LOCAL SHTUKAS AND THE LANGLANDS PROGRAM

Abstract. These are notes for the course given by Jared Weinstein in 2022 Summer School on the Langlands Program at IHES. In the Langlands program over number fields, automorphic representations and Galois representations are placed into correspondence, using the cohomology of Shimura varieties as an intermediary. Over a function field, the appropriate intermediary is a moduli space of shtukas. We introduce the shtukas and their local analogues, which play a similar role in the local Langlands program. Along the way we construct the Fargues–Fontaine curve and discuss perfectoid spaces and diamonds. This survey may be seen as preparatory for the lectures of Fargues–Scholze.

Contents

1. The Langlands correspondence, Shimura varieties, and stacks of shtukas

Let F be a local field, and the G/F be a reductive algebraic group. One of the aims of the Langlands program is to generalize class field theory for F. It predicts a correspondence between the following two sets:

- **The automorphic side:** Algebraic[1] automorphic representations of $G(\mathbb{A}_F)$ with complex coefficients.
- **The Galois side:** L-parameters, which are conjugacy classes of continuous homomorphisms $\varphi : \mathrm{Gal}(\overline{F}/F) \to {}^LG(\overline{\mathbb{Q}}_\ell)$, where ${}^LG$ is the Langlands dual group. (As part of the setup, we choose an auxiliary prime ℓ different from the characteristic of F, and also an isomorphism between $\mathbb{C}$ and $\overline{\mathbb{Q}}_\ell$.)

The correspondence should be a finite-to-one map, with fibers $\Phi_\varphi(G)$ (known as L-packets) relating to the phenomenon of endoscopy. In the case $G = \mathrm{GL}_n$, the L-packets are singletons. In that case, *cuspidal* automorphic representations π should correspond to irreducible φ.

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[1] We refer here to the notion of L-algebraicity [BG14]. It is a condition on the archimedean components of π.

There is also a local version of the conjectural Langlands correspondence for each place v of F, which matches irreducible admissible representations of $G(F_v)$ with local L-parameters $\varphi_v : W_{F_v} \to {}^L G(\overline{\mathbb{Q}}_\ell)$. The local correspondence should be compatible with the global one, in the sense that the component of π at each place v of F corresponds with the restriction φ_v of φ to the Weil group at v. We note right away that is easy to characterize and prove the unramified local correspondence (which matches unramified π_v with unramified φ_v), using Satake parameters. This allows us to characterize the global correspondence $\pi \mapsto \varphi$ by demanding that π_v match φ_v for all unramified places v. Whereas, it is much harder to characterize local correspondence in general.

Let us discuss the state of the global Langlands correspondence. Despite enormous progress, the correspondence remains wide open in the number field case. Already in the case $F = \mathbb{Q}$ and $G = \mathrm{GL}_2$, it is not currently known whether an algebraic Maass form always has a corresponding 2-dimensional Galois representation. Regarding GL_n for general n, powerful results such as [Sch13, Theorem I.4] construct the correspondence for a large class of automorphic representations over a real or CM field F, but there remains a regularity hypothesis which excludes the Maass forms and their higher rank analogues. Nor does there seem to be an approach over number fields that is uniform across all groups G.

The picture is much rosier in the case that F is a global field of characteristic $p > 0$, i.e. a function field. In that case, Drinfeld [Dri80] and L. Lafforgue [Laf02] established the complete Langlands correspondence for all GL_n. And then V. Lafforgue [Laf12] constructed the "automorphic to Galois" direction of the correspondence for general G over function fields.

Why have we come so much further in the function field case? A simple answer is that essentially all known instances of the global Langlands correspondence are mediated by geometric objects — the Shimura varieties and their analogues — and that function fields are simply "closer to geometry" than number fields. In this section, we will review the geometric objects relevant to the Langlands story which existed before [SW20].

1.1. **Modular curves.** It may be helpful to review the case $F = \mathbb{Q}$ and $G = \mathrm{GL}_2$. Recall [Gel75] that there is a bijection between cuspidal newforms of weight $k \geqslant 2$ on the one hand, and cuspidal automorphic representations $\pi = \pi_f \otimes \pi_\infty$ of $G(\mathbb{A}_\mathbb{Q})$ whose archimedean component π_∞ is the discrete series representation of weight k. The construction of φ from π is due to Deligne, and the local-global compatibility of $\pi \mapsto \varphi$ (even at the ramified places) is due to Carayol [Car86]. In this situation, the relevant Shimura varieties are the modular curves Y_K, as K ranges through open compact subgroups of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. These classified elliptic curves are with level structure and are defined over $\mathbb{Q}$. Let X_K be the compactification of Y_K, and let $X_{K,\overline{\mathbb{Q}}}$ be its base change to $\overline{\mathbb{Q}}$.

Deligne constructs a $\overline{\mathbb{Q}}_\ell$-adic local system ξ_k on X_K for each $k \geqslant 2$ (for $k = 2$ this is the constant local system). Let $H^1(\xi_k) = \varinjlim_K H^1(X_{K,\overline{\mathbb{Q}}}, \xi_k)$, a representation of $\mathrm{GL}_2(\mathbb{A}_f) \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Theorem 1.1 (Deligne, Carayol)**.** *As a representation of* $\mathrm{GL}_2(\mathbb{A}_f) \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*, the cohomology* $H^1(\xi_k)$ *contains a summand (the cuspidal part) isomorphic to*

$$\bigoplus_\pi \pi_f \boxtimes \varphi_\pi,$$

where $\pi = \pi_f \otimes \pi_\infty$ *runs over cuspidal automorphic representations for which* π_∞ *is discrete series of weight* k*, and* φ_π *is* 2*-dimensional and irreducible. Furthermore, for each prime* p*, the local component* π_p *determines the restriction of* φ_π *to* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$*.*

The fact that the cuspidal part of $H^1(\xi_k)$ has the form as in Theorem 1.1 for some unspecified 2-dimensional φ_π comes via comparison with a similar formula for singular cohomology; this is a special case of Matsushima's formula. To show that φ_π matches π at all unramified places is a matter of applying the Eichler–Shimura relation. There is also a derivation of this fact using the "Langlands–Kottwitz method" [Sch11], ultimately a matter of counting points on X_K over finite fields. The proof that φ_π matches π even at the ramified primes requires a detailed study of the bad reduction of X_K. Crucial to that study is the deformation space of a p-divisible formal group of height 2, which appears at the completion of an integral model of X_K at one of the supersingular points modulo p. This is Lubin–Tate space, and it and its generalizations will be the focus of much of this article.

Regarding automorphic representations of $\mathrm{GL}_2/\mathbb{Q}$ not covered by Theorem 1.1: In [DS74], congruences are used to construct the Galois representations attached to holomorphic modular forms of weight 1; these correspond to those π for which π_∞ is a "limit of discrete series". On the other hand, the cohomology of modular curves does not seem to make contact with Maass forms at all.

1.2. **The formalism of Shimura varieties.** The general formalism of Shimura varieties is described in [Del71], and is rooted in Hodge theory. Let us give a hurried account. A smooth manifold X has singular cohomology $H^i_{\mathrm{sing}}(X,\mathbb{Z})$ and complex de Rham cohomology $H^i_{\mathrm{dR}}(X)$, which gets identified with $H^i_{\mathrm{sing}}(X,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}$ under the de Rham isomorphism (via integration).

When X is a projective variety over $\mathbb{C}$ (or even just a Kähler manifold), then $V = H^1_{\mathrm{sing}}(X,\mathbb{Z})$ is an integral Hodge structure of weight i, meaning that there is a Hodge decomposition $V\otimes\mathbb{C}\cong H^i_{\mathrm{dR}}(X/\mathbb{C})\cong\bigoplus_{p+q=i}V^{pq}$, where $V^{pq}=H^{p,q}(X)$ satisfies $\overline{V}^{pq}=V^{qp}$. The data of the Hodge structure is captured by a morphism of real groups $\mu:\mathbb{C}^\times\to\mathrm{GL}(V\otimes\mathbb{R})\cong\mathrm{GL}_n(\mathbb{R})$. The conjugacy classes of such μ may be recorded by a nonincreasing tuple of integers $(k_1,\dots,k_n)$, where the multiplicity of p among the k_i is the dimension of V^{pq}.

Recall the case of abelian varieties over $\mathbb{C}$: if $A/\mathbb{C}$ is an abelian variety of dimension n, then $H^1(A,\mathbb{Z})$ is a polarizable Hodge structure of type $(1,\dots,1,0,\dots,0)$ (with n 1s and n 0s). Passing to duals, we get the classification of complex abelian varieties:

Theorem 1.2. *The following categories are equivalent:*

(1) *Abelian varieties over* $\mathbb{C}$.
(2) *Polarizable weight* -1 *integral Hodge structures of type* $(0,\dots,0,-1,\dots,-1)$.

Theorem 1.2 leads directly to the construction of the Siegel modular varieties: principally polarized abelian varieties are in equivalence with polarizable Hodge structures of this type, which are in turn equivalent to points on Siegel upper-half plane modulo $\mathrm{Sp}_{2n}(\mathbb{Z})$.

In the general formalism, one begins with a *Shimura datum*: a pair (G,μ) consisting of a reductive group $G/\mathbb{Q}$ and a morphism of real groups $\mu:\mathbb{C}^\times\to G(\mathbb{R})$, up to $G(\mathbb{R})$-conjugation. This pair is required to satisfy axioms which ensure that the space $\mathcal{H}_\mu$ of conjugacy classes of μ is a complex manifold (a *Hermitian symmetric domain*). The tower of Shimura varieties is

$$\mathrm{Sh}(G,\mu)_K = G(\mathbb{Q})\backslash(\mathcal{H}_\mu\times G(\mathbb{A}_f)/K)$$

as K varies through compact open subgroups of $G(\mathbb{A}_f)$. The main theorem of [Del71] is that $\mathrm{Sh}(G,\mu)_K$ is a quasi-projective variety over $\mathbb{C}$, which admits a canonical model over a number field E, namely the field of definition of the conjugacy class μ.

The ℓ-adic cohomology of the tower $\mathrm{Sh}(G,\mu)_K$ admits an action of $G(\mathbb{A}_f)\times\mathrm{Gal}(\overline{E}/E)$. Kottwitz [Kot90] proposed that the Langlands correspondence should appear in this cohomology, in the following sense. (To avoid burdening the reader we will be a bit vague; we refer to the introduction to [SS13] for an in-depth discussion.) We assume that $\mathrm{Sh}(G,\mu)_K$ is compact. To an algebraic representation of G there corresponds a local system ξ on $\mathrm{Sh}(G,\mu)_K$ and we define

$$H^i(\xi)=\varinjlim_K H^i(\mathrm{Sh}(G,\mu)_{K,\overline{\mathbb{Q}}},\overline{\mathbb{Q}}_\ell),$$
$$H^*(\xi)=\bigoplus_i(-1)^iH^i(\xi),$$

so that $H^*(\xi)$ lies in the Grothendieck group of representations of $G(\mathbb{A}_{\mathbb{Q}})\times\mathrm{Gal}(\overline{\mathbb{Q}}/E)$.

Conjecture 1.3. *In this Grothendieck group, we have*

$$H^*(\xi)=\sum_\varphi\sum_{\pi\in\Pi_\varphi(G)}a(\pi_\infty,\xi)\pi_f\boxtimes(R_\mu\circ\varphi|_{\mathrm{Gal}(\overline{\mathbb{Q}}/E)}),$$

where φ runs over L-parameters, $a(\pi_\infty,\xi)$ is the multiplicity appearing in Matsushima's formula, and $R_\mu:{}^LG\to\mathrm{GL}(N)$ is the representation of highest weight μ.

Essentially all approaches to constructing the Langlands correspondence [HT01, SS13] run through the cohomology of Shimura varieties, using Conjecture 1.3 as a guide for how that cohomology should behave. These attacks have limitations. Shimura varieties only exist for certain groups G. They do not exist for GL_n with $n\geqslant 3$. Even when they do exist, the cohomology of Shimura varieties can only directly access those automorphic representations π for which $a(\pi_\infty,\xi)\neq 0$ for some local system ξ (these π are called "cohomological"). For example, in the case of $G=\mathrm{GL}_2/\mathbb{Q}$, we mentioned above that [Del74] considers the ℓ-adic cohomology of a modular curve. This cohomology contains data relevant to the holomorphic modular forms of weight $\geqslant 2$; these correspond to those automorphic representations π for which π_∞ is discrete series.

1.3. **Drinfeld modules and Drinfeld modular varieties.** In the case where F is a function field, one has analogues of the Shimura varieties; these are known as Drinfeld modular varieties. (See [Gos96] for an introduction to function field arithmetic, and [Gek92] for the development of Drinfeld modular varieties.) We give a brief summary of the story. Let $\mathbb{F}_q$ be the field of constants of F. Let ∞ be a place of F, and let $A \subset F$ be the ring of functions regular away from ∞. Let F_∞ be the completion; thus F_∞ is isomorphic to a field of Laurent series over the residue field of ∞. Finally, let C_∞ be a complete algebraically closed field containing F_∞.

