Lectures on Mod p Langlands Program for GL_2 (1/4)

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<u>References</u>: Breuil - Herzig - Hu - Morra - Schraen (1), (2)
(1) Gelfand-Kirillov dim
(2) (φ, Γ)-mod
Hu - Wang

} $\bar{\rho}$ semi-simple.

§ Introduction

$L/\mathbb{Q}_p$ finite ext'n.

$\bar{\rho}: G_L \longrightarrow GL_2(\overline{\mathbb{F}}_p) \overset{?}{\longleftrightarrow} \pi(\bar{\rho})$: sm adm rep'n of $GL_2(L)$.

$\overline{\mathbb{F}}_p \supset \mathbb{F}$

Known for $GL_2(\mathbb{Q}_p)$: Breuil, Colmez, Emerton

$\bar{\rho}$: irred. $\longleftrightarrow$ $\pi(\bar{\rho})$ supersingular (Breuil)

$\bar{\rho}$: reducible $\longleftrightarrow$ $\pi(\bar{\rho})$: $PS_1 \longrightarrow PS_2$.

When $L \neq \mathbb{Q}_p$, no classification for <u>s.s.</u>

- Breuil - Paskunas (2007) (infinite) family of s.s. rep.
- Hu, Schraen, Wu : s.s. are not of finite rep'n

$$0 \to \underset{\text{(not finite type)}}{\mathrm{Ker}} \longrightarrow \text{c-Ind}_{GL_2(\mathcal{O})}^{GL_2(L)} \sigma \longrightarrow \underset{GL_2(L)\text{-rep'n.}}{\pi} \to 0$$

- Le : $\exists$ non-adm smooth irred. s.s. rep.

<u>Candidate</u>: for $\pi(\bar{\rho})$, F tot real field.

D/F quaternion alg split above p;

at ∞: either non-split or split at only one place above ∞

$\bar{r} : G_F \to GL_2(\mathbb{F})$ cont. odd,

$\bar{r}|_{G_{F_v}} \simeq \bar{\rho}$, $U^v \subseteq (D \otimes \mathbb{A}_F^{\infty, v})^\times$ compact open

($F_v \simeq L$, $v | p$)

$\rightsquigarrow$ $\pi_v^D(\bar{r}) := \{ f : D^\times \backslash (D \otimes_F \mathbb{A}_F^\infty)^\times / U^v \to \mathbb{F} \text{ cont} \} [m_{\bar{r}}]$.

$\circlearrowleft$ $GL_2(L)$? $\|$

eigenspace for $m_{\bar{r}}$ $\pi(\bar{\rho})^{\oplus d}$ (assume $d = 1$)

(max'l ideal of Hecke alg).

<u>Goal</u> · property of $\pi(\bar{\rho})$? (finite length).

· "locality" of $\pi(\bar{\rho})$?

<u>Conjectural properties of $\pi(\bar{\rho})$</u>?

(1) As $K := GL_2(\mathcal{O}_L)$ - rep'n $\twoheadrightarrow GL_2(\mathbb{F}_q)$, $\mathcal{O}_L/(p) = \mathbb{F}_q$.

· Buzzard - Diamond - Jarvis (weight part for Serre conj).

$\mathrm{Soc}_K \pi(\bar{\rho}) = \bigoplus_{\sigma \in W(\bar{\rho})} \sigma$. $W(\bar{\rho}) :=$ set of irred Γ-rep.

· [BP] Let $K_1 = \ker(K \twoheadrightarrow \Gamma)$,

unramified case

$\overset{?}{\Rightarrow}$ $\pi(\bar{\rho})^{K_1} = D_0(\bar{\rho})$ fin. dim Γ-rep'n.

proved by Gee, Emerton - Gee - Savitt, Le

(ver. with multiplicities) all n_σ's are equal),

(by using a patching functor)

(2) As $GL_2(L)$-rep'n:

[BP] [· $\pi(\bar{\rho})$ is generated by $\pi(\bar{\rho})^{K_1} = D_0(\bar{\rho})$ (i.e. finitely generated).

· $\pi(\bar{\rho})$ has finite length: { 1 if $\bar{\rho}$ irred, ok!

$f+1$ if $\bar{\rho}$ is reducible (generic)

(Emerton's Conjecture (?) :

f.g. + adm. $\Rightarrow$ fin length.)

$f = 2$, ok ; $\underset{PS}{\pi_0} - \underbrace{\pi_1 - \cdots}_{S.S.} - \underset{PS}{\pi_f}$ (?)

- $\pi(\bar{\rho})$ has Gelfand-Kirillov dim f.

<u>Recall</u> $K_n = 1 + p^n M_2(\mathcal{O}_L)$, $\dim_{\mathbb{F}} \pi(\bar{\rho})^{K_n}$.

$$\exists\ 0 \le c \le \dim K \ ,\quad a \ge b > 0$$

c integer, $\dim K = 4f$

$$\rightsquigarrow\ b p^{nc} + O(p^{n(c-1)}) \le \dim_{\mathbb{F}} \pi(\bar{\rho})^{K_n} \le a p^{nc} + O(p^{n(c-1)})$$

$$c := GK\text{-}(\pi(\bar{\rho})).$$

<u>E.g.</u> · PS $\dim (\mathrm{Ind}_{B(L)}^{GL_2(L)} \chi)^{K_n} = p^{(n-1)f}(p^f+1) = p^{nf} + p^{(n-1)f}$. $GK = f$.

- $GL_2(\mathbb{Q}_p)$ (Morra):

$$\dim \pi^{K_n} = (p+1)(2p^{n-1}+1) + \begin{cases} p-3 \\ p-2 \end{cases} \longleftarrow r \in \{0, p-1\}, \quad \pi \text{ s.s.}$$

$$\Rightarrow GK(\pi) = 1.$$

- $GK\text{-}\dim = 0$ iff $\dim_{\mathbb{F}} \pi < \infty$.

<u>Rmk</u> $\mathbb{F}[\![K_1]\!]$ local ring. $M = \pi(\bar{\rho})^\vee$ is f.g. $\mathbb{F}[\![K_1]\!]$-mod

$$\pi(\bar{\rho})^{K_1} \longleftrightarrow M/\mathfrak{m}_{\mathbb{F}[\![K_1]\!]} \subseteq M/\mathfrak{m}^2_{\mathbb{F}[\![K_1]\!]}$$

<u>Rmk</u> The importance of GK-dim: ($\Rightarrow M_\infty$ is flat over K_∞).

patching $\rightsquigarrow M_\infty$ / over $R_\infty = R_{\bar{\rho}}[\![x_1, \cdots, x_g]\!]$ univ. def ring.

satisfies $M_\infty / \mathfrak{m}_{R_\infty} \simeq \pi(\bar{\rho})^\vee$.

$\forall\ x : R_\infty \to \overline{\mathbb{Q}}_p$

$$0 \overset{?}{\neq} (\underbrace{M_\infty \otimes_{R_\infty, x} \mathcal{O}_E}_{p\text{-torsion}})^d [\tfrac{1}{p}] \xleftrightarrow{p\text{-adic}} \rho_x^{\text{univ}}.$$

$(-)^d = \mathrm{Hom}^{\text{cont}}_{\mathcal{O}_E}(-, \mathcal{O}_E)$ duality.

§ Serre Weight

$$\begin{cases} \text{(i) Serre weight: } \pi(\bar\rho)^{K_1}, \dots \\ \text{(ii) GK-}\dim(\pi(\bar\rho)). \end{cases}$$

$\bar\rho \rightsquigarrow$ define the modular weight

$$W^{?}(\bar\rho) = \{\sigma \text{ irred } \Gamma\text{-rep or } K\text{-rep} \mid \mathrm{Hom}_K(\sigma, \pi(\bar\rho)) \neq 0\}$$

(all $\mathrm{soc}_K(\pi(\bar\rho))$.)

[BDJ] construct $W^{\mathrm{expl}}(\bar\rho)$.

Notation: irred σ has the form $\quad (\Gamma = GL_2(\mathbb{F}_q))$

$$(r_0, \dots, r_{f-1}) \otimes \det{}^a := \mathrm{Sym}^{r_0}\mathbb{F}^2 \otimes (\mathrm{Sym}^{r_1})^{\mathrm{Frob}} \otimes \cdots \otimes (\mathrm{Sym}^{r_{f-1}})^{\mathrm{Frob}^{f-1}} \otimes \det{}^a$$

$$0 \le r_i \le p-1, \quad 0 \le a \le p^f - 2.$$

(i) If $\bar\rho$ is reducible,

$$\bar\rho|_{I_L} \simeq \begin{pmatrix} \omega_f^{\sum_{i=0}^{f-1} p^i(r_i+1)} & * \\ 0 & \underbrace{1}_{\text{quotient}} \end{pmatrix} \text{ up to twist.}$$

generic condition: $0 \le r_i \le p-3$ but not all 0 or $p-3$

ω_f : Serre's fundamental character at level f.

Define (formally)

$$W^{\mathrm{expl}}(\bar\rho) := \left\{ (s_0, \dots, s_{f-1}) \otimes \theta \;\middle|\; \begin{array}{l} \exists J \subseteq \{0,1,\dots,f-1\} \text{ s.t.} \\ \bar\rho|_{I_L} \simeq \begin{pmatrix} \omega_f^{\sum_{j\notin J} p^j(s_j+1)} & * \\ 0 & \omega_f^{\sum_{j\in J} p^j(s_j+1)} \end{pmatrix} \otimes \theta. \\ \text{and comes from a Fontaine-Laffaille mod.} \end{array} \right\}$$

E.g. $f=2$:

$$\bar\rho|_{I_L} \simeq \underbrace{\begin{pmatrix} \omega_2^{(r_0+1)+p(r_1+1)} & * \\ 0 & 1 \end{pmatrix}}_{\text{split}} \simeq \begin{pmatrix} \omega_2^{r_0+2} & * \\ 0 & \omega_2^{p(p-1-r_1)} \end{pmatrix} \otimes \omega_2^{(p-1)+pr_1}$$

$J = \{1\}$

$$\simeq \begin{pmatrix} \omega_2^{p(r_1+2)} & * \\ 0 & \omega_2^{(p-1-r_0)} \end{pmatrix} \otimes \omega_2^{r_0+p(p-1)} \simeq \boxed{\begin{pmatrix} 1 & * \\ 0 & \omega_2^{(p-2-r_0)+p(p-2+r_1)} \end{pmatrix} \otimes \omega_2^{r_0+1+p(r_1+1)}}$$

comes from FL iff $* = 0$.

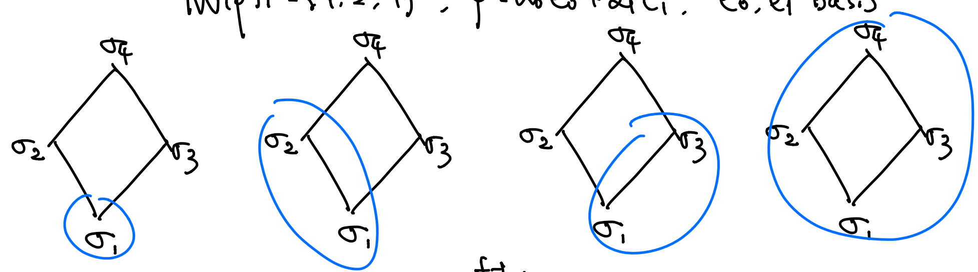
$\exists$ crystalline lift with HT weights $((r_0+2,0),(0,p-1+r_1))$.

