Koszul duality and an exercise in linear algebra

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(In discussion w/ Dmitry Kubrak)

<u>Exercise</u> (V, ω) symplectic space

$$\psi_i : V^{\otimes n} \to V^{\otimes(n-2)} \quad \Big(v_1 \otimes \cdots \otimes \underbrace{v_i \otimes v_{i+1}}_{\text{delete this}} \otimes \cdots \otimes v_n \Big).$$

$$\rightsquigarrow \quad \begin{array}{c} V^{\otimes n} \\ \cup \\ W_n \end{array} := \bigcap_{i=1}^{n-1} \ker\left(V^{\otimes n} \xrightarrow{\psi_i} V^{\otimes(n-2)}\right)$$

$$\begin{tikzcd} W_n \otimes V \arrow[r, "\text{surj?}"] \arrow[d, phantom, "\cap"] & W_{n-1} \arrow[d, phantom, "\cap"] \\ V^{\otimes(n+1)} \arrow[r, "\psi_n"] & V^{\otimes(n-1)} \end{tikzcd}.$$

<u>Construction</u> (1) Pre-dual:

<u>Input</u> $\mathrm{Alg}_k^{\mathrm{aug}} \ni A \rightsquigarrow A \otimes A \to A, \; A \to k$

E_1-alg w/ cohom.

$$\rightsquigarrow k \otimes_A k \in \mathrm{coAlg}_k^{\mathrm{aug}} \qquad k \otimes_A k =: \boxed{\mathrm{Bar}(A)}.$$

Consider

$$\begin{tikzcd} \Omega(X, x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \text{“}\mathrm{Spec}\, k\text{”} \arrow[d, "x"] \\ \mathrm{Spec}\, k \arrow[r, "x"] & X = \mathrm{Spec}\, A \end{tikzcd}$$

loop space equipped w/ assoc multi.

$$\begin{tikzcd} k \otimes_A k \arrow[r] \arrow[d, equal] & (k \otimes_A k) \otimes_k (k \otimes_A k) \arrow[d, equal] \\ k \otimes_A A \otimes_A k \arrow[r] & k \otimes_A k \otimes_A k \end{tikzcd}$$

So ... $\rightsquigarrow$ Co-multi str.

(2) $D(A) := \mathrm{Bar}(A)^\vee = \mathrm{Hom}_k(k \otimes_A k, k) = \mathrm{Hom}_k(k, k).$

This two constr'ns are related.

N.B. A may have higher cohom, i.e. deg $\neq 0$.

But $\mathrm{Bar}(A)^\vee$ & $\mathrm{Bar}(A)$ have no higher str.

Motivation Question

For some "common" A,

is $\mathrm{Bar}(A)$ or $D(A)$ concentrated in coh deg 0 ?

E.g. · $R\Gamma_{sing}(\Sigma ; \mathbb{Z}$ or $\mathbb{F}_p)$. Σ = pointed Riemann surface

· $R\Gamma(X, \mathcal{O}_X)$ or $R\Gamma_{dR}(X/k)$.

X pointed curve

· $R\Gamma_{dR}(X/k)$. X = abelian sch.

Lem (EKMM spectral seq)

Assume

$$H^*(A) = \begin{cases} 0, & * < 0 \\ k, & * = 0 \\ \text{do not care}, & * > 0 \end{cases}$$

Then $\mathrm{Tor}_i^{H^*(A)}(k,k)$ has degree i (only nondeg at diag)

$\overset{(*)_i}{\Rightarrow}$ $\mathrm{Bar}(A)$ & $D(A)$ concentrated in deg 0.

Prop (Beilinson-Ginzberg-Soergel)

k field,

$$A^i = \begin{cases} 0, & i < 0 \\ k, & i = 0 \\ \cdots, & i > 0 \end{cases}$$ & A^i is a f.g. k-mod, $\forall i$.

Then $(*) \iff \exists$ graded proj resol'n of k

$$\cdots \to A^{\cdot} \otimes_k P_1 \to A^{\cdot} \otimes_k P_0 \to k$$

w/ P_i = graded k-mod in deg i.

(Indeed, $P_i = \mathrm{Tor}_i^{H^*(A^{\cdot})}(k, k)$.)

<u>Def</u> $A^{\cdot}$ called Koszul alg if satisfies this property $(*)_i$, $\forall i$.

<u>Lem</u> (BGS) $(*)_1 \iff A^{\cdot}$ is given by A^1 over A^0

$(*)_1 + (*)_2 \iff A^{\cdot}$ is given by A^1 over A^0 + rel'n given in deg 2.

