Geometric local class field theory

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(Loo-Keng Hua lectures)

Lecture 1: Classical geometric class field theory

Let X/k sm proj curve, $k = \bar{k}$ alg closed.

Thm $\quad \pi_1(X)^{ab} \simeq \pi_1(\mathrm{Jac}_X)$.

Example $\quad X/\mathbb{C}$, $\quad \mathrm{Jac}_X^{an} \simeq \underbrace{H^1(X, \mathcal{O}_X)}_{} / \Lambda$

$\qquad$ ↳ tangent space at 0 of Jac_X

where $\Lambda = \pi_1(\mathrm{Jac}_X^{an}) \subset H^1(X, \mathcal{O}_X)$

$$\begin{array}{ccc} \pi_1(\mathrm{Jac}_X^{an}) & \subset & H^1(X,\mathcal{O}_X) \\ \| & & \Big| \wr \text{ Serre duality} \\ \pi_1(X^{an})^{ab} & & \\ \| & & \\ H_1(X^{an}, \mathbb{Z}) & \subset & H^0(X, \Omega^1_X)^* \end{array}$$

This phenomenon has a point of view from geom Langlands:

$\mathcal{E}/X$ rk 1, étale $\bar{\mathbb{Q}}_\ell$-local system.

⇝ For $d \geq 1$, $\quad \mathrm{Div}^d = X^d / S_d$

$\qquad$ ↳ Hilb sch of deg d effective divisors $/X$.

So $\quad \mathrm{Div}^1 = X \quad$ as schemes.

$$\left(\text{c.f.} \quad \begin{array}{rcl} \Sigma : X^d & \longrightarrow & \mathrm{Div}^d \\ (x_1, \dots, x_d) & \longmapsto & \sum_{i=1}^d [x_i] \end{array} \quad \text{symmetrization morphism.} \right)$$

$\rightsquigarrow$ Define $\quad \mathrm{Pic} = \coprod_{d\in\mathbb{Z}} \mathrm{Pic}^d$

$\llcorner$ coarse moduli space of line bdls on X.

$(\mathrm{Pic}^0 = \mathrm{Jac}_X)$.

Abel-Jacobi map $\quad AJ^d : \mathrm{Div}^d \longrightarrow \mathrm{Pic}^d$

$$D \longmapsto \mathcal{O}(D).$$

<u>Construction</u> For $\mathcal{E}$ line bdl $/X$,

$$\rightsquigarrow \quad \mathcal{E}^{(d)} := (\underbrace{\Sigma^d_* \mathcal{E}^{\boxtimes d}})^{S_d}$$

S_d-equiv loc sys on Div^d

This $\mathcal{E}^{(d)}$ = rk 1 $\bar{\mathbb{Q}}_\ell$-loc sys on Div^d.

<u>Rmk</u> If rk $\mathcal{E} > 1$ in general,

$\Sigma^d_* \mathcal{E}^{\boxtimes d}$ is a sheaf w/ abstract action of S_d on Div^d

$\mathcal{E}^{(d)}$ is a perverse sheaf on Div^d

<u>Facts</u> (1) For X/k alg var with char $k = p > 0$, $\underline{\underline{d \geq 2}}$,

$\pi_1(X^d/S_d) = \pi_1(X)^{ab}$ (see SGA1, Ch. IX, Rem 5.8).

(2) For X CW complex, $\underline{\underline{d\geq 2}}$.

$\pi_1(X^d/S_d) = \pi_1(X)^{ab}$

$\rightsquigarrow$ <u>Slogan</u> Symmetrization abelianizes the π_1.

For $d > 2g_X - 2$, AJ^d is a locally trivial fibration in $\mathbb{P}^{d-g_X}$

$\llcorner$ genus of X $\qquad$ simply connected)

$\Rightarrow$ $\mathcal{E}^{(d)}$ descends to a rk 1 $\bar{\mathbb{Q}}_\ell$-loc sys $\mathcal{F}^{(d)}$ on Pic^d.

i.e. $\mathcal{E}^{(d)} = (AJ^d)^* \mathcal{F}^{(d)}$.

Let $\mathcal{F}^{>2g-2} = \coprod_{d>2g-2} \mathcal{F}^{(d)}$ on $\mathrm{Pic}^{>2g-2} = \coprod_{d>2g-2} \mathrm{Pic}^d$.

Then $\mathcal{F}^{>2g-2}$ is equivariant on the monoid $\mathrm{Pic}^{>2g-2}$.

$\rightsquigarrow$ $m : \mathrm{Pic}^{2g-2} \times \mathrm{Pic}^{2g-2} \longrightarrow \mathrm{Pic}^{2g-2}$

s.t. $m^* \mathcal{F}^{>2g-2} \simeq \mathcal{F}^{>2g-2} \boxtimes \mathcal{F}^{>2g-2}$

+ Compatibility relations.

<u>Fact</u> This monoid $\mathrm{Pic}^{>2g-2}$ generates the grp Pic.

$\Rightarrow$ $\mathcal{F}^{>2g-2}$ extends naturally to a rk 1 equivariant $\bar{\mathbb{Q}}_\ell$-loc sys on Pic.

$\rightsquigarrow$ Take $\mathcal{F} = \coprod_{d\in\mathbb{Z}} \underbrace{\mathcal{F}^{(d)}}_{\text{on } \mathrm{Pic}^d}$ on $\mathrm{Pic} = \coprod_{d\in\mathbb{Z}} \mathrm{Pic}^d$.

<u>Note</u> $\mathcal{F}^{(0)}$ = rk 1 $\bar{\mathbb{Q}}_\ell$-loc sys on $\mathrm{Pic}^0 = \mathrm{Jac}_X$.

<u>En résumé</u>

$$\begin{tikzcd}
\mathcal{E} \arrow[r, squiggly, "\text{symm powers}"] \arrow[d, dashed] & \mathcal{E}^{(d)} \arrow[d, dashed, "\text{descents via } AJ^d"] \\
\mathcal{F}^{(0)} & \mathcal{F}^{(d)} \arrow[l, squiggly, "\text{extension}"', "\text{via grp law on Pic}"]
\end{tikzcd}$$

for $d \gg 0$

<u>Rmk</u> Let $\mathcal{P}ic$ = the Picard stack

$\rightsquigarrow$ $\mathcal{P}ic \longrightarrow \mathrm{Pic}$ is a $\mathbb{G}_m$-gerbe.

connected

(this is trivial after finding a k-rat'l pt, so $\mathcal{P}ic \simeq [\mathrm{Pic}/\mathbb{G}_m]$).

$\Rightarrow$ $\bar{\mathbb{Q}}_\ell$- loc sys on Pic $\xrightarrow{\sim}$ $\bar{\mathbb{Q}}_\ell$- loc sys on $\mathcal{P}ic$.

