Lecture 1: Satake isomorphism for p-adic groups

Oct 7

<u>Setups</u> F non-arch local field
$\cup$
$\mathcal{O}$ ring of integers
$\in$
ϖ uniformizer $\rightsquigarrow$ $k = \mathbb{F}_q$ residue field, $\ell \neq p$ prime, $q = p^f$.

$G \supset B \supset T$ all split $/F$

$K = G(\mathcal{O}) \subset G(F)$. $\quad \mathrm{vol}_{dg}(K) = 1$.

Define $\mathcal{H}_K(G) := C_c(K \backslash G(F) / K, \bar{\mathbb{Q}}_\ell)$

equipped with $*$:

$$f_1 * f_2(g) = \int_{G(F)} f_1(g x^{-1}) f_2(x)\, dx$$

<u>Thm</u> (Satake for split grp)

$$\left(\mathcal{H}_{T(\mathcal{O})}(T(F)) =\right) \mathcal{H}_K(G) \cong \bar{\mathbb{Q}}_\ell[X_*(T)]^{W_0} \quad (\text{Comm f.t. } \bar{\mathbb{Q}}_\ell\text{-alg})$$
$$f \longmapsto S(f)$$

where $W_0 = N_G T(F) / T(F)$.

- Here to define $S(f)$:

take $d\mu$ on $U(F)$ w/ $\mathrm{vol}_{d\mu}(U(\mathcal{O})) = 1$.

$$\rightsquigarrow S(f)(t) = \delta_B^{1/2}(t) \int_{U(F)} f(tu)\, du$$

$$\delta_B : T(F) \longrightarrow \bar{\mathbb{Q}}_\ell^\times \ni q^{1/2} \text{ (fixed)}$$
$$t \longmapsto |\det \mathrm{Ad}(t) ; \mathrm{Lie}\, U(F)|_F$$
$$q^{\widehat{\mathbb{Z}}} \subset \bar{\mathbb{Q}}_\ell^\times .$$

More general grps

$G \supset P = MN$ min F-parabolic

A = max F-split torus. $M = C_G(A)$ = min F-levi.

Some Bruhat-Tits theory

Kottwitz hom: $\breve{F} \cong \widehat{F^{un}}$, $\kappa_G : G(\breve{F}) \twoheadrightarrow \pi_1(G)_I$.

where $I \subset \Gamma = Gal(\bar{F}/F)$ inertia

$\pi_1(G) = X_*(T)/Q^\vee$, $Q^\vee$ = abs coroot lattice.

Then $\kappa_{GL_n} = val_F \circ \det : GL_n(\breve{F}) \longrightarrow \mathbb{Z}$.

Write $G(\breve{F})_1 := \ker \kappa_G$; $G(F)_1 = G(\breve{F})_1 \cap G(F)$.

if G/F, set $\kappa_G : G(F) \twoheadrightarrow \pi_1(G)_I^\sigma$

with σ = any Frob $\in \Gamma$.

Let $\mathcal{A}$ = apartment in BT building corr to A

Let $o \in \mathcal{A}$ special vertex.

$\rightsquigarrow$ BT grp sch $\mathcal{G} = \mathcal{G}_o / \mathcal{O}$

- $\mathcal{G}(\mathcal{O}) = \mathrm{Fix}_{G(F)}\, o \cap G(F)_1$
- $\mathcal{G}$ sm affine geom conn $/\mathcal{O}$.

Take $K := \mathcal{G}(\mathcal{O})$ special max parabolic of $G(F)$.

$\Lambda_M = M(F)/M(F)_1$ f.g. abelian grp

$M(F)_1 = K \cap M(F)$ parabolic subgrp in $M(F)$.

$W_0 = N_G A(F)/M(F)$ rel Weyl grp of (G, A).

Thm (Haines-Rostorin, 2009)

$$\mathcal{H}_K(G) \cong \bar{\mathbb{Q}}_\ell[\Lambda_M]^{W_0}.$$

Rmk Cartier (Corvallis) proved

$$\mathcal{H}_{\tilde{K}}(G) = \bar{\mathbb{Q}}_\ell[\tilde{\Lambda}_M]^{W_0} \quad \text{(a little weaker)}$$

$\tilde{K} \supset K$ special max compact

$\tilde{\Lambda}_M = M(F)/M(F)^1, \quad \underset{\substack{\uparrow \\ \text{max cpt in } M(F).}}{M(F)^1} \supset M(F)_1$

<u>A special case</u> G qs $/F$.

$$G \supset \underset{\substack{\| \\ TU}}{B} \supset T = C_G(A)$$

K again $\mathcal{G}(\mathcal{O})$ special max parahoric

$$\Lambda_M = \Lambda_T = X_*(T)_I^\sigma = X^*(\hat{T}^I)^\sigma.$$

Get $\mathcal{H}_K(G) \cong \bar{\mathbb{Q}}_\ell[X^*(\hat{T}^I)^\sigma]^{W_0}$

$$\text{Cartier} \Rightarrow \mathcal{H}_{\tilde{K}}(G) \cong \bar{\mathbb{Q}}_\ell[\underbrace{X^*(\hat{T}^{I,\overset{\text{conn component}}{\circ}})^\sigma}_{(X_*(T)_I/\text{tor})^\sigma}]^{W_0}.$$

Let $\hat{G}$ = Langlands dual grp w/ I-action fixes splitting $\hat{G} \supset \hat{B} \supset \hat{T}$.

<u>Root datum</u> $(X^*(\hat{T}) \supset \Phi^\vee,\ X_*(\hat{T}) \supset \Phi,\ \Delta^\vee) \circlearrowleft I$

Φ = abs roots for (G, B, T)

$\Phi^\vee$ = abs coroots for (G, B, T).

Thm The $\bar{\mathbb{Q}}_\ell$-grp $\hat{G}^I$ is reductive

$\hat{G}^{I,\circ} \supset \hat{T}^{,\circ}$ max'l torus of conn comp with root system $[(\Phi^{\vee})^{\diamond}]_{red}$ ⟵ if $(a, 2a) \in R^{\diamond}$, discard $2a$.

(R any root system w/ I-action
⇝ $R^{\diamond}$ = set of I-averages.)

Geometric setup

Change notations $k = \bar{k}$ any field w/ char $k \neq \ell$.

$F = k((t)) \supset \mathcal{O} = k[[t]]$

G/F conn red (Steinberg $\Rightarrow$ q.s.)

$G \supset B = TU \supset T = C_G(A)$.

Choose $\mathcal{G}$ = special max parahoric grp sch $/\mathcal{O}$

Construct ind sch $L^+\mathcal{G}$ $/k$

Main Thm

(1) $(P_{L^+\mathcal{G}}(Gr_{\mathcal{G}}, \bar{\mathbb{Q}}_\ell), \star)$ is a Tannakian cat w/ ⟵ certain convolution product

fiber functor $\mathcal{F} \longmapsto R^*\Gamma(Gr_{\mathcal{G}}, \mathcal{F}) = \bigoplus_{i \in \mathbb{Z}} R^i\Gamma(Gr_{\mathcal{G}}, \mathcal{F})$.

(2) $(P_{L^+\mathcal{G}}(Gr_{\mathcal{G}}, \bar{\mathbb{Q}}_\ell), \star) \simeq (Rep(\hat{G}^I), \otimes)$.

2 major setups

(1) Construct $(P_{L^+\mathcal{G}}(Gr_{\mathcal{G}}, \bar{\mathbb{Q}}_\ell), \star)$

& show it is Tannakian.

(2) Identify Tannakian grp on $\hat{G}^I$.

Xinwen Zhu Main thm w/ G/F tamely ramified

Timo Richarz Removed tameness assumption

Earlier Timo proved split case in a novel way,
using Larsen-Kazhdan-Varshavsky char of H
in terms of Grothendieck semiring of rep'ns.

<u>Idea</u> Know set of irred objects $\{V_\mu\}$ in Rep_H.
and $V_\mu \otimes V_\lambda = \bigoplus_\nu V^\nu_{\mu,\lambda} \otimes V_\nu$
allows one to recover H.

