Fargues-Fontaine curve and vector bundles

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Recap For $S\in \mathrm{Perf}_{\overline{\mathbb{F}}_p}$, define X_S relative FF curve,

$\rightsquigarrow$ $\mathcal{E}/X_S$ $\rightsquigarrow$ $\mathcal{BC}(\mathcal{E}) : T/S \longmapsto H^0(X_T, \mathcal{E}|_{X_T})$

(Banach-Colmez functor).

Geometry of $\mathcal{BC}(\mathcal{E})$ is crucial for study v.b. on X_S.

Take an isocrystal $D\in \mathrm{Isoc}_k$.

$\rightsquigarrow$ $S\in \mathrm{Perf}_{\overline{\mathbb{F}}_p}$, $\mathcal{O}(D)/X_S$

$\rightsquigarrow$ $\mathcal{BC}(D) : S \longmapsto$ "$H^{0(1)}(X_S, \mathcal{O}_{X_S}(D))$" absolute BC-space

$\rightsquigarrow$ $\mathcal{BC}(D)\setminus\{0\} : S \longmapsto H^{0(1)}(X_S, \mathcal{O}_{X_S}(D))\setminus\{0\}$ punctured ver.

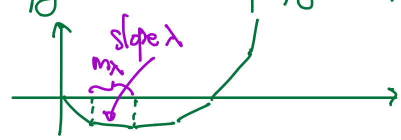
§1 Cohomology of vector bundles on relative FF curves

Prop (A little strength of Kedlaya-Liu)

Fix $S\in \mathrm{Perf}_{\overline{\mathbb{F}}_p}$, $\mathcal{E}$ v.b. over X_S s.t. $\boxed{\mathrm{NH}}(\mathcal{E}_x)\equiv 0$ everywhere.

C curve. $\mathcal{E}$ (over X_C) $\simeq \bigoplus \mathcal{O}(\lambda)^{m_\lambda}$ by classification theorem

$\rightsquigarrow$ NH polygon (Newton polygon of $\mathcal{E}$)



Then analytic locally on S, $\exists$ an exact sequence of vbs

$$0 \to \mathcal{O}_{X_S}(-1)^d \to \mathcal{F} \to \mathcal{E} \to 0$$

s.t. $\mathcal{F}$ is ss of deg 0 everywhere.

Proof $S=\mathrm{Spa}(R,R^+)$, assumed to be of $\deg = d$

$\mathrm{rank}\,\mathcal{E} = \mathrm{const}$ on S.

Pick d untilts of S

$\rightsquigarrow S_i^\# = \mathrm{Spa}(R_i^\#, R_i^{+\#}) \hookrightarrow X_S$

Denote $W_i := \mathcal{E}|_{S_i^\#}$, then

$$0 \to \mathcal{O}_{X_S}(-1)^d \to \mathcal{O}_{X_S}^d \to \bigoplus_{i=1}^d \mathcal{O}_{S_i^\#} \to 0 \quad \text{exact.}$$

Pull-back to get

$$\mathcal{E} \to \bigoplus_{i=1}^d \mathcal{E} \otimes_{\mathcal{O}_{X_i}} \mathcal{O}_{S_i^\#} \simeq \bigoplus_{i=1}^d W_i \otimes_{R_i^\#} \mathcal{O}_{S_i^\#} \to \bigoplus_{i=1}^d \mathcal{O}_{S_i^\#}$$

take a rank 1 quotient for each i.

$$\Rightarrow \quad 0 \to \mathcal{O}_{X_S}(-1)^d \to \mathcal{E}' \to \mathcal{E} \to 0.$$

<u>Claim</u> Can choose rank 1 quotients of W_i s.t.

$\mathcal{E}'$ is s.s. of slope 0 everywhere.

It reduces equivalently to

find $\mathcal{E}_i$, $i = 1, \cdots, d$ s.t.

$$0 \to \mathcal{O}_{X_S}(-1)^i \to \mathcal{E}_i \to \mathcal{E} \to 0$$

where $\forall i$. slope of $\mathcal{E}_i \geq 0$.

Equivalently, it's enough to find $\mathcal{E}_1$ only.

