2022 Summer School on the Langlands Program

THE LOCAL LANGLANDS CONJECTURE (1/3)

OLIVER TAÏBI
(NOTES BY WENHAN DAI)

ABSTRACT. We formulate the local Langlands conjecture for connected reductive groups over local fields, including the internal parametrization of L-packets.

1. Introduction

The goal of this talk is to state the local Langlands conjectures (which consists of several conjectures) and tell you what is know about them. Our plan for the consecutive three lectures is as follows.

A Tentative Plan.

(1) Introduce the local Langlands conjectures, by going from smooth irreducible representations of $G(F)$, where G is a connected reductive group over a local field F, to two Langlands parameters.
(2) Discuss about the refined local Langlands conjecture, by describing the fibers of this map for quasi-split groups.
(3) Generalize (2) to the general case (with difficulties) with the so-called Galois gerbs.

2. Representation Theory of Connected Reductive Groups

This section is for preliminaries on representation theory. Including some basic notions and three different classification results for representations of (connected) reductive groups over local fields.

We start by recalling notions about the smooth representations of connected reductive groups[1].

2.1. Smooth Representations of Reductive Groups.

Let us begin with setups.

- Let F be a local field (especially, we will have a bias towards a finite extension of $\mathbb{Q}_p$). Its ring of integers $\mathcal{O}_F$ obtains a maximal ideal generated by some chosen uniformizer ϖ_F. Denote q the cardinality of the residue field $\mathcal{O}_F/(\varpi_F)$.
- Let $\|\cdot\| : F^\times \to \mathbb{R}_{>0}$ be a non-archimedean normalized form such that $\|\varpi_F\|^{-1} = q$.
- Suppose C is an algebraically closed field of characteristic 0. In practice, it is going to be either $\mathbb{C}$ (for almost all times) or $\overline{\mathbb{Q}}_\ell$.
- Let G be a connected reductive group over F. For example, $G = \mathrm{GL}_n$, SL_n, Sp_{2n}, Spin_{2n+1}, or E_8. Denote $G(F)$ by the group of F-rational points.

Then the smooth representations are morphisms like

$$\pi : G(F) \to \mathrm{GL}(V),$$

for which V is a C-vector space that is typically infinite-dimensional. And the action $G(F) \times V \to V$ on V is continuous with respect to the discrete topology on V.

We will be mainly interested in the irreducible smooth representations. Recall the following nontrivial fact one cannot prove at the very beginning of the process.

Date: July 11, 2022.

[1] In some sense, not everything we are going to recall is essential to state the local Langlands correspondence, but these make things into a more natural sense.

Proposition 2.1. *If (V,π) is an irreducible smooth representation, then it is automatically admissible. That is, for any open compact subgroup $K \subset G(F)$, the K-invariant vectors of V form a finite dimensional C-vector space, i.e.,* $\dim_C V^K < \infty$.

An Interlude about $(\mathfrak{g}, K)$-modules. Keep the notations as before except for assuming F is *archimedean* instead. In this case, for making the classification being simpler in some sense, one may concern about $(\mathfrak{g}, K)$-modules rather than smooth representations (which morally makes the basic notions being more complicated). Here

- $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \operatorname{Lie} G(F)$ is the complexified Lie algebra for $G(F)$, and
- K is a maximal compact subgroup of $G(F)$.

Loosely, the $(\mathfrak{g}, K)$-module is an algebraic version of topological representations. Yet it is not an actual representation of $G(F)$.

From now on, **assume F is non-archimedean** unless otherwise stated.

2.2. **Algebraic Notions.** Continuing with the irreducible representation (V,π), there is a central character (by some variation of Schur's lemma)

$$\omega_\pi : Z(G(F)) \to C^\times = \mathrm{GL}_1(C)$$

say. There's also a contragredient representation, denoted by $(\widetilde{V}, \widetilde{\pi})$.

2.2.1. *Parabolic Induction.*

Notation 2.2. Fix a Borel subgroup $B(F)$ of $G(F)$. Let $P \supset B$ be a parabolic subgroup P of G containing B. If $N \subset P$ is the unipotent radical of P, it is a normal subgroup and we have the Levi decomposition $P = MN$ say, where M is a reductive group that contains a fixed maximal split torus in B. And M is called a Levi subgroup.

