Work over field k char $k = p > 0$

Thm 1. $f : X \to Y$ isogeny of AVs w/ kernel K.

$\hat{f} : \hat{Y} \to \hat{X}$ w/ kernel K'

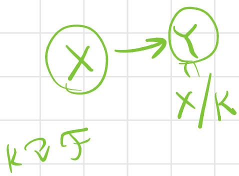
$\exists$ canonical isom. of $K' \cong \hat{K}$

Pf: $0 \to K' \to \hat{Y} \to \hat{X}$

$\Rightarrow \forall$ scheme S

$$K'(S) = \ker\left(\mathrm{Hom}(S, \hat{Y}) \longrightarrow \mathrm{Hom}(S, \hat{X})\right)$$

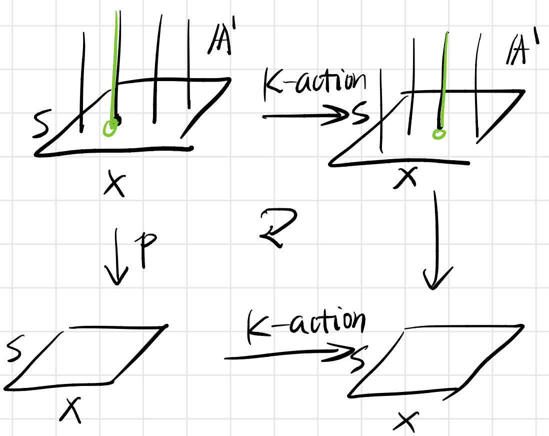
$$= \ker\left\{\begin{pmatrix}\text{l.b on } Y\times S, \\ \text{trivial on } 0\times S\end{pmatrix} \to \begin{pmatrix}\text{l.b on } X\times S \text{ trivial} \\ 0\times S\end{pmatrix}\right\}$$



$$= \ker\left\{\underline{\begin{pmatrix}\text{l.b. on } Y\times S\end{pmatrix} \to \begin{pmatrix}\text{l.b on} \\ X\times S\end{pmatrix}}\right\}$$

$Y\times S$ is quotient of $X\times S$ w/ free action of K

$$= \left\{\begin{matrix}K\text{-equivariant str} \\ \text{on } \mathcal{O}_{X\times S}\end{matrix}\right\}$$



$$\begin{tikzcd}
K\times S\times X\times \mathbb{A}^1 \arrow[r, "\mu"] \arrow[d, "p_{123}"'] & S\times X\times \mathbb{A}^1 \arrow[d, "p_{12}"] \\
K\times S\times X \arrow[r, "\mu"] & S\times X
\end{tikzcd}$$

T-points $\quad$ μ-action

$$\underset{\substack{\cap \\ K(T)}}{k} \cdot (s, x, t) \longmapsto (s,\ k+x,\ \eta(k,s,x,t))$$

$$\eta(k,s,x,t) = t \cdot \eta(k,s,x,1)$$

Fix k, s, then $\eta : X \longrightarrow \mathbb{A}^1 \Rightarrow \eta$ independ of $x \in X$

$$(k_1 k_2) \cdot (s, x, t) = k_1 \cdot (k_2 \cdot (s, x, t))$$

$$\Rightarrow \quad \eta(k_1+k_2, s, 0, 1) = \eta(k_1, s, 0, 1) \cdot \eta(k_2, s, 0, 1)$$

$$\Rightarrow \eta \in \underline{\mathrm{Hom}_S(K, \mathbb{G}_m)}$$

$$\begin{tikzcd} K\times S \arrow[r] \arrow[dr] & \mathbb{G}_m \times S \arrow[d] \\ & S \end{tikzcd}$$

$$\Rightarrow K'(S) \cong \mathrm{Hom}_S(K, \mathbb{G}_m) \overset{\text{上次}}{=} \widehat{K}(S)$$

$$\Rightarrow K' \cong \widehat{K}$$

<u>DEF</u>: $f : X \to Y$ isogeny of AVs is said to be of ht 1 if $K(X)^p \subseteq K(Y)$

<u>PROP</u>. f ht 1 $\Leftrightarrow$ $\ker f =: K$ is gp scheme of ht 1.

