Basics of the Moduli Spaces of Curves

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§1 Introduction

Work on $k = \bar{k}$. Functor $h = \mathcal{M}_g : \underline{Sch} \longrightarrow \underline{Sets}$

of finite type

- For every scheme X, $h(X)$ is the set of families $f: Y \to X$ morphism between flat schemes over X s.t. all geom fibers are proj. smooth curves of genus $\boxed{g \geq 2}$, modulo the isom. of Y's.
- For every morphism $f: X' \to X$ $\exists$ a map $h(f): h(X) \to h(X')$ satisfying (1) $h(id_X) = id_{h(X)}$, pull-back (contravariant).

  (2) for $X'' \xrightarrow{f'} X' \xrightarrow{f} X$, have $h(X) \xrightarrow{h(f)} h(X') \xrightarrow{h(f')} h(X'')$.

<u>Def'n</u> Let $\mathcal{M}$ be a moduli functor. If there is a scheme M which represents $\mathcal{M}$, then M is called a <u>fine moduli space</u> of $\mathcal{M}$.

<u>Recall</u> Representability: $\exists$ natural isom $\eta: \mathcal{M} \to h_M$

where $h_M : S \mapsto \mathrm{Hom}_{Sch}(M, S)$

s.t. $\eta: \mathcal{M}(X) \to h_M(X)$ is a bijection in $\mathrm{Mor}(Sets)$.

$\Rightarrow \mathcal{M}$ is rep'd by M.

Furthermore, the bijection is compatible: $\forall f: X' \to X$,

$$\begin{tikzcd} \mathcal{M}(X) \arrow[r, "\eta"] \arrow[d, "\mathcal{M}(f)"'] \arrow[dr, phantom, "\circlearrowright"] & h_M(X) \arrow[d, "h(f)"] \\ \mathcal{M}(X') \arrow[r, "\eta"] & h_M(X') \end{tikzcd}$$

But $\mathcal{M}_g$ does NOT admit a fine moduli space.

Reason: If M is fine, $\exists$ univ family.

Def'n A coarse moduli space for a moduli functor $\mathcal{M}$ is a scheme M and a natural transformation $\eta : \mathcal{M} \to h_M$, $h_M : S \mapsto \mathrm{Hom}_{Sch}(M, S)$.

(a) $\eta_{\mathrm{Spec}\,k} : \mathcal{M}(\mathrm{Spec}\,k) \longrightarrow h_M(\mathrm{Spec}\,k)$ bijection. $\leftarrow$ preserving closed pts.

(b) For any scheme N and natural transform $\nu : \mathcal{M} \to h_M$
$\exists !$ morphism $f : M \to N$
s.t. $\nu = h_f \circ \eta$, where $h_f : h_M \to h_N$.

univ property

Rmk A coarse moduli space for M is unique $/\sim$.

Theorem There exists a coarse moduli space for $\mathcal{M}_g$ $(g \geq 2)$.

Approach: construct using geom. invariant theory.

§2 Construction of coarse moduli space

Fix an integer $n \geq 5$. Consider

$K_g := \{ (C, \varphi : C \to \mathbb{P}^r) \}$ where

- C smooth proj curve of genus g
- φ non-degenerate embedding s.t. $\varphi^* \mathcal{O}_{\mathbb{P}^r}(1) = \omega_C^{\otimes n}$.

i.e. $\nexists$ $C \to \mathbb{P}^{r-1}$ proj.

abs cotangent bundle.

and $r = \dim_k \boxed{\langle \omega_C^{\otimes n} \rangle} - 1$ corrs. complete linear system.

turns out to be $\to = (2n-1)(g-1) - 1$ (computing dim (glob sec) by Riemann-Roch)

Now K_g is a locally closed subsch of $\mathrm{Hilb}(\mathbb{P}^r)$.

$\rightsquigarrow$ $\boxed{K_g} = \{(C, [\underbrace{s_0, \cdots, s_r}_{\text{characterizing the way of embedding.}}]) \mid C$ as before, $[s_0, \cdots, s_r]$ form a basis of $H^0(C, \omega_C^{\otimes n})\}$.

$\circlearrowleft$ PGL_{r+1} naturally

Keynote Define K_g / PGL_{r+1} correctly.

(And then $K_g / PGL_{r+1} = M_g$, coarse moduli).

Def'n A stable curve is a complete connected curve that

has only nodes at singularities, and

has only finitely many automorphisms.

Meaning C stable. Let $\tilde{C}_v$ be an irred comp of the normalization of C.

- $g \geq 2$: no more condition.
- $g = 1$: $\tilde{C}_v$ contains the preimage of at least 1 node on C
- $g = 0$: $\tilde{C}_v$ contains the preimage of at least 3 nodes on C

Fact For a stable curve C, ω_C is a line bundle.

$\omega_C^{\otimes n}$ $(n \geq 5)$ induces an embedding $C \hookrightarrow \mathbb{P}^r$.

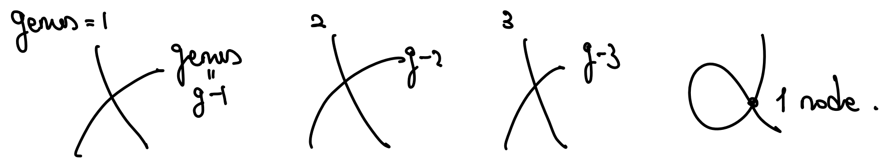
Rough Picture $K_g \subseteq K_g^{ss} \subseteq \overline{K_g}$ $\rightsquigarrow$ $K_g^{ss} / PGL_{r+1} = \overline{M}_g$ proj.

(ss: semi-stable)

Question What is $\overline{M}_g \setminus M_g$? (some geom structure involved.)

Fact $\Delta = \overline{M}_g \setminus M_g$ is a divisor.

Δ is not irred. Each irred comp is the closure of the curves with 1 node.



We have boundary divisors $\Delta_0, \Delta_1, \dots, \Delta_{[\frac{g}{2}]}$.

The ~~other~~ two important divisors: κ, λ

"Suppose" $\pi : \mathcal{C}_g \to \overline{M}_g$ is a universal curve.

$\mathcal{C}_g = \overline{M}_{g,1}$

κ-class : $\pi_*(c_1(\omega_{\mathcal{C}_g/\overline{M}_g})^2)$

λ-class : $c_1(\underbrace{\pi_* \omega_{\mathcal{C}_g/\overline{M}_g}}_{\text{rank } g.})$ (Hodge bundle)

The relation (or Mumford relation):

$$12\lambda - \kappa - [\Delta_0] - \frac{1}{2}[\Delta_1] - [\Delta_2] - \cdots - [\Delta_{[\frac{g}{2}]}] \sim 0 \in \mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}.$$

(pf: using noetherian formula on families of curves over a surface with one parameter.)