A CRASH INTRODUCTION ON EISENSTEIN SERIES

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The readers are assumed to acquire basic notions of representation theory of compact groups and harmonic analysis.

The upcoming context was completed in the time I took to fit it into the tight schedule of the IHES Summer School, so the time available for writing was extremely limited. It may therefore contain a large number of errors in detail. Please use at your own risk as I will never read or revise it.

Langlands' Spectral Decomposition. The classical theory on Fourier series dictates that, the space of square-integrable functions $L^2(\mathbb{R}/\mathbb{Z})$ on $\mathbb{R}/\mathbb{Z}$ has a orthogonal basis $e^{2n\pi it}$, where $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. On the other hand, these $e^{2n\pi it}$ can be viewed as the unitary characters of $\mathbb{R}/\mathbb{Z}$ whose corresponding 1-dimensional representations are denoted by χ_n. Then there is a decomposition of representations, say

$$L^2(\mathbb{R}/\mathbb{Z}) \simeq \widehat{\bigoplus_{n\in\mathbb{Z}}} \chi_n.$$

Alternatively, by considering the space $L^2(\mathbb{R})$ over $\mathbb{R}$, the Fourier transform gives an interpretation of the direct integral decomposition

$$L^2(\mathbb{R}) \simeq \int_{\mathbb{R}} \chi_s ds.$$

The theory of automorphic representation concerns about generalizations of these phenomena over some ad hoc types of non-abelian groups. Also, as one would expect, the two antipodal cases, read as the discrete case and the continuous case, will occur in a mixed appearance.

Let G be the reductive group over $\mathbb{Q}$ with a fixed Haar measure on it. Denote T_0 the maximal $\mathbb{Q}$-split torus of $Z(G)$, the center of G. Denote A_G the unit connected component of $T_0(\mathbb{R})$. Then there is an action[1] by right multiplications on the following quotient:

$$[G] := G(\mathbb{Q})A_G\backslash G(\mathbb{A}) \circlearrowleft G(\mathbb{A})$$

where $\mathbb{A}$ is the Adele of $\mathbb{Q}$. Then the space $L^2([G])$ can be realized as the unitary representation of $G(\mathbb{A})$. From rudimentary theory on unitary representation, there is a Borel measure μ on the unitary dual $\widehat{G(\mathbb{A})}$ of $G(\mathbb{A})$, together with a Borel measurable function

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[1]The principal reason to introduce $[G]$ is that it has a finite volume with respect to the fixed Haar measure. Hence it is relatively easy to tackle with the analytic problems.

$m(\pi) : \widehat{G(\mathbb{A})} \to \mathbb{Z}_{\geqslant 0} \cap \{+\infty\}$ such that we obtain the direct integral decomposition:[2]

$$L^2([G]) \simeq \int_{\widehat{G(\mathbb{A})}}^{\oplus} \widehat{\bigoplus}^{m(\pi)} V_\pi d\mu(\pi).$$

Also, there is a *unique* decomposition for the measure

$$\mu = \mu_{\text{disc}} + \mu_{\text{cont}},$$

where the support of μ_{disc} is countable, and μ_{cont} has zero measure on arbitrary countable set. Take their corresponding direct integrals, we get

$$L^2([G]) = L^2_{\text{disc}}([G]) \oplus L^2_{\text{cont}}([G]),$$

in which the two direct summands are called the discrete spectrum and the continuous spectrum of $G(A_{\mathbb{F}})$, respectively.

Questions.

(1) A natural curiosity is to wonder that which representations would appear in the discrete (resp. continuous) spectrum.
(2) Also, for some π appearing in $L^2_{\text{disc}}([G])$, what is the multiplicity $m(\pi)$?

For general reductive groups, we are still far from the answer to question (2). However, Langlands' important work dictates for (1) that:

- ⋄ The discrete spectrum of Levi subgroups of G determines the continuous spectrum of G. More precisely, let P be a parabolic subgroup of a reductive group G and take the Levi decomposition $P = M \rtimes N$, then

$$L^2_{\text{disc}}([N]) \quad \rightsquigarrow \quad L^2_{\text{cont}}([G]).$$

This is the so-called *Langlands Spectral Decomposition Theorem.* The primary tool at work for Langlands is basically Eisenstein series.

Eisenstein Series. The original motivation of Eisenstein is extremely simple. By taking the sum through the orbits under actions of discrete subgroups, one attains the periodic functions. Its earliest form was proposed by the nineteenth century German mathematician Eisenstein in his study of elliptic functions, and the corresponding version of the modular form began to be studied by mathematicians in the same period. After World War II, the German mathematician Maass began to systematically study non-holomorphic Eisenstein series on the complex upper half-plane, based on the work of Hecke, Rankin and others.

