Serre Duality

§1 Preliminaries on Ext & $\mathcal{E}xt$

Setup X ringed sp. $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$.

$\rightsquigarrow \mathrm{Hom}(\mathcal{F}, -)$, $\mathcal{H}om(\mathcal{F}, -)$ left-exact.

$\mathrm{Hom}(\mathcal{F},-)$: im $\in \mathrm{Mod}_{\Gamma(X, \mathcal{O}_X)}$; $\mathcal{H}om(\mathcal{F},-)$: im $\in \mathrm{Mod}_{\mathcal{O}_X}$, as sh.

$\mathrm{Mod}_{\mathcal{O}_X}$ w/ enough inj. obj.

$\rightsquigarrow$ right derived fun $\begin{cases} \mathrm{Ext}^i \text{ (glo ext'n).} \\ \mathcal{E}xt^i \text{ (sh ext'n)} \end{cases}$

compute via $I^\bullet$

Prop $\mathcal{E}$ loc. free $\Rightarrow$ $\mathcal{E}xt^i(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee, \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{E})$

$\cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{E}$.

& $\mathrm{Ext}^i(\mathcal{F} \otimes \mathcal{E}^\vee, \mathcal{G}) \cong \mathrm{Ext}^i(\mathcal{F}, \mathcal{G} \otimes \mathcal{E})$

can tensor as sh

Prop $U \subset X$ open, $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})(U) = \mathcal{E}xt^i(\mathcal{F}|_U, \mathcal{G}|_U)$

i.e. $\underline{\mathcal{E}xt^i \text{ is local}}$.

Caution: $\mathcal{E}xt$, Ext genuinely diff.

e.g. $\mathrm{Ext}^i(\mathcal{O}_X, \mathcal{F}) = H^i(X, \mathcal{F})$

whereas $\mathcal{E}xt^i(\mathcal{O}_X, \mathcal{F}) = \begin{cases} \mathcal{F}, & i = 0, \\ 0, & \text{otherwise} \end{cases}$

Thm (Loc-glo spec'l seq'ce).

$$E_2^{p,q} = \boxed{H^p(X, \underbrace{\mathcal{E}xt^q(\mathcal{F}, \mathcal{G})}_{\text{loc.}}) \Rightarrow \underbrace{\mathrm{Ext}^{p+q}(\mathcal{F}, \mathcal{G})}_{\text{glo.}}}$$

$\Uparrow$

Grothendieck spec'l seq'ce,

i.e. $\mathrm{Hom} = \Gamma \circ \mathcal{H}om$. $\in$ Ab Fun,

e.g. $\mathrm{Hom}_{coh}(\mathcal{F}, \mathcal{G}) = \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))$.

In Abgp: enough proj.s $\rightsquigarrow P^\bullet \rightsquigarrow \mathrm{Ext}$. (✓)

In Sh: (✗). not enough (e.g. $\mathcal{E}$ free & not proj.).

But loc. free res $\mathcal{E}^\bullet \rightsquigarrow \mathcal{E}xt$.

by acyc - $\mathcal{O}_X$ ($\Rightarrow$ acyc of $\mathcal{E}$) & spec'l seq'ce.

(Cor X loc. nt, $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}_X \Rightarrow \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \in \mathrm{Coh}_X$)

(✗) for Qcoh. even for $i = 0$.

§2 Thm (Serre duality). X CM + proj., $\mathrm{pdim}_k X = n$ (pure)

Yoneda $\Rightarrow$ $\exists!$ $\omega_X \in \mathrm{Coh}_X$ s.t.

(a) $\mathcal{F}$ vec. bundle $\rightsquigarrow$ $H^i(X,\mathcal{F}) \times H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X) \to k$
perfect.

(b) $\mathcal{F} \in \mathrm{Coh}_X$. $\mathrm{Ext}^i(\mathcal{F}, \omega_X) \times H^{n-i}(X,\mathcal{F}) \longrightarrow k$
perfect.

$\Downarrow i=0$

(b') $\mathcal{F} \in \mathrm{Coh}_X$. $\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^n(X,\mathcal{F}) \longrightarrow k$ perfect

(CM not needed).

