Homological Algebra

§1 Abelian Categories

ab cat = ab grp with composition + more restrictions.

<u>Recall</u> (1) Biproducts: $\forall X_1, \cdots, X_n \in \mathrm{Obj}(\mathcal{C})$, $\exists Y$ &

$$\iota_i : X_i \to Y, \quad \pi_i : Y \to X_i, \quad i = 1, \cdots, n$$

s.t. $Y = \prod X_i$ w.r.t. π_i, $Y = \coprod X_i$ w.r.t. ι_i.

and $\sum \iota_i \circ \pi_i = 1$.

(2) ker and coker:

$$\ker f = \text{limit of } \begin{tikzcd} X \arrow[dr, "f"'] & & 0 \arrow[dl] \\ & Y & \end{tikzcd}$$

$$\mathrm{coker} f = \text{colim of } \begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr] & \\ Y & & 0 \end{tikzcd}$$

E.g. of Ab cats: AbGrp, Mod_R, $\mathrm{Sh}(X)^{ab} = \{\mathcal{F} : \underline{X} \to \mathcal{C}, \mathcal{C} \text{ abelian}\}$.

<u>Recommendation</u> Just thinking over ab grps.

Freyd-Mitchell embedding thm: can reduce ab cat to AbGrp. (using diagram-chasing).

§2 Complexes and Exact Sequences

Complex: $\underbrace{\cdots \to C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1}}_{\text{increasing: cohomology grading}} \to \cdots$, $d^i \circ d^{i-1} = 0 \ (\forall i)$.

resp. $\to C_i \xrightarrow{d_i} C_{i-1} \to$: homo $\sim$.

i-th cohom: $h^i(C) = \ker d^i / \mathrm{im}\, d^{i-1}$. Say exact if $h^i(C) = 0$, $\forall i$.

$f^\cdot : C^\cdot \to D^\cdot \rightsquigarrow f^i : h^i(C^\cdot) \to h^i(D^\cdot)$, $\forall i$.

Say f quasi-isom if f^i isom, $\forall i$

$\Leftarrow f \approx 0$ (homotopy).

Say $f^\cdot, g^\cdot$ homotopy ($f^\cdot \approx g^\cdot$) if $\exists k^i : C^i \to D^{i-1}$

s.t.

$$\begin{tikzcd}
\to C^{i-1} \arrow[r, "d^{i-1}"] \arrow[d, "f^{i-1}"'] & C^i \arrow[r, "d^i"] \arrow[d, "f^i"] \arrow[dl, "k^i"'] & C^{i+1} \to \arrow[d, "f^{i+1}"] \arrow[dl, "k^{i+1}"'] \\
\to D^{i-1} \arrow[r, "d^{i-1}"] & D^i \arrow[r] & D^{i+1} \to
\end{tikzcd}$$

$$f^i - g^i = k^{i+1} \circ d^i + d^{i-1} \circ k^i.$$

$\Rightarrow f$ & g induce the same map on cohoms.

<u>Important Comparison</u> (Homotopy equiv) versus (quasi-isom)

Def'n	arrow-theoretic	algebraic
Applying Functors	stable	not stable
$\rightsquigarrow$ Sequences	stable: being complex	unstable: being exact

<u>Philosophy</u> (1) Cohom theory losses information.

(like filtered obj $\rightsquigarrow$ associated graded obj)

(2) Derived Cat: a better environment

s.t. all q-isoms are invertible (formally).

