Moduli stacks of p-adic shtukas and
integral models of Shimura varieties
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<u>§1 Motivation</u> (function field)

$X/\mathbb{F}_p$ sm proj curve, geom conn.

S $\mathbb{F}_p$-scheme.

Def (Drinfeld) A (X-)shtuka of rk n over S with m legs is $((x_i : S \to X), \mathcal{E}, \varphi_{\mathcal{E}})$ $(i = 1, \cdots, m)$

where
- $\mathcal{E}$ rk n vec bundle on $X \times S$
- $\varphi_{\mathcal{E}} : \varphi_S^* \mathcal{E}|_{X_S \setminus \Gamma_{x_i}} \xrightarrow{\sim} \mathcal{E}|_{X_S \setminus \Gamma_{x_i}}$, $\varphi_S = id \times \mathrm{Frob}_S$

Let $S = \mathrm{Spec}\, k$, $k = \bar{k}$.

For each leg x $\rightsquigarrow$ cochar $\mu(t) = (t^{k_1}, \ldots, t^{k_n})$ which measures the failure of $\varphi_{\mathcal{E}}$ to be an isom on $\Gamma_x \subset X_S$.

If $t \in \hat{\mathcal{O}}_{X,x} \cong k[\![t]\!]$,

$\widehat{\varphi_S^* \mathcal{E}_x} \cong k[\![t]\!]^{\oplus n}$, $\varphi_{\mathcal{E}}^{-1}(\hat{\mathcal{E}}_x) \cong \bigoplus_{i=1}^{n} t^{k_i} k[\![t]\!]$

$\rightsquigarrow (\mu_1, \cdots, \mu_m)$.

For $G/\mathbb{F}_p$ $\rightsquigarrow$ G-shtukas of level K bounded by $(\mu_1, \cdots, \mu_m)$

$$\mathrm{Sht}_{G,(\mu_1,\cdots,\mu_m),K} \longrightarrow X^m \longleftarrow \bar{\eta}$$

with η generic pt, $\bar{\eta}$ geometric pt (specialized).

Can consider fiber of moduli $\mathrm{Sht}_{G,(\mu_1,\cdots,\mu_m),K,\bar{\eta}}$ at $\bar{\eta}$

$\rightsquigarrow$ its étale cohom carries autom + Gal actions

So can construct Langlands correspondence over $K(X)$
(GL_2 : Drinfeld ; GL_n : L. Lafforgue ;
G : autom $\Rightarrow$ Gal by V. Lafforgue).

§2 p-adic shtukas (Scholze, number field)

(char $= p$).

Perf_v site of perf'd spaces in char p + v-top.

w/ basis $\mathrm{Spa}(R, R^+)$, $R^+ = \prod_i C_i^+$

where $C_i = \bar{C_i}$ complete non-arch fields.

"product of geometric pts".

e.g. $\cdot$ $\mathrm{Spd}\,\mathbb{Z}_p : S \longmapsto \{S^\# \to \mathrm{Spa}\,\mathbb{Z}_p\}$

$\cdot$ $X/\mathbb{Z}_p$ scheme,

$X^\diamond : S = \mathrm{Spa}(R, R^+) \longmapsto \{(S^\#, \mathrm{Spec}\,R^+ \to X)\}$ $/ \mathrm{Spd}\,\mathbb{Z}_p$.

FF curve For $S = \mathrm{Spa}(R, R^+)$, $\mathcal{Y}_S = \mathrm{Spa}\, W(R^+) \setminus V([\varpi])$

ϖ p.u. $\updownarrow$ (analogues)

$S \times \mathrm{Spa}\,\mathbb{Z}_p \longleftrightarrow S \times \hat{X}_x$.

note Closed Cartier divisors of $\mathcal{Y}_S$ classifies untilts $S^\#$.

Def A p-adic shtuka of rk n over S with m legs is

$$((x_i : S \to \mathrm{Spd}\,\mathbb{Z}_p)_{i=1,\dots,m}, \mathcal{E}, \varphi_{\mathcal{E}})$$

- $\mathcal{E}$ rk n vec bundle on $\mathcal{Y}_S$
- $\varphi_{\mathcal{E}} : \varphi_S^* \mathcal{E}|_{\mathcal{Y}_S \setminus \bigcup_{i=1}^m S^{\# i}} \xrightarrow{\sim} \mathcal{E}|_{\mathcal{Y}_S \setminus \bigcup_{i=1}^m S^{\# i}}$ meromorphic

<u>Examples</u> (1) no legs. $(\mathcal{E}|_{\mathcal{Y}_S},\ \varphi_S^*\mathcal{E} \xrightarrow{\sim} \mathcal{E})$.

$\rightsquigarrow$ $\mathcal{Y}_S \setminus V(p) \circlearrowleft \varphi_S$

$\rightsquigarrow$ $X_S := (\mathcal{Y}_S \setminus V(p)) / \varphi_S^{\mathbb{Z}}$ FF curve

$\rightsquigarrow$ $(\mathcal{E}, \varphi_{\mathcal{E}})|_{\mathcal{Y}_S \setminus V(p)}$ descends to a vec bundle

$\updownarrow$ $\bar{\mathcal{E}}$ on FF curve, geom fiberwise trivial

$\mathbb{Q}_p$-local system on S.

