Cohomology of Quasicoherent Sheaves

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§1 A Fundamental Theorem about Affine Schemes

(and a Bogus Proof).

4th fund thm : X affine sch, $\mathcal{F} \in \mathrm{Qcoh}(X)$.

$\Rightarrow$ $H^i(X, \mathcal{F}) = 0$, $\forall i > 0$ i.e. $\mathcal{F}$ acyclic.

sheaf cohom.

(!) Bogus Proof. $X = \mathrm{Spec} A$, $\mathcal{F} = \widetilde{M}$, $M \in \mathrm{Mod}_A$.

enough inj's

What's wrong?

$\rightsquigarrow$ $M \to I$ mono s.t. I inj A-mod.

$$0 \to \widetilde{M} \to \widetilde{I} \to \widetilde{I/M} \to 0$$

($\rightsquigarrow$ $0 \to M \to I \to I/M \to 0$ taking $\Gamma(X, -)$)

$\Rightarrow$ in cohom long exact seq. $\delta^i = 0$, $i \geq 0$.

Also, $H^i(X, \widetilde{I}) = 0$, $\forall i > 0$ $\Rightarrow$ $H^1(X, \widetilde{M}) = 0$.

Moreover, $\forall i > 1$, $H^i(X, \widetilde{M}) \cong H^{i-1}(X, \widetilde{I/M})$

$\rightsquigarrow$ proved by dim shifting. $\square$

I inj. in Mod_A $\Rightarrow$ $\widetilde{I}$ inj. in $\mathrm{Qcoh}(\mathrm{Mod}_{\mathcal{O}_X})$

$\not\Rightarrow$ $\widetilde{I}$ inj. in $\mathrm{Sh}(\mathrm{Mod}_{\mathcal{O}_X})$

In particular: I inj. $\overset{?}{\Rightarrow}$ $\widetilde{I}$ flasque

Two Ways to Fix

(1) in nt rings, inj. $\Rightarrow$ flasque. (c.f. Hartshorne Prop II.5.6)

(2) (EGA) compute $\check{H}$ instead of H.

Lemma $X = \operatorname{Spec} A$, $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$ exact $/\mathrm{Mod}_{\mathcal{O}_X}$.

s.t. $\mathcal{F}_1 \in \mathrm{Qcoh}$, $\mathcal{F}$ & $\mathcal{F}_2$ arbitrary.

$\Rightarrow \quad 0 \to \Gamma(X, \mathcal{F}_1) \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}_2) \to 0$ exact.

This implies that $\delta^0 : H^0(X, \mathcal{F}_2) \to H^1(X, \mathcal{F}_1)$ is zero

$\Rightarrow \quad 0 \to H^1(X, \mathcal{F}_1) \to H^1(X, \mathcal{F})$ inj.

If $\mathcal{F}$ inj. $\Rightarrow H^1(X, \mathcal{F}_1) = 0$.

§2 Applications

Cor $X \in \mathrm{Sch}$, $\mathcal{U} = \{U_i\}_{i \in I}$ open cover. $\forall J \subseteq I$ finite, $U_J = \bigcap_{i \in J} U_i$ affine.

$\Rightarrow \forall \mathcal{F} \in \mathrm{Qcoh}(X)$, sh cohom of $\mathcal{F}$ is given by Čech cohom:

$$H^i(X, \mathcal{F}) = \check{H}^i(\mathcal{U}, \mathcal{F}).$$

Recall X separate $\Rightarrow \operatorname{Spec} A_i \cap \operatorname{Spec} A_j = \operatorname{Spec} B$.

aff $\cap$ aff $=$ aff (opens)

Cor X sep sch. $\mathcal{U} = \{U_i\}_{i \in I}$ open cover.

$\Rightarrow \forall \mathcal{F} \in \mathrm{Qcoh}(X)$, $H^i(X, \mathcal{F}) = \check{H}^i(\mathcal{U}, \mathcal{F})$

Useful in computing $\boxed{H^i(\mathbb{P}^r, \mathcal{O}(n))}$. (next notes)

Useless Cor $f_1, \cdots, f_n \in A$ cring, $(1) = (f_1, \cdots, f_n)$.

