



Minimal surfaces

Xin Zhou

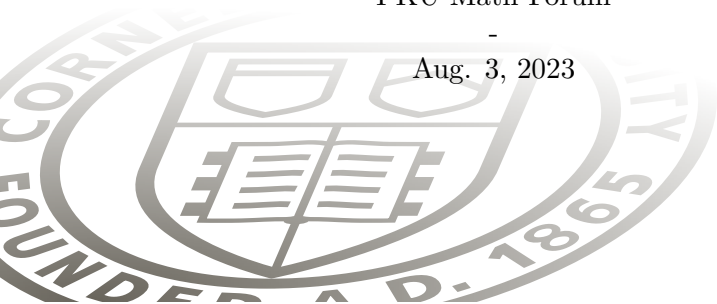
Cornell University

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PKU Math Forum

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① Introduction

② Volume Spectrum and Multiplicity One

③ Minimal Spheres

1. Introduction

Minimal surfaces

Minimal surfaces are surfaces in an equilibrium position and they are defined as critical points for the area functional.

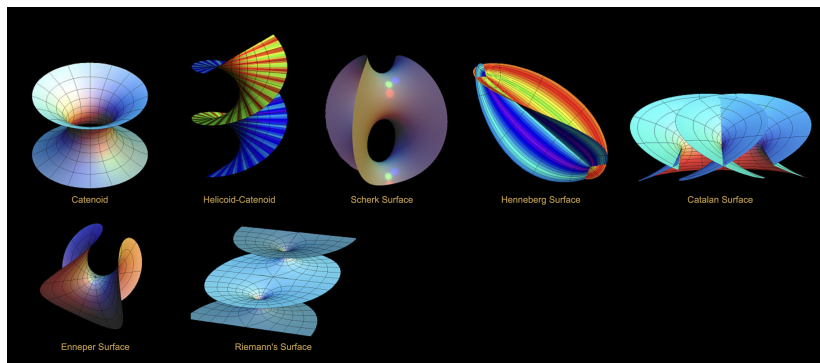
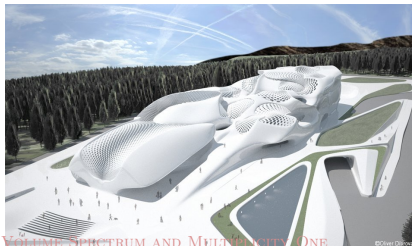
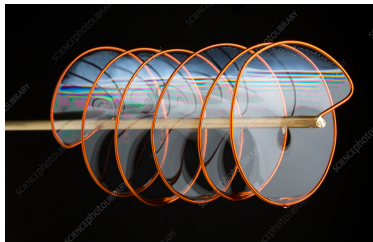


Figure: www.virtualmathmuseum.org

Minimal surfaces

Minimal surfaces are physical objects and appear naturally in applied science as soap films, outermost horizons in Relativity, and architecture structures.



Minimal surfaces

Minimal surfaces have been used to solve several open problems in geometry.

- Schoen-Yau: Positive Mass Theorem in General Relativity.
- Siu-Yau: Frankel conjecture in Algebraic Geometry.
- Micallef-Moore: Sphere Theorem in Riemannian Geometry.
- Marques-Neves: Willmore conjecture in surface theory.

First variation

Let (M^{n+1}, g) be a closed Riemannian manifold of dimension $(n + 1)$. Denote by $\Sigma^n \subset M$ an embedded hypersurface.

Σ is a minimal hypersurface if Σ locally minimizes area.

The first variation of Area of Σ along any vector field $X \in \mathfrak{X}(M)$ is given by

$$\delta \text{Area}_\Sigma(X) = \left. \frac{d}{dt} \right|_{t=0} \text{Area}(\Sigma_t) = \int_\Sigma H \langle \mathbf{n}, X \rangle,$$

where \mathbf{n} is a unit normal of Σ , and H is the mean curvature of Σ .

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where \mathbf{n} is a unit normal of Σ , and H is the mean curvature of Σ .