Take $z = x + iy \in \mathbb{H}$ in the upper half plane. The Eisenstein series considered by Maass is the formal sum

$$E_s(z) = \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1}} \frac{y^s}{|mz+n|^{2s}}.$$

This infinite sum keeps invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, absolutely converges when $\mathrm{Re}(s) > 1$, and is holomorphic with respect to s in the region defined by $\mathrm{Re}(s) > 1$.

[2] The Borel measure is with respect to the Fell topology on $\widehat{G(\mathbb{A})}$ say. This topology is T_0, which is the necessary condition to have the direct integral decomposition.

Maass proved that $E_s(z)$ admits a meromorphic continuation on s to the whole plane $\mathbb{C}$. Moreover, it satisfies the function equation

$$E_s(z) = c(s)E_{1-s}(z),$$

where $c(s)$ is a meromorphic simple factor that can be expressed in terms of $\Gamma(s)$ and $\zeta(s)$. Maass's work was the beginning of the analytical theory of Eisenstein series, and after him mathematicians such as Roelcke and Selberg studied Eisenstein series on more general spaces and were able to prove the meromorphic continuity and function equation in some cases. At almost the same time, mathematicians such as Gelfand and Harish-Chandra began to introduce representationist ideas into the study of automorphic forms. This reveals the representation-theoretical meanings of Eisenstein series: it provides the homomorphism from parabolically induced representation to the space of automorphic forms on G. Building on previous work, Langlands developed the most general Eisenstein series theory[3] in his article *On the functional equations satisfied by Eisenstein series* in the 1960s.

The Meromorphic Continuation.

Setups. In order to describe the work of Langlands, more notations and definitions are required[4]. Let P_0 be the minimal parabolic subgroup of G whose corresponding Levi subgroup is M_0 say. For any parabolic subgroup P containing P_0, denote M_P the Levi subgroup of P containing M_0. Take

$$\mathfrak{a}_P^* = \mathrm{Hom}(P, \mathbb{G}_m) \otimes \mathbb{R} = X^*(P) \otimes \mathbb{R}.$$

Then for each $\lambda \in \mathfrak{a}_P^* \otimes \mathbb{C}$, we have the parabolic induction

$$(\pi, V) = (I(P, \lambda), \mathrm{Ind}_P^G(L^2_{\mathrm{disc}}([M_P])))$$

of $G(\mathbb{A})$. When $\lambda \in i\mathfrak{a}_P^*$, the parabolic induction is morally a unitary representation. The linear space $V = \mathrm{Ind}_P^G(L^2_{\mathrm{disc}}([M_P]))$ equipped with an action of $I(P, \lambda)$ can be realized as a subspace of the space of functions on $N(\mathbb{A})M(\mathbb{Q})A_{M_P}\backslash G(\mathbb{A})$. In fact, one can check that this space is independent of the choice of λ. For any $\phi \in \mathrm{Ind}_P^G(L^2_{\mathrm{disc}}([M_P]))$ and $\lambda \in \mathfrak{a}_P^* \otimes \mathbb{C}$, its corresponding Eisenstein series is defined to be the formal sum

$$E(x, \phi, \lambda) := \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi(\delta x) e^{\langle H_P(\delta x), \lambda + \rho_P \rangle}.$$

Results. Then the theorem on meromorphic continuation can be summarized as follows.

- ◇ If ϕ is automorphic[5], then the formal sum which defines the Eisenstein series converges for $\lambda \gg 0$ and admits a meromorphic continuation to $\mathfrak{a}_P^* \otimes \mathbb{C}$.

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[3]The initial version of theory of Langlands was in terms of the classical language using Lie groups and discrete subgroups. But he later realized the number-theoretical meaning of introducing Adele groups. Now the theory of Langlands are usually given in terms of Adele groups.

[4]A detailed explanation of the full details would require a great deal of preliminaries, and we introduce only the necessary concepts for the context. For more undefined notations and details, see [GH22] and [MW95].

[5]By definition, ϕ is automorphic, if it is K-finite for the maximal compact subgroup K of $G(\mathbb{A})$, and for any x, the map $m \to \phi(mx)$ outputs automorphic forms on M.

The Function Equation.