§3 The Long Exact Sequence in Cohomology

Short exact: $0 \to C^\cdot \to D^\cdot \to E^\cdot \to 0$ of complexes

$\rightsquigarrow$ Long exact:

$$\begin{tikzcd}
\cdots \arrow[r] & h^i(C^\cdot) \arrow[r] & h^i(D^\cdot) \arrow[r] & h^i(E^\cdot) \arrow[dll, "\delta^i"] \\
& h^{i+1}(C^\cdot) \arrow[r] & h^{i+1}(D^\cdot) \arrow[r] & h^{i+1}(E^\cdot) \to \cdots
\end{tikzcd}$$

Def'n of δ^i: $x \in E^i$ representing a class in $h^i(E^\cdot)$

$\rightsquigarrow$ lifting to $y \in D^i$ by exactness

$\rightsquigarrow$ $d^i(y) \in E^i \Leftrightarrow d^i(x) = 0$

$\rightsquigarrow$ $d^i(y)$ lifts to $z \in C^{i+1}$

Check: $d^{i+2}(d^{i+1}(z)) = 0$

$\Rightarrow$ z rep's a class in $h^{i+1}(C^\cdot)$

$$\begin{tikzcd}
 & & y \arrow[d, phantom, "\cap" description] & x \arrow[d, phantom, "\cap" description] \arrow[ddll, bend left=60] & \\
0 \arrow[r] & C^i \arrow[r] \arrow[d, "d^i"'] & D^i \arrow[r] \arrow[d, "d^i"'] & E^i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^{i+1} \arrow[r] & D^{i+1} \arrow[r] & E^{i+1} \arrow[r] & 0 \\
 & z \arrow[u, phantom, "\cup" description] & d^i(y) \arrow[u, phantom, "\cup" description] & &
\end{tikzcd}$$

§4 Cohomological Functors

$F: \mathcal{C}_1 \to \mathcal{C}_2$ additive <u>covariant</u> b/w ab cats.

<u>Recall</u> · F left-exact: $0 \to A_1 \to A_2 \to A_3$ exact

$\rightsquigarrow$ $0 \to F(A_1) \to F(A_2) \to F(A_3)$ exact.

· F right-exact: $A_1 \to A_2 \to A_3 \to 0$ exact

$\rightsquigarrow$ $F(A_1) \to F(A_2) \to F(A_3) \to 0$ exact

· F exact = left + right exact.

$\Leftrightarrow$ F preserves exact sequences (any length).

E.g. (1) $\mathrm{Hom}_{\mathcal{C}}(X, -)$ ($\mathcal{C}$ ab cat) is left-exact.

(2) In Mod_R, $X \otimes_R (-)$ left exact on $0 \to Y_1 \to Y_2 \to Y_3 \to 0$

Y_3 flat mod $\Rightarrow$ exact. (note: free $\Rightarrow$ flat)

Idea of Resolutions

$$\underset{\substack{\uparrow \\ \text{bad single obj}}}{X} \rightsquigarrow 0 \to X \to 0 \rightsquigarrow \underbrace{0 \to \overbrace{X^0 \to X^1}^{\text{nice objects}} \to \cdots}_{\text{quasi-isom. complex}}$$

"nice" = (e.g. flat) induces exact sequences

Construction: $0 \to A \to B \to C \to 0$

$\rightsquigarrow \quad 0 \to A^\cdot \to B^\cdot \to C^\cdot \to 0$ resolution

$A^\cdot, B^\cdot, C^\cdot$ sufficiently nice. F <u>left</u>-exact

$\rightsquigarrow \quad 0 \to F(A^\cdot) \to F(B^\cdot) \to F(C^\cdot) \to 0$

$\rightsquigarrow \quad 0 \to h^0(F(A^\cdot)) \to h^0(F(B^\cdot)) \to h^0(F(C^\cdot)) \xrightarrow{\delta^0} h^1(F(A^\cdot)) \to \cdots$

What we really want: $h^0(F(A^\cdot)) = A$.

s.t. $0 \to F(A) \to F(B) \to F(C) \underbrace{\to \cdots}_{\text{fills the gaps here}}$

Define δ-functor (or cohomological functor)

b/w ab cats $T^i : \mathcal{C}_1 \longrightarrow \mathcal{C}_2 \quad (i = 0, 1, \cdots)$

& $\delta^i : T^i(C) \longrightarrow T^{i+1}(A)$ connector

for each $0 \to A \to B \to C \to 0$ in $\mathcal{C}_1$.

