Notes for Berkeley Lectures on p-adic Geometry

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2014

Lecture 1: Introduction

Motivation Drinfeld's work on global Langlands correspondence.

Fix $C/\mathbb{F}_q$ smooth proj geom conn curve.

Definition A shtuka of rank n over $\mathbb{F}_q$-scheme S is a vector bundle $\mathcal{E}/C\times_{\mathbb{F}_q} S$ + meromorphic isom $\varphi_E : \mathrm{Frob}_S^* \mathcal{E} \to \mathcal{E}$ (defined on $\mathcal{U} \subset C\times S$ which is fibrewise dense in C).

Geometric Langlands Stack Bun_{GL_n} of rank n vector bundle on C.

Arithmetic Langlands Bun_{GL_n} + its Frobenius map

$\rightsquigarrow$ moduli space of shtukas

($\approx$ Frobenius fixed points).

Data attached to shtukas over $S = \mathrm{Spec}\, k$, $k = \bar{k}$:

(1) points $x_1, \cdots, x_m \in C(k)$ points of indeterminancy of φ_E "zeros/poles" or "legs"

(2) For each $i = 1, \cdots, m$, the relative position of $(\mathrm{Frob}_S^* \mathcal{E})^\wedge_{x_i}$ and $\hat{\mathcal{E}}_{x_i}$.

These are free rank n modules over $\hat{\mathcal{O}}_{C,x_i} \cong k[\![t_i]\!]$ with same generic fibre.

i.e. the $k[\![t_i]\!]$-lattice in same n-dim $k(\!(t_i)\!)$-v.s.

Elementary divisors:

$\exists$ basis $e_1, \dots, e_n$ of $\hat{\mathcal{E}}_{x_i}$ s.t. $t_i^{k_1} e_1, \dots, t_i^{k_n} e_n$

is a basis of $(\mathrm{Frob}_S^* \mathcal{E})^{\wedge}_{x_i}$,

with $k_1 \geq \dots \geq k_n$

well-defined set of integers.

The data $\{k_1, \dots, k_n\} \cong$ Conj class of cocharacters $\mu_i : \mathbb{G}_m \to GL_n$

(for a fixed i) $\quad \cong$ highest weight reprs r_i of $\widehat{GL_n} = GL_n$.

Fix discrete data : m, $\{\mu_1, \dots, \mu_m\}$.

$\Rightarrow$ $\exists$ moduli space $\mathrm{Sht}_{m, \{\mu_1, \dots, \mu_m\}}$ of such shtukas

(Deligne-Mumford stack, not quasi-compact!)

$$\begin{array}{ccc} \mathrm{Sht}_{m, \{\mu_1, \dots, \mu_m\}} & \ni & \text{shtuka} \\ f \downarrow & & \downarrow \\ C^m & \ni & \text{legs} \end{array}$$

form equal char analogue of Shimura varieties.

Let $d = \dim \mathrm{Sht}_{m, \{\mu_1, \dots, \mu_m\}} - m$.

Consider $R^d f_! \overline{\mathbb{Q}}_\ell$ (not constructible, as f not finite type)

ignore this.

$(R^d f_! \mathbb{Q}_\ell)_{\bar{\eta}^m}$ $\quad \bar{\eta}^m$ geom gen point of C^n

$\circlearrowleft$

$\mathrm{Gal}(\bar{\eta}^m / \eta^m)$

$=$ Gal grp of gen point of C^n.

(bigger than $\mathrm{Gal}(\bar{\eta}/\eta)^m$.)

· Partial Frobenii: for $i = 1, \dots, m$, $F_i : C^m \to C^m$

$(id, id, \dots, \underbrace{\mathrm{Frob}_C}_{i\text{-th spot}}, \dots)$

There are canonical (commuting) isom

$$F_i^*(R^d f_! \bar{\mathbb{Q}}_\ell) \cong R^d f_! \bar{\mathbb{Q}}_\ell .$$

<u>Lemma</u> (Drinfeld) For $U \subset C$ open,

$$\pi_1(U^m/\text{partial Frobenii}) \cong \pi_1(U) \times \cdots \times \pi_1(U) \quad (m \text{ copies})$$

" classifies finite étale covers of U^m equipped with partial Frobenii.

$$\Rightarrow \quad (R^d f_! \mathbb{Q}_\ell)_{\bar{\eta}^m} \circlearrowleft \mathrm{Gal}_F \times \cdots \times \mathrm{Gal}_F, \quad F = \text{func field of } C/\mathbb{F}_q.$$

(Can introduce level structures on $\mathrm{Sht}_{m,\{\mu_1,\cdots,\mu_m\}}$.)

<u>Rough Conjecture</u>

$$\varinjlim_{\text{levels}} R^d f_! \bar{\mathbb{Q}}_\ell = \bigoplus_{\substack{\pi \text{ autom repr} \\ \text{of } GL_n(\mathbb{A}_F)}} \pi \otimes (r_1 \circ \sigma(\pi)) \otimes \cdots \otimes (r_m \circ \sigma(\pi))$$

$$\circlearrowleft \; GL_n(\mathbb{A}_F) \times \underbrace{\mathrm{Gal}_F \times \cdots \times \mathrm{Gal}_F}_{m}$$

where each $\pi \rightsquigarrow \sigma(\pi) : \mathrm{Gal}_F \longrightarrow \widehat{GL}_n(\bar{\mathbb{Q}}_\ell)$ assoc Gal repr (L-parameter)

$$r_i \downarrow \text{ the repr of } \widehat{GL}_n \text{ corr to } \mu_i$$

$$GL_{n_i}(\bar{\mathbb{Q}}_\ell)$$

- Drinfeld: $n=2$.
- L. Lafforgue: general n, $m=2$.

  $\{\mu_1, \mu_2\} = \{(1,0,\cdots,0), (0,\cdots,0,-1)\}$.

  $r_1 : GL_n \xrightarrow{=} GL_n$.

  $r_2 : GL_n \xrightarrow[\text{dual}]{\sim} GL_n$.

  Should get $\bigoplus_{\pi} \pi \otimes \sigma(\pi) \otimes \sigma(\pi)^\vee$.
- V. Lafforgue: Considered general reductive group G in place of GL_n.

+ Constructed L-parameters for all cusp autom repr of G using crucially <u>all</u> $\mathrm{Sht}_{m,\{y_1,\dots,y_m\}}$ simultaneously.
(varying m & $\{y_i\}$)

If trying to do this over number fields,
What is $\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}$?
(fibre product "over $\mathbb{F}_1$").

<u>First aim of this course</u>
Describe the completion at (p,p) of $\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}$.

<u>Open problem</u> Describe completions of $\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}$ along diagonal.

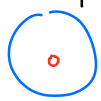
<u>Remark</u> Can probably <u>not</u> hope for more.
If $\ell\neq p$, $\operatorname{Spec}\mathbb{F}_p\times\operatorname{Spec}\mathbb{F}_\ell\subset\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}$
does not seem to make sense.

One description: of $\underbrace{\operatorname{Spa}\mathbb{Q}_p}\times\operatorname{Spa}\mathbb{Q}_p$
adic spaces: in world of nonarch analytic geometry.

First, $\operatorname{Spa}\mathbb{F}_p((t))\times_{\operatorname{Spa}\mathbb{F}_p}\operatorname{Spa}\mathbb{F}_p((t))$: This exists as adic space.
$\downarrow$ $\operatorname{Spa}\mathbb{F}_p((t))$: formal punctured unit disc.
For all nonarch fields $K/\mathbb{F}_p$,
$$\operatorname{Spa}\mathbb{F}_p((t))\times_{\operatorname{Spa}\mathbb{F}_p}\operatorname{Spa}K=\mathbb{D}_K^*=\{x\mid 0<|x|<1\}$$
punctured open unit disc.

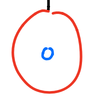
$$\Rightarrow \quad \underbrace{\operatorname{Spa}\mathbb{F}_p((t)) \times_{\operatorname{Spa}\mathbb{F}_p} \operatorname{Spa}\mathbb{F}_p((u))}$$

$= \mathbb{D}^*_{\mathbb{F}_p((t))}$



t parameter
u const

$= \mathbb{D}^*_{\mathbb{F}_p((u))}$

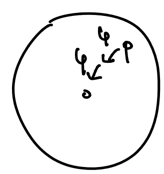


u parameter
t const

<u>Local Drinfeld's lemma</u> $(n=2)$

$$\operatorname{Spa}\mathbb{F}_p((t)) \times_{\operatorname{Spa}\mathbb{F}_p} \operatorname{Spa}\mathbb{F}_p((u)) = \mathbb{D}^*_{\mathbb{F}_p((t))} \circlearrowleft \varphi : t \mapsto t,\ u \mapsto u^p$$

<u>Seen</u> φ-action is totally discontinuous.



Let $X := \mathbb{D}^*_{\mathbb{F}_p((t))} / \varphi^{\mathbb{Z}}$: adic space $/ \mathbb{F}_p((t))$.

<u>Lemma</u> (local Drinfeld lemma, Fargues-Fontaine, Weinstein)

$$\pi_1(X) \cong G_{\mathbb{F}_p((t))} \times G_{\mathbb{F}_p((t))}.$$

<u>Analogue for $\mathbb{Q}_p$</u> (Fargues-Fontaine, Weinstein)

(One model for) $\quad \operatorname{Spa}\mathbb{Q}_p \times \operatorname{Spa}\mathbb{Q}_p = \tilde{\mathbb{D}}^*_{\mathbb{Q}_p} / \mathbb{Z}_p^*$

quotient taken in "formal sense".

Here. $\mathbb{D}_{\mathbb{Q}_p} = \{ x \mid |x| < 1 \} \hookrightarrow \mathbb{G}_m$ is a group, $\mathbb{Z}_p$-module.

$$x \longmapsto 1 + x$$

$\tilde{\mathbb{D}}_{\mathbb{Q}_p} := \varprojlim_{x \mapsto (1+x)^p - 1} \mathbb{D}_{\mathbb{Q}_p}$ "(pre) perfectoid space".

& $\mathbb{Q}_p$-vector space object

$\tilde{\mathbb{D}}^*_{\mathbb{Q}_p} := \tilde{\mathbb{D}}_{\mathbb{Q}_p} \setminus \{0\} \circlearrowleft \mathbb{Q}_p^*$ scaling

<u>NOT</u> an adic space (even if it is "punctured").

<u>Note</u> Have $\tilde{\mathbb{D}}^*_{\mathbb{Q}_p} / \mathbb{Z}_p^* \circlearrowleft \varphi \cong p \in \mathbb{Q}_p^*$ totally discontinuous.

Let $X := (\widetilde{\mathbb{D}}^*_{\mathbb{Q}_p} / \mathbb{Z}_p^*) / \varphi^{\mathbb{Z}} = \widetilde{\mathbb{D}}^*_{\mathbb{Q}_p} / \mathbb{Q}_p^*$.

<u>Thm</u> $\pi_1(X) \cong G_{\mathbb{Q}_p} \times G_{\mathbb{Q}_p}$.

<u>First part of the course</u>

Explain theory of "diamonds" where these objects live.

+ definition of $\mathrm{Spa}\,\mathbb{Q}_p \times \mathrm{Spa}\,\mathbb{Q}_p$ becomes proposition.

<u>Second aim of the course</u>

Define "spaces" of local shtukas in this setup.

<u>function fields</u>	<u>number fields</u>
moduli space of shtukas	Shimura varieties
assoc with $(G, \{\mu_1, \dots, \mu_m\})$	assoc with (G, μ)
· G reductive group	· G reductive group $/\mathbb{Q}$
· $\{\mu_i\}$ conj classes of cochar of G	· μ conj class of <u>minuscule</u> cochar
These live over m copies of a curve.	These live over one copy of "curve" $\mathrm{Spec}\,\mathbb{Z}$.
	moduli space of abelian vars with extra structures.

<u>This course</u> Look at local analogues of this spaces.

moduli space of local shtukas (Pink, Hartl, Viehmann)	Rapoport-Zink spaces moduli spaces of p-div grps.

Rapoport-Viehmann: should exist "local Shimura varieties" (still for (G, μ), μ minuscule.)

<u>Goal</u> (1) Define notion of local Shtuka in mixed characteristic

(2) Construct moduli spaces as "diamonds" of these for any $(G, \{\mu_i\})$, $\{\mu_i\}$ system of <u>any</u> cochars, living on m copies of $\mathrm{Spa}\, \mathbb{Q}_p$.

(3) Show that these generalized RZ spaces and specialize to local Shimura varieties (should be classical rigid spaces) i.e. relate local shtukas + p-div grps.

<u>Open problem</u> Define version / $\mathrm{Spec}\, \mathbb{Z}$ of shtukas + moduli spaces of such.

Lecture 2: Adic spaces (1/2)

Want category that contains
schemes, formal schemes, and rigid spaces
as full subcategories.

If $X \longrightarrow \mathrm{Spf}\, \mathbb{Z}_p$ formal scheme (not necessarily p-adic,
e.g. $X = \mathrm{Spf}\, \mathbb{Z}_p[\![T]\!]$, (p,T)-adic).

generic fibre $X_\eta = X^{ad} \times_{(\mathrm{Spf}\,\mathbb{Z}_p)^{ad}} \{\eta\}$
where $(\mathrm{Spf}\, \mathbb{Z}_p)^{ad} = \{s, \eta\}$, η generic point
s special point.

e.g. if $X = \mathrm{Spf}\, \mathbb{Z}_p[\![T]\!]$, then X_η open unit ball / $\mathbb{Q}_p$.

On the other hand, can also take $X^{ad} \times_{(\mathrm{Spf}\,\mathbb{Z}_p)^{ad}} \{s\}$
if X is p-adic
this is the usual special fibre.

