THE HODGE–TATE PERIOD MAP ON PERFECTOID SHIMURA VARIETY

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1. Introduction

1.1. **Course description.** About a decade ago Scholze discovered that the Siegel modular varieties (i.e. moduli spaces of polarized abelian varieties with level structure) become perfectoid when passing to infinite level, and that there is a Hodge–Tate period map from the resulting perfectoid space to a flag variety [Sch15]. He used the geometry of the period map to prove important p-adic properties of torsion cohomology classes of locally symmetric spaces, and deduced the existence of Galois representations attached to such classes; this was a major breakthrough in the Langlands program. Later, Caraiani and Scholze used their refinement of the Hodge–Tate period map to prove important vanishing properties of torsion cohomology classes of certain Shimura varieties. The proof was simplified by Koshikawa [Kos21] by relating the problem to the cohomology of local Shimura varieties. Even more recently, people have got a clearer understanding of the relationship between the Hodge–Tate period map and the geometrization of local Langlands via the Fargues–Fontaine curve, in particular the groundbreaking work of Fargues–Scholze. From this point of view, more general vanishing results for torsion cohomology classes are being proved.

In this course we will start with the proof of Scholze's theorem that the Siegel modular varieties become perfectoid at infinite level. We will then discuss the important geometric properties of the Hodge–Tate period map as in the works mentioned above. When we move to more recent developments and more advanced topics, we will still try to give as many proofs as possible.

The main references are [Sch12, Sch13, Sch15] by Scholze, and [CS17, CS19] by Caraiani–Scholze. Scholze's expository articles (his CDM report and the two ICM talks), together with the Berkeley Lectures (2014) by Scholze–Weinstein [SW20] are recommended. Moreover, the notes by Caraiani–Shin at IHÉS 2022 Summer School [CS22], and the notes for the Arizona Winter School 2017 on perfectoid spaces [Bha17, Car17, Ked17, Wei17] can be useful.

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1.2. **The idea of a period map.** It is natural to investigate the period maps in general. Let $\{X_s\}$ be a family of "spaces" parametrized by points s in a "base space" S. One may associate to each X_s a linear algebraic invariant $L(X_s)$, and hence get the map of sets

$$\begin{aligned} S &\longrightarrow \{\text{all linear algebraic structures}\} \\ s &\longmapsto L(X_s). \end{aligned}$$

This can be a very rough construction of a *period map*.

To make this more concrete, consider $\{X_s\}$ a family of algebraic varieties parametrized by $s \in S$, where S is the base algebraic variety. By applying a suitable cohomology theory $\mathbf{H}$ to X_s, we would obtain $H_s := \mathbf{H}(X_s)$, regarded as a vector space over some field k. Hodge theory dictates that each H_s admits a filtration structure $\mathrm{Fil}^\bullet H_s$.

However, it is rarely possible to trivialize the system $\{H_s\}_s$ over S. For instance, S may be a complex manifold, and $\{H_s\}_s$ may be a local system on S which is non-trivial due to $\pi_1(S)$ being non-trivial. To remedy this, we morally consider measuring the extent that the trivialization fails to be effective, by inserting the morphism $q\colon \widetilde{S} \to S$, where $\widetilde{S}$ parametrizes all trivializations $H_s \xrightarrow{\sim} H$, with H a fixed constant k-vector space. Namely, for each $\tilde{s} \in \widetilde{S}$ with $q(\tilde{s}) = s$, the point $\tilde{s}$ gives a trivialization $i_{\tilde{s}}\colon H_s \xrightarrow{\sim} H$. (In the complex analytic setting, one can for instance take $\widetilde{S}$ to be the universal covering of S (assumed to be connected); then any local system on S would pull back to a constant local system on $\widetilde{S}$ since the latter is simply connected.) In this situation, we are able to define the period map

$$\begin{aligned} \pi\colon \widetilde{S} &\longrightarrow \{\text{all filtrations on } H\} \\ \tilde{s} &\longmapsto \mathrm{Fil}^\bullet H_s, \end{aligned}$$

where $s = q(\tilde{s})$ and $\mathrm{Fil}^\bullet H_s$ is defined as a filtration on H_s, passing to that on H via the isomorphism $i_{\tilde{s}}$. On the other hand, by definition, the set of all filtrations on H turns out to be a flag variety $\mathscr{F}\ell_H$. We thus obtain the following diagram:

$$\begin{tikzcd} & \widetilde{S} \arrow[dl, "q"'] \arrow[dr, "\pi"] & \\ S & & \mathscr{F}\ell_H. \end{tikzcd}$$

Recall that this diagram arises from an algebraic family of algebraic varieties $\{X_s\}_{s\in S}$ over S. However, $\widetilde{S}$ is often not an algebraic variety but instead some analytic space (i.e. the geometric object defined by convergent power series); depending on the context, "analytic" will mean complex analytic or p-adic analytic. It turns out that both q and π are analytic morphisms. We will be primarily interested in the geometry of π from the arithmetic point of view.

1.3. **A complex analytic example: modular curves.** Consider the complex manifold $\mathcal{H}^{\pm} = \mathbb{C} - \mathbb{R}$ with the $\mathrm{GL}_2(\mathbb{R})$-action via the Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}, \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}),\ z \in \mathcal{H}^{\pm}.$$

Let $N \geqslant 3$ be an integer and define

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \bmod N \right\},$$

which is a discrete subgroup of $\mathrm{GL}_2(\mathbb{R})$. The quotient of $\mathcal{H}^{\pm}$ by the action of $\Gamma(N)$ defines a smooth complex manifold

$$Y(N) \coloneqq \Gamma(N)\backslash\mathcal{H}^{\pm},$$

called the (non-compact) modular curve. It in fact has a unique structure of an affine algebraic variety over $\mathbb{C}$. It is well known that the following map is a bijection:

$$\begin{aligned} Y(N)(\mathbb{C}) &\xrightarrow{\quad\sim\quad} \{(E,\gamma)\}/\cong \\ \tau \in \mathcal{H}^{\pm} &\longmapsto (\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau), ((a,b) \overset{\gamma}{\longmapsto} (a+b\tau)/N)). \end{aligned}$$

Here on the right hand side we consider the set of pairs (E,γ), where

- E is an elliptic curve over $\mathbb{C}$, and
- γ is a full level-N structure, which is equivalent to the choice of an isomorphism $(\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} E[N]$ of groups,

and we mod out by the natural notion of isomorphism. In fact, $Y(N)$ as an algebraic variety is the moduli space over $\mathbb{C}$ of the moduli problem of elliptic curves with level-N structures. There is thus a universal family $\{E_s\}_{s\in Y(N)}$ of elliptic curves over $Y(N)$.

We then discuss the Hodge structure arising from homology for any elliptic curve E over $\mathbb{C}$. The Betti homology $H_1(E,\mathbb{Z})$ is non-canonically isomorphic to $\mathbb{Z}^2$, and it is well known that it admits a Hodge structure of the form

$$H_1(E,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = F^{-1,0} \oplus F^{0,-1}$$

satisfying the complex conjugation condition $\overline{F^{-1,0}} = F^{0,-1}$. (Here complex conjugation is defined with respect to the $\mathbb{R}$-structure $H_1(E,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$.)

To describe this Hodge structure, we first make a general observation in linear algebra. Assume that V is a real vector space of even dimension. Then giving a Hodge decomposition

$$V \otimes_{\mathbb{R}} \mathbb{C} = F^{-1,0} \oplus F^{0,-1}$$

satisfying $\overline{F^{-1,0}} = F^{0,-1}$ is equivalent to having a complex structure on V, i.e., some $\mathsf{J} \in \mathrm{End}_{\mathbb{R}}(V)$ such that $\mathsf{J}^2 = -1$, representing the scalar multiplication by i. To explain this equivalence, note that given such a J, we can construct $F^{-1,0}$ (resp. $F^{0,-1}$) to be the eigenspace of J corresponding to eigenvalue i (resp. $-i$).

Now the Hodge structure on $H_1(E,\mathbb{Z})$ is equivalent to a $\mathbb{C}$-structure on $H_1(E,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$, and the latter is given by the canonical isomorphism

$$H_1(E,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{Lie}\, E,$$

where the right-hand side is a $\mathbb{C}$-vector space. Note that $H_1(E,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C}$ is 2-dimensional over $\mathbb{C}$. Using the Hodge structure, we define the one-step filtration on it by

$$\mathrm{Fil}^0(H_1(E,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C}) \coloneqq F^{0,-1},$$

which is a 1-dimensional subspace of $H_1(E,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C}$.

Note that the quotient map $q\colon \mathcal{H}^{\pm}\to Y(N)$ parametrizes all trivializations of $H_1(E_s,\mathbb{Z})$. Namely, for each $\tau\in\mathcal{H}^{\pm}$ with $s=q(\tau)$, we have the presentation $E_s=\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$, and this presentation gives an isomorphism $H_1(E_s,\mathbb{Z})\cong\mathbb{Z}+\mathbb{Z}\tau$ and hence a trivialization $H_1(E_s,\mathbb{Z})\cong\mathbb{Z}^2$ by using the basis $\{1,\tau\}$ of $\mathbb{Z}+\mathbb{Z}\tau$. From the construction above, we obtain the period map

$$\begin{aligned}\pi\colon \mathcal{H}^{\pm} &\longrightarrow \mathrm{Gr}(2,1)_{\mathbb{C}}=\mathbb{P}^1_{\mathbb{C}}\\ \tau &\longmapsto (\mathrm{Fil}^0\subset H_1(E_s,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C}\xrightarrow{\sim}\mathbb{C}^2),\end{aligned}$$

where $s=q(\tau)\in Y(N)$, and the isomorphism towards $\mathbb{C}^2$ is defined by τ. Also recall that $\mathrm{Gr}(2,1)_{\mathbb{C}}=\mathbb{P}^1_{\mathbb{C}}$ parametrizes all choices of a 1-dimensional $\mathbb{C}$-vector subspace in $\mathbb{C}^2$. As an exercise, check that

$$\pi(\tau)=[\tau:1],$$

for a suitable choice of coordinates on $\mathbb{P}^1_{\mathbb{C}}$.

In summary, we have the diagram

$$\begin{tikzcd} & \mathcal{H}^{\pm} \arrow[dl, "q"'] \arrow[dr, "\pi"] & \\ \Gamma(N)\backslash\mathcal{H}^{\pm} & & \mathbb{P}^1_{\mathbb{C}} \end{tikzcd}$$

Note that π is in fact independent of the level N. Moreover, it is $\mathrm{GL}_2(\mathbb{R})$-equivariant, where $\mathrm{GL}_2(\mathbb{R})$ acts naturally on $\mathbb{P}^1_{\mathbb{C}}$. Finally, note that $\mathcal{H}^{\pm}=\mathbb{C}-\mathbb{R}$ is not an algebraic variety but a $\mathbb{C}$-manifold, with q and π being analytic.

1.4. **The p-adic setup.** In this subsection we consider a p-adic analogue for the complex period map for the modular curves. Let p be a prime and write $\mathbb{C}_p=\widehat{\overline{\mathbb{Q}}}_p$. Fix a field isomorphism $\mathbb{C}\cong\overline{\mathbb{Q}}_p$, so we have an embedding $\mathbb{C}\hookrightarrow\mathbb{C}_p$. Let E be any elliptic curve over $\mathbb{C}_p$. Take the étale cohomology group

$$H^1_{\mathrm{et}}(E,\mathbb{Z}_p)\cong\mathbb{Z}_p^2,$$

which is dual to the Tate module $T_pE\coloneqq\varprojlim_n E[p^n]$. By general p-adic Hodge theory, there is a Hodge–Tate filtration on $H^1_{\mathrm{et}}(X,\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\mathbb{C}_p$ for any smooth projective variety X over $\mathbb{C}_p$ (or more generally, any smooth proper rigid analytic variety X over $\mathbb{Z}_p$). In the case of elliptic curves, this is defined as the image of a natural injection

$$\mathrm{Lie}\,E^*\longhookrightarrow H^1_{\mathrm{et}}(E,\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\mathbb{C}_p.$$

Here E^* is the dual elliptic curve, and the above injection is an injection from a 1-dimensional $\mathbb{C}_p$-vector space into a 2-dimensional $\mathbb{C}_p$-vector space.

Fix an integer $N\geqslant 3$ satisfying $p\nmid N$. We base change the modular curve $Y(Np^k)$ over $\mathbb{C}$ along $\mathbb{C}\hookrightarrow\mathbb{C}_p$, and obtain $Y(Np^k)_{\mathbb{C}_p}$, an algebraic curve over $\mathbb{C}_p$. This construction is in fact independent of the choice of $\mathbb{C}\to\mathbb{C}_p$; in fact $Y(Np^k)$ has a canonical model over $\mathbb{Q}$ representing the same moduli problem over $\mathbb{Q}$. For $s\in Y(Np^k)_{\mathbb{C}_p}$, we have the canonical trivialization

$$H^1_{\mathrm{et}}(E_s,\mathbb{Z}/p^k\mathbb{Z})\cong(\mathbb{Z}/p^k\mathbb{Z})^2,$$

arising from the trivialization $E[p^k]\cong(\mathbb{Z}/p^k\mathbb{Z})^2$ given by the level structure. Varying the level, it is reasonable to consider the inverse limit

$$\varprojlim_k Y(Np^k)\coloneqq\varprojlim(\cdots\to Y(Np^2)\to Y(Np)\to Y(N)).$$

Here each map $Y(Np^k)\to Y(Np^{k-1})$ is finite étale. If we imagine that the inverse limit exists in a suitable category, then on the inverse limit we would imagine that there is a trivialization of

$$H^1_{\mathrm{et}}(E_s,\mathbb{Z}_p)\coloneqq\varprojlim_k H^1_{\mathrm{et}}(E_s,\mathbb{Z}/p^k\mathbb{Z}).$$

The following theorem asserts that this inverse limit exists as a *perfectoid space.*

Theorem 1.4.1 (Scholze, a moral version, cf. [Sch15])**.**

(1) *There is a unique perfectoid space* $\widetilde{Y}_N$ *over* $\mathbb{C}_p$ *such that*

$$\widetilde{Y}_N \sim \varprojlim_k Y(Np^k)_{\mathbb{C}_p},$$

where $\sim$ *can be defined precisely in Definition 3.1.10.*

(2) *There is a Hodge–Tate period map between adic spaces, written as*

$$\pi_{\mathrm{HT}} \colon \widetilde{Y}_N \longrightarrow (\mathbb{P}^1_{\mathbb{C}_p})^{\mathrm{ad}},$$

encoding the Hodge–Tate filtration

$$\mathrm{Lie}\, E_s^* \hookrightarrow H^1_{\mathrm{et}}(E_s, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{C}_p \cong \mathbb{C}_p^2$$

at least for $\mathbb{C}_p$*-valued points* $s \in \widetilde{Y}_N$*. Here the last isomorphism comes from the trivialization* $H^1_{\mathrm{et}}(E_s, \mathbb{Z}_p) \cong \mathbb{Z}_p^2$ *for* $s \in \widetilde{Y}_N$*, as one may expect. The target of* π_{HT} *is the adic space associated to* $\mathbb{P}^1$*, often understood as the "analytification" of* $\mathbb{P}^1$*.*

(3) *The Hodge–Tate period map has interesting geometry, and its fibers can be described (in terms of Igusa varieties).*

1.5. **Perfectoid spaces.** A perfectoid space is a highly special kind of adic space. The category of adic spaces is very robust within the notion of p-adic analytic spaces. Perfectoid spaces do not satisfy any usual finiteness properties, yet can be very nice from other perspectives.

Definition 1.5.1. A *perfectoid field* is a complete topological field K whose topology is defined by a non-archimedean absolute value $|\cdot| \colon K \to \mathbb{R}_{\geqslant 0}$ such that

(1) $|\cdot|$ is non-discrete, and

(2) if we write p as the residue characteristic, then the ring homomorphism

$$\mathcal{O}_K/p \longrightarrow \mathcal{O}_K/p, \quad x \longmapsto x^p$$

is surjective.

Non-Example 1.5.2. Any extension of $\mathbb{Q}_p$ of finite ramification cannot be perfectoid.

Example 1.5.3. The p-adic completions $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ and $(\mathbb{Q}_p(p^{1/p^\infty}))^\wedge$, together with the t-adic completion $(\mathbb{F}_p(t^{1/p^\infty}))^\wedge$, are all perfectoid fields.

Construction 1.5.4 (Tilting)**.** We can elementarily construct a functor

$$\begin{array}{ccc} (\text{Perfectoid fields with residue characteristic } p) & & K \\ \downarrow & & \downarrow \\ (\text{Perfectoid fields of characteristic } p) & & K^\flat \end{array}$$

called *tilting.* This in particular sends perfectoid fields of mixed characteristic $(0, p)$ to perfectoid fields of characteristic p.

Theorem 1.5.5 (Fontaine–Wintenberger, Scholze, cf. [FW79, Sch12])**.**

(1) *Let* K *be a perfectoid field and* L *be a finite extension of* K*. Then, with the natural topology,* L *is a perfectoid field.*

(2) *Tilting induces an equivalence of categories*

$$\begin{array}{ccc} (\textit{Finite extensions of } K) & \longrightarrow & (\textit{Finite extensions of } K^\flat) \\ L & \longmapsto & L^\flat \end{array}.$$

(It follows from (1) *that this functor is well defined.) In particular,* $\mathrm{Gal}_K \cong \mathrm{Gal}_{K^\flat}$*.*

Example 1.5.6. Under tilting, the perfectoid field $(\mathbb{Q}_p(p^{1/p^\infty}))^\wedge$ goes to the perfectoid field $(\mathbb{F}_p(t^{1/p^\infty}))^\wedge$.

Perfectoid spaces are higher dimensional geometric generalizations of perfectoid fields.

Theorem 1.5.7. *Let K be a perfectoid field. Then there is an equivalence of categories*

$$\begin{aligned} (\textit{Perfectoid spaces over } K) &\longrightarrow (\textit{Perfectoid spaces over } K^\flat) \\ X &\longmapsto X^\flat. \end{aligned}$$

Moreover, we have an isomorphism of the underlying topological spaces $|X| \cong |X^\flat|$, and an equivalence of the étale sites $X_{\mathrm{et}} \cong (X^\flat)_{\mathrm{et}}$.

Remark 1.5.8. The tilting equivalence serves as a main player in Scholze's proof of weight-monodromy conjecture, which helps him to reduce the conjecture in certain cases to the equal characteristic case.

Example 1.5.9. Here comes a toy example for an inverse limit of algebraic varieties becoming perfectoid. Let K be a perfectoid field, with tilt $K^\flat$. Consider the projective system

$$(\cdots \longrightarrow \mathbb{A}^1_K \longrightarrow \mathbb{A}^1_K \longrightarrow \cdots \longrightarrow \mathbb{A}^1_K)$$

in which each map is given by $x \mapsto x^p$ on the coordinate x. There is a unique perfectoid space X over K such that

$$X \sim \varprojlim \mathbb{A}^1_K,$$

where $\sim$ is in the same sense as in Theorem 1.4.1. Also, as in Theorem 1.5.7, we can consider $X^\flat$, which is a perfectoid space over $K^\flat$, and we have

$$|X| \cong |X^\flat| \cong |\mathbb{A}^{1,\mathrm{ad}}_{K^\flat}|, \quad X_{\mathrm{et}} \cong (X^\flat)_{\mathrm{et}} \cong (\mathbb{A}^{1,\mathrm{ad}}_{K^\flat})_{\mathrm{et}}.$$

Here $\mathbb{A}^{1,\mathrm{ad}}_{K^\flat}$ is the adic space over $K^\flat$ attached to $\mathbb{A}^1$. It is not a perfectoid space, but it turns out that the perfectoid space $X^\flat$ has the same underlying topological space and étale site as $\mathbb{A}^{1,\mathrm{ad}}_{K^\flat}$.

1.6. **Generalizations and applications.** More generally, one can replace the modular curve $Y(N)$ in the previous discussion by Siegel modular varieties (i.e. moduli space of polarized abelian varieties with level structures), or even Hodge-type Shimura varieties. Our first main goal in the course is to understand Theorem 1.4.1 and the generalization to these Shimura varieties. The application lies in using the geometry of π_{HT} to understand the cohomology of Shimura varieties [CS17, CS19]. We remark that the geometry of π_{HT} has relation with cohomology of Scholze's local Shimura varieties, and the geometry of LLC by Fargues–Scholze.

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1.7. **Rough description of π_{HT} for modular curve.** We resume on the setup of §1.4 and only consider $\mathbb{C}_p$-valued points s on $\widetilde{Y}_N$ (cf. Theorem 1.4.1(2)), namely the classical points. For each classical point s, we obtain E_s over $\mathbb{C}_p$ with the trivialization $H^1_{\mathrm{et}}(E_s, \mathbb{Z}_p) \cong \mathbb{Z}_p^2$. In the following discussion we only consider s in the *good reduction locus*, which in concrete terms means that E_s over $\mathbb{C}_p$ has good reduction, i.e., it extends to an elliptic curve over $\mathcal{O}_{\mathbb{C}_p}$. In this case, we denote by $\overline{E}_s$ the reduction over $\overline{\mathbb{F}}_p$.

Recall that the p^k-torsion $\mathcal{G}_k := \overline{E}_s[p^k]$ is a finite flat group scheme over $\overline{\mathbb{F}}_p$ of rank p^{2k}. Furthermore, each $\overline{E}_s$ is either *ordinary* or *supersingular*. The distinction is as follows: We write $\mathcal{G}_k^\circ$ for the identity component of the group scheme $\mathcal{G}_k$.

- When $\overline{E}_s$ is ordinary, namely $s \in \widetilde{Y}_N^{\mathrm{ord}}$, the rank of $\mathcal{G}_k^0$ is p^k. There is a short exact sequence

  $$0 \longrightarrow \mathcal{G}_k^\circ \longrightarrow \mathcal{G}_k \longrightarrow \mathcal{G}_k/\mathcal{G}_k^\circ \longrightarrow 0,$$

  in which the quotient is étale and has rank p^k. Also, we have

  $$\mathcal{G}_k(\overline{\mathbb{F}}_p) \cong (\mathcal{G}_k/\mathcal{G}_k^\circ)(\overline{\mathbb{F}}_p),$$

  and this group is abstractly isomorphic to $\mathbb{Z}/p^k\mathbb{Z}$. In particular, the surjective reduction map

  $$E_s[p^k](\mathbb{C}_p) \longrightarrow \overline{E}_s[p^k](\overline{\mathbb{F}}_p)$$

  is isomorphic to a surjection $(\mathbb{Z}/p^k\mathbb{Z})^2 \to \mathbb{Z}/p^k\mathbb{Z}$. We write C_k for its kernel. Then C_k is an order p^k cyclic subgroup of $E_s(\mathbb{C}_p)$, known as the *canonical subgroup*.

  These constructions are compatible with the varying of k. More precisely, the following diagram commutes:

$$\begin{tikzcd} E_s[p^{k+1}] \arrow[r, "{[p]}"] & E_s[p^k] \\ C_{k+1} \arrow[u, hook] \arrow[r, two heads] & C_k. \arrow[u, hook] \end{tikzcd}$$

- When $\overline{E}_s$ is supersingular, namely $s \in \widetilde{Y}_N^{\mathrm{ss}}$, we have
$$\mathcal{G}_k = \mathcal{G}_k^\circ, \qquad \mathcal{G}_k(\overline{\mathbb{F}}_p) = \{0\}.$$

The ordinary case. Suppose $\overline{E}_s$ is ordinary. In the Tate module,
$$T_p(E_s) := \varprojlim_k E_s[p^k] \cong \mathbb{Z}_p^2,$$
the canonical subgroups $\{C_k\}_k$ define a rank-1 $\mathbb{Z}_p$-submodule $C = \varprojlim_k C_k \subset T_p(E_s)$. By duality, we obtain a 1-dimensional $\mathbb{Q}_p$-subspace of
$$H^1_{\mathrm{et}}(E_s, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$
Recall that the above is canonically trivialized to $\mathbb{Q}_p^2$. We thus obtain a line in $\mathbb{Q}_p^2$, i.e., a point $u_s \in \mathbb{P}^1(\mathbb{Q}_p)$.

Proposition 1.7.1. *For s in the ordinary locus, we have*
$$\pi_{\mathrm{HT}}(s) = u_s \in \mathbb{P}^1(\mathbb{Q}_p).$$
Also, π_{HT} shrinks every connected component of $\widetilde{Y}_N^{\mathrm{ord}}$ to a point in $\mathbb{P}^1(\mathbb{Q}_p)$.

Morally, Proposition 1.7.1 dictates that the restriction of π_{HT} to $\widetilde{Y}_N^{\mathrm{ord}}$ measures the position of the canonical subgroup $C = \varprojlim_k C_k$ in $T_p(E_s)$. Moreover it is "locally constant".

The supersingular case. Up to isomorphism there are only finitely many supersingular elliptic curves over $\overline{\mathbb{F}}_p$ with additional level-N structure, listed as $E_1, \cdots, E_m$. Denote by $\widetilde{Y}_{N,i}$ the locus in $\widetilde{Y}_N$ (or rather, the good reduction locus) on which $\overline{E}_s \simeq E_i$ together with the level-N structure. We shall describe π_{HT} on $\widetilde{Y}_{N,i}$. Firstly, we have
$$\widetilde{Y}_{N,i} \sim \varprojlim_k \mathcal{M}_k,$$
where $(\cdots \to \mathcal{M}_k \to \cdots \to \mathcal{M}_1 \to \mathcal{M}_0)$ is the *Lubin–Tate tower* to be discussed below, and $\sim$ is in the same sense as in Theorem 1.4.1.

Construction 1.7.2 (The Lubin–Tate tower and the Drinfeld tower)**.** We start with the formal scheme $\mathfrak{M}$ over $\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p})$ representing the moduli problem of connected p-divisible groups of height 2 and dimension 1. (We omit the formulation of the precise moduli problem.) Abstractly, $\mathfrak{M}$ is isomorphic to $\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p}[\![X]\!])$. Let $\mathcal{M}_0 = \mathfrak{M}_\eta^{\mathrm{ad}}$ be the adic generic fiber of $\mathfrak{M}$, which is an adic space over $\mathbb{C}_p$ (isomorphic to the open unit disc over $\mathbb{C}_p$). By adding more and more level structures, we get a tower
$$(\cdots \longrightarrow \mathcal{M}_k \longrightarrow \cdots \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_0 = \mathfrak{M}_\eta^{\mathrm{ad}}), \tag{LT}$$
called the *Lubin–Tate tower.*

Similarly, we can start with the formal scheme $\mathfrak{M}^\vee$ over $\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p})$ representing the moduli problem of certain p-divisible groups of height 4 and dimension 2, together with an $\mathcal{O}_D$-action. Here D is the quaternion algebra over $\mathbb{Q}_p$. By passing to the adic generic fiber and adding more and more level structures, we get the *Drinfeld tower*
$$(\cdots \longrightarrow \mathcal{M}_k^\vee \longrightarrow \cdots \longrightarrow \mathcal{M}_1^\vee \longrightarrow \mathcal{M}_0^\vee = (\mathfrak{M}^\vee)_\eta^{\mathrm{ad}}). \tag{Dr}$$

Thanks to the classification by Scholze–Weinstein of p-divisible groups [SW13], the Lubin–Tate tower and Drinfeld tower become isomorphic at infinite level, i.e., there is a unique perfectoid space that is $\sim \varprojlim_k \mathcal{M}_k$ *and* $\sim \varprojlim_k \mathcal{M}_k^\vee$. Therefore,
$$\widetilde{Y}_{N,i} \sim \varprojlim_k \mathcal{M}_k^\vee.$$

In particular, for any $k \in \mathbb{N}$ we have a canonical map

$$\pi_k : \widetilde{Y}_{N,i} \longrightarrow \mathcal{M}_k^\vee .$$

For $k = 0$, we have $\mathcal{M}_0^\vee$ being the complement of $\mathbb{P}^1(\mathbb{Q}_p)$ in the adic space $(\mathbb{P}^1_{\mathbb{C}_p})^{\mathrm{ad}}$. This is denoted by Ω and called the *Drinfeld upper-half space*, analogous to $\mathcal{H}^\pm = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$.

Proposition 1.7.3. *The map*

$$\pi_0 : \widetilde{Y}_{N,i} \longrightarrow \mathcal{M}_0^\vee = \Omega \subset (\mathbb{P}^1_{\mathbb{C}_p})^{\mathrm{ad}}$$

is equal to π_{HT}.

Summary on geometry of π_{HT}.

$$\begin{array}{ccccc} \widetilde{Y}_N & \supset & \widetilde{Y}_N^{\mathrm{ss}} & \sqcup & \widetilde{Y}_N^{\mathrm{ord}} \\ \downarrow{\scriptstyle \pi_{\mathrm{HT}}} & & \downarrow{\scriptstyle \pi_{\mathrm{HT}}^{\mathrm{ss}}} & & \downarrow{\scriptstyle \pi_{\mathrm{HT}}^{\mathrm{ord}}} \\ (\mathbb{P}^1_{\mathbb{C}_p})^{\mathrm{ad}} & = & \Omega & \sqcup & \mathbb{P}^1(\mathbb{Q}_p) \end{array}$$

Here in the first row we only have $\supset$ instead of $=$ because we omitted the points of bad reduction.

Remark 1.7.4. (1) Here we have only described π_{HT} on classical points. The case can be more subtle on more general types of points.

(2) The modular curve $Y(N)$ has a canonical smooth model over $\mathbb{Z}_p$ (representing the same moduli problem over $\mathbb{Z}_p$), and the special fiber has a stratification

$$Y(N)_{\mathbb{F}_p} = Y(N)^{\mathrm{ord}}_{\mathbb{F}_p} \sqcup Y(N)^{\mathrm{ss}}_{\mathbb{F}_p}.$$

Here the ordinary and supersingular loci are Zariski open and closed, respectively. In contrast, inside $(\mathbb{P}^1_{\mathbb{C}_p})^{\mathrm{ad}}$, Ω is open and $\mathbb{P}^1(\mathbb{Q}_p)$ is closed.

2. Adic spaces

2.1. **The rough idea.** As a rough idea, adic spaces are glued from affinoid adic spaces, resembling the situation that schemes are glued from affine schemes. For a *Huber pair* (A, A^+), which is a pair of topological rings $A^+ \subset A$ satisfying certain conditions, we can construct the affinoid adic spectrum

$$\mathrm{Spa}(A, A^+) := \{\text{continuous valuations } |\cdot| : A \to \Gamma \cup \{0\} \text{ such that } |A^+| \leqslant 1\}/\sim,$$

where Γ is a totally ordered abelian group written multiplicatively, such as $(\mathbb{R}_{>0}, \times)$. The set $Y = \mathrm{Spa}(A, A^+)$ is equipped with the so-called adic topology and it has a structure presheaf $\mathcal{O}_Y$. It is in general a subtle question as to determine when $\mathcal{O}_Y$ is a sheaf. When it is a sheaf, we say that the Huber pair (A, A^+) is *sheafy*. Roughly, adic spaces are glued from $\mathrm{Spa}(A, A^+)$ where (A, A^+) are Huber pairs.

Definition 2.1.1. An *adic space* is an object $(X, \mathcal{O}_X)$ in the category of locally topologically ringed spaces, equipped with choice of (equivalence class of) continuous valuations on the local rings $\mathcal{O}_{X,x}$ for all $x \in X$, which is locally isomorphic to $\mathrm{Spa}(A, A^+)$ for a sheafy Huber pair (A, A^+).

2.2. **Huber rings.** We now discuss the conditions we impose on the pair (A, A^+).

Definition 2.2.1. Let A be a topological ring.

(1) We say A is *Huber* if there is an open subring $A_0 \subset A$ such that the topology on A_0 is I-adic, for some finitely generated ideal I of A_0. Recall that the I-adic topology on A_0 is the unique topology such that $\{I^n\}$ is an open neighborhood basis of $0 \in A_0$. If A is Huber, then any choice of A_0 as above (which is non-unique) is called a *ring of definition*, and I is called an *ideal of definition*.

(2) We say A is *Tate* if it is Huber and there is $\varpi \in A^\times$ which is topologically nilpotent, i.e., $\varpi^n \to 0$ as $n \to \infty$. Such a choice of ϖ is called a *pseudo-uniformizer*.

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Here are some simple observations about Tate rings. For a Tate ring A, we choose a pseudo-uniformizer ϖ and a ring of definition $A_0 \subset A$. Then $\varpi^n \in A_0$ for $n \gg 0$ since A_0 is open. In this case, we may replace ϖ by ϖ^n and assume $\varpi \in A_0$. Then we have the following.

Lemma 2.2.2. *The topology on A_0 is ϖA_0-adic. Moreover, we have $A = A_0[\varpi^{-1}]$.*

Proof. For any open neighborhood U of 0 in A_0, we show that $(\varpi A_0)^m \subset U$ for all sufficiently large m. we may assume $U = I^n$ for some ideal of definition $I \subset A_0$. We see from Definition 2.2.1(2) that $\varpi^m \to 0$, and hence $\varpi^m \in U = I^n$ for $m \gg 0$. Since I^n is an ideal, this further implies $(\varpi A_0)^m \subset U$. On the other hand, for any $m \geqslant 0$, we need $(\varpi A_0)^m = \varpi^m A_0$ to be open. But this is true because the multiplication-by-ϖ^m map is a homeomorphism $A \xrightarrow{\sim} A$. □

Conversely, start with any ring A_0 and the nonzero-divisor $\varpi \in A_0$, we define $A = A_0[\varpi^{-1}]$ and put topology on A by declaring that A_0 is open, where the topology on A_0 is ϖ-adic. In this case, A is a priori a topological group and we can check that the multiplication is continuous. Then A is Tate with A_0 is being ring of definition and ϖ_0 being a pseudo-uniformizer.

Definition 2.2.3. A *non-archimedean field* is a field K equipped with the absolute value $|\cdot|\colon K \to \mathbb{R}_{\geqslant 0}$, such that

- $|\cdot|$ takes at least 3 different values, and
- the topology on K induced by $|\cdot|$ is non-archimedean and complete. (We do not require that $|\cdot|$ is discretely valued.)

Example 2.2.4 (Tate rings)**.** Let K be a non-archimedean field.

(1) The ring K is Tate with a ring of definition $\mathcal{O}_K = \{x \in K\colon |x| \leqslant 1\}$. Moreover, for any $\varpi \in K^\times$, it is a pseudo-uniformizer if and only if $|\varpi| < 1$.
(2) Let A be a topological K-algebra, where we require that the map $K \to A$ is a homeomorphism to its image. Then A is Tate if and only if there is an open subring $A_0 \subset A$ such that $\{xA_0\colon x \in K^\times\}$ is a neighborhood basis of 0 in A. (Note that each xA_0 is automatically open in A, since multiplication by x is a homeomorphism.) In this case, any pseudo-uniformizer ϖ in K is also a pseudo-uniformizer in A.
(3) Let $(A, \|\cdot\|)$ be a normed K-algebra, namely A is a K-algebra and $\|\cdot\|\colon A \to \mathbb{R}_{\geqslant 0}$ is a map satisfying that
   - $\|x\| = 0$ if and only if $x = 0$,
   - $\|x + y\| \leqslant \max(\|x\|, \|y\|)$,
   - $\|\lambda x\| = |\lambda| \cdot \|x\|$ for any $\lambda \in K$ and $x \in A$,
   - $\|xy\| \leqslant \|x\| \cdot \|y\|$, and
   - $\|1\| \leqslant 1$.

   We define the topology on A by the metric $(x, y) \mapsto \|x - y\|$. Then A is Tate by the criterion in (2). Indeed, we can take A_0 as in (2) to be $\{a \in A\colon \|a\| \leqslant 1\}$.

Definition 2.2.5 (Power-bounded elements)**.**

(1) In a topological ring A, a subset $S \subset A$ is *bounded* if for any neighborhood U of 0, there is a neighborhood V of 0 such that $S \cdot V \subset U$.
(2) An element $a \in A$ is called *power-bounded* if $\{a^n : n \geqslant 1\}$ is a bounded subset of A. We denote

$$A^0 := \{\text{power-bounded elements}\},$$
$$A^{00} := \{\text{topologically nilpotent elements}\}.$$

Exercise 2.2.6. Prove the following assertions.

(1) For any topological ring A, we always have $A^0 \supset A^{00}$.
(2) If A is Huber, then A^0 is a subring of A.
(3) If A is discrete, then every subset is bounded, and $A^0 = A$.
(4) If A is Tate with a ring of definition A_0 and a pseudo-uniformizer ϖ, then $S \subset A$ is a bounded subset if and only if $S \subset \varpi^{-n} A_0$ for some $n \geqslant 1$.

(5) Let K be a non-archimedean field, and let $(A, \|\cdot\|)$ be a normed K-algebra. Then $S \subset A$ is a bounded subset if and only if $\sup_{s\in S}\|s\| < \infty$. If $\|\cdot\|$ is further multiplicative (i.e. $\|xy\| = \|x\|\cdot\|y\|$ for all $x, y \in A$), then we have $A^0 = \{a \in A \colon \|a\| \leqslant 1\}$.
(6) (Gauss Lemma.) Let $(A, \|\cdot\|)$ be a normed K-algebra. Let $B = \{a \in A \colon \|a\| \leqslant 1\}$, and $J = \{a \in A \colon \|a\| < 1\}$. Then B is a subring, and J is an ideal of B. Assume that the image of $\|\cdot\| \colon A \to \mathbb{R}_{\geqslant 0}$ is equal to the image of $|\cdot| \colon K \to \mathbb{R}_{\geqslant 0}$. Then the ideal J is prime if and only if $\|\cdot\|$ is multiplicative.

Proposition 2.2.7. *Suppose A is a Huber ring and $B \subset A$ is a subring. Then the following are equivalent:*

(1) *B is a ring of definition in A;*
(2) *B is open and bounded.*

Proof. We leave the implication (1) $\Rightarrow$ (2) as an exercise. For (2) $\Rightarrow$ (1), we need that the topology of B is J-adic for some finitely generated ideal $J \subset B$. Choose $A_0 \subset A$ a ring of definition and $I \subset A_0$ an ideal of definition. Take a finite set $T \subset A_0$ generating I as an ideal in A_0. For each integer $n \geqslant 1$, we write T^n for the set of all monomials of degree n formed by elements of T.

Since B is assumed to be open, we have $I^k \subset B$ for some $k \gg 0$ as $\{I^k\}_k$ is a neighborhood basis of 0 in A. (Here I^k denotes the k-th power of I as an ideal of A_0.) We fix such a k and define $J \coloneqq T^kB$, which is a finitely generated ideal of B. It remains to prove that the topology on B is J-adic.

For any $n \geqslant 1$, we have $J^n = T^{nk}B$. (Here J^n denotes the n-th power of J as an ideal of B.) Since $B \supset I^n$, we have $J^n \supset T^{nk}I^n = I^{nk+n}$, which implies that J^n is open.

Conversely, for any neighborhood U of 0 in B, we need to show that a sufficiently high power of J is contained in U. As B is assumed to be bounded, there is a neighborhood V of 0 in A such that $V \cdot B \subset U$ by Definition 2.2.5(1). There is an integer n such that $I^n \subset V$, and hence $I^n \cdot B \subset U$. But $I^n \cdot B \supset J^n$, which completes the proof. $\square$

Corollary 2.2.8. *Suppose A is a Huber ring. Then A is bounded if and only the topology on A is adic defined by some finitely generated ideal.*

Note that if A is Huber with a ring of definition A_0, then A_0 is bounded (which is one direction of Proposition 2.2.7). For any $a \in A_0$, the set $\{a^n\}$ is contained in A_0, and hence bounded. It follows that $A_0 \subset A^0$. In particular, A^0 is an open subring of A. Moreover, we have the following.

Exercise 2.2.9. When A is a Huber ring, A^0 is the union of all possible rings of definition in A. Moreover, this union is filtered, i.e., the union of any two rings of definition is contained in a third.

Definition 2.2.10. Let A be a Huber ring. We call A *uniform* if the subring of power-bounded elements A^0 is bounded, i.e., A^0 itself is a ring of definition.

Example 2.2.11. Any normed K-algebra $(A, \|\cdot\|)$ with $\|\cdot\|$ multiplicative is uniform since $A^0 = \{a \in A \colon \|a\| \leqslant 1\}$ is bounded (cf. Exercise 2.2.6(5)).

Example 2.2.12. The following are some examples and non-examples of Huber and Tate rings.

(1) A discrete ring A is always Huber with the ring of definition A and ideal of definition (0).
(2) Any non-archimedean field K is Tate. Note that $K^0 = \mathcal{O}_K$ and $K^{00} = \mathfrak{m}_K$. The ring of integers $\mathcal{O}_K$ is also Huber *but not Tate*.
(3) Continue with (2). The ring of formal power series $\mathcal{O}_K[\![T_1, \cdots, T_n]\!]$ is equipped with the adic topology defined by $(\varpi, T_1, \cdots, T_n)$, where $\varpi \in \mathfrak{m}_K$. It is Huber and bounded, *but not Tate.*
(4) Similarly, for any discrete ring R, the ring of formal power series $R[\![T_1, \cdots, T_n]\!]$ is equipped with the $(T_1, \cdots, T_n)$-adic topology. It is Huber and bounded, *but not Tate.*

Example 2.2.13 (Affinoid algebra)**.** For any non-archimedean field $(K, |\cdot|)$, consider

$$A = K\langle T_1, \cdots, T_n\rangle = \{f(T_1, \cdots, T_n) \in K[\![T_1, \cdots, T_n]\!] \colon \text{coefficients of } f \text{ tend to } 0\}.$$

It is equipped with the Gauss norm

$$\|\cdot\| \colon A \longrightarrow \mathbb{R}_{\geqslant 0}, \qquad \sum_{\underline{i}\in\mathbb{N}^n} a_{\underline{i}} T^{\underline{i}} \longmapsto \sup_{\underline{i}} |a_{\underline{i}}|_K.$$

Then $(A, \|\cdot\|)$ is a normed K-algebra, and hence Tate. Moreover, in the notation of Exercise 2.2.6(6), we have

$$B = \{f \in K\langle T_1, \cdots, T_n\rangle \colon \text{all coefficients of } f \text{ are in } \mathcal{O}_K\}$$

and

$$J = \{f \in K\langle T_1, \cdots, T_n\rangle \colon \text{all coefficients of } f \text{ are in } \mathfrak{m}_K\}.$$

Note that $B/J \cong (\mathcal{O}_K/\mathfrak{m}_K)[T_1, \cdots, T_n]$, which is an integral domain. Hence by that exercise, $\|\cdot\|$ is multiplicative. Then by Exercise 2.2.6(5) and Example 2.2.11, A is uniform and we have

$$A^0 = B = \{f \in K\langle T_1, \cdots, T_n\rangle \colon \text{all coefficients of } f \text{ are in } \mathcal{O}_K\}.$$

This ring is also denoted by $\mathcal{O}_K\langle T_1, \cdots, T_n\rangle$. Note that this is the ϖ-adic completion of $\mathcal{O}_K[T_1, \cdots, T_n]$ for any pseudo-uniformizer $\varpi \in \mathfrak{m}_K$.

The objects in Example 2.2.12 and 2.2.13 are all uniform. The following comes a non-uniform example.

Example 2.2.14. Let $A = \mathbb{Q}_p[T]/T^2$. Define Gauss norm on A as before. Then A is a normed $\mathbb{Q}_p$-algebra and hence Tate. However, $A^0 = \mathbb{Z}_p + \mathbb{Q}_p T$ is unbounded. So A is non-uniform. (Note that the Gauss norm on A is no longer multiplicative.)

2.3. **Definition of adic spectra.**

Definition 2.3.1. Suppose A is a Huber ring.

(1) A subring $A^+ \subset A$ is called a *ring of integral elements* if it is open and integrally closed in A and if it satisfies $A^+ \subset A^0$.
(2) When $A^+ \subset A$ is a ring of integral elements, we call the pair (A, A^+) a *Huber pair*.

Our goal is now to define $\mathrm{Spa}(A, A^+)$ for a Huber pair (A, A^+).

Definition 2.3.2. Let A be a topological ring. By a *valuation* on A we mean a pair (v, Γ), where

- Γ is a totally ordered abelian group written multiplicatively.
- $v \colon A \to \Gamma \cup \{0\}$ is a map such that
  - $v(ab) = v(a) \cdot v(b)$,
  - $v(a + b) \leqslant \max(v(a), v(b))$, and
  - $v(1) = 1$, $v(0) = 0$.

  Here 0 is a formal symbol outside Γ (whose neutral element is denoted by 1), and we keep the conventions $\gamma > 0$ and $\gamma \cdot 0 = 0 \cdot \gamma = 0$ for all $\gamma \in \Gamma$.

We call v *continuous* if for any $a \in A$ the set $\{b \in A \colon v(b) < v(a)\}$ is open.

Exercise 2.3.3. Given a valuation (v, Γ) on A as above, we have the following.

(1) Γ is torsion-free, and thus for any $a \in A$, the condition $v(a^n) = 1$ for some $n \geqslant 1$ implies $v(a) = 1$. In particular, $v(-1) = 1$.
(2) For any $a, b \in A$ such that $v(a) \neq v(b)$, we always have $v(a + b) = \max(v(a), v(b))$.
(3) The continuity of v is equivalent to the following condition: For any converging sequence (a_n) in A with limit a,
   (a) If $v(a) \neq 0$, then $v(a) = v(a_n)$ for $n \gg 0$;
   (b) If $v(a) = 0$, then for any $b \in A$ such that $v(b) \neq 0$, we have $v(a_n) < v(b)$ for $n \gg 0$.

Fact 2.3.4. Let (v, Γ) and (v', Γ') be two valuations on A. Then the following are equivalent:

(1) For any $a, b \in A$, $v(a) \leqslant v(b)$ if and only if $v'(a) \leqslant v'(b)$.
(2) There exist subgroups $\Gamma_1 < \Gamma$ and $\Gamma_1' < \Gamma'$ such that $\Gamma_1 \cup \{0\}$ and $\Gamma_1' \cup \{0\}$ contain the images of v and v' respectively, and there exists an isomorphism $\Gamma_1 \cong \Gamma_1'$ such that the resulting bijection $\Gamma_1 \cup \{0\} \cong \Gamma_1' \cup \{0\}$ takes v to v'.

Definition 2.3.5. Say v and v' are *equivalent* via $\sim$ in the case (1) or (2) of Fact 2.3.4.

Definition 2.3.6 (Adic spectrum)**.** Let (A, A^+) be a Huber pair. Define its *adic spectrum* to be

$$\mathrm{Spa}(A, A^+) = \{(v, \Gamma) \text{ continuous valuation on } A\colon v(A^+) \leqslant 1\}/\sim.$$

The topology on $\mathrm{Spa}(A, A^+)$ is generated by sets of form

$$U\left(\frac{f}{g}\right) = \{v \in \mathrm{Spa}(A, A^+)\colon v(f) \leqslant v(g) \neq 0\}$$

for some fixed $f, g \in A$.

For any finite subset $T \subset A$ and $g \in A$, we write

$$U\left(\frac{T}{g}\right) \coloneqq \bigcap_{t \in T} U\left(\frac{t}{g}\right).$$

This is an open set in $\mathrm{Spa}(A, A^+)$. If $T = \{t_1, \cdots, t_n\}$, we also write

$$U\left(\frac{t_1, \cdots, t_n}{g}\right) \coloneqq U\left(\frac{T}{g}\right).$$

Definition 2.3.7. A subset of $\mathrm{Spa}(A, A^+)$ is called *rational* if it is of form

$$U\left(\frac{T}{g}\right)$$

where T is a finite subset of A generating an open ideal in A, namely such that TA is open. (If A is Tate, any open ideal must be equal to A.)

Lemma 2.3.8. *Rational subsets form a basis of topology of* $\mathrm{Spa}(A, A^+)$ *and the basis is stable under finite intersections.*

Proof. From Definition 2.3.6, we see the topology of $\mathrm{Spa}(A, A^+)$ is generated by open subsets of form $U(f/g)$. We need to show that $U(f/g)$ is a union of rational subsets. For this, let A_0 be a ring of definition, $I \subset A_0$ be an ideal of definition, and T be a finite set of generators of I as an ideal of A_0. If $v \in U(f/g)$, then $v(g) \neq 0$, and hence by the continuity of v we know that $\{h \in A\colon v(h) < v(g)\}$ is an open neighborhood of 0. Then for sufficiently large k we have $T^k \subset I^k \subset \{h \in A\colon v(h) < v(g)\}$. In other words, $v \in U(T^k/g)$. Hence we have

$$U\left(\frac{f}{g}\right) = \bigcup_{k \geqslant 1} U\left(\frac{\{f\} \cup T^k}{g}\right).$$

It remains to show that (f, T^k) is an open ideal of A for each fixed k. This is clear as it contains I^k.

We now check that the intersection of two rational subsets is rational. Let T_1, T_2 be finite subsets of A generating open ideals, and let $g_1, g_2 \in A$. Then we need to check that $U(T_1/g_1) \cap U(T_2/g_2)$ is rational. This set is equal to

$$U\left(\frac{\{t_1t_2, t_1g_2, t_2g_1\colon t_1 \in T_1, t_2 \in T_2\}}{g_1g_2}\right).$$

Here in the "numerator" we could have deleted the elements t_1t_2, but then the remaining elements may not generate an open ideal of A. The point here is that the elements t_1t_2 for $t_1 \in T_1$ and $t_2 \in T_2$ already generate an open ideal of A, the verification of which we leave as an exercise. $\square$

Remark 2.3.9. An open subset of $\mathrm{Spa}(A, A^+)$ of the form

$$U\left(\frac{T}{g}\right)$$

for a finite set $T \subset A$ may or may not be a rational subset, if we do not require that T generates an open ideal. (It could still be rational, since it may have some other presentation $U(T'/g')$ where T' is a finite set generating an open ideal.) When it is not rational, it could be non-quasi-compact.

2.4. Geometry of adic spectra.

Theorem 2.4.1 (Huber, cf. [Hub93, Theorem 3.5(i)]). *Let (A, A^+) be a Huber pair. The adic spectrum $X = \mathrm{Spa}(A, A^+)$ is a spectral topological space, i.e., it satisfies*

(a) *X has basis of topology $\{T_i\}_{i\in I}$ satisfying that each T_i is quasi-compact and that for any $i, j \in I$ there is $k \in I$ such that $T_i \cap T_j = T_k$;*
(b) *X is quasi-compact;*
(c) *each irreducible closed subset of X has a unique generic point. (This property is called "sober".)*

Moreover, the rational subsets of X form such a basis and in particular every rational subset is quasi-compact.

Remark 2.4.2. A topological space is spectral in the sense of Theorem 2.4.1 if and only if it is homeomorphic to $\mathrm{Spec}\, B$ for some ring B.

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Fact 2.4.3. A topological space $(X, \mathcal{T})$ is spectral if

- $(X, \mathcal{T})$ is T_0, and
- there exists a different topology $\mathcal{T}'$ on the same underlying set X, such that $(X, \mathcal{T}')$ is quasi-compact and there is a collection $\{U_i\}$ of clopens[1] in $\mathcal{T}'$ generating $\mathcal{T}$.

Moreover, in this case, every U_i is quasi-compact for the topology induced from $\mathcal{T}$.

Proof Sketch of Huber's Theorem 2.4.1. Let $\mathrm{Cont}(A)$ be the set of equivalence classes of continuous valuations on A, and let $\mathrm{Spv}(A)$ be the set of equivalence classes of all valuations on A. We then have

$$\mathrm{Spa}(A, A^+) \subset \mathrm{Cont}(A) \subset \mathrm{Spv}(A).$$

For any ideal I of A such that $\sqrt{I}$ is equal to $\sqrt{J}$ for some finitely generated ideal J, Huber defines a subset $\mathrm{Spv}(A, I) \subset \mathrm{Spv}(A)$, which is independent of the topology on A. This definition is technical and we omit it. We take $I := A^{00} \cdot A$ to be the ideal generated by A^{00}. In this case, we have

$$\mathrm{Cont}(A) \subset \mathrm{Spv}(A, I).$$

We define the topology on $\mathrm{Spv}(A)$ and $\mathrm{Spv}(A, I)$ in the same way as in Definition 2.3.6. On $\mathrm{Spv}(A, I)$, one can further define rational subsets

$$\widetilde{U}\left(\frac{T}{g}\right),$$

in the same way as in Definition 2.3.7, except that one replaces the condition "TA is open" by the condition "$I \subset \sqrt{TA}$". (For our particular I, these two conditions are equivalent, but the second condition is in terms of only the pair (A, I), not the topology on A.)

