LIE ALGEBRAS AND REPRESENTATION THEORY

NOTES BY WENHAN DAI

Abstract. These are prepared notes for a mini-course given by Jinpeng An at the invitation of Tianyuan Mathematical Center in Southwest China in February 2022. The course closely follows [Hum12] and [Car05]. We introduce the basics of finite-dimensional complex Lie algebras, with emphasis on the structure and classification of complex semisimple Lie algebras, and will also briefly discuss the basic properties of the representations.

Contents

1. Introduction 1
1.1. Basic Notions 1
1.2. The Main Classification Theorem of Simple Lie Algebras 4
2. Abelian, Nilpotent, and Solvable Lie Algebras 6
2.1. Ad-semisimple and Ad-nilpotent Elements 6
2.2. A Characterization of Abelian Lie Algebras 6
2.3. Engel's Theorem for Nilpotent Lie Algebras 8
2.4. Lie's Theorem for Linear Solvable Lie Algebras 9
3. Invariant Bilinear Forms and Applications 11
3.1. An Application of the Trace Form 11
3.2. Jordan Decomposition 12
3.3. Cartan's Criterions 13
3.4. Structure of Semisimple Lie Algebras 14
3.5. Abstract Jordan Decomposition 15
4. Root Spaces and Root Systems 17
4.1. Root Space Decompositions 17
4.2. Root Systems 23
4.3. Conjugacy of Cartan Subalgebras 24
4.4. Simple Lie Algebras and Irreducible Root Systems 27
5. Classification of Root Systems 29
6. Representations 33
6.1. Basic Notions 33
6.2. Weyl's Theorem on Complete Reducibility 34
6.3. Application of Weyl's Theorem: Jordan Decomposition 35
6.4. Representations of $\mathfrak{sl}_2(\mathbb{C})$ 37
References 38

1. Introduction

1.1. Basic Notions.

Definition 1.1 (Lie algebra)**.** Let L be a vector space over a field F. Suppose an operation (called **Lie bracket**)

$$L \times L \to L, \quad (x, y) \mapsto [x, y]$$

is given and satisfies

Date: January 17, 2022.

- (Bilinearity) for all $x, y, z \in L$ and $a, b \in F$,

$$\begin{cases} [ax + by, z] = a[x, z] + b[y, z], \\ [x, ay + bz] = a[x, y] + b[x, z]; \end{cases}$$

- (Alternativity) $[x, x] = 0$ for all $x \in L$;
- (Jacobi identity) for all $x, y, z \in L$,

$$[[x, y], z] + [[y, z], x] + [z, x], y] = 0.$$

Then L is called a **Lie algebra** over F.

The alternativity and bilinearity imply

- (Anticommutativity) for all $x, y \in L$,

$$[x, y] = -[y, x].$$

In fact, we see $0 = [x + y, x + y] = [x, x] + [y, y] + [x, y] + [y, x] = [x, y] + [y, x]$.

Remark 1.2. The motivation to define Lie algebras turns out to be "linearization" of Lie groups. Let G be a real Lie group, and $x, y \in T_eG$. Let $g, h : (-\varepsilon, \varepsilon) \to G$ be smooth curves such that

$$g(0) = h(0) = e, \quad g'(0) = x, \quad h'(0) = y.$$

Then

$$[x, y] := \left.\frac{\partial^2}{\partial s \partial t}\right|_{s=t=0} g(s)h(t)g(s)^{-1}h(t)^{-1}$$

is independent of the choices of the curves g, h, and defines a Lie bracket on T_eG.

Example 1.3 (Abelian Lie algebra)**.** On any F-vector space L, one can define a trivial Lie bracket by

$$[x, y] = 0, \quad \forall x, y \in L.$$

Then L becomes a Lie algebra, called an **abelian Lie algebra**.

Example 1.4 (General linear Lie algebra)**.**
(1) Let $\mathfrak{gl}_n(F)$ be the space of all $n \times n$ matrices over F, and define

$$[x, y] = xy - yx, \quad \forall x, y \in \mathfrak{gl}_n(F).$$

Then $\mathfrak{gl}_n(F)$ becomes a Lie algebra.
(2) Let V be a finite-dimensional F-vector space, and $\mathfrak{gl}_n(V)$ be the space of all linear maps $V \to V$. Define

$$[x, y] = xy - yx, \quad \forall x, y \in \mathfrak{gl}(V).$$

Then $\mathfrak{gl}(V)$ becomes a Lie algebra.

Both $\mathfrak{gl}_n(F)$ and $\mathfrak{gl}(V)$ are called **general linear Lie algebras**.

Definition 1.5 (Homomorphism, isomorphism)**.** Let L and L' be Lie algebras over F.

(1) A linear map $\phi : L \to L'$ is called a **homomorphism** if

$$\phi([x, y]) = [\phi(x), \phi(y)], \quad \forall x, y \in L.$$

(2) A homomorphism $\phi : L \to L'$ is called an **isomorphism** if it is bijective.
(3) L and L' are said to be **isomorphic** if there exists an isomorphism $L \to L'$, denoted $L \cong L'$.

Naively, isomorphic Lie algebras can be identified in the most sense.

Example 1.6. If $\dim_F V = n$, then $\mathfrak{gl}(V) \cong \mathfrak{gl}_n(F)$.

Definition 1.7 (Representation)**.** Let L be a Lie algebra over F. A **representation** of L is a homomorphism $\phi : L \to \mathfrak{gl}(V)$, where V is some finite-dimensional F-vector space.

Example 1.8 (Adjoint representation)**.** Let L be a Lie algebra over F. Define a linear map $\mathrm{ad} : L \to \mathfrak{gl}(L)$ by

$$\mathrm{ad}(x)(y) = [x, y], \quad \forall x, y \in L.$$

We claim that it is a representation, called the **adjoint representation** of L. In fact, it follows from the Jacobi identity that for any $x, y, z \in L$,

$$\begin{aligned}\operatorname{ad}([x,y])(z) &= [[x,y],z] \\ &= [x,[y,z]] - [y,[x,z]] \\ &= (\operatorname{ad}(x)\operatorname{ad}(y) - \operatorname{ad}(y)\operatorname{ad}(x))(z) \\ &= [\operatorname{ad}(x),\operatorname{ad}(y)](z).\end{aligned}$$

Thus, for any $x, y \in L$,

$$\operatorname{ad}([x,y]) = [\operatorname{ad}(x),\operatorname{ad}(y)].$$

Namely, the linear representation ad commutes with the Lie bracket.

Definition 1.9 (Subalgebra, ideal, quotient algebra)**.** Let L be a Lie algebra over F.

(1) If $S, T \subset L$ are subspaces, write

$$[S,T] := \operatorname{Span}\{[x,y] : x \in S, y \in T\}.$$

(2) A subspace $K \subset L$ is a **subalgebra** if $[K,K] \subset K$, denoted $K < L$.
(3) A subspace $I \subset L$ is an **ideal** if $[I,L] \subset I$, denoted $I \lhd L$.
(4) Let $I \lhd L$. On the quotient space L/I, we introduce the Lie bracket

$$[x+I, y+I] := [x,y] + I, \quad \forall x, y \in L.$$

Then L/I becomes a Lie algebra, called the **quotient algebra** of L by I.

Example 1.10. (1) Let $\phi : L \to L'$ be a homomorphism. Then

$$\operatorname{Ker}(\phi) \lhd L, \quad \operatorname{im}(\phi) < L, \quad \operatorname{im}(\phi) \cong L/\operatorname{Ker}(\phi).$$

(2) The **center** of L is defined as

$$Z(L) := \{x \in L : [x,y] = 0, \forall y \in L\}.$$

We have $Z(L) = \operatorname{Ker}(\operatorname{ad})$. So $Z(L) \lhd L$, and $L/Z(L) \cong \operatorname{ad}(L)$.

Definition 1.11 (Direct sum)**.** Let $L_1, \dots, L_r$ be Lie algebras over F. On the (external) vector space direct sum $L_1 \oplus \cdots \oplus L_r$, we introduce the Lie bracket

$$[(x_1,\dots,x_r),(y_1,\dots,y_r)] = ([x_1,y_1],\dots,[x_r,y_r]), \quad \forall x_k, y_k \in L_k,\ 1 \leqslant k \leqslant r.$$

This makes $L_1, \dots, L_r$ a Lie algebra, called the **(external) Lie algebra direct sum** of $L_1, \dots, L_r$.

We always make the natural identification

$$L_k \cong \{(x_1,\dots,x_r) : x_j = 0, \forall j \neq k\}.$$

Then each L_k is an ideal of $L_1 \oplus \cdots \oplus L_r$.

Remark 1.12. (1) If a Lie algebra L is the internal vector space direct sum of ideals $I_1, \dots, I_r$, then L is isomorphic to external Lie algebra direct sum $I_1 \oplus \cdots \oplus I_r$.
(2) But this is not true if some I_k is only a subalgebra that is not an ideal.

Definition 1.13 (Linear Lie algebra)**.** Subalgebras of $\mathfrak{gl}_n(F)$ and $\mathfrak{gl}(V)$ are called **linear Lie algebra**.

We obtain the following deep result.

Theorem 1.14 (Ado-Iwasawa)**.** *All finite-dimensional Lie algebras over F are isomorphic to linear Lie algebras.*

Here comes some important type of linear Lie algebras.

Example 1.15 (Special linear Lie algebra)**.** Set

$$\begin{aligned}\mathfrak{sl}_n(F) &= \{x \in \mathfrak{gl}_n(F) : \operatorname{tr}(x) = 0\}, \\ \mathfrak{sl}(V) &= \{x \in \mathfrak{gl}(V) : \operatorname{tr}(x) = 0\},\end{aligned}$$

where V is a finite-dimensional F-vector space. Then

$$\mathfrak{sl}_n(F) \lhd \mathfrak{gl}_n(F), \quad \mathfrak{sl}(V) \lhd \mathfrak{gl}(V).$$

Example 1.16 (The Lie algebra $L(V,f)$)**.** Let V be a finite-dimensional F-vector space, and $f: V\times V\to F$ be a bilinear form. For $x\in\mathfrak{gl}(V)$, we say that f is **invariant under x (in the infinitesimal sense)** if

$$f(xv,w)+f(v,xw)=0,\quad \forall v,w\in V.$$

Let $L(V,f)\subset\mathfrak{gl}(V)$ be the subspace of all $x\in\mathfrak{gl}(V)$ that leave f invariant, namely

$$L(V,f)=\{x\in\mathfrak{gl}(V): f(xv,w)+f(v,xw)=0,\forall v,w\in V\}.$$

We claim that $L(V,f)<\mathfrak{gl}(V)$. In fact, if $x,y\in L(V,f)$, then for any $v,w\in V$,

$$\begin{aligned}f([x,y]v,w)+f(v,[x,y]w)&=f(xyv,w)-f(yxv,w)+f(v,xyw)-f(v,yxw)\\&=-f(yv,xw)+f(xv,yw)-f(xv,yw)+f(yv,xw)\\&=0.\end{aligned}$$

This implies $[x,y]\in L(V,f)$.

Remark 1.17 (Meaning of "invariance in the infinitesimal sense"). Suppose $F=\mathbb{R}$ or $\mathbb{C}$, and $g(t):V\to V$ (with $-\varepsilon<t<\varepsilon$) is a smooth curve of linear maps with $g(0)=\mathrm{id}$ and $g'(0)=x$, such that

$$f(g(t)v,g(t)w)=f(v,w)$$

for any $v,w\in V$ and $t\in(-\varepsilon,\varepsilon)$. Then taking $\frac{d}{dt}|_{t=0}$ attains

$$f(g'(0)v,g(0)w)+f(g(0)v,g'(0)w)=f(xv,w)+f(v,xw)=0.$$

Example 1.18 (Orthogonal and symplectic Lie algebras)**.** Let us consider two special cases of $L(V,f)$.

(1) Let $V=F^n$ (the space of column vectors), and f be the symmetric from given by

$$f(v,w)=v^tw,\quad\forall v,w\in F^n.$$

Then $\mathfrak{o}_n(F):=L(F^n,f)$ is called the **orthogonal Lie algebra**. Under the identification $\mathfrak{gl}(F^n)\cong\mathfrak{gl}_n(F)$, we have

$$\begin{aligned}\mathfrak{o}_n(F)&=\{x\in\mathfrak{gl}_n(F):(xv)^tw+v^txw=0,\forall v,w\in F^n\}\\&=\{x\in\mathfrak{gl}_n(F):x^t+x=0\}.\end{aligned}$$

(2) Let $V=F^{2n}$, and f be the symplectic form given by

$$f(v,w)=v^t\begin{pmatrix}0&I_n\\-I_n&0\end{pmatrix}w,\quad\forall v,w\in F^{2n}.$$

Then $\mathfrak{sp}_{2n}(F):=L(F^{2n},f)$ is called the **symplectic Lie algebra**. Under the identification $\mathfrak{gl}(F^{2n})\cong\mathfrak{gl}_{2n}(F)$, we have

$$\begin{aligned}\mathfrak{sp}_{2n}(F)&=\left\{x\in\mathfrak{gl}_{2n}(F):(xv)^t\begin{pmatrix}0&I_n\\-I_n&0\end{pmatrix}w+v^t\begin{pmatrix}0&I_n\\-I_n&0\end{pmatrix}xw=0\right\}\\&=\left\{x\in\mathfrak{gl}_{2n}(F):x^t\begin{pmatrix}0&I_n\\-I_n&0\end{pmatrix}+\begin{pmatrix}0&I_n\\-I_n&0\end{pmatrix}x=0\right\}\\&=\left\{\begin{pmatrix}x&y\\z&-x^t\end{pmatrix}:x,y,z\in\mathfrak{gl}_n(F),y^t=y,z^t=z\right\}.\end{aligned}$$

1.2. **The Main Classification Theorem of Simple Lie Algebras.** Suppose $I\lhd L$. In the roughest sense, the information of L is implied by I and L/I. This motivates the following.

Definition 1.19 (Simple Lie algebra, semisimple Lie algebra)**.** Let L be a finite-dimensional Lie algebra over F.

(1) L is **simple** if it is nonabelian and has no nontrivial ideals.
(2) L is **semisimple** if it is nonzero and has no nonzero abelian ideals.

Clearly, a simple Lie algebra is semisimple. One of our main purposes is to explain the proof of the following classification theorem.

Theorem 1.20 (Main theorem, the classification of complex simple Lie algebras)**.** *Let L be a finite-dimensional Lie algebra over $\mathbb{C}$.*

(1) *L is semisimple if and only if it is isomorphic to the direct sum of finitely many simple Lie algebras.*

(2) *L is simple if and only if it is isomorphic to one of the following Lie algebras:*
- ⋄ $\mathfrak{sl}_n(\mathbb{C})$, *$n \geqslant 2$;*
- ⋄ $\mathfrak{o}_n(\mathbb{C})$, *$n \geqslant 7$;*
- ⋄ $\mathfrak{sp}_{2n}(\mathbb{C})$, *$n \geqslant 2$;*
- ⋄ *one of the 5 exceptional complex simple Lie algebras, denoted by $\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2$, respectively.*

Remark 1.21. In the classification of simple Lie algebras, the condition $n \geqslant 7$ for $\mathfrak{o}_n(\mathbb{C})$ is deduced from the following fact. It can be shown that

$$\begin{aligned}
&\mathfrak{o}_2(\mathbb{C}) \cong \mathbb{C}, \\
&\mathfrak{o}_3(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sp}_2(\mathbb{C}), \\
&\mathfrak{o}_4(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}), \\
&\mathfrak{o}_5(\mathbb{C}) \cong \mathfrak{sp}_4(\mathbb{C}), \\
&\mathfrak{o}_6(\mathbb{C}) \cong \mathfrak{sl}_4(\mathbb{C}).
\end{aligned}$$

2. Abelian, Nilpotent, and Solvable Lie Algebras

From now on, we will only consider finite-dimensional complex Lie algebras.

Notation 2.1. Let us make the following conventions:

- L always denotes a finite-dimensional complex Lie algebra,
- V always denotes a nonzero finite-dimensional complex vector space.

2.1. Ad-semisimple and Ad-nilpotent Elements.

Recall that for $x \in \mathfrak{gl}(V)$,

- x is said to be **semisimple** if it is diagonalizable;
- x is said to be **nilpotent** if $x^r = 0$ for some $r \geqslant 1$.

Definition 2.2 (Ad-semisimple and ad-nilpotent elements)**.** Let L be a (finite-dimensional complex) Lie algebra. We say that

1. $x \in L$ is **ad-semisimple** if $\operatorname{ad}(x) \in \mathfrak{gl}(L)$ is semisimple;
2. $x \in L$ is **ad-nilpotent** if $\operatorname{ad}(x) \in \mathfrak{gl}(L)$ is nilpotent.

Proposition 2.3. *Let $L < \mathfrak{gl}(V)$, $x \in L$.*

1. *If x is semisimple, then it is ad-semisimple.*
2. *If x is nilpotent, then it is ad-nilpotent.*

Proof. Consider $T : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$, $y \mapsto xy - yx$. Then $\operatorname{ad}(x) = T|_L$. It suffices to prove:

- x is semisimple $\Longrightarrow$ T is semisimple;
- x is nilpotent $\Longrightarrow$ T is nilpotent.

(1) Suppose x is semisimple. Let $\mathcal{B}$ be a basis of V such that $[x]_{\mathcal{B}} = \operatorname{diag}(a_1, \ldots, a_n)$. Let $e_{ij} \in \mathfrak{gl}(V)$ be such that the (i,j)-entry of $[e_{ij}]_{\mathcal{B}}$ is 1 and all other entires are 0. Then $\{e_{ij}\}$ is a basis of $\mathfrak{gl}(V)$. Since

$$[Te_{ij}]_{\mathcal{B}} = [xe_{ij} - e_{ij}x]_{\mathcal{B}} = [x]_{\mathcal{B}}[e_{ij}]_{\mathcal{B}} - [e_{ij}]_{\mathcal{B}}[x]_{\mathcal{B}} = (a_i - a_j)[e_{ij}]_{\mathcal{B}},$$

we have $Te_{ij} = (a_i - a_j)e_{ij}$. So T is semisimple.
(2) Suppose x is nilpotent. Define $T_1, T_2 : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ as $T_1(y) = xy$, $T_2(y) = yx$. Then $T = T_1 - T_2$ and $T_1T_2 = T_2T_1$. The nilpotency of x implies that T_1 and T_2 are nilpotent. So also is T. $\square$

Remark 2.4. If $L < \mathfrak{gl}(V)$ is semisimple, then the converse of Proposition 2.3 also holds.