<u>Group-theoretic preliminaries</u>

[HR08] Given (G, A),

<u>Def's</u> Iwahori-Weyl grp $W := N_G T(F)/T(F)_1$
finite Weyl grp $W_0 := N_G T(F)/T(F)$

Known $T(F)/T(F)_1 \xrightarrow[\sim]{\kappa_T} X_*(T)_I$.
$W_0 = (N_G T(F) \cap K)/T(F)_1$.
$\rightsquigarrow$ $W \cong X_*(T)_I \rtimes W_0$

Take cochar lattice $V = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R} \circlearrowleft I$
$\overset{\cup}{V^I} = V_I$

<u>Bruhat-Tits theory</u> provides set Φ_{aff} of affine roots for (G, A).
$\overset{\shortparallel}{\{\alpha + r \mid \alpha \in X^*(A) \text{ rel root, some } r \in \mathbb{R}\}}$.

Φ_{aff} affine-linear functionals on V^I
$\rightsquigarrow$ affine hyperplanes $H_{\alpha+r} = \{v \in V^I \mid \alpha(v) + r = 0\}$.

V^I = v.s. underlying apartment A for (G,A) in $\mathcal{B}(G,F)$

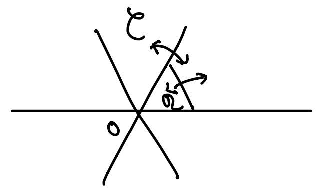
$\Phi_{\mathrm{aff}}, \{H_{\alpha+r}\} \rightsquigarrow$ Coxeter complex. $\mathfrak{f}$ facets, $\mathfrak{a}$ alcoves.

Fix dominant Weyl chamber $\mathcal{C} \subset \mathcal{A}$

$$\mathcal{C} = \{ v \in V^I \mid \langle \alpha, v \rangle > 0, \ \forall \alpha \in \operatorname{Lie} B \}.$$

Choose origin in $\mathcal{A}$ to corr to choice of $\mathcal{G}_0 = \mathcal{G}$.

require base alcove $\mathfrak{a} \subset \mathcal{C}$,



$0 \rightsquigarrow \mathcal{G}_0$, $\mathfrak{a} \rightsquigarrow \mathcal{G}_{\mathfrak{a}}$ BT Iwahori grp sch.

$(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ = Coxeter system given by simple affine reflections in V^I through walls of $\mathfrak{a}$.

$= Q^\vee \rtimes W_0 \quad (Q^\vee = Q^\vee(\Sigma))$

<u>note</u> (1) $V^I \curvearrowleft W = X_*(T)_I \rtimes W_0$

with W_0 = grp gen'd by reflection through walls of $\mathfrak{a}$ containing 0.

$\lambda \in X_*(T)_I$ acts by translation by $-\lambda$.

(2) W permutes Φ_{aff}, $\{H_{\alpha+r}\}$, $\{\text{alcoves}\}$.

$\overset{\cup}{W_{\mathrm{aff}}}$ acts simply transitively on set of alcoves.

$\Rightarrow$ Having fixed $\mathfrak{a}$, get canonical decomp

$$W = W_{\mathrm{aff}} \rtimes \underset{\overset{\cap}{\hat{W}}}{\Omega_{\mathfrak{a}}} \quad \text{stabilizer of } \mathfrak{a}.$$

(3) W_{aff} has length fcn $\ell : W_{\mathrm{aff}} \longrightarrow \mathbb{Z}_{\geq 0}$

Bruhat order $\leq$.

Extends these to W:

- require $\Omega_{\mathfrak{a}}$ = length 0 elts in W
- require $w_1\omega_1 \le w_2\omega_2 \iff w_1 \le w_2$ in W_{aff}, $\omega_1 = \omega_2$ in $\Omega_{\mathfrak{a}}$.

So W has the str of quasi-Coxeter grp.

Prop $\exists!$ reduced root system $\Sigma \subset V^I$
s.t. Φ_{aff} consists of $\alpha + n$, $\alpha \in \Sigma$, $n \in \mathbb{Z}$

Thus $$W_{aff} = W_{aff}(\Sigma \times \mathbb{Z})$$

Prop The following statements on Σ hold:

(1) Σ described in terms of I-action on abs roots $(\Phi \supset \Delta)$, Δ simple as follows: $\forall \alpha \in \Delta$, set

$$N'_I \alpha = \begin{cases} \sum_{\beta \in I\alpha} \beta, & \text{if } \{\beta \in I\alpha\} \text{ pairwise orthogonal} \\ 2\sum_{\beta \in I\alpha} \beta, & \text{otherwise.} \end{cases}$$

$N'_I \Delta$ = set of roots in a root system in V^I which is $\cong$ to set of simple roots in Σ.

(2) $\exists$ identification of (based) root system
$$\Sigma^\vee \cong (\Phi^{\vee,\diamond})_{red}$$
i.e. Σ is of type dual to $\Phi(\hat{G}^{I,\circ}, \hat{T}^{I,\circ})$

3 decompositions of $G(F)$

(1) Bruhat-Tits: $K_{\mathfrak{a}} = \mathcal{G}_{\mathfrak{a}}(\mathcal{O}) \subset K \subset G(F)$, $\mathfrak{a}$ fixed basic alcove

$\forall w \in W$, fix any lift $\mathring{w} \in N_G T(F)$.

Then $$G(F) = \coprod_{w \in W} K_\alpha \mathring{w} K_\alpha .$$

Rmk $G = GL_n$. can prove by hand using row + column operators.

(2) <u>Cartan</u>: $\forall \lambda \in X_*(T)_I$,

choose any Kottwitz lift $t^\lambda \in T(F)$

$$\kappa_T(t^\lambda) = \lambda \in X_*(T)_I .$$

Then $$G(F) = \coprod_{\lambda \in \underbrace{X_*(T)_I^+}_{}} K t^\lambda K , \quad K = \mathcal{G}(\mathcal{O}).$$

dominant for Σ roots

Rmk Can be deduced formally from (1)

using BN-pair relations ($+\varepsilon$).

(3) <u>Iwasawa</u>: $$G(F) = \coprod_{\lambda \in X_*(T)_I} \mathcal{U}(F) \cdot t^\lambda \cdot K .$$

Rmk $\forall \lambda \in X_*(T)_I$, $\mu \in X_*(T)_I^+$,

$$\mathcal{U}(F) t^\lambda K \cap K t^\mu K \neq \phi \Rightarrow \mu - \lambda = (\text{sum of pos coroots in } \Sigma^\vee).$$

Prop (See notes)

$$X_*(T)^+ \twoheadrightarrow X_*(T)_I^+ \quad \text{surjective}.$$

dom for abs simple roots — dom for $\Delta(\Sigma) = N_T' \Delta$

Lecture 2: Representability of affine Grassmanians

Oct 9

2 Corrections for Lec 1

(1) $\Phi_{aff} \neq \Sigma \times \mathbb{Z}$, they just define the same hyperplanes in A

(2) $X_*(T)^+ \longrightarrow X_*(T)_I^+$

So far can only prove surjective when $Z(G)$ conn.

Announcement Focus on split case.

- Final ver of notes will cover general case
- Remarks at the end of lectures about general case.

$(X^*(T) \supset \Phi,\ X_*(T) \supset \Phi^\vee, \Delta) \circlearrowleft I$

$\rightsquigarrow X_*(T)_I,\ \Sigma,\ \Sigma^\vee,\ (\Phi^{\vee I})_{red}$ ($= \Phi^\vee$ if split)

In split case, I acts trivially, $\Sigma = \Phi$.