We will need auxiliary topologies on each of the three spaces $\mathrm{Spa}(A, A^+)$, $\mathrm{Spv}(A, I)$, $\mathrm{Spv}(A)$. Let $\mathrm{Spa}(A, A^+)'$ denote the new topology on $\mathrm{Spa}(A, A^+)$ which is the coarsest such that every rational subset is clopen. (This is more refined than the original topology on $\mathrm{Spa}(A, A^+)$.) Similarly, we define $\mathrm{Spv}(A, I)'$ using the notion of rational subsets of $\mathrm{Spv}(A, I)$. In addition, we define a new topology on $\mathrm{Spv}(A)$, denoted by $\mathrm{Spv}(A)'$, to be generated by subsets of the form $\{v\colon v(f) \leqslant v(g)\}$ or $\{v\colon v(f) < v(g)\}$ for $f, g \in A$. (Note that $\mathrm{Spv}(A)'$ is more refined than the original topology on $\mathrm{Spv}(A)$, since the latter is generated by $\{v\colon v(f) \leqslant v(g)\} \cap \{v\colon v(0) < v(g)\}$.)

It is easily checked that $\mathrm{Spv}(A)$ and all its subspaces are T_0. Thus in view of Lemma 2.3.8 and Fact 2.4.3, in order to prove the theorem it suffices to prove that $\mathrm{Spa}(A, A^+)'$ is quasi-compact. This is proved in the following three steps.

Step I. Reduce to showing that $\mathrm{Spv}(A, I)'$ is quasi-compact.

It turns out that for any $v \in \mathrm{Spv}(A, I)$, v is continuous if and only if $v(f) < 1 = v(1)$ for all $f \in A^{00}$. (This equivalence is not true for a general element $v \in \mathrm{Spv}(A)$.) Such elements v form the complement of the rational subset $\widetilde{U}(1/f)$ in $\mathrm{Spv}(A, I)$. Also, for v to lie in $\mathrm{Spa}(A, A^+)$, we have the condition $v(f) \leqslant 1 = v(1)$ imposed for every $f \in A^+$. For each f, this condition defines the rational subset $\widetilde{U}(\{f, 1\}/1) \subset \mathrm{Spv}(A, I)$. Consequently, $\mathrm{Spa}(A, A^+)$ is the intersection of certain subsets of $\mathrm{Spv}(A, I)$ which are either rational subsets or complements of rational subsets of $\mathrm{Spv}(A, I)$. Thus the image of

[1] A clopen is a subset that is simultaneously closed and open.

the inclusion $\mathrm{Spa}(A, A^+)' \to \mathrm{Spv}(A, I)'$ is closed. Moreover, this inclusion is a homeomorphism onto its image, since every rational subset of $\mathrm{Spa}(A, A^+)$ is the intersection of $\mathrm{Spa}(A, A^+)$ with a rational subset of $\mathrm{Spv}(A, I)$.

Step II. Reduce to showing that $\mathrm{Spv}(A)'$ is quasi-compact.

This is done through constructing and studying a natural retraction $r\colon \mathrm{Spv}(A) \to \mathrm{Spv}(A, I)$ for the inclusion $\mathrm{Spv}(A, I) \hookrightarrow \mathrm{Spv}(A)$. We need the following property: For any rational subset $U = \widetilde{U}(T/g) \subset \mathrm{Spv}(A, I)$ with $\sqrt{TA} \supset I$, we have $r^{-1}(U) = \{v \in \mathrm{Spv}(A)\colon \forall f \in T,\ v(f) \leqslant v(g) \neq 0\}$. (This seemingly harmless statement is not true without the condition $\sqrt{TA} \supset I$.) It easily follows that r induces a continuous map $\mathrm{Spv}(A)' \to \mathrm{Spv}(A, I)'$. Since r is surjective and $\mathrm{Spv}(A)'$ is quasi-compact, we conclude that $\mathrm{Spv}(A, I)'$ is quasi-compact.

Step III. Prove that $\mathrm{Spv}(A)'$ is quasi-compact.

Each $v \in \mathrm{Spa}(A)$ gives rise to a binary relation $|$ on A, namely $f|g$ if and only if $v(f) \geqslant v(g)$. The resulting map

$$\mathrm{Spv}(A) \longrightarrow \{0,1\}^{A\times A}$$

is injective with closed image. Here we equip the target with the product topology of the discrete topology on $\{0,1\}$, which is quasi-compact by Tychonoff's theorem. Clearly the subspace topology on $\mathrm{Spv}(A)$ defined by the above injection is exactly $\mathrm{Spv}(A)'$. $\square$

Remark 2.4.4. Similar methods of proof also show that $\mathrm{Cont}(A)$ and $\mathrm{Spv}(A)$ are spectral.

Theorem 2.4.5 (Huber's reconstruction theorem)**.** *Suppose (A, A^+) is a Huber pair.*

(1) $\mathrm{Spa}(A, A^+) = \emptyset$ *if and only if* $A = \overline{\{0\}}$, *i.e. the maximal Hausdorff quotient of A is* 0.
(2) *Given any $f \in A$, we have $f \in A^+$ if and only if $v(f) \leqslant 1$ for all $v \in \mathrm{Spa}(A, A^+)$.*

Remark 2.4.6. We have the following strengthening of Theorem 2.4.5(2). For any subset S of A, define

$$X_S = \{v \in \mathrm{Cont}(A)\colon v(f) \leqslant 1 \text{ for all } f \in S\}.$$

Then we have inverse bijections

$$\begin{Bmatrix}\text{all possible rings } A^+ \subset A \\ \text{of integral elements}\end{Bmatrix} \longleftrightarrow \begin{Bmatrix}\text{all possible subsets } X \subset \mathrm{Cont}(A) \text{ of form} \\ X = X_S,\ \text{where } S \text{ is any subset of } A\end{Bmatrix}$$

$$A^+ \longmapsto \mathrm{Spa}(A, A^+)$$

$$\{f \in A\colon v(f) \leqslant 1 \text{ for all } v \in X\} \longleftarrow\!\shortmid X.$$

Example 2.4.7. We analyze $\mathrm{Spa}(A, A^+)$ when A is a field.

(1) Let K be a field. Recall that a subring $R \subset K$ is called a *valuation subring* if we have $\forall f \in K^\times, f \in R$ or $f^{-1} \in R$. Then there are inverse bijections

$$\mathrm{Spv}(K) = \{\text{all valuations } v\colon K \to \Gamma \cup \{0\}\}/\sim \ \longleftrightarrow \{\text{valuation subrings } R \subset K\}$$

$$v \longmapsto R_v = \{f \in K\colon v(f) \leqslant 1\}$$

$$(v_R, \Gamma_R) \longleftarrow\!\shortmid R.$$

Here, as for the image of R, we write
- $\Gamma_R = K^\times/R^\times$ as an abelian group, together with the total order defined by the rule $f \leqslant g$ if and only if $f/g \in R$.
- $v_R\colon K \to (K^\times/R^\times) \cup \{0\}$, the natural projection map.

(2) Consider the Huber pair (K, K^+) for a discrete field K. Here K^+ is any integrally closed subring of K. Then the bijection in (1) restricts to a bijection

$$\mathrm{Spa}(K, K^+) \longleftrightarrow \{R \subset K \text{ valuation subring such that } R \supset K^+\}.$$

Now we work with the special case that K^+ is itself a valuation ring. As an exercise, check that any valuation subring of K is integrally closed in K. We have the following facts:

(a) Any subring R of K containing K^+ is automatically a valuation subring. All valuation subrings of K are local.

(b) There are inverse bijections

$$\begin{array}{ccc} \{R \text{ valuation subring of } K \colon R \supset K^+\} & \longleftrightarrow & \operatorname{Spec} K^+ \\ R & \longmapsto & \mathfrak{m}_R \\ (K^+)_{\mathfrak{p}} & \longleftarrow\!\shortmid & \mathfrak{p} \end{array}$$

Here $\mathfrak{m}_R$ denotes the unique maximal ideal of R.

In this special case where $K^+ \subset K$ is a valuation subring, we have

$$\operatorname{Spa}(K, K^+) \cong \operatorname{Spec} K^+.$$

This is in fact a homeomorphism.

(3) Let K be a non-archimedean field with a ring of integral elements K^+. Further, assume K^+ is a valuation subring, i.e. K^+ is an arbitrary open bounded valuation subring of K. If v is any valuation on K such that $v(K^+) \leqslant 1$, corresponding to the valuation ring R with the unique maximal ideal $\mathfrak{m}_R$, then v is continuous if and only if $\mathfrak{m}_R \supset K^{00}$. Hence

$$\begin{aligned} \operatorname{Spa}(K, K^+) &\cong \{R \text{ valuation subring of } K \colon R \supset K^+,\ \mathfrak{m}_R \supset K^{00}\} \\ &\cong \operatorname{Spec}(K^+/K^{00}). \end{aligned}$$

Note that this formula is also true in case where K is discrete with the valuation subring K^+, by the discussion in (2).

Definition 2.4.8 (Affinoid field)**.** A Huber pair (A, A^+) is called an *affinoid field* if A is a field whose topology is either discrete or non-archimedean (induced by $|\cdot|\colon A \to \mathbb{R}_{\geqslant 0}$) and if A^+ is a valuation subring.

Our previous discussion shows the following:

Proposition 2.4.9. *Let* (A, A^+) *be an affinoid field. Then* $\operatorname{Spa}(A, A^+)$ *is homeomorphic to* $\operatorname{Spec}(A^+/A^{00})$.

Example 2.4.10. (1) By Example 2.4.7(3),

$$\operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \cong \operatorname{Spec}(\mathbb{Z}_p/p\mathbb{Z}_p) = \{*\}.$$

More precisely, in this case, the single point is the class of the usual absolute value $|\cdot|_p$.

(2) Consider $v \in \operatorname{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$. We see $v^{-1}(0) \subset \mathbb{Z}_p$ is always a prime ideal $\operatorname{supp}(v)$.
- ⋄ If $\operatorname{supp}(v) = (0)$, then v extends to a continuous valuation of $\mathbb{Q}_p$. This forces v to be $|\cdot|_p$ as in (1).
- ⋄ If $\operatorname{supp}(v) = (p)$, then v factors through $\overline{v} : \mathbb{F}_p \to \Gamma \cup \{0\}$, where $\overline{v} \in \operatorname{Spa}(\mathbb{F}_p, \mathbb{F}_p) = \{*\}$. (Here $\mathbb{F}_p$ has the discrete topology. As an exercise, check that $\operatorname{Spv}(\mathbb{F}_p) = \operatorname{Spa}(\mathbb{F}_p, \mathbb{F}_p)$.) If we write v_0 for the single point of $\operatorname{Spa}(\mathbb{F}_p, \mathbb{F}_p)$, then $v_0(0) = 0$ and $v_0(x) = 1$ for $x \neq 0$.

Combining these, we see

$$\operatorname{Spa}(\mathbb{Z}_p, \mathbb{Z}_p) = \{|\cdot|_p, v_0\}$$

where $|\cdot|_p$ is the usual p-adic absolute value and

$$v_0(x) = \begin{cases} 1, & p \nmid x, \\ 0, & \text{otherwise.} \end{cases}$$

2.5. **The adic closed unit disc.** Let $(K, |\cdot|_K)$ be an algebraically closed non-archimedean field. Consider the uniform Tate ring

$$A = K\langle T\rangle = \left\{ \sum_{n \geqslant 0} a_n T^n \colon a_n \in K,\ a_n \to 0 \text{ as } n \to \infty \right\},$$

whose topology is given by the multiplicative Gauss norm $\|f\| = \max_{n \geqslant 1} |a_n|_K$. We have

$$A^0 = \mathcal{O}_K\langle T\rangle = \left\{ \sum_{n \geqslant 0} a_n T^n \colon a_n \in \mathcal{O}_K,\ a_n \to 0 \text{ as } n \to \infty \right\}.$$

(See Example 2.2.13 for details.)

Exercise 2.5.1. In any normed K-algebra $(A, \|\cdot\|)$ such that $\|\cdot\|$ is multiplicative, the set $\{a \in A \colon \|a\| \leqslant 1\}$ is an integrally closed subring.

By the exercise $(A, A^+) = (K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$ is a Huber pair. We then consider $X = \mathrm{Spa}(A, A^+)$. The goal is to describe its points. For any $x \in \mathcal{O}_K$ and $r \in [0,1]$, we introduce the notations

$$\begin{aligned} B_{x,r} &= \{y \in K \colon |y - x|_K \leqslant r\}, \\ B^\circ_{x,r} &= \{y \in K \colon |y - x|_K < r\}. \end{aligned}$$

Note that $B_{x,r} \subset B_{0,1}$ always holds. For any $f \in A$ and any $x \in B_{0,1}$, we can evaluate $f(x) \in B_{0,1}$.

Exercise 2.5.2 (Maximum modulus principle)**.** Fix an element $f \in A$. Let $x \in B_{0,1}$ and $r \in [0,1]$.

(1) If $r \in |K^\times|_K$, then the function $B_{x,r} \to \mathbb{R}_{\geqslant 0}$ via $y \mapsto |f(y)|_K$ has a maximum, and it is reached on $B_{x,r} - B^0_{x,r}$.

(2) For all $r \in [0,1]$, define
$$v_{x,r}(f) \coloneqq \sup_{y \in B_{x,r}} |f(y)|_K.$$
Check that $v_{x,r} \in X$ is a continuous valuation on A such that $v_{x,r}(A^+) \leqslant 1$. Further, if we have $f(T) = \sum_n a_n (T-x)^n$, then
$$v_{x,r}(f) = \max_n |a_n|_K \cdot r^n.$$

(3) For any $f_1, f_2 \in A$, we always have
$$v_{x,r}(f_1 f_2) = v_{x,r}(f_1) \cdot v_{x,r}(f_2).$$

(4) We have $v_{x,r} \in \mathrm{Spa}(A, A^+)$.

Lect.5, Oct 17

We are going to give a complete classification of points in $X = \mathrm{Spa}(A, A^+)$, with $A = K\langle T\rangle \supset A^+ = A^0 = \mathcal{O}_K\langle T\rangle$, into certain types. It easily follows from the above exercise that for each $x \in \mathcal{O}_K$ and $r \in [0,1]$, the function $v_{x,r} \colon A \to \mathbb{R}_{\geqslant 0}$ is a continuous valuation such that $v_{x,r}(f) \leqslant 1$ for all $f \in A^+$. Thus $v_{x,r}$ can be viewed as an element of X.

Construction 2.5.3 (Points in the adic closed unit disc)**.** Assume $\Gamma = \mathbb{R}_{>0}$ for (I)–(IV) in the following.

(I) Classical points: $v_{x,0} \colon f \mapsto |f(x)|_K$ for $x \in \mathcal{O}_K$.

(II) $v_{x,r}$ with $x \in \mathcal{O}_K$ and $r \in (0,1] \cap |K^\times|_K$.

(III) $v_{x,r}$ with $x \in \mathcal{O}_K$, for $r \in (0,1]$ but $r \notin |K^\times|_K$. Note that for $r = 1$, the point $v_{x,r}$ is independent of x, and is nothing but the Gauss norm $\|\cdot\|$ on A. We denote this point by β, called the Gauss point.

(IV) The points of this type only exist when K is not spherically complete. In this case, we can have a family $\mathcal{F} = (B_{x_1,r_1} \supset B_{x_2,r_2} \supset \cdots)$ of closed balls with descending radii r_i such that $\bigcap_n B_{x_n,r_n} = \emptyset$. (It follows that $\lim_i r_i > 0$.) Then we can construct $v_{\mathcal{F}} \in X$ by
$$v_{\mathcal{F}}(f) = \inf_n v_{x_n,r_n}(f).$$
Note that this can happen when $K = \mathbb{C}_p$.

In contrast with $\Gamma = \mathbb{R}_{>0}$ before, we now assume $\Gamma = \gamma^{\mathbb{Z}} \times \mathbb{R}_{>0}$ for some generator γ such that $r < \gamma < 1$ for all $r \in (0,1)$. That is, we equip the abelian group $\Gamma \cong \mathbb{Z} \times \mathbb{R}_{>0}$ with the total order such that $(\gamma^n, r) \leqslant (\gamma^m, s)$ if and only if $r < s$ or ($r = s$ and $n \geqslant m$).

(V) Rank 2 points: Start with a point $v_{x,r} \in X$ of type (II) or (III) for $x \in \mathcal{O}_K$ and $r \in (0,1)$. We have
$$v_{x,r}\left(\sum_n a_n (T-x)^n\right) = \sup_n |a_n| \cdot r^n.$$
We are to modify this point by $\varepsilon \in \{\pm 1\}$, to get $v_{x,r,\varepsilon} \in X$ defined by
$$v_{x,r,\varepsilon}\left(\sum_n a_n (T-x)^n\right) = \sup_n |a_n| \cdot (r \cdot \gamma^\varepsilon)^n \in (\gamma^{\mathbb{Z}} \times \mathbb{R}_{>0}) \cup \{0\}.$$
More precisely, the new points of type (V) arises from the following recipe:

(i) If $r \notin |K^\times|_K$, i.e. $v_{x,r}$ is of type (III), then we do not get new points because $v_{x,r,\varepsilon} \sim v_{x,r}$, i.e., they are equivalent valuations.
(ii) If $r \in |K^\times|_K$, then $v_{x,r,\varepsilon} \not\sim v_{x,r}$. We see $v_{x,r,+1}$ depends on (x,r) only via $B^\circ_{x,r}$, and $v_{x,r,-1}$ depends on (x,r) via $B_{x,r}$.

Now for the type (II) point $v_{x,1} = \beta$ (i.e., the Gauss point), we have a new type (V) point $v_{x,1,+1}$ by the same recipe as above, for each $x \in \mathcal{O}_K$. The point $v_{x,r,+1}$ depends on x only via $B^\circ_{x,1}$. Note that the valuation $v_{x,1,-1}$ sends $T \in A^+$ to γ^{-1}, which is > 1. This violates the defining condition for $\mathrm{Spa}(A, A^+)$. Hence we do not have points in $\mathrm{Spa}(A, A^+)$ of the form $v_{x,1,-1}$.

In fact, for $v_{x,r}$ of type (II), the closure of $\{v_{x,r}\}$ in X consists of $v_{x,r}$ and certain type (V) points. More precisely:

- When $r \in (0, 1)$,

$$\overline{\{v_{x,r}\}} = \{v_{x,r}\} \sqcup \{v_{x',r,+1} \colon B_{x,r} = B_{x',r}, \text{ i.e. } x' \in B_{x,r}\} \sqcup \{v_{x,r,-1}\}.$$

Moreover, for $x', x'' \in B_{x,r}$, we have $v_{x',r,+1} = v_{x'',r,+1}$ if and only if $B^\circ_{x',r} = B^\circ_{x'',r}$, if and only if $|x' - x''| < r$. Thus if we choose an element $u \in K$ such that $|u| = r$ and denote by l the residue field of K, then the set $\{v_{x',r,+1} \colon x' \in B_{x,r}\}$ is in bijection with $\mathbb{A}^1(l)$ under the map $v_{x',r,+1} \mapsto$ the residue class of $(x' - x)/u \in \mathcal{O}_K$. This bijection extends to a bijection

$$\overline{\{v_{x,r}\}} \xrightarrow{\sim} \mathbb{P}^1_l, \tag{2.5.1}$$

where we send $v_{x,r}$ to the generic point and send $v_{x,r,-1}$ to the closed point ∞. The above bijection turns out to be a homeomorphism.

- When $r = 1$, we have $v_{x,r} = \beta$, and

$$\overline{\{\beta\}} = \{\beta\} \sqcup \{v_{x',1,+1} \colon x' \in \mathcal{O}_K\} \cong \mathbb{A}^1_l.$$

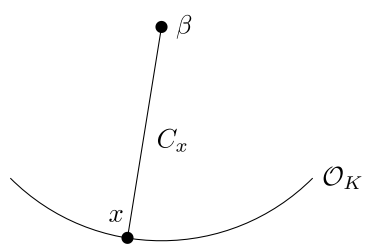
Such points in Construction 2.5.3 form $\mathrm{Spa}(A, A^+)$ whose geometry can be illustrated as a tree. A more precise description is as follows.

Construction 2.5.4 (Visualizations of points of each type in Construction 2.5.3)**.** Let $x \in \mathcal{O}_K$ and consider the curve

$$C_x \coloneqq \{v_{x,r} \colon r \in [0, 1]\} \subset X.$$

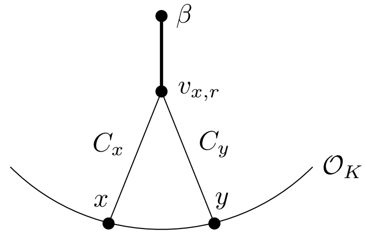
For $r = 0$, $v_{x,r}$ is a classical point; for $r = 1$, $v_{x,r} = \beta$ is the Gauss point. We thus regard C_x as a curve going from β to the classical point $v_{x,0}$ as r decreases. However, we caution that the natural map $[0, 1] \xrightarrow{\sim} C_x$ is not a homeomorphism.

(I) In the following picture, the curve below denotes the set of all classical points.



Each $x \in \mathcal{O}_K$ corresponds to a point $v_{x,1}$ of type (I) via C_x. Hence the classical points are "endpoints" of the tree X, and the set of them can be identified with $\mathcal{O}_K$.

(II) We consider $v_{x,r}$ by fixing $0 \neq r \in |K^\times|_K$ and say $r = |x - y|$ with $x \neq y$. Then $B_{x,r} = B_{y,r}$ and hence $v_{x,r} = v_{y,r}$. Thus the two curves C_x and C_y meet at $v_{x,r}$. Moreover, for $r' \in [0, r)$, we have $B_{x,r'} \neq B_{y,r'}$ and so $v_{x,r'} \neq v_{y,r'}$, which means that C_x and C_y actually depart each other at $v_{x,r}$ when they go downward. Thus the point $v_{x,r}$ of type (II) is a *branching point.*



Note that the branching dos not happen along the upward direction, i.e., $v_{x,r'} = v_{y,r'}$ for all $r' \in [r,1]$. In other words, the segment of C_x from β to $v_{x,r}$ is equal to the segment of C_y from β to $v_{x,r}$.

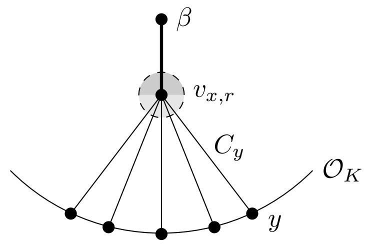
(III) Any point $v_{x,r}$ of type III is not a branching point. Therefore, the points of type (II) and (III) are seen as "limbs" of the tree X.

(IV) By Construction 2.5.3, each point $v_{\mathcal{F}}$ of type (IV) is represented by a path on the tree starting at β, passing through infinitely many branching points, but never reaching an end point. We can write this path as

$$(\beta \xrightarrow{C_{x_1}} v_{x_1,r_1} \xrightarrow{C_{x_2}} v_{x_2,r_2} \longrightarrow \cdots)$$

where the symbol C_{x_i} on the i-th edge means that the part of the path from $v_{x_{i-1},r_{i-1}}$ to v_{x_i,r_i} agrees with the corresponding segment of the curve C_{x_i}. Therefore, the point $v_{\mathcal{F}}$ in the tree X looks like "the limiting point of ramification points on limbs".

(V) Again, start with a point $v_{x,r}$ of type (II) for $r<1$. For each distinct ray C_y passing through $v_{x,r}$ downward, we uniquely have $v_{y,r,+1}$. On the other hand, from $v_{x,r}$ we have a unique direction going upward, corresponding to $v_{y,r,-1}$.



Therefore, we see points of type (V) of rank 2 lie in the closure of points of type (II), or more heuristically, they are "infinitesimally close to the limbs" of the tree X. For $r=1$, namely for $v_{x,r} = \beta$, the visualization is similar, except that we do not have the "upward direction", corresponding to the fact that we do not have the point $v_{x,r,-1}$.

Proposition 2.5.5. *All points of X are given by Construction 2.5.3.*

Proof. **Step I.** Use the completion containment

$$(K[T], \mathcal{O}_K[T]) \subset (K[T]^\wedge, \mathcal{O}_K[T]^\wedge) = (K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$$

to get a natural map (by restricting the valuations)

$$\mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle) \longrightarrow \mathrm{Spa}(K[T], \mathcal{O}_K[T]).$$

It is a general fact about completions of Huber pairs that the above is a homeomorphism. We will discuss this general fact later in Proposition 2.6.8.

Step II. Classify points of rank 1.

The rank-one points are by definition continuous valuations $v\colon K[T] \to \mathbb{R}_{\geqslant 0}$ such that $v(\mathcal{O}_K[T]) \leqslant 1$. Fix such a v. We define $r_x := v(T-x)$ for any $x \in \mathcal{O}_K$. Since $T - x \in \mathcal{O}_K[T]$, we have $r_x \in [0,1]$. Thus we can form the ball $B_{x,r_x} \subset B_{0,1}$. Using axioms for v, one can show for any $x, y \in \mathcal{O}_K$ that

$$r_x \leqslant r_y \implies B_{x,r_x} \subset B_{y,r_y}.$$

One can also show that for any $f \in K[T]$,

$$v(f) = \inf_{x \in \mathcal{O}_K} v_{x,r_x}(f) = \inf_{x \in \mathcal{O}_K} \sup_{y \in B_{x,r_x}} |f(y)|_K$$

by reducing to the case where f is a linear polynomial. Combining these two observations, we can find a decreasing family $(B_{x_1,r_1} \supset B_{x_2,v_2} \supset \cdots)$ such that

$$v(f) = \inf_n v_{x_n,r_n}(f).$$

This forces such v to be one of the points of type (I)–(IV).

Step III. Classify the rest of the point of X.

In general, for any Huber pair (A, A^+) and any $(v : A \to \Gamma \cup \{0\}) \in \operatorname{Spa}(A, A^+)$, the subset

$$\operatorname{supp}(v) \coloneqq v^{-1}(0)$$

of A is a prime ideal. (Moreover, the map $\operatorname{supp}\colon \operatorname{Spa}(A, A^+) \to \operatorname{Spec} A, v \mapsto \operatorname{supp}(v)$ is continuous.) We define the *residue field* of v to be the residue field of $\operatorname{supp}(v)$, namely

$$k(v) \coloneqq \operatorname{Frac}(A/\operatorname{supp}(v)).$$

This v induces a valuation $k(v) \to \Gamma \cup \{0\}$, along which the inverse image $k(v)^+$ of $\{\gamma \in \Gamma \colon \gamma \leqslant 1\} \cup \{0\}$ is a valuation subring of $k(v)$. Consequently, $k(v)^+$ is a local ring with residue field

$$\kappa(v) = \{x \in k(x) \colon v(x) \leqslant 1\}/\{x \in k(x) \colon v(x) < 1\} = k(v)^+/\{x \in k(x) \colon v(x) < 1\}.$$

Note that the natural image of A^+ in $k(v)$ is contained in $k(v)^+$.

Fact 2.5.6. Suppose for $X = \operatorname{Spa}(A, A^+)$, the Huber pair (A, A^+) is Tate. Then:

(1) If $v, w \in X$ such that $v \rightsquigarrow w$, i.e. $w \in \overline{\{v\}}$, then $\operatorname{supp}(w) = \operatorname{supp}(v)$.

(2) Fix a point v. Then

$$\begin{aligned} \overline{\{v\}} &\xrightarrow{\sim} \{R \text{ valuation subring of } k(v) \colon \operatorname{im}(A^+ \to k(v)^+) \subset R \subset k(v)^+\} \\ &\xrightarrow{\sim} \{\text{valuation subrings of } \kappa(v) \text{ containing } \operatorname{im}(A^+ \to \kappa(v))\} \\ &\xrightarrow{\sim} \operatorname{Spa}(\kappa(v), \operatorname{im}(A^+ \to \kappa(v))). \end{aligned}$$

Here the first isomorphism sends w to $k(w)^+$, which is a valuation subring of $k(w)$ and we have $k(w) = k(v)$ by part (1). The second isomorphism is induced by the projection $k(v)^+ \to \kappa(v)$. In the third line, we equip $\kappa(v)$ with the discrete topology. Moreover, the composition of the above three bijections is a homeomorphism.

(3) For each $w \in X$ there exists a unique point of rank 1, say $v \in X$, such that $w \in \overline{\{v\}}$.

Back to considering the adic closed unit disc $\operatorname{Spa}(K[T], \mathcal{O}_K[T])$. We have already classified all rank 1 points. By Fact 2.5.6 (3), in order to classify all points, we only need to compute the closure of every rank 1 point. This closure can be computed using Fact 2.5.6 (2). Now all points of rank 1 are closed except for those of type (II). Take $v_{x,r}$ of type (II) with $r < 1$. Then we have $\kappa(v_{x,r}) = l(T)$ and $\operatorname{im}(A^+ \to \kappa(v_{x,r})) = l$. Here l is the residue field of K. Thus we have

$$\overline{\{v_{x,r}\}} \cong \operatorname{Spa}(l(T), l) \cong \mathbb{P}^1_l.$$

By unwrapping the constructions it is not hard to see that the type (V) points actually realize this bijection, in the way described in (2.5.1). Similarly, for $r = 1$, we have $\kappa = l(T)$ and $\operatorname{im}(A^+) = l[T]$. The situation is different because

$$\operatorname{Spa}(l(T), l[T]) \cong \mathbb{A}^1_l.$$

$\square$

2.6. Completion.

Definition 2.6.1. Let A be a topological ring. Say A is *complete* if A is Hausdorff and every Cauchy filter basis has a limit in A.

Fact 2.6.2 (Universal property of completions)**.** There is a continuous map $A \to \widehat{A}$ such that $\widehat{A}$ is complete and this map is universal for all continuous maps $A \to B$ for complete B. The A-algebra $\widehat{A}$ is called the completion of A; it is unique up to unique isomorphism as a topological A-algebra.

Example 2.6.3. If A has I-adic topology for an ideal I, then $\widehat{A}$ is the I-adic completion of A, i.e., $\widehat{A} = \varprojlim_n A/I^n$. However, we caution that $\widehat{A}$ may not be I-adically complete as an A-module, that is, we may not have $\widehat{A} \cong \varprojlim_n \widehat{A}/I^n\widehat{A}$. This is not a contradiction with the fact that $\widehat{A}$ is complete. When we say that $\widehat{A}$ is complete, it is with respect to some intrinsic topology on $\widehat{A}$ on account of $\widehat{A}$ being a completion of A; in the current case the topology of $\widehat{A}$ is the inverse limit topology coming from the discrete topology on A/I^n for all n. This topology is in general different from the I-adic topology on $\widehat{A}$. All these problems go away when I is finitely generated, as we will see below.

The following facts can be more important.

Fact 2.6.4. If A has I-adic topology for a finitely generated ideal I, then the topology on $\widehat{A}$ is $(I \cdot \widehat{A})$-adic.

This can be further extended to the case where A is Huber with a ring of definition A_0 and an ideal of definition I. Denote by $\psi\colon A \to \widehat{A}$ the completion map, and let $\widehat{A}_0$ be the closure of $\psi(A_0)$ in $\widehat{A}$, equipped with the subspace topology inherited from $\widehat{A}$. The notation $\widehat{A}_0$ clashes with the notation for the completion of A_0, but the two are actually isomorphic as stated below.

Fact 2.6.5. The map $\psi\colon A_0 \to \widehat{A}_0$ identifies the topological ring $\widehat{A}_0$ with the completion of A_0. In particular, by Fact 2.6.4, the topology on $\widehat{A}_0$ is $(\psi(I) \cdot \widehat{A}_0)$-adic. Here note that $\psi(I) \cdot \widehat{A}_0$ is a finitely generated ideal in $\widehat{A}_0$. Moreover, $\widehat{A}_0$ is an open subring of $\widehat{A}$. In particular, $\widehat{A}$ is Huber with a ring of definition $\widehat{A}_0$ and an ideal of definition $\psi(I) \cdot \widehat{A}_0$.

Remark 2.6.6. Notation as above, we have a natural isomorphism

$$\widehat{A} \cong \widehat{A}_0 \otimes_{A_0} A.$$

One can actually use the right hand side to *construct* $\widehat{A}$, and then show that it satisfies the characterizing universal property.

Fact 2.6.7. If A is Huber with completion map $\psi : A \to \widehat{A}$, then we have a bijection.

$$\begin{aligned} \{\text{open subrings of } A\} &\xleftrightarrow{\sim} \{\text{open subrings of } \widehat{A}\} \\ R &\longmapsto \overline{\psi(R)} \end{aligned}$$

Moreover, $\overline{\psi(R)}$ is identified with the completion of R itself. Under this bijection, A^0 is sent to $(\widehat{A})^0$. This bijection also restricts to the following bijections:

$$\{\text{rings of definition of } A\} \xleftrightarrow{\sim} \{\text{rings of definition of } \widehat{A}\}$$

and

$$\{\text{rings of integral elements of } A\} \xleftrightarrow{\sim} \{\text{rings of integral elements of } \widehat{A}\}.$$

By virtue of the last bijection, starting with any Huber pair (A, A^+) we can take its completion $(\widehat{A}, \widehat{A}^+)$ which is another Huber pair, where $\widehat{A}$ is the completion of A, and $\widehat{A}^+$ is the closure of the image of A^+ in $\widehat{A}$, itself identified with the completion of A^+. By restricting the valuations, we get from the process above a natural map

$$\varphi\colon \operatorname{Spa}(\widehat{A}, \widehat{A}^+) \longrightarrow \operatorname{Spa}(A, A^+).$$

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The following proposition claims the homeomorphism which appeared in the proof of Proposition 2.5.5 before.

Proposition 2.6.8. *The map* $\varphi\colon \operatorname{Spa}(\widehat{A}, \widehat{A}^+) \longrightarrow \operatorname{Spa}(A, A^+)$ *is a homeomorphism that preserves rational subsets.*

To prove Proposition 2.6.8, we need the following two lemmas.

Lemma 2.6.9 (Minimum modulus principle)**.** *Suppose* (A, A^+) *is a Huber pair and take a quasi-compact subset* $C \subset \operatorname{Spa}(A, A^+)$. *Suppose* $f \in A$ *is such that* $v(f) \neq 0$ *for all* $v \in C$. *Then there is a neighborhood* U *of* 0 *in* A *such that for all* $g \in U$ *we have*

$$v(f) > v(g), \quad \forall v \in C.$$

Proof. Choose a ring of definition $A_0 \subset A$ and an ideal of definition $I \subset A_0$. Write $I = T \cdot A_0$ for some finite set $T \subset A_0$. For each integer $n \geqslant 1$, the set

$$X_n = \{v \in \operatorname{Spa}(A, A^+) \colon v(t) \leqslant v(f) \neq 0 \text{ for all } t \in T^n\}$$

is open. For any $v \in C$, since v is continuous, we see $v(f) \neq 0$ implies that $\{g \in A \colon v(g) < v(f)\}$ is an open neighborhood of 0, and hence contains $I^n \supset T^n$ for some $n \gg 0$. This means that $v \in X_n$. Thus we have

$$C \subset \bigcup_{n \geqslant 1} X_n.$$

Note each $t \in T$ satisfies $t^n \to 0$, so for any continuous valuation v on A we must have $v(t) < 1$. (Indeed, for sufficiently large n, t^n will lie in $\{f \colon v(f) < v(1) = 1\}$, which is an open neighborhood of 0 in A. Hence $v(t^n) < 1$ and so $v(t) < 1$.) Thus $X_n \subset X_{n+1} \subset \cdots$. It then follows from the quasi-compactness of C that $C \subset X_n$ for some $n \gg 0$. If so, we can take $U = I^{n+1}$ and then $v(g) < v(f)$ for all $g \in U$ and all $v \in C$. (For the strict inequality again use $v(t) < 1$ for any $t \in T$.) $\square$

Example 2.6.10. To illustrate the situation of Lemma 2.6.9, we consider the adic closed unit disc. Let $(K, |\cdot|)$ be an algebraically closed non-archimedean field. Take

$$X = \operatorname{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle).$$

Then the following subset is open in X:

$$C = X - \{v_{0,0}\} = \{v \in X \colon v(T) \neq 0\}.$$

Take $f = T$. We have $v_{0,|p^{n+1}|} \in C$ for all $n \in \mathbb{N}$. Also note that

$$v_{0,|p^{n+1}|}(p^n) = |p^n| > v_{0,|p^{n+1}|}(T) = |p^{n+1}|.$$

Since $p^n \to 0$ in A, this means there does not exist an open neighborhood U of 0 in A satisfying the conclusion of Lemma 2.6.9 for $f = T$. Thus C is not quasi-compact. This C is an example of an open subset of $\operatorname{Spa}(A, A^+)$ which is not quasi-compact; in particular it is not a rational subset.

Lemma 2.6.11. *Suppose* (A, A^+) *is a complete Huber pair. Take*

$$U = U\left(\frac{f_1, \cdots, f_n}{g}\right) \subset \operatorname{Spa}(A, A^+)$$

as a rational subset with open ideal $(f_1, \ldots, f_n)A$. *Then there is a neighborhood* V *of* 0 *in* A *satisfying the following condition: For all* $f_i' \in f_i + V$ *with* $i = 1, \ldots, n$, *and for all* $g' \in g + V$, *we still have*

$$U = U\left(\frac{f_1', \cdots, f_n'}{g'}\right)$$

and $(f_1', \ldots, f_n')A$ *is an open ideal as well.*

Proof. We need to make use of the following

> *Claim.* Let B be a ring, and $J = r_1 B + \cdots + r_m B$ a finitely generated ideal. Assume that B is J-adically complete. Then for any $r_1' \in r_1 + J^2, \cdots, r_m' \in r_m + J^2$, we have $J = r_1' B + \cdots + r_m' B$.

To prove the claim, we need to show that the B-module map $\phi : B^m = B^{\oplus m} \to J, (b_1, \cdots, b_m) \mapsto \sum b_i r'_i$ is surjective. Both sides are J-adically complete, so it suffices to show that ϕ induces a surjection between the graded modules associated to the J-adic filtrations on the two sides. (See [Bou64, III.2, No 8, Cor.2].) Thus we need to show for each $i \geqslant 0$ that $\phi : J^i B^m / J^{i+1} B_m \to J^i J / J^{i+1} J$ is surjective. For this, clearly it is enough to show that $\{r'_1, \cdots, r'_m\} \cup J^2$ generates J as a B-module. This is true since the ideal generated by $\{r'_1, \cdots, r'_m\} \cup J^2$ is contained in the ideal generated by $\{r_1, \cdots, r_m\}$, which is J. The claim is proved.

We now start the proof of the lemma.

Choose a ring of definition A_0 in A and an ideal of definition $I = (r_1, \ldots, r_m)A_0 \subset A_0$. Since $(f_1, \cdots, f_n)A$ is open, we may assume that $I \subset (f_1, \cdots, f_n)A$. We then write

$$r_j = \sum_{i=1}^n a_{ij} f_i, \quad a_{ij} \in A.$$

Let V_0 be a neighborhood of 0 in A such that $a_{ij} V_0 \subset I^2$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then for arbitrary $f'_i \in f_i + V_0$, if we define

$$r'_j = \sum_{i=1}^n a_{ij} f'_i,$$

we get $r'_j \in r_j + I^2$ and so $(r'_1, \ldots, r'_m)A_0 = I$ by the claim. But this implies that $(f'_1, \ldots, f'_n)A$ is again an open ideal in A.

We now show that if we further shrink V_0, then we can ensure that $U = U(\frac{f'_1, \ldots, f'_n}{g'})$ for arbitrary $f'_i, g' \in V_0$.

Set $f_0 = g$, and $U_i = U(\frac{f_0, f_1 \ldots, f_n}{f_i})$ for $i = 0, \cdots, n$. Thus $U = U_0$. Since U_i is rational, it is quasi-compact. Since f_i is non-zero on U_i, using Minimum modulus principle (Lemma 2.6.9) we find a neighborhood V of 0 in A such that for any $f \in V$, any $0 \leqslant i \leqslant n$, and any $v \in U_i$, we have $v(f_i) > v(f)$. We may and shall assume that $V \subset V_0 \cap A^{00}$.

We now show that for arbitrary $f'_0 \in f_0 + V, \cdots, f'_n \in f_n + V$, we have

$$U\left(\frac{f'_0, \cdots, f'_n}{f'_0}\right) = U_0.$$

Clearly this will imply the lemma. Denote the left hand side by U'_0.

We prove $U'_0 \supset U_0$. Let $v \in U_0$. Then $v(f_0) > v(f_i - f'_i)$ for all $i = 0, \cdots, n$, since $f_i - f'_i \in V$. We then have

$$v(f'_i) \leqslant \max(v(f_i), v(f_i - f'_i)) \leqslant v(f_0) = v(f'_0 - f'_0 + f_0) = v(f'_0).$$

Hence $v \in U'_0$.

Finally, we prove $U'_0 \subset U_0$. Let $v \in U'_0$ and suppose $v \notin U_0$. We analyze the situation in two cases. Case one, $v(f_i) = 0$ for all $i = 0, \cdots, n$. In this case, $\mathrm{supp}(v)$ is an ideal containing all f_i and hence it is open. Since $f_0 - f'_0 \in V \subset A^{00}$, we have $f_0 - f'_0 \in \sqrt{\mathrm{supp}(v)} = \mathrm{supp}(v)$. Thus $v(f'_0) = 0$, a contradiction with the assumption that $v \in U'_0$.

Case two, $v(f_i) \neq 0$ for some i. We may assume that i is such that $v(f_i) = \max(v(f_0), \cdots, v(f_n))$. Since $v \notin U_0$, we have either $v(f_0) = 0$, or $v(f_j) > v(f_0)$ for some j. In all cases, we have $v(f_i) > v(f_0)$. In particular $i \neq 0$. Then $v(f'_0) \leqslant \max(v(f_0), v(f_0 - f'_0)) < v(f_i) = v(f'_i)$, a contradiction. □

Now we are ready to prove that $\varphi\colon \mathrm{Spa}(\widehat{A}, \widehat{A}^+) \to \mathrm{Spa}(A, A^+)$ is a homeomorphism which preserves rational subsets.

Proof of Proposition 2.6.8. It is easy to show that φ is bijective. Then it remains to show that φ preserves rational subsets, since after that it will automatically follow that φ is a homeomorphism. It is not hard to show that φ^{-1} maps a rational subset of $\mathrm{Spa}(A, A^+)$ to a rational subset of $\mathrm{Spa}(\widehat{A}, \widehat{A}^+)$. Thus it remains to show the following:

- If $U = U(\{f_1, \ldots, f_n\}/g)$ is rational in $\mathrm{Spa}(\widehat{A}, \widehat{A}^+)$ and $(f_1, \ldots, f_n)\widehat{A}$ is open, then $\varphi(U)$ is rational in $\mathrm{Spa}(A, A^+)$.

The natural map $\iota\colon A \to \widehat{A}$ has dense image, so by Lemma 2.6.11, without loss of generality, we may assume $f_i = \iota(h_i)$ and $g = \iota(k)$ with $k \in A$ and $h_i \in A$ for $i = 1, \ldots, n$. This further implies

$$\varphi(U) = U\left(\frac{h_1, \cdots, h_n}{k}\right).$$

So the proof would be complete if $(h_1, \ldots, h_n)$ is an open ideal in A.

In general, we observe that U is quasi-compact since it is rational, and $\iota(k) = g$ is non-zero on U. By Lemma 2.6.9, there is a neighborhood V of 0 in $\widehat{A}$ such that for all $y \in V$ we have $v(y) < v(g)$ for all $v \in U$.

Recall from Fact 2.6.5 that if we take a ring of definition $A_0 \subset A$ and an ideal of definition $I \subset A_0$, then $\widehat{A}_0$ is a ring of definition in $\widehat{A}$ whose topology is $(\iota(I) \cdot \widehat{A}_0)$-adic. Therefore, up to replacing I by a power of itself, we can choose I such that $\iota(I) \subset V$. Thus for each $y \in I$, $v(\iota(y)) < v(g)$ holds for any $v \in U$. This implies that $v(y) < v(k)$ for all $v \in \varphi(U)$. Consequently,

$$\varphi(U) = U\left(\frac{h_1, \cdots, h_n}{k}\right) = U\left(\frac{h_1, \cdots, h_n, T}{k}\right),$$

in which T is a finite set of generators of the ideal I in A_0. Then $\{h_1, \cdots, h_n\} \cup T$ generate an open ideal in A, and so $\varphi(U)$ is rational. $\square$

Proposition 2.6.12. *Let $U \subset \mathrm{Spa}(A, A^+)$ be a rational subset. Then there is*

- *a complete Huber pair (A_U, A_U^+) associated to U, and*
- *a map of Huber pairs $\varphi_0\colon (A, A^+) \to (A_U, A_U^+)$,*

such that

(1) *The map $\mathrm{Spa}(A_U, A_U^+) \to \mathrm{Spa}(A, A^+)$ induced by φ_0 is a homeomorphism $\mathrm{Spa}(A_U, A_U^+) \xrightarrow{\sim} U$, and this homeomorphism preserves rational subsets. (Here rational subsets of U refer to those rational subsets of $\mathrm{Spa}(A, A^+)$ that are contained in U.)*

(2) *For each complete Huber pair (B, B^+) together with a map $\varphi\colon (A, A^+) \to (B, B^+)$, if the induced map $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ factors through U, then φ factors uniquely through φ_0, i.e.,*

$$\begin{tikzcd} (A, A^+) \arrow[r, "\varphi"] \arrow[dr, "\varphi_0"'] & (B, B^+) \\ & (A_U, A_U^+) \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

Proof. We only sketch the construction of (A_U, A_U^+). It is not *a priori* clear that the construction is independent of various choices. However, one proves that the construction indeed satisfies the universal property as in the proposition, from which uniqueness follows.

Choose a ring of definition $A_0 \subset A$ and an ideal of definition $I \subset A_0$. We construct (A_U, A_U^+) as follows. Write

$$U = U\left(\frac{T}{g}\right)$$

where T is a finite subset of A generating an open ideal. We admit the following claim.

> *Claim.* On $A[1/g]$, there exists a unique ring topology such that $A_0[T/g]$ is open with $(I \cdot A_0[T/g])$-adic topology.

Here existence and uniqueness of a *group topology* (for addition) satisfying the same conditions is clear. The key point is to show that this group topology is also a ring topology, i.e., that multiplication is continuous. Granting the claim, we see $A[1/g]$, equipped with the prescribed topology, is a Huber ring; we then take its completion and denote the completion by $A\langle T/g\rangle$. The integral closure of $A^+[T/g]$ in $A[1/g]$ is a ring of integral elements in the Huber ring $A[1/g]$. Hence the completion of it is a ring of integral elements in $A\langle T/g\rangle$. We denote this ring of integral elements by $A\langle T/g\rangle^+$. In conclusion we have obtained a complete Huber pair $(A\langle T/g\rangle, A\langle T/g\rangle^+)$, and this is our desired (A_U, A_U^+). $\square$

Remark 2.6.13 (Universal property of the A-algebra $A\langle T/g\rangle$). The map $A \to A\langle T/g\rangle$ is universal among the continuous maps $\varphi\colon A \to B$ with complete B and such that $\varphi(g)$ is invertible and $\varphi(T)/\varphi(g)$ is power-bounded[2] as a set in B.

2.7. **The structure presheaf and adic spaces.** Fix a Huber pair (A, A^+) and let $X = \operatorname{Spa}(A, A^+)$.

Definition 2.7.1. Let $W \subset X = \operatorname{Spa}(A, A^+)$ be an open subset. Define

$$\mathcal{O}_X(W) \coloneqq \varprojlim_{\substack{U\subset W\\ \text{rational}}} A_U, \qquad \mathcal{O}_X^+(W) \coloneqq \varprojlim_{\substack{U\subset W\\ \text{rational}}} A_U^+.$$

Here in both projective limits, the transition maps in the projective system arise from the fact that, for two rational sets $U' \subset U$, the Huber pair $(A_{U'}, A_{U'}^+)$ is an algebra over (A_U, A_U^+).

We equip $\mathcal{O}_X(W)$ and $\mathcal{O}_X^+(W)$ with the inverse limit topology coming from the topologies on A_U and A_U^+. Inverse limits of complete rings are complete. Hence $\mathcal{O}_X(W)$ and $\mathcal{O}_X^+(W)$ are complete rings. Thus $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are presheaves sending open sets in X to complete topological rings. However, they are not sheaves in general.

For a rational subset $U \subset X$, we have $\mathcal{O}_X(U) = A_U$ and $\mathcal{O}_X^+(U) = A_U^+$. Since rational subsets form a basis of the topology, the stalks of $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are given by

$$\mathcal{O}_{X,x} \coloneqq \varinjlim_{\substack{x\in U\subset X\\ U \text{ rational}}} \mathcal{O}_X(U) = \varinjlim_{U} A_U, \qquad \mathcal{O}_{X,x}^+ \coloneqq \varinjlim_{U} A_U^+.$$

Take $x \in X$ with corresponding valuation $v\colon A \to \Gamma \cup \{0\}$. For each rational subset U of X containing x, one checks that v extends uniquely to a continuous valuation $v\colon A_U \to \Gamma \cup \{0\}$. Thus we obtain a canonical valuation

$$v_x\colon \mathcal{O}_{X,x} \longrightarrow \Gamma \cup \{0\}.$$

Therefore, for a general open subset $W \subset X$, and for any $f \in \mathcal{O}_X(W)$, the valuation $v(f)$ for $v \in W$ makes sense. Namely, we define $v(f)$ to be the canonical valuation on $\mathcal{O}_{X,v}$ applied to the image of f in $\mathcal{O}_{X,v}$.

Fact 2.7.2. Consider $X = \operatorname{Spa}(A, A^+)$. We naturally view $\mathcal{O}_X^+$ as a sub-presheaf of $\mathcal{O}_X$.

(1) For any open subset W, we have
$$\mathcal{O}_X^+(W) = \{f \in \mathcal{O}_X(W)\colon v(f) \leqslant 1 \text{ for all } v \in W\}.$$
In particular, $\mathcal{O}_X^+$ is determined by $\mathcal{O}_X$ together with the canonical valuations on the stalks of $\mathcal{O}_X$.

(2) For each $x \in X$, $\mathcal{O}_{X,x}$ is a local ring whose residue field is isomorphic to
$$k(x) = \operatorname{Frac}(A/\operatorname{supp}(x)).$$

(3) We have
$$\mathcal{O}_{X,x}^+ = \{f \in \mathcal{O}_{X,x}\colon v_x(f) \leqslant 1\}.$$
Moreover, it is a local ring with maximal ideal $\{f \in \mathcal{O}_{X,x}^+\colon v_x(f) < 1\}$. Its residue field is identified with the residue field of $k(x)^+$. (Recall that $k(x)^+$ is the valuation subring of $k(x)$ corresponding to the valuation on $k(x)$ induced by v_x.)

Definition 2.7.3. A Huber pair (A, A^+) is called *sheafy* if the presheaf $\mathcal{O}_X$ on $X = \operatorname{Spa}(A, A^+)$ is a sheaf.

Theorem 2.7.4. *The Huber pair (A, A^+) is sheafy in the following cases.*

(1) $\widehat{A}$ *is discrete.*

(2) $\widehat{A}$ *has a noetherian ring of definition.*

(3) A *is Tate and strongly noetherian, i.e., for each $n \geqslant 1$,*
$$\widehat{A}\langle T_1, \dots, T_n\rangle = \{f \in \widehat{A}[\![T_1, \dots, T_n]\!]\colon \textit{coefficients of } f \textit{ tend to } 0 \textit{ in } \widehat{A}\}$$
is noetherian.

[2] A subset $S \subset B$ is called power-bounded if $\bigcup_n S^n$ is bounded.

(4) (A, A^+) *is perfectoid, cf.* [SW20, Theorem 7.1] *and see Definition 3.1.1.*

Definition 2.7.5. Let $f\colon A \to B$ be a continuous map between Huber rings A and B. It is of *topologically finite type* (TFT) if

(i) B is complete,
(ii) there are rings of definition $A_0 \subset A$ and $B_0 \subset B$ such that $f(A_0) \subset B_0$,
(iii) B is a finitely generated algebra over $f(A) \cdot B_0$, and
(iv) there is $n \in \mathbb{N}$ and a surjective open continuous map

$$\widehat{A}_0\langle x_1, \dots, x_n\rangle \longrightarrow B_0$$

of A_0-algebras.

Proposition 2.7.6. *If $f\colon A \to B$ is of topological finite type, and A satisfies* (2) *or* (3) *of Theorem 2.7.4, then B also satisfies* (2) *or* (3) *respectively.*

Proposition 2.7.7. *Every non-archimedean field K is (Tate and) strongly noetherian.*

Definition 2.7.8 (Adic spaces)**.** We define a category $\mathscr{V}$ as follows.

- Each object of $\mathscr{V}$ is a tuple $(X, \mathcal{O}_X, \{v_x\}_{x\in X})$, where
  - X is a topological space,
  - $\mathcal{O}_X$ is a sheaf of topological rings such that $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$, and
  - v_x for each $x \in X$ is an equivalence class of valuations on $\mathcal{O}_{X,x}$ such that $\mathrm{supp}(v_x)$ is the maximal ideal of $\mathcal{O}_{X,x}$.
- Each morphism of $\mathscr{V}$ is the data consisting of
  - a map $f\colon X \to Y$ of topological spaces, together with
  - a map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of topological rings, satisfying the property that the induced maps between stalks of $\mathcal{O}_X$ and $\mathcal{O}_Y$ are compatible with the canonical valuations.

