On the X^{Δ} beyond X^{HT}

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Let $X/\mathcal{O}_K$ sm p-adic formal sch.

$\mathcal{O}_K$ CDVR of mixed char.

<u>Motivation</u> Understand arith info of X via prismatic crystals.

- X_η $\mathbb{Z}_p$-local system
- X_Δ

Expect: prism crys can unify many coeffs.

<u>Def</u> X_Δ the abs prismatic sites of X.

(1) objs: a $\underbrace{\text{prism } (A, I)}$ + $\operatorname{Spf} \bar{A} = \operatorname{Spf} A/I \to X$

- $A = \delta$-ring,
  $\delta : A \to A$ s.t. $\varphi(x) = x^p + p\delta(x)$ is a ring homo
- $I \subseteq$ a Cartier divisor on $\operatorname{Spec} A$
  locally gen'd by d s.t. $\delta(d)$ is a unit.

(2) Covers:

(p, I)-adically complete faithfully flat morphisms.

Various str sheaves: $\mathcal{O}_\Delta : (A, I) \longmapsto A$

$\mathcal{O}_\Delta / I^n : (A, I) \longmapsto A/I^n$

Hodge – Tate: $\bar{\mathcal{O}}_\Delta := \mathcal{O}_\Delta / I : (A, I) \longmapsto A/I$

$\mathbb{B}^+_{dR} : (A, I) \longmapsto (A[\frac{1}{p}])^\wedge_I$.

Def (prismatic crystals) For $* \in \{\mathcal{O}_{\Delta}, \mathcal{O}_{\Delta}/I^n, \bar{\mathcal{O}}_{\Delta}, B^+_{dR}\}$,

$$\mathrm{Vect}(X_{\Delta}, *) := \varprojlim_{(A,I)\in X_{\Delta}} \mathrm{Vect}(*(A,I)).$$

e.g. $X = \mathrm{Spf}\, \mathcal{O}_K$, $k = \mathcal{O}_K/m$.

$$(\sigma = W(k)[\![u]\!], E(u)) \in ((\mathcal{O}_K)_{\Delta}, \qquad \sigma/E(u) \cong \mathcal{O}_K.$$
$$+ \ \varphi(u) = u^p.$$

Thm (Bhatt-Scholze) $\mathrm{Vect}^{\varphi}((\mathrm{Spf}\,\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}) \cong \mathrm{Rep}^{crys}_{\mathbb{Z}_p}(G_K)$.

Goal Understand $\mathrm{Perf}(X_{\Delta}, \mathcal{O}_{\Delta})$ on $\mathrm{Perf}(X_{\Delta}, \mathcal{O}_{\Delta}/I^n)$.

Drinfeld, Bhatt-Lurie's stacky approach

Main $\underset{sm}{X} \rightsquigarrow X^{\Delta} : p\text{-Nilp} \to \mathrm{Set}$

$$R \in p\text{-Nilp}, \quad X^{\Delta}(R) := \left\{ \begin{array}{l} \alpha : I \to W(R) \\ \beta : \mathrm{Spf}\, \overline{W(R)/^{L} I} \to X \end{array} \right\}.$$

Here · I an invertible $W(R)$-mod,

· $\alpha : I \to W(R)$ s.t.

$I \xrightarrow{\alpha} W(R) \xrightarrow{\delta} W(R)$ generates the unit ideal.

$+ \ I \xrightarrow{\alpha} W(R) \to R$

(top'ly nilp.

