Hecke orbits on Shimura varieties of Hodge type

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(Joint with D'Addezio)

§ Introduction

Let k alg closed.

$\mathcal{A}_g$ moduli of ppav.s $(A, \lambda : A \xrightarrow{\sim} {}^tA)/k$ of dim g.

$\rightsquigarrow$ $\mathcal{A}_g$ sm of dim $\frac{1}{2}g(g+1)$.

Def We say $x, y \in \mathcal{A}_g(k)$ are <u>isogeneous</u> if there is a surjective homomorphism

$$A_x \xrightarrow{f} A_y$$

s.t.

$$\begin{tikzcd} A_x \arrow[r, two heads, "f"] \arrow[d, "c\cdot\lambda_x"'] \arrow[dr, phantom, "\circlearrowright"] & A_y \arrow[d, "\lambda_y"] \\ A_x^t & A_y^t \arrow[l, "f^\vee"] \end{tikzcd}$$

for some $c \in \mathbb{Q} \setminus \{0\}$.

$\rightsquigarrow$ Get equiv classes $I_x \subset \mathcal{A}_g(k)$ (countable).

Question (Oort, 93)

What does I_x look like?

How does it lie inside $\mathcal{A}_g(k)$?

[What's its Zariski closure?]

<u>$g=1$, $k=\mathbb{C}$</u> $\quad \mathcal{A}_1(\mathbb{C}) \leftarrow GL_2(\mathbb{C}) \backslash \mathbb{H}^{\pm}$ $\quad$ here $\mathbb{H}^{\pm} = \mathbb{C} - \mathbb{R}$.

$GL_2(\mathbb{R})$ acts as $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = (a\tau + b)\cdot(c\tau+d)^{-1}$.

& $\tau \in \mathbb{C} \rightsquigarrow$ ell curve $E_\tau = \mathbb{C}/\Lambda_\tau$, $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$.

<u>Exercise</u> For $x \in \mathcal{A}_1(\mathbb{C})$ with lift $\tau \in \mathbb{H}^\pm$,

we have $\mathcal{I}_x = GL_2(\mathbb{Q}) \cdot \tau / GL_2(\mathbb{Z})$.

<u>Consequence</u> $\mathcal{I}_x \subset \mathcal{A}_1(\mathbb{C})$ is dense in analytic top.

($GL_2(\mathbb{Q}) \subset GL_2(\mathbb{R})$ dense).

Similar considerations apply to general g for $k = \mathbb{C}$.

<u>Lefschetz principle</u>:

$\mathcal{I}_x$ Zariski dense if char $k = 0$.

<u>$g = 1$, $k = \overline{\mathbb{F}_p}$</u> Curve $\mathcal{A}_{1,\overline{\mathbb{F}_p}} = \underbrace{\mathcal{A}^{ord}_{1,\overline{\mathbb{F}_p}}}_{\text{open}} \cup \underbrace{\mathcal{A}^{ss}_{1,\overline{\mathbb{F}_p}}}_{\text{closed (finite)}}$

for $x \in \mathcal{A}^?_1(\overline{\mathbb{F}_p})$ we have $\mathcal{I}_x \subset \mathcal{A}^?_1(\overline{\mathbb{F}_p})$.

Moreover, $\mathcal{A}^{ss}_1(\overline{\mathbb{F}_p}) \subset \mathcal{I}_x$ (finite set) for $x \in \mathcal{A}^{ss}_1(\overline{\mathbb{F}_p})$,

$x \in \mathcal{A}^{ord}_1(\overline{\mathbb{F}_p})$ has $\mathcal{I}_x$ infinite, thus dense.

<u>General g</u> $\exists$ a stratification

$$\mathcal{A}_{g,\overline{\mathbb{F}_p}} = \bigcup_{\substack{b \text{ Symm} \\ \text{Newton polygon}}} \mathcal{A}^b_{g,\overline{\mathbb{F}_p}}$$

For $x \in \mathcal{A}_g(\mathbb{F}_q)$, we have

$$\mathrm{Frob}_q \curvearrowright H^1_{\text{ét}}(\mathcal{A}_{g,\overline{\mathbb{F}_p}}, \mathbb{Q}_\ell), \quad \ell \neq p$$

and char poly $\in \mathbb{Q}[T]$ determines b_x.

$\Rightarrow$ ($x \in \mathcal{A}^b_g(\overline{\mathbb{F}_p}) \Rightarrow \mathcal{I}_x \subseteq \mathcal{A}^b_g(\overline{\mathbb{F}_p})$).

Can also look at isogeny class of $A_x[p^\infty]$,
the p-div grp associated to A_x.

<u>Conj</u> (Oort, 1995)
For $x \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$, we have $\mathcal{I}_x \subset \mathcal{A}_g^{b_x}(\overline{\mathbb{F}}_p)$ Zariski dense.

<u>Thm</u> (Chai, 1995) True for b_x ordinary.

<u>Thm</u> (dAvH) If $p \geq 2g$, then the conj holds.
($p \geq$ # slopes of distinct b_x).

<u>Rmk</u> Chai-Oort have announced a proof for all p.

<u>Rmk</u> We also have results for Shimura vars of Hodge type at good primes $p \geq$ Coxeter number.

