Integral p-adic Hodge theory
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Reality

<u>Problem</u> Let $K/\mathbb{Q}_p$ fin ext'n,

$\mathfrak{X}/\mathcal{O}_K$ proper sm formal sch.

Goal: Relate $H^i_{cris}(X_s/W)$ & $H^i_{et}(X_{\bar{K}}, \mathbb{Z}_p)$ integrally

If $K/\mathbb{Q}_p$ unram, i small: Fontaine-Laffaille theory.

Let $C/\mathbb{Q}_p$ alg closed complete w/ res field k.

Let $A_{inf} = W(\mathcal{O}_{C^\flat})$, $\mathcal{O}_{C^\flat} = \varprojlim_{\Phi} \mathcal{O}_C/p$.

$\circlearrowleft$
φ $\quad + \quad A_{inf} \xrightarrow{\theta} \mathcal{O}_C$.

<u>Fact</u> $\ker\theta$ gen'd by $\xi = \dfrac{[\varepsilon^p]-1}{[\varepsilon]-1}$

where $\varepsilon = (\zeta_p, \zeta_{p^2}, \cdots) \in \mathcal{O}_{C^\flat}$.

<u>Def</u> (Fargues) A Breuil-Kisin module $/A_{inf}$ is a finite free A_{inf}-mod M with

$$\varphi_M : (\varphi^* M)[\xi^{-1}] \xrightarrow{\sim} M[\xi^{-1}].$$

<u>Example</u> $M = A_{inf}(-1)$: Tate twist in B-K world.

$$\varphi_{A_{inf}(-1)} = \xi \varphi_{A_{inf}}.$$

<u>Rem</u> A B-K module has several "realizations".

(1) étale: $T(M) := (M \underset{A_{inf}}{\otimes} W(C^\flat))^{\varphi=1}$

finite free $\mathbb{Z}_p$-mod (no Gal action)

Then $M[\frac{1}{\mu}] \simeq T(M) \underset{\mathbb{Z}_p}{\otimes} A_{inf}[\frac{1}{\mu}]$

where $\mu = [\varepsilon] - 1 = $ "$\underbrace{\varphi^{-1}(\xi)\varphi^{-2}(\xi)}_{\text{infinite product}} \cdots$" .

(2) de Rham: $(\varphi^* M[\frac{1}{p}])^\wedge_\xi$ and $M[\frac{1}{p}]^\wedge_\xi = T(M) \otimes_{\mathbb{Z}_p} B_{dR}^+$

are two $\underset{A_{inf}[\frac{1}{p}]^\wedge_\xi}{\overset{\parallel}{B_{dR}^+}}$-lattices in the same B_{dR}-v.s.

(3) crystalline: $\underset{\varphi_M}{\overset{\circlearrowright}{M}} \otimes_{A_{inf}} W(k)$ F-crystal $/k$.

<u>Thm</u> (Fargues) The functor

$$\left\{ \begin{array}{c} \text{B-K modules} \\ / A_{inf} \end{array} \right\} \xrightarrow{\sim} \left\{ (T, \Xi) \;\middle|\; \begin{array}{l} T \text{ fin free } \mathbb{Z}_p\text{-mod} \\ \Xi \subseteq T \otimes B_{dR},\ B_{dR}^+\text{-lattice} \end{array} \right\}$$

is an equiv of cats.

<u>Thm</u> (Scholze, Conrad-Gabber)

Let X/C proper sm rigid analytic variety.

Then

$H^i_{ét}(X, \mathbb{Z}_p)$ f.g. $\mathbb{Z}_p$-mod (so fin free)

& there is a nat'l B_{dR}^+-lattice $\Xi \subseteq H^i_{et}(X, \mathbb{Z}_p) \otimes B_{dR}$

s.t. $\Xi \otimes_{B_{dR}^+} C = H^i_{dR}(X)$.

If $X = X_K \hat{\otimes}_K C$, $K / \mathbb{Q}_p$ finite.

then $\Xi = H^i_{dR}(X_K) \otimes_K B_{dR}^+$.

<u>Cor</u> For any X/C, get B-K module

$H^i_{A_{inf}}(X)$ assoc with $(T = H^i_{et}(X, \mathbb{Z}_p)/\text{torsion}, \Xi)$.

<u>Thm</u> (Bhatt - Morrow - Scholze)

Let $\mathfrak{X} / \mathcal{O}_C$ proper sm formal sch. Then

(1) If $H^i_{crys}(\mathfrak{X}_k / W(k))$ is torsion-free, then

$H^i_{et}(\mathfrak{X}_C, \mathbb{Z}_p)$ is torsion-free.

(rigid-analytic generic fibre).

(2) If H^i_{crys}, H^{i+1}_{crys} are torsion-free, then

(*) $H^i_{crys}(\mathfrak{X}_k / W(k)) \cong H^i_{A_{inf}}(X) \otimes_{A_{inf}} W(k)$.

<u>Rem</u> (1) New for K3 surfaces over 2-adic fields.

(2) If $\mathfrak{X}$ already defined $/ \mathcal{O}_K$, $K / \mathbb{Q}_p$ finite,

then $H^i_{A_{inf}}$ can be recovered from $H^i_{et} \circlearrowleft G_K$.

<u>Strategy</u> Define new cohom theory $R\Gamma_{A_{inf}}(\mathfrak{X})$:

perfect complex of A_{inf}-mod + φ-action.

which compare well w/ all other cohom theories.