An *adic space* is an object of $\mathscr{V}$ that is locally isomorphic to $\mathrm{Spa}(A, A^+) \in \mathscr{V}$ for a sheafy Huber pair (A, A^+). Morphisms between adic spaces are by definition morphisms in $\mathscr{V}$.

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Remark 2.7.9. The functor

$$\begin{aligned}(\text{Sheafy Huber pairs})^{\mathrm{op}} &\longrightarrow (\text{Adic spaces})\\ (A, A^+) &\longmapsto \mathrm{Spa}(A, A^+).\end{aligned}$$

is in fact fully faithful.

In the sequel, we shall write $\mathrm{Spa}(A)$ for $\mathrm{Spa}(A, A^+)$ in the occasion where $A^+ = A^0$.

2.8. **Adic spaces arising from usual algebraic geometry.**

2.8.1. *Locally noetherian formal schemes.* Locally, such a formal scheme is of the form $\mathrm{Spf}\, A$, where A is a noetherian topological ring which is complete with respect to the I-adic topology for an ideal $I \subset A$. Then (A, A) is a sheafy Huber pair by Theorem 2.7.4(2). We obtain a functor

$$(\text{Locally noetherian formal schemes}) \longrightarrow (\text{Adic spaces}), \quad \mathfrak{X} \longmapsto \mathfrak{X}^{\mathrm{ad}}.$$

by gluing the local constructions

$$\mathrm{Spf}\, A \longmapsto (\mathrm{Spf}\, A)^{\mathrm{ad}} := \mathrm{Spa}(A, A).$$

This functor turns out to be fully faithful.

We caution the reader that the underlying topological spaces of $\mathfrak{X}$ and $\mathfrak{X}^{\mathrm{ad}}$ are typically different. For instance, $\mathrm{Spf}\, \mathbb{Z}_p$ has one point, and $\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$ has two points.

The following situation is typical. Let K be a non-archimedean field. Assume that $\mathcal{O}_K$ is noetherian, and pick a pseudo-uniformizer $\varpi \in K^\times$. Let X be an $\mathcal{O}_K$-scheme that is locally of finite type. We can think of X as an integral model of X_K. By completing X along its special fiber, we get a locally noetherian formal scheme $\mathfrak{X}$. (For example, one can take $X = \mathrm{Spec}\, A$ and see $\mathfrak{X} = \mathrm{Spf}(\widehat{A})$ where $\widehat{A}$ is the ϖ-adic completion $\varprojlim_n A/\varpi^n$ equipped with the ϖ-adic topology.) We then obtain an adic space $\mathfrak{X}^{\mathrm{ad}}$ over $\mathrm{Spa}(\mathcal{O}_K)$ (short hand notation for $\mathrm{Spa}(\mathcal{O}_K, \mathcal{O}_K)$).

In this situation, $\mathfrak{X}^{\mathrm{ad}}$ is locally of finite type over $\mathrm{Spa}(\mathcal{O}_K)$. The meaning of "locally finite type" will be defined later, in Definition 2.9.3. This condition allows us to take the generic fiber, i.e., the fiber product $\mathfrak{X}^{\mathrm{ad}} \times_{\mathrm{Spa}(\mathcal{O}_K)} \mathrm{Spa}(K)$ exists in the category of adic spaces. (Recall that $\mathrm{Spa}(K)$ is shorthand notation for $\mathrm{Spa}(K, \mathcal{O}_K)$.) We denote this fiber product by $\mathfrak{X}^{\mathrm{ad}}_\eta$, and call it the *adic generic fiber* of $\mathfrak{X}^{\mathrm{ad}}$ or of $\mathfrak{X}$. Locally, if $\mathfrak{X}^{\mathrm{ad}}$ (or any adic space locally of finite type over $\mathrm{Spa}(\mathcal{O}_K)$) is of the form $\mathrm{Spa}(A, A^+)$, then the generic fiber is nothing but the rational open set in $\mathrm{Spa}(A, A^+)$ consisting of v such that $v(\varpi) \neq 0$.

Example 2.8.1. Let K and $\mathcal{O}_K$ be as above. Consider the affine line $X = \mathrm{Spec}\, \mathcal{O}_K[T]$. Its completed noetherian formal scheme is

$$\mathfrak{X} = \mathrm{Spf}(\mathcal{O}_K\langle T\rangle),$$

where we recall that

$$\mathcal{O}_K\langle T\rangle = \left\{ \sum_{n=0}^{\infty} a_n T^n \colon a_n \in \mathcal{O}_K,\ a_n \to 0 \right\}$$

is the ϖ-adic completion of $\mathcal{O}_K[T]$. Accordingly,

$$\mathfrak{X}^{\mathrm{ad}}_\eta = \mathrm{Spa}(\mathcal{O}_K\langle T\rangle[1/p], \mathcal{O}_K\langle T\rangle) = \mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle),$$

which is the closed unit disc. Here $\mathcal{O}_K\langle T\rangle[1/p]$ is equipped with the topology such that $\mathcal{O}_K\langle T\rangle$ is open. This can be regarded as the "*good reduction locus* in X_K".

2.8.2. *Base changing a scheme to an adic space.* We have an obvious forgetful functor

$$\begin{array}{ccc} (\text{Adic spaces}) & \xrightarrow{\text{forget}} & (\text{Locally ringed spaces}) \\ S & \longmapsto & \underline{S}. \end{array}$$

Namely, we view the structure sheaf of topological rings as a sheaf of abstract rings, and we forget the distinguished equivalence class of valuations on each stalk of the structure sheaf.

Proposition 2.8.2. *Let $f\colon Y \to Z$ be a morphism of schemes. Assume that f is locally of finite type. Assume S is an adic space covered by $\mathrm{Spa}(A, A^+)$ such that*

- ◦ *either A contains a noetherian ring of definition,*
- ◦ *or A is both Tate and strongly noetherian,*

implying that (A, A^+) is a sheafy Huber pair from Theorem 2.7.4. Let $g\colon \underline{S} \to Z$ be a morphism of locally ringed spaces. Then there exists an adic space R together with a morphism $R \to S$ of adic spaces, equipped with a morphism $\underline{R} \to Y$ of locally ringed spaces, satisfying the universal property below.

- *The following square diagram commutes, and for any adic space $R' \to S$ equipped with a morphism $\underline{R'} \to Y$ making a similar diagram commutative, there exists a unique map $R' \to R$ of adic spaces fitting into the following commutative diagram*

$$\begin{tikzcd} \underline{R'} \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \underline{R} \arrow[r] \arrow[d] & Y \arrow[d] \\ & \underline{S} \arrow[r] & Z \end{tikzcd}$$

Moreover, $R \to S$ is locally of finite type (see Definition 2.9.3). For fixed $\underline{S} \to Z$, the formation of the adic space R over S is functorial with respect to the scheme Y over Z. That is, if we have two Z-schemes Y_1, Y_2 and a Z-scheme map $Y_1 \to Y_2$, then we obtain two adic spaces R_1, R_2 over S together with a map $R_1 \to R_2$ over S.

As an application of Proposition 2.8.2, we consider $Z = \mathrm{Spec}\, K$, where K is a non-archimedean field. For $S = \mathrm{Spa}(K) \coloneqq \mathrm{Spa}(K, \mathcal{O}_K)$ and $\underline{S} \to Z$ the obvious map (in fact, $\underline{S} = Z$), and any K-scheme Y locally of finite type, we get the fiber product

$$Y \times_{\mathrm{Spec}\, K} \mathrm{Spa}(K),$$

which is an adic space over $\mathrm{Spa}(K)$, still locally of finite type. We thus obtain a functor from schemes locally of finite type over K to adic spaces locally of finite type over $\mathrm{Spa}(K)$. Here the proposition is applicable since K is Tate and strongly noetherian.

Similarly, assuming that $\mathcal{O}_K$ is a DVR, we can take $Z = \mathrm{Spec}\,\mathcal{O}_K$, $S = \mathrm{Spa}(\mathcal{O}_K)$, and take $g\colon \underline{S} \to Y$ to be the "identity map". That is, each of $\underline{S}$ and Z consists of two points, a generic point η and a special point s. At the level of topological spaces we let g send s to s and send η to η. At the level of sheaves, we define $\mathcal{O}_Z(\{\eta\}) = K \to \mathcal{O}_{\underline{S}}(\{\eta\}) = K$ to be the identity map, and define $\mathcal{O}_Z(Z) = \mathcal{O}_K \to \mathcal{O}_{\underline{S}}(\underline{S}) = \mathcal{O}_K$ to be the identity map. We thus obtain a functor from schemes locally of finite type over $\mathcal{O}_K$ to adic spaces locally of finite type over $\mathrm{Spa}(\mathcal{O}_K)$.

We will sketch the construction of the fiber product in Proposition 2.8.2 at least in the affine case. For this we need some preparations.

2.9. **Power series with convergence speed.** Fix a Huber ring A. A subset $M \subset A$ is called *admissible* if M is finite and MA is an open ideal. (This is non-standard terminology.) Let $n \geqslant 1$ be an integer and $M_1, \dots, M_n$ be admissible subsets. Define

$$A\langle X\rangle_M = A\langle X_1, \dots, X_n\rangle_{M_1,\dots,M_n},$$

to be the ring of formal power series

$$\sum_\nu a_\nu X^\nu \in \widehat{A}[\![X]\!]$$

satisfying the following "converging speed" condition: For any neighborhood U of 0 in $\widehat{A}$, we have for all but finitely many indices $\nu = (\nu_1, \dots, \nu_n)$ that $a_\nu \in M_1^{\nu_1} M_2^{\nu_2} \cdots M_n^{\nu_n} \cdot U$. In the sequel, we abbreviate $M_1^{\nu_1} M_2^{\nu_2} \cdots M_n^{\nu_n} \cdot U$ as $M^\nu \cdot U$. This condition here can be understood in the following way. Suppose $M_1 = \cdots = M_n = \{\varpi\}$ for some $\varpi \in A^{00}$. Then our requirement is that $a_\nu \to 0$ in a way faster than $\varpi^{\nu_1 + \cdots + \nu_n} \to 0$.

The topology of $A\langle X\rangle_M$ is given as follows: For each neighborhood U of 0 in $\widehat{A}$, consider the subset

$$\left\{\sum_\nu a_\nu X^\nu \colon a_\nu \in M^\nu \cdot U \text{ for all } \nu\right\} \subset A\langle X\rangle_M.$$

We require these subsets for all U form a neighborhood basis of 0 in $A\langle X\rangle_M$.

Fact 2.9.1. The ring $A\langle X\rangle_M$ with the above topology is a complete Huber ring, and the natural map $A \to A\langle X\rangle_M$ is continuous.

Fact 2.9.2. Let $A \to B$ be a map of Huber rings. Then it is of topological finite type (cf. Definition 2.7.5) if and only if B is complete and there exists a continuous surjective open map of A-algebras $A\langle X\rangle_M \to B$ for suitable n and $M_1, \cdots, M_n$.

Now fix a Huber pair (A, A^+). Take $M_1, \dots, M_n$ as before. Then we get a Huber pair

$$(A\langle X\rangle_M, A\langle X\rangle_M^+),$$

where $A\langle X\rangle_M^+$ is the integral closure in $A\langle X\rangle_M$ of the subset $\{\sum_\nu a_\nu X^\nu \colon a_\nu \in M^\nu \cdot \widehat{A}^+ \text{ for all } \nu\}$.

Definition 2.9.3. We say that a map between Huber pairs $\varphi\colon (A, A^+) \to (B, B^+)$ is of *topologically finite type* if there are admissible $M_1, \dots, M_n$ in A together with a continuous surjective open map $\widetilde{\varphi}\colon A\langle X\rangle_M \to B$ extending $\varphi\colon A \to B$, and such that B^+ exactly equals the integral closure in B of $\widetilde{\varphi}(A\langle X\rangle_M^+)$. We say that a map between adic spaces $X \to Y$ is locally of finite type, if for each $x \in X$ we can find an open $\mathrm{Spa}(B, B^+) \subset X$ containing x, an open $\mathrm{Spa}(A, A^+)$ containing the image of $\mathrm{Spa}(B, B^+)$, such that the map $(A, A^+) \to (B, B^+)$ corresponding to the map $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ is of topologically finite type.

Sketch of Proposition 2.8.2. We are now in a position to give the construction of the fiber product in Proposition 2.8.2, at least in the affine case:

$$\begin{tikzcd} ? \arrow[r, dashed] \arrow[d, dashed] & Z = \operatorname{Spec} B[X_1, \ldots, X_n]/I \arrow[d] \\ \underline{\operatorname{Spa}(A, A^+)} \arrow[r] & \operatorname{Spec} B \end{tikzcd}$$

The idea to do this is to use $(A\langle X\rangle_M, A\langle X\rangle_M^+)$ to construct larger and larger "polydiscs" in $\mathbb{A}^n$. Namely, for varying M corresponding to stricter and stricter converging speed conditions, elements of $A\langle X\rangle_M$ have larger and larger "polydiscs of convergence" in $\mathbb{A}^n$. More precisely, these polydiscs are given by $\operatorname{Spa}(A\langle X\rangle_M, A\langle X\rangle_M^+)$. If we take a suitable union of them, then we would obtain the desired base change in the situation $I = 0$ with $Z = \mathbb{A}_B^n$. For general I, we need to take the quotient of each $(A\langle X\rangle_M, A\langle X\rangle_M^+)$ by I in a suitable way.

Note that it is possible to fix a finite subset $E \subset A^{00}$ such that $E \cdot A$ is open. For each integer $k \geqslant 1$, E^k is an admissible subset and we write

$$A(k) \coloneqq A\langle X_1, \ldots, X_n\rangle_{E^k, \ldots, E^k}$$

and naturally get a Huber pair $(A(k), A(k)^+)$ from (A, A^+) as explained before.

If $I = (0)$, the case would be easy, because we obtain $\operatorname{Spec} B[X_1, \ldots, X_n]/I = \mathbb{A}_B^n$ and it suffices to take

$$\mathbb{A}_B^n \times_{\operatorname{Spec} B} \operatorname{Spa}(A, A^+) \coloneqq \varinjlim_k \operatorname{Spa}(A(k), A(k)^+).$$

Here for any $h \geqslant k$, we obtain a map $(A(h), A(h)^+) \to (A(k), A(k)^+)$. After taking $\operatorname{Spa}(-)$, the image turns out to be a rational subset. Hence the direct limiting makes sense in the category of adic spaces.

In general, we have the map

$$\begin{aligned} g_k \colon B[X_1, \ldots, X_n] &\longrightarrow A(k) \\ X_i &\longmapsto X_i \end{aligned}$$

extending $B \to A$, which comes from

$$\underline{\operatorname{Spa}(A, A^+)} \longrightarrow \operatorname{Spec} B.$$

We write $A(k)/I$ for the quotient of $A(k)$ by the ideal generated by $g_k(I)$. There is a unique topology on $A(k)/I$ as a Huber ring such that the canonical quotient map $A(k) \to A(k)/I$ is continuous and open. Define $(A(k)/I)^+$ to be the integral closure in $A(k)/I$ of the image of $A(k)^+$. It turns out that $(A(k)/I, (A(k)/I)^+)$ is sheafy, as $A(k)/I$ either has a noetherian ring of definition or is Tate and strongly noetherian, depending on which of the two properties the original A has. For each $h \geqslant k$, the natural morphism

$$R_k \coloneqq \operatorname{Spa}(A(k)/I, (A(k)/I)^+) \longrightarrow \operatorname{Spa}(A(h)/I, (A(h)/I)^+) \eqqcolon R_h$$

is an isomorphism onto a rational subset of R_h. Then the desired base change is taken as

$$(\operatorname{Spec} B[x_1, \ldots, x_n]/I) \times_{\operatorname{Spec} B} \operatorname{Spa}(A, A^+) \coloneqq \varinjlim_k R_k.$$

□

Example 2.9.4. Take $A = B = \mathbb{Q}_p$ and $I = (0)$. Then we can take $E = \{p\}$. We aim to compute the base change

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{Q}_p[X] \arrow[d] \\ \operatorname{Spa} \mathbb{Q}_p \arrow[r] & \operatorname{Spec} \mathbb{Q}_p \end{tikzcd}$$

Following the recipe above, we get

$$A(k) = \left\{ \sum_n a_n X^n \colon a_n \in \mathbb{Q}_p,\ a_n/p^{nk} \to 0 \right\}$$

and $A(k)^+$ is the integral closure in $A(k)$ of

$$A(k)^{+,\prime} = \left\{ \sum_n a_n X^n : a_n/p^{nk} \to 0,\ a_n/p^{nk} \in \mathbb{Z}_p \right\}.$$

Note that we have an isomorphism

$$\begin{aligned} (A(k), A(k)^{+,\prime}) &\xrightarrow{\sim} (\mathbb{Q}_p\langle X\rangle, \mathbb{Z}_p\langle X\rangle) \\ X &\longmapsto p^k X \end{aligned}$$

which in particular shows that $A(k)^{+,\prime}$ is in fact already integrally closed in $A(k)$, so $A(k)^+ = A(k)^{+,\prime}$. By this isomorphism, it is reasonable to call $R_k = \mathrm{Spa}(A(k), A(k)^+)$ the closed disc centered at origin of radius p^k. It is also easy to see that for $k \leqslant h$, the natural map $R^k \to R^h$ is an isomorphism onto the rational subset $\{v : v(X) \leqslant v(p^{-k}) \neq 0\}$, which is another justification for this terminology.

Remark 2.9.5. Let K be a non-archimedean field such that $\mathcal{O}_K$ is noetherian. Let $\mathscr{X}$ be a locally finite-type scheme over $\mathcal{O}_K$. Then we have seen two ways of obtaining an adic space over $\mathrm{Spa}(K)$. First: $\mathfrak{Z}_\eta^{\mathrm{ad}}$, where $\mathfrak{Z}$ is the formal scheme obtained by completing $\mathscr{X}$ along the special fiber. Second, $\mathscr{X}_K^{\mathrm{ad}} := \mathscr{X}_K \times_{\mathrm{Spec}\, K} \mathrm{Spa}\, K$. The first is naturally an open in the second. Indeed, in the affine case $\mathscr{X} = \mathrm{Spec}\, \mathcal{O}_K[X_1, \cdots, X_n]/I$ and following the notation in the Sketch of Proposition 2.8.2 (applied to $A = B = K$ with $Z = \mathscr{X}_K = \mathrm{Spec}\, K[X_1, \cdots, X_n]/I$), we have

$$\mathfrak{Z}_\eta^{\mathrm{ad}} \cong R_0 = \mathrm{Spa}(A(0), A(0)^+) \subset \varinjlim_k R_k = \mathscr{X}_K^{\mathrm{ad}}.$$

3. Perfectoid Shimura varieties

3.1. **Perfectoid spaces.**

Definition 3.1.1. Fix a non-archimedean field $(K, |\cdot|)$ with residue characteristic p.

(1) Say K is a *perfectoid field* if $|\cdot|$ is non-discrete (i.e., $|K^\times|$ is not a discrete subgroup of $\mathbb{R}_{>0}$) and the Frobenius map
$$\Phi := \mathcal{O}_K/p \longrightarrow \mathcal{O}_K/p$$
sending x to x^p is surjective.

(2) Fix a perfectoid field K as in (1). Let A be a Huber K-algebra, i.e., A is a Huber ring equipped with a continuous homomorphism $K \to A$. We say that A is *perfectoid* if it is uniform (i.e. A^0 is bounded), complete, and such that $\Phi : A^0/p \to A^0/p$ is surjective.

(3) Let (A, A^+) be a Huber pair such that A is a perfectoid K-algebra in the sense of (2). Then we call (A, A^+) a *perfectoid Huber pair over K*. Note that we do not require that (A, A^+) should be over $(K, \mathcal{O}_K)$.

The following lemma collects some basic facts about Huber algebras over a non-archimedean field.

Lemma 3.1.2. *Let K be a non-archimedean field, A be a Huber ring, and $\varphi : K \to A$ be a continuous homomorphism. Assume that A is Hausdorff. The following statements hold.*

(1) *$\varphi(\mathcal{O}_K)$ is bounded, and in particular it is contained in A^0.*
(2) *For any ring of definition $A_0 \subset A$, we have $\varphi^{-1}(A_0) \subset \mathcal{O}_K$.*
(3) *φ induces a homeomorphism $K \xrightarrow{\sim} \varphi(K)$.*

Proof. If $\varpi \in K^\times$ is a pseudo-uniformizer, then so is $\varphi(\varpi) \in A$. Statement (1) then follows from Lemma 2.2.2 (cf. also Exercise 2.2.6 (4)). Suppose (2) is false. Then there exists a pseudo-uniformizer $\varpi \in K^\times$ such that $\varpi^{-1} \in \varphi^{-1}(A_0)$. On the other hand $\varphi^{-1}(A_0)$ is open and hence it contains $\varpi^n \mathcal{O}_K$ for some $n \geqslant 0$. We conclude that $\varphi(K) \subset A_0$. By Lemma 2.2.2, any neighborhood of 0 in A_0 contains $\varphi(\varpi)^k A_0$ for sufficiently large k, which contains $\varphi(K)$. In particular $1 \in A_0$ is in the closure of 0, contradicting with the assumption that A is Hausdorff. Having proved (2), we see that $\varphi(\mathcal{O}_K)$ is open in $A \cap \varphi(K)$. It then follows from Lemma 2.2.2 that $\varphi : K \to \varphi(K)$ is an open map, and hence a homeomorphism. □

Remark 3.1.3. There exist more general notions of perfectoid Huber rings without reference to perfectoid fields. We will not need them.

Exercise 3.1.4. Let K be a non-archimedean field, and let A be a Huber K-algebra that is Hausdorff and uniform. Then A is a normed K-algebra with norm satisfying $\|1\| = 1$.

> *Hint.* Fix a pseudo-uniformizer $\varpi \in K$. Show that
> $$\|x\| \coloneqq \inf\{|\lambda| \colon \lambda \in K^\times, \lambda^{-1}x \in A^0\}$$
> is a norm on A (satisfying all axioms in Example 2.2.4) and defines the same topology as the original topology on A. Then show that $\|1\| = 1$ using the fact $A^0 \cap K = \mathcal{O}_K$, which follows from Lemma 3.1.2. (Alternatively, if $\|1\| < 1$, then $\|1\| = 0$ and A is not Hausdorff.)

Thus for every perfectoid algebra A over a perfectoid field K, we know that A is a complete normed K-algebra (= Banach K-algebra).

Fix K a perfectoid field.

Fact 3.1.5. Let (A, A^+) be a perfectoid Huber pair over K. Then it is sheafy. Moreover, for any rational subset U in $X = \mathrm{Spa}(A, A^+)$, the Huber K-algebra $\mathcal{O}_X(U)$ is perfectoid.

Definition 3.1.6. A perfectoid space over K is an adic space covered by $\mathrm{Spa}(A, A^+)$ such that each (A, A^+) is a perfectoid Huber pair over K. (Again we do not require that $\mathcal{O}_K \subset A^+$, so a perfectoid space over K may not admit a map to $\mathrm{Spa}(K, \mathcal{O}_K)$.)

Caution 3.1.7. The subtlety lying in Definition 3.1.6 is that when (A, A^+) is a Huber pair, with A a K-algebra, such that $\mathrm{Spa}(A, A^+)$ is a perfectoid space over K, it is not known whether A is a perfectoid K-algebra.

Example 3.1.8. The following are perfectoid fields:
$$\mathbb{Q}_p(\mu_{p^\infty})^\wedge, \quad \mathbb{Q}_p(p^{1/p^\infty})^\wedge.$$
Let us verify the first one. Call the field K. The extension $\mathbb{Q}_p(\mu_{p^r})/\mathbb{Q}_p$ is totally ramified of degree $\phi(p^r)$, which is unbounded as $r \to +\infty$. Hence the absolute value on K is non-discrete. Let $x \in \mathcal{O}_K/p$. Then x can be lifted to some element of $\mathcal{O}_{\mathbb{Q}_p(\mu_{p^r})} = \mathbb{Z}_p[\mu_{p^r}]$ for some finite r, say $\tilde{x} = \sum_{i=1}^{p^r} a_i\zeta^i$, where $a_i \in Z_p$ and ζ is a primitive p^r-th root of unity. Let ξ be a p^{r+1}-th root of unity such that $\xi^p = \zeta$. Then $y = \sum_i a_i\xi^i \in \mathcal{O}_K$ satisfies that $y^p = x$ in $\mathcal{O}_K/p$. (Note that $a_i^p \equiv a_i \bmod p\mathbb{Z}_p$.)

Example 3.1.9 (Perfectoid unit disc)**.** Let K be a perfectoid field. Consider
$$A = K\langle T^{1/p^\infty}\rangle,$$
which collects all formal power series $\sum_{\nu\in\mathbb{Z}[1/p]_{\geqslant 0}} a_\nu T^\nu$, satisfying that for any neighborhood U of 0 in K, we have $a_\nu \in U$ for almost all ν. The Gauss norm $\|\cdot\|$ on A is defined by
$$\|\cdot\| \colon \sum_{\nu\in\mathbb{Z}[1/p]_{\geqslant 0}} a_\nu T^\nu \longmapsto \sup_\nu |a_\nu|.$$
Then $(A, \|\cdot\|)$ is a complete normed K-algebra. We have
$$\{a \in A \colon \|a\| \leqslant 1\} = \mathcal{O}_K\langle T^{1/p^\infty}\rangle = \left\{\sum a_\nu T^\nu \in A \colon a_\nu \in \mathcal{O}_K \text{ for all } \nu\right\},$$
Since $\|\cdot\|$ shares the same image as $|\cdot|$, and since
$$\{a \in A \colon \|a\| \leqslant 1\}/\{a \in A \colon \|a\| < 1\} = k[T^{1/p^\infty}]$$
is an integral domain (where k is the residue field of K), we conclude by Exercise 2.2.6 and Example 2.2.11 that $\|\cdot\|$ is multiplicative, that A is uniform, and that $A^0 = \mathcal{O}_K\langle T^{1/p^\infty}\rangle$. Since $\|\cdot\|$ is multiplicative and since $A^0 = \{x \colon \|x\| \leqslant 1\}$, we know that A^0 is integrally closed by Exercise 2.5.1. Thus (A, A^0) is a complete, uniform, and Huber pair over $\mathrm{Spa}(K, \mathcal{O}_K)$.

(Note that A^0 is also isomorphic to the ϖ-adic completion of $\mathcal{O}_K[T^{1/p^\infty}]$, where ϖ is a pseudo-uniformizer in $K^\times$.)

We check that (A, A^0) is a perfectoid Huber pair. It remains to check that $\Phi \colon A^0/p \to A^0/p$ is surjective. This is obvious since $A^0/p \cong (\mathcal{O}_K/p)[T^{1/p^\infty}]$, and since $\Phi \colon \mathcal{O}_K/p \to \mathcal{O}_K/p$ is surjective by K being a perfectoid field.

The perfectoid space $\mathrm{Spa}(A, A^0)$ is referred to as the perfectoid closed unit disc.

Fix a perfectoid field K and a perfectoid space X over K. Lect.8, Oct 26

Definition 3.1.10. Let $(X_i)_i$ be an inverse system of adic spaces over K, whose transition maps are assumed to be quasi-compact and quasi-separated. Let $\varphi = (\varphi_i)_i$ be a system of compatible maps $\varphi_i \colon X \to X_i$. We say φ induces

$$X \sim \varprojlim_i X_i$$

(where the right side is merely a formal symbol but not necessarily an adic space) if

(i) φ induces a homeomorphism

$$|X| \xrightarrow{\sim} \varprojlim_i |X_i|,$$

where $|X|$ and $|X_i|$ are underlying topological spaces of X and X_i, respectively, and the projective limit is equipped with the inverse limit topology.

(ii) X is covered by opens $\mathrm{Spa}(A, A^+)$ with each A being perfectoid and such that the subset

$$\bigcup_i \bigcup_{U \subset X_i} \mathrm{im}(\mathcal{O}_{X_i}(U) \to A)$$

of A is dense. Here the second union is over all affinoid opens $U = \mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ in X_i containing the image of $\mathrm{Spa}(A, A^+) \hookrightarrow X \xrightarrow{\varphi_i} X_i$.

The above definition makes precise the sense of $\sim$ in Theorem 1.4.1(1).

Example 3.1.11. We have

$$\mathrm{Spa}(K\langle T^{1/p^\infty}\rangle, \mathcal{O}_K\langle T^{1/p^\infty}\rangle) \sim \varprojlim_{T \mapsto T^p} \mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle).$$

Here all objects in the inverse system are copies of $\mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$, indexed by $\mathbb{N}$. The transition maps between adjacent copies are given by $K\langle T\rangle \to K\langle T\rangle, \sum a_n T^n \mapsto \sum a_n T^{pn}$.

The following is a useful fact.

Proposition 3.1.12. *If we have $X \sim \varprojlim_i X_i$ induced by φ as in Definition 3.1.10, then the pair (X, φ) represents the functor*

$$\begin{aligned} (\textit{Perfectoid spaces over } K) &\longrightarrow \mathsf{Sets} \\ Y &\longmapsto \varprojlim_i \mathrm{Hom}_K(Y, X_i). \end{aligned}$$

In particular, (X, φ) is unique up to a unique isomorphism.

3.2. **Review of abelian schemes.**

Definition 3.2.1. Let S be a scheme. An *abelian scheme* $X \to S$ is a proper smooth group scheme that is geometrically connected.

As a group scheme, an abelian scheme X is equipped with group operations

$$m \colon X \times_S X \longrightarrow X, \quad e \colon S \longrightarrow X, \quad i \colon X \longrightarrow X,$$

which respectively serve as the roles of multiplication, identity, and inverse operations. We also call e the *neutral section.*

Fact 3.2.2. The group structure on an abelian scheme is automatically commutative. Further, any S-scheme map $X \to Y$ between abelian schcmes preserving neutral sections is automatically a group homomorphism.

Definition 3.2.3. An *isogeny* between abelian schemes is a homomorphism $f \colon X \to Y$ that is surjective and quasi-finite. (It follows that f is finite flat.)

From now on, in all our discussions of abelian schemes, we tacitly assume that the base scheme S is noetherian and that the abelian scheme $X \to S$ is projective, i.e., X admits a closed embedding into $\mathbb{P}(\mathcal{E})$ for a coherent sheaf $\mathcal{E}$ over S. (This assumption turns out to be automatic if S is the spectrum of a field, but not in general.)

Fact 3.2.4. Consider the functor

$$\underline{\mathrm{Pic}}_{X/S}\colon T \longmapsto \{(L,\iota)\}/\cong,$$

where on the right side,

- L is a line bundle over $X_T = X \times_S T$, and
- $\iota\colon e_T^* L \xrightarrow{\sim} \mathcal{O}_T$ is an isomorphism of $\mathcal{O}_T$-modules. Here e_T is the neutral section $e_T\colon T \to X_T$.

Then this functor is represented by an S-scheme $\mathrm{Pic}_{X/S}$ which is smooth and locally of finite type over S. Moreover, $\mathrm{Pic}_{X/S}$ is a group scheme with respect to the tensor product of line bundles.

Fact 3.2.5. There exists a unique clopen subscheme $X^\vee$ of $\mathrm{Pic}_{X/S}$ such that fiberwise it is the neutral connected component (i.e., the connected component of the group scheme containing the neutral section). Moreover, the following are true.

1. $X^\vee$ is projective over S. In particular, it is an abelian scheme over S, called the dual abelian scheme of X.
2. $X^{\vee\vee} \cong X$ canonically.
3. For each isogeny $\varphi\colon X \to Y$, we can canonically obtain the dual isogeny $\varphi^\vee\colon Y^\vee \to X^\vee$ by pulling back line bundles along φ.

Construction 3.2.6 (Mumford line bundle and Mumford homomorphism)**.** Let $X \to S$ be an abelian scheme. Starting with a line bundle L on X, we consider

$$m\colon X \times_S X \longrightarrow X, \quad \mathrm{pr}_i\colon X \times_S X \longrightarrow X \quad (i = 1, 2)$$

and define

$$\mathfrak{M}(L) = (m^* L) \otimes (\mathrm{pr}_1^* L^{-1}) \otimes (\mathrm{pr}_2^* L^{-1}),$$

which is a line bundle on $X \times_S X = X_X$. Also, restricted to the neutral section $X \to X_X$, $\mathfrak{M}(L)$ is canonically trivialized. Hence we get an X-valued point of $\mathrm{Pic}_{X/S}$, namely a map $X \to \mathrm{Pic}_{X/S}$. Because X is fiberwise connected, the map $X \to \mathrm{Pic}_{X/S}$ factors through $X^\vee$. This leads to an S-scheme morphism

$$\Lambda(L)\colon X \longrightarrow X^\vee.$$

One easily checks that $\Lambda(L)$ preserves the neutral sections, and hence $\Lambda(L)$ is a homomorphism.

Fact 3.2.7. Let $S = \operatorname{Spec} k$ for some field k. If L is ample, then $\lambda = \Lambda(L)$ is an isogeny. Moreover, in this case it is *symmetric*, meaning that $\lambda\colon X \to X^\vee$ and $\lambda^\vee\colon X^{\vee\vee} = X \longrightarrow X^\vee$ are the same morphism.

Remark 3.2.8. Suppose k is algebraically closed and $S = \operatorname{Spec} k$. Take a closed point $x \in X(k)$. Then

$$\Lambda(L)(x) \in X^\vee(k) \subset \mathrm{Pic}_{X/k}(k) = \mathrm{Pic}(X)$$

is given by the isomorphism class of the line bundle $t_x^* L \otimes L^{-1}$ on X, where $t_x\colon X \to X$ is the translation-by-x map $y \mapsto x + y$.

Definition 3.2.9 (Polarization)**.** Let $X \to S$ be an abelian scheme.

1. A *polarization* of X is a homomorphism
$$\lambda\colon X \longrightarrow X^\vee$$
such that for any geometric point $\bar{s}$ of S, the fiber morphism
$$\lambda_{\bar{s}}\colon X_{\bar{s}} \longrightarrow X_{\bar{s}}^\vee$$
is of form $\Lambda(L_{\bar{s}})$, where $L_{\bar{s}}$ is an *ample* line bundle on $X_{\bar{s}}$. Note that any polarization λ is always a symmetric isogeny.
2. We say a polarization λ is *principal* if it is an isomorphism.

Remark 3.2.10. If $\lambda\colon X \to X^\vee$ is a polarization, then there exists a line bundle L on X which is relatively ample over S, such that $[2]\lambda = \Lambda(L)$.

3.3. **Frobenius and Verschiebung.** We now consider abelian schemes specifically over characteristic p. Let S be the base scheme, assumed to have characteristic $p > 0$. Let $X \to S$ be an abelian scheme. Then we have the *absolute Frobenius* $\mathrm{Fr}\colon S \to S$, which is the identity on the underlying topological space and sends each $f \in \mathcal{O}_S$ to f^p. Similarly, we have $\mathrm{Fr}\colon X \to X$. We define $X^{(p)}$ by the pullback diagram below, which is an abelian scheme over S as well. We also define $F\colon X \to X^{(p)}$ to make the following diagram commute, and call it the *relative Frobenius*. Note that F is a morphism of S-schemes, unlike $\mathrm{Fr}\colon X \to X$ which does not preserve the structure morphism $X \to S$.

$$\begin{tikzcd} X \arrow[dr, dashed, "F"] \arrow[drr, bend left, "\mathrm{Fr}"] \arrow[ddr, bend right] & & \\ & X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\ & S \arrow[r, "\mathrm{Fr}"] & S. \end{tikzcd}$$

Fact 3.3.1. The relative Frobenius $F\colon X \to X^{(p)}$ is an isogeny of degree p^g, where g is the relative dimension of X over S. If $S = \operatorname{Spec} k$ with k a field, then F is purely inseparable.

There is also a naturally defined homomorphism $V\colon X^{(p)} \to X$, called the *Verschiebung*. It is in fact defined for any flat commutative group scheme X over S. The definition is a bit complicated, and we omit it. We have

$$V \circ F = [p]\colon X \longrightarrow X$$

in this generality. In the current case with an abelian scheme $X \to S$, the above relation uniquely characterizes V, and implies that V is an isogeny of degree p^g. (Recall that $\deg[n] = n^{2g}$.)

Remark 3.3.2. We have $F \circ V = [p]\colon X^{(p)} \longrightarrow X^{(p)}$. Indeed, we have $F \circ V \circ F = F \circ [p] = [p] \circ F$. Since F is an isogeny, we can cancel it from the right and get $F \circ V = [p]$.

3.4. **Siegel modular varieties.** Fix a vector space V over $\mathbb{Q}$ of dimension $2g$, together with a symplectic pairing (i.e., non-degenerate alternating form) ψ. We can and do fix an identification $V \cong \mathbb{Q}^{2g}$ such that ψ is defined by

$$\psi(x, y) = x^{\mathrm{t}} \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} y.$$

Note that the natural lattice $\mathbb{Z}^{2g} \subset \mathbb{Q}^{2g}$ is self-dual with respect to ψ. The *symplectic similitude group* $\mathrm{GSp}_{2g} = \mathrm{GSp}(V, \psi)$ is the reductive group scheme over $\mathbb{Z}$ whose points in a commutative ring R are given by

$$\mathrm{GSp}_{2g}(R) = \left\{ x \in \mathrm{GL}_{2g}(R)\colon \exists \nu(x) \in R^\times,\ x^{\mathrm{t}} \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} x = \nu(x) \cdot \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \right\}.$$

We write G for GSp_{2g}. The scalar $\nu(x)$ defines a map $\nu\colon G \to \mathbb{G}_m$, and thus there is a short exact sequence

$$1 \longrightarrow \mathrm{Sp}(V, \psi) \longrightarrow G \xrightarrow{\nu} \mathbb{G}_m \longrightarrow 1.$$

Fix a prime p. Suppose K_p is *maximal*, and choose K^p to be a *small* compact open subgroup of $G(\widehat{\mathbb{Z}}^{(p)})$. This means

$$\mathsf{K}_p = G(\mathbb{Z}_p), \quad \mathsf{K}^p \subset \Gamma(N)^{(p)} = \{g \in G(\widehat{\mathbb{Z}}^{(p)})\colon g \equiv 1 \bmod N\}$$

for some integer $N \geqslant 3$ with $p \nmid N$. We then write

$$\mathsf{K} = \mathsf{K}^p \mathsf{K}_p \subset G(\mathbb{A}_f).$$

Theorem 3.4.1 (Mumford)**.** *There is a smooth quasi-projective scheme X_{K} over $\operatorname{Spec} \mathbb{Z}_{(p)}$, of dimension $g(g+1)/2$, representing the functor*

$$S \longmapsto \{(A, \lambda, \eta)\}/\cong,$$

where

- *A is a projective abelian scheme of relative dimension g over S,*
- *λ is a principal polarization of A, and*

- *η is a K^p-level structure on A.*

We omit the general definition of the notion of a K^p-level structure for now. In the special case $\mathsf{K}^p = \Gamma(N)^{(p)}$, such a structure is equivalent to the choice of an isomorphism

$$\eta\colon A[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^{2g}$$

of group schemes such the alternating form on $A[N]$ given by λ corresponds to $\psi \bmod N$ on
Lect.9, Oct 31 $(\mathbb{Z}/N\mathbb{Z})^{2g}$ up to scalar.

We give more explanations about the isomorphism above. On $A[N]$, the alternating form given by λ refers to the Weil pairing

$$e^{\lambda}\colon A[N] \times A[N] \xrightarrow{(\mathrm{id},\lambda)} A[N] \times A^{\vee}[N] \xrightarrow{e} \mu_N,$$

where e is a canonical map. We explain e in the special case where $S = \operatorname{Spec} k$ for an algebraically closed field k. Let $x \in A[N]$ and $L \in A^{\vee}[N]$. Then L is a line bundle on A such that $L^{\otimes N} \cong \mathcal{O}_A$. Since $L \in A^{\vee}$, we have for each integer $n \geqslant 1$ that $L^{\otimes n} \cong [n]^*L$, where $[n]\colon A \to A$ is the multiplication-by-n map. Thus we can fix an isomorphism between line bundles $\varphi\colon [N]^*L \xrightarrow{\sim} \mathcal{O}_A$. Then we have a commutative diagram

$$\begin{tikzcd}
t_x^*[N]^*L \arrow[r, "\varphi", "\sim"'] \arrow[d, "\cong"', "\mathrm{can}"] & t_x^*\mathcal{O}_A \arrow[r, "\cong"] & \mathcal{O}_A. \arrow[d, dashed] \\
{[N]^*L} \arrow[rr, "\varphi", "\sim"'] & & \mathcal{O}_A
\end{tikzcd}$$

Here the left vertical isomorphism comes from the fact that x is of N-torsion and hence $[N] = [N] \circ t_x$. The right vertical map is an automorphism of $\mathcal{O}_A$ whose N-th power is trivial, i.e., a section of μ_N.

We have explained that the alternating form on $A[N]$ given by λ refers to e^{λ}. The requirement that the isomorphism $\eta\colon A[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^{2g}$ should take e^{λ} to $\psi \bmod N$ up to scalar really means that we are able to relate a μ_N-valued alternating form on $(\mathbb{Z}/N\mathbb{Z})^{2g}$, i.e. $e^{\lambda}(\eta(\cdot), \eta(\cdot))$, to a $\mathbb{Z}/N\mathbb{Z}$-valued alternating form, i.e. $\psi \bmod N$, after *some choice of isomorphism* $\mu_N \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z}$ of group schemes over S. Thus the precise meaning of a level $\Gamma(N)^{(p)}$-structure is as below.

Definition 3.4.2. A *level-$\Gamma(N)^{(p)}$ structure* is a pair (η, t), where

- η is an isomorphism of group schemes $A[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^{2g}$ over S, and
- $t\colon \mu_N \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z}$ is an isomorphism of group schemes over S

such that η takes $t \circ e^{\lambda}$ to $\psi \bmod N$

We observe that the functor $S \mapsto \{t\colon \mu_N \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z} \text{ over } S\}$ on $\mathbb{Z}_{(p)}$-schemes is represented by $\operatorname{Spec} \mathbb{Z}_{(p)}[\zeta_N]$ together with a particular choice of ζ_N. Hence there is a morphism

$$X_{\Gamma(N)^{(p)}\mathsf{K}_p} \longrightarrow \operatorname{Spec} \mathbb{Z}_{(p)}[\zeta_N].$$

In the sequel, for general K^p (including $\mathsf{K}^p = \Gamma(N)^{(p)}$), we still denote a K^p-level structure by a single letter η.

Theorem 3.4.3 (Shimura and Deligne)**.** *The $\mathbb{C}$-valued points have a uniformization*

$$X_{\mathsf{K},\mathbb{Q}}(\mathbb{C}) \cong G(\mathbb{Q})\backslash \mathcal{H}_{2g}^{\pm} \times G(\mathbb{A}_f)/\mathsf{K}.$$

Moreover, the $\mathbb{Q}$-scheme $X_{\mathsf{K},\mathbb{Q}}$ is the Shimura variety associated to the data $(G, \mathcal{H}_{2g}^{\pm}, \mathsf{K})$.

About the last point, here the difficulty lies in what one means by "the" Shimura variety. Shimura and Deligne discovered a way to uniquely and abstractly characterize a canonical model over $\mathbb{Q}$ for the $\mathbb{C}$-variety $G(\mathbb{Q})\backslash \mathcal{H}_{2g}^{\pm} \times G(\mathbb{A}_f)/\mathsf{K}$. The characterization is modeled on the theory of complex multiplication, but it does not refer to any moduli problem involving abelian schemes. In the current case, $X_{\mathsf{K},\mathbb{Q}}$ turns out to satisfy that characterization.

For smaller compact open subgroups $\mathsf{U} \subset \mathsf{K}_p = G(\mathbb{Z}_p)$, we still have smooth quasi-projective $\mathbb{Q}$-schemes

$$X_{\mathsf{K}^p\mathsf{U},\mathbb{Q}} \xrightarrow{\text{finite étale}} X_{\mathsf{K}^p\mathsf{K}_p,\mathbb{Q}}.$$

These are also Shimura varieties, and we have

$$X_{\mathsf{K}^p\mathsf{U},\mathbb{Q}}(\mathbb{C}) \cong G(\mathbb{Q})\backslash\mathcal{H}_{2g}^{\pm} \times G(\mathbb{A}_f)/\mathsf{K}^p\mathsf{U}.$$

Moreover, one can easily describe the moduli problem over $\mathbb{Q}$ represented by $X_{\mathsf{K}^p\mathsf{U},\mathbb{Q}}$ for the following choices:

$$\begin{aligned}
&\mathsf{U} = \Gamma(p^n) = \{g \in G(\mathbb{Z}_p) \colon g \equiv 1 \bmod p^n\},\\
&\mathsf{U} = \Gamma_0(p^n) = \left\{g \in G(\mathbb{Z}_p) \colon g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod p^n \text{ (with size } g \text{ blocks)}\right\}\\
&\mathsf{U} = \Gamma_{\mathrm{s}}(p^n) = \{g \in \Gamma_0(p^n) \colon \nu(g) \equiv 1 \bmod p^n\}.
\end{aligned}$$

(Here the subscript "s" stands for "special" or "Scholze".) Thus we have

$$\mathsf{K}_p = G(\mathbb{Z}_p) \supset \Gamma_0(p^n) \supset \Gamma_{\mathrm{s}}(p^n) \supset \Gamma(p^n).$$

Our definition of $\Gamma_{\mathrm{s}}(p^n)$ refers to [Sch15, Definition III.1.1], but note that there is possibly a typo in Scholze's paper (namely he considers the condition $\det(g) \equiv 1$ as opposed to $\nu(g) \equiv 1$).

We now describe the moduli problems over $\mathbb{Q}$ for the above three cases of U.

(1) For $\mathsf{U} = \Gamma(p^n)$, $X_{\mathsf{K}^p\mathsf{U},\mathbb{Q}}$ represents

$$S \longmapsto \{(A, \lambda, \eta, \eta_p, t_p)\}/\cong,$$

where
- $(A, \lambda, \eta) \in X_{\mathsf{K}}(S)$, and
- (η_p, t_p) is a level-p^n structure, which has the same meaning as in Definition 3.4.2 with N replaced by p^n.

In this case, we get a natural morphism of $\mathbb{Q}$-schemes

$$X_{\mathsf{K}^p\Gamma(p^n),\mathbb{Q}} \longrightarrow \operatorname{Spec}\mathbb{Q}(e^{2\pi i/p^n}), \quad (A, \lambda, \eta, \eta_p, t_p) \longmapsto t_p$$

as the latter represents the moduli over $\mathbb{Q}$ of isomorphisms $\mu_{p^n} \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$. We shall denote this morphism by t_p.

(2) For $\mathsf{U} = \Gamma_0(p^n)$, $X_{\mathsf{K}^p\mathsf{U},\mathbb{Q}}$ represents

$$S \longmapsto \{(A, \lambda, \eta, D)\}/\cong,$$

where
- $(A, \lambda, \eta) \in X_{\mathsf{K}}(S)$, and
- D is a subgroup of order p^{ng} in $A[p^n]$ (whose order is p^{2ng}), which is *totally isotropic* with respect to e^{λ}, i.e., $e^{\lambda}(D, D) = 0$.

(3) For $\mathsf{U} = \Gamma_0(p^n)$, $X_{\mathsf{K}^p\mathsf{U},\mathbb{Q}}$ represents

$$S \longmapsto \{(A, \lambda, \eta, D, t_p)\}/\cong,$$

where $(A, \lambda, \eta, D) \in X_{\mathsf{K}^p\Gamma_0(p^n)}$, and t_p is an isomorphism $\mu_{p^n} \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$. In particular, we have

$$t_p \colon X_{\mathsf{K}^p\Gamma_{\mathrm{s}}(p^n),\mathbb{Q}} \longrightarrow \operatorname{Spec}\mathbb{Q}(e^{2\pi i/p^n}).$$

We have

$$\begin{array}{ccccccc}
X_{\mathsf{K}^p\Gamma(p^n),\mathbb{Q}} & \longrightarrow & X_{\mathsf{K}^p\Gamma_{\mathrm{s}}(p^n),\mathbb{Q}} & \longrightarrow & X_{\mathsf{K}^p\Gamma_0(p^n),\mathbb{Q}} & \longrightarrow & X_{\mathsf{K},\mathbb{Q}}\\
(A, \lambda, \eta, \eta_p, t_p) & \longmapsto & (A, \lambda, \eta, D, t_p) & \longmapsto & (A, \lambda, \eta, D) & \longmapsto & (A, \lambda, \eta).
\end{array}$$

Here $\eta_p \colon A[p^n] \xrightarrow{\sim} (\mathbb{Z}/p^n)^{2g}$ leads to $D = \eta_p^{-1}(D_0)$, where D_0 is the subgroup $(\mathbb{Z}/p^n\mathbb{Z})^g$ of $(\mathbb{Z}/p^n\mathbb{Z})^{2g}$ given by the embedding of the first g coordinates.

Varying the level, we have a commutative diagram of Shimura varieties over $\mathbb{Q}$: (we omit writing the subscripts $\mathbb{Q}$)

$$\begin{tikzcd}
X_{\mathsf{K}^p\Gamma(p^{n+1})} \arrow[r] \arrow[d] & X_{\mathsf{K}^p\Gamma(p^n)} \arrow[d] \\
X_{\mathsf{K}^p\Gamma_{\mathrm{s}}(p^{n+1})} \arrow[r] \arrow[d] & X_{\mathsf{K}^p\Gamma_{\mathrm{s}}(p^n)} \arrow[d] \\
X_{\mathsf{K}^p\Gamma_0(p^{n+1})} \arrow[r] & X_{\mathsf{K}^p\Gamma_0(p^n)}.
\end{tikzcd}$$

e.g., the top horizontal map sends (η_p, t_p) to their restrictions to p^n-torsions, and the bottom horizontal map sends D to $D[p^n] = D \cap A[p^n]$. They all map to $X_{\mathsf{K},\mathbb{Q}}$, and all the maps between the Shimura varieties of different levels are finite étale. Moreover, the following diagram commutes, and the square is Cartesian:

$$\begin{tikzcd}
X_{\mathsf{K}^p\Gamma(p^n)} \arrow[dr] \arrow[drr, bend left, "t_p"] \arrow[ddr] & & \\
& X_{\mathsf{K}^p\Gamma_{\mathrm{s}}(p^n)} \arrow[r, "t_p"] \arrow[d] & \operatorname{Spec} \mathbb{Q}(e^{2\pi i/p^n}) \arrow[d] \\
& X_{\mathsf{K}^p\Gamma_0(p^n)} \arrow[r] & \operatorname{Spec} \mathbb{Q}.
\end{tikzcd}$$

Notation 3.4.4. Since K^p is always fixed, we can simplify the notation by denoting $X_{\mathsf{K}_p} := X_{\mathsf{K}^p\mathsf{K}_p,\mathbb{Z}_p}$ over $\mathbb{Z}_p$. For each $\mathsf{U} \in \{\Gamma(p^n), \Gamma_0(p^n), \Gamma_{\mathrm{s}}(p^n)\}$, denote $X_{\mathsf{U}} := X_{\mathsf{K}^p\mathsf{U},\mathbb{Q}_p}$ over $\mathbb{Q}_p$.

We will also fix a compatible system of p^n-th roots of unity $\zeta_{p^n} \in \overline{\mathbb{Q}}_p$. Then we have a commutative diagram

$$\begin{tikzcd}
X_{\Gamma(p^n)} \arrow[rr] \arrow[dr, "t_p"'] & & X_{\Gamma_{\mathrm{s}}(p^n)} \arrow[dl, "t_p"] \\
& \operatorname{Spec} \mathbb{Q}_p(\zeta_{p^n}). &
\end{tikzcd}$$

Moreover the change-of-level maps, from level p^{n+1} to level p^n, is compatible with $\operatorname{Spec} \mathbb{Q}_p(\zeta_{p^{n+1}}) \to \operatorname{Spec} \mathbb{Q}_p(\zeta_{p^n})$.

3.5. **The sheaf of translation-invariant differentials.** Let $\pi\colon A \to S$ be an abelian scheme. Let g be the relative dimension, and let $e\colon S \to A$ be the neutral section. Then we have an isomorphism

$$e^*\Omega^1_{A/S} \simeq \pi_*\Omega^1_{A/S}$$

of vector bundles of rank g on A. Intuitively, the left side is the cotangent space of A at e, which is isomorphic to the space of all translation-invariant global differentials on A; the right side is the space of all global differentials on A, which are all automatically translation-invariant since A is proper.

More precisely, we have a natural isomorphism $u\colon \pi^*e^*\Omega^1_{A/S} \xrightarrow{\sim} \Omega^1_{A/S}$ for all group schemes A over S; see [EvdGM, (3.15)]. The geometric meaning of this isomorphism is that regular functions times translation-invariant differentials give rise to all differentials (up to taking a sum). Now for an abelian scheme A over S, a standard application of cohomology and base change yields that the natural map $\mathcal{O}_S \to \pi_*\mathcal{O}_A$ is an isomorphism. (Here we use that π is proper, flat, surjective, and all its geometric fibers are connected and reduced.) It then easily follows that for any locally finite free $\mathcal{O}_S$-module L, the adjunction map $L \to \pi_*\pi^*L$ is an isomorphism. Applying this to $L = e^*\Omega^1_{A/S}$, we obtain $e^*\Omega^1_{A/S} \cong \pi_*\pi^*e^*\Omega^1_{A/S} \cong \pi^*\Omega^1_{A/S}$ as desired. Here the last isomorphism is $\pi_*(u)$.