From the Levi decomposition, we have $M \cap N = \{1\}$ and hence

$$P = M \rtimes N, \quad M \cong P/N.$$

Fix a the smooth representation (V,σ) of $M(F)$ (we are to use σ for representations of Levi subgroups and π for representations of G). Via the surjective group homomorphism $P \to M$, our (V,σ) can be view as a representation of P through

$$\begin{tikzcd} P(F) \arrow[r, two heads] \arrow[rr, bend left] & M(F) \arrow[r, "\sigma"] & \mathrm{GL}(V). \end{tikzcd}$$

Definition 2.3. The **normalized parabolic induction** with respect to σ is a functor

$$\begin{aligned} i_P^G : \mathsf{Rep}_{M(F)} &\longrightarrow \mathsf{Rep}_{G(F)} \\ \sigma &\longrightarrow \delta_P^{1/2} \otimes \operatorname{Ind}_P^G(\sigma). \end{aligned}$$

The underlying space of the induced representation is

$$\left\{ f : G(F) \to V \;\middle|\; \begin{array}{c} f \text{ is smooth and } f(pg) = \delta_P^{1/2}(p)\sigma(p)f(g) \\ \text{for all } p \in P(F) \text{ and } g \in G(F) \end{array} \right\}$$

in which the second condition is read as f being left equivariant for P. Also, δ_P is the module character of the group P, i.e., for all $p \in P$, $p^*(\mu_P) = \delta_P(p) \cdot \mu_P$ for any left-invariant Haar measure μ_P on P.

The point of introducing the square root $\delta_P^{1/2}$ here is to preserve the unitarizability. So if σ has an invariant Hermitian inner product, then so does the parabolic induction. Note that $\delta_P^{1/2}$ need to choose some $\sqrt{q} \in C$; if $C = \mathbb{C}$, then definitely $\sqrt{q} > 0$.

2.2.2. *Jacquet Functor.* It turns out that the parabolic induction i_P^G is a functor that is adjoint to the Jacquet functor. Again we take $P = MN$ and let (V, π) be a smooth representation of $G(F)$. Take

$$\begin{array}{l} \boxed{V_N} := \text{coinvariant vectors in } V \text{ under the action of } N(F). \\ \circlearrowleft \ \pi_N \\ M(F) \end{array}$$

Then V_N is a quotient of V and there is a surjective map of vector spaces $V \to V_N$. (Caution: do not be confused by the notation π_N, which is a representation of $M(F)$ that depends on the choice of N.)

Definition 2.4. The **Jacquet functor** is

$$\begin{aligned} r_P^G : \mathsf{Rep}_{G(F)} &\longrightarrow \mathsf{Rep}_{M(F)} \\ \pi &\longrightarrow \delta_P^{1/2} \otimes \pi_M \end{aligned}$$

Definition 2.5. Let (V, π) be an irreducible representation of $G(F)$. It is **supercuspidal** if for any proper parabolic subgroup $P \subset G$, we have $V_N = 0$.

Remark 2.6. Equivalently, for arbitrary $v \in V$ and $\widetilde{v} \in \widetilde{V}$, the matrix coefficients of

$$\begin{aligned} G(F) &\longrightarrow C \\ g &\longmapsto \langle \pi(g)v, \widetilde{v} \rangle \end{aligned}$$

have compact supports modulo $Z(G(F))$. Note that this constructed map coincides with the central character ω_π while restricting to $Z(G(F))$ (hence this makes sense).

Alternatively, π is supercuspidal if for any proper Levi subgroup M and irreducible F-representation σ of $M(F)$, π does not appear as a subquotient of $i_P^G(\sigma)$.

2.3. **Classification via Supercuspidal Supports.** We then give a rough classification of irreducible representations of $G(F)$ by their supercuspidal supports.

Jargon Watch. For a supercuspidal representation σ of F-points $M(F)$ in a Levi subgroup, the pair (M, σ) is said to belongs to the supercuspidal support of π, if π is a subquotient of $i_P^G \sigma$.