Examples (of $K, K_{\mathbb{I}}$ in split case)

- $\mathcal{G} = G/\mathcal{O}$, $K = \mathcal{G}(\mathcal{O})$
- $\mathcal{G}_{\mathbb{I}}$, $K_{\mathbb{I}} = \mathcal{G}_{\mathbb{I}}(\mathcal{O}) = \{g \in \mathcal{G}(\mathcal{O}) : g \bmod \varpi \in B(k)\}$

$$\begin{array}{ccc} \mathcal{G}(\mathcal{O}) & \longrightarrow & G(k) \\ \cup & & \cup \\ \mathcal{G}_{\mathbb{I}}(\mathcal{O}) = K_{\mathbb{I}} & \longrightarrow & B(k) \end{array}$$

e.g. $G = GL_n$, our convention gives

$$K_{\mathbb{I}} = \begin{pmatrix} \mathcal{O}^\times & & \cdots & \mathcal{O} \\ & \mathcal{O}^\times & & \vdots \\ & & \ddots & \\ \varpi\mathcal{O} & & & \mathcal{O}^\times \end{pmatrix}.$$

Ind-scheme k field.

Def An ind-sch is a $\operatorname{colim}_{i \in I} F_i$ ← in cat of preshv.

with $F_i : \mathrm{Aff}_k \longrightarrow (\mathrm{Sets})$

each $F_i \simeq k$-sch X_i, $\forall i \in I$ (directed set).

⇝ $\operatorname{colim}_{i \in I} F_i \in \mathrm{Preshv}(\mathrm{Aff}_k, \mathrm{Sets})$.

also $\in \mathrm{Shv}_{\mathcal{C}}(\mathrm{Aff}_k, (\mathrm{Sets}))$

Fact Filtered colim commutes with finite lims (such as equalizers)

so if F_i are $\mathcal{C}$-sheaves for some Groth top $\mathcal{C}$ on Aff_k,

then so is $\operatorname{colim}_{i \in I} F$, and

Recall Presheaf $F : \mathrm{Aff}_k \longrightarrow (\mathrm{Sets}), (\mathrm{Grps}), \cdots$

is a $\mathcal{C}$-sheaf if $\forall$ $\mathcal{C}$-corr $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R$,

$$F(R) \longrightarrow F(R') \rightrightarrows F(R' \otimes_R R')$$

is an equalizer in target cat.

- $\mathcal{C}$ Zar, fppf, étale, fpqc :

X sch $\Rightarrow$ X is fpqc sheaf

$\Rightarrow$ any ind-sch is also a fpqc sheaf.

Strictly ind-sch : $X_i \to X_j$ closed immersion.

Sheafification $F \in \mathrm{Preshv}(\mathrm{Aff}_k, \mathrm{Sets})$, $\mathcal{C}$ Groth top.

$\exists$ sheafification F^{++} & canonical $F \to F^{++}$

with univ property : $\forall$ sheaf F',

$$\mathrm{Hom}_{\mathrm{Preshv}}(F, F') = \mathrm{Hom}_{\mathrm{Shv}}(F^{++}, F') .$$

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F' \\ F^{++} \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

Loop groups & positive loop groups

$G / k((t))$, $\mathcal{G} / k[[t]]$

$$\mathcal{G} \times_{\operatorname{Spec} k[[t]]} \operatorname{Spec} k((t)) = G.$$

Define presheaves:

$$LG : R \longmapsto G(R((t)))$$
$$\cup$$
$$L^+\mathcal{G} : R \longmapsto \mathcal{G}(R[[t]])$$

<u>Exercise</u> Prove:

(1) LG is rep'd by ind-aff grp ind-sch

(2) $L^+\mathcal{G}$ is a grp k-sch (of infinite type $/k$)

same as fpqc sheafification.

<u>Def</u> $\mathrm{Gr}_{\mathcal{G}}$ is the étale sheafification of the presheaf

$$R \longmapsto LG(R)/L^+\mathcal{G}(R)$$

<u>Rmk</u> NOT always the case that $\mathrm{Gr}_{\mathcal{G}}(R) = LG(R)/L^+\mathcal{G}(R)$.

However, it is true when G split + R local.

<u>Goal</u> (At least) When G reductive $/k((t))$, $\mathcal{G} = G_{\mathcal{O}}$,

$\mathrm{Gr}_{\mathcal{G}}$ is rep'd by an ind-proj ind-sch $/k$.

Start with $G = GL_n$, $\mathcal{G} = GL_{n,\mathcal{O}}$.

Starting point: $LG(R) = GL_n(R((t)))$

acts on set $\mathrm{Latt}_n(R) = \{R[[t]]\text{-lattices } L \subset R((t))^n\}$

<u>Def</u> R any ring, $\Lambda = R[[t]]^n$.

An $R[[t]]$-lattice is an $R[[t]]$-submod $L \subset R((t))^n$

s.t. $\exists N$ with $t^N \Lambda \subset L \subset t^{-N}\Lambda$ & L is $R[[t]]$-proj as a mod.

Will turn out that

$$\mathrm{Gr}_{\mathcal{G}}(R) = \mathrm{Latt}_n(R) = \operatorname{colim}_N \underbrace{\mathrm{Latt}_{n,N}(R)}_{\text{proj } R\text{-sch.}}$$

<u>Main Prop</u> R any ring. $\Lambda = R[\![t]\!]^n$.

TFAE as conditions on $R[\![t]\!]$-mod L w/ $t^N\Lambda \subset L \subset \Lambda$ for $N \in \mathbb{N}$:

(1) L is $R[\![t]\!]$-proj

(2) Λ/L (and hence $L/t^N\Lambda$) are R-proj.

<u>Lemma</u> R any ring, M f.p. R-mod,

then M is R-flat $\Leftrightarrow$ R-proj.

In particular, if R noetherian,

then for M finite $/R$, flat $\Leftrightarrow$ proj.

<u>Proof</u> <u>(1) $\Rightarrow$ (2)</u>: $t^N\Lambda \subset L \subset \Lambda$

<u>Claim</u> L f.g. $/R_t = R[\![t]\!]$,

This is a local prop: ETS for $L_\mathfrak{p}$ f.g. $/R_{t,\mathfrak{p}}$. $\forall$ prime $\mathfrak{p}$.

$t^N\Lambda_\mathfrak{p} \subset L_\mathfrak{p} \subset \Lambda_\mathfrak{p}$ + $R_t \longrightarrow R(\!(t)\!)$ flat

$\Rightarrow$ $L_\mathfrak{p} \otimes_{R_{t,\mathfrak{p}}} R(\!(t)\!)_\mathfrak{p} = R(\!(t)\!)^n_\mathfrak{p}$

OTOH $L_\mathfrak{p}$ is proj $/R_{t,\mathfrak{p}}$ $\leftarrow$ local

so $L_\mathfrak{p}$ is $R_{t,\mathfrak{p}}$-free (Kaplansky's thm)

$\circledast \Rightarrow$ finite type. $\Rightarrow$ claim.

Now L $R[\![t]\!]$-proj:

$t^{-1}L/L$ $(R = R[\![t]\!]/tR[\![t]\!])$-proj

$t^{-m}L/L$ R-proj, $\forall m \geq 1$,

& R-flat.

$R((t))^n/L = \bigcup_m t^{-m}L/L \quad R\text{-flat}$

Have SES

$$0 \to L/t^N\Lambda \to \underset{R\text{-flat}}{R((t))^n/t^N\Lambda} \to \underset{R\text{-flat}}{R((t))^n/L} \to 0$$

Tor-vanishing $\Rightarrow$ $L/t^N\Lambda$ R-flat

Λ/L R-flat

$$0 \to \underset{\text{finite}}{L/t^N\Lambda} \to \Lambda/t^N\Lambda \to \underset{\text{f.p. as } R\text{-mod \& } R\text{-flat.}}{\Lambda/L} \to 0$$

(2) $\Rightarrow$ (1) $\quad t^N\Lambda \subset L \subset \Lambda$, assume Λ/L, $L/t^N\Lambda$ R-proj.

Want L R_t-proj.

Claim ETS for R noeth. $R = \operatorname{colim}_i R_i$, R_i noeth

$Gr_N^f(R)$: Λ/L R-proj

$Gr_N(R)$: L R_t-proj

- $Gr_N \hookrightarrow Gr_N^f$
- $Gr_N(R_i) \xrightarrow{\sim} Gr_N^f(R_i)$
- $Gr_N^f(\operatorname{colim} R_i) = \operatorname{colim} Gr_N^f(R_i)$.

