# Sheaves

## §1 Presheaves

Fix $\mathcal{C}$ cat, $X$ top space, $\underline{X}$ = cat of opens of $X$.

Define a presheaf $\mathcal{F} : \underline{X} \longrightarrow \mathcal{C}$ by

objs: $\forall\, U \in \underline{X}$, $\mathcal{F}(U) \in \mathcal{C}$

Mors: $\forall\, V \subseteq U$, $Res_{U,V} = Res_{U,V}(\mathcal{F}) : \mathcal{F}(U) \to \mathcal{F}(V)$ s.t.

(i) $\forall\, U \in \underline{X}$, $Res_{U,U} = id_{\mathcal{F}(U)}$.

(ii) $\forall\, W \subseteq V \subseteq U$, $Res_{V,W} \circ Res_{U,V} = Res_{U,W} \in Hom_{\underline{X}}(U, W)$.

<u>Ambiguity</u>. What is $\mathcal{F}(\phi)$? (This does not matter)

EGA avoids this by omitting the "presheaves".

$\mathcal{F}(U) :=$ section of $\mathcal{F}$ on $U$ = $\underbrace{\Gamma(U, \mathcal{F})}_{\text{contravariant functor } \Gamma(U, \cdot)}$.

## §2 Sheaves

E.g. Presheaf $\mathcal{F} : \underline{X} \longrightarrow Sets$ by fixing $Y \in Top$.

$U \longmapsto Hom_{\underline{Top}}(U, Y)$

Special Feature: a conti. func. can be specified locally

<u>Motivation</u> $\left\{ \begin{matrix} \mathcal{F}(V_i),\ i \in I \\ \text{agrees on intersections} \end{matrix} \right\} \xleftrightarrow{\ \approx\ } \left\{ \mathcal{F}(U).\ U \subseteq \bigcup V_i \right\}$

Axiom for sheaves:

(i) $s_1, s_2 \in \mathcal{F}(U)$ s.t. $s_1|_{V_i} = s_2|_{V_i}$, $\forall i \Rightarrow s_1 = s_2$. (when $\mathcal{C} = \underline{Ab}$, just check for $s_2 = 0$). } uniqueness.

(ii) $s_i \in \mathcal{F}(V_i)$ s.t. $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, $\forall i \neq j \Rightarrow \exists\, s \in \mathcal{F}(U)$ s.t. $s|_{V_i} = s_i$.

E.g. for sheaves:

① On manifolds: $\mathcal{F}(U) = \{\text{conti func } f: U \to Y\}$, $Y \in \mathcal{T}op$.
 ⇝ $Y = C$ discrete ⇒ $\mathcal{F}$ = loc. const sheaf.

② On differentiable mfld: $\mathcal{F}(U) = \{\text{diff func on } U\}$.

③ On cplx mflds: $\mathcal{F}(U) = \{\text{holo func on } U\}$.

④ On $X \in Var_{alg}(\bar{k})$. $\mathcal{F}$ = reg funcs = $\mathcal{O}_X$ or $\mathcal{F} = \Omega_{X/\bar{k}}$.

⇒ locally ringed spaces.

(E-Hypothesis) ∃ forgetful functor $\mathcal{C} \to \underline{\text{Sets}}$ reflecting small limits & colimits.

## <u>§3 Defining Sheaves on a Basis</u>

$\mathcal{B}$ = basis of $X$, i.e. $\forall U \in X$, $U = \cup V_i$, $V_i \in \mathcal{B}$.

<u>Vakil's Definition</u>: $\mathcal{B}$ nice ⟺ $\forall U_1, U_2 \in \mathcal{B}$, $U_1 \cap U_2 \in \mathcal{B}$.

⇝ Basic Lemma:

$$\begin{tikzcd} \mathcal{F}_{\mathcal{B}} : \underline{\mathcal{B}} \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \underline{X} \arrow[ur, dashed, "\mathcal{F}"'] & \end{tikzcd}$$

& $\mathcal{F}_{\mathcal{B}} \to \mathcal{F}'_{\mathcal{B}}$ also extends to $\mathcal{F} \to \mathcal{F}'$.

<u>Philosophy</u> "extending sections $s_i$ ⇝ $s$" ⇝ "extending sheaves $\mathcal{F}_i$ ⇝ $\mathcal{F}$".

⇒ we can glue sheaves.

⇒ "Sheaf of sheaves is a sheaf".

i.e. $\forall i \in I$, $\mathcal{F}_i : \underline{U_i} \to \mathcal{C}$ satisfying <u>axioms</u>, $X \subseteq \cup U_i$

⇒ $\mathcal{F} : \underline{X} \to \mathcal{C}$ s.t. $\mathcal{F}|_{U_i} = \mathcal{F}_i$.

