Higgs bundles in arithmetic geometry

Jinbang Yang

(Joint with Kang Zuo)

Let $X/\mathbb{C}$ sm proj var.

$\rightsquigarrow$ · dR bdl (V,∇), $\nabla: V \to V \otimes \Omega^1_X$ s.t. $\nabla \cdot \nabla = 0$

· Higgs bdl (E,θ), $\theta: E \to E \otimes \Omega^1_X$ s.t. $\theta \circ \theta = 0$

$\mathcal{O}_X$-linear

<u>Thm</u> $\{(V,\nabla) \text{ on } X\} \overset{RH}{\simeq} \{\mathbb{L} \text{ loc sys on } X\}$

$\overset{\text{Simpson}}{\simeq} \{(E,\theta) \text{ on } X, \text{ semistable w/ triv Chern classes}\}$

i.e. $c_i(E) = 0$, $\forall i$.

<u>log version</u> $\nabla: V \longrightarrow V \otimes \Omega^1_X(\log D)$

$\theta: E \longrightarrow E \otimes \Omega^1_X(\log D)$.

<u>Parabolic ver</u> Replace V, E w/ parab bdls.

<u>Example</u> $f: Y \to X$ family of sm proj var

$\rightsquigarrow$ $\mathbb{L}_{Y/X} = R^n_B f_* \mathbb{C}_Y$ loc sys $/X$

$V_{Y/X} = R^n_{dR} f_* (\Omega^\bullet_{Y/X}, d_{Y/X})$.

$\nabla_{Y/X}$ Gauß-Manin conn.

$Fil_{Y/X}$ Hodge filtration.

Taking gradings

$\rightsquigarrow$ $(E_{Y/X}, \theta_{Y/X}) := Gr(V_{Y/X}, \nabla_{Y/X}, Fil_{Y/X})$.

<u>Def</u> Subquotients of $\mathbb{L}_{Y/X}$ / $(V_{Y/X}, \nabla_{Y/X})$ / $(E_{Y/X}, \theta_{Y/X})$ are called motivic or geometric objects.

Question How to determine the motivicity of a given obj ?

Rmk (1) $/\mathbb{C}$, $\mathbb{L} \triangleq (V,\nabla) \triangleq (E,\theta)$

· $\mathbb{L}$ motivic $\Leftrightarrow$ (V,∇) motivic $\Leftrightarrow$ (E,θ) motivic.

(2) rk 1. motivic $\Leftrightarrow$ torsion.

Motivic local systems & motivic de Rham bundle

Conj (Simpson) Rigid loc sys must be motivic.

Known cases (1) $X = \mathbb{P}^1$, $D = \{x_1, \cdots, x_n\}$,

$\mathbb{L}/(X \setminus D)$ rigid $\Rightarrow$ motivic.

(Katz : "middle convolution").

(2) rigid , rk 2 (Corlette - Simpson '08).

(3) rk 3 , Coh rigid (Langer - Simpson '16, Gröchnig - Esnault '19)

· Coh rigid $\Rightarrow$ integral ($\mathbb{L}/\mathcal{O}_K$).

· integral rigid loc sys of rk 3 $\Rightarrow$ motivic.

Conj (Grothendieck - Katz) $(V,\nabla)/X$, X/K = # field

(V,∇) has finite monodromy

$\Leftrightarrow$ $\psi_\mathfrak{p} = 0$, for "almost all" $\mathfrak{p}$

(p-curvature of $(V,\nabla) \otimes_{\mathcal{O}_K[\frac{1}{N}]} k_\mathfrak{p}$

Conj (André 1989) $(V,\nabla)/X/K$,

(V,∇) motivic $\Leftrightarrow$ $\psi_\mathfrak{p}$ is nilpotent for almost all $\mathfrak{p}$.

