Torsion in the homology of locally symmetric spaces

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Lecture 1: Introduction

Galois theory

Let $\bar{\mathbb{Q}} \subseteq \mathbb{C}$ algebraic numbers

$\shortparallel$ $\{\alpha \in \mathbb{C} \mid \exists P(x) = X^n + a_{n-1}X^{n-1} + \dots + a_0,\ a_i \in \mathbb{Q},\ P(\alpha) = 0\}$.

<u>Basic point of Galois theory</u>:

There is no algebraic way of distinguishing $\sqrt{2}$ & $-\sqrt{2}$.

(sol'ns of $x^2 - 2 = 0$).

Use absolute Galois grp $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) = \mathrm{Aut}(\bar{\mathbb{Q}})$

$\shortparallel$ set of bijections comm with $+, \cdot$ on $\bar{\mathbb{Q}}$.

For any $\alpha \in \bar{\mathbb{Q}}$, $P(\alpha) = 0$,

the orbit $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \cdot \alpha \subseteq \{x \mid P(x) = 0\}$ is a finite set:

$\forall \gamma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, $P(\gamma \cdot \alpha) = \gamma \cdot P(\alpha) = 0$.

Make $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ a topological grp

by asking that for all $\alpha \in \bar{\mathbb{Q}}$,

$\gamma \mapsto \gamma \cdot \alpha$ is continuous

($\Leftrightarrow$ $\forall \alpha \in \bar{\mathbb{Q}}$, $\mathrm{Stab}(\alpha) \subseteq \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is an open subgrp).

This makes $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ a Cantor set

(totally disconnected).

<u>Def'n</u> A finite ext'n $K/\mathbb{Q}$, $K \subseteq \bar{\mathbb{Q}}$ is <u>Galois</u> if it is stable under $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action.

Define $\mathrm{Gal}(K/\mathbb{Q}) = \mathrm{Aut}(K)$.

<u>Examples</u> (i) $K = \mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$.

Galois: $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \cdot \sqrt{2} = \{\sqrt{2}, -\sqrt{2}\} \subseteq \mathbb{Q}(\sqrt{2})$.

$\mathrm{Gal}(K/\mathbb{Q}) \cong S_2$ via action on $\{\sqrt{2}, -\sqrt{2}\}$.

(ii) $K = \mathbb{Q}(\sqrt[3]{2}) = \{a + b\sqrt[3]{2} + c\sqrt[4]{2} \mid a, b, c \in \mathbb{Q}\}$

not Galois: $\sqrt[3]{2}$ sol'n to $X^3 - 2 = 0$

all sol'ns are $\{\sqrt[3]{2}, \zeta_3\sqrt[3]{2}, \zeta_3^2\sqrt[3]{2}\} \not\subseteq \mathbb{Q}(\sqrt[3]{2}) \subseteq \mathbb{R}$.

(iii) $K = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$, $\zeta_3 = e^{2\pi i/3}$

Galois: $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3$

via action on $\{\sqrt[3]{2}, \zeta_3\sqrt[3]{2}, \zeta_3^2\sqrt[3]{2}\}$.

<u>Thm</u> (Galois) (1)

$$\left\{\begin{array}{c} K \subseteq \bar{\mathbb{Q}}, \\ K/\mathbb{Q} \text{ finite} \end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{c} H \subseteq \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \\ \text{open subgrp} \end{array}\right\}$$

$$K \longmapsto \{\gamma \mid \forall \alpha \in K,\ \gamma \cdot \alpha = \alpha\}$$

$$\bar{\mathbb{Q}}^H \longleftarrow\!\shortmid\ H$$

$$\bar{\mathbb{Q}}^H = \{\alpha \mid \forall \gamma \in H,\ \gamma\alpha = \alpha\}.$$

(2) K Galois $\Leftrightarrow$ H normal;

then $\mathrm{Gal}(K/\mathbb{Q}) = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) / H$.

$$\overline{\mathbb{Q}} = \bigcup_{\substack{K/\mathbb{Q} \\ \text{Galois}}} K, \qquad \underbrace{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}_{\text{profin grp}} = \varprojlim \underbrace{\mathrm{Gal}(K/\mathbb{Q})}_{\text{finite grp}}$$

Can try to understand $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ via maps to finite grps of Lie type, e.g. $\mathrm{GL}_n(\mathbb{F}_p)$.

<u>Def'n</u> A finite mod p Galois repr is a continuous map

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_n(\mathbb{F}_q)$$

<u>Note</u> $H = \ker \rho \leq \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ open normal subgrp

so $K(\rho) = \overline{\mathbb{Q}}^H$ finite normal ext'n $/\mathbb{Q}$.

$\rightsquigarrow \quad \rho : \mathrm{Gal}(K(\rho)/\mathbb{Q}) \hookrightarrow \mathrm{GL}_n(\mathbb{F}_p)$.

<u>Diagram of main story</u>

$$\left\{\begin{matrix}\text{alg vars} \\ /\mathbb{Q}\end{matrix}\right\} \xrightarrow[\text{cohom}]{\text{étale}} \left\{\begin{matrix}\text{finite Galois} \\ \text{repr}\end{matrix}\right\} \overset{\text{A}}{\underset{\text{B}}{\leftrightarrows}} \left\{\begin{matrix}\text{torsion in the homology} \\ \text{of loc symm spaces}\end{matrix}\right\}$$

$\parallel$ systems of polynomial eq'ns with $\mathbb{Q}$-coeffs

<u>Two conjectures</u>

(A) Torsion in the homology of loc symm spaces gives rise to finite Gal reprs.

(B) ("Serre's conj") (Almost) all finite Galois rep'ns appear in this way.

This lecture series is about (A).

