Ihara's lemma for definite unitary groups

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<u>Background</u> $X_0(N)$ mod curve, $p \nmid N$.

$$X_0(Np) = \{E \to E' \text{ } p\text{-isog}\}$$

$$\begin{tikzcd} X_0(Np) \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] \\ X_0(N) \end{tikzcd} \qquad \begin{tikzcd} {[}E \to E'{]} \arrow[d] \arrow[d] \\ E \quad E' \end{tikzcd}$$

$\ell \neq p$, $\Lambda = \overline{\mathbb{F}}_\ell$.

$\rightsquigarrow$ $H^1(X_0(Np), \Lambda) \circlearrowleft \mathbb{T}$ global Hecke alg

<u>Ihara's lem</u> For $\mathfrak{m} \subseteq \mathbb{T}$ non-Eisenstein,

$$\pi_{1*} + \pi_{2*} : H^1(X_0(Np), \Lambda)_\mathfrak{m} \xrightarrow{\text{surj}} H^1(X_0(N), \Lambda)_\mathfrak{m}^{\oplus 2}.$$

$$\pi_1^* + \pi_2^* : H^1(X_0(N), \Lambda)_\mathfrak{m}^{\oplus 2} \xhookrightarrow{\text{inj}} H^1(X_0(Np), \Lambda)_\mathfrak{m}.$$

<u>A variant</u> $B/\mathbb{Q}$ quaternion alg.

s.t. $B \otimes \mathbb{R} = \mathbb{H}$ Hamil, $B \otimes \mathbb{Q}_p \simeq M_2(\mathbb{Q}_p)$.

Then $(B \otimes \mathbb{Q}_p)^\times \simeq GL_2(\mathbb{Q}_p) \supset K = GL_2(\mathbb{Z}_p) \supset I = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$.

Fix $K^p \subset (B \otimes \mathbb{A}_f^p)^\times$

$\rightsquigarrow$ $\mathcal{A} = H^0(B^\times \backslash (B \otimes \mathbb{A}_f)^\times / K^p, \Lambda) \circlearrowleft GL_2(\mathbb{Q}_p) \times \mathbb{T}$.

Let $\mathfrak{m} \subseteq \mathbb{T}$ non-Eisenstein. Then

$$\mathcal{A}_\mathfrak{m}^I \twoheadrightarrow \mathcal{A}_\mathfrak{m}^{K, \oplus 2} \text{ surj } \& \ \mathcal{A}_\mathfrak{m}^{K, \oplus 2} \hookrightarrow \mathcal{A}_\mathfrak{m}^I \text{ inj}.$$

<u>Setup</u> E/F CM field, B/E division alg,

$*$ involution on B.

$$G = \mathcal{U}(B,*) = \{ x \in B^\times \mid x^* x = 1 \} \ / \mathbb{Q}.$$

Assume $G(\mathbb{R})$ cpt.

Take

$$\begin{array}{ccc} E & & v \quad v^c \\ | & & \diagdown \ \diagup \\ F & & \wp \\ | & & |\ \text{unr inert} \\ \mathbb{Q} & & p \end{array}$$

Assume B splits at v & v^c

$$\Downarrow$$

$$G(\mathbb{Q}_p) \cong GL_n(E_v).$$

Fix $K^p \subseteq G(\mathbb{A}_f^p)$

Def $\mathcal{A} = H^0(G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / K^p, \Lambda) \circlearrowleft GL_n(E_v) \times \mathbb{T}$.

Take $\mathfrak{m} \subseteq \mathbb{T}$ s.t. $\mathcal{A}_{\mathfrak{m}}^K \neq 0$,

$$\underset{\text{hyperspecial}}{K = GL_n(\mathcal{O}_{E_v})} \supset \underset{\text{Iwahori}}{I} = \{ g \in K \mid g \bmod v \text{ upper-triangular} \}.$$

$\rightsquigarrow$ $\rho_{\mathfrak{m}} : Gal_E \longrightarrow GL_n(\Lambda)$ unr at v.

<u>Assumptions</u> (1) $\ell \nmid |GL_n(\mathbb{F}_q)|$, $q = q_v$.

