Abelian Sheaves

§1 Abelian Groups

Fix $\mathcal{C} = \mathrm{AbGrp}$ (but this can be generalized to any ab cat).

$f: A \to B \rightsquigarrow \ker f, \operatorname{im} f, \operatorname{coker} f$.

For $\cdots \to A_{i-1} \to A_i \to A_{i+1} \to \cdots$ (finite / infinite)

exact $\Leftrightarrow \operatorname{im}(A_{i-1} \to A_i) = \ker(A_i \to A_{i+1})$, $\forall i$

complex $\Leftrightarrow \operatorname{im}(A_{i-1} \to A_i) \subseteq \ker(A_i \to A_{i+1})$ i.e. $A_{i-1} \to A_i \to A_{i+1}$ is zero $\Big\}\ \forall i$

Diagram chasing:

(1) Five Lemma:

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d, "f_0"'] & A_1 \arrow[r] \arrow[d, "f_1"'] & A_2 \arrow[r] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"'] & A_4 \arrow[d, "f_4"] \\
B_0 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4
\end{tikzcd}$$

(a) f_1, f_3 mono & f_0 epi $\Rightarrow f_2$ mono

(b) f_1, f_3 epi & f_4 mono $\Rightarrow f_2$ epi.

opposite

(2) Snake Lemma:

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r] \arrow[d, "f_1"'] & A_2 \arrow[r] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"] & 0 & \text{exact} \\
0 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & 0 & \text{exact}
\end{tikzcd}$$

$\rightsquigarrow \exists\, \delta: \ker f_3 \to \operatorname{coker} f_1$ s.t. exact:

$$0 \to \ker f_1 \to \ker f_2 \to \ker f_3 \xrightarrow{\delta} \operatorname{coker} f_1 \to \operatorname{coker} f_2 \to \operatorname{coker} f_3 \to 0$$

(3) (Corollary) Short Five Lemma: same statement as in (2).
& diagrams commute.

$\rightsquigarrow f_2$ mono/epi $\Leftrightarrow f_1, f_3$ both are mono/epi.

§2 Exact Functors

(1) Additive functor: $F: \mathcal{C}_1 \to \mathcal{C}_2$ commutes w/ addition on $\mathrm{Mor}(\mathcal{C}_1)$.

$\rightsquigarrow$ preserves complexes & split-exactness

(but not exactness)

(2) Left-exact: $0 \to A_1 \to A_2 \to A_3$

$\rightsquigarrow$ $0 \to F(A_1) \to F(A_2) \to F(A_3)$

E.g. $\forall X \in \mathcal{C}$, $\mathrm{Hom}(X,-)$ covariant.

$\mathrm{Hom}(-,X)$ contravariant.

(3) Right-exact.

E.g. $\forall X \in \mathcal{C}$, $X \otimes (\cdot)$ covariant

$(\cdot) \otimes X$ contravariant

Prop $f^*: \mathcal{C}_1 \to \mathcal{C}_2$, $f_*: \mathcal{C}_2 \to \mathcal{C}_1$ covariant adj pair

$\Rightarrow$ f^* right-exact & f_* left-exact.

§3 Abelian Sheaves

$\mathcal{F} \in Sh_{Ab}(X)$. subsheaf: subgrp.

quotient sheaf: $(U \mapsto \mathcal{F}(U)/\mathcal{G}(U))^+$

stalk: $\mathcal{F}_x/\mathcal{G}_x$

also: ker / im / coker sheaf for $\phi: \mathcal{F} \to \mathcal{G}$.

Prop $\forall x \in X$, $(\ker\phi)_x = \ker(\phi_x)$, $(\mathrm{im}\,\phi)_x = \mathrm{im}(\phi_x)$, $(\mathrm{coker}\,\phi)_x = \mathrm{coker}\,\phi_x$

$\Rightarrow$ $\mathrm{im}\,\phi \cong \mathcal{F}/\ker\phi$, $\mathrm{coker}\,\phi \cong \mathcal{G}/\mathrm{im}\,\phi$.

Define $\Gamma(X,-) : Sh_{Ab}(X) \longrightarrow Abgrp$ global section functors

s.t. $\Gamma(X,\mathcal{F}) = \mathcal{F}(X)$

Prop left-exact ($\rightsquigarrow R^i\Gamma(X,-) = H^i(X,-)$ later)

§4 Abelian Categories

Construction Preadditive cat : $Hom(X,Y)$ to be ab grp.

$\rightsquigarrow$ Additive cat : $\oplus = \Pi$ on finite ones

$$\begin{tikzcd} \ker f \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & 0 \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d] \\ Y \arrow[r] & \operatorname{coker} f \end{tikzcd}$$

$\rightsquigarrow$ Preab cat : every morphism admits ker & coker.

$\rightsquigarrow$ Ab cat : f mono, $f = \ker(\operatorname{coker} f)$

g epi, $g = \operatorname{coker}(\ker g)$.

Freyd-Mitchell embedding thm:

$\mathcal{C}$ Small ab $\rightsquigarrow F : \mathcal{C} \to Mod_R$ (R not necessarily comm.)

fully faithful

i.e. can reduce ab cat to Mod_R.