s.t.
$$\left.\begin{tikzcd} 0 \arrow[r] & T^0(A) \arrow[r] & T^0(B) \arrow[r] & T^0(C) \arrow[dll, "\delta^0"'] \\ & T^1(A) \arrow[r] & T^1(B) \arrow[r] & T^1(C) \xrightarrow{\delta^1} \cdots \end{tikzcd}\right\} \text{exact.}$$

[Say δ-functor $T = \{T^i\}$ is <u>universal</u> if

$\forall$ δ-fun $S = \{S^i\}$, given $f^0 : T^0 \to S^0$, then

$\exists!$ sequence of natural trans $f^i : T^i \to S^i$

s.t.
$$\begin{tikzcd} T^i(C) \arrow[r, "\delta^i"] \arrow[d, "f^i"'] \arrow[dr, phantom, "\circlearrowright"] & T^{i+1}(A) \arrow[d, "f^{i+1}"] \\ S^i(C) \arrow[r, "\delta^i"] & S^{i+1}(A) \end{tikzcd}.$$
]

<u>Criterion for universality</u>

Say $F : \mathcal{C}_1 \to \mathcal{C}_2$ is <u>effaceable</u> if $\forall A \in \mathcal{C}_1$, $\exists (A \xrightarrow{u} B) \in \mathrm{Mor}(\mathcal{C}_1)$

u monomorphism & $Fu = 0$.

How to think about this?

Kedlaya: We mostly deal with "monotonic" functors:

larger input $\longleftrightarrow$ larger output.

But effaceable functors are the <u>opposite</u>.

<u>Thm</u> (Grothendieck)

$T^\cdot : \mathcal{C}_1 \to \mathcal{C}_2$ δ-functor s.t. T^i effaceable, $\forall \underline{\underline{i>0}}$.

$\Rightarrow$ T universal.

Typically: $A \in \mathcal{C}_1 \rightsquigarrow A \xrightarrow{u} B$ mono, B acyclic $\Big\}$ "nice"

i.e. $T^i(B) = 0$, $\forall \underline{\underline{i>0}}$.

<u>Thm</u> (Acyclic Resolution) $T^\cdot : \mathcal{C}_1 \to \mathcal{C}_2$ univ δ-functor. $J \in \mathcal{C}_1$.

$0 \to A^0 \to A^1 \to \cdots$ in $\mathcal{C}_1$. each A^i acyclic $\Big\}$ acyclic resolution of J.

& $h^0(A^\cdot) \cong J$, $h^i(A^\cdot) = 0$, $\forall i > 0$.

$\Rightarrow$ $\forall \underline{\underline{i \geq 0}}$, $\exists\, T^i(h^0(A^\cdot)) \cong h^i(T^0(A^\cdot))$ functorial.

i.e. $T^0(A^\cdot) \rightsquigarrow T^i(J)$.

<u>§5 Derived Functors</u>

While making some univ-δ:

$$\begin{tikzcd} \delta\text{-functor} \arrow[rr, bend right] & & \text{acyclic resolution} \arrow[ll, bend right] \end{tikzcd}$$

- A vicious circle?
- Get out by identifying always-acyclic objs.

<u>Def'n</u> (1) $X \in \mathcal{C}$ injective if $\underbrace{\mathrm{Hom}(-,X)}_{\text{already left-exact}} : \mathcal{C}^{op} \to \underline{\mathbf{Ab}}$ exact.

$0 \to Y \to Z$ mono, $\forall Y \to X$, $\exists Z \to X$ s.t.

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[dr] & Z \arrow[d, dashed, "\exists"] \\ & & X \end{tikzcd}$$

i.e. everything injects into X.

(2) $X \in \mathcal{C}$ projective if $\underbrace{Hom(X,-)}_{\text{also left-exact naturally}} : \mathcal{C} \to \underline{Ab}$ exact

$$\begin{tikzcd} Y \arrow[r] & Z \arrow[r] & 0 \\ & X \arrow[u] \arrow[ul, dashed, "\exists"] & \end{tikzcd}$$

i.e. X big enough to projects to everything

E.g. (1) Mod_R : free $\Rightarrow$ proj.

