# Sheaf Cohomology

## §1 Having Enough Injectives

**Lemma** Abgrp has enough injs. (by elementary arguments).

Rmk Grothendieck's criterion: so does $\mathrm{Mod}_R$.

Categories of sheaves have enough injs:

$X$ loc ringed, $\mathcal{C}$ ab cat., $\mathcal{D} = \{ \mathcal{F} \in Sh(X) : \mathcal{F} : X \to \mathcal{C} \} = Sh_{\mathcal{C}}(X)$.

ab cat. again.

(Need $\mathcal{D}$ to have enough injs)

Method: constructing a large class of inj objs by using skyscraper sheaves

$$i_x(G) = i_* G, \quad i : \{x\} \hookrightarrow X.$$

$$\text{section: } (i_x(G))(U) = \begin{cases} G, & x \in U \\ 0, & \text{otherwise} \end{cases}$$

$\rightsquigarrow$ $\mathrm{Hom}_{Sh_{\mathcal{C}}(X)}(\mathcal{F}, i_x G) = \mathrm{Hom}_{\mathcal{C}}(\underset{\underset{i_x^{-1}\mathcal{F}.}{\|}}{\mathcal{F}_x}, G)$ by adjointness

$\rightsquigarrow$ mono $\mathcal{F}_x \longrightarrow G_x$, $\mathcal{F} \hookrightarrow \prod_{x \in X} i_x(G_x)$.

$\forall\, U \subseteq X$ open,

$$\mathcal{F}(U) \longrightarrow \Big(\prod_{x \in X} i_x(G_x)\Big)(U) = \prod_{x \in U} G_x = \prod_{x \in U} \mathcal{F}_x$$

$$s \longmapsto (s_x)_{x \in U} \quad \text{germs.}$$

**Prop** $Sh_{\mathrm{Mod}_{\mathcal{O}_X}}(X)$ has enough injs ($X$ ringed space).

Caution: $X$ loc. ringed $\not\Rightarrow$ $\mathrm{Qcoh}(\mathrm{Mod}_{\mathcal{O}_X})$ has enough injs
but true for $X$ affine b/c $\mathrm{Mod}_R$ does.

§2 Grothendieck's Criterion

Thm $\mathcal{C}$ ab cat s.t.

(a) $\mathcal{C}$ admits arbitrary (small) direct sums

weak condition

(b) $X \to Y$, $I$ totally ordered, $Y_i \hookrightarrow Y$ increasing family (i.e. $Y_i \to Y_j \to Y$, $\forall i \le j$ in $I$). Then

$$(\sum Y_i) \cap X = (\sum (Y_i \cap X)).$$

in other words, forming direct lim of $Y_i$ commutes with taking the fibred product $(\cdot) \times_Y X$.

(c) $\exists\, U \in \mathcal{C}$ s.t. $X \to Y$ not epi

$\Rightarrow$ $\mathrm{Hom}(U, X) = \mathrm{Hom}(U, Y)$ not epi

i.e. $\exists\, U \to Y$ not through $X$.

Also, the class of isom classes of monos in $U$ is small.

($\mathcal{C}$ admits a forgetful additive fun to $\underline{\mathrm{Ab}}$ implies this)

Then $\mathcal{C}$ has enough injs.

$\mathrm{Mod}_R$ satisfies (a)(b) easily. To check (c):

$U = \bigoplus_{V \in \underline{X}} j_{V*}(\mathbb{Z}_V)$, $\mathbb{Z}_V$ = const sheaf over $V$ w/ values in $\mathbb{Z}$.

Upshot
$$\begin{aligned}\mathrm{Hom}(\bigoplus_V j_{V*}(\mathbb{Z}_V), \mathcal{G}) &= \bigoplus_V \mathrm{Hom}(j_{V*}(\mathbb{Z}_V), \mathcal{G}) \\ &= \bigoplus_V \mathrm{Hom}(\mathbb{Z}_V, \mathcal{G}|_V) \quad \text{adj.} \\ &= \bigoplus_V \Gamma(V, \mathcal{G}) \quad \text{by def.}\end{aligned}$$

## §3 Sheaf Cohomology for Top & Ringed Spaces

Define $H^i : Sh_{\mathcal{C}}(X) \longrightarrow \mathcal{C}$ as

$$R^i\Gamma(X,-) = H^i(X,-).$$

effaceable $\delta$-functor as well.

