Intersection cohomology is useless

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Explaination on "useless"

$H^*(\text{Compact Sh var}) \rightsquigarrow$ Same Galois

$H_c^*(\text{noncompact Sh var}) \rightsquigarrow$ repr.

§1 Introduction

Thm G reductive $/\mathbb{Q}$, X_G = Symmetric space of $G(\mathbb{R})$

$\Gamma\backslash X_G$, $\Gamma \subset G(\mathbb{Q})$ arith subgrp.

V alg rep of G $(/\mathbb{Q}$ or other num fields)

$\rightsquigarrow \mathcal{F}(V)$ locally const on $\Gamma\backslash X_G \leftarrow X_G$.

V^2

If $\Gamma\backslash X_G$ is a Shimura var,

then $\mathcal{F}(V)$ has an ℓ-adic version.

Now just look at Siegel case for simplicity.

$G = GSp_{2g}$, $g \geq 0$, $n \geq 3$.

$M_{g,n}$ = moduli space of principally polarized AVs with n level str $/\mathbb{Q}$

$\rightsquigarrow M_{g,n}$ (canonical model of) Shimura var for G.

non-cpt unless $G=0$ $(M_{g,n} = pt)$.

Fix V. $\rightsquigarrow H_c^*(M_{g,n,\bar{\mathbb{Q}}}, \mathcal{F}(V)) = H_c^*(M_{g,n}, V)$

$\circlearrowleft$ $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

$j : M_{g,n} \hookrightarrow \overline{M_{g,n}}$ minimal compactification.

$$IC(V) := (\underbrace{j_{!*}F(V)[\cdot]}_{\text{perv sheaf}})[\cdot] \leftarrow \text{s.t. concentrated in deg} \le 0.$$

self-dual

$$\text{where } \begin{tikzcd} j_!F(V) \arrow[r] \arrow[rr, bend right, leftrightarrow, "\text{dual}"'] & j_{!*}F(V) \arrow[r] & Rj_*F(V) \end{tikzcd}$$

Write $IC(V) = j_{!*}F(V)$ for cheating.

$\rightsquigarrow \; IH^*(M_{g,n}, V) := \mathbb{H}^*(\overline{M_{g,n,\bar{\mathbb{Q}}}}, IC(V))$.

Want to know $\overline{M_{g,n}} - M_{g,n}$ = the boundary

(in Siegel case) $= \coprod_{0 \le g' \le g-1} \coprod_{\text{finite}} M_{g',n}$. "Satake cpt"

$$\updownarrow$$

$$G \overset{\max}{\underset{\text{Levi}}{\supset}} \begin{pmatrix} \square_{g} & & 0 \\ & \square_{2g'} & \\ 0 & & \square_{g} \end{pmatrix} = GSp_{2g'} \subset GSp_{2g'} \times GL_{2g-2g'}$$

"Thm" In the Groth grp of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$-rep,

$$[IH^*(M_{g,n}, V)] - [H_c^*(M_{g,n}, V)] = \sum_{0 \le g' \le g-1} [H_c^*(M_{g',n}, V_{g'})]$$

virtual alg reps of $GSp_{2g'}$.

Actually, $[IC(V)] - [j_!F(V)] = \displaystyle\sum_{0 \le g' \le g-1} \sum_{\substack{i : M \hookrightarrow \overline{M}_{g,n} \\ \text{is } M_{g',n}}} [i_! F(W_M)]$

<u>Rmk</u> (1) Commutative diagram

$$\begin{tikzcd}
IC(V) \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & * \arrow[r, "+1"] \arrow[d] & {} \\
* \arrow[r] \arrow[d] & j_!F(V) \arrow[r] \arrow[d] & \square \arrow[r, "+1"] \arrow[d] & {} \\
\# \arrow[r] \arrow[d, "+1"'] & \square \arrow[r] \arrow[d, "+1"'] & \square \arrow[r, "+1"] \arrow[d, "+1"'] & {} \\
{} & {} & {} &
\end{tikzcd} \qquad (g=2)$$

$\square$ = some $\mathcal{F}(W)$, W = complex of repns.
(in dim g)

