

Serre Duality for Projective Spaces

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§1 Ext Groups

$$M, N \in \text{Mod}_R. \quad \text{Ext}^i(M, N) := \underbrace{R^i \text{Hom}_R(M, N)}_{= (R^i \text{Hom}_R(M, -))(N)} \text{ (more strictly)}$$

Recall: $\text{Hom}_R(M, -)$ left exact covariant : $\text{Mod}_R \rightarrow \text{Ab}$.

Also, $\text{Ext}^i(M, N) := (R^i \text{Hom}_R(-, N))(M)$ on Mod_R^op .

$\Rightarrow \text{Ext}^i(\mathcal{F}, -), \text{Hom}(\mathcal{F}, -)$ sheaf-version.

E.g. \exists natural isom $\text{Ext}^i(\mathcal{O}_X, \mathcal{F}) \cong H^i(X, \mathcal{F})$

b/c both are derived functors of $\text{Hom}(\mathcal{O}_X, \mathcal{F}) \cong H^0(X, \mathcal{F})$.

Also, $\text{Hom}(\mathcal{O}_X, \mathcal{F}) = \text{id}$ functor

$$\Rightarrow \text{Ext}^0(\mathcal{O}_X, \mathcal{F}) \cong \mathcal{F}, \quad \text{Ext}^i(\mathcal{O}_X, \mathcal{F}) = 0 \quad (i > 0).$$

Lemma \mathcal{I} inj. $\mathcal{O}_X\text{-mod} \Rightarrow \forall U \subseteq X \text{ open}, \mathcal{I}|_U \text{ inj } \mathcal{O}_U\text{-mod}$.

Proof. $j: U \hookrightarrow X$. Given $\mathcal{F} \rightarrow \mathcal{G}$ mono, $\mathcal{F} \rightarrow \mathcal{I}|_U \xrightarrow{\exists} \mathcal{G} \rightarrow \mathcal{I}|_U$

j^* denotes ext'n by 0 s.t. $(j^*\mathcal{F})_x = 0$ unless $x \in U$.

$\Rightarrow j^*\mathcal{F} \rightarrow j^*\mathcal{G}$ mono.

Moreover, $\exists j^* \mathcal{I}|_U \rightarrow \mathcal{I}$ by adjunction.

$\& j^*\mathcal{F} \rightarrow j^* \mathcal{I}|_U \rightarrow \mathcal{I}$ extends to $j^*\mathcal{G} \rightarrow \mathcal{I}$.

$\Rightarrow \mathcal{G} \rightarrow \mathcal{I}|_U$ by adj . \square

Cor $\forall U \subseteq X \text{ open}, \text{Ext}^i(\mathcal{F}, \mathcal{G})|_U \cong \text{Ext}^i(\mathcal{F}|_U, \mathcal{G}|_U)$ naturally.

In particular, $\text{Ext}^i(\mathcal{F}|_U, \mathcal{G}|_U) \in \text{Sh}(U)$.

e.g. $\text{Ext}^i(\mathcal{F}, \mathcal{G}) = 0$, $\forall i > 0$ whenever \mathcal{F} be free of rank $< \infty$.

Proof. Both sides are δ -functor in \mathcal{G} .

& higher terms vanish on injectives by Lemma.
 \Rightarrow effaceable \Rightarrow univ. \square

Cor \mathcal{I} inj Gr-mod . $\text{Hom}_{\mathcal{I}}(\cdot, \mathcal{I})$, $\text{Hom}(\cdot, \mathcal{I})$ are exact.
 by def'n of inj. by Lemma.

Prop $\mathcal{F} \in \text{Mod}_{\mathcal{O}_X}$, $\text{Ext}^i(-, \mathcal{F})$, $\text{Ext}^i(-, \mathcal{F})$ δ -functors on $\text{Mod}_{\mathcal{O}_X}^{\text{op}}$.

Proof. $0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^+$ inj resolution. Given

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0 \text{ in } \text{Sh}(\text{Mod}_{\mathcal{O}_X}).$$

$\Rightarrow \text{Hom}(-, \mathcal{I}) \& \text{Hom}(-, \mathcal{I})$ short exact.