<u>Issue</u> $\mathcal{A}_g^b$ are quite singular,
so don't want to use them.
Instead, work with central leaves.

$$\begin{array}{ccc} \mathcal{I}_x & \subset & \mathcal{A}_g^{b_x} \\ \cup\!\vert & & \cup\!\vert \\ \mathcal{I}_x^p & \subset & C_x \end{array}$$

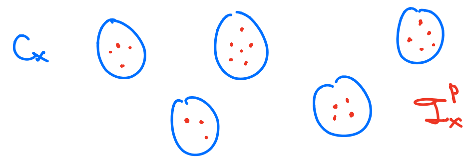
where $C_x(\overline{\mathbb{F}}_p) = \{ y \in \mathcal{A}_g^{b_x}(\overline{\mathbb{F}}_p) \mid (A_y[p^\infty], \lambda_y) \xrightarrow{\sim} (A_x[p^\infty], \lambda_x) \}$.

<u>Conj</u> (Oort 95) $\mathcal{I}_x^p \subset C_x$ is Zariski dense.

<u>Thm</u> If $p \geq \#(\text{slopes of } b_x)$, then this holds.

<u>Same remarks apply</u>

I_x^p is genuinely discrete (I_x contains $\mathbb{P}^1(\overline{\mathbb{F}}_p)$).



<u>Thm</u> (Kret-Shin) $I_x^p \twoheadrightarrow \pi_0(C_x)$, $\forall x$ surjective.

§ <u>Strategy</u> (due to Chai-Oort)

Take $x \in A_g(\overline{\mathbb{F}}_p)$, $Z := \overline{I_x^p} \subset C_x$.

(1) Prove "big monodromy results" for $A \to Z$.

- ℓ-adic monodromy (Chai, vH):
- Can use results of d'Addezio to get big p-adic monodromy.

$\rightsquigarrow$ using d'Addezio + dAvH

$\Rightarrow$ For smooth $z \in Z$, the p-adic monodromy of $A[p^\infty]$ over $\mathrm{Spf}\, \hat{\mathcal{O}}_{Z,z}$ is "big"

(as big as the monodromy over $\hat{\mathcal{O}}_{C_x,z}$.)

(2) Show that for $z \in Z$ as above,

we have that

$$\mathrm{Spf}\, \hat{\mathcal{O}}_{Z,z} \subset \mathrm{Spf}\, \hat{\mathcal{O}}_{C_x,z}.$$

is "strongly Tate-linear".

(e.g. if z is ordinary then $\mathrm{Spf}\, \hat{\mathcal{O}}_{C_x,z}$ is a

Serre-Tate torus, and we prove $\mathrm{Spf}\,\hat{\mathcal{O}}_{Z,z}$
is a subtorus $\hat{\mathbb{G}}_m^N \subset \hat{\mathbb{G}}_m^{\frac{g(g+1)}{2}}$.)

<u>Pf of this</u> Uses the "infinitesimal" Hecke action.

Stabilizer of z in $GSp_{2g}(\mathbb{A}_f^p)$ acts on $\mathrm{Spf}\,\hat{\mathcal{O}}_{Z,z} \subset \mathrm{Spf}\,\hat{\mathcal{O}}_{C_x,z}$.

(3) Prove strongly Tate-linear subvarieties have small monodromy.
Here we use Cartier-Witt stacks of Bhatt-Lurie.

*<u>Meaning of strongly Tate linear</u>

$$\begin{array}{ccc} QISoG(A_x[p^\infty], \lambda) & \curvearrowright & \mathrm{Spf}\,\hat{\mathcal{O}}_{Ig,\tilde{z}} \\ & & \downarrow \\ & & \mathrm{Spf}\,\hat{\mathcal{O}}_{C_x,z} \end{array}$$

Identifies $\varprojlim_{\varphi} \mathrm{Spf}\,\hat{\mathcal{O}}_{C_x,z} \xrightarrow{\sim} QISoG(A_x[p^\infty], \lambda)$

Can ask $\varprojlim_{\varphi} \mathrm{Spf}\,\hat{\mathcal{O}}_{Z,z}$ to be a subgrp

<u>Input</u> $QISoG(-)^\diamond \xrightarrow{\sim} \widetilde{GSp}_{b_x}^{>0\,\diamond}$ (Fargues-Scholze).

& STL $\Leftrightarrow$ $(\varprojlim_{\varphi} \hat{\mathcal{O}}_{Z,z})^\diamond \subset \widetilde{GSp}_{b_x}^{>0}$.
is a "sub-unipotent grp diamond".

&
$$\begin{array}{ccc} & & \mathrm{Spf}\,\hat{\mathcal{O}}_{Ig,\tilde{z}} \\ & & \downarrow \\ \mathrm{Spf}\,\hat{\mathcal{O}}_{Z,z} & \hookrightarrow & \mathrm{Spf}\,\hat{\mathcal{O}}_{C_x,z} \end{array}$$

can also be interpreted as the fpqc torsor $\mathrm{Isom}^\circ((A[p^\infty],\lambda), (A_x[p^\infty], \lambda_x))$.

"reduction of structure".

e.g.
$$\begin{array}{ccc} \widetilde{\hat{\mathbb{G}}}_m^{\oplus 2} & \longrightarrow & \widetilde{\hat{\mathbb{G}}}_m^{\oplus 4} \\ \downarrow & & \downarrow \; \mathrm{Hom}((\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus 2}, \mu_{p^\infty}^{\oplus 2}). \\ \hat{\mathbb{G}}_m^{\oplus 2} & \subseteq & \mathrm{Def}((\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus 2} \oplus \mu_{p^\infty}) \cong \hat{\mathbb{G}}_m^{\oplus 4} \end{array}$$