Consider the following line bundle on S:

$$\omega_{A/S} := \wedge^g(e^*\Omega^1_{A/S}).$$

We describe the functoriality of $\omega_{A/S}$. Given a pullback diagram

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d] & A \arrow[d] \\ S' \arrow[r, "g"] & S, \end{tikzcd}$$

we have $\Omega^1_{A'/S'} \cong f^*\Omega^1_{A/S}$. The neutral section $e' \colon S' \to A' = A \times_S S'$ is equal to $(e \circ g, \mathrm{id})$. Hence $e'^*\Omega_{A'/S'} \cong g^*(e^*\Omega_{A/S})$. Thus,

$$\omega_{A'/S'} \cong g^*\omega_{A/S}.$$

In particular, if S is of characteristic p and we take $g = \mathrm{Fr}$ to be the absolute Frobenius on S, then

$$\omega_{A^{(p)}/S} \cong \mathrm{Fr}^*\omega_{A/S} \cong \omega_{A/S}^{\otimes p}.$$

In fact, we have $\mathrm{Fr}^*L \cong L^{\otimes p}$ for any line bundle L on S.

Similarly, if $f \colon A' \to A$ is a homomorphism of abelian schemes over S, then there is a canonical map $f^*\Omega^1_{A/S} \to \Omega^1_{A'/S}$. Pulling back along the neutral section $S \to A'$ we obtain a canonical map

$$\omega_{A/S} \longrightarrow \omega_{A'/S}.$$

3.6. **Hasse invariant in characteristic** p**.** Let $A \to S$ be an abelian scheme, and assume that S has characteristic p. Recall from §3.3 that we have Frobenius and Verschiebung maps

$$F \colon A \longrightarrow A^{(p)}, \quad V \colon A^{(p)} \longrightarrow A$$

with the composition $V \circ F = [p]$. Here F, V, and $[p]$ are isogenies of degrees p^g, p^g, and p^{2g}, respectively.

Definition 3.6.1. By functoriality, the homomorphism $V \colon A^{(p)} \to A$ induces a map

$$\omega_{A/S} \longrightarrow \omega_{A^{(p)}/S} \cong \omega_{A/S}^{\otimes p}.$$

This gives a canonical section of $\omega_{A/S}^{\otimes(p-1)}$, called the *Hasse invariant* of A/S, written as

$$\mathrm{Ha}(A/S) \in \Gamma(S, \omega_{A/S}^{\otimes(p-1)}).$$

Proposition 3.6.2 (Equivalent conditions for ordinary)**.** *Let k be an algebraically closed field of characteristic p. Let A be an abelian scheme over $S = \operatorname{Spec} k$ of dimension g. Then the following are equivalent:*

(1) $\#A[p](k) = p^g$.
(2) $A[p](k) \cong \mathbb{Z}/p^g\mathbb{Z}$.
(3) *The maximal étale quotient of $A[p]$ has order p^g.*
(4) *Same as (1) or (2) or (3) but with p replaced by p^n for any $n \in \mathbb{N}$.*
(5) *The p-divisible group $A[p^\infty]$ is isomorphic to $(\mu_{p^\infty})^g \times (\mathbb{Q}_p/\mathbb{Z}_p)^g$.*
(6) $\mathrm{Ha}(A/k) \neq 0$.
(7) $\mathrm{Ker}(V \colon A^{(p)} \to A)$ *is an étale group scheme.*

Definition 3.6.3. When one of the conditions in Proposition 3.6.2 holds, we say A is *ordinary*.

Remark 3.6.4. In (1) and (3), p^g is the maximal possible number. Similarly, if we consider p^n-torsion, then p^{ng} is the maximal possible number.

Partial Proof of Proposition 3.6.2. The equivalence between (1)–(5) can be given by standard arguments. We consider (3)(6)(7) below. Since $F \colon A \to A^{(p)}$ is an isogeny, we have an isomorphism $A/\mathrm{Ker}(F) \cong A^{(p)}$ induced by F. The relation $V \circ F = [p]$ then implies the following short exact sequence

$$0 \longrightarrow \mathrm{Ker}(F) \longrightarrow A[p] \xrightarrow{F} \mathrm{Ker}(V) \longrightarrow 0,$$

where $\mathrm{Ker}(F)$ and $\mathrm{Ker}(V)$ are group schemes of order p^g and $\mathrm{Ker}(F)$ is connected. Thus the maximal étale quotient of $A[p]$ is isomorphic to that of $\mathrm{Ker}(V)$. Hence (3) is equivalent to (7). Moreover,

$$(7) \iff V \text{ is étale} \iff V \text{ induces an isomorphism on cotangent space at } e \iff (6).$$

This completes the proof. □

Definition 3.6.5. Let A be an abelian scheme over S, where S is of characteristic p. We say A is *ordinary* if for any geometric point $\bar{s} \to S$, the abelian scheme $A_{\bar{s}}$ is ordinary.

Corollary 3.6.6. *Keeping the notations above, A is ordinary if and only if* $\mathrm{Ha}(A/S) \in \Gamma(S, \omega_{A/S}^{\otimes(p-1)})$ *is invertible. In particular, if this holds, then*

$$\omega_{A/S}^{\otimes(p-1)} \simeq \mathcal{O}_S.$$

Notation 3.6.7. We write $A^{(p^n)} := (A^{(p^{n-1})})^{(p)}$. Define

$$F^n \colon A \longrightarrow A^{(p^n)}, \quad V^n \colon A^{(p^n)} \longrightarrow A$$

by iteration. For instance, F^{n+1} is the composition of $F^n \colon A \to A^{(p^n)}$ followed by the relative Frobenius for $A^{(p^n)}$:

$$F \colon A^{(p^n)} \to (A^{(p^n)})^{(p)} = A^{(p^{n+1})}.$$

It is easy to see that $V^n \circ F^n = [p^n] \colon A \to A$. We observe that for algebraically closed k, since $A^{(p)} \cong A/\operatorname{Ker}(F)$, we have

$$A^{(p)}[p](k) \cong A[p](k)/(\operatorname{Ker}(F))(k) \cong A[p](k).$$

Consequently, if A is ordinary, then so also is $A^{(p)}$, and then by induction so is $A^{(p^n)}$.

Using the short exact sequence

$$0 \longrightarrow \operatorname{Ker}(F^n) \longrightarrow A[p^n] \xrightarrow{F^n} \operatorname{Ker}(V^n) \longrightarrow 0,$$

by the same argument as before we see that A is ordinary if and only if $\operatorname{Ker}(V^n)$ is étale, generalizing Proposition 3.6.2(7). (Indeed, when the base is an algebraically closed field, F^n is purely inseparable since it is a composition of such maps. Hence $\operatorname{Ker}(F^n)$ is connected.)

3.7. **Lifting to characteristic 0.** Let R be a p-adically complete flat $\mathbb{Z}_p$-algebra. Let A be an abelian scheme over $\operatorname{Spec} R$. Consider $A_1 := A \otimes_R R/p$.[3] Assume A_1 is ordinary. Then for $n \in \mathbb{N}$, we have the short exact sequence

$$(*) \qquad 0 \longrightarrow \operatorname{Ker}(F^n) \longrightarrow A_1[p^n] \longrightarrow \operatorname{Ker}(V^n) \longrightarrow 0.$$

Here $F \colon A_1 \to A_1^{(p)}$ and $V \colon A_1^{(p)} \to A_1$ are the Frobenius and Verschiebung maps. From the following proposition, this short exact sequence lifts canonically from R/p to R.

Proposition 3.7.1. *Assuming A_1 is ordinary, there exists a unique closed subgroup scheme $C_n = C_n(A)$ of $A[p^n]$, which is flat over R and lifts* $\operatorname{Ker}(F^n)$.

Definition 3.7.2. The unique subgroup scheme C_n in Proposition 3.7.1 is called the *n-th canonical subgroup of A*.

Using Proposition 3.7.1, the sequence $(*)$ has a canonical lift to R, read as

$$0 \longrightarrow C_n \longrightarrow A[p^n] \longrightarrow A[p^n]/C_n \longrightarrow 0.$$

By definition, the reduction of A/C_n modulo p is exactly $A_1/\operatorname{Ker}(F^n) \cong A_1^{(p^n)}$, and the natural map

$$f \colon A \longrightarrow A/C_n$$

is a canonical lift of $F^n \colon A_1 \to A_1^{(p^n)}$. Recall that $A_1^{(p^n)}$ is still ordinary. Hence we can consider canonical subgroups of A/C_n. We denote the first canonical subgroup by

$$C_1' = C_1(A/C_n).$$

Clearly, C_n is an order p^{ng} subgroup scheme in $A[p^n]$, the latter having order p^{2ng}. It follows that $A[p^n]/C_n$ also has order p^{ng}. Using f, we identify

$$A[p^n]/C_n \cong f(A[p^n]) \subset (A/C_n)[p^n].$$

In other words, the left hand side is an order p^{ng} subgroup of $(A/C_n)[p^n]$.

[3]Here we use the notation A_1 that refers to $A_n := A \otimes_R R/p^n$.

Example 3.7.3. For $n = 1$, the order of $(A/C_n)[p]$ is p^{2g}, and we get two subgroups of order p^g in it, namely $f(A[p])$ (which is isomorphic to the quotient $A[p]/C_1(A)$) and $C_1' = C_1(A/C_1(A))$.

Lemma 3.7.4. *Inside $(A/C_n)[p^n]$, the intersection*

$$f(A[p^n]) \cap C_1' = \{0\}.$$

Proof. As usual we identify the reduction of A/C_n mod p with $A_1^{(p^n)}$. Notice that $f(A[p^n])$ is an étale lifting of $\mathrm{Ker}(V^n \colon A_1^{(p^n)} \to A_1)$, while C_1' is a lifting of $\mathrm{Ker}(F \colon A_1^{(p^n)} \to A_1^{(p^{n+1})})$. The latter is a connected group scheme over R/p. It follows that $f(A[p^n]) \cap C_1'$ is an étale lift over R of some connected group scheme over R/p, and hence it must be trivial. □

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Abusing language, we say that A is ordinary over R if A_1 is ordinary over R/p. Observe that when this is the case, the canonical subgroups C_n of A satisfy $C_1 \subset C_2 \subset \cdots$. So if some subgroup of $A[p^n]$ is disjoint from C_1, then it is also disjoint from all C_i.

Suppose A is ordinary over R and admits a principal polarization $\lambda \colon A \xrightarrow{\sim} A^\vee$. Then $A^\vee$ is also ordinary. There is a perfect pairing

$$e \colon A[p^n] \times A^\vee[p^n] \longrightarrow \mu_{p^n}$$

of group schemes over R.

For $C_n(A) \subset A[p^n]$ we define $C_n(A)^\perp$ to be the annihilator of $C_n(A)$ under this perfect pairing. Its order is $p^{2ng}/p^{ng} = p^{ng}$, equal to that of $C_n(A)$.

Claim. $C_n(A)^\perp = C_n(A^\vee)$.

For the claim, it indeed suffices to prove that $C_n(A)^\perp$ lifts $\mathrm{Ker}(F^n \colon A_1^\vee \to (A_1^\vee)^{(p^n)})$. On the other hand, we already know that $C_n(A)$ lifts $\mathrm{Ker}(F^n \colon A_1 \to A_1^{(p^n)})$. We need that these two kernels are annihilators of each other under the reduced perfect pairing

$$e \colon A_1[p^n] \times A_1^\vee[p^n] \longrightarrow \mu_{p^n},$$

which we still denote by e through an abuse of notation.

The general picture is described as follows. Given morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ of abelian schemes, we naturally obtain

$$A^\vee \xleftarrow{f^\vee} B^\vee \xleftarrow{g^\vee} C^\vee.$$

There are canonical pairings

$$e_g \colon \mathrm{Ker}\, g \times \mathrm{Ker}\, g^\vee \longrightarrow \mathbb{G}_m$$

and

$$e_{g\circ f} \colon \mathrm{Ker}(g \circ f) \times \mathrm{Ker}(f^\vee \circ g^\vee) \longrightarrow \mathbb{G}_m,$$

which are compatible with the maps $f \colon \mathrm{Ker}(g \circ f) \to \mathrm{Ker}(g)$ and $\mathrm{Ker}(g^\vee) \hookrightarrow \mathrm{Ker}(f^\vee \circ g^\vee)$. It follows that $\mathrm{Ker}(f) \subset \mathrm{Ker}(g \circ f)$ and $\mathrm{Ker}(g^\vee) \subset \mathrm{Ker}(f^\vee \circ g^\vee)$ annihilate each other under $e_{g\circ f}$. In fact, they are annihilators of each other.

We apply this to

$$f = F^n \colon A_1 \to A_1^{(p^n)}, \quad g = V^n \colon A_1^{(p^n)} \to A_1.$$

We finish the argument by the following fact: $(F^n)^\vee \colon (A_1^{(p^n)})^\vee \to A_1^\vee$ is equal to V^n for $A_1^\vee$, where $(A_1^{(p^n)})^\vee$ is canonically identified $(A_1^\vee)^{(p^n)}$.

Each principal polarization $\lambda \colon A \xrightarrow{\sim} A^\vee$ must take $C_n(A)$ to $C_n(A^\vee)$ by functoriality, and this is equal to $C_n(A)^\perp$ by the above claim. Hence

$$e^\lambda \colon A[p^n] \times A[p^n] \xrightarrow{(\mathrm{id},\lambda)} A[p^n] \times A^\vee[p^n] \xrightarrow{e} \mu_{p^n}$$

restricts trivially to $C_n(A) \times C_n(A)$, i.e., $C_n(A)$ is a totally isotropic subgroup; it is also maximal since its order is p^{ng}.

3.8. **Idea of anti-canonical tower.** We now pass to the local rational picture and consider

$$X_{\mathsf{K}_p,\mathbb{Q}_p} = X_{\mathsf{K}_p} \times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec}\mathbb{Q}_p.$$

Associated to this, the adic space $(X_{\mathsf{K}_p,\mathbb{Q}_p})^{\mathrm{ad}}$ contains the open good reduction locus, namely the adic space attached to the p-adic formal completion of the $\mathbb{Z}_p$-scheme X_{K_p}.

The following discussion is only heuristic and informal.

Definition 3.8.1. Inside the locus of good reduction in $(X_{\mathsf{K}_p,\mathbb{Q}_p})^{\mathrm{ad}}$, let $\mathcal{X}^{\mathrm{ord}}_{\mathsf{K}_p}$ denote the open that is the ordinary locus, parameterizing (A,λ,η) such that A has a good and ordinary reduction. Let $\mathcal{X}^{\mathrm{ord}}_{\Gamma_0(p^n)}$ denote the inverse image of $\mathcal{X}^{\mathrm{ord}}_{\mathsf{K}_p}$ under $X^{\mathrm{ad}}_{\Gamma_0(p^n),\mathbb{Q}_p} \to (X_{\mathsf{K}_p,\mathbb{Q}_p})^{\mathrm{ad}}$. Informally, $\mathcal{X}^{\mathrm{ord}}_{\Gamma_0(p^n)}$ parameterizes (A,λ,η,D), where A has a good and ordinary reduction, and D is a $\Gamma_0(p^n)$-level structure on (the generic fiber of) A.

Construction 3.8.2 (Canonical section)**.** The natural map $\mathcal{X}^{\mathrm{ord}}_{\Gamma_0(p^n)} \to \mathcal{X}^{\mathrm{ord}}_{\mathsf{K}_p}$, $(A,\lambda,\eta,D) \to (A,\lambda,\eta)$ has the *canonical section* $(A,\lambda,\eta) \mapsto (A,\lambda,\eta,C_n(A))$.

Construction 3.8.3 (Anti-canonical embedding)**.** There is another morphism

$$i_{\mathrm{a}} \colon \mathcal{X}^{\mathrm{ord}}_{\mathsf{K}_p} \longrightarrow \mathcal{X}^{\mathrm{ord}}_{\Gamma_0(p^n)}$$

defined by

$$(A,\lambda,\eta) \longmapsto (A/C_n(A), \lambda', \eta', A[p^n]/C_n(A)).$$

It turns out to be an open embedding. We call it the *anti-canonical embedding*, and denote its image by

$$\mathcal{X}^{\mathrm{ord,a}}_{\Gamma_0(p^n)} \subset \mathcal{X}^{\mathrm{ord}}_{\Gamma_0(p^n)},$$

(with superscript "a").

Here we used that C_n is totally isotropic, and therefore A/C_n inherits a principal polarization λ' from λ on A; further, $A[p^n]/C_n$ is totally isotropic with respect to this inherited polarization. Also, the K^p-level structure η on A induces a K^p-level structure η' on $A/C_n(A)$.

Definition 3.8.4. The *anti-canonical tower* is the projective system on the following right side:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \\
\mathcal{X}^{\mathrm{ord}}_{\mathsf{K}_p} \arrow[r, "\sim"] \arrow[d] & \mathcal{X}^{\mathrm{ord,a}}_{\Gamma_0(p^{n+1})} \arrow[d] & (n+1)\text{-st} \\
\mathcal{X}^{\mathrm{ord}}_{\mathsf{K}_p} \arrow[r, "\sim"] \arrow[d] & \mathcal{X}^{\mathrm{ord,a}}_{\Gamma_0(p^{n})} \arrow[d] & n\text{-th} \\
\vdots \arrow[d] & \vdots \arrow[d] & \\
\mathcal{X}^{\mathrm{ord}}_{\mathsf{K}_p} \arrow[r, "="] & \mathcal{X}^{\mathrm{ord,a}}_{\mathsf{K}_p}. & 0\text{-th}
\end{tikzcd}$$

Here the horizontal maps are i_{a}, the vertical maps on the right side are the natural ones, and the vertical maps on the left are given as follows:

(a) The map from the $(n{+}1)$-st $\mathcal{X}^{\mathrm{ord}}_{\mathsf{K}_p}$ to the n-th $\mathcal{X}^{\mathrm{ord}}_{\mathsf{K}_p}$ is given by the canonical quotient $A \mapsto A/C_1$.

(b) The map from the n-th $\mathcal{X}^{\mathrm{ord}}_{\mathsf{K}_p}$ to the 0-th $\mathcal{X}^{\mathrm{ord}}_{\mathsf{K}_p}$ is given by $A \mapsto A/C_n$.

Note that A/C_1 is a lift of $A_1^{(p)}$ where A_1 is the mod p reduction of A. Thus the transition maps on the left look like some kind of characteristic zero lift of the Frobenius. Essentially using this idea, one can show that

$$\varprojlim_n \mathcal{X}^{\mathrm{ord,a}}_{\Gamma_0(p^n)} \sim (\text{Certain perfectoid space}),$$

at least after the base change to a perfectoid field.

Recall that our ultimate goal is to show that

$$\varprojlim_n (X_{\Gamma(p^n),\mathbb{Q}_p})^{\mathrm{ad}} \sim \text{(Certain perfectoid space)}.$$

To go from the first statement to the second, we need to pass from the $\Gamma_0(p^n)$-tower to the $\Gamma(p^n)$-tower (this step is highly non-trivial because the $\Gamma(p^n)$ are cofinal among all compact open subgroups of $G(\mathbb{Q}_p)$, while the $\Gamma_0(p^n)$ are not cofinal), and to extend from the ordinary anti-canonical locus to the full Shimura variety (and even its Baily–Borel compactification).

3.9. **Neighborhood of anti-canonical tower.** We need to extend the construction of canonical subgroups slightly outside the ordinary locus. That is, we want to extend to an ε-neighborhood of the ordinary locus, in the sense of p-adic geometry. For now our discussion will be purely algebraic for a while.

Notation 3.9.1. We write

$$\mathbb{Z}_p^{\mathrm{cycl}} = \mathbb{Z}_p[\mu_{p^\infty}]^\wedge \subset \mathbb{Q}_p^{\mathrm{cycl}} = \mathbb{Q}_p(\mu_{p^\infty})^\wedge.$$

For each $r \in \mathbb{N}$, $\mathbb{Q}_p(\mu_{p^r})$ is a totally ramified extension over $\mathbb{Q}_p$ of degree $\varphi(p^r)$. Thus p-adic valuations of elements of $\mathbb{Z}_p^{\mathrm{cycl}}$ are rational numbers of the form $a/\varphi(p^r)$ for $a, r \in \mathbb{Z}_{\geqslant 0}$. (Note that $\varphi(p^r) \to \infty$ as $r \to \infty$.) Let ε be such a rational number, and assume that $0 \leqslant \varepsilon < 1/2$. We write p^ε for any element of $\mathbb{Z}_p^{\mathrm{cycl}}$ whose p-adic valuation is ε.

3.9.1. *Canonical subgroup.* Fix R to be a p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra.

Theorem 3.9.2 (Illusie, see [Sch15, Cor.III.2.2])**.** *In the following we understand each "group" as a finite locally free commutative group scheme over the base.*[4] *Also a "subgroup" means a closed subgroup scheme which is also locally free. Let G be a group over R and take $G_1 := G \otimes_R R/p$. Let $C_1 \subset G_1$ be a subgroup with quotient $H_1 = G_1/C_1$.*

Assume the multiplication-by-p^ε map on the Lie complex of H_1 is null-homotopic for some $0 \leqslant \varepsilon < 1/2$. Then there exists a subgroup $C \subset G$ over R such that

$$C \otimes_R R/p^{1-\varepsilon} = C_1 \otimes_{R/p} R/p^{1-\varepsilon}$$

as subgroups of $G \otimes_R R/p^{1-\varepsilon}$.

In particular, if we can take $\varepsilon = 0$ (which is equivalent to H_1 being étale), then C_1 lifts to R.

Remark 3.9.3. We explain the Lie complex of H_1 in Theorem 3.9.2. Let S be any ring, and let H be a finite locally free commutative group scheme over S. The Lie complex of H over S, denoted by $\check{\ell}_H$, is represented by a perfect complex of S-modules with amplitude in $[0,1]$. The 0-th homology of $\check{\ell}_H$ is exactly

$$\operatorname{Lie} H = \operatorname{Hom}_S(e^*\Omega_{H/S}, S).$$

The full $\check{\ell}_H$ is defined by the similar formula as above, replacing Hom with $R\operatorname{Hom}$, e^* with derived e^*, and $\Omega_{H/S}$ by the cotangent complex $L_{H/S}$. In practice, if H sits in a short exact sequence $0 \to H \to A \to B \to 0$ where A and B are *smooth* commutative group schemes (not necessarily finite), then $\check{\ell}_H$ is represented by the complex $(\operatorname{Lie} A \to \operatorname{Lie} B)$.

Example 3.9.4. If $H_1 = \mu_n$ for some n invertible in R/p, then the short exact sequence $1 \to \mu_n \to \mathbb{G}_m \xrightarrow{[n]} \mathbb{G}_m \to 1$ gives that

$$\check{\ell}_{H_1} = (R/p \xrightarrow{n} R/p).$$

In this case the Lie complex is null-homotopic. One can take $\varepsilon = 0$ to fit in Theorem 3.9.2.

Lemma 3.9.5. *Let X be a scheme over R such that $\Omega^1_{X/R}$ is killed by p^ε for some $\varepsilon \geqslant 0$. Then for each $\delta > \varepsilon$ such that p^δ makes sense in R, the map*

$$X(R) \longrightarrow X(R/p^\delta)$$

is injective.

[4] Note that our base scheme, for instance R, is not necessarily noetherian here, and finite locally free is stronger than finite flat.

Proof. See [Sch15, Lemma III.2.4]. □

We now consider the application of Theorem 3.9.2. Let A be an abelian scheme over R. For any $\delta \geqslant 0$ such that p^δ makes sense, we write R_δ for R/p^δ, and write A_δ for $A \otimes_R R_\delta$. **In the following we always assume** $0 \leqslant \varepsilon < 1/2$**.**

Definition 3.9.6. We say A or A_1 is $O(n, \varepsilon)$ for an integer $n \in \mathbb{Z}_{\geqslant 1}$ and $0 \leqslant \varepsilon < 1/2$ if

$$\mathrm{Ha}(A_1/R_1)^{(p^n-1)/(p-1)} \in \Gamma(R_1, \omega_{A_1/R_1}^{\otimes(p^n-1)}) \cong R_1$$

divides p^ε as elements in R_1. Here we pick an arbitrary trivialization of the line bundle ω_{A_1/R_1}, which does not affect the condition.

Our goal is to define a notion that is more general than *ordinary*, and in that context define (weak) canonical subgroups. The following result clarifies the existence and uniqueness of such subgroups.

Theorem 3.9.7 (Weak canonical subgroup)**.** *Suppose the abelian scheme A over R is $O(n, \varepsilon)$ for some $0 \leqslant \varepsilon < 1/2$. Then there exists a unique closed subgroup $C_n \subset A[p^n]$, called the n-th weak canonical subgroup, which is locally free over R and lifts the group scheme* $\mathrm{Ker}(F^n \colon A_1 \to A_1^{(p^n)})$ *over R/p modulo $p^{1-\varepsilon}$, i.e.,*

$$C_n \otimes_R R/p^{1-\varepsilon} = \mathrm{Ker}(F^n) \otimes_{R/p} R/p^{1-\varepsilon}.$$

Moreover,

(1) *For $H := A[p^n]/C_n$, the multiplication-by-p^ε map kills $\Omega_{H/R}$.*
(2) *Given any other p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra R' with $R \to R'$, we have*

$$C_n(R') = \{s \in A[p^n](R') \colon s \equiv 0 \bmod p^{(1-\varepsilon)/p^n}\}.$$

Proof. **Step I.** For the existence, we want to lift $\mathrm{Ker}(F^n \colon A_1 \to A_1^{(p^n)})$ whose cokernel in $A_1[p^n]$ is exactly $H_1 = \mathrm{Ker}(V^n \colon A_1^{(p^n)} \to A_1)$. To apply Illusie's Theorem 3.9.2 we must check if multiplication by p^ε on $\check{\ell}_{H_1}$ is null-homotopic. For this, by Remark 3.9.3 we have

$$\check{\ell}_{H_1} = (\mathrm{Lie}\, A_1^{(p^n)} \xrightarrow{\mathcal{V}^n} \mathrm{Lie}\, A_1).$$

Here $\mathcal{V}$ denotes the induced map of V between Lie algebras. As a direct consequence of definitions, we have

$$\det \mathcal{V}^n = \mathrm{Ha}(A_1/R_1)^{(p^n-1)/(p-1)} \in \omega_{A_1/R_1}^{\otimes(p^n-1)}.$$

Note. Suppose L is finite free R/p-module, and consider a 2-term complex $L \xrightarrow{f} L$. Let $A(f)$ be the adjugate of f, so that $f \circ A(f) = A(f) \circ f = \det f$. The diagram

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "\det f"'] & L \arrow[d, "\det f"] \arrow[dl, "A(f)"'] \\ L \arrow[r, "f"] & L \end{tikzcd}$$

exhibits that multiplication by $\det f$ on the complex $L \xrightarrow{f} L$ is null-homotopic.

Using the note above, we see that multiplication by $\mathrm{Ha}(A_1/R_1)^{(p^n-1)/(p-1)}$is null-homotopic on $\check{\ell}_{H_1}$. In particular, multiplication by p^ε is null-homotopic on $\check{\ell}_{H_1}$. So we are in the same case as described by Theorem 3.9.2 with $G = A[p^n]$, and the desired existence follows. Then we have

$$C_n \otimes_R R/p^{1-\varepsilon} = \mathrm{Ker}(F^n) \otimes_{R/p} R/p^{1-\varepsilon}.$$

But it still remains to check (1) and (2) and to check uniqueness.

Lect.11, Nov 14 **Step II.** We prove (1). For simplicity, write $\Omega = \Omega_{H/R}$. Suppose p^ε kills $\Omega/p^{1-\varepsilon}\Omega$. Then

$$p^\varepsilon\Omega \subset p^{1-\varepsilon}\Omega = p^{1-2\varepsilon} \cdot p^\varepsilon\Omega \subset p^{1-2\varepsilon} \cdot p^{1-\varepsilon}\Omega = p^{2(1-2\varepsilon)+\varepsilon}\Omega \subset \cdots \subset p^{k(1-2\varepsilon)+\varepsilon}\Omega,$$

where $1 - 2\varepsilon > 0$ since $\varepsilon < 1/2$. Since Ω is p-adically complete, it follows that $p^\varepsilon\Omega = 0$. Thus we only need to check that p^ε kills $\Omega/p^{1-\varepsilon}\Omega$. But the latter is isomorphic to

$$\Omega_{H \otimes_R R_{1-\varepsilon}/R_{1-\varepsilon}}.$$

To check that $\Omega_{H\otimes_R R_{1-\varepsilon}/R_{1-\varepsilon}}$ is killed by p^ε, it suffices to prove that p^ε already kills Ω_{H_1/R_1}, since we have $H \otimes_R R_{1-\varepsilon} = H_1 \otimes_{R_1} R_{1-\varepsilon}$. But recall that p^ε is null-homotopic on $\check{\ell}_{H_1}$ (see Step I). So $p^\varepsilon = 0$ on $(\operatorname{Lie} H_1)^\vee$, the dual Lie algebra of H_1, which is $e^*\Omega_{H_1/R_1}$, where e is the neutral section of $\pi\colon H_1 \to \operatorname{Spec} R_1$. On the other hand, we have

$$\Omega_{H_1/R_1} = \pi^* e^* \Omega_{H_1/R_1}.$$

Hence p^ε, already killing $e^*\Omega_{H_1/R_1}$, also kills Ω_{H_1/R_1}, as desired. This finishes the proof of (1).

Step III. The uniqueness of C_n follows from (2). This is because C_n is locally of form $\operatorname{Spec} R'$ for such R', and then one can apply Yoneda's lemma to determine it. To prove (2), we may assume $R = R'$ without loss of generality. For any $s \in C_n(R)$, it is known that

$$C_n \otimes_R R_{1-\varepsilon} = \operatorname{Ker}(F^n) \otimes_{R_1} R_{1-\varepsilon}.$$

So the image of s in $A(R_{1-\varepsilon}) = A_1(R_{1-\varepsilon})$, denoted by $s_{1-\varepsilon}$, lies in the kernel of F^n. By the definition of F^n we have the commutative diagram:

$$\begin{tikzcd}
A_1(R_{1-\varepsilon}) \arrow[r, "F^n"] \arrow[dr] & A_1^{(p^n)}(R_{1-\varepsilon}) \arrow[d, equal] \\
& A_1(\mathrm{Fr}^n_* R_{1-\varepsilon})
\end{tikzcd}$$

where $\mathrm{Fr}^n_* R_{1-\varepsilon}$ is the ring $R_{1-\varepsilon}$ viewed as an R_1-algebra via $R_1 \to R_{1-\varepsilon},\ x \mapsto x^{p^n}$; the slant arrow is induced by $\mathrm{id}\colon A_1 \to A_1$ and the R_1-algebra map $g : R_{1-\varepsilon} \to \mathrm{Fr}^n_* R_{1-\varepsilon},\ x \mapsto x^{p^n}$. To see that the above diagram indeed commutes, using definition of Fr^n in §3.3 together with the fact that Fr^n commutes with any morphism $t\colon \operatorname{Spec} R_{1-\varepsilon} \to A_1$ in characteristic p, there are commutative diagrams:

$$\begin{tikzcd}
\operatorname{Spec} R_{1-\varepsilon} \arrow[d, "t"'] \arrow[drr, "\mathrm{Fr}^n"] & & \\
A_1 \arrow[d] \arrow[dr, "F^n"'] \arrow[drr, "\mathrm{Fr}^n"] & & \operatorname{Spec} R_{1-\varepsilon} \arrow[d, "t"] \\
\operatorname{Spec} R_1 \arrow[dr, "\mathrm{id}"'] & A_1^{(p^n)} \arrow[r] \arrow[d] & A_1 \arrow[d] \\
& \operatorname{Spec} R_1 \arrow[r, "\mathrm{Fr}^n"] & \operatorname{Spec} R_1.
\end{tikzcd}$$

Beginning with $t\colon \operatorname{Spec} R_{1-\varepsilon} \to A_1$ on the top, the image of t under $F^n\colon A_1(R_{1-\varepsilon}) \to A_1(\mathrm{Fr}^n_* R_{1-\varepsilon})$ should be given by the composition $\mathrm{Fr}^n \circ t = t \circ \mathrm{Fr}^n$. Thus the first diagram indeed commutes.

Since $s_{1-\varepsilon}$ is a point of the affine scheme $A_1[p^n]$, its having trivial image under the induced map $g\colon A_1[p^n](R_{1-\varepsilon}) \to A_1[p^n](\mathrm{Fr}^n_* R_{1-\varepsilon})$ is equivalent to its being congruent to zero modulo $\operatorname{Ker}(g)$. But from the construction

$$\operatorname{Ker}(g) = \{x \in R' \colon x^{p^n} \in p^{1-\varepsilon}\} = (p^{(1-\varepsilon)/p^n}),$$

or in other words

$$R_{1-\varepsilon}/\operatorname{Ker}(g) = R_{(1-\varepsilon)/p^n}.$$

This proves the inclusion LHS $\subset$ RHS.

For the converse inclusion, we start with $s \in$ RHS and then by the same argument as above we have $F^n(s_{1-\varepsilon}) = 0$. Thus $s_{1-\varepsilon}$ is a point of $C_n(R/p^{1-\varepsilon})$ that has zero image in $H(R/p^{1-\varepsilon})$, where $H = A[p^n]/C_n$ as before. By the conclusion in (1) and by Lemma 3.9.5 applied to $\delta = 1 - \varepsilon$, the map

$$H(R) \longrightarrow H(R/p^{1-\varepsilon})$$

is injective. So the image of $s \in A[p^n](R)$ in $(A[p^n]/C_n)(R)$ is zero. Hence $s \in C_n(R)$, which proves that LHS $\supset$ RHS. This also completes the proof of uniqueness for the above-mentioned reason. $\square$

Definition 3.9.8. For $0 \leqslant \varepsilon < 1/2$, say A or A_1 is *strongly* $O(n, \varepsilon)$ if $\mathrm{Ha}(A_1)^{p^n}$ divides p^ε in the same sense as in Definition 3.9.6. In this case, we call the weak canonical subgroup C_n in Theorem 3.9.7 the *canonical subgroup*. We say (strongly) $O(n)$, when we mean (strongly) $O(n, \varepsilon)$ for some unspecified $0 \leqslant \varepsilon < 1/2$.

Note that the strong $O(n, \varepsilon)$ is equivalent to the strong $O(1, \varepsilon/p^n)$. Also, we have the following trivial observations:

(a) Any ordinary abelian scheme A is $O(n, 0)$ for all $n \in \mathbb{Z}_{\geqslant 1}$.
(b) The $O(n, \varepsilon)$ implies the $O(m, \varepsilon')$ for all $1 \leqslant m \leqslant n$ and $\varepsilon' \geqslant \varepsilon$.

In particular, when $O(n, \varepsilon)$ holds we have a sequence of weak canonical subgroups $C_n, C_{n-1}, \cdots$. By Theorem 3.9.7(2) we have $C_n \supset C_{n-1} \supset \cdots$.

Proposition 3.9.9. *Suppose A is strongly $O(n)$. In this case A possesses a canonical subgroup $C_n(A)$ and we write $B \coloneqq A/C_n(A)$. Then B is strongly $O(m)$ if and only if A is strongly $O(n+m)$. (Here ε is deliberately suppressed because the values of it needed for A and B can differ.) If so, there is a short exact sequence*

$$0 \longrightarrow C_n(A) \longrightarrow C_{n+m}(A) \longrightarrow C_m(B) \longrightarrow 0,$$

where the latter map is induced from the group quotient map $A \to B$.

Proof. The only nontrivial part is to check that $A \to B$ really maps $C_{n+m}(A)$ onto $C_m(B)$. It suffices to prove for R' as in Theorem 3.9.7(2) that $C_{n+m}(A)(R') \to C_m(B)(R')$. Again, we may assume that $R' = R$. Let $H \coloneqq B[p^m]/C_m(B)$. So it further reduces to proving that each $s \in C_{n+m}(A)(R)$ has zero image in H.

Here comes the key idea. Let ε be such that B is (strongly) $O(m, \varepsilon)$. As in the proof of Theorem 3.9.7, we know that $H(R) \to H(R/p^{1-\varepsilon})$ is injective. So it is enough to check that s is mapped to 0 in $H(R/p^{1-\varepsilon})$. Modulo $p^{1-\varepsilon}$ we respectively have congruences between A and A_1, and between B and $A_1^{(p^n)}$. Then H is congruent to $B_1^{(p^m)}$, which is congruent to $A_1^{(p^{m+n})}$. So the natural maps $A \to B \to H$ are congruent to

$$A_1 \xrightarrow{F^n} A_1^{(p^n)} \xrightarrow{F^m} A_1^{(p^{m+n})}.$$

Hence we only need to check that modulo $p^{1-\varepsilon}$, s is killed by F^{m+n}. But this follows from the definition of $C_{n+m}(A)$. □

Proposition 3.9.10. *Assume that A is strongly $O(n)$ and its unique canonical subgroup is C_n. Then for any geometric point $\bar{x}$ of* $\operatorname{Spec} R[1/p]$, *we have*

$$C_n(\bar{x}) \cong (\mathbb{Z}/p^n)^g,$$

for $g = \dim A_{\bar{x}}$.

Recall that we always have $A_{\bar{x}}[p^n] \cong (\mathbb{Z}/p^n)^{2g}$ in this case.

Proof. Note that the group scheme $A[p^n]$ over $\operatorname{Spec} R[1/p]$ is étale. So the statement holds for all geometric points of some connected component if it is true for one geometric point in it. On the other hand, for each connected component $\mathcal{Z}$ of $\operatorname{Spec} R[1/p]$, by a standard specialization argument there exists a map $\operatorname{Spec} \mathcal{O}_K \to \operatorname{Spec} R$ whose image lies in $\mathcal{Z}$, where K is a non-archimedean algebraically closed field. Thus we can reduce to the case where $R = \mathcal{O}_K$, and it suffices to show $C_n(K) \cong (\mathbb{Z}/p^n)^g$. For this, we already know that

$$C_n(K) \subset A[p^n](K) \cong (\mathbb{Z}/p^n)^{2g}$$

and that $C_n(K)$ has order p^{ng}. Suppose $C_n(K) \ncong (\mathbb{Z}/p^n)^g$ for the sake of contradiction. Write $C_n(K) \cong \mathbb{Z}/p^{r_1} \oplus \cdots \oplus \mathbb{Z}/p^{r_k}$ with integers $r_1, \ldots, r_k \leqslant n$. Note that not all of r_i's are equal to n by our assumption, and this forces $k > g$. It follows that $(C_n(K) \cap A[p])(K)$ has more than $p^g = \#C_1(K)$
Lect.12, Nov 16 elements. So there exists $s \in (C_n(K) \cap A[p])(K) - C_1(K)$. Let t be the image of s in $H = A[p]/C_1$. Then $t \in H(K) - \{0\}$. Since C_n is a finite group scheme over $\mathcal{O}_K$, we can extend s to $\tilde{s} \in C_n(\mathcal{O}_K)$. Similarly, we get an extended image $\tilde{t} \in H(\mathcal{O}_K)$ of $\tilde{s}$. Then $\tilde{s}$ is mapped to 0 in $C_n(\mathcal{O}_K/p^{(1-\varepsilon)/p^n})$ by Theorem 3.9.7(2), and thus $\tilde{t}$ is mapped to 0 in $H(\mathcal{O}_K/p^{(1-\varepsilon)/p^n})$.

From the fact that A is strongly $O(n, \varepsilon)$, which is equivalent to being strongly $O(1, \varepsilon/p^n)$, we see applying Theorem 3.9.7(1) to $n = 1$ renders that p^{ε/p^n} kills $\Omega_{H/\mathcal{O}_K}$. It follows that $H(R) \to H(R/p^\delta)$ is injective for all $\delta > \varepsilon/p^n$ by Lemma 3.9.5. So taking $\delta = (1-\varepsilon)/p^n$ we get $\tilde{t} = 0$. Hence $t = 0$ in $H(K)$, a contradiction. □

Remark 3.9.11. In applications, A over R will be equipped with a principal polarization. Then by the uniqueness of C_n, the same argument as that preceding §3.8 shows that C_n is totally isotropic with respect to that polarization. It follows that the subgroup ($= \mathbb{F}_p$-vector subspace) $C_n(\bar{x})[p] \subset A[p](\bar{x})$ is totally isotropic, and hence its order cannot exceed p^g. Thus we obtain a simpler proof of the above proposition.

3.9.2. *Canonical Frobenius lifts.* We keep the settings on Shimura varieties as in §3.4. Recall that $X \coloneqq X_{\mathsf{K}^p G(\mathbb{Z}_p)}$ is an $\mathbb{Z}_{(p)}$-scheme, which is the moduli for principally polarized abelian varieties with extra K^p-level structure.

Construction 3.9.12. Let $\mathfrak{X}$ be the p-adic completion of $X \times_{\operatorname{Spec} \mathbb{Z}_{(p)}} \mathbb{Z}_p^{\mathrm{cycl}}$, which is a formal scheme over $\operatorname{Spf} \mathbb{Z}_p^{\mathrm{cycl}}$. We then take the generic fiber of the associated adic space

$$\mathcal{X} \coloneqq \mathfrak{X}_\eta^{\mathrm{ad}} = \mathfrak{X}^{\mathrm{ad}} \times_{\operatorname{Spa}(\mathbb{Z}_p^{\mathrm{cycl}}, \mathbb{Z}_p^{\mathrm{cycl}})} \operatorname{Spa}(\mathbb{Q}_p^{\mathrm{cycl}}, \mathbb{Z}_p^{\mathrm{cycl}}).$$

Recall that $\mathfrak{X}$ locally looks like $\operatorname{Spf} R$ for some R equipped with the p-adic topology. Correspondingly, $\mathfrak{X}^{\mathrm{ad}}$ is covered by $\operatorname{Spa}(R, R)$, and $\mathcal{X}$ is covered by the rational opens in $\operatorname{Spa}(R, R)$ defined by $v(p) \neq 0$.

Consider for $\mathsf{K}_p \in \{\Gamma_{\mathrm{s}}(p^n), \Gamma(p^n)\}$ that

$$\begin{tikzcd} X_{\mathsf{K}_p, \mathbb{Q}} \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{Q}(\mu_{p^n}) \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & \operatorname{Spec} \mathbb{Q}. \end{tikzcd}$$

Based on this, we make the following definition.

Definition 3.9.13. Define X^{ad} to be the adification of the finite-type $\mathbb{Q}_p^{\mathrm{cycl}}$-scheme $X_{\mathbb{Q}_p^{\mathrm{cycl}}} = X_{\mathbb{Q}} \times_{\operatorname{Spec} \mathbb{Q}} \operatorname{Spec} \mathbb{Q}_p^{\mathrm{cycl}}$, i.e.,

$$X^{\mathrm{ad}} \coloneqq X_{\mathbb{Q}_p^{\mathrm{cycl}}} \times_{\operatorname{Spec} \mathbb{Q}_p^{\mathrm{cycl}}} \operatorname{Spa} \mathbb{Q}_p^{\mathrm{cycl}},$$

which is a finite-type adic space over $\operatorname{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})$. Also, for $\mathsf{K}_p \in \{\Gamma_{\mathrm{s}}(p^n), \Gamma(p^n)\}$, define $X_{\mathsf{K}_p}^{\mathrm{ad}}$ to be the adification of $X_{\mathsf{K}_p, \mathbb{Q}} \times_{\operatorname{Spec} \mathbb{Q}(\mu_{p^n})} \operatorname{Spec} \mathbb{Q}_p^{\mathrm{cycl}}$, i.e.,

$$X_{\mathsf{K}_p}^{\mathrm{ad}} \coloneqq (X_{\mathsf{K}_p, \mathbb{Q}} \times_{\operatorname{Spec} \mathbb{Q}(\mu_{p^n})} \operatorname{Spec} \mathbb{Q}_p^{\mathrm{cycl}}) \times_{\operatorname{Spec} \mathbb{Q}_p^{\mathrm{cycl}}} \operatorname{Spa} \mathbb{Q}_p^{\mathrm{cycl}},$$

which is also a finite-type adic space over $\operatorname{Spa}(\mathbb{Q}_p^{\mathrm{cycl}})$. Then $\mathcal{X}$ admits an open embedding into X^{ad}, and $\mathcal{X}$ is exactly the *good reduction locus* in X^{ad}. The finite étale map $X_{\mathsf{K}_p, \mathbb{Q}} \to X_{\mathbb{Q}}$ induces a finite étale map (in the category of adic spaces) $X_{\mathsf{K}_p}^{\mathrm{ad}} \to X^{\mathrm{ad}}$. Denote by $\mathcal{X}_{\mathsf{K}_p}$ the preimage of $\mathcal{X}$ along this map. Thus we have the following diagram:

$$\begin{tikzcd} X_{\mathsf{K}_p}^{\mathrm{ad}} \arrow[d] & \supset & \mathcal{X}_{\mathsf{K}_p} \arrow[d] & & \\ X^{\mathrm{ad}} & \supset & \mathcal{X} & = & \text{good reduction locus.} \end{tikzcd}$$

Remark 3.9.14. Note that $\mathcal{X}$ has an integral model, i.e., a formal scheme over $\operatorname{Spf} \mathbb{Z}_p^{\mathrm{cycl}}$ such that $\mathcal{X}$ is the adic generic fiber of that. However, $\mathcal{X}_{\mathsf{K}_p}$ do not have such integral models.

We say a few words about the "functor of points" of $\mathfrak{X}$. Let R be a p-adically complete $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Then $\operatorname{Spf} R$ is a formal scheme over $\operatorname{Spf} \mathbb{Z}_p^{\mathrm{cycl}}$. We claim that giving a map

$$\operatorname{Spf} R \longrightarrow \mathfrak{X}$$

over $\operatorname{Spf}\mathbb{Z}_p^{\mathrm{cycl}}$ is equivalent to giving a map $\operatorname{Spec} R \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$ over $\operatorname{Spec}\mathbb{Z}_p^{\mathrm{cycl}}$. The latter can be interpreted by the moduli description, that is, a triple (A, λ, η) of principally polarized abelian variety with K_p-level structure over $\operatorname{Spec} R$. Our claim is true because if $X_{\mathbb{Z}_p^{\mathrm{cycl}}}$ is locally of the form $\operatorname{Spec} S$, then $\mathfrak{X}$ is of the form $\operatorname{Spf} S_p^\wedge$, where $S_p^\wedge$ denotes the p-adic completion of S. Then $\operatorname{Hom}_{\operatorname{Spf}\mathbb{Z}_p^{\mathrm{cycl}}}(\operatorname{Spf} R, \operatorname{Spf} S_p^\wedge) = \operatorname{Hom}_{\mathrm{cont}, \mathbb{Z}_p^{\mathrm{cycl}}}(S_p^\wedge, R) = \operatorname{Hom}_{\mathbb{Z}_p^{\mathrm{cycl}}}(S, R)$.

Over $X_{\mathbb{Z}_p^{\mathrm{cycl}}}$ we have the universal abelian scheme coming from the moduli problem. We can p-adically complete it, and get a formal abelian scheme $\mathfrak{A}$ over $\mathfrak{X}$. In combination with the previous paragraph, we see that $\mathfrak{A}$ has the following property: For any $\operatorname{Spf} R \to \mathfrak{X}$ as above, the pullback of $\mathfrak{A}$ over $\operatorname{Spf} R$ is automatically algebraizable, i.e., it comes from an abelian scheme over $\operatorname{Spec} R$.

Definition–Proposition 3.9.15. Fix $0 \leqslant \varepsilon < 1$ such that $p^\varepsilon \in \mathbb{Z}_p^{\mathrm{cycl}}$ makes sense. There exists a morphism $\mathfrak{X}(\varepsilon) \to \mathfrak{X}$ of formal schemes over $\operatorname{Spf}\mathbb{Z}_p^{\mathrm{cycl}}$ such that $\mathfrak{X}(\varepsilon)$ is flat over $\mathbb{Z}_p^{\mathrm{cycl}}$. It represents the following moduli problem: For any p-adically complete flat $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra R, the set of morphisms $\operatorname{Spf} R \to \mathfrak{X}(\varepsilon)$ is in bijection with the set of pairs $(f, [u])$, where

- f is a map $\operatorname{Spf} R \to \mathfrak{X}$, and
- $[u]$ is an extra datum as follows. The datum of f is equivalent to the datum of a map $f\colon \operatorname{Spec} R \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$. Pulling back the universal abelian scheme we then obtain an abelian scheme A over R. Consider the Hasse invariant $\operatorname{Ha}(A_1) \in \omega_{A/R}^{\otimes p-1} \otimes_R R/p$ of $A_1 := A \otimes_R (R/p)$. (Note that $\omega_{A/R}$ is a free R-module of rank 1.) Take $u \in \omega_{A/R}^{\otimes(1-p)}$ such that modulo p,
$$u \cdot \operatorname{Ha}(A_1) = p^\varepsilon \in R/p.$$
Define $[u]$ to be the equivalence class of u under the relation that $u \sim u'$ if $u' = u(1 - p^{1-\varepsilon}h)$ for some $h \in R$.

Proof. It suffices to prove the existence as the uniqueness would follow from Yoneda's lemma. Given $f\colon \operatorname{Spf} R \to \mathfrak{X}$ together with the data for defining u as above, we fix a lift $\widetilde{\operatorname{Ha}} \in \omega_{A/R}^{\otimes(p-1)}$ of $\operatorname{Ha} := \operatorname{Ha}(A_1/R_1) \in \omega_{A_1/R_1}^{\otimes(p-1)}$ with $R_1 := R/p$. Suppose we have u as in the moduli problem. Then
$$u \cdot \widetilde{\operatorname{Ha}} = p^\varepsilon - ph = p^\varepsilon(1 - p^{1-\varepsilon}h)$$
for some $h \in R$. Thus we may change u by the factor $1 - p^{1-\varepsilon}h$, and arrange that $u \cdot \widetilde{\operatorname{Ha}} = p^\varepsilon$. Conversely, if we have u and u' such that $u \cdot \widetilde{\operatorname{Ha}} = u' \cdot \widetilde{\operatorname{Ha}} = p^\varepsilon$ and such that $[u] = [u']$, then we claim that $u = u'$. Indeed, suppose $u = u'(1 - p^{1-\varepsilon}h) \in R$. Then, without modulo p,
$$p^\varepsilon = u \cdot \widetilde{\operatorname{Ha}} = u' \cdot \widetilde{\operatorname{Ha}}(1 - p^{1-\varepsilon}h) = p^\varepsilon(1 - p^{1-\varepsilon}h) = p^\varepsilon - ph.$$
Since R is p-torsion-free, we have $h = 0$, i.e. $u = u'$. We conclude that the moduli problem over the fixed f is equivalent to the classification of $u \in R$ such that $u \cdot \widetilde{\operatorname{Ha}} = p^\varepsilon$. So if we cover $\mathfrak{X}$ by $\operatorname{Spf} R$ and for each $\operatorname{Spf} R$ we pick a lift $\widetilde{\operatorname{Ha}}$ as above, then the moduli problem of $\mathfrak{X}(\varepsilon)$ is represented by the formal scheme glued from
$$\operatorname{Spf} R\langle u\rangle/(u \cdot \widetilde{\operatorname{Ha}} - p^\varepsilon).$$
The above formal scheme is indeed flat over $\mathbb{Z}_p^{\mathrm{cycl}}$. ∎

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From the proof above we make the following observation. It is clear that, in the sense of §2.8.1, taking the adic generic fiber of $\operatorname{Spf} R$ gives $\operatorname{Spa}(A, A^+)$ where $A = R[1/p]$ and A^+ is the integral closure of R in A. The topology on A is such that R is an open subring with restricted topology the same as the usual p-adic topology. Further, taking the adic generic fiber of $\operatorname{Spf} R\langle u\rangle/(u \cdot \widetilde{\operatorname{Ha}} - p^\varepsilon)$, we get a rational subset
$$U\left(\frac{p^\varepsilon}{\widetilde{\operatorname{Ha}}}\right) = \{v \in \operatorname{Spa}(A, A^+)\colon v(p^\varepsilon) \leqslant v(\widetilde{\operatorname{Ha}}) \neq 0\} \subset \operatorname{Spa}(A, A^+).$$

The map $\mathfrak{X}(\varepsilon) \hookrightarrow \mathfrak{X}$ of formal schemes is not an open embedding. However, the argument above shows that at the level of adic generic fibers we get an open embedding. Recall the notation from Construction 3.9.12 that $\mathcal{X} = \mathfrak{X}_\eta^{\mathrm{ad}}$, and take $\mathcal{X}(\varepsilon) := \mathfrak{X}(\varepsilon)_\eta^{\mathrm{ad}}$. The upshot is as follows:

(a) $\mathcal{X}(\varepsilon) \hookrightarrow \mathcal{X}$ is an open embedding even if $\mathfrak{X}(\varepsilon) \hookrightarrow \mathfrak{X}$ is not.