Theorem 2.7. (1) *Any irreducible representation π of $G(F)$ embeds in a parabolic induction $i_P^G(\sigma)$ of some supercuspidal irreducible Levi representation $\sigma \in \mathsf{Rep}_{M(F)}$.*
(2) *If π further occurs as a subquotient of some other parabolic induction, say $i_{P'}^G(\sigma')$, where σ' is again supercuspidal, then*

$$(M, \sigma) \sim (M', \sigma')$$

by $G(F)$-conjugations.

From the theorem, one can then partition irreducible representations of $G(F)$ by the supercuspidal support which is a $G(F)$-conjugacy class.

2.4. **Asymptotic/Topological Notions.** Set $C = \mathbb{C}$. The upcoming things are actually topological properties (even though the representation don't use the topology of the coefficient field).

Definition 2.8. Fix an irreducible representation (V, π) of $G(F)$. If the corresponding central character ω_π is unitary, say that π is **essentially square-integrable** (or abbreviated by **essentially** L^2), if for all $v \in V$ and $\widetilde{v} \in \widetilde{V}$,

$$\int_{G(F)/Z(G(F))} |\langle \pi(g)v, \widetilde{v} \rangle|^2 dg < \infty$$

after choosing some Haar measure on the group $G(F)/Z(G(F))$.

By Remark 2.6, since the supercuspidal representations are compactly supported modulo the center, we have:

- ⋄ if π is essentially L^2, then π embeds into some equivariant L^2 space of $G(F)$, denoted by $L^2(G(F), \omega_\pi)$.

In general, without assuming that the central character ω_π is unitary, there is a *unique*[2] continuous character

$$\chi : G(F) \to \mathbb{R}_{>0}$$

such that the twist $\chi \otimes \pi$ has unitary central character $\omega_{\chi\otimes\pi}$. Say that π is **essentially** L^2 if so also $\chi \otimes \pi$ is. From this, one can always reduce to the case of unitary central characters.

This property can be checked on Jacquet modules. More generally, when we are outside the compact subgroup of $G(F)$, the matrix coefficients are given by those of Jacquet modules.

Let M be a Levi subgroup of G. Denote A_M the maximal central split torus in M. Take

$$\mathfrak{a}_M^* := X^*(A_M) \otimes_{\mathbb{Z}} \mathbb{R},$$

where $X^*(A_M)$ is the group of characters. Then

$$\begin{aligned} \mathfrak{a}_M^* &\xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{cont}}(A_M(F), \mathbb{R}_{>0}) \\ \chi \otimes s &\longmapsto (x \mapsto \|\chi(x)\|^s). \end{aligned}$$

Via this isomorphism, we are going to consider $\mathfrak{a}_M^*$ as an additive group.

Proposition 2.9. *Let (V, π) be an irreducible representation of $G(F)$. Assume that the central character ω_π is unitary. Then the following are equivalent.*

(1) *π is essentially L^2.*
(2) *For any parabolic subgroup $P = MN$ and any character $\chi : A_M(F) \to \mathbb{C}^\times$ occurring in the Jacquet module $r_P^G(\pi)$, $|\chi|$ is a linear combination with positive coefficients (as $\mathfrak{a}_M^*$ can be realized as an additive group) of the simple roots*[3] *of A_M acting on N.*

Definition 2.10. An irreducible representation (V, π) is called **tempered** if condition (ii) in the proposition holds with "non-negative" instead of "positive".

The definition is also equivalent to some growth condition on the coefficients in the matrix (namely, some L^2 condition). But we do not really need it.

Remark 2.11. These tempered representations are exactly the ones occurring in (a variance of) the Plancherel formula:

$$f(1) = \int_{\pi \in \mathsf{Rep}_{G(F)}} \mathrm{tr}(\pi(f))\, d\mu(\pi).$$

Summary. Suppose ω_π is unitary. Then

$$\begin{aligned} & \pi \text{ is supercuspidal} \\ \Longrightarrow\ & \pi \text{ is essentially } L^2 \\ \Longrightarrow\ & \pi \text{ is tempered} \\ \Longrightarrow\ & \pi \text{ is unitary.} \end{aligned}$$

The notion of unitary representation is natural but contains more subtlety to make things more complicated. So we do not take care of this.

