Notes for a Short Talk at Tianyuan Mathematical Center

ARITHMETIC OF QUADRATIC TWISTS OF ELLIPTIC CURVES

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(NOTES BY WENHAN DAI)

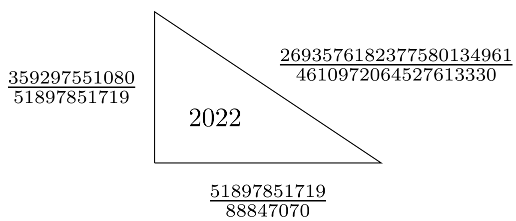
1. Motivation: the congruent number problem

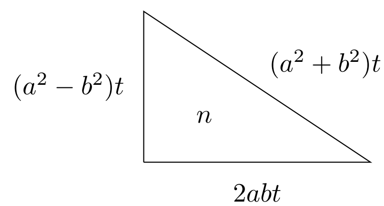
Definition 1.1 (Congruent number)**.** A positive integer is called a **congruent number** if it is the area of a right angled triangle with rational side lengths.

For example,

- 5, 6, 7 are congruent numbers (Fibonacci),
- 1, 2, 3 are non-congruent numbers (Fermat).

Example 1.2. The number 2022 is congruent with the "simplest" triangle having side lengths





For the triangle on the right hand side, require that $\gcd(a,b)=1$, $2 \nmid a+b$, and $t \in \mathbb{Q}_{>0}^{\times}$. Plugging in with $(a,b)=(5,4),(2,1),(16,9)$ outputs the triple $(5,6,7)$.

The Congruent Number Problem. The congruent number problem is to determine whether or not a given positive integer is congruent number.

Theorem 1.3 (Heegner 1952)**.** *Any positive* $q \equiv 5,6,7 \bmod 8$*, a prime or twice of a prime, is a congruent number.*

Theorem 1.4 (Tian 2012)**.** *Let* $n = qp_1 \cdots p_k$ *with odd part* n_0*, with*

- q*: as in Heegner case;*
- $p_i :\equiv 1 \bmod 8$ *distinct primes*

such that $\mathbb{Q}(\sqrt{-n_0})$ *has no ideal class of order 4. Then* n *is a congruent number.*

Theorem 1.5 (Smith, Yuan-Zhang-Tian 2014)**.** *At least half of square-free positive integers* $\equiv 5,6,7 \bmod 8$ *are congruent numbers.*

Main ingredients:

(1) Heegner points construction: complex multiplication;
(2) Gross-Zagier and Waldspurger formulae: special L-values;
(3) Positive density: imaginary quadratic fields with no order 4 ideal class;
(4) Induction methods: relation among quadratic twist Heegner points.

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1.1. **The θ-congruent number problem.** Consider triangles with rational side lengths with an angle θ fixed, called θ-rational triangles. (Note that $\cos\theta$ must be rational). The θ-congruent number problem is stated as follows.

Question 1.6. *For which integers $n, n\sin\theta$ is the area of a θ-rational triangle?*

It turns out that the problem is essentially to ask which quadratic twists

$$ny^2 = x(x-a)(x-b), \quad a, b \in \mathbb{Q}$$

have positive Mordell-Weil ranks. Here $\cos\theta = \frac{a+b}{a-b}$ if $ab < 0$ (we may always assume this). In particular, the congruent number problem is about the family

$$ny^2 = x^3 - x.$$

In general, an elliptic curve A over $\mathbb{Q}$ is a smooth curve given by

$$y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Q}.$$

The set of all the rational points (together with ∞), denoted by $A(\mathbb{Q})$, has an abelian group structure: Theorem (Mordell 1921)

$$A(\mathbb{Q}) \cong A(\mathbb{Q})_{\text{tor}} \oplus \mathbb{Z}^r, \quad r \geq 0.$$

