Lecture 4: Algebraic Theory (I) - Definition of Abelian Varieties

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(Always work over $k=\bar{k}$.)

<u>Def'n</u> An <u>abelian variety</u> X is a complete alg var $/k$
with grp law $m: X\times X \to X$ & $i: X\to X$ morphisms.

<u>Note</u> complete : $X\times Y \to Y$ closed map. $f: X\to Y$ b/w affine vars
$\rightsquigarrow f(X)=\{y_0\}$.

<u>More stories</u> char $k=p$, $g=\dim X$.

<u>Property 1</u> As an abstract grp, X is commutative & $\boxed{\text{divisible}}$

For $[n]: X\to X$ with $X[n]=\ker[n]$,

$$X[n]\cong\begin{cases}(\mathbb{Z}/n\mathbb{Z})^{2g}, & p\nmid n\\ (\mathbb{Z}/p^m\mathbb{Z})^{i}, & n=p^m\end{cases},\quad \text{where } i=p\text{-rank}.$$

<u>Question 2</u> Calculate $H^p(X,\Omega^q)$.

$\rightsquigarrow H^q(X,\Omega^p)\cong\wedge^p H^0(X,\Omega^1)\otimes\wedge^q H^1(X,\mathcal{O}_X)$ (by Serre duality (?))

with $\dim H^0(X,\Omega^1)=\dim H^1(X,\mathcal{O}_X)=g$

Consider $G \curvearrowright Y \xrightarrow{f} X=Y/G$

$\rightsquigarrow \exists g: X\to Y$ s.t. $f\circ g=[n]$.

<u>Question 3</u> What is $\mathrm{Pic}(X)$?

Have $0\to \mathrm{Pic}^0(X)\to \mathrm{Pic}(X)\to NS(X)\to 0$

$\mathrm{Pic}^0(X)$: an ab var; $NS(X)\cong\mathbb{Z}^{\oplus r}$, r = base number.

<u>Question 4</u> Classification of line bundles?

<u>Prop</u> As a grp variety, X is non-singular.

<u>Proof</u> Consider $x\in X \rightsquigarrow T_{y\cdot x^{-1}}: X\to X$ left multi.

where y = a fixed non-sing pt.

Thm X is an abelian grp.

Proof. Consider the conjugation by fixing $x \in X$:

$$\begin{aligned} C_x : \ & X \to X \\ & y \mapsto xyx^{-1} \end{aligned}$$

$$C_x^* : \mathcal{O}_{X,e} \to \mathcal{O}_{X,e} \rightsquigarrow C_{x,n}^* : \mathcal{O}_{X,e}/m_{X,e}^n \to \mathcal{O}_{X,e}/m_{X,e}^n \quad C_{x,n}^* \in \mathrm{Aut}(\mathcal{O}_{X,e}/m_{X,e}^n).$$

$$\begin{aligned} \rightsquigarrow \gamma : X & \longrightarrow \mathrm{Aut}(\mathcal{O}_{X,e}/m_{X,e}^n) \\ x & \longmapsto C_{x,n}^* \end{aligned}$$

For $n \gg 0$, as $\bigcap m^n = (0)$, we get $C_{x,n}^* = \mathrm{id}$. $\square$

$T_{X,0} = (m_{X,e}/m_{X,e}^2)^*$. $\Omega_0 = (T_{X,0})^*$ with $\Omega_0 \otimes_k \mathcal{O}_X \cong \Omega^1$.

Prop When $p \nmid n$, $[n]$ is surjective.

Proof Descend $m : X \times X \to X$ to the level of tangent spaces:

$$\rightsquigarrow dm : T \oplus T \to T$$

$$(t_1, t_2) \mapsto t_1 + t_2$$

$$\rightsquigarrow X \longrightarrow X \times X \xrightarrow{pr_1} X \quad \text{identity}$$

Induction on n: $(d[n])_0 = 0 \cdot n$

$p \nmid n \Rightarrow (d[n])_0 : T \xrightarrow{\simeq} T'$. $\dim [n]^{-1}(0) > 0$.

For $t \in T$, $(d[n])_0(t) = 0$. $\square$

Lemma (Rigidity) X complete, Y, Z alg var.

$f : X \times Y \to Z$, $\exists y_0$ s.t. $f(X \times \{y_0\}) = \{z_0\} \subseteq Z$

Then $\exists g : Y \to Z$ s.t. $f = g \circ p_2$, $p_2 : X \times Y \to Y$.

Corollary (1) X, Y AVs. $f: X \to Y$ morphism.

Then $f(x) = h(x) + a$, h homo.

Pf. Assume $f(0) = 0$. $\psi: X \times X \longrightarrow Y$

def'd by $\psi(x, x') = f(x + x') - f(x) - f(x')$

$\Rightarrow \psi(x, 0) = \psi(0, x) = 0 \Rightarrow \psi \equiv 0$.

(2) X commutative.

Pf. $i: X \to X$, $i(e) = e \leadsto i$ homomorphism.

Thm X complete var. $e \in X$ pt.

$\exists\ m: X \times X \to X$ morphism s.t. $m(x, e) = m(e, x) = x$

$\Rightarrow X$ is an AV.

Proof Denote $m(x, y) = xy$, $\leadsto \psi: X \times X \longrightarrow X \times X$

$(x, y) \longmapsto (xy, y)$

s.t. $\psi^{-1}(e, e) = (e, e)$. Then

$\dim(X \times X) = \dim(\operatorname{im} \psi) \Rightarrow \psi$ surjective.

$\forall x, \exists x'$ s.t. $x'x = e$. Consider the graph

$\Gamma' = \{(x, y) \mid xy = e\} \Rightarrow pr_2(\Gamma') = X \quad (i = 1, 2)$.

Take $\Gamma \subseteq \Gamma'$ irred component. $\leadsto pr_i' = pr_i|_\Gamma$.

Also, $pr_i'^{-1}(e) = (e, e)$. $\dim(\operatorname{im} pr_i') = \dim \Gamma \geq \dim X \Rightarrow pr_i'$ surj.

Set $\phi: \Gamma \times X \longrightarrow X$ $\Rightarrow \phi(\Gamma \times \{e\}) = e$.

$((x', x), y) \longmapsto x'(xy)$, $\Rightarrow x'(xy) = y$. $\forall (x', x) \in \Gamma$.

Choose $(x'', x') \in \Gamma$ again to get the associativity.