The Fargues-Fontaine curve and local Langlands

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Reminder on geom CFT

X sm proj + geom conn curve $/\mathbb{F}_q$.

$E = \mathbb{F}_q(X)$ function field.

Let $\mathbb{A} = \prod'_{x \in |X|} \hat{E}_x$, $\mathbb{O} = \prod_{x \in |X|} \hat{\mathcal{O}}_x$.

<u>Unramified CFT</u>: the map

$$\begin{aligned} \mathbb{A}^\times &\longrightarrow (\mathrm{Gal}_E^{\mathrm{unr}})^{ab} = \mathrm{Gal}(E^{\mathrm{unr}}/E)^{ab} \\ (a_x)_{x \in |X|} &\longmapsto \prod_{x \in |X|} \mathrm{Frob}_x^{\mathrm{ord}_x(a_x)} . \end{aligned}$$

induces an isom

$$\underbrace{(E^\times \backslash \mathbb{A}^\times / \mathbb{O}^\times)^\wedge}_{\text{profinite completion}} = (\mathrm{Gal}_E^{\mathrm{unr}})^{ab}$$

<u>Goal</u> Would like to reformulate & prove this statement geometrically.

<u>Note</u> (1) $\mathrm{Gal}_E^{\mathrm{unr}} = \pi_1(X)$

(2) (Weil) $E^\times \backslash \mathbb{A}^\times / \mathbb{O}^\times \simeq \underbrace{\mathrm{Pic}_X}_{\text{Picard sch of } X.}(\mathbb{F}_q)$

Can reformulate as follows: $\exists$ natural bijection

$$\left\{ \begin{matrix} \text{continuous characters} \\ \pi_1(X) \longrightarrow \overline{\mathbb{Z}}_\ell^\times \end{matrix} \right\} \overset{\simeq}{\longleftrightarrow} \left\{ \begin{matrix} \text{characters} \\ \mathrm{Pic}_X(\mathbb{F}_q) \to \overline{\mathbb{Z}}_\ell^\times \end{matrix} \right\}$$

$$\text{s.t.} \quad (\rho : \pi_1(X) \longrightarrow \overline{\mathbb{Q}}_\ell^\times) \triangleq \begin{pmatrix} \chi_\rho : \mathrm{Pic}_X(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times \\ \mathcal{O}(x) \longmapsto \rho(\mathrm{Frob}_x) \end{pmatrix}.$$

To go further, observe:

- LHS $\triangleq$ rk 1 $\overline{\mathbb{Q}}_\ell$-local systems on X
- RHS $\triangleq$ "character local systems" on Pic_X

  i.e. $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{F}$ on Pic_X

  s.t. $m^* \mathcal{F} \simeq p_1^* \mathcal{F} \otimes p_2^* \mathcal{F}$

  for $m : \mathrm{Pic}_X \times \mathrm{Pic}_X \longrightarrow \mathrm{Pic}_X$ & p_1, p_2 two projections.

For this new corresp. LHS $\longleftrightarrow$ RHS,

"$\longleftarrow$": Take the tr of Frob

"$\longrightarrow$": Use the Lang isogeny, regarding Pic_X as an étale torsor over itself with Gal grp $\mathrm{Pic}_X(\mathbb{F}_q)$.

<u>Second (and final) reformulation</u>

Let $AJ : X \longrightarrow \mathrm{Pic}_X$ Abel-Jacobi map.

$\qquad x \longmapsto \mathcal{O}(x)$

Then AJ^* induces an equiv of cats:

$$\left\{\begin{array}{c}\text{character loc sys} \\ \text{on } \mathrm{Pic}_X\end{array}\right\} \simeq \left\{\begin{array}{c}\text{rk 1 loc sys} \\ \text{on } X\end{array}\right\}.$$

<u>Rmk</u> (1) Reformulation makes sense over any base field (e.g. $\mathbb{C}$).

(2) In the above, used the Pic scheme Pic_X $(X/\mathbb{F}_q)$ instead of Pic stack (a $\mathbb{G}_m$-gerbe / Pic_X).

ℓ-adic sheaves on these two are the same.
(**But** The opposite will happen in the local setting
c.f. F-S setting).

(3) This geom approach can be extended
to allow ramification.

<u>pf sketch of AJ^*</u> (Deligne)

Can work over $\overline{\mathbb{F}}_q$.