$\Leftarrow$ semicontinuity of Kedlaya-Liu on $S = \mathrm{Spa}(K, K^+)$.

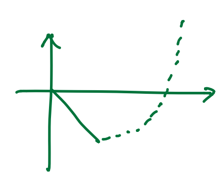
- $\mathcal{E}$ has a natural filtration on slopes.

$\mathcal{E}^{\geq \lambda} \subseteq \mathcal{E}$ of maximal slope λ ($\lambda > 0$).

$$\begin{tikzcd}
\rightsquigarrow 0 \arrow[r] & \mathcal{O}_{X_S}(-1) \arrow[r] & \mathcal{E}_1 \arrow[r] & \mathcal{E} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_{X_S}(-1) \arrow[r] \arrow[u, equal] & \mathcal{G} \arrow[r] \arrow[u] & \mathcal{E}^{\geq\lambda} \arrow[r] \arrow[u, "\text{pull back}"'] & 0
\end{tikzcd} \qquad (*)$$

Then $\mathcal{E}_1$ has slopes $\geq 0 \iff \mathcal{G}$ has slopes ≥ 0.

$\iff (*)$ is non-split.



Consider

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(-1) \arrow[r] & \mathcal{G} \arrow[r] \arrow[dd] & \mathcal{E}^{\geq\lambda} \arrow[r] \arrow[l, bend left] \arrow[ddl, "0"'] \arrow[dd, "0"] \arrow[dr] & & 0 \\
& & & & W_1' \arrow[dl, "0"] & \\
0 \arrow[r] & \mathcal{O}_X(-1) \arrow[r] & \mathcal{O}_{X_S} \arrow[r] & \mathcal{O}_{S_i^\#} \arrow[rr] & & 0
\end{tikzcd}$$

$(\lambda > 0)$

where $W_1' := \mathcal{E}^{\geq\lambda}|_{S_i^\#}$.

We choose $W_1 \xrightarrow{f} R_1^\#$ s.t. $f|_{W_1'} \neq 0$. $\square$

<u>Prop</u> Fix $S \in \mathrm{Perf}_{\bar{\mathbb{F}}_p}$. $\mathcal{E}/X_S$ s.t. slope of $\mathcal{E} > 0$ everywhere.

Then $\underbrace{\text{étale locally}}_{\text{can be strengthen indeed}}$ on S,

$$0 \to \mathcal{G} \to \mathcal{O}_{X_S}^m \to \mathcal{E} \to 0 \quad \text{exact}$$

& $\mathcal{G}$ is s.s. of slope -1 everywhere

<u>Proof</u> $\mathcal{E}$ constant of deg d, rank r, $m = d + r$.

Consider $\mathcal{BC}(\mathcal{E})^m = \mathcal{H}om_{\mathcal{O}_{X_S}}(\mathcal{O}_{X_S}^m, \mathcal{E})$

$\qquad\qquad \cup\!|$

$\qquad\qquad U = \mathrm{Surj}_{\mathcal{O}_{X_S}}(\mathcal{O}_{X_S}^m, \mathcal{E})$ space of surj morphisms.

<u>Claim</u> U is an open subdiamond.

pf. $\mathcal{F} \to \mathcal{G}$ of small v-sheave, $\mathcal{G}$ diamond, open

if $\forall T \in \mathrm{Perf}_{\bar{\mathbb{F}}_p}$, $T \to \mathcal{G}$,

$\mathcal{F} \times_{\mathcal{G}} T$ is rep'ble by perf'd space T'

s.t. $T' \hookrightarrow T$ open immersion

Then over X_T, $\exists$ universal $\mathrm{univ}_T : \mathcal{O}_{X_T}^m \to \mathcal{E}|_{X_T}$.

$\mathrm{Supp}(\mathrm{Coker}(\mathrm{univ}_T))$ is closed $\subseteq |X_T|$

$$|X_T| \simeq |\mathrm{Div}^1 \times T| \longrightarrow |T|,$$

$$\begin{tikzcd}
{|X_T|} \arrow[r, bend right, "\text{closed}"'] & {|T|}
\end{tikzcd}$$

Recall $\mathrm{Div}^1 \to *$ is proper.