(c) When X sm. $\omega_X \cong K_X := \wedge^n \Omega_X$.

note CM : to construct $X \to \mathbb{P}^n$ flat.

by Miracle flatness.

counter e.g. $X = \mathbb{P}^2 \amalg \mathbb{P}^2 \subseteq \mathbb{P}^4$. Serre fails.

fact (b) $\Rightarrow$ X CM.

(b') $\rightsquigarrow$ $t = tr : H^n(X, \omega_X) \to k \in H^n(X,\omega_X)^\vee$

$\cong$ $\mathrm{id} \in \mathrm{Hom}(\omega_X, \omega_X)$

by $\mathcal{F} = \omega_X$.

$$\rightsquigarrow \begin{tikzcd} \mathrm{Hom}(\mathcal{F},\omega_X) \times H^n(X,\mathcal{F}) \arrow[rr] \arrow[dr, dashed] & & k. \\ & H^n(X,\omega_X) \arrow[ur, "t"'] & \end{tikzcd}$$

Fact $H^i(X,\mathcal{F}) \times H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X) \dashrightarrow H^n(X,\omega_X) \xrightarrow{tr} k$ (a)

$\mathrm{Ext}^i(\mathcal{F},\omega_X) \times H^{n-i}(X,\mathcal{F}) \dashrightarrow H^n(X,\omega_X) \xrightarrow{tr} k$ (b)

Strategy of pf. ① $X = \mathbb{P}^n$ by cal.

② $\pi : X \to \mathbb{P}^n$ fin. flat $\rightsquigarrow$ (b).

$$\begin{array}{ccc} X & \to & \mathbb{P}^n \\ | & & | \\ \omega_X & & \pi^! \omega_{\mathbb{P}^n} \end{array}$$

③ $i : X \hookrightarrow \mathbb{P}^N$ reg. emb. $\rightsquigarrow$ (c)

ω_X ✓ K_X ✓.

④ Prop (b) $\Rightarrow$ (a)

pf. $H^i(X, \mathcal{F})^\vee \cong \mathrm{Ext}^{n-i}(\mathcal{F}, \omega_X)$ (b)

$\cong \mathrm{Ext}^{n-i}(\mathcal{O}_X, \mathcal{F}^\vee \otimes \omega_X)$

$\cong H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X)$.

§3 Serre duality holds for proj. sp.

Goal (b) w/ $\omega_{\mathbb{P}^n} = \mathcal{O}(-n-1)$ $\rightsquigarrow$ (c). $\rightsquigarrow$ (a).

Lem $X = \mathbb{P}^n$, $\omega_X = \mathcal{O}(-n-1)$ $\Rightarrow$ (b').

pf. $\exists$ canonical $\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X)$.

$H^n(X, \omega_X) \cong k$ ← choose

① if $\mathcal{F} \cong \mathcal{O}(m)$ line bun.

$\rightsquigarrow \mathrm{Hom}(\mathcal{O}(m), \mathcal{O}(-n-1)) \times H^n(\mathcal{O}(m)) \to H^n(\mathcal{O}(-n-1)) \cong k$.

$\mathrm{Hom}(\mathcal{O}(m), \mathcal{O}(-n-1)) = H^0(\mathcal{O}(-m-n-1)) = \langle \prod_{i=0}^{n} x_i^{n_i}, \sum n_i = -m-n-1 \rangle^{\exp \geq 0}$

$H^n(\mathcal{O}(m)) = \langle \prod_{i=0}^{n+1} x_i^{n_i}, \sum n_i = m \rangle^{\exp < 0}$

$H^n(\mathcal{O}(-n-1)) = \langle \cdots, = -n-1 \rangle^{\exp < 0}$

$\Rightarrow$ pairing map = multi. $\Rightarrow$ perfect.

$\Rightarrow$ (b') w.r.t. $\mathcal{F} = \mathcal{E}^{\otimes m}$.

② Arbitrary $\mathcal{F}$.

