Shimura varieties with infinite level,
and torsion in the cohomology of locally symmetric spaces

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$/\mathbb{C}$: $\mathfrak{h} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$.

$\Gamma \subseteq SL_2(\mathbb{Z}) \rightsquigarrow$ modular curve $M_\Gamma = \mathfrak{h}/\Gamma$.

param ell curves w/ level str.

$$\rightsquigarrow \varprojlim_{\Gamma} \mathfrak{h}/\Gamma \approx \mathfrak{h} = \left\{ (E,\alpha) \;\middle|\; \begin{array}{l} E \text{ ell curve } + \\ \alpha : H_1(E,\mathbb{Z}) \underset{\text{orientation-preserving}}{\simeq} \mathbb{Z}^2 \end{array} \right\}$$

$$\mathfrak{h} \to \mathbb{P}^1(\mathbb{C})$$

by $(E,\alpha) \mapsto \mathbb{C}^2 \underset{\alpha}{\cong} H_1(E,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \twoheadrightarrow \operatorname{Lie} E$.

Hodge fil'n

<u>Want</u> A similar picture $/\mathbb{C}_p$.

<u>Recall</u> Hodge fil'ns $/\mathbb{C}_p$:

Let $C/\mathbb{Q}_p$ alg closed + complete

X/C proper sm.

<u>Thm</u> (Hodge) $\exists$ Hodge-de Rham spectral seq

$$E_1^{ij} = H^j(X, \Omega^i_X) \Rightarrow H^{i+j}_{dR}(X)$$

degenerates at $E_1 \Rightarrow$ Hodge-de Rham fil'n.

<u>Note</u> Unlike $/\mathbb{C}$, $H^i_{\acute{e}t}(X,\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \neq H^i_{dR}(X)$.

(not even after $\otimes_C B_{dR}$: $C \not\hookrightarrow B_{dR}$.).

<u>Thm</u> (Scholze) $\exists$ Hodge-Tate spectral seq

$$E_2^{ij} = H^i(X, \Omega_X^j)\underbrace{(-j)}_{\text{Tate twist}} \Rightarrow H^{i+j}_{et}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C$$

degenerate at E_2 $\Rightarrow$ HT fil'n.

<u>Example</u> Let E/C ell curve. Get 2 fil'ns:

$$0 \to (\mathrm{Lie}\, E^*)^* \to H_1^{dR}(E) \to \mathrm{Lie}\, E \to 0.$$

$\cong$ universal vec ext'n.

$$0 \to (\mathrm{Lie}\, E)(1) \to T_p E \otimes_{\mathbb{Z}_p} C \to (\mathrm{Lie}\, E^*)^* \to 0.$$

<u>Back to modular curve</u>

Let $\mathcal{M}_\Gamma^{ad}/C$ be the adic space

($\approx$ rigid-analytic variety)

over C assoc with $\mathcal{M}_\Gamma$ (say $\mathbb{C} \supseteq \bar{\mathbb{Q}}_p \hookrightarrow C$).

Let $\Gamma(p^m) \subseteq \Gamma$ be the principal congr subgrps.

<u>Thm</u> (Scholze) (1) $\exists!$ perfectoid space $\mathcal{M}_{\Gamma(p^\infty)}/C$

s.t. $\mathcal{M}_{\Gamma(p^\infty)} \sim$ "$\varprojlim_m \mathcal{M}^{ad}_{\Gamma(p^m)}$"

$GL_2(\mathbb{Q}_p)$ acts on $\mathcal{M}_{\Gamma(p^\infty)}$; ($\sim$: similar to (as cat varies)).

(i.e., homeo on top spaces

+ $\varinjlim_m \mathcal{O}_{\mathcal{M}_{\Gamma(p^m)}} \to \mathcal{O}_{\mathcal{M}_{\Gamma(p^\infty)}}$ has dense image.)

(2) $\exists$ Hodge-Tate period map

$$\pi_{HT}: \mathcal{M}_{\Gamma(p^\infty)} \longrightarrow (\mathbb{P}^1)^{ad}$$

- affine & $GL_2(\mathbb{Q}_p)$-equiv ($\sim$ Hecke corresp.)
- equiv for Hecke operators away from p w.r.t. trivial action on $\mathbb{P}^1$.
- $\omega \cong \pi_{HT}^* \mathcal{O}(1)$.
- extends to min cpt'n

$$\pi_{HT}^*: \mathcal{M}^*_{\Gamma(p^\infty)} \longrightarrow (\mathbb{P}^1)^{ad}.$$

Also, $\pi_{HT}((E, \alpha: T_pE \cong \mathbb{Z}_p^2)) \longmapsto (\mathrm{Lie}\, E)(1) \hookrightarrow T_pE \otimes_{\mathbb{Z}_p} C \overset{\alpha}{\cong} C^2$.

<u>Rem</u> (1) <u>Thm</u> (Scholze-Weinstein)

$C/\mathbb{Q}_p$ alg closed + complete, $\mathcal{O}_C \subseteq C$.

Then

$$\{p\text{-div grps} / \mathcal{O}_C\} \cong \left\{ (\Lambda, W) \,\middle|\, \begin{array}{l} \Lambda \text{ fin free } \mathbb{Z}_p\text{-mod} \\ W \subseteq \Lambda \otimes_{\mathbb{Z}_p} C \text{ sub-}C\text{-v.s.} \end{array} \right\}$$

$$G \longmapsto (T_pG, \text{HT fil'n}).$$

(Think of Λ = Tate mod at p.)

