A new method for overconvergence of étale (φ,Γ)-modules

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(Joint with Tong Liu.)

Let $K = \mathbb{Q}_p$ for simplicity.

Study $\quad \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ free $\mathbb{Z}_p$-lattices

via φ-mods w/ additional str.

For $\mathbb{C}_p = \hat{\bar{K}}$, v_p on K extends to $\mathbb{C}_p$.

$\mathcal{O}_{\mathbb{C}_p}$ unit ball.

Define $\quad \tilde{A}^+ := A_{\inf}(\mathcal{O}_{\mathbb{C}_p}) = W(\mathcal{O}_{\mathbb{C}_p}^\flat)$.

$$\mathcal{O}_{\mathbb{C}_p}^\flat = \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p}, \quad p^\flat = (p, p^{1/p}, p^{1/p^2}, \dots),$$
$$\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots).$$

$\rightsquigarrow \quad \varphi \circlearrowright \tilde{A}^+, \tilde{A}$.

<u>Thm</u> (Katz) $\quad \{\text{free } \mathbb{Z}_p\text{-lattices}\} \xrightarrow{\sim} \varphi\text{-mod}^{\text{ét}}_{\tilde{A}}$

$$\varphi\text{-mod}^{\text{ét}}_{\tilde{A}} = \left\{ (\mathcal{M}, \varphi_{\mathcal{M}}) \;\middle|\; \begin{array}{l} \mathcal{M} \text{ fin free } \tilde{A}\text{-mod}, \\ \varphi_{\mathcal{M}} : \mathcal{M} \otimes_{\tilde{A},\varphi} \tilde{A} \xrightarrow{\sim} \mathcal{M} \end{array} \right\}$$

<u>Fact</u> For k perfect,

$$\mathbb{L}\mathrm{oc}_{\mathbb{Z}_p}(k) \simeq \varphi\text{-mod}^{\text{ét}}_{W(k)}$$

implying Katz's thm in this case.

<u>Def</u> $\quad \tilde{A}^\dagger \subset \tilde{A}^{\dagger,r} \subset \tilde{A}, \quad \forall r > 0$

$$\tilde{A}^{\dagger,r} = \left\{ x = \sum_{n\geq 0} [a_n] p^n \;\middle|\; a_n \in \mathbb{C}_p^\flat,\ r v_{p^\flat}(a_n) + n \to \infty \right\}.$$

$$\tilde{A}^{\dagger} := \bigcup_{r>0} \tilde{A}^{\dagger,r} \rightsquigarrow \tilde{A}^{\dagger} \circlearrowleft \varphi\text{-stable}$$

(but each $\tilde{A}^{\dagger,r}$ is not).

Have $\tilde{A}^{\dagger} \longrightarrow \tilde{A}$.

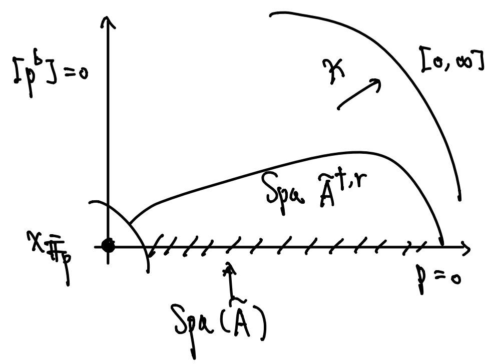
Thm $\varphi\text{-mod}^{\text{ét}}_{\tilde{A}^{\dagger}} \xrightarrow{\sim} \varphi\text{-mod}^{\text{ét}}_{\tilde{A}} \xrightarrow{\sim} \{\mathbb{Z}_p\text{-lattices}\}$

$$T \otimes_{\mathbb{Z}_p} \tilde{A}^{\dagger} \longleftarrow\!\shortmid\; T \otimes_{\mathbb{Z}_p} \tilde{A} \longleftarrow\!\shortmid\; T$$

For full faithfulness, $\forall M^{\dagger} \in \varphi\text{-mod}^{\text{ét}}_{\tilde{A}^{\dagger}}$.

$$(M^{\dagger} \otimes \tilde{A})^{\varphi=1} = (M^{\dagger})^{\varphi=1}.$$

Classical picture of $\operatorname{Spa}(\tilde{A}^{\dagger}) \setminus \{x_{\bar{\mathbb{F}}_p}\}$:



$\tilde{A}^{\dagger,r} = \mathcal{O}(\mathcal{Y}_{[0,r]})$

where $\mathcal{Y}_{[0,r]} = \{ |p(\tilde{x})|^{r} \le |[p^{b}](\tilde{x})| \}$,

$x \in \operatorname{Spa}(\tilde{A}^{\dagger}) \setminus \{x_{\bar{\mathbb{F}}_p}\}$.

