$\mathcal{D}_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)$

Jiahao Niu

Jan 6

§1 On six functor formalism

(i) Λ prime-to-p torsion ring,

· $\mathcal{D}$: {Small v-stacks} $\longrightarrow \mathrm{Cat}_\infty$

$X \longmapsto \mathcal{D}_{\acute{e}t}(X, \Lambda)$

extends to a 6-functor formalism

$\otimes$, $R\mathcal{H}om$, Rf_*, f^*, $Rf_!$, $f^!$. ← def'd for all nice f we will encounter.

A v-hypersheaf,

can glue $\mathcal{D}_{\acute{e}t}(X,\Lambda)$ from $\mathcal{D}_{\acute{e}t}(S,\Lambda) = D(S_{\acute{e}t},\Lambda)$

S = std perf'd space.

· X locally spatial diamond,

$\rightsquigarrow \mathcal{D}^+_{\acute{e}t}(X,\Lambda) \simeq D^+(X_{\acute{e}t},\Lambda)$.

(ii) Λ (relative) discrete $\mathbb{Z}_\ell$-alg,

$\rightsquigarrow$ $\mathcal{D}_{lis}$: {Artin v-stacks} $\longrightarrow \mathrm{Cat}_\infty$

$X \longmapsto \mathcal{D}_{lis}(X,\Lambda)$

admitting 5-functors : $\otimes$, $R\mathcal{H}om$, Rf_*, f^*, $f_\natural$

always left adj to f^*.

(iii) (Lucas-Mann)

Λ nuclear $\mathbb{Z}_\ell$-alg

$\rightsquigarrow$ $\mathcal{D}_{nuc}$: {small v-stacks} $\longrightarrow \mathrm{Cat}_\infty$

$X \longmapsto \mathcal{D}_{nuc}(X,\Lambda)$

v-hypersheaf extends to 6-functor formalism.

<u>Notation</u> Today: $D(X,\Lambda) = D_{\acute{e}t}(X,\Lambda)$

<u>§2 Recap on Bun_G</u>

- $|\mathrm{Bun}_G| \simeq B(G)$, $\mathrm{Bun}_G = \text{"}\bigcup_{b\in B(G)}\text{"} [*/\tilde{G}_b] \longleftarrow \mathrm{Bun}_G^{\text{HN-split}}$

  $S \mapsto \mathcal{E}/X_S$, grading given by HN-filt'n.

  Ref. [FS] III.4.7: $\mathcal{E} = \bigoplus_{\lambda\in\mathbb{Q}} \mathcal{E}^\lambda$, λ = slope (given), $\mathcal{E}^\lambda$ s.s.

- Also, $\mathrm{Bun}_G^{\text{HN-split}} \simeq \coprod_{b\in B(G)} [*/G_b(E)]$

  $\rightsquigarrow \quad |\mathrm{Bun}_G| \xrightarrow{\nu} (X_*^+(T)_{\mathbb{Q}})^\Gamma$

  $|\mathrm{Bun}_G| \xrightarrow{\kappa} \pi_1(G)_\Gamma \rightsquigarrow B(G)_{bas} = \pi_0(\mathrm{Bun}_G)$

- Normalization: $\nu^*: b \mapsto w_0(\nu(-b))$,

  w_0 = largest element in Weyl grp.

- Specialization make slope positive.

  GL_2: $\underset{(\frac{1}{2},\frac{1}{2})}{\mathcal{O}(\frac{1}{2})} \rightsquigarrow \underset{(1,0)}{\mathcal{O}(1)\oplus\mathcal{O}} \rightsquigarrow \mathcal{O}(2)\oplus\mathcal{O}(1)$

  semistable $B(G)_{bas}$ specialization

- $\tilde{G}_b^{\geq\lambda}$ s.t. $\tilde{G}_b^{>0} \rtimes G_b(E) \cong \tilde{G}_b$

  $\tilde{G}_b^{>0}$ = successive ext'n of $\mathcal{BC}(\langle\alpha,\nu_b\rangle)$,

  all α positive roots s.t. $\langle\nu_b,\alpha\rangle > 0$.

<u>Fact</u> $D(\mathrm{Bun}_G,\Lambda)$ admits a "smooth" decomposition by Bun_G^b:

$$D(\mathrm{Bun}_G,\Lambda) \underset{①}{\xrightarrow{\sim}} D([*/\underline{G_b(E)}],\Lambda) \underset{②}{\cong} \underbrace{D(G_b(E),\Lambda)}_{\text{der cat of sm } G_b(E)\text{-rep'ns.}}$$

We will omit the details about ② and only prove ①

Have $[*/\tilde{G}_b] = \mathrm{Bun}_G^b \rightleftarrows [*/\underline{G_b(E)}]$.

