2022 Spring – Selected Topic in Algebraic Topology

SOLUTION TO FINAL PROBLEMS

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Problem 1. Show that a retract of a contractible space is contractible.

Proof. A space X is said to be contractible if it is homotopy equivalent to a point. Let $A \subset X$ be a retract of X, so there is a homotopy

$$\begin{aligned} h : X \times I &\to X \\ (x, 0) &\mapsto x \\ (a, t) &\mapsto a \end{aligned}$$

for all $x \in X$, $a \in A$, and $t \in I$. Then for each $t \in I$ we get the induced map

$$\begin{aligned} h_t : X &\to X \\ x &\mapsto h(x, t). \end{aligned}$$

Since X is constructible, there is a deformation

$$\begin{aligned} H : X \times I &\to X \\ (x, 0) &\mapsto x \\ (x, 1) &\mapsto x_0 \end{aligned}$$

for the fixed point $x_0 \in X$. Note that

$$f = (h_1 \circ H)|_{A \times I}$$

is a homotopy of A such that $f(a, 0) = h_1(a) = a$ for all $a \in A$ and that $f(a, 1) = h_1(x_0)$ is a constant point of A. Thus A is constructible. □

Problem 2. If G is a simplicial group, considered as a fibrant simplicial set, show that any two choices of basepoint lead to naturally isomorphic $\pi_n(G)$.

Proof. First note that by definition, the homotopic equivalence relation can be rewritten as follows. Any two n-simplices, say g and $g' \in G_n$, are homotopic if

(i) $g, g' \in Z_n$, i.e., $d_i(g) = d_i(g')$ for $0 \leqslant i \leqslant n$;
(ii) there is a simplex $h \in G_{n+1}$ such that $d_n(h) = g$, $d_{n+1}(h) = g'$, and $d_i(h) = s_{n-1}d_i(g) = s_{n-1}d_i(g')$ for $0 \leqslant i \leqslant n-1$.

So the definition of $\sim$ is independent of the choice of $*$. Consider the following left (resp. right) action of G_0 on G via

$$(*, g) \mapsto *g := s_0^n(*)g \quad (\text{resp. } g* := gs_0^n(*))$$

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for $* \in G_0$ and $g \in G_n$. The constant map $\Delta[n] \to *$ (which we will also denote by $*$) represents the identity element in $\pi_n(G)$. Indeed, given $g \in G$ representing an element $[g]$ of $\pi_n(G, *)$, the $(n+1)$-simplex $s_n(g)$ will have

$$d_{n+1}s_n(g) = d_n s_n(g) = g,$$

while for $i < n$,

$$d_i s_n(g) = s_{n-1}d_i(g) = *.$$

This realizes $g = *g$. Similarly, consideration of $s_{n-1}(g)$ gives $g = g*$. Hence the multiplicative action of G_0 on G induces a group automorphism of $\pi_n(G)$ (we will prove that it is actually a group). More precisely, the multiplication by the vertex g defines a group homomorphism

$$\pi_n(G, *) \to \pi_n(G, *g) \simeq \pi_n(G, g*)$$

with inverse defined by multiplication by g^{-1}. So any two choices of $*$ lead to naturally isomorphic $\pi_n(G)$. □

Remark 1. Since G is considered as a fibrant simplicial set, it satisfies the Kan condition. This is necessary for homotopy to be an equivalence relation. But we choose to omit the details here.

Problem 3. Show that B_n is a normal subgroup of Z_n, so that $\pi_n(G)$ is a group for all $n \geqslant 0$. Then show that $\pi_n(G)$ is abelian for $n \geqslant 1$.

Proof. Following the hint. For all $x \in Z_n$ and $y = s_n(z) \in N_{n+1}$ (with some $z \in G_n$), note that

$$d_n((s_{n-1}(x))y(s_{n-1}(x))^{-1}) = x d_n(y) x^{-1},$$

because of $d_n s_{n-1} = 1$. So $B_n = \operatorname{im} d_{n+1}$ is normal in Z_n. Thus for all $n \geqslant 0$ there are isomorphisms

$$\frac{\ker d_n}{\operatorname{im} d_{n+1}} \simeq \pi_n(G).$$

We are to show that this group structure coincides with the group structure on $\pi_n(G)$ induced from the multiplication on G. Recall the product on the homotopy groups is defined as follows. For $[x], [y] \in \pi_n(G)$ with $x, y \in G_n$, we can choose some $z \in G_{n+1}$ such that $d_i(z) = *$ for $0 \leqslant i < n-1$, $d_{n-1}(z) = x$, and $d_{n+1}(z) = y$. Then $[x][y] := [d_n(z)]$ is well-defined, which is independent of the choice of z. It can be seen in the following table.