There are two important objects to study the Mordell-Weil groups:

$$\boxed{\text{Selmer Groups}}, \qquad \boxed{L\text{-functions}}.$$

2. Selmer groups

2.1. **2-Selmer groups.** Let $A : y^2 = (x-c_1)(x-c_2)(x-c_3)$ be an elliptic curve over $\mathbb{Q}$ with $A[2] \subseteq A(\mathbb{Q})$. For $(x, y) \in A(\mathbb{Q})$, write

$$x - c_i = m_i z_i^2, \quad \text{with } m_i \in \mathbb{Z} \text{ square-free and } z_i \in \mathbb{Q},$$

then $m_1m_2m_3$ is a square. This motivates to consider double covers of A. For $m = (m_1, m_2, m_3) \in (\mathbb{Q}^\times/\mathbb{Q}^{\times 2})^{\oplus 3, \mathrm{Nm}=1}$, define the double cover $C_m \subset \mathbb{P}^3$ of A:

$$C_m : \quad m_i z_i^2 - m_j z_j^2 = (c_j - c_i)t^2, \quad \forall 1 \leq i < j \leq 3.$$

and 2-Selmer group $\mathrm{Sel}_2(A/\mathbb{Q})$ by

$$A(\mathbb{Q})/2A(\mathbb{Q}) \cong \{m \mid C_m(\mathbb{Q}) \neq \emptyset\} \subseteq \mathrm{Sel}_2(A/\mathbb{Q}) := \{m \mid C_m(\mathbb{Q}_v) \neq \emptyset \text{ for all } v\}.$$

We obtain

(1) $\mathrm{rank}_{\mathbb{Z}} A(\mathbb{Q}) \leq s(A) := \dim_{\mathbb{F}_2} \mathrm{Sel}_2(A/\mathbb{Q})/A(\mathbb{Q})[2]$.
(2) if $s(A) = 0$ then $A(\mathbb{Q})$ is finite..
(3) if $s(A) = 1$, then it is conjectured that $\mathrm{rank}_{\mathbb{Z}} A(\mathbb{Q}) = 1$.

A basic arithmetic question is as follows.

Question 2.1. *Find the distribution of $s(A)$ when A runs over a quadratic twist family $\mathscr{A}$.*

2.2. **General Selmer groups.** In general, if $A/\mathbb{Q}$ is an elliptic curve and p is a prime, the p-Selmer group and p^∞-Selmer group of A can be defined in term of cohomology

$$\mathrm{Sel}_p(A/\mathbb{Q}) = \mathrm{Ker}(H^1(\mathbb{Q}, A[p]) \longrightarrow \prod_v H^1(\mathbb{Q}_v, A)/A(\mathbb{Q}_v)/pA(\mathbb{Q}_v)),$$
$$\mathrm{Sel}_{p^\infty}(A/\mathbb{Q}) = \mathrm{Ker}(H^1(\mathbb{Q}, A[p^\infty]) \longrightarrow \prod_v H^1(\mathbb{Q}_v, A)/A(\mathbb{Q}_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p)).$$

They fit into short exact sequences

$$0 \to A(\mathbb{Q})/pA(\mathbb{Q}) \to \mathrm{Sel}_p(A/\mathbb{Q}) \to Ш(A/\mathbb{Q})[p] \to 0$$

and

$$0 \to A(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}_p(A/\mathbb{Q}) \to Ш(A/\mathbb{Q})[p^\infty] \to 0.$$

Conjecture 2.2. *The Shafarevich-Tate group* $Ш(A/\mathbb{Q})$ *is finite and*

$$\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_p(A/\mathbb{Q}) = \mathrm{rank}_{\mathbb{Z}} A(\mathbb{Q}).$$

A. Smith gives a method to study the distribution of 2^∞-Selmer groups $\mathrm{Sel}_{2^\infty}(A/\mathbb{Q})$ of A in a quadratic twist family $\mathscr{A}$ starting from distribution of 2-Selmer groups.

3. L-functions

Conjecture 3.1 (Birch and Swinnerton-Dyer)**.** *$A/\mathbb{Q}$ elliptic curve, $r \geqslant 0$ integer, p prime. Then the following are equivalent:*

(1) $\mathrm{ord}_{s=1} L(A, s) = r$*;*
(2) $\mathrm{rank}_{\mathbb{Z}} A(\mathbb{Q}) = r$ *and* $\Pi(A/\mathbb{Q})$ *finite;*
(3) $\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_p \infty(A/\mathbb{Q}) = r$.

