Introduction (II)

Yichao Tian Jan 2

§1 ULA-sheaves (continued)

Thm An object $A \in D(\mathrm{Bun}_G, \Lambda)$ is ULA (for $\mathrm{Bun}_G \to *$)

iff $\forall b \in B(G)$, $i_b^* A \in D(G_b(\mathbb{Q}_p), \Lambda)$ is <u>admissible</u>

i.e. $\forall K \subseteq G_b(\mathbb{Q}_p)$ pro-p subgroup, open compact, $(i_b^* A)^K$ is perfect.

<u>Rmk</u> $D(\mathrm{Bun}_G, \Lambda)$ is relatively easy to construct for $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$.

For $\Lambda = \mathbb{Z}_\ell$ or $\bar{\mathbb{Q}}_\ell$ it is more difficult

$\rightsquigarrow$ they used ideas from condensed mathematics.

<u>Key ingredients</u> local charts for Bun_G, denoted by $\mathcal{M}_b$.

$G = GL_n$, $X_*(T)^+_{\mathbb{Q}} = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Q}^n$

with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ ($\rightsquigarrow$ upper-triangular Borel).

$$\lambda_1 = \cdots = \lambda_{i_1} > \lambda_{i_1+1} = \cdots = \lambda_{i_2} > \cdots > \lambda_{i_{k-1}+1} = \cdots = \lambda_n = \lambda_{i_k}$$

$\rightsquigarrow$ $\mathcal{M}_b : \mathrm{Perf}_{\bar{\mathbb{F}}_p} \longrightarrow \mathrm{Groupoids}$

$$S \longmapsto \left\{ (\mathcal{E}, \mathrm{Fil}^{\bullet}\mathcal{E}) \;\middle|\; \begin{array}{l} \mathcal{E} \text{ vector bundle of rk } n \text{ on } X_S \\ \mathrm{Fil}^{\bullet}\mathcal{E} \text{ an increasing exhaustive filtration} \end{array} \right\}$$

with $\mathrm{gr}_j \mathcal{E}$ semi-simple of slope $\lambda_{i_{k+1-j}}$.

$$\begin{tikzcd}
 & (\mathcal{E}, \mathrm{Fil}^{\bullet}\mathcal{E}) \arrow[r, mapsto] & \mathcal{E} \\
 & \mathcal{M}_b \arrow[r, "\pi_b"] \arrow[d, "q_b"] & \mathrm{Bun}_G \\
\mathrm{gr}^{\bullet}\mathcal{E} & {[*/G_b(\mathbb{Q}_p)]} &
\arrow[from=1-2, to=3-1, bend right, mapsto]
\end{tikzcd}$$

G_b: inner form of a Levi of G.

$$\mathrm{gr}^{\bullet}\mathcal{E} = \bigoplus_{i=1}^{k} \mathrm{Fil}_i\mathcal{E}/\mathrm{Fil}_{k}\mathcal{E}$$

Thm (1) π_b, q_b are partially proper, representable in loc spatial diamonds and coh smooth of rel dim $\langle 2\rho, \nu_b \rangle$

(2) $\mathrm{Im}(\pi_b)$ consists of the generalizations of $b \in B(G) = |\mathrm{Bun}_G|$.

(cf. [FS] Prop V.3.5, Thm V.3.7)

To prove the smoothness of π_b,

they use a certain "Jacobian criterion".

Example $G = GL_2$, $b = \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} \longleftrightarrow \mathcal{E}_b = \mathcal{O} \oplus \mathcal{O}(1)$.

$$\rightsquigarrow \mathcal{M}_b : S \longmapsto \left\{ \begin{array}{l} 0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}' \to 0 \\ \text{extension of v.b. on } X_S \\ \deg \mathcal{L} = 0,\ \deg \mathcal{L}' = 1 \end{array} \right\}$$

$\rightsquigarrow \widetilde{\mathcal{M}}_b \to \mathcal{M}_b$ the $\underline{\mathbb{Q}_p^\times} \times \underline{\mathbb{Q}_p^\times}$-torsors

on parametrizing $\mathcal{L} \cong \mathcal{O}$, $\mathcal{L}' \cong \mathcal{O}(1)$ over $\widetilde{\mathcal{M}}_b$:

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(1) \to 0.$$

$\widetilde{\mathcal{M}}_b = BC(\mathcal{O}(-1)[1]) : S \longmapsto H^1(X_S, \mathcal{O}_{X_S}(-1))$ Banach-Colmez.

