Geometric Satake equivalence (I)

Xu Shen $\qquad\qquad$ Jan 9

§1 Classical settings

k field, G/k conn reductive grp.

$\rightsquigarrow$ affine Grassmannian $Gr_G = LG/L^+G$

$\forall$ k-alg R,

$Gr_G(R) = \{(\mathcal{E},\beta) \mid \mathcal{E}/D_R$ G-torsor, $\beta: \mathcal{E}[\frac{1}{t}] \xrightarrow{\sim} \mathcal{E}_0[\frac{1}{t}]$ triv $/D_R^*\}/\simeq$

where $D_R = \operatorname{Spec} R[\![t]\!] \supseteq D_R^* = \operatorname{Spec} R(\!(t)\!)$.

ind-scheme, ind-proj$/k$, $L^+G \curvearrowright Gr_G$.

For $\ell \neq \operatorname{char} k$, the Satake set

$Sat_G := P_{L^+G}(Gr_G) :=$ the cat of L^+G-equi $\bar{\mathbb{Q}}_\ell$-perverse sheaves $/Gr_G$.

$= \varinjlim_{O} P_{L^+G}(\bar{O})$ $\qquad$ O runs through L^+G-orbits

The case k alg closed

Thm (Ginzberg, Mirković-Vilonen, etc.)

(1) For $A_1, A_2 \in Sat_G$, $\exists$ convolution prod $A_1 * A_2 \in Sat_G$

$(Sat_G, *)$ forms $\mathfrak{S}$ symmetric monoidal cat s.t.

$\omega := \bigoplus_i R^i\Gamma(Gr_G, -): Sat_G \longrightarrow Vect_{\bar{\mathbb{Q}}_\ell}$ is sym monoidal.

(2) $(Sat, *, \omega)$ is neutralized Tannakian cat,

and $Sat_G \simeq Rep(\hat{G})$, $IC_\mu \longmapsto V_\mu$, $\mu \in X^*(\hat{T})^+ = X_*(T)^+$.

$\hat{G}$ = Langlands dual of $G/\bar{\mathbb{Q}}_\ell$.

Construction

$$\begin{tikzcd} & LG \times Gr_G \arrow[dl, "p"'] \arrow[dr, "q"] & & \\ Gr_G \times Gr_G & & LG \overset{L^+G}{\times} Gr_G \arrow[r, "m"] & Gr_G \end{tikzcd}$$

$Gr_G \tilde{\times} Gr_G$ convolution aff Grass. $= LG \overset{L^+G}{\times} Gr_G$

- p, q L^+G-torsors,
- m induced by the LG-action on Gr_G.

$A_1, A_2 \in Sat_G \rightsquigarrow A_1 \boxtimes A_2 \in Sat_G$ perverse sheaf on $Gr_G \times Gr_G$

$\rightsquigarrow \exists!$ perverse sheaf $A_1 \tilde{\boxtimes} A_2$ on $Gr_G \tilde{\times} Gr_G$

s.t. $p^*(A_1 \boxtimes A_2) \simeq q^*(A_1 \tilde{\boxtimes} A_2)$.

Also, $A_1 * A_2 := Rm_*(A_1 \tilde{\boxtimes} A_2)$ (Key: $A_1 * A_2 \in Sat_G$)

$\rightsquigarrow$ Beilinson-Drinfeld Grass'n

X/k smooth conn curve,

$\Sigma :=$ moduli of rel effective Cardiv $X = \coprod_{d \geq 1} \underbrace{\boxed{X^d/S_d}}_{= \Sigma_d}$

$$\begin{array}{c} Gr_G^{BD} \\ \downarrow \\ \Sigma \end{array} \quad \forall\, k\text{-alg } R,\quad Gr_G^{BD}(R) := \left\{ (D, \mathcal{E}, \beta) \;\middle|\; \begin{array}{l} D \in \Sigma(R) \text{ Cardiv of } X_R,\ \mathcal{E}/X_R \text{ } G\text{-torsor}, \\ \beta : \mathcal{E}|_{X_R \setminus D} \xrightarrow{\sim} \mathcal{E}_0|_{X_R \setminus D} \text{ triv}/(X_R \setminus D) \end{array} \right\} / \simeq$$