Now we know $G_K, G, \tilde{A}, \tilde{A}^{+}, \tilde{A}^{\dagger}, \tilde{A}^{\dagger,r}$.

Def $(\varphi, G_K)\text{-mod}^{\text{ét}}_{\tilde{A}^{\dagger}} := \{(\mathcal{M}, \varphi_{\mathcal{M}}, \rho_{G_K})\}$,

- $(\mathcal{M}, \varphi_{\mathcal{M}})$ ét φ-mod,
- ρ_{G_K} semilinear G_K-action, comm w/ φ.

$$\rightsquigarrow \operatorname{Rep}_{\mathbb{Z}_p}(G_K) \xrightarrow{\sim} (\varphi, G_K)\text{-mod}^{\text{ét}}_{\tilde{A}^{\dagger}}$$

$$\xrightarrow{\sim} (\varphi, G)\text{-mod}^{\text{ét}}_{\tilde{A}}.$$

Imperfect period ring ($K = \mathbb{Q}_p$)

$$A_K^+ = \mathbb{Z}_p[[T-1]] \longrightarrow \tilde{A}^+ \circlearrowleft \varphi, G_K$$

$$T \longmapsto [\varepsilon],$$

$$A_K := \left(\mathbb{Z}_p[[T-1]]\left[\tfrac{1}{T-1}\right]\right)^\wedge_p \longrightarrow \tilde{A}$$

note A_K^+, A_K stable under φ, G_K-action.

G_K acts on A_K via $G_K \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) =: \Gamma \overset{\chi}{\simeq} \mathbb{Z}_p^\times$.

$H_K := \ker(G_K \twoheadrightarrow \Gamma)$.

$$\begin{tikzcd}
A_K^+ \arrow[r, hook] \arrow[d] & (\tilde{A}^+)^{H_K} =: \tilde{A}_K^+ \arrow[r, hook] \arrow[d] & \tilde{A}^+ \arrow[d] \\
A_K \arrow[r, hook] & (\tilde{A})^{H_K} =: \tilde{A}_K \arrow[r, hook] & \tilde{A}
\end{tikzcd}$$

$A_K^\dagger := A_K \cap \tilde{A}_K^+$ in $\tilde{A}$.

Thm (Overconvergence)

(1)

$$\begin{tikzcd}
A_K^\dagger \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \tilde{A}_K^\dagger \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond'"] & \tilde{A}^\dagger \arrow[d] \\
A_K \arrow[r] & \tilde{A}_K \arrow[r] & \tilde{A}
\end{tikzcd}$$

Define $(\varphi,\Gamma)\text{-mod}^{\text{ét}}_{\tilde{A}_K^\dagger} := \{(M, \varphi, \rho_\Gamma)\}$

where $\cdot$ $(M,\varphi) \in \varphi\text{-mod}^{\text{ét}}_{\tilde{A}_K^\dagger}$

$\cdot$ ρ_Γ semilinear action of Γ on M comm w/ φ.

Then $\diamond$, $\diamond'$ define equivalences of

ét (φ,Γ)-mods (or (φ, G_K)-mods)

over the corresponding base rings.

(2) $\diamond$ defines

$$\begin{tikzcd}
\varphi\text{-}\mathrm{mod}^{\acute{e}t}_{A_K^+} \arrow[r, "(i)"] \arrow[d, "(ii)"'] & \varphi\text{-}\mathrm{mod}^{\acute{e}t}_{A_K} \arrow[d, "(iii)"] \\
\varphi\text{-}\mathrm{mod}^{\acute{e}t}_{\tilde{A}_K^+} \arrow[r, "(iv)"] & \varphi\text{-}\mathrm{mod}^{\acute{e}t}_{\tilde{A}_K}
\end{tikzcd}$$

we have (iii) (iv) equivalence

(i) (ii) fully faithful but not ess surj.

<u>Note</u> (1) due to Cherbonner-Colmez, Kedlaya.

(2) (iv) Faltings almost purity

(iii) $A_K \longrightarrow \bigcup_n \varphi^{-n}(A_K) \longrightarrow \tilde{A}_K \circlearrowleft \varphi, \varphi^{-1}$

has dense image.