Under equivalent conditions, the BSD formula holds:

$$\frac{L^{(r)}(A,1)}{r! \cdot R \cdot \Omega} = \frac{\#Ш(A/\mathbb{Q}) \cdot \prod_\ell c_\ell(A)}{(\#A(\mathbb{Q})_{\mathrm{tor}})^2}.$$

Theorem 3.2 (Gross-Zagier, Kolyvagin)**.** *Let A be an elliptic curve over $\mathbb{Q}$ and $\mathrm{ord}_{s=1} L(A,s) \leqslant 1$, then*

$$\mathrm{ord}_{s=1} L(A, s) = \mathrm{rank}_{\mathbb{Z}} A(\mathbb{Q}), \quad \#\Pi(A/\mathbb{Q}) < \infty.$$

Theorem 3.3 (Tunnell 1983)**.** *Let E be the elliptic curve $y^2 = x^3 - x$. Let n be a positive square-free integer, $a = 1$ for n odd and $a = 2$ for n even. Then*

$$\frac{L(E^{(n)}, 1)}{\Omega/\sqrt{n}} = \frac{a}{16} \left(\sum_{2ax^2+y^2+8z^2=\frac{n}{2}} (-1)^z \right)^2, \quad \Omega = \int_1^\infty \frac{dx}{\sqrt{x^3 - x}}.$$

3.1. **Quadratic twists of elliptic curves over $\mathbb{Q}$.** In general, for an elliptic curve over $\mathbb{Q}$ given by: $y^2 = x^3 + ax + b$, let $\mathscr{A}$ denote the set of all isomorphism classes of its quadratic twists:

$$ny^2 = x^3 + ax + b, \quad n \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2}.$$

As $A \in \mathscr{A}$ varies, we are interested in the distribution of

- $\mathrm{rank}_{\mathbb{Z}} A(\mathbb{Q})$, $\#Ш(A/\mathbb{Q})[p^\infty]$, $\dim_{\mathbb{F}_p} \mathrm{Sel}_p(A/\mathbb{Q})$, and $\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_p \infty(A/\mathbb{Q})$.
- leading term of $L(A, s)$, i.e. $\mathrm{ord}_{s=1} L(A, s)$, and $Ш^{\mathrm{an}}(A/\mathbb{Q})$.

We now start with **L-function side**, some of the discussions are related to Selmer groups via the BSD conjecture.

3.2. conjectures on leading terms of L-series under quadratic twists.

Conjecture 3.4 (Goldfeld). *Let $\mathscr{A}$ be a quadratic twist family of elliptic curves over $\mathbb{Q}$. Then*

- (*Even parity*) $\mathrm{Prob}(\mathrm{ord}_{s=1} L(A,s) = 0 \mid A \in \mathscr{A}, \epsilon(A) = +1) = 1$.
- (*Odd parity*) $\mathrm{Prob}(\mathrm{ord}_{s=1} L(A,s) = 1 \mid A \in \mathscr{A}, \epsilon(A) = -1) = 1$.

The behavior for analytic Sha is subtle. However, Kolyvagin proposed the following:

Conjecture 3.5 (Kolyvagin). *Let $\mathscr{A}$ be a quadratic twist family of elliptic curves over $\mathbb{Q}$ and p any prime. There exists $A \in \mathscr{A}$ such that*

- $\mathrm{ord}_{s=1} L(A,s) = 0$ *(resp. 1), and*
- $p \nmid \text{Ш}^{\mathrm{an}}(A/\mathbb{Q})$.

3.3. Goldfeld conjecture for CM families.

Theorem 3.6. *For quadratic twist families of CM elliptic curves over $\mathbb{Q}$, we have the following*

(1) *the even parity Goldfeld conjecture holds if the CM field is not $\mathbb{Q}(\sqrt{-2})$;*
(2) *the odd parity Goldfeld conjecture holds if $p = 2$ is an ordinary prime.*

Thus the Goldfeld conjecture holds for the family containing the conductor 49 curve.

The proof of the result consists of two parts.

Theorem 3.7 (Burungale-Tian, Burungale-Castella-Skinner-Tian). *Let A be a CM elliptic curve over $\mathbb{Q}, p$ a prime and $r = 0, 1$. Then the rank r p-converse holds:*

$$\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_p(A/\mathbb{Q}) = r \quad \Longrightarrow \quad \mathrm{ord}_{s=1} L(A,s) = r,$$

provided that p is ordinary for $r = 1$.

