Gromov-Hausdorff convergence of Kähler manifolds and applications

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Real

Theorem (Cheeger, 70')

Given $K, v, d, n > 0$, consider a class of compact Riemannian n-manifolds with $|sec| \leq K$, $diam < d$, $vol > v$. Then such class is precompact in $C^{1,\alpha}$-topology. In other words, given a suquence of manifolds in this class, there exists a subsequence convergent in Cheeger-Gromov sense to a smooth manifold M (metric is $C^{1,\alpha}$). As a corollary, this class contains only finite diffeomorphism types.

There are many subsequent works. 1. Consider only sectional curvature lower bound, this is related to Alexandrov geometry; 2. Remove the volume lower bound condition. This is the collapsing theory. 3. Replace sectional curvature by Ricci curvature.

Real

Usually, when the volume of a unit ball has a lower bound (uniform), we call it noncollapsing. We shall only consider the noncollapsing case in this talk.

Theorem (Anderson, 90')

Given $C, i, d, n > 0$, consider a class of compact Riemannian n-manifolds with $|Ric| \leq C$, $inj > i$, $diam < d$. Then the previous theorem holds for such class.

Remark: 1. Anderson's theorem satisfies the noncollapsing condition. 2. one cannot replace the injectivity radius bound by noncollapsing condition as in Cheeger's theorem. Otherwise, the limit may not be smooth.

Real

In order to obtain to get precompactness when the limit is not smooth, one has to consider weaker convergence. An important notion is Gromov-Hausdorff distance. This defines a distance between two compact metric spaces. We say a sequence of metric spaces converge in the Gromov-Hausdorff sense, if the Gromov-Hausdorff distance is approaching zero.

Theorem (Gromov)

Given $C, d, n > 0$, consider a class of compact Riemannian n-manifolds with $Ric \geq -C$, $diam < d$. Then this class is precompact in the Gromov-Hausdorff sense (note the Gromov-Hausdorff limit may be far from smooth).

Real

For noncompact manifolds, one can consider manifold with a base pointed. Then the notion of pointed-Gromov-Hausdorff convergence makes sense, i.e., first consider the Gromov-Hausdorff convergence in a geodesic ball of fixed radius, then let the radius go to infinity (diagonal sequence).

A basic problem in metric differential geometry is to study the regularity of the Gromov-Hausdorff limit of manifolds with Ricci curvature lower bound (noncollapsed). There are fundamental contributions by Cheeger, Colding, Tian, Naber, etc.

Given a limit space X and a point $p \in X$, we can consider a blow up of X at p. A blow up limit is called a tangent cone at p (note the tangent cone at p need not be unique).

Real

Definition

A point $p \in X$ is called regular, if a tangent cone is isometric to a Euclidean space $\mathbb{R}^m$. A point is singular, it is not regular.

Theorem (Cheeger-Colding)

Given $n, v > 0$, let (X, o) be the pointed-Gromov-Hausdorff limit of a sequence of n-manifolds (M_i, p_i) with $Ric \geq -(n-1)$ and $vol(B(p_i, 1)) > v$. Then

- *X is metric length space of Hausdorff dimension n. The Hausdorff measure is equal to the limit of volume element on M_i.*
- *The singular set has Hausdorff codimension at least 2 (sharp).*
- *Any tangent cone is a metric cone.*

Real

The theorem above tells us that most points on X are regular. However, note the following:

1. Regular points are not so "regular". For example, consider a doubled disk. Then all points are regular. Near the boundary of the disk, the metric is only Lipschitz. It is a conjecture that near a regular point on X, the metric is bilipschitz to a Euclidean ball. Currently the best known regularity is biholder.

2. Regular set is not necessarily open. In other words, the singular set need not be closed. In fact, it could be dense. This already appears in the real two dimensional case (the singular set in this case is countable). In higher dimensions, Li-Naber constructed a limit space so that the singular set is given by a fat Cantor set. In other words, the topology of singular set could be very complicated.

Kähler

In the above, we considered the Gromov-Hausdorff limit of Riemannian manifolds with Ricci curvature lower bound and noncollapsed volume. What if these manifolds are all Kähler? Can we get extra results? Observe all two dimensional Riemannian manifolds (oriented) are Kähler. So we cannot expect too much from the extra Kähler assumption. For simplicity, let us call the Gromov-Hausdorff limit of Kähler manifolds with Ricci curvature lower bound Kähler Ricci limit space.