2.2. A Characterization of Abelian Lie Algebras.

Theorem 2.5. *A Lie algebra L is abelian if and only if it consists of ad-semisimple elements.*

Proof. $\Longrightarrow$: Suppose L is abelian. Then for every $x \in L$, we have $\operatorname{ad}(x) = 0$, so x is ad-semisimple.

$\Longleftarrow$: Suppose L consists of ad-semisimple elements. To prove L is abelian, it suffices to prove $\operatorname{ad}(x) = 0$ for every $x \in L$. Since $\operatorname{ad}(x)$ is semisimple, it suffices to prove the only eigenvalue of $\operatorname{ad}(x)$ is 0. Let a be an eigenvalue of $\operatorname{ad}(x)$. Let $y \in L\backslash\{0\}$ be such that

$$\operatorname{ad}(x)(y) = ay.$$

Then

$$\operatorname{ad}(y)(x) = -ay \quad \Longrightarrow \quad \operatorname{ad}(y)^2(x) = 0.$$

Since $\operatorname{ad}(y)$ is semisimple, this implies

$$\operatorname{ad}(y)(x) = 0 \quad \Longrightarrow \quad a = 0.$$

$\square$

For a Lie algebra L, we define two sequences of ideals

$$L = L^0 \supset L^1 \supset \cdots, \quad L = L^{(0)} \supset L^{(1)} \supset \cdots$$

by

$$L^0 = L^{(0)} = L, \quad L^k = [L, L^{k-1}], \quad L^{(k)} = [L^{(k-1)}, L^{(k-1)}], \quad k \geqslant 1.$$

Definition 2.6 (Nilpotent and solvable Lie algebras)**.** Keep the notations as above.

1. L is said to be **nilpotent** if $L^k = 0$ for some k.
2. L is said to be **solvable** if $L^{(k)} = 0$ for some k.

The definition immediately renders two observations.

- Note that $L^1 = L^{(1)} = [L, L]$. So
$$L \text{ is abelian} \quad \Longrightarrow \quad L \text{ is nilpotent.}$$
- It is easy to see that $L^k \supset L^{(k)}$ for every k. So
$$L \text{ is nilpotent} \quad \Longrightarrow \quad L \text{ is solvable.}$$

Example 2.7. We define
$$\begin{aligned}\mathfrak{b}_n(\mathbb{C}) &:= \{\text{upper triangular matrices in } \mathfrak{gl}_n(\mathbb{C})\},\\ \mathfrak{n}_n(\mathbb{C}) &:= \{\text{strictly upper triangular matrices in } \mathfrak{gl}_n(\mathbb{C})\}.\end{aligned}$$
It is easy to see that they are subalgebras of $\mathfrak{gl}_n(\mathbb{C})$. We claim that
- ◇ $\mathfrak{n}_n(\mathbb{C})$ is nilpotent;
- ◇ $\mathfrak{b}_n(\mathbb{C})$ is solvable, but is not nilpotent if $n \geqslant 2$.

In fact, the claims are verified as follows.

(1) It is easy to verify: if $x \in \mathfrak{n}_n(\mathbb{C})^k$, then
$$j \leqslant i + k \quad \Longrightarrow \quad \text{the } (i,j)\text{-entry of } x \text{ is } 0.$$
So $\mathfrak{n}_n(\mathbb{C})^{n-1} = 0$. Thus $\mathfrak{n}_n(\mathbb{C})$ is nilpotent.

(2) We have
$$\mathfrak{b}_n(\mathbb{C}) \subset \mathfrak{n}_n(\mathbb{C}) \quad \Longrightarrow \quad \mathfrak{b}_n(\mathbb{C})^{(k+1)} \subset \mathfrak{n}_n(\mathbb{C})^{(k)} \subset \mathfrak{n}_n(\mathbb{C})^k.$$
It follows that $\mathfrak{b}_n(\mathbb{C})^{(n)} = 0$. So $\mathfrak{b}_n(\mathbb{C})$ is solvable.

(3) Note that in $\mathfrak{b}_2(\mathbb{C})$,
$$\left[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right] = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \in \mathfrak{b}_2(\mathbb{C})^k, \ \forall k \geqslant 0.$$
So $\mathfrak{b}_2(\mathbb{C})$ is not nilpotent.

(4) For $n \geqslant 2$, $\mathfrak{b}_n(\mathbb{C})$ has a subalgebra which is isomorphic to $\mathfrak{b}_2(\mathbb{C})$.

Proposition 2.8. *If L is nilpotent (resp. solvable), then so are its subalgebras and quotient algebras.*

Proof. Let $K < L$. Then
$$K^k \subset L^k, \quad K^{(k)} \subset L^{(k)}$$
for all k. Hence L is nilpotent (resp. solvable) implies that K is nilpotent (resp. solvable).

Again, let $I \lhd L$. Then
$$(L/I)^k = (L^k + I)/I, \quad (L/I)^{(k)} = (L^{(k)} + I)/I$$
for all k. Hence L is nilpotent (resp. solvable) implies that L/I is nilpotent (resp. solvable). □

Proposition 2.9. *Let L be a nonzero Lie algebra. Then the following statements are equivalent.*

(1) *L is semisimple, namely, it has no nonzero abelian ideals;*
(2) *L has no nonzero nilpotent ideals;*
(3) *L has no nonzero solvable ideals.*

Proof. (2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (2) are obvious. As for (1) $\Longrightarrow$ (3), suppose (3) is not true, i.e., L has a nonzero solvable ideal I. Let $k \geqslant 0$ be the largest integer such that $I^{(k)} \neq 0$. Then $I^{(k)}$ is a nonzero abelian ideal of L, contradicting to (1). □

Remark 2.10. It can be proved that if I, J are nilpotent ideals (resp. solvable ideals) of a Lie algebra L, then so is $I + J$. This implies:

- L has a unique maximal nilpotent ideal, called the **nilradical** of L, denoted by $\mathrm{Nil}(L)$;
- L has a unique maximal solvable ideal, called the **radical** of L, denoted by $\mathrm{Rad}(L)$.

Clearly,
$$\mathrm{Nil}(L) \subset \mathrm{Rad}(L).$$
It turns out that $\mathrm{Rad}(L)$ is more important.

(1) The quotient algebra $L/\mathrm{Rad}(L)$ is always semisimple.
(2) (Levi's Decomposition Theorem) There exists a semisimple subalgebra S of L such that $S \cap \mathrm{Rad}(L) = 0$ and $L = S + \mathrm{Rad}(L)$.
(3) We must have $S \cong L/\mathrm{Rad}(L)$. Such S is called a **Levi subalgebra** of L.

2.3. Engel's Theorem for Nilpotent Lie Algebras.

Theorem 2.11 (Engel). *Let $L < \mathfrak{gl}(V)$ be a linear Lie algebra consisting of nilpotent transformations. Then the following statements hold:*

1. *There exists $v \in V\backslash\{0\}$ such that $Lv = 0$.*
2. *V has a basis such that the matrices of all $x \in L$ are strictly upper triangular. In particular, L is a nilpotent Lie algebra.*

Proof. The proof is divided into 3 steps.

(I) We first assume (1) and prove (2) by induction on $\dim V$. The case where $\dim V = 1$ is trivial. Suppose $\dim V = n \geqslant 2$ and (2) holds for spaces of dimension $n-1$. By (1), we can choose $v_1 \in V\backslash\{0\}$ such that $Lv_1 = 0$. Consider the representation

$$\phi : L \to \mathfrak{gl}(V/\mathbb{C}v_1), \quad \phi(x)(v + \mathbb{C}v_1) = xv + \mathbb{C}v_1.$$

Then $\phi(L) < \mathfrak{gl}(V/\mathbb{C}v_1)$ consists of nilpotent transformations. By the induction hypothesis, $V/\mathbb{C}v_1$ has a basis $\mathcal{B} = \{v_2 + \mathbb{C}v_1, \ldots, v_n + \mathbb{C}v_1\}$ such that for every $x \in L$, the matrix $[\phi(x)]_{\mathcal{B}}$ is strictly upper triangular. For the basis $\{v_1, \ldots, v_n\}$ of V, the matrix of $x \in L$ has the form

$$\begin{pmatrix} 0 & * \\ 0 & [\phi(x)]_{\mathcal{B}} \end{pmatrix},$$

which is strictly upper triangular. This proves (1) $\Longrightarrow$ (2).

(II) It remains to prove (1), namely

$$\forall x \in L < \mathfrak{gl}(V),\ x \text{ is nilpotent} \quad \Longrightarrow \quad \exists v \in V\backslash\{0\} \text{ such that } Lv = 0.$$

We proceed by induction on $\dim L$. The case where $\dim L = 1$ is trivial. Suppose $\dim L \geqslant 2$ and (1) holds for Lie algebras of smaller dimensions. Let $K < L$ be a maximal proper subalgebra. We first prove

$$(*) \qquad \exists y \in L\backslash K \text{ such that } L = K + \mathbb{C}y \text{ and } [K, y] \subset K;$$

namely, K is a codimension-one ideal. Consider the representation

$$\psi : K \to \mathfrak{gl}(L/K), \quad \psi(x)(y + K) = [x, y] + K.$$

For all $x \in K$, $\psi(x) : L/K \to L/K$ is induced from $\operatorname{ad}(x) : L \to L$. Note that

$$x \text{ is nilpotent} \quad \Longrightarrow \quad \operatorname{ad}(x) \text{ is nilpotent} \quad \Longrightarrow \quad \psi(x) \text{ is nilpotent.}$$

So $\psi(K) < \mathfrak{gl}(L/K)$ consists of nilpotent transformations. By induction hypothesis, there exists $y \in L\backslash K$ such that $\psi(K)(y + K) = K$, i.e., $[K, y] \subset K$. This implies $K + \mathbb{C}y < L$. As K is maximal, one deduces that $L = K + \mathbb{C}y$. This proves $(*)$.

(III) We set

$$W = \{w \in V : Kw = 0\}.$$

From the induction hypothesis, we see $W \neq 0$. The claim is that $yW \subset W$. To verify this, say for all $w \in W$, we are to show $yw \in W$. Yet $K(yw) = 0$ is given by

$$x \in K \quad \Longrightarrow \quad [x, y] \in K \quad \Longrightarrow \quad x(yw) = y(xw) + [x, y]w = 0.$$

On the other hand, $y|_W$ is nilpotent implies that there is $v \in W\backslash\{0\}$ such that $yv = 0$. Thus $L = K + \mathbb{C}y$ leads to $Lv = 0$.

These complete the proof. □

The following theorem is parallel to Theorem 2.5.

Theorem 2.12 (Engel). *A Lie algebra L is nilpotent if and only if it consists of ad-nilpotent elements.*

Proof. $\Longrightarrow$: Suppose L is nilpotent. Let $k \geqslant 1$ be such that $L^k = 0$. For all $x \in L$,

$$[L, L^{\ell-1}] = L^{\ell}, \quad \Longrightarrow \quad \operatorname{ad}(x)(L^{\ell-1}) \subset L^{\ell} \quad (1 \leqslant \ell \leqslant k).$$

So

$$\operatorname{ad}(x)^k(L) = \operatorname{ad}(x)^k(L^0) \subset \operatorname{ad}(x)^{k-1}(L^1) \subset \cdots \subset L^k = 0.$$

Thus $\operatorname{ad}(x)^k = 0$, hence x is ad-nilpotent.

$\Longleftarrow$: Suppose L consists of ad-nilpotent elements. The above Engle's Theorem 2.11 implies that $\operatorname{ad}(L) < \mathfrak{gl}(L)$ is nilpotent. Also, $L/Z(L) \cong \operatorname{ad}(L)$, which is nilpotent as well. Let $m \geqslant 0$ be such that

$(L/Z(L))^m = (L^m + Z(L))/Z(L) = 0$. Then $L^m \subset Z(L)$. This implies $L^{m+1} = [L, L^m] \subset [L, Z(L)] = 0$. So L is nilpotent. □

2.4. **Lie's Theorem for Linear Solvable Lie Algebras.**

Theorem 2.13 (Lie's Theorem)**.** *Let $L < \mathfrak{gl}(V)$ be a solvable linear Lie algebra. Then the following statements hold:*

1. *L has a common eigenvector, i.e., there exists $v \in V\backslash\{0\}$ such that $Lv \subset \mathbb{C}v$.*
2. *V has a basis such that the matrices of all $x \in L$ are upper triangular.*

Proof. We first claim that (1) and (2) are equivalent. The (1) $\Longrightarrow$ (2) direction is similar to the case of Engel's Theorem 2.11. And the converse direction is obvious. It remains to prove (1) by induction on $\dim L$.

The case where $\dim L = 1$ is trivial[1]. Suppose $\dim L \geqslant 2$ and (1) holds for Lie algebras of smaller dimensions. The condition that L is solvable naively implies that $[L, L] \neq L$ by definition. Let $K \supset [L, L]$ be a codimension-one subspace of L, and let $y \in L\backslash K$. Then $K \triangleleft L$ and $L = K + \mathbb{C}y$. By the induction hypothesis, there exists some $w \in V\backslash\{0\}$ such that $Kw \subset \mathbb{C}w$. This namely means that for all $x \in K$, there is $\lambda(x) \in \mathbb{C}$ such that $xw = \lambda(x)w$. Thus we obtain a linear function $\lambda : K \to \mathbb{C}$.

In the upcoming context, we will prove

$$\lambda([x, y]) = 0, \quad \forall x \in K. \tag{$*$}$$

First, we assume the truth of $(*)$ and proceed the proof of (1). Consider the weight space

$$V_\lambda = \{v \in V : xv = \lambda(x)v, \forall x \in K\}.$$

Note that $w \in V_\lambda$ and $V_\lambda \neq 0$. We claim

$$yV_\lambda \subset V_\lambda.$$

To verify this with assuming $(*)$, say for all $v \in V_\lambda$, it suffices to notice

$$\begin{aligned} x \in K \quad &\Longrightarrow \quad x(yv) = y(xv) + [x, y]v = \lambda(x)yv + \lambda([x, y])v \overset{(*)}{=} \lambda(x)yv \\ &\Longrightarrow \quad yv \in V_\lambda. \end{aligned}$$

Therefore, any eigenvector of $y|_{V_\lambda}$ is a common eigenvector of L. This proves (1).

Now it remains to prove $(*)$. Denote

$$W_0 = 0, \quad W_k = \operatorname{Span}\{w, yw, \dots, y^{k-1}w\} \text{ for } 1 \leqslant k \leqslant m.$$

Then $yW_m \subset W_m$. We prove that for any $k \in \{0, 1, \dots, m-1\}$,

$$xy^k w \in \lambda(x)y^k w + W_{k\prime} \quad \forall x \in K. \tag{$**$}$$

When $k = 0$, this means $xw \in \lambda(x)w + W_0$, which is obvious. Suppose $1 \leqslant k \leqslant m-1$ and $(**)$ holds for $k-1$. Then for every $x \in K$, we have

$$\begin{aligned} xy^k w &= y(xy^{k-1}w) + [x, y]y^{k-1}w \\ &\in y(\lambda(x)y^{k-1}w + W_{k-1}) + (\lambda([x, y])y^{k-1}w + W_{k-1}) \\ &= \lambda(x)y^k w + yW_{k-1} + \lambda([x, y])y^{k-1}w + W_{k-1} \\ &\subset \lambda(x)y^k w + W_k. \end{aligned}$$

This proves $(**)$. It follows from $(**)$ that for any $x \in K$, we have $xW_m \subset W_m$, and the matrix of $x|_{W_m}$ for the basis $\{w, yw, \dots, y^{m-1}w\}$ is upper triangular with diagonal entries $\lambda(x)$. So $\operatorname{tr}(x|_m) = m\lambda(x)$. Therefore, for every $x \in K$, we have

$$m\lambda([x, y]) = \operatorname{tr}([x, y]|_{W_m}) = \operatorname{tr}([x|_{W_m}, y|_{W_m}]) = 0.$$

This proves $(*)$. □

Corollary 2.14. *A Lie algebra L is solvable if and only if $[L, L]$ is nilpotent.*

[1]Do remember that we are working over $\mathbb{C}$; the same statement fails to be true on non-algebraically closed fields.

Proof. $\Longleftarrow$: Suppose $[L, L]$ is nilpotent. Then it is solvable as well. Also,

$$L^{(k+1)} = [L, L]^{(k)},\ \forall k \geqslant 0 \quad \Longrightarrow \quad \exists k \text{ such that } L^{(k+1)} = 0.$$

So L is solvable.

$\Longrightarrow$: Suppose L is solvable. Then $\mathrm{ad}(L) < \mathfrak{gl}(L)$ is solvable by Proposition 2.3. Apply Lie'e Theorem 2.13 to $\mathrm{ad}(L)$, we see L has a basis such that the matrix of any $T \in \mathrm{ad}(L)$ is upper triangular. Therefore, the matrix of any T in $\mathrm{ad}_L([L, L]) = [\mathrm{ad}(L), \mathrm{ad}(L)]$ is strictly upper triangular. And consequently, for any $x \in [L, L]$ which is ad_L-nilpotent, it must be $\mathrm{ad}_{[L,L]}$-nilpotent. Finally, by Engel's Theorem 2.12, $[L, L]$ is nilpotent. $\square$

3. Invariant Bilinear Forms and Applications

Caution. In what follows, we will only consider *symmetric* bilinear forms on L.

Recall that a bilinear form f on V is said to be invariant under $x \in \mathfrak{gl}(V)$ (in the infinitesimal sense) if

$$f(xv, w) + f(v, xw) = 0, \quad \forall v, w \in V.$$

Definition 3.1. Let L be a Lie algebra. A bilinear form f on L is said to be **invariant** if it is invariant under every $\mathrm{ad}(x)$ (in the infinitesimal sense), namely,

$$f([x, y], z) + f(y, [x, z]) = 0, \quad \forall x, y, z \in L.$$

Note that the definition is equivalent to say

$$f([x, y], z) = f(x, [y, z]), \quad \forall x, y, z \in L.$$

So invariant bilinear forms are also called **associative**.

Proposition 3.2. *Let f be a symmetric invariant bilinear form on L, and let $I \triangleleft L$. Then*

$$I^{\perp} := \{x \in L : f(x, y) = 0, \forall y \in I\}$$

is an ideal of L.

Proof. Let $x \in I^{\perp}$, $y \in L$. To verify $[x, y] \in I^{\perp}$, it suffices to notice:

$$\forall z \in I \quad \Longrightarrow \quad f([x, y], z) = f(x, [y, z]) = 0.$$

□

Remark 3.3. We call $I^{\perp}$ the **orthogonal ideal** of I relative to f. If f is nondegenerate, then

$$\dim I + \dim I^{\perp} = \dim L.$$

However, even in this case, it may happen that $I \cap I^{\perp} \neq 0$ and $I + I^{\perp} \neq L$.

