Fargues-Fontaine curve and vector bundles

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§2 Vector bundles on FF curves

§2.1 Examples of vbs on FF curves

Setups
- $k/\mathbb{F}_q$ a fixed alg closure of $\mathbb{F}_q$
- $\breve{E} = W_{\mathcal{O}_E}(k)[\frac{1}{\pi}]$
- $S \in \mathrm{Perf}_k$, $k = \overline{\mathbb{F}}_q$.
- $\exists$ natural exact $(-)\otimes_{\breve{E}}(-)$ functor

$$\mathrm{Isoc}_{\overline{\mathbb{F}}_q} \longrightarrow \mathrm{Bun}(X_S)$$

$$(D,\varphi) \longmapsto \mathcal{E}(D,\varphi) := \text{the quotient of the v.b. } D\otimes_{\breve{E}}\mathcal{O}_{Y_S} \text{ by the action } \varphi\otimes\varphi.$$

- $\mathcal{O}_{X_S}(n) := \mathcal{E}(\breve{E}, \pi^{-n}\sigma) = \breve{E}\otimes_{\breve{E}}\mathcal{O}_{Y_S}/(\pi^{-n}\sigma)\otimes(\pi^{-n}\sigma)$
$$= \mathcal{O}_{Y_S}/(\pi^{-n}\sigma)\otimes(\pi^{-n}\sigma).$$

More generally, $\mathcal{E}(\boxed{D_\lambda,\varphi_\lambda}) =: \mathcal{O}_{X_S}(-\lambda)$

the simple isocrystal of slope λ.

In particular, $\mathcal{O}_{X_S}(1) = \mathcal{E}(\breve{E}, \pi^{-1}\sigma)$.

Main Thm C = complete alg closed non-arch field $/\mathbb{F}_q$.
$\Rightarrow$ every v.b. on $X_C = X_{\mathrm{Spa}\,C}$ is $\bigoplus \mathcal{O}_{X_C}(\lambda_i)$ $\lambda_i\in\mathbb{Q}$.

either infin or fin.

Remark The natural functor $\mathrm{Isoc}_{\overline{\mathbb{F}}_q} \to \mathrm{Bun}(X_C)$
is <u>NOT</u> an equivalence,
But $\mathrm{Isoc}_{\overline{\mathbb{F}}_q}^{ss,-\lambda} \xrightarrow{\sim} \mathrm{Bun}(X_C)^{ss,\lambda}$.

§2.2 v-Topology and v-Descent

Perfd := the category of all perf'd spaces.

Def'n The $\underline{v\text{-top}}$ on perf'd is the top gen'd by open covers, and all surj maps of affinoids.

In other words, $\{f_i : X_i \to Y\}_{i\in I}$ is a cover

$\Leftrightarrow$ $\forall$ qc open subset $V \subseteq Y$,

$\exists$ finite subset $I_V \subseteq I$ & qc open $U_i \subseteq X_i$, $\forall i \in I_V$,

s.t. $\quad V = \bigcup_{i\in I_V} f_i(U_i)$.

$\rightsquigarrow$ Can define v-sheaves & v-stacks.

Prop Let S be a perf'd space $/\mathbb{F}_q$. $\mathcal{E}$ a v.b. on X_S.

(1) The functor

$$\mathrm{Perf}_S \ni T \longmapsto R\Gamma(X_T, \mathcal{E}|_{X_T}) \left(\underset{\substack{\uparrow \\ \text{if } T = \text{affinoid space } (\Rightarrow Y_T \text{ is "Stein"}).}}{:=} H^0(Y_T, \mathcal{E}|_{X_T}) \xrightarrow{1-\varphi} H^0(Y_T, \mathcal{E}|_{X_T}) \right)$$

is a v-sheaf of complexes.

(2) The functor

$$\begin{aligned} \mathrm{Perf}_S &\longrightarrow \mathrm{Groupoid} \\ T &\longmapsto \{\text{vbs on } X_S\} \end{aligned}$$

is a v-stack.