$\rightsquigarrow$ ind-scheme, ind-proper over Σ,

$$Gr_G^{BD}(R) = \mathcal{L}G/\mathcal{L}^+G, \quad \mathcal{L}G : R \mapsto \{(D, s) \mid D \in \Sigma(R),\ s \in G(\hat{D}^\circ)\}$$
$$\mathcal{L}^+G : R \mapsto \{(D, s) \mid D \in \Sigma(R),\ s \in G(\hat{D})\}.$$

And $\hat{X}_{R,D} = \mathrm{Spf}\, \hat{\mathcal{O}}_{X,D}$, $\hat{D} = \mathrm{Spec}\, \hat{\mathcal{O}}_{X,D} \supseteq D$, $\hat{D}^\circ = \hat{D} \setminus D$.

Consider $x \in X(k) \rightsquigarrow D_x \in \Sigma(k)$.

$$\hat{D}_x = \mathrm{Spec}\, k[\![t]\!],\ \mathcal{L}G_x \cong LG,\ \mathcal{L}^+G_x \cong L^+G,\ Gr_{G,x}^{BD} \cong Gr_G.$$

Set $gr = Gr_G^{BD}$, $d \geq 1$,

$$\begin{tikzcd} & \widetilde{\mathcal{L}G}_d \arrow[dl, "p_d"'] \arrow[dr, "q_d"] & & \\ gr^d & & \widetilde{gr}_d \arrow[r, "m_d"] & gr \end{tikzcd}$$

$$\widetilde{gr}_d(R) = \left\{ (D_i, \mathcal{E}_i, \beta_i)_{1 \leq i \leq d} \;\middle|\; \begin{array}{l} D_i \in \Sigma(R),\ \mathcal{E}_i/X_R \text{ } G\text{-torsors} \\ \beta_i : \mathcal{E}_i|_{X_R \setminus D_i} \xrightarrow{\sim} \mathcal{E}_{i-1}|_{X_R \setminus D_i},\ 1 \leq i \leq d \end{array} \right\} / \simeq.$$

$\rightsquigarrow \widetilde{gr}_d(R) \xrightarrow{m_d} gr(R)$

$(D_i, \mathcal{E}_i, \beta_i) \longmapsto (D = D_1 \cup \cdots \cup D_d, \mathcal{E}_d, \beta_1|_{X_R \setminus D} \circ \cdots \circ \beta_d|_{X_R \setminus D})$.

I finite set, $|I| = d$,

then $X^I \twoheadrightarrow \Sigma^d \subseteq \Sigma$,

$gr_I := gr \times_\Sigma X^I$, if $I = \{*\}$, $gr_X := gr_{\{*\}}$.

(can define similarly $P_{\mathcal{L}^+G}(gr_I)$)

$$\begin{tikzcd} & \widetilde{\mathcal{L}G}_I \arrow[dl, "p_I"'] \arrow[dr, "q_I"] & & \\ \coprod_{i \in I} gr_{X,i} & & \widetilde{gr}_I \arrow[r, "m_I"] & gr_I \end{tikzcd}$$

$Sat_{X^I} := P_{\mathcal{L}^+G}(gr_I)^{ULA}$

$=$ cat of $\mathcal{L}^+G$-equiv perverse sheaves on gr_I which are ULA w.r.t. $gr_I \to X^I$.

Take $A_i \in Sat_X \rightsquigarrow A_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} A_d$ perverse sheaf on $\widetilde{gr}_I$.

$A_1 \star \cdots \star A_d := R(m_I)_* (A_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} A_d)$.

Prop $A_1 \star \cdots \star A_d \in Sat_{X^I}$.

$\rightsquigarrow$ fusion product $A_1 * \cdots * A_d = \Delta^\circ(\underbrace{A_1 \star \cdots \star A_d}_{\in Sat_X})$

where $\Delta : X \to X^I$ diagonal, $\Delta^\circ = \Delta^*[-(d-1)]$.

$\rightsquigarrow$ can define similarly $*$ on Sat_{X^I}.

Prop (1) $(Sat_{X^I}, *)$ is a symmetric monoidal cat.

(2) $Sat_G \longrightarrow Sat_X$ is fully faithful.

