mathfrak{q}^{(4|2,1)} & \mathfrak{q}^{(4|2,2)} \end{pmatrix}$$

$$= \begin{pmatrix} {}^{\natural}\mathfrak{q}^{((4))} + \frac{p^{2r}-1}{p^2-1} & p^{2r-3}(p^2+1) & 0 \\ 1 & p^2\, {}^{\natural}\mathfrak{q}^{((4))} + (p-1) + \frac{p^{2r}-1}{p^2-1} & p^{2r-1} \\ 0 & p^2+1 & p^4\, {}^{\natural}\mathfrak{q}^{((4))} + (p-1)(p^2+1) + \frac{p^{2r}-1}{p^2-1} \end{pmatrix}$$

$$\Rightarrow \det\left( \mathfrak{q}^{(4)} - l^{1}_{N,p} - \frac{p^{2r}-1}{p^2-1} \right)$$

$$= p^6 \Big( \underbrace{{}^{\natural}\mathfrak{q}^{((4))} - l^{1}_{N-4,p}}_{\substack{\wr \\ \Omega^{((4))}_{N,L}}} \Big) \cdot \left( {}^{\natural}\mathfrak{q}^{((4))} - l^{-1}_{N-4,p} \right) \cdot \left( {}^{\natural}\mathfrak{q}^{((4))} - l^{-3}_{N-4,p} \right) .$$

<u>Sketch of (B)</u>

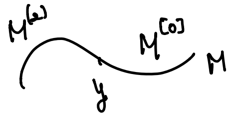
Recall $gr^F_0 R\Psi(\Omega^{\eta}_{N,L})|_{M^{(4)}} = j_{(-p)^{-1}} \cdot IC(\bar{M}, \Omega^{((2))}_{N,p})|_{M^{(4)}}$.

$\cap$

$(f^{\Delta}_* L \oplus f^{\nabla}_* L)|_{M^{(4)}} \leftarrow$ local system.

Suffices to show

$$IC(\bar{M}, \Omega^{((2))}_{N,L})|_y \cong \Omega^{((4))}_{N,L}|_y , \quad \forall\, y \in M^{[4]} \text{ generic pt.}$$




$\hat{\mathcal{O}}_{M,y}$ CDVR, $\bar{\eta}_y \in \operatorname{Spec} \hat{\mathcal{O}}_{M,y}$ generic pt.

$$D_y := \mathrm{Gal}(\mathrm{Fr}(\hat{\mathcal{O}}_{M,y})) \supseteq I_y.$$

Then $\quad IC(\bar{M}, \Omega^{((2,1))}_{N,L})_y = (\Omega^{((2))}_{N,L,\bar{\eta}_y})^{I_y} \underset{\substack{\uparrow \\ \text{Technical part.}}}{=} \Omega^{((4))}_{N,L}.$ $\square$

Lem 1 $\Rightarrow$ $\mathrm{gr}^F_{-1} R\Psi(\Omega^{\eta}_{N,L}) \cap IC(M^{[4]}, \Omega^{((4))}_{N,L}) = 0$

$$\rightsquigarrow \begin{tikzcd} \mathrm{gr}^F_{-1} R\Psi(\Omega^{\eta}_{N,L}) \arrow[r, hook] \arrow[dr, hook] & \mathrm{gr}^F_{-1} R\Psi(f^{\eta}_* L) \arrow[d, "\gamma"] \\ & L_{M^{(1)}}(-r)[1-2r] \end{tikzcd}$$

**Prop** The map above is surjective.

$$0 \to \mathrm{gr}^F_0 R\Psi(f^{\eta}_* L) \to F_{\geq 0} R\Psi(f^{\eta}_* L) \to \mathrm{gr}^F_1 R\Psi(f^{\eta}_* L) \to 0.$$

$$\rightsquigarrow \begin{array}{ccc} \mathrm{gr}^F_1 R\Psi(f^{\eta}_* L) & \longrightarrow & \mathrm{gr}^F_0 R\Psi(L) \\ \| & & \| \\ Rf^{\dagger}_* L(-1)[-2] & & Rf^{\Delta}_* L \oplus Rf^{\nabla}_* L \end{array}$$

Applying $m^{(1),!}$:

$$\begin{tikzcd} P^{\langle 1\rangle} \subseteq P^{(1)} \arrow[r, hook] \arrow[d, "f^{(1)}"'] \arrow[dr, phantom, "\square"] & P \arrow[d, "f"] \\ M^{(1)} \arrow[r, hook, "m^{(1)}"] & M \end{tikzcd}$$

with $f^{\langle 1\rangle}: P^{\langle 1\rangle} \to M^{(1)}$.