For $d \geq 1$, consider the deg d Abel-Jacobi map

$$AJ^d : X^{(d)} \longrightarrow \mathrm{Pic}_X^d$$
$$D \longmapsto \mathcal{O}(D)$$

d-th symm product of $X := X^d / S_d$.

(so $AJ = AJ^1$).

Let $\mathcal{L}$ be a rk 1 local system on X.

Because $\mathcal{L}$ has rk 1,

$\mathcal{L}^{\boxtimes d}$ (étale sheaf on X^d) descends to a rk 1
local system on $X^{(d)} = X^d / S_d$.

(Higher rk: end up with some perv sheaf on $X^{(d)}$.)

$\rightsquigarrow$ Fibres of AJ^d: linear system
$$|D| = \mathbb{P}(H^0(X, \mathcal{O}(D))).$$

By Riemann-Roch, as long as $d > 2g(X) - 2$,
AJ is a fibration in proj spaces over Pic_X^d.

<u>Fact</u> $\pi_1(\mathbb{P}^n_{\overline{\mathbb{F}}_q}) = 0$, for $n \geq 1$.

So $\mathcal{L}^{(d)}$ descends to a local system $\mathrm{Aut}_{\mathcal{L}}^d$ on Pic_X^d

for each $d > 2g(X)-2$.

Since $\mathrm{Aut}_{\mathcal{L}}$ is obtained by descending $\mathcal{L}^{\boxtimes d}$, one checks that

$\forall\, d, d' > 2g(X)-2$, if $m : \mathrm{Pic}_X^d \times \mathrm{Pic}_X^{d'} \longrightarrow \mathrm{Pic}_X^{d+d'}$,

then $m^* \mathrm{Aut}_{\mathcal{L}}^{d+d'} = \mathrm{Aut}_{\mathcal{L}}^d \boxtimes \mathrm{Aut}_{\mathcal{L}}^{d'}$.

This property allows to define $\mathrm{Aut}_{\mathcal{L}}^d$ for all d

& to show that the loc sys $\mathrm{Aut}_{\mathcal{L}}$ so constructed

is a character local system.

For general d,
$$X \times \mathrm{Pic}_X^d \xrightarrow{\ m\ } \mathrm{Pic}_X^{d+1}$$
$$(x, \mathcal{L}) \longmapsto \mathcal{L}(x)$$

$\rightsquigarrow$ $m^* \mathrm{Aut}_{\mathcal{L}}^{d+1} \simeq \mathrm{Aut}_{\mathcal{L}}^d \boxtimes \mathcal{L}$

Repeat this to get result for $d \leq 2g(X)-2$. $\square$

Local CFT

<u>Setup</u> Assume E nonarch local field

($E = \mathbb{F}_q((t))$ or $E/\mathbb{Q}_p$ finite).

Fix C complete alg closed nonarch field

containing $\mathbb{F}_q$ res field of E.

$\rightsquigarrow$ Can define FF curve

$X_{C,E} \longrightarrow \mathrm{Spa}\, E$ adic space.

Fact $\pi_1(X_{C,E}) = \mathrm{Gal}_E$.

(Hint: If $\mathcal{E}$ is a finite étale $\mathcal{O}_{X_{C,E}}$-algebra, use the classification thm + étale alg to check $\mathcal{E}$ is semi-stable of slope 0.)

Q Can one replace the curve $X/\mathbb{F}_q$ in the geom CFT setting before with $X_{C,E}/\mathrm{Spa}\, E$ to say anything about local CFT geom'ly?

<u>Two difficulties</u>

- $X_{C,E}$ depends on the auxiliary choice of C
- $X_{C,E}$ is an adic space / $\mathrm{Spa}\, E$.

Consider "$\mathrm{Pic}_{C,E}$": $\{\text{adic spaces} / E\} \longrightarrow \text{Grpoids}$

$$T \longmapsto \{\text{line bdls on } X_{C,E} \times_{\mathrm{Spa}\, E} T\}.$$

<u>But</u> not many interesting ℓ-adic sheaves on "$\mathrm{Pic}_{C,E}$"!

$\rightsquigarrow$ This is not the right thing.

(But still an interesting moduli problem, c.f. Emerton-Gee, Hellmann).

