HIDA THEORY ON p-ADIC MODULAR FORMS

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ABSTRACT. We give an introduction to Hida's construction of analytic families of ordinary p-adic modular forms and their associated Galois representations. We will explain Hida's control theorems for ordinary p-adic modular forms and show how these theorems have been useful in relating certain Hecke algebras with universal Galois deformation rings. We will also explain examples and open problems on these topics.

CONTENTS

Part 1. Introduction.

We begin with some setups. For convenience, we fix throughout the course an odd prime p (whereas the Hida theory is valid for $p = 2$ as well). Fix field embeddings $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ together with $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. The p-adic valuation and norm on $\mathbb{C}_p$ are normalized such that $v_p(p) = 1$ and $|p|_p = p^{-1}$.

1. THE p-ADIC FAMILY OF EISENSTEIN SERIES

Let $k > 2$ be an *even* integer and consider the Eisenstein series $E_k(z) \in M_k(\mathrm{SL}_2(\mathbb{Z}); \mathbb{C})$ of weight k defined as

$$E_k(z) = \frac{G(k)}{2\zeta(k)}, \quad G_k(z) = \sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(cz+d)^k}.$$

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For any point z lying in the upper-half complex plane, the Eisenstein series admits a q-expansion in $q = e^{2\pi i z}$ as follows:

$$E_k(z) = \frac{\zeta(1-k)}{2} + \sum_{n \geqslant 1} G_{k-1} q^n,$$

where the coefficients are given by

$$\sigma_{k-1}(n) = \sum_{d>0, d|n} d^{k-1}.$$

Note that all coefficients in the q-expansion of $E_k(z)$ are rational, and we can view these σ's as p-adic numbers for prime p.

Keynote Goal. We aim to interpolate the coefficients above in the p-adic sense for various k.

Naively, one can attempt to extend the function $k \mapsto \sigma_{k-1}(n)$ and $k \mapsto \zeta(1-k)/2$ to continuous functions $\mathbb{Z}_p \to \mathbb{Z}_p$. For this, consider first the nonconstant coefficients and it suffices to consider the associations $k \mapsto d^k$ for a positive integer d. But in a sequel, it is clearly impossible to interpolate the map $k \mapsto d^k$ when $p \mid d$. (More detailedly, fix $k_1 \in \mathbb{Z}$ and vary $k_2 > k_1$; observe that $v_p(p^{k_1} - p^{k_2}) = k_1$. So the coefficient function depends badly on k.) Such a phenomenon indicates us to remove those divisor d's of n divisible by p in the expansion of $\sigma_{k-1}(n)$. This leads us to consider the modified coefficients

$$\sigma_{k-1}^{(p)}(n) = \sum_{\substack{d>0 \\ p \nmid d|n}} d^{k-1}.$$

To make these coefficients meaningful, we consider the p-stabilization of the Eisenstein series $E_k(z)$,

$$E_k^{(p)}(z) = E_k(z) - p^{k-1} E_k(pz) = \frac{\zeta_p(k)}{2} + \sum_{n \geqslant 1} \sigma_{k-1}^{(p)}(n) q^n \in M_k(\Gamma_0(p); \mathbb{C}),$$

where $\zeta_p(k) = (1 - p^{k-1})\zeta(1-k)$. It is a general phenomenon that p-adic family of modular forms interpolate p-stabilized Hecke eigenforms. For example, the following result explains how the modified coefficients $\sigma_{k-1}^{(p)}(n)$ depends p-adic continuously on the weight k.

- ⋄ If $k_1 \equiv k_2 \pmod{p^{\alpha-1}(p-1)}$ for some $\alpha \geqslant 1$, then $\sigma_{k_1-1}^{(p)}(n) \equiv \sigma_{k_2-1}^{(p)}(n) \pmod{p^\alpha}$ for each n.

It turns out that the correct track to step in the world of p-adic modular forms is to view the weight k as a continuous homomorphism $\kappa_k : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ via $z \mapsto z^k$.

Definition 1.1. A *p-adic weight* is a continuous homomorphism $\kappa : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$. The *weight space* $\mathcal{W}$ is the rigid analytification of the Iwasawa algebra $\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$; that is, when $L \subseteq \mathbb{C}_p$ is a closed subfield, we have the L-valued points

$$\mathcal{W}(L) = \{\kappa : \mathbb{Z}_p^\times \to L^\times \text{ continuous homomorphism}\}.$$

Remark 1.2. Here comes a geometric interpretation of the weight space $\mathcal{W}$. Consider the natural factor isomorphism

$$\mathbb{Z}_p^\times \cong \Delta \times (1 + p\mathbb{Z}_p)$$

where Δ is the torsion subgroup of $\mathbb{Z}_p^\times$, and $1 + p\mathbb{Z}_p$ is a topologically cyclic subgroup (and hence we fix a topological generator $u = 1 + p \in 1 + p\mathbb{Z}_p$ of it). Denote $\omega_\kappa : \Delta \to \mathbb{Z}_p^\times$ the Teichmüller character induced by the isomorphism above. There exists a bijection

$$\begin{aligned} \mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times) &\longleftrightarrow \Delta^\vee \times \mathbb{D}_1 = \textstyle\bigsqcup_{\chi \in \Delta^\vee} \mathbb{D}_1 \\ \kappa &\longmapsto (\kappa|_\Delta, w_\kappa). \end{aligned}$$

Here we define $w_\kappa := \kappa(u) - 1$. On the right-hand set, $\Delta^\vee = \mathrm{Hom}(\Delta, \mathbb{Z}_p^\times)$ is the character group of Δ, and $\mathbb{D}_1 = \{z \in \mathbb{C}_p : |z|_p < 1\}$ is the rigid open unit disc in $\mathbb{C}_p$.

Therefore, in the rigid-geometric sense, $\mathcal{W}$ is the disjoint union of $p-1$ copies of open unit disc indexed by the characters of Δ. The condition $k_1 \equiv k_2 \pmod{p^{\alpha-1}(p-1)}$ means that the weights κ_{k_1} and κ_{k_2} belong to the same disc and their coordinates are close to each other.

Let k be a positive integer and denote $w_k = w_{\kappa_k} = u^k - 1$ the p-adic weight of k. Consider the p-adic logarithm function $\log : 1 + p\mathbb{Z}_p \to p\mathbb{Z}_p$, which induces an analogue

$$\varphi : 1 + p\mathbb{Z}_p \longrightarrow \mathbb{Z}_p, \quad x \mapsto \frac{\log x}{\log u}$$

Fix $x \in \Delta^\vee$ an even character, i.e. a character such that $\chi(-1) = 1$. Then $\chi = w^a$ for some even $0 \leqslant a \leqslant p-2$. Let $k \geqslant 4$ be an even integer such that $k \equiv a \pmod{p-1}$. Take any $d \in \mathbb{Z}_p^\times \cong \Delta \times (1 + p\mathbb{Z}_p)$, whose image is (d_0, d_1) via this isomorphism. Then

$$d^k = d_0^k \cdot d_1^k = \chi(d_0) \cdot (u^{\varphi(d_1)})^k = \chi(d_0) \cdot (1 + w_k)^{\varphi(d_1)},$$

and therefore

$$\sigma_{k-1}^{(p)}(n) = \sum_{\substack{d>0 \\ p\nmid d \mid n}} d^{k-1} = \sum_{p\nmid d\mid n} \frac{1}{d}\chi(d_0)(1+w_k)^{\varphi(d_1)}.$$

Recall that for each $\alpha \in \mathbb{Z}_p$ we have a power series expansion

$$(1+X)^\alpha = \sum_{n \geqslant 0} \binom{\alpha}{n} X^n \in \mathbb{Z}_p[\![X]\!].$$

Define

$$A_\chi(n; X) := \sum_{p\nmid d\mid n} \frac{\chi(d_0)}{d}(1+X)^{\varphi(d_1)} \in \mathbb{Z}_p[\![X]\!].$$

Then we particularly obtain

$$A_\chi(n; w_k) = \sigma_{k-1}^{(p)}(n)$$

for each admissible k. Also, recall that the L-series for character χ is defined as

$$L(s, \chi) = \sum_{n \geqslant 1} \chi(n) n^{-s}.$$

Proposition 1.3 ([Hid93, §3.6])**.** *For any (even) character $\chi : \Delta \to \mathbb{Z}_p^\times$, there exists a power series $\Phi_\chi(X) \in \mathbb{Z}_p[\![X]\!]$ such that for any $k \geqslant 2$,*

$$\Phi_\chi(w_k) = \Phi_\chi(u^k - 1) = \begin{cases} (1 - (\chi w^{-k})(p) \cdot p^{k-1}) \cdot L(1-k, \chi w^{-k}), & \chi \neq \mathrm{id}, \\ (u^k - 1)(1 - (\chi w^{-k})(p) \cdot p^{k-1}) \cdot L(1-k, \chi w^{-k}), & \chi = \mathrm{id}. \end{cases}$$

Granting this proposition, by definition we get

$$A_\chi(0; X) = \begin{cases} \Phi_\chi(X)/2, & \chi \neq \mathrm{id}, \\ \Phi_\chi(X)/2X, & \chi = \mathrm{id}. \end{cases}$$

The *generalized Eisenstein series* with respect to χ is defined as

$$\mathbb{E}_\chi(X) = \sum_{n=0}^{\infty} A_\chi(n; X) q^n \in \mathrm{Frac}(\mathbb{Z}_p[\![X]\!])[\![q]\!].$$

The relation ship with the p-adic Eisenstein series defined before is given by

$$E_\chi(u^k - 1) = E_k^{(p)}(z)$$

for all $k \geqslant 4$ such that $k \equiv a \pmod{p-1}$.

Fix now a finite extension K over $\mathbb{Q}_p$. Denote $\mathcal{O}$ its ring of integers and $\mathbb{F}$ its residue field (by choosing a uniformizer). Let $\Lambda = \mathcal{O}[\![1 + p\mathbb{Z}_p]\!]$ be a choice of an $\mathcal{O}$-lattice. A priori there is a natural isomorphism $\Lambda \cong \mathcal{O}[\![X]\!]$ by corresponding $[u]$ to $1 + X$.

Definition 1.4 (First definition of adic form)**.** A formal series

$$F(X) = \sum_{n=0}^{\infty} A(n; X) q^n \in \Lambda[\![q]\!]$$

is a Λ-*adic form* of character χ (of tame level 1) if its q-expansion $F(u^k-1)$ gives the q-expansion of a modular form in $M_k(\Gamma_0(p), \chi w^{-k}; \mathcal{O})$ for almost all positive integers k. Also, $F(X)$ is further called a Λ-*adic cusp form* if $F(u^k-1)$ is a cusp form for almost all k.

In fact, there would be three aspects to generalize Definition 1.4:

(a) May allow another tame level;
(b) May allow higher conductor at p;
(c) Can replace Λ by a finite free Λ-algebra, namely, the Hecke algebra for Λ.

Let I be the integral closure of Λ in a finite field extension of $\mathrm{Frac}(\Lambda)$. For any $k \geqslant 1$ with a finite character $\varepsilon : 1+p\mathbb{Z}_p \to \overline{\mathbb{Q}}_p^\times$, define an $\mathcal{O}$-algebra homomorphism $\varphi_{k,\varepsilon} : \Lambda = \mathcal{O}[\![1+p\mathbb{Z}_p]\!] \to \overline{\mathbb{Q}}_p$ corresponding to the character $1+p\mathbb{Z}_p \to \overline{\mathbb{Q}}_p^\times$ that sends z to $z^k \cdot \varepsilon(z)$.

- A point $\varphi \in \mathrm{Spec}(I)(\overline{\mathbb{Q}}_p) = \mathrm{Hom}_{\mathcal{O}}(I, \overline{\mathbb{Q}}_p)$ is called *arithmetic* if $\varphi|_\Lambda = \varphi_{k,\varepsilon}$ for some k, ε.
- When $k \geqslant 2$, φ is called a *classical* point.

Definition 1.5 (Generalized definition of adic form)**.** A formal series $F(q) = \sum_{n\geqslant 0} A_n q^n$ with $A_n \in I$ is called an I-*adic form* of tame level N with respect to Dirichlet character $\chi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \overline{\mathbb{Q}}_p^\times$, if for almost all classical points $\varphi : I \to \overline{\mathbb{Q}}_p$ with $\varphi|_\Lambda = \varphi_{k,\varepsilon}$, the image $\varphi(F(q)) \in \overline{\mathbb{Q}}_p[\![q]\!]$ gives the q-expansion of a modular form in $M_k(\Gamma_0(Np^r), \varepsilon\chi w^{-k}; \overline{\mathbb{Q}}_p)$.

Example 1.6. (1) The generalized Eisenstein series $\mathbb{E}_\chi(X) \in \Lambda[\![q]\!]$ is a Λ-adic form when $\chi : \Delta \to \mathbb{Z}_p^\times$ is nontrivial. Else when $\chi = \mathrm{id}$, we have $X \cdot \mathbb{E}_\chi(X)$ being a Λ-adic form.
(2) For $f \in M_k(\Gamma_0(p), \chi_0; \mathcal{O})$, the product $f \cdot \mathbb{E}_\chi(X) \in \Lambda[\![q]\!]$ is a Λ-adic form when f is a cusp form. Indeed, this $f \cdot \mathbb{E}$ is a Λ-adic cusp form.

We are really interested in those modular forms carrying some arithmetic information. In particular, we mostly care about p-adic family of Hecke eigenforms.

2. The ordinary part of spaces of modular forms

Let $N > 0$ be an integer with an associated Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$. For a $\mathbb{Z}[\chi]$-subalgebra A in $\mathbb{C}$, we consider the space

$$M_k(\Gamma_0(N), \chi; A) = \left\{ f = \sum_{n=0}^{\infty} a(n; f) q^n : a(n; f) \in A \right\} \subseteq M_k(\Gamma_0(N), \chi; \mathbb{C}).$$

One can define $M_k(\Gamma_1(N); A)$ and $S_k(\Gamma_1(N); A)$ similarly for A.

Proposition 2.1. *The space $M_k(\Gamma_1(N); \mathbb{C})$ (resp. $M_k(\Gamma_0(N), \chi; \mathbb{C})$) has a $\mathbb{C}$-basis in $M_k(\Gamma_1(N); \mathbb{Z})$ (resp. $M_k(\Gamma_0(N), \chi; \mathbb{Z}[\chi])$).*

Proof. Refer to [Hid00, III, Proposition 3.1.1] for a geometric proof and to [Hid93, §5.4] for an explicit construction when N is a power of p. $\square$

We introduce some notations as follows. For a subring A of $\mathbb{C}_p$, we define

$$M_k(\Gamma_1(N); A) = M_k(\Gamma_1(N); \mathbb{Z}) \otimes_{\mathbb{Z}} A,$$
$$S_k(\Gamma_1(N); A) = S_k(\Gamma_1(N); \mathbb{Z}) \otimes_{\mathbb{Z}} A.$$

For a $\mathbb{Z}[\chi]$-subalgebra A of $\mathbb{C}_p$, we define

$$M_k(\Gamma_0(N), \chi; A) = M_k(\Gamma_0(N), \chi; \mathbb{Z}[\chi]) \otimes_{\mathbb{Z}} A,$$
$$S_k(\Gamma_0(N), \chi; A) = S_k(\Gamma_0(N), \chi; \mathbb{Z}[\chi]) \otimes_{\mathbb{Z}} A.$$

Definition 2.2. When A is a $\mathbb{Z}[\chi]$-subalgebra of $\mathbb{C}$, define

$$H_k(\Gamma_0(N), \chi; A) = A[T(n)]_{n\in\mathbb{N}} \subseteq \mathrm{End}_A(M_k(\Gamma_0(N), \chi; A)),$$

and similarly for $h_k(\Gamma_1(N); A) \subseteq \mathrm{End}_A(S_k(\Gamma_1(N); A))$.

When A is a $\mathbb{Z}[\chi]$-subalgebra of $\mathbb{C}_p$, define

$$H_k(\Gamma_0(N),\chi;A) = H_k(\Gamma_0(N),\chi;\mathbb{Z}[\chi]) \otimes_{\mathbb{Z}[\chi]} A,$$
$$h_k(\Gamma_1(N);A) = h_k(\Gamma_1(N);\mathbb{Z}[\chi]) \otimes_{\mathbb{Z}[\chi]} A.$$

In both cases, define

$$m_k(\Gamma_0(N),\chi;A) = \{f \in M_k(\Gamma_0(N),\chi;\mathrm{Frac}(A)) : a(n;f) \in A \text{ for all } n \geqslant 1\},$$
$$s_k(\Gamma_1(N);A) = \{f \in S_k(\Gamma_1(N);\mathrm{Frac}(A)) : a(n;f) \in A \text{ for all } n \geqslant 1\}.$$

Theorem 2.3 ([Hid93, §5.3, Theorem 1]). *Let A be a $\mathbb{Z}[\chi]$-subalgebra of $\mathbb{C}$ or $\mathbb{C}_p$. Then the pairing*

$$\begin{aligned}\langle\cdot,\cdot\rangle : H_k(\Gamma_0(N),\chi;A) \times m_k(\Gamma_0(N),\chi;A) &\longrightarrow A \\ (h,f) &\longmapsto a(1,f/h)\end{aligned}$$

is perfect and induced isomorphisms

$$\mathrm{Hom}_A(H_k(\Gamma_0(N),\chi;A),A) \cong m_k(\Gamma_0(N),\chi;A)$$
$$\mathrm{Hom}_A(h_k(\Gamma_1(N);A),A) \cong s_k(\Gamma_1(N);A).$$

Recall that K is a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$.