In the next lectures, in Fargues's context, this will be different:

$\mathbb{Q}_p^\times$ gerbe

└ totally disconnected

$\Rightarrow$ will have to work with $\mathcal{P}ic$ (stack) as opposed to Pic (coarse moduli).

<u>Example</u> $X/\mathbb{F}_q$ sm proper geom-conn'd curve.

The proceeding <u>geom constr'n</u> for $X_{\bar{\mathbb{F}}_q}$ is compatible w/ Frob and induces rk 1 $\bar{\mathbb{Q}}_\ell$- Weil loc systems / X

$\Big\downarrow$ arith constr'n

equiv rk 1 $\bar{\mathbb{Q}}_\ell$- Weil loc systems / Pic.

$\Big\downarrow$ trace of Frob

character $\mathrm{Pic}(\mathbb{F}_q) \longrightarrow \bar{\mathbb{Q}}_\ell^\times$

(c.f. Grothendieck's faisceau-functions dictionary).

$\mathcal{F}$ equivariant $\Rightarrow$ $\mathrm{tr}(\mathcal{F})$ character.

<u>Note</u> $\mathrm{Pic}(\mathbb{F}_q) \simeq F^\times \backslash \mathbb{A}_F^\times / \prod_v \mathcal{O}_{F_v}^\times$ for $F = \mathbb{F}_q(X)$.

So this defines dually the isom of everywhere-unram'd local CFT:

$$F^\times \backslash \mathbb{A}_F^\times / \prod_v \mathcal{O}_{F_v}^\times \simeq W^{ab}_{F^{ur}/F}.$$

- F^{ur}/F max unram in F^{sep},
- Weil grp def'd by

$$\begin{tikzcd} W_{F^{ur}/F} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \pi_1(X) \arrow[d] \\ \mathrm{Frob}^{\mathbb{Z}} \arrow[r, hook] & \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \end{tikzcd}.$$

To deduce the isom: use $A/\mathbb{F}_q$ ab var

with $\quad W_A^{ab} \simeq \underbrace{A(\mathbb{F}_q)}_{\text{Lang}} \times \mathbb{Z}$.

In fact, $\quad \pi_1(A_{\overline{\mathbb{F}}_q}) \simeq \prod_{\ell \neq p} T_\ell(A)$

$$\Rightarrow \quad W_A^{ab} \simeq \underbrace{\mathrm{Coker}\Big(\prod_{\ell\neq p} T_\ell(A) \xrightarrow{F-\mathrm{Id}} \prod_{\ell\neq p} T_\ell(A)\Big)}_{A(\mathbb{F}_q)} \times \mathbb{Z}.$$

Next, prove local CFT:

$$E^\times \xrightarrow{\sim} W_E^{ab}$$

for $[E:\mathbb{Q}_p] < \infty$ or $E = \mathbb{F}_q((\pi))$,

by proving that for $d \gg 0$,

$$\mathrm{Div}^d \xrightarrow{AJ^d} \mathrm{Pic}^d$$

Picard stack of deg d line bdls on the curve

is a pro-étale locally trivial fibration in simply connected diamonds.

Lecture 2: The fundamental curve of p-adic Hodge theory

(Defined & studied by Fargues-Fontaine.)
Will replace the proper sm curve X before to geometrize the CFT for local field E.

$$E \begin{cases} \text{either } \mathbb{F}_q((\pi)) \\ \text{or } [E:\mathbb{Q}_p]<\infty, \ \mathcal{O}_E/\pi = \mathbb{F}_q. \end{cases}$$

Let $F/\mathbb{F}_q$ perfectoid field
(i.e. Complete w.r.t. nontriv $|\cdot|: F \to \mathbb{R}_+$ + perfect)
e.g. $F = \mathbb{F}_q((T^{1/p^\infty}))$, $\widehat{\overline{\mathbb{F}_q((T))}}$.

Define holomorphic fcts with variable π & coeffs in F.
(1) equal char: $E = \mathbb{F}_q((\pi))$

$\rightsquigarrow$ $Y_F := \underbrace{\mathbb{D}_F^*}_{\text{punctured unit disk } /F} = \{0<|\pi|<1\} \subset \mathbb{A}^1_F$

as an adic space / $\mathrm{Spa}\, E$.

$\rightsquigarrow$ $\mathcal{O}(Y_F) = \{\sum_{m\in\mathbb{Z}} a_m \pi^m \mid a_m \in F,\ \forall \rho \in (0,1),\ \lim_{|m|\to\infty} |a_m|\cdot \rho^m = 0\}$.

Get

$$\begin{tikzcd} & \mathbb{D}_F^* \arrow[dl] \arrow[dr] & \\ \mathrm{Spa}\, F & & \mathbb{D}_{\mathbb{F}_q}^* = \mathrm{Spa}\, E \end{tikzcd}$$

$\triangleq$ $E = \mathbb{F}_q((\pi)) \subset \mathcal{O}(Y_F)$

standard str morphism

the structural morph we are interested in (not locally of fin type)

(2) mixed char: $E/\mathbb{Q}_p$ finite,

$$\mathbb{A} = W_{\mathcal{O}_E}(\mathcal{O}_F) = W(\mathcal{O}_F) \otimes_{\mathbb{F}_q} \mathcal{O}_E \qquad \text{(Fontaine's ring } A_{\inf}\text{)}.$$
$$= \left\{ \sum_{n \geq 0} [x_n] \pi^n \;\middle|\; x_n \in F \right\}.$$

Take pseudo-unif $\varpi \in F$ s.t. $0 < |\varpi| < 1$.

$$\rightsquigarrow \mathbb{A}\left[\tfrac{1}{\pi}, \tfrac{1}{[\varpi]}\right] = \left\{ \sum_{n \gg -\infty} [x_n] \pi^n \;\middle|\; x_n \in F,\ \sup_n |x_n| < +\infty \right\}.$$

holo fcts that are mero along (π) & $([\varpi])$.

<u>Def'n</u> For $\rho \in (0,1)$, $\left| \sum_n [x_n] \pi^n \right|_\rho := \sup_n |x_n| \rho^n$,

$|\cdot|_\rho$ = Gauss norm of radius ρ.

$\rightsquigarrow$ $\mathcal{O}(Y_F) :=$ Frechet E-algebra completion of $\mathbb{A}\left[\tfrac{1}{\pi}, \tfrac{1}{[\varpi]}\right]$ w.r.t. $(|\cdot|_\rho)_{\rho \in (0,1)}$.

So define $Y_F := \mathrm{Spa}(\mathbb{A}, \mathbb{A}) \setminus V(\pi[\varpi])$.

equipped with $(\pi, [\varpi])$-adic top.

E-adic space, "Stein",

completely determined by $\mathcal{O}(Y_F)$.

(The fact this is sheafy is a thm.).