$A \longmapsto A_X := \overline{\mathbb{Q}}_\ell \tilde{\boxtimes} A[1]$, $\quad (gr_X \simeq X \tilde{\times} Gr_G)$

Moreover, $(A_1 * \cdots * A_d)_X \simeq A_{1,X} * \cdots * A_{d,X}$

$\Rightarrow (Sat_G, *)$ symm monoidal.

For general k $\Gamma_k := \mathrm{Gal}(\bar{k}/k)$, $\omega : \mathrm{Sat}_G \longrightarrow \mathrm{Rep}^c_{\bar{\mathbb{Q}}_\ell}(\Gamma_k)$

$$A \longmapsto \bigoplus_i R^i\Gamma(\mathrm{Gr}_{G,\bar{k}}, A).$$

$$\Gamma_k \overset{\text{geom}}{\curvearrowright} \underline{\mathrm{Aut}}^\otimes(\omega) \simeq \hat{G} \overset{\text{alg}}{\curvearrowleft} \Gamma_k$$

$\rightsquigarrow$ $\mathrm{act}^{\text{geom}} = \mathrm{act}^{\text{alg}} \circ \mathrm{Ad}_{\chi}$, where $\chi : \underbrace{\Gamma_k \to \mathbb{Z}_\ell^\times}_{\ell\text{-adic cyclotomic char}} \xrightarrow{\rho_{\mathrm{ad}}} \hat{G}_{\mathrm{ad}}(\bar{\mathbb{Q}}_\ell)$.

$\rightsquigarrow$ ${}^LG^{\text{geo}}$, ${}^LG^{\text{alg}}$, if χ lifts to a cont. homomorphism $\tilde{\chi} : \Gamma_k \to \hat{G}(\bar{\mathbb{Q}}_\ell)$,
then ${}^LG^{\text{geo}} \simeq {}^LG^{\text{alg}}$.

Theorem $\omega : \mathrm{Sat}_G \longrightarrow \mathrm{Rep}^c_{\bar{\mathbb{Q}}_\ell}(\Gamma_k)$ lifts to a canonical equiv

s.t.
$$\begin{tikzcd}
\mathrm{Sat}_G \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Rep}^c_{\bar{\mathbb{Q}}_\ell}({}^LG^{\text{geo}}) \arrow[d, "\mathrm{Res}."] \\
\mathrm{Sat}_{G_{\bar{k}}} \arrow[r, "\sim"] & \mathrm{Rep}^c_{\bar{\mathbb{Q}}_\ell}(\hat{G})
\end{tikzcd}$$

Relative case $(\mathrm{Sat}_{X^I}, *)$ symm monoidal cat
$\qquad\qquad \mathrel{\|}$
$\qquad P_{L^+G}(\mathrm{Gr}_I)^{\mathrm{ULA}}$, I finite set.

$\rightsquigarrow$ $\mathrm{LocSys}(X^I) =$ Cat of $\bar{\mathbb{Q}}_\ell$-loc. systems on X^I.

Thm $\exists$ canonical equiv $\mathrm{Sat}_{X^I} \xrightarrow{\sim} \mathrm{Rep}(\hat{G}^I, \mathrm{LocSys}(X^I))$.

$\rightsquigarrow$ For more information, see X. Zhu "An Intro to aff Grass'n and geom. Satake".

§2 p-adic settings

E non-arch local field with res field $\mathbb{F}_q/\mathbb{F}_p$.

$d \geq 1$. $\mathrm{Div}^d = (\mathrm{Div}^1)^d / S_d$ (p-adic analogue of Σ_d).

$\quad\downarrow \qquad$ where $\mathrm{Div}^1_X = \mathrm{Div}^1 = \mathrm{Spd}\, E/\varphi^{\mathbb{Z}}$.

$\mathrm{Spd}\, \mathbb{F}_q$

$\rightsquigarrow$ Div^d = small v-sheaf of deg d effective Cardiv on the FF curve.

For any $C/\mathbb{F}_q$ alg closed perf'd field,

Div^1_C = the mirror FF curve.

$(\mathrm{Div}^1_\eta = (\mathrm{Spd}\, E)^d/S_d,\ \mathrm{Div}^1_{\mathrm{sy}} = (\mathrm{Spd}\, \mathcal{O}_E)^d/S_d)$.

G/E red grp, $\mathrm{Gr}_{G,\mathrm{Div}^d} \longrightarrow \mathrm{Div}^d$ BD Grass.

