COURSEWORK FOR TOPICS IN LANGLANDS PROGRAM (SPRING 2024)

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This document is about the course *Topics in Langlands Program* by Yihang Zhu at Qiuzhen College, Tsinghua University, during the Spring 2024 semester. The following contains all homework problems with solutions by W.D., who is responsible for any mistakes in this document.

The course roughly covers the first half of the book [BH06] of Bushnell–Henniart which is about the (smooth) representation theory of a p-adic group G, with an emphasis on $G = \mathrm{GL}_2$.

Contents

Homework 1

Exercise 1.1 (General properties of topological groups)**.** Let G be a topological group.

(1) All open subgroups of G are closed.
(2) A subgroup H of G is open if and only if the coset space G/H with quotient topology is discrete.
(3) If H and H' are subgroups of G such that $H' \subset H$ and H' is open in G, then H is open in G.
(4) G is Hausdorff if and only if $\{e\}$ is closed. Here e is the identity element.

Solution. (1) Let H be an open subgroup of G. Apply the set-theoretical decomposition

$$G = H \sqcup ((G - H)H).$$

Note that $G - H$ is closed and H is open, and thus $(G - H)H$ is open. This follows from the fact that the product of an open subset of G with any subset of G is open. Therefore, H is also closed.

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(2) If H is open in G then every coset gH is open. It follows that every single point in G/H is open, which is the definition that G/H is discrete. The converse is proved by reversing the argument.

(3) Since H' is open, we have from (2) that G/H' is a discrete quotient. On the other hand, by assumption G/H is a quotient of G/H', which is again discrete. It again follows from (2) that H is open in G.

(4) For the "if" part, suppose $\{e\}$ is closed. Let $x, y \in G$ be such that $x \neq y$. Then $x(G - \{e\})y^{-1}$ is open, so its complement $x\{e\}y^{-1} = \{xy^{-1}\}$ is closed, where $xy^{-1} \neq e$ by assumption. Let U be an open neighborhood of 1 such that $U^2 \subset G - \{xy^{-1}\}$.[1] We check the Hausdorff property by showing that $Ux \cap Uy = \emptyset$, as Ux and Uy are open neighborhoods of x and y in G, respectively. If the intersection is non-empty, then we can find $u_1, u_2 \in U$ such that $u_1 x = u_2 y$. But then $xy^{-1} = u_1^{-1} u_2 \in \{xy^{-1}\} \cap U^2$, which is not possible.

For the "only if" part, suppose G is Hausdorff, and thus the diagonal $\Delta \coloneqq \{(x,x) \colon x \in G\}$ is closed in $G \times G$. On the other hand, given the continuous action map $\gamma \colon G \times G \to G,\ (x,y) \mapsto xy^{-1}$, we get $\gamma^{-1}(\{e\}) = \Delta \subset G \times G$, which is closed. It follows that $\{e\}$ is closed in G. $\square$

Exercise 1.2. Consider $\mathrm{GL}_n(\mathbb{C})$ equipped with the natural topology induced from that on $\mathbb{C}$. Show that there exists a neighborhood U of 1 in $\mathrm{GL}_n(\mathbb{C})$ such that no non-trivial subgroup of $\mathrm{GL}_n(\mathbb{C})$ is contained in U. (Hint: Use the exponential map.) Use this to show that any continuous homomorphism from a profinite group to $\mathrm{GL}_n(\mathbb{C})$ must have finite image.

Solution. For the first statement, consider $\exp \colon \mathrm{M}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ which is locally a diffeomorphism at $0 \in \mathrm{M}_n(\mathbb{C})$, i.e., there exists an open neighborhood V of $0 \in \mathrm{M}_n(\mathbb{C})$ such that $\exp|_V$ is a diffeomorphism. Take $U = \exp(V/2) \subset \mathrm{GL}_n(\mathbb{C})$, which is an open neighborhood of $1 \in \mathrm{GL}_n(\mathbb{C})$. For any $g \in U$ there is $v \in V/2$ such that $\exp(v) = g$; thus, $g^k = \exp(kv)$ with $kv \notin V/2$ for each $k \geqslant 2$. It follows that $g^k \notin U$, namely U does not contain any subgroup of $\mathrm{GL}_n(\mathbb{C})$.

For the second statement, let G be a profinite group and $f \colon G \to \mathrm{GL}_n(\mathbb{C})$ be a continuous homomorphism. Taking $U \subset \mathrm{GL}_n(\mathbb{C})$ as before, the preimage $f^{-1}(U)$ must be an open neighborhood of e. Since G is profinite, for e the identity element in G, there is a neighborhood basis consisting of compact open subgroups of G (c.f. Exercise 1.3(2)). It follows that there is an open subgroup $H \subset f^{-1}(U)$ of G. Since U contains no non-trivial subgroup of $\mathrm{GL}_n(\mathbb{C})$, we get a subgroup $f(H) \subset f(f^{-1}(U)) = U$. This leads to a contradiction unless $f(H) = \{1\}$. Again since G is profinite, it is by definition compact; this together with $f(H) = \{1\}$ force f to have finite image. $\square$

Exercise 1.3. Let X be a Hausdorff, compact, totally disconnected topological space. We show that X is homeomorphic to an inverse limit of finite sets, as follows. Let I be the set of maps $f \colon X \to \mathbb{Z}$ such that $\mathrm{im}(f)$ is finite and each $f^{-1}(n)$ is open. Informally, I is the set of ways of partitioning X into a finite disjoint union of open subsets. For $f, g \in I$, define $f \leq g$ if there exists a (necessarily unique) map $\phi_{f,g} \colon \mathrm{im}(g) \to \mathrm{im}(f)$ such that $f = \phi_{f,g} \circ g$ (i.e., the partition of X corresponding to g refines that corresponding to f). Then $(\mathrm{im}(f))_{f \in I}$ is a projective system of finite sets with transition maps $\phi_{f,g}$, and we have a natural continuous map

$$\begin{aligned} \Phi \colon X &\longrightarrow \varprojlim_{f \in I} \mathrm{im}(f) \\ x &\longmapsto (f(x))_f \end{aligned}$$

[1] Such U exists for the following reason. Let V be an open neighborhood of 1 in G and $m \colon G \times G \to G,\ (x,y) \mapsto xy$ be the continuous G-action on itself. Then $W \coloneqq m^{-1}(V)$ is open. We have $(1,1) \in W$ because $1^2 = 1 \in V$. By definition of the product topology on $G \times G$, there exists an open subset $\Omega \ni 1$ of G such that $\Omega \times \Omega \subset W$. We have $\Omega^2 \subset V$ by definition of W. Let $U = \Omega \cap \Omega^{-1}$. We know that Ω^{-1} is open from that Ω is open together with the fact that the inverse map $\iota \colon G \to G,\ x \mapsto x^{-1}$ is continuous. So U is open and it is symmetric by construction. We clearly have $1 \in U$ and $U^2 \subset \Omega^2 \subset V$.

Here, the right hand side is equipped with the inverse limit topology coming from the discrete topology on each $\mathrm{im}(f)$ (which is a finite set).

(1) Prove that Φ is a homeomorphism. Hint: Use the following facts (which you can also try and prove yourself):
  (a) Any bijective continuous map from a compact space to a Hausdorff space is a homeomorphism.
  (b) In a compact Hausdorff space, the connected component containing a point is the intersection of all clopen sets (i.e. sets that are simultaneously closed and open) containing that point.

(2) Use (1) to show that the topology on X has a basis consisting of compact open sets.

Solution. (1) Note that Φ is naturally continuous. Using Hint (a), since X is compact by assumption and $\varprojlim_{f\in I} \mathrm{im}(f)$ is Hausdorff, it suffices to check that Φ is bijective.

- To show Φ is surjective, let $\underline{n} = (n_f)_f \in \varprojlim_{f\in I} \mathrm{im}(f)$. We need to find a preimage of $\underline{n}$. For each $f \in I$, let $C_f = f^{-1}(n_f)$. Then C_f is clopen in X, and we have $C_g \subset C_f$ whenever $f \le g$. Note that $\Phi^{-1}(\underline{n}) = \bigcap_{f\in I} C_f$. If this were empty then by compactness of X we know that a finite sub-intersection is empty. Finding a common upper bound of the indices, we get some $f \in I$ such that $C_f = \emptyset$, a contradiction. This shows that Φ is surjective.
- To show Φ is injective, by Hint (b) above, we have for any $x \in X$ that $\{x\}$ is the intersection of all clopens containing x. If $\Phi(x) = \Phi(y)$ then for any clopen U containing x it also contains y, since the characteristic function $\mathbb{1}_U$ of U is an element of I. Taking the intersections of all such clopens we conclude $x = y$.

This completes the proof that Φ is a homeomorphism.

(2) By (1) we see X is profinite; it is the subset of $\prod_{f\in I} \mathrm{im}(f)$ consisting of $(x_f)_f$ such that $\phi_{f,g}(x_g) = x_f$ for all $f, g \in I$ with $f \le g$. We endow it with the subspace topology inherited from the product topology on $\prod_{f\in I} \mathrm{im}(f)$, where each $\mathrm{im}(f)$ having the discrete topology. Fix a finite subset $I_0 \subset I$ and for each $f \in I_0$ we fix $X_f \subset \mathrm{im}(f)$. Let

$$U := \prod_{f\in I-I_0} \mathrm{im}(f) \times \prod_{g\in I_0} X_g.$$

Then U is clopen in $\prod_{f\in I} \mathrm{im}(f)$. Varying I_0 and $(X_g)_{g\in I_0}$ the U's form a clopen basis of the topology. Also, each such U is a closed subset of X, and thus U is Hausdorff, compact, and totally disconnected. It follows that U is profinite. Therefore, X has a basis of topology consisting of compact open profinite subsets. □

Exercise 1.4 (van Dantzig's Theorem)**.** We are to prove the following statement: Let G be a topological group which is Hausdorff, locally compact, and totally disconnected. Then G is locally profinite in the sense of the lecture, i.e., 1 has a neighborhood basis consisting of compact open subgroups of G. (Assuming Hausdorff, the converse is also true, and is easier to prove.) Prove this theorem in the following steps:

(1) Using Exercise 1.3, show that the compact open neighborhoods of 1 in G form a neighborhood basis. Let K be an arbitrary compact open neighborhood of 1. In the remaining part it suffices to construct a compact open subgroup H of G contained in K.
(2) For every $x \in K$ there is an open neighborhood V_x of 1 such that $xV_x^2 \subset K$.
(3) There is an open neighborhood V of 1 such that $KV \subset K$. In particular, $V \subset K$ and $V^2 \subset K$.
(4) We may take V to satisfy $V^{-1} = V$.
(5) Let H be the subgroup of G generated by V. Then H is an open subgroup of G, and $H \subset K$.
(6) H is compact since it is closed and contained in K.

Solution. Since G is locally compact, there is a compact open neighborhood of 1 in G, say K. Then K is a Hausdorff, compact, and totally disconnected topological space. It follows from Exercise 1.3 that K has a basis consisting of compact open sets. It suffices to construct a compact open subgroup H of G contained in K. Since the left translation $G \to G,\ g \mapsto xg$ is continuous at 1, there is an open neighborhood U_x of 1 with $xU_x \subset K$. Since the multiplication $m\colon G \times G \to G$ is continuous at $(1,1)$, there is an open neighborhood V_x of 1 such that $V_x^2 \subset U_x$ (see the footnote on page 2 for details). So we get $xV_x^2 \subset K$.

Notice that K admits an open cover $\bigcup_{x\in K} xV_x$. Then the compactness of K renders a finite subcover, say $\bigcup_{i=1}^n x_iV_{x_i} \supset K$. Let $V = V_{x_1} \cap \cdots \cap V_{x_n}$. Then

$$KV \subset \bigcup_{i=1}^n x_iV_{x_i}V \subset \bigcup_{i=1}^n x_iV_{x_i}V_{x_i} \subset \bigcup_{i=1}^n x_iU_{x_i} \subset K.$$

The inclusion $V \subset KV \subset K$ implies $V^2 \subset K$. As the inverse map $G \to G,\ g \mapsto g^{-1}$ is also continuous, we may take V to satisfy $V^{-1} = V$, i.e. $g \mapsto g^{-1}$ induces a bijection while restricting on $V \subset G$. Let H be the subgroup of G generated by V. Then H is an open subgroup of G, and $H \subset K$. By Exercise 1.1(1), H is also closed, and hence compact. This completes the proof. □

Exercise 1.5. Let G be a locally profinite group.

(1) A representation (π, V) of G is smooth if and only if the action map $G\times V \to V,\ (g, v) \mapsto \pi(g)v$ is continuous, where V is equipped with the discrete topology.
(2) Let (π, V) be a smooth representation. Then any subrepresentation or quotient representation of (π, V) is again smooth.
(3) For an arbitrary representation (π, V) of G, the subspace $V^\infty = \bigcup_{K \text{ cos of } G} V^K$ is a subrepresentation, and it is the maximal smooth subrepresentation.

Solution. (1) Since the target V is equipped with the discrete topology, each singleton $\{\pi(g)v\}$ is open. Then the given action map is continuous if and only if $\{g \in G\colon \pi(g)v = v\}$ for all fixed $v \in V$ are open in G. This is equivalent to the definition that (π, V) is smooth.

(2) By (1), if (π, V) is smooth then $G \times V \to V,\ (g, v) \mapsto \pi(g)v$ is continuous. Let W be a subrepresentation of V. Then $G \times W \to V,\ (g, w) \mapsto \pi(g)w$ is also continuous (here W is a subrepresentation implies that the image of G-action again lands in W), which implies that $\{g \in G\colon \pi(g)w = w\}$ is open in G, and thus W is smooth. For the case of quotient representation, the result is implied by the continuity of $G \times (V/W) \to V/W$.

(3) By construction, each element $v^\infty \in V^\infty$ is stabled by certain compact open subgroup K, and hence $V^\infty \subset \{v \in V : \mathrm{Stab}_vG \text{ is open}\}$. Since G is locally profinite, there is a neighborhood basis of $1 \in G$ consisting compact open subgroups. It follows that each $v \in V$ with open Stab_vG is stabilized by some open compact subgroup, and hence $V^\infty = \{v \in V : \mathrm{Stab}_vG \text{ is open}\}$. Note that the latter set is G-stable, and so also is V^∞. This shows that V^∞ is a subrepresentation, which is smooth by definition. Moreover, if there is $v \in V$ that is not stabilized by any compact open subgroup, i.e. $v \notin V^\infty$, then Stab_vG cannot be open, and hence v is not smooth. Therefore, V^∞ exactly consists of all smooth vectors in V, and hence is the maximal smooth subrepresentation. □

Exercise 1.6. Let F be a finite extension of $\mathbb{Q}_p$. Show that the natural map $\mathrm{GL}_n(\mathcal{O}_F) \to \varprojlim_i \mathrm{GL}_n(\mathcal{O}_F/\mathfrak{m}_F^i)$ is an isomorphism of topological groups. Here on the right hand side each $\mathrm{GL}_n(\mathcal{O}_F/\mathfrak{m}_F^i)$ is finite and equipped with the discrete topology.

Solution. It suffices to check the compatibility of the topologies on both $\varprojlim_i \mathrm{GL}_n(\mathcal{O}_F/\mathfrak{m}_F^i)$ and $\mathrm{GL}_n(\mathcal{O}_F)$. Since the former is a projective limit of finite discrete groups, we are to show that $\mathrm{GL}_n(\mathcal{O}_F)$ is profinite. By Exercise 1.3, it boils down to showing that $\mathrm{GL}_n(\mathcal{O}_F)$ is a Hausdorff, compact, and totally disconnected topological space. For this, identify $\mathrm{M}_n(\mathcal{O}_F)$ with $\mathcal{O}_F^{n^2}$, and consider $\mathrm{GL}_n(\mathcal{O}_F) \hookrightarrow \mathrm{M}_n(\mathcal{O}_F)\times\mathcal{O}_F \simeq \mathcal{O}_F^{n^2+1}$ via $g \mapsto (g, \det(g^{-1}))$; also, note that $\mathrm{GL}_n(\mathcal{O}_F)$ is cut out from $\mathcal{O}_F^{n^2+1}$ by polynomial equations, and is thus a closed subset with induced topology from $\mathcal{O}_F^{n^2+1}$. This proves the desired result as $\mathcal{O}_F$ itself is a Hausdorff, compact, and totally disconnected topological space. $\square$

Homework 2

Exercise 2.1. Let V be a possibly infinite-dimensional vector space over a field F.

(1) Use Zorn's lemma to show that V has a basis.
(2) For any subspace $W \subset V$, there exists a complement, i.e., a subspace $W' \subset V$ such that $V = W \oplus W'$.
(3) If $0 \to W \to V \to U \to 0$ is a short exact sequence of vector spaces, then the dual sequence $0 \to U^* \to V^* \to W^* \to 0$ is also exact.

Solution. (1) Let $(U_i)_{i\in I}$ be a totally ordered collection of linearly independent subsets in V, i.e. for each U_i and finitely many vectors $v_1, \dots, v_n \in U_i$ we have $f_1 v_1 + \dots + f_n v_n = 0 \in V$ implies $f_1 = \dots = f_n = 0$ in F; here the order is given by the set-theoretical inclusion, that is, $U_i \leq U_j$ if and only if $U_i \subset U_j$. From this construction, $(U_i)_{i\in I}$ forms a chain of linearly independent subsets in V, which obtains a maximal element $\bigcup_{i\in I} U_i$. It follows from Zorn's lemma that among all linearly independent subsets in V there is a set-theoretically maximal subset, whose elements form a basis of V.

(2) Let W be a subspace of V. Pick a basis $\{e_i\}_{i\in I}$ of V so that $V = \bigoplus_{i\in I} Fe_i$. Consider the set $\mathcal{J}$ of subsets J of I for which $W \cap \sum_{j\in J} Fe_j = 0$. Then $\mathcal{J} \neq \emptyset$ and $\mathcal{J}$ is inductively ordered by inclusion. If J is a maximal element of $\mathcal{J}$ then the sum $X := W + \bigoplus_{j\in J} Fe_j$ is direct. If $X \neq V$ there is $i \in I$ with $Fe_i \not\subset X$, so the sum $X + Fe_i$ is direct, and $J \cap \{i\} \in \mathcal{J}$, contrary to hypothesis. Thus $W' = \bigoplus_{j\in J} Fe_j$ is the desired complement of W.

(3) Given an injective map $s\colon W \to V$, it induces the dual map $s^*\colon V^* \to W^*$, $\phi \mapsto \phi \circ s$. If s is surjective and $\phi \circ s = 0$, then $\phi = 0$ follows; in this case s^* is injective. Similarly, whenever $t\colon V \to U$ is injective, we have that $t^*\colon U^* \to V^*$ is surjective. Therefore, the dual functor $(-)^*$ on vector spaces is exact. (Here we omit to check the exactness condition $\ker s^* = \operatorname{im} t^*$.) $\square$

Exercise 2.2. Show that a countable-dimensional vector space over $\mathbb{Q}$ (the rational numbers) is countable as a set. Show that the dual of such a vector space is uncountable as a set. (Here, countable means infinite countable.)

Solution. Let V be a countable-dimensional $\mathbb{Q}$-vector space with basis $\{e_i\}_{i\in I}$. Then $\operatorname{card} I \leqslant \operatorname{card} \mathbb{Z}$. Fix an isomorphism $V = \bigoplus_{i\in I} \mathbb{Q}e_i \simeq \mathbb{Q}^I$ of vector spaces. Then by set theory, $\operatorname{card} V = \operatorname{card} \mathbb{Q}^I = \operatorname{card} I \cdot \operatorname{card} \mathbb{Q} = \operatorname{card} I \cdot \operatorname{card} \mathbb{Z} \leqslant \operatorname{card} \mathbb{Z} \cdot \operatorname{card} \mathbb{Z} = \operatorname{card} \mathbb{Z}$. It follows that V is countable as a set.

As for the dual space V^*, it suffices to show that

$$\dim_{\mathbb{Q}} V < \dim_{\mathbb{Q}} V^*.$$

Let $\{e_i\}_{i\in I}$ be a basis of V. Each element of V^* is determined by its values on $\{e_i\}_{i\in I}$. For each $i \in I$, define $\phi_i \in V^*$ by $\phi_i(e_i) = 1$ and $\phi_i(e_j) = 0$ for $j \neq i$; then $\{\phi_i\}_{i\in I}$ is a linearly independent subset of V^*. It follows that V^* is infinite-dimensional, and thus $\dim_{\mathbb{Q}} V^* \geqslant \operatorname{card} \mathbb{Q}$.

> *Lemma.* If W is a vector space over a field K, then
>
> $$\operatorname{card} W = \max(\operatorname{card} K, \dim_K W).$$
>
> *Proof of Lemma.* For a non-zero $w \in W$, the set Kw is a sunset of W with size $\operatorname{card} K$, so $\operatorname{card} K \leqslant \operatorname{card} W$. A basis of W is a subset of W with size $\dim_K W$, so $\dim_K W \leqslant \operatorname{card} W$. Next we show $\operatorname{card} W \leqslant \max(\operatorname{card} K, \dim_K W)$. Pick a basis $\{f_i\}_{i\in I}$ of W. The elements of $w \in W$ are unique finite linear combinations $\sum_{i\in I} c_i f_i$ where all but finitely many $c_i \in K$ are non-zero; so we get an embedding of W into the finite subsets of $K \times I$ by $w \mapsto \{(c_i, f_i)\colon c_i \neq 0\}$. (Note when $w = 0$ we get $\emptyset \subset K \times I$.) Since I is infinite and $K \neq \emptyset$, we have $K \times I$ is infinite, and the cardinality of the finite subsets of an infinite set equals the cardinality of the set. Thus $\operatorname{card} W \leqslant \operatorname{card}(K \times I)$. It follows that

$\operatorname{card}(W) \leqslant \max(\operatorname{card}(K), \operatorname{card}(I)) = \max(\operatorname{card} K, \dim_K W)$. Thus the equality follows.

Using the lemma, we see that $\operatorname{card} V^* = \max(\operatorname{card} \mathbb{Q}, \dim_{\mathbb{Q}} V^*) = \dim_{\mathbb{Q}} V^*$. Now to prove $\dim_{\mathbb{Q}} V < \dim_{\mathbb{Q}} V^*$, it remains to show that $\operatorname{card} V^* > \dim_{\mathbb{Q}} V$. We know that $\dim_{\mathbb{Q}} V = \operatorname{card} I$. Elements of V^* are determined by their values on that basis, and those values can be arbitrary, so as a set V^* is in bijection with $\prod_{i \in I} \mathbb{Q}$. Denote $\mathcal{P}(I)$ the power set of I, i.e. the set of all subsets of I. Then $\operatorname{card} V^* \geqslant \operatorname{card}(\prod_{i \in I} \mathbb{Q}) \geqslant \operatorname{card} \mathcal{P}(I) > \operatorname{card} I = \dim_{\mathbb{Q}} V$. This finishes the proof that V^* is uncountable. □

In the following, fix a locally profinite group G, and all representations are smooth representations of G.

Exercise 2.3. Let $V = \bigoplus_{i \in I} U_i$, where each U_i is an irreducible representation. Let $W \subset V$ be a subrepresentation. Use Zorn's lemma to show that there is a maximal subset $J \subset I$ such that $W \cap \bigoplus_{j \in J} U_j = 0$. For such J, show that $V = W \oplus \bigoplus_{j \in J} U_j$.

Solution. The situation is trivial when $W = V$. If $V \neq W$ then by Exercise 2.1(2) there is a proper subspace $0 \neq W' \subsetneq V$ such that $V = W \oplus W'$; in particular, $W \cap W' = 0$. It follows that there exists $i \in I$ such that U_i is a subspace of W' with that $W \cap U_i = 0$. Again if $W \oplus U_i \neq V$ then we get $j \in I$ with $j \neq i$ and $W \cap (U_i \oplus U_j) = 0$. Repeating this process iteratively we get an increasing chain of index sets $\{i\} \subset \{i, j\} \subset \cdots \subset I$. By Zorn's lemma there is a maximal subset J in this chain with the desired property. Also, the equality $V = W \oplus \bigoplus_{j \in J} U_j$ follows from that J is maximal. □

Exercise 2.4. Let V be a finitely generated representation. That is, there exist finitely many elements $v_1, \ldots, v_n$ such that V is spanned by $\bigcup_{i=1}^n Gv_i$. Prove that there exists a maximal proper subrepresentation $W \subset V$. (Proper means $W \neq V$, but we allow W to be zero.) In general, for any representation V, show that a proper subrepresentation W is maximal if and only if V/W is irreducible.

Solution. By the finitely-generated assumption, each proper subrepresentation of V does not contain the subset $\{v_1, \ldots, v_n\}$ of V. (Otherwise the G-orbit of $\{v_1, \ldots, v_n\}$ must span the whole V as it contains $Gv_1 \cup \cdots \cup Gv_n$. Let S be the set of all proper subrepresentations of V, which is non-empty since $0 \in S$. Since $n < \infty$, any totally ordered union of elements in S is still proper. Hence by Zorn's lemma there is a maximal element in S.

In general, if V is any (smooth) representation, it is clear that V/W is irreducible implies that W is maximal. Conversely, if V/W is reducible, then for non-zero element $vW \in V/W$ we have that $W \subsetneq W + Gv \subsetneq V$, which implies that W is not maximal as a subrepresentation. □

Exercise 2.5. Show that a sequence of representations $0 \to U \to V \to W \to 0$ is exact if and only if $0 \to U^K \to V^K \to W^K \to 0$ is exact for every compact open subgroup $K \subset G$.

Solution. The "if" part is clear and we omit the discussion. For the "only if" part, since taking invariants $(-)^K \colon V \mapsto V^K$ is always left exact, it suffices to show that whenever $b \colon V \to W$ is surjective, so also would $b^K \colon V^K \to W^K$ be for all compact open subgroups K. By the hypothesis, given $w \in W^K \subset W$ there exists $v \in V$ such that $b(v) = w$. Since W^K is a subrepresentation of W, we have a natural projection map $p_W \colon W \to W^K$; similarly we define $p_V \colon V \to V^K$. From the commutative diagram

$$\begin{tikzcd} V \arrow[r, "b"] \arrow[d, "p_V"'] & W \arrow[d, "p_W"] \\ V^K \arrow[r, "b^K"] & W^K \end{tikzcd}$$

we have that $w = p_W(w) = p_W(b(v)) = b^K(p_V(v))$. This proves the surjectivity of b^K and thus completes the proof. $\square$

Exercise 2.6 ([BH06, p.17, Exercise (2)(3)])**.**

(1) Let (π, V) be a smooth representation of G and (σ, W) an abstract representation. Let $f\colon V \to W$ be a G-homomorphism. Show that $f(V) \subset W^\infty$, and hence $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_G(V, W^\infty)$.

(2) Let $0 \to U \to V \to W \to 0$ be an exact sequence of G-homomorphisms of abstract G-spaces. Show that the induced sequence $0 \to U^\infty \to V^\infty \to W^\infty$ is exact. Show by example that the map $V^\infty \to W^\infty$ need not be surjective.

Solution. (1) Since V is a smooth representation we have $V = V^\infty$; in particular, each $v \in V$ lies in some V^K with K a compact open subgroup of G. Fix $v \in V^K \subset V$. As f is a G-homomorphism, we have that $\sigma(g)f(v) = f(\pi(g)v)$ for all $g \in G$. Whenever $g \in K$, we get $\sigma(g)f(v) = f(\pi(g)v) = f(v)$, that is, $f(v) \in W^K$. It follows that $f(V^K) \subset W^K$, and then $f(V) = f(V^\infty) \subset W^\infty$; therefore, $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_G(V, W^\infty)$.