Get $\quad m^{(1),!} Rf^{\dagger}_* L(-1)[-2] \xrightarrow{\begin{pmatrix}\text{natural} - \text{another} \\ \text{Gysin} \quad\; \text{Gysin}\end{pmatrix}} m^{(1),!}(Rf^{\Delta}_* L \oplus Rf^{\nabla}_* L)$

$$\begin{array}{cc} \| & \| \\ Rf^{\langle 1\rangle}_* L_{P^{\langle 1\rangle}}(-r)[-2r] & (Rf^{\langle 1\rangle}_* L_{P^{\langle 1\rangle}}(-r)[-2r]) \\ & \oplus (-p^{-1}) Rf^{\langle 1\rangle}_* (j|_{P^{\langle 1\rangle}}) L_{P^{\langle 1\rangle}}(-r)[-2r] \end{array}$$

(note proper BC $\Rightarrow f_*^{\nabla} = f_*^{\Delta} \circ j$.)

<u>Lem</u> $j|_{P^{\langle 1\rangle}}$ is purely inseparable of deg $p^{2r-2}$

(recall: $j : P^{\nabla} \to P^{\Delta}$ purely inseparable of deg $p^{2r-1}$!)

<u>Recall</u> $gr_0^F R\psi(\Omega^{\eta}_{N,L}) = j_{(-p)^{-1}} IC(\bar{M}, \Omega^{((2))})$.

<u>Lecture 8</u>

Known so far: $c, gr_0^F R\psi(f_*^{\eta} L) = f_*^{\Delta} L \oplus f_*^{\nabla} L$

$gr_0^F R\psi(\Omega^{\eta}_{N,L}) \simeq IC(M, \Omega^{(2)}_{N,L})_{\bar{M}} = j_{(-p)^{-1}} IC(M, \Omega^{((2))}_{N,L})_{\bar{M}}$.

$gr_1^F R\psi(\Omega^{\eta}_{N,L}) \simeq L_{M^{(1)}}(-r)[1-2r]$.

$$
\begin{array}{l|ccccc}
q \geq 2r+1 & 0 & \longrightarrow & H^0(\bar{M}, IC(M, \Omega^{(2)}_{N,L}(r)) & \longrightarrow & 0 \\
q = 2r & H^0(\bar{M}^{(1)}, L) & \longrightarrow & H^{2r}(\bar{M}, IC(M, \Omega^{(2)}_{N,L}(r)) & \longrightarrow & 0 \\
q = 2r-1 & 0 & \longrightarrow & H^{2r-1}(\bar{M}, IC(M, \Omega^{(2)}_{N,L}(r)) & \longrightarrow & 0 \\
q = 2r-2 & 0 & \longrightarrow & H^{2r-2}(\bar{M}, IC(M, \Omega^{(2)}_{N,L}(r)) & \longrightarrow & H^0(\bar{M}^{(1)}, L(1)) \\
q = 2r-3 & 0 & \longrightarrow & H^{2r-3}(\bar{M}, IC(M, \Omega^{(2)}_{N,L}(r)) & \longrightarrow & 0 \\
\hline
 & p=-1 & & p=0 & & p=1
\end{array}
\qquad {}^{\Omega}E_1^{p,q}
$$

${}^{\Omega}E_1^{p,q} \Rightarrow H^{2r-1}(\bar{M}, R\psi(\Omega^{\eta}_{N,L})(r))$.

$$
\begin{array}{ccc}
H^0(\bar{M}^{(1)}, L) & \xrightarrow{d_1^{-1,2r}} & H^{2r}(\bar{M}, IC(M, \Omega^{(2)}_{N,L}(r)) \\
\wr\| & & \cap\| \\
H^0(\bar{P}^{\langle 1\rangle}, L) & & H^{2r}(P^{\Delta}, L(1)) \oplus H^{2r}(P^{\nabla}, L(1)) \\
 & & \cup
\end{array}
$$

im = (class of $\bar{P}^{\langle 1\rangle}$ in $P^{\Delta}$, $(-1)\cdot$ class of $\bar{P}^{\langle 1\rangle}$ in $P^{\nabla}$)

$$
\begin{aligned}
\vartheta : H^{2r-1}(\bar{M}, F_{\geq 0} R\Psi(\Omega^{\eta}_{N,L})(r)) &\hookrightarrow H^{2r-1}(\bar{M}, F_{\geq 0} R\Psi(f^{\eta}_{*} L)(r)) \\
&\xrightarrow{\sim} H^{2r-1}(\bar{M}, Rf_{*} F_{\geq 0} R\Psi L(r)) \\
&= H^{2r-1}(\bar{P}, F_{\geq 0} R\Psi L(r)) \\
&\xrightarrow{i_! \circ f_!} H^{2r-1}(\bar{M}, F_{\geq 0} R\Psi L(r)) \\
&= H^{2r-1}(\bar{M}, L(r)) \xleftarrow{\ \vartheta\ }
\end{aligned}
$$

$$\theta : H^{2r-1}(\bar{M}, gr^F_0 R\Psi(\Omega^{\eta}_{N,L})(r)) \xrightarrow{\tau} H^{2r-1}(\bar{M}, F_{\geq 0} R\Psi(\Omega^{\eta}_{N,L})(r))$$

$$\beta : {}^{\Omega}E_2^{-1,2r} \to \operatorname{coker} \theta$$

Let $\phi_M$: $p^2$-absolute Frob morph of $M$.

