Serre Duality for Projective Spaces

§1 Ext Groups

$M, N \in \mathrm{Mod}_R$. $\mathrm{Ext}^i(M,N) := \underbrace{R^i\mathrm{Hom}_R(M,N)}$

$= (R^i\mathrm{Hom}_R(M,-))(N)$ (more strictly)

Recall: $\mathrm{Hom}_R(M,-)$ left exact covariant : $\mathrm{Mod}_R \to \mathrm{Ab}$.

Also, $\mathrm{Ext}^i(M,N) := (R^i\mathrm{Hom}_R(-,N))(M)$ on Mod_R^{op}.

$\rightsquigarrow$ $\mathcal{E}xt^i(\mathcal{F},-)$, $\mathcal{H}om(\mathcal{F},-)$ sheaf-version.

E.g. $\exists$ natural isom $\mathrm{Ext}^i(\mathcal{O}_X,\mathcal{F}) \cong H^i(X,\mathcal{F})$

b/c both are derived functors of $\mathrm{Hom}(\mathcal{O}_X,\mathcal{F}) \cong H^0(X,\mathcal{F})$.

Also, $\mathcal{H}om(\mathcal{O}_X,\mathcal{F})$ = id functor

$\Rightarrow$ $\mathcal{E}xt^0(\mathcal{O}_X,\mathcal{F}) \cong \mathcal{F}$, $\mathcal{E}xt^i(\mathcal{O}_X,\mathcal{F}) = 0$ $(i>0)$.

Lemma $\mathcal{I}$ inj. $\mathcal{O}_X$-mod $\Rightarrow$ $\forall\, U \subseteq X$ open, $\mathcal{I}|_U$ inj $\mathcal{O}_U$-mod.

Proof. $j: U \hookrightarrow X$. Given $\mathcal{F} \to \mathcal{G}$ mono, $\mathcal{F} \to \mathcal{I}|_U \overset{?}{\rightsquigarrow} \mathcal{G} \to \mathcal{I}|_U$

j_* denotes ext'n by 0 s.t. $(j_*\mathcal{F})_x = 0$ unless $x \in U$.

$\Rightarrow$ $j_*\mathcal{F} \to j_*\mathcal{G}$ mono.

Moreover, $\exists\, j_*\mathcal{I}|_U \to \mathcal{I}$ by adjunction.

& $j_*\mathcal{F} \to j_*\mathcal{I}|_U \to \mathcal{I}$ extends to $j_*\mathcal{G} \to \mathcal{I}$.

$\rightsquigarrow$ $\mathcal{G} \to \mathcal{I}|_U$ by $(\cdot)|_U$. $\square$

Cor $\forall\, U \subseteq X$ open, $\mathcal{E}xt^i(\mathcal{F},\mathcal{G})|_U \cong \mathcal{E}xt^i(\mathcal{F}|_U,\mathcal{G}|_U)$ naturally.

In particular, $\mathcal{E}xt^i(\mathcal{F}|_U,\mathcal{G}|_U) \in \mathrm{Sh}(U)$.

e.g. $\mathcal{E}xt^i(\mathcal{F},\mathcal{G}) = 0$, $\forall i>0$ whenever $\mathcal{F}$ be free of rank $< \infty$.

Proof. Both sides are δ-functor in $\mathcal{G}$.

& higher terms vanish on injectives by Lemma.

$\Rightarrow$ effaceable $\Rightarrow$ univ. $\square$

<u>Cor</u> $\mathcal{I}$ inj $\mathcal{O}_X$-mod. $\underset{\substack{\uparrow \\ \text{by def'n of inj.}}}{\mathrm{Hom}(\cdot,\mathcal{I})}$, $\underset{\substack{\uparrow \\ \text{by Lemma.}}}{\mathcal{H}om(-,\mathcal{I})}$ are exact.

<u>Prop</u> $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$, $\mathrm{Ext}^i(-,\mathcal{F})$, $\mathcal{E}xt^i(-,\mathcal{F})$ δ-functors on Mod_X^{op}.