(2) As in Exercise 2.5, taking invariants $(\textbf{–})^K$ is always left exact, and thus the induced sequence $0 \to U^\infty \to V^\infty \to W^\infty$ is exact. For the counterexample, let V, W be two abstract G-spaces such that there is a G-homomorphism $V \to W$ factoring through $V/V^\infty \to W$ and $V/V^\infty \to W$ is surjective; such a construction exists because V/V^∞ is isomorphic to a sub-representation of V when V is semi-simple and sufficiently large. On the other hand, we have $(V/V^\infty)^\infty = 0$, which cannot map onto W^∞ unless $W^\infty = 0$. $\square$

Exercise 2.7 ([BH06, p.25, Exercise])**.** Let (π, V) and (σ, W) be smooth representations of G. Let $\wp(\pi, \sigma)$ be the space of G-invariant bilinear pairings $V \times W \to \mathbb{C}$. Show that there are canonical isomorphisms

$$\mathrm{Hom}_G(\pi, \check{\sigma}) \cong \wp(\pi, \sigma) \cong \mathrm{Hom}_G(\sigma, \check{\pi}).$$

Solution. It suffices to show the first isomorphism, and the second one follows from swapping π and σ. Consider the map $\wp(\pi, \sigma) \to \mathrm{Hom}_G(\pi, \check{\sigma})$ that sends $f\colon V \times W \to \mathbb{C}$ to $\mathrm{R}_f\colon V \to W^\vee$, where $\mathrm{R}_f(v) = f(v, \textbf{–})$. As a map between vector spaces, it suffices to show that the map is bijective. For the injectivity, note that $\mathrm{R}_f(v)$ is a trivial linear form on W if and only if f is degenerate. For the surjectivity, it suffices to define the inverse of R_f; given $\psi\colon V \to W^\vee$ we have $\psi(v)\colon w \mapsto \psi(v)(w) \in \mathbb{C}$, so $(v, w) \mapsto \psi(v)(w)$ is the desired inverse map. This completes the proof. $\square$

Homework 3

Exercise 3.1. Show that in a profinite group, the compact open normal subgroups form a neighborhood basis of 1.

Solution. From the result of Exercise 1.3, if K is an arbitrary compact open neighborhood of $1 \in G$, where G is a profinite group, then K has a basis consisting of compact open sets, and we aim to show that there is a compact open normal subgroup H of G contained in K. By the proof of van Dantzig's theorem (c.f. Exercise 1.4), there is a compact open subgroup H_0 of G contained in K. We then check that the desired H is given by

$$\bigcap_{g \in G/H_0} gH_0g^{-1}.$$

This is clearly an open normal subgroup of G, and hence is closed in G by Exercise 1.1(1); and as a closed subset of the compact H_0, it is compact as well. This completes the proof by following the guideline of Exercise 1.4. ☐

Exercise 3.2. Prove that the two maps specified in the proof of Frobenius Reciprocity (for Ind_G^H) are indeed inverse to each other.

Solution. Let H be a closed subgroup of G. Consider smooth representations π and σ of G and H, respectively. To prove the Frobenius reciprocity of Ind_H^G and Res_H^G, we have defined the following maps:

$$\begin{aligned} r\colon \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) &\longrightarrow \mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \sigma) \\ \Phi &\longmapsto \alpha_\sigma \circ \Phi \end{aligned}$$

together with

$$\begin{aligned} i\colon \mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \sigma) &\longrightarrow \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) \\ \Psi &\longmapsto (v \mapsto (f\colon G \to \sigma,\ g \mapsto \Phi(gv))). \end{aligned}$$

Here $\alpha_\sigma\colon \mathrm{Ind}_H^G \sigma \to \sigma$ is an H-map defined via $(f\colon G \to \sigma) \mapsto f(1)$. We first check that $r(i(\Psi)) = \Psi$. For each $v \in \pi$, its image along $r(\Psi)$ is $f\colon g \mapsto \Psi(gv)$, and thus after the pre-composition by α_σ we have $f(1) = \Psi(v) \in \sigma$, which is the same as the image of $v \in \mathrm{Res}_H^G \pi$ along $\Psi\colon \mathrm{Res}_H^G \pi \to \sigma$. We then check that $i(r(\Phi)) = \Phi$. For this we compute $i(\alpha_\sigma \circ \Phi)$ by considering $\pi \to \mathrm{Ind}_H^G \sigma \to \sigma$, $v \mapsto (g \mapsto \Phi(gv) \mapsto \Phi(v))$, which is the same as $v \mapsto \Phi(v)$. ☐

Exercise 3.3 ([BH06, p.19, Exercise 1–2])**.**

(1) Show that the functor c-Ind_H^G is additive and exact.
(2) Let G be a locally profinite group. Suppose H is open in G. Let $\phi\colon G \to W$ be a function, compactly supported modulo H, such that $\phi(hg) = \sigma(h)\phi(g)$ for $h \in H$ and $g \in G$. Show that $\phi \in X_c$.

Solution. (1) Recall that for a smooth H-representation σ, c-$\mathrm{Ind}_H^G\sigma$ consists of functions $f \in \mathrm{Ind}_H^G \sigma$ whose supports have compact projection image in $H\backslash G$. So the additivity of c-Ind_H^G follows from that of Ind_H^G, which is proved in [BH06, §2.4, Proposition]. For the exactness[2], by definition each function $f \in$ c-$\mathrm{Ind}_H^G\sigma$ satisfies $f(hgk) = \sigma(h)f(g)$, where $h \in H, g \in G$ and $k \in K$ for any open compact subgroup K. Thus if K is fixed then $f^K\colon G/K \to \sigma$ is determined by all $f(g)$'s whenever g runs through $H\backslash G/K$. (The double coset $H\backslash G/K$ is not necessarily finite unless $H\backslash G$ is compact.) This indeed proves that there is a natural isomorphism

[2]If we further assume H is open in G, then we have the Frobenius reciprocity $\mathrm{Hom}_G(\text{c-}\mathrm{Ind}_H^G\sigma, \pi) \cong \mathrm{Hom}_H(\sigma, \mathrm{Res}_H^G \pi)$. In this case c-$\mathrm{Ind}_H^G$ has a right adjoint Res_H^G; it follows that c-Ind_H^G is right exact.

$$\begin{aligned}(\text{c-Ind}_H^G\sigma)^K &\xrightarrow{\ \sim\ } \bigoplus_{g\in H\backslash G/K} \sigma^{gKg^{-1}\cap H}\\ f &\longmapsto (f(g))_{g\in H\backslash G/K}.\end{aligned}$$

On the other hand, to show the exactness of $\text{c-Ind}_H^G(-)$, it suffices to show for an arbitrary open compact subgroup K that $(\text{c-Ind}_H^G(-))^K$ is exact. Applying the result of Exercise 2.5, the functor $\sigma \mapsto \sigma^{gKg^{-1}\cap H}$ is exact as $gKg^{-1}\cap H$ is a compact open subgroup of H. So the functor sending σ to the target direct sum of the above isomorphism is exact as well. This finishes the proof that $\sigma \mapsto \text{c-Ind}_H^G\sigma$ is exact.

(2) Let (σ, W) be a smooth H-representation. By construction, X_{c} is the space of functions $f\colon G \to W$ satisfying that

- (i) f is compactly supported modulo H, i.e. $\operatorname{supp}(f)$ has compact image along the projection $G \to H\backslash G$,
- (ii) $f(hg) = \sigma(h)f(g)$ for $h \in H$ and $g \in G$, and
- (iii) f is right G-smooth, i.e. there is a compact open subgroup K of G (depending on f) such that $f(gk) = f(g)$ for $g \in G$ and $k \in K$.

So the point is to show that the right G-smooth condition for ϕ is automatic. Here the condition (i) is equivalent to that $\operatorname{supp}(f) \subset HC$ for some compact subset $C \subset G$. The condition that H is open in G implies that H is a closed subgroup and that HC is an open subset. Since G is locally profinite, H is compact, and HC is thus an open compact subset. Again as G is locally profinite, $1 \in G$ has a neighborhood basis consisting of open compact subgroups. As a consequence, there are finitely many sufficiently small open compact subgroups $K_1, \dots, K_n$ such that HC admits a finite cover by $\{g_iK_i\}_{i=1}^n$ for distinct $g_i \in G$. It follows that $\operatorname{supp}(f)$ admits a finite cover by $\{Hg_iK_i\}_{i=1}^n$. Note that G is totally disconnected and K is a sufficiently small neighborhood of 1. Therefore, $K \coloneqq K_1 \cap \cdots \cap K_n$ is the desired open compact subgroup such that $f(gk) = f(g)$ for all $g \in G$ and $k \in K$. □

Exercise 3.4. Let H, H_0 be closed subgroups of a locally profinite group G with $H_0 \subset H$. Show that $\operatorname{Ind}_H^G \circ \operatorname{Ind}_{H_0}^H$ is canonically isomorphic to $\operatorname{Ind}_{H_0}^G$.

Solution. We have the natural isomorphism between functors that $\operatorname{Res}_{H_0}^H \circ \operatorname{Res}_H^G \cong \operatorname{Res}_{H_0}^G$. For any smooth H_0-representation σ, we aim to show $\operatorname{Ind}_H^G \operatorname{Ind}_{H_0}^H \sigma \cong \operatorname{Ind}_{H_0}^G \sigma$. By applying Yoneda's lemma in an opposite sense, it suffices to show there is a canonical isomorphism

$$\operatorname{Hom}_G(\pi, \operatorname{Ind}_H^G \operatorname{Ind}_{H_0}^H \sigma) \cong \operatorname{Hom}_G(\pi, \operatorname{Ind}_{H_0}^G \sigma)$$

for any smooth G-representation π. Since H and H_0 are closed, we have from Frobenius reciprocity that

$$\begin{aligned}\operatorname{Hom}_H(\operatorname{Res}_H^G \pi, \operatorname{Ind}_{H_0}^H \sigma) &\cong \operatorname{Hom}_{H_0}(\operatorname{Res}_{H_0}^H \operatorname{Res}_H^G \pi, \sigma)\\ &\cong \operatorname{Hom}_{H_0}(\operatorname{Res}_{H_0}^G \pi, \sigma).\end{aligned}$$

This proves the desired isomorphism $\operatorname{Ind}_H^G \circ \operatorname{Ind}_{H_0}^H \cong \operatorname{Ind}_{H_0}^G$ of functors. □

Exercise 3.5. Complete the proof of [BH06, §2.7, Lemma (2)].

Solution. Let G be a locally profinite group, and let $H \subset G$ be an open subgroup of finite index. If (σ, W) is a semi-simple smooth representation of H, we aim to prove that $\operatorname{Ind}_H^G \sigma$ is semi-simple as a representation of G. In class we proved it assuming H is normal. More precisely, when H is normal, we have concluded that

$$\operatorname{Res}_H^G \operatorname{Ind}_H^G \sigma = \bigoplus_{g\in G/H} \sigma^g$$

which is semi-simple. It follows that $\operatorname{Ind}_H^G \sigma$ is also semi-simple by [BH06, §2.7, Lemma (1)].

For general H, we set $H_0 \coloneqq \bigcap_{g \in G/H} gHg^{-1}$. Here the intersection is finite since $[G : H] < \infty$, so H_0 is an open normal subgroup in G (as well as in H) of finite index. Motivated by the argument above, we aim to show that $\mathrm{Res}_H^G \mathrm{Ind}_H^G \sigma$ is semi-simple; it suffices to show that $\mathrm{Res}_{H_0}^H \mathrm{Res}_H^G \mathrm{Ind}_H^G \sigma = \mathrm{Res}_{H_0}^G \mathrm{Ind}_H^G \sigma$ is semi-simple. For this, note that since σ is semi-simple and H_0 is normal in H, there exists a smooth semi-simple representation σ_0 of H_0 such that $\sigma = \mathrm{Ind}_{H_0}^H \sigma_0$. Thus we have to show $\mathrm{Res}_{H_0}^G \mathrm{Ind}_H^G \mathrm{Ind}_{H_0}^H \sigma_0 = \mathrm{Res}_{H_0}^G \mathrm{Ind}_{H_0}^G \sigma_0$ is semi-simple; but this is clear from the result that assumes the normality. $\square$

Exercise 3.6. Let G be a locally profinite group. Prove the following statements. Every element of $C_c^\infty(G)$ is a finite linear combination of elements of the form $\mathbb{1}_{gK}$, for $g \in G$ and K compact open subgroups of G. Also, every element of $C_c^\infty(G)$ is a finite linear combination of elements of the form $\mathbb{1}_{KgK}$, for $g \in G$ and K compact open subgroups of G.

Solution. For $f \in C_c^\infty(G)$, it is locally constant and compactly supported. Since G is locally profinite, 1 has a neighborhood basis consisting of compact open subgroups of G; let K be one of these subgroups, then the local constancy of f implies that $f(1) = f(k)$ for all $k \in K$. Similarly, taking the left and right translations by $g \in G$, we get $f(kg) = f(g) = f(gk')$ for all $k, k' \in K$ for K sufficiently small. On the other hand, since $\mathrm{supp}(f)$ is compact, it admits a finite open cover of form $\{g_iU\}_{i=1}^n$, where U is an open neighborhood of 1. By shrinking U to a sufficiently small subset if necessary, we have that $KU = UK = U$. This together with the argument for Exercise 1.4 dictates that we may assume $U = K$ without loss of generality. Since f is locally constant on U, the same is true for g_iU, say $f(g_iU) \equiv a_i \in \mathbb{C}$. Then

$$f = \sum_{i=1}^n a_i \mathbb{1}_{g_iU} = \sum_{i=1}^n a_i \mathbb{1}_{g_iK}.$$

Using a similar argument with $\{g_iU\}_{i=1}^n$ replaced by $\{Ug_i\}_{i=1}^n$, we can show that f is a finite linear combination of elements of the form $\mathbb{1}_{Kg_i}$; combining these two results together deduces the desired proof for $\mathbb{1}_{KgK}$. $\square$

Homework 4

Throughout G is a locally profinite group.

Exercise 4.1. Let μ be a left Haar measure on G. Show that the functional $C_c^\infty(G) \to \mathbb{C}$ sending f to

$$\int_G f(g)\delta_G(g^{-1})\mathrm{d}\mu(g)$$

is well-defined and is a right Haar measure.

Solution. We first check that the functional is a right Haar measure, i.e. the image of f and $r(x)f$ are the same, where $r(x)$ denotes the right translation by $x \in G$. For this, compute

$$\begin{aligned}\int_G f(gx)\delta_G(g^{-1})\mathrm{d}\mu(g) &= \delta_G(x)\int_G f(gx)\delta_G(x^{-1}g^{-1})\mathrm{d}\mu(g)\\ &= \delta_G(x)\int_G f(gx)\delta_G((gx)^{-1})\mathrm{d}\mu(g)\\ &= \int_G f(g)\delta_G(g^{-1})\mathrm{d}\mu(g).\end{aligned}$$

It follows that $\mathrm{d}\mu(g^{-1}) = \delta_G(g^{-1})\mathrm{d}\mu(g)$ is a right Haar measure whenever $\mathrm{d}\mu(g)$ is a left Haar measure. To show the functional is well-defined, note that by Exercise 3.6, $f \in C_c^\infty(G)$ is a finite linear combination of $\mathbb{1}_{xK}$'s, where $x \in G$ and K is an open compact subgroup in G. Then the image of f is a finite linear combination of integrals of form

$$\int_G \mathbb{1}_{xK}(g)\delta_G(g^{-1})\mathrm{d}\mu(g) = \int_G \mathbb{1}_{xK}(g^{-1})\mathrm{d}\mu(g),$$

which is non-zero only when $g \in xK \cap (xK)^{-1}$. Therefore, the integral $\int_G \mathbb{1}_{xK}(g^{-1})\mathrm{d}\mu(g)$ is controlled by $\int_G \mathbb{1}_{xK}(g)\mathrm{d}\mu(g) = \mu(xK)$. So the well-definedness follows from the choice of $\mu(gK)$, which makes sense from the assumption that μ is already a left Haar measure. $\square$

Exercise 4.2. Let μ be a left Haar measure on G. Let K be a compact open subgroup. Show that $\mu(gKg^{-1}) = \delta_G(g)^{-1}\mu(K)$. Thus we have the formula

$$\delta_G(g) = [K : K \cap gKg^{-1}][gKg^{-1} : K \cap gKg^{-1}]^{-1}.$$

Solution. For the first formula, compute

$$\begin{aligned}\int_G \mathbb{1}_{gK}(xg)\mathrm{d}\mu(x) &= \delta_G(g)^{-1}\int_G \mathbb{1}_{gK}(x)\mathrm{d}\mu(x)\\ &= \delta_G(g)^{-1}\int_G \mathbb{1}_{gK}(gx)\mathrm{d}\mu(x).\end{aligned}$$

Note that the first equality above follows from definition of modulus character δ_G, and the second equality is because μ is a left Haar measure (and hence $\mathrm{d}\mu(gx) = \mathrm{d}\mu(x)$). Since we have $\mathbb{1}_{gK}(xg) = \mathbb{1}_{gKg^{-1}}(x)$ and $\mathbb{1}_{gK}(gx) = \mathbb{1}_K(x)$, we deduce

$$\mu(gKg^{-1}) = \int_G \mathbb{1}_{gKg^{-1}}(x)\mathrm{d}\mu(x) = \delta_G(g)^{-1}\int_G \mathbb{1}_K(x)\mathrm{d}\mu(x) = \delta_G(g)^{-1}\mu(K).$$

For the second formula, notice that both gKg^{-1} and $K \cap gKg^{-1}$ are compact open subgroups of G. Then

$$\delta_G(g) = \frac{\mu(K)}{\mu(gKg^{-1})} = \frac{[K : K \cap gKg^{-1}] \cdot \mu(K \cap gKg^{-1})}{[gKg^{-1} : K \cap gKg^{-1}] \cdot \mu(K \cap gKg^{-1})} = \frac{[K : K \cap gKg^{-1}]}{[gKg^{-1} : K \cap gKg^{-1}]}.$$

$\square$

Exercise 4.3. Let F be a non-archimedean local field, and let

$$G \coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F) \;\middle|\; c = 0 \right\}.$$

Let

$$K \coloneqq \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G \;\middle|\; a, d \in \mathcal{O}_F^\times,\ b \in \mathcal{O}_F \right\}.$$

Show that K is a compact open subgroup of G. Then use the previous exercise to show that $\delta_G\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = |d/a|$, where $|\cdot|$ is the absolute value on F normalized so that a uniformizer has absolute value $|k_F|^{-1}$.

Solution. Such K is a subgroup of G because of

$$\begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ 0 & \mathcal{O}_F^\times \end{pmatrix}^{-1} \subset \begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ 0 & \mathcal{O}_F^\times \end{pmatrix}, \quad \begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ 0 & \mathcal{O}_F^\times \end{pmatrix}^{2} \subset \begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ 0 & \mathcal{O}_F^\times \end{pmatrix}.$$

As for the topology, note that K is homeomorphic to $\mathcal{O}_F^\times \times \mathcal{O}_F^\times \times \mathcal{O}_F$ and G is equipped with the induced topology from $\mathrm{M}_2(F)$. We know that $\mathcal{O}_F$ is compact; to show K is compact it suffices to show $\mathcal{O}_F^\times$ is closed in $\mathcal{O}_F$, but this is implied by the definition that $\mathcal{O}_F^\times$ consists of elements $x \in \mathcal{O}_F$ with $|x| = 1$. Also, K contains the subgroup $\mathrm{R_u}(G) \cap \mathrm{GL}_2(\mathcal{O}_F)$ that is homeomorphic to $\mathcal{O}_F$, where $\mathrm{R_u}(G)$ is the unipotent subgroup of G consisting of matrices of form $\left(\begin{smallmatrix} 1 & t \\ & 1 \end{smallmatrix}\right)$ with $t \in F$; as $\mathcal{O}_F$ is the open unit ball in F, we deduce from Exercise 1.1(3) that K is open. So we have shown that K is an open compact subgroup. For $g = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$, we compute

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1} = \begin{pmatrix} x & ayd^{-1} + \cdots \\ 0 & z \end{pmatrix}.$$

It then follows from Exercise 4.2 that

$$\delta_G(g) = \frac{\mu(K)}{\mu(gKg^{-1})} = |\det(\mathrm{Ad}(g))| = |a| \cdot |d|^{-1} = |ad^{-1}|.$$

In particular, as δ_G is non-trivial while restricting to K, we conclude that G is not unimodular. □

Exercise 4.4. Let $H \leqslant G$ be a closed subgroup, and let μ_H be a left Haar measure on H. Let $\theta = \delta_G|_H \cdot \delta_H^{-1} \colon H \to \mathbb{R}_{>0}$. In class we defined the G-map

$$\begin{aligned} \mathscr{A} \colon (C_c^\infty(G), \rho) &\longrightarrow (C_c^\infty(H\backslash G, \theta), \rho) \\ f &\longmapsto \tilde{f}, \end{aligned}$$

where

$$\tilde{f}(g) = \int_H \delta_G(h)^{-1} f(hg) \mathrm{d}\mu_H(h).$$

Fix a compact open subgroup K of G. Write $\ker(\mathscr{A})^K$ for the set of right K-invariant elements of $\ker(\mathscr{A})$.

(1) Show that every element of $\ker(\mathscr{A})^K$ is a linear combination of elements $f_i \in \ker(\mathscr{A})^K$ such that each f_i satisfies $\mathrm{supp}(f_i) \subset Hg_iK$ for some $g_i \in G$.

(2) Let $f \in \ker(\mathscr{A})^K$, and assume that f is supported on HgK. Show that f is a finite linear combination

$$f = \sum_{i=1}^{n} c_i \mathbb{1}_{h_i gK},$$

with $c_i \in \mathbb{C}$ and $h_i \in H$, satisfying

$$\sum_{i=1}^{n} c_i \delta_G(h_i)^{-1} = 0.$$

(3) Show that every element of $\ker(\mathscr{A})$ is killed by the right Haar measure on G. Therefore we obtain a non-zero, right G-invariant map $C_{\mathrm{c}}^{\infty}(H\backslash G,\theta) \to \mathbb{C}$.

Solution. Let δ_G be the fixed Haar measure on H appearing in the definition of θ.

(1) Recall the conclusion from class that for a fixed $g \in G$, the image $\mathscr{A}(\mathbb{1}_{gK})$ is the unique element in $C_{\mathrm{c}}^{\infty}(H\backslash G,\theta)$ exactly supported on HgK that is right K-invariant and has the value $\mu_H(gKg^{-1} \cap H)$ at g. On the other hand, by Exercise 3.6, $f \in \ker(\mathscr{A})^K \subset C_{\mathrm{c}}^{\infty}(G)$ is a finite linear combination of $f_i := \mathbb{1}_{g_iK}$'s for all $i = 1,\dots,n$, and therefore $\mathscr{A}(f)$ is a finite linear combination of $\mathscr{A}(\mathbb{1}_{g_iK})$'s. Now we automatically have $\operatorname{supp}(f_i) \subset Hg_iK$ because of the prescribed conclusion, and it remains to check $f_1,\dots,f_n \in \ker(\mathscr{A})^K$. But this is implied from the linear independence of f_i's and the assumption that $\mathscr{A}(f) = 0$.

(2) By assumption each element of $\operatorname{supp}(f)$ has form hgk for some $h \in H$ and $k \in K$. And by (1) it always lies in certain compact open g_iK. It follows that there is a fixed element $g \in G$ together with $h_1,\dots,h_n \in H$ so that $g_i = h_ig$. So f is a finite linear combination

$$f = \sum_{i=1}^{n} c_i \mathbb{1}_{h_igK},$$

with $c_i \in \mathbb{C}$ and $h_i \in H$. To check the desired equality, it suffices to show that if $\mathscr{A}(\mathbb{1}_{h_igK}) = \widetilde{\mathbb{1}}_{h_igK} = 0$ then $\delta_G(h_i)^{-1} = 0$; this assumption means that $\int_G \delta_G(h)^{-1}\mathbb{1}_{h_igK}(hx)\mathrm{d}\mu_H(h) = 0$ for all $x \in G$, and in particular this holds for all $x = h^{-1}h_ig$, where h runs through all elements of H. Since δ_G is a homomorphism with target $\mathbb{R}_{\geqslant 0}$, this deduces $\delta_G(h)^{-1} = 0$ for all h as desired (otherwise the integral of $\delta_G(h)^{-1}\mathbb{1}_{h_igK}(hx)$ would be positive). Therefore, we get

$$\sum_{i=1}^{n} c_i\delta_G(h_i)^{-1} = 0.$$

(3) We are to show that $\tilde{f}(g) = \int_H \delta_G(h)^{-1}f(hg)\mathrm{d}\mu_H(h) = 0$ implies $\int_G f(g)\mathrm{d}\mu_G(g) = 0$ for any fixed $g \in G$. By assumption, we have for any $\phi \in C_{\mathrm{c}}^{\infty}(G)$ that

$$\begin{aligned}
0 &= \int_G \phi(g)\tilde{f}(g)\mathrm{d}\mu_G(g) \\
&= \int_G \phi(g) \int_H \delta_G(h)^{-1}f(hg)\mathrm{d}\mu_H(h)\mathrm{d}\mu_G(g) \\
&= \int_H \int_G \phi(g)\delta_G(h)^{-1}f(hg)\mathrm{d}\mu_G(g)\mathrm{d}\mu_H(h) \\
&= \int_H \int_G \phi(h^{-1}g)\delta_G(h)^{-1}f(h\cdot h^{-1}g)\mathrm{d}\mu_G(g)\mathrm{d}\mu_H(h),
\end{aligned}$$

where the second-last equality is by changing the order of integrals and the last equality is by replacing g with $h^{-1}g$. Here we have used the property that μ_G is a left Haar measure and hence $\mathrm{d}\mu_G(g) = \mathrm{d}\mu_G(h^{-1}g)$. Changing the integral order again, we have that

$$\begin{aligned}
0 &= \int_G f(g) \int_H \delta_G(h)^{-1}\phi(h^{-1}g)\mathrm{d}\mu_H(h)\mathrm{d}\mu_G(g) \\
&= \int_G f(g)\tilde{\phi}(g)\mathrm{d}\mu_G(g).
\end{aligned}$$

Recall from class that $\mathscr{A}$ is surjective. It follows that if C is a compact subset containing $\operatorname{supp}(f)$ then there exists $\phi \in C_{\mathrm{c}}^{\infty}(G)$ such that $\tilde{\phi}(x)$ is a non-zero constant for all $x \in HC$ (to guarantee this we need the conclusion at the beginning of (1)). This shows that $\int_G f(g)\mathrm{d}\mu_G(g) = 0$ and we complete the proof. $\square$

Homework 5

Exercise 5.1. Let F be a non-archimedean local field, $G = \mathrm{GL}_2(F)$, and let B, N, T be the usual subgroups. Let $K = \mathrm{GL}_2(\mathcal{O}_F)$. Suppose the Haar measures on G, $T \cong F^\times \times F^\times$, $N \cong F$ are all induced, as explained in class, by the same Haar measure on F normalized by $\mu(\mathcal{O}_F) = C$. (See also [BH06, 7.4–7.6].) Suppose the Haar measure μ_B on B is normalized such that $\int_B f(b)\mathrm{d}\mu_B(b) = \int_T \int_N f(tn)\mathrm{d}\mu_T(t)\mathrm{d}\mu_N(n)$, and suppose the Haar measure on K is the restriction of that on G. In class, we showed that there exists a non-zero constant D such that

$$\forall f \in C_c^\infty(G), \quad \int_G f(g)\mathrm{d}\mu_G(g) = D \cdot \int_K \int_B f(bk)\mathrm{d}\mu_B(b)\mathrm{d}\mu_K(k).$$

Compute the constant D (which possibly depends on C).

Solution. As K is an open compact subgroup of G, we have $\mathbb{1}_K \in C_c^\infty(G)$. From the assumption, μ_K is restricted from μ_G, so that

$$\mu_K(K) = \mu_G(K).$$

To compute the integral with respect to $\mathrm{d}\mu_B \cdot \mathrm{d}\mu_K$, the only ambiguity lies in $H \coloneqq \{(x, x) \colon x \in B \cap K\}$. Put the normalized Haar measure μ_H on H by restricting μ_B from B to H. Since μ_B is normalized so that $\mathrm{d}\mu_B = \mathrm{d}\mu_T \cdot \mathrm{d}\mu_N$, and both μ_T, μ_N are compatible with μ_K, we see μ_H is compatible with the restriction of μ_K. Now we have

$$\int_K \int_B \mathbb{1}_K(bk)\mathrm{d}\mu_B(b)\mathrm{d}\mu_K(k) = \mu_H(H) \cdot \mu_K(K),$$

$$\int_G \mathbb{1}_K(g)\mathrm{d}\mu_G(g) = \mu_G(K).$$

Thus it suffices to compute $\mu_H(H)$, where $H \cong B(\mathcal{O}_F) \cong \mathcal{O}_F^\times \times \mathcal{O}_F^\times \times \mathcal{O}_F$. On the other hand, we have the formula $|\det a| \cdot \mathrm{d}\mu_{F^\times}(ag) = \mathrm{d}\mu_F(g) = \mathrm{d}\mu(g)$, with $|\det a| = 1$ for $a \in \mathcal{O}_F^\times = \mathrm{GL}_1(\mathcal{O}_F)$. Combining these deduces $\mu_{F^\times} = \mu$; and then $\mu(\mathcal{O}_F^\times) = \mu(\mathcal{O}_F) = C$ because of $\mathcal{O}_F^\times = \mathcal{O}_F \cap F^\times$ and $\mathcal{O}_F = \mathcal{O}_F \cap F$. It follows that

$$D = \frac{1}{\mu_H(H)} = \frac{1}{\mu(\mathcal{O}_F^\times)^2 \cdot \mu(\mathcal{O}_F)} = \frac{1}{C^3}.$$

□

Exercise 5.2. In the previous notation, show that for given Haar measures μ_T, μ_N on T and N, the formula

$$\int_B f(b)\mathrm{d}\mu_B(b) \coloneqq \int_T \int_N f(tn)\mathrm{d}\mu_N(n)\mathrm{d}\mu_T(t)$$

indeed defines a left Haar measure on B. Explain why this does not work if we replace $f(tn)$ by $f(nt)$ in the above formula.