$\mathrm{Gr}_{G,\mathrm{Div}^d} = L_{\mathrm{Div}^d} G / L^+_{\mathrm{Div}^d} G$.

$$\begin{tikzcd} S \arrow[dr] & & \mathrm{Gr}_{G,\mathrm{Div}^d} \arrow[dl] \\ & \mathrm{Div}^d & \end{tikzcd}$$

S affinoid $\in \mathrm{Perf}_{\mathbb{F}_q}$

Define $\mathrm{Gr}_{G,\mathrm{Div}^d}(S) = \left\{ (\mathcal{E}, \beta) \;\middle|\; \begin{array}{l} \mathcal{E}\ G\text{-bundle} / B^+_{\mathrm{Div}^d}(S) \\ \beta : \text{triv of } \mathcal{E}/B_{\mathrm{Div}^d}(S) \end{array} \right\} / \simeq$.

$\mathrm{Hck}_{G,\mathrm{Div}^d} = [L^+_{\mathrm{Div}^d} G \backslash \mathrm{Gr}_{G,\mathrm{Div}^d}] \hookrightarrow \mathrm{Div}^d$ Hecke stack.

- I finite set with $|I| = d$.

$\rightsquigarrow$ $(\mathrm{Div}^1)^I \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Div}^d$,

Λ coeff ring, killed by some n and $p \nmid n$, e.g. $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$.

$\rightsquigarrow$ $D_{\acute{e}t}(\mathrm{Hck}^I_G, \Lambda)$

$\cup$

$D_{\acute{e}t}(\mathrm{Hck}^I_G, \Lambda)^{bd}$ = full subcat of obj with supp over $(\mathrm{Div}^1)^I$.

$\cup$

$\mathrm{Sat}^I_G(\Lambda)$ = full subcat of obj which are flat perverse over Λ and ULA w.r.t. $\mathrm{Hck}^I_G \xrightarrow{\pi_G} (\mathrm{Div}^1)^I$.

$R\pi_{G,*} \downarrow$

$D_{\acute{e}t}((\mathrm{Div}^1)^I, \Lambda)$

$\forall i$, $\mathcal{H}^i(R\pi_{G,*} A) \in \mathrm{LocSys}((\mathrm{Div}^1)^I, \Lambda)$

$\cong$ Drinfeld Lemma. Prop IV.7.3

$\mathrm{Rep}_{W_E^I}(\Lambda)$ = cat of cont. rep'n of W_E^I on finite proj Λ-mod.

$\rightsquigarrow$ $F^I := \bigoplus_i \mathcal{H}^i(R\pi_{G,*}) : \mathrm{Sat}^I_G(\Lambda) \longrightarrow \mathrm{Rep}_{W_E^I}(\Lambda)$.

Tannakian duality $\rightsquigarrow \exists\, \mathcal{H}_G^I$ Hopf alg

s.t. $\mathrm{Sat}_G^I(\Lambda) \cong \mathrm{Rep}(\boxed{\mathcal{H}_G^I}, \mathrm{Rep}_{W_E^I}(\Lambda))$.

$\mathcal{H}_G^I = \bigotimes_{*\in I} \mathcal{H}_G$.

$\mathcal{H}_G \rightsquigarrow$ grp scheme $\boxed{\check{G}/\Lambda} \circlearrowleft W_E$.

<u>Main thm (Fargues-Scholze)</u>

$\exists$ canonical isom $\check{G} \simeq \hat{G}$, s.t. the actions of W_E on two sides agree up to an explicit cyclotomic twist.

If $\sqrt{q} \in \Lambda$, then

$$\mathrm{Sat}_G^I(\Lambda) \simeq \mathrm{Rep}(({}^LG)^I, \Lambda).$$

<u>§3 Affine flag varieties</u>

G/E split $\rightsquigarrow G/\mathcal{O}_E$.

Fix C/E alg closed perf'd $\rightsquigarrow A_{inf} = W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat})$

$\rightsquigarrow G_{A_{inf}}$ and $\mathcal{I} \longrightarrow G_{A_{inf}}$ Iwahori grp scheme

$$\begin{array}{ccc} S & \longrightarrow & \mathrm{Div}_{\mathcal{Y}}^1 \\ \downarrow & & \downarrow \\ \mathrm{Spd}\, \mathcal{O}_C & \longrightarrow & \mathrm{Spd}\, \mathcal{O}_E \end{array}$$

small v-stack

$\rightsquigarrow$ $\mathcal{F}\ell_{G,S} \longrightarrow S$, $\mathcal{F}\ell_{G,S} = LG/L^+\mathcal{I}$, $L^+\mathcal{I}(R,R^+) = \mathcal{I}(B_{dR}^+(R^\sharp))$.

