Igusa stacks and local-global compatibility

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(Joint with Daniels – van Hoften – Kim & with Caraiani)

G conn red / F number field, ($F=\mathbb{Q}$ this talk)

Fix $\iota: \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$

Conj (Buzzard–Gee)

$$\left\{ \begin{array}{c} \text{autom rep of } G(\mathbb{A}) \\ + \ L\text{- or } C\text{-algebraic} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{cont Galois rep of} \\ \mathrm{Gal}_F \to {}^{L/C}G(\overline{\mathbb{Q}}_\ell) \end{array} \right\}$$

Compatible with LLC at all places of F.

e.g. $G = GL_2 / \mathbb{Q}$, $K \subset G(\mathbb{A}_f)$ compact subgrp

$$\mathcal{E} \xrightarrow{\pi} S_K$$

$k \geq 2$, $\mathcal{V}_k := \mathrm{Sym}^{k-2} R^1 \pi_* \overline{\mathbb{Q}}_\ell$

$$H^1(\mathcal{V}_k) := \varinjlim_K H^1_{par}(S_{K,\overline{\mathbb{Q}}}, \mathcal{V}_k) \circlearrowleft G(\mathbb{A}_f) \times \mathrm{Gal}_{\mathbb{Q}}.$$

Thm (Eichler–Shimura, Deligne, Igusa, Carayol, Saito, ...)

$$H^1(\mathcal{V}_k) \cong \bigoplus_{\substack{f \text{ cusp new} \\ \text{form of wt } k}} \underbrace{\pi^\infty(f)}_{\text{autom}} \boxtimes_{\overline{\mathbb{Q}}_\ell} \underbrace{\rho_f}_{\text{Gal}}$$

- $\pi^\infty(f) = \bigotimes_p{}' \, LLC(\rho_f|_{D_p})$
  - explicitly constructed
- ρ_f a.e. unr, char by Eichler–Shimura relations at unr places.

For LLC — Fargues-Scholze's construction
GLC — cohomology of Shimura varieties.

<u>Goal</u> (1) link between them (Igusa stacks)
$\rightsquigarrow$ applications to cohomologies.
(2) geometric form of local-global compatibility (LGC).

<u>§1 Local Langlands of Fargues-Scholze</u>

p, $G/\mathbb{Q}_p$ split, $\Lambda = \overline{\mathbb{F}}_\ell$ ($\ell \neq p$)
$\hat{G}/\Lambda$, $W_{\mathbb{Q}_p}$.

LLC:
$$\left\{\begin{matrix}\text{irred sm reps of}\\ G(\mathbb{Q}_p) \text{ as } \Lambda\text{-mod}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{continuous 1-cocycles}\\ W_{\mathbb{Q}_p} \to \hat{G}(\Lambda)\end{matrix}\right\}$$
$$\downarrow$$

$D(\mathrm{Bun}_G) := D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)$ $\qquad\qquad$ $X_{\hat{G}} := [Z^1(W_{\mathbb{Q}_p}, \hat{G})/\hat{G}]$

moduli stack of G-torsors on the Fargues-Fontaine curve $\qquad$ moduli stack of L-parameters.

<u>Thm</u> (Fargues-Scholze) $\exists$ linear action
$$\mathrm{Perf}(X_{\hat{G}}) \circlearrowright D(\mathrm{Bun}_G).$$

<u>Example 1</u> $\mathcal{O}_{X_{\hat{G}}} * (-) = \mathrm{id}_{D(\mathrm{Bun}_G)}$
$$\mathcal{O}(X_{\hat{G}}) = \mathrm{End}_{X_{\hat{G}}}(\mathcal{O}) \longrightarrow \mathrm{End}(\mathrm{id}_{D(\mathrm{Bun}_G)}) \xrightarrow{ev_A} \mathrm{End}(A) \simeq \Lambda.$$
$$\underset{A}{\rotatebox{90}{$\in$}}$$

$\rightsquigarrow$ $\varphi_A \in X^{\text{coarse}}_{\hat{G}}(\Lambda)$ semi-simple L-param
given by a max ideal of $\mathcal{O}(X_{\hat{G}})$.