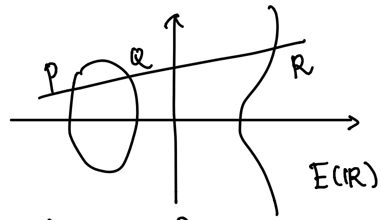
An example

Let $E/\mathbb{Q}$ elliptic curve. $\exists$ zero section $0 \in E$,

$\rightsquigarrow$ $E \setminus \{0\}$: $y^2 = x^3 + ax + b$, $a, b \in \mathbb{Q}$

s.t. $\Delta = 4a^3 + 27b^2 \neq 0$.

$E(\mathbb{R})$:



$P + Q + R = 0$.

For any char 0 field K,

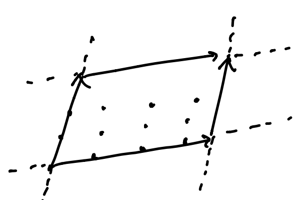
$E(K)$ is an abelian grp with zero 0.

$E(\mathbb{C})$: $E(\mathbb{C}) \cong$  $\cong \mathbb{C}/\Lambda$,

$\Lambda \cong \mathbb{Z}^2 \subseteq \mathbb{C}$ lattice

<u>Note</u> Fix $n \geq 1$. Let $E(K)[n] = \ker(E(K) \xrightarrow{[n]} E(K))$.

Then $E(\mathbb{C})[n] = (\frac{1}{n}\Lambda)/\Lambda \cong (\mathbb{Z}/n\mathbb{Z})^2$



and $E(\overline{\mathbb{Q}})[n] = E(\mathbb{C})[n]$.

$\circlearrowleft$ $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Take $n = p$ prime.

$\rightsquigarrow$ get $\rho_{E,p} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}(\underbrace{E(\overline{\mathbb{Q}})[p]}_{= \mathbb{F}_p^2}) = \mathrm{GL}_2(\mathbb{F}_p)$.

Let $p \geq 5$, $a^p + b^p = c^p$, $a, b, c \in \mathbb{Z}_{>0}$.

Frey Look at $E: \; y^2 = x(x-a^p)(x+b^p)$

difference at zeros of RHS are $a^p, b^p, a^p+b^p = c^p$.

$\Rightarrow \quad \Delta = (abc)^{2p}$ p-th power disc.

In this case, for $\rho_{E,p}: \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\mathbb{F}_p)$:

Fact $\rho_{E,p}$ unramified outside $2, p$

& with very little ramification at $2, p$.

Conj (Serre) $\rho_{E,p}$ "comes from" modular form

of wt 2 + level $\Gamma_0(2)$. (Khare-Winterberger).

But there are no such forms! (Wiles, Taylor Wiles).

$\Rightarrow$ Fermat's last thm is true.

Rmk Everything should work over arbitrary finite ext'n $K/\mathbb{Q}$.

$$\left\{ \begin{array}{c} \text{alg vars} \\ /K \end{array} \right\} \xrightarrow[\text{cohom"}]{\text{"étale}} \left\{ \begin{array}{c} \text{finite Gal repr} \\ \text{of } \mathrm{Gal}(\bar{\mathbb{Q}}/K) \end{array} \right\} \underset{\textcircled{B}}{\overset{\textcircled{A}}{\rightleftarrows}} \{\text{torsion}\}.$$

Ⓐ + Ⓑ = "mod p global Langlands corresp".

Consider two cases

(a) $n=2, \; K=\mathbb{Q}$

(b) $n=2, \; K=\mathbb{Q}(i), \quad i^2=-1$ (imaginary quad field).

Rmk $n>2, \; K=\mathbb{Q}$: very similar to (b).

Case (a) "modular curves"

$$\mathbb{H}^2 = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\} = \mathbb{R} \times \mathbb{R}_{>0}$$

$PSL_2(\mathbb{R}) \curvearrowright \mathbb{H}^2$ $\qquad x+iy \longleftrightarrow (x,y)$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \dfrac{a\tau+b}{c\tau+d}$

$\rightsquigarrow$ $\mathbb{H}^2$ 2-dim'l hyperbolic space

w/ metric $ds^2 = \frac{1}{y^2}(dx^2+dy^2)$.

Take $\Gamma \subseteq SL_2(\mathbb{Z})$ congruence subgrp

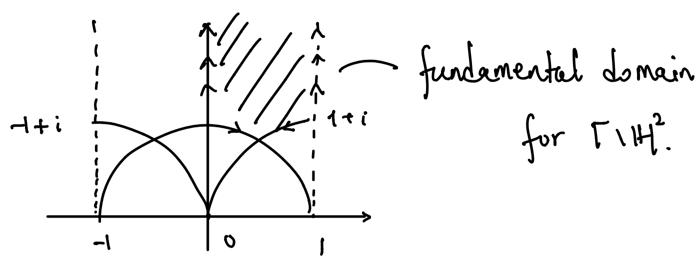
i.e. $\Gamma \supseteq \{\gamma \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N\} = \Gamma(N)$.

In fact, assume stronger condition:

$$\Gamma \supseteq \Gamma_1(N) = \{\gamma \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod N\}.$$

Def'n modular curve = locally symmetric space $\Gamma \backslash \mathbb{H}^2$.

e.g. $\Gamma = SL_2(\mathbb{Z})$



If Γ suff. small, then $\Gamma\backslash\mathbb{H}^2$ real hyperbolic mfd

& in fact, 1-dim'l complex mfd ("curve").

Thm $\Gamma\backslash\mathbb{H}^2$ carries a unique structure as an alg curve $/\mathbb{C}$,

in fact, it is defined $/\mathbb{Q}$.

Homology $H_i(\Gamma\backslash\mathbb{H}^2, \mathbb{Z})$ torsion-free for $i=0,1$, trivial else.

($H_0 = \mathbb{Z}$, boring).