(2) Abgrp : divisible $\Leftrightarrow$ inj.

<u>Lemma</u> I inj $\Rightarrow$ $0 \to I \to B \to C \to 0$ splits

i.e. $\exists\, C \to B$ s.t. $(C \to B \to C) = id_C$.

Proof.
$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[dr] & B \arrow[d, dashed, "\exists"] \\ & & I \end{tikzcd}$$
$\rightsquigarrow$ $ker(B \to I) \cong C$.

$I \to B \to I$ identity. $\square$

<u>Note</u> Additive functor preserves $\begin{cases} \text{no general exactness} \\ \text{split-exactness.} \end{cases}$

<u>Prop</u> $T^{\cdot}$ δ-fun s.t. T^i effaceable, $\forall i > 0$. ($\Rightarrow$ univ).

$\Rightarrow \forall I$ inj obj. $T^i(I) = 0$, $\forall i > 0$.

Proof. $\exists\, I \xrightarrow{u} B$ mono s.t. $T^i(u) = 0,\ \forall i > 0$.

& $0 \to I \xrightarrow{u} B \to C \to 0$ splits

$C \cong \operatorname{coker}(u)$

$\Rightarrow \forall j \geq 0,\ 0 \to T^j(I) \xrightarrow{T^j(u) = 0} T^j(B) \to T^j(C) \to 0$ exact.

$\Rightarrow \delta^i : T^i(C) \to T^{i+1}(I),\ \delta^i = 0,\ \forall i \geq 0$.

$\rightsquigarrow T^{i-1}(C) \xrightarrow[0]{\delta^i} T^i(I) \xrightarrow[0]{T^i(u)} T^i(B)$. $\square$

$T^i(I) = 0$

$\mathcal{C}$ has enough injs $\Rightarrow \forall X \in \mathcal{C},\ \exists X \to I$ mono & I inj.

$\rightsquigarrow$ inj resolution to compute univ-δ.

Better yet: $\forall X \to Y$ & $X \to I^\cdot$ inj resolution,

$\exists\, Y \to I'^{\cdot}$ (another) inj resolution.

Define right derived functors of F (left-exact):

$\forall X \in \mathcal{C}$, $I^\cdot$ inj resolution, $0 \to X \to I^\cdot$

put $\boxed{R^i F(X) = h^i(F(I^\cdot))}$.

Thm $\mathcal{C}$ has enough injs. well def'd effaceable δ-functor

($\Rightarrow$ universal).

§6 Examples

$X \in \mathrm{Mod}_R$, $X \otimes (\cdot)$ right-exact $\mathrm{Mod}_R \to \mathrm{Mod}_R$

($\Leftrightarrow$ left-exact $\mathrm{Mod}_R^{op} \to \mathrm{Mod}_R^{op}$).

$R^i = \mathrm{Tor}^i(X, -)$.

Prop $\forall X \in \mathrm{Mod}_R$, TFAE:

(a) X flat

(b) $\mathrm{Tor}^i(X,Y) = 0$, $\forall i > 0$ & $Y \in \mathrm{Mod}_R$

(c) $\mathrm{Tor}^1(X,Y) = 0$, $\forall Y \in \mathrm{Mod}_R$.

(d) $X \otimes (-)$ right exact.

Note $\otimes$ is symmetry $\Rightarrow \mathrm{Tor}^i(X,Y) = \mathrm{Tor}^i(Y,X)$.

But def'n of Tor is asymmetric

(need proj. resolution & homology to compute $\mathrm{Tor}^i(Y,X)$).