Note (1) $X^{HT} \hookrightarrow X^{\Delta}$,

$$X^{HT}(R) = \left\{ \alpha, \beta \text{ s.t. } \begin{tikzcd} I \arrow[r, "\alpha"] \arrow[dr] & W(R) \arrow[d] \\ & R \end{tikzcd}, \text{ i.e. } \alpha(I) \subseteq V\,W(R) \right\}.$$

(2) $\forall (A, I) \in X_{\Delta}$, $\quad \mathrm{Spf}\, A \to X^{\Delta}$

$$(A \xrightarrow{f} R) \longmapsto \left\{ \begin{tikzcd} & W(R) \arrow[d] \\ A \arrow[ur, "\exists!\, \tilde{f}"] \arrow[r, "f"'] & R \end{tikzcd} + \begin{array}{l} \alpha : I \otimes_{A, \tilde{f}} W(R) \to W(R) \\ \beta : A/I \longrightarrow \mathrm{Cone}\, \alpha \end{array} \right\}.$$

Define
$$\begin{tikzcd} \mathrm{Spf}\,(A/I) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Spf}\, A \arrow[d] \\ X^{HT} \arrow[r, hook] & X^{\Delta}. \end{tikzcd}$$

Thm (Bhatt - Lurie) $\quad \mathrm{Perf}\,(X^{\Delta}) \xrightarrow{\sim} \mathrm{Perf}\,(X_{\Delta}, \mathcal{O}_{\Delta})$.

$\mathrm{Perf}\,(X^{HT}) \xrightarrow{\sim} \mathrm{Perf}\,(X_{\Delta}, \overline{\mathcal{O}}_{\Delta})$.

In general, $X/\mathcal{O}_K$ sm p-adic formal sch.

<u>To understand $\mathrm{Perf}\,(X^{\Delta})$</u>:

(1) Understand $\mathrm{Perf}\,(X^{\Delta}/\sigma) = \mathrm{Perf}((X/\sigma)_{\Delta}, \mathcal{O}_{\Delta})$

(2) Understand $(\mathrm{Spf}\, \mathcal{O}_K)^{\Delta}$.

$$\begin{tikzcd} X^{\Delta}/\sigma \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X^{\Delta} \arrow[d] \\ \mathrm{Spf}\, \sigma \arrow[r] & (\mathrm{Spf}\, \mathcal{O}_K)^{\Delta} \end{tikzcd}$$

<u>Understand $(\mathrm{Spf}\, \mathcal{O}_K)^{\Delta}$</u>

$X = \mathrm{Spf}\, \mathcal{O}_K$, $\quad X^{HT} \hookrightarrow X^{\Delta}$.

Thm (Bhatt - Lurie, Anschütz - Heuer - Le Bras).

(1) X^{HT}/X is a BG_π - gerbe

i.e. $X^{HT} \cong BG_\pi$, $\quad G_\pi = \mathbb{G}_a^{\#} = \mathrm{Spf}\left(\mathcal{O}_K[\frac{u^n}{n!}]_{n \geq 1}\right)^{\wedge}_p$.

(2)
$$\begin{tikzcd} D(X^{HT}) \arrow[r, "\simeq"] \arrow[d, phantom, "\wr\wr"] & D(BG_\pi) \arrow[r, "\text{Cartier duality}", "\sim"'] & \mathrm{Mod}_{G_\pi^\vee}(\mathcal{O}_K) \arrow[d, phantom, "\shortparallel"] \\ D(X_{\Delta}, \overline{\mathcal{O}}_{\Delta}) & & D_{nil}(\mathcal{O}_K[\sigma]). \end{tikzcd}$$

$\lightning$ <u>Warning</u> $X^{\Delta} \not\longrightarrow X$ no map.

<u>Construction / Notation</u> $X^{HT} = X_1^{\Delta} \hookrightarrow \cdots \hookrightarrow \underset{\text{n-truncated prism of } X.}{X_n^{\Delta}} \cdots \hookrightarrow X^{\Delta}.$

$$X_n^{\Delta}(R) = \left\{ (\alpha, \beta) \text{ s.t. } \begin{tikzcd} I^{\otimes n} \arrow[r, "\alpha^{\otimes n}"] \arrow[dr] & W(K) \arrow[d] \\ & R \end{tikzcd} \right\}.$$

$\rightsquigarrow \operatorname{Perf}(X_n^{\Delta}) \simeq \operatorname{Perf}(X_{\Delta}, \mathcal{O}_{\Delta}/I^n).$

