Lecture 1: Analytic Theory (I) - Abelian Varieties & Complex Tori

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§1 Abelian Varieties as Group Varieties

"Abelian variety $/k$ = complete group variety".

<u>Def'n</u> (1) A <u>group variety</u> is a variety X/k equipped with morphisms

$$m: X \times_k X \longrightarrow X, \quad i: X \longrightarrow X.$$

& $\exists$ point $e \in X(k)$ satisfying the grp law $(e: \operatorname{Spec} \bar{k} \to \operatorname{Spec} k \xrightarrow{e} X)$.

(2) An <u>Abelian variety</u> is a complete (i.e. proper) grp var.

<u>Facts</u> (1) Every grp var is nonsingular.

(2) AVs are projective and <u>commutative</u>.

(3) Elliptic curves = AVs of dim 1.

(4) $AV/\mathbb{C}$ = polarizable complex tori (which admits a $\underbrace{\text{Riemann form}}_{\text{positive def'd Hermitian form}}$).

<u>Prop'n</u> The group law on A/k is automatically commutative.

<u>Proof</u>. $X = A(\mathbb{C})$ connected compact complex Lie group.

$\rightsquigarrow$ $V = T_e(X) = \mathrm{Lie}(X)$ its Lie algebra with $\exp: V \to X$ holomorphic.

<u>Adjoint rep'n</u>: $C: G \longrightarrow \mathrm{Aut}(G)$

$$g \longmapsto (C_g : h \longmapsto ghg^{-1}).$$

$\rightsquigarrow$ $\mathrm{Ad}: G \longrightarrow \mathrm{Aut}(\mathfrak{g})$

$$g \longmapsto (\mathrm{Ad}_g : h \longmapsto (dC_g)_e(h))$$

$\rightsquigarrow$ $\mathrm{ad}: \mathfrak{g} \longrightarrow \mathrm{Der}(\mathfrak{g}) := \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$

$$x \longmapsto \mathrm{ad}_x := d(\mathrm{Ad})_e(x)$$

note: $\mathrm{Ad}_g(Y) = gYg^{-1}$, $Y \in \mathfrak{g}$ $\Rightarrow$ For $g = \exp(tX)$, $\mathrm{Ad}_{e^{tX}}(Y) = e^{tX} Y e^{-tX}$

$\Rightarrow$ $\mathrm{ad}_X Y = XY - YX$ by taking derivation at $t = 0$.

Consider $\mathrm{Ad}: X \longrightarrow \mathrm{Aut}(V) \subseteq \mathrm{End}(V)$ s.t. $\mathrm{Ad}(a) = dC_a(e)$ (with $C_a(x) = axa^{-1}$), holo.

Here X = connected compact manifold / $\mathbb{C}$

$\mathrm{End}(V)$ = fin dim'l $\mathbb{C}$-v.s., $\mathrm{Ad}(X)$ = open submanifold in $\mathrm{End}(V)$.

Fact Global holomorphic functions on X are const.

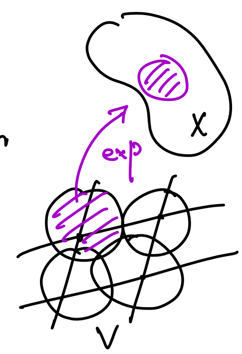
So Ad = const & $\mathrm{ad} = 0 \Rightarrow V = \mathrm{Lie}(X)$ comm Lie algebra. $\square$

§2 Compact Complex Lie Groups

On the same statement $X = A(\mathbb{C})$, $V = \mathrm{Lie}(X)$.

(1) $\exp: V \to X$ homomorphism (by commutativity of X).

(2) $\exp: V \twoheadrightarrow X$ surjective.



b/c $\exp(V) \subseteq X$ open subgrp as exp is a local homeomorphism

Also, $X \setminus \exp(V) = \bigcup_{x \in X\setminus \exp(V)} x\cdot\exp(V)$ union of opens

$\Rightarrow \exp(V)$ open & closed $\Rightarrow \exp(V) = X$.

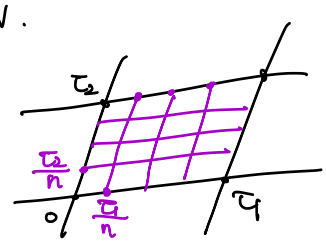
(3) $\ker(\exp) \subseteq V$ is a lattice

(Recall subgrp $U \subseteq V$ is a lattice $\Leftrightarrow$ U discrete with compact quotient.)

b/c $\ker(\exp)$ is discrete with $V/\ker(\exp) \approx \exp(V) = X$ compact.