$\omega_2^{p^2-1}=1$.

$$\underset{\substack{\uparrow\\ \text{split}}}{W^{\text{expl}}(\bar\rho)} = \Big\{ \overset{\sigma_1}{(r_0,r_1)},\ \overset{\sigma_2}{(r_0+1,\ p-2-r_1)\otimes \det{}^{p-1+pr_1}},$$
$$\underset{\sigma_3}{(p-2-r_0,\ r_1+1)\otimes \det{}^{r_0+p(p-1)}},\ \underset{\sigma_4}{(p-3-r_0,\ p-3-r_1)\otimes \det{}^{(r_0+1)+p(r_1+1)}} \Big\}$$

In general, $W^{\text{exp}}(\bar\rho) \subseteq W^{\text{expl}}(\bar\rho^{ss})$. $\sigma_1 \in W^{\text{expl}}(\bar\rho)$, $\sigma_4 \in W^{\text{expl}}(\bar\rho)$ iff $\bar\rho$ splits

$|W(\bar\rho)| = \{1,2,4\}$, $\bar\rho = \alpha_0 e_0 + \alpha_1 e_1$. e_0, e_1 basis



When $\bar\rho$ irred, $f=2$, $\bar\rho|_{I_2} \simeq \begin{pmatrix} \omega_{2f}^{\sum_{i=0}^{f-1} p^i(r_i+1)} & 0 \\ 0 & \omega_{2f}^{p^f\cdot\sum_{i=0}^{f-1} p^i(r_i+1)} \end{pmatrix}$

with $J \subseteq \{0,1,\dots,2f-1\} \xrightarrow{\text{mod } f} \{0,\dots,f-1\}$ up to twist.

Generic condition: $1 \le r_0 \le p-2$, $0 \le r_i \le p-3$, $i \ne 0$.

$$f=2 \rightsquigarrow W^{\text{expl}}(\bar\rho) = \left\{ \begin{array}{l} (r_0,r_1),\ (r_0-1,\ p-2-r_1)\otimes \det{}^?, \\ (p-1-r_0,\ p-3-r_1)\otimes \det{}^?,\ (p-2-r_0,\ r_1+1)\otimes\det{}^? \end{array} \right\}.$$

§ Concerning about $D_0(\bar\rho)$

Thm (BP) $\exists$ unique f-dim Γ-rep $= GL_2(\mathbb{F}_q)$ $D_0(\bar\rho)$ s.t.

(i) $\mathrm{soc}_\Gamma D_0(\bar\rho) = \bigoplus_{\sigma\in W^{\text{expl}}(\bar\rho)} \sigma$ (Fact: $W^?(\bar\rho) = W(\bar\rho)$.)

(ii) any weight of $W(\bar\rho)$ appears once in $D_0(\bar\rho)$

(iii) $D_0(\bar\rho)$ is max'l w.r.t. (i)(ii).

Moreover, $D_0(\bar\rho)$ is multip one, and $D_0(\bar\rho) = \bigoplus_{\sigma\in W(\bar\rho)} D_{0,\sigma}(\bar\rho)$.

E.g. $f=1$: $\bar\rho = \begin{pmatrix} \omega_1^{r+1} & * \\ 0 & 1 \end{pmatrix}$ non-split,

$$W(\bar\rho) = \{\mathrm{Sym}^r \mathbb{F}^2\} \qquad (r_0)$$

(i) $\Rightarrow D_0(\bar\rho) \hookrightarrow \mathrm{Inj}_\Gamma \mathrm{Sym}^r \mathbb{F}^2$ (injective envelop)

$$\begin{tikzcd}
\mathrm{Sym}^r \mathbb{F}^2 \arrow[r, dash] \arrow[dr, dash] & \mathrm{Sym}^{p-1-r} \otimes \det^a \arrow[dr, dash] & \\
& \mathrm{Sym}^{p-3-r} \otimes \det^{a+1} \arrow[r, dash] & \cancel{\mathrm{Sym}^r \mathbb{F}^2}
\end{tikzcd}$$

with $D_0(\bar\rho) = \mathrm{Sym}^r \mathbb{F}^2$ and $\mathrm{Sym}^r\mathbb{F}^2 \subset \mathrm{Inj}_\Gamma \mathrm{Sym}^r\mathbb{F}^2$.

Rmk $\pi(\bar\rho) \hookrightarrow \underbrace{\mathrm{Inj}_K \Big(\bigoplus_{\sigma \in W(\bar\rho)} \sigma\Big)[m_F]}$. $\quad K = GL_2(\mathbb{Q}_p)$

of ∞-dim'l with GK-dim $= 4f$.

If $\pi(\bar\rho)^{K_1} \subsetneq (\mathrm{Inj}_K(\cdots))^{K_1} = \mathrm{Inj}_\Gamma \mathrm{Sym}^r \mathbb{F}^2 \rightsquigarrow$ a control of GK $(\pi(\bar\rho))$.

E.g. When irred with $f=1$, $\bar\rho|_{I_L} = \begin{pmatrix} \omega_2^{r+1} & 0 \\ 0 & \omega_2^{p(r+1)} \end{pmatrix}$.

$\rightsquigarrow W(\bar\rho) = \{\mathrm{Sym}^r \mathbb{F}^2, \mathrm{Sym}^{p-1-r} \otimes \det^r\}$

$$\begin{tikzcd}
& & \cancel{\mathrm{Sym}^{p-1-r}} \arrow[dr, dash] & \\
& \mathrm{Sym}^r \arrow[ur, dash] \arrow[dr, dash] & & \cancel{\mathrm{Sym}^r} \\
D_0(\bar\rho) \arrow[r, hook] & \oplus & \mathrm{Sym}^{p-3-r} \arrow[ur, dash] & \\
& & \cancel{\mathrm{Sym}^r} \arrow[dr, dash] & \\
& \mathrm{Sym}^{p-1-r} \arrow[ur, dash] \arrow[dr, dash] & & \cancel{\mathrm{Sym}^{p-1-r}} \\
& & \mathrm{Sym}^{r-2} \arrow[ur, dash] &
\end{tikzcd}$$

§ Patching module / functor

$R_{\bar\rho}[[X_1, \cdots, X_g]]$.

Def A patching module M_∞ is f.g. $R_\infty[GL_2(L)]$-module s.t.

(a) $M_\infty / m_{R_\infty} \simeq \pi(\bar\rho)^\vee \longleftarrow$ (minimal). M_∞ is projective $S_\infty[[K]]$-mod.

(b) $\forall$ type (w, τ) $\quad w$: of HT wts $(a_i, b_i)_{0 \le i \le f-1} \rightsquigarrow R_{\bar\rho}(w, \tau)$ Kisin.

$\updownarrow \qquad \tau : I_L \to GL_2(E)$

$\sigma(w, \tau) =$ K-rep fin-dim $\supseteq \mathcal{O}$ lattice

Define $M_\infty(\Theta) := \mathrm{Hom}_K(M_\infty, \Theta^\vee)^\vee$ is max'l Cohen-Macaulay (f.g.) $R_\infty(w,\tau)$-module.

$$\circlearrowleft$$
$$R_\infty(w,\tau) = R_\infty \otimes_{R_{\bar\rho}} R_{\bar\rho}(w,\tau).$$

Fact If $R_{\bar\rho}(w,\tau)=0$, then $M_\infty(\Theta)=0$.

- $M_\infty(-)$: $\mathcal{O}_E[\![K]\!]$-mod (finite generated over $\mathcal{O}$) $\longrightarrow$ f.g. R_∞-mod.

  is an exact functor.

Cor For $\Theta \subseteq \sigma(w,\tau)$ lattice,

$$M_\infty(\Theta) \neq 0 \iff M_\infty(\Theta/p\Theta) \neq 0,$$
$$\iff \exists\, \sigma \in \underset{\substack{\uparrow \\ \text{Jordan-Hodge factors}}}{JH(\Theta/p\Theta)},\ M_\infty(\sigma) \neq 0$$

Thm $W^?(\bar\rho) = W(\bar\rho)$.

Pf. ① $W^?(\bar\rho) \subseteq W(\bar\rho)$.

Fact (a) in def'n $\Rightarrow M_\infty(\Theta)/\mathfrak{m}_{R_\infty} \simeq \mathrm{Hom}_K(\underbrace{\Theta, \pi(\bar\rho)}_{})^\vee$

typically, as $(\pi(\bar\rho)^\vee, \Theta^\vee)$.

$\rightsquigarrow$ $\sigma \in W^?(\bar\rho) \iff \mathrm{Hom}_K(\sigma, \pi(\bar\rho)) \neq 0 \iff M_\infty(\sigma) \neq 0$.

Assume $\exists\, w=(0,1)$, τ = tame type.

$\sigma \in JH(\overline{\sigma(\tau)})$, $R_{\bar\rho}((0,1),\tau)=0 \Rightarrow M_\infty(\sigma)=0$.

Lemma If $\sigma \notin W(\bar\rho)$ then such a τ always exists.

tame type
- PS type $\rightarrow \sigma(\tau) = \mathrm{Ind}_{B(\mathbb{F}_q)} \tilde\chi_1 \otimes \tilde\chi_2$ (char 0, $(q+1)$-dim)
- cusp $\rightarrow \sigma(\tau)$ = cusp rep'n $(q-1)$-dim.

$\overline{\sigma(\tau)} = \left(\sigma(\tau)^\circ / p\right)^{ss}$:

$f=1$: $\overline{\sigma(\tau)} = \mathrm{Ind}_B^\Gamma(\omega^a \otimes 1) \rightsquigarrow \overbrace{\mathrm{Sym}^a \mathbb{F}^2 \otimes \mathrm{Sym}^{p-1-a} \otimes \det^a}^{W(\bar\rho) \text{ for } \bar\rho \text{ irred.}}$

$\mathrm{Sym}^b \notin W(\bar\rho)$,

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<u>Recall</u> $\pi(\bar\rho)$ = rep'n of $GL_2(L)$

<u>Thm</u> $\mathrm{Soc}_{\underbrace{GL_2(\mathcal{O}_L)}_{= K}} \pi(\bar\rho) = \bigoplus_{\sigma\in W(\bar\rho)} \sigma$ (minimal)

<u>Lemma</u> σ = irred K-rep'n = irred $GL_2(\mathbb{F}_q)$-rep

(i) If $\sigma \in W(\bar\rho)$ then $\exists$ tame type

$\tau : I_L \longrightarrow GL_2(E) \longleftrightarrow \sigma(\tau)$: rep'n of K

s.t. $\sigma \in JH(\overline{\sigma(\tau)})$, $JH(\overline{\sigma(\tau)}) \cap W(\bar\rho) = \{\sigma\}$.

(ii) If $\sigma \notin W(\bar\rho)$, then $\exists\ \tau$

s.t. $\sigma \in JH(\overline{\sigma(\tau)})$, $JH(\overline{\sigma(\tau)}) \cap W(\bar\rho) = \phi$.

Recall that τ is either of principal series type (PS).

or of cusp type

$\rightsquigarrow \sigma(\tau)$ on $GL_2(\mathbb{F}_q)$: either PS or cusp

$$\underset{q+1}{\uparrow} \qquad \underset{q-1}{\uparrow}.$$

M_∞ = patching module.

<u>Recall</u> $M_\infty(\sigma(\tau)) \neq 0 \iff M_\infty(\overline{\sigma(\tau)}) \neq 0$

$\iff \exists\ \sigma \in JH(\overline{\sigma(\tau)})$ s.t. $M_\infty(\sigma) \neq 0$.

Proof of Lemma. (i) If $\sigma \notin W(\bar\rho)$, need $\mathrm{Hom}_K(\sigma, \pi(\bar\rho)) = 0$ (Gee, 2004: (i) without Kisin, with the same philosophy)

$\iff M_\infty(\sigma) = 0$.

$\rightsquigarrow$ find τ as in (ii), compute $R_{\bar\rho}(\underline{(0,1)}, \tau) = 0$.

(ii) If $\sigma \in W(\bar\rho)$, need $M_\infty(\sigma) \neq 0$, and

$\dim M_\infty(\sigma)/\mathfrak{m}_{R_\infty} = 1 \iff M_\infty(\sigma)$ is cyclic as an R_∞-mod.

$\rightsquigarrow$ find τ as in (i)

Compute $R_{\bar\rho}((0,1),\tau) \neq 0$, a power series ring (regular).

