Integral Hodge filtration and integral Sen theory

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<u>Motivation</u> (1) F global field.

automorphic Galois rep $r_\pi : \mathrm{Gal}_F \to GL_d(\mathbb{Z}_p)$

$$\rightsquigarrow \bar{r}_\pi \leftarrow \text{-- -} \; r_{\pi'}$$

Serre wt Conj : possible wts of π' s.t. $\bar{r}_{\pi'} = \bar{r}_\pi$?

(2) Local analogue :

Crys rep $\rho : \mathrm{Gal}_K \to GL_d(\mathbb{Z}_p)$

$$\rightsquigarrow \bar{\rho} \leftarrow \text{-- -} \; \rho'$$

How about HT wts of ρ' s.t. $\bar{\rho} = \bar{\rho}'$?

<u>Notation</u> k char p field, $W(k)$,

$K_0 = W(k)[\frac{1}{p}]$, K/K_0 fin, tot ram.

Fix unif $\pi \in \mathcal{O}_K$. $E = \mathrm{Irr}(\pi, K_0) \in \mathfrak{S} = W(k)[[u]]$.

Let $T : G_K \to GL_d(\mathbb{Z}_p)$ semistable,

HT wts $0 \le r_1 \le \cdots \le r_d$.

$\rightsquigarrow$ $\exists$ a Breuil-Kisin mod $\mathfrak{M}$ fin free $/\mathfrak{S}$

$+ \varphi : \mathfrak{M} \to \mathfrak{M}$ an E-isog, semilin w.r.t. $\varphi_{\mathfrak{S}}$.

s.t. $\mathfrak{S}[\frac{1}{E}] \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \xrightarrow[\sim]{1 \otimes \varphi} \mathfrak{S}[\frac{1}{E}] \otimes_{\mathfrak{S}} \mathfrak{M}$ is an isom.

Denote $\mathfrak{M}^* := \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \overset{1 \otimes \varphi}{\hookrightarrow} \mathfrak{M}$ regarded as a submod of $\mathfrak{M}$.

Define Nygaard filn $\mathrm{Fil}^i \mathfrak{M}^* := \mathfrak{M}^* \cap E^i \mathfrak{M}$, $i \in \mathbb{Z}$.

Let $\mathfrak{M}_{dR} := \mathfrak{M}^* / E\mathfrak{M}^*$.

$\mathfrak{M}^* \twoheadrightarrow \mathfrak{M}_{dR}$ induces the Hodge filn $\mathrm{Fil}^i \mathfrak{M}_{dR}$.

Check: $\mathrm{Fil}^i \mathfrak{M}_{dR} = \mathrm{Fil}^i \mathfrak{M}^* / E\, \mathrm{Fil}^{i-1} \mathfrak{M}^*$.

<u>Facts</u> (1) $(\mathrm{Fil}^\bullet \mathfrak{M}_{dR})[\frac{1}{p}] = \mathrm{Fil}^\bullet D_{dR}(T[\frac{1}{p}])$.

(2) The map $\iota : \mathrm{Fil}^\bullet \mathfrak{M}_{dR} \to \mathrm{Fil}^\bullet \mathfrak{M}_{dR}[\frac{1}{p}]$

is not strict in general.

(3) For $\bar{\mathfrak{M}} = \mathfrak{M} \pmod p$,

$\mathrm{Fil}^\bullet \mathfrak{M}^* \rightsquigarrow \mathrm{Fil}^\bullet \bar{\mathfrak{M}}^* \rightsquigarrow \mathrm{Fil}^\bullet \bar{\mathfrak{M}}_{dR}$

Then $\mathrm{Fil}^\bullet \mathfrak{M}_{dR} \to \mathrm{Fil}^\bullet \bar{\mathfrak{M}}_{dR}$ is not strict in general.

<u>Facts</u> TFAE: (i) L_{dR} is strict.

(ii) $\mathfrak{M}$ has a basis $e_1, \dots, e_d$

s.t. $\varphi(e_1, \dots, e_d) = (e_1, \dots, e_d) X \begin{pmatrix} E^{r_1} & & \\ & \ddots & \\ & & E^{r_d} \end{pmatrix} Y$. for some $X, Y \in GL_d(\mathfrak{S})$.

(iii) If T crys, the (reflexive) F-gauge

attached to T is a VB on $\mathcal{O}_K^{\mathrm{syn}}$.

Pf (i) $\Leftrightarrow$ (ii) Gee-Liu-Savitt '14

(i) $\Leftrightarrow$ (iii) : Stacky pf by BL

sheaf theoretic: Gao-Li.