Example 3.4 (Trace Form)**.** Suppose $L < \mathfrak{gl}(V)$. The symmetric bilinear form

$$\tau : L \times L \to \mathbb{C}, \quad \tau(x, y) = \mathrm{tr}(xy)$$

is called the **trace form** of L. It is invariant: for all $x, y, z \in L$, we have

$$\tau([x, y], z) + \tau(y, [x, z]) = \mathrm{tr}([x, y]z) + \mathrm{tr}(y[x, z]) = \mathrm{tr}(xyz - yxz + yxz - yzx) = 0.$$

Example 3.5 (Killing Form)**.** For a general L, we can compose a representation $\phi : L \to \mathfrak{gl}(V)$ with the trace form τ on $\mathfrak{gl}(V)$. Let

$$f_{\phi} : L \times L \to \mathbb{C}, \quad f_{\phi}(x, y) = \tau(\phi(x), \phi(y)) = \mathrm{tr}(\phi(x)\phi(y)).$$

Note that f_{ϕ} is invariant. For $x, y, z \in L$, we have

$$\begin{aligned} f_{\phi}([x, y], z) + f_{\phi}(y, [x, z]) &= \tau(\phi([x, y]), \phi(z)) + \tau(\phi(y), \phi([x, z])) \\ &= \tau([\phi(x), \phi(y)], \phi(z)) + \tau(\phi(y), [\phi(x), \phi(z)]) \\ &= 0. \end{aligned}$$

When $\phi = \mathrm{ad}$, we call $\kappa := f_{\mathrm{ad}}$ the **Killing form** of L, namely,

$$\kappa(x, y) := \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)), \quad \forall x, y \in L.$$

3.1. **An Application of the Trace Form.**

For $x \in \mathfrak{gl}_n(\mathbb{C})$, denote $x^* = (\overline{x})^t$, the transposition of complex conjugacy.

Proposition 3.6. *Suppose $L < \mathfrak{gl}_n(\mathbb{C})$ is nonzero and satisfies two conditions:*

- $x \in L \Longrightarrow x^* \in L$;
- $Z(L) = 0$.

Then L is semisimple.

Proof. Firstly, the trace form of L is nondegenerate[2]. It is because for $x \in L\backslash\{0\}$, we have $x^* \in L$ and $\operatorname{tr}(xx^*) \neq 0$. As $I \triangleleft L$, we claim that

$$L = I^* \oplus L^\perp,$$

where $I^* := \{x^* : x \in L\}$. It can be checked as follows.

- I^* is a complex subspace. Because for $x, y \in I^*$ and $a, b \in \mathbb{C}$, we have $x^*, y^* \in I$, and $\overline{a}x^* + \overline{b}y^* \in I$. Thus $ax + by = (\overline{a}x^* + \overline{b}y^*)^* \in I^*$.
- $I^* \triangleleft L$. Because for $x \in I^*$ and $y \in L$, $x^* \in I$ and $[y^*, x^*] \in I$. Hence $[x, y] = [y^*, x^*]^* \in I^*$.
- Since the trace form is nondegenerate and $\dim I^* = \dim I$, we have
$$\dim I^* + \dim I^\perp = \dim L.$$
- $I^* \cap I^\perp = 0$. Because for $x \in I^* \cap I^\perp$, we have $x^* \in I$ and $x \in I^\perp$. Then $\operatorname{tr}(xx^*) = 0$, and therefore $x = 0$.

It suffices to show that any abelian ideal $I \triangleleft L$ must be 0. Fix $x \in I$. For arbitrary $y \in L$, write $y = y_1^* + y_2$ with $y_1 \in I$ and $y_2 \in I^\perp$, then

$$[x^*, y] = [y_1, x]^* + [x^*, y_2] = [x^*, y_2] \in I^* \cap I^\perp = 0.$$

This implies $x^* \in Z(L) = 0$. So $x = 0$. □

Corollary 3.7. *The Lie algebras $\mathfrak{sl}_n(\mathbb{C})$ $(n \geqslant 2)$, $\mathfrak{o}_n(\mathbb{C})$ $(n \geqslant 3)$, and $\mathfrak{sp}_{2n}(\mathbb{C})$ $(n \geqslant 1)$ are semisimple.*

Proof. Recall that

$$\begin{aligned}
\mathfrak{sl}_n(\mathbb{C}) &= \{x \in \mathfrak{gl}_n(\mathbb{C}) : \operatorname{tr}(x) = 0\}, && n \geqslant 2;\\
\mathfrak{o}_n(\mathbb{C}) &= \{x \in \mathfrak{gl}_n(\mathbb{C}) : x + x^t = 0\}, && n \geqslant 3;\\
\mathfrak{sp}_{2n}(\mathbb{C}) &= \left\{\begin{pmatrix} x & y \\ z & -x^t \end{pmatrix} : x, y, z \in \mathfrak{gl}_n(\mathbb{C}), y^t = y, z^t = z\right\}, && n \geqslant 1.
\end{aligned}$$

These L satisfy $L = L^*$ (i.e., $x \in L \Longrightarrow x^* \in L$), and $Z(L) = 0$ (exercise). □

Remark 3.8. These L are in fact simple except for $\mathfrak{o}_4(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.

3.2. **Jordan Decomposition.**

Theorem 3.9 (Jordan Decomposition)**.** *Every $x \in \mathfrak{gl}(V)$ can be uniquely decomposed as*

$$x = x_s + x_n$$

such that x_s is semisimple, x_n is nilpotent, and $[x_s, x_n] = 0$. Moreover, there exist polynomials $p(t), q(t) \in \mathbb{C}[t]$ (depending on x) such that $x_s = p(x)$ and $x_n = q(x)$.

Proof. The proof goes into 3 steps.

(I) Existence of decomposition. Fix a basis $\mathcal{B}$ of V such that $[x]_\mathcal{B}$ is a Jordan matrix. Let $x_s, x_n \in \mathfrak{gl}(V)$ be such that $x = x_s + x_n$, with $[x_s]_\mathcal{B}$ being diagonal and $[x_n]_\mathcal{B}$ being strictly upper triangular. Then x_s is semisimple, and x_n is nilpotent. Also, $[x_s, x_n] = 0$.

(II) Construction of $p, q \in \mathbb{C}[t]$. Let $a_1, \ldots, a_r$ be the distinct eigenvalues of x. By the Chinese Remainder Theorem, we can choose $p(t) \in \mathbb{C}[t]$ such that

$$p(t) \equiv a_k \bmod (t - a_k)^d, \quad 1 \leqslant k \leqslant r, \quad d = \dim V.$$

Note that if J is a Jordan block in $[x]_\mathcal{B}$ with eigenvalue a_k, then $(J - a_k I)^d = 0$, and hence $p(J) = a_k I$. This implies

$$[p(x)]_\mathcal{B} = p([x]_\mathcal{B}) = [x_s]_\mathcal{B}.$$

So $p(x) = x_s$. Let $q(t) = t - p(t)$. Then $q(x) = x - x_s = x_n$.

(III) Uniqueness of decomposition. Suppose there is another decomposition $x = x'_s + x'_n$ such that x'_s is semisimple, x'_n is nilpotent, and $[x'_s, x'_n] = 0$. Then

$$x_s - x'_s = x'_n - x_n.$$

Note that

$$x'_s, \quad x'_n, \quad x_s = p(x), \quad x_n = q(x)$$

commute pairwise. So $x_s - x'_s$ is semisimple, and $x_n - x'_n$ is nilpotent. Therefore, both sides of the formula are 0, namely, $x'_s = x_s$ and $x'_n = x_n$.

[2]We will see this is morally equivalent to the semisimplicity by Cartan's criterion (c.f. Theorem 3.12).

□

x_s and x_n are called the **semisimple part** and **nilpotent part** of x, respectively.

Proposition 3.10. *Let* $\mathrm{ad} = \mathrm{ad}_{\mathfrak{gl}(V)}$. *For every* $x \in \mathfrak{gl}(V)$, *we have*

$$\mathrm{ad}(x_s) = \mathrm{ad}(x)_s, \quad \mathrm{ad}(x)_n = \mathrm{ad}(x_n).$$

Proof. The decomposition $\mathrm{ad}(x) = \mathrm{ad}(x_s) + \mathrm{ad}(x_n)$ satisfies that $\mathrm{ad}(x_s)$ is semisimple, $\mathrm{ad}(x_n)$ is nilpotent, and

$$[\mathrm{ad}(x_s), \mathrm{ad}(x_n)] = \mathrm{ad}([x_s, x_n]) = 0.$$

□

3.3. Cartan's Criterions.

Theorem 3.11. *Suppose* $L < \mathfrak{gl}(V)$ *has trace form* $\tau \equiv 0$. *Then* L *is solvable.*

Proof. It suffices to prove $[L, L]$ is nilpotent. By Engel's Theorem 2.11, it suffices to prove that every $x \in [L, L]$ is a nilpotent transformation. Let $\mathcal{B}$ be a basis of V such that $[x]_{\mathcal{B}}$ is a Jordan matrix. Suppose $[x]_{\mathcal{B}} = \mathrm{diag}(a_1, \ldots, a_n)$. Let $\overline{x}_s \in \mathfrak{gl}(V)$ be such that $[\overline{x}_s]_{\mathcal{B}} = \mathrm{diag}(\overline{a}_1, \ldots, \overline{a}_n)$. We claim $[\overline{x}_s, L] \subset L$ and verify this as follows.

- Denote $\mathrm{ad} = \mathrm{ad}_{\mathfrak{gl}(V)}$. Then $\mathrm{ad}(x)(L) \subset L$.
- Since $\mathrm{ad}(x_s) = \mathrm{ad}(x)_s$ is a polynomial of $\mathrm{ad}(x)$, we have $\mathrm{ad}(x_s)(L) \subset L$.
- Let $p \in \mathbb{C}[t]$ be such that $p(a_i - a_j) = \overline{a}_i - \overline{a}_j$ for all i, j. Then $\mathrm{ad}(\overline{x}_s) = p(\mathrm{ad}(x_s))$. Hence $\mathrm{ad}(\overline{x}_s) = (L) \subset L$.

Suppose $x = \sum_{k=1}^{r} [y_k, z_k]$, where $y_k, z_k \in L$. Then

$$\begin{aligned}\sum_{i=1}^{n} |a_i|^2 = \mathrm{tr}(\overline{x}_s x) = \sum_{k=1}^{r} \mathrm{tr}(\overline{x}_s, [y_k, z_k]) = \sum_{k=1}^{r} \mathrm{tr}([\overline{x}_s, y_k] z_k) \\ = \sum_{k=1}^{r} \tau([\overline{x}_s, y_k], z_k) = 0.\end{aligned}$$

It follows that $a_1 = \cdots = a_n = 0$. Thus $x_s = 0$. Hence x is nilpotent. □

Theorem 3.12. *Suppose* $L < \mathfrak{gl}(V)$ *is semisimple. Then its trace form* τ *is nondegenerate.*

Proof. Since the trace form of $L^{\perp}$ is zero, the above theorem implies that $L^{\perp}$ is solvable. So $L^{\perp} = 0$, namely τ is nondegenerate. □

Theorem 3.13 (Cartan's Criterion for Solvability)**.** *For a Lie algebra* L *with Killing form* κ, *the following statements are equivalent:*

(1) *L is solvable;*
(2) *$\kappa([x, y], z) = 0$ for any $x, y, z \in L$;*
(3) *$\kappa|_{[L,L]} = 0$.*

Proof. (1) $\Longrightarrow$ (2): Suppose L is solvable. Then $\mathrm{ad}(L) < \mathfrak{gl}(V)$ is solvable. Using Lie's Theorem 2.13, there is a basis $\mathcal{B}$ of L such that $[\mathrm{ad}(x)]_{\mathcal{B}}$ is upper triangular for all $x \in L$. Thus for all $x, y \in L$, $[\mathrm{ad}([x, y])]_{\mathcal{B}}$ is strictly upper triangular. Consequently, for all $x, y, z \in L$,

$$\kappa([x, y], z) = \mathrm{tr}([\mathrm{ad}([x, y])]_{\mathcal{B}} [\mathrm{ad}(z)]_{\mathcal{B}}) = 0.$$

(2) $\Longrightarrow$ (3): Obvious.

(3) $\Longrightarrow$ (1): Suppose $\kappa|_{[L,L]} = 0$. Then the trace form of $\mathrm{ad}_L([L, L]) < \mathfrak{gl}(L)$ is zero. By Theorem 3.11, this shows that $\mathrm{ad}_L([L, L]) = [\mathrm{ad}_L(L), \mathrm{ad}_L(L)]$ is solvable. Then $\mathrm{ad}_L(L) \cong L/Z(L)$ is solvable. Then L is solvable as well. □

Theorem 3.14 (Cartan's Criterion for Simplicity)**.** *A Lie algebra* $L \neq 0$ *is semisimple if and only if its Killing form* κ *is nondegenerate.*

Proof. $\Longrightarrow$: Suppose L is semisimple. Then $\mathrm{ad}(L) \cong L$ is semisimple as semisimple Lie algebras have no nonzero abelian ideals and $Z(L)$ is abelian if it is nontrivial. Then the trace form of $\mathrm{ad}(L)$ is nondegenerate by Theorem 3.12. Hence κ is nondegenerate.

$\Longleftarrow$: Suppose κ is nondegenerate. To prove L is semisimple, it suffices to show

$$(*) \qquad \text{If } I \triangleleft L \text{ is an abelian ideal, then } \kappa(x,y) = 0, \quad \forall x \in I, y \in L.$$

Once $(*)$ is valid, we see $x = 0$ from the nondegeneracy. For every $z \in L$, we have

$$\mathrm{ad}(x)(z) \in I \implies \mathrm{ad}(y)\,\mathrm{ad}(x)(z) \in I \implies \mathrm{ad}(x)\,\mathrm{ad}(y)\,\mathrm{ad}(x)(z) = 0.$$

So

$$\begin{aligned} \mathrm{ad}(x)\,\mathrm{ad}(y)\,\mathrm{ad}(x) = 0 &\implies (\mathrm{ad}(x)\,\mathrm{ad}(y))^2 = 0 \\ &\implies \mathrm{ad}(x)\,\mathrm{ad}(y) \text{ is nilpotent} \\ &\implies \kappa(x,y) = \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0. \end{aligned}$$

This completes the proof of $(*)$. □

Remark 3.15. Indeed, for a solvable Lie algebra L, its Killing form need not to be zero. Conversely, the useful fact at work is that once L enjoys a degenerate Killing form κ, it must be solvable (c.f. Theorem 3.13).

3.4. **Structure of Semisimple Lie Algebras.** In this subsection we prove the first statement in our Main Theorem 1.20:

- ◇ *A finite dimensional complex Lie algebra is semisimple if and only if it is isomorphic to the direct sum of finitely many simple Lie algebras.*

Note that the $\Longleftarrow$ direction can be proved from the definition (as follows).

Proposition 3.16. *Let $L_1, \dots, L_r$ be semisimple Lie algebras. Then $\bigoplus_{i=1}^r L_i$ is semisimple.*

Proof. Let $I \triangleleft \bigoplus_{i=1}^r L_i$ be an abelian ideal. Then for each i,

$$[I, L_i] \subset I \cap L_i \implies [I, L_i] \text{ is an abelian ideal of } L_i \implies [I, L_i] = 0.$$

Let $x = \sum_{i=1}^r x_i \in I$, where $x_i \in L_i$. Then

$$[x_i, L_i] = [x, L_i] = 0 \implies x_i \in Z(L_i) = 0 \implies x = 0.$$

So $I = 0$. □

To prove the $\Longrightarrow$ direction, let us notice the following.

Lemma 3.17. *Let L be a Lie algebra, and $I \triangleleft L$. Then $\kappa_L|_I = \kappa_I$.*

Proof. Let $\mathcal{B}_I$ be a basis of I, and extend it to a basis $\mathcal{B}_L$ of L. Then

$$x \in I \implies \mathrm{ad}_L(x)(L) \subset I \implies [\mathrm{ad}_L(x)]_{\mathcal{B}_L} = \begin{pmatrix} [\mathrm{ad}_I(x)]_{\mathcal{B}_I} & * \\ 0 & 0 \end{pmatrix}.$$

Thus, for $x, y \in I$,

$$\kappa_L(x,y) = \mathrm{tr}([\mathrm{ad}_L(x)]_{\mathcal{B}_L}[\mathrm{ad}_L(y)]_{\mathcal{B}_L}) = \mathrm{tr}([\mathrm{ad}_I(x)]_{\mathcal{B}_I}[\mathrm{ad}_I(y)]_{\mathcal{B}_I}) = \kappa_I(x,y).$$

Then $\kappa_L|_I = \kappa_I$ as required. □

Lemma 3.18. *Let L be semisimple, and $I \triangleleft L$. Then*

1. *$L = I \oplus I^\perp$, where $I^\perp$ is the orthogonal ideal of I relative to κ_L.*
2. *If $J \triangleleft I$, then $J \triangleleft L$.*
3. *I and L/I are semisimple.*

Proof. (1) By the above Lemma 3.17, $\kappa_{I\cap I^\perp} = \kappa_L|_{I\cap I^\perp} = 0$. So Cartan's Criterion yields to the solvability of $I \cap I^\perp \triangleleft L$. And then $I \cap I^\perp = 0$. Since κ_L is nondegenerate, we have $L = I \oplus I^\perp$.

(2) By (1), we have $[J, L] = [J, I] \oplus [J, I^\perp] \subset J \oplus (I \cap I^\perp) = J$.

(3) By (2), any abelian ideal of I is an abelian ideal of L, hence is 0. Thus I is semisimple. Similarly, $I^\perp$ is semisimple. So $L/I \cong I^\perp$ is semisimple as well. □

Now we are ready to prove the $\Longrightarrow$ direction of Main Theorem.

Theorem 3.19. *Let L be semisimple. Then there are simple ideals $L_1, \dots, L_r$ of L such that*

$$L = \bigoplus_{i=1}^{r} L_i.$$

Proof. If L is simple, there is nothing to prove. Suppose L is not simple. Let $I \triangleleft L$ be a nontrivial ideal. Then by Lemma 3.18 (1) above, one factors $L = I \oplus I^\perp$ with $I, I^\perp$ semisimple. Using induction, we may assume I and $I^\perp$ are direct sums of their simple ideals. Again, by Lemma 3.18 (2)(3), these simple ideals are also simple ideals of L. Therefore, it is clear that L is their direct sum. □

Corollary 3.20. *Let L be a semisimple Lie algebra. Then*

$$[L, L] = L.$$

Proof. Suppose $L = \bigoplus_{i=1}^{r} L_i$, where $L_i \triangleleft L$ are simple ideals. Then

$$[L, L] \supset \bigoplus_{i=1}^{r} [L_i, L_i] = \bigoplus_{i=1}^{r} L_i = L.$$

□

Remark 3.21. The converse of the corollary is not true, whereas it provides the insolvability of L.