First variation

$$\begin{aligned}\Sigma \text{ is minimal} &\iff \delta \text{Area}_\Sigma = 0 \\ &\iff H \text{ (mean curvature)} \equiv 0.\end{aligned}$$

Examples

Let $M = \mathbb{S}^3 = \{(x, y, z, w) : x^2 + y^2 + z^2 + w^2 = 1\} \subset \mathbb{R}^4$.

- Equator: $\mathbb{S}^2 = \{(x, y, z, 0) : x^2 + y^2 + z^2 = 1\} \subset \mathbb{S}^3$;
- Clifford torus: $T_c^2 = \{(x, y, z, w) : x^2 + y^2 = z^2 + w^2 = \frac{1}{2}\} \subset \mathbb{S}^3$;
- Lawson surfaces: genus- g embedded minimal surfaces in \mathbb{S}^3 .

Plateau's problem

According to Wiki:

"In mathematics, Plateau's problem is to show the existence of a minimal surface with a given boundary, a problem raised by Joseph-Louis Lagrange in 1760. However, it is named after Joseph Plateau who experimented with soap films."

It was solved in 1930 independently by Jesse Douglas and Tibor Radó, and this initiated systematically investigations on the existence theory of minimal surfaces.

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Two conjectures of Yau

In his 1982 problem list, S. T. Yau posed the following two problems:

Problem 88

Prove that any 3-dimensional (closed) manifold must contain an infinite number of immersed (closed) minimal surfaces.

Problem 89

Prove that there are four distinct embedded minimal spheres in any manifold diffeomorphic to S^3 .

In this talk, we will survey recent progress of these two conjectures.

Major results

Theorem (Marques-Neves 13, A. Song 18)

Any closed manifold (M^{n+1}, g) with $3 \leq n+1 \leq 7$ admits infinitely many closed embedded minimal hypersurfaces.

Theorem (Z. Wang - Z., 23)

Assume that g is a bumpy metric or a metric with positive Ricci curvature on S^3 . Then there exist at least four distinct embedded minimal two-spheres in (S^3, g) .

A metric g is bumpy on M^{n+1} if every closed minimal hypersurface is **non-degenerate** as a critical point of the area functional. White proved that the set of bumpy metrics is generic in the Baire sense in 1991.

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History on Yau's 1st conjecture

Theorem (Almgren 65, Pitts 81, Schoen-Simon 81)

Every closed Riemannian manifold (M^{n+1}, g) contains one closed minimal hypersurface Σ^n , which is smoothly embedded away from a singular set of co-dimension 7.

- Yau's 1st conjecture was motivated by this result and the existence results of infinitely many closed geodesics on surfaces.
- If (M^{n+1}, g) has nontrivial n -dim homology under \mathbb{Z} or \mathbb{Z}_2 , the existence of Σ follows from a minimization procedure using standard tools in Geometric Measure Theory (GMT).
- When the n -dim homology vanishes, this follows from a min-max procedure based on GMT (to be introduced later).

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- When the n -dim homology vanishes, this follows from a min-max procedure based on GMT (to be introduced later).

Now let us assume $3 \leq n + 1 \leq 7$, so the min-max hypersurface Σ is everywhere smoothly embedded.

- Marques-Neves, 13 confirmed Yau's 1st conjecture under $\text{Ric}_g > 0$ condition.
- Ire-Marques-Neves, 17 confirmed Yau's 1st conjecture for a generic set of smooth metrics.
- A. Song, 18 fully confirmed Yau's 1st conjecture based on Marques-Neves' work and an ingenious contradiction argument.
- Z. 19 provided a direct proof of Yau's 1st conjecture for all bumpy metrics by solving the Multiplicity One Conjecture.

History on Yau's 2nd conjecture

The use of GMT and homological relations makes it hard to prescribe the topology of Σ , even when in dimension $n + 1 = 3$.