As the notation suggests, C_∞ is the function field analogue to the complex number $\mathbb{C}$. As complex elliptic curves arise as quotients $\mathbb{C}/\Lambda$ for a lattice $\Lambda \subset \mathbb{C}$, we might seek to define their function field analogues by examining similar quotients of C_∞.

Let $\Lambda \subset C_\infty$ be a locally free A-submodule of rank n. Drinfeld observed that there is an analytic isomorphism of $\mathbb{F}_q$-vector spaces $C_\infty/\Lambda \cong C_\infty$. (Certainly, this phenomenon is unique to the positive characteristic setting. The isomorphism is represented by a power series of the form $\sum_{i \geqslant 0} a_i z^{q^i}$, with $a_i \in C_\infty$. In the case $\Lambda = A = \mathbb{F}_q[t]$, this series is called the Carlitz exponential.) Through this isomorphism, C_∞ inherits an exotic A-module structure, in which each $\alpha \in A$ acts on $x \in C_\infty$ through a polynomial $\psi_a(x) = \alpha x + \alpha_1 x^q + \cdots \alpha_m x^{q^m}$, where $m = -n \operatorname{ord}_\infty \alpha$. The collection of polynomials ψ_α for $\alpha \in A$ constitutes a *Drinfeld module* ψ of rank n.

The notion of a Drinfeld module is entirely algebraic; one can talk about a Drinfeld module ψ of rank n over R, where R is any A-algebra. For a nonzero ideal $N \subset A$, the equations $\psi_\alpha(x) = 0$ for $\alpha \in N$ define an (A/N)-module scheme $\psi[N]$ which is finite flat of rank n. If $NR = R$ then $\psi[N]$ is étale. For a subgroup $\Gamma \subset \mathrm{GL}_2(A)$ containing the principal congruence subgroup $\Gamma(N)$, let $\mathsf{DrMod}_A^n(\Gamma)$ be the moduli space of Drinfeld modules ϕ together with a Γ-orbit of trivializations $(A/N)^{\oplus d} \to \phi[N]$. Then for Γ small enough, $\mathsf{DrMod}_A^n(\Gamma)$ is an affine variety over $\operatorname{Spec} A \backslash \operatorname{Supp} N$ which is smooth of relative dimension $n-1$.

As with Shimura varieties, the C_∞-points of Drinfeld modular varieties have analytic descriptions. For instance if $n = 2$, the C_∞-points of $\mathsf{DrMod}_A^2(\Gamma)$ may be identified with $\mathcal{H}_\infty/\Gamma$, where $\mathcal{H}_\infty = C_\infty \backslash F_\infty$ is Drinfeld's upper-half plane. Drinfeld showed [Dri74] that the ℓ-adic H^1 of the curve $\mathsf{DrMod}_A^2(\Gamma)$ (as Γ ranges through congruence subgroups) realizes the Langlands correspondence for all cuspidal automorphic representations π of GL_2/F for which π_∞ is a Steinberg (special) representation. The general result for GL_n is due to Laumon [Lau96].

1.4. **From Drinfeld module to shtuka.** The theory of Drinfeld modular varieties is already more flexible than that of Shimura varieties, at least for the reason that the former exists for all GL_n. But from the point of view of the Langlands correspondence, we are still limited by the condition that π_∞ must be a Steinberg representation, as noted above. (This is analogous to the phenomenon that the cohomology of Shimura varieties sees only those automorphic representations which are discrete series at infinite places.) It might be preferable to have a theory that treats all places of F equally. Such a theory was also introduced by Drinfeld, in the form of *moduli stacks of shtukas.*

Definition 1.4. Let $X/\mathbb{F}_q$ be the smooth projective curve whose function field is F. Let $S/\mathbb{F}_q$ be a scheme. An *X-shtuka of rank n over S* (or just *shtuka* if the context is clear) consists of the following data:

- Morphisms $x_1, \dots, x_r : S \to X$, called the *legs* of the shtuka,
- Vector bundles $\mathcal{E}_0, \dots, \mathcal{E}_r = \mathrm{Fr}_S^* \mathcal{E}_0$ of rank n over $X \times S$,
- For $i = 1, \dots, r$, an isomorphism $f_i : \mathcal{E}_{i-1}|_{(X\times S)\backslash \Gamma_{x_i}} \xrightarrow{\sim} \mathcal{E}_i|_{(X\times S)\backslash\Gamma_{x_i}}$, where Γ_{x_i} denotes the graph of morphism x_i lying in $X \times S$.

For each geometric point $s \in S$, the failure of f_i to be an isomorphism at $X \times s$ is measured by a tuple of integers $\mu = (k_1, \dots, k_n)$ with $k_1 \geqslant \cdots \geqslant k_n$, as we now explain. Suppose X/k is a curve over a field, $x \in X(k)$ is a point with uniforming parameter t, and $f : \mathcal{E}|_{X\backslash\{x\}} \cong \mathcal{E}'|_{X\backslash\{x\}}$ is an isomorphism of vector bundles of rank n. Then the completed stalks $\hat{\mathcal{E}}_x$ and $\hat{\mathcal{E}}'_x$ are free module over $\hat{\mathcal{O}}_{X,x} \cong k[\![t]\!]$, whose generic fibers are identified via f. If we choose bases $\hat{\mathcal{E}}_x \cong k[\![t]\!]^{\oplus n}$ and $\hat{\mathcal{E}}'_x \cong k[\![t]\!]^{\oplus n}$, then we may represent f by an element of $\mathrm{GL}_n(k((t)))$, whose class in the double coset space

$$\mathrm{GL}_n(k[\![t]\!])\backslash \mathrm{GL}_n(k((t)))/\mathrm{GL}_n(k[\![t]\!]) \tag{1.4.1}$$

is independent of our choices. A complete set of double coset representatives is given by those matrices in Smith normal form $\operatorname{diag}(t_1^{k_1},\dots,t_n^{k_n})$, with $k_1 \geqslant \cdots \geqslant k_n$; this is the tuple of integers associated with f. We put a partial order on (1.4.1); $(k_1,\dots,k_n) \leqslant (k_1',\dots,k_n')$ means $\sum_{i=1}^j k_i \leqslant \sum_{i=1}^j k_i'$ for each $j=1,\dots,n-1$, and also $\sum_{i=1}^n k_i = \sum_{i=1}^n k_i'$.

For instance, an f of type $(1,0,\dots,0)$ is a map $\mathcal{E}\to\mathcal{E}'$ whose kernel is the pushforward from x of a 1-dimensional vector space. An f of type $(0,\dots,0,-1)$ is a map $\mathcal{E}'\to\mathcal{E}$ with the same property. Both of these types are minuscule, meaning minimal under the partial order.

There is a notion of the level structure of such shtukas: for each effective divisor $D\subset X$ not meeting the legs of the shtuka, a level D structure is a collection of trivializations $\mathcal{E}_i|_D \cong \mathcal{O}_D^{\oplus n}$ for $i=0,\dots,r-1$ which are compatible with the f_i.

Let $\mu_1,\dots,\mu_r$ be r elements of (1.4.1). Let us write $\mathrm{Sht}(\mathrm{GL}_n;\mu_1,\dots,\mu_r)_D$ for the moduli stack of r-legged shtukas of rank n with level D structure, such that the modification at the ith leg is $\leqslant \mu_i$.

Example 1.5 (Shtukas of rank 1, and geometric class field theory)**.** See [Laf18, Section 3] for more details concerning this example. The Picard group $\mathrm{Pic}(X)$ classifies rank 1 vector bundles (line bundles) on X. An everywhere unramified automorphic representation π of GL_1/F is the same as a character of $\mathrm{Pic}(X)$. The (unramified) local Langlands correspondence for GL_1/F is a bijection between such π and 1-dimensional everywhere unramified characters φ of W_F. If π is finite-order, then φ extends to a character of $\pi_1(X)$, the étale fundamental group of X.

A rank 1 vector bundle (or line bundle) is the same thing as an invertible sheaf. Let Pic_X be the Picard scheme of X, whose S-points classify line bundles on $X\times S$. Then Pic_X is a group variety under the tensor product, whose neutral component Pic_X° is the Jacobian variety of X. Its $\mathbb{F}_q$-points $\mathrm{Pic}_X(\mathbb{F}_q)=\mathrm{Pic}(X)$ form the Picard group of X.

Observe that for a morphism $x\colon S\to X$, and line bundles $\mathcal{E},\mathcal{E}'$ on $X\times S$, an isomorphism $f\colon \mathcal{E}|_{(X\times S)\backslash\Gamma_x} \xrightarrow{\sim} \mathcal{E}'|_{(X\times S)\backslash\Gamma_x}$ of type (±1) is the same as an isomorphism $\mathcal{E}\otimes\mathcal{O}_{X\times S}(\pm\Gamma_x)\xrightarrow{\sim}\mathcal{E}'$.

A shtuka of rank 1 with two legs $x_1,x_2\colon S\to X$ of types $\mu_1=(1)$ and $\mu_2=(-1)$ respectively would thus be a line bundle $\mathcal{E}$ on $X\times S$ together with an isomorphism $\mathcal{E}\otimes\mathcal{O}_{X\times S}(\Gamma_{x_1}-\Gamma_{x_2})\cong \mathrm{Fr}_S^*(\mathcal{E})$. Ignoring automorphisms, the moduli space of such shtukas appears as a cartesian product

$$\begin{tikzcd}
\mathrm{Sht}(\mathrm{GL}_1;\mu_1,\mu_2) \arrow[r] \arrow[d] & \mathrm{Pic}_X \arrow[d] \\
X^2 \arrow[r] & \mathrm{Pic}_X^\circ
\end{tikzcd}$$

where the lower horizontal arrow sends $x_1,x_2\in X(S)$ to $\mathcal{O}_X(\Gamma_{x_1}-\Gamma_{x_2})$, and $\mathcal{L}(\mathcal{E})=\mathrm{Fr}_S^*(\mathcal{E})\otimes\mathcal{E}^\vee$ is the Lang isogeny. Since $\mathcal{L}$ is an étale torsor for the group $\mathrm{Pic}_X(\mathbb{F}_q)=\mathrm{Pic}(X)$, the same is true for $\mathrm{Sht}(\mathrm{GL}_1;\mu_1,\mu_2)\to X^2$. We thus obtain a surjective homomorphism from the étale fundamental group $\pi_1(X^2)$ to any finite quotient of $\mathrm{Pic}(X)$. In fact this map factors through $\pi_1(X^2)\to\pi(X)^2$, as $\mathcal{M}^1$ admits a "partial Frobenius structure" [Laf18, Lemma 1.1]. As a result, any finite-order character π of $\mathrm{Pic}(X)$ can be pulled back to a character of $\pi_1(X)^2$, necessarily of the form $\varphi\boxtimes\varphi^{-1}$. Then $\pi\mapsto\varphi$ is the (unramified) local Langlands correspondence for GL_1/F.

Drinfeld's proof of the Langlands correspondence for GL_2 [Dri80] makes use of the moduli stacks of shtukas of rank 2 with 2 legs, one of type $\mu_1=(1,0)$ and the other of type $(0,-1)$. Drinfeld shows that $\mathrm{Sht}(\mathrm{GL}_2;\mu_1,\mu_2)_D$ is a Deligne–Mumford stack, and that the morphism $\mathrm{Sht}(\mathrm{GL}_2;\mu_1,\mu_2)_D\to (X\backslash \operatorname{Supp} D)^2$ is smooth of relative dimension 2. The tower of stacks $\mathrm{Sht}(\mathrm{GL}_2;\mu_1,\mu_2)_D$ (as D varies through all effective divisors) admits an action of $\mathrm{GL}_2(\mathbb{A}_F)$. Drinfeld considers the cohomology

$$V_\ell = \varinjlim_D H^2(\mathrm{Sht}(\mathrm{GL}_2;\mu_1,\mu_2)_D\times\overline{\eta}_2,\overline{\mathbb{Q}}_\ell), \tag{1.4.3}$$

where $\eta_2=\operatorname{Spec}F_2$ is the generic point of the surface X^2, and $\overline{\eta}_2=\operatorname{Spec}\overline{F}_2$. Then $\mathrm{GL}_2(\mathbb{A}_F)$ acts on V_ℓ. If π is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with $\overline{\mathbb{Q}}_\ell$ coefficients, Drinfeld shows that the representation of $\mathrm{Gal}(\overline{F}_2/F_2)$ on $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{A}_F)}(\pi,V_\ell)$ factors through a representation of $W_F\times W_F$ of the form $\varphi\boxtimes\check{\varphi}$, where φ is irreducible and 2-dimensional. Then $\pi\mapsto\varphi$ is the Langlands correspondence for GL_2/F.

L. Lafforgue [Laf02] establishes the Langlands correspondence for GL_n/F for all n, using moduli stacks of rank n shtukas (still with two legs).

