SOME GEOMETRY ABOUT HILBERT MODULAR FORMS

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This expository writing is adapted from Jan Hendrik Bruinier's paper *Hilbert modular forms and their applications* and a talk given by Shervin Shahrokhi Tehrani at University of Toronto in 2011. The author will be very sorry if someone's work is in display without mentioning the reference.

1. Hilbert modular surfaces

1.1. **Setups.** Let F be a totally real quadratic field over $\mathbb{Q}$. Let $\mathfrak{a}$ be a fractional ideal in $\mathcal{O}_F$, which is the ring of integers in F. Denote $\mathrm{Cl}(F)$ and $\mathrm{Cl}(F)^+$ the ideal class group and the arrow ideal class group of F, respectively. For simplicity, we use $\mathbb{H}$ to denote the complex upper-half plane, and $\mathbb{P}^1(F) = F \cup \{\infty\}$ the projective line. Define $e(w) = e^{2\pi i w}$ for any $w \in \mathbb{H}$. If $M \subseteq F$ is a $\mathbb{Z}$-module of rank 2, then there is a natural perfect trace pairing $\mathrm{tr} : M \times M \to \mathbb{Z}$. The dual module is thus taken as

$$M^\vee = \{\lambda \in F \mid \forall \mu \in M,\ \mathrm{tr}(\mu\lambda) \in \mathbb{Z}\},$$

which is as well a $\mathbb{Z}$-module of rank 2. Denote $\mathbb{A} = \mathbb{A}_\infty \mathbb{A}_f$ the adelic ring over F where $\mathbb{A}_f$ is the finite part of $\mathbb{A}$.

1.2. **Hilbert modular group and Hilbert modular surface.** Since F is a real quadratic field, there is a natural embedding of groups

$$\mathrm{SL}_2(F) \hookrightarrow \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}),$$

where $\mathrm{SL}_2(F)$ acts on $\mathbb{H} \times \mathbb{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z_1, z_2) = \left(\frac{az_1 + b}{cz_1 + d}, \frac{a'z_2 + b'}{c'z_2 + d'} \right).$$

Definition 1.1. Define the following congruence subgroup

$$\Gamma(\mathcal{O}_F \oplus \mathfrak{a}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) : a, d \in \mathcal{O}_F,\ b \in \mathfrak{a}^{-1},\ c \in \mathfrak{a} \right\}.$$

In particular, when $\mathfrak{a} = \mathcal{O}_F$, the *Hilbert full modular group* is

$$\Gamma_F = \Gamma(\mathcal{O}_F \oplus \mathcal{O}_F) = \mathrm{SL}_2(\mathcal{O}_F).$$

Recall that any subgroup of $\mathrm{SL}_2(F)$ which is commensurable with Γ_F is called an *arithmetic subgroup*. Let Γ be an arithmetic subgroup. It acts properly discontinuous on $\mathbb{H}^2$, i.e., if $W \subseteq \mathbb{H}^2$ is compact, then $\{\gamma \in \Gamma : \gamma W \cap W \neq \emptyset\}$ is a finite set.

Definition 1.2. Let Γ be any arithmetic subgroup of $\mathrm{SL}_2(F)$. The space

$$X'_\Gamma = \Gamma \backslash \mathbb{H}^2$$

is called the *modular surface*.

We will further denote the compactification of X'_Γ by X_Γ in Subsection 2.1.

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1.3. **Singularities on the modular surface.** Fix a modular surface X'_Γ as above.

(1) For any $a \in \mathbb{H}^2$, its stabilizer $\Gamma_a = \{\gamma \in \Gamma : \gamma a = a\}$ is a finite subgroup in Γ. Then a is called an *elliptic fixed point* if $\overline{\Gamma}_a = \Gamma_a/\{\pm 1\}$ is not trivial. It turns out that there are finite number of elliptic fixed points, and these are the only singularities of X'_Γ.

(2) The X'_Γ is not compact in general, therefore, there are points at infinity. Recall that $\mathrm{SL}_2(F)$ acts on $\mathbb{P}^1(F)$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\alpha, \beta) = \frac{a\alpha + b\beta}{c\alpha + d\beta}.$$

We call Γ-classes of $\mathbb{P}^1(F)$ the *cusp points* of X'_Γ. The punchline of this definition lies in the following result: the map

$$\begin{aligned} \varphi : \Gamma_F \backslash \mathbb{P}^1(F) &\longrightarrow \mathrm{Cl}(F) \\ (\alpha : \beta) &\longmapsto \alpha\mathcal{O}_F + \beta\mathcal{O}_F \end{aligned}$$

is bijective; namely, the cusp points are in bijection with ideal classes of $\mathcal{O}_F$. In particular, for a counting-points argument, the number of cusp points of X'_{Γ_F} is the class number of F.