Lemma 2.4. *Let A be a commutative $\mathcal{O}$-algebra which is free of finite rank over $\mathcal{O}$ with the p-adic topology. For any $x \in A$, the limit $\lim_{n\to\infty} x^{n!}$ exists in A and gives an idempotent of A.*

Proof. Assume first $A = \mathcal{O}_L$ for some finite extension L/K, with q equal to the cardinality of the residue field k_L. Then for each $a \in \mathcal{O}_L^\times$,

$$a^{q^r(q-1)} \equiv 1 \pmod{\mathfrak{m}_L^{r+1}} \implies \lim_{n\to\infty} a^{n!} = 1.$$

For each $a \in \mathfrak{m}_L$, it follows that $a^{n!} \to 0$ as $n \to \infty$.

Now assume $A \otimes_{\mathcal{O}} K$ is semisimple. Then it is isomorphic to a finite product of L_i's, where each L_i is a finite extension over K. For any $x \in A$, the image of x via the isomorphism $A \otimes_{\mathcal{O}} K \simeq \prod_{i=1}^k L_i$ lands in $\prod_{i=1}^k \mathcal{O}_{L_i}$. The assertion follows from the argument when $A = \mathcal{O}_L$.

In general, suppose $A \otimes_{\mathcal{O}} K$ is a finite-dimensional K-algebra. By Wedderburn theorem, $A \otimes_{\mathcal{O}} K = N \oplus S$ for N the nilradical of $A \otimes K$, and S a semisimple K-subalgebra of $A \otimes_{\mathcal{O}} K$. For each $x \in A$ we write $x = m + s$, where m is nilpotent and $s \in S$. From the previous argument we see the limit exists:

$$\lim_{n\to\infty} s^{n!} = \lim_{n\to\infty} s^{p^r(q-1)}.$$

Assume $m^j = 0$ for some integer $j > 0$. Then

$$(s+m)^{p^r(q-1)} = s^{p^r(q-1)} + \sum_{i=1}^{j-1} \binom{p^r(q-1)}{i} s^{p^r(q-1)-i} m^i$$

and for each $1 \leqslant i \leqslant j-1$,

$$v_p\left(\binom{p^r(q-1)}{i}\right) \geqslant r - v_p(j!).$$

This further deduces $\lim_{n\to\infty} x^{n!} = \lim_{n\to\infty} s^{n!}$ is an idempotent. $\square$

Let N be a prime-to-p integer and $r \geqslant 0$. Consider the Dirichlet character

$$\chi : (\mathbb{Z}/Np^{r+1}\mathbb{Z})^\times \to \mathcal{O}^\times$$

and apply Lemma 2.4 to $u(p) \in H_k(\Gamma_0(Np^{r+1},\chi;\mathcal{O}))$ (or alternatively, $h_k(\Gamma_1(Np^{r+1};\mathcal{O}))$, the limit

$$e = \lim_{n\to\infty} u(p)^{n!}$$

which is called the *Hida ordinary projector*, exists in $H_k(\Gamma_0(Np^{r+1},\chi;\mathcal{O}))$ and is idempotent. Take any Hecke eigenform $f \in M_k(\Gamma_0(Np^{r+1}),\chi;\mathcal{O})$ and a $u(p)$-eigenvalue a_p. Then

$$f|_e = \begin{cases} f, & |a_p|_p = 1, \\ 0, & |a_p|_p < 1. \end{cases}$$

When $f|_e = f$, we say f is *p-ordinary*.

Definition 2.5. We define the *ordinary part* of Hecke algebra and spaces of modular forms and cusp forms as

$$\begin{aligned} H_k^{\mathrm{ord}}(\Gamma_0(Np^{r+1},\chi;\mathcal{O})) &= eH_k(\Gamma_0(Np^{r+1},\chi;\mathcal{O})),\\ M_k^{\mathrm{ord}}(\Gamma_0(Np^{r+1},\chi;\mathcal{O})) &= M_k(\Gamma_0(Np^{r+1},\chi;\mathcal{O}))|_e,\\ S_k^{\mathrm{ord}}(\Gamma_0(Np^{r+1},\chi;\mathcal{O})) &= S_k(\Gamma_0(Np^{r+1},\chi;\mathcal{O}))|_e. \end{aligned}$$

And define similarly for h_k, m_k, and s_k.

Lemma 2.6. *The perfect pairing of Theorem 2.3*

$$\langle\cdot,\cdot\rangle : H_k(\Gamma_0(Np^{r+1}),\chi;\mathcal{O}) \times m_k(\Gamma_0(Np^{r+1}),\chi;\mathcal{O}) \longrightarrow \mathcal{O}$$

restricts to the ordinary part, and stays perfect as well.

Remark 2.7. For the ordinarily restricted pairing, we get an idempotent factorization

$$H_k(\Gamma_0(Np^{r+1}),\chi;\mathcal{O}) = H_k^{\mathrm{ord}}(\Gamma_0(Np^{r+1}),\chi;\mathcal{O}) \times (1-e)H_k(\Gamma_0(Np^{r+1}),\chi;\mathcal{O}).$$

Here the image of $u(p)$ in the first factor is a unit, and that in the second factor is a topological nilpotent element.

Similarly, for a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathcal{O}^\times$, we can use the Hecke $T(p)$-operator (see Definition 4.3(2)) in $H_k(\Gamma_0(N),\chi;\mathcal{O})$ (resp. $h_k(\Gamma_0(N),\chi;\mathcal{O})$) to define an idempotent e_0 in $H_k(\Gamma_0(N),\chi;\mathcal{O})$ (resp. $h_k(\Gamma_0(N),\chi;\mathcal{O})$), and then define the ordinary part of Hecke algebras and spaces of modular forms $H_k^{\mathrm{ord}}(\Gamma_0(N),\chi;\mathcal{O})$, etc..

Note that we have a natural bijection

$$\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times,\mathcal{O}^\times) \cong \mathrm{Hom}_{\mathcal{O}\text{-alg}}(\mathcal{O}[\![\mathbb{Z}_p^\times]\!],\mathcal{O}).$$

Let $\nu : \mathbb{Z}_p^\times \to \mathcal{O}^\times$ be the character defined via inclusion. For any $k \geqslant 1$, we also use ν^k to denote the corresponding $\mathcal{O}$-algebra homomorphism $\mathcal{O}[\![\mathbb{Z}_p^\times]\!] \to \mathcal{O}$ corresponding to the character ν^k. Let $\Lambda = \mathcal{O}[\![1+p\mathbb{Z}_p]\!]$ be the Iwasawa algebra as before, and fix an isomorphism $\Lambda \cong \mathcal{O}[\![X]\!]$ by sending $[u] = [1+p]$ to $1+X$. We may also view ν^k as a character of $1+p\mathbb{Z}_p$ by restriction. Then $\nu^k(\Psi(X)) = \Psi(u^k-1) \in \mathcal{O}$ for all $\Psi(X) \in \mathcal{O}[\![X]\!]$. The main theorem is the following.

Theorem 2.8 (Hida)**.** *Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathcal{O}^\times$ be the Dirichlet character. Then there is a finitely generated projective $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$-module $S(\Gamma_0(N),\chi;\Lambda)$, called the space of ordinary Λ-adic cusp forms of level Np^∞, which carries an action of Hecke operators such that we have the following Hecke equivariant isomorphisms:*

$$S(\Gamma_0(N),\chi;\Lambda) \otimes_{\mathcal{O}[\![\mathbb{Z}_p^\times]\!],\nu^k} \mathcal{O} \cong S_k^{\mathrm{ord}}(\Gamma_0(Np),\chi;\mathcal{O}) \cong S_k^{\mathrm{ord}}(\Gamma_0(N),\chi;\mathcal{O}), \quad k \geqslant 3,$$

$$S(\Gamma_0(N),\chi;\Lambda) \otimes_{\Lambda,\nu^k} \mathcal{O} \cong \bigoplus_{i=0}^{p-2} S_k^{\mathrm{ord}}(\Gamma_0(Np),\chi\omega^i;\mathcal{O}), \quad k \geqslant 2.$$

Remark 2.9. One subtlety of this main result lies in that it fails in general for $k \geqslant 1$. Also, the reader should be careful that the isomorphism

$$S_k^{\mathrm{ord}}(\Gamma_0(Np),\chi;\mathcal{O}) \cong S_k^{\mathrm{ord}}(\Gamma_0(N),\chi;\mathcal{O})$$

for the first case above is not induced naively by the inclusions

$$S_k^{\mathrm{ord}}(\Gamma_0(N),\chi;\mathcal{O}) \hookrightarrow S_k(\Gamma_0(N),\chi;\mathcal{O}) \hookrightarrow S_k(\Gamma_0(Np),\chi;\mathcal{O}).$$

This is because the Hecke operator action on $S_k(\Gamma_0(N),\chi;\mathcal{O})$ is by $T(p)$, whereas it is by $U(p)$ on $S_k(\Gamma_0(Np),\chi;\mathcal{O})$. In fact, newforms in $S_k(\Gamma_0(Np),\chi;\mathcal{O})$ has $U(p)$-eigenvalues $\pm\sqrt{\chi(p)}\cdot p^{(k-2)/2}$.

Part 2. Cohomological approach to modular forms

3. Preliminaries on group cohomology

The presented cohomological approach is good at proving the following fundamental theorem on ordinary families.

Theorem 3.1. *Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{Q}}_p^\times$ be a Dirichlet character. Suppose there is a field K containing $\mathbb{Q}_p(\chi)$. Then for any $k \geqslant 2$ we have*

$$\mathrm{rank}_{\mathcal{O}}(S_k^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-k}; \mathcal{O})) = \mathrm{rank}_{\mathcal{O}}(S_2(\Gamma_0(Np), \chi\omega^{-2}; \mathcal{O})),$$
$$\mathrm{rank}_{\mathcal{O}}(M_k^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-k}; \mathcal{O})) = \mathrm{rank}_{\mathcal{O}}(M_2(\Gamma_0(Np), \chi\omega^{-2}; \mathcal{O})).$$

Let Γ be a torsion-free congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, for example, $\Gamma = \Gamma_1(N)$ for some $N \geqslant 4$. Let $\mathfrak{H}$ denote the upper-half complex plane. Consider the modular curve $Y(\Gamma) = \Gamma\backslash\mathfrak{H}$ as well as its compactification $X(\Gamma)$. Take the set $S = X(\Gamma)\backslash Y(\Gamma)$. Then for any point $s \in S$ we define Γ_s to be the stabilizer of s in Γ. The assumptions above imply that Γ_s is cyclic, and we take a generator π_s of it. Finally, let P be the set of all Γ-conjugates of π_s for all $s \in S$.

Definition 3.2. Let R be a commutative ring, G a group, and M a left $R[G]$-module.

(1) Define $H^i(G, -) = R^i(-)^G$ to be the ith derived functor of $(-)^G$, where $(-)^G$ is the functor from the category of $R[G]$-modules to that of R-modules by sending M to M^G. In particular, $H^0(G, M) = M^G$. We list the cocycle conditions below:
- $Z^1(G, M) = \{f : G \to M \mid \forall g, g' \in G,\ f(gg') = f(g) + gf(g')\}$,
- $B^1(G, M) = \{f : G \to M \mid \exists m \in M,\ \forall g \in G,\ f(g) = gm - m\}$,
- $Z^2(G, M) = \{f : G \times G \to M \mid \forall g_1, g_2, g_3 \in G,\ g_1 f(g_2, g_3) - f(g_1g_2, g_3) + f(g_1, g_2g_3) - f(g_1, g_2) = 0\}$,
- $B^2(G, M) = \{f : G \times G \to M \mid \exists h : G \to M,\ \forall g_1, g_2 \in G,\ f(g_1, g_2) = d^1(h)(g_1, g_2) = g_1h(g_2) - h(g_1g_2) + h(g_1)\}$,

and $H^i(G, M) = Z^i(G, M)/B^i(G, M)$ for $i = 1, 2$. Also, the differential map of the corresponding complex $d^i : C^i(G, M) \to C^{i+1}(G, M)$ is given as

$$d^0(m)(g) = gm - m,$$

and for $i \geqslant 1$,

$$d^i(f)(g_1, \dots, g_{i+1}) = g_1 \cdot f(g_2, \dots, g_{i+1}) + (-1)^{i+1} f(g_1, \dots, g_i) + \sum_{j=1}^{i} (-1)^j f(g_1, \dots, g_jg_{j+1}, \dots, g_{i+1}).$$

(2) For $G = \Gamma$, we define the parabolic cohomological cycles and boundaries as

$$Z_P^1(\Gamma, M) = \{f \in Z^1(G, M) \mid \forall \pi \in P,\ f(\pi) \in (\pi - 1)M\} \subseteq Z^1(\Gamma, M),$$
$$B_P^2(\Gamma, M) = \{f = d^1(h) \mid h : \Gamma \to M,\ \forall \pi \in P,\ h(\pi) \in (\pi - 1)M\} \subseteq B^2(\Gamma, M).$$

Then define the parabolic cohomology via

$$H_P^1(\Gamma, M) = Z_P^1(\Gamma, M)/B^1(\Gamma, M),$$
$$H_P^2(\Gamma, M) = Z^2(\Gamma, M)/B_P^2(\Gamma, M).$$

In fact, there is an exact sequence (see [Hid93, Appendix, Proposition 2])

$$0 \to H_P^1(\Gamma, M) \to H^1(\Gamma, M) \to \bigoplus_{s \in S} H^1(\Gamma_s, M) \to H_P^2(\Gamma, M) \to H^2(\Gamma, M) \to 0.$$

As a consequence, if $R \to A$ is a flat homomorphism, then

$$H_P^1(\Gamma, M \otimes_R A) = H_P^1(\Gamma, M) \otimes_R A,$$
$$H^i(\Gamma, M \otimes_R A) = H^i(\Gamma, M) \otimes_R A.$$

Proposition 3.3 ([Hid93, §6.1, Proposition 1])**.** *Recall that* Γ *is a torsion-free congruence subgroup of* $\mathrm{SL}_2(\mathbb{Z})$*. For any* Γ*-module* M*, we have*

$$H_P^2(\Gamma, M) \cong M/DM, \quad H^2(\Gamma, M) = 0.$$

Here $DM = \sum_{r \in \Gamma}(r-1)M$*.*

Let $\varphi : H \to G$ be a group homomorphism. For any R-module M, there is an induced $R[H]$-module from M via φ. We consider

$$\begin{array}{ccc} H^0(G, M) & \longrightarrow & H^0(H, M) \\ \| & & \\ M^G & \hookrightarrow & H^G. \end{array}$$

This gives a natural transformation called *restriction*, written as

$$\mathrm{Res}^n : H^n(G, -) \longrightarrow H^n(H, -).$$

Again, if $\varphi : H \to G$ is a subgroup of finite index d in G. Let $\{r_1, \dots, r_d\}$ be a set of representatives of G/H. then there is a norm map

$$N_{G/H}(m) = \sum_{i=1}^{d} r_i(m) : M^H \longrightarrow M^G.$$

Formally, this again induces the *corestriction* functor

$$\mathrm{Cores}^n : H^n(H, -) \longrightarrow H^n(G, -).$$

Lemma 3.4. *The composite*

$$\mathrm{Cores}^n \circ \mathrm{Res}^n : H^n(G, -) \longrightarrow H^n(G, -)$$

is the multiplication-by-d *map.*

Proof. See [Ser79, VII, §7, Proposition 6]. □

4. On Eichler–Shimura isomorphism

For any nonnegative integer n and commutative ring R, define

$$L(n; R) := \{P(X, Y) \in R[X, Y] : P(X, Y) \text{ is homogeneous of degree } n\}.$$

We define a left action of the semigroup $\mathrm{M}_2(\mathbb{Z})_{\neq 0} = \mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (P(X, Y)) = P\left(\begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = P(aX + cY, bX + dY).$$

Let N be a positive integer. Consider its associated Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to R^\times$. Denote $L(n, \chi; R)$ for the R-module $L(n; R)$ but in additional with an action of $\Gamma_0(N)$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (P(X, Y)) = \chi(d) \cdot P(aX + cY, bX + dY).$$

This action extends to

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z})_{\neq 0} : c \equiv 0 \bmod N,\ (d, N) = 1 \right\}.$$

Now for $z_0 \in \mathfrak{H}$ with $f \in M_k(\Gamma; \mathbb{C})$, we define a map $\varphi_{z_0}(f) : \Gamma \to L(k-2; \mathbb{C})$ via

$$\varphi_{z_0}(f)(\gamma) := \int_{z_0}^{\gamma(z_0)} f(z)(Xz + Y)^{k-2} dz \in L(k-2; \mathbb{C}), \quad \gamma \in \Gamma.$$

This map admits the following properties:

- $\varphi_{z_0}(f) \in Z^1(\Gamma, L(k-2; \mathbb{C}))$;
- $\varphi_{z_0}(f) - \varphi_{z_0'}(f) \in B^1(\Gamma, L(k-2; \mathbb{C}))$ for another $z_0' \in \mathfrak{H}$;
- $\varphi_{z_0}(f)(\pi_s) \in (\pi_s - 1)L(\Gamma, L(k-2; \mathbb{C}))$ when $f \in S_k(\Gamma; \mathbb{C})$.