[2]This is obtain from the character of N normalized by M. By considering $\sigma \otimes \chi$, the parabolic induction has the underlying set defined by $f(nmg) = \chi(n)\delta_N^{1/2}(m)\sigma(m)f(g)$ for all $p = mn \in P$.

[3]These roots do not form a root system. But they can be even regarded as roots of A_M acting on G. So these simple roots truly came from some root system.

2.5. **Classification of Tempered Representations.** Here comes a non-complete classification of tempered representations in terms of essentially L^2 irreducible (global) representations of Levi subgroups.

Proposition 2.12. (1) *For $P = MN$, assume σ is an essentially L^2 irreducible representation of $M(F)$ attached with a unitary central character ω_π. Then its parabolic induction $i_P^G(\sigma)$ is semisimple and has finite length. Moreover, any constituent of it is tempered.*

(2) *Once given two such inducing data (P, σ) and (P', σ') as in (1), their induced representations $i_P^G(\sigma)$ and $i_{P'}^G(\sigma')$ have a common irreducible subrepresentation if and only if*

$$(M, \sigma) \sim (M', \sigma'),$$

conjugated via $G(F)$. And then

$$i_P^G(\sigma) \simeq i_{P'}^G(\sigma');$$

namely, the same constituent have the same multiplicity.

(3) *Any tempered irreducible representation π of $G(F)$ occurs in some irreducible $i_P^G(\sigma)$ as in (1).*

Actually, we point out that this gives a much simpler classification because the induced representations such as $i_P^G(\sigma)$ are always irreducible when σ's are unitary. On the other hand, the Bernstein-Zelevinsky classification of the essentially L^2 ones in terms of supercuspidal ones precisely makes everything to be reduced[4].

2.6. **Langlands' Classification.** (The result over $\mathbb{R}$ is given by Langlands and is due to Silberger for non-archimedean local fields.) Be careful that the following statements look a bit the same as before but they are subtly different.

Theorem 2.13. (1) *Let $P = MN$ and σ be a tempered (and hence unitary) irreducible representation of $M(F)$. Also take a continuous character*

$$\nu : M(F) \to \mathbb{R}_{>0}.$$

*Then (vaguely) under a certain positivity condition on ν, $i_P^G(\sigma \otimes \nu)$ has a unique irreducible quotient[5] (which is the so-called **Langlands quotient**), denoted by $J(P, \sigma, \nu)$.*

(2) *Conversely, for any irreducible representation π of $G(F)$, up to $G(F)$-conjugations, there exists a unique triple (P, σ, ν) such that*

$$\pi \simeq J(P, \sigma, \nu).$$

2.7. **Harish-Chandra Character.** Although this will not come up right now, it shall be an essential notion for further use. Fix a Haar measure on $G(F)$. Take

$$\mathcal{C}_c^\infty(G(F)) := \{f : G(F) \to \mathbb{C} \mid f \text{ smooth and compactly supported function}\}.$$

We are to search for functions that are both left and right K-invariant on some open compact subgroup K of $G(F)$ (depending on F). By the admissibility, for $f \in \mathcal{C}_c^\infty(G(F))$,

$$\pi(f) : V \to V, \quad \pi(f)v = \int_{G(F)} f(g)\pi(g)v dg$$

is actually a finite sum and has a finite range. In particular, it factors as

[4]Note that in general, these $i_P^G(\sigma)$'s are not irreducible.

