Frobenius structure on theta connections and epipelagic Langlands parameters

Daxin Xu

Automorphic side: Epipelagic reps

Setup F non-arch local field, t uniformizer,

$k = \mathbb{F}_q$ res field.

G split, simple, simply conn grp $/ F$.

$T \subset B \subset G$. Φ roots $\supseteq \Delta$ simple roots.

Simple supercusp reps of $G(F)$ (Gross – Reeder).

$$\begin{tikzcd} G(F) & G(\mathcal{O}_F) \arrow[l, phantom, "\supseteq"] \arrow[d, "\pi"', "\text{mod } t"] & I \arrow[l, phantom, "\supseteq"] \arrow[d] & I(1) = \pi^{-1}(U(k)) \arrow[l, phantom, "\triangleright"] \arrow[d] \\ & G(k) & B(k) \arrow[l, phantom, "\supseteq"] & U(k) \arrow[l, phantom, "\supseteq"] \end{tikzcd}$$

$$I(1)/I(2) = [I(1), I(1)] \simeq \bigoplus_{\substack{\alpha \text{ affine} \\ \text{simple roots}}} U_\alpha$$

$$\begin{tikzcd} I(1) \arrow[r] \arrow[d] & \mu_p \subseteq \mathbb{C}^\times \\ \bigoplus_{\substack{\alpha \text{ affine} \\ \text{simple roots}}} k^+ \arrow[r, "\phi"] & k \arrow[u, "\psi : k \xrightarrow{\mathrm{Tr}} \mathbb{F}_p \to \mu_p"'] \end{tikzcd}$$

linear fct s.t. $\phi|_{U_\alpha} \neq 0$, $\forall \alpha$.

Thm-Def $c\mathrm{Ind}_{I(1)}^{G(F)} \psi \circ \phi \simeq \bigoplus_{\text{finite}}$ (supercusp irreps)

simple cusp reps.

E.g. At Iwahori level: In SL_n,

$$I = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} & \cdots & \mathcal{O} \\ t\mathcal{O} & \mathcal{O}^\times & & \vdots \\ \vdots & & \ddots & \mathcal{O} \\ t\mathcal{O} & \cdots & t\mathcal{O} & \mathcal{O}^\times \end{pmatrix}, \quad I(1) = \begin{pmatrix} 1+t\mathcal{O}^\times & \mathcal{O} & \cdots & \mathcal{O} \\ t\mathcal{O} & \ddots & & \vdots \\ \vdots & & \ddots & \mathcal{O} \\ t\mathcal{O} & \cdots & t\mathcal{O} & 1+t\mathcal{O}^\times \end{pmatrix},$$

$$I(2) = \begin{pmatrix} 1+t\mathcal{O}^\times & t\mathcal{O} & \mathcal{O} & \mathcal{O} \\ t\mathcal{O} & \ddots & \ddots & \mathcal{O} \\ & \ddots & \ddots & t\mathcal{O} \\ \underbrace{t^2\mathcal{O}}_{\text{changed}} & & t\mathcal{O} & 1+t\mathcal{O}^\times \end{pmatrix}$$

<u>epipelagic reps</u> (Reeder - J.K. Yu)

$I \supseteq I(1) \supseteq I(2) \rightsquigarrow P \supseteq P(1) \supseteq P(2)$ some parahoric + Moy-Prasad fil'n.

such a $P \xleftrightarrow{1-1} m$ = regular elliptic number of G

$\xleftrightarrow{1-1}$ stable inner grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_i$ (*)

assume $p \gg 0$ & $p \nmid m$

$\rightsquigarrow$ (*) is defined by an order m.

<u>Constr'n</u> $\theta \in \mathrm{Aut}^{\mathrm{inner}}(\mathfrak{g})$

s.t. $\mathfrak{g}_i = \zeta_m^i$ - eigenspace of θ, $\theta^m = id$

mth root of 1.

$\theta = \mathrm{Ad}_g$, $g \in G$, $g = \lambda(\zeta_m)$ for $\lambda \in X_*(T)$

If $x = \frac{\lambda}{m} \in X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, $P = \{ T(\mathcal{O}_F), U_\alpha(t^{-\lfloor \alpha(x) \rfloor}), \alpha \in \Phi \}$

Now $\mathfrak{g}_0 \subset \mathfrak{g} \rightsquigarrow P/P(1) \simeq G_0 \subseteq G$

$\circlearrowleft$

$P(1)/P(2) \simeq \mathfrak{g}_1$.

$\phi : \mathfrak{g}_1 \to \mathbb{A}^1$ <u>stable</u> linear functor

$G_0 \cdot \phi$ closed & $C_{G_0}(\phi)$ finite.

e.g. $m = h$ = Coxeter number of G

(e.g. A_n : $h = n+1$, B_n, C_n : $h = 2n$, D_n : $h = 2n-2$).