$(\widetilde{\mathcal{M}}_{b,C} \cong (\mathbb{A}^1_C)^\diamond / \underline{\mathbb{Q}_p}.)$

$\forall x \in \widetilde{\mathcal{M}}_b(C)$, $0 \to \mathcal{O}_{X_C} \to \mathcal{E}_x \to \mathcal{O}_{X_C}(1) \to 0$

$$\mathcal{E}_x \cong \left\{ \begin{array}{l} \boxed{\mathcal{O}_{X_C}(1) \oplus \mathcal{O}_{X_C}} \\ \boxed{\mathcal{O}_{X_C}(\tfrac{1}{2})} \end{array} \right.$$

$\pi_b^{-1}(\mathcal{O}_{X_C}(\tfrac{1}{2})) = BC(\mathcal{O}_{X_C}(\tfrac{1}{2}) \setminus \{0\}) / \underline{\mathbb{Q}_p^\times}$

$\pi_b^{-1}(\mathcal{O}_{X_C}(1) \oplus \mathcal{O}_{X_C}) = BC(\mathcal{O}_{X_C} \oplus \mathcal{O}_{X_C}(1) \setminus \{0\} / \underline{\mathbb{Q}_p^\times}$

$= (\underline{\mathbb{Q}_p} \times BC(\mathcal{O}_{X_C}(1) \setminus \{0\}) / \underline{\mathbb{Q}_p^\times}$.

For any $K \subseteq G_b(\mathbb{Q}_p)$ open compact pro-p subgrp,

$f_{b,K} : [\widetilde{\mathcal{M}}_b / K] \longrightarrow \mathrm{Bun}_G$,

$$A_K^b := Rf_{b,K,!}(Rf_{b,K}^{!}(\Lambda)).$$

Thm $(A_K^b)_{K,b}$ form a set of compact generators of $D(\mathrm{Bun}_G, \Lambda)$.

§2 Story on Galois side

Def'n Let Λ be a $\mathbb{Z}_\ell$-alg. An L-parameter for G with coeff in Λ is a "condensed" 1-cocycle

$$\varphi : W_{\mathbb{Q}_p} \longrightarrow \hat{G}(\Lambda).$$

Here "condensed" means φ take values in $\hat{G}(L)$ for some finite $\mathbb{Z}_\ell$-subalg $L \subseteq \Lambda$, and is const.

Thm $\exists$ scheme $Z^1(W_{\mathbb{Q}_p}, \hat{G})/\mathbb{Z}_\ell$ parametrizing L-parameters.

It is a disjoint union of affine subschemes of finite type $/\mathbb{Z}_\ell$, flat and lci, of rel dim = dim G.

$$Z^1(W_{\mathbb{Q}_p}, \hat{G}) \circlearrowleft \hat{G}.$$

implying cat LLC, LLC, etc.

Main conjecture (refined ver of categorical LLC)

G quasi-split. Fix a Whittaker datum

$$G \supseteq B \supseteq \boxed{U} \leftarrow \text{unipotent radical of } B$$

$$\psi : U(\mathbb{Q}_p) \longrightarrow \mathcal{O}_L^\times, \quad [L : \mathbb{Q}_\ell^\times] < \infty.$$

Then $\exists$ a canonical equivalence of stable ∞-cat's

$$D(\mathrm{Bun}_G, \mathcal{O}_L)^{\boxed{\omega}} \xrightarrow{\sim} D^{b,qc}_{\mathrm{Coh,nilp}}(Z^1(W_{\mathbb{Q}_p}, \hat{G})_{\mathcal{O}_L}/\hat{G}).$$

Compact object $\qquad\qquad$ $\mathrm{Perf}(Z^1(W_{\mathbb{Q}_p}, \hat{G})/\hat{G})$

$$j_!^1(c\text{-}\mathrm{Ind}_{U(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\psi) \longleftarrow \mathcal{O} \qquad \text{perfect complexes}$$

where $j^1 : \mathrm{Bun}_G^1 \hookrightarrow \mathrm{Bun}_G$.

Main theorem (Fargues - Scholze)

Assume ℓ is "very good" for G

(e.g. if $G = GL_n$, any ℓ is very good;

if G is classical, $\ell > 2$ very good;

if G is split & semisimple, then $\ell > 5$ very good.)

Then $\exists$ canonical spectral action

$$\mathrm{Perf}(Z^1(W_{\mathbb{Q}_p}, \hat{G})/\hat{G}) \curvearrowright D(\mathrm{Bun}_G, \mathbb{Z}_\ell)^\omega$$

Rmk If work with $\mathbb{Q}_\ell$, then any ℓ will be allowed.