Def'n If $(\mathcal{E}_1 \to \mathcal{E}_0)$ is a complex of vb on X_S,

s.t. $\quad H^0(X_T, \mathcal{E}_1) = 0, \ \forall T \in \mathrm{Perf}_S$

Set the sheaf

$$\mathcal{BC}([\mathcal{E}_1 \to \mathcal{E}_0]) : \mathrm{Perf}_S \ni T \longmapsto \mathbb{H}^0(X_T, [\mathcal{E}_1 \to \mathcal{E}_0])$$

called the $\underline{\text{Banach-Colmez space}}$ associated with $[\mathcal{E}_1 \to \mathcal{E}_0]$.

§2.3 $\mathcal{O}(1)$

Prop Let $\mathrm{Spa}(R, R^+)$ be a perf'd space $/\mathbb{F}_q$,
with untilt $(R^\#, R^{\#,+})/\mathcal{O}_E$.

(1) The map
$$R^{\circ\circ} \longrightarrow W_{\mathcal{O}_E}(R^+)$$
$$\varepsilon \longmapsto t_\varepsilon := \sum_{i\in\mathbb{Z}} [\varepsilon^{q^{-i}}]\pi^i$$
induces a natural bijection
$$R^{\circ\circ} \xrightarrow{\sim} H^0(X_{\mathrm{Spa}(R,R^+)}, \mathcal{O}(1)) = H^0(Y_{\mathrm{Spa}(R,R^+)}, \mathcal{O})^{\varphi=\pi}.$$

In fact, $R^{\circ\circ}$ is the $R^{\#,+}$-point of the universal cover $\tilde{\mathcal{G}}$
of a certain $\mathcal{O}_E$-formal group $\mathcal{G}$,
and with this structure of $\mathcal{O}_E$-module on $R^{\circ\circ}$,
the two maps above are also $\mathcal{O}_E$-linear.

(2) If the untilt $S^\#$ is defined $/E_\infty$
$\rightsquigarrow$ $\exists$ short exact sequence
$$0 \to \mathcal{O}_{X_S} \to \mathcal{O}_{X_S}(1) \xrightarrow{\mathrm{ev}} i_*\mathcal{O}_{S^\#} \to 0 \qquad (i: S^\# \hookrightarrow X_S).$$

Taking cohomology $\rightsquigarrow$ a map of $\mathcal{O}_E$-modules
$$\tilde{\mathcal{G}}(R^{\#,+}) \longrightarrow R^\# = \widehat{\mathbb{G}_a^{ad}}(R^\#, R^{\#,+}) \qquad (*)$$
which can be identified with the logarithm morphism
attached to $\tilde{\mathcal{G}}$.

In particular, $(*)$ surjective for pro-étale top
& has kernel $= \underline{E}$ $\quad (\Rightarrow \mathcal{BC}(\mathcal{O}) \simeq \underline{E})$

Cor $\exists$ a well-def'd map $\mathcal{BC}(\mathcal{O}(1))\setminus\{0\} \longrightarrow \mathrm{Div}^1$ which induces an isom
$$\underbrace{(\mathcal{BC}(\mathcal{O}(1))\setminus\{0\})/\underline{E}^\times}_{\text{proper, coh smooth, spatial diamond.}} \cong \mathrm{Div}^1$$

§2.4 Some absolute Banach-Colmez spaces

Prop $\lambda \in \mathbb{Q}$. $k = \overline{\mathbb{F}}_q$

(1) $\lambda < 0$: $H^0(X_S, \mathcal{O}_{X_S}(\lambda)) = 0$, $\forall S \in \mathrm{Perf}_{\mathbb{F}_q}$.

Moreover, let $\mathcal{BC}(\mathcal{O}(\lambda)[1]) : S \longmapsto H^1(X_S, \mathcal{O}_{X_S}(\lambda))$.