$\rightsquigarrow \mathcal{U} = \{D(f_i)\}$ open cover of $X = \operatorname{Spec} A$

$\Rightarrow \forall M \in \mathrm{Mod}_A$, $\check{H}^0(\mathcal{U}, \widetilde{M}) = M$, $\check{H}^i(\mathcal{U}, M) = 0$ $(i > 0)$.

§3 A Correct Proof

Step 1 Show that $0 \to M \to \check{C}^0(\mathcal{U}, \widetilde{M}) \to \check{C}^1(\mathcal{U}, \widetilde{M}) \to \cdots$ exact.

$\overset{\Gamma(X,-)}{\longleftarrow\!\!\!\sim\!\!\!\sim}$ $0 \to \widetilde{M} \to \check{\mathcal{C}}^0(\mathcal{U}, \widetilde{M}) \to \check{\mathcal{C}}^1(\mathcal{U}, \widetilde{M}) \to \cdots$ (as Qcohs).

the 2nd seq'ce is exact by computing at stalks.
Moreover, constituent sheaves are quasicoh.

b/c $\check{C}^i(\mathfrak{U},\tilde{M}) = \bigoplus j_{U*}(\tilde{M}|_U) = \tilde{M_g}$.

$U = \bigcap_{i\in J} U_i$ for some $J \subseteq I$ finite.
$= D(g)$, $g \in A$

Step 2 $0 \to M \to \check{C}^0(\mathfrak{U},\tilde{M}) \to \check{C}^1(\mathfrak{U},\tilde{M}) \to \cdots$ exact.

$\Rightarrow$ $\check{H}^0(\mathfrak{U},\tilde{M}) = M$, $\check{H}^i(\mathfrak{U},\tilde{M}) = 0$ $(i>0)$.

$\Rightarrow$ by taking $\varinjlim_{\mathfrak{U}}$ under all opens

$\check{H}^0(X,\tilde{M}) = M$, $\check{H}^i(X,\tilde{M}) = 0$ $(i>0)$.

(every $\mathfrak{U}$ can be refined to a finite cover by distinguished opens).

[Caveat X not Hausdorff here
(can't use $\varinjlim_{\mathfrak{U}\neq} \check{H}^i(\mathfrak{U},\mathcal{F}) = H^i(X,\mathcal{F})$)
lim on refinements]

$\Rightarrow$ $H^0(X,\tilde{M}) = M$, $H^i(X,\tilde{M}) = 0$ $(i>0)$

by the following thm of Cartan.

Thm (Cartan) $X \in \mathrm{Top}$. $\mathcal{B}$ a basis of X, $U_i \cap U_j \in \mathcal{B}$ for $U_i, U_j \in \mathcal{B}$.
$\mathcal{F} \in \mathrm{Sh}_{Ab}(X)$ s.t. $\check{H}^i(U,\mathcal{F}) = 0$, $\forall U \in \mathcal{B}$.
$\Rightarrow$ $\check{H}^i(X,\mathcal{F}) \cong H^i(X,\mathcal{F})$, $\forall i \geq 0$.

§4 Comparison of Čech and Sheaf Cohomology

On flasque sheaves:

Lemma $X \in \mathrm{Top}$. $\mathcal{F} \in \mathrm{Sh}_{Ab}(X)$ s.t. $\check{H}^1(X,\mathcal{F}) = 0$. Then
for any $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ exact,

$$0 \to \Gamma(X,\mathcal{F}) \to \Gamma(X,\mathcal{G}) \to \Gamma(X,\mathcal{H}) \to 0 \text{ exact.}$$

Proof. Check right surjectivity. $\forall s \in \Gamma(X,\mathcal{H})$,

$\exists\, \mathcal{U} = \{U_i\}_{i\in I}$ s.t. $\forall i \in I$, $t_i \longmapsto s|_{U_i}$

$\Gamma(U_i,\mathcal{G}) \longrightarrow \Gamma(U_i,\mathcal{H})$.