(2) $$q^{\frac{1-n}{2}} \rho_{\mathfrak{m}}(\mathrm{Frob}_v) \sim \begin{pmatrix} q^{n-1}\alpha & & & \\ & q^{n-2}\alpha & & \\ & & \ddots & \\ & & & \alpha \end{pmatrix} =: x_{\mathfrak{m}}$$

for some $u \in \Lambda^\times$.

<u>Rmk</u> (2) $\Leftarrow$ $x_{\mathfrak{m}}$ regular semisimple.

$\rightsquigarrow$ $\mathcal{H} = C_c(I \backslash GL_n(E_v) / I, \Lambda) \circlearrowleft \mathcal{A}_{\mathfrak{m}}^I$.

Def (Bernstein) $\lambda \in X_*^-(T)$, $\theta_\lambda := q^{\langle \lambda, \rho \rangle} \cdot \mathbb{1}_{I \varpi^\lambda I} \in \mathcal{H}$

<u>Note</u> $\lambda \in X_*(T)$, write $\lambda = \mu - \nu$, $\mu, \nu \in X_*^-(T)$.

$$\theta_\lambda = \theta_\mu \cdot \theta_\nu^{-1}.$$

Let $R \subset \mathcal{H}$ subalg gen'd by θ_λ.

$$R \cong \Lambda[X_*(T)] \cong \mathcal{O}(\hat{T})$$

$$\rightsquigarrow \quad Z(\mathcal{H}) := R^W \cong \mathcal{O}(\hat{T}/W).$$

Local-global compatibility:

$$A_m^I \circlearrowleft Z(\mathcal{H}) \text{ supported at } [x_m] \in \hat{T}/W.$$

<u>Def</u> $\mathrm{Rep}_{[x_m]}(GL_n(E_v)) := \left\{ \begin{array}{l} \text{adm unip } GL_n(E_v)\text{-rep } \pi / \Lambda \\ \text{s.t. } \pi^I \circlearrowleft Z(\mathcal{H}) \text{ is supp'd at } [x_m] \end{array} \right\}$.

<u>Prop</u> $\forall \pi \in \mathrm{Rep}_{[x_m]}(GL_n(E_v))$, $\exists$ a canonical decomp

$$\pi^I = \bigoplus_{w \in W} \pi^{I \cdot w} \quad \text{s.t.} \quad \pi^{I \cdot w} \circlearrowleft R \text{ supp'd at } w^{-1}(x_m) \in \hat{T}.$$

<u>Thm</u> (Yang) $w \in W$. $P_w = \{ i \in \{1, \dots, n-1\} \mid w^{-1}(i) < w^{-1}(i+1) \}$.

(1) If $w, w' \in W$, $P_w \supseteq P_{w'}$, then $\exists$ functorial maps

$$\alpha_{w,w'} : \pi^{I \cdot w} \longrightarrow \pi^{I \cdot w'}$$

$$\beta_{w,w'} : \pi^{I \cdot w'} \longrightarrow \pi^{I \cdot w}$$

(2) If $E_v \neq \mathbb{Q}_p$, ρ_m generic (c.f. CS, K, HL), then

$$A_m^{I \cdot w} \overset{\alpha}{\twoheadrightarrow} A_m^{I \cdot w'} \text{ surj} \quad \& \quad A_m^{I \cdot w'} \hookrightarrow A_m^{I \cdot w} \text{ inj}.$$

<u>note</u> ρ_m generic : $\exists w$ s.t. $\rho_m(\mathrm{Frob}_v) = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$, $x_i \neq x_j$ or $q_v x_j$

<u>Rmk</u> Compare w/ original Ihara lem:

$$n = 2 \ (GL_2): \quad A_m^I = A_m^{I,1} \oplus A_m^{I,s}$$

Can show $A_m^{I,s} \cong A_m^K$.

Then Thm $\iff$ original Ihara.

<u>Cor</u> (Conj of Clozel - Harris - Taylor)

Under above ass'ns, any irred $GL_n(F_v)$-submod of A_m is Whittaker generic.

<u>Pf of Cor</u> $\pi \subset A_m$, $\pi \simeq St_J \otimes \chi$, $J \subset \{1, \dots, n-1\}$

χ: some char

St_J: unique irred quotient of $\mathrm{Ind}^G_{P_{\{1,\dots,n-1\}-J}} \Lambda$

e.g. $St_\phi = \Lambda$, $St_{\{1,\dots,n-1\}}$ is Whittake generic.