Proof. $0 \to \mathcal{F} \to \mathcal{I}^\cdot$ inj resolution. Given

$$0 \to \mathcal{E} \to \mathcal{G} \to \mathcal{H} \to 0 \quad \text{in } \mathrm{Sh}(\mathrm{Mod}_{\mathcal{O}_X}).$$

$\rightsquigarrow$ $\mathrm{Hom}(-,\mathcal{I})$ & $\mathcal{H}om(-,\mathcal{I})$ short exact.

$\rightsquigarrow$ cohom long exact ...

By a pushout constr. it suffices to compare $\mathcal{I}^\cdot$ & $\mathcal{J}^\cdot$

with $\mathcal{I}^\cdot \longrightarrow \mathcal{J}^\cdot$ quasi-isom. $\square$

<u>Fact</u> $\mathrm{Sh}(\mathrm{Mod}_{\mathcal{O}_X})$ has enough projs $\rightsquigarrow$ acyclic resolutions

$\Rightarrow$ $\mathrm{Ext}^i(-,\mathcal{F})$, $\mathcal{E}xt^i(-,\mathcal{F})$ effaceable.

<u>Prop</u> Suppose $\cdots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0$ exact in $\mathrm{Sh}(\mathrm{Mod}_{\mathcal{O}_X})$

where $\mathcal{L}_i$ be free of rank $< \infty$.

(Say: $\mathcal{L}_\cdot$ be free resolution of $\mathcal{F}$).

<u>Then</u> $\mathcal{E}xt^i(\mathcal{F},\mathcal{G}) \cong h^i(\underset{\substack{\uparrow \\ \text{bi-functorial in } \mathcal{G} \,\&\, \mathcal{L}_\cdot}}{\mathcal{H}om(\mathcal{L}_\cdot,\mathcal{G})})$

Proof. $\mathcal{L}_i$ loc free of fin rank

$\Rightarrow \mathcal{E}xt^i(\mathcal{I},-)=0 \Rightarrow \mathcal{H}om(\mathcal{L}_i,-)$ exact

(even though $\mathcal{H}om(\mathcal{L}_i|_U,-)$ not exact, $\forall U$).

$\Rightarrow h^i(\mathcal{H}om(\mathcal{L}_i,-))$ δ-functor:

$$\left(\begin{array}{ll} & 0\to\mathcal{G}_1\to\mathcal{G}\to\mathcal{G}_2\to 0 \quad \text{exact} \\ \rightsquigarrow & 0\to\mathcal{H}om(\mathcal{L},\mathcal{G}_1)\to\mathcal{H}om(\mathcal{L}_\cdot,\mathcal{G})\to\mathcal{H}om(\mathcal{L},\mathcal{G}_2)\to 0 \\ \rightsquigarrow & \text{long exact in cohom.}\end{array}\right)$$

Now both sides:

$\left.\begin{array}{l}\text{(i) } \delta\text{-functors in } \mathcal{G} \\ \text{(ii) higher der} = 0 \text{ on injs}\end{array}\right\} \Rightarrow$ effaceable $\Rightarrow$ univ. $\square$

<u>Note</u> loc free resolutions are much easier to write down than inj resolutions.

$\boxed{\text{E.g.}}$ $X=\mathbb{P}^n_k$, k any field, $\mathcal{F}$ coherent

$\rightsquigarrow \exists\, \mathcal{E}\twoheadrightarrow\mathcal{F}$, $\mathcal{E}=\bigoplus$(twisting sheaf)

$\mathcal{O}_X(n)$, $n\ll 0$.

by Serre's finiteness thm.

(not just loc free, but free).

<u>Prop</u> $\forall\, \mathcal{F},\mathcal{G}\in Coh(\mathbb{P}^n_k)$, k field, $\mathcal{E}xt^i(\mathcal{F},\mathcal{G})\in Coh$ again.

Proof. $\exists\, \mathcal{L}_\cdot\to\mathcal{F}\to 0$ free resolution

Need to check $\underbrace{h^i(\mathcal{H}om(\mathcal{L}_\cdot,\mathcal{F}))\in Coh}$

true b/c $\mathcal{L}_\cdot,\mathcal{G}\in Coh \Rightarrow \mathcal{H}om(\mathcal{L}_\cdot,\mathcal{G})\in Coh$ $\square$

Lemma $\mathcal{F}, \mathcal{G}, \mathcal{L} \in Sh(Mod_{\mathcal{O}_X})$, $\mathcal{L}$ be free of fin rank.