§3 From spectral action to L-parameters

Def'n (1) The spectral Berstein center

$$Z^{spec}(G, \mathbb{Z}_\ell) := \mathcal{O}(Z^1(W_{\mathbb{Q}_p}, \hat{G})/\hat{G})$$
$$= \mathcal{O}(Z^1(W_{\mathbb{Q}_p}, \hat{G}))^{\hat{G}}.$$

(2) $Z^{geom}(G, \mathbb{Z}_\ell) := \pi_0(\mathrm{End}(\mathrm{id}_{D(\mathrm{Bun}_G, \Lambda)}))$

(3) $Z(G(\mathbb{Q}_p), \mathbb{Z}_\ell) := \varprojlim_{\underbrace{K \subseteq G(\mathbb{Q}_p)}_{\text{open compact pro-}p\text{ subgrp}}} Z(\mathbb{Z}_\ell[K \backslash G(\mathbb{Q}_p)/K])$.

Cor $\exists$ canonical map

$$\begin{array}{ccccc} Z^{spec}(G, \mathbb{Z}_\ell) & \longrightarrow & Z^{geom}(G, \mathbb{Z}_\ell) & \longrightarrow & Z(G(\mathbb{Q}_p), \mathbb{Z}_\ell). \\ \| & & \| & & \\ \mathrm{End}(\mathcal{O}_{Z^1(W_{\mathbb{Q}_p}, \hat{G})/\hat{G}}) & \to & \pi_0(\mathrm{End}(\mathrm{id}_{D(\mathrm{Bun}_G, \mathbb{Z}_\ell)^\omega})) & & \end{array}$$

$$\begin{array}{ccccccc} \mathrm{Bun}_G^1 & \xhookrightarrow{j^1} & \mathrm{Bun}_G & \rightsquigarrow & D(\mathrm{Bun}_G, \mathbb{Z}_\ell)^\omega & \xrightarrow{j^{1*}} & D(\mathrm{Bun}_G^1, \mathbb{Z}_\ell)^\omega \\ \| & & & & & & \wr\| \\ [*/G(\mathbb{Q}_p)] & & & & & & D(G(\mathbb{Q}_p), \mathbb{Z}_\ell)^\omega. \end{array}$$

Def'n An L-parameter $\varphi \in Z^1(W_{\mathbb{Q}_p}, \hat{G})$ is <u>semi-simple</u>

if whenever φ factors through a parabolic $\hat{P} \subseteq \hat{G}$,

it then factors through a Levi $\hat{M} \subseteq \hat{P}$.

Remark For $\varphi \in Z^1(W_{\mathbb{Q}_p}, \hat{G})(\mathbb{Z}_\ell)$,

φ semi-simple $\Leftrightarrow$ φ is continuous for the adic top on $\hat{G}(\mathbb{Z}_\ell)$.

Cor (Construction of semi-simple L-parameters)

$K = \bar{\mathbb{F}}_\ell$ or $\bar{\mathbb{Q}}_\ell$, π irred smooth K-rep'n of $G(\mathbb{Q}_p)$.

$\Rightarrow$ $\exists$ a semi-simple

$\varphi_\pi : W_{\mathbb{Q}_p} \longrightarrow \hat{G}(K)$ up to conj by $\hat{G}$.

proof $Z^{\mathrm{Spec}} \longrightarrow Z(G(\mathbb{Q}_p), K) \longrightarrow \mathrm{End}_K(\pi) = K$.

$\varphi_\pi \in \mathrm{Spec}(Z^{\mathrm{Spec}})(K) = (Z^1(W_{\mathbb{Q}_p}, \hat{G}) /\!/ \hat{G})(K)$

categorical quotient. $\square$

§4 Construction of the spectral action

Fix a finite quotient Q of $W_{\mathbb{Q}_p}$ s.t.

the action of $W_{\mathbb{Q}_p}$ on $\hat{G}$ factors through this Q.

Thm Assume ℓ is "very good". Λ is a $\mathbb{Z}_\ell$-alg.

Let $\mathcal{C}$ be a small $\mathbb{Z}_\ell$-linear independent compact stable ∞-cat.

Then giving a "compactly supported" Λ-linear action

of $\mathrm{Perf}(Z^1(W_{\mathbb{Q}_p}, \hat{G})/\hat{G})$ on $\mathcal{C}$

$\Leftrightarrow$ giving for any finite set I,

an exact $\mathrm{Rep}_\Lambda(Q^I)$-linear monoidal functor

$\mathrm{Rep}^{\mathrm{fp}}_\Lambda(\hat{G} \rtimes Q)^I \longrightarrow \mathrm{End}_\Lambda(\mathcal{C})^{BW^I_{\mathbb{Q}_p}}$

finite presentation

that is functorial in I

i.e. $\forall V \in \mathrm{Rep}^{fp}_{\Lambda}(\hat{G} \rtimes Q)^I$,

$\exists T_V : \mathcal{C} \to \mathcal{C}^{BW^I_{\mathbb{Q}_p}} := \{X \in \mathcal{C}$, with action by $W^I_{\mathbb{Q}_p}$ on $X\}$,

satisfying certain compatibilities

(e.g. $T_{V\otimes W} = T_V \circ T_W$).

by Nadler-Yun (2019), Gaitsgory-Kazhdan-Rozenblyum-Varshavsky (≥ 2020).

