# Lecture 9: The case $k=\mathbb{C}$; the theorem of the cube (II).

by Shun Yin, Nov 11.

**Goal** (1) To compute $T^*_{\pi(a)}(L(H,\alpha))$

(2) To describe divisoral correspondence & to compute $\hat{X}$ analytically.

(3) The theorem of cube (II)

**(A) Recall** $k=\mathbb{C}$, $X=V/U$, where $V$ complex v.s., $U$ lattice ($\rightsquigarrow V = U\otimes_{\mathbb{Z}}\mathbb{R}$) with the line bundle on $X$, $L(H,\alpha)$,

$$\begin{cases} H = \text{Hermitian form}, \\ E = \operatorname{Im} H : U\times U \to \mathbb{Z} \end{cases}$$

A $\mathbb{C}\times V/U$ - action

$$u \mapsto \left(\phi_u(\lambda, v) = (\lambda\cdot\alpha(u)\, e^{\pi H(v,u)+\frac{\pi}{2}H(u,u)}, z+u\right).$$

Here $\alpha(u_1+u_2) = \alpha(u_1)\cdot\alpha(u_2)\cdot e^{i\pi E(u_1,u_2)}$, $\forall u_1,u_2\in U$.

To simplify the cocycle:

- take $g(z)$ = nonzero holo function, $g(z) = e^{\pi H(z,a)}$.

$\rightsquigarrow$ The action on $\mathbb{C}\times V_2/U$:

$$\phi'_u(\lambda, v) = (\lambda e_u(Tv), v+u)$$

The action on $\mathbb{C}\times V_1/U$:

$$(\lambda, v) \longmapsto (\lambda\cdot g(v), v)$$

$$\begin{tikzcd} \mathbb{C}\times V_1 \arrow[r, "T"] \arrow[d, "\phi'"'] & \mathbb{C}\times V_2 \arrow[d, "\phi"] \\ X_1 \arrow[r, "\bar{T}"] & X_2 \end{tikzcd}$$

via $\phi'_u(\lambda, v) = (\lambda\cdot e_u(Tv)\, g(u+v)\, g(v)^{-1}, v)$.

When $T = T_a : V_1 \longrightarrow V_2$ with $U$-actions on $L_1, L_2$, resp.

$$\phi_u(\lambda, v) = (\lambda\cdot\alpha(u)\, e^{\pi H(v+a,u)+\frac{1}{2}\pi H(u,u)}, v+u).$$

**Prop** $T^*_{\pi(\alpha)}[L(H,\alpha)] \cong L(H, \alpha\cdot\gamma_\alpha)$ where $\gamma_\alpha(u) = e^{2\pi i E(\alpha,u)}$.

<u>Immediate consequences</u>

(1) $\phi_{L(H,\alpha)}(\pi(\alpha))$ has image pt. $L(0,\gamma_\alpha) \in \mathrm{Pic}^0(X)$.

Recap: $\phi_L(x) = T_x^* L \otimes L^{-1}$,

$\rightsquigarrow \phi_{L(H,\alpha)}(x) = L(H, \alpha\delta_x) \otimes L^{-1}(H,\alpha) = (0, \delta_x)$.

(2) $\phi_{L(H,\alpha)}$ is a homomorphism,

b/c $\gamma_x \cdot \gamma_y = \gamma_{x+y}$ $\leftarrow$ $(T_x^* L \otimes L^{-1}) \otimes (T_y^* L \otimes L^{-1}) \cong T_{x+y}^* L \otimes L^{-1}$

(note this is the thm of square.)

(3) $\mathrm{Ker}\,\phi_L = K(L)$ $\quad (\delta_x = 1 \Leftrightarrow E(x,u) \in \mathbb{Z},\ \forall u \in U)$

Then $K(L(H,\alpha)) = U^\perp / U \subseteq V/U = X$

where $U^\perp = \{x \in V : E(x,u) \in \mathbb{Z},\ \forall u \in U\}$.

(4) By (3), $L(H,\alpha) \in \mathrm{Pic}^0(X) = \ker(L \mapsto \phi_L)$.

$\Leftrightarrow K(L(H,\alpha)) = X \Leftrightarrow U^\perp = V \Leftrightarrow E = 0$ (algebraic side)

$\Leftrightarrow H = 0$, i.e. $L(H,\alpha) \in \mathrm{Pic}^0(X)$ (analytic side).

b/c $\mathrm{Pic}^0(X) = \{E : E(v,u) \in \mathbb{Z},\ u \in U,\ v \in V\}$

(5) $K(L(H,\alpha))$ finite $\Leftrightarrow U^\perp/U$ finite

$\Leftrightarrow U^\perp$ finite

$\Leftrightarrow E \neq 0 \Leftrightarrow H \neq 0$ by (4).