Then $\mathcal{BC}(\mathcal{O}(\lambda)[1]) \longrightarrow *$ relatively rep'ble

by a locally spatial diamond,

partially proper, and coh smooth.

(2) $\lambda = 0$: the natural map

$$\underline{E} \xrightarrow{\sim} \mathcal{BC}(\mathcal{O}) \; (:= \mathrm{Perf}_{\mathbb{F}_q} \ni S \mapsto H^0(X_S, \mathcal{O}_{X_S}))$$

isom of pro-étale sheaves,

and the pro-étale sheafification of $S \mapsto H^1(X_S, \mathcal{O}_{X_S})$ vanishes.

(3) $\lambda > 0$: $H^1(X_S, \mathcal{O}_{X_S}(\lambda)) = 0$, $\forall S \in \mathrm{Perf}_{\mathbb{F}_q}$.

The projection $\mathcal{BC}(\mathcal{O}(\lambda)) \longrightarrow *$ relatively rep'ble

in loc spatial diamonds,

partially proper, and coh smooth.

(4) $0 < \lambda \leq [E : \mathbb{Q}_p]$ (resp. $\forall \lambda$, if $\mathrm{char}\, E = p$)

$$\mathcal{BC}(\mathcal{O}(\lambda)) \simeq \mathrm{Spd}(k[\![x_1^{1/p^\infty}, \cdots, x_r^{1/p^\infty}]\!]) \leftarrow \text{Connected}$$

as functors on Perf_k

where $\lambda = r/s$, with coprime $r, s > 0$.

Example (1) $S = \mathrm{Spa}(R, R^+)/\mathbb{F}_q \rightsquigarrow H^0(X_S, \mathcal{O}_{X_S}(1)) \simeq R^{\circ\circ} = \mathrm{Spa}(\underbrace{\mathbb{F}_q[\![t^{1/p^\infty}]\!]}_{\text{perf'd}})(S)$

$\Rightarrow \mathcal{BC}(\mathcal{O}(1)) \simeq \mathrm{Spa}(\mathbb{F}_q[\![t^{1/p^\infty}]\!]) = \mathrm{Spd}(\mathcal{O}_{E_\infty})$.

$\mathcal{O}_{E_\infty}^\flat \cong \mathbb{F}_q[\![t^{1/p^\infty}]\!]$.

(2) $x \in |X_C|^{cl} \xleftrightarrow{1-1}$ short exact sequence

$$0 \to \mathcal{O}_{X_C} \to \mathcal{O}_{X_C}(1) \to C^\# \to 0$$

$$\Rightarrow \quad 0 \to \mathcal{O}_{X_C}(-1) \to \mathcal{O}_{X_C} \to C^{\#} \to 0$$

$$\Rightarrow \quad 0 \to \underline{E} \to \mathbb{A}_{C^\#}^{1,\diamond} \to \mathcal{BC}(\mathcal{O}_{X_C}(-1)[1]) \to 0 \text{ as functors on } \mathrm{Perf}_{C^\#}.$$

$$\Rightarrow \quad \boxed{\mathcal{BC}(\mathcal{O}_{X_C}(-1)[1]) \cong \mathbb{A}_{C^\#}^{1,\diamond}/\underline{E}}\ .$$

§2.5 The algebraic FF curve and GAGA

Thm (Kedlaya-Liu, ampleness of $\mathcal{O}(1)$)

$S = \mathrm{Spa}(R, R^+)$ affinoid perf'd $/\mathbb{F}_q$.

$\mathcal{E}$ = any v.b. over X_S.

Then $\exists\, n_0 \in \mathbb{Z}$ s.t. $\forall n \geq n_0$, (i) $\mathcal{E}(n)$ globally generated, and

(ii) $H^1(X_S, \mathcal{E}(n)) = 0$.

