Sheafified and stacky approaches of p-adic Riemann-Hilbert correspondence

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Geometric p-adic RH by Liu-Zhu

Thm $K/\mathbb{Q}_p$ local field, X/K sm rigid, $C = \hat{\bar{K}}$.

$$\mathcal{X} = (X_C, \nu_* \mathcal{O}\mathbb{B}_{dR,X}), \quad \nu_* : \mathrm{Shv}(X_{C,\text{proét}}) \longrightarrow \mathrm{Shv}(X_{C,\text{ét}}).$$

Then $\exists$ a rk-preserving functor

$$RH : \{\mathbb{Q}_p\text{-local systems}\} \longrightarrow \left\{ (\mathcal{V}, \nabla) \,\middle|\, \begin{array}{l} \mathcal{V} \text{ VB} / \mathcal{X}, \\ \nabla \text{ int conn} \end{array} \right\}$$

Recall the constr'n

$$\mathbb{L} : \quad RH(\mathbb{L}) = \nu_*(\mathcal{O}\mathbb{B}_{dR} \otimes_{\mathbb{Q}_p} \hat{\mathbb{L}}), \qquad \nabla : \nu_*(d \otimes \mathrm{id}_{\hat{\mathbb{L}}})$$

Indeed, assume $\mathcal{O}\mathbb{B}_{dR}^{[0,\infty)} = \mathrm{Fil}^0(\mathcal{O}\mathbb{B}_{dR})$

$$\Rightarrow RH^{[0,\infty)}(\mathbb{L}) = \nu_*(\mathcal{O}\mathbb{B}_{dR}^{[0,\infty)} \otimes \hat{\mathbb{L}})$$

$$\text{VB} / (X_{C,\text{ét}}, \mathbb{B}_{dR}^+(C) \otimes \mathcal{O}_X).$$

Consider $\mathbb{L} / \mathbb{B}_{dR}^+$ proét vect bdl

$$\text{“} \underbrace{RH(\mathbb{L})}_{\text{behaves badly!}} = \nu_*(\mathcal{O}\mathbb{B}_{dR}^{[0,\infty)} \otimes_{\mathbb{B}_{dR}^+} \mathbb{L}) \text{”}$$

Point If $\mathbb{L}$ comes from $\mathbb{Q}_p$, on each toric tower,

$$\begin{array}{ccccc} \mathcal{U}_\infty & \xrightarrow{\Gamma} & \mathcal{U} = \mathrm{Spa}(A, A^+) & \longrightarrow & X_C \\ \downarrow & \lrcorner & \downarrow & & \\ \mathrm{Spa}(C\langle T^{\pm 1/p^\infty}\rangle) & \longrightarrow & \mathrm{Spa}(C\langle T^{\pm 1}\rangle) & & \end{array}$$

the action of Γ on $\mathbb{L} \otimes \hat{\mathcal{O}}|_{\mathcal{U}_\infty}$ is nilpotent.

<u>Generalize</u> Nil $\rightsquigarrow$ small assumption (by Yupeng Wang, et. al)

If consider $RH^{[0,\infty)}(\mathbb{L}) \otimes_{B_{dR}^+,\theta} C \rightsquigarrow \underbrace{\mathcal{H}(\mathbb{L})}_{\text{p-adic Simpson of } \mathbb{L}}$.

This tells us

Geometric RH = deformation of p-adic Simpson corresp.

<u>Today's Goal</u> $(K, K^+) / \mathbb{Q}_p$ perfectoid,

$$B_\alpha := B_{dR}^+(K)/(\ker\theta)^\alpha.$$

(I) <u>Thm</u> X/B_α sm adic space.

$\rightsquigarrow$ $\exists$ a canonical sheaf isom

$$RH_X : R^1\nu_*(GL_n(B_{\alpha,X})) \xrightarrow{\sim} MIC(X).$$

MIC = sheafification of the étale presheaf

$$\left[(U \xrightarrow{et} X) \longmapsto \begin{array}{c} \text{the isom class of} \\ \left\{ (M, \nabla) \;\middle|\; \begin{array}{l} M/U \text{ vB of rk } n, \\ \nabla : M \to M \otimes_{\mathcal{O}} \Omega\{-1\} \end{array} \right\} \end{array} \right].$$

<u>Note</u> If $\alpha = 1$, sheafified p-adic Simpson corresp

proved by Ben Heuer.

$$d : \mathcal{O} \to \Omega \to \Omega\{-1\}, \quad \{-1\} = (-) \otimes_{B_{dR}^+} (\ker\theta)^+ B_{dR}^+.$$

(II) Stacks : $(K, K^+) = (C, C^+)$

X / B_α smooth.