Basic building blocks: affinoid adic spaces $\underbrace{\mathrm{Spa}}_{\text{adic spectrum}}(A, A^+)$

Today Define which pairs (A, A^+) are allowed,
Spa as top space.

Definition A topological ring A is f-adic (f = "finite")
if A admits an open subring $A_0 \subset A$
which is adic with finite gen'd ideal of def'n.

(i.e. $\exists$ f.g. ideal $I \subset A_0$ s.t. $\{I^n \mid n \geq 0\}$ forms basis of open nbhds of 0.)

<u>Reference</u>: [Huber, "Cont valuations"].

<u>Remark</u> A f-adic $\Rightarrow$ $\hat{A}$ f-adic

$\hat{A}_0$ = closure of $A_0 \subset \hat{A}$.

$\quad$ = completion of $A_0 = \varprojlim_n A_0/I^n$

open subring with f.g. ideal of def'n $I \cdot \hat{A}_0$.

Nothing will change if replacing A by $\hat{A}$.

Any such $A_0 \subset A$ is called <u>ring of definition</u> of A.

<u>Examples</u> (1) ("schemes")

Any discrete ring A is f-adic.

All subrings $A_0 \subset A$ are rings of def'n with ideal of def'n $= (0)$.

(2) ("formal schemes")

An adic ring A is f-adic iff it has a f.g. ideal of def'n.

In that case $A_0 = A$ ring of def'n.

(3) ("rigid spaces")

Let A_0 any ring, $g \in A_0$ a nonzero-divisor.

Let $A = A_0[g^{-1}]$ equipped with the topology making $\{g^n A_0 \mid n \geq 0\}$ basis of open nbhds of 0.

Then A is f-adic with ring of def'n A_0 ideal of def'n $I = (g)$.

e.g. If A Banach alg / nonarch field K,
let $A_0 \subseteq A$ unit ball,
$g \in K^*$ with $|g| < 1$.

<u>Definition</u> Subset $S \subseteq A$ is <u>bounded</u> if for all open nbhds $\mathcal{U}$ of 0
$\exists$ open nbhd V of 0 s.t. $V \cdot S \subset \mathcal{U}$.
(namely, S doesn't spread too wide beyond $\mathcal{U}$)
(In checking this, you are allowed to shrink $\mathcal{U}$
+ assume $\mathcal{U}$ stable under addition
b/c $\{I^n\}$ forms basis of open nbhds of 0.)

<u>Lemma</u> Let A an adic ring.
A subring $A_0 \subseteq A$ is a ring of def'n
iff it is open and bounded.

<u>Proof</u> If A_0 ring of def'n, it is open.
Let $\mathcal{U}$ open nbhd of 0 in A. wlog $\mathcal{U} = I^n$, $n \gg 0$.
But then can take $V = I^n$:
$$V \cdot A_0 = I^n \cdot A_0 = I^n \subseteq \mathcal{U} = I^n.$$
Converse: easy exercise. $\square$

<u>Definition</u> An f-adic ring A is <u>Tate</u> if it has a topologically nilpotent unit $g \in A$.

<u>Prop</u> (i) If $A = A_0[g^{-1}]$ as in Ex (3), then A is Tate.

(2) If A is Tate, $A_0 \subseteq A$ any ring of def'n,
then A_0 is g^n-adic ($n \gg 0$, $g^n \in A_0$)
and $A = A_0[(g^n)^{-1}]$
(thus arises as in Ex (3)).

(3) If A is Tate,
$S \subset A$ is bounded $\iff$ $S \subset g^{-n} A_0$ for some n
(A_0 any fixed ring of def'n.)

<u>Proof</u> (1) $g \in A$ top-nilpotent $\Rightarrow$ A is Tate.

(2) Let $I \subseteq A_0$ ideal of def'n. wlog $g \in I$
(replace g by g^n for $n \gg 0$).

g invertible $\Rightarrow$ gA_0 = preimage of A_0 under $g^{-1} : A \to A$
gA_0 is open
$\Rightarrow$ $\exists n$ s.t. $I^n \subseteq (g)$
$\Rightarrow$ $(g^n) \subseteq I^n \subseteq (g)$
$\Rightarrow$ A_0 is g-adic.

Remains to show $A = A_0[g^{-1}]$.
Clearly, $A_0[g^{-1}] \hookrightarrow A$.
If $x \in A$, then $g^n x \to 0$ as $n \to \infty$ (as g top-nilp)
$\Rightarrow$ $\exists n$, $g^n x \in A_0$ $\Rightarrow$ $x \in A_0[g^{-1}]$.

(3) Exercise $\square$

<u>Remark</u> Let A complete Tate ring, fix $A_0 \subseteq A$ ring of def'n, $g \in A_0 \cap A^*$ top-nilp
Then can define a norm $|\cdot| : A \to \mathbb{R}_{\geq 0}$ on A

$$|a| = \inf_{\{n \in \mathbb{Z} \mid g^n a \in A_0\}} 2^n .$$

- $a \in A_0 \Rightarrow |a| \leq 1$.
- $|g| = \frac{1}{2}$, $|g^{-1}| = 2$.

$\Rightarrow$ If A complete then A becomes Banach ring with unit ball A_0.

Prop This gives equiv. of categories

$$\left\{ \begin{array}{c} \text{complete Tate rings} \\ (+ \text{ cont homomorphisms}) \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Banach rings } A \text{ which admits} \\ g \in A^*,\ |g| < 1 \text{ s.t. } |g| \cdot |g^{-1}| = 1 \\ \text{(and bounded homomorphisms)} \end{array} \right\}$$

<u>Definition</u> A Huber ring (or f-adic ring).
An element $x \in A$ is <u>power-bounded</u> if $\{x^n \mid n \geq 0\} \subset A$ is bounded.
Let $A^\circ \subset A$ subring of power-bounded elements.

<u>Example</u> If $A = \mathbb{Q}_p[T]/T^2$ then $A^\circ = \mathbb{Z}_p \oplus \mathbb{Q}_p \cdot T$
This is not bounded, thus not a ring of def'n.

Prop (1) Any ring of def'n $A_0 \subseteq A$ is contained in A°.
(2) A° is the filtered direct limit of the rings of def'n $A_0 \subseteq A$.
any 2 subrings of def'n contained in a third.

Proof (1) For any $x \in A_0$, $\{x^n\} \subseteq A_0$ bounded $\Rightarrow x \in A^\circ$.
(2) Filtered: If $A_0, A_0' \subseteq A$ rings of def'n, let

$$A_0'' = \left\{ \sum_{i=1}^{n} x_i y_i \mid x_i \in A,\ y_i \in A_0' \right\} \subseteq A$$

ring gen'd by A_0, A_0'.

To show A_0'' bounded (clearly open).

If $\mathcal{U} \subseteq A$ open nbhd of 0, have to find V s.t. $V \cdot A_0'' \subseteq \mathcal{U}$.

wlog $\mathcal{U}$ closed under addition.

Pick $\mathcal{U}_1$ s.t. $\mathcal{U}_1 \cdot A_0 \subseteq \mathcal{U}$.

Pick V s.t. $V \cdot A_0' \subset \mathcal{U}_1$.

Then for all $\sum_{i=1}^{n} x_i y_i \in A_0''$,

$$\left(\sum x_i y_i\right) V \subseteq \sum (x_i \underbrace{y_i V}_{\subseteq A_0'}) \subseteq \sum (\underbrace{x_i \cdot \mathcal{U}_1}_{\subseteq A_0}) \subseteq \sum \mathcal{U} = \mathcal{U}.$$

Clearly, $\varinjlim_{A_0} A_0 \hookrightarrow A^\circ$.

Pick any $x \in A^\circ$, any A_0.

Then $A_0[x]$ is still ring of def'n, i.e. still bounded.

Same argument, using $\{x^n\}$ bounded.

<u>Definition</u> An f-adic ring A is <u>uniform</u> if A° is bounded

($\Leftrightarrow$ A° ring of def'n).

(A Tate + uniform $\Rightarrow$ A reduced.)

<u>Definition</u> Let A f-adic.

(1) A ring of integral elements $A^+ \subseteq A$ is an open + integral closed subring $A^+ \subseteq A^\circ$.

(2) An affinoid ring (A, A^+) is a pair of f-adic ring A + $A^+ \subset A$ ring of integrals.

<u>Remark</u> often, $A^+ = A^\circ$.

$A^\circ \subseteq A$ integrally closed.

Adic spectra

Definition (A top ring)

A continuous valuation is a map
$$|\cdot| : A \longrightarrow \Gamma_{|\cdot|} \cup \{0\}$$
where $\Gamma_{|\cdot|}$ totally ordered multiplicative group, written multiplicatively, e.g. $\Gamma = \mathbb{R}_{>0}$,

or e.g. $\Gamma = \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$, $r < \gamma < 1$ for all reals $r < 1$.

s.t. $|ab| = |a| \cdot |b|$

$|a+b| \leq \max(|a|, |b|)$,

$|1| = 1$,

$|0| = 0$,

+ Continuity s.t. for all $\gamma \in \Gamma_{|\cdot|}$,

$\{a \in A \mid |a| < \gamma\} \subset A$ open.

Two cont. valuations $|\cdot|$ & $|\cdot|'$ are <u>equivalent</u> if $|a| \geq |b| \iff |a|' \geq |b|'$.

(In that case, after replacing $\Gamma_{|\cdot|}$ by subgroup gen by image of A,

$\exists$ order-preserving isom $\Gamma_{|\cdot|} \cong \Gamma_{|\cdot|'}$

s.t.
$$\begin{tikzcd} & \Gamma_{|\cdot|} \cup \{0\} \arrow[dd, "\wr"] \\ A \arrow[ur, "|\cdot|"] \arrow[dr, "|\cdot|'"'] \arrow[r, phantom, "\circlearrowright"] & \\ & \Gamma_{|\cdot|'} \cup \{0\}. \end{tikzcd}$$

Definition (A, A^+) affinoid ring.

$$\mathrm{Spa}(A, A^+) := \left\{ \begin{array}{l} \text{equiv classes of cont valuations } |\cdot| \text{ on } A \\ \qquad\qquad \text{s.t. } |A^+| \leq 1 \end{array} \right\}.$$

Notation For $x \in \mathrm{Spa}(A, A^+)$, write $g \mapsto |g(x)|$ for (choice of) corresponding valuation

topology generated by open subsets of the form

$$\{x \mid |f(x)| \leq |g(x)| \neq 0\}, \quad f, g \in A.$$

Want $\{x \mid |f(x)| \neq 0\}$ open + $\{x \mid |f(x)| \leq 1\}$.

Thm (Huber) $\mathrm{Spa}(A, A^+)$ as a spectral space.

Recall **Definition** (Hochster)

A top. space T is spectral if the following equiv. conditions are satisfied.

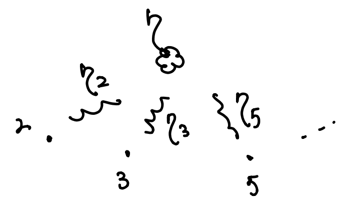
(1) $T \cong \mathrm{Spec}\, R$ for some ring R

(2) $T \cong \varprojlim T_i$ where T_i finite T_0-spaces.

(3) $\exists$ basis for top of qc open subsets, stable under finite intersection, and T is sober (i.e. every irred closed subset has unique generic point).

Example Let R discrete ring. Then

$$\mathrm{Spa}(R, R) = \{\text{valuations on } R \text{ s.t. } |R| \leq 1\}.$$



$$|\cdot|_\eta : \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \{0,1\}$$
$$0 \longmapsto 0$$
$$x \neq 0 \longmapsto 1$$

$$p : \mathbb{Z} \longrightarrow \mathbb{F}_p \longrightarrow \{0,1\}$$

$\eta_p = |\cdot|_p : \mathbb{Z} \longrightarrow \mathbb{Z}_p \longrightarrow p^{\mathbb{Z}} \cup \{0\}$ usual p-adic value.

In general, $\mathfrak{p} \longmapsto (R \to \mathrm{Frac}(R/\mathfrak{p}) \to \{0,1\})$

$$\begin{tikzcd}
\mathrm{Spec}\, R \arrow[r] \arrow[dr, "id"'] & \mathrm{Spa}(R,R) \arrow[d] & \ni |\cdot| \arrow[d, mapsto] \\
& \mathrm{Spec}\, R & \ni \mathrm{Ker}(|\cdot|) = \{f \in R \mid |f| = 0\}
\end{tikzcd}$$

The functor $\mathrm{Spec}\, R \longrightarrow \mathrm{Spa}(R,R)$

will become fully faithful functor

$$\{\text{schemes}\} \longrightarrow \{\text{adic spaces}\}$$

Lecture 3: Adic spaces (2/2)

Let A ring, M A-module, $I \subseteq A$ f.g. ideal.
Then, for $\hat{M} = \varprojlim_n M/I^n M$ I-adic completion,
one has $\hat{M}/I\cdot\hat{M} = M/I\cdot M$
($\hat{M}$: I-adically complete).
<u>Reference</u> Stacks project, Section 10.91.

<u>Today</u> Define structure (pre)sheaf $\mathcal{O}_X$ on $X = \mathrm{Spa}(A, A^+)$.
(Reference: [Huber, "A generalization of formal schemes and rigid-analytic varieties"])

<u>Recall</u> (1) A Huber ring (= f-adic ring) is a top ring A which admits an open subring $A_0 \subset A$ which is adic with f.g. ideal of def'n.

(2) A Huber pair (= affinoid ring) is a pair (A, A^+) where $A^+ \subset A$ open + int. closed.

$\rightsquigarrow$ $X = \mathrm{Spa}(A, A^+) = \{$equiv. cl. of cont val on A s.t. $|A^+| \leq 1\}$.