Here $\bar\rho$ has a potential BT type τ

$\Rightarrow$ $\bar r$ has a lift at p which is of potential BT type τ

By "automorphic lifting"

$\Rightarrow M_\infty(\sigma(\tau)) \neq 0$

$\Rightarrow \exists\, \sigma' \in JH(\overline{\sigma(\tau)})$ s.t. $M_\infty(\sigma') \neq 0$

but σ is the only possibility so that $\sigma = \sigma'$, $M_\infty(\sigma) \neq 0$.

The "minimal" assumption: (= of multiplicity one).

$R_\infty(\tau) = R_\infty \otimes_{R_{\bar\rho}^{univ}} R_{\bar\rho}((0,1),\tau)$ is a regular local ring.

& $M_\infty(\sigma)$ is a Cohen-Macaulay module.

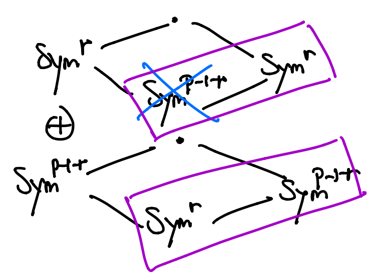
(Auslender-Buchsbaum thm:

$\text{depth}\, m + \text{proj}\dim M = \dim R$.)

Thm $\pi(\bar\rho)^{K_1} = D_0(\bar\rho)$.

- $f=1$, $\bar\rho$ irreducible. (the case of $\mathbb{Q}_p$).

$$D_0(\bar\rho) = \begin{pmatrix} \mathrm{Sym}^r & - & \mathrm{Sym}^{p-3-r} \otimes \det^{r-1} \\ \oplus & & | \\ \mathrm{Sym}^{p-1-r} \otimes \det^r & - & \mathrm{Sym}^{r-2} \end{pmatrix} \hookrightarrow$$



Choose PS type τ s.t.

$\sigma(\tau)^\circ/\varpi_E \simeq \boxed{\mathrm{Sym}^r - \mathrm{Sym}^{p-1-r}}$

To prove that $\mathrm{Hom}_K(\sigma(\tau)^\circ/\varpi_E, \pi(\bar\rho))$ is 1-dim'l

$\Leftarrow$ $M_\infty(\sigma(\tau)^\circ/\varpi_E)$ is a cyclic R_∞-module.

$\mathcal{O}_E[[X,Y]]/(XY-p)$ is a regular local ring if $E/\mathbb{Q}_p$ unramified.

[EGS] work with unramified coeff. field

$\Rightarrow$ $M_\infty(\sigma(\tau)^\circ)$ is a cyclic R_∞-mod

multitype deformation ring.

E.g. When $f=1$, reducible non-split, $W(\bar\rho)=\{\mathrm{Sym}^r\}$

$$\bar\rho|_{I_{\mathbb{Q}_p}} = \begin{pmatrix} \omega^{r+1} & * \\ 0 & 1 \end{pmatrix}, \quad D_0(\bar\rho) = \left(\begin{tikzcd}[row sep=small] & \mathrm{Sym}^{p-3-r} \\ \mathrm{Sym}^r \arrow[ur, no head] \arrow[dr, no head] & \\ & \mathrm{Sym}^{p-1-r} \end{tikzcd}\right) \quad \begin{tikzcd}[row sep=small] {} \arrow[dr, no head] & \\ & \cancel{\mathrm{Sym}^r} \\ {} \arrow[ur, no head] & \end{tikzcd}$$

Consider τ_1: PS type, reduction Sym^r & Sym^{p-1-r}

τ_2: cusp type, reduction Sym^r & Sym^{p-3-r}.

<u>Lemma</u> $\exists\ \Theta \subseteq \sigma(\tau_1)\oplus\sigma(\tau_2)$ s.t.

$$\Theta/\varpi_E\Theta \simeq \begin{tikzcd}[row sep=small] & \bullet \arrow[dr, no head] & \\ \bullet \arrow[ur, no head] \arrow[dr, no head] & & \bullet \\ & \bullet \arrow[ur, no head] & \end{tikzcd}$$

<u>Proof</u> $\big(\underbrace{\mathrm{Proj}_{\mathcal{O}[GL_2(\mathbb{F}_q)]}\sigma}_{2p}\big)[\tfrac{1}{p}] = \underset{\substack{\uparrow \\ p+1}}{\sigma(\tau_1)} \oplus \underset{\substack{\uparrow \\ p-1}}{\sigma(\tau_2)}$.

<u>Recall</u> H finite grp, σ = irred rep of $H/\mathbb{F}$.

$\rightsquigarrow$ $(\mathrm{Proj}_{\mathcal{O}[H]}\sigma)[\tfrac{1}{p}]/\varpi = \mathrm{Proj}\ \mathbb{F}[H]^{\sigma}$

$(\mathrm{Proj}_{\mathcal{O}[H]}\sigma)[\tfrac{1}{p}]$ (semi-simple) $= \bigoplus_i V_i^{a_i}$

where $a_i = \dim_{\mathbb{F}[H]}(\mathrm{Proj}\,\sigma, \bar V_i) = [\bar V_i : \sigma]$.

Now $\Theta \subseteq \sigma(\tau_1)\oplus\sigma(\tau_2)$

$\rightarrow \Theta_1 \subseteq \sigma(\tau_1)$ as $\mathcal{O}$-lattices

$\rightarrow \Theta_2 \subseteq \sigma(\tau_2)$

$\rightsquigarrow$ $0 \to \Theta \hookrightarrow \Theta_1\oplus\Theta_2 \twoheadrightarrow \underset{\substack{\| \\ \sigma}}{\overset{p\text{-torsion mod}}{\boxed{\mathrm{Sym}^r}}} \to 0$ (check).

Taking $M_\infty(-)$:

$$0 \to M_\infty(\Theta) \to M_\infty(\Theta_1) \oplus M_\infty(\Theta_2) \to M_\infty(\sigma) \to 0$$

cyclic modules

Also, $M_\infty(\Theta)$ cyclic $\Leftrightarrow$ $\mathrm{Ann}(M_\infty(\Theta_1)) + \mathrm{Ann}(M_\infty(\Theta_2)) = \mathrm{Ann}(M_\infty(\sigma))$

$$\begin{array}{ccc} \mathrm{Ann}(M_\infty(\Theta_1)) & \mathrm{Ann}(M_\infty(\Theta_2)) & \mathrm{Ann}(M_\infty(\sigma)) \\ \not\Downarrow & \not\Downarrow & \Downarrow \\ p & p & p. \end{array}$$

Ex $R_{\bar p}((\underline{0,1}), \tau_1) = \mathcal{O}[X,Y]/(X)$, $(\mathcal{O}[X,Y] = R_{\bar p}^{univ})$

$R_{\bar p}((\underline{0,1}), \tau_2) = \mathcal{O}[X,Y]/(X-p)$

cannot be written as $\mathcal{O}[\![Y]\!]$. need the coordinate.

When $f=2$, need to consider 2 PS types + 2 cusp types

say τ_1, τ_2 & τ_1', τ_2'.

such that $\sigma_1 \in JH(\overline{\sigma(\tau_i)}) \cap W(\bar p)$, $\sigma_1 \in JH(\overline{\sigma(\tau_i')}) \cap W(\bar p)$.

Assume $\bar p$ reducible non-split

$W(\bar p) = \{\sigma_1, \sigma_2\}$, $\sigma_1 = (r_0, r_1)$, $\sigma_2 = (p-2-r_0, r_1+1)$

$\rightsquigarrow$ $JH(\overline{\sigma(\tau_1)}) \cap W(\bar p) = \{\sigma_1\}$,

$JH(\overline{\sigma(\tau_2)}) \cap W(\bar p) = \{\sigma_1, \sigma_2\}$

$JH(\overline{\sigma(\tau_1')}) \cap W(\bar p) = \{\sigma_1, \sigma_2\}$

$JH(\overline{\sigma(\tau_2')}) \cap W(\bar p) = \{\sigma_1\}$, (by B-M).

E.g. $R_{\bar p}^{univ} = \mathcal{O}[\![X_0, Y_0, X_1, Y_1]\!][\![\text{others}]\!]$, $R_{\bar p}(?) = R_{\bar p}^{univ}/I_?$

$R_{\bar p}(\tau_1)$	$R_{\bar p}(\tau_2)$	$R_{\bar p}(\tau_1')$	$R_{\bar p}(\tau_2')$
(X_0, X_1)	(X_0-p, X_1Y_1-p)	(X_0, X_1Y_1-p)	(X_0-p, X_1)

Take $\Gamma = \mathbb{F}[GL_2(\mathbb{F}_q)] = \mathbb{F}[\![K]\!]/m_{K_1}$, $\tilde\Gamma := \mathbb{F}[\![K]\!]/m_{K_1}^2$

with $\underbrace{\mathbb{F}[\![K_1]\!]}_{\text{pro-}p\text{ grp}} \subseteq \mathbb{F}[\![K]\!]$, local ring $m_{K_1} = (k_1 - 1)$.

$$\rightsquigarrow \quad \pi(\bar\rho)^{K_1} = \pi(\bar\rho)[\mathfrak{m}_{K_1}] \quad (\text{killed by } \mathfrak{m}_{K_1})$$

$$\pi(\bar\rho)[\mathfrak{m}_{K_1}^2] \subsetneq \pi(\bar\rho)^{K_2}.$$

$$\hookrightarrow (\mathrm{Inj}_K \mathrm{Sym}^r)[\mathfrak{m}_{K_1}^2].$$

- Motivation: $f=1$, $\bar\rho$ = red non-split

$$\pi(\bar\rho)^{K_1} = D_0(\bar\rho) = \mathrm{Sym}^r \begin{array}{c} \diagup\ \mathrm{Sym}^{p-3-r}\ \diagdown \\ \diagdown\ \mathrm{Sym}^{p-1-r}\ \diagup \end{array} \cancel{\mathrm{Sym}^r}$$

$$\pi(\bar\rho)|_K \hookrightarrow \mathrm{Inj}_K \mathrm{Sym}^r \longrightarrow \mathrm{Sym}_K \mathrm{Sym}^r.$$

- Dual: $\mathrm{Proj}_K(\mathrm{Sym}^r)^\vee \xrightarrow{X} \mathrm{Proj}_K(\mathrm{Sym}^r)^\vee \longrightarrow \pi(\bar\rho)^\vee \longrightarrow 0$

$$\rightsquigarrow \quad \underbrace{\mathrm{Proj}_K(\mathrm{Sym}^r)^\vee/X}_{\substack{\uparrow \\ \mathrm{Gk}\dim = 3 \ (\text{fix control character})}} \twoheadrightarrow \pi(\bar\rho)^\vee \quad (\text{expect } \dim = 1)$$

- Aim: control $\mathrm{Gk}(\pi(\bar\rho))$

$$\Rightarrow \mathrm{Gk}(\pi(\bar\rho)^\vee) \le \mathrm{Gk}(\mathrm{Proj}(\mathrm{Sym}^r)^\vee/X).$$

To prove: $\mathrm{Gk}(\mathrm{Proj}_K(\mathrm{Sym}^r)^\vee/(x,y)) = 1.$

<u>Recall</u> $\quad 0 \to V^{K_1} \to V \to V/V^{K_1} \to 0$

$$\Rightarrow \quad 0 \to V^{K_1} \xrightarrow{=} V^{K_1} \to (V/V^{K_1})^{K_1} \xrightarrow{=} H^1(K_1, V^{K_1}) \to 0 \qquad (f=1)$$

$$\underbrace{(\mathrm{Sym}^2\mathbb{F}^2 \otimes \det^{-1})}_{\mathrm{SL}_2} \otimes V^{K_1} = H^1(K_1,\mathbb{F}) \otimes V^{K_1}$$

$$H^1(K_1,\mathbb{F}) = \underbrace{\mathrm{Hom}(K_1/Z_1, \mathbb{F})}_{3\text{-dim'l } \mathbb{F}\text{-v.s.}} \circlearrowleft \Gamma\text{-action}$$

Get $\quad 0 \to \underset{(2<r<p-3)}{\mathrm{Inj}_\Gamma \mathrm{Sym}^r} \to (\mathrm{Inj}_K \mathrm{Sym}^r)[\mathfrak{m}_{K_1}^2] \to \underbrace{(\mathrm{Sym}^2 \otimes) \otimes \mathrm{Inj}_\Gamma \mathrm{Sym}^r}_{\circlearrowleft\ \Gamma} \to 0$

<u>Thm</u> $\pi(\bar\rho)[\mathfrak{m}_{K_1}^2]$ is of "multiplicity one".