(6) If $H \neq 0$ (e.g. $L(H,\alpha)$ ample say), then

$\forall\, U \xrightarrow{f} \mathbb{R}$ homomorphism, $\exists\, \alpha \in V$ s.t. $f : u \mapsto E(\alpha, u)$,

$\Rightarrow$ every $U \to \mathbb{C}_1^\times$ is given by $u \mapsto e^{2\pi i E(\alpha,u)}$, $\alpha \in V$.

$\Rightarrow \forall\, L \in \mathrm{Pic}^0(X),\ \exists\, \alpha \in V$ s.t. $L = L(0, \gamma_\alpha)$.

this proves the main thm in §8 when $k = \mathbb{C}$.

(classifying the $\mathrm{Pic}^0(X)$-elements).

(B) $Q = L(H,\alpha)$ divisoral correspondence if $Q$ trivial on $0 \times \{X_2\}$ and $\{X_1\} \times 0$.

Let $B : V_1 \times V_2 \longrightarrow \mathbb{C}$

$$(x,y) \longmapsto H((x,0),(0,y))$$

Get $H((x_1,y_1),(x_2,y_2)) = H((x_1,0),(0,y_2)) + H((0,y_1),(x_2,0))$.

$$= B(x_1,y_2) + \overline{B(x_2,y_1)}.$$

$$\alpha(u_1,u_2) = \alpha((u_1,0),(0,u_2)) \cdot e^{i\pi E((u_1,0),(0,u_2))} = e^{i\pi\beta(u_1,u_2)}, \quad \beta = \mathrm{Im} B.$$

Need $Q \rightsquigarrow$ a map $X_2 \to \hat{X}_1$:

to calculate $Q|_{X_1 \times \{\pi_2(a_2)\}}$, $a_2 \in V_2$.

Let $Q' = (\mathbb{C} \times V_1 \times V_2)/U_1 = (id \times \pi_2)^* Q$, $Q'|_{X_1 \times \{a_2\}} = Q|_{X_1 \times \pi_2(a_2)}$.

where $id \times \pi_2 : X_1 \times V_2 \to X_1 \times X_2$.

$$\phi'_{u_1}(\lambda, v_1, v_2) = (\lambda \cdot e^{\pi B(u_1,v_2)}, v_1 + u, v_2)$$

Have $Q|_{X_1 \times \{\pi_2(a_2)\}} = L(0, \gamma_{a_2})$. $\gamma_{a_2}(u) = e^{2\pi i \beta(a_2,u)}$

$\rightsquigarrow$ $Q|_{X_1 \times \{\pi_2(a_2)\}}$ trivial $\Leftrightarrow \gamma_{a_2} \equiv 1$

$\Leftrightarrow \beta(a_2,u) \in \mathbb{Z}$. $\forall u \in U_1$.

So $X_1 \cong \hat{X}_2 \Leftrightarrow$ (i) $B$ non-degenerate

(ii) $U_2 = \{a_2 \in V : \beta(a_2,u_1) \subseteq \mathbb{Z}\}$.

- Begin with $0 \to \mathbb{Z} \to \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$

$\rightsquigarrow$ $\mathrm{Pic}^0(X) = H^1(X, \mathcal{O}_X)/H^1(X,\mathbb{Z})$.

$\bar{T} = H^1(X,\mathcal{O}_X)$, $\bar{T} \longrightarrow \mathrm{Pic}^0(X)$

$\qquad l \longmapsto L(0,\alpha_l)$.

$\rightsquigarrow$ $\bar{T} \simeq H^1(X,\mathcal{O}_X) \xrightarrow{\exp(2\pi i \cdot)} \mathrm{Pic}(X)$

Commutative diagram

nonzero

$$\begin{tikzcd}
\mathrm{Hom}(U,\mathbb{C}) \arrow[r, equal] \arrow[d, equal] & H^1(U,\mathbb{C}) \arrow[r, "e^{2\pi i}"] \arrow[dd, "\cong"] & H^1(U,H^{\times}) \arrow[dd, "\cong"] \\
\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) \arrow[d, "\cong"] & & \\
\bar{T} \leq T \oplus \bar{T} \arrow[r, "\sim"] & H^1(X,\mathbb{C}) \arrow[r, "e^{2\pi i}"] & \mathrm{Pic}(X)
\end{tikzcd}$$

## § The theorem of cubes (II)

**Prop** $X$ complete var., $Y$ $k$-sch., $L$ line bundle on $X \times Y$.

Then $\exists!\ Y_1 \hookrightarrow Y$ closed subsch s.t.

