Characters

§1 Definitions about characters

Def (1) $f: G \to k$ <u>central</u> if $\forall g, h \in G$,

$$f(gh) = f(hg).$$

$\rightsquigarrow$ $\mathcal{C}(G,k) = k$-Alg of central fcns $G \to k$.

(2) $(V, \rho) \in R_k(G)$. The <u>character</u> of V is

$$\chi_V : G \to k, \quad g \longmapsto \mathrm{Tr}(\rho(g)).$$

Prop For $(V, \rho_V), (W, \rho_W) \in R_k(G)$,

(1) $\chi_V(1) = \dim_k V$, (3) $\chi_{V \oplus W} = \chi_V + \chi_W$,

(2) $\chi_V \in \mathcal{C}(G,k)$, (4) $\chi_{V \otimes_k W} = \chi_V \cdot \chi_W$.

$G \curvearrowright V \otimes_k W$ via $g(v \otimes w) = (gv) \otimes (gw)$.

Proof Only (4) is nontrivial.

(4) $(e_1, \cdots, e_n), (f_1, \cdots, f_m)$ k-basis of V & W.

$\rightsquigarrow$ $\rho_V(g) = (x_{ij}), \ \rho_W(g) = (y_{ij})$ for a fixed $g \in G$.

$\Rightarrow$ $(e_i \otimes f_j)$ k-basis of $V \otimes_k W$,

$$\rho_V(g) \otimes \rho_W(g) = (x_{ii'} y_{jj'})_{1 \le i, i' \le n,\ 1 \le j, j' \le m}.$$

$$\Rightarrow \quad \chi_{V \otimes_k W}(g) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ii} y_{jj} = \Big(\sum_{i=1}^{n} x_{ii}\Big)\Big(\sum_{j=1}^{m} x_{jj}\Big) = \chi_V(g) \cdot \chi_W(g). \quad \square$$

Cor $V \longmapsto \chi_V$ $\rightsquigarrow$ morph of rings $R_k(G) \longrightarrow \mathcal{C}(G,k)$.

Rmk (1) For K/k & $V \in R_k(G)$, can get $V \otimes_k K \in R_K(G)$

with

$$\begin{tikzcd}
{[V]} \arrow[d, mapsto] & R_k(G) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{C}(G,k) \arrow[d, "\text{inclusion.}"] \\
{[V \otimes_k K]} & R_K(G) \arrow[r] & \mathcal{C}(G,K)
\end{tikzcd}$$

(2) For $V \in R_k(G)$, have $k[G] \to \mathrm{End}_k(G)$ of k-Algs.

$\rightsquigarrow$ can extend χ_V to $\chi_V : k[G] \to k$ with same properties.

<u>Def</u> $V, W \in R_k(G)$.

(1) $G \curvearrowright \mathrm{Hom}_k(V, W) \in R_k(G)$ via

$$(g \cdot f)(v) = g f(g^{-1} v), \quad \forall f \in \mathrm{Hom}_k(V, W),\ v \in V.$$

(2) $G \curvearrowright V^* = \mathrm{Hom}_k(V, k) \in R_k(G)$ via

$$(g.f)(v) = f(g^{-1} v), \quad \forall f \in \mathrm{Hom}_k(V, W),\ v \in V.$$

<u>Def</u> $V^G := \{ v \in V : \forall g \in G,\ gv = v \}$, the space of invariants of G.

<u>Rmk</u> Have $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_k(V, W)^G$.

<u>Prop</u> Let $V, W \in R_k(G)$. Then
$$\begin{aligned} V^* \otimes_k W &\xrightarrow{\varphi} \mathrm{Hom}_k(V, W) \\ f \otimes w &\longmapsto (v \mapsto f(v) \cdot w) \end{aligned}$$
is a G-equivariant isom.

<u>Proof</u> φ k-bilinear $\Rightarrow$ φ well-def'd.

- <u>G-equivariant</u>: $f \in V^*$, $w \in W$, $g \in G$, $v \in V$.

$$\left.\begin{aligned} \Rightarrow \varphi(g(f \otimes w))(v) &= \varphi((gf) \otimes (gw))(v) \\ &= (gf)(v) \cdot (gw) \\ &= f(g^{-1} v) \cdot (gw) \\ \& \ (g \varphi(f \otimes w))(v) &= g(\varphi(f \otimes w)(g^{-1} v)) \\ &= g f(g^{-1} v) \cdot w \end{aligned}\right\} \text{equal.}$$

- <u>φ bijective</u>: (e_i) k-basis of W.

