Cohomology of Projective Spaces

§1 Finitely Generated Sheaves

$(X, \mathcal{O}_X)$ loc. ringed space. $\mathcal{F} \in Sh(Mod_{\mathcal{O}_X})$

<u>Def'n</u> $\mathcal{F}$ fin. gen'd if $\forall x \in X$, $\exists x \in U \subseteq X$, $n \geq 0$ s.t. $\mathcal{O}_X^{\oplus n}|_U \to \mathcal{F}|_U$.

namely: loc. gen'd by fin. collection of sections.

<u>Lemma</u> $M \in Mod_A$. M f.g. as A-mod $\Leftrightarrow$ $\tilde{M} \in \mathcal{O}coh$ f.g.

<u>Caution</u> Hartshorne: "f.g. + q coh = coh" (in general wrong)

but valid when X = loc. nt sch.

§2 Odds and Ends

<u>Lemma</u> $f: Z \to X$ closed immersion. $\mathcal{F} \in Sh_{Ab}(Z)$

$\Rightarrow$ canonical $H^i(Z, \mathcal{F}) \xrightarrow{\cong} H^i(X, f_*\mathcal{F})$ $(i \geq 0)$

Proof. f_* preserves flasqueness.

$$\& \quad (f_*\mathcal{F})_x = \begin{cases} \mathcal{F}_x, & x \in Z. \\ 0, & x \notin Z \end{cases}$$

Note that $H^0(Z, \mathcal{F}) \cong H^0(X, f_*\mathcal{F})$ (as $\Gamma(\cdot, \cdot)$) canonically.

A flasque resolution of $\mathcal{F}$

$\overset{\Gamma(Z, \cdot)}{\rightsquigarrow}$ same complexes on X $\Rightarrow$ same H^i. $\square$

<u>Lemma</u> X noetherian. top space. $(\mathcal{F}_j)$ direct system of ab sh.

$$\Rightarrow \mathcal{F}_j \to \varinjlim_j \mathcal{F}_j \rightsquigarrow \varinjlim_j H^i(X, \mathcal{F}_j) \xrightarrow{\cong} H^i(X, \varinjlim_j \mathcal{F}_j).$$

§3 Main Result

$\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(\mathrm{Mod}_{\mathcal{O}_X})$, $\mathcal{F}$ q-coh & flat (as $\mathcal{O}_X$-mod).

$\rightsquigarrow$ natural homomorphism

$$H^0(X, \mathcal{F}) \otimes_{\mathcal{O}_X} H^r(X, \mathcal{G}) \longrightarrow H^r(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \quad (\forall r \geq 0)$$

of $\Gamma(X, \mathcal{O}_X)$-mods.

can be computed using Čech complexes

It comes from the facts:

- both sides are δ-functors in $\mathcal{G}$ (note: $\mathcal{F}$ flat on $\mathcal{F} \otimes_{\mathcal{O}_X}(\cdot)$. indeed, $H^0(X, \mathcal{F})$ flat over $\mathcal{O}_X$).
- Left functor: effaceable ($\Rightarrow$ univ).
- $\exists$ natural map for $r = 0$.

Thm (Serre) $\forall r \geq 1$, put $X = \mathbb{P}^r_A$, $S = A[x_0, \dots, x_r]$.

(a) The natural isom of graded S-mods:

$$S \xrightarrow{\cong} \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(n)).$$

(b) $\forall 0 < i < r$, $n \in \mathbb{Z}$,

$$H^i(X, \mathcal{O}_X(n)) = 0.$$

(c) $H^r(X, \mathcal{O}_X(-r-1)) \cong A$.

(d) Perfect pairing (as a natural A-linear map)

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \longrightarrow H^r(X, \mathcal{O}_X(-r-1)) \cong A.$$

(of f.g. A-mods, $\forall n \in \mathbb{Z}$, via $\mathrm{Hom}(-, A)$).

(e) $\forall i > r$, $H^i(X, \mathcal{O}_X(n)) = 0$ $(n \in \mathbb{Z})$.