<u>Rmk</u> When $E = \mathbb{F}_q((\pi))$,

$$Y_F = \mathrm{Spa}\, F \times_{\mathrm{Spa}\, \mathbb{F}_q} \mathrm{Spa}\, E \qquad \text{categorical product.}$$

When $E/\mathbb{Q}_p$, can give a meaning to this

$$Y_F^\diamond = \mathrm{Spa}\, F \times_{\mathrm{Spa}\, \mathbb{F}_q} (\mathrm{Spa}\, E)^\diamond$$

via Scholze's theory of diamonds.

The adic curve

$\varphi = \mathrm{Frob}_F$ induces an automorphism of Y_F

via $\varphi\left(\sum_m [r_m]\pi^m\right) = \sum_m [r_m^q]\pi^m$.

Note $\varphi \curvearrowright Y_F$ properly discontinuous without fixed pt.

For $\rho \in (0,1)$, $|\varphi(f)|_\rho = |f|^q_{\rho^{1/q}}$.

$\Rightarrow$ $\varphi(\text{annulus } \{|\pi| = \rho\}) = \text{annulus } \{|\pi| = \rho^{1/q}\}$.

Def'n $X_F := Y_F / \varphi^{\mathbb{Z}}$ quasi-cpt $\underbrace{\text{E-adic space}}$

(but not of finite type / $\mathrm{Spa}\, E$).

Classical points

Def'n $\xi = \sum_{n \geq 0} [x_n]\pi^n \in \mathbb{A}$ is $\underline{\text{primitive of deg } d \geq 1}$

if $x_0 \neq 0$, $x_0, \cdots, x_{d-1} \in \mathfrak{m}_F$ & $x_d \in \mathcal{O}_F^\times$.

(deg d $\times$ deg d' = deg $d+d'$).

This notion comes from Weierstrass factorization theory.

Thm $\xi \in \mathbb{A}$ irred & primitive of deg d.

(1) $K = \mathcal{O}(Y_F)/\xi$ is a perf'd field / E

with $F \hookrightarrow K^\flat$, $x \longmapsto ([x^{p^{-n}}] \bmod \xi)_{n \geq 0}$

satisfying $[K^\flat : F] = d$.

(2) If $F/\mathbb{F}_q$ alg closed, then $d = 1$.

($\leadsto$ Weierstrass factorization: $\forall\ \xi$ primitive,

$$\xi = \underbrace{u}_{\in A^\times} \times (\pi - [a_1]) \times \cdots \times (\pi - [a_d])$$

not unique if $E/\mathbb{Q}_p$.)

(3) For any $I = [\rho_1, \rho_2] \subset (0,1)$ compact,

let Y_I be the compact annulus $\{|\pi| \in I\}$.

Then $\mathcal{O}(Y_I)$ is a PID with

$$\mathrm{Spm}(\mathcal{O}(Y_I)) = \{\xi \text{ irred prim s.t. } \|\xi\| \in I\} / A^\times.$$

where $\|\xi\| = |x_0|^{1/d}$.

Also have good theory of Newton polygons for elts of $\mathcal{O}(Y_F)$.

<u>Def'n</u> The classical Tate pts:

$$|Y_F|^{cl} := \{V(\xi) \mid \xi \text{ prim irred}\} \subset |Y_F|.$$

$$|X_F|^{cl} := |Y_F|^{cl} / \varphi^{\mathbb{Z}}.$$

<u>The schematical curve</u>

Let $\mathcal{O}(1)$ line bdl $/ X_F$,

trivial on Y_F w.r.t. automorphy factor $\pi^{-1}\varphi$.

(So $H^0(X_F, \mathcal{O}(1)) = H^0(Y_F, \mathcal{O})^{\varphi = \pi}$).

<u>Fact</u> For $d \in \mathbb{Z}$, $H^0(X_F, \mathcal{O}(d)) = \begin{cases} 0, & d < 0 \\ E, & d = 0, \\ \mathcal{O}(Y_F)^{\varphi = \pi^d}, & d > 0. \end{cases}$

When $d>0$, this is an ∞-dim'l Banach E-space.

<u>Def'n</u> $\mathcal{X}_F = \mathrm{Proj}\left(\underbrace{\bigoplus_{d\geq 0} \mathcal{O}(Y_F)^{\varphi=\pi^d}}_{\text{graded alg of Fontaine's periods.}}\right)$ scheme $/E$.

Again, $\mathcal{X}_F/E$ is <u>not</u> of finite type.

<u>Thm</u> (1) $\mathcal{X}_F$ is a Dedekind scheme ("curve").

(2) $\exists$ morphism of ringed spaces

$$X_F \longrightarrow \mathcal{X}_F \quad \text{(of GAGA type)}$$

inducing $|X_F|^{cl} \xrightarrow{\sim} |\mathcal{X}_F| =$ closed pts.

s.t. if $\tilde{x} \longmapsto x$ along GAGA map above,

then $k(\tilde{x}) = k(x)$ perfectoid field $/E$,

& $\widehat{\mathcal{O}}_{\mathcal{X}_F,x} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X_F,\tilde{x}} = \underbrace{B_{dR}^+(k(x))}_{\text{Fontaine's period ring.}}$ DVR

(3) $\forall f \in E(\mathcal{X}_F)^\times$, $\deg(\mathrm{div} f) = 0$.

Here for $x \in X_F$, $\deg(x) := [k(x)^\flat : F]$.

<u>Slogan</u> "The curve is complete".

$\rightsquigarrow$ good notion of degree of a line bdl

+ Harder–Narasimhan fil'n for v.b. $/X_F$.

(4) Have a moduli interpretation:

$$|\mathcal{X}_F|^{\deg=1} \xrightarrow{\sim} \{\text{untilts of } F \text{ over } E\} / \mathrm{Frob}_F$$
$$x \longmapsto k(x) + \text{isom } F \xrightarrow{\sim} k(x)^\flat.$$

(5) If F alg closed, then

$$\forall\, x \in |\mathcal{X}_F|, \quad \deg x = 1.$$

$$\rightsquigarrow \quad H^0(\mathcal{O}(1)) \setminus \{0\} \xrightarrow{\sim} |\mathcal{X}_F| \quad (\text{"}= \mathrm{Div}^1\text{"})$$
$$t \longmapsto V^+(t)$$

Fontaine's t

Moreover,

$$\mathcal{X}_F \setminus \{\infty\} = \mathrm{Spec}(\underbrace{\mathcal{O}(Y_F)[\tfrac{1}{t}]^{\varphi=\pi}}_{\text{P.I.D.}})$$

But $\quad \mathcal{X}_F \setminus \{\infty\} \triangleq (\mathcal{O}(Y_F)[\tfrac{1}{t}]^{\varphi=\pi}, -\mathrm{ord}_\infty)$

not an Euclidean domain

$\rightsquigarrow$ $H^1(X_F, \mathcal{O}(-1)) \neq 0$, contrary to $H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$.

$(\mathbb{P}^1 \triangleq (k[T], \deg))$.

Upshot The curve shares some similarities with $\mathbb{P}^1$, but very different here.

