THE LANDAU-SIEGEL ZERO PROBLEM IN NUMBER THEORY

YITANG ZHANG
(NOTES BY WENHAN DAI)

These are notes from a short public lecture given by Yitang Zhang (UCSB) after receiving the Future Science Prize at the China Future Forum in November 2019.[1]

1. The Generalized Riemann Hypothesis

The Riemann zeta function is originally defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},\quad \mathrm{Re}(s)>1.$$

This series is not convergent in case $\mathrm{Re}(s)\leqslant 1$. It is Riemann who proves that $\zeta(s)$ can be analytically continued to the whole complex plane with a simple pole at $s=1$ (whose residue is exactly 1), and satisfies a function equation

$$\zeta(s)=(\text{simple factor})\cdot\zeta(1-s).$$

The Riemann zeta function is closely related to the prime numbers via the Euler identity

$$\zeta(s)=\prod_p(1-\frac{1}{p^s})^{-1},\quad \mathrm{Re}(s)>1,$$

where p runs through prime numbers. For example, the famous prime number theorem

$$\lim_{x\to\infty}\frac{\ln x}{x}\cdot\sum_{p\leqslant x}1=1,$$

proved by Hadamard and de la Vallée Poussin in 1896, is equivalent to the assertion that $\zeta(s)\neq 0$ if $\mathrm{Re}(s)=1$. More explicit information about the distribution of prime numbers depends on further studies of the zeros of $\zeta(s)$. The famous **Riemann Hypothesis** (RH) asserts that all non-real zeros of $\zeta(s)$ has the real part $1/2$. This is equivalent to the assertion that

$$\zeta(s)\neq 0,\quad \mathrm{Re}(s)>\frac{1}{2}.$$

Right now, however, the known results on the location of ζ-zeros are much weaker than the Riemann Hypothesis.

There are other functions analogous to the Riemann zeta function which play important roles in number theory. Let $D\geqslant 1$. A **Dirichlet character** to the modulus D is a function

$$\chi:\mathbb{Z}\longrightarrow\mathbb{C}$$

that is not identically zero and such that

(1) $\chi(n+D)=\chi(n)$ for all $n\in\mathbb{Z}$;
(2) $\chi(mn)=\chi(m)\chi(n)$ for all $m,n\in\mathbb{Z}$; namely, χ is multiplicative;

Date: October 17, 2022.

[1]In mid-October 2022, Yitang Zhang implicitly admitted (at a Peking University alumni gathering in the New York area) that he had solved the Landau-Siegel zero problem, and an preprint copy of the thesis is scheduled to be released in November.

(3) $\chi(n) = 0$ if n is not coprime to D.

The **Dirichlet L-function** for χ is given by

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \mathrm{Re}(s) > 1,$$

which admits the analytic continuation and function equation like $\zeta(s)$.

The **Generalized Riemann Hypothesis** (GRH) asserts that, for any Dirichlet character χ,

$$L(s,\chi) \neq 0, \quad \mathrm{Re}(s) > \frac{1}{2}.$$

Right now, no one is able to prove this hypothesis for a single χ. Since the end of the 19th century, mathematicians have developed a number of analytic tools to study the zeros of $\zeta(s)$ and $L(s,\chi)$, and obtained many important results. However, in comparison with the GRH, these results seem to be much weaker. For example, we can only prove that $\zeta(s)$ and $L(s,\chi)$ do not vanish when the real part of s is near to 1, using elementary inequalities like

$$3 + 4\cos\theta + \cos 2\theta \geqslant 0.$$

2. The Landau-Siegel Zero Problem

Let χ mod D be a nontrivial real character (this means $\chi(n)$ takes the value 1, -1, and 0 only). It is known that the functions $L(s,\chi)$ has at most one simple, real zero β such that

$$1 - \frac{c}{\log D} < \beta < 1,$$

where $c > 0$ is a constant. Such a real zero β, if exists, is called the **Landau-Siegel zero of** $L(s,\chi)$.

We are interested in the case that D is sufficiently large only. Then the bound $1 - c/\log D$ is close to 1. Thus the existence of β means that $L(s,\chi)$ vanishes at some s near to 1. This obviously contradicts the GRH. The Landau-Siegel zero problem is to show that such a β does not exist!

The Landau-Siegel zero problem remains open; the typical methods relying on elementary inequalities are useless here. This problem is regards as a "bottle neck" in number theory, since its solution will result in many important consequences in both analytic and algebraic number theory. Since the early 20th century, many number theorists have attempted to attack this problem without success. Some experts even predict that the solution of the Landau-Siegel zero problem would be more difficult than the ordinary RH. In many researches, number theorists are forced to discuss two cases depending on the Landau-Siegel zero exists or not.

3. If the Landau-Siegel Zero Exists?

One may ask what can happen if the Landau-Siegel zero exists for some χ. It is interesting that, in such a situation, many consequences can be obtained, some of which seem to be too strong. This phenomenon is, roughly speaking, based on the following fact:

- ⋄ *the function $\zeta(2s)\zeta(s)^{-1}$ has a simple zero at $s = 1$.*

If $L(s,\chi)$ has a simple zero near to 1, a careful analysis shows that $\zeta(2s)\zeta(s)^{-1}$ behaves like $L(s,\chi)$ in a certain region. Now, suppose that the "certain region" can be taken as the half-plane $\mathrm{Re}(s) > 1/2$. Since $L(s,\chi)$ is analytic in this region, the function $\zeta(2s)\zeta(s)^{-1}$ is also analytic for $\mathrm{Re}(s) > 1/2$. This leads to the classical RH.

We give two examples to show what can happen if the Landau-Siegel zero exists.

(I) **Assumption:** there is an infinite set of real characters such that, for each χ belonging to this set, the corresponding Landau-Siegel zero exists.

Conclusion: there are infinitely many pairs of twin primes.

(II) **Assumption:** there is a real character χ mod D such that the corresponding Landau-Siegel zero exists.
Conclusion: there is a set Ψ of characters ψ and a complex region Ω, both depending on D, such that, whenever

$$\psi \in \Psi, \quad s \in \Omega, \quad \mathrm{Re}(s) > \frac{1}{2},$$

one has

$$L(s, \psi) \neq 0.$$

4. Distribution of Zeros on $\mathrm{Re}(s) > 1/2$

Resuming with the Landau-Siegel zero problem, it should be remarked that, in Example (II) above, information about the distribution of zeros of $L(s, \psi)$ on the vertical line $\mathrm{Re}(s) = 1/2$ in Ω is also obtained. However, such information contradicts some known conjectures on the distribution of zeros (on assuming GRH). This provides an opportunity to derive a contradiction from the existence of the Landau-Siegel zero! Here the analytic tools used to study the simple zeros may apply.

5. A New Attempt

Currently, Zhang's approach to the Landau-Siegel zero problem consists in reducing it to evaluating certain sums over the zeros of $L(s, \psi)$ as mentioned above. Although there are still many technical details to fulfill, it is possible to find functions $\Phi(s, \psi)$ and show that the existence of the Landau-Siegel zero implies

$$\sum_{\psi \in \Phi} \sum_{\rho} |\Phi(\rho, \psi)| < \left| \sum_{\psi \in \Phi} \sum_{\rho} \Phi(\rho, \psi) \right|,$$

where ρ runs through the zeros of $L(s, \psi)$ in Ω. This contradiction would solve the problem.

School of Mathematical Sciences, Peking University, 100871, Beijing, China
Email address: daiwenhan@pku.edu.cn