In particular: $H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F}) = \mathcal{F}(X)$.

Let $(X, \mathcal{O}_X)$ ringed space, can define $H^i : Sh(Mod_{\mathcal{O}_X}) \to \mathcal{C}$.

Remember we can compute $H^i$ by resolutions.

Say $\mathcal{F} \in Sh(X)$ is flasque (flabby) if

$$\forall\, U \subseteq V \subseteq X \text{ opens}, \quad \mathcal{F}(V) \twoheadrightarrow \mathcal{F}(U)$$

e.g. $X$ irred. $\Rightarrow$ $\underline{C}_X$ const sheaf are flasque.

<u>Reminder</u> $\forall\, C \in \mathcal{C}$, $\underline{C}_X = (U \mapsto C)^+$.

However: $X = \mathbb{R}$ w/ usual top

<u>Lemma</u> $(X, \mathcal{O}_X)$ ringed, any inj $\mathcal{O}_X$-mod is flasque.

In particular, any inj sheaf of ab grps on $X$ is flasque.

$\uparrow$ $\mathcal{O}_X = \underline{\mathbb{Z}}_X$

(c.f. Hartshorne Lem III.2.4)

Proof. $\mathcal{I}$ inj. $\mathcal{O}_X$-mod. $\forall\, U \in \underline{X}$, $\mathcal{O}_U =$ ext by $0$ of $\mathcal{O}_X|_U$ to $X$.

i.e. $$\mathcal{O}_U = \left( V \longmapsto \begin{cases} \mathcal{O}_X(V), & V \subseteq U \\ 0, & \text{otherwise} \end{cases} \right)^+$$

$\rightsquigarrow$ $\mathcal{O}_{U,x} = \mathcal{O}_{X,x}$, $x \in U$; $\mathcal{O}_{U,x} = 0$, $x \notin U$.

(differs from $i_*\mathcal{O}_U$, $i : U \to X$,
$i_*\mathcal{O}_U(W) \neq 0$ when $W \cap V \neq \phi$.)

Take $V \subseteq U$ $\rightsquigarrow$ $\mathcal{O}_V \xrightarrow{\text{mono}} \mathcal{O}_U$ / $Sh(Mod_{\mathcal{O}_X})$.

$\mathcal{I}$ inj. $\Rightarrow$ $\operatorname{Hom}(\mathcal{O}_U, \mathcal{I}) \twoheadrightarrow \operatorname{Hom}(\mathcal{O}_V, \mathcal{I})$

$$\begin{array}{ccc} \operatorname{Hom}(\mathcal{O}_U,\mathcal{I}) & \twoheadrightarrow & \operatorname{Hom}(\mathcal{O}_V,\mathcal{I}) \\ \| & & \| \\ \mathcal{I}(U) & \longrightarrow & \mathcal{I}(V) \end{array} \Rightarrow \mathcal{I} \text{ flasque. } \square$$

**Prop** $\mathcal{F}$ flasque $: X \to \mathrm{AbGrp}$. $\Rightarrow H^i(X, \mathcal{F}) = 0$, $\forall \boxed{i \geq 0}$.

Proof. AbGrp has enough injs

$$\Rightarrow \quad 0 \to \mathcal{F} \to \underset{\substack{\uparrow \\ \text{inj.}}}{\mathcal{I}} \to \mathcal{G} = \underset{\substack{\uparrow \\ \text{coker.}}}{\mathcal{I}/\mathcal{F}} \to 0$$

$\mathcal{F}$ flasque $\Rightarrow 0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{I}) \to H^0(X, \mathcal{G}) \to 0$ exact

$$\left.\begin{array}{l} \Rightarrow H^1(X,\mathcal{F}) = 0 \\ \mathcal{I} \text{ acyclic (since } H^i \text{ is effaceable)} \\ \Rightarrow H^i(X, \mathcal{I}) = 0 \ (i > 0) \end{array}\right\} \Rightarrow \underset{(i>1)}{H^i(X,\mathcal{F}) \cong H^{i-1}(X,\mathcal{G})}$$

Also, $\mathcal{G}$ is flasque ($\mathcal{F}$, $\mathcal{I}$ flasque)

$\Rightarrow H^i(X, \mathcal{F}) \cong H^{i-1}(X, \mathcal{G}) = 0$ by induction. $\square$

Watch: Dimension Shifting Recipe!