<u>Thm</u> (Huber) $X = \mathrm{Spa}(A, A^+)$ spectral as top space,
i.e. X quasicompact, Sober (every int closed subset has unique gen point).
and there is a basis for top consisting of qc open subsets, stable under intersection.
(open: $\{x \in X \mid |f(x)| \leq |g(x)| \neq 0\}$ for $f, g \in A$, but not qc in general.)

Prop Let $(\hat{A}, \hat{A}^+)$ completion of (A, A^+). Then

$$\mathrm{Spa}(\hat{A}, \hat{A}^+) \xrightarrow{\cong} \mathrm{Spa}(A, A^+).$$

Today All Huber pairs are complete.

Prop $\mathrm{Spa}(A, A^+)$ is "large enough" in the following sense.

(i) $A \neq 0$ (+ complete), then $\mathrm{Spa}(A, A^+) \neq \emptyset$.

(ii) $A^+ = \{ f \in A \mid \forall x \in \mathrm{Spa}(A, A^+),\ |f(x)| \leq 1 \}$.

(iii) $f \in A$ invertible $\iff \forall x,\ |f(x)| \neq 0$.

Rational subsets

Definition Let $s_1, \cdots, s_n \in A$ and $T_1, \cdots, T_n \subseteq A$ finite subsets s.t. $T_i A \subseteq A$ is open for all i.

Then $\mathcal{U}(\{\frac{T_i}{s_i}\}) := \mathcal{U}(\frac{T_1}{s_1}, \cdots, \frac{T_n}{s_n})$

$$= \{ x \mid \forall i,\ t_i \in T_i,\ |t_i(x)| \leq |s_i(x)| \neq 0 \}$$

$$\subseteq \mathrm{Spa}(A, A^+) \text{ open.}$$

is a <u>rational subset</u>.

Note Intersection of rat'l subsets is rat'l.

Thm Let $\mathcal{U} \subseteq \mathrm{Spa}(A, A^+)$ be a rat'l subset.

Then there exists a complete Huber pair

$$(A, A^+) \longrightarrow (\mathcal{O}_X(\mathcal{U}), \mathcal{O}_X^+(\mathcal{U}))$$

s.t. $\mathrm{Spa}(\mathcal{O}_X(\mathcal{U}), \mathcal{O}_X^+(\mathcal{U})) \to \mathrm{Spa}(A, A^+)$ factors over $\mathcal{U}$

and is universal for such.

Moreover, this is a homeomorphism onto $\mathcal{U}$.

<u>Proof</u> (Sketch) Choose s_i, T_i s.t. $\mathcal{U} = \mathcal{U}(\{\frac{T_i}{s_i}\})$.

Choose $A_0 \subseteq A$ ring of def'n, $I \subset A_0$ f.g. ideal of def'n.

Take $(A, A^+) \to (B, B^+)$ s.t. $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ factors through $\mathcal{U}$.

Then (1) s_i are invertible in B

$\rightsquigarrow$ get map $A[\{\frac{1}{s_i}\}] \to B$

(2) All t_i/s_i are $|\cdot| \le 1$ everywhere on $\mathrm{Spa}(B, B^+)$ $\Rightarrow$ $\frac{t_i}{s_i} \in B^+ \subseteq \underset{\varinjlim B_0}{\underset{\parallel}{B^\circ}}$

wlog all $t_i/s_i \in B_0$.

$\rightsquigarrow$ get map $\underset{\underset{A[\{\frac{1}{s_i}\}]}{\cap}}{A_0}[\frac{t_i}{s_i} \mid i = 1, \dots, n,\ t_i \in T_i] \to B_0$

and A_0 maps into B_0.

$A[\{\frac{t_i}{s_i}\}]$ equipped with $I \cdot A_0[\{\frac{t_i}{s_i}\}]$-adic topology.

<u>Lemma</u> This defines a ring topology on $A[\{\frac{1}{s_i}\}]$ making $A[\{\frac{t_i}{s_i}\}]$ open subring.

<u>Crucial point</u> to show $\exists\, n$ s.t. $\frac{1}{s_i} I^n \subseteq A_0[\{\frac{t_i}{s_i}\}]$.

enough $I^n \subseteq T_i \cdot A_0$, follows from

<u>Lemma</u> If $T \subseteq A$ subset s.t. $T \cdot A \subseteq A$ open, then $T \cdot A_0$ open.

<u>Proof</u> wlog $\underset{(f_1, \dots, f_k)}{\overset{\parallel}{I}} \subseteq T \cdot A$. I f.g. $\Rightarrow$ $\exists$ finite set R s.t. $f_1, \dots, f_k \in T \cdot R$.

I top-nilp $\Rightarrow$ $\exists\, m$ s.t. $R \cdot I^m \subseteq A_0$.

Then for all $i=1,\dots,k$, $f_i \cdot I^m \subseteq T\cdot R \cdot I^m \subseteq I \cdot A_0$.

Sum over all i $\rightsquigarrow$ $I^{m+1} \subseteq T \cdot A_0$. $\square$

Back to the proof of Thm.

We have $A[\{\frac{1}{s_i}\}]$ a (non-complete) Huber ring.

Let $A\langle\{\frac{1}{s_i}\}\rangle^+$ = integral closure of image of $A^+[\{\frac{t_i}{s_i}\}]$ in $A[\{\frac{t_i}{s_i}\}]$.

Let $(A\langle\{\frac{T_i}{s_i}\}\rangle, A\langle\{\frac{T_i}{s_i}\}\rangle^+)$ = completion of this Huber pair.

This has desired universal property.

Its Spa $\xrightarrow{\sim} \mathcal{U}$: Use that Spa doesn't change under completion.

$\square$

<u>Definition</u> presheaf $\mathcal{O}_X$ on $\mathrm{Spa}(A,A^+)$:

If $\mathcal{U} \subseteq X$ rational, $\mathcal{O}_X(\mathcal{U})$ as in Theorem.

$$\mathcal{O}_X(W) = \varprojlim_{\substack{\mathcal{U}\subseteq W \\ \text{rational}}} \mathcal{O}_X(\mathcal{U}).$$

Same for $\mathcal{O}_X^+$.

<u>Prop</u> For all $\mathcal{U} \subseteq \mathrm{Spa}(A,A^+)$,

$$\mathcal{O}_X^+(\mathcal{U}) = \{ f \in \mathcal{O}_X(\mathcal{U}) \mid |f(x)| \le 1,\ \forall x \in \mathcal{U}\}.$$

In particular, $\mathcal{O}_X^+$ is a sheaf if $\mathcal{O}_X$ is.

<u>Thm</u> (Huber) $\mathcal{O}_X$ is a sheaf of top rings in the following situations:

(1) ("scheme") A is discrete.

(2) ("formal scheme") A is f.g. algebra over a noetherian ring of def'n.

(3) ("rigid space") A is Tate and strongly noetherian.

meaning that
$$A\langle T_1, \cdots, T_n\rangle = \left\{ \sum_{\underline{i}=(i_1,\cdots,i_n)\geq 0} a_{\underline{i}} T^{\underline{i}} \;\middle|\; a_{\underline{i}} \in A,\ a_{\underline{i}} \to 0 \right\}$$
is noetherian for all $n \geq 0$.

Example $A = \mathbb{C}_p$: (2) fails as $\mathcal{O}_{\mathbb{C}_p}$ is not noetherian

$(\,(p^{1/n}) \subseteq \mathcal{O}_{\mathbb{C}_p}\,)$.

But (3) applies b/c $\mathbb{C}_p\langle T_1, \cdots, T_n\rangle$ is noeth

(so $\mathbb{C}_p$ strongly noetherian).

Example $X = \mathrm{Spa}(\mathbb{Z}_p[\![T]\!], \mathbb{Z}_p[\![T]\!])$

$\mathbb{Z}_p[\![T]\!]$ with (p,T)-adic topology (nice by (2)).

The topological space

There is a unique point $s \in X$ s.t. $\ker(s)$ is open,

given by $|\cdot|_s : \mathbb{Z}_p[\![T]\!] \twoheadrightarrow \mathbb{F}_p \to \{0,1\}$.

Let $Y = X \setminus \{s\}$: All points have non-open kernel,

i.e. are "analytic".

Definition A point $x \in \mathrm{Spa}(A, A^+)$ is non-analytic

if $\ker(|\cdot(x)|)$ is open

(i.e. it comes from A/I which is discrete)

Otherwise, call it analytic.

Note If x analytic, $I \subseteq A_0$ as usual, then

$\exists f \in I$ s.t. $|f(x)| \neq 0$.

Let $\gamma = |f(x)| \in \Gamma = \Gamma_x$. Then for all $\gamma' \in \Gamma$,

$\exists\, n \gg 0$ s.t. $\gamma^n < \gamma'$.

Indeed, $\{g \in A \mid |g(x)| < \gamma'\} \subseteq A$ open,

so contains f^n for $n \gg 0$, then $\gamma^n = |f^n(x)| < \gamma'$.

<u>Prop</u> Let Γ totally ordered abelian group,

$\gamma < 1$ s.t. $\forall \gamma' \in \Gamma$, $\exists\, n \gg 0$ s.t. $\gamma^n < \gamma'$.

Then $\exists$ unique order-preserving map $\Gamma \to \mathbb{R}_{>0}$ s.t. $\gamma \mapsto 1/2$.

(kernel = elements "infinitesimally closed to 1".)

Thus, any analytic point x gives rise to $\tilde{x} : A \longrightarrow \mathbb{R}_{\geq 0}$

(equiv class of $\tilde{x}$ well-def'd).

Then $\tilde{x}$ is maximal generalization of x.

(<u>Ex</u> If x value group $\mathbb{R}_{>0} \times \delta^{\mathbb{Z}}$

where $r < \delta < 1$ for all $r \in \mathbb{R}$, $r < 1$

then $\tilde{x}$: projection to $\mathbb{R}_{>0}$.)

<u>Proof</u> If $U = \{|f(y)| \leq |g(y)| \neq 0\}$ contains x,

then it also contains $\tilde{x}$ $\Rightarrow$ $\tilde{x}$ is a gen. of x.

$\tilde{x}$ maximal: exercise. $\square$

In particular, if x analytic point,

$K(x)$ = completion of $\mathrm{Frac}(A/\ker(|\cdot(x)|))$ w.r.t. $|\cdot(x)|$,

is a nonarch field.

Definition A nonarch field is a non-discrete complete top field K whose top is induced by a nonarch norm $|\cdot|: K \to \mathbb{R}_{\geq 0}$.

(At non-analytic points x, $K(x)$ is discrete.)

Back to $\mathcal{Y} = X \setminus \{s\}$ $\quad$ all points are analytic.

$$= \mathrm{Spa}(\mathbb{Z}_p[\![T]\!], \mathbb{Z}_p[\![T]\!])$$

Prop There is a unique, continuous surjective map

$$\kappa : \mathcal{Y} \longrightarrow [0, \infty]$$

characterized by the following property:

$$\kappa(x) = r \quad \text{iff} \quad \text{for all } \frac{m}{n} > r, \ |p(x)|^n \geq |T(x)|^m$$
$$\& \ \text{for all } \frac{m}{n} < r, \ |p(x)|^n \leq |T(x)|^m.$$

__Sketch__ well-defined: Any $x \in \mathcal{Y}$ analytic, defines

$$|\cdot(\tilde{x})| : \mathbb{Z}_p[\![T]\!] \longrightarrow \mathbb{R}_{\geq 0}$$

with either $|p(\tilde{x})| \neq 0$ or $|T(\tilde{x})| \neq 0$.

If $|p(\tilde{x})| \neq 0$ then $\kappa(x) = (\log_{|p(\tilde{x})|} |T(\tilde{x})|)^{-1} \in [0, \infty)$.

If $|T(\tilde{x})| \neq 0$ then $\kappa(x) = \log_{|T(\tilde{x})|} |p(\tilde{x})| \in (0, \infty]$.

$(\kappa(x) = 0 \iff |T(\tilde{x})| = 0 \iff |T(x)| = 0$

$\iff x$ factors through $\mathbb{Z}_p[\![T]\!] \to \mathbb{Z}_p \xrightarrow{|\cdot|_p} \mathbb{R}_{\geq 0})$

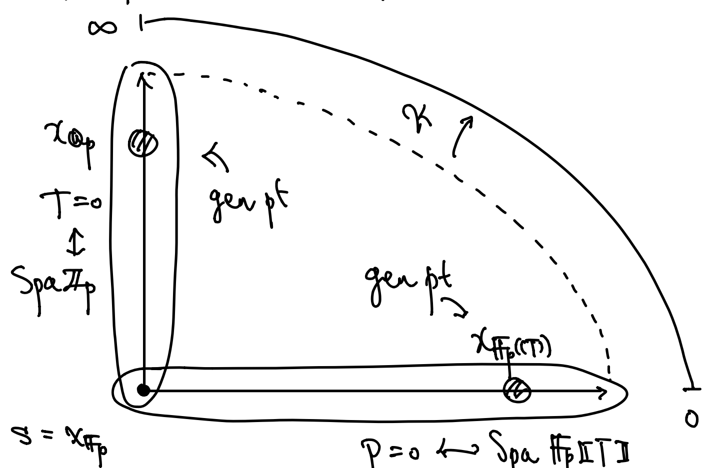
(Similarly for $\kappa(x) = \infty$.)

Continuous: preimage of $(\frac{m}{n}, \infty]$:

$\{x \in \mathcal{Y} \mid |T^m(x)| \leq |p^n(x)|\} = \{x \in \mathcal{Y} \mid |T^m(x)| \leq |p^n(x)| \neq 0\}$ is open.

Similarly for $[0, \frac{m}{n})$. $\square$

A picture of $\mathrm{Spa}(A, A)$, $A = \mathbb{Z}_p[\![T]\!]$:



For $I \subseteq [0,\infty]$ let $\mathcal{Y}_I = \kappa^{-1}(I)$.