Solution. To show the left B-invariance, take an arbitrary $b \in B$ and write $b = t'n'$ for some $t' \in T$ and $n' \in N$. Since for $T \cong F^\times \times F^\times$ and $N \cong F$, the Haar measures μ_T and μ_N are induced from $\mu_F = \mu_{F^\times}$ by Exercise 5.1, and $F^\times = \mathbb{G}_\mathrm{m}(F)$ is unimodular, we see in particular that μ_N is left N-invariant and μ_T is right T-invariant. Note that b and t commutes. So we compute

$$\begin{aligned}\int_T \int_N f(btn)\mathrm{d}\mu_N(n)\mathrm{d}\mu_T(t) &= \int_T \int_N f(tbn)\mathrm{d}\mu_N(n)\mathrm{d}\mu_T(t)\\ &= \int_T \int_N f(tt'n'n)\mathrm{d}\mu_N(n)\mathrm{d}\mu_T(t)\\ &= \int_T \int_N f(tn)\mathrm{d}\mu_N(n)\mathrm{d}\mu_T(t).\end{aligned}$$

It follows that μ_B is a left Haar measure on B. Note that n does not commute with $b \in B$ in general. Thus if we replace $f(tn)$ by $f(nt)$ then we are unable to deduce $f(tnb) = f(tbn)$. Instead, there always holds $f(tnb) = \delta_B(b)^{-1} f(tn)$ with non-trivial δ_B, because B is not unimodular by Exercise 4.3. $\square$

Exercise 5.3. In the previous notation, prove the Iwasawa decomposition $G = BK$ by explicit matrix manipulation. Generalize it to GL_n.

Solution. We directly prove the case where $G = \mathrm{GL}_n(F)$ (as opposed to dealing with GL_2 first). It suffices to show that given $g = (g_{ij})_{i,j} \in \mathrm{GL}_n(F)$ we are able to transform it into an element of B via taking the right multiplication by matrices in $K = \mathrm{GL}_2(\mathcal{O}_F)$. The algorithm (which is an analogue of Gram–Schmidt orthonormalization over $\mathbb{C}$) is as follows:

(i) Let $|\cdot|$ be the standard normalized absolute value on F. Among $a_{n1}, \dots, a_{nn}$ there is an entry a_{nj} such that $|a_{nj}| \leqslant |a_{nk}|$ for $k \neq j$. Then swap the jth and the nth columns of g.
(ii) Apply Gauss–Jordan algorithm to eliminate the first $n-1$ columns by the nth column so that we may assume $a_{ni} = 0$ for $1 \leqslant i \leqslant n-1$.
(iii) Iterate the operation above for the submatrix in g consisting of the first $n-1$ rows and columns.

Note that each step can be realized by an element of K multiplied from the right side of g. Thus the direct product factorization $G = BK$ follows. $\square$

Exercise 5.4. For F a non-archimedean local field with uniformizer π, use Smith normal form to prove the Cartan decomposition:

$$\mathrm{GL}_n(F) = \bigsqcup_{\substack{a_1 \geqslant \cdots \geqslant a_n \\ a_i \in \mathbb{Z}}} \mathrm{GL}_n(\mathcal{O}_F) \begin{pmatrix} \pi^{a_1} & & \\ & \ddots & \\ & & \pi^{a_n} \end{pmatrix} \mathrm{GL}_n(\mathcal{O}_F).$$

Then show that $\mathrm{GL}_n(F)$ satisfies the condition "countable at infinity", i.e. $\mathrm{GL}_n(F)/K$ is countable for any compact open subgroup K.

Solution. Since $\mathcal{O}_F$ is a PID, using the fundamental theorem of finitely generated modules over PIDs, any matrix $g \in \mathrm{GL}_n(F)$ can be turned into Smith normal form with elementary divisors $\pi^{a_1}, \dots, \pi^{a_n}$ such that π^{a_j} divides π^{a_i} for $i \leqslant j$, namely $a_1 \geqslant \cdots \geqslant a_n$ with $a_i \in \mathbb{Z}$. Similar to the argument of Exercise 5.3, this process can be done by multiplying by elementary matrices in $\mathrm{GL}_n(\mathcal{O}_F)$ from left and right, and hence we have proved the (set-theoretical) Cartan decomposition.

We then prove that for $G = \mathrm{GL}_n$, $G(F)$ is countable at infinity. Let K be any compact open subgroup. Then $K \cap G(\mathcal{O}_F)$ is compact, open, and of finite index in $G(\mathcal{O}_F)$. Therefore, the surjection $\pi\colon G(F)/(K \cap G(\mathcal{O}_F)) \twoheadrightarrow G(F)/G(\mathcal{O}_F)$ has finite fibres. Consequently, if the target of π is countable then so also is the source, and hence $G(F)/K$ is countable as well. Therefore, it suffices to show that $G(F)/G(\mathcal{O}_F)$ is countable. Indeed, by Cartan decomposition, the homogeneous space $G(\mathcal{O}_F)\backslash G(F)/G(\mathcal{O}_F)$ is countable, and each double coset contains only finitely many right cosets of form $gG(\mathcal{O}_F)$, implying that $G(F)/G(\mathcal{O}_F)$ is countable. $\square$

Exercise 5.5 ([BH06, p.51, Exercise])**.** Let F be a non-archimedean local field. Let K be a compact subgroup of $G = \mathrm{GL}_2(F)$. Show that $gKg^{-1} \subset \mathrm{GL}_2(\mathcal{O}_F)$ for some $g \in G$. Deduce that, up to G-conjugacy, $\mathrm{GL}_2(\mathcal{O}_F)$ is the unique maximal compact subgroup of G.

Solution. Note that $\mathrm{GL}_2(\mathcal{O}_F)$ is exactly the set of $g \in G$ such that $g\mathcal{O}_F^2 = \mathcal{O}_F^2$. Because $G = \mathrm{GL}_2(F)$ acts transitively on the set of bases in F^2, it also acts transitively on the set of lattices in F^2 (defined as $\mathcal{O}_F$-submodules of finite type and maximal rank 2). We obtain an identification of $\mathrm{GL}_2(F)/\mathrm{GL}_2(\mathcal{O}_F)$ with the set of lattices in F^2. Note that the quotient topology on $\mathrm{GL}_2(F)/\mathrm{GL}_2(\mathcal{O}_F)$ is the discrete topology because $\mathrm{GL}_2(\mathcal{O}_F)$ is open in $\mathrm{GL}_2(F)$.

The statement is equivalent to the existence of a lattice $\Lambda \subset F^2$ such that $k(\Lambda) = \Lambda$ for all $k \in K$. Because K is compact, the image of $\det\colon K \to F^\times$ is compact, and so for any $k \in K$

such that $k(\Lambda) \subset \Lambda$ we actually have the equality; so it suffices to show there is a K-stable lattice Λ. Let Λ_0 be any lattice, then there is an open compact subgroup $K' \subset K$ such that Λ_0 is stable under K' (if $L_0 = \mathcal{O}_F^2$ we can take $K' = K \cap \mathrm{GL}_2(\mathcal{O}_F)$). Then we complete the proof by taking $L = \sum_{g \in K/K'} g(L_0)$. □

Alternative Solution. We refer to [Tit79, §3.2] and [Bor11, Chapter VII, Theorem 1.2]. Consider the G-action on $F \oplus F$; up to scalar there is a unique $\mathcal{O}_F$-lattice Λ_0 spanned by $(1,0)$ and $(0,1)$ whose stabilizer is exactly $\mathrm{GL}_2(\mathcal{O}_F)$. Let v_0 be the corresponding vertex of Λ_0 in the associated Bruhat–Tits tree $\mathbf{T}_G$ of G. Recall that G acts on $\mathbf{T}_G$ transitively. Thus from the prescribed observation there is a bijection between $\mathrm{GL}_2(F)/\mathrm{GL}_2(\mathcal{O}_F)$ and the set of vertices of $\mathbf{T}_G$. It follows that $gKg^{-1} \subset \mathrm{GL}_2(\mathcal{O}_F)$ if and only if gKg^{-1} stabilizes v_0 in $\mathbf{T}_G$, which is further equivalent to that K stabilizes gv_0 for some g. Again, since G acts on $\mathbf{T}_G$ transitively, it suffices to find out a lattice of $F \oplus F$ that is stabilized by K. For this, let Λ be a lattice such that $\mathfrak{p}\Lambda_0 \subset \Lambda \subset \Lambda_0$, where $\mathfrak{p}$ is the place of F above $p \in \mathbb{Q}_p$, or equivalently Λ is (up to $\mathcal{O}_F^\times$-scalars) a vertex of $\mathbf{T}_G$ in the neighborhood of Λ_0; then $K\Lambda$ is the desired lattice stabilized by K. This completes the proof. □

Exercise 5.6 (Iwahori decomposition, [BH06, (7.3.1)])**.** Show that we have $I = (I \cap N')(I \cap T)(I \cap N)$. More precisely, the product map

$$\varphi\colon (I \cap N') \times (I \cap T) \times (I \cap N) \longrightarrow I$$

is bijective, and a homeomorphism, for any ordering of the factors on the left hand side.

Solution. Denote by ϖ the uniformizer of $\mathfrak{p}_F$. To show the map is bijective, we construct the inverse map $\psi\colon I \to (I \cap N') \times (I \cap T) \times (I \cap N)$ as

$$\begin{pmatrix} 1+\varpi a & b \\ \varpi c & 1+\varpi d \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 \\ \frac{\varpi c}{1+\varpi a} & 1 \end{pmatrix} \begin{pmatrix} 1+\varpi a & 0 \\ 0 & 1+\varpi d - \frac{\varpi bc}{1+\varpi a} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{1+\varpi a} \\ 0 & 1 \end{pmatrix},$$

where $a, b, c, d \in \mathcal{O}_F$. Here the image makes sense because $1 + \mathfrak{p}_F$ is invertible in $\mathcal{O}_F$. Clearly, ψ is indeed the inverse of φ, and hence φ is bijective. By definition,

$$I = \begin{pmatrix} 1+\varpi\mathcal{O}_F & \mathcal{O}_F \\ \varpi\mathcal{O}_F & 1+\varpi\mathcal{O}_F \end{pmatrix},$$

and thus

$$I \cap N' \cong \varpi\mathcal{O}_F, \quad I \cap T \cong (1+\varpi\mathcal{O}_F)^2, \quad I \cap N \cong \mathcal{O}_F.$$

It follows that both I and $(I \cap N') \times (I \cap T) \times (I \cap N)$ are compact and Hausdorff. In this case, as a continuous bijection, φ must be a homeomorphism. Finally, note that both the set-theoretical bijection and the homeomorphism do not depend on the order of product, as there are only finitely many components on the source of φ. □

Exercise 5.7 ([BH06, p.55, Exercise])**.**

(1) Let I be the standard Iwahori subgroup; let $\mathrm{d}n', \mathrm{d}t, \mathrm{d}n$ be Haar measures on the groups $I \cap N'$, $I \cap T$, $I \cap N$, respectively. Show that the functional

$$f \longmapsto \iiint f(n'tn)\mathrm{d}n'\mathrm{d}t\mathrm{d}n, \quad f \in C_{\mathrm{c}}^\infty(I),$$

is a Haar integral on I.

(2) Let $C = N'TN$. Show that C is open and dense in G, and that the product map $N' \times T \times N \to C$ is a homeomorphism.

(3) Let $\mathrm{d}g$ be a Haar measure on G. Show that there are Haar measures $\mathrm{d}n', \mathrm{d}t, \mathrm{d}n$ on N', T, N such that

$$\int_G f(g)\mathrm{d}g = \iiint f(n'tn)\delta_B(t)^{-1}\mathrm{d}n'\mathrm{d}t\mathrm{d}n, \quad f \in C_{\mathrm{c}}^\infty(G).$$

Solution. (1) Since $\mathrm{d}n', \mathrm{d}t, \mathrm{d}n$ are already Haar measures, the positivity of the functional (i.e. the image of $f \geqslant 0$ is always non-negative) is clear. To check $n_0' n_0 n' t n$

(2) For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d - bc/a \end{pmatrix} \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix} \in N'TN$$

and the only condition for g is that $a \in F^\times$. Since $F^\times$ is dense in F and $N \cong F$ is open in G, we have that C is open and dense in G. To show that $\iota \colon N' \times T \times N \to C$ is a homeomorphism, note that ι is continuous in matrix entries □

Exercise 5.8. Let G be a locally profinite group. Show that for all $g \in G$ and $f_1, f_2 \in C_c^\infty(G)$, we have $(\lambda(g) f_1) * f_2 = \lambda(g)(f_1 * f_2)$. Here $\lambda(g)$ denotes the left translation of a function by g.

Solution. By definition we have $(\lambda(g) f)(x) = f(g^{-1}x)$ for all $f \in C_c^\infty(G)$. Also, from the definition of convolution product,

$$\begin{aligned} ((\lambda(g) f_1) * f_2)(x) &= \int_G (\lambda(g) f_1)(y) f_2(y^{-1}x) \mathrm{d}\mu(y) \\ &= \int_G f_1(g^{-1}y) f_2(y^{-1}x) \mathrm{d}\mu(y). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (\lambda(g)(f_1 * f_2))(x) &= \int_G f_1(y) f_2(y^{-1}g^{-1}x) \mathrm{d}\mu(y) \\ &= \int_G f_1(g^{-1}y) f_2(y^{-1}x) \mathrm{d}\mu(y). \end{aligned}$$

Here the second equality above is by replacing y by $g^{-1}y$, according to $\mathrm{d}\mu(y) = \mathrm{d}\mu(g^{-1}y)$. □

Exercise 5.9. Let G be an abelian locally profinite group. Show that any Haar measure is invariant under the automorphism $g \mapsto g^{-1}$. Then show that the Hecke algebra $\mathcal{H}(G)$ of G is commutative.

Solution. Let μ be a left Haar measure on G. Recall from Exercise 4.1 that we have $\mathrm{d}\mu(g^{-1}) = \delta_G(g^{-1}) \mathrm{d}\mu(g)$. On the other hand, since G is abelian, we have $\mathrm{d}\mu(gx) = \mathrm{d}\mu(xg) = \mathrm{d}\mu(g)$ for all $x \in G$. Thus μ is also a right Haar measure. It follows that δ_G is trivial, and thus $\mathrm{d}\mu(g^{-1}) = \mathrm{d}\mu(g)$, namely μ is invariant under $g \mapsto g^{-1}$. To show that $\mathcal{H}(G)$ is commutative, consider

$$\begin{aligned} (f_1 * f_2)(x) &= \int_G f_1(y) f_2(y^{-1}x) \mathrm{d}\mu(y) \\ &= \int_G f_1(y^{-1}x) f_2(x^{-1}yx) \mathrm{d}\mu(y^{-1}x) \\ &= \int_G f_1(y^{-1}x) f_2(y) \mathrm{d}\mu(y) \\ &= (f_2 * f_1)(x). \end{aligned}$$

In the second line above, we replace y by $y^{-1}x$; in the third line above, we use the assumption that G is abelian to deduce $f_2(x^{-1}yx) = f_2(y)$ and use the previous result to deduce $\mathrm{d}\mu(y^{-1}x) = \mathrm{d}\mu(y)$. □

Exercise 5.10. Let G be a locally profinite group with a fixed a Haar measure μ. Suppose the Hecke algebra $\mathcal{H}(G)$ is commutative. Show that G is abelian.

Solution. Note that we have $\mathbb{1}_{gK} = \lambda(g) \mathbb{1}_K$ for all $g \in G$. Recall that $e_K := \mu(K)^{-1} \mathbb{1}_K$ satisfies the property $e_K * e_K = e_K$. Take $g, h \in G$ arbitrarily and compute

$$(\lambda(g) e_K) * (\lambda(h) e_K) = \lambda(g)(e_K * \lambda(h) e_K) = \lambda(g)(\lambda(h) e_K * e_K) = \lambda(gh) e_K.$$

Here the first equality is by Exercise 5.8, the second equality is because of the assumption that $\mathcal{H}(G)$ is commutative, and the last equality is again an application of Exercise 5.8. Again, since $\mathcal{H}(G)$ is commutative, we have

$$\lambda(gh)e_K = (\lambda(g)e_K) * (\lambda(h)e_K) = (\lambda(h)e_K) * (\lambda(g)e_K) = \lambda(hg)e_K.$$

It then follows that

$$\mathbb{1}_{ghK} = \mathbb{1}_{hgK}.$$

Therefore, we get $ghK = hgK$, which implies the desired result $gh = hg$ to show that G is abelian. $\square$

Homework 6

Let G be a locally profinite group, and K be a compact open subgroup. Write $\mathcal{H}$ for the Hecke algebra $\mathcal{H}(G)$, and write $\mathcal{H}(G,K)$ for $e_K * \mathcal{H} * e_K$.

Exercise 6.1. For any $f \in \mathcal{H}$, show that $e_K * f$ is the average of the K-orbit of f with respect to the left translation action, and that $f * e_K$ is the average of the K-orbit of f with respect to the right translation action.

Solution. For the first assertion we need to show that

$$(e_K * f)(x) = (\lambda(e_K) \cdot f)(x)$$

for all $x \in G$ and any $f \in \mathcal{H}$. Here λ denotes the left translation. By definition of convolution product on $C_{\mathrm{c}}^{\infty}(G)$, we have

$$(e_K * f)(x) = \int_G e_K(g) f(g^{-1}x) \mathrm{d}\mu(g).$$

On the other hand,

$$\begin{aligned}(\lambda(e_K) \cdot f)(x) &= \frac{1}{\mu(K)} \cdot \int_G \mathbb{1}_K(g) \cdot \lambda(g) f(x) \mathrm{d}\mu(g) \\ &= \frac{1}{\mu(K)} \cdot \int_G \mathbb{1}_K(g) \cdot f(g^{-1}x) \mathrm{d}\mu(g).\end{aligned}$$

So the desired equality follows from the known formula that $\mu(K) \cdot e_K = \mathbb{1}_K$. Also, the second assertion $f * e_K = \rho(e_K) \cdot f$ follows from a similar argument. $\square$

Exercise 6.2. Show that the inclusion $e_K * \mathcal{H} \to \mathcal{H}$ has a splitting. Moreover, this splitting is a map of right $\mathcal{H}$-modules if we view $e_K * \mathcal{H}$ and $\mathcal{H}$ both as right $\mathcal{H}$-modules.

Solution. Consider the map $\psi\colon \mathcal{H} \to e_K * \mathcal{H},\ f \mapsto e_K * f$. For ψ to be a splitting we need to check that the composite $e_K * \mathcal{H} \hookrightarrow \mathcal{H} \to e_K * \mathcal{H}$ is the identity; but this follows from the property that $e_K * e_K = e_K$, because an element of form $e_K * f_0$ in the source is mapped to $e_K * e_K * f_0 = e_K * f_0$ in the target.

Both $e_K * \mathcal{H}$ and $\mathcal{H}$ can be viewed as right $\mathcal{H}$-modules in the sense that $\mathcal{H}$ acts by $(\boldsymbol{-}) * f$ for some $f \in \mathcal{H}$. By construction of ψ it is clearly compatible with the $\mathcal{H}$-actions from the right. $\square$

Exercise 6.3. Let W be a left unital $\mathcal{H}(G,K)$-module, and we write $*$ for the multiplication of $\mathcal{H}(G,K)$ on W. Then the $\mathcal{H}(G,K)$-module given by the K-fixed points of the smooth G-representation $\mathcal{H} \otimes_{\mathcal{H}(G,K)} W$ is isomorphic, as an $\mathcal{H}(G,K)$-module, to $(e_K * \mathcal{H}) \otimes_{\mathcal{H}(G,K)} W$. Moreover, there is a well-defined map $(e_K * \mathcal{H}) \otimes_{\mathcal{H}(G,K)} W \to W$ sending $(e_K * f) \otimes w$ to $(e_K * f * e_K) * w$.

Solution. Recall that if V is a smooth G-representation then $V^K = e_K * V$. Since W is a left unital $\mathcal{H}(G,K)$-module, it is invariant under the left action by e_K, i.e. $e_K * W = W$. It follows that

$$(\mathcal{H} \otimes_{\mathcal{H}(G,K)} W)^K = e_K * (\mathcal{H} \otimes_{\mathcal{H}(G,K)} W) = (e_K * \mathcal{H}) \otimes_{\mathcal{H}(G,K)} W.$$

Now consider the map $(e_K * \mathcal{H}) \otimes_{\mathcal{H}(G,K)} W \to W$. By Exercise 6.2 we see $\mathcal{H} \to e_K * \mathcal{H}$ via $f \mapsto e_K * f$ is surjective, and so also is the map $\mathcal{H} \otimes_{\mathcal{H}(G,K)} W \to (e_K * \mathcal{H}) \otimes_{\mathcal{H}(G,K)} W$ between $\mathcal{H}(G,K)$-modules. Moreover, this surjection admits a splitting as well, and hence we get an injection $(e_K * \mathcal{H}) \otimes_{\mathcal{H}(G,K)} W \hookrightarrow \mathcal{H} \otimes_{\mathcal{H}(G,K)} W$. On the other hand, we have the well-defined map

$$\mathcal{H} \otimes_{\mathcal{H}(G,K)} W \longrightarrow W, \quad f \otimes w \longmapsto w.$$

So it suffices to recognize $(e_K * f) \otimes w$ in terms of $(e_K * f * e_K) * w$. For this, we compute

$$\begin{aligned}(e_K * f) \otimes w &= (e_K * f) \otimes (e_K * w)\\ &= (e_K * f * e_K) \otimes w\\ &= e_K * (e_K * f * e_K) \otimes w\\ &= e_K \otimes ((e_K * f * e_K) * w).\end{aligned}$$

Here we have used the invariance property of $\mathcal{H}(G,K)$-modules with respect to $e_K * (\textbf{—}) * e_K$ for several times. $\square$

Exercise 6.4. Let F be a field. Prove the Bruhat decomposition

$$\mathrm{GL}_2(F) = B \sqcup BwB.$$

Here, as usual, B is the subgroup of upper triangular matrices, and $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Solution. We prove by elementary computation (as opposed to using the theory of Tits systems). If $b \in B \cap BwB$ then there are $b_1, b_2 \in B$ such that $b = b_1 w b_2$, and thus $w = b_1^{-1} b b_2^{-1} \in B$, which is impossible. So $B \cap BwB = \emptyset$. It suffices to show that for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(F)$, if $g \notin B$ then there are $g_1 = \left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)$ and $g_2 = \left(\begin{smallmatrix} r & s \\ 0 & t \end{smallmatrix}\right) \in B$ such that $g_1 w g_2 = g$. For this, compute

$$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} r & s \\ 0 & t \end{pmatrix} = \begin{pmatrix} yr & xt + ys \\ zr & zs \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note from the assumption that $z, r \in F^\times$. So we may take $z = 1$ and $r = c$; this makes sense because $g \notin B$ implies $c \in F^\times$. Then we can solve the matrix equation above and express x, y, s, t in terms of a, b, c, d. $\square$

Exercise 6.5. Classify and count the number of conjugacy classes in $\mathrm{GL}_3(\mathbb{F}_q)$.

Solution. Since $\mathbb{F}_q$ is not algebraically closed, we use rational canonical forms to determine the conjugacy classes. For this, we need to consider the characteristic polynomial $f(X)$ as well as elementary divisors of any $X \in \mathrm{GL}_3(\mathbb{F}_q)$.

(1) $f(X) = X^3 + aX^2 + bX + c$ is irreducible of degree 3 over $\mathbb{F}_q$. Then the arithmetic Frobenius map permutes the 3 roots of $f(X)$ in $\mathbb{F}_{q^3}$. Hence $f(X)$ is exactly determined by an element of $\mathbb{F}_{q^3} - \mathbb{F}_q$ up to Frobenius. Thus there are in total

$$\frac{1}{3}(q^3 - q)$$

conjugacy classes of $\mathrm{GL}_3(\mathbb{F}_q)$ in correspondence. In this case, the only elementary divisor is $f(X)$ itself, and the rational canonical form of X is

$$\begin{pmatrix} 0 & 0 & -c \\ 1 & 0 & -b \\ 0 & 1 & -a \end{pmatrix}.$$

(2) $f(X) = (X^2 + aX + b)(X - c)$ with $c \in \mathbb{F}_q^\times$ and $X^2 + aX + b$ an irreducible of degree 2 over $\mathbb{F}_q$. Similarly, we have $q^2 - q$ choices of $X^2 + aX + b$ to be the minimal polynomial of some elements of $\mathbb{F}_{q^2} - \mathbb{F}_q$ and $q - 1$ choices of $c \in \mathbb{F}_q^\times$. Hence there are there are in total

$$\frac{1}{2}(q^2 - q) \cdot (q - 1)$$

conjugacy classes of $\mathrm{GL}_3(\mathbb{F}_q)$ in correspondence. In this case, the elementary divisors are $X - c$ and $f(X)$, and the rational canonical form of X is

$$\begin{pmatrix} 0 & -b & 0 \\ 1 & -a & 0 \\ 0 & 0 & c \end{pmatrix}.$$

(3) $f(X) = (X-a)(X-b)(X-c)$ with $a, b, c \in \mathbb{F}_q^\times$. In this case the elementary divisors are possibly as follows.

(i) When a, b, c are mutually distinct, up to permuting a, b, c, there is essentially one case for elementary divisors, which is $f(X)$ itself. Choosing a, b, c modulo permutation, we see the number of conjugacy classes is
$$\frac{1}{6}(q-1)(q-2)(q-3).$$

(ii) When $a = b \neq c$, the elementary divisors can be $X - a, (X-a)(X-c)$ or $f(X)$ itself. In both cases there are $q-1$ choices of $a = b$ and $q-2$ choices of c. Up to permutation, the number of conjugacy classes is
$$\frac{1}{3}((q-1)+(q-1))(q-2).$$

(iii) When $a = b = c$, the elementary divisors can be either of $(X-a, X-a, X-a)$ or $(X-a, (X-a)^2)$ or $f(X)$ itself. It follows that the number of conjugacy classes is
$$3(q-1).$$

To sum up, the number of all conjugacy classes in $\mathrm{GL}_3(\mathbb{F}_q)$ is
$$\frac{1}{3}(q-1)(3q^2-q+8).$$
□

Exercise 6.6. Let G be a finite group, and H, K subgroups. Let $W \in \mathsf{Rep}(K)$. State and prove Mackey's formula, which describes $\mathrm{Res}_H^G \mathrm{Ind}_K^G W$.

Solution. Let $g_1, \dots, g_r$ be a system of representatives of the double classes in $H \backslash G / K$. In other words, we have
$$G = \bigsqcup_{i=1}^r H g_i K.$$
For every $i \in \{1, \dots, r\}$, let $K_i := g_i K g_i^{-1} \cap H$, and let (W_i, ρ_i) be the representation of K_i on W given by $\rho_i(k) = \rho(g_i^{-1} k g_i)$ for $k \in K$.

