Cohomology of algebraic varieties

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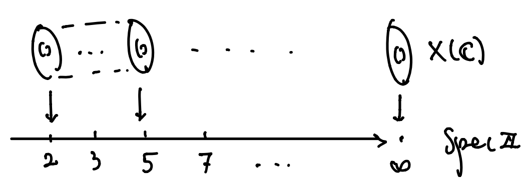
<u>Object of interest</u> X projective sm var / Spec $\mathbb{Z}$
(or / Spec $\mathbb{Z}[\frac{1}{N}]$, etc.)

Then $X(\mathbb{C}) \subset \mathbb{CP}^n$
compact complex mfd def'd by polynomial eq'ns
with integral coefficients.

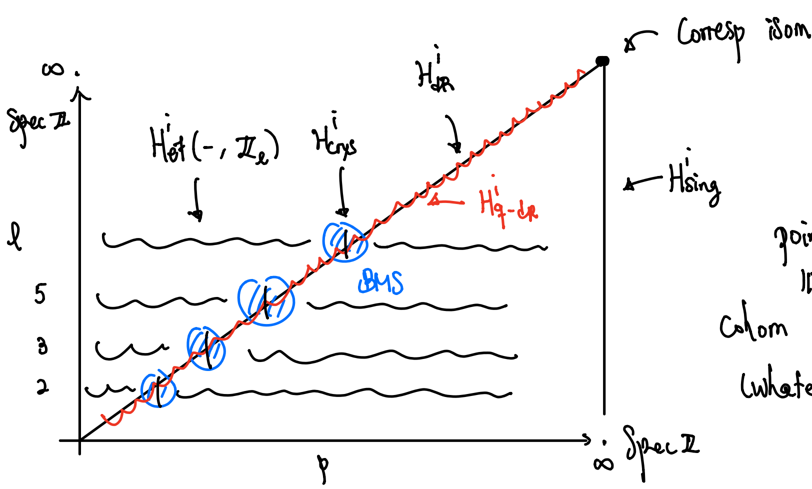
<u>Goal</u> Understand "the" cohom of X.

<u>Note</u>
- Can take cohom with coeff in $\mathbb{Z}$, $\mathbb{F}_\ell$, $\mathbb{Z}_\ell$, $\mathbb{C}$, ...
- $X \to \text{Spec}\,\mathbb{Z}$ has various fibres $X_{\mathbb{F}_p}$ / Spec $\mathbb{F}_p$.



p prime.

<u>Picture</u>



point (p, ℓ)
$\leftrightarrow$
cohom $H^*(X_{\mathbb{F}_p}, \mathbb{F}_\ell)$
(whatever this is).

(I) Singular cohom

Have $H^i_{sing}(X(\mathbb{C}), \mathbb{Z})$ sing cohom groups of top space $X(\mathbb{C})$

- f.g. abelian grps. Poincaré duality, ...

e.g. $X = \mathbb{P}^1$, $X(\mathbb{C}) = \mathbb{CP}^1 \simeq S^2$

$$H^i_{sing}(\mathbb{CP}^1, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0, 2 \\ 0, & o/w \end{cases}$$

Look at Mayer-Vietoris sequence $\mathbb{P}^1 = \underset{0}{\overset{\mathbb{A}^1}{\cup}} \cup \underset{\infty}{\overset{\mathbb{A}^1}{\cup}}$

w/ intersection $\underset{\mathbb{A}^1 \setminus pt.}{\overset{\mathbb{G}_m}{\shortparallel}}$, $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$.

$\mathbb{A}^1(\mathbb{C}) = \mathbb{C}$ contractible

$$\Rightarrow \quad \underset{\mathbb{Z}}{\overset{H^1_{sing}(\mathbb{C}^*, \mathbb{Z})}{\shortparallel}} \xrightarrow{\sim} H^2_{sing}(\mathbb{CP}^1, \mathbb{Z}).$$

related to univ covering $\exp : \mathbb{C} \longrightarrow \mathbb{C}^*$

Note This is not algebraic! But any finite covering

$$\begin{aligned} \mathbb{C}^* &\longrightarrow \mathbb{C}^* \\ x &\longmapsto x^n \end{aligned}$$ is algebraic.

(II) Étale cohom

Motivating question If $Y / \operatorname{Spec} \mathbb{F}_p$ (e.g. $Y = X_{\mathbb{F}_p}$),

Can one define an analogue of "$H^i_{sing}(Y(\mathbb{C}), \mathbb{Z})$" ?

Answer (Grothendieck) Yes, but only with $\mathbb{Z}_\ell$-coeffs with $\ell \neq p$.