Moreover, $\ker(\exp) \approx \mathbb{Z}^{2g}$, $g = \dim_{\mathbb{C}} X = \frac{1}{2}\dim_{\mathbb{R}} V$.

(4) $X = V/\mathbb{Z}^{2g}$ is a complex torus (by (2)(3)).



(5) $X[n] := \ker(X \xrightarrow{[n]} X) \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$.

b/c $X[n] \approx \frac{1}{n}\ker(\exp)/\ker(\exp)$.

Theorem There are canonical isom's for $X = V/L$ ($L = \mathbb{Z}^{2g}$)

$$\Lambda^r H^1(X,\mathbb{Z}) \longrightarrow H^r(X,\mathbb{Z}) \longrightarrow \mathrm{Hom}(\Lambda^r L, \mathbb{Z}).$$

Proof. The cup product defines $\Lambda^r H^1(X,\mathbb{Z}) \xrightarrow{\varphi} H^r(X,\mathbb{Z})$

$$a_1\wedge\cdots\wedge a_r \longmapsto a_1\cup\cdots\cup a_r.$$

Moreover, by $\boxed{\text{Künneth formula}}$, as $X \approx (S^1)^{2g}$, to check φ is an isom, it boils down to show

$$\Lambda^r H^1(S^1,\mathbb{Z}) \simeq H^r(S^1,\mathbb{Z}).$$

If $\forall r,s\geq 0$, $H^r(X,\mathbb{Z})$ and $H^s(Y,\mathbb{Z})$ are free $\mathbb{Z}$-mods, then

$$H^m(X\times Y,\mathbb{Z}) = \underbrace{\bigoplus_{r+s=m} H^r(X,\mathbb{Z})\otimes H^s(Y,\mathbb{Z})}$$

$$p^*a\cup q^*b \longleftarrow a\otimes b$$

where

$$\begin{tikzcd} & X\times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y. \end{tikzcd}$$

This is b/c $H^1(S^1,\mathbb{Z}) \simeq \mathrm{Hom}(\pi_1(S^1,x),\mathbb{Z})$

$$= \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) = \mathbb{Z}.$$

(alternatively, c.f. Hartshorne Ex III.2.7.)

Again, note that $\pi_1(X,x) = L$.

So $H^1(X,\mathbb{Z}) \simeq \mathrm{Hom}(L,\mathbb{Z}) = L^\vee$. And by taking Λ^r:

$$\begin{tikzcd} \Lambda^r H^1(X,\mathbb{Z}) \arrow[r, phantom, "\simeq"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\Downarrow"] & \Lambda^r \mathrm{Hom}(L,\mathbb{Z}) \arrow[d, "\simeq"] \\ H^r(X,\mathbb{Z}) \arrow[r, phantom, "\simeq"] & \mathrm{Hom}(\Lambda^r L,\mathbb{Z}). \end{tikzcd}$$

$\square$

§3 Cohomology of AVs

Goal Given X complex torus, to compute $H^q(X,\Omega^p)$ for all $p,q\geq 0$.

where $\Omega^p :=$ sheaf of holomorphic p-forms on X.

Ⓐ Dolbeault cohomology theory

(1) Complex vector bundles on complex manifold X.

$$\Omega_X = T^* = \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})^* \quad \text{tangent vector bundle}, \quad \Omega_X^k = \Lambda^k\Omega_X.$$

Decompose $T_{\mathbb{C}} = T\otimes\mathbb{C} = T^{1,0}\oplus T^{0,1}$

by dualizing $\Omega^k = \Omega_X^k\otimes\mathbb{C} = \bigoplus_{p+q=k}\Omega^{p,q}$ where $\Omega^{p,q} = \Lambda^p(T^{1,0})^*\otimes\Lambda^q(T^{0,1})^*$.

Notation

Complex vector bundles	Sheaves of sections
$\Lambda^k_{\mathbb{C}} X := \Lambda^k(T_{\mathbb{C}}X)^*$	$C^k_{X,\mathbb{C}} = C^k$
$\Lambda^{p,q}X := \Lambda^p(T^{1,0}X)^*\otimes\Lambda^q(T^{0,1}X)^*$	$C^{p,q}_{X,\mathbb{C}} = C^{p,q}$

sheaf of C^∞ complex-valued diff forms of type (p,q).

$$\begin{tikzcd} C^{p,q+1} \arrow[r, "\partial"] & C^{p+1,q+1} \\ C^{p,q} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & C^{p+1,q} \arrow[u, "\bar\partial"'] \end{tikzcd}$$