(i) $Ext^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong Ext^i(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G})$;

(ii) $\mathcal{E}xt^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \otimes \mathcal{L}^\vee$.

In particular, $\mathcal{F} = \mathcal{L}^\vee$, $\mathcal{G} = \mathcal{L}$

$\Rightarrow$ (i) $Ext^i(\mathcal{O}_X, \mathcal{L}) \cong Ext^i(\mathcal{L}^\vee, \mathcal{O}_X)$;

(ii) $\mathcal{E}xt^i(\mathcal{O}_X, \mathcal{L}) \cong \mathcal{E}xt^i(\mathcal{L}^\vee, \mathcal{O}_X)$.

Proof. Again: $i>0$, check effaceable δ-functor in $\mathcal{G}$ ($\Rightarrow$ univ).

$i=0$: Hartshorne Prop III.6.7. $\square$

Final Note Relationship b/w Ext & $\mathcal{E}xt$?

General fact: F, G left exact functors

$\Rightarrow \exists$ spectral sequence relating D^iF, D^iG & $D^i(F \circ G)$ ($D^i :=$ derived functors).

In our case: Given $\mathcal{F} \in Sh(X)$, take

$F = H^0(X, -)$, $G = \mathcal{H}om(\mathcal{F}, -)$, $F \circ G = Hom_X(\mathcal{F}, -)$.

§2 Duality on Projective Spaces

For the rest, we work over k (any field).

Big Thm (Serre Duality) $X = \mathbb{P}^n_k$. $\mathcal{F} \in Coh(X)$.

Recall $\dim_k H^n(X, \mathcal{O}_X(-n-1)) = 1$.

(a) Perfect pairing

$$Hom_X(\mathcal{F}, \mathcal{O}_X(-n-1)) \times H^n(X, \mathcal{F}) \longrightarrow H^n(X, \mathcal{O}_X(-n-1)).$$

of fin-dim'l k-vector spaces.

(i.e. $A = \mathrm{Hom}_k(B, H^n(X, \mathcal{O}_X(-n-1)))$ & vice versa).

(b) $V \in \mathrm{Vec}_k$, put

$$V' = \mathrm{Hom}_k(V, H^n(X, \mathcal{O}_X(-n-1)).$$

$\forall i \geq 0$, $\exists$ natural isom

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{O}_X(-n-1)) \longrightarrow H^{n-i}(X, \mathcal{F})'$$

which $i=0 \Rightarrow$ (a),

and which is compatible w/ short exact sequences.

Proof. (a) We have $\mathrm{Hom}(\mathcal{F}, \mathcal{O}_X(-n-1)) \to H^n(X, \mathcal{F})'$ $(*)$

of left covariant functors on $\mathrm{Sh}(\mathrm{Mod}_{\mathcal{O}_X})$.

When $\mathcal{F} = \mathcal{O}_X(m)$, we want

$$H^0(X, \mathcal{O}_X(-m-n-1)) \cong \mathrm{Hom}(H^n(X, \mathcal{O}_X(m)), H^n(X, \mathcal{O}_X(-n-1))).$$

$$\mathrm{Hom}(\mathcal{O}_X, \mathcal{F}) = H^0(X, \mathcal{F})$$

Likewise, when $\mathcal{F} = \bigoplus \mathcal{O}_X(m)$, $(*)$ isom.

In general, $\underset{\bigoplus \mathcal{O}_X(m)}{\underset{\parallel}{\mathcal{E}_1}} \to \underset{\bigoplus \mathcal{O}_X(n)}{\underset{\parallel}{\mathcal{E}_0}} \to \mathcal{F} \to 0$

and the things we're computing are left-exact.

$$\rightsquigarrow \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \boxed{\mathrm{Hom}(\mathcal{F}, \mathcal{O}_X(-n-1)) \longrightarrow H^n(X, \mathcal{F})'} & & \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\mathcal{E}_0, \mathcal{O}_X(-n-1)) & \xrightarrow{\sim} & H^n(X, \mathcal{E}_0)' \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\mathcal{E}_1, \mathcal{O}_X(-n-1)) & \xrightarrow{\sim} & H^n(X, \mathcal{E}_1)' \end{array}$$

isom by 5-lemma

(b) Two δ-functors (agree at $i=0$) on $\mathrm{Coh}(\mathrm{Mod}_{\mathcal{O}_X})$

To check: both are effaceable.