Apply Hodge theory :

$$H_1(\Gamma \backslash \mathbb{H}^2, \mathbb{C}) = H_1(\Gamma \backslash \mathbb{H}^2, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$$

can be computed in terms of harmonic fcts.

Here weight 2 mod forms (of level Γ)

$$f : \mathbb{H}^2 \longrightarrow \mathbb{C} \quad \text{s.t.} \quad f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 f(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

On the other hand, as $\Gamma \backslash \mathbb{H}^2$ alg var $/\mathbb{Q}$,

$$\Rightarrow \quad H_1(\Gamma \backslash \mathbb{H}^2, \mathbb{F}_p) \circlearrowleft \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \qquad \text{"torsion homology"}$$

$$\text{"} \; H_1(\Gamma \backslash \mathbb{H}^2, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p$$

("étale cohom".

This gives Ⓐ in case (a) ($n=2$, $K = \mathbb{Q}$).

Ⓑ Serre's Conj $\Longleftrightarrow$ $\rho_{E,p}$ appears as a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-subquotient of $H_1(\Gamma_0(2) \backslash \mathbb{H}^2, \mathbb{F}_p)$

Lecture 2: An example

F number field, either totally real or CM.

$\rightsquigarrow$ locally symm space X_K for GL_n/F, K level.

<u>Thm</u> $H_i(X_K, \overline{\mathbb{F}}_q) \neq 0 \Rightarrow \exists$ Gal repr $\rho: \mathrm{Gal}(\overline{\mathbb{Q}}/F) \longrightarrow GL_n(\overline{\mathbb{F}}_p)$.

s.t. Frob traces = Hecke eigenvals.

<u>Case (b)</u> $n=2$, $F = \mathbb{Q}(i)$.

$$\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_{>0} \ni (x_1 + i x_2, y)$$

$$PSL_2(\mathbb{C}) = \{\text{orientation-preserving isometries}\}$$

$\rightsquigarrow$ $\mathbb{H}^3$ 3-dim'l hyperbolic space w/ metric $ds^2 = \frac{1}{y^2}(dx_1^2 + dx_2^2 + dy^2)$.

Let $\Gamma \leq SL_2(\mathbb{Z}[i])$ congr subgrp.

s.t. $\Gamma \supseteq \Gamma_1(N) = \{\gamma \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod N\}$.

$\Rightarrow$ $\Gamma \backslash \mathbb{H}^3$ Bianchi manifold (1892).

<u>Picture</u>

<u>Note</u> real dim 3 $\Rightarrow$ no complex structure

$\Rightarrow$ no algebraic structure.

$\Rightarrow$ On $H_i(\Gamma \backslash \mathbb{H}^3, \mathbb{F}_p)$, no Galois action.

But $H_i(\Gamma\backslash\mathbb{H}^3, \mathbb{F}_p)$ has lots of torsion!

e.g. For $\Gamma = \Gamma_1(\mathfrak{p})$, $\mathfrak{p} \in F$ with $Nm\,\mathfrak{p} = 4969$.

then $\underbrace{867272937 1087}_{\text{prime}} \mid \# H_1(\Gamma\backslash\mathbb{H}^3, \mathbb{Z})_{tors}$ (c.f. Sengün).

One expects exponential growth of torsion subgrp w.r.t. $\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)$.

(c.f. Bergeron - Venkatesh).

Thm $\Rightarrow$ existence of big sporadic ext'n of F with little ramification.

<u>Hecke operators</u>

Fix $\mathfrak{p} \subseteq \mathcal{O}_F$, $\mathfrak{p} \nmid N$ prime ideal ($F = \mathbb{Q}(i)$).

$\mathfrak{p} = (\varpi)$, $\varpi \in \mathcal{O}_F$ generator

$\rightsquigarrow$ $\Gamma_{\mathfrak{p}} = \Gamma \cap \begin{pmatrix} \varpi^{-1} & \\ & 1 \end{pmatrix} \Gamma \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}$.

Hecke corresp.

$$\begin{tikzcd}[column sep=small] & \Gamma_{\mathfrak{p}}\backslash\mathbb{H}^3 \arrow[r, "\overset{\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}}{\cong}"] \arrow[dl, "\pi_1"'] & \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} \Gamma_{\mathfrak{p}} \begin{pmatrix} \varpi^{-1} & \\ & 1 \end{pmatrix} \backslash\mathbb{H}^3 \arrow[dr, "\pi_2"] & \\ \Gamma\backslash\mathbb{H}^3 & & & \Gamma\backslash\mathbb{H}^3 \end{tikzcd}$$

π_1, π_2 finite covering maps

$\rightsquigarrow$ $T_{\mathfrak{p}} := \pi_{2*}\pi_1^* : H_i(\Gamma\backslash\mathbb{H}^3, \mathbb{F}_p) \longrightarrow H_i(\Gamma\backslash\mathbb{H}^3, \mathbb{F}_p)$.

Fact The $T_{\mathfrak{p}}$'s commute ($\Rightarrow \exists$ simultaneous eigenvectors).
called Hecke eigenforms.

Thm Let $\alpha = (\alpha_{\mathfrak{p}})_{\mathfrak{p} \nmid pN}$ be a system of Hecke eigenvals appearing in $H_i(\Gamma \backslash \mathbb{H}^3, \overline{\mathbb{F}}_p)$.

Then $\exists\ \rho_\alpha : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \longrightarrow GL_2(\overline{\mathbb{F}}_p)$
unramified outside $p \cdot N$,
s.t. $\forall\ \mathfrak{p} \nmid pN$, $\mathrm{tr}(\rho_\alpha(\mathrm{Frob}_{\mathfrak{p}})) = \alpha_{\mathfrak{p}}$.

Example (Figueiredo) $n = 2$, $F = \mathbb{Q}(i)$.