Perf_C = v-stack of ~~aff'd~~ perf'd spaces $/C$

$\mathrm{Spa}(R, R^+)$, R perf'd alg.

Def $\mathbb{M}_{dR,n} := (R,R^+) \longmapsto$ groupoid of

$$\{(M,\nabla)/X_{\mathbb{B}_\alpha(R)} : M \text{ rk } n \text{ VB},\ \nabla: M \to M \otimes \Omega\{-1\}\}$$

$\mathbb{M}_{\mathbb{B}_\alpha\text{-loc},n} := (R,R^+) \longmapsto$ groupoid of $\{\mathbb{B}_\alpha\text{-VB of rk } n\}$.

Proof to sheafified RH

Assume $X = \mathrm{Spa}(R,R^+)/\mathbb{B}_\alpha \xrightarrow{\text{std ét}} \mathbb{T}$ dim $=1$

$$\mathbb{T} = \mathrm{Spa}\ \mathbb{B}_\alpha^+\langle T^{\pm 1}\rangle.$$

__Warning__ $\mathbb{B}_\alpha^+\langle T^{\pm1}\rangle$ is __not__ uniform, unless $\alpha=1$.

$R_n = R[T^{1/p^n}]$ finite ét $/R$. $\quad \bar{R}_n = R_n/\ker\theta$

$R_\infty = \bigcup_{n\geq 0} R_n \quad \bar{R}_\infty = R_\infty/\ker\theta$

$\hat{R}_\infty = \mathbb{B}_\alpha(\hat{\bar{R}}_\infty)$

$\quad \hat{\bar{R}}_\infty =$ Completion of $\bar{R}_\infty$.

__E.g.__ $R = \mathbb{B}_\alpha\langle T^{\pm1}\rangle$, $\quad \bar{R} = C\langle T^{\pm1}\rangle$,

$R_n = \mathbb{B}_\alpha\langle T^{\pm1/p^n}\rangle$, $\quad \bar{R}_n = C\langle T^{\pm 1/p^\infty}\rangle$,

$R_\infty = \bigcup_n \mathbb{B}_\alpha\langle T^{\pm1/p^n}\rangle \quad \bar{R}_\infty = \bigcup_n C\langle T^{\pm1/p^n}\rangle$

$\hat{R}_\infty = \mathbb{B}_\alpha\langle [T^{\flat\, \pm 1/p^\infty}]\rangle$, $\quad \hat{\bar{R}}_\infty = C\langle T^{\pm1/p^\infty}\rangle$.

$\quad T^\flat = (T, T^{1/p}, \dots)$

$R_\infty \to \hat{R}_\infty$ via $T^{1/p^n} \to [T^\flat, T^{1/p^n}]$.

Let $\Gamma := \mathrm{Gal}(R_\infty/R)$. Fix a choice of $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots)$

$$t = \log[\varepsilon] \rightsquigarrow \Gamma \simeq \mathbb{Z}_p.$$

$\Gamma \circlearrowright \hat{R}_\infty = \mathbb{B}_\alpha(\hat{\bar{R}}_\infty)$:

action of γ on $\mathbb{B}_\alpha(\hat{\bar{R}}_\infty)$ by $\gamma_{dR}: T^{1/p^n} \longmapsto [\varepsilon^{1/p^n}]\, T^{1/p^n}$.

$\Gamma \circlearrowright R_\infty$: $\gamma : T^{1/p^n} \mapsto \zeta_{p^n} T^{1/p^n}$, denote by γ_ζ.

<u>Prop</u> $R_n = \hat{R}_\infty^{\,p^n\Gamma\text{-an}}$, $R_\infty = \hat{R}_\infty^{\,\Gamma\text{-la}}$.