<u>Lem</u>

$$
\begin{tikzcd}
\operatorname{coker}\theta \arrow[rr, two heads] & & H^{2r}(\bar{M}, L(r))_{\phi_M^{*} = p^{2r}} = H^1(\mathbb{F}_{p^2}, H^{2r-1}(\bar{M}, L(r))) \\
& H^{2r-1}(\bar{M}, L(r)) \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

<u>Rmk</u> $N = 2$, $\operatorname{coker}\theta = H^1(\mathbb{F}_{p^2}, H^{2r-1}(\bar{M}, L(r)))$.

$N \geq 4$, conjecturally true.

Suffices to show:

$$
\begin{array}{ccc}
f_! \circ i_! : H^{2r-1}(\bar{P}, gr^F_0 R\Psi L) & \longrightarrow & H^{2r-1}(\bar{M}, L) \\
\cup\!\!| & & \cup\!\!| \\
H^{2r-1}(\bar{M}, gr^F_0 R\Psi(\Omega^{\eta}_{N,L})) & \longrightarrow & (1 - p^{-2r}\phi_M^{*}) \cdot H^{2r-1}(\bar{M}, L).
\end{array}
$$

Note that every elt in $H^{2r-1}(\bar{M}, gr^F_0 R\Psi(\Omega^{\eta}_{N,L}))$ is of the form $j_{(-p)^{-1}}(c)$ for some $c \in H^{2r-1}(\bar{P}^{\Delta}, L)$.

<u>Key</u> $\forall\, \ell \in L$,

$$(*) \qquad f_! i_! j_\ell(c) = (1 + p^{1-2r}\ell\, \phi_M^{*}) f^{\nabla}_! i_! c.$$

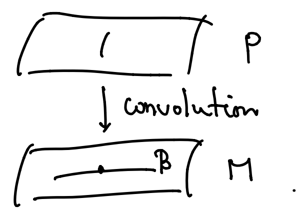
(in particular, $f_! i_! j_{(-p)^{-1}}(c) = (1 - p^{-2r}\phi_M^{*}) f^{\nabla}_! i_! c$.)

$${}^{\Omega}E_1^{-1,2r} = H^0(\bar{P}^{\langle 1\rangle}, L)^{\diamond}$$
$$= \ker\Big((p_!^{\langle 1\rangle,\Delta} - p_!^{\langle 1\rangle,\nabla}) : H^0(\bar{P}^{\langle 1\rangle}, L) \to H^{2r}(\bar{P}^{\Delta}, L(r)) \oplus H^{2r}(\bar{P}^{\nabla}, L(r))\Big)$$

Need that
$$\begin{tikzcd} H^0(\bar{P}^{\langle 1\rangle}, L)^{\diamond} \arrow[r, "\beta"] \arrow[d, dashed, "?"'] \arrow[dr, phantom, "\circlearrowright"] & \operatorname{coker}\theta \arrow[d, two heads] \\ H^0(\bar{B}, L)^{\diamond} \arrow[r, "\alpha"] & H^1(\mathbb{F}_{p^2}, H^{2r-1}(\bar{M}, L(r))) \end{tikzcd}$$

$\rightsquigarrow$ $\beta$ surj (known by Ihara) $\Rightarrow$ $\alpha$ surj (ultimate goal)

$\exists\, i_B$ s.t.
$$\begin{tikzcd} B \arrow[r, "i_B", "\sim"'] \arrow[dr, "\iota"'] & \bar{P}^{\langle 1\rangle} \arrow[d, "f\circ i"] \\ & M \end{tikzcd}$$




Have $\mathrm{gr}_0^F R\Psi(\Omega^{\eta}_{N,L})|_{\bar{M}^{]2r-1[}} = F_{\geq 0} R\Psi(\Omega^{\eta}_{N,L})|_{M^{]2r-1[}}$.