So assume R noeth, $R_t = R[[t]]$ noeth,

L finite R_t-mod,

$$\begin{array}{ccc} \Lambda_0 = R[t]^n & \longleftrightarrow & \Lambda = R[[t]]^n \\ \cup & & \cup \\ L_0 & \longleftrightarrow & L \\ \cup & & \cup \\ t^N\Lambda_0 & \longleftrightarrow & t^N\Lambda \end{array}$$

$L = L_0 \otimes_{R[t]} R[[t]]^n$ (use $R[t] \to R[[t]]$ is flat)

ETS L_0 is $R[t]$-proj.

Since $R[t]$ noeth, L_0 fin/$R[t]$,

ETS L_0 $R[t]$-flat.

$\rightsquigarrow$ ETS: $\forall$ max $\mathfrak{q} \subset R[t]$,

$L_{0,\mathfrak{q}}$ $R[t]_{\mathfrak{q}}$-flat, $\mathfrak{q}$ lies over $\mathfrak{p} \subset R$.

(<u>Lemma</u> (See notes)

$\exists\, a \in R - \mathfrak{p}$ s.t. $R_a/\mathfrak{p}_a$ is a field and

$R_a[t]_{\mathfrak{q}_a} = R[t]_{\mathfrak{q}}$.)

Further, since $R_a/\mathfrak{p}_a \hookrightarrow R_a[t]/\mathfrak{q}_a$

See $\mathfrak{q}_a$ lies over $\mathfrak{p}_a$ max'l

$\Rightarrow$ can replace R with R_a

assume $\mathfrak{p} = \mathfrak{m}$ max in R, $k = R/\mathfrak{m}$.

Want $L_{0,\mathfrak{q}}$ is a free $R[t]_{\mathfrak{q}}$-mod.

Apply following to $R_{\mathfrak{m}} \longrightarrow R[t]_{\mathfrak{q}}$, $M = L_{0,\mathfrak{q}}$

(<u>Lemma</u> $\underset{\mathfrak{m}}{\overset{R}{\cup}} \longrightarrow S$ map of noeth local rings

M finite S-mod s.t.

(a) $M/\mathfrak{m}M$ free $S/\mathfrak{m}S$-mod.

(b) M flat / R

Then M free / S (and S flat / R).)

<u>Check (b)</u> $R[t]_{\mathfrak{q}} = S^{-1}R[t]_{\mathfrak{m}}$ (some S)

$L_{0,\mathfrak{q}} = L_{0,\mathfrak{m}} \otimes_{R[t]_{\mathfrak{m}}} R[t]_{\mathfrak{q}}$

L_0 is R-flat (recall Λ_0/L_0, Λ_0 R-proj)

Want $L_{0,\mathfrak{q}}$ to be $R_{\mathfrak{m}}$-flat:

$N \hookrightarrow P$ $R_{\mathfrak{m}}$-mods, $L_{0,\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} N \hookrightarrow L_{0,\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} P$

also $R[t]_{\mathfrak{m}}$-linear.

ETS: $\forall$ max $\mathfrak{q} \subset R[t]$, $L_{0,\mathfrak{q}}$ $R[t]_{\mathfrak{q}}$-flat, $\mathfrak{q}$ over $\mathfrak{p} \subset R$.

$$(R[t]_{\mathfrak{q}} \otimes_{R[t]_{\mathfrak{m}}} L_{0,\mathfrak{m}}) \otimes_{R_{\mathfrak{m}}} N \hookrightarrow (R[t]_{\mathfrak{q}} \otimes_{R[t]_{\mathfrak{m}}} L_{0,\mathfrak{m}}) \otimes_{R_{\mathfrak{m}}} P \Rightarrow (b).$$

Check (a) Start with $L_{0,\mathfrak{q}}/\mathfrak{m}L_{0,\mathfrak{q}} = (L_0/\mathfrak{m}L_0)_{\mathfrak{q}}$

is free $R[t]_{\mathfrak{q}}/\mathfrak{m}R[t]_{\mathfrak{q}} = (R[t]/\mathfrak{m}R[t])_{\mathfrak{q}}$ -mod.

ETS $L_0/\mathfrak{m}L_0$ is free $\underset{\substack{\uparrow \\ \text{PID}}}{k[t]}$-mod

$$0 \to \underbrace{L_0 \otimes R/\mathfrak{m}}_{\text{has no } k[t]\text{-torsion}} \to \Lambda_0 \otimes_R R/\mathfrak{m} \to (\underbrace{\Lambda_0/L_0}_{R\text{-proj}}) \otimes_R R/\mathfrak{m} \to 0$$

$\Rightarrow$ (a) $\square$

note We used: if $X = \mathrm{Gr}^f_N$, then $\mathrm{colim}\, R_i = R$

& $X(\mathrm{colim}\, R_i) = \mathrm{colim}\, X(R_i)$.

$(t^N \Lambda \subset L \subset t^{-N}\Lambda)$

$\mathrm{Gr}^f_N \overset{\text{closed}}{\subset} \coprod_r \mathrm{Gr}(2nN, r)$.

being t-stable is a closed condition.

$\Rightarrow$ Gr^f_N is a proj k-sch.

Lemma A morph of schs

$X \to S$ is locally of finite presentation

$\Leftrightarrow$ $\forall$ directed set I & inverse system

of affine schs $\{T_i\}$ / S

$$\mathrm{Hom}_S(\varprojlim_i T_j, X) = \varinjlim_i \mathrm{Hom}_S(T_i, X).$$

Lemma $L \in \mathrm{Gr}_N(R)$. Then $\exists$ Zariski cover $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R$

s.t. $L \otimes_{R[[t]]} R'[[t]]$ is $R[[t]]$-free.

Proof $R_t = R[[t]]$. We know L finite / R_t.

$\exists\, g_1, \dots, g_r \in R_t,\ (g_1, \dots, g_r)_{R_t} = (1)$

s.t. L_{g_i} is $(R_t)_{g_i}$-free, $\forall i$.

Set $f_i := g_i(0) \in R$

$\Rightarrow (f_1, \dots, f_r)_R = (1)$, $f_i \in R_{t,g_i}^\times$, $f_i \neq 0$.

$L \otimes_{R_t} R_{f_i}[[t]] = (L \otimes_{R_t} R_{t,g_i}) \otimes_{R_{t,g_i}} R_{f_i}[[t]]$ $\quad R_{f_i}[[t]]$-free. $\square$

Define $\mathrm{Latt}_n = \operatorname{colim}_N Gr_N^f$ inj & proj ind-sch $/k$.

and hence a fpqc sheaf on Aff_k.

Note $LG(R)/L^+\mathcal{G}(R)$ identifies with R_t-free L to $R((t))^n$.

mono & locally epi (w.r.t. Zar top)

$$\begin{tikzcd}
(LG/L^+\mathcal{G})_{\mathrm{presh}} \arrow[r, "a"] \arrow[d, "b"'] & \mathrm{Latt}_n \\
LG/L^+\mathcal{G} \arrow[ur, dashed, "\exists ! c"']
\end{tikzcd}$$

mono + epi $\Downarrow$ isom.

$LG/L^+\mathcal{G} = (LG/L^+\mathcal{G})_{\mathrm{presh}}^{++}$

Used:

<u>Lemma</u> (1) $F \longmapsto F^{++}$ preserves finite lim

(2) In any cat with fiber products,

$F \to F'$ to mono $\Leftrightarrow F \xrightarrow{\Delta} F \times_{F'} F \rightrightarrows F$

is an equalizer.

Thus, · $F \mapsto F^{++}$ preserves monos.

· morph of sheaves is an isom $\Leftrightarrow$ mono + locally epi

Can propose a torsor description of $Gr_{\mathcal{G}}$.

Lecture 3:

Oct 11

<u>Summary of last time</u>

$Gr_{GL_n,\mathcal{O}} = LGL_n / L^+GL_{n,\mathcal{O}}$

is isomorphic as étale sheaf to $Latt_n = \operatorname{colim}_N Latt_{n,N}$

$\Rightarrow Gr_{GL_n,\mathcal{O}}$ is rep'd by an ind-proj ind-sch $/k$.