(a) if $L|_{X \times Y_1} = L_1$, then $\exists\ M_1$ line bundle on $Y_1$ with $p_2^* M_1 \cong L_1$ on $X \times Y_1$ } triviality

(b) if $f : Z \to Y$ any morphism s.t. $\exists\ K$ line bundle on $Z$ with $p_2^* K \cong (1_X \times f)^*(L)$ on $X \times Z$, then $f : Z \to Y_1 \hookrightarrow Y$ factorization. (maximality).

- $f : X \to Y$ flat, proper b/w noetherian schemes.

$\dim H^0(X_y, \mathcal{O}_{X_y}) = 1$, $\forall y \in Y \Rightarrow \mathcal{O}_Y = f_* \mathcal{O}_X$.

if $L_{Y_1} \cong p_{Y_1}^* M$, $p_{Y_1,*} L_{Y_1} \cong p_{Y_1,*}(p_{Y_1}^* M \otimes \mathcal{O}_X) \cong M \otimes p_{Y_1,*}\mathcal{O}_X \cong M$.

To prove that ① $p_{Y_1,*} L_{Y_1}$ line bundle

② $p_{Y_1}^* p_{Y_1,*} L_{Y_1} \xrightarrow{\sim} L_{Y_1}$ isom.

**Proof** The question is local on $Y$. Take $Y = \mathrm{Spec}\, A$.

Free resolution from §5: $0 \to A^{r_0} \xrightarrow{d} A^{r_1} \to \cdots$

$\mathcal{F}$ coherent on $X$, $H^i(X, \mathcal{F}) = H^i(K^\bullet)$

Via $f: \operatorname{Spec} B \to \operatorname{Spec} A$, $H^i(X_B, \mathcal{F}_B) = H^i(K^\cdot \otimes_A B)$.

$\rightsquigarrow (\cdot) \otimes B : \; B^{r_1} \xrightarrow{d^r \otimes B} B^{r_0} \longrightarrow M \otimes B \longrightarrow 0.$

$\rightsquigarrow \operatorname{Hom}(\cdot, B) : \; 0 \to \operatorname{Hom}_B(M \otimes B, B) \to B^{r_0} \to B^{r_1}$

Take $X_B := X \times_{\operatorname{Spec} A} \operatorname{Spec} B$

$\rightsquigarrow p_B^*((1_X \times f)^* L) \cong \operatorname{Hom}_B(M \otimes_A B, B) \cong \operatorname{Hom}_A(M, B)$.

Consider $F = \{ y \text{ closed pts} \mid L_y = L|_{X \times \operatorname{Spec} k(y)} \text{ trivial} \}$ closed set

$H^0(X_y, L_y) = \operatorname{Hom}_{k(y)}(M \otimes k(y), k(y))$

$\Rightarrow \; 1 = \dim_{k(y)} M/m_y M \Rightarrow \exists f \in A$ with $y \in A_f$, $A_f \twoheadrightarrow M_f$.

By Nakayama, for $R$-mods $M, N$ & $I \subseteq R$ ideal,

$N \to M$, $N/IN \twoheadrightarrow M/IM$ ($M$ finite)

$\Rightarrow \exists f \in 1 + I$, $M_f \twoheadrightarrow N_f$.

$\Rightarrow \exists f \in 1 + m_y$, $M/m_y M \longrightarrow A/m_y A$, $A_f \twoheadrightarrow M_f$.

Assume $M \cong A/I$, $Z = V(I) \subseteq Y = \operatorname{Spec} A$.

For $f: Z \to Y$, $f_{Z,*}(\underbrace{(1_X \times f)^* L}_{\substack{\| \\ L_Z \text{ line bundle}}}) \cong \operatorname{Hom}_A(M, A/I) \cong A/I$.

Adjoint $\lambda: f_Z^* f_{Z,*} L_Z \to L_Z$.

Also note that $f: T \to Y$ factors through $Z$

$\Leftrightarrow f_{T,*} L_T$ line bundle

$T = \operatorname{Spec} B$, $f_{T,*} L_T = \mathcal{O}_T$,

$B \simeq \operatorname{Hom}_A(A/I, B)$

$\Leftrightarrow I \subseteq \ker(A \to B) \Leftrightarrow A \to A/I \to B$.

Then $L_y$ trivial $\Leftrightarrow \lambda_{x_y}$ surjective.

(... to be continued next time).