Artin v-stacks and ULA sheaves

Weizhe Zheng Jan 4

§1 Artin stacks

<u>Def'n</u> The <u>v-stack</u> is a stack $X: Sch^{op} \to$ Groupoid
on cat of schemes (which is fppf.)
s.t. (a) $\Delta_X: X \to X \times X$ rep'ble in alg spaces
(b) (smooth atlas)
$\exists$ surj map $f: U \to X$, f smooth. U alg space.

<u>Idea</u> · Alg spaces $\rightsquigarrow$ locally spatial diamonds.
· Smoothness $\rightsquigarrow$ coh smooth

<u>Def'n</u> An <u>Artin v-stack</u> is a v-stack on Perf s.t.
(a) $\Delta_X: X \to X \times X$ rep'ble by locally spatial diamond.
(b) (atlas) $\exists f: U \to X$. U locally spatial diamond,
and f separated & coh smooth

<u>Remark</u> $\forall U$ locally spatial diamond,
any $U \to X$ is rep'ble by locally spatial diamond.

<u>Def'n</u> $f: X' \to X$ separated map of small v-stacks
rep'ble in locally spatial diamonds.
(1) f is ℓ-cohom smooth if
· f compactifiable,
· f locally of $\dim = \operatorname{trdeg} f < \infty$.

(2) $Rf^! : D_{et}(X, \mathbb{F}_\ell) \longrightarrow D_{et}(X', \mathbb{F}_\ell)$

($Rf^! \simeq Rf^!\Lambda \otimes_{\mathbb{F}_\ell} f^*$, $Rf^!\Lambda$ invertible $(\Lambda = \mathbb{F}_\ell)$.
and the same holds after base change.)

Rmk If $\Lambda = \ell$-power torsion, the same will hold for $\mathbb{F}_\ell \to \Lambda$.

Also, Δ_X qsep (X qsep). E.g. Bun_G not qsep, $[*/\underline{G(E)}]$ not qsep.

Example (1) Any locally spatial diamond is an Artin v-stack.

(2) $Y \to X$ rep'ble in locally spatial diamonds.
Then X Artin v-stack $\Rightarrow$ Y Artin v-stack.

(3) A fiber prod of Artin v-stacks is an Artin v-stack.

(4) $*$ is an Artin v-stack, with $\mathrm{Spd}(E) \to *$ atlas.

(5) G locally profinite grp sch, $G \hookrightarrow GL_n(E)$ admissible.
$\Rightarrow$ $[*/\underline{G}]$ is an Artin v-stack.

Pf of (5): $H/\underline{G} \longrightarrow [\mathrm{Spd}\, E/\underline{G}] \longrightarrow [*/\underline{G}]$
(Take $H = GL_{n,E}^\diamond$.) cohom sm atlas.

Warning $* \longrightarrow [*/\underline{G}]$ is not cohom sm.

Prop $Y'' \xrightarrow{g} Y' \xrightarrow{f} Y$. f, g separated of small v-stacks
on locally spatial diamonds.

f, g coh sm $\Rightarrow$ fg coh sm.

g, fg coh sm, g surj $\Rightarrow$ f coh sm.

Def'n $f : Y \to X$ map of Artin v-stack.
Assume $s : V \to Y$ sep, coh sm, surj.

V locally spatial diamond,
s.t. fg is separated.
Say f is <u>coh sm</u> if for one such g, fg is coh sm (or for any g).

<u>Six functors formalism</u>

- $Rf^! \Lambda \in D_{\acute{e}t}(Y, \Lambda)$ invertible,
  equipped with $Rg^!(Rf^!\Lambda) = R(fg)^!\Lambda$
  & compatibilities with compositions.
- $Rf^! := Rf^!\Lambda \overset{\mathbb{L}}{\otimes}_{\Lambda} f^*$.
- f is of ℓ-dimension $d \in \frac{1}{2}\mathbb{Z}$ if $R^i f^! \mathbb{F}_\ell = 0$, $i \neq 2d$.