<u>Example 2</u> $V \in \mathrm{Rep}^{\mathrm{alg}}_{\Lambda}(\hat{G})$.
Consider $\mathcal{O}_{Z(W_{\mathbb{Q}_p}, \hat{G})} \otimes V$ + diagonal $\hat{G}$-action.
$\rightsquigarrow$ $\mathcal{V}$ on $X_{\hat{G}}$ $\rightsquigarrow$ $\mathcal{V} * (-)$

$$\mathrm{Hck}_G = \{(\overbrace{D}^{\text{deg 1 divisor}}, \mathcal{E}_1 \dashrightarrow \mathcal{E}_2)\}$$

$$\begin{tikzcd} & \mathrm{Hck}_G \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \end{tikzcd}$$

$$V \xrightarrow[\text{Satake}]{\text{geom}} \underbrace{\mathcal{S}_V}_{\text{kernel of Fourier integral trans.}}, \quad \mathcal{V} * (-) = T_V = Rp_{2!}(p_1^*(-) \otimes^{\mathbb{L}}_{\Lambda} \mathcal{S}_V)$$

<u>§2 Relation to Shimura varieties</u>

(G, X), μ: $K^p \subset G(\mathbb{A}_f^p)$, $E / \mathbb{Q}_p$, $\mu_{\bar{\mathbb{Q}}_p}$, $G = G_{\mathbb{Q}_p}$.

$$\mathcal{S}_{K^p} := \varprojlim_{K_p} \mathcal{S}^{\diamond}_{K^p K_p, E} \circlearrowleft G(\mathbb{Q}_p)$$

<u>Conj</u> (Scholze)

$$\exists \quad \begin{tikzcd} {[\mathcal{S}_{K^p}/G(\mathbb{Q}_p)]} \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\ulcorner"] & & B\overbrace{G(\mathbb{Q}_p)}^{\mathrm{Aut(triv}\ G\text{-bdl)}} \times \mathrm{Div}^1 \arrow[d] \\ & \mathrm{Hck}_{G, \leq \mu} \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathrm{Bun}_G \times \mathrm{Div}^1 \approx [*/W_E]. \\ \text{"}\mathrm{Igs}_{K^p}\text{"} \arrow[r, "\bar{\pi}"] & \mathrm{Bun}_G & \end{tikzcd}$$

Thm (D-vH-K-Z, 2024)

Conj holds for (good reduction locus) of Hodge-type Shimura varieties.

& $\ell \neq p$, Λ ℓ-torsion,

$\rightsquigarrow$ Igs_{K^p} ℓ-cohom smooth of ℓ-dim 0, whose dualizing complex $\cong \Lambda[0]$.

§3 Application to cohomologies

$\Lambda = \overline{\mathbb{F}}_\ell$, $\ell \neq p$. $d = \dim S_K$, μ, E ($E = \mathbb{Q}_p$, G split)

$\mathcal{F}_! := R\overline{\pi}_! \Lambda$, $\mathcal{F}_* := R\overline{\pi}_* \Lambda$.

$$\begin{tikzcd} {[S_{K^p}/\underline{G(\mathbb{Q}_p)}]} \arrow[r, "q"] & B\underline{G(\mathbb{Q}_p)} \arrow[d, "i_1"] \\ & \mathrm{Bun}_G \end{tikzcd}$$

$r_\mu : \hat{G} \to GL(V_\mu) \rightsquigarrow \mathcal{V}_\mu$ vec bdl on $X_{\hat{G}}$,

T_μ corresponding Hecke operator.

Obs $R\Gamma_c(S_{K^p}, \Lambda) = Rq_!\Lambda \cong i_1^* T_\mu(\underbrace{\mathcal{F}_![-d](-\tfrac{d}{2})}_{=:\tilde{\mathcal{F}}_!}) \cong i_1^* \mathcal{V}_\mu * (\tilde{\mathcal{F}}_!)$.

b/c $S_{V_\mu} \cong \Lambda[d](\tfrac{d}{2})$

(o) Rmk $|\mathrm{Bun}_G| \cong B(G)$

supp filtration $\Rightarrow$ Spectral sequence computing $R\Gamma_c(S_{K^p}, \Lambda)$.

$\rightsquigarrow$ Mantovan's product formula.

