Ultraproduct cohomology and decomposition theorem

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(Joint w/ A. Cadoret)

$X / k = \bar{k}$ var, $\quad H^i(X, \mathbb{Z}_\ell)$ fin gen $\mathbb{Z}_\ell$-mod, $\ell \neq \mathrm{char}\, k = p$.

- $\mathrm{char}\, k = 0$, $\quad k = \mathbb{C}$,

  Then $\quad \underbrace{H^i(X(\mathbb{C}), \mathbb{Z})}_{\text{f.g. } \mathbb{Z}\text{-mod}} \otimes \mathbb{Z}_\ell \cong H^i(X, \mathbb{Z}_\ell)$.

  Consequences: $\cdot$ $\dim H^i(X, \mathbb{Q}_\ell)$ indep of ℓ

  $\cdot$ $H^i(X, \mathbb{Z}_\ell)$ torsion-free for $\ell \gg 0$.

- $\mathrm{char}\, k = p > 0$.

  <u>Thm</u> $\quad X$ proper sm

  $\cdot$ (Deligne) $\dim H^i(X, \mathbb{Q}_\ell)$ indep of ℓ

  $\cdot$ (Gabber) $H^i(X, \mathbb{Z}_\ell)$ torsion-free for $\ell \gg 0$.

  <u>Thm 1</u> $\quad A$ proper

  $\cdot$ (Gabber) $\dim IH^i(X, \mathbb{Q}_\ell)$ indep of ℓ

  $\cdot$ (Cadoret-Zheng) $IH^i(X, \mathbb{Z}_\ell)$ torsion-free for $\ell \gg 0$.

<u>Reformulation</u> ($\mathrm{char}\, k = p > 0$)

$$\exists \text{ fin gen } \mathbb{Z}\text{-mod } M \text{ s.t. } IH^i(X, \mathbb{Z}_\ell) \simeq M \otimes \mathbb{Z}_\ell$$

non-canonical

Let $A := A_{\mathbb{Q}}^{p,\infty} = \prod_{\ell \neq p} \mathbb{Z}_\ell \otimes \mathbb{Q} = \underbrace{\hat{\mathbb{Z}}^{(p)}}_{= \varprojlim_{p \nmid n} \mathbb{Z}/n} \otimes \mathbb{Q}$

Then $\quad IH^i(X, \mathbb{A}) = \prod_{\ell \neq p} IH^i(X, \mathbb{Z}_\ell) \otimes \mathbb{Q}$

f.g. <u>free</u> $\mathbb{A}$-mod of finite rk.

<u>Thm 1$^+$</u> For $X \xrightarrow[\text{curve}]{\text{proper}} C$, Same statements for IH^* & IH_c^*

For $\underset{\text{sm}}{X} \xrightarrow[\text{curve}]{\text{proper}} C$, Same statements for H^* & H_c^*

<u>Standard Conj</u> $\quad \sim_{\text{num}} = \sim_{\ell\text{-hom}}$.

van Dobben - de Bruyn:

$\sim_{\ell\text{-hom}}$ indep of $\ell \iff \dim H^i(X, \mathbb{Q}_\ell)$ indep of ℓ,

$\forall X$ sm quasi-proj.

<u>Decomposition thm</u> Let $f: X \to Y$.

Leray spectral seq:

$$(*) \quad E_2^{p,q} = H^p(Y, R^q f_* \mathbb{Q}_\ell) \Rightarrow H^{p+q}(X, \mathbb{Q}_\ell).$$

<u>Thm</u> (Deligne) f proj sm & Y sm.

Then $(*)$ degenerates at E_2,

$$H^n(X, \mathbb{Q}_\ell) = \bigoplus_{p+q=n} H^p(Y, R^q f_* \mathbb{Q}_\ell)$$

w/ $\quad R f_* \mathbb{Q}_\ell \simeq \bigoplus_q R^q f_* \mathbb{Q}_\ell[-q]$.

and each $R^q f_* \mathbb{Q}_\ell$ semisimple $\mathbb{Q}_\ell$-local system.

