Projective Morphisms (II)

Wenhan Dai

Eisenbud-Harris is particularly recommended in this section.

§1 Relative Proj

X sch. $\mathcal{S} = \bigoplus_{n=0}^{\infty} \mathcal{S}_n$ graded qcoh $\mathcal{O}_X$-alg.

$\forall U \in \underline{X}$, $\operatorname{Proj}(\mathcal{S}(U)) \longrightarrow U$

$\rightsquigarrow$ glue to give $\underbrace{\operatorname{Proj}\mathcal{S}}_{\text{relative proj of } \mathcal{S}} \longrightarrow X$.

Assume $\mathcal{S}_1$ f.g. (local notion)

& $\mathcal{S} = \mathcal{S}_0[\mathcal{S}_1]$ (loc) gen'd as $\mathcal{S}_0$-alg.

Also assume: $\mathcal{S}_0 = \mathcal{O}_X/?$ quotient.

(Hartshorne: assume in fact $\mathcal{S}_0 = \mathcal{O}_X$).

Pick open affine U & $\mathcal{O}_U^{\oplus(m+1)} \twoheadrightarrow \mathcal{S}_1|_U$

$\rightsquigarrow$ $\operatorname{Sym}\mathcal{O}_U^{\oplus(m+1)} \twoheadrightarrow \mathcal{S}|_U$.

$\operatorname{Sym}\mathcal{O}_U^{\oplus(m+1)} = \bigoplus_{n=0}^{\infty} \operatorname{Sym}^n \mathcal{O}_U^{\oplus(m+1)}$.

$\rightsquigarrow$ $\operatorname{Proj}\mathcal{S} \to \underbrace{\operatorname{Proj}\operatorname{Sym}\mathcal{O}_U^{\oplus(m+1)}}_{=\mathbb{P}^m_U} \Rightarrow \operatorname{Proj}\mathcal{S} \to X$ loc proj.

Def'n $f: Y \to X$ proj iff it occurs as a relative Proj.

i.e. $Y = \operatorname{Proj}\mathcal{S}$ for some $\mathcal{S}$.

note: globally proj $\Rightarrow$ this proj $\Rightarrow$ loc proj

but no coincidence.

Following EGA: $\mathbb{P}(\mathcal{F}) = \operatorname{Proj}\operatorname{Sym}\mathcal{F}$, $\mathcal{F} \in \operatorname{Qcoh}(\operatorname{Mod}_{\mathcal{O}_X})$ f.g.

§2 Very Ample Sheaves

<u>Def'n</u> An immersion $f : Y \to X$ is

on Top : $Y \approx \underset{\text{closed}}{V} \subseteq \underset{\text{open}}{U} \subseteq X$ (V loc closed)

s.t. $\forall y \in Y$, $y \mapsto x$, $f^{\#} : \mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{Y,y}$ surj.

<u>Note</u> (1) Any composition of closed imm & open imm is an imm

(2) Any imm factors as (1) : open $\circ$ closed.

<u>Def'n</u> $f : Y \to X$, $\mathcal{F} \in \mathrm{Qcoh}(Y)$ very ample rel to f

if $\exists\, Y \xrightarrow{g} \mathbb{P}(S_1)$ <u>imm</u> for $S_1 \in \mathrm{Qcoh}(\mathrm{Mod}_{\mathcal{O}_X})$ f.g.

under which $\mathcal{F} \cong g^* \mathcal{O}(1)$.

<u>Philosophy</u> Saying "$\mathcal{F}$ very ample rel $f : Y \to X$"

= saying all rel information of Y is recorded in $\mathcal{F}$

namely, one can recover Y from X via $\mathcal{F}$.

<u>Rmk</u> Unlike projectivity, this notion is indeed <u>local</u> on the base.

<u>Lemma</u> $f : Y \to X$ proj. $\Leftrightarrow$ f proper & $\exists\, \mathcal{F} \in \mathrm{Qcoh}(Y)$ very ample rel to f.

Proof. See Hartshorne Rmk II.5.16.1.

<u>Note</u> $\mathcal{F} = j^* \mathcal{O}(1) \overset{\text{retrive}}{\rightsquigarrow} Y \to \mathbb{P}(S_1)$.

namely, S_1 glob f.g., $S_1 = (\bigoplus A s_i)$

$\downarrow$ pull back

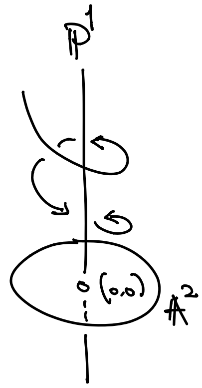
sections of $j^* \mathcal{O}(1)$.

morphism to $\mathbb{P}^n_X$. (see Hartshorne II.7.1).

§3 Blowups

A neat class of rel Proj:

$\mathcal{I}$ f.g. ideal $\in Sh(X)$, $Y = \mathbb{P}(\mathcal{I})$ = blowup of X along $\mathcal{I}$.