Motivic Higgs bundles

A baby example: $X = \mathbb{P}^1$, $D_\lambda = \{0, 1, \lambda, \infty\}$

$\rightsquigarrow \quad \mathcal{O}(1) \xrightarrow[\cong]{\theta} \mathcal{O}_{\mathbb{P}^1}(1) \otimes \Omega^1_X(\log D_\lambda)$.

$$E_\lambda^{univ} := \mathcal{O}(1) \oplus \mathcal{O}(-1)$$

$$\theta_\lambda^{univ} := \begin{pmatrix} 0 & \theta \\ 0 & 0 \end{pmatrix} : E_\lambda \longrightarrow E_\lambda \otimes \Omega^1_{\mathbb{P}^1}(\log D_\lambda).$$

$\overset{\text{Simpson}}{\rightsquigarrow} \quad \mathbb{H}_\lambda^{univ} / (\mathbb{P}^1 \setminus D_\lambda)$.

<u>Thm</u> $(E_\lambda^{univ}, \theta_\lambda^{univ})$ motivic $\iff \lambda \sim \zeta_6, -1, -8, \left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right)^5$

$\iff \mathbb{P}^1 \setminus D_\lambda$ is a modular curve $\cong X(\Gamma)$, $\Gamma \subset SL_2(\mathbb{Z})$.

Known by Viehweg - Zuo, Beauville.

<u>Note</u> $D_\lambda \sim D_{\lambda'} \Rightarrow \lambda \sim \lambda'$

Have an analogue of André Conj for Higgs bdls.

- Fontaine - Faltings mod
- Higgs - de Rham flow

Let $X / W(k)$ sm proj, $D \subset X$ relative NC divisor.

<u>Def</u> An FF module tuple $(V, \nabla, Fil, \varphi) / X$

- $\tilde{V} := \sum_i \frac{Fil^i V}{p^i} \subseteq V \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.
- $\tilde{\nabla} := p\nabla|_{\tilde{V}}$.
- $\Phi_X^*(\tilde{V}, \tilde{\nabla})$ de Rham bdl.
- $\varphi : \Phi_X^*(\tilde{V}, \tilde{\nabla}) \xrightarrow{\cong} (V, \nabla)$.

<u>Note</u>
$$\begin{tikzcd} (V, \nabla, Fil) \arrow[dr, "\widetilde{(\cdot)}"'] & & \Phi^*(\tilde{V}, \tilde{\nabla}) \arrow[ll, "\cong"', "\varphi"] \\ & (\tilde{V}, \tilde{\nabla}) \arrow[ur, "\Phi^*"'] & \end{tikzcd}$$

$\rightsquigarrow$ mod p graded Higgs bdl.

Def Higgs - de Rham flow $/X_k$ is a diagram

$$\begin{tikzcd}[column sep=small]
 & (\bar{V},\bar{\nabla})_1 + \bar{Fil}_1 \arrow[dr, "Gr"] & & \cdots \arrow[dr] & & \cdots \\
(\bar{E},\bar{\theta})_0 / X_k \arrow[ur, "C^{-1}", "\Phi^*"'] & & (\bar{E},\bar{\theta})_1 \arrow[ur, "C^{-1}"'] & & (\bar{E},\bar{\theta})_f \arrow[ur] &
\end{tikzcd}$$

Called f-periodic $\overset{\text{Def}}{\Longleftrightarrow}$ $\exists f$ s.t. $(\bar{E},\bar{\theta})_f \simeq (\bar{E},\bar{\theta})_0$.

Note f-period Higgs - de Rham flow $/X_W$ is like

$$\begin{tikzcd}[column sep=small]
 & (V,\nabla)_1 + Fil \arrow[dr, "Gr"] & & \cdots \arrow[dr] & \\
(E,\theta)_0 / X_W \arrow[ur, "C^{-1}"] & & \cdots \arrow[ur] & & (E,\theta)_f \arrow[llll, "\simeq"', "\varphi"]
\end{tikzcd}$$

From HDF to FF mod:

$$(V,\nabla,Fil) := \bigoplus_{i=1}^{f} (V,\nabla,Fil)_i$$

periodicity $\varphi \rightsquigarrow \varphi : \Phi^*(\tilde{V},\tilde{\nabla}) \xrightarrow{\sim} (V,\nabla)$.

f-periodic $\rightsquigarrow \iota : \mathbb{Z}_{p^f} \hookrightarrow \mathrm{End}((V,\nabla,Fil,\varphi))$.