Define $P := \bigoplus_{n \geq 0} H^0(X_S, \mathcal{O}_{X_S}(n))$, $X^{alg} := \mathrm{Proj}\ P$.

Prop (GAGA) $\exists$ natural morphism of locally ringed spaces

$$(X, \mathcal{O}_X) \longrightarrow X^{alg}.$$

Moreover, $\exists$ equiv of cats

$$\mathrm{Bun}(X^{alg}) \xrightarrow{\sim} \mathrm{Bun}((X, \mathcal{O}_X))$$

& $\forall$ v.b. $\mathcal{E}^{alg}$ on X^{alg}, with pullback $\mathcal{E}$ on X,

$$H^*(X^{alg}, \mathcal{E}^{alg}) \xrightarrow{\sim} H^*(X, \mathcal{E}).$$

Prop C = complete alg closed non-arch field $/\mathbb{F}_q$.

$\Rightarrow$ X_C^{alg} connected regular noetherian scheme of Krull dim 1.

And $|X_C| \longrightarrow |X_C^{alg}| \rightsquigarrow$ bijection

$$|X_C|^{cl} \xrightarrow{\sim} \text{closed pts of } |X_C^{alg}|.$$

Moreover, P is graded factorial.

& $\forall x \in |X_C|$ closed pt, $X_C^{alg} \setminus \{x\} = \mathrm{Spa}\, B$, $B = \mathrm{PID}$.

§2.6 Classification of vector bundles

$S = \mathrm{Spa}(C)$, C = complete alg closed non-arch field / $\mathbb{F}_q$.

Prop (Classification of line bundles on X_C)

The map is bijective: $\mathbb{Z} \longrightarrow \mathrm{Pic}(X_C)$

$$n \longmapsto \mathcal{O}_{X_C}(n).$$

Proof Also need to classify $\mathrm{Pic}(X_C^{alg})$.

Take $x \in |X_C^{alg}|$ a closed pt

$$\longleftrightarrow x \in |X_C|^{cl} \longleftrightarrow \text{untilt } C^\# \in \mathrm{Div}^1.$$

$\rightsquigarrow$ $X_C^{alg} \setminus \{x\}$ = the spectrum of a PID

$\Rightarrow$ every line bun on X_C^{alg} is of the form $\mathcal{O}_{X_C^{alg}}(nx)$

On the other hand, $x \in |X_C|^{cl} \Rightarrow \exists$ short exact sequence

$$0 \to \mathcal{O}_{X_C} \to \mathcal{O}_{X_C}(1) \xrightarrow{ev} \mathcal{O}_{C^\#} \to 0$$

$$\Rightarrow \quad 0 \to \mathcal{O}_{X_C^{alg}} \to \mathcal{O}_{X_C}(1)^{alg} \to \mathcal{O}_{C^\#} \to 0$$

$$\Rightarrow \quad \mathcal{O}_{X_C}(1)^{alg}(-x) = \mathcal{O}_{X_C^{alg}}$$

i.e. $\mathcal{O}_X(1)^{alg} \cong \mathcal{O}_{X_C^{alg}}(x)$.

Hence every line bundle of X_C is a power of $\mathcal{O}_{X_C}(1)$. $\square$

Def'n Let $\mathcal{E}$ = vb on X_C. Define

- $\deg \mathcal{E}$ = the image of $\det \mathcal{E}$ via the bijection $\mathrm{Pic}\, X_C \xrightarrow{\sim} \mathbb{Z}$
- $\mu(\mathcal{E}) = \deg(\mathcal{E})/\mathrm{rank}(\mathcal{E})$ the slope of $\mathcal{E}$ (provided $\mathcal{E} \neq 0$).