Thus, $\mathcal{Y}_{[0,\infty)}$ is the locus $\{p \neq 0\}$.
"generic fibre (over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ of $\mathrm{Spa}(\mathbb{Z}_p[\![T]\!], \mathbb{Z}_p[\![T]\!])$).

<u>Note</u> $\mathcal{Y}_{[0,\infty)}$ is not affine, not even qc.
(Any qc open subset contained in $\mathcal{Y}_{[0,r]}$ for a compact interval $[0,r]$.)
In particular, $\mathcal{Y}_{[0,\infty)}$ is not affinoid.

Might think $\mathcal{Y}_{[0,\infty)} =$ "$\mathrm{Spa}(\underbrace{\mathbb{Z}_p[\![T]\!][\tfrac{1}{p}]}_{\text{but not Huber}}, \mathbb{Z}_p[\![T]\!])$"

$\mathbb{Z}_p[\![T]\!][\frac{1}{p}]$ with direct limit topology of $\underbrace{p^{-n}\mathbb{Z}_p[\![T]\!]}_{(p,T)\text{-adic}}$

$\mathbb{Z}_p[\![T]\!] \subseteq \mathbb{Z}_p[\![T]\!][\frac{1}{p}]$ is <u>not</u> open (as $p^{-1}T^n \xrightarrow{n\to\infty} 0$)

Also, $\mathbb{Z}_p[\![T]\!][\frac{1}{p}]$ would have to be Tate
(as p top-nilp unit)

Thus, any ring of def'n has an ideal of def'n (p).

To get affinoid subsets of $\mathcal{Y}_{[0,\infty)}$,

need to impose some condition $|T^n| \leq |p|$.

$(+\ |p| \leq |p| \Rightarrow (p, T^n)$ open ideal$)$

$\Longleftrightarrow$ exhausting $\mathcal{Y}_{[0,\infty)}$ by all $\mathcal{Y}_{[0,n]} = \mathrm{Spa}(\mathbb{Q}_p\langle T, \frac{T^n}{p}\rangle, \mathbb{Z}_p\langle T, \frac{T^n}{p}\rangle)$.

Lecture 4: General adic spaces

So far Huber pairs (A, A^+)

$\rightsquigarrow$ $X = \mathrm{Spa}(A, A^+)$ equipped with presheaves $\mathcal{O}_X$, $\mathcal{O}_X^+ \subseteq \mathcal{O}_X$.

often sheaves of top rings.

Moreover, for each $x \in X$, have equiv class of cont val $|\cdot(x)|$ on $\mathcal{O}_{X,x}$.

Remark There are examples (Rost) where $\mathcal{O}_X$ not a sheaf.

Definition A Huber pair (A, A^+) is sheafy if

$\mathcal{O}_X$ is a sheaf of top rings ($\Rightarrow$ $\mathcal{O}_X^+$ sheaf).

Definition (1) Category (V) of loc top ringed top spaces equipped with valuations

objects $(X, \mathcal{O}_X, (|\cdot(x)|$ equiv class of cont val on $\mathcal{O}_{X,x}$, $x \in X))$

X top space, $\mathcal{O}_X$ sheaf of (complete) top rings

($\rightsquigarrow$ subsheaf $\mathcal{O}_X^+ : \mathcal{O}_X^+(\mathcal{U}) = \{f \in \mathcal{O}_X(\mathcal{U}) \mid \forall x \in \mathcal{U}, |f(x)| \leq 1\}$.)

morphisms maps of loc ringed top spaces, continuous on $\mathcal{O}_X$.

for all $x \in X$, one can choose vals s.t.

with $f: X \to Y$,
$$\begin{array}{ccc} \mathcal{O}_{Y,f(x)} & \longrightarrow & \mathcal{O}_{X,x} \\ \downarrow & \circlearrowleft & \downarrow \\ \Gamma_{f(x)} \cup \{0\} & \longrightarrow & \Gamma_x \cup \{0\} . \end{array}$$

(2) An honest adic space (= adic space in sense of Huber)

is an object $(X, \cdots)$ of (V) s.t.

$\exists$ Cover $X = \bigcup \mathcal{U}_i$ s.t.

$$(\mathcal{U}_i, \mathcal{O}_X|_{\mathcal{U}_i}, (|\cdot(x)| \text{ for } x \in \mathcal{U}_i)) = \mathrm{Spa}(A_i, A_i^+)$$

for <u>sheafy</u> Huber pair (A_i, A_i^+).

(3) $\mathrm{Spa}(A_i, A_i^+) + (\mathcal{O}_i, \mathcal{O}_i^+)$ called affinoid adic space.

Prop The functor

$$\{\text{complete sheafy Huber pairs}\}^{op} \longrightarrow \{\text{adic spectrum}\}$$
$$(A, A^+) \longmapsto \mathrm{Spa}(A, A^+)$$

is fully faithful.

<u>Sketch</u> Can recover $(A, A^+) = H^0(\mathrm{Spa}(A, A^+), (\mathcal{O}, \mathcal{O}^+))$

Map on H^0 determines map of top spaces.

ok, as valuation part of data.

What to do for non-sheafy rings?

(1) Not care about them

(2) Replace all completions by henselizations ($\rightsquigarrow \mathcal{O}_X$ sheaf)

(3) Use "functor of points" approach (Scholze-Weinstein).

<u>Abstract setup</u> C category endowed with notion of "C-étale" morphisms. (e.g. rat'l embeddings).

s.t. (0) isoms are C-étale.

(1) pullbacks of C-étale morphisms exist and are C-étale.

(2) Composites of C-étale morphs are C-étale

(3) If
$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "h"'] & & A' \\ & B \arrow[ur, "g"'] & \end{tikzcd}$$
, h, g C'-étale $\Rightarrow$ f C-étale

C endowed with structure of site s.t.

(0) isoms are coverings.

(1) Coverings of A are certain valuations of $\{A_i \to A\}$ s.t. all $A_i \to A$ are C-étale

(2) pullbacks of coverings are coverings

(3) composites of coverings are coverings.

<u>Examples</u> (A) $C = \{\text{comm rings}\}^{op}$

- C-étale morphisms are maps $(A \to A[f^{-1}])^{op}$, $f \in A$.
- Coverings = Zariski coverings.

(B) $C = \{\text{Comm rings}\}^{op}$

- C-étale = usual étale.
- coverings = étale coverings.

(C) $C = \{\text{complete Huber pairs}\}^{op}$

- C-étale morphisms are maps $((A, A^+) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U)))^{op}$ for rational subset $U \subseteq X = \mathrm{Spa}(A, A^+)$
- Coverings = collection of maps $\{(A, A^+) \to (\mathcal{O}_X(U_i), \mathcal{O}_X^+(U_i))\}^{op}$ for cover $X = \bigcup U_i$ by rat'l subsets $U_i \subseteq Y$.

(D) $C = (v)$,

- C-étale morphs are open embeddings
- Coverings = open covers.

Define C-spaces in the following way.

<u>Definition</u> (1) For $A \in C$, let $\mathrm{Space}(A)$ be the sheafification of
$$B \mapsto \mathrm{Hom}_C(B, A).$$
(In Ex: (A)(B)(D) this is already a sheaf.)

(2) A map $f_i : \mathcal{F} \to \mathcal{G}$ of sheaves on C is C-étale if for all $A \in C$, $\mathrm{Space}(A) \to \mathcal{G}$
(= sections $\mathcal{G}(A)$ by Yoneda lem)
$\mathcal{F} \times_{\mathcal{G}} \mathrm{Space}(A) \to \mathrm{Space}(A)$ can be written as the colimit (= direct limit) of $\mathrm{Space}(B) \to \mathrm{Space}(A)$ for diagram $\{B_i \to A : C\text{-étale}\}$.

(3) A sheaf $\mathcal{F}$ on C is called <u>C-space</u> if $\mathcal{F}$ can be written as the colim of diagram $\{\mathrm{Space}(A_i) \to \mathcal{F} : C\text{-étale}\}$.

<u>In example</u> (A) C-space = scheme.

(B) C-space = algebraic space.

(C) C-space =: adic space.

(D) (v)-space = (v).

<u>Lemma</u> If $B \to A$ is C-étale, then $\mathrm{Space}(B) \to \mathrm{Space}(A)$ is C-étale.

<u>Proof</u> Let $\mathrm{Space}(A') \to \mathrm{Space}(A)$ maps of sheaves
$= $ section $\mathrm{Space}(A)(A')$
This is given by a cover $\{A'_i \to A' : C\text{-étale}\}$
maps $f_i : A \to A'_i$

s.t. each $A'_{ij} = A'_i \times_{A'} A'_j$ has covering $\{A'_{ijk} \to A'_{ij}\}$ s.t.

$$\begin{tikzcd} & A'_i \arrow[dr, "f_i"] & \\ A'_{ijk} \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\circlearrowright"] & & A \\ & A'_j \arrow[ur, "f_j"'] & \end{tikzcd}$$

Then $\mathrm{Space}(A') = \text{equalizer}\left(\coprod \mathrm{Space}(A'_{ijk}) \to \coprod \mathrm{Space}(A'_i)\right)$
= colimit of diagram of things C-étale / A'

$\mathrm{Space}(B) \times_{\mathrm{Space}(A)} \mathrm{Space}(A')$
$= \text{equalizer}\left(\coprod \mathrm{Space}(B \times_A A'_{ijk}) \to \coprod \mathrm{Space}(B \times_A A'_i)\right)$
= colimit of diagram of things C-étale / A' "$\square$"

<u>Lemma</u> C, C' categories with étale structure as above.
Let $F : C \to C'$ functor s.t.
$F(C\text{-étale})$ is C'-étale
and $F(C\text{-cover})$ is a C'-cover.
Then one gets a canonical functor
$$\{C\text{-spaces}\} \xrightarrow{\tilde{F}} \{C'\text{-spaces}\}$$
s.t. $\tilde{F}(\mathrm{Space}(A)) = \mathrm{Space}(F(A))$.

<u>Proof</u> F is a morphism of sites and can pullback sheaves
$$F^* : \{\text{sheaves on } C\} \to \{\text{sheaves on } C'\}$$
$F^*(\mathcal{F})$ is the sheafification of
$$A' \longmapsto \varinjlim_{A' \to F(A)} \mathcal{F}(A) \qquad \text{for } A' \in C'.$$
<u>Check</u> (1) If $A \in C$, then $F^*(\mathrm{Space}(A)) = \mathrm{Space}\, F(A)$.
(2) If $f : \mathcal{F} \to \mathcal{G}$ C-étale,
then $F^*(f) : F^*(\mathcal{F}) \to F^*(\mathcal{G})$ is C'-étale.

(3) F^* maps C-spaces to C'-spaces. "$\square$"

<u>Example</u> Consider the functor

$$\{\text{Complete Huber pairs}\}^{op} \longrightarrow (V)$$
$$(A, A^+) \longmapsto \mathrm{Spa}(A, A^+) \text{ with sheafified } \mathcal{O}_X$$

This induces a functor

$$\{\text{adic spaces}\} \longrightarrow (V)$$
$$X \longmapsto (|X|, \cdots)$$

In particular, any adic space has an underlying top space.

Analytic adic spaces

<u>Definition</u> An adic space is analytic if all its points are analytic.

<u>Prop</u> There is a fully faithful functor
$$\{\text{honest adic spaces}\} \to \{\text{adic spaces}\}.$$

<u>Prop</u> An adic space is analytic iff it is the colimit of a diagram of spaces $\mathrm{Spa}(A, A^+)$ for A <u>Tate</u>.

<u>Proof</u> Let $X = \mathrm{Spa}(A, A^+)$, $x \in X$ analytic point.

<u>To prove</u> $\exists$ rat'l nbhd $U \ni x$ s.t. $\mathcal{O}_X(U)$ is Tate.

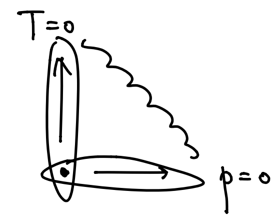
Proof: Let $I \subseteq A_0$ as usual. Take $f \in I$ s.t. $|f(x)| \neq 0$.

Then $\exists n$, $\underset{(g_1, \cdots, g_k)}{\overset{}{I^n}} \subseteq \{g \in A \mid |g(x)| < |f(x)|\}$ open

$\Rightarrow\ \mathcal{U} = \{ y \mid |g_i(y)| \le |f(y)| \ne 0 \}$ rat'l subset.

On $\mathcal{U}$, f is a unit and top-nilp (as $f \in I$). $\square$

<u>Example</u> $X = \mathrm{Spa}(\mathbb{Z}_p[\![T]\!], \mathbb{Z}_p[\![T]\!])$



- nbhd of $T=0$ & $p \ne 0$:

  $s \in X$ unique non-analytic point.

  $|\cdot(s)| : \mathbb{Z}_p[\![T]\!] \longrightarrow \mathbb{F}_p \longrightarrow \{0,1\}$

  $\mathcal{U} = \{ |T(x)| \le |p(x)| \ne 0 \}$

  $+\ |p(x)| \le |p(x)| \ne 0.$

  (p,T) open $\Rightarrow \mathcal{U}$ rational.

  $\mathcal{O}_X(\mathcal{U})$ = Completion of $\mathbb{Z}_p[\![T]\!][\frac{1}{p}]$

  w.r.t. $(p,T) = (p, p\cdot(\frac{T}{p})) = (p)$-adic top on $\mathbb{Z}_p[\![T]\!][\frac{T}{p}]$

  $= \mathbb{Q}_p\langle \frac{T}{p} \rangle$ Banach alg $/\mathbb{Q}_p$.