Therefore, we get a well-defined map

$$\varphi : M_k(\Gamma;\mathbb{C}) \longrightarrow H^1(\Gamma, L(k-2;\mathbb{C}))$$

which induces

$$\varphi : S_k(\Gamma;\mathbb{C}) \longrightarrow H^1_P(\Gamma, L(k-2;\mathbb{C})).$$

Theorem 4.1 (Eichler–Shimura). *For any $k \geqslant 2$ and a torsion-free congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, the Eichler–Shimura map*

$$\begin{aligned} M_k(\Gamma;\mathbb{C}) \oplus \overline{S_k(\Gamma;\mathbb{C})} &\longrightarrow H^1(\Gamma, L(k-2;\mathbb{C})) \\ (f,\overline{g}) &\longmapsto \varphi(f) + \overline{\varphi(g)} \end{aligned}$$

is an isomorphism of $\mathbb{C}$-vector spaces. It restricts to an isomorphism

$$S_k(\Gamma;\mathbb{C}) \oplus \overline{S_k(\Gamma;\mathbb{C})} \longrightarrow H^1_P(\Gamma, L(k-2;\mathbb{C})).$$

Corollary 4.2. *Fix an integer $N \geqslant 4$.*

(1) *The map*

$$\begin{aligned} S_k(\Gamma_1(N);\mathbb{C}) &\longrightarrow H^1_P(\Gamma_1(N), L(k-2;\mathbb{R})) \\ f &\longmapsto \mathrm{Re}(\varphi(f)) \end{aligned}$$

is an isomorphism of $\mathbb{R}$-vector spaces.

(2) *For the Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, we have isomorphisms of $\mathbb{C}$-vector spaces:*

$$\begin{aligned} M_k(\Gamma_0(N),\chi;\mathbb{C}) \oplus \overline{S_k(\Gamma_0(N),\chi;\mathbb{C})} &\cong H^1(\Gamma_0(N), L(k-2,\chi;\mathbb{C})), \\ S_k(\Gamma_0(N),\chi;\mathbb{C}) \oplus \overline{S_k(\Gamma_0(N),\chi;\mathbb{C})} &\cong H^1_P(\Gamma_0(N), L(k-2,\chi;\mathbb{C})). \end{aligned}$$

For a matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z})_{\neq 0}$, we denote $\alpha^\iota = \det(\alpha)\cdot\alpha^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Definition 4.3. Let R be a commutative ring and $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup.

(1) Fix some $\alpha \in \mathrm{M}_2(\mathbb{Z})_{\neq 0}$. Let M be an $R[\Gamma]$-module whose Γ-action extends to the semigroup generated by Γ and α. Fix a decomposition $\Gamma\alpha\Gamma = \bigsqcup_{i=1}^n \Gamma\alpha_i$. Define a map

$$\tau_\alpha : H^1(\Gamma, M) \to H^1(\Gamma, M)$$

as follows:

$$\tau_\alpha(f)(\gamma) = \sum_{i=1}^n \alpha_i^\iota \cdot f(\gamma_i), \quad f \in Z^1(\Gamma, M),\ \gamma \in \Gamma.$$

Here γ_i is defined via the equality $\alpha_i\gamma = \gamma_i\alpha_{j_i}$ for some (unique) $j_i \in \{1,\dots,m\}$.

(2) Assuming $\Gamma = \Gamma_1(N)$ for $N \geqslant 4$. Fix another positive integer n. Set

$$\Delta = \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}) : a \equiv 1 \bmod N,\ c \equiv 0 \bmod N,\ \det(\alpha) = n \right\}.$$

Then M has an action of Δ^ι. Fix a decomposition $\Delta = \bigsqcup_{j=1}^m \Gamma\alpha_j\Gamma$. We define the Hecke T-operator

$$T_n : H^1(\Gamma, M) \longrightarrow H^1(\Gamma, M), \quad T(n) = \sum_{i=1}^m \tau_{\alpha_j}.$$

In particular, when $n = p$ is a prime, we have

$$T(p) = \tau_{\alpha_p}, \quad \alpha_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}.$$

One can easily check that the map

$$\varphi : M_k(\Gamma;\mathbb{C}) \longrightarrow H^1(\Gamma_1(N), L(k-2;\mathbb{C}))$$

is Hecke equivariant, i.e. it is compatible with Hecke operators on the source and target. To verify this, we can show that

$$\gamma^\iota\left(\int_{z_1}^{z_2} f(z)(Xz+Y)^{k-2}dz\right)=\int_{z_1}^{z_2} f|_\gamma(z)(Xz+Y)^{k-2}dz$$

for $\gamma\in \mathrm{M}_2(\mathbb{Z})$ with $\det(\gamma)>0$ and

$$f|_\gamma(z)=\det(\gamma)^{k-1}(cz+d)^{-k}f\left(\frac{az+b}{cz+d}\right).$$

Remark 4.4. Indeed, one can define the $T(n)$-operator on $H^i(\Gamma,M)$ for all $i\geqslant 0$. However, we almost work for $i=1$ (and hardly for $i=2$).

Corollary 4.5. *Let L and L' be the quotients of $H^1_P(\Gamma,L(n;\mathbb{Z}))$ and $H^1(\Gamma,L(n;\mathbb{Z}))$ by their maximal torsion subgroups, respectively. Then*

$$L\cong \mathrm{im}(H^1_P(\Gamma,L(n;\mathbb{Z}))\longrightarrow H^1_P(\Gamma,L(n;\mathbb{R})),$$
$$L'\cong \mathrm{im}(H^1(\Gamma,L(n;\mathbb{Z}))\longrightarrow H^1(\Gamma,L(n;\mathbb{R})),$$

and

$$L\otimes_{\mathbb{Z}}\mathbb{R}=H^1_P(\Gamma,L(n;\mathbb{R})),$$
$$L'\otimes_{\mathbb{Z}}\mathbb{R}=H^1(\Gamma,L(n;\mathbb{R})).$$

Thus we can identify L as a $\mathbb{Z}$-lattice of the $\mathbb{R}$-vector space $S_k(\Gamma;\mathbb{C})$ with $k=n+2$.

Definition 4.6. (1) Let A be a subring of $\mathbb{C}$. Define $h_k(\Gamma_1(N);A)$ to be the A-subalgebra of $\mathrm{End}_A(L\otimes_{\mathbb{Z}}A)$ generated by Hecke operators $T(n)$.

(2) Let A be a subring of $\mathbb{C}_p$. Define $h_k(\Gamma_1(N);A)=h_k(\Gamma_1(N);\mathbb{Z})\otimes_{\mathbb{Z}}A$.

It turns out that when $A=\mathcal{O}$, we have $T(p)\in h_k(\Gamma_1(N);\mathcal{O})$, and then the ordinary projector e. Under the above definition, we have

$$H_k(\Gamma_1(N);A)=H_k(\Gamma_1(N);\mathbb{Z})\otimes_{\mathbb{Z}}A,$$
$$h_k(\Gamma_1(N);A)=h_k(\Gamma_1(N);\mathbb{Z})\otimes_{\mathbb{Z}}A.$$

Also, there are natural homomorphisms between A-algebras:

$$H_k(\Gamma_1(N);A)\longrightarrow H_k(\Gamma_0,\chi;A),$$
$$h_k(\Gamma_1(N);A)\longrightarrow h_k(\Gamma_0,\chi;A).$$

5. Proof of Theorem 3.1

Theorem 5.1. *For all primes $p\geqslant 3$ such that $(N,p)=1$, and integers $r\geqslant 1$,*

$$\mathrm{rank}_{\mathbb{Z}_p} S_k^{\mathrm{ord}}(\Gamma_1(Np^r);\mathbb{Z}_p)<C(N,p,r)$$

are bounded, and the boundaries $C(N,p,r)$ are independent of $k\geqslant 2$.

Proof. Write $\Gamma=\Gamma_1(Np^r)$ and $n=k-2$ for simplicity. Let L' be the image of $H^1(\Gamma,L(n;\mathbb{Z}))$ in $H^1(\Gamma,L(n;\mathbb{R}))$, then $L=L'\cap H^1_P(\Gamma,L(n;\mathbb{R}))$. Then L is a lattice in $H^1_P(\Gamma,L(n;\mathbb{R}))$. We set $L_p=L\otimes_{\mathbb{Z}}\mathbb{Z}_p$. So

- $h_k(\Gamma;\mathbb{Z})$ is a $\mathbb{Z}$-subalgebra of $\mathrm{End}_{\mathbb{Z}}(L)$, and
- $h_k(\Gamma;\mathbb{Z}_p)$ is a $\mathbb{Z}_p$-subalgebra of $\mathrm{End}_{\mathbb{Z}_p}(L_p)$.

It follows that $h_k^{\mathrm{ord}}(\Gamma;\mathbb{Z}_p)$ is a subalgebra of $\mathrm{End}_{\mathbb{Z}_p}(eL_p)$. Therefore, it suffices to show that $\mathrm{rank}_{\mathbb{Z}_p} eL_p$ has a bound which is independent of $k\geqslant 2$.

A priori there is a short exact sequence of $\mathbb{Z}_p[\Gamma]$-modules

$$0\longrightarrow L(n;\mathbb{Z}_p)\xrightarrow{p} L(n;\mathbb{Z}_p)\longrightarrow L(n;\mathbb{F}_p)\longrightarrow 0.$$

It induces a long exact sequence

$$\cdots\longrightarrow H^1(\Gamma,L(n;\mathbb{Z}_p))\xrightarrow{p} H^1(\Gamma,L(n;\mathbb{Z}_p))\longrightarrow H^1(\Gamma,L(n;\mathbb{F}_p))\longrightarrow\cdots$$

and hence

$$\mathrm{Ker}(H^1(\Gamma,L(n;\mathbb{Z}_p))\otimes_{\mathbb{Z}_p}\mathbb{F}_p\hookrightarrow H^1(\Gamma,L(n;\mathbb{F}_p)))=0.$$

Note that $L \hookrightarrow L'$ and $L/pL \hookrightarrow L'/pL'$. Then $\Gamma, L(n;\mathbb{Z}_p)) \otimes_{\mathbb{Z}_p} \mathbb{F}_p \twoheadrightarrow L'/pL'$ is surjective. It boils down to show that $\dim_{\mathbb{F}_p} eH^1(\Gamma, L(n;\mathbb{F}_p))$ is independent of k. For this, we aim to establish an isomorphism

$$eH^1(\Gamma, L(n;\mathbb{F}_p)) \cong eH^1(\Gamma, L(0;\mathbb{F}_p)) = eH^1(\Gamma, \mathbb{F}_p).$$

Define

$$\varphi : L(n;\mathbb{F}_p) \longrightarrow \mathbb{F}_p, \quad P(X,Y) \longmapsto P(0,1).$$

It can be checked that φ is $\Gamma = \Gamma_1(Np^r)$-equivariant. Moreover, $\operatorname{Ker}\varphi$ is generated by monomials of forms $X^{n-i}Y^i$ for all $i = 0, \dots, n-1$. Then there is a short exact sequence of $\mathbb{F}_p[\Gamma]$-modules:

$$0 \longrightarrow \operatorname{Ker}(\varphi) \longrightarrow L(n;\mathbb{F}_p) \longrightarrow \mathbb{F}_p \longrightarrow 0.$$

It induces a long exact sequence

$$\cdots \longrightarrow H^1(\Gamma, \operatorname{Ker}(\varphi)) \longrightarrow H^1(\Gamma, L(n;\mathbb{F}_p)) \longrightarrow H^1(\Gamma, \mathbb{F}_p) \longrightarrow H^2(\Gamma, \operatorname{Ker}(\varphi)) \longrightarrow \cdots.$$

Note that the operators $\alpha_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ and $\alpha_p^\iota = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ acts on $X^{n-i}Y^i$ via

$$\alpha_p^\iota(X^{n-i}Y^i) = (pX)^{n-i}Y^i,$$

and then $\alpha_p^\iota \cdot \operatorname{Ker}(\varphi) = 0$. Thus, for $i = 1, 2$,

$$T(p) \cdot H^i(\Gamma, \operatorname{Ker}(\varphi)) = 0.$$

Finally, by applying e to the long exact sequence above, we see

$$eH^1(\Gamma, L(n;\mathbb{F}_p)) \cong eH^1(\Gamma, \mathbb{F}_p).$$

This completes the proof. □

Lemma 5.2. *Fix a prime $p \geqslant 5$. For any $\mathcal{O}[\Gamma_0(Np)]$-module M, we have $H^2(\Gamma_0(Np), M) = 0$.*

Proof. Take $\Gamma = \Gamma_0(Np) \cap \Gamma_1(p)$, which is a torsion-free congruence subgroup with index $[\Gamma_0(Np) : \Gamma] = p-1$. By Lemma 3.4, the following composition or restriction and corestriction is the multiplication-by-$(p-1)$ map, and hence an isomorphism.

$$\begin{tikzcd}
H^2(\Gamma_0(Np), M) \arrow[r, "\mathrm{Res}^2"] \arrow[rr, bend left, "\underset{\simeq}{\times(p-1)}"] \arrow[d, equal] & H^2(\Gamma, M) \arrow[r, "\mathrm{Cores}^2"] \arrow[d, equal] & H^2(\Gamma_0(Np), M). \\
0 & 0 &
\end{tikzcd}$$

So it follows that $H^2(\Gamma_0(Np), M) = H^2(\Gamma, M) = 0$ by Proposition 3.3. □

Set $n = k-2$ as before. Let $\psi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \mathcal{O}^\times$ be a Dirichlet character. From the exact sequence of $\mathcal{O}[\Gamma_0(Np)]$-modules

$$0 \longrightarrow L(n,\psi;\mathcal{O}) \xrightarrow{\pi} L(n,\psi;\mathcal{O}) \longrightarrow L(n,\psi;\mathbb{F}) \longrightarrow 0,$$

we have a long exact sequence:

$$\begin{aligned}
&\cdots \longrightarrow H^0(\Gamma_0(Np), L(n,\psi;\mathbb{F})) \\
&\longrightarrow H^1(\Gamma_0(Np), L(n,\psi;\mathcal{O})) \xrightarrow{\pi} H^1(\Gamma_0(Np), L(n,\psi;\mathcal{O})) \longrightarrow H^1(\Gamma_0(Np), L(n,\psi;\mathbb{F})) \\
&\longrightarrow H^2(\Gamma_0(Np), L(n,\psi;\mathcal{O})) = 0.
\end{aligned}$$

Hence $H^1(\Gamma_0(Np), L(n,\psi;\mathcal{O})) \otimes_{\mathcal{O}} \mathbb{F} \cong H^1(\Gamma_0(Np), L(n,\psi;\mathbb{F}))$, and by applying the idempotent e associated to the $T(p)$-operator, we get an isomorphism

$$eH^1(\Gamma_0(Np), L(n,\psi;\mathcal{O})) \otimes_{\mathcal{O}} \mathbb{F} \cong eH^1(\Gamma_0(Np), L(n,\psi;\mathbb{F})).$$

From the short exact sequence, there is also a surjective map (after applying e again)

$$eH^0(\Gamma_0(Np), L(n,\psi;\mathbb{F})) \twoheadrightarrow eH^1(\Gamma_0(Np), L(n,\psi;\mathcal{O}))[\pi].$$

By definition, the $T(p)$-operator on $H^0(\Gamma_0(Np), L(n,\psi;\mathbb{F}))$ is given by

$$T(p)(X^{n-j}Y^j) = \sum_{i=0}^{p-1} \begin{pmatrix} 1 & i \\ 0 & p \end{pmatrix}^{\iota} \cdot (X^{n-j}Y^j) = \sum_{i=0}^{p-1} (pX)^{n-j}(-iX+Y)^j.$$

In particular, we have $T(p)(X^{n-j}Y^j) = 0$ for $0 \leqslant j < n$. On the other hand, note that $T(p)(Y^n)$ has no Y^n-term as we are in $L(n,\psi;\mathbb{F})$. So we have

$$\begin{aligned} T(p)^2 H^0(\Gamma_0(Np), L(n,\psi;\mathbb{F})) = 0 &\implies eH^0(\Gamma_0(Np), L(n,\psi;\mathbb{F})) = 0 \\ &\implies eH^1(\Gamma_0(Np), L(n,\psi;\mathcal{O}))[\pi] = 0. \end{aligned}$$

It follows that $eH^1(\Gamma_0(Np), L(n,\psi;\mathcal{O}))$ is torsion-free as an $\mathcal{O}$-module. Combining this with the isomorphism on eH^1's, we attain

$$\operatorname{rank}_{\mathcal{O}} eH^1(\Gamma_0(Np), L(n,\psi;\mathcal{O})) = \dim_{\mathbb{F}} eH^1(\Gamma_0(Np), L(n,\psi;\mathbb{F})).$$