(b) $\mathcal{X}(\varepsilon) \subset \mathcal{X}(\varepsilon')$ whenever $\varepsilon \leqslant \varepsilon'$.

Clearly $\mathcal{X}(\varepsilon)$ is an open neighborhood of the good ordinary reduction locus in $\mathcal{X}$. We think of ε as measuring how large this neighborhood is.

Notation 3.9.16. Let $\mathfrak{Y}$ be a formal scheme over $\operatorname{Spf}\mathbb{Z}_p^{\mathrm{cycl}}$. Write

$$\mathfrak{Y}/p := \mathfrak{Y} \times_{\operatorname{Spf}\mathbb{Z}_p^{\mathrm{cycl}}} \operatorname{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p).$$

We make the following observation.

Proposition 3.9.17. *Over* $\operatorname{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p)$*, there is a natural isomorphism*

$$(\mathfrak{X}(p^{-1}\varepsilon)/p)^{(p)} := (\mathfrak{X}(p^{-1}\varepsilon)/p) \times_{\operatorname{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p),\mathrm{Fr}} \operatorname{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p) \xrightarrow{\sim} \mathfrak{X}(\varepsilon)/p.$$

Proof. Let $\operatorname{Spf} R$ be a test formal scheme over $\mathbb{Z}_p^{\mathrm{cycl}}/p$. Then

$$(\mathfrak{X}(p^{-1}\varepsilon)/p)^{(p)}(R) = (\mathfrak{X}(p^{-1}\varepsilon)/p)(\mathrm{Fr}_*R).$$

Here Fr_*R is equal to R as a ring but seen as a $(\mathbb{Z}_p^{\mathrm{cycl}}/p)$-algebra by pre-composing the structure map $\mathbb{Z}_p^{\mathrm{cycl}}/p \to R$ with $\mathrm{Fr}\colon \mathbb{Z}_p^{\mathrm{cycl}}/p \to \mathbb{Z}_p^{\mathrm{cycl}}/p,\ x \mapsto x^p$. The Fr_*R-points of $\mathfrak{X}(p^{-1}\varepsilon)/p$ are by definition pairs $(f,[u])$, where

$$f\colon \operatorname{Spec}(\mathrm{Fr}_*R) \longrightarrow X_{\mathbb{Z}_p^{\mathrm{cycl}}} = X_{\mathbb{Z}_p} \times_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}}$$

is the map of $\mathbb{Z}_p^{\mathrm{cycl}}$-schemes. But the datum of f is equivalent to the datum of a map of $\mathbb{Z}_p$-schemes

$$f\colon \operatorname{Spec} R = \operatorname{Spec}(\mathrm{Fr}_*R) \longrightarrow X_{\mathbb{Z}_p}.$$

Here we are crucially using the fact that $X_{\mathbb{Z}_p^{\mathrm{cycl}}}$ is the base change of $X_{\mathbb{Z}_p}$, i.e., the moduli space of abelian schemes is already defined over $\mathbb{Z}_p$. So the map f above admits a moduli interpretation, namely an abelian scheme A over R with a principal polarization and K^p-level structure. Since R is of characteristic p, the element u is an element of $\Omega_{A/R}^{\otimes(1-p)}$ such that $u\mathrm{Ha}(A/R) = p^{\varepsilon/p} \in \mathrm{Fr}_*R$. (Here a priori $p^{\varepsilon/p} \in \mathbb{Z}_p^{\mathrm{cycl}}$ but we multiplicate it with $1 \in \mathrm{Fr}_*R$ to get $p^{\varepsilon/p} = p^{\varepsilon/p}\cdot 1 \in \mathrm{Fr}_*R$.) On the other hand, as $\mathbb{F}_p$-algebras there is an identity $\mathrm{Fr}_*R = R$ under which $p^{\varepsilon/p} \mapsto p^{\varepsilon}$. So the condition is also read as

$$u \cdot \mathrm{Ha}(A/R) = p^{\varepsilon} \in R.$$

Therefore, by comparing this with Definition–Proposition 3.9.15, we can interpret $(f,[u])$ as an R-point of $\mathfrak{X}(\varepsilon)/p$. This proves the desired canonical isomorphism. $\square$

By Proposition 3.9.17 above we have the following commutative diagram

$$\begin{tikzcd}
(\mathfrak{X}(p^{-1}\varepsilon)/p)^{(p)} \arrow[r, "\sim"] \arrow[d] & \mathfrak{X}(\varepsilon)/p \arrow[d] \\
(\mathfrak{X}/p)^{(p)} \arrow[r, "\sim"] & \mathfrak{X}/p.
\end{tikzcd}$$

The lower horizontal is an isomorphism because $\mathfrak{X}$ is defined over $\mathbb{Z}_p$.

Definition 3.9.18. Define $\mathfrak{A}(\varepsilon)$ to be the pullback of the universal family $\mathfrak{A}$ along $\mathfrak{X}(\varepsilon) \to \mathfrak{X}$.

Construction 3.9.19. Modulo p, we work on $\mathbb{Z}_p^{\mathrm{cycl}}/p$ and obtain the commutative diagrams

$$\begin{tikzcd}
\tilde{\varphi}\colon & \mathfrak{A}(p^{-1}\varepsilon)/p \arrow[r, "F"] \arrow[d] & (\mathfrak{A}(p^{-1}\varepsilon)/p)^{(p)} \arrow[r, "\cong"] \arrow[d] & \mathfrak{A}(\varepsilon)/p \arrow[d] \\
\varphi\colon & \mathfrak{X}(p^{-1}\varepsilon)/p \arrow[r, "F"] \arrow[d] & (\mathfrak{X}(p^{-1}\varepsilon)/p)^{(p)} \arrow[r, "\cong"] \arrow[d] & \mathfrak{X}(\varepsilon)/p \arrow[d] \\
\varphi_0\colon & \mathfrak{X}/p \arrow[r, "F"] & (\mathfrak{X}/p)^{(p)} \arrow[r, "\cong"] & \mathfrak{X}/p.
\end{tikzcd}$$

where the F's are the relative Frobenius maps over $\mathbb{Z}_p^{\mathrm{cycl}}/p$, and the twistings $(\cdot)^{(p)}$ mean $(\cdot)\times_{\operatorname{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p),\mathrm{Fr}} \operatorname{Spf}(\mathbb{Z}_p^{\mathrm{cycl}}/p)$. The following are moduli interpretations of the three horizontal maps above.

- The map $\varphi_0\colon \mathfrak{X}/p \to \mathfrak{X}/p$ sends an R-point $f\colon \operatorname{Spf} R \to \mathfrak{X}/p$ to $\varphi_0(f)\colon \operatorname{Spf} R \xrightarrow{\mathrm{Fr}} \operatorname{Spf} R \xrightarrow{f} \mathfrak{X}/p$. Here Fr denotes the absolute Frobenius on R. In terms of the moduli of abelian schemes, this is equivalent to taking the relative Frobenius twist over R, that is
$$\varphi_0\colon A/R \longmapsto A^{(p)}/R,$$
where $A^{(p)} := A \times_{\operatorname{Spf} R,\mathrm{Fr}} \operatorname{Spf} R$. (This notation is *different* from the notation used in the diagram.)
- The map $\varphi\colon \mathfrak{X}(p^{-1}\varepsilon)/p \to \mathfrak{X}(\varepsilon)/p$ begins with the pair $(f,[u])$. From the argument above we can identify f with an abelian scheme A/R, and we have $u \in \omega_{A/R}^{\otimes(1-p)}$. Then
$$\varphi\colon (A,[u]) \longmapsto (A^{(p)},[u^p]).$$
This construction makes sense for the following reason. Since $\omega_{A/R}^{\otimes p} = \omega_{A^{(p)}/R}$ we have $u^p \in \omega_{A/R}^{\otimes p(1-p)} = \omega_{A^{(p)}/R}^{\otimes(1-p)}$. Also $\mathrm{Ha}(A^{(p)}) \in \omega_{A^{(p)}/R}^{\otimes(p-1)}$ is equal to $\mathrm{Ha}(A)^p \in \omega_{A/R}^{\otimes p(1-p)}$.
- When pulled back along an R-point of $\mathfrak{X}(p^{-1}\varepsilon)/p$ corresponding to A/R as above, the map $\tilde{\varphi}$ becomes a map $A \to A^{(p)}$ between abelian schemes over R. This is nothing but the relative Frobenius for A over R.

Moreover, the diagrams between $\varphi_0, \varphi, \tilde{\varphi}$ can be lifted to those before modulo p. We first need some preparation.

Construction 3.9.20. We discuss two maps of $\mathbb{Q}$-schemes
$$\begin{tikzcd} & X_{\Gamma_{\mathrm{s}}(p^n)} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_{\mathbb{Q}} & & X_{\mathbb{Q}} \end{tikzcd}$$
Recall that a (necessarily characteristic 0) point of $X_{\Gamma_{\mathrm{s}}(p^n)}$ corresponds to the tuple (A,λ,η,D) over a ring R over $\mathbb{Q}$. Here $\lambda\colon A \xrightarrow{\sim} A^\vee$ is the principal polarization, η is the K^p-level structure, and D is the $\Gamma_{\mathrm{s}}(p^n)$-level structure. More precisely, the datum of D consists of two parts: one is a choice of a p^n-th root of unity that we omit to discuss, and the other is the totally isotropic subgroup $D \subset A[p^n]$ such that $D \cong (\mathbb{Z}/p^n)^g$ over each geometric fiber. We have
$$\pi_1\colon (A,\lambda,\eta,D) \longmapsto (A,\lambda,\eta),$$
$$\pi_2\colon (A,\lambda,\eta,D) \longmapsto (A/D,\lambda',\eta').$$
This π_1 is simply the forgetful map, and we think of it as the standard map from $X_{\Gamma_{\mathrm{s}}(p^n)}$ to $X_{\mathbb{Q}}$. As for the definition of π_2, we say a few words on the definition of λ', a principal polarization on A/D. Consider
$$A/D \xleftarrow{q} A \xrightarrow{\lambda} A^\vee \xleftarrow{q^\vee} (A/D)^\vee,$$
where q is the canonical projection of group schemes. Here comes the explanation of this. By functoriality of the Weil pairing, $D \subset A[p^n]$ is annihilated by $q^\vee((A/D)^\vee[p^n]) \subset A^\vee[p^n]$; also, since D is totally isotropic with respect to e^λ (see (2) on page 35), it is annihilated by $\lambda(D)$. Recall that if A has relative dimension g over R, then $A[p^n]$ is finite of order p^{2ng}; further, both D and $D^\perp = \lambda(D)$ are of order p^{ng}. It follows that $q^\vee((A/D)^\vee[p^n]) = D^\perp = \lambda(D)$. So $q \circ \lambda^{-1} \circ q^\vee\colon (A/D)^\vee \to A/D$ kills the full p^n-torsion, i.e., it factors through the multiplication-by-p^n map. Then we can define
$$\mu := \frac{1}{[p^n]} \circ q \circ \lambda^{-1} \circ q^\vee.$$
For degree reason, μ is an isomorphism. One can check that $\lambda' := \mu^{-1}$ is really a polarization.

Proposition 3.9.21. *Fix $0 \leqslant \varepsilon < 1/2$ such that $p^\varepsilon \in \mathbb{Z}_p^{\mathrm{cycl}}$ makes sense.*

(1) *There exists a unique diagram lifting the top two rows in the diagram in Construction 3.9.19 modulo $p^{1-\varepsilon}$, of the form*

$$\begin{tikzcd} \mathfrak{A}(p^{-1}\varepsilon) \arrow[r, "\widetilde{F}_{\mathfrak{A}}"] \arrow[d] & \mathfrak{A}(\varepsilon) \arrow[d] \\ \mathfrak{X}(p^{-1}\varepsilon) \arrow[r, "\widetilde{F}_{\mathfrak{X}}"] & \mathfrak{X}(\varepsilon). \end{tikzcd}$$

(2) *For any integer $m \in \mathbb{Z}_{>0}$, the formal abelian scheme*

$$\mathfrak{A}(p^{-m}\varepsilon) \longrightarrow \mathfrak{X}(p^{-m}\varepsilon)$$

is strongly $O(m,\varepsilon)$, in the sense that for any $(\operatorname{Spf} R)$-point of $\mathfrak{X}(p^{-m}\varepsilon)$ we get a strongly $O(m,\varepsilon)$ abelian scheme A over $\operatorname{Spec} R$. In a similar sense, we get the canonical subgroup C_m of $\mathfrak{A}(p^{-m}\varepsilon)[p^m]$ of level m. This induces a morphism on the adic generic fiber

$$\iota_m \colon \mathcal{X}(p^{-m}\varepsilon) \longrightarrow \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)},$$

which "maps a point corresponding to A to a point corresponding to $A/C_m(A)$ equipped with $\Gamma_{\mathrm{s}}(p^m)$-level structure $A[p^m]/C_m(A)$". The rigorous definition of ι_m will be explained in the proof. This ι_m is an open immersion of adic spaces, and we have commutative Cartesian diagram

$$\begin{tikzcd} \mathcal{X}(p^{-m-1}\varepsilon) \arrow[r, hook, "\iota_{m+1}"] \arrow[d, "(\widetilde{F})^{\mathrm{ad}}_{\eta}"'] & \mathcal{X}_{\Gamma_{\mathrm{s}}(p^{m+1})} \arrow[d] \\ \mathcal{X}(p^{-m}\varepsilon) \arrow[r, hook, "\iota_m"] & \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}, \end{tikzcd}$$

where $\widetilde{F}$ is as in (1) and the right vertical map is induced by the natural scheme morphism $X_{\Gamma_{\mathrm{s}}(p^{m+1})} \to X_{\Gamma_{\mathrm{s}}(p^m)}, (A, D) \mapsto (A, D[p^m])$. In particular, $(\widetilde{F})^{\mathrm{ad}}_{\eta}$ is finite étale since the right vertical map is finite étale.

(3) *The universal family $\mathfrak{A}(\varepsilon)$ over $\mathfrak{X}(\varepsilon)$ is weakly $O(1,\varepsilon)$. Hence we get the weakly canonical subgroup C_1 in the sense of (2). Define $\mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon)$ to be the pullback via the Cartesian diagram*

$$\begin{tikzcd} \mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon) \arrow[r, hook] \arrow[d] & \mathcal{X}_{\Gamma_{\mathrm{s}}(p)} \arrow[d, "\pi_1"] \\ \mathcal{X}(\varepsilon) \arrow[r, hook] & \mathcal{X}. \end{tikzcd}$$

Then the following is a commutative diagram:

$$\begin{tikzcd} \mathcal{X}(p^{-1}\varepsilon) \arrow[r, hook, "\iota_1"] \arrow[d, "\widetilde{F}"'] & \mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon) \arrow[d] \\ \mathcal{X}(\varepsilon) \arrow[r, "\mathrm{id}"] & \mathcal{X}(\varepsilon). \end{tikzcd}$$

The image of ι_1 is open and closed in $\mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon)$, equal to the anti-canonical locus $\mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon)_{\mathrm{a}}$.

Lect.14, Nov 23

We give an informal explanation of Proposition 3.9.21. Note that in Proposition 3.9.21, (3) is about increasing the level from $m = 0$ to $m = 1$, and (2) is about increasing the level from $m \geqslant 1$ to $m+1$. In the good reduction locus $\mathcal{X}$ of X^{ad}, we have open subsets

$$\cdots \subset \mathcal{X}(p^{-m}\varepsilon) \subset \cdots \subset \mathcal{X}(p^{-1}\varepsilon) \subset \mathcal{X}(\varepsilon) \subset \mathcal{X}.$$

These are all open neighborhoods of the "ordinary locus" $\mathcal{X}(0)$. Roughly speaking, the open $\mathcal{X}(p^{-m}\varepsilon)$ (resp. $\mathcal{X}(\varepsilon)$) has the property that "the abelian scheme A over it" (in an informal sense; to make this rigorous we have to consider the formal model $\mathfrak{X}(p^{-m}\varepsilon)$ of $\mathcal{X}(p^{-m}\varepsilon)$) is strongly $O(m,\varepsilon)$ (resp. weakly $O(1,\varepsilon)$). On the other hand, between these $\mathcal{X}(p^{-m}\varepsilon)$, in addition to the inclusion maps we have lifted Frobenius maps $\widetilde{F} \colon \mathcal{X}(p^{-(m+1)}\varepsilon) \to \mathcal{X}(p^{-m}\varepsilon)$. These maps are finite étale as opposed to open immersions, and they form the *Frobenius tower*

$$\cdots \xrightarrow{\widetilde{F}} \mathcal{X}(p^{-m}\varepsilon) \xrightarrow{\widetilde{F}} \cdots \xrightarrow{\widetilde{F}} \mathcal{X}(p^{-1}\varepsilon) \xrightarrow{\widetilde{F}} \mathcal{X}(\varepsilon).$$

Moreover, this tower is isomorphic to the *anti-canonical tower*, which is obtained from the natural tower $\cdots \to \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)} \to \cdots \to \mathcal{X}_{\Gamma_{\mathrm{s}}(p)} \to \mathcal{X}$ first restricted to $\mathcal{X}(\varepsilon) \subset \mathcal{X}$ and then restricted to $\mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon)_{\mathrm{a}} \subset \mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon)$, that is

$$\cdots \longrightarrow \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}} \longrightarrow \cdots \longrightarrow \mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon)_{\mathrm{a}} \longrightarrow \mathcal{X}(\varepsilon).$$

The key point of Proposition 3.9.21 is that these two towers are isomorphic via $(\iota_m)_{m\geqslant 1}$.

The anti-canonical tower is said to be "overconvergent as a whole tower" in the sense that at each level $\Gamma_{\mathrm{s}}(p^m)$, the member $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}}$ is determined as the inverse image of $\mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon)_{\mathrm{a}} \subset \mathcal{X}_{\Gamma_{\mathrm{s}}(p)}$ along $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)} \to \mathcal{X}_{\Gamma_{\mathrm{s}}(p)}$, where $\mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon)_{\mathrm{a}}$ is an "ε-neighborhood of the anti-canonical ordinary locus in $\mathcal{X}_{\Gamma_{\mathrm{s}}(p)}$"; we do not need to shrink ε when m increases.

Proof of Proposition 3.9.21. As usual, let R be a p-adically complete $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. If the map $\operatorname{Spf} R \to \mathfrak{X}(p^{-m}\varepsilon)$ for $m \geqslant 1$ defines a pair $(f, [u])$ that corresponds to an abelian scheme A/R (together with principal polarization and away-from-p level structure) and $u \in \omega_{A/R}^{\otimes(1-p)}$, with

$$u \cdot \mathrm{Ha}(A_1/R_1) = p^{p^{-m}\varepsilon} \in R_1,$$

then Ha^{p^m} must divide p^{ε}, namely A is strongly $O(m,\varepsilon)$. Similarly, if $m = 0$, then $\mathrm{Ha}^{(p-1)/(p-1)} = \mathrm{Ha}$ divides p^{ε}, namely A is $O(1,\varepsilon)$.

(1) Define the following map by moduli

$$\begin{aligned} \widetilde{F}_{\mathfrak{X}} \colon \mathfrak{X}(p^{-1}\varepsilon) &\longrightarrow \mathfrak{X}(\varepsilon) \\ (A/R, [u]) &\longmapsto ((A/C_1)/R, [u^p]), \end{aligned}$$

where on the source $u \in \omega_{A/R}^{\otimes(1-p)}$ and the image A/C_1 makes sense because A is $O(1,\varepsilon)$. We then define $\widetilde{F}_{\mathfrak{A}} \colon A \to A/C_1$ to be the natural projection. Then the desired diagram exists and commutes. It is unique from uniqueness of C_1.

(2) The difficulty lies in figuring out the definition of ι_m. We use the following argument to reduce it to figuring out certain level structure on A. Let R be a p-adically complete $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. We fix $f \colon \operatorname{Spf} R \to \mathfrak{X} \coloneqq (X_{\mathbb{Z}_p^{\mathrm{cycl}}})^{\wedge}$. We know that f is induced by a scheme map $\operatorname{Spec} R \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$ which we still denote by f, and we denote by A the corresponding abelian scheme over $\operatorname{Spec} R$. Since $X_{\mathbb{Z}_p^{\mathrm{cycl}}}$ is of finite type over $\mathbb{Z}_p^{\mathrm{cycl}}$, there exist R_0 and f_0, where R_0 is a finite-type $\mathbb{Z}_p^{\mathrm{cycl}}$-subalgebra of R, and f_0 is a map of $\mathbb{Z}_p^{\mathrm{cycl}}$-schemes $f_0 \colon \operatorname{Spec} R_0 \to X_{\mathbb{Z}_p^{\mathrm{cycl}}}$ inducing f. At the level of adic generic fibers, we want to lift the map $f_\eta^{\mathrm{ad}} \colon (\operatorname{Spf} R)_\eta^{\mathrm{ad}} \to X^{\mathrm{ad}}$ along $X_{\Gamma_{\mathrm{s}}(p^m)}^{\mathrm{ad}} \to X^{\mathrm{ad}}$. On the other hand, there are bijections

$$\begin{gathered} \varinjlim_{(f_0,R_0)} \{\text{liftings of } f_0[1/p]^{\mathrm{ad}} \colon (\operatorname{Spec} R_0[1/p])^{\mathrm{ad}} \to X^{\mathrm{ad}} \text{ along } X_{\Gamma_{\mathrm{s}}(p^m)}^{\mathrm{ad}} \to X^{\mathrm{ad}}\} \\ \updownarrow \\ \varinjlim_{(f_0,R_0)} \{\text{liftings of } f_0[1/p] \colon \operatorname{Spec} R_0[1/p] \to X_{\mathbb{Q}_p^{\mathrm{cycl}}} \text{ along } X_{\Gamma_{\mathrm{s}}(p^m),\mathbb{Q}_p^{\mathrm{cycl}}} \to X_{\mathbb{Q}_p^{\mathrm{cycl}}}\} \\ \updownarrow \\ \{\text{liftings of } f[1/p] \colon \operatorname{Spec} R[1/p] \to X_{\mathbb{Q}_p^{\mathrm{cycl}}} \text{ along } X_{\Gamma_{\mathrm{s}}(p^m),\mathbb{Q}_p^{\mathrm{cycl}}} \to X_{\mathbb{Q}_p^{\mathrm{cycl}}}\} \\ \updownarrow \\ \{\Gamma_0(p^m)\text{-level structures on } A \otimes_R R[1/p]\} \end{gathered}$$

Here the direct limits are over the set of pairs (f_0, R_0) as above. The first set admits an obvious map to the set of liftings of $f_\eta^{\mathrm{ad}} \colon (\operatorname{Spf} R)_\eta^{\mathrm{ad}} \to X^{\mathrm{ad}}$ along $X_{\Gamma_{\mathrm{s}}(p^m)}^{\mathrm{ad}} \to X^{\mathrm{ad}}$, since we can first restrict to the open subspace $(\operatorname{Spf} R_0^{\wedge})_\eta^{\mathrm{ad}} \subset (\operatorname{Spec} R_0[1/p])^{\mathrm{ad}}$ (here $R_0^{\wedge}$ denotes the p-adic completion of R_0) and then pre-compose with the adic generic fiber of the natural map $\operatorname{Spf} R \to \operatorname{Spf} R_0^{\wedge}$. To explain the three bijections, we need the general fact:

> *Fact.* If $Y \to Z$ is a finite étale map of finite-type schemes over $\mathbb{Q}_p^{\mathrm{cycl}}$ then there is a natural bijection between the set of sections of $Y \to Z$ and the set of sections of $Y^{\mathrm{ad}} \to Z^{\mathrm{ad}}$.

In practice, take $Y = X_{\Gamma_{\mathrm{s}}(p^m),\mathbb{Q}_p^{\mathrm{cycl}}} \times_{\mathbb{Q}_p^{\mathrm{cycl}}} \operatorname{Spec} R_0[1/p]$ and $Z = X_{\mathbb{Q}_p^{\mathrm{cycl}}} \times_{\mathbb{Q}_p^{\mathrm{cycl}}} \operatorname{Spec} R_0[1/p]$ to get the first bijection. The second bijection is by the finite type of $X_{\mathbb{Q}_p^{\mathrm{cycl}}}$ and $X_{\Gamma_{\mathrm{s}}(p^m),\mathbb{Q}_p^{\mathrm{cycl}}}$. The third bijection follows from the isomorphism

$$X_{\Gamma_{\mathrm{s}}(p^m),\mathbb{Q}_p^{\mathrm{cycl}}} \coloneqq X_{\Gamma_{\mathrm{s}}(p^m),\mathbb{Q}_p} \times_{\operatorname{Spec}\mathbb{Q}_p(\mu_{p^m})} \operatorname{Spec}\mathbb{Q}_p^{\mathrm{cycl}} \cong X_{\Gamma_0(p^m),\mathbb{Q}_p} \times_{\operatorname{Spec}\mathbb{Q}_p} \operatorname{Spec}\mathbb{Q}_p^{\mathrm{cycl}} \tag{3.9.1}$$

in view of $X_{\Gamma_{\mathrm{s}}(p^m),\mathbb{Q}} \cong X_{\Gamma_0(p^m),\mathbb{Q}} \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\mathbb{Q}(\mu_{p^m})$.

Granting the bijections, we now aim to define ι_m to fit in the diagram

$$\begin{tikzcd} & \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)} \arrow[d] \\ \mathcal{X}(p^{-m}\varepsilon) \arrow[ur, dashed, "\iota_m"] \arrow[r, "f_\eta^{\mathrm{ad}}"] & \mathcal{X}. \end{tikzcd}$$

For this, one first defines $f\colon \mathfrak{X}(p^{-m}\varepsilon) \to \mathfrak{X}$ via the moduli interpretation $A \mapsto A/C_m$. (Here, the definition of the principal polarization on A/C_m is similar to the definition of π_2 in Construction 3.9.20, using that C_m is totally isotropic with respect to the polarization on A, which follows from the uniqueness of C_m, cf. the discussion preceding §3.8.) We then lift f_η^{ad} to ι_m by taking

$$(A[p^m]/C_m) \otimes_R R[1/p]$$

as $\Gamma_0(p^m)$-level structure on $A \otimes_R R[1/p]$. (One checks that the above is indeed totally isotropic with respect to the principal polarization on A/C_m, using the fact that C_m is totally isotropic with respect to the principal polarization on A.)

Next we check that ι_m is an open embedding. As in Construction 3.9.20, define $\pi_1, \pi_2\colon X_{\Gamma_{\mathrm{s}}(p^m)} \to X$ via $\pi_1(A, D) = A$ and $\pi_2(A, D) = A/D$, where D is the totally isotropic subgroup given by the level structure. Consider the composition

$$\mathcal{X}(p^{-m}\varepsilon) \xrightarrow{\iota_m} \mathcal{X}_{\Gamma_{\mathrm{s}}(p^n)} \xrightarrow{\pi_2} \mathcal{X}.$$

Informally, this composition admits a moduli interpretation $(A, [u]) \mapsto (A/C_m, A[p^m]/C_m) \mapsto A/A[p^m]$. Formally, the composition is the adic generic fiber of the map $\mathfrak{X}(p^{-m}\varepsilon) \to \mathfrak{X}, (A, [u]) \mapsto A/A[p^m]$. Note that the map $\theta\colon X_{\mathbb{Q}_p} \to X_{\mathbb{Q}_p}, A \mapsto A/A[p^m]$ is an isomorphism. Now $\pi_2 \circ \iota_m$ is the restriction to the open $\mathcal{X}(p^{-m}\varepsilon) \subset X^{\mathrm{ad}}$ of the adification $\theta^{\mathrm{ad}} : X^{\mathrm{ad}} \xrightarrow{\sim} X^{\mathrm{ad}}$ of θ. Hence $\pi_2 \circ \iota_m$ is an open embedding. Since π_2 is finite étale, ι_m is an open embedding.

To check the commutativity of the given diagram in (2), recall that A/R is $O(m+1)$ and we have the short exact sequence

$$0 \longrightarrow C_1(A) \longrightarrow C_{m+1}(A) \longrightarrow C_m(A/C_1(A)) \longrightarrow 0.$$

The original given diagram can be written as the moduli interpretation

$$\begin{tikzcd} A \arrow[r, mapsto, "\iota_{m+1}"] \arrow[d, mapsto] & (A/C_{m+1}(A), A[p^{m+1}]/C_{m+1}(A)) \arrow[d, mapsto, dashed] \\ B \arrow[r, mapsto, "\iota_m"] & (B/C_m(B), B[p^m]/C_m(B)), \end{tikzcd}$$

where $B \coloneqq A/C_1(A)$. From the short exact sequence and definition, $B/C_m(B) \cong A/C_{m+1}(A)$ and $C_m(B) = C_{m+1}(A)/C_1(A)$. So it suffices to show that the right vertical map sends the source level structure to its p^m-torsion. But this follows from

$$B[p^m]/C_m(B) = \{x \in A\colon [p^m]x \in C_1(A)\}/C_{m+1}(A).$$

Here the condition on $x \in A$ implies that $x \in A[p^{m+1}]$.

It remains to check the Cartesian property. Observe that $\widetilde{F}_{\mathfrak{X}}$ is finite and locally free of rank $p^{\dim X} = p^{g(g+1)/2}$, because it is a lift of relative Frobenius modulo $p^{1-\varepsilon}$. So $(\widetilde{F})_\eta^{\mathrm{ad}}$ is finite étale (it is étale because the two horizontal maps are open embeddings and the right vertical map is finite étale) of degree $p^{g(g+1)/2}$. It then suffices to prove that the finite étale map

$$X_{\Gamma_0(p^{m+1}),\mathbb{Q}_p} \longrightarrow X_{\Gamma_0(p^m),\mathbb{Q}_p}$$

has degree $p^{g(g+1)/2}$. (Here we have Γ_0 instead of Γ_{s}, in view of (3.9.1).) This will be proved in Lemma 3.9.22 below. This finishes the proof of (2).

(3) The commutativity can be shown in the same way as in (2). The image of ι_1 is clearly open in $\mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon)$. For checking it to be closed, note that we have a finite map $\widetilde{F}\colon \mathcal{X}(p^{-1}\varepsilon) \to \mathcal{X}(\varepsilon)$ coming from a finite locally free map $\mathfrak{X}(p^{-1}\varepsilon) \to \mathfrak{X}(\varepsilon)$ between formal schemes, and a finite étale map $\mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon) \to \mathcal{X}(\varepsilon)$. Also, ι_1 is an open immersion. So ι_1 is a closed immersion.

We have already obtained the clopen adic subspace $\mathrm{im}(\iota_1) \subset \mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon)$, but we also need to identify it with the "anti-canonical locus" $\mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon)_{\mathrm{a}}$. We do not offer the complete proof, and we do not even give the rigorous definition of $\mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon)_{\mathrm{a}}$. Rather, we are contented in checking only the following. Let K be an algebraically closed non-archimedean extension of $\mathbb{Q}_p^{\mathrm{cycl}}$, and let $\operatorname{Spf}\mathcal{O}_K \to \mathfrak{X}(p^{-1}\varepsilon)$ be a point corresponding to $A/\mathcal{O}_K$. Then A has a canonical subgroup C_1 of level 1, and in shorthand

$$(B, D) \coloneqq ((A/C_1)_K, (A[p]/C_1)_K).$$

We shall check that (B, D) is anti-canonical, i.e. $C_1(B) \cap D = 0$. Take a geometric point $\bar{s} \in (C_1(B)\cap D)(K)$. It first lifts to some $s \in A[p](K)$ and then extends to $s \in A[p](\mathcal{O}_K)$. Since $s \in C_1(B)$, by the formula for C_1 in Theorem 3.9.7(2), $s \equiv 0 \bmod p^{(1-\varepsilon)/p}$ in $B = A/C_1(A)$. Set $H \coloneqq A[p]/C_1(A)$ and then s is mapped to zero in $H(\mathcal{O}_K/p^{(1-\varepsilon)/p})$. Also, $p^{\varepsilon/p}$ kills $\Omega_{H/\mathcal{O}_K}$, so s is mapped to zero in $H(\mathcal{O}_K)$. Then $s \in C_1(A)(\mathcal{O}_K)$ and hence $\bar{s} = 0$. This finishes the proof. □

Lect.15, Nov 28

Lemma 3.9.22. *For $m \geqslant 1$, we have*

$$[\Gamma_0(p^m) : \Gamma_0(p^{m+1})] = p^{g(g+1)/2}.$$

Proof. Recall that $\Gamma_0(p^m)$ consists of matrices $g \in G(\mathbb{Z}_p)$ such that $g \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod p^m$ with size g blocks. Note that both $\Gamma_0(p^m)$ and $\Gamma_0(p^{m+1})$ contain $\Gamma(p^{m+1}) = \operatorname{Ker}(G(\mathbb{Z}_p) \to G(\mathbb{Z}_p/p^{m+1}))$. Let U and U' be the images of $\Gamma_0(p^m)$ and $\Gamma_0(p^{m+1})$ in $G(\mathbb{Z}_p/p^{m+1})$, respectively. Then we shall compute $[\mathsf{U} : \mathsf{U}']$.

For any ring R and any $A, B, C, D \in \mathrm{M}_g(R)$, we have $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G(R)$ if and only if there exists (unique) $\nu \in R^\times$ such that

$$\begin{pmatrix} A^{\mathrm{t}} & C^{\mathrm{t}} \\ B^{\mathrm{t}} & D^{\mathrm{t}} \end{pmatrix} \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \nu \cdot \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

Indeed, the above implies that $\det\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)^2 = \nu^{2g} \in R^\times$, and hence $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GL}_g(R)$. The above equality is equivalent to

$$A^{\mathrm{t}} C = C^{\mathrm{t}} A, \quad A^{\mathrm{t}} D - C^{\mathrm{t}} B = \nu \cdot I_g.$$

Let V' consist of matrices of the form $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ in $G(\mathbb{Z}_p/p^{m+1})$. Then by definition we have $\mathsf{U}' \subset \mathsf{V}'$. We claim that $\mathsf{U}' = \mathsf{V}'$. Indeed, if $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in G(\mathbb{Z}_p/p^{m+1})$, then we have $A^{\mathrm{t}} D = \nu \cdot I_g$ for $\nu \in (\mathbb{Z}_p/p^{m+1})^\times$. In particular, A and D are invertible. Choose arbitrary lifts $\tilde{A} \in \mathrm{M}_g(\mathbb{Z}_p)$ and $\tilde{\nu} \in \mathbb{Z}_p$ of A and ν. Then $\tilde{A} \in \mathrm{GL}_g(\mathbb{Z}_p)$, and $\tilde{\nu} \in \mathbb{Z}_p^\times$. Define $\tilde{D} = \tilde{\nu} \cdot (\tilde{A}^{\mathrm{t}})^{-1}$. Then $\left(\begin{smallmatrix} \tilde{A} & \tilde{B} \\ 0 & \tilde{D} \end{smallmatrix}\right) \in G(\mathbb{Z}_p)$ is a lift of $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in G(\mathbb{Z}_p/p^{m+1})$. We conclude that $\mathsf{U}' = \mathsf{V}'$.

We now calculate the cardinality of U' by counting the choices of A, B, D, ν. We can choose ν to be an arbitrary element of $(\mathbb{Z}_p/p^{m+1})^\times$, and choose A to be an arbitrary element of $\mathrm{GL}_g(\mathbb{Z}_p/p^{m+1})$. Then D is determined as $\nu \cdot (A^{\mathrm{t}})^{-1}$. In addition, we can choose B to be an arbitrary element of $\mathrm{M}_g(\mathbb{Z}_p/p^{m+1})$. Write R for $\mathbb{Z}_p/p^{m+1}$. We conclude that

$$\#\mathsf{U}' = \#R^\times \cdot \#\mathrm{GL}_g(R) \cdot \#\mathrm{M}_g(R).$$

Now let V consist of matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G(\mathbb{Z}_p/p^{m+1})$ such that $C \equiv 0 \bmod p^m$. Clearly we have $\mathsf{U} \subset \mathsf{V}$. We claim that they are equal. Indeed, let $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathsf{V}$. Then by the equation $A^{\mathrm{t}} D = C^{\mathrm{t}} B + \nu I_g$ we have $A, D \in \mathrm{GL}_g(\mathbb{Z}_p/p^{m+1})$, because the right hand side is congruent to νI_g modulo p^m and hence invertible over $\mathbb{Z}_p/p^{m+1}$. Find an arbitrary lift $\tilde{A} \in \mathrm{GL}_g(\mathbb{Z}_p)$ of A, and find an arbitrary symmetric matrix $E \in \mathrm{M}_g(\mathbb{Z}_p)$ lifting the symmetric matrix $A^{\mathrm{t}}C \in \mathrm{M}_g(\mathbb{Z}_p/p^{m+1})$. Set $\tilde{C} = (\tilde{A}^{\mathrm{t}})^{-1}E$. Then we have $\tilde{A}^{\mathrm{t}}\tilde{C} = \tilde{C}^{\mathrm{t}}\tilde{A}$, and $\tilde{C}$ is a lift of C. In particular, $\tilde{C} \equiv 0 \bmod p^m$. Fix a lift $\tilde{\nu} \in \mathbb{Z}_p^\times$ of ν, fix a lift $\tilde{B} \in \mathrm{M}_g(\mathbb{Z}_p)$ of B, and set $\tilde{D} = (\tilde{A}^{\mathrm{t}})^{-1}(\tilde{C}^{\mathrm{t}}\tilde{B} + \tilde{\nu}I_g)$. Then $\tilde{D}$ lifts D. We have thus found a lift $\left(\begin{smallmatrix}\tilde{A} & \tilde{B}\\ \tilde{C} & \tilde{D}\end{smallmatrix}\right) \in \Gamma_0(p^m) \subset G(\mathbb{Z}_p)$ of $\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right) \in \mathsf{V}$. We conclude that $\mathsf{U} = \mathsf{V}$.

Similarly as before, we calculate the cardinality of U by counting the choices of A, B, C, D, ν. The condition $C \equiv 0 \bmod p^m$ implies that $C = p^m C_0$ for some $C_0 \in \mathrm{M}_g(\mathbb{Z}_p/p^{m+1})$. Moreover, only the image of C_0 in $\mathrm{M}_g(\mathbb{F}_p)$ is well defined, and this element and C determine each other. We shall thus think of C_0 as in $\mathrm{M}_g(\mathbb{F}_p)$. We have seen that A and D are invertible. Thus whenever A, B, C, ν are fixed with A invertible, D is uniquely determined by the second equation. Now B can be arbitrarily chosen from $\mathrm{M}_g(\mathbb{Z}/p^{m+1})$. From the equation $A^{\mathrm{t}}C = C^{\mathrm{t}}A$, we see that after fixing A, C can be determined by the symmetric matrix $A^{\mathrm{t}}C$ as A^{t} is always invertible. Now this matrix is symmetric if and only if the image of $A^{\mathrm{t}}C_0$ in $\mathrm{M}_g(\mathbb{F}_p)$ is symmetric, and the latter matrix and C_0 determine each other. Hence the choice of C amounts to the choice of a symmetric matrix over $\mathbb{F}_p$. Thus,

$$\#\mathsf{U} = \#R^\times \cdot \#\mathrm{GL}_g(R) \cdot \#\mathrm{M}_g(R) \cdot \#\mathrm{Sym}_g(\mathbb{F}_p),$$

where $R = \mathbb{Z}_p/p^{m+1}$ and $\mathrm{Sym}_g(\mathbb{F}_p)$ is the set of symmetric matrices over $\mathbb{F}_p$.

To conclude, we have

$$[\Gamma_0(p^m) : \Gamma_0(p^{m+1})] = [\mathsf{U} : \mathsf{U}'] = \#\mathrm{Sym}_g(\mathbb{F}_p) = p^{g(g+1)/2}.$$

□

Summary of Proposition 3.9.21. Fix $0 \leqslant \varepsilon < 1/2$ and $p^\varepsilon \in \mathbb{Z}_p^{\mathrm{cycl}}$. Then set $\mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon) \subset \mathcal{X}_{\Gamma_{\mathrm{s}}(p)}$ to be the inverse image of $\mathcal{X}(\varepsilon) \subset \mathcal{X}$ along $\mathcal{X}_{\Gamma_{\mathrm{s}}(p)} \to \mathcal{X}$. We have the open embedding $\iota_1 \colon \mathcal{X}(p^{-1}\varepsilon) \hookrightarrow \mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon)$, $A \mapsto (A/C_1, A[p]/C_1)$, whose image is the anti-canonical locus $\mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon)_{\mathrm{a}}$, a clopen in $\mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon)$. More generally, for each $m \geqslant 1$, we can define the clopen anti-canonical locus $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}} \subset \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}$ to be the inverse image of $\mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon)_{\mathrm{a}}$ along the natural map $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)} \to \mathcal{X}_{\Gamma_{\mathrm{s}}(p)}$. Then we get two towers

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathcal{X}(p^{-m}\varepsilon) & \xrightarrow[\sim]{\iota_m} & \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}} \\
{\scriptstyle (\widetilde{F})^{\mathrm{ad}}_\eta}\downarrow & & \downarrow \\
\mathcal{X}(p^{-m+1}\varepsilon) & \xrightarrow[\sim]{\iota_{m-1}} & \mathcal{X}_{\Gamma_{\mathrm{s}}(p^{m-1})}(\varepsilon)_{\mathrm{a}} \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathcal{X}(p^{-1}\varepsilon) & \xrightarrow[\sim]{\iota_1} & \mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon)_{\mathrm{a}} \\
{\scriptstyle (\widetilde{F})^{\mathrm{ad}}_\eta}\downarrow & & \downarrow \\
\mathcal{X}(\varepsilon) & \xrightarrow{\mathrm{id}} & \mathcal{X}(\varepsilon).
\end{array}$$

Corollary 3.9.23. *There exists a unique perfectoid space $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}}$ over $\mathbb{Q}_p^{\mathrm{cycl}}$ such that*

$$\mathcal{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}} \sim \varprojlim_m \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}}$$

in the sense of Definition 3.1.10.

Proof. We replace the tower in question by the isomorphic tower $(\mathcal{X}(p^{-m}\varepsilon))_m$. By construction we regard the formal scheme $\mathfrak{X}(p^{-m}\varepsilon)$ as the integral model for $\mathcal{X}(p^{-m}\varepsilon)$. The transition morphisms

$$\widetilde{F} \colon \mathfrak{X}(p^{-m-1}\varepsilon) \longrightarrow \mathfrak{X}(p^{-m}\varepsilon)$$

are finite. So we can take inverse limit in the category of formal schemes to get

$$\mathfrak{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}} \coloneqq \varprojlim_{m,\tilde{F}} \mathfrak{X}(p^{-m}\varepsilon).$$

Note that if for a fixed integer m_0 we have an open $\operatorname{Spf} R_{m_0} \subset \mathfrak{X}(p^{-m_0}\varepsilon)$, then the inverse image of it in $\mathfrak{X}(p^{-m}\varepsilon)$ for $m \geqslant m_0$ (resp. in $\mathfrak{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}}$) is also affine, say $\operatorname{Spf} R_m$ (resp. $\operatorname{Spf} R_\infty$). Here R_∞ is the p-adic completion of $\varinjlim_m R_m$, where the injective limit is taken over transition maps $R_m \to R_{m+1}$ that lift relative Frobenii modulo the pseudo-uniformizer $p^{1-\varepsilon}$ of $\mathbb{Z}_p^{\mathrm{cycl}}$. Thus R_∞ is a perfectoid almost $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. In particular, $R_\infty[1/p]$ is a perfectoid algebra over $\mathbb{Q}_p^{\mathrm{cycl}}$, with the topology such that $R_\infty \subset R_\infty[1/p]$ is open and p-adic. Using this, we get the desired perfectoid space by taking the adic generic fiber

$$\mathcal{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}} := (\mathfrak{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}})^{\mathrm{ad}}_{\eta}.$$

The uniqueness was discussed in Proposition 3.1.12. □

3.9.3. *Tilting perfectoid anti-canonical neighborhood.* In this course we have not yet defined the tilt of a perfectoid space, but for now we describe the tilt of the perfectoid space $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}}$ without proof, to get some impression. Consider the scheme $X_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$, and base change it to $\operatorname{Spec} \mathcal{O}_{\mathbb{K}}$ along the natural map $\mathbb{Z}_p \to \mathbb{F}_p \to \mathcal{O}_{\mathbb{K}}$. Here $\mathbb{K} = \mathbb{F}_p(\!(t^{1/(p-1)p^\infty})\!)$, which is the fraction field of

$$\mathcal{O}_{\mathbb{K}} = \mathbb{F}_p[\![t^{1/(p-1)p^\infty}]\!] = \left(\varinjlim_n \mathbb{F}_p[t^{1/(p-1)p^n}]\right)^\wedge_t.$$

We then formally t-adically complete the scheme $X_{\mathcal{O}_{\mathbb{K}}}$, to get a formal scheme $\mathfrak{X}'$ over $\operatorname{Spf} \mathcal{O}_{\mathbb{K}}$. Hence we have an adic generic fiber $\mathcal{X}'$, as an adic space over $\operatorname{Spa} \mathbb{K}$. There exists a unique perfectoid space

$$\mathcal{X}'^{\mathrm{perf}} \sim \varprojlim_{\Phi} \mathcal{X}' = \varprojlim(\cdots \xrightarrow{\Phi} \mathcal{X}' \xrightarrow{\Phi} \mathcal{X}'),$$

where Φ is the relative Frobenius over $\mathbb{K}$. Such $\mathcal{X}'^{\mathrm{perf}}$ can be easily constructed as follows. Each local chart $\operatorname{Spa}(R, R^+) \subset \mathcal{X}'$ corresponds to a local chart $\operatorname{Spa}(R^{\mathrm{perf}}, R^{+,\mathrm{perf}})$ in $\mathcal{X}'^{\mathrm{perf}}$. Here

$$R^{+,\mathrm{perf}} := \varinjlim_{\Phi} R^+, \quad R^{\mathrm{perf}} = R^{+,\mathrm{perf}}[1/p].$$

The perfectoid space $\mathcal{X}'^{\mathrm{perf}}$ is over the perfectoid field $\mathbb{K}$, and the latter is the tilt of $\mathbb{Q}_p^{\mathrm{cycl}}$.

Remark 3.9.24. Similarly, one has the following perfect flat t-adic formal scheme over $\operatorname{Spf} \mathcal{O}_{\mathbb{K}}$:

$$\mathfrak{X}'^{\mathrm{perf}} = \varprojlim_{\Phi} \mathfrak{X}'.$$

Fact 3.9.25. The tilt of $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}}$ is the open subspace in $\mathcal{X}'^{\mathrm{perf}}$ cut out by the condition $|\mathrm{Ha}| \geqslant |t^\varepsilon|$. That is, given a local chart $\mathcal{X}' \supset \operatorname{Spa}(R, R^+)$, we look at the inverse image of the open locus $\{v \in \operatorname{Spa}(R, R^+) \colon v(\mathrm{Ha}) \geqslant v(t^\varepsilon)\}$ in $\mathcal{X}'^{\mathrm{perf}}$. Here Ha is defined on $\mathcal{X}'$ because it is in characteristic p.

Notation 3.9.26. For $m \geqslant 1$, let $\mathcal{X}_{\Gamma(p^m)}(\varepsilon)$ (resp. $\mathcal{X}_{\Gamma(p^m)}(\varepsilon)_{\mathrm{a}}$) be the inverse image of $\mathcal{X}(\varepsilon)$ (resp. $\mathcal{X}_{\Gamma_{\mathrm{s}}(p)}(\varepsilon)_{\mathrm{a}}$) in $\mathcal{X}_{\Gamma(p^m)}$.

Now we obtain a new anti-canonical tower of level $\Gamma(p^m)$, read as

$$(\cdots \longrightarrow \mathcal{X}_{\Gamma(p^m)}(\varepsilon)_{\mathrm{a}} \longrightarrow \mathcal{X}_{\Gamma(p^{m-1})}(\varepsilon)_{\mathrm{a}} \longrightarrow \cdots \longrightarrow \mathcal{X}_{\Gamma(p)}(\varepsilon)_{\mathrm{a}}).$$

Note that as subgroups in $G(\mathbb{Z}_p)$ the level $\Gamma(p^m)$ tends to be trivial as $m \to \infty$, whereas $\Gamma_{\mathrm{s}}(p^m)$ does not. So the towers should not be expected to be isomorphic, i.e., we expect

$$\varprojlim_m \mathcal{X}_{\Gamma(p^m)}(\varepsilon)_{\mathrm{a}} \ncong \varprojlim_m \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}}.$$

Nevertheless, the perfectoidness for ε-neighborhood is preserved when we pass to the full level $\Gamma(p^m)$.

Corollary 3.9.27. *Fix $0 \leqslant \varepsilon < 1/2$. There exists a unique perfectoid space*

$$\mathcal{X}_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}} \sim \varprojlim_m \mathcal{X}_{\Gamma(p^m)}(\varepsilon)_{\mathrm{a}}.$$

Proof. The map

$$\mathcal{X}_{\Gamma(p^m)}(\varepsilon)_{\mathrm{a}} \longrightarrow \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}}$$

is finite étale. Pulling back along $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}} \to \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}}$, we obtain a finite étale map

$$\mathcal{Y}_m \longrightarrow \mathcal{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}},$$

where the right hand side is perfectoid by Corollary 3.9.23. The essence of proof lies in the almost purity result by Faltings and Scholze [Sch12, Theorem 7.9(iii)].

> *Almost purity.* Let K be any perfectoid field and R be any perfectoid K-algebra. Let R' be a finite étale R-algebra. Then the subring R'^0 of power-bounded elements in R' is a finite étale almost R^0-algebra, and R' is also perfectoid.

Applying this, we see $\mathcal{Y}_m$ is a perfectoid space. Moreover, applying this again, we see that if in $\mathcal{Y}_m$ we have an affinoid open $\mathrm{Spa}(R, R^+)$ with R perfectoid, then the inverse image of it in any $\mathcal{Y}_{m'}$ with $m' \geqslant m$ is of the form $\mathrm{Spa}(R', R'^+)$ with R' perfectoid. (The inverse image is always affinoid, since the transition map is finite.) Using this observation and the fact that a direct limit of perfectoid algebras is perfectoid, we obtain a perfectoid space

$$\mathcal{X}_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}} \sim \varprojlim_m \mathcal{Y}_m.$$

□

3.10. **Baily–Borel compactification.** Recall that our ultimate goal is to show that

$$\varprojlim_m \mathcal{X}_{\Gamma(p^m)}$$

is perfectoid. So we need to get beyond the ε-neighborhood and the anti-canonical locus. Moreover, we want to show that

$$\varprojlim_m \mathcal{X}^*_{\Gamma(p^m)}$$

is perfectoid, where $*$ denotes the minimal compactification (aka. Baily–Borel compactification). As the first step toward these goals, we explain how to extend perfectoidness of $\varprojlim_m \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}}$ to the analogue of it for the minimal compactification.

Fix an integer $g \geqslant 1$ and set $G = \mathrm{GSp}_{2g}$ as before. For any (sufficiently small) compact open subgroup $\mathsf{K} \subset G(\mathbb{A}_f)$, we get a Siegel modular variety X_{K} over $\mathbb{Q}$. It is smooth, quasi-projective, but not proper. We have a canonical minimal compactification $X_{\mathsf{K}} \hookrightarrow X^*_{\mathsf{K}}$, where X^*_{K} is a projective normal variety over $\mathbb{Q}$. However, X^*_{K} is *not smooth* as long as $g \geqslant 2$. The boundary $X^*_{\mathsf{K}} - X_{\mathsf{K}}$ has a stratification, where each stratum is isomorphic to a certain Siegel modular variety for $\mathrm{GSp}_{2g'}$ with $g' < g$. Each $g' < g$ will show up, and in general for more than one strata. Thus the boundary has dimension $(g-1)g/2$ and codimension g.

From now on we assume $g \geqslant 2$. All conclusions are valid when $g = 1$ but need different approaches to prove. When $\mathsf{K} = \mathsf{K}^p\mathsf{K}_p$ with $\mathsf{K}_p = G(\mathbb{Z}_p)$, i.e. when there is no level at p, X_{K} has an integral model $X_{\mathbb{Z}_{(p)}}$ over $\mathrm{Spec}\,\mathbb{Z}_{(p)}$, and X^*_{K} has a canonical integral model $X^*_{\mathbb{Z}_{(p)}}$ that is still normal and projective over $\mathbb{Z}_{(p)}$. The boundary strata of $X^*_{\mathbb{Z}_{(p)}}$ are canonical integral models of smaller Siegel modular varieties with no level at p. Lect.16, Nov 30

Notation 3.10.1. Let $\mathfrak{X}^*$ be the formal p-adic completion of $X^*_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p^{\mathrm{cycl}}$. Take $\mathcal{X}^* = (\mathfrak{X}^*)^{\mathrm{ad}}_\eta$ be the generic adic fiber. For any compact open subgroup $\mathsf{K}_p \subset G(\mathbb{Z}_p)$, let $\mathcal{X}^*_{\mathsf{K}_p} \subset (X^*_{\mathsf{K}_p, \mathbb{Q}_p^{\mathrm{cycl}}})^{\mathrm{ad}}$ be the inverse image of $\mathcal{X}^* \subset (X^*_{\mathbb{Q}_p^{\mathrm{cycl}}})^{\mathrm{ad}}$.