$\Rightarrow \exists$ k-linear isoms $u: \bigoplus_{i\in I} \mathrm{Hom}_k(V,k) \xrightarrow{\sim} \mathrm{Hom}_k(V,W)$

$$(f_i) \longmapsto (v \mapsto \sum f_i(v)\cdot e_i).$$

& $v: \bigoplus_{i\in I} V^* \xrightarrow{\sim} V^* \otimes_k W$

$$(f_i) \longmapsto (\sum f_i \otimes e_i)$$

notice: $\varphi = u \circ v^{-1}$. $\square$

<u>Prop</u> $V, W \in R_k(G)$. Then

(1) $\chi_{V^*}(g) = \chi_V(g^{-1})$,

(2) $\chi_{\mathrm{Hom}_k(V,W)}(g) = \chi_V(g^{-1})\cdot \chi_W(g)$.

<u>Proof</u> Note that (1) $\Rightarrow$ (2) essentially.

(1) $\rho: G \to \mathrm{End}_k(V)$, $\mathcal{B}$ k-basis of V.

$M = [\rho(g)^{-1}]_{\mathcal{B}} \rightsquigarrow g$ acts by ${}^tM$ on V^*.

Then $\mathrm{Tr}(M) = \chi_V(g^{-1}) = \mathrm{Tr}({}^tM) = \chi_{V^*}(g)$. $\square$

<u>§2 Orthogonality of characters</u>

Denote $S_k(G)$ set of rep'tives of isom classes of irreps of G/k.

<u>Thm</u> (Schur's lemma)

$V, W \in R_k(G)$ irred. Then

- $\mathrm{Hom}_k(V,W) = 0$ unless $V \simeq W$.
- $\mathrm{End}_k(V)$ fin-dim'l k-division alg.
- $k = \bar{k} \Rightarrow \mathrm{End}_k(V) = k$.

<u>Thm</u> $\frac{1}{|G|}\sum_{g\in G} \chi_V(g^{-1})\chi_W(g) = \frac{1}{|G|}\sum_{g\in G} \chi_{V^*}(g)\chi_W(g) = \dim_k \mathrm{Hom}_G(V,W)$.

<u>Lem A</u> $k = \bar{k}$. (V, ρ) irrep. Then

$$\sum_{g \in G} \chi_V(g) = \begin{cases} 0, & \text{if } V \not\simeq \mathbb{1} \\ |G|, & \text{if } V \simeq \mathbb{1}. \end{cases}$$

<u>Proof</u> $\rho : G \to \mathrm{End}_k(V) \rightsquigarrow \rho : k[G] \to \mathrm{End}_k(V)$

Let $c = \sum_{g \in G} g \in k[G] \Rightarrow c \in Z(k[G])$

as $hc = ch = c$, $\forall h \in G$.

$\Rightarrow \rho(c) \in \mathrm{End}_k(V)$ G-equivariant.

By Schur's lem, $\exists \lambda \in k$ s.t. $\rho(c) = \lambda \cdot \mathrm{id}_V$

$$\Rightarrow \sum_{g \in G} \chi_V(g) = \chi_V(c) = \mathrm{Tr}(\rho(c)) = \lambda \cdot \dim V.$$

Moreover,

$$\lambda \cdot \rho(h) = \rho(c)\rho(h) = \rho(ch) = \rho(c) = \lambda \cdot \mathrm{id}_V, \quad \forall h \in G.$$

So $\lambda \neq 0 \Rightarrow V \simeq \mathbb{1}$

$$\Rightarrow \sum_{g \in G} \chi_V(g) = \sum_{g \in G} 1 = |G|. \qquad \square$$

<u>Lem B</u> $V \in R_k(G)$. Then

$$\sum_{g \in G} \chi_V(g) = |G| \cdot \dim(V^G).$$

<u>Proof</u> Can assume $k = \bar{k}$.

Write $V = \bigoplus_{W \in S_k(G)} W^{\oplus n_W}$. Then

$$\text{Lem A} \Rightarrow \sum_{g \in G} \chi_V(g) = \sum_{W \in S_k(G)} n_W \sum_{g \in G} \chi_W(g) = n_{\mathbb{1}} \cdot |G|.$$

On the other hand,

$W \in S_k(G)$ & $W \not\simeq \mathbb{1} \Rightarrow W^G = 0$ (b/c $W^G \subseteq W$ subrep)

$\Rightarrow V^G = \mathbb{1}^{n_{\mathbb{1}}}$ & $\dim V^G = n_{\mathbb{1}}$. $\qquad \square$

