Hodge filtrations and p-adic Langlands programs

Yiwen Ding

$E/\mathbb{Q}_p$ finite. Classically,

$$\left\{\begin{matrix}n\text{-dim'l WD rep'ns}\\ \text{of } \mathbb{Q}_p \ /E\end{matrix}\right\} \overset{LL}{\longleftrightarrow} \left\{\begin{matrix}\text{Some sm rep of } GL_n(\mathbb{Q}_p)\\ /E\end{matrix}\right\}$$

Fontaine's theory $\Updownarrow$ $\qquad\qquad\qquad\qquad \cap$

$$\left\{\begin{matrix}\text{de Rham } n\text{-dim'l}\\ \text{cont rep of } \mathrm{Gal}_{\mathbb{Q}_p} /E\end{matrix}\right\} \overset{?}{\longleftrightarrow} \left\{\begin{matrix}\text{Some loc an rep}\\ \text{of } GL_n(\mathbb{Q}_p) \ /E\end{matrix}\right\}$$

loses info of Hodge fil.

How to see Hodge fil on autom side?

- Known for $GL_2(\mathbb{Q}_p)$. But mysterious for $n \geq 3$.
- Patching argument:

$$\rho \overset{\text{CEGGPS}}{\rightsquigarrow} \pi(\rho) \circlearrowleft GL_n(\mathbb{Q}_p)$$

- may depend on global setup
- $\pi(\rho) \neq 0$?

Let ρ de Rham of regular HT wts $\underline{h} = (h_1 > h_2 > \cdots > h_n)$

$$\rho \rightsquigarrow r \longrightarrow \pi_{sm}(r) \circlearrowleft GL_n(\mathbb{Q}_p).$$

$$\underline{h} \rightsquigarrow \lambda = (h_1 - (n-1), \dots, h_n) \rightsquigarrow L(\lambda) \text{ alg.}$$

s.t. $\pi_{alg}(r, \underline{h}) := \pi_{sm}(r) \otimes L(\lambda)$.

- $\pi_{alg}(r, \underline{h}) \hookrightarrow \pi(\rho)$, $\quad \rho \rightsquigarrow (r, \underline{h})$

Hodge fil.

$\underline{GL_2(\mathbb{Q}_p)}$ $\exists$ $\pi_{s_1}(r, \underline{h})$ depending on $r, \underline{h}$

$\quad\circlearrowleft$

$GL_2(\mathbb{Q}_p)$

s.t. $\pi(\rho)$ has the form $\pi_{alg}(r,\underline{h}) - \pi_{s_1}(r,\underline{h})$

hidden info about $\mathrm{Fil}_{\mathrm{Hodge}}$.

(with a pattern like $M(\lambda)^\vee = L(\lambda) - L(s_1\cdot\lambda)$).

$\underline{GL_n(\mathbb{Q}_p)}$

$$M(\lambda)^\vee = L(\lambda) \begin{cases} - L(s_1\cdot\lambda) \\ \quad\vdots \\ - L(s_{n-1}\cdot\lambda) \end{cases}$$

$(r, \underline{h}, \mathrm{Fil}^{\max} D_{pst}(\rho))$

$$\left.\pi_{alg}(r,\underline{h}) \begin{cases} \pi_{s_1}(r,\underline{h}) \\ \quad\vdots \\ \pi_{s_{n-1}}(r,\underline{h}). \end{cases}\right\} \hookrightarrow \pi(\rho)$$

$\underline{\text{Conj}}$ (Breuil) For $r, \underline{h}$, any s_i, $\exists$ a loc an $\pi_{s_i}(r,\underline{h})$ depending only on $r, \underline{h}$,

together with a bijection

$$\mathrm{rec}_i : \mathrm{Ext}^1(\pi_{s_i}(r,\underline{h}), \pi_{alg}(r,\underline{h})) \xrightarrow{\sim} \underbrace{\Lambda^i r}_{\dim = \binom{n}{i}} \supseteq \mathrm{Fil}^{\max}\Lambda^i D_{pst}(\rho)$$

s.t. $\forall$ ρ overlying $(r, \underline{h})$, define

$$\mathrm{rec}_i^{-1}(\mathrm{Fil}^{\max}\Lambda^i D_{pst}(\rho)) =: [\pi_{s_i}(\rho)],$$

then $\pi_{s_i}(\rho) \hookrightarrow \pi(\rho)$

local $\qquad\qquad$ global.