$\rightsquigarrow$ formalism of stable / semistable vbs on X_C

and the Harder-Narasimhan filtration of a vb on X_C

i.e. for any vb $\mathcal{E}$ on X_C,

$\exists!$ exhaustive separating $\mathbb{Q}$-indexed filtration by subbundles

$$\mathcal{E}^{\geq\lambda} \subseteq \mathcal{E}, \quad \lambda \in \mathbb{Q}$$

called the HN filtration, s.t.

$$\mathcal{E}^{\lambda} := \mathcal{E}^{\geq\lambda}/\mathcal{E}^{>\lambda}, \text{ where } \mathcal{E}^{>\lambda} = \bigcup_{\lambda'>\lambda} \mathcal{E}^{\geq\lambda'} \subseteq \mathcal{E}$$

is semistable of slope λ.

<u>Example</u> For $\lambda \in \mathbb{Q}$, $\mathcal{O}_{X_C}(\lambda)$ is stable of slope λ.

<u>Prop</u> The HN filtration of a vb $\mathcal{E}$ is compatible with a field extension C'/C and E'/E.

(1) C'/C, then $(\mathcal{E} \otimes_C C')^{\geq\lambda} \simeq$ "$\mathcal{E}^{\geq\lambda} \otimes_C C'$"

(2) E'/E unramified. Let $r = [E':E]$.

Then $(\mathcal{E} \otimes_E E')^{\geq r} = \mathcal{E}^{\geq\lambda/r} \otimes_E E'$

$E \longleftrightarrow \varphi$

$E' \longleftrightarrow \varphi^r$.

<u>Thm</u> (Classification of vbs on X_C)

(1) Any vb $\mathcal{E}$ on X_C is isom to a direct sum of vb of the form $\mathcal{O}_{X_C}(\lambda)$, $\lambda \in \mathbb{Q}$.

(2) If $\mathcal{E}$ semistable of slope λ, then

$$\mathcal{E} \simeq \mathcal{O}_{X_C}(\lambda)^m.$$

for some $m \geq 0$.

<u>Proof</u>. Induction on the $n = \operatorname{rank} \mathcal{E}$.

Only need to check: if $\mathcal{E}$ semistable of slope λ, then $\exists$ nontrivial map

$$\mathcal{O}_{X_C}(\lambda) \longrightarrow \mathcal{E}.$$

Some reductions

(a) We may assume $\lambda = 0$

Indeed, write $\lambda = \frac{s}{r}$, $(s,r)=1$, $r \geq 1$,

E'/E unramified of deg r $\rightsquigarrow X_{C,E'} \xrightarrow{f} X_{C,E}$

$\Rightarrow \mathcal{O}_{X_{C,E}}(\lambda) = f_* \mathcal{O}_{X_{C,E'}}(s)$.

$\mathcal{O}_{X_C}(\lambda) \longleftrightarrow \mathcal{E} \longleftrightarrow \mathcal{O}_{X_{C,E'}}(s) \longleftrightarrow f^* \mathcal{E}$.

So can also assume $\lambda \in \mathbb{Z}$.

By appropriate twisting, we reduce to $\lambda = 0$.

(b) We may enlarge C.

Indeed, consider $\underline{\mathrm{Isom}}(\mathcal{E}, \mathcal{O}_{X_C}^n)$ v-sheaf on Perf_C.

If for some ext'n C'/C, $\exists$ a nonzero map

$$\mathcal{O}_{X_{C'}} \longrightarrow \mathcal{E}|_{X_{C'}},$$

inductive hypothesis $\Rightarrow \mathcal{E}|_{X_{C'}} = \mathcal{O}_{X_{C'}}^n$.

$\Rightarrow$ $\underline{\mathrm{Isom}}(\mathcal{E}, \mathcal{O}_{X_C}^n)$ is a v-torsor under the grp $GL_n(E)$.

$\Rightarrow$ so also a torsor for the pro-étale topology.

$\Rightarrow$ $\mathcal{E} \cong \mathcal{O}_{X_C}^n$.

So we are free to replacing C by

a complete alg closed non-arch field ext'n C'/C.

We start the proof.