<u>Thm</u>

$$\begin{tikzcd} \operatorname{Spf} \sigma/E^n \arrow[r] \arrow[d, "\rho"] & \operatorname{Spf} \sigma \arrow[d, "\text{faithfully flat.}"] \\ X_n^{\Delta} \arrow[r, hook] & X^{\Delta} \end{tikzcd}$$

(1) If $X = \operatorname{Spf} W(K)$ & $n \leq p$, then pullback along ρ induces $\mathcal{D}(X_n^{\Delta}) \xhookrightarrow{\simeq} \mathcal{D}_{\mathrm{Nilp}}(\mathrm{MIC}(\sigma/E^n))$

$$\mathcal{E} \longmapsto (\rho^*\mathcal{E}, \theta_{\mathcal{E}}).$$

RHS: (M, θ), M a f.p. σ/E^n-mod (VB)

$\theta : M \to M$

s.t. $\theta(fx) = \theta(f)x + f\,\theta(x)$

$\forall f \in \sigma/E^n, \ x \in M.$

(2) For $X = \operatorname{Spf} \mathcal{O}_K$, $n \in \mathbb{N} \cup \{\infty\}$,

$$\begin{array}{ccc} \mathcal{D}(\tilde{X}_n^{\Delta}) & \xrightarrow{\simeq} & \mathcal{D}_{\mathrm{Nilp}}(\mathrm{MIC}(\sigma[\![E/p]\!]/(E/p)^n)) \\ \text{\rotatebox{90}{$\simeq$}} & & \\ \mathcal{D}(X_{\Delta}, \mathcal{O}_{\Delta}[\![\frac{I}{p}]\!]/(I/p)^n). & & \end{array}$$

<u>Rmk</u> (1) $X = \mathbb{Z}_p \rightsquigarrow (\mathbb{Z}_p)^{\Delta} = \mathrm{WCart}$, $\mathrm{WCart}_n = (\mathbb{Z}_p)^{\Delta}_n$

$(\mathbb{Z}_p)^{HT} = \mathrm{WCart}^{HT}.$

& $\mathrm{WCart} = \varinjlim \mathrm{WCart}_n$

$\lambda = n-p.$ $\quad \mathbb{Z}_p[[\lambda]] \xrightarrow{\theta} \mathbb{Z}_p[[\lambda]]$

$\qquad \lambda^i \longmapsto i\cdot\lambda^i$

but $H^1((\mathbb{Z}_p)_{\Delta}, \mathcal{O}_{\Delta}) = \prod_{n\in\mathbb{N}} \mathbb{Z}_p.$

(2) When $n<p$, $X = \mathbb{Z}_p$,

Petrov's obs: $\mathrm{WCart}_n \xrightarrow{\sim} \mathrm{Sym}^{<n}_{\mathrm{WCart}^{HT}}(\mathcal{O}\{1\})$

the rel stack / WCart^{HT}
formed by the coherent sheaf $\mathrm{Sym}^{<n}(\mathcal{O}\{1\})$,
which is the quotient of the sym alg of $\mathcal{O}\{1\}$
by the ideal of elts of deg $\geq n$.

<u>Idea of pf</u> $\quad \mathcal{D}(X_n^{\Delta}) = \lim_{\mathrm{Spec}\,R \to X_n^{\Delta}} \mathcal{D}(R).$

Work with $R = (\mathfrak{S}/E^n)[\varepsilon]/\varepsilon^2$.

$$\rightsquigarrow \quad \begin{array}{ccc} \mathrm{Spf}\,R & \xrightarrow[u\,\mapsto\, u+\varepsilon E(u)]{\delta} & \mathrm{Spf}\,R \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X_n^{\Delta} & \xrightarrow{=} & X_n^{\Delta} \end{array}$$

$\forall\, \mathcal{E} \in \mathcal{D}(X_n^{\Delta})$, $\quad \gamma : \delta^*\pi^*\mathcal{E} \cong \pi^*\mathcal{E} = (p^*\mathcal{E})[\varepsilon]/\varepsilon^2$

$\Rightarrow \quad \mathrm{Id} + \varepsilon\Theta_{\mathcal{E}} : p^*\mathcal{E} \longrightarrow (p^*\mathcal{E})[\varepsilon]/\varepsilon^2$

$\Rightarrow \quad \Theta_{\mathcal{E}} : p^*\mathcal{E} \longrightarrow p^*\mathcal{E}.$

Unfortunately, such γ exists on the locus adjoining I/p.

