Shimura varieties (2/3)

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<u>Recap</u> $\mathfrak{h}_d^+ \overset{?}{\simeq} \{(A,\lambda,\eta_{\mathbb{Z}})\}/\sim$

$\| \{X \in M_d(\mathbb{C}) \mid {}^tX = X,\ \mathrm{Im}\, X > 0\}$

- (A,λ) p.p.a.v. over $\mathbb{C}$ of dim d,
- $\eta_{\mathbb{Z}} : H_1(A,\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}^{2d}$ symplectic isom.

Via $\iota(X) = (\mathbb{C}^d / \underbrace{\mathbb{Z}^d + X\mathbb{Z}^d}_{\Lambda_X},\ \underbrace{(\mathrm{Im}\, X)^{-1}}_{\text{pos-def}},\ \Lambda_X \xrightarrow{\sim} \mathbb{Z}^{2d})$.

Also recall $Sp_{2d}(\mathbb{R}) \circlearrowright \mathfrak{h}_d^+$.

(How does it act on $\eta_{\mathbb{Z}}$? Will remember a little bit of $\eta_{\mathbb{Z}}$ only.)

<u>Def'n</u> $n \geq 1$, (A,λ) p.p.a.v. scheme $/S$ of rel dim d

A level structure on (A,λ) is a pair (η,φ)

where $\left.\begin{array}{l} \eta : A[n] \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z}_S)^{2d} \\ \varphi : \mathbb{Z}/n\mathbb{Z}_S \xrightarrow{\sim} \mu_{n,S} \end{array}\right\}$ s.t. $\quad$ ($\varphi \in \mu_n(S)$ primitive).

for this,

$$\begin{array}{ccc} A[n] \times A[n] & \xrightarrow[\eta\times\eta]{\sim} & ((\mathbb{Z}/n\mathbb{Z}_S)^{2d})^2 \\ \text{Weil pairing for } \lambda \downarrow & \circlearrowleft & \downarrow \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \\ \mu_{n,S} & \xleftarrow[\varphi]{\sim} & \mathbb{Z}/n\mathbb{Z}_S \end{array}$$

<u>Cor</u> Let $\Gamma(n) = \ker(Sp_{2d}(\mathbb{Z}) \twoheadrightarrow Sp_{2d}(\mathbb{Z}/n\mathbb{Z}))$.

Then $\Gamma(n)\backslash \mathfrak{h}_d^+ \xrightarrow[\iota]{\sim} \{(A,\lambda,\eta)\}/\sim$,

$\eta : A[n] \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^{2d}$ s.t. $(\eta, e^{-\frac{2\pi i}{n}})$ is a level structure.

<u>Rmk</u> If $A = \mathbb{C}^d/\Lambda_X$, $A[n] = \frac{1}{n}\Lambda_X/\Lambda_X \xrightarrow[\eta_{\mathbb{Z}}]{\sim} (\mathbb{Z}/n\mathbb{Z})^{2d}$

§4 Moduli problems (depending on n).

Let $\mathcal{O}_n = \mathbb{Z}[\frac{1}{n}][T]/(T^n-1) \hookrightarrow \mathbb{C}$

$\zeta_n = [T] \longmapsto e^{\frac{2\pi i}{n}}$.

Def'n $\mathcal{M}'_{d,n} : Sch/\mathcal{O}_n \longrightarrow Sets$

$S \longmapsto \{(A, \lambda, \eta)\}/\sim$

Here (A,λ) p.p.a.v. over S of rel dim d

(η, ζ_n) is a level structure.

Thm (Mumford) If $n \geq 3$ then $\mathcal{M}'_{d,n}$ is representable by a smooth quasi-proj scheme $/\mathbb{Z}[\frac{1}{n}]$ of rel dim $\boxed{\frac{1}{2}d(d+1)}$.

Consequence: $\Gamma(n)\backslash \mathfrak{h}_d^+$ has a model over $\underbrace{\mathbb{Q}(e^{-\frac{2\pi i}{n}})}$ and over $\underbrace{\mathcal{O}_n}$.

Problem: depend on n

(for canonical models we can do better).

