ABELIAN VARIETIES

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Contents

Part 1. Analytic Theory of Abelian Varieties.

1. Motivation

Let C be a complex smooth projective curve of genus g. Define

$$\Omega^1 := \text{sheaf of holomorphic 1-forms on } C.$$

Then we have the following theorem.

Theorem 1.1 (Abel-Jacobi). *Denote $H^0(C,\Omega^1)^*$ the complex dual of $H^0(C,\Omega^1)$ as a $\mathbb{C}$-vector space. The map*

$$\begin{aligned} H_1(C,\mathbb{Z}) &\longrightarrow H^0(C,\Omega^1)^* \\ \gamma &\longmapsto (\omega \mapsto \int_\gamma \omega) \end{aligned}$$

is injective and identifies $H_1(C,\mathbb{Z})$ with a lattice of $H^0(C,\Omega^1)^*$.

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The notes follow the context of [Mum85, §1–§14] closely. They are adapted from Bin Zhao's non-public course at Tsinghua University in 2020; any mistake is due to my negligence.

Assuming the theorem, one may define a complex torus by taking the quotient.

Definition 1.2 (Jacobian)**.** The **Jacobian variety** associated to C is

$$\mathrm{Jac}(C) := H^0(C,\Omega^1)^*/H^1(C,\mathbb{Z}).$$

It turns out that $\mathrm{Jac}(C)$ has some extra geometric structure beyond the structure of complex torus. When $g=1$, it is well known that for a fixed lattice $\Lambda \subset \mathbb{C}$, the complex torus $\mathbb{C}/\Lambda$ admits the structure of an elliptic curve. Moreover, it is projective as an algebraic variety via the morphism

$$\begin{aligned} \mathbb{C}/\Lambda &\longrightarrow \mathbb{P}^2_{\mathbb{C}} \\ z &\longmapsto [\wp_\Lambda(z), \wp'_\Lambda(z), 1]. \end{aligned}$$

Here $\wp_\Lambda(\cdot)$ denotes the Weierstrass $\wp$-function associated to Λ. To be more precise,

$$\wp_\Lambda(z) := \frac{1}{z^2} + \sum_{w\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

Through some more complicated computation, it can be verified that for each $z \in \mathbb{C}/\Lambda$, its image $[\wp_\Lambda(z), \wp'_\Lambda(z), 1]$ lies on an **elliptic curve**, which is defined to be a smooth projective algebraic curve of genus 1, on which there is a specified point O satisfying some group law[1].

Proposition 1.3. *For each* $z \in \mathbb{C}/\Lambda$,

$$\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2\wp_\Lambda(z) - g_3,$$

where the coefficients are read as

$$g_2 = 60G_4(\Lambda), \quad g_3 = 140G_6(\Lambda),$$

given by Eisenstein series

$$G_{2k}(\Lambda) := \sum_{w\in\Lambda\setminus\{0\}} w^{-2k}.$$

In general, in case when $g>1$, there is a significant difference: for an arbitrary lattice $\Lambda \subset \mathbb{C}^g$, the quotient $\mathbb{C}^g/\Lambda$ is not automatically projective. However, we are primarily to concern about projective complex tori.

For this, we need to study (ample/very ample) line bundles on the complex tori. Also, we pay attention to the construction of abelian varieties, which can be viewed as the correct generalization of elliptic curves to higher dimensional sense, by using the language of algebraic geometry.

2. Line Bundles on a Complex Torus

Setups. Let X be a complex torus (equivalently, a compact connected complex Lie group) of dimension g. Let $V = \mathrm{Lie}\, X$ be the corresponding Lie algebra.[2] This naturally induces an exponential map $\exp : V \to X$ which is a surjective homomorphism of complex Lie groups. Take $U := \mathrm{Ker}(\exp)$ to be a lattice of V. (Recall: it means by saying $U \subset V$ is a lattice that U is a discrete subgroup with the maximal rank whose quotient is compact; namely, U is a free abelian group of rank $2g$ in V.)

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[1] Possibly with at least one rational point on it, i.e., $E(\mathbb{Q}) \neq \emptyset$ for an elliptic curve E.

[2] In [Mum85], X is supposedly a group variety equipped with two morphisms $m : X \times X \to X$ and $i : X \to X$ that correspond to the multiplication and the inversion as group operations. Also, by definition there is a (closed) point $e \in X$ playing the role of the group identity. Then an equivalent definition of V is to say $V = T_eX$, the tangent space of X at some point $e \in X$.

Proposition 2.1. $(V, \exp)$ *is the universal covering of* X*. Moreover,*

$$\pi_1(X) = \pi_1(X, e) \cong U = \mathrm{Ker}(\exp).$$

Definition 2.2. The **Picard group** of an algebraic variety X over $\mathbb{C}$ is

$$\mathrm{Pic}(X) := \{\text{isomorphism classes of holomorphic line bundles on } X\}$$

In fact, this group can be computed explicitly via some sheaf cohomology

$$\boxed{\mathrm{Pic}(X) \cong H^1(X, \mathscr{O}_X^*)}$$

and this sheaf cohomology can be computed by some group cohomology. Here

$$\mathscr{O}_X := \text{the sheaf of holomorphic functions on } X,$$
$$\mathscr{O}_X^* := \text{the sheaf of invertible holomorphic functions on } X.$$

Let X be a nice topological space and G be a discrete group (i.e., equipped without any nontrivial topology) acting freely and discontinuously on X.[3] Then

$$X \circlearrowleft G \quad \rightsquigarrow \quad \begin{array}{c} X \\ \big\downarrow{\scriptstyle \pi} \\ X/G \end{array}$$

Let $\mathscr{F}$ be a sheaf of abelian groups on $Y = X/G$. Our goal is to compare for $p \geqslant 0$ that

$$H^p(G, \Gamma(X, \pi^*\mathscr{F})) \quad \text{and} \quad H^p(Y, \mathscr{F}).$$

2.1. **Group Cohomology.** Let G be as above and M be a left G-module (i.e., M is an abelian group with a left G-action). Define a **cochain complex** $C^\bullet = (C^p, \delta^p)_{p \geqslant 0}$ as follows.

(a) $C^p := \mathrm{Map}(G^p, M)$;
(b) $\delta^p : C^p \to C^{p+1}$ is defined by

$$(\delta^p f)(\sigma_0, \dots, \sigma_p) = \sigma_0(f(\sigma_1, \dots, \sigma_p)) + \sum_{i=0}^{p-1} (-1)^{i+1} f(\sigma_0, \dots, \sigma_i\sigma_{i+1}, \dots, \sigma_p)$$
$$+ (-1)^{p+1} f(\sigma_0, \dots, \sigma_{p-1}).$$

When $p = 0$, we have $G^p = \{e\}$, and $C^0 = M$, $C^1 = \mathrm{Map}(G, M)$. Hence

$$\begin{aligned} \delta^0 : M &\longrightarrow \mathrm{Map}(G, M) \\ m &\longmapsto (g \mapsto gm - m) \end{aligned}$$

2.1.1. *Definition of Group Cohomology.* Let (C^p, δ^p) be as above. Then define

$$Z^p(G, M) = \mathrm{Ker}(\delta^p), \quad B^p(G, M) = \mathrm{im}(\delta^{p-1}).$$

There are two equivalent types of definitions for the group cohomology $H^p(G, M)$.

(1) Define $H^p(G, M) := H^p(C^\bullet) = Z^p(G, M)/B^p(G, M)$ as the cohomology of the complex.
(2) Define $H^p(G, M)$ to be the p-th right derived functor of the functor

$$\begin{aligned} \mathsf{Mod}_G &\longrightarrow \mathsf{Ab} \\ M &\longmapsto M^G = \{m \in N \mid gm = m,\ \forall g \in G\} = \mathrm{Hom}_{\mathsf{Mod}_G}(\mathbb{Z}, M). \end{aligned}$$

[3] By definition, the discontinuity means that for all $x \in X$ there is some neighborhood U_x of x such that $U_x \cap \sigma(U_x) = \emptyset$ for all $\sigma \in G \backslash \{e\}$.

Example 2.3. (1) Consider the groups of 1-cocycles and 1-coboundaries:

$$Z^1(G,M)=\{f:G\to M\mid \underbrace{f(gg')=f(g)+gf(g')}_{\text{1-cocycle condition}},\ \forall g,g'\in G\},$$

$$B^1(G,M)=\{f:G\to M\mid \exists a\in M \text{ such that } f(g)=ga-a,\ \forall g\in G\}.$$

In particular, when G acts trivially on M, we obtain $ga-a=0$ for all $g\in G$, $a\in M$. Then $B^1(G,M)$ is trivial and hence

$$Z^1(G,M)\cong H^1(G,M)\cong \mathrm{Hom}(G,M).$$

(2) One can compute the 2-cocycle condition by definition:

$$g(f(g',g''))-f(gg',g'')+f(g,g'g'')-f(g,g')=0. \tag{$*$}$$

Consequently, the group of 2-cocycles is given by

$$Z^2(G,M)=\{f:G\times G\to M\mid \forall g,g',g''\in G,\ (*)\text{ holds}\}.$$

2.1.2. *Cup Product of Group Cohomology.* Take $M,N,P\in\mathsf{Mod}_G$. Given a bilinear map

$$\varphi:M\times N\to P\quad\text{such that}\quad \varphi(gm,gn)=g\varphi(m,n),$$

we can define a **cup product** as

$$\begin{aligned}\cup:H^p(G,M)\times H^q(G,N)&\longrightarrow H^{p+q}(G,P)\\ (f,g)&\longmapsto f\cup g.\end{aligned}$$

For all $f\in H^p(G,M)$ and $g\in H^q(G,N)$,

$$(f\cup g)(\sigma_1,\ldots,\sigma_{p+q})=f(\sigma_1,\ldots,\sigma_p)\cdot(\sigma_1,\ldots,\sigma_p)g(\sigma_{p+1},\ldots,\sigma_{p+q}).$$

2.2. **Čech Cohomology.** Let Y be a topological space and $\mathscr{F}$ be a sheaf of abelian groups on Y. Let $\mathcal{U}=\{V_i\}_{i\in I}$ be an open covering of Y. Define the **Čech chain complex**

$$C^\bullet(\mathcal{U},\mathscr{F})=(C^p(\mathcal{U},\mathscr{F}),d^p)_{p\geqslant 0}$$

as follows.

(a) Denote $V_{i_0\cdots i_p}=V_{i_0}\cap\cdots\cap V_{i_p}$ (or more generally, $V_{i_0\cdots i_p}=V_{i_0}\times_Y\cdots\times_Y V_{i_p}$). Then

$$C^p(\mathcal{U},\mathscr{F})=\prod_{(i_0,\ldots,i_p)\in I^{p+1}}\mathscr{F}(V_{i_0\cdots i_p}).$$

(b) Define $\delta^p:C^p(\mathcal{U},\mathscr{F})\to C^{p+1}(\mathcal{U},\mathscr{F})$ as follows. For the coordinate of d^pf corresponding to $(i_0,\ldots,i_{p+1})\in I^{p+2}$,

$$(d^pf)_{i_0\cdots i_{p+1}}=\sum_{j=0}^{p+1}\mathrm{res}_j(f_{i_0\cdots\hat{i}_j\cdots i_{p+1}}).$$

Here $\mathrm{res}_j:\mathscr{F}(V_{i_0\cdots\hat{i}_j\cdots i_{p+1}})\to\mathscr{F}(V_{i_0\cdots i_{p+1}})$ is induced by the inclusion $V_{i_0\cdots i_{p+1}}\subset V_{i_0\cdots\hat{i}_j\cdots i_{p+1}}$. The **Čech cohomology** of $\mathscr{F}$ with respect to $\mathcal{U}$ is defined to be

$$\check{H}^p(\mathcal{U},\mathscr{F}):=H^p(C^\bullet(\mathcal{U},\mathscr{F}),d^\bullet).$$

Remark 2.4. Some comments about Čech cohomology.

(1) Čech cohomology can be computed in terms of **alternating cochains**. Say $f\in C^p(\mathcal{U},\mathscr{F})$ is alternating if
   (i) $f_{i_0\cdots i_p}=0$ when there are $r\neq s$ such that $i_r=i_s$, and
   (ii) $f_{\sigma(i_0)\cdots\sigma(i_p)}=\mathrm{sgn}(\sigma)f_{i_0\cdots i_p}$ for $\sigma\in S_{p+1}$.

If we use $C'^p(\mathcal{U},\mathscr{F})$ to denote the subgroup of $C^p(\mathcal{U},\mathscr{F})$ consisting of alternating cochains, then

$$\check{H}^p(\mathcal{U},\mathscr{F}) \cong H^p(C'^\bullet(\mathcal{U},\mathscr{F}), d^\bullet).$$

(2) Čech cohomology $\check{H}^p(\mathcal{U},\mathscr{F})$ is not always isomorphic to sheaf cohomology $H^p(Y,\mathscr{F})$. However, they are related by spectral sequences. See [Har13, III, Thm 4.5] for an example.

2.3. **A Comparison Between Sheaf Cohomologies and Group Cohomologies.** We state the main result in [Mum85, Appendix of §2]. See also [Mil08, Chap.III, Example 2.6].

Theorem 2.5 (Comparison)**.** *For any sheaf $\mathscr{F}$ on Y, there is a natural group homomorphism*

$$\phi_p : H^p(G,\Gamma(X,\pi^*\mathscr{F})) \to H^p(Y,\mathscr{F})$$

with the following properties:

(1) *If*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

is an exact sequence of sheaves on Y, and

$$0 \to \Gamma(X,\pi^*\mathscr{F}') \to \Gamma(X,\pi^*\mathscr{F}) \to \Gamma(X,\pi^*\mathscr{F}'') \to 0$$

is exact, then we get a homomorphism from the cohomology sequence of $H^p(G,\cdot)$ to that of $H^p(Y,\cdot)$; i.e., the following diagram commutes:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H^p(G,\Gamma(X,\pi^*\mathscr{F}')) \arrow[r] \arrow[d] & H^p(Y,\mathscr{F}') \arrow[d] \\
H^p(G,\Gamma(X,\pi^*\mathscr{F})) \arrow[r] \arrow[d] & H^p(Y,\mathscr{F}) \arrow[d] \\
H^p(G,\Gamma(X,\pi^*\mathscr{F}'')) \arrow[r] \arrow[d] & H^p(Y,\mathscr{F}'') \arrow[d] \\
H^{p+1}(G,\Gamma(X,\pi^*\mathscr{F}')) \arrow[r] \arrow[d] & H^{p+1}(Y,\mathscr{F}') \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

(2) *For any $p \geqslant 0$, ϕ_p is compatible with the cup products.*[4]

(3) *If for all $i \geqslant 1$,*

$$H^i(X,\pi^*\mathscr{F}) = 0,$$

then

$$\phi_p : H^p(G,\Gamma(X,\pi^*\mathscr{F})) \to H^p(Y,\mathscr{F})$$

is an isomorphism.

Remark 2.6 (Important subtlety for beginners). In algebraic geometry, the functor

[4]For some historical reason, Mumford meant to say by (2) that ϕ_p commutes with the cup product. However, it is difficult to define the cup product of sheaf cohomologies. Fortunately, the case that will be at work may assume $\mathscr{F}$ is a constant sheaf.

$$\begin{aligned}\mathsf{Ring} &\longrightarrow \mathsf{Sch}\\ A &\longmapsto \mathrm{Spec}(A)\end{aligned}$$

is contravariant. On the other hand, in complex geometry, the functor

$$\begin{aligned}\mathsf{Manifold}_{\mathbb{C}} &\longrightarrow \mathsf{Ring}\\ X &\longmapsto \Gamma(X, \mathscr{O}_X)\end{aligned}$$

is contravariant as well. As a consequence, if we have a left action of G on X, it induces a right action of G on $\Gamma(X, \pi^*\mathscr{F})$.

Following the convention of Mumford, we define a left action of G on $\Gamma(X, \pi^*\mathscr{F})$ by composing with the inverse. In the case of complex torus, U acts on $H = \Gamma(V, \mathscr{O}_V)$ (resp., $H^* = \Gamma(V, \mathscr{O}_V^*)$) via the formula $(uh)(z) = h(z-u)$ for $u \in U$, $h \in H$ (resp., $h \in H^*$), and $z \in V$.

Now we come to the definition of the maps ϕ_p. Choose an open covering $\mathcal{U} = \{V_i\}_{i\in I}$ of Y such that:

(1) For all $p \geqslant 0$,
$$\check{H}^p(\mathcal{U}, \mathscr{F}) \cong H^p(Y, \mathscr{F}).$$

(2) Along with π, we obtain
$$\pi^{-1}(V_i) = \bigsqcup_{\sigma \in G} \sigma(U_i),$$
where $U_i \subset X$ are open such that $\pi|_{U_i} : U_i \to V_i$ are all isomorphisms. (Recall that the action of G is discontinuous, hence we take the disjoint union.)

(3) For all i, j, there is at most one $\sigma_{ij} \in G$ such that $U_i \cap \sigma_{ij} U_j \neq \emptyset$.

Note that in (3), σ_{ij} exists if and only if $V_i \cap V_j \neq \emptyset$. Also, $\sigma_{ij}^{-1} = \sigma_{ji}$. If $V_i \cap V_j \cap V_k \neq \emptyset$, then $\sigma_{ik} = \sigma_{ij}\sigma_{jk}$. Now we are ready to construct the map ϕ_p. For $i \in I$, define α_i to be the composite

$$\begin{tikzcd} \Gamma(X, \pi^*\mathscr{F}) \arrow[r, "\mathrm{res}"] \arrow[rr, bend left, "\alpha"] & \Gamma(U_i, \pi^*\mathscr{F}) \arrow[r, "\cong"] & \Gamma(V_i, \mathscr{F}). \end{tikzcd}$$

Define ϕ_p as the group homomorphism

$$\phi_p : C^p(G, \Gamma(X, \pi^*\mathscr{F})) \to C^p(\mathcal{U}, \mathscr{F}),$$

for which

$$(\phi_p f)_{i_0 \cdots i_p} = \mathrm{res} \circ \alpha_{i_0}(f(\sigma_{i_0 i_1}, \sigma_{i_1 i_2}, \ldots, \sigma_{i_{p-1} i_p}))$$

for all $(i_0, \ldots, i_p) \in I^{p+1}$. Note that res is basically the restriction map

$$\Gamma(V_{i_0}, \mathscr{F}) \to \Gamma(V_{i_0 \cdots i_p}, \mathscr{F}).$$

Exercise 2.7. Check that ϕ_p induces a morphism of cohomology groups.

2.4. **Geometric Description of Holomorphic Line Bundles.** We concern about the case of complex torus $X = V/U$ say. Here X is a connected compact complex Lie group, $V = \mathrm{Lie}\, X$, and U is a fixed lattice in V. There is a natural projection

$$\pi = \exp : V \to X = V/U.$$

Denote $H^* := \Gamma(V, \pi^*\mathscr{O}_X^*) = \Gamma(V, \mathscr{O}_V^*)$. Theorem 2.5 (3) dictates that

$$H^p(U, \Gamma(V, \pi^*\mathscr{O}_V^*)) \xrightarrow[\cong]{\phi_p} H^p(V, \mathscr{O}_V^*).$$

At the level of line bundles (when $p = 1$), here comes another construction of the isomorphism

$$H^1(X, \mathscr{O}_X^*) \xrightarrow{\cong} H^1(U, H^*).$$

Theorem 2.8. *For $p \geqslant 1$, we have*

$$H^p(\mathbb{C}^g, \mathscr{O}) \cong H^p(V, \mathscr{O}_V) = 0$$

by viewing $\mathbb{C}^g = \mathbb{C}^{\dim X} \approx V$ as an algebraic variety over $\mathbb{C}$. In particular,

$$H^p(\mathbb{C}^g, \mathscr{O}^*) \cong H^p(V, \mathscr{O}_V^*) \cong \mathrm{Pic}(V) = 0$$

for $p \geqslant 1$.

Let $\mathscr{L}$ be a (holomorphic) line bundle on X. By the theorem, $\pi^*\mathscr{L} \in \mathrm{Pic}(V)$ is trivial on V. So we can choose and fix an isomorphism

$$\chi : \pi^*\mathscr{L} \xrightarrow{\cong} \mathbb{C} \times V.$$

Conversely, to define a line bundle $\mathscr{L}$ on X, it suffices to define an action of U on $\mathbb{C}\times V$ that covers the translation action of U on V. Recall that H^* is the group of nowhere vanishing holomorphic functions on V. We define the action of U on H^* by

$$(uf)(z) = f(z+u).$$

The U-action on $\mathbb{C} \times V$ is of the following form:

$$u(\alpha, z) = (e_u(z)\cdot\alpha, z+u), \quad \alpha \in \mathbb{C},\ z \in V,\ u \in U,\ e_u \in H^*.$$

Exercise 2.9. Check that the association

$$\varphi : U \to H^*, \quad e \mapsto e_u$$

is indeed a 1-cocycle, i.e., an element in $Z^1(U, H^*)$.

If we modify the trivialization χ by a function $f \in H^*$, each of e_u will be replaced by

$$e'_u(z) = e_u(z) f(z+u) f(z)^{-1}.$$

This is because

$$\begin{tikzcd}
(\alpha, z) \arrow[r, mapsto] \arrow[d, mapsto, "u"] & (\alpha f(z), z) \arrow[d, mapsto, "u"] \\
(e_u(z)\alpha, z+u) \arrow[r, mapsto] & (e_u(z)\alpha f(z+u), z+u)
\end{tikzcd}$$

in which the right vertical arrow denotes the action under the new trivialization. So we have a well-defined map

$$H^1(X, \mathscr{O}_X^*) \to H^1(U, H^*),$$

which is compatible with the isomorphism established before. The upshot here is that if $u \mapsto e_u$ is a 1-cocycle under the action $(uf)(z) = f(z+u)$, then $u \mapsto f_u := e_{-u}$ is also a 1-cocycle under the action $(uh)(z) = h(z-u)$.

Proposition 2.10. *For any complex manifold Y, we have an exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathscr{O}_Y \xrightarrow{\exp(2\pi i(\cdot))} \mathscr{O}_Y^* \longrightarrow 0.$$

Applying the proposition to X, it induces a long exact sequence of cohomology:

$$\begin{tikzcd}
 & & \mathrm{Pic}(X) \arrow[d, phantom, "\parallel"] & \\
\cdots \arrow[r] & H^1(X,\mathbb{Z}) \arrow[r] & \boxed{H^1(X, \mathscr{O}_X)} \arrow[r] & H^1(X, \mathscr{O}_X^*) \arrow[dll, out=-10, in=170] \\
 & H^2(X,\mathbb{Z}) \arrow[r] & H^2(X, \mathscr{O}_X) \arrow[r] & H^2(X, \mathscr{O}_X^*) \arrow[r] & \cdots
\end{tikzcd}$$

On the other hand, one shall notice that the following is an exact sequence

$$
\begin{array}{ccccccccc}
0 \longrightarrow & H^0(V,\mathbb{Z}) & \longrightarrow & H^0(V,\pi^*\mathscr{O}_X) & \xrightarrow{\exp(2\pi i(\cdot))} & H^0(V,\pi^*\mathscr{O}_X^*) & \longrightarrow 0. \\
& \parallel & & \parallel & & \parallel & \\
& \mathbb{Z} & & H & & H^* &
\end{array}
$$

For the middle and the right terms above, use an open covering of V to see the equality to H and H^*, respectively. This sequence terminates on the right side because

$$H^i(V,\mathbb{Z}) = 0, \quad \forall i > 0.$$

Then by Theorem 2.5 (3) again, we have isomorphisms

$$H^p(U,\mathbb{Z}) \xrightarrow{\cong} H^p(X,\mathbb{Z})$$

for all $p \geqslant 0$. Here on the left side, we make the abelian group U acts trivially on $\mathbb{Z}$; on the right side, we regard $\mathbb{Z}$ as a constant sheaf on X. Moreover, the following diagram commutes:

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & & \\
 & H^1(U,H) \arrow[d, "\exp(2\pi i(\cdot))"'] & H^1(X,\mathscr{O}_X) \arrow[d] & & \\
 & H^1(U,H^*) \arrow[r, "\cong"] \arrow[d, "\delta"'] & \boxed{H^1(X,\mathscr{O}_X^*)} \cong \mathrm{Pic}(X) \ni [\mathscr{L}] \arrow[d] \arrow[dr, bend left] & & \\
\wedge^2 \mathrm{Hom}_{\mathbb{Z}}(U,\mathbb{Z}) \arrow[d, equal] & & & & \\
\mathrm{Hom}_{\mathbb{Z}}(\wedge^2 U,\mathbb{Z}) \cong & \boxed{H^2(U,\mathbb{Z})} \arrow[r, "\cong"] \arrow[d] & H^2(X,\mathbb{Z}) \arrow[d] & \ni & c_1(\mathscr{L}) \\
 & \vdots & \vdots & &
\end{tikzcd}
$$

Here $c_1(\mathscr{L})$ denotes the first Chern class of $\mathscr{L}$.

Lemma 2.11. *The map*

$$
\begin{aligned}
A : Z^2(U,\mathbb{Z}) &\longrightarrow \mathrm{Hom}(\wedge^2 U,\mathbb{Z}) \\
F &\longmapsto AF(u_1,u_2) := F(u_1,u_2) - F(u_2,u_1)
\end{aligned}
$$

induces an isomorphism

$$A : H^2(U,\mathbb{Z}) \longrightarrow \mathrm{Hom}(\wedge^2 U,\mathbb{Z}) \xrightarrow{\cong} \wedge^2 \mathrm{Hom}(U,\mathbb{Z}).$$

Moreover, we have the commutative diagram

$$
\begin{tikzcd}
H^1(U,\mathbb{Z}) \times H^1(U,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^2(U,\mathbb{Z}) = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[d, "\cong"', "A"] \\
\mathrm{Hom}(U,\mathbb{Z}) \times \mathrm{Hom}(U,\mathbb{Z}) \arrow[r, "\wedge"] & \wedge^2 \mathrm{Hom}(U,\mathbb{Z})
\end{tikzcd}
$$

Remark 2.12. In fact we have isomorphism of graded rings

$$H^*(U,\mathbb{Z}) \xrightarrow{\cong} H^*(X,\mathbb{Z}).$$

Given $[\mathscr{L}] \in H^1(X, \mathscr{O}_X^*)$ that corresponds to the element $[\{e_u\}]$ in $H^1(U, H^*)$, let $E \in \operatorname{Hom}(\wedge^2 U, \mathbb{Z})$ be the alternating form obtained by

$$H^1(U, H^*) \xrightarrow{\delta} H^2(U, \mathbb{Z}) \xrightarrow{\cong} \operatorname{Hom}(\wedge^2 U, \mathbb{Z})$$

along the isomorphism given by Lemma 5.11.

Lemma 2.13. *If we $\mathbb{R}$-linearly extend E to a map $E : V \times V \to \mathbb{R}$, then this extended E satisfies the identity*

$$E(ix, iy) = E(x, y), \quad \forall x, y \in V.$$

Proof. The proof requires some Hodge theory that is explained in the following two commutative diagrams.

(1) Regarding $\mathbb{C}$ as a constant sheaf over X, we have by Theorem 2.5 that $H^1(U, \mathbb{C}) \cong H^1(X, \mathbb{C})$. Also, by the Hodge decomposition, $H^1(X, \mathbb{C}) \cong H^{1,0}(X) \oplus H^{0,1}(X)$. Therefore,

$$\begin{tikzcd}
H^1(U, \mathbb{C}) \arrow[r, "\cong"] \arrow[d, "\cong"] & \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C}) \arrow[r, "\cong"] & \Omega \oplus \overline{\Omega} \arrow[d] \\
H^1(X, \mathbb{C}) \arrow[rr, "\text{Hodge Decomposition}", "\cong"'] & & H^{1,0}(X) \oplus H^{0,1}(X).
\end{tikzcd}$$

(2) Again, the Hodge decomposition for $H^2(X, \mathbb{C})$ leads to a natural projection $H^2(X, \mathbb{C}) \to H^{0,2}(X)$.

$$\begin{tikzcd}
 & H^2(U, \mathbb{Z}) \arrow[d, phantom, "\shortparallel"] & \\
H^1(X, \mathscr{O}_X^*) \arrow[r, "c_1"] & H^2(X, \mathbb{Z}) \arrow[r] \arrow[d, hook] & H^2(X, \mathscr{O}_X) \arrow[d, equal] \\
 & H^2(X, \mathbb{C}) \arrow[r, two heads, "\text{Hodge}"] \arrow[d, "\cong"] & H^{0,2}(X) \arrow[d, "\cong"] \\
 & \wedge^2\Omega \oplus (\Omega \times \overline{\Omega}) \oplus \wedge^2\overline{\Omega} \arrow[r] & \wedge^2\overline{\Omega}.
\end{tikzcd}$$

Then the image of E in $H^2(X, \mathbb{C})$ lies in the mixed part $\Omega \times \overline{\Omega}$, i.e., it is a Hodge class, and the lemma follows. ☐

Upshot. In the proof of Lemma 2.13, the idea is to pretend E to be the first Chern class of some line bundle representative. Then chase along the diagram to find out how it can be realized as a Hodge class.

Lemma 2.14. *Let V be a complex vector space. We have a bijective correspondence*

$$\begin{aligned}
\{\text{Hermitian forms } H \text{ on } V\} &\longleftrightarrow \left\{\begin{matrix}\text{real skew-symmetric form } E \text{ on } V \text{ such that} \\ E(iu, iv) = E(u, v) \text{ for all } u, v \in V\end{matrix}\right\} \\
H &\longmapsto \operatorname{Im} H \\
H(x, y) = E(x, y) + iE(x, y) &\longleftarrow E
\end{aligned}$$

Definition 2.15 (Néron-Severi group)**.** Define the Néron-Severi group of $X = V/U$ to be

$$\operatorname{NS}(X) := \{H \text{ Hermitian form on } V \mid \operatorname{Im} H|_{U \times U} \subset \mathbb{Z}\}.$$

Loosely, it consists of the Hermitian forms that take integral imaginary part on the lattice. Seriously,

$$\mathrm{NS}(X) := \mathrm{im}(H^1(X, \mathscr{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z})).$$

Also, $\mathrm{NS}(X)$ has a natural abelian group structure under addition.

Lemma 2.16. *Fix $H \in \mathrm{NS}(X)$ and let $E = \mathrm{Im}\, H$.*

(1) *There exists (but not necessarily unique) a map*

$$\alpha : U \longrightarrow \mathbb{C}_1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

such that

$$\alpha(u_1 + u_2) = e^{i\pi E(u_1,u_2)}\alpha(u_1)\alpha(u_2), \quad \forall u_1, u_2 \in U.$$

(2) *For such an α as in (1), define*

$$e_u(z) := \alpha(u)e^{\pi H(z,u)+\frac{1}{2}\pi H(u,u)}, \quad u \in U,\ z \in V.$$

Then $u \mapsto e_u$ defines an element in $H^1(U, H^*)$ and its Chern class of the associated line bundle is $E \in H^2(U, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$.

Now we define

$$P(X) := \{(H, \alpha) \mid H \in \mathrm{NS}(X),\ \alpha \text{ be as in (1)}\}.$$

Then $P(X)$ has an abelian group structure as well: say

$$(H_1, \alpha_1)(H_2, \alpha_2) = (H_1 + H_2, \alpha_1\alpha_2).$$

And therefore,

$$\mathscr{L}(H_1, \alpha_1) \otimes \mathscr{L}(H_2, \alpha_2) = \mathscr{L}(H_1 + H_2, \alpha_1\alpha_2).$$

We infer that there is an exact sequence of abelian groups

$$\begin{tikzcd}
 & \alpha \arrow[r, mapsto] & (0,\alpha) & & \\
0 \arrow[r] & \mathrm{Hom}_{\mathsf{Grp}}(U, \mathbb{C}_1) \arrow[r] \arrow[d, dashed, "\lambda"] & P(X) \arrow[r] \arrow[d, "(2)"] & \mathrm{NS}(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] & H^1(X, \mathscr{O}_X^*) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathrm{NS}(X) \arrow[r] & 0 \\
 & & \mathrm{Pic}(X) & &
\end{tikzcd}$$

where $\mathrm{Pic}^0(X) := \mathrm{Ker}(H^1(X, \mathscr{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}))$. And the middle vertical map above is given by Lemma 2.16 (2).

