B_{dR} affine Grassmanian

Jingren Chi

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§1 Introduction to B_{dR}^+

__Setups__ $(E, \mathcal{O}_E, \pi)$ non-arch local field, $\mathcal{O}_E/(\pi) \cong \mathbb{F}_q$.

$G/\mathcal{O}_E$ reductive split.

$\mathrm{Perf} = \{\text{perf'd spaces in char } p\}$.

$\mathrm{Div}^1_{Y} = \mathrm{Spd}\, \mathcal{O}_E$:

$$\mathrm{Perf} \ni S = \mathrm{Spa}(R, R^+) \longrightarrow \mathrm{Spd}\, \mathcal{O}_E$$

$$\rightsquigarrow \quad S^\# = \mathrm{Spa}(R^\#, R^{\#+}) / \mathrm{Spa}\, \mathcal{O}_E$$

$$\downarrow$$

$$\mathcal{Y}_S = \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \setminus V([\varpi]),$$

$\varpi \in R$ pseudo-uniformizer.

Define $\cdot$ $B^+(S) = B_{dR}^+(R^\#) = \xi\text{-adic completion of } W_{\mathcal{O}_E}(R^+)(\frac{1}{[\varpi]})$,

where $(\xi) = \ker\left(W_{\mathcal{O}_E}(R^+)[\frac{1}{[\varpi]}] \longrightarrow R^\#\right)$.

$\cdot$ $B(S) = B_{dR}(R^\#) = B_{dR}^+(R^\#)[\frac{1}{\xi}]$.

__E.g.__ (1) $S = \mathrm{Spa}(C, C^+)$, $C/\mathbb{F}_p$ perf'd field, alg closed.

$$\begin{array}{l} S \\ \downarrow \\ \mathrm{Spd}\, \mathcal{O}_E \end{array} \quad \rightsquigarrow (C^\#, C^{\#,+}), \quad \underbrace{B_{dR}^+(C^\#) \cong C^\#[\![\xi]\!]}_{\text{Complete DVR, of char 0.}}$$

(2) If $S^\# = S = \mathrm{Spa}(R, R^+)$, $\xi = \pi \in \mathcal{O}_E$ the uniformizer.

then $\quad B^+(S) = W_{\mathcal{O}_E}(R)$, $B(S) = W_{\mathcal{O}_E}(R)[\frac{1}{\pi}]$.

__Def'n__ $L^+G / \mathrm{Spd}\, \mathcal{O}_E : S \longmapsto L^+G(S) := G(B^+(S))$. positive loop grp

$LG / \mathrm{Spd}\, \mathcal{O}_E : S \longmapsto LG(S) := G(B(S))$. loop grp.

§2 The Grassmanian scheme

We have the Grassmanian

$$\mathrm{Gr}_G/\mathrm{Spd}\,\mathbb{Q}_E : S \longmapsto \mathrm{Gr}_G(S) := \left\{ (\mathcal{E}, \iota) \;\middle|\; \begin{array}{l} \mathcal{E} = G\text{-torsor on } \mathrm{Spec}\, B^+(S), \\ \iota = \text{trivialization of } \mathcal{E}|_{\mathrm{Spec}\, B(S)} \end{array} \right\}$$

$$\mathrm{Hck}_G/\mathrm{Spd}\,\mathbb{Q}_E : S \longmapsto \left\{ (\mathcal{E}_1, \mathcal{E}_2, \varphi) \;\middle|\; \begin{array}{l} \mathcal{E}_1, \mathcal{E}_2 = G\text{-torsors on } \mathrm{Spec}\, B^+(S) \\ \varphi : \mathcal{E}_1|_{\mathrm{Spec}\, B(S)} \xrightarrow{\sim} \mathcal{E}_2|_{\mathrm{Spec}\, B(S)} \end{array} \right\}.$$

Prop (VI.1.9) Gr_G is a small v-sheaf.

$$\mathrm{Gr}_G \cong LG/L^+G \text{ as étale sheaves over } \mathrm{Spd}\,\mathbb{Q}_E.$$

Similarly, $\mathrm{Hck}_G \cong L^+G\backslash LG/L^+G$ is a small v-stack.