Theorem (Cheeger-Colding-Tian)

Let X be a noncollapsed Kähler Ricci limit space. Then any tangent cone splits even dimensional Euclidean factor. In other words, the splitting lines must come in pairs.

Kähler

Theorem (Tian/Donaldson-Sun)

Let X be the Gromov-Hausdorff limit of a sequence of polarized Kähler manifolds M_i with $|Ric| \leq C$ and $diam < d$, $vol > v$. Then X is homeomorphic to a normal projective variety.

Such result is a key to the existence of Kähler-Einstein metrics on Fano manifolds.

Kähler

Theorem (Donaldson-Sun)

Let X be as above, and also assume the sequence is Kähler-Einstein. Then

- *the metric singularity of X is the same as the complex analytic singularity of X.*
- *for any $p \in X$, the tangent cone is homeomorphic to a normal affine algebraic variety.*
- *the tangent cone is unique.*

Now we consider a different setting in the Kähler case. Let us assume that the bisectional curvature has a lower bound. Note
$sec \geq 0 => BK \geq 0 => Ric \geq 0$.

let $\mathcal{O}_d(M)$ be the space of holomorphic functions of polynomial growth with order at most d. Below is a very remarkable result:

Theorem (Ni, Chen-Fu-Le-Zhu)

Let M^n be a complete noncompact Kähler manifold of nonnegative bisectional curvature ($BK \geq 0$). Then $dim(\mathcal{O}_d(M)) \leq dim(\mathcal{O}_d(\mathbb{C}^n))$ for all $d > 0$. If the equality holds for some integer $d > 0$, then M is biholomorphic and isometric to $\mathbb{C}^n$.

Remark

Ni first proved the theorem under the assumption that the manifold has maximal volume growth. Later, by Ni's method, Chen-Fu-Le-Zhu removed the volume condition. The key in Ni's method is a monotonicity formula for heat flow on Kähler manifold with nonnegative bisectional curvature.

Later on, we discovered a three circle theorem, which gave an alternative proof to the sharp dimension estimate. Let us state the three circle theorem, as well as two corollaries:

Theorem

Let M be a complete noncompact Kähler manifold with nonnegative holomorphic sectional curvature, i.e., $H \geq 0$. Let f be a holomorphic function on M. Then $\log M(r)$ is a convex function of $\log r$, where $M(r) = \sup\limits_{B(p,r)} |f(x)|$.

Remark

Note $BK \geq 0$ implies $H \geq 0$. However, in general, there is no relation between $H \geq 0$ and $Ric \geq 0$.

Corollary

Let M be a complete noncompact Kähler manifold with nonnegative holomorphic sectional curvature. Let $f \in \mathcal{O}_d(M)$. Then $\frac{M(r)}{r^d}$ is monotonic nonincreasing.

Corollary

Let M be a complete noncompact Kähler manifold with nonnegative holomorphic sectional curvature. Then $dim(\mathcal{O}_d(M)) \leq dim(\mathcal{O}_d(\mathbb{C}^n))$ for all $d > 0$.

Note in the estimates above, the existence of polynomial growth holomorphic function is not known. Ni proposed the following conjecture:

Conjecture (Ni)

Let M^n be a complete noncompact Kähler manifold of positive bisectional curvature. Then the following three conditions are equivalent:
(a) M has maximal volume growth;
(b) There exists a nonconstant polynomial growth holomorphic function on M;
(c) There exists $C > 0$ such that $r^{-2n+2} \int_{B(p,r)} S < \infty$ for all $r > 0$.

Remark

This is a qualitative conjecture. Essentially it says when the volume gets smaller, the orders of polynomial growth holomorphic functions get higher, and the average of scalar curvature gets larger.

Earlier, Ni and Ni-Tam solved the equivalence of (b) and (c). Their approach uses Poincare-Lelong equation, heat flow, as well as solving $dd^c u = Ric$. Later, by using the Gromov-Hausdorff convergence theory developed by Cheeger-Colding-Tian and the three circle theorem, we managed to prove

Theorem

(a) is equivalent to (b).