Pf: "$\Rightarrow$" $f : X \to Y$. $\rightsquigarrow$ $\underline{K(Y)} \hookrightarrow K(X)$

- $\mathcal{O}_{Y,0}$ is integrally closed.

  $\mathcal{O}_{X,0}$ is integral closure of $\mathcal{O}_{Y,0}$ in $K(X)$

$$\ker f = \mathrm{Spec}\,(\underline{\mathcal{O}_{X,0} / m_{Y,0}\mathcal{O}_{X,0}})$$

$$\begin{aligned} \{ f^p \mid f \in \mathcal{O}_{X,0} \} &\subseteq K(X)^p \cap \mathcal{O}_{X,0} \\ &\subseteq K(Y) \cap \mathcal{O}_{X,0} \\ &= \mathcal{O}_{Y,0} \end{aligned}$$

$\Rightarrow \underline{\{f^p \mid f \in \mathfrak{m}_{X,0}\} \subseteq \mathfrak{m}_{Y,0}}$

$\Big($ if $f^p \in \mathcal{O}_{Y,0}^\times \Rightarrow (f^{-1})^p \in \mathcal{O}_{Y,0}^\times$

$\Rightarrow f^{-1} \in \mathcal{O}_{X,0}$

$\Rightarrow f^{-1} \in \mathcal{O}_{X,0}^\times$ Contradiction! $\Big)$

"$\Leftarrow$" If K is of ht 1. let $K = \operatorname{Spec} R$ $R = \operatorname{Spec} R^*$

$\operatorname{Spec} A =: \mathcal{U}$ non-empty affine open set in X.

then $Y = \operatorname{Spec} A^K$

K action on $\operatorname{Spec} A$

$\Leftrightarrow$ a k-linear map $\alpha \longmapsto D_\alpha$ from $\underline{\operatorname{Lie} K}$ to derivation

$\{1\}$ $D_\alpha : \mathcal{O}_X \to \mathcal{O}_X$ s.t. $[D_\alpha, D_\beta] = 0$. $D_{\alpha^{(p)}} = (D_\alpha)^p$

$\Leftrightarrow D : \underset{U(\operatorname{Lie} K)}{\overset{\shortparallel}{R^*}} \xrightarrow{\text{homo}} \operatorname{Diff}(\mathcal{O}_A)$ s.t. a set of generators of

R^* maps to $\underline{\text{vector fields}}$ on $\operatorname{Spec} A$

$\underline{A^K} = \{a \in A \mid a$ is killed by the above vector fields$\}$

$\supseteq A^{(p)} := \{f^p \mid f \in A\}$

$\Rightarrow K(Y) = \operatorname{frac}(A^K) \supseteq K(X)^p$

$\underline{\text{Thm 2}}$ $\{f : X \to Y$ isogeny of ht $1\} / \text{isom} \overset{1:1}{\longleftrightarrow} \{$sub p-Lie algebra of $\underline{\operatorname{Lie} X}\}$

$f \downarrow \qquad \updownarrow 1:1$

$\ker f$

$$\left\{ K \le X \text{ subgp-scheme}, K \text{ is of ht } 1 \right\} \Big/ \text{isom}$$

$$\updownarrow 1:1$$

$$\{ K \le \underline{X_p} \text{ subgp scheme} \} \overset{1:1}{\longleftrightarrow} \{ \text{p-Lie subalgebras of } \underline{\mathrm{Lie}\, X_p} \}$$

$\underline{X_p} = \ker P_x$

Application: $\exists$ abelian variety X admitting infinite many distint isogenies $X \to Y$ of ht 1.