[5]This is also viewed as a unique irreducible subrepresentation of $i_{\overline{P}}^G(\sigma \otimes \nu)$, where $\overline{P}$ is the opposite parabolic.

$$\begin{tikzcd} & V \arrow[dl, dashed] \arrow[d, "\pi(f)"] \\ V^K \subset & V \end{tikzcd}$$

By the representation theory of finite-dimensional associative algebras,

$$\Theta_\pi : \mathcal{C}_c^\infty(G(F)) \to \mathbb{C}$$

determines a finite length π up to semisimplification. Thus, there is no problem to define

$$\operatorname{tr}(\pi(f)) := \Theta_\pi(f).$$

Theorem 2.14 (Harish-Chandra)**.** *Let F be a finite extension of $\mathbb{Q}_p$. There exists a unique $\Theta_\pi \in L^1_{\mathrm{loc}}(G(F))$ such that for all $f \in \mathcal{C}_c^\infty(G(F))$,*

$$\Theta_\pi(f) = \int_{G(F)} f(g)\Theta_\pi(g)dg.$$

Moreover, Θ_π is invariant under conjugations by $G(F)$ and represented by a unique smooth function on the regular semisimple locus $G_{\mathrm{rs}}(F)$, hence uniquely determined by this function because $\operatorname{vol}(G(F)\backslash G_{\mathrm{rs}}(F)) = 0$.

Remark 2.15. It is essential to work on the fields of characteristic 0. The ingredient of Theorem 2.14 is the same when $F = \mathbb{R}$ or $\mathbb{C}$. But the explicit statement gets to be annoying. The general case with characteristic $p > 0$ for local fields is unknown so far.

3. Langlands Dual Groups

The Langlands dual ${}^LG$ of a reductive algebraic group G (also called the L-group of G) is a group that controls the representation theory of G. Suppose F is a general local field of coefficients. Fix a separable closure $\overline{F}$ and take $\Gamma = \operatorname{Gal}(\overline{F}/F)$ to be the absolute Galois group.

3.1. Based Root Data and the Groupoid Fiber.

There exists a finite subextension E/F of $\overline{F}/F$ and a Killing pair (or Borel pair) (B, T) in G_E. This defines a based root datum $(X, R, R^\vee, \Delta)$.

- $X = X^*(T)$ is the group of characters;
- $R \subset X$ is the set of roots of T in G_E;
- $R^\vee \subset X^\vee = \operatorname{Hom}(X, \mathbb{Z}) = X_*(T)$ is the set of coroots of T;
- $\Delta \subset R$ are simple roots for B.

Also note that G_E acts transitively on all Killing pairs, and the stabilizer of (B, T) is just T. Hence the G_E leaves a root datum invariant and there is a canonical way to define $(X, R, R^\vee, \Delta)$ with (B, T) fixed. Moreover, for the same reason, it has a smooth finite absolute Galois action of Γ. We obtain a functor

$$\operatorname{Brd}_F : \textsf{Groupoid of Conn Red Groups} \longrightarrow \underset{\substack{\circlearrowleft \\ \Gamma}}{\textsf{Based Root Data}},$$

where the object of the left category are connected reductive groups. In the formulation of local Langlands, we will be particularly interested in the groupoid fibers of Brd_F.

Definition 3.1 (Inner Twists)**.** As a category, the **groupoid of inner twists of** G, denoted by $\mathsf{IT}(G)$, is defined as follows[6].

[6]Sorry for these (possibly unreadable) fussy formulas. But they will no longer be at actual use.

- $\mathrm{Ob}(\mathsf{IT}(G))$: consisting of pairs (G', ψ) where
$$\psi : G_{\overline{F}} \xrightarrow{\sim} G'_{\overline{F}}$$
such that for all $\sigma \in \Gamma$,
$$\psi^{-1}\sigma(\psi) \in G_{\mathrm{ad}}(\overline{F}) := \mathrm{Inn}(G_{\overline{F}}),$$
the group of inner automorphisms.
- $\mathrm{Mor}(\mathsf{IT}(G))$: take $\mathrm{Hom}_{\mathsf{IT}(G)}((G_1, \psi_1), (G_2, \psi_2))$ to be the set
$$\{g \in G_{\mathrm{ad}}(\overline{F}) \mid \forall \sigma \in \Gamma,\ \psi_2^{-1}\sigma(\psi_2) = \mathrm{Ad}(g)\psi_1^{-1}\sigma(\psi_1)\,\mathrm{Ad}(\sigma(g))^{-1}\}.$$