<u>Better</u> Keep the "E-direction" fixed & consider the "C-direction" as variable.

For any $S \in \mathrm{Perf}_{\mathbb{F}_q} = \{\text{perfectoid spaces} / \mathbb{F}_q\}$,
can define $X_{S,E}$ analytic adic space / $\mathrm{Spa}\, E$,

and consider the following moduli problem:

$$\mathrm{Div}^1 := \mathrm{Div}^1_E : S \longmapsto \left\{ \begin{array}{l} \text{relative Cartier divisors} \\ \text{of deg } 1 \text{ on } X_{S,E} \end{array} \right\}$$

$$\mathrm{Pic} := \mathrm{Pic}_E : S \longmapsto \{ \text{line bdls on } X_{S,E} \}$$

$\shortparallel$ Bun_1 (Bun_G for $G = GL_1 = \mathbb{G}_m$).

<u>What is the geometry of these objects?</u>

(1) Pic is a stack for the v-top on $\mathrm{Perf}_{\mathbb{F}_q}$,

with a simple str:

$$\mathrm{Pic} = \coprod_{d\in\mathbb{Z}} \mathrm{Pic}^d \simeq \coprod_{d\in\mathbb{Z}} [\mathrm{Spa}\,\mathbb{F}_q / \underline{E}^\times]$$

Here isom b/c any line bdl of deg d

is v-locally on $S \in \mathrm{Perf}_{\mathbb{F}_q}$ isom to $\mathcal{O}(d)$

& $\underline{\mathrm{Aut}}(\mathcal{O}(D)) = \underline{E}^\times$ ($\underline{E}^\times : S \mapsto \mathrm{Cont}(|S|, E^\times)$).

In particular, for $\ell \neq p$,

$$D_{\acute{e}t}(\mathrm{Pic}^d_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell) \simeq D(\text{sm } \overline{\mathbb{Q}}_\ell\text{-rep of } E^\times)$$

<u>Rmk</u> Really important to work with Picard stack to see reps of $E^\times$!

(2) Div^1:

Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_q}$,

$S^\sharp$ untilt of S / E, $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$.

Fontaine's θ-map: $W_{\mathcal{O}_E}(R^+) = W(R^+) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E \longrightarrow R^{\sharp+}$

$$\sum [x_n]\pi^n \longmapsto \sum x_n^\sharp \pi^n$$

surjective w/ $\ker\theta = (\xi)$,

ξ a non-zero divisor, primitive of deg 1.

This defines a map

$$\{\text{untilts} / E\} \longrightarrow \{\text{closed Cartier divisors on } Y_{S,E}\}$$

$$\rightsquigarrow \{\text{untilts} / E\}/\phi \longrightarrow \{\text{closed Cartier divisors on } X_{S,E}\}$$

$$S^{\#} = \mathrm{Spa}(R^{\#}, R^{\#+}) \longmapsto V(\xi).$$

Say that a closed Cartier divisor is <u>relative of deg 1</u> if it is in the image of this map (i.e. looks like $V(\xi)$).

<u>Thm</u> $$\mathrm{Div}^1 \simeq \underbrace{(\mathrm{Spa}\, E)^{\diamond}}_{\text{untilts over } E} / \phi^{\mathbb{Z}}.$$

$\rightsquigarrow$ Div^1 is a nice diamond (but not spatial)

proper, ℓ-cohom sm ($\ell \neq p$), etc.

<u>Next</u> Want to give another description of Div^1.

Let G be a Lubin-Tate formal grp law $/ \mathcal{O}_E$

$\circlearrowleft$ $\mathcal{O}_E$-action involved.

G^{ad} adic generic fibre.

Choose coord to represent

$$G \simeq \mathrm{Spf}(\mathcal{O}_E[\![T]\!]).$$

$\rightsquigarrow$ $G^{ad} \simeq \mathbb{D}_E$ rigid-analytic open unit disc $/E$

Form the universal cover

$$\tilde{G} = \varprojlim_{\times\pi} G \simeq \mathrm{Spf}(\mathcal{O}_E[\![x^{1/p^\infty}]\!]).$$

Let A $\mathcal{O}_E$-algebra, π-adically complete.

