Interview Questions on Algebra

Wenhan Dai

Set IX: Categories and Functors

1. Which is the connection between Hom and tensor product? What is this called in representation theory?

   Answer. They are a pair of adjoint functors, i.e., in some small category $\mathcal{C}$,

   $$\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{Hom}(Y, Z)).$$

   This is called **Frobenius reciprocity** in representation theory, which states tensor product as the functor for induced representations and Hom as the functor of restrictions, respectively.

2. Can you get a long exact sequence from a short exact sequence of abelian groups together with another abelian group?

   Answer. This is just the long exact sequence of group cohomology. For a (discrete) group G, which is not necessarily abelian, acting on another abelian group M (with discrete topology), which is called a G-module, we can define $H^i(G, M) := \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, M)$. This theory is covariant in M and contravariant in G.

3. Do you know what the Ext functor of an abelian group is? Do you know where it appears? What is $\mathrm{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$? What is $\mathrm{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$? How about $\mathrm{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q})$?

   Answer. The Ext functor is a derived functor that measures the failure of a short exact sequence of modules to split. Also, $\mathrm{Ext}(A, B)$ classifies abelian extensions of A by B. It appears in Cartan–Eilenberg's 1956 book *Homological Algebra*.

   Solution. For any $\mathbb{Z}$-module M, the homomorphism $\mathbb{Z} \to M$ is defined by the image of 1 in H. So

   $$\mathrm{Hom}(\mathbb{Z}, M) = M, \quad \mathrm{Ext}(\mathbb{Z}, M) = 0$$

   because $\mathbb{Z}$ is a projective module (c.f. Set 7, Question 8). We begin the computation with a projective resolution of $\mathbb{Z}/m\mathbb{Z}$ as follows:

   $$0 \longrightarrow \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0.$$

   Taking the contravariant functor $\mathrm{Hom}(-, M)$, we get

   $$\begin{tikzcd} & & M \arrow[d, equal] & M \arrow[d, equal] \\ 0 \arrow[r] & \mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, M) \arrow[r] & \mathrm{Hom}(\mathbb{Z}, M) \arrow[r, "\times m"] & \mathrm{Hom}(\mathbb{Z}, M) \arrow[dll, rounded corners] \\ & \mathrm{Ext}(\mathbb{Z}/m\mathbb{Z}, M) \arrow[r] & \mathrm{Ext}(\mathbb{Z}, M) = 0. & \end{tikzcd}$$

   It follows that

   $$\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, M) = \mathrm{Ker}\, m = M[m], \quad \mathrm{Ext}(\mathbb{Z}/m\mathbb{Z}, M) = M/mM.$$

   In particular, for $d = (m, n)$, $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) = 0$ and $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$. Moreover,

   $$\mathrm{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}, \quad \mathrm{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}.$$

   Also, we have $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q}) = \mathrm{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q}) = 0$.