Theorem 3.8 (Smith). *The 2^∞-Selmer analogue Goldfeld conjecture holds for families $\mathscr{A}$ over $\mathbb{Q}$ satisfying the following assumption S.*

Assumption S: There is $A \in \mathscr{A}$ such that one of the following holds:

- $A(\mathbb{Q})[2] = 0$; or
- $A(\mathbb{Q})[2] \cong \mathbb{Z}/2\mathbb{Z}$ and for the unique $\mathbb{Q}$-degree 2 isogeny $A \to A_0$, $\mathbb{Q}(A_0[2]) \neq \mathbb{Q}, \mathbb{Q}(A[2])$; or
- $A(\mathbb{Q})[2] \cong (\mathbb{Z}/2\mathbb{Z})^2$ and A has no cyclic degree 4 isogeny over $\mathbb{Q}$.

Actually, in many cases Smith proved the Selmer analogue Goldfeld conjecture via establishing that the distribution of 2^∞-Selmer groups in $\mathscr{A}$ follows the same principle in the BKLPR conjecture for $p = 2$.

Conjecture 3.9 (Bhargava-Kane-Lenstra-Poonen-Rains). *Let $\mathfrak{A}_F$ be the set of all isomorphism classes of elliptic curves over a fixed number field F, ordered by height. For $r = 0, 1$ and any G finite symplectic p-group,*

$$\mathrm{Prob}(\mathrm{Sel}_{p^\infty}(A/F) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^r \oplus G \mid A \in \mathfrak{A}_F, \epsilon(A) = (-1)^r) = \frac{(\#G)^{1-r}}{\#\mathrm{Sp}(G)} \cdot \prod_{i \geqslant r} (1 - p^{1-2i}).$$

In particular, the average of $\#\operatorname{Sel}_2(A/F)$ *is* 3 *and*

$$\operatorname{Prob}(\operatorname{rank}_{\mathbb{Z}} A(F) = r \mid A \in \mathfrak{A}_F,\ \epsilon(A) = (-1)^r) = 1.$$

4. On quadratic twist families

4.1. **Equivalence relation for quadratic twist families.** For general quadratic twist families of elliptic curves over $\mathbb{Q}$, the distribution of Selmer groups does not follow the BKLPR's principle.

For $\mathscr{A}$ a quadratic twist family of elliptic curves over $\mathbb{Q}$, let Σ be a finite set of places $\Sigma \supseteq \{p \mid \text{any } A \in \mathscr{A} \text{ has bad reduction at } p\} \cup \{2, \infty\}$.

Definition 4.1. $A_1, A_2 \in \mathscr{A}$ are called **Σ-equivalent** if $A_1 \cong A_2$ over $\mathbb{Q}_v$ for any $v \in \Sigma$.

The root numbers of elliptic curves in a fixed class $\mathfrak{X}$ are the same, denoted by $\epsilon(\mathfrak{X})$.

4.2. **Elliptic curves with full rational 2-torsion.** Recall that we have seen that θ-congruent number problem is essentially about quadratic twist families of elliptic curves over $\mathbb{Q}$ with full rational 2-torsions.

Families $\mathscr{A}$ over $\mathbb{Q}$ with full rational 2-torsion points are divided into three types:

(A) $\mathscr{A}$ does not have a rational cyclic 4-isogeny, e.g. the congruent number curves $ny^2 = x^3 - x$.
(B) $\mathscr{A}$ has a rational cyclic 4-isogeny, and $A[4] \not\subseteq A(\mathbb{Q}(\sqrt{-1}))$ for any $A \in \mathscr{A}$, e.g. the tiling number curves $ny^2 = x(x-3)(x+1)$.
(C) $\mathscr{A}$ has a rational cyclic 4-isogeny, and $A[4] \subseteq A(\mathbb{Q}(\sqrt{-1}))$ for some $A \in \mathscr{A}$. e.g. $ny^2 = x(x-9)(x-25)$.

We now discuss the distribution of $s(A) := \dim_{\mathbb{F}_2} \operatorname{Sel}_2(A/\mathbb{Q})/A(\mathbb{Q})[2]$.

4.3. **Distribution of 2-Selmer groups for type (A).** For type (A), the distribution of 2^∞-Selmer groups is independent of equivalence classes $\mathfrak{X} \subset \mathscr{A}$.