<u>Def</u> $A^{\cdot}$ called quadratic alg if satisfies $(*)_1 + (*)_2$.

<u>Example</u> (Koszul) $A^1 = V$

(1) $\mathrm{Sym}_k^{\cdot}(V) = A^{\cdot}$ forms a finite resol'n.

$$\mathrm{Sym}_k^{\cdot}(V) \otimes_k \Lambda^{top}(V) \to \cdots \to \mathrm{Sym}_k^{\cdot}(V) \otimes_k V \to \mathrm{Sym}_k^{\cdot}(V) \to k.$$

$\rightsquigarrow$ $D(A) \cong \Lambda_k^{\cdot}(V^{\vee})$ classical Koszul duality.

(2) $A^{\cdot} = \Lambda^{\cdot} V$

$$\cdots \to \underbrace{\Lambda^{\cdot}(V) \otimes \Gamma^2(V)}_{\text{Symm tensor}} \to \Lambda^{\cdot}(V) \otimes V \to \Lambda^{\cdot}(V) \to k.$$

$\rightsquigarrow$ $D(A) = D(\Lambda^{\cdot} V) \cong \mathrm{Sym}^{\cdot}(V^{\vee})$.

the other direction of classical Koszul duality.

(3) $A^{\cdot} = T_k^{\cdot}(V)$ quad alg

$$T^\cdot(V) \otimes V \to T^\cdot(V) \to k \qquad 2 \text{ terms}$$

$$\begin{array}{lll} 0 = V^{\otimes 3} & & \\ V^{\otimes 2} & V^{\otimes 2} & \\ V & V & \\ & k \longrightarrow & k \end{array}$$

$\rightsquigarrow \ D(T^\cdot_k(V)) \cong k \oplus V^\vee.$

(4) $k \oplus V = A^\cdot$

$$\cdots \to A^\cdot \otimes (V^{\otimes 2}) \to A^\cdot \otimes V \to A^\cdot \to k.$$

$\rightsquigarrow \ D(A^\cdot) \cong T^\cdot_k(V^\vee)$

(5) $H^*_{sing}(\Sigma, \overset{k}{\overset{\shortparallel}{\mathbb{F}_q}}) = A^\cdot$

$$\begin{array}{l} W_3 \otimes V \xrightarrow{\psi_3} W_2 \\ W_3 \to W_2 \otimes V \xrightarrow{\psi_2} V \\ 0 \to W_2 \to V \otimes V \xrightarrow{w} k \\ \qquad\qquad V \longrightarrow V \\ \qquad\qquad\qquad k \longrightarrow k \end{array}$$

Lie tensor

$\rightsquigarrow D(A^\cdot) \cong T^\cdot_k(V^\vee)/(\eta)$

$\llcorner$ dual of symplectic pairing.

Prop (BGS) $(A^\cdot)^! := T^\vee_k(V^\vee)/(R^\vee)^\perp.$

$$R \to V \otimes V \twoheadrightarrow A^2 \rightsquigarrow R^\vee \leftarrow V^\vee \otimes V^\vee \leftarrow R^\perp$$

Then $A^\cdot$ Koszul $\iff (A^\cdot)^!$ Koszul

& $D(A^\cdot) = ((A^\cdot)^!)^{op}.$

Example (Continued)

(6) $A^{\bullet} = T_k^{\bullet}(V)/(\eta)$ $\qquad \eta := \sum_i [a_i, b_i]$.

(do not know)

$$\begin{array}{ccccc}
V\otimes\eta & \xrightarrow{\text{inj}} & \dfrac{V\otimes V\otimes V}{\eta\otimes V} & \twoheadrightarrow & \dfrac{V\otimes V\otimes V}{\eta\otimes V+V\otimes\eta} & & \\
k\cdot\eta & \longrightarrow & V\otimes V & \twoheadrightarrow & \dfrac{V\otimes V}{\eta} & & \\
 & & V & \longrightarrow & V & & \\
 & & & & k & \longrightarrow & k
\end{array}$$

Claim: $A^{\bullet}\otimes k\eta \xrightarrow{?} A^{\bullet}\otimes V^{\bullet} \longrightarrow A^{\bullet} \longrightarrow k$

This is inj

pf. $A^{\bullet}$ is secretly the universal enveloping alg of Lie alg

(Apply PBW : nonzero $\Rightarrow$ inj).

i.e. $\eta \in \mathfrak{g}$, $T_k^{\bullet}(V) = \mathcal{U}(\mathfrak{g})$, $A^{\bullet} = \mathcal{U}(\mathfrak{g}/\eta)$.