(1) Eichler-Shimura relations

(d'après Xiao-Zhu, Wu, Koshikawa).

$$W_E \dashrightarrow \mathrm{End}_{X_{\hat{G}}}(\mathcal{V}_\mu) \longrightarrow \mathrm{End}_{D(G(\mathbb{Q}_p),\Lambda)}(i_1^* \mathcal{V}_\mu * \tilde{\mathcal{F}}_!)$$

$$W_E \xrightarrow{\varphi^{univ}} \hat{G} \xrightarrow{r_\mu} GL(\mathcal{V}_\mu)$$

$$\gamma \longmapsto r_0 \circ \varphi^{univ}(\gamma).$$

$$H_{G,\mu,\gamma}(X) := \sum_i (-1)^i \operatorname{tr}\left(r_\mu \circ \varphi^{univ}(\gamma) \,\middle|\, \Lambda^{\dim r_\mu - i} \mathcal{V}_\mu\right) X^i \in \mathcal{O}(X_{\hat{G}})[X].$$

Hamilton-Cayley thm $\Rightarrow$ it annihilates γ.

Choosing γ to be a lift of Frob_q, $\mathbb{F}_q$ = res field of E.

& passing to hyperspecial level.

$$\rightsquigarrow \quad \underbrace{\mathcal{O}(X_{\hat{G}}^{un})}_{\cong \mathcal{H}_p^{sph}}[X] / H_{G,\mu,\gamma}(X) \longrightarrow \mathrm{End}_{\mathcal{H}_p^{sph}}(R\Gamma_c(S_K, \Lambda))$$

spherical Hecke algebra

(2) Torsion vanishing

(d'après Caraiani-Scholze, Koshikawa, Santos, Hamann-Lee)

For generic part of cohom of compact Hodge-type Shimura varieties concentrated in middle degree,

(3) LGC (jt w/ Caraiani)

$$\mathrm{Igs}_{K^p} \xrightarrow{\bar{\pi}} \mathrm{Bun}_G$$

$\mathcal{F}_! := R\bar{\pi}_! \Lambda$, $\mathcal{F}_* := R\bar{\pi}_* \Lambda$.

<u>Q</u> How to construct global Gal reps from $\mathcal{F}_!$ & $\mathcal{F}_*$? Compatibilities?

$S \ni \ell, p$ + all ramified places

$\rightsquigarrow \mathbb{T}^S \curvearrowright \mathcal{F}_!, \mathcal{F}_*$

$\mathbb{T}^S = \bigotimes_{v \notin S} \Lambda[G(\mathbb{Q}_v) /\!/ K_v] \supset \mathfrak{m}$ fixed max ideal.

$\updownarrow$

Hecke eigensystem

<u>Expect</u> (1) If $(\mathcal{F}_!)_\mathfrak{m}$ or $(\mathcal{F}_*)_\mathfrak{m} \neq 0$, then

$\exists$ L-param $\bar{\rho}_\mathfrak{m} : \mathrm{Gal}_\mathbb{Q} \longrightarrow {}^cG(\overline{k(\mathfrak{m})})$

unr outside S

+ matching Frob eigenval w/ Satake params.

(2) $\varphi := (\bar{\rho}_\mathfrak{m}|_{W_{\mathbb{Q}_p}})^{ss}$, $(\mathcal{F}_*)_{\mathfrak{m},\varphi} \cong (\mathcal{F}_*)_\mathfrak{m}$ (similarly for $\mathcal{F}_!$)

two localizations

(3) suitable condition on $\mathfrak{m}$ (e.g. non-Eisenstein)

$(\mathcal{F}_*)_\mathfrak{m} \cong \mathcal{G}_\varphi^{\oplus a_\mathfrak{m}}$ an explicit multiplicity

Hecke eigensheaf

(4) $\mathcal{F}_! \in {}^pD^{\leq 0}(\mathrm{Bun}_G)$, $\mathcal{F}_* \in {}^pD^{\geq 0}(\mathrm{Bun}_G)$

If $\mathfrak{m}$ is non-Eisenstein then $(\mathcal{F}_!)_\mathfrak{m} \cong (\mathcal{F}_*)_\mathfrak{m}$ perverse.

Thm (Caraiani-Zhang, in progress)

In the setup of [CS19],

$(F, F^+ \neq \mathbb{Q}, V \cong F^{2n}$

$\rightsquigarrow S_K$ quasi-split unitary Shimura var)

(1)(2)(4) hold; (3) holds if φ is generic in the sense of [HL].

<u>Rmk</u> Can remove $F^+ \neq \mathbb{Q}$ condition using a congruence tech developed by Fintzen & Shin.