(2) The W_M are explicit

§2 Intersection complex

$\underline{\tau_{\le a}}$ K "complex of sheaves" $(D^b_c(X, \mathbb{Q}_\ell))$

$$\tau_{\le a}K \longrightarrow K \text{ s.t. } H^k(\tau_{\le a}K) \xrightarrow{\sim} H^k(K) \text{ if } k \le a$$
$$\& \ H^k(\tau_{\le a}K) = 0 \text{ if } k > a.$$

$$M_{g'} := \coprod (\text{strata} \simeq M_{g',n} \text{ in } M_{g,n})$$

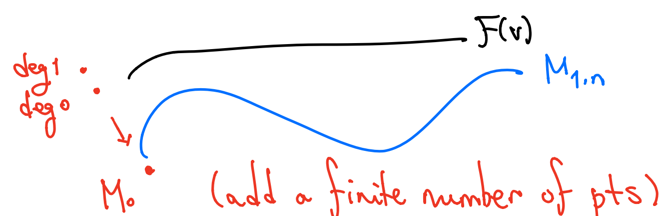
$$\rightsquigarrow \underbrace{M_{g,n} \amalg M_{g-1} \amalg \cdots \amalg M_{g'}}_{\overset{\shortparallel}{M_g}} \xrightarrow{j_{g'}} M_{g,n} \amalg \cdots \amalg M_{g'} \amalg M_{g'-1}$$
$$(1 \le g' \le g)$$

Write $j = j_1 \circ \cdots \circ j_g$.

$\underline{\text{Thm}}$ $IC(V) = \tau_{\le c_0} Rj_{1,*} \cdots \tau_{\le c_{g-1}} Rj_{g,*} \mathcal{F}(V)$

where $c_{g'} = \operatorname{codim} M_{g'} - 1$

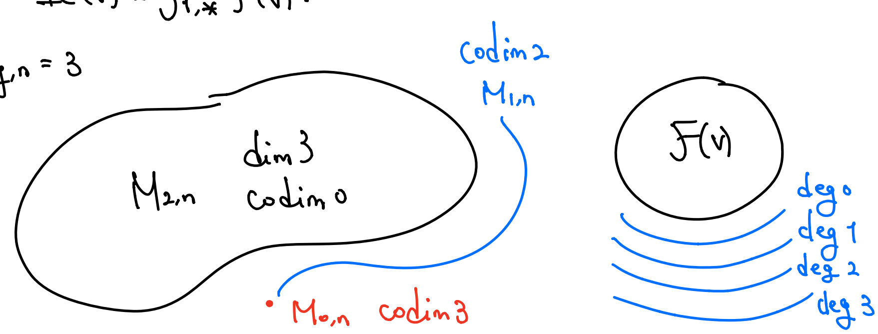
E.g. (1) $g = 1$, modular curve $M_{1,n}$:



Formula $\Rightarrow$ to get rid of deg 1.

$\Rightarrow IC(V) = j_{1,*}\mathcal{F}(V)$.

(2) $g = 2$, $\dim M_{g,n} = 3$



$$IC(V) = \tau_{\leq 2} Rj_{1,*} \underbrace{\tau_{\leq 0} Rj_{2,*} \mathcal{F}(V)}_{j_{2,*}\mathcal{F}(V)}$$

$$\tau_{\geq 1} Rj_{2,*}\mathcal{F}(V) \longrightarrow Rj_{2,*}\mathcal{F}(V) \longrightarrow \mathcal{F}(W) \xrightarrow{+1} \cdots$$

(may apply $Rj_{1,*}$ on it).

§3 Weighted cohomology

$$\underset{sm}{j : U} \longrightarrow \underset{\substack{sm\\ proper}}{X} \quad \rightsquigarrow \quad j_{!*}\mathbb{Q}_{\ell,U} = \mathbb{Q}_{\ell,X}.$$

<u>Q</u> Is there a resolution of singularities of $\overline{M_{g,n}}$

$$\begin{tikzcd} & \widetilde{M_{g,n}} \arrow[d, "\pi"] \\ M_{g,n} \arrow[ur, hook] \arrow[r, hook] & \overline{M_{g,n}} \end{tikzcd}$$

s.t. $IC(\mathbb{Q}_\ell) = R\pi_*\mathbb{Q}_\ell$?