<u>Outer space</u>

<u>Topological Hochschild homology</u>

Given (usual) ring A, there is a

"cyclotomic spectrum" $\mathrm{THH}(A)$.

$\circlearrowleft$

S^1.

For each finite subgrp $C \subset S^1$,

have "genuine C-fixed pt spectrum"

$$TR^C(A) := THH(A)^C.$$

If $C = C_{p^{r-1}}$, write

$$TR^r(A;p) := TR^{C_{p^{r-1}}}(A).$$

There are maps

$$\begin{tikzcd} TR^{r+1}(A;p) \arrow[d, bend right, "R"'] \arrow[d, "F"] \\ TR^r(A;p) \arrow[u, bend right, "V"'] \end{tikzcd}$$

F = inclusion of fixed pts "Frobenius"

V = "trace" = "Verchiebung"

R has to do w/ geom fixed pts.

<u>Thm</u> (Hesselholt-Madson)

$$\pi_0 TR^r(A;p) = W_r(A) \quad \text{ring of Witt vects.}$$

Comp. w/ F, R, V.

Let $TF^\bullet(A;p)$ = system of $TR^r(A;p)$ w/ F as transition map.

$$\left(A_{\inf} = \varprojlim_F W_r(\mathcal{O}_C). \right)$$

$$\underset{\circlearrowleft\, R}{TF(A;p)} := \varprojlim_F TF^\bullet(A;p)$$

$\rightsquigarrow$ $\underset{\circlearrowleft\, R}{TF(A;p,\mathbb{Z}_p)}$ = its p-adic completion

"topological Frob homology".

<u>Thm</u> (Hesselholt)

$$\underset{\circlearrowleft\, R}{\pi_i TF(\mathcal{O}_{\mathbb{C}_p};p,\mathbb{Z}_p)} = \begin{cases} \overset{\varphi^{-1}\circlearrowleft}{A_{\inf}(\tfrac{i}{2})}, & i \text{ even} \\ 0, & i \text{ odd}. \end{cases}$$

can replace w/ general C.

<u>Conj</u> There should be an E_2-spectral seq.

$$\varphi^{-1} \circlearrowright H^i_{A_{\inf}}(\mathfrak{X})(-\tfrac{j}{2}) \Rightarrow \pi_{-i-j} TF(\mathfrak{X};p,\mathbb{Z}_p) \circlearrowleft R.$$

$(i \geq 0,\ j \leq 0 \text{ even})$ $\qquad$ $\mathfrak{X}/\mathcal{O}_C$ as before.

(at least in large degrees).

<u>Fact</u> $TR^r(A;p,\mathbb{Z}_p)$ for A a smooth $\mathcal{O}_C$-alg

can be computed in terms of de Rham-Witt grps

$W\Omega^i_{A/\mathcal{O}_C}$ (Langer-Zink).

"twisted de Rham"

<u>Conj</u> For a smooth $\mathcal{O}_C$-alg A, there is a natural complex $\widetilde{\Omega}^\bullet_{A/\mathcal{O}_C}$

of A-modules

s.t. $H^i(\widetilde{\Omega}^\bullet_{A/\mathcal{O}_C}) = (\Omega^i_{A/\mathcal{O}_C})^\wedge_p$ (mod Tate twist). (*)

Same for $\widetilde{W_r\Omega}^\bullet_{A/\mathcal{O}_C}$.

Moreover, $\widetilde{\Omega}^\bullet_{A/\mathcal{O}_C} \otimes_{\mathcal{O}_C} k \cong \Omega^\bullet_{A_k/k}$

considered as complex of A_k-mods via F

s.t. (*) becomes Cartier isom.

<u>Back to reality</u>

<u>Thm</u> $\widetilde{W_r\Omega}^\bullet_{A/\mathcal{O}_C}$ exists w/ all desired properties.

(<u>Then</u> $R\Gamma_{A_{inf}}(\mathfrak{X}) = R\varprojlim_{r,F} R\Gamma(\mathfrak{X}, \widetilde{W_r\Omega}^\bullet)$.

pf relies on 2 observations:

(1) (Recall: know the complex for special fibres $R\Gamma_{A_{inf}}(\mathfrak{X}_k)$.)

If $\nu: X_{pro\acute{e}t} \to \mathfrak{X}$ projection, then

(Scholze) $R^i\nu_*\widehat{\mathcal{O}}_X = \Omega^i_{\mathfrak{X}}[\frac{1}{p}](-i)$.

Thus, $R\nu_*\widehat{\mathcal{O}}_X$ does job rationally.

<u>But</u> $R^i\nu_*\widehat{\mathcal{O}}_X^+$ has some junk torsion killed by $\zeta_p - 1$.

(2) Let $L\eta_{\zeta_p-1}K^\bullet$ for complex $K^\bullet$ of flat $\mathcal{O}_C$-mods

be $(L\eta_{\zeta_p-1}K^\bullet)^i = \{x \in (\zeta_p-1)^i K^i \mid dx \in (\zeta_p-1)^{i+1}K^{i+1}\}$.

This kills the junk torsion!

Take

$$\widetilde{\Omega}^{\cdot} = L\eta_{\zeta_p - 1} R\nu_* \hat{\mathcal{O}}_X^{+}.$$

(almost, but can make it completely integral).