We remark here that there is a (far from obvious) connection between Drinfeld modules and shtukas, see [Mum78]. Drinfeld modules of rank n are in equivalence with certain kinds of 2-legged shtukas, where one leg is fixed at the point ∞.

Varshavsky [Var04] defined a generalization of Definition 1.4 in the context of a reductive group $G/\mathbb{F}_q$. The idea is to replace the vector bundles $\mathcal{E}_i$ in that definition to G-torsors. For simplicity let us assume that G is split, with maximal torus T. The double coset space in (1.4.1) becomes $G(k[\![t]\!])\backslash G(k(\!(t)\!))/G(k[\![t]\!])$. This last set may be identified with the partially ordered set of dominant cocharacters $\mu : \mathbb{G}_m \to T$.

Varshavsky's definition may be used to define a morphism of stacks

$$\lambda : \mathrm{Sht}(G; \mu_1, \dots, \mu_r)_D \longrightarrow (X\backslash \operatorname{Supp} D)^r$$

classifying G-shtukas (he calls them principal F-bundles) with level D structure, where the shtuka is bounded by μ_i on the ith leg.

We summarize the conjectural link between Varshavsky's stacks of G-shtukas and the Langlands correspondence. As in (1.4.3), one creates a $\overline{\mathbb{Q}}_\ell$-vector space V_ℓ admitting an action of $G(\mathbb{A}_F) \times W_F^r$, using the ℓ-adic cohomology of the $\mathrm{Sht}(G; \mu_1, \dots, \mu_r)$ base changed to $\overline{\eta}_r$. (To be precise, one must use the intersection cohomology, as these stacks fail to be smooth in general.) For $i = 1, \dots, r$, let $R_i : \hat{G} \to \mathrm{GL}(V_i)$ be the algebraic representation with highest weight $\mu_i \in X_*(T) \cong X^*(\hat{T})$.

Conjecture 1.6. *Let π be a cuspidal automorphic representation of $G(\mathbb{A}_F)$ with $\overline{\mathbb{Q}}_\ell$-coefficients. Let $\varphi : W_F \to \hat{G}(\overline{\mathbb{Q}}_\ell)$ be the Langlands correspondence of π. The representation of W_F^r on $\mathrm{Hom}_{G(\mathbb{A}_F)}(\pi, V_\ell)$ is isomorphic to*

$$(R_1 \circ \varphi) \boxtimes \cdots \boxtimes (R_r \circ \varphi).$$

1.5. **Interlude: geometric Langlands.** For the exposition of this topic, see [Fre05].

The unramified automorphic forms on GL_n over a global field F are smooth functions on the double coset space

$$\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)/\prod_v \mathrm{GL}_n(\mathcal{O}_{F_v}) \tag{1.5.1}$$

where v runs over finite places of F. When F is a function field corresponding to a curve $X/\mathbb{F}_q$, (1.5.1) is discrete; it is the set of isomorphism classes of rank n vector bundles on X (that is, $\mathcal{O}_X$-modules which are locally free of rank n). In the geometric Langlands program, one replaces (1.5.1) with the stack $\mathrm{Bun}_n/\mathbb{F}_q$, which assigns to a scheme $S/\mathbb{F}_q$ the groupoid of rank n vector bundles on $X \times S$, so that automorphic forms are simply functions on $\mathrm{Bun}_n(\mathbb{F}_q)$. Then Bun_n is a smooth Artin stack. Following Grothendieck's philosophy of the function-sheaf correspondence, in geometric Langlands, one seeks "automorphic sheaves" on Bun_n, whose Frobenius traces will recover automorphic forms. These sheaves will be objects in a triangulated category $D(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ of ℓ-adic étale shaves on Bun_n.

One also geometrizes the Hecke operators acting on automorphic forms, which arise from the Hecke correspondences on (1.5.1). For $i = 1, \dots, n-1$, we have a diagram of stacks over $\mathbb{F}_q$:

$$\begin{tikzcd} & \mathrm{Hecke}_i \arrow[dl, "h_1"'] \arrow[dr, "\mathrm{supp}\times h_2"] & \\ \mathrm{Bun}_n & & X \times \mathrm{Bun}_n. \end{tikzcd}$$

For a scheme $S/\mathbb{F}_q$, the S-points of Hecke_i classify tuples $(x, \mathcal{E}_1, \mathcal{E}_2, f)$, where $x : S \to X$, the $\mathcal{E}_1$ and $\mathcal{E}_2$ are rank n vector bundles on $X \times S$, and $f : \mathcal{E}_1 \to \mathcal{E}_2$ is an injective map whose cokernel is a vector bundle of rank i on the graph Γ_x. The last condition means that f has type $(1, \dots, 1, 0, \dots, 0)$ (with i 1s) at every point of S. The maps h_1, h_2, and supp indicated in the diagram above send this tuple to $\mathcal{E}_1$, $\mathcal{E}_2$, and x, respectively. Note that the fiber of $\mathrm{supp} \times h_2$ of a pair $(x, \mathcal{E}_2)$ is the Grassmannian of i-dimensional quotients of $(\mathcal{E}_2)_x$; it has dimension $d = i(n-i)$,

Now define the Hecke operator $\mathcal{H}_i$:

$$\begin{aligned} \mathcal{H}_i : D(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell) &\longrightarrow D(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell) \\ A &\longmapsto R(\mathrm{supp} \times h_2)_* \left(h_1^* A \left(\frac{d}{2}\right)[-d] \right). \end{aligned}$$

Here the $(d/2)$ is a Tate twist (requiring a choice of $\sqrt{q} \in \overline{\mathbb{Q}}_\ell$) and $[-d]$ is a cohomological shift.

The (unramified) geometric Langlands correspondence for GL_n/F is the following statement.

Theorem 1.7 (Drinfeld for $n=2$, Frenkel–Gaitsgory–Vilonen in general)**.** *Let E be an irreducible $\overline{\mathbb{Q}}_\ell$-local system on X, i.e., an irreducible representation of $\mathrm{Gal}(\overline{F}/F)$ of dimension n with $\overline{\mathbb{Q}}_\ell$-coefficients which is unramified everywhere. Then there exists a (nonzero) perverse sheaf Aut_E in $D(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ which is a Hecke eigensheaf with value E, meaning that*

$$\mathcal{H}_i(\mathrm{Aut}_E) \cong \bigwedge^i E \boxtimes \mathrm{Aut}_E\,.$$

1.6. **The local picture: local class field theory, Lubin–Tate spaces.** Now let F be a nonarchimedean local field with residue field $\mathbb{F}_q$: thus either $F \cong \mathbb{F}_q[\![t]\!]$ or else F is a finite extension of $\mathbb{Q}_p$. Class field theory for F is encoded by the reciprocity map

$$\mathrm{rec}_F : F^\times \xrightarrow{\sim} W_F^{\mathrm{ab}}.$$

A choice of uniformizer π of F induces a factorization $F^\times \cong \mathbb{Z} \times \mathcal{O}_F^\times$, which in turn induces via rec_F a description of the maximal abelian extension F^{ab}/F: it is the compositum of the maximal unramified extension F^{nr} with a totally ramified extension F_π/F with $\mathrm{Gal}(F_\pi/F) \cong \mathcal{O}_F^\times$.

Historically, rec_F was constructed as a byproduct of the global reciprocity map, but the theory of Lubin–Tate formal groups [LT65] provides an explicit description. We briefly review the story. Let A be an $\mathcal{O}_F$-algebra, with structure map $\iota : \mathcal{O}_F \to A$. A 1*-dimensional formal $\mathcal{O}_F$-module law* over A consists of power series $\mathcal{G}(X,Y) = X + Y + \cdots \in A[\![X,Y]\!]$ and $[\alpha]_{\mathcal{G}}(X) = \iota(\alpha)X + \cdots \in A[\![X]\!]$ for each $\alpha \in \mathcal{O}_F$, which together obey the axioms of an $\mathcal{O}_F$-module. (The ellipses mean terms of degree > 1.) Thus for instance $\mathcal{G}(X,Y) = \mathcal{G}(Y,X)$ and $\mathcal{G}([\alpha]_{\mathcal{G}}(X), [\alpha]_{\mathcal{G}}(Y)) = [\alpha]_{\mathcal{G}}(\mathcal{G}(X,Y))$. There is an evident notion of a morphism of $\mathcal{O}_F$-module laws $\psi : \mathcal{G} \to \mathcal{G}'$: this is a power series $\psi(X) \in A[\![X]\!]$ interlacing $\mathcal{G}(X,Y)$ with $\mathcal{G}'(X,Y)$ and $[\alpha]_{\mathcal{G}}(X)$ with $[\alpha]_{\mathcal{G}'}(X)$.

If $A = k$ is a perfect field with $\iota(\pi) = 0$, then a 1-dimensional formal $\mathcal{O}_F$-module law $\mathcal{G}$ over k falls into one of two classes: either $[\pi]_{\mathcal{G}}(X) = 0$, in which case $\mathcal{G}$ is isomorphic to the formal additive group $\hat{\mathbb{G}}_a(X,Y) = X + Y$, or else $[\pi]_{\mathcal{G}}(X)$ is a nonzero power series in X^{q^h} for some maximal h, in which case we say $\mathcal{G}$ is π-divisible of height h. For example, the formal multiplicative group $\mathbb{G}_m(X,Y) = X + Y + XY$ is a 1-dimensional formal $\mathbb{Z}_p$-module law over $\mathbb{F}_p$ of height 1, since $[p]_{\mathbb{G}_m}(X) = X^p$. When k is algebraically closed, all 1-dimensional π-divisible $\mathcal{O}_F$-module laws of height h are isomorphic.

Lubin and Tate considered the problem of deforming a π-divisible $\mathcal{O}_F$-module law from $\mathbb{F}_q$ to $\mathcal{O}_F$. In the case that the height h equals 1, there exists a unique deformation. More precisely, they show that if $f(X) = \pi X + \cdots \in \mathcal{O}_F[X]$ satisfies $f(X) \equiv X^q \pmod{\pi}$, then there exists a unique formal $\mathcal{O}_F$-module law $\mathcal{G}_f$ over $\mathcal{O}_F$ with $[\pi]_{\mathcal{G}_f}(X) = f(X)$, and also that two such f give isomorphic $\mathcal{G}_f$. We might therefore assume that $f(X) = \pi X + X^q$. Finally, they show that the filed F_π predicted by local class field theory is obtained by adjoining to F a system of primitive roots λ_n of $f^{(n)}(X) = [\pi^n]_{\mathcal{G}_f}(X)$ for $n \geqslant 1$. That is, $\lambda_1 \neq 0$ is a root of $f(X)$ and $f(\lambda_n) = \lambda_{n-1}$ for all $n > 1$. The reciprocity law is realized by the relation $[\alpha]_{\mathcal{G}_f}(\lambda_n) = \mathrm{rec}(\alpha^{-1})(\lambda_n)$ for all $\alpha \in \mathcal{O}_F^\times$.

For a general π-divisible 1-dimensional $\mathcal{O}_F$-module $\mathcal{G}$ over $\overline{\mathbb{F}}_q$ of height h, Lubin and Tate show [LT66] that the moduli space of deformations of $\mathcal{G}$ is a formal open polydisc of dimension $h-1$ over $\mathcal{O}_{\hat{F}^{\mathrm{nr}}}$; i.e., it is isomorphic to $\mathrm{Spf}\,\mathcal{O}_{\hat{F}^{\mathrm{nr}}}[\![T_1, \ldots, T_{h-1}]\!]$. The generic fiber of this formal scheme over $\mathrm{Spf}\,\mathcal{O}_{\hat{F}^{\mathrm{nr}}}$ is a rigid-analytic space $\mathcal{M}_0$ over $\hat{F}^{\mathrm{nr}}$, and then adjoining π^n-torsion points of the universal deformation of $\mathcal{G}$ produces a tower of étale covers $\mathcal{M}_n$ known as the *Lubin–Tate tower*. A suitably modified version of the tower with $\mathbb{Z}$ connected components admits an action of the group $\mathrm{GL}_h(F)$.

Nonabelian Lubin–Tate theory [Car90] is the idea that the cohomology of the Lubin–Tate tower could be used to establish the local Langlands correspondence for this group. This program was executed by Deligne and Carayol for $h=2$ and Harris–Taylor [HT01] for all h, ultimately using the global theory of automorphic forms.

The theorem of Harris–Taylor fits into a yet more general program known as the Kottwitz conjecture, which we review in the next section.

1.7. p**-divisible groups, Rapoport–Zink spaces, and local Shimura varieties.** In the case that F is a p-adic field, the 1-dimensional formal $\mathcal{O}_F$-module laws we have described are examples of p-*divisible groups*. The p-divisible groups are the local analogues of abelian varieties. Just as moduli spaces of abelian varieties are instances of Shimura varieties, the moduli spaces of p-divisible groups (known as *Rapoport–Zink spaces*) are instances of geometric objects relevant to the local Langlands correspondence.