Then $\tilde{G}(A) = \mathrm{Hom}_{\mathcal{O}_E}(\mathcal{O}_E[\![x^{1/q^\infty}]\!], A)$

$$= \varprojlim_{x \mapsto x^q} A^{\circ\circ} = A^{\flat,\circ\circ}.$$

top nilp. elts

<u>Claim</u> $S = \mathrm{Spa}(R, R^+)$ aff'd perf'd $/ \mathbb{F}_q$

$S^\# = \mathrm{Spa}(R^\#, R^{\#+})$ aff'd perf'd $/ E$.

Then the map

$$\tilde{G}(R^{\#+}) \simeq R^{\circ\circ} \longrightarrow H^0(Y_{S,E}, \mathcal{O}) \circlearrowleft \phi_S$$

$$x \longmapsto \sum_{i \in \mathbb{Z}} \pi^i [x^{q^{-i}}].$$

induces $\tilde{G}(R^{\#+}) \simeq H^0(Y_{S,E}, \mathcal{O})^{\varphi = \pi}$

$$= H^0(X_{S,E}, \mathcal{O}(1)).$$

<u>Recall</u>: $Y_{S,E} = \mathrm{Spa}(W_{\mathcal{O}_E}(R^+), W_{\mathcal{O}_E}(R^+)) \setminus V(\pi[\varpi])$, ϖ p.u. of R.

some Stein space.

If char $E = p$, $Y_{S,E} = \mathbb{D}_S^*$

punctured open unit disc $/ S$.

In fact, if $S^\# / E_\infty$ = Completion of the ext'n of E obtained by adding π^∞-torsion pts in $\bar{E}$,

then have a short exact seq.:

$$0 \to \mathcal{O}_{X_{S,E}} \to \mathcal{O}_{X_{S,E}}(1) \to \mathcal{O}_{S^\#} \to 0.$$

$(S^\# \hookrightarrow X_{S,E})$

which on global sections identifies with

$$0 \to G^{ad}[\pi^\infty] \to \tilde{G}^{ad} \xrightarrow{\log} \mathbb{A}^1_{S^\#} \to 0.$$

$\rightsquigarrow$ Get a map

$$(BC(\mathcal{O}(1)) \setminus \{0\}) / E^\times \longrightarrow \mathrm{Div}^1$$
$$f \longmapsto V(f)$$

Here $BC(\mathcal{O}(1)) : S \longmapsto H^0(X_{S,E}, \mathcal{O}(1))$.

$\rightsquigarrow$ $BC(\mathcal{O}(1)) \setminus \{0\} \simeq \mathrm{Spa}\, \mathbb{F}_q((x^{1/p^\infty})) \simeq \mathrm{Spa}\, E_\infty^b \simeq (\mathrm{Spa}\, E_\infty)^\diamond$.

So the map above identifies with

$$(\mathrm{Spa}\, E_\infty)^\diamond \xrightarrow[\mathcal{O}_E^\times\text{ - torsor}]{} (\mathrm{Spa}\, E)^\diamond \longrightarrow (\mathrm{Spa}\, E)^\diamond / \phi^{\mathbb{Z}} = \mathrm{Div}^1.$$

$\rightsquigarrow$ $\mathrm{Div}^1 \simeq (BC(\mathcal{O}(1)) \setminus \{0\}) / E^\times$.

More generally, define a deg d relative Cartier divisor as
- a line bdl $\mathcal{I}$ on $X_{S,E}$ of deg d,

together with
- an injective map $\mathcal{I} \to \mathcal{O}_{X/S}$ which stays injective after pullback to any generic pt of S s.t. $\mathcal{I}$ still has deg d.

$\rightsquigarrow$ $\mathrm{Div}^d = (BC(\mathcal{O}(d)) \setminus \{0\}) / E^\times$.

Prop The surj map $(\mathrm{Div}^1)^d \longrightarrow \mathrm{Div}^d$ induces
$$\mathrm{Div}^d \simeq (\mathrm{Div}^1)^d / S_d \quad \text{as } v\text{-sheaves}.$$

Rmk On generic pt, this prop is the fundamental thm of new algebra.

<u>Geometric proof of local CFT</u> (Fargues) Let $k = \overline{\mathbb{F}}_q$.

Let $\rho: W_E \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ conti char.

