Cohomology of Quasicoherent Sheaves

§1 A Fundamental Theorem about Affine Schemes

(and a Bogus Proof).

<u>4th fund thm</u>: X affine sch, $\mathcal{F} \in \mathrm{Qcoh}(X)$.

$\Rightarrow H^i(X, \mathcal{F}) = 0$, $\forall \underline{\underline{i>0}}$ i.e. $\mathcal{F}$ acyclic.

$\uparrow$ sheaf cohom.

<u>(!) Bogus Proof</u>. $X = \mathrm{Spec}\,A$, $\mathcal{F} = \tilde{M}$, $M \in \mathrm{Mod}_A$.

enough injs

What's wrong?

$\rightsquigarrow M \to I$ mono s.t. I inj A-mod.

$0 \to \tilde{M} \to \tilde{I} \to \widetilde{I/M} \to 0$

($\rightsquigarrow$ $0 \to M \to I \to I/M \to 0$ taking $\Gamma(X, -)$)

$\Rightarrow$ in cohom long exact seq. $\delta^i = 0$, $i \geq 0$.

Also, $H^i(X, \tilde{I}) = 0$, $\forall i > 0$ $\Rightarrow$ $H^1(X, \tilde{M}) = 0$.

Moreover, $\forall i > 1$, $H^i(X, \tilde{M}) \cong H^{i-1}(X, \widetilde{I/M})$

$\rightsquigarrow$ proved by dim shifting. $\square$

I inj. in Mod_A $\Rightarrow$ $\tilde{I}$ inj. in $\mathrm{Qcoh}(\mathrm{Mod}_{\mathcal{O}_X})$

$\not\Rightarrow$ $\tilde{I}$ inj. in $\mathrm{Sh}(\mathrm{Mod}_{\mathcal{O}_X})$

In particular: I inj. $\overset{?}{\Rightarrow}$ $\tilde{I}$ flasque

<u>Two Ways to Fix</u>

(1) in nt rings, inj. $\Rightarrow$ flasque. (c.f. Hartshorne Prop II.5.6)

(2) (EGA) compute $\check{H}$ instead of H.

Lemma $X = \operatorname{Spec} A$, $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$ exact $/\mathrm{Mod}_{\mathcal{O}_X}$.
s.t. $\mathcal{F}_1 \in \mathrm{Qcoh}$, $\mathcal{F}$ & $\mathcal{F}_2$ arbitrary.

$$\Rightarrow \quad 0 \to \Gamma(X, \mathcal{F}_1) \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}_2) \to 0 \quad \text{exact.}$$

This implies that $\delta^0 : H^0(X, \mathcal{F}_2) \to H^1(X, \mathcal{F}_1)$ is zero

$$\Rightarrow \quad 0 \to H^1(X, \mathcal{F}_1) \to H^1(X, \mathcal{F}) \quad \text{inj.}$$

If $\mathcal{F}$ inj. $\Rightarrow H^1(X, \mathcal{F}_1) = 0$.

§2 Applications

Cor $X \in \mathrm{Sch}$, $\mathcal{U} = \{U_i\}_{i \in I}$ open cover. $\forall J \subseteq I$ finite, $U_J = \bigcap_{i \in J} U_i$ affine.
$\Rightarrow \forall \mathcal{F} \in \mathrm{Qcoh}(X)$, sh cohom of $\mathcal{F}$ is given by Čech cohom:

$$H^i(X, \mathcal{F}) = \check{H}^i(\mathcal{U}, \mathcal{F}).$$

Recall X separate $\Rightarrow \operatorname{Spec} A_i \cap \operatorname{Spec} A_j = \operatorname{Spec} B$.
aff $\cap$ aff $=$ aff (opens)

Cor X sep sch. $\mathcal{U} = \{U_i\}_{i \in I}$ open cover.
$\Rightarrow \forall \mathcal{F} \in \mathrm{Qcoh}(X)$, $H^i(X, \mathcal{F}) = \check{H}^i(\mathcal{U}, \mathcal{F})$

Useful in computing $\boxed{H^i(\mathbb{P}^r, \mathcal{O}(n))}$.
(next notes)

Useless Cor $f_1, \cdots, f_n \in A$ crings. $(1) = (f_1, \cdots, f_n)$.
$\rightsquigarrow \mathcal{U} = \{D(f_i)\}$ open cover of $X = \operatorname{Spec} A$
$\Rightarrow \forall M \in \mathrm{Mod}A$, $\check{H}^0(\mathcal{U}, \widetilde{M}) = M$. $\check{H}^i(\mathcal{U}, M) = 0$ $(i > 0)$.