(6) GAGA: $\quad VB(\mathcal{X}_F) \simeq VB(X_F)$.

Picard grp $\cdot$ For F alg closed,

$$\deg : \underset{\langle \mathcal{O}(1) \rangle}{\overset{\|}{\mathrm{Pic}(X_F)}} \xrightarrow{\sim} \mathbb{Z}$$

$\cdot$ For any F, this generalizes to

$$\mathrm{Pic}^\circ(X_F) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Gal}(\bar{F}/F), E^\times)$$
$$[\mathcal{L}] \longmapsto \chi_{\mathcal{L}}.$$

This isom is a Sen-type descent result from $\hat{\bar{F}}$ to F.

Precisely: if $\beta : X_{\hat{\bar{F}}} \xrightarrow{\sim} X_F$,

$u : \mathcal{O}_{X_{\hat{\bar{F}}}} \xrightarrow{\sim} \beta^*\mathcal{L}$,

then $\chi_{\mathcal{L}}(\sigma) = u^{-\sigma} \circ u \in \Gamma(X_{\hat{\bar{F}}}, \mathcal{O})^\times = E^\times$.

Rmk More general Harder-Narasimhan type statement:

$$\left\{ \begin{array}{c} \text{slope 0 semi-stable} \\ \text{vect bdl} / X_F \end{array} \right\} \xrightarrow{\sim} \mathrm{Rep}_E(\mathrm{Gal}(\bar{F}/F)).$$

For F alg closed $/ \mathbb{F}_q$, E_n / E unram of deg n.

$$Y_F / \varphi^{n\mathbb{Z}} = X_{F, E_n} = X_{F,E} \otimes_E E_n$$

$$\rightsquigarrow \quad \pi_n \downarrow \mathbb{Z}/n\mathbb{Z}\text{-cover}$$

$$(\quad Y_F / \varphi^{\mathbb{Z}} = X_{F,E}.$$

nonsplit, partially cover $Y \longrightarrow Y/\varphi^{\mathbb{Z}}$.

Def'n For $\lambda = \frac{d}{n} \in \mathbb{Q}$ with $(d, n) = 1$,

$$\mathcal{O}(\lambda) := \pi_{n,*} \mathcal{O}_{X_{F,E_n}}(d)$$

(stable bdl of slope λ.

Thm (Classification of VBs)

(i) Any slope λ semi-stable v.b. $/X$ is isom to $\mathcal{O}(\lambda)^m$ for some m.

(2) The HN fil'n of a v.b. $/X$ is split.

(3) $\{\lambda_1 \geq \cdots \geq \lambda_n \mid n \in \mathbb{N}, \lambda_i \in \mathbb{Q}\} \xrightarrow{\sim} \mathrm{Bun}_X/\sim$

$$(\lambda_1, \cdots, \lambda_n) \longmapsto \left[\bigoplus_{i=1}^{n} \mathcal{O}(\lambda_i)\right].$$

Plenty of other results

(1) Classification of G-bdls on X, (G reductive grp $/E$).

$\rightsquigarrow$ link with Kottwitz set $B(G)$.

(2) F alg closed $\Rightarrow$ $X_{F,E}$ is geom simply connected,

i.e. $\mathrm{Gal}(\bar{E}/E) \xrightarrow{\sim} \pi_1(X_{F,E})$.

Moreover, if $\cdot$ M = finite ab grp with $\circlearrowleft$ $\mathrm{Gal}(\bar{E}/E)$ discrete action,

$\cdot$ $\mathcal{F}$ = associated local system on X_F,

s.t. $R\Gamma_{\acute{e}t}(E, M) \xrightarrow{\sim} R\Gamma_{\acute{e}t}(X_F, \mathcal{F})$,

then $H^2(E, \mu_n) \xrightarrow{\sim} H^2(X_F, \mu_n) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$

$\begin{pmatrix} \text{fundamental class} \\ \text{of local CFT} \end{pmatrix} \triangleq \begin{pmatrix} \eta_* = c_1(\mathcal{O}(1)) \text{ fundamental class} \\ \text{of the curve} \end{pmatrix}$

Lecture 3: The Picard stack of the curve

The relative curve

Last time $E = \mathbb{F}_q((\pi))$ or $E/\mathbb{Q}_p$ finite, $\mathbb{F}_q = \mathcal{O}_E/\pi$.
F perf'd field $/\mathbb{F}_q$.
$\rightsquigarrow$ X_F adic space $/\operatorname{Spa} E$.

Now $\mathrm{Perf}_{\mathbb{F}_q} \ni S \rightsquigarrow X_S$ adic space $/\operatorname{Spa} E$.
$\quad$ perf'd spaces $/\mathbb{F}_q$.

Morally, X_S = family of "curves" $(X_{k(s)})_{s\in S}$.

Construction of the curve

- If $E = \mathbb{F}_q((\pi))$,

$$\begin{array}{c} Y_S = \mathbb{D}_S^* = \{0<|\pi|<1\} \subset \mathbb{A}_S^1 \\ \circlearrowleft\ \varphi = \mathrm{Frob}_S \quad \big\downarrow \\ \mathbb{D}_{\mathbb{F}_q}^* = \operatorname{Spa} E \end{array}$$

$\rightsquigarrow$ $X_S := Y_S/\varphi^{\mathbb{Z}}$.

- If $E/\mathbb{Q}_p$ finite,
$S = \operatorname{Spa}(R, R^+)$ affinoid perfectoid,
$\rightsquigarrow$ $Y_S = \operatorname{Spa}(W_{\mathcal{O}_E}(R^+), W_{\mathcal{O}_E}(R^+)) \setminus V(\pi[\varpi])$.
($\varpi \in R^{\circ\circ} \cap R^\times$ pseudo-unif, i.e. top nilp unit)
$X_S = Y_S/\varphi^{\mathbb{Z}}$.

Here φ induced by

$$\sum_{n\geq 0} [x_n]\pi^n \longmapsto \sum_{n\geq 0} [x_n^q]\pi^n \quad \text{on } W_{\mathcal{O}_E}(R^+).$$

It stabilizes the subset $V(\pi[\varpi])$.

Can glue this constr'n for any $S \in \mathrm{Perf}_{\mathbb{F}_q}$ $\rightsquigarrow$ X_S.

<u>Rmk</u> There is NO structural morphism

$$\underset{\text{char } 0 \text{ when } E/\mathbb{Q}_p}{X_S} \not\longrightarrow \underset{\text{always char } p.}{S} \quad \text{of adic spaces}$$

<u>But</u> Nevertheless, can construct a conti map

$$|X_S| \xrightarrow[\text{open \& closed}]{} |S|$$

(namely, "X_S/S is proper smooth".)

<u>Background on the pro-étale topology</u>

<u>Goal</u> Étale cohomology of diamond.

Given $(\mathrm{Spa}(R_i, R_i^+))_{i\in I}$ filtered proj system of affinoid perf'd spaces.