We refer to [Tat67] for precise definitions and the basics of the theory. Let R be a p-adically complete ring. A p-divisible group H of height h over R is an inductive system of group schemes $H[p^n]$, each locally free over R of rank p^{nh}, which satisfy axioms meant to mimic an abstract p-divisible group. The neutral component $H^\circ \subset H$ is the p-power torsion in a p-divisible formal group $\mathcal{G}$, and indeed there is an equivalence of categories between connected p-divisible groups and p-divisible formal groups. By definition $\operatorname{Lie} H = \operatorname{Lie} \mathcal{G}$ and $\dim H = \dim \mathcal{G}$. If A/R is an abelian variety of dimension d, then $A[p^\infty]$ is a p-divisible group of height p^{2d} and dimension d.

Over a perfect field k, Dieudonné theory (see [Fon77]) gives a complete classification of p-divisible groups. Let $W = W(k)$ be the ring of Witt vectors, and let $\sigma \in \operatorname{Aut} W$ be the Frobenius automorphism. A *Dieudonné module* is a free $W(k)$-module of finite rank admitting σ-linear and σ^{-1}-linear endomorphisms F, V satisfying $FV = p$.

Theorem 1.8 (Cartier–Dieudonné). *Let k be a perfect field. There is an anti-equivalence of categories $H \mapsto D(H)$ between p-divisible groups and Dieudonné modules. It satisfies the properties:*

1. *If H has height h, then $D(H)$ has rank h.*
2. *There is a functorial isomorphism of k-vector spaces $D(H)/FD(H) \cong (\operatorname{Lie} H)^*$, where $*$ means k-linear dual.*
3. *H is connected if and only if F is topologically nilpotent on $D(H)$. H is étale if and only if F is bijective on $D(H)$.*

We remark that if A/k is an abelian variety, then we have the relation

$$D(A[p^\infty]) \cong H^1_{\mathrm{cris}}(A/W).$$

See [Kat81] for more details. An *isocrystal* is a vector space V over $W[\frac{1}{p}]$ equipped with a σ-linear automorphism σ_V. If k is algebraically closed, then the category of isocrystals over k is semisimple, with simple objects V_λ classified by rational numbers λ, which are called the *slopes*. To write down an isocrystal of dimension n is to write down an element $b \in \mathrm{GL}_n(W[\frac{1}{p}])$ (the effect of σ_V on a basis for V) up to the relation $b \sim \sigma(x) b x^{-1}$ of σ-conjugacy. Written this way, the isocrystal V_λ for $\lambda = m/n$, with $n \geqslant 1$ and m, n relatively prime, corresponds to the $n \times n$ matrix

$$b = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ p^m & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

From Theorem 1.8 we see that the category of p-divisible groups over k up to isogeny is anti-equivalent to the category of isocrystals with slopes in the interval $[0, 1]$, via $H \mapsto D(H)[\frac{1}{p}]$.

Example 1.9. As isocrystals, we have

1. $D(\mathbb{Q}_p/\mathbb{Z}_p)[\frac{1}{p}] \cong V_0$.
2. $D(\mu_{p^\infty})[\frac{1}{p}] \cong V_1$.
3. For E/k an ordinary elliptic curve: $H^1_{\mathrm{cris}}(E/W)[\frac{1}{p}] \cong V_0 \oplus V_1$.
4. For E/k a supersingular elliptic curve: $H^1_{\mathrm{cris}}(E/W)[\frac{1}{p}] \cong V_{1/2}$.

Keep the assumption that k is an algebraically closed field of characteristic p. Let H_0 be a p-divisible group over k. Rapoport–Zink [RZ96] defined a moduli space of deformations of H_0 as follows.

Theorem 1.10 (Rapoport, Zink). *Consider the functor $\mathcal{M}_{H_0}$ which assigns to a p-adically complete W-algebra R the set of isomorphism classes of pairs (H, ρ), where*

- *H/R is a p-divisible group, and*
- *$\rho : H_0 \otimes_k R/p \to H \otimes_R R/p$ is a quasi-isogeny, i.e., an isomorphism in the isogeny category).*

Then $\mathcal{M}_{H_0}$ is representable by a formal scheme (which we continue to call $\mathcal{M}_{H_0}$) that locally admits a finitely generated ideal of definition.

Consequently the generic fiber $\mathcal{M}_{H_0,\eta}$ over $\eta = \operatorname{Spec} W[\frac{1}{p}]$ is a rigid analytic space. Trivializing the p^n-torsion in the universal deformation over $\mathcal{M}_{H_0,\eta}$, one obtains a tower of rigid-analytic spaces $\mathcal{M}_{H_0,\eta,n}$. The tower $\mathcal{M}_{H_0,\eta,\infty}$ admits commuting actions of two groups: the group $\mathrm{GL}_n(\mathbb{Q}_p)$, which acts on level structures, and the group $\operatorname{Aut} D(H_0)[\frac{1}{p}]$, which acts on the quasi-isogeny ρ. The latter group is always an inner form of a Levi subgroup of GL_n.

In [RZ96] there appear variations of the deformation problem of "EL type", meaning endomorphisms and level, parallel to the situation of Shimura varieties of EL type, which parametrize abelian varieties with endomorphisms.

But just as there are Shimura varieties that do not parametrize abelian varieties at all, there ought to be "local Shimura varieties" which do not parametrize p-divisible groups.

Conjecture 1.11 (Rapoport–Viehmann, [RV14])**.** *Let F be a finite extension of $\mathbb{Q}_p$. Let G/F be a reductive group, let $b \in B(G)$ be a σ-conjugacy class, and let μ be a conjugacy class of minuscule cocharacters $\mathbb{G}_m \to G$ defined over $\overline{F}$. Assume that $b \in B(G,\mu)$ is a "neutral acceptable class". There exists a tower of rigid-analytic spaces*

$$\mathrm{Sh}(G,b,\mu) = \{\mathrm{Sh}(G,b,\mu)_K\},$$

the so-called local Shimura varieties, as K runs through compact open subgroups of $G(F)$, satisfying the following properties:

(1) *The dimension of $\mathrm{Sh}(G,b,\mu)$ equals $d = \langle 2\rho_G, \mu\rangle$, where $2\rho_G$ is the sum of all the positive roots of G.*
(2) *Each $\mathrm{Sh}(G,b,\mu)_K$ admits an action of $G_b(F)$.*
(3) *The group $G(F)$ operates on the tower through Hecke correspondences.*
(4) *Let E/F be the field of definition of the conjugacy class μ. Each $\mathrm{Sh}(G,b,\mu)_K$ is defined over $\breve{E}$, and there is a Weil descent datum for the tower down to E. (The Weil descent datum extends the action of the inertia group of E on the cohomology of $\mathrm{Sht}(G,b,\mu)_K$ to the Weil group W_E.)*
(5) (The Kottwitz conjecture). *Assume that b is basic. Let $\varphi : W_F \to \hat{G}(\overline{\mathbb{Q}}_\ell)$ be a discrete Langlands parameter. For ρ lying in the L-packet $\Pi_\varphi(G_b)$, let*

$$R\Gamma(G,b,\mu)[\rho] = \varinjlim_{K\subset G(F)} R\operatorname{Hom}_{G_b(F)}(R\Gamma_c(\mathrm{Sh}(G,b,\mu)_{K,\hat{\breve{E}}}, \overline{\mathbb{Q}}_\ell[d](d/2)), \rho).$$

Then $R\Gamma(G,b,\mu)$ is a finite-length W_E-equivariant object in the derived category of admissible representations of $G(F)$. Let $\mathrm{Mant}(G,b,\mu)[\rho]$ be its image in the Grothendieck group of the category of finite-length admissible representations of $G(F)$ with $\overline{\mathbb{Q}}_\ell$-coefficients, which come equipped with a continuous action of W_E commuting with the $G(F)$-action. Then

$$\mathrm{Mant}(G,b,\mu)[\rho] = \sum_{\pi\in\Pi_\phi(G)} \pi \boxtimes \operatorname{Hom}_{S_\phi}(\delta_{\pi,\rho}, R_\mu \circ \varphi_E).$$

Let $H_0/\overline{\mathbb{F}}_p$ be a p-divisible group of height n and dimension m. The tower of local Shimura varieties $\mathrm{Sh}(G,b,\mu)$ coincides with the Rapoport–Zink tower with base $\mathcal{M}_{H_0,\eta}$, when $G = \mathrm{GL}_n$, when b is the isocrystal of slope m/n as above, and when $\mu = (1,\dots,1,0,\dots,0)$ with m 1s and $n-m$ 0s.

We explain the notation in the Kottwitz conjecture. The group G and G_b have the same Langlands dual group, so to the L-parameter φ there are associated L-packets $\Pi_\varphi(G)$ and $\Pi_\varphi(G_b)$. Let S_φ be the centralizer of $\varphi(W_F)$. Expected endoscopic character relations among the constituents of an L-packet allow for the definition of a pairing

$$\begin{aligned} \Pi_\varphi(G)\times\Pi_\varphi(G_b) &\longrightarrow \mathsf{Rep}_{S_\varphi} \\ (\pi,\rho) &\longmapsto \delta_{\pi,\rho}. \end{aligned}$$

Finally, R_μ is the algebraic representation of $\hat{G}$ with highest weight μ, and φ_E is the restriction of φ to W_E, so that $R_\mu \circ \varphi_E$ is a representation of $S_\varphi \times W_E$, and finally $\operatorname{Hom}_{S_\phi}(\delta_{\pi,\rho}, R_\mu\circ\varphi_E)$ is a representation of W_E. We remark that if $\Pi_\varphi(G) = \{\pi\}$ is a singleton, then the formula in the Kottwitz conjecture reduces to the simple identity

$$\mathrm{Mant}(G,b,\mu)[\rho] = \pi \boxtimes (R_\mu \circ \varphi_E).$$

2. p-divisible groups and shtukas over $\mathcal{O}_C$

The material in this section was adapted from [SW20], especially Lectures 12–14.

2.1. **The possibility of local shtuka spaces.** Given the successes of the Langlands program in the function field setting, we might dream of moduli spaces of shtukas over $\mathbb{Z}$, which specialize to the Shimura varieties in the case of one leg. Such a space would be fibered over products like

$$\text{“}\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}\times\cdots\times\operatorname{Spec}\mathbb{Z}\text{”}$$

where the product is considered over “the field with one element”. We do not yet know what this product is. Scholze's Berkeley lectures [SW20] rigorously define the completion of such a product at $(p,p,\dots,p)$, which is to say a sort of product:

$$\text{“}\operatorname{Spa}\mathbb{Z}_p\times\operatorname{Spa}\mathbb{Z}_p\times\cdots\times\operatorname{Spa}\mathbb{Z}_p\text{”}.$$

Here Spa refers to the adic spectrum. One of the main accomplishments of [SW20] is to define a moduli space of r-legged *mixed-characteristic local shtukas*

$$\operatorname{Sht}(G,b;\mu_1.\dots,\mu_r)$$

lying over the open subset of $\operatorname{Spa}\mathbb{Z}_p\times\operatorname{Spa}\mathbb{Z}_p\times\cdots\times\operatorname{Spa}\mathbb{Z}_p$:

$$\text{“}\operatorname{Spa}\mathbb{Q}_p\times\operatorname{Spa}\mathbb{Q}_p\times\cdots\times\operatorname{Spa}\mathbb{Q}_p\text{”}.$$

Here $G/\mathbb{Q}_p$ is a reductive group, $b\in B(G)$ is a σ-conjugacy class, and the μ_i are *arbitrary* conjugacy classes of cocharacters of $G_{\overline{\mathbb{Q}}_p}$. (These local shtukas have legs taking values in the generic point $\operatorname{Spa}\mathbb{Q}_p\subset\operatorname{Spa}\mathbb{Z}_p$. The final chapters of [SW20] are devoted to extending $\operatorname{Sht}(G,b;\mu_1,\dots,\mu_r)$ over the product spectrum of $\operatorname{Spa}\mathbb{Z}_p$; we do not discuss this here.) Similar self-products of the adic space $\operatorname{Spa}\mathbb{Q}_p$ appear in the construction by Fargues–Scholze [FS21] of the automorphic-to-Galois direction of local Langlands for an arbitrary reductive group $G/\mathbb{Q}_p$.

Our desiderata for $\operatorname{Sht}(G,b;\mu_1,\dots,\mu_r)$ are as follows:

(1) It should recover the Lubin–Tate tower in the case $G=\mathrm{GL}_n$, b isoclinic of slope $1/n$, $r=1$, and $\mu_1=(1,0,\dots,0)$.
(2) More generally it should recover the Rapoport–Zink spaces in situations relevant to p-divisible groups.
(3) More generally still, it should specialize to the local Shimura varieties of [RV14] in the case of one leg and μ_1 minuscule.
(4) Its cohomology (with suitable coefficients) should give representations of the product group $G(F)\times G_b(F)\times\prod_i W_{E_i}$, subject to a Kottwitz conjecture in the case of b basic. Here E_i is the field of definition of the conjugacy class μ_i.
(5) It should make contact with a geometric Langlands story concerning vector bundles on a curve.