<u>Goal</u> Prove $Gr_{\mathcal{G}}$ ind-proj ind-sch $/k$

$\forall$ (special max) parahoric $\mathcal{G}/\mathcal{O}$.

Assuming $\mathcal{G}/\mathcal{O}$ split for simplicity.

<u>Torsor description of $Gr_{\mathcal{G}}$</u>

Let $G \to X$ any aff grp sch

Let τ be a Grothendieck top of Aff_k.

<u>Def</u> A (right) <u>G-torsor</u> $\mathcal{E}$ on X is a τ-sheaf on Aff_X

w/ right action $\mathcal{E} \times_X G \to \mathcal{E}$

s.t. $\forall R$, $\mathcal{E}(R) \circlearrowleft G(R)$ is simply transtive

& s.t. $\forall R$, $\exists \tau$-cover $\operatorname{Spec} R' \to \operatorname{Spec} R$ with $\mathcal{E}(R') \neq \phi$.

k any field, $\mathbb{D}_R = \operatorname{Spec} R[[t]]$, $\quad R \in Aff_k$.

$\cup$

$\mathbb{D}_R^* = \operatorname{Spec} R((t))$.

Assume $\mathcal{G} \to \mathbb{D}_k$ aff grp sch of fin type.

Def $\mathrm{Gr}^{tor}_{\mathcal{G}} : \mathrm{Aff}_k \longrightarrow \mathrm{Sets}$

$R \longmapsto \{(\mathcal{E}, \alpha)\}/\cong$

- $\mathcal{E} \to \mathbb{D}_R$ right $\mathcal{G} \times_{\mathbb{D}_R} \mathbb{D}_R$ - torsor ("$\mathcal{G}$-torsor")
- $\alpha \in \mathcal{E}(\mathbb{D}^{\times}_R)$ i.e. isom of $\mathbb{D}$-torsors $\mathcal{E}|_{\mathbb{D}^*_R} \xrightarrow{\sim} \mathcal{E}_0|_{\mathbb{D}^*_R}$

Here $\mathcal{E}_0$ = triv $\mathcal{G}$-torsor

(An isom $(\mathcal{E}, \alpha) \xrightarrow{\sim} (\mathcal{E}', \alpha')$ is a map $\mathcal{E} \xrightarrow{\pi} \mathcal{E}'$ of $\mathcal{G}$-torsors $\Downarrow \cong$ s.t. $\alpha = \alpha' \circ \pi$.)

Lem (0) $\mathcal{E}$ is rep'd by an aff sch + $\mathcal{E} \to \mathbb{D}_k$

(eff α-étale descent of aff sch.)

(1) $\mathrm{Gr}^{tor}_{\mathcal{G}}$ has base pt $(\mathcal{E}_0, id)$

(2) $L\mathcal{G}$ acts on left on $\mathrm{Gr}^{tor}_{\mathcal{G}} : (g, (\mathcal{E}, \alpha)) \longmapsto (\mathcal{E}, g \circ \alpha)$.

(3) $\mathcal{G} \longmapsto \mathrm{Gr}^{tor}_{\mathcal{G}}$ is functorial in $\mathcal{G}$ (can "pushout" torsors)

(4) $\mathrm{Gr}_{GL_n, \mathcal{O}} \longrightarrow \mathrm{Gr}^{tor}_{GL_n, \mathcal{O}}$ a natural map

$\uparrow$ use LGL_n-action on base pt.

$\mathrm{Gr}^{tor}_{GL_n, \mathcal{O}} \cong \mathrm{Latt}_n$ = vec bun on $\mathbb{D}_R$.

Given GL_n-torsor $\mathcal{E}$ on $\mathbb{D}_R$, get $\mathcal{E} \times^{GL_n, \mathcal{O}} \mathcal{O}^n$ a v.b.

Given v.b. $\mathcal{E}$, $\mathcal{E} = \underline{\mathrm{Isom}}(\mathcal{O}^n, \mathcal{E})$.

$\underline{\underline{\text{So}}}$ $\mathrm{Gr}^{tor}_{GL_n, \mathcal{O}}$ is étale fpqc sheaf

$\mathrm{Gr}_{GL_n, \mathcal{O}} \xrightarrow{\alpha} \mathrm{Gr}^{tor}_{GL_n, \mathcal{O}}$

$g \in LGL_n, \mathcal{O} \longrightarrow (\mathcal{E}_0, g)$.

Thm $\mathcal{G}$ sm aff grp sch / $\mathbb{D}_k$. Then

(1) $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Spec}\, k$ is rep'd by a separated ind-sch.

(2) If $\mathcal{G}$ <u>reductive</u> (i.e. $\mathcal{G} = G_0$ split)
then $Gr_{\mathcal{G}}$ is ind-proj / k.

<u>Prop</u> (Key, Xinwen Zhu)
Let $\mathcal{G} \hookrightarrow \mathcal{H}$ be a closed imm of f.t. aff grp schs / $\mathbb{D}_k$
s.t. the fppf $\mathcal{H}/\mathcal{G}$ is rep'd by quasi-aff (aff) sch / $\mathbb{D}_k$.
Then $Gr_{\mathcal{G}}^{tor} \longrightarrow Gr_{\mathcal{H}}^{tor}$ is rep'd by a qc intersection (closed imm).

<u>Proof</u> Take pullback

$$\begin{tikzcd} \boxed{F \arrow[r] \arrow[d] & \operatorname{Spec} R} \arrow[d, "(E,\alpha)"] \\ Gr_{\mathcal{G}}^{tor} \arrow[r] \arrow[ur, phantom, "\perp"] & Gr_{H}^{tor} \end{tikzcd}$$

$\longrightarrow$ need this in a locally closed qc intersection (closed imm)

$\mathcal{E} \xrightarrow{\pi} \mathbb{D}_R$ str. $\mathcal{E}/\mathcal{G} \xrightarrow{\tilde{\pi}} \mathbb{D}_R$

By eff of descent for quasi-aff schs,
$\exists\, W$ aff of f.p. / $\mathbb{D}_R$ and qc open embedding $\mathcal{E}/\mathcal{G} \hookrightarrow W$
Def of $\mathbb{D}_R$ of cat $[\mathcal{E}/\mathcal{G}]$ yields identification.

$$\begin{tikzcd} \mathcal{E}/\mathcal{G} \arrow[r, hook] \arrow[dr, "\tilde{\pi}"] & W \arrow[d, "p"] \\ & \mathbb{D}_R \arrow[ul, bend left, dashed, "\tilde{\alpha}"] \arrow[u, bend right, dashed, "\tilde{\alpha}"'] \end{tikzcd}$$

$\exists\, (R \to R') = \{ \text{section } \beta \text{ of } \tilde{\pi} \text{ over } \mathbb{D}_{R'} : \beta|_{\mathbb{D}_{R'}^*} \simeq \tilde{\alpha}|_{\mathbb{D}_{R'}^*} \}$.

<u>Lem</u> $V \xrightarrow{p} \mathbb{D}_R$ aff sch of fin presentation
and s section on $\mathbb{D}_R^*$. Then
$(R \to R') \longmapsto \{ \text{sections } s' \text{ of } p \text{ over } \mathbb{D}_{R'} : s'|_{\mathbb{D}_{R'}^*} = s|_{\mathbb{D}_{R'}^*} \}$
is rep'd by a closed subsch of Spec R.

<u>Pf of Lem</u> $V \hookrightarrow \mathbb{A}^N_{\mathbb{D}_R}$, $N \gg 0$, $s = (s_1(f), \cdots, s_N(f))$,
$s_i = \sum_{j \gg -\infty} s_{ij} f^j \in R((f))$.

Presheaf is rep'd by $\operatorname{Spec} A$, $A = R/\langle s_{ij} : j<0\rangle$. $\Rightarrow$ Lem.