§3 A Correct Proof

Step 1 Show that $0 \to M \to \check{C}^0(\mathcal{U}, \widetilde{M}) \to \check{C}^1(\mathcal{U}, \widetilde{M}) \to \cdots$ exact.

$$\xleftarrow{\Gamma(X,-)} \quad 0 \to \widetilde{M} \to \check{\mathcal{C}}^0(\mathcal{U}, \widetilde{M}) \to \check{\mathcal{C}}^1(\mathcal{U}, \widetilde{M}) \to \cdots \quad (\text{as Qcohs}).$$

the 2nd seq'ce is exact by computing at stalks.
Moreover, constituent sheaves are quasicoh.

b/c $\check{C}^i(\mathfrak{U}, \tilde{M}) = \bigoplus j_{U*}(\tilde{M}|_U) = \tilde{M_g}$.

$U = \bigcap_{i \in J} U_i$ for some $J \subseteq I$ finite.
$= D(g)$, $g \in A$

<u>Step 2</u> $0 \to M \to \check{C}^0(\mathfrak{U}, \tilde{M}) \to \check{C}^1(\mathfrak{U}, \tilde{M}) \to \cdots$ exact.

$\Rightarrow$ $\check{H}^0(\mathfrak{U}, \tilde{M}) = M$, $\check{H}^i(\mathfrak{U}, \tilde{M}) = 0$ $(i > 0)$.

$\Rightarrow$ by taking $\varinjlim_{\mathfrak{U}}$ under all opens

$\check{H}^0(X, \tilde{M}) = M$, $\check{H}^i(X, \tilde{M}) = 0$ $(i > 0)$.

(every $\mathfrak{U}$ can be refined to a finite cover by distinguished opens).

[<u>Caveat</u> X not Hausdorff here
(can't use $\varinjlim_{\mathfrak{U}} \check{H}^i(\mathfrak{U}, \mathcal{F}) = H^i(X, \mathcal{F})$)
lim on refinements]

$\Rightarrow$ $H^0(X, \tilde{M}) = M$, $H^i(X, \tilde{M}) = 0$ $(i > 0)$

by the following thm of Cartan.

<u>Thm</u> (Cartan) $X \in \mathrm{Top}$. $\mathcal{B}$ a basis of X, $U_i \cap U_j \in \mathcal{B}$ for $U_i, U_j \in \mathcal{B}$.
$\mathcal{F} \in \mathrm{Sh}_{Ab}(X)$ s.t. $\check{H}^i(U, \mathcal{F}) = 0$, $\forall U \in \mathcal{B}$.
$\Rightarrow$ $\check{H}^i(X, \mathcal{F}) \cong H^i(X, \mathcal{F})$, $\forall i \geq 0$.

<u>§4 Comparison of Čech and Sheaf Cohomology</u>

On flasque sheaves:

<u>Lemma</u> $X \in \mathrm{Top}$. $\mathcal{F} \in \mathrm{Sh}_{Ab}(X)$ s.t. $\check{H}^1(X, \mathcal{F}) = 0$. Then
for any $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ exact,

$$0 \to \Gamma(X,\mathcal{F}) \to \Gamma(X,\mathcal{G}) \to \Gamma(X,\mathcal{H}) \to 0 \quad \text{exact.}$$

Proof. Check right surjectivity. $\forall s \in \Gamma(X,\mathcal{H})$,

$\exists\ \mathcal{U} = \{U_i\}_{i\in I}$ s.t. $\forall i \in I$, $t_i \longmapsto s|_{U_i}$

$$\Gamma(U_i,\mathcal{G}) \longrightarrow \Gamma(U_i,\mathcal{H}).$$

$\forall i,j \in I$, put $u_{ij} = t_i|_{U_i\cap U_j} - t_j|_{U_i\cap U_j} \in \Gamma(U_i\cap U_j, \mathcal{G})$.