Proposition 2.17. *The map $P(X) \to \mathrm{Pic}(X)$ induces an isomorphism*

$$\lambda : \mathrm{Hom}_{\mathsf{Grp}}(U, \mathbb{C}_1) \xrightarrow{\cong} \mathrm{Pic}^0(X).$$

Proof. First we prove that λ is injective. For $\alpha \in \mathrm{Hom}_{\mathsf{Grp}}(U, \mathbb{C}_1)$, if $\lambda(\alpha) = 1$, the 1-cocycle

$$\alpha \in \mathrm{Hom}_{\mathsf{Grp}}(U, \mathbb{C}_1) \subset \mathrm{Hom}_{\mathsf{Grp}}(U, \mathbb{C}^*) = H^1(U, \mathbb{C}^*)$$

will become trivial in $H^1(U, H^*) \cong H^1(X, \mathscr{O}_X^*)$, which renders that we can find $g(z) \in H^*$ such that

$$\frac{g(z+u)}{g(z)} = \alpha(u), \quad \forall z \in X.$$

Since $|\alpha(u)| = 1$ for all $u \in U$, we see $|g(z+u)| = |g(z)|$. As $V/U = X$ is compact, $g(z)$ should be bounded on V. Then $g(z)$ is a constant function, and then $\alpha \equiv 1$. Therefore, λ is injective. It suffices to prove that it is surjective as well. Let us consider the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^1(X,\mathbb{Z}) \arrow[r] & H^1(X,\mathscr{O}_X) \arrow[r, "\exp(2\pi i(\cdot))"] & H^1(X,\mathscr{O}_X^*) \arrow[r, "c_1"] & H^2(X,\mathbb{Z}) \\
& & H^1(X,\mathbb{C}) \arrow[u, two heads, "\text{Hodge Decomp.}"', "p"] \arrow[r, "\exp(2\pi i(\cdot))"] & H^1(X,\mathscr{O}_X^*) \arrow[u, equal] & \\
& & H^1(U,\mathbb{C}) \arrow[u, equal, "\text{Theorem 2.5}"] \arrow[r, "\exp(2\pi i(\cdot))"] & H^1(U,H^*) \arrow[u, equal, "\text{by construction}"'] &
\end{tikzcd}$$

By the Hodge decomposition, $p : H^1(X,\mathbb{C}) \to H^1(X,\mathscr{O}_X)$ is surjective. This gives rise to

$$\operatorname{im}(H^1(X,\mathbb{C}) \to H^1(X,\mathscr{O}_X^*)) = \operatorname{im}(H^1(X,\mathscr{O}_X) \to H^1(X,\mathscr{O}_X^*)) = \operatorname{Pic}^0(X).$$

It follows that any line bundle $\mathscr{L}$ corresponds to a 1-cocycle $e_u(z) = e^{2\pi i f(u)}$ for some $f \in \operatorname{Hom}_{\mathsf{Grp}}(U,\mathbb{C})$. We extend $\mathbb{R}$-linearly that

$$\operatorname{Im} f : U \to \mathbb{R} \quad \rightsquigarrow \quad \operatorname{Im} f : V \to \mathbb{R}.$$

Also define

$$\begin{aligned} l : V &\longrightarrow \mathbb{C} \\ v &\longmapsto \operatorname{Im}(f(iv)) + i\operatorname{Im}(f(v)) \end{aligned}$$

together with

$$e'_u(z) := e_u(z)\cdot e^{2\pi i(l(z)-l(u+z))} = e^{2\pi i(f(u)+l(z)-l(u+z))}.$$

Notice that e'_u equals to e_u in $H^1(U,H^*)$ and

$$f(u)+l(z)-l(u+z) = \operatorname{Re}(f(u)) - \operatorname{Im}(f(iu)) \in \mathbb{R}$$

does not depend on z. Hence $e'_u(z) \in H^1(U,\mathbb{C}_1)$, which completes the proof. $\square$

3. Algebraizability of Tori

Goal. We have previously mentioned that the complex torus $\mathbb{C}/\Lambda$ can be realized as a projective algebraic curve. Or more generally, we are to realize the complex Lie group $X = V/U$ as a projective algebraic variety over $\mathbb{C}$, via the existence of a positive definite Hermitian forms.

3.1. **On Projective Morphisms.** The prototypical reference of this is [Har13, II, §7].

(1) To give a projective morphism $\varphi : X \to \mathbb{P}^n_k$ is equivalent to giving an invertible sheaf $\mathscr{L} = \varphi^*(\mathscr{O}(1))$ and sections $s_i = \varphi^*(x_i)$ $(i = 0,\dots,n)$ that generate $\mathscr{L}$.
(2) Under the notations in (1), φ is a closed immersion if and only if for the subspace $V = \operatorname{Span}\{s_i\} \subset \Gamma(X,\mathscr{L})$ the following two conditions hold:
   - elements of V separate points, i.e., for all $P,Q \in X$, there is $s \in V$ such that $s \in \mathfrak{m}_P\mathscr{L}_P$, $s \notin \mathfrak{m}_Q\mathscr{L}_Q$, or vice versa;
   - elements of V separate tangent vectors, i.e., for each $P \in X$, the set $\{s \in V \mid s_P \in \mathfrak{m}_P\mathscr{L}_P\}$ spans the k-vector space $\mathfrak{m}_P\mathscr{L}_P/\mathfrak{m}_P^2\mathscr{L}_P$.

Remark 3.1. (1) Sections of $\Gamma(X,\mathscr{L})$ cannot simultaneously vanish (resp., non-vanish) on some subvariety of X.
(2) φ induces injective maps on tangent spaces at all closed points.

3.2. **Global Sections of a Line Bundle.** Therefore, it is important to study the global sections of a line bundle $\mathscr{L} \in \operatorname{Pic}(X)$. We now explain why it is necessary to assume that H is positive definite and non-degenerate.

(a) Suppose that $\mathscr{L} = \mathscr{L}(H, \alpha)$, where H is a Hermitian form on V such that $E = \operatorname{Im} H$ being integral on $U \times U$ and $\alpha : U \to \mathbb{C}_1$ satisfying
$$\alpha(u_1 + u_2) = e^{i\pi E(u_1,u_2)}\alpha(u_1)\alpha(u_2).$$
It turns out that
$$\begin{aligned}&\Gamma(X, \mathscr{L}(H, \alpha))\\ &= \{\text{sections of } \mathbb{C} \times V \to V \text{ that are invariant under the action of } U\}\\ &= \{\theta : V \to \mathbb{C} \text{ theta function} \mid \theta(z + u) = e_u(z)\theta(z),\ \forall z \in V,\ u \in U\}.\end{aligned}$$

(b) Suppose H is degenerate. Let
$$N := \{x \in V \mid H(x, y) = 0,\ \forall y \in V\} = \{x \in V \mid E(x, y) = 0,\ \forall y \in V\}.$$
Then $N \subset V$ is a complex subspace and $N \cap U$ is a lattice in N as $E|_{U\times U} \subset \mathbb{Z}$. For a theta function $\theta \in \Gamma(X, \mathscr{L}(H, \alpha))$ such that $\theta(z + u) = \alpha(u)\theta(z)$ for all $u \in N \cap U$,
$$\theta(z + z') = \theta(z), \quad \forall z \in V,\ z' \in N.$$
Consider the complex subtorus $X' = N/N \cap U$ of X. Elements in $\Gamma(X, \mathscr{L}(H, \alpha))$ cannot separate points in X'. Therefore, $\mathscr{L}(H, \alpha)$ can never ever be ample.

(c) When H is non-degenerate but not positive definite, one can show that $\Gamma(X, \mathscr{L}(H, \alpha)) = 0$ and hence $\mathscr{L}(H, \alpha)$ cannot be ample.

Therefore, in the upcoming context, we **always assume H is positive definite** to make all of the constructions to be reasonable.

Proposition 3.2. *When H is positive definite, we have*
$$\dim H^0(X, \mathscr{L}(H, \alpha)) = \sqrt{\det(E)}.$$

Sketch of Proof. By the construction, U is a lattice in V of rank $2g$. We choose a sublattice U' of U or rank g such that $E|_{U'\times U'} \equiv 0$, and if $W = U' \otimes_{\mathbb{Z}} \mathbb{R}$ then $W \cap U = U'$ (this is to ensure that U is the maximal sublattice with respect to the previous condition).

Define a map
$$\begin{aligned}\beta : U &\longrightarrow \widehat{U} := \operatorname{Hom}_{\mathbb{Z}}(U, \mathbb{Z})\\ u &\longmapsto (u' \mapsto E(u, u')).\end{aligned}$$
By some complex analysis calculation, one can prove that
$$\dim H^0(X, \mathscr{L}(H, \alpha)) = \#\operatorname{Coker}(U \xrightarrow{\beta} \widehat{U} \to \widehat{U}').$$
Then the proposition follows from the following commutative diagram:
$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & U' \arrow[r] \arrow[d, "\alpha"] & U \arrow[r] \arrow[d, "\beta"] & U/U' \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & \widehat{U/U'} \arrow[r] & \widehat{U} \arrow[r] & \widehat{U}' \arrow[r] & 0 \end{tikzcd}$$

and the fact that

$$\#\,\mathrm{Coker}\,\beta = (\det E)^2, \quad \#\,\mathrm{Coker}\,\alpha = \#\,\mathrm{Coker}\,\beta.$$

$\square$

3.3. **Dual Complex Tori.**

Definition 3.3. The **$\mathbb{C}$-vector space of $\mathbb{C}$-antilinear maps** is defined to be

$$\overline{V} := \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C}).$$

We have a canonical $\mathbb{R}$-linear isomorphism

$$\begin{aligned} \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C}) &\xrightarrow{\cong} \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \\ l &\longmapsto \mathrm{Im}\, l \\ l(v) = -k(iv) + ik(v) &\longleftarrow k. \end{aligned}$$

This leads to a non-degenerate bilinear map

$$\begin{aligned} \langle \cdot, \cdot \rangle : \overline{V} \times V &\longrightarrow \mathbb{R} \\ (l, v) &\longmapsto \mathrm{Im}\, l(v). \end{aligned}$$

Define

$$\widehat{U} = \{ l \in \overline{V} \mid \langle l, U \rangle \subset \mathbb{Z} \},$$

which is a lattice of $\overline{V}$. The complex torus

$$\widehat{X} := \overline{V} / \widehat{U}$$

is called the dual complex torus of X.

Proposition 3.4. *We list out the following basic facts about $\widehat{X}$.*

(1) *The map*

$$\begin{aligned} \overline{V} &\longrightarrow \mathrm{Hom}_{\mathsf{Grp}}(U, \mathbb{C}_1) = \mathrm{Pic}^0(X) \\ l &\longmapsto e^{2\pi i \langle l, \cdot \rangle} \end{aligned}$$

induces an isomorphism $\widehat{X} \cong \mathrm{Pic}^0(X)$.

(2) *For every $\mathscr{L} = \mathscr{L}(H, \alpha) \in \mathrm{Pic}(X)$, the map*

$$\begin{aligned} V &\longrightarrow \overline{V} \\ v &\longmapsto H(v, \cdot) \end{aligned}$$

induces a homomorphism of complex torus $\phi_{\mathscr{L}} : X \to \widehat{X}$.

(3) *Moreover, as in* (2),

$$\phi_{\mathscr{L}} \textit{ is surjective as an isogeny} \quad \Longleftrightarrow \quad H \textit{ is non-degenerate.}$$

And in this case,

$$\deg(\phi_{\mathscr{L}}) = \det \mathrm{Im}(H).$$

When H is positive definite, we see that

$$\deg(\phi_{\mathscr{L}}) = (\dim H^0(X, \mathscr{L}))^2.$$

Theorem 3.5 (Lefschetz)**.** *Let X be a complex torus and $\mathscr{L} = \mathscr{L}(H, \alpha)$ be a line bundle on X. Then the following are equivalent:*

(1) *H is positive definite;*
(2) *$\mathscr{L}$ is ample;*

(3) *$\mathscr{L}^{\otimes n}$ is very ample for all $n \geqslant 3$.*

We will give an algebraic proof of the results above.

Corollary 3.6. *If $\mathscr{L} = \mathscr{L}(H, \alpha)$ is ample and* $\det E = 1$*, i.e., $\mathscr{L}$ gives a principal polarization of X, then X admits a closed immersion into some projective space:*

$$X \hookrightarrow \mathbb{P}^{3^g - 1}, \quad g = \dim_{\mathbb{C}} X.$$

In the future, this corollary is the key to construct the moduli space of abelian varieties.

Part 2. Algebraic Theory via Varieties.

4. Definition of Abelian Varieties

Let k be an algebraically closed field.

Definition 4.1. An **abelian variety** X is a complete algebraic variety over k (that is, X is an integral scheme proper and of finite type over k) with a group law induced by the morphisms

$$m : X \times X \to X, \quad e : \mathrm{Spec}(k) \to X, \quad i : X \to X$$

such that m and i are both morphisms of varieties.

Remark 4.2. (1) As we will see later, an abelian variety is automatically projective. This is not true for abelian schemes.

(2) In most of the cases, Mumford worked over an algebraically closed field. This makes the discussion much simpler in some cases. In practice, one should be aware of whether this assumption really affects the statement. For example, over a general field k, the correct definition of an abelian variety should be the same as the above definition except that one replaces "integral" with "geometric integral".

Exercise 4.3. Let X be a variety over a field k. Show that

$$X \text{ is projective} \iff X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \overline{k} \text{ is projective.}$$

Now we give some basic properties of abelian varieties.

Lemma 4.4. (cf. [Har13, II, §8]) *An abelian variety X is everywhere nonsingular (i.e., smooth) when k is algebraically closed.*

Proof. It suffices to check on closed points. Since k is algebraically closed, it is known that there is an open dense subset U of X which is nonsingular. For $x_0 \in U$ and $x \in X$, the left translation $T_{xx_0^{-1}}$ induces an isomorphism $\mathscr{O}_{X,x} \cong \mathscr{O}_{X,x_0}$. Hence x is nonsingular. $\square$

Next, we will prove that X is commutative as a group.

Lemma 4.5 (Rigidity)**.** *Let X be a complete variety. Let Y and Z be any varieties. Assume $f : X \times Y \to Z$ is a morphism such that there exists a closed point y_0 of Y with*

$$f(X \times \{y_0\}) = \{z_0\},$$

a single closed point z_0 of Z. Then there exists a morphism $g : Y \to Z$ such that $f = g \circ p_2$:

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \times Y \arrow[ur, "p_2"] \arrow[rr, "f"] & & Z \end{tikzcd}$$

where $p_2 : X \times Y \to Y$ is the second projection to Y.

Proof. Fix a closed point x_0 of X and define a morphism $g : Y \to Z$ to be the composite

$$Y \xrightarrow{\cong} \{x_0\} \times Y \hookrightarrow X \times Y \xrightarrow{f} Z.$$

It suffices to prove that f and $g \circ p_2$ agree on a nonempty open subscheme of $X \times Y$. We choose an open affine neighborhood U of z_0 in Z. Since $f^{-1}(U)$ is open in $X \times Y$, its complement $W = X \times Y \backslash f^{-1}(U)$ is closed in $X \times Y$.

Since p_2 is proper, $p_2(W)$ is closed in Y. By assumption, $W \cap (X \times \{y_0\}) = \emptyset$ and then $y_0 \notin p_2(W)$. We can find an open neighborhood V of y_0 in Y such that $V \cap p_2(W) = \emptyset$. Then the restriction $f|_{X\times V}$ factors through $U \subset Z$, and hence $g|_V$ factors through $U \subset Z$. For any closed point $y \in V$, $f|_{X\times\{y\}}$ is constant as $X \times \{y\}$ is proper and U is affine. It shows that for any closed point x of X,

$$f(x,y) = f(x_0,y) = (g \circ p_2)(x_0,y).$$

In other words, $f|_{X\times V}$ and $(g\circ p_2)|_{X\times V}$ agree on all closed points. Therefore, they agree on $X \times V$. This extends to

$$f = g \circ g_2 : X \times Y \to Z,$$

which completes the proof. □

Corollary 4.6. *If X and Y are abelian varieties and $f : X \to Y$ is a morphism, then $f(x) = h(x) + a$ where $h : X \to Y$ is a homomorphism and $a \in Y(k)$.*

Proof. Up to translations, it suffices to show that if $f(e_X) = e_Y$, then f is a homomorphism. Consider the morphism $\phi : X \times X \to Y$ defined by

$$\phi(x,y) = f(xy) - f(x) - f(y) := f(xy) + i(f(x)) + i(f(y)).$$

Since $\phi(X \times \{e_X\}) = \phi(\{e_X\} \times X) = \{e_Y\}$, it follows from Lemma 4.5 that $\phi(x,x') \equiv e_Y$ on $X \times X$. Hence f is always a homomorphism. □

Corollary 4.7. *X is a commutative group.*

Proof. Apply the previous corollary to show the morphism attached to the group variety

$$i : X \to X, \quad x \mapsto x^{-1}$$

is a group morphism. □

Corollary 4.8. *Let X be an abelian variety with base point e_X. Then on the category of complete varieties with base point, the functor*

$$S \longmapsto \mathrm{Hom}(S,X)$$

is linear, i.e., for S,T in this category, the natural map

$$\mathrm{Hom}(S,X) \times \mathrm{Hom}(T,X) \longrightarrow \mathrm{Hom}(S\times T,X), \quad (f,g) \longmapsto h$$

such that

$$h(s,t) = f(s) + g(t)$$

is a bijection.

Proof. If we use s_0 to denote the base point then

$$h(s_0,t) = g(t), \quad h(s,t_0) = f(s), \quad \forall s \in S,\ t \in T.$$

Then the map is injective. Now given $h \in \mathrm{Hom}(S\times T,X)$, define $f : S \to X$ and $g : T \to X$ by

$$f(s) = h(s,t_0), \quad g(t) = h(s_0,t)$$

for some fixed $s_0 \in S$ and $t_0 \in T$. Then the morphism

$$k : S \times T \to X, \quad k(s,t) = h(s,t) - g(t) - f(s)$$

satisfies

$$k(S \times \{t_0\}) = k(\{s_0\} \times T) = \{e_X\} \quad \Longrightarrow \quad k(s,t) \equiv e_X$$

by Lemma 4.5. □

Now let e_X : $\operatorname{Spec} k \to X$ be the identity element and $\Omega^1_X = \Omega^1_{X/k}$ be the sheaf of relative differentials of X over k. On $\operatorname{Spec} k$, the coherent sheaf

$$\omega_X = e_X^* \Omega^1_X$$

corresponds to the Zariski tangent space Ω_e of X at e (see [Har13, II, Prop 8.7]).

Proposition 4.9. *There is a natural isomorphism $\Omega^1_X \cong \pi^*\omega_X$ of coherent $\mathscr{O}_X$-modules.*

Proof. We regard the product $X \times X$ as an X-scheme via the second projection p_2. The morphism

$$\tau = (m, p_2) : X \times X \to X \times X, \quad (x, y) \mapsto (x + y, y)$$

is an automorphism of $X \times X$ over X, and it induces an isomorphism

$$\psi : \tau^*\Omega^1_{X\times X/X} \xrightarrow{\cong} \Omega^1_{X\times X/X}.$$

Since the following diagram is Cartesian,

$$\begin{tikzcd} X \times X \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"] & \operatorname{Spec} k \end{tikzcd}$$

we have $\Omega^1_{X\times X/X} \cong p_1^*\Omega^1_X$. Under this isomorphism, ψ becomes

$$\psi : m^*\Omega^1_X \xrightarrow{\cong} p_1^*\Omega^1_X.$$

We pull this isomorphism back along the morphism

$$(e_X \circ \pi, \mathrm{id}_X) : X \to X \times X, \quad x \mapsto (e_X, X),$$

where $\pi : X \to \operatorname{Spec} k$ is the natural section. It gives rise to the desired isomorpism

$$\Omega^1_X \xrightarrow{\cong} \pi^*(e_X^*\Omega^1_X) = \pi^*\omega X.$$

□

Remark 4.10. The above result holds for arbitrary group scheme $\pi : G \to S$ over S that is separated and of finite type.

Proposition 4.11. *For every n that is not divisible by* $\operatorname{char}(k)$*, the endomorphism*

$$n_X : X \to X, \quad x \mapsto nx$$

is surjective.

Proof. We impose T to denote the Zariski tangent space of X at e_X.

Claim. The addition morphism $m : X \times X \to X$ induces the tangent map at (e_X, e_X), say

$$d(m) : T_{X\times X,(e_X,e_X)} \cong T \oplus T \to T, \quad (t_1, t_2) \mapsto t_1 + t_2.$$

For this, note that the composite

$$X \xrightarrow{(\mathrm{id}_X, e_X)} X \times X \xrightarrow{m} X$$

is the identity map. One infers that $d(m)(t_1, 0) = t_1$ for all $t_1 \in T$; and similarly, $d(m)(0, t_2) = t_2$ for all $t_2 \in T$. The claim follows from the fact that $d(m)$ is additive.
Granting the claim, we are to prove the proposition. Take $K := \operatorname{Ker}(n_X)$ that sits in the left Cartesian diagram below. And consider the tangent maps on the right hand side below.

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathrm{Spec}(k) \arrow[d, "e_X"] \\ X \arrow[r, "n_X"] & X \end{tikzcd} \qquad \begin{tikzcd} T_{K,e_X} \arrow[r] \arrow[d, hook] & 0 \arrow[d] \\ T \arrow[r, "(\cdot)\times n"] & T \end{tikzcd}$$

Since n is not divisible by $\mathrm{char}(k)$, we see that

$$T_{K,e_X} = 0 \implies \dim \mathscr{O}_{K,e_X} = 0 \implies \dim K = 0.$$

Then the dimension formula implies that $\dim(\mathrm{im}(n_X)) = \dim X$. Hence n_X is surjective. □

Remark 4.12. We can actually show that n_X is a finite flat étale morphism (recall that the finiteness is implied by quasi-finiteness and properness). When $\mathrm{char}(k) \mid n$, n_X is still finite flat but no longer étale.

Corollary 4.13. *For all $x \in X$, $\mathscr{O}_{X,x}$ is regular and hence a UFD. So we identity the Weil divisor classes to the line bundle classes over X, say*

$$\mathrm{Cl}(X) \cong \mathrm{Pic}(X).$$

5. Cohomology and Base Change

The references for this section is [Har13, III, §12] and Conrad's lecture notes [Con00, §9].

Setups. Let $f : X \to Y$ be a proper morphism of noetherian schemes and $\mathscr{F}$ be a coherent sheaf of $\mathscr{O}_X$-modules. Assume that $\mathscr{F}$ is flat over Y, i.e., for any $x \in X$, $\mathscr{F}_x$ is flat as an $\mathscr{O}_{Y,f(x)}$-module. For any $y \in Y$, we denote

$$X_y := X \times_Y \mathrm{Spec}(k(y))$$

and $\mathscr{F}_y$ the inverse image of $\mathscr{F}$ via the morphism $X_y \to X$.

Goal: For any $i \geqslant 0$, we want to understand the fiber cohomology $H^i(X_y, \mathscr{F}_y)$ as a function of $y \in Y$. And the idea is to find relations between the sheaf $R^i f_* \mathscr{F}$ and the cohomology groups $H^i(X_y, \mathscr{F}_y)$.

We assume the following result.

Theorem 5.1 (Proper base change). *If $f : X \to Y$ is a proper morphism of locally noetherian schemes and $\mathscr{F}$ a coherent sheaf of $\mathscr{O}_X$-modules on X, then the direct image sheaves $R^p f_* \mathscr{F}$ are coherent sheaves of $\mathscr{O}_Y$-modules for all $p \geqslant 0$.*

When f is projective, this follows from [Har13, III, Thm 8.8]. As for the general case, it follows from EGA III, see [GD66, III, 3.2.1].

Theorem 5.2. *Let $f : X \to Y$ be a proper morphism of noetherian schemes with $Y = \mathrm{Spec}\, A$ affine, and $\mathscr{F}$ be a coherent sheaf of $\mathscr{O}_X$-module that is flat over Y. Then there exists a finite complex $K^\bullet$, say*

$$0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$$

of finitely generated projective A-modules and equivalences of functors

$$H^p(X \times_Y \mathrm{Spec}(\cdot), \mathscr{F} \otimes_A (\cdot)) = H^p(K^\bullet \otimes_A (\cdot)), \quad p \geqslant 0$$

on the category of A-algebras. Hence for any $B \in \mathsf{Alg}_A$,

$$H^p(X \times_Y \mathrm{Spec}\, B, \mathscr{F} \otimes_A B) \cong H^p(K^\bullet \otimes_A B), \quad p \geqslant 0.$$

Problem 5.3. Here the sheaf $\mathscr{F} \otimes_A B$ is the inverse image sheaf of $\mathscr{F}$ under the projection $X \times_Y \operatorname{Spec} B \to X$. How to give the association $B \mapsto H^p(X \times_Y \operatorname{Spec} B, \mathscr{F} \otimes_A B)$ rise to be a functor on the category of A-algebras? (To remedy this, one can use Čech cohomology, but how to make it formal?)

Remark 5.4.
(1) Since $\mathscr{F}$ is flat over $Y = \operatorname{Spec} A$, for any affine open subset $U \subset X$, $\mathscr{F}(U)$ is flat as an A-module.
(2) Since X is separated and noetherian, the coherent cohomology $H^*(X, \mathscr{F})$ can be computed by Čech cohomology with respect to finite affine open coverings, for any quasi-coherent sheaf $\mathscr{F}$ on X. The same is true for $X \times_Y \operatorname{Spec} B$.
(3) As for $H^p(K^\bullet \otimes_A B)$, it is generally not a finitely generated algebra over A, and the cohomology does not commute with $(\cdot) \otimes_A B$ in most cases.

Proof of Theorem 5.2. Let $\mathcal{U} = \{U_i\}_{i=0,\dots,n}$ be a finite affine open covering of X and $(C^\bullet(\mathcal{U}, \mathscr{F}), d^\bullet)$ be the Čech cochain complex of alternating cochains with respect to the open covering $\mathcal{U}$ and the sheaf $\mathscr{F}$. In particular,

$$C^p(\mathcal{U}, \mathscr{F}) = \bigoplus_{0 \leqslant i_0 < \cdots < i_p \leqslant n} \mathscr{F}(U_{i_0 \cdots i_p})$$

is a free A-module for all p (being nonzero only when $0 \leqslant p \leqslant n$), and the Čech cohomology groups $H^\bullet(\mathcal{U}, \mathscr{F})$ are isomorphic to $H^\bullet(X, \mathscr{F})$.

Moreover, for any A-algebra B, $\{U_i \times_Y \operatorname{Spec} B\}_{i=0,\dots,n}$ is an affine open covering of $X \times_Y \operatorname{Spec} B$, and $C^\bullet(\mathcal{U}, \mathscr{F}) \otimes_A B$ is the Čech cochain complex for this open covering and the sheaf $\mathscr{F} \otimes_A B$ on $X \times_Y \operatorname{Spec} B$. Therefore,

$$H^p(X \times_Y \operatorname{Spec} B, \mathscr{F} \otimes_A B) \cong H^p(C^\bullet(\mathcal{U}, \mathscr{F}) \otimes_A B), \quad p \geqslant 0,$$

and this isomorphism is functorial for B. □

Lemma 5.5. *Let $C^\bullet$ be a cochain complex of A-modules (but each C^p may not be finitely generated over A) such that $H^i(C^\bullet)$ are finitely generated A-modules for all $i \geqslant 0$, and such that $C^\bullet$ is bounded on $[0, n]$.*[5] *Then there exists a complex $K^\bullet$ of finitely generated A-modules, bounded on $[0, n]$ and such that K^p is free for all $1 \leqslant p \leqslant n$, and a homomorphism of cochain complexes $\phi : K^\bullet \to C^\bullet$ such that ϕ induces isomorphisms $H^i(K^\bullet) \to H^i(C^\bullet)$ for all i; namely, ϕ is a quasi-isomorphism.*

Moreover, if all the C^p's are A-flat, then K^0 will be A-flat as well.

Proof. We will use descending induction on m to construct the following diagram

$$\begin{tikzcd}
 & K^m \arrow[r, dashed, "d_K^m"] \arrow[d, dashed, "\phi_m"] & K^{m+1} \arrow[r, "d_K^{m+1}"] \arrow[d, "\phi_{m+1}"] & K^{m+2} \arrow[r] \arrow[d, "\phi_{m+2}"] & \cdots \\
\cdots \arrow[r] & C^m \arrow[r, "d_C^m"] & C^{m+1} \arrow[r, "d_C^{m+1}"] & C^{m+2} \arrow[r] & \cdots
\end{tikzcd}$$

with the following properties:
(1) $d_K^{p+1} \circ d_K^p = 0$ for $p \geqslant m+1$;
(2) $\phi_{p+1} \circ d_K^p = d_C^p \circ \phi_p$ for $p \geqslant m+1$;
(3) ϕ_p induces an isomorphism of cohomology groups $H^p(K^\bullet) \to H^p(C^\bullet)$ for $p \geqslant m+2$ and a surjective homomorphism $\operatorname{Ker}(d_K^{m+1}) \to H^{m+1}(C^\bullet)$;
(4) K^p is a finite free A-module for $p \geqslant m+1$.

[5] This is not a standard notation to say that $C^p \neq 0$ implies $0 \leqslant p \leqslant n$. Indeed, using the truncation functor, one may replace $C^\bullet$ with $\tau^{\geqslant 0}\tau^{\leqslant n} C^\bullet$.

We are going to construct K^m, d_K^m, ϕ_m with the above properties. One can find finite free A-modules $(K')^m$ and $(K'')^m$, and surjective maps of A-modules:

$$\begin{aligned}(K')^m &\twoheadrightarrow \operatorname{Ker}(\operatorname{Ker}(d_K^{m+1}) \to H^{m+1}(C^\bullet)),\\ (K'')^m &\twoheadrightarrow H^m(C^\bullet).\end{aligned}$$

Roughly speaking, the first surjection is to make ϕ_{m+1} into an isomorphism between cohomology groups; and the second surjection is to force ϕ_m to satisfy the desired property.

By construction, we have an inclusion $i'_m : (K')^m \to (K'')^{m+1}$ that factors through $\operatorname{Ker}(d_K^{m+1})$. Define

$$K^m := (K')^m \oplus (K'')^m, \quad d_K^m = (i'_m, 0) : K^m \to K^{m+1}.$$

Then property (1) and (4) hold for $p = m$, and ϕ_{m+1} induces an isomorphism $H^{m+1}(K^\bullet) \to H^{m+1}(C^\bullet)$. Since $(K'')^m$ is projective, we can lift the map $(K'')^m \to H^m(C^\bullet)$ to a map

$$\phi''_m : (K'')^m \to \operatorname{Ker}(d_C^m) \to C^m.$$

On the other hand, the composite

$$\begin{tikzcd}
(K')^m \arrow[r, "i'_m"] \arrow[dr, "i'_m"'] & K^{m+1} \arrow[r, "\phi_{m+1}"] & C^{m+1} \\
& \operatorname{Ker}(d_K^{m+1}) \arrow[u, hook] \arrow[r, "\phi_{m+1}"] & \operatorname{Ker}(d_C^{m+1}) \arrow[u, hook]
\end{tikzcd}$$

lies in $\operatorname{Ker}(d_C^{m+1})$ and is 0 in $H^{m+1}(C^\bullet)$. Then

$$(K')^m \xrightarrow{i'_m} \operatorname{Ker}(d_K^{m+1}) \xrightarrow{\phi_{m+1}} \operatorname{Ker}(d_C^{m+1})$$

factors through $\operatorname{im}(d_C^m)$. Since $(K')^m$ is projective, we can lift the map $(K')^m \to \operatorname{im}(d_C^m)$ to a map $\phi'_m : (K')^m \to C^m$ by the universal property. Finally we define

$$\phi_m = (\phi'_m, \phi''_m) : K^m \longrightarrow C^m.$$

It is straightforward to verify that $\phi_{m+1} \circ d_K^m = d_C^m \circ \phi_m$ and ϕ_m induces a surjective map

$$\operatorname{Ker}(d_K^m) = (K'')^m \longrightarrow H^m(C^\bullet).$$

This finishes the construction for m. Now we have the following diagram

$$\begin{tikzcd}
& K^0 \arrow[r, "d_K^0"] \arrow[d, "\phi_0"] & K^1 \arrow[r, "d_K^1"] \arrow[d, "\phi_1"] & \cdots \\
0 \arrow[r] & C^0 \arrow[r, "d_C^0"] & C^1 \arrow[r, "d_C^1"] & \cdots
\end{tikzcd}$$

that satisfies (1)-(4) above. We replace K^0 by $K^0/(\operatorname{Ker}(d_K^0) \cap \operatorname{Ker}(\phi_0))$ and d_K^0, ϕ_0 by their induced maps. Then the new diagram satisfies all the properties (1)-(4) except that K^0 is no longer free.