Even if we were only interested in one-legged shtukas, we would be still be forced to consider products like $\operatorname{Spa}\mathbb{Q}_p\times\operatorname{Spa}\mathbb{Q}_p\times\cdots\times\operatorname{Spa}\mathbb{Q}_p$. For example, let C be an algebraically closed non-archimedean field containing $\mathbb{Q}_p$, with rig of integers $\mathcal{O}_C$. A local shtuka with one leg at $\operatorname{Spa}\mathcal{O}_C\to\operatorname{Spa}\mathbb{Z}_p$ should be a vector bundle $\mathcal{E}$ over the product

$$Y=\text{“}\operatorname{Spa}\mathbb{Z}_p\times\operatorname{Spa}\mathcal{O}_C\text{”}$$

together with an isomorphism

$$\phi_{\mathcal{E}}:\mathcal{E}|_{Y\backslash\Gamma}\xrightarrow{\sim}\phi_C^*\mathcal{E}|_{Y\backslash\Gamma}$$

defined away from the “graph” $\Gamma\subset Y$ of the map $\operatorname{Spa}\mathcal{O}_C\to\operatorname{Spa}\mathbb{Z}_p$. It is not immediately clear how to define the object Y, or how to define the Frobenius automorphism ϕ_C on it.

2.2. **Inspiration from equal characteristic local fields.** When we replace $\mathbb{Q}_p$ with an equal characteristic local field $F=\mathbb{F}_q(\!(\pi)\!)$, the way forward is clearer. We think of $\hat{X}=\operatorname{Spf}\mathcal{O}_F$ as the completion of a curve X at a point. For a formal scheme $S/\mathbb{F}_q$, we can form the product $\hat{X}\times S$ in the category of formal schemes over $\mathbb{F}_q$. From there we can give a definition of $\hat{X}$-shtuka over S.

We only make this precise in the case $S=\operatorname{Spf}\mathcal{O}_C$, where $C/\mathbb{F}_q$ is an algebraically closed non-archimedean field. In that case, we have

$$\hat{X}\times S\cong\operatorname{Spf}\mathcal{O}_C[\![\pi]\!].$$

The leg of an $\hat{X}$-shtuka will be a morphism $x : S \to \hat{X}$, i.e. a continuous homomorphism $\iota : \mathbb{F}_p[\![\varpi]\!] \to \mathcal{O}_C$. Let $\varpi = \iota(\pi)$, so that ϖ belongs to the maximal ideal of $\mathcal{O}_C$. Then the graph of x is the vanishing locus of the ideal $(x - \varpi)$ in $\mathcal{O}_C[\![\pi]\!]$. Let $\phi_C \in \operatorname{Aut} \mathcal{O}_C$ be the qth power Frobenius map; we extend ϕ_C to $\mathcal{O}_C[\![\pi]\!]$ by having it act trivially on π.

Definition 2.1. A (1-legged) $\hat{X}$-*shtuka* of rank n over S with leg x is a pair (M, ϕ_M), where:

- M is a free $\mathcal{O}_C[\![\pi]\!]$-module of finite rank.
- $\phi_M : M[\frac{1}{\pi-\varpi}] \xrightarrow{\sim} \phi_C^* M[\frac{1}{\pi-\varpi}]$ is an isomorphism of $\mathcal{O}_C[\![\pi]\!][\frac{1}{\pi-\varpi}]$-modules.

This kind of equal characteristic shtuka is discussed in [HS19]. (The theory of G-shtukas for a group scheme G over $\hat{X}$ is developed in [Vie18].) Such shtukas are closely related to π-divisible formal $\mathcal{O}_F$-module laws, as the following example demonstrates.

Example 2.2. Consider the $\hat{X}$-shtuka (M, ϕ_M), where M is a free $\mathcal{O}_C[\![\pi]\!]$-module of rank 1 with basis e, and $\phi_M(\alpha e) = \mathrm{Fr}_C(\alpha)(\pi - \varpi)^{-1} e$. This local shtuka corresponds to the Lubin–Tate formal $\mathcal{O}_F$-module law $\mathcal{G}$ over $\mathcal{O}_C$ defined by $\mathcal{G}(X, Y) = X + Y$ and $[\pi]_{\mathcal{G}}(X) = \varpi X + X^q$.

Consider now the subset $M^{\phi_M = 1}$ of elements fixed by ϕ_M. By inspection, every such element is of the form $\sum_{n \geqslant 0} \lambda_n \pi^n$, where $\lambda_0, \lambda_1, \dots \in C$ is a compatible system of roots of $[\pi]_{\mathcal{G}}(T), [\pi^2]_{\mathcal{G}}(T), \dots$. Let $T_\pi \mathcal{G} = \varinjlim_n \mathcal{G}[\pi^n]$ be the π-adic Tate module of $\mathcal{G}$. Then we have an isomorphism of $\mathbb{F}_q[\![\pi]\!]$-modules:

$$T_\pi \mathcal{G} \cong M^{\varphi_M = 1}.$$

2.3. **Fontaine's ring A_{inf}, and Breuil–Kisin–Fargues modules.** We now return to the mixed-characteristic setting, and the challenge of defining products like "$\operatorname{Spf} \mathbb{Z}_p \times \operatorname{Spf} \mathbb{Z}_p$". The naive interpretation $\operatorname{Spf} \mathbb{Z}_p \times \operatorname{Spf} \mathbb{Z}_p \cong \operatorname{Spf}(\mathbb{Z}_p \otimes \mathbb{Z}_p)$ (with the completed tensor product) returns $\operatorname{Spf} \mathbb{Z}_p$ again, whereas we were hoping for the product to be 2-dimensional. The problem of course is that $p \otimes 1 = 1 \otimes p$ in $\mathbb{Z}_p \otimes \mathbb{Z}_p$, and this is because p is not an indeterminant, but rather

$$p = 1 + \cdots + 1.$$

Somehow one wants an additional avatar of p which can act as a variable. This theme appears throughout p-adic Hodge theory, especially in [FF18]. The key to all is to use Witt vectors.

For an $\mathbb{F}_p$-algebra R, we propose the definition

$$\text{“}R \otimes \mathbb{Z}_p\text{”} = W(R)$$

where $W(R)$ is the ring of (p-crystal) Witt vectors. Each element of $W(R)$ may be represented by a power series in p:

$$\sum_{n=0}^{\infty} [a_n] p^n, \quad a_n \in R.$$

Here $[a_n] \in W(R)$ is the Teichmüller representative. The definition of $W(R)$ by $R \otimes \mathbb{Z}_p$ is reasonable because $W(R)$ admits natural maps $R \to W(R)$ (namely $a \mapsto [a]$) and $\mathbb{Z}_p \to W(R)$ (from $\mathbb{Z}_p \cong W(\mathbb{F}_p) \to W(R)$). The second of these maps is a ring homomorphism, while the first is only multiplicative.

Let C be an algebraically closed non-archimedean field containing $\mathbb{Q}_p$. To define "$C \otimes \mathbb{Z}_p$" we will convert C to a perfect field $C^\flat$ and then apply $W(R) = R \otimes \mathbb{Z}_p$. This conversion is the famous *tilting process*, defined by

$$C^\flat = \varprojlim_{x \mapsto x^p} C.$$

A priori this is only topological space (even a multiplicative monoid), but it can be given a ring structure where the addition law is

$$(x_0, x_1, \dots) + (y_0, y_1, \dots) = (z_0, z_1, \dots),$$

where

$$z_i = \lim_{n \to \infty} (x_{n+i} + y_{n+i})^{p^n}.$$

In fact, $C^\flat$ is an algebraically closed non-archimedean field of characteristic p, with the ring of integers

$$\mathcal{O}_{C^\flat} = \varprojlim_{x \mapsto x^p} \mathcal{O}_C / p$$

and pseudo-uniformizer $[p^\flat] = (p, p^{1/p}, p^{1/p^2}, \dots)$.

Definition 2.3. Fontaine's ring A_{inf} is the 2-dimensional local ring

$$A_{\mathrm{inf}} = \text{“}\mathcal{O}_C \otimes \mathbb{Z}_p\text{”} = W(\mathcal{O}_{C^\flat}).$$

We give A_{inf} the topology generated by the ideal $(p, [p]^\flat)$. Let $\phi \in \operatorname{Aut} A_{\mathrm{inf}}$ be the Frobenius automorphism $\phi([x]) = [x^p]$.

Let $a \mapsto a^\sharp$ denote the map $C^\flat \to C$ projecting a sequence onto its initial coordinate; thus for instance $[p^\flat]^\sharp = p$. Then there is a surjective ring homomorphism

$$\begin{aligned} \theta : A_{\mathrm{inf}} &\longrightarrow \mathcal{O}_C \\ \sum_{n\geqslant 0}[a_n]p^n &\longmapsto \sum_{n\geqslant 0} a_n^\sharp p^n. \end{aligned}$$

The kernel of θ is a principal ideal (ξ). One possible choice of generator is $\xi = p - [p^\flat]$. (An element of $\sum_{n\geqslant 0}[a_n]p^n$ is *primitive of degree* 1 if a_0 is a pseudo-uniformizer and a_1 is a unit. Any element of $\ker\theta$ which is primitive of degree 1 is a generator for that ideal. For proofs of these claims, see [Fon13].)

The vanishing locus of (ξ), considered as a formal subscheme of $\operatorname{Spf} A_{\mathrm{inf}} = \text{“}\operatorname{Spf}\mathbb{Z}_p \times \operatorname{Spf}\mathcal{O}_C\text{”}$, is the “graph” of $\operatorname{Spf}\mathcal{O}_C \to \operatorname{Spf}\mathbb{Z}_p$. In analogy with Definition 2.1, we are led to the following definition of a (one-legged) *mixed-characteristic shtuka* over $\mathcal{O}_C$, also known as a *Breuil–Kisin–Fargues module*.

Definition 2.4 (Fargues, [Far])**.** A *Breuil–Kisin–Fargues module* is a pair (M, ϕ_M), where M is a finite free A_{inf}-module, and

$$\phi_M : \phi^* M\left[\frac{1}{\phi(\xi)}\right] \xrightarrow{\sim} M\left[\frac{1}{\phi(\xi)}\right]$$

is an isomorphism. Here ξ generates the kernel of $\theta : A_{\mathrm{inf}} \to \mathcal{O}_C$.

Remark 2.5. The appearance of $\phi_C(\xi)$ instead of ξ is necessary for the comparison theorems of integral p-adic Hodge theory [BMS18]. Also note that ϕ_M may be recast as a ϕ-linear bijection $M[\frac{1}{\xi}] \xrightarrow{\sim} M[\frac{1}{\phi(\xi)}]$.

Note that a Breuil-Kisin-Fargues module (M, ϕ_M) has the following invariants:

- (The crystalline realization) After base changing (M, ϕ_M) along $A_{\mathrm{inf}} \to W(k)[\frac{1}{p}]$, we obtain an isocrystal.
- (The étale realization) After base changing (M, ϕ_M) along $A_{\mathrm{inf}} \to W(k)[\frac{1}{p}]$, we obtain a pair (N, ϕ_N), where N is a fine free $W(C^\flat)$-module, and $\phi_N : \phi^* N \xrightarrow{\sim} N$ is an isomorphism. By Artin–Schreier–Witt theory, these are in equivalence with finite free $\mathbb{Z}_p$-modules, via the functor $N \mapsto N^{\phi_N = 1}$ and its quasi-inverse $T \mapsto (T \otimes_{\mathbb{Z}_p} W(C^\flat), 1 \otimes \phi)$.

As with Drinfeld shtukas, we can measure the failure of the ϕ_M in a Breuil–Kisin–Fargues module to be an isomorphism near the leg by means of a tuple of weights.

Definition 2.6. Let B_{dR}^+ be the ξ-adic completion of $A_{\mathrm{inf}}[\frac{1}{p}]$, and let $B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[\frac{1}{\xi}]$. Then B_{dR}^+ is a complete discrete valuation ring with uniformizer ξ, residue field C, and fraction field B_{dR}. The automorphism ϕ of A_{inf} allows us to identify the $\phi^n(\xi)$-adic completion of $A_{\mathrm{inf}}[\frac{1}{p}]$ with B_{dR}^+, for any $n \in \mathbb{Z}$.

Given a Breuil–Kisin–Fargues module (M, φ_M), base change along $A_{\mathrm{inf}} \to B_{\mathrm{dR}}^+$ (completion at $\phi(\xi)$) produces two B_{dR}^+-modules which get identified over B_{dR}, and isomorphism classes of such data are in bijection with tuples of weights $\mu = (k_1 \geqslant \cdots \geqslant k_n)$. We can thus talk about a Breuil–Kisin–Fargues module being of type μ, or bounded by μ.