$\Rightarrow U$ is an open subdiamond.

with $V \subseteq U$, $\ker(V \hookrightarrow U)$ s.s. of slope -1 everywhere
and is open by semi-continuity.

The rest of proof is done by some analytic arguments.
(To find étale opens of S in application.) $\square$

Rmk A variant of this: [FS] Cor II.3.3.
If $\mathcal{E}$ has slope $> \frac{1}{r}$ everywhere,
then étale-locally on S,
$$0 \to \mathcal{G} \to \mathcal{O}_{X_S}(\tfrac{1}{r})^n \to \mathcal{E} \to 0$$
s.t. $\mathcal{G}$ has slope $= \frac{1}{2r}$ everywhere.
(pf: Apply Prop'n to $\pi_{2r}^* \mathcal{E}(-2)$.)

Prop'n $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_p}$.
(1) If (slope of $\mathcal{E}$) < 0 everywhere,
then $H^0(X_S, S) = 0$.
(2) If (slope of $\mathcal{E}$) ≥ 0 everywhere,
then $\exists\, \tilde{S} \to S$ pro-étale s.t. $H^1(X_{\tilde{S}}, \mathcal{E}|_{X_{\tilde{S}}}) = 0$.
(3) If (slope of $\mathcal{E}$) > 0 everywhere,
then $\exists\, S' \to S$ étale, s.t. $T \to S'$, T affinoid,
and $H^1(X_T, \mathcal{E}|_{X_T}) = 0$.

Proof (1) $H^0(X_S, \mathcal{E}) \neq 0 \Rightarrow H^0(X_C, \mathcal{E}|_{X_C}) \neq 0$.
Reduces to $S = \mathrm{Spa}(C, C^+)$, the classical [FF] case.
(2) Have $0 \to \mathcal{O}(-1)^n \to \mathcal{G} \to \mathcal{E} \to 0$
where $\mathcal{G}$ is semi-stable of slope -1.
$\exists\, \tilde{S} \to S$ pro-étale, $\mathcal{G} \simeq \mathcal{O}^{n'}$ over $\tilde{S}$,

and $H^1(X_S, \mathcal{O}_{X_S}) \simeq H^1_{\text{proét}}(S, \underline{E}) = 0$ vanishes

$\underline{E}$ = a proét local system

(3) $0 \to \mathcal{G} \to \mathcal{O}_{X_S}(\tfrac{1}{r})^m \to \mathcal{E} \to 0$ étale locally on S

$\Rightarrow H^1(X_T, \mathcal{O}_{X_T}(\tfrac{1}{r})^m) = 0$ when T is affinoid. □

§2 Family of Banach-Colmez spaces

$[\mathcal{E}_1 \to \mathcal{E}_0] / X_S$

$\rightsquigarrow \mathcal{BC}([\mathcal{E}_1 \to \mathcal{E}_0]) := (T \in \mathrm{Perf}_S \mapsto \mathbb{H}^0(X_T, [\mathcal{E}_1 \to \mathcal{E}_0]_{X_T}))$.

Prop (1) $\mathcal{BC}([\mathcal{E}_1 \to \mathcal{E}_0])$ is a locally spatial diamond & partially proper $/S$.

(2) The projectivized BC space

$$(\mathcal{BC}([\mathcal{E}_1 \to \mathcal{E}_0]) \setminus \{0\}) / \underline{E}^\times$$

is a locally spatial diamond & proper $/S$.

Proof S affinoid. $\mathcal{O}_{X_S}^m(d) \twoheadrightarrow \mathcal{E}_0$.

$\mathcal{E}_1' := \ker(\mathcal{O}_{X_S}^m(-d) \oplus \mathcal{E}_1 \to \mathcal{E}_0)$

Then $[\mathcal{E}_1' \to \mathcal{O}_{X_S}^m(-d)] \xrightarrow{f} [\mathcal{E}_1 \to \mathcal{E}_0] \to \mathrm{Cone}(f) = 0$.

Moreover, $\mathcal{E}_1'$ has slope <0 everywhere.

