Cycles on products of elliptic curves and a conjecture of Beilinson

Wei Zhang

Local to global principle:

<u>Last time</u> $f_i(x_1, \dots, x_m) = 0, \quad \forall\, 1 \le i \le m$

w/ $\mathbb{Q}$-coefficients.

$\{ y^2 = g(x) \} \supseteq \{ x = a \}, \quad a \in \mathbb{Q}$

Subvarieties

$\rightsquigarrow$ a family of deg 2 divisors $/\mathbb{Q}$.

Study $CH^i(X) = (\{\text{alg cycles of codim } i\} / \text{rat'l equiv relation}) \otimes \mathbb{Q}$

(abelian grp

Birch, Swinnerton-Dyer:

$X: \ y^2 = f(x) = x^3 + Ax + B$ cubic w/ no repeated roots.

$A, B \in \mathbb{Z}$.

For $p \gg 0$ (avoiding fin many primes)

$$\prod_{p<N} \frac{\# X(\mathbb{F}_p)}{p} \overset{\text{conj}}{\sim} c(\log N)^{r_X},$$

$r_X = \operatorname{rank} X(\mathbb{Q})$.

(Kind of local-to-global compatibility.

Goldfeld: This is too strong.

Hasse-Weil L-function :

$X/\mathbb{Q} \rightsquigarrow X(\mathbb{F}_{p^n})$

Have $\exp\left(\sum_{n=1}^{\infty} \frac{\# X(\mathbb{F}_{p^n})}{n} \cdot p^{-ns}\right)$

$\parallel$ zero at p

$\zeta_p(X, s)$, $s \in \mathbb{C}$.

& $\prod_p \zeta_p(X, s) = \zeta(X, s)$

$$= \prod_{i=0}^{2\dim X} (-1)^i L(\underbrace{H^i(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)}_{\overset{!!}{H^i(X)}}, s).$$

When X elliptic curve,

$$\zeta_X = \frac{\text{zeta functions}}{L(H^1(X), s)}.$$

For p good, $\# X(\mathbb{F}_p) = p+1-a_p$

$$L_p(H^1(X), s) = \frac{1}{(1-\alpha_p \cdot p^{-s})(1-\beta_p \cdot p^{-s})}$$

with $\begin{cases} \alpha_p + \beta_p = a_p/\sqrt{p} \\ \alpha_p \cdot \beta_p = 1. \end{cases}$

Reformulate B-SD conj :

For X = elliptic curve,

$$\operatorname{ord}_{s=1/2} L(H^1(X), s) = \operatorname{rank}_{\mathbb{Z}} X(\mathbb{Q}).$$

<u>note</u> It implies Riemann hypothesis for L-fcts.

Generally, $Ch^1(X) \xrightarrow{\deg} \mathbb{Z}$ w/ $\ker(\deg) \simeq X(\mathbb{Q})$.

Let $X/\mathbb{Q}$ sm proj.

$$X(\mathbb{C}) \rightsquigarrow \underset{\text{cycle class map}}{CH^i(X)_{\mathbb{Q}} \longrightarrow} H^{2i}(X(\mathbb{C}), \mathbb{Z}) \cap \underset{\text{part of dR cohom}}{H^{i,i}_{dR}}$$

Hodge conj: this should be surjective.

<u>Tate</u> $\underset{\text{cycle class map}}{CH^i(X) \longrightarrow} H^{2i}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)(i)^{Gal(\bar{\mathbb{Q}}/\mathbb{Q})}$

Tate conj: (i) Image $\otimes_{\mathbb{Q}} \mathbb{Q}_p$ spans all Gal-inv target

i.e. surjective when $\otimes_{\mathbb{Q}} \mathbb{Q}_p$.

(ii) $\mathrm{ord}_{s=1+i} L(H^{2i}(X), s) = -$ rank of image.

<u>Known</u> $X = E^n = E \times \cdots \times E$,

where E = ell curve $/\mathbb{Q}$ with non-CM.

$\Rightarrow$ Tate conj (i) (ii) hold for $X = E^n$ in all codim cycles (i.e. in all CH^i).

Consider Abel-Jacobi map

$$\begin{array}{l} CH^i(X)_0 := \ker(\text{cycle class map}) \\ \big\downarrow AJ \\ H^1(\mathbb{Q}, \underbrace{H^{2i-1}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)(i)}_{=:V^{2i-1}}) \end{array}$$

Conj (Beilinson–Bloch, Kato)

(i) A p-adic analogue of AJ:

$$CH^i(X) \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq H^1_f(\mathbb{Q}, V^{2i-1}) \leftarrow \text{B-K Selmer grp}$$

$$\| $$

$$\ker\left(H^1(\mathbb{Q}, V^{2i-1}) \to H^1(\mathbb{Q}_p, V^{2i-1} \otimes B_{cris})\right)$$

(ii) $\operatorname{ord}_{s=1/2} L(H^{2i-1}(X), s) = \dim_{\mathbb{Q}_p}\left(H^1_f(\mathbb{Q}, V^{2i-1})\right)$.

<u>Rankin–Selberg Case</u>

Let E_1, E_2 ell curves $/ F_0 \neq \mathbb{Q}$ totally real

non-CM, non-isogeneous $/ F$,

F/F_0 CM quadratic

Suppose $\operatorname{Sym}^n H^1(E_1)$, $\operatorname{Sym}^{n+1} H^1(E_2)$ are automorphic.

<u>Thm</u> (LTXZZ) $\operatorname{ord}_{s=1/2} L(\operatorname{Sym}^n H^1(E_1/F) \otimes \operatorname{Sym}^{n+1}(E_2/F), s) = 0$

$$\Rightarrow H^1_f(F, H^{2i-1}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)(i)) = 0$$

with $X = E_1^n \times E_2^{n-1}$.

<u>Rmk</u> $F_0 = \mathbb{Q}$: by Newton–Throne

<u>Pf sketch</u> (I) Special value formula

$F/F_0 \rightsquigarrow W_n \hookrightarrow W_{n+1}$ Herm spaces

(definite at ∞)

$\rightsquigarrow$ $G = U(W_n) \times U(W_{n+1}) \curvearrowleft \pi_n \boxtimes \pi_{n+1}$
$\quad\quad\quad\quad \cup$ $\quad\quad\quad\quad\quad\quad\quad\quad$ auto rep of G.
$\quad\quad$ $H = U(W_n)$.

part of GGP Conj:

$$\underset{\substack{\not\parallel \\ 0}}{L(\pi_n \times \pi_{n+1}, \tfrac{1}{2})} = \Big|\int_{[H]} \varphi(h)\, dh\Big|^2. \quad \varphi \in \pi$$

$$[H] = H(\mathbb{Q})\backslash H(\mathbb{A})$$

(II) Congruence of cycles on Shimura vars.

W_n, W_{n+1} indefinite

of sgns $(n-1, 1)$, $(n, 1)$

$\rightsquigarrow$ $Sh_H \hookrightarrow Sh_G$ $\quad /F$.

Arithmetic diagonal cycle:

(choose certain parahoric level)