$\boxed{\exists!\ \text{gluing}}$

§4 Stalks

Look in the nbhd of a point.

Direct system = contravariant functor $P \xrightarrow{F} \mathcal{C}$

directed set.

[ Say $S, T \in P$, $x \in F(S)$, $y \in F(T)$. $x \sim y$ if

$$\begin{tikzcd} S \arrow[dr, "f"'] & & T \arrow[dl, "g"] \\ & U & \end{tikzcd}$$

s.t. $Ff(x) = Fg(y) \in F(U)$.

$\rightsquigarrow \varinjlim F = \bigcup_{S \in P} F(S) / \sim$ ]

e.g. $R$ integral, $\operatorname{Frac} R = \varinjlim_{f \in R \setminus \{0\}} R[x]/(xf-1)$.

Here $P = R \setminus \{0\}$ (ordered under divisibility)

$R[x]/(xf-1) \mapsto R[x]/(xfg-1) \leftrightsquigarrow x \mapsto xg$.

Def'n $\mathcal{F}_x = \varinjlim \mathcal{F}$, $\mathcal{F} : P_x \to \mathcal{C}$. $\mathcal{F}_x$ = stalk at $x = \coprod$ germs

$f, g \in \Gamma(U, \mathcal{F})$ defines the same germ at $x \in U$ if $f(U) = g(U)$

Caution In the case of (cplx) manifolds:

germs contain more info than values.

Also define stalk at $Z \subseteq X$ (any subset) by $\mathcal{F}_Z = \varinjlim_{U \supseteq Z} \mathcal{F}(U)$.

§5 Stalks and Morphisms

Prop $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$. $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$.

① $\phi_x$ inj./bij., $\forall x \iff \phi(U)$ inj./bij. $\forall U$

② $\phi_x$ surj. $\forall x$ $\not\Rightarrow$ $\phi(U)$ surj. $\forall U$.

Eg. $X = \mathbb{C} \setminus \{0\}$. $\mathcal{F}_1$ = sheaf of holo on $X$

$\mathcal{F}_2$ = sheaf of holo (nowhere vanishing)

$\rightsquigarrow \phi : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$

$(f : U \to \mathbb{C}) \mapsto (\exp\circ f : U \to \mathbb{C})$

<u>Then</u> (i) $\phi$ is surj b/c $\log(\exp\circ f)$ always exists <u>locally</u>

(ii) but not globally : $z \in \Gamma(X, \mathcal{F}_2)$, $z \notin \operatorname{im}\phi(X)$.

## <u>§6 Sheafification</u>

$\mathcal{F} : \underline{X} \longrightarrow \mathcal{C}$ presheaf. Define $\mathcal{F}^+ : \underline{X} \to \mathcal{C}$

$$\mathcal{F}^+(U) = \left\{ s = \prod_{x\in U} s_x,\ s_x \in \mathcal{F}_x \ \middle|\ \begin{array}{l} \forall x \in U,\ \exists\, x \in V \subseteq U \ \& \\ t \in \mathcal{F}(V) \text{ s.t. } s_y = t_y,\ \forall y \in V \end{array} \right\}$$

$\rightsquigarrow \mathcal{F}^+$ sheaf, with $\mathcal{F}^+_x = \mathcal{F}_x$.

<u>Note</u> $(\cdot)^+$ & forgetful functor are adjoint.

## <u>§7 Direct and Inverse Image</u>

$f : X \longrightarrow Y$ conti. $\mathcal{F} \in Sh(X)$, $\mathcal{G} \in Sh(Y)$.

$\rightsquigarrow (f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ defines $f_*\mathcal{F} \in Sh(Y)$.

And presheaf $(f^{-1}_-)\mathcal{G} \in Sh(X)$ : $(f^{-1}_-\mathcal{G})(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$, $U \in \underline{X}$.

$\rightsquigarrow (f^{-1}_-)^+ = f^{-1}$ 'inverse image' functor

<u>Prop</u> $f_*, f^{-1}$ are adjoint, i.e.

$$\mathrm{Hom}_{Sh(Y)}(f_*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{Sh(X)}(\mathcal{F}, f^{-1}\mathcal{G})$$

Define the <u>restriction</u> of $\mathcal{F}$ : $Z \subseteq X$ arbitrary,

$\mathcal{F}|_Z = i^{-1}\mathcal{F}$, $i : Z \hookrightarrow X$.