In fact, by properness, we have $\mathcal{X}^* = (X^*_{\mathbb{Q}_p^{\mathrm{cycl}}})^{\mathrm{ad}}$ and $\mathcal{X}^*_{\mathsf{K}_p} = (X^*_{\mathsf{K}_p, \mathbb{Q}_p^{\mathrm{cycl}}})^{\mathrm{ad}}$.

Fact 3.10.2.
(1) For the universal abelian variety A^{univ} on $X_{\mathbb{Z}_p}$, the line bundle $\omega_{A^{\mathrm{univ}}/X_{\mathbb{Z}_p}}$ extends canonically to an *ample* line bundle ω on $X^*_{\mathbb{Z}_p}$.
(2) The Hasse invariant $\mathrm{Ha} \in \Gamma(X_{\mathbb{F}_p}, \omega^{\otimes(p-1)})$ extends uniquely to a section $\mathrm{Ha} \in \Gamma(X^*_{\mathbb{F}_p}, \omega^{\otimes(p-1)})$.

(3) For $0 \leqslant \varepsilon < 1/2$ such that p^ε makes sense in $\mathbb{Z}_p^{\mathrm{cycl}}$, the morphism $\mathfrak{X}(\varepsilon) \to \mathfrak{X}$ extends to a morphism $\mathfrak{X}^*(\varepsilon) \to \mathfrak{X}^*$, which is still locally of form $\operatorname{Spf} R\langle u\rangle/(u \cdot \widetilde{\mathrm{Ha}} - p^\varepsilon) \to \operatorname{Spf} R$, where $\widetilde{\mathrm{Ha}} \in \omega^{\otimes(p-1)}$ is a lift of $\mathrm{Ha} \in \omega^{\otimes(p-1)}$ modulo p.

Notation 3.10.3. Set $\mathcal{X}^*(\varepsilon) \coloneqq (\mathfrak{X}^*(\varepsilon))^{\mathrm{ad}}_\eta$, which admits an open embedding $\mathcal{X}^*(\varepsilon) \hookrightarrow \mathcal{X}^*$.

3.10.1. *Hartog's extension principle.* We will apply repeatedly Hartog's extension principle when work with canonical Frobenius lifts on minimal compactifications. Loosely, Hartog's result from multi-variable complex analysis dictates that, in nice situations, one can extend functions through some closed subset of codimension $\geqslant 2$. We need algebro-geometric versions of this principle.

Proposition 3.10.4 (Hartog's extension principle, classical version)**.** *Let R be a normal ring (i.e., for any $\mathfrak{p} \in \operatorname{Spec} R$ the localization $R_\mathfrak{p}$ is an integrally closed domain). Assume R is noetherian. Let $Z \subset \operatorname{Spec} R$ be a closed subscheme of codimension at least 2 everywhere, i.e. all $\mathfrak{p} \in Z$ has height $\geqslant 2$. Then*

$$H^0(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) \xrightarrow{\sim} H^0((\operatorname{Spec} R)\backslash Z, \mathcal{O}_{\operatorname{Spec} R})$$

is an isomorphism.

Proposition 3.10.5 (Hartog's extension principle, more technical version)**.** *Let R be a topologically finitely generated, flat, and p-adically complete $\mathbb{Z}_p$-algebra, such that $\bar{R} = R/p$ is normal. Fix $f \in R$ such that its reduction $\bar{f} \in \bar{R} = R/p$ is not a zero-divisor. Take some $0 < \varepsilon \leqslant 1$ such that $p^\varepsilon \in \mathbb{Z}_p^{\mathrm{cycl}}$ makes sense.*[5] *Consider the algebra*

$$S = (R \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})\langle u\rangle/(u \cdot f - p^\varepsilon).$$

Then S is p-adically complete and flat over $\mathbb{Z}_p^{\mathrm{cycl}}$. Fix a closed subscheme $Y \subset \operatorname{Spec} \bar{R}$ of codimension $\geqslant 2$ everywhere, and let Z be its inverse image in $\operatorname{Spf} S$. Then for $U \coloneqq |\operatorname{Spf} S|\backslash Z$,

$$S = H^0(\operatorname{Spf} S, \mathcal{O}_{\operatorname{Spf} S}) \xrightarrow{\sim} H^0(U, \mathcal{O}_{\operatorname{Spf} S}).$$

From now on if the context is clear we will write $H^0(X) \coloneqq H^0(X, \mathcal{O}_X)$ instead. Here comes the explanation for applications of Hartog's extension principle.

- Use Proposition 3.10.4 to extend functions on $X_{\mathbb{Z}_p}$ or on $\mathfrak{X}$ to those on $X^*_{\mathbb{Z}_p}$ or $\mathfrak{X}^*$, respectively. For instance, it follows that, on the special fiber, $\mathrm{Ha} \in \Gamma(X_{\mathbb{F}_p}, \omega^{p-1})$ indeed extends from $X_{\mathbb{F}_p}$ to $X^*_{\mathbb{F}_p}$.
- Use Proposition 3.10.5 to extend functions on $\mathfrak{X}(\varepsilon)$ to $\mathfrak{X}^*(\varepsilon)$.

By Proposition 3.10.5, when $m \geqslant 0$,

$$\begin{tikzcd}
\mathfrak{X}(p^{-m-1}\varepsilon) \arrow[r, hook] \arrow[d, "\widetilde{F}"'] & \mathfrak{X}^*(p^{-m-1}\varepsilon) \arrow[d, dashed, "\exists! \widetilde{F}"] \\
\mathfrak{X}(p^{-m}\varepsilon) \arrow[r, hook] & \mathfrak{X}^*(p^{-m}\varepsilon).
\end{tikzcd}$$

Namely, $\widetilde{F}$ has a unique extension to the minimal compactification. Moreover, the extended $\widetilde{F}$ (i.e., the right vertical arrow) still lifts the relative Frobenius modulo $p^{1-\varepsilon}$. When $m \geqslant 1$, the open embeddings ι_m also extend

$$\begin{tikzcd}
\mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)} \arrow[r, phantom, "\subset"] & \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)} \\
\mathcal{X}(p^{-m}\varepsilon) \arrow[u, hook, "\iota_m"] \arrow[r, phantom, "\subset"] & \mathcal{X}^*(p^{-m}\varepsilon) \arrow[u, hook, dashed, "\iota_m"']
\end{tikzcd}$$

and we denote the extension still by ι_m.

Notation 3.10.6. Denote the image of $\iota_m \colon \mathcal{X}^*(p^{-m}\varepsilon) \to \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}$ by $\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}}$.

Similarly as in Proposition 3.9.21, we obtain the Cartesian diagram

[5] Note that the proof of [Sch15, Lemma III.2.10] makes sense only when $\varepsilon \neq 0$.

$$\begin{tikzcd} \mathcal{X}^*(p^{-m-1}\varepsilon) \arrow[r, hook, "\iota_{m+1}"] \arrow[d, "\widetilde{F}"'] & \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^{m+1})} \arrow[d, "\pi"] \\ \mathcal{X}^*(p^{-m}\varepsilon) \arrow[r, hook, "\iota_m"] & \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}. \end{tikzcd}$$

Extending to the minimal compactification, we see the following new result which we haven't seen before.

Lemma 3.10.7. *Fix $0 \leqslant \varepsilon < 1/2$. Then for $m \gg 0$ the adic space $\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}}$ is affinoid.*

Proof. Since ω is ample, there exists $m \gg 0$ such that

$$H^i(X^*_{\mathbb{Z}_p}, \omega^{\otimes(p-1)p^m}) = 0, \quad \forall i \geqslant 1.$$

Then we get a global lifting ξ of Ha^{p^m} (caution that Ha itself does not lift), where ξ is a global section of the ample line bundle $\omega^{\otimes(p-1)p^m}$. Consider

$$\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}} \xrightarrow[\sim]{\iota_m^{-1}} \mathcal{X}^*(p^{-m}\varepsilon) \subset \mathcal{X}^*,$$

where $\mathcal{X}^*(p^{-m}\varepsilon)$ is the open locus in $\mathcal{X}^*$ cut out by $|\xi| \geqslant |p^\varepsilon|$.

We first consider the following. Write $K = \mathbb{Q}_p^{\mathrm{cycl}}$. The open sub-adic space V defined by $|x_0| \geqslant |p^\varepsilon|$ in $\mathbb{P}^{n,\mathrm{ad}}_K$ is affinoid, where x_0 is from the coordinate $(x_0 : x_1 : \cdots : x_n)$ on $\mathbb{P}^n_K$. Indeed, $V \subset \mathbb{A}^{n,\mathrm{ad}}_K$, where $\mathbb{A}^n_K = \{x_0 \neq 0\} \subset \mathbb{P}^n_K$. Moreover, if we identify $\mathbb{A}^n_K$ with $\operatorname{Spec} K[t_1, \cdots, t_n]$ with $t_i = x_i/x_0$, then $V \subset \mathbb{A}^{n,\mathrm{ad}}_K$ is the closed adic ball defined by $|t_i| \leqslant 1/|p^\varepsilon|$.

In our case, $\mathcal{X}^*(p^{-m}\varepsilon)$ is the intersection of the Zariski closed $\mathcal{X}^* \subset \mathbb{P}^{n,\mathrm{ad}}_K$ with an open in $\mathbb{P}^{n,\mathrm{ad}}_K$ of the above form. But we have seen that the latter open is affinoid from the observation above. It easily follows that $\mathcal{X}^*(p^{-m}\varepsilon)$ is affinoid. ☐

Remark 3.10.8. Lemma 3.10.7 is not true without the compactification.

Using the same argument as before, we get:

Corollary 3.10.9. *There exists a unique perfectoid space $\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}}$ over $\mathbb{Q}_p^{\mathrm{cycl}}$, such that*

$$\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}} \sim \varprojlim_m \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}}.$$

The tilt of $\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}}$ over $\mathbb{Q}_p^{\mathrm{cycl},\flat} = \mathbb{K} = \mathbb{F}_p(\!(t^{1/p^\infty(p-1)})\!)$ is the locus $|\mathrm{Ha}| \geqslant |t^\varepsilon|$ in $(\mathcal{X}'^*)^{\mathrm{perf}}$, where $(\mathcal{X}'^*)^{\mathrm{perf}}$ is the analogue of $\mathcal{X}'^{\mathrm{perf}}$ as before. That is, we first consider

$$(X^*_{\mathbb{Z}_p} \otimes_{\mathbb{F}_p} \mathcal{O}_{\mathbb{K}})^\wedge_t = \mathfrak{X}'^*.$$

Write $\mathcal{X}'^* = (\mathfrak{X}'^*)^{\mathrm{ad}}_\eta$ over $\operatorname{Spa}\mathbb{K}$. Then $(\mathcal{X}'^*)^{\mathrm{perf}}$ is the perfectoid space over $\mathbb{K}$ such that

$$(\mathcal{X}'^*)^{\mathrm{perf}} \sim \varprojlim_\Phi \mathcal{X}'^*.$$

Also, Ha extends from $\mathcal{X}'$ to $\mathcal{X}'^*$.

Recall that we have used Almost Purity together with the fact that $\mathcal{X}_{\Gamma(p^m)} \to \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}$ is finite étale, in order to prove that there is a unique perfectoid space

$$\mathcal{X}_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}} \sim \varprojlim_m \mathcal{X}_{\Gamma(p^m)}(\varepsilon)_{\mathrm{a}}.$$

However, while passing to compactifications, the natural map

$$X^*_{\Gamma(p^m)} \longrightarrow X^*_{\Gamma_{\mathrm{s}}(p^m)}$$

has ramifications at boundary and hence no longer finite étale, unless X is a modular curve. Because of this new difficulty, we cannot directly prove that $\varprojlim_m \mathcal{X}^*_{\Gamma(p^m)}(\varepsilon)_{\mathrm{a}}$ is perfectoid by the same argument.

Proof of Proposition 3.10.4. Recall the following relationship between local cohomology and depth from [Gro68, Exposé III, Prop. 3.3].

Fact. Let X be a locally noetherian scheme and $Z \subset X$ a closed subscheme. Let $\mathscr{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Let $n \geqslant 1$ be an integer. Then the following are equivalent:

(i) For any open subscheme $V \subset X$, the map

$$H^i(V, \mathscr{F}) \longrightarrow H^i(V \backslash Z, \mathscr{F})$$

is bijective for $i \leqslant n-2$ and injective for $i = n-1$.

(ii) For any open subscheme $V \subset X$, the local cohomology

$$H^i_{V \cap Z}(V, \mathscr{F}) = 0$$

for all $i \leqslant n-1$.

(iii) For any $x \in Z$ the depth of $\mathscr{F}_x$ as an $\mathcal{O}_{X,x}$-module is at least n.

Recall that the depth of a finite generated module M over a noetherian local ring $(R, \mathfrak{m}_R)$ is defined as the maximal length of an M-regular sequence in $\mathfrak{m}_R$. On the other hand we have another fact.

Fact (Serre's criterion). A noetherian ring R is normal if and only if $R_{\mathfrak{p}}$ is regular for any $\mathfrak{p}$ of height $\leqslant 1$, and simultaneously $R_{\mathfrak{p}}$ has depth $\geqslant 2$ for any $\mathfrak{p}$ of height $\geqslant 2$.

Granting these facts, we apply the first with $n = 2$ and $\mathscr{F} = \mathcal{O}_X$. In order to verify (iii) we need that for any $x \in Z$ we have $\operatorname{depth} \mathcal{O}_{X,x} \geqslant 2$. But by assumption R is normal and each point $x \in Z$ has height $\geqslant 2$. So the desired result follows from the second fact. □

Proof of Proposition 3.10.5. Fix $0 < \varepsilon \leqslant 1$.

Step I. We first show that $S \to H^0(U, \mathcal{O}_{\mathrm{Spf}\, S})$ is injective. Since the source is p-adically separated and the target is flat over $\mathbb{Z}_p^{\mathrm{cycl}}$, we reduce to checking the injectivity of $H^0(\operatorname{Spec} S_\varepsilon) \to H^0((\operatorname{Spec} S_\varepsilon) \backslash Z_\varepsilon)$, where $S_\varepsilon = S \otimes_{\mathbb{Z}_p^{\mathrm{cycl}}} (\mathbb{Z}_p^{\mathrm{cycl}}/p^\varepsilon)$ and Z_ε is the inverse image of Y in $\operatorname{Spec} S_\varepsilon$. By the definition of S we have

$$S_\varepsilon = (R \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})\langle u \rangle / (uf, p^\varepsilon) = R_\varepsilon[u]/(uf_\varepsilon),$$

where $R_\varepsilon \coloneqq R \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p^{\mathrm{cycl}}/p^\varepsilon)$. Consider the decomposition

$$\begin{aligned} \operatorname{Spec} S_\varepsilon = N \cup W &\coloneqq \operatorname{Spec} R_\varepsilon[u]/(u) \cup \operatorname{Spec} R_\varepsilon[u]/(f_\varepsilon) \\ &\cong \operatorname{Spec} R_\varepsilon \cup \operatorname{Spec}(R_\varepsilon/f_\varepsilon)[u] \\ &\cong \operatorname{Spec} R_\varepsilon \cup (\operatorname{Spec}(\bar{R}/\bar{f}) \times_{\mathbb{F}_p} \mathbb{A}^1_{\mathbb{Z}_p^{\mathrm{cycl}}/p^\varepsilon}). \end{aligned}$$

Take $V \coloneqq \operatorname{Spec} \bar{R}/\bar{f} \subset \operatorname{Spec} \bar{R}$. Also take $V_\varepsilon \coloneqq \operatorname{Spec}(\bar{R}/\bar{f} \otimes_{\mathbb{F}_p} \mathbb{Z}_p^{\mathrm{cycl}}/p^\varepsilon)$ that is closed inside $N \cong \operatorname{Spec} R_\varepsilon$ defined by $f_\varepsilon = 0$. We have

$$W \coloneqq V \times_{\mathbb{F}_p} \mathbb{A}^1_{\mathbb{Z}_p^{\mathrm{cycl}}/p^\varepsilon} = V_\varepsilon \times_{\mathbb{Z}_p^{\mathrm{cycl}}/p^\varepsilon} \mathbb{A}^1_{\mathbb{Z}_p^{\mathrm{cycl}}/p^\varepsilon}.$$

Consider the closed embedding $V_\varepsilon \hookrightarrow W, x \mapsto (x, 0)$. Then $W \cap N = V_\varepsilon$. We then describe $H^0(\operatorname{Spec} S_\varepsilon)$ and $H^0((\operatorname{Spec} S_\varepsilon) \backslash Z_\varepsilon)$ as follows.

- Each section in $H^0(\operatorname{Spec} S_\varepsilon)$ is a pair (g_1, g_2) such that $g_1 \in H^0(N) \cong R_\varepsilon$, $g_2 \in H^0(W) \cong H^0(V_\varepsilon)[u]$, satisfying $g_1|_{N \cap W} = g_2|_{N \cap W}$, i.e., $(g_1 \bmod f) = (g_2 \bmod u) \in H^0(V_\varepsilon)$.
- Each section in $H^0((\operatorname{Spec} S_\varepsilon) \backslash Z_\varepsilon)$ is a pair (g_1, g_2) such that $g_1 \in H^0(N \backslash Z_\varepsilon)$, $g_2 \in H^0(W \backslash Z_\varepsilon) = H^0((V \backslash Y) \times_{\mathbb{F}_p} \mathbb{A}^1_{\mathbb{Z}_p^{\mathrm{cycl}}/p^\varepsilon})$, satisfying $g_1 = g_2$ on $(N \cap W) \backslash Z_\varepsilon$.

We have $H^0(N \backslash Z_\varepsilon) \cong H^0(N)$ by Proposition 3.10.4 applied to $Y \subset \operatorname{Spec} \bar{R}$. Indeed, the two sides are the base change $(\cdot) \otimes_{\mathbb{F}_p} \mathbb{Z}_p^{\mathrm{cycl}}/p^\varepsilon$ of $H^0((\operatorname{Spec} \bar{R}) \backslash Y)$ and $H^0(\operatorname{Spec} \bar{R})$ respectively. The proof is reduced to the injectivity of

$$H^0(V) \otimes_{\mathbb{F}_p} (\mathbb{Z}_p^{\mathrm{cycl}}/p^\varepsilon)[u] = H^0(W) \longhookrightarrow H^0(W \backslash Z) = H^0(V \backslash Y) \otimes_{\mathbb{F}_p} (\mathbb{Z}_p^{\mathrm{cycl}}/p^\varepsilon)[u].$$

It then suffices to prove the injectivity of

$$H^0(V) \longhookrightarrow H^0(V \backslash Y)$$

where both V and $V\backslash Y$ are $\mathbb{F}_p$-schemes. Since $V \subset \operatorname{Spec} \bar{R}$ is defined by the vanishing of the non-zero divisor $\bar{f} \in \bar{R}$, we have

$$\operatorname{depth} \mathcal{O}_{V,y} = \operatorname{depth} \bar{R}_y - 1$$

for any $y \in V$ (see [Gro68, Exposé III, Cor. 2.5]). In particular, for $y \in V \cap Y$, the above is $\geqslant 2-1=1$ by Serre's criterion for the normal ring $\bar{R}$ and the fact that Y has codimension $\geqslant 2$ in $\operatorname{Spec} \bar{R}$. Hence by the first fact recalled in the Proof of Proposition 3.10.4 we have the desired injectivity.

Step II. For the surjectivity, we first prove it after u-adic completion. Recall that Lect.17, Dec 5

$$S = (R \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})\langle u \rangle / (u \cdot f - p^{\varepsilon}).$$

Let S' be the u-adic completion of S equipped with (p,u)-adic topology. This topology is the same as u-adic topology, because u divides p^{ε} in S. Thus the underlying topological space $|\operatorname{Spf} S'|$, by definition, equals $|\operatorname{Spec} S'/(u)| = |\operatorname{Spec} S/(u)| = |\operatorname{Spec} R_{\varepsilon}|$, which is a closed topological subspace in $|\operatorname{Spf} S|$. The u-adic completion of $H^0(U, \mathcal{O}_{\operatorname{Spf} S})$ is $H^0(U \cap |\operatorname{Spf} S'|, \mathcal{O}_{\operatorname{Spf} S'})$. The goal of this step is to show that

$$S' \longrightarrow H^0(U, \mathcal{O}_{\operatorname{Spf} S'})$$

is surjective. It suffices to show the surjectivity modulo u. Thus we need to show the surjectivity of the map

$$\bar{R} \otimes_{\mathbb{F}_p} \mathbb{Z}_p^{\mathrm{cycl}}/p^{\varepsilon} = R_{\varepsilon} \longrightarrow H^0(U \cap \operatorname{Spec} R_{\varepsilon}, \mathcal{O}_{\operatorname{Spec} R_{\varepsilon}}) = H^0(U \cap \operatorname{Spec} \bar{R}, \mathcal{O}_{\operatorname{Spec} \bar{R}}) \otimes_{\mathbb{F}_p} \mathbb{Z}_p^{\mathrm{cycl}}/p^{\varepsilon}.$$

It then suffices to show the surjectivity of

$$\bar{R} \longrightarrow H^0(U \cap \operatorname{Spec} \bar{R}, \mathcal{O}_{\operatorname{Spec} \bar{R}}) = H^0((\operatorname{Spec} \bar{R})\backslash Y, \mathcal{O}_{\operatorname{Spec} \bar{R}}).$$

But this is just Proposition 3.10.4.

Step III. Suppose $\xi \in S'$ is such that its image in $H^0(U \cap \operatorname{Spf} S', \mathcal{O}_{\operatorname{Spf} S'})$ comes from $H^0(U, \mathcal{O}_{\operatorname{Spf} S})$. We aim to show that $\xi \in S \subset S'$.

> *Claim.* Suppose $\gamma \in S'/p^{\varepsilon}$ is such that its image in $H^0(U \cap \operatorname{Spf}(S'/p^{\varepsilon}), \mathcal{O}_{\operatorname{Spf}(S'/p^{\varepsilon})})$ comes from $H^0(U, \mathcal{O}_{\operatorname{Spf}(S/p^{\varepsilon})})$. Then $\gamma \in S/p^{\varepsilon}$.

The claim implies our goal for the reason below. We have $S' = (R \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{cycl}})[\![u]\!]/(u \cdot f - p^{\varepsilon})$. By the claim, $\xi \in S'$ must be of the form $\xi_0(u) + p^{\varepsilon}(\xi_1(u) + p^{\varepsilon}(\xi_2(u) + \cdots))$, where $\xi_i(u)$ are polynomials in u. Then $\xi \equiv \xi_0 \in S'/p^{\varepsilon}$. Thus $\xi \in S$.

> *Proof of Claim.* We have $S'/p^{\varepsilon} = R_{\varepsilon}[\![u]\!]/(u \cdot f)$ and $S/p^{\varepsilon} = R_{\varepsilon}[u]/(u \cdot f)$. It suffices to show the claim modulo f, because if a power series in $R_{\varepsilon}[\![u]\!]$ is congruent to a polynomial modulo f, then it is congruent to a polynomial modulo fu.
>
> Thus we need: Given an element $\gamma \in (R_{\varepsilon}/f)[\![u]\!]$ such that its image in $H^0(U \cap \operatorname{Spf}(R_{\varepsilon}/f)[\![u]\!], \mathcal{O}_{\operatorname{Spf}(R_{\varepsilon}/f)[\![u]\!]})$ comes from $H^0(U \cap \operatorname{Spec}(R_{\varepsilon}/f)[u], \mathcal{O}_{\operatorname{Spec}(R_{\varepsilon}/f)[u]})$, show that $\gamma \in (R_{\varepsilon}/f)[u]$. By construction $R_{\varepsilon}/f = (\bar{R}/\bar{f}) \otimes_{\mathbb{F}_p} \mathbb{Z}_p^{\mathrm{cycl}}/p^{\varepsilon}$. Then it is enough to show that for any $\gamma \in (\bar{R}/\bar{f})[\![u]\!]$ whose image in $H^0((\operatorname{Spec} \bar{R}/\bar{f})\backslash Y, \mathcal{O}_{\operatorname{Spec}(\bar{R}/\bar{f})[\![u]\!]}) = H^0((\operatorname{Spec} \bar{R}/\bar{f})\backslash Y)[\![u]\!]$ lies in $H^0(U', \mathcal{O}_{\operatorname{Spec}(\bar{R}/\bar{f})[u]}) = H^0((\operatorname{Spec} \bar{R}/\bar{f})\backslash Y)[u]$, we have $\gamma \in (\bar{R}/\bar{f})[u]$. Here U' is the inverse image of $(\operatorname{Spec} \bar{R})\backslash Y$ under the natural map $\operatorname{Spec}(\bar{R}/\bar{f})[u] \to \operatorname{Spec} \bar{R}$. But this just requires the injectivity of the map $\bar{R}/\bar{f} \to H^0((\operatorname{Spec} \bar{R}/\bar{f})\backslash Y)$, which is already done in Step I.

Here comes a brief summary. Now we have shown that

- The ring homomorphism $S' \twoheadrightarrow H^0(U \cap \operatorname{Spf} S', \mathcal{O}_{\operatorname{Spf} S'})$ is surjective.
- If $\xi \in S'$ is such that its image in $H^0(U \cap \operatorname{Spf} S', \mathcal{O}_{\operatorname{Spf} S'})$ comes from $H^0(U, \mathcal{O}_{\operatorname{Spf} S})$, then $\xi \in S$.

We want the following surjection

$$S \longrightarrow H^0(U, \mathcal{O}_{\operatorname{Spf} S}).$$

Step IV. We are to show the injectivity of

$$H^0(U, \mathcal{O}_{\operatorname{Spf} S}) \longrightarrow H^0(U \cap \operatorname{Spf} S', \mathcal{O}_{\operatorname{Spf} S'}).$$

We first reduce this to the case modulo p^ε. Then we use the decomposition as in Step I, to reduce to showing that the map from $H^0((V\backslash Y)\times_{\mathbb{F}_p}\mathbb{A}^1_{\mathbb{Z}_p^{\mathrm{cycl}}/p^\varepsilon})$ to its u-adic completion is injective. Namely, setting $A := H^0(V\backslash Y)\otimes_{\mathbb{F}_p}\mathbb{Z}_p^{\mathrm{cycl}}/p^\varepsilon$, we need to show that the map $A[u]\to A[\![u]\!]$ is injective. This is clear. □

3.11. **Classical setting of Tate's normalized trace.** We will need yet another version of Hartog's extension principle, which allows us, roughly speaking, to "extend perfectoidness to the boundary". As preparation, we need a technical machinery, called Tate's normalized trace.

For motivation, we recall the classical setting of Tate's normalized trace, which is about certain totally ramified extensions of local fields. Let K be a finite extension of $\mathbb{Q}_p$. Let K_∞ be a totally ramified $\mathbb{Z}_p$-extension of K, i.e. it is a totally ramified extension of K and there is an isomorphism $\psi\colon \mathrm{Gal}(K_\infty/K)\xrightarrow{\sim}\mathbb{Z}_p$ of topological groups.

> (For instance, here is one way to obtain such an extension: Start with a finite extension $E/\mathbb{Q}_p$. Choose a uniformizer $\pi\in E$, and then obtain the Lubin–Tate extension E_π/E which is the union of all finite abelian extensions E'/E such that $\pi\in N_{E'/E}E'^\times$. Then $\mathrm{Gal}(E_\pi/E)\cong\mathcal{O}_E^\times$ via the Artin map. Choose a positive integer n. Then the open subgroup $1+\pi^n\mathcal{O}_E\subset\mathcal{O}_E^\times$ corresponds to an intermediate extension $E_\pi/E'/E$ such that E'/E is finite and $\mathrm{Gal}(E_\pi/E')\cong 1+\pi^n\mathcal{O}_E$. When n is large enough, this group is isomorphic to $\mathcal{O}_E$, and we assume this is the case. But $\mathcal{O}_E\cong\mathbb{Z}_p^{\oplus[E:\mathbb{Q}_p]}$, so we can find, in multiple ways in general, a closed subgroup $H\subset\mathrm{Gal}(E_\pi/E')$ such that $\mathrm{Gal}(E_\pi/E')/H\cong\mathbb{Z}_p$. Then $\mathrm{Gal}(E_\pi^H/E')\cong\mathbb{Z}_p$, and the extension E_π^H/E' is totally ramified because E_π/E is so. We have already seen that $E'/\mathbb{Q}_p$ is finite, so we can take $K=E'$ and $K_\infty=E_\pi^H$.)

For each $n\geqslant 0$ we write

$$K_n := (K_\infty)^{\psi^{-1}(p^n\mathbb{Z}_p)}.$$

Then K_n/K is a totally ramified extension with Galois group $\mathbb{Z}_p/p^n\mathbb{Z}_p=\mathbb{Z}/p^n$. This construction forms a tower

$$K=K_0\subset K_1\subset K_2\subset\cdots\subset K_\infty=\bigcup_n K_n.$$

Definition 3.11.1. For any finite extension of fields E/F, we define the *normalized trace* as

$$\overline{\mathrm{tr}}_{E/F} := \frac{1}{[E:F]}\,\mathrm{tr}_{E/F}.$$

The first property of this normalized trace is that if $E'/E/F$ are finite extensions then

$$(\overline{\mathrm{tr}}_{E'/F})|_E=\overline{\mathrm{tr}}_{E/F}.$$

Thus for any algebraic extension E/F, not necessarily of finite degree, we can define $\overline{\mathrm{tr}}_{E/F}\colon E\to F$.

In particular, on $K_\infty=\bigcup_n K_n$ we have an additive homomorphism

$$\overline{\mathrm{tr}}=\overline{\mathrm{tr}}_{K_\infty/K}\colon K_\infty\longrightarrow K$$

such that $(\overline{\mathrm{tr}})|_{K_n}=\overline{\mathrm{tr}}_{K_n/K}$. Also, for $n\geqslant m$, we have

$$\overline{\mathrm{tr}}_m|_{K_n}=\frac{1}{[K_n:K_m]}\overline{\mathrm{tr}}_{K_n/K_m}.$$

Theorem 3.11.2 (Tate)**.** *The map $\overline{\mathrm{tr}}\colon K_\infty\to K$ is continuous, and hence extends to $\overline{\mathrm{tr}}\colon\widehat{K}_\infty\to K$, where $\widehat{K}_\infty$ denotes the p-adic completion of K_∞. More precisely, there exists a real constant $A>0$ such that for any $x\in K_\infty$,*

$$|\,\overline{\mathrm{tr}}(x)|\leqslant A\cdot|x|,$$

which implies the continuity of $\overline{\mathrm{tr}}$.

Proof Idea. One uses ramification theory to compute the valuation of the different ideal $\mathscr{D}_{K_{n+1}/K_n}$ in terms of higher ramification groups. The calculation yields that there exists a universal constant $a > 0$, independent of n, such that

$$v_K(\mathscr{D}_{K_{n+1}/K_n}) \geqslant v_K(p)(1 - a/p^n)$$

for all n. Here v_K denotes the standard valuation on K, and for any ideal $\mathscr{I} \subset \mathcal{O}_{K_{n+1}}$ such as $\mathscr{D}_{K_{n+1}/K_n}$, we write $v_K(\mathscr{I})$ for $v_K(b)$ for any generator b of $\mathscr{I}$.

> In fact one calculates $v_K(\mathscr{D}_{K_{n+1}/K_n})$ by using $v_K(\mathscr{D}_{K_{n+1}/K_n}) = v_K(\mathscr{D}_{K_{n+1}/K}) - v_K(\mathscr{D}_{K_n/K})$ and calculating $v_K(\mathscr{D}_{K_n/K})$. For the latter, the key input is the formula (which holds for an arbitrary finite Galois extension of local fields in place of K_n/K)
>
> $$v_K(\mathscr{D}_{K_n/K}) = \int_{-1}^{\infty} (1 - |G_n^\nu|^{-1}) d\nu,$$
>
> where G_n^ν are the upper-indexed ramification subgroups of $\mathrm{Gal}(K_n/K)$. Recall that $G_n^\nu = G^\nu \, \mathrm{Gal}(K_\infty/K_n)/\,\mathrm{Gal}(K_\infty/K_n)$, where G^ν are the upper-indexed ramification subgroups of $\mathrm{Gal}(K_\infty/K)$. Another key observation is that there exists a positive integer i_0 such that for all $i > i_0$, we have G^ν jumps from $p^i\mathbb{Z}_p \subset \mathrm{Gal}(K_\infty/K) = \mathbb{Z}_p$ to $p^{i+1}\mathbb{Z}_p$ precisely after ν increases by $v_K(p)$. It is then not hard to see that
>
> $$v_K(\mathscr{D}_{K_n/K}) = v_K(p)n + c + p^{-n}a_n$$
>
> where c is a constant independent of n, and $(a_n)_n$ is a bounded sequence. From this we immediately get the desired lower bound for $v_K(\mathscr{D}_{K_{n+1}/K_n})$.

Thus (and in fact equivalently) for all $x \in K_{n+1}$,

$$|\operatorname{tr}_{K_{n+1}/K_n}(x)| \leqslant |p|^{1-a/p^n} \cdot |x|.$$

Here $|\cdot|$ is the absolute value on $\overline{K}$ corresponding to v_K. As for the normalized trace, we have the following key estimation

$$|\overline{\operatorname{tr}}_{K_{n+1}/K_n}(x)| = \left|\frac{\operatorname{tr}_{K_{n+1}/K_n}(x)}{p}\right| \leqslant |p^{-1}|^{a/p^n} \cdot |x|,$$

with $|p^{-1}| > 1$. Moreover, since $\overline{\operatorname{tr}}_{K_n/K} = \overline{\operatorname{tr}}_{K_n/K_{n-1}} \circ \overline{\operatorname{tr}}_{K_{n-1}/K_{n-2}} \circ \cdots \circ \overline{\operatorname{tr}}_{K_1/K}$, we have, for all $x \in K_n$,

$$|\overline{\operatorname{tr}}_{K_n/K}(x)| \leqslant |p^{-1}|^{a(1+p^{-1}+\cdots+p^{-(n-1)})} \cdot |x| \leqslant |p^{-1}|^C \cdot |x|,$$

with the constant $C \coloneqq \sum_{i=0}^{\infty} a/p^i < \infty$ that is independent of n. □

The above proof clearly gives the following result: For any integers $n \geqslant m \geqslant 0$ and $x \in K_n$,

$$|\overline{\operatorname{tr}}_{K_n/K_m}(x)| \leqslant |p^{-1}|^{a(p^{-m}+\cdots+p^{-(n-1)})} \cdot |x| \leqslant |p^{-1}|^{C_m} \cdot |x|, \tag{3.11.1}$$

where $|\cdot|$ is still the absolute value corresponding to v_K, and $C_m = \sum_{i=m}^{\infty} a/p^i$. Thus for each $m \geqslant 0$, we know that the normalized trace $\overline{\operatorname{tr}}_m = \overline{\operatorname{tr}}_{K_\infty/K_m} : K_\infty \to K_m$ satisfies $(\overline{\operatorname{tr}}_m)|_{K_n} = \overline{\operatorname{tr}}_{K_n/K_m}$ for all $n \geqslant m$ and $|\overline{\operatorname{tr}}_m(x)| \leqslant |p^{-1}|^{C_m} \cdot |x|$ for all $x \in K_\infty$. In particular, $\overline{\operatorname{tr}}_m$ extends by continuity to a map $\overline{\operatorname{tr}}_m : \widehat{K}_\infty \to K_m$ satisfying the same bound.

The following result says that any element of $\widehat{K}_\infty$ can be written as the limit of a canonical sequence in K_∞.

Corollary 3.11.3. *For all $x \in \widehat{K}_\infty$, we have*

$$x = \lim_{n\to\infty} \overline{\operatorname{tr}}_n(x),$$

for $\overline{\operatorname{tr}}_n(x) \in K_n \subset K_\infty$.

Proof. Let $y_n = \overline{\operatorname{tr}}_n(x)$. Then we want that $y_n \to x$. For any $\varepsilon > 0$ there exists $y \in K_{n_0} \subset K_\infty$ such that $|x - y| < \varepsilon/2$. By construction for all $n \geqslant n_0$ we have $y = \overline{\operatorname{tr}}_{n_0}(y) = \overline{\operatorname{tr}}_n(y)$. Also, by definition $y_n = \overline{\operatorname{tr}}_n(x)$. It follows that whenever $n \geqslant n_0$ we have $y - y_n = \overline{\operatorname{tr}}_n(y - x)$, and then

$$|y - y_n| \leqslant |p^{-1}|^{C_n} \cdot |y - x|.$$

Since $C_n \to 0$, there exists an integer $N > n_0$ such that $|p^{-1}|^{C_n} \leqslant 2$ for all $n > N$. For such n, we have $|y - y_n| \leqslant 2|y - x|$. Therefore,

$$|x - y_n| \leqslant \max(|x - y|, |y - y_n|) \leqslant 2|x - y| < \varepsilon.$$

It follows that $y_n \to x$. □

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3.12. **Tate's normalized trace for the anti-canonical tower.** Notice that Theorem 3.11.2, (3.11.1), and Corollary 3.11.3 all rely on following key estimate: For all $x \in K_{m+1}$ with $m \in \mathbb{N}$,

$$\left|\frac{\mathrm{tr}_{K_{m+1}/K_m}(x)}{[K_{m+1} : K_m]}\right| \leqslant |p^{-1}|^{a/p^m} \cdot |x|,$$

where a is a universal constant independent of n and x.

Let $m \geqslant 1$ and fix $0 \leqslant \varepsilon < 1/2$. We shall draw an analogy between the tower

$$(K = K_0 \longrightarrow K_1 \longrightarrow K_2 \longrightarrow \cdots).$$

and the tower in Proposition 3.9.21:

$$(\cdots \longrightarrow \mathfrak{X}(p^{-2}\varepsilon) \longrightarrow \mathfrak{X}(p^{-1}\varepsilon) \longrightarrow \mathfrak{X}(\varepsilon)).$$

Here the transition maps are the lifted Frobenii. Write $n = \dim X = g(g+1)/2$. Then, we have the following analogous key estimate: Along the map

$$\overline{\mathrm{tr}} = \frac{1}{p^n}\,\mathrm{tr}\colon \mathcal{O}_{\mathfrak{X}(p^{-(m+1)}\varepsilon)}[1/p] \longrightarrow \mathcal{O}_{\mathfrak{X}(p^{-m}\varepsilon)}[1/p],$$

the image of $\mathcal{O}_{\mathfrak{X}(p^{-(m+1)}\varepsilon)}$ is contained in $p^{-a/p^m} \cdot \mathcal{O}_{\mathfrak{X}(p^{-m}\varepsilon)}$ for some constant $a > 0$ independent of m. (We will prove this for $a = (g^2 + g + 1)\varepsilon$.) Here, p^n is the degree of $\mathcal{O}_{\mathfrak{X}(p^{-(m+1)}\varepsilon)}[1/p]$ over $\mathcal{O}_{\mathfrak{X}(p^{-m}\varepsilon)}[1/p]$.

Corollary 3.12.1 (Tate's normalized trace)**.** *Fix an integer $m \geqslant 1$ and $0 \leqslant \varepsilon < 1/2$. We have the normalized trace map*

$$\overline{\mathrm{tr}}_m\colon \varinjlim_{m' \geqslant m} \mathcal{O}_{\mathfrak{X}(p^{-m'}\varepsilon)}[1/p] \longrightarrow \mathcal{O}_{\mathfrak{X}(p^{-m}\varepsilon)}[1/p]$$

characterized by the condition that on the m'-th part, one has

$$\overline{\mathrm{tr}}_m = \frac{1}{p^{(m'-m)n}}(\mathrm{tr}\colon \mathcal{O}_{\mathfrak{X}(p^{-m'}\varepsilon)}[1/p] \longrightarrow \mathcal{O}_{\mathfrak{X}(p^{-m}\varepsilon)}[1/p]).$$

The image of $\varinjlim_{m' \geqslant m} \mathcal{O}_{\mathfrak{X}(p^{-m'}\varepsilon)}$ is contained in $p^{-C_m} \cdot \mathcal{O}_{\mathfrak{X}(p^{-m}\varepsilon)}$ for some constant $C_m > 0$, satisfying that $C_m \to 0$ as $m \to \infty$. So we have an extension of $\overline{\mathrm{tr}}_m$ by continuity to a map

$$\left(\varinjlim_{m'} \mathcal{O}_{\mathfrak{X}(p^{-m'}\varepsilon)}\right)^{\wedge}_p [1/p] = \mathcal{O}_{\mathfrak{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}}}[1/p] \longrightarrow \mathcal{O}_{\mathfrak{X}(p^{-m}\varepsilon)}[1/p],$$

satisfying that the image of $\mathcal{O}_{\mathfrak{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}}}$ is contained in $p^{-C_m} \cdot \mathcal{O}_{\mathfrak{X}(p^{-m}\varepsilon)}$. Moreover, for any $x \in \mathcal{O}_{\mathfrak{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}}}[1/p]$,

$$x = \lim_{m \to \infty} \overline{\mathrm{tr}}_m(x).$$

Assuming the key estimate, the proof of the above corollary is completely analogous to the previous arguments in the classical setting. We now prepare to prove the key estimate. Note that the key estimate is equivalent to the following: The trace map

$$\mathrm{tr}\colon \mathcal{O}_{\mathfrak{X}(p^{-(m+1)}\varepsilon)}[1/p] \longrightarrow \mathcal{O}_{\mathfrak{X}(p^{-m}\varepsilon)}[1/p]$$

sends $\mathcal{O}_{\mathfrak{X}(p^{-(m+1)}\varepsilon)}$ into $p^{n-(2n+1)\varepsilon} \cdot \mathcal{O}_{\mathfrak{X}(p^{-m}\varepsilon)}$.

Lemma 3.12.2. *Let R be a p-adically complete flat $\mathbb{Z}_p$-algebra. Fix $Y_1, \ldots, Y_n \in R$. Let $P_1, \ldots, P_n \in R\langle X_1, \ldots, X_n\rangle$ be topologically nilpotent elements, or equivalently, each P_i has topologically nilpotent coefficients in R. (Here "topological" always means with respect to the p-adic topology.) Take*

$$S = R\langle X_1, \ldots, X_n\rangle/(X_1^p - Y_1 - P_1, \ldots, X_n - Y_n - P_n).$$

Then we have

(1) *The ring S is a finite free R-module of rank p^n, whose basis is $X_1^{i_1}\cdots X_n^{i_n}$ with $0 \leqslant i_1,\ldots,i_n \leqslant p-1$.*
(2) *If $I \subset R$ is the ideal generated by p together with all coefficients of all P_i's, then $\mathrm{tr}_{S/R}\colon S \to R$ sends S into I^n.*

We first discuss an example to illustrate Lemma 3.12.2.

Example 3.12.3. Let $p = 2$ and $R = \mathbb{Z}_p$. Take

$$S = \mathbb{Z}_p\langle X, Y, Z\rangle/(X^p - p(X+Y), Y^p - p(X+Y), Z^p - p(X+Y)).$$

Following Lemma 3.12.2, we want to check that $\mathrm{tr}_{S/R}(X) \in I^3 = (p^3)$. By Lemma 3.12.2(1), we have the basis $X^aY^bZ^c$ with $0 \leqslant a,b,c \leqslant p-1 = 1$. To compute $\mathrm{tr}_{S/R}(X)$, we need to compute $X\cdot(X^aY^bZ^c)$, and $\mathrm{tr}_{S/R}(X)$ is given by the sum (running through $a,b,c \in \{0,1\}$) of coefficients of the monomial $X^aY^bZ^c$ in the expansion of $X\cdot(X^aY^bZ^c)$. If $a = 0$, then $X\cdot(X^aY^bZ^c) = XY^bZ^c$, which does not contribute to the trace since XY^bZ^c is already another element of the basis. So it suffices to compute with $a = 1$. For this, we have

$$\begin{aligned} X\cdot X &= \boxed{2X} + 2Y,\\ X\cdot XY &= (2X+2Y)\cdot Y = 2XY + 2Y^2 = \boxed{2XY} + 4X + 4Y,\\ X\cdot XZ &= (2X+2Y)\cdot Z = \boxed{2XZ} + 2YZ,\\ X\cdot XYZ &= (2X+2Y)\cdot YZ = 2XYZ + 2Y^2Z = \boxed{2XYZ} + 2YZ + 4XZ + 4YZ.\end{aligned}$$

Then $\mathrm{tr}_{S/R}(X) = 2+2+2+2 = 2^3 \in (p^3)$.

Proof of Lemma 3.12.2. **(1)** We need the following general fact.

> *Fact.* Let R be a ring and $I \subset R$ be an ideal such that R is I-complete. Take elements $F_1,\ldots,F_n \in R[X_1,\ldots,X_n]^\wedge_I$ in the I-adic completion that form a regular sequence in $(R/I)[X_1,\ldots,X_n]$. Suppose the images of $\boldsymbol{e}_1,\ldots,\boldsymbol{e}_l \in R[X_1,\ldots,X_n]$ in $R[X_1,\ldots,X_n]/(I,F_1,\ldots,F_n)$ form a basis of the latter as an R/I-module. Then $\boldsymbol{e}_1,\ldots,\boldsymbol{e}_l$ is a basis of $R[X_1,\ldots,X_n]^\wedge_I/(F_1,\ldots,F_n)$.

The idea to prove this fact is to use induction on $k \geqslant 1$ to show the following: The Koszul complex of $F_1,\ldots,F_n$, acting on $B_k \coloneqq (R/I^k)[X_1,\ldots,X_n]$, read as

$$\wedge^n(B_k^{\oplus n}) \longrightarrow \cdots \longrightarrow \wedge^2(B_k^{\oplus n}) \longrightarrow B_k^{\oplus n} \xrightarrow{(F_1,\ldots,F_n)} B_k,$$

is acyclic at nonzero degrees, and moreover the 0-th homology $(R/I^k)[X_1,\ldots,X_n]/(F_1,\ldots,F_n)$ has an R/I^k-basis $\boldsymbol{e}_1,\ldots,\boldsymbol{e}_l$.

To apply this fact to our setting, we need to check the conditions:

(i) The p-adic topology and I-adic topology on R are the same. In particular, R is I-adically complete, and we have $R\langle X_1,\ldots,X_n\rangle \simeq R[X_1,\ldots,X_n]^\wedge_I$.
(ii) All $F_i = X_i^p - Y_i - P_i$ with $i = 1,\ldots,n$ form a regular sequence in $(R/I)[X_1,\ldots,X_n]$.
(iii) The elements $X_1^{i_1}\cdots X_n^{i_n}$ for $0 \leqslant i_j \leqslant p-1$ form a basis of $R[X_1,\ldots,X_n]/(I,F_1,\ldots,F_n)$ over R/I.

For (i), note that I is a finitely generated ideal, generated by p and those finitely many coefficients of the P_i's which are not divisible by p. Thus $pR \subset I = (p,r_1,\ldots,r_t)$ for finitely many topologically nilpotent elements $r_i \in R$. Then $I^m \subset pR$ for $m \gg 0$. For (ii), in the ring $(R/I)[X_1,\ldots,X_n]$, we have $F_i = X_i^p - Y_i$, and one directly checks that they form a regular sequence. For (iii), it is clear for the reason that $R[X_1,\ldots,X_n]/(I,F_1,\ldots,F_n) = (R/I)[X_1,\ldots,X_n]/(X_1^p - Y_1,\ldots,X_n^p - Y_n)$ with $Y_i \in R/I$. Hence we are able to apply the general fact and obtain (1).

(2) We illustrate the essence of proof by working with $n = 2$ exclusively, for simplicity. By some slightly non-trivial reduction arguments, one reduces the proof to showing just that $\mathrm{tr}_{S/R}(X_1) \in I^2$. (Note that *a priori*, one must show that $\mathrm{tr}_{S/R}(X_1^aX_2^b) \in I^2$ for all $0 \leqslant a,b \leqslant p-1$. The point of this reduction is that for each fixed (a,b), there is a way to change to new coordinates X_1', X_2' such that $X_1' = X_1^aX_2^b$ and such that the form of the problem remains the same for the new coordinates.)

Moreover, one reduces to the case where $Y_1 = Y_2 = 0$ and P_1, P_2 are polynomials whose degrees in each one of X_1, X_2 are at most $p-1$. One further reduces to the "universal case" where

$$R = \mathbb{Z}_p[\![a_{1,\underline{i}}, a_{2,\underline{i}}]\!], \qquad \underline{i} = (i_1, i_2) \in \{0, \dots, p-1\}^2$$

and

$$P_1 = \sum_{\underline{i}} a_{1,\underline{i}} X_1^{i_1} X_2^{i_2}, \quad P_2 = \sum_{\underline{i}} a_{2,\underline{i}} X_1^{i_1} X_2^{i_2}.$$

The subtlety here lies in that R is no longer p-adically complete; but R is still I-adically complete for $I = (p, a_{1,\underline{i}}, a_{2,\underline{i}})$, so the conclusion of (1) still holds in the current setting, i.e., as an R-module,

$$S = R\langle X_1, X_2\rangle/(X_1^p - P_1, X_2^p - P_2)$$

still has an R-basis $X_1^{i_1}X_2^{i_2}$ with $0 \leqslant i_1, i_2 \leqslant p-1$. Let $I_1 = (a_{1,\underline{i}})_{\underline{i}}$ and $I_2 = (a_{2,\underline{i}})_{\underline{i}}$ be ideals generated by coefficients of P_1 and P_2, respectively, so that $I = (p, I_1, I_2)$.

Claim. $I_1 \cap (pI_1 + I_2) \subset I^2$.

Proof of Claim. Suppose there exists some $r \in I_1 \cap (pI_1 + I_2)$ such that $r \notin I^2$. Then r is a power series in $a_{1,\underline{i}}$ and $a_{2,\underline{i}}$ with coefficients in $\mathbb{Z}_p$, with $r \notin I^2$.

There are only three cases in which $r \notin I^2$. In the first case, the constant term of r is not divisible by p^2; we observe that it is impossible because $r \in I_1$. In the second case, r contains a term $c \cdot a_{2,\underline{i}}$ with prime-to-p coefficient c; this is also impossible because $r \in I_1$. In the third case, r contains a term $c \cdot a_{1,\underline{i}}$ with prime-to-p coefficient c; then r does not lie in $pI_1 + I_2$, which is still a contradiction.

Granting the claim, for showing that $\mathrm{tr}_{S/R}(X_1) \in I^2$, it suffices to show $\mathrm{tr}_{S/R}(X_1) \in I_1 \cap (pI_1 + I_2)$. For it lying in I_1, note that if a basis element $X_1^{i_1}X_2^{i_2}$ contributes to $\mathrm{tr}_{S/R}(X_1)$ then $i_1 = p-1$. (As in Example 3.12.3, if $i_1 < p-1$ then $X_1 \cdot X_1^{i_1}X_2^{i_2}$ is another element of basis and contains no contribution of $X_1^{i_1}X_2^{i_2}$.) Thus, to compute the trace, we only have to consider for $0 \leqslant i_2 \leqslant p-1$ that

$$X_1 \cdot X_1^{p-1}X_2^{i_2} = X_1^p X_2^{i_2} = P_1 X_2^{i_2} = \sum_{\underline{j}} a_{1,\underline{j}} X_1^{j_1} X_2^{j_2+i_2}.$$

Note that the expression on the right hand side is not necessarily an expansion into the basis $\{X_1^{k_1}X_2^{k_2} \mid 0 \leqslant k_1, k_2 \leqslant p-2\}$; we may have $j_2 + i_2 \geqslant p$ and in that case we need to use $X_2^p = P_2$ to further simplify. However, it is clear that eventually the coefficient of $X_1^{i_1}X_2^{i_2}$ is an R-linear combination of $a_{1,\underline{k}} \in I_1$ and hence itself lies in I_1. Finally, it remains to prove that $\mathrm{tr}_{S/R}(X_1) \in pI_1 + I_2$. As before, we have

$$\mathrm{tr}_{S/R}(X_1) = \sum_{i_2=0}^{p-1} C(i_2)$$

where $C(i_2)$ is the coefficient of $X_1^{p-1}X_2^{i_2}$ in $P_1X_2^{i_2}$. Now there are two ways that $X_1^{p-1}X_2^{i_2}$ can appear in $P_1X_2^{i_2}$. First, P_1 contains the term $a_{1,(p-1,0)}X_1^{p-1}$, so $P_1X_2^{i_2}$ contains the term $a_{1,(p-1,0)}X_1^{p-1}X_2^{i_2}$. Second, for each term $a_{1,(u,v)}X_1^uX_2^v$ in P_1 such that $v + i_2 \geqslant p$, we have a contribution $X_1^uX_2^vX_2^{i_2}$ in $P_1X_2^{i_2}$, and this contribution may contain $X_1^{p-1}X_2^{i_2}$ since we need to apply $X_2^p = P_2$ to simplify $X_2^{v+i_2}$. Note that in the second way, the coefficient of $X_1^{p-1}X_2^{i_2}$ lies in I_2 because the simplification $X_2^p = P_2$ must be applied for at least once. Hence modulo I_2 we can ignore the second way. Thus modulo I_2 we have

$$\mathrm{tr}_{S/R}(X_1) \equiv \sum_{i_2=0}^{p-1} a_{1,(p-1,0)} = pa_{1,(p-1,0)}$$

and this lies in $pI_1 + I_2$ as desired. This finishes the proof. □

Lect.19, Dec 14

Our next goal is to prove the key estimate for the tower $\mathfrak{X}(p^{-m}\varepsilon)$ of interest. For $m \geqslant 1$ and $0 \leqslant \varepsilon < 1/2$, we need to establish that the trace map

$$\mathrm{tr}\colon \mathcal{O}_{\mathfrak{X}(p^{-(m+1)}\varepsilon)}[1/p] \longrightarrow \mathcal{O}_{\mathfrak{X}(p^{-m}\varepsilon)}[1/p]$$

sends $\mathcal{O}_{\mathfrak{X}(p^{-(m+1)}\varepsilon)}$ into $p^{n-(2n+1)\varepsilon} \cdot \mathcal{O}_{\mathfrak{X}(p^{-m}\varepsilon)}$. Here $n = \dim X = g(g+1)/2$.