3.5. **Abstract Jordan Decomposition.** This subsection works for the following statement:

- ◇ (Abstract Jordan Decomposition) *Let L be semisimple. Then every $x \in L$ can be uniquely decomposed as $x = x_{(s)} + x_{(n)}$ such that $x_{(s)}$ is ad-semisimple, $x_{(n)}$ is ad-nilpotent, and $[x_{(s)}, x_{(n)}] = 0$.*

To prove this, we use the notion of derivation.

Definition 3.22 (Derivation)**.** Let L be a Lie algebra. A **derivation** of L is a linear map $D : L \to L$ such that

$$D[x, y] = [Dx, y] + [x, Dy], \quad \forall x, y \in L.$$

Example 3.23 (Inner Derivation)**.** For any $x \in L$, $\mathrm{ad}(x)$ is a derivation of L because for all $y, z \in L$,

$$\mathrm{ad}(x)[y, z] = [x, [y, z]] = [[x, y], z] + [y, [x, z]] = [\mathrm{ad}(x)(y), z] + [y, \mathrm{ad}(x)(z)]$$

by the Jacobi identity. Such derivations are called **inner derivations**.

Lemma 3.24. *Let L be a Lie algebra and D be a derivation. Then its semisimple and nilpotent parts, denoted by D_s and D_n, are derivations.*

Proof. For fixed D and $a \in \mathbb{C}$, let

$$L_a := \{x \in L : (D - a)^n x = 0 \text{ for some } n \geqslant 1\}.$$

Then $L = \bigoplus_{a \in \mathbb{C}} L_a$ and $D_s|_{L_a} = a \cdot \mathrm{id}$. Note that $a \in \mathbb{C}$ need not be any eigenvalue. Using induction, it is straightforward to verify

$$(D - a - b)^n [x, y] = \sum_{k=0}^{n} \binom{n}{k} [(D - a)^k x, (D - b)^{n-k} y], \quad \forall a, b \in \mathbb{C},\ n \geqslant 1.$$

And this implies

$$[L_a, L_b] \subset L_{a+b}.$$

So for all $x \in L_a$ and $y \in L_b$,

$$D_s[x, y] = (a + b)[x, y] = [ax, y] + [x, by] = [D_s x, y] + [x, D_s y].$$

By linearity, D_s is a derivation. Therefore, so also is $D_n = D - D_s$. □

Lemma 3.25. *Let L be a Lie algebra, D be a derivation, and $x \in L$. Then*

$$\mathrm{ad}(Dx) = [D, \mathrm{ad}(x)],$$

Proof. For any $y \in L$, we have

$$\begin{aligned}\mathrm{ad}(Dx)(y) &= [Dx, y] \\ &= D[x, y] - [x, Dy] \\ &= (D \circ \mathrm{ad}(x) - \mathrm{ad}(x) \circ D)(y) \\ &= [D, \mathrm{ad}(x)](y).\end{aligned}$$

So $\mathrm{ad}(Dx) = [D, \mathrm{ad}(x)]$. □

Lemma 3.26. *Let L be semisimple. Then every derivation D of L is inner.*

Proof. Cartan's criterion dictates that κ is nondegenerate on L. While x running through all elements in L, $\kappa(x, \cdot)$ can be realized as an arbitrary linear map. Particularly, there is some $x \in L$ such that

$$\kappa(x, \cdot) = \mathrm{tr}(D \circ \mathrm{ad}(\cdot)).$$

It suffices to show that for all $y, z \in L$,

$$\kappa(Dy, z) = \kappa(\mathrm{ad}(x)(y), z).$$

Yet this is straightforward, because of

$$\begin{aligned}\kappa(Dy, z) &= \mathrm{tr}(\mathrm{ad}(Dy) \circ \mathrm{ad}(z)) \\ &= \mathrm{tr}([D, \mathrm{ad}(y)] \circ \mathrm{ad}(z)) && \text{by Lemma 3.25} \\ &= \mathrm{tr}(D \circ [\mathrm{ad}(y), \mathrm{ad}(z)]) && \text{as } \kappa \text{ is invariant (associative)} \\ &= \mathrm{tr}(D \circ \mathrm{ad}([y, z])) \\ &= \kappa(x, [y, z]) && \text{by assumption} \\ &= \kappa([x, y], z) = \kappa(\mathrm{ad}(x)(y), z).\end{aligned}$$

Therefore, $D = \mathrm{ad}(x)$. □

Proposition 3.27. *Let L be semisimple. Then for every $x \in L$, we have $\mathrm{ad}(x)_s, \mathrm{ad}(x)_n \in \mathrm{ad}(L)$.*

Proof. By definition, $\mathrm{ad}(x)$ is a derivation and so also is $\mathrm{ad}(x)_s$ by linearity. From Lemma 3.26, every derivation on L is inner. Hence $\mathrm{ad}(x)_s$ is always an inner derivation. This shows that $\mathrm{ad}(x)_s \in \mathrm{ad}(L)$. Similarly, $\mathrm{ad}(x)_n \in \mathrm{ad}(L)$. □

Remark 3.28. The proposition is a special case of a more general result. Say if $L < \mathfrak{gl}(V)$ is semisimple, then for every $x \in L$, we have $x_s, x_n \in L$.

Now we are ready to understand the abstract Jordan decomposition.

Theorem 3.29 (Abstract Jordan Decomposition)**.** *Let L be semisimple. Then every $x \in L$ can be uniquely decomposed as $x = x_{(s)} + x_{(n)}$ such that $x_{(s)}$ is ad-semisimple, $x_{(n)}$ is ad-nilpotent, and $[x_{(s)}, x_{(n)}] = 0$.*

Proof. The abstract Jordan decomposition is deduced from the Jordan decomposition for linear Lie algebras.

(I) Existence. Let $x \in L$. The above proposition implies that $\mathrm{ad}(x)_s, \mathrm{ad}(x)_n \in \mathrm{ad}(L)$, i.e., there are $x_{(s)}, x_{(n)} \in L$ such that

$$\mathrm{ad}(x_{(s)}) = \mathrm{ad}(x)_s, \quad \mathrm{ad}(x_{(n)}) = \mathrm{ad}(x)_n.$$

Note that $x_{(s)}$ is ad-semisimple, $x_{(n)}$ is ad-nilpotent, and

$$\mathrm{ad}([x_{(s)}, x_{(n)}]) = [\mathrm{ad}(x)_s, \mathrm{ad}(x)_n] = 0 \quad \Longrightarrow \quad [x_{(s)}, x_{(n)}] = 0.$$

(II) Uniqueness. Suppose $x = x_{(s)} + x_{(n)} = x'_{(s)} + x'_{(n)}$, where $x_{(s)}, x'_{(s)}$ are ad-semisimple, $x_{(n)}, x'_{(n)}$ are ad-nilpotent, and $[x_{(s)}, x_{(n)}] = [x'_{(s)}, x'_{(n)}] = 0$. Then

$$\mathrm{ad}(x) = \mathrm{ad}(x_{(s)}) + \mathrm{ad}(x_{(n)}) \quad \text{and} \quad \mathrm{ad}(x) = \mathrm{ad}(x'_{(s)}) + \mathrm{ad}(x'_{(n)})$$

are both the Jordan decomposition of $\mathrm{ad}(x)$. So $\mathrm{ad}(x_{(s)}) = \mathrm{ad}(x'_{(s)})$, which implies $x_{(s)} = x'_{(s)}$. Similarly, $x_{(n)} = x'_{(n)}$.

□

Remark 3.30. When $L < \mathfrak{gl}(V)$ (and semisimple), it can be proved that $x_{(s)} = x_s, x_{(n)} = x_n$.

4. Root Spaces and Root Systems

This section starts the classification theory of complex simple Lie algebras.

Definition 4.1 (Toral subalgebra, Cartan subalgebra)**.** Let L be a semisimple Lie algebra. A subalgebra of L is called

- a **toral subalgebra**, if it consists of ad_L-semisimple elements;
- a **Cartan subalgebra**, if it is a maximal toral subalgebra.

Proposition 4.2. *Let L be semisimple and $H < L$ be a Cartan subalgebra. Then $H \neq 0$ and is abelian.*

Proof. Note that all $x \in H$ is ad_L-semisimple, so is ad_H-semisimple. So H is abelian by Theorem 2.5. To see $H \neq 0$, it suffices to show L contains nonzero ad-semisimple elements. Suppose not, then for all $x \in L$, $x = x_{(s)} + x_{(n)}$ is ad-nilpotent. However, by Engel's theorem, this implies that L is nilpotent, which is a contradiction. □

Remark 4.3. For a general Lie algebra L, a subalgebra $H < L$ is called a **Cartan subalgebra** if H is nilpotent and $N_L(H) = H$ (namely, H is self-normal). If L is semisimple, the two definitions coincide.

4.1. Root Space Decompositions.

Fix a semisimple Lie algebra L and a Cartan subalgebra $H < L$. Take the construction as follows.

- $\{\mathrm{ad}(h) : h \in H\}$ is a commuting family of diagonalizable linear transformations on L, hence its elements are simultaneously diagonalizable.
- This means there is a basis $\{x_1, \dots, x_n\}$ of L consisting of common eigenvectors.
- For $1 \leqslant i \leqslant n$, let $\alpha_i(h)$ be the eigenvalue of $\mathrm{ad}(h)$ corresponding to x_i, namely,
$$\mathrm{ad}(h)(x_i) = \alpha_i(h)x_i, \quad \forall h \in H.$$
Then each $\alpha_i : H \to \mathbb{C}$ is a linear function.

This can be interpreted as follows. For every $\alpha \in H^* = \mathrm{Hom}(H, \mathbb{C})$, consider the weight space
$$L_\alpha = \{x \in L : [h, x] = \alpha(h)x, \forall h \in H\}.$$
Denote
$$\Phi = \{\alpha \in H^* \backslash \{0\}, L_\alpha \neq 0\}.$$
Then $\Phi \subset H^*$ is finite because of
$$(*) \qquad L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha.$$
Namely, since L is a finite-dimensional Lie algebra, there are only finitely many α such that $L_\alpha \neq 0$. Note that $(*)$ is equivalent to say elements in $\{\mathrm{ad}(h) : h \in H\}$ are simultaneously diagonalizable. Since H is abelian, we have
$$H \subset L_0 = C_L(H) := \{x \in L : [x, H] = 0\}.$$
An element $\alpha \in \Phi$ is called a **root**; L_α is called the corresponding **root space**.

Example 4.4. Let $L = \mathfrak{sl}_n(\mathbb{C})$ with $n \geqslant 2$. Then
$$H = \{\mathrm{diag}(a_1, \dots, a_n) : a_i \in \mathbb{C}, \sum a_i = 0\}$$
has ad-semisimple elements and is a maximal abelian subalgebra, hence a Cartan subalgebra. Let $e_i \in H^*$ be[3]
$$e_i : \mathrm{diag}(a_1, \dots, a_n) \mapsto a_i, \quad 1 \leqslant i \leqslant n.$$
Then
$$\Phi = \{e_i - e_j : i \neq j\}.$$
We have $L_0 = H$ and $L_{e_i - e_j} = \mathbb{C}E_{ij}$.

[3] Caution: these e_i's DO NOT form a basis of H^* because of $\dim H = n - 1$.

Example 4.5. Let $L = \mathfrak{o}_{2n}(\mathbb{C})$ with $n \geqslant 2$. Then

$$H = \{\mathrm{diag}(a_1 J, \dots, a_n J) : a_i \in \mathbb{C}\}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is a Cartan subalgebra. Let $e_i \in H^*$ be

$$e_i : \mathrm{diag}(a_1 J, \dots, a_n J) \mapsto \sqrt{-1} a_i, \quad 1 \leqslant i \leqslant n.$$

Then

$$\Phi = \{\pm e_i \pm e_j : i \neq j\}.$$

Example 4.6. Let $L = \mathfrak{o}_{2n+1}(\mathbb{C})$ with $n \geqslant 1$. Then

$$H = \{\mathrm{diag}(a_1 J, \dots, a_n J, 0) : a_i \in \mathbb{C}\}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is a Cartan subalgebra. Let $e_i \in H^*$ be

$$e_i : \mathrm{diag}(a_1 J, \dots, a_n J) \mapsto \sqrt{-1} a_i, \quad 1 \leqslant i \leqslant n.$$

Then

$$\Phi = \{\pm e_i \pm e_j : i \neq j\} \cup \{\pm e_i : 1 \leqslant i \leqslant n\}.$$

Example 4.7. Let $L = \mathfrak{sp}_{2n}(\mathbb{C})$ with $n \geqslant 1$. Then

$$H = \{\mathrm{diag}(a_1, \dots, a_n, -a_1, \dots, -a_n) : a_i \in \mathbb{C}\}$$

is a Cartan subalgebra. Let $e_i \in H^*$ be

$$e_i : \mathrm{diag}(a_1, \dots, a_n, -a_1, \dots, -a_n) \mapsto a_i, \quad 1 \leqslant i \leqslant n.$$

Then

$$\Phi = \{\pm e_i \pm e_j : i \neq j\} \cup \{2e_i : 1 \leqslant i \leqslant n\}.$$

Theorem 4.8. *Let L be a Lie algebra and H be its Cartan subalgebra.*

1. *$L_0 = H$. In particular, H is a maximal abelian subalgebra of L.*
2. *Let κ be the Killing form of L. Then $\kappa|_H$ is nondegenerate.*
3. *For every $\alpha \in \Phi$, we have $\kappa(H, L_\alpha) = \kappa(L_0, L_\alpha) = 0$.*

Proof. The recipe is to verify the following claims one by one.

(I) If $\alpha \in \Phi$, then $\kappa(L_0, L_\alpha) = 0$.
Choose $h \in H$ such that $\alpha(h) \neq 0$. Then for all $x \in L_0$ and $y \in L_\alpha$,

$$\alpha(h)\kappa(x, y) = \kappa(x, \alpha(h)y) = \kappa(x, [h, y]) = \kappa([x, h], y) = 0.$$

Then the assumption on α shows that $\kappa(x, y) = 0$. Namely L_0 is orthogonal to any other L_α with $\alpha \in \Phi$.

(II) $\kappa|_{L_0}$ is nondegenerate.
Let $x \in L_0$ be such that $\kappa(x, L_0) = 0$. We also have $\kappa(x, L_\alpha) = 0$ for all $\alpha \in \Phi$. So $\kappa(x, L) = 0$. The degeneracy of κ yields to $x = 0$.

(III) For $x \in L_0$ we have $x_{(s)} \in H$ and $x_{(n)} \in L_0$.
As $x \in L_0$ we see $[x, H] = 0$. A computation shows

$$\mathrm{ad}_L[x, H] = [\mathrm{ad}_L(x), \mathrm{ad}_L(H)] = 0 \quad \Longrightarrow \quad [\mathrm{ad}_L(x_{(s)}), \mathrm{ad}_L(H)] = 0.$$

Because the commutativity between $\mathrm{ad}_L(x)$ and $\mathrm{ad}_L(H)$ is inherited after taking a polynomial on $\mathrm{ad}_L(x)$. Recall that for semisimple Lie algebra L, we have $\mathrm{ad}_L(L) \cong L$. Thus $[x_{(s)}, H] = 0$. On the other hand, since $x_{(s)}$ is already ad-semisimple,

$$H + \mathbb{C}x_{(s)} \text{ is a toral subalgebra} \quad \Longrightarrow \quad x_{(s)} \in H \quad \Longrightarrow \quad x_{(n)} \in L_0.$$

(IV) L_0 is nilpotent.
By Engel's theorem, it suffices to show that all elements in L_0 are ad_{L_0}-nilpotent. For all $x \in L_0$,

$$\mathrm{ad}_{L_0}(x) = \mathrm{ad}_{L_0}(x_{(s)}) + \mathrm{ad}_{L_0}(x_{(n)}) = \mathrm{ad}_L(x_{(n)})|_{L_0}$$

is nilpotent, namely x is ad_{L_0}-nilpotent.

(V) L_0 contains no nonzero ad_L-nilpotent elements.
Let $x \in L_0$ be ad_L-nilpotent. Then $\mathrm{ad}_L(L_0) < \mathfrak{gl}(L)$ is nilpotent, hence is solvable. By Lie's theorem, there is a basis $\mathcal{B}$ of L such that for all $y \in L_0$, $[\mathrm{ad}_L(y)]_{\mathcal{B}}$ is upper triangular. Moreover, as $[\mathrm{ad}_L(y)]_{\mathcal{B}}$ is nilpotent, it is strictly upper triangular. Hence

$$\kappa(x, y) = \mathrm{tr}(\mathrm{ad}_L(x)\,\mathrm{ad}_L(y)) = 0, \quad \forall y \in L_0.$$

Also, since $\kappa|_{L_0}$ is nondegenerate, we have $x = 0$.

(VI) $L_0 \subset H$.
For all $x \in L_0$, we have $x = x_{(s)} + x_{(n)} = x_{(s)} \in H$.

These arguments complete the proof of (1)-(3). □

From the theorem, the root space decomposition becomes

$$L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha.$$

Again, as $\kappa|_H$ is nondegenerate, there exists a *unique* linear isomorphism

$$H^* \xrightarrow{\cong} H, \quad \alpha \longmapsto t_\alpha,$$

such that

$$\alpha = \kappa(t_\alpha, \cdot)|_H.$$

Namely, all linear maps on H are defined by some Killing form. This induces a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ on H^*:

$$(\alpha, \beta) := \kappa(t_\alpha, t_\beta) = \alpha(t_\beta), \quad \forall \alpha, \beta \in H^*.$$

Theorem 4.9. *The set of roots $\Phi \subset H^*\backslash\{0\}$ satisfies the following properties.*

(1) *The real subspace $E := \mathrm{Span}_{\mathbb{R}}(\Phi)$ of H^* satisfies*

$$H^* = E \oplus \sqrt{-1}E,$$

and the restriction $(\cdot,\cdot)|_E$ is a (real and positive definite) inner product.