① Find the unique elliptic curve E s.t. p^{th} power map in $\mathrm{Lie}\, E$ is 0

② $X := E \times E$ $\qquad \mathrm{Lie}\, X = \mathrm{Lie}\, E \oplus \mathrm{Lie}\, E$

$\forall$ 1-dim subspace of $\mathrm{Lie}\, X$ is stable under the p^{th} power map $\Rightarrow$ is p-Lie subalgebra of $\mathrm{Lie}\, X$

The p-rank

X AV of dim g

$X_n = \ker n_X$ $\qquad$ let $n = p^r m$, $\quad (p, m) = 1$.

$$X_m \times X_{p^r} \xrightarrow{\sim} \underline{X_n}$$

$X_m = (X_n)_{(r,r)}$ $\qquad \underline{(X_n)_{(r,r)}} \times \underline{(X_n)_{(r,l)} \times (X_n)_{(l,r)} \times (X_n)_{(l,l)}}$

$X_m \cong (\mathbb{Z}/m\mathbb{Z})^{2g}$ $\qquad \# X_n = (n)^{2g} = m^{2g} \cdot (p^r)^{2g}$

$G_n := X_{p^n}$

① $\# X_{p^n} = (p^n)^{2g}$

$$\begin{tikzcd} 0 \arrow[r] & X_p \arrow[r] \arrow[d, "p"] & X \arrow[r, "p"] \arrow[d, "p^{n+1}"] & X \arrow[r] \arrow[d, "p^n"] & 0 \\ 0 \arrow[r] & X_p \arrow[r] & X \arrow[r, "id"] & X \arrow[r] & 0 \end{tikzcd}$$

$\because X$ is divisible

snake lem $\rightsquigarrow$ $0 \to (G_1)_{red} \to (G_{n+1})_{red} \xrightarrow{p} (G_n)_{red} \to 0$

Let $(G_1)_{red} = (\mathbb{Z}/p\mathbb{Z})^r$

By induction, $(G_n)_{red} = (\mathbb{Z}/p^n\mathbb{Z})^r$

$G_n = (G_n)_{(r,l)} \times (G_n)_{(l,r)} \times (G_n)_{(l,l)}$

$\hat{G_n} =$

- $\hat{G_n} \overset{\text{thm1}}{=} \ker((p^n)_{\hat{X}})$

$\exists s$ s.t. $(\hat{G_n})_{red} = (\mathbb{Z}/p^n\mathbb{Z})^s$

(r,l) type $\|$ $\widehat{(G_n)_{(l,r)}}$

$\Rightarrow (G_n)_{(l,r)} = \widehat{(\mathbb{Z}/p^n\mathbb{Z})^s} = \mu_{p^n}^s$

$\Rightarrow G_n = (\mathbb{Z}/p^n\mathbb{Z})^r \times \mu_{p^n}^s \times G_n^\circ$

(l,l) type.

$(p^n)^{2g} = (p^n)^r \cdot (p^n)^s \cdot \# G_n^\circ$

$\Rightarrow$ let $t = 2g - r - s$, then $\# G_n^\circ = p^{t \cdot n}$

- If $X \simeq Y$ are isogenous AVs, $(r_X, s_X, t_X) = (r_Y, s_Y, t_Y)$

$\hat{X} \simeq \hat{Y}$ $\quad s_X = r_{\hat{X}}$

Reduce to prove $r_X = r_Y$

$$0 \to K \to X \xrightarrow{f} Y \to 0 \qquad \leftarrow \text{isogeny}$$

$$\Rightarrow \quad \underbrace{\#(X_{p^n})_{red}}_{= p^{n \cdot r_X}} \le \underbrace{k \cdot \#(Y_{p^n})_{red}}_{= k \cdot p^{n \cdot r_Y}} \qquad \forall n$$

$$\Rightarrow \quad r_X \le r_Y$$

K is finite $\Rightarrow \exists N > 0$ annihilate K.

$\Rightarrow \ker f = K \subseteq \ker(N_X)$

By $Y \cong X/K$ & property of quotient

$$\begin{tikzcd} X \arrow[r, "N_X"] \arrow[d, "f"'] & X \\ Y \arrow[ur, "\exists g"'] & \end{tikzcd}$$

N_X surj.