\Rightarrow cohom long exact ...

By a pushout constr. it suffices to compare \mathcal{I}^+ & \mathcal{G}^+

with $\mathcal{I}^+ \rightarrow \mathcal{G}^+$ quasi-isom. \square

Fact $\text{Sh}(\text{Mod}_{\mathcal{O}_X})$ has enough projs \Rightarrow acyclic resolutions

$\Rightarrow \text{Ext}^i(-, \mathcal{F})$, $\text{Ext}^i(-, \mathcal{F})$ effaceable.

Prop Suppose $\cdots \rightarrow \mathcal{L}_i \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0$ exact in $\text{Sh}(\text{Mod}_{\mathcal{O}_X})$

where \mathcal{L}_i be free of rank $< \infty$.

(Say: \mathcal{L}_i be free resolution of \mathcal{F}).

Then $\text{Ext}^i(\mathcal{F}, \mathcal{G}) \cong h^i(\text{Hom}(\mathcal{L}_i, \mathcal{G}))$
 bi-functorial in \mathcal{G} & \mathcal{L}_i .

Proof. L_i loc free of fin rank
 $\Rightarrow \text{Ext}^i(\mathcal{F}, -) = 0 \Rightarrow \text{Hom}(L_i, -)$ exact
 (even though $\text{Hom}(L_{i+1}, -)$ not exact, H^i).
 $\Rightarrow h^i(\text{Hom}(L_i, -))$ δ -functor:

$$\left(\begin{array}{l} 0 \rightarrow \mathcal{G}_1 \rightarrow \mathcal{G} \rightarrow \mathcal{G}_2 \rightarrow 0 \text{ exact} \\ \hookrightarrow 0 \rightarrow \text{Hom}(L_i, \mathcal{G}_1) \rightarrow \text{Hom}(L_i, \mathcal{G}) \rightarrow \text{Hom}(L_i, \mathcal{G}_2) \rightarrow 0 \\ \hookrightarrow \text{long exact in cohom.} \end{array} \right)$$

Now both sides:

- (i) δ -functors in \mathcal{G}
 - (ii) higher der = 0 on injs
- $\left. \begin{array}{l} \text{(i) } \delta\text{-functors in } \mathcal{G} \\ \text{(ii) higher der = 0 on injs} \end{array} \right\} \Rightarrow \text{effaceable} \Rightarrow \text{univ. } \square$

Note loc free resolutions are much easier to write down than inj resolutions.

E.g. $X = \mathbb{P}_k^n$, k any field, \mathcal{F} coherent
 $\hookrightarrow \exists \mathcal{E} \rightarrow \mathcal{F}$, $\mathcal{E} = \bigoplus (\text{twisting sheaf})$
 $\uparrow \mathcal{O}_{X(n)}, n < \infty$.

by Serre's finiteness thm.
 (not just loc free, but free).

Prop $\forall \mathcal{F}, \mathcal{G} \in \text{Coh}(\mathbb{P}_k^n)$, k field, $\text{Ext}^i(\mathcal{F}, \mathcal{G}) \in \text{Coh}$ again.

Proof. $\exists L \rightarrow \mathcal{F} \rightarrow 0$ free resolution

Need to check $\underline{h^i(\text{Hom}(L, \mathcal{G}))} \in \text{Coh}$

true b/c $L, \mathcal{G} \in \text{Coh} \Rightarrow \text{Hom}(L, \mathcal{G}) \in \text{Coh} \quad \square$

Lemma $\mathcal{F}, \mathcal{G}, \mathcal{L} \in \text{Sh}(\text{Mod}(X))$. \mathcal{L} be free of fin rank.

$$(i) \text{Ext}^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \text{Ext}^i(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G});$$

$$(ii) \text{Ext}^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \text{Ext}^i(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G}) \cong \text{Ext}^i(\mathcal{F}, \mathcal{G}) \otimes \mathcal{L}^\vee.$$

In particular, $\mathcal{F} = \mathcal{L}^\vee, \mathcal{G} = \mathcal{L}$

$$\Rightarrow (i) \text{Ext}^i(\mathcal{O}_X, \mathcal{L}) \cong \text{Ext}^i(\mathcal{L}^\vee, \mathcal{O}_X);$$

$$(ii) \text{Ext}^i(\mathcal{O}_X, \mathcal{L}) \cong \text{Ext}^i(\mathcal{L}^\vee, \mathcal{O}_X).$$

Proof. Again: $i > 0$, check effaceable δ -functor in \mathcal{G}
 $(\Rightarrow \text{univ})$.