As in the proof of Theorem 5.1, we consider the following map

$$\varphi : L(n,\psi;\mathbb{F}) \longrightarrow L(0,\psi\omega^n;\mathbb{F}), \quad P(X,Y) \longmapsto P(0,1).$$

Again it is straightforward to check that φ is $\Gamma_0(Np)$-equivariant. So we have an exact sequence of $\mathbb{F}[\Gamma_0(Np)]$-modules:

$$0 \longrightarrow \operatorname{Ker}(\varphi) \longrightarrow L(n,\psi;\mathbb{F}) \xrightarrow{\varphi} L(0,\psi\omega^n;\mathbb{F}) \longrightarrow 0.$$

It induces a long exact sequence of cohomology groups:

$$\begin{aligned} \cdots \longrightarrow H^1(\Gamma_0(Np), \operatorname{Ker}(\varphi)) \longrightarrow H^1(\Gamma_0(Np), L(n,\psi;\mathbb{F})) \longrightarrow H^1(\Gamma_0(Np), L(0,\psi\omega^n;\mathbb{F})) \\ \longrightarrow H^2(\Gamma_0(Np), \operatorname{Ker}(\varphi)) = 0. \end{aligned}$$

Moreover, as in the proof of Theorem 5.1, we have

$$T(p) \cdot H^1(\Gamma_0(Np), \operatorname{Ker}(\varphi)) = 0.$$

Applying the idempotent e to the above exact sequence, we obtain an isomorphism

$$eH^1(\Gamma_0(Np), L(n,\psi;\mathbb{F})) \cong eH^1(\Gamma_0(Np), L(0,\psi\omega^n;\mathbb{F})).$$

Consequently, the rank-dimension formula above together with this isomorphism render that

$$\operatorname{rank}_{\mathcal{O}} eH^1(\Gamma_0(Np), L(n,\psi;\mathcal{O})) = \operatorname{rank}_{\mathcal{O}} eH^1(\Gamma_0(Np), L(0,\psi\omega^n;\mathcal{O})).$$

Now let $\psi = \chi\omega^{-k}$. By Eichler–Shimura isomorphism, we have

$$\begin{aligned} &\operatorname{rank}_{\mathcal{O}} M_k^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-k};\mathcal{O}) + \operatorname{rank}_{\mathcal{O}} S_k^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-k};\mathcal{O}) \\ &= \operatorname{rank}_{\mathcal{O}} eH^1(\Gamma_0(Np), L(n,\chi\omega^{-k};\mathcal{O})) \\ &= \operatorname{rank}_{\mathcal{O}} eH^1(\Gamma_0(Np), L(0,\chi\omega^{-2};\mathcal{O})) \\ &= \operatorname{rank}_{\mathcal{O}} M_2^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-2};\mathcal{O}) + \operatorname{rank}_{\mathcal{O}} S_2^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-2};\mathcal{O}). \end{aligned}$$

On the other hand, it follows from the proof of [Hid86a, Lemma 5.3] that the rank of the space

$$\mathcal{E}_k^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-k};\mathcal{O}) := M_k^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-k};\mathcal{O}) / S_k^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-k};\mathcal{O})$$

is independent of k. In fact, [Hid86a, §5] gives an explicit basis of $e\mathcal{E}_k(\Gamma_1(Np^r);\mathbb{Q})$ consisting of Eisenstein series and we can deduce the results for $\mathcal{E}_k^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-k};\mathcal{O})$ by computing these Eisenstein series.

Therefore for $k \geqslant 2$, we conclude that

$$\begin{aligned} \operatorname{rank}_{\mathcal{O}} S_k^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-k};\mathcal{O}) &= \operatorname{rank}_{\mathcal{O}} S_2^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-2};\mathcal{O}), \\ \operatorname{rank}_{\mathcal{O}} M_k^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-k};\mathcal{O}) &= \operatorname{rank}_{\mathcal{O}} M_2^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-2};\mathcal{O}). \end{aligned}$$

So we complete the proof of Theorem 3.1.

Part 3. Geometric approach to modular forms

There are at least three approached for the interpolation result of Hida.

(1) Computations on group cohomology [Hid86a].
(2) Geometric interpretation of (p-adic) modular forms (which will be defined later) [Hid86b].
(3) Λ-adic forms (due to Wiles).

This course follows an adopted version of (2) as it has the potential to generalization to automorphic forms on some Shimura varieties. The philosophy is to view the modular forms as global sections of the structure sheaf on some modular curves.

6. Geometric modular forms

6.1. Basic definitions and properties.

Definition 6.1 (Level structure)**.** Let $\pi : E \to S$ be an elliptic curve. For any integer $N \geqslant 1$, denote $E[N]$ the kernel of multiplication-by-N morphism $[N]_E : E \to E$.

(1) Assume that N is invertible on S, i.e. S is a $\mathbb{Z}[\frac{1}{N}]$-scheme.
  (a) A $\Gamma_(N)$-level structure on E/S is an isomorphism
$$\phi_{\Gamma(N)} : \underline{(\mathbb{Z}/N\mathbb{Z})^2}_S \to E[N]$$
  of finite flat group schemes over S. Here $\underline{G}_S$ is the constant group scheme on S defined by some abelian group G.
  (b) A $\Gamma_1(N)$-level structure on E/S is an injective morphism
$$\phi_{\Gamma_1(N)} : \underline{\mathbb{Z}/N\mathbb{Z}}_S \hookrightarrow E[N].$$
  (c) A $\Gamma_0(N)$-level structure on E/S is a subgroup scheme C/S of $E[N]$ which is cyclic of order N, i.e. C becomes isomorphic to $\underline{\mathbb{Z}/N\mathbb{Z}}_S$ after a finite étale extension of S.

(2) Let p be a prime (not necessarily invertible on S). A $\Gamma_0(p)$-level structure on E/S is a finite flat subgroup scheme H/S in E/S of rank p.

Remark 6.2. If S is a $\mathbb{Z}[\frac{1}{N}, \zeta_N]$-scheme, and if we denote
$$e_N : E[N] \times E[N] \longrightarrow \mu_N$$
the canonical Weil pairing on E/S, then we say that a $\Gamma(N)$-structure $\phi_{\Gamma(N)}$ on E/S has determinant ζ_N whenever
$$e_N(\phi_{\Gamma(N)}(1,0), \phi_{\Gamma(N)}(0,1)) = \zeta_N \in \mu_N(S).$$
An arithmetic $\Gamma(N)$-level structure on E/S is an isomorphism
$$\psi_{\Gamma(N)} : \underline{\mu_{N,S}} \times \underline{\mathbb{Z}/N\mathbb{Z}}_S \xrightarrow{\sim} E[N]$$
of finite flat group schemes over S such that the pairing induced by Cartier duality on the left hand side corresponds to the Weil pairing e_N on the right hand side.

Definition 6.3 (Tate curve)**.** The affine equation
$$\mathrm{Tate}(q) : y^2 + xy = x^3 + a_4(q)x + a_6(q), \quad a_4(q), a_6(q) \in \mathbb{Z}[\![q]\!]$$
defines an elliptic curve over the ring of finite-tailed Laurent series $\mathbb{Z}(\!(q)\!)$, which is called the *Tate curve.*

We list some properties of the Tate curve in the following proposition and refer to [KM16, §8.8] (and even the references given there) for more details.

Proposition 6.4. (1) *The Tate curve* $\mathrm{Tate}(q)$ *has a nowhere vanishing invariant* 1*-form*
$$\omega_{\mathrm{can}} = \frac{dx}{2y + x},$$
which is called the canonical differential. It is a basis of $\underline{\omega}_{\mathrm{Tate}(q)/\mathbb{Z}(\!(q)\!)}$.[1]

[1]For any elliptic curve $f : E \to S$ over S, the canonical sheaf of invariant differentials is $\underline{\omega}_{E/S} = f_*\Omega^1_{E/S}$.

(2) *The Tate curve has discriminant and j-invariant as follows*

$$\Delta = q \prod_{n \geqslant 1} (1 - q^n)^{24}, \quad j = \frac{1}{q} + 744 + \cdots .$$

(3) *There is a unique isomorphism of formal groups over $\mathbb{Z}(\!(q)\!)$, written as*

$$\phi_{\text{can}} : \widehat{\text{Tate}}(q) \xrightarrow{\sim} \widehat{G}_m, \quad \omega_{\text{can}} \longmapsto \frac{dx}{x}.$$

(4) *There is a unique short exact sequence for every $N \geqslant 1$ in the category of sheaves of abelian groups on the fppf site of* $\operatorname{Spec} \mathbb{Z}(\!(q)\!)$

$$0 \longrightarrow \mu_N \xrightarrow{\alpha_N} \text{Tate}(q)[N] \xrightarrow{\beta_N} \mathbb{Z}/N\mathbb{Z} \longrightarrow 0$$

of finite flat group schemes over $\mathbb{Z}(\!(q)\!)$. Moreover, for any $\mathbb{Z}(\!(q)\!)$-algebra R and $\zeta \in \mu_N(R)$, $x \in \text{Tate}(q)[N](R)$, we have

$$\phi_{\text{can}}(\alpha_N(\zeta)) = \zeta, \quad e_N(\alpha_N(\zeta), x) = \zeta^{\beta_N(x)}.$$

(5) *When $K = \mathbb{C}$ or any local field, for $q_0 \in K$ such that $|q_0| < 1$, there exists an isomorphism*

$$K^\times / q_0^{\mathbb{Z}} \xrightarrow{\sim} \text{Tate}(q)(K).$$

(6) *Let $N \geqslant 1$ be an integer. Define*

$$\text{Tate}(q^N) : y^2 + xy = x^3 + a_4(q^N)x + a_6(q^N).$$

Then all N-torsion points in $\text{Tate}(q^N)$ *are defined over* $\mathbb{Z}[\zeta_N][\![q]\!] \subset \mathbb{Z}[\zeta_N, \frac{1}{N}][\![q]\!]$.

Definition 6.5 (Geometric modular forms)**.** Fix $N \geqslant 1$ and $\Gamma \in \{\Gamma(N), \Gamma_1(N), \Gamma_0(N)\}$. Let R_0 be a $\mathbb{Z}[\frac{1}{N}, \zeta_N]$-algebra. A (*meromorphic*) *modular form* f of weight $k \in \mathbb{Z}$ and level Γ over R_0 is a rule which assigns an element $f(E/R, \omega, \phi_\Gamma) \in R$ to any triple $(E/R, \omega, \phi_\Gamma)$, consisting of

- E/R, an elliptic curve, where R is an R_0-alebra,
- ω, a basis of $\underline{\omega}_{E/R} = f_* \Omega^1_{E/R}$, and
- ϕ_Γ, a Γ-level structure on E/R.

Moreover, f is required to satisfy the following conditions:

(a) f only depends on the isomorphism class of the triple $(E/R, \underline{\omega}, \phi_\Gamma)$.
(b) f is homogeneous of degree $-k$ in ω, i.e.,

$$f(E/R, \lambda\omega, \phi_\Gamma) = \lambda^{-k} f(E/R, \omega, \phi_\Gamma)$$

for any $\lambda \in R^\times$.
(c) f commutes with base changes; in other words, for any homomorphism $g : R \to R'$ between R_0-algebras, we have

$$f(E/R', \omega_{R'}, \phi_{\Gamma,R'}) = g(f(E/R, \omega, \phi_\Gamma)).$$

For a modular form f as above, the evaluation $f((\text{Tate}(q), \omega_{\text{can}})_{R_0}) \in \mathbb{Z}(\!(q)\!) \otimes_{\mathbb{Z}} R_0$ of f on the pair $(\text{Tate}(q), \omega_{\text{can}})_{R_0}$ is called the *q-expansion* of f. Here the index R_0 means the base change of the Tate curve and its canonical differential from $\mathbb{Z}(\!(q)\!)$ to $\mathbb{Z}(\!(q)\!) \otimes_{\mathbb{Z}} R_0$. Also, a meromorphic modular form f is called *holomorphic* if its q-expansion lies in the subring $Z[\![q]\!] \otimes_{\mathbb{Z}} R_0$ of $\mathbb{Z}(\!(q)\!) \otimes_{\mathbb{Z}} R_0$.

Definition 6.6 (Modular scheme)**.** Let $\Gamma \in \{\Gamma(N), \Gamma_1(N), \Gamma_0(N)\}$ as before. Consider the following functor

$$\begin{aligned} \mathcal{P}_\Gamma : \mathsf{Sch} &\longrightarrow \mathsf{Sets} \\ S &\longmapsto \left\{ (E/S, \phi_\Gamma) \;\middle|\; \begin{array}{c} E/S \text{ is an elliptic curve, and} \\ \phi_\Gamma \text{ is a } \Gamma\text{-level structure on } E/S \end{array} \right\} / \cong . \end{aligned}$$

This modular functor obtains the following representabilities:

- When $N \geqslant 3$, the functor $\mathcal{P}_{\Gamma(N)}$ is represented by an affine smooth scheme $Y_{\Gamma(N)}/\mathbb{Z}[\frac{1}{N}]$.
- When $N \geqslant 4$, the functor $\mathcal{P}_{\Gamma_1(N)}$ is represented by an affine smooth scheme $Y_{\Gamma_1(N)}/\mathbb{Z}[\frac{1}{N}]$.

However, the functor $\mathcal{P}_{\Gamma_0(N)}$ is never representable as it is not rigid due to the element $-I_2 \in \Gamma_0(N)$. More unfortunately, this is not the only obstruction to the representability of $\mathcal{P}_{\Gamma_0(N)}$. For example, if we consider for a prime $p \geqslant 5$ that

$$\begin{aligned} \mathcal{P}'_{\Gamma_0(N)} : \mathsf{Sch}_{\mathbb{Z}_p} &\longrightarrow \mathsf{Sets} \\ S &\longmapsto \left\{ (E/S, \phi_{\Gamma_0(N)}, \omega) \;\middle|\; \begin{array}{c} (E/S, \phi_{\Gamma_0(N)}) \in \mathcal{P}_{\Gamma_0(N)}(S) \\ \omega \text{ is a basis of } \underline{\omega}_{E/S} \end{array} \right\} / \cong, \end{aligned}$$

then its representability will depends on the number N mod 12. The main reason is that over a field of character 2 or 3, elliptic curves have large automorphism groups. We refer to [Kat73, Chapter 1] for a treatment to handle small levels and the $\Gamma_0(p)$-level structure.

We give a description of the modular scheme $Y_{\Gamma(N)}$ for $N \geqslant 3$ following [Kat73, §1.4–1.5]. The case for the modular scheme $Y_{\Gamma_1(N)}$ for $N \geqslant 4$ is similar and can be found in [KM16]. The modular scheme $Y_{\Gamma(N)}$ is finite and flat of degree $\#(\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\})$ over the affine j-line $\mathbb{Z}[\frac{1}{N}][j]$ (the coarse moduli scheme for the functor $\mathcal{P}_{\Gamma(1)}$), and étale over the open subscheme $\mathbb{Z}[\frac{1}{N}][j, (j(j-1728))^{-1}]$. The normalization of the projective j-line $\mathbb{P}^1/\mathbb{Z}[\frac{1}{N}]$ in $Y_{\Gamma(N)}$ is a proper smooth curve $X_{\Gamma(N)}$ over $\mathbb{Z}[\frac{1}{N}]$, and is called the natural compactification of $Y_{\Gamma(N)}$. We have decompositions

$$\begin{aligned} Y_{\Gamma(N)} \times_{\mathbb{Z}[\frac{1}{N}]} \mathbb{Z}\left[\frac{1}{N}, \zeta_N\right] &= \bigsqcup_{\zeta} Y^{\zeta}_{\Gamma(N)}, \\ X_{\Gamma(N)} \times_{\mathbb{Z}[\frac{1}{N}]} \mathbb{Z}\left[\frac{1}{N}, \zeta_N\right] &= \bigsqcup_{\zeta} X^{\zeta}_{\Gamma(N)} \end{aligned}$$

of the base changes on the left-hand side of $Y_{\Gamma(N)}$ (resp. $X_{\Gamma(N)}$) into disjoint unions of $\varphi(N)$ affine (resp. proper) smooth geometrically connected curves $Y^{\zeta}_{\Gamma(N)}$ (resp. $X^{\zeta}_{\Gamma(N)}$) over $\mathbb{Z}[\frac{1}{N}, \zeta_N]$. These curves are bijective with the primitive Nth roots of unity, and for any primitive Nth root of unity ζ, the corresponding curve $Y^{\zeta}_{\Gamma(N)}$ represents the functor

$$\begin{aligned} \mathcal{P}^{\zeta}_{\Gamma(N)} : \mathsf{Sch} &\longrightarrow \mathsf{Sets} \\ S &\longmapsto \{(E/S, \phi_{\Gamma(N)})\}/\cong, \end{aligned}$$

where E/S is an elliptic curve and $\phi_{\Gamma(N)}$ is a level $\Gamma(N)$-structure on E/S with determinant ζ.