Def'n $\mathcal{M}_{d,n}$ is $Sch/\mathbb{Z}[1/n] \longrightarrow Set$

$S \longmapsto \{(A, \lambda, \boxed{\eta, \varphi})\}/\sim$

level str on (A,λ).

Question: $\mathcal{M}_{d,n}(\mathbb{C}) = ?$

$GSp_{2d} := \{g \in GL_{2d} \mid {}^t g \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} g = c(g) \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix},\ c(g) \in GL_1\}$.

$\rightsquigarrow c : GSp_{2d} \longrightarrow GL_1$, $Sp_{2d} = \ker c$.

Analogue of $\mathbb{C}\backslash\mathbb{R}$: $\mathfrak{h}_d = \mathfrak{h}_d^+ \cup \boxed{-\mathfrak{h}_d^+}$ $\{X \in M_d(\mathbb{C}) \mid {}^t X = X,\ \operatorname{Im} X < 0\}$

$GSp_{2d}(\mathbb{R})$ transitive.

$$\rightsquigarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot X = (AX+B)(CX+D)^{-1}.$$

$$\mathrm{Stab}_{GSp_{2d}(\mathbb{R})}(i I_d) = GO(2d) \cap GSp_{2d}(\mathbb{R}) = \mathbb{R}_{>0} \cdot K_\infty$$

Prop $\mathcal{M}_{d,n}(\mathbb{C}) \simeq GSp_{2d}(\mathbb{Q}) \backslash (\mathfrak{H}_d \times GSp_{2d}(\mathbb{A}_f)/K(n)) =: M^{GSp_{2d}}_{K(n)}$

where $K(n) = \ker(GSp_{2d}(\hat{\mathbb{Z}}) \twoheadrightarrow GSp_{2d}(\mathbb{Z}/n\mathbb{Z}))$.

$$\simeq \underbrace{GSp_{2d}(\mathbb{Q})^+}_{\text{the part with } c>0} \backslash \mathfrak{H}_d^+ \times GSp_{2d}(\mathbb{A}_f)/K(n))$$

On the other hand,

$$\begin{tikzcd} GSp_{2d}(\mathbb{Q})^+ \backslash GSp_{2d}(\mathbb{A}_f)/K(n) \arrow[r, "c", "\sim"'] & \mathbb{Q}_{>0} \backslash \mathbb{A}_f^\times / c(K(n)). \arrow[d, bend left, "\simeq"] \\ (\mathbb{Z}/n\mathbb{Z})^\times & \hat{\mathbb{Z}}^\times/(1+n\hat{\mathbb{Z}}) \arrow[l] \end{tikzcd}$$

by strong approximation theorem

Punchline $\mathcal{M}_{d,n}(\mathbb{C})$ is def'd over $\mathbb{Q}$ rather than sth depending on n.

(e.g. over $\mathbb{Z}[\frac{1}{n}]$, $\mathcal{O}_n$, etc.)

§5 Shimura data

Def'n (Deligne) A Shimura datum is a pair (G, h),

where G connected reductive grp $/\mathbb{Q}$

$h : \mathbb{C}^\times \longrightarrow G(\mathbb{R})$ is a morphism of $\mathbb{R}$-algebraic groups

s.t. (a) $h(\mathbb{R}^\times)$ is central

(b) The characters of $\mathbb{C}^\times$ on $\mathrm{Lie}(G_{\mathbb{C}})$ are acting by $\mathrm{Ad} \circ h$.

among $1, z\bar{z}^{-1}, z^{-1}\bar{z}$.

(c) $\mathrm{Int}(h(i))$ is a Cartan involution on $G^{der}(\mathbb{R})$.

($u : U(1) \to G_{ad}(\mathbb{R})$ and h are related by $u(z) = h(\sqrt{z})$.

Example $G = GSp_{2d}$. $h : \mathbb{C}^\times \longrightarrow G(\mathbb{R})$

$$a+ib \longmapsto \begin{pmatrix} aI_d & -bI_d \\ bI_d & aI_d \end{pmatrix}.$$

with $u(a+ib) = h(\sqrt{a+ib})$.

