Multivariable (φ,Γ)-mod and local-global compatibility

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§ Introduction

Mod p Langlands: $K/\mathbb{Q}_p$ finite

$$\left\{\begin{array}{c}\text{mod } p \text{ Galois rep} \\ \bar\rho : G_K \to GL_n(\bar{\mathbb{F}}_p)\end{array}\right\} \xleftrightarrow{?} \left\{\begin{array}{c}\text{adm sm rep} \\ \text{of } GL_n(K) / \bar{\mathbb{F}}_p\end{array}\right\}.$$

Known case $n=1$. $n=2$ & $K=\mathbb{Q}_p$ (Breuil, Colmez, Emerton, etc.)

Other cases: unknown (even for $K=\mathbb{Q}_{p^2}$).

When $n=2$: has complete classification of RHS.

Two main results in $GL_2(\mathbb{Q}_p)$ case:

(i) Colmez (05): exact functor

$$\begin{tikzcd} \{\text{Gal rep}\} \arrow[rr] \arrow[dr, leftrightarrow, "\text{Fontaine}"'] & & \{\text{rep of } GL_2(\mathbb{Q}_p)\} \arrow[dl, dashed, "\text{Colmez}"] \\ & \{\text{étale } (\varphi,\Gamma)\text{-mod} / \mathbb{F}_p((T))\} & \end{tikzcd}$$

For rep of $GL_2(\mathbb{Q}_p)$:

$$\begin{pmatrix} \mathbb{Z}_p - \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \qquad \mathbb{F}[\![\mathbb{Z}_p]\!] = \mathbb{F}[\![T]\!] \text{ for } \mathbb{F} = \bar{\mathbb{F}}_p.$$

$$\cap \qquad\qquad \mathbb{Z}_p \setminus \{0\} = \mathbb{Z}_p^\times \times p^{\mathbb{N}}, \quad \Gamma = \mathbb{Z}_p^\times.$$

$$\begin{pmatrix} \mathcal{O}_K^\times \times p^{\mathbb{N}} & \mathcal{O}_K \\ 0 & 1 \end{pmatrix} \qquad p \text{ mimic of } \varphi\text{-action}.$$

when $K/\mathbb{Q}_p$ unramified.

(ii) Emerton's local-global compatibility:

$$\bar\rho \longleftrightarrow \pi(\bar\rho) \quad \text{rep of } GL_2(\mathbb{Q}_p)$$

Let $\bar r: G_{\mathbb{Q}} \to GL_2(\mathbb{F})$

$$\rightsquigarrow \quad \mathrm{Hom}_{G_{\mathbb{Q}}}\Big(\bar r, \varinjlim_{U_p \subset GL_2(\mathbb{Q}_p)} H^1_{et}(X_{U_p U^p} \times \bar{\mathbb{Q}}, \mathbb{F})\Big) \cong \pi(\bar r|_{G_{\mathbb{Q}_p}})^{(\oplus d)}.$$

$$\circlearrowleft \; G_{\mathbb{Q}} \times GL_2(\mathbb{Q}_p)$$

Upgrade: modular curve $\rightsquigarrow$ Shimura curve

Start with $\bar\rho: G_K \to GL_2(\mathbb{F})$.

Choose $F/\mathbb{Q}$ tot real, s.t. $F_v \cong K$ for some $v|p$.

$$\rightsquigarrow \quad \bar r: G_F \longrightarrow GL_2(\mathbb{F})$$

$$\text{s.t. } \bar\rho = \bar r|_{G_{F_v} = G_K}.$$

$\rightsquigarrow$ Construct $\pi(\bar\rho)$ adm sm rep of $GL_2(K)$.

<u>Goal</u> Study $\pi(\bar\rho)$.

<u>Question</u> Does $\pi(\bar\rho)$ only depends on $\bar\rho$ locally ?

(i.e. not on F or $\bar r$, globally).

Today: Generalize Colmez's functor (exact)

$$D_A: \{\text{adm sm rep of } GL_2(K)\} \longrightarrow \{\text{étale multivar } (\varphi, \mathcal{O}_K^\times)\text{-mod} / A\}$$

$$\pi(\bar\rho) \longmapsto D_A(\pi(\bar\rho))$$

Thm (BHHMS)

(i) Can explicitly compute $D_A(\pi(\bar\rho))$

$\Rightarrow$ it only depends on $\bar\rho$.