- $\check{\rho} = \frac{1}{2}\sum(\text{pos coroots})$, $\theta = \mathrm{Ad}_{\check{\rho}(\zeta_h)}$
- $\mathfrak{g}(i) = \bigoplus_{ht(\alpha)=i} \mathfrak{g}_\alpha$, $\alpha = \sum n_i \alpha_i \Rightarrow ht(\alpha) = \sum n_i$.
- $\mathfrak{g}_i = \mathfrak{g}(i) \oplus \mathfrak{g}(i-h)$.

For $P = I$, $\mathfrak{g}_0 = \mathfrak{t}$, $\mathfrak{g}_1 = \underbrace{\Big(\bigoplus_{\alpha \text{ simple}} \mathfrak{g}_{\alpha_i}\Big)}_{\mathfrak{g}(1)} \oplus \underbrace{\mathfrak{g}_{-\theta}}_{\mathfrak{g}(1-h)}$ (θ max root).

$\phi : \mathfrak{g}_1 \to \mathbb{A}^1$.

e.g. B_n, C_n types : $m = \frac{2n}{k}$ $(k \mid n)$

G_2 type : $m = 2, 3, 6$

epipelagic rep $\rightsquigarrow$ $\rho : W_F \to \check{G}(\mathbb{C})$ L-param

$\circlearrowright$ $\check{\mathfrak{g}}$ ($\check{G}$ dual grp of G, $\check{\mathfrak{g}} = \mathrm{Lie}\,\check{G}$).

<u>Conj</u> (Reeder-Yu, inspired by formal deg conj)

(1) $\check{\mathfrak{g}}^{I_F} = 0$.

(2) $\mathrm{Sw}(\check{\mathfrak{g}}) = \#\Phi/m$.

<u>Global-local method on function field</u>

Heinloth-Ngô-Yun, Yun:

- $F = k(x)$, $t = x^{-1}$, f autom fct
  - unram on $\mathbb{G}_m$, f is P^{op}-invariant.
  - at ∞, $t = x^{-1}$, $f \in c\mathrm{Ind}_{P(1)}^{G(F)}(\psi \circ \phi)$.

· Geometrize f as Aut_ϕ on $\mathrm{Bun}_{\mathcal{G}} / \mathbb{P}^1$

$$\mathcal{G}|_{\mathbb{G}_m} = G \times \mathbb{G}_m, \quad \mathcal{G}|_0 = P^{op}, \quad \mathcal{G}|_\infty = P(1).$$

$\rightsquigarrow$ $\mathcal{E}$ Hecke eigenval of Aut_ϕ, $\check{G}$-local sys $/ \mathbb{G}_m$.

<u>Explicit L-params / $\mathbb{C}$</u>

Alg D-mods $\rightsquigarrow$ $\mathcal{E}$ $\check{G}$-connection $/ \mathbb{G}_m$.

$m \rightsquigarrow$ stable grading on $\check{\mathfrak{g}} = \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} \check{\mathfrak{g}}_i$.

<u>Lem</u>

$$\begin{tikzcd}
\mathfrak{g}^* /\!/ G \simeq \mathfrak{t}^* /\!/ W = \check{\mathfrak{t}} /\!/ W \simeq \check{\mathfrak{g}} /\!/ \check{G} & & \\
\check{\mathfrak{g}}_1 /\!/ G_0 \arrow[u, hook] \arrow[rr, "\sim"] & & \check{\mathfrak{g}}_1 /\!/ \check{G}_0 \arrow[u, hook]
\end{tikzcd}$$

$$\begin{array}{ccc} \check{\mathfrak{g}}_1 /\!/ G_0 & & \check{\mathfrak{g}}_1 /\!/ \check{G}_0 \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \phi & & X \end{array}$$

$\exists\, X$ on the dual side corresp to ϕ.

$X = X_1 + X_{1-m} \in \check{\mathfrak{g}}_1 = \check{\mathfrak{g}}(1) \oplus \check{\mathfrak{g}}(1-m)$.

Stable vect $\in \check{\mathfrak{g}}_1 \Rightarrow$ regular semisimple.

<u>Thm-Def</u> (θ-connection) $\mathcal{E} \simeq d + (X_1 + X_{1-m} \cdot x) \frac{dx}{x}$

as $\check{G}$-connections on $\mathbb{G}_m$.

(Zhu, Chen-Yi) This is rigid & is an oper.

e.g. $m = h$, $X_1 = \sum_{\alpha \text{ simple root}} E_\alpha$, $E_\alpha \in \check{\mathfrak{g}}_\alpha$,

$X_{1-h} = E_{-\theta} \in \check{\mathfrak{g}}_{-\theta}$.

$\check{G} = SL_n$, $\mathrm{Bes}_n = d + \begin{pmatrix} 0 & & & x \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \frac{dx}{x}$.

$\rightsquigarrow$ Bessel diff eq'n $\delta^n(f) = x \cdot f$, $\delta = x \frac{d}{dx}$.

$/ \mathbb{F}_q$, p-adic coeffs,

arith $\mathcal{D}$-mods $\rightsquigarrow$ $\mathcal{E}^\dagger$ $\check{G}$-overconv F-isocrystal $/ \mathbb{G}_m$.

global L-param by Yun in p-adic setting.