- nbhd of $T \ne 0$ & $p = 0$:

  pt: $\mathbb{Z}_p[\![T]\!] \longrightarrow \mathbb{F}_p((T)) \xrightarrow{|T(\cdot)|} \mathbb{R}_{\ge 0}$

  $V = \{ |p(x)| \le |T(x)| \ne 0 \}$ rat'l subset.

  $\mathcal{O}_X(V)$ = Completion of $\mathbb{Z}_p[\![T]\!][\frac{1}{T}]$

  w.r.t. $(p,T) = (\frac{p}{T}\cdot T, T) = (T)$-adic top on $\mathbb{Z}_p[\![T]\!][\frac{p}{T}]$

  $= \mathbb{Z}_p[T]\langle \frac{p}{T} \rangle$ Tate ring (T top-nilp).

  but not over a nonarch field.

Complete Tate rings are "as good as" Banach alg / nonarch fields.

<u>Prop</u> (Huber, "Cont val", Lemma 2.4 (i))

If A complete Tate ring, M, N are complete Banach A-mods,

Then any cont surj map $f: M \to N$ is <u>open</u>.

(Banach's open mapping thm).

Lecture 5: Complements on adic spaces

Next plan Perfectoid spaces, diamonds, shtukas (finally).

Preview $C = $ (perf spaces of char p)

C-étale = pro-étale

$\rightsquigarrow$ C-spaces = diamonds.

Correction to last time

Rational subsets are not stable under pullback.

e.g. $\mathrm{Spa}(\mathbb{Z}_p[\![T]\!], \mathbb{Z}_p[\![T]\!]) \to \mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p) = \{s, \eta\} \supset \{\eta\}$ rational.

preimage $= \mathcal{Y}_{[0,\infty)}$ not even quasicompact.

Can be repaired, c.f. lecture notes.

For the lectures, we only need honest adic spaces!

Definition A morphism $f: A \to B$ of Huber rings is adic if for (one, hence any) choice of rings of def'n $A_0 \to B_0$, $I \subseteq A_0$ ideal of def'n, $I \cdot B_0 \subseteq B_0$ is an ideal of def'n.

Remark If A is Tate, any $f: A \to B$ is adic.

Prop (i) If $(A, A^+) \to (B, B^+)$ is adic (i.e. $A \to B$ is),

then pullback along $\mathrm{Spa}(B,B^+) \to \mathrm{Spa}(A,A^+)$
preserves rat'l subsets.

(2) If $(A,A^+) \to (B,B^+)$, $(A,A^+) \to (C,C^+)$ are adic,

then $D = B \otimes_A C$ with ring topology
making image D_0 of $B_0 \otimes_{A_0} C_0$ in D open
with ideal of def'n $I \cdot D_0$
(where $I \subseteq A_0$ ideal of def'n).

This D is a Huber ring, (D, D^+) is the pushout of diagram,
where D^+ = integral closure of the image of $B^+ \otimes_{A^+} C^+$.
(If working with complete Huber pair, it is complete.)

Prop Let (A,A^+) Huber pair. Then any rat'l subset $U \subseteq \mathrm{Spa}(A,A^+)$
is of form $U = \{x \mid |f_i(x)| \le |g(x)| \neq 0\}$
for $f_1, \dots, f_n, g \in A$ s.t. $(f_1, \dots, f_n)A \subseteq A$ is open.

Proof Any such U rational.
Conversely, any rational U is finite intersection of such.
But $\{x \mid |f_i(x)| \le |g(x)| \neq 0\} \cap \{x \mid |f_j'(x)| \le |g'(x)| \neq 0\}$
$= \{x \mid |f_i f_j'(x)|, |f_i g'(x)|, |f_j' g(x)| \le |g g'(x)| \neq 0\}$
The $f_i f_j$'s generate open ideal. $\square$

For Huber ring A,
$\mathrm{Cont}(A) := \{$equiv. classes of cont val'ns on $A\}$.
top gen by $\{|f(x)| \le |g(x)| \neq 0\}$, $(f, g \in A)$.

Prop (1) $\mathrm{Cont}(A)$ is spectral space

(2) There is a 1-1 correspondence

$$\left\{\begin{array}{l}\text{intersections of subsets}\\ \text{of } \mathrm{Cont}(A) \text{ of given by}\\ \text{Condition } |f|\le 1,\ f\in \underset{\substack{\uparrow\\ \text{power-bounded}}}{A^\circ}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{open integrally closed}\\ \text{subrings } A^+\subset A^\circ\end{array}\right\}$$

(𝒰 might not be open) $\quad \mathcal{U} \longmapsto \{f \mid |f(x)|\le 1,\ \forall x\in\mathcal{U}\}$

$\{x \mid |f(x)|\le 1,\ \forall f\in A^+\} \longleftarrow A^+.$

So A^+ keeps track of precisely which inequalities have been enforced.

<u>note</u> $\mathrm{Spa}(K,K^+)$, K^+ open bdd valuation subring $\overset{\ne}{K^\circ} = \mathcal{O}_K$

$\uparrow$ points of an adic space.

<u>Examples</u> (1) Final objects $\mathrm{Spa}(\mathbb{Z},\mathbb{Z})$.

(2) $\mathrm{Spa}(\mathbb{Z}[T],\mathbb{Z}[T])$ represents $X\mapsto \mathcal{O}_X^+(X)$:

(discrete top)

$$\mathrm{Hom}(\mathrm{Spa}(R,R^+),\ \mathrm{Spa}(\mathbb{Z}[T],\mathbb{Z}[T]))$$
$$= \mathrm{Hom}(\mathbb{Z}[T],R^+) = R^+.$$

(3) $\mathrm{Spa}(\mathbb{Z}[T],\mathbb{Z})$ represents $X\longmapsto \mathcal{O}_X(X)$: same completion.

If K any nonarch field. (in honest adic spaces)

(1) (closed unit disc / K)

$$\mathrm{Spa}(\mathbb{Z}[T],\mathbb{Z}[T])\times \mathrm{Spa}(K,\mathcal{O}_K) = \mathrm{Spa}(K\langle T\rangle,\mathcal{O}_K\langle T\rangle)$$

b/c $\quad \mathrm{LHS} = \big(\underbrace{\mathrm{Spa}(\mathbb{Z}[T], \mathbb{Z}[T]) \times \mathrm{Spa}(\mathcal{O}_K, \mathcal{O}_K)}_{= \mathrm{Spa}(\mathcal{O}_K\langle T\rangle, \mathcal{O}_K\langle T\rangle)}\big) \times_{\mathrm{Spa}(\mathcal{O}_K, \mathcal{O}_K)} \mathrm{Spa}(K, \mathcal{O}_K)$

& $\mathrm{Spa}(\hat{A}, \hat{A}^+) \xrightarrow{\sim} \mathrm{Spa}(A, A^+)$.

(2) (Adic affine line).

$$\mathrm{Spa}(\mathbb{Z}[T], \mathbb{Z}) \times \mathrm{Spa}(K, \mathcal{O}_K) = \bigcup_{n \geq 1} \mathrm{Spa}(K\langle \varpi^n T\rangle, \mathcal{O}_K\langle \varpi^n T\rangle).$$

(Check universal property!) = increasing union of closed discs (as $|T| \leq |\varpi|^{-n}$).

Here $\varpi \in K$ pseudo uniformizer (= top-nilp unit).

(3) (Fibre products do not exist)

$\mathrm{Spa}(\mathbb{Z}[T], \mathbb{Z}) \times \mathrm{Spa}(\mathcal{O}_K, \mathcal{O}_K)$ does not exists.

$$\mathrm{Spa}(\mathbb{Z}[T_1, T_2, \cdots], \mathbb{Z}) \times \mathrm{Spa}(K, \mathcal{O}_K)$$

$$= \varinjlim_{(n_i) \to \infty} \mathrm{Spa}(K\langle \varpi^{n_i} T_1, \cdots\rangle, \mathcal{O}_K\langle \varpi^{n_i} T_1, \cdots\rangle)$$

transition maps not open

given by ∞-many inequalities $|T_i| \leq |\varpi|^{-n_i}$!

Nonetheless, fibre products of <u>analytic</u> adic spaces exist.

(But do not necessarily stay honest.)

(4) (Universal formal open disc)

Let $\mathbb{D} = \mathrm{Spa}(\mathbb{Z}[\![T]\!], \mathbb{Z}[\![T]\!])$ "formal disc"

$$\mathbb{D} \times \mathrm{Spa}(K, \mathcal{O}_K) = \big(\underbrace{\mathbb{D} \times \mathrm{Spa}(\mathcal{O}_K, \mathcal{O}_K)}_{\mathrm{Spa}(\mathcal{O}_K[\![T]\!], \mathcal{O}_K[\![T]\!])}\big) \times_{\mathrm{Spa}(\mathcal{O}_K, \mathcal{O}_K)} \mathrm{Spa}(K, \mathcal{O}_K)$$

not clear that this is honest!

$$= \{\varpi \neq 0\} \subseteq \mathrm{Spa}(\mathcal{O}_K[\![T]\!], \mathcal{O}_K[\![T]\!])$$

open unit disc $= \bigcup_{n \geq 1} \mathrm{Spa}(K\langle T, \frac{T^n}{\varpi}\rangle, \mathcal{O}_K\langle T, \frac{T^n}{\varpi}\rangle) =: \mathbb{D}_K$.

(K-points $= \{x \in K \mid |x| < 1\}$.)

Let $\mathbb{D}^* = \mathrm{Spa}(\mathbb{Z}((T)), \mathbb{Z}[[T]])$ "formal punctured disc".

$\mathbb{Z}((T)) = \mathbb{Z}[[T]][\frac{1}{T}]$.

$\mathbb{D}^* \times \mathrm{Spa}(K, \mathcal{O}_K) = \mathbb{D}_K \setminus \{T=0\}$ punctured unit disc $/K$.

Look up Example of $\mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$, "Perfectoid spaces, 2.20".

<u>Analytic adic spaces</u>

(results of Buzzard-Verberkmoes, Mihara, Kedlaya-Liu.)

Let (A, A^+) Tate Huber pair, $X = \mathrm{Spa}(A, A^+)$.

<u>Question</u> When is $\mathcal{O}_X$ sheaf?

<u>Recall</u> <u>Definition</u> A is <u>uniform</u> if $A^\circ \subseteq A$ bounded.

<u>Remark</u> A uniform $\Rightarrow$ A reduced

If $x \in A$ nilpotent, then $\varpi^{-n} x \in A^\circ$ for all n.

<u>Thm</u> (Berkovich) For A uniform, $A \longrightarrow \prod_{x \in \mathrm{Spa}(A,A^+)} K(x)$

($K(x)$: complete res field)

is strictly injective.

($\Rightarrow$ This is a homeomorphism onto its image.)

$A^\circ = \{f \in A \mid f \in K(x)^\circ, \forall x \in X\}$.

also follows from $A^+ = \{f \in A \mid |f(x)| \leq 1, \forall x \in X\}$.

<u>Cor</u> $A \to H^0(X, \mathcal{O}_X^{sh})$ is injective

(b/c $H^0 \to \prod_x k(x)$.)

<u>Definition</u> (A, A^+) is <u>stably uniform</u> if $\mathcal{O}_X(\mathcal{U})$ uniform $\forall\, \mathcal{U} \subseteq \mathrm{Spa}(A, A^+)$ rational.

<u>Thm</u> (Buzzard-Verberkmoes, Mihara)
If (A, A^+) stably uniform, then (A, A^+) sheafy.

<u>Thm</u> (Kedlaya-Liu) If (A, A^+) sheafy, then $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.

<u>Strategy of proof</u> Using combinatorial arguments (going back to Tate), can reduce to checking anything for a simple Laurent covering

$$X = \underbrace{\{|f| \le 1\}}_{= \mathcal{U}} \cup \underbrace{\{|f| \ge 1\}}_{= V}.$$

In this process, may need to replace X with rat'l subset.

$f \in A$, $\mathcal{O}_X(\mathcal{U}) = A\langle T\rangle / \overline{(T-f)}$

$\mathcal{O}_X(V) = A\langle S\rangle / \overline{(Sf-1)}$

$\mathcal{O}_X(\mathcal{U} \cap V) = A\langle T, T^{-1}\rangle / \overline{(T-f)}$.

Reduce to:

<u>Lemma</u> (1) $A\langle T\rangle/\overline{(T-f)} \oplus A\langle S\rangle / \overline{\langle Sf-1\rangle} \longrightarrow A\langle T, T^{-1}\rangle / \overline{(T-f)}$

$$S \longmapsto T^{-1}$$

is surjective.

(2) If A uniform, then ker in (1) equals A.

<u>pf of lem</u> (1) clear

(2) hard part: ideals closed.

Berkovich $\Rightarrow$ norm on $A = \sup_{x \in \mathrm{Spa}(A,A^+)} |\cdot|_{K(x)}$

norm on $A\langle T\rangle = \sup_{x \in \mathrm{Spa}(A,A^+)} |\cdot|_{K(x)\langle T\rangle}$ ← "Gauß norm" multiplicative

$\left(|\sum a_n T^n|_{A\langle T\rangle} = \sup_n |a_n|_A \right)$

<u>Claim</u> For all $g \in A\langle T\rangle$,

$$|(T\cdot f)g|_{A\langle T\rangle} \geq |g|_{A\langle T\rangle}.$$

$$\begin{aligned}|(T\cdot f)g|_{A\langle T\rangle} &= \sup_x |(T-f)g|_{K(x)\langle T\rangle} \\ &= \sup_x |T-f|_{K(x)\langle T\rangle} \cdot |g|_{K(x)\langle T\rangle} \\ &\geq \sup_x |g|_{K(x)\langle T\rangle} = |g|_{A\langle T\rangle}.\end{aligned}$$

$\square$

<u>Examples</u> (B-V, M) In general, $\mathcal{O}_X$ not a sheaf.