Remark

Since our proof is by contradiction, it is not straightforward to extract a quantitative version of the estimate.

Theorem

Let M be a complete noncompact Kähler manifold with nonnegative bisectional curvature and maximal volume growth. Then $\lim_{r\to\infty} r^2 \fint_{B(p,r)} S$ exists (independent of p).

Remark

This answers a question of Ni. In fact, by combining Ni's earlier works, we can remove the volume condition.

Now we come to the quantitative version. Recall in complex dimension one, we have

Proposition

Let Σ be a noncompact oriented Riemann surface with a conformal metric of nonnegative curvature, assume the asympototic volume ratio is $\frac{\alpha}{2\pi}$, then $\int_\Sigma K = 2\pi - \alpha$. On Σ, the least order of polynomial growth holomorphic functions (nonconstant) is equal to $\frac{2\pi}{\alpha}$.

We managed to generalize the (sharp) estimates to higher dimensions: Let d be the smallest degree of polynomial growth holomorphic functions on M. Also let

$$v = \lim_{r\to\infty} \frac{Vol(B(p,r))}{Vol(B_{\mathbb{C}^n}(o,r))},$$

$$\alpha = \lim_{r\to\infty} r^2 \fint_{B(p,r)} S.$$

Theorem

(1)

$$4n^2(v^{-\frac{1}{n}} - 1) \le \alpha \le 4n(v^{-1} - 1),$$

(2)

$$1 \le d \le v^{-\frac{1}{n}},$$

(3)

$$\lim_{k\to\infty} \frac{dim(\mathcal{O}_k(M))}{k^n/n!} = v \le 1 = \lim_{k\to\infty} \frac{dim(\mathcal{O}_k(\mathbb{C}^n))}{k^n/n!}.$$

Remark

- *When $n = 1$, we recovered the proposition.*
- *When the metric is unitary symmetric, the left side of (1) and rightside of (2) become equality. However, the metric is not necessarily unique in this case.*
- *The right inequality of (1) becomes an equality if and only if M splits off $\mathbb{C}^{n-1}$.*
- (3) *essentially says that dimension estimate is just volume comparison.*

Recall Yau's uniformization conjecture:

Conjecture

Let M be a complete noncompact Kähler manifold with positive bisectional curvature. Then M is biholomorphic to a complex Euclidean space.

Theorem

Let M be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Then the ring of polynomial growth holomorphic functions is finitely generated. If in addition, M has maximal volume growth, i.e., $vol(B(p,r)) \geq cr^{2n}$ for all $r > 0$, then M is biholomorphic to an affine algebraic variety.

Theorem

Let X be the Gromov-Hausdorff limit of a sequence of complete Kähler manifolds with bisectional curvature lower bound and local noncollapsed volume. Then X is homeomorphic to a normal complex analytic space.

Remark: Later, by using flow techniques, Lee-Tam/Bamler-Wilking proved that X is in fact smooth in the complex analytic sense, i.e., X is a complex manifold. Note we do not assume polarization.

We also obtained

Theorem

Let M be a complete noncompact Kähler manifold with nonnegative Ricci curvature. Assume M has maximal volume growth and that $BK \geq -C/r^{2+\epsilon}$, then M is biholomorphic to an resolution of an affine algebraic variety.

We obtained more results in this direction:

Theorem

Let M be a complete noncompact Kähler manifold of nonnegative Ricci curvature and maximal volume growth. Assume $|Rm| \leq C/r^2$. Then

- *If M is Ricci flat, then M is biholomorphic to a crepant resolution of an affine algebraic variety.*
- *If M has positive Ricci curvature, then M is quasi-projective.*

Remark: This theorem is related with many previous results, such as The first part generalized a result of Tian, where he considered the case when $|Rm|$ has faster than quadratic decay. The second part generalized a result of Mok, where he assumed an additional condition: $\int Ric^n < +\infty$.

Remark: Very recently, we were able to show that under the condition of the theorem, $\int Ric^n < +\infty$ is valid, i.e., Mok's assumption $\int Ric^n < +\infty$ always holds.

With more efforts, we were able to prove

Theorem (Lee-Tam/L)

Let M be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Assume M has maximal volume growth, then M is biholomorphic to $\mathbb{C}^n$.