The complement $X_{\Gamma(N)} \backslash Y_{\Gamma(N)}$ is finite étale over $\mathbb{Z}[\frac{1}{N}]$ and after base change to $\mathbb{Z}[\frac{1}{N}, \zeta_N]$, it is a disjoint union of sections, which are called cusps of $X_{\Gamma(N)}$. The cusps are bijective with the isomorphism classes of level $\Gamma(N)$-structure on the Tate curve $\mathrm{Tate}(q^N)$ over $\mathbb{Z}(\!(q)\!) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{N}, \zeta_N]$. Moreover, the completion of $X_{\Gamma(N)} \times_{\mathbb{Z}[\frac{1}{N}]} \mathbb{Z}[\frac{1}{N}, \zeta_N]$ at any cusp is isomorphic to $\mathbb{Z}[\frac{1}{N}, \zeta_N][\![q]\!]$.

Let $\mathbb{E}/Y_{\Gamma(N)}$ be the universal elliptic curve, which extends to a generalized elliptic curve $\mathbb{E}/X_{\Gamma(N)}$ in the sense of [DR, II, Definition 1.12]. The invertible sheaf $\underline{\omega}_{\mathbb{E}/Y_{\Gamma(N)}}$ extends uniquely to an invertible sheaf $\underline{\omega}_{\mathbb{E}/X_{\Gamma(N)}}$ such that its sections over the complement $\mathbb{Z}[\frac{1}{N}, \zeta_N][\![q]\!]$ at each cusp are $\mathbb{Z}[\frac{1}{N}, \zeta_N][\![q]\!]\omega_{\mathrm{can}}$ (recall that ω_{can} is the canonical differential on the Tate curve). A holomorphic modular form of weight $k \in \mathbb{Z}$ and level $\Gamma(N)$ over a $\mathbb{Z}[\frac{1}{N}]$-algebra R_0 is precisely a section in

$$\Gamma(X_{\Gamma(N)} \times_{\mathbb{Z}[\frac{1}{N}]} R_0, \underline{\omega}^{\otimes k}_{\mathbb{E}/X_{\Gamma(N)}}) = \Gamma(X_{\Gamma(N)}, (\underline{\omega}^{\otimes k}_{\mathbb{E}/X_{\Gamma(N)}}) \otimes_{\mathbb{Z}[\frac{1}{N}]} R_0).$$

7. Hasse invariant and ordinary loci of modular schemes

7.1. **Hasse invariant.** Let R be an $\mathbb{F}_p$-algebra and E/R be an elliptic curve. We consider the absolute Frobenius map on the structure sheaf $F_{\mathrm{abs}} : \mathcal{O}_E \to \mathcal{O}_E$, which induces a p-linear endomorphism $F^*_{\mathrm{abs}} : H^1(E, \mathcal{O}_E) \to H^1(E, \mathcal{O}_E)$ (i.e. F^*_{abs} is additive and $F^*_{\mathrm{abs}}(\lambda\eta) = \lambda^p F^*_{\mathrm{abs}}(\eta)$ for $\lambda \in R$ and $\eta \in H^1(E, \mathcal{O}_E)$). If ω is an R-basis of $\underline{\omega}_{E/R}$, let $\eta \in H^1(E, \mathcal{O}_E)$ be the dual basis under Serre duality. Then there exists an element $A(E/R, \omega) \in R$ such that

$$F^*_{\mathrm{abs}}(\eta) = A(E/R, \omega) \cdot \eta.$$

For $\lambda \in R^\times$, if we replace ω by $\lambda\omega$, the dual basis in $H^1(E, \mathcal{O}_E)$ should be replaced by $\lambda^{-1}\omega$. Then we have

$$A(E/R, \lambda\omega) \cdot (\lambda^{-1}\eta) = F_{\text{abs}}^*(\lambda^{-1}\eta) = \lambda^{-p} F_{\text{abs}}^*(\eta) = \lambda^{-p} A(E/R, \omega) \cdot \eta,$$

and hence

$$A(E/R, \lambda\omega) = \lambda^{1-p} A(E/R, \omega).$$

In particular, the association $(E/R, \omega) \mapsto A(E/R, \omega)$ defines a meromorphic modular form of weight $p-1$ and level 1 over $\mathbb{F}_p$, which is called the *Hasse invariant*. We list some properties of Hasse invariant in the following proposition.

Proposition 7.1. (1) *For any $\mathbb{F}_p$-algebra R, we have*

$$A((\mathrm{Tate}(q), \omega_{\mathrm{can}})_R) = 1.$$

In particular, $A(E/R, \omega)$ is a holomorphic modular form;

(2) *Let k be an algebraically closed field of characteristic p and E/k be an elliptic curve. Then $A(E/k, \omega) = 0$ if and only if E is supersingular, i.e. $E[p](k) = (0)$;*

(3) *For any prime $p \geqslant 5$, the Hasse invariant can be lifted to $\mathbb{Q} \cap \mathbb{Z}_p$ in the sense that there is a modular form of weight $p-1$ and level 1 over $\mathbb{Q} \cap \mathbb{Z}_p$ whose q-expansion modulo p equals 1. In fact, we can choose the Eisenstein series*

$$\mathcal{E} = \mathcal{E}_{p-1} = 1 - \frac{2(p-1)}{B_{p-1}} \sum_{n \geqslant 1} \sigma_{p-2}(n) q^n$$

as such a lifting.

7.2. **Ordinary loci of modular schemes.** Fix a prime $p \geqslant 5$. Let W be the ring of integers of a finite extension of $\mathbb{Q}_p$ and $\pi \in W$ be a uniformizer. Fix a positive integer N and let Y_Γ/W (resp. X_Γ/W) be the base change of the modular curve Y_Γ (resp. X_Γ) to W for $\Gamma \in \{\Gamma(N), \Gamma_1(N)\}$. Here we assume that N is large enough to guarantee the representability of the corresponding moduli problem. When $\Gamma = \Gamma(N)$, we consider the arithmetic $\Gamma(N)$-level structure and we assume that W contains a primitive Nth root of unity. Fix a lift $\mathcal{E}$ of the Hasse invariant as in Proposition 7.1(3). We may regard $\mathcal{E}$ as a global section in $\Gamma(Y_\Gamma/W, \underline{\omega}_{\mathbb{E}/Y_\Gamma}^{\otimes(p-1)})$ or $\Gamma(X_\Gamma/W, \underline{\omega}_{\mathbb{E}/X_\Gamma}^{\otimes(p-1)})$. Let $f : \mathbb{E} \to Y_\Gamma$ be the universal elliptic curve that extends to a semistable curve on X_Γ, which is still denoted by $\mathbb{E}$. Define $Y_\Gamma^{\mathrm{ord}}/W$ (resp. $X_\Gamma^{\mathrm{ord}}/W$) to be the open subscheme of Y_Γ/W (resp. X_Γ/W) where the section $\mathbb{E}$ is invertible. For $m \geqslant 1$, set

$$W_m = W/\pi^m W, \quad S_m^\circ = Y_\Gamma^{\mathrm{ord}} \otimes_W W_m, \quad S_m = X_\Gamma^{\mathrm{ord}} \otimes_W W_m.$$

Then S_m° and S_m are affine smooth curves over W_m with geometrically connected fibers. For any $n \geqslant 1$, we have a connected-étale exact sequence of finite flat group schemes over S_m°:

$$0 \to \mathbb{E}^\circ[p^n] \to \mathbb{E}[p^n] \to \mathbb{E}^{\text{ét}}[p^n] \to 0,$$

where $\mathbb{E}^\circ[p^n]$ is the kernel of $[p^n] : \hat{\mathbb{E}} \to \hat{\mathbb{E}}$ of formal group of $\mathbb{E}$, as well as the connected component of the identity section of $\mathbb{E}[p^n]$; also, $\mathbb{E}^{\text{ét}}[p^n]$ is the maximal étale quotient of $\mathbb{E}[p^n]$ and the Cartier dual of $\mathbb{E}^\circ[p^n]$. Locally on the étale topology on S_m°,

$$\mathbb{E}^\circ[p^n] \simeq \mu_{p^n}, \quad \mathbb{E}^{\text{ét}}[p^n] \simeq (\mathbb{Z}/p^n\mathbb{Z})_{S_m^\circ}.$$

By [Kat73, Theorem 4.2.2], the connected-étale exact sequence extends to S_m. Then $\mathbb{E}^{\text{ét}}[p^n]$ extends to a finite flat group scheme over S_m, which is isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$ after a finite etale extension of S_m. We also call such an object a twisted version of $\underline{\mathbb{Z}/p^n\mathbb{Z}}_{S_m}$.

Definition 7.2. The functor

$$T_{m,n} = \mathbf{Isom}_{S_m}(\mathbb{Z}/p^n\mathbb{Z}, \mathbb{E}^{\text{ét}}[p^n])$$

is represented by an affine scheme $T_{m,n} = \mathrm{Spec}(V_{m,n}$. For the universal curve $f : \mathbb{E} \to X_\Gamma$, we write $\underline{\omega} = \underline{\omega}_{\mathbb{E}/X_\Gamma}$ and $\underline{\omega}_m$ for the restriction of $\underline{\omega}$ to S_m° and S_m. We make the following definitions.

(1) Define the space of (true) *holomorphic modular forms* of weight k as

$$H^0(X_\Gamma/W, \underline{\omega}^{\otimes k}).$$

(2) Define the space of *false modular forms* of weight k as

$$H^0(S_\infty, \underline{\omega}^{\otimes k}) := \varprojlim_m H^0(S_m, \underline{\omega}^{\otimes k}).$$

(3) Define the following V to be the space of *p-adic modular forms*:

$$V_{m,\infty} = \varinjlim_n V_{m,n}, \quad V = \varprojlim_m V_{m,\infty}.$$

We will see that there are inclusions

$$\{\text{true modular forms}\} \subset \{\text{false modular forms}\} \subset \{p\text{-adic modular forms}\}$$

and the images of these inclusions are dense with respect to the p-adic topology. We will state the results in a more general setting.

8. "False" modular forms á la Deligne

Let k be a finite field of characteristic p and consider $W = W(k)$. As before, we take $W_m = W/\pi^m W$ for any $m \geqslant 1$, where $\pi \in W$ is a choice of the uniformizer. Let $\{S_m/W_m\}$ be a family of flat affine schemes, such that $S_m = S_{m+1} \times_{W_{m+1}} W_m$. Fix $S = X_\Gamma^{\mathrm{ord}}/W$. Let $\mathcal{P}$ be a rank 1 p-adic étale sheaf on S_m, i.e., an inverse system

$$\mathcal{P} = (\mathcal{P}_n = \mathcal{P}/p^n\mathcal{P})_{n\geqslant 1},$$

such that each $\mathcal{P}_n$ is a (finite flat) sheaf on S_m such that $\mathcal{P}_n$ becomes isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})_{S_m}$ for all $m \geqslant 1$, after a finite étale base change of S_m. For example, over S we consider the short exact sequence

$$0 \longrightarrow \mathbb{E}^\circ[p^n] \longrightarrow \mathbb{E}[p^n] \longrightarrow \mathbb{E}^{\text{ét}}[p^n] = \mathcal{P}_n \longrightarrow 0$$

of sheaves on S_m. Let $\underline{\omega}_m = \mathcal{P} \otimes_{\mathbb{Z}_p} \mathcal{O}_{S_m}$ be an invertible coherent sheaf on S_m satisfying $\varphi_m^*(\underline{\omega}_{m+1}) = \underline{\omega}_m$, where $\varphi_m : S_m \to S_{m+1}$ is the natural base change morphism. (For example, one may take $\underline{\omega}_m = f_*\Omega + \mathbb{E}/S_m = \underline{\omega}_{\mathbb{E}/S_m}$.)

We define two graded rings

$$R'_m = \bigoplus_{k\geqslant 0} H^0(S_m, \underline{\omega}^{\otimes k}),$$

$$R'_\infty = \bigoplus_{k\geqslant 0} \varprojlim_m H^0(S_m, \underline{\omega}^{\otimes k}).$$

For this, recall that $H^0(S_m, \underline{\omega}^{\otimes k})$ denotes the space of Deligne's false modular forms. For any $m, n \geqslant 1$ we also define the functor

$$T_{m,n} : \mathsf{Sch}_{S_m} \longrightarrow \mathsf{Sets}$$

$$(X \xrightarrow{\gamma} S_m) \longmapsto \mathbf{Isom}_{S_m}((\mathbb{Z}/p^n\mathbb{Z})_X, \gamma^*\mathcal{P}_n).$$

By definition there is a natural Igusa tower

$$\begin{array}{ccccccc}
T_{m+1,n+1} & \longrightarrow & T_{m+1,n} & \longrightarrow & \cdots & \longrightarrow & S_{m+1} \\
\downarrow & & \downarrow & & & & \downarrow \\
T_{m,n+1} & \longrightarrow & T_{m,n} & \longrightarrow & \cdots & \longrightarrow & S_m \\
\| & & \| & & & & \\
\operatorname{Spec}(V_{m,n+1}) & \longrightarrow & \operatorname{Spec}(V_{m,n}) & \longrightarrow & \cdots & &
\end{array}$$

and the group $(\mathbb{Z}/p^n\mathbb{Z})^\times$ acts on $T_{m,n}$ via the formula $\alpha(\psi_n) = \alpha^{-1}\psi_n$.

Claim A. $T_{m,n}$ is represented by a finite étale S_m-scheme $\operatorname{Spec}(V_{m,n})$ and $(\mathbb{Z}/p^n\mathbb{Z})^\times$ acts freely on $\operatorname{Spec}(V_{m,n})$ with the quotient S_m.

- ⋄ To show this, we may assume that S_m is connected. When $\mathcal{P}_n$ is the constant sheaf $(\mathbb{Z}/p^n\mathbb{Z})_{S_m}$, the functor is represented by the scheme

$$\coprod_{\alpha\in(\mathbb{Z}/p^n\mathbb{Z})^\times} S_m,$$

and the other statements follow immediately. For a general $\mathcal{P}_n$, we can find a finite étale Galois covering $S'_m \to S_m$ with Galois group G such that $\mathcal{P}_n|_{S'_m}$ is constant. Let $T'_{m,n}$ be the restriction of the functor $T_{m,n}$ to $\mathsf{Sch}_{S'_m}$. By the above discussion, $T'_{m,n}$ is represented by a finite étale S'_m-scheme $\mathrm{Spec}(V'_{m,n})$. Since $\mathcal{P}_n$ is defined over S_m, the sheaf $\mathcal{P}'_n := \mathcal{P}_n \times_{S_m} S'_m$ carries an action of the Galois group G which induces an action of G on the functor $T'_{m,n}$ and hence on the scheme $\mathrm{Spec}(V'_{m,n})$. The claim follows from standard results in Galois descent.

We define a homomorphism

$$\beta(m) : R'_m \to V_{m,m} = \Gamma(T_{m,m}, \mathcal{O}) \hookrightarrow V_{m,\infty}$$

as follows. On $\gamma_m : T_{m,m} \to S_m$ we have a "universal isomorphism" $\psi_m : (\mathbb{Z}/p^m\mathbb{Z})_{T_{m,m}} \to \gamma_m^*(\mathcal{P}_m)$ of constant sheaves. The element $1 \in \mathbb{Z}/p^m\mathbb{Z}$ gives a section in $\Gamma(T_{m,m}, \mathcal{P}_m)$, and hence an invertible section in $\Gamma(T_{m,m}, \omega_m)$, which is denoted by $\omega_{\mathrm{can}}(m)$. Define

$$\beta(m)\left(\sum_i f_i\right) = \sum_i \frac{f_i}{\omega_{\mathrm{can}}(m)^{\otimes i}}, \quad f_i \in \Gamma(S_m, \underline{\omega}_m^{\otimes i}).$$

This construction leads to homomorphisms

$$\begin{aligned} \beta(m) :& R'_m = \bigoplus_{k \geqslant 0} H^0(S_m, \underline{\omega}_m^{\otimes k}) \to V_{m,m} \hookrightarrow V_{m,\infty}, \\ \beta(\infty) :& R'_\infty = \bigoplus_{k \geqslant 0} \varinjlim_m H^0(S_m, \underline{\omega}_m^{\otimes k}) \to \varinjlim_m V_{m,\infty} = V. \end{aligned}$$

Claim B. $\beta(m)$ is not injective for each $m \geqslant 1$, whereas $\beta(\infty)$ is injective.