Theorem 4.2 (Heath-Brown, Swinnetron-Dyer, Kane)**.** *Let $\mathscr{A}$ be a quadratic twist family of type (A) and $\mathfrak{X} \subset \mathscr{A}$ an equivalence class. Let $t \in \{0,1\}$ with $(-1)^t = \epsilon(\mathfrak{X})$. Then for any $d \in \mathbb{Z}_{\geqslant 0}$,*

$$\operatorname{Prob}(s(A) = d \mid A \in \mathfrak{X}) = \lim_{k \to \infty} P^{\mathrm{Alt}}_{k;t}(d),$$

where $P^{\mathrm{Alt}}_{k,t}(d)$ is the ratio of corank d matrices in alternating $(2k+t) \times (2k+t)$ matrices of coefficient $\mathbb{F}_2$.

The above result is the starting point of Smith's work on 2^∞-Selmer groups.

Corollary 4.3. *The average of $\#\operatorname{Sel}_2(A/\mathbb{Q})/A(\mathbb{Q})[2]$ for $A \in \mathfrak{X}$ of type (A) is always 3.*

4.4. **Distribution of 2-Selmer groups for type** (B) **and (C).** To describe the distribution of 2-Selmer groups for type (B) and (C), we introduce the following model of alternating matrices.

Let $t \in \{0,1\},\ s \in \{0,1,2\},\ \vec{t} = (t_1, \cdots, t_s) \in \mathbb{Z}^s$ such that $t_i \equiv t \bmod 2$ for all i. When $s = 2$ we require that $t_1 + t_2 \leqslant 0$.

Let $P^{\mathrm{Alt}}_{k;t,\vec{t}}(d)$ be the ratio of corank d matrices in alternating $(2k+t)\times(2k+t)$ matrices of coefficient $\mathbb{F}_2$ of form

$$\begin{pmatrix} 0 & B_{12} & \cdots & B_{1,s+1} \\ B_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & B_{s,s+1} \\ B_{s+1,1} & \cdots & B_{s+1,s} & B_{s+1,s+1} \end{pmatrix}$$

such that for $1\leqslant i\leqslant s, 1\leqslant j\leqslant s$, the B_{ij} is of size

$$(k+\frac{t_i+t}{2})\times(k+\frac{t_j+t}{2}).$$

Theorem 4.4 (Pan-Tian). *Let $\mathscr{A}$ be a family of type (B) or (C) and $\mathfrak{X}\subset\mathscr{A}$ an equivalence class. Then there exists $\vec{t}\in\mathbb{Z}^s$ only dependent on $\mathfrak{X}$, with $s=1$ for type (B), $s=2$ for type (C), such that for any $d\in\mathbb{Z}_{\geqslant 0}$,*

$$\mathrm{Prob}(s(A)=d\mid A\in\mathfrak{X})=\lim_{k\to\infty}P^{\mathrm{Alt}}_{k;t,\vec{t}}(d),$$

in particular, it is positive if and only if $d\geqslant\max_i t_i$ and $d\equiv t \bmod 2$. Moreover, if $\Sigma\subset\Sigma'$, then any Σ'-equivalence class $\mathfrak{X}'\subset\mathfrak{X}$ has the same $\vec{t}$.

We expect to establish the distribution of 2^∞-Selmer groups starting from this result.

Corollary 4.5. *The average of $\#\,\mathrm{Sel}_2(A/\mathbb{Q})/A[2]$ for $A\in\mathfrak{X}$ is equal to $3+\sum_i 2^{t_i}$.*

Corollary 4.6. *Let $A_0\in\mathfrak{X}$ be any curve which has a bad prime outside Σ. Then*

$$\mathrm{Prob}(s(A)=s(A_0)\mid A\in\mathfrak{X})>0.$$

4.5. **Kolyvagin's Question.** Kolyvagin's conjecture is displayed as

$$\min_{A\in\mathscr{A},\mathrm{sign}(A)=(-1)}\dim_{\mathbb{F}_2}\mathrm{Sel}_2(A)/A[2]=r,\quad r=0,1.$$

Note: Distribution has nicer behavior when restricted in equivalence classes.

Conjecture 4.7. *Given $\mathscr{A}$, exists equivalent class $\mathfrak{X}$ such that* $\vec{t}\leqslant\begin{cases}\vec{0}, & \text{if } \epsilon(\mathfrak{X})=+1;\\ \vec{1}, & \text{if } \epsilon(\mathfrak{X})=-1.\end{cases}$

For some classes $\mathfrak{X}, \min_{A\in\mathfrak{X}}\dim_{\mathbb{F}_2}\mathrm{Sel}_2(A)/A[2]$ may not reach minimal.