Then $\varprojlim_{i\in I} \mathrm{Spa}(R_i, R_i^+) = \mathrm{Spa}(R_\infty, R_\infty^+)$

exists in Perf = cat of perf'd spaces.

with
$$R_\infty^+ = \left(\varinjlim_i R_i^+\right)^\wedge_\varpi$$
$$R_\infty = R_\infty^+\left[\tfrac{1}{\varpi}\right].$$

(ϖ = image of any p.u. $\varpi_i \in R_i^+$ for some i).

<u>Def'n</u> (1) $X \to Y$ in Perf is <u>pro-étale</u> if locally on X & Y, it is of the form $\varprojlim_{i \geq i_0} S_i \to S_{i_0}$ with étale transition morphisms.

(2) A pro-étale morphism $X \xrightarrow{f} Y$ is a <u>covering</u> if $\forall\, V \subset Y$ open & qc, $\exists\, U \subset X$ open & qc, s.t. $f(U) = V$.

This covering def'n gives rise to the pro-étale top on Perf.

<u>Example</u> $x \in X$, $\varprojlim_{x \in U} U = \mathrm{Spa}(k(x), k(x)^+) \xrightarrow{\text{pro-étale}} X$

set of generalizations of x.

(This is very different from schs: $\varprojlim_{x \in U} U = \mathrm{Spec}(\mathcal{O}_{X,x})$ if $X \in \mathrm{Sch}$).

<u>Example</u> $\exists$ fully faithful cat embedding

$$\left\{ \begin{array}{c} \text{normal rigid-analytic} \\ \text{spaces } / \mathbb{Q}_p \end{array} \right\} \hookrightarrow \left\{ \begin{array}{c} \text{Pro-étale sheaves} \\ \text{on } \mathrm{Perf}_{\mathbb{Q}_p} \end{array} \right\}.$$

Moreover, $\forall\, X \in$ LHS, $\exists\, \tilde{X}$ — perfectoid

$\downarrow$ — pro-ét cover like before

X

$\rightsquigarrow$ $\underbrace{\tilde{X} \times_X \tilde{X}}_{\text{again perf'd}} \rightrightarrows X$ is a pro-ét equiv relation.

both pro-ét

fibre product in $\widetilde{\mathrm{Perf}}$ (not in the cat of adic spaces).

$\Rightarrow$ $X = \tilde{X}/R$ as a pro-ét sheaf.

Such X = algebraic space for the pro-ét top.

<u>Notation</u> A = locally profinite set.

$$\underline{A}(S) := \mathcal{C}^0(|S|, A)$$

$\hookleftarrow$ $\underline{A}$ = Const sheaf with value A.

defines a pro-ét sheaf on Perf.

<u>Examples</u> (of $X = \tilde{X}/R$, $\tilde{X}$ perf'd).

(1) $\mathrm{Spa}(\mathbb{Q}_p) = \underbrace{\mathrm{Spa}(\mathbb{Q}_p^{cycl})}_{\text{perf'd}} / \mathbb{Z}_p^{\times}$

(2) $\mathrm{Spa}(\mathbb{Q}_p\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle)$

$$= \mathrm{Spa}(\mathbb{Q}_p\langle T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}\rangle) / \underline{\mathbb{Z}_p^{\times}(1) \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}.$$

(3) $\mathring{\mathbb{B}}^1(1,1)$

$$\begin{tikzcd} \mathring{\mathbb{B}}^1(1,1) \arrow[d, "\log"'] \\ \mathbb{A}^1_{\mathbb{Q}_p} \end{tikzcd}$$ $\mathbb{Q}_p/\mathbb{Z}_p(1)$ - pro-ét loc sys of pro-ét sheaves.

(4) $\mathfrak{X} \simeq \mathrm{Spf}(W(\overline{\mathbb{F}}_p)[[h_1, \dots, h_{n-1}]])$

Lubin-Tate space of deformations of a formal p-div grp $/\overline{\mathbb{F}}_p$ with dim 1 & ht n.

($\mathfrak{X}$ = Def (supersing ell curve $/\overline{\mathbb{F}}_p$) if $n=2$).

$$\rightsquigarrow \quad \begin{tikzcd} X_k \arrow[d] \\ \mathfrak{X}_\eta \simeq \mathring{\mathbb{B}}^{n-1} \end{tikzcd} \; GL_n(\mathbb{Z}/p^k\mathbb{Z}) \text{ - Galois cover}$$

generic fibre

Here X_k = moduli space of level-k structures on the universal deformation

$$(\mathbb{Z}/p^k\mathbb{Z})^n \xrightarrow{\sim} H^{\mathrm{univ}}[p^k]_\eta .$$

$$\begin{tikzcd}
\varprojlim_k X_k \arrow[d, "\rightsquigarrow\ GL_n(\mathbb{Z}_p)\text{-pro-ét torsor}"'] \arrow[dd, bend left, "GL_n(\mathbb{Q}_p)^1\text{-pro-ét torsor.}"] \\
\mathfrak{X}_\eta \arrow[d, "\pi_{GH}"'] \\
\mathbb{P}^{n-1}
\end{tikzcd}$$

Gross-Hopkins period map

given by the Hodge fil'n of univ deformation.

<u>Note</u> fibres of π_{GH} = Hecke orbits.

<u>The Picard stack</u>

$\mathrm{Perf}_{\mathbb{F}_p}$ + pro-ét top

⇝ $\widetilde{\mathrm{Perf}_{\mathbb{F}_p}}$ pro-ét topos.

<u>Warning</u> The final object of $\widetilde{\mathrm{Perf}_{\mathbb{F}_p}}$ = $\mathrm{Spa}\,\mathbb{F}_p$

But this is not rep'ble b/c not perf'd.

<u>Def'n</u> $S \in \mathrm{Perf}_{\mathbb{F}_q}$, $\mathcal{P}ic(S) :=$ groupoid of line bdls / X_S.

<u>Fact</u> (Not easy) $\mathcal{P}ic$ is a stack on $\mathrm{Perf}_{\mathbb{F}_q}$.

<u>Fact</u> (Not easy) Let $S \in \mathrm{Perf}_{\mathbb{F}_q}$, $\mathcal{L}$ line bdl / X_S.

Then $|S| \longrightarrow \mathbb{Z}$ is locally const.

$$s \longmapsto \deg(\mathcal{L}|_{X_{\mathrm{Spa}(k(s),k(s)^+)}})$$

$$\Rightarrow \qquad \mathrm{Pic} = \coprod_{d\in\mathbb{Z}} \underbrace{\mathrm{Pic}^d}_{\text{open \& closed substack}}.$$

As a consequence of Kedlaya-Liu, one can prove:

Thm
$$\begin{aligned} \mathrm{Pic}^d &\xrightarrow{\sim} [\mathrm{Spa}\,\mathbb{F}_q / \underline{E}^\times] \\ \mathcal{L} &\longmapsto \mathrm{Isom}(\mathcal{O}(d), \mathcal{L}) \end{aligned}$$

classifying stack of pro-ét $\underline{E}^\times$-torsors.