Even: uniform $\not\Rightarrow$ sheafy.

In particular, uniform $\not\Rightarrow$ stably uniform.

<u>Thm</u> (Kedlaya-Liu) Let (A, A^+) sheafy Tate-Huber pair.

$X = \mathrm{Spa}(A, A^+) = \bigcup \mathcal{U}_i$,

$\mathrm{Spa}(A_i, A_i^+) = \mathcal{U}_i$ rational, $\mathcal{U}_i \cap \mathcal{U}_j = \mathcal{U}_{ij}$.

Then

$$\{\text{fin proj } A\text{-mod } M\} \xrightarrow{\sim} \left\{ (M_i, \beta_{ij}) \;\middle|\; \begin{array}{l} M_i \text{ fin proj } A\text{-mod}, \\ \beta_{ij} : M_j \otimes_{A_i} A_{ij} \to M_i \otimes_{A_j} A_{ij} \\ \text{s.t. } \beta_{ij} \circ \beta_{jk} = \beta_{ik} \text{ over } A_{ijk} \end{array} \right\}$$

<u>Cor</u> On honest analytic adic space X,

have category $VB(X)$ of vector bundles on X

(consisting of certain sheaves of $\mathcal{O}_X$-mods)

s.t. $VB(\mathrm{Spa}(\underbrace{A, A^+}_{\text{Tate sheafy}})) \cong \{\text{fin proj } A\text{-mod}\}$

+ one can glue.

~~Strategy of proof~~ Reduce to simple Laurent covering,

+ imitate proof of Beauville – Laszlo.

<u>Lemma</u> (B-L) R ring, $f \in R$ nonzero-divisor,

$\hat{R} = f$-adic completion of R

$R \to \hat{R},\ R[f^{-1}]$.

Then $\{\text{flat } R\text{-mod } M \text{ s.t. } f \text{ not zero-divisor in } M\}$

$$\downarrow \cong$$

$$\left\{ (M_{\hat{R}}, M_{R[f^{-1}]}, \beta) \;\middle|\; \begin{array}{l} M_{\hat{R}}\ \hat{R}\text{-mod, s.t. } f \text{ not zero-divisor} \\ M_{R[f^{-1}]}\ R[f^{-1}]\text{-module,} \\ \beta : M_{\hat{R}}[f^{-1}] = M_{R[f^{-1}]} \otimes_R \hat{R} \ \text{ isom} \end{array} \right\}.$$

Does not follow from fpqc descent!

($R \to \hat{R}$ not flat)

(and as no descent datum on $\hat{R} \otimes_R \hat{R}$).

Lecture 6: Perfectoid rings

Correction *Exercise* $\mathrm{Spa}(\mathbb{Z}[T], \mathbb{Z}) \times \mathrm{Spa}(\mathcal{O}_K, \mathcal{O}_K)$ exists.
If A is fin gen over ring of def'n $A_0 \subseteq A^+$, fibre products should exist.

Fix p prime.

Definition (Fontaine) A complete Tate ring R is *perfectoid* if R uniform (i.e. $R^\circ \subset R$ bounded) and $\exists\ \varpi \in R^\circ$ pseudo-uniformizer (i.e. top-nilp unit) s.t. $\varpi^p \mid p$ in R° & s.t.

$$\Phi : R^\circ/\varpi \xrightarrow{\sim} R^\circ/\varpi^p$$
$$x \longmapsto x^p.$$

Question Is there a more general def'n of perfectoid Huber rings? (cf. Gabber-Ramero).

Example $\mathbb{Q}_p^{cycl} = \mathbb{Q}_p(\mu_{p^\infty})^\wedge$, $\mathbb{C}_p$, $(\mathbb{F}_p((t))(p^{1/p^\infty}))^\wedge$, $\mathbb{Q}_p^{cycl}\langle T^{1/p^\infty}\rangle$.
(need to be complete)

$\mathbb{Z}_p^{cycl}\langle (\frac{T}{p})^{1/p^\infty}\rangle[\frac{1}{T}]$: take $\varpi = T^{1/p}$, $\varpi^p = T = p \cdot \frac{T}{p}$ (dividing p).

$\mathbb{Z}_p^{cycl}[\![T^{1/p^\infty}]\!]$ should be perfectoid as Huber ring.
$\quad = (p,T)$-adic completion of $\mathbb{Z}_p^{cycl}[T^{1/p^\infty}]$.

Prop $\{\text{perfectoid rings } R \text{ s.t. } pR=0\}$

$$= \{\text{perfect uniform complete Tate rings}\}$$

$\uparrow\ \Phi : R \xrightarrow{\sim} R$ (can take $(-)^{1/p}$).

Proof Let R uniform complete Tate.

If R perfect, then take ϖ any pseudo uniformizer,

$\Rightarrow \varpi^p \mid 0 = p$

See: ($x \in R$ power-bounded $\Leftrightarrow x^p \in R$ is)

$\Rightarrow \Phi : R^\circ \xrightarrow{\sim} R^\circ$

$\Rightarrow \Phi : R^\circ/\varpi \xrightarrow{\sim} R^\circ/\varpi^p$.

Conversely, if R perfectoid.

$$\Phi : R^\circ/\varpi \xrightarrow{\sim} R^\circ/\varpi^p$$

By induction on n,

$$\Phi : R^\circ/\varpi^n \xrightarrow{\sim} R^\circ/\varpi^{np}$$

$\xrightarrow{\varprojlim_n}$ $\Phi : R^\circ \xrightarrow{\sim} R^\circ \overset{[\frac{1}{\varpi}]}{\Longrightarrow} R$ perfect. $\square$

Definition A perfectoid field is a perfectoid ring R which is a nonarch field.

Prop Let K nonarch field. Then

K perfectoid $\Leftrightarrow$ K not discretely valued, $|p|<1$, and $\Phi : \mathcal{O}_K/p \twoheadrightarrow \mathcal{O}_K/p$.

Remark Not clear that perfectoid ring + field $\overset{?}{\Rightarrow}$ perfectoid field.

Prop Let R complete uniform Tate ring.

Then R perfectoid $\iff \exists\, \varpi$ p.u. s.t. $\varpi^p \mid p$ and $\Phi : R^\circ/p \twoheadrightarrow R^\circ/p$.

Proof Take $\pi \in K$ p.u. s.t. $\varpi^p \mid p$, $\Phi : \mathcal{O}_K/\varpi \xrightarrow{\sim} \mathcal{O}_K/\varpi^p$.

(Lemma Let R perfectoid, $\varpi \in R^\circ$ any p.u. s.t. $\varpi^p \mid p$.
Then $R^\circ/\varpi \xrightarrow{\sim} R^\circ/\varpi^p$.)

If R perfectoid. $\Phi : R^\circ/\varpi \xrightarrow{\sim} R^\circ/\varpi^p$.

$\Rightarrow \quad \Phi : R^\circ/(\varpi^n, p) \twoheadrightarrow R^\circ/(\varpi^{np}, p)$

$\Rightarrow \quad \Phi : R^\circ/p \twoheadrightarrow R^\circ/p$ by taking $\varprojlim_n$.

Conversely, if $R^\circ/p \twoheadrightarrow R^\circ/p$ then $R^\circ/\varpi \twoheadrightarrow R^\circ/\varpi^p$.

Injectivity: If $x \in R^\circ$ s.t. $\left(\frac{x}{\varpi}\right)^p \in R^\circ$, then $\frac{x}{\varpi} \in R^\circ$. $\square$

Theorem (Scholze, Kedlaya-Liu)

If (R, R^+) Huber pair s.t. R perfectoid,
then (R, R^+) sheafy.

Question Does this extend to more general perfectoid Huber rings?

enough to prove If $\mathcal{U} \subseteq X = \mathrm{Spa}(R, R^+)$ rational.
then $\mathcal{O}_X(\mathcal{U})$ perfectoid ($\Rightarrow$ uniform).
(Thus (R, R^+) stably uniform $\Rightarrow$ sheafy.)

Unfortunately, proof convoluted.
It makes essential use of tilting.

Definition Let R perfectoid Tate ring.

The tilt of R is (constr'd by Fontaine)

$$R^\flat = \varprojlim_{x \mapsto x^p} R \quad \text{(as topological multiplicative monoids).}$$

with addition

$$(x^{(0)}, x^{(1)}, \dots) + (y^{(0)}, y^{(1)}, \dots) = (z^{(0)}, z^{(1)}, \dots)$$

where $z^{(i)} = \lim_{n \to \infty} (x^{(i+n)} + y^{(i+n)})^{p^n} \in R$.

<u>Lemma</u> This defines a perfect uniform complete Tate ring $R^\flat$ with power-bounded elements

$$\begin{tikzcd} R^{\flat,\circ} = \varprojlim_{x \mapsto x^p} R^\circ \arrow[r, "\sim"] \arrow[r, bend right, "\sim"', "\text{as ring}"] & \varprojlim_{\Phi} R^\circ/\pi \arrow[r, "\sim"] & \varprojlim_{\Phi} R^\circ/\pi \end{tikzcd}$$

There exists a p.u. $\varpi \in R^\circ$, $\varpi^p \mid p$, which admits sequence of p-power roots ϖ^{1/p^n}

$\rightsquigarrow \varpi^\flat = (\varpi, \varpi^{1/p}, \dots) \in R^\flat$.

Then $R^\flat \simeq R^{\flat,\circ}[(\varpi^\flat)^{-1}]$.

<u>Proof</u> First, check

$$\varprojlim_{x \mapsto x^p} R^\circ \xrightarrow{\sim} \varprojlim_{\Phi} R^\circ/p \xrightarrow{\sim} \varprojlim_{\Phi} R^\circ/\varpi.$$

<u>Essential point</u>:

Any $(\bar{x}_0, \bar{x}_1, \dots) \in \varprojlim_{\Phi} R^\circ/\varpi$ lifts uniquely to $(x^{(0)}, x^{(1)}, \dots) \in \varprojlim_{x \mapsto x^p} R^\circ$.

where $x^{(i)} = \lim_{n \to \infty} x_{n+i}^{p^n}$,

where $x_j \in R^\circ$ any lift of $\bar{x}_j$.

<u>Use</u> If $x \equiv y \mod \varpi^n \Rightarrow x^p \equiv y^p \mod \varpi^{n+1}$.

$\rightsquigarrow$ Get well-defined ring $R^{\flat,\circ}$.

Construct ϖ with p-power roots

$$R^{\flat,\circ} = \varprojlim_{\Phi} R^\circ/\varpi_0^p \twoheadrightarrow R^\circ/\varpi_0^p$$
$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \varpi^\flat & \longmapsto & \varpi_0 \end{array}$$

Then $\varpi^\flat = (\varpi, \varpi^{1/p}, \dots) \in \varprojlim_{x \mapsto x^p} R^\circ$, with $\varpi \equiv \varpi_0 \mod \varpi_0^p$.

Then $R^\flat = R^{\flat,\circ}[(\varpi^\flat)^{-1}]$ is the desired ring. $\square$

In particular, get map

$$\begin{array}{ccccc} R^\flat & \longrightarrow & \varprojlim R & \longrightarrow & R \\ \rotatebox{90}{$\in$} & & & & \rotatebox{90}{$\in$} \\ f & & \longmapsto & & f^\# \end{array}$$
(project to last coordinate)

continuous & multiplicative.

Also, $R^{\flat,\circ}/\varpi^\flat \xrightarrow{\sim} R^\circ/\varpi$.

<u>Lemma</u> $\left\{ \begin{array}{c} R^+ \subseteq R^\circ \text{ rings of} \\ \text{integral elements} \end{array} \right\} \xleftrightarrow{1:1} \{ R^{\flat,+} \subseteq R^{\flat,\circ} \}$

with $R^{\flat,+} = \varprojlim_{x \mapsto x^p} R^+$.

Also, $R^{\flat,+}/\varpi^\flat \xrightarrow{\sim} R^+/\varpi$.

<u>Thm</u> (Kedlaya, Scholze)

Let (R, R^+) perf'd Tate Huber pair, with tilt $(R^\flat, R^{\flat,+})$.

Then $\mathrm{Spa}(R, R^+) \xrightarrow{\sim} \mathrm{Spa}(R^\flat, R^{\flat,+})$ homeomorphically

$x \longmapsto x^\flat$, $|f(x^\flat)| = |f^\#(x)|$.

preserving rational subsets.

~~Strategy for proving~~

$\mathcal{O}_X(\mathcal{U})$ perfectoid. if $\mathcal{U} \subseteq X = \mathrm{Spa}(R, R^+) = \mathrm{Spa}(R^\flat, R^{\flat,+})$ rational.

First, show $\mathcal{O}_{X^\flat}(\mathcal{U})$ perfect uniform. (easy).
Then deduce $\mathcal{O}_X(\mathcal{U})$ perfectoid,
with $\mathcal{O}_X(\mathcal{U})^\flat = \mathcal{O}_{X^\flat}(\mathcal{U})$.

<u>Thm</u> (Scholze) Let R perf'd ring, with tilt $R^\flat$.
Then $\{\text{perf'd } R\text{-alg}\} \xrightarrow{\cong} \{\text{perf'd } R^\flat\text{-alg}\}$
$S \longmapsto S^\flat$.

<u>Description of inverse functor</u>
More generally, given perf'd R of char p,
what are <u>all</u> untilts $R^\sharp$ of R ?