It seems natural to consider the following variation of Kolyvagin's problem:

Question 4.8. *For an equivalence class $\mathfrak{X}$ of quadratic twists of elliptic curves over $\mathbb{Q}$ and a prime p, let $r\in\{0,1\}$ with $(-1)^r=\epsilon(\mathfrak{X})$, what is the behavior of*

$$\min_{A\in\mathcal{X},r_A=r}\mathrm{ord}_p\,\text{Ш}^{\mathrm{an}}(A),$$

as $\mathfrak{X}\subseteq\mathscr{A}$ varies?

Suitable constructed (arithmetic) theta series on $\widetilde{\mathrm{SL}_2}$ have Fourier coefficients basically $\mathrm{W}^{\mathrm{an}}(A)$ exactly for $A\in\mathfrak{X}$.

4.6. **Arithmetic Rallis inner product formula.** Fix $\mathbb{H}_v$-invariant pairings $(\ ,\ )_v$ on π_v such that for any pure tensors $f_i = (f_{i,U})_U$,

$$\Pi_v(f_{1,v}, f_{2,v})_v \doteq f_{1,u} \circ f_{2,U}^N \quad (\text{fixed} \quad \pi_{A,C} \cong \otimes \pi_v\ ).$$

Theorem 4.9 (He-Xiong-Tian)**.** *For pure tensors $f_1, f_2 \in \pi_A$ and $\phi_1, \phi_2 \in \mathscr{S}(\mathbb{V})$, the following equality holds (with standard measures):*

$$(\vartheta_{\phi_1}^{f_1}, \vartheta_{\phi_2}^{f_2})_{NT} = \frac{L'(1/2, \pi_A)}{L(2, 1_Q)} \cdot \prod_v Z_v(\phi_{1,v}, \phi_{2,v}, f_{1,v}, f_{2,v}),$$

where

$$Z_v(\phi_{1,v}, \phi_{2,v}, f_{1,v}, f_{2,v}) = \frac{L(2, 1_v)}{L(1/2, \pi_v)} \cdot \int_{\mathbb{H}_v} (h\phi_{1,v}, \phi_{2,v})_v (hf_{1,v}, f_{2,v})_v dh.$$

Remark 4.10.

(1) Previous work on RI were established by (arith.) Siegel-Weil formula and doubling method. Our approach does not involve doubling method, but
   (i) a decomposition formula of Fourier-Whittaker periods, and
   (ii) Gross-Zagier formula of Yuan-Zhang-Zhang.
(2) Certain form of ARI was first conjectured by Kudla, and proved by Kudla-Rapoport-Yang etal via an arithmetic Siegel-Weil over $\mathbb{Q}$ in certain case.
(3) In the rank 0 case, we have the parallel results and proofs. Our results have application to Kolyvagin's problem.

4.7. **Tunnell Type result.** As a byproduct of the rank 0 case Fourier-Whittaker period formula, we have:

Theorem 4.11. *Tunnell Type result holds for general quadratic twist family of elliptic curves over $\mathbb{Q}$.*

Remark 4.12.

(1) For CN curves $E^{(n)} : ny^2 = x^3 - x$, we get new formula: For $n > 0$ square-free
$$\sum_{x^2+2y^2+8z^2=n} (-1)^z = \pm \sum_{x^2+8y^2+16z^2=n} (-1)^{y+z}, \quad 0 < n \equiv 1 \pmod 8,$$
whose square is essentially $L(E^{(n)}, 1)$. Similar for other congruent class.
(2) In general, there may be local obstructions due to Atkin-Lehner involutions: Consider $A = X_0(14) : y^2 + xy + y = x^3 + 4x - 6$ and $\mathfrak{X}$ the class of negative fundamental discriminants $n \equiv -3 \bmod 56$. Let
$$Q(x, y, z) = (x + 14y + 4z)^2 + (x - 14y - 2z)^2 + x^2,$$
for each $n \in \mathfrak{X}$,
$$\frac{L(A^{(n)}, 1)}{\Omega(A^{(-1)})/\sqrt{|n|}} = 2 \left(\sum_{\substack{Q(x,y,z)=|n|,\\ 3x+2z\equiv 3 \bmod 4,\\ 3x+2z\equiv 3 \bmod 7}} 1 - \sum_{\substack{Q(x,y,z)=|n|,\\ 3x+2z\equiv 3 \bmod 4,\\ 3x+2z\equiv -3 \bmod 7}} 1 \right)^2.$$

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