Comments for *Algebraic Geometry* with EGA & SGA

DEFINITION OF SCHEMES

This sheet refers to [EGA I, §2.1].

By definition, the *scheme* is a ringed space $(X, \mathcal{O}_X)$ in which for any point $x \in X$ there exists an open neighborhood $U \ni x$, such that $(U, \mathcal{O}_X|_U)$ is isomorphic to some ringed space $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$. Here:

- $\operatorname{Spec} A$ is the spectrum space of some commutative ring A, equipped with the Zariski topology. Moreover, all basic open subsets of the form $D(a)$ with $a \in A$ generate a basis of the Zariski topology.
- Define a presheaf $D(a) \mapsto Aa$ over the basic open sets. We thus obtain the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ over $\operatorname{Spec} A$.
- The isomorphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ between ringed spaces can be understood naively. That is, a homeomorphism between underlying topological spaces $X \simeq Y$, together with ring isomorphisms over each open subset, which is compatible with restriction. (c.f. [EGA I, §1.3] for more details.)

Hopefully, one may notice that other geometric objects share a similar algebraic expression. For example,

(1) **Smooth manifolds**.

Each point on a smooth manifold X has an open neighborhood U such that $(U, \mathcal{O}_X|_U)$ is isomorphic to $(Q, \mathcal{O}_Q)$, where Q is the unit open ball in some Euclidean space, and $\mathcal{O}_Q$ is the sheaf of real-valued smooth functions. Moreover, one can get the notion of *analytic manifold* by the same argument.

(2) **Complex manifolds.**

We must modify the construction of $\mathcal{O}_Q$. Let Q be the unit open ball in $\mathbb{C}^n$, i.e. $Q = \{(z_i) \in \mathbb{C}^n : |z_i| < 1\}$. Then take $\mathcal{O}_Q$ as the sheaf of complex-analytic functions on Q.

(3) **Complex analytic spaces.**

The notion of *complex analytic space* is a generalized version to that of complex manifold. It can be defined by replacing $(Q, \mathcal{O}_Q)$ with $(Q', \mathcal{O}_{Q'})$ as follows. Given finitely many functions $f_1, \dots, f_m \in \mathcal{O}_Q(Q)$ on Q, we can redefine a new ringed space $(Q', \mathcal{O}_{Q'})$ say, where Q', a subset of Q, is the intersection of zero sets of $f_1, \dots, f_m$, and $\mathcal{O}_{Q'} = (\mathcal{O}_Q/(f_1, \dots, f_m))|_{Q'}$. Unlike complex manifolds, a complex analytic space can possibly obtain very complicated singularities, such as singularities of Schubert cycles. On the other hand, granting GAGA, we can somehow regard the complex manifolds as "smooth" complex analytic spaces.

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