Mackey's formula. We have an isomorphism of H-representations
$$\mathrm{Res}_H^G \mathrm{Ind}_K^G W \simeq \bigoplus_{i=1}^r \mathrm{Ind}_{K_i}^K W_i.$$

To prove it, for every $j \in \{1, \dots, r\}$, let $(x_i)_{i \in I_j}$ be a system of representatives of H/K_j. Then $\{x_i g_j : 1 \leqslant j \leqslant r,\ i \in I_j\}$ is a system of representatives of G/K. Indeed,
$$G = \bigsqcup_{j=1}^r H g_j K = \bigsqcup_{j=1}^r \bigsqcup_{i \in I_j} x_i K_j g_j K = \bigsqcup_{j=1}^r \bigsqcup_{i \in I_j} x_j g_j (g_j^{-1} K_j g_j) K = \bigsqcup_{j=1}^r \bigsqcup_{i \in I_j} x_j g_j K$$
because $g_j^{-1} K_j g_j \subset K$ for every j. Thus,
$$V := \mathrm{Ind}_K^G W = \bigoplus_{j=1}^r V_j,$$
where $V_j = \bigoplus_{i \in I_j} x_i g_j \overline{F}[K] \otimes_{\overline{F}[K]} V$. Note that $V_j \subset V$ is stabilized by H. Fix $j \in \{1, \dots, r\}$. It suffices to show the following isomorphism of $\overline{F}[H]$-modules:
$$V_j \simeq \mathrm{Ind}_{K_j}^H W_j.$$
But for this, we can construct the $\overline{F}$-linear maps
$$\varphi\colon\ \mathrm{Ind}_{K_j}^H W_j \longrightarrow V_j, \quad \sum_{i \in I_j} x_i \otimes v_i \longmapsto \sum_{i \in I_j} (x_i g_j) \otimes v_i$$

and

$$\psi\colon V_i \longrightarrow \operatorname{Ind}_{K_j}^{H} W_j, \qquad \sum_{i\in H\cdot I_j}(x_i g_j)\otimes v_i \longmapsto \sum_{i\in I_j} x_i\otimes v_i.$$

It is clear that φ and ψ are inverse of each other. This completes the proof. $\square$

Exercise 6.7. Prove in detail the lemma in [BH06, §6.3].

Solution. Let π be an irreducible representation of $G = \mathrm{GL}_2(F)$, where F is a non-archimedean local field. We are to prove that π is contained in $\operatorname{Ind}_B^G \chi$ for some character $\chi\colon T \to \mathbb{C}^\times$ if and only if $\operatorname{Res}_N^G \pi$ contains the trivial representation of N. Here the character χ of T can be viewed as a representation of B for the following reason.

Note that given the canonical projection $B \to B/N$ and the isomorphism $B/N \cong T$, the data of the character $T \to \mathbb{C}^\times$ is equivalent to the data of the character $B \to B/N \xrightarrow{\sim} T \to \mathbb{C}^\times$ that is trivial on N. Therefore, $\operatorname{Res}_N^G \pi$ contains the trivial representation of N if and only if $\operatorname{Res}_B^G \pi$ contains an irreducible representation σ of B such that σ contains a trivial representation of N. Further, this is equivalent to $\sigma \cong \operatorname{Ind}_T^B \chi$ for some character $\chi\colon T \to \mathbb{C}^\times$. By Frobenius reciprocity, we have that

$$\operatorname{Hom}_G(\pi, \operatorname{Ind}_B^G \sigma) \cong \operatorname{Hom}_B(\operatorname{Res}_B^G \pi, \sigma) \cong \operatorname{Hom}_B(\operatorname{Res}_B^G \pi, \operatorname{Ind}_T^B \chi).$$

Therefore, the multiplicity of $\operatorname{Ind}_T^B \chi$ in $\operatorname{Res}_B^G \pi$ is the same as that of $\operatorname{Ind}_B^G \sigma$ in π, which proves the desired equivalence. $\square$

Homework 7

Exercise 7.1. Let k be a field, and let l/k be a Galois quadratic extension, with Galois group $\{1, \sigma\}$. Fix a k-linear isomorphism $l \cong k^2$. Then the left multiplication action of $l^\times$ on l defines an injective group homomorphism $i\colon l^\times \to \mathrm{GL}_2(k)$. Denote the image by E. Let N be the normalizer of E in $\mathrm{GL}_2(k)$.

(1) Show that there exists g in N such that the automorphism $\mathrm{Int}(g)\colon E \to E$ is the same as σ on $E \cong l^\times$.
(2) Show that for any $g \in N$, the automorphism $\mathrm{Int}(g)\colon E \to E$ must either be the identity, or σ as above. Conclude that the centralizer of E in $\mathrm{GL}_2(k)$ is of index 2 in N.
(3) Show that the centralizer of E in $\mathrm{GL}_2(k)$ is E.
(4) Let $g \in E - Z$, where Z denotes the center of $\mathrm{GL}_2(k)$. Show that the centralizer of g in $\mathrm{GL}_2(k)$ is E. Then show that if $h \in \mathrm{GL}_2(k)$ is such that $hgh^{-1} \in E$, then h normalizes E.

Solution. (1) Consider $l \cong k^2$ as an l-module via $l \times l \to l,\ (a, b) \mapsto \sigma(a)b$. Also consider l as the standard l-module via $l \times l \to l,\ (a, b) \mapsto ab$. Then $\sigma\colon l \to l$ is an isomorphism of l-modules because it is a bijective homomorphism in the sense that $\sigma(ab) = \sigma(a)\sigma(b)$. Further, it descends to an automorphism $\sigma\colon l^\times \to l^\times$ of $l^\times$-modules; through the isomorphism $l^\times \cong E$ induced by i, we get an automorphism $E \to E$ of E-modules induced from σ. Using the compatibility with E-action, it must be inner, written as $\mathrm{Int}(g)\colon E \to E$.

(2) Notice that $i\colon l^\times \to \mathrm{GL}_2(k)$ maps $k^\times$ to $k^\times \cdot \mathrm{id} \subset \mathrm{GL}_2(k)$, and $k^\times \cdot \mathrm{id}$ is clearly invariant under $\mathrm{Int}(g)$. On the other hand, via the isomorphism $E \cong l^\times$ of groups, each $\mathrm{Int}(g)$ gives rise to an automorphism of $l^\times$. Thus it must keep $k^\times$ invariant, and it is thus an element of $\mathrm{Aut}_k(l^\times)$. Moreover, such an action extends thereof to an element in $\mathrm{Gal}(l/k)$ uniquely because l is a field, so the description in need follows. As each element $n \in N$ normalizes E, or namely defines an automorphism $\mathrm{Int}(n)\colon E \to E$, the set $\mathrm{Inn}(E)$ is in bijection with $\{1, \sigma\}$, meaning that $N/Z_G(E)$ for $G = \mathrm{GL}_2(k)$ contains a unique non-trivial element, and therefore $Z_G(E)$ is of index 2 in N.

(3) As E is isomorphic to the abelian group $l^\times$, it is clear that E centralizes itself, or equivalently E is a subgroup of $Z_G(E)$; it suffices to show the converse set-theoretical inclusion. Using part (2), any $g \in Z_G(E)$ can be regarded as $1 \in \mathrm{Gal}(l/k)$, and hence defines $\mathrm{id}\colon l^\times \to l^\times$ up to isomorphism, meaning that it is an automorphism of $l^\times$ admitting a trivialization, which can only be scalar multiplication of $l^\times$. Thus the data of g-action is equivalent to the data of an abstract map $E \times E \xrightarrow{\sim} l^\times \times E \to E$. So we conclude that $E = Z_G(E)$.

(4) Denote by $Z_G(g)$ the centralizer of g. Then from (3) we have $E \subset Z_G(g)$ and the subgroup $\langle g \rangle$ generated by g is normal in E, since $g \notin Z$ and any $x \in E$ commutes with g. Then

$$Z_G(g) = E = Z_G(E),$$

where the first equality follows from the definition and the second equality is by part (3). Now it remains to show such h normalizes E. For this, we only need that $h^{-1}xh \in E = Z_G(g)$ for all $x \in E$. Note that by considering G-conjugacy action on itself, we can identify $Z_G(g)$ with $\mathrm{Stab}_G(g)$. So the desired inclusion is implied by $h^{-1}xh \cdot g = g \cdot h^{-1}xh$; but this is equivalent to $hgh^{-1} \cdot x = x \cdot hgh^{-1}$, i.e. $x \in Z_G(hgh^{-1})$ for all $x \in E$. Hence we finish the proof, because this is implied by part (3) that $E = Z_G(E) \subset Z_G(hgh^{-1})$. □

Exercise 7.2. Keep the same notation as in [BH06, p.48] and in addition let χ_1 be the character of $\mathrm{Ind}_{ZN}^G \theta_\psi$, and let χ_2 be the character of $\mathrm{Ind}_E^G \theta$.

(1) Verify the character formula (6.4.1).
(2) Verify that

$$\langle \chi_1, \chi_1 \rangle = q, \quad \langle \chi_1, \chi_2 \rangle = q - 1.$$

Solution. (1) The desired formula (6.4.1) computes $\operatorname{tr}\pi_\theta = \chi_1 - \chi_2$. By construction and character formulae for inductions we have

$$\chi_1(g) = \sum_{\substack{h\in G/ZN \\ h^{-1}gh\in ZN}} \theta_\psi(h^{-1}gh), \qquad \chi_2(g) = \sum_{\substack{h\in G/E \\ h^{-1}gh\in E}} \theta(h^{-1}gh).$$

Here $\theta_\psi\colon zu \mapsto \theta(z)\psi(u)$ for $z \in Z$ and $u \in N$ is a character of ZN.

(i) Suppose $g = z \in Z$. Then $h^{-1}zh = z \in ZN \cap E$, and hence

$$\begin{aligned}\operatorname{tr}\pi_\theta(g) &= |G/ZN| \cdot \theta_\psi(z) - |G/E| \cdot \theta(z) \\ &= (q^2-1)\cdot\theta(z) - (q^2-q)\cdot\theta(z) \\ &= (q-1)\cdot\theta(z).\end{aligned}$$

Here we have used the observations $|Z| = |\mathbb{F}_q^\times| = q-1$, $|N| = |\mathbb{F}_q| = q$, and $|E| = |\mathbb{F}_{q^2}^\times| = q^2-1$, as well as the known fact that $|G| = |\mathrm{GL}_2(\mathbb{F}_q)| = (q^2-1)(q^2-q)$.

(ii) Suppose $g = zu$ for $z \in Z$ and $u \in N\backslash\{1\}$. We first claim that g is not conjugate to E. Indeed, if this is not true, then $h^{-1}uh \in E$ for some $h \in G$ with $\det(h^{-1}uh) = \det(u) = 1$; so the image of $h^{-1}uh$ in $l^\times$ is an element with norm 1. On the other hand, as l/k is a quadratic extension, there are at most 2 candidates for the image of $h^{-1}uh$ in $l^\times$, but the condition $\operatorname{tr}(h^{-1}uh) = \operatorname{tr}(u) > 0$ forces $h^{-1}uh$ to be trivial in $l^\times$. This contradicts to $u \in N\backslash\{1\}$ and proves the claim. Consequently, we have

$$\chi_2(zu) = 0,$$

so it remains to compute $\chi_1(zu)$. For this, notice that $\theta_\psi(h^{-1}zuh) = \theta_\psi(z\cdot h^{-1}uh) = \theta(z)\psi(h^{-1}uh) = \theta(z)\psi(u)$. So we have

$$\operatorname{tr}\pi_\theta(g) = \chi_1(zu) = \theta(z)\sum_{u'\sim u}\psi(u'),$$

where the sum runs through elements $u' \in N\backslash\{1\}$ that are G-conjugate to u. Since $N\backslash\{1\}$ is in bijection with $F^\times$, if we fix an element $u = \left(\begin{smallmatrix}1 & x \\ 0 & 1\end{smallmatrix}\right) \in N\backslash\{1\}$ then any other $u' \in N\backslash\{1\}$ is of form $\left(\begin{smallmatrix}1 & ax \\ 0 & 1\end{smallmatrix}\right)$ for some $a \in F^\times$; according to the relation $huh^{-1} = u'$ with $h = \left(\begin{smallmatrix}a & 0 \\ 0 & 1\end{smallmatrix}\right) \in G$, any two elements of $N\backslash\{1\}$ are G-conjugate. On the other hand, $\psi\colon N \to \mathbb{C}^\times$ gives rise to a class function

$$\Psi\colon N \longrightarrow \mathbb{C}^\times, \quad u \longmapsto \sum_{u'\sim u}\psi(u').$$

Because both θ and $\operatorname{tr}\pi_\theta$ are group homomorphisms, so also is Ψ. Combining the two observations above, the image of Ψ consists of two values that form a subgroup of $\mathbb{C}^\times$, namely $\Psi(N) = \{\pm 1\}$. It follows that $\Psi(N\backslash\{1\}) = \{-1\}$, and we deduce

$$\operatorname{tr}\pi_\theta(g) = -\theta(z).$$

(iii) Suppose $g = y \in E\backslash Z$. For a similar reason as in proving the claim in (ii), we see g is not conjugate to ZN, and thus $\chi_1(g) = 0$. By Exercise 7.1 all the automorphisms of E of form $\mathrm{Int}(g)\colon E \to E$ with $g \in N_G(E)$ are exactly elements of $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q) = \{1, \sigma\}$. It follows that the orbit of θ under $N_G(E)$ is exactly $\{\theta, \theta\circ\sigma\} = \{\theta, \theta^q\}$. Therefore,

$$\operatorname{tr}\pi_\theta(g) = -\chi_2(y) = -(\theta(y) + \theta^q(y)).$$

So far we have verified the desired formula about $\operatorname{tr}\pi_\theta$.

(2) We begin with verifying $\langle\chi_1,\chi_1\rangle = q$. We have from part (1) that

$$\begin{aligned}|G|\cdot\langle\chi_1,\chi_1\rangle &= \sum_{g\in G}\chi_1(g)\overline{\chi_1(g)}\\ &= \sum_{g\in Z}|(q^2-1)\theta(g)|^2 + \sum_{\substack{g\sim uz\\ uz\in NZ\backslash Z}}|-\theta(z)|^2\\ &= (q^2-1)^2\cdot|Z| + \sum_{\substack{g\sim uz\\ uz\in NZ\backslash Z}}1.\end{aligned}$$

Here in the last equality we have used $|\theta(g)|^2 = \theta(g)\cdot\overline{\theta(g)} = \theta(g)\cdot\theta(g)^{-1} = 1$. As for the second sum, modulo the center, the number of ways that uz is G-conjugate to some $g\in G$ is the same as that for u conjugate to g. Now the only ambiguity lies in the choice of $u\in N$. Thus we have

$$\sum_{\substack{g\sim uz\\ uz\in NZ\backslash Z}}1 = \frac{|G/Z|}{|N\backslash\{1\}|\cdot|N|}\sum_{uz\in NZ\backslash Z}1 = (q+1)\cdot|NZ\backslash Z|.$$

Therefore,

$$\langle\chi_1,\chi_1\rangle = \frac{1}{|G|}((q^2-1)^2\cdot|Z| + (q+1)\cdot|NZ\backslash Z|) = q.$$

We then verify that $\langle\chi_1,\chi_2\rangle = q-1$. Again by the formulae of part (1),

$$\begin{aligned}|G|\cdot\langle\chi_1,\chi_2\rangle &= \sum_{g\in G}\chi_1(g)\overline{\chi_2(g)}\\ &= \sum_{g\in Z}(q^2-1)(q^2-q)|\theta(g)|^2 + \sum_{\substack{g\sim g'\\ g'\in(NZ\cap E)\backslash Z}}\chi_1(g)\overline{\chi_2(g)}\\ &= (q^2-1)(q^2-q)\cdot|Z|.\end{aligned}$$

Here the second sum in the second line vanishes simply because $(NZ\cap E)\backslash Z = \emptyset$. So we get

$$\langle\chi_1,\chi_2\rangle = \frac{|Z|}{|G|}\cdot(q^2-1)(q^2-q) = q-1.$$

□

Exercise 7.3 ([BH06, p.48, Exercise]). Let ψ be a non-trivial character of N, and consider the representation

$$\mathcal{F} = \mathrm{Ind}_N^G\,\psi.$$

Let σ be an irreducible representation of G. Show that:

(1) If $\sigma = \phi\circ\det$, for a character ϕ of $k^\times$, then σ does not occur in $\mathcal{F}$.
(2) Otherwise, σ occurs in $\mathcal{F}$ with multiplicity one.

Solution. (1) Note that for unipotent radical N we always have $\det(n) = 1\in k^\times$ for all $n\in N$. It follows that $\sigma(N) = \phi(\det(N))$ is trivial. If σ occurs in $\mathcal{F}$ then there is $f\in\sigma$ such that

$$f(ng) = \psi(n)\cdot f(g)$$

for all $n\in N$ and $g\in G$. Since ψ is non-trivial on N, we may assume $\psi(n)$ is non-trivial for a specified n and $g = n'\in N$; then the equality above breaks. This leads to a contradiction.

(2) It suffices to show the space of G-maps $\mathrm{Hom}_G(\mathcal{F},\sigma)$ is exactly 1-dimensional. By Frobenius reciprocity we need to look at $\mathrm{Hom}_G(\mathrm{Ind}_N^G\,\psi,\sigma) = \mathrm{Hom}_N(\psi,\sigma|_N)$. Here $\sigma|_N$ is necessarily identified with a character because of the isomorphism $N\cong F$. As ψ is another non-trivial character of N, we see $\dim\mathrm{Hom}_N(\psi,\sigma|_N) = 1$ as desired. □

Exercise 7.4 ([BH06, pp.62–63, Exercises 1,2]).

(1) Let (π, V) be an irreducible smooth representation of G with a non-trivial $\pi(N)$-fixed vector. Show that $\pi = \phi \circ \det$, for some character ϕ of $F^\times$.

(2) Let (π, V) be an irreducible smooth representation of G such that $\dim V$ is finite. Show that V has a non-zero $\pi(N)$-fixed vector. Deduce that $\dim V = 1$ and π is of the form $\phi \circ \det$, for some character ϕ of $F^\times$.

Solution. (1) Since (π, V) is irreducible, we have either $V(N) = 0$ or $V(N) = V$. But the existence of non-trivial $v \in V$ such that $v = \pi(n)v$ for all $n \in N$ forces $V(N)$ to be 0. It follows that $V = V_N$ as vector spaces, and in particular $V_N \neq 0$, which means π is non-cuspidal. So π is contained in $\mathrm{Ind}_B^G \chi$ for some character $\chi\colon T \to \mathbb{C}^\times$ at the level of B-representations, and the Jordan–Hölder theory implies that V is 1-dimensional from $V(N) = 0$. To show that $\pi = \phi \circ \det$, note that $\ker \pi$ contains the commutator subgroup $\mathrm{SL}_2(F)$ of G because of $\dim(\pi, V) = 1$, where $\phi\colon F^\times \to \mathbb{C}^\times$ is a group homomorphism. Since $\det\colon G = \mathrm{GL}_2(F) \to F^\times$ is surjective and open, such ϕ is a character. This finishes the proof.

(2) If $V(N) \neq 0$ then $V(N) = V$ and $V_N = 0$, by the irreducibility, and therefore $V_\vartheta \neq 0$ for all non-trivial characters $\vartheta\colon N \to \mathbb{C}^\times$; it implies that $\dim V$ is infinite, which contradicts to the assumption. Thus $V(N) = 0$ and then N acts trivially on V, so that V has a non-zero $\pi(N)$-fixed vector. Again by $V_N \neq 0$ and the Jordan–Hölder theory of non-cuspidal representation, we have $\dim V = 1$ and $\pi = \phi \circ \det$ for some character $\phi\colon F^\times \to \mathbb{C}^\times$. $\square$

Homework 8

In the following F denotes a non-archimedean local field, and $\mathfrak{m}$ denotes the maximal ideal of $\mathcal{O}_F$. All characters $F \to \mathbb{C}^\times$ are assumed to be smooth.

Exercise 8.1. Show that if $\theta, \theta' \colon F \to \mathbb{C}^\times$ are two non-trivial characters, then there exists $a \in F^\times$ such that $\theta'(x) = \theta(ax)$ for all $x \in F$.

Solution. It suffices to show that if we fix a non-trivial character θ of F then the function $a\theta \colon F \to \mathbb{C}^\times,\ x \mapsto \theta(ax)$ ranges over the characters of F as a ranges over F. If this is the case then $a \in F^\times$ follows from the non-triviality of θ'. As θ is fixed, we are to show that there is a group isomorphism

$$\iota \colon F \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont}}(F, \mathbb{C}^\times), \quad a \longmapsto a\theta.$$

It is clear that ι is an injective group homomorphism. So it remains to show the surjectivity.

Recall that for non-trivial $\psi \in \mathrm{Hom}_{\mathrm{cont}}(F, \mathbb{C}^\times)$ the level of ψ is defined as the least integer d such that $\mathfrak{m}^d \subset \ker \psi$; if we fix d then the subset of $\mathrm{Hom}_{\mathrm{cont}}(F, \mathbb{C}^\times)$ of level at most d is the subgroup consisting of those ψ such that $\psi|_{\mathfrak{m}^d} = \mathbb{1}$.

Let ϖ be a chosen uniformizer of F and $u \in \mathcal{O}_F^\times$. Let the levels of θ and θ' be l and l' respectively. Then the character $u\varpi^{l-l'}\theta$ has level l' and so agrees with θ' on $\mathfrak{m}^{l'}$; moreover, the characters $u\varpi^{l-l'}\theta$ and $u'\varpi^{l-l'}\theta$ with $u, u' \in \mathcal{O}_F^\times$ agree on $\mathfrak{m}^{l'-1}$ if and only if $u \equiv u' \bmod \mathfrak{m}$. The group $\mathfrak{m}^{l'-1}$ has $q-1$ non-trivial characters which are trivial on $\mathfrak{m}^{l'}$. (Here q denotes the residual cardinality of ϖ.) As u ranges over $\mathcal{O}_F^\times/(1+\mathfrak{m})$, the $q-1$ characters $u\varpi^{l-l'}\theta|_{\mathfrak{m}^{l'-1}}$ are distinct, non-trivial, but trivial on $\mathfrak{m}^{l'}$. Therefore one of them, say $u_1\varpi^{l-l'}\theta|_{\mathfrak{m}^{l'-1}}$, equals $\theta'|_{\mathfrak{m}^{l'-1}}$. Iterating this procedure, we find a sequence of elements $u_n \in \mathcal{O}_F^\times$ such that $u_n\varpi^{l-l'}\theta$ agrees with θ' on $\mathfrak{m}^{l'-n}$ and $u_{n+1} \equiv u_n \bmod \mathfrak{m}^n$. Then the Cauchy sequence $\{u_n\}$ converges to some $u \in \mathcal{O}_F^\times$ and we have $\theta' = u\varpi^{l-l'}\theta = \iota(u\varpi^{l-l'})$. This completes the proof that ι is an isomorphism. $\square$

Exercise 8.2. Let $\theta \colon F \to \mathbb{C}^\times$ be a non-trivial character. Let $j \geqslant 1$ be a positive integer, and let $a \in F^\times$ be such that $a \notin 1 + \mathfrak{m}^j$. Show that there exists $x \in F$ such that the map $1 + \mathfrak{m}^j \to \mathbb{C},\ u \mapsto \theta(aux) - \theta(x)$ is a non-zero constant.

Solution. Referring to the proof of Exercise 8.1, let $k \geqslant 1$ be the level of θ, so that $\theta|_{\mathfrak{m}^k}$ is trivial whilst $\theta|_{\mathfrak{m}^{k-1}}$ is non-trivial. Again by Exercise 8.1 we may assume $k = j$ by replacing θ with $\varpi^{k-j}\theta$ if necessary. Then there is some $y \in \mathfrak{m}^{j-1}$ such that $\theta(y) \in \mathbb{C}\backslash\{0\}$. Fix such y and let $x = y/a \in F$, so we get

$$\theta(aux) = \theta(uy) = \theta(y + (u-1)y) = \theta(y) \cdot \theta|_{\mathfrak{m}^j}((u-1)y).$$

As $\theta|_{\mathfrak{m}^j}$ is trivial by hypothesis, we have $\theta(aux)$ equal to the constant $\theta(y)$. Also, as x is independent of u, we see $\theta(aux) - \theta(x) \equiv \theta(y) - \theta(x) \in \mathbb{C}$ is a constant. To show this is non-zero under the assumption $a \notin 1 + \mathfrak{m}^j$, it suffices to show $\theta(y) = \theta(x)$ implies $a \in 1 + \mathfrak{m}^j$. But for this, we notice that $\theta(ax) = \theta(x) \cdot \theta((a-1)x) = \theta(x)$ implies $a - 1 \in \mathfrak{m}^j$, which is as desired. $\square$

Exercise 8.3. Let $(V_i)_{i \in I}$ be a family of smooth representations of $N \cong F$. Show that there are natural isomorphisms

$$\bigoplus_{i \in I} V_i(N) \cong \left(\bigoplus_{i \in I} V_i\right)(N), \qquad \bigoplus_{i \in I} V_{i,N} \cong \left(\bigoplus_{i \in I} V_i\right)_N.$$

Solution. It suffices to show the first isomorphism, and the second one can be deduced from the same argument with replacing V_i by their contragredients V_i^*. Note that N acts on the direct sum (π, V) for $V = \bigoplus_{i \in I} V_i$ diagonally, i.e. $\pi(n)((v_i)_{i \in I}) = (\pi_i(n)v_i)_{i \in I}$. Then $\pi(n)((v_i)_{i \in I}) -$

$(v_i)_{i\in I} = (\pi_i(n)v_i)_{i\in I} - (v_i)_{i\in I} = (\pi_i(n)v_i - v_i)_{i\in I}$, and thus by definition,

$$\begin{aligned} V(N) &= \operatorname{span}\{(\pi_i(n)v_i - v_i)_{i\in I} \colon n \in N,\ v_i \in V_i \text{ for all } i \in I\} \\ &\cong \bigoplus_{i\in I} \operatorname{span}\{\pi_i(n)v_i - v_i \colon n \in N,\ v_i \in V_i\}. \end{aligned}$$

This proves the desired result. □

Exercise 8.4. Let G be a locally profinite group. A smooth representation V of G is said to be of *finite length*, if there exists a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ where each V_i is a subrepresentation, and V_{i+1}/V_i is non-zero irreducible. The integer n is called the length of V, and the isomorphism classes of the irreducible representations V_{i+1}/V_i are called the Jordan–Hölder factors (or *composition factors*) of V (counting multiplicities).

(1) Show that both n and the set of Jordan–Hölder factors (counting multiplicities) depend only on V, and are independent of the choice of the filtration.
(2) Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be a short exact sequence of smooth representations. Show that V_2 is of finite length if and only if V_1 and V_3 are of finite length.
(3) Let B be a closed subgroup of G. Let V be a smooth representation of G such that it is of finite length as a B-representation. Show that it is of finite length as a G-representation, and that the length is at most equal to the length as a B-representation.

Solution. (1) We implicitly suppose V is of finite length n. It suffices to show that any two Jordan–Hölder filtrations are equivalent, in the sense that for $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ and $0 = V'_0 \subset V'_1 \subset \cdots \subset V'_m = V$ we have $m = n$ and $V_{i+1}/V_i \cong V'_{\sigma(i)+1}/V'_{\sigma(i)}$ for all $0 \leqslant i \leqslant m-1$ with some $\sigma \in S_n$. We will prove that if the statement is true for any subrepresentation of V then it is true for V. If V is irreducible then the statement is trivial; otherwise we only need to consider the case where $V_{n-1} \neq V'_{m-1}$, which implies $V_{n-1} + V'_{m-1} = V$, and hence

$$V/V_{n-1} \cong V'_{m-1}/(V'_{m-1} \cap V_{n-1}), \quad V/V'_{m-1} \cong V_{n-1}/(V'_{m-1} \cap V_{n-1}).$$

By assumption both V/V_{n-1} and V/V'_{m-1} are irreducible, so that

$$\begin{aligned} 0 = W_0 \subset W_1 \subset \cdots \subset W_{r-1} = V'_{m-1} \cap V_{n-1} \subset V_{n-1} \subset V_n = V \\ 0 = W_0 \subset W_1 \subset \cdots \subset W_{r-1} = V'_{m-1} \cap V_{n-1} \subset V'_{m-1} \subset V_m = V \end{aligned}$$

are two Jordan–Hölder filtrations of V; they are clearly equivalent and the desired result follows from the inductive assumption on $V'_{m-1} \cap V_{n-1}$.

(2) Suppose V_2 is of finite length of length k with Jordan–Hölder filtration $0 = V_{2,0} \subset \cdots \subset V_{2,k} = V_2$. Then for each $0 \leqslant i \leqslant k-1$, since $V_1 \subset V_2$, the quotient $(V_1 \cap V_{2,i+1})/(V_1 \cap V_{2,i})$ is a subrepresentation of the non-zero irreducible representation $V_{2,i+1}/V_{2,i}$, so it is either 0 or $V_{2,i+1}/V_{2,i}$. It follows that after deleting some steps to get rid of zero quotients, the sequence

$$0 = V_1 \cap V_{2,0} \subset \cdots \subset V_1 \cap V_{2,k} = V_1$$

is a Jordan–Hölder filtration for V_1 with length at most k. As for V_3, the proof is similar by first considering the images of all $V_{2,i}$ in V_3 and then deleting some indices.

