P-adic Borel hyperbolicity of A_g

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$$D(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}, \quad D(z_0, r)^\times = D(z_0, r) - \{z_0\}.$$

Theorem (Great Picard theorem)

Let $f : D(0, r)^\times \to \mathbb{C}$ be a holomorphic function with essential singularity at 0, then $\sharp(\mathbb{C} - f(D(0, r)^\times)) \leq 1$.

Note that the exponential function $\exp : \mathbb{C} \twoheadrightarrow \mathbb{C}^\times := \mathbb{C} - \{0\}$ shows that Picard's theorems are sharp.

Hyperbolic Riemann surfaces

Recall the Riemann sphere, also known as 1-dim. projective space, is

$$\mathbb{P}^1 = \mathbb{C} \cup \{\infty\} = (\mathbb{C}^2 - \{(0,0)\})/(z_1, z_2) \sim (\lambda z_1, \lambda z_2).$$

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1. *Every holomorphic map* $f : \mathbb{C} \to \mathbb{P}^1 - \{0, 1, \infty\}$ *is constant.*
2. *Every holomorphic map* $f : D(0,r)^\times \to \mathbb{P}^1 - \{0, 1, \infty\}$ *extends to a holomorphic map* $\widetilde{f} : D(0,r) \to \mathbb{P}^1$.

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The above form of Picard's theorem holds for all hyperbolic Riemann surfaces.

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- $\mathbb{C}^\times = \{(x,y) \in \mathbb{C}^2 \mid xy = 1\}$, also denoted by $\mathbb{G}_m$.
- (Partial) flag varieties $\mathcal{F}\ell$ parameterizing chains of subspaces in $\mathbb{C}^n$ with given dimensions. A special case is the projective n-space $\mathbb{P}^n = (\mathbb{C}^{n+1} - \{(0,\ldots,0)\})/\mathbb{G}_m$, parameterizing lines in $\mathbb{C}^{n+1}$.
- $\Sigma \backslash \{z_1,\ldots,z_r\}$ with Σ compact Riemann surfaces, called algebraic curves. E.g. $(\mathbb{C}-\Lambda)/\Lambda \cong \{(x,y)\in\mathbb{C}^2 \mid 4y^2 = x^3 - g_2x - g_3\}$.
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The notion of algebraic varieties makes sense by replacing $\mathbb{C}$ by any field k. In addition, if $k \to k'$ is a field extension, an algebraic variety over k gives an algebraic variety over k'.

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This algebraicity theorem in turn implies the little Picard theorem. (Hint: if both $f : \mathbb{C} \to X$ and $f \circ \exp$ are polynomial maps, then f is constant.)

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Conjecture (Green-Griffith-Lang)

Assume that X is compact. Then TFAE:

- *X is Brody hyperbolic;*
- *Every closed subvariety of X is of general type;*
- *There is no non-constant rational map from an abelian variety to X.*

Arithmetic manifolds

Let D be a bounded symmetric domain in a complex vector space, also known as hermitian symmetric domain. It is of the form

$$D = G/K$$

where G is the group of holomorphic automorphism of D, which is a real (semisimple) Lie group and K is a maximal compact subgroup of G.

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- $D(0,1) \cong \mathfrak{H} = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\} \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2$;
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Theorem (Baily-Borel)

For an arithmetic subgroup $\Gamma \subset G$, $X = \Gamma\backslash D$ has a natural algebraic variety structure. Indeed, X admits a canonical compactification X^*, usually called the Bailey-Borel (or minimal) compactification of X, and X^* can be embedded into some projective space $\mathbb{P}^n$.

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  This means that a map $f : S \to A_g$ corresponds to a family of abelian varieties $A \to S$.
- For general D and Γ congruent subgroup of G, the space $\Gamma \backslash D$ is a (connected component of a) Shimura variety.

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The algebraic structure on X is unique.

This is extremely important for the arithmetic theory of Shimura varieties.

A little bit about p-adics

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- $\mathbb{Q}_p$, $\dim_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p = \infty$. The completion $\mathbb{C}_p$ of $\overline{\mathbb{Q}}_p$ is algebraically closed.

In each case, the absolute value $|\cdot|_v$ extends uniquely to these fields.

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- The (Berkovich space of) $D(0, r)^\times$ is contractible.

A little bit p-adic analysis

"Analytic/holomorphic" functions $\mathcal{O}(D^+(z_0, r))$ on the closed disc are

$$\mathbb{C}_p\langle z\rangle := \{f(z) = \sum a_i(z - z_0)^i \mid a_i \in \mathbb{C}_p, |a_i|_p r^i \to 0 \text{ as } i \to \infty\}.$$

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Some pathologies:

- In general, the anti derivative of an analytic function on $D^+(0, r)$ is only an analytic function on a smaller disc.
- The exponential function $\exp(z) = \sum_{i\geq 0} z^n/n!$ only converges for $|z|_p < \frac{1}{p^{1/(p-1)}}$.

Non-archimedean Picard's theorem

Theorem (non-archimedean little Picard theorem)

Every non-constant analytic function $f : \mathbb{C}_p \to \mathbb{C}_p$ *is surjective.*

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As before, we can reformulate the above results as

Theorem

1. *Every analytic function $\mathbb{C}_p \to \mathbb{C}_p^\times$ is constant.*
2. *Every analytic function $f : D(0, r)^\times \to \mathbb{C}_p^\times$ extends to an analytic function $\widetilde{f} : D(0, r) \to \mathbb{P}^1$.*
3. *Every analytic map $S \to \mathbb{G}_m$ from an algebraic curve S is algebraic.*

P-adic Brody hyperbolicity

We do not want to claim $\mathbb{G}_m$ to be hyperbolic. For this reason,

Definition (Javanpeykar-Vezzani)

A p-adic variety X is called Brody hyperbolic if every analytic map $f : G \to X$ is constant, where G is an algebraic group over $\mathbb{C}_p$.