$p = 3$. $\mathbb{F}_3 = \{0, \pm 1\}$.

$$\Gamma = \Gamma_1(3) \cap \Gamma_0(61) = \left\{ \gamma \,\middle|\, \gamma \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod 3 \ \&\ \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod 61 \right\}$$

$$\cap\!\!\!\mid$$

$$\tilde{\Gamma} = \Gamma_0(3) \cap \Gamma_0(61)$$

$\rightsquigarrow \quad \tilde{\Gamma}/\Gamma = \mathbb{F}_3^\times \xrightarrow[=]{\chi} \mathbb{F}_3^\times$ tautological char
acts on $H_1(\Gamma \backslash \mathbb{H}^3, \mathbb{F}_3)$.

Fact $H_1(\Gamma \backslash \mathbb{H}^3, \mathbb{F}_3)_{\text{Nebentypus} = \chi}$ has dim $= 2$.
& has two Hecke eigenval systems α_1, α_2.

$N\mathfrak{p}$	$\mathfrak{p}$	$\alpha_{1,\mathfrak{p}}$	$\alpha_{2,\mathfrak{p}}$	$\chi_3(\mathrm{Frob}_{\mathfrak{p}})$	$X^4 - 7X^2 - 3X + 1 \bmod \mathfrak{p}$
2	$1+i$	1	-1	-1	$X = 1$ sol'n
5	$1+2i,\ 2+i$	1	-1	-1	$X = -2$ sol'n
13	$3+2i,\ 2+3i$	1	1	1	$X = 6$ sol'n
17	$1+4i,\ 4+i$	1	-1	-1	$X = 2$ sol'n

$N\mathfrak{p}$	$\mathfrak{p}$	$\alpha_{1,\mathfrak{p}}$	$\alpha_{2,\mathfrak{p}}$	$\chi_3(\mathrm{Frob}_{\mathfrak{p}})$	$X^4-7X^2-3X+1 \bmod \mathfrak{p}$
29	$\vdots$	-1	1	-1	$X=8$ sol'n
37	$\vdots$	0	0	-1	no sol'n
41		0	0	-1	no sol'n
49	$\vdots$	-1	-1	1	$X=-3$ sol'n
53		0	0	-1	no sol'n

Can write a program to compute $\alpha_{1,\mathfrak{p}}$, $\alpha_{2,\mathfrak{p}}$.

<u>Observations</u> (1) $\alpha_{i,\mathfrak{p}} = \alpha_{i,\bar{\mathfrak{p}}}$ for $i=1,2$, all $\mathfrak{p}$.

(2) $\alpha_{1,\mathfrak{p}} = 0 \iff \alpha_{2,\mathfrak{p}} = 0$.

Both (1)(2) can be proved.

Thm $\Rightarrow$ $\exists\, \rho_i : \mathrm{Gal}(\bar{\mathbb{Q}}/F) \longrightarrow SL_2(\mathbb{F}_3)$ (can replace $GL_2(\mathbb{F}_3)$).

unramified outside $3, 61$.

$\det \rho_i = 1$ ($\Leftarrow$ Nebentypus $= \chi$).

<u>Rmk</u> Let $\chi_3 : \mathrm{Gal}(\bar{\mathbb{Q}}/F) \longrightarrow \mathbb{F}_3^\times$ be mod 3 cycl char (ramified only at 3).

$\Rightarrow$ $\rho_i \otimes \chi_3$ satisfies all the same statements.

<u>Question</u> $\rho_2 = \rho_1 \otimes \chi_3$?

Equiv'ly, $\alpha_{2,\mathfrak{p}} = \alpha_{1,\mathfrak{p}} \cdot \chi_3(\mathrm{Frob}_{\mathfrak{p}})$ $\longrightarrow \begin{cases} 1, & N\mathfrak{p} \equiv 1 \pmod 3 \\ -1, & N\mathfrak{p} \equiv -1 \pmod 3 \end{cases}$

(can be proved directly)

Concentrate on α_1, $\rho = \rho_1 : \mathrm{Gal}(\bar{\mathbb{Q}}/F) \longrightarrow SL_2(\mathbb{F}_3)$.

$\alpha_{1,\mathfrak{p}} = \alpha_{1,\bar{\mathfrak{p}}} \Rightarrow \rho$ extends to $\tilde{\rho} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\mathbb{F}_3)$.

Consider $\bar{\tilde{\rho}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow PGL_2(\mathbb{F}_3) \cong S_4$.

$\rightsquigarrow$ gives rise to deg-4-ext'n $L/\mathbb{Q}$, unram outside 3, 61.

(a priori not normal)

<u>Fact</u> $\exists!$ A_4-ext'n of $\mathbb{Q}$, unram outside 3, 61.

(splitting field of $X^4 - 7X^2 - 3X + 1$).

<u>Lemma</u> TFAE: for $\mathfrak{p} \nmid 3, 61$,

(i) P has a root mod $\mathfrak{p}$:

$\exists\, X \in \mathbb{Z}[i]$ s.t. $P(X) \equiv 0 \mod \mathfrak{p}$.

(ii) $\rho(\mathrm{Frob}_{\mathfrak{p}}) \in SL_2(\mathbb{F}_3)$ has $\mathrm{tr}(\rho(\mathrm{Frob}_{\mathfrak{p}})) \in \{\pm 1\}$.

<u>But</u> This can't be proved directly!

<u>Rmks</u> Actually, $\tilde{\rho} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow SL_2(\mathbb{F}_3)$ is an <u>even</u> rep'n.

$\Rightarrow$ does not arise from homology of modular curve

(only odd ones do arise).

But it does after restriction to $F/\mathbb{Q}$ from $\bar{\mathbb{Q}}/\mathbb{Q}$

$\rightsquigarrow$ even/odd distinction goes away.

Can lift $\tilde{\rho}$ to $r : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow SL_2(\mathbb{C})$.