	Input	$\cdots$	$d_{n-2}(-)$	$d_{n-1}(-)$	$d_n(-)$	$d_{n+1}(-)$	Output
(0)	z	$*$	$*$	x	$d_n(z)$	y	$[x][y] = [d_n(z)]$

We now fix $x, y, z, w, t, u \in G_n$. By definition of the multiplication, we obtain the following.

	Input	$\cdots$	$d_{n-2}(-)$	$d_{n-1}(-)$	$d_n(-)$	$d_{n+1}(-)$	Output
(1)	v_{n+1}	$*$	w	x	y	$*$	$[y][w] = [x]$
(2)	v_{n-1}	$*$	$*$	t	x	w	$[t][w] = [x]$
(3)	v_n	$*$	$*$	t	y	$*$	$[t][*] = [t] = [y]$

Here the relation (1) is deduced from (2) and (3). It can be checked that $v_n \in G_{n+1}$ satisfies the Kan condition. Again:

	Input	$\cdots$	$d_{n-2}(-)$	$d_{n-1}(-)$	$d_n(-)$	$d_{n+1}(-)$	Output
(4)	v_n	$*$	w	$*$	y	z	$[w][y]=[z]$
(5)	v_{n-1}	$*$	w	$*$	t	$*$	$[w][t]=[*]$
(6)	v_{n+1}	$*$	$*$	t	y	z	$[t][z]=[y]$

Here the relation (4) is deduced from (5) and (6). It can be checked that $v_{n+1} \in G_{n+1}$ satisfies the Kan condition. Combining these to get the following result.

	Input	$\cdots$	$d_{n-2}(-)$	$d_{n-1}(-)$	$d_n(-)$	$d_{n+1}(-)$	Output
(7)	v_{n+2}	$*$	w	x	y	z	$[w][y]=[x][z]$
(8)	v_{n-2}	$*$	t	$*$	$*$	w	$[t][*]=[w]$
(9)	v_{n-1}	$*$	t	$*$	u	x	$[t][u]=[x]$
(10)	v_{n+1}	$*$	$*$	u	y	z	$[u][z]=[y]$

Here (8), (9) are deduced from (4) and (10) is by definition of the multiplication. And finally, (7) is implied by (4), (8), (9), and (10). Therefore, for any fixed $x, y, z, w \in G_n$, we have $[w][y] = [x][z]$. In particular, by letting $[z] = [*]$, the conditions (1) and (7) are read as

$$[y][w] = [x], \quad [w][y] = [x].$$

So $[y][w] = [w][y]$ and $\pi_n(G)$ is abelian for all $n \geqslant 1$ (with $d_{-1} = d_{n-1}$). $\square$

Problem 4. If $G \to G$ is a surjection of simplicial groups with kernel G. Show that there is a short exact sequence of (not necessarily abelian) chain complex

$$1 \to NG' \to NG \to NG'' \to 1.$$

Proof. We will show that the functor $G \mapsto NG$ over simplicial groups is exact. By definition, $N_n(G) = \cap_{i \neq n} \ker(d_i : G_n \to G_{n-1})$. So the functor N exactly preserves kernels and is left-exact. It suffices to show the right-exactness, which boils down to prove the associated chain complex map $NG \to NG''$ is surjective in all degrees.

Given a commutative diagram of simplicial set maps

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha"] \arrow[d, hook] & G \arrow[d] \\ \Delta[n] \arrow[r, "\beta"] & G'' \end{tikzcd}$$

Then there is a simplex $x \in G_n$ such that the image of x via $G \to G''$ is β. Also note that

$$x|_{\Lambda^n_k} - \alpha : \Lambda^n_k \longrightarrow G$$

factors through $G' = \ker(G \to G'')$, and so $x|_{\Lambda^n_k} - \alpha$ extends to an n-simplex $\widetilde{x} \in G'$ in the sense that there is a commutative diagram of simplicial set maps

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "x|_{\Lambda^n_k} - \alpha"] \arrow[d, hook] & G' \\ \Delta[n] \arrow[ur, "\widetilde{x}"'] & \end{tikzcd}$$

Then $(x - \widetilde{x})|_{\Lambda^n_k} = \alpha$ and the image of $x - \widetilde{x}$ via $G \to G''$ is β. Therefore, $G \to G''$ is a fibration of simplicial groups. So the induced map $N_n(G) \to N_n(H)$ is surjective for all $n \geqslant 1$. Therefore, the functor $N(-)$ preserves epimorphisms. There is a short exact sequence

$$1 \to NG' \to NG \to NG'' \to 1.$$

This completes the proof. □

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