Apply lem to $(*)$,

$\{\beta$ section of $\tilde{\pi}$ (mod p) : $\beta|_{D_R^*} = \tilde{\alpha}|_{D_R^*}\}$

$=$ section of p landing in open $\mathcal{E}/\mathcal{G} \subset W$.

$$\mathcal{E} \underset{\alpha}{\overset{\pi}{\rightleftarrows}} \mathbb{D}_R \underset{\tilde{\alpha}}{\overset{\tilde{\pi}}{\leftrightarrows}} \mathcal{E}/\mathcal{G}$$

So we get open in $\operatorname{Spec} A$. $\square$

<u>Lem</u> $\mathcal{G}$ flat aff grp sch $/\mathbb{D}_R$.

Then (a) (Pappas-Rapoport)

$\exists$ closed imm $\mathcal{G} \hookrightarrow GL_{n,\mathcal{O}} \times GL_{1,\mathcal{O}}$

s.t. $GL_{n,\mathcal{O}} \times GL_{1,\mathcal{O}}/\mathcal{G}$ is quasi-aff

(b) (J. Alper)

If $\mathcal{G}$ reductive (e.g. $\mathcal{G} = G_{\mathcal{O}}$ split)

then $GL_{n,\mathcal{O}} \times GL_{1,\mathcal{O}}/\mathcal{G}$ is affine.

<u>Cor</u> $\forall$ flat aff $\mathcal{G}/\mathbb{D}_R$, $Gr_{\mathcal{G}}^{tor}$ is rep'ble.

If $\mathcal{G}$ reductive, $Gr_{\mathcal{G}}^{tor}$ is ind-proper & hence ind-proj.

Now that we know $Gr_{\mathcal{G}}^{tor}$ is fpqc-sheaf, get

<u>Cor</u> If $\mathcal{G} \to \mathbb{D}_R$ <u>smooth</u> and affine,

then $Gr_{\mathcal{G}} \longrightarrow Gr_{\mathcal{G}}^{tor}$ is an isom.

<u>Proof</u> (Xinwen Zhu) $L\mathcal{G}(R)/L^+\mathcal{G}(R) \longrightarrow Gr_{\mathcal{G}}^{tor}(R)$

$g \longmapsto (\mathcal{E}, g)$.

stratifies to $\mathrm{Gr}_{\mathcal{G}} \longrightarrow \mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}}$ which is a mono.

So ETS étale-locally an epi.

Given $(\mathcal{E}, \alpha) \in \mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}}(R)$, need to show

$\exists$ étale cover $R \to R'$ s.t. $\mathcal{E}_{\mathbb{D}_{R'}} \simeq \mathcal{E}_{0,\mathbb{D}_{R'}}$.

For $\mathbb{D}_R \longrightarrow \operatorname{Spec} R \rightsquigarrow R[\![t]\!] \to R \quad (t \mapsto 0)$

$$\rightsquigarrow \begin{tikzcd} & \mathcal{E} \times_{\mathbb{D}_R} \operatorname{Spec} R \arrow[d] \\ \operatorname{Spec} R' \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} R \end{tikzcd}$$

$\mathcal{G}_R$ smooth $/ \operatorname{Spec} R$ ($\mathcal{G} \times_{\mathbb{D}_R} \operatorname{Spec} R$ - torsor)

Then inf lifting, the section lifts to $\operatorname{Spf} R[\![t]\!] \longrightarrow \mathcal{E} \times_{\mathbb{D}_R} \mathbb{D}_{R'}$

i.e. to $\operatorname{Spec} R'[\![t]\!] \longrightarrow \underbrace{\mathcal{E} \times_{\mathbb{D}_R} \mathbb{D}_{R'}}_{\text{affine sch.}}$ $\square$

These ingredients prove Theorems ($\mathcal{G}$ red)

$$\mathrm{Gr}_{\mathcal{G}} \cong \mathrm{Gr}_{\mathcal{G}}^{\mathrm{tor}} \xrightarrow{\text{closed}} \mathrm{Gr}_{GL_n}^{\mathrm{tor}} = \mathrm{Latt}_n.$$

$\Rightarrow$ $\mathrm{Gr}_{\mathcal{G}}$ ind-proj $/k$ $\Rightarrow$ the previous thm. $\square$

Consider $(P_{L^+\mathcal{G}}(\mathrm{Gr}_{\mathcal{G}}, \bar{\mathbb{Q}}_\ell), \star)$

k field ($k = \bar{k}$ or $k = \mathbb{F}_q$), ℓ prime, $\ell \neq \operatorname{char} k$.

X/k f.t. separated k-sch.

$\rightsquigarrow$ $D_c^b(X, \Lambda)$, Λ ℓ-torsion abelian grp.

(BBD) (e.g. $\mathbb{Z}/\ell^n\mathbb{Z}$)

$$D_c^b(X, \mathbb{Q}_\ell) = D_c^b(X, \mathbb{Z}_\ell) \otimes \mathbb{Q} = \varprojlim_n D_c^b(X, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes \mathbb{Q}$$

& $D_c^b(X, \bar{\mathbb{Q}}_\ell)$ + 6-functor formalism $Rf_*, Rf_!, f^*, f^!, R\mathcal{H}om(-,-), \otimes$.

$\cup$

$P(X, \bar{\mathbb{Q}}_\ell)$ cat of perv shv.

<u>Fact</u> $K_X = f^! \bar{\mathbb{Q}}_{\ell,S}$, $f : X \to \operatorname{Spec} k = S$.

$\Rightarrow D_X(f) = R\mathcal{H}om(\bar{\mathbb{Q}}_\ell, K_X)$.

<u>Def/Thm</u> ${}^p D^{\leq 0}(X, \bar{\mathbb{Q}}_\ell) = \{ f \in D^b_c \mid \dim \operatorname{supp} H^i \mathcal{F} \leq -i, \ \forall i \in \mathbb{Z} \}$

${}^p D^{\geq 0}(X, \bar{\mathbb{Q}}_\ell) = \{ D_X f , \ f \in {}^p D^{\leq 0} \}$.

$\Rightarrow {}^p D^{\leq 0}(X, \bar{\mathbb{Q}}_\ell) \cap {}^p D^{\geq 0}(X, \bar{\mathbb{Q}}_\ell) = P(X, \bar{\mathbb{Q}}_\ell)$

is an abelian cat whose objs have finite length.

- $j : \mathcal{U} \hookrightarrow X$ open.

$j_{!*} : P(\mathcal{U}) \longrightarrow P(X)$

$\mathcal{U} \xhookrightarrow{j} \bar{\mathcal{U}} \xhookrightarrow{i} X$

Write $IC(\mathcal{U}, \mathcal{L}) = i_* j_{!*} \underbrace{\mathcal{L}[d]}_{\text{perverse on } \mathcal{U}}$, which is simple if $\mathcal{L}$ irred.

<u>Def/Facts</u> $f : X \to Y$ stratified schs, proper.

$X = \coprod X_\alpha$, $Y = \coprod Y_\beta$

f semismall : $Rf_* \mathcal{F} \in P(Y)$ if $\mathcal{F} \in P(X)$

f small : $Rf_* IC(X, \mathcal{L}) = IC(Y, (f|_{X})_* \mathcal{L})$

Lecture 4: Convolution properties of perverse sheaves

Oct 16

Lecture 5

Oct 18

Assume for simplicity G split, $\mathfrak{g} = G/\sigma$.

$P_{L^+G}(Gr_G)$:

- simple objs : IC_μ, $\mu \in X_*(T)^+$
- semisimplicity follows from

  $\forall \lambda, \mu \in X_*(T)^+$, $\mathrm{Hom}_{D(Gr_G)}(IC_\lambda, IC_\mu[1]) = 0$

  i.e. given ext'n in $P(Gr_G)$,

  $$0 \to IC_\mu \to F \to IC_\lambda \to 0$$

  gives dist triangle in $D(Gr_G)$ and hence exact seq.