Use this as a blackbox.

<u>Want</u> $\forall V \in \mathrm{Rep}^{f.p}_{\Lambda}(\hat{G} \rtimes Q)^I$, $\exists$ a functor

$$T_V : D(\mathrm{Bun}_G, \Lambda)^{\omega} \longrightarrow D(\mathrm{Bun}_G, \Lambda)^{\omega, BW^I_{\mathbb{Q}_p}}.$$

(Use geom Satake + Hecke correspondence.)

$$\mathrm{Div}^1 := \mathrm{Spd}(\breve{\mathbb{Q}}_p)/\varphi^{\mathbb{Z}} \ : \ S \mapsto \{S^{\#} : \text{untilt of } S\}/\varphi^{\mathbb{Z}}$$

$$= \{\text{degree 1 Cartier divisors on } X_S\}$$

$\rightsquigarrow$ $LG : S \longmapsto \{(S^{\#}, g) \mid S^{\#} \in \mathrm{Div}^1(S),\ g \in G(B_{dR}(S^{\#}))\}$.

$LG = L_{\mathrm{Div}^1}(G)$, $S = \mathrm{Spec}\, C$

$\rightsquigarrow$ $LG \supseteq L^+G : S \mapsto \{(S^{\#}, g) \mid S^{\#} \in \mathrm{Div}^1(S),\ g \in G(B^+_{dR}(S^{\#}))\}$.

<u>Define</u> $\mathrm{Gr}_G := LG/L^+G \longrightarrow \mathrm{Div}^1$

$\mathcal{H}\mathrm{k}_G := L^+G \backslash LG / L^+G = L^+G \backslash \mathrm{Gr}_G$.

Now for each finite set I, have $\mathcal{H}\mathrm{k}^I_G \longrightarrow (\mathrm{Div}^1)^I$.

<u>Upshot</u> $\exists$ a good notion of "perverse sheaves" on $\mathcal{H}\mathrm{k}^I_G$.

$$\rightsquigarrow \mathrm{Sat}^I_G(\Lambda) := \left\{ \begin{array}{c} \Lambda\text{-flat perverse sheaves on } \mathcal{H}\mathrm{k}^I_G \\ \underline{\text{ULA}} \text{ over } (\mathrm{Div}^1)^I \end{array} \right\}.$$

Thm If $\sqrt{p}\in\Lambda$ is fixed, then $\exists$ equivalence of cats

$$\begin{array}{ccc} \mathrm{Sat}_G^I(\Lambda) & \xrightarrow{\ \sim\ } & \mathrm{Rep}^{fp}_{\Lambda}(\hat{G}\rtimes W_{\mathbb{Q}_p})^I . \\ S_V & \longleftarrow\!\!\shortmid & V \end{array}$$

Construction of T_V:

$$\widetilde{HK}_G^I := \left\{ (\mathcal{E}_1, \mathcal{E}_2, (x_i)_{i\in I}, f) \ \middle|\ \begin{array}{l} \cdot\ \mathcal{E}_i \text{ } G\text{-bundles} \\ \cdot\ (x_i)_{i\in I} \in (\mathrm{Div}^1)^I \\ \cdot\ f: \mathcal{E}_1 \to \mathcal{E}_2 \text{ modifications at } (x_i)_{i\in I} \end{array} \right\}$$

$\rightsquigarrow$ $\exists$ obvious maps

$$\begin{tikzcd} & \widetilde{HK}_G^I \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times (\mathrm{Div}^1)^I . \end{tikzcd}$$

Define operator

$$\begin{tikzcd} T_V := Rh_{2,*}(h_1^*(-) \overset{\mathbb{L}}{\otimes}_{\Lambda} S_V) : D(\mathrm{Bun}_G, \Lambda) \arrow[r] \arrow[dr, dashed, "T_V"'] & D(\mathrm{Bun}_G \times (\mathrm{Div}^1)^I, \Lambda) \\ & D(\mathrm{Bun}_G, \Lambda)^{BW_{\mathbb{Q}_p}^I} . \arrow[u] \end{tikzcd}$$

$\rightsquigarrow$ $\pi_1(\mathrm{Div}^1) = \pi_1(\mathrm{Spd}\, \breve{\mathbb{Q}}_p / \varphi^{\mathbb{Z}}) = W_{\mathbb{Q}_p}$.

$\rightsquigarrow$ $\mathrm{Im}\, T_V$ lies in the subset of objects that are locally const in $(\mathrm{Div}^1)^I$-direction.

Thm T_V preserves compact objects and ULA sheaves.