Note $\forall\, \mathcal{L} \in \mathrm{Pic}^d(S)$, $\exists\, \tilde{S} \to S$ pro-ét cover

s.t. $\mathcal{L}|_{X_{\tilde{S}}} \simeq \mathcal{O}(d)$ + $H^0(X_S, \mathcal{O}_{X_S}) = \underline{E}(S)$

$\Rightarrow$ $\underline{\mathrm{Aut}}(\mathcal{O}(d)) = \underline{E}^\times$ by Thm.

Thus, the coarse moduli space of Pic^d is a pt

$\Rightarrow$ no geometry on Pic^d.

Next step $d \geq 1$, Div^d sheaf on $\mathrm{Perf}_{\mathbb{F}_q}$ def'd by

$$\mathrm{Div}^d(S) = \left\{ (\mathcal{L}, u) \;\middle|\; \begin{array}{l} \mathcal{L} \in \mathrm{Pic}^d(S),\ u \in H^0(X_S, \mathcal{L}) \\ \text{s.t. } \forall s \in S,\ u|_{X_{\mathrm{Spa}(k(s), k(s)^+)}} \neq 0 \end{array} \right\} / \sim .$$

Fact (1) This is a diamond (alg space for pro-ét top)

$\leadsto$ Hilbert diamond of deg d effective divisors on the curve.

(2) Diamond formula:

$$\underbrace{(\mathrm{Spa}\,E)^\diamond}_{\text{Scholze's diamond of untilts}} / \varphi^{\mathbb{Z}} \simeq \mathrm{Div}^1.$$

$\rightsquigarrow$ Link between X_S (not def'd / $\mathbb{F}_q$)
and Div^1 / $\mathbb{F}_q$.

$\rightsquigarrow$ $\bar{\mathbb{Q}}_\ell$-loc sys / $\mathrm{Div}^1_{\mathbb{F}_p}$ $\simeq$ $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(W_E)$

$\rightsquigarrow$ To study $\Sigma^d : (\mathrm{Div}^1)^d \longrightarrow \mathrm{Div}^d$

$$((\mathcal{L}_i, u_i))_{i=1}^d \longmapsto (\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_d, u_1 \otimes \cdots \otimes u_d).$$

$\rightsquigarrow$ Study $AJ^d : \mathrm{Div}^d \longrightarrow \mathrm{Pic}^d$.

Lecture 4: Structure of AJ^d and geometric local CFT

<u>$(\mathrm{Spa}\, E)^\diamond$</u> Sheaf of untilts over E on $\mathrm{Perf}_{\mathbb{F}_q}$.

$\forall\, S \in \mathrm{Perf}_{\mathbb{F}_q}$,

$$(\mathrm{Spa}\, E)^\diamond(S) = \{ (S^\#, \iota) \mid S^\# \in \mathrm{Perf}_E,\ \iota : S \xrightarrow{\sim} S^{\#,\flat} \} / \sim.$$

(quickly said: "$S^\#$ is an untilt of S".)

$(\mathrm{Spa}\, E)^\diamond$ is a pro-ét sheaf via Scholze's equiv

$$(-)^\flat : \mathrm{Perf}_{S^\#} \xrightarrow{\sim} \mathrm{Perf}_S \quad \text{of pro-ét sites.}$$

Moreover, it induces

$$\widetilde{\mathrm{Perf}}_E \xrightarrow{\sim} \widetilde{\mathrm{Perf}}_{\mathbb{F}_q} / (\mathrm{Spa}\, E)^\diamond.$$

<u>Example</u> $E = \mathbb{Q}_p$, $\widetilde{\mathrm{Perf}}_{\mathbb{Q}_p} = \widetilde{\mathrm{Perf}}_{\mathbb{F}_p} / (\mathrm{Spa}\, \mathbb{Q}_p)^\diamond$

as char 0 pro-ét topos

($=$ localization of char p pro-ét topos).

<u>Construction of map $(\mathrm{Spa}\, E)^\diamond \to \mathrm{Div}^1$</u>

<u>An adjunction</u> $E / \mathbb{Q}_p$.

$$\text{Perfect } \mathbb{F}_q\text{-alg} \underset{(-)^\flat}{\overset{W_{\mathcal{O}_E}(-)}{\rightleftarrows}} \pi\text{-adic } \mathcal{O}_E\text{-algebras}$$

$W_{\mathcal{O}_E}(-)$ is fully faithful, and

$$\mathrm{Id} \xrightarrow{\sim} W_{\mathcal{O}_E}((-)^\flat)$$

$$x \longmapsto ([x^{p^{-n}}])_{n \geq 0}.$$

$$\theta : W_{\mathcal{O}_E}((-)^\flat) \xrightarrow{\sim} \mathrm{Id}$$

$$\Big(\sum_{n\geq 0} [x_n]\pi^n \longmapsto \sum_{n \geq 0} x_n^\# \pi^n.$$

Fontaine's θ-map (= adjunction map).

<u>Recall</u> Given R perfectoid $\mathbb{F}_q$-alg,

$$\xi = \sum_{n\geq 0}[x_n]\pi^n \in W_{\mathcal{O}_E}(R^\flat)$$

is said to be primitive of deg 1

if $x_0 \in R^{\circ\circ} \cap R^\times$, $x_1 \in R^\circ \cap R^\times$.

<u>Prop</u> (1) $R^\#$ untilt of R over E, $R = R^{\#,\flat}$.

Then $\theta : W_{\mathcal{O}_E}(R^\circ) \twoheadrightarrow R^{\#,\circ}$ is surj

with $\ker\theta = (\xi)$, ξ primitive of deg 1.

(2) Reciprocally, if $\xi \in W_{\mathcal{O}_E}(R^\circ)$ is primitive of deg 1,

then $W_{\mathcal{O}_E}(R^\circ)/(\xi)[\frac{1}{\pi}]$ = an untilt $R^\#$ of R.

<u>So</u> $(\mathrm{Spa}\, E)^\diamond(R, R^+) = \{\text{deg 1 primitive}\} / W_{\mathcal{O}_E}(R^\circ)^\times$.

Given $\xi \in \mathcal{O}(Y_{R,R^+})$ and $\times\xi : \mathcal{O}_{Y_{R,R^+}} \hookrightarrow \mathcal{O}_{Y_{R,R^+}}$

$\rightsquigarrow$ $\mathrm{Spa}(R^\#, R^{\#+}) \hookrightarrow Y_{R,R^+}$

This gives a Cartier divisor $V(\xi)$.