We still need to prove that K^0 is A-flat. Let $C[-1]^\bullet$ be the complex shifted by -1 of the cochain complex $C^\bullet$, i.e.,

$$C[-1]^p := C^{p-1}, \quad d^p_{C[-1]} := -d_C^{p-1}.$$

Consider the mapping cone of the morphism $\phi : K^\bullet \to C^\bullet$, which is defined as follows:

$$\operatorname{Cone}(\phi)^p := K^p \oplus C^{p-1} = K^p \oplus C[-1]^p,$$

together with[6]

$$
\begin{aligned}
d^p_{\mathrm{Cone}(\phi)} : K^p \oplus C^{p-1} &\longrightarrow K^{p+1} \oplus C^p \\
(x,y) &\longmapsto (d^p_K(x), \phi_p(x) - d^{p-1}_C(y)).
\end{aligned}
$$

One can easily check that $(\mathrm{Cone}(\phi)^p, d^p_{\mathrm{Cone}(\phi)})_p$ is a cochain complex. Moreover, we have an exact sequence of cochain complexes for each p, say

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C[-1]^p & \longrightarrow & K^p \oplus C[-1]^p & \longrightarrow & K^p & \longrightarrow & 0 \\
 & & y & \longmapsto & (0,y) & & & & \\
 & & & & (x,y) & \longmapsto & x & &
\end{array}
$$

And we have a long exact sequence of cohomology groups

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & H^p(C[-1]^\bullet) & \longrightarrow & H^p(\mathrm{Cone}(\phi)^\bullet) & \longrightarrow & H^p(K^\bullet) & \xrightarrow{\delta^p} & H^{p+1}(C[-1]^\bullet) & \longrightarrow \cdots \\
 & \parallel & & & & & & \parallel & \\
 & H^{p-1}(C^\bullet) & & & & & & H^p(C^\bullet) &
\end{array}
$$

Again, it is easy to verify that under the isomorphism $H^{p+1}(C[-1]^\bullet) \cong H^p(C^\bullet)$, the corresponding homomorphism δ^p is the one induced by the morphism ϕ_p^*, which is an isomorphism as well. Hence

$$H^p(\mathrm{Cone}(\phi)^\bullet) = 0, \quad \forall p.$$

So the cochain complex

$$\mathrm{Cone}(\phi)^\bullet : \quad 0 \to K^0 = \mathrm{Cone}(\phi)^0 \to \mathrm{Cone}(\phi)^1 \to \cdots \to \mathrm{Cone}(\phi)^{n+1} = C^n \to 0$$

is exact, in which $\mathrm{Cone}(\phi)^p$ is A-flat for all $p \geqslant 1$. Also, $\mathrm{Cone}(\phi)^\bullet$ breaks into n short exact sequences

$$0 \to \mathrm{Ker}(d^p_{\mathrm{Cone}(\phi)}) \to \mathrm{Cone}(\phi)^p \to \mathrm{Ker}(d^{p+1}_{\mathrm{Cone}(\phi)}) \to 0, \quad p = 1, \ldots, n.$$

Since $\mathrm{Ker}(d^{n+1}_{\mathrm{Cone}(\phi)}) = C^n$ is A-flat, so also is $\mathrm{Ker}(d^n_{\mathrm{Cone}(\phi)})$. We use descending induction and conclude that $\mathrm{Ker}(d^0_{\mathrm{Cone}(\phi)}) = K^0$ is A-flat. This proves the lemma. ☐

We apply Lemma 5.5 to the Čech cochain complex $C^\bullet = C^\bullet(\mathcal{U}, \mathscr{F})$ and obtain a cochain complex $K^\bullet$ and a cochain map $\phi : K^\bullet \to C^\bullet$ such that

(1) $K^\bullet$ is bounded on $[0,n]$;
(2) K^0 is finite and A-flat, and K^p are finite free A-modules for $p \geqslant 1$;
(3) ϕ is a quasi-isomorphism, i.e., for all p, $\phi_p : H^p(K^\bullet) \to H^p(C^\bullet)$ is an isomorphism.

Granting these conditions, we see K^p is projective as A-module for each $p \geqslant 0$. It remains to prove that for any A-algebra B,

$$\phi_B : H^p(K^\bullet \otimes_A B) \longrightarrow H^p(C^\bullet \otimes_A B)$$

is an isomorphism for each $p \geqslant 0$.

In fact, recall that the mapping cone $\mathrm{Cone}(\phi)^\bullet$ of ϕ breaks into short exact sequences

$$0 \to \mathrm{Ker}(d^p_{\mathrm{Cone}(\phi)}) \to \mathrm{Cone}(\phi)^p \to \mathrm{Ker}(d^{p+1}_{\mathrm{Cone}(\phi)}) \to 0, \quad p = 1, \ldots, n$$

[6]There is an alternative (and decorated) way to write the differential map as

$$d^p_{\mathrm{Cone}(\phi)} : K^p \oplus C[-1]^p \to K^{p+1} \oplus C[-1]^{p+1}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} d^p_K & 0 \\ \phi_p & d^p_{C[-1]} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

and all the three terms are flat A-modules. Consequently, for each $p = 1, \dots, n$,

$$0 \to \mathrm{Ker}(d^p_{\mathrm{Cone}(\phi)}) \otimes_A B \to \mathrm{Cone}(\phi)^p \otimes_A B \to \mathrm{Ker}(d^{p+1}_{\mathrm{Cone}(\phi)}) \otimes_A B \to 0$$

is also exact due to the flatness. In particular, the cochain complex $\mathrm{Cone}(\phi)^\bullet \otimes_A B$ is exact as well. On the other hand, $\mathrm{Cone}(\phi)^\bullet \otimes_A B$ is the mapping cone of $\phi_B = \phi \otimes_A B : K^\bullet \otimes_A B \to C^\bullet \otimes_A B$. So we have a long exact sequence of cohomology groups

$$\begin{array}{c} \cdots \longrightarrow H^p(\mathrm{Cone}(\phi)^\bullet \otimes_A B) \longrightarrow H^p(K^\bullet \otimes_A B) \xrightarrow{\phi_B} H^{p+1}((C^\bullet \otimes_A B)[-1]) \longrightarrow \cdots \\ \hphantom{\cdots \longrightarrow H^p(\mathrm{Cone}(\phi)^\bullet \otimes_A B) \longrightarrow H^p(K^\bullet \otimes_A B) \xrightarrow{\phi_B}} \| \hphantom{\longrightarrow \cdots} \\ \hphantom{\cdots \longrightarrow H^p(\mathrm{Cone}(\phi)^\bullet \otimes_A B) \longrightarrow H^p(K^\bullet \otimes_A B) \xrightarrow{\phi_B}} H^p(C^\bullet \otimes_A B) \end{array}$$

Therefore, ϕ_B is an isomorphism for each p.

Now let $f : X \to Y$ be a proper morphism of noetherian schemes and $\mathscr{F}$ a coherent sheaf of $\mathscr{O}_X$-module on X that is flat over Y. Recall that for $y \in Y$, we define the fiber $X_y = X \times Y \, \mathrm{Spec}(k(y))$ and $\mathscr{F}_y$ the inverse image of $\mathscr{F}$ on X_y. (Caution: Y is not necessarily affine.)

Corollary 5.6. *Under the above notations, we have*

(1) *For every $p \geqslant 0$, the function*

$$Y \to \mathbb{Z}, \quad y \mapsto \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$$

is upper semicontinuous on Y. A function $h : Y \to \mathbb{Z}$ is, by definition, upper semicontinuous, if for all $n \in \mathbb{Z}$ the set $\{y \in Y \mid h(y) \geqslant n\}$ is a closed subset of Y.

(2) *The function*

$$Y \to \mathbb{Z}, \quad y \mapsto \chi(\mathscr{F}_y) = \sum_{p=0}^{\infty} (-1)^p \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$$

is locally constant on Y.

Proof. The question is local on Y so one may assume that $Y = \mathrm{Spec}\, A$ is affine. We apply the pervious Theorem 5.2 to the morphism $f : X \to Y$ and the sheaf $\mathscr{F}$, and obtain a cochain complex $K^\bullet$ such that

$$H^p(X_y, \mathscr{F}_y) \cong H^p(K^\bullet \otimes_A k(y)), \quad \forall p \geqslant 0, \ y \in Y.$$

Shrinking Y if necessary, we can assume that K^p is free for all p (the idea is to pretend K^p to be the pth Čech complex). For $p \geqslant 0$, we define

$$W^p := \mathrm{Coker}(d_K^{p-1} : K^{p-1} \to K^p).$$

So we have an exact sequence

$$W^p \xrightarrow{d_K^p} K^{p+1} \longrightarrow W^{p+1} \longrightarrow 0.$$

Applying the functor $(\cdot) \otimes_A k(y)$, we get

$$0 \to H^p(K^\bullet \otimes_A k(y)) \to W^p \otimes_A k(y) \to K^{p+1} \otimes_A k(y) \to W^{p+1} \otimes_A k(y) \to 0.$$

This is basically because the cokernel commutes with base changes, and so we have

$$W^p \otimes_A k(y) \cong \mathrm{Coker}(d_K^{p-1} \otimes_A k(y) : K^{p-1} \otimes_A k(y) \to K^p \otimes_A k(y)).$$

Therefore,

$$\begin{aligned} \dim_{k(y)} H^p(K^\bullet \otimes_A k(y)) = \dim_{k(y)} W^p \otimes_A k(y) &- \dim_{k(y)} K^{p+1} \otimes_A k(y) \\ &+ \dim_{k(y)} W^{p+1} \otimes_A k(y). \end{aligned}$$

Since the function

$$y \mapsto \dim_{k(y)} K^{p+1} \otimes_A k(y)$$

is (locally) constant, it suffices to prove that the function

$$y \mapsto \dim_{k(y)} W^p \otimes_A k(y)$$

is upper semicontinuous.

Claim. For any finitely generated A-module M, the function

$$Y \to \mathbb{Z}, \quad y \mapsto \dim_{k(y)} M \otimes_A k(y)$$

is upper semicontinuous.

The proof of the claim is leave as an exercise. Granting the claim, (2) follows by taking alternating sum of the dimension equation above. □

Corollary 5.7. *Under the above notations, assume further that Y is reduced and connected. Then for all p, the following are equivalent.*

(1) *The function*

$$y \mapsto \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$$

is constant.

(2) $R^p f_* \mathscr{F}$ *is a locally free sheaf on Y, and for all $y \in Y$, the natural map*

$$R^p f_* \mathscr{F} \otimes_{\mathscr{O}_Y} k(y) \to H^p(X_y, \mathscr{F}_y)$$

is an isomorphism.

If any one of (1)(2) *hold, we also have that*

$$R^{p-1} f_* \mathscr{F} \otimes_{\mathscr{O}_Y} k(y) \cong H^{p-1}(X_y, \mathscr{F}_y)$$

for all $y \in Y$.

We can assume that $Y = \operatorname{Spec} A$ is affine and let $K^\bullet$ be the cochain complex in Theorem 5.2. Then (2) $\Longrightarrow$ (1) is obvious. So it boils down to prove (1) $\Longrightarrow$ (2).

Lemma 5.8. *Let Y be a reduced affine scheme and $\mathscr{F}$ be a coherent sheaf on Y. If*

$$\dim_{k(y)} \mathscr{F} \otimes_{\mathscr{O}_Y} k(y) = r$$

for all $y \in Y$ (as $k(y)$-vector spaces), then $\mathscr{F}$ is a locally free $\mathscr{O}_Y$-module of rank r.

Proof. Let $Y = \operatorname{Spec} A$ and $\mathscr{F} = \widetilde{M}$. Fix $y \in Y$ that correspond to $\mathfrak{p} \in \operatorname{Spec} A$. We choose $x_1, \ldots, x_r \in M_\mathfrak{p}$ such that the images of x_i's in $M \otimes_A k(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ form a basis of this $k(\mathfrak{p})$-vector space. By Nakayama's lemma, the $A_\mathfrak{p}$-linear homomorphism $\phi_\mathfrak{p} : A_\mathfrak{p}^r \to M_\mathfrak{p}$ determined by $x_1, \ldots, x_r$ is surjective. Then there exists $a \in A \backslash \mathfrak{p}$ such that $\phi_\mathfrak{p}$ extends to a surjective A_a-linear homomorphism $A_a^r \to M_a$. Replacing A by A_a, we can assume that there exists a surjective A-linear map

$$\phi : A^r \longrightarrow\!\!\!\!\!\rightarrow M.$$

For any $\mathfrak{q} \in \operatorname{Spec} A$, $\phi \otimes_A k(\mathfrak{q})$ is a surjective $k(\mathfrak{q})$-linear map of $k(\mathfrak{q})$-vector spaces of dimension r. Then $\phi \otimes_A k(\mathfrak{q})$ is an isomorphism. Let $K = \operatorname{Ker}(\phi)$, and hence

$$K_\mathfrak{q} \subset (\mathfrak{q}A_\mathfrak{q})^r, \quad \forall \mathfrak{q} \in \operatorname{Spec} A.$$

Since A is reduced, we have $K = 0$, and then ϕ is an isomorphism. So M is free. □

Lemma 5.9. *Let Y be a reduced noetherian affine scheme, and $\phi : \mathscr{F} \to \mathcal{G}$ be a morphism of finite and locally free $\mathscr{O}_Y$-modules. If*

$$\dim_{k(y)} \operatorname{im}(\phi \otimes_{\mathscr{O}_Y} k(y))$$

is locally constant, then we can find a decomposition of finite and locally free $\mathscr{O}_Y$-modules

$$\mathscr{F} = \mathscr{F}_1 \otimes \mathscr{F}_2, \quad \mathcal{G} = \mathcal{G}_1 \otimes \mathcal{G}_2$$

such that ϕ factors through $\mathcal{G}_1$, $\phi|_{\mathscr{F}_1} = 0$, and $\phi : \mathscr{F}_2 \to \mathcal{G}_1$ is an isomorphism.

Proof. Write $Y = \operatorname{Spec} A$ and $\mathscr{F} = \widetilde{M}$, $\mathcal{G} = \widetilde{N}$ for locally free A-modules M, N of finite rank; $\phi : M \to N$ is an A-linear map. For any $\mathfrak{p} \in \operatorname{Spec} A$,

$$\dim_{k(y)} \operatorname{Coker}(\phi \otimes_A k(y)) = \dim_{k(y)} N \otimes_A k(y) - \dim_{k(y)} \operatorname{im}(\phi \otimes_A k(y))$$

is locally constant. By Lemma 5.8, $\operatorname{Coker} \phi$ is a locally free A-module of finite rank. Define

$$N_1 := \operatorname{Ker}(N \to \operatorname{Coker} \phi) = \operatorname{im} \phi.$$

So we have an exact sequence

$$0 \to N_1 \to N \to \operatorname{Coker} \phi \to 0.$$

We see that N_1 is locally free of finite rank, and there is a decomposition

$$N = N_1 \oplus N_2$$

such that $N_2 \cong \operatorname{Coker} \phi$ under the natural map $N \to \operatorname{Coker} \phi$. Also define $M_1 = \operatorname{Ker} \phi$. We have an exact sequence

$$0 \to M_1 \to M \xrightarrow{\phi} N_1 \to 0.$$

This shows that M_1 is locally free of finite rank. Moreover, notice that the exact sequence splits at M. So there is a decomposition $M = M_1 \oplus M_2$ such that $\phi|_{M_2} : M_2 \to N_1$ is an isomorphism. $\square$

Now we are ready to prove the corollary.

Proof of Corollary 5.7. Applying Theorem 5.2 to $f : X \to Y$ and $\mathscr{F}$, we attain a cochain complex $K^\bullet$ such that for each $p \geqslant 0$,

$$H^p(X_y, \mathscr{F}_y) = H^p(K^\bullet \otimes_A k(y)).$$

Therefore,

$$\begin{aligned} &\dim_{k(y)} H^p(X_y, \mathscr{F}_y) \\ =&\dim_{k(y)} \operatorname{Ker}(d_K^p \otimes_A k(y)) - \dim_{k(y)} \operatorname{im}(d_K^{p-1} \otimes_A k(y)) \\ =&\dim_{k(y)} K^p \otimes_A k(y) - \dim_{k(y)} \operatorname{im}(d_K^p \otimes_A k(y)) - \dim_{k(y)} \operatorname{im}(d_K^{p-1} \otimes_A k(y)) \end{aligned}$$

is constant. Hence

$$\underbrace{\dim_{k(y)} \operatorname{im}(d_K^p \otimes_A k(y))}_{=\phi_1(y)} - \underbrace{\dim_{k(y)} \operatorname{im}(d_K^{p-1} \otimes_A k(y))}_{=\phi_2(y)}$$

is locally constant. Shrinking Y if necessary, we can assume that $\phi_1(y) + \phi_2(y) = C$ (constant) on Y. Since $\phi_1(y)$ and $\phi_2(y)$ are lower semicontinuous, there is a natural stratification on Y, read as

$$\begin{aligned} Y &= \bigsqcup_{n=0}^{c} \{y \in Y \mid \phi_1(y) = n,\ \phi_2(y) = c - n\} \\ &= \bigsqcup_{n=0}^{c} \{y \in Y \mid \phi_1(y) \leqslant n,\ \phi_2(y) \leqslant c - n\}. \end{aligned}$$

Since Y is connected, ϕ_1 and ϕ_2 are constant on Y. Now we can apply Lemma 5.9 to $d_K^p : K^p \to K^{p+1}$ and $d_K^{p-1} : K^{p-1} \to \mathrm{Ker}(d_K^p)$, to see there is a decomposition of locally free A-modules of finite rank:

$$\begin{array}{ccccccccc} & & Z^{p-1} \oplus (K')^{p-1} & & B^p \oplus H^p \oplus (K')^p & & B^{p+1} \oplus (K')^{p+1} & & \\ & & \| & & \| & & \| & & \\ \cdots & \longrightarrow & K^{p-1} & \xrightarrow{\quad d_K^{p-1} \quad} & K^p & \xrightarrow{\quad d_K^p \quad} & K^{p+1} & \longrightarrow & \cdots \end{array}$$

such that

$$Z^{p-1} = \mathrm{Ker}(d_K^{p-1}), \qquad d_K^{p-1} : (K')^{p-1} \xrightarrow{\cong} B^p = \mathrm{im}(d_K^{p-1});$$
$$B^p \oplus H^p = \mathrm{Ker}(d_K^p), \qquad d_K^p : (K')^p \xrightarrow{\cong} B^{p+1} = \mathrm{im}(d_K^p).$$

Therefore, for any A-algebra B,

$$H^p(K^\bullet \otimes_A B) \cong H^p \otimes_A B \cong H^p(K^\bullet) \otimes_A B.$$

Since $R^p f_* \mathscr{F}$ corresponds to the A-module

$$H^p(X, \mathscr{F}) \cong H^p(K^\bullet) \cong H^p,$$

we have that $R^p f_* \mathscr{F}$ is a locally free A-module of finite rank, and

$$(R^p f_* \mathscr{F}) \otimes_A B \cong H^p \otimes_A B \cong H^p(K^\bullet \otimes_A B) \cong H^p(X_y, \mathscr{F}_y).$$

This proves (2). Moreover, in this case,

$$\begin{aligned} (R^{p-1} f_* \mathscr{F}) \otimes_A k(y) &\cong H^{p-1}(X, \mathscr{F}) \otimes_A k(y) \\ &\cong \mathrm{Ker}(d_K^{p-1}) \otimes_A k(y) / \mathrm{im}(d_K^{p-1}) \otimes_A k(y) \\ &\cong Z^{p-1} \otimes_A k(y) / \mathrm{im}(d_K^{p-1} \otimes_A k(y)) \\ &\cong H^{p-1}(K^\bullet \otimes_A k(y)). \end{aligned}$$

Therefore,

$$(R^{p-1} f_* \mathscr{F}) \otimes_A k(y) \cong H^{p-1}(X_y, \mathscr{F}_y)$$

for all $y \in Y$. □

Corollary 5.10. *Under the above notations (Y may not be reduced or connected), assume that $H^p(X_y, \mathscr{F}_y) = 0$ for some p and all $y \in Y$. Then the rational map*

$$R^{p-1} f_* \mathscr{F} \otimes_{\mathscr{O}_Y} k(y) \xrightarrow{\cong} H^{p-1}(X_y, \mathscr{F}_y)$$

is an isomorphism for all $y \in Y$.

Proof. Let $K^\bullet$ be the cochain complex by Theorem 5.2. Fix $y \in Y$ such that

$$H^p(X_y, \mathscr{F}_y) \cong H^p(K^\bullet \otimes_A k(y)) = 0.$$

Then the sequence

$$K^{p-1} \otimes_A k(y) \xrightarrow{d_K^{p-1} \otimes_A k(y)} K^p \otimes_A k(y) \xrightarrow{d_K^p \otimes_A k(y)} K^{p+1} \otimes_A k(y)$$

is exact. We can decompose the $k(y)$-vector space $K^p \otimes_A k(y)$ as $\overline{W}_1 \oplus \overline{W}_2$ such that

$$\overline{W}_1 = \mathrm{im}(d_K^{p-1} \otimes_A k(y))$$

and $d_K^p \otimes_A k(y)|_{\overline{W}_2}$ is injective. Let $\{\overline{x}_1, \ldots, \overline{x}_r\}$ be a basis of $\overline{W}_1$ and $\{\overline{y}_1, \ldots, \overline{y}_s\}$ be a basis of $\overline{W}_2$. For $i = 1, \ldots, s$, denote

$$\overline{z}_i = d_K^p \otimes_A k(y)(\overline{y}_i) \in K^{p+1} \otimes_A k(y),$$

and extend $\{\overline{z}_1,\dots,\overline{z}_s\}$ to a basis $\{\overline{z}_1,\dots,\overline{z}_n\}$ of $K^{p+1}\otimes_A k(y)$. We choose lifting $x_i\in\operatorname{im}(d_K^{p-1})$ of $\overline{x}_i$ for $i=1,\dots,r$, $y_i\in K^p$ of $\overline{y}_j$ for $j=1,\dots,s$, and $z_i\in K^{p+1}$ of $\overline{z}_l$ for $l=1,\dots,s$. Shrinking A by a localization A_a at a such that $a(y)\neq 0$, one may assume that $\{x_1,\dots,x_r,y_1,\dots,y_r\}$ is a basis of K^p, and $\{z_1,\dots,z_n\}$ is a basis of K^{p+1}. Let W_1, W_2 be the free modules generated by $x_1,\dots,x_r$ and $y_1,\dots,y_s$, respectively. Then $K^p=W_1\oplus W_2$, where $W_1\subset\operatorname{im}(d_K^{p-1})$ and $d_K^p|_{W_2}$ is injective. Hence $W_1=\operatorname{im}(d_K^{p-1})$. As W_1 is free, it is projective. So there is a decomposition $K^{p-1}=W_1\oplus\operatorname{Ker}(d_K^{p-1})$. Now we have two exact sequences

$$K^{p-2}\xrightarrow{d_K^{p-2}}\operatorname{Ker}(d_K^{p-1})\longrightarrow H^{p-1}(K^\bullet)\cong H^{p-1}(X,\mathscr{F})\longrightarrow 0,$$

and

$$\begin{array}{ccccccc} K^{p-2}\otimes_A k(y) & \xrightarrow{d_K^{p-2}\otimes_A k(y)} & \operatorname{Ker}(d_K^{p-1}\otimes_A k(y)) & \longrightarrow & H^{p-1}(K^\bullet\otimes_A k(y)) & \longrightarrow & 0. \\ & & \| & & \| & & \\ & & \operatorname{Ker}(d_K^{p-1})\otimes_A k(y) & & H^{p-1}(X_y,\mathscr{F}_y) & & \\ & & \text{(by } K^{p-1}=W_1\oplus\operatorname{Ker}(d_K^{p-1})\text{)} & & & & \end{array}$$

Since the cokernel is stable under base changes, we have an isomorphism

$$\begin{array}{ccc} \boxed{H^{p-1}(X,\mathscr{F})}\otimes_A k(y) & \xrightarrow{\cong} & H^{p-1}(X_y,\mathscr{F}_y). \\ \| & & \\ R^{p-1}f_*\mathscr{F} & & \end{array}$$

This completes the proof. ∎

Corollary 5.11. *If $R^kf_*\mathscr{F}=0$ for $k\geqslant k_0$, then*
$$H^k(X_y,\mathscr{F}_y)=0,\quad \forall y\in Y,\ k\geqslant k_0.$$

Corollary 5.12 (Flat base change)**.** *If B is a flat A-algebra, then*
$$H^p(X\times_Y\operatorname{Spec}B,\mathscr{F}\otimes_A B)\cong H^p(X,\mathscr{F})\otimes_A B.$$

Corollary 5.13 (Seesaw's theorem)**.** *Let X be a complete*[7] *variety and T be any variety. Choose a line bundle $\mathscr{L}\in\operatorname{Pic}(X\times T)$. Then the set*
$$T_1:=\{t\in T\mid \mathscr{L}|_{X\times\{t\}}\text{ is trivial on }X\times\{t\}\}$$
is closed in T, and $\mathscr{L}|_{X\times T_1}\cong p_2^*\mathscr{M}$ for some $\mathscr{M}\in\operatorname{Pic}(T_1)$, where $p_2:X\times T_1\to T_1$ is the second projection.

Lemma 5.14. *A line bundle (i.e., an invertible sheaf) $\mathscr{M}$ on a complete variety X is trivial if and only if*
$$\dim H^0(X,\mathscr{M})>0,\quad \dim H^0(X,\mathscr{M}^{-1})>0.$$

Proof. Exercise. ∎

Proof of Seesaw's Theorem. It follows from Lemma 5.14 that
$$\begin{aligned} T_1&=\{t\in T\mid \mathscr{L}|_{X\times\{t\}}\text{ is trivial on }X\times\{t\}\}\\ &=\left\{t\in T\;\middle|\; \begin{array}{c}\dim_{k(t)}H^0((X\times T)\times_T\operatorname{Spec}(k(t)),\mathscr{L}\otimes_{\mathscr{O}_T}k(t))>0,\text{ and}\\ \dim_{k(t)}H^0((X\times T)\times_T\operatorname{Spec}(k(t)),\mathscr{L}^{-1}\otimes_{\mathscr{O}_T}k(t))>0\end{array}\right\}. \end{aligned}$$

[7] Can be replaced with properness.

By the semicontinuity theorem (Corollary 5.6), T_1 is closed in T. We regard T_1 as a reduced closed subscheme of T, and $p_2 : X \times T_1 \to T_1$ is a proper morphism of noetherian schemes. Denote for simplicity that $\mathscr{L}_1 = \mathscr{L}|_{X\times T_1}$. By definition of T_1, for any $t \in T_1$,

$$\dim_{k(t)} H^0((X \times T_1) \times_{T_1} \operatorname{Spec}(k(t)), \mathscr{L}_1 \otimes_{\mathscr{O}_{T_1}} k(t)) > 0$$

By Corollary 5.7, $\mathscr{M} := p_{2,*}\mathscr{L}_1$ is an invertible sheaf on T_1 and the natural map

$$p_{2,*}\mathscr{L}_1 \otimes_{\mathscr{O}_{T_1}} k(t) \longrightarrow H^0(X \times \{t\}, \mathscr{L}_1|_{X\times\{t\}})$$

is an isomorphism for any $t \in T_1$.

We prove that the natural morphism $p_2^*\mathscr{M} \to \mathscr{L}_1$ is an isomorphism. In fact, for any $t \in T_1$, the sheaf $p_2^*\mathscr{M}|_{X\times\{t\}}$ is the inverse image of $\mathscr{M}$ under

$$X \times \{t\} \hookrightarrow X \times T_2 \xrightarrow{p_2} T_2.$$

It is the trivial invertible sheaf on $X\times\{t\}$ and is the pullback of the $k(t)$-vector space $p_{2,*}\mathscr{L}_1\otimes_{\mathscr{O}_{T_1}} k(t)$ under $X \times \{t\} \to \{t\} = \operatorname{Spec}(k(t))$. On the other hand, $\mathscr{L}_1|_{X\times\{t_1\}}$ is also trivial and the restriction of $p_2^*\mathscr{M} \to \mathscr{L}_1$ on $X \times \{t\}$ corresponds to the morphism

$$p_{2,*}\mathscr{L}_1 \otimes_{\mathscr{O}_{T_1}} k(t) \longrightarrow H^0(X \times \{t\}, \mathscr{L}_1|_{X\times\{t\}})$$

of global sections. Therefore, the restriction of $p_2^*\mathscr{M} \to \mathscr{L}_1$ on $X \times \{t\}$ is an isomorphism for each $t \in T_1$. This is enough to show that $p_2^*\mathscr{M} \to \mathscr{L}_1$ is itself an isomorphism. □

Remark 5.15. We can assume that T is a (reduced) scheme of finite type over an algebraically closed field k.

6. The Theorem of the Cube (I)

All varieties live on an algebraically closed field k.

6.1. **Statement and the Primary Ingredients.**

Theorem 6.1 (The theorem of the cube)**.** *Let X, Y be complete normal varieties, and Z be any normal variety. Take x_0, y_0, and z_0 as base (closed) points on X, Y, and Z, respectively. If $\mathscr{L}$ is any line bundle on $X \times Y \times Z$ where restrictions to $\{x_0\} \times Y \times Z$, $X \times \{y_0\} \times Z$, and $X \times Y \times \{z_0\}$ are all trivial, then $\mathscr{L}$ is trivial.*

This section is primarily dedicated to the proof of this theorem. We begin with introducing two lemmas.

Lemma 6.2 (Arcwise connectedness of complete varieties)**.** *Let X be a complete variety and x_0, x_1 be two closed points of X. Then there exists an irreducible curve C on X containing x_0 and x_1.*

Proof. Use induction on $\dim_k X$. May assume $\dim X > 1$, and by Chow's lemma[8] X can be taken as a projective variety. Now we can find a birational morphism $f : X' \to X$ with X' projective, satisfying $\dim f^{-1}(x_i) \geqslant 1$ for $i = 0, 1$. For example, we can take X' to be the blow-up of X along the closed subscheme $\{x_0, x_1\}$.

[8] The topological completeness is interpreted as the properness in an algebraic sense. And Chow's lemma is a machine to turn the conditions for projective varieties into those for simply proper varieties. More precisely, if X itself is not projective but complete, then there is a projective variety X' and a birational morphism $X' \to X$. This is moreover surjective as a map.

Another way of construction is in [Mum85]. Choose a rational function h on X with indeterminacies at x_1 and x_2. Let X' be the graph[9] of h. Then X' is projective and the first projection $X \times \mathbb{P}^1 \to X$ induces a birational morphism $f : X' \to X$. If $\dim f^{-1}(x_i) = 0$ for $i = 0, 1$, by dimension theory, there are open neighborhoods V_i of x_i in X such that

$$g = f|_{f^{-1}(V_i)} : f^{-1}(V_i) \longrightarrow V_i$$

is quasi-finite. As g is proper, by Zariski Main Theorem, we infer that g is finite. Suppose V_i is normal without loss of generality (otherwise one can replace X by its normalization), and g is birational, we see g must be an isomorphism. Therefore, h is well-defined at x_i, which is a contradiction. Therefore,

$$\dim f^{-1}(x_i) \geqslant 1, \quad i = 0, 1.$$

Now we choose a projective embedding $X' \hookrightarrow \mathbb{P}^N$ for some N. By Bertini's theorem, there is a hyperplane of $\mathbb{P}^N$, say H, that does not contain X', such that $Y' := H \cap X'$ is irreducible. Again, since $\dim f^{-1}(x_i) \geqslant 1$, we see $H \cap f^{-1}(x_i) \neq \emptyset$.

Let $Y = f(Y') \subset X$ be with the reduced irreducible closed subscheme structure that contains x_0, x_1 and $\dim Y = \dim X - 1$. By induction, we can find an irreducible curve $C \subset Y$ containing x_0, x_1. So we are done. □

Lemma 6.3. *Let X be a smooth projective curve with a fixed line bundle $\mathscr{L}$ on it. For any divisor D on X with*

$$h^0(D) := \dim H^0(X, \mathscr{L}_X(D)) > 0,$$

we have

$$h^0(D - P) = h^0(D) - 1$$

for all but finitely many closed points P on X.

Proof. In fact, we have an exact sequence of sheaves on X:

$$0 \to \mathscr{L}(D - P) \to \mathscr{L}(D) \to k(P) \to 0.$$

This induces the left-exact cohomological sequence

$$0 \to H^0(X, \mathscr{L}_X(D - P)) \to H^0(X, \mathscr{L}_X(D)) \xrightarrow{\varphi} k(P).$$

Pick a nonzero section $f \in H^0(X, \mathscr{L}_X(D))$, the following set is finite:

$$\#\{P \in X(k) \mid f_P \in \mathfrak{m}_P \mathscr{L}_X(D)_P\} < \infty.$$

For those P landing outside this set, $\varphi : H^0(X, \mathscr{L}_X(D)) \to k(P)$ is surjective (so that the second sequence is right-exact). Hence $h^0(D - P) = h^0(D) - 1$. □

6.2. **Proof of the Theorem of Cube.**

Proof of Theorem 6.1. We begin with some reductions.