<u>even</u> Artin repr.

Langlands–Tunnell: This comes from Maaß form of Laplacian eigenvalue $1/4$ on modular curve.

(not known for general even Artin rep'ns).

Again, such forms are not seen by homology of modular curve.

Expect

$$
\begin{tikzcd}
\left\{ \begin{array}{c} \text{Maaß form of } \Delta\text{-eigenval} \\ 1/4 \text{ on } \Gamma(N)\backslash \mathbb{H}^2 \end{array} \right\} \arrow[rr, dashed, leftrightarrow, "??"] \arrow[dd, dashed, "\approx"'] & & \left\{ \begin{array}{c} \text{mod } p \text{ homology} \\ \text{of } \Gamma(Np)\backslash \mathbb{H}^3 \end{array} \right\} \\
 & & \left\{ \bar\rho|_{\mathrm{Gal}(\bar{\mathbb{Q}}/F)} : \mathrm{Gal}(\bar{\mathbb{Q}}/F) \to GL_2(\bar{\mathbb{F}}_p) \right\} \arrow[u, dashed, "\text{Serre's conj}"'] \\
\left\{ \begin{array}{c} \rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{C}) \\ \text{even Artin} \end{array} \right\} \arrow[r, "\text{reduce}", "\text{mod } p"'] & \left\{ \bar\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\bar{\mathbb{F}}_p) \right\} \arrow[ur, "\text{restrict to imag quad } F"'] &
\end{tikzcd}
$$

Question: Is there an analytic way to show that a Maaß form of Δ-eigenval $1/4$ on $\Gamma(N)\backslash \mathbb{H}^2$ forces p-torsion in homology of $\Gamma(pN)\backslash \mathbb{H}^3$.

Lecture 3: Construction of Galois representation

F totally real or CM, $n \geq 1$.

$K \subseteq GL_n(\mathbb{A}_{F,f})$ compact open

$\rightsquigarrow$ $X_K = G(F) \backslash \left(GL_n(F \otimes_{\mathbb{Q}} \mathbb{R}) / K_\infty \mathbb{R}_{>0} \times GL_n(\mathbb{A}_{F,f}) / K \right)$.

$K_\infty \subset GL_n(F \otimes_{\mathbb{Q}} \mathbb{R})$ max compact subgrp.

Thm (Scholze) For any Hecke EV system $\alpha = (\alpha_{\mathfrak{p}})$ appearing in $H_i(X_K, \overline{\mathbb{F}}_p)$,

$\exists$ Galois repr

$$\rho_\alpha : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \longrightarrow GL_n(\overline{\mathbb{F}}_p)$$

s.t. $\forall$ unram prime $\mathfrak{p}$,

Char poly of $\rho_\alpha(\mathrm{Frob}_{\mathfrak{p}})$ is given in terms of Hecke operators.

Case $F = \mathbb{Q}$

What was known ($g := n$).

Let $\mathcal{A}_n$ = Siegel moduli space of genus n.

$= \Gamma \backslash S_n$.

where $\Gamma \subseteq Sp_{2n}(\mathbb{Z})$ congr subgrp.

$S_n = \{ A \in M_n(\mathbb{C}) \mid A = A^t \ \& \ \mathrm{Im}\, A \text{ pos def} \}$.

$\circlearrowleft$

$Sp_{2n}(\mathbb{R})$

Rmk $\mathcal{A}_1$ = modular curve.

All $\mathcal{A}_n$ are algebraic / some fin ext'n of $\mathbb{Q}$.
└ moduli space of p.p.a.v.

<u>Thm</u> Let f cuspidal Hecke eigenform of regular weight on $\mathcal{A}_n$.
Then for any p. $\bar{\mathbb{Q}}_p \cong \mathbb{C}$,

$$\exists\, \rho_{f,p} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_{2n+1}(\bar{\mathbb{Q}}_p) \quad \text{assoc with } f.$$

(self dual)

factors through $\widehat{Sp_{2n}}(\bar{\mathbb{Q}}_p) = SO_{2n+1}(\bar{\mathbb{Q}}_p)$.

Arthur, Ngô, Waldspurger : endoscopic transfer

$$\left\{\begin{matrix}\text{discrete autom}\\ \text{repr of } Sp_{2n}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{self-dual autom}\\ \text{repr of } GL_{2n+1}\end{matrix}\right\}$$

$$\downarrow$$

known by Clozel, Kottwitz, Labesse, Harris-Taylor

$$\left\{\begin{matrix}\text{Galois repr}\\ \rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_{2n+1}(\bar{\mathbb{Q}}_p)\end{matrix}\right\}$$

<u>Rmk</u> One finds these Gal repr (usually) in Cohom of $U(1,2n)$- Shimura variety.

<u>Borel-Serre compactification</u>

$$\mathcal{A}_n \overset{j}{\hookrightarrow} \mathcal{A}_n^{BS}$$

alg var — real manifold with corners (not algebraic)

- Such j is a homotopy equiv.

- Boundary strata $\mathcal{A}_{n,P}^{BS}$ of $\mathcal{A}_n^{BS}$ are param by
  rat'l parabolic subgrps $P \subseteq Sp_{2n}$.
- $\mathcal{A}_{n,P}^{BS}$ torus bdl over loc symm space for Levi M of P.

e.g. $P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \begin{matrix} n \\ n \end{matrix} \subseteq Sp_{2n}$ Siegel parabolic

Levi $M = GL_n$.

$\Rightarrow$ torsion homology of loc symm space for GL_n

contributes to $H_i(\mathcal{A}_n^{BS}, \overline{\mathbb{F}}_p)$

<u>Thm</u> (Scholze) For any system α of Hecke EVs appearing in $H_i(\mathcal{A}_n, \overline{\mathbb{F}}_p)$,

$\exists$ <u>cuspidal</u> Hecke eigenform f (of reg wt) $(\in H^0(\mathcal{A}_n, \omega^{\otimes k})_{cusp})$

s.t. Hecke EV of $f \equiv \alpha \mod p$.