<u>A</u> No (probably) : indeed unknown.

<u>But</u> $\exists$

$$\begin{tikzcd} & \widetilde{M_{g,n}(\mathbb{C})} \arrow[d, "\pi"] \\ M_{g,n}(\mathbb{C}) \arrow[ur, hook, "\tilde{j}"] \arrow[r, hook] & \overline{M_{g,n}(\mathbb{C})} \end{tikzcd}$$

"partial" resolution of singularities

$\exists$ truncation of $R\tilde{j}_*\mathcal{F}(V)$, $W(V)$

s.t. $R\pi_* W(V) = IC(V)$.

Here, $\widetilde{M_{g,n}(\mathbb{C})}$ = reductive Borel-Serre compactification (Zucker)

$W(V)$ = weighted Cohom complex (Goresky-Harder-MacPherson)

$$\rightsquigarrow \widetilde{M_{g,n}(\mathbb{C})} - M_{g,n}(\mathbb{C}) = \coprod_{\substack{M \text{ Levi}\\ \text{of } G/\mathbb{N}}} \coprod_{\text{finite}} \Gamma_M \backslash X_M$$

not smooth in any sense

but it has nice property

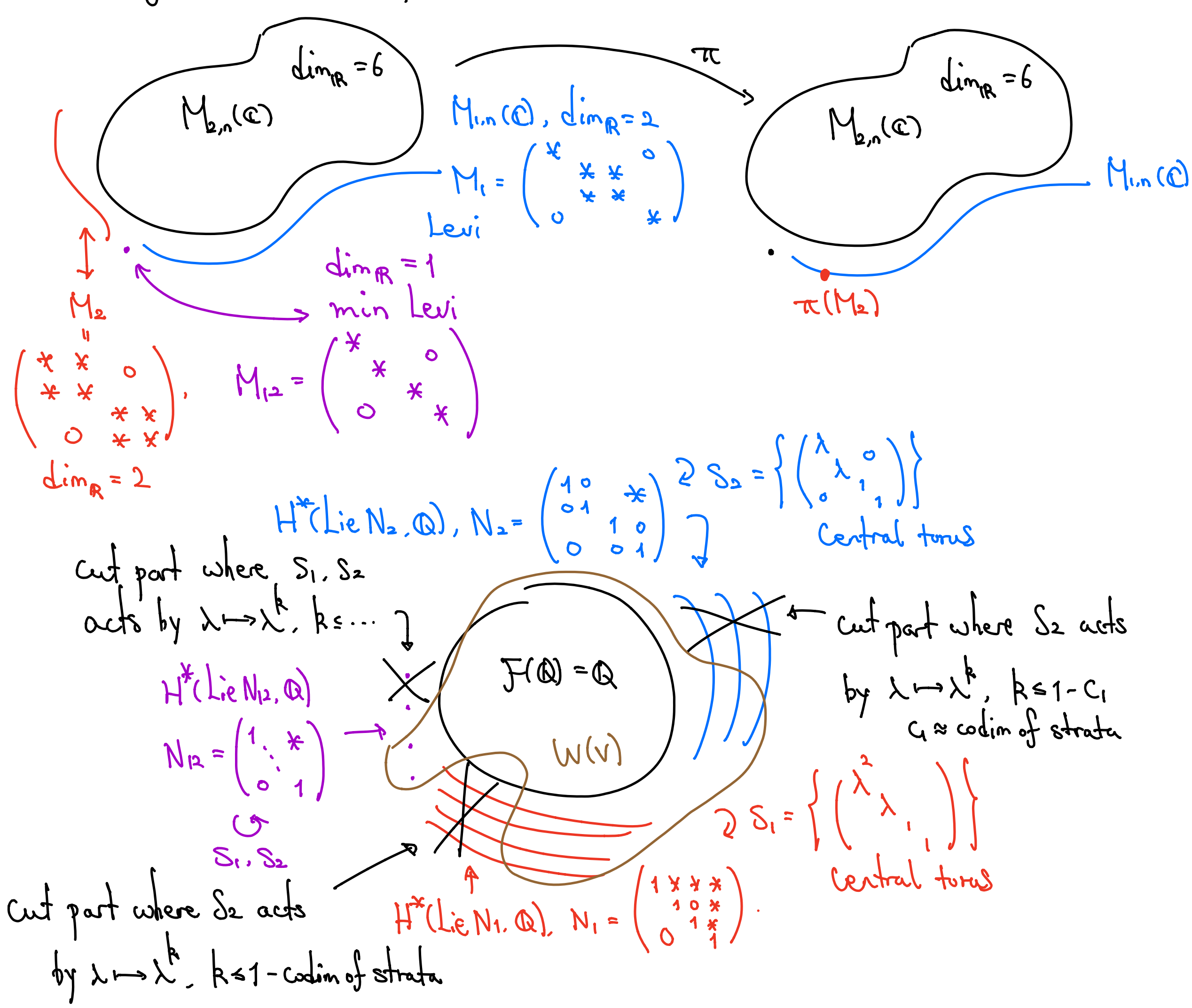
— sings are not too bad.