<u>Observation</u> $H^1(\mathbb{G}_m, \mathbb{Z}_\ell) = \varprojlim_n H^1(\mathbb{G}_m, \mathbb{Z}/\ell^n\mathbb{Z})$

related to system of covers

$$\mathbb{G}_m \longrightarrow \mathbb{G}_m$$
$$x \longmapsto x^{\ell^n}.$$

Deck transformations given by $x \longmapsto \zeta_{\ell^n} x$,
ℓ^n-th root of unity

But non-triv ℓ^n-th roots of unity exist $/\mathbb{F}_p$ only when $\ell \neq p$.

$$(x^p = 1 \underset{\text{char } p}{\Longleftrightarrow} (x-1)^p = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1).$$

$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ ($\bar{\mathbb{Q}} \subseteq \mathbb{C}$ field of alg numbers)

In our situation, have $H^i_{\text{ét}}(X_{\bar{\mathbb{Q}}}, \mathbb{Z}_\ell) \cong H^i_{\text{sing}}(X(\mathbb{C}), \mathbb{Z}_\ell)$.
$\cong H^i_{\text{ét}}(X_{\bar{\mathbb{F}}_p}, \mathbb{Z}_\ell)$. $(\ell \neq p)$

<u>(III) de Rham cohom</u> (intermediate theory)

<u>Idea</u> Holes in your $\underset{\text{complex mfd}}{\text{space}}$ = obstructions to integrating (locally) diff'l forms

(Sing cohom)

e.g. On $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. have $\frac{dx}{x}$;

integrate to $\log(x)$ but $2\pi i \mathbb{Z}$ ambiguity.

$$(2\pi i = \int_{S^1} \frac{dx}{x}).$$

<u>Note</u> $\frac{dx}{x}$ algebraic!

Starting with X / Spec $\mathbb{Z}$, can build an alg de Rham complex

$$\Omega_X^\bullet : [0 \longrightarrow \underset{\substack{\| \\ \text{sheaf of regular fcts on } X}}{\mathcal{O}_X} \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \longrightarrow \cdots \longrightarrow \Omega_X^d \longrightarrow 0]$$

$(d = \dim X)$

Then $H^i_{dR}(X) := \mathbb{H}^i(X, \Omega_X^\bullet)$ hyper cohom.

f.g. ab groups + Poincaré duality.

Thm $H^i_{dR}(X) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^i_{dR}(X(\mathbb{C}))$

$\cong H^i_{sing}(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ Poincaré lemma

Rmk · H^i_{dR} + H^i_{sing} are very different $\mathbb{Z}$-lattices

In example above, differed by $2\pi i = \int_{S^1} \frac{dx}{x}$

Periods !

Fact $H^i_{dR}(X) \otimes_{\mathbb{Z}} \mathbb{F}_p \cong H^i_{dR}(X_{\mathbb{F}_p})$

(IV) Crystalline cohom

Problem If $Y / \mathbb{F}_p$. have no $H^*(Y, \mathbb{Z}_p)$ yet

Have: $H^*(Y, \mathbb{F}_p) := H^*_{dR}(Y)$, $Y / \mathbb{F}_p$.

Answer (Grothendieck) There are "crystalline" cohom grps

$H^*_{crys}(Y / \mathbb{Z}_p)$ f.g. $\mathbb{Z}_p$-mod

+ Poincaré duality.

"defining property": If $\tilde{Y} / \mathbb{Z}_p$ sm lift of Y,

then $H^*_{crys}(Y / \mathbb{Z}_p) \cong H^*_{dR}(\tilde{Y})$

(Can always find such lift locally, but this is "indep" of $\tilde{Y}$.)

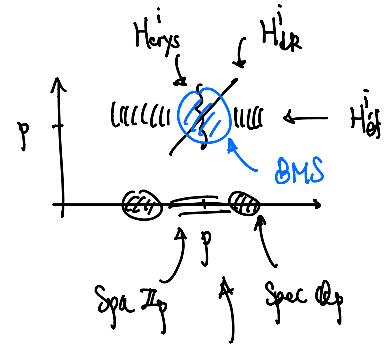
Crystalline cohom is related to deforming from $\mathbb{F}_p$ to $\mathbb{Z}_p$.

(v) More ?

Question: Can one fill in more of this diagram by explicit cohom theories ?

For example: What happens as étale cohom "degenerates" into crystalline cohom ?

Zoom in (p, p) :



Work on "punctured disc" at p

Thm (Bhatt - Morrow - Scholze)

One can fill this diagram by new cohom theory with values in "2-dim'l complete local ring" (For experts, this is Fontaine's ring A_{inf}).

Consequence: $\# H^i_{dR}(X)_{tor} \geq \# H^i_{sing}(X(\mathbb{C}), \mathbb{Z})_{tor}$.

Conj: There exists "q-deformation of dR cohom".

More precisely, a $\mathbb{Z}[[q^{\pm}]]$-valued cohom theory $H^i_{q\text{-}dR}(X)$ s.t.

$$H^i_{q\text{-}dR}(X) \otimes_{\mathbb{Z}[[q-1]],\, q\mapsto 1} \mathbb{Z} = H^i_{dR}(X)$$

(in derived sense)

interpolating $\underbrace{\text{B-M-S}}$ among all p.

$\llcorner$ $H^i_{q\text{-}dR}(X) \otimes_{\mathbb{Z}[[q-1]]} \mathbb{Z}_p[[q-1]]$.

<u>Example</u> $H^*_{q\text{-}dR}(\mathbb{A}^1)$ computed by

$$\mathbb{Z}[X][[q-1]] \xrightarrow{d_q} \mathbb{Z}[X][[q-1]]\,dX$$
$$X^n \longmapsto [n]_q X^{n-1}\,dX$$

Better: $(d_q f)(x) = \dfrac{f(qx) - f(x)}{qx - x} \cdot dX$

"Jackson q-derivative".

This depends on coordinates !!

(Q Is this related to deforming X from $\mathbb{Z}$ to somewhere ?)