<u>Thm</u> (GLS 14) For K unram, T crys, $HT \subseteq [0,p]$,

the conditions (i) (ii) (iii) hold.

<u>Thm A</u> (Gao-Liu) K unram, HT $0 \le r_1 \le \dots \le r_d$.

(1) If $n \notin \{ r_i + kp \mid k \ge 0 \} \cap [0, r_d]$, then $\mathrm{gr}^n \mathfrak{M}_{dR} = 0$.

Also, $\forall n$, $(\mathrm{gr}^n \mathfrak{M}_{dR})_{tor}$ is killed by $n!$

& $(\mathrm{gr}^{r_d} \mathfrak{M}_{dR})_{tor} = 0$,

$(\mathrm{gr}^n \mathfrak{M}_{dR}) = 0, \quad \forall n > r_d$

$\Rightarrow (\mathrm{gr}^n \mathfrak{M}_{dR})_{tor}$ uniformly killed by $(r_d - 1)$!

Also (easy fact): # generators of $\mathrm{gr}^n \mathfrak{M}_{dR}$ is $\le d$, $\forall n$

($\Rightarrow$ (k finite) then torsion is always finite).

(2) (Gee-Kisin)

If $n \notin \{r_i + kp \mid k \in \mathbb{Z}\} \cap [0, r_d]$, then $\mathrm{gr}^n \overline{\mathfrak{M}}_{dR} = 0$.

More precisely, let $b_1, \dots, b_d :=$ jumps of $\mathrm{Fil}^{\cdot} \overline{\mathfrak{M}}_{dR}$,

then $\{b_1, \dots, b_d\} \equiv \{r_1, \dots, r_d\} \pmod p$.

i.e. both sides depend on unordered set of elts in $\mathbb{Z}/p\mathbb{Z}$ with some multiplicities.

<u>Ex</u> $HT = \{0, \dots, 2p\}$. Then

(1) $\mathrm{gr}^n \mathfrak{M}_{dR} = 0$, if $n \notin \{0, p, 2p\}$.

Also knows $\cdot$ $(\mathrm{gr}^p \mathfrak{M}_{dR})$ is killed by p!

$\cdot$ $\mathrm{gr}^0 \mathfrak{M}_{dR}$, $\mathrm{gr}^{2p} \mathfrak{M}_{dR}$ are free of rk 1.

(2) Jumps of $\mathrm{Fil}^{\cdot} \overline{\mathfrak{M}}_{dR}$ could be $\{0, 2p\}$ or $\{p, p\}$.

<u>Rmk</u> In Gee-Kisin's IAS talk, they considered φ on $\overline{\mathfrak{M}}/k[\![u]\!]$ with matrix $X \begin{pmatrix} u^{a_1} & & \\ & \ddots & \\ & & u^{a_d} \end{pmatrix} Y$.

Then $\{a_1, \dots, a_d\} \equiv \{r_1, \dots, r_d\} \pmod p$.

pf Explicitly compute $\mathrm{Fil}^{\cdot} \overline{\mathfrak{M}}^* \rightsquigarrow \mathrm{Fil}^{\cdot} \overline{\mathfrak{M}}_{dR}$ jumps at $\{a_1, \dots, a_d\}$.

<u>Key tool</u> (for Thm A) Conjugate Fil on $\mathfrak{M}_{HT} := \mathfrak{M}/E\mathfrak{M}$.

The map $\mathrm{Fil}^i \mathfrak{M}^* = \mathfrak{M}^* \cap E^i \mathfrak{M} \xhookrightarrow{E^{-i}} \mathfrak{M}$

$\rightsquigarrow \mathrm{Fil}^i \mathfrak{M}^* / \mathrm{Fil}^i \xhookrightarrow{E^{-i}} \mathfrak{M}/E\mathfrak{M} = \mathfrak{M}_{HT}$.

Define $\mathrm{Fil}^i_{conj} \mathfrak{M}_{HT} :=$ image of above map.

<u>Facts</u> (1) $\mathrm{Fil}^i_{\mathrm{conj}} M_{HT}$ is increasing.

(2) $\mathrm{gr}^i M_{HT} = \mathrm{Fil}^i M_{HT} / \mathrm{Fil}^{i-1} M_{HT}$

$$\simeq \mathrm{gr}^i M_{dR} = \mathrm{Fil}^i M_{dR} / \mathrm{Fil}^{i+1} M_{dR}.$$

(essentially b/c $(A/B)/(C/D) \simeq (A/C)/(B/D)$.)

<u>Upshot</u> $\mathrm{Fil}^* M_{HT}$ admits extra "symmetry",

i.e. it admits a Sen operator.

