Stacks of p-isogenies with G-structure

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Let $p>2$ prime.

<u>Motivation</u> Construct an action of spherical Hk alg $\mathcal{H}(G)$ on local / global Shimura vars at hyperspecial level at p.

<u>Coherent automorphic setting</u> (Fakhruddin-Pilloni).

Shimura var of abelian type:

$Sh_K(G,X)$ / E = reflex field

$\rightsquigarrow$ $\mathcal{S}_K^{(tor)}(G,X)$ / $\mathcal{O}_{E_v}$ integral model at $v|p$.

$\mathcal{V}_K$ automorphic bdl on $Sh_K(G,X)$

(extends over $\mathcal{S}_K^{tor}(G,X)$).

Consider cohom $R\Gamma(\mathcal{S}_K^{tor}(G,X), \mathcal{V}_K)$,

want an action of $\mathbb{Z}$-integral spherical Hecke alg.

($\mathbb{Z}[T_{x_1}, \dots, T_{x_n}] \subset \mathcal{H}(G)$).

<u>K_0 setting</u> (Li-Rapoport-Zhang)

Rapoport-Zink space $RZ(G,b,\mu)$

$\llcorner$ sm formal sch.

Want an action on $K_0^{\sigma}(RZ(G,b,\mu)) \circlearrowleft \mathcal{H}(G)$.

$\llcorner$ Gillet-Soulé K-theory.

__First observation__ For $\mathbb{1}_{K_p g K_p} \in \mathcal{H}(G)$, want a diagram

$$\begin{tikzcd} & C_{[K_p g K_p]} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X \end{tikzcd}$$

$\mathcal{V}_K$ on each X.

algebraic corresp.

p_1, p_2 are proper.

Here $X = RZ(G, b, \mu)$ or $\mathscr{S}_K^{\text{for}}(G, X)$

(locally) (globally)

C = natural object obtained by changing level.

__But__ If we want to extend this integrally and get an action as desired, need that p_1, p_2 to be <u>lci</u> maps.

__Want__ (1) Cohomological corresp.

$$p_2^* \mathcal{V}_K \longrightarrow p_1^! \mathcal{V}_K \quad \text{(all derived functors)}$$

which should be (normalized) composition:

$$p_2^* \mathcal{V}_K \longrightarrow p_1^* \mathcal{V}_K \longrightarrow p_1^! \mathcal{V}_K$$

naturally comes from description of C

given by $p_1^* \mathcal{O}_X \to p_1^! \mathcal{O}_X$ (fundamental class)

(i.e. $\mathcal{O}_C \to \omega_{C/X}$ for lci p_1)

__Note__ In lci situation, can always construct this fundamental class.

(2) Moreover, composition is given by a derived tensor product.

$$\begin{tikzcd} & & C \times^{\mathbb{L}} D \arrow[dl] \arrow[dr] \arrow[d, phantom, "\wedge"] & & \\ & C \arrow[dl] \arrow[dr] & {} & D \arrow[dl] \arrow[dr] & \\ X & & X & & X \end{tikzcd}$$

Need "coherent base change".

Consider examples of C which are derived schemes.

<u>Goal</u> Produce quasi-smooth derived schemes.

Want to construct moduli functor on animated comm rings.

- Does not work naively,
  work instead with local construction.
- Work over the stack of p-divisible grps BT_∞.

<u>Thm</u> (Gardner-Madapusi-Mathew)

Describe this BT_∞ as moduli functor on animated comm rings.

i.e. given $R \longrightarrow R^{syn}$, (syntomification <u>stack</u> of Bhatt-Lurie

moduli functor of GL_n-torsors of R^{syn} with HT wts $\mu = (1, \cdots, 1, 0, \cdots, 0)$.

<u>Want</u> Construct stacks of p-isogenies over $BT_\infty = BT_\infty^{G,\mu}$ (G-torsors on R^{syn}, HT wts given by μ).