$\rightsquigarrow$ Diagram in derived cat
$$\begin{tikzcd} \mathrm{gr}_0^F R\Psi(\Omega^{\eta}_{N,L}) \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{gr}_0^F R\Psi(\Omega^{\eta}_{N,L}) \arrow[d] \\ F_{\geq 0} R\Psi(\Omega^{\eta}_{N,L}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \iota_*\iota^*\mathrm{gr}_0^F R\Psi(\Omega^{\eta}_{N,L}) \arrow[d] \\ \mathrm{gr}_1^F R\Psi(\Omega^{\eta}_{N,L}) \arrow[r] & \mu_!\mu^!\mathrm{gr}_0^F R\Psi(\Omega^{\eta}_{N,L}) \end{tikzcd} \Bigg\} \text{Gysin seq}$$

with $\mu : M^{[2r-1]} \longrightarrow M$,
$\iota : M^{]2r-1[} \hookrightarrow M$.

Construct
$$\begin{tikzcd} \mathrm{gr}_0^F R\Psi(\Omega^{\eta}_{N,L}) \arrow[rr, bend left, "\sigma"] \arrow[d, phantom, "\cap"] & & \\ f_*^{\Delta}L \oplus f_*^{\nabla}L \arrow[r, "pr_1"] & f_*^{\Delta}L \arrow[r, "f_!^{\nabla}\circ i_!"] & L_{\bar{M}} \end{tikzcd}$$

Chaising the $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_{p^2})$-equivariant diagram,

$$\begin{tikzcd}
H^{2r-1}(\bar M, \mathrm{gr}^F_0 R\Psi(\Omega^\ell_{N,L})(r)) \arrow[r, "="] \arrow[d] & H^{2r-1}(\bar M, \mathrm{gr}^F_0 R\Psi(\Omega^\ell_{N,L})(r)) \arrow[r, "\sigma"] \arrow[d] & H^{2r-1}(\bar M, L(r)) \arrow[d] \\
H^{2r-1}(\bar M, F_{\geq 0} R\Psi(\Omega^\ell_{N,L})(r)) \arrow[r, "\rho"] \arrow[d] \arrow[rrd, bend right] & H^{2r-1}(\bar M^{]2r-1[}, \mathrm{gr}^F_0 R\Psi(\Omega^\ell_{N,L})(r)) \arrow[r, "\sigma"] \arrow[d] & H^{2r-1}(\bar M^{]2r-1[}, L(r)) \arrow[d] \\
H^0(\bar P^{<1>}, L) \arrow[r, "(f\circ i)_!"] \arrow[d, bend right] & H^{2r}(\bar M^{[2r-1]}, \mu^! \mathrm{gr}^F_0 R\Psi(\Omega^\ell_{N,L})(r)) \arrow[r, "\sigma"] & H^{2r}(\bar M^{[2r-1]}, \mu^! L(r)) \\
H^0(\bar B, L) & &
\end{tikzcd}$$

Revisit AJ map:

$\delta \in H^0(B, L) = H^0(\bar B, L) \rightsquigarrow [\delta] \in H^{2r}(M^{[2r-1]}, \mu^! L(r))$.

refined cycle class w/ cpt supp

If $\delta \in H^0(B, L)^\diamond$ cohom triv,

then $[\delta]$ maps to $0$ in $H^{2r}(\bar M, L(r))$

$\exists\, \tilde\delta \in H^{2r-1}(M^{]2r-1[}, L(r))$ mapping to $[\delta]$.

$\rightsquigarrow \alpha(\delta) = (1 - \mathrm{Fr}^*)\tilde\delta = (1 - p^{-2r}\phi_M^*)\tilde\delta$.

Final step: show that

$$\begin{tikzcd}
H^{2r-1}(\bar M, F_{\geq 0} R\Psi(\Omega^\ell_{N,L})(r)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & H^{2r}(\bar M^{[2r-1]}, \mu^! L(r)) \\
H^0(\bar P^{<1>}, L) \arrow[r, "(\iota_B^{-1})_!"] & H^0(\bar B, L) \arrow[u, "z_!"', "\text{abs cycle}"]
\end{tikzcd}$$

$b \mapsto i_B^* b = (\iota_B^{-1})_! b$.

$\forall\, b \in H^0(\bar P^{<1>}, L)^\diamond$, $\exists\, \tilde b \in H^{2r-1}(\bar M, F_{\geq 0} R\Psi(\Omega^\ell_{N,L})(r))$ as its preimage.

Compute $\rho(\tilde b) = j_{-p^{-1}}(c)$ for some $c \in H^{2r-1}(P^{]2r-1[} \cap \bar P^\Delta, L(r))$

$\Rightarrow \beta(b) = f_! i_! \rho(\tilde b) = f_! i_! j_{-p^{-1}}(c)$

$$\overset{(*)}{=} (1-p^{-2r}\phi_M^*)\, f_!^{\nabla} \underbrace{i_! C}_{=\,\sigma\rho(\tilde{b}),\ \text{trivial by exactness}} = (1-p^{-2r}\phi_M^*).$$

$\square$