(f may have deeper level at p).

<u>Rmk</u> Works for any Shimura variety of Hodge type.

When $\mathcal{A}_n$ = Siegel Shimura var for Sp_{2n}:

$$\begin{array}{ccc}
\{\text{torsion homology of } GL_n\} & \longrightarrow & \{\text{torsion homology of } \mathcal{A}_n\} \\
& & \Big\downarrow \text{lifting} \\
& & \{\text{cuspidal eigenform on } \mathcal{A}_n\} \\
& & \Big\downarrow \\
\{\bar\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_{2n+1}(\overline{\mathbb{F}}_p)\} & \xleftarrow[\text{mod } p]{\text{reduce}} & \{\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_{2n+1}(\overline{\mathbb{Q}}_p)\}
\end{array}$$

<u>Final step</u> $\bar\rho \cong \rho_0 \oplus \rho_0^\vee \oplus \mathbb{1}$, where ρ_0 sought-after repr to GL_n.

Need to prove lifting thm:

<u>Note</u> Demand congr mod p for cusp form f

$\rightsquigarrow$ work with $\mathcal{A}_n \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

Proof relies on new results on p-adic geometry of $\mathcal{A}_n \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

This is new even for $n=1$ (modular curve for $Sp_2 = Sl_2$).

$/\mathbb{C}$: $\mathbb{H}^2 = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$

$\rightsquigarrow \mathcal{M}_\Gamma = \Gamma \backslash \mathbb{H}^2$, $\Gamma \subseteq Sl_2(\mathbb{Z})$ congr subgrp.

param elliptic curves + level-Γ str.

$\rightsquigarrow$ <u>Diagram</u>

$$\begin{tikzcd} \varprojlim_\Gamma \mathcal{M}_\Gamma \approx \mathbb{H}^2 = \left\{ (E,\beta) \;\middle|\; \begin{array}{l} E \text{ ell curve } /\mathbb{C} \\ \beta: H_1(E,\mathbb{Z}) \simeq \mathbb{Z}^2 \\ \quad \text{orientation preserving} \end{array} \right\} \arrow[d] \arrow[dr] & \\ \mathbb{P}^1(\mathbb{C}) & \mathbb{C}^2 \underset{\beta}{\cong} H_1(E,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \twoheadrightarrow \operatorname{Lie} E \quad \text{Hodge fil'n.} \end{tikzcd}$$

Want a similar picture $/\mathbb{C}_p$.

$\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ alg closed, complete ext'n of $\mathbb{Q}_p$.

<u>Hodge fil'ns $/\mathbb{C}_p$</u> Let $X/\mathbb{C}_p$ proj sm.

(Everything works for proper sm rigid-analytic vars).

<u>Thm</u> (Hodge) The Hodge-de Rham spectral sequence

$$E_1^{ij} = H^j(X, \Omega_X^i) \Rightarrow H_{dR}^{i+j}(X)$$

degenerates at E_1.

$\rightsquigarrow$ So get Hodge (-de Rham) fil'n on $H_{dR}^*(X)$.

Note $\quad H^i_{dR}(X) \not\cong \underbrace{H^i_{et}(X, \mathbb{Q}_p)}\otimes_{\mathbb{Q}_p} \mathbb{C}_p$ not canonically.

get trivialized in terms of mod curves if X ell curve.

Thm (Scholze) $\exists$ Hodge-Tate spectral sequence

$$E_2^{ij} = H^i(X, \Omega^j_X)\underbrace{(-j)}_{\text{Tate twist}} \Rightarrow H^{i+j}_{et}(X, \mathbb{Q}_p)\otimes_{\mathbb{Q}_p}\mathbb{C}_p.$$

degenerates at E_2

$\rightsquigarrow$ So get Hodge-Tate fil'n on $H^i_{et}(X, \mathbb{Q}_p)\otimes_{\mathbb{Q}_p}\mathbb{C}_p$.

Example Let $E/\mathbb{C}_p$ elliptic curve.

Hodge-de Rham:

$$0 \to (\mathrm{Lie}\, E)^* \to H_1^{dR}(E) \to \mathrm{Lie}\, E \to 0$$

Hodge-Tate:

$$0 \to (\mathrm{Lie}\, E)(1) \to T_p E \otimes_{\mathbb{Z}_p} \mathbb{C}_p \to (\mathrm{Lie}\, E^*)^* \to 0.$$

Both are not canonically split.

Back to modular curves

Fix Γ. Set $\Gamma(p^n) = \{\gamma \in \Gamma \mid \gamma \equiv \begin{pmatrix}1&0\\0&1\end{pmatrix} \bmod p^n\}$.

$(M_{\Gamma(p^n)} \otimes_{\mathbb{Q}} \mathbb{C}_p)^{ad}$ adic space $/\mathbb{C}_p$

(some kind of p-adic analytic space).

Thm (Scholze) (1) $\exists!$ perfectoid space $M_{\Gamma(p^\infty)}/\mathbb{C}_p$.

full subcat of adic spaces

s.t. $\mathcal{M}_{\Gamma(p^\infty)} \sim \varprojlim_{\Gamma} (\mathcal{M}_{\Gamma(p^n)} \otimes_{\mathbb{Q}} \mathbb{C}_p)^{ad}$

$\circlearrowleft$

$GL_2(\mathbb{Q}_p)$

Cat of perf'd spaces doesn't admit $\varprojlim$.