$\underline{\text{Aside}}$ $\rho \rightsquigarrow D_{pst}(\rho) \cong r +$ flag on r

$\rightsquigarrow \Lambda^i D_{pst}(\rho) = \lambda^i r + \mathrm{Fil}.$

increasing

with $\mathrm{Fil}^{max} \lambda^i r$ = non-trivial fil'n w/ max index

$\rightsquigarrow \dim \mathrm{Fil}^{max} \lambda^i r = 1.$

<u>Wall-crossing</u> $\pi(\rho) \circlearrowleft \mathfrak{gl}_n$, λ regular wt.

Let μ_i be an integer wt, only singular at s_i.

(E.g. $\lambda = (0, \dots, 0)$, $\mu_i = (0, \dots, 0, \underset{\substack{\uparrow \\ i\text{-th}}}{1}, \dots, 1)$)

$\pi(\rho) \otimes V$. $T_\lambda^{\mu_i} \pi(\rho)$ = the translation of $\pi(\rho)$ to the μ_i-block.

f.d. alg rep

Should just lose the info on $\mathrm{Fil}^{-h_i} D_{pst}(\rho)$.

$\rightsquigarrow \Theta_i \pi(\rho) := T_{\mu_i}^{\lambda} T_\lambda^{\mu_i} \pi(\rho)$.

a direct summand of $\pi(\rho) \otimes V \otimes V^*$.

$\rightsquigarrow z_i : \pi(\rho) \longrightarrow \Theta_i \pi(\rho)$.

$$\underset{\substack{\cup\!\!\mid \\ \ker z_i}}{\pi(\rho)}$$

Conj $I \subseteq \{1, \dots, n-1\}$,

- $\pi(\rho) \supseteq \bigcap_{i \in I} \ker z_i$ determines $(r, \mathfrak{b}, \{\mathrm{Fil}^{-h_j}\}_{j \notin I})$

  (only depend on ?)

- $\pi_{s_i}(\rho)$ determines $(r, \mathfrak{b}, \mathrm{Fil}^{max} \Lambda^i D_{pst}(\rho))$.

  $\updownarrow$

  $L(s_i \lambda)$

- $\pi_{s_i}(\rho)$ should be contained in $\bigcap_{j \neq i} \ker z_j$.

Rmk Breuil's conj is known for crystalline case.

Note $(r, \underline{h}, \mathrm{Fil}^{-h_i} D_{pst}(\rho)) \rightsquigarrow (r, \underline{h}, \mathrm{Fil}^{max} \wedge^i D_{pst}(\rho))$

i-dim

If r irred, both sides carry the same info.

(The conj is implied by Breuil's conj.)

Think of $\mathrm{Aut}(r) \backslash GL_n / P_i \longrightarrow \mathrm{Aut}(\wedge^i r) \backslash GL_{\binom{n}{i}} / P$

$$P = \begin{pmatrix} GL_1 & & * \\ & \ddots & \\ & & GL_{\binom{n}{i}-1} \end{pmatrix}.$$

inj when $\mathrm{Aut}(r) = 1$

(but not inj in general).

Thm If ρ is crystalline, non-crit, then the conj is true.

Example for $GL_4(\mathbb{Q}_p)$ ρ crystalline, $\alpha_1, \cdots, \alpha_4$ φ-eigenvals.

A candidate of $\pi_{S_3}(r, \underline{h})$ is $\bigoplus_{i \neq j} C(S_2, \alpha_i \alpha_j)$

Ker $Z_1 \cap$ Ker Z_3 contains a rep of form

$$\pi_{alg}(r, \underline{h}) - \bigoplus_{i \neq j} C(S_2, \alpha_i \alpha_j) - \pi_{alg}(r, \underline{h})$$

determines & depends only on $(r, \underline{h}, \mathrm{Fil}^{-h_i})$.

$$T \backslash GL_4 / \begin{pmatrix} GL_2 & * \\ 0 & GL_2 \end{pmatrix} \longrightarrow \underbrace{T \backslash GL_6 / \begin{pmatrix} GL_1 & * \\ 0 & GL_5 \end{pmatrix}}_{\text{more info here.}}$$