Pf of thm Now we have

$$\frac{1}{|G|}\sum_{g\in G}\chi_V(g^{-1})\chi_W(g) = \frac{1}{|G|}\sum_{g\in G}\underbrace{\chi_{V^*}(g)\chi_W(g)}$$

$$= \chi_{V^*\otimes_k W}(g) = \chi_{\mathrm{Hom}_k(V,W)}(g).$$

Lem A, B $\Rightarrow$ $\sum_{g\in G}\chi_{\mathrm{Hom}_k(V,W)}(g) = |G|\cdot \dim_k \mathrm{Hom}_k(V,W)^G$

$$= |G|\cdot \dim_k \mathrm{Hom}_G(V,W). \qquad \square$$

Cor $V, W \in S_k(G)$, $V \not\cong W$, then

$$\sum_{g\in G}\chi_{V^*}(g)\cdot\chi_W(g) = 0.$$

Cor $k=\bar{k}$, $V\in R_k(G)$. Then

$$V\in S_k(G) \text{ (irred) } \Longleftrightarrow \sum_{g\in G}\chi_{V^*}(g)\cdot\chi_V(g) = |G|.$$

Proof Thm $\Rightarrow$ $\sum_{g\in G}\chi_{V^*}(g)\chi_V(g) = |G|\cdot\dim_k(\mathrm{End}_G(V))$.

V irred $\Rightarrow$ $\mathrm{End}_G(V)=k$ by Schur.

Suppose the formula holds. $V=\bigoplus V_i^{\oplus n_i}$.

irred, $V_j\not\cong V_i$.

$$\Rightarrow \sum_{g\in G}\chi_{V^*}(g)\chi_V(g) = \sum_{i,j} n_i n_j \sum_{g\in G}\chi_{V_i^*}(g)\chi_{V_j}(g)$$

$$= |G|\cdot\sum_i n_i^2. \qquad \square$$

Rmk When $k\neq\bar{k}$, the same pf shows "$\Leftarrow$" side.

Cor The family $(\chi_V)_{V\in S_k(G)}$ linearly indep in $\mathcal{C}(G,k)$.

Proof $\sum_{W\in S_k(G)}\alpha_W\chi_W = 0$, $\alpha_W\in k$

$$\Rightarrow \forall\, V\in S_k(G),\quad 0=\sum_{g\in G}\Big(\sum_{W\in S_k(G)}\alpha_W\chi_W(g)\Big)\chi_{V^*}(g)$$

$$= \sum_{W\in S_k(G)}\alpha_W\sum_{g\in G}\chi_W(g)\chi_{V^*}(g) = \alpha_V\cdot|G|.$$

$\Rightarrow \alpha_V = 0.$ $\square$

Cor $V, V' \in R_k(G)$. $V \simeq V' \Leftrightarrow \chi_V = \chi_{V'}$.

In particular, $\begin{array}{ccc} R_k(G) & \longrightarrow & \mathcal{C}(G,k) \\ V & \longmapsto & \chi_V \end{array}$ is injective.

Proof Write $V = \bigoplus_{W \in S_k(G)} W^{\oplus n_W}$, $V' = \bigoplus_{W \in S_k(G)} W^{\oplus n'_W}$.

$\Rightarrow \chi_V = \sum n_W \chi_W$, $\chi_{V'} = \sum n'_W \chi_W$.

$\rightsquigarrow \chi_V = \chi_{V'} \Leftrightarrow n_W = n'_W$ for each $W \in S_k(G)$

$\Leftrightarrow V \simeq V'$. $\square$

§3 Characters & rep ring

Thm $k = \bar{k}$. Then $(\chi_W)_{W \in S_k(G)}$ is a basis of $\mathcal{C}(G,k)$.

Lem If $f \in \mathcal{C}(G,k)$ s.t. $\sum_{g \in G} f(g)\chi_{W^*}(g) = 0$, $\forall W \in S_k(G)$,

then $f = 0$.

Proof $\rho : G \to \mathrm{End}_k(V) \rightsquigarrow$ Set $\rho(f) := \sum_{g \in G} f(g)\rho(g) \in \mathrm{End}_k(V)$.

$\Rightarrow \forall g \in G$, $\rho(g)\rho(f) = \sum_{h \in G} f(h)\rho(gh) = \sum_{h \in G} f(h)\rho(ghg^{-1}g)$

$= \sum_{h \in G} f(ghg^{-1})\rho(ghg^{-1}) \cdot \rho(g) = \rho(f) \cdot \rho(g)$

(b/c f is central.)

$\Rightarrow$ if $V \in S_k(G)$ then $\rho(f) \in \mathrm{End}_G(V) = k$ (by Schur)

$\rightsquigarrow$ can write $\rho(f) = \lambda \cdot \mathrm{id}_V$, $\lambda \in k$

& $\lambda \cdot \dim V = \mathrm{Tr}(\rho(f)) = \sum_{g \in G} f(g)\chi_V(g) = 0$

$\Rightarrow \rho(f) = 0$.