## §4 Sheaf Cohomology & Top Cohomology

Pretend that $H^i_{sing}(X, \mathbb{Z}) \cong H^i(X, \underline{\mathbb{Z}}_X)$ (and skip this sec) if you're not familiar with $H^*_{sing}$.

**Thm** $X$ loc. contractible top space.

$\Rightarrow H^i(X, \mathbb{Z}_X) \cong H^i_{sing}(X, \mathbb{Z})$ canonically.

(loc. contractible = each pt has a basis of contractible nbhds).

## §5 Čech Cohomology

$X$ top space. $\mathfrak{U} = \{U_i\}$ open cover of $X$ $(i \in I)$

(i.e. $x \in X$ appears in only finitely many $U_i$'s).

$J \subseteq I$ finite $\rightsquigarrow$ $U_J = \bigcap_{i \in J} U_i$, $U_\phi = X$.

$\mathcal{F} \in Sh_{Ab}(X)$. Define Čech complex of $\mathcal{F}$ w.r.t. $\{U_i\}$:

$\forall j \geq 0$, $\check{C}^j(\mathfrak{U}, \mathcal{F}) = \prod_J \Gamma(U_J, \mathcal{F})$

over all $(j+1)$-element subset of $I$

$d^j : \check{C}^j(\mathfrak{U}, \mathcal{F}) \to \check{C}^{j+1}(\mathfrak{U}, \mathcal{F})$

$\alpha = (\alpha_J) \longmapsto (d^j(\alpha)_J)$

where $d^j(\alpha)_J = \sum_{k=0}^{j+1} (-1)^k \operatorname{Res}_{U_{J-\{i_k\}}, J}(\alpha_{J-\{i_k\}})$

$J = \{i_0 \leq \cdots \leq i_{j+1}\}$ ← $j+2$ elements.

E.g. $\mathfrak{U} = \{U_1, U_2\}$:

$$0 \to \Gamma(U_1, \mathcal{F}) \oplus \Gamma(U_2, \mathcal{F}) \xrightarrow{d^0} \Gamma(U_1 \cap U_2, \mathcal{F}) \to 0.$$

**Lemma** $0 \to \mathcal{F} \to \check{C}^0(\mathfrak{U}, \mathcal{F}) \to \check{C}^1(\mathfrak{U}, \mathcal{F}) \to \cdots$ exact.

(c.f. Hartshorne Lemma III.4.2)

$\rightsquigarrow$ Write $\check{H}^i(\mathfrak{U}, \mathcal{F}) = h^i(\check{C}^\cdot(\mathfrak{U}, \mathcal{F}))$.

NOT a $\delta$-functor if $\mathfrak{U}$ is fixed.

(See Hartshorne Caution 4.0.2)

- If $\mathfrak{U} = \{X\}$, then $\Gamma(X, -)$ is not exact.

**Lemma** $\mathcal{F}$ flasque $\Rightarrow$ $\check{H}^i(\mathfrak{U}, \mathcal{F}) = 0$, $\forall i > 0$

Refinement limit: $\check{H}^i(X, \mathcal{F}) = \varinjlim_{\mathfrak{U}} \check{H}^i(\mathfrak{U}, \mathcal{F})$.

<u>Thm</u> $X$ paracpt (i.e. Hausdorff + every $\mathcal{U}$ refines to a loc. finite subcovering).

$\Rightarrow \check{H}^{\cdot}(X,\mathcal{F})$ effaceable $\Rightarrow$ universal-$\delta$

$\Rightarrow \check{H}^{\cdot}(X,\mathcal{F}) \cong H^{i}(X,\mathcal{F})$

$\rightsquigarrow \forall \mathcal{U}$ particular covering, $\check{H}^{\cdot}(\mathcal{U},\mathcal{F}) \to H^{\cdot}(X,\mathcal{F})$.

<u>Thm</u> (Leray) $\mathcal{U}$ good for $\mathcal{F}$ (i.e. $\forall J \subseteq I$, $\mathcal{F}|_{U_J}$ acyclic)

$\Rightarrow \check{H}^{i}(\mathcal{U},\mathcal{F}) \cong H^{i}(X,\mathcal{F})$, $\forall i \geq 0$

namely: $\check{C}^{i}(\mathcal{U},\mathcal{F})$ computes the sheaf cohom.

<u>Rmk</u> Analogues: (1) contractible open $\longleftrightarrow$ affine

(2) good covers $\longleftrightarrow$ quasi-coherent sheaves