<u>Perverse sheaves</u> $\quad K \in D_c^b(X, \mathbb{Q}_\ell)$,

$\forall x \in X, \quad i_x : x \to X, \quad d_x = \dim \overline{\{x\}}$,

get $\quad i_x^* K \in D^{\leq -d_x}, \quad R i_x^! K \in D^{\geq d_x}$.

Intersection Complex: U sm $\hookrightarrow X$ w/ $\mathcal{V}$ loc sys on U.

$\rightsquigarrow IC_X(\mathcal{V})$ on X

s.t. $IC_X(\mathcal{V})|_U = \mathcal{V}[\dim U]$

& $\forall x \in X \setminus U$, $i_x^* K \in D^{\leq -d_x - 1}$, $R i_x^! K \in D^{\geq d_x + 1}$.

Thm (BBDG) $f: X \to Y$ proper.

Then $Rf_* \mathbb{Q}_\ell \simeq \bigoplus_q {}^pR^q f_* \mathbb{Q}_\ell[-q]$.

& ${}^pR^q f_* \mathbb{Q}_\ell \simeq \bigoplus_\alpha IC_{\bar{Y}_\alpha}(\mathcal{V}_\alpha)$. $Y = \coprod_\alpha Y_\alpha$ stratification.

where each $\mathcal{V}_\alpha$ semisimple $\mathbb{Q}_\ell$-loc sys.

Thm 2 (Cadoret-Zheng) $X \xrightarrow{f} Y$ proper, $\ell \gg 0$.

Then $Rf_* \mathbb{Z}_\ell = \bigoplus_q R^q f_* \mathbb{Z}_\ell[-q]$,

$R^q f_* \mathbb{Z}_\ell \simeq \bigoplus_\alpha IC_{\bar{Y}_\alpha}(\mathcal{M}_\alpha)$,

torsion-free perverse sheaf.

where $\mathcal{M}_\alpha \simeq \bigoplus_j S_{\alpha j}$, $S_{\alpha j}$ torsion-free $\mathbb{Z}_\ell$-loc sys.

s.t. mod ℓ, $S_{\alpha j} \otimes \mathbb{F}_\ell$ is simple.

Cor Let $X \xrightarrow{f} Y$ alteration, X sm.

Then IC_Y is a direct summand of $Rf_* \mathbb{Z}_\ell$.

$$0 \to H^i(X, \mathbb{Z}_\ell) \otimes \mathbb{F}_\ell \to H^i(X, \mathbb{F}_\ell) \to \mathrm{Tor}_1(\mathbb{F}_\ell, H^{i+1}(X, \mathbb{Z}_\ell)) \to 0$$

Fact For $\ell \gg 0$, $\dim H^i(X, \mathbb{F}_\ell) = \dim H^i(X, \mathbb{Q}_\ell)$

$\iff H^i(X, \mathbb{Z}_\ell)$, $H^{i+1}(X, \mathbb{Z}_\ell)$ torsion-free.

Logics : Some existence for $\ell \gg 0$ $\rightsquigarrow$ ultraproduct $\mathcal{L} = \{\ell \neq p\}$.

<u>Recall</u> Ultrafilter on $\mathcal{L}$: $\mathcal{U} \subsetneq \mathrm{PowerSet}(\mathcal{L})$

s.t. $\cdot$ $A \in \mathcal{U}$, $B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$.

$\cdot$ $A \in \mathcal{U}$, $A \subset B \subset \mathcal{L} \Rightarrow B \in \mathcal{U}$.

$\cdot$ $\forall A \subset \mathcal{L}$, $A \in \mathcal{U}$ or $\mathcal{L} \setminus A \in \mathcal{U}$.

$\rightsquigarrow$ can assign $\mathcal{L} \longrightarrow \{0,1\}^{\mathcal{L}}$

to know if $A \in \mathcal{U}$ or not.