(1) **First reduction:** *by symmetry, it suffices to show that for any closed point $(x, z) \in X \times Z$, the invertible sheaf $\mathscr{L}|_{\{x\} \times Y \times \{z\}}$ is trivial.*

    In fact, notice that $X \times Z$ is Jacobson and hence closed points are dense in $X \times Z$. By Seesaw's theorem (Corollary 10.7), $\mathscr{L}|_{\{x\} \times Y \times \{z\}}$ is trivial for any point (x, z) of $X \times Z$ and hence there is a line bundle $\mathscr{M}$ on $X \times Z$ such that $\mathscr{L} \cong \pi^* \mathscr{M}$ along the projection $\pi : X \times Y \times Z \to X \times Z$. Since $\mathscr{L}|_{X \times \{y_0\} \times Z}$ is trivial, we see $\mathscr{M}$ is trivial. This deduces the triviality of $\mathscr{L}$ itself.

[9] Let U be the maximal open subvariety of X on which h is well-defined. Then the graph of h is defined to be the image of the morphism $(i, h) : U \to X \times \mathbb{P}^1$, where $i : U \hookrightarrow X$ is the open immersion.

(2) **Second reduction:** *it suffices to prove the theorem under the assumption that X is a smooth projective curve and Y, Z are normal varieties.*

By Step (1), we need to show that $\mathscr{L}|_{\{x\}\times Y\times\{z\}}$ is trivial for any closed point $(x, y) \in X \times Z$. By Lemma 6.2 we can find an irreducible curve C of X that contains x_0 and x. Let $\pi : C' \to C$ be the normalization of C. By assumption, C' is a smooth projective curve. Pick a closed point $x' \in \pi^{-1}(x)$. We also denote $\pi : C' \times Y \times Z \to C \times Y \times Z$. So that

$$(\pi^*\mathscr{L})|_{\{x'\}\times Y\times\{z\}} \cong \mathscr{L}|_{\{x\}\times Y\times\{z\}}.$$

So we can assume that X is a smooth projective curve. Consider the normalizations of Y and Z. By a similar argument as above, they are assumed to be normal.

(3) **Third reduction:** *it boils down to find a nonempty open subset Z' of Z such that $\mathscr{L}|_{X\times Y\times Z'}$ is trivial.*

If so, $\mathscr{L}|_{X\times Y\times\{z\}}$ is trivial for any point $z \in Z'$. Since Z' is dense in Z, we see that $\mathscr{L}|_{X\times Y\times\{z\}}$ is trivial for all $z \in Z$ by Seesaw's theorem (Corollary 10.7). Therefore, $\mathscr{L}|_{\{x\}\times Y\times\{z\}}$ is trivial for any closed point $(x, z) \in X \times Z$.

Now we are ready to prove the theorem of cubes. Let Ω^1_X be the sheaf of differentials of X/k, and $g := \dim H^0(X, \Omega^1_X)$ be the genus of X. We can find g closed points $p_1, \dots, p_g$ such that

$$H^0(X, \Omega^1_X \otimes \mathscr{L}_X(-D)) = 0,$$

where $D = \sum_{i=1}^g P_i$. This follows from Lemma 6.3. For such a divisor D, we define

$$\mathscr{L}' = \mathscr{L} \otimes p_1^*\mathscr{L}_X(D),$$

where $p_1 : X \times Y \times Z \to X$ is the first projection. For any point $y \in Y$, we have

$$\mathscr{L}'|_{X\times\{y\}\times\{z_0\}} \cong \mathscr{L}_X(D),$$

and

$$\dim H^1(X, \mathscr{L}_X(D)) = \dim H^0(X, \Omega^1_X \otimes \mathscr{L}_X(-D)) = 0$$

by Serre duality. If one uses F to denote the closed subset

$$\{(y, z) \in Y \times Z \mid \dim H^1(X, \mathscr{L}'_{(y,z)}) \geqslant 1\} \subset Y \times Z,$$

where $\mathscr{L}'_{(y,z)} = \mathscr{L}'|_{X\times\{y\}\times\{z\}}$, we see that $F \cap (Y \times \{z_0\}) = \emptyset$. Since $Y \times Z \to Z$ is proper, we can find an open subset Z' of Z such that $F \cap (Y \times Z') = \emptyset$. By Step (3) above, it suffices to prove that $\mathscr{L}|_{X\times Y\times Z'}$ is trivial. Replacing Z by Z', we can assume that for all points $(y, z) \in Y \times Z$, $F \cap (Y \times Z) = \emptyset$, i.e., $H^1(X, \mathscr{L}'_{(y,z)}) = 0$. Consequently, by Corollary 5.6 in the previous lecture,

$$\begin{aligned}\dim H^0(X, \mathscr{L}'_{(y,z)}) &= \chi(\mathscr{L}'_{(y,z)}) \overset{(5.6)}{=} \chi(\mathscr{L}'_{(y_0,z_0)}) = \chi(\mathscr{L}_X(D)) \\ &= \deg D + 1 - g = 1.\end{aligned}$$

Consider the natural projection $p_{23} : X \times Y \times Z \to Y \times Z$. By Corollary 5.7, we see that $p_{23,*}\mathscr{L}'$ is an invertible sheaf on $Y \times Z$ and

$$p_{23,*}\mathscr{L}' \otimes k(y, z) \longrightarrow H^0(X, \mathscr{L}'_{(y,z)})$$

is an isomorphism for all points (y, z) of $Y \times Z$.

Denote $\mathscr{M} = p_{23,*}\mathscr{L}'$.[10] We define an effective Cartier divisor $\widetilde{D}$ on $X \times Y \times Z$ that corresponds to the invertible sheaf $\mathscr{L}' \in \mathrm{Pic}(X \times Y \times Z)$ as follows: for any open subset U of $Y \times Z$ such that $\mathscr{M}|_U$ is trivial, we choose a generating section $\sigma_U \in \Gamma(U, \mathscr{M})$. Since

$$\Gamma(U, \mathscr{M}) \cong \Gamma(U, p_{23,*}\mathscr{L}') \cong \Gamma(X \times U, \mathscr{L}'),$$

we obtain a nonzero section $f_U \in \Gamma(X \times U, \mathscr{L}')$. Let $\widetilde{D}_U$ be the effective Weil divisor on $X \times U$ associated to f_U (i.e., the divisor of zeros of f_U, see [Har13, II, §7]). Note that two different generating sections of $\mathscr{M}|_U$ in $\Gamma(U, \mathscr{M})$ are differed by an element in $\Gamma(U, \mathscr{O}_U^*)$.

It follows that the collections $\{U, \widetilde{D}_U\}_U$ (where U runs through open subsets of $Y \times Z$) defines an effective Weil divisor $\widetilde{D}$ (by abuse of notation) that correspond to $\mathscr{L}'$ under the natural isomorphism

$$\mathrm{CaCl}(X) \cong \mathrm{Pic}(X)$$

(see [Har13, II, §6]).

A key property for $\widetilde{D}$ is in the following. For any closed point $(y, z) \in Y \times Z$, $\widetilde{D}|_{X \times \{y\} \times \{z\}}$ is the effective Cartier divisor associated to a nonzero section of $H^0(X, \mathscr{L}'_{(y,z)})$. The condition $\dim H^0(X, \mathscr{L}'_{(y,z)}) = 1$ implies that the linear system for $\mathscr{L}'_{(y,z)}$ consists of a single effective divisor, i.e., there is a unique effective divisor E with $\mathscr{L}'_{(y,z)} = \mathscr{L}_X(E)$. This fact is implicitly but crucially used in the argument below.

Fix a closed point P of X such that $P \neq P_i$ for $i = 1, \dots, g$. Let S be the support of $\widetilde{D}|_{\{P\} \times Y \times Z}$ which is a closed subset of $\{P\} \times Y \times Z$, and all the irreducible components of S have codimension one in $\{P\} \times Y \times Z$.[11]

Since $\widetilde{D}|_{X \times \{y\} \times \{z_0\}} \cong \mathscr{L}_X(D)$, we have

$$S \cap \{P\} \times Y \times \{z_0\} = \emptyset.$$

Hence the image of S under $Y \times Z \to Z$ is a proper closed subset of Z. Thus,

$$S = \bigcup (\{P\} \times Y \times T_j)$$

where $T_j \subset Z$ are closed irreducible subvarieties of codimension 1. However, $S \cap (\{P\} \times \{y_0\} \times Z) = \emptyset$ for $P \neq P_i$, $i = 1, \dots, g$. Denote D' the Weil divisor on $X \times Y \times Z$ associated to $\widetilde{D}$. It follows that

$$D' = \sum_{i=1}^{g} n_i \{P_i\} \times Y \times Z.$$

Restricting to $X \times \{y_0\} \times \{z_0\}$, we see each $n_i = 1$. Then

$$\mathscr{L}'_{(y,z)} \cong \mathscr{L}_X(D), \quad \forall (y, z) \in Y \times Z.$$

Consequently, $\mathscr{L}|_{X \times \{y\} \times \{z\}}$ is trivial for all $(y, z) \in Y \times Z$. Finally, we apply Step (1) to $Y \times Z$ to see that $\mathscr{L}$ is trivial on $X \times Y \times Z$. This finishes the proof. $\square$

[10] A priori we have this, whereas the natural pushforward map $p_{23}^*\mathscr{M} \to \mathscr{L}'$ is NOT an isomorphism in general. In fact, compared with the proof of Seesaw's theorem (Corollary 10.7), we need to assume that $\mathscr{L}'|_{X \times \{y\} \times \{z\}}$ is trivial for all points $(y, z) \in Y \times Z$. But this is not the case in our discussion. As a consequence, $\mathscr{L}'|_{X \times U}$ is NOT trivial in general.

[11] In fact, this is Krull's Hauptidealsatz in a general case.

6.3. **Consequences of the Main Theorem.**

Corollary 6.4. *Let X, Y, and Z be the same as in Theorem 6.1. Then any line bundle on $X \times Y \times Z$ is isomorphic to*
$$\pi_1^*\mathscr{L}_1 \otimes \pi_2^*\mathscr{L}_2 \otimes \pi_3^*\mathscr{L}_3,$$
where $\pi_1 : X \times Y \times Z \to Y \times Z$, $\pi_2 : X \times Y \times Z \to X \times Z$, $\pi_3 : X \times Y \times Z \to X \times Y$ are natural projections; $\mathscr{L}_1 \in \operatorname{Pic}(Y \times Z)$, $\mathscr{L}_2 \in \operatorname{Pic}(X \times Z)$, and $\mathscr{L}_3 \in \operatorname{Pic}(X \times Y)$.

Proof. Using the same method as in the proof of Theorem 6.1. Define
$$\sigma_1 : Y \times Z \to X \times Y \times Z, \quad (y, z) \mapsto (x_0, y, z),$$
and also σ_2, σ_3 in respective similar ways. And
$$\pi_X : X \times Y \times Z \to X, \quad \pi_Y : X \times Y \times Z \to Y, \quad \pi_Z : X \times Y \times Z \to Z.$$
Again, define
$$\sigma_X : X \to X \times Y \times Z, \quad x \mapsto (x, y_0, z_0)$$
and also σ_Y, σ_Z in respective similar ways. Define
$$\mathscr{L}_? = \sigma_?^*\mathscr{L}, \quad ? = 1, 2, 3, X, Y, Z,$$
and
$$\begin{aligned}\mathscr{M} &= \mathscr{L} \otimes \pi_1^*\mathscr{L}_1^{-1} \otimes \pi_2^*\mathscr{L}_2^{-1} \otimes \pi_3^*\mathscr{L}_3^{-1} \otimes \pi_X^*\mathscr{L}_X^{-1} \otimes \pi_Y^*\mathscr{L}_Y^{-1} \otimes \pi_Z^*\mathscr{L}_Z^{-1} \\ &\in \operatorname{Pic}(X \times Y \times Z).\end{aligned}$$
It is straightforward to verify that $\sigma_i^*\mathscr{M}$ is trivial for $i = 1, 2, 3$. By theorem of cube (Theorem 6.1), $\mathscr{M}$ is trivial and hence
$$\mathscr{L} \cong \pi_1^*\mathscr{L}_1 \otimes \pi_2^*\mathscr{L}_2 \otimes \pi_3^*\mathscr{L}_3 \otimes \pi_X^*\mathscr{L}_X^{-1} \otimes \pi_Y^*\mathscr{L}_Y^{-1} \otimes \pi_Z^*\mathscr{L}_Z^{-1},$$
which is of the form in the corollary. □

Corollary 6.5. *Let X be any variety and Y be an abelian variety. Let $f, g, h : X \to Y$ be morphisms. Then for each $\mathscr{L} \in \operatorname{Pic}(Y)$, we have*
$$(f + g + h)^*\mathscr{L} \cong (f + g)^*\mathscr{L} \otimes (f + h)^*\mathscr{L} \otimes (g + h)^*\mathscr{L} \otimes f^*\mathscr{L}^{-1} \otimes g^*\mathscr{L}^{-1} \otimes h^*\mathscr{L}^{-1}.$$

Proof. Let $p_i : Y \times Y \times Y \to Y$ be the projection to the ith factor for $i = 1, 2, 3$. Also, denote $m_{ij} = p_i + p_j$, and $m = p_1 + p_2 + p_3 : Y \times Y \times Y \to Y$. Define
$$\begin{aligned}\mathscr{M} &= m^*\mathscr{L} \otimes m_{12}^*\mathscr{L}^{-1} \otimes m_{23}^*\mathscr{L}^{-1} \otimes m_{13}^*\mathscr{L}^{-1} \otimes p_1^*\mathscr{L} \otimes p_2^*\mathscr{L} \otimes p_3^*\mathscr{L} \\ &\in \operatorname{Pic}(Y \times Y \times Y).\end{aligned}$$
One can verify that $\mathscr{M}|_{\{e_Y\} \times Y \times Y}$, $\mathscr{M}|_{Y \times \{e_Y\} \times Y}$, and $\mathscr{M}|_{Y \times Y \times \{e_Y\}}$ are all trivial. By Theorem 6.1, $\mathscr{M}$ itself is trivial. We pull $\mathscr{M}$ back along $(f, g, h) : X \to Y \times Y \times Y$ and get the desired isomorphism. □

Corollary 6.6. *If X is an abelian variety and $n \in \mathbb{Z}$, then for all $\mathscr{L} \in \operatorname{Pic}(X)$,*
$$n_X^*\mathscr{L} \cong \mathscr{L}^{\frac{n^2+n}{2}} \otimes (-1)_X^*\mathscr{L}^{\frac{n^2-n}{2}}.$$

Proof. In Corollary 6.5, take $f = (n + 1)_X$, $g = 1_X$, and $h = (-1)_X$ to deduce
$$(n + 1)_X^*\mathscr{L} \otimes n_X^*\mathscr{L}^{-2} \otimes (n - 1)_X^*\mathscr{L} \cong \mathscr{L} \otimes (-1)_X^*\mathscr{L},$$
and hence
$$n_X^*\mathscr{L} \otimes (n - 1)_X^*\mathscr{L}^{-1} \cong \mathscr{L}^n \otimes ((-1)_X^*\mathscr{L})^{n-1}.$$

We can infer from it that

$$n_X^*\mathscr{L} \cong \mathscr{L}^{\frac{n^2+n}{2}} \otimes ((-1)_X^*\mathscr{L})^{\frac{n^2-n}{2}}.$$

□

Corollary 6.7 (Theorem of square)**.** *For any $\mathscr{L} \in \mathrm{Pic}(X)$ and closed points $x, y \in X$, where X is an abelian variety, we have*

$$T_{x+y}^*\mathscr{L} \otimes \mathscr{L} \cong T_x^*\mathscr{L} \otimes T_y^*\mathscr{L}.$$

Here $T_x : X \to X$ is the translation by x. In other words, the map

$$\phi_{\mathscr{L}} : X(k) \to \mathrm{Pic}(X), \quad x \mapsto T_x^*\mathscr{L} \otimes \mathscr{L}^{-1}$$

is a group homomorphism.

Proof. In Corollary 6.5, we take $X = Y$ and

$$f : X \to k \xrightarrow{x} X, \quad g : X \to k \xrightarrow{y} X, \quad h = \mathrm{id}_X : X \to X$$

to complete the proof. □

Remark 6.8. The map $\phi_{\mathscr{L}} : X(k) \to \mathrm{Pic}(X)$ defined above has the following properties:

(1) $\phi_{\mathscr{L}_1 \otimes \mathscr{L}_2} = \phi_{\mathscr{L}_1} +_{\mathrm{Pic}(X)} \phi_{\mathscr{L}_2}$;
(2) $\phi_{T_x^*\mathscr{L}} = \phi_{\mathscr{L}}$.

Definition 6.9. Let X be an abelian variety as above. For $\mathscr{L} \in \mathrm{Pic}(X)$, define

$$K(\mathscr{L}) := \mathrm{Ker}(\phi_{\mathscr{L}}) = \{x \in X(k) \mid T_x^*\mathscr{L} \cong \mathscr{L}\}.$$

Proposition 6.10. *$K(\mathscr{L})$ is a Zariski closed subset of X (here we view X as an algebraic variety over k).*

Proof. Consider the line bundle $\mathscr{M} = m^*\mathscr{L} \otimes p_2^*\mathscr{L}^{-1}$ on $X \times X$, where m is the addition map and p_2 is the second projection. By Seesaw's theorem 10.7,

$$F := \{x \in X \mid \mathscr{M}|_{\{x\}\times X} \text{ is trivial}\}$$

is a closed subset of X. When $x \in X(k)$, namely, x is a closed point, we have

$$\mathscr{M}|_{\{x\}\times X} \cong T_x^*\mathscr{L} \otimes \mathscr{L}^{-1},$$

then $K(\mathscr{L})$ is the set of closed points of F. Therefore, $K(\mathscr{L})$ is Zariski closed in the algebraic variety X. □

6.4. Some Further Applications.

The following theorem is the first explicit application of theorem of cube and its corollaries.

Theorem 6.11. *Let D be an effective divisor on an abelian variety X, and $\mathscr{L} \cong \mathscr{L}_X(D)$ be the associated invertible sheaf. Then the following are equivalent:*

(1) *The (complete) linear system $|2D|$ has no base point and defines a finite morphism $X \to \mathbb{P}^N$ with $N = \dim H^0(X, \mathscr{L}(2D)) - 1$.*
(2) *$\mathscr{L}$ is ample on X.*
(3) *$K(\mathscr{L})$ is finite.*
(4) *The subgroup $H = \{x \in X(k) \mid T_x^*(D) = D\}$ of $X(k)$ is finite. Here $T_x^*(D) = D$ is an equality of divisors rather than divisor classes.*

Proof. We first prove (1) ⇒ (2) ⇒ (3) ⇒ (4). After this, we setup a lemma in order to prove (4) ⇒ (1).

(1) ⇒ (2) This follows from the fact that under a finite morphism of complete varieties, the inverse image of an ample line bundle is again ample (cf. [Har13, III, Exer 5.7]).

(2) ⇒ (3) Suppose that $K(\mathscr{L})$ is a positive dimensional k-scheme[12] and let Y be the connected component of e_X in $K(\mathscr{L})$. Then Y is an abelian variety of positive dimension. Since $Y \hookrightarrow X$ is a closed immersion, $\mathscr{L}_Y := \mathscr{L}|_X$ is ample on Y. Since $\mathscr{L}_Y$ is stable under the translation T_y for any $y \in Y(k)$, we have $T_y^*\mathscr{L}_y \cong \mathscr{L}_y$ (now we view T_y as the translation by y on Y rather than on X). Hence the line bundle

$$\mathscr{M} = m^*\mathscr{L}_Y \otimes p_1^*\mathscr{L}_Y^{-1} \otimes p_2^*\mathscr{L}_Y^{-1}$$

on $Y \times Y$ such that $\mathscr{M}|_{\{y\}\times Y}$ and $\mathscr{M}|_{Y\times\{y\}}$ are trivial for all $y \in Y(k)$. Here m is addition and p_i is the ith projection. By Seesaw's theorem (Corollary 10.7), $\mathscr{M}$ is trivial. We pull back $\mathscr{M}$ along the morphism $(1_Y, (-1)_Y) : Y \to Y \times Y$ and see that $\mathscr{L}_Y \otimes (-1)_Y^*\mathscr{L}_Y$ is trivial on Y. Since $\mathscr{L}_Y$ is ample and $(-1)_Y$ is an automorphism of Y, $(-1)_Y^*\mathscr{L}_Y$ is ample. Then $\mathscr{L}_Y \otimes (-1)_Y^*\mathscr{L}_Y$ is ample, i.e., $\mathscr{O}_Y$ itself is ample on Y. But this is impossible when $\dim Y > 0$. Then $\dim K(\mathscr{L}) = 0$ and $K(\mathscr{L})$ is finite.

(3) ⇒ (4) This is obvious as $H \subset K(\mathscr{L})$.

(4) ⇒ (1) By the theorem of the square (Corollary 6.7), the (complete) linear system $|2D|$ contains the divisor $T_x^*(D) + T_{-x}^*(D)$ for all $x \in X(k)$. For any $P \in X(k)$, we can find $x \in X(k)$ such that $P \pm x \notin \mathrm{Supp}(D)$ if and only if $P \notin \mathrm{Supp}(T_x^*(D) + T_{-x}^*(D))$. It follows that $|2D|$ is base-point-free and any basis of $H^0(X, \mathscr{L}_X(2D))$ gives a morphism $\phi : X \to \mathbb{P}^N$. Since ϕ is proper, it follows from Zariski Main Theorem that to prove ϕ is finite, it suffices to prove that ϕ is quasi-finite. Suppose it is not the case for the sake of contradiction. Then we can find an irreducible curve C on X such that $\phi(C)$ is a single closed point of $\mathbb{P}^N$. It follows that for any (Weil) divisor D' in $|2D|$, we have either $C \cap D' = \emptyset$ or $C \subset D'$.[13]

We now introduce the following lemma.

Lemma 6.12. *If C is an irreducible curve and E is a prime divisor on X such that $C \cap E = \emptyset$, then E is invariant under translations defined by $x_1 - x_2$ for all $x_1, x_2 \in C(k)$.*

Proof of Lemma. We use $\mathscr{L}$ to denote the invertible sheaf $\mathscr{L}_X(E)$ associated to E. Since $E \cap C = \emptyset$, $\mathscr{L}|_C$ is trivial and hence $\deg(\mathscr{L}|_C) = 0$. Let $\mathscr{M}$ be the pullback of $\mathscr{L}$ along the morphism $X \times C \hookrightarrow X \times X \xrightarrow{m} X$. We infer that $\mathscr{M}|_{\{x\}\times C} \cong (T_x^*\mathscr{L})|_C$.

For any invertible sheaf $\mathscr{N}$ on $X \times C$, since $p_1 : X \times C \to X$ is proper and flat, by Corollary 5.6 (2), we see that the function $x \mapsto \chi(\mathscr{N}|_{\{x\}\times C})$ is constant, i.e., $\chi(\mathscr{N}|_{\{x\}\times C})$ is independent of $x \in X(k)$. Replacing $\mathscr{N}$ by $\mathscr{N}^n$ for all $n \in \mathbb{Z}_{>0}$, we get $x \longmapsto \chi(\mathscr{N}^n|_{\{x\}\times C})$ is independent of $x \in X(k)$ as a function in n. However, C is a curve and it is well-known that

$$n \longmapsto \chi(\mathscr{N}^n|_{\{x\}\times C})$$

is a linear function on n with the linear coefficient $\deg(\mathscr{N}|_{\{x\}\times C})$. Therefore, the function

$$x \mapsto \deg(\mathscr{N}|_{\{x\}\times C})$$

is constant. To summarize, we have $\deg(T_x^*\mathscr{L})|_C = 0$ for all $x \in X(k)$. Then E and $T_x(C)$ cannot intersect at only finitely many (but not empty) closed points. Thus, either $E \cap T_x(C) = \emptyset$ or $T_x(C) \subset E$. Fix $y \in E(k)$ and $x_1, x_2 \in C(k)$. Since $y \in T_{y-x_2}(C) \cap E$, we get $T_{y-x_2}(C) \subset E$. Therefore, $y - x_2 + x_1 \in E$. This proves the lemma. □

[12] See [Har13, II, Prop 2.6] for the functor $t : \mathsf{Var}_k \to \mathsf{Sch}_k$ from the category of varieties over k to schemes over k.

[13] In fact, D' corresponds to a nonzero section in $\Gamma(X, \mathscr{L}_X(2D))$ and hence D' is the preimage of a hyperplane under ϕ.

Resume on. Now we are to finish the proof of (4) $\Rightarrow$ (1).

(4) $\Rightarrow$ (1) Fix $P \in C(k)$. By our previous discussion, there exists $x \in X(k)$ such that $P \notin \operatorname{Supp}(T_x^*(D) + T_{-x}^*(D))$. Then $C \cap \operatorname{Supp}(T_x^*(D) + T_{-x}^*(D)) = \emptyset$. Let $C' = T_x(C)$ and then $C' \cap D = \emptyset$. If $D = \sum_i n_i D_i$ with $n_i > 0$ and D_i prime divisors, then $C' \cap D_i = \emptyset$ follows. By the lemma, D_i must be stable under translations by $x_1 - x_2$ for all $x_1, x_2 \in C(k)$. This contradicts with the condition that H is finite. Hence ϕ is finite.

This completes the proof of the theorem. $\square$

Corollary 6.13. *An abelian variety X is projective.*

Proof. It suffice to find an effective Weil divisor D on X such that $H = \{x \in X(k) \mid T_x^*(D) = D\}$ is finite. We first prove the following lemma.

Lemma 6.14. *Let X be a noetherian, separable, normal, and integral scheme; let $U \subset X$ be a nonempty affine open subset and $U \neq X$. Then every irreducible component of $X \backslash U$ is of codimension 1 in X.*

Proof of Lemma. By the noetherian condition, $X \backslash U$ has only finitely many irreducible components. Let ξ be a generic point of $X \backslash U$. We can find an affine open neighborhood V of ξ in X such that ξ is the only generic point of $X \backslash U$ in V. It suffices to prove that $\dim \mathscr{O}_{X,\xi} = 1$.

Suppose not, then $V \backslash (U \cap V)$ has codimension $\geqslant 2$ in V. Recall the following result: if A is an normal integral domain of dimension $\geqslant 1$, then we have the following equality in $\operatorname{Frac}(A)$:

$$A = \bigcap_{\substack{\mathfrak{p} \in \operatorname{Spec}(A), \\ \operatorname{ht}(\mathfrak{p}) = 1}} A_{\mathfrak{p}}.$$

As a result, the restriction map $\Gamma(V, \mathscr{O}_X) \to \Gamma(V \cap U, \mathscr{O}_X)$ is an isomorphism (here we use the fact that $U \cap V$ is affine as X is separated). Therefore, that $V \cap U \to V$ is an isomorphism and $V \backslash V \cap U = \emptyset$ are both valid, which is a contradiction. Then $\dim \mathscr{O}_{X,\xi} = 1$. $\square$

Resume on. For the abelian variety X, we choose an affine open neighborhood U of e_X and the above lemma implies that the irreducible component $D_1, \ldots, D_n$ of $X \backslash U$ are all of codimension one. So $D = \sum_{i=1}^n D_i$ is a Weil divisor on X. Also,

$$H = \{x \in X(k) \mid T_x^*(D) = D\}$$

is a closed subgroup of $X(k)$.[14] In particular, H is proper.

On the other hand, for each $x \in H$, U is stable under T_x. As $e_x \in U$, we have $x \in U$ and then $H \subset U$. However, H itself is proper and U is affine. This forces H to be finite. $\square$

Here comes the second application of the result.

Proposition 6.15. *An abelian variety is a divisible group, and for each $n \geqslant 1$, $X_n = \operatorname{Ker}(n_X : X \to X)$ is finite over k.*

Proof. By dimension theory, it suffices to prove that X_n is finite. We view X_n as a (reduced) closed subscheme of X. Let $\mathscr{L} \in \operatorname{Pic}(X)$ be an ample line bundle.[15] Clearly $(n_X^* \mathscr{L})|_{X_n}$ is trivial. On the

[14] Alternatively, use the description

$$H = \{x \in X(k) \mid T_x^*(D) \subset D\} = \bigcap_{d \in D(k)} (d - D).$$

[15] Indeed, one may be able to prove that $n_X : X \to X$ is flat (cf. [Har13, III, Exer 10.9]).

other hand,

$$n_X^*\mathscr{L} \cong \mathscr{L}^{\frac{n(n+1)}{2}} \otimes ((-1)_X^*\mathscr{L})^{\frac{n(n-1)}{2}}.$$

Since $(-1)_X^*\mathscr{L}$ is also ample, the pullback $n_X^*\mathscr{L}$ is ample. In particular, so also is $(n_X^*\mathscr{L})|_{X_n}$ and hence X_n is finite. □

It is known tat $n_X : X \to X$ is a finite surjective homomorphism; in particular, n_X is dominated. Also, n_X induces a field embedding $n_X^* : k(X) \to k(X)$. Denote $\deg(n_X)$ the degree of this field extension, which is called the *degree of* n_X. One can similarly define the separable degree and the inseparable degree.

By intersection theory, we have

$$(n_X^*D_1, \dots, n_X^*D_g) = \deg(n_X)(D_1, \dots, D_g), \quad g = \dim X$$

for arbitrary Cartier divisors $D_1, \dots, D_g$ on X. We take D to be an ample symmetric divisor on X, i.e., $(-1)_X^*D = D$.[16] Consequently, $n_X^*D \sim n^2 D$ as a linear equivalence. So $\deg(n_X) = n^{2g}$.

When $p \mid n$ we have seen before that the induced map on tangent spaces by n_X, say $dn_X : T_{X,e_X} \to T_{X,e_X}$ is 0. Recall that $\omega_X = e_X^*\Omega_X^1$ can be identical with the cotangent space of X at e_X and $\pi^*\omega_X \cong \Omega_X^1$, where $\pi : X \to k$ is the structure map. So the canonical map $n_X^* : \Omega_X^1 \to \Omega_X^1$ is the zero map, and so also is $n_X^*\Omega^1_{k(X)/k} \to \Omega^1_{k(X)/k}$ under the field extension $k \to k(X) \xrightarrow{n_X^*} k(X)$. In particular, the composition of n_X^* with the canonical derivation d is zero:

$$\begin{tikzcd} k(X) \arrow[r, "n_X^*"] \arrow[rr, bend left, "0"] & k(X) \arrow[r, "d"] & \Omega^1_{k(X)/k}. \end{tikzcd}$$

Therefore,

$$n_X^*(k(X) \subset \operatorname{Ker}(d) = k(X)^p \subset k(X),$$

for which the proof of the fact $\operatorname{Ker}(d) = k(X)^p$ is leave as an exercise.

Fact. $k(X)/k(X)^p$ is a purely inseparable extension of degree p^g. (We use the fact that k is algebraically closed and $\operatorname{trdeg}_k k(X) = g$.)[17]

Proposition 6.16. *Keep the notations as above. We obtain the following.*

(1) $\deg(n_X) = n^{2g}$.
(2) n_X *is separable if and only if* $p \nmid n$. *In fact,* n_X *is separable if and only if it is étale as a morphism.*
(3) *If* $p \nmid n$, *then* $X_n(k) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$.
(4) *There exists* $0 \leqslant i \leqslant g$ *such that for each* $m \geqslant 1$, $X_{p^m}(k) \cong (\mathbb{Z}/p^m\mathbb{Z})^i$.

7. Dividing Varieties by Finite Groups

Definition 7.1 (Étale morphism)**.** Let $f : X \to Y$ be a morphism of algebraic varieties over an algebraically closed field k. Then f is called **étale** if

(1) f is flat;
(2) f is unramified, i.e., for each closed point $x \in X$, let $y = f(x) \in Y$ and $\mathfrak{m}_x$ (resp. $\mathfrak{m}_y$) be the maximal ideal of $\mathscr{O}_{X,x}$ (resp. $\mathscr{O}_{Y,y}$), then $f^*(\mathfrak{m}_y)\mathscr{O}_{X,x} = \mathfrak{m}_x$ for $f^* : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$. (In general, we also need to assume that $k(y) \to k(x)$ is separable.

Or equivalently (cf. [Har13, III, Exercise 10.4]),

[16] This is possible because one may choose an ample divisor D' on X and then let $D = D' + (-1)X^*D'$.