$\rightsquigarrow \quad 0 \to \underset{\text{ker}}{\mathcal{G}} \to \underset{\bigoplus \mathcal{O}(m_i) \text{ sum of line bds.}}{\mathcal{E}} \to \mathcal{F} \to 0$

$$\Rightarrow \quad \begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(\mathcal{F}, \omega_X) \arrow[r] \arrow[d, "\text{inj.}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}(\mathcal{E}, \omega_X) \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}(\mathcal{G}, \omega_X) \arrow[d] \\ 0 \arrow[r] & H^n(X, \mathcal{F})^\vee \arrow[r] & H^n(X, \mathcal{E})^\vee \arrow[r] & H^n(X, \mathcal{G})^\vee \end{tikzcd}$$

left-exact

note $\dim X = \dim \mathbb{P}^n = n \Rightarrow H^{n+1}(X, \mathcal{G}) = 0$.

nice! $0 \to \mathrm{Hom}(\mathcal{G}, \omega_X) \xrightarrow{} H^n(X, \mathcal{G})^\vee$
inj.

By diag. chasing $\rightsquigarrow \mathrm{Hom}(\mathcal{F}, \omega_X) \cong H^n(X, \mathcal{F})^\vee$

Prop $X = \mathbb{P}^n$, $\omega_X = \mathcal{O}(-n-1) \Rightarrow$ (b)

pf. To show $\mathrm{Ext}^i(-, \omega_X)$ & $H^{n-i}(X, -)^\vee$
univ. δ-functors.

(b'): $i = 0 \rightsquigarrow$ isom.

- left-exact at $i=0$ ✓
- long exact seq'ce ✓
- coefficiability (!)

<u>Check</u> $\forall \mathcal{F} \in \mathrm{Coh}\, X$,

$$\mathcal{O}(-m)^{\oplus N} \longrightarrow \mathcal{F} \longrightarrow 0 \quad , \; m \gg 0 .$$

indeed: $H^{n-i}(\mathbb{P}^n, \mathcal{O}(-m)^{\oplus N}) = 0$, $\forall i > 0$

$$\mathrm{Ext}^i(\mathcal{O}(-m)^{\oplus N}, \omega) = H^i(\mathbb{P}^n, \mathcal{O}(-m-n-1))^N = 0$$

$\forall i > 0$. □

§4 The dualizing sheaf exists in general

<u>Lem</u> X equi-dim'l proj. k-sch.

$\rightsquigarrow \exists\, \pi : X \longrightarrow \mathbb{P}^n$ fin.

& X CM $\Rightarrow \pi$ flat.

Pf. $X \hookrightarrow \mathbb{P}^N$

$\cup$

H codim 1, linear sub, $H \cap X = \phi$.

$\rightsquigarrow \mathbb{P}^N \setminus H \xrightarrow{pr} X \rightsquigarrow \pi$.

X CM $\Rightarrow$ apply miracle flatness. □

⚠ only works for $\# k = \infty$.

but can be modified to $\# k < \infty$.

Def $\pi : X \to Y$ fin. $\mathcal{G} \in \mathrm{Qcoh}_{\mathcal{O}_Y} \overset{?}{\rightsquigarrow} \pi^! \mathcal{G} \in \mathrm{Qcoh}_{\mathcal{O}_X}$.

Step 1 $Y = \mathrm{Spec} B$, $X = \mathrm{Spec} A$, $\mathcal{G} = \tilde{N}$, $N \in \mathrm{Mod}_B$.

$\rightsquigarrow M = \mathrm{Hom}_B(A, N) \in \mathrm{Mod}_B \rightsquigarrow {}_A M$.

$b \in B$, M_b ok

$\Downarrow$

$\tilde{M} \in \mathrm{Qcoh}_X$.

Step 2 Globally: $\boxed{\pi^! \mathcal{G} := \mathcal{H}om_{\mathcal{O}_Y}(\pi_* \mathcal{O}_X, \mathcal{G})}$. $\mathcal{F}$ coh.

Fact $\pi^! \mathcal{G} \in \mathrm{Coh}_{\mathcal{O}_Y}$ or ${}_{\mathcal{O}_X}\pi^! \mathcal{G} \in \mathrm{Coh}_{\mathcal{O}_X}$

when f fin. + $\mathcal{G} \in \mathrm{Coh}$.