Theorem (Simon-Smith, 82)

Every Riemannian 3-sphere (S^3, g) contains at least one embedded minimal 2-sphere.

- This combined Almgren-Pitts with the works of Almgren-Simon 79 and Meeks-Simon-Yau 82 on isotopy minimizing problems.
- Yau's 2nd conjecture was motivated by the fact (by Hatcher's proof of Smale Conjecture) that the space of embedded $S^2 \hookrightarrow S^3$ deformation retracts to the space of great spheres in S^3 , which is homeomorphic to \mathbb{RP}^3 , with cup-length equal to 4.

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- White, 91 proved that, using a degree theory argument, every (S^3, g) with $\text{Ric}_g > 0$ contains at least Two embedded minimal 2-spheres.
- In the same paper, White also proved that if the metric g is sufficiently close to the round metric, then (S^3, g) contains at least Four embedded minimal 2-spheres.
- Halshofer-Ketover, 17 proved that (S^3, g) , with g a bumpy metric, contains at least Two embedded minimal 2-spheres.
- Sacks-Uhlenbeck, 81 proved the existence of a branched immersed minimal 2-sphere in any (M^n, g) when $\pi_k(M)$ is nontrivial for some $k \geq 2$.

A common major challenge

The major challenge of using the GMT version of min-max theory to prove the existence of multiple solutions is the [Existence of Integer Multiplicity](#) (to be specified later).

We will discuss two general Multiplicity One Theorems in this talk.

2. Volume Spectrum and Multiplicity One Conjecture

Spectrum – a toy model

Example: critical points for quadratic forms

Let A be an $n \times n$ symmetric matrix. Its k -th eigenvalue is given by

$$\lambda_k = \min_{P \subset \mathbb{R}^n} \max_{x \in P, x \neq 0} Q_A(x), \text{ where } Q_A(x) = \frac{\langle Ax, x \rangle}{\langle x, x \rangle},$$

where P is a k -dimensional linear subspace.

Since $Q_A(x)$ is invariant under \mathbb{R}^* -action, we can take the quotient of P, \mathbb{R}^n under this action:

$$P \setminus \{0\} \rightarrow \mathbb{RP}^{k-1}, \quad \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{RP}^{n-1}.$$

$$\text{Then } \lambda_k = \min_{\mathbb{RP}^{k-1} \subset \mathbb{RP}^{n-1}} \max_{x \in \mathbb{RP}^{k-1}} Q_A(x).$$

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Volume Spectrum – space of cycles

Theorem (Almgren 1961)

The space of all closed separating hypersurfaces $\Sigma^n \subset M^{n+1}$, modulo the \mathbb{Z}_2 -action on identifying the two orientations, satisfies:

$$\mathcal{Z}_n(M, \mathbb{Z}_2) = \{\Sigma = \partial\Omega \sim \partial(M \setminus \Omega)\} \simeq \mathbb{RP}^\infty.$$

Therefore, the \mathbb{Z}_2 -cohomological ring is:

$$\mathcal{H}^*(\mathcal{Z}_n(M, \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2[\lambda].$$

Volume Spectrum – k -sweepouts

Definition: k -sweepout

A k -sweepout ($k \in \mathbb{N}$) is a continue map: $\Phi : X \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2)$, such that

$$\Phi^*(\lambda^k) \neq 0 \in H^k(X, \mathbb{Z}_2).$$

X is an arbitrary finite dimensional parameter space.

Roughly speaking, Φ is a k -sweepout, if given any k points $\{p_1, \dots, p_k\}$ in M , there exists a parameter $x \in X$, such that

the hypersurface $\Phi(x)$ contains all p_1, \dots, p_k .