Given $\mathcal{F} \in \mathrm{Coh}(\mathrm{Mod}_{\mathcal{O}_X})$, $q \gg 0$,

$$\mathcal{E} = \mathcal{O}_X(-q)^{\oplus m} \longrightarrow \underset{\substack{\uparrow \\ \text{quotient}}}{\mathcal{F}} \longrightarrow 0$$

It suffices to check $\forall i > 0$,

$$\mathrm{Ext}^i(\mathcal{O}_X(-q), \mathcal{O}_X(-n-1)) = 0 \quad \& \quad \underbrace{H^{n-i}(X, \mathcal{O}_X(-q)) = 0}_{\text{Serre vanishing}}, \quad q \gg 0.$$

$$\mathrm{Ext}^i(\mathcal{O}_X(-q), \mathcal{O}_X(-n-1)) = \mathrm{Ext}^i(\mathcal{O}_X, \mathcal{O}_X(q-n-1)) \cong H^i(X, \mathcal{O}_X(q-n-1)).$$

$\square$

§3 Differentials and Duality

(This is not the right way to view the duality thm b/c it doesn't generalize well.)

$X = \mathbb{P}^n_k$. $\Omega_{X/k}$ Kähler diffs, $\omega_X = \Lambda^n \Omega_{X/k}$ (canonical sheaf)

<u>Lemma</u> $X = \mathbb{P}^n_k$. $\omega_X \cong \mathcal{O}_X(-n-1)$.

Proof. $0 \to \Omega_{X/k} \longrightarrow \mathcal{O}_X(-1)^{\oplus n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0$.

corresponding to $\bigoplus_{i=0}^{n} S(-1) e_i$, $S = k[x_0, \dots, x_n]$

& $\left.\begin{array}{l} S(-1)^{n+1} \longrightarrow S \\ e_i \longmapsto x_i \end{array}\right\}$ Hartshorne Thm 8.13.

$\rightsquigarrow \quad 0 \to \Omega^i_{X/k} \to \Lambda^i_k \mathcal{O}_X(-1)^{\oplus n+1} \longrightarrow \Omega^{i-1}_{X/k} \to 0$, $\forall i$.

For $i = n+1$, $\Omega^{n+1}_{X/k} = 0 \Rightarrow \Lambda^{n+1}_k \mathcal{O}_X(-1)^{\oplus n+1} = \mathcal{O}_X(-n-1) \cong \Omega^n_{X/k}$.

<u>Alternatively</u> Finding glob generator of $\omega_X(n+1)$.

E.g. Define $\alpha \in H^0(D_+(x_0, \dots, x_n), \omega_X)$ by

$$\alpha = \frac{x_0^{n+1}}{x_0 \cdots x_n} d(x_1/x_0) \wedge \cdots \wedge d(x_n/x_0)$$

$x_0 \cdots x_n \alpha$ generates $\omega_X(n+1)$ over $D_+(x_0)$.

$$\alpha = \frac{1}{x_0 \cdots x_n} \sum_{i=0}^{n} (-1)^i x_i \left(dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\right).$$

$\Rightarrow$ performing an aut of X: swap x_i, x_j

& changes α by a sign

the same is true of the aut of X

which swaps x_0, x_n

$\Rightarrow$ $x_0 \cdots x_n \alpha$ generates $\omega_X(n+1)$ over $D_+(x_i)$, $i = 1, \cdots, n$. $\square$

<u>Summary</u> In any case, we can use ω_X to replace $\mathcal{O}_X(-n-1)$ when $X = \mathbb{P}^n_k$ in Serre Duality.

$\rightsquigarrow$ $\mathrm{Ext}^i(\mathcal{F}, \omega_X) \times H^{n-i}(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X)$, $\forall \mathcal{F} \in \mathrm{Coh}(X)$.