Reduce ① to:

<u>Prop</u> ([FS], V.2.1) $f: S' \to S$ of small v-stacks which is a $\mathcal{BC}(\mathcal{E})$-torsor

$\mathcal{E}$ vb on X_S with positive slopes everywhere.

$\Rightarrow \quad f^*: D_{et}(S,\Lambda) \longrightarrow D_{et}(S',\Lambda)$ fully faithful.

<u>Proof</u> v-locally: torsor to be split, $S' = \mathcal{BC}(\mathcal{E})/S$

$\rightsquigarrow \quad 0 \to \mathcal{O}_{X_S}(\frac{1}{2r})^m \longrightarrow \mathcal{O}_{X_S}(\frac{1}{r})^m \longrightarrow \mathcal{E} \to 0.$

Reduce to $\mathcal{E} = \mathcal{O}_{X_S}(\frac{1}{n})$.

$$\rightsquigarrow \quad \begin{tikzcd} 0 \arrow[r] & \mathcal{BC}(\mathcal{O}(\frac{1}{2r})^{m'}) \arrow[r] \arrow[dr] & \mathcal{BC}(\mathcal{O}(\frac{1}{r})^m) \arrow[r] \arrow[d] & \mathcal{BC}(\mathcal{E}) \arrow[r] \arrow[dl] & 0 \\ & & S & & \end{tikzcd}$$

note: $\mathcal{BC}(\mathcal{E})$ = 1-dim'l perf'd ball.

Then f^* fully faithful $\Leftrightarrow$ so is $f^!$ $\quad$ (f coh sm)

$\Leftrightarrow \underbrace{Rf_! f^!(A)}_{\text{Need this.}} \xrightarrow{\sim} A$ by base change

$\rightsquigarrow$ Reduce to $\boxed{S = \mathrm{Spa}(C, C^+)}$ $\leftarrow$ both sides commutes with fil'd colimts

(enough to check by applying $R\Gamma$)

$\Rightarrow$ constructible $j_!(\text{const})$, where $j: U \to S$ qc. étale. $\square$

<u>§3 $D_{et}(\mathrm{Bun}_G, \Lambda)$</u>

compact admissible reflexion

· To describe compact objects, ULA objects, reflexive objects, and Beristein Zelevisky duality $\longleftrightarrow$ B-Z.

Need nice charts. To understand $|\mathrm{Bun}_G|$ via $Gr_G \longrightarrow \mathrm{Bun}_G$.

To construct $D(\mathrm{Bun}_G, \Lambda)$ (need $\mathcal{M}_b$).

<u>Def'n</u> ([FS], V.3.2) $\mathcal{M}$: Perf $\longrightarrow$ Groupoid $\quad$ "Splitting of HN-fil'n"

an analogue of $\mathrm{Bun}_G^{\text{HN-split}}$ $\quad S \longmapsto (\mathcal{E}/X_S, \{(p_*\mathcal{E})^{\le\lambda} \subseteq p_*\mathcal{E}\}_{\lambda\in\mathbb{Q}})$

$\mathrm{Bun}_G(S)$ $\quad$ $\mathrm{Rep}_E(G) \longleftrightarrow \mathrm{Bun}_G(X)$.

where $\rho \in \mathrm{Rep}_E G$.

Let $(\rho_*\mathcal{E})^{\lambda} := \bigcup_{\lambda'<\lambda} (\rho_*\mathcal{E})^{\leq\lambda'}$, $(\rho_*\mathcal{E})^{\lambda} = (\rho_*\mathcal{E})^{\leq\lambda}/(\rho_*\mathcal{E})^{<\lambda}$,

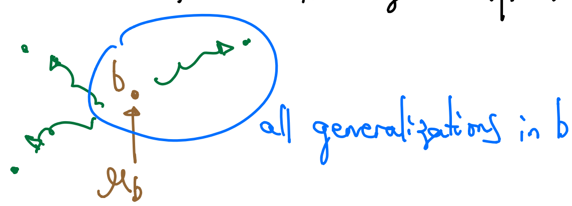
stable ob of slope λ, ($\forall \lambda \in \mathbb{Q}$ & $\rho \in \mathrm{Rep}_E G$).