- $\mathrm{Hom}_{D(Gr_G)}(IC_\lambda, IC_\mu) \to \mathrm{Hom}_{D(Gr_G)}(IC_\lambda, F)$

  $$\to \mathrm{Hom}_{D(Gr_G)}(IC_\lambda, IC_\lambda) \to \underbrace{\mathrm{Hom}_{D(Gr_G)}(IC_\lambda, IC_\lambda[1])}_{=\,0} \to \cdots$$

- Then similar argument shows any $F \in P_{L^+G}(Gr_G)$

  is ss by induction on length.

<u>Tech lemma</u> If $i : \bar{O}_\lambda \hookrightarrow \bar{O}_\mu$ $(\lambda < \mu)$

then $i^* IC_\mu$ in deg $p \leq -2$.

<u>Case 1</u> $\lambda = \mu$ (ok)

<u>Case 2</u> $\lambda \neq \mu$, $\lambda < \mu$ or $\mu < \lambda$ ok.

$\mu < \lambda$: $i : \bar{O}_\mu \hookrightarrow \bar{O}_\lambda$

$$\mathrm{Hom}(IC_\lambda, i_* IC_\mu[1]) = \mathrm{Hom}(\underbrace{i^* IC_\lambda}_{p \leq -2}, \underbrace{IC_\mu[1]}_{p \geq -1}) = 0$$

Case 3 $\lambda \nleq \mu$, $\mu \nleq \lambda$:

$\pi_0(Gr_G) = \pi_1(G) := X_*(T)/Q^\vee$

WLOG λ, μ "in same conn comp".

$\mu - \lambda \in Q^\vee$, $\nu \in X_*(T)^+$, $\lambda < \nu$, $\mu < \nu$.

$$\begin{tikzcd} \bar{\mathcal{O}}_\lambda \times_{\bar{\mathcal{O}}_\nu} \bar{\mathcal{O}}_\mu \arrow[r, hook, "\lambda_1"] \arrow[d, hook, "\lambda_2"'] & \bar{\mathcal{O}}_\mu \arrow[d, hook, "i_2"] \\ \bar{\mathcal{O}}_\lambda \arrow[r, hook, "i_1"] & \bar{\mathcal{O}}_\nu \end{tikzcd}$$

$\mathrm{Hom}(i_{1*} IC_\lambda, i_{2*} IC_\mu[1])$

$= \mathrm{Hom}(i_2^* i_{1*} IC_\lambda, IC_\mu[1])$

$= \mathrm{Hom}(i_{1*} i_2^* IC_\lambda, IC_\mu[1])$

$= \mathrm{Hom}(\underbrace{i_2^* IC_\lambda}_{p \leq -1}, \underbrace{i_1^! IC_\mu[1]}_{p \geq 0}) = 0.$

<u>Pf of tech lemma</u> (Use facts about Schubert var)

Know $i^* IC_\mu \in {}^pD^{\leq -1}$

Need to show $i^* IC_\mu \in {}^pD^{\leq -2}$.

Will show $\forall$ odd j, $i^* IC_\mu \in {}^pD^{\leq j} \overset{(*)}{\Rightarrow} i^* IC_\mu \in {}^pD^{\leq j-1}$.

<u>Use 2 facts</u> ($d_\mu = \dim Gr_{\leq \mu} = \langle 2\rho, \mu \rangle$.)

(I) $H^i IC_\mu$ vanish unless $i \equiv d_\mu \bmod 2$

(parity vanishing)

(II) $\dim Gr_{\leq \lambda} \equiv \dim Gr_{\leq \mu} \bmod 2$, $\forall \lambda \leq \mu$

pf. (II) clear, $\mu - \lambda = \sum_{\text{some } \alpha} \alpha^\vee$, $\langle 2\rho, \alpha^\vee \rangle$ = even.

(I) Use $\underbrace{Fl_G = LG/L^+\mathcal{G}_{\mathbb{Z}}}_{\text{Iwahori}} \xrightarrow{?} LG/L^+G = Gr_G$

sm rel dim d with fiber $\cong G/B$.

$x \longmapsto y$

$p^{-1}(Gr_{\leq \mu}) = Fl_{\leq t_\mu w_0 = w_\mu}$

[BBD] $IC_{w_\mu} = p^*[d] IC_\mu$.

$\mathcal{H}^{i-d}_y \underbrace{IC_{w_\mu}}_{j_{!*}\bar{\mathbb{Q}}_\ell[d+d_\mu]} = \mathcal{H}^i_x IC_\mu$

Take dom resolution

$$\pi : \widetilde{Fl}_{w_\mu} \longrightarrow Fl_{w_\mu}$$

(fibers are paved by affine spaces.)

Smooth $IC_{w_\mu} \xhookrightarrow{\oplus} \pi_* \bar{\mathbb{Q}}_\ell[d+d_\mu]$

$\Rightarrow$ $\mathcal{H}^{i-d}_y IC_{w_\mu} = 0$ unless $i-d-(d+d_\mu)$ even

i.e. unless $\mu - d_\mu$ even $\Rightarrow$ (I).

Claims (I)(II) $\Rightarrow$ $\circledast$. $\square$

Lem $K \in D(Gr_{\leq \mu})$ s.t. (i) $\mathcal{H}^i K = 0$ unless $i \equiv d_\mu \mod 2$.

(ii) $\forall i$, dim supp $\underbrace{\mathcal{H}^i K}_{\text{equiv of } K = i^* IC_\mu}$ $\equiv d_\mu \mod 2$.

Then $K \in {}^pD^{\leq j} \Rightarrow K \in {}^pD^{\leq j-1}$ if j odd.

Proof $K \in {}^pD^{\leq j} \Leftrightarrow$ dim supp $\mathcal{H}^i K[j] \leq -i$. $\forall i \in \mathbb{Z}$.

$\Leftrightarrow$ dim supp $\mathcal{H}^i K \leq -i+j$. $\forall i \in \mathbb{Z}$

(WLOG $i = 2m + d_\mu$)

$\Leftrightarrow$ dim supp $\mathcal{H}^i K \leq -2m - d_\mu + j$

$\leq -2m - d_\mu + j - 1$. $\square$

Up to now, have proved $P_{L^+G}(Gr_G)$ semisimple cat w/ simple objs IC_μ.

Need Construct $*$ convolution product on $P_{L^+G}(Gr_G)$ ($\rightsquigarrow$ $(\mathcal{H}_K(G), *)$.)

For f_1, f_2, $\quad f_1 * f_2(g) = \int_G f_1(x) f_2(x^{-1}g)\, dx$.

Can take $f_1 = \mathbb{1}_{Kt^{\mu_1}K}$, $f_2 = \mathbb{1}_{Kt^{\mu_2}K}$.

$$\#\{xK : xK \in Kt^{\mu_1}K,\ x^{-1}gK \in Kt^{\mu_2}K\} = ?$$

Consider $K \xrightarrow{\mu_1} xK \xrightarrow{\mu_2} x^{-1}gK$.

$$Gr_G \tilde{\times} Gr_G := LG \overset{L^+G}{\times} LG/L^+G$$
$$= (LG \times LG /_{a_2} L^+G \times L^+G)^{\text{ét}}$$

Define $\quad a_2 : (x, y)\cdot(h_1, h_2) = (xh_1, h_1^{-1}yh_2)$

$\qquad\quad a_1 : (x, y)\cdot(h_1, h_2) = (xh_1, yh_2)$.

$\rightsquigarrow \quad LG \times LG /_{a_1} L^+G \times L^+G = Gr_G \times Gr_G$

with $\quad Gr_G \tilde{\times} Gr_G \xrightarrow{m} Gr_G$

$\qquad\qquad (x, y) \longmapsto xy$.

<u>Convolution diagram</u>

$$\underset{(\mathcal{F}_1, \mathcal{F}_2)}{Gr_G \times Gr_G} \xleftarrow[a_1\text{-quot}]{p} LG \times LG \xrightarrow[a_2\text{-quot}]{q} Gr_G \tilde{\times} Gr_G \xrightarrow[\text{ind-proper}]{m} Gr_G$$

<u>Thm</u> The following hold:

(a) p, q are "smooth of same relative dim".

(b) m locally triv (in stratified sense) and semi-small.