[17] On separating the transcendental basis: let K/k be a finitely generated field extension and k is perfect; then there exists a transcendental basis $x_1, \dots, x_m$ of K/k such that $K/k(x_1, \dots, x_m)$ is an algebraic separable extension.

(2') for any $x \in X$, let $y = f(x) \in Y$ and let $\widehat{\mathscr{O}}_{X,x}$ (resp. $\widehat{\mathscr{O}}_{Y,y}$) be the completion of $\mathscr{O}_{X,x}$ (resp. $\mathscr{O}_{Y,y}$). Then f^* insudes an isomorphism $\widehat{f}^* : \widehat{\mathscr{O}}_{Y,y} \to \widehat{\mathscr{O}}_{X,x}$.

7.1. **The Quotient along an Étale Morphism.**

Theorem 7.2. *Let X be an algebraic variety and G be a finite group of automorphisms of X. Suppose that for any $x \in X$, the orbit G_x of x is contained in an affine open subset of X. Then there is a pair (Y, π) where Y is a variety and $\pi : X \to Y$ is a morphism with the following conditions.*

(1) *As a topological space, (Y, π) is the quotient of X under the G-action.*
(2) *Denote $\pi_*(\mathscr{O}_X)^G$ the subsheaf of G-invariants of $\pi_*\mathscr{O}_X$ for the action of G on $\pi_*\mathscr{O}_X$ deduced from (1), the natural homomorphism $\mathscr{O}_Y \to \pi_*(\mathscr{O}_X)G$ is an isomorphism.*

Moreover, the pair is uniquely determined (up to isomorphisms) by (1) and (2). The morphism π is finite, surjective, and separable. If X is affine then so also is Y. If further G acts freely on X, i.e., $gx \neq x$ for all $x \in X$ for any $g \in G \backslash \{e\}$, π shall be étale.

Remark 7.3. (1) We essentially use the language of varieties instead of schemes in this lecture.
(2) G acts on X from the left and acts on $\mathscr{O}_X$ (the sheaf of regular functions on X) via the formula
$$(g(f))(x) = f(g^{-1}x), \quad \forall f \in \mathscr{O}_X(U),\ x \in U,\ g \in G,$$
where $U \subset X$ is an open subset. This is also viewed as a left action.
(3) When X is quasi-projective, any finitely many points of X is contained in an affine open subset of X, and hence the assumption in the theorem is satisfied. To be more explicit, by definition, we have
$$X \overset{j}{\hookrightarrow} \overline{X} \overset{f}{\hookrightarrow} \mathbb{P}^N,$$
where j is an open immersion and f is a closed immersion. We know that, by the prime avoidance, if I is an ideal and $\mathfrak{p}_1, \ldots, p_r$ are prime ideals such that $I \not\subset \mathfrak{p}_i$ for each i, then $I \not\subset \bigcup \mathfrak{p}_i$. Thus there is a homogeneous element $f \in k[x_0, \ldots, x_N]$ such that $\overline{X} \backslash X \subset V_+(f)$ and $x_i \notin V_+(f)$ for all i; $D_+(f) \cap X = D_+(f) \cap \overline{X}$ is affine and does not contain the points $x_1, \ldots, x_r$.

Proof of Theorem 7.2. Note that (1) determines the topology on Y and (2) determines the structure sheaf on Y, so the uniqueness follows. We are to show that if one takes $Y = X/G$ as a topological space[18] then the pair $(Y, (\pi_*\mathscr{O}_X)^G)$ is an algebraic variety. First we reduce to the affine case. For any closed point x of X and an affine open neighborhood U of x in X, the intersection $\bigcap_{g \in U} gU$ is an affine open neighborhood of x, and is G-stable. So we can find an affine open G-stable subset U containing x. This renders that $pi^{-1}(\pi(U)) = U$ where $\pi : X \to Y$ is the quotient map and $\pi(U)$ is open in Y.

So it is harmless to assume that $X = \operatorname{Spec} A$ affine. Since A is a finitely generated k-algebra, we let $\{x_1, \ldots, x_n\}$ be a set of generators of A over k. Let $v = |G|$ with $G = \{g_1, \ldots, g_v\}$. For each $f \in A$ and $1 \leqslant k \leqslant v$ we use $\sigma_k(f)$ to denote the elementary symmetric function of degree k in $\{g_1(f), \ldots, g_v(f)\}$. Let B' be the k-subalgebra of A generated by $\{\sigma_k(x_i) \mid i = 1, \ldots, n\ k = 1, \ldots, v\}$ and $B = A^G$. Then we have $B' \subset B \subset A$. For each $1 \leqslant i \leqslant n$, the x_i satisfies the monic equation over B', say
$$x^v - \sigma_1(x_i)x^{v-1} + \cdots + (-1)^v \sigma_v(x_i) = 0.$$

[18] Strictly, it should be written as $Y = G \backslash X$.

Hence x_i is integral over B' and then A is integral over B'. Again, since A is a finitely generated k-algebra, it is finite over B'. As B' is noetherian, B is finite over B' and so also is A over B. In particular, B is a finitely generated k-algebra.

Let $Y = \operatorname{Spec} B$ and let $\pi : X \to Y$ be the morphism corresponding to the inclusion $B \hookrightarrow A$. Then Y is an algebraic variety and π is finite surjective. Let K (resp. L) be the quotient field of B (resp. A). The G-action on A extends to L in the obvious way. Clearly, we have $K \subset L^G$. On the other hand, if $a/b \in L^G$, one can verify that

$$a \cdot \prod_{g \in G \backslash \{e\}} g(b) \in B, \quad \prod_{g \in G} g(b) \in B.$$

Thus $a/b \in K$, and then actually $K = L^G$; so L/K is a Galois extension. This shows that π is separable.

Since π is finite, $\pi_* \mathscr{O}_X$ is a coherent sheaf on Y. Note that the G-invariant part is

$$(\pi_* \mathscr{O}_X)^G = \operatorname{Ker}(\pi_* \mathscr{O}_X \overset{(g_1,\dots,g_v)}{\longrightarrow} \prod_{i=1}^{v} \pi_* \mathscr{O}_X),$$

which is coherent on Y. Since the natural morphism $\mathscr{O}_Y \to (\pi_* \mathscr{O}_X)^G$ induces an isomorphism of global sections, and Y is itself affine, we see

$$\mathscr{O}_Y \cong (\pi_* \mathscr{O}_X)^G.$$

Now we check both the set-theoretical and the topological properties. Let x_1, x_2 be two closed points of X such that $Gx_1 \cap Gx_2 = \emptyset$. By the Chinese remainder theorem, there is $f \in A$ such that $f(gx_1) = 1$ and $f(gx_2) = 0$ for each $g \in G$. Let

$$\phi = \prod_{g \in G} g(f) \in A^G = B,$$

and $\phi(\pi(x_1)) = 1$, $\phi(\pi(x_2)) = 0$. Then $\pi(x_1) \neq \pi(x_2)$. Thus, the equality $Y = X \backslash G$ holds set-theoretically. Again, note that $\pi : X \to Y$ is continuous and finite, and hence a closed map; we see that $Y \approx X/G$ as topological spaces.

We still need to check that when G acts on X freely, the morphism π is étale. Fix a closed point $x \in X$ and let $y = \pi(x)$. Let $\mathfrak{m} \in \operatorname{Spec} A$ (resp. $\mathfrak{n} \in \operatorname{Spec} B$) be the maximal ideal that corresponds to the point x (resp. y). Let $\widehat{A}$ and $\widehat{B}$ be the $\mathfrak{n}$-adic completions of A and B, respectively. Then

$$\widehat{B} = \widehat{\mathscr{O}}_{Y,y}, \quad \widehat{A} \cong \widehat{B} \otimes_B A$$

as A is finite over B. Also note that elements in the form $g\mathfrak{m}$ for $g \in G$ are exactly all the prime ideals of A lying over $\mathfrak{n} \in \operatorname{Spec} B$. (In fact, these $g\mathfrak{m}$'s are all distinct as the G-action is free.) Using the Chinese remainder theorem, we have

$$\widehat{B} \otimes_B A \cong \widehat{A} \xrightarrow{\cong} \prod_{g \in G} \widehat{\mathscr{O}}_{X,gx}.$$

Since $\mathfrak{n}$ is stable under G, the G-action on A induces a G-action on $\widehat{A}$. Under the isomorphism $\widehat{B} \otimes_B A \cong \widehat{A}$, this action is given by $g(\hat{b} \otimes a) = \hat{b} \otimes g(a)$. The fact that $B = A^G$ can be expressed as the following exact sequence of B-modules:

$$\begin{array}{ccccccc} 0 & \longrightarrow & B & \longrightarrow & A & \longrightarrow & \prod_{g \in G} A \\ & & & & a & \longmapsto & (ga - a)_{g \in G}. \end{array}$$

Since $\widehat{B}$ is flat over B, we have the exactness of

$$0 \longrightarrow \widehat{B} \longrightarrow \widehat{B} \otimes_B A \longrightarrow \textstyle\prod_{g\in G} \widehat{B} \otimes_B A.$$

Therefore, $\widehat{B} = \widehat{A}^G$. On the other hand, for any $g \in G$, the action of g on A induces an isomorphism

$$\widehat{\mathscr{O}}_{X,x} \xrightarrow{\cong} \widehat{\mathscr{O}}_{X,gx}.$$

So we can identify the products of them over $g \in G$:

$$\prod_{g\in G} \widehat{\mathscr{O}}_{X,x} \xrightarrow{\cong} \prod_{g\in G} \widehat{\mathscr{O}}_{X,gx}.$$

Again, the freeness condition is essential to infer that

$$\left(\prod_{g\in G} \widehat{\mathscr{O}}_{X,x}\right)^G = \widehat{\mathscr{O}}_{X,x};$$

note that the right hand side is viewed as the diagonal elements in the product. Thus, we finally attain the isomorphism

$$\widehat{\mathscr{O}}_{Y,y} = \widehat{B} \xrightarrow{\cong} \widehat{\mathscr{O}}_{X,x}.$$

□

Notation 7.4. We call the pair (Y, π) the quotient of X by G and it is denoted by X/G (again, this is strictly $G\backslash X$).

7.2. **Coherent Sheaves under Group Actions.** Now the goal is to study and understand how (coherent) sheaves behave under the group actions.

Let G, X and (Y, π) be as before and $\mathscr{F}$ be a coherent sheaf on Y. Fix a $g \in G$. From the following commutative diagram

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "\pi"'] & & X \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

we obtain an $\mathscr{O}_X$-linear isomorphism[19]

$$\phi_g : g^*(\pi^*\mathscr{F}) \xrightarrow{\cong} \pi^*\mathscr{F}.$$

The isomorphism $\{\phi_g \mid g \in G\}$ satisfy the cocycle condition[20]

$$\phi_{gh} = \phi_h \circ h^*(\phi_g) : (gh)^*(\pi^*\mathscr{F}) \to \pi^*\mathscr{F}.$$

Definition 7.5. Let $\mathfrak{g}$ be a coherent sheaf on X. Then $\mathfrak{g}$ is called a **coherent G-sheaf on X** if for each $g \in G$, we have an isomorphism of $\mathscr{O}_X$-modules $\phi_g : g^*\mathfrak{g} \to \mathfrak{g}$ satisfying the above cocycle conditions.

Remark 7.6. To understand this sheaf, let us see what happens in the affine case. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and $\mathscr{F}$ a sheaf of $\mathscr{O}_Y$-module that corresponds to a B-module M. Then

$$\pi^*\mathscr{F} = (M \otimes_B A)^\sim.$$

[19]This can be equivalently translated to $\phi_g : (g^{-1})^*(\pi^*\mathscr{F}) \to \pi^*\mathscr{F}$ if one prefers to consider the left action. In practice, to make it into a left action, g should be replaced by g^{-1}. Possibly it is better to understand this notation via vector bundles.

[20]Similarly, if we use the left action, this becomes $\phi_{gh} = \phi_h \circ (g^{-1})^*(\phi_h) : ((gh)^{-1})^*(\pi^*\mathscr{F}) \to \pi^*\mathscr{F}$.

For $g \in G$, we use $g^* : A \to A$ to denote the action of g on $\Gamma(X, \mathscr{O}_X)$. Then we have a map

$$g^* : M \otimes_B A \to M \otimes_B A, \quad m \otimes a \mapsto m \otimes g^*(a).$$

Unfortunately, this is NOT A-linear. More precisely, we have

$$g^*((m \otimes a)b) = g^*(m \otimes a) \cdot g^*(b).$$

Then g^* induces an A-linear isomorphism

$$\begin{aligned} \phi_g : (M \otimes_B A) \otimes_{A,g^*} A &\longrightarrow M \otimes_B A \\ (m \otimes a) \otimes b &\longmapsto g * (m \otimes a) \cdot b. \end{aligned}$$

Note that the left hand side gives the A-module structure.

Definition 7.7. For a finitely generated A-module N, we say that N is a (G, A)**-module** if we have an additive map $\psi_g : N \to N$ and $\psi_g(an) = g^*(a)\psi_g(n)$ for all $g \in G$, $a \in A$, and $n \in N$.

The condition in the definition makes ψ_g to be B-linear. And it satisfies the cocycle condition of ϕ_g, read as $\psi_{hg} = \psi_h \circ \psi_g$.

Proposition 7.8. *Let G acts freely on X and $Y = X/G$. Then the functor $\mathscr{F} \mapsto \pi^*\mathscr{F}$ is an equivalence between the category of coherent $\mathscr{O}_Y$-modules and that of coherent G-sheaves on X, whose inverse in given by $\mathfrak{g} \mapsto \pi_*\mathfrak{g}^G$. The locally free sheaves correspond to the locally free sheaves of the same rank.*

Proof. We can reduce to the affine case. Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. We need to show that the functors

$$\begin{aligned} \mathsf{Mod}_B &\longleftrightarrow \mathsf{Mod}_{(G,A)} \\ M &\longmapsto M \otimes_B A \\ N^G &\longleftarrow\!\shortmid N \end{aligned}$$

are truly inverses of each other. In other words, we are to prove that, for each B-module M,

$$S(M) : M \to (M \otimes_B A)^G, \quad m \mapsto m \otimes I,$$

and for each (G, A)-module N,

$$T(N) : N^G \otimes_B A \to N, \quad n \otimes a \mapsto an$$

are isomorphisms. Since the composite

$$M \otimes_B A \xrightarrow{S(M) \otimes_B A} (M \otimes_B A)^G \otimes_B A \xrightarrow{T(M \otimes_B A)} M \otimes_B A$$

is the identity map and $B \to A$ is faithfully flat, $T(M \otimes_B A)$ is an isomorphism and hence $S(M)$ is an isomorphism. So it suffices to show that $T(N)$ is an isomorphism for all (G, A)-module N. Regard $T(N) : N^G \otimes_B A \to N$ as a homomorphism of B-modules. Then $T(N)$ is an isomorphism if and only if for all maximal ideal $\mathfrak{n}$ of B, the localization $T(N)_{\mathfrak{n}} : (N^G \otimes_B A) \otimes_B \widehat{B}_n \to N \otimes_B \widehat{B}$ is an isomorphism.

In the following discussion we write $\widehat{B} = \widehat{B}_n$, $\widehat{A} = A \otimes_B \widehat{B}$, $\widehat{N} = N \otimes_B \widehat{B}$ for simplicity. As we have seen before,

$$\prod_{g \in G} \widehat{\mathscr{O}}_{X,gx} \cong \widehat{A} \cong \prod_{g \in G} \widehat{B}$$

and the G-action on $\widehat{A}$ is simply a permutation of the product factors.

Since N is a (G,A)-module, $\widehat{N}=N\otimes_A(A\otimes_B\widehat{B})$ is a $(G,\widehat{A})$-module. Under the isomorphism $\widehat{A}\cong\prod_{g\in G}\widehat{B}$, we see that $\widehat{N}\cong\prod_{g\in G}\widehat{N}_1$ for some $\widehat{B}$-module $\widehat{N}_1$ and G acts on $\widehat{N}$ via the permutation of factors. As

$$N^G=\mathrm{Ker}(N\xrightarrow{\psi_g-1}\prod_{g\in G}N),$$

and $B\to\widehat{B}$ is flat, we have $N^G\otimes_B\widehat{B}\cong(N\otimes_B\widehat{B})^G$. Under the above notations, the morphism $\widehat{T(N)}$ becomes

$$\begin{tikzcd}
(N\otimes_B A)^G\otimes_B\widehat{B} \arrow[r, "\cong"] \arrow[d, "\widehat{T(N)}"'] & (N\otimes_B\widehat{B})^G\otimes_{\widehat{B}}\widehat{A} \arrow[r, "\cong"] & \widehat{N}^G\otimes_{\widehat{B}}\widehat{A} \arrow[d] \\
\widehat{N} \arrow[rr, "="] & & \widehat{N}
\end{tikzcd}$$

Here the right vertical map is clearly an isomorphism, hence $\widehat{T(N)}$ is an isomorphism. This completes the proof. $\square$

In the following discussion, we assume that X is complete and G acts freely on X. Denote

$$\widehat{G}:=\mathrm{Hom}_{\mathsf{Grp}}(G,k^*).$$

Proposition 7.9. *For all $\alpha\in\widehat{G}$, define*

$$\mathscr{L}_\alpha=\{a\in\pi_*\mathscr{O}_X\mid g(a)=\alpha(g)\cdot a,\ \forall g\in G\}.$$

Then $\mathscr{L}_\alpha$ is an invertible sheaf on Y and the multiplication in $\pi_*\mathscr{O}_X$ induces an isomorphism[21]

$$\mathscr{L}_\alpha\otimes\mathscr{L}_\beta\xrightarrow{\sim}\mathscr{L}_{\alpha+\beta}.$$

The association $\alpha\mapsto\mathscr{L}_\alpha$ defines an isomorphism

$$\widehat{G}\xrightarrow{\sim}\mathrm{Ker}(\mathrm{Pic}\,Y\to\mathrm{Pic}\,X).$$

Proof. It follows from the previous Proposition 7.8 that

$$\mathrm{Ker}(\mathrm{Pic}\,Y\xrightarrow{\pi^*}\mathrm{Pic}\,X)\longleftrightarrow\{\text{coherent } G-\text{sheaf structure on } \mathscr{O}_X\}.$$

Given a G-action on the coherent sheaf $\mathscr{O}_X$,[22] for any $g\in G$ and $\Gamma(X,\mathscr{O}_X)\cong k$ say,

$$g:\Gamma(X,\mathscr{O}_X)\to\Gamma(X,\mathscr{O}_X)$$

is determined by $g(1)\in k^*$. We define $\alpha(g):=g(1)^{-1}$. Then $\alpha:G\to k^*$ is a group homomorphism. Conversely, given $\alpha\in G\to k^*$, we define an action of G on $\mathscr{O}_X$ via $g(f)=\alpha^{-1}(g)\cdot f\circ g^{-1}$. Then $g(af)=g(a)\cdot g(f)$ for all $g\in G$, $f\in\mathscr{O}_X$ ($\mathscr{O}_X$ as a coherent sheaf), and $a\in\mathscr{O}_X$ ($\mathscr{O}_X$ as the structure sheaf) such that $g(a)=a\circ g^{-1}$. This makes $\mathscr{O}_X$ a coherent G-sheaf.

In this way we establish an isomorphism

$$\mathrm{Ker}(\mathrm{Pic}\,Y\xrightarrow{\pi^*}\mathrm{Pic}\,X)\xrightarrow{\sim}\widehat{G}.$$

Fix $\alpha\in\widehat{G}$, we use σ to denote the G-action on $\mathscr{O}_X$ corresponding to α. This comes from

$$\sigma(g)(f)=\alpha^{-1}(g)\cdot g(f)$$

[21] We use Mumford's notation. However, it is better to replace $\mathscr{L}_{\alpha+\beta}$ by $\mathscr{L}_{\alpha\beta}$.

[22] Caution: in case this coincides with the G-action on the structure sheaf $\mathscr{O}_X$ but in general they differ from each other.

where $g(f) = f \circ g^{-1}$ is the action of G on the structure sheaf $\mathscr{O}_X$. Then

$$\mathscr{L}_\alpha = (\pi_* \mathscr{O}_X)^\sigma = \{a \in \pi_* \mathscr{O}_X \mid g(a) = \alpha(g) \cdot a\}.$$

For two $\alpha, \beta \in \widehat{G}$, we have a natural map $\mathscr{L}_\alpha \otimes \mathscr{L}_\beta \to \mathscr{L}_{\alpha+\beta}$. For $U \subset Y$ an open subset such that $\mathscr{L}_\alpha|_U$ is trivial. We can find a generating section $f \in \Gamma(U, \mathscr{L}_\alpha) \subset \Gamma(\pi^{-1}(U), \mathscr{O}_X)$, which vanishes nowhere on $\pi^{-1}(U)$. Therefore, f^{-1} is a well-defined nowhere-vanishing section on $\Gamma(\pi^{-1}(U), \mathscr{O}_X)$, and for any $g \in \Gamma(U, \mathscr{L}_{\alpha+\beta})$, $f^{-1}g \in \Gamma(U, \mathscr{L}_\beta)$. Thus, the map is an isomorphism: $\mathscr{L}_\alpha \otimes \mathscr{L}_\beta \cong \mathscr{L}_{\alpha+\beta}$. ∎

Remark 7.10. If G is commutative and $\mathrm{char}(k) \nmid |G|$, we have the decomposition

$$\pi_* \mathscr{O}_X \cong \bigoplus_{\alpha \in \widehat{G}} \mathscr{L}_\alpha.$$

Theorem 7.11. *Let X be an abelian variety. Then there is a one-to-one correspondence between the following two sets of objects:*

(1) *finite subgroups $K \subset X$;*[23]
(2) *separable isogenies, i.e., finite separable (surjective) homomorphisms $f : X \to Y$, where two isogenies $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ are considered equal if there is an isomorphism $h : Y_1 \to Y_2$ such that $f_2 = h_2 \circ f_1$.*

Explicitly, these maps are given by $K \mapsto (\pi : X \to X/K)$ and $(f : X \to Y) \mapsto K = \mathrm{Ker}(f)$.

Sketchy Proof. Given a finite subgroup $K \subset X(k)$, K acts on X via translation, and this is a free action. Let $(X/K, \pi)$ be the quotient. The multiplication map $m : X \times X \to X$ induces a morphism m' as

$$\begin{tikzcd}
X \times X \arrow[r, "m"] \arrow[d, "\pi\times\pi"'] & X \arrow[d, "\pi"] \\
X/K \times X/K \arrow[r, "m'"] \arrow[d, equal] & X/K \\
(X \times X)/(K \times K) &
\end{tikzcd}$$

Moreover, m' makes X/K an abelian variety and $\pi : X \to X/K$ is a separable isogeny. Conversely, given a separable isogeny $f : X \to Y$ with $K = \mathrm{Ker}(f)$, the condition that f is separable implies $\#K(k) = \deg(f)$. Let $g : X \to X/K$ be the natural morphism. Then f factors through g, i.e.,

$$\begin{tikzcd}
& X \arrow[dl, "g"'] \arrow[dr, "f"] & \\
X/K \arrow[rr, dashed, "h"] & & Y
\end{tikzcd}$$

where h is bijective on points. Note that when f is separable, so also is h. So that

$$\deg(h) \cdot \deg(g) = \deg(f).$$

However, on the other hand, $\deg(g) = \deg(f) = \#K(k)$. This forces $\deg(h)$ to be 1, namely, h is birational. Finally, via the Zariski Main Theorem, h is an isomorphism (cf. [Har13, III, Cor 11.4]). ∎

Corollary 7.12. *A separable isogeny $f : X \to Y$ is an étale morphism.*

[23]Strictly, finite subgroups $K \subset X(k)$.

Corollary 7.13. *Let $f : X \to Y$ be an isogeny of order prime to $p = \mathrm{char}(k)$. Then the kernel of f and the kernel of $f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ are dual as finite abelian groups.*

For this, by the previous Theorem 7.11, $f : X \to Y$ can be identified with $f : X \to X/K$ with $\#K$ prime to $\mathrm{char}(k)$. Then one can apply Proposition 7.9 to the morphism $f : X \to X/K$.

8. The Dual Abelian Variety in Characteristic 0

Goal. This section is to construct dual abelian variety over k with $\mathrm{char}(k) = 0$. Being out of tune, the characteristic assumption will be used at the end of our discussion.

Recall that in the theorem of the square (Corollary 6.7) we have defined the map

$$\phi_{\mathscr{L}} : X(k) \to \mathrm{Pic}(X), \quad x \mapsto T_x^*\mathscr{L} \otimes \mathscr{L}^{-1}.$$

Definition 8.1. The **principal Picard group** is defined as the subgroup

$$\mathrm{Pic}^0(X) := \{\mathscr{L} \in \mathrm{Pic}(X) \mid \phi_{\mathscr{L}} \equiv 0\}.$$

in $\mathrm{Pic}(X)$.

One can check that the map $\phi_{\mathscr{L}}$ takes values in $\mathrm{Pic}^0(X)$. Moreover, we get an exact sequence of abelian groups

$$\begin{array}{ccccccc} & & & & & & \text{finitely generated free } \mathbb{Z}\text{-module} \\ 0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \boxed{\mathrm{Hom}(X(k), \mathrm{Pic}^0(X))} \\ & & & & \mathscr{L} & \longmapsto & \phi_{\mathscr{L}}. \end{array}$$

In a natural sense one may want to endow $\mathrm{Pic}^0(X)$ with a structure of abelian varieties; after this, it will be shown that $\mathrm{Pic}^0(X)$ is isomorphic to another abelian variety that is called **the dual of** X and denoted by $\widehat{X}$.

8.1. General Observations on $\mathrm{Pic}^0(X)$.

(1) *By definition, $\mathscr{L} \in \mathrm{Pic}^0(X)$ if and only if $T_x^*\mathscr{L} \cong \mathscr{L}$ for all $x \in X(k)$. This is also equivalent to say on $X \times X$ that*

$$m^*\mathscr{L} \cong p_1^*\mathscr{L} \otimes p_2^*\mathscr{L}.$$

Here $m : X \times X \to X$ is the group operation.

Proof. By Seesaw (Corollary 10.7), the sheaf

$$\mathscr{M} := m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^*\mathscr{L}^{-1}$$

is trivial on $X \times X$. Equivalently, for all $x \in X(k)$, $\mathscr{M}|_{X \times \{x\}} \cong T_x^*\mathscr{L} \otimes \mathscr{L}^{-1}$ is trivial and $\mathscr{M}|_{\{e_X\} \times X}$ is trivial (the latter is always true). Therefore, it is to say $\mathscr{L} \in \mathrm{Pic}^0(X)$. □

(2) *If $\mathscr{L} \in \mathrm{Pic}^0(X)$, then for all schemes S and all morphisms $f, g : S \to X$, we have*

$$(f+g)^*\mathscr{L} \cong f^*\mathscr{L} \otimes g^*\mathscr{L}.$$

Proof. Pull back the isomorphism $m^*\mathscr{L} \cong p_1^*\mathscr{L} \otimes p_2^*\mathscr{L}$ along the morphism $(f, g) : S \to X \times X$. □

(3) *If $\mathscr{L} \in \mathrm{Pic}^0(X)$ then $n_X^*\mathscr{L} \cong \mathscr{L}^{\otimes n}$.*

Proof. Use (2) and the induction on n. □

(4) *For any $\mathscr{L} \in \mathrm{Pic}(X)$, we have*

$$n_X^*\mathscr{L} \cong \mathscr{L}^{n^2} \otimes \mathscr{L}_1$$

with $\mathscr{L}_1 \in \mathrm{Pic}^0(X)$.

Proof. Recall that

$$n_X^* \mathscr{L} \cong \mathscr{L}^{\frac{n^2+n}{2}} \otimes ((-1)_X^* \mathscr{L})^{\frac{n^2-n}{2}} \cong \mathscr{L}^{n^2} \otimes (\mathscr{L} \otimes (-1)_X^* \mathscr{L}^{-1})^{-\frac{n^2-n}{2}}.$$

So it suffices to show that $\mathscr{L} \otimes (-1)_X^* \mathscr{L}^{-1} \in \mathrm{Pic}^0(X)$. For any $x \in X(k)$,

$$\begin{aligned}
&T_x^*(\mathscr{L} \otimes (-1)_X^* \mathscr{L}^{-1}) \otimes (\mathscr{L} \otimes (-1)_X^* \mathscr{L}^{-1})^{-1} \\
\cong\ & T_x^* \mathscr{L} \otimes (-1)_X^* T_{-x}^* \mathscr{L}^{-1} \otimes \mathscr{L}^{-1} \otimes (-1)_X^* \mathscr{L} \\
\cong\ & T_x^* \mathscr{L} \otimes \mathscr{L}^{-1} \otimes (-1)_X^* (T_x^* \mathscr{L}^{-1} \otimes \mathscr{L}) \\
\cong\ & T_x^* \mathscr{L} \otimes \mathscr{L}^{-1} \otimes (T_x^* \mathscr{L}^{-1} \otimes \mathscr{L})^{-1} \qquad \text{by (3)} \\
\cong\ & T_x^* \mathscr{L} \otimes T_{-x}^* \mathscr{L} \otimes \mathscr{L}^{-2}.
\end{aligned}$$

Hence it is trivial by the theorem of the square (Corollary 6.7). □

(5) *If $\mathscr{L} \in \mathrm{Pic}(X)$ has finite order, then $\mathscr{L} \in \mathrm{Pic}^0(X)$.*

Proof. Let n be the order of $\mathscr{L}$, then by definition we have $\phi_{\mathscr{L}^n}(x) \equiv 0$ for each $x \in X(k)$. But also

$$\phi_{\mathscr{L}^n}(x) = \underbrace{\phi_{\mathscr{L}}(x) + \cdots + \phi_{\mathscr{L}}(x)}_{n \text{ terms}} = \phi_{\mathscr{L}}(nx),$$

and $X(k)$ is a divisible group. This forces $\phi_{\mathscr{L}}(x) \equiv 0$ and $\mathscr{L} \in \mathrm{Pic}^0(X)$. □

(6) *For any variety S over k and any line bundle $\mathscr{L}$ on $X \times S$, if we denote $\mathscr{L}_s := \mathscr{L}|_{X \times \{s\}}$ for $s \in S(k)$, then for all $s_0, s_1 \in S(k)$,*

$$\mathscr{L}_{s_1} \otimes \mathscr{L}_{s_0}^{-1} \in \mathrm{Pic}^0(X).$$

Proof. Since S is irreducible, the question is local on S. So we can assume that $\mathscr{L}|_{\{e_X\} \times S}$ is trivial. Fix $s_0 \in S(k)$. Replacing $\mathscr{L}$ by $\mathscr{L} \otimes p_1^* \mathscr{L}_{s_0}^{-1}$, we can assume that $\mathscr{L}_{s_0}$ is trivial. We need to show that $\mathscr{L}_{s_1} \in \mathrm{Pic}^0(X)$ for each $s_1 \in S(k)$. By (1), it further boils down to show that

$$m^* \mathscr{L}_{s_1} \otimes p_1^* \mathscr{L}_{s_1}^{-1} \otimes p_2^* \mathscr{L}_{s_1}^{-1}$$

is trivial on $X \times X$. We view this as a family of line bundles on $X \times X \times S$. More precisely, define

$$\mathscr{M} := m^* \mathscr{L} \otimes p_{13}^* \mathscr{L}^{-1} \otimes p_{23}^* \mathscr{L}^{-1},$$

where p_{13}, p_{23} are the natural projections and

$$m : X \times X \times S \to X \times S, \quad (x, y, s) \mapsto (x +_X y, s).$$

Then for each $s \in S(k)$,

$$\mathscr{M}|_{X \times X \times \{s\}} \cong m^* \mathscr{L}_s \otimes p_1^* \mathscr{L}_s^{-1} \otimes p_2^* \mathscr{L}_s^{-1}.$$

In particular, $\mathscr{M}|_{X \times X \times \{s_0\}}$ is trivial.