Since $\mathrm{Div}^1 = (\mathrm{Spa}\, E)^\diamond / \phi^{\mathbb{Z}}$,

$$\mathrm{Div}^1_k = (\mathrm{Spa}\, \breve{E})^\diamond / \phi^{\mathbb{Z}} = (\mathrm{Spa}\, \hat{\overline{E}})^\diamond / W_E.$$

$\rightsquigarrow$ ρ can be seen as an ℓ-adic local system $\mathcal{L}$ of rk 1 on Div^1_k.

$\rightsquigarrow$ $AJ: \mathrm{Div}^1_k = (BC_k(\mathcal{O}(1)) \setminus \{0\}) / \underline{E}^\times \longrightarrow \mathrm{Pic}^1_k = [\mathrm{Spa}\, k / \underline{E}^\times]$.

<u>Want</u> $\mathcal{L}$ comes via pullback from a rk 1 loc sys on Pic^1_k.

(If this is true, compatibility with Artin reciprocity will follow from $BC(\mathcal{O}(1)) \setminus \{0\} \simeq (\mathrm{Spa}\, E_\infty)^\diamond$.

As before, can also introduce for $d \geq 1$ that

$$AJ^d: \mathrm{Div}^d_k \longrightarrow \mathrm{Pic}^d_k$$

fibres are $BC_k(\mathcal{O}(d)) \setminus \{0\}$.

<u>Thm</u> $\forall d \geq 3$, $BC_k(\mathcal{O}(d)) \setminus \{0\}$ simply connected.

<u>pf of thm</u> Only in char p ($E = \mathbb{F}_q((\pi))$):

$$BC(\mathcal{O}(d)) \simeq \mathrm{Spa}\, \mathbb{F}_q[[x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]]$$

$$BC(\mathcal{O}(d)) \setminus \{0\} \simeq (\mathrm{Spa}\, \mathbb{F}_q[[x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]]) \setminus V(x_1, \dots, x_d)$$

(perfectoid space (for $E = \mathbb{F}_q((\pi))$).

$\rightsquigarrow$ $\mathrm{FEt}(BC(\mathcal{O}(d)) \setminus \{0\}) = \varinjlim_n \mathrm{FEt}((\mathrm{Spa}\, \mathbb{F}_q[[x_1^{1/p^n}, \dots, x_d^{1/p^n}]]) \setminus V(x_1, \dots, x_d))$.

And ϕ is purely insep

$\Rightarrow$ ETS that any fin ét cover of

$$\mathrm{Spa}(k[\![x_1,\dots,x_d]\!]) \setminus V(x_1,\dots,x_d) \quad \text{splits.}$$

For this,
$$\mathrm{F\acute{E}t}(\mathrm{Spa}(k[\![x_1,\dots,x_d]\!]) \setminus V(x_1,\dots,x_d))$$
$$= \mathrm{F\acute{E}t}(\mathrm{Spec}\, k[\![x_1,\dots,x_d]\!] \setminus V(x_1,\dots,x_d)).$$
by GAGA
$$= \mathrm{F\acute{E}t}(\mathrm{Spec}\, k[\![x_1,\dots,x_d]\!])$$
by Zariski-Nagata purity (for $d \geq 2$).
$$\simeq \mathrm{F\acute{E}t}(\mathrm{Spec}\, k).$$
by Hensel's lemma $\square$

<u>Lubin-Tate theory & Artin rec</u>

As before, ρ comes from a char $\chi : E^\times \to \overline{\mathbb{Q}}_\ell^\times$

by composition with the inverse of
$$\mathrm{Art} : E^\times \xrightarrow{\sim} W_E^{ab}.$$

$\rightsquigarrow$ $\mathrm{Art}^{-1}(g) = \chi_{LT}^{-1}(g)$, for $g \in I_E^{ab} \simeq \mathcal{O}_E^\times$.

$\mathrm{Art}^{-1}(\sigma) = \pi$.

$\rightsquigarrow$ Get the $E^\times$-torsor
$$BC(\mathcal{O}(1)) \setminus \{0\} \longrightarrow \mathrm{Div}_k^1.$$

This identifies with
$$(\mathrm{Spa}\, E_\infty)^\diamond \longrightarrow (\mathrm{Spa}\, E)^\diamond \longrightarrow (\mathrm{Spa}\, E)^\diamond/\phi$$

Lubin-Tate ext'n,

$\mathcal{O}_E^\times$ - torsor