Conversely, suppose both V_1 and V_3 are of finite length. Let $0 = V_{1,0} \subset \cdots \subset V_{1,m} = V_1$ and $0 = V_{3,0} \subset \cdots \subset V_{3,n} = V_3$ be Jordan–Hölder filtrations of V_1 and V_3, respectively. Note that each subrepresentation $V_{3,i}$ of V_3 corresponds to a unique subrepresentation $\tilde{V}_{3,i}$ of V_2 containing V_1; here we can take $\tilde{V}_{3,i}$ to be the inverse image of $V_{3,i}$ in V_2. Then

$$0 = V_{1,0} \subset \cdots \subset V_{1,m} = V_1 = \tilde{V}_{3,0} \subset \cdots \subset \tilde{V}_{3,n} = V_2$$

is a Jordan–Hölder filtration of V_2 of finite length $m + n$.

(3) Let $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ be the Jordan–Hölder filtration of B-representations. Since V is a priori a G-representation we have $V^G = V$. On the other hand, provided the

Iwasawa decomposition $G = BK$, we see $V_i^G = V_i^K$ because each V_i is a B-representation. Thus we have an induced filtration

$$0 = V_0^K \subset V_1^K \subset \cdots \subset V_n^K = V$$

of G-representations. By Exercise 2.5, it follows from the exactness of $(\mathbf{-})^K$ that $V_{i+1}^K/V_i^K = (V_{i+1}/V_i)^K$ for all $0 \leqslant i \leqslant n-1$. Consequently, similar to part (2), it turns out that either $(V_{i+1}/V_i)^K = 0$ or $(V_{i+1}/V_i)^K = V_{i+1}/V_i$. Therefore, by deleting the zero Jordan–Hölder factors in G-filtration above we get the desired Jordan–Hölder filtration with length at most n. $\square$

Homework 9

Exercise 9.1. Let F be a non-archimedean local field, with valuation v. Let $\phi\colon F^\times \to \mathbb{C}^\times$ be a homomorphism. Assume one of the following two conditions holds:

(i) There exists $n \in \mathbb{Z}$ such that ϕ is constant on $\{x \in F^\times : v(x) \geqslant n\}$.
(ii) There exists $n \in \mathbb{Z}$ such that ϕ is constant on $\{x \in F^\times : v(x) \leqslant n\}$.

Show that ϕ is trivial.

Solution. Choose ϖ to be a uniformizer of $\mathcal{O}_F$. We claim that $\phi(\varpi) = 1$. Indeed, in either of cases (i) or (ii), there is $m \in \mathbb{Z}$ with $|m| \gg 0$ such that $\phi(\varpi^k) \equiv C \in \mathbb{C}^\times$ whenever $|k| \geqslant |m|$. We may assume $k \geqslant m \geqslant 0$ and then $\phi(\varpi^{k+1}) = C = \phi(\varpi^k)$, which implies the claim, because ϕ is a homomorphism of multiplicative groups.

For general $x \in F^\times$, again by assumption there is $m \in \mathbb{Z}$ with $|m| \gg 0$ such that $v(\varpi^m x)$ satisfies either (i) or (ii). It follows that $C = \phi(\varpi^m x) = \phi(\varpi)^m \phi(x) = \phi(x)$. This proves that ϕ is a constant on the whole $F^\times$, and hence equal to $\phi(\varpi) = 1$, meaning that ϕ is trivial. $\square$

Exercise 9.2. Let G be a locally profinite group, and K a compact open subgroup. Fix a left Haar measure on G, and equip $G \times G$ with the product Haar measure. Show that there is a natural algebra isomorphism

$$\mathcal{H}(G \times G, K \times K) \cong \mathcal{H}(G, K) \otimes_{\mathbb{C}} \mathcal{H}(G, K).$$

Solution. Note that $K \times K$ is a compact open subgroup of $G \times G$. Consider the natural map

$$\Phi\colon \mathcal{H}(G, K) \otimes_{\mathbb{C}} \mathcal{H}(G, K) \longrightarrow \mathcal{H}(G \times G, K \times K), \quad f_1 \otimes f_2 \longmapsto f_1 f_2,$$

where $\Phi(f_1 \otimes f_2)(g_1, g_2) = (f_1 f_2)(g_1, g_2) = f_1(g_1) f_2(g_2)$. Notice that the compact support condition is automatic. To check the image of Φ is bi-$(K \times K)$-invariant, compute for any $(k_1, k_2), (k_1', k_2') \in K \times K$ that

$$\Phi(f_1 \otimes f_2)((k_1, k_2)(g_1, g_2)(k_1', k_2')) = f_1(k_1 g_1 k_1') f_2(k_2 g_2 k_2') = f_1(g_1) f_2(g_2),$$

because f_1 and f_2 are bi-K-invariant. To check Φ is a homomorphism, we compute

$$\begin{aligned}
\Phi((f_1 \otimes f_2) * (f_1' \otimes f_2'))(g_1, g_2) &= \int_{G \times G} f_1(x) f_2(y) f_1'(x^{-1} g_1) f_2'(y^{-1} g_2) \mathrm{d}(x, y) \\
&= \int_G f_1(x) f_1'(x^{-1} g_1) \mathrm{d}x \cdot \int_G f_2(y) f_2'(y^{-1} g_2) \mathrm{d}y \\
&= (f_1 * f_1')(g_1) \cdot (f_2 * f_2')(g_2),
\end{aligned}$$

which matches the convolution $*$ in $\mathcal{H}(G \times G, K \times K)$. The second equality above uses the condition that $G \times G$ is equipped with the normalized product Haar measure. Now it remains to check the bijectivity. By Exercise 3.6, every function in $\mathcal{H}(G \times G, K \times K)$ can be written as finite linear combination of functions of form

$$\mathbb{1}_{(g, g')(K \times K)} = \mathbb{1}_{gK} \otimes \mathbb{1}_{g'K} \in \mathcal{H}(G, K) \otimes_{\mathbb{C}} \mathcal{H}(G, K),$$

and it follows that Φ is surjective; also, the injectivity is clear because $f_1(g_1) f_2(g_2) = 0$ for all $g_1, g_2 \in G$ implies $f_1 = 0$ or $f_2 = 0$, and thus $f_1 \otimes f_2 = 0$. $\square$

Exercise 9.3. Prove [BH06, (9.5.2)].

Solution. Suppose $\chi = \chi_1 \otimes \chi_2$ is a character of T and ϕ is a character of $F^\times$. We need to prove that

$$\operatorname{Ind}_B^G(\phi \cdot \chi) \cong \phi \operatorname{Ind}_B^G \chi.$$

For this, inflate $\phi \cdot \chi \coloneqq \phi\chi_1 \otimes \phi\chi_2$ to a B-representation that is trivial on N, and then

$$\phi \cdot \chi = (\phi \circ \det|_B) \otimes \chi.$$

To complete the proof, check for each $g \in G$ that

$$\begin{aligned}(\phi \operatorname{Ind}_B^G \chi)(g) &= \phi(\det g) \cdot (\operatorname{Ind}_B^G \chi)(g) \\ &= ((\phi \circ \det) \otimes (\operatorname{Ind}_B^G \chi))(g) \\ &= (\operatorname{Ind}_B^G((\phi \circ \det)|_B \otimes \chi))(g) \\ &= (\operatorname{Ind}_B^G((\phi \circ \det|_B) \otimes \chi))(g) \\ &= (\operatorname{Ind}_B^G(\phi \cdot \chi))(g).\end{aligned}$$

The third equality above is due to the projection formula (or the associativity of tensor product of $\mathbb{C}[B]$-modules). $\square$

Exercise 9.4. The *Burnside's theorem for matrix algebras* is stated as follows:

> *Theorem.* Let C be an algebraically closed field, and V a finite-dimensional C-vector space. Let A be a unital subalgebra of $\operatorname{End}_C(V)$ (i.e., a subalgebra containing 1) such that V is a simple A-module. Then $A = \operatorname{End}_C(V)$.

Show that the finite-dimensionality assumption is necessary by constructing a counter-example when this assumption is dropped.

Solution. Let $C = \mathbb{C}$ and $V = \mathbb{C}[x]$ as an infinite-dimensional vector space over $\mathbb{C}$. Note that the derivative $D \colon p(x) \mapsto p'(x)$ is an element of $\operatorname{End}_{\mathbb{C}}(V)$. Define A as the subalgebra of $\operatorname{End}_{\mathbb{C}}(V)$ generated by D over $\mathbb{C}$. We check the desired properties on A and V as follows.

(i) It is clear that A is unital. Also, if $p(x) \in V$ is such that $D(p(x)) = p(x)$, then $p(x) = 0$. It follows that there is no non-trivial A-invariant subspace in V, and thus V is a simple A-module.

(ii) Note that the shift operator $T \colon p(x) \mapsto xp(x)$ is also an element of $\operatorname{End}_{\mathbb{C}}(V)$ but is not contained in A. So $A \neq \operatorname{End}_{\mathbb{C}}(V)$.

$\square$

Homework 10

No class in this week.

Homework 11

Exercise 11.1. Let (π, V) be a smooth representation of a locally profinite group G.

(1) Show that $\langle \check{v}, v\rangle = \langle \check{\pi}(g)\check{v}, \pi(g)v\rangle$ for all $g \in G$ and $v \in \pi,\ \check{v} \in \check{\pi}$.
(2) Let K be a compact open subgroup of G. Show that for $v \in \pi^K$ and $\check{v} \in \check{\pi}^K$, the function $\gamma\colon G \to \mathbb{C},\ g \mapsto \langle \check{v}, \pi(g)v\rangle$ is left and right K-invariant.
(3) For arbitrary $\check{v} \in \check{\pi}$ and $v \in \pi$, show that $\langle \check{v}, \pi(e_K)v\rangle = \langle \check{\pi}(e_K)\check{v}, v\rangle$.

Solution. (1) By definition of contragredient we have $\check{\pi} = (\pi^*)^\infty\colon G \to \mathrm{Aut}_{\mathbb{C}}(\check{V})$. For $v^* \in \pi^*$, recall that $\langle \pi^*(g)v^*, v\rangle = \langle v^*, \pi(g^{-1})v\rangle$ for all $g \in G$, which still holds by taking the smooth subrepresentation; so we deduce the equality

$$\langle \check{\pi}(g)\check{v}, v\rangle = \langle \check{v}, \pi(g^{-1})v\rangle$$

for all $g \in G$ with $v \in \pi$ and $\check{v} \in \check{\pi}$. It follows that

$$\langle \check{\pi}(g)\check{v}, \pi(g)v\rangle = \langle \check{v}, \pi(g^{-1})\pi(g)v\rangle = \langle \check{v}, v\rangle.$$

(2) The image of γ under K-action on γ is $g \mapsto \langle \check{v}, \pi(k_1)\pi(g)\pi(k_2)v\rangle$ with $k_1, k_2 \in K$. Since $v \in \pi^K$ and $\check{v} \in \check{\pi}^K$, we have

$$\langle \check{v}, \pi(k_1)\pi(g)\pi(k_2)v\rangle = \langle \check{v}, \pi(k_1)\pi(g)v\rangle = \langle \check{\pi}(k_1^{-1})\check{v}, \pi(g)v\rangle = \langle \check{v}, \pi(g)v\rangle.$$

and hence γ is bi-K-invariant. Here the first and third equalities respectively follow from $v \in \pi^K$ and $\check{v} \in \check{\pi}^K$; the second equality is because of the formula in (1).

(3) By the preparation work in (1), we get

$$\langle \check{v}, \pi(e_K)v\rangle = \langle \check{\pi}(e_K^{-1})\check{v}, v\rangle = \langle \check{\pi}(e_K)\check{v}, v\rangle.$$

□

Exercise 11.2. Let F be a non-archimedean local field, and we write $\mathcal{O}$ for $\mathcal{O}_F$, and $\mathfrak{p}$ for the maximal ideal. Let n be a positive integer. Let $K_n = 1 + \mathrm{M}_2(\mathfrak{p}^n)$, and $T_n = T \cap K_n$, $N_n = N \cap K_n,\ N'_n = N' \cap K_n$. Here T is the subgroup of $\mathrm{GL}_2(F)$ consisting of diagonal matrices, and N (resp. N') is the subgroup of $\mathrm{GL}_2(F)$ consisting of upper triangular (resp. lower triangular) matrices with eigenvalues 1.

(1) Show that the map $N_n \times T_n \times N'_n \to K_n,\ (x, t, y) \mapsto xty$ is a homeomorphism. Previously, we showed the similar statement with K_n replaced by the Iwahori subgroup $I = \left(\begin{smallmatrix} \mathcal{O}^\times & \mathcal{O} \\ \mathfrak{p} & \mathcal{O}^\times \end{smallmatrix}\right)$.
(2) Let $t = \left(\begin{smallmatrix} \pi & \\ & 1 \end{smallmatrix}\right)$ where $\pi \in F$ is a uniformizer. Let a be an integer. Show that $t^{-a}K_nt^a = N_{n-a}T_nN'_{n+a}$.
(3) Let (π, V) be a smooth representation of $\mathrm{GL}_2(F)$, and let $v \in V^{K_n}$. Let a be a non-negative integer. Show that the $t^{-a}K_nt^a$-average of v is a non-zero constant times the N_{n-a}-average of v.

Solution. (1) Since K_n is compact and $N_n \times T_n \times N'_n$ is Hausdorff as spaces, to show they are homeomorphism it suffices to show that $N_n \times T_n \times N'_n \to K$ is bijective. Note that $K_n = N_nT_nN'_n$ as a group, so the map $N_n \times T_n \times N'_n \to N_nT_nN'_n,\ (x, t, y) \mapsto xty$ is clearly surjective. For the injectivity, we have $N_n \cap T_n = T_n \cap N'_n = N_n \cap N'_n = \{1\}$, and hence $xty = 1$ implies $x = t = y = 1$. To conclude, we get a homeomorphism $N_n \times T_n \times N'_n \simeq K_n$.

(2) By (1) we get an identity $K_n = N_nT_nN'_n$ between topological groups for $n \in \mathbb{N}$. Then we notice that

$$t^{-a}K_nt^a = (t^{-a}N_nt^a)(t^{-a}T_nt^a)(t^{-a}N'_nt^a).$$

So it reduces to the following computation. We have

$$t^{-a}N_nt^a = \begin{pmatrix}\pi^{-a} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & \mathfrak{p}^n\\ 0 & 1\end{pmatrix}\begin{pmatrix}\pi^{a} & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & \mathfrak{p}^{n-a}\\ 0 & 1\end{pmatrix} = N_{n-a},$$
$$t^{-a}N'_nt^a = \begin{pmatrix}\pi^{-a} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \mathfrak{p}^n & 1\end{pmatrix}\begin{pmatrix}\pi^{a} & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ \mathfrak{p}^{n-a} & 1\end{pmatrix} = N'_{n-a}.$$

Also, it is immediate that

$$t^{-a}T_nt^a = T_n$$

since all the matrices are diagonal. So we conclude that

$$t^{-a}K_nt^a = N_{n-a}T_nN'_{n+a}.$$

(3) From (2) there is a non-zero constant $C_0 \neq 0$ such that the Haar measure on $t^{-a}K_nt^a = N_{n-a}T_nN'_{n+a}$ is normalized as

$$\mathrm{d}\mu_{t^{-a}K_nt^a}(g) = \mathrm{d}\mu_{N_{n-a}T_nN'_{n+a}}(xty) = C_0\cdot \mathrm{d}\mu_{N_{n-a}}(x)\mathrm{d}\mu_{T_n}(t)\mathrm{d}\mu_{N'_{n+a}}(y).$$

On the other hand, as both T_n and N'_{n+a} are subgroups of K_n, we see $\pi(t)\pi(y)v = v$ for all $t\in T_n$ and $y \in N'_{n+a}$ because of $v \in V^{K_n}$. Thus, we compute the K_n-average of v as

$$\begin{aligned}\int_{t^{-a}K_nt^a}\pi(g)v\mathrm{d}g &= C_0\cdot\int_{N_{n-a}T_nN'_{n+a}}\pi(xty)v\mathrm{d}x\mathrm{d}t\mathrm{d}y\\ &= C_0\cdot\int_{N_{n-a}}\pi(x)v\mathrm{d}x\cdot\int_{T_n}\pi(t)v\mathrm{d}t\int_{N'_{n-a}}\pi(y)v\mathrm{d}y\\ &= C\cdot\int_{N_{n-a}}\pi(x)v\mathrm{d}x,\end{aligned}$$

where C is the product of C_0 with T_n-average and N'_{n-a}-average of v, which is non-zero. □

Exercise 11.3. Let G be a locally profinite group. Let K be a closed subgroup of G. Let (ρ, W) be an irreducible smooth representation of K.

(1) Let $\phi\colon W \to \text{c-Ind}_K^G W$ be a K-map. Fix $g \in G$. Show that the map $f\colon W \to W,\ w \mapsto \phi(w)(g)$ is an element of $\mathrm{Hom}_{g^{-1}Kg\cap K}(\rho, \rho^g)$. Here ρ^g is the representation of $g^{-1}Kg$ given by
$$g^{-1}Kg \xrightarrow{\mathrm{Int}(g)} K \xrightarrow{\rho} \mathrm{Aut}_{\mathbb{C}}(W).$$
(2) Assume that K is compact open, and let $\alpha\colon W \to \text{c-Ind}_K^G W$ be the canonical K-map sending w to the unique $f \in \text{c-Ind}_K^G W$ that is supported on K and satisfies $f(1) = w$. Show that $\text{c-Ind}_K^G W$ is spanned by the G-translates of the image of α.

Solution. (1) Since ϕ is K-equivariant, we have by definition that, for each $k \in K$,

$$\phi(\rho(k)w)(x) = ({}^k\phi(w))(x) = \phi(w)(xk), \quad \forall x \in G.$$

It follows that f is K-equivariant. Also, by definition of compact induction, we have

$$\rho^g(g^{-1}kg)\phi(w)(g) = \rho(k)\phi(w)(g) = \phi(w)(kg).$$

This means $f(W)$ admits the interior action of $g^{-1}Kg$ and hence f can be regarded as a K-map $f\colon \rho\to\rho^g$. Moreover, in the above argument, replace K with gKg^{-1}, we see such f can also be $g^{-1}kg$-equivariant for those $k\in K$ such that $g^{-1}kg \in K$. To conclude, we see f is compatible with the action of $g^{-1}Kg\cap K$, and hence an element of $\mathrm{Hom}_{g^{-1}Kg\cap K}(\rho,\rho^g)$.

(2) From the construction of α we see that $\alpha(w)(k) = \rho(k)w$ for all $k \in K$ and $w \in W$, and $\mathrm{supp}(\alpha(w)) = K$. Using this, notice that $\mathrm{im}(\alpha) = \{f \in \text{c-Ind}_K^G W\colon \mathrm{supp}(f) = K\}$. Take an

arbitrary $f \in \text{c-Ind}_K^G W$, there are finitely many $g_1, \dots, g_m \in G$ such that $\operatorname{supp}(f) = \bigcup_{i=1}^m g_i K$, and hence we may write

$$f = \sum_{i=1}^m f_i, \quad f_i = f|_{g_i K} \in \text{c-Ind}_K^G W.$$

For our purpose we aim to show there exists $g \in G$ such that $\lambda_g f \in \operatorname{im}(\alpha)$. It suffices to check this for each f_i in replace of f. But by construction we have $\operatorname{supp}(f_i) = g_i K$, and then the left translation $\lambda_{g_i^{-1}} f_i \colon x \mapsto f_i(g_i x)$ is supported on K; consequently, we have $f_i \in \lambda_{g_i} \operatorname{im}(\alpha)$, so

$$\text{c-Ind}_K^G W = \sum_{g \in G} \lambda_g \operatorname{im}(\alpha).$$

□

Exercise 11.4. Let G be a locally profinite group. Let Z be a closed subgroup of G such that it is central in G and G/Z is compact. Let (π, V) be a smooth representation of G such that Z acts on V by a smooth character. Show that (π, V) is semi-simple.

Solution. Suppose Z acts by the central character $\omega \colon Z \to \mathbb{C}^\times$ so that for all $v \in V$, we have $\pi(z)v = \omega(z)v$. Fix a vector $v \in V$, then v is fixed by some open compact subgroup K of G by smoothness assumption, i.e. $v \in V^K$; it implies $\omega(z)v \in V^K$ as well. Since G/Z is compact, it is clear that G/KZ is finite and equipped with discrete topology, so KZ is an open subgroup of G of finite index. Thus we can apply the following result from [BH06, §2.7, Lemma].

> *Lemma.* For any open and finite index subgroup H of G, the semi-simplicity of π is equivalent to the semi-simplicity of $\operatorname{Res}_H^G \pi$.

Using this, it suffices to show that π is semi-simple as a KZ-representation. Note that in our case, the space spanned by

$$\pi(kz)v = \pi(k)\omega(z)v = \omega(z)v$$

with varying $k \in K$ and $z \in Z$ is one-dimensional and equal to an irreducible G-space. It follows that $\operatorname{Res}_{KZ}^G \pi$ is semi-simple, and so also is π as a G-representation. □

Exercise 11.5. Let G be a locally profinite group with center Z. Let K be a closed subgroup of G. Let (ρ_1, W_1) and (ρ_2, W_2) be irreducible smooth representations of K. Show that for $g \in G$, the property $\operatorname{Hom}_{g^{-1}Kg \cap K}(\rho_1^g, \rho_2) \neq 0$ depends only on the image of g in $K \backslash G / KZ$. (In other words, for any $g' \in KgKZ$, either g and g' both have this property, or they both do not have this property.)

Solution. To simplify the notation, we denote by $\mathtt{P}_g$ the property $\operatorname{Hom}_{g^{-1}Kg\cap K}(\rho_1^g, \rho_2) \neq 0$ for $g \in G$. Fix $g \in G$ and consider $g' = k_1 g k_2 z \in KgKZ$ for some $k_1, k_2 \in K$ and $z \in Z$. It suffices to show that $\mathtt{P}_g$ implies $\mathtt{P}_{g'}$, and the converse direction can be verified by a similar argument. Since $\mathtt{P}_g$ is true by assumption, there is a non-zero map $f \colon W_1 \to W_2$ such that

$$f \circ \rho_1^g(y) = \rho_2(y) \circ f, \qquad \forall y \in g^{-1}Kg \cap K.$$

To prove $\mathtt{P}_{g'}$, we need a non-zero map $f^\flat_{g,g'} \colon W_1 \to W_2$ in terms of g, g' and f, such that

$$f^\flat_{g,g'} \circ \rho_1^{g'}(x) = \rho_2(x) \circ f^\flat_{g,g'}, \qquad \forall x \in {g'}^{-1}Kg' \cap K.$$

For this, define that

$$f^\flat_{g,g'} \coloneqq \rho_2(k_2^{-1}) \circ f \circ \rho_1(k_1^{-1}),$$

and it then remains to verify the desired intertwining property, i.e. $(g'^{-1}Kg \cap K)$-equivariance. By definition, if $x \in g^{-1}kg$ we have $\rho_1^g(x) = \rho_1(gxg^{-1})$, and the similar holds for g'. Thus, given $x \in {g'}^{-1}Kg' \cap K$, there is some $k \in K$ so that

$$x = {g'}^{-1}kg' = z^{-1}k_2^{-1}g^{-1}k_1^{-1}kk_1gk_2z = k_2^{-1}g^{-1}k_1^{-1}kk_1gk_2.$$

From this, we see $k_2xk_2^{-1} \in g^{-1}Kg$, and hence $k_2xk_2^{-1} \in g^{-1}Kg \cap K$ as $x \in K$. Thus, we can apply the intertwining property of f to get

$$\rho_2(k_2xk_2^{-1}) \circ f = f \circ \rho_1^g(k_2xk_2^{-1}) = f \circ \rho_1(gk_2xk_2^{-1}g^{-1}).$$

Finally, to check the desired intertwining property of $f^\flat_{g,g'}$, we compute for $x \in g'^{-1}Kg \cap K$ that

$$\begin{aligned} f^\flat_{g,g'} \circ \rho_1^{g'}(x) &= \rho_2(k_2^{-1}) \circ f \circ \rho_1(k_1^{-1}) \circ \rho_1(k_1gk_1kk_2^{-1}g^{-1}k_1^{-1}) \\ &= \rho_2(k_2^{-1}) \circ (f \circ \rho_1(gk_1kk_2^{-1}g^{-1})) \circ \rho_1(k_1^{-1}) \\ &= \rho_2(k_2^{-1}) \circ (\rho_2(k_2xk_2^{-1}) \circ f) \circ \rho_1(k_1^{-1}) \\ &= \rho_2(x) \circ (\rho_2(k_2^{-1}) \circ f \circ \rho_1(k_1^{-1})) \\ &= \rho_2(x) \circ f^\flat_{g,g'}. \end{aligned}$$

It means that $f^\flat_{g,g'}$ is an element of $\mathrm{Hom}_{g'^{-1}Kg' \cap K}(\rho_1^{g'}, \rho_2)$, and it is non-zero if and only if f is non-zero, because both ρ_1 and ρ_2 are non-zero irreducible representations of K. This completes the proof. $\square$

Homework 12

In the following F is a local non-archimedean field, with ring of integers $\mathcal{O}$, uniformizer π, and maximal ideal $\mathfrak{p} = \pi\mathcal{O}$ in $\mathcal{O}$.

Exercise 12.1. Let $\mathcal{L} = \{L_i\}_{i\in\mathbb{Z}}$ be a collection of lattices in F^2 such that $L_{i+1} \subsetneq L_i$ for all i. Show that there exists $e \in \mathbb{Z}$ such that $L_{i+e} = \pi L_i$ for all $i \in \mathbb{Z}$ if and only if $\mathcal{L}$ is stable under multiplication by $F^\times$. (These are two equivalent definitions of a lattice chain.)

Solution. Suppose $\mathcal{L}$ is stable under multiplication by $F^\times$. In particular, for each $L_i \in \mathcal{L}$ and $\pi \in F^\times$, we have $\pi L_i \in \mathcal{L}$. Since $\pi L_i \subset L_i$ always holds, for each $i \in \mathbb{Z}$ there exists some $0 \leqslant e(i) < \infty$ such that $\pi L_i = L_{i+e(i)}$. Note that the assumption $L_{i+1} \subsetneq L_i$ for all $i \in \mathbb{Z}$ forces such $e(i)$ to be uniquely determined by i. Then the following map of ordered sets

$$\iota\colon (\mathbb{Z}, \leqslant) \longrightarrow (\mathbb{Z}, \leqslant), \quad i \longmapsto i + e(i)$$

is a bijection. Thus $e(i)$ must be a constant $e \in \mathbb{Z}$. This proves that $L_{i+e} = \pi L_i$ for all $L_i \in \mathcal{L}$.

Conversely, given $e \in \mathbb{Z}_{\geqslant 0}$ such that $L_{i+e} = \pi L_i$ for all $i \in \mathbb{Z}$, then for each $a \in F^\times$ there exists $u \in \mathcal{O}^\times$ and $n \in \mathbb{Z}$ such that $a = u\pi^n$. Note that each $L_i \in \mathcal{L}$ is an $\mathcal{O}$-module so that $L_i = uL_i$. Thus it suffices to show π^n stabilizes $\mathcal{L}$. But this is clear because the assumption implies that π stabilizes $\mathcal{L}$. To conclude, we get $\{aL_i\}_{i\in\mathbb{Z}} \subset \mathcal{L}$ for all $a \in F^\times$. □

Exercise 12.2. Let $\mathfrak{A}_\mathcal{L}$ be the chain order associated with a lattice chain $\mathcal{L}$. Show that $\mathfrak{A}_\mathcal{L}$ is indeed an order in $A = \mathrm{M}_2(F)$, i.e., a subring which is also an $\mathcal{O}$-lattice.

Solution. By definition we have $\mathfrak{A}_\mathcal{L} = \{x \in A\colon xL_i \subset L_i \text{ for all } i \in \mathbb{Z}\}$. It is clear that $\mathfrak{A}_\mathcal{L}$ is closed under matrix addition and multiplication, and hence a subring of A. By classification of chain orders in A (c.f. [BH06, (12.1.2)]), there exists some $g \in \mathrm{GL}_2(F)$ such that

$$g\mathfrak{A}_\mathcal{L}g^{-1} \in \{\mathfrak{M}, \mathfrak{I}\} = \left\{ \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}, \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix} \right\}.$$

Since the G-translation of an $\mathcal{O}$-lattice is still an $\mathcal{O}$-lattice (see Exercise 12.7(4)), and both $\mathfrak{M}$ and $\mathfrak{I}$ are clearly $\mathcal{O}$-lattices, we have proved that $\mathfrak{A}_\mathcal{L}$ is also an $\mathcal{O}$-lattice. □

Exercise 12.3. Let $\mathfrak{A}_\mathcal{L}$ be as above. Show that a lattice L in F^2 is a member of $\mathcal{L}$ if and only if it is stabilized by $\mathfrak{A}_\mathcal{L}$.