<u>Def of convolution</u> $\mathcal{F}_1, \mathcal{F}_2 \in P_{L^+G}(Gr_G)$

$\rightsquigarrow \quad \mathcal{F}_1 \boxtimes \mathcal{F}_2 \in P_{equiv}(Gr_G \times Gr_G)$

$p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$, "perverse" up to shifts, a_1-equiv

Autom a_2-action (uses $\mathcal{F}_1, \mathcal{F}_2$ L^+G-equiv).

Descent Lemma $\Rightarrow p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ descends, i.e.

$\exists!$ perverse sheaf $\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2$ on $Gr_G \tilde{\times} Gr_G$

s.t. $p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \cong q^*(\mathcal{F}_1 \hat{\boxtimes} \mathcal{F}_2)$.

Def $\mathcal{F}_1 * \mathcal{F}_2 = Rm_*(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2)$.

This is perverse since m is semismall & L^+G-equiv.

Get well-def'd map

$$P_{L^+G}(Gr_G) \times P_{L^+G}(Gr_G) \xrightarrow{\quad * \quad} P_{L^+G}(Gr_G).$$

Finite-dim'l version: $Gr_{\leq\mu} =: \bar{\mathbb{O}}_\mu$, $Gr_\mu =: \mathbb{O}_\mu$.

<u>Convolution diagram</u> $p_1 : LG \to Gr_G$

$$\begin{array}{ccccccc}
Gr_G \times Gr_G & \xleftarrow[a_1\text{-quot}]{p} & LG \times LG & \xrightarrow[a_2\text{-quot}]{q} & Gr_G \tilde{\times} Gr_G & \xrightarrow{m} & Gr_G \\
\cup & & & & \cup & & \\
\bar{\mathbb{O}}_\lambda \times \bar{\mathbb{O}}_\mu & \xleftarrow{a_{1,n}} & p_1^{-1}(\bar{\mathbb{O}}_\lambda)/L^{\geq n}G \times p_1^{-1}(\bar{\mathbb{O}}_\mu)/L^{\geq n}G & \xrightarrow{a_{2,n}} & \bar{\mathbb{O}}_\lambda \tilde{\times} \bar{\mathbb{O}}_\mu & \xrightarrow{m} & \bar{\mathbb{O}}_{\lambda+\mu}
\end{array}$$

- $n \gg 0$, action of L^+G on $\bar{\mathbb{O}}_\lambda$, $\bar{\mathbb{O}}_\mu$, $\bar{\mathbb{O}}_\lambda \tilde{\times} \bar{\mathbb{O}}_\mu$, $\bar{\mathbb{O}}_{\lambda+\mu}$

  factor through $G_n = L^+G/\underbrace{L^{\geq n}G}_{\ker(G(R[[t]]) \to G(R[t]/t^n))}$.

Why p, q (resp. p_n, q_n) smooth of same rel dim?

- p smooth. q Zar-locally isom to p.

$$\begin{array}{c} Gr_G \tilde{\times} Gr_G \\ \Big\downarrow pr_1 \\ Gr_G \end{array} \quad \text{Zar-locally triv on base.}$$

- m locally triv: $\bar{\mathbb{O}}_\lambda \tilde{\times} \bar{\mathbb{O}}_\mu = \coprod_{\substack{\mu' \leq \mu, \\ \lambda' \leq \lambda}} \mathbb{O}_{\mu'} \tilde{\times} \mathbb{O}_{\lambda'} \xrightarrow{m} \bar{\mathbb{O}}_{\lambda+\mu} = \coprod_{\nu \leq \lambda+\mu} \mathbb{O}_\nu$

$$\& \quad \mathbb{O}_{\mu'} \times \mathbb{O}_{\lambda'} \xrightarrow{m|_{(-)}} \coprod_{\text{Cartan } \nu} \mathbb{O}_\nu \ni y$$

· Property: given $y \in \mathcal{O}_\nu \subset \mathrm{im}(m|_{(-)})$,

$\exists$ open V, $y \in V \subset \mathcal{O}_\nu$ and $m|_{(-)}^{-1}(V) \cong V \times m|_{(-)}^{-1}(y)$.

Def of semismallness here is:

r-fold convolution morphisms

$(\mu_1, \cdots, \mu_r) = \mu$, $\quad |\mu| = \sum \mu_i$

$\lambda < |\mu|$, $\; y = t^\lambda e_0 \in \mathcal{O}_\lambda$.

$m_\mu : \bar{\mathcal{O}}_{\mu_1} \tilde{\times} \cdots \tilde{\times} \bar{\mathcal{O}}_{\mu_r} \longrightarrow \bar{\mathcal{O}}_{|\mu|} \supset \mathcal{O}_\lambda \ni y$.

with $\dim m_\mu^{-1}(y) \leq \langle \rho, |\mu| - \lambda \rangle$.

By local triviality, equiv to

$$\dim m_\mu^{-1}(\mathcal{O}_\lambda) \leq \langle \rho, |\mu| + \lambda \rangle.$$

<u>Strategy</u> $\dim(S_\lambda \cap \bar{\mathcal{O}}_\mu) \leq \langle \rho, \mu + \lambda \rangle$, $\mu \in X_*(T)^+$, $\lambda \in \Omega(\mu)$.

$\Omega(\mu)$ = wt space of $V_\mu \in \mathrm{Rep}(\hat{G})$

$= \{\lambda \in X_*(T) \mid w\lambda \leq \mu, \; \forall w \in W_0\}$.

Write $S_\lambda := LU t^\lambda e_0 \subset \mathrm{Gr}_G$ (locally closed sub-ind sch).

Sketched red: $\mu := |(\mu_1, \cdots, \mu_r)|$.

$$m_\mu^{-1}(S_\lambda \cap \bar{\mathcal{O}}_\mu) = \bigcup_{\substack{\nu \in X_*(T) \\ |\nu| = \lambda}} (S_{\nu_1} \times \bar{\mathcal{O}}_{\mu_1}) \tilde{\times} \cdots \tilde{\times} (S_{\mu_r} \times \bar{\mathcal{O}}_{\mu_r}).$$

$\dim \mathrm{LHS} \leq \max_\nu \sum_i \langle \rho, \nu_i + \mu_i \rangle = \langle \rho, \lambda + \mu \rangle$.

Need to show

$$\dim(S_\lambda \cap \bar{\mathcal{O}}_\mu) \leq \langle \rho, \lambda + \mu \rangle. \qquad \text{[MV07], [NP01].}$$

$\rightsquigarrow$ ETS for $k = \bar{\mathbb{F}}_q$, (fix $\mathbb{F}_q$ when q varies)

$K = G(\mathbb{F}_q[\![t]\!])$, $\; \mathcal{U} = U(\mathbb{F}_q[\![t]\!])$,

$T_0 = T(\mathbb{F}_q[\![t]\!])$, $\; G = G(\mathbb{F}_q(\!(t)\!))$.

By Weil conj, ETS

$$\lim_{q \to \infty} \frac{\#(Ut^{\lambda}K/K \cap Kt^{\lambda}K/K)}{q^{\langle \rho, \mu+\lambda \rangle}} = m_{\mu}(\lambda)$$

multi of λ in V_{μ} of $\hat{G}$.

$$\mathrm{LHS} = (1_{Kt^{-w_0/\mu}K})^{\vee}(t^{-w_0\lambda})\, q^{\langle \rho, \lambda \rangle}.$$

By MacDonalds' formula, this is the coeff of $t^{-w_0\lambda}$ in

$$\frac{q^{\langle \rho, \lambda+\mu \rangle}}{W_{-w_0,\mu}(q^{-1})} \sum_{w \in W_0} w\left(\prod_{\alpha>0} \frac{1-q^{-1}t^{-\alpha^{\vee}}}{1-t^{-\alpha^{\vee}}}\right) t^{-ww_0\mu}.$$

by Weyl

$$\mathrm{LHS} = m_{-w_0\lambda}(-w_0,\mu) = m_{\lambda}(\mu). \qquad \square$$

Lecture 6 : Tannakian Categories

Oct 23