(In particular, $\forall \sigma \in W(\bar\rho)$, $\mathrm{Sym}^r \begin{array}{c} \diagup \cdot - \cancel{\mathrm{Sym}^r} \\ \diagdown \cdot - \cancel{\mathrm{Sym}^r} \end{array}$

σ occurs once in $\pi(\bar\rho)[\mathfrak{m}_{K_1}^2]$)

Thus,

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & \mathrm{Sym}^{p+2} \arrow[r, dash] & \mathrm{Sym}^{p-1-r} \\
 & \mathrm{Sym}^{p-3-r} \arrow[ur, dash] \arrow[dr, dash] & & \\
\pi(\bar{p})[\mathfrak{m}_{K_1}^2] = \mathrm{Sym}^r \arrow[ur, dash] \arrow[dr, dash] & & \cancel{\mathrm{Sym}^r \oplus \mathrm{Sym}^r} & \\
 & \mathrm{Sym}^{p-1-r} \arrow[ur, dash] \arrow[dr, dash] & & \\
 & & \mathrm{Sym}^{p-2} \arrow[r, dash] & \mathrm{Sym}^{p+1-r}
\end{tikzcd}
$$

<u>Proof</u>. $(\mathrm{Sym}^2\mathbb{F}^2 \otimes \det^{-1}) \otimes \mathrm{Proj}_\Gamma \mathrm{Sym}^r$

$$\xrightarrow{\text{lift to char } 0} (\underbrace{\mathrm{Sym}^2\mathcal{O}^2 \otimes \det^{-1}}_{\text{Alg rep'n}}) \otimes \underbrace{(\mathrm{Proj}_{\mathcal{O}[\Gamma]} \mathrm{Sym}^r)[\tfrac{1}{p}]}_{= \text{tame type direct sum.}}$$

Use deformation ring $(-1, 2)$ of tame type, multitype.

$\rightsquigarrow$ $(\mathrm{Proj}_{\mathcal{O}[\tilde{\Gamma}]} \sigma)[\frac{1}{p}]$ smooth $\tilde{\Gamma}$-rep'n, $\tilde{\Gamma}$ finite algebra.

Attempt: $GL_2(\mathbb{Q}_p)$ + Paškūnas:

$$
\begin{tikzcd}[column sep=small]
 & & \mathrm{Proj}(\mathrm{Sym}^r)^\vee \arrow[dr, "x"] & & & \\
0 \arrow[r] & \mathrm{Proj}_K(\mathrm{Sym}^r)^\vee \arrow[ur, "-y"] \arrow[dr, "x"'] & & \mathrm{Proj}_K(\mathrm{Sym}^r)^\vee \arrow[r] & \mathrm{coker} \arrow[r] & 0 \\
 & & \mathrm{Proj}(\mathrm{Sym}^r)^\vee \arrow[ur, "y"'] & & \pi(\bar{p})^\vee \arrow[u] &
\end{tikzcd}
$$

§ <u>Iwahori subgroup</u>

$I = \begin{pmatrix} * & * \\ p* & * \end{pmatrix} \supseteq I_1 =$ Sylow pro-p Iwahori.

$\rightsquigarrow$ $\mathbb{F}[\![I_1/Z_1]\!]$ Iwasawa algebra,

a $3f$-dim'l noetherian domain.

I_1/Z_1 is <u>not uniform</u>: $\mathrm{Hom}(I_1/Z_1, \mathbb{F}) = 2f$. ($K_1/Z_1$ unramified)

corresp. to poly ring of $3f$ variables.

$\Rightarrow$ $\mathrm{gr}_{\mathfrak{m}_{I_1}}(\mathbb{F}[\![I_1/Z_1]\!])$ not commutative.

Thm $\mathrm{gr}_{m_{I_1}}(\mathbb{F}[\![I_1/Z_1]\!]) = U(\mathfrak{g})$

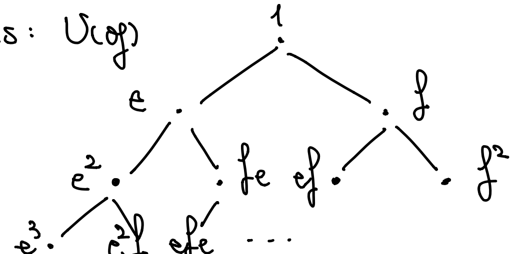
where $\mathfrak{g} = \bigoplus_{i=0}^{f-1} \mathbb{F}e_i^{(1)} \oplus \mathbb{F}f_i^{(1)} \oplus \mathbb{F}h_i^{(2)}$, $h_i \in Z(\mathfrak{g})$.

s.t. $[e_i, f_i] = h_i$, $[e_i, h_i] = [f_i, h_i] = 0$, $[e_i, f_j] = 0$ $(i \neq j)$.

When $f = 1$, $\mathfrak{g} = \mathbb{F}e \oplus \mathbb{F}f \oplus \mathbb{F}h$, $h = ef - fe$.

and $e(ef - fe) = (ef - fe)e$.

PBW basis: $U(\mathfrak{g})$

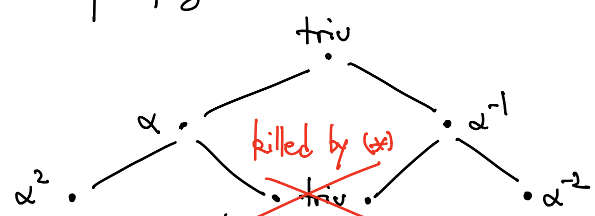


Take $H \curvearrowright \mathfrak{g}$, $H \ni \alpha_i : \begin{pmatrix} [a] & 0 \\ 0 & [d] \end{pmatrix} \longmapsto ad^{-1}$

$\rightsquigarrow$ $g \cdot e_i = \alpha_i(g) \cdot e_i$, $g \cdot f_i = \alpha_i^{-1}(g) \cdot f_i$, $g \cdot h_i = h_i$.

with $e_i \longleftrightarrow \begin{pmatrix} 1 & [x_i] \\ & 1 \end{pmatrix} - 1$.

On the H-rep'n page,



Facts (i) h_i is central in $U(\mathfrak{g})$, $U(\mathfrak{g})/(h_0, \dots, h_{f-1}) \simeq \mathbb{F}[e_i, f_i]_{0 \le i \le f-1}$.

poly ring of Krull dim $= 2f$.

(ii) $U(\mathfrak{g})/(e_i f_i, f_i e_i)$ is a commutative noetherian ring of Krull dim $= f$.

Thm Let π = adm smooth rep'n of I/Z_1.

Assume $\forall \chi$ in $\mathrm{soc}_I(\pi)$, $[\pi[m_{I_1}^3] : \chi] \overset{(*)}{=} [\pi[m_{I_1}] : \chi]$ then $\mathrm{GK}\text{-}\dim(\pi) \le f$.

Further condition (*): $\pi^\vee$ is f.g. $\mathbb{F}[\![I_1/Z_1]\!]$-mod $\circlearrowleft$ H-action

$\mathrm{gr}(\pi^\vee)$ is f.g. $\mathrm{gr}(I_1/Z_1)$-mod.

$\Rightarrow$ $(e_if_i, f_ie_i) \in \mathrm{Ann}_{U(\mathfrak{g})}(\mathrm{gr}(\pi^\vee))$.

i.e. $\mathrm{gr}(\pi^\vee)$ is f.g. module over $\underbrace{\mathbb{F}[e_i, f_i]/(e_if_i)}_{\text{Krull-dim} = f}$.

$\Rightarrow$ $\mathrm{GK}\text{-}\dim(\mathrm{gr}(\pi^\vee)) \le f$

$\Rightarrow$ $\mathrm{GK}(\pi^\vee) \le f$.

Thm $\mathrm{GK}(\pi(\bar\rho)) \le f$

Proof Known: $\pi(\bar\rho)[\mathfrak{m}^2_{K_1}] \circlearrowleft K$ is of multiplicity one.

$\Downarrow$ Frob-reciprocity.

I-rep'n. $\pi(\bar\rho)[\mathfrak{m}^3_{I_1}]$ is of multiplicity one. (*)

$\Downarrow$

$\mathrm{GK}(\pi(\bar\rho)) \le f$.

Lectures on Mod p Langlands Program for GL_2 (3/4)

Yongquan Hu

Recap $\pi(\bar\rho)$: $GL_2(L)$-rep'n

with GK-dim of $\pi(\bar\rho) \leq f$ (+ Gee-Newton) $\Rightarrow$ GK dim $= f$

Take $\mathbb{F}[\![I_1/Z_1]\!]$ enveloping algebra.

$$\rightsquigarrow \mathrm{gr}_{m_{I_1}}(\mathbb{F}[\![I_1/Z_1]\!]) \simeq U(\mathfrak{g}), \quad \mathfrak{g} = \langle e_i, f_i, h_i \rangle_{0 \leq i \leq f-1}$$

$$\cup\!\!\!\mid \qquad J = (e_i f_i, f_i e_i)_{0 \leq i \leq f-1}.$$

Theorem $\mathrm{gr}(\pi(\bar\rho)^\vee)$ is annihilated by J.

Def'n Category $\mathcal{C} := \{\pi : \text{adm sm s.t. } \mathrm{gr}_{m_{I_1}}(\pi^\vee) \text{ is killed by } J^n, \text{ for some } n\}$.

$\rightsquigarrow \forall \pi \in \mathcal{C}$, $GK(\pi) \leq f$.

Remark: $\pi(\bar\rho) = S(U^v, \mathbb{F})[m_{\bar r}]$, $\mathbb{J}$ = Hecke algebra.

$\rightsquigarrow$ looking at $S(U^v, \mathbb{F})[m_{\bar r}^k] \in \mathcal{C}$.

Today To define generalized Colmez's functor for $\pi \in \mathcal{C}$.

§ Colmez's functor

$\mathbb{D} : \{\text{adm rep'n of } GL_2(\mathbb{Q}_p) \text{ of finite length}\} \longrightarrow \{\text{étale } (\varphi, \Gamma)\text{-mod}\}$

(torsion coeffs) (torsion coeffs)

Recall $\pi \in$ LHS over $\mathbb{F}$, $P^+ = \begin{pmatrix} \mathbb{Z}_p \setminus \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ semigroup

$= p^{\mathbb{N}} \times \mathbb{Z}_p^\times$

$\cup\!\!\!\mid$

W $\qquad\updownarrow$

$\|$ $\qquad (\varphi, \Gamma)\text{-mod} / \mathbb{F}[\![X]\!] = \mathbb{F}[\![\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}]\!]$.

fin-dim'l $GL_2(\mathbb{Z}_p)$-rep'n, generating π.

Define $I_w^+(\pi) := \langle P^+, w\rangle \subseteq \pi$

$\circlearrowleft \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \Gamma$

$I_w^-(\pi) := \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \langle P^+, w\rangle \subseteq \pi$

$\circlearrowleft \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \Gamma$

$\rightsquigarrow G = P^+ \cdot GL_2(\mathbb{Z}_p) \cup \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} P^+ GL_2(\mathbb{Z}_p)$

$\rightsquigarrow 0 \to I^+(\pi) \cap I^-(\pi) \to I^+(\pi) \oplus I^-(\pi) \to \pi \to 0$

<u>Dual</u> $\mathbb{D}(\pi) := \underbrace{I_w^+(\pi)^\vee}_{(\psi,\Gamma)\text{-mod}} \otimes_{\mathbb{F}[\![X]\!]} \mathbb{F}((x))$

(ψ,Γ)-mod, with $\pi^\vee \twoheadrightarrow I_w^+(\pi)^\vee$

Colmez: proved that $\mathbb{D}(\pi)$ is an étale (φ,Γ)-mod.