For every $K \subset G(\mathbb{A}_f)$ compact subgroup, set

$$\boxed{M_K(G,h)(\mathbb{C}) = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f)/K)}$$

where $X = G(\mathbb{R})/\mathrm{Cent}_{G(\mathbb{R})}(h)$ (i.e. $G(\mathbb{R})$-conj classes of h).

Complex alg var, quasi-proj.

Rmk (Last time) $M_K(G,h)(\mathbb{C}) = \coprod_{\text{finite}} \Gamma_i \backslash X$ ← finite union of HSDs.

- Morphisms: $(G_1,h_1) \longrightarrow (G_2,h_2)$ is $u : G_1 \to G_2$ s.t.

  $u \circ h_1 \sim h_2$ via $G_2(\mathbb{R})$-conj.

  $\rightsquigarrow u(K_1,K_2) : M_{K_1}(G_1,h_1)(\mathbb{C}) \to M_{K_2}(G_2,h_2)(\mathbb{C})$.

- Reflex field

  $\mathbb{C}^\times \xrightarrow{h} G(\mathbb{R})$ induces $\mathbb{S}(\mathbb{C}) \xrightarrow{h_{\mathbb{C}}} G(\mathbb{C})$

  $\mathbb{C}^\times = \mathbb{S}(\mathbb{R})$, $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} GL_1$, $\mathbb{S}(\mathbb{C}) = (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^\times = \mathbb{C}^\times \times \mathbb{C}^\times$

  $\mathbb{C}^\times \to \mathbb{C}^\times \times \mathbb{C}^\times$, $z \mapsto (z,1)$

Def'n The reflex field $E = E(G,h) \subset \mathbb{C}$ is the field of def'n of the conjugacy class of μ_h.

E.g. If $G = T$ is a torus, for any $h : \mathbb{C}^\times \to T(\mathbb{R})$, (T,h) is a Shimura datum.

Trivially in case, $X = *$,

$M_K(T,h)(\mathbb{C}) = T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/K$ is a finite set.

$\rightsquigarrow$ $E = E(T,h) =$ field of def'n of μ_h.

$$\begin{tikzcd} \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_{m,E} \arrow[r, "\mu_h"] \arrow[rr, bend right, "r"'] & \mathrm{Res}_{E/\mathbb{Q}} T_E \arrow[r, "N_{E/\mathbb{Q}}"] & T \end{tikzcd}$$

reciprocity map.

Global class field theory

$$\pi_0(E^\times \backslash \mathbb{A}_E^\times) \xrightarrow{r} \pi_0(T(\mathbb{Q}) \backslash T(\mathbb{A}))$$
$$\mathrm{Gal}(\bar{E}/E)^{ab} \text{ (is) } \curvearrowright \; T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/K.$$

Hence a model of $M_K(T,h)(\mathbb{C})$ over E.

- <u>General case</u> (G,h), $E = E(G,h)$.

<u>Def'n</u> A canonical model of $(M_K(G,h)(\mathbb{C}))_K$ is a proj system $(M_K(G,h))_K$ of varieties over E,

with a smooth $G(\mathbb{A}_f)$-action,

with $(M_K(G,h) \otimes_E \mathbb{C})_K \cong (M_K(G,h)(\mathbb{C}))_K$.

$G(\mathbb{A}_f)$-equivariant

s.t. $\forall\, m : T \hookrightarrow G$ injective morphism,

$(T,h') \mapsto (G,h)$

the morphisms $M_{K_T}(T,h')(\mathbb{C}) \to M_K(G,h)(\mathbb{C})$

are all defined over $E(T,h') \supseteq E(G,h)$

<u>Prop</u> (Deligne) Canonical models are unique up to unique isom.

<u>Thm</u> (Deligne) $(\mathcal{A}_{d,n}/\mathbb{Q})_n$ form a canonical model for $(G,h) = (GSp_{2d}, h_d)$

<u>Thm</u> (Milne/Mooren, based on Borovoi, Kazhdan)

Canonical model exists for <u>any</u> Shimura variety.