Recall Diff operators (above) with $d: C^k \to C^{k+1}$ $\mathbb{C}$-lin ext'n

satisfying
(i) (decomposition) $d = \partial + \bar\partial$
(ii) (complex differential) $\partial^2 = \bar\partial^2 = 0$
(iii) (anti-commutative) $\partial\bar\partial = -\bar\partial\partial$
(iv) (Leibniz rule) $\partial(x\wedge y) = \partial x \wedge y + (-1)^{p+q} x \wedge \partial y$

$\bar\partial(x\wedge y) = \bar\partial x \wedge y + (-1)^{p+q} x \wedge \bar\partial y.$

(2) Dolbeault cohom of complex manifold X.

Def'n $H^{p,q}(X) := H^q(C^{p,\cdot}, \bar\partial) := \dfrac{\ker(C^{p,q} \xrightarrow{\bar\partial} C^{p,q+1})}{\operatorname{im}(C^{p,q-1} \xrightarrow{\bar\partial} C^{p,q})}$.

Fact (1) Dolbeault resolution $0 \to \Omega^p \to C^{p,1} \to C^{p,2} \to \cdots$ exact sequence

(2) All $C^{p,q}_X$ are flasque sheaves.

Think about: Over $\mathbb{R}$, $\mathcal{A}^k$ = sheaf of C^∞ diff forms $\curvearrowright d: \mathcal{A}^k \to \mathcal{A}^{k+1}$

i.e. sections of the bundle $\Omega^k_{X,\mathbb{R}}$.

Poincaré's lemma: a closed form of deg $k>0$ is locally exact

$\Rightarrow$ $\mathcal{A}^{k-1} \xrightarrow{d} \mathcal{A}^k \xrightarrow{d} \mathcal{A}^{k+1}$ $(k\geq 1)$ exact at $\mathcal{A}^k$.

Also, $\ker(d: \mathcal{A}^0 \to \mathcal{A}^1) = \{\text{locally const functions}\}$.

So $0 \to \mathcal{A}^0 \to \mathcal{A}^1 \to \cdots \to \mathcal{A}^n \to 0$ $(n = \dim_{\mathbb{R}} X)$

is a resolution for the const sheaf $\mathbb{R}$ on X.

Apply similar argument to $\bar\partial: C^{0,q} \to C^{0,q+1}$.

Now we have $\boxed{H^{p,q}(X) \cong H^q(X, \Omega^p)}$.

(E.g. note that $\underbrace{H^0(X, \Omega^p)}_{\text{the space of holomorphic p-forms}} = \{s \in C^{p,0} : \bar\partial(s) = 0\} = H^{p,0}(X)$)

Ⓑ <u>Main Theorem</u> Set $T = \mathrm{Hom}_{\mathbb{C}\text{-lin}}(V, \mathbb{C})$, $\bar{T} = \mathrm{Hom}_{\mathbb{C}\text{-antilin}}(V, \mathbb{C})$, where $V = T_0(X)$.

Then $H^q(X, \Omega^p) \simeq \Lambda^q \bar{T} \otimes \Lambda^p T$.

<u>Step 1</u> For all $p, q \geq 0$, $H^q(X, \Omega^p) \simeq H^q(X, \mathcal{O}_X) \otimes \Lambda^p T$

with $T = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) = (T_0^{1,0})^*$ cotangent space at 0.

Fix $x \in X$ to define $T_x : X \longrightarrow X$, $y \longmapsto y + x$. Choose $\alpha \in \Lambda^p T$ p-covector.

⇝ ω_α translation invariant holomorphic p-form, $(\omega_\alpha)_x := T_{-x}^*(\alpha)$

⇝ Morphism of sheaves $\mathcal{O}_X \otimes \Lambda^p T \longrightarrow \Omega^p$ ← b/c by dualizing,
$\alpha \longmapsto \omega_\alpha$. $T_x X \simeq m_{-x}^* T_e X \simeq V$.

(An isom in local coordinates, and hence an isom of sheaves).

$\Rightarrow$ $H^q(X, \Omega^p) \simeq H^q(X, \mathcal{O}_X \otimes \Lambda^p T) = H^q(X, \mathcal{O}_X) \otimes \Lambda^p T$.

<u>Step 2</u> Take $\mathfrak{a} = \Gamma(X, C^{0,0})$ and $\Lambda^\cdot = \bigoplus_{q \geq 0} \Lambda^q \bar{T}$ (the exterior algebra of $\bar{T}$, regarded as a chain complex with 0 differentials).