$\Rightarrow g$ surj. $\ker g$ finite

$\Rightarrow g$ is isogeny $\Rightarrow r_Y \le r_X$

COR. $X \overset{\text{isogenous}}{\simeq} \hat{X} \Rightarrow r_X = r_{\hat{X}} = s_X =: r$ ← p-rank of X

$r + r + t = 2g$ $\Rightarrow r \le g$

pth power in Lie $\hat{X}$

- $\mathrm{Lie}\, X = \mathrm{Lie}\, X_p \cancel{\oplus\, \mathrm{Lie}(G_1)} = \mathrm{Lie}\left((\mathbb{Z}/p\mathbb{Z})^r \times (\mu_p)^r \times G_1^0\right)$

$\leftarrow$ induced by P_X

$\mathrm{Lie}\, X \xrightarrow{P} \mathrm{Lie}\, X$ is zero

$\Rightarrow \mathrm{Lie}\, X_p = \mathrm{Lie}\, X$

$$= \boxed{\mathrm{Lie}(\mu_p)^r} \oplus \mathrm{Lie}\, G_1^0$$

↑ §14 semi-simple

↑ nilpotent w.r.t. pth power map.

p-rank of X = dim of semi-simple part of Lie X w.r.t. pth power map.

• $\mathrm{Lie}\,\hat{X} \overset{\S 13}{\cong} H^1(X,\mathcal{O}_X)$

$F: \mathcal{O}_X \to \mathcal{O}_X$ Frobenius map

$\overset{\text{induces}}{\rightsquigarrow}$ p-linear map $H^1(X,\mathcal{O}_X) \xrightarrow{F} H^1(X,\mathcal{O}_X)$

Thm3 $\mathrm{Lie}\,\hat{X} \cong H^1(X,\mathcal{O}_X)$

pth power map $\longleftrightarrow$ Frobenius map on $H^1(X,\mathcal{O}_X)$

Pf: $\Lambda = k[\varepsilon]/(\varepsilon)^2$

D: vector field on X $\longleftrightarrow$

$$\begin{tikzcd} \underline{D}: X\times\operatorname{Spec}\Lambda \arrow[rr, "\text{automorphism}"] \arrow[dr] & & X\times\operatorname{Spec}\Lambda \arrow[dl] \\ & \operatorname{Spec}\Lambda & \end{tikzcd}$$

$$M := k[\varepsilon_1,\dots,\varepsilon_p]/(\varepsilon_1^2,\dots,\varepsilon_p^2)$$

$$\eta_i: \Lambda \longrightarrow M, \quad \varepsilon \longmapsto \varepsilon_i$$

Assume $X = \operatorname{Spec} A$

$$D_i: A\otimes_k M \longrightarrow A\otimes_k M$$
$$a \longmapsto a + D(a)\varepsilon_i = (1+\varepsilon_i D)a$$

$$D': A\otimes_k M \longrightarrow A\otimes_k M$$
$$a \longmapsto \prod_{i=1}^{p}(1+\varepsilon_i D)a = (1+s_1 D + s_2 D^2 + \cdots + s_p D^p)a$$

s_i: elementary sym. functions

$$D'': A\otimes_k M' \longrightarrow A\otimes_k M'$$
$$a \longmapsto \underline{(1+s_1 D + \cdots + s_p D^p)a}$$

$\underline{M' \hookrightarrow M}$

$\underline{\phi_i}: \operatorname{Spec} M \to \operatorname{Spec}\Lambda$

D base change using ϕ_i

$D_i: X\times\operatorname{Spec}M \to X\times\operatorname{Spec}M$

$\underline{D'} = D_1\circ\cdots\circ D_p: X\times\operatorname{Spec}M \to X\times\operatorname{Spec}M$

$M' := k[s_1,\dots,s_p]$ subalg of M

$\exists\ \underline{D''}: X\times\operatorname{Spec}M' \to X\times\operatorname{Spec}M'$ s.t. base change using ψ induces D'