(2) $\exists$ Hodge-Tate period map

$$\pi_{HT}: \mathcal{M}_{\Gamma(p^\infty)} \longrightarrow (\mathbb{P}^1)^{ad}$$

$$(E, \beta: T_pE \cong \mathbb{Z}_p^2) \longmapsto (\mathbb{C}_p^2 \overset{\beta}{\simeq} T_pE \otimes_{\mathbb{Z}_p} \mathbb{C}_p \twoheadrightarrow (\mathrm{Lie}\, E^*)^*)$$

that is $GL_2(\mathbb{Q}_p)$-equiv

& equiv for Hecke operators prime to p

w.r.t. trivial action on $\mathbb{P}^1$.

$\rightsquigarrow$ contracts $GL_2(\mathbb{A}_f^p)$-orbits.

<u>Lecture 4: Proof of the lifting theorem</u>

Fix $g \geq 1$ $\rightsquigarrow$ Siegel moduli space

$$\mathcal{A}_{g,\Gamma} = \Gamma \backslash \mathfrak{S}_g \quad \text{of level } \Gamma \leq Sp_{2g}(\mathbb{Z}).$$

<u>Lifting thm</u> Any Hecke EV system appearing in $H^{?}_c(\mathcal{A}_{g,\Gamma}, \mathbb{F}_p)$ lift to Hecke EV system of cusp form.

Use p-adic geometry of $\mathcal{A}_{g,\Gamma}$:

universal compactification $\mathcal{A}^*_{g,\Gamma}$ (= Satake / Baily-Borel).

Let $\Gamma(p^n) = \{\gamma \in \Gamma \mid \gamma \equiv 1 \mod p^n\}$.

<u>Thm</u> (1) $\exists !$ perfectoid space $\mathcal{A}^*_{g,\Gamma(p^\infty)}$ $/\mathbb{C}_p$

s.t. $\mathcal{A}^*_{g,\Gamma(p^\infty)} \sim \varprojlim_n (\mathcal{A}^*_{g,\Gamma(p^n)} \otimes_{\mathbb{Q}} \mathbb{C}_p)^{ad}$

$\circlearrowleft Sp_{2g}(\mathbb{Q}_p)$

(2) $\exists$ Hodge-Tate period map

$$\pi_{HT} : \mathcal{A}^*_{g,\Gamma(p^\infty)} \longrightarrow \mathcal{F}\ell$$

param totally isotropic subspaces of $\mathbb{C}_p^{2g}$.

- $Sp_{2g}(\mathbb{Q}_p)$-equiv
- equiv for Hecke operators prime to p w.r.t. trivial actions on $\mathcal{F}\ell$.

Moreover, this map is affinoid b/w adic spaces (but very far from being finite).

<u>Example</u> $g=1$, $\mathcal{M}^*_{\Gamma(p^\infty)} := \mathcal{A}^*_{1,\Gamma(p^\infty)}$ "cpt modular curve"

$$\rightsquigarrow \quad \begin{array}{ccccc} \mathcal{M}^{*}_{\Gamma(p^\infty)} & = & \mathcal{M}^{\mathrm{ord}}_{\Gamma(p^\infty)} & \sqcup & \mathcal{M}^{\mathrm{ss}}_{\Gamma(p^\infty)} \\ {\scriptstyle \pi_{HT}}\downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^1 & = & \mathbb{P}^1(\mathbb{Q}_p) & \sqcup & \Omega^2 \\ & & & & \| \\ & & & & \mathbb{P}^1_{\mathbb{C}_p} \setminus \mathbb{P}^1(\mathbb{Q}_p) \end{array}$$

Drinfeld's upper half space.

- On ordinary locus,

  π_{HT} measures position of canonical subgrp.

- $$\begin{tikzcd} \mathcal{M}^{ss}_{\Gamma(p^\infty)} \arrow[r, phantom, "\cong"] \arrow[ddr, bend right, "\pi^{ss}_{HT}"'] & \coprod_{\text{finite}} \mathcal{M}_{LT,\infty} \\ \arrow[r, phantom, "\overset{\text{Faltings}}{\cong}"] & \coprod_{\text{finite}} \mathcal{M}_{Dr,\infty} \arrow[d, "\text{pro-finite \& étale}"] \\ & \Omega^2 \end{tikzcd}$$

  $\mathcal{M}_{LT,\infty}$ — Lubin-Tate space at ∞ level

  $\mathcal{M}_{Dr,\infty}$ — Drinfeld's space at ∞ level

<u>Thm</u> (Scholze-Weinstein)

Let $C/\mathbb{Q}_p$ alg closed + Complete (e.g. $C = \mathbb{C}_p$).

$$\{p\text{-div grps} / \mathcal{O}_C\} \xrightarrow{\cong} \left\{ (\Lambda, W) \;\middle|\; \begin{array}{l} \Lambda \text{ finite free } \mathbb{Z}_p\text{-mod} \\ + W \subset \Lambda \otimes_{\mathbb{Z}_p} C \text{ sub v.s.} \end{array} \right\}$$

$$G \longmapsto (T_pG, \text{Hodge-Tate fil'ns}).$$

<u>Rmk</u> (1) Analogue of classification of complex tori.

(2) Thus, on geom points of locus of good reduction,

π_{HT} is the map

$$(A + \text{extra str}) \longmapsto (A[p^\infty] + \text{extra str}).$$

And $\mathcal{F}l$ "=" moduli space of p-div grps.

(3) Prove first the case that

$$\left.\begin{array}{l} |\cdot| : C \longrightarrow \mathbb{R}_{\geq 0} \text{ surjective} \\ + \quad C \text{ spherically complete} \end{array}\right\rangle (\mathbb{C}_p \text{ fails both !})$$

$+$ descent.

(requires C really large).

(A) Implies "image of Rapoport-Zink period morphism".