<u>Rmk</u>

$$\begin{array}{c} V(\pi) \\ \cup\!\!| \\ \mathrm{Spa}\,(R, R^+) \end{array} \xhookrightarrow{\pi = 0} \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \setminus V([\varpi])$$

deformation of the Cartier divisor outside $V(\pi)$ $\rightsquigarrow$ $\mathrm{Spa}(R^\#, R^{\#+}) = V(\xi)$ $\xrightarrow{\xi = 0}$ $\mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \setminus V([\varpi])$

(note: $\varpi \in R$, $\pi \in E$ $\rightsquigarrow$ $\varpi, \pi \in A = W_{\mathcal{O}_E}(R^+)$.)

For $S \in \mathrm{Perf}_{\mathbb{F}_q}$, $S^\#$ untilt of S over E,

get
$$\begin{tikzcd} S^\# \arrow[r, hook] \arrow[dr] & Y_S \arrow[d] \\ & \mathrm{Spa}\, E \end{tikzcd} \quad \overset{\text{easy}}{\rightsquigarrow} \quad \begin{tikzcd} S^\# \arrow[r, hook] \arrow[dr, hook] & Y_S \arrow[d] \\ & X_S \end{tikzcd}$$

Cartier divisor locally on S def'd by $V(\xi)$

ξ primitive of deg 1.

This defines a morphism
$$\begin{tikzcd} (\mathrm{Spa}\, E)^\diamond \arrow[r] \arrow[d] & \mathrm{Div}^1 \\ (\mathrm{Spa}\, E)^\diamond/\varphi^{\mathbb{Z}} \arrow[ur, dashed] & \end{tikzcd}$$

— factorization

φ: Frob of $(\mathrm{Spa}\, E)^\diamond$.

<u>Thm</u> $(\mathrm{Spa}\, E)^\diamond/\varphi^{\mathbb{Z}} \xrightarrow{\sim} \mathrm{Div}^1$.

<u>About Div^d</u>

<u>Thm</u> $\Sigma^d : (\mathrm{Div}^1)^d \longrightarrow \mathrm{Div}^d$

induces an isom $(\mathrm{Div}^1)^d/S_d \xrightarrow{\sim} \mathrm{Div}^d$

quotient of pro-ét sheaves.

<u>So</u> $\forall d \geq 1$, Div^d is a diamond

(alg space for pro-ét top).

<u>Pf technics</u> Use the notion of quasi-pro-ét map by Scholze.

At the end this is reduced to prove that:

if $F/\mathbb{F}_q$ alg closed, then

$$\mathrm{Div}^1(F)^d / S_d \simeq \mathrm{Div}^d(F).$$

Deduced from the following factorization thm w/ Fontaine:

<u>Thm</u> $$P = \bigoplus_{d\geq 0} \mathcal{O}(Y_F)^{\varphi = \pi^d}$$

is graded factorial w/ deg 1 irred elts.

$\rightsquigarrow$ $\forall x \in P_d \setminus \{0\}$, $x = t_1 \cdots t_d$, $t_i \in P_1$

$(t_1, \ldots, t_d)$ well-def'd up to permutation & multiplication by an elt of $E^\times$. $\square$

<u>$\overline{\mathbb{Q}}_\ell$-local systems on $\mathrm{Div}^1_{\overline{\mathbb{F}}_q}$</u>

Let $\breve{E} = \widehat{E^{ur}}$. $\rightsquigarrow$ $(\mathrm{Spa}\, E)^\diamond \times_{\mathrm{Spa}\,\mathbb{F}_q} \mathrm{Spa}\,\overline{\mathbb{F}}_q = (\mathrm{Spa}\,\breve{E})^\diamond$

$\circlearrowleft$ $\sigma = \mathrm{Frob}$ $\circlearrowleft$ $\left.\begin{matrix} \varphi_{E^\diamond} \times \mathrm{id} \\ + \\ \mathrm{id} \times \varphi_{\overline{\mathbb{F}}_q} \end{matrix}\right\} \hat{=} \sigma.$

<u>Note</u> Composition of both partial Frobenii $=$ absolute Frob σ.

$\rightsquigarrow$ acts trivially on the étale topos.

Have $\mathrm{Div}^1_{\overline{\mathbb{F}}_q} \simeq (\mathrm{Spa}\,\breve{E})^\diamond / \varphi^{\mathbb{Z}}$, "$\varphi$" $\hat{=} \sigma$.

$\Rightarrow$ $\overline{\mathbb{Q}}_\ell$-loc sys $/ \mathrm{Div}^1_{\overline{\mathbb{F}}_q}$ $= (\varphi_{E^\diamond} \times \mathrm{id})$-equiv loc sys on $(\mathrm{Spa}\,E)^\diamond \times_{\mathrm{Spa}\,\mathbb{F}_q} \mathrm{Spa}\,\overline{\mathbb{F}}_q$

$= \sigma$-equiv loc sys on $(\mathrm{Spa}\,\breve{E})^\diamond$

$= \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_E).$

The Abel-Jacobi morphism

For $d \geq 1$, $\qquad AJ^d : \mathrm{Div}^d \longrightarrow \mathcal{P}ic^d$.

Another formula for Div^d

Let $\mathbb{B}$ = pro-ét sheaf of rings on $\mathrm{Perf}_{\mathbb{F}_q}$,

s.t. $\mathbb{B}(S) = \mathcal{O}(Y_S)$

sheaf of Fontaine's type of ring.

$\rightsquigarrow \quad \mathbb{B}^{\varphi = \pi^d}$ = sheaf of relative global sections of $\mathcal{O}(d)$.

s.t. $\mathbb{B}^{\varphi=\pi^d}(S) = H^0(X_S, \mathcal{O}(d))$.

Pro-étale locally, any $\mathcal{L} / X_S$ of deg d is isom to $\mathcal{O}(d)$

$$\Rightarrow \quad \mathrm{Div}^d = (\mathbb{B}^{\varphi=\pi^d} \setminus \{0\}) / \underbrace{\underline{E}^\times}_{\underline{\mathrm{Aut}}(\mathcal{O}(d))}.$$

Thus, $\qquad AJ^d : \underbrace{(\mathbb{B}^{\varphi=\pi^d} \setminus \{0\}) / \underline{E}^\times}_{\mathrm{Div}^d} \longrightarrow \underbrace{[\mathrm{Spa}\, \mathbb{F}_q / \underline{E}^\times]}_{\mathcal{P}ic^d}$.

One deduces:

Thm $\forall d \geq 1$, AJ^d is pro-étale locally a trivial fibration in $\underbrace{\mathbb{B}^{\varphi=\pi^d} \setminus \{0\}}_{\text{diamond}}$.

Geometric Langlands

Let us admit a difficult fact:

Thm $\forall\, d \geq 2$ (resp. $d \geq 3$ if $E/\mathbb{Q}_p$),

$\mathbb{B}_{\overline{\mathbb{F}}_q}^{\varphi = \pi^d} \setminus \{0\}$ is simply connected

(i.e. any finite étale cover $\underbrace{\text{has a section}}_{\text{"splits"}}$).