Setups. Denote $\mathfrak{a}_P$ the dual space of $\mathfrak{a}_P^*$. Let W that acts on $\mathfrak{a}P_0$ be the Weyl group of T_0 in G. Note that for each parabolic subgroup P containing P_0, $\mathfrak{a}_P$ is a subspace of $\mathfrak{a}_{P_0}$ in a natural sense. Therefore, for any parabolic subgroups $P, Q \supset P_0$, define

$$W(P,Q) = \{w \in W \mid w\mathfrak{a}_P = \mathfrak{a}_Q\} \subset W.$$

When $W(P,Q) \neq \emptyset$, we say P, Q are adjoint. For each $w \in W(P,Q)$ and $\phi \in \mathrm{Ind}_P^G(L^2_{\mathrm{disc}}([M]))$, there is an explicit integral to define $M(w,\lambda)\phi$ (due to Langlands). If ϕ is automorphic, then this integral converges when $\lambda \gg 0$. Moreover, in this case, $M(w,\lambda)\phi \in \mathrm{Ind}_P^G(L^2_{\mathrm{disc}}([M_Q]))$ admits a meromorphic continuation onto $\mathfrak{a}_P^* \otimes \mathbb{C}$, and is holomorphic at those $\lambda \in i\mathfrak{a}_P^*$; for these λ, it gives the unitary operators

$$\mathrm{Ind}_P^G(L^2_{\mathrm{disc}}([M_P])) \to \mathrm{Ind}_P^G(L^2_{\mathrm{disc}}([M_Q])).$$

Results. The function equation for Eisenstein series is read as

$$E(x,\phi,\lambda) = E(x, M(w,\lambda)\phi, w\lambda)$$

when ϕ is automorphic.

Langlands' Description for $L^2([G])$. Let $\mathcal{P}$ be the adjoint class of some parabolic subgroup. Consider the family of functions $\{F_P\}_{P\in\mathcal{P}}$, where each $F_P : i\mathfrak{a}_P \to \mathrm{Ind}_P^G(L^2_{\mathrm{disc}}([M_P]))$ is a measurable function such that

(i) for all $P, Q \in \mathcal{P}$ and $w \in W(P,Q)$, we have $M(w,\lambda)F_P(\lambda) = F_Q(w\lambda)$;
(ii) also,
$$\frac{1}{n_{\mathcal{P}}} \sum_{P\in\mathcal{P}} \int_{i(\mathfrak{a}_P/\mathfrak{a}_G)^*} \|F_P(\lambda)\|^2 d\lambda < +\infty.$$

Here (ii) defines an inner product on $\mathcal{L}_{\mathcal{P}}$. Denote $\widehat{\mathcal{L}_{\mathcal{P}}}$ the completion of it. For elements $\{F_P\}_{P\in\mathcal{P}} \subset \mathcal{L}_{\mathcal{P}}$, consider

$$\frac{1}{n_{\mathcal{P}}} \sum_{P\in\mathcal{P}} \int_{i(\mathfrak{a}_P/\mathfrak{a}_G)^*} E(x, F_P(\lambda), \lambda) d\lambda.$$

In a dense subspace of $\mathcal{L}$, this expression renders the elements in $L^2([G])$ together with an isometry map. Therefore, it extends to the embedding

$$\widehat{\mathcal{L}_{\mathcal{P}}} \longrightarrow L^2([G]),$$

whose image space is a closed subspace of $L^2([G])$, denoted by $\widehat{\mathcal{L}_{\mathcal{P}}}$ again, by abuse of notation.

Langlands proved that when $\mathcal{P}$ runs through all adjoint classes of all parabolic subgroups, the subspace above deduces the direct sum decomposition

$$L^2([G]) \simeq \bigoplus \widehat{\mathcal{L}_{\mathcal{P}}}.$$

When the adjoint class consists G only, $\widehat{\mathcal{L}_{\mathcal{P}}} = L^2_{\mathrm{disc}}([G])$. As for other adjoint classes, $\widehat{\mathcal{L}_{\mathcal{P}}}$ is the subspace of $L^2_{\mathrm{cont}}([G])$. Hence this decomposition describes continuous spectra explicitly by Eisenstein series for discrete spectra of Levi subgroups. Furthermore, Langlands imposed a finer classification of discrete spectra by Eisenstein series[6]. The work of Gelfand and Piatski-Shapiro tells us the cusp forms of G are square-integrable, and the space spanned by these cusp forms is contained in $L^2_{\mathrm{disc}}([G])$, whose closure is called the *cuspidal spectrum* of G (denoted by $L^2_{\mathrm{cusp}}([G])$). By the way, Langlands proved that the orthogonal complement of $L^2_{\mathrm{cusp}}([G])$ in $L^2_{\mathrm{disc}}([G])$ can be spanned by the residues of Eisenstein series. This orthogonal complement can be called the *residue spectrum* of G.

Epilogue. Langlands' work on the Eisenstein series was so difficult that he himself considered some parts of his proof to be "almost impenetrable", but the impact of this work is still self-evident. After completing this work, Langlands computed the constant terms of the Eisenstein series, and it was in this process that Langlands realized the importance of L-groups, gave the definition of self-consistent L-functions and formulated the famous functionality conjecture – the core of Langlands' program. Although some progress has been made, the Langlands program is still a long way from completion and needs to be further investigated and explored by mathematicians.

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[6]The original article by Langlands contains a patchwork on the proof of properties of Eisenstein series and the decomposition of discrete spectrum.