$[\mathcal{E}_1' \to \mathcal{O}_X(-d)^m] \to \mathcal{E}_1^{\circ}[1] \to \mathcal{O}_X(-d)^m[1] \xrightarrow{+1} \cdots$ dist triangle.

$\Rightarrow 0 \to \mathcal{BC}([\mathcal{E}_1' \to \mathcal{O}_X(-d)^m]) \to \underset{\mathcal{G}}{\underset{\parallel}{\mathcal{BC}(\mathcal{E}_1^{\circ}[1])}} \to \underset{\mathcal{H}}{\underset{\parallel}{\mathcal{BC}(\mathcal{O}_X(-d)^m[1])}}$

(Δ is closed) separated.

$$\Rightarrow \begin{tikzcd} \mathcal{H} \arrow[r, "\Delta"', "\text{closed}"] & \mathcal{H} \times_S \mathcal{H} \\ S \arrow[u, "e", "\text{closed}"'] \arrow[r, "e"] & \mathcal{H} \arrow[u] \end{tikzcd} \longleftrightarrow \begin{tikzcd} \ker \arrow[r] \arrow[d, "\text{closed}"'] & S \arrow[d, "e"] \\ \mathcal{G} \arrow[r] & \mathcal{H} \end{tikzcd}$$

Reduce to prove for $\mathcal{BC}(\mathcal{E}_1'[1])$. □

§3 Punctured absolute BC space

Div^d : $D \subseteq X_S$ closed Cartier divisor.

$\iff \mathcal{I}$ line subbundle $\hookrightarrow \mathcal{O}_{X_S}$ with closed image

$\rightsquigarrow$ locally closed on S.

$\mathcal{I} \simeq \mathcal{O}_{X_S}(-d) \hookrightarrow \mathcal{O}_{X_S}$ $(d>0)$

Construct Div^d "$:=$" $(\mathcal{BC}(\mathcal{O}(d)) \setminus \{0\}) / E^\times \xrightarrow{\text{proper}} *$.

Prop $(\mathrm{Div}^1)^d \longrightarrow \mathrm{Div}^d$

$(D_1, \cdots, D_d) \longmapsto D_1 + \cdots + D_d$ induces $(\mathrm{Div}^1)^d / \Sigma_d \longrightarrow \mathrm{Div}^d$

$\Rightarrow \mathrm{Div}^d$ is a diamond.

Proof $(\mathrm{Div}^1)^d \to (\mathrm{Div}^1)^d / \Sigma_d$ is quasi-pro-étale

in the sense of Scholze.

$\Rightarrow (\mathrm{Div}^1)^d / \Sigma_d$ is a diamond.

Hard part is to show

$(\mathrm{Div}^1)^d / \Sigma_d \simeq \mathrm{Div}^d$ as v-sheaves

(while the injectivity is easy).

- $(\mathrm{Div}^1)^d$ & Div^d are proper of $*$.

$\Rightarrow (\mathrm{Div}^1)^d \to \mathrm{Div}^d$ is proper $\Rightarrow$ quasi-compact.

(Lemma 17.4.9 from Scholze's Berkeley notes).

$\mathcal{G} \to \mathcal{F}$ q.c. morphism of small v-sheaves

$\forall T = \mathrm{Spa}(C, C^+)$, $\mathcal{G}(T) \twoheadrightarrow \mathcal{F}(T)$.

$\Rightarrow \mathcal{G} \twoheadrightarrow \mathcal{F}$ surj.

The rest is the original result by FF. $\square$

Prop $D \in \mathrm{Isoc}_k$ with negative (resp. positive) slope

$$\mathcal{BC}(D) := \mathcal{BC}(\mathcal{O}(D)) \quad (\text{resp. } \mathcal{BC}(\mathcal{O}(D)[1]))$$

Then $\mathcal{BC}(D)\setminus\{0\}$ (resp. $\mathcal{BC}(D[1])\setminus\{0\}$) is a spatial diamond.

Proof Spatialness

$\mathcal{BC} = \mathcal{BC}(D)$ or $\mathcal{BC}(D[1])$ $\rightsquigarrow$ $\mathcal{BC}\setminus\{0\}$ def'd over $\mathbb{F}_q$.