Remark 3.2.
(1) For any inner twist $(G', \psi) \in \mathrm{Ob}(\mathsf{IT}(G))$, we have a natural isomorphism
$$\mathrm{Brd}_F(G) \simeq \mathrm{Brd}_F(G').$$
(2) There is a group homomorphism
$$\begin{aligned} \Gamma &\longrightarrow G_{\mathrm{ad}}(\overline{F}) \\ \sigma &\longmapsto \psi^{-1}\sigma(\psi) \end{aligned}$$
which outputs a 1-cocycle. It measures the difference between two Galois actions induced by g and g' say.
(3) The group $\mathrm{Hom}_{\mathsf{IT}(G)}((G_1, \psi_1), (G_2, \psi_2))$ induces an isomorphism
$$\psi_2\,\mathrm{Ad}(g)\psi_1^{-1} : G_{1,\overline{F}} \xrightarrow{\sim} G_{2,\overline{F}}$$
between groups over F.
(4) It is not difficult to compute the automorphism group as the rational points of the adjoint group,
$$\mathrm{Aut}_{\mathsf{IT}(G)}(G', \psi) = G'_{\mathrm{ad}}(F).$$
In particular, the automorphism group is (strictly) larger than the quotient $G'(F)/Z(G'(F))$ (which is the source of a series of problems).

We then concern about the description of groupoid fibers. Namely, we are considering to trace back along the functor Brd_F. The following proposition dictates some way of classifying all connected reductive groups in a fixed fiber.

Proposition 3.3. *Fix a based root datum b with an action of Γ. Let CRG_b be the groupoid of pairs (G, α) determined by b. Here G is a connected reductive group over F and α is some fixed isomorphism*
$$\alpha : b \xrightarrow{\simeq} \mathrm{Brd}_F(G).$$
(1) *Up to isomorphisms, there is a unique pair $(G^*, \alpha^*) \in \mathrm{Ob}(\mathsf{CRG}_b)$ such that G^* is quasi-split (i.e., it has a Borel subgroup defined over F).*
(2) *Choose $(G, \alpha) \in \mathrm{Ob}(\mathsf{CRG}_b)$. There is a natural equivalence of categories:*
$$Z^1(F, G_{\mathrm{ad}}) \xleftarrow{\ \sim\ } \mathsf{IT}(G) \xrightarrow{\ \sim\ } \mathsf{CRG}_b.$$
Here $Z^1(F, G_{\mathrm{ad}})$ is the category whose objects are all 1-cocycles, and the equivalence between two cocycles that defines H^1 gives a morphism between two cocycles[7].

From this we have some classification[8] read as:

[7]The language is fancy but the upshot lies in that the inner twists $\mathsf{IT}(G)$ can be described by Galois cohomologies. Here $Z^1(F, G_{\mathrm{ad}})$ is viewed as another groupoid (rather than simply a group) using the definition that describes what it means for two cocycles to be in the same class. This definition leads to the isomorphisms between two cocycles.

[8]**Joke.** *Following this recipe, one can get isomorphism classes of a given connected reductive group more than once. But this is still a good classification.*

(i) enumerate all isomorphism classes of based root data with an action of Γ;
(ii) for an arbitrary one of this kind of datum b say, we have a unique quasi-split group whose based root datum exactly coincides with b;
(iii) the other connected reductive groups who have the same based root datum b are exactly the inner twists of G; on the other hand, these are parametrized by the isomorphism classes in $H^1(F, G_{\mathrm{ad}})$ that take values in G_{ad}.

3.2. **Definition of Langlands Dual Groups.** Given a connected reductive group G, take

$$\mathrm{Brd}_F(G) = (X, R, R^\vee, \Delta) \quad \rightsquigarrow \quad (X^\vee, R^\vee, R, \Delta^\vee),$$

both with actions of Γ. These form the **pinned connected reductive group** over C:

$$(\widehat{G}, \mathcal{B}, \mathcal{T}, (X_\alpha)_{\alpha\in\Delta^\vee}).$$

Here $\widehat{G}$ is defined by the dual based root datum $(X^\vee, R^\vee, R, \Delta^\vee)$. Also[9], each X_α is a basis of an eigenspace for $\mathbb{T}$ acting on the Lie algebra of B.