Define $D^+(\pi) := \{\mu \in \pi^\vee \mid \mu|_{I^-(\pi)} = 0\} \hookrightarrow \pi^\vee$

$\rightsquigarrow D^+(\pi) \otimes_{\mathbb{F}[\![X]\!]} \mathbb{F}((x)) \hookrightarrow \pi^\vee \otimes_{\mathbb{F}[\![X]\!]} \mathbb{F}((x)) \to I_w^+(\pi)^\vee \otimes_{\mathbb{F}[\![X]\!]} \mathbb{F}((x))$

$\circlearrowleft (\varphi,\Gamma), \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$

<u>Generally</u> $D^+(\pi) \hookrightarrow \pi^\vee \twoheadrightarrow D^\natural(\pi)$ (D-nature)

$(-) \otimes \mathbb{F}((x))$ becomes an isom (difference is $(\underbrace{I^+(\pi) \cap I^-(\pi)}_{\text{fin-dim'l only for } GL_2(\mathbb{Q}_p)}{}^\vee)$.

<u>Fact</u> $I^+(w)^{\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}}$ is finite dim'l $\mathbb{F}$-v.s. (only for $GL_2(\mathbb{Q}_p)$)

$\Rightarrow I^+(w)^\vee$ is a f.g. $\mathbb{F}[\![X]\!]$-module

$\Rightarrow \mathbb{D}(\pi)$ is a finite-rank (φ,Γ)-module.

<u>Key</u> All irred rep'ns of $GL_2(\mathbb{Q}_p)$ are of finite presentation!

(FALSE for s.s. $GL_2(L)$, $L \neq \mathbb{Q}_p$).

<u>Theorem</u> $\mathbb{D}$ is an exact functor $\xrightarrow{\text{Fontaine}}$ $\mathbb{V}: \pi \mapsto$ Galois rep'n. s.t.

(1) $\mathbb{V}(\chi \circ \det) = 0$,

(2) $\mathbb{V}(\mathrm{Ind}_B^G \chi_1 \otimes \chi_2 \omega^{-1}) = \chi_2$,

(3) $\mathbb{V}(\text{s.s.}) = $ 2-dim'l irred $\bar{\rho}$.

<u>Generalization</u> problem lies in $I^+(\pi)^{\begin{pmatrix}1 & \mathcal{O}_L\\ 0 & 1\end{pmatrix}}$ is ∞-dim'l
(if $L \neq \mathbb{Q}_p$, π is s.s.).

<u>Breuil's version (2015)</u>

$\mathrm{tr}_F : \mathcal{O}_L \to \mathbb{Z}_p$ trace map, $N_0 = \begin{pmatrix}1 & \mathcal{O}_L\\ 0 & 1\end{pmatrix} \trianglerighteq N_1 = \begin{pmatrix}1 & \ker(\mathrm{tr})\\ 0 & 1\end{pmatrix}$.
$\rightsquigarrow N_0/N_1 \simeq \mathbb{Z}_p$.

π : adm. rep'n of $GL_2(L)$.

$\pi^{N_1} \subseteq \pi$ carries $\begin{cases} \text{action of } \begin{pmatrix}1 & \mathbb{Z}_p\\ 0 & 1\end{pmatrix} \\ \text{action of } \begin{pmatrix}\mathbb{Z}_p\setminus\{0\} & 0\\ 0 & 1\end{pmatrix}\end{cases}$.

$$x\cdot v = \sum_{n_1 \in N_1/\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix} N_1 \begin{pmatrix}x^{-1} & 0\\ 0 & 1\end{pmatrix}} n_1 \begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix} v \in \pi^{N}, \quad x \in \mathbb{Z}_p\setminus\{0\},\ v \in \pi^{N_1}.$$

<u>Fact</u> tr is $\mathbb{Z}_p$-linear

<u>Def'n</u> A subspace $W \subseteq \pi^{N_1}$ is called <u>admissible</u> if W
is stable under P^+-action,
and $W^{\begin{pmatrix}1 & \mathbb{Z}_p\\ 0 & 1\end{pmatrix}}$ is of finite dim.

An analogue of $I^+_W(\pi)$.

$\Rightarrow W^\vee$ is f.g. $\mathbb{F}[\![X]\!]$-mod, with (φ,Γ)-mod structure

<u>Lemma</u> If W is admissible, then $\mathbb{F}((X)) \otimes_{\mathbb{F}[\![X]\!]} W^\vee$ is étale (φ,Γ)-module.

<u>Def'n</u> (Breuil) $\mathbb{D}(\pi) := \varprojlim_{\substack{W \subseteq \pi^{N_1}\\ \text{adm}}} (\mathbb{F}((X)) \otimes_{\mathbb{F}[\![X]\!]} W^\vee)$ pro-étale (φ,Γ)-mod.

<u>Breuil's work</u> (1) compute $\mathbb{D}(\pi)$ if π is not ss.
(2) $\mathbb{D}$ left-exact, and exact on non-ss rep'ns.

Not known (1) $\{w \text{ adm}\}$ is non-empty?

(2) $\mathbb{D}(\pi)$ is finite generated $/ \mathbb{F}((x))$?

(3) $\mathbb{D}(-)$ is exact?

Theorem (Hu-Wang) On category $\mathcal{C}$,

$\mathbb{D}$ is exact & $\mathbb{D}(\pi)$ is f.g. over $\mathbb{F}((x))$.

· For $\pi(\bar{\rho}) \in \mathcal{C}$, $\mathbb{D}(\pi(\bar{\rho}))$ is explicitly determined.

§ The ring A

$$\mathbb{F}[\![N_0]\!] \simeq \mathbb{F}[\![Y_0, \cdots, Y_{f-1}]\!], \text{ where } Y_i = \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{p^i} \begin{pmatrix} 1 & (\tilde{\lambda}) \\ 0 & 1 \end{pmatrix} \in m_{N_0}.$$

$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ $\quad m_{N_0}$

Teichmüller lifting

with $\sum_{\lambda \in \mathbb{F}_q} \lambda^{-p^i} = 0$.

m_{N_0}-adic filtration

$$\forall a \in N_0, \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} - 1 \in m_{N_0}.$$

(Y_i eigenvector for the action of $\begin{pmatrix} [a] & 0 \\ 0 & [d] \end{pmatrix}$).

Let $S := \{(Y_0, \cdots, Y_{f-1})^n : n \geq 0\}$ multip subset

$\mathbb{F}[\![N_0]\!]_S$: extend a filtration.

$$\deg \frac{h}{(Y_0 \cdots Y_{f-1})^n} = \deg h - nf.$$

Def'n $A = \widehat{\mathbb{F}[\![N_0]\!]_S}$ (in general, $\hat{M} = \varprojlim_n M/\mathrm{Fil}_n M$).

$$\rightsquigarrow gr(A) = gr(\mathbb{F}[\![N_0]\!]_S) \simeq \underbrace{gr(\mathbb{F}[\![N_0]\!])}_{\mathbb{F}[y_0, \cdots, y_{f-1}]}[(y_0 \cdots y_{f-1})^{-1}]$$

y_i = principal part of Y_i.

On A: $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ on N_0

$\Rightarrow \varphi : \mathbb{F}[\![N_0]\!] \longrightarrow \mathbb{F}[\![N_0]\!]$ finite flat of degree q.

Check: $\varphi(Y_i) = Y_{i-1}^p$, φ extends to $\mathbb{F}[\![N_0]\!]_S$, and then extends to A.

$\mathcal{O}_L^\times$-action: $\gamma_a : \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ on N_0

doesn't preserve S.

But γ_a extends to A, i.e. $\gamma(y_0, \cdots, y_{f-1}) \notin \mathbb{F}[\![N_0]\!]_S$, $\in A^\times$

<u>Def'n</u> A $(\varphi, \mathcal{O}_L^\times)$-mod over A is a f.g. A-mod with commuting $(\varphi, \mathcal{O}_L^\times)$-action.

<u>Prop</u> If M is f.g. A-mod with an $\mathcal{O}_L^\times$-semilinear action, then M is finite free as an A-module. (Colmez: $A = \mathbb{F}((X))$).

<u>Proof</u> <u>Step 1</u> An ideal of A is stable under $\mathcal{O}_L^\times$ is either 0 or A itself.

$\Rightarrow$ if M is a torsion A-mod, then $(\mathrm{Ann}_A(M) \neq 0 \Rightarrow M = 0)$.

<u>Step 2</u> (Double duality spectral sequence):

$$E_2^{p,q} = \mathrm{Ext}_A^p(\mathrm{Ext}_A^q(M, A), A) \Rightarrow H^{p-q}(M) = \begin{cases} M, & p-q=0 \\ 0, & p-q \neq 0 \end{cases}.$$

If $p \neq 0$ or $q \neq 0$, $E_2^{p,q}$ is A-torsion. (Auslander regular)

$\Rightarrow E_2^{p,q} = 0$

We obtain
$$\begin{cases} \mathrm{Hom}(\mathrm{Hom}(M, A), A) \simeq M & ① \\ \mathrm{Ext}_A^p(\underbrace{\mathrm{Hom}(M, A)}_{M'}, A) = 0, \quad \forall p > 0 & ② \end{cases}$$

② $\Rightarrow$ M' is a projective module $\Rightarrow$ M proj.

<u>Step 3</u> Lütkebohmert (1977)

§ The functor $\mathcal{C} \to \{$étale $(\varphi, \mathcal{O}_K^\times)$-mod$\}$

$\pi \in \mathcal{C}$, $\pi^\vee$ is f.g. $\mathbb{F}[\![I_1(z_1)]\!]$-module (but not f.g. over $\mathbb{F}[\![N_0]\!]$).

$$(\pi^\vee)_S = \mathbb{F}[\![N_0]\!]_S \otimes_{\mathbb{F}[\![N_0]\!]} \pi^\vee.$$

Key give "tensor product filtration on $(\pi^\vee)_S$"

- $\mathbb{F}[\![N_0]\!]_S$ the usual one — but $\exists\, \mathbb{F}[\![N_0]\!] \to \mathbb{F}[\![I_1]\!]$.

To kill the "negative" part →

- $\pi^\vee$: $\mathfrak{m}_{I_1}$-adic filtration (!) NOT $\mathfrak{m}_{N_0}$-adic filtration,

Define $D_A(\pi) := \widehat{(\pi^\vee_S)} = \varprojlim_n \pi^\vee_S / \mathrm{Fil}^n \pi^\vee_S$.

Rmk $\mathbb{Q}_p$ case: $\mathbb{F}((x)) \hat{\otimes}_{\mathbb{F}[\![x]\!]} \pi^\vee = \underbrace{\mathbb{F}((x)) \otimes_{\mathbb{F}[\![x]\!]} I^+(\pi)^\vee}_{\text{Colmez's def'n}}$ $\circlearrowleft$ I_1-action, positive.

(Obstruction: $I(\pi)^\vee$ is not f.g. over $\mathbb{F}((x))$).

Reason: $\mathbb{F}((x)) \hat{\otimes}_{\mathbb{F}[\![x]\!]} I(\pi)^\vee = 0$ b/c $\mathrm{Fil}^n M = \mathrm{Fil}^{n+1} M = \cdots$ from some n.

$$v \otimes w = \frac{v}{y} \otimes yw$$

deg $=0$, deg $=-1$, deg "$p-1$"

$$= \frac{v}{y^2} \otimes y^2 w$$

on $I(\pi)^\vee$: deg $yw > 1 + \deg w$ under $\mathfrak{m}_{I_1}$-adic filtration

on $I^+(\pi)^\vee$: deg $(y/w) = 1 + \deg w$.