<u>Construction</u>

$$\begin{array}{ccl} \mathbb{A} & \longrightarrow & \underbrace{\prod_{\ell \neq p} \mathbb{F}_\ell \otimes \mathbb{Q}}_{\text{ring, Krull dim} = 0} \\ \downarrow & & \\ \mathbb{Q}_\ell & & \end{array}$$

$\rightsquigarrow$ each max ideal of it gives

$$\prod_{\ell \neq p} \mathbb{F}_\ell \otimes \mathbb{Q} \twoheadrightarrow \underset{\text{res field}}{\mathbb{Q}_{\mathcal{U}} = \prod_\ell \mathbb{F}_\ell / \sim_{\mathcal{U}}}$$

where $(a_\ell) \sim (b_\ell) \iff \{\ell \mid a_\ell = b_\ell\} \in \mathcal{U}$.

e.g. $\mathcal{U} = \{A \mid \ell \in A \subseteq \mathcal{L}\}$, $\mathcal{L} \subset \mathcal{L} \cup \underset{\{\text{non-principal ultrafilter}\}}{\underset{\shortparallel}{\mathcal{U}}}$ $\quad \mathrm{Card}(\mathcal{U}) = 2^{c}$.

<u>note</u> For V_ℓ an $\mathbb{F}_\ell$-v.s., $\prod_\ell V_\ell / \sim_{\mathcal{U}}$ $\mathbb{Q}_{\mathcal{U}}$-v.s.

& $\dim_{\mathbb{Q}_{\mathcal{U}}} \prod_\ell V_\ell / \sim_{\mathcal{U}} = d$, $\forall \mathcal{U}$.

$\iff \dim_{\mathbb{F}_\ell} V_\ell = d$, $p \gg 0$.

<u>Def</u> Ultraproduct cohom:

$$H^i(X, \mathbb{Q}_{\mathcal{U}}) := \prod_\ell H^i(X, \mathbb{F}_\ell) / \sim_{\mathcal{U}}.$$

Thm 1' $\dim_{\mathbb{Q}_t} IH^i(X, \mathbb{Q}_t)$ indep of $t \in \mathcal{L} \cup \mathcal{U}$.

Consider $\prod_\ell Sh_c(X, \mathbb{F}_\ell)/\sim$, $\mathcal{F}_\ell \in Sh_c(X, \mathbb{F}_\ell)$.

quasi-tame condition of $\mathcal{F}_\ell$:

$$\begin{array}{ccc} Y & \overset{j}{\hookrightarrow} & \bar{Y} \\ f \big\downarrow \text{alteration} & & \\ X & - \mathcal{F}_\ell & \end{array}$$

require that $j_! f^* \mathcal{F}_\ell$ tame, $\forall \ell$.

Def $D^b_c(X, \mathbb{Q}_u) := D^b_{c,qt}(X, \mathbb{F}_\ell)/\sim_u$.

Thm (Orgogozo) stable under 6 functors.

$X/\mathbb{F}_q$ sm. $\pi_1^{geom}(X) \longrightarrow GL_n(\bar{\mathbb{Q}}_\ell) \rightsquigarrow G_{geom}$.

Thm (Grothendieck) $\mathrm{rad}(G^\circ_{geom})$ unipotent.

$$\begin{array}{ccccc} \pi_1^{geom}(X') & \longrightarrow & G^\circ_{geom}(\bar{\mathbb{Q}}_\ell) & & X' \\ \big\downarrow & & \big\downarrow & & \big\downarrow \text{ finite étale.} \\ \pi_1^{geom}(X) & \longrightarrow & G_{geom}(\bar{\mathbb{Q}}_\ell) & & X \end{array}$$

Thm (Drinfeld) $\pi_1^{geom,t}(X)$ f.g. profin grp.

Thm (Nikolov – Segal)

π f.g. profin grp

$\cup$

H, $[\pi : H] < \infty \Rightarrow H \subset \pi$ open.

(pf need classification of fin simple grps).