Remark: Lee-Tam's proof used Chern-Ricci flow, while ours used Gromov-Hausdorff convergence theory.

Theorem (Szekelyhidi-L)

Let X be the Gromov-Hausdorff limit of a sequence of polarized Kähler manifolds M_i with $Ric \geq -C$ and $diam < d$, $vol > v$. Then X is homeomorphic to a normal projective variety. Moreover, the metric singularity is equal to a countable union of complex analytic subvarieties.

Remark: The first part is a straightforward generalization of Tian/Donaldson-Sun. The second part is different: In Donaldson-Sun case, the metric singularity is equal to the complex singularity (metric singular set is always closed when $|Ric|$ is bounded), while in our case, the complex singularity is only a subset of metric singularity (recall in Ric lower bound case, the metric singular set need not be closed. So countable union is optimal). Li-Naber's example tells us that Riemannian case is very different from Kähler case.

The theorem above tells us that both the limit space and metric singularity has strong rigidity, comparing with the Riemannian case.

Theorem (Szekelyhidi-L)

Let X be a noncollapsed Kähler Ricci limit space. Then any tangent cone is homeomorphic to an affine algebraic variety. The affine coordinate ring is given by polynomial growth holomorphic functions. We can also solve $\overline{\partial}$ on the tangent cone.

Remark: This is a generalization of Donaldson-Sun. Note we do not assume polarization, and such result is new even for Ricci flat case.

Corollary

Let M be a Ricci flat Kähler manifold of maximal volume. Then the asymptotic volume ratio is an algebraic number.

Theorem (Szekelyhidi-L)

There exists $\epsilon > 0$, depending on the dimension n with the following property. Suppose that $B(p, \epsilon^{-1})$ is a relatively compact ball in a (not necessarily complete) Kähler manifold (M^n, p, ω), satisfying $\mathrm{Ric}(\omega) > -\epsilon\omega$, and

$$d_{GH}\Big(B(p, \epsilon^{-1}), B_{\mathbb{C}^n}(0, \epsilon^{-1})\Big) < \epsilon.$$

Then there is a holomorphic chart $F : B(p, 1) \to \mathbb{C}^n$ which is a $\Psi(\epsilon|n)$-Gromov-Hausdorff approximation to its image. In addition on $B(p, 1)$ we can write $\omega = i\partial\bar{\partial}\phi$ with $|\phi - r^2| < \Psi(\epsilon|n)$, where r is the distance from p.

Corollary

There exists $\epsilon > 0$ depending on n, so that if M^n is a complete noncompact Kähler manifold with $\mathrm{Ric} \geq 0$ and $\lim_{r\to\infty} r^{-2n}\mathrm{vol}(B(p,r)) \geq \omega_{2n} - \epsilon$, then M is biholomophic to $\mathbb{C}^n$. Here ω_{2n} is the volume of the Euclidean unit ball.

Corollary

Let (M_i^n, ω_i, p_i) be a sequence of complete Kähler manifolds with $\mathrm{Ric} > -1$. Assume $(M_i^n, p_i) \to (M^{2n}, p)$ in the pointed Gromov-Hausdorff sense, where M^{2n} is a smooth Riemannian manifold. Then the scalar curvature S_i of M_i converges to the scalar curvature S of M in the measure sense. That is to say, for any points $M_i \ni q_i \to q \in M$, and any $r > 0$, we have $\int_{B(q_i,r)} S_i\omega_i^n \to \int_{B(q,r)} S\omega^n$ as $i \to \infty$.

Corollary

Given any $\epsilon > 0$, there is a $\delta > 0$ depending on ϵ, n satisfying the following. Let $B(p, 1)$ be a relatively compact unit ball in a Kähler manifold (M, ω) satisfying $\mathrm{Ric} > -1$, and $d_{GH}(B(p, 1), B_{\mathbb{C}^n}(0, 1)) < \delta$. Then $|\int_{B(p,\frac{1}{2})} S| < \epsilon$, where S is the scalar curvature of ω.

Theorem

Let M be a complete noncompact Kähler manifold of nonnegative Ricci curvature and maximal volume growth. Then $dim(\mathcal{O}_d(M)) \leq cd^n$, where c depends only on n, α (α is the asymptotic volume ratio). Note the power is sharp.

Thank you!