The pinning is used to split the exact sequence[10]

$$\begin{tikzcd} 1 \arrow[r] & \boxed{\mathrm{Inn}(\widehat{G})} \arrow[r] & \mathrm{Aut}(\widehat{G}) \arrow[r] & \mathrm{Out}(\widehat{G}) \arrow[r] \arrow[l, bend left, "\text{induced by pinning}"] & 1 \\ & = \widehat{G}_{\mathrm{ad}} & & & \end{tikzcd}$$

The section $\mathrm{Out}(\widehat{G}) \to \mathrm{Aut}(\widehat{G})$ can be taken from the pinning. This is because the automorphisms of the whole pinning $(\widehat{G}, \mathcal{B}, \mathcal{T}, (X_\alpha)_{\alpha\in\Delta^\vee})$ map bijectively onto $\mathrm{Aut}(\widehat{G})$. Consequently, as Γ acts on the based root datum, it then acts on $\widehat{G}$ via this section. Moreover,

$$(\widehat{G}, \mathcal{B}, \mathcal{T}, (X_\alpha)_{\alpha\in\Delta^\vee}) \quad \circlearrowleft \quad \Gamma$$

Note that the Γ-action on $\widehat{G}$ is the transpose of the Γ-action on G.

3.2.1. *The Basic Construction.*

Definition 3.4. The **dual Langlands group** is defined to be

$${}^LG := \widehat{G} \rtimes \Gamma.$$

Example 3.5. We list out some of pinned connected reductive groups.

G	GL_n	SL_n	$\mathrm{SO}_{2n+1}^?$	$\mathrm{Spin}_{2n}^?$
$\widehat{G}$	GL_n	PGL_n	Sp_{2n}	PSO_{2n}

There is some ambiguity in defining SO_{2n+1} and Spin_{2n}, for which some decoration is in need[11].

Proposition 3.6. (1) *G is semisimple and simply connected if and only if $\widehat{G}$ is adjoint.*
(2) *G_{der} is simply connected if and only if $Z(\widehat{G})$ is a torus; and then it is isomorphic to the dual of another torus:*

$$Z(\widehat{G}) \simeq \widehat{G/G_{\mathrm{der}}}.$$

(3) *${}^LG_1 \simeq {}^LG_2$ if and only if both G_1 and G_2 are inner twists of each other.*

[9] The note taker did not know what this sentence means. The lecturer is supposedly meant to say about the unipotent radical.

[10] By abuse of notations, for groups over C, we are not really going to distinguish between the algebraic groups and the group of points.

[11] The reason is that there should be more than one SO_{2n+1} or Spin_{2n}, which are different quadratic spaces. We also have not specified the Γ-actions anyway. This example is just to give some ideas of how this behaves with respect to isogenies.

3.2.2. *The Functoriality of* ${}^L(\cdot)$. If $G = G_1 \times G_2$, then we have the fiber product

$$ {}^LG \simeq {}^LG_1 \times_\Gamma {}^LG_2. $$

Moreover, let $\theta : G \to H$ be any central isogeny. Then it induces the dual central isogeny

$$ {}^L\theta : {}^LH \longrightarrow {}^LG. $$

Fix a maximal torus T of G. Choose B to be the Borel subgroup of $G_{\overline{F}}$ containing $T_{\overline{F}}$. Then we get

$$ \widehat{T} \simeq \mathcal{T}, $$

where $\mathcal{T}$ is the maximal torus in $\widehat{G}$.

Caution. This is not Γ-equivariant unless T and B were divisible, or unless we are in a quasi-split situation where B itself is defined over F. So the Γ-actions differed by a 1-cocycle take values in some Weyl group; here G and $\widehat{G}$ have the same Weyl group.

As a remark for further use, we morally have an embedding $Z(\widehat{G}) \hookrightarrow \widehat{T}$ that is Γ-equivariant because the 1-cocycle is not going to do much on the center.

$\boxed{\text{To be continued in Lecture 2/3.}}$

School of Mathematical Sciences, Peking University, 100871, Beijing, China
Email address: daiwenhan@pku.edu.cn