<u>Why?</u>

$$\begin{tikzcd} \sigma \arrow[r] \arrow[rr, bend left, "\exists\ \delta\text{-ring map}", "\tilde{f}"'] & (\sigma/E^n)[\varepsilon]/\varepsilon^2 \arrow[r, "f"] \arrow[d, "\delta"'] & R \\ & (\sigma/E^n)[\varepsilon]/\varepsilon^2 \arrow[ur, "f"'] & \end{tikzcd}$$

with $W(R) \to R$.

$$\begin{tikzcd} (E) \arrow[r] \arrow[d, shift right, "\tilde{f}"'] \arrow[d, shift left, "\tilde{g}"] & \sigma \arrow[d, shift right, "\tilde{f}"'] \arrow[d, shift left, "\tilde{g}"] \arrow[dl, dashed, "\gamma"'] \\ (E) \otimes_{\sigma, \tilde{f}} W(R) \arrow[r] & W(R) \end{tikzcd}$$

$$\pi(f) = \left\{ \begin{array}{l} \alpha : (E) \otimes_{\sigma,\tilde{f}} W(R) \longrightarrow W(R) \\ \quad + \eta : \sigma/E \to \mathrm{Cone}\ \alpha \end{array} \right\}$$

$$\pi \circ \delta(f) = \left\{ \begin{array}{l} \alpha' : (E) \otimes_{\sigma,\tilde{f}} W(R) \longrightarrow W(R) \\ \quad + \eta' : \sigma/E \to \mathrm{Cone}\ \alpha' \end{array} \right\}.$$

$$\gamma \triangleq \Big(\underbrace{\gamma_b : \alpha \cong \alpha'}_{\cong} + \text{homotopy } \gamma_c : \tilde{f}(w) - \tilde{g}(w) = \tilde{f}(\lambda)\cdot c \Big).$$

b s.t. $\tilde{f}(\lambda) b = \tilde{g}(\lambda)$ (add $\frac{I^p}{p}$) ; (add $\frac{I}{p}$).

<u>Relative prismatization</u>

(A, I) $\quad X/\bar{A}$ sm p-adic formal sch,

$$X^{HT}_{/A} \hookrightarrow X^{\mathbb{A}}_{/A,n} \hookrightarrow X^{\mathbb{A}}_{/A}.$$

Thm (Bhatt - Lurie) $X^{HT}_{/A}$ is a gerbe banded by $T^{\#}_{X/\bar{A}}\{1\}$.

$\Rightarrow$ Can describe $\mathrm{Perf}(X^{HT}_{/A})$ using Higgs mods when it splits.

Thm Assume $n \in \mathbb{N} \cup \{\infty\}$, and

(i) X lifts to a sm p-adic formal $X_n / (A/I^{n+1})$.

(ii) X_1/p^2 admits a Frob lifting that on $X/(p)$,

& compatible w/ Frob str of $A/(I^2, p^2)$.

Then (1) If $(A, I) = (A, (p))$.

$$\mathcal{D}(X^{\Delta}_{A,n}) \xrightarrow{\sim} \mathcal{D}_{Nil_p}(MIC(X_{n,et}))$$

Rmk This was obtained by Wang, using prismatic site, when working w/ VB.

(2) If (A, I) is a transversal prism, then

$$\begin{array}{c} \mathcal{D}(\tilde{X}^{\Delta}_{A,n}) \xrightarrow{\sim} \mathcal{D}_{Nil_p}(MIC(X_{n,et}, \mathcal{O}_{X_{n,et}}[[I/p]]/(I/p)^n)) \\ \| \\ \mathcal{D}((X/A)_{\Delta}, \mathcal{O}_{\Delta}[[I/p]]/(I/p)^n) . \end{array}$$