FINAL EXAM
SELECTED TOPIC IN ALGEBRAIC TOPOLOGY

Exercise 1. Show that a retract of a contractible space is contractible.

Definition 2. Given a fibrant simplicial set X and a base point $* \in X_0$, we define $\pi_n(X)$ as follows. By abuse of notation, we write $*$ for the element $s_0^n(*)$ of X_n and set $Z_n = \{x \in X_n : d_i(x) = * \text{ for all } i = 0, \dots, n\}$. We say that two elements $x, x' \in Z_n$ are homotopic, and write $x \sim x'$, if there is a $y \in X_{n+1}$ such that

$$d_i(y) = \begin{cases} * \text{ if } i < n \\ x \text{ if } i = n \\ x' \text{ if } i = n+1. \end{cases}$$

We define $\pi_n(X) = Z_n/\sim$.

Exercise 3. If G is a simplicial group, considered as a fibrant simplicial set, show that any two choices of basepoint lead to naturally isomorphic $\pi_n(G)$. (Hint: G_0 acts on G)

If G is a simplicial group, considered as a fibrant simplicial set with base point $* = 1$, it is helpful to consider the subgroups

$$N_n(G) = \{x \in G_n : d_i x = 1 \text{ for all } i \neq n\}.$$

Then $Z_n = \ker(d_n : N_n \to N_{n-1})$ and the image of the homomorphism $d_{n+1} : N_{n+1} \to N_n$ is $B_n = \{x : x \sim 1\}$. Hence $\pi_n(G)$ is the homology group Z_n/B_n of the (not necessarily abelian) chain complex N_*

$$1 \leftarrow N_0 \leftarrow N_1 \leftarrow N_2 \leftarrow \cdots$$

Exercise 4. Show that B_n is a normal subgroup of Z_n, so that $\pi_n(G)$ is a group for all $n \geq 0$. Then show that $\pi_n(G)$ is abelian for $n \geq 1$. Hint: Consider $(s_{n-1}x)(s_n y)$ and $(s_n x)(s_{n-1} y)$ for $x, y \in G_n$.

Exercise 5. If $G \to G''$ is a surjection of simplicial groups with kernel G'. Show that there is a short exact sequence of (not necessarily abelian) chain complex

$$1 \to NG' \to NG \to NG'' \to 1.$$

(This induces the long exact sequence of homotopy groups)

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