Peking Online International Number Theory Seminar

THE p-ADIC BOREL HYPERBOLICITY OF $\mathcal{A}_g$

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(NOTES BY WENHAN DAI)

Abstract. A theorem of Borel says that any holomorphic map from a smooth complex algebraic variety to a smooth arithmetic variety is automatically an algebraic map. The key ingredient is to show that any holomorphic map from the punctured disc to the arithmetic variety has no essential singularity. I will discuss some work towards a p-adic analogue of this theorem for Shimura varieties of Hodge type. Joint with Abhishek Oswal and Ananth Shankar.

1. Motivation: Generalizing Picard's Big Theorem

1.1. **Background.** We start with a well-known fact.

Proposition 1.1 (Picard's big theorem). *Let $f : \Delta^\times \to \mathbb{C}$ be a holomorphic function with essential singularity at the origin 0, where Δ is an open connected disc. Then*

$$\#(\mathbb{C} - f(\Delta^\times)) \leqslant 1.$$

This is equivalent to the following. Every holomorphic map

$$f : \Delta^\times \to \mathbb{P}^1 - \{0, 1, \infty\}$$

extends to a holomorphic $f : \Delta \to \mathbb{P}^1$. Actually, one properly obtains the same extension for $\mathbb{P}^1 - \{0, 1, \infty\} \hookrightarrow \mathbb{P}^1$, replaced by any hyperbolic curve $Y \subset \overline{Y}$. Therefore, we attain the following result.

Proposition 1.2. *Let $f : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ be a holomorphic map of smooth algebraic curves over $\mathbb{C}$, with Y hyperbolic, then f is induced by an algebraic map*

$$f^{\mathrm{alg}} : X \to Y.$$

Proof. Consider $X \hookrightarrow \overline{X} \hookrightarrow \{P_1, \dots, P_n\}$ around each P_i one can choose a small disc around $P_i \in \Delta_i$, say

$$\begin{tikzcd} \Delta_i^\times \arrow[r] \arrow[d, hook] & Y^{\mathrm{an}} \arrow[d, hook] \\ \Delta_i \arrow[r, dashed] & \overline{Y}^{\mathrm{an}} \end{tikzcd}$$

This gives $\widetilde{f} : \overline{X}^{\mathrm{an}} \to \overline{Y}^{\mathrm{an}}$ that identifies with $(\widetilde{f}^{\mathrm{alg}})^{\mathrm{an}}$, satisfying $f^{\mathrm{alg}} = \widetilde{f}^{\mathrm{alg}}|_X$. □

1.2. **A theorem of Borel.** We begin with the following setups. Let $\mathcal{D}$ be a hermitian symmetric domain that is isomorphic to G/K, where G is a real Lie group. Let $\Gamma \subset G$ be an arithmetic subgroup (can be choose to be torsion-free). Then $\Gamma \backslash \mathcal{D}$ admits a structure of complex manifold.

Theorem 1.3 (Baily-Borel, Borel). *There exists a unique quasi-projective algebraic variety structure on $\Gamma \backslash \mathcal{D}$.*

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The existence is proved by Baily-Borel, by basically considering the embedding

$$\Gamma\backslash\mathcal{D} \overset{\text{open}}{\hookrightarrow} \Gamma\backslash\mathcal{D}^* \overset{\text{closed}}{\hookrightarrow} \mathbb{P}^N.$$

And the uniqueness is prove by Borel. Note that every holomorphic map as follows has an extension:

$$\begin{tikzcd}
(\Delta^\times)^a \times \Delta^b \arrow[r, "f"] \arrow[d, hook] & \Gamma\backslash\mathcal{D} \arrow[d, hook] \\
\Delta^{a+b} \arrow[r, dashed, "\widetilde{f}"] & \Gamma\backslash\mathcal{D}^*
\end{tikzcd}$$

Hence for any $f : V^{\mathrm{an}} \to \Gamma\backslash\mathcal{D}$ where V is any algebraic variety, it is algebraizable.

2. The p-adic analogue of this phenomenon

The slogan is there exists, as well, a non-archimedean version of Picard's big theorem. Say any analytic map

$$f : \Delta^\times \to \mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$$

with essential singularity at $0 \in \Delta$ is surjective.

Theorem 2.1 (Oswal-Shankar-Zhu). *Let $f : \Delta^\times \to \mathcal{A}_g^{\mathrm{an}}$ be an analytic map defined over some p-adic field k (morally, we concentrate on finite extensions of $\mathbb{Q}_p$).*

Suppose the image of f does not contain bad reduction points (or equivalently, $f : \Delta^\times \to \widehat{\mathcal{A}_g}^{\mathrm{rig}}$). Then f extends to an analytic map

$$\widetilde{f} : \Delta \to \mathcal{A}_g^{\mathrm{an}}.$$

Corollary 2.2. *Let X be a compact Shimura variety of Hodge type. Then for every algebraic variety V/k, any analytic map*

$$f : V^{\mathrm{an}} \to X^{\mathrm{an}}$$

is automatically algebraic.