$\rightsquigarrow \mathcal{F}\ell_{G,S}$

$\downarrow$ torsor under $(G/B)^\diamond$.

$\mathrm{Gr}_{G,S/\mathrm{Div}_{\mathcal{Y}}^1} = \mathrm{Gr}_{G,\mathrm{Div}_{\mathcal{Y}}^1} \times_{\mathrm{Div}_{\mathcal{Y}}^1} S$

Let $\tilde{W}$ be Iwahori-Weyl grp. Then

$$0 \longrightarrow X_*(T) \longrightarrow \tilde{W} \longrightarrow W \longrightarrow 0$$

$$1 \longrightarrow W_{aff} \longrightarrow \tilde{W} \longrightarrow \Omega \longrightarrow 1.$$

Also, $|L^+\mathcal{I}\backslash \mathrm{Fl}_{G,\mathrm{Spd}\,C}| \cong \widetilde{W} \ni w$.

$\rightsquigarrow \mathrm{Fl}_{G,w,S} \subset \mathrm{Fl}_{G,\leq w,S} \subset \mathrm{Fl}_{G,S}$.

<u>Thm VI.5.5</u> $\mathrm{Fl}_{G,\leq w,S} \subset \mathrm{Fl}_{G,S}$ closed.

$\mathrm{Fl}_{G,\leq w,S} \longrightarrow S$ proper, rep'ble in spatial diamonds of fin-dim'l trg.

$\mathrm{Fl}_{G,w,S} \subset \mathrm{Fl}_{G,\leq w,S}$ open dense.

<u>Proof</u> Use the Demazure resolution. $w = w'\alpha$,

$w' = \prod_{j=1}^{l} s_{i_j} \in W_{\mathrm{aff}}$, $\alpha \in \Omega$.

s_i affine simple reflexions $\rightsquigarrow \mathcal{P}_{s_i}$ parahoric grp.

$L^+\mathcal{P}_{s_i}/L^+\mathcal{I} \cong \mathbb{P}^{1,\diamond}$.

$$\begin{tikzcd}
\mathrm{Dem}_{\dot w,S} := L^+\mathcal{P}_{s_{i_1}} \overset{L^+\mathcal{I}}{\times} L^+\mathcal{P}_{s_{i_2}} \overset{L^+\mathcal{I}}{\times} \cdots \overset{L^+\mathcal{I}}{\times} L^+\mathcal{P}_{s_{i_l}}/L^+\mathcal{I} \arrow[r] \arrow[dr, two heads] & \mathrm{Fl}_{G,S} \\
& \mathrm{Fl}_{G,\leq w,S} \arrow[u, bend right, "\subseteq"']
\end{tikzcd}$$

$(p_1,\dots,p_l) \longmapsto p_1\cdots p_l\cdot\alpha$. $\square$

successive $\mathbb{P}^{1,\diamond}$-fibration $/S$

$\rightsquigarrow$ spatial diamond proper $/S$

<u>Prop VI.5.7</u> $j_w : \mathrm{Fl}_{G,w,S} \hookrightarrow \mathrm{Fl}_{G,\leq w,S}$ open embedding.

then $j_{w!}\Lambda \in D_{\acute{e}t}(\mathrm{Fl}_{G,\leq w,S},\Lambda)$ ULA over S.

<u>Proof</u> $\tilde{j}_w : \mathrm{Fl}_{G,w,S} \hookrightarrow \mathrm{Dem}_{\dot w,S}$ open embedding,

it suffices to prove the statement for $\tilde{j}_{w!}\Lambda$.

$\tilde{j}_{w!}$ can be resolved by Λ and all $i_{\dot w',\dot w,*}\Lambda$ for

$i_{\dot w',\dot w} : \mathrm{Dem}_{\dot w',S} \longrightarrow \mathrm{Dem}_{\dot w,S}$,

$\dot w'$ subword of $\dot w$. $\square$