Let $d = \min\{\underbrace{d \in \mathbb{Z} \mid \exists C'/C \text{ and inclusion } \mathcal{O}_{X_{C'}}(-d) \hookrightarrow \mathcal{E}_{C'}}_{\neq \phi \text{ by ampleness of } \mathcal{O}(1)}\}$

Need to show $d = 0$.

(a) Assume $d > 0$. Minimality of $d \Rightarrow \mathcal{O}_{X_C}(-d) \hookrightarrow \mathcal{E}$ is a subbundle,

i.e. $\mathcal{F} := \mathcal{E}/\mathcal{O}_{X_C}(-d)$ is a vb on X_C.

Inductive hyp $\Rightarrow$ the classification thm holds for $\mathcal{F}$.

(b) If $d \geq 2$, then we can find an injection by induction

$$\mathcal{O}_{X_C}(-d+2) \hookrightarrow \mathcal{F} = \mathcal{E}/\mathcal{O}_{X_C}(-d).$$

Taking pullbacks $\rightsquigarrow$ $\exists$ ext'n

$$0 \to \mathcal{O}_{X_C}(-d) \to \mathcal{G} \to \mathcal{O}_{X_C}(-d+2) \to 0.$$

By twisting, $\exists$ exact sequence

$$0 \to \mathcal{O}_{X_C}(-1) \to \boxed{\mathcal{G}(d-1)} \to \mathcal{O}_{X_C}(1) \to 0.$$

$H^0(X_C, \mathcal{G}(d-1)) \neq 0.$

Key lemma below $\Rightarrow$ after enlarging C,

can find an injection $\mathcal{O}_{X_C} \hookrightarrow \mathcal{G}(d-1)$.

$\rightsquigarrow$ get an injection $\mathcal{O}_{X_C}(-d+1) \hookrightarrow \mathcal{F}$ (impossible).

(c) Assume $d = 1$. $\Rightarrow$ $\mathcal{F}$ of deg 1 & rank $n-1$.

- If $\mathcal{F}$ is not s.s.

  $\Rightarrow$ $\mathcal{F}$ has a subbun $\mathcal{F}' \subsetneq \mathcal{F}$ of $\deg \geq 1$, rank $\leq n-2$.

  $\rightsquigarrow$ get an ext'n $0 \to \mathcal{O}_{X_C}(-1) \to \mathcal{E}' \to \mathcal{F}' \to 0$.

<u>Key lemma</u> $n \geq 1$. Consider ext'n

$$0 \to \mathcal{O}_{X_C}(-1) \to \mathcal{E} \to \mathcal{O}_{X_C}(\tfrac{1}{n}) \to 0 \quad \text{of vbs on } X_C.$$

Then $\exists C'/C$ s.t. $H^0(X_{C'}, \mathcal{E}|_{X_{C'}}) \neq 0$.

Key lem $\Rightarrow$ $\mathcal{E}'$ has no nonzero global section (after enlarging C).

$\Rightarrow$ $\mathcal{F} \supseteq \mathcal{E}'$ has a nonzero global section after enlarging C

(impossible.)

- $\mathcal{F}$ s.s. $\Rightarrow$ $\mathcal{F} = \mathcal{O}_{X_C}(\tfrac{1}{n})$.

  Key lem $\Rightarrow$ $\mathcal{E}$ has a nonzero global section (after enlarging C)

  (also impossible) $\square$

Proof of Key lemma Assume the contrary.

$\Rightarrow \mathcal{BC}(\mathcal{E}) = H^0(X_C, \mathcal{E}|_{X_C}) = 0.$

$\Rightarrow$ get an injection of v-sheaves on Perf_C

$$f : \mathcal{BC}(\mathcal{O}_{X_C}(\tfrac{1}{n})) \hookrightarrow \mathcal{BC}(\mathcal{O}_{X_C}(-1)[1]) \qquad (*).$$

But $\mathcal{BC}(\mathcal{O}_{X_C}(\frac{1}{n})) = \mathrm{Spd}(C[\![x^{1/p^\infty}]\!])$ connected,

while $\mathcal{BC}(\mathcal{O}_{X_C}(-1)[1]) = (\mathbb{A}^1_{C^\#})^\diamond/\underline{E}$.