$\Rightarrow$ $\mathcal{BC}\setminus\{0\} \times_{\mathbb{F}_q} \mathrm{Spa}(\mathbb{F}_q((t^{1/p^\infty}))) \longrightarrow \mathrm{Spa}(\mathbb{F}_q((t^{1/p^\infty})))$

$\Rightarrow$ $\mathcal{BC}\setminus\{0\} \times_{\mathbb{F}_q} \mathrm{Spa}(\mathbb{F}_q((t^{1/p^\infty})))/\varphi^N \times id$ is a spatial diamond.

Note : $(\varphi \times \varphi)^N \curvearrowright |\mathcal{BC}\setminus\{0\} \times_{\mathbb{F}_q} \mathrm{Spa}(\mathbb{F}_q((t^{1/p^\infty})))|$ trivially.

$\Rightarrow$ $|\mathcal{BC}\setminus\{0\} \times_{\mathbb{F}_q} \mathrm{Spa}(\mathbb{F}_q((t^{1/p^\infty})))/\varphi^N \times id|$ is spatial.

Lem II.3.8 $\Rightarrow$ $\mathcal{BC}\setminus\{0\}$ spatial

since $\mathrm{Spa}(\mathbb{F}_q((t^{1/p^\infty})))/(\varphi^N)^{\mathbb{Z}}$ proper & coh smooth.

$$|X_S| = |X_S^\diamond| = |S \times \mathrm{Spd}\, E/\varphi^{\mathbb{Z}} \times id|$$
$$= |S \times \mathrm{Spd}\, E/ id \times \varphi^{\mathbb{Z}}| \longrightarrow |S|$$

$\rightsquigarrow$ $\mathrm{Div}^1 := \mathrm{Spd}\, E/\varphi^{\mathbb{Z}}$.

To show it is a diamond,

reduce to D simple and of rank 1.

Then slope of $D \geq 0$.

$\mathcal{BC}(D)\setminus\{0\} \simeq \mathcal{BC}(\mathcal{O}(d))\setminus\{0\}$ is a $\underline{E}^\times$-torsor over Div^d (diamond).

So $\mathcal{BC}(D)\setminus\{0\}$ is also a diamond

when $D = (E, \pi^n\varphi)$, $n > 0$.

Then $D := \mathcal{BC}(D[1])\setminus\{0\}$ classifies ext'ns

$$0 \to \mathcal{O}_{X_S}(-n) \to \mathcal{E} \to \mathcal{O}_{X_S} \to 0$$

that is not split.

Fiberwise, $\mathcal{E} \simeq \begin{cases} \mathcal{O}_{X_S}(-n+i) \oplus \mathcal{O}_{X_S}(-i) & 0 < i \leq \frac{n}{2} \\ \mathcal{O}_{X_S}(-\frac{n}{2}), & \text{otherwise.} \end{cases}$

Via NH strata, $D_\alpha \hookrightarrow D$ (highly nontrivial, so called "miracle lemma").

It reduces to show D_α is a diamond.

$\exists$ quasi-pro-étale cover $\tilde{D}_\alpha \to D_\alpha$ given by trivialize the lowest slope part of $\mathcal{E}$.

$\rightsquigarrow$ $\boxed{\tilde{D}_\alpha(S)} \to \mathcal{O}_{X_S}(-n) \to \mathcal{E} \to$ "lowest part" $= \begin{cases} \mathcal{O}_{X_S}(-n+i), & 0 < i \leq \frac{n}{2}. \\ \mathcal{O}_{X_S}(-\frac{n}{2}), & \text{otherwise} \end{cases}$

get from

$X = \mathcal{BC}(\mathcal{O}(i)) \setminus \{0\}$ or $\mathcal{BC}(\mathcal{O}(\frac{n}{2})) \setminus \{0\}$, both are diamonds.

To show the product $(\tilde{D}_\alpha \hookrightarrow) \tilde{D}_\alpha \times X$ is a diamond.

But $\tilde{D}_\alpha \longmapsto *$ is rep'ble in diamonds.

$\Rightarrow$ $\tilde{D}_\alpha$ is a diamond. $\square$