(2) *For any $\alpha \in \Phi$, we have*

$$\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}.$$

(3) *For any $\alpha, \beta \in \Phi$, we have*

$$\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}, \quad \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi.$$

Theorem 4.10. *The root spaces L_α satisfy the following properties.*

(1) *For any $\alpha \in \Phi$, we have $\dim L_\alpha = 1$.*

(2) *For any $\alpha, \beta \in \Phi$, we have $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$. Moreover,*

$$\begin{aligned} \alpha, \beta, \alpha+\beta \in \Phi \quad &\Longrightarrow \quad [L_\alpha, L_\beta] = L_{\alpha+\beta};\\ \alpha \in \Phi \quad &\Longrightarrow \quad [L_\alpha, L_{-\alpha}] = \mathbb{C}t_\alpha. \end{aligned}$$

(3) *For $\alpha, \beta \in \Phi$, we have*

$$\alpha + \beta \neq 0 \quad \Longleftrightarrow \quad \kappa(L_\alpha, L_\beta) = 0.$$

Proof of Theorem 4.9 and 4.10. All details are listed below[4].

(I) $\mathrm{Span}_{\mathbb{C}}(\Phi) = H^*$.
It suffices to verify $\bigcap_{\alpha\in\Phi} \mathrm{Ker}(\alpha) = 0$. For this,

$$h \in \bigcap_{\alpha\in\Phi} \mathrm{Ker}(\alpha) \quad \Longrightarrow \quad [h, L_\alpha] = 0,\ \forall \alpha \in \Phi \cup \{0\} \quad \Longrightarrow \quad h \in Z(L) = 0.$$

[4]These theorems are the most important sort for the classification of complex semisimple Lie algebras. Some result occurring in the proof can also be useful.

(II) For any $\alpha, \beta \in \Phi$, we have $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$.
Let $x \in L_\alpha$ and $y \in L_\beta$. For all $h \in H$ we have

$$\begin{aligned}\operatorname{ad}(h)([x,y]) &= [\operatorname{ad}(h)(x), y] + [x, \operatorname{ad}(h)(y)] \\ &= [\alpha(h)x, y] + [x, \alpha(h)y] \\ &= (\alpha+\beta)(h)[x,y].\end{aligned}$$

This means $[x,y] \in L_{\alpha+\beta}$.

(III) For any $\alpha, \beta \in \Phi$, if $\alpha + \beta \neq 0$, then $\kappa(L_\alpha, L_\beta) = 0$.
Choose $h \in H$ such that $(\alpha+\beta)(h) \neq 0$. Then for all $x \in L_\alpha$ and $y \in L_\beta$,

$$\begin{aligned}0 &= \kappa([h,x], y) + \kappa(x, [h,y]) \\ &= \kappa(\alpha(h)x, y) + \kappa(x, \beta(h)y) \\ &= (\alpha+\beta)(h)\kappa(x,y).\end{aligned}$$

Thus $\kappa(x,y) = 0$.

(IV) If $\alpha \in \Phi$, then $-\alpha \in \Phi$ and $\kappa(L_\alpha, L_{-\alpha}) \neq 0$.
Suppose $\kappa(L_\alpha, L_{-\alpha}) = 0$. Then for all $\beta \in \Phi \cup \{0\}$, $\kappa(L_\alpha, L_\beta) = 0$. Thus,

$$\kappa(L_\alpha, L) = 0 \quad \Longrightarrow \quad \text{contradiction},$$

because κ is nondegenerate. Again, we see if $-\alpha \notin \Phi$, there should be $\kappa(L_\alpha, L_{-\alpha}) = 0$, which is impossible.

(V) For $\alpha \in \Phi$, we have $[L_\alpha, L_{-\alpha}] = \mathbb{C}t_\alpha$.
Let $x \in L_\alpha$ and $y \in L_{-\alpha}$. Then $[x,y] \in L_0 = H$ by (IV). On the other hand, for all $h \in H$, we have

$$\kappa(h, [x,y]) = \kappa([h,x], y) = \alpha(h)\kappa(x,y) = \kappa(h, t_\alpha)\kappa(x,y) = \kappa(h, \kappa(x,y)t_\alpha).$$

As $\kappa|_H$ is nondegenerate and the equation above holds for all $h \in H$, we see $[x,y] = \kappa(x,y)t_\alpha \in \mathbb{C}t_\alpha$. Therefore, $[L_\alpha, L_{-\alpha}] \subset \mathbb{C}t_\alpha$. As for the inverse direction, note that

$$\kappa(L_\alpha, L_{-\alpha}) \neq 0 \quad \Longrightarrow \quad [L_\alpha, L_{-\alpha}] \neq 0 \quad \Longrightarrow \quad [L_\alpha, L_{-\alpha}] = \mathbb{C}t_\alpha.$$

For any $\alpha \in \Phi$ we fix $u_\alpha \in L_\alpha$ and $v_\alpha \in L_{-\alpha}$ such that $[u_\alpha, v_\alpha] = t_\alpha$ (see (V)). Let

$$S_\alpha = \operatorname{Span}\{t_\alpha, u_\alpha, v_\alpha\}.$$

Indeed, we can check that $S_\alpha \cong \mathfrak{sl}_2(\mathbb{C})$ (which is not in need at this moment).

(VI) For any subspace $V \subset L$ with $[S_\alpha, V] \subset V$, we have $\operatorname{tr}(\operatorname{ad}(t_\alpha)|_V) = 0$.
In fact,

$$\operatorname{tr}(\operatorname{ad}(t_\alpha)|_V) = \operatorname{tr}(\operatorname{ad}([u_\alpha, v_\alpha])|_V) = \operatorname{tr}([\operatorname{ad}(u_\alpha)|_V, \operatorname{ad}(v_\alpha)|_V]) = 0.$$

We will take various such V.

(VII) For any $\alpha \in \Phi$ we have $\alpha(t_\alpha) \neq 0$. (Comment: recall that before this, we have claimed $(\cdot,\cdot)|_{\operatorname{Span}_{\mathbb{R}}(\Phi)}$ is a real and positive definite inner product.)
As $\operatorname{Span}_{\mathbb{C}}(\Phi) = H^*$, we see there is some $\beta \in \Phi$ with $\beta(t_\beta) \neq 0$. Let

$$V = \bigoplus_{k \in \mathbb{Z}} L_{\beta + k\alpha}.$$

Note that there are only finitely many nonzero factors in the direct sum, i.e., for almost all $k \in \mathbb{Z}$, $\beta + k\alpha$ is neither zero nor a root. As $[S_\alpha, V] \subset V$ by assumption, we have

$$\operatorname{tr}(\operatorname{ad}(t_\alpha)|_V) = 0.$$

Suppose to the contrary that $\alpha(t_\alpha) = 0$. Then

$$\operatorname{ad}(t_\alpha)|_{L_{\beta+k\alpha}} = (\beta + k\alpha)(t_\alpha) \cdot \mathrm{id} = \beta(t_\alpha) \cdot \mathrm{id}.$$

As $\beta(t_\alpha)$ is independent of the choice of k, it renders that

$$\operatorname{ad}(t_\alpha)|_V = \beta(t_\alpha) \cdot \mathrm{id} \quad \Longrightarrow \quad \operatorname{tr}(\operatorname{ad}(t_\alpha)|_V) \neq 0.$$

But this contradicts to (VI).

(VIII) For $\alpha \in \Phi$, we have $\dim L_\alpha = 1$ and $\Phi \cap \mathbb{Z}\alpha = \{\pm\alpha\}$.
For $v_\alpha \in L_\alpha$ that we have fixed before, let

$$V = \mathbb{C}v_\alpha \oplus \mathbb{C}t_\alpha \oplus \bigoplus_{k=1}^{\infty} L_{k\alpha}.$$

Then $[S_\alpha, V] \subset V$. Hence

$$\begin{aligned} 0 &= \operatorname{tr}(\operatorname{ad}(t_\alpha)|_V) \\ &= -\alpha(t_\alpha) + \sum_{k=1}^{\infty} k\alpha(t_\alpha) \dim L_{k\alpha} \\ &= \alpha(t_\alpha)(-1 + \sum_{k=1}^{\infty} k \dim L_{k\alpha}). \end{aligned}$$

This further implies $\dim L_\alpha = 1$, and for $k \geqslant 2$,

$$k\alpha \notin \Phi \quad \Longrightarrow \quad -k\alpha \notin \Phi \quad \Longrightarrow \quad \Phi \cap \mathbb{Z}\alpha = \{\pm\alpha\}.$$

(IX) For $\alpha \in \Phi$, we have $\Phi \cap \mathbb{C}\alpha = \{\pm\alpha\}$.
Suppose to the contrary that there is $c \in \mathbb{C}\backslash\{\pm 1\}$ such that $c\alpha \in \Phi$. Then $c \notin \mathbb{Z}$ by the previous step. Let $p, q \in \mathbb{Z}$ with $p \leqslant 0 \leqslant q$ be such that

$$\begin{aligned} k \in \mathbb{Z},\ p \leqslant k \leqslant q \quad &\Longrightarrow \quad (c+k)\alpha \in \Phi; \\ k \in \{p-1, q+1\} \quad &\Longrightarrow \quad (c+k)\alpha \notin \Phi. \end{aligned}$$

Again, we construct V as follows to use (VI). Say

$$V = \bigoplus_{k=p}^{q} L_{(c+k)\alpha}.$$

Then $[S_\alpha, V] \subset V$. Hence

$$\begin{aligned} 0 = \operatorname{tr}(\operatorname{ad}(t_\alpha)|_V) &= \sum_{k=p}^{q} (c+k)\alpha(t_\alpha) \\ &= \frac{1}{2}(q-p+1)(2c+p+q)\alpha(t_\alpha) \quad \Longrightarrow \quad 2c = -(p+q) \in \mathbb{Z}. \end{aligned}$$

As $c \notin \mathbb{Z}$, we see $p+q$ must be odd. On the other hand,

$$p \leqslant \frac{p+q+1}{2} \leqslant q \quad \Longrightarrow \quad (c + \frac{p+q+1}{2})\alpha = \frac{\alpha}{2} \in \Phi.$$

Therefore, $\Phi \cap \mathbb{Z}(\alpha/2) = \{\pm\alpha/2\}$, and then $\alpha \notin \Phi$. This is a contradiction.

(X) For $\alpha, \beta \in \Phi$, we have

$$\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi, \quad \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}.$$

Namely, the reflection image of β with respect to the orthogonal space of α lies in Φ.
This is clear if $\beta = \pm\alpha$. Suppose $\beta \neq \pm\alpha$. Then $\beta \notin \mathbb{C}\alpha$. Let $p, q \in \mathbb{Z}$ with $p \leqslant 0 \leqslant q$ be such that

$$\begin{aligned} k \in \mathbb{Z},\ p \leqslant k \leqslant q \quad &\Longrightarrow \quad \beta + k\alpha \in \Phi; \\ k \in \{p-1, q+1\} \quad &\Longrightarrow \quad \beta + k\alpha \notin \Phi. \end{aligned}$$

Again, we construct V as follows to use (VI). Say

$$V = \bigoplus_{k=p}^{q} L_{\beta+k\alpha}.$$

Then $[S_\alpha, V] \subset V$. Hence

$$\begin{aligned} 0 = \operatorname{tr}(\operatorname{ad}(t_\alpha)|_V) &= \sum_{k=p}^{q} (\beta + k\alpha)(t_\alpha) \\ &= \frac{1}{2}(q-p+1)(2\beta(t_\alpha) + (p+q)\alpha(t_\alpha)). \end{aligned}$$

Therefore,

$$\frac{2(\beta,\alpha)}{(\alpha,\alpha)} = \frac{2\beta(t_\alpha)}{\alpha(t_\alpha)} = -(p+q) \in \mathbb{Z}.$$

Also,

$$p \leqslant p+q \leqslant q \quad \Longrightarrow \quad \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha = \beta + (p+q)\alpha \in \Phi.$$

(XI) For $\alpha, \beta \in \Phi$, if in case $\alpha+\beta \in \Phi$, then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.
Let $p \leqslant 0 \leqslant q$ be as above, namely, they satisfy

$$\begin{aligned} k \in \mathbb{Z},\ p \leqslant k \leqslant q \quad &\Longrightarrow \quad \beta + k\alpha \in \Phi; \\ k \in \{p-1, q+1\} \quad &\Longrightarrow \quad \beta + k\alpha \notin \Phi. \end{aligned}$$

Suppose to the contrary that $[L_\alpha, L_\beta] = 0$. Then

$$V' := \bigoplus_{k=p}^{0} L_{\beta+k\alpha}$$

satisfies $[S_\alpha, V'] \subset V'$. So

$$\begin{aligned} 0 = \operatorname{tr}(\operatorname{ad}(t_\alpha)|_{V'}) &= \sum_{k=p}^{q} (\beta + k\alpha)(t_\alpha) \\ &= \frac{1}{2}(-p+1)(2\beta(t_\alpha) + p\alpha(t_\alpha)). \end{aligned}$$

This deduces

$$\frac{2(\beta,\alpha)}{(\alpha,\alpha)} = \frac{2\beta(t_\alpha)}{\alpha(t_\alpha)} = -p.$$

On the other hand, by comparison,

$$\frac{2(\beta,\alpha)}{(\alpha,\alpha)} = -(p+q) \quad \Longrightarrow \quad q = 0 \quad \Longrightarrow \quad \alpha+\beta \notin \Phi.$$

This is a contradiction.

(XII) For $\alpha, \beta \in \Phi$, we have $(\beta, \alpha) \in \mathbb{R}$.
For any $\lambda \in H^*$, we have

$$(\lambda,\lambda) = \kappa(t_\lambda, t_\lambda) = \operatorname{tr}(\operatorname{ad}(t_\lambda)^2) = \sum_{\gamma\in\Phi} \gamma(t_\lambda)^2 = \sum_{\gamma\in\Phi} (\gamma,\lambda)^2. \tag{$*$}$$

On the other hand,

$$\frac{2(\gamma,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z},\ \forall \gamma \in \Phi \quad \Longrightarrow \quad \frac{1}{(\alpha,\alpha)} = \sum_{\gamma\in\Phi} \frac{(\gamma,\alpha)^2}{(\alpha,\alpha)} \in \mathbb{R} \quad \Longrightarrow \quad (\alpha,\alpha) \in \mathbb{R}.$$

And in particular,

$$\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z} \quad \Longrightarrow \quad (\beta,\alpha) = (\alpha,\alpha)\cdot\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{R}.$$

(XIII) $E := \operatorname{Span}_{\mathbb{R}}(\Phi)$ satisfies $H^* = E \oplus \sqrt{-1}E$.
Let $\{\alpha_1, \dots, \alpha_n\} \subset \Phi$ be a basis of H^*. Let $E_0 := \operatorname{Span}_{\mathbb{R}}\{\alpha_1, \dots, \alpha_n\}$. Then

$$H^* = E_0 \oplus \sqrt{-1}E_0.$$

The claim goes to $E = E_0$. It suffices to prove that $\Phi \subset E_0$. Let $\beta = \sum_{i=1}^n c_i\alpha_1 \in \Phi$ with $c_i \in \mathbb{C}$. We need to prove $c_i \in \mathbb{R}$. We obtain

$$(\beta, \alpha_j) = \sum_{i=1}^{n} c_i(\alpha_i, \alpha_j), \quad 1 \leqslant j \leqslant n,$$

or equivalently,

$$((\beta,\alpha_1)\ \dots, (\beta,\alpha_n)) = (c_1, \dots, c_n)\begin{pmatrix} (\alpha_1,\alpha_1) & \cdots & (\alpha_1,\alpha_n) \\ \vdots & & \vdots \\ (\alpha_n,\alpha_1) & \cdots & (\alpha_n,\alpha_1) \end{pmatrix}.$$

Also, $(\cdot,\cdot)$ is nondegenerate, hence the above matrix is invertible. Then

$$(c_1,\ldots,c_n) = ((\beta,\alpha_1)\ \ldots,(\beta,\alpha_n)) \begin{pmatrix} (\alpha_1,\alpha_1) & \cdots & (\alpha_1,\alpha_n) \\ \vdots & & \vdots \\ (\alpha_n,\alpha_1) & \cdots & (\alpha_n,\alpha_1) \end{pmatrix}^{-1} \in \mathbb{R}^n.$$

(XIV) $(\cdot,\cdot)|_E$ is real.
Let $\lambda,\lambda' \in E$. Suppose

$$\lambda = \sum_{\alpha\in\Phi} c_\alpha \alpha, \quad \lambda' = \sum_{\beta\in\Phi} c'_\beta \beta, \quad \text{where } c_\alpha, c'_\beta \in \mathbb{R}.$$

Then

$$(\lambda,\lambda') = \sum_{\alpha,\beta\in\Phi} c_\alpha c'_\beta (\alpha,\beta) \in \mathbb{R}.$$

(XV) $(\cdot,\cdot)|_E$ is positive definite.
Let $\lambda \in E\backslash\{0\}$. By $(*)$ in (XII),

$$(\lambda,\lambda) = \sum_{\gamma\in\Phi} (\gamma,\lambda)^2.$$

Again, since $(\cdot,\cdot)$ is nondegenerate, there exists $\gamma \in \Phi$ such that $(\gamma,\lambda) \neq 0$, which implies $(\lambda,\lambda) > 0$.
This completes the proof of both theorems. □

Remark 4.11. The subalgebra $S_\alpha = \mathrm{Span}\{t_\alpha, u_\alpha, v_\alpha\}$ constructed in the proof satisfies

$$S_\alpha = \mathbb{C}t_\alpha \oplus L_\alpha \oplus L_{-\alpha} \cong \mathfrak{sl}_2(\mathbb{C}).$$

In fact, the condition $\dim L_\alpha = \dim L_{-\alpha} = 1$ immediacy implies $S_\alpha = \mathbb{C}t_\alpha \oplus L_\alpha \oplus L_{-\alpha}$. Let $h_\alpha \in \mathbb{C}t_\alpha$ be the unique element such that $\alpha(h_\alpha) = 2$, namely $h_\alpha = 2t_\alpha/(\alpha,\alpha)$. Then for any $x \in L_\alpha$ and $y \in L_{-\alpha}$, we have

$$[h_\alpha, x] = 2x, \quad [h_\alpha, y] = -2y.$$

Fix $x_\alpha \in L_\alpha$ and $y_\alpha \in L_{-\alpha}$ such that $[x_\alpha, y_\alpha] = h_\alpha$. Then $\{h_\alpha, x_\alpha, y_\alpha\}$ is a basis of S_α, and the linear map $S_\alpha \to \mathfrak{sl}_2(\mathbb{C})$ determined by

$$h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y_\alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is a Lie algebra isomorphism.

4.2. **Root Systems.** The above theorem on Φ motivates the following.