$\forall i,j \in I$, put $u_{ij} = t_i|_{U_i\cap U_j} - t_j|_{U_i\cap U_j} \in \Gamma(U_i\cap U_j,\mathcal{G})$.

Čech 1-cocycle of $\mathcal{F}$.

(also view u_{ij} as elt in $\Gamma(U_i\cap U_j,\mathcal{F})$
since $u_{ij} \mapsto 0 \in \Gamma(U_i\cap U_j,\mathcal{G})$.)

Now $\check{H}^1(X,\mathcal{F}) = 0 \Rightarrow \mathcal{U}$ refinments

s.t. u_{ij} becomes a Čech cobundary

i.e. $v_i|_{U_i\cap U_j} - v_j|_{U_i\cap U_j} = u_{ij}$ $(\forall i,j\in I)$

$(v_i \in \Gamma(U_i,\mathcal{F}),\ \forall i)$

$\Rightarrow$ $w_i = t_i - v_i \in \Gamma(U_i,\mathcal{G})$.

$\Rightarrow$ $w_i|_{U_i\cap U_j} - w_j|_{U_i\cap U_j} = 0$ (by computation)

$\Rightarrow$ $w \in \Gamma(X,\mathcal{G})$ lifting $s \in \Gamma(X,\mathcal{H})$.

<u>Proof of Cartan's Comparison</u>

Induction on i & dim shifting.

① $i=0$: given by sheaf axioms.

② Fix $i>0$ & assume $\check{H}^k(X,\mathcal{F}) \cong H^k(X,\mathcal{F})$, $\forall k<i$.

$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$, $\mathcal{G}$ flasque

$\Rightarrow$ $0 \to \Gamma(U,\mathcal{F}) \to \Gamma(U,\mathcal{G}) \to \Gamma(U,\mathcal{H}) \to 0$ exact

($\forall U \in B$ by Lemma).

Let $\mathcal{U} = \{U_i\}_{i\in I}$ cover of X by $U_i \in B$.

$\mathcal{B}$ is closed under $(\cdot)\cap(\cdot)$:

$\Rightarrow \quad 0 \to \check{C}^{\cdot}(\mathcal{U},\mathcal{F}) \to \check{C}^{\cdot}(\mathcal{U},\mathcal{G}) \to \check{C}^{\cdot}(\mathcal{U},\mathcal{H}) \to 0$ exact.

<u>Note</u> any open cover refines to basis opens

$\Rightarrow$ [taking $\lim$ on all opens

$\Leftrightarrow$ taking $\lim$ on all basic opens]

$\rightsquigarrow \quad 0 \to \check{C}^{\cdot}(X,\mathcal{F}) \to \check{C}^{\cdot}(X,\mathcal{G}) \to \check{C}^{\cdot}(X,\mathcal{H}) \to 0$ exact.

$\mathcal{G}$ flasque $\Rightarrow$ $\check{H}^i(X,\mathcal{G}) = H^i(X,\mathcal{G}) = 0 \quad (i>0)$

$$\begin{array}{ccccccccc}
\to & \check{H}^{i-1}(X,\mathcal{G}) & \longrightarrow & \check{H}^{i-1}(X,\mathcal{H}) & \longrightarrow & \check{H}^{i}(X,\mathcal{F}) & \longrightarrow & \check{H}^{i}(X,\mathcal{G}) & \to \\
 & \downarrow = & & \downarrow \cong & \xRightarrow{\text{5-Lem}} & \cong \downarrow & & \downarrow = & \\
\to & H^{i-1}(X,\mathcal{G}) & \longrightarrow & H^{i-1}(X,\mathcal{H}) & \longrightarrow & H^{i}(X,\mathcal{F}) & \longrightarrow & H^{i}(X,\mathcal{G}) & \to \\
 & \| & & \text{inductive hyp.} & & & & \| & \\
 & 0 \ (i>1) & & & & & & 0 &
\end{array}$$

$\square$