Let's work with pairs (R, R^+).
<u>Lemma</u> (Fontaine, Fargues-Fontaine, Kedlaya-Liu).
Let $R^\sharp$ be any untilt of R, $R^{\sharp,+} \subseteq R^\sharp$.
i.e. perfectoid ring $R^\sharp$ + isom $(R^\sharp)^\flat = R$.
(1) There is a canonical <u>surj</u> ring homo

$$\theta : W(R^+) \longrightarrow R^{\sharp,+}.$$

$$\sum_{n\geq 0} [r_n] p^n \longmapsto \sum_{n\geq 0} r_n^\sharp p^n.$$

(2) The kernel of θ is generated by a nonzero-divisor ξ
of the form $\xi = p + [\varpi]\alpha$,
$\varpi \in R^+$ some p.u., $\alpha \in W(R^+)$.

<u>Definition</u> An ideal $I \subseteq W(R^+)$ is <u>primitive of deg 1</u>

if I is generated by an element of form $\xi = p + [\varpi]\alpha$,

$\varpi \in R^+$ p.u., $\alpha \in W(R^+)$.

(necessarily nonzero-divisor).

<u>Proof of lemma</u> (1) $W(R^+)$ is universal p-adically complete ring A

+ conti multiplicative map $R^+ \longrightarrow A$ (Deninger).

$$\begin{pmatrix} R^+ \longrightarrow W(R^+) \\ r \longmapsto [r] \end{pmatrix}$$

<u>Surjectivity</u> $R^+ \twoheadrightarrow R^{\#,+}/p$

θ mod p surj $\Rightarrow$ θ surj.

(2) Fix $\varpi \in R^+$ p.u. s.t. $\varpi^\# \in R^{\#,+}$ satisfies $(\varpi^\#)^p | p$.

<u>Claim</u> $\exists f \in \varpi R^+$ s.t. $f^\# \equiv p \mod p\varpi^\# R^{\#+}$.

pf. Consider $\alpha = p/\varpi^\# \in R^{\#+}$, $\exists \beta \in R^+$ s.t. $\beta^\# \equiv \alpha \mod pR^{\#+}$.

$$(\varpi\beta)^\# \equiv \varpi^\#\alpha = p \mod p\varpi^\# R^{\#+}.$$

Take $f = \varpi\beta$.

Thus, $p = f^\# + p\varpi^\# \sum_{n\geq 0} r_n^\# p^n$, $r_n \in R^+$,

$\Rightarrow$ $\xi = p - [f] - [\varpi] \sum_{n\geq 0} [r_n] p^{n+1} \in \ker\theta$

of desired form.

[<u>Lemma</u> Any ξ of form $\xi = p + [\varpi]\alpha$ is a nonzero-divisor.

<u>Proof</u> Assume $\xi \cdot \sum_{n\geq 0} [c_n] p^n = 0$.

Mod $[\varpi]$: $\sum [c_n] p^{n+1} = 0 \mod \varpi$

$\Rightarrow$ all $c_n \equiv 0 \mod \varpi$

Divide by $[\varpi]$ + induct.]

It remains to prove ξ generates $\ker(\theta)$.

$$W(R^+)/(\xi) \xrightarrow{f} R^{\#+}.$$

enough to show f isom modulo $[\varpi]$
(b/c everything $[\varpi]$-torsion-free
& $[\varpi]$-adically complete.)

Now $\quad W(R^+)/(\xi, [\varpi]) = W(R^+)/(p, [\varpi])$
$$= R^+/[\varpi] = R^+/\varpi \xrightarrow{\sim} R^{\#+}/\varpi^{\#}. \qquad \square$$

<u>Cor</u> (Kedlaya-Liu, Fontaine)

There is an equiv of categories

$$\{\text{perfectoid Tate Huber pairs } (S, S^+)\}$$
$$\updownarrow \text{ 1-1}$$
$$\left\{ (R, R^+, J) \;\middle|\; \begin{array}{c} (R, R^+) \text{ perf Tate-Huber pair of char } p \\ J \subseteq W(R^+) \text{ premitive of deg 1} \end{array} \right\}$$

via

$$\begin{array}{ccc} (S, S^+) & & \big((W(R^+)[[\varpi]^{-1}]/J, \ W(R^+)/J\big) \\ \big\downarrow & & \big\uparrow \\ (S^\flat, S^{\flat,+}, \ker\theta) & & (R, R^+, J). \end{array}$$

Lecture 7: Perfectoid spaces

<u>Recall</u> <u>Definition</u> A perfectoid Tate ring R
= Complete, uniform, $\exists$ a pseudo-uniformizer ϖ
s.t. $\varpi^p | p$ in R° & s.t.
$$\Phi : R^\circ/\varpi \xrightarrow{\sim} R^\circ/\varpi^p.$$
Gabber-Ramero: define perfectoid Huber ring
(+ show they are sheafy)
e.g. $\mathbb{Z}_p^{cycl}[[T^{1/p^\infty}]]$.

<u>Tilt</u> (R, R^+) with R perfectoid,
$R^\flat := \varprojlim_{x \mapsto x^p} R$ perfectoid of char p
$\leadsto$ map $R^\flat \to R$, $f \mapsto f^\#$.
$R^{\flat +} := \varprojlim_{x \mapsto x^p} R^+$

<u>Theorem</u> (R, R^+) with R perfectoid is sheafy.
Let $X = \mathrm{Spa}(R, R^+)$, $X^\flat = \mathrm{Spa}(R^\flat, R^{\flat +})$
then have homeom
$$\begin{aligned} X &\xrightarrow{\sim} X^\flat \\ x &\longmapsto x^\flat \end{aligned}$$
where $|f(x^\flat)| = |f^\#(x)|$.
Moreover for U perfectoid $\subset X$,
$\mathcal{O}_X(U)$ is perfectoid with tilt $\mathcal{O}_{X^\flat}(U)$.

<u>Definition</u> A <u>perfectoid space</u> is an adic space

Covered by $\mathrm{Spa}(R, R^+)$ with R perf'd.

Tilting extends to spaces : $X \longmapsto X^\flat$.

<u>Remark</u> If $\mathrm{Spa}(R, R^+)$ is a perf'd space,
not clear that R perf'd.
But true if R of char p. (Buzzard - Verberkmoes).

<u>Thm</u> Tilting induces an equivalence

$$\left\{ \begin{matrix} (S, S^+), \\ S \text{ perf'd} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} (R, R^+, J), \\ (R, R^+) \text{ perf'd of char } p, \\ J \subseteq W(R^+) \text{ primitive ideal of deg } 1 \end{matrix} \right\}$$

* <u>Why perfectoid spaces?</u>

(1) Any adic space / $\mathbb{Q}_p$ is pro-étale locally perf'd (Colmez).
i.e. if A Banach $\mathbb{Q}_p$-alg, there exists a filtered system
of finite étale A-alg A_i s.t.

$$A_\infty = \widehat{\varinjlim A_i} \text{ is perfectoid.}$$

<u>Example</u> $X = \mathrm{Spa}(\mathbb{Q}_p\langle T^{\pm 1}\rangle, \mathbb{Z}_p\langle T^{\pm 1}\rangle)$ "unit circle"
has a pro-étale covering

$$\tilde{X} = \mathrm{Spa}(\mathbb{Q}_p^{cycl}\langle T^{\pm 1/p^\infty}\rangle, \mathbb{Z}_p^{cycl}\langle T^{\pm 1/p^\infty}\rangle)$$

$\overset{\shortparallel}{\text{inverse limit of}}$ $X_n = \mathrm{Spa}(\mathbb{Q}_p(\zeta_{p^n})\langle T^{\pm 1/p^n}\rangle, \mathbb{Z}_p(\zeta_{p^n})\langle T^{\pm 1/p^n}\rangle)$.

(2) If X perf'd space, all topological information
(e.g. $|X|$, even $X_{\acute{e}t}$).

can be recovered from tilt $X^\flat$.

But $X^\flat$ forgets structure morphism $X \to \mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$.

The following can be made precise (next lect.):

$$\{\text{perf'd spaces}/\mathbb{Q}_p\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{perf'd spaces } X \text{ of char } p \\ + \text{"structure morphism} \\ X \to \mathbb{Q}_p\text{"} \end{array} \right\}.$$

<u>This lecture</u> $X_{\acute{e}t} \cong X^\flat_{\acute{e}t}$.

<u>Case of perf'd fields</u>

<u>Thm 1</u> (Fontaine-Weintenberger, Kedlaya-Liu, Scholze)

Let K perf'd field with tilt $K^\flat$.

(1) If L/K finite, L perfectoid.

(2)
$$\begin{array}{ccc} \{\text{fin ext'ns of } K\} & \xrightarrow{\sim} & \{\text{fin ext'ns of } K^\flat\} \\ L & \longmapsto & L^\flat \end{array}$$

equivalence, preserving degrees.

(<u>Upshot</u> $\mathrm{Gal}_K \cong \mathrm{Gal}_{K^\flat}$.)

<u>Thm 2</u> (Tate, Gabber-Ramero)

Let K perf'd field, L/K finite.

Then $\mathcal{O}_L/\mathcal{O}_K$ almost finite étale.

<u>Example</u> $K = \mathbb{Q}_p(p^{1/p^\infty})^\wedge$, $L = K(p^{1/2})$ $(p \neq 2)$.

$K_n = \mathbb{Q}_p(p^{1/p^n})$, $L_n = K_n(p^{1/2}) = \mathbb{Q}_p(p^{1/2p^n})$.

$\rightsquigarrow$ $\mathcal{O}_{L_n} = \mathcal{O}_{K_n}[x]/(x^2 - p^{1/p^n})$. $f(x) = x^2 - p^{1/p^n}$.

Differential ideal $\delta_{L_n/K_n} = f'(p^{1/2p^n}) \cdot \mathcal{O}_{L_n} = (p^{1/2p^n})$.

$\rightsquigarrow$ $v_p(\delta_{L_n/K_n}) = 1/2p^n \to 0$ as $n \to \infty$.

So L_n/K_n getting less ramified.

<u>Key</u> Can almost get rid of ramification along the special fibre by passing to a tower with perf'd limit.

<u>Thm 2 $\Rightarrow$ Thm 1</u> (Choose $\varpi, \varpi^\flat$ s.t. $\mathcal{O}_K/\varpi \simeq \mathcal{O}_{K^\flat}/\varpi^\flat$.)

$$\begin{array}{ccc}
\{\text{fin étale } K\text{-algs}\} & \xrightarrow{\text{Thm 2}} & \{\text{almost fin étale } \mathcal{O}_K^a\text{-algs}\} \\
 & & \downarrow \simeq \ \text{(lift over nilpotents)} \\
\{\text{almost fin étale } (\mathcal{O}_{K^\flat}/\varpi^\flat)^a\text{-algs}\} & \xleftarrow{=} & \{\text{almost fin étale } (\mathcal{O}_K/\varpi)^a\text{-algs}\} \\
\downarrow \ \text{(lift over nilpotents)} & & \\
\{\text{almost fin étale } \mathcal{O}_{K^\flat}^a\text{-algs}\} & \xrightarrow{\text{Thm 2}} & \{\text{fin étale } K^\flat\text{-algs}\}.
\end{array}$$

<u>Almost mathematics d'après Faltings</u>

<u>Definition</u> An R°-module M is <u>almost zero</u> if $\varpi M = 0$ for all pseudo-uniformizers ϖ

($\Leftrightarrow$ $\varpi^{1/p^n} M = 0$ for all n, if ϖ p.u. with chosen p-power roots.)

(Similarly for R^+-modules.)

<u>Examples</u> (1) If K perf'd field, then $\mathcal{O}_K/\mathfrak{m}_K$ is almost zero.

(General almost-zero mod = direct sum of such.)

(2) If R perf'd, $R^+ \subseteq R^\circ$ any ring of integral elts.

then R°/R^+ is almost zero.

(b/c $\forall \varpi$ p.u., $\forall x \in R^\circ$, ϖx top nilp

$$\Rightarrow (\varpi x)^n \in R^+ \underset{\substack{\uparrow \\ \text{by integral closedness.}}}{\Rightarrow} \varpi x \in R^+ \Rightarrow \varpi R^\circ \subset R^+.)$$

Note Ext's of almost zero modules are almost zero.

Definition $R^{\circ a}$-mod := {almost R°-modules}

:= {R°-modules} / {almost zero modules}.

Same for R^{+a}-mod $\simeq R^{\circ a}$-mod.

Thm Let (R, R^+) pefectoid Tate Huber pair, $X = \mathrm{Spa}(R, R^+)$.

By Kedlaya-Liu $\Rightarrow H^i(X, \mathcal{O}_X) = 0$, for $i > 0$

$H^i(X, \mathcal{O}_X^+)$ almost zero mod for $i > 0$.

$(H^0(X, \mathcal{O}_X^+) = R^+.)$

Proof To show: for any finite rational covering $X = \bigcup U_i$,

all cohom grp of

$$(*) \qquad 0 \to R^+ \to \prod \mathcal{O}_X^+(U_i) \longrightarrow \prod \mathcal{O}_X^+(U_i \cap U_j) \to \cdots$$

are almost zero.

We know

$$(*)[\varpi^{-1}] \quad 0 \to R \to \prod \mathcal{O}_X(U_i) \to \prod \mathcal{O}_X(U_i \cap U_j) \to \cdots \text{ is exact.}$$

Apply Banach's open mapping theorem:

each cohom grp of $(*)$ is killed by a power of ϖ.

R perfect $\Rightarrow$ Frob acts as isom of $(*)$.

So Frob induces isom of all cohom grps.

If killed by ϖ^n, then by all ϖ^{n/p^k} $\Rightarrow$ almost zero. $\square$

<u>Almost purity Thm</u> (Faltings, Scholze)

Let R perfectoid with tilt $R^\flat$.

(1) For any finite étale R-alg S, S is perf'd.