Solution. The "only if" part is clear by definition of $\mathfrak{A}_\mathcal{L}$. As for the "if" part, suppose $x \in A$ stabilizes L together with all $L_i \in \mathcal{L}$. Up to G-conjugacy, we may assume $\mathfrak{A}_\mathcal{L} \in \{\mathfrak{M}, \mathfrak{I}\}$. In both cases that $\mathfrak{A}_\mathcal{L} = \mathfrak{M}$ or $\mathfrak{I}$, we have idempotent elements $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathfrak{A}_\mathcal{L}$, and that

$$L = e_1L \oplus e_2L.$$

This is because each vector $v \in L$ has the expression $v = e_1v + e_2v$. It follows that there are $a, b \in \mathbb{Z}$ such that $e_1L = \mathfrak{p}^a$ and $e_2L = \mathfrak{p}^b$, and then $L = \mathfrak{p}^a \oplus \mathfrak{p}^b$. Also, in both cases we have

$$\begin{pmatrix} 1 & \mathcal{O} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathfrak{p}^a \\ \mathfrak{p}^b \end{pmatrix} = \begin{pmatrix} \mathfrak{p}^a + \mathcal{O}\mathfrak{p}^b \\ \mathfrak{p}^b \end{pmatrix} \subset \begin{pmatrix} \mathfrak{p}^a \\ \mathfrak{p}^b \end{pmatrix},$$

which forces $\mathfrak{p}^a + \mathfrak{p}^b \subset \mathfrak{p}^a$, namely $b \geqslant a$. We then split the argument into two cases as follows.

(i) Whenever $\mathfrak{A}_\mathcal{L} = \mathfrak{M}$, by a symmetric argument, we have

$$\begin{pmatrix} 1 & 0 \\ \mathcal{O} & 1 \end{pmatrix} \begin{pmatrix} \mathfrak{p}^a \\ \mathfrak{p}^b \end{pmatrix} = \begin{pmatrix} \mathfrak{p}^a \\ \mathcal{O}\mathfrak{p}^a + \mathfrak{p}^b \end{pmatrix} \subset \begin{pmatrix} \mathfrak{p}^a \\ \mathfrak{p}^b \end{pmatrix},$$

which deduces $a \geqslant b$, and hence $L = \mathfrak{p}^a \oplus \mathfrak{p}^a$. In this case, $e_\mathcal{L} = 1$ and

$$\mathcal{L} = \{\pi^i\mathcal{O} \oplus \pi^i\mathcal{O}\}_{i\in\mathbb{Z}} = \{\mathfrak{p}^i \oplus \mathfrak{p}^i\}_{i\in\mathbb{Z}},$$

so we have proved that $L \in \mathcal{L}$.

(ii) Whenever $\mathfrak{A}_{\mathcal{L}} = \mathfrak{I}$, the stability condition implies

$$\begin{pmatrix} 1 & 0 \\ \mathfrak{p} & 1 \end{pmatrix} \begin{pmatrix} \mathfrak{p}^a \\ \mathfrak{p}^b \end{pmatrix} = \begin{pmatrix} \mathfrak{p}^a \\ \mathfrak{p}^{a+1} + \mathfrak{p}^b \end{pmatrix} \subset \begin{pmatrix} \mathfrak{p}^a \\ \mathfrak{p}^b \end{pmatrix}.$$

This gives $\mathfrak{p}^{a+1} + \mathfrak{p}^b \subset \mathfrak{p}^b$, and hence $a+1 \geqslant b$. On the other hand we have $b \geqslant a$, so $b = a$ or $a+1$. In this case, $e_{\mathcal{L}} = 2$ and

$$\mathcal{L} = \{\pi^i(\mathfrak{p} \oplus \mathfrak{p}), \pi^i(\mathcal{O} \oplus \mathfrak{p})\}_{i \in \mathbb{Z}},$$

so we have proved that $L = \mathfrak{p}^a \oplus \mathfrak{p}^b \in \mathcal{L}$.

This completes the proof. □

Exercise 12.4. Give an example of an order in A which is not a chain order.

Solution. Fix $r, s \in \mathcal{O}$ and consider the order

$$B = \begin{pmatrix} r\mathcal{O} & \mathcal{O} \\ \mathcal{O} & s\mathcal{O} \end{pmatrix} \subset A.$$

It is clear that B is a subring of A and we have the equality $B \otimes_{\mathcal{O}} F = A$ between $\mathcal{O}$-modules. So B is an order in A. If B is a chain order then there exists $g \in \mathrm{GL}_2(F)$ such that $gBg^{-1} \in \{\mathfrak{M}, \mathfrak{I}\}$. However, in both $\mathfrak{M}$ and $\mathfrak{I}$ the trace of any matrix can be arbitrarily chosen in $\mathcal{O}$, whereas gBg^{-1} only have matrices of trace divisible by $r + s$. This shows B is not a chain order. □

Exercise 12.5. Let $\mathfrak{A}$ be a chain order.

(1) Show that each $U_{\mathfrak{A}}^n$ with $n \in \mathbb{Z}_{\geqslant 0}$ is a compact open subgroup of $G = \mathrm{GL}_2(F)$, and normal in $U_{\mathfrak{A}}^0$. Moreover, show that $\{U_{\mathfrak{A}}^n\}_n$ is a neighborhood basis of 1 in G.
(2) Show that for $2m \geqslant n > m \geqslant 1$, there is a group isomorphism

$$\vartheta \colon \mathfrak{P}_{\mathfrak{A}}^m / \mathfrak{P}_{\mathfrak{A}}^n \xrightarrow{\sim} U_{\mathfrak{A}}^m / U_{\mathfrak{A}}^n, \quad x \longmapsto 1 + x,$$

where $\mathfrak{P}_{\mathfrak{A}}$ is the radical of $\mathfrak{A}$.

Solution. (1) Since $\mathfrak{A}$ is a chain order, we have $\mathfrak{A} = \mathfrak{A}_{\mathcal{L}}$ for some $\mathcal{L} = \{L_i\}_{i \in \mathbb{Z}}$. By definition,

$$U_{\mathfrak{A}}^n = 1 + \mathfrak{P}_{\mathfrak{A}}^n = \{1 + x^n \colon x \in \mathfrak{A} \text{ such that } xL_i \subset L_{i+1} \text{ for all } i \in \mathbb{Z}\}.$$

Notice that each $g \in U_{\mathfrak{A}}^n$ can be written as $g = 1 + y$ for some $y \in \mathfrak{A}$ such that $yL_i \subset L_{i+n}$. It follows that $f(y) \in \mathfrak{P}_{\mathfrak{A}}^n$ for all $f(y) \in \mathbb{Z}[\![y]\!]/\mathbb{Z}$. In particular, for $f(y) = \sum_{i=1}^{\infty} (-1)^i y^i$, we get $g^{-1} = 1 + f(y) \in U_{\mathfrak{A}}^n$. So $U_{\mathfrak{A}}^n$ is a subgroup of G. To show that $U_{\mathfrak{A}}^n$ is compact open, it suffices to show $\mathfrak{P}_{\mathfrak{A}}$ is compact open. For this, note that $\mathfrak{P}_{\mathfrak{A}}$ is a subring in $\mathfrak{A}$; Exercise 12.2 dictates that $\mathfrak{A}$ is an order in A, so $\mathfrak{P}_{\mathfrak{A}}$ is an $\mathcal{O}$-lattice in A as well. Thus, $\mathfrak{P}_{\mathfrak{A}}$ is an $\mathcal{O}$-lattice in finite-dimensional vector space $A \simeq F^4$, and hence an open compact subgroup in A. Since the topology of $G = \mathrm{GL}_2(F)$ and that of A are both induced from non-archimedean topology of F, we see $\mathfrak{P}_{\mathfrak{A}}$ is open compact in G.

We have proved that $U_{\mathfrak{A}}^n$ is an open compact subgroup of G, and then we check it is normal in $U_{\mathfrak{A}}^0$. For each $x \in U_{\mathfrak{A}}^n$ and $y \in U_{\mathfrak{A}}^0$, we have for all $i \in \mathbb{Z}$ that $xL_i \subset L_{i+n}$ and $yL_i \subset L_{i+1}$. The latter condition implies $y^{-1}L_{i+1} \subset L_i$. Thus $(y^{-1}xy)L_j \subset y^{-1}xL_{j+1} \subset y^{-1}L_{j+1+n} \subset L_{j+n}$ for any $j \in \mathbb{Z}$, and hence $y^{-1}xy \in U_{\mathfrak{A}}^n$, implying the normality of $U_{\mathfrak{A}}^n$ in $U_{\mathfrak{A}}^0$. To show that $\{U_{\mathfrak{A}}^n\}_n$ form a neighborhood basis of 1 in G, we apply the following fact based on Zorn's lemma.

> *Filter Criterion.* Let $\mathcal{B}$ be a collection of open subsets in a topological space X containing a special point $x \in X$. If for any $U_1, U_2 \in \mathcal{B}$ there exists $V \in \mathcal{B}$ such that $x \in V \subset U_1 \cap U_2$, then $\mathcal{B}$ is an open neighborhood basis of x in X.

Provided this, we see from definition that each $U_{\mathfrak{A}}^n$ contains $1 \in G$; also, if $m \geqslant n$ then $\mathfrak{P}_{\mathfrak{A}}^m \subset \mathfrak{P}_{\mathfrak{A}}^n$, and hence $U_{\mathfrak{A}}^m \subset U_{\mathfrak{A}}^n$. So the condition of filter criterion holds because $U_{\mathfrak{A}}^{\max(i,j)} \subset U_{\mathfrak{A}}^i \cap U_{\mathfrak{A}}^j$ for all $i, j \in \mathbb{Z}_{\geqslant 0}$. To conclude, $\{U_{\mathfrak{A}}^n\}_n$ is a compact open neighborhood basis of 1 in G.

(2) To check ϑ is a group homomorphism, note that the source of ϑ is an additive group and the target is a multiplicative group. For each $x, y \in \mathfrak{P}_{\mathfrak{A}}^m$, we have in $U_{\mathfrak{A}}^m$ that

$$\begin{aligned}\vartheta(x+y)^{-1} \cdot \vartheta(x)\vartheta(y) &= (1+x+y)^{-1}(1+x)(1+y) \\ &= 1 + (1+x+y)^{-1}xy.\end{aligned}$$

So we need to show $(1+x+y)^{-1}xy \in \mathfrak{P}_{\mathfrak{A}}^n$. Since $2m \geqslant n > m \geqslant 1$, we see $xy \in \mathfrak{P}_{\mathfrak{A}}^{2m}$ implies $xy \in \mathfrak{P}_{\mathfrak{A}}^n$. On the other hand, if $x, y \in \mathfrak{P}_{\mathfrak{A}}^m/\mathfrak{P}_{\mathfrak{A}}^n$ then $1+x+y \in U_{\mathfrak{A}}^n$ as well, which proves that ϑ is a group homomorphism. Moreover, if $\vartheta(x) = 1+x = 1$ then $x = 0$, so ϑ is injective; for each $y \in U_{\mathfrak{A}}^m/U_{\mathfrak{A}}^n$ we have $y = \vartheta(y-1)$, so ϑ is surjective. The above operations make sense in the ring $A = \mathrm{M}_2(F)$, so ϑ is an isomorphism of groups. $\square$

Exercise 12.6 ([BH06, p.89, Exercises])**.** Define

$$\mathcal{K}_{\mathfrak{A}} = \{g \in G \colon g\mathfrak{A}g^{-1} = \mathfrak{A}\}.$$

(1) Let $g \in G$; show that $g \in \mathcal{K}_{\mathfrak{A}}$ if and only if $g\mathfrak{A} = \mathfrak{P}^m$ for some $m \in \mathbb{Z}$.
(2) Show that $\mathcal{K}_{\mathfrak{A}}$ is the normalizer $N_G(U_{\mathfrak{A}})$ of $U_{\mathfrak{A}}$ in G.
(3) Show that $\mathcal{K}_{\mathfrak{A}}$ is the G-normalizer of $U_{\mathfrak{A}}^m$ for any $m \geqslant 0$.

Solution. (1) We first introduce the following result at work.

Lemma. If $\mathfrak{A} = \mathfrak{A}_{\mathcal{L}}$, then

$$\mathcal{K}_{\mathfrak{A}} = \mathrm{Aut}_{\mathcal{O}}(\mathcal{L}) := \{g \in G \colon gL_i \in \mathcal{L} \text{ for all } i \in \mathbb{Z}\}.$$

Proof of Lemma. It suffices to show the set-theoretical equality; we begin with the following observation. Let $g \in G$ be satisfying that $gL_i \in \mathcal{L}$ for all $i \in \mathbb{Z}$. Then there exists $f(g,i) \in \mathbb{Z}$ such that $gL_i = L_{f(g,i)}$, which gives rise to a map

$$f \colon G \times \mathbb{Z} \longrightarrow \mathbb{Z}, \quad (g,i) \longmapsto f(g,i).$$

Fix such $g \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{L})$ in the following. Since g is an $\mathcal{O}$-automorphism on $\mathcal{L}$, it preserves the partially ordered set $(\mathcal{L}, \subsetneq)$, and so also $(\mathbb{Z}, \leqslant)$; in other words, $i \leqslant j$ implies $f(g,i) \leqslant f(g,j)$, namely f is monotonely increasing in its second variable. Again as $g \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{L})$, we see $f(g, -)$ is a bijection $\mathbb{Z} \to \mathbb{Z}$. It follows that there exists a constant $m_g \in \mathbb{Z}$ (depending on g) such that $f(g,i) = i + m_g$ for all $i \in \mathbb{Z}$. In summary, we have

$$gL_i = L_{i+m_g}, \quad \forall i \in \mathbb{Z}.$$

Now we need to prove that $g\mathfrak{A}g^{-1} = \mathfrak{A}$ whenever $g \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{L})$. Using the observation above, for each $x \in \mathfrak{A}$, we have $(gxg^{-1})L_i = gxL_{i-m_g} \subset gL_{i-m_g} = L_i$ by definition of $\mathfrak{A}$. Then $gxg^{-1} \in \mathfrak{A}$, and hence $g\mathfrak{A}g^{-1} \subset \mathfrak{A}$. Conversely, note that $g \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{L})$ implies $g^{-1} \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{L})$, so the same argument applies (with $m_{g^{-1}} = -m_g$) to see $g^{-1}\mathfrak{A}g \subset \mathfrak{A}$, which implies $g\mathfrak{A}g^{-1} = \mathfrak{A}$. This gives the first inclusion $\mathrm{Aut}_{\mathcal{O}}(\mathcal{L}) \subset \mathcal{K}_{\mathfrak{A}}$ and it remains to prove the converse. Suppose g is such that $g\mathfrak{A}g^{-1} = \mathfrak{A}$, and also $g^{-1}\mathfrak{A}g = \mathfrak{A}$ holds. Then for each $i \in \mathbb{Z}$, if $xL_i \subset L_i$ then $g^{-1}xgL_i \subset L_i$ for all $x \in \mathfrak{A}$, and hence $x(gL_i) \subset gL_i$ for all $x \in \mathfrak{A}$. By Exercise 12.3, this then deduces $gL_i \in \mathcal{L}$ because $\mathfrak{A} = \mathfrak{A}_{\mathcal{L}}$, and thus $g \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{L})$ as desired. This proves the lemma.

We tackle with part (1). Suppose $g \in \mathcal{K}_{\mathfrak{A}}$, then there exists $m_g \in \mathbb{Z}$ as in the proof of lemma, and $gL_i = L_{i+m_g}$ for each $i \in \mathbb{Z}$ because $\mathcal{K}_{\mathfrak{A}} = \mathrm{Aut}_{\mathcal{O}}(\mathcal{L})$. Then for all $x \in \mathfrak{A}$, we have $(gx)L_i \subset gL_i = L_{i+m_g}$, and hence $gx \in \mathfrak{P}^{m_g}$ with $m_g \in \mathbb{Z}$. This proves $g\mathfrak{A} \subset \mathfrak{P}^{m_g}$. On the other hand, if $y \in \mathfrak{P}^{m_g}$ then $yL_i \subset L_{i+m_g}$ for all $i \in \mathbb{Z}$, and further $(g^{-1}y)L_i \subset g^{-1}L_{i+m_g} = L_i$; this shows $g^{-1}y \in \mathfrak{A}$, so $g\mathfrak{A} \subset \mathfrak{P}^{m_g}$ since y lies in $g\mathfrak{A}$.

Now we have proved the "only if" part and are remained to the "if" part. Suppose $g \in G$ satisfies $g\mathfrak{A} = \mathfrak{P}^m$ for some $m \in \mathbb{Z}$; a priori we cannot assign $m = m_g$ at this point. We need to show $gL_i \in \mathcal{L}$ for each $i \in \mathbb{Z}$. But by Exercise 12.3 this is equivalent to showing that gL_i is stabilized by $\mathfrak{A} = \mathfrak{A}_{\mathcal{L}}$, i.e. $xgL_i \subset gL_i$ for all $x \in \mathfrak{A}$. Note that there exists $p \in \mathfrak{P}^m$ such that $pL_i = L_{i+m}$[3]. From the assumption $g\mathfrak{A} = \mathfrak{P}^m$, there exists $\alpha \in \mathfrak{A}$ such that $p = g\alpha$. Then $g^{-1}L_{i+m} \subset \alpha L_i \subset L_i$ for all $i \in \mathbb{Z}$, and it follows that $g^{-1}xgL_i \subset g^{-1}L_{i+m} \subset L_i$, which is the desired relation to deduce $g \in \mathcal{K}_{\mathfrak{A}}$.

(2) Recall the definition that $U_{\mathfrak{A}} = \mathfrak{A}^\times$, so $N_G(U_{\mathfrak{A}}) = \{g \in G\colon g\mathfrak{A}^\times g^{-1} = \mathfrak{A}^\times\}$. It is clear that $\mathcal{K}_{\mathfrak{A}} \subset N_G(U_{\mathfrak{A}})$, because $x \in \mathfrak{A}^\times$ if and only if $gxg^{-1} \in \mathfrak{A}^\times$. For the converse inclusion, we need to show $g\mathfrak{A}^\times g^{-1} = \mathfrak{A}^\times$ implies $g\mathfrak{A}g^{-1} = \mathfrak{A}$. For this purpose, we only need to show that $\mathfrak{A}$ is contained in the $\mathcal{O}$-algebra generated by $\mathfrak{A}^\times$. Observe that if $x \in \mathfrak{A}^\times$ then $x^{-1} \in \mathfrak{A}^\times$, and by definition for all $i \in \mathbb{Z}$ we have $xL_i \subset L_i$ and $x^{-1}L_i \subset L_i$, so $L_i \subset xL_i \subset L_i$. It follows that $xL_i = L_i$ for all $i \in \mathbb{Z}$, and consequently $x \in \mathcal{O}^\times$. From this we see $\mathcal{O}^\times$ is a multiplicative subgroup of $U_{\mathfrak{A}}$. Consequently, $\mathcal{O}[\![\mathcal{O}^\times \mathbb{1}_2]\!] \subset \mathcal{O}[\![\mathfrak{A}^\times]\!]$ as $\mathcal{O}$-subalgebras of $\mathrm{M}_2(\mathcal{O})$. So it suffices to show $\mathfrak{A} \subset \mathcal{O}[\![\mathcal{O}^\times \mathbb{1}_2]\!]$ as subalgebras of $\mathrm{M}_2(\mathcal{O})$, where $\mathbb{1}_2$ is the identity matrix. Since the desired condition is invariant up to G-conjugation, we may replace $\mathfrak{A}$ with either $\mathfrak{M} = \left(\begin{smallmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{smallmatrix}\right)$ or $\mathfrak{I} = \left(\begin{smallmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{smallmatrix}\right)$. Then the inclusion clearly holds in both cases[4].

(3) Recall from proof of part (1) that there exists $\Pi \in G$ such that $m_\Pi = 1$, where the map $G \to \mathbb{Z},\ g \mapsto m_g$ is defined in the proof of lemma. It follows that for each $m \in \mathbb{Z}$, there exists $g \in G$ such that $m = m_g$. Again by part (1), for $\mathfrak{P}$ the radical of $\mathfrak{A}$, the condition $g\mathfrak{A}g^{-1} = \mathfrak{A}$ is equivalent to $g\mathfrak{A} = \mathfrak{P}^{m_g}$. So $g \in \mathcal{K}_{\mathfrak{A}}$ if and only if $g\mathfrak{A}g^{-1} = \mathfrak{A} = \mathfrak{P}^m g^{-1} = g^{-1}\mathfrak{P}^m$, namely $g\mathfrak{P}^m g^{-1} = \mathfrak{P}^m$ for some $m \in \mathbb{Z}$, and it further makes sense to take $m = m_g$ for the prescribed reason. This proves that $\mathcal{K}_{\mathfrak{A}} = \{g \in G\colon g\mathfrak{P}^{m_g}g^{-1} = \mathfrak{P}^{m_g}\} = \{g \in G\colon gU_{\mathfrak{A}}^{m_g}g^{-1} = U_{\mathfrak{A}}^{m_g}\}$; here the second equality is because of $U_{\mathfrak{A}}^m = 1 + \mathfrak{P}^m$ for $m \geqslant 0$. Therefore, $\mathcal{K}_{\mathfrak{A}}$ is the G-normalizer of $U_{\mathfrak{A}}^m$ for any $m \geqslant 0$. $\square$

Exercise 12.7. Fix a non-trivial character $\psi\colon F \to \mathbb{C}^\times$. Then for every lattice P in $A = \mathrm{M}_2(F)$, we have defined $P^* = \{x \in A\colon \psi_A(xy) = 1 \text{ for all } y \in P\}$.

1. Show that P^* is also a lattice, and we have $(P^*)^* = P$.
2. Let Q be another lattice in A. Show that $P \cap Q$ and $P + Q$ are both lattices in A.
3. Show that $(P+Q)^* = P^* \cap Q^*$ and $(P \cap Q)^* = P^* + Q^*$.
4. Let $g \in \mathrm{GL}_2(F)$. Show that gP and Pg are lattices in A, and $(gP)^* = P^*g^{-1}$. Here $gP = \{gx\colon x \in P\}$, where gx is the matrix product.

Solution. (1) Recall that for $x \in A$, we define $\psi_A(x) \coloneqq \psi(\mathrm{tr}(x))$; it follows that for $x_1, x_2 \in P^*$, we have $\psi((x_1 + x_2)y) = \psi(x_1y)\psi(x_2y) = 1$ for all $y \in P$ and hence $x_1 + x_2 \in P^*$. Also, if $x \in P^*$ and $a \in \mathcal{O}$, then for all $y \in P$ we have $\psi_A((ax)y) = \psi(a \cdot \mathrm{tr}(xy)) = \psi_A(xy)^a = 1$; here the second last equality makes sense because $a \in \mathcal{O}$ and ψ_A is $\mathcal{O}$-linear. Finally, P^* is clearly discrete because $\psi_A(x) = 1$ leads to discrete $\mathcal{O}$-module structure. So P^* is also a lattice.

We then check $(P^*)^* = P$. Note that $\psi_A(xy) = \psi_A(yx)$ for all $x, y \in A$. For any $y \in P$ and $x \in P^*$, by definition we have $\psi_A(xy) = 1$, and then $y \in (P^*)^*$. This proves $P \subset (P^*)^*$. For the converse inclusion, assume $x \in (P^*)^*$; this means $\psi_A(xy) = 1$ for all $y \in P^*$. Suppose $x \notin P$ then there must be some $y_0 \in P^*$ such that $\psi_A(y_0x) \neq 1$, which is a contradiction, and thus $x \in P$. This gives $(P^*)^* \subset P$.

(2) By assumption $P \cap Q$ is clearly a discrete and compactly generated $\mathcal{O}$-module in A, and hence a lattice. As for $P + Q$, it is closed under addition and $\mathcal{O}$-multiplication, equipped with the discrete topology inherited from P and Q. So $P + Q$ is also a lattice in A.

(3) For $x \in (P+Q)^*$ with $p \in P$ and $q \in Q$, we have $\psi_A(x(p+q)) = \psi_A(xp)\psi_A(xq) = 1$. Since this holds for arbitrary p, q, we see $\psi_A(xp) = \psi_A(xq) = 1$ must hold individually, and hence

[3] To see the existence of such p, recall that up to G-conjugacy we can identify $\mathfrak{A}$ with either $\mathfrak{M}$ or $\mathfrak{I}$. In both cases we have $\mathfrak{P}^m = \Pi^m\mathfrak{A}$ for some $\Pi \in \mathrm{GL}_2(\mathcal{O})$. In particular, we have $\Pi L_i = L_{i+1}$ and in this case we can take $p = \Pi^m$. Note that this is the only step of the proof that relies on the special feature of $G = \mathrm{GL}_2(F)$.

[4] Indeed, for $\mathfrak{A} \in \{\mathfrak{M}, \mathfrak{I}\}$ we have $\mathfrak{A} = \mathcal{O}[\![\mathcal{O}^\times \mathbb{1}_2]\!] = \mathcal{O}[\![\mathfrak{A}^\times]\!]$ unless when $\mathfrak{A} = \mathfrak{I}$ and $\mathcal{O}/\mathfrak{p} = \mathbb{F}_2$.

$x \in P^* \cap Q^*$. This proves $(P+Q)^* \subset P^* \cap Q^*$. The converse inclusion $P^* \cap Q^* \subset (P+Q)^*$ is clear because $\psi_A = \psi_A(xq) = 1$ always implies $\psi_A(x(p+q)) = 1$.

Now we prove the second equality. By part (1) it suffices to show $(P^* + Q^*)^* = P \cap Q$. But for this, using the first equality of (3), we get $(P^* + Q^*)^* = (P^*)^* \cap (Q^*)^* = P \cap Q$, where we used part (1) again. This proves $(P \cap Q)^* = P^* + Q^*$.

(4) Note that gP is a translation of P, and hence they have the same discrete topology. Also, for each $a \in \mathcal{O}$, $a(gP) = g(aP)$ holds and then gP is an $\mathcal{O}$-lattice. The same argument works for Pg. For each $x \in (gP)^*$ we have $\psi_A(xgp) = 1$ for all $p \in P$, and hence $xg \in P^*$, which means $x \in P^* g^{-1}$; this proves $(gP)^* \subset P^* g^{-1}$. The converse inclusion also follows from $\psi_A(x(gp)) = \psi_A((xg)p)$. Then we have done. □

Exercise 12.8. In the setting of [BH06, §12.8, Proposition], the following are equivalent:

(1) The coset $a + \mathfrak{P}^{1-n}$ contains a nilpotent element of $\mathfrak{A}$.
(2) There is an integer $r \geqslant 1$ such that $a^r \in \mathfrak{P}^{1-rn}$.

Show that each of the two conditions does not change if we replace $(\mathfrak{A}, n, a)$ by $(\mathfrak{A}, n - e_{\mathfrak{A}}, \pi a)$; that is, condition (1) is satisfied by $(\mathfrak{A}, n, a)$ if and only if it is satisfied by $(\mathfrak{A}, n - e_{\mathfrak{A}}, \pi a)$, and similarly for condition (2).

Solution. In the following we do not a priori admit the proposition that (1) and (2) are equivalent. So we need to check both conditions respectively.

(1) Suppose condition (1) holds for $(\mathfrak{A}, n, a)$ then there exists $t \in G$ such that $a+t$ is nilpotent in $a + \mathfrak{P}^{1-n}$; for this we need to require that $tL_i \subset L_{i+1-n}$ for all $i \in \mathbb{Z}$. Recall from periodicity that $\pi L_{i-e_{\mathfrak{A}}} = L_i$ for all $i \in \mathbb{Z}$. So we equivalently have $t(\pi L_{i-e_{\mathfrak{A}}}) \subset L_{i+1-n}$ for all $i \in \mathbb{Z}$, and hence $(\pi t)L_i \subset L_{i+(1-n+e_{\mathfrak{A}})}$. It is clear that this means $\pi t \in \mathfrak{P}^{1-n+e_{\mathfrak{A}}}$. Also, $a+t$ is nilpotent if and only if $\pi a + \pi t$ is nilpotent. So the condition (1) for $(\mathfrak{A}, n + e_{\mathfrak{A}}, \pi a)$ follows. The converse verification is by simply reverse the argument above.