On the other hand, since $\mathscr{L}|_{\{e_X\} \times S}$ is trivial, we have $\mathscr{M}|_{\{e_X\} \times X \times S}$ and $\mathscr{M}|_{X \times \{e_X\} \times S}$ both being trivial. By the theorem of the cube (Theorem 6.1), $\mathscr{M}$ is trivial. Thus, $\mathscr{M}|_{X \times X \times \{s_1\}} \cong m^* \mathscr{L}_{s_1} \otimes p_1^* \mathscr{L}_{s_1}^{-1} \otimes p_2^* \mathscr{L}_{s_1}^{-1}$ is trivial and $\mathscr{L}_{s_1} \in \mathrm{Pic}^0(X)$. □

(7) *If $\mathscr{L} \in \mathrm{Pic}^0(X)$ and $\mathscr{L}$ is not trivial, then $H^i(X, \mathscr{L}) = 0$ for all $i \geqslant 0$.*[24]

[24] We will see later that

$$\dim_k H^i(X, \mathscr{O}_X) = \binom{g}{i}, \quad g = \dim_k X.$$

Proof. Suppose $H^0(X, \mathscr{L}) \neq 0$, then we can find an effective Weil devisor D such that $\mathscr{L} \cong \mathscr{O}_X(D)$. As $\mathscr{L} \in \operatorname{Pic}^0(X)$, by (3) it turns out that $(-1)_X^*\mathscr{L} \cong \mathscr{L}^{-1}$. Since $(-1)_X^*\mathscr{L} \cong \mathscr{O}_X((-1)_X^* D)$, we get

$$\mathscr{O}_X \cong \mathscr{L} \otimes \mathscr{L}^{-1} \cong \mathscr{O}_X(D + (-1)_X^* D).$$

On the other hand, recall that

$$H^i(X, \mathscr{O}_X) = \begin{cases} k, & i = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the condition $H^0(X, \mathscr{O}_X) = k$ implies $D + (-1)_X^* D = 0$. (This is an equality of divisors rather than divisor classes.) Since D is effective, we have $D = 0$ and then $\mathscr{L} = \mathscr{O}_X(D) = \mathscr{O}_X$. This contradicts to the assumption that $\mathscr{L}$ is not trivial. Then $H^0(X, \mathscr{L}) = 0$.

Assume the claim is not true and there exists $k > 0$ such that $H^k(X, \mathscr{L}) \neq 0$. We may choose k to be the smallest index. Then the morphisms

$$\begin{tikzcd} X \arrow[r, "s_1"] \arrow[rr, bend left, "\mathrm{id}_X"] & X \times X \arrow[r, "m"] & X \\ x \arrow[r, mapsto] & (x, e_X) & \end{tikzcd}$$

which induces morphisms of cohomology groups:

$$\begin{tikzcd} H^k(X, \mathscr{L}) & H^k(X, \times X, m^*\mathscr{L}) \arrow[l, "s_1^*"'] & H^k(X, \mathscr{L}) \arrow[l, "m^*"'] \arrow[ll, bend right, "\mathrm{id}"'] \end{tikzcd}$$

Since $\mathscr{L} \in \operatorname{Pic}^0(X)$, by (1), $m^*\mathscr{L} \cong p_1^*\mathscr{L} \otimes p_2^*\mathscr{L}$. Applying the Künneth formula to $X \times X$, we obtain

$$H^k(X \times X, m^*\mathscr{L}) \cong H^k(X \times X, p_1^*\mathscr{L} \otimes p_2^*\mathscr{L}) \cong \bigoplus_{i+j=k} H^i(X, \mathscr{L}) \otimes H^j(X, \mathscr{L}).$$

Since $H^i(X, \mathscr{L}) = 0$ for all $0 \leqslant i < k$, $H^k(X \times X, m^*\mathscr{L}) = 0$ and therefore $H^k(X, \mathscr{L}) = 0$. This leads to a contradiction. □

8.2. **The Key Theorem.**

Theorem 8.2. *Let $\mathscr{L}$ be an ample line bundle on X, and $\mathscr{M} \in \operatorname{Pic}^0(X)$. Then there exists $x \in X(k)$ such that*

$$\mathscr{M} \cong T_x^*\mathscr{L} \otimes \mathscr{L}^{-1}.$$

In other words, the map

$$\phi_{\mathscr{L}} : X(k) \longrightarrow \operatorname{Pic}^0(X)$$

is surjective.

Proof. We consider the following line bundle on $X \times X$:

$$K = m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^*(\mathscr{L}^{-1} \otimes \mathscr{M}^{-1}).$$

It enjoys the property that

$$K|_{\{x\} \times X} \cong T_x^*\mathscr{L} \otimes \mathscr{L}^{-1} \otimes \mathscr{M}^{-1}, \quad K|_{X \times \{x\}} \cong T_x^*\mathscr{L} \otimes \mathscr{L}^{-1}.$$

For the two projections $p_1, p_2 : X \times X \to X$, we have the Leray spectral sequences

$$E_2^{p,q} := H^p(X, R^q p_{1,*}K) \Rightarrow H^{p+q}(X \times X, K), \tag{1}$$

and

$$E_2^{p,q} := H^p(X, R^q p_{2,*}K) \Rightarrow H^{p+q}(X \times X, K). \tag{2}$$

May assume that the statement in the theorem does not hold, i.e., $T_x^*\mathscr{L}\otimes\mathscr{L}^{-1}\ncong\mathscr{M}$ for all $x\in X(k)$. Then $K|_{\{x\}\times X}$ is a nonzero element in $\mathrm{Pic}^0(X)$. By (7), considering an arbitrary fiber $\{x\}\times X$ of p_1,

$$H^q(\{x\}\times X, K|_{\{x\}\times X})=0,\quad q\geqslant 0.$$

By Corollary 5.7 (2), $R^q p_{1,*}K=0$ while restricting to $X\backslash K(\mathscr{L})$. Therefore $H^n(X\times X,K)=0$ by spectral sequence (1) for all $n\geqslant 0$.

On the other hand, $K|_{X\times\{x\}}\cong T_x^*\mathscr{L}\otimes\mathscr{L}^{-1}$ is a nonzero element in $\mathrm{Pic}^0(X)$ if and only if $x\notin K(\mathscr{L})$, which is a finite closed subgroup of $X(k)$. For $x\in X(k)\backslash K(\mathscr{L})$, by (7) again, we have

$$H^q(X\times\{x\}, K|_{X\times\{x\}})=0,\quad q\geqslant 0.$$

Similarly, $R^q p_{2,*}K=0$ while restricting to $X\backslash K(\mathscr{L})$. Hence

$$\mathrm{Supp}(R^q p_{2,*}K)\subset K(\mathscr{L}).$$

For a coherent sheaf $\mathscr{F}$ on X with $\mathrm{Supp}(\mathscr{F})\subset K(\mathscr{L})$, we have

$$H^p(X,\mathscr{F})=0,\quad p\geqslant 1.$$[25]

By the spectral sequence (2), we have for all n that

$$H^0(X,R^n p_{2,*}K)=\bigoplus_{x\in K(\mathscr{L})}(R^n p_{2,*}K)_x\cong H^n(X\times X,K).$$

But $H^n(X\times X,K)=0$ for all n and hence $(R^n p_{2,*}K)_x=0$. This shows that $R^n p_{2,*}K$ vanishes for all n. By Corollary 5.11 before, it turns out that

$$H^n(X\times\{x\},K_{X\times\{x\}})=0,\quad n\geqslant 0.$$

In particular, we take $n=0$ and $x=e_X$ to get

$$H^0(X,\mathscr{O}_X)=0.$$

But this is a contradiction. □

8.3. **Characterizing the Dual Abelian Variety.** The key theorem implies that as an abstract group, $\mathrm{Pic}^0(X)$ is isomorphic to $X(k)/K(\mathscr{L})$, and we can endow $\mathrm{Pic}^0(X)$ with an abelian variety structure, and is called the dual abelian variety of X, denoted by $\widehat{X}$. But this definition depends on the choice of $\mathscr{L}$. How to characterize $\widehat{X}$? The following answer can be interpreted as the definition of the dual abelian variety in characteristic 0.

Theorem 8.3. *The* ***dual abelian variety*** $\widehat{X}$ *of* X *is an abelian variety* $\widehat{X}$ *with an isomorphism of groups* $\widehat{X}(k)\cong\mathrm{Pic}^0(X)$ *satisfying the following conditions.*

(1) *There is a line bundle* P *on* $X\times\widehat{X}$ *called the Poincaré bundle, such that for any* $\alpha\in\widehat{X}(k)$, *the line bundle* $P|_{X\times\{\alpha\}}$ *represents the line bundle in* $\mathrm{Pic}^0(X)$ *given by* α *under the above isomorphism* $\widehat{X}(k)\cong\mathrm{Pic}^0(X)$; *moreover,* $P|_{\{e_X\}\times X}$ *is trivial.*[26]
(2) *For every normal variety* S *and a line bundle* K *on* $X\times S$, *suppose that*
   (i) $K_s:=K|_{X\times\{s\}}\in\mathrm{Pic}^0(X)$ *for one* $s\in S$ *– and hence for all by (6);*
   (ii) $K|_{\{e_X\}\times S}$ *is trivial.*
   Then the map of sets $f:S\to\widehat{X}$ *satisfying* $K_s\cong P_{f(s)}$ *is a morphism of varieties and* $K\cong(\mathrm{id}_X\times f)^*P$.

[25] There are several ways to see this. For example, if we use U to denote the open subvariety $X\backslash K(\mathscr{L})$, then $\mathscr{F}|_U=0$ and so for all $p\geqslant 0$, $H^p_{K(\mathscr{L})}(X,\mathscr{F})\cong H^0(X,\mathscr{F})$. Then use excision to reduce to the case for X affine. Then $H^p_{K(\mathscr{L})}(X,\mathscr{F})\cong H^0(X,\mathscr{F})=0$ for all $p\geqslant 1$.

[26] Note that by Seesaw's theorem, these properties characterize P.

Proof. The above two properties imply that $(\widehat{X}, P)$, if it exists, is unique up to canonical isomorphism. Fix an ample line bundle on X. We define $\widehat{X}$ to be the quotient $X/K(\mathscr{L})$ and let $\pi : X \to \widehat{X}$ be the natural morphism. We apply property (2) to the line bundle $K = m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^*\mathscr{L}^{-1}$ on $X \times X$, we see that the Poincaré bundle P on $X \times \widehat{X}$ satisfies the property that $(\mathrm{id}_X \times \pi)^*P \cong K$ on $X \times X$. Hence it suffices to define an $\{e_X\} \times K(\mathscr{L})$-action on K hat is compatible with its action on $X \times X$. For simplicity, we use $T_{0,a}$ to denote the translation

$$T_{0,a} : X \times X \to X \times X, \quad (x, y) \mapsto (x, y + a).$$

By a direct computation we obtain

$$\begin{aligned} T_{0,a}^* K &\cong T_{0,a}^* m^*\mathscr{L} \otimes T_{0,a}^* p_1^*\mathscr{L}^{-1} \otimes T_{0,a}^* p_2^*\mathscr{L}^{-1} \\ &\cong m^* T_a^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^* T_a^*\mathscr{L}^{-1} \\ &\cong m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^*\mathscr{L}^{-1} = K. \end{aligned}$$

This induces an isomorphism $\phi_a : T_{0,a}^* K \xrightarrow{\sim} K$, once the isomorphism $T_a^*\mathscr{L} \cong \mathscr{L}$ is fixed. But we still need to choose ϕ_a's carefully to make the qualities $\phi_{a+b} = \phi_a \circ T_{0,a}^*(\phi_b)$ hold for all $a, b \in K(\mathscr{L})$.

Since X is complete, if $\mathscr{L}_1, \mathscr{L}_2$ are two line bundles on $X \times X$ and $\phi, \psi : \mathscr{L}_1 \to \mathscr{L}_2$ are two isomorphisms, then ϕ and ψ are differed by a scalar in $k^\times$. To remedy this, we must kill the ambiguity of this scalar. Consider the restriction $K|_{\{e_X\}\times X}$. If we denote $\mathscr{L}^{-1}(0)$ for $e_X^*(\mathscr{L}^{-1})$ on $\operatorname{Spec} k$, then $K|_{\{e_X\}\times X} \cong p_1^*(\mathscr{L}^{-1}(0))$.

Fix such an isomorphism, we get a canonical isomorphism

$$\psi_a : T_a^*(K|_{\{e_X\}\times X}) \to K|_{\{e_X\}\times X}.$$

We require that $\phi_a|_{\{e_X\}\times X} = \psi_a$ for all $a \in K(\mathscr{L})$. Then we get a well-defined $\{e_X\} \times X$-action on K and hence obtain a line bundle P on $X \times \widehat{X}$ with $(\mathrm{id}_X \times \pi)^*P \cong K$.

Let us verify the pair $(X \times \widehat{X}, P)$ satisfies (1) and (2). For $\alpha \in \widehat{X}$, $\alpha = \pi(x)$ for some $x \in X(k)$. Then

$$P_\alpha = P|_{X\times\{\alpha\}} \cong \pi^*(P)|_{X\times\{x\}} \cong T_x^*\mathscr{L} \otimes \mathscr{L}^{-1} = \phi_{\mathscr{L}}(x) \in \mathrm{Pic}^0(X).$$

Since $K|_{\{e_X\}\times X} \cong p_1^*(\mathscr{L}^{-1}(0))$, by our construction above, $P|_{\{e_X\}\times X}$ is trivial. This proves (1). As for (2), we consider the line bundle $E = p_{12}^*(K) \otimes p_{13}^*(P^{-1})$ on $X \times S \times \widehat{X}$. Thus,

$$E|_{X\times\{s,\alpha\}} \cong K|_{X\times\{s\}} \times P^{-1}|_{X\times\{\alpha\}}.$$

By Seesaw, the set

$$\Gamma = \{(s, \alpha) \mid E|_{X\times\{\alpha,s\}} \text{ is trivial}\}$$

is Zariski closed in $S \times \widehat{X}$. We see $\Gamma(k)$ is the set-theoretic graph of the map $f : S(k) \to \widehat{X}(k)$ such that $K_s \cong P_{f(s)}$. Then the first projection $\Gamma \to S$ is bijective on the closed points.

Now we use the assumption that $\mathrm{char}(k) = 0$, which infers that $\Gamma \to S$ is birational. Since

$$\Gamma \to \widehat{X} \times S \to S$$

is proper, quasi-finite, and birational, and S is normal, we see $\Gamma \to S$ is an isomorphism. Γ induces a morphism of algebraic varieties $f : S \to \widehat{X}$ and by Seesaw again, $(\mathrm{id}_X \times f)^*(P) \cong K$ as desired. ∎

Remark 8.4. (1) For any $\mathscr{L} \in \mathrm{Pic}(X)$ the map $\phi_{\mathscr{L}} : X(k) \to \widehat{X}(k)$ is a morphism of varieties. To see this, apply property (2) in Theorem 8.3 to the line bundle

$$K = m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^*\mathscr{L}^{-1}$$

on $X \times X$.

(2) Let $f : X \to Y$ be a homomorphism of abelian varieties. The map $f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ maps $\mathrm{Pic}^0(Y)$ to $\mathrm{Pic}^0(X)$ (check this), and induces a map
$$\widehat{f}(k) : \widehat{Y}(k) \to \widehat{X}(k).$$
I claim that this is a morphism. In fact, if we use Q to denote the Poincaré bundle on $Y \times \widehat{Y}$ and let $Q' = (f \times \mathrm{id}_{\widehat{Y}})^* Q \in \mathrm{Pic}(X \times \widehat{Y})$, then
$$Q'|_{X \times \{\widehat{y}\}} \cong f^*(Q|_{Y \times \{\widehat{y}\}}) \in \mathrm{Pic}^0(X)$$
for all $\widehat{y} \in \widehat{Y}(k)$, and $Q'|_{\{e_X\} \times \widehat{Y}} \cong Q|_{\{e_Y\} \times \widehat{Y}}$ is trivial. Then there is a morphism $\widehat{f} : \widehat{Y} \to \widehat{X}$ such that $Q' \cong (\mathrm{id}_X \times \widehat{f})^* P$, where P is the Poincaré bundle on $X \times \widehat{X}$. The morphism of $\widehat{f}$ on k-points is just the $\widehat{f}(k)$ defined above.

(3) If $f : X \to Y$ is an isogeny, so is $\widehat{f} : \widehat{Y} \to \widehat{X}$; and there exists a canonical duality of abelian groups between $\mathrm{Ker}(f)$ and $\mathrm{Ker}(\widehat{f})$. In fact, by Proposition 7.9 and Corollary 7.13, we have a canonical duality between $\mathrm{Ker}(f)$ and $\mathrm{Ker}(f^* : \mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(X))$. Since $\mathrm{Ker}(f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X))$ is finite, any $\mathscr{L}$ in $\mathrm{Ker}(f^*)$ is of finite order. Thus, $\mathscr{L} \in \mathrm{Pic}^0(Y)$ and $\mathrm{Ker}(f^*) = (\mathrm{Ker}(\widehat{f}))(k)$. By the dimension argument, we see $\widehat{f}$ is also an isogeny.

Part 3. Algebraic Theory via Schemes.

9. Basic Theory of Group Schemes

9.1. **Categorical Perspective of Schemes.** Fix an algebraically closed field k. We use Sch_k to denote the category of schemes of finite type over k. For two objects X, S in Sch_k, we define an S-valued point of X to be a morphism $S \to X$ in Sch_k. Denote

$$\underline{X}(S) := \mathrm{Hom}_k(S, X).$$

The association $S \mapsto \underline{X}(S)$ defines a contravariant functor $\underline{X} : \mathsf{Sch}_k^{\mathrm{op}} \to \mathsf{Sets}$ from the opposite category of Sch_k. When X varies, we get a functor $\mathsf{Sch}_k \to \mathrm{Fun}(\mathsf{Sch}_k^{\mathrm{op}}, \mathsf{Sets})$, which is fully faithful. Granting this, we can view Sch_k as a full subcategory of $\mathrm{Fun}(\mathsf{Sch}_k^{\mathrm{op}}, \mathsf{Sets})$. Similarly, if we use Alg_k to denote the category of finitely generated k-algebras, then any $X \in \mathsf{Sch}_k$ defines a covariant functor

$$\underline{X} : \mathsf{Alg}_k \to \mathsf{Sets}, \quad \underline{X}(R) := \underline{X}(\mathrm{Spec}\, R) = \mathrm{Hom}_k(\mathrm{Spec}\, R, X).$$

The functor

$$\mathsf{Sch}_k \to \mathrm{Fun}(\mathsf{Alg}_k, \mathsf{Sets}), \quad X \mapsto \underline{X}$$

is fully faithful and we can view Sch_k as a full subcategory of $\mathrm{Fun}(\mathsf{Alg}_k, \mathsf{Sets})$.

Definition 9.1 (Group scheme)**.** A **group scheme** is a scheme G of finite type over k together with

- a multiplication morphism $m : G \times G \to G$,
- an identity point $e : \mathrm{Spec}\, k \to G$, and
- an inverse morphism $i : G \to G$,

such that the following axioms hold.

(1) (Associativity) The diagram is commutative:

$$\begin{tikzcd} G \times G \times G \arrow[r, "m \times 1_G"] \arrow[d, "1_G \times m"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

(2) (Axiom of the identity section) The diagram is commutative:

$$\begin{tikzcd} G \times \mathrm{Spec}\, k \arrow[r, "\mathrm{id}_G \times e"] \arrow[d, "\cong"'] & G \times G \arrow[d, "m"] \\ G \arrow[r, "\mathrm{id}_G"] \arrow[d, "\cong"'] & G \\ G \times \mathrm{Spec}\, k \arrow[r, "e \times \mathrm{id}_G"] & G \times G \arrow[u, "m"'] \end{tikzcd}$$

(3) The diagram is commutative:

$$\begin{tikzcd} & G \times G \arrow[dr, "m"] & \\ G \arrow[ur, "(\mathrm{id}_G, i)"] \arrow[r] \arrow[dr, "(i, \mathrm{id}_G)"'] & \mathrm{Spec}\, k \arrow[r, "e"] & G \\ & G \times G \arrow[ur, "m"'] & \end{tikzcd}$$

Remark 9.2. (1) We use $\underline{G}$ to denote the functor $\mathsf{Sch}_k^{\mathrm{op}} \to \mathsf{Sets}$ associated to G. Then G is a group scheme if and only if $\underline{G}$ factors through the forgetful functor $\mathsf{Grp} \to \mathsf{Sets}$, i.e., $\underline{G}$ is the composite

$$\underline{G} : \mathsf{Sch}_k^{\mathrm{op}} \to \mathsf{Grp} \to \mathsf{Sets}.$$

(2) For a closed point $x \in G(k)$, we can define the right translation R_x to be the composite

$$G \cong G \times \operatorname{Spec} k \xrightarrow{(\mathrm{id}_G, x)} G \times G \xrightarrow{m} G$$

and we define the left translation L_x similarly. In general, for a k-scheme $S \in \mathsf{Sch}_k$ and an S-point $x \in \underline{G}(S)$, we define the right translation R_x to be the S-morphism

$$G \times S \xrightarrow{(m \circ (\mathrm{id}_G \times x), p_2)} G \times S.$$

One can check that $R_{xy} = R_y \circ R_x$ and define L_x in a similar way.

9.2. **Lie Algebras.** Let $X \in \mathsf{Sch}_k$ and $\Omega_X = \Omega^1_{X/k}$ be the sheaf of relative differentials in X/k.

Definition 9.3. (1) A **vector field** D on X is a k-linear map $D : \mathscr{O}_X \to \mathscr{O}_X$ such that for any open subset U of X,

$$(D(U) : \mathscr{O}_X(U) \to \mathscr{O}_X(U)) \in \mathrm{Der}_k(\mathscr{O}_X(U), \mathscr{O}_X(U)).$$

In other words, $D : \mathscr{O}_X \to \mathscr{O}_X$ is the composite

$$\mathscr{O}_X \xrightarrow{d} \Omega_X \xrightarrow{f} \mathscr{O}_X$$

where d is the canonical derivation and f is an $\mathscr{O}_X$-linear map.

(2) A **tangent vector** d of X at a closed point $x \in X$ (cf. [Har13, II, Exer 2.8]) is a k-derivation $d : \mathscr{O}_{X,x} \to k \in \mathrm{Der}_k(\mathscr{O}_{X,x}, k)$, for which

$$\begin{aligned} \mathrm{Der}_k(\mathscr{O}_{X,x}, k) &\cong \mathrm{Hom}_{\mathscr{O}_{X,x}}(\Omega^1_{\mathscr{O}_{X,x}/k}, k) \\ &\cong \mathrm{Hom}_{\mathscr{O}_{X,x}}(\Omega_{X,x}, k) \\ &\cong \mathrm{Hom}_k(\Omega_{X,x} \otimes_{\mathscr{O}_{X,x}} k, k) \\ &\cong \mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k), \end{aligned}$$

where $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$ is the maximal ideal. The last isomorphism is from [Har13, II, Prop 8.7]. Therefore, to give a tangent vector at x, say $d : \mathscr{O}_{X,x} \to k$, is equivalent to giving a k-linear map $t : \mathfrak{m}_x/\mathfrak{m}_x^2 \to k$.

(3) For a vector field $D : \mathscr{O}_X \to \mathscr{O}_X$ we define its **value** at $x \in X$ to be the tangent vector $\mathscr{O}_{X,x} \xrightarrow{D_x} \mathscr{O}_{X,x} \to k$.

For two schemes $X, Y \in \mathsf{Sch}_k$, we have a canonical isomorphism $\Omega_{X\times Y} \cong p_1^*\Omega_X \oplus p_2^*\Omega_Y$ where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are natural projections. Let

$$D : \mathscr{O}_X \xrightarrow{d} \Omega_X \xrightarrow{f} \mathscr{O}_X$$

be a vector field on X, we define a vector field $D \otimes 1$ on $X \times Y$ that corresponds to the $\mathscr{O}_{X\times Y}$-linear map

$$\Omega_{X\times Y} \xrightarrow{\sim} p_1^*\Omega_X \oplus p_2^*\Omega_Y \xrightarrow{(p_1^*(f), 0)} \mathscr{O}_{X\times Y}$$

Definition 9.4. Let G be a group scheme over k. A vector field D on G is called **left invariant** if the following diagram commutes:

$$\begin{tikzcd} \mathscr{O}_G \arrow[r, "D"] \arrow[d, "m^*"'] & \mathscr{O}_G \arrow[d, "m^*"] \\ \mathscr{O}_{G\times G} \arrow[r, "1\otimes D"] & \mathscr{O}_{G\times G} \end{tikzcd}$$

Proposition 9.5. *For any tangent vector t at e_G to G, there is a unique left invariant vector field on G whose value at e_G is exactly t.*

Proof. First we give another expression of tangent vectors and vector fields. Let $\Lambda = k[\varepsilon]/(\varepsilon)^2$. Let A be a k-algebra and B be an A-algebra. Then we have a bijection between sets, say

$$D \longmapsto \varphi(a) = a\cdot 1_B + D(a)\varepsilon$$

$$\mathrm{Der}_k(A,B) \longleftrightarrow \left\{ \varphi : A \to B\otimes_k \Lambda \;\middle|\; \begin{array}{c} \varphi \text{ is a } k\text{-algebra homomorphism} \\ \text{such that } \overline{\varphi} : A \to B \text{ is the structure map} \\ \text{read as } \overline{\varphi} : A \xrightarrow{\varphi} B\otimes_k \Lambda \xrightarrow{\mathrm{mod}\,\varepsilon} B \end{array} \right\}$$

$$\updownarrow$$

$$\left\{ \varphi' : A\otimes_k \Lambda \to B\otimes_k \Lambda \;\middle|\; \begin{array}{c} \varphi' \text{ is a } \Lambda\text{-algebra homomorphism such that} \\ \varphi' \otimes_k \Lambda : A \to B \text{ is the structure map} \end{array} \right\}.$$

Under the above bijections:

(i) a tangent vector t to X at $x \in X$ corresponds to a morphism $\tilde{t} : \operatorname{Spec}\Lambda \to X$ such that the composite $\operatorname{Spec} k \to \operatorname{Spec}\Lambda \xrightarrow{\tilde{t}} X$ is basically the point $x \in X$.

(ii) a vector field D on X corresponds to a morphism over $\operatorname{Spec}\Lambda$:

$$\begin{tikzcd} X\times \operatorname{Spec}\Lambda \arrow[rr, "\widetilde{D}"] \arrow[dr] & & X\times\operatorname{Spec}\Lambda \arrow[dl] \\ & \operatorname{Spec}\Lambda & \end{tikzcd}$$

such that $\widetilde{D}\times_{\operatorname{Spec}\Lambda} \operatorname{Spec} k : X \to X$ is id_X.

(iii) for a vector field D on X, and t_x the value of D at $x\in X$, the morphism $\tilde{t}_x$ corresponds to the morphism

$$\operatorname{Spec}\Lambda \xrightarrow{\cong} \operatorname{Spec} k \times \operatorname{Spec}\Lambda \xrightarrow{(x,\mathrm{id}_X)} X\times\operatorname{Spec}\Lambda \xrightarrow{\widetilde{D}} X\times\operatorname{Spec}\Lambda \xrightarrow{p_1} X$$

Under the above expressions, we see that a vector field D on G is left invariant if and only if the following diagram is commutative:

$$\begin{tikzcd} G\times G\times\operatorname{Spec}\Lambda \arrow[r, "\mathrm{id}_G\times\widetilde{D}"] \arrow[d, "m\times\mathrm{id}_\Lambda"'] & G\times G\times\operatorname{Spec}\Lambda \arrow[d, "m\times\mathrm{id}_\Lambda"] \\ G\times\operatorname{Spec}\Lambda \arrow[r, "\widetilde{D}"] & G\times\operatorname{Spec}\Lambda \end{tikzcd}$$

Here all arrows are morphisms over $\operatorname{Spec}\Lambda$. We use $\widetilde{D}_1$ to denote the composite

$$\widetilde{D}_1 : G\times\operatorname{Spec}\Lambda \xrightarrow{\widetilde{D}} G\times\operatorname{Spec}\Lambda \xrightarrow{p_1} G.$$

Then D is left invariant if and only if for any $S \in \mathsf{Sch}_k$, $x, y \in \underline{G}(S)$, and $l \in \operatorname{Spec}\Lambda(S)$, $\widetilde{D}_1(xy, l) = x\widetilde{D}(y, l)$. (Caveat: one should be very careful about the order.) Alternatively, this is equivalent to say

$$\widetilde{D}_1(x, l) = x\widetilde{D}_1(e_G(S), l),$$

where $e_G(S) \in \underline{G}(S)$ is the identity element; note that $(e_G(S), l)$ is the value of $\widetilde{D}$ at e_G.

Now given a tangent vector t of G at e_G, we define a vector field D on X that corresponds to the following with $\tilde{t} : \operatorname{Spec}\Lambda \to G$,

$$\widetilde{D} : G \times \operatorname{Spec}\Lambda \xrightarrow{(\mathrm{id}_G, \tilde{t}, \mathrm{id}_\Lambda)} G \times G \times \operatorname{Spec}\Lambda \xrightarrow{(m, \mathrm{id}_\Lambda)} G \times \operatorname{Spec}\Lambda.$$

In other words, $\widetilde{D}_1$ satisfies $\widetilde{D}_1(x, l) = x\widetilde{D}_1(e_G(S), l)$ for each $S \in \mathsf{Sch}_k$, $x \in \underline{G}(S)$, and $l \in (\operatorname{Spec}\Lambda)(S)$. It renders that $\widetilde{D}$ is left invariant and has value t at e_G. Hence the uniqueness follows obviously. $\square$

Let D_1, D_2 be two vector fields on X. Their **Poisson bracket**

$$[D_1, D_2] = D_1D_2 - D_2D_1$$

is also a vector field on X. When $\operatorname{char}(k) = p > 0$, D_1^p is a vector field on X. When $X = G$ is a group scheme, the above two operators preserve left invariant vector fields.

Definition 9.6. The **Lie algebra of a group scheme** G is the k-vector space of left invariant vector fields, together with the operation of Poisson bracket (and the pth power operation if $\operatorname{char}(k) = p > 0$).

