Divisors, Linear Systems, and Projective Embeddings

Why we'll be interested in cohom of Qcohs?

§1 Weil Divisors

Hypothesis (*) (by Hartshorne):

X noe, int, sep, reg in codim 1.

i.e. $\forall x \in X$, $\mathcal{O}_{X,x}$ regular, Krull dim = 1.

Lemma A noe local ring of dim 1. TFAE:

(a) A regular

(b) A normal

(c) A DVR.

$\rightsquigarrow$ normalizing a 1-dim'l noe ring produces a regular ring.

Warning noe int domain A. A normal $\Rightarrow$ A reg in codim 1

but no converse!

$\Leftarrow$: add Serre's S_2: codim $(0) = 0$, and

$\forall a \in A$, codim $(a) = 1$ when a is not a zero div.

Def'n prime (Weil) div on X = closed int subsch of codim 1.

int = irred + red

Weil div = $\mathbb{Z}$-formal comb of primes.

effective = coefficients ≥ 0.

E.g. $K(X)$ = func field of X = local ring of X at η gen pt.

= Frac $(\mathcal{O}(U))$, $\forall \phi \neq U \subseteq X$ open affine sub

$\forall\, 0 \neq f \in K(X) \rightsquigarrow$ principal div ass to f:

$\forall\, Z \subseteq X$ prime div, $\mathcal{O}_{X,\eta_Z}$ DVR with v_Z valuation.

$\rightsquigarrow\ (f) = \sum_Z v_Z(f) \cdot Z$.

only fin many $v_Z(f) \neq 0$ (by noe).

b/c $f|_U$ regular func for some $\phi \neq U \subseteq X$

$v_Z(f) = 0$ whenever $Z \nsubseteq X - U$.

$\mathrm{Div}\, X$ = grp of Weil Divs on X.

subgrp: principal divs, $(f) + (g) = (fg)$. $\rightsquigarrow \mathrm{PDiv}(X)$.

$\rightsquigarrow\ \mathrm{Div}(X)/\mathrm{Div}^0(X) = \mathrm{Cl}(X)$.

E.g. $X = \mathrm{Spec}\, A$, A DDK domain (e.g. $A = \mathcal{O}_K$ for K field)

$\Rightarrow \mathrm{Cl}(X)$ = ideal class group

$\mathrm{Div}(X)$ = grp of fractional ideals.

$\sim$ (lin equiv): $\mathfrak{a} \sim \mathfrak{b} \Leftrightarrow \mathfrak{a} = \lambda \mathfrak{b}$

(differed by a principal div).

<u>Example</u> (Elliptic Curve) $k = \bar{k}$. $P(x,y,z) \in k[x,y,z]$ homo, $\deg = 3$.

$\rightsquigarrow$ P defines a nonsing sub<u>var</u> $C \subseteq \mathbb{P}^2_k$.

Pick $O \in C(k)$, $\exists$ surj $\deg: \mathrm{Div}\, X \twoheadrightarrow \mathbb{Z}$

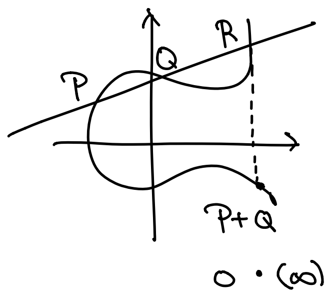
$P \longmapsto 1$, P prime.

note: $P \in \mathrm{Div}^0(X) \Rightarrow \deg P = 0$

$\Rightarrow \deg$ factors through $\mathrm{Div}\, X \to \mathrm{Cl}\, X \to \mathbb{Z}$.

$\ker(\mathrm{Cl}\, X \to \mathbb{Z}) = ((P) - (O))_{P \in C(k)}$.

In fact: $(P) - (O)$ can be computed by the grp law.



$(P) + (Q) + (R) \sim (R) + (P+Q) + (O)$

$\Rightarrow (P) - (O) + (Q) \sim (P+Q)$

$\Rightarrow \big((P) - (O)\big) + \big((Q) - (O)\big) \sim (S) - (O).$

§2 Cartier Divisors

X not regular. A more restrictive notion.

$\mathcal{K}$ loc const sheaf ass to $K(X)$.

Def'n A Cartier div on X : section of the sheaf $K(X)/\mathcal{O}^\times$.

$\rightsquigarrow$ CaDiv $\longrightarrow$ WDiv

D. rep'd by $f \in K(X)$ on $U \longmapsto \bar{D}$, $\bar{D}|_U = \mathrm{Div}(f)$.

components of primes that meet U.

Fact X normal $\Rightarrow$ CaDiv $\hookrightarrow$ WDiv inj.

b/c int closed noe domain $A = \bigcap A_{\mathfrak{p}}$

$\mathfrak{p}$ minimal prime.

Prop (Hartshorne Prop II.6.11)

X loc factorial (i.e. $\forall x \in X$, $\mathcal{O}_{X,x}$ UFD)

$\Rightarrow \mathrm{CaDiv}(X) \xrightarrow{\cong} \mathrm{WDiv}(X)$.