$i = 0$: Hartshorne Prop II.6.7. \square

Final Note Relationship b/w Ext & Ext^i ?

General fact: F, G left exact functors

$\Rightarrow \exists$ spectral sequence relating $D^i F, D^i G$ & $D^i(F \circ G)$
 $(D^i := \text{derived functors})$.

In our case: Given $\mathcal{F} \in \text{Sh}(X)$, take

$$F = H^0(X, -), G = \text{Hom}(\mathcal{F}, -), F \circ G = \text{Hom}_X(\mathcal{F}, -).$$

Ex Duality on Projective Spaces

For the rest, we work over k (any field).

Big Thm (Serre Duality) $X = \mathbb{P}^n_k, \mathcal{F} \in \text{Coh}(X)$.

Recall $\dim_k H^n(X, \mathcal{O}_X(-n-1)) = 1$.

(a) Perfect pairing

$$\text{Hom}_X(\mathcal{F}, \mathcal{O}_X(-n-1)) \times H^n(X, \mathcal{F}) \rightarrow H^n(X, \mathcal{O}_X(-n-1)).$$

of fin-dim'l k -vector spaces.

(i.e. $A = \text{Hom}_k(B, H^0(X, \mathcal{O}_X(-n-i)))$ & vice versa).

(b) $V \in \text{Vec}_k$, put

$$V' = \text{Hom}_k(V, H^n(X, \mathcal{O}_X(-n-i))).$$

$\forall i \geq 0$, \exists natural isom

$$\text{Ext}^i(F, \mathcal{O}_X(-n-i)) \rightarrow H^{n-i}(X, F)$$

which $i=0 \Rightarrow (a)$.

and which is compatible w/ short exact sequences.

Proof. (a) We have $\text{Hom}(F, \mathcal{O}_X(-n-i)) \rightarrow H^n(X, F)$ (**)
of left covariant functors on $\text{Sh}(\text{Mod}_{\mathcal{O}_X})$.

When $F = \mathcal{O}_X(m)$, we want

$$H^0(X, \mathcal{O}_X(-m-n-i)) \cong \text{Hom}(H^0(X, \mathcal{O}_X(m)), H^n(X, \mathcal{O}_X(-n-i))).$$

\uparrow

$$\text{Hom}(\mathcal{O}_X, F) = H^0(X, F)$$

Likewise, when $F = \bigoplus \mathcal{O}_X(n)$, (***) isom.

In general, $\begin{matrix} \xi_i \rightarrow \xi_0 \rightarrow F \rightarrow 0 \\ \uparrow \quad \uparrow \quad \uparrow \\ \bigoplus \mathcal{O}_X(n) \quad \bigoplus \mathcal{O}_X(n) \end{matrix}$

and the things we're computing are left-exact.

$$\begin{array}{ccc}
 0 & \xrightarrow{\quad} & 0 \\
 \downarrow & & \downarrow \\
 0 & \xrightarrow{\quad} & 0 \\
 \downarrow & & \downarrow \\
 \boxed{\text{Hom}(F, \mathcal{O}_X(-n-i)) \rightarrow H^n(X, F)} & & \text{isom by 5-lemma} \\
 \downarrow & & \downarrow \\
 \text{Hom}(\xi_0, \mathcal{O}_X(-n-i)) & \xrightarrow{\sim} & H^n(X, \xi_0) \\
 \downarrow & & \downarrow \\
 \text{Hom}(\xi_i, \mathcal{O}_X(-n-i)) & \xrightarrow{\sim} & H^n(X, \xi_i)
 \end{array}$$

(b) Two δ -functors (agree at $i=0$) on $\text{Coh}(\text{Mod}_{\mathcal{O}_X})$

To check: both are effaceable.