1.4. **Adelic interpretation of Hilbert modular surface.** Let $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2(F)$ be the reductive algebraic group over $\mathbb{Q}$. Therefore,

$$G(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})^2.$$

We take $K_\infty = (\mathrm{SO}(2) \cdot \mathbb{R}_{>0}) \times (\mathrm{SO}(2) \cdot \mathbb{R}_{>0})$. The quotient $G(\mathbb{R})/K_\infty$ is homeomorphic with $\mathbb{H}^\pm \times \mathbb{H}^\pm$. Let K^∞ be an open compact subgroup of $G_f = G(\mathbb{A}_f)$. Using the Strong Approximation Theorem, we have the following.

Theorem 1.3. *There is an identification*

$$G(\mathbb{Q}) \backslash G(\mathbb{A})/K_\infty K^\infty = \bigcup_{j=1}^{m} \Gamma_j \backslash \mathbb{H}^2$$

with $\Gamma_j = G(\mathbb{Q}) \cap g_j G(\mathbb{R})^0 K_f g_j^{-1}$, *where* $G(\mathbb{R})^0$ *denotes the connected component of* $G(\mathbb{R})$ *containing the identity element.*

In particular, by taking $K^\infty = K_0 = \prod_{\nu \in S_f} \mathrm{GL}_2(\mathcal{O}_{F_\nu})$ with S_f a set of collections of finite places in F, we have:

Corollary 1.4. *There is an adelic interpretation of union of modular surfaces:*

$$G(\mathbb{Q}) \backslash G(\mathbb{A})/K_\infty K_0 = \bigcup_{\mathfrak{a}} \Gamma(\mathcal{O}_F \oplus \mathfrak{a}) \backslash \mathbb{H}^2,$$

where $\mathfrak{a}$ *runs over a complete set of representatives of* $\mathrm{Cl}(F)^+$.

1.5. **Further properties.** Moreover, there is a fundamental domain for action of Γ on $\mathbb{H}^2$ in terms of Siegel domains. Also, the form $\omega = \omega_1 \wedge \omega_2$ is a volume form on X'_Γ, where

$$\omega_1 = \frac{1}{2\pi} \cdot \frac{dx_1 \wedge dy_1}{y_1^2}, \quad \omega_2 = \frac{1}{2\pi} \cdot \frac{dx_2 \wedge dy_2}{y_2^2}.$$

2. Compactification

2.1. **Baily–Borel compactification.** Since F is a quadratic totally real field, we have a natural embedding

$$\mathbb{P}^1(F) \hookrightarrow \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R}).$$

Let $(\mathbb{H}^2)^* = \mathbb{H}^2 \cup \mathbb{P}^1(F)$ be the compactification of $\mathbb{H}^2$. Then the group Γ acts on $(\mathbb{H}^2)^*$. Consider the (compact) Hilbert modular surface

$$X_\Gamma = \Gamma \backslash (\mathbb{H}^2)^*.$$

Theorem 2.1 (Baily–Borel)**.** *On $(\mathbb{H}^2)^*$ there is a unique topology such that the $\Gamma\backslash(\mathbb{H}^2)^*$ with quotient topology is a compact Hausdorff space. Moreover, there is a sheaf of functions $\mathcal{O}_{X_\Gamma}$ on X_Γ such that $(X_\Gamma, \mathcal{O}_{X_\Gamma})$ is a complex normal space.*

In fact, one can construct a very ample line bundle using that of modular forms in sufficiently large weights on X_Γ, which gives an embedding in to some projective space. Therefore, the compact Hilbert modular surface X_Γ is a projective algebraic variety and X'_Γ is quasi-projective.

2.2. **Toroidal compactification and de singularization.** There is smooth compactification of X'_Γ using **Toroidal theory**. Therefore, we can resolve the singularities at boundary of Baily–Borel compactification model. Also, by using the theory of Hironaka, we are able to resolve the singularities caused by elliptic fixed points. We are going to use the adelic version and fix the following spaces. Consider

$$X'_{K^\infty} = G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})/K_\infty K^\infty.$$

Let X_{K^∞} be its Baily–Borel compactification. Denote Y_{K^∞} (resp. Z_{K^∞}) the minimal resolution of all singularities (resp. all the cusps).