Let $(\chi : W_E \longrightarrow \overline{\mathbb{Q}}_\ell^\times) \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_E)$ $(\ell \neq p)$.

Recall that $\chi \triangleq \mathcal{E}$ rk 1 $\overline{\mathbb{Q}}_\ell$-loc sys on $\mathrm{Div}^1_{\overline{\mathbb{F}}_q}$

$\rightsquigarrow$ $\mathcal{E}^{(d)} = (\Sigma^d_* \mathcal{E}^{\boxtimes d})^{S_d}$ rk 1 $\overline{\mathbb{Q}}_\ell$-loc sys on $\mathrm{Div}^d_{\overline{\mathbb{F}}_q}$

$\rightsquigarrow$ $\mathcal{F}^{(d)}$ on Pic^d for $d \gg 0$.

$\llcorner$ s.t. $\mathcal{E}^{(d)} = (AJ^{(d)})^* \mathcal{F}^{(d)}$.

Then grp law on Pic

$\rightsquigarrow$ $\mathcal{F}^{(d)}$ extends to all d's

to a local system on Pic.

In particular,

* $\mathcal{E}$ descends along AJ^1.

<u>What does it mean?</u>

$$AJ^1 : \mathrm{Div}^1_{\overline{\mathbb{F}}_q} \longrightarrow [\,\mathrm{Spa}\, \overline{\mathbb{F}}_q / E^\times\,]$$

"$\pi_1(AJ^1) : W_E \xrightarrow{\ f\ } E^\times$".

$\llcorner$ no good theory of π_1 à priori,

but makes sense dually at the level of pullback

of $\overline{\mathbb{Q}}_\ell$-local systems.

Construct $\chi_{LT} : W_E \longrightarrow \mathcal{O}_E^\times$

Lubin-Tate char assoc to $\pi \in E$.

$\rightsquigarrow$ "$\pi_L(AJ)(\tau)$" $= f(\tau) := \chi_{LT}(\tau)^{-1}\pi^{v(\tau)}$

$(\tau \bmod \pi) = \mathrm{Frob}_q^{v(\tau)}$.

Thus, $\forall\, \chi : W_E \longrightarrow \bar{\mathbb{Q}}_\ell^\times$, it factors through

$f : W_E^{ab} \xrightarrow{\sim} E^\times$ (f is easily surj)

$\rightsquigarrow$ get local CFT.

<u>Proof of simply-conn thm</u>

<u>Need</u> $B_{\bar{\mathbb{F}}_q}^{\varphi=\pi^d} \setminus \{0\}$ is simply conn when $d \geq 2$ for $E = \mathbb{F}_q((\pi))$.

Let (R, R^+) $\bar{\mathbb{F}}_q$-alg, affinoid perfectoid,

with $\|\cdot\| : R \longrightarrow \mathbb{R}_+$ multiplicative norm defining its top.

Recall $\underset{\varphi}{\overset{\circlearrowleft}{\mathcal{O}(Y_{R,R^+})}} = \left\{ \sum_{n\in\mathbb{Z}} [x_n]\pi^n \;\middle|\; x_n \in R,\ \forall\, \rho \in (0,1),\ \lim_{|n|\to\infty} \|x_n\|\rho^n = 0 \right\}$.

$$\rightsquigarrow \quad (R^{\circ\circ})^d \xrightarrow{\ \sim\ } \mathcal{O}(Y_{R,R^+})^{\varphi=\pi^d}$$

$$(x_0, \cdots, x_{d-1}) \longmapsto \sum_{i=0}^{d-1} \sum_{k\in\mathbb{Z}} x_i^{q^{-k}} \pi^{kd+i}.$$

$\Rightarrow \quad B_{\bar{\mathbb{F}}_q}^{\varphi=\pi^d} \simeq \mathrm{Spa}(\bar{\mathbb{F}}_q[\![x_0^{1/p^\infty}, \cdots, x_{d-1}^{1/p^\infty}]\!])$.

$\Rightarrow \quad B_{\bar{\mathbb{F}}_q}^{\varphi=\pi^d} \setminus \{0\} \simeq \mathrm{Spa}(\bar{\mathbb{F}}_q[\![x_0^{1/p^\infty}, \cdots, x_{d-1}^{1/p^\infty}]\!]) \setminus V(x_0, \cdots, x_{d-1})$

a perfectoid quasi-cpt space.

$$= \bigcup_{i=0}^{d-1} D(x_i) \quad \text{open perf'd ball}$$

$$\downarrow$$

$$\mathrm{Spa}\, \bar{\mathbb{F}}_q((x_i^{1/p^\infty})).$$

$\rightsquigarrow$ union of d open perf'd balls / different perf'd fields.

<u>GAGA Thm</u> A noetherian I-adic. Then

$$\text{finite étale} / \operatorname{Spec} A \setminus V(I) \simeq \text{finite étale} / \operatorname{Spa} A \setminus V(I).$$

By Elkik decomp.

$$\begin{array}{c} 2\text{-}\varinjlim_n \text{ fin ét} / \operatorname{Spa}(\overline{\mathbb{F}}_q[\![x_0^{1/p^n}, \cdots, x_{d-1}^{1/p^n}]\!]) \setminus V(x_0, \cdots, x_{d-1}) \\ \downarrow \sim \\ \text{fin ét} / \operatorname{Spa}(\overline{\mathbb{F}}_q[\![x_0^{1/p^\infty}, \cdots, x_{d-1}^{1/p^\infty}]\!]) \setminus V(x_0, \cdots, x_{d-1}). \end{array}$$

Then suffices to prove that for $d \geq 2$,

$$\operatorname{Spa}(\overline{\mathbb{F}}_q[\![x_0, \cdots, x_{d-1}]\!]) \setminus V(x_0, \cdots, x_{d-1}) \text{ is simply conn.}$$

<u>GAGA</u> This is equiv to

$$\operatorname{Spec}(\overline{\mathbb{F}}_q[\![x_0, \cdots, x_{d-1}]\!]) \setminus V(x_0, \cdots, x_{d-1}) \text{ is simply conn.}$$

Zariski – Nagata purity:

$$\begin{array}{c} \text{fét} / \operatorname{Spec}(\overline{\mathbb{F}}_q[\![x_0, \cdots, x_{d-1}]\!]) \\ \downarrow \wr \\ \text{fét} / \operatorname{Spec}(\overline{\mathbb{F}}_q[\![x_0, \cdots, x_{d-1}]\!]) \setminus V(x_0, \cdots, x_{d-1}) \end{array}$$

But $\operatorname{Spec}(\overline{\mathbb{F}}_q[\![x_0, \cdots, x_{d-1}]\!])$ simply conn (Hensel's lem). $\square$

<u>Rmk</u> When $E/\mathbb{Q}_p$, $\mathbb{B}_{\overline{\mathbb{F}}_q}^{\varphi = \pi^d} \setminus \{0\}$ is a diamond and the pf of simple connectedness is much more involved.