2.4. **The Fargues–Fontaine curve $\mathcal{X}_{\mathrm{FF},C}$.** We have posited that the concept of a “one-legged $\mathbb{Z}_p$-shtuka over $\mathcal{O}_C$” might be reasonably interpreted as a Breuil–Kisin–Fargues module. The latter is an A_{inf}-module — which may be interpreted as a vector bundle on $\operatorname{Spec} A_{\mathrm{inf}}$ — together with a ϕ-linear operator which is an isomorphism away from a single point $x_C \in \operatorname{Spec} A_{\mathrm{inf}}$, where x_C is the vanishing locus of ξ.

Our goal in this section is do define an object like “$(\operatorname{Spec} A_{\mathrm{inf}})/\phi^{\mathbb{Z}}$”, and then in retrospect interpret $\mathbb{Z}_p$-shtukas in terms of objects (more precisely, modifications of vector bundles) living on this quotient. This shift in perspective is necessary to connect $\mathbb{Z}_p$-shtukas to something resembling geometric

Langlands — in fact this quotient will play the role of the curve in the geometric Langlands. We are referring here to the Fargues–Fontaine curve introduced in [FF18].

Nothing so naive as a scheme quotient "$(\operatorname{Spec} A_{\text{inf}})/\phi^{\mathbb{Z}}$" can exist. The automorphism ϕ fixes the prime ideals (p) and $\mathfrak{m}_C$ (where $\mathfrak{m}_C$ is the maximal ideal of $\mathcal{O}_C$), suggesting that we should discard those primes and try instead to construct "$(\operatorname{Spec} A_{\text{inf}}[\frac{1}{p}, \frac{1}{[p^\flat]}])/\phi^{\mathbb{Z}}$". But this cannot be well-behaved either, as the $\phi^{\mathbb{Z}}$ orbit of x_C is dense. Equivalently, there is no nonzero $f \in A_{\text{inf}}$ with a zero at $\phi^n(x_C)$ for all $n \in \mathbb{Z}$. The point is illustrated by the following example.

Example 2.7. Let $\epsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}_{C^\flat}$ be a sequence of primitive pth power roots of 1. Then $\theta(\phi^n(\epsilon)) = 1$ for all $n \geqslant 0$, and therefore $[\epsilon] - 1 \in A_{\text{inf}}$ has a zero at $\phi^n(x_C)$ for all $n \in \mathbb{Z}$, we might try to form the logarithm

$$\log[\epsilon] = \lim_{n\to\infty} \frac{[\epsilon]^{p^n} - 1}{p^n},$$

the idea being that $\log \zeta_{p^n} = 0$. The problem of course is that the limit above does not make sense in $A_{\text{inf}}[\frac{1}{p}, \frac{1}{[p^\flat]}]$.

We note here that

$$\xi = \frac{[\epsilon] - 1}{[\epsilon^{1/p}] - 1} = 1 + [\epsilon^{1/p}] + \cdots + [\epsilon^{(p-1)/p}]$$

is a primitive element of degree 1, with exactly one zero at x_C; this is sometimes a more convenient choice for ξ.

The above example demonstrate the need to pass to an analytic setting where limits like $\log[\epsilon]$ make sense. In [FF18] we encounter many extension rings of A_{inf} consisting of "fonctions holomorphes de la variable p". It is convenient to express some of these rings in terms of adic spaces. We refer the reader to our volume in [BCKW19] for an introduction to adic spaces. For the moment we recall that points of $\operatorname{Spa} A_{\text{inf}}$ are continuous valuations on A_{inf} valued in an ordered abelian group.

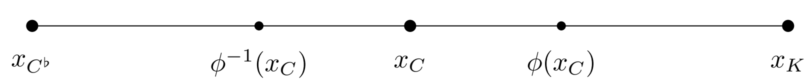
We give names to four special points of $\operatorname{Spa} A_{\text{inf}}$, labeled by their residue fields:

- x_k, the unique non-analytic point, which corresponds to $A_{\text{inf}} \to k$, where k is the residue field of C.
- $x_{C^\flat}$, which corresponds to $A_{\text{inf}} \to \mathcal{O}_{C^\flat} \to \mathbb{C}^\flat$ (the étale point).
- x_C, which corresponds to $A_{\text{inf}} \xrightarrow{\theta} \mathcal{O}_C \to C$.
- x_K, which corresponds to $A_{\text{inf}} \to W(k) \to W(k)[\frac{1}{p}] = K$ (the crystalline point).

For each r in the interval $(0, \infty)$ we also define a point η_r in $\operatorname{Spa} A_{\text{inf}}$ as the Gauss norm:

$$\left| \sum_{n\geqslant 0} [x_n] p^n \right|_r = \sup_n \frac{|x_n|^r}{p}.$$

The complement of x_k in $\operatorname{Spa} A_{\text{inf}}$ is an analytic space over $\operatorname{Spa} \mathbb{Z}_p$. This locus can be pictured as lying along a continuum from $x_{C^\flat}$ to x_K, like this:



As a justification for the picture, we note there exists a unique continuous surjective map

$$\kappa : (\operatorname{Spa} A_{\text{inf}}) \backslash \{x_k\} \longrightarrow [0, \infty]$$

which takes the value

$$\kappa(|\cdot|) = \frac{\log |[p^\flat]|}{\log |p|}$$

on all real-valued continuous valuations $|\cdot|$. The map κ carries the special points $x_{C^\flat}, x_C, x_K$ to $0, 1, \infty$, respectively, and it carries η_r to r. It transforms under ϕ this way: $\kappa(\phi(x)) = p\kappa(x)$. Thus $x_{C^\flat}$ (respectively, x_K) is a repulsive (respectively, attractive) fixed point of ϕ. Removing those fixed points leaves an adic space on which the action of ϕ is properly discontinuous.

Definition 2.8. Let

$$\begin{aligned} \mathcal{Y}_{\text{FF}} &= (\operatorname{Spa} A_{\text{inf}}) \backslash \{|p[p^\flat]| = 0\} \\ &= (\operatorname{Spa} A_{\text{inf}}) \backslash \{x_k, x_{C^\flat}, x_K\}. \end{aligned}$$

Then we can form the quotient

$$\mathcal{X}_{\mathrm{FF}} = \mathcal{Y}_{\mathrm{FF}}/\phi^{\mathbb{Z}}.$$

This is the (absolute) *adic Fargues–Fontaine curve*. It is an analytic adic space over $\mathbb{Q}_p$. We keep the notation x_C for the image of that point in $\mathcal{X}_{\mathrm{FF}}$.

Of special interest is the ring of analytic functions defined on $\mathcal{Y}_{\mathrm{FF}}$:

$$B^+ = H^0(\mathcal{Y}_{\mathrm{FF}}, \mathcal{O}_{\mathcal{Y}_{\mathrm{FF}}}).$$

It is the Fréchet completion of $A_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[p^\flat]}]$ with respect to the Gauss norms η_r for all $r \in (0,\infty)$. That is, a sequence in $A_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[p^\flat]}]$ converges in B^+ if it converges in all η_r. Note that ϕ extends to an automorphism of B^+ which we continue to call ϕ.

With these definitions, the limit

$$\log[\epsilon] = \lim_{n\to\infty} \frac{[\epsilon]^{p^n} - 1}{p^n}$$

converges to an element $t \in B^+$ which has a simple zero at each $\phi^n(x_C)$ for $n \in \mathbb{Z}$ and which satisfies $\phi(t) = pt$. Now that we have constructed the Fargues–Fontaine curve $\mathcal{X}_{\mathrm{FF}}$, we turn our attention towards its theory of vector bundles, as evidently these are related to Breuil–Kisin–Fargues modules. A description of the category of vector bundles on $\mathcal{X}_{\mathrm{FF}}$ is one of the main results of [FF18]. To state it, let us define a functor

$$\begin{aligned} (\text{Isocrystals}) &\longrightarrow (\text{Vector bundles on } \mathcal{X}_{\mathrm{FF}}) \\ (N,\sigma_N) &\longmapsto \mathcal{E}(N,\sigma_N), \end{aligned}$$

where $\mathcal{E}(N,\sigma_N)$ denotes the descent of $N \otimes_K \mathcal{O}_{\mathcal{Y}_{\mathrm{FF}}}$ via $\sigma_N \otimes \phi$. The global sections of $\mathcal{E}(N,\sigma_N)$ are:

$$H^0(\mathcal{X}_{\mathrm{FF}}, \mathcal{E}(N,\sigma_N)) = (N \otimes_K B^+)^{\sigma_N\otimes\phi}.$$

If (N,σ_N) is the basic isocrystal with slope $\lambda = m/n \in \mathbb{Q}$ as in Page 8, it is customary to notate $\mathcal{E}(N,\sigma_N)$ by $\mathcal{O}(-\lambda)$. Thus for instance the element t belongs to $H^0(\mathcal{X}_{\mathrm{FF}}, \mathcal{O}(1))$ because $\phi(t) = pt$.

Theorem 2.9. *Every vector bundle on $\mathcal{X}_{\mathrm{FF}}$ is isomorphic to $\mathcal{E}(N,\sigma_N)$ for some isocrystal (N,σ_N).*

It is by no means the case that $(N,\sigma_N) \mapsto \mathcal{E}(N,\sigma_N)$ is an equivalence of categories. The element t represents a morphism of vector bundles $\mathcal{O} \to \mathcal{O}(1)$ which does not arise from a morphism of isocrystals. It is the case however that $\mathrm{Aut}(N,\sigma_N) \to \mathrm{Aut}\,\mathcal{E}(N,\sigma_N)$ is an isomorphism when (N,σ_N) is basic.

Theorem 2.9 admits the following generalization.

Theorem 2.10 (c.f. [Far20])**.** *Let $G/\mathbb{Q}_p$ be a reductive group. There is a natural bijection $b \mapsto \mathcal{E}^b$ between the Kottwitz set $B(G)$ and the set of isomorphism classes of G-torsors over $\mathcal{X}_{\mathrm{FF}}$.*

Theorem 2.11 (Fargues)**.** *The following categories are equivalent:*

(1) *Breuil–Kisin–Fargues modules (M,ϕ_M) with a leg at $\phi^{-1}(x_C)$, that is, M is a finite free A_{inf}-module, and $\phi_M : \phi^* M[\frac{1}{\phi(\xi)}] \to M[\frac{1}{\phi(\xi)}]$.*
(2) *Pairs (T,Ξ), where*
   - *T is a free $\mathbb{Z}_p$-module of finite rank,*
   - *$\Xi \subset T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$ is a B_{dR}^+-lattice.*
(3) *Triples $(T,\mathcal{E},\beta)$, where*
   - *T is a free $\mathbb{Z}_p$-module of finite rank,*
   - *$\mathcal{E}$ is a vector bundle on $\mathcal{X}_{\mathrm{FF}}$,*
   - *$\beta : T \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{X}_{\mathrm{FF}}\setminus\{x_C\}} \xrightarrow{\sim} \mathcal{E}_{\mathcal{X}_{\mathrm{FF}}\setminus\{x_C\}}$ is an isomorphism of vector bundles which is meromorphic at x_C.*

Under these equivalences, T is the étale realization of M, and $\mathcal{E} \cong \mathcal{E}(N,\sigma_N)$, where (N,σ_N) is the crystalline realization of M.

The proof is given in [SW20, Lecture 14]; we give a brief discussion. Suppose we have a Breuil–Kisin–Fargues module (M,ϕ_M) with leg at $\phi^{-1}(x_C)$. We are supposed to produce a $\mathbb{Z}_p$-module T and a vector bundle $\mathcal{E}$. In brief, the trivial vector bundle $T \otimes \mathcal{O}_{\mathcal{X}_{\mathrm{FF}}}$ will appear to the left of x_C in our picture, and the vector bundle $\mathcal{E}$ will appear to the right of x_C. Throughout, we think of M as as vector bundle on $\mathrm{Spa}\,A_{\mathrm{inf}}$.

First we consider the part of the diagram to the left of x_C. The completed local ring of $\operatorname{Spa} A_{\inf}\backslash\{x_k\}$ at $x_{C^\flat}$ is $W(C^\flat)$. As we have seen, the étale realization $T=(M\otimes_{A_{\inf}}W(C^\flat))^{\phi_M=1}$ gives a trivialization of the ϕ=module $M\otimes_{A_{\inf}}W(C^\flat)$, in the sense that there is an isomorphism of ϕ-modules $M\otimes_{A_{\inf}}W(C^\flat)\cong T\otimes_{\mathbb{Z}_p}W(C^\flat)$. This trivialization extends some small distance to the right of $x_{C^\flat}$, but then using ϕ_M, we can extend the trivialization to all of $\mathcal{Y}_{\mathrm{FF}}$, excluding the points $x_C,\phi(x_C),\dots$:

$$\iota\colon M|_{\mathcal{Y}_{\mathrm{FF}}\backslash\bigcup_{n\geqslant 0}\phi^n(x_C)}\xrightarrow{\sim}T\otimes\mathcal{O}_{\mathcal{Y}_{\mathrm{FF}}\backslash\bigcup_{n\geqslant 0}\phi^n(x_C)}.$$

Localizing at x_C, we obtain an isomorphism

$$\iota_{x_C}\colon M\otimes_{A_{\inf}}B_{\mathrm{dR}}\cong T\otimes_{\mathbb{Z}_p}B_{\mathrm{dR}}.$$

Let Ξ be the image of $M\otimes_{A_{\inf}}B_{\mathrm{dR}}$ under ι_{x_C}. Then (T,Ξ) is the pair in Theorem 2.11(2).