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Volume Spectrum – Definition

Volume spectrum: Gromov 88, Guth 10, Marques-Neves 13

The k -th volume spectrum is

$$\omega_k(M, g) = \inf_{\Phi: k\text{-sweepout}} \max_{x \in X = \text{dom}(\Phi)} \text{Area}(\Phi(x)).$$

Weyl Law: Liokumovich-Marques-Neves 16

$$\omega_k(M, g) \sim a_n \text{Vol}(M)^{\frac{n}{n+1}} k^{\frac{1}{n+1}}.$$

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Min-max Theorem

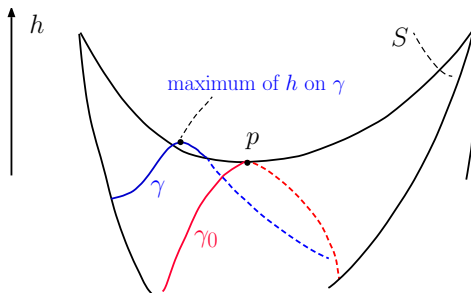
Theorem (Almgren-Pitts, Schoen-Simon, Marques-Neves)

$\{\omega_k(M, g)\}$ are achieved by areas of closed minimal hypersurfaces (smoothly embedded when $2 \leq n \leq 6$) *counted with multiplicity*, i.e.

$$\omega_k = \sum_{i=1}^{l_k} m_{k,i} \text{Area}(\Sigma_{k,i}), \quad m_{k,i} \in \mathbb{N}_{>0}.$$

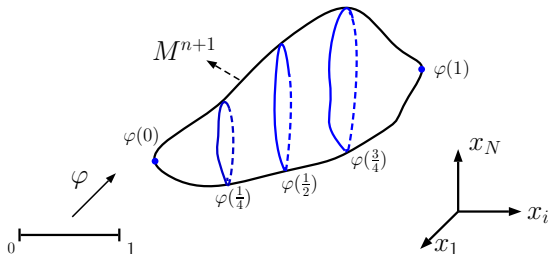
Min-max method – a toy model

Multivariable Calculus: h is the height function on S , and p is a saddle point.



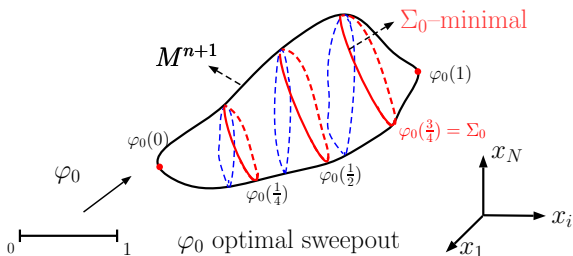
$$h(p) = \max_{t \in [0,1]} h(\gamma_0(t)) = \min_{\gamma \in [\gamma_0]} \max_{t \in [0,1]} h(\gamma(t)).$$

Min-max theorem – 1-sweepouts



- $\varphi : [0, 1] \rightarrow \text{space of hypercycles}$, – “1-sweepout”;
- Min-max value —“width”:

$$L = \inf \left\{ \max_{t \in [0, 1]} \text{Area}(\phi(t)) : \phi \text{ is a 1-sweepout} \right\}.$$



Theorem (Almgren 1961, Pitts 1981, Schoen-Simon 1981)

The width L is achieved by the area of some closed minimal hypersurface Σ_0 counted with integer multiplicity (with a codim-7 singular set).

Integer multiplicity/density may appear as compactness/convergence of minimal hypersurfaces were essentially used.

Yau's 1st conjecture

Conjecture (Yau 1982)

Prove that any 3-dimensional (closed) manifold must contain an infinite number of immersed (closed) minimal surfaces.

In 2013, Marques and Neves initiated a program toward this conjecture using volume spectrum and the min-max theorem. This conjecture was finally proved by Song in 2018 via this program.

The key challenge was that existence of multiplicities may cause re-occurrence of minimal hypersurfaces when applying the min-max theorem to higher ω_k , so one may not produce genuine new solutions!

Marques and Neves in 2014 raised the Multiplicity One Conjecture, which asserted that for a bumpy metric (which is a generic notion), the multiplicities are always one.