(2) Construct another functor

$$D_A^{\otimes} : (\bar{\rho} : G_K \to GL_2(\mathbb{F})) \longmapsto \text{Some étale } (\varphi, \mathcal{O}_K^{\times})\text{-mod } / A.$$

(3) $D_A(\pi(\bar{\rho})) = D_A^{\otimes}(\bar{\rho})$.

§ $(\varphi, \mathcal{O}_K^{\times})$-mod over ring A

$K / \mathbb{Q}_p$ unram of deg f. $q = p^f$.

$\Rightarrow$ $\mathbb{F}[\![\mathcal{O}_K]\!]$ = power series ring of f variables.

* Ring A

Roughly $A \rightsquigarrow \mathbb{F}(\!(T)\!)$ $(\mathbb{F}[\![\mathbb{Z}_p]\!] = \mathbb{F}[\![T]\!])$.

Define $Y_{\sigma} := \sum_{\lambda \in \mathbb{F}_q^{\times}} \sigma(\lambda)^{-1} \underbrace{[\tilde{\lambda}]}$, $\sigma : \mathbb{F}_q \hookrightarrow \mathbb{F}$.

(Teichmuller

Then φ-action $\longleftrightarrow$ p-action

by formula $\varphi(Y_{\sigma}) = Y_{\sigma \circ \varphi^{-1}}^{p}$.

Construct A as

$$A = \left(\mathbb{F}[\![(Y_{\sigma})_{\text{all } \sigma}]\!]\left[\left(\tfrac{1}{Y_{\sigma}}\right)_{\text{all } \sigma}\right]\right)^{\wedge} \circlearrowleft \varphi, \mathcal{O}_K^{\times}.$$

Completion w.r.t. (Y_{σ}, σ)-adic top.

Def $(\varphi, \mathcal{O}_K^{\times})$-mod $/A \ni M$, fin proj A-mod

w/ semi-linear comm $(\varphi, \mathcal{O}_K^{\times})$-actions.

Called <u>étale</u> if $M \otimes_{A, \varphi} A \xrightarrow{\sim} M$.

Rmk If M = f.g. A-mod w/ semi-linear $\mathcal{O}_K^{\times}$-action,

then M is free $/ A$.

* <u>Functor D_A</u>

$$D_A : \{\text{adm sm reps of } GL_2(K)\} \longrightarrow \{\text{ét } (\varphi, \mathcal{O}_K^\times)\text{-mod} / A\}$$

$$\pi \longmapsto \left(A \otimes_{\mathbb{F}[\![\begin{pmatrix} 1 & \mathcal{O}_K \\ 0 & 1\end{pmatrix}]\!]} \pi^\vee\right)^\wedge_{\mathfrak{m}_{I_1}\text{-adic top}}$$

$$\pi^\vee = \mathrm{Hom}_{\mathbb{F}}(\pi, \mathbb{F}).$$

($\mathfrak{m}_{I_1}$ = max ideal of $\mathbb{F}[\![I_1]\!]$.)

Let I_1 = pro-p Iwahori subgrp = $\begin{pmatrix} 1+p\mathcal{O}_K & \mathcal{O}_K \\ p\mathcal{O}_K & 1+p\mathcal{O}_K \end{pmatrix}$

<u>Issue</u> (if we do not use I_1) $\qquad \mathcal{O}_K \longleftrightarrow \begin{pmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{pmatrix}$.

- $\pi^{\begin{pmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{pmatrix}} = \infty$-dim'l
- $\pi^\vee$ is not f.g. mod over $\mathbb{F}[\![\begin{pmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{pmatrix}]\!]$.

<u>Thm</u> (BHHMS)

$$\pi \longmapsto D_A(\pi) \text{ is exact.}$$

<u>Rmk</u> In general, don't know if $\mathrm{rk}_A D_A(\pi) < \infty$).
We do know $D_A(\pi(\bar\rho))$ is of fin rk $/A$ as an étale $(\varphi, \mathcal{O}_K^\times)$-mod.

