A new approach to isogenies and Hecke correspondences in mixed characteristic

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<u>Notion of isogenies</u>

- For elliptic curves:

  $f : E_1 \to E_2$ non-const grp homomorphism

- For abelian varieties:

  $f : A_1 \to A_2$ surjective grp homomorphism with finite kernel.

- For p-divisible grps:

  $f : \mathcal{G}_1 \to \mathcal{G}_2$ same as isog of AVs.

<u>Moduli interpretation for Hecke corresp.</u>

$$\mathrm{Isog} = \{(S_1, S_2, f) \mid \mathcal{G}_1 \xrightarrow{f} \mathcal{G}_2\}$$

$$\begin{tikzcd} & & \mathrm{Isog} \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & & \\ \mathcal{G}_1 \arrow[r, bend right] & \mathcal{G} & & \mathcal{G} & \mathcal{G}_2 \arrow[l, bend left] \end{tikzcd}$$

* <u>"p-div grps with additional structure"</u>

<u>EL case</u> $\mathcal{G} \circlearrowleft \mathcal{O}_D$

$$\mathrm{Isog}_{\mathcal{O}_D} := \{(S_1, S_2, f) \mid f : \mathcal{G}_1 \to \mathcal{G}_2 \ \mathcal{O}_D\text{-equivariant}\}$$

<u>PEL case</u> $\mathcal{G} \circlearrowleft \mathcal{O}_D$ + principal polarization $\mathcal{G} \xrightarrow{\sim} \mathcal{G}^\vee$

Q How to define the correct notion of "isogenies" in this context?

Hodge-type case Use that we can somehow "embed" p-div grps into a tensor category using Dieudonné theory.

e.g. Over perfect field K of char p,

$$p\text{-div}_K \longleftrightarrow \left\{ \begin{array}{c|c} \text{F-crystals} / K & M \ W(K)\text{-module} \\ \text{i.e. } (M, \varphi) & \varphi : M \to M \text{ Frob-semilin} \end{array} \right\}$$

by Dieudonné-Manin.

Abelian-type case Objects that can be assoc on the Dieudonné side via constructions arising from Dieudonné mods of p-div grps.

Q Can we do Dieudonné theory without talking about p-div grps?

A Yes! (at least for conn p-div grps).

Bültel-Pappas Let R p-adically complete ring.

B-P define a stack of (G, μ)-displays

- G red grp / $\mathbb{Z}_p$
- μ minuscule cochar def'd / $\mathbb{Z}_p$.

$\rightsquigarrow$ $R \longmapsto$ Groupoid of G-torsors on $W(R)$ equipped w/ some Frob structure.

Zink & Lau $\rightsquigarrow$ BP stack recovers stack of connected p-div grps when $G = GL_n$.

To access the rest of the picture,

can use the new stacks of Bhatt-Lurie & Drinfeld.

R p-adically complete ring.

$\rightsquigarrow$

R^{Δ} prismatization (as a stack)

R^{N} (Nygaard) filtered prismatization,

R^{syn} Syntomification.

<u>Upshot</u> $R^{\Delta} \longleftrightarrow$ (absolute) prismatic cohom of R

"ring of functions" $\quad \|$ if $R/\mathbb{F}_p$

crystalline cohom of R

$\|$ if R sm w/ a sm lift $\tilde{R}/\mathbb{Z}_p$

de Rham cohom of $\tilde{R}$.

$R^{syn} \longleftrightarrow$ prismatic cohom + Nygaard fil'n

+ Frob w/ its divisibility properties.

<u>Drinfeld</u> Defined some stack

$$BT^{G,\mu} := \left(R \longmapsto \begin{array}{l} \text{Groupoid of } G\text{-torsors on } R^{syn} \\ \text{that are bounded by } \mu. \end{array} \right)$$

<u>Rmk</u> Quite close to def'n of equal char shtukas.

<u>Conj A</u> $BT^{G,\mu}$ is pro-algebraic smooth

(i.e. $\varprojlim$ (sm alg stack)).

(This is true for stack of p-div grps by a thm of Grothendieck.)

<u>Conj B</u> For $G = GL_n$, $\mu = \mu_d := (\underbrace{1, \cdots, 1}_{d}, \underbrace{0, \cdots, 0}_{n-d})$,

$$BT^{GL_n, \mu_d} \simeq \{\text{p-div grps of ht } n \text{ \& dim } d\}.$$

<u>Thm</u> Conj A & B hold.

(A) Gardner - Madapusi - Mathew

(B) Dieudonné - Manin

Gabber - Lau (perfect rings)

Anschütz - le Bras (Guo-Li) (syntomic rings)

Gardner - Madapusi - Mathew (in general)

<u>General meaning of isogenies</u>

Let V_1, V_2 two VBs / some flat p-adic base.

- A candidate notion of an isogeny is
  a map $f: V_1 \to V_2$ of vec bdls
  s.t. $f[\frac{1}{p}]$ is an isom.

