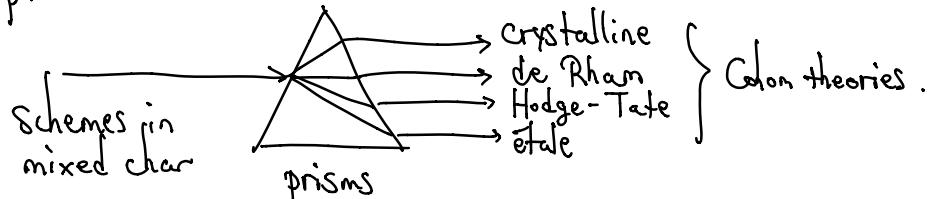


Prismatic cohomology (1/4)

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Fix prime p .



X/\mathbb{Z}_p sm p-adic formal sch $\hookrightarrow R\Gamma_{\text{crys}}(X_S/\mathbb{Z}_p)$ on special fiber

$$\left. \begin{array}{c} \{ \\ \{ \\ \downarrow \\ \end{array} \right\} \quad \left. \begin{array}{c} X_S := X \times_{\mathbb{Z}_p} \text{Spec } \mathbb{F}_p \\ R\Gamma_{\text{dR}}(X/\mathbb{Z}_p) = R\Gamma(X, \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/\mathbb{Z}_p} \xrightarrow{d} \dots) \\ R\Gamma_{\text{HT}}(X) \text{ twisted form of } R\Gamma(X, \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/\mathbb{Z}_p} \xrightarrow{d} \dots) \end{array} \right.$$

$R\Gamma_{\text{ét}}(X_\eta, \mathbb{Z}_p)$, X_η = rigid geom. fiber.

§ Prismatic cohom in char 0

k perfect field of char 0. $A := k[[t]]$. R k -alg.

Def'n (prismatic site in char 0)

$(R/A)_A^{\text{op}}$ with • objects $(R \xrightarrow{\imath} S/t \leftarrow S)$

with $S = t$ -complete t -torsion-free A -alg

• $R \xrightarrow{\imath} S/t$ map of k -algs.

• morphs $(R \xrightarrow{\imath} S/t \leftarrow S) \rightarrow (R \xrightarrow{\imath'} S'/t \leftarrow S')$

given by $\alpha: S \rightarrow S'$ map of A -algs

s.t. $\alpha \circ \imath = \imath'$.

• covers given by isoms.

Set $\mathcal{O}_\Delta(R \rightarrow S/t \leftarrow S) = S$, $\overset{\uparrow}{\mathcal{O}_\Delta}(R \rightarrow S/t \leftarrow S) = S/t$.
reduced ver.

$R\Gamma((R/A)_\Delta, \mathcal{O}_\Delta) = \underset{(R \rightarrow S/t \leftarrow S)}{\lim^{\text{R}}_{\text{inv}}} S$ derived inverse limit.

Theorem (Wassmann) R sm over k .

(1) $H^i((R/A)_\Delta, \bar{\mathcal{O}}_\Delta) \simeq \Omega_{R/k}^i$, $i \geq 0$ canonically

(2) Assume \tilde{R} t -complete t -torsion-free lift of R to A .

Then $R\Gamma((R/A)_\Delta, \mathcal{O}_\Delta) = [\tilde{R} \xrightarrow{t \cdot d} \tilde{\Omega}_{\tilde{R}/A}^1 \xrightarrow{t \cdot d} \tilde{\Omega}_{\tilde{R}/A}^2 \rightarrow \dots]$

Sketch of pf Step 1 Comparison map in (1).

Have ses $0 \rightarrow \bar{\mathcal{O}}_\Delta \xrightarrow{t} \bar{\mathcal{O}}_\Delta/t^2 \rightarrow \bar{\mathcal{O}}_\Delta \rightarrow 0$

\leadsto get $H^0((R/A)_\Delta, \bar{\mathcal{O}}_\Delta) \xrightarrow{\beta} H^1((R/A)_\Delta, \bar{\mathcal{O}}_\Delta)$

\uparrow \mathbb{Z}_{abs} (a canonical map by def'n)
 R

Now $H^*((R/A)_\Delta, \bar{\mathcal{O}}_\Delta)$ is a commutative graded alg

& $\beta \circ \mathbb{Z}_{\text{abs}}$ is a k -linear derivation.