$\diamond$ In fact, we may regard $V_{m,m}$ (resp. V) as the ring of functions that associates the pair

$$(\gamma : X \to S_m, \psi_m : (\mathbb{Z}/p^m\mathbb{Z})_X \xrightarrow{\sim} \gamma^*\mathcal{P}_m) \longrightarrow$$

to some section in $\Gamma(X, \mathcal{O}_X)$, being compatible with the base change. The action of $(\mathbb{Z}/p^m\mathbb{Z})^\times$ on the isomorphisms ψ_m's (resp. the action of $\mathbb{Z}_p^\times$ on the isomorphisms ϕ's) induces an action of $(\mathbb{Z}/p^m\mathbb{Z})^\times$ (resp. $\mathbb{Z}_p^\times$) on the ring $V_{m,m}$ (resp. V). Explicitly, for $\alpha_m \in (\mathbb{Z}/p^m\mathbb{Z})^\times$ and $f_m \in V_{m,m}$, we obtain

$$\alpha_m(f_m)(\gamma : X \to S_m, \psi_m) = f_m(\gamma : X \to S_m, \alpha_m^{-1}\psi_m).$$

And for $\alpha \in \mathbb{Z}_p^\times$ and $f \in V$,

$$\alpha(f)(\gamma : X \to S, \psi) = f(\gamma : X \to S, \alpha^{-1}\psi).$$

Under the above interpretations, the homomorphism $\beta(m)$ identifies $H^0(S_m, \underline{\omega}^{\otimes k})$ with the subspace of functions in $V_{m,m}$ on which $(\mathbb{Z}/p^m\mathbb{Z})^\times$ acts via the character $\alpha \mapsto \alpha^k$. In particular,

$$\beta(m)(H^0(S_m, \underline{\omega}^{\otimes k})) = \beta(m)(H^0(S_m, \underline{\omega}^{\otimes (k+(p-1)p^{m-1})})) \subset V_{m,m},$$

and hence $\beta(m)$ is not injective.

On the other hand, $\beta(\infty)$ identifies $\varinjlim_m H^0(S_m, \underline{\omega}^{\otimes k})$ with the subspace of functions in V on which $\mathbb{Z}_p^\times$ acts via the character $\alpha \mapsto \alpha^k$ for all $k \geqslant 0$. Recall that $T_{m,n} = \mathrm{Spec}(V_{m,n})$ is finite étale over S_m, and hence $V_{m,n}$ is a flat W_m-algebra. Therefore, $V_{m,\infty} = \varinjlim_n V_{m,n}$ is also a flat W_m-algebra. Since $V = \varprojlim_m V_{m,\infty}$ we see that V is p-adically complete, i.e.,

$$V \cong \varinjlim_m V/\pi^m V, \quad V/\pi^m V \cong V_{m,\infty}.$$

Also, V is flat as a W-algebra, i.e. V is π-torsion-free. Combining with the above interpretation of $\beta(\infty)$, we see that $\beta(\infty) : R'_\infty \to V$ is injective.

Since V and R'_∞ are flat W-algebras, we have the following commutative diagram of inclusions:

$$\begin{tikzcd} R'_\infty \arrow[r, hook, "\beta(\infty)"] \arrow[d, hook] & V \arrow[d, hook] \\ R'_\infty[\frac{1}{p}] \arrow[r, hook, "\beta(\infty)"] & V[\frac{1}{p}], \end{tikzcd}$$

Theorem 8.1. *Define $D' = \beta(\infty)(R'_\infty[\frac{1}{p}]) \cap V$. Then the natural inclusion $\iota : D' \to V$ induces an isomorphism*

$$\iota_m : D'/\pi^m D' \xrightarrow{\sim} V/\pi^m V, \quad m \geqslant 1.$$

In other words, V is the p-adic completion of D'.

Proof. It follows from the definition of D' that V/D' is W-flat. Hence the exact sequence

$$0 \to D' \to V \to V/D' \to 0$$

remains exact when tensoring with W_m over W. Hence $\iota_m : D'/\pi^m D' \to V/\pi^m V$ is injective. To show the surjectivity, by Nakayama lemma, it suffices to show that

$$\iota_1 : D'/\pi D' \longrightarrow V/\pi V \cong V_{1,\infty}$$

is surjective. Pike some $f \in V_{1,n} \subset V_{1,\infty}$ and choose one of its lift $F \in \varinjlim_l V_{l,n}$ of $f \in V_{1,n}$. Here the integer m is sufficiently large such that $m > n$ and $\pi^{m-1}/(p^n-1)! \in W$ (namely, m has a non-negative p-adic valuation). It boils down to show that

$$\pi^{m-1} \cdot F \in \beta(\infty)(R'_\infty) + \pi^m V$$

as this implies that $F \in D' + \pi V$. Let F_m be the image of F under the morphism

$$\varprojlim_l V_{l,n} \to V_{m,n} \hookrightarrow V_{m,m}.$$

So it is enough to show that $\pi^{m-1} F_m \in \beta(m)(R'_m)$, or equivalently, the inclusion

$$\pi^{m-1} V_{m,n} \subset \beta(m)(R'_m) \subset V_{m,m}.$$

For this, we first assume that $\mathcal{P}_m$ is the constant sheaf given by $\mathbb{Z}/p^m\mathbb{Z}$ on $S_m = \operatorname{Spec} A$. Then, due to the representability argument before,

$$T_{m,m} = \bigsqcup_{\alpha \in (\mathbb{Z}/p^m\mathbb{Z})^\times} S_m, \quad V_{m,m} = \bigsqcup_{\alpha \in (\mathbb{Z}/p^m\mathbb{Z})^\times} A_m = \mathbf{Maps}((\mathbb{Z}/p^m\mathbb{Z})^\times, A_m),$$

where $A_m = \Gamma(S_m, \mathcal{O})$ is a $\mathbb{Z}/p^m\mathbb{Z}$-algebra. Note that the invertible sheaf $\underline{\omega}_m$ is trivial on S_m, and hence

$$R'_m = \bigoplus_{k \geqslant 0} H^0(S_m, \underline{\omega}^{\otimes k}) \cong A_m[X].$$

Under this isomorphism, the map $\beta(m) : R'_m \to V_{m,m}$ is just to interpret the polynomials over A_m as functions on $(\mathbb{Z}/p^m\mathbb{Z})^\times$ with values in A_m (this makes sense since $p^m A_m = 0$ as A_m is a $W_m = W/\pi^m W$-algebra), and the inclusion $V_{m,n} \hookrightarrow V_{m,m}$ becomes the natural inclusion

$$\mathbf{Maps}((\mathbb{Z}/p^n\mathbb{Z})^\times, A_m) \longrightarrow \mathbf{Maps}((\mathbb{Z}/p^m\mathbb{Z})^\times, A_m), \quad m > n.$$

We need to show that

$$\pi^{m-1}\mathbf{Maps}((\mathbb{Z}/p^n\mathbb{Z})^\times, A_m) = \mathbf{Maps}((\mathbb{Z}/p^n\mathbb{Z})^\times, \pi^{m-1}A_m) \subset \operatorname{Im}(\beta(m)).$$

Notice that $\pi^{m-1}A_m$ is an $\mathbb{F}$-vector space. The space $\mathbf{Maps}((\mathbb{Z}/p^n\mathbb{Z})^\times, \mathbb{F})$ admits a Mahler basis, whose elements are of the form $\binom{x}{i}$ with $0 \leqslant i \leqslant p^n - 1$. Since

$$\mathbf{Maps}((\mathbb{Z}/p^n\mathbb{Z})^\times, \pi^{m-1}A_m) = \mathbf{Maps}((\mathbb{Z}/p^n\mathbb{Z})^\times, \mathbb{F}) \otimes_{\mathbb{F}} \pi^{m-1}A_m,$$

we see that any element of $\pi^{m-1}\mathbf{Maps}((\mathbb{Z}/p^n\mathbb{Z})^\times, A_m)$ is of the form

$$\sum_{i=0}^{p^n-1} a_i \binom{z}{i}, \quad a_i \in \pi^{m-1}A_m.$$

It follows from our choice of m that

$$\pi^{m-1}\binom{z}{i} \in W[X], \quad 0 \leqslant i \leqslant p^n - 1,$$

and hence $\pi^{m-1}F_m \in \beta(m)(R'_m)$.

Now we consider the general case. We can find a finite étale covering $\operatorname{Spec} A \to S_m = \operatorname{Spec} A_m$ such that $\mathcal{P}|_{\operatorname{Spec} A}$ is constant. From the above discussion, we have

$$\pi^{m-1} \cdot V_{m,n} \otimes_{A_m} A \subset \beta(m)(R'_m \otimes_{A_m} A),$$

i.e. the map

$$\pi^{m-1} : V_{m,n} \otimes_{A_m} A \longrightarrow V_{m,m} \otimes_{A_m} A/\beta(m(R'_m \otimes_{A_m} A))$$

is the zero map. Since A is faithfully flat over A_m, we see that

$$\pi^{m-1} : V_{m,n} \longrightarrow V_{m,m}/\beta(m)(R'_m)$$

is the zero map, i.e. $\pi^{m-1} \cdot V_{m,n} \subset \beta(m)(R'_m)$. □

Now assume that M is a proper smooth scheme over W, whose fibers are geometrically connected curves. For example, one may take $M = M_\Gamma/W$. Denote $M_m = M \otimes_W W_m$. Suppose there exists a finite set Ω of closed points of M_1, such that $S_1 = M_1 \backslash \Omega$ is still affine. We obtain an invertible sheaf $\underline{\omega}$ on M and a p-adic étale sheaf $\mathcal{P} = (\mathcal{P}/p^n\mathcal{P})_{n \geqslant 1}$ on S_m of rank 1 such that $\underline{\omega}$ induces the sheaf $\mathcal{P} \otimes_{\mathbb{Z}_p} \mathcal{O}_{S_m}$ on S_m. Notice that $\underline{\omega}^{\otimes(p-1)}$ is trivial on S_1 as we have canonical isomorphisms

$$(\mathcal{P} \otimes_{\mathbb{Z}_p} \mathcal{O}_{S_1})^{\otimes(p-1)} \cong (\mathcal{P}/p\mathcal{P})^{\otimes(p-1)} \otimes_{\mathbb{Z}/p\mathbb{Z}} \mathcal{O}_{S_1}, \quad (\mathcal{P}/p\mathcal{P})^{\otimes(p-1)} \cong \mathbb{Z}/p\mathbb{Z}.$$

Let $A \in H^0(S_1, \underline{\omega}^{\otimes(p-1)})$ be the section corresponding to $1 \in \mathbb{Z}/p\mathbb{Z}$ under this isomorphism.

Theorem 8.2. *Suppose that A extends to a section $A \in H^0(M, \underline{\omega}^{\otimes(p-1)})$ which vanishes at points of Ω. Define*

$$R_\infty = \bigoplus_{k \geqslant 0} H^0(M, \omega^{\otimes k}).$$

Then we have

$$R_\infty \subset R'_\infty \subset V.$$

Set $D = R_\infty[\frac{1}{p}] \cap V$. The inclusions $D \subset D' \subset V$ induces isomorphisms

$$D/\pi^m D \cong D'/\pi^m D' \cong V/\pi^m V, \quad m \geqslant 1.$$

Proof. The inclusion $D \subset D'$ follows from

$$H^0(M, \underline{\omega}^{\otimes k}) = \varprojlim_m H^0(M_m, \underline{\omega}_m^{\otimes k})$$

and the injection $H^0(M_m, \underline{\omega}_m^{\otimes k}) \hookrightarrow H^0(S_m, \underline{\omega}_m^{\otimes k})$ for every $m \geqslant 1$. By the same argument as in the proof of Theorem 8.1, we see that $D/\pi^m D \to V/\pi^m V$ is injective for all $m \geqslant 1$. Again by Nakayama lemma, it suffices to show $D/\pi^D \to D'/\pi D'$ is surjective. The invertible sheaf $\underline{\omega}$ on M_1 has positive degree since $H^0(M_1, \underline{\omega}^{\otimes(p-1)})$ has a nonzero section A which has zeros. We choose an integer $\nu > 0$ large enough so that

$$\deg(\underline{\omega}^{\otimes \nu(p-1)}|_{M_1}) > 2g - 2,$$

where g is genus of M_1. Then the section $A^\nu \in H^0(M_1, \underline{\omega}^{\otimes \nu(p-1)})$ lifts to a section $\mathcal{E} \in H^0(M, \underline{\omega}^{\otimes \nu(p-1)})$ (the obstruction of this lifting lies in $H^1(M_1, \underline{\omega}^{\otimes \nu(p-1)})$, which is zero by our assumption on ν). We view $\mathcal{E} \in R_\infty$ as an element in V under the inclusion $R_\infty \subset R'_\infty \subset V$, and we have $\mathcal{E} \in 1 + \pi V$. In fact, it is enough to show $\mathcal{E} \bmod \pi \equiv 1$ in $V/\pi V = V_{1,\infty}$, and this follows from the construction of the section A which shows that the image of A in $V_{1,1}$ is the function 1. The open subscheme S_m of M_m is defined by the open subset where the global section $\mathcal{E} \in H^0(M, \underline{\omega}^{\otimes \nu(p-1)})$ is invertible. So we have

$$H^0(S_m, \underline{\omega}_m^{\otimes k}) = \varinjlim_n \frac{H^0(M_m, \underline{\omega}^{\otimes k + n\nu(p-1)})}{\mathcal{E}^n}.$$

Now we can prove the surjectivity of the map $D/\pi D \to D'/\pi D'$. Given an element $f \in R'_\infty \cap \pi^m V$ (or equivalently, $\frac{1}{p^m} f \in R'_\infty[\frac{1}{p}] \cap V = D'$), we need to show that there exits $g \in D$ such that

$$f \equiv g \bmod \pi^{m+1} V.$$

Clearly we may assume that

$$f = f_i \in \varprojlim_m H^0(S_m, \underline{\omega}_m^{\otimes i})$$

for some $i \geqslant 0$. By the formula above, we can find

$$g \in H^0(M_m, \underline{\omega}^{\otimes(i+Np^m\nu(p-1))}), \quad N \gg 0,$$

such that $f_i \equiv g/\mathcal{E}^{Np^m} \bmod \pi^{m+1}R'_\infty$. Since $\mathcal{E} \in 1+\pi V$, we have $1/\mathcal{E}^{Np^m} \in 1+\pi^{m+1}V$. It follows $f_i - g \in \pi^{m+1}V$, i.e. $f \in R_\infty + \pi^{m+1}V$ and hence $D/\pi D \to D'/\pi D'$ is surjective. $\square$

Now we give a rough idea on how to construct the desired Λ-adic forms under the above abstract setting. Recall that

$$V = \varprojlim_m V/\pi^m V = \varprojlim_m V_{m,\infty}$$

is a flat W-algebra and $W_{m,\infty} = V/\pi^m V$ is a flat W_m-algebra. We set

$$\mathcal{V} = \varinjlim_m V/\pi^m V = \varinjlim_m \frac{1}{\pi^m} V/V,$$

which is a π-divisible W-module. We have defined an action of $\mathbb{Z}_p^\times$ on V and it induces actions of $\mathbb{Z}_p^\times$ on $V/\pi^m V$ for $m \geqslant 1$. We regard $\mathcal{V}$ as a discrete W-module and its Pontryagin dual

$$\mathcal{V}^* := \mathrm{Hom}_W(\mathcal{V}, K/W) = \varprojlim_m \mathrm{Hom}_W(\mathcal{V}[\pi^m], \frac{1}{\pi^m}W/W)$$

is equipped with the profinite topology and then a compact $W[\![\mathbb{Z}_p^\times]\!]$-module. Also, recall that $\nu : \mathbb{Z}_p^\times \to W^\times$ is the inclusion character. For a W-module M with a W-linear action of $\mathbb{Z}_p^\times$ and $k \geqslant 1$, we denote by $M[\nu^k]$ the submodule of M on which $\mathbb{Z}_p^\times$ acts via the character ν^k. Under this notation, we have

$$\mathcal{V}[\nu^k] = \varprojlim_m V_{m,\infty}[\nu^k] = \varprojlim_m V_{m,m}[\nu^k],$$

here the last equality uses the fact that $V_{m,\infty}$ is a W_m-module. Note that we have shown that the image of the homomorphism $\beta(m) : H^0(S_m, \underline{\omega}_m^{\otimes k}) \to V_{m,m}$ equals $V_{m,m}[\nu^k]$.

Now assume that we have a W-linear idempotent map $e : V \to V$ which is compatible with the action of $\mathbb{Z}_p^\times$. We make the following two assumptions on e:

(1) $e(Af) = A \cdot e(f)$ for all $f \in H^0(S_1, \underline{\omega}_1^{\otimes k})$ and all k;
(2) $\dim_K eH^0(M/K, \underline{\omega}^{\otimes k})$ is bounded independently of k.

The following (2') is impulsed as a stronger version of (2) which will be used to make Theorem 8.4 more transparent.

(2′) $\dim_K eH^0(M/K, \underline{\omega}^{\otimes k})$ depends only on $k \pmod{p-1}$ for $k \geqslant k_0$, with some given positive integer k_0.

Recall that we have an isomorphism $\beta(m) : H^0(S_m, \underline{\omega}_m^{\otimes k}) \to V_{m,m}[\nu^k]$. Obtaining this, we claim:

- ⋄ $\dim_{\mathbb{F}} eH^0(S_1, \underline{\omega}_1^{\otimes k})$ is finite and bounded independently of $k \leqslant 1$.