Definition 4.12. Let E be a Euclidean space (i.e., a finite-dimensional real inner product space). A finite subset $\Phi \subset E\backslash\{0\}$ is called a (**reduced**) **root system** in E if

(1) $\mathrm{Span}(\Phi) = E$;
(2) for all $\alpha \in \Phi$, $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$;
(3) for all $\alpha,\beta \in \Phi$, $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ and $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi$.

Remark 4.13. For $\alpha \in E\backslash\{0\}$, the orthogonal reflection $\sigma_\alpha : E \to E$ with respect to the hyperplane $\alpha^\perp$ is given by

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha, \quad \forall \beta \in E.$$

So condition (3) in the definition implies $\sigma_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$. All integers of the form

$$\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$$

are called **Cartan integers**.

Given a complex vector space V, a real subspace $E \subset V$ is called a **real form** of V if $V = E \oplus \sqrt{-1}E$. The above theorem on Φ can be restated as follows.

Theorem 4.14. *There exists a real form E of H^* such that*

- ⋄ $(\cdot,\cdot)|_E$ *is a (real and positive definite) inner product;*
- ⋄ Φ *is a root system in the Euclidean space E.*

Remark 4.15. One can also view $\Phi \subset H$ via the identification $H^* \cong H$, $\alpha \mapsto t_\alpha$. More precisely,

- $H_0 := \mathrm{Span}_{\mathbb{R}}\{t_\alpha : \alpha \in \Phi\}$ is a real form of H;
- $\kappa_L|_{H_0}$ is a (real and positive definite) inner product;
- $\{t_\alpha : \alpha \in \Phi\}$ is a root system in H_0.

From our construction above, note that a semisimple Lie algebra L, together with a Cartan subalgebra $H < L$, gives a root system $\Phi(L, H)$. We will prove that the isomorphism class of $\Phi(L, H)$ is independent of H. This gives a map

$$\{\text{isom classes of semisimple Lie algebras}\} \longrightarrow \{\text{isom classes of root systems}\}.$$

It can be proved that this map is bijective. Therefore,

◇ *Classifying semisimple Lie algebras is reduced to classfifying root systems.*

Here, isomorphism relation between roots systems is defined as follows.

Definition 4.16. Two root systems $\Phi \subset E$ and $\Phi' \subset E'$ are said to be *isomorphic* if there is a linear isomorphism $\iota : E \to E'$ such that $\iota(\Phi) = \Phi'$ and

$$\frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \frac{2(\iota(\beta), \iota(\alpha))}{(\iota(\alpha), \iota(\alpha))}, \quad \forall \alpha, \beta \in \Phi.$$

Caution 4.17. To make the classification problem easier, we do not require ι to be an isometry.

By abuse of notation, we denote the isomorphism class of Φ again by Φ.

4.3. **Conjugacy of Cartan Subalgebras.**

Theorem 4.18 (Conjugacy Theorem)**.** *Let H, H' be Cartan subalgebras of a semisimple Lie algebra L. Then there exists an automorphism $\sigma \in \mathrm{Aut}(L)$ such that $\sigma(H) = H'$.*

The common dimension of Cartan subalgebras is called the **rank** of L.

Corollary 4.19. $\Phi(L, H) \cong \Phi(L, H')$.

Proof. Let $\sigma \in \mathrm{Aut}(L)$ be such that $\sigma(H) = H'$. Consider the linear isomorphism

$$\iota : H^* \to (H')^*, \quad \iota(\alpha)(h') = \alpha(\sigma^{-1}(h')), \quad \forall \alpha \in H^*, h' \in H'.$$

The ι maps $\Phi(L, H)$ onto $\Phi(L, H')$, and it restricts to a linear isomorphism between $E := \mathrm{Span}_{\mathbb{R}}(\Phi)$ and $E' := \mathrm{Span}_{\mathbb{R}}(\Phi')$. Then $\iota|_E$ is an isometry for the inner products on E and E' induced from κ_L. Thus $\Phi(L, H) \cong \Phi(L, H')$. □

For convenience, we denote the isomorphism class of $\Phi(L, H)$ by $\Phi(L)$. Now we derive Conjugacy Theorem 4.18 from the following.

Proposition 4.20 (Open Dense)**.** *Let H be a Cartan subalgebra of a semisimple Lie algebra L, and let*

$$H_{\mathrm{reg}} := H \setminus \bigcup_{\alpha \in \Phi(L,H)} \mathrm{Ker}(\alpha).$$

Then the set

$$\bigcup_{\sigma \in \mathrm{Aut}(L)} \sigma(H_{\mathrm{reg}})$$

contains an open dense subset of L.

Proof of "Open Dense" $\Longrightarrow$ Conjugacy Theorem. Given the Cartan subalgebras $H, H' < L$, the proposition implies

$$\bigcup_{\sigma \in \mathrm{Aut}(L)} \sigma(H_{\mathrm{reg}}) \quad \text{and} \quad \bigcup_{\sigma \in \mathrm{Aut}(L)} \sigma(H'_{\mathrm{reg}})$$

both contain open dense subsets of L. So their intersection is nonempty.

Now let $h \in H_{\mathrm{reg}}$, $h' \in H'_{\mathrm{reg}}$, and $\sigma_1, \sigma_2 \in \mathrm{Aut}(L)$ be such that $\sigma_1(h) = \sigma_2(h')$. Let $\sigma = \sigma_2^{-1}\sigma_1$. Then $\sigma(h) = h'$. It follows that

$$\sigma(H) = \sigma(C_L(h)) = C_L(\sigma(h)) = C_L(h') = H'.$$

□

To prove Proposition 4.20, recall

- for a (finite-dimensional complex) vector space V, the exponential map
$$\exp : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$$
is defined as
$$\exp(x) = e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

It satisfies the following properties:

(1) the series converges uniformly on compact sets;
(2) the map exp is analytic;
(3) $\frac{d}{dt} e^{tx} = x e^{tx}$;
(4) if x is nilpotent, then e^x is a polynomial (in finitely many terms) of x.

Lemma 4.21. *Let D be a derivation of L. Then $e^D \in \mathrm{Aut}(L)$. In particular, for arbitary $x \in L$, $e^{\mathrm{ad}(x)} \in \mathrm{Aut}(L)$.*

Proof. Let $x, y \in L$. To prove $e^D[x,y] = [e^D x, e^D y]$, consider the curve
$$\gamma : \mathbb{R} \to L, \quad \gamma(t) = e^{-tD}[e^{tD}x, e^{tD}y].$$
Then
$$\begin{aligned}
\frac{d}{dt}\gamma(t) &= \left(\frac{d}{dt}e^{-tD}\right)[e^{tD}x, e^{tD}y] + e^{-tD}\left[\left(\frac{d}{dt}e^{tD}\right)x, e^{tD}y\right] + e^{-tD}\left[e^{tD}x, \left(\frac{d}{dt}e^{tD}\right)y\right] \\
&= -De^{-tD}[e^{tD}x, e^{tD}y] + e^{-tD}[De^{tD}x, e^{tD}y] + e^{-tD}[e^{tD}x, De^{tD}y] \\
&= -e^{-tD}D[e^{tD}x, e^{tD}y] + e^{-tD}D[e^{tD}x, e^{tD}y] \\
&= 0.
\end{aligned}$$
It follows that $\gamma = \text{const}$. In particular,
$$e^{-D}[e^{tD}x, e^{tD}y] = \gamma(1) = \gamma(0) = [x,y].$$
So $e^D[x,y] = [e^D x, e^D y]$. □

We also need the following fact from algebraic geometry.

Theorem 4.22. *Let V be a finite-dimensional complex vector space, and let $P : V \to V$ be a polynomial map. Suppose the tangent map $T_{v_0}P : V \to V$ is nonsingular at some point $v_0 \in V$. Then, for any nonzero polynomial function $f : V \to \mathbb{C}$, the image of the set*
$$\{v \in V : f(v) \neq 0\}$$
under P contains an open dense subset of V.

Proof. See [Car05, Corollary 3.11]. □

Example 4.23. Let $f, g, h \in \mathbb{C}[x,y,z]$ be polynomials without constant and first order terms. Then the polynomial map
$$P : \mathbb{C}^3 \to \mathbb{C}^3, \quad (x,y,z) \mapsto (x + f(x,y,z), y + g(x,y,z), z + h(x,y,z))$$
satisfies $T_0 P = \mathrm{id}$. It follows that the system of equations
$$\begin{cases} x + f(x,y,z) = a \\ y + g(x,y,z) = b \\ z + h(x,y,z) = c \end{cases}$$
has solutions $(x,y,z) \in \mathbb{C}^3$ for every (a,b,c) in an open dense subset of $\mathbb{C}^3$.

Proof of the "Open Dense". We want to prove Proposition 4.20 which claims that
$$\bigcup_{\sigma \in \mathrm{Aut}(L)} \sigma(H_{\mathrm{reg}})$$

contains an open dense subset of L, where

$$H_{\mathrm{reg}} := H \setminus \bigcup_{\alpha\in\Phi(L,H)} \mathrm{Ker}(\alpha).$$

(I) Let $\alpha \in \Phi := \Phi(L,H)$, then all $x \in L_\alpha$ are ad-nilpotent.
In fact, let $k > 0$ be such that $\beta \in \Phi \cup \{0\}$. So $\beta + k\alpha \notin \Phi \cup \{0\}$, then

$$\mathrm{ad}(x)^k(L) \subset \sum_{\beta\in\Phi\cup\{0\}} \mathrm{ad}(x)^k(L_\beta) \subset \sum_{\beta\in\Phi\cup\{0\}} L_{\beta+k\alpha} = 0.$$

Suppose $\Phi = \{\alpha_1, \dots, \alpha_m\}$. Consider the map $P : L \to L$ defined by

$$P(h + \sum_{i=1}^m x_i) = e^{\mathrm{ad}(x_1)} \circ \cdots \circ e^{\mathrm{ad}(x_m)} h, \quad \text{where } h \in H,\ x_i \in L_{\alpha_i}.$$

(II) P is a polynomial map.
It suffices to notice

$$\mathrm{ad}(x_i) \text{ is nilpotent} \quad \Longrightarrow \quad e^{\mathrm{ad}(x_i)} \text{ is a polynomial in } \mathrm{ad}(x_i).$$

(III) If $h_0 \in H_{\mathrm{reg}}$ then $T_{h_0}P$ is nonsingular.
If $h \in H$, then

$$(T_{h_0}P)(h) = \frac{d}{dt}\bigg|_{t=0} P(h_0 + th) = \frac{d}{dt}\bigg|_{t=0} (h_0 + th) = h;$$

again, if $x_i \in L_{\alpha_i}$, then

$$(T_{h_0}P)(x_i) = \frac{d}{dt}\bigg|_{t=0} P(h_0 + tx_i) = \frac{d}{dt}\bigg|_{t=0} e^{t\,\mathrm{ad}(x_i)}(h_0) = \mathrm{ad}(x_i)(h_0) = -\alpha_i(h_0)x_i$$

with $\alpha_i(h_0) \neq 0$. So $\mathrm{im}(T_{h_0}P) = L$.

Consider the polynomial function $f : L \to \mathbb{C}$ given by

$$f(h + \sum x_i) = \prod \alpha_i(h).$$

Then $f(h + \sum x_i) \neq 0$ if and only if $h \in H_{\mathrm{reg}}$. On the other hand, by the algebraic geometry fact, the set

$$P(\{x \in L : f(x) \neq 0\})$$

contains an open dense subset of L. Since $e^{\mathrm{ad}(x_i)} \in \mathrm{Aut}(L)$, we have

$$\begin{aligned} P(\{x \in L : f(x) \neq 0\}) &= \{P(h + \sum x_i) : h \in H_{\mathrm{reg}}, x_i \in L_{\alpha_i}\} \\ &\subset \{\sigma(h) : h \in H_{\mathrm{reg}}, \sigma \in \mathrm{Aut}(L)\} \\ &= \bigcup_{\sigma\in\mathrm{Aut}(L)} \sigma(H_{\mathrm{reg}}). \end{aligned}$$

This completes the proof. □

The following theorem is important but we omit the proof.

Theorem 4.24. *The assignment $L \mapsto \Phi(L)$ induces a bijective map*

$$\{\textit{isom classes of semisimple Lie algebras}\} \overset{1-1}{\longleftrightarrow} \{\textit{isom classes of root systems}\}.$$

More precisely, we have the following.

1. *Let L_1, L_2 be semisimple Lie algebras. Suppose $\Phi(L_1) \cong \Phi(L_2)$. Then $L_1 \cong L_2$.*
2. *For any root system Φ, there exists a semisimple Lie algebra L such that $\Phi(L) \cong \Phi$.*

4.4. **Simple Lie Algebras and Irreducible Root Systems.**

Definition 4.25. Let Φ_i be a root system in E_i for $1 \leqslant i \leqslant r$. We view $E_i \subset \bigoplus_{i=1}^r E_i$. Then

$$\bigoplus_{i=1}^r \Phi_i := \bigcup_{i=1}^r \Phi_i$$

is a root system in $\bigoplus_{i=1}^r E_i$, called the **direct sum** of $\Phi_1, \dots, \Phi_r$.

Definition 4.26. A root system Φ in E is said to be

- **reducible** if there exists a nontrivial orthogonal decomposition $E = E_1 \oplus E_2$ such that $\Phi \subset E_1 \cup E_2$;
- and **irreducible** otherwise.

If Φ is reducible and E_1, E_2 are as in the definition, then $\Phi_i := \Phi \cap E_i$ is a root system in E_i, and $\Phi \cong \Phi_1 \oplus \Phi_2$. It follows that any root system Φ is isomorphic to the direct sum of finitely many irreducible ones, called the **irreducible components** of Φ.

Proposition 4.27. *Let L be a semisimple Lie algebra.*

(1) *L is simple if and only if $\Phi(L)$ is irreducible.*
(2) *Let $L = \bigoplus_{i=1}^r L_i$ be the simple ideal decomposition. Then*

$$\Phi(L) \cong \bigoplus_{i=1}^r \Phi(L_i).$$

Note that (1) gives a bijective map

$$\{\text{isom classes of simple Lie algebras}\} \overset{1-1}{\longleftrightarrow} \{\text{isom classes of irreducible root systems}\}.$$

Thus, the problem of classifying simple Lie algebras is reduced to classifying irreducible root systems.

Also, (2) gives a one-to-one correspondences between simple ideals of L and irreducible components of $\Phi(L)$.

Proof. We first prove (2), namely,

$$L = \bigoplus_{i=1}^r L_i \quad \Longrightarrow \quad \Phi(L) \cong \bigoplus_{i=1}^r \Phi(L_i).$$

For each i, let H_i be a Cartan subalgebra of L_i. Then $H := \bigoplus H_i$ is a Cartan subalgebra of L. We view $H_i^* \subset H$ by identifying $\lambda \in H_i^*$ with its extension to H such that $\lambda(\bigoplus_{i \neq j} H_j) = 0$. Then

$$H^* = \bigoplus_{i=1}^r H^*.$$

Claim: $\Phi = \bigcup_{i=1}^r \Phi_i$.

- "$\supset$": for all i and all $\alpha_i \in \Phi_i$, the root subspace $L_{\alpha_i} \subset L_i$ is also a root space for (L, H), whose corresponding root is (the extension of) α_i. So $\alpha_i \in \Phi$.
- "$\subset$": note that

$$L_i = H_i \oplus \bigoplus_{\alpha_i \in \Phi_i} L_{\alpha_i} \quad \Longrightarrow \quad L = H \oplus \bigoplus_{1 \leqslant i \leqslant r,\ \alpha_i \in \Phi_i} L_{\alpha_i}.$$

So the α_i's are all roots Φ.

Let $E_i = \mathrm{Span}_{\mathbb{R}}(\Phi_i)$ and $E = \mathrm{Span}_{\mathbb{R}}(\Phi)$. Then $E = \bigoplus_{i=1}^r E_i$ orthogonally. This proves (2). We now tackle to (1), namely, L is simple if and only if $\Phi(L)$ is irreducible. The $\Longleftarrow$ direction follows from (2).

Now we suppose L is simple and $\Phi(L)$ is reducible. Let $H < L$ be a Cartan subalgebra $\Phi := \Phi(L, H)$. Then $E := \mathrm{Span}_{\mathbb{R}}(\Phi)$ has a nontrivial orthogonal decomposition $E = E_1 \oplus E_2$ such that $\Phi \subset E_1 \cup E_2$. Let

$$\Phi_1 = \Phi \cap E_1, \quad H_1 = \bigcap_{\lambda \in E_2} \mathrm{Ker}(\lambda), \quad L_1 = H_1 \oplus \bigoplus_{\alpha \in \Phi_1} L_\alpha.$$

We claim that $L_1 \triangleleft L$. Note that

$$\alpha \in \Phi_1 \quad \Longrightarrow \quad t_\alpha \in H_1.$$

Also,

$$\alpha \in \Phi_1, \beta \in \Phi \backslash \Phi_1 \quad \Longrightarrow \quad \alpha + \beta \notin \Phi \cup \{0\} \quad \Longrightarrow \quad [L_\alpha, L_\beta] \subset L_{\alpha+\beta} = 0.$$

It follows that

$$[L_1, L] \subset [L_1, H] + \sum_{\beta \in \Phi_1} [L_1, L_\beta] + \sum_{\beta \in \Phi \backslash \Phi_1} [L_1, L_\beta] \subset L_1 + L_1 + 0 = L_1.$$

However, L is not simple since $L_1 \notin \{0, L\}$, which gives a contradiction. $\square$

5. Classification of Root Systems

Recall Definition 4.12 for root systems and the reflection images.

Definition 5.1. Let $\Phi \subset E$ be a root system.

- A subset $\Phi^+ \subset \Phi$ is a **set of positive roots** if there exists a hyperplane $P \subset E$ with $P \cap \Phi = \emptyset$ and a connected component E^+ of $E \backslash P$ such that $\Phi^+ = \Phi \cap E^+$.
- A subset $\Delta \subset \Phi$ is a **base** of Φ (or a **set of simple roots**) if Δ is a basis of E and

$$\Phi \subset \mathrm{Span}_{\mathbb{Z}_{\geqslant 0}}(\Delta) \cup \mathrm{Span}_{\mathbb{Z}_{\leqslant 0}}(\Delta).$$

One can prove the following properties.

