

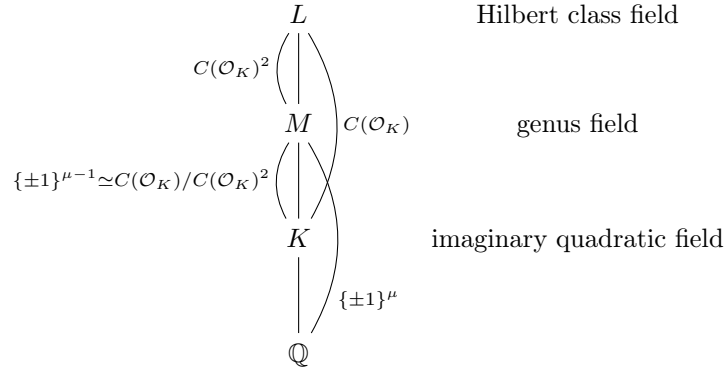
BASIC NUMBER THEORY: LECTURE 12

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This lecture explains the solution to the mid-term exam and uses the rest of the lesson to teach the following.

GENUS FIELD REVISITED

Recall the whole picture of genus field and Hilbert class field of any imaginary quadratic field K as follows.



Let p_1, \dots, p_r be all odd prime divisors of d_K . Then we obtain

$$M = K(\sqrt{p_1^*}, \dots, \sqrt{p_r^*}) = \begin{cases} \mathbb{Q}(\sqrt{p_1^*}, \dots, \sqrt{p_r^*}) & d_K \equiv 1 \pmod{4}, \\ \mathbb{Q}(\sqrt{-1}, \sqrt{p_1^*}, \dots, \sqrt{p_r^*}) & \frac{d_K}{4} \equiv 1 \pmod{4}, \\ \mathbb{Q}(\sqrt{-2}, \sqrt{p_1^*}, \dots, \sqrt{p_r^*}) & \frac{d_K}{4} \equiv 2 \pmod{8}, \\ \mathbb{Q}(\sqrt{2}, \sqrt{p_1^*}, \dots, \sqrt{p_r^*}) & \frac{d_K}{4} \equiv 6 \pmod{8}. \end{cases}$$

Also, we have defined the Artin map

$$\left(\frac{M/K}{\cdot} \right) : I_K \rightarrow \text{Gal}(M/K) \rightarrow \prod_{i=1}^r \text{Gal}(K_i/K) \xrightarrow{\sim} \{\pm 1\}^r,$$

where $\Phi_K : \text{Gal}(M/K) \rightarrow \{\pm 1\}^r$ and $K_i = K(\sqrt{p_i^*})$. For each fractional ideal $\mathfrak{a} \subseteq \mathcal{O}_K$,

$$\Phi_K(\mathfrak{a}) = \left(\left(\frac{N(\mathfrak{a})}{p_1} \right), \dots, \left(\frac{N(\mathfrak{a})}{p_r} \right) \right).$$

Claim: Denote P the group of principal genus in $C(\mathcal{O}_K)$. Then the following diagram commutes:

$$\begin{array}{ccccc}
C(d_K)/C(d_K)^2 & \longrightarrow & C(\mathcal{O}_K)/P & \longrightarrow & \{\pm 1\}^\mu \\
\downarrow \sim & & & & \downarrow \\
C(\mathcal{O}_K)/C(\mathcal{O}_K)^2 & \xrightarrow[\sim]{(\chi_1, \dots, \chi_r)} & & & \{\pm 1\}^r.
\end{array}$$

To be continued in Lecture 13.

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