<u>Colmez's constr'n</u> For $GL_2(\mathbb{Q}_p)$, $D_{\mathbb{F}((T))}(\pi(\bar\rho)) = D(\bar\rho')$.

* <u>Structure of $D_A(\pi(\bar\rho))$?</u>

Relation to classical $(\varphi, \mathbb{Z}_p^\times)$-mod $/ \mathbb{F}((T))$:

$$\mathrm{tr} : \mathcal{O}_K \twoheadrightarrow \mathbb{Z}_p \quad \mathbb{Z}_p\text{-linear}$$

$$\rightsquigarrow \quad \mathbb{F}[\![\mathcal{O}_K]\!][(\tfrac{1}{y_\sigma})] \twoheadrightarrow \mathbb{F}[\![\mathbb{Z}_p]\!] = \mathbb{F}[\![T]\!][\tfrac{1}{T}]$$

$$\mathbb{F}[\![\mathcal{O}_K]\!][(\tfrac{1}{y_\sigma})] \longrightarrow \mathbb{F}((T))$$

$$\mathbb{F}[\![\mathbb{Z}_p]\!][\tfrac{1}{T}] = \mathbb{F}((T))$$

$$\rightsquigarrow \quad A \twoheadrightarrow \mathbb{F}((T)) \circlearrowleft \varphi, \mathbb{Z}_p^\times.$$

again $p \leftrightarrow \varphi$, $\mathbb{Z}_p^\times \hookrightarrow \mathcal{O}_K^\times$.

On the other hand,

$$\begin{array}{ccccc} \pi \longmapsto D_A(\pi) \longmapsto & D_A(\pi)\otimes_A \mathbb{F}((T)) & & (\varphi, \mathbb{Z}_p^\times)\text{-mod} & \\ & \updownarrow & & \updownarrow & \text{Fontaine} \\ & ? & & \text{Gal rep} & \end{array}$$

Thm (BHHMS) $\pi = \pi(\bar\rho)$, $\bar\rho$ 2-dim'l semisimple.

Then $D_A(\pi(\bar\rho)) \otimes_A \mathbb{F}((T)) \xleftrightarrow[\sim]{\text{Fontaine}} \mathrm{ind}_{G_K}^{\otimes G_{\mathbb{Q}_p}} \bar\rho$.

$\circlearrowleft$ $(\varphi, \mathbb{Z}_p^\times)$

<u>Tensor induction</u> $H < G$, $[G:H] = d$. $\rho \in \mathrm{Rep}(H)$.

Define $\mathrm{ind}_H^G \rho = \bigoplus_{g \in G/H} (g \otimes_H \rho)$ $\quad \dim = d \cdot \dim \rho$

$\mathrm{ind}_H^{\otimes G} \rho = \bigotimes_{g \in G/H} (g \otimes_H \rho)$ $\quad \dim = (\dim \rho)^d$.

<u>Cor</u> $\mathrm{rank}_A D_A(\pi(\bar\rho)) = 2^f$.

Let $\sigma_0 : \mathbb{F}_q \hookrightarrow \mathbb{F}$, $\sigma_i = \sigma_0 \circ \varphi^i$.

$(\varphi, \mathbb{Z}_p^\times)$-mod assoc to $\mathrm{ind}_{G_K}^{\otimes G_{\mathbb{Q}_p}} \bar\rho$ is $D_{\sigma_0}(\bar\rho) \otimes_{\mathbb{F}((T))} \cdots \otimes D_{\sigma_{f-1}}(\bar\rho)$.

Thm $D_A(\pi(\bar\rho)) = D_{A,\sigma_0}(\bar\rho) \otimes_A \cdots \otimes_A D_{A,\sigma_{f-1}}(\bar\rho)$

$\circlearrowleft$ A, $(\varphi^f, \mathcal{O}_K^\times)$

Can lift $D_{A,\sigma_0}(\bar\rho)$ to $D_{\sigma_0}(\bar\rho)$, a $(\varphi^f, \mathbb{Z}_p^\times)$-mod

& $D_{A,\sigma_0}(\bar\rho) \longleftrightarrow$ Lubin-Tate $(\varphi, \mathcal{O}_K^\times)$-mod / $\mathbb{F}((T_K))$.

(K_∞ / K LT ext'n w/ Gal grp $\mathcal{O}_K^\times$.)