For this we need to consider the transition map from $\mathcal{O}_{\mathfrak{X}(p^{-m}\varepsilon)}[1/p]$ to $\mathcal{O}_{\mathfrak{X}(p^{-(m+1)}\varepsilon)}[1/p]$, which we recall is the "lifting of relative Frobenius". Recall from the proof of Definition–Proposition 3.9.15 that $\mathfrak{X}(p^{-m}\varepsilon)$ locally looks like

$$\mathrm{Spf}(R\,\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p^{\mathrm{cycl}})\langle u\rangle/(u\cdot f-p^{p^{-m}\cdot\varepsilon}),$$

where $\mathrm{Spf}\,R\subset(X_{\mathbb{Z}_p})^{\wedge}_p$, and $f\in R$ is a local lifting of Ha such that $\bar f\in R/p$ is a nonzero-divisor.

Proposition 3.12.4 (Key estimate)**.** *Let R be a p-adically complete flat $\mathbb{Z}_p$-algebra, which is topologically of finite type and formally smooth of dimension n over $\mathbb{Z}_p$. Let $f\in R$ be such that its modulo p reduction $\bar f\in R/p$ is a nonzero-divisor. Take $0\leqslant\varepsilon<1/2$, and define*

$$S_\varepsilon:=(R\,\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p^{\mathrm{cycl}})\langle u_\varepsilon\rangle/(u_\varepsilon\cdot f-p^\varepsilon).$$

Here u_ε is seen as a variable. Also define $S_{\varepsilon/p}$ in the same way.

Assume we have a map[6]

$$\varphi\colon S_\varepsilon\longrightarrow S_{\varepsilon/p},$$

such that modulo $p^{1-\varepsilon}$, φ is the relative Frobenius over $\mathbb{Z}_p^{\mathrm{cycl}}/p^{1-\varepsilon}$. In other words, it can be given as follows. Write $R_{1-\varepsilon}:=R\otimes_{\mathbb{Z}_p}(\mathbb{Z}_p^{\mathrm{cycl}}/p^{1-\varepsilon})$. Then $\varphi \bmod p^{1-\varepsilon}$ is the map

$$R_{1-\varepsilon}[u_\varepsilon]/(f\cdot u_\varepsilon-p^\varepsilon)\longrightarrow R_{1-\varepsilon}[u_{\varepsilon/p}]/(f\cdot u_{\varepsilon/p}-p^{\varepsilon/p})$$

which sends u_ε to $u_{\varepsilon/p}^p$ and which restricts to $\mathrm{Fr}_{\bar R}\otimes\mathrm{id}$ on $R_{1-\varepsilon}=\bar R\otimes_{\mathbb{F}_p}\mathbb{Z}_p^{\mathrm{cycl}}/p^{1-\varepsilon}$. Then

1. *The map*
$$\varphi[1/p]\colon S_\varepsilon[1/p]\longrightarrow S_{\varepsilon/p}[1/p]$$
is finite and flat of degree p^n.
2. *The trace map*
$$\mathrm{tr}\colon S_{\varepsilon/p}[1/p]\longrightarrow S_\varepsilon[1/p]$$
sends $S_{\varepsilon/p}$ into $p^{n-(2n+1)\varepsilon}S_\varepsilon$. Here the trace map is defined by viewing $S_{\varepsilon/p}[1/p]$ as an $S_\varepsilon[1/p]$-algebra via $\varphi[1/p]$ by (1).

Proof Sketch. Define

$$S':=(R\,\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p^{\mathrm{cycl}})\langle v\rangle/(f^p\cdot v-p^\varepsilon).$$

Note that there is an $R\,\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p^{\mathrm{cycl}}$-algebra map

$$\begin{aligned}\tau\colon S'&\longrightarrow S_{\varepsilon/p}\\ v&\longmapsto(u_{\varepsilon/p})^p.\end{aligned}$$

We make the following claims:

(i) After inverting p we get an isomorphism $\tau[1/p]\colon S'[1/p]\xrightarrow{\sim}S_{\varepsilon/p}[1/p]$.
(ii) The map τ is injective.
(iii) The cokernel of τ is annihilated by p^ε.

We prove (i) by directly spotting the inverse. To write down $(\tau[1/p])^{-1}$, formally we expect that $(\tau[1/p])^{-1}\colon u_{\varepsilon/p}\mapsto v^{1/p}$. However, we must make sense of $v^{1/p}$ in $S'[1/p]$. For this, we formally compute

$$v^{1/p}=\left(\frac{p^\varepsilon}{f^p}\right)^{1/p}=\frac{p^{\varepsilon/p}}{f}=\frac{f^{p-1}\cdot p^{\varepsilon/p}}{f^p}=\frac{f^{p-1}\cdot p^{\varepsilon/p}}{p^\varepsilon/v}=f^{p-1}\cdot p^{\varepsilon/p-\varepsilon}\cdot v.$$

Here $p^{\varepsilon/p-\varepsilon}$ makes sense in $S'[1/p]$. Thus, this formula gives $(\tau[1/p])^{-1}\colon u_{\varepsilon/p}\mapsto f^{p-1}\cdot p^{\varepsilon/p-\varepsilon}\cdot v$ as expected. The claim (ii) is implied by (i) together with the fact that S' is p-torsion-free (using Proposition 3.10.5, we see S' is flat over $\mathbb{Z}_p^{\mathrm{cycl}}$, which can be checked by hand directly). As for (iii), it is enough to note that the formula for $(\tau[1/p])^{-1}$ involves only p^ε on the denominator.

Next we show that $\varphi\colon S_\varepsilon\to S_{\varepsilon/p}$ factors through S' as

[6] Recall that it took non-trivial work to establish the existence of the map $\mathfrak{X}(p^{-(m+1)}\varepsilon)\to\mathfrak{X}(p^{-m}\varepsilon)$ lifting the relative Frobenius. (We used the theory of canonical subgroups.) In the current setting, the existing of $\phi:S_\varepsilon\to S_{\varepsilon/p}$ lifting the relative Frobenius is really a non-automatic assumption.

$$S_\varepsilon \xrightarrow{\psi} S' \xhookrightarrow{\tau} S_{\varepsilon/p}.$$

It suffices to check this modulo $p^{1-\varepsilon}$. Indeed, if we know the factorization modulo $p^{1-\varepsilon}$, then for each $x \in S_\varepsilon$ we have $\varphi(x) \in \mathrm{Im}(\tau) + (p^{1-\varepsilon})$. Since $1-\varepsilon > \varepsilon$, we have $(p^{1-\varepsilon}) \subset \mathrm{Im}(\tau)$ since $\mathrm{Coker}(\tau)$ is killed by p^ε. Now to check the factorization modulo $p^{1-\varepsilon}$, we know that φ is the relative Frobenius by assumption, so we can directly check that φ factors through $\mathrm{Im}(\tau)$.

(1) Now it remains to show that $\psi[1/p]\colon S_\varepsilon[1/p] \to S'[1/p]$ is finite flat. For this it is natural to first recognize ($\psi \bmod p^{1-\varepsilon}$). Consider the composition $(\tau \bmod p^{1-\varepsilon}) \circ (\psi \bmod p^{1-\varepsilon}) = (\phi \bmod p^{1-\varepsilon})$, which is the same as the relative Frobenius. The idea is that if ($\tau \bmod p^{1-\varepsilon}$) is injective then the above condition uniquely determines ($\psi \bmod p^{1-\varepsilon}$). However, this may not be the case. We can control the kernel of ($\tau \bmod p^{1-\varepsilon}$) as follows. The short exact sequence

$$0 \longrightarrow S' \xrightarrow{\tau} S_{\varepsilon/p} \longrightarrow \mathrm{Coker}(\tau) \longrightarrow 0$$

induces a long exact sequence after $\otimes_{\mathbb{Z}_p^{\mathrm{cycl}}} \mathbb{Z}_p^{\mathrm{cycl}}/p^{1-\varepsilon}$, and then produces a quotient map

$$\mathrm{Tor}_{1,\mathbb{Z}_p^{\mathrm{cycl}}}(\mathrm{Coker}(\tau), \mathbb{Z}_p^{\mathrm{cycl}}/p^{1-\varepsilon}) \twoheadrightarrow \mathrm{Ker}(\tau \bmod p^{1-\varepsilon}).$$

By (iii), the left hand side is annihilated by p^ε, and so is $\mathrm{Ker}(\tau \bmod p^{1-\varepsilon})$. Now consider the map

$$\psi' : R_{1-\varepsilon}[u_\varepsilon]/(fu_\varepsilon - p^\varepsilon) \longrightarrow R_{1-\varepsilon}[v]/(f^p v - p^\varepsilon)$$

sending u_ε to v and equal to $\mathrm{Fr}_{\bar{R}} \otimes \mathrm{id}$ on $R_{1-\varepsilon} = \bar{R} \otimes_{\mathbb{F}_p} \mathbb{Z}_p^{\mathrm{cycl}}/p^{1-\varepsilon}$. We directly check that

$$(\tau \bmod p^{1-\varepsilon}) \circ \psi' = (\phi \bmod p^{1-\varepsilon}) = (\tau \bmod p^{1-\varepsilon}) \circ (\psi \bmod p^{1-\varepsilon}).$$

Write $\bar{\psi}$ for ($\psi \bmod p^{1-\varepsilon}$). Therefore, for any $x \in S_\varepsilon/p^{1-\varepsilon}$, we have $\bar{\psi}(x) - \psi'(x) \in \mathrm{Ker}(\tau \bmod p^{1-\varepsilon})$. We have seen that this kernel is contained in $\mathrm{Ker}(p^\varepsilon : S'/p^{1-\varepsilon} \to S'/p^{1-\varepsilon})$. Since S' is p^ε-torsion free, we conclude that the pre-image of $\bar{\psi}(x) - \psi'(x)$ along $S' \to S'/p^{1-\varepsilon}$ is contained in $p^{1-2\varepsilon}S'$. Therefore, $\bar{\psi} \equiv \psi' \bmod p^{1-2\varepsilon}$.

The upshot is that we have obtained an explicit description of $\psi \bmod p^{1-2\varepsilon} : R_{1-2\varepsilon}[u_\varepsilon]/(fu_\varepsilon - p^\varepsilon) \to R_{1-2\varepsilon}[v]/(f^p v - p^\varepsilon)$. Namely, it sends u_ε to v and equals $\mathrm{Fr}_{\bar{R}} \otimes \mathrm{id}$ on $R_{1-2\varepsilon}$.

Using the assumption on R, after a suitable localization, we may assume $\mathrm{Fr}_{\bar{R}}\colon \bar{R} \to \bar{R}$, $x \mapsto x^p$ makes $\bar{R}$ a finite free $\bar{R}$-module with basis $Y_1^{i_1} \cdots Y_n^{i_n}$ with $0 \leqslant i_1, \ldots, i_n \leqslant p-1$, for some fixed elements $Y_1, \ldots, Y_n \in \bar{R}$. (For instance, if $R = \mathbb{Z}_p\langle Y_1, \ldots, Y_n\rangle$, then this is evident.) It follows that the map ($\psi \bmod p^{1-2\varepsilon}$) makes $R_{1-2\varepsilon}[v]/(f^p \cdot v - p^\varepsilon)$ a finite free module over $R_{1-2\varepsilon}[u_\varepsilon]/(f \cdot u_\varepsilon - p^\varepsilon)$, with the same basis. It then follows that the S_ε-algebra S' (with structure map ψ) is isomorphic to

$$S' \cong S_\varepsilon\langle X_1, \ldots, X_n\rangle/(X_1^p - Y_1 - P_1, \ldots, X_n^p - Y_n - P_n), \tag{3.12.1}$$

with $Y_1, \ldots, Y_n \in S_\varepsilon$ and $P_1, \ldots, P_n \in S_\varepsilon\langle X_1, \ldots, X_n\rangle$. Here all coefficients of P_i's are in $p^{1-2\varepsilon}S_\varepsilon$. By Lemma 3.12.2 (1), we know that $\psi[1/p]$ is finite flat.

(2) Since $\mathrm{Coker}\,\tau$ is annihilated by p^ε from (iii), we have the inclusion

$$\mathrm{tr}_{S_{\varepsilon/p}/S'}(S_{\varepsilon/p}) \subset p^{-\varepsilon}S'.$$

Therefore, it suffices to show that

$$\mathrm{tr}_{S'/S_\varepsilon}(S') \subset p^{n-2n\varepsilon}S_\varepsilon.$$

By (3.12.1) and 3.12.2 (2), we have

$$\mathrm{tr}_{S'/S_\varepsilon}(S') \subset I^n \subset p^{(1-2\varepsilon)n}S_\varepsilon,$$

where I is the ideal generated by p together with all coefficients of P_i's. This finishes the proof. $\square$

Lect.20, Dec 19 3.13. **Constructing perfectoid anti-canonical neighborhood.** We now introduce an important criterion Proposition 3.13.1 of perfectoidness in the limit. The following preparation work will be useful in its proof.

Recall Tate's normalized trace from Corollary 3.12.1: For $m \geqslant 1$ and $0 \leqslant \varepsilon < 1/2$, we have normalized trace map

$$\overline{\mathrm{tr}}_m \colon \varinjlim_{m' \geqslant m} \mathcal{O}_{\mathfrak{X}(p^{-m'}\varepsilon)}[1/p] \longrightarrow \mathcal{O}_{\mathfrak{X}(p^{-m}\varepsilon)}[1/p].$$

The image of $\varinjlim_{m' \geqslant m} \mathcal{O}_{\mathfrak{X}(p^{-m'}\varepsilon)}$ along $\overline{\mathrm{tr}}_m$ is contained in $p^{-C_m} \cdot \mathcal{O}_{\mathfrak{X}(p^{-m}\varepsilon)}$ for some constant $C_m > 0$, satisfying that $C_m \to 0$ as $m \to \infty$. So we have an extension of $\overline{\mathrm{tr}}_m$ by continuity to a map

$$\overline{\mathrm{tr}}_m \colon \mathcal{O}_{\mathfrak{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}}}[1/p] \longrightarrow \mathcal{O}_{\mathfrak{X}(p^{-m}\varepsilon)}[1/p],$$

Here recall that $\mathfrak{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}}$ is the formal scheme representing the inverse limit $\varprojlim_{m'} \mathfrak{X}(p^{-m'}\varepsilon)$. Moreover, since $C_m \to 0$, all functions at infinite level are canonical limits of functions at finite levels, i.e. for any $f \in \mathcal{O}_{\mathfrak{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}}}[1/p]$ one has

$$f = \lim_m \overline{\mathrm{tr}}_m(f),$$

where $\overline{\mathrm{tr}}_m(f) \in \mathcal{O}_{\mathfrak{X}(p^{-m}\varepsilon)}[1/p]$ is identified with its image in $\mathcal{O}_{\mathfrak{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}}}[1/p]$.

Consider the anti-canonical neighborhood $\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}}$. Recall that

$$\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}} \cong \mathcal{X}^*(p^{-m}\varepsilon)$$

is affinoid for $m \gg 0$, and this would fail to be true without compactification (see Lemma 3.10.7 and Remark 3.10.8). Also recall from Corollary 3.10.9 that we have constructed the perfectoid space

$$\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}} \sim \varprojlim_m \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}}.$$

Our goal now is to construct the perfectoid space

$$\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}} \sim \varprojlim_m \mathcal{X}^*_{\Gamma(p^m)}(\varepsilon)_{\mathrm{a}}$$

of the full level. In order to achieve this we need the following.

Proposition 3.13.1. *Fix m such that $\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}} \cong \mathcal{X}^*(p^{-m}\varepsilon)$ is affinoid. Let*

$$\mathcal{Y}^*_m \longrightarrow \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}}$$

be a finite morphism such that it is étale away from boundary of compactifications, namely it is étale on $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}} = \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}} - \partial$. Let $\mathcal{Y}_m$ be the preimage of $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}}$ in $\mathcal{Y}^*_m$, so that $\mathcal{Y}_m \to \mathcal{X}_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}}$ is finite étale. Assume $\mathcal{Y}^*_m$ is normal and none of its irreducible component is mapped entirely into the boundary of $\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}}$.

For any $m' \geqslant m$, define

$$\mathcal{Y}^*_{m'} \longrightarrow \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^{m'})}(\varepsilon)_{\mathrm{a}}$$

to be the pullback of $\mathcal{Y}^*_m$ along $\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^{m'})}(\varepsilon)_{\mathrm{a}} \to \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}}$. Also define

$$\mathcal{Y}^*_\infty \longrightarrow \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}}$$

to be the pullback of $\mathcal{Y}^*_m$ along $\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}} \to \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}}$. Similarly, define $\mathcal{Y}_{m'}$ and $\mathcal{Y}_\infty$ by pulling back $\mathcal{Y}_m = \mathcal{Y}^*_m - \partial$ to $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^{m'})}(\varepsilon)_{\mathrm{a}}$ and $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}}$, respectively.

Since $\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}}$ is affinoid, each $\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^{m'})}(\varepsilon)_{\mathrm{a}}$ is affinoid for $m' \geqslant m$, and so also is $\mathcal{Y}^*_{m'}$; we write $\mathcal{Y}^*_{m'} = \mathrm{Spa}(S_{m'}, S^+_{m'})$. Then

(1) *For all $m' \geqslant m$,*

$$S^+_{m'} = H^0(\mathcal{Y}_{m'}, \mathcal{O}^+_{\mathcal{Y}_{m'}}).$$

(2) *The map*

$$\varinjlim_{m'} S^+_{m'} \longrightarrow H^0(\mathcal{Y}_\infty, \mathcal{O}^+_{\mathcal{Y}_\infty})$$

is injective and has dense image.

(3) *Assume* $S_\infty \coloneqq H^0(\mathcal{Y}_\infty, \mathcal{O}_{\mathcal{Y}_\infty})$ *is a perfectoid* $\mathbb{Q}_p^{\mathrm{cycl}}$*-algebra. Then*
$$\mathcal{Y}_\infty^* = \mathrm{Spa}(S_\infty, S_\infty^\circ)$$
is an affinoid perfectoid space over $\mathbb{Q}_p^{\mathrm{cycl}}$*, and*
$$\mathcal{Y}_\infty^* \sim \varprojlim_{m'} \mathcal{Y}_{m'}^*.$$

Before the proof we have some explanations of statements in Proposition 3.13.1.

(1) Each $\mathcal{Y}_{m'}^*$ is determined by its interior $\mathcal{Y}_{m'}$.
(3) The tower $(\mathcal{Y}_{m'}^*)_{m'}$ is the same as base change of the tower $(\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^{m'})}(\varepsilon)_{\mathrm{a}})_{m'}$ along $\mathcal{Y}_m^* \to \mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^m)}(\varepsilon)_{\mathrm{a}}$. The original tower $(\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^{m'})}(\varepsilon)_{\mathrm{a}})_{m'}$ is known to have perfectoid limit. In order for the new tower $(\mathcal{Y}_{m'}^*)_{m'}$ to still have perfectoid limit, we just need $S_\infty \coloneqq H^0(\mathcal{Y}_\infty, \mathcal{O}^+_{\mathcal{Y}_\infty})$ to be perfectoid. But S_∞ only depends on the interior. So one can essentially work on interior and the global sections away from boundary.

As an important remark, since we know $\mathcal{Y}_\infty$ is finite étale over $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}}$, the almost purity result (see [Sch12, Theorem 7.9(iii)] or proof of Corollary 3.9.27) implies that $\mathcal{Y}_\infty$ is perfectoid, since $\mathcal{X}_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}}$ is known to be perfectoid. However, caution that this does not directly imply the perfectoidness of S_∞.

Proof Sketch. **(1)** Begin with $\mathcal{Y}_{m'}^* = \mathrm{Spa}(S_{m'}, S_{m'}^+)$ we need to show $S_{m'}^+ = H^0(\mathcal{Y}_{m'}, \mathcal{O}^+_{\mathcal{Y}_{m'}})$. It is enough to show that $S_{m'} = H^0(\mathcal{Y}_{m'}, \mathcal{O}_{\mathcal{Y}_{m'}})$. Set
$$R \coloneqq H^0(\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^{m'})}(\varepsilon)_{\mathrm{a}}, \mathcal{O}_{\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^{m'})}(\varepsilon)_{\mathrm{a}}}).$$
Using classical Hartog's principle (Proposition 3.10.4), one shows that the map $R \to S_{m'}$ is finite étale. In particular we obtain the trace map
$$\mathrm{tr}_{R/S_{m'}} : S_{m'} \longrightarrow R$$
such that the trace pairing induces an isomorphism $S \xrightarrow{\sim} \mathrm{Hom}_R(S, R), x \mapsto (y \mapsto \mathrm{tr}_{R/S_{m'}}(xy))$. Using this, the proof is reduced to the analogous statement about R, i.e. it reduces to showing
$$R = H^0(\mathcal{X}_{\Gamma_{\mathrm{s}}(p^{m'})}(\varepsilon)_{\mathrm{a}}, \mathcal{O}_{\mathcal{X}_{\Gamma_{\mathrm{s}}(p^{m'})}(\varepsilon)_{\mathrm{a}}}).$$
But this is just our second version of Hartog's extension principle (Proposition 3.10.5).

(3) This directly follows from (2) together with the construction.

(2) We only explain that the given map has dense image using Tate's normalized trace. Injectivity follows from Tate's normalized trace as well. By (1), $S_{m'} = H^0(\mathcal{Y}_{m'}, \mathcal{O}^+_{\mathcal{Y}_{m'}})$. So we need the map
$$\varinjlim_{m'} S_{m'} = \varinjlim_{m'} H^0(\mathcal{Y}_{m'}, \mathcal{O}^+_{\mathcal{Y}_{m'}}) \longrightarrow H^0(\mathcal{Y}_\infty, \mathcal{O}^+_{\mathcal{Y}_\infty}) = S_\infty$$
to have dense image.

We first explain why this is not obvious. Note that $\mathcal{Y}_\infty$ is the perfectoid space with $\mathcal{Y}_\infty \sim \varprojlim_{m'} \mathcal{Y}_{m'}$. This means after localizing at one fixed m by replacing $\mathcal{Y}_m$ with some affinoid open $\mathrm{Spa}(R_m, R_m^+)$, the tower becomes
$$(\mathrm{Spa}(R_{m'}, R_{m'}^+))_{m'}$$
and $\mathcal{Y}_\infty$ becomes $\mathrm{Spa}(R_\infty, R_\infty^+)$ such that R_∞ is the completed direct limit of $R_{m'}$'s. In particular, the map $\mathrm{inj}\lim_{m'} R_{m'} \to R_\infty$ has dense image. However, before localization, it is *a priori* unclear that the map between global sections $\varinjlim_{m'} H^0(\mathcal{Y}_{m'}, \mathcal{O}^+_{\mathcal{Y}_{m'}}) \to H^0(\mathcal{Y}_\infty, \mathcal{O}^+_{\mathcal{Y}_\infty})$ has dense image.

What we want is true because by Tate's normalized trace (Corollary 3.12.1 and the discussion at the beginning of this lecture), each local section of $\mathcal{O}_{\mathcal{Y}_\infty}$ is *canonically* the limit of a canonical sequence of local sections of $\mathcal{O}_{\mathcal{Y}_{m'}}$ and the local section
$$f = \varinjlim_{m'} \overline{\mathrm{tr}}_{m'}(f)$$
is compatible with gluing. Therefore, each global section $f \in H^0(\mathcal{Y}_\infty, \mathcal{O}_{\mathcal{Y}_\infty})$ is still a limit of $\overline{\mathrm{tr}}_{m'}(f) \in H^0(\mathcal{Y}_{m'}, \mathcal{O}_{\mathcal{Y}_{m'}})$. ∎

Next we want to apply the above criterion to the following choice of $\mathcal{Y}_m^* \to \mathcal{X}^*_{\Gamma_s(p^m)}(\varepsilon)_a$. Take

$$\mathcal{Y}_m^* = \mathcal{X}^*_{\Gamma_1(p^m)}(\varepsilon)_a$$

which is the inverse image of $\mathcal{X}^*_{\Gamma_s(p^m)}(\varepsilon)_a$ (as a locally closed in $\mathcal{X}^*_{\Gamma_s(p^m)} = (\mathcal{X}^*_{\Gamma_s(p^m),\mathbb{Q}_p^{\mathrm{cycl}}})^{\mathrm{an}}$) along the map

$$(X^*_{\Gamma_1(p^m),\mathbb{Q}_p^{\mathrm{cycl}}})^{\mathrm{an}} = \mathcal{X}^*_{\Gamma_1(p^m)} \longrightarrow \mathcal{X}^*_{\Gamma_s(p^m)}.$$

Here the level

$$\Gamma_1(p^m) := \left\{ g \in G(\mathbb{Z}_p) \colon g \equiv \begin{pmatrix} I_g & * \\ 0 & I_g \end{pmatrix} \bmod p^m \text{ (with size } g \text{ blocks)} \right\} \subset \Gamma_s(p^m).$$

Now we need to show $S_\infty = H^0(\mathcal{Y}_\infty, \mathcal{O}_{\mathcal{Y}_\infty})$ is perfectoid. Scholze uses really delicate arguments to show this. We give a very rough outline of the ideas.

(a) The first step is to guess the tilt of S_∞, denoted by S'_∞, which is a perfectoid algebra over $\mathbb{Q}_p^{\mathrm{cycl},\flat} = \mathbb{F}_p[\![t^{1/(p-1)p^\infty}]\!]$.
(b) The perfectoid algebra S'_∞ is constructed by hand by using the Siegel modular variety over characteristic p with $\Gamma_1(p^m)$-level structure. In general, $\Gamma_1(p^m)$-level structure does not make sense in characteristic p. But the construction of S'_∞ involves only the ordinary locus of the Siegel modular variety in characteristic p, and in this case the $\Gamma_1(p^m)$-level structure makes sense.
(c) Then, show that the untilt of S'_∞ over $\mathbb{Q}_p^{\mathrm{cycl}}$, which is a perfectoid algebra over $\mathbb{Q}_p^{\mathrm{cycl}}$, has to be isomorphic to S_∞. It follows that S_∞ is perfectoid. The main tool at work is Riemann's Hebbarkeitssatz (removable singularity theorem) for perfectoid spaces in characteristic p. One uses this to sandwich the almost algebra $(S_\infty/p)^a$ from two sides by
   - Computing tilt of the perfectoid space $\mathcal{Y}_\infty$, and
   - Computing tilt of base change of $\mathcal{Y}_m$ to the perfectoid space $\mathcal{X}^*_{\Gamma_s(p^\infty)}(\varepsilon)_a$.

The upshot of the above argument is the existence of a perfectoid space with m fixed:

$$\mathcal{X}^*_{\Gamma_1(p^m)\cap\Gamma_s(p^\infty)}(\varepsilon)_a \sim \varprojlim_{m'} \mathcal{X}^*_{\Gamma_1(p^m)\cap\Gamma_s(p^{m'})}(\varepsilon)_a.$$

The new tower with perfectoid limit is more refined in the sense that the intersection level group is smaller than $\Gamma_s(p^{m'})$. Further, the limit of LHS over m (by first passing to affinoid perfectoid covering, and then over each chart taking completed limit of the Huber pairs over m, as apposed to directly taking the limit in the category of adic spaces) leads to a perfectoid space

$$\mathcal{X}^*_{\Gamma_1(p^\infty)}(\varepsilon)_a \sim \varprojlim_{m} \mathcal{X}^*_{\Gamma_1(p^m)}(\varepsilon)_a.$$

Finally, from this, we deduce the existence of perfectoid space

$$\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_a \sim \varprojlim_{m} \mathcal{X}^*_{\Gamma(p^m)}(\varepsilon)_a.$$

This step uses the almost purity together with the amazing fact that the level-changing map

$$\mathcal{X}^*_{\Gamma(p^m)}(\varepsilon)_a \longrightarrow \mathcal{X}^*_{\Gamma_1(p^m)}(\varepsilon)_a$$

is finite étale even over the boundary. Caution that this only holds for anti-canonical ε-neighborhood and fails to hold for $\mathcal{X}^*_{\Gamma(p^m)} \to \mathcal{X}^*_{\Gamma_1(p^m)}$.

Remark 3.13.2. The last step above uses almost purity. Recall that we encountered this argument before: We used the perfectoidness of the tower

$$\varprojlim_{m} \mathcal{X}_{\Gamma_s(p^m)}(\varepsilon)_a$$

to prove perfectoidness of full-level tower $\varprojlim_m \mathcal{X}_{\Gamma(p^m)}(\varepsilon)_a$ by almost purity and the finite étaleness of the map $\mathcal{X}_{\Gamma(p^m)} \to \mathcal{X}_{\Gamma_s(p^m)}$.

Conclusion. We have now proved that for $0 \leqslant \varepsilon < 1/2$ there is a perfectoid space over $\mathbb{Q}_p^{\mathrm{cycl}}$:

$$\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}} \sim \varprojlim_m \mathcal{X}^*_{\Gamma(p^m)}(\varepsilon)_{\mathrm{a}}.$$

What remains to do is as follows.

(1) Construct the perfectoid space

$$\mathcal{X}^*_{\Gamma(p^\infty)} \sim \varprojlim_m \mathcal{X}^*_{\Gamma(p^m)} = \varprojlim_m (X^*_{\Gamma(p^m),\mathbb{Q}_p^{\mathrm{cycl}}})^{\mathrm{an}}.$$

(2) Construct the Hodge–Tate period map

$$\pi_{\mathrm{HT}} \colon \mathcal{X}^*_{\Gamma(p^\infty)} \longrightarrow \mathscr{F}\!\ell,$$

where $\mathscr{F}\!\ell$ is the adic space over $\mathbb{Q}_p^{\mathrm{cycl}}$ associated to some flag variety.

These two things will be done simultaneously. Moreover, if time permits, we will explain the application of these, that is, to understand the cohomology by using the geometry of $\mathcal{X}^*_{\Gamma(p^\infty)}$ and π_{HT}. We come to the outline to do (1) and (2) in the following.

Lect.21, Dec 26 3.14. **Hodge–Tate filtration.** For the purposes (1)(2) above, we discuss Hodge–Tate exact sequence and Hodge–Tate filtration for an abelian variety over some suitable p-adic field.

Let L be a discretely valued complete non-archimedean field (such as a finite extension of $\mathbb{Q}_p$ or $(\mathbb{Q}_p^{\mathrm{un}})^\wedge$). Let $C = \widehat{\overline{L}}$. Let A be an abelian variety over C. Given these, we have a *canonical* one-step filtration on $H^1_{\mathrm{et}}(A, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C = \mathrm{Hom}_{\mathbb{Z}_p}(T_p(A), C)$, read as

$$0 \subsetneq \mathrm{Fil}^1 \subsetneq H^1_{\mathrm{et}}(A, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C = \mathrm{Fil}^0$$

of C-vector spaces. If we suppose the abelian variety A has dimension g over C, then

$$\dim_C \mathrm{Fil}^1 = g, \quad \dim_C \mathrm{Fil}^0 = 2g.$$

The associated graded pieces have the following canonical identifications:

$$\begin{aligned} \mathrm{gr}^1 &= \mathrm{Fil}^1 \cong H^1(A, \mathcal{O}_A), \\ \mathrm{gr}^0 &= \mathrm{Fil}^0/\mathrm{Fil}^1 \cong H^0(A, \Omega_A)(-1). \end{aligned}$$

As a remark, in the isomorphism of gr^0, the Tate twist (-1) on $H^0(A, \Omega_A)$ is used to record the following: When A happens to be defined over L, both gr^0 and gr^1 are not only C-vector spaces but also Galois representations of $\mathrm{Gal}(\overline{L}/L)$; the isomorphisms are equivariant with respect to Galois actions only after taking the Tate twist. More precisely, if A comes from A_0 over L, then

- the $\mathrm{Gal}(\overline{L}/L)$ action on $\mathrm{Hom}_{\mathbb{Z}_p}(T_p(A), C)$ is via the natural action on $T_p(A)$ and the natural action on C (which extends the tautological action on $\overline{L}$ by continuity.)
- the filtration Fil^1 is stable under $\mathrm{Gal}(\overline{F}/F)$.
- the action of $\mathrm{Gal}(\overline{F}/F)$ on $H^1(A, \mathcal{O}_A) = H^1(A_0, \mathcal{O}_{A_0}) \otimes_L C$ is via the trivial action on the first factor and the natural action on the second.
- The action of $\mathrm{Gal}(\overline{F}/F)$ on $H^0(A, \Omega_A)(-1) = H^0(A_0, \Omega_A) \otimes_L C(-1)$ is via the trivial action on the first factor and the natural action on $C(-1)$, i.e., the natural action on C twisted by the inverse of the p-adic cyclotomic character $\mathrm{Gal}(\overline{F}/F) \to \mathbb{Z}_p^\times$.

Further, one may also assume that A is defined over L by enlarging L if necessary (as A is always defined over a subfield of C which is of finite transcendence degree over L, and hence discretely valued). If this is the case, the filtration has a unique $\mathrm{Gal}(\overline{L}/L)$-equivariant splitting, i.e. the quotient $\mathrm{gr}^0 = \mathrm{Fil}^0/\mathrm{Fil}^1$ is canonically a direct summand, and hence we have the *Hodge–Tate decomposition*

$$H^1_{\mathrm{et}}(A, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C = H^0(A, \Omega_A)(-1) \oplus H^1(A, \mathcal{O}_A).$$

This canonical splitting is due to the fact that $H^0(\mathrm{Gal}(\overline{L}/L), C) = L$ and $H^1(\mathrm{Gal}(\overline{L}/L), C(r)) = 0$ for $r \neq 0$ (Tate).

The upshot lies in that we have a short exact sequence of C-vector spaces

$$0 \longrightarrow H^1(A, \mathcal{O}_A) \longrightarrow H^1_{\mathrm{et}}(A, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \longrightarrow H^0(A, \Omega_A)(-1) \longrightarrow 0.$$

For convenience we take the dual of C-vector spaces, and obtain that

$$0 \longrightarrow (\operatorname{Lie} A)(1) \longrightarrow T_p(A) \otimes_{\mathbb{Z}_p} C \longrightarrow (\operatorname{Lie} A^\vee)^* \longrightarrow 0, \tag{HT}$$

where $A^\vee$ is the dual abelian variety. This (HT) is called the *Hodge–Tate exact sequence*, and we have seen that it splits canonically if A is defined over L.

First comments. Assume A has good reduction, i.e. it extends to an abelian scheme $\mathcal{A}$ over the ring of integers $\mathcal{O}_C$. Let G be the p-divisible group associated to $\mathcal{A}$. So G is a p-divisible group over $\mathcal{O}_C$. Then (HT) becomes in terms of the p-divisible group, written as

$$0 \longrightarrow (\operatorname{Lie} G) \otimes_{\mathcal{O}_C} C(1) \longrightarrow T_p(G_C) \otimes_{\mathbb{Z}_p} C \longrightarrow \operatorname{Hom}_{\mathcal{O}_C}((\operatorname{Lie} G^\vee), C) \longrightarrow 0.$$

Here $\operatorname{Lie} G$ is seen as a finite free $\mathcal{O}_C$-module and so is $\operatorname{Lie} G^\vee$. The two maps above have elementary definitions in terms of the p-divisible group G, see e.g. [SW20, §12.1]. However, just using these elementary definitions, it would be hard to show we really get an exact sequence. Actually, for any p-divisible group G over $\mathcal{O}_C$, these elementary definitions will always give an exact sequence as above, and furthermore this construction enters the following theorem.

Theorem 3.14.1 (Scholze–Weinstein, [SW13, Theorem B])**.** *There is an equivalence between*

- *The category of p-divisible groups over $\mathcal{O}_C$, and*
- *The category of pairs (T, W), where T is a finite free $\mathbb{Z}_p$-module and W is a C-vector subspace of $T \otimes_{\mathbb{Z}_p} C$.*

For one direction, beginning with a p-divisible group G over $\mathcal{O}_C$, we can take $T = T_p(G_C)$ and W as the image of the first map $(\operatorname{Lie} G) \otimes_{\mathcal{O}_C} C(1) \to T_p(G_C) \otimes_{\mathbb{Z}_p} C$ in the Hodge–Tate exact sequence for G. (We no longer require that G comes from an abelian scheme, so the numbers $\operatorname{rank} T$ and $\dim W$ are arbitrary, not necessarily satisfying $\operatorname{rank} T = 2 \dim W$.)

The point to takeaway is that:

> *Main idea.* The process of extracting the Hodge–Tate exact sequence from the abelian scheme $\mathcal{A}$ over $\mathcal{O}_C$ is like the process of associating to $\mathcal{A}$ a p-divisible group over $\mathcal{O}_C$.

Remark 3.14.2. We are mostly working in the case where A has good reduction for simplicity. But even in the bad reduction case, we can still look at the p-divisible group G attached to the connected Néron model of A over $\mathcal{O}_C$. In this situation G still has its own Hodge–Tate exact sequence. The Hodge–Tate filtration for A is determined by that for G by the following fact: There is a canonical injection $T_p(G) \otimes_{\mathbb{Z}_p} C \hookrightarrow T_p(A) \otimes_{\mathbb{Z}_p} C$, because the p^m-torsion points of G are exactly those p^m-torsion points of A which have "good reduction", i.e., extend along the connected Néron model. Under this injection, the nontrivial piece $\operatorname{Fil}^1 \subset T_p(G) \otimes_{\mathbb{Z}_p} C$ is mapped isomorphically to $\operatorname{Fil}^1 \subset T_p(A) \otimes_{\mathbb{Z}_p} C$.

Second comments. The Hodge–Tate exact sequence for abelian varieties recalled above was established by Tate, and it was later generalized to arbitrary smooth proper algebraic varieties by Faltings. For our purposes, we will take the point of view that the Hodge–Tate exact sequence and Hodge–Tate filtration come from the *Hodge–Tate spectral sequence*, which was established by Scholze in a purely p-adic analytic setting (as opposed to the setting of algebraic varieties).

Let X be any smooth proper rigid analytic variety (equivalently, an adic space of finite type which is smooth and proper) over C. We have the Hodge–Tate spectral sequence:

$$E_2^{i,j} = H^i(X, \Omega^j_{X/C})(-j) \implies H^{i+j}_{\text{et}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C.$$

Moreover, we also have the following.

Fact 3.14.3. The Hodge–Tate spectral sequence above degenerates if

- either X comes from an algebraic variety via analytification,
- or X is defined over $L \subset C$.

3.15. **The topological Hodge–Tate period map.** For the construction of the perfectoid space $\mathcal{X}^*_{\Gamma(p^\infty)}$, we begin with some preparations. First define the topological space

$$|\mathcal{X}^*_{\Gamma(p^\infty)}| \coloneqq \varprojlim_m |\mathcal{X}^*_{\Gamma(p^m)}|.$$

Similarly, take

$$|\mathcal{X}_{\Gamma(p^\infty)}| \coloneqq \varprojlim_m |\mathcal{X}_{\Gamma(p^m)}|.$$

Also define the boundary at level $\Gamma(p^m)$ as

$$\mathcal{Z}_{\Gamma(p^m)} \coloneqq \mathcal{X}^*_{\Gamma(p^m)} \backslash X^{\mathrm{ad}}_{\Gamma(p^m),\mathbb{Q}_p^{\mathrm{cycl}}}$$

together with its limiting topological space

$$|\mathcal{Z}_{\Gamma(p^\infty)}| \coloneqq \varprojlim_m |\mathcal{Z}_{\Gamma(p^m)}|.$$

Recall that $\mathcal{X}_{\Gamma(p^m)}$ is the good reduction locus. Thus, note that $\mathcal{X}_{\Gamma(p^m)}$ is an open subspace of $X^{\mathrm{ad}}_{\Gamma(p^m),\mathbb{Q}_p^{\mathrm{cycl}}}$ but they are not equal. In particular, $|\mathcal{X}^*_{\Gamma(p^\infty)}| \backslash |\mathcal{Z}_{\Gamma(p^\infty)}|$ is actually larger than $|\mathcal{X}_{\Gamma(p^\infty)}|$.

Definition 3.15.1. A *non-discrete affinoid field* is a Huber pair (L, L^+), where L is a non-archimedean field whose topology is not discrete, and L^+ is assumed to be a valuation subring of L (i.e. L^+ is an arbitrary open bounded valuation subring of L).

Fact 3.15.2. Let (R, R^+) be a Tate Huber pair. Then there exists a bijection of sets

$$|\operatorname{Spa}(R, R^+)| \longleftrightarrow \{(L, L^+, \varphi)\}/\cong,$$

where on RHS, for each triple (L, L^+, φ),

- (L, L^+) is a non-discrete affinoid field, and
- $\varphi\colon (R, R^+) \to (L, L^+)$ is a map of Huber pairs such that $\varphi(R)$ is dense in L.

We point out a subtlety in Fact 3.15.2. The set $|\operatorname{Spa}(L, L^+)|$ is not always a singleton. In fact, it is in bijection with the set of all valuation subrings $U \subset L$ such that $L^+ \subset U \subset L^0$. Moreover, if U, U' are two such valuation subrings corresponding to $y, y' \in |\operatorname{Spa}(L, L^+)|$, then we have $U \subset U'$ if and only if y is a specialization of y', i.e., $y \in \overline{\{y'\}}$. Also observe that for any non-discrete affinoid field (L, L^+), there is always a natural map $\operatorname{Spa}(L, \mathcal{O}_L) \to \operatorname{Spa}(L, L^+)$. The left hand side is a singleton, and the image of this map is the unique generic point. Compare the discussion in Example 2.4.7.

If $x \in |\operatorname{Spa}(R, R^+)|$ corresponds to the map $\varphi\colon \operatorname{Spa}(L, L^+) \to \operatorname{Spa}(R, R^+)$ through the bijection in Fact 3.15.2, then φ sends the unique closed point of $|\operatorname{Spa}(L, L^+)|$ (corresponding to the valuation subring $U = L^+$) to x, and sends the unique generic point of $|\operatorname{Spa}(L, L^+)|$ (corresponding to the valuation subring $U = \mathcal{O}_L$) to the unique maximal generalization of x. Conversely, starting with x, one gets the corresponding (L, L^+, φ) by taking L to be the completion of the residue field $k(x)$ of x, and taking L^+ to be the completion of $k(x)^+$ (cf. the discussion around Fact 2.5.6).

Construction 3.15.3 (Flag variety)**.** Let $G = \mathrm{GSp}_{2g}$, the similitude symplectic group as before. Consider

$$\mathrm{Fl} = G/P,$$

where P is the parabolic subgroup of G consisting of block upper-triangular $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in G$ with size-g blocks. Then Fl is a projective smooth algebraic variety over $\mathbb{Q}$, which classifies all Lagrangian subspaces of a $2g$-dimensional standard vector space. More precisely, we have the following moduli interpretation of it. For any $\mathbb{Q}$-algebra R,

$$\mathrm{Fl}(R) = \left\{ \begin{array}{c} R\text{-submodules } L \subset R^{2g} \text{ which is a locally direct summand of rank } g \\ \text{and isotropic with respect to the symplectic form } \left(\begin{smallmatrix} 0 & I_g \\ -I_g & 0 \end{smallmatrix}\right) \end{array} \right\}.$$

This is an analogue of the Grassmannian $\mathrm{Gr}(2g, g)$ for GSp_{2g}. Let

$$\mathscr{F}\!\ell \coloneqq (\mathrm{FL}_{\mathbb{Q}_p^{\mathrm{cycl}}})^{\mathrm{ad}}.$$

This is an adic space over $\mathbb{Q}_p^{\mathrm{cycl}}$.

Construction 3.15.4. We construct a map

$$|\pi_{\mathrm{HT}}|\colon |\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}^*_{\Gamma(p^\infty)}| \longrightarrow |\mathscr{F}\ell|.$$

Let $x \in |\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}^*_{\Gamma(p^\infty)}|$. Let $x_m \in |\mathcal{X}^*_{\Gamma(p^m)}|\backslash|\mathcal{Z}^*_{\Gamma(p^m)}|$ be the image of x. By Fact 3.15.2, the information carried by x_m is equivalent to a morphism $\varphi_m\colon \operatorname{Spa}(L_m, L_m^+) \to \mathcal{X}^*_{\Gamma(p^m)}\backslash\mathcal{Z}_{\Gamma(p^m)} = X^{\mathrm{ad}}_{\Gamma(p^m),\mathbb{Q}_p^{\mathrm{cycl}}}$, where (L_m, L_m^+) is a non-discrete affinoid field and L_m is taken to be the completion of the residue field of x_m. We observe that L_{m+1}/L_m is finite, and thus there exists a minimal non-discrete affinoid field (L, L^+) that contains $\bigcup_m (L_m, L_m^+)$. It follows from minimality that (L, L^+) is unique up to isomorphism. We note that by the minimality requirement, we do not have the freedom of enlarging L^+, and in particular it may not be equal to $\mathcal{O}_L$.

The φ_m we obtained before induces a morphism

$$\varphi_m\colon \operatorname{Spa}(L, L^+) \longrightarrow X^{\mathrm{ad}}_{\Gamma(p^m),\mathbb{Q}_p^{\mathrm{cycl}}}$$

that factors through $\operatorname{Spa}(L_m, L_m^+)$. Consider

$$\widetilde{\varphi}_m\colon \operatorname{Spa}(L, \mathcal{O}_L) \longrightarrow \operatorname{Spa}(L, L^+) \xrightarrow{\varphi_m} X^{\mathrm{ad}}_{\Gamma(p^m),\mathbb{Q}_p^{\mathrm{cycl}}},$$

i.e., the restriction of φ_m to the unique generic point of $\operatorname{Spa}(L, L^+)$. By the universal property of $X^{\mathrm{ad}}_{\Gamma(p^m),\mathbb{Q}_p^{\mathrm{cycl}}}$, a map from $\operatorname{Spa}(L, \mathcal{O}_L)$ to it is equivalent to a map of locally ringed topological spaces $\underline{\operatorname{Spa}(L, \mathcal{O}_L)} \to X_{\Gamma(p^m),\mathbb{Q}_p^{\mathrm{cycl}}}$, where the source is the underlying locally ringed topological space of $\operatorname{Spa}(L, \mathcal{O}_L)$. But $\underline{\operatorname{Spa}(L, \mathcal{O}_L)} = \operatorname{Spec} L$, so this datum exactly determines an L-point of the scheme $X_{\Gamma(p^m),\mathbb{Q}_p^{\mathrm{cycl}}}$.

From the moduli interpretation, this gives rise to a principally polarized abelian variety $(A, \lambda, \eta^p, \eta_p)$ over L of dimension g, together with its level structure $\eta_p\colon A[p^m](\overline{L}) \xrightarrow{\sim} (\mathbb{Z}/p^m)^{2g}$ at p. We have such a structure compatibly for all m, and hence have a trivialization $\eta_p : T_p(A_{\overline{L}}) \xrightarrow{\sim} \mathbb{Z}_p^{2g}$. Let C be the completion of $\overline{L}$. We obtain from the Hodge–Tate exact sequence for A that

$$0 \longrightarrow \operatorname{Lie} A_C \longrightarrow T_p(A_C) \otimes_{\mathbb{Z}_p} C \xrightarrow[\sim]{\eta_p} C^{2g}.$$

Thus, we obtain a g-dimensional C-vector subspace of C^{2g}, which can be checked to be isotropic. It is a fact that this subspace comes from L via base change (since A is defined over L). We thus get an L-point of Fl by the moduli interpretation of Fl, and similarly as before an L-point of Fl is equivalent to an $(L, \mathcal{O}_L)$-point of $\mathscr{F}\ell$. Since FL is proper, we have the partial properness of $\mathscr{F}\ell$, which implies that any map $\operatorname{Spa}(L, \mathcal{O}_L) \to \mathscr{F}\ell$ extends uniquely to $\operatorname{Spa}(L, L^+) \to \mathscr{F}\ell$, with the following commutative diagram.

$$\begin{tikzcd} \operatorname{Spa}(L, \mathcal{O}_L) \arrow[r] \arrow[d, hook] & \mathscr{F}\ell \\ \operatorname{Spa}(L, L^+) \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

In conclusion, we started with x and arrived at a morphism $\operatorname{Spa}(L, L^+) \to \mathscr{F}\ell$. We define $|\pi_{\mathrm{HT}}|(x) \in |\mathscr{F}\ell|$ to be the image of the unique closed point of $\operatorname{Spa}(L, L^+)$ under the morphism $\operatorname{Spa}(L, L^+) \to \mathscr{F}\ell$ constructed above. We thus obtain a map

$$|\pi_{\mathrm{HT}}|\colon |\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}^*_{\Gamma(p^\infty)}| \longrightarrow |\mathscr{F}\ell|.$$

The following theorem is difficult.

Theorem 3.15.5. *The map $|\pi_{\mathrm{HT}}|$ is continuous.*

The proof of Theorem 3.15.5 uses relative p-adic Hodge theory à la Scholze. We will discuss this in §3.18 later.

3.16. **The $G(\mathbb{Q}_p)$-action.** For $G = \mathrm{GSp}_{2g}$, note that $|\mathcal{X}^*_{\Gamma(p^\infty)}|$ carries a canonical $G(\mathbb{Q}_p)$-action. An element $g \in G(\mathbb{Q}_p)$ acts on the tower $(X^*_{\Gamma(p^m),\mathbb{Q}})_m$ in the following sense. For $m, n \geqslant 1$ such that $g\Gamma(p^m)g^{-1} \subset \Gamma(p^n)$ as subgroups of $G(\mathbb{Q}_p)$, we have a map $g\colon X^*_{\Gamma(p^m),\mathbb{Q}} \to X^*_{\Gamma(p^n),\mathbb{Q}}$. These maps are compatible with the multiplication in $G(\mathbb{Q}_p)$. Then by functoriality we have a $G(\mathbb{Q}_p)$-action on the Lect.22, Jan 2 tower of topological spaces $(|\mathcal{X}^*_{\Gamma(p^m)}|)_m$ in the similar sense. This structure induces the $G(\mathbb{Q}_p)$-action on $|\mathcal{X}^*_{\Gamma(p^\infty)}| = \varprojlim_m |\mathcal{X}^*_{\Gamma(p^m)}|$.

In fact, the $G(\mathbb{Q}_p)$-action on the tower $(X^*_{\Gamma(p^m),\mathbb{Q}})_m$ (which is equivalent to a $G(\mathbb{Q}_p)$-action on the projective limit $\varprojlim_m X^*_{\Gamma(p^m),\mathbb{Q}}$ in the category of schemes, where the limit exists because the transition maps are finite) is uniquely determined by a natural $G(\mathbb{Q}_p)$-action on $(X_{\Gamma(p^m),\mathbb{Q}})_m$. (In particular, the $G(\mathbb{Q}_p)$-action on $|\mathcal{X}^*_{\Gamma(p^\infty)}$ stabilizes $|\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}|$.) To explain the latter action, it is convenient to use a different version of the moduli problem for $X_{\Gamma(p^m),\mathbb{Q}}$.

Instead of looking at principally polarized abelian schemes up to isomorphisms, we look at polarized abelian schemes up to isogenies. Since this process identifies two isogenous abelian schemes as one point of the moduli space, it requires a new meaning of K^p-level structures (otherwise we would lose much more information about the moduli) together with a new meaning of $\Gamma(p^m)$-level structures. Let S be a test scheme and A a polarized abelian scheme up to isogeny over S. (An "abelian scheme up to isogeny" is by definition an object in the isogeny category of abelian schemes, i.e., the category whose objects are abelian schemes and whose Hom groups are given by $\mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}$. There is also a suitable notion of polarization for an abelian scheme up to isogeny.) For simplicity we assume S is connected and choose a geometric point s. Recall that $\Gamma(p^m)$ denotes a compact open subgroup of $G(\mathbb{Z}_p)$.