$$\leadsto \begin{tikzcd} \mathcal{M}_b \arrow[r, "q_b"] & {[*/\underline{G_b(E)}]} \\ \widetilde{\mathcal{M}}_b \arrow[u] \arrow[r] & * \arrow[u] \end{tikzcd}$$

$\widetilde{\mathcal{M}}_b : S \longmapsto (\mathcal{E}/X_S, \{((\rho_*\mathcal{E})^{\leq\lambda}, (\rho_*\mathcal{E})^{\lambda})\}_{\lambda\in\mathbb{Q}})$

trivialization $(\rho_*\mathcal{E})^{\lambda} \simeq (\mathcal{E}_b)^{\lambda}$, i.e. $\mathcal{E}^{\lambda} = (\mathcal{E}_b)^{\lambda}$.

Picture:



E.g. $G = GL_2$, $b = \mathcal{O} \oplus \mathcal{O}(1)$,

$\leadsto \mathcal{M}_b$ classifies $\langle 0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}' \to 0 \rangle \longleftrightarrow \mathrm{Bun}_{GL_2}$.

(deg 0) (deg 1)

in $\widetilde{\mathcal{M}}_b$ it is trivial: $\mathcal{L} \simeq \mathcal{O}$, $\mathcal{L}' \simeq \mathcal{O}(1)$

$\widetilde{\mathcal{M}}_b = \mathcal{BC}(\mathcal{O}(-1)[1])$ with ext'ns $\mathcal{O} \oplus \mathcal{O}(1)$, $\mathcal{O}(\frac{1}{2})$.

Prop (V.3.5) $b \in B(G)$, $q_b : \mathcal{M}_b \longrightarrow [*/\underline{G_b(E)}]$ partially proper,

rep'd in locally spatial diamonds, coh. sm of dim $\langle 2\rho, \nu_b \rangle$.

Also, $\widetilde{\mathcal{M}}_b \longrightarrow *$ succesive torsor under negative BC space.

(III.5.1) $\widetilde{G}_b \leadsto$ alg grp $H \supseteq H^{\geq 0}$. $H^{\geq 0}$ parabolic with fil'n.

Idea fil'n $\longleftrightarrow \mathfrak{P}$-structure, where $\mathfrak{P}$ parabolic

$\updownarrow$

$\mathfrak{p}^{\geq 0}$ opositive of $H^{\geq 0}$ used in III.5.1.

$\widetilde{\mathcal{M}}_b$ classifies set of $\mathfrak{P}$-torsors

$\longleftrightarrow$ "$H^1(X_S, \mathfrak{p}^{\leq 0})$", $\mathfrak{p}^{\leq 0}$ fil'd by obs

Thm (V.3.7) $\pi_b : \mathcal{M}_b \to \mathrm{Bun}_G$ partially proper, rep'd in locally spatial diamonds, coh sm of ℓ-dim $\langle 2\rho, \nu_b \rangle$.

Proof Consider $\pi_b : \mathcal{M}_b \to \mathrm{Bun}_G$, embed it into $Z = \mathcal{E}/P$ (flag var)

$$\begin{tikzcd} \mathcal{M}_b \arrow[r] & \mathrm{Bun}_G \\ \mathcal{M}_b \times_{\mathrm{Bun}_G} S \arrow[u] \arrow[r] \arrow[ur, phantom, "\lrcorner"] & S \arrow[u] \end{tikzcd}$$

$b \longleftrightarrow$ positive $\mathbb{Q}$-fil'ns on $\mathcal{E}$ with associated grading.

$\rightsquigarrow$ $(\mathcal{E}, (\rho_* \mathcal{E})^{\leq \lambda}) \longleftrightarrow$ a P-structure

- Moduli problem $\mathcal{M}_b \times_{\mathrm{Bun}_G} S$ lies in $\mathcal{M}_b^{sm}$
- The pull back tan bundles of $Z = \mathcal{E}/P$ will have positive slopes.

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}' \to 0$$
$$\mathcal{L} \cong \mathcal{O}, \quad \mathcal{L}' \cong \mathcal{O}(1)$$

tangent bundle $\longleftrightarrow$ $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(1))$.

$\longleftrightarrow$ $\mathrm{Hom}(\mathcal{E}, \mathcal{E}/\mathcal{E}_1)$ $\quad$ $(\mathrm{Gr}: 0 \subseteq \mathcal{E}_1 \subseteq \mathcal{E}.)$ $\quad \square$

a machine from lower slope to higher slope.

Prop (V.3.6)