- ◇ If Φ^+ is set of positive roots, then

$$\Delta(\Phi^+) := \Phi^+ \backslash (\Phi^+ + \Phi^+)$$

  is a base. Here $\Phi^+ + \Phi^+ := \{\alpha + \beta : \alpha \in \Phi^+, \beta \in \Phi^+\}$.
- ◇ If Δ is a base, then

$$\Phi^+(\Delta) := \Phi \cap \mathrm{Span}_{\mathbb{Z}_{\geqslant 0}}(\Delta)$$

  is a set of positive roots.
- ◇ The assignments $\Phi^+ \mapsto \Delta(\Phi^+)$ and $\Delta \mapsto \Phi^+(\Delta)$ are inverses of each other.
- ◇ This gives a bijection

$$\{\text{sets of positive roots}\} \longleftrightarrow \{\text{bases}\}.$$

Example 5.2 (Root system A_n)**.** Let $n \geqslant 1$. Endow $\mathbb{R}^{n+1}$ with the standard inner product, and let $\{e_1, \dots, e_{n+1}\}$ be the standard basis. Let

$$E = \left\{ \sum_{i=1}^{n+1} x_i e_i : \sum_{i=1}^{n+1} x_i = 0 \right\}.$$

Then

$$\Phi_{A_n} = \{e_i - e_j : i \neq j\}$$

is a root system in E, called the root system of type A_n. A base can be chosen as

$$\Delta_{A_n} = \{e_1 - e_2, \dots, e_n - e_{n+1}\}.$$

Example 5.3 (Root systems B_n, C_n, and D_n)**.** Let $n \geqslant 1$. Endow $\mathbb{R}^n$ with the standard inner product, and let $\{e_1, \dots, e_n\}$ be the standard basis.

(1) The set

$$\Phi_{B_n} = \{\pm e_i \pm e_j : i \neq j\} \cup \{\pm e_i\}$$

is a root system in E, called the root system of type B_n. A base can be chosen as

$$\Delta_{B_n} = \{e_1 - e_2, \dots, e_{n-1} - e_n, e_n\}.$$

(2) The set

$$\Phi_{C_n} = \{\pm e_i \pm e_j : i \neq j\} \cup \{\pm 2e_i\}$$

is a root system in E, called the root system of type C_n. A base can be chosen as

$$\Delta_{C_n} = \{e_1 - e_2, \dots, e_{n-1} - e_n, 2e_n\}.$$

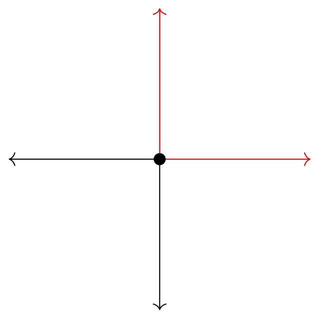
(3) When $n \geqslant 2$, the set

$$\Phi_{D_n} = \{\pm e_i \pm e_j : i \neq j\}$$

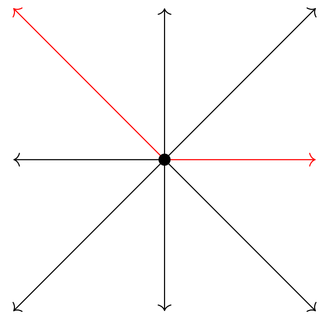
is a root system in E, called the root system of type D_n. A base can be chosen as

$$\Delta_{D_n} = \{e_1 - e_2, \dots, e_{n-1} - e_n, e_{n-1} + e_n\}.$$

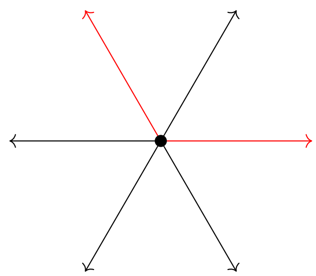
$A_1 \cong B_1 \cong C_1$



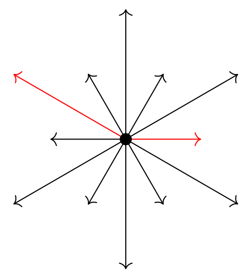
$A_1 \oplus A_1 \cong D_2$



$B_2 \cong C_2$



A_2



G_2

Note that

$$\begin{aligned}
A_n &\cong \text{root system of } \mathfrak{sl}_{n+1}(\mathbb{C}), \quad n \geqslant 1;\\
B_n &\cong \text{root system of } \mathfrak{o}_{2n+1}(\mathbb{C}), \quad n \geqslant 1;\\
C_n &\cong \text{root system of } \mathfrak{sp}_{2n+1}(\mathbb{C}), \quad n \geqslant 1;\\
D_n &\cong \text{root system of } \mathfrak{o}_{2n}(\mathbb{C}), \quad n \geqslant 2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
A_1 \cong B_1 \cong C_1 \quad &\Longrightarrow \quad \mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{o}_3(\mathbb{C}) \cong \mathfrak{sp}_2(\mathbb{C}),\\
A_1 \oplus A_1 \cong D_2 \quad &\Longrightarrow \quad \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{o}_4(\mathbb{C}),\\
B_2 \cong C_2 \quad &\Longrightarrow \quad \mathfrak{o}_5(\mathbb{C}) \cong \mathfrak{sp}_4(\mathbb{C}),\\
A_3 \cong D_3 \quad &\Longrightarrow \quad \mathfrak{sl}_4(\mathbb{C}) \cong \mathfrak{o}_6(\mathbb{C}).
\end{aligned}$$

Also,

(1) A_n $(n \geqslant 1)$, B_n $(n \geqslant 1)$, C_n $(n \geqslant 1)$, D_n $(n \geqslant 3)$ are irreducible;
(2) D_2 is reducible.

Correspondingly,

(1) $\mathfrak{sl}_{n+1}(\mathbb{C})$ $(n \geqslant 1)$, $\mathfrak{sp}_{2n}$ $(n \geqslant 1)$, $\mathfrak{o}_n(\mathbb{C})$ $(n = 3$ or $n \geqslant 5)$ are simple;
(2) $\mathfrak{o}_4(\mathbb{C})$ is not simple.

For $\alpha, \beta \in \Phi$, denote $c_{\alpha\beta} = 2(\beta, \alpha)/(\alpha, \alpha)$.

Proposition 5.4. *Let $\Delta \subset \Phi$ be a base and assume $\alpha, \beta \in \Delta$ are distinct with $|\alpha| \geqslant |\beta|$. Then*

(1) $(\alpha, \beta) \leqslant 0$;

(2) $c_{\alpha\beta}c_{\beta\alpha} \in \{0,1,2,3\}$*; moreover,*

$$\begin{aligned}
c_{\alpha\beta}c_{\beta\alpha} = 0 &\iff \alpha \perp \beta, \\
c_{\alpha\beta}c_{\beta\alpha} = 1 &\iff \angle(\alpha,\beta) = \frac{2\pi}{3},\ |\alpha| = |\beta|, \\
c_{\alpha\beta}c_{\beta\alpha} = 2 &\iff \angle(\alpha,\beta) = \frac{3\pi}{4},\ |\alpha| = \sqrt{2}|\beta|, \\
c_{\alpha\beta}c_{\beta\alpha} = 3 &\iff \angle(\alpha,\beta) = \frac{5\pi}{6},\ |\alpha| = \sqrt{3}|\beta|.
\end{aligned}$$

Proof. (1) By definition, the condition $\beta - c_{\alpha\beta}\alpha \in \Phi$ implies that $c_{\alpha\beta} \leqslant 0$, and hence $(\alpha,\beta) \leqslant 0$.

(2) Let $\theta = \angle(\alpha,\beta)$. Then

$$c_{\alpha\beta}c_{\beta\alpha} = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = 4\cos^2\theta \in \{0,1,2,3\}.$$

The other statements are easy to check. □

Definition 5.5. Let $\Delta \subset \Phi$ be a base of a root system. The **Dynkin diagram** $\mathcal{D} = \mathcal{D}(\Phi,\Delta)$ is defined to be

- the graph with vertex set Δ,
- in which α and β ($\alpha \neq \beta$) are joined by $c_{\alpha\beta}c_{\beta\alpha} \in \{0,1,2,3\}$ edges,
- with an arrow pointing to β if $c_{\alpha\beta}c_{\beta\alpha} \in \{2,3\}$ and $|\alpha| > |\beta|$.

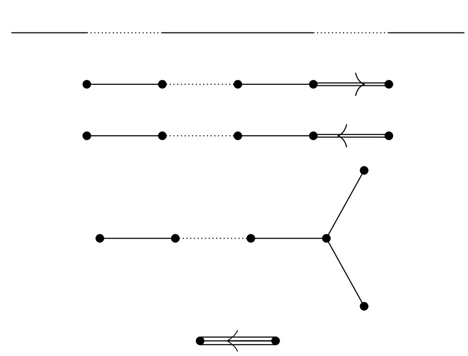
A_n ($n \geqslant 1$)

B_n ($n \geqslant 1$)

C_n ($n \geqslant 1$)

D_n ($n \geqslant 2$)

G_2



The isomorphism class of $\mathcal{D}(\Phi,\Delta)$ is independent of Δ. To explain this, we introduce the following definition.

Definition 5.6. The subgroup of $O(E)$ generated by orthogonal reflections $\{\sigma_\alpha \mid \alpha \in \Phi\}$ is called the **Weyl group** of Φ, denoted by $W = W(\Phi)$.

By regarding W as a permutation group on Φ, we see $|W| < \infty$. It can be proved that W acts simply transitively on the set of bases. In particular, if Δ_1 and Δ_2 are bases, then there exists $\sigma \in O(E)$ such that $\sigma(\Delta_1) = \Delta_2$. It follows by definition that $\mathcal{D}(\Phi,\Delta_1) = \mathcal{D}(\Phi,\Delta_2)$. We denote (the isomorphism class of) $\mathcal{D}(\Phi,\Delta)$ by $\mathcal{D}(\Phi)$, called the **Dynkin diagram** of Φ.

Theorem 5.7. *Root systems and Dynkin diagrams are in a one-to-one correspondence. That is,*

- *two root systems Φ_1 and Φ_2 are isomorphic if and only if $\mathcal{D}(\Phi_1) \cong \mathcal{D}(\Phi_2)$.*
- *Φ is irreducible if and only if $\mathcal{D}(\Phi)$ is connected.*

The exceptional isomorphisms between low dimensional Lie algebras can be seen from Dynkin diagrams:

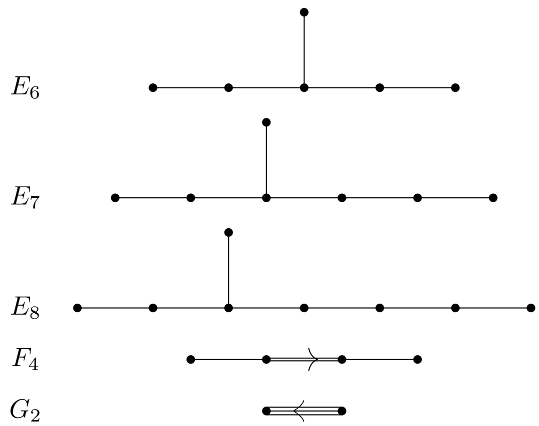
Dynkin Diagrams	Root Systems	Lie Algebras
$\mathcal{D}(A_1) \cong \mathcal{D}(B_1) \cong \mathcal{D}(C_1)$	$A_1 \cong B_1 \cong C_1$	$\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{o}_3(\mathbb{C}) \cong \mathfrak{sp}_2(\mathbb{C})$
$\mathcal{D}(A_1 \oplus A_1) \cong \mathcal{D}(D_2)$	$A_1 \oplus A_1 \cong D_2$	$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{o}_4(\mathbb{C})$
$\mathcal{D}(B_2) \cong \mathcal{D}(C_2)$	$B_2 \cong C_2$	$\mathfrak{o}_5(\mathbb{C}) \cong \mathfrak{sp}_4(\mathbb{C})$
$\mathcal{D}(A_3) \cong \mathcal{D}(D_3)$	$A_3 \cong D_3$	$\mathfrak{sl}_4(\mathbb{C}) \cong \mathfrak{o}_6(\mathbb{C})$

By classifying connected Dynkin diagrams, one can prove the classification theorem.

Theorem 5.8. *Any irreducible root system is isomorphic to one of the following:*

- A_n ($n \geqslant 1$)*;*

- *B_n ($n \geqslant 2$);*
- *C_n ($n \geqslant 3$);*
- *D_n ($n \geqslant 4$);*
- *one of the 5 exceptional root systems, denoted E_6, E_7, E_8, F_4, G_2 respectively.*



6. Representations

Let L be a (finite-dimensional complex) Lie algebra. Recall that:

- a **representation** of L on a (finite-dimensional complex) vector space V is a homomorphism $\phi : L \to \mathfrak{gl}(V)$.

It will be convenient to also use the language of L-module.

Definition 6.1. A (finite-dimensional complex) vector space V is called an L**-module** if a bilinear operation

$$L \times V \to V, \quad (x, v) \mapsto xv$$

is given and satisfies

$$[x, y]v = x(yv) - y(xv), \quad \forall x, y \in L,\ v \in V.$$

A representation $\phi : L \to \mathfrak{gl}(V)$ gives an L-module structure on V by $xv = \phi(x)v$. Conversely, an L-module structure on V gives a representation $\phi : L \to \mathfrak{gl}(V)$ by $\phi(x)v = xv$.

6.1. Basic Notions.

Definition 6.2. Let $\phi : L \to \mathfrak{gl}(V)$ be a representation, namely, V is an L-module.

(1) A subspace $W \subset V$ is called an **invariant subspace** if $\phi(L)W \subset W$. In this case,
   - the representation
   $$\phi_W : L \to \mathfrak{gl}(W), \quad \phi_W(x) = \phi(x)|_W$$
   is called a **subrepresentation** of ϕ;
   - W (endowed with the restricted module structure) is called a **submodule** of V.

(2) Let $W \subset V$ be an invariant subspace.
   - The representation
   $$\phi_{V/W} : L \to \mathfrak{gl}(V/W), \quad \phi_{V/W}(x)(v + W) = \phi(x)v + W$$
   is called a **quotient representation** of ϕ;
   - V/W (endowed with the induced module structure) is called a **quotient module** of V.

Example 6.3. Let V be an L-module. Then

$$V^L := \{v \in V : xv = 0, \forall x \in L\}$$

is a submodule.

Definition 6.4. Let $\phi : L \to \mathfrak{gl}(V)$ and $\psi : L \to \mathfrak{gl}(W)$ be representations.

(1) A linear map $f : V \to W$ is **equivariant**, or a **homomorphism of** L**-modules**, if
$$f(xv) = x(fv), \quad \forall x \in L,\ v \in V.$$
Here $xv = \phi(x)v$ and $xw = \psi(x)v$.

(2) A bijective equivariant linear map is called an **equivalence** between ϕ and ψ, also called an **isomorphism of** L**-modules**.

(3) If there exists an equivalence $V \to W$, we say that ϕ and ψ are **equivalent**, or the L-modules V and W are **isomorphic**.

Denote

$$\mathrm{Hom}(V, W) := \{\text{linear maps } V \to W\},$$
$$\mathrm{Hom}_L(V, W) := \{L\text{-module homomorphisms } V \to W\}.$$

If $f \in \mathrm{Hom}_L(V, W)$, then $\mathrm{Ker}(f)$ is a submodule of V, and $\mathrm{im}(f)$ is a submodule of W. A natural L-module structure on $\mathrm{Hom}(V, W)$ can be defined by

$$(xf)v = x(fv) - f(xv), \quad \forall x \in L,\ f \in \mathrm{Hom}(V, W),\ v \in V.$$

The following fact is clear.

Proposition 6.5. *Let V and W be L-modules. Then*

$$\mathrm{Hom}(V, W)^L = \mathrm{Hom}_L(V, W).$$

Definition 6.6. Let $\phi : L \to \mathfrak{gl}(V)$ be a representation.

- ϕ is said to be **irreducible** if V is nonzero and has no nontrivial invariant subspaces. (In this case, we also say the L-module V is **irreducible** or **simple**.)
- ϕ is said to be **completely reducible** if for any invariant subspace $W \subset V$, there exists an invariant subspace $W' \subset V$ such that $V = W \oplus W'$.
(In this case, we also say the L-module V is **completely reducible** or **semisimple**.)

Note that all irreducible representations are completely reducible by definition. Also, V is completely reducible if and only if V is the direct sum of finitely many irreducible submodules.

Theorem 6.7 (Schur's lemma). *Let V be an irreducible L-module. Then*

$$\mathrm{Hom}_L(V, V) = \mathbb{C}\mathrm{id}_V.$$

Proof. Let $f \in \mathrm{Hom}_L(V, V)$. Let $a \in \mathbb{C}$ be an eigenvalue of f. Then

$$\begin{aligned} \mathrm{Ker}(f - a \cdot \mathrm{id}_V) \text{ is a nonzero submodule} \quad &\Longrightarrow \quad \mathrm{Ker}(f - a \cdot \mathrm{id}_V) = V \\ &\Longrightarrow \quad f = a \cdot \mathrm{id}_V. \end{aligned}$$

□

6.2. **Weyl's Theorem on Complete Reducibility.**

Theorem 6.8 (Weyl). *Any representation of a semisimple Lie algebra is completely reducible.*

Lemma 6.9. *Let $L < \mathfrak{gl}(V)$ be (nonzero and) semisimple. Then there exists $c \in \mathfrak{gl}(V)$ such that*

$$[c, L] = 0, \quad \mathrm{tr}(c) \neq 0, \quad \textit{and } \mathrm{im}(c) \subset \sum_{x \in L} \mathrm{im}(x).$$

Proof. Let $\{x_1, \dots, x_n\}$ be a basis of L. Since the trace form of L is nondegenerate, there exists a basis $\{y_1, \dots, y_n\}$ of L such that $\mathrm{tr}(x_i x_j) = \delta_{ij}$. We prove that $c := \sum_{i=1}^n x_i y_i$ satisfies the requirements. The requirement on $\mathrm{im}(c)$ is clear. Also,

$$\mathrm{tr}(c) = \sum_{i=1}^n \mathrm{tr}(x_i y_i) = n \neq 0.$$

It remains to verify $[c, L] = 0$. Note that

$$z = \sum_{j=1}^n \mathrm{tr}(z y_j) x_j = \sum_{j=1}^n \mathrm{tr}(x_j z) y_j, \quad z \in L.$$

So for all $w \in L$, we have

$$\begin{aligned} &= \sum_{i=1}^n [w, x_i] y_i + \sum_{i=1}^n x_i [w, y_i] \\ &= \sum_{i=1}^n \left(\sum_{j=1}^n \mathrm{tr}([w, x_i] y_j) x_j \right) y_i + \sum_{i=1}^n x_i \left(\sum_{j=1}^n \mathrm{tr}(x_j [w, y_i]) y_j \right) \\ &= \sum_{i,j=1}^n \mathrm{tr}([w, x_j] y_i) x_i y_j + \sum_{i,j=1}^n \mathrm{tr}(x_j [w, y_j]) x_i y_j \\ &= \sum_{i,j=1}^n (\mathrm{tr}([w, x_j] y_i) + \mathrm{tr}(x_j [w, y_j]) x_i y_j) = 0. \end{aligned}$$

□

Remark 6.10. The element c constructed above is called the **Casimir operator** of L. It is independent of the choice of the basis $\{x_1, \dots, x_n\}$.