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There is the analogue of GGL conjecture. Instead giving the formulation, we mention some evidences (even for the original GGL conjecture).

Theorem (Cherry, Kawamata, Ueno)

Let $K = \overline{K}$ with $\mathrm{char} K = 0$. Let X be a closed subvariety of an abelian variety A over K. Then the following are equivalent.

1. *X does not contain the translate of a positive-dimensional abelian subvariety of A.*
2. *Every closed integral subvariety of X is of general type.*
3. *If $K = \mathbb{C}$ or $\mathbb{C}_p$, the projective variety is Brody hyperbolic.*

Main theorem

Theorem (Oswal-Shankar-Z.)

Every analytic map $f : D(0,r)^\times \to A_g$ *defined over some finite extension* $K/\mathbb{Q}_p$ *can be extended to an analytic map* $\widetilde{f} : D(0,r) \to A_g^*$.

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Every analytic map $f : S \to A_g$ *defined over some finite extension* $K/\mathbb{Q}_p$ *with* S *an algebraic variety is algebraic.*

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- Our proof requires $K/\mathbb{Q}_p$ as above. Unfortunately, $K = \mathbb{C}_p$ is currently not allowed.
- By some standard arguments, the results hold with A_g replaced by Shimura varieties of abelian type.

Step I

The minimal compactification of A_g looks like

$$A_g^* = A_g \sqcup \partial A_g^* = \sqcup_{g' \leq g} A_{g'}.$$

Here $A_{g'}$ could appear in several boundary components and are equipped with appropriate level structures.

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E.g. the tubular neighborhood of $A_{g,\mathbb{F}_p}$ is contained in $A_{g,\mathbb{Q}_p}$, parameterizing those abelian varieties with (potentially) good reduction. It is the rigid analytic variety $\mathcal{A}_g^{\mathrm{rig}}$ associated to the formal completion of $A_g/\mathbb{Z}_p$ (along $p = 0$).

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Proposition

Every analytic map $D(0,r)^\times \to A_g^*$ is contained in one of the above tubular neighborhoods.

Step II

The variety $A_{g,\mathbb{F}_p}$ parameterizes abelian varieties (with polarization) over $\mathbb{F}_p$. It admits a decomposition

$$A_{g,\mathbb{F}_p} = \bigsqcup_{\phi} S_\phi,$$

where each S_ϕ is locally closed and two points $x, y \in A_g(\overline{\mathbb{F}}_p)$ belong to the same S_ϕ if the corresponding abelian varieties (with polarization) A_x and A_y are (quasi-)isogenous.

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The proof uses the Tate conjecture for abelian varieties over global function fields.

Step III

Fixing a point $x \in \,]S_\phi[$, Rapoport-Zink constructed a uniformzation map (i.e. an analytic map which is a topological covering)

$$\mathrm{RZ}_x \to \,]S_\phi[\ ,$$

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We may summarize the last (and the main) step of the proof as filling out dotted arrows in the following commutative

$$\begin{tikzcd}
 & D(0,r)^\times \arrow[r] \arrow[dl] \arrow[d, dotted, "(1)"'] & D(0,r) \arrow[dl, dotted, "(3)"'] \arrow[d, dotted, "(2)"] \\
\mathcal{A}_g^{\mathrm{rig}} & \mathrm{RZ}_x \arrow[l] \arrow[r, "\pi_{GM}"] & \mathcal{F}\ell.
\end{tikzcd}$$

Cohomology of (algebraic) variety

Let X be a smooth (projective) algebraic variety over a field k $(\mathrm{char} k = 0)$. There are various cohomology theory attached to X

- $\sigma : k \subset \mathbb{C}$, the singular/Betti cohomology $H^*_{\mathrm{B}}(\sigma X, \mathbb{Z})$;
- The étale cohomology $H^*_{\mathrm{et}}(X_{\overline{k}}, \mathbb{Q}_p)$;
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Now let $f : X \to S$ be a family of smooth projective varieties. Then various cohomology theories of $\{X_s\}_s$ also vary in family, giving:

- a $\mathbb{Z}$-local system on σS;
- a étale local system on S_{et};
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There are various comparison isomorphisms between different cohomology theories.

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Now we have $D(0,r)^\times \to \mathrm{RZ}_x \to \mathcal{F}\ell$. The following theorem, which can be regarded as the p-adic analogue of Schimd's theorem on limit Hodge structure, implies that this map a meromorphic and therefore extends to an analytic map $D(0,r) \to \mathcal{F}\ell$.

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Theorem (Diao-Lan-Liu-Zhu)

Let $\mathbb{L}$ be a de Rham p-adic local system on $D(0,r)^\times$, with the associated filtered connection $(\mathcal{E}, \nabla, \mathrm{Fil}^\bullet)$. Then $(\mathcal{E}, \nabla)$ admits a canonical extension (in the sense of Deligne) to a vector bundle $\overline{\mathcal{E}}$ on $D(0,r)$ with logarithmetic pole. In addition, the filtration $\mathrm{Fil}^\bullet$ extends to a filtration of $\overline{\mathcal{E}}$ by vector bundles.

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That $D(0,r) \to \mathcal{F}\ell$ lifts to $D(0,r) \to \mathrm{RZ}_x$ uses some theory of crystalline representations.

Thank You!