<u>Lem</u> $$(St_J)^{I,1} = \begin{cases} 0, & \text{if } J \neq \{1, \dots, n-1\} \\ \text{1-dim'l}, & \text{if } J = \{1, \dots, n-1\}. \end{cases}$$

Thm $\Rightarrow$ $\forall w$,
$$\begin{tikzcd} \pi^{I.w} \arrow[r, hook, "\beta"] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \pi^{I,1} \arrow[d] \\ A_m^{Iw} \arrow[r, hook, "\beta"] & A_m^{I,1} \end{tikzcd}$$

$\Rightarrow$ $\forall w$, $\pi^{I.w} \hookrightarrow \pi^{I,1}$ $\Rightarrow$ $\pi^{I,1} \neq 0$. $\square$

<u>Categorical Local Langlands</u>

$G/\mathbb{Q}_p$ reductive.

<u>Conj</u> $\exists$ equiv of cats
$$Shv(Isoc_G, \Lambda)^\omega \simeq Coh^{fc}(Loc_{\hat{G}}).$$

Here $Loc_{\hat{G}} = Z^1(W_{\mathbb{Q}_p}, \hat{G})/\hat{G} = \{\phi : W_{\mathbb{Q}_p} \to {}^LG\}/\hat{G}$ $/\Lambda$.

$k = \overline{\mathbb{F}}_p$. $LG :=$ perfect ind-sch $/k$, $R \mapsto G(W(R)[\frac{1}{p}])$.

$$Isoc_G = \left[\frac{LG}{Ad_\sigma LG}\right], \quad Ad_\sigma(g)(h) = gh\sigma(g)^{-1}.$$

Then $|Isoc_G| \cong B(G) \ni b$, $Isoc_G^b \cong [*/G_b(\mathbb{Q}_p)]$.

$\Rightarrow$ $Shv(Isoc_G, \Lambda)$ has semi-orthogonal decomp into $Rep(G_b(\mathbb{Q}_p), \Lambda)$.

<u>Thm</u> (Hemo-Zhu) G unram, l not too small.

$$\Rightarrow \quad \mathbb{L} : Shv^{unip}(Isoc_G, \Lambda)^{\omega} \cong Coh(\underbrace{Loc^{unip}_{\hat{G}}}_{\text{stack of unip L-params}}).$$

- $\mathcal{F} \in Shv^{unip}(Isoc_G, \Lambda)$ if $i_b^* \mathcal{F}$ is unip in the sense of Lusztig.
- $\mathbb{L}(i_{1*} \, c\mathrm{Ind}_I^{G(\mathbb{Q}_p)} \Lambda) \cong Spr := \pi_* \mathcal{O}_{Loc^{unip}_{\hat{B}}}$

$$\pi : Loc^{unip}_{\hat{B}} \longrightarrow Loc^{unip}_{\hat{G}}.$$

For $G = Res_{E_r/\mathbb{Q}_p} GL_n$.

$$\begin{array}{ccl} Loc^{unip}_{\hat{G}} & = & \{(x, y) \in GL_n \times \mathcal{N}_{GL_n} \mid ad(x)(y) = qy\} / GL_n . \\ \downarrow & & \quad \ \downarrow \\ \hat{T}/W & & \quad [x] \end{array}$$

Define $V_{[x_m]} := Loc^{unip}_{\hat{G}} \times_{\hat{T}/W} (\hat{T}/W)^{\wedge}_{[x_m]}$.

$$\pi \in Rep_{[x_m]}(GL_n(E_r)) \rightsquigarrow \mathbb{L}(i_{1*}\pi) \in Coh(V_{[x_m]}) \subset Coh(Loc^{unip}_{\hat{G}}).$$

<u>Prop</u> $V_{[x_m]} \simeq Spf \dfrac{\Lambda[\![x_1, \cdots, x_n]\!][y_1, \cdots, y_{n-1}]}{(x_1 y_1, \cdots, x_{n-1} y_{n-1})} / \hat{T}$, $|x_i| = 0$, $|y_i| = \alpha_i$.

<u>Prop</u> $Spr|_{V_{[x_m]}} = \bigoplus_{w \in W} \mathcal{O}_{Z_{P_w}}$, $Z_{P_w} = \{y_j = 0, j \notin P_w\} \subset V_{[x_m]}$.