(2) Unwinding the definition, for the stratum $(\mathfrak{A}, n, a)$, condition (2) (i.e. $a^r \in \mathfrak{P}^{1-rn}$ for some $r \geqslant 1$) means $L_i \subset a^r L_{i+rn-1}$ for all $i \in \mathbb{Z}$. For the same r in the following, condition (2) for the stratum $(\mathfrak{A}, n - e_{\mathfrak{A}}, \pi a)$ (i.e. $(\pi a)^r \in \mathfrak{P}^{1-r(n-e_{\mathfrak{A}})}$) means $L_i \subset (\pi a)^r L_{i+r(n-e_{\mathfrak{A}})-1}$ for all $i \in \mathbb{Z}$. To show the two conditions above are equivalent, we compute

$$(\pi a)^r L_{i+r(n-e_{\mathfrak{A}})-1} = a^r(\pi^r L_{(i+rn-1)-re_{\mathfrak{A}}}) = a^r L_{i+rn-1}.$$

Here in the last equality, we recursively used the property $\pi L_j = L_{j+e_{\mathfrak{A}}}$ for all $j \in \mathbb{Z}$ to deduce $\pi^r L_{i-re_{\mathfrak{A}}} = L_i$. □

Exercise 12.9. Let $\mathfrak{I} = \left(\begin{smallmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{smallmatrix}\right)$ be the standard period-2 chain order. Let $\mathfrak{P}$ be its radical.

(1) Show that
$$\mathfrak{P} = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix} \mathfrak{I} = \mathfrak{I} \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix} = \begin{pmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix}$$
using the definition $\mathfrak{P} = \{x \in \mathfrak{I} \colon xL_i \subset L_{i+1} \text{ for all } i \in \mathbb{Z}\}$. Here $\{L_i\}_{i \in \mathbb{Z}}$ is the lattice chain corresponding to $\mathfrak{I}$.
(2) Let $a \in \mathfrak{I}$ be such that some positive power of a lies in $\mathfrak{P}$. Show that $a + \mathfrak{P}$ contains a nilpotent element.
(3) Let $a \in \mathfrak{P}$ be such that $a^r \in \mathfrak{P}^{1+r}$ for some positive integer r. Show that $a + \mathfrak{P}^2$ contains a nilpotent element.

Solution. (1) The equalities $\left(\begin{smallmatrix} 0 & 1 \\ \pi & 0 \end{smallmatrix}\right)\mathfrak{I} = \mathfrak{I}\left(\begin{smallmatrix} 0 & 1 \\ \pi & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right)$ are immediate from direct computation. So it suffices to check $\mathfrak{P} = \left(\begin{smallmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right)$. Let $g \in \mathfrak{P}$ with the condition $gL_i \subset L_{i+1}$ for all $i \in \mathbb{Z}$. Since $\mathcal{L} = \{L_i\}_{i \in \mathbb{Z}}$ is a period-2 chain order, for each fixed $i \in \mathbb{Z}$, this condition can be written as

$$\begin{aligned} &\text{either} \qquad gL_i = g(\mathfrak{p}^{n-1} \oplus \mathfrak{p}^n) \subset L_{i+1} = \mathfrak{p}^n \oplus \mathfrak{p}^n, \\ &\text{or} \qquad gL_i = g(\mathfrak{p}^n \oplus \mathfrak{p}^n) \subset L_{i+1} = \mathfrak{p}^n \oplus \mathfrak{p}^{n+1}, \end{aligned}$$

for some $n \in \mathbb{Z}$. If we write $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then it satisfies the two relations above simultaneously for i varying, i.e., for all $n \in \mathbb{Z}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \mathfrak{p}^{n-1} \\ \mathfrak{p}^n \end{pmatrix} \subset \begin{pmatrix} \mathfrak{p}^n \\ \mathfrak{p}^n \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \mathfrak{p}^n \\ \mathfrak{p}^n \end{pmatrix} \subset \begin{pmatrix} \mathfrak{p}^n \\ \mathfrak{p}^{n+1} \end{pmatrix}.$$

The first inclusion implies $a, c \in \mathfrak{p}$ and $b, d \in \mathcal{O}$; the second inclusion together with $c \in \mathfrak{p}$ deduce $a, b \in \mathcal{O}$ and $d \in \mathfrak{p}$. Combining these, we have $g \in \left(\begin{smallmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right)$. Conversely, following the computation above it is clear that any element $x \in \left(\begin{smallmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right)$ satisfies $xL_i \subset L_{i+1}$ for all $i \in \mathbb{Z}$, and hence lies in $\mathfrak{P}$. This proves the desired expression of $\mathfrak{P}$.

(2) Write $a = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_4 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & a_2 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ a_3 & 0 \end{smallmatrix}\right) \in \left(\begin{smallmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{smallmatrix}\right)$ and suppose that $a^r \in \mathfrak{P}$ for some $r \geqslant 1$. Note that the latter two matrices are nilpotent in $\mathrm{GL}_2(\mathcal{O})$. Using part (1), the assumption means

$$a^r = \begin{pmatrix} a_1^r & * \\ * & a_4^r \end{pmatrix} + b \in \begin{pmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix} = \mathfrak{P},$$

where the non-diagonal entries can be computed in terms of a_2 and a_3, and $b \in \mathrm{M}_2(\mathcal{O})$ has diagonal entries with degrees in a_1, a_4 no more than $r-1$. It now follows that $a_1^r, a_4^r \in \mathfrak{p}$, and hence $a_1, a_4 \in \mathfrak{p}$. Therefore, we see

$$a \in \begin{pmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix} = \mathfrak{P},$$

so $-a \in \mathfrak{P}$ as well; then $a + \mathfrak{P}$ contains a nilpotent element because it contains $0 = a + (-a)$.

(3) Based on the proof of part (2), we directly copy the argument of [BH06, §12.8, Proposition]. Note that if $a = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right) \in \mathfrak{P}$, then, by replacing a with its $U_{\mathfrak{J}}$-conjugate if necessary, we have

$$a + \mathfrak{P}^2 = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix} + \mathfrak{P}^2.$$

By a similar computation, the condition $a^r \in \mathfrak{P}^{1+r}$ for some $r \geqslant 1$ implies $a_1 a_4 \in \mathfrak{p}$. This can be done through the matrix computation on condition $a^r L_i \subset L_{i+1+r}$ for all $i \in \mathbb{Z}$ as in part (1), with $L_i \in \{\mathfrak{p}^{n-1} \oplus \mathfrak{p}^n, \mathfrak{p}^n \oplus \mathfrak{p}^n\}$. In this case, as $a_3 \in \mathfrak{p}$, there exists $\jmath \in \mathfrak{P}^2$ such that

$$a + \jmath = a + \begin{pmatrix} -a_1 & 0 \\ -a_3 & -a_4 \end{pmatrix} = \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix}, \quad a_2 \in \mathcal{O}.$$

This is clearly a nilpotent element so we are done. □

Exercise 12.10. Classify the lattice chains in F^3. More specifically, for each $e \geqslant 1$, classify the $\mathrm{GL}_3(F)$-orbits of lattice chains in F^3 with period e.

Solution. Let $\mathcal{L} = \{L_i\}_{i \in \mathbb{Z}}$ be a lattice chain in F^3 with period $e \geqslant 1$, so we have $L_{i+e} = \pi L_i$ for all $i \in \mathbb{Z}$. Consider the chain $L_i \supsetneq L_{i+1} \supsetneq \cdots \supsetneq L_{i+e}$, in which each subquotient of lattices is viewed as an $\mathcal{O}$-submodule of $L_i/L_{i+e} = L_i/\pi L_i$. Applying $(-) \otimes_{\mathcal{O}} k$ where k is the residue field of $\mathfrak{p}$, we can identify each subquotient L_{i+m}/L_{i+m+n} with $0 \leqslant m < m+n \leqslant e$ with some k-subspace in k^3 of dimension either of $1, 2, 3$. In particular, we have $\dim_k(L_i/L_{i+e}) \otimes_{\mathcal{O}} k \leqslant 3$ but $\dim_k(L_{i+j}/L_{i+j+1}) \otimes_{\mathcal{O}} k \geqslant 1$ for all $j \in \{0, \ldots, e-1\}$. This forces $e \leqslant 3$.

We then split the case into cases with $e = 1, 2, 3$ respectively. Before we start, fix $i \in \mathbb{Z}$ and note that the lattice $L_i \subset F^3$ is in the G-orbit of the standard lattice $\mathcal{O}^3$, i.e. there exists $g \in G = \mathrm{GL}_3(F)$ such that $L_i = g\mathcal{O}^3 \subset F^3$. Thus, up to G-translate, we may assume $L_i = \mathcal{O}^3$ without loss of generality.

Case I. Suppose $e = 1$. Then $L_{i+1} = \pi L_i = \pi \mathcal{O}^3$. which further implies that $L_j = \pi^{i-j} L_i = \pi^{i-j}\mathcal{O}^3$ for each $j \in \mathbb{Z}$. Therefore, in this case the lattice chain is read as

$$\mathcal{L} = \{L_j\}_{j \in \mathbb{Z}} = \{\pi^{i-j}\mathcal{O}^3\}_{j \in \mathbb{Z}} = \{\mathfrak{p}^n \oplus \mathfrak{p}^n \oplus \mathfrak{p}^n\}_{n \in \mathbb{Z}}.$$

Case II. Suppose $e = 2$. Then $L_{i+2} = \pi L_i = \pi\mathcal{O}^3$ and then $\pi\mathcal{O}^3 = L_{i+2} \subsetneq L_{i+1} \subsetneq L_i = \mathcal{O}^3$. For the prescribed reason L_i/L_{i+2} corresponds to a 2-dimensional k-subspace of k^3. This forces L_i/L_{i+1} to correspond to a subspace of k^3 of dimension 1 or 2, and L_{i+1}/L_{i+2} then corresponds to the direct sum complement of L_i/L_{i+1}. To conclude, in this case the chain lattice can be either

$$\mathcal{L} = \{\cdots \subsetneq (\mathfrak{p}^n)^3 \subsetneq \mathfrak{p}^{n-1} \oplus (\mathfrak{p}^n)^2 \subsetneq (\mathfrak{p}^{n-1})^3 \subsetneq \cdots\}$$

or

$$\mathcal{L} = \{\cdots \subsetneq (\mathfrak{p}^n)^3 \subsetneq (\mathfrak{p}^{n-1})^2 \oplus \mathfrak{p}^n \subsetneq (\mathfrak{p}^{n-1})^3 \subsetneq \cdots\}.$$

Case III. Suppose $e = 3$. Then $L_{i+3} = \pi L_i = \pi\mathcal{O}^3$. Similar to Case II above, for $l \in \{0, 1, 2\}$, L_{i+l}/L_{i+l+1} must correspond to a 1-dimensional subspace of k^3, and the direct sum of these three subspaces equals to k^3. So in this case, the chain lattice must look like

$$\mathcal{L} = \{\cdots \subsetneq (\mathfrak{p}^n)^3 \subsetneq \mathfrak{p}^{n-1} \oplus (\mathfrak{p}^n)^2 \subsetneq (\mathfrak{p}^{n-1})^2 \oplus \mathfrak{p}^n \subsetneq (\mathfrak{p}^{n-1})^3 \subsetneq \cdots\}.$$

$\square$

Homework 13

Exercise 13.1. Let π be an irreducible representation of $G = \mathrm{GL}_2(F)$ with $\ell(\pi) = 0$. Show that π contains a stratum of the form $\left(\mathfrak{I}, 1, \left(\begin{smallmatrix} 0 & 0 \\ a_0 & 0 \end{smallmatrix}\right)\right)$, where $a_0 \in \mathcal{O}_F$.

Solution. By definition, $\ell(\pi) = 0$ means π contains a trivial character on $U^1_{\mathfrak{M}}$. Observe that

$$U^2_{\mathfrak{I}} \subset U^1_{\mathfrak{M}} \subset U^1_{\mathfrak{I}},$$

so π contains a character χ on $U^1_{\mathfrak{I}}$ that is trivial on $U^1_{\mathfrak{M}}$, and hence χ is trivial on $U^2_{\mathfrak{I}}$. In other words, π contains a trivial character on $U^2_{\mathfrak{I}}$. This further means π contains some stratum $(\mathfrak{I}, 1, \alpha)$ with $\alpha \in U^1_{\mathfrak{I}}/U^2_{\mathfrak{I}}$. In order to determine α, note that

$$U^1_{\mathfrak{I}}/U^2_{\mathfrak{I}} = \frac{1+\mathfrak{P}_{\mathfrak{I}}}{1+\mathfrak{P}^2_{\mathfrak{I}}} = \begin{pmatrix} 1+\mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix} \Big/ \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{pmatrix} \simeq \begin{pmatrix} 0 & \boldsymbol{k} \\ \boldsymbol{k} & 0 \end{pmatrix},$$

where $\boldsymbol{k} = \mathcal{O}/\mathfrak{p}$; the last isomorphism above is between a multiplicative group and an additive group. It follows that the image of α in $\mathrm{M}_2(\boldsymbol{k})$ must be of form $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ with $b, c \in \boldsymbol{k}$. On the other hand, if we fix $\psi\colon F \to \mathbb{C}^\times$ such that $\psi(0) = 1$, then as π contains $\psi_\alpha|_{U^2_{\mathfrak{I}}} = \mathbb{1}$, we have that $\psi(\mathrm{tr}(\alpha(x-1))) = 1 \in \mathbb{C}^\times$ for all $x \in U^2_{\mathfrak{I}}$, or equivalently the trace of $\alpha(x-1)$ modulo $\mathfrak{P}^2_{\mathfrak{I}} = \left(\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} \end{smallmatrix}\right)$ must be zero. For this, we identify $b, c \in \boldsymbol{k}$ with their liftings to $\mathcal{O}$ and compute

$$\alpha(x-1) \in a\mathfrak{P}^2_{\mathfrak{I}} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\begin{pmatrix} \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} \end{pmatrix} = \begin{pmatrix} b\mathfrak{p}^2 & 0 \\ 0 & c\mathfrak{p} \end{pmatrix} \equiv \begin{pmatrix} b\mathfrak{p}^2 & 0 \\ 0 & 0 \end{pmatrix} \bmod \mathfrak{P}^2_{\mathfrak{I}}.$$

This forces $b = 0$, and therefore, there exists $a_0 \in \mathcal{O}$ that is congruent with a lift of c modulo $\mathfrak{I}$, such that $\alpha = \left(\begin{smallmatrix} 0 & 0 \\ a_0 & 0 \end{smallmatrix}\right) \in \mathrm{M}_2(\mathcal{O})$. So we have proved that π contains the stratum $(\mathfrak{I}, 1, \alpha)$. $\square$

Exercise 13.2. Consider a stratum of the form $(\mathfrak{I}, -1, \alpha)$. Check that it is fundamental if and only if $\alpha\mathfrak{I} = \mathfrak{P}_{\mathfrak{I}}$.

Solution. Recall from definition of stratum that $(\mathfrak{I}, -1, \alpha)$ is fundamental if and only if $\alpha + \mathfrak{P}^2_{\mathfrak{I}}$ contains no nilpotent element of A; we aim to prove this is equivalent to $\alpha\mathfrak{I} \neq \mathfrak{P}_{\mathfrak{I}}$. Also by definition of stratum, it is automatic that $\alpha \in U^1_{\mathfrak{I}}/U^2_{\mathfrak{I}}$, so α can be viewed as the lift of an element of the additive group $\left(\begin{smallmatrix} 0 & \boldsymbol{k} \\ \boldsymbol{k} & 0 \end{smallmatrix}\right)$, using the argument in the proof of Exercise 13.1; without loss of generality, we may assume $\alpha = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ with $b, c \in \mathcal{O}$, and see

$$\alpha\mathfrak{I} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix} = \begin{pmatrix} b\mathfrak{p} & b\mathcal{O} \\ c\mathcal{O} & c\mathcal{O} \end{pmatrix} \neq \begin{pmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix} = \mathfrak{P}_{\mathfrak{I}}$$

unless $b = 1$ and $c = \pi$ up to scalar by $\mathcal{O}^\times$. Further, since $\mathfrak{P}_{\mathfrak{I}}$ is closed under addition, modulo $\mathfrak{P}_{\mathfrak{I}}$ we may assume $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ c & 0 \end{smallmatrix}\right)$; this makes sense because we only concern about the condition $\alpha\mathfrak{I} = \mathfrak{P}_{\mathfrak{I}}$. Therefore, we only need to consider $\alpha \in A = \mathrm{M}_2(\mathcal{O})$ of form $\left(\begin{smallmatrix} 0 & 1 \\ c & 0 \end{smallmatrix}\right)$, where c is a lift of some element in $\mathcal{O}/\mathfrak{p}$. On the other hand, reducing everything from $\mathrm{M}_2(\mathcal{O})$ to $\mathrm{M}_2(\boldsymbol{k})$, we observe that

$$\alpha\mathfrak{I} \otimes_{\mathcal{O}} \boldsymbol{k} = \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{k} & \boldsymbol{k} \\ 0 & \boldsymbol{k} \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{k} \\ c\boldsymbol{k} & c\boldsymbol{k} \end{pmatrix} \neq \begin{pmatrix} 0 & \boldsymbol{k} \\ 0 & 0 \end{pmatrix} = \mathfrak{P}_{\mathfrak{I}} \otimes_{\mathcal{O}} \boldsymbol{k}$$

unless the image of c is $0 \in \boldsymbol{k}$, or equivalently π divides c in $\mathcal{O}$. Again, as c is lifted from $\mathcal{O}/\mathfrak{p}$, we see $c \neq 0$ if and only if $c = \pi$ under the prescribed situation.

To proceed on, suppose $\alpha + \mathfrak{P}^2_{\mathfrak{I}}$ contains no nilpotent element of A. In particular, $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ c & 0 \end{smallmatrix}\right)$ is not nilpotent, and hence $c \neq 0$; but this is equivalent to $c = \pi$, implying that $\alpha\mathfrak{I} = \mathfrak{P}_{\mathfrak{I}}$. Conversely, suppose there is a nilpotent element in $\alpha + \mathfrak{P}^2_{\mathfrak{I}}$, then $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ c & 0 \end{smallmatrix}\right) \in A$ is nilpotent modulo $\mathfrak{P}^2_{\mathfrak{I}}$, and thus $c = 0$; if this is true then $\alpha\mathfrak{I} \neq \mathfrak{P}_{\mathfrak{I}}$. This proves the desired equivalence. $\square$

Exercise 13.3. Let $(\mathfrak{A}, n, \alpha)$ be a stratum with $e_{\mathfrak{A}} = 1$. Define $\tilde{f}_\alpha(t)$ as in [BH06, §13.2].

(1) Show that $\tilde{f}_\alpha(t)$ depends only on the G-conjugacy class of $(\mathfrak{A}, n, \alpha)$.
(2) Show that $(\mathfrak{A}, n, \alpha)$ is fundamental if and only if $\tilde{f}_\alpha(t) \neq t^2$.

Solution. Recall from definition that for $\alpha = \pi^{-n}\alpha_0$ with $\alpha_0 \in \mathfrak{A}$, the polynomial $\tilde{f}_\alpha(t) \in \boldsymbol{k}[t]$ is defined as the characteristic polynomial of image of $\alpha_0 \in \mathfrak{A}$ in $\mathfrak{A}/\mathfrak{P}_{\mathfrak{A}} \cong \mathrm{M}_2(\boldsymbol{k})$.

(1) The conjugation by $g \in G$ maps the stratum $(\mathfrak{A}, n, \alpha)$ to $(g\mathfrak{A}g^{-1}, n, g\alpha g^{-1})$. If we write $\alpha = \pi^{-n}\alpha_0$ with $\alpha_0 \in \mathfrak{A}$ then we have $g\alpha g^{-1} = \pi^{-n}(g\alpha_0 g^{-1})$ with $g\alpha_0 g^{-1} \in g\mathfrak{A}g^{-1}$. Thus, $\tilde{f}_{g\alpha g^{-1}}(t)$ is the characteristic polynomial of image of $g\alpha_0 g^{-1}$ in $\mathfrak{A}/\mathfrak{P}_{\mathfrak{A}}$, which is the same as that of α_0. This proves $\tilde{f}_{g\alpha g^{-1}}(t) = \tilde{f}_\alpha(t)$.

(2) By part (1) we can replace $(\mathfrak{A}, n, \alpha)$ by its G-conjugation $(\mathfrak{M}, n, g\alpha g^{-1})$, where $g \in G$ is such that $\mathfrak{M} = g\mathfrak{A}g^{-1}$. By [BH06, (12.8.1)], this stratum is non-fundamental if and only if $g\alpha g^{-1} = \pi^{-n}\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Therefore, if $(\mathfrak{A}, n, \alpha)$ is non-fundamental then

$$\tilde{f}_\alpha(t) = \tilde{f}_{g\alpha g^{-1}}(t) = t^2.$$

Conversely, recall from Exercise 13.1 that we can view α as a lift of an element in $\left(\begin{smallmatrix} 0 & \boldsymbol{k} \\ \boldsymbol{k} & 0 \end{smallmatrix}\right)$ with $\boldsymbol{k} = \mathcal{O}/\mathfrak{p}$. If $\tilde{f}_\alpha(t) = t^2$ then α must be nilpotent in $\mathfrak{A}$ up to G-conjugacy, so there exists $g \in G$ such that

$$g\mathfrak{A}g^{-1} = \mathfrak{M}, \quad g\alpha g^{-1} = \pi^{-n}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This is equivalent to $(\mathfrak{A}, n, \alpha)$ being fundamental. $\square$

Exercise 13.4 ([BH06, p.98, Exercise])**.** Let E/F be a quadratic field extension, let $\alpha \in E^\times$, and write $n = -v_E(\alpha)$. Show that α is minimal over F if and only if $\alpha + \mathfrak{p}_E^{1-n} \cap F = \emptyset$.

Solution. Choose a uniformizer π of F. Recall from definition that α minimal means the sub-algebra $E = F[\alpha]$ of $\mathrm{M}_2(\mathcal{O}_F)$ is a field and

- If E/F is totally ramified, then $n = -v_E(x)$ is odd.
- If E/F is unramified, then the coset $\pi^n\alpha + \mathfrak{p}_E$ generates the field extension $\boldsymbol{k}_E/\boldsymbol{k}$.

For the "if" part, assume $\alpha + \mathfrak{p}_E^{1-n} \cap F \neq \emptyset$. In this case, if E/F is totally ramified, then there exists $x \in \mathcal{O}_E$ such that $\alpha + \pi_E^{1-n}x \in F$. In particular,

$$v_E(\alpha + \pi_E^{1-n}x) = \min(v_E(\alpha), v_E(\pi_E^{1-n}x)) = \min(-n, 1-n+v_E(x)) \in \mathbb{Z}$$

must be even because $\pi_E^2 = \pi$. On the other hand, we have $v_E(x) \geqslant 0$ so this valuation must equal to $-n$, which proves that n is even and contradicts to the definition of minimality. So it suffices to consider the case where E/F is unramified, for which we can identify π_E with π. By assumption, there exists $x \in \mathcal{O}_E$ such that $\alpha + \pi^{1-n}x \in F$, or equivalently $\pi^n\alpha + \pi x \in F$. Notice that $v_E(\pi^n\alpha) = v_F(\pi^n\alpha) = 0$ and $\pi^n\alpha \in \mathcal{O}_E$. We then assume that $E = F[\alpha]$ as well as α is minimal to deduce the contradiction. Under this assumption, $\mathcal{O}_E = \mathcal{O}_F[\pi^n\alpha]$ by [BH06, §13.4, Lemma], and $\alpha \notin F$; the latter further implies $\pi x \notin F$ and hence $x \notin F$. Since α is minimal, $\boldsymbol{k}_E/\boldsymbol{k}$ is generated by $\pi^n\alpha + \mathfrak{p}_E$, then $x \in \mathcal{O}_E$ but $x \notin \mathcal{O}_F$, namely $x \in \mathcal{O}_F[\pi^n\alpha]$ but $x \notin \mathcal{O}_F$. So there exists $s, t \in \mathcal{O}_E$ such that $x = s + t(\pi^n\alpha)$ because $\pi^n\alpha \in \mathcal{O}_E$ must be a root of a quadratic polynomial over $\mathcal{O}_F$. This forces

$$\pi^n\alpha + \pi x = s\pi + (t+1)(\pi^n\alpha) = s\pi + \alpha \cdot (t+1)\pi^n \in \mathcal{O}_F + \alpha\mathcal{O}_F,$$

and it follows that $(\mathcal{O}_F + \alpha\mathcal{O}_F) \cap F \neq \emptyset$. This is a contradiction to $\alpha \notin F$. So we have proved that α cannot be minimal.

For the "only if" part, assume $\alpha + \mathfrak{p}_E^{1-n} \cap F = \emptyset$ and we need to show the minimality of α. By assumption we have $\alpha \notin F$, and hence $E = F[\alpha]$ holds. When E/F is totally ramified, by assumption $\pi_E^{1-n}x + \alpha \notin F$ for all $x \in \mathcal{O}_E$. Notice that $\alpha \notin F$ implies $\pi_E^{1-n}x \notin F$. In this case, if $n = -v_E(\alpha)$ is even, then $\pi_E x \notin F$ for all $x \in \mathcal{O}_E$, which fails to be true; for a counter-example, one can take $x = \pi_E \in \mathcal{O}_E$ so that $\pi_E x = \pi_E^2 = \pi \in \mathcal{O}_F$. This proves that n is odd. Now it remains to consider the case when E/F is unramified with $\pi_E = \pi$. To show the extension $\boldsymbol{k}_E/\boldsymbol{k}$ of degree 2 is generated by $\pi^n\alpha + \mathfrak{p}_E$, it suffices to show $\pi^n\alpha \in \mathcal{O}_E - \mathcal{O}_F$. For this we only need $\pi^n\alpha \notin \mathcal{O}_F$. But for all $x \in \mathcal{O}_E$, note that $\pi^{1-n}x + \alpha \notin F$ if and only if $\pi x + \pi^n\alpha \notin F$. On the other hand, whenever $\pi^n\alpha \in \mathcal{O}_F$ we have $\pi x \notin F$ and hence $x \notin F$ for all $x \in \mathcal{O}_E$, which is again impossible. This completes the proof. $\square$

Exercise 13.5. Prove [BH06, §14.2, Proposition (2)(3)]. Here the level of a character of $F^\times$ is defined in [BH06, §1.8], and the terminology *essentially scalar fundamental stratum* is defined in [BH06, §13.2].

Solution. Let $\chi = \chi_1 \otimes \chi_2$ be a character of T and set $\Sigma = \mathrm{Ind}_B^G \chi$. Let n_i be the level of χ_i. We need to prove the following:

(2) If $n_1 = n_2 = n \neq 0$ and $\chi_1\chi_2^{-1}|_{U_F^n}$ is the trivial character, then Σ contains an essentially scalar fundamental stratum.

(3) If $n_1 = n_2 = 0$, then Σ contains the trivial character of $U_{\mathfrak{J}}^1$.

In the following, fix a character $\psi\colon F \to \mathbb{C}^\times$.

To prove (2), for $i = 1, 2$ there exists $a_i \in \mathfrak{p}^{-n}$ such that $\chi_i(1 + x) = \psi(a_i x)$ for all $x \in \mathfrak{p}^n$. By definition of level, n_i is the least integer such that

$$\psi(a_i x) = \chi_i(1 + x) = 1, \quad \forall x \in \mathfrak{p}^{n_i+1}.$$

Since $n_1 = n_2 = n \neq 0$, we see $v_F(a_1) = v_F(a_2) = -n$. Moreover, since $\chi_1\chi_2^{-1}|_{U_F^n}$ is trivial, we see $\chi_1(1 + x) = \chi_2(1 + x)$ for all $x \in \mathfrak{p}^n$, or equivalently $\psi(a_1x) = \psi(a_2x)$ for all $x \in \mathfrak{p}^n$. This implies $a_1 \equiv a_2 \bmod \mathfrak{p}^{1-n}$. In particular, we have $\pi^n a_1 \equiv \pi^n a_2 \bmod \mathfrak{p}$, namely $\pi^n a_1$ and $\pi^n a_2$ have the same image in $\boldsymbol{k} = \mathcal{O}/\mathfrak{p}$. Thus,

$$(\mathfrak{M}, n, a) = \left(\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}, n, \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\right)$$

must be an essentially scalar stratum, because $\tilde{f}_a(t)$ has doubled root in $\boldsymbol{k}^\times$.