Frob reciprocity $\mathrm{Hom}_A(M, \underbrace{\mathrm{Hom}_B(A, N)}_{N \to B}) \cong \mathrm{Hom}_B(M_B, N)$

$M \to A$.

$M_B \to B$

glo. $\rightsquigarrow \pi_* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \pi^! \mathcal{G}) \cong \mathcal{H}om_{\mathcal{O}_Y}(\pi_* \mathcal{F}, \mathcal{G})$.

$\Rightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \pi^! \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\pi_* \mathcal{F}, \mathcal{G})$.

$\qquad \parallel \qquad\qquad\qquad \parallel$

$\quad \Gamma(X, \cdots) \qquad\qquad \Gamma(Y, \cdots)$.

Lem $\pi : X \to Y = \mathbb{P}^n$, $\omega_X = \pi^! \omega_Y$. $\Rightarrow$ (b)'.

Pf. Now $\mathrm{Hom}_X(\mathcal{F}, \pi^! \omega_Y) \times H^n(X, \mathcal{F})$

$\updownarrow \qquad \text{\rotatebox{90}{$\cong$}} \qquad \updownarrow$ $\mathcal{F} \in \mathrm{Coh}_X$.

$\mathrm{Hom}_Y(\pi_* \mathcal{F}, \omega_Y) \times H^n(Y, \pi_* \mathcal{F})$

$$\begin{tikzcd}
\mathrm{Hom}_X(\mathcal{F}, \pi^! \omega_Y) \times H^n(X,\mathcal{F}) \arrow[r, "\text{perfect}"] \arrow[d, "\cong"'] & H^n(X, \pi^! \omega_Y) \arrow[d] \arrow[rr, bend left, "\mathrm{tr}_X"] & & k \\
& H^n(Y, \pi_* \pi^! \omega_Y) \arrow[d] & & \\
\mathrm{Hom}_Y(\pi_*\mathcal{F}, \omega_Y) \times H^n(Y, \pi_*\mathcal{F}) \arrow[uu, Rightarrow] \arrow[r, "\text{perfect}"'] & H^n(Y, \omega_Y) \arrow[rruu, bend right, "\mathrm{tr}_Y"'] & &
\end{tikzcd}$$

$\square$

Rmk "π flat" is not used $\rightsquigarrow$ $\boxed{(b') \not\Rightarrow X \text{ CM}.}$

Lem $\pi : X \longrightarrow Y = \mathbb{P}^n$, X CM & $\omega_X = \pi^! \omega_Y \Rightarrow (b)$.

pf. Check $\mathrm{Ext}^i_X(-, \pi^!\omega_Y)$, $H^{n-i}(X,-)^\vee$ coeffaceable.

($\Rightarrow$ univ. δ-fun's).

Again: $\mathcal{O}_X(-m)^{\oplus N} \longrightarrow \mathcal{F} \to 0$, $m \gg 0$.
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \in \mathrm{Coh}$.

$X, Y \underset{\text{emb}}{\overset{\text{pr}}{\hookrightarrow}} \mathbb{P}^N \Rightarrow \pi^* \mathcal{O}_Y(1) \cong \mathcal{O}_X(1)$.

Now $\forall i > 0$, $\mathrm{Ext}^i_X(\mathcal{O}_X(-m), \pi^!\omega_Y) \cong \mathrm{Ext}^i(\mathcal{O}_X, \pi^!\omega_Y(m)) = 0$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \wr\wr$

$m \gg 0$, Serre vanishing $\longrightarrow$ $H^i(X, \pi^!\omega_Y(m))$

Also $H^{n-i}(X, \mathcal{O}_X(-m))^\vee \cong H^{n-i}(Y, \pi_*(\underbrace{\mathcal{O}_X \otimes \pi^*\mathcal{O}_Y(-m)}_{\pi^! \mathcal{O}_Y(-m)}))^\vee$

(a) of $Y = \mathbb{P}^n$ i.e. $\quad\quad \cong H^{n-i}(Y, \pi_* \mathcal{O}_X \otimes \mathcal{O}_Y(-m))^\vee$

$H^i(Y,\mathcal{F})^\vee \cong H^{n-i}(Y, \mathcal{F}^\vee \otimes \omega_Y) \longrightarrow \cong H^i(Y, (\pi_*\mathcal{O}_X)^\vee \otimes \mathcal{O}_Y(m) \otimes \omega_Y)$.