For $A/\mathcal{O}_C$ ab var, then

$$\text{HT fil'n of } A = \text{HT fil'n of } A[p^\infty].$$

Thus, on geom pts of locus of good reduction,

π_{HT} is the map $(E, \alpha) \mapsto (E[p^\infty], \alpha)$

$\rightsquigarrow$ equiv properties of π_{HT}.

(2) Explicit description of π_{HT}:

$$\begin{tikzcd}
\mathcal{M}_{\Gamma(p^\infty)} \arrow[d, "\pi_{HT}"'] & = & \mathcal{M}^{ord}_{\Gamma(p^\infty)} \arrow[d, "\pi_{HT}"] & \sqcup & \mathcal{M}^{ss}_{\Gamma(p^\infty)} \arrow[d, "\pi_{HT}"] \\
\mathbb{P}^1_{/\mathbb{C}_p} & = & \mathbb{P}^1(\mathbb{Q}_p) & \sqcup & \Omega^2
\end{tikzcd}$$

Drinfeld's upper-half space.

π_{HT} on $\mathcal{M}^{ord}_{\Gamma(p^\infty)}$ measures the position of canonical subgrp.

Lubin-Tate space at ∞ level

Faltings (+ Fargues, Scholze-Weinstein)

$$\begin{tikzcd}
\mathcal{M}^{ss}_{\Gamma(p^\infty)} \arrow[r, phantom, "\cong"] \arrow[rrd, bend right, "\pi_{HT}"'] & \coprod_{\text{finite}} \mathcal{M}_{LT,\infty} \arrow[r, phantom, "\cong"] & \coprod_{\text{finite}} \mathcal{M}_{Dr,\infty} \arrow[d] \\
 & & \Omega^2
\end{tikzcd}$$

— tower of fin ét covers of Ω^2

(3) Everything works for Shimura varieties of Hodge type.

<u>Applications</u> Thm (Scholze)

Let $Sh_K / \bar{\mathbb{Q}}$ Shimura var of Hodge type.

Any system of Hecke eigenvals in $H^i_{c,\acute{e}t}(Sh_K, \bar{\mathbb{F}}_p)$ can be lifted to system of Hecke eigenvals of classical cusp form.

<u>Cor 1</u> Let F tot real or CM,

X_K locally symm space for GL_n / F.

Then for any system of Hecke eigenval appearing in $H^i(X_K, \bar{\mathbb{F}}_p)$,

$\exists$ cont Gal repr $\rho: Gal(\bar{F}/F) \to GL_n(\bar{\mathbb{F}}_p)$ assoc with it.

Cor 2 For any regular alg cusp autom repr π of GL_n / F. $\mathbb{C} \cong \bar{\mathbb{Q}}_p$,

$\exists$ cont Gal repr $\rho_\pi : \mathrm{Gal}(\bar{F}/F) \longrightarrow GL_n(\bar{\mathbb{Q}}_p)$

assoc with it.

(also proved by Harris-Lan-Taylor-Throne).

Proof of Thm Let $Sh_K / \mathbb{C}$ assoc adic spaces.

Step 1 (Comp. result for torsion coeff., Scholze)

$$H^i_{c,\acute{e}t}(Sh_K, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \xrightarrow[\text{(almost)}]{\cong} H^i_{\acute{e}t}(Sh_K^*, I^+/p)$$

where $I^+ = I \cap \mathcal{O}^+ \subseteq \mathcal{O}$

sheaf of cusp forms (I) ; fcts of $\mathcal{O}$ w/ $|\cdot| \le 1$ ($\mathcal{O}^+$)

Let $K = K_p K^p$.

$$\rightsquigarrow \varinjlim_{K_p} H^i_{c,\acute{e}t}(Sh_{K_pK^p}, \mathbb{F}_p) \otimes \mathcal{O}_C/p \xrightarrow[\text{almost}]{\cong} H^i_{\acute{e}t}(Sh_{K^p}^*, I^+/p).$$

Step 2 $$\underbrace{H^i_{an}(Sh_{K^p}^*, I^+/p)} \xrightarrow[\text{(almost)}]{\cong} H^i_{\acute{e}t}(Sh_{K^p}^*, I^+/p)$$

can be computed by Čech complex.

terms are $\Gamma(\mathcal{U}, I^+)/p$, $\mathcal{U}$ affinoid

cusp forms of ∞ level.

This relies on:

Thm (S.) Let (R, R^+) perf'd affinoid C-alg, $X = \mathrm{Spa}(R, R^+)$.

Then

$$H^i_{\acute{e}t}(X, \mathcal{O}_X^+) \overset{\text{almost}}{=} \begin{cases} R^+, & i = 0 \\ 0, & i > 0. \end{cases}$$

<u>Step 3</u> Approximate by classical cusp forms (of finite level).

<u>Usually</u> $\mathcal{U}$ = ord locus,

<u>Sol'n</u> Multiply by Hasse invariant.

Commutes w/ Hecke operator away from p.

(doesn't mess up eigenvals).

<u>Solution</u> Pullback any section of $\mathcal{O}(1)/\mathbb{P}^1$ via π_{HT} to get "fake Hasse inv".