Proof of lifting thm

Step 1 $\underbrace{H^i_{c,\text{ét}}((\mathcal{A}_{g,\Gamma} \otimes_{\mathbb{Q}} \mathbb{C}_p)^{ad}, \mathbb{F}_p)}_{= H^i_c(\mathcal{A}_{g,\Gamma}, \mathbb{F}_p)} \otimes_{\mathbb{F}_p} \mathcal{O}_C/p$. $\quad (C = \mathbb{C}_p)$.

$$\cong \Big\downarrow \text{(almost)}$$

(p-adic Hodge theory isom, for torsion coeff).

$H^i_{\text{ét}}((\mathcal{A}^*_{g,\Gamma} \otimes_{\mathbb{Q}} \mathbb{C}_p)^{ad}, I^+/p)$ — coherent cohom

(by Scholze, after result of Faltings).

Here $I^+ = I \cap \mathcal{O}^+ \subseteq \mathcal{O}$ — sheaf of bounded fcts w.r.t. $|\cdot|$.

I: sheaf of cusp forms; $\mathcal{O}^+$: sheaf of fcts f s.t. $|f| \leq 1$

Meaning of "almost":

Faltings's "Almost Math" (c.f. Gabber-Ramero).

An $\mathcal{O}_C$-module M is <u>almost zero</u> if

$$p^{\varepsilon} M = 0 \text{ for all } \varepsilon > 0.$$

$f: M \to N$ almost isom if $\ker f$ & $\operatorname{coker} f$ almost zero.

<u>Problem</u> Need to take étale cohom.

<u>Step 2</u> Pass to infinite level.

$$\underbrace{\left(\varinjlim_n H_c^i(\mathcal{A}_{g,\Gamma(p^n)}, \mathbb{F}_p)\right)}_{\text{Completed cohom mod } p} \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \xrightarrow[\sim]{\text{almost}} H^i_{\acute{e}t}(\mathcal{A}^*_{g,\Gamma(p^\infty)}, \mathcal{I}^+/p)$$

(c.f. Emerton's theory)

<u>Step 3</u> $\mathcal{A}^*_{g,\Gamma(p^\infty)}$ perfectoid

$\Rightarrow$ For $\mathcal{U} \subseteq \mathcal{A}^*_{g,\Gamma(p^\infty)}$ affinoid,

$$H^i_{\acute{e}t}(\mathcal{U}, \mathcal{I}^+/p) \overset{\text{almost}}{=} \begin{cases} \mathcal{I}^+(\mathcal{U})/p, & i=0 \\ 0, & i>0. \end{cases}$$

(Nothing similar would be true at finite level!)

$\Rightarrow$ Can compute $H^i_{\acute{e}t}(\mathcal{A}^*_{g,\Gamma(p^\infty)}, \mathcal{I}^+/p)$ via Čech complex with terms $\mathcal{I}^+(\mathcal{U})/p$ for affinoid $\mathcal{U} \subseteq \mathcal{A}^*_{g,\Gamma(p^\infty)}$.

$\Rightarrow$ Can realize Hecke EV by "cusp form on $\mathcal{U}$".

<u>Step 4</u> Approximate by globally defined cusp form, without messing up Hecke EV (Hecke-equiv).

<u>Usually</u> $\mathcal{U}$ = ordinary locus, multiply by large power of Hasse invariant.

- vanishes outside ordinary locus
- Commutes with Hecke operators away from p.

<u>Solution</u> Any function (or section of autom VB) that comes via pullback along π_{HT} from $\mathcal{F}\ell$ will commute with Hecke operators away from p. $\rightsquigarrow$ "fake Hasse invariant".

T_{HT} affinoid $\Rightarrow$ these are enough of them.

<u>Why is $\mathcal{M}^*_{\Gamma(p^\infty)}$ perfectoid?</u>

<u>Def</u> A Banach $\mathbb{C}_p$-alg R is <u>perfectoid</u>

if (i) $R^\circ \subseteq R$ (subring of power-bounded elts) is bounded

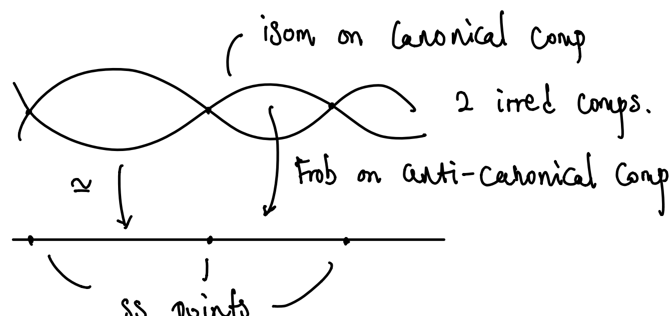
(ii) $\Phi : R^\circ/p \longrightarrow R^\circ/p$ surjective.

$$x \longmapsto x^p$$

e.g. $R = C\langle T^{1/p^\infty}\rangle = \mathcal{O}_C[T^{1/p^\infty}]^\wedge_p[\frac{1}{p}]$.

<u>Geometry</u> $\mathcal{M}_{\Gamma_0(p)}$ mod p

$\mathcal{M}_\Gamma$ mod p



Generally, $\mathcal{M}_{\Gamma_0(p^n)}$ mod p has $n+1$ irred comps.

precisely 1 of them maps to anti-can comp (does so via Frob).

$\Rightarrow$ $\mathcal{M}_{\Gamma_0(p^\infty)}$ perfectoid above anti-can ordinary locus.

$\Rightarrow$ so does $\mathcal{M}_{\Gamma(p^\infty)}$.

almost purity thm

(R perf'd, S/R finite étale $\Rightarrow$ S perf'd).

Theory of overconvergence of canonical subgrp

$\Rightarrow$ Same is true in strict nbhd of anti-can ordinary locus

(this is enough)

<u>Note</u> $GL_2(\mathbb{Q}_p) \circlearrowright \mathcal{M}_{\Gamma(p^\infty)}$

locus where perfectoid is stable under $GL_2(\mathbb{Q}_p)$-action.