<u>Example</u>

$$\begin{tikzcd}
{[*/\underline{G}]} \arrow[r] & * \\
{[\mathrm{Spd}\, E/\underline{G}]} \arrow[r] \arrow[u] & \mathrm{Spd}\, E \arrow[u] \\
H/\underline{G} \arrow[u, "a"] \arrow[ur, "b"'] &
\end{tikzcd}$$

cohom smooth of ℓ-dim 0

a, b cohom smooth of ℓ-dim n^2.
$H = GL_n$.

<u>Thm</u> Bun_G is a cohom smooth Artin v-stack of ℓ-dim 0.

<u>Proof</u> (a)
$$\begin{tikzcd}
\mathrm{Isom}(\mathcal{E}_1, \mathcal{E}_2) \arrow[r] \arrow[d] & S \arrow[d, "(\mathcal{E}_1, \mathcal{E}_2)"] \\
\mathrm{Bun}_G \arrow[r] & \mathrm{Bun}_G \times \mathrm{Bun}_G
\end{tikzcd}$$

$\rightsquigarrow \mathrm{Isom}(\mathcal{E}_1, \mathcal{E}_2) \underset{\text{open}}{\subseteq} \mathcal{BC}(\mathcal{E}_1^\vee \otimes \mathcal{E}_2)$ locally spatial diamond

$\Rightarrow \mathrm{Isom}(\mathcal{E}_1, \mathcal{E}_2)$ coh sm Artin v-stack

$\Rightarrow \Delta_{\mathrm{Bun}_G}$ is rep'ble by locally spatial diamonds.

(b) Beauville-Laszlo uniformization map $\mathrm{Gr}_G \longrightarrow \mathrm{Bun}_G$,

$(\underset{\text{triv}}{\mathcal{E}'} \longleftrightarrow \mathcal{E}) \longmapsto \mathcal{E}$

Triviality on geom fibers: $[\underline{G(E)}\backslash \mathrm{Gr}_G] \longrightarrow \mathrm{Bun}_G$.

$(\xi' \hookleftarrow \xi) \longmapsto \xi$

<u>Notations</u> For $\bar{\mu} \in X_*(T)^+/\Gamma$, $\mathrm{Gr}_{G,\bar{\mu}} \times_{\mathrm{Spd}\,E} \mathrm{Spd}\,E' = \coprod_{\mu' \equiv \mu} \mathrm{Gr}_{G,\mu',E'}$.

$\rightsquigarrow$
$$\begin{tikzcd} & \coprod_{\bar{\mu} \in X_*(T)^+/\Gamma} [\underline{G(E)}\backslash \mathrm{Gr}_{G,\bar{\mu}}] \arrow[r, "\pi"] \arrow[dr, "\pi"] & \mathrm{Bun}_G \\ T \arrow[ur] \arrow[r] & S \arrow[r, "{(\xi, S)}"] & \mathrm{Bun}_G \times \mathrm{Spd}\,E \arrow[u] \end{tikzcd}$$

where
$$\begin{tikzcd} T \arrow[d] & \xi' \hookleftarrow \xi \\ T' & \xi' \leftarrow\text{-}\text{-}\rightarrow \xi \end{tikzcd}$$
by dropping the open condition

open with trivial geom fibers

Then, v-locally, $T' \simeq \coprod_{\bar{\mu}} \mathrm{Gr}_{G,\bar{\mu}} \times_{\mathrm{Spd}\,E} S$.

(<u>Prop</u> ([FS], VI.2.4) $\forall \mu$, $\mathrm{Gr}_{G,\mu}/\mathrm{Spd}\,E'$ coh sm of ℓ-dim $\langle 2\rho, \mu\rangle$)

$\Rightarrow$ π coh sm of ℓ-dim $\langle 2\rho, \mu\rangle$.

$[\underline{G(E)}\backslash \mathrm{Gr}_{G,\bar{\mu}}] \longrightarrow [\underline{G(E)}\backslash \mathrm{Spd}\,E] \longrightarrow \mathrm{Spd}\,E$

coh sm of ℓ-dim $\langle 2\rho, \mu\rangle$ $\qquad$ coh sm of ℓ-dim 0.