<u>Example</u> $X = \operatorname{Spec} k[x,y]$, k any field.

$\mathcal{I} = (x,y)$ ideal sheaf. $U = D(x) \cup D(y)$

$\rightsquigarrow \mathcal{I}|_U \cong \mathcal{O}_X|_U = \mathcal{O}_U$. $Y = \mathbb{P}(\mathcal{I})$.

$\rightsquigarrow Y \times_X U \xrightarrow{\cong} U$

But the fibre of $Y \to X$ over $(0,0)$ looks like $\mathbb{P}^1$ (with homo coord x, y).

<u>Note</u> The blowup $\mathbb{P}(\mathcal{I})$ carries <u>less</u> information than $\mathcal{I}$.

e.g. $\mathcal{I}$ loc principal, $\mathbb{P}(\mathcal{I}) = X$.

<u>Special Property</u> For $f: Y \to X$ & $\mathcal{I} \in Sh_{ideal}(X)$

may compose $\mathcal{I} \to \mathcal{O}_X$ with $f^\#: \mathcal{O}_X \to f_* \mathcal{O}_Y$

$\rightsquigarrow \underbrace{f^*\mathcal{I} \to \mathcal{O}_Y}$ by performing adjunction.

im = ideal sheaf on Y

"inverse image ideal sheaf of $\mathcal{I}$ under f".

<u>Thm</u> $f: Y \to X$ blowup def'd by f.g. $\mathcal{I}$ on X.

$\Rightarrow f^{-1}\mathcal{I} = \operatorname{im}(f^*\mathcal{I} \to \mathcal{O}_Y)$ $\underbrace{\text{loc principal}}$

($\Rightarrow$ has trivial blowups)

Proof. $Y = \operatorname{Proj} S$, $S_n = \operatorname{Sym}^n \mathcal{I}$

$\rightsquigarrow f^{-1}\mathcal{I} = \mathcal{O}_Y(1)$ loc free. $\square$

Fact (Univ property of blowups)

$f: Y \to X$ blowup. $\forall\, Z \to X$, s.t. $g^{-1}\mathcal{I}$ loc principal on Z:

$$\begin{tikzcd} g^{-1}\mathcal{I} \quad Z \arrow[r, dashed, "\exists!"] \arrow[dr, "g"'] & Y \quad f^{-1}\mathcal{I} \arrow[d, "f"] \\ & X \quad \mathcal{I} \end{tikzcd}$$

(Hartshorne Prop II.7.14).

More concrete description of the standard example:

blowup of $\operatorname{Spec} k[x,y]$ at (x,y):

$Y = \mathbb{P}(\mathcal{I})$ covered by $\operatorname{Spec} k[x, \frac{y}{x}]$ & $\operatorname{Spec} k[y, \frac{x}{y}]$

glued along $\operatorname{Spec} k[x, y, \frac{y}{x}, \frac{x}{y}]$.

In fact: any blowup can be described analogously:

$\mathbb{P}(\mathcal{I}) \to \operatorname{Spec} A$, $\mathcal{I} \mapsto I = (r_0, \cdots, r_m)$

covered by $m+1$ charts. a typical one of which is

$\operatorname{Spec} A[t_1, \cdots, t_m]/(t_1 r_0 - r_1, \cdots, t_m r_0 - r_m)$.

Upshot $f^{-1}\mathcal{I}$ is supposed to be loc principal.

$\rightsquigarrow$ must force $\underbrace{r_i/r_j}$

different choices $\rightsquigarrow$ different charts.

Blowup = special e.g. of modification

$f: Y \twoheadrightarrow X$ proper, surj, birational

(i.e. $f|_U$ isom, $\forall$ open dense $U \subseteq Y$).

(Hartshorne Thm II.7.17: under certain circumstance, every modif can be a blowup.)

<u>Catch</u> $\mathcal{I}$ is not unique, e.g. $\mathbb{P}(x,y) \to \operatorname{Spec} k[x,y]$
$\mathbb{P}(x^2, xy, y^2) \to \operatorname{Spec} k[x,y]$
defines the same blowup.

<u>§4 Chow's Lemma</u>

Modification: good at proper $\underset{\text{into}}{\overset{\text{turn}}{\rightsquigarrow}}$ proj sch / some base.

<u>Thm</u> (Chow's Lemma)

$f: X \to S$ finite type.

Assume: Either S noe

or S q-cpt & X fin. many irred components.

$\Rightarrow \exists$ quasi-proj X'/S & $f: X' \to X$ proj surj

s.t. $f|_U$ isom & $f^{-1}(U) \cong U$ dense in X'.

Moreover, X red/irred/int $\Rightarrow$ so is X'.

Reference: EGA 2, Lemma 5.6.1 (weaker result).

Hartshorne Ex II.4.10.