Heights of special points on quaternionic Shimura varieties

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André-Oort conjecture

Conj Let S be a Shimura var
$\cup\!|$
V irred subvar.

If the special pts (or CM pts) of S are Zariski dense in V,
then V = Hecke translate of a Sh subvar.

Proven by Pila, Shankar, Tsimerman.

Sketch (1) O-minimality: Pila-Wilkie pt counting thm ('06).
(2) Functional transcendence Ax-Schanuel
(Pila, Tsimerman '14)
(3) Large Galois orbit conj.

If $P \in S$ is a special pt on a Shimura var,
can associate a pair (E, ϕ)

- E/F imag quad ext'n
  (tot real
- $\phi \subseteq \mathrm{Hom}(E, \mathbb{C})$ s.t. $\phi \cap \bar{\phi}$ = empty set.

Call (E, ϕ) a "partial CM type".

Thm (Binyamini-Schmidt-Yafaev '23)
Suppose that $h: S(\bar{\mathbb{Q}}) \to \mathbb{R}$ is a Weil ht fct s.t.
$(\forall \varepsilon > 0,\ h(P) = O(d_E^{\varepsilon})) \Rightarrow$ André-Oort Conj.

<u>Thm</u> (Pila, Shankar, Tsimerman)

$\exists$ a canonical ht of a partial CM type $h(\phi)$

depends only on CM type

(but not pts on Sh vars)

This is compatible w/ their canonical hts on Sh vars.

<u>Thm</u> (Zhao) Say $[E:\mathbb{Q}] = 2g$

Then $ht(\phi) \text{ "=" } \frac{1}{2^{g-|\phi|}} \sum_{\Phi \supseteq \phi} h(\Phi) - \frac{g-|\phi|}{g 2^g} \sum_{\Phi'} h(\Phi') + \log D$

over all CM types $/E$

D involves disc of E, ϕ, etc.

bdd in terms of disc E.

<u>Quaternionic Sh vars</u>

Let $\Sigma := \phi|_F \subseteq \mathrm{Hom}(F, \mathbb{R})$.

B/F a quat alg, w/ ramification at ∞ is exactly Σ.

i.e. $B \otimes_{\mathbb{Q}} \mathbb{R} = \underbrace{M_2(\mathbb{R})^{\Sigma}}_{\text{inside } \Sigma} \times \underbrace{\mathbb{H}^{\Sigma^c}}_{\text{outside } \Sigma}$

$\rightsquigarrow$ Construct a quat Sh var X_U s.t.

$X_U(\mathbb{C}) = B^\times \backslash (\mathcal{H}^{\pm})^{\Sigma} \times B(\mathbb{A}_f)^\times / U$, $\dim X_U = |\phi|$.

Given (E, ϕ) s.t. $E \hookrightarrow B$, get a special pt $P \in X_U$.

Let $\mathcal{L}_U :=$ Hodge bdl on X_U.

<u>Thm</u> (Zhao) $\frac{1}{2} h_{\hat{\mathcal{L}}}(P_U) = \frac{1}{2^{|\Sigma^c|}} \sum_{\Phi \supseteq \phi} h(\Phi) - \frac{|\Sigma^c|}{g 2^g} \sum_{\Phi'} h(\Phi') + \log D$.

up to $O(\log(\mathrm{disc}\, E))$.

Let $B' = B \otimes_F E$. Let $\phi' \subseteq \mathrm{Hom}(E, \mathbb{C})$ a complimentary partial CM type to ϕ.

i.e. $\phi \sqcup \phi'$ is a CM type.

The Shimura var assoc to B' classifies ab vars

$A_1 \times A_2$ with A_1 of CM type $(E, \phi \sqcup \phi')$

A_2 of CM type $(E, \bar{\phi} \sqcup \phi')$

<u>Yuan-Zhang</u> decomposed the Faltings ht into

$$h(\Phi) = \sum_{\sigma \in \Phi} h(\Phi, \sigma)$$

Define $h(\Phi, \phi) := \sum_{\sigma \in \phi} h(\Phi, \sigma)$

What we show is

$$\frac{1}{2} h(\beta) = h(\phi \sqcup \phi', \phi) + h(\bar{\phi} \sqcup \phi', \bar{\phi}).$$

Now, we sum the formula for all choices of complementary CM types ϕ'.

Simplify the remaining terms by using Yuan-Zhang.

<u>Thm</u> (Yuan-Zhang)

If $|\Phi \cap \Phi'| = g-1$, and differ at $\tau \in \Phi \setminus \Phi'$, $\tau' \in \Phi' \setminus \Phi$,

then $h(\Phi, \tau) + h(\Phi', \tau') = \frac{1}{g^{2g}} \sum_{\Phi} h(\Phi)$.