\Rightarrow get a unique ext'n $\alpha_{\text{HT}}: \Omega_{R/k}^* \rightarrow H^*((R/A)_\Delta, \bar{\mathcal{O}}_\Delta)$.

\leadsto reduce (1) to (2).

Step 2 A formula for $R\Gamma((R/A)_\Delta, \mathcal{O}_\Delta)$.

lem $(R/A)_\Delta$ has fiber products (via $\hat{\otimes}$) and non-empty product.

To get products:

Def'n T t -complete, t -torsion-free A -alg, $\mathcal{J} \subseteq T$ any ideal, $t \in \mathcal{J}$.

The prismatic envelope $Dg(T)$ of T in \mathcal{J} is $T[\frac{j}{t} \mid j \in \mathcal{J}]_t^\wedge$.

(t -adic completion).

Note S t -complete, t -torsion-free A -alg.

$$\text{Then } \text{Hom}((T, \mathcal{T}), (S, (t))) \simeq \text{Hom}(D_{\mathcal{T}}(T), (t)), (S, (t))).$$

Now, let $(R \xrightarrow{\iota} S/t \leftarrow S)$, $(R \xrightarrow{\iota'} S'/t \leftarrow S')$.

$$\text{Set } T := S \hat{\otimes}_R S', \quad \mathcal{T} = \ker(T \rightarrow T/t = S/t \otimes_R S'/t \rightarrow S/t \otimes_R S'/t).$$

Then $(R \xrightarrow{\iota \otimes \text{Id}} T/t \rightarrow D_{\mathcal{T}}(T)/t \leftarrow D_{\mathcal{T}}(T))$ represents the product.

Thus, $R\Gamma((R/A)_{\Delta}, \mathcal{O}_{\Delta})$ can be calculated as follows.

- Pick P to be any t -complete smooth A -alg

with swj $P \rightarrow R$ & kernel = \mathcal{T} .

- Set $D := D_{\mathcal{T}}(P)$. Then $(R \rightarrow D/t \leftarrow D) \in (R/A)_{\Delta}$

Covers the final object of $(R/A)_{\Delta}$.

$\Rightarrow R\Gamma((R/A)_{\Delta}, \mathcal{O}_{\Delta}) = (D(0) \rightarrow D(1) \rightarrow D(2) \rightarrow \dots)$ complex of rings
where $D(i) = (i+1)\text{th fold product of } D$ in $(R/A)_{\Delta}$.

Note $D(n) = \text{prism envelope of } P^{\hat{\otimes}_A^{(n+1)}}$

$$\text{and } \mathcal{T}_n = \ker(P^{\hat{\otimes}^{n+1}} \rightarrow R) = (t, x_1, \dots, x_r)$$

locally generated by regular sequence.

$$\Rightarrow D(n) = P^{\hat{\otimes}^{(n+1)}} \left[\frac{x_1}{t}, \dots, \frac{x_r}{t} \right]_t^{\wedge}.$$

$$\& D(n)/t \simeq R \left[\frac{x_1}{t}, \dots, \frac{x_r}{t} \right].$$

Step 3 t -de Rham complexes.

$$\text{Indeed, } P \xrightarrow{t \cdot d} \Omega_{P/A} \hat{\otimes}_P D$$

$$\begin{array}{ccc} \downarrow & \curvearrowright & \uparrow \\ D & \dashrightarrow & \exists! \nabla^t \end{array} \quad \nabla^t \left(\frac{j}{t} \right) := d_j \text{ for } j \in J.$$

Step 4 Finish the argument.