(i) full faithfulness : N. Tuzaki.

not ess surj: Tuzuki, Cherbonner thesis.

<u>Rmk</u> (1) Most important application:

Berger : p-adic monodromy thm.

(φ,Γ)-mod $\rightsquigarrow$ (φ,∂)-mod.

(2) Generalizations:

- $K(\zeta_{p^\infty})/K$ replaced by other perfectoid towers

  $K(\varpi^{1/p^\infty})/K$ Kummer tower

  $\rightsquigarrow$ (φ,τ)-mods (by Camargo).

- overcon of ét (φ,τ)-mods, proved by Gao-Liu, Gao-Poyeton.

  [GP] under the formulation of Berger-Colmez for

  K_∞/K p-adic Lie ext'n.

(3) For geometric family of Gal rep:

- Kedlaya-Liu $X \xrightarrow{\text{ét}} \mathbb{T}^d$
  can formulate & prove the overconvergence
  $\rightsquigarrow$ Liu-Zhu, DLLZ.
- Pan, Camargo
  (generalized Sen theory of Berger-Colmez into family.)

* <u>New formulation</u>

X has sm formal model over Spf $\mathcal{O}_K$.

<u>Goal</u> State over foundation when $X = \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$.

$(\mathbb{Z}_p)^{\circ}_{\Delta}$ transversal absolute prismatic site.
$\ni (A, I, \varphi_A)\ (= (A, I))$

with
- A ring $\supseteq I$, $\varphi_A : A \to A$ s.t. $\varphi_A \equiv \mathrm{Fr} \bmod p$.
- I locally gen'd by nonzero divisors f
  s.t. $(\varphi_A(f) - f^p)/p$ is unit.
- A (p, I)-adically complete,
  A/I p-torsion free.
- $(A, I) \to (B, J)$ maps compatible w/ all strs.
  Called <u>covering</u> if $B \otimes^{\mathbb{L}}_A A/(p, I)$ is discrete
  & $H^0(B \otimes^{\mathbb{L}}_A A/(p, I))$ is fin flat $/ A/(p, I)$.

Define functors

$$\mathcal{O}_{\Delta} : (A, I) \longmapsto A$$
$$\mathcal{O}_{\mathcal{E}, \Delta} : (A, I) \longmapsto A\left[\tfrac{I}{p}\right]^{\wedge}_{p}$$

$$\widetilde{\mathcal{O}}_{\Delta} : (A, I) \longmapsto \Big(\varinjlim_{\varphi_A} A\Big)^{\wedge}_{(p,I)} =: A_{perf}$$

$$\widetilde{\mathcal{O}}_{\mathcal{E},\Delta} : (A, I) \longmapsto A_{perf}\Big[\frac{1}{I}\Big]^{\wedge}_{p}.$$

<u>Thm</u> (1)
$$\left.\begin{aligned} \mathcal{O}^{+}_{\mathcal{E},\Delta} &= \operatorname{colim}_n \mathcal{O}_{\Delta}\Big\langle \frac{p}{I_{\Delta}^{n}} \Big\rangle \Big[\frac{1}{I_{\Delta}}\Big] \\ \widetilde{\mathcal{O}}^{+}_{\mathcal{E},\Delta} &= \operatorname{colim}_n \widetilde{\mathcal{O}}_{\Delta}\Big\langle \frac{p}{I_{\Delta}^{n}} \Big\rangle \Big[\frac{1}{I_{\Delta}}\Big] \end{aligned}\right\rangle \text{ as sheaves}$$

(2)
$$\begin{tikzcd} \mathcal{O}^{+,w}_{\mathcal{E},\Delta} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{O}_{\mathcal{E},\Delta} \arrow[d] \\ \widetilde{\mathcal{O}}^{+}_{\mathcal{E},\Delta} \arrow[r] & \widetilde{\mathcal{O}}_{\mathcal{E},\Delta}. \end{tikzcd}$$

(3) Have an equivalence
$$\mathrm{Vect}\big((\mathbb{Z}_p)^{\circ}_{\Delta}, \mathcal{O}^{+}_{\mathcal{E},\Delta}\big)^{\varphi=1} \xrightarrow{\sim} \mathrm{Vect}\big((\mathbb{Z}_p)^{\circ}_{\Delta}, \mathcal{O}_{\mathcal{E},\Delta}\big).$$

<u>Rmk</u> Evaluate LHS of (3) at a sheaf

$\rightsquigarrow$ recovers ét (φ,Γ)-mod in Berger-Colmez's classical context.