In fact, one can consider the injective map by multiplying A^s as follows:

$$A^s : H^0(S_1, \underline{\omega}_1^{\otimes k}) \longrightarrow H^0(S_1, \underline{\omega}_1^{\otimes k+s(p-1)}).$$

Let $\{\overline{f}_1, \ldots, \overline{f}_l\}$ be a set of linearly independent sections in $eH^0(S_1, \underline{\omega}^{\otimes k})$ that can be lifted to $f_i \in H^0(S_1, \underline{\omega}^{\otimes k})$ for each i. By assumption (1) above, the sections $\{e(\mathcal{E}^s \cdot f_i)\}_{i=1,\ldots,l}$ are linearly independent over K in $H^0(S, \underline{\omega}^{k+s(p-1)})$. On the other hand, we know that

$$\mathcal{E}^s \cdot f_i \in H^0(M_{/K}, \underline{\omega}^{\otimes k+s(p-1)}) quads \gg 0.$$

It follows that

$$\dim_{\mathbb{F}} eH^0(S_1, \underline{\omega}_1^{\otimes k}) \leqslant D$$

if D satisfies for all k that $\dim_K eH^0(M_{/K}, \underline{\omega}^{\otimes k}) \leqslant D$. We also observe that $\dim_K eH^0(M_{/K}, \underline{\omega}^{\otimes k})$ only depends on k modulo $p-1$ for $k \gg 0$.

The idempotent map $e : V \to V$ induces an idempotent map $e : V/\pi^m V \to V/\pi^m V$ for every $m \geqslant 1$ and hence induces idempotent maps e on $\mathcal{V}$ and $\mathcal{V}^*$. Set $\mathcal{V}_{\mathrm{ord}} = e \cdot \mathcal{V}$ and $\mathcal{V}^*_{\mathrm{ord}} = e \cdot \mathcal{V}^*$. For any $k \geqslant 1$, we observe that

(a) $\mathcal{V}^*_{\mathrm{ord}} \otimes_{W[\![\mathbb{Z}_p^\times]\!],\nu^k} W$ is the Pontryagin dual of $\mathcal{V}_{\mathrm{ord}}[\nu^k]$

(b) $\mathcal{V}^*_{\mathrm{ord}} \otimes_{W[\![\mathbb{Z}_p^\times]\!],\nu^k} W \otimes_W W/\pi W$ is the Pontryagin dual of $\mathcal{V}_{\mathrm{ord}}[\nu^k][\pi]$ (this is isomorphic to a finite-dimensional $\mathbb{F}$-vector space $eH^0(S_1, \underline{\omega}_1^{\otimes k})$ via $\beta(1)$).

Lemma 8.3 (Topological Nakayama lemma). *Let $(A, \mathfrak{m})$ be a complete local ring and M be a profinite A-module. If there exists a closed ideal $\mathfrak{a}$ of A such that $M/\mathfrak{a}M$ is finitely generated over A/f, then M is a finitely generated A-module, and the minimal number of generators of M over A is equal to that of $M/\mathfrak{a}M$ over $A/\mathfrak{a}$. In particular, if $M/\mathfrak{a}M = 0$, then $M = 0$.*

Since $\mathcal{V}^*_{\mathrm{ord}}$ is a profinite $W[\![\mathbb{Z}_p^\times]\!]$-module, we write

$$\nu^*_{\mathrm{ord}} = \bigoplus_{\chi \in \Delta^\vee} \mathcal{V}^*_{\mathrm{ord}}[\chi],$$

where $\mathcal{V}^*_{\mathrm{ord}}[\chi]$ is the $\Lambda = W[\![1+p\mathbb{Z}_p]\!]$-submodule of $\mathcal{V}^*_{\mathrm{ord}}$ on which Δ acts via the character $\chi : \Delta \to W^\times$. For each integer $k \geqslant 1$ that corresponds to weight κ_k such that $\kappa_k|_\Delta = \chi$, we have a finite-dimensional $\mathbb{F}$-vector space $(\mathcal{V}^*_{\mathrm{ord}}[\chi] \otimes_{\Lambda,\nu^k} W) \otimes_W \mathbb{F} \simeq \mathcal{V}^*_{\mathrm{ord}}[\chi] \otimes_\Lambda \mathbb{F}$. By topological Nakayama lemma (c.f. Lemma 8.3), $\mathcal{V}^*_{\mathrm{ord}}[\chi]$ is finitely generated as a Λ-module, and $\dim_{\mathbb{F}} eH^0(S_1, \underline{\omega}_1^{\otimes k})$ only depends on $\chi := \kappa_k|_\Delta$. Suppose it is generated over Λ by $d_\chi = \dim_{\mathbb{F}} \mathcal{V}^*_{\mathrm{ord}}[\chi] \otimes_\Lambda \mathbb{F}$ elements. Then there is a surjective Λ-linear $\varphi : \Lambda^{d_\chi} \twoheadrightarrow \mathcal{V}^*_{\mathrm{ord}}[\chi]$, and φ specializes to weight κ_k via $\nu^k : \Lambda \to W$, written as

$$\varphi \otimes_{\Lambda,\nu^k} W : W^{d_\chi} \xrightarrow{\sim} \mathcal{V}^*_{\mathrm{ord}}[\chi] \otimes_{\Lambda,\nu^k} W.$$

It turns out that this is an isomorphism for all $k \geqslant 1$. Then so also is φ. Hence $\mathcal{V}^*_{\mathrm{ord}}[\chi]$ is a free Λ-module of finite rank d_χ.

Since $\dim_{\mathbb{F}} eH^0(S_1, \underline{\omega}_1^{\otimes k})$ is finite, we see

$$eH^0(S_1, \underline{\omega}_1^{\otimes k+s(p-1)}) = eH^0(M_1, \underline{\omega}_1^{\otimes k+s(p-1)}), \quad s \gg 0.$$

Also, every element in $eH^0(S_1, \underline{\omega}_1^{\otimes k})$ lands in $H^0(M_1, \underline{\omega}_1^{\otimes k+s(p-1)})$ after multiplying a sufficiently large power $\mathcal{E}^s$. Hence we have

$$eH^0(S, \underline{\omega}^{\otimes k} \otimes K/W) = eH^0(M_{\Gamma/W}, \underline{\omega}^{\otimes k} \otimes K/W) = \mathcal{V}_{\mathrm{ord}}[\nu^k] = \varinjlim_m eV_{m,m}[\nu^k], \quad k \gg 0.$$

For such a k, we summarize the above discussion in the following theorem:

Theorem 8.4. *Suppose the existence of the idempotent e and the assumptions* (1) *and* (2) *before. Then $\mathcal{V}^*_{\mathrm{ord}}$ is a finitely generated projective $W[\![\mathbb{Z}_p^\times]\!]$-module which satisfies*

$$\mathcal{V}^*_{\mathrm{ord}} \otimes_{W[\![\mathbb{Z}_p^\times]\!],\nu^k} W \cong \mathrm{Hom}_W(eH^0(M_{/W}, \underline{\omega}^{\otimes k}), W).$$

For every character $\chi \in \Delta^\vee$, $\mathcal{V}^*_{\mathrm{ord}}[\chi]$ is a free Λ-module of rank d_χ. Under the stronger assumption $(2')$, the same result holds for all $k \geqslant k_0$.

Also, notice that with $M = M_\Gamma$, $eH^0(M_{/W}, \underline{\omega}^{\otimes k})$ is the ordinary part of Hecke algebra.

9. p-adic families of ordinary modular forms

We introduce some notations. Denote $p\mathsf{Alg}_{/W}$ the category of p-ordinary complete W-algebras. Define the functor for $n \geqslant 0$ that

$$\mathcal{E}^{\mathrm{ord}}_{\Gamma,n} : p\mathsf{Alg}_{/W} \longrightarrow \mathsf{Sets}$$

$$R \longmapsto \left\{ (E/R, \phi_\Gamma, \phi_{p^n}) \,\middle|\, \begin{array}{c} E \text{ is an elliptic curve over } R, \\ \phi_\Gamma \text{ is the } \Gamma\text{-level structure,} \\ \text{and } \phi_{p^n} : \mu_{p^n} \hookrightarrow E \end{array} \right\} / \cong .$$

For $n = 0$, we obtain a universal elliptic curve $\mathcal{E}^{\mathrm{ord}}_{\Gamma,0}/W = Y^{\mathrm{ord}}_{\Gamma/W}$ as well as a finite étale covering

$$\mathcal{E}^{\mathrm{ord}}_{\Gamma,n} \longrightarrow \mathcal{E}^{\mathrm{ord}}_{\Gamma,n} \cong Y^{\mathrm{ord}}_{\Gamma/W},$$

which further extends to the smooth compactification

$$\mathcal{E}^{\mathrm{ord}}_{\Gamma,n} \longrightarrow X^{\mathrm{ord}}_{\Gamma/W}.$$

Now let $\mathbb{E} \to S := X^{\mathrm{ord}}_{\Gamma/W}$ be the generalized elliptic curve. We obtain a natural exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{E}^{\circ}[p^n] \arrow[r] \arrow[d, equal] & \mathbb{E}[p^n] \arrow[r] & \mathbb{E}^{\text{ét}}[p^n] \arrow[r] \arrow[d, equal] & 0 \\ & \mathcal{C}_n & & \mathcal{P}_n & \end{tikzcd}$$

as group schemes over $X_\Gamma^{\text{ord}} = S$. We denote $S_m = S \times_W W_m$. Then points of $\operatorname{Spec}(V_{m,n})$ can be interpreted as isomorphisms over S_m between $\mathcal{P}_n$ and $\mathbb{Z}/p^n\mathbb{Z}$.

Theorem 9.1. *For each $m \geqslant 1$, we obtain*

$$\mathcal{E}_{\Gamma,n}^{\text{ord}} \cong \operatorname{Spec}(V_{m,n}) = \mathbf{Isom}_{S_m}(\mathcal{P}_n, \mathbb{Z}/p^n\mathbb{Z}).$$

This further gives an isomorphism $\psi : \mathcal{P}_n \xrightarrow{\sim} \mathbb{Z}/p^n\mathbb{Z}$ over A.

Proof. This follows from the properties of Cartier duality. If $\operatorname{Spec} A$ is an affine scheme over S_m such that there exists an isomorphism $\psi : \mathcal{P}_{n,A} \xrightarrow{\sim} (\mathbb{Z}/p^n\mathbb{Z})_{\operatorname{Spec} A}$, we can take the Cartier dual of the morphism $\mathbb{E}[p^n]_A \mathcal{P}_{n,A} \cong (\mathbb{Z}/p^n\mathbb{Z})_{\operatorname{Spec} A}$ and get an injective morphism $\mu_{p^n} \hookrightarrow \mathbb{E}[p^n]$ over A. This gives an A-valued point of $\operatorname{Spec}(V_{\Gamma,m,n})$. Since Cartier duality is perfect, we can reverse the above process and get the desired isomorphism. □

By post-composing ψ with $\mathbb{E}[p^n] \to \mathcal{P}_n$ and taking the Cartier duality, we can recover the morphism $\mu_{p^n} \hookrightarrow E[p^n]$ over A.

Corollary 9.2. *Let $f : \mathbb{E} \to S_m$ be the generalized elliptic curve defined above. Denote $0 : S_m \to \mathbb{E}$ the zero section of f. Then we obtain a canonical isomorphism*

$$f_*\Omega_{\mathbb{E}/S_m} = \underline{\omega}_{\mathbb{E}/S_m} \cong \mathcal{P}_m \otimes_{\mathbb{Z}_p} \mathcal{O}_{S_m}.$$

Proof. By definition we write $f_*\Omega_{\mathbb{E}/S_m} \cong \mathbf{0}^*\Omega_{\mathbb{E}/S_m}$. On the other hand, $\mathcal{C}_m$ is an S_m-subscheme of $\mathbb{E}[p^m]$. We are to use the following essential fact at work:

- ◇ Let A be a $\mathbb{Z}/p^m\mathbb{Z}$-algebra and set $\mu_{p^m} = \operatorname{Spec} A[T]/((1+T)^{p^m} - 1)$ as a scheme over A. Then $\Omega_{\mu_{p^m}/A}$ is an invertible sheaf on μ_{p^m}/A.

Consequently, $\Omega_{\mathcal{C}_m/S_m}$ is an invertible sheaf on $\mathcal{C}_m$, with a surjection

$$\Omega_{\mathbb{E}/S_m} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathcal{C}_m} \longrightarrow \Omega_{\mathcal{C}_m/S_m}.$$

Then we get a surjection between pullbacks along the zero section, read as

$$\mathbf{0}^*\Omega_{\mathbb{E}/S_m} \longrightarrow \mathbf{0}^*\Omega_{\mathcal{C}_m/S_m}$$

which is further an isomorphism. By Cartier duality, we have $\mathcal{P}_m = \operatorname{Hom}(\mathcal{C}_m, \mathbb{G}_m)$. For any morphism $\phi : \mathcal{C}_m \to \mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$, we get a section $\phi^*(dt/t) \in \Omega_{\mathcal{C}_m/S_m}$ by writing $\mathbb{G}_m = \operatorname{Spec}(\mathbb{Z}[T, T^{-1}]) \times_{\mathbb{Z}} S_m$. As a mimic of Hodge–Tate map, we get a surjective morphism

$$\begin{aligned} \mathcal{P}_m \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathcal{O}_{S_m} &\xrightarrow{\cong} \Omega_{\mathcal{C}_m/S_m} \\ \phi &\longmapsto \phi^*\left(\tfrac{dt}{t}\right) \end{aligned}$$

that is indeed an isomorphism. Therefore, as desired,

$$\underline{\omega}_{\mathbb{E}/S_m} \cong f_*\Omega_{\mathbb{E}/S_m} \cong \mathbf{0}^*\Omega_{\mathbb{E}/S_m} \mathcal{P}_m \otimes_{\mathbb{Z}_p} \mathcal{O}_{S_m}.$$

□

Let $A \in p\mathsf{Alg}_{/W}$. We set

$$V_{\Gamma/A} = \varprojlim_n V_{\Gamma/W} \otimes_W A/p^n A.$$

Each $f \in V_{\Gamma/W}$ is a rule which assigns an element $f(E/R, \phi_\Gamma, \phi_{p^\infty})$ to every triple $(E/R, \phi_\Gamma, \phi_{p^\infty})$ defined over a p-adically complete A-algebra R satisyfing:

(1) $f(E/R, \phi_\Gamma, \phi_{p^\infty})$ only depends on the isomorphism class of the triple $(E/R, \phi_\Gamma, \phi_{p^\infty})$;
(2) $f(E/R, \phi_\Gamma, \phi_{p^\infty})$ commutes with arbitrary base change;
(3) For any level Γ-structure ϕ_Γ on the Tate curve $\operatorname{Tate}(q^N)$, if we let $\phi_{p^\infty}^{\text{can}} : \mu_{p^\infty} \to \operatorname{Tate}(q^N)$ be the natural morphism, then we have

$$f(\operatorname{Tate}(q^N), \phi_\Gamma, \phi_{p^\infty}^{\text{can}}) \in A \otimes_{\mathbb{Z}} \mathbb{Z}[\![q]\!].$$

If the element $f(\mathrm{Tate}(q^N), \phi_\Gamma, \phi_{p^\infty}^{\mathrm{can}})$ further belongs to $q \cdot A \otimes_{\mathbb{Z}} \mathbb{Z}[\![q]\!]$, then f is called a *p-adic cusp form* of level Γ.

Theorem 9.3 (Density theorem)**.** *For any W-algebra A, we define*

$$M(\Gamma; A) := \bigoplus_{k \geqslant 0} H^0(M_{\Gamma/A}, \underline{\omega}^{\otimes k}).$$

Then the following subset of $V_\Gamma[\frac{1}{p}]$:

$$M(\Gamma; K) \cap V_{\Gamma/W}$$

is dense in $V_{\Gamma/W}$ with respect to the p-adic topology. This p-adic topology coincides with the topology induced by the notion $|f| = \sup_n |a(n; f)|_p$ for each $f \in V_{\Gamma/W}$. Also,

$$f(\mathrm{Tate}(q^N), \phi_\Gamma, \phi_{p^\infty}) = \sum_{n \geqslant 0} a(n; f) q^n$$

for any fixed level-Γ structure on $\mathrm{Tate}(q^N)$*.*

Proof. Following the recipe provided by [Kat73], the second statement follows from the fact that $\mathrm{Spec}(V_{m,n})$ is an irreducible smooth scheme over S_m, as well as the following fact. If f is a global section of a coherent sheaf on an irreducible smooth curve M over W_m, which is 0 on the formal completion at a closed point of M, then $f \equiv 0$. □

We now give an explicit expression of the inclusion

$$\beta : M(\Gamma; A) = \bigoplus_{k \geqslant 0} H^0(M_{\Gamma/A}, \underline{\omega}^{\otimes k}) \longrightarrow V_{\Gamma/A}$$

for a p-adically complete W-algebra A.