Exactness: $0 \to \pi_1 \to \pi \to \pi_2 \to 0$

$$0 \to \pi_2^\vee \to \pi^\vee \to \pi_1^\vee \to 0$$

Check: $\widehat{(-)_S}$ ⇝ exactness of $D_A(-)$.

Finiteness: $D_A(\pi)$ is a finite (free) A-mod for $\pi \in \mathcal{C}$

Pf. It suffices to show $\mathrm{gr}(D_A(\pi))$ is f.g. $\mathrm{gr}(A)$-module

$$\mathrm{gr}(D_A(\pi)) = \mathrm{gr}_{\mathfrak{m}_{I_1}}(\pi^\vee)[(y_0, \cdots, y_{f-1})^{-1}]. \qquad \mathrm{gr}(A) = \mathbb{F}[y_0, \cdots, y_{f-1}][(y_0, \cdots, y_{f-1})^{-1}]$$

Assume it is killed by J (WLOG)

$\Rightarrow$ f.g. module over $\mathbb{F}[y_i, z_i]/(y_i z_i)$

Key $(\mathbb{F}[y_i, z_i]/(y_i z_i))[(y_0, \cdots, y_{f-1})^{-1}] \simeq \mathbb{F}[y_i^{\pm 1}, 0 \le i \le f-1] = \mathrm{gr}(A)$. $\square$

Action $\mathcal{O}_K^\times$ acts on $D_A(\pi)$

φ-action $\circlearrowleft \widehat{\pi}_S$

(small issue) $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$-action on π will become a ψ-action on $\pi^\vee$.

Define $\psi : \pi^\vee \to \pi^\vee$, $\mu \longmapsto \mu \circ \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.

which satisfies $v \in \pi^\vee$, $a \in \mathbb{F}[\![N_0]\!]$, $\psi(\varphi(a)\mu) = a \cdot \psi(\mu)$.

ψ extends to $(\pi^\vee)_S$: $\frac{\mu}{(Y_0, \cdots, Y_{f-1})^{pn}} \longmapsto \frac{\psi(\mu)}{(Y_0, \cdots, Y_{f-1})^n}$

$\rightsquigarrow$ ψ extends to $D_A(\pi) \to D_A(\pi)$

- How to get an action of φ ?

Rmk M over (A, φ). Then

$\exists$ étale φ-action on M $\Leftrightarrow$ $A \otimes_{A,\varphi} M \xrightarrow{\sim} M$ isom
$a \otimes m \longmapsto a\varphi(m)$.

It is equivalent to : $\exists$ ψ-action $\psi : M \to M$ satisfying

(i) $\psi(\varphi(a)m) = a \cdot \psi(m)$

(ii) an isom $M \xrightarrow{(*)} A \otimes_{\varphi,A} M$

$m \longmapsto \sum_{\underbrace{n \in N_0/N_0^p}_{\text{a basis of } A \text{ over } \varphi(A).}} \delta_n \otimes \psi(\delta_n^{-1} m)$, $\delta_n \in [n] \in \mathbb{F}[\![N_0]\!] \subseteq A$.

But In general, not able to prove the map

$D_A(\pi) \xrightarrow{(*)} A \otimes_{A,\varphi} D_A(\pi)$ is an isom.

Prop $\exists$ a max'l quotient of $D_A(\pi)$, denoted by $D_A(\pi)^{\text{ét}}$

s.t. $D_A(\pi)^{\text{ét}}$ is stable under ψ, $\mathcal{O}_K^\times$, and $D_A(\pi)^{\text{ét}} \xrightarrow{\sim} A \otimes_{\varphi,A} D_A(\pi)^{\text{ét}}$

is an isom.

$\Rightarrow D_A(\pi)^{\text{ét}}$ is étale $(\varphi, \mathcal{O}_L^\times)$-mod.

(This $(-)^{\text{ét}}$ is formal, doesn't depend on $D_A(\pi)$).

$\rightsquigarrow$ Get a functor $D_A(-)^{\text{ét}} = (\widehat{\pi^\vee_S})^{\text{ét}}$, with exactness.

§ Computation on $D_A(\pi(\bar\rho))$.

<u>Relation to Breuil's version</u> $(\varphi, \mathbb{Z}_p^\times)$-module

$$\mathbb{F}[\![N_0]\!] \xrightarrow{tr} \mathbb{F}[\![\mathbb{Z}_p]\!] : \ Y_0, \dots, Y_{f-1} \mapsto X$$

$$\rightsquigarrow \ \widehat{\mathbb{F}[\![N_0]\!]_S} \longrightarrow \mathbb{F}((X))$$

Given $\pi \in \mathcal{C}$, this gives $D_A(\pi) \to \mathbb{D}_{Breuil}(\pi)$

In fact $D_A(\pi)^{et} / (Y_i - Y_0)_{1 \le i \le f-1} \simeq \mathbb{D}_{Breuil}(\pi)$. $\leftarrow$ need étaleness (but freeness doesn't need ét). $\Updownarrow$ with $\mathbb{Q}_L^\times$-semilin action,

<u>Thm</u> $\mathbb{D}_{Breuil}$ is exact.

<u>Proof</u>. $D_A(\pi)^{et}$ is a free A-module. $\square$

Compute: $\mathbb{D}_{Breuil}(\pi(\bar\rho))$ finite étale (φ, Γ)-module.

<u>Thm</u> (Conjecturally) $\mathbb{V}_{Breuil}(\pi(\bar\rho)) = \boxed{\mathrm{ind}_L^{\otimes \mathbb{Q}_p} \bar\rho}$ tensor induction $= \mathrm{ind}_{G_L}^{\otimes G_{\mathbb{Q}_p}} \bar\rho$, of $\dim = 2^f$.

<u>Recall</u> $H < G$ of index n, $G = \coprod_{1 \le i \le n} g_i H$

$(\rho, V) =$ fin-dim'l rep'n of H

$\rightsquigarrow$ ① $\mathrm{ind}_H^{\otimes G} \rho = \bigotimes_{i=1}^n (g_i \otimes V)$

② $g'(g_i \otimes v_i) := g_{i'} \otimes (g_{i'}^{-1} g' g_i) v_i$, if $g' g_i \in g_{i'} H$.

with $\dim = (\dim V)^n$.

<u>Recall</u> $\bar\rho$ semisimple $\Rightarrow \# W(\bar\rho) = 2^f$

$(2^f = \# \mathrm{Soc}_{GL_2(\mathcal{O}_L)} \pi(\bar\rho).)$

<u>Step 1</u> upper bound of $\dim \mathbb{V}_{Breuil}(\pi(\bar\rho)) = \mathrm{rank}_A (D_A(\pi(\bar\rho))^{et}) \le \mathrm{rank}_A D_A(\pi(\bar\rho))$.

Back to proof of $D_A(\pi(\bar\rho))$ f.g.

$\Rightarrow \mathrm{gr}(D_A(\pi(\bar\rho)))$ f.g. over $\mathrm{gr}(A) = \mathbb{F}[y_i^{\pm 1}]$.

<u>Generally</u> # of generators of $D_A(\pi)$

$\le$ # of generators of $\mathrm{gr}(D_A(\pi))$.

Over $\mathbb{F}[y_i, z_i]/(y_i z_i)$: minimal prime ideal $\mathfrak{p}$

$$\mathfrak{p}_0 = (y_0, \dots, y_{f-1})$$

then $\mathrm{rank}_A D_A(\pi(\bar\rho)) \le m_{\mathfrak{p}_0}(\mathrm{gr}\, D_A(\pi(\bar\rho)))$ multiplicity

$$m_{\mathfrak{p}_0}(M) = \mathrm{length}_{A_{\mathfrak{p}_0}}(M_{\mathfrak{p}_0})$$

Can compute $\mathrm{gr}(\pi(\bar{\rho})^\vee)$ with multiplicity at $\mathfrak{p}_0$.

<u>Prop</u> $m_{\mathfrak{p}_0}(\mathrm{gr}(\pi(\bar{\rho}))^\vee) \leq 2^f$.

<u>Step 2</u> Find inside π^{N_1}, an adm $W \subseteq \pi^{N_1}$ s.t.

$$\mathbb{F}((X)) \otimes_{\mathbb{F}[[X]]} W^\vee \simeq (\varphi, \mathbb{Z}_p^\times)\text{-mod of } \mathrm{ind}_L^{\otimes \mathbb{Q}_p} \bar{\rho}.$$

<u>Recall</u> $D_{\mathrm{Breuil}}(\pi) = \varinjlim_{\substack{W \subseteq \pi^{N_1} \\ \mathrm{adm}}} \mathbb{F}((X)) \otimes_{\mathbb{F}[[X]]} W^\vee$

get $\dim D_{\mathrm{Breuil}}(\pi(\bar{\rho})) \geq 2^f$

Byproduct: $D_A(\pi(\bar{\rho})) = D_A^{\text{ét}}(\pi(\bar{\rho}))$.

Lectures on Mod p Langlands Program for GL_2 (4/4)

Yongquan Hu

<u>Recall</u> Defined generalized Colmez's functor.

Also Breuil's version:

$$\mathbb{D}_B : \pi \longmapsto \text{proétale } (\varphi,\Gamma)\text{-module}$$

If $\pi \in \mathcal{C}$, then obtain étale (φ,Γ)-module killed by J^n.

$$\left(\text{as } \dim \mathbb{V}_B(\pi) \le m_{\mathfrak{p}_0}(gr(\pi^\vee)).\quad \mathfrak{p}_0 = (y_0, \cdots, y_{f-1})\right).$$

<u>Theorem</u> $\mathbb{V}_B(\pi(\bar{\rho})) = \mathrm{Ind}_1^{\otimes \mathbb{Q}_p} \bar{\rho}$ $\quad (\dim = 2^f)$.

E.g. For $\dim \mathbb{V}_B(\pi(\bar{\rho})) \le 2^f$ i.e. $m_{\mathfrak{p}_0}(gr(\pi^\vee)) \le 2^f$.

- $f=1$, $\bar{\rho}$ = reducible split, $\pi(\bar{\rho}) = \pi_0 \oplus \pi_1$ (principal series)

$\pi_0^{I_1} = 2$-dim'l, $\chi_0 \oplus \chi_0^s$ conjugation by $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$.

$\pi_1^{I_1} = 2$-dim'l, $\chi_1 \oplus \chi_1^s$

Serre weights: $\mathrm{Sym}^r \mathbb{F}^2 = \sigma_0$, $\mathrm{Sym}^{p-3-r} \otimes \det^{r+1} = \sigma_1$. $\leftrightarrow \chi_1^s : \mathrm{Sym}^{r+2} \otimes \det^{-1}$.

<u>Fact</u> $\chi_0 = \chi_1^s \alpha^{-1}$, $\quad \alpha : \begin{pmatrix} [a] & 0 \\ 0 & [d] \end{pmatrix} \longmapsto ad^{-1}$.

<u>Known Key Property</u> $\pi[\mathfrak{m}_{I_1}^3]$ is multiplicity free:

$$\begin{array}{ccccccc} 0 & e_0\ \chi_0^\vee & \overset{\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}}{\longleftrightarrow} & e_0'\ (\chi_0^s)^\vee & & e_1\ \chi_1^\vee & e_1'\ (\chi_1^s)^\vee \end{array}$$

$$gr(\pi(\bar{\rho})^\vee) \qquad \chi_0^\vee \alpha \cdot \overset{y}{\diagup} \cdot \overset{z}{\diagdown} \cdot \underset{\substack{\| \\ (\chi_1^s)^\vee}}{\chi_0^\vee \alpha^{-1}}$$

$z \cdot e_0 = 0 \Leftrightarrow y \cdot e_0' = 0$ $\qquad\qquad U(\mathfrak{g})/J = \mathbb{F}[y,z]/(yz)$.