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Multiplicity One Conjecture

Theorem (Z. 19)

Let M^{n+1} be a closed manifold with $3 \leq n+1 \leq 7$. For a bumpy metric g , for each $k \in \mathbb{N}$, we have

$$\omega_k = \sum_{i=1}^{l_k} \text{Area}(\Sigma_{k,i}).$$

That is, all multiplicities are exactly 1.

Theorem (Z. 19)

Let (M^{n+1}, g) be a closed Riemannian manifold with $3 \leq n+1 \leq 7$. For each $k \in \mathbb{N}$, if the multiplicity $m_{k,i} > 1$, then

- either $\Sigma_{k,i}$ is weakly stable,*
- or $\Sigma_{k,i}$ is 1-sided, and its 2-sided double cover is weakly stable.*

Multiplicity One Conjecture

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Some remarks

The second variation of Area of Σ along any vector field $X = f\mathbf{n}$ with $f \in C_c^1(\Sigma)$ is given by

$$\delta^2 \text{Area}_\Sigma(f, f) = \left. \frac{d^2}{dt^2} \right|_{t=0} \text{Area}(\Sigma_t) = \int_\Sigma f L_\Sigma f d\mathcal{H}^n,$$

where the **Jacobi operator** $L_\Sigma f = -\Delta_\Sigma f + Qf$ is linear and elliptic.

- Σ is said to be **stable** if the first eigenvalue $\lambda_1(L_\Sigma) \geq 0$.
- Σ is said to be **weakly stable** if $\lambda_1(L_\Sigma) = 0$, that is, there exists some $\varphi \geq 0$ s.t.

$$L_\Sigma \varphi = 0.$$

Proof: PMC approximations

- Perturb Area to $\mathcal{A}^{\epsilon h}(\Omega) = \text{Area}(\partial\Omega) - \epsilon \int_{\Omega} h \, d\text{Vol}$, with $\epsilon \rightarrow 0$. Here $h \in C^{\infty}(M)$.
- Since \mathcal{A}^h is only defined on $\mathcal{C}(M)$, consider the double cover

$$\pi : \mathcal{C}(M) \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2),$$

and interpret ω_k , a min-max value on $\mathcal{Z}_n(M, \mathbb{Z}_2)$, as a relative min-max value of on $\mathcal{C}(M)$.

- When h is chosen in a generic manner, the min-max value $\omega_{k,\epsilon}$ of $\mathcal{A}^{\epsilon h}$ is achieved by some $\Omega_{k,\epsilon} \subset M$ with smooth, multiplicity one boundary by the PMC min-max theory of Z. - Zhu 18.
- If h is carefully chosen, and if $\partial\Omega_{k,\epsilon}$ converges to a smooth limit Σ_k with higher multiplicity, then Σ_k has to be a degenerate minimal hypersurface, contradicting with bumpyness of the metric.

3. Minimal Spheres

Yau's 2nd conjecture

Conjecture (Yau 1982)

Prove that there are four distinct embedded minimal spheres in any manifold diffeomorphic to S^3 .

Topology of the space of 2-spheres

By Smale Conjecture, the closure of the space \mathcal{X} of smoothly embedded S^2 in S^3 (including degenerate embeddings) is homotopic to \mathbb{RP}^4 minus a ball. Therefore

$$H^*(\overline{\mathcal{X}}, \partial\mathcal{X}, \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/[\alpha^5].$$

Let $[-1, 1] \times \mathbb{RP}^3$ to denote the twisted $[-1, 1]$ -bundle over \mathbb{RP}^3 , and $[a_0, a_1, a_2, a_3, a_4]$ to denote a point in $[-1, 1] \times \mathbb{RP}^3$. When $a_0 \neq \pm 1$, let

$$\mathcal{G}([a_0, a_1, a_2, a_3, a_4]) := \{a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = a_0\} \cap \mathbb{S}^3;$$

