EXAMS OF ALGEBRAIC GEOMETRY I (2022 FALL)

Take-home Quiz

Supposedly, the following 5 problems are to be finished in consecutive 4 hours.

Problem 1. *First order deformations.*

(1) (10 points) Let X be a k-scheme. Show that

$$\mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}\, k[\epsilon]/\epsilon^2, X) \cong \{(x, v) \mid x \in X \text{ such that } \kappa(x) \cong k \text{ and } v \in \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)\},$$

where $\mathfrak{m} \subset \mathcal{O}_{X,x}$ is the maximal ideal and $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}$.

(2) (5 points) Prove that this description uniquely characterizes the scheme Spec $k[\epsilon]/\epsilon^2$ among all k-schemes.

Problem 2. *Prime avoidance.*

(1) (10 points) Let $X = \mathrm{Spec}\, A$ be an affine scheme and $Z \subset X$ be a closed subset. Prove that for any finite number of points $x_1, \dots, x_n$ (not necessarily closed), there is an element f of A such that $D(f)$ is disjoint from Z and contains all the points $x_1, \dots, x_n$.

(2) (5 points) Translate this into a statement about rings and ideals. (If you want to reduce Problem 1 to this theorem, you have to prove this algebraic theorem as well.)

Hint: *use induction. Even though the geometric form and algebraic form are completely equivalent to each other, you are encouraged to think and prove this in the geometric form.*

Problem 3. *Open dense dominance in reduced-separated case.*

(20 points) Let X, Y be schemes over S. Assume that X is reduced, Y is separated over S. Let U be an open dense subset of X. Prove that

$$\mathrm{Hom}_S(X, Y) \to \mathrm{Hom}_S(U, Y)$$

is injective.

Problem 4. *Finite type versus finiteness.*

(20 points) Let $f : X \to Y$ be a generically finite, dominant, finite type morphism between integral schemes. Prove that there is an open subscheme $V \subset Y$ such that $f : f^{-1}(V) \to V$ is a finite morphism.

Problem 5. *Restrictive representability.*

Let L/K be a finite field extension and X a scheme over L. Define a functor

$$\mathrm{Res}_{L/K}(X) : \{\text{Schemes of finite type over } K\}^{\mathrm{op}} \longrightarrow \mathbf{Sets}$$

sending a finite type affine K-scheme T to $\mathrm{Hom}_{\mathbf{Sch}_L}(T \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L, X)$. The image of morphisms are defined in the obvious way.

(1) (10 points) Prove that if $X = \mathbb{A}^1_L$, then this functor is representable.

(2) (10 points) Prove that if X is a finite type affine L-scheme, $\mathrm{Res}_{L/K}(X)$ is representable.

(3) (10 points) Let X be a group scheme over L (i.e. a group object in the category of L-schemes, or an L-scheme such that all the group operations are L morphisms). Prove that if $\mathrm{Res}_{L/K}(X)$ is representable, it is a group scheme over K. If $X = \mathrm{SL}(n)_L$, the special linear group scheme over L. What is the Zariski tangent space of $\mathrm{Res}_{L/K}(X)$ at the identity?

Date: February 24, 2023.

Midterm Exam

Students are required to solve the following 2 problems in consecutive 2 hours.

Problem 6. Let X be a noetherian scheme and F a coherent sheaf on X. For each point $x \in X$, define

$$\phi(x) = \dim_{\kappa(x)} F_x \otimes_{\mathcal{O}_{X,x}} \kappa(x).$$

(1) Show that ϕ is upper-semi-continuous on X.
(2) Assume that X is reduced and that ϕ is constant. Prove F is locally free.
(3) Give an example of non-reduced scheme X and a coherent sheaf F such that ϕ is constant but F is not locally free.

Problem 7. Let X be a noetherian scheme. Prove that X is affine if and only if every integral closed subscheme Y of X is affine. (Here Y could equal X if X is itself an integral scheme.

Final Exam

Students are required to solve the following 2 problems in consecutive 2 hours.

Problem 8. Let $f : X \to Y$ be a finite surjective morphism between noetherian separated schemes. Assume that X is affine. Prove that Y is affine.

Problem 9. Let k be an algebraically closed field. Let $X \subset \mathbb{P}^4_k$ be defined by the homogeneous ideal $(X_0X_1 - X_2X_3)$, where $(X_0 : X_1 : X_2 : X_3)$ is the projective coordinate. Compute $\mathrm{Cl}(X)$ and $\mathrm{Pic}(X)$.

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