Set $P(n) = P^{\widehat{\otimes}(n+1)}$. Set $K^{ab} := \widehat{\Omega}_{P(A)/A}^b \widehat{\otimes}_{P(A)} D(a)$.

$$\begin{array}{c} \vdots \\ \uparrow \\ \widehat{\Omega}_{P(A)/A}^1 \widehat{\otimes}_P D \longrightarrow \dots \\ \downarrow \nabla^t \quad \uparrow \\ D(0) \longrightarrow D(1) \longrightarrow D(2) \end{array}$$

Rather formal: $K^{*,b}$ acyclic for $b > 0$.

$$\hookrightarrow \text{Tot}(K^{*,*}) \simeq (D(0) \rightarrow D(1) \rightarrow \dots).$$

Claim each of the cosimplicial str maps $D(0) \rightarrow D(n)$
induces a quasi-isom $K^{0,*} \rightarrow K^{n,*}$.

$$\hookrightarrow \text{Tot}(K^{*,*}) \simeq R \lim_{\Delta} K^{n,*} = K^{0,*} \quad D$$

$\xrightarrow{\simeq}$

(assume $P = \tilde{R}$ & $\tilde{R}/t = R$.)

$$\hookrightarrow (\tilde{R} \xrightarrow{t \cdot d} \widehat{\Omega}_{\tilde{R}/A}^1 \xrightarrow{t \cdot d} \dots)$$

lem S t -complete, t -torsion-free A -alg.

$$P = S[x_1, \dots, x_r]_t^\wedge, \quad \mathcal{T} = (t, x_1, \dots, x_r), \quad D := D_{\mathcal{T}}(P).$$

$$\Rightarrow 0 \rightarrow S \rightarrow D \xrightarrow{\nabla^t} \widehat{\Omega}_{P/S}^1 \widehat{\otimes}_P D \rightarrow \dots \text{ exact.}$$

Proof Reduce mod t .

$$D/t \simeq S/t[x_1/t, \dots, x_r/t], \quad \nabla^t(x_i/t) = d x_i.$$

\hookrightarrow apply usual Poincaré lem for poly rings in char 0. \square

8 Prismatic coh in char p

Fix a prime p . All rings are p -adic complete.

Def'n A \mathfrak{s} -ring is a ring A with a map

$$\mathfrak{s} := \mathfrak{s}_A : A \longrightarrow A \text{ (of rfs) s.t.}$$

- (i) $\delta(0) = 0, \delta(1) = 0,$
- (ii) $\delta(x \cdot y) = x^p \delta(y) + y^p \delta(x) + p \delta(x) \delta(y).$
- (iii) $\delta(x+y) = \delta(x) + \delta(y) + \frac{1}{p} (x^p + y^p - (x+y)^p).$

Given δ -ring A , set $\varphi := \varphi_A : A \rightarrow A, x \mapsto x^p + p \cdot \delta(x).$

Lemma (1) A δ -ring $\Rightarrow \varphi_A$ is a lift of Frobenius.

(2) A p -torsion, $\varphi : A \rightarrow A$ lift of Frobenius.

$\delta(x) := \frac{1}{p} (\varphi(x) - x^p)$ is a δ -str on A .

Fix a perfect field k , R k -alg.

Defn (prismatic site in char p)

$(R/W(k))_{\Delta}$ has

• objects $(R \xrightarrow{\imath} A/p \leftarrow A)$

with A p -complete p -torsion-free δ -ring over $W(k)$.

• morphs: morphs of δ -rings $/ W(k)$ compatible with \imath

• Covering: topology $A \rightarrow B$ cover if

$A/p \rightarrow B/p$ is faithfully flat.

Set $\mathcal{O}_A(R \rightarrow A/p \leftarrow A) = A$, $\bar{\mathcal{O}}_A(R \rightarrow A/p \leftarrow A) = A/p$.

Theorem (Bhatt-Scholze, Ogus) R sm / k .

(1) $\varphi_{W(k)}^* R\Gamma((R/W(k))_{\Delta}, \mathcal{O}_A) \simeq R\Gamma_{\text{crys}}(R/W(k)).$

(2) $H^*((R/W(k))_{\Delta}, \bar{\mathcal{O}}_A) \simeq \Omega_{R/k}^*$ canonically.