- A bit more refined:

$$\left.\begin{tikzcd} V_1 \arrow[r, "f"] \arrow[dr, "p^r"'] & V_2 \arrow[d, "g"] \arrow[dr, "p^r"] & \\ & V_1 \arrow[r, "f"] & V_2 \end{tikzcd}\right\} (*)$$

(for some r)

Say $V_i \triangleq \mathcal{P}_i$ GL_n-torsors whose sections are basis of V_i

Then $(*)$ above $\triangleq$ $X := \{(A, B) \in M_n \times M_n : AB = BA = p^r I_n\}$

as an abstract scheme.

& $\quad GL_n \curvearrowright X \curvearrowleft GL_n$

by $\quad g_1(A,B)\, g_2 = (g_1 A g_2^{-1},\, g_2 B g_1^{-1})$.

Define $\quad X(\mathcal{P}_1,\mathcal{P}_2) :=$ "twist" of X by $\mathcal{P}_1, \mathcal{P}_2$ as a $GL_n \times GL_n$-torsor.

<u>Note</u> Data of diagram $(*)$ $\triangleq$ Data of a section of $X(\mathcal{P}_1,\mathcal{P}_2)$.

<u>Defn</u> An <u>isogeny model</u> for G is a $G\times G$-equivariant scheme

$$X \longrightarrow \operatorname{Spec} \mathbb{Z}_p$$

equipped with an isom / $\mathbb{Q}_p$

$$X_{\mathbb{Q}_p} \xrightarrow{\simeq} G_{\mathbb{Q}_p}$$

that is $G\times G$-equivariant.

<u>Thm</u> (Lee-Madapusi) $\exists$ correspondence

$$\begin{tikzcd} & \mathrm{Isog}_X \arrow[dl, bend right, "s"'] \arrow[dr, bend left, "t"] & \\ \mathcal{P}_S \in BT^{G,\mu} & & BT^{G,\mu} \ni \mathcal{P}_T \end{tikzcd}$$

Here s, t are finitely presented maps of stacks

& $\mathrm{Isog}_X(\mathcal{P}_s, \mathcal{P}_t)$ = sections of $X(\mathcal{P}_s, \mathcal{P}_t)$.

<u>Rmk</u> For $\{$B-P displays of GL_n-types$\}$, this is a result by Bartling-Hoff.

<u>Note</u> If X smooth, then Isog_X is flat & lci / $\mathbb{Z}_p$.

By a result of Néron (Busch-Lathelihnert-Raynard),

for any isogeny model X,

$\exists$ "smoothening" $\tilde{X} \longrightarrow X$

s.t. $\tilde{X}$ sm $/ \mathbb{Z}_p$ & $\tilde{X}(\mathbb{Z}_p^{ur}) \xrightarrow{\simeq} X(\mathbb{Z}_p^{ur})$.

Using this, can draw some global conclusions.

<u>Thm</u> (Madapusi, Imai-Kato-Youcis)

Let $\mathscr{S}$ Shimura variety of ab type assoc with $(G, \{\mu\})$,

$K \le G(\mathbb{A}_f)$, p prime w/ $K_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$,

at hyperspecial level ($G_{\mathbb{Z}_{(p)}}$ red grp)

Then $\exists!$ formally étale, surjective map

$$\mathscr{S} \longrightarrow\!\!\!\!\!\rightarrow BT^{G^c, \mu^{-1}}$$

G^c = some quotient of G.

Take "globalizations" of Isog_X:

$$\begin{tikzcd} s^*\mathrm{Isog}_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Isog}_X \arrow[d, "s"] \\ \mathscr{S} \arrow[r] & BT^{G^c, \mu^{-1}} \end{tikzcd} \qquad \begin{tikzcd} t^*\mathrm{Isog}_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Isog}_X \arrow[d, "t"] \\ \mathscr{S} \arrow[r] & BT^{G^c, \mu^{-1}} \end{tikzcd}$$

<u>Thm</u> (Lee-Madapusi) $\exists$ canonical isom

$$s^*\mathrm{Isog}_X \xrightarrow{\cong} t^*\mathrm{Isog}_{X^\vee}$$

$$(s, \mathcal{G}_s \to \mathcal{G}') \longmapsto (t, \mathcal{G} \mapsto \mathcal{G}_t)$$

(but not as schemes $/\mathscr{S}$!)

<u>Rmk</u> Using this, can recover the construction of Kisin & Chen-Kisin-Viehmann to the map

$$\text{ADLVs} \longrightarrow \mathscr{S}(\overline{\mathbb{F}}_p).$$

Precisely, do this by

$$\begin{tikzcd} & s^*\mathrm{Isog}_X = p\text{-}\mathrm{Isog}_X = t^*\mathrm{Isog}_X \arrow[dl] \arrow[dr] & \\ \mathcal{S} & & \mathcal{S}. \end{tikzcd}$$

<u>Observation</u> $X(\mathbb{Z}_p) = \coprod_{\lambda \in \Phi^+(X)} G(\mathbb{Z}_p)\,\lambda(p)\,G(\mathbb{Z}_p)$ by Cartan decomp.

So $p\text{-}\mathrm{Isog}_X$ parametrizes "isogenies" bounded by cochars in $\Phi^+(X)$.