On the right side of the diagram, (M,ϕ_M) descends to a vector bundle $\mathcal{E}$ on $\mathcal{X}_{\mathrm{FF}}$. We can now descend ι to $\mathcal{X}_{\mathrm{FF}}$ to obtain an isomorphism between $T\otimes_{\mathbb{Z}_p}\mathcal{O}_{\mathcal{X}_{\mathrm{FF}}}$ and $\mathcal{E}$ away from x_C; this is the triple $(T,\mathcal{E},\beta)$ in Theorem 2.11(3). Note that the completed local ring of $\mathcal{X}_{\mathrm{FF}}$ at x_C is once again B_{dR}^+, and we can recover Ξ from $(T,\mathcal{E},\beta)$ as the preimage under β of the completed stalk $\hat{\mathcal{E}}_{x_C}$.

2.5. **Example: The Tate twist.** We illustrate Theorem 2.11 with an important example. Let $\epsilon=(1,\zeta_p,\zeta_{p^2},\dots)\in\mathcal{O}_{C^\flat}$ and $\xi=([\epsilon]-1)([\epsilon^{1/p}]-1)$ be as in Example 2.7.

In the setting of Breuil–Kisin–Fargues modules, the *Tate twist* $A_{\inf}\{1\}$ is the pair (M,ϕ_M), where M is a free $A_{\inf}$ module of rank 1 with basis element e_M, and $\phi_M(e_M)=\frac{1}{\phi(\xi)}e_M$. This is a Breuil–Kisin–Fargues module of type (-1). Its realizations are as follows:

(1) The crystalline realization is the pair (N,σ_N), where $N=Ke_N$ and $\sigma_N(e_N)=\frac{1}{p}e_N$; i.e. it is the 1-dimensional isocrystal of slope -1.
(2) The étale realization is $T=(M\otimes_{A_{\inf}}W(C^\flat))^{\phi_M=1}$; this is a free $\mathbb{Z}_p$-module of rank 1 generated by the element $([\epsilon]-1)e_M$.

We claim that the equivalences in Theorem 2.11 relate the following objects:

- The Tate twist $\mathbb{A}_{\inf}\{1\}$,
- The pair $(\mathbb{Z}_p,\xi^{-1}B_{\mathrm{dR}}^+)$,
- The triple $(\mathbb{Z}_p,\mathcal{O}(1),\beta)$, where $\beta\colon\mathcal{O}\to\mathcal{O}(1)$ corresponds to the element t.

On the étale side, the natural inclusion $T=\mathbb{Z}_p([\epsilon]-1)\to M$ induces a map of ϕ-modules

$$\mathcal{O}_{\mathcal{Y}}\xrightarrow{[\epsilon]-1}M|_{\mathcal{Y}}$$

with simple zeros at $x_C,\phi(x_C),\phi^2(x_C),\dots$, and an isomorphism everywhere else. Combining the étale and crystalline sides, we arrive at a map of ϕ-modules

$$\mathcal{O}_{\mathcal{Y}_{\mathrm{FF}}}\xrightarrow{t}N\otimes\mathcal{O}_{\mathcal{Y}_{\mathrm{FF}}}$$

with a simple zero at each point of $\phi^{\mathbb{Z}}(x_C)$. This map descends to the map $\beta\colon\mathcal{O}\to\mathcal{O}(1)$ in term (3) above.

2.6. **p-divisible groups over $\mathcal{O}_C$.** So far we have seen that a (one-legged) $\mathbb{Z}_p$-shtuka over $\mathcal{O}_C$ is a Breuil–Kisin–Fargues module, and that these are in equivalence with modifications of a trivial vector bundle on the Fargues–Fontaine curve $\mathcal{X}_{\mathrm{FF}}$ at its special point x_C. It will now be instructive to connect this story with p-divisible groups.

For a p-divisible group $H/\mathcal{O}_C$ of height h and dimension d, we have the following three naturally associated linear algebra objects:

(1) The Dieudonné module $D(H_k)$ of the special fiber.
(2) The Tate module $T_pH=\varinjlim H[p^n]$, a free $\mathbb{Z}_p$-module of rank h.
(3) The Lie algebra $\operatorname{Lie}H\otimes_{\mathcal{O}_C}C$, a C-vector space of dimension d.

Remark 2.12. In case $H=A[p^\infty]$ for an abelian scheme $A/\mathcal{O}_C$, these three objects have cohomological interpretations:

(1) $D(H_k)=H^1_{\mathrm{cris}}(A_k/W)$ is crystalline cohomology.
(2) $T_pH=H^1_{\mathrm{\acute{e}t}}(A,\mathbb{Z}_\ell)^*$ is the dual of étale cohomology.
(3) $\operatorname{Lie}H\otimes_{\mathcal{O}_C}C$ is the dual of $H^1(A,\Omega^1_{A_C/C})$, which appears in the Hodge filtration of $H^1_{\mathrm{dR}}(A_C/C)$.

The objects (2) and (3) above are related by means of the following theorem.

Theorem 2.13 (The Hodge–Tate exact sequence, [Far08])**.** *There is a natural short exact sequence*

$$0 \longrightarrow \operatorname{Lie} H \otimes_{\mathcal{O}_C} C(1) \xrightarrow{\alpha^*_{H^*}} T_p H \otimes_{\mathbb{Z}_p} C \xrightarrow{\alpha_H} (\operatorname{Lie} H^*)^* \otimes_{\mathcal{O}_C} C \to 0.$$

Let us give the construction of α_H: An element $t \in T_p H$ is exactly the same as a homomorphism of p-divisible groups $(\mathbb{Q}_p/\mathbb{Z}_p)_{\mathcal{O}_C} \to H$, whose Cartier dual is a homomorphism $H^* \to \mu_{p^\infty,\mathcal{O}_C}$, whose derivative is a linear map $\operatorname{Lie} H^* \to \operatorname{Lie} \mu_{p^\infty,\mathcal{O}_C} = \mathcal{O}_C$, i.e. an element of $(\operatorname{Lie} H^*)^*$. This is $\alpha_H(t)$.

Theorem 2.14 (c.f. [SW13])**.** *The functor*

$$H \longmapsto (T_p H, \operatorname{Lie} H \otimes_{\mathcal{O}_C} C(1))$$

is an equivalence between the categories:

- *p-divisible groups H over $\mathcal{O}_C$;*
- *Pairs (T, W), where T is a finite free $\mathbb{Z}_p$-module, and $W \subset T \otimes_{\mathbb{Z}_p} C$ is a C-subspace.*

A natural question arises: Can we determine the special fiber H_k (or at least its isocrystal) directly from the pair (T, W)? The (far from explicit) answer runs through Theorem 2.11. Such pairs (T, W) are in bijection with pairs (T, Ξ), where Ξ is a lattice lying between $T \otimes_{\mathbb{Z}_p} B^+_{\mathrm{dR}}$ and $\xi^{-1}(T \otimes_{\mathbb{Z}_p} B^+_{\mathrm{dR}})$. Theorem 2.11 now supplies a Breuil–Kisin–Fargues module, whose crystalline realization is the (covariant) isocrystal of H_k.

We can assume the situation this way:

Theorem 2.15. *The following categories are equivalent:*

1. *p-divisible groups H over $\mathcal{O}_C$.*
2. *Breuil–Kisin–Fargues modules (M, ϕ_M) with a leg at $\phi^{-1}(x_C)$ of type $(-1, \dots, -1, 0, \dots, 0)$.*

Compare to Theorem 1.2. (In contrast to that situation, in Theorem 2.15 the numbers of (-1)s and 0s need not be equal, and there is no Riemann form.)

There is a wider story of Breuil–Kisin–Fargues modules associated to proper smooth formal schemes over $\mathcal{O}_C$, see [BMS18].

3. Perfectoid spaces, diamond, and the geometric of $\operatorname{Spa} \mathbb{Q}_p$

For a complete algebraically closed field $C/\mathbb{Q}_p$, we have seen how to give meaning to the product "$\operatorname{Spa} \mathbb{Z}_p \times \operatorname{Spa} \mathcal{O}_C$": this is the adic space $\operatorname{Spa} W(\mathcal{O}_{C^\flat})$. It is not clear how to extend this definition to the self-product "$\operatorname{Spa} \mathbb{Z}_p \times \operatorname{Spa} \mathbb{Z}_p$", at least for the reason that there is no tilted version of $\mathbb{Z}_p$ that produces anything interesting. One can however pass from $\mathbb{Z}_p$ to a larger ring like $\mathbb{Z}_p^{\mathrm{cycl}}$, which is a ring obtained from $\mathbb{Z}_p$ by adjoining all pth power roots of 1 and completing), which does have a tilt, so long as one remembers the action of the Galois group. This is roughly the idea behind Scholze's category of diamonds, to which such self-products belong.

3.1. A crash course in perfectoid spaces.

We quickly review some notions from [Sch12].

Definition 3.1. Let A be a topological ring. A is a *perfectoid ring* if the following conditions hold:

1. A is a Huber ring, meaning that it contains an open subring A^+ and a finitely-generated ideal $I \subset A^+$ such that $\{I^n\}$ forms a basis of neighborhoods of 0.
2. A is Tate, meaning it contains a pseudo-uniformizer (a topologically nilpotent unit).
3. A is uniform, meaning that the subset of power-bounded elements $A^\circ \subset A$ is bounded.
4. There exists a pseudo-uniformizer ϖ such that $\varpi^p \mid p$ in A°, and such that the pth power map $A/\varpi \to A/\varpi^p$ is an isomorphism.

A non-archimedean field which is a perfectoid ring is called a *perfectoid field*.

In characteristic p, a perfectoid field is exactly the same as a perfect non-archimedean field.

Theorem 3.2. *Let (A, A^+) be a Huber pair, with A perfectoid. Then (A, A^+) is sheafy, so that $X = \operatorname{Spa}(A, A^+)$ is an adic space. Furthermore, $\mathcal{O}_X(U)$ is a perfectoid ring for every rational subset $U \subset X$.*

Definition 3.3. A *perfectoid space* is an adic space which is covered by affinoids of the form $\operatorname{Spa}(A, A^+)$, where A is perfectoid.

The tilting operation we discussed in §2.3 extends to perfectoid spaces. For a perfectoid ring A with pseudo-uniformizer ϖ, we define its tilt by

$$A^\flat = \left(\varprojlim_{x\mapsto x^p} A/\varpi\right)\left[\frac{1}{\varpi^\flat}\right],$$

where $\varpi^\flat = (\varpi, \varpi^{1/p}, \dots)$ is a system of pth power roots modulo ϖ. Then $A^\flat$ is a perfectoid ring of characteristic p. The tilting construction globalizes to give a functor $S \mapsto S^\flat$ from all perfectoid spaces to perfectoid spaces in characteristic p.

Theorem 3.4. *Let S be a perfectoid space. The categories of perfectoid spaces over S and $S^\flat$ are equivalent, via $X \mapsto X^\flat$.*

Definition 3.5. A morphism $f : X \to Y$ of perfectoid spaces is *pro-étale* if locally on X it is of the form $\operatorname{Spa}(A_\infty, A_\infty^+) \to \operatorname{Spa}(A, A^+)$, where A and A_∞ are perfectoid rings, and

$$(A_\infty, A_\infty^+) = \left(\varinjlim_i (A_i, A_i^+)\right)^\wedge$$

is a filtered colimit of pairs (A_i, A_i^+), such that $\operatorname{Spa}(A_i, A_i^+) \to \operatorname{Spa}(A, A^+)$ is étale.

Definition 3.6. Consider the category Perf of perfectoid spaces of characteristic p. We endow this with the structure of a site by declaring that a collection of morphisms $\{f_i : X_i \to X\}$ is a covering (a *pro-étale cover*) if the f_i are pro-étale, and if for all quasi-compact open $U \subset X$, there exists a finite subset $I_U \subset I$, and a quasi-compact open $U_i \subset X$ for $i \in I_U$, such that

$$U = \bigcup_{i\in I_U} f_i(U_i).$$

If X is a perfectoid space of characteristic p, we have the representable presheaf h_X defined by $h_X(Y) = \operatorname{Hom}(Y, X)$.

Proposition 3.7 (c.f. [SW20, Proposition 8.2.7])**.** *The presheaf h_X is a sheaf.*

The functor $X \mapsto h_X$ exhibits Perf as a full subcategory of the category of sheaves on Perf, so henceforth we will confuse X with the sheaf on Perf that it represents.