The image of $(*)$ must contain some non-classical pt.

$\Rightarrow$ after enlarging C (if needed), we deduce that

f contains a $\phi \neq U \subseteq \mathcal{BC}(\mathcal{O}_{X_C}(-1)[1])$ open subset.

$\Rightarrow$ By translation, using $\underline{E}$-action,

$$f : \mathcal{BC}(\mathcal{O}_{X_C}(\tfrac{1}{n})) \xrightarrow{\sim} \mathcal{BC}(\mathcal{O}(-1)[1]) \quad \text{isom.}$$

But $\mathcal{BC}(\mathcal{O}_X(\frac{1}{n}))$ perf'd $\Rightarrow \mathcal{BC}(\mathcal{O}(-1)[1])$ perf'd.

$$\overset{\parallel}{(\mathbb{A}^1_{C^\#})^\diamond/\underline{E}}.$$

On the other hand,

$$(\mathbb{A}^1_{C^\#})^\diamond \longrightarrow (\mathbb{A}^1_{C^\#})^\diamond/\underline{E} \quad \text{pro-étale.}$$

$\Rightarrow (\mathbb{A}^1_{C^\#})^\diamond$ perf'd. $\rightsquigarrow$ Absurd if char $E = 0$ as $\mathbb{A}^{1,\diamond}_{C^\#}$ is "not" perf'd.

If char $E = p > 0$, $(\mathbb{A}^1_{C^\#})^\diamond = \mathbb{A}^1_C$ indeed perf'd.

$\rightsquigarrow$ need a more careful argument. which can be done for any E. $\square$

§2.5 Family of vector bundles

Prop Let S = perf'd space $/\mathbb{F}_q$. $\mathcal{E}$ vb on X_S.

Then $\mathcal{BC}(\mathcal{E}) : T \longmapsto H^0(X_T, \mathcal{E}|_{X_T})$

is a locally spatial diamond, partially proper $/S$.

Moreover, $(\mathcal{BC}(\mathcal{E}) \setminus \{0\})/\underline{E}^\times \longrightarrow S$

is a locally spatial diamond, proper $/S$.

<u>Thm</u> (Kedlaya–Liu) S perf'd space $/\mathbb{F}_q$, $\mathcal{E}$ v.b. $/X_S$, of rank n.

(1) The function taking a geom pt $\mathrm{Spa}\,C \to S$ to the HN polygon of $\mathcal{E}|_{X_C}$ is upper semiconti.

(2) Assume the HN polygon of $\mathcal{E}$ is constant. Then $\exists$ a global HN fil'n $\mathcal{E}^{\geq\lambda} \subseteq \mathcal{E}$, $\lambda \in \mathbb{Q}$, specializing to the HN fil'n at each pt.
Moreover, after replacing S by a pro-étale cover, the HN fil'n can be split and $\exists$ isoms $\mathcal{E}^{\lambda} \cong \mathcal{O}_{X_S}(\lambda)^{n_\lambda}$, $n_\lambda \geq 0$.

<u>Cor</u> (Kedlaya–Liu) S perf'd space.
Then $\exists$ cat equiv

$$\left\{\begin{array}{c}\text{pro-étale } \underline{E}\text{-local} \\ \text{system}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\text{vbs on } X_S \text{ whose HN} \\ \text{polygon is const} = 0\end{array}\right\}$$

$$\mathbb{L} \longmapsto \mathbb{L} \otimes_{\underline{E}} \mathcal{O}_{X_S}.$$

$$\mathcal{BC}(\mathcal{E}) \longleftarrow\!\!\shortmid\ \mathcal{E}$$