<u>Upshot</u>

$$\{\text{shtukas } /S \text{ with no legs}\}$$
$$\updownarrow 1\text{-}1$$
$$\{\mathbb{Z}_p\text{-local systems on } S\}.$$

(2) 1-leg.

Consider $(S^{\#}, \mathcal{E}, \varphi_{\mathcal{E}} : \varphi_S^*\mathcal{E} \dashrightarrow \mathcal{E})$

iso away from $S^{\#}$.

① $S^{\#}/\mathbb{Q}_p$: $\{\text{shtukas } /S \text{ with } 1 \text{ leg at } S^{\#}\}$

$$\updownarrow 1\text{-}1$$

$$\left\{ (\mathbb{T}, \Xi \subset \mathbb{T} \otimes_{\mathbb{Z}_p} \mathbb{B}_{dR,S^{\#}}) \;\middle|\; \begin{array}{l} \mathbb{T} : \mathbb{Z}_p\text{-local system on } S^{\#} \\ \Xi : \mathbb{B}^{+}_{dR,S^{\#}}\text{-lattice} \end{array} \right\}$$

② $S = \mathrm{Spa}(R, R^{\circ})$, $R^{\circ} = \prod_i \mathcal{O}_{C_i}$, $C_i = \bar{C_i}$ complete non-arch field.

$$\{\text{shtukas } /S \text{ with } 1 \text{ leg at } S^{\#}\}$$
$$\updownarrow 1\text{-}1$$
$$\{(M, \varphi_M : (\varphi_S^* M)[\tfrac{1}{\xi}] \xrightarrow{\sim} M[\tfrac{1}{\xi}]) \mid M : \text{finite free } W(R^{+})\text{-mod}\}$$

$(\xi) := \ker(W(R^{+}) \xrightarrow{\theta} R^{+\#})$

"Breuil-Kisin-Fargues mod over S with 1 leg at $S^{\#}$."

The equivalence is due to Fargues for $S = \mathrm{Spa}(C, \mathcal{O}_C)$ & Gleason-Ivanov (2023) for general S.

Can restrict both sides to full subcats

$$\left\{ \begin{array}{c} \text{shtukas } /S \text{ with 1 leg bounded by} \\ \text{minuscule cochar } \mu \text{ at } S^\# \end{array} \right\}$$

$$\Big\updownarrow \text{1-1}$$

$$\{ M \subset \varphi_M(M) \subset \tfrac{1}{\xi} \cdot M \}$$

$\Big\updownarrow$ Further: Scholze-Weinstein + prism Dieudonne theory

$$\{ p\text{-div grps over } \operatorname{Spf} R^{\#+} \}.$$

Both cases are useful in moduli of p-adic shtukas w/ 1 leg.

§3 Moduli of p-adic shtukas

$G/\mathbb{Q}_p$. For simplicity, $\mathcal{G}/\mathbb{Z}_p$ reductive

Def $\mathrm{Sht}_{\mathcal{G}} : \mathrm{Perf}/\operatorname{Spd}\mathbb{Z}_p \longrightarrow \text{Groupoids}$

$$(S \to \operatorname{Spd}\mathbb{Z}_p) \longmapsto \left\{ \begin{array}{l} \text{ob}: (\mathcal{E}, \varphi_{\mathcal{E}}) \ \mathcal{G}\text{-shtukas with} \\ \qquad\qquad \text{1 leg at } S^\# \\ \text{Mor}: \mathcal{E} \xrightarrow[\sim]{f} \mathcal{E}', \ \varphi_{\mathcal{E}'} \circ \mathrm{Fr}_S^* f = f \circ \varphi_{\mathcal{E}} \end{array} \right\}.$$

Prop (1) $\mathrm{Sht}_G \longrightarrow \operatorname{Spd}\mathbb{Z}_p$ has proper diagonal

$\mu \in X_*^+(T)$, $T \subset B \subset G_{\bar{\mathbb{Q}}_p}$, $\underbrace{[\mu]}/\, E \ (/\mathbb{Q}_p)$

$\left(\begin{array}{c} G_{\bar{\mathbb{Q}}_p}\text{-conj class of } \mu \text{ def'd on reflex} \\ \text{field } E \text{ finite } /\mathbb{Q}_p \end{array} \right)$

$\rightsquigarrow \mathrm{Sht}_{\mathcal{G}, \le \mu} \longrightarrow \operatorname{Spd}\mathcal{O}_E$ quasi-compact.