Proposition 9.7. *If G is a commutative group scheme, then its Lie algebra $\mathfrak{g}$ is abelian, i.e., $[D_1, D_2] = 0$ for all $D_1, D_2 \in \mathfrak{g}$.*

Proof. We first make the following observation. Let $X \in \mathsf{Sch}_k$, D_1, D_2 be vector fields on X, and $D_3 = [D_1, D_2]$. Let $\widetilde{D}_i : X \times \operatorname{Spec}\Lambda \to X \times \operatorname{Spec}\Lambda$ be the morphism corresponding to D_i for $i = 1, 2, 3$. I claim that $x_3 = x_1x_2x_1^{-1}x_2^{-1}$. The question is local on X so we can assume that $X = \operatorname{Spec} A$ is affine. The automorphism χ_i of $X \times \operatorname{Spec}\Lambda'$ over $\operatorname{Spec}\Lambda'$ corresponds to the Λ'-algebra automorphism

$$f_i : A[\varepsilon, \varepsilon']/(\varepsilon^2, \varepsilon'^2) \longrightarrow A[\varepsilon, \varepsilon']/(\varepsilon^2, \varepsilon'^2)$$

that is defined as follows:

$$\begin{aligned}
f_1 &: a_1 + a_2\varepsilon + a_3\varepsilon' + a_4\varepsilon\varepsilon' \longmapsto a_1 + (D_1(a_1) + a_2)\varepsilon + a_3\varepsilon' + (D_1(a_3) + a_4)\varepsilon\varepsilon',\\
f_2 &: a_1 + a_2\varepsilon + a_3\varepsilon' + a_4\varepsilon\varepsilon' \longmapsto a_1 + a_2\varepsilon + (D_2(a_1) + a_3)\varepsilon' + (D_2(a_2) + a_4)\varepsilon\varepsilon',\\
f_3 &: a_1 + a_2\varepsilon + a_3\varepsilon' + a_4\varepsilon\varepsilon' \longmapsto a_1 + a_2\varepsilon + a_3\varepsilon' + (D_3(a_1) + a_4)\varepsilon\varepsilon';
\end{aligned}$$

also, f_i^{-1} is given by replacing D_i by $-D_i$ in the above formulas. Hence

$$\begin{aligned}
&f_3^{-1} = f_2^{-1} \circ f_1^{-1} \circ f_2 \circ f_1 : A \otimes_k \Lambda' \to A \otimes_k \Lambda',\\
\rightsquigarrow\quad &\chi_3^{-1} = \chi_1 \circ \chi_2 \circ \chi_1^{-1} \circ \chi_2^{-1} : X \times \operatorname{Spec}\Lambda' \to X \times \operatorname{Spec}\Lambda'.
\end{aligned}$$

Now let D_1, D_2 be two left invariant vector fields on a commutative group scheme G that corresponds to the tangent vector $t_i : \operatorname{Spec}\Lambda \to G$ with $i = 1, 2$. Define $D_3 = [D_1, D_2]$ and $\tilde{t}_3$ that corresponds to D_3. For $i = 1, 2, 3$, define $T_i \in \underline{G}(\Lambda')$ to be the composite

$$T_i : \operatorname{Spec}\Lambda' \xrightarrow{\sigma_i} \operatorname{Spec}\Lambda \xrightarrow{\tilde{f}_i} G.$$

Thus, $\chi_i : G \times \operatorname{Spec}\Lambda' \to G \times \operatorname{Spec}\Lambda'$ be the right translation by $T_i \in \underline{G}(\Lambda')$. Then

$$\chi_3^{-1} = \chi_1 \circ \chi_2 \circ \chi_1^{-1} \circ \chi_2^{-1},$$

but $\chi_1 \circ \chi_2 \circ \chi_1^{-1} \circ \chi_2^{-1}$ is the right translation by $T_2^{-1} \cdot T_1^{-1} \cdot T_2 \cdot T_2 \in \underline{G}(\Lambda')$, which is $e_G(\Lambda')$ as G is commutative. It follows that χ_3^{-1} (and hence χ_3) is the identity morphism so that $D_3 = [D_1, D_2] = 0$. $\square$

Theorem 9.8. *Any group scheme over a field k of characteristic 0 is automatically smooth (and in particular reduced).*

Proof. We can assume that k is algebraically closed and it suffices to prove the group scheme G is smooth at $e \in G$, the identity point. For simplicity, denote $\mathscr{O} = \mathscr{O}_{G,e}$, $\mathfrak{m} \subset \mathscr{O}$ the maximal ideal, and $\widehat{\mathscr{O}}$ the $\mathfrak{m}$-adic completion of $\mathscr{O}$. Denote $n = \dim_k \mathfrak{m}/\mathfrak{m}^2$. The multiplication map $m : G \times G \to G$ induces a continuous homomorphism

$$m^* : \widehat{\mathscr{O}} \longrightarrow \widehat{\mathscr{O}} \widehat{\otimes}_k \widehat{\mathscr{O}}.$$

Here $\widehat{\otimes}$ is the complete tensor product; that is, the $(1 \otimes \widehat{\mathfrak{m}} + \widehat{\mathfrak{m}} \otimes 1)$-adic completion of $\widehat{\mathscr{O}} \otimes_k \widehat{\mathscr{O}}$. Since the two composites

$$\begin{tikzcd} G \arrow[r, shift left, "(\mathrm{id}_G, e)"] \arrow[r, shift right, "(e, \mathrm{id}_G)"'] & G \times G \arrow[r, "m"] & G \end{tikzcd}$$

are both identity, the composites

$$\widehat{\mathscr{O}} \xrightarrow{m^*} \widehat{\mathscr{O}} \widehat{\otimes}_k \widehat{\mathscr{O}} \longrightarrow \widehat{\mathscr{O}} \widehat{\otimes}_k k \cong \widehat{\mathscr{O}},$$

$$\widehat{\mathscr{O}} \xrightarrow{m^*} \widehat{\mathscr{O}} \widehat{\otimes}_k \widehat{\mathscr{O}} \longrightarrow k \widehat{\otimes}_k \widehat{\mathscr{O}} \cong \widehat{\mathscr{O}}$$

are both identity as well. Thus, for any $a \in \widehat{\mathfrak{m}}$,

$$m^*(a) \in 1 \otimes a + a \otimes 1 + \widehat{\mathfrak{m}} \widehat{\otimes}_k \widehat{\mathfrak{m}}.$$

Claim. *For any k-linear map $f : \widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2 \to k$, there is a k-derivation $D : \widehat{\mathscr{O}} \to \widehat{\mathscr{O}}$ such that $f = D|_{\widehat{\mathfrak{m}}} \bmod \widehat{\mathfrak{m}}$.*

Since we have a decomposition of k-vector spaces, $\widehat{\mathscr{O}} = k \oplus \widehat{\mathfrak{m}}$. A k-linear map $F : \widehat{\mathscr{O}} \to k$ could be found such that $F|_k = 0$ and $F|_{\widehat{\mathfrak{m}}}$ is the composite $\widehat{\mathfrak{m}} \to \widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2 \xrightarrow{f} k$. We define D to be the composite

$$\widehat{\mathscr{O}} \xrightarrow{m^*} \widehat{\mathscr{O}} \widehat{\otimes}_k \widehat{\mathscr{O}} \xrightarrow{1 \otimes F} \widehat{\mathscr{O}} \widehat{\otimes}_k k \xrightarrow{\cong} \widehat{\mathscr{O}}.$$

Clearly D is k-linear and $D(k) \equiv 0$ as $F(k) \equiv 0$. For $a \in \widehat{\mathfrak{m}}$,

$$D(a) = (1 \otimes F)(1 \otimes a + a \otimes 1 + b) = F(a) + (1 \otimes F)(b)$$

for some $b \in \widehat{\mathfrak{m}} \widehat{\otimes}_k \widehat{\mathfrak{m}}$. Consequently, $D(a) \bmod \widehat{\mathfrak{m}} = F(a) = f(a \bmod \mathfrak{m}^2)$. This proves the claim that $f = D|_{\widehat{\mathfrak{m}}} \bmod \widehat{\mathfrak{m}}$.

We still need to verify that D is a derivation, i.e., for all $a, b \in \widehat{\mathfrak{m}}$, we have $D(ab) = aD(b) + bD(a)$. By a direct computation,

$$\begin{aligned} m^*(ab) =& (a \otimes 1) m^*(b) + (b \otimes 1) m^*(a) \\ &+ (1 \otimes ab - ab \otimes 1 + S(1 \otimes a) + R(1 \otimes b) + RS). \end{aligned}$$

If we write $m^*(a) = 1 \otimes a + a \otimes 1 + R$, $m^*(b) = 1 \otimes b + b \otimes 1 + S$. In particular, $R, S \in \widehat{\mathfrak{m}} \widehat{\otimes}_k \widehat{\mathfrak{m}}$ and

$$T = 1 \otimes ab - ab \otimes 1 + S(1 \otimes a) + R(1 \otimes b) + RS \in \widehat{\mathscr{O}} \otimes 1 + \widehat{\mathscr{O}} \otimes \widehat{\mathfrak{m}}^2.$$

We infer that

$$\begin{aligned} D(ab) &= (1 \otimes F)(m^*(ab)) \\ &= a(1 \otimes F)(m^*(b)) + b(1 \otimes F)(m^*(a)) + \underbrace{(1 \otimes F)(T)}_{=0} \\ &= aD(b) + bD(a). \end{aligned}$$

Here $(1 \otimes F)(T) = 0$ because of $T \in \widehat{\mathscr{O}} \otimes 1 + \widehat{\mathscr{O}} \otimes \widehat{\mathfrak{m}}^2$. Choose $x_1, \ldots, x_n \in \widehat{\mathfrak{m}}$ such that $\{\overline{x}_1, \ldots, \overline{x}_n\}$ forms a k-basis of $\widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2$ and $\{f_i : \widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2 \to k \mid i = 1, \ldots, n\}$ be the dual basis. Let $D_i : \widehat{\mathscr{O}} \to \widehat{\mathscr{O}}$ be the k-derivation such that $D_i|_{\widehat{\mathfrak{m}}} \bmod \widehat{\mathfrak{m}} = f_i$ for each i. In particular, we have $D_i(x_j) \bmod \widehat{\mathfrak{m}} = \delta_{ij}$ for all $1 \leqslant i, j \leqslant n$.

Define a k-algebra homomorphism

$$\alpha : k[\![t_1, \ldots, t_n]\!] \to \widehat{\mathscr{O}}, \quad t_i \mapsto x_i, \quad i = 1, \ldots, n.$$

Since $\{x_1, \ldots, x_n\}$ generates $\widehat{\mathfrak{m}}$ by Nakayama's lemma, α is surjective. Define another k-algebra homomorphism

$$\beta : \widehat{\mathscr{O}} \to k[\![t_1, \ldots, t_n]\!], \quad f \mapsto \sum_{\alpha \in \mathbb{Z}^n_{\geqslant 0}} \overline{\left(\frac{D^\alpha f}{\alpha!}\right)} \cdot t^\alpha.$$

Here the operator $\overline{(\cdot)}$ means modulo $\widehat{\mathfrak{m}}$ (so that the coefficients are elements in k) the power series is defined through

$$\alpha! = \alpha_1! \cdots \alpha_n!, \quad D^\alpha f = D_1^{\alpha_1} \cdots D_n^{\alpha_n} f, \quad t^\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}.$$

By Leibniz's formula, one can check that β is a continuous homomorphism. Moreover,

$$\beta(x_i) \equiv t_i \mod (t_1, \ldots, t_n)^2, \quad i = 1, \ldots, n.$$

Hence β is surjective. The composite $\beta \circ \alpha : k[\![t_1, \ldots, t_n]\!] \to k[\![t_1, \ldots, t_n]\!]$ is onto and satisfies $\beta \circ \alpha \equiv \mathrm{id} \bmod (t_1, \ldots, t_n)^2$. Therefore, $\beta \circ \alpha$ is an isomorphism.[27] So α is injective and hence an isomorphism as well. This shows

$$\widehat{\mathscr{O}} \cong k[\![t_1, \ldots, t_n]\!],$$

which implies that $\widehat{\mathscr{O}}$ is regular, and so also is $\mathscr{O}$ itself. This proves G is smooth at e. □

10. Quotients by Finite Group Schemes

10.1. The Group Scheme Action on Scheme.

Definition 10.1 (Left action of schemes). A **left action of a group scheme G on a scheme** X is a morphism $\mu : G \times X \to X$ such that

(1) the composite

$$X \xrightarrow{\cong} \operatorname{Spec} k \times X \xrightarrow{e_G \times 1_X} G \times X \xrightarrow{\mu} X$$

is the identity morphism;

(2) the diagram

[27] For this implication, see [Eis13, §7].

$$\begin{tikzcd} G \times G \times X \arrow[r, "m\times 1_X"] \arrow[d, "1_G\times\mu"'] & G \times X \arrow[d, "\mu"] \\ G \times X \arrow[r, "\mu"] & X \end{tikzcd}$$

is commutative;

Remark 10.2. Indeed, we have the following equivalent characterization of a G-action on X.

(1) For any affine[28] scheme S we have a (left) $\underline{G}(S)$-action on $\underline{X}(S)$, which is functorial in S.
(2) More explicitly, for any $x \in \underline{G}(S)$, we have an automorphism over S; say the diagram

$$\begin{tikzcd} X \times S \arrow[rr, "T_x"] \arrow[dr, "p_2"'] & & X \times S \arrow[dl, "p_2"] \\ & S & \end{tikzcd}$$

commutes and is such that
 (i) $T_x \circ T_y = T_{xy}$ for all $x, y \in \underline{G}(S)$;
 (ii) for any morphism $f : S \to S'$ in Sch_k and $x \in \underline{G}(S')$, $x \circ f \in G(S)$.

We have another commutative diagram

$$\begin{tikzcd} X \times S \arrow[r, "T_{x\circ f}"] \arrow[d, "1_X\times f"'] & X \times S \arrow[d, "1_X\times f"] \\ X \times S' \arrow[r, "T_x"] & X \times S' \end{tikzcd}$$

For $x \in \underline{G}(S)$, the morphism $T_x : X \times S \to X \times S$ is given by (T'_x, p_2) where $T'_x : X \times S \to X$ is the composite

$$X \times S \cong S \times X \xrightarrow{f\times 1_X} G \times X \xrightarrow{\mu} X.$$

Conversely, the morphism μ can be recovered from the above datum. Take $S = G$ and $x = \mathrm{id}_G \in \underline{G}(G)$. Then μ is the composite

$$G \times X \xrightarrow{\cong} X \times G \xrightarrow{T_x} X \times G \xrightarrow{p_1} X.$$

Definition 10.3. A morphism $f : X \to Y$ is called G**-invariant** if the following diagram is commutative[29]

$$\begin{tikzcd} G \times X \arrow[r, "\mu"] \arrow[d, "p_2"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

More explicitly, for each $S \in \mathsf{Sch}_k$, $g \in \underline{G}(S)$, $x \in \underline{X}(S)$, we have $f(\mu(g, x)) = f(x)$.

The action of G on X is **free** if the morphism $(\mu, p_2) : G \times X \to X \times X$ is a closed immersion.

Definition 10.4. Let $\mathscr{F}$ be a coherent sheaf on X. A **lift of the action** μ **to** $\mathscr{F}$ is an isomorphism

$$\lambda : p_2^*\mathscr{F} \xrightarrow{\sim} \mu^*\mathscr{F}$$

of sheaves on $G \times X$ such that the following diagram of sheaves on $G \times G \times X$ is commutative:

---

[28]Note that the problem is local.

[29]When $Y = \mathbb{A}^1 = \operatorname{Spec} k[T]$, one can talk about the G-invariant sections.

$$\begin{tikzcd}
p_3^*\mathscr{F} \arrow[rr, "(p_2{,}p_3)^*(\lambda)"] \arrow[dr, "(m\times 1_X)^*(\lambda)"'] & & \xi^*\mathscr{F} \arrow[dl, "(1_G\times\mu)^*(\lambda)"] \\
& \eta^*\mathscr{F} &
\end{tikzcd}$$

Here p_1, p_2, p_3 are natural projections from $G \times G \times X$ and

$$\xi : G \times G \times X \xrightarrow{(p_2,p_3)} G \times X \xrightarrow{\mu} X$$

$$\begin{tikzcd}
\eta : G \times G \times X \arrow[r, "1_G\times\mu"] \arrow[dr, bend right, "m\times 1_X"'] & G \times X \arrow[r, "\mu"] & X \\
& G \times X \arrow[ur, "\mu"'] &
\end{tikzcd}$$

10.2. **Classification of Quotients.**

Theorem 10.5 (Quotients by finite group schemes, somehow tedious)**.**

(A) *Let G be a finite group scheme acting on a scheme X such that the orbit of any point in contained in an affine open subset of X. Then there is a pair (Y, π) where Y is a scheme and $\pi : X \to Y$ a morphism, satisfying the following conditions:*
  (1) *as a topological space, (Y, π) is the quotient of X for the action of the underlying finite group;*
  (2) *the morphism $\pi : X \to Y$ is G-invariant, and if $\pi_*(\mathscr{O}_X)^G$ denotes the subsheaf of $\pi_*\mathscr{O}_X$ of G-invariant functions, the natural homomorphism $\mathscr{O}_Y \to \pi_*(\mathscr{O}_X)^G$ is an isomorphism.*
  The pair (Y, π) is uniquely determined up to isomorphism by these conditions. The morphism π is finite and surjective; Y will be denoted X/G, and it has the functorial property that for any G-invariant morphism $f : X \to Z$, there is a unique morphism $g : Y \to Z$ such that $f = g \circ \pi$.

$$\begin{tikzcd}
X \arrow[rr, "\pi"] \arrow[dr, "f"'] & & Y \arrow[dl, dashed, "\exists!"', "g"] \\
& Z &
\end{tikzcd}$$

(B) *Suppose further that the action of G is free and $G = \operatorname{Spec} R$, $n = \dim_k R$. Then π is a flat morphism of degree n, i.e., $\pi_*\mathscr{O}_X$ is a locally free $\mathscr{O}_Y$-module of rank n, and the subscheme of $X \times X$ defined by the closed immersion*

$$(\mu, p_2) : G \times X \to X \times X$$

is equal to the subscheme $X \times_Y X \subset X \times X$. Finally, if $\mathscr{F}$ is a coherent $\mathscr{O}_Y$-module, $\pi_*\mathscr{F}$ has a natural defined G-action lifting that on X, and $\mathscr{F} \mapsto \pi^*\mathscr{F}$ is an equivalence between the category of coherent $\mathscr{O}_Y$-modules (resp. locally free $\mathscr{O}_Y$-modules of finite rank) and the category of coherent $\mathscr{O}_X$-modules with G-action (resp. locally free $\mathscr{O}_X$-modules of finite rank with G-action).

Remark 10.6. (1) The assumption that the orbit of any point in contained in an affine open subset of X can be expressed as follows: for any closed point $x \in X(k)$, the morphism

$$G \cong G \times \operatorname{Spec} k \xrightarrow{1_G \times x} G \times X \xrightarrow{\mu} X$$

factors through an open affine subset of X, i.e., we obtain $G \to U \subset X$. This holds for X quasi-projective over k.

(2) $G_{\mathrm{red}} = \operatorname{Spec}(R_{\mathrm{red}})$ is a closed subgroup scheme of G, and the action of G on X induces an action

$$\mu_{\mathrm{red}} : G_{\mathrm{red}} \times X \to X$$

of G_{red} on X. As a scheme over k,

$$G_{\mathrm{red}} \cong \coprod_{g \in G(k)} \operatorname{Spec} k,$$

and we are in the situation of varieties. Theorem 10.5 (A)(1) says that as a topological space, (Y, π) only depends on the action μ_{red} of G_{red} on X.

(3) Under the assumption of (B), we have an isomorphism

$$(\mu, p_2) : G \times X \xrightarrow{\sim} X \times_Y X,$$

and $\pi : X \to Y$ is faithfully flat. Let $\mathscr{F}$ be a coherent sheaf on X. Under the above isomorphism, a lift of the action μ to $\mathscr{F}$ becomes an isomorphism $\lambda : p_2^*\mathscr{F} \xrightarrow{\sim} p_1^*\mathscr{F}$ (sheaves on $X \times_Y X$) such that the diagram

$$\begin{tikzcd} p_3^*\mathscr{F} \arrow[rr, "p_{23}^*(\lambda)"] \arrow[dr, "p_{13}^*(\lambda)"'] & & p_2^*\mathscr{F} \arrow[dl, "p_{12}^*(\lambda)"] \\ & p_1^*\mathscr{F} & \end{tikzcd}$$

commutes.[30]

10.3. Proof of Theorem (A).

Proof. We can reduce to the case for $X = \operatorname{Spec} A$ affine. Recall that $G = \operatorname{Spec} R$ and $n = \dim_k R$. Consider the k-algebra homomorphisms in the following correspondences:

Algebraic Homomorphisms	Geometric Morphisms
$\varepsilon : R \to k$	$e : \operatorname{Spec} k \to G$
$m^* : R \to R \otimes_k R$	$m : G \times G \to G$
$\mu^* : A \to R \otimes_k A$	$\mu : G \times X \to X$

For any k-algebra S, $R \otimes_k S$ is a free S-module of rank n. We have a norm map $\mathrm{Nm}_S : R \otimes_k S \to S$, i.e., for any $x \in R \otimes_k S$ the multiplication by x defines an S-linear map $l_x : R \otimes_k S \to R \otimes_k S$, and $\mathrm{Nm}_S(x) = \det l_x$. Also,

$$\mathrm{Nm}_S(ax) = a^n \, \mathrm{Nm}_S(x), \quad \forall a \in S,\ x \in R \otimes_k S,$$

and Nm_S is multiplicative.

Also define $B = \{a \in A \mid \mu^*(a) = 1 \otimes a\} \subset A$ to be the k-subalgebra of A consisting of G-invariant sections; that is, for $a \in A$, the morphism $X \to \mathbb{A}^1$ corresponding to $k[T] \to A$, $T \mapsto G$ is G-invariant if and only if $a \in B$. Define the composite

$$N : A \xrightarrow{\mu^*} R \otimes_k A \xrightarrow{\mathrm{Nm}_A} A.$$

Note that N is multiplicative and k-homogeneous of degree n.

Claim. *$N(A) \subset B$, i.e., $\mu^*(N(a)) = 1 \otimes N(a)$ for each $a \in A$.*

Proof of Claim. We define two k-algebra homomorphisms (with their geometric correspondences) as follows:

$$\phi : A \to R \otimes_k A, \quad a \mapsto 1 \otimes a$$

corresponding to

$$p_2 : G \times X \to X, \quad (g, x) \mapsto x;$$

[30] This is the standard descent theory. See [MA67, Chap VII].

and
$$\psi : R \otimes_k R \otimes_k A \to R \otimes_k R \otimes_k A, \quad x \otimes y \otimes a \mapsto (m^*(x) \otimes 1) \cdot (1 \otimes y \otimes a)$$
corresponding to
$$\operatorname{Spec} \psi : G \times G \times X \to G \times G \times X, \quad (g, h, x) \mapsto (gh, h, x).$$
Firstly, we make a remark. If $f : S_1 \to S_2$ is a k-algebra homomorphism, then the diagram commutes, namely, $f \circ \mathrm{Nm}_{S_1} = \mathrm{Nm}_{S_2} \circ (1_R \otimes f)$.

$$\begin{tikzcd} R \otimes_k S_1 \arrow[r, "\mathrm{Nm}_{S_1}"] \arrow[d, "1_R \otimes f"'] & S_1 \arrow[d, "f"] \\ R \otimes_k S_2 \arrow[r, "\mathrm{Nm}_{S_2}"] & S_2 \end{tikzcd}$$

So, by the above remark, we obtain that
$$\mu^* \circ N = \mu^* \circ \mathrm{Nm}_A \circ \mu^* = \mathrm{Nm}_{R \otimes_k A} \circ (1_R \otimes \mu^*) \circ \mu^*.$$
Moreover,
$$\mathrm{Nm}_{R \otimes_k A} \circ (1_R \otimes \mu^*) \circ \mu^* = \mathrm{Nm}_{R \otimes_k A} \circ (m^* \otimes 1_A) \circ \mu^* = \mathrm{Nm}_{R \otimes_k A} \circ \psi \circ (1_R \otimes \phi) \circ \mu^*$$
because of the two diagrams are commutative:

$$\begin{tikzcd} G \times G \times X \arrow[r, "1_G \times \mu"] \arrow[d, "m \times 1_X"'] & G \times X \arrow[d, "\mu"] \\ G \times X \arrow[r, "\mu"] & X \end{tikzcd} \qquad \begin{tikzcd} G \times G \times X \arrow[r, "1_G \times \mu"] \arrow[d, "m \times 1_X"'] & G \times G \times X \arrow[r, "\mathrm{id}_G \times p_2"] & G \times X \arrow[dl, "\mu"] \\ G \times X \arrow[r, "\mu"] & X & \end{tikzcd}$$

Let us regard $R \otimes_k (R \otimes_k A)$ as an $R \otimes_k A$-algebra via the last two factors, i.e., via the k-algebra homomorphism
$$R \otimes_k A \to R \otimes_k R \otimes_k A, \quad r \otimes a \mapsto 1 \otimes r \otimes a.$$
Then $\psi : R \otimes_k R \otimes_k A \to R \otimes_k R \otimes_k A$ is an $R \otimes_k A$-algebra automorphism; equivalently, we need to verify that

$$\begin{tikzcd} G \times G \times X \arrow[rr, "\operatorname{Spec} \psi"] \arrow[dr, "p_{23}"'] & & G \times G \times X \arrow[dl, "p_{23}"] \\ & G \times X & \end{tikzcd}$$

is an automorphism. Thus,
$$\mathrm{Nm}_{R \otimes_k A} \circ \psi = \mathrm{Nm}_{R \otimes_k A}.$$
And therefore,
$$\mu^* \circ N = \mathrm{Nm}_{R \otimes_k A} \circ (1_R \otimes \phi) \circ \mu^* = \phi \circ \mathrm{Nm}_A \circ \mu^* = 1 \otimes N.$$
This proves our claim. □

We extend the G-action on X to $X \times \mathbb{A}^1$ such that G acts trivially on $\mathbb{A}^1$ with $\mu \times \mathrm{id}_{\mathbb{A}^1} : G \times X \times \mathbb{A}^1 \to X \times \mathbb{A}^1$. Correspondingly, $\mu^* : A \to R \otimes_k A$ can be extended to a k-algebra homomorphism $A[T] \to R \otimes_k A[T]$. So we can extend the map $N : A \to A$ to $N : A[T] \to A[T]$. For $a \in A$, we set $\chi_a(T) = N(T - a)$ and we can extend χ_a to a k-algebra homomorphism $k[T] \to A[T]$ (and hence determines a morphism $X \times \mathbb{A}^1 \to \mathbb{A}^1$).

It is straightforward to verify that $\chi_a(T) \in A[T]$ is the characteristic polynomial of the A-linear map

$$l_{\mu^*(a)} : R \otimes_k A \to R \otimes_k A$$

that is induced by the multiplication by $\mu^*(a)$, and $\chi_a(T)$ is G-invariant, i.e., the morphism $X \times \mathbb{A}^1 \to \mathbb{A}^1$ determined by $\chi_a(T)$ is G-invariant. So

$$\chi_a(T) = T^n + s_1 T^{n-1} + \cdots + s_n \in A[T]$$

is monic of degree n, and $s_i \in B$ for all i; namely, $\chi_a(T) \in B[T]$.

Fix $a \in A$. The map $\varepsilon : R \to k$ corresponding to the section $e : \operatorname{Spec} k \to G$ extends to an A-linear map $\varepsilon \otimes 1_A : R \otimes_k A \to A$ such that the composite $A \xrightarrow{\mu^*} R \otimes_k A \xrightarrow{\varepsilon \otimes 1_A} A$ is nothing but id_A. Thus the A-linear map $l_{\mu^*(a)-a} : R \otimes_k A \to R \otimes_k A$ induces the zero map on the quotient $\varepsilon \otimes 1_A : R \otimes_k A \to A$. It follows that

$$\chi_a(a) = \det(l_{a-\mu^*(a)}) = 0,$$

namely, a is integral over B. Hence A is integral over B. Since A is a finitely generated k-algebra, there exists a finitely generated k-subalgebra $B' \subset B$ such that A is integral and finite over B'. Then B is finite over B'. Hence B is a finitely generated k-algebra. If we use $\pi : X \to Y = \operatorname{Spec} B$ to denote the morphism corresponding to the inclusion $B \hookrightarrow A$, then π is definitely finite and surjective.

Now we prove that π separates orbits, i.e. for two closed points $x_1, x_2 \in X(k)$, if $G_{\mathrm{red}}(k) = G'$ and $G' \cdot x_1 \cap G' \cdot x_2 = \emptyset$, then $\pi(x_1) \neq \pi(x_2)$. Define

$$N_{\mathrm{red}} : A \xrightarrow{\mu^*_{\mathrm{red}}} R_{\mathrm{red}} \otimes_k A \xrightarrow{\mathrm{Nm}} A.$$

By the argument in the previous lectures for Chapter II, we can find $a \in A$ such that $a(g'x_1) = 1$, $a(g'x_2) = 0$ for all $g' \in G'$. Granting this, we obtain

$$N_{\mathrm{red}}(a)(x_1) = 1, \quad N_{\mathrm{red}}(a)(x_2) = 0.$$

From the commutative diagrams (with $\alpha \in R \otimes_k A$ arbitrarily fixed):

$$\begin{tikzcd}
A \arrow[r, "\mu^*"] \arrow[dr, "\mu^*_{\mathrm{red}}"'] & R \otimes_k A \arrow[r, "l_\alpha"] \arrow[d] & R \otimes_k A \arrow[d] \\
& R_{\mathrm{red}} \otimes_k A \arrow[r, "l_{\overline{\alpha}}"] & R_{\mathrm{red}} \otimes_k A
\end{tikzcd}$$

We can verify that $N(a)(x_1) \neq 0$, $N(a)(x_2) = 0$. Their implications are that $l_{\overline{\alpha}}(x_1)$ is an isomorphism (hence is surjective), and $l_{\overline{\alpha}}(x_2)$ is not surjective, respectively. One can actually show

$$N_{\mathrm{red}}(T-a)(x_1) = (T-1)^n, \quad N_{\mathrm{red}}(T-a)(x_2) = T^n.$$

On the other hand, since $N(a) \in B$, it forces $\pi(x_1) \neq \pi(x_2)$.

By definition, $\pi_*(\mathscr{O}_X)^G$ is the kernel of the $\mathscr{O}_Y$-linear map

$$\pi_*\mathscr{O}_X \to \pi_*\mathscr{O}_X \otimes_k R, \quad f \mapsto \mu^*(f) - f \otimes 1.$$

Then $\pi_*(\mathscr{O}_X)^G$ is coherent on Y, and $\mathscr{O}_Y \cong \pi_*(\mathscr{O}_X)^G$. Finally, the universal property of (Y, π) naturally follows from the construction. This finishes the proof of (A). $\square$

10.4. **Proof of Theorem (B).**

Proof. Given a coherent sheaf $\mathscr{F}$ on Y, we have a canonical isomorphism

$$\lambda : p_2^*(\pi^*\mathscr{F}) \to \mu^*(\pi^*\mathscr{F})$$

as the two composites

$$\begin{tikzcd} G \times X \arrow[r, bend left, "\mu"] \arrow[r, bend right, "p_2"'] & X \arrow[r, "\pi"] & Y \end{tikzcd}$$

are equal. One can verify that this defined a lift of μ to $\pi^*\mathscr{F}$, i.e., can check the diagram is commutative:

$$\begin{tikzcd} p_3^*\pi^*\mathscr{F} \arrow[rr] \arrow[dr] & & \xi^*\pi^*\mathscr{F} \arrow[dl] \\ & \eta^*\pi^*\mathscr{F} & \end{tikzcd}$$

Conversely, let $\mathscr{G}$ be a coherent sheaf on X and we have a lift of μ to $\mathscr{G}$. In case when $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$ are affine, $\mathscr{G} = \widetilde{N}$ for some A-module N. We define $\pi_*(\mathscr{G})^G$ to be the coherent $\mathscr{O}_Y$-module corresponding to the B-module

$$N^G = \{n \in N \mid \lambda(\underbrace{1 \otimes n}_{p_2^*(n)}) = \mu^*(n) = n \otimes_{A,\mu^*} 1 \in N \otimes_{A,\mu^*} (R \otimes_k A)\}.$$

We run this construction for all open affine G-stable subsets of X, and can define $\pi_*(\mathscr{G})^G$ in general.

Now we assume that the action of G on X is free. The requirement is to prove:

(1) π is flat; alternatively, $B \to A$ is flat;
(2) $G \times X \xrightarrow{\sim} X \times_Y X$ is an isomorphism;
(3) the functors
$$\mathsf{Mod}_{\mathscr{O}_Y} \to \mathsf{Mod}_{(G,\mathscr{O}_X)}, \quad \mathscr{F} \mapsto \pi^*\mathscr{F}$$
and
$$\mathsf{Mod}_{(G,\mathscr{O}_X)} \to \mathsf{Mod}_{\mathscr{O}_Y}, \quad \mathscr{G} \mapsto \pi_*(\mathscr{G})^G$$
are inverses to each other. For this, it suffices to show $T(\mathscr{G}) : \pi^*\pi_*(\mathscr{G})^G \to \mathscr{G}$ is an isomorphism for each $(G, \mathscr{O}_X)$-module $\mathscr{G}$.

Now we assume $X = \operatorname{Spec} A$ is affine. As the G-action is free, $(\mu, p_2) : G \times X \to X \times X$ is a closed immersion. Since it factors through $X \times_Y X$, we get a surjective k-algebra homomorphism

$$\varphi : A \otimes_B A \to R \otimes_k A, \quad a_1 \otimes a_2 \mapsto \mu^*(a_1)(1 \otimes a_2).$$

Then it boils down to prove that

(1') A is flat over $B = A^G$, and φ is injective;
(2') for each coherent (G, A)-module M, the natural map $M^G \otimes_B A \to M$ is an isomorphism;
(3') if M is a projective A-module, M^G is projective as a B-module.

We first explain that $(1')(2')$ imply $(3')$. It suffices to show that M^G is flat as a B-module, or equivalently, the functor

$$(\cdot) \otimes_B M^G : \mathsf{Mod}_B \to \mathsf{Mod}_B$$

is exact. Since $B \to A$ is faithfully flat by (1'), this is to prove that the functor

$$((\cdot) \otimes_B M^G) \otimes_B A : \mathsf{Mod}_B \to \mathsf{Mod}_A$$

is exact. For any B-module N, we have

$$\begin{aligned}(N \otimes_B M^G) \otimes_B A &\cong (N \otimes_B A) \otimes_A (A \otimes_B M^G) \\ &\cong (N \otimes_B A) \otimes_A M \qquad \text{by granting (2')} \\ &\cong N \otimes_B M.\end{aligned}$$

And since M is A-flat with A being B-flat, the functor is morally exact. Therefore, we are left to prove (1')(2').