Rmk Also have versions of Gr_G, Hck_G over $\mathrm{Div}^d_{\mathcal{Y}} = (\mathrm{Div}^1_{\mathcal{Y}})^d/S_d$

Geometric fibers: $S \to \mathrm{Spd}\,\mathbb{Q}_E$, $\mathrm{Gr}_{G,S} := \mathrm{Gr}_G \times_{\mathrm{Spd}\,\mathbb{Q}_E} S$.

Take $S = \mathrm{Spa}(C, C^+) \longrightarrow \mathrm{Spd}\,\mathbb{Q}_E$ with $S^\# = \mathrm{Spa}(C^\#, C^{\#+})$.

- If $C^\#$ has char 0, then $B^+_{dR}(C^\#) \cong C^\#[\![\xi]\!]$.

$$\mathrm{Gr}_{G,S} \cong \big(\underbrace{\mathrm{Gr}^{cl}_{G,C^\#}}_{\text{classical affine Gr}}\big)^\diamond, \quad \mathrm{Gr}^{cl}_{G,C^\#}(C^\#) = G(C^\#(\!(\xi)\!))/G(C^\#[\![\xi]\!])$$

- If $C^\# = C$ of char p, $B^+_{dR}(C^\#) = W_{\mathbb{Q}_E}(C)$, $\xi = \pi$.

$$\mathrm{Gr}_{G,S} \cong (\mathrm{Gr}^{\mathrm{Witt}}_{G,C})^\diamond, \quad \mathrm{Gr}^{\mathrm{Witt}}_{G,C}(C) = G(W_{\mathbb{Q}_E}(C)[\tfrac{1}{\pi}])/G(W_{\mathbb{Q}_E}(C)).$$

Schubert varieties $G(B_{dR}(C^\#)) = \coprod_{\mu \in X_*(T)^+} G(B^+_{dR}(C^\#))\,\mu(\xi)\,G(B^+_{dR}(C^\#))$.

by Cartan decomposition.

$$\mathrm{Gr}_G(\mathrm{Spa}(C, C^+)) = \coprod_{\mu \in X_*(T)^+} L^+G(\mathrm{Spa}(C, C^+)) \cdot \mu(\xi).$$
$$\downarrow$$
$$\mathrm{Spd}\,\mathbb{Q}_E$$

Def'n $\mathrm{Gr}_{G, \le \mu} \subseteq \mathrm{Gr}_G$ subfunctor of maps $S \to \mathrm{Gr}_G$

s.t. $\forall$ geom pt $S' = \mathrm{Spa}(C, C^+) \to S$ lies in L^+G-orbit of some $\mu' \le \mu$.

Similarly, $\mathrm{Gr}_{G,\mu} \subset \mathrm{Gr}_{G,\le\mu}$.

<u>Prop</u> $\mathrm{Gr}_{G,\le\mu} \subset \mathrm{Gr}_G$ is closed, $\mathrm{Gr}_{G,\mu} \subset \mathrm{Gr}_{G,\le\mu}$ open.

$\mathrm{Gr}_{G,\le\mu}$ is a spatial diamond, proper over $\mathrm{Spd}\, \mathcal{O}_E$.

<u>Idea</u> Reduce to the case $G = GL_n$, $\mu = (N, 0, \dots, 0)$

$$\widetilde{\mathrm{Gr}}_N / \mathrm{Spd}\,\mathcal{O}_E : \underset{\mathrm{Spa}(R,R^+)}{S \,\|} \longmapsto \left\{ \begin{array}{l} B_{dR}^+(R^\#)\text{-lattices } 0 \subset M_1 \subset \cdots \subset M_N = B_{dR}^+(R^\#)^n \\ \text{where } M_{i+1}/M_i \text{ invertible } R^\#\text{-modules} \end{array} \right\}$$

successive $(\mathbb{P}^{n-1})^\diamond$-bundle over $\mathrm{Spd}\,\mathcal{O}_E$

$\Rightarrow$ $\widetilde{\mathrm{Gr}}_N$ is proper, spatial diamond.

<u>Proof</u> Consider
$$\pi : \widetilde{\mathrm{Gr}}_N \longrightarrow \mathrm{Gr}_{\le\mu}$$
$$(M_1 \subset \cdots \subset M_N) \longmapsto M_1$$

- π is surj on geom pts (for char 0 pts, easy.)
  For char p pts, see [Bhatt-Scholze]
- π is qc since $\widetilde{\mathrm{Gr}}_N$ is qc
  $\Rightarrow$ $\mathrm{Gr}_{\le\mu}$ is qc

$\Rightarrow$ π is a v-cover (cf. [Sch17, Lem 12.11].).