$\psi: \operatorname{Spec}M \to \operatorname{Spec}M'$

$D'' \otimes (M' \hookrightarrow M) = D'$

$M' = k[s_1, \dots, s_p]$

$s_1 \cdot s_i = i\, s_{i+1} \qquad 1 \le i \le p-1$

$\Rightarrow \underline{M' = k[s_1, s_p]}$

$s_1^p = 0$

$s_p^2 = 0$

$$\eta : M' \to \Lambda, \quad s_1 \mapsto 0, \quad s_p \mapsto \varepsilon$$

$$\begin{array}{llcl} D'' : & A \otimes_k \Lambda & \longrightarrow & A \otimes_k \Lambda \\ \| & \quad \| & & \\ D^{(p)} & (A \otimes_k M') \otimes_{M'} \Lambda & & \\ & a & \longmapsto & (1+\varepsilon D^p)a \\ & & & \quad \| \\ & & & a + \varepsilon D^p(a) \end{array}$$

$\underline{\Phi} : \operatorname{Spec}\Lambda \to \operatorname{Spec} M'$

$\rightsquigarrow$

$D'' : X \times \operatorname{Spec}\Lambda \longrightarrow X \times \operatorname{Spec}\Lambda$

$\|$

$\underline{D^{(p)}}$

Now $\forall t \in \operatorname{Lie}\hat{X}$, $\underline{t} : \underline{\operatorname{Spec}\Lambda \to \hat{X}} \rightsquigarrow \underline{D} : \hat{X} \times \operatorname{Spec}\Lambda \xrightarrow{\text{translation by } t} \hat{X} \times \operatorname{Spec}\Lambda$

$$\begin{tikzcd} \operatorname{Spec}\Lambda \arrow[r, "t"] & \hat{X} \\ \operatorname{Spec}M \arrow[u, "\Phi_i"] \arrow[ur, "t_i"'] \arrow[ur, phantom, shift left=3, "\circlearrowleft"] & \end{tikzcd}$$

$$\begin{tikzcd} \operatorname{Spec}\Lambda \arrow[d] \arrow[rr, "t'' = t^{(p)}"] & & \hat{X} \\ \operatorname{Spec}M' \arrow[rr, "t''"] \arrow[urr, phantom, "\circlearrowleft"] & & \\ \operatorname{Spec}M \arrow[u, "\psi"] \arrow[uurr, "t' = \prod t_i"'] \arrow[ur, phantom, "\circlearrowleft"] & & \end{tikzcd}$$

$\mathcal{O}_X \qquad S := \operatorname{Spec}\Lambda$

$\left(\S 13 \quad 0 \to \underline{1+\varepsilon\mathcal{O}_X} \to \mathcal{O}^*_{X\times S} \to \mathcal{O}^*_X \to 0\right) \qquad X \times S$

(with $\mathcal{O}_X \cong 1+\varepsilon\mathcal{O}_X$)

$$\begin{tikzcd} H^1(X, \mathcal{O}_X) \arrow[r, hook] & H^1(X\times S, \mathcal{O}^*_{X\times S}) \arrow[d] & \operatorname{Hom}_0(S, \hat{X}) = \operatorname{Lie}\hat{X} \arrow[l] \arrow[d, "p\text{th power}"] \\ & H^1(X\times \operatorname{Spec}M, \mathcal{O}^*_{X\times\operatorname{Spec}M}) & \end{tikzcd}$$

$(\mathfrak{U}, f_{ij}) \longmapsto (\mathfrak{U}, 1+\varepsilon f_{ij}) \longleftrightarrow t$

$t_k = (\mathfrak{U}, 1+\varepsilon_k f_{ij})$

F

$$(\mathcal{U},\ \prod_{k=1}^{p}(1+\varepsilon_k f_{ij})\cdot \bar{\varepsilon}) \longleftrightarrow t'$$

$$t'' \to t$$

$$\begin{tikzcd}
H^1(X, \mathcal{O}_X) \arrow[r, hook] & H^1(X\times S, \mathcal{O}^*_{X\times S}) & t^{(p)}\ \mathrm{Lie}\,\hat{X} \arrow[l, hook']
\end{tikzcd}$$

$$(\mathcal{U},\ f_{ij}^{p}) \longmapsto (\mathcal{U},\ 1+\varepsilon f_{ij}^{p}) \longleftrightarrow t^{(p)}$$