3.2. **Untilts and the functor $X \mapsto X^\diamondsuit$.** For a perfectoid space S of characteristic p, an *untilt* of S is a pair $(S^\sharp, \iota)$, where $S^\sharp$ is a perfectoid space and $\iota : S^{\sharp\flat} \to S$ is an isomorphism. (We often leave off ι when referring to the pair.) We are interested in the question of describing all untilts of S.

On the other hand, there is an explicit description of untilts in terms of Witt vectors. Suppose $S = \operatorname{Spa}(R, R^+)$ is affinoid. Then we have the Witt ring $W(R^\circ)$. A *primitive ideal of degree* 1 in $W(R^\circ)$ is a principal ideal generated by an element of the form $\xi = \sum_{n=0}^\infty [x_n]p^n$, where $x_0 \in R$ is topologically nilpotent and $x_1 \in R^\circ$ is a unit.

Theorem 3.8 (c.f. [Fon13])**.** *We obtain the following bijection:*

$$\begin{aligned}(\textit{Ideals in } W(R^\circ) \textit{ which are primitive of degree } 1) &\longleftrightarrow (\textit{Untilts of } R)\\ I &\longmapsto (W(R^\circ)/I)[\tfrac{1}{p}].\end{aligned}$$

On the other hand, we might want to consider the functor of untilts as a geometric object in its own right.

Definition 3.9. Let X be an analytic adic space over $\operatorname{Spa}\mathbb{Z}_p$. Define a presheaf $X^\diamondsuit$ on Perf as follows. For a perfectoid space S in characteristic p, let $X^\diamondsuit(S)$ be the set of isomorphism classes of pairs $(S^\sharp, S^\sharp \to X)$, where $S^\sharp$ is an untilt of S, and $S^\sharp \to X$ is a morphism.

Theorem 3.10. *$X^\diamondsuit$ is a pro-étale sheaf.*

A special case is $\operatorname{Spd}\mathbb{Z}_p = (\operatorname{Spa}\mathbb{Z}_p)^\circ$, the functor of all untilts, and its subfunctor $\operatorname{Spd}\mathbb{Q}_p = (\operatorname{Spa}\mathbb{Q}_p)^\diamondsuit$, the functor of untilts to characteristic 0. If X is already a perfectoid space (of whatever characteristic), then $X^\diamondsuit = X^\flat$ by Theorem 3.4.

Theorem 3.11. *As sheaves on* Perf *we have an isomorphism*

$$\operatorname{Spd}\mathbb{Q}_p \cong \operatorname{Spa}\mathbb{Q}_p^{\mathrm{cycl},\flat}/\underline{\mathbb{Z}}_p^\times,$$

where the constant sheaf of groups $\underline{\mathbb{Z}}_p^\times$ acts on $\operatorname{Spa}\mathbb{Q}_p^{\mathrm{cycl},\flat}$ through the canonical isomorphism $\mathbb{Z}_p^\times \cong \operatorname{Gal}(\mathbb{Q}_p^{\mathrm{cycl}}/\mathbb{Q}_p)$.

Proof. Suppose $S^\sharp \to \operatorname{Spa}\mathbb{Q}_p$ is an untilt of S to characteristic 0, then the base change $\tilde{S}^\sharp = S^\sharp \times_{\operatorname{Spa}\mathbb{Q}_p} \operatorname{Spa}\mathbb{Q}_p^{\mathrm{cycl}}$ is a pro-étale $\underline{\mathbb{Z}}_p^\times$-torsor of perfectoid spaces over $S^\sharp$, with tilt $S \to S$; this last object constitutes a section of $\operatorname{Spa}\mathbb{Q}_p^{\mathrm{cycl},\flat}/\underline{\mathbb{Z}}_p^\times$ over S.

Conversely, suppose we are given a pro-étale $\underline{\mathbb{Z}}_p^\times$-torsor $S \to S$ and a morphism $\tilde{S} \to \operatorname{Spa}\mathbb{Q}_p^{\mathrm{cycl},\flat}$. Then again by Theorem 3.4, we get an untilt $\tilde{S}^\sharp \to \operatorname{Spa}\mathbb{Q}_p^{\mathrm{cycl}}$, equipped with a descent datum to $\operatorname{Spa}\mathbb{Q}_p$. By Theorem 3.10, the descent datum is effective, so we obtain an untilt $S^\sharp$ over $\operatorname{Spa}\mathbb{Q}_p$ as required. □

Theorem 3.11 prompts the following definition.

Definition 3.12. A *diamond* is a sheaf on Perf of the form X/R, where X is an object of Perf, and $R \subset X \times X$ is a pro-étale equivalence relation.

Theorem 3.11 shows that $\operatorname{Spd}\mathbb{Q}_p$ is a diamond. (The sheaf $\operatorname{Spd}\mathbb{Z}_p$ is not a diamond however; the argument in the proof of Theorem 3.11 breaks down because $\mathbb{Z}_p^\times$ does not act freely on the sheaf $\operatorname{Spa}\mathbb{Z}_p^{\mathrm{cycl},\flat}$.) More generally:

Theorem 3.13. *Let X be an analytic adic space over $\operatorname{Spa}\mathbb{Z}_p$. Then $X^\diamondsuit$ is a diamond. The functor $X \mapsto X^\diamondsuit$ is fully faithful when restricted to the category of seminormal rigid-analytic spaces.*

3.3. **The relative Fargues-Fontaine curve.**

Definition 3.14. Let $S = \operatorname{Spa}(R, R^+)$ be an affinoid perfectoid space in characteristic p, with pseudo-uniformizer ϖ. The adic space $\mathcal{Y}_S$ is defined as

$$\mathcal{Y}_{\mathrm{FF},S} = (\operatorname{Spa} W(R^+))\backslash\{|p[\varpi]| = 0\}.$$

Let ϕ denote the automorphism of $\mathcal{Y}_{\mathrm{FF},S}$ induced by the Frobenius on R. The relative Fargues–Fontaine curve is

$$\mathcal{X}_{\mathrm{FF},S} = \mathcal{Y}_{\mathrm{FF},S}/\phi.$$

Gluing, we may define $\mathcal{X}_{\mathrm{FF},S}$ for any object S of Perf.

There is a bijection $S^\sharp \mapsto D_{S^\sharp}$ between the following two sets:

- Untilts $S^\sharp$ of S over $\mathbb{Q}_p$,
- Cartier divisors of $\mathcal{Y}_{\mathrm{FF},S}$ of degree 1.

We also note the relation:

$$(\operatorname{Spa} W(R^+)\backslash\{|[\varpi]| = 0\})^\diamondsuit \cong \operatorname{Spd}\mathbb{Z}_p \times S.$$

A morphism $x : S \to \operatorname{Spd}\mathbb{Q}_p$ is an untilt $S^\sharp$ over $\mathbb{Q}_p$. We may now interpret the "graph" of x as the divisor $D_{S^\sharp}$ of $\mathcal{Y}_{\mathrm{FF},S}$. Despite 3.14 there is no morphism of adic spaces from $\mathcal{Y}_{\mathrm{FF},S}$ or $\mathcal{X}_{\mathrm{FF},S}$ to S

Our aim is to define, for an object S of Perf, a "mixed-characteristic local shtuka over S" with leges at finitely many morphisms $x_i : S \to \operatorname{Spd}\mathbb{Q}_p$. For simplicity let us discuss the case of one leg x, corresponding to an untilt $S^\sharp$. In light of 3.14, such a shtuka should be a pair $(\mathcal{M}, \phi_{\mathcal{M}})$, where $\mathcal{M}$ is a vector bundle on $\operatorname{Spa} W(R^+)\backslash\{|[\varpi]| = 0\}$, and $\phi_{\mathcal{M}}$ is a ϕ-linear automorphism of $\mathcal{M}$ away from the divisor $D_{S^\sharp}$.

By an appropriate generalization of Theorem 2.11, such a pair corresponds to a modification of vector bundles on $\mathcal{X}_{\mathrm{FF},S}$. So it will be important to understand the category of such vector bundles.

Theorem 3.15.
1. *Let $\mathcal{E}$ be a vector bundle on $\mathcal{X}_{\mathrm{FF},S}$. The locus of points $s \in S$ over which $\mathcal{E}$ is trivial is open.*
2. *The following categories are equivalent:*
   - *Vector bundles $\mathcal{E}$ which are trivial over all $s \in S$.*
   - *$\underline{\mathbb{Q}}_p$-local systems on the pro-étale site of S.*

The ideal is that if $\mathcal{V}$ is a $\underline{\mathbb{Q}}_p$-local system on S, then $\mathcal{V} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathcal{X}_{\mathrm{FF},S}}$ descends to a vector bundle on $\mathcal{X}_{\mathrm{FF},S}$.

We finally arrive at the following definition.

Definition 3.16. Let S be a perfectoid space of characteristic p. A *mixed-characteristic local shtuka* of rank n over S with leg at $S^\sharp$ is a triple $(\mathcal{T}, \mathcal{E}, \beta)$, where

- $\mathcal{T}$ is a $\underline{\mathbb{Z}}_p$-local system of rank n on S,
- $\mathcal{E}$ is a vector bundle on $\mathcal{X}_{\mathrm{FF},S}$,
- $\beta : \mathcal{T} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{X}_{\mathrm{FF},S} \setminus D_{S^\sharp}} \xrightarrow{\sim} \mathcal{E}|_{\mathcal{X}_{\mathrm{FF},S} \setminus D_{S^\sharp}}$ is an isomorphism which is meromorphic along $D_{S^\sharp}$.

If μ is a tuple $k_1 \geqslant \cdots \geqslant k_n$, it makes sense to talk about such a shtuka being bounded by μ.

Definition 3.17. Let $b \in B(\mathrm{GL}_n)$ be an isocrystal, and let $\mathcal{E}^b$ be the associated vector bundle on the Fargues–Fontaine curve. Let μ be a tuple $k_1 \geqslant \cdots \geqslant k_n$. The *infinite-level* local shtuka space $\mathrm{Sht}(\mathrm{GL}_n, b, \mu) \to \mathrm{Spd}\, \mathbb{Q}_p$ is the sheaf on Perf which assigns to an object S the set of isomorphisms

$$\beta : \mathcal{O}_{\mathcal{X}_{\mathrm{FF},S} \setminus D_{S^\sharp}} \xrightarrow{\sim} \mathcal{E}^b|_{X_{\mathrm{FF},S} \setminus D_{S^\sharp}}.$$

Then $\mathrm{Sht}(\mathrm{GL}_n, b, \mu)$ admits an action of $\mathrm{GL}_n(\mathbb{Q}_p) \times G_b(\mathbb{Q}_p)$. For each compact open subset $K \subset \mathrm{GL}_n(\mathbb{Z}_p)$, let $\mathrm{Sht}(\mathrm{GL}_n, b, \mu)_K$ be the quotient of $\mathrm{Sht}(\mathrm{GL}_n, b, \mu)$ by K.

It is possible to generalize the definition to an arbitrary reductive group $G/\mathbb{Q}_p$, and multiple legs; the result is a tower $\mathrm{Sht}(G, b; \mu_1, \ldots, \mu_r)_K$ fibered over $(\mathrm{Spa}\, \mathbb{Q}_p)^r$.

Theorem 3.18. $\mathrm{Sht}(G, b; \mu_1, \ldots, \mu)$ *is a locally spatial diamond.*

3.4. Moduli spaces of local shtukas, and the Kottwitz conjecture.

Conjecture 3.19. *Assume the refined local Langlands correspondence. Let $\phi : W_F \to {}^L G$ be a supercuspidal Langlands parameter with $\overline{\mathbb{Q}}_\ell$-coefficients, and let $\rho \in \Pi_\phi(G_b)$ be a member of its L-packet. We have an equality in* $\mathrm{Groth}(G(\mathbb{Q}_p) \times W_E)$*:*

$$\mathrm{Mant}(b, \mu)[\rho] = \sum_{\pi \in \Pi_\phi(G)} \pi \boxtimes \mathrm{Hom}_{S_\phi}(\delta_{\pi,\rho}, r_\mu \circ \phi_E).$$

Theorem 3.20 (c.f. [HKW22])**.** *In the context of Conjecture 3.19, we have (after ignoring the W_E-action) an equality in* $\mathrm{Groth}\, G(\mathbb{Q}_p)$*:*

$$\mathrm{Mant}(b, \mu)[\rho] = [\dim \mathrm{Hom}_{S_\phi}(\delta_{\pi,\rho}, r_\mu)]\pi + \mathrm{err},$$

where $\mathrm{err} \in \mathrm{Groth}\, G(\mathbb{Q}_p)$ *is a virtual representation whose character vanishes on the locus of elliptic elements of $G(F)$. If in addition the semisimple Fargues–Scholze parameter φ_ρ is supercuspidal, then in fact* $\mathrm{err} = 0$.

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