It is partially proper by def'n.

So $\mathrm{Gr}_{\le\mu}$ is proper over $\mathrm{Spd}\,\mathcal{O}_E$.

- $\mathrm{Gr}_{\le\mu}$ is spatial [SW, §19, p.175] (Berkeley lecture).
  $\Rightarrow$ $\mathrm{Gr}_{\le\mu}$ is diamond by [Sch17, Thm 12.18]. $\square$

Define a section $[\mu] : \mathrm{Spd}\,\mathcal{O}_E \longrightarrow \mathrm{Gr}_{G,\mu}$
$$S \longmapsto \mu(\xi) \in G(S).$$

Let $(L^+G)_\mu \subset L^+G$ denote the stabilizer of $[\mu]$.

Prop $Gr_{G,\mu} \cong L^+G/(L^+G)_\mu$ is coh smooth of l-dim $\langle 2\rho, \mu\rangle$ / $\mathrm{Spd}\, O_E$.

Proof $\forall m \geq 1$, $(L^+G)^{\geq m}(S) = \mathrm{Ker}(\underset{\substack{\| \\ L^+G(S)}}{G(B^+(S))} \longrightarrow G(B^+(S)/\xi^m))$

Let $(L^+G)^{\geq m}_\mu = (L^+G)_\mu \cap (L^+G)^{\geq m}$.

Then we have

- $L^+G/(L^+G)^{\geq 1} \cong G^\diamond$,
- $(L^+G)^{\geq m}/(L^+G)^{\geq m+1} \cong (\mathrm{Lie}\, G)^\diamond\{m\}$ (use exponential)

Here $(\mathrm{Lie}\, G)^\diamond\{m\}(S) = (\boxed{\mathrm{Lie}\, G} \otimes_{O_E} \mathcal{I}_S^m/\mathcal{I}_S^{m+1})(D_S)$

as an O_E-mod

& $D_S \hookrightarrow \mathcal{Y}_S$ closed Cartier divisor.

$\forall m$, let $(\mathrm{Lie}\, G)_{\mu \leq m} \subseteq \mathrm{Lie}\, G$ subspace

where μ acts by weight $\leq m$ under adjoint action.

$P_\mu^- \subset G$ parabolic subgrp with Lie alg $(\mathrm{Lie}\, G)_{\mu \leq 0}$.

Then

$$(L^+G)_\mu/(L^+G)^{\geq 1}_\mu \cong (P_\mu^-)^\diamond \subset L^+G/(L^+G)^{\geq 1} \cong G^\diamond.$$

$\forall m \geq 1$, $(L^+G)^{\geq m}_\mu/(L^+G)^{\geq m+1}_\mu \cong (\mathrm{Lie}\, G)^\diamond_{\mu\leq m}\{m\} \subset (\mathrm{Lie}\, G)^\diamond\{m\}$.

$\rightsquigarrow$ exact sequence

$$1 \longrightarrow (L^+G)^{\geq 1} + (L^+G)_\mu/(L^+G)_\mu \longrightarrow \underset{\substack{\| \\ Gr_\mu}}{L^+G/(L^+G)_\mu} \longrightarrow \underset{\substack{\wr\| \\ (G/P_\mu^-)^\diamond}}{L^+G/(L^+G)^{\geq 1} + (L^+G)_\mu} \longrightarrow 1.$$

$$\rightsquigarrow \frac{(L^+G)^{\geq 1} + (L^+G)_\mu}{(L^+G)_\mu} \supseteq \frac{(L^+G)^{\geq 2} + (L^+G)_\mu}{(L^+G)_\mu} \supseteq \cdots \supseteq \frac{(L^+G)^{\geq N} + (L^+G)_\mu}{(L^+G)_\mu} = 1.$$

and $$\frac{(L^+G)_\mu + (L^+G)^{\geq i}}{(L^+G)_\mu + (L^+G)^{\geq i+1}} \cong \frac{(L^+G)^{\geq i}}{(L^+G)^{\geq i}_\mu + (L^+G)^{\geq i+1}} \cong \frac{(L^+G)^{\geq i}/(L^+G)^{\geq i+1}}{(L^+G)^{\geq i}_\mu/(L^+G)^{\geq i+1}_\mu}.$$

$$\Rightarrow \dim \mathrm{Gr}_\mu = \langle 2\rho, \mu \rangle. \qquad \square$$

<u>Example</u> $G = GL_n$, $\mu = (k_1, \dots, k_n)$, $k_1 \geq \dots \geq k_n$.

$$\rightsquigarrow k_1 = \dots = k_{n_1} > k_{n_1+1} = \dots = k_{n_1+n_2} > \dots > k_{n-n_r+1} = \dots = k_n.$$

with $n = n_1 + \dots + n_r$.