Lemma 6.11. *Let L be a semisimple Lie algebra, let V and W be L-modules, and let $f \in \mathrm{Hom}_L(V, W)$. Then*

$$f(V)^L = f(V^L).$$

Proof. We induct on $\dim \mathrm{Ker}(f)$. The $\mathrm{Ker}(f) = 0$ case is trivial. Suppose $\dim \mathrm{Ker}(f) > 0$ and the lemma holds for smaller $\dim \mathrm{Ker}(f)$. We divide the proof into two cases.

Case 1. Suppose the L-module $\mathrm{Ker}(f)$ is reducible. Let $U \subset \mathrm{Ker}(f)$ be a nontrivial submodule. Then there is a natural commutative diagram

$$\begin{tikzcd} V \arrow[r, "f_1"] \arrow[rr, bend right, "f"'] & V/U \arrow[r, "f_2"] & W \end{tikzcd}$$

of L-module homomorphisms. Note that

$$\dim \mathrm{Ker}(f_i) < \dim \mathrm{Ker}(f), \quad i = 1, 2.$$

By the induction hypothesis,

$$f(V^L) = f_2(f_1(V^L)) = f_2(f_1(V)^L) = f_2(f_1(V))^L = f(V)^L.$$

Case 2. Suppose $\mathrm{Ker}(f)$ is irreducible. Clearly, one obtains $f(V^L) \subset f(V)^L$. We need to prove $f(V)^L \subset f(V^L)$. Replacing W and V with $f(V)^L$ and $f^{-1}(f(V)^L)$ respectively, we may assume $W^L = W = f(V)$. It suffices to prove $f(V^L) = W$, that is,

$$V = \mathrm{Ker}(f) + V^L.$$

The $V^L = V$ case is trivial. Suppose $V^L \neq V$. Let $\phi : L \to \mathfrak{gl}(V)$ denote the representation corresponding to the L-module V. Then $\phi(L) < \mathfrak{gl}(V)$ is nonzero and semisimple. By the above lemma, there exists $c \in \mathfrak{gl}(V)$ such that

$$[c, \phi(L)] = 0, \quad \mathrm{tr}(c) \neq 0, \quad \mathrm{im}(c) \subset \sum_{x \in L} \mathrm{im}(\phi(x)).$$

From the condition $[c, \phi(L)] = 0$, we see $c \in \mathrm{Hom}_L(V, V)$. Also,

$$W^L = W \quad \Longrightarrow \quad \mathrm{im}(\phi(x)) \subset \mathrm{Ker}(f), \quad \forall x \in L \quad \Longrightarrow \quad \mathrm{im}(c) \subset \mathrm{Ker}(f).$$

One can show that $c|_{\mathrm{Ker}(f)} \neq 0$ (if not, then $c^2 = 0$ and hence $\mathrm{tr}(c) = 0$, a contradiction). By the irreducibility of $\mathrm{Ker}(f)$, $c|_{\mathrm{Ker}(f)}$ can be nothing but a nonzero scalar by Schur's lemma (Theorem 6.7). Therefore, $V = \mathrm{Ker}(f) \oplus \mathrm{Ker}(c)$. For all $x \in L$,

$$\begin{aligned} \phi(x)(\mathrm{Ker}(c)) \subset \mathrm{Ker}(c) \cap \mathrm{Ker}(f) = 0 \quad &\Longrightarrow \quad \mathrm{Ker}(c) \subset V^L \\ &\Longrightarrow \quad V = \mathrm{Ker}(f) \oplus \mathrm{Ker}(c) \subset \mathrm{Ker}(f) + V^L. \end{aligned}$$

This completes the proof. □

Proof of Weyl's Theorem 6.8. Let L be a semisimple Lie algebra, V an L-module, and $W \subset V$ a submodule. We need to prove that there exists a submodule $W' \subset V$ such that $V = W \oplus W'$.

Consider the L-modules $\mathrm{Hom}(V, W)$ and $\mathrm{Hom}(W, W)$. The map

$$\mathrm{Hom}(V, W) \to \mathrm{Hom}(W, W), \quad f \mapsto f|_W$$

is a surjective L-module homomorphism. Note that $\mathrm{id}_W \in \mathrm{Hom}_L(W, W) = \mathrm{Hom}(W, W)^L$. The above lemma deduces that there exists some $f \in \mathrm{Hom}(V, W)^L = \mathrm{Hom}_L(V, W)$ such that

$$f|_W = \mathrm{id}_W.$$

Finally, the submodule $\mathrm{Ker}(f) \subset V$ satisfies $V = W \oplus \mathrm{Ker}(f)$. □

6.3. Application of Weyl's Theorem: Jordan Decomposition.

Theorem 6.12. *Let $L < \mathfrak{gl}(V)$ be semisimple. Then for every $x \in L$, we have $x_s, x_n \in L$.*

Proof. By Weyl's theorem, the L-module V is completely reducible. Suppose $V = \bigoplus_{i=1}^r V_i$, where each V_i is an irreducible submodule. For all $x \in L$, we see $xV_i \subset V_i$. By the classical Jordan-Chevalley decomposition, x_n is a polynomial of x, so that $x_n V_i \subset V_i$. In particular, $x_n|_{V_i}$ is nilpotent, and $\mathrm{tr}(x_n|_{V_i}) = 0$.

We denote $\mathrm{ad} = \mathrm{ad}_{\mathfrak{gl}(V)}$. Then $\mathrm{ad}(x)(L) \subset L$. Since $\mathrm{ad}(x_n) = \mathrm{ad}(x)_n$ is a polynomial of $\mathrm{ad}(x)$, we get $\mathrm{ad}(x_n)L \subset L$. Note that $\mathrm{ad}(x_n)|_L$ is a derivation of L, which must be inner. Hence there exists some $y \in L$ such that $\mathrm{ad}(x_n)|_L = \mathrm{ad}(y)|_L$. Therefore,

$$[x_n - y, L] = 0 \quad \Longrightarrow \quad x_n - y \in \mathrm{Hom}_L(V, V) \quad \Longrightarrow \quad x_n|_{V_i} - y|_{V_i} \in \mathrm{Hom}_L(V_i, V_i).$$

Now by Schur's lemma, $x_n|_{V_i} - y|_{V_i} \in \mathbb{C}\mathrm{id}_{V_i}$. On the other hand, for all $y \in L = [L, L]$, $\mathrm{tr}(y|_{V_i}) = 0 = \mathrm{tr}(x_n|_{V_i})$, which implies $x_n|_{V_i} - y|_{V_i} = 0$. So $x_n = y \in L$, and $x_s = x - x_n \in L$ as well. □

Corollary 6.13. *Let $L < \mathfrak{gl}(V)$ be semisimple, and let $\phi : L \to \mathfrak{gl}(W)$ be a representation.*

1. *For any $x \in L$, we have $\phi(x_s) = \phi(x)_s$ and $\phi(x_n) = \phi(x)_n$.*
2. *If $x \in L$ is semisimple (resp. nilpotent), then so is $\phi(x)$.*

Proof. (1) Consider the graph of ϕ, namely

$$\widetilde{L} := \{(x, \phi(x)) : x \in L\} < \mathfrak{gl}(V \oplus W).$$

It turns out that $\widetilde{L} \cong L$, hence is semisimple. By the above theorem, for all $x \in L$,

$$(x_s, \phi(x)_s) = (x, \phi(x))_s \in \widetilde{L}.$$

This implies $\phi(x_s) = \phi(x)_s$. Similarly, $\phi(x_n) = \phi(x)_n$.

(2) $x \in L$ is semisimple, so $\phi(x) = \phi(x_s) = \phi(x)_s$ (by (1)) is semisimple. Similarly, when $x \in L$ is nilpotent, so also is $\phi(x)$. □

For a general semisimple L, there are embeddings $L \hookrightarrow \mathfrak{gl}(V)$. For example, $\mathrm{ad} : L \to \mathfrak{gl}(L)$ is an embedding. If $\phi : L \to \mathfrak{gl}(V)$ is an embedding, one can pull back the Jordan decomposition on $\phi(L)$ to get a decomposition on L. Such a decomposition on L is independent of ϕ, as the following corollary states.

Corollary 6.14. *Let $\phi : L \to \mathfrak{gl}(V)$ and $\psi : L \to \mathfrak{gl}(W)$ be two embeddings, and let $x \in L$.*

1. *We have $\phi^{-1}(\phi(x)_s) = \psi^{-1}(\psi(x)_s)$ and $\phi^{-1}(\phi(x)_n) = \psi^{-1}(\psi(x)_n)$.*
2. *$\phi(x)$ is semisimple (resp. nilpotent) if and only if so is $\psi(x)$.*

Proof. (1) Consider the representation $\psi\phi^{-1} : \phi(L) \to \mathfrak{gl}(W)$. By the previous corollary,

$$(\psi\phi^{-1})(\phi(x)_s) = (\psi\phi^{-1})(\phi(x))_s = \psi(x)_s.$$

Taking ψ^{-1} on both sides, we get the first formula. The second one is similar.

(2) Suppose $\phi(x)$ is semisimple or nilpotent. By the previous corollary,

$$\phi(x) = (\psi\phi^{-1})(\phi(x))$$

has the same property. Similarly, if $\psi(x)$ is semisimple or nilpotent, then so is $\phi(x)$. □

Let us redefine the "abstract Jordan decomposition" on L.

Definition 6.15. Let L be a semisimple Lie algebra. Choose an embedding $\phi : L \to \mathfrak{gl}(V)$.

- $x \in L$ is said to be **semisimple**/**nilpotent** if $\phi(x)$ has the same property.
- For $x \in L$, denote $x_s = \phi^{-1}(\phi(x)_s)$ and $x_n = \phi^{-1}(\phi(x)_n)$. The decomposition $x = x_s + x_n$ is called the **abstract Jordan decomposition** of x.

By the above Corollary, these notions are independent of the choice of ϕ.

Remark 6.16. (1) If $L < \mathfrak{gl}(V)$, the inclusion map $L \hookrightarrow \mathfrak{gl}(V)$ is an embedding. So
   - ⋄ the abstract Jordan decomposition on L coincides with the usual one;
   - ⋄ it is safe to use the notations x_s and x_n.

(2) Previously, we defined the abstract Jordan decomposition $x = x_{(s)} + x_{(n)}$ using the adjoint representation. Clearly, the two definitions coincide.

Corollary 6.17. *Let L and K be semisimple Lie algebras, and let $\phi : L \to K$ be a homomorphism.*

1. *For any $x \in L$, we have $\phi(x_s) = \phi(x)_s$ and $\phi(x_n) = \phi(x)_n$.*
2. *If $x \in L$ is semisimple (resp. nilpotent), then so also is $\phi(x)$.*

Proof. By taking embeddings $L \hookrightarrow \mathfrak{gl}(V)$ and $K \hookrightarrow \mathfrak{gl}(W)$, we may assume that L and K are linear. Then the results follow from a previous corollary. □

6.4. **Representations of $\mathfrak{sl}_2(\mathbb{C})$.** Let us classify representations of $\mathfrak{sl}_2(\mathbb{C})$. By Weyl's theorem, it is enough to classify irreducible ones.

In this subsection, denote

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

One can check that $\{h, x, y\}$ is a basis of $\mathfrak{sl}_2(\mathbb{C})$, and

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h.$$

Let $\phi : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$ be a representation. Then $\phi(h)$ is semisimple. For $\lambda \in \mathbb{C}$, denote

$$V_\lambda = \{v \in V : \phi(h)v = \lambda v\}.$$

If $V_\lambda \neq 0$, then λ is called a **weight**, and V_λ is called a **weight space**. Denote the (finite) set of weights by

$$\Lambda := \{\lambda \in \mathbb{C} : V_\lambda \neq 0\}.$$

Then decomposition

$$V = \bigoplus_{\lambda \in \Lambda} V_\lambda$$

is called the **weight space decomposition**.

Example 6.18. For $m \geqslant 0$, identify $\mathfrak{gl}(\mathbb{C}^{m+1}) \cong \mathfrak{gl}_{m+1}(\mathbb{C})$, and denote

$$h_m = \begin{pmatrix} m & & & & \\ & m-2 & & & \\ & & \ddots & & \\ & & & -(m-2) & \\ & & & & -m \end{pmatrix},$$

$$x_m = \begin{pmatrix} 0 & m & & & \\ & 0 & m-1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \quad y_m = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 2 & \ddots & & \\ & & \ddots & 0 & \\ & & & m & 0 \end{pmatrix}.$$

It is straightforward to check

$$[h_m, x_m] = 2x_m, \quad [h_m, y_m] = -2y_m, \quad [x_m, y_m] = h_m.$$

Therefore, the linear map $\phi_m : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(\mathbb{C}^{m+1})$ determined by

$$\phi_m(h) = h_m, \quad \phi_m(x) = x_m, \quad \phi_m(y) = y_m$$

is an $(m+1)$-dimensional representation. We have $\phi_0 = 0$, $\phi_1 = \mathrm{id}$, and $\phi_2 \cong \mathrm{ad}$. The weights for ϕ_m are $\{m, m-2, \ldots, -(m-2), -m\}$, namely the diagonal elements of h_m. For $0 \leqslant k \leqslant m$, the weight space $(\mathbb{C}^{m+1})_{m-2k} = \mathbb{C}e_{k+1}$. In particular, $(\mathbb{C}^{m+1})_0 \oplus (\mathbb{C}^{m+1})_1$ is 1-dimensional.

Theorem 6.19. *Keep the notations as above.*

(1) *The representations $\phi_0, \phi_1, \ldots$ are all irreducible.*
(2) *Any irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ is equivalent to some ϕ_m.*

Proof. We begin with proving (2) first. Let $\phi : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$ be an irreducible representation, namely, V is an irreducible $\mathfrak{sl}_2(\mathbb{C})$-module. We want to prove $\phi \cong \phi_m$ for some $m \geqslant 0$.

(I) For all $\lambda \in \mathbb{C}$, $xV_\lambda \subset V_{\lambda+2}$ and $yV_\lambda \subset V_{\lambda-2}$.
If $v \in V_\lambda$, then

$$h(xv) = [h, x]v + x(hv) = 2xv + \lambda xv = (\lambda + 2)xv \quad \Longrightarrow \quad xv \in V_{\lambda+2},$$
$$h(yv) = [h, x]v + y(hv) = -2yv + \lambda xv = (\lambda - 2)xv \quad \Longrightarrow \quad yv \in V_{\lambda-2}.$$

Since the set of weights Λ is finite, there is a weight $\lambda \in \mathbb{C}$ such that $\lambda + 2 \notin \Lambda$.

(II) We prove the following identities:
  (a) $hv_k = (\lambda - 2k)v_k$ $(k \geqslant 0)$;

(b) $xv_k = (\lambda - k + 1)v_{k-1}$ $(k \geqslant 0)$;
(c) $yv_k = (k+1)v_{k+1}$ $(k \geqslant -1)$.

(a) follows from Step 1. (c) follows from the definition of v_k. We prove (b) by induction. For $k = 0$, we have $xv_0 \in xV_\lambda \subset V_{\lambda+2} = 0$. So (b) holds for $k = 0$. Suppose $k \geqslant 1$ and (b) holds for $k-1$. Then

$$\begin{aligned} kxv_k &\overset{\text{(c)}}{=} x(yv_{k-1}) = [x,y]v_{k-1} + y(xv_{k-1}) \\ &\overset{\text{(b)}}{=} hv_{k-1} + (\lambda - k + 2)yv_{k-2} \\ &\overset{\text{(a)}}{=} (\lambda - 2k + 2)v_{k-1} + (\lambda - k + 2)(k-1)v_{k-1} \\ &= k(\lambda - k + 1)v_{k-1}. \end{aligned}$$

This implies (b) for k.

(III) (a) shows that nonzero v_k are linearly independent, hence there is $m \geqslant 0$ such that $v_m \neq 0$ and $v_{m+1} = 0$. Also, (b) with $k = m+1$ dictates that $\lambda = m$, and then (a)-(c) become

$$(*) \qquad \begin{cases} hv_k = (m-2k)v_k, \\ xv_k = (m-k+1)v_{k-1}, \quad 0 \leqslant k \leqslant m. \\ yv_k = (k+1)v_{k+1}, \end{cases}$$

Moreover, V is irreducible and $\bigoplus_{k=0}^m \mathbb{C}v_k$ is an invariant subspace. Then

$$V = \bigoplus_{k=0}^{m} \mathbb{C}v_k.$$

Consequently, $\mathcal{B} = \{v_0, \ldots, v_m\}$ is a basis of V. It follows from $(*)$ that

$$[\phi(h)]_{\mathcal{B}} = h_m, \quad [\phi(x)]_{\mathcal{B}} = x_m, \quad [\phi(y)]_{\mathcal{B}} = y_m.$$

So ϕ is equivalent to ϕ_m.

This proves (2). The following is the proof of (1) that each ϕ_m is irreducible.

Recall that $(\mathbb{C}^{m+1})_0 \oplus (\mathbb{C}^{m+1})_1$ is 1-dimensional. Let $\mathbb{C}^{m+1} = \bigoplus_{i=1}^r W_i$ be an irreducible submodule decomposition. By (2), the subrepresentation on each W_i is equivalent to some ϕ_{m_i}, so $(W_i)_0 \oplus (W_i)_1$ is 1-dimensional. For $\lambda \in \mathbb{C}$,

$$(\mathbb{C}^{m+1})_\lambda = \bigoplus_{i=1}^{r} (W_i)_\lambda .$$

So

$$(\mathbb{C}^{m+1})_0 \oplus (\mathbb{C}^{m+1})_1 = \left(\bigoplus_{i=1}^{r} (W_i)_0 \right) \oplus \left(\bigoplus_{i=1}^{r} (W_i)_1 \right) = \bigoplus_{i=1}^{r} ((W_i)_0 \oplus (W_i)_1).$$

Taking dimensions on both sides, we get $1 = r$. So ϕ_m is irreducible. This completes the proof. □

References

[Car05] Roger W. Carter. *Lie algebras of finite and affine type*. Number 96. Cambridge University Press, 2005.

[Hum12] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9. Springer Science & Business Media, 2012.

School of Mathematical Sciences, Peking University, 100871, Beijing, China
Email address: daiwenhan@pku.edu.cn