Serre $\longrightarrow$ $= 0$ $\quad m \gg 0$. $\square$

Rmk (a): $\pi_* \mathcal{O}_X$ vec. bud. $\Leftarrow$ π fin. flat $\Leftarrow$ CM.

$\boxed{\S 5}$ The dualizing sheaf is the canonical bundle for sm. var.

X sm. ($\Rightarrow$ CM) proj. var, p $\dim_k X = n$

$\Rightarrow X \to \mathbb{P}^N = Y$ reg. emb. w/ codim $N-n=r$.

$\boxed{\text{Step 1}}$ Goal $\omega_X \cong \mathcal{E}xt^r_Y(\pi_* \mathcal{O}_X, \omega_Y)$.

$\qquad\qquad\qquad\qquad\qquad\qquad \in \mathrm{Coh}_{\mathcal{O}_X}$.

Subtlety $X \xrightarrow{\pi} Y$, $\mathcal{G} \in \mathrm{Qcoh}_Y$.

$\rightsquigarrow \mathcal{H}om_Y(\pi_* \mathcal{O}_X, \mathcal{G}) \in \mathrm{Qcoh}_{\mathcal{O}_Y}$ w/ $\mathcal{O}_X$-str.

$\rightsquigarrow {}_{\mathcal{O}_X}\mathcal{H}om_Y(\pi_* \mathcal{O}_X, \mathcal{G})$ need.

b/c if $\mathcal{G} \notin \mathrm{Qcoh}$, inj. sol. may not be qcoh.

- $X \to Y$ closed emb $\Rightarrow \exists\, \mathcal{O}_X$-str. $\forall \mathcal{G}$.

top reason: $\forall\, U \underset{\text{open}}{\subseteq} X$, $U = \pi^{-1}(V)$, $V \subseteq Y$ open

$\Rightarrow$ data to define $\mathcal{O}_X$-action,

$\rightsquigarrow \pi^! \mathcal{G}$ ok. (recover $(\pi_*, \pi^!)$).

& ${}_{\mathcal{O}_X}\mathcal{E}xt^i_Y(\pi_* \mathcal{O}_X, \omega_Y)$ makes sense.

$\boxed{\text{Step 2}}$ <u>Prop</u> $\mathrm{Hom}_X(\mathcal{F}, \mathcal{E}xt^r_Y(\pi_*\mathcal{O}_X, \omega_Y)) \cong \mathrm{Ext}^r_Y(\pi_*\mathcal{F}, \omega_Y)$.

$\Downarrow$

<u>Goal</u> b/c By (b), (b'): $\quad Y = \mathbb{P}^N = \mathbb{P}^{n+r}$

$$\mathrm{Hom}_X(\mathcal{F}, \mathcal{E}xt^r_Y(\pi_*\mathcal{O}_X, \omega_Y)) \cong \mathrm{Ext}^r_Y(\pi_*\mathcal{F}, \omega_Y)$$
$$\overset{(b)}{\cong} H^n(Y, \pi_*\mathcal{F})^\vee \cong H^n(X, \mathcal{F})^\vee$$
$$\overset{(b')}{\cong} \mathrm{Hom}_X(\mathcal{F}, \omega_X).$$

Yoneda $\Rightarrow \omega_X \cong \mathcal{E}xt^r_Y(\pi_*\mathcal{O}_X, \omega_Y)$

<u>Recall</u> (b) $\mathrm{Ext}^i_X(\mathcal{F}, \omega_X) \times H^{n-i}(X, \mathcal{F}) \longrightarrow k$.

(b') $\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \longrightarrow k$.