$\leadsto$ get $\left.\begin{array}{l}(\chi_0^\vee \otimes \mathbb{F}[y]) \oplus ((\chi_0^s)^\vee \otimes \mathbb{F}[z]) \\ \oplus (\chi_1^\vee \otimes \mathbb{F}[y]) \oplus (\alpha_1^s 1^\vee \otimes \mathbb{F}[z])\end{array}\right| \xrightarrow[\text{Nakayama}]{\text{graded}} gr(\pi(\bar{\rho})^\vee) \Rightarrow m_{\mathfrak{p}_0}(gr(\pi(\bar{\rho})^\vee)) \le 2$

E.g. $f=2$, $\bar{\rho}$ irred,

then $\pi(\bar{\rho})^{I_1} = 8$-dim'l. So $\mathrm{gr}(\pi^\vee)$ is $\boxed{U(\mathfrak{g})/J}$-mod with 8 generators

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\mathbb{F}[y_0, y_1, z_0, z_1]/(y_0 z_0, y_1 z_1)$.

Serre weights

$\sigma_1 = \boxed{(r_0, r_1)}$ — $(r_0+1, p-2-r_0) = \sigma_4'$, $(p-1-r_0, r_1-1)$ — $(p-3-r_0, p-1-r_1)$

$\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$

$\sigma_2 = (r_0-1, p-2-r_1)$ — $\boxed{(p-1-r_0, p-1-r_1)} = \sigma_1'$, (r_0-2, r_1) — $(p-r_0, r_1-1)$ $\quad e_1'$

$\sigma_3 = (p-1-r_0, p-3-r_1)$ — $(p-r_0, r_1+1) = \sigma_2'$, $(r_0-1, p-4-r_1)$ — (r_0-2, r_1+2)

χ_α

$\sigma_4 = (p-2-r_0, r_1+1)$ — $(r_0, r_1+2) = \sigma_3'$, $(p-3-r_0, p-3-r_1)$ — $(r_0+1, p-4-r_1)$

e.g. when $f=2$ & $\bar{\rho}$ reducible.

$\boxed{\sigma_1 \to \sigma_1' \to \cdots}$ PS π_0

$\sigma_2 \leftrightarrow \sigma_3$ } supersingular. π_1

$\boxed{\sigma_4 \to \sigma_4' \to \cdots}$ PS π_2

Dually,

$$\begin{array}{c} \chi_{\sigma_1}^\vee \longleftrightarrow e_1 \\ \swarrow^{y_0} \quad \swarrow^{y_1} \quad \searrow^{z_0} \quad \searrow^{z_1} \\ \chi_0^\vee \alpha_0 \qquad \chi^\vee \alpha_1 \qquad \chi_0^\vee \alpha_0^{-1} \qquad \chi^{-1}\alpha_1^{-1} = (\chi_{\sigma_3}^s)^\vee \end{array}$$

$\xRightarrow{\text{multi one}}$ $z_1 \cdot e_1 = 0$, etc. $\xleftrightarrow{\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}}$ $y_1 \cdot e_1' = 0$

$\Rightarrow \left(\chi_1^\vee \otimes \mathbb{F}[y_0, y_1, z_0]/(y_0 z_0)\right) \oplus \left((\chi_1^s)^\vee \otimes \mathbb{F}[y_0, y_1, z_1]/(y_1)\right) \oplus \cdots$

$\qquad \twoheadrightarrow \mathrm{gr}(\pi(\bar{\rho}))^\vee$.

$\Rightarrow m_{\mathfrak{p}_0}(\mathrm{gr}\,\pi(\bar{\rho})^\vee) \leq 4$

§ Results in finite length

$\pi(\bar{\rho})$: length as $GL_2(L)$-rep'n.

Recall Expectation : $\pi(\bar{\rho})$ is irred. + ss. if $\bar{\rho}$ is irred.

(generic) $\cdot\ \pi(\bar\rho)$ is of length $f+1$ if $\bar\rho$ is reducible.

$$\simeq \underset{\uparrow \atop PS}{\pi_0} \oplus \underbrace{\pi_1 \oplus \cdots}_{s.s.} \oplus \underset{\uparrow \atop PS}{\pi_f} \quad \leftarrow \text{if } \bar\rho = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

then (supposedly)

$\pi_0 = \mathrm{Ind}_B^{GL_2}(\chi_2 \otimes \chi_1\omega^{-1})$

$\pi_f = \mathrm{Ind}_B^{GL_2}(\chi_1 \otimes \chi_2\omega^{-1})$.

<u>Thm</u> (1) $\bar\rho$ irred $\Rightarrow$ Expectation

(2) $\bar\rho$ reducible $\Rightarrow$ Expectation for $f=2$.

$\begin{cases} \text{split : [BHHMS2]} \\ \text{non-split : [Hu-Wang].} \end{cases}$

§ Self-duality of $\pi(\bar\rho)$

<u>Recall</u> complex smooth rep'n π

$$\pi^\vee := \mathrm{Hom}_{\mathbb{C}}(\pi, \mathbb{C})^\infty \quad \text{smooth dual.}$$

$$\rho : G_L \to GL_2(\mathbb{C}) \xleftrightarrow{LLC} \pi$$

$$\rho^\vee \simeq \rho \otimes (\det\rho)^{-1} \longleftrightarrow \pi^\vee$$

<u>But</u> for mod p rep'n:

<u>Problem</u> $\mathrm{Hom}_{\mathbb{F}}(\pi_1, \mathbb{F})^\infty = 0$ (most of the time).

Need a new version of duality ("smooth dual").

$$\underset{\text{adm.}}{GL_2(L) \hookrightarrow \pi} \rightsquigarrow \underset{\substack{\text{f.g. over} \\ \Lambda = \mathbb{F}[\![I_1/Z_1]\!]}}{\pi^\vee} \rightsquigarrow \underset{\substack{\text{f.g. } \Lambda\text{-mod} \\ \circlearrowleft \\ GL_2(L)}}{\mathrm{Ext}^i_\Lambda(\pi^\vee, \Lambda)} \rightsquigarrow \underset{\text{sm adm rep'n.}}{(\mathrm{Ext}^i_\Lambda(\pi^\vee, \Lambda))^\vee}$$

Grade: $j_\Lambda(M) := \min\{i \geq 0,\ \mathrm{Ext}^i(M, \Lambda) \neq 0\}$, M = f.g. Λ-mod.

<u>Remark</u> Have $GK(\pi) + j_\Lambda(\pi^\vee) = \dim\Lambda\ (=3f)$.

So $j_\Lambda(M) \nearrow \Rightarrow M$ size $\searrow$.

Auslander condition on Λ: $\forall N \subseteq \mathrm{Ext}^i_\Lambda(M, \Lambda)$, $j_\Lambda(N) \geq i$.

satisfied by $\mathbb{F}[\![I_1/Z_1]\!]$ (cf. Venjakob 02).

(i.e. $i \nearrow \Rightarrow \mathrm{Ext}^i_\Lambda(M, \Lambda)$ size $\searrow$).

Def'n M is Cohen-Macaulay if $\exists$ only one i, s.t. $\mathrm{Ext}^i_\Lambda(M,\Lambda) \neq 0$.

Def'n M = f.g. Λ-mod & compatible with $GL_2(L)$-action

Say M is self-dual if $\mathrm{Ext}_\Lambda^{j_\Lambda(M)}(M,\Lambda) \simeq M$,

M is essentially self-dual if $\mathrm{Ext}^{j(M)}(M,\Lambda) \simeq M \otimes$ (up to twist) (determined by central characters).

E.g. $\cdot$ $\pi = \mathrm{Ind}_B^G \chi$, PS. ($\dim \Lambda = 3f$) $\Rightarrow j_\Lambda(\pi^\vee) = 2f$.

Kohlhaase: $\mathrm{Ext}^{2f}(\pi^\vee, \Lambda) = (\mathrm{Ind}_B^{GL_2} \chi^{-1} \cdot \alpha_B)^\vee$.

$\alpha_B = \omega \otimes \omega^{-1}$ modulo char.

Moreover, $\pi^\vee$ is CM.

$\Rightarrow GL_2(\mathbb{Q}_p)$, $\pi(\bar\rho) = (\pi_0 - \pi_1)$

So $\pi(\bar\rho)^\vee$ is essentially self-dual.

- π = ss. for $GL_2(\mathbb{Q}_p)$

  $\mathrm{Ext}^{2f}(\pi^\vee, \Lambda) \simeq \pi^\vee \otimes \zeta \cdot \det$

  (not complete proof yet. see Paskunas' student's master thesis).

- Complete cohomology $\tilde H^1$(Shimura curve)

  $\tilde H^0$(Shimura set)

  Emerton: $E_2^{i,j} = \mathrm{Ext}^i_\Lambda(\tilde H_j, \Lambda) \Rightarrow \tilde H_{d-(i+j)}$.

Thm1 $\pi(\bar\rho)^\vee$ is ess. self-dual

Proof $GK(\pi(\bar\rho)) = f \Rightarrow M_\infty$ is flat mod over $R_\infty = \mathcal{O}[\![x_1,\dots,x_g]\!]$

S_∞

patched module (assume $\bar\rho$ generic)

and $\pi(\bar\rho)^\vee \simeq M_\infty / \mathfrak{m}_{R_\infty}$.

i.e. M_∞ defines a Koszul complex resolution of $\pi(\bar\rho)^\vee$.

ok, if one knows M_∞ is self-dual.

Solve: $\begin{cases} M_\infty / m_{S_\infty} \simeq \tilde{H}_0 & \text{complete homology} \\ R_\infty / m_{S_\infty} \simeq \tilde{\mathbb{T}}_m & \text{Hecke algebra; completed intersection ring.} \end{cases}$

Fact $\begin{cases} M \text{ over } A, \ A \text{ complete inter'n} \Rightarrow \text{Gorenstein} \\ M \text{ ess self-dual} \qquad \text{Some self-dual property.} \end{cases}$

$\Rightarrow M/m_A M$ is ess self-dual.

§ The semisimple case in $\bar{\rho}$

Thm 2 $\pi(\bar{\rho})$ is generated by its K-socle $= \bigoplus_{\sigma \in W(\bar{\rho})} \sigma$, as $GL_2(L)$-rep'n.

($\Rightarrow$ generated by $\pi(\bar{\rho})^{K_1}$).

Caution: f.g. $\not\Rightarrow$ finite length.

Lemma 3 Let π' be a subquotient of $\pi(\bar{\rho})$ $\Rightarrow$ $\mathbb{V}_B(\pi')$ is a subquot of $\mathbb{V}_B(\pi(\bar{\rho}))$

(i) $\dim \mathbb{V}_B(\pi') = m_{\mathfrak{p}_0}(\pi')$ where $\dim \mathbb{V}_B(\pi(\bar{\rho})) = 2^f$

(ii) if π' is subrep of $\pi(\bar{\rho})$, then $= \text{length } soc_K(\pi)$

$\dim \mathbb{V}_B(\pi') = \text{length}(soc_K(\pi'))$ $= m_{\mathfrak{p}_0}(\pi(\bar{\rho}))$

(iii) if $\pi' \neq 0$ is a quotient of $\pi(\bar{\rho})$, then $\mathbb{V}_B(\pi') \neq 0$.

Rmk A priori: don't know length of $\pi(\bar{\rho})$

Can happen that $\exists$ ∞-many subquotient of $\pi(\bar{\rho})$, $\mathbb{V}_B(-) = 0$.

Lemma 3 $\Rightarrow$ Thm 2

$\pi' :=$ generated by $soc_K(\pi(\bar{\rho})) \subseteq \pi(\bar{\rho}) \twoheadrightarrow \pi''$.

$\Rightarrow \dim \mathbb{V}_B(\pi') = \text{length } soc_K(\pi(\bar{\rho})) = \dim \mathbb{V}_B(\pi(\bar{\rho}))$

So $\dim \mathbb{V}_B(\pi'') = 0$

$\overset{(ii)}{\Rightarrow} \pi'' = 0$.