<u>So</u> $\forall \pi \in Rep_{[x_m]}(GL_n(E_r))$,

$$\begin{aligned} \pi^I &= RHom(c\mathrm{Ind}_I^G \Lambda, \pi) \\ &= RHom(Spr, \mathbb{L}(\pi)) \\ &= \bigoplus_{w \in W} \underbrace{RHom(\mathcal{O}_{Z_{P_w}}, \mathbb{L}(\pi))}_{= \pi^{I_w}}. \end{aligned}$$

If $P_{w'} \subseteq P_w$,

$$\mathcal{O}_{Z_{P_w}} \twoheadrightarrow \mathcal{O}_{Z_{P_{w'}}} \rightsquigarrow \pi^{I\cdot w'} \xrightarrow{\beta_{w,w'}} \pi^{I\cdot w}$$

$$\prod_{j \in P_w - P_{w'}} x_j : \mathcal{O}_{Z_{P_{w'}}} \hookrightarrow \mathcal{O}_{Z_{P_w}} \rightsquigarrow \pi^{I\cdot w} \xrightarrow{\alpha_{w,w'}} \pi^{I\cdot w'}$$

A Cohomology formula of Sh vars

Thm (DvHKZ) (G, X) shimura datum of Hodge type.

$$\begin{tikzcd} Sh_{K^pI}(G,X)^{perf}_k \arrow[r, "\pi"] \arrow[d] & Sht_{I,\mu} \arrow[d, "N_I"] \\ Igs_{K^p} \arrow[r, "\bar{\pi}"] & Isoc_G \end{tikzcd}$$

Sempliner - van Hoften: Igs_{K^p} depends only on $K^p \subset G(\mathbb{A}_f^p)$.

Now $\quad R\mathcal{H}om(\underbrace{N_{I*} Z_\mu}_{\text{Gaitsgory's central sheaf}}, \bar{\pi}_* \omega_{Igs_{K^p}}) \cong R\mathcal{H}om(\underbrace{\pi^* Z_\mu}_{\text{nearby cycle on Sh}}, \omega_{Sh})$

$\cong R\Gamma(Sh, \Lambda)[\dim]$.

Prop $\quad \mathbb{L}(N_{I*} Z_\mu) = V_\mu \otimes Spr$

$\Rightarrow \quad R\mathcal{H}om(V_\mu \otimes Spr, \underbrace{\mathcal{I}_m}_{}) = R\Gamma(Sh, \Lambda)[\dim]$.

Denote $\mathcal{I} := \mathbb{L}((\bar{\pi}_* \omega)^{unip})$.

Xiao - Zhu $\quad \exists!$ inner form G' of G s.t. $G'(\mathbb{A}_f) \cong G(\mathbb{A}_f)$

& $\quad G'(\mathbb{R}) = U(1, n-1) \times U(n-1, 1) \times U(0, n)^{2[F:\mathbb{Q}]}$

Then $\quad R\mathcal{H}om(V_\mu \otimes Spr, \mathcal{I}_m) = R\Gamma(Sh_{G'})_m[\dim] \leftarrow$ sits in deg 0.

Also, $$\mathcal{V}_{\mu'} \otimes \mathcal{S}_{pr}\big|_{Y_{[x_m]}} = \bigoplus_{1\le i,j\le n} \bigoplus_{w\in W} \mathcal{O}_{Z_{P_w}}(\varepsilon_i - \varepsilon_j)$$

$$\Rightarrow \quad H^{\dim}(Sh_G)_m = \bigoplus_{1\le i,j\le n} \bigoplus_{w\in W} H^{w}_{i,j}.$$

If $w, w' \in W$ s.t. $P_w = P_{w'} \sqcup \{j\}$,

$$\mathrm{Cone}\big(\mathcal{A}_m^{I_{w'}} \xrightarrow{\beta} \mathcal{A}_m^{I_w}\big) \simeq \mathrm{Cone}\big(H^{w}_{j,j+1} \xrightarrow{\alpha} H^{w'}_{j,j+1}\big)[1]$$

$\deg \in [-1,0]$ $\qquad\qquad$ $\deg \in [0,1]$

$\Rightarrow$ injective. $\qquad \square$