Suppose that we are given a triple $(E/A, \phi_\Gamma, \phi_{p^\infty} : \mu_{p^\infty} \hookrightarrow E)$. The invariant differential dt/t on $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[T, T^{-1}])$ induces an invariant differential ω_{can} on μ_{p^∞}. For $m \geqslant 1$, we set $A_m = A/p^m A$, and let $\mathcal{C}_m/A$ be the identity component of the finite flat group scheme $E[p^m]/A$. The morphism ϕ_{p^∞} induces an isomorphism $mu_{p^m} \xrightarrow{\sim} E^\circ[p^m]$ over A. As we see in the proof of Corollary 9.2 before, we have isomorphisms $\mathbf{0}^*\Omega_{\mu_{p^m}/A_m} \cong \mathbf{0}^*\Omega_{E/A_m} \cong \underline{\omega}_{E/A_m}$. The invariant differential ω_{can} on μ_{p^∞} then gives a basis of the A_m-module $\underline{\omega}_{E/A_m}$. Then taking inverse limit for m, we get a basis of $\underline{\omega}_{E/A}$, which is denoted by $\phi_{p^\infty,*}\omega_{\mathrm{can}}$. Then β is defined as

$$\beta(f)(E/A, \phi_\Gamma, \phi_{p^\infty}) = f(E/A, \phi_\Gamma, \phi_{p^\infty,*}\omega_{\mathrm{can}}).$$

In particular, if $f \in H^0(M_{/A}, \underline{\omega}^{\otimes k})$ then we have $\beta(f) \in V_{\Gamma/A}[\nu^k]$.

10. Hecke operators

Let ℓ be a prime number and R be a ring such that $\ell^{-1} \in R$. Fix a prime-to-ℓ integer N. Let E be an elliptic curve over R. Suppose there exists a finite étale ring homomorphism $R \to R'$ such that over R', we have $E[\ell]_{R'} \cong (\mathbb{Z}/\ell\mathbb{Z})^2_{R'}$. It turns out that $E[\ell]_{R'}$ has $\ell + 1$ finite flat subgroup schemes of rank ℓ over R'. Let $H \subset E[\ell]$ be such a group scheme. Let $\pi : E \to E/H$ be the natural projection and $\pi^t : E/H \to E$ be its dual map.

For a triple $(E/R, \omega, \phi_{\Gamma(N)})$ consisting of an elliptic curve E/R, a basis ω of $\underline{\omega}_{E/R}$, and a level-$\Gamma(N)$ structure $\phi_{\Gamma(N)}$ over R, we define another triple $(E_{R'}/H, \underline{\omega}'_H, \phi_{\Gamma(N),H})$, where $E_{R'}/H$ is the quotient of $E_{R'}$ by H, $\underline{\omega}'_H$ is the basis of $\underline{\omega}_{(E_{R'}/H)/R'}$ defined by $\pi^*(\omega'_H) = \omega$, and $\phi_{\Gamma(N),H} : (E_{R'}/H)[N] \cong (\mathbb{Z}/N\mathbb{Z})^2_{R'}$ is the level structure on $E_{R'}/H$ defined over R' via the commutative diagram:

$$\begin{tikzcd} E_{R'}[N] \arrow[rr, "\pi"] \arrow[dr, "\phi_{\Gamma(N)}"'] & & (E_{R'}/H)[N] \arrow[dl, "\phi_{\Gamma(N),H}"] \\ & (\mathbb{Z}/N\mathbb{Z})^2_{/R'}. & \end{tikzcd}$$

One can also define a $\Gamma_1(N)$-level structure on $E_{R'}/H$ from a level $\Gamma_1(N)$-structure on E/R by a similar construction.

Let f be a modular form of weight k and level $\Gamma = \Gamma(N)$ or $\Gamma_1(N)$ over R. We define

$$(f|T(\ell))(E/R, \omega, \phi_\Gamma) = \frac{1}{\ell} \sum_{H \subset E[N]_{/R'}} f(E_{R'}/H, , \underline{\omega}'_H, \phi_{\Gamma,H}),$$

where the sum is taken over all finite flat subgroup schemes of rank ℓ of $E[\ell]_{/R'}$. We can deduce from the q-expansion principle that $(f|T(\ell))(E/R, \omega, \phi_\Gamma) \in R$ and it is independent of the choice of R'. Hence $(f|T(\ell))$ is a modular form of weight k and level Γ over R. This $T(\ell)$ is called the *Hecke operator* on the space of modular forms of weight k and level Γ over R.

More generally, if ℓ is not invertible in R yet not a zero divisor, we may define first $T(\ell)$ on modular forms of weight k and level Γ over $R[\frac{1}{\ell}]$, and then show that $T(\ell)$ leaves modular forms over R stable. Moreover, if $\ell \mid N$ and $\Gamma = \Gamma_1(N)$, notate the image of level-Γ structure by C via $\phi_\Gamma : \mathbb{Z}/N\mathbb{Z} \to E$, and henceforth

$$(f|U(\ell))(E/R, \omega, \phi_\Gamma) = \frac{1}{\ell} \sum_{H \subset E[N]_{/R'}} f(E_{R'}/H, , \underline{\omega}'_H, \phi_{\Gamma,H}),$$

where H runs over all finite flat subgroup schemes of $E[\ell]_{R'}$ such that $H \cap C = (0)$.

Let $p \geqslant 5$ and A be a p-adic W-algebra. We define the Hecke $U(p)$-operator on $V_{\Gamma/A}$ as follows. Consider the short exact sequence of finite flat group schemes over R:

$$0 \longrightarrow \mu_p \xrightarrow{\phi_p} E[p] \xrightarrow{\psi_p} E^{\text{ét}}[p] \longrightarrow 0$$

as well as the triple $(E/R, \phi_\Gamma, \phi_{p^\infty} : \mu_{p^\infty} \to E)$. Suppose these are defined over a finite étale extension R' of R such that over R', there are exactly p finite flat subgroup schemes H of $E[p]$, just so $\psi_p : H \to E^{\text{ét}}[p]$ is an isomorphism. For such an H, let $\phi_{p^\infty,H}$ be the composite

$$\phi_{p^\infty,H} : \mu_{p^\infty,R'} \xrightarrow{\phi_{p^\infty}} E_{R'} \longrightarrow E_{R'}/H.$$

For each $f \in V_{\Gamma/A}$, we define

$$(f|U(p))(E/R, \phi_\Gamma, \phi_{p^\infty}) = \frac{1}{p} \sum_{H \subset E[p]/R'} f(E/R', \phi_{\Gamma,H}, \phi_{p^\infty,H}).$$

However, the following map for $\Gamma = \Gamma(N)$ is not Hecke-equivariant:

$$\beta^{\text{naive}} : \bigoplus_{k \geqslant 0} H^0(M_{\Gamma/W}, \underline{\omega}^{\otimes k}) \longrightarrow V_{\Gamma/W},$$

because the left-hand side carries the $T(p)$-operator whereas the right-hand side carries the $U(p)$-operator. Instead, one should replace the map above with

$$\beta : \bigoplus_{k \geqslant 0} H^0(M_{\Gamma \cap \Gamma_1(p)/W}, \underline{\omega}^{\otimes k}) \longrightarrow V_{\Gamma/W},$$

where the left-hand side is a subgroup of $\bigoplus_{k \geqslant 0} H^0(M_{\Gamma/W}, \underline{\omega}^{\otimes k})$. Now both sides carry the $U(p)$-operator, and β is Hecke-equivariant.

Proposition 10.1. *The Hecke operators $T(\ell)$ for $\ell \nmid Np$, and $U(\ell)$ for $\ell \mid Np$ on $V_{\Gamma/W}$, give continuous endomorphisms on $V_{\Gamma/W}$ and $V_{\Gamma,m,n/W_m}$ for all m, n.*

Proof. The statement follows from the q-expansion principle and a careful computation of Hecke operators on Tate curves. We refer to [Kat73, §1.12] for a careful discussion. We give a list of formulas of the Hecke operators on p-adic modular forms in the case of $\Gamma_1(N)$-level structure. Fix a triple $(\mathrm{Tate}(q^N), \phi_{\Gamma_1(N)}, \phi_{p^\infty})$. For $f \in V_{\Gamma/W}$, we denote by $a(n; f)$ the q^n-coefficient of the q-expansion to $f(\mathrm{Tate}(q^N), \phi_{\Gamma_1(N)}, \phi_{p^\infty})$. For $\ell \nmid Np$, we define

$$(f|\ell)(\mathrm{Tate}(q^N), \phi_{\Gamma_1(N)}, \phi_{p^\infty}) = f(\mathrm{Tate}(q^N), \ell \cdot \phi_{\Gamma_1(N)}, \ell^{-1} \cdot \phi_{p^\infty}).$$

Under the above notations, we have for $\ell \nmid Np$ that

$$a(n; (f|T(\ell))) = a(\ell n; f) + \ell^{-1} a(n/\ell; (f|\ell)),$$

and for $\ell \mid Np$ that

$$f(n, (f|U(\ell))) = a(\ell n; f).$$

$\square$

Corollary 10.2. *The limit* $e = \lim_{n\to\infty} U(p)^{n!}$ *exists and gives an idempotent in* $V_{\Gamma/W}$.

Proof. It follows from the fact that the limit $\lim_{n\to\infty} U(p)^{n!}$ exists in

$$M(\Gamma; K) = \bigoplus_{k\geqslant 0} H^0(M_{\Gamma/K}, \underline{\omega}^{\otimes k})$$

and Theorem 9.3. $\square$

11. Analytic family of p-adic ordinary modular forms

The space $V_{\Gamma/W}$ of p-adic modular forms is a flat W-module with an action of $\mathbb{Z}_p^\times$. A p-adic modular form $f \in V_{\Gamma/W}$ is of weight $s \in \mathbb{Z}_p$ if $z(f) = z^s \cdot f$ for all $z \in 1 + p\mathbb{Z}_p$. Recall that we fix a topological generator $u = 1 + p$ of $1 + p\mathbb{Z}_p$ and an isomorphism $\Lambda = W[\![1 + p\mathbb{Z}_p]\!] \to W[\![X]\!]$, sending $[u]$ to $1 + X$.

Definition 11.1. A formal series

$$\Phi(X, q) = \sum_{n\geqslant 0} a(n; \Phi)(X)q^n \in \Lambda[\![q]\!]$$

is called an Λ-*adic modular form* if $\Phi(u^s - 1, q)$ is a p-adic modular form of weight s for all $s \in \mathbb{Z}_p$. The set of p-adic modular forms $\{\Phi(u^s - 1, q)\}_{s\in\mathbb{Z}_p}$ is called a family of p-adic modular forms. A Λ-adic modular form Φ (or the family $\{\Phi(u^s - 1, q)\}_{s\in\mathbb{Z}_p}$) is called *arithmetic* if $\Phi(u^k - 1, q)$ is a (true) modular form for almost all positive integer k. We call Φ a Λ-adic cusp form if $\Phi(u^s - 1, q)$ is a p-adic cusp form for all $s \in \mathbb{Z}_p$.

Recall that $\Lambda = W[\![X]\!]$ can be identified with the space of bounded measures

$$\mathrm{Meas}(\mathbb{Z}_p; W) := \mathrm{Hom}_W(\mathcal{C}(\mathbb{Z}_p, W), W)$$

over $\mathbb{Z}_p$ with values in W via the formula

$$\int_{\mathbb{Z}_p} (1 + X)^s d\phi(z) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \binom{z}{n} d\phi(z) \cdot X^n.$$

Denote the right-hand side by $\phi(X)$. Here saying a measure ϕ is bounded means that

$$\left| \int_{\mathbb{Z}_p} f(x)d\phi(z) \right|_p \leqslant |f|_p = \sup_{z\in\mathbb{Z}_p} |f(z)|_p.$$

Under the homeomorphism $\mathbb{Z}_p \to 1 + p\mathbb{Z}_p$ via $z \mapsto u^z$, we can identify Λ with $\mathrm{Meas}(1 + p\mathbb{Z}_p; W)$. In particular, let $X = u^s - 1$ in the above formula, we have

$$\int_{1+p\mathbb{Z}_p} t^s d\phi(t) = \phi(u^s - 1).$$

We can generalize the above notion to bounded measures on $1 + p\mathbb{Z}_p$, with values in $V_{\Gamma/W} \hookrightarrow W[\![q]\!]$ as the letter spaces are equipped with the norm $|f|_p = \sup_{n\geqslant 0} |a(n; f)|_p$ for $f = \sum_{n\geqslant 0} a(n; f)q^n$. Then a Λ-adic form Φ can be identified with a bounded measure on $1 + p\mathbb{Z}_p$ with values in $V_{\Gamma/W}$ and it satisfies

$$\int_{1+p\mathbb{Z}_p} x^s d\Phi = \sum_{n\geqslant 0} a(n; \Phi)(u^s - 1) \cdot q^n, \quad \forall s \in \mathbb{Z}_p.$$

If $T : V_{\Gamma/W} \to V_{\Gamma/W}$ is a bounded W-linear map, then for any Λ-adic form Φ, $T \circ d\Phi$ is also a bounded measure on $1 + p\mathbb{Z}_p$ with values in $V_{\Gamma/W}$, and it corresponds to a Λ-adic form $(\Phi|T)$. In particular, we get a $\mathbb{Z}_p^\times$-action and Hecke operators on the space of Λ-adic forms.

Definition 11.2. Let A be a p-adically complete W-algebra. A p-adic modular form $f \in V_{\Gamma/A}$ is called p-*ordinary* if $f \in eV_{\Gamma/A}$. A Λ-adic form Φ is p-*ordinary* if $\Phi(u^s - 1, q) \in V_{\Gamma/W}$ is p-ordinary for all $s \in \mathbb{Z}_p$.

We denote by $M^{\text{ord}}(\Gamma,\Lambda)$ (resp. $S^{\text{ord}}(\Gamma,\Lambda)$) for the space of p-ordinary Λ-adic forms (p-ordinary Λ-adic cusp forms) with level Γ. We also denote by $H^{\text{ord}}(\Gamma,\Lambda)$ (resp. $h^{\text{ord}}(\Gamma,\Lambda)$) for the Λ-subalgebra of $\text{End}_\Lambda(M^{\text{ord}}(\Gamma,\Lambda))$ (resp. $\text{End}_\Lambda(S^{\text{ord}}(\Gamma,\Lambda))$) generated by all the Hecke operators.

Theorem 11.3 (Vertical control theorem)**.** *Under the above notations, we have*

(1) $H^{\text{ord}}(\Gamma,\Lambda)\cong\mathcal{V}^*_{\text{ord}}$, *and* $H^{\text{ord}}(\Gamma,\Lambda)\otimes_{\Lambda,\nu^k}W\cong H_k^{\text{ord}}(\Gamma\cap\Gamma_1(p),W)$ *for* $k\geqslant 2$.
(2) $M^{\text{ord}}(\Gamma,\Lambda)\cong\text{Hom}_\Lambda(\mathcal{V}^*_{\text{ord}},\Lambda)$.
(3) $M^{\text{ord}}(\Gamma,\Lambda)$ *is free of finite rank over* Λ.
(4) *For* $k\geqslant 2$, *the specialization* $\Phi\mapsto\Phi(u^k-1,q)$ *induces an isomorphism* $M^{\text{ord}}(\Gamma,\Lambda)\otimes_{\Lambda,\nu^k}W\cong M_k^{\text{ord}}(\Gamma\cap\Gamma_1(p),W)$.
(5) *All* p-*ordinary* Λ-*adic forms are arithmetic.*

Proof. For any $n\geqslant 0$, the map $a(n):\mathcal{V}_{\text{ord}}\to K/W$ which assigns $f\in eV_{\Gamma/W}$ to its coefficient of q^n is an element of $\mathcal{V}^*_{\text{ord}}$. Consider the map

$$(*)\qquad \text{Hom}_\lambda(\mathcal{V}^*_{\text{ord}},\Lambda)\longrightarrow\Lambda[\![q]\!],\quad \phi\longmapsto\sum_{n\geqslant 0}\phi(a(n))q^n=:\Phi(X,q).$$

Since $\mathcal{V}^*_{\text{ord}}$ is a finite free Λ-module, we have

$$\begin{aligned}\text{Hom}_\Lambda(\mathcal{V}^*_{\text{ord}},\Lambda)\otimes_{\Lambda,\nu^k}W&\cong\text{Hom}_W(\mathcal{V}^*_{\text{ord}}\otimes_{W,\nu^k}W,W)\\&\cong\text{Hom}_W(\text{Hom}_W(M_k^{\text{ord}}(\Gamma\cap\Gamma_1(p)),W),W)\\&\cong M_k^{\text{ord}}(\Gamma\cap\Gamma_1(p)).\end{aligned}$$

If we write down the above isomorphism explicitly, we see that the power series $\Phi(X,q)$ defined in ($*$) satisfies $\Phi(u^k-1,q)\in M_k^{\text{ord}}(\Gamma\cap\Gamma_1(p),W)$ for all $k\geqslant 2$. Hence t he map in ($*$) gives an isomorphism

$$\text{Hom}_\Lambda(\mathcal{V}^*_{\text{ord}},\Lambda)\xrightarrow{\sim}M^{\text{ord}}(\Gamma,\Lambda).$$

This proves (2). The other statements follow from the duality between the space of ordinary Λ-adic forms and its Hecke algebra. □

Remark 11.4. We may consider the subspace $\mathcal{V}_{\text{ord,cusp}}\subset\mathcal{V}_{\text{ord}}$ of elements whose constant terms of the q-expansions are 0. The above theorem holds if we replace $M(\Gamma,\Lambda)$ (resp. $\mathcal{V}^*_{\text{ord}}$) by $S(\Gamma,\Lambda)$ (resp. $\mathcal{V}^*_{\text{ord,cusp}}$).

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