Look at one stratum M:

$$\begin{tikzcd} M \arrow[r, hook, "i"] \arrow[dr] & \widetilde{M_{g,n}(\mathbb{C})} & M_{g,n}(\mathbb{C}) \arrow[l, hook', "\tilde{j}"'] \\ & \bar{M} \arrow[u, hook, "\bar{i}"'] & \end{tikzcd}$$

Have $\bar{i}^* R\tilde{j}_* F(v) \simeq R i_* i^* R\tilde{j}_* F(v)$.

<u>When $g=2$</u>: 3 boundary strata



§4 Weighted cohomology (algebraic version)

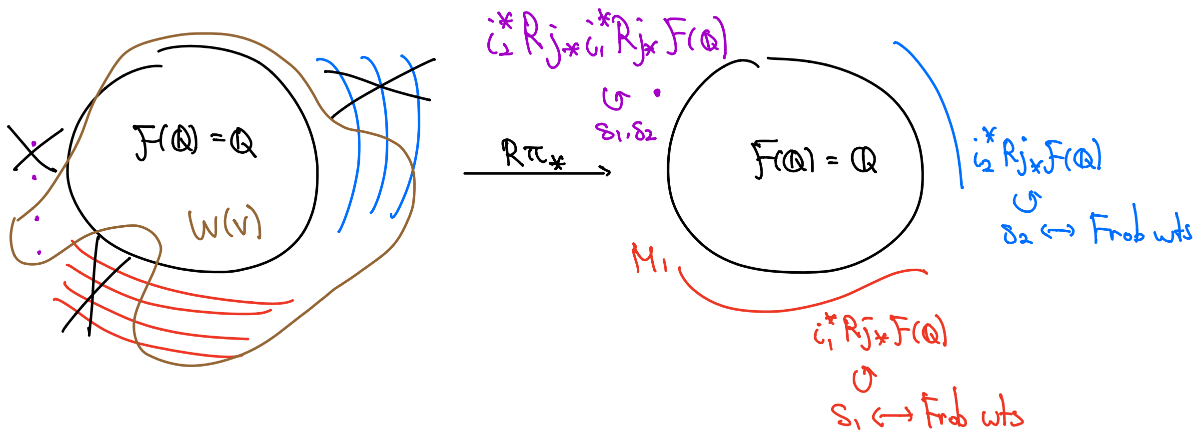
Notation (With $g=2$):

$$i_1 : M_{1,n}(\mathbb{C}) \hookrightarrow \overline{M_{2,n}}(\mathbb{C})$$

$$i_2 : \pi(M_2)(\mathbb{C}) \hookrightarrow \overline{M_{2,n}}(\mathbb{C})$$

0-dim stratum.

Idea To send everything about $W(v)$ above to $\overline{M_{2,n}}$



General philosophy

3 truncations: ① by degrees (not this at work).

② by tori weights

③ by Frobenius weights.