Thm (Lan - Sheng - Zuo)

$$\{ f\text{-periodic HDF} \} \xrightarrow{\simeq} \left\{ (M,\iota) \;\middle|\; \begin{array}{l} M \text{ FF mod} \\ \iota : \mathbb{Z}_{p^f} \hookrightarrow \mathrm{End}(M) \end{array} \right\}.$$

Conj Let $(E,\theta)/X/K$, $K/\mathbb{Q}$. Then

(E,θ) motivic $\Longleftrightarrow$ $\exists f>0$ s.t. (E,θ) mod $\mathfrak{P}$ is f-periodic, $\forall\forall \mathfrak{P}$.

Example $X = \mathbb{P}^1$, $D_\lambda = \{0, 1, \lambda, \infty\}$ (λ fixed)

$$\mathcal{O} \xrightarrow[\neq 0]{\theta} \mathcal{O}(-1) \otimes \Omega^1_{\mathbb{P}^1}(\log D_\lambda).$$

$a := \mathrm{Supp}(\mathrm{coker}\,\theta) \in \mathbb{P}^1$

(*) $$E_\lambda = \mathcal{O} \oplus \mathcal{O}(-1),$$

$$\theta_{\lambda,a} := \begin{pmatrix} 0 & \theta \\ 0 & 0 \end{pmatrix} : E_\lambda \longrightarrow E_\lambda \otimes \Omega^1_{\mathbb{P}^1}(\log D_\lambda).$$

<u>Lem</u> $\mathcal{M}^*_{\mathrm{Hig}} \simeq \mathbb{P}^1$, $(E_\lambda, \theta_{\lambda a}) \mapsto a$.

<u>Thm</u> (Yang-Zuo '23) Conj holds for $(E_\lambda, \theta_{\lambda a})$

Also, $(E_\lambda, \theta_{\lambda a})$ motivic $\Longleftrightarrow$ $\pi^{-1}(a)$ torsion

$$\begin{array}{ccc} (x,y) & & C_\lambda : y^2 = x(x-1)(x-\lambda) \\ \big\downarrow & & \pi \big\downarrow 2:1 \\ x & & \mathbb{P}^1 \end{array}$$

<u>Idea of pf</u> Special case: λ s.t. C_λ mod p supersingular

$(E_\lambda, \theta_{\lambda a})$ s.t. $\pi^{-1}(a)$ torsion

$\Rightarrow$ $(E_\lambda, \theta_{\lambda a})$ mod p f-periodic (will take $f = [\mathbb{Q}(\zeta_\ell)^+ : \mathbb{Q}]$).

$\Rightarrow$ $(E_\lambda, \theta_{\lambda a})$ f-periodic $/ X_W$.

$\overset{\text{LSZ}}{\rightsquigarrow}$ FF module $((V, \nabla, \mathrm{Fil}, \varphi), 2)$

$\overset{\text{forget Fil}, \otimes \mathbb{Q}_p}{\rightsquigarrow}$ convergent F-isocrystal.

$\overset{p\text{-}\ell \text{ comparison}}{\rightsquigarrow}$ ℓ-adic loc sys of rk 2.

$\overset{\text{Drinfeld theory}}{\rightsquigarrow}$ family of ab var $/ X_k$

$\overset{\text{log ver of GMK}}{\rightsquigarrow}$ family $/ X_W$

(GMK = Grothendieck - Messing - Kato).

lifting family of AVs $\Longleftrightarrow$ lifting Hodge fil'n.