$K = \mathbb{Q}_p(\pi)$, $\pi^{p-1} = -p$

$\{\text{such } \pi\} \xleftrightarrow{1-1} \{\psi_\pi : \mathbb{F}_p \to K^\times\}$.

<u>Thm</u> (Xu-Zhu, Xu-Yi) $\exists$ a Frob str $\varphi \in \check{G}(A^\dagger)$ on $\mathcal{E}_\pi$

$$\mathcal{E}_\pi = d + (X_1 + \pi^m X_{1-m}\, x)\frac{dx}{x} \quad \text{s.t. } \mathcal{E}^\dagger \simeq (\mathcal{E}_\pi, \varphi)$$

where $A^\dagger = \bigcup_{r>1} K\langle \frac{x}{r} \rangle$, $|x|_p < r$.

<u>Local monodromy of $\mathcal{E}^\dagger$</u>

$\mathcal{E}^\dagger \in \check{G}\text{-}F\text{-}\mathrm{Iso}^\dagger(\mathbb{G}_m, \mathbb{F}_p) \rightsquigarrow \mathcal{E}^\dagger|_x \in \check{G}\text{-}(\varphi,\nabla)\text{-mods} /_{\mathcal{R}}$, $x = 0, \infty$.

$\xrightarrow[\text{monodromy}]{p\text{-adic local}}$ WD rep $(\rho_x, \mathcal{N}_x)$ assoc to $\mathcal{E}_x^\dagger$

$\rho : W_F \to \check{G}(K)$, $\mathcal{N} \in \check{\mathfrak{g}}$, $\mathrm{Ad}_{\rho(F_r)}\mathcal{N} = q^{-1}\mathcal{N}$.

$\mathcal{E}_0^\dagger$ unip monodromy with $\mathcal{N}_0 = X_1$, $\rho_0(I_F)$ trivial

At ∞, $\mathcal{N}_\infty = 0 \rightsquigarrow$ understand $\rho_\infty(I_F)$.

* Local monodromy of $\mathcal{E}_\infty$ as formal connections:

$\langle \mathcal{E}_\infty \rangle$ Tannakian subcat gen'd by subquotients of $\mathcal{E}_\infty(V)$, $\forall V \in \mathrm{Rep}(\check{G})$.

$\downarrow$

Diff-mods $/ (\bar{K}((t))/\bar{K}) \rightsquigarrow G_{\text{formal}}(\mathcal{E}_\infty)$ Tannakian grp.

<u>Prop</u> Let $\check{T} = C_{\check{G}}(X)$, $X = X_1 + X_{1-m}$

a max torus $\subseteq \check{G}$.

Then
$$\begin{tikzcd} 1 \arrow[r] & P \arrow[r] \arrow[d] & I = G_{formal}(\mathcal{E}_\infty) \arrow[r] \arrow[d] & G^t \simeq \mathbb{Z}/m\mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 1 \arrow[r] & \hat{T} \arrow[r] & N_{\check{G}}(\check{T}) \arrow[r] & W \arrow[r] & 1 \end{tikzcd}$$
$1 \mapsto \sigma$ = reg ell elt of order m

where P = wild part (P is a torus).

Pf Babbit - van Dasjan's formal reduction.

$$t = u^m, \quad [t \mapsto u^m]^* \mathcal{E}_\infty \simeq d + X \frac{du}{u^2} \quad \left(\simeq \bigoplus \exp(\not\exists / u)\right)$$

via $Q \in \check{G}(K((u)))$. $\square$

Let $\rho : W_F \to \check{G}(\bar{K})$ Weil rep assoc to $\mathcal{E}_\infty^t$.

Thm (Conj of Reeder - Yu)

(1) Suppose $(p, m) = 1$. $\check{G}$ adjoint.

$$\begin{tikzcd} 1 \arrow[r] & \rho(P_F) \arrow[r] \arrow[d, hook] & \rho(I_F) \arrow[r] \arrow[d, hook] & \mathbb{Z}/m\mathbb{Z} \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & P \arrow[r] \arrow[d, hook] & I \arrow[r] \arrow[d, hook] & \mathbb{Z}/m\mathbb{Z} \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \check{T} \arrow[r] & N_G(\check{T}) \arrow[r] & W \arrow[r] & 1 \end{tikzcd}$$

(2) $\rho : I_F \to \check{G} \circlearrowright \check{\mathfrak{g}}$.

$$\check{\mathfrak{g}}^{I_F} = 0, \quad \mathrm{Sw}(\check{\mathfrak{g}}) = \frac{\#\Phi}{m}.$$

Key ingredient $Q \in \check{G}(K((u)))$ Converges on $0 < |u| < 1$.

Rmk (a) For $p \mid m = h$, Conj (1) fails.

(b) $F^\#$ = mixed char. Apply Deligne - Kazhdan philosophy

$\rightsquigarrow$ obtain similar results for $F^\#$

under Fargues - Scholze LLC.