2.1. Idea of proofs.

Theorem 2.3. *Let $f : \Delta^\times \to \widehat{\mathcal{A}_g}^{\mathrm{rig}}$ be as in Theorem 2.1. Then there is a lifting*

$$\begin{tikzcd}
 & \widehat{\mathrm{RZ}_b}^{\mathrm{rig}} \arrow[d] \\
\Delta^\times \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"'] & \widehat{\mathcal{A}_g}^{\mathrm{rig}}
\end{tikzcd}$$

where RZ_b is some Rapoport-Zink space over $\mathrm{Spf}\,\breve{\mathbb{Z}}_p = W(\overline{\mathbb{F}}_p)$. Here b corresponds to some principally polarized abelian variety (A_0, λ_0) over $\overline{\mathbb{F}}_p$. More precisely,

$$\mathrm{RZ}_b : \mathsf{Nilp}_{\breve{\mathbb{Z}}_p} \longrightarrow \mathsf{Sets}$$

$$R \longmapsto \left\{ (A,\lambda,\iota) \;\middle|\; \begin{array}{c} (A,\lambda) \textit{ principally polarized abelian variety over } R, \\ \iota : (A,\lambda)|_{R/p} \to (A_0,\lambda_0) \otimes_{\overline{\mathbb{F}}_p} R/p \textit{ is a } p\textit{-quasi-isogeny} \end{array} \right\}$$

Theorem 2.4. *Every $f : \Delta^\times \to \widehat{\mathrm{RZ}_b}^{\mathrm{rig}}$ over k extends to*

$$\widetilde{f} : \Delta \to \widehat{\mathrm{RZ}_b}^{\mathrm{rig}}.$$

Proof of Theorem 2.4. Over $\widehat{\mathrm{RZ}}_b^{\mathrm{rig}}$, there exists a p-adic étale $\mathbb{Z}_p$-local system $\mathbb{L}$ consists of the following data:

- $(\mathcal{V}, \nabla, F^\bullet)$, a vector bundle with a flat connection and a decreasing filtration. Moreover,
$$(\mathcal{V}, \nabla) \simeq (\mathbb{D}(A_0) \otimes \mathcal{O}, 1 \otimes d)$$
where $\mathbb{D}$ denotes the rational Diéudonne module.

There is a p-adic period map equipped with filtration on $\mathcal{V} = \mathcal{O}^r$ to a flag variety, say
$$\widehat{\mathrm{RZ}}_b^{\mathrm{rig}} \longrightarrow \mathcal{FL}^a \subset \mathcal{FL},$$
composing with the latter open embedding.

Ingredient: the p-adic Riemann hypothesis
$$\left\{\begin{array}{c} \mathbb{Z}_p\text{-étale de Rham} \\ \text{local systems on } \Delta^\times \end{array}\right\}$$
$$\Big\downarrow \,{\scriptstyle (\mathrm{DLLZ})\ \mathrm{RH}_{\log}}$$
$$\left\{ (\mathcal{V}, \nabla, F^\bullet) \;\middle|\; \begin{array}{c} (\mathcal{V}, F^\bullet) \text{ a filtered vector bundle on } \Delta \\ \nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega_\Delta^{\log}(0) \text{ with residue } \mathrm{res}(\nabla) \\ \text{having eigenvalues in } \mathbb{Q} \cap [0,1] \end{array} \right\}$$

Then there is $f : \Delta^\times \to \widehat{\mathrm{RZ}}_b^{\mathrm{rig}}$ such that $f^*(\mathcal{O}^r, d, F^\bullet)$ extends to Δ and $f^*(\mathcal{O}^r_{\Delta^\times}, d) \cong (\mathcal{O}^r_\Delta, d)$. Hence $\mathrm{res}(\nabla^{\log}) = 0$. Consequently, $\mathbb{Z}$ also extends to a $\mathbb{Z}_p$-étale local system on Δ. See the diagram below.

$$\begin{tikzcd} \Delta^\times \arrow[r, "f"] \arrow[dd, hook] & \widehat{\mathrm{RZ}}_b^{\mathrm{rig}} \arrow[d] \\ & \mathcal{FL}^a \arrow[d, hook] \\ \Delta \arrow[r, "\widetilde{f}"] & \mathcal{FL} \end{tikzcd}$$

What we are desiring is $\widetilde{\widetilde{f}}(0) \in \mathcal{FL}^a$, which is implied by the fact that $\mathbb{L}|_0$ is crystalline.

Ingredient 2: the following theorem is in need.

> **Theorem 2.5** (Koji Shimizu)**.** *Let $\mathbb{L}$ be a horizontal de Rham $\mathbb{Z}_p$-local system on some V. If $\mathbb{L}$ is crystalline at one classical point, then $\mathbb{L}$ is crystalline at every point.*

Then the theorem of Koji completes the argument. □

The proof of Theorem 2.1 relies on the following ingredient:

Theorem 2.6 (Anand Patel)**.** *Let $f : \{r_1 \leqslant |z| \leqslant r_2\} \to \mathcal{X}^{\mathrm{rig}}$ where $\mathcal{X}$ is a "nice" formal scheme over $\mathcal{O}_k$. Then f is induced by some $F : \mathcal{R} \to \mathcal{X}$, where $\mathcal{R}$ is some formal model of $\{r_1 \leqslant |z| \leqslant r_2\}$ such that the reduced special fiber of $\mathcal{R}$ is a tree of $\mathbb{A}^1$ or $\mathbb{P}^1$.*

Note that Theorem 2.6 gives the construction of $\mathcal{R} \to \widehat{\mathcal{A}_g}$.

Conjecture 2.7 (Tate conjecture for function fields)**.** *$\mathcal{R} \otimes \overline{\mathbb{F}}_p$ maps to a single isogeny class.*

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