(3) If \tilde{R} p -completely sm δ -lift of R to $W(k)$,

then $R\Gamma((R/W(k))_{\Delta}, \mathcal{O}_A) \simeq [\tilde{R} \xrightarrow{P \cdot \delta} \hat{\mathcal{S}}_{\tilde{R}/W(k)}^1 \xrightarrow{P \cdot \delta} \dots]$ φ -equiv.

Prop $\varphi: \Omega_{R/k}^* \rightarrow H^*(C_R/W(k))_a, \bar{G}_a$ can be constructed as before.

In (2),

$$\begin{aligned} & R\Gamma((R/W(k))_a, G_a) \\ & \downarrow \quad \simeq [\tilde{R} \xrightarrow{d} \frac{1}{p} \hat{\Omega}_{R/W(k)}^1 \xrightarrow{d} \frac{1}{p^2} \hat{\Omega}_{R/W(k)}^2 \xrightarrow{d} \dots] \\ & \varphi_{W(k), *} R\Gamma((R/W(k))_a, G_a) \quad \varphi_{\tilde{R}} \downarrow \quad \varphi_{\tilde{R}} \downarrow \\ & \simeq \varphi_{W(k), *} [\tilde{R} \xrightarrow{d} \hat{\Omega}_{R/W(k)}^1 \xrightarrow{d} \dots]. \end{aligned}$$

Modulo $p \rightsquigarrow$ reduces to Cartier isom

$$\Omega_{R^{(p)}/k}^1 \xrightarrow{\sim} H^i(\Omega_{R/k}^*)$$

for $R^{(p)} = R \otimes_{R,p} k$.

Some ingredients in the pf of (1)

Lemme $H: \{\delta\text{-rings}\} \longrightarrow \{\text{rings}\}$

has a left adjoint F & a right adjoint G .

In fact, $G(R) = W(R) = p\text{-typical Witt vectors of } R$.

$$F: \mathbb{Z}\{x\} \hookrightarrow F(\mathbb{Z}[x]) = \mathbb{Z}[x, \delta(x), \delta^2(x), \dots].$$

Prop Given p -complete, p -torsion-free δ - $W(k)$ -alg P

and $\mathcal{T} = (p, x_1, \dots, x_r)$ via a regular seq,

then $D := P\left\{ \frac{x_1}{p}, \dots, \frac{x_r}{p} \right\}_p$ is the prism envelope of P in \mathcal{T} .

\hookrightarrow Hom of δ -pairs.

$$\text{Also, } \text{Hom}((P, \mathcal{T}), (A, (p))) = \text{Hom}(D, (p)) \cong A/(p)$$

\uparrow
 p -complete p -torsion-free

key lemma A p -torsion-free δ - \mathbb{Z}_p -alg, $x \in A$ s.t. $\varphi(x)/p \in A$.

Then x has all divided powers.

In fact, if (p, x_1, \dots, x_r) is regular,

then $A\{\frac{\varphi(x_1)}{p}, \dots, \frac{\varphi(x_r)}{p}\} = \text{PD-envelope of } (x_1, \dots, x_r) \text{ in } A$.

Sketch pf $\frac{\varphi(x)}{p} = \frac{x^p}{p} + \delta(x) \Rightarrow \gamma_p(x) = \frac{x^p}{p!} \in A$.

Claim $\gamma_{p^2}(x) = \text{unif. } \frac{x^{p^2}}{p^{p+1}} \in A$

$$\text{Thus } \underset{A}{\delta\left(\frac{x^p}{p}\right)} = \frac{1}{p} \left(\frac{\varphi(x)}{p} - \frac{x^{p^2}}{p^p} \right) = \frac{1}{p^2} \underbrace{\left(x + p\delta(x) \right)^p}_{\in A} - \frac{x^{p^2}}{p^{p+1}}.$$

$$\text{b/c } p^{p-2} \left(\frac{x^p}{p} + \delta(x) \right)^p \in A.$$

Finish by induction using

$$\gamma_{p,p}(\eta) = \text{unif. } \gamma_p(\gamma_p(\eta)).$$

□