(In particular, it holds for X reg.
b/c reg loc $\Rightarrow$ factorial by comm alg).

E.g. $X = \mathrm{Spec}\, k[x,y,z]/(xy - z^2)$, $I = (x, z)$ defines a WDiv

but it's not a CaDiv.

Principal CaDiv : def'd by single $f \in K(X)$.

$\rightsquigarrow \mathrm{CaCl}(X) := \mathrm{CaDiv}(X)/\mathrm{CaDiv}^0(X)$.

§3 The Picard Group

$\underbrace{\mathrm{Pic}(X)}$:= ($\mathcal{L}$ invertible ; $\otimes$) $\in$ Grp.

usually the same as $\mathrm{CaCl}(X)$.

Namely, $D \in \mathrm{CaDiv}(X)$, $\mathcal{L}(D) \subseteq \mathcal{K}$ subsheaf s.t.

$$\mathcal{L}(D)(U) = \{ f \in \mathcal{K}(X) : ((f) + (D))|_U \geq 0 \}.$$

Assume X normal $\Rightarrow$ $\mathcal{L}(D)$ loc free of rank 1

$\Rightarrow$ $\mathcal{L}(D)$ inv

$\rightsquigarrow$ $\mathrm{CaDiv}(X) \longrightarrow \mathrm{Pic}(X)$ homo, kills $\mathrm{CaDiv}^{\circ}(X)$.

$\rightsquigarrow$ $\mathrm{CaCl}(X) \longrightarrow \mathrm{Pic}(X)$ always injective

- X integral $\Rightarrow$ surj. (Hartshorne Prop II.6.15).

Note D effective $\Rightarrow$ $1 \in \mathcal{K}(X)$ defines global sec on $\mathcal{L}(D)$.

$\mathcal{L}$ loc principal $\Rightarrow$ $\mathcal{L}|_U \cong \mathcal{O}_U$ locally

When we do so,

subsh of $\mathcal{L}(D)$ gen'd by $1 \longleftrightarrow \mathcal{I} \subseteq \mathcal{O}_X$ ideal sheaf defining $D \subseteq X$ closed.

i.e. D = zero locus of certain section of $\mathcal{L}(D)$.

More generally, (even if $D \geq 0$),

can view $D = \mathrm{Zero}(f)$, f mero sec of $\mathcal{L}(D)$.

(i.e. the zero locus of $\mathcal{L}(D) \otimes_{\mathcal{O}_X} \mathcal{K}(X)$).

Indeed: $\forall f \in \mathcal{L}(D)$ mero, $\mathrm{Zero}(f) \sim D$ linearly.

§4 Linear Systems

X int sep sch of fin type $/k$ any field.
$\mathcal{L}$ inv sheaf on X.

Def'n A linear system def'd by $\mathcal{L}$

$$= \{ \mathrm{Zero}(f),\ f \in H : H \ k\text{-subspace of } H^0(X,\mathcal{L}) \}.$$

if $H = H^0(X,\mathcal{L})$: complete lin system.

$h \in H \rightsquigarrow X \to \mathbb{P}^n_k$, $n = \dim_k H - 1$

- H may have a <u>base pt</u> $\rightsquigarrow$ this constr fails
  i.e. $P \in \bigcap_{f \in H} \mathrm{Zero}(f)$

In fact, we get $X \subseteq \mathbb{P}^n_k \Leftrightarrow H$ has no base pt.

Suppose now $k = \bar{k}$. X proj, 1-dim'l, irred, nonsing (i.e. curve)
Consider $\exists D \in \mathrm{Div}(X)$, $\mathcal{L}(D) \rightsquigarrow$ complete lin system.

(a) Can get $X \to \mathbb{P}^n_k \iff \forall x \in X(k)$, $\dim_k H^0(X, \mathcal{L}(D-x)) = \dim_k H^0(X,\mathcal{L}(D)) - 1$.

namely, $\exists f \in \mathcal{L}(D)$ s.t. $f(x) \neq 0$.

(b) $X \xrightarrow{i} \mathbb{P}^n_k$ injective $\iff \forall x \neq y \in X(k)$,
(as sets) $\dim_k H^0(X, \mathcal{L}(D-x-y)) = \dim_k H^0(X,\mathcal{L}(D)) - 2$

namely, $\exists f \in \mathcal{L}(D)$ s.t. $f(x) \neq 0$, $f(y) = 0$ and vice versa.

(c) $X \xrightarrow{i} \mathbb{P}^n_k$ in (b) closed imm $\iff \forall x \in X(k)$,
$\dim_k H^0(X, \mathcal{L}(D-2x)) = \dim_k H^0(X,\mathcal{L}(D)) - 2$

namely, $\exists f, g \in \mathcal{L}(D)$ s.t. $f(x) \neq 0$, $g(x) = 0$ of order 1.

<u>Rmk</u> (c) ensures $T_x X \hookrightarrow T_{i(x)} \mathbb{P}^n_k$.

when X embeds into $\mathbb{P}^n_k$

$\rightsquigarrow$ compute $\dim_k H^0(X, \mathcal{L}(D))$, $\forall D \in \mathrm{Div}\, X$

(abetted by Riemann-Roch).