Definition 2.2. A holomorphic function $f : \mathbb{H}^2 \to \mathbb{C}$ is called a *Hilbert modular form* of weight $k = (k_1, k_2) \in \mathbb{Z}^2$ on Γ if

$$\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad f(\gamma z) = (cz_1 + d)^{k_1}(c'z_2 + d')^{k_2} f(z).$$

If $k = k_1 = k_2$ then k is called the *parallel weight* of f.

2.3. **Fourier expansion.** If f is a Hilbert modular form then it has Fourier expansion at the cusp ∞ as follows. There is a $\mathbb{Z}$-module $M \subset F$ of rank 2, such that $f(z + \mu) = f(z)$ for each $\mu \in M$, and

$$f = \sum_{\nu \in M^\vee} a_\nu e(\mathrm{tr}(\nu z)),$$

where

$$a_\nu = \frac{1}{\mathrm{vol}(\mathbb{R}/M))} \int_{\mathbb{R}^2/M} f(z) e(-\mathrm{tr}(\nu z)) dx_1 dx_2.$$

In contrast to the 1-dimensional case, we have:

Theorem 2.3 (Koecher principle)**.** *Let $f : \mathbb{H}^2 \to \mathbb{C}$ be a Hilbert modular form. Then $a_\nu \neq 0$ implies $\nu = 0$ or $\nu \gg 0$.*

We denote the space of all Hilbert modular forms of weight k by M_k. This has an interpretation as global section of line bundles over Hilbert modular surface, and by using sheaf cohomology, M_k is finite dimensional.

Definition 2.4. A Hilbert modular form is called a *cusp form* if it vanishes at all cusps of Γ. Denote S_k the space of all cusp forms of weight k.

2.4. **Non-vanishing conditions.**

Theorem 2.5. *Let f be a Hilbert modular form of weight $k = (k_1, k_2)$ for Γ. Then f vanishes identically unless k_1, k_2 are both positive or $k_1 = k_2 = 0$. In latter case f is constant.*

Corollary 2.6. *If $\pi : Z_{K_f} \to X_{K_f}$ is the natural map, then any holomorphic one-form on Z_{K_f} vanishes identically, i.e.*

$$H^1(Z_{K_f}, \mathcal{O}_{K_f}) = 0.$$

Proof. Let ω be a 1-form and η be the pullback on regular points of X_{K_f}. We have

$$\eta = f_1(z)dz_1 + f_2(z)dz_2,$$

where f_1, f_2 are Hilbert modular form of weight $(2,0)$ and $(0,2)$, respectively. Using the theorem, we can say η vanishes. Therefore, $\omega \equiv 0$. □

As a remark, using the Hodge theory, one can show the vanishment of $H^1(Z_{K_f}, \mathbb{C})$. This means that the interesting part of cohomology of Hilbert modular surfaces is in degree 2.

3. Cohomology stories

Due to the vanishing reason of H^1, we are looking at $H^2(Z_\Gamma, \mathbb{Q})$. Using Poincaré duality, we have a non-degenerate pairing

$$H_2(Z_\Gamma) \times H_2(Z_\Gamma) \longrightarrow \mathbb{Q}.$$

Let E_σ be the subspace of $H_2(Z_\Gamma, \mathbb{Q})$ generated by the classes of the curves S_σ in the resolving of cusp σ. We have the decomposition

$$H_2(Z_\Gamma, \mathbb{Q}) = \operatorname{Im}(j_* : H_2(X'_\Gamma, \mathbb{Q}) \to H_2(Z_\Gamma, \mathbb{Q})) \oplus \bigoplus_\sigma E_\sigma,$$

where $j : X'_\Gamma \to Z_\Gamma$. By duality for cohomology we have

$$H^2(Z_\Gamma, \mathbb{Q}) = \pi^* H^2(X_\Gamma, \mathbb{Q}) \oplus \bigoplus_\sigma E_\sigma^\vee,$$

where $\pi : Z_\Gamma \to X_\Gamma$. There is an exact sequence

$$0 = H^1(X_\Gamma - X'_\Gamma, \mathbb{Q}) \to H^2_c(X'_\Gamma, \mathbb{Q}) \to H^2(X_\Gamma, \mathbb{Q}) \to H^2(X_\Gamma - X'_\Gamma, \mathbb{Q}) = 0.$$

Proposition 3.1. *We have a canonical isomorphism*

$$H^2(X_\Gamma, \mathbb{Q}) \cong H^2_c(X'_\Gamma, \mathbb{Q}).$$

3.1. **Mixed Hodge structure.** Since X'_Γ is a quasi-projective algebraic variety, by Deligne–Hodge theory, there is a mixed Hodge structure $\{(F_p, W_k) : (p,k) \in \mathbb{Z}^2\}$ as follows.