$\Rightarrow$ Bun_G coh sm of ℓ-dim 0. $\square$

<u>Prop</u> $\forall b \in B(G)$, Bun_G^b coh sm Artin v-stack of ℓ-dim $-\langle 2\rho, \nu_b\rangle$

<u>Proof</u>
$$\begin{tikzcd} \mathrm{Bun}_G^b = [*/\tilde{G}_b] \arrow[r] & [*/\underline{G_b(E)}] \\ {[*/\tilde{G}_b^{>0}]} \arrow[u] \arrow[r] & * \arrow[u] \end{tikzcd}$$

$\rightsquigarrow$ $\tilde{G}_b^{>0}$ locally spatial diamond, coh sm of ℓ-dim $\langle 2\rho, \nu_b\rangle$. $\square$

<u>Cor</u> The Kottwitz map $\kappa : \pi_0(\mathrm{Bun}_G) \longrightarrow \pi_1(G)_\Gamma$ is bijective.

$B(G)_{\mathrm{triv}}$ $\sim$

<u>Sketch of Pf</u> Suffices to show $\forall\, U \subseteq \mathrm{Bun}_G$ open,

U contains a basic pt,

$\exists\, b \in B(G)$, $\mathrm{Bun}_G^b \subset U \subset \mathrm{Bun}_G$ open $\Rightarrow -\langle 2\rho, \nu_b\rangle = 0$, b basic. $\square$

§2 ULA sheaves

Schemes <u>Def'n</u> $f: X \to S$ <u>acyclic</u> if $\Lambda \xrightarrow{\sim} Rf_*\Lambda$.

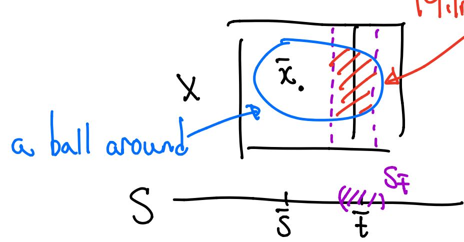
$f: X \to S$ <u>locally acyclic</u> if

$$\forall \begin{array}{ccc} \bar{x} & \longrightarrow & X \\ \downarrow & \square & \downarrow \\ \bar{s} & \longrightarrow & S \end{array}, \quad f_{\bar{x}}: X_{\bar{x}} \longrightarrow S_{\bar{s}} \text{ acyclic, i.e. } \Lambda \simeq Rf_{\bar{x}*}\Lambda.$$

strict Henselization.

$\Leftrightarrow$ $\forall\, \bar{t} \rightsquigarrow \bar{s}$ specialization,

$$\Lambda \simeq (Rf_{\bar{s},*}\Lambda)_{\bar{t}} = R\Gamma(\underbrace{X_{\bar{x}} \times_{S_{\bar{s}}} S_{\bar{t}}}_{\text{Milnor's tube}}, \Lambda)$$



- $A \in D_{\acute{e}t}(X, \Lambda)$, A is <u>f-LA</u> if $Rf_{\bar{x},*}A$ is const

$\Leftrightarrow$ $A_{\bar{x}} \simeq R\Gamma(X_{\bar{x}}, A) \xrightarrow{\sim} R\Gamma(X_{\bar{x}} \times_{S_{\bar{s}}} S_{\bar{t}}, A)$.

Diamonds <u>Def'n</u> $f: X \to S$ compactifiable map of locally spatial diamonds with locally $\dim = \mathrm{trdeg}\, f < +\infty$.

$A \in D_{\acute{e}t}(X, \Lambda)$.

(1) Say A is <u>f-LA</u> if

(a) For $\bar{t} \rightsquigarrow \bar{s} \xleftarrow{f} \bar{x}$,

$$R\Gamma(X_{\bar{x}}, A) \xrightarrow{\sim} R\Gamma(X_{\bar{x}} \times_{S_{\bar{s}}} S_{\bar{t}}, A).$$

(b) $\forall j : U \to X$ sep étale. s.t. (fj) qc, and $R(fj)_! j^* A \in D_{\acute{e}t}(pc)$

perfect-constructible.