Do decomposition on $\hat{R}_\infty$:

<u>Recall</u> $\Gamma_{lc} \subseteq \Gamma_{la} \subseteq \Gamma_{et}$. $\Gamma_* = \text{An-Spec}(C_*(\Gamma, \mathbb{Q}_p))$

$$\Gamma_\dagger \longrightarrow \Gamma_{la}$$

$C_\dagger$ = stalk of an fct at 0.

$X_\infty = \mathrm{Spa}(\hat{R}_\infty, \hat{R}_\infty^+)$, $X_\infty^{la} = (|X_\infty|, \hat{\mathcal{O}}_{X_\infty}^{la})$

$\circlearrowright \Gamma_{et}$ $\quad$ $\circlearrowright \Gamma_{la}$ $\circlearrowright \Gamma_{lc}$

$\mathcal{O}_{X_\infty}^{la} := (U \subseteq X_\infty) \mapsto \Gamma\text{-la}$ v.s. of $\hat{\mathcal{O}}(U)$.

<u>Decompletion</u> $\exists$ an equiv

$$\begin{tikzcd} \mathrm{VB}(X_\infty^{la}/\Gamma_{la}) \arrow[rr, "\sim"] \arrow[rr, bend left, "\text{taking completion}"] & & \mathrm{VB}(X_\infty/\Gamma_{et}) \arrow[ll, bend left, "\text{taking la vectors}"] \\ \mathrm{VB}(X_\infty^{la}/\Gamma_\dagger) \arrow[u] & & \end{tikzcd}$$

<u>Thm</u> $\exists$ an equiv.

$$t\text{-MIC}^{\text{stack}}(X) \simeq \mathrm{VB}(X_\infty^{la}/\Gamma_\dagger) + \text{loc const action by } \Gamma.$$

$$\ni (M, \nabla), \quad \nabla : M \longrightarrow M \otimes \Omega.$$

pf RHS = M_X^{la}-action of $\mathrm{Lie}\,\Gamma \circlearrowright M_\infty^{la}$ + loc const action by Γ.

$= (M/X, \ \mathrm{Lie}\,\Gamma \circlearrowright M)$.

Define t-conn of M by defining the action of $\underbrace{T\frac{d}{dT}}_{\parallel\ \partial} = \gamma$-action.

<u>Lem</u> $\exists$ map $\left(\mathrm{VB}(X_\infty^{la}/\Gamma_\dagger)/\simeq\right)^{\overset{\text{Gal action}}{\gamma_\zeta}} \longrightarrow t\text{-MIC}^{\text{sheaf}}(X)$.

for M/R_N on LHS, $\gamma_\xi^* M \simeq M$.

<u>Thm</u> $\forall\ M \in VB(X_\infty^{la}/\Gamma_{la})$, $\exists$ étale covering $\mathcal{U} \twoheadrightarrow X$ s.t. $M|_{\mathcal{U}_\infty^{la}/\Gamma_\dagger}$ is γ_ξ-inv (as isom class).

pf $\tilde{\nabla} = T\dfrac{d}{dT}$.

<u>Step 1</u> $\gamma_{exp} = \exp(t\tilde{\nabla}) : f \longmapsto \sum_{n=0}^{\infty} \dfrac{t^n \cdot \tilde{\nabla} f}{n!}$.

Then $\forall\ f \in R_\infty$, $\gamma_\xi \circ \gamma_{exp} = \gamma_{exp} \cdot \gamma_\xi = \gamma_{dR}$.

e.g. $\gamma_\xi(T^{1/p^n}) = \zeta_{p^n} T^{1/p^n}$.

$\gamma_{exp}(\zeta_{p^n} T^{1/p^n}) := \zeta_{p^n} \cdot \exp(\frac{t}{p^n}) \cdot T^{1/p^n} = [\varepsilon^{1/p^n}]\ T^{1/p^n}$.

<u>Step 2</u> M stabilized by γ_{dR}

$\rightsquigarrow$ need to find $\mathcal{U} \twoheadrightarrow X$ s.t. $M|_{\mathcal{U}_\infty^{la}/\Gamma_\dagger}$ stabilized by γ_{exp}.

<u>Step 3</u> Pass to étale stalk.

Can do $\exp \gamma_{exp}$, $\log \gamma_{exp}$, etc. $\square$