(a) $\{F_p\}_{p\in\mathbb{Z}}$ is a decreasing filtration on $H^i(X'_\Gamma, \mathbb{Q}) \otimes \mathbb{C}$;
(b) $\{W_k\}_{k\in\mathbb{Z}}$ is an increasing weight filtration on $H^2(X'_\Gamma, \mathbb{Q})$, such that
  - $W_k H^2(X'_\Gamma, \mathbb{Q}) = 0$ for $k \leqslant 1$,
  - $W_2 H^2(X'_\Gamma, \mathbb{Q}) = W_3 H^2(X'_\Gamma, \mathbb{Q}) = j^* H^2(Z_\Gamma, \mathbb{Q})$, and
  - $W_k H^2(X'_\Gamma, \mathbb{Q}) = H^2(X'_\Gamma, \mathbb{Q})$ for $k \geqslant 4$.

In fact, there exists a pure Hodge structure of weight k on $W_k H^2(X'_\Gamma, \mathbb{Q}) / W_{k-1} H^2(X'_\Gamma, \mathbb{Q})$.

3.2. **Pure Hodge structure.** Recall that for $X'_\Gamma \xrightarrow{j} Z_\Gamma \xrightarrow{\pi} X_\Gamma$, we have

$$j^*H^2(Z_\Gamma,\mathbb{Q}) = j^*\left(\pi^*H^2(X_\Gamma,\mathbb{Q}) \oplus \bigoplus_\sigma E_\sigma^\vee\right).$$

Since $j^*E_\sigma^\vee = 0$ for all cusps, there is a pure Hodge structure of weight 2 on

$$\mathbb{H}^2(X_\Gamma,\mathbb{C}) := j^*\pi^*H^2(X_\Gamma,\mathbb{C}).$$

This is (formally) compact, since

$$\mathbb{H}^2(X_\Gamma,\mathbb{Q}) = \mathrm{Im}(H_c^2(X'_\Gamma,\mathbb{Q}) \to H^2(X'_\Gamma,\mathbb{Q})).$$

Now let $f = (\ldots, f_j, \ldots)$ be a Hilbert modular form of weight 2. This defines a 2-form ω_f on $\Gamma_j\backslash\mathbb{H}^2$ by

$$\omega_f = (2\pi i)^2 f_j(z)dz_1 \wedge dz_2.$$

We obtain the following fact as a lemma:

$$F^2\mathbb{H}^2(X_\Gamma,\mathbb{C}) = \{\omega_f : f \in S_2\}.$$

Suppose there are actions on $\mathbb{H}^\pm \times \mathbb{H}^\pm$ via

$$\varepsilon_1 : (z_1,z_2) \mapsto (\overline{z}_1, z_2), \quad \varepsilon_2 : (z_1,z_2) \mapsto (z_1, \overline{z}_2),$$

where

$$\varepsilon_1 = \left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, 1\right), \quad \varepsilon_2 = \left(1, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right).$$

Define further that

$$\eta_f = \varepsilon_2^*\omega_f, \quad \eta'_f = \varepsilon_1^*\omega_f.$$

Theorem 3.2. *There is a (pure) Hodge factorization up to the signatures: $\mathbb{H}^2(X_\Gamma,\mathbb{C})$ is the direct sum of*

- *its $(2,0)$-component $\{\omega_f : f \in S_2\}$,*
- *its $(0,2)$-component $\{\overline{\omega}_f : f \in S_2\}$, and*
- *its $(1,1)$-component $\{\eta_f + \eta'_g : f, g \in S_2\} \oplus W$, where W is the space generated by the forms ω_1, ω_2 on all components of X_Γ.*

Remark 3.3. We introduce a terminology at usual work. Define the *cuspidal cohomology* $\mathbb{H}^2_{\mathrm{cusp}}(X_\Gamma,\mathbb{C})$ to be the orthogonal complement of W in $\mathbb{H}^2(X_\Gamma,\mathbb{C})$.

4. The action of Hecke operators

4.1. **Hecke ring.** In this section we define

$$B = G(\mathbb{A}_\mathbb{Q}) \cap \left(G(\mathbb{R})^0 \times \prod_{v\in S_f} \mathrm{GL}_2(\mathcal{O}_{F_v})\right)$$

and

$$R = G(\mathbb{R})^0 \times K_0.$$

Let H_K be the algebra over $\mathbb{Q}$ generated by

$$T(\mathfrak{m}) = \sum_b RbR,$$

where $\mathfrak{m}$ is an integral ideal and the sum runs through those b such that $\det(b)\mathcal{O}_F = \mathfrak{m}$.