<u>(Integral) filtered Sen theory</u>

<u>Thm</u> (Kisin) $\mathrm{Rep}^{st, \geq 0}_{G_K}(\mathbb{Q}_p) \simeq \mathrm{Mod}^{\varphi, N_\nabla, 0}_{\mathcal{O}} \ni \mathcal{M}$

where $\mathcal{O} = \left\{ \begin{array}{l} f(u) = \sum_{i \geq 0} a_i u^i, \ a_i \in K_0, \\ f(u) \text{ converges on open unit disc} \end{array} \right\}.$

Let $\lambda = \prod_{n=0}^{\infty} \varphi^n \left(\frac{E(u)}{E(0)} \right) \in \mathcal{O}$, $N_\nabla = -u\lambda \frac{d}{du} : \mathcal{O} \to \mathcal{O}$

$\varphi : \mathcal{O} \to \mathcal{O}$, $u \longmapsto u^p$.

Have $\mathcal{M}$ fin free $\mathcal{O}$-mod $+ \ \varphi : \mathcal{M} \to \mathcal{M}$ E-isog

$+ \ N_\nabla : \mathcal{M} \to \mathcal{M}$, $\varphi \longmapsto \frac{pE(u)}{E(0)} \varphi N_\nabla$

and $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{R}$ is pure of slope 0.

Let K any field,

T semistable. $T[\frac{1}{p}] \rightsquigarrow \mathcal{M} \circlearrowleft N_\nabla$

$\rightsquigarrow N_\nabla \circlearrowright \mathcal{M}_{HT} = \mathcal{M}/E\mathcal{M} = (\mathfrak{M}/E\mathfrak{M})[\frac{1}{p}].$

Let $a = \begin{cases} E'(\pi), & \text{if } T \text{ crys} \\ \pi E'(\pi), & \text{if } T \text{ semist}. \end{cases}$

<u>Thm B</u> (Gao–Liu) Let $c = \frac{1}{\theta_{\mathrm{Fm}}(u\lambda')} \in K$, denote $\theta = cN_\nabla : \mathcal{M}_{HT} \to \mathcal{M}_{HT}$.

(1) Θ semistable with eigenvals $r_1, \cdots, r_d$

(2) $\mathrm{Fil}^i_{\mathrm{conj}} \mathcal{M}_{HT} = (\mathrm{Fil}^i \mathfrak{M}_{HT})[\frac{1}{p}] = \bigoplus_{j \le i} \mathcal{M}_{HT}^{\Theta = j}$

(3) The map $\Theta = a\Theta : \mathcal{M}_{HT} \to \mathcal{M}_{HT}$ is integral,

i.e. $\Theta(\mathfrak{M}_{HT}) \subseteq \mathfrak{M}_{HT}$.

(4) $\Theta - an$ satisfies "Griffith trans" on $\mathrm{Fil}^n \mathfrak{M}_{HT}$,

i.e. $(\Theta - an)(\mathrm{Fil}^n_{\mathrm{conj}} \mathfrak{M}_{HT}) \subseteq \mathrm{Fil}^{n-1}_{\mathrm{conj}} \mathfrak{M}_{HT}$.

pf Uses a "locally analytic" interpretation $N_\nabla = \frac{\lambda}{t} \log \tau$

where τ = generator of $\mathrm{Gal}(L/K_\infty) \simeq \mathbb{Z}_p$.

$K_\infty = K(\pi, \pi^{1/p}, \pi^{1/p^2}, \dots)$, $L = K_\infty(\zeta_{p^n})$.

Rmk When K unram, T crys,

Thm B is known by Bhatt–Lurie.

$$(\mathcal{O}_K^{N})_{t=0} \hookrightarrow \mathcal{O}_K^{N} \hookrightarrow \mathcal{O}_K^{syn}$$

$$\mathcal{E}_{HT} \qquad\qquad \mathcal{E}$$

a mod / $W(k)\{x, D\}/(Dx - xD - 1)$

roughly, "$D = \Theta$".

<u>Thm B $\Rightarrow$ Thm A</u> : For simplicity, suppose $n \not\equiv r_i \pmod p$

Then we prove $\mathrm{Fil}^n \mathfrak{M}_{HT} \xrightarrow[\Theta - n]{\Theta - an \,=} \mathrm{Fil}^{n-1} \mathfrak{M}_{HT} \hookrightarrow \mathrm{Fil}^n$.

is $\underbrace{\text{bijective}}$ $\Rightarrow$ $\mathrm{gr}^n = 0$

eigenvals are $r_i - n$'s.