Given $F \in \text{Coh}(\text{Mod } \mathcal{O}_X)$, $g \gg 0$,

$$\mathcal{E} = (\mathcal{O}_X(-g))^{\oplus m} \xrightarrow{\quad} \bigoplus \xrightarrow{\quad} 0$$

↑
quotient

It suffices to check $H_i = 0$.

$$\begin{aligned} \text{Ext}^i(\mathcal{O}_X(-g), \mathcal{O}_X(-n-i)) &= 0 \quad \& \quad H^{n-i}(X, \mathcal{O}_X(-g)) = 0. \quad g \gg 0. \\ \text{Ext}^i(\mathcal{O}_X, \mathcal{O}_X(n-i)) &\stackrel{\parallel}{\quad} \quad \text{Serre vanishing} \\ &\stackrel{\text{IS}}{\quad} \\ H^i(X, \mathcal{O}_X(n-i)) &. \end{aligned}$$

\square

§3 Differentials and Duality

(This is not the right way to view the duality thm)
 b/c it doesn't generalize well.

$X = \mathbb{P}^n_k$. $\Omega_{X/k}$ Kähler diff's, $\omega_X = \lambda^n \Omega_{X/k}$ (canonical sheaf)

Lemma $X = \mathbb{P}^n_k$. $\omega_X \cong \mathcal{O}_X(-n-1)$.

Proof. $0 \rightarrow \Omega_{X/k} \rightarrow (\mathcal{O}_X(-1))^{\oplus n+1} \rightarrow \mathcal{O}_X \rightarrow 0$.

corresponding to $\bigoplus_{i=0}^n S(-1) e_i$, $S = k[x_0, \dots, x_n]$

$$\begin{cases} S(-1)^{n+1} \rightarrow S \\ e_i \mapsto x_i \end{cases} \quad \left\{ \text{Hartshorne Thm 8.13.} \right.$$

$$\hookrightarrow 0 \rightarrow \Omega_{X/k}^i \rightarrow \Lambda_k^i (\mathcal{O}_X(-1))^{\oplus n+1} \rightarrow \Omega_{X/k}^{i-1} \rightarrow 0, \quad \forall i.$$

$$\text{For } i=n+1, \quad \Omega_{X/k}^{n+1} = 0 \Rightarrow \Lambda_k^{n+1} (\mathcal{O}_X(-1))^{\oplus n+1} = \mathcal{O}_X(-n-1) \cong \Omega_{X/k}^n.$$

Alternatively finding glob generator of $\omega_X(n+1)$.

E.g. Define $\alpha \in H^0(D^+(x_0, \dots, x_n), \omega_X)$ by

$$\alpha = \frac{x_0^{n+1}}{x_0 \cdots x_n} dx_1/x_0 \wedge \cdots \wedge dx_n/x_0$$

$x_0 \cdots x_n \alpha$ generates $\omega_{X(n+1)}$ over $D_+(x_0)$.

$$\alpha = \frac{1}{x_0 \cdots x_n} \sum_{i=0}^n (-1)^i x_i (dx_0 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n).$$

\Rightarrow performing an aut of X : swap x_i, x_j

& changes α by a sign

the same is true of the aut of X

which swaps x_0, x_n

$\Rightarrow x_0 \cdots x_n \alpha$ generates $\omega_{X(n+1)}$ over $D_+(x_i)$, $i=1, \dots, n$. \square

Summary In any case, we can use ω_X to replace $\mathcal{O}(d-n)$
when $X = \mathbb{P}_k^n$ in Serre Duality.

$\hookrightarrow \text{Ext}^i(\mathcal{F}, \omega_X) \times H^{n-i}(X, \mathcal{F}) \rightarrow H^n(X, \omega_X), \quad \forall \mathcal{F} \in \text{Coh}(X).$