Then $(L^+G)_\mu(S) = GL_n(B^+(R^\#)) \cap \mu(\xi)\, GL_n(B^+(R^\#))\, \mu(\xi)^{-1}$

where $\mu(\xi) = \begin{pmatrix} \xi^{k_1} & & \\ & \ddots & \\ & & \xi^{k_n} \end{pmatrix}$

$$= \{(a_{ij}) \in GL_n(B^+(R^\#)) \mid \forall i<j,\ a_{ij} \in \xi^{k_i - k_j} B^+(R^\#)\}.$$

$P_\mu^- =$ 

<u>Semi-infinite orbits</u>

$\lambda : \mathbb{G}_m \longrightarrow T \subset G \rightsquigarrow M_\lambda$ with Lie alg $(\mathrm{Lie}\, G)_{\lambda=0}$

$U_\lambda \subseteq P_\lambda = P_\lambda^+$ with Lie alg $(\mathrm{Lie}\, G)_{\lambda \geq 0}$,

P_λ^- with Lie alg $(\mathrm{Lie}\, G)_{\lambda \leq 0}$.

$\mathbb{G}_m$ acts on Gr_G via $\mathbb{G}_m \xrightarrow{[\cdot]} L^+\mathbb{G}_m \xrightarrow{L^+\lambda} L^+G \circlearrowright \mathrm{Gr}_G$.

$\mathbb{G}_m(\mathrm{Spa}(R, R^+)) = R^+$,

$$\begin{array}{ccc} \mathrm{Gr}_{P_\lambda} \longrightarrow \mathrm{Gr}_{M_\lambda} \longrightarrow & \mathrm{Gr}_{\bar{M}_\lambda}, & \bar{M}_\lambda = M_\lambda / [M_\lambda, M_\lambda] \\ \| & \| & \\ \displaystyle\coprod_{\nu \in X_*(\bar{M}_\lambda)} \mathrm{Gr}_{P_\lambda}^\nu \longrightarrow & X_*(\bar{M}_\lambda) \times \mathrm{Spd}\, \mathcal{O}_E. & \\ & \cup & \\ & \nu. & \end{array}$$

<u>Prop</u> $\mathrm{Gr}_{P_\lambda} \longrightarrow \mathrm{Gr}_G$ is bijective on geom pts,

locally closed immersion on each $\mathrm{Gr}_{P_\lambda}^\nu$.

$\left(\bigcup_{\nu' \leq \nu} \mathrm{Gr}_{P_\lambda}^{\nu'} \text{ has closed image.} \right)$

- $\mathbb{G}_m$-action via $L^+\lambda$ on Gr_{P_λ} extends to $\mathbb{A}^1$
- $\mathbb{G}_m$-fixed pts on Gr_G is Gr_{M_λ}.

<u>Cor</u> $S \longrightarrow \mathrm{Div}_X^d$ small v-stack

$$\begin{tikzcd} & \mathrm{Gr}_{P_\lambda,S} \arrow[dl, "q^+"'] \arrow[dr, "p^+"] & \\ \mathrm{Gr}_G \arrow[dr, "q^-"'] & & \mathrm{Gr}_{M_\lambda} \arrow[dl, "p^-"] \\ & \mathrm{Gr}_{P_\lambda^-} & \end{tikzcd}$$

$CT_{P_\lambda} := R(p^-)_* R(q^-)^! \xrightarrow{\cong} R(p^+)_!(q^+)^*$:

$$D_{et}(\mathrm{Gr}_G, \Lambda)^{\mathbb{G}_m\text{-mon}, bd} \longrightarrow D_{et}(\mathrm{Gr}_{M_\lambda}, \Lambda)^{bd}$$

a $\mathbb{G}_m$-monodromy condition (cf. Def IV.6.11)

CT_{P_λ} commutes with any base change in S

and preserves ULA property with respect to S.