Then $H^q(X, \mathcal{O}_X) \simeq H^q(\mathfrak{a} \otimes \Lambda^\cdot)$.

Apply Dolbeault resolution for Ω^p:

$$0 \to \Omega^p \to C^{p,1} \to C^{p,2} \to \cdots \qquad (*)$$

"Lifting" the isomorphism $\mathcal{O}_X \otimes \Lambda^p T \longrightarrow \Omega^p$ along the resolution we get $\phi_{p,q} : C^{0,0} \otimes \Lambda^p T \otimes \Lambda^q \bar{T} \xrightarrow{\simeq} C^{p,q}$.

$$\sum f_i \otimes \alpha_i \longmapsto \sum f_i \omega_{\alpha_i}, \quad f_i \in C^{0,0}, \ \alpha_i \in \Lambda^p T \otimes \Lambda^q \bar{T}.$$

In particular,

$(*) \overset{p=0}{\Rightarrow}$ $H^q(X, \mathcal{O}_X) = H^{0,q}(X) \simeq \ker(C^{0,q} \xrightarrow{\bar\partial} C^{0,q+1}) / \mathrm{im}(C^{0,q-1} \xrightarrow{\bar\partial} C^{0,q})$.

Also, $\phi_{0,q}$ corresponds to $\underbrace{\mathfrak{a} \otimes \Lambda^q}_{} \simeq \Gamma(X, C^{0,q})$.

the q-th term in complex $\mathfrak{a} \otimes \Lambda^\cdot$ with $\bar\partial : \mathfrak{a} \otimes \Lambda^q \longrightarrow \mathfrak{a} \otimes \Lambda^{q+1}$, $f \otimes a \longmapsto \bar\partial f \wedge a$

⇝ get isoms $\mathfrak{a} \otimes_{\mathbb{C}} \Lambda^\cdot \simeq \Gamma(X, C^{0,\cdot})$.

$\Rightarrow$ $H^q(X, \mathcal{O}_X) \simeq H^{0,q}(X) \simeq H^q(\mathfrak{a} \otimes \Lambda^q)$, $q \geq 0$.

<u>Step 3</u> The inclusion $\Lambda^\cdot \to \mathscr{E}\otimes_{\mathbb{C}}\Lambda^\cdot$ of complexes defines grp isoms.

$$\Lambda^q \simeq H^q(X, \mathscr{E}\otimes_{\mathbb{C}}\Lambda^\cdot)$$

This is proved by Fourier analysis (details are omitted).

<u>§4 The Hodge Decomposition</u>

Same method to compute de Rham cohomology.

<u>Observation</u> For $C^k = \bigoplus_{p+q=k} C^{p,q}$ (C^∞-diff k-forms), $\ker(C^0 \xrightarrow{d} C^1) \simeq \mathbb{C}$.

By Poincaré lemma, $\exists$ acyclic resolution

$$0 \to \underline{\mathbb{C}} \to C^0 \xrightarrow{d} C^1 \xrightarrow{d} \cdots$$

$\rightsquigarrow$ to compute $\underbrace{H^n(X,\mathbb{C})}_{\text{cohom with } \mathbb{C}\text{-coeff}} = \underbrace{H^n(X,\underline{\mathbb{C}})}_{\text{cohom of sheaf}}$.

In particular, $H^n(X,\mathbb{C}) = \dfrac{\ker(d: C^n \to C^{n+1})}{\operatorname{im}(d: C^{n-1}\to C^n)} = \dfrac{\{d\text{-closed } n\text{-forms}\}}{d\{(n-1)\text{-forms}\}}$.

Recall in Step 2, for $(0,q)$-forms, $\forall \omega \in \{d\text{-closed } n\text{-form}\}$,

$\exists!$ translation invariant n-form ω_α ($\alpha \in \Lambda^n \operatorname{Hom}(V,\mathbb{C})$)

s.t. $\omega = \omega_\alpha + d\eta$, $\eta \in \{(n-1)\text{-forms}\}$.

Therefore,
$$\begin{aligned} H^n(X,\mathbb{C}) &\simeq \Lambda^n \operatorname{Hom}(V,\mathbb{C}), \quad n \geq 0. \\ &\simeq \Lambda^n(T \oplus \overline{T}) \quad \text{(linear + anti-linear)} \\ &\simeq \bigoplus_{p+q=n} \Lambda^p T \oplus \Lambda^q \overline{T} \quad \text{(by def'n)} \\ &\simeq \bigoplus_{p+q=n} H^q(X,\Omega^p) \quad \text{(by Main Thm)}. \end{aligned}$$