(2) Say A is <u>f-ULA</u> if the same holds after BC by $S' \to S$ locally spatial diamonds.

<u>Rmk</u> $X_{\bar{x}} = \mathrm{Spa}(C, C^+)$, C alg closed

$S_{\bar{t}} \hookrightarrow S_{\bar{s}}$, $X_{\bar{x}} \times_{S_{\bar{s}}} S_{\bar{t}} \hookrightarrow X_{\bar{x}}$ strictly local of closed pt $\bar{y}$.

(a) $A_{\bar{x}} \xrightarrow{\sim} A_{\bar{y}}$ overconvergent along $\bar{y} \rightsquigarrow \bar{x}$.

<u>Fact</u> LA descends along v-covers of the tangent.

<u>Prop</u> (a) universally $\iff$ A overconvergent, i.e. $A_{\bar{x}} \xrightarrow{\sim} A_{\bar{y}}$, $\forall \bar{y} \to \bar{x}$.

<u>Example</u> $f : S \xrightarrow{id} S$, then

A is f-LA $\iff$ A locally const with perfect stalks

pf of "$\Rightarrow$": (a) A overconvergent

(b) A perfect constructible. $\square$

We have the following generalization.

<u>Prop</u> $f : X \to S$ sep & ℓ-coh sm.

A locally const with perfect stalk $\Rightarrow$ A is f-ULA.

<u>Proof</u> (a) overconvergence, ok

(b) <u>Claim</u> f qc, sep, coh sm $\Rightarrow$ $Rf_!(D_{\acute{e}t}, pc) \subseteq D_{\acute{e}t}, pc$.

pf. We may assume S std perf.

$A \in D_{et}(X,\Lambda)$ perfect constructible $\Leftrightarrow$ compact.

$Rf^!(-) \simeq Rf^!\Lambda \overset{\mathbb{L}}{\otimes}_\Lambda f^*(-)$ preserves direct sums

$\Rightarrow$ $Rf_!$ preserves compact objects. $\square$

<u>Properties</u>

$$\begin{tikzcd} Y \arrow[d, "g"'] \\ X \arrow[d, "f"'] \\ S \end{tikzcd}$$

· If g proper, $A \in D_{et}(Y,\Lambda)$ is (fg)-LA, then $Rg_*A \in D_{et}(X,\Lambda)$, f-LA.

· If g sep & coh sm, $B \in D_{et}(X,\Lambda)$, then B f-ULA $\Rightarrow$ g^*B (fg)-ULA

g^*B (fg)-ULA, g surj $\Rightarrow$ B f-ULA.

§3 ULA sheaves and duality

S locally spatial diamond.

$\mathcal{C}_S$ 2-cat, obj $= X \xrightarrow{f} S$ compactifiable of locally spatial diamonds with locally $\dim.\mathrm{trdeg} f < \infty$.

$X, Y \in \mathcal{C}_S \rightsquigarrow \mathrm{Fun}_{\mathcal{C}_S}(X,Y) = D_{et}(X \times_S Y, \Lambda)$

$\rightsquigarrow \mathrm{Fun}_{\mathcal{C}_S}(X,Y) \times \mathrm{Fun}_{\mathcal{C}_S}(Y,Z) \longrightarrow \mathrm{Fun}_{\mathcal{C}_S}(X,Z)$

$(A,B) \longmapsto A \star B = R\pi_{13,!}(\pi_{12}^*A \overset{\mathbb{L}}{\otimes}_\Lambda \pi_{23}^*B)$.

$\mathrm{Fun}_{\mathcal{C}_S}(X,X) \ni R\Delta_!\Lambda$ is the identity.