The pf of Prop: first need

<u>Lem</u> $\forall i < r$, $\mathcal{E}xt^i_Y(\pi_*\mathcal{O}_X, \omega_Y) = 0$.

pf. $\mathcal{E}xt^i_Y(\pi_*\mathcal{O}_X, \omega_Y) \in \mathrm{Coh}_Y$

$\rightsquigarrow$ <u>To show</u> $m \gg 0$:

$$0 = H^0(Y, \mathcal{E}xt^i_Y(\pi_*\mathcal{O}_X, \omega_Y)(m)).$$
$$\parallel$$
$$H^0(Y, \mathcal{E}xt^i_Y(\pi_*\mathcal{O}_X, \omega_Y(m)))$$

Serre vanishing: $H^j(Y, \cdots) = 0, \ \forall j > 0$.

$\Rightarrow$ loc-glo. spec'l seq'ce degen. at $E_2^{p,q}$ $(m \gg 0)$.

$$\Rightarrow \; H^0(Y, \mathcal{E}xt^i_Y(\pi_*\mathcal{O}_X, \omega_Y(m))) \longleftarrow H^j(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G})) = E_2^{i,j}$$

$$\begin{array}{ccc} \mathrm{Ext}^i_Y(\pi_*\mathcal{O}_X, \omega_Y(m)). & & \mathrm{Ext}^{i+j}(\mathcal{F}, \mathcal{G}). \end{array}$$

$$\cong$$

$$\mathrm{Ext}^i_Y(\pi_*\mathcal{O}_X(-m), \omega_Y) \overset{(b)}{\cong} H^{N-i}(Y, \pi_*\mathcal{O}_X(-m))^\vee$$

$$\mathrm{Ext}^i_X(\mathcal{F}, \omega_X) \cong H^{n-i}(X, \mathcal{F})^\vee$$

$$i < r \Rightarrow N-i > N-r = n \Rightarrow H^{N-i}(\cdots) = 0. \qquad \square$$

dim supp $\pi_*\mathcal{O}_X(-m)$.

<u>Pf of Prop</u> Grothendieck spec'l seq'ce

on $\mathcal{H}om_Y(\pi_*\mathcal{O}_X, -)$ & $\mathrm{Hom}_X(\mathcal{F}, -)$.

$\mathcal{G}$ inj. $\Rightarrow \mathcal{H}om_Y(\pi_*\mathcal{O}_X, \mathcal{G}) = \pi^!\mathcal{G}$ acyc. for $\mathrm{Hom}_X(\mathcal{F}, -)$.

$\Rightarrow \mathrm{Ext}^i_X(\mathcal{F}, \mathcal{E}xt^j_Y(\pi_*\mathcal{O}_X, \omega_Y)) = 0, \; \forall j < r.$

$\rightsquigarrow E_2^{i,j}$ degen.

$\Rightarrow \mathrm{Ext}^r_Y(\pi_*\mathcal{F}, \omega_Y) \cong \mathrm{Hom}_X(\mathcal{F}, \mathcal{E}xt^i_Y(\pi_*\mathcal{O}_X, \omega_Y)).$

$\boxed{\text{Step 3}}$ <u>Lem</u> $\mathcal{E}xt^r_Y(\pi_*\mathcal{O}_X, \omega_Y) \cong \det \mathcal{N}_{X/Y} \otimes_{\mathcal{O}_X} \omega_Y|_X$.

natural.

$\rightsquigarrow$ (c) ✓. b/c K_X satisfies same adjunction.

<u>note</u> up to now, haven't used X sm.

pf: (GTM52, Thm III.7.11).

Ext: compute locally.

use Koszul cplx: $Y \xrightarrow[\text{by } \underbrace{x_1, \dots, x_r}_{\text{reg. seq'ce}}]{\text{cut out}} X$.

$\rightsquigarrow$ free resol. of $\pi_* \mathcal{O}_X$.

$\mathrm{Hom}(-, \omega_Y) \rightsquigarrow \omega_Y / (x_1, \dots, x_r) \omega_Y = \omega_Y|_X$

(looking loc.) has dim r cohom.

& 0 otherwise.

<u>Rmk</u> A feature of the whole proof:

I generally push the hardest ingredients to the end.