Pf of lem 3 (i) easy: $\mathbb{V}_B$, $m_{\beta_0}(-)$ exact

$\Rightarrow \dim \mathbb{V}_B(\pi(\bar\rho)) = m_{\beta_0}(\pi(\bar\rho)^\vee)$ ok

$\Rightarrow$ ok for any subquot π'.

(ii) (*) $\dim \mathbb{V}_B(\pi') \le \text{length}\ \mathrm{soc}_K(\pi')$

To prove $m_{\beta_0}(\pi') \le \text{length}\ \mathrm{soc}_K(\pi')$

Find a graded module $N' \twoheadrightarrow \mathrm{gr}(\pi')$ satisfying

$m_{\beta_0}(N') = \text{length}(\mathrm{soc}_K \pi')$.

(iii) Consider $0 \to \pi'' \to \pi(\bar\rho) \to \pi' \to 0$

$0 \to \pi'^\vee \to \pi(\bar\rho)^\vee \to \pi''^\vee \to 0$

$\mathrm{Ext}^i(-,\Lambda)$ contravariant

$$0 \to \mathrm{Ext}^{2f}(\pi''^\vee,\Lambda) \to \mathrm{Ext}^{2f}(\pi(\bar\rho)^\vee,\Lambda) \xrightarrow{\gamma} \mathrm{Ext}^{2f}(\pi'^\vee,\Lambda) \to \mathrm{Ext}^{2f+1}(\pi''^\vee,\Lambda) \to \mathrm{Ext}^{2f+1}(\pi(\bar\rho)^\vee,\Lambda) = 0$$

CM property

Define $\tilde\pi' := (\mathrm{Im}\,\gamma \otimes \zeta^{-1})^\vee$ sm adm rep'n of $GL_2(L)$.

So $\tilde\pi' \hookrightarrow \pi(\bar\rho)$.

Key $m_{\beta_0}(\pi') = m_{\beta_0}(\tilde\pi')$ + (ii) $\Rightarrow \mathbb{V}_B(\pi') \neq 0$.

By definition, $\exists$ sequence

$$0 \to \mathrm{Im}\,\gamma \to \underbrace{\mathrm{Ext}^{2f}(\pi'^\vee,\Lambda)}_{j(-)=2f} \to \mathrm{Ext}^{2f+1}(\pi''^\vee,\Lambda) \to 0$$

(Auslander condition $j(-) \ge 2f+1 \Rightarrow m_{\beta_0}(-) = 0$).

(Fact $j(\mathrm{Ext}^{j(M)}(M,\Lambda)) = j(M)$).

$\Rightarrow m_{\beta_0}(\mathrm{Im}(\gamma)) = m_{\beta_0}(\mathrm{Ext}^{2f}(\pi'^\vee,\Lambda)) = m_{\beta_0}(\pi'^\vee)$. $\square$

a general fact.

Rmk Have proved $\mathbb{V}_B(\pi') \neq 0 \iff \mathbb{V}_B(\tilde\pi') \neq 0$

i.e. $m_{\beta_0}(\pi') \neq 0 \iff m_{\beta_0}(\tilde\pi') \neq 0$ (only this).

Proof of Thm 1 (i) $\bar{\rho}$ irred $\Rightarrow$ $\pi(\bar{\rho})$ irred

(Breuil-Paškūnas) MAMS: $\pi(\bar{\rho})$ generated by $\mathrm{soc}_K(\pi(\bar{\rho}))$ & $\bar{\rho}$ irred

$\Rightarrow$ $\pi(\bar{\rho})$ irred.

Recall Irreducibility creterion:

if $\forall \sigma \in \mathrm{soc}_K(\pi(\bar{\rho}))$, σ generates $\pi(\bar{\rho})$,

then $\pi(\bar{\rho})$ is irred.

(Used a fact: $\forall\, 0 \neq \pi' \subseteq \pi(\bar{\rho})$, $\pi' \cap \mathrm{soc}_K(\pi(\bar{\rho})) \neq 0$)

It suffices to prove: $\forall \sigma \hookrightarrow \pi(\bar{\rho})$, σ generates $\mathrm{soc}_K \pi(\bar{\rho})$.

Rmk $f=3$, $\bar{\rho}$ irred, a bit more complicated (but still valid).

When $\bar{\rho}$ reducible split ($f=2$):

Conj $\pi(\bar{\rho}) = \underset{\substack{\uparrow \\ PS}}{\pi_0} \oplus \underbrace{\cdots}_{s.s.} \oplus \underset{\substack{\uparrow \\ PS}}{\pi_f}$

Easy to prove: $\pi_0 \hookrightarrow \pi(\bar{\rho}) \hookleftarrow \pi_f$.

Claim: $\pi(\bar{\rho}) = \pi_0 \oplus \pi_f \oplus \underset{\substack{\| \\ \pi'}}{(sth)}$ (need self-duality).

$\rightsquigarrow$ $\pi_0 \oplus \pi_f \hookrightarrow \pi(\bar{\rho}) \twoheadrightarrow \pi_0 \oplus \pi_f$ $\Rightarrow$ isom.

- $f=2$ $\Rightarrow$ π' is irred and supersingular.

Why $f=3$ fails to be valid?

$$\left.\begin{array}{ll} \pi_0 : & \boxed{\sigma_1} \\ \pi_1 : & \boxed{\sigma_2 \leftrightarrow \sigma_3 \hookleftarrow \sigma_4} \\ \pi_2 : & \boxed{\sigma_5 \leftrightarrow \sigma_6 \hookleftarrow \sigma_7} \\ \pi_3 : & \boxed{\sigma_8} \end{array}\right\} \hookrightarrow \pi(\bar{\rho}).$$

§ Reducible non-split case

Thm 4 $\bar\rho = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$, $* \neq 0$. Assume $f=2$.

Then $\pi(\bar\rho)$ has length 3 : $\pi_0 - \pi_1 - \pi_2$
$\qquad$ PS $\quad$ s.s. $\quad$ PS

Thm 5 $\pi(\bar\rho)$ is generated by $D_0(\bar\rho) = \pi(\bar\rho)^{K_1}$ as a $GL_2(L)$-rep'n.

Proof idea of Thm 5

- $\pi_0 \hookrightarrow \pi(\bar\rho)$: because $\pi_0 = $ PS (use Serre wt + Hecke action).
- self-duality of $\pi(\bar\rho) \Rightarrow \pi(\bar\rho) \twoheadrightarrow \pi_2$

  Moreover, $\pi_0 = \mathrm{soc}_{GL_2(L)} \pi(\bar\rho)$ b/c $\bar\rho$ non-split

  $\Rightarrow \pi_2 = \mathrm{cosoc}_{GL_2(L)}(\pi(\bar\rho))$.
- Lemma Let $\tau \subseteq \pi(\bar\rho)|_K$, if for some σ (irred rep'n of K), some i,

$$\begin{tikzcd} & \mathrm{Ext}^i_K(\sigma,\tau) \arrow[d] \arrow[dl, dashed] \arrow[dr, dashed, "\beta"] & \\ \mathrm{Ext}^i_K(\sigma,\pi') \arrow[r] & \mathrm{Ext}^i_K(\sigma,\pi(\bar\rho)) \arrow[r] & \mathrm{Ext}^i_K(\sigma,\pi_2) \end{tikzcd}$$

composite is nonzero

Then $\pi(\bar\rho)$ is generated by τ as $GL_2(L)$-rep'n.

Pf Let $\pi' = \langle GL_2(L).\tau \rangle \subseteq \pi(\bar\rho) \twoheadrightarrow \pi_2$
then $\pi' \neq \pi(\bar\rho)$ iff the composite is 0. $\}$ cosocle

Assume $\pi' \neq \pi(\bar\rho)$, then the composite $\beta = 0 \rightsquigarrow$ contradiction

Choice of i in Lemma: $\boxed{i=2f}$. $\square$

Thm 5 $\Rightarrow$ Thm 4 (when $f=2$).

Step 1 Known: $\pi_0 \hookrightarrow \pi(\bar\rho)$, study $\pi(\bar\rho)/\pi_0$: what is its $GL_2(L)$-socle ?

Fact ($f \geq 2$) If π' irred rep'n of $GL_2(L)$, π' non-s.s.

Assume $\mathrm{Ext}^1_{GL_2}(\pi', \pi_0) \neq 0$. Then $\pi' \cong \pi_0$.

$\pi(\bar\rho) = (\pi_0 - \pi')$. Here $\overset{\text{irred}}{\pi'} \hookrightarrow \pi(\bar\rho)/\pi_0$.

Fact $\Rightarrow$ either π' s.s. or $\pi' = \pi_0$.

<u>Step 2</u> π' must be s.s.

Use ordinary part of $\pi(\bar\rho)$:

<u>Recall</u> Emerton defines $\mathrm{ord}_B : G\text{-rep} \to \overset{T}{\overset{\parallel}{\text{Torus}}}\text{-rep}$

$\rightsquigarrow \mathrm{Hom}(\mathrm{Ind}_{\bar B}^{G} U, V) \simeq \mathrm{Hom}_T(U, \mathrm{ord}_B V)$.

$\Rightarrow \mathrm{ord}_B(\mathrm{Ind}_{\bar B} U) = U$.

On the other hand $\mathrm{ord}_B(\text{s.s.}) = 0$.

<u>Key</u> If $0 \to \pi_0 \to \mathcal{E} \to \pi_0 \to 0$ self-ext'n.

then $0 \to \underset{\substack{\updownarrow \\ \chi\text{-char}}}{\mathrm{ord}_B(\pi_0)} \to \mathrm{ord}_B(\mathcal{E}) \to \underset{\substack{\parallel \\ \chi}}{\mathrm{ord}_B(\pi_0)} \to 0$ exact

<u>Thm</u> $\mathrm{ord}_B(\pi(\bar\rho))$ is semi-simple (Hu-Breuil-Ding)

$\Rightarrow \pi'$ is a supersingular rep'n;

$(\pi_0 - \pi') \hookrightarrow \pi(\bar\rho)$.

<u>Step 3</u> Study $\pi(\bar\rho)/(\pi_0 - \pi')$ (?)

Expectation: it is π_2 (PS).

Apply Thm 5 $\Rightarrow \pi(\bar\rho)$ is generated by $D_0(\bar\rho)$.

So does $\pi(\bar\rho)/(\pi_0 - \pi')$: generated by $D_0(\bar\rho)/(D_0(\bar\rho) \cap (\pi_0 - \pi'))$

Computation: this quotient is generated by σ_4.

Frob reciprocity $\Rightarrow$ c-$\mathrm{Ind}_{GL_2(\mathcal{O}_L) L^\times}^{GL_2(L)} \sigma_4 \twoheadrightarrow \underset{\substack{\parallel \\ Q}}{\pi(\bar\rho)/(\pi_0 - \pi')}$

satisfy $\mathrm{cosoc}_G Q = \pi_2$, PS.

($f = 2$: $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \longleftrightarrow \mathrm{soc}_K \pi_2$).

<u>Lemma</u> If Q = quotient of c-$\mathrm{Ind}\, \sigma_4$, s.t.

$\mathrm{cosoc}_G Q \simeq PS$, $\pi_2 \simeq$ c-Ind $\sigma_4/(T-\lambda)$, $\lambda \neq 0$.

by Barthel-Livné.

Then $Q \simeq$ c-Ind $\sigma_4/(T-\lambda)^n$ for some n.

i.e. $Q \simeq (\underbrace{\pi_2 - \pi_2 - \cdots - \pi_2}_{n \text{ copies}})$.

$\rightsquigarrow$ Left to prove $n=1$:

self-duality:

$\exists (\underbrace{\pi_0 - \cdots - \pi_0}_{n \text{ copies}}) \hookrightarrow \pi(\bar{\rho})$ $\Rightarrow$ so $n=1$.

$\Rightarrow \pi(\bar{\rho}) = (\pi_0 - \pi_1 - \pi_2')$. $\square$