$\mathcal{C}_S$ admits a triangularization $\mathcal{C}_S \longrightarrow \mathrm{TrCat}$

$X \longmapsto D_{et}(X,\Lambda)$

$A \longmapsto \begin{pmatrix} D_{et}(X,\Lambda) \longrightarrow D_{et}(Y,\Lambda) \\ B \longmapsto R\pi_{2,!}(A \overset{\mathbb{L}}{\otimes}_\Lambda \pi_1^*B) \end{pmatrix}$

$$\begin{tikzcd} & X \times_S Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

· $X \xrightarrow{f} Y$ left adjoint to $Y \xrightarrow{g} X$ if $\exists\ \mathrm{id}_X \xrightarrow{\alpha} gf$, $fg \xrightarrow{\beta} \mathrm{id}_Y$

s.t. $f \xrightarrow{\alpha} fgf \xrightarrow{\beta} f$, $g \xrightarrow{\alpha} gfg \xrightarrow{\beta} g$ are identities.

<u>Main theorem</u> $X \in \mathcal{C}_S$, $A \in D_{ct}(X, \Lambda)$. TFAE:

Hard part

(1) A is f-ULA

(2) $p_1^* \mathbb{D}_{X/S}(A) \overset{\mathbb{L}}{\otimes}_\Lambda p_2^* A \xrightarrow{\sim} R\underline{Hom}_\Lambda(p_1^* A, Rp_2^! A)$

where

$$\begin{tikzcd} & X \times_S X \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] & \\ X & & X. \end{tikzcd}$$

(3) $A \in \mathrm{Fun}_{\mathcal{C}_S}(X, S)$ is a left adjoint in $\mathcal{C}_S$ whose right adjoint is given by $\mathbb{D}_{X/S}(A) \in \mathrm{Fun}_{\mathcal{C}_S}(S, X)$

Verdier duality,

$\mathbb{D}_{X/S} := R\underline{Hom}_\Lambda(-, Rf^! \Lambda)$.

<u>Cor</u> A f-ULA $\Rightarrow$ $\mathbb{D}_{X/S}(A)$ f-ULA, and $A \xrightarrow{\sim} \mathbb{D}_{X/S}\mathbb{D}_{X/S}(A)$.

<u>Proof</u> $\mathbb{D}_{X/S}(A) \in \mathrm{Fun}(S, X)$ right adj to $A \in \mathrm{Fun}(X, S)$

$(\mathcal{C}_S \simeq \mathcal{C}_S^{op})$

or $\mathbb{D}_{X/S}(A) \in \mathrm{Fun}(S, X)$ left adj to $A \in \mathrm{Fun}(X, S)$. $\square$

<u>Cor</u> $A_i \in D_{ct}(X_i, \Lambda)$ f_i-ULA $\Rightarrow$ $A_1 \boxtimes \cdots \boxtimes A_n$ is $f_1 \times_S \cdots \times_S f_n$-ULA.

<u>Cor</u> f is ℓ-coh sm $\Leftrightarrow$ $\mathbb{F}_\ell$ is f-ULA and $Rf^! \mathbb{F}_\ell$ invertible.

<u>Proof</u> "$\Rightarrow$" already done

"$\Leftarrow$" $\left.\begin{array}{l} Rf^! \Lambda \overset{\mathbb{L}}{\otimes}_\Lambda f^*(-) \xrightarrow{\sim} Rf^! \\ Rf^! \Lambda \text{ commutes with base change} \end{array}\right\} \Rightarrow f$ ℓ-coh sm. $\square$

<u>Rmk</u> (2) ⇒ (3) of main thm:

$$\begin{array}{l} \underset{\underset{f}{\rotatebox{90}{$\in$}}}{A} \star \underset{\underset{g}{\downarrow}}{\mathbb{D}_{X/S}(A)} = Rf_!\left(\mathbb{D}_{X/S}(A) \overset{\mathbb{L}}{\otimes}_{\Lambda} A\right) \longrightarrow \Lambda \\ \qquad\qquad\qquad\qquad\quad\ \mathbb{D}_{X/S}(A) \overset{\mathbb{L}}{\otimes}_{\Lambda} A \longrightarrow Rf^!\Lambda . \end{array}$$

For $S \backslash \ell s$, $(X, A) \in D_{ct}(X, \Lambda)$ is dualizable.