We claim that $(\mathfrak{M}, n, a)$ is a fundamental stratum. Indeed, it suffices to check that a modulo $\mathfrak{P}_{\mathfrak{M}}^{1-n}$ is not nilpotent in $\mathrm{M}_2(F)$, for which we only need to consider $\pi^n a$ modulo $\mathfrak{P}_{\mathfrak{M}} = \left(\begin{smallmatrix} \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} \end{smallmatrix}\right)$. Then the claim is clear because $v_F(\pi^n a_i) = 0$, meaning that $\pi^n a_i \in \mathcal{O} - \mathfrak{p}$.

Now it remains to verify that $\Sigma = \mathrm{Ind}_B^G \chi$ contains the character

$$\psi_a\colon U_{\mathfrak{M}}^n = 1 + \mathfrak{P}_{\mathfrak{M}}^n \longrightarrow \mathbb{C}^\times, \quad 1 + x \longmapsto \psi(\mathrm{tr}(ax)).$$

For this, let $f \in \Sigma$ be supported on $BU_{\mathfrak{M}}^n = BN_n'$, where $N_n' = \left(\begin{smallmatrix} 1 & 0 \\ \mathfrak{p}^n & 1 \end{smallmatrix}\right)$ as in Exercise 11.2. Then for any $u \in U_{\mathfrak{M}}^n$, we can write $u = bn'$ for some $b \in B$ and $n' \in N_n'$. Note that

$$a(n' - 1) \in \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \mathfrak{p}^n & 0 \end{pmatrix} \subset \begin{pmatrix} 0 & 0 \\ a_2\mathfrak{p}^n & 0 \end{pmatrix}$$

and in particular it has trace zero. Thus, for our purpose we may assume f is fixed by N_n', so $u = bn'$ acts on f by $\chi(b)$. By definition of smooth induction, we get

$$b \cdot f = \chi(b)f = \psi(\mathrm{tr}(a(b - 1)))f = \psi_a(b)f,$$

where the second equality is because of the construction $\chi = \chi_1 \otimes \chi_2$ and $\chi_i(1 + x) = \psi(a_i x)$. So Σ contain ψ_a when restricted to $U_{\mathfrak{M}}^n$. This gives the conclusion that Σ contains an essentially scalar fundamental stratum $(\mathfrak{M}, n, a)$.

To prove (3), we only need to modify the argument for part (2). Note that when $n_1 = n_2 = 0$ we have $\chi_1(1 + x) = \chi_2(1 + x) = 1$ for all $x \in \mathfrak{p}$. So for any $u \in U_{\mathfrak{J}}^1 = \left(\begin{smallmatrix} 1+\mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & 1+\mathfrak{p} \end{smallmatrix}\right)$, with the same assumption as above, we have $\mathrm{tr}(a(u - 1)) = 0$, and hence $\psi_a(u)$ is trivial. So Σ contains the trivial character of $U_{\mathfrak{J}}^1$. $\square$

Homework 14

Exercise 14.1. Let F be a local field. Let $n \geqslant 1$, and let $\chi\colon U_F^n = 1 + \mathfrak{p}^n \to \mathbb{C}^\times$ be a smooth character. Show that χ can be extended to a smooth character $F^\times \to \mathbb{C}^\times$.

Solution. Choose π to be a uniformizer of $\mathcal{O}_F$. Since $F^\times \simeq \pi^{\mathbb{Z}} \times \mathcal{O}_F^\times = \pi^{\mathbb{Z}} \times U_F^0$, if χ extends to a smooth character of U_F^0, it then further extends to a smooth character of $F^\times$ by defining $\chi(\pi) = 1$. Since χ is smooth and U_F^m's form a neighborhood basis of $1 \in \mathcal{O}_F^\times$, there exists $m \gg 0$ such that χ is constant on each coset of $\mathcal{O}_F^\times / U_F^m$. In the case where $m < n$ there is nothing to prove because χ is a constant modulo U_F^n. As for the case where $m \geqslant n$, we can extend χ to $\widetilde{\chi}\colon \mathcal{O}_F^\times \to \mathbb{C}^\times$ by defining $\widetilde{\chi}(tU_F^n) = 1$ for each $t \in \mathcal{O}_F^\times / U_F^n$. This clearly admits the group homomorphism property on χ and is smooth (for which we only need the locally constant property). Thus $\widetilde{\chi}$ further extends to $F^\times$ by assigning $\widetilde{\chi}(\pi) = 1$. $\square$

In the following exercises, let G be a unimodular locally profinite group. Fix a Haar measure μ, and let $\mathcal{H}$ be the Hecke algebra of G.

Exercise 14.2. Let $K \subset G$ be a compact open subgroup. Let (ϕ, W) be a smooth irreducible representation of K (which is necessarily of finite dimension). Define a function $e_\phi\colon G \to \mathbb{C}$ by

$$e_\phi(g) \coloneqq \begin{cases} \mu(K)^{-1} \dim W \cdot \operatorname{tr}(\phi(g)^{-1}), & g \in K, \\ 0, & g \notin K. \end{cases}$$

(1) Show that $e_\phi \in \mathcal{H}$.
(2) Show that for each smooth representation (π, V) of G, the operator $\pi(e_\phi)\colon V \to V$ is the projection to the ϕ-isotypic component V^ϕ[5].
(3) Show that e_ϕ is idempotent, i.e., $e_\phi * e_\phi = e_\phi$.
(4) Show that $\mathcal{H}_\phi \coloneqq e_\phi * \mathcal{H} * e_\phi$ is a subalgebra of $\mathcal{H}$ with multiplicative identity e_ϕ.

Solution. (1) By definition e_ϕ is supported on the open compact subset K of G. Since (ϕ, W) is a smooth representation of G and K is an open compact subgroup, ϕ is locally constant on K. Thus $\mu(K)^{-1} \dim W \cdot \operatorname{tr}(\phi(g)^{-1})$ is a locally constant function in $g \in K$. It follows that $e_\phi\colon G \to \mathbb{C}$ is an element of $\mathcal{H}$.

(2) We copy the argument in [BH06, §4.4] here. Let K' be the kernel of ϕ, then K' is an open compact subgroup of K. Also, e_ϕ is constant on cosets gK' and $K'g$, so $e_{K'} * e_\phi = e_\phi * e_{K'} = e_\phi$. Thus, e_ϕ lies in the subalgebra $e_{K'} * \mathcal{H} * e_{K'}$ of $\mathcal{H}$. However, $e_{K'} \mapsto 1$ induces an algebra isomorphism $e_{K'} * \mathcal{H} * e_{K'} \to \mathbb{C}[K/K']$. This isomorphism takes e_ϕ to the idempotent of the group algebra corresponding to the irreducible representation ϕ of the finite group K/K'. It follows that $\pi(e_\phi)V = V^\phi$, where V^ϕ is a module over the subalgebra $e_\phi * \mathcal{H} * e_\phi$.

(3) It suffices to show for any smooth representation (π, V) of G that $\pi(e_\phi)$ is idempotent, because we always have $\pi(e_\phi * e_\phi) = \pi(e_\phi)^2$ (this is given by the general formula $\pi(f * g) = \pi(f)\pi(g)$ in $\operatorname{End}_{\mathbb{C}}(V)$ for $f, g \in \mathcal{H}$). But it is clear from part (2) that $\pi(e_\phi)^2 = \pi(e_\phi)$, namely $\pi(e_\phi)$ is idempotent.

(4) To check that $\mathcal{H}_\phi$ is a subalgebra of $\mathcal{H}$, we only need for $f, g \in \mathcal{H}$ that $(e_\phi * f * e_\phi) * (e_\phi * g * e_\phi) = e_\phi * (f * e_\phi * g) * e_\phi$ with $f * e_\phi * g \in \mathcal{H}$; here we have used the result of (3). Also, by part (3) and the associativity of convolution, for each $e_\phi * f * e_\phi \in \mathcal{H}_\phi$ with $f \in \mathcal{H}$, we have $e_\phi * (e_\phi * f * e_\phi) = (e_\phi * e_\phi) * f * e_\phi = e_\phi * f * e_\phi$, and similarly for e_ϕ acting on the right. So e_ϕ is the identity of $\mathcal{H}_\phi$. $\square$

Exercise 14.3. Assume that G is countable at infinity. Prove the following theorem in steps.

[5] Recall that the K-representation V is semi-simple, so we have the isotypic decomposition $V = \bigoplus_\rho V^\rho$, where ρ runs through isomorphism classes of irreducible smooth representations of K, and V^ρ is the sum of all K-subrepresentations of V which are isomorphic to ρ.

Theorem. For every non-zero $f \in \mathcal{H}$, there exists an irreducible smooth representation (π, V) of G such that the operator $\pi(f)\colon V \to V$ is non-zero.

(1) Let R be a unital $\mathbb{C}$-algebra of countable $\mathbb{C}$-dimension. (We do not assume that R is commutative.) Let $r \in R$ be a non-nilpotent element. Show that there exists $\lambda \in \mathbb{C}^\times$ such that $r - \lambda \in R$ is not invertible.
(2) Show that there exists a non-zero simple left unital R-module M such that $rM \neq 0$. Here unital means that $1 \in R$ acts as the identity.
(3) Show that for every compact open subgroup K of G, the algebra $\mathcal{H}_K = e_K * \mathcal{H} * e_K$ is of countable dimension over $\mathbb{C}$.
(4) Define $f^* \in \mathcal{H}$ by $f^*(g) := \overline{f(g^{-1})}$. Let $h = f * f^* \in \mathcal{H}$. Show that $h^* = h$, and $h(1) > 0$.
(5) Show that h is not nilpotent.
(6) Show that there exists a compact open subgroup $K \subset G$ such that $h \in \mathcal{H}_K$ and there exists a simple $\mathcal{H}_K$-module on which h is non-zero.
(7) Show that there exists an irreducible smooth representation π of G such that $\pi(f) \neq 0$.

Solution. (1) Suppose R is a unital $\mathbb{C}$-algebra of countable $\mathbb{C}$-dimension. Then there exists a homomorphism $\rho\colon R \to D$ of $\mathbb{C}$-algebras, where D is a matrix algebra with identity element $1 \in D$; note that there is a natural determinant map $\det\colon D \to \mathbb{C}$. Given r non-nilpotent, $\rho(r)$ is non-nilpotent neither, and hence there exists $\lambda \in \mathbb{C}^\times$ such that $\det(\rho(r) - t \cdot 1) \in \mathbb{C}[\![t]\!]$ vanishes at λ. It follows that $\det(\rho(r - \lambda) - t \cdot 1)$ vanishes at 0, meaning that $\rho(r - \lambda)$ is not invertible in D. This proves that $r - \lambda$ is not invertible in R.

(2) Keep the notations in part (1). Let $I \subsetneq R$ be a maximal ideal of R containing $r - \lambda$. Then $M = R/I$ is simple left R-module. Moreover, it is unital because I does not contain $R^\times$. If $rM = 0$ then $r \in I$, which further implies $\lambda \in I$ as $r - \lambda \in I$. However, this cannot be true while assuming $\lambda \in \mathbb{C}^\times \subset R^\times$. So we have $rM \neq 0$.

(3) Since G is countable at infinity, G/K is countable for any open subgroup K. For each element $f \in \mathcal{H}_K$ that is stabilized by K, its G-orbit has size at most G/K, which is countable. Thus, $\mathcal{H}_K$ is generated by at most countably many elements as a $\mathbb{C}$-algebra, and hence has a countable dimension over $\mathbb{C}$.

(4) We first show that $h^* = h$, i.e. $(f * f^*)^*(x) = (f * f^*)(x)$ for all $x \in G$. For this, it suffices to show that $\overline{(f * f^*)(x^{-1})} = (f * f^*)(x)$, which can be verified through

$$\begin{aligned}
\overline{\int_G f(g) f^*(g^{-1}x^{-1}) \mathrm{d}\mu(g)} &= \int_G \overline{f(g)} \cdot \overline{f^*(g^{-1}x^{-1})} \mathrm{d}\mu(g) \\
&= \int_G f^*(g^{-1}) f(xg) \mathrm{d}\mu(g) \\
&= \int_G f(g) f^*(g^{-1}x) \mathrm{d}\mu(g).
\end{aligned}$$

Here the last equality is by replacing g with $x^{-1}g$ in the integral.

As for the assertion $h(1) = (f * f^*)(1) > 0$, we compute

$$(f * f^*)(1) = \int_G f(g) f^*(g^{-1}) \mathrm{d}\mu(g) = \int_G f(g)\overline{f(g)} \mathrm{d}\mu(g) = \int_G |f(g)|^2 \mathrm{d}\mu(g).$$

Since f is assumed to be non-zero, we have $|f(g)|^2 > 0$ for some $g \in G$, and hence $h(1) > 0$ follows.

(5) Consider $h^{*2} := h * h \in \mathcal{H}$. By part (4), we have $h * h = h * h^*$ where h is a nonzero element of $\mathcal{H}$. So we can apply the same argument before with f replaced by h, to deduce that $(h * h^*)^* = h * h^*$ and $(h * h^*)(1) > 0$. Thus, for each $n \geqslant 1$, the self-convolution of h on itself in 2^n times, denoted by $h^{*(2^n)}$, must be non-zero. This shows that h is not nilpotent.

(6) Since G is assumed to be locally profinite, $1 \in G$ has a neighborhood basis $\{K_i\}_{i\in I}$ consisting of compact open subgroups of G. It follows that $\mathcal{H} = \bigcup_{i\in I} \mathcal{H}_{K_i}$, and hence there is some $K = K_i$ such that $h \in \mathcal{H}_K$. By parts (3) and (5), we know $h \in \mathcal{H}_K$ is non-nilpotent and $\mathcal{H}_K$ is a unital $\mathbb{C}$-algebra of countable dimension. So part (2) implies there exists a non-zero simple (left) unital $\mathcal{H}_K$-module W on which $hW \neq 0$, namely h is non-zero on W.

(7) Continuing with part (6), the simple $\mathcal{H}_K$-module W satisfies $\mathcal{H}_K W \subset W$, and hence $\mathcal{H}_K W$ is either 0 or W; but $hW \neq 0$ leads to $\mathcal{H}_K W = W$. Thus, W can be regarded as an irreducible smooth G-representation (π, W), such that $\pi(h) \neq 0$. From the construction above, it follows that $\pi(f * f^*) = \pi(f)\pi(f^*) \neq 0$, and in particular $\pi(f) \neq 0$. □

Exercise 14.4. Assume that G is countable at infinity and such that every irreducible smooth representation of G is admissible (e.g. $G = \mathrm{GL}_2(F)$). Let $e \in \mathcal{H}$ be a non-zero idempotent.

(1) Show that for each smooth representation (π, V) of G, we have a canonical decomposition $V = \ker(\pi(e)) \oplus \mathrm{im}(\pi(e))$. Moreover, $\mathrm{im}(\pi(e))$ is finite-dimensional if (π, V) is admissible.

(2) Suppose $f_1, f_2 \in e * \mathcal{H} * e$ satisfies $f_1 * f_2 = e$. Show that $f_2 * f_1 = e$.

Solution. (1) Since e is non-zero idempotent, $\pi(e)$ is idempotent as well. So we have the canonical decomposition

$$V = (1 - \pi(e))V \oplus \pi(e)V = \ker(\pi(e)) \oplus \mathrm{im}(\pi(e)).$$

Further, as G is countable at infinity, by the proof of Exercise 14.3(6), there is an open compact subgroup K of G such that $e \in \mathcal{H}_K$, and hence $e = e_{K'}$ for some open compact subgroup K' of K. So we have $\mathrm{im}(\pi(e)) = \pi(e)V = \pi(e_{K'})V = V^{K'}$. By assumption (π, V) is admissible, and hence $\dim V^{K'} < \infty$, which is as desired.

(2) We have $\pi(e) = \pi(f_1 * f_2) = \pi(f_1)\pi(f_2)$. On the other hand, $\pi(e)V = \mathrm{im}(\pi(e))$ must be finite-dimensional by part (1). As $\pi(e)$ is non-zero idempotent, both $\pi(f_1)$ and $\pi(f_2)$ can be regarded as linear operators on $\pi(e)V$, so $\pi(e) = \pi(f_1)\pi(f_2)$ implies $\pi(e) = \pi(f_2)\pi(f_1) = \pi(f_2 * f_1)$. By assumption on G, such (π, V) can be replaced by any other irreducible smooth representation, and hence $f_2 * f_1 = e$ holds. □

Homework 15

Exercise 15.1. Let $I = U_{\mathfrak{J}}$ be the Iwahori subgroup of $G = \mathrm{GL}_2(F)$, and let ϕ be a character of I trivial on $U^1_{\mathfrak{J}}$.

(1) Let $f \in \mathcal{H}_\phi$ be a function supported on ItI, and let $h \in \mathcal{H}_\phi$ be a function supported on $It^{-1}I$. Assume that $f(1) \neq 0$ and $h(1) \neq 0$. Compute $(f * h)(1)$ in terms of $f(1)$ and $h(1)$.

(2) Let (π, V) be a smooth representation of G. Let $w \in V$ be a vector satisfying $\pi(y)w = \phi(y)w$ for all $y \in N'_1 T^0 N_1$. (Here $N_j := \left(\begin{smallmatrix} 1 & \mathfrak{p}^j \\ 0 & 1 \end{smallmatrix}\right)$ and $N'_j := \left(\begin{smallmatrix} 1 & 0 \\ \mathfrak{p}^j & 1 \end{smallmatrix}\right)$.) Show that w is fixed by N_1, and that $\pi(e_\phi)w$ is equal to a non-zero scalar times $\sum_{g \in N_0/N_1} \pi(g)w$.

Solution. (1) By definition the function $f*h$ has support contained in $ItIt^{-1}I = IN'_0I$. Suppose $z \in N'_0$ lies in the support of $f * h$. Then by [BH06, §11.2, Lemma], z intertwines ϕ. However, recall that $U_{\mathfrak{M}} = \mathrm{GL}_2(\mathcal{O})$ with $\mathfrak{M} = \mathrm{M}_2(\mathcal{O})$, so we clearly have $z \in U_{\mathfrak{M}}$, and the image $\bar{z}$ of z in $\mathrm{GL}_2(\boldsymbol{k})$ therefore intertwines the character $\bar{\phi}$ of the Borel group of upper triangular matrices in $\mathrm{GL}_2(\boldsymbol{k})$. However, the character $\bar{\phi}$ induces irreducibly to $\mathrm{GL}_2(\boldsymbol{k})$ by [BH06, §6.3, Proposition], so $\bar{z} = 1$ and $z \in N'_1$. Thus, the support of $f * h$ is contained in I. Recall that for each $g \in G$ together with $F \in \mathcal{H}$, we have

$$\int_G F(x)\mathbb{1}_{IgI}(x)\mathrm{d}\mu(x) = \mu(IgI)F(g).$$

In particular, letting $F(x) = f(x)h(x^{-1})$ and $g = 1$, it thus follows that

$$(f * h)(1) = \int_G f(x)h(x^{-1})\mathbb{1}_I(x)\mathrm{d}\mu(x) = \mu(I)f(1)h(1).$$

(2) For any $n_1 \in N_1$, it is clear that $n_1 \in N'_1T^0N_1$, and hence $\pi(n_1)w = \phi(n_1)w$. Since ϕ is trivial on $U^1_{\mathfrak{J}} = 1 + \mathfrak{P}_{\mathfrak{J}} = \left(\begin{smallmatrix} 1+\mathfrak{p} & \mathcal{O}_F \\ \mathfrak{p} & 1+\mathfrak{p} \end{smallmatrix}\right)$, it is in particular trivial on $N_1 \subset U^1_{\mathfrak{J}}$, and thus $\pi(n_1) = 1$. So w is fixed by N_1. To compute $\pi(e_\phi)w$ in terms of a sum running through $g \in N_0/N_1$, recall from Exercise 14.1 that for the character ϕ on I,

$$e_\phi(g) = \mu(I)^{-1} \dim\phi \cdot \mathrm{tr}(\phi(g)^{-1}) = \mu(I)^{-1}\phi(g)^{-1}$$

for any $g \in G$. Thus,

$$\begin{aligned}\pi(e_\phi)w &= \int_G e_\phi(g)\pi(g)w\mathrm{d}\mu(g) \\ &= \mu(I)^{-1}\int_I \phi(g)^{-1}\pi(g)w\mathrm{d}\mu(g) \\ &= \mu(I)^{-1}\mu(N_1)\sum_{x \in I/N_1}\phi(x)^{-1}\pi(x)w.\end{aligned}$$

Here to deduce the last equality, note that w is fixed by N_1 and so also is e_ϕ (as ϕ is trivial on N_1). We then notice

$$N'_1T^0N_0 = \begin{pmatrix} 1 & 0 \\ \mathfrak{p} & 1 \end{pmatrix}\begin{pmatrix} \mathcal{O}_F^\times & 0 \\ 0 & \mathcal{O}_F^\times \end{pmatrix}\begin{pmatrix} 1 & \mathcal{O}_F \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ \mathfrak{p} & \mathcal{O}_F^\times \end{pmatrix} = I.$$

So in the last row of formula of $\pi(e_\phi)w$ above, we may write $x = n'_1t^0g$ for some $n'_1 \in N'_1$, $t^0 \in T^0$, and $g \in N_0/N_1$. Again as ϕ is trivial on $U^1_{\mathfrak{J}}$, it is trivial on both N'_1 and N_0; thus, we have $\phi(x)^{-1} = \phi(n'_1t^0g)^{-1} = \phi(t^0)^{-1}$. It now remains to compute the sum

$$\sum_{x \in I/N_1}\phi(x)^{-1}\pi(x)w = \sum_{n'_1t^0 \in N'_1T^0}\phi(t^0)^{-1}\pi(n'_1t^0)\sum_{g \in N_0/N_1}\pi(g)w.$$

By assumption of w, it is clear that $N'_1T^0N_1$ acts on $\sum_{g \in N_0/N_1}\pi(g)w$ via ϕ, so the sum above becomes

$$\sum_{n'_1t_0 \in N'_1T^0}\phi((t^0)^{-1}n'_1t^0)\sum_{g \in N_0/N_1}\pi(g)w$$

where $(t^0)^{-1}n_1't^0 \in N_0$. Therefore, $\pi(e_\phi)w$ is a non-zero scalar times $\sum_{g\in N_0/N_1}\pi(g)w$. $\square$

Exercise 15.2. Let I be the Iwahori subgroup of G, and let $K = \mathrm{GL}_2(\mathcal{O}_F) \subset G$. Let k denote the residue field of F.

(1) Show that reduction modulo $\mathfrak{p}$ induces surjective homomorphisms $K \to \mathrm{GL}_2(k)$ and $I \to B_k$, where B_k is the subgroup of $\mathrm{GL}_2(k)$ consisting of upper triangular matrices.
(2) Use the Bruhat decomposition $\mathrm{GL}_2(k) = B_k \cup B_k w B_k$, where $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, to show that
$$K = I \cup IwI.$$
(3) Use the Iwasawa decomposition $G = BK$ to show that
$$G = BI \cup BwI.$$
(4) Show that the Steinberg representation $\mathrm{St}_G = (\pi, V)$ satisfies $\dim V^I = 1$ and $V^K = 0$.
(5) Show that every double coset in $I\backslash G/I$ has a representative of the form either $\left(\begin{smallmatrix} \pi^a & 0 \\ 0 & \pi^b \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & \pi^a \\ \pi^b & 0 \end{smallmatrix}\right)$ for some $a, b \in \mathbb{Z}$.

Solution. (1) The construction of homomorphism $K \to \mathrm{GL}_2(k)$ is clear. As for the Iwahori subgroup, we have $I = \left(\begin{smallmatrix} U_F & \mathcal{O}_F \\ \mathfrak{p} & U_F \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} k^\times & k \\ 0 & k^\times \end{smallmatrix}\right) \bmod \mathfrak{p}$, and hence the map $I \to B_k$ follows. This is also a group homomorphism.

(2) We only need to show the set-theoretical equality. If $K \neq I \cup IwI$, then along the maps in (1), modulo $\mathfrak{p}$ we have $\mathrm{GL}_2(k) \neq B_k \cup B_k w B_k$. But this contradicts to the Bruhat decomposition, so $K = I \cup IwI$.

(3) The Iwasawa decomposition together with part (2) shows
$$G = BK = B(I \cup IwI) = BI \cup BIwI = BI \cup BwI.$$
Here the last equality is deduced as follows. It is automatic that $B \subset BI$ and hence $BwI \subset BIwI$; but in G both BwI and $BIwI$ have only trivial intersection with BI, so we must have $BIwI = BwI$.

(4) By definition, St_G sits in the exact sequence
$$0 \longrightarrow \mathbb{C} \longrightarrow \mathrm{Ind}_B^G \delta_B^{-1/2} \longrightarrow \mathrm{St}_G \longrightarrow 0.$$
By part (3) we have $G = BI \cup BwI$, and then
$$\dim(\mathrm{Ind}_B^G(\delta^{-1/2}))^I = 2,$$
because it has the same property when viewed as a B-representation. As for the subrepresentation $\mathbb{C}$, we have $\dim \mathbb{C} = \dim \mathbb{C}^I = 1$. It follows that $\dim \mathrm{St}_G^I = 1$. Again, if we view $\mathrm{Ind}_B^G \delta_B^{-1/2}$ as a B-representation, then $G = BK$ acts on it with $\mathrm{St}_G^K = 0$.

(5) By Cartan decomposition (Exercise 5.4), we have
$$G = \mathrm{GL}_2(F) = \bigsqcup_{a \geqslant b} K \begin{pmatrix} \pi^a & 0 \\ 0 & \pi^b \end{pmatrix} K.$$
So there are representatives $\left(\begin{smallmatrix} \pi^a & 0 \\ 0 & \pi^b \end{smallmatrix}\right)$ for $K\backslash G/K$. Modulo the center of G, it suffices to consider the cosets $Kt^{-a}K$ for $a \geqslant 0$, where $t = \left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right)$. If $a = 0$ then the coset is K itself, and $K = I \cup IwI$ is already known in part (2). Now we assume $a \geqslant 1$. Again, since $K = I \cup IwI$, each $Kt^{-a}K$ decomposes into four double cosets:
$$It^{-a}I, \quad It^{-a}IwI, \quad IwIt^{-a}I, \quad IwIt^{-a}IwI.$$
It is clear that they respectively contain double cosets of form
$$It^{-a}I, \quad It^{-a}wI, \quad Iwt^{-a}I, \quad Iwt^{-a}wI,$$
and all of $t^{-a}, t^{-a}w, wt^{-a}, wt^{-a}$ are of the form either $\left(\begin{smallmatrix} \pi^a & 0 \\ 0 & \pi^b \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & \pi^a \\ \pi^b & 0 \end{smallmatrix}\right)$ as desired.

To show these four are all double cosets in $Kt^{-a}K$, choose a Haar measure μ such that $\mu(I) = 1$, and it suffices to show that

$$\mu(Kt^{-a}K) = \mu(It^{-a}I) + \mu(It^{-a}wI) + \mu(Iwt^{-a}I) + \mu(Iwt^{-a}wI).$$

Note that $\mu(K) = q + 1$ and then

$$\mu(Kt^{-a}K) = (q + 1)^2 q^{a-1}.$$

On the other hand, to compute $\mu(It^{-a}I)$, we have $t^{-a}It^a = N_{-a}T^0N'_{a+1}$, and hence $I \cap t^{-a}It^a = N_0T^0N'_{a+1}$. This group has index q^a in I, so

$$\mu(It^{-a}I) = [It^{-a}I : I] \cdot \mu(I) = [It^{-a}It^a : It^a] = [I : I \cap t^{-a}It^a] = q^a.$$

By replacing t^{-a} above with either of $t^{-a}w$, wt^{-a}, or $wt^{-a}w$ and applying the similar argument, we also deduce

$$\mu(It^{-a}wI) = q^{a-1}, \quad \mu(Iwt^{-a}I) = q^{a+1}, \quad \mu(Iwt^{-a}wI) = q^a.$$

So the measures of the four double cosets sum up to $\mu(Kt^{-a}K)$ as desired. Finally, the only remaining ambiguity lies in $It^{-a}I \cap Iwt^{-a}wI$. If this is non-empty then $It^{-a}I = Iwt^{-a}wI$, which means there are $x, y \in I$ such that $xt^{-a} = wt^{-a}wy$; but this turns out to be impossible by testing the matrix entries. $\square$

PHOTOGRAPH — JUNE 8, 2024; AT GETTY CENTER, LOS ANGELES, CALIFORNIA. *The sculpture for Christ and Mary Magdalene has been rendered with subtlety on different faces, so that the observer can hardly imagine the appearance on the other side.*

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