<u>Prop</u> (VI.4.2) λ = dominant regular. $P_\lambda = B$ Borel,

$S \longrightarrow \mathrm{Div}_X^d$ small v-sheaf.

Let $A \in D_{et}(\mathrm{Hck}_{G,S}, \Lambda)$ with support qc over S.

If $CT_B(A) = 0$, then $A = 0$.

<u>Proof</u> $\mathrm{Gr}_{T,S} = X_*(T) \times S$, $S_\nu = \mathrm{Gr}_B^\nu$, $\nu \in X_*(T)$.

$CT_B(A) = \bigoplus_{\nu \in X_*(T)} R(P_\nu)_!(A|_{S_\nu})$,

$P_\nu : S_\nu \times_{\mathrm{Spd}\, O_E} S \longrightarrow S \xhookrightarrow{\nu} \mathrm{Gr}_{T,S}$.

If $A \neq 0$, let μ be max'l s.t. $A|_{\mathrm{Gr}_\mu} \neq 0$.

Take $\nu = w_0(\mu)$, $w_0 \in W$ largest element.

$$S_{w_0(\mu)} \cap \mathrm{Gr}_\mu = [w_0(\mu)].$$

$$0 = R(P_{w_0(\mu)})_! A|_{S_{w_0(\mu)}} = A|_{S_{w_0(\mu)} \cap \mathrm{Gr}_\mu}.$$

$\rightsquigarrow$ contradict to L^+G-equivariance of A. $\square$

<u>Comments</u> $Gr_G \times_{\mathrm{Spd}\, \mathcal{O}_E} \mathrm{Spd}\, \mathbb{F}_q = (Gr_G^{Witt})^\diamond$,

$$Gr_G^{Witt}(R) = \left\{ (\mathcal{E}, \iota) \;\middle|\; \begin{array}{l} \mathcal{E} = G\text{-torsor on } \mathrm{Spec}\, W_{\mathcal{O}_E}(R) \\ \iota = \text{trivialization of } \mathcal{E}|_{\mathrm{Spec}\, W_{\mathcal{O}_E}(R)[\frac{1}{\pi}]} \end{array} \right\}$$

$\forall$ perf'd $\mathbb{F}_p$-alg R.

<u>Prop'n</u> $\lambda \in X_*(T)$, $S_\lambda = Gr_B^\lambda \subset Gr_B^{Witt}$.

$S_\lambda \cap Gr_{\le\mu}^{Witt}$ is rep'ble by an affine scheme

equi-dim'l of dim $\langle \rho, \mu+\lambda \rangle$.

<u>Pf sketch</u> $\mathcal{L}$ = ample line bun Gr_G^{Witt} constructed by Bhatt-Scholze

- $\mathcal{L}|_{S_\lambda}$ is naturally trivialized.
- $\mathcal{O}_{S_\lambda} \simeq \mathcal{L}|_{S_\lambda}$ extends to section of $\mathcal{L}$ on $\overline{S}_\lambda := \bigcup_{\lambda' \le \lambda} S_{\lambda'}$

  vanishes on $\overline{S}_\lambda \setminus S_\lambda$.

$\Rightarrow Gr_{\le\mu}^{Witt} \cap S_\lambda$ is affine.

Let $S_d = \bigcup_{\langle 2\rho, \lambda \rangle \le d} S_\lambda \cap Gr_{\le\mu}^{Witt} \subseteq Gr_{\le\mu}^{Witt}$ closed.

$\rightsquigarrow$ $d = \langle 2\rho, \mu \rangle$, $S_{\langle 2\rho, \mu \rangle} = Gr_{\le\mu}^{Witt}$ with $\dim = \langle 2\rho, \mu \rangle$

$d = -\langle 2\rho, \mu \rangle$, $S_{-\langle 2\rho, \mu \rangle} = [w_0(\mu)]$, $\dim = 0$.

And $S_d \setminus S_{d-2} = \coprod_{\langle 2\rho, \lambda \rangle = d} S_\lambda \cap \underbrace{Gr_{\le\mu}}_{\text{affine}}$

$\dim S_d - \dim S_{d-2} \le 1$.

$\Rightarrow \dim S_\lambda \cap Gr_{\le\mu} = \langle \rho, \lambda+\mu \rangle$, equi-dim'l. $\square$