4.2. **The action of Hecke operators.** There is a version of action of Hecke operators on modular forms, we are going to use this fact that S_k (the space of cusp forms) has a basis of eigenforms for all Hecke operators. Moreover, the Hecke action on modular forms follows the **multiplicity-one principle**, which dictates that two eigenforms with the same eigenvalue must be multiples of each other. Namely, the Hecke action is morally "simple".

As for the Hecke action on cohomology groups, we have

$$\mathbb{H}^2(X_{K_0}, \mathbb{Q}) = \mathbb{H}^2_{\mathrm{cusp}}(X_{K_0}, \mathbb{Q}) \oplus (\mathbb{Q}(-1))^{h+},$$

where $\mathbb{Q}(-1)$ is the rational Hodge structure of type $(1,1)$ of $(2\pi i)\mathbb{Q}$. Recall that $\mathbb{H}^2_{\mathrm{cusp}}(X_{K_0}, \mathbb{Q})$ has Hodge decomposition where each term is isomorphic to a space of cusp forms. Therefore, H_K acts on $\mathbb{H}^2_{\mathrm{cusp}}(X_\Gamma, \mathbb{C})$, and the action is compatible with that towards modular forms.

Theorem 4.1. *For any $T \in H_K$ we can attach T^*, an endomorphism of $\mathbb{H}^2(X_{K_0}, \mathbb{Q})$ preserving the Hodge decomposition, such that*

$$\int_{T_* c} \omega = \int_c T^* \omega,$$

for any homological cycle $c \in H_2(X'_{K_0}, \mathbb{Q})$ and $\omega \in H^2(X', \mathbb{Q})$. Here T_* is the dual endomorphism of T^* on $\mathbb{H}_2(X_{K_0}, \mathbb{Q})$. Also,

$$\langle \omega_1, T^* \omega_2 \rangle = \langle T^* \omega_1, \omega_2 \rangle.$$

4.3. **The H_K-module structure on S_2.** Here comes a decomposition of H_K.

Proposition 4.2. *As a $\mathbb{Q}$-algebra, H_K is semisimple and finite-dimensional; also, S_2 is an $H_K \otimes \mathbb{C}$-module of rank one. Moreover,*

$$H_K = \bigoplus k_i,$$

where each k_i is a finite field extension of $\mathbb{Q}$.

We can choose a set of primitive idempotents $\{e_1, e_2, \ldots, e_n\}$ such that $k_i = e_i H_K$. If f is a normalized eigenform in $e_i S_2$, then there is an embedding $\sigma_i : k_i \to \mathbb{C}$ such that $t(f) = \sigma_i(t) f$ for all $t \in k_i$. Therefore, $k_i \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_\sigma \mathbb{C}$. Using this embedding, we see

$$e_i S_2 = \bigoplus_\sigma \mathbb{C} f^\sigma,$$

where the f^σ's are the normalized eigenforms with $t(f^\sigma) = \sigma(t) f^\sigma$. In some materials, f^σ is called the *companion* of f.

4.4. **Decomposition of the cusp cohomology.** Let $F = \{f_1, \ldots, f_n\}$ be a set of normalized eigenforms, such that $f_i \in e_i S_2$ for each i. If $f \in e_i S_2$, we shall write $e_i = e_f$ and $k_i = k_f$. Under the projector action, we let

$$H^2(M_f, \mathbb{Q}) := e_f \mathbb{H}^2_{\mathrm{cusp}}(X, \mathbb{Q}).$$

Theorem 4.3. *There is a decomposition of polarized Hodge structure on $\mathbb{H}^2_{\mathrm{cusp}}(X, \mathbb{Q})$ as*

$$\mathbb{H}^2_{\mathrm{cusp}}(X, \mathbb{Q}) = \bigoplus_{f \in F} H^2(M_f, \mathbb{Q}).$$

Consider $\varepsilon_1, \varepsilon_2$ as involutions on $\mathbb{H}^2(X, \mathbb{Q})$. Because the actions of $\varepsilon_1, \varepsilon_2$ commutes with Hecke operators, we see

$$H^2(M_f, \mathbb{Q}) = \bigoplus_{s, s' \in \{+,-\}} H^2(M_f, \mathbb{Q})_{ss'},$$

where ε_1 and ε_2 act on $H^2(M_f, \mathbb{Q})_{ss'}$ as Id and $-\,$Id, respectively.

Proposition 4.4. *For every normalized eigenform $f \in S_2$, we have*

$$\mathrm{rank}_{k_f} H^2(M_f, \mathbb{Q})_{ss'} = 1.$$

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