The category of representations of p-adic groups
Jessica Fintzen

p prime, $F/\mathbb{Q}_p$, or $F = \mathbb{F}_q((t))$, $F \supset \mathcal{O} \ni \varpi$, $\mathcal{O} \twoheadrightarrow \mathbb{F}_q$.
G conn red grp $/F$, splits over a tamely ramified ext of F.

<u>Goal</u> Want to explicitly describe the cat $R(G)$ of smooth complex rep'ns of $G(F)$.

- building blocks = supercuspidal rep'ns

<u>Fact</u> π irred rep of $G(F)$,
then $\pi \hookrightarrow \mathrm{Ind}_{M(F)N(F)}^{G(F)} \sigma \otimes \mathbb{1}$ for some σ,
where σ = sc rep of $M(F)$.

<u>Bernstein decomp</u> $\quad R(G) = \prod_{(M,\sigma)/\sim} R(G)_{[M,\sigma]}$

$R(G)_{[M,\sigma]}$: Bernstein block,
contains all irred subquots of $\mathrm{Ind}_{M(F)N(F)}^{G(F)} \sigma$.

<u>Vague theorem</u> (in progress, Adler-Fintzen-Mishra-Ohara,
Fintzen-Schwein (input for small p)).

$$R(G)_{[M,\sigma]} \simeq \underbrace{R(G^0)_{[M^0,\sigma_0]}}_{\text{rather well-understood.}} \qquad \sigma_0 = \text{depth-zero sc rep}$$

E.g. $G = SL_2$:

(a) $M = G$, σ sc rep of $M(F) = G(F)$

$\rightsquigarrow$ $R(G)_{[G,\sigma]} = \{\sigma, \sigma\oplus\sigma, \sigma\oplus\sigma\oplus\sigma, \dots\}$

exactly arbitrary direct sums.

(b) $M = T = \begin{pmatrix} * & \\ & * \end{pmatrix} \subseteq SL_2 = G$. $\sigma = \mathrm{triv}$.

$\rightsquigarrow$ $R(G)_{[T, \underset{\mathrm{triv}}{\overset{\shortparallel}{\sigma}}]}$ = principal block (complicated).

$\ni$ triv, $\mathrm{Ind}_{B(F)}^{G(F)}$ triv, St.

$$(1 \to \mathrm{triv} \longrightarrow \mathrm{Ind}_{\underset{\begin{pmatrix} * & * \\ & * \end{pmatrix}}{\overset{\shortparallel}{B(F)}}}^{G(F)} \mathrm{triv} \to \mathrm{St} \to 1)$$

Def A pair (K, ρ), with K compact open subgrp $\subset G(F)$,

ρ irrep of K

is an $[M,\sigma]$-type if for all irreps π of $G(F)$, TFAE:

(i) $\pi \in R(G)_{[M,\sigma]}$

(ii) $\rho \hookrightarrow \pi|_K$, i.e. $\mathrm{Hom}_K(\rho, \pi) \neq \{0\}$.

E.g. <u>Depth-zero types</u>

(1) $G = SL_2$,

(a) $M = G$, $K = SL_2(\mathcal{O})$,

$$\rho : K \to SL_2(\mathcal{O}) / \begin{pmatrix} 1+\varpi\mathcal{O} & \varpi\mathcal{O} \\ \varpi\mathcal{O} & 1+\varpi\mathcal{O} \end{pmatrix}_{\det = 1} \simeq SL_2(\mathbb{F}_q)$$

$$\downarrow \rho \ \text{csp rep}$$

$$\mathrm{End}(V_\rho).$$

$\Rightarrow (K, \rho)$ is a $[G, \underbrace{\text{c-ind}_K^{G(F)} \rho}_{\text{irred sc}}]$-type

(b) $M = T$, $\left(K = I_W = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O} \end{pmatrix}, \mathrm{triv}\right)$ is a $[T, \mathrm{triv}]$-type.

(2) G ss simply connected.

$x \in$ Bruhat-Tits building. $K = G_x = \mathrm{Stab}_{G(F)}(x)$.

$$\begin{array}{c} K = G_x \\ \downarrow \\ \mathbb{F}_q\text{-pts of some reductive} = G_x / \underbrace{G_{x,0,+}} \leftarrow \text{pro-}p \text{ unip radical.} \\ \downarrow \text{ cusp rep} \\ \mathrm{End}(V_\rho) \end{array}$$

$\rightsquigarrow \rho : K \to \mathrm{End}(V_\rho)$

Then (K, ρ) is a depth-zero type.

Thm (Bushnell-Kutzko, 1998)

Suppose (K, ρ) is an (M, σ)-type, then

$$R(G)_{[M,\sigma]} \xrightarrow{\sim} \mathcal{H}(K,\rho)\text{-mod}$$

where $\mathcal{H}(K,\rho) := \left\{ f : G(F) \to \mathrm{End}(V_\rho), \text{ Compactly supp'd s.t. } f(kgk') = \rho(k) f(g) \rho(k') \right\}$

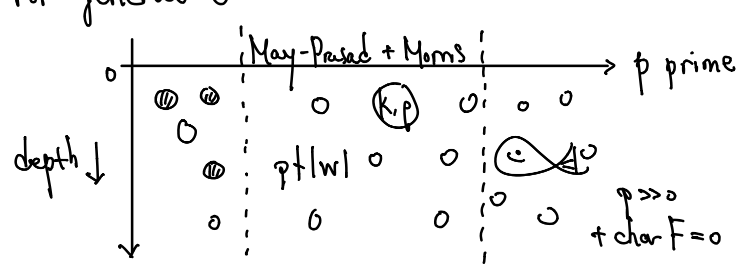
$\mathcal{H}(K,\rho) = \mathcal{H}(G,K,\rho)$

<u>Question</u> (1) When do type exist?

(2) What is $\mathcal{H}(G, K, \rho)$?

<u>Answers</u> (1) Types exist for GL_n, classical grps ($p \neq 2$), inner forms of GL_n.

For general G:



<u>Expected result</u> (in progress, Fintzen-Schwein)

Provide a constr'n of new types for small p.

(2) <u>Thm</u> (in progress, Adler-Fintzen-Ohara)

Let (K, ρ) be a Kim-Yu type for $[M, \sigma]$.

Then $\exists\, G^\circ \subset G$, $M^\circ \subset G^\circ$, σ_0 a depth-zero sc rep of $M^\circ(F)$,

and (K°, ρ_0) an $[M^\circ, \sigma_0]$-type,

s.t. $\mathcal{H}(G, K, \rho) \simeq \mathcal{H}(G^\circ, K^\circ, \rho_0)$.

<u>Thm</u> (Morris (+ AFMO))

$$\mathcal{H}(G^\circ, K^\circ, \rho_0) \simeq \mathbb{C}[\Omega(\sigma_0), \mu] \ltimes \mathcal{H}_{aff}(W_{aff}(\sigma_0), q_S)$$

- Roche 1998: $M = T$ max split torus
- Ohara 2021: $M = G$.