(1') Replacing B by $B_{\mathfrak{m}}$ where $\mathfrak{m} \subset B$ is the maximal ideal and A by $A \otimes_B B_{\mathfrak{m}}$, we may assume, without loss of generality, that B is local and A is semi-local. Regard $A \otimes_B A$ and $R \otimes_k A$ as A-algebras via the second factor. The map

$$\varphi : A \otimes_B A \to R \otimes_k A, \quad a_1 \otimes a_2 \mapsto \mu^*(a_1)(1 \otimes a_2)$$

is a homomorphism of A-algebras. Since φ is onto, $R \otimes_k A$ is generated by $\mu^*(a)$ with $a \in A$ as an A-algebra. Since A is semi-local one can find $\{a_1, \ldots, a_n\}$ in A such that $\{\mu^*(a_i) \mid 1 \leqslant i \leqslant n\}$ form a basis of $R \otimes_k A$ as an A-module.[31]

Claim. For $a, \lambda_1, \ldots, \lambda_n \in A$,

$$(*) \qquad \mu^*(a) = \sum_{i=1}^n (1 \otimes \lambda_i) \cdot \mu^*(a_i) \quad \Longleftrightarrow \quad a = \sum_{i=1}^n \lambda_i \cdot a_i \text{ with } \lambda_1, \ldots, \lambda_n \in B.$$

For ($\Leftarrow$), apply μ^* to $a = \sum_{i=1}^n \lambda_i \cdot a_i$ and use the fact that $\mu^*(\lambda_i) = 1 \otimes \lambda_i$ as $\lambda_i \in B$. For ($\Rightarrow$), since $(1_R \otimes \mu^*)(\mu^* a) = (m^* \otimes 1_A)(\mu^* a)$ in $R \otimes_k R \otimes_k A$, we have

$$\begin{aligned}&\sum_{i=1}^n (1 \otimes \mu^*(\lambda_i))(1_R \otimes \mu^*)(\mu^*(a_i)) \\ =&\sum_{i=1}^n (1 \otimes 1 \otimes \lambda_i)(m^* \otimes 1_A)(\mu^*(a_i)) \\ =&\sum_{i=1}^n (1 \otimes 1 \otimes \lambda_i)(1_R \otimes \mu^*)(\mu^*(a_i)).\end{aligned}$$

Since $\{\mu(a_i) \mid 1 \leqslant i \leqslant n\}$ is a basis of $R \otimes_k A$ as an A-module, $(1_R \otimes \mu^*)(\mu^*(a_i))$ is a basis of $R \otimes_k R \otimes_k A$ as an $R \otimes_k A$-module via the last two factors. (Here we have used $(R \otimes_k A) \otimes_{A,\mu^*} (R \otimes_k A) \cong R \otimes_k R \otimes_k A$.) Thus, in $R \otimes_k R \otimes_k A$,

$$1 \otimes \mu^*(\lambda_i) = 1 \otimes 1 \otimes \lambda_i.$$

So all λ_i's land in B. Apply $\varepsilon \otimes 1$ to $\mu^*(a) = \sum_{i=1}^n (1 \otimes \lambda_i)(\mu^*(a_i)$, we have $a = \sum_{i=1}^n \lambda_i \cdot a_i$. So we have proved $(*)$. However, $(*)$ implies that A is a free B-module with basis $\{a_1, \ldots, a_n\}$.

[31] Here are more details about this step of argument. Since $R \otimes_k A$ is a free A-module of rank n, it suffices to show that $\{\mu^*(a_i) \mid 1 \leqslant i \leqslant n\}$ generates $R \otimes_k A$ as an A-module for some suitable $\{a_i\}_{1 \leqslant i \leqslant n}$. By Nakayama's lemma, it reduces to the case where $A = \prod_{i=1}^m k$. We are to prove the following: if M is a free A-module of rank n, and a k-subspace $\Sigma \subset M$ is a set of generators of M, then there exist $x_1, \ldots, x_n \in \Sigma$ such that $\{x_1, \ldots, x_n\}$ is a basis of M as an A-module. To see this, one can use induction on $n = \operatorname{rank}_A M$. When $n = 1$, it suffices to find an element $x \in \Sigma$ such that if $x = (x^1, \ldots, x^m)$ then $x^i \neq 0$ for all $i = 1, \ldots, m$. We can prove this by induction on m and use the fact that $k = \overline{k}$ is algebraically closed. And hence k is infinite. In general, suppose the statement holds for n and M is a free A-module of rank $n+1$. Then one may find $x_1 \in M$ if we write $x_1 = (x_1^1, \ldots, x_1^m)$ under the decomposition $A = \prod_{i=1}^m k$. Thus $x_1^i \neq 0$ for each $i = 1, \ldots, m$, i.e., $Ax_1 \subset M$ is a free A-submodule of rank 1. Since A is isomorphic to m-copies of k, any (finitely generated) A-module is locally free and hence projective. Therefore, there exists $M_1 \subset M$ that is free of rank n such that $M = Ax_1 \oplus M_1$. Apply the inductive hypothesis to M_1 and get the desired $\{x_1, \ldots, x_{n+1}\}$.

This shows A is flat over B. Moreover, the A-linear map $\varphi : A \otimes_B A \to R \otimes_k A$ is a map between free A-modules of rank n and takes a basis $\{a_i \otimes 1\}$ to a basis $\{\mu^*(a_i)\}$. So φ is an isomorphism.

(2') Morally, this follows from the general descent theory. We only list out a sketch. View $M \otimes_B A$ and $A \otimes_B M$ as $A \otimes_B A$-modules in the obvious way. Then a G-action on M is an isomorphism of $A \otimes_B A$-modules $\tau : A \otimes_B M \to M \otimes_B A$ such that

$$\begin{tikzcd} & A \otimes A \otimes M \arrow[dl, "1_A \otimes \tau"'] \arrow[dd, "(p_1 \circ \tau) \otimes 1_A \otimes (p_2 \circ \tau)"] \\ A \otimes M \otimes A \arrow[dr, "\tau \otimes 1_A"'] & \\ & M \otimes A \otimes A \end{tikzcd}$$

Note that the right vertical map is given by τ on the first and the third factors together with 1_A on the second factor.

Define

$$N = \{m \in M \mid \tau(1 \otimes m) = m \otimes 1\}.$$

We need to show that $N \otimes_B A \to M$ is an isomorphism. Notice that

$$N = \operatorname{Ker}(\phi : M \to M \otimes_B A), \quad m \mapsto m \otimes 1 - \tau(1 \otimes m)$$

and $B \to A$ is flat, we have

$$N \otimes_B A = \left\{ \sum_i m_i \otimes a_i \in M \otimes_B A \,\middle|\, \sum_i m_i \otimes 1 \otimes a_i = \sum_i \tau(1 \otimes m_i) \otimes a_i \right\}.$$

Applying the commutative diagram above to $1 \otimes 1 \otimes m \in A \otimes A \otimes M$, we have

$$\tau(1 \otimes M) \subset N \otimes_B A$$

as subsets in $M \otimes_B A$. We view $M \otimes_B A$ as an A-module via the second factor, then $\tau(1 \otimes M)$ and $N \otimes_B A$ are A-submodules of $M \otimes_B A$; and $N \otimes_B A$ is generated by those $n \otimes 1$ with $n \in N$. Since $n \otimes 1 = \tau(1 \otimes n) \in \tau(1 \otimes M$, we have $M \otimes_B A = \tau(1 \otimes M)$. On the other hand, as $B \to A$ is faithfully flat, the map $M \to A \otimes_B M$ is injective and we have an isomorphism

$$\begin{tikzcd} M \arrow[r] \arrow[rr, bend left, "\sim"] & 1 \otimes M \arrow[r, "\tau"] & \tau(1 \otimes M) \\ m \arrow[r, mapsto] & 1 \otimes m & \end{tikzcd}$$

so we get a canonical isomorphism $N \otimes_B A \cong M$.

We have accomplished the proof of (B). ☐

Interlude: On Seesaw's Theorem

This is a preliminary part of the upcoming lecture which recalls and generalizes the classical Seesaw's theorem we have mentioned in Chapter II.

Theorem 10.7 (Seesaw)**.** *Let X be a complete variety, T any variety, and $\mathscr{M}$ a line bundle on $X \times T$. Then the set*

$$T_1 = \{t \in T \mid \mathscr{M}|_{X \times \{t\}} \text{ is trivial on } X \times \{t\}\}$$

is closed in T, and if $p_2 : X \times T_1 \to T_1$ is the second projection, then $\mathscr{M}|_{X\times T_1} \cong p_2^*\mathscr{N}$ for some line bundle $\mathscr{N}$ on T_1. Also, T_1 has the reduced closed subscheme structure.

Proposition 10.8 (Generalized Seesaw)**.** *Let X be a complete variety, Y a scheme, and $\mathscr{M}$ a line bundle on $X \times Y$. Then there exists a unique and closed subscheme $Y_1 \hookrightarrow Y$ with the following properties.*

1. *If $\mathscr{M}_1 = \mathscr{M}|_{X\times Y_1}$, there is a line bundle $\mathscr{N}_1$ on Y_1 and an isomorphism $p_2^*\mathscr{N}_1 \cong \mathscr{M}_1$ on $X \times Y_1$; or alternatively, if $i_1 : Y_1 \hookrightarrow Y$ denotes the first natural closed immersion, we obtain $p_2^*\mathscr{N}_1 \cong (1_X \times i_1)^*\mathscr{M}$.*
2. *If $f : Z \to Y$ is a morphism such that there exists a line bundle $\mathscr{K}$ on Z and an isomorphism $p_2^*\mathscr{K} \cong (1_X \times f)^*\mathscr{M}$ on $X \times Z$, then f factors as*
$$f : Z \to Y_1 \hookrightarrow Y.$$

Remark 10.9. For any closed point $y_1 \in Y_1(k)$, we have
$$\mathscr{M}|_{X\times\{y_1\}} \cong \mathscr{M}_1|_{X\times\{y_1\}} \cong (p_2^*\mathscr{N}_1)|_{X\times\{y_1\}}$$
being trivial. So the closed subvarieties given by the above two Seesaw's are homeomorphic as topological spaces. But the closed subscheme Y_1 in the second proposition may have nonreduced closed subscheme structure so that the universal property (2) holds.

We refer the closed subscheme Y_1 of Y in Proposition 10.8 as the maximal closed subscheme of Y over which $\mathscr{M}$ is trivial. (Caveat: the notation is a little misleading as $\mathscr{M}|_{X\times Y_1}$ is NOT a trivial line bundle in general. Sorry for this!)

11. The Dual Abelian Variety in Any Characteristic

In Chapter II, we defined a reduced closed subscheme $K(\mathscr{L})$ of X, for every line bundle $\mathscr{L}$ on an abelian variety X, i.e.,
$$K(\mathscr{L}) = \{x \in X(k) \mid T_x^*\mathscr{L} \cong \mathscr{L}\}.$$
We want to make $K(\mathscr{L})$ as a (nonreduced) closed subgroup scheme of X.

Definition 11.1. Consider the line bundle $\mathscr{M} = m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^*\mathscr{L}^{-1}$ on $X \times X$. We define $K(\mathscr{L})$ to be the maximal closed subscheme of X such that $\mathscr{M}|_{X\times K(\mathscr{L})}$ is trivial.

Remark 11.2. We apply the generalized Seesaw theorem (Proposition 10.8) to $\mathscr{M} \in \mathrm{Pic}(X \times X)$ and get a line bundle $\mathscr{N}_1$ on $K(\mathscr{L})$ and an isomorphism $p_2^*\mathscr{N}_1 \cong \mathscr{M}|_{X\times K(\mathscr{L})}$. But $\mathscr{N}_1 \cong (p_2^*\mathscr{N}_1)|_{\{e_X\}\times K(\mathscr{L})} \cong \mathscr{M}|_{\{e_X\}\times K(\mathscr{L})}$ is trivial as $\mathscr{M}|_{\{e_X\}\times K(\mathscr{L})}$ is trivial. So that $\mathscr{M}|_{X\times K(\mathscr{L})}$ is trivial as well.

In the upcoming context we are to verify that $K(\mathscr{L})$ is a *closed subgroup scheme* of X. Recall we have defined the "translation by f" morphism, say T_f, as an automorphism over S as follows:

$$\begin{tikzcd} X \times S =: X_S \arrow[rr, "T_f"] \arrow[dr] & & X_S \arrow[dl] \\ & S & \end{tikzcd}$$

Also, $p_1 \circ T_f : X_S \to X$ is the composite
$$X \times S \xrightarrow{1_X\times f} X \times X \xrightarrow{m} X.$$
Here X is a commutative group scheme, so there is no difference between left and right translations.

Lemma 11.3. *Set $\mathscr{L}_S = p_1^*\mathscr{L} \in \operatorname{Pic}(X_S)$. Then $f \in K(\mathscr{L})(S)$ if and only if $T_f^*\mathscr{L}_S \cong \mathscr{L}_S \otimes p_2^*\mathscr{N}$ for some $\mathscr{N} \in \operatorname{Pic}(S)$.*

Proof. By direct computation, we have

$$T_f^*\mathscr{L}_S = T_f^* p_1^*\mathscr{L} \cong (1_X \times f)^*(m^*\mathscr{L}),$$

and hence $T_f^*\mathscr{L}_S|_{\{e_X\}\times S} \cong f^*\mathscr{L}$; the restriction $\mathscr{L}_S|_{\{e_X\}\times S}$ is trivial. So if $T_f^*\mathscr{L}_S \cong \mathscr{L}_S \otimes p_2^*\mathscr{N}$ for some $\mathscr{N} \in \operatorname{Pic}(S)$, by restricting to $\{e_X\} \times S$, we have $\mathscr{N} \cong f^*\mathscr{L}$. Hence

$$\begin{aligned}
& T_f^*\mathscr{L}_S \cong \mathscr{L}_S \otimes p_2^*\mathscr{N} \\
\iff\ & (1_X \times f)^* m^*\mathscr{L} \cong p_1^*\mathscr{L} \otimes p_2^* f^*\mathscr{L} \\
\iff\ & (1_X \times f)^* m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^*(f^*\mathscr{L})^{-1} \cong (1_X \times f)^*\mathscr{M} \text{ is trivial on } X \times S \\
\iff\ & f \text{ factors through } K(\mathscr{L}).
\end{aligned}$$

This is equivalent to say $f \in K(\mathscr{L})(S)$. ☐

It follows from Lemma 11.3 that $K(\mathscr{L})(S)$ is a subgroup of $X(S)$.

Hence $K(\mathscr{L})$ is a subgroup scheme of X. Now we are ready to construct the dual abelian variety over any characteristic. Fix an ample line bundle $\mathscr{L}$ on X. Then $K(\mathscr{L})$ is a closed finite subgroup scheme of X. Define $\widehat{X} = X/K(\mathscr{L})$ where $K(\mathscr{L})$ acts on X via translation and $\pi : X \to \widehat{X}$ is the natural morphism. One can verify that $\widehat{X}$ is also an abelian variety and π is an isogeny of abelian varieties, i.e., a finite surjective homomorphism. Consequently,

$$\widehat{X}(k) \xrightarrow{\ \sim\ } X(k)/K(\mathscr{L})(k) \xrightarrow[\phi_{\mathscr{L}}]{\ \sim\ } \operatorname{Pic}^0(X).$$

There is another isomorphism of abelian groups $\widehat{X}(k) \cong \operatorname{Pic}^0(X)$. As before, we want to define the Poincaré bundle $P \in \operatorname{Pic}(X \times \widehat{X})$ such that $(1_X \times \pi)^*(P) = \mathscr{M}$, via the isogeny $1_X \times \pi : X \times X \to X \times \widehat{X}$ with its kernel $K = 1 \times K(\mathscr{L})$. So it suffices to define a lift of the action of K on $X \times X$ to $\mathscr{M}$. More precisely, we need to find an isomorphism

$$\lambda : p_2^*\mathscr{M} \to \mu^*\mathscr{M}$$

where $p_2 : K \times (X \times X) \to X \times X$ is the natural projection. (Also recall that $\mu : K \times (X \times X) \to X \times X$ is the translation morphism.

In general, for a scheme S and an S-valued point $(e, x) : S \to K = 1 \times K(\mathscr{L})$ of K (so $x \in K(\mathscr{L})(S)$), let

$$T_{(e,x)} : X_S \times_S X_S \to X_S \times_S X_S$$

be the translation by $(e, x) \in K(S) \subset (X \times X)(S)$ and $T_x : X_S \to X_S$ be the translation by $x \in X(S)$. Let $\mathscr{M}_S$ be the inverse image of $\mathscr{M}$ under the projection $X_S \times_S X_S \cong S \times X \times X \to X \times X$ and $\mathscr{L}_S$ the image of $\mathscr{L}$ under $X_S = X \times S \to X$. Then we have $T_{(e,x)}^*\mathscr{M}_S \cong m_S^* T_x^*\mathscr{L}_S \otimes p_{1,S}^*\mathscr{L}_S^{-1}$. Since $x \in K(\mathscr{L})(S)$ we have an isomorphism

$$T_x^*\mathscr{L}_S \cong \mathscr{L}_S \otimes p_S^*\mathscr{N}, \qquad \text{for some } \mathscr{N} \in \operatorname{Pic}(S).$$

Here $p_S : X_S = X \times S \to S$ is the natural projection. Fix such an isomorphism and we obtain an isomorphism on $X_S \times_S X_S$:

$$\lambda_S : \mathscr{M}_S \xrightarrow{\ \sim\ } T_{(e,x)}^*\mathscr{M}_S.$$

In particular we take $S = 1 \times K(\mathscr{L}) = K$ and $(e, x) \in K(S)$ to be the identity morphism. We get an isomorphism

$$\lambda : p_2^*\mathscr{M} \to \mu^*\mathscr{M}$$

as before. Here λ cannot be chosen arbitrarily as it must satisfy some extra condition. In general, we want to have a "canonical" isomorphism $\lambda_S : \mathscr{M}_S \to T^*_{(e,x)}\mathscr{M}_S$ on $X_S \times_S X_S$ for all S. Fortunately, this can be done by restricting to $e_S \times_S S \hookrightarrow X_S \times_S X_S$. (Check this; as an exercise).

As a consequence, we obtain a Poincaré bundle P on $X \times \widehat{X}$ such that $P|_{\{e_X\}\times\widehat{X}}$ is trivial and for all $\alpha \in \widehat{X}(k)$, $P|_{X\times\{\alpha\}}$ corresponds to the element in $\mathrm{Pic}^0(X)$ under the isomorphism $\widehat{X}(k) \cong \mathrm{Pic}^0(X)$, i.e., $(\widehat{X}, P)$ satisfies the first property of Theorem 8.3, in Chapter II, that characterizes $\widehat{X}$. But some modification towards the second property is required. It generalizes as follows.

Theorem 11.4. *Let S be any scheme. Let $\mathscr{L} \in \mathrm{Pic}(X \times S)$ be such that $\mathscr{L}|_{\{e_X\}\times S}$ is trivial and $\mathscr{L}|_{X\times\{s\}} \in \mathrm{Pic}^0(X)$ for each closed point $s \in S(k)$. Then there exists a unique morphism $\phi : S \to \widehat{X}$ such that $\mathscr{L} \cong (1_X \times \phi)^*P$.*

Proof. As before, we consider the line bundle $\mathscr{M} = p_{13}^*P \otimes p_{12}^*\mathscr{L}^{-1}$ on $X \times S \times \widehat{X}$ and let Γ_S be the maximal closed subscheme of $S \times \widehat{X}$ over which $\mathscr{M}$ is trivial.[32] The goal is to show

$$f : \Gamma_S \hookrightarrow S \times \widehat{X} \xrightarrow{p_1} S$$

is an isomorphism. We know f is a homeomorphism on the underlying topological spaces. It suffices to show that for any closed subscheme S_0 of S such that $|S_0|$ is a single point of S. Then the morphism

$$f \times_S S_0 : \Gamma_S \times_S S_0 \to S_0$$

is an isomorphism. In fact, if this is valid, then f is bijective on closed points, and hence f is quasi-finite. Since f is proper, we have f being finite by the Zariski Main Theorem.

The statement follows from the fact. let $(A, \mathfrak{m})$ be a local ring and B a finite A-algebra. If $A/\mathfrak{m}^n \to B/\mathfrak{m}^nB$ is an isomorphism for any m, then $A \to B$ is an isomorphism. So we may assume $S = \mathrm{Spec}\, B$ where B is a finite local k-algebra and $S = \{s\}$ a single point. Moreover, we can assume that $\mathscr{L}|_{X\times\{s\}}$ is trivial. Consider the line bundle $\mathscr{M} = p_{13}^*P \otimes p_{12}^*\mathscr{L}^{-1}$ on $X \times S \times \widehat{X}$. Since $\mathscr{M}|_{\{e_X\}\times\{s\}\times\widehat{X}} \cong P|_{\{e_X\}\times\widehat{X}}$ is trivial (and hence belongs to $\mathrm{Pic}^0(\widehat{X})$, we have $\mathscr{M}|_{\{x\}\times\{s\}\times\widehat{X}} \in \mathrm{Pic}^0(\widehat{X})$ for all $x \in X(k)$. On the other hand,

$$\pi^*(\mathscr{M}|_{\{x\}\times\{s\}\times\widehat{X}}) \cong (T_x^*\mathscr{L}_a) \otimes \mathscr{L}_a^{-1},$$

where $\mathscr{L}_a$ is the ample line bundle on X we have chosen before to construct $\widehat{X}$. So there are only finitely many $x \in X(k)$ such that $\mathscr{M}|_{\{x\}\times\{s\}\times\widehat{X}}$ is trivial.

Since $H^i(\widehat{X}, \mathscr{L}_{\widehat{X}}) = 0$ for all $i \geqslant 0$ and $0 \neq \mathscr{L}_{\widehat{X}} \in \mathrm{Pic}^0(\widehat{X})$, the support of $R^ip_{12,*}\mathscr{M}$ on $X \times S$ is the disjoint union of finitely many closed points. So

$$H^n(X \times S, R^ip_{12,*}\mathscr{M}) = 0, \quad n \geqslant 1.$$

By the Leray spectral sequence

$$H^i(X \times S, R^jp_{12,*}\mathscr{M}) \Rightarrow H^{i+j}(X \times S \times \widehat{X}, \mathscr{M})$$

we have the canonical isomorphisms

$$H^n(X \times S \times \widehat{X}, \mathscr{M}) \cong H^0(X \times S, R^np_{12,*}\mathscr{M}).$$

[32]Here recall the fact at work that $\mathscr{M}|_{\{e_X\}\times S\times\widehat{X}}$ is trivial.

Now apply the projection formula (cf. [Har13, III, Exer 8.3]),

$$\begin{aligned} R^n p_{12,*}\mathscr{M} &= R^n p_{12,*}(p_{13}^* P \otimes p_{12}^* \mathscr{L}^{-1}) \\ &\cong R^n p_{12,*} p_{13}^* P \otimes \mathscr{L}^{-1} \\ &\cong R^n p_{12,*} p_{13}^* P. \end{aligned}$$

The last step above uses the fact that $R^n p_{12,*} p_{13}^* P$ has support on finitely many closed points. Therefore, in summary,

$$H^n(X \times S \times \widehat{X}, \mathscr{M}) \cong H^n(X \times S \times \widehat{X}, p_{13}^* P) \quad \underset{\substack{\uparrow \\ \text{by flat base change theorem}}}{\boxed{\cong}} \quad H^n(X \times \widehat{X}, P) \otimes_k B.$$

In particular, $H^n(X \times S \times \widehat{X}, \mathscr{M})$ are free B-modules for all $n \geqslant 0$. Similarly, we consider the projection $p_{23} : X \times S \times \widehat{X} \to S \times \widehat{X}$. As $\mathscr{L}|_{X \times \{s\}}$ is trivial by our assumption, we get

$$\mathscr{M}|_{X \times \{s\} \times \{\alpha\}} \cong P|_{X \times \{\alpha\}} \otimes \mathscr{L}^{-1}|_{X \times \{s\}} \cong P|_{X \times \{\alpha\}},$$

which further implies that $M|_{X \times \{s\} \times \{\alpha\}} \in \operatorname{Pic}^0(X)$ for all $\alpha \in \widehat{X}(k)$ and it is trivial if and only if $\alpha = e_{\widehat{X}}$. We infer that $R^i p_{23,*}\mathscr{M}$ is supported at the point $(s, e_{\widehat{X}})$ of $S \times \widehat{X}$. Then

$$H^n(X \times S \times \widehat{X}, \mathscr{M}) \cong H^0(S \times \widehat{X}, R^n p_{23,*}\mathscr{M}) = (R^n p_{23,*}\mathscr{M})_{(s, e_{\widehat{X}})},$$

the stalk of the sheaf at the closed point $(s, e_{\widehat{X}})$. For simplicity, we use $\mathscr{O}$ to denote the stalk $\mathscr{O}_{\widehat{X}, e_{\widehat{X}}}$ of $\widehat{X}$ at $e_{\widehat{X}}$. Then the stalk A of $S \times \widehat{X}$ at $(s, e_{\widehat{X}})$ is given by $B \otimes_k \mathscr{O}$. Consider the following Cartesian diagram

$$\begin{tikzcd} \mathscr{M}_A \arrow[d, dash] & & \mathscr{M} \arrow[ll] \arrow[d, dash] \\ X \times \operatorname{Spec} A \arrow[rr] \arrow[d, "p"'] & & X \times S \times \widehat{X} \arrow[d, "p_{23}"] \\ \operatorname{Spec} A \arrow[rr] & & S \times \widehat{X} \end{tikzcd}$$

and we have $R^i p_{23,*}\mathscr{M}|_{(s, e_{\widehat{X}})} \cong R^i p_* \mathscr{M}_A$. Since $p : X \times \operatorname{Spec} A \to \operatorname{Spec} A$ is proper and flat, and $\mathscr{M}_A$ is a line bundle on $X \times \operatorname{Spec} A$, by the main theorem (Theorem 5.2) in *Cohomology and base change*, there is a finite complex

$$K^\bullet : \quad 0 \to K^0 \to K^1 \to \cdots \to K^g \to 0$$

of finitely generated free A-modules such that

$$H^i(K^\bullet) \cong R^i p_{23,*}\mathscr{M}|_{(s, e_{\widehat{X}})} \cong R^i p_* \mathscr{M}_A = H^i(X \times \operatorname{Spec} A, \mathscr{M}_A),$$

where $g = \dim X = \dim \mathscr{O}$. This is crucial in the computation of cohomology groups of P and $\mathscr{O}_X$.[33]

Lemma 11.5. *Let $\mathscr{O}$ be a regular local ring of* $\dim g$*, and*

$$0 \to K^0 \to \cdots \to K^g \to 0$$

be a complex of finitely generated free $\mathscr{O}$-modules. If $H^i(K^\bullet)$ are artinian $\mathscr{O}$-modules, we have $H^i(K^\bullet) = 0$ for each $0 \leqslant i < g$.

[33]Exercise: we only know $K^\bullet$ should be bounded by the theorem. Why is it bounded on $[0, g]$?

Resume on. By this lemma, we see that $R^i p_{23,*}\mathscr{M} = 0$ for each $0 \leqslant i < g$, and we get an exact sequence of A-modules:

$$0 \to K^0 \to \cdots \to K^g \to N \to 0$$

such that $N = (R^g p_{23,*}\mathscr{M})_{(s,e_{\widehat{X}})} \cong H^g(X \times S \times \widehat{X}, \mathscr{M})$ which is a free B-module.

Now we apply $\mathrm{Hom}_A(\cdot, A)$ to the complex $K^\bullet$, and get another complex

$$\widehat{K}^\bullet : \quad 0 \to \widehat{K}^g \to \cdots \to \widehat{K}^0 \to 0$$

and by the lemma above, we get an exact sequence

$$0 \to \widehat{K}^g \to \cdots \to \widehat{K}^0 \to K \to 0$$

of A-modules. Since

$$\begin{aligned} H^0(K^\bullet \otimes_A k) &\cong H^0(X \times \{s\} \times \{e_{\widehat{X}}\}, \mathscr{M}|_{X \times \{s\} \times \{e_{\widehat{X}}\}}) \\ &\cong k = \mathrm{Ker}(K^0 \otimes_A k \to K^1 \otimes_A k), \end{aligned}$$

we see $K \otimes_A k = \mathrm{Ker}(\widehat{K}^1 \otimes_A k \to \widehat{K}^0 \otimes_A k)$ is 1-dimensional over k. Thus, for some ideal I, there is an isomorphism of A-modules $K \cong A/I$. Then one can show the closed subscheme Γ_S of $S \times \widehat{X}$ is the closed subscheme of $\mathrm{Spec}\, A$ defined by the ideal I and the map $B \to B \otimes_k \mathscr{O} = A \to A/I$ is an isomorphism. In other words, the composite $\Gamma_S \hookrightarrow S \times \widehat{X} \xrightarrow{p_1} S$ is an isomorphism. So we get a morphism

$$\phi : S \xrightarrow{\sim} \Gamma_S \xrightarrow{p_2} \widehat{X}$$

which is the unique morphism we need. $\square$

The importance of the proof is that it helps us to compute the cohomology groups of P and $\mathscr{O}_X$.

Corollary 11.6. *As for the cohomology groups of P, we have*

$$H^i(X \times \widehat{X}, P) = \begin{cases} 0, & i \neq g = \dim X; \\ k, & i = g = \dim X. \end{cases}$$

Proof. In the previous proof, we take $S = \mathrm{Spec}\, k$ and $\mathscr{L}$ is trivial. So that

$$H^n(X \times \widehat{X}, P) \cong H^n(K^\bullet), \quad n \geqslant 0.$$

In this case $\Gamma_S = \mathrm{Spec}\, k$ and $\phi : S \to \widehat{X}$ is given by $e_{\widehat{X}}$. Thus, $K \cong k$ and we have an exact sequence of A-modules:

$$0 \to \widehat{K}^g \to \widehat{K}^{g-1} \to \cdots \to \widehat{K}^0 \to k \to 0,$$

i.e., $\widehat{K}^\bullet$ is a free resolution of k. Since $\mathscr{O} = \mathscr{O}_{\widehat{X}, e_{\widehat{X}}}$ is a regular local ring of dimension g, the $\mathscr{O}$-module k has a standard resolution by free $\mathscr{O}$-modules that is called the Koszul complex $L_\bullet$ and is defined as follows.

Let $(x_1, \ldots, x_g)$ be a system of generators of $\mathscr{O}$. Take

$$L_k := \text{free } \mathscr{O}\text{-modules with basis } \{e_{i_1 \cdots i_k} \mid 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant g\},$$

and the differentials

$$d_k : L_k \to L_{k-1}, \quad e_{i_1 \cdots i_k} \mapsto \sum_{l=1}^{k} (-1)^l \chi_{i_l} e_{i_1 \cdots \widehat{i_l} \cdots i_g}.$$

Then we have a resolution

$$0 \to L_g \to L_{g-1} \to \cdots \to L_0 \to k \to 0$$

of k. Hence $L_\bullet$ is homotopic to $\widehat{K}^\bullet$ as chain complexes. Therefore,

$$H^i(X \times \widehat{X}, P) \cong H^i(K^\bullet) \cong H_{g-i}(L_\bullet) = \begin{cases} 0, & i \neq g; \\ k, & i = g. \end{cases}$$

For more details, see [Mat80, §18]. □

Corollary 11.7. *Let $g = \dim X$. Then*

$$\dim_k H^p(X, \mathscr{O}_X) = \binom{g}{p}.$$

Proof. Using the same notation as in the proof of Corollary 11.6 above. We have

$$H^p(X, \mathscr{O}_X) \cong H^p(K^\bullet \otimes_A k) \cong H^p(L_\bullet \otimes_A k) = L_{g-p}.$$

Hence

$$\dim_k H^p(X, \mathscr{O}_X) = \binom{g}{g-p} = \binom{g}{p}.$$

□

Corollary 11.8. *There is a canonical isomorphism between the tangent space at $e_{\widehat{X}}$ on $\widehat{X}$ and $H^1(X, \mathscr{O}_X)$. That is,*

$$\mathrm{Lie}\, \widehat{X} \cong H^1(X, \mathscr{O}_X).$$

References

[Con00] B. Conrad. Grothendieck duality and base change. *Lecture notes in mathematics*, 1750:1, 2000.

[Eis13] David Eisenbud. *Commutative algebra: with a view toward algebraic geometry*, volume 150 of *Graduate Text in Mathematics*. Springer Science & Business Media, 2013.

[GD66] A. Grothendieck and J. Dieudonné. *Éléments de Géométrie Algébrique*, volume 28. 1966.

[Har13] Robin Hartshorne. *Algebraic geometry*, volume 52. World Publishing Co., Beijing, China, 2013.

[MA67] Jacob P. Murre and Siva Anantharaman. Lectures on an introduction to Grothendieck's theory of the fundamental group. 1967.

[Mat80] Hideyuki Matsumura. *Commutative algebra*. Benjamin/Cummings Pub. Co, Reading, Mass, 2nd edition, 1980.

[Mil08] James S. Milne. Abelian varieties, 2008. Available at `www.jmilne.org/math/`.

[Mum85] David Mumford. *Abelian varieties*. Published for the Tata Institute of Fundamental Research, Bombay by Oxford University Press, London, 2nd edition, 1985.

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