$P^\cdot, Q^\cdot$ proj resolutions for X, Y

i.e. $P^\cdot \to X \to 0$, $Q^\cdot \to Y \to 0$

$\rightsquigarrow$ double complex

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d, "y \text{ s.t. } \tilde{y} \in \mathrm{Tor}^1(Y,X)"] & & \\
\cdots \arrow[r] & P_1 \otimes Q_1 \arrow[r] \arrow[d] & P_1 \otimes Q_0 \arrow[r] \arrow[d] & P_1 \otimes Y \arrow[r] \arrow[d] & 0 & \\
\cdots \arrow[r] & P_0 \otimes Q_1 \arrow[r] \arrow[d] & P_0 \otimes Q_0 \arrow[r] \arrow[d] & P_0 \otimes Y \arrow[r] \arrow[d] & 0 & \text{exact} \\
\cdots \arrow[r] & X \otimes Q_1 \ni x \arrow[r] \arrow[d] \arrow[uurr, bend left] & X \otimes Q_0 \arrow[r] \arrow[d] & X \otimes Y \arrow[r] \arrow[d] & 0 \arrow[r, squiggly, "H_i"] & \mathrm{Tor}^i(X,Y) \\
\tilde{x} \in \mathrm{Tor}^1(X,Y) & 0 & 0 & 0 & & \\
 & \underbrace{\qquad\qquad}_{\text{exact}} & & \mathrm{Tor}^j(Y,X) \arrow[rruu, bend right, leftrightarrow, "\cong \text{ canonical}"'] & &
\end{tikzcd}$$

$$\text{columns: } 0 \ \Big\}\, H^j \ \mathrm{Tor}^j(Y,X)$$

Construction X regular excellent sch, Y, Z sub of $\underbrace{\mathcal{I}, \mathcal{J}}_{\text{sheaves of ideals}}$

x gen pt of a component in $Y \cap Z$.

naive intersection multiplicity of Y, Z:

$$\mathcal{O}_{Y\cap Z, x} = \mathcal{O}_{X,x}/(\mathcal{I}\mathcal{J})_x \cdot \Big\} \rightsquigarrow \text{correct answer}$$

e.g. $\dim X = 2$, $\dim Y = \dim Z = 1$

but incorrect in general!

Serre: the right version

$$\sum_i (-1)^i \operatorname{length}_{\mathcal{O}_{X,x}} \underbrace{\operatorname{Tor}^i_{\mathcal{O}_{Z,x}}}(\mathcal{O}_{X,x}/\mathcal{I}_x, \mathcal{O}_{X,x}/\mathcal{J}_x).$$

geom. interpretation?

Given by Jacob Lurie et al. by derived geometry.

Roughly: Replace R by $R^{\otimes\infty}$ (some top ring)

$\rightsquigarrow \operatorname{Spec} R^{\otimes\infty}$ instead of $\operatorname{Spec} R$.

Right derived of $\operatorname{Hom}(-, Y)$: $\operatorname{Ext}^i(-, Y)$.

$\operatorname{Hom}(X, -)$: $\operatorname{Ext}^i(X, -)$.

<u>Important Example</u> (one more)

G grp w/ discrete top. $\mathbb{Z}[G] = \bigoplus_{g\in G} \mathbb{Z}[g]$, $[g]\cdot[h] = [gh]$

① $(\cdot)^G : \operatorname{Mod}_{\mathbb{Z}[G]} \longrightarrow \operatorname{Mod}_{\mathbb{Z}}$ covariant G-inv.

& left-exact.

$\rightsquigarrow R^i(\cdot)^G = H^i(G, \cdot)$ grp cohom.

② $(\cdot)_G : \operatorname{Mod}_{\mathbb{Z}[G]} \longrightarrow \operatorname{Mod}_{\mathbb{Z}}$ contravariant G-coinv.

$M \longmapsto M/(g(m)-m)$, $\forall g\in G$, $m \in M$.

& right-exact.

$\rightsquigarrow L^i(\cdot)_G = H_i(G, -)$ grp homo.

Namely, $H^i(G, M) = \operatorname{Ext}^i_{\operatorname{Mod}_{R[G]}}(R, M)$, $H_i(G, M) = \operatorname{Tor}_i^{\operatorname{Mod}_{R[G]}}(R, M)$.