And $k[G]$ semisimple $\Rightarrow \rho(f)=0$, $\forall \rho$.

$\Rightarrow \quad 0 = \rho_{reg}(f)\cdot 1 = \sum_{g\in G} f(g)\cdot g \in k[G]$

$\Rightarrow \quad f(g)=0,\ \forall g\in G \quad \Rightarrow f=0.$ $\square$

<u>Pf of thm</u> Known: linearly indep.

$f\in \mathcal{C}(G,k)$, $\forall W\in S_k(G)$, set

$$\alpha_W = \frac{1}{|G|}\sum_{g\in G} f(g)\chi_{W^*}(g).$$

If $(\chi_W)_{W\in S_k(G)}$ basis of $\mathcal{C}(G,k)$

then α_W = coeff of χ_W in f.

$\rightsquigarrow$ Set $f' = f - \sum_{W\in S_k(G)} \alpha_W \chi_W$ & try to prove $f'=0$.

Cor $\Rightarrow \forall W\in S_k(G)$,

$$\sum_{g\in G} f'(g)\chi_{W^*}(g) = \sum_{g\in G} f(g)\chi_{W^*}(g) - \alpha_W \sum_{g\in G}\chi_W(g)\cdot\chi_{W^*}(g) = 0.$$ $\square$

<u>Cor</u> $k=\bar{k}$. Then $|S_k(G)| = \dim_k \mathcal{C}(G,k)$

& this equals #(Conjugacy classes of G).

<u>Cor</u> $k=\bar{k}$. $R_k(G)\longrightarrow \mathcal{C}(G,k)$, $[V]\longmapsto \chi_V$

induces an isom $R_k(G)\otimes_{\mathbb{Z}} k \xrightarrow{\sim} \mathcal{C}(G,k)$ of k-Algs.

<u>Rmk</u> (1) Many results still true for $k\neq\bar{k}$ with char $k\nmid |G|$.

(2) But the key needs $k=\bar{k}$:

$(\chi_W)_{W\in S_k(G)}$ spans whole $\mathcal{C}(G,k)$.

§4 The case $k = \mathbb{C}$

Prop k any field. $(V, \rho) \in R_k(G)$.

Then $\forall g \in G$, all eigenvalues of $\rho(g)$

are $|G|$th root of unity in $\bar{k}$.

Proof $g \in G$, $\rho(g^{|G|}) = \rho(g)^{|G|} = \mathrm{id}_V$

$\Rightarrow$ char poly $(\rho(g)) \mid X^{|G|} - 1$. $\square$

Cor $\forall g \in G$, $\chi_{V^*}(g) = \overline{\chi_V(g)}$ over $\mathbb{C}$.

Proof $g \in G$, $\lambda_1, \dots, \lambda_n$ = all eigenvals of $\rho(g)$.

$\Rightarrow$ $\chi_{V^*}(g) = \chi_V(g^{-1}) = \lambda_1^{-1} + \cdots + \lambda_n^{-1} = \bar{\lambda}_1 + \cdots + \bar{\lambda}_n$. $\square$

Cor Define Herm inner product on the fin-dim'l $\mathbb{C}$-v.s. $\mathcal{C}(G, \mathbb{C})$

by $\quad f_1 \cdot f_2 = \frac{1}{|G|} \sum_{g \in G} f_1(g) \cdot \overline{f_2(g)}$.

Then $(\chi_W)_{W \in S_{\mathbb{C}}(G)}$ = an orthonormal basis of $\mathcal{C}(G, \mathbb{C})$.

§5 Rep of a product of groups

$V_1 \in R_k(G_1)$, $V_2 \in R_k(G_2)$.

$\Rightarrow$ $V_1 \otimes_k V_2 \in R_k(G_1 \times G_2)$

with $(g_1, g_2)(v_1 \otimes v_2) = (g_1 v_1) \otimes (g_2 v_2)$.

Easy to show:

$$\chi_{V_1 \otimes_k V_2}(g_1, g_2) = \chi_{V_1}(g_1) \cdot \chi_{V_2}(g_2).$$

Thm $k = \bar{k}$. $V_1 \in R_k(G_1)$, $V_2 \in R_k(G_2)$.

(1) $V_1 \otimes_k V_2 \in S_k(G_1 \times G_2)$ irred

$\iff$ $V_1 \in S_k(G_1)$, $V_2 \in S_k(G_2)$ irred.

(2) Every irrep of $G_1 \times G_2$ is of form $V_1 \otimes_k V_2$.

<u>Proof</u> See irrep of symmetric grps / $\mathbb{C}$.

(using Young tableaux).