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Consider the four maps:

$$\Psi_1 : [-1, 1] \times \mathbb{RP}^0 \rightarrow \overline{\mathcal{X}}, \quad a_0 \mapsto \mathcal{G}(a_0, 1, 0, 0, 0);$$

$$\Psi_2 : [-1, 1] \times \mathbb{RP}^1 \rightarrow \overline{\mathcal{X}}, \quad [a_0, a_1, a_2] \mapsto \mathcal{G}(a_0, a_1, a_2, 0, 0);$$

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We can consider min-max values associated with Ψ_i :

$$\mathbf{L}_i = \inf_{\Phi \sim \Psi_i} \sup_{x \in \text{dom}(\Phi)} \mathcal{H}^2(\Phi(x)).$$

By the Lusternik-Schnirelmann theory, we have that if (S^3, g) contains only finitely many embedded minimal 2-spheres, then

$$0 < \mathbf{L}_1 < \mathbf{L}_2 < \mathbf{L}_3 < \mathbf{L}_4.$$

Multiplicity One Theorem

- Let $\Phi_0 : X \rightarrow \mathcal{X}$ be a fixed continuous map.
- Let Π be the homotopy class of Φ_0 relative to $\Phi_0 : Z \rightarrow \mathcal{X}$, $Z \subset X$.

Define

$$\mathbf{L}(\Pi) = \inf_{\Phi \in \Pi} \sup_{x \in X} \mathcal{H}^2(\Phi(x)).$$

Theorem (Wang- Z. 23)

If $\mathbf{L}(\Pi) > \sup_{x \in Z_0} \mathcal{H}^2(\Phi_0(x)) > 0$, then

$$\mathbf{L}(\Pi) = m_1 \mathcal{H}^2(\Gamma_1) + \cdots m_l \mathcal{H}^2(\Gamma_l),$$

where $\{\Gamma_j\}$ is a disjoint collection of embedded minimal 2-spheres, so that

- if $m_j > 1$, then Γ_j is stable.

Proof of generic four spheres theorem

Theorem (Z. Wang - Z., 23)

Assume that g is a bumpy metric or $\text{Ric}_g > 0$. Then there exist at least four distinct embedded minimal two-spheres in (S^3, g) .

- If there is no stable minimal surfaces in (S^3, g) , e.g. $\text{Ric}_g > 0$, this follows directly the previous Multiplicity One Theorem.
- If g is bumpy and there exists one strictly stable minimal $S^2 \hookrightarrow S^3$, we used a cutting argument as follows:

Proof of Multiplicity One Theorem

Idea: approximate Area by $\mathcal{A}^{\epsilon h}(\Omega) = \mathcal{H}^2(\partial\Omega) - \epsilon \int_{\Omega} h d\text{Vol}$, where $\partial\Omega$ is an embedded 2-sphere.

- Develop a PMC min-max theory for the $\mathcal{A}^{\epsilon h}$ functional in the space of embedded 2-spheres.
- Choose suitable $h \in C^\infty(S^3)$ such that min-max solutions Σ_ϵ of $\mathcal{A}^{\epsilon h}$ has “multiplicity one” when $\epsilon \ll 1$.
- Show the the limit of Σ_ϵ when $\epsilon \rightarrow 0$ has to be a stable minimal 2-sphere if the multiplicity is greater than 1.

PMC min-max theory: why $C^{1,1}$?

Theorem (Wang - Z. 23)

The min-max solution of \mathcal{A}^h is $C^{1,1}$ closed, almost embedded, strongly \mathcal{A}^h -stationary surface Σ .

- $C^{1,1}$ regularity is natural for the isotopy problem:
- We introduced a new scheme of proving min-max regularity as $C^{1,1}$ solutions may not satisfy Unique Continuation property.
- We may assume the min-max solutions Σ_{ϵ_k} associated with $\mathcal{A}^{\epsilon_k h}$ converges to a smoothly embedded minimal 2-sphere Σ_∞ .

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Choose h to change sign along the limit minimal surface Σ_∞ :

- The strong \mathcal{A}^h -property implies that Σ has to be an “almost” multiplicity one surface:
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Thank you for your attention!