(2) Tilting induces an equiv

$$\begin{array}{ccc} \{\text{fin ét } R\text{-alg}\} & \xrightarrow{\sim} & \{\text{fin ét } R^\flat\text{-alg}\} \\ S & \longmapsto & S^\flat \end{array}$$

(3) (Almost purity)

For any fin ét R-alg S, S° almost fin ét over R°.

i.e. S° almost self-dual under trace pairing.

Line of argument: Prove (2), deduce (1) + (3).

(by proving (1) + (3) in char p).

<u>Sketch of proof of (2)</u> Reduce to case of perf'd fields.

Let $x \in X = \mathrm{Spa}(R, R^+)$ $\rightsquigarrow$ $\underbrace{K(x),\ K(x^\flat)}$ (res field of $x^\flat \in X^\flat$)

(any R^+, say $R^+ = R^\circ$). perf'd fields, $K(x)^\flat = K(x^\flat)$.

<u>Lemma</u> $$\begin{array}{ccc} \text{2-}\varinjlim_{x \in U} \{\text{fin ét } \mathcal{O}_X(U)\text{-algs}\} & \xrightarrow{\sim} & \{\text{fin ét } K(x)\text{-algs}\} \\ & & \wr\vert \\ \text{2-}\varinjlim_{x \in U} \{\text{fin ét } \mathcal{O}_{X^\flat}(U)\text{-algs}\} & \xrightarrow{\sim} & \{\text{fin ét } K(x^\flat)\text{-algs}\} \end{array}$$

(admitting lemma to get the second row).

Thus, get equiv locally at any point. Then glue together.

Lecture 8: Diamonds (1/2)

Idea There should be a functor

$$\{\text{adic spaces} / \mathbb{Z}_p\} \longrightarrow \{\text{diamonds}\}$$

forgetting the structure morphism to $\mathbb{Q}_p$

(not fully faithful).

Obs For perfectoid spaces $/ \mathbb{Z}_p$, $X \longmapsto X^\flat$ has this property.

In general, X adic space $/ \mathbb{Q}_p$.

then X is pro-étale locally perfectoid:

$$X = \mathrm{Coeq}(\tilde{\tilde{X}} \rightrightarrows \tilde{X})$$

w/ $\tilde{X} \to X$ pro-étale perfectoid cover,

$\tilde{\tilde{X}} \to \tilde{X} \times_X \tilde{X}$ perfectoid.

The functor is $X \longmapsto \mathrm{Coeq}(\underbrace{\tilde{\tilde{X}}^\flat \rightrightarrows \tilde{X}^\flat}_{\text{proét \underline{under} perf'd space } \tilde{X}^\flat \text{ of char } p.})$

Q In which category?

Example $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \longmapsto \mathrm{Spa}((\mathbb{Q}_p^{cycl})^\flat, (\mathbb{Z}_p^{cycl,\flat})) / \mathbb{Z}_p^\times$

action of Gal grp.

Definition A morphism $f : X \to Y$ of perf'd spaces is <u>pro-étale</u> if locally (on X & Y) of form

$$\mathrm{Spa}(A_\infty, A_\infty^+) \longrightarrow \mathrm{Spa}(A, A^+)$$

where A, A_∞ perfectoid,

& $(A_\infty, A_\infty^+) = \left(\varinjlim (A_i, A_i^+)\right)^\wedge$ filtered colim of (A_i, A_i^+)

A_i perfectoid,

s.t. $\mathrm{Spa}(A_i, A_i^+) \to \mathrm{Spa}(A, A^+)$ étale.

<u>Lemma</u> Let

$$\begin{tikzcd} \mathrm{Spa}(A_\infty, A_\infty^+) \arrow[dr] & & \mathrm{Spa}(B_\infty, B_\infty^+) \arrow[dl] \\ & \mathrm{Spa}(A, A^+) & \end{tikzcd}$$

w/ $\mathrm{Spa}(A_\infty, A_\infty^+) = \left(\varinjlim_{i \in I} (A_i, A_i^+)\right)^\wedge$

$\mathrm{Spa}(B_\infty, B_\infty^+) = \left(\varinjlim_{j \in J} (B_j, B_j^+)\right)^\wedge$ as in Def'n.

Then

$$\mathrm{Hom}_{/\mathrm{Spa}(A,A^+)}(\mathrm{Spa}(A_\infty, A_\infty^+), \mathrm{Spa}(B_\infty, B_\infty^+))$$
$$= \varprojlim_J \varinjlim_I \mathrm{Hom}_{/\mathrm{Spa}(A,A^+)}(\mathrm{Spa}(A_i, A_i^+), \mathrm{Spa}(B_j, B_j^+)).$$

<u>Proof</u> Wlog $J = \{1\}$, with $(B, B^+) = (B_\infty, B_\infty^+)$.

To check:

$$\mathrm{Hom}(\mathrm{Spa}(A_\infty, A_\infty^+), \mathrm{Spa}(B, B^+))$$
$$= \varinjlim_I \mathrm{Hom}(\mathrm{Spa}(A_i, A_i^+), \mathrm{Spa}(B, B^+))$$

(can be checked locally on $\mathrm{Spa}(B, B^+)$.

Wlog $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ composition of rat'l embeddings + fin. étale morphs.

even either rat'l emb or fin étale.

(1) f rat'l embedding.

$U = \mathrm{Spa}(B, B^+) \hookrightarrow \mathrm{Spa}(A, A^+)$.

To show: $\mathrm{Spa}(A_\infty, A_\infty^+) \to \mathrm{Spa}(A, A^+)$ factors over U

$\Rightarrow \exists i : \mathrm{Spa}(A_i, A_i^+) \to \mathrm{Spa}(A, A^+)$ factors over $\mathcal{U}$

But topologically, know

$$\mathrm{Spa}(A_\infty, A_\infty^+) = \varprojlim \mathrm{Spa}(A_i, A_i^+)$$

$\Rightarrow$ $\mathrm{Spa}(A_\infty, A_\infty^+) \setminus \{\text{preimage of } \mathcal{U}\}$

$$= \varprojlim \big(\underbrace{\mathrm{Spa}(A_i, A_i^+) \setminus \{\text{preimage of } \mathcal{U}\}} \big)$$

(closed in a spectral space

$\Rightarrow$ spectral $\Rightarrow$ compact for constructible top

$\Rightarrow$ $\mathrm{Spa}(A_i, A_i^+) \setminus \{\text{preimage of } \mathcal{U}\} \neq \emptyset$.)

(2) f finite étale.

<u>Recall</u> $\{\text{fin ét } A_\infty\text{-alg}\} = 2\text{-}\varinjlim \{\text{fin ét } A_i\text{-alg}\}$.

So $$\mathrm{Hom}_A(B, A_\infty) = \mathrm{Hom}_{A_\infty}(B \otimes A_\infty, A_\infty)$$
$$= \varinjlim_i \mathrm{Hom}_{A_i}(B \otimes A_i, A_i)$$
$$= \varinjlim_i \mathrm{Hom}_A(B, A_i). \qquad \square$$

Prop (1) Compositions of pro-étale maps are pro-étale.

(2)
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$
g, h pro-étale $\Rightarrow$ f pro-étale.

(3) Pullbacks of pro-étale maps are pro-étale.

<u>Note</u> Category of affinoid perf'd spaces has all conn limits

$\Rightarrow$ fibre products of perf'd spaces exist.

<u>Definition</u> (big pro-étale site) Consider category:

Perf of perf'd spaces of char p.

Endow with str of site by saying that

$\{f_i : X_i \to X\}_{i\in I}$ is a pro-ét covering

if all f_i are pro-ét,

and for all qc open $U \subseteq X$,

$\exists$ finite set $I_U \subseteq I$, qc open $U_i \subseteq X$, $\forall i \in I_U$,

s.t. $U = \bigcup_{i\in I_U} f_i(U_i)$ (as for fpqc covers).

Prop For perf'd space X of char p,

$$h_X : Y \longmapsto \mathrm{Hom}(Y, X)$$

is a sheaf on pro-ét site.

(Namely, this site is subcanonical.)

Proof Essential point: $X \longmapsto \mathcal{O}_X(X)$ sheaf for pro-étale top.

Wlog $X = \mathrm{Spa}(R, R^+)$ perf'd affinoid.

Fix $\varpi \in R^+$ p.u.

Finite cover by $X_i = \mathrm{Spa}(R_{\infty,i}, R^+_{\infty,i})$,

$$(R^+_{\infty,i}, R^+_{\infty,i}) = \left(\varinjlim_{j\in J} (R_{ji}, R^+_{ji})\right)^\wedge.$$

Claim $0 \to R^+/\varpi \to \prod_i R^+_{\infty i}/\varpi \to \cdots$ is almost exact.

Pf For all j,

$$0 \to R^+/\varpi \to \prod_i R^+_{ji}/\varpi \to \quad \text{is almost exact}$$

as $H^i(X_{\text{ét}}, \mathcal{O}^+_X/\varpi) \overset{\text{almost}}{=} \begin{cases} R^+/\varpi, & i=0 \\ 0, & i>0. \end{cases}$

$\xrightarrow{\varinjlim_j}$ $0 \to R^+/\varpi \to \prod_i R^+_{\infty i}/\varpi \to \cdots$ is almost exact.

$\Rightarrow$ Claim.

In claim, all terms are ϖ-torsion-free + ϖ-adically complete.

$\overset{[\varpi^{-1}]}{\Rightarrow}$ essential pt.

Now wlog $X = \mathrm{Spa}(R, R^+)$ affinoid,

$Y = \mathrm{Spa}(S, S^+)$ affinoid, Cover $\{Y_i \to Y\}_{i \in I}$,

wlog I finite, Y_i qc.

wlog all $Y_i = \mathrm{Spa}(S_\infty, S_\infty^+) \to \mathrm{Spa}(S, S^+)$

as in def'n of pro-étale maps.

$\rightsquigarrow$ get map as desired:

$$R \longrightarrow H^0(Y_{\text{proét}}, \mathcal{O}_Y) = S. \qquad \square$$

Definition (1) A map $f: \mathcal{F} \to \mathcal{G}$ of sheaves on Perf

is pro-étale if for all perf'd spaces X,

maps $h_X \to \mathcal{G}$ ($\triangleq$ sections of $\mathcal{G}(X)$),

$h_X \times_{\mathcal{G}} \mathcal{F}$ repr'd by perf'd space Y,

and $Y \to X$ is pro-étale, ($\triangleq h_Y \to h_X$).

possibly after pro-étale localization on X.

(2) A diamond is a sheaf on Perf

s.t. $\exists$ pro-ét maps $h_{X_i} \to \mathcal{F}$

s.t. $\coprod_i h_{X_i} \twoheadrightarrow \mathcal{F}$ surjective.

Notation If (R, R^+) perf'd, write

$\mathrm{Spd}(R, R^+) := h_{\mathrm{Spa}(R,R^+)} \in \{\text{diamonds}\}$.

(will be extended to more general Tate-Huber pairs later).

Have fully faithful functor

$$X \longmapsto X^{\diamond} = h_X$$

from perf'd spaces of char p.

<u>Example</u> Def'n $\mathrm{Spd}\, \mathbb{Q}_p := \mathrm{Spd}(\mathbb{Q}_p, \mathbb{Z}_p)$

$$= \mathrm{Spd}(\mathbb{Q}_p^{cycl}, \mathbb{Z}_p^{cycl}) / \mathbb{Z}_p^{\times}.$$

$$= h_{\mathrm{Spa}(\mathbb{F}_p((t^{1/p^\infty})),\, \mathbb{F}_p[[t^{1/p^\infty}]])} / \mathbb{Z}_p^{\times}.$$

where $(\mathbb{Q}_p^{cycl})^{\flat} = \mathbb{F}_p((t^{1/p^\infty})) \circlearrowleft \mathbb{Z}_p^{\times}$ freely

Prop If $X = \mathrm{Spd}(R, R^+)$, R char p, then

$$(\mathrm{Spd}\, \mathbb{Q}_p)(X) = \{ (R \to \tilde{R}),\ t \in \tilde{R} \}$$

where
- $R \to \tilde{R}$ $\mathbb{Z}_p^{\times}$-torsor ($\tilde{R} = \widehat{\varinjlim} R_n$, R_n/R fin ét, w/ Gal grp $\mathbb{Z}/p^n\mathbb{Z}$)

  $+$ $\mathbb{Z}_p^{\times}$-equiv map $(\mathbb{Q}_p^{cycl})^{\flat} \to \tilde{R}$
- $t \in \tilde{R}$ top nilp unit

  s.t. $\forall \gamma \in \mathbb{Z}_p^{\times}$, $\gamma(t) = (1+t)^{\gamma} - 1$.

<u>Note</u> $(\mathrm{Spd}\, \mathbb{Q}_p)(X)$: $\exists$ Cover $\{ f_i : X_i \longrightarrow X \}$

$$X_i = \mathrm{Spa}(R_{\infty i}, R_{\infty i}^+), \quad X = \mathrm{Spa}(R, R^+)$$

as in Def'n of pro-étale map.

$+$ maps $(\mathbb{Q}_p^{cycl})^{\flat} \longrightarrow R_{\infty i}$

$\triangleq$ $t_i \in R_{\infty i}$ top nilp unit.

over $R_{\infty i} \hat{\otimes}_R R_{\infty j}$, $\exists\, \gamma_{ij} \in \mathbb{Z}_p^{\times}$

with $\gamma_{ij}(t_i) \otimes 1 = 1 \otimes t_j$

$+$ cocycle condition.

__Thm__ $\{\text{perf'd spaces } /\mathbb{Q}_p\} \xrightarrow{\sim} \left\{\begin{matrix}\text{perf'd spaces } X \text{ of char } p \\ + \text{ str morph } X^\diamond \to \operatorname{Spd} \mathbb{Q}_p\end{matrix}\right\}.$

(Will prove this next time.)

Due to the restriction on the maximal size of this file, I have to truncate the notes at this point.