Čech 1-cocycle of $\mathcal{F}$.

$\left(\begin{array}{l}\text{also view } u_{ij} \text{ as elt in } \Gamma(U_i\cap U_j,\mathcal{F}) \\ \text{since } u_{ij} \mapsto 0 \in \Gamma(U_i\cap U_j,\mathcal{G}).\end{array}\right)$

Now $\check{H}^1(X,\mathcal{F}) = 0 \Rightarrow \mathcal{U}$ refinements

s.t. u_{ij} becomes a Čech coboundary

i.e. $v_i|_{U_i\cap U_j} - v_j|_{U_i\cap U_j} = u_{ij}$ $(\forall i,j\in I)$

$(v_i \in \Gamma(U_i,\mathcal{F}),\ \forall i)$

$\Rightarrow$ $w_i = t_i - v_i \in \Gamma(U_i,\mathcal{G})$.

$\Rightarrow$ $w_i|_{U_i\cap U_j} - w_j|_{U_i\cap U_j} = 0$ (by computation)

$\Rightarrow$ $w \in \Gamma(X,\mathcal{G})$ lifting $s \in \Gamma(X,\mathcal{H})$.

<u>Proof of Cartan's Comparison</u>

Induction on i & dim shifting.

① $i=0$: given by sheaf axioms.

② Fix $i>0$ & assume $\check{H}^k(X,\mathcal{F}) \cong H^k(X,\mathcal{F})$, $\forall k<i$.

$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$, $\mathcal{G}$ flasque

$\Rightarrow$ $0 \to \Gamma(U,\mathcal{F}) \to \Gamma(U,\mathcal{G}) \to \Gamma(U,\mathcal{H}) \to 0$ exact

($\forall U \in B$ by Lemma).

Let $\mathcal{U} = \{U_i\}_{i\in I}$ cover of X by $U_i \in B$.

$\mathcal{B}$ is closed under $(\cdot)\cap(\cdot)$:

$\Rightarrow \quad 0 \to \check{C}^\cdot(\mathcal{U},\mathcal{F}) \to \check{C}^\cdot(\mathcal{U},\mathcal{G}) \to \check{C}^\cdot(\mathcal{U},\mathcal{H}) \to 0$ exact.

<u>Note</u> any open cover refines to basis opens

$\Rightarrow$ [taking $\varinjlim$ on all opens $\Leftrightarrow$ taking $\varinjlim$ on all basic opens]

$\leadsto \quad 0 \to \check{C}^\cdot(X,\mathcal{F}) \to \check{C}^\cdot(X,\mathcal{G}) \to \check{C}^\cdot(X,\mathcal{H}) \to 0$ exact.

$\mathcal{G}$ flasque $\Rightarrow \check{H}^i(X,\mathcal{G}) = H^i(X,\mathcal{G}) = 0 \quad (i>0)$

$$
\begin{tikzcd}
\to \check{H}^{i-1}(X,\mathcal{G}) \arrow[r] \arrow[d, "="] & \check{H}^{i-1}(X,\mathcal{H}) \arrow[r] \arrow[d, "\cong"] \arrow[r, phantom, "\overset{5\text{-Lem}}{\Longrightarrow}"] & \check{H}^{i}(X,\mathcal{F}) \arrow[r] \arrow[d, "\cong"] & \check{H}^{i}(X,\mathcal{G}) \to \arrow[d, "="] \\
\to H^{i-1}(X,\mathcal{G}) \arrow[r] & H^{i-1}(X,\mathcal{H}) \arrow[r] & H^{i}(X,\mathcal{F}) \arrow[r] & H^{i}(X,\mathcal{G}) \to
\end{tikzcd}
$$

$H^{i-1}(X,\mathcal{G}) = 0 \quad (i>1)$; $H^{i-1}(X,\mathcal{H})$: inductive hyp.; $H^{i}(X,\mathcal{G}) = 0$ $\square$