- The new meaning of a $\Gamma(p^m)$-level structure for A is a $\pi_1^{\mathrm{et}}(S, s)$-stable $\Gamma(p^m)$-orbit of isomorphisms
$$T_p(A_s) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \mathbb{Q}_p^{2g}$$
that preserve symplectic forms up to scalar. The condition of $\pi_1^{\mathrm{et}}(S, s)$-stability makes the structure independent of the choice of s. (Note that we do not require any single isomorphism to be $\pi_1^{\mathrm{et}}(S, s)$-equivariant.) Here $\Gamma(p^m)$ acts on $\mathbb{Q}_p^{2g}$ via the natural action of $G(\mathbb{Q}_p)$, and $\pi_1^{\mathrm{et}}(S, s)$ acts on $T_p(A_s)$.
- The new meaning of a K^p-level structure is defined similarly, which is a $\pi_1^{\mathrm{et}}(S, s)$-stable K^p-orbit of isomorphisms
$$T^p(A_s) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} (\mathbb{A}_f^p)^{2g},$$
that preserves symplectic forms up to scalar. For more details, see [GN06, §2.6]. (In this reference the authors only consider the passage to the prime-to-p-isogeny category and the corresponding change of meaning of a K^p-level structure, but the spirit is the same.)

Now $G(\mathbb{Q}_p)$ acts on the tower $(X_{\Gamma(p^m),\mathbb{Q}})_m$ by the above moduli interpretation and the natural $G(\mathbb{Q}_p)$-action on $\mathbb{Q}_p^{2g}$, which is used to change the level structure at p. We caution the reader that this action is not defined for a single level $\Gamma(p^m)$. Indeed, if $\{\eta\}$ is a $\Gamma(p^m)$-orbit of isomorphisms $T_p(A_s) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \mathbb{Q}_p^{2g}$, then for a general $g \in G(\mathbb{Q}_p)$, the set $\{g \circ \eta\}$ is not a $\Gamma(p^m)$-orbit of such isomorphisms, but rather a $g\Gamma(p^m)g^{-1}$-orbit. In the case where $g\Gamma(p^m)g^{-1} \subset \Gamma(p^n)$, we indeed get a new $\Gamma(p^n)$-level structure, and thus a morphism $g : X_{\Gamma(p^m),\mathbb{Q}} \to X_{\Gamma(p^n),\mathbb{Q}}$.

On the flag variety $\mathrm{Fl}_{\mathbb{Q}_p}$ we also have a $G(\mathbb{Q}_p)$-action. This action has the following moduli interpretation. Recall that for any $\mathbb{Q}_p$-algebra S,

$$\mathrm{Fl}_{\mathbb{Q}_p}(S) = \left\{ \begin{matrix} S\text{-submodules } L \subset S^{2g} \text{ which is a locally direct summand of rank } g \\ \text{and isotropic with respect to the symplectic form } \left(\begin{smallmatrix} 0 & I_g \\ -I_g & 0 \end{smallmatrix}\right) \end{matrix} \right\}.$$

The $G(\mathbb{Q}_p)$-action on the above set is via the natural $G(\mathbb{Q}_p)$-action on $S^{2g} = \mathbb{Q}_p^{2g} \otimes_{\mathbb{Q}_p} S$ (via the first factor), which indeed permutes isotropic locally direct summands.

The following is easy to see:

Proposition 3.16.1. *The map $|\pi_{\mathrm{HT}}|$ is $G(\mathbb{Q}_p)$-equivariant.*

3.17. **Outline of the proof of the main theorem.** We now roughly describe the whole argument that shows the existence of the perfectoid space

$$\mathcal{X}^*_{\Gamma(p^\infty)} \sim \varprojlim_m \mathcal{X}^*_{\Gamma(p^m)}.$$

Now we have already constructed the underlying topological space $|\mathcal{X}^*_{\Gamma(p^\infty)}|$.

Definition 3.17.1. An open subset $U \subset |\mathcal{X}^*_{\Gamma(p^\infty)}|$ is called *affinoid perfectoid* if there exists some $m \geqslant 1$ such that U is the inverse image of an open in $|\mathcal{X}^*_{\Gamma(p^m)}|$ coming from an affinoid open subspace $\mathrm{Spa}(R_m, R_m^+) \subset \mathcal{X}^*_{\Gamma(p^m)}$, and moreover if we write the inverse image of $\mathrm{Spa}(R_m, R_m^+)$ in $\mathcal{X}^*_{\Gamma(p^{m'})}$ for $m' \geqslant m$ as $\mathrm{Spa}(R_{m'}, R_{m'}^+)$ (this is necessarily affinoid), then the completed direct limit (R_∞, R_∞^+) of the pairs $(R_{m'}, R_{m'}^+)$ is such that R_∞ is a perfectoid $\mathbb{Q}_p^{\mathrm{cycl}}$-algebra and $R_\infty = R_\infty^+[1/p]$.

Definition 3.17.2. A union of affinoid perfectoid open subsets in $|\mathcal{X}^*_{\Gamma(p^\infty)}|$ is called *perfectoid.*

Our goal now is to show that $|\mathcal{X}^*_{\Gamma(p^\infty)}|$ is perfectoid. Once this is achieved, the argument would be done because we can equip the topological space $|\mathcal{X}^*_{\Gamma(p^\infty)}|$ with the structure of a perfectoid space $\mathcal{X}^*_{\Gamma(p^\infty)}$ coming from "charts" provided by the affinoid perfectoid open subsets that cover it, and moreover it directly follows that $\mathcal{X}^*_{\Gamma(p^\infty)} \sim \varprojlim_m \mathcal{X}^*_{\Gamma(p^m)}$.

By our previous work, we already know that

- The open subset $|\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}}| \subset |\mathcal{X}^*_{\Gamma(p^\infty)}|$ is (affinoid) perfectoid.

The main idea is to use the $G(\mathbb{Q}_p)$-action to "spread out" this open. See Proposition 3.19.1 and 3.19.3 later. The proof will use the topology of $|\pi_{\mathrm{HT}}|$ in an essential way.

Conclusion. The topological space $|\mathcal{X}^*_{\Gamma(p^\infty)}|$ is equipped with the structure of perfectoid space $\mathcal{X}^*_{\Gamma(p^\infty)}$ and $\mathcal{X}^*_{\Gamma(p^\infty)} \sim \varprojlim \mathcal{X}^*_{\Gamma(p^m)}$. After this, some more work shows that $|\pi_{\mathrm{HT}}|$ can be promoted to a map of adic spaces

$$\pi_{\mathrm{HT}} \colon \mathcal{X}^*_{\Gamma(p^\infty)} \longrightarrow \mathscr{F}\ell.$$

3.18. **Relative Hodge–Tate filtration.** In order to prove Theorem 3.15.5, we need the relative version of the Hodge–Tate filtration. For our purpose, we only need the first step of the filtration for H^1.

To setup the theory, let X be a locally noetherian adic space over $\mathbb{Q}_p$, namely X has local charts of form $\mathrm{Spa}(R, R^+)$ with R either being strongly noetherian (i.e. $R\langle X_1, \ldots, X_n\rangle$ is noetherian for all $n \geqslant 1$) or containing a noetherian ring of definition. In [Sch13] and its erratum, Scholze defines the pro-étale site X_{proet} of X. The objects of X_{proet} are certain cofiltered projective systems $(U_i)_{i\in I}$ where each U_i is an adic space equipped with an étale map to X. Only those projective systems which are *pro-étale* over X are objects of X_{proet}, and we omit the formal definition of this notion. Lect.23, Jan 4

The key idea is that *a pro-étale object over X is a tower of étale objects where transition maps are eventually finite étale surjective.* For this phenomenon, there is a natural morphism of sites

$$\nu \colon X_{\mathrm{proet}} \longrightarrow X_{\mathrm{et}}$$

induced by the functor of converse direction, along which the preimage of U with an étale map to X is the system $(U)_{i\in\{*\}}$. We illustrate the idea in a more precise sense below:

- *A typical object* of X_{proet} is the cofiltered projective system $(U_i)_{i\in I}$, where each U_i is an adic space equipped with an étale map $U_i \to X$ and where transition maps are eventually finite étale surjective, i.e. for all sufficiently large integers $i > j$ the transition maps $U_i \to U_j$ are finite étale surjective. (Caution: X_{proet} indeed has more objects than these "typical" ones.)
- The morphisms in X_{proet} are the usual morphisms between projective systems, namely,
$$\mathrm{Hom}((U_i)_{i\in I}, (V_j)_{j\in J}) = \varprojlim_j \varinjlim_i \mathrm{Hom}(U_i, V_j).$$
- The covers in X_{proet} are quite technical to define that we omit here, and we refer to the erratum of [Sch13].

It turns out that X_{proet} has nice basis (as a site) consisting of affinoid perfectoid objects. Inspired by Definition 3.17.1, we make the following definition.

Definition 3.18.1. A cofiltered projective system $U = (U_i)_{i\in I}$ of étale adic spaces over X is called *affinoid perfectoid* if

- transition maps are surjective and eventually finite étale (in the sense we described in the typical example before),
- we have $U_i = \mathrm{Spa}(R_i, R_i^+)$ for all $i \in I$, and

- denoting by R^+ the p-adic completion of $\varinjlim_i R_i^+$, the $\mathbb{Q}_p^{\mathrm{cycl}}$-algebra $R \coloneqq R^+[1/p]$ is perfectoid.

(For simplicity, we may assume that X and all U_i's are over some perfectoid field.)

An affinoid perfectoid $U = (U_i)_{i\in I}$ is an object of X_{proet}. A basis of the site X_{proet} is formed by all objects of this form. In the situation of Definition 3.18.1, $\widehat{U} \coloneqq \mathrm{Spa}(R, R^+)$ is an affinoid perfectoid space. Moreover, one can see that

$$\widehat{U} \sim \varprojlim_i U_i$$

in the usual sense.

With the morphism $\nu \colon X_{\mathrm{proet}} \to X_{\mathrm{et}}$, we consider some sheaves on X_{proet} as follows.

- $\mathcal{O}_X \coloneqq \nu^*\mathcal{O}_{X_{\mathrm{et}}}$ and $\mathcal{O}_X^+ = \nu^*\mathcal{O}_{X_{\mathrm{et}}}^+$ (recall that the $\mathcal{O}^+$-sheaf can be defined on every adic space, and in this case $\mathcal{O}_{X_{\mathrm{et}}}^+ \colon U \mapsto \mathcal{O}_U^+(U)$ as a functor).
- $\widehat{\mathcal{O}}_X^+$, the p-adic completion of $\mathcal{O}_X^+$ (at the level of sheaves $\widehat{\mathcal{O}}_X^+(U)$ may not be the same as $\varprojlim_n \mathcal{O}_X^+(U)/p^n$), and $\widehat{\mathcal{O}}_X \coloneqq \widehat{\mathcal{O}}_X^+[1/p]$.
- $\widehat{\mathbb{Z}}_p = \varprojlim_n \underline{\mathbb{Z}/p^n}$, where the limit is taken in the category of sheaves on X_{proet}.

Fact 3.18.2. Let $g \colon X \to Y$ be a proper smooth morphism between smooth locally noetherian adic spaces X and Y over $\mathbb{Q}_p$.

(1) We have an isomorphism in the category of sheaves on Y_{proet}:

$$(R^1 g_*\widehat{\mathbb{Z}}_p) \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathcal{O}}_Y \cong R^1 g_*\widehat{\mathcal{O}}_X.$$

(2) Obtaining (1), the natural map

$$(R^1 g_*\mathcal{O}_X) \otimes_{\mathcal{O}_Y} \widehat{\mathcal{O}}_Y \longrightarrow R^1 g_*\widehat{\mathcal{O}}_X \cong (R^1 g_*\widehat{\mathbb{Z}}_p) \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathcal{O}}_Y$$

relativizes the Hodge–Tate filtration $H^1(X, \mathcal{O}_X) \hookrightarrow H^1_{\mathrm{et}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C$ on H^1 in the absolute case (i.e. $Y = \mathrm{Spa}(C, \mathcal{O}_C)$ with $C = \widehat{\overline{\mathbb{Q}}}_p$).

(3) If $U = (U_i)_{i\in I}$ is an affinoid perfectoid object in Y_{proet}, then

$$\widehat{\mathcal{O}}_Y(U) = R, \quad \widehat{\mathcal{O}}_Y^+(U) = R^+,$$

where (R, R^+) is as in Definition 3.18.1 with affinoid perfectoid space $\widehat{U} = \mathrm{Spa}(R, R^+)$.

Remark 3.18.3. (1) For (1) and (3) of Fact 3.18.2, they ultimately come from the following theorem of Scholze in his thesis:

Theorem 3.18.4 (Scholze, [Sch12]). *For any affinoid perfectoid space W, we have*

$$H^i_{\mathrm{et}}(W, \mathcal{O}_W^{+,\mathrm{a}}) = 0, \quad \forall i > 0.$$

Here $\mathcal{O}_W^{+,\mathrm{a}}$ is the almost $\mathcal{O}_W^+$-module.

(2) To get complete Hodge–Tate filtration in arbitrary degrees of cohomology (in contrast to H^1 exclusively), a better point of view is the relative comparison theorem between p-adic étale and de Rham cohomologies (see [CS17]).

Proof of Theorem 3.15.5. We are to prove the continuity of

$$|\pi_{\mathrm{HT}}| \colon |\mathcal{X}^*_{\Gamma(p^\infty)}| \backslash |\mathcal{Z}^*_{\Gamma(p^\infty)}| \longrightarrow |\mathscr{F}\ell|.$$

At $\Gamma(1)$-level we take $S = \mathcal{X}^* \backslash \mathcal{Z} = X^{\mathrm{ad}}_{\mathbb{Q}_p^{\mathrm{cycl}}}$. Let A_S be the universal abelian variety over S with the proper morphism $g \colon A_S \to S$. One can apply Fact 3.18.2 about relative Hodge–Tate filtration to g to deduce the map

$$(R^1 g_*\mathcal{O}_{A_S}) \otimes_{\mathcal{O}_S} \widehat{\mathcal{O}}_S \longrightarrow R^1 g_*\widehat{\mathbb{Z}}_p \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathcal{O}}_S \tag{3.18.1}$$

between sheaves over S_{proet}. Under the usual topology, S is covered by affinoid opens U_0 such that U_0 is the bottom level of an affinoid perfectoid object $U = (U_i)_{i\in I}$ in S_{proet}, where in the projective system U the limit $\widehat{U}$ is affinoid perfectoid.

For each integer $m \geqslant 1$ let $\widehat{U}_m$ be the base change of $\widehat{U}$ to $\mathcal{X}^*_{\Gamma(p^m)} \backslash \mathcal{Z}_{\Gamma(p^m)}$. Similar to the argument in page 69, the almost purity result from the proof of Corollary 3.9.27 implies that $\widehat{U}_m$ is an affinoid perfectoid space, since it is finite étale over $\widehat{U}$ and $\mathcal{X}^*_{\Gamma(p^m)}\backslash\mathcal{Z}_{\Gamma(p^m)}$ is finite étale over S. As usual, we can construct an affinoid perfectoid space

$$\widehat{U}_\infty \sim \varprojlim_m \widehat{U}_m,$$

where the projective limit is induced by taking the complete direct limit of Huber pairs. Note that $\widehat{U}_\infty$ is induced by the perfectoid space associated to an affinoid perfectoid object in S_{proet}, which is the base change of $(U_i)_{i\in I}$ to $\mathcal{X}^*_{\Gamma(p^m)}\backslash\mathcal{Z}_{\Gamma(p^m)}$. We denote by $V \in S_{\mathrm{proet}}$ this new projective system indexed by (i, m) and then evaluate (3.18.1) at V. This leads to a map

$$(\mathrm{Lie}\, A_S) \otimes_{\mathcal{O}_S} R \longrightarrow R^{2g}$$

where $\mathrm{Lie}\, A_S$ is the relative Lie algebra, as a vector bundle on S, and $R = \mathcal{O}_{\widehat{U}_\infty}(\widehat{U}_\infty)$ with $\widehat{U}_\infty = \mathrm{Spa}(R, R^+)$. It follows that there is a map of adic spaces

$$\widehat{U}_\infty \xrightarrow{\varphi} \mathscr{F}\!\ell,$$

constructed by first getting an R-point of Fl and then extending it to $\widehat{U}_\infty = \mathrm{Spa}(R, R^+) \to \mathscr{F}\!\ell$ using partial properness (cf. Construction 3.15.4). We have the commutative diagram

$$\begin{tikzcd}
{|\widehat{U}_\infty|} \arrow[rr, "|\varphi|"] \arrow[d, "\psi"'] & & {|\mathscr{F}\!\ell|} \\
{|U_0| \times_{|S|} (\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}|)} \arrow[rr, hook, "\text{open}"] & & {|\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}|.} \arrow[u, "|\pi_{\mathrm{HT}}|"']
\end{tikzcd}$$

Here the map ψ is given as follows. By construction $\widehat{U}_\infty$ is the limit of base changes of U_i's to $\mathcal{X}^*_{\Gamma(p^m)}\backslash\mathcal{Z}_{\Gamma(p^m)}$ with respect to the index pair (i, m), so the underlying topological space of the base change admits a natural map to $|\mathcal{X}^*_{\Gamma(p^m)}\backslash\mathcal{Z}_{\Gamma(p^m)}|$; on the other hand the transition maps with respect to $i \in I$ gives $|\widehat{U}_\infty| \to |U_i| \to |U_0|$.

Moreover, ψ is continuous, and is open and surjective by pro-étale property. To conclude, we see the restriction of $|\pi_{\mathrm{HT}}|$ to the open

$$|U_0| \times_{|S|} (\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}|) \subset \mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}|$$

is continuous. On the other hand, different choices of U_0 altogether cover S, so such opens cover $\mathcal{X}^*_{\Gamma(p^\infty)}|\backslash|\mathcal{Z}_{\Gamma(p^\infty)}|$. Then we get the continuity of $|\pi_{\mathrm{HT}}|$. □

3.19. **The Hodge–Tate period map on adic spaces.** Fix $0 < \varepsilon \leqslant 1/2$. For our ultimate purpose (i.e. the existence of perfectoid space $\mathcal{X}^*_{\Gamma(p^\infty)} \sim \varprojlim_m \mathcal{X}^*_{\Gamma(p^m)}$) it remains to spreading out the perfectoidness via $G(\mathbb{Q}_p)$-action by proving the following.

Proposition 3.19.1. *We have*

$$G(\mathbb{Z}_p) \cdot |\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}}| = |\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)|.$$

Here the right side space is defined as the inverse image of the affinoid open $\mathcal{X}^*(\varepsilon) \subset \mathcal{X}^*$ (see Notation 3.10.3). More precisely, there are finitely many $\gamma_1, \ldots, \gamma_k \in G(\mathbb{Z}_p)$ such that

$$\bigcup_{i=1}^{k} \gamma_i \cdot |\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}}| = |\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)|.$$

Proof Sketch. By moduli interpretation, a point of $\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)$ is the triple of abelian variety (A, λ, η) where $\eta\colon T_p(A) \xrightarrow{\sim} \mathbb{Z}_p^{2g}$ (cf. Construction 3.15.4). Assume A has good reduction. The existence of anti-canonical subgroup $C_1 \subset A[p]$ is equivalent to that, modulo p,

$$(\eta \bmod p)\colon A[p] \xrightarrow{\sim} \mathbb{F}_p^{2g}$$

is an isomorphism sending C_1 to a subgroup of $\mathbb{F}_p^{2g}$ that is disjoint from the standard subgroup $\mathbb{F}_p^g \subset \mathbb{F}_p^{2g}$. But one can always achieve this up to moving $(\eta \bmod p)$ by an element of $G(\mathbb{F}_p)$, because the image of C_1 under $(\eta \bmod p)$ is a Lagrangian subspace in $\mathbb{F}_p^{2g}$ and $G(\mathbb{F}_p)$ acts transitively on the set of all

Lagrangian subspaces (so there is an element of $G(\mathbb{F}_p)$ moving any Lagrangian subspace to the last g coordinates). Note that $G(\mathbb{F}_p)$ is finite and take all γ_i's to be lifts of all elements of $G(\mathbb{F}_p)$ along $G(\mathbb{Z}_p) \twoheadrightarrow G(\mathbb{F}_p)$. □

Corollary 3.19.2. *$|\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)|$ is quasi-compact.*

Proof. For any $m \geqslant 1$, $|\mathcal{X}^*_{\Gamma(p^m)}(\varepsilon)_{\mathrm{a}}| \cong |\mathcal{X}^*(p^{-m}\varepsilon)|$ is quasi-compact. Taking the limit, so also is the finite union $|\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)|$. □

Proposition 3.19.3. *We have*

$$G(\mathbb{Q}_p) \cdot |\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)| = |\mathcal{X}^*_{\Gamma(p^\infty)}|.$$

More precisely, there are finitely many $\gamma_1, \dots, \gamma_k \in G(\mathbb{Q}_p)$ such that

$$\bigcup_{i=1}^{k} \gamma_i \cdot |\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)| = |\mathcal{X}^*_{\Gamma(p^\infty)}|.$$

Lect.24, Jan 9

The proof essentially uses the geometry of $|\pi_{\mathrm{HT}}|$. The key arithmetic nature lies in the following lemma. Before stating it, we write for simplicity that

$$\mathcal{Y}_\infty := |\mathcal{X}^*_{\Gamma(p^\infty)}| \backslash |\mathcal{Z}_{\Gamma(p^\infty)}|$$

and then write

$$\pi := |\pi_{\mathrm{HT}}| \colon \mathcal{Y}_\infty \longrightarrow |\mathscr{F}\ell|.$$

With $0 \leqslant \varepsilon < 1$ fixed, we define

$$\mathcal{Y}_\infty(\varepsilon) := |\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)| \backslash |\mathcal{Z}_{\Gamma(p^\infty)}(\varepsilon)|.$$

Lemma 3.19.4. *The preimage of $\mathrm{Fl}(\mathbb{Q}_p) = \mathscr{F}\ell(\mathbb{Q}_p) \subset |\mathscr{F}\ell|$ along $\pi\colon \mathcal{Y}_\infty \to |\mathscr{F}\ell|$ is the closure of $\mathcal{Y}_\infty(0)$ in $\mathcal{Y}_\infty$.*

Proof. We input two observations at work:

- Inside $|\mathscr{F}\ell|$, $\mathscr{F}\ell(\mathbb{Q}_p)$ is stable under generalization and specialization, i.e. for any $x, y \in |\mathscr{F}\ell|$ such that $x \rightsquigarrow y$ (i.e. $y \in \overline{\{x\}}$), $y \in \mathscr{F}\ell(\mathbb{Q}_p)$ if and only if $x \in \mathscr{F}\ell(\mathbb{Q}_p)$.
- Note that for $\varepsilon = 0$, $\mathcal{Y}_\infty(0)$ looks like the tubular neighborhood of the ordinary locus. Inside $\mathcal{Y}_\infty$, the closure of $\mathcal{Y}_\infty(0)$ precisely contains all specializations of points of $\mathcal{Y}_\infty(0)$, or equivalently all points $x \in \mathcal{Y}_\infty$ such that the (unique) maximal generalization $\widetilde{x}$ of x lies in $\mathcal{Y}_\infty(0)$. Here $\widetilde{x}$ corresponds to the value of $\mathrm{Spa}(L, \mathcal{O}_L)$ in $\mathcal{Y}_\infty$ for some non-discrete affinoid field $(L, \mathcal{O}_L)$.

Given these two observations, we reduce to proving that for any maximal generalization $x \in \mathcal{Y}_\infty$ we have $x \in \mathcal{Y}_\infty(0)$ if and only if $\pi(x) \in \mathscr{F}\ell(\mathbb{Q}_p)$. For this, in fact, we can consider $x\colon \mathrm{Spa}(C, \mathcal{O}_C) \to \mathcal{Y}_\infty$ with C a complete algebraically closed non-archimedean field over $\mathbb{Q}_p$, which corresponds by moduli problem to the triple (A, λ, η) over C, where (A, λ) is a principally polarized abelian variety and $\eta\colon T_p(A) \xrightarrow{\sim} \mathbb{Z}_p^{2g}$ is the level structure.

We first deal with the case of good reduction, in which A extends to an "integral" abelian scheme $\mathcal{A}$ over $\mathcal{O}_C$. Let $H := \mathcal{A}[p^\infty]$ be the p-divisible group over $\mathcal{O}_C$ associated to $\mathcal{A}$. It is known from Corollary 3.6.6 that $x \in \mathcal{Y}_\infty(0)$ if and only if H is ordinary (in which case the Hasse invariant is invertible), or equivalently $H \cong H_0 := (\mathbb{Q}_p/\mathbb{Z}_p)^g \times \mu_{p^\infty}^g$. So it suffices to show the equivalence between this and $\pi(x) \in \mathscr{F}\ell(\mathbb{Q}_p)$. This is done in the following.

- ◦ Suppose H is ordinary. Then $\pi(x)$, regarded as the Hodge–Tate filtration, measures the position of the canonical subgroups C_m of $\eta\colon \mathcal{A}[p^m] \xrightarrow{\sim} (\mathbb{Z}_p/p^m)^{2g}$. In particular, this proves $\pi(x) \in \mathscr{F}\ell(\mathbb{Q}_p)$.
- ◦ Conversely, suppose $\pi(x) \in \mathscr{F}\ell(\mathbb{Q}_p)$. By the classification result (Theorem 3.14.1) for the p-divisible group H over $\mathcal{O}_C$, it is uniquely characterized by the pair (T, W), where $T = T_p(H)$ and $W = \mathrm{Fil}^1 \subset T \otimes_{\mathbb{Z}_p} C$ is the Hodge–Tate filtration as a C-vector subspace. For a similar reason, H_0 corresponds to a unique pair (T_0, W_0). Next, we insert the following:

  Fact. $G(\mathbb{Z}_p)$ acts transitively on $\mathscr{F}\ell(\mathbb{Q}_p)$.

By the assumption that $\pi(x) \in \mathscr{F}\ell(\mathbb{Q}_p)$, we see the Hodge-Tate filtration for A is $\mathbb{Q}_p$-rational. We also identify T with $\mathbb{Z}_p^{2g}$ and hence $G(\mathbb{Z}_p)$ acts on $T \otimes_{\mathbb{Z}_p} C = C^{2g}$; moreover, this action clearly stabilize T. Also, there exists an element of $G(\mathbb{Z}_p)$ sending W to W_0. It follows that $(T, W) \cong (T_0, W_0)$, and hence $H \cong H_0$ as desired.

Now we deal with the case of bad reduction, in which x lies in the ordinary neighborhood $\mathcal{Y}_\infty(0)$ if and only if the p-divisible group H attached to Néron model of A over $\mathcal{O}_C$ is ordinary. On the other hand, the Hodge–Tate filtration on $T_p(A) \otimes_{\mathbb{Z}_p} C$ is determined by that on $T_p(H) \otimes_{\mathbb{Z}_p} C \subset T_p(A) \otimes_{\mathbb{Z}_p} C$. Then the same argument applies as in the good reduction case. This finishes the proof. $\square$

The following lemma is a key technical input, meaning that to understand the Hodge–Tate map for $\mathscr{F}\ell(\mathbb{Q}_p)$ it suffices to understand that restricted on some neighborhood $\mathcal{Y}_\infty(\varepsilon)$ of good reduction.

Lemma 3.19.5. *Fix $0 < \varepsilon < 1$. There is an open subset $U \subset \mathscr{F}\ell$ containing $\mathscr{F}\ell(\mathbb{Q}_p)$ such that*

$$\pi^{-1}(U) \subset \mathcal{Y}_\infty(\varepsilon).$$

Proof. The first step reduces the given statement to a similar one for $\tau := \pi|_{|\mathcal{X}_{\Gamma(p^\infty)}|}$, i.e. there exists an open neighborhood U of $\mathscr{F}\ell(\mathbb{Q}_p)$ in $\mathscr{F}\ell$ such that $\tau^{-1}(U) \subset |\mathcal{X}_{\Gamma(p^\infty)}(\varepsilon)|$. The idea is that when A has bad reduction, its Hasse invariant (defined by Hartog's extension principle) and Hodge–Tate filtration behave similarly to the case where A' has good reduction for another abelian variety A' with $\dim A' < g = \dim A$. In this case we take care of such points by induction on g.

Next, we are to prove the reduced statement $\tau^{-1}(U) \subset |\mathcal{X}_{\Gamma(p^\infty)}(\varepsilon)|$. This can be done with the following claims.

(a) The $\mathbb{Q}_p$-points $\mathscr{F}\ell(\mathbb{Q}_p)$ equals the intersection of all open neighborhoods U of $\mathscr{F}\ell(\mathbb{Q}_p)$.
(b) The quasi-compact open neighborhoods of $\mathscr{F}\ell(\mathbb{Q}_p)$ are cofinal among all open neighborhoods.

To prove (a), first note that $\mathscr{F}\ell(\mathbb{Q}_p)$ is definitely contained in the given intersection. For the inverse containment, also note that $\mathscr{F}\ell(\mathbb{Q}_p)$ is stable under generalization. It follows that if there exists some point $\xi \in U - \mathscr{F}\ell(\mathbb{Q}_p)$ for some open neighborhood $U \supset \mathscr{F}\ell(\mathbb{Q}_p)$, then the generalization $\{\xi\}$, as a closed subset, lies outside $\mathscr{F}\ell(\mathbb{Q}_p)$. Thus the complement of $\{\xi\}$ is another open neighborhood of $\mathscr{F}\ell(\mathbb{Q}_p)$, which makes it impossible for ξ lying in the intersection; so we attain a contradiction. As for (b), the topology on $\mathscr{F}\ell(\mathbb{Q}_p)$ inherited from $\mathscr{F}\ell$ is the same as the usual p-adic topology on $\mathbb{Q}_p$-points of any algebraic variety over $\mathbb{Q}_p$. In particular, since $\mathscr{F}\ell_{\mathbb{Q}_p}$ is projective, $\mathscr{F}\ell(\mathbb{Q}_p)$ itself is quasi-compact. But the topology of $\mathscr{F}\ell$ is generated by quasi-compact open sets.

Now it follows from (a consequence of) Lemma 3.19.4 that

$$\tau^{-1}(U) = \bigcap_{U \supset \mathscr{F}\ell(\mathbb{Q}_p)} \tau^{-1}(U) = \overline{|\mathcal{X}_{\Gamma(p^\infty)}(0)|} \subset |\mathcal{X}_{\Gamma(p^\infty)}(\varepsilon)|$$

where the intersection takes through quasi-compact open neighborhoods U of $\mathscr{F}\ell(\mathbb{Q}_p)$ in $|\mathscr{F}\ell|$, the closure is taken inside $|\mathcal{X}_{\Gamma(p^\infty)}|$, and the containment is due to $\varepsilon > 0$.

Our goal now is to extract a single $U \supset \mathscr{F}\ell(\mathbb{Q}_p)$ such that $\tau^{-1}(U) \subset |\mathcal{X}_{\Gamma(p^\infty)}(\varepsilon)|$. Indeed, τ is quasi-compact (this can be proved by using the same reduction to affinoid perfectoid case as in the proof of continuity of π, cf. Theorem 3.15.5), so each $\tau^{-1}(U)$ is quasi-compact and open by continuity. Also, recall that for any spectral topological space X, the constructible topology X_{cons} on X is the coarsest topology on X such that every quasi-compact open subset of X in original topology is open and closed. In fact,

- The constructible topology X_{cons} is more refined than the spectral topology X.
- The constructible topology X_{cons} is Hausdorff and quasi-compact.

In X_{cons}, $\tau^{-1}(U)$ is closed. Also, the complement of $|\mathcal{X}_{\Gamma(p^\infty)}(\varepsilon)|$ in $|\mathcal{X}_{\Gamma(p^\infty)}|$ is closed and therefore quasi-compact in X_{cons}. Therefore, there is some $U \subset \mathscr{F}\ell$ as desired. $\square$

Lemma 3.19.6. *For any open subset $U \subset \mathscr{F}\ell$ such that $U(\mathbb{Q}_p) \neq \emptyset$, we have*

$$G(\mathbb{Q}_p) \cdot U = \mathscr{F}\ell.$$

Proof. In general, one uses Plücker embedding of $\mathscr{F}\ell$ to some projective space. For simplicity, here we only illustrate the case where $g = 1$, $\mathscr{F}\ell = \mathbb{P}^1$, and $G = \mathrm{GL}_2$. Then each point in $\mathscr{F}\ell$ admits a projective coordinates $(x:y)$. Let V be the set of all points $(x:y)$ such that $|x|_p \geqslant |y|_p$, where $|\cdot|_p$ is the p-adic valuation. Then V is an affinoid open in $(\mathbb{P}^1_{\mathbb{Q}_p})^{\mathrm{ad}}$ (cf. $\mathcal{X}^*_{\Gamma(p^m)}(\varepsilon)$ is affinoid). In this case we want to show $G(\mathbb{Q}_p)\cdot V = |\mathscr{F}\ell|$. It suffices to prove that $V \subset G(\mathbb{Q}_p)\cdot U$ for any open subset $U \subset \mathscr{F}\ell$ such that $U(\mathbb{Q}_p) \neq \emptyset$. Indeed, we know $G(\mathbb{Q}_p)\cdot U$ contains $\mathscr{F}\ell(\mathbb{Q}_p)$. In particular, we can check whether an arbitrary point of $\mathscr{F}\ell(\mathbb{Q}_p)$ is the image of $G(\mathbb{Q}_p)$-action on V. For example, take $t = (1:0) \in \mathscr{F}\ell(\mathbb{Q}_p)$, and notice that for $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ we have $\gamma^n \cdot (x:y) \to t$ as $n \to \infty$ for all $(x:y) \in V$. Since V is quasi-compact, there is $n \in \mathbb{N}$ such that $\gamma^n \cdot V \subset G(\mathbb{Q}_p)\cdot U$ uniformly. This proves the lemma. $\square$

Proof of Proposition 3.19.3. Take U as in Lemma 3.19.5, such that $\pi^{-1}(U) \subset \mathcal{Y}_\infty(\varepsilon)$. By Lemma 3.19.6, $G(\mathbb{Q}_p)\cdot U = \mathscr{F}\ell$ as quasi-compact topological spaces. Then there are $\gamma_1, \dots, \gamma_k \in G(\mathbb{Q}_p)$ such that

$$\bigcup_{i=1}^{k} \gamma_i \cdot U = \mathscr{F}\ell.$$

On the other hand,

$$\mathcal{Y}_\infty = \pi^{-1}(\mathscr{F}\ell) = \bigcup_{i=1}^{k} \pi^{-1}(\gamma_i \cdot U) = \bigcup_{i=1}^{k} \gamma_i \cdot \pi^{-1}(U) \subset \bigcup_{i=1}^{k} \gamma_i \cdot \mathcal{Y}_\infty(\varepsilon).$$

Then

$$\mathcal{Y}_\infty = \bigcup_{i=1}^{k} \gamma_i \cdot \mathcal{Y}_\infty(\varepsilon).$$

Recall that our goal is to show the following:

$$|\mathcal{X}^*_{\Gamma(p^\infty)}| = \bigcup_{i=1}^{k} \gamma_i \cdot |\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)|.$$

For this, denote by V the right-hand side. Then V is a quasi-compact open in $|\mathcal{X}^*_{\Gamma(p^\infty)}|$ containing $\mathcal{Y}_\infty$. In fact, any quasi-compact open neighborhood of $\mathcal{Y}_\infty$ in $|\mathcal{X}^*_{\Gamma(p^\infty)}|$ is the whole space $|\mathcal{X}^*_{\Gamma(p^\infty)}|$ itself. Again, since V is quasi-compact, there exists m such that V is the preimage of some $V_m \subset |\mathcal{X}^*_{\Gamma(p^m)}|$, where V_m is a quasi-compact open neighborhood of $\mathcal{X}^*_{\Gamma(p^m)} \backslash \mathcal{Z}_{\Gamma(p^m)}$, for the constructible topology. Then it suffices to show the equality $V_m = |\mathcal{X}^*_{\Gamma(p^m)}|$. If this fails to hold, then there is a nonempty open in $\mathcal{X}^*_{\Gamma(p^m)}$ disjoint from $\mathcal{X}^*_{\Gamma(p^m)} \backslash \mathcal{Z}_{\Gamma(p^m)}$, which is impossible. $\square$

Lect.25, Jan 11

Recall that we have known the perfectoid space

$$\mathcal{X}^*_{\Gamma(p^\infty)} \sim \varprojlim_m \mathcal{X}^*_{\Gamma(p^m)}.$$

Now we are ready to assemble everything to upgrade the Hodge–Tate period map

$$\pi = |\pi_{\mathrm{HT}}| \colon |\mathcal{X}^*_{\Gamma(p^\infty)}| \backslash |\mathcal{Z}_{\Gamma(p^\infty)}| \longrightarrow |\mathscr{F}\ell|$$

on topological spaces to

$$\pi_{\mathrm{HT}} \colon \mathcal{X}^*_{\Gamma(p^\infty)} \longrightarrow \mathscr{F}\ell.$$

For this, the first (and the easiest) step is to construct the map $\pi_{\mathrm{HT}} \colon \mathcal{X}^*_{\Gamma(p^\infty)} \backslash \mathcal{Z}_{\Gamma(p^\infty)} \to \mathscr{F}\ell$ of adic spaces. It suffices to upgrade $\pi = |\pi_{\mathrm{HT}}|$ on anti-canonical neighborhood $\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}} \backslash \mathcal{Z}_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}}$. Note that this can be covered by affinoid perfectoid spaces U such that U is the perfectoid space attached to some affinoid perfectoid object in pro-étale site of $\mathcal{X}^*_{\Gamma(p^\infty)} \backslash \mathcal{Z}_{\Gamma(p^\infty)}$. As in the proof of continuity of π (Theorem 3.15.5), we can define $\pi_{\mathrm{HT}} \colon U \to \mathscr{F}\ell$ by relative Hodge–Tate filtration.

The main difficulty lies in extending the interior map $\pi_{\mathrm{HT}} \colon \mathcal{X}^*_{\Gamma(p^\infty)} \backslash \mathcal{Z}_{\Gamma(p^\infty)} \to \mathscr{F}\ell$ to the whole $\mathcal{X}^*_{\Gamma(p^\infty)}$ with the boundary.

- (*Uniqueness of extension*). In fact, for any affinoid perfectoid open $U \subset \mathcal{X}^*$, $\pi \colon U \backslash \mathcal{Z}_{\Gamma(p^\infty)} \to \mathscr{F}\ell$ has at most one extension to U.

- (*Existence of extension*). Recall that the adic space structure of $\mathcal{X}^*_{\Gamma(p^\infty)}$ is given by covering $|\mathcal{X}^*_{\Gamma(p^\infty)}|$ by $G(\mathbb{Q}_p)$-translates of $\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}}$ for any choice of $0 < \varepsilon < 1/2$. For the existence it reduces to showing that $\pi_{\mathrm{HT}} \colon \mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}} \backslash \mathcal{Z}_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}} \to \mathscr{F}\ell$ extends to $\mathcal{X}^*(\varepsilon)_{\mathrm{a}}$ for some fixed $0 < \varepsilon < 1/2$.

We discuss about the existence in details. The main step is to show that

$$\operatorname{im}(\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}} \backslash \mathcal{Z}_{\Gamma(p^\infty)} \xrightarrow{\pi_{\mathrm{HT}}} \mathscr{F}\ell) \subset D$$

for some affinoid open $D \subset \mathscr{F}\ell$. To explain why this is enough for our purpose, write $D = \operatorname{Spa}(R, R^+)$ with $R = R^+[1/p]$; the elements of R^+ pull back to global sections of $\mathcal{O}_D^+$ on $\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}} \backslash \mathcal{Z}_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}}$, and the same for those of R up to multiplying power of p. On the other hand, π on $\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}} \backslash \mathcal{Z}_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}}$ is defined by bounded functions on its source. Earlier, when we proved perfectoidness of $\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}}$ we went from $\mathcal{X}^*_{\Gamma_{\mathrm{s}}(p^\infty)}(\varepsilon)_{\mathrm{a}}$ to $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\varepsilon)_{\mathrm{a}}$. For this, we need two ingredients:

- ∘ Tate's normalized trace, and
- ∘ Riemann's Hebbarkeitssatz (removable singularity theorem) for perfectoid space of characteristic p (to a priori determine the tilt of $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\varepsilon)_{\mathrm{a}}$).

There is a byproduct of the second ingredient, that is, Riemann's Hebbarkeitssatz of characteristic 0, for $(\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}}, \mathcal{Z}_{\Gamma(p^\infty)}, \mathcal{X}_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}})$ ♠♠♠ Wenhan: [We haven't introduced a precise statement of Hebbarkeitssatz yet.], implying that

$$\begin{tikzcd}
\{\text{bounded functions on } \mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}} \backslash \mathcal{Z}_{\Gamma(p^\infty)}\} \arrow[dr, hook] \arrow[dd, leftrightarrow, "\sim"'] & \\
& \{\text{bounded functions on } \mathcal{X}_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}}\}. \\
\{\text{bounded functions on } \mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}}\} \arrow[ur, hook] &
\end{tikzcd}$$

Granting this, since π_{HT} on $\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}} \backslash \mathcal{Z}_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}}$ is defined using bounded functions, it extends to $\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}}$ as desired.

For the main step concerning about the image of π_{HT} on $\mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}} \backslash \mathcal{Z}_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}}$, we are to write down the affinoid D such that $\pi \colon \mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}} \backslash \mathcal{Z}_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}} \to D$.

At the level of algebraic varieties over $\mathbb{Q}$, we have

$$\mathrm{Fl} \hookrightarrow \mathrm{Gr}(2g, g) \hookrightarrow \mathbb{P}^{\binom{2g}{g}-1} = \mathbb{P}(\wedge^g \mathbb{Q}^{2g}).$$

Here Fl is the moduli of Lagrangian subspaces of the standard $2g$-dimensional symplectic space, so it naturally embeds into $\mathrm{Gr}(2g, g)$. The second map is the Plücker embedding, sending a subspace $L \subset \mathbb{Q}^{2g}$ of dimension g to $\wedge^g L$, and the image is a line in $\wedge^g \mathbb{Q}^{2g}$.

Fix a basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{2g}\}$ of $\mathbb{Q}^{2g}$. Then $\wedge^g \mathbb{Q}^{2g}$ has basis $\{\boldsymbol{e}_{\mathtt{J}}\}_{\mathtt{J}}$, where each index $\mathtt{J}$ is a subset of $\{1, \ldots, 2g\}$ of size g, and $\boldsymbol{e}_{\mathtt{J}} \coloneqq \boldsymbol{e}_{j_1} \wedge \cdots \wedge \boldsymbol{e}_{j_g}$ when $\mathtt{J} = \{j_1 < \cdots < j_g\}$. This induces a homogenous coordinate system $x_{\mathtt{J}}$ on $\mathbb{P}^{\binom{2g}{g}-1}$.

In general, suppose the points in projective space $\mathbb{P}^N$ over $\mathbb{Q}_p$ are encoded by the homogeneous coordinates $(x_0 : \cdots : x_N)$. Then for each fixed $0 \leqslant j \leqslant N$ we have an affinoid open in $\mathbb{P}^{N,\mathrm{ad}}$ given by points with $|x_i| \leqslant |x_j|$ for all $0 \leqslant i \leqslant N$. In fact, this forms the closed unit disk in $\mathbb{A}^{N,\mathrm{ad}}$ where $\mathbb{A}^N$ is defined by points with $x_j \neq 0$ for fixed j. Then $\mathbb{P}^{N,\mathrm{ad}}$ is covered by these closed unit disks indexed by $0 \leqslant j \leqslant N$. Moreover, $(\mathbb{P}^{\binom{2g}{g}-1})^{\mathrm{ad}}$ is covered by the closed unit disks indexed by $\mathtt{J} \subset \{1, \ldots, 2g\}$ of size g; the inverse image of such a disk in $(\mathbb{P}^{\binom{2g}{g}-1})^{\mathrm{ad}}$ inside $\mathscr{F}\ell$ is denoted by $\mathscr{F}_{\mathtt{J}}$, which is an affinoid open in $\mathscr{F}\ell$.

From now on, for simplicity we assume $\mathtt{J} = \{g+1, \ldots, 2g\}$. We have the following easy fact.

Fact 3.19.7. Taking a point $L \in \mathscr{F}\ell(\mathbb{Q}_p)$, it is a Lagrangian subspace in $\mathbb{Z}_p^{2g}$, i.e. it is a totally isotropic direct summand of rank g. For $\mathtt{J} = \{g+1, \ldots, 2g\}$, we have $L \in \mathscr{F}_{\mathtt{J}}$ if and only if $(L \bmod p)$, as an $\mathbb{F}_p$-vector subspace in $\mathbb{F}_p^{2g}$, is disjoint from $\mathbb{F}_p^g \oplus \mathbf{0}^g \subset \mathbb{F}_p^{2g}$. (Here $\mathbb{F}_p^g \oplus \mathbf{0}^g$ denotes the subspace that the last g coordinates are all zeros.)

As before we fix $0 < \varepsilon < 1/2$. It now remains to show that $\pi_{\mathrm{HT}}\colon \mathcal{X}^*_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}}\backslash\mathcal{Z}_{\Gamma(p^\infty)}(\varepsilon)_{\mathrm{a}} \to \mathscr{F}\ell$ factors through $\mathscr{F}_{\mathtt{J}}$ for $\mathtt{J} = \{g+1,\dots,2g\}$ (and also for all other $\mathtt{J}$'s). Using the topological argument, it reduces to showing the following. (Cf. Lemma 3.19.4, dictating that the preimage of $\mathscr{F}\ell(\mathbb{Q}_p)$ in $\mathcal{X}^*_{\Gamma(p^\infty)}\backslash\mathcal{Z}_{\Gamma(p^\infty)}$ under π_{HT} is the closure of $\mathcal{X}^*_{\Gamma(p^\infty)}(0)\backslash\mathcal{Z}_{\Gamma(p^\infty)}(0)$.)

Lemma 3.19.8. *The preimage of $\mathscr{F}_{\mathtt{J}}(\mathbb{Q}_p)$ along $\pi_{\mathrm{HT}}\colon \mathcal{X}^*_{\Gamma(p^\infty)}\backslash\mathcal{Z}_{\Gamma(p^\infty)} \to \mathscr{F}\ell$ equals the closure of $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_{\mathrm{a}}\backslash\mathcal{Z}_{\Gamma(p^\infty)}(0)_{\mathrm{a}}$.*

Proof. **Step I.** In the case of good reduction, the preimage of $\mathscr{F}_{\mathtt{J}}(\mathbb{Q}_p)$ in $\mathcal{X}_{\Gamma(p^\infty)}$ is the closure of $\mathcal{X}_{\Gamma(p^\infty)}(0)_{\mathrm{a}}$ inside $\mathcal{X}_{\Gamma(p^\infty)}$. Indeed, we only need to check it for $(C,\mathcal{O}_C)$-points $x\colon \operatorname{Spa}(C,\mathcal{O}_C) \to \mathcal{X}_{\Gamma(p^\infty)}$ that $\pi_{\mathrm{HT}}(x) \in \mathscr{F}_{\mathtt{J}}(\mathbb{Q}_p)$ if and only if $x \in \mathcal{X}_{\Gamma(p^\infty)}(0)_{\mathrm{a}}$. It is already known from the proof of Lemma 3.19.4 that $\pi_{\mathrm{HT}}(x) \in \mathscr{F}\ell(\mathbb{Q}_p)$ if and only if $x \in \mathcal{X}_{\Gamma(p^\infty)}(0)$. Thus, we only need to translate the condition imposed by $\mathtt{J}$ onto the anti-canonical locus. However, $\pi(x)$ measures the position of canonical subgroups of $\mathcal{A}_x$ over $\mathcal{O}_C$, so we are done when $\mathcal{A}_x$ has good reduction.

Step II. Next we show that $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_{\mathrm{a}}\backslash\mathcal{Z}_{\Gamma(p^\infty)}(0)_{\mathrm{a}}$ is mapped to $\mathscr{F}_{\mathtt{J}}(\mathbb{Q}_p)$. Suppose this is not the case. Then there is a clopen in $\mathscr{F}\ell(\mathbb{Q}_p)$ disjoint from $\mathscr{F}_{\mathtt{J}}(\mathbb{Q}_p)$ that intersects with the image of $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_{\mathrm{a}}\backslash\mathcal{Z}_{\Gamma(p^\infty)}(0)_{\mathrm{a}}$. It follows that we can take a nonempty clopen in $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_{\mathrm{a}}\backslash\mathcal{Z}_{\Gamma(p^\infty)}(0)_{\mathrm{a}}$ whose image in $\mathscr{F}\ell(\mathbb{Q}_p)$ is disjoint from $\mathscr{F}_{\mathtt{J}}(\mathbb{Q}_p)$. By Riemann's Hebbarkeitssatz, any clopen of $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_{\mathrm{a}}\backslash\mathcal{Z}_{\Gamma(p^\infty)}(0)_{\mathrm{a}}$ is the restriction of a unique clopen of $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_{\mathrm{a}}$. Consequently, $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_{\mathrm{a}}$ contains a clopen V such that $\pi_{\mathrm{HT}}(V\backslash\mathcal{Z}_{\Gamma(p^\infty)}(0)_{\mathrm{a}}) \subset \mathscr{F}\ell(\mathbb{Q}_0)\backslash\mathscr{F}_{\mathtt{J}}(\mathbb{Q}_p)$. Again by Riemann's Hebbarkeitssatz, the space of bounded functions on $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_{\mathrm{a}}$ injects into that of bounded functions on $\mathcal{X}_{\Gamma(p^\infty)}(0)_{\mathrm{a}}$. It follows that $V \cap \mathcal{X}_{\Gamma(p^\infty)}(0)_{\mathrm{a}} \neq \emptyset$, and then one can take $x \in V \cap \mathcal{X}_{\Gamma(p^\infty)}(0)_{\mathrm{a}}$ just so $x \in V\backslash\mathcal{Z}_{\Gamma(p^\infty)}(0)_{\mathrm{a}}$. This forces x to be mapped outside $\mathscr{F}_{\mathtt{J}}$, which is a contradiction to Step I.

Step III. For any $(C,\mathcal{O}_C)$-point $x \in \mathcal{X}^*_{\Gamma(p^\infty)}\backslash\mathcal{Z}_{\Gamma(p^\infty)}$, we want to prove that $\pi_{\mathrm{HT}}(x) \in \mathscr{F}_{\mathtt{J}}(\mathbb{Q}_p)$ if and only if $x \in \mathcal{X}^*_{\Gamma(p^\infty)}(0)_{\mathrm{a}}\backslash\mathcal{Z}_{\Gamma(p^\infty)}(0)_{\mathrm{a}}$. Note that the "if" direction is implied by Step II. Suppose the "only if" direction is not true. With this assumption, fixing an element $\gamma \in G(\mathbb{Z}_p)$, there exists $x \in \mathcal{X}^*_{\Gamma(p^\infty)}(0) = G(\mathbb{Z}_p)\cdot\mathcal{X}^*_{\Gamma(p^\infty)}(0)_{\mathrm{a}}$ (applying Proposition 3.19.1) such that $\gamma\cdot x \in \mathcal{X}^*_{\Gamma(p^\infty)}(0)_{\mathrm{a}}\backslash\gamma\cdot\mathcal{X}^*_{\Gamma(p^\infty)}(0)_{\mathrm{a}}$. Similar to the proof in Step II, it turns out that if $\mathcal{X}^*_{\Gamma(p^\infty)}(0)_{\mathrm{a}}\backslash\gamma\cdot\mathcal{X}^*_{\Gamma(p^\infty)}(0)_{\mathrm{a}}$ contains an element y that is mapped into $\gamma\cdot\mathscr{F}_{\mathtt{J}}(\mathbb{Q}_p)$. Then $\mathcal{X}_{\Gamma(p^\infty)}(0)_{\mathrm{a}}\backslash\gamma\cdot\mathcal{X}_{\Gamma(p^\infty)}(0)_{\mathrm{a}}$ contains some y' that is mapped into $\gamma\cdot\mathscr{F}_{\mathtt{J}}(\mathbb{Q}_p)$. Note that such y can be taken as $\gamma\cdot x$, and in this case the corresponding y' satisfies $\gamma^{-1}y' \in \gamma^{-1}\cdot\mathcal{X}_{\Gamma(p^\infty)}(0)_{\mathrm{a}}\backslash\mathcal{X}_{\Gamma(p^\infty)}(0)_{\mathrm{a}} \subset \mathcal{X}_{\Gamma(p^\infty)}(0)\backslash\mathcal{X}_{\Gamma(p^\infty)}(0)_{\mathrm{a}}$, and hence $\gamma^{-1}\cdot y'$ is mapped into $\mathscr{F}_{\mathtt{J}}(\mathbb{Q}_p)$. This contradicts to Step I again. ∎

Now we have completed the proof of the main theorem.

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