<u>Idea</u> Construct isogeny models.

<u>Def'n</u> An <u>m-isogeny model</u> is a sch $Z / \mathbb{Z}_p$ w/ action of $\mathcal{G}^{m+1}$ (red model of $G/\mathbb{Z}_p$). and an isom $j: Z_{\mathbb{Q}_p} \xrightarrow{\sim} \mathcal{G}^m_{\mathbb{Q}_p}$, equivariant for G^{m+1}-action

Example $G = GL_d$, 1-isogeny model Z^r is

$$Z^r(R) = \{(A, B) \in \mathrm{Mat}_d(R) \times \mathrm{Mat}_d(R) : AB = BA = p^r I_d\}$$

+ choice $j = j_s : Z^r_{\mathbb{Q}_p} \xrightarrow{\sim} GL_{d,\mathbb{Q}_p}$

$$(A, B) \longmapsto p^s A.$$

Upshot Given an m-isogeny model Z,

can construct 1-bounded stack (GMM)

by taking $(X^\diamond, X^\circ)$ over $B\mathbb{G}_m$:

$$\begin{array}{ccc} X^\diamond = [Z/\mathcal{G}^{m+1}], & X^\circ = [Z'/M_\mu^{m+1}] & \\ \downarrow & & \text{(where } Z' = \text{fixed pt locus).} \\ [*/\mathcal{G}^{m+1}]. & & \end{array}$$

1-bounded stack $\mathcal{X}_Z = (X^\diamond, X^\circ) \rightsquigarrow \mathrm{Isog}_Z \longrightarrow BT_\infty^{G,\mu}$.

Roughly, Isog_Z "=" syntomic sections of $X^\diamond$

w/ HT cochar determined by X°.

Assuming $\pi_1, \ldots, \pi_m : \mathrm{Isog}_Z \longrightarrow BT_\infty^{G,\mu}$ proper projection maps.

Thm (Imai-Kato-Youcis) $\exists$ formally étale syntomic realization map

$$\hat{\mathcal{S}}_K(G, X) \longrightarrow BT_\infty^{G,\mu}.$$

$\rightsquigarrow$ Define

$$\begin{tikzcd} \widehat{p\text{-}\mathrm{Isog}_Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Isog}_Z \arrow[d, "\pi_1"] & \leftarrow \text{choose this map.} \\ \hat{\mathcal{S}}_K(G, X) \arrow[r, two heads] & BT_\infty^{G,\mu} & \text{(quasi-sm)} \end{tikzcd}$$

as derived fibre product (so $\widehat{p\text{-}\mathrm{Isog}_Z}$ not a sch / $\hat{\mathcal{S}}_K$!)

<u>Expect</u> If Z is 1-isog model & $G = GL_d$,

$\widehat{p\text{-Isog}_Z}$ is the Conn components completion of $\underbrace{p\text{-isog}}$ along special fibre

(Faltings–Chai : moduli space of p-power isogenies / $\mathcal{S}_K(G, X)$.

In this case, Isog_Z was classical formal scheme.

For $m \geq 2$, expect genuinely derived schs.

<u>Prop</u> For any 1-isog model Z, Isog_Z is classical.

<u>So</u> $\widehat{p\text{-Isog}_Z} \longrightarrow \hat{\mathcal{S}}_K(G, X)$

is a map of classical formal schs that is lci.

Can algebraize this formal sch.

For $m > 1$: should think of these as compositions.

$\rightsquigarrow$ reduced to studying m-isogeny models :

$$\underset{\text{2-isog model}}{(Z^{r_1} \times Z^{r_2})} \longrightarrow \underset{\text{1-isog model.}}{Z^{r_1 + r_2}}$$

<u>Claim</u> Set of isog models should produce a geometrization of spherical Hecke algebras !

$$(Z_1) \cdot (Z_2) \longrightarrow (Z_1 \times Z_2).$$