Namely, the information in S is completely determined by

$$\begin{cases} H^0 \longrightarrow \Gamma_*(X) = \bigoplus H^0(X, \mathcal{O}_X(n)). \\ H^i \ (0<i<r) : \text{vanishing} \\ H^r \longrightarrow \text{"dual" to } H^0. \\ H^i \ (i>r) : \text{vanishing} \end{cases}$$

Proof. To compute $\mathcal{F} = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(n)$.

(a) Checked earlier (see e.g. Hartshorne Prop II.5.13).

(e) Čech : $\mathfrak{U} = \{D_+(x_i)\}$ $i=0,\dots,r$. $\check{C}^i(\mathfrak{U}, \mathcal{O}_X(n)) = 0$, $\forall i>r$

$\rightsquigarrow H^i(X, \mathcal{O}_X(n)) = 0$, $\forall i>r$.

c.f. Hartshorne Thm III.2.7.

(c) To compute $H^i(X,\mathcal{F})$, we need

$$\operatorname{coker}\left(d^{r-1} : \prod_{k=0}^{r} S_{x_0\cdots x_{k-1}x_{k+1}\cdots x_r} \to S_{x_0\cdots x_r}\right).$$

View $S_{x_0\cdots x_r} = A[x_0^{\pm},\dots,x_r^{\pm}]$ as a free A-mod.

(gen'd by $x_0^{e_0}\cdots x_r^{e_r}$ monomials).

$\Rightarrow \operatorname{im} d^{r-1} = \operatorname{span}_A\{x_0^{e_0}\cdots x_r^{e_r},\ e_i \geq 0\}$.

$\Rightarrow H^r(X,\mathcal{F})$ free A-mod gen'd by $x_0^{e_0}\cdots x_r^{e_r}$, $e_i<0$.

$\Rightarrow$ In deg $-r-1$, exactly one monomial

$$(x_0^{-1}\cdots x_r^{-1}) = H^r(X, \mathcal{O}_X(-r-1)) \simeq A,$$

(d) $n<0$: $H^0(X, \mathcal{O}_X(n)) = S_n = 0$

$\Rightarrow H^0(X, \mathcal{O}_X(-n-r-1)) = 0$

b/c $\nexists$ monomials of deg $>-r-1$ as $x_0^{e_0}\cdots x_r^{e_r}$, $e_i<0$.

$\rightsquigarrow$ nothing to check.

May assume $n\geq 0$ thereafter.

$$H^0(X, \mathcal{O}_X(n)) = \mathrm{span}_A\{ x_0^{e_0}\cdots x_r^{e_r} : e_0+\cdots+e_r = n,\ e_i \geq 0\}$$

$$\Rightarrow H^r(X, \mathcal{O}_X(-n-r-1)) = \mathrm{span}_A\{ x_0^{e_0}\cdots x_r^{e_r} : e_0+\cdots+e_r = -n-r-1,\ e_i < 0\}.$$

$\rightsquigarrow$ Description of the pairing:

multiplying together and throw away $x_0^{-1}\cdots x_r^{-1}$.

<u>note</u> this also implies (c).

(b) Induction on r.

① $r=1$: nothing to check (never $0<i<r$)

② Localize: $\check{C}^i(X, \mathcal{O}_X(n))_{x_r} = \check{C}^i(D_+(x_r), \mathcal{O}_X(n)|_{D_+(x_r)})$

affine. Qcoh on affine

$\Rightarrow \check{C}^i(X, \mathcal{O}_X(n))_{x_r}$ acyclic.

On the other hand, $(\cdot)_{x_r}$ is exact.

$\rightsquigarrow (\cdot)_{x_r}$ commutes /w $H^i(\cdot)$.

i.e. $\forall i>0$, $H^i(X, \mathcal{O}_X(n))_{x_r} = 0$.

(in other words, $\forall s \in H^i(X, \mathcal{O}_X(n))$, $x_r^d \cdot s = 0$, $d \in \mathbb{N}$).

It suffices to show $\forall 0<i<r$,

$(\cdot)\times x_r : H^i(X, \mathcal{O}_X(n)) \to H^i(X, \mathcal{O}_X(n))$ injective.