(2) Have an isom

$$\mathrm{Sht}_{\mathcal{G}, \mathbb{Q}_p} \xrightarrow{\ \sim\ } [\mathrm{Gr}_G / \underline{\mathcal{G}(\mathbb{Z}_p)}]$$

where $\left(S = \mathrm{Spa}(R,R^+) \mapsto G(B_{dR}(R^{\#}))/G(B^+_{dR}(R^{\#}))\right)^{\text{sheaf'n}}$

$$\begin{array}{ccc} S = \mathrm{Spa}(R,R^+) & \longmapsto & G(B_{dR}(R^{\#}))/G(B^+_{dR}(R^{\#})) \\ \downarrow & & \downarrow \ \mathrm{Gr}_{G/\mathrm{Spd}\,\mathbb{Q}_p} \\ \mathrm{Spd}\,\mathbb{Q}_p & & \end{array}$$

$\{\mathcal{E}_{triv} \dashrightarrow \mathcal{E} : \text{mero \& iso away from } S^{\#}\}$.

(3) Have stack of G-torsors on X_{FF}:

$$\mathrm{Sht}_{\mathcal{G}} \longrightarrow \mathrm{Bun}_{G,\mathbb{F}_p}$$

$$(S^{\#}, \mathcal{E}, \varphi_{\mathcal{E}}) \longmapsto (\mathcal{E}, \varphi_{\mathcal{E}})|_{\mathcal{Y}_{[r,\infty)}(S) \subset \mathcal{Y}_S} \quad (\rightsquigarrow \bar{\mathcal{E}})$$

where $\mathcal{Y}_{[r,\infty)}(S) := \{0 < |[\varpi]| \leq |p|^r,\ r \gg 0\}$,

s.t. $\mathcal{Y}_{[r,\infty)}(S) \cap S^{\#} = \phi$.

Rmk fibres on $\mathrm{Sht}_{\mathcal{G},\mu}$ are (integral model of) local Shimura varieties.

§4 Integral models of Shimura varieties

$(G/\mathbb{Q},\ X = \{G(\mathbb{R})\text{-conj classes of } \mathrm{Res}^{\mathbb{C}}_{\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}\})$

+ axioms.

Example (1) Siegel moduli:

$\mathcal{A}_{g,K}$ principally polarized abelian schemes of dim g with K-level / isom $(GSp_{2g}, \mathcal{H}_g^{\pm})$, $K \subset GSp_{2g}(\mathbb{A}_f)$.

(2) Hodge type $(G,X) \hookrightarrow (GSp_{2g}, \mathcal{H}_g^{\pm})$.

$GSp_{2g} \curvearrowright V^{std} =: V$, $T \subset V^{\otimes}$ tensors

$G = \bigcap_{t \in T} GL_t(V)$.

$(G,X) \rightsquigarrow [\mu]/E_0$ reflex field, $K \subset G(\mathbb{A}_f)$

$\mathrm{Sh}_{K,E_0} = \{(\underset{\substack{\uparrow \\ \text{p.p.a.v.}}}{A}, \underset{\substack{\uparrow \\ \text{Hodge tensors}}}{(t_\alpha)}, K)\}$ / isom.

But this moduli interpretation only works in char 0.

For $v | p$, $E = E_{0,v}$, $\mathrm{Sh}_{K,E} \rightsquigarrow \mathscr{S}_K / \mathcal{O}_E$.

Let K_p hyperspecial / parahoric.

Kisin, Kisin-Pappas $\rightsquigarrow \mathscr{S}_K$.

<u>Observation</u> $(\mathrm{Sh}_{K,E})^\diamond$ supports a $G_{\mathbb{Q}_p}$-shtuka $\longleftrightarrow (\mathbb{T}, \Xi)$

$$\mathbb{T} = T_p A$$

$$\rightsquigarrow \quad (\mathrm{Sh}_{K,E})^\diamond \xrightarrow{\mathcal{P}_K} [\mathrm{Gr}_G / \underline{K_p}].$$

<u>Thm</u> (Pappas-Rapoport)

K_p parahoric with $\mathcal{G} / \mathbb{Z}_p$,

$\exists \quad \mathscr{S}_K^\diamond \xrightarrow{\pi_{crys}} \mathrm{Sht}_{\mathcal{G},\mu}$ extending $\mathcal{P}_K$.

("univ" in formal nbhd of closed pts on $\widehat{\mathscr{S}}_K$.)

<u>Conj</u> (Scholze) $\exists\ \mathrm{Igs}_{K^p} / \mathbb{F}_p$ + maps sitting in a Cartesian diagram

$$\begin{tikzcd} \mathscr{S}_K^\diamond \arrow[r, "\pi_{crys}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sht}_{\mathcal{G},\mu} \arrow[d] \\ \mathrm{Igs}_{K^p} \arrow[r, "\overline{\pi}_{HT}"] & \mathrm{Bun}_G. \end{tikzcd}$$

<u>Partial results</u>

(1) (Zhang) PEL type AC, K_p hyperspecial

+ minimal cpt'n.

(2) (Joint with Daniels - van Hoften - Kim, in progress)

Hodge type, K_p parahoric,

- functoriality.
- version of perfect schemes
- torsion vanishing for compact Shimura vars