$\Leftarrow\ 0 \to S(-1) \xrightarrow{x_r} S \to S/(x_r) \to 0$ exact

(of graded S-mods)

Consider $H \cong \mathbb{P}_A^{r-1}$ hyperplane $x_r = 0$, $j: H \hookrightarrow X$

$\rightsquigarrow 0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \bigoplus_{n\in\mathbb{Z}} (j_*\mathcal{O}_H)(n) \to 0.$

$\rightsquigarrow \delta^0 : H^0(X, \oplus\cdots) \to H^1(X, \mathcal{O}_X(-1))$ zero.

(ind hyp) $\Rightarrow H^i(X, \oplus(j_*\mathcal{O}_H)(n)) \cong H^i(H, \oplus\,\mathcal{O}_H(n))$, $\forall 0<i<r-1$

$\Rightarrow$ $(\cdot)\times x_r$ bijective, $\forall 0 \leq i \leq r-2$.
& $(\cdot)\times x_r$ inj. $(i = r-1)$
$\Bigg\} \Rightarrow H^i(X, \mathcal{O}_X(n)) = 0.$ $(0<i<r)$ □

§4 Finiteness of Cohomology on Projective Schemes

(The truely powerful vanishing thm by Serre)

Little Thm (Hartshorne II.5.17, without nt hyp).

$\varphi : X \longrightarrow \mathbb{P}^r_A$ closed imm, $r \geq 1$.

Let $\mathcal{O}_X(1) = \varphi^{-1}\mathcal{O}_{\mathbb{P}^r_A}(1)$. $\mathcal{F} \in \mathrm{Qcoh}(X)$ f.g.

$\Rightarrow$ $\mathcal{F}(n)$ gen'd by finitely many global sections, $n \gg 0$.

Cor $\exists n \ll 0$ s.t. $\bigoplus_{i=1}^{m} \mathcal{O}_X(n) \twoheadrightarrow \mathcal{F}$.

Big Thm (Serre vanishing) A (nt). $X \xrightarrow{f} \mathbb{P}^r_A$ closed imm ($r \geq 1$).

$\mathcal{O}_X(1) := f^{-1}\mathcal{O}_{\mathbb{P}}(1)$. $\mathcal{F} \in \mathrm{Qcoh}(X)$ f.g.

Then (a) $H^i(X, \mathcal{F})$ (as A-mods) f.g. $\forall i \geq 0$.

(b) $H^i(X, \mathcal{F}(n)) = 0$, $n \gg 0$. $\forall i > 0$.

Proof. (Descending induction on i).

X admits a good cover by $\leq r+1$ open affines

$\Rightarrow$ $H^i(X, \mathcal{F}) = 0$, $\forall i > r$.

Cor $\Rightarrow$ $\mathcal{F} = \mathcal{E}/\mathcal{G}$, $\mathcal{E} = \bigoplus_{i=1}^{m} \mathcal{O}_X(n)$.

Nt hyp $\Rightarrow$ $\mathcal{G}$ f.g.

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$$

$$\rightsquigarrow \cdots \to \underset{\bigoplus_{k=1}^{m} H^i(X, \mathcal{O}_X(n_i))}{\underset{\cong}{H^i(X, \mathcal{E})}} \to H^i(X, \mathcal{F}) \to \underset{\substack{\text{f.g. as A-mod.} \\ \text{(ind hyp)}}}{\underset{\uparrow}{H^{i+1}(X, \mathcal{G})}} \to \cdots$$

$\Rightarrow$ $H^i(X, \mathcal{F})$ f.g. by nt. $\Rightarrow$ (a).

For (b): twisting by $n \gg 0$

$$\rightsquigarrow \quad \cdots \to \underset{\substack{\| \\ 0 \\ \text{(by explicit calculation)}}}{H^i(X, \mathcal{E}(n))} \longrightarrow H^i(X, \mathcal{F}(n)) \longrightarrow \underset{\substack{\| \\ 0 \\ \text{(ind hyp)}}}{H^{i+1}(X, \mathcal{G}(n))} \longrightarrow \cdots$$

$$\Rightarrow \quad H^i(X, \mathcal{F}(n)) = 0 \,. \qquad \square$$