Sparsity of rational and algebraic points

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Motivation

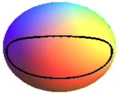
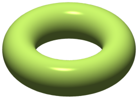
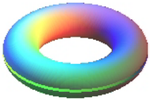
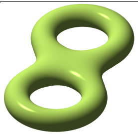
It is a fundamental question in mathematics to solve equations.

For example:
$f(X, Y)$= polynomial in X and Y with coefficients in $\mathbb{Q}$.
What can we say about the $\mathbb{Q}$-solutions to $f(X, Y) = 0$?

$\rightsquigarrow$ Diophantine problem

Motivation

Some examples:

$f(X,Y)$	X^2+Y^2-1	Y^2-X^3-X	Y^2-X^3-2	$Y^2-X^6-X^2-1$
$\mathbb{Q}$-solutions	$(3/5,4/5)$, $(5/13,12/13)$, $(8/17,15/17)$, *etc.* infinitely many	$(0,0)$, $(\pm 1,0)$. finitely many	$(-1,1)$, $(34/8,71/8)$, $(2667/9261,13175/9261)$, *etc.* infinitely many	$(0,\pm 1)$, $(\pm 1/2,\pm 9/8)$. finitely many
Associated Algebraic Curve				
genus of the associated curve	0	1	1	2

Setup

In what follows,

- $g \geq 0$ and $d \geq 1$ integers;
- K= number field of degree d;
- $C =$ irreducible smooth projective curve of genus g defined over K.

As usual, we use $C(K)$ to denote the set of K-points on C.

C is the set of zeros of some polynomials; C is defined over $K \leftrightsquigarrow$ these polynomials have coefficients in K; $C(K) :=$ the set of K-solutions.

Genus 0 and 1

- ✎ If $g = 0$, then either $C(K) = \emptyset$ or $C \cong \mathbb{P}^1$ over K.
- ✎ If $g = 1$ and $C(K) \neq \emptyset$, then $C(K)$ has a structure of abelian groups with an identity element $O \in C(K)$. $\rightsquigarrow$ Elliptic curve $E/K := (C, O)$.

Theorem (Mordell–Weil)

$E(K)$ is a finitely generated abelian group. Namely,

$$E(K) \cong \mathbb{Z}^{\rho} \oplus E(K)_{\mathrm{tor}}$$

with $\rho < \infty$ and $E(K)_{\mathrm{tor}}$ finite.

Genus 1: infinite part

- In general, no effective method to calculate ρ.

Conjecture (Birch and Swinnerton–Dyer)

$\rho = \mathrm{ord}_{s=1} L(E, s)$.

Coates–Wiles, Gross–Zagier, Kolyvagin, Rubin, Breuil–Conrad–Diamond–Taylor, Darmon, Bhargava–Shankar, Nevokar, Dokchitser–Dokchitser, Skinner–Urban...

- Upper bound:
  - Ooe–Top '89: $\rho \leq c_1 \log |N_{K/\mathbb{Q}}\mathcal{N}_{E/K}| + c_2$, where $\mathcal{N}_{E/K}$ is the conductor of E, and c_1 and c_2 depend on K in an explicit way.
  - Is ρ bounded for fixed K? Divergent opinions, already for $K = \mathbb{Q}$!
    * Park–Poonen–Voight–Wood ('19): heuristic which suggests that $\rho \leq 21$ except for at most finitely many $E/\mathbb{Q}$.
    * Elkies (2006): $E/\mathbb{Q}$ with $\rho \geq 28$ ($=$ under GRH).

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Genus 1: finite part

Theorem (Mazur '77 for $K = \mathbb{Q}$, Merel '96)

$\#E(K)_{\mathrm{tor}}$ is uniformly bounded above in terms of $[K : \mathbb{Q}]$.

Mazur proved this result by establishing the following theorem:

Theorem (Mazur '77)

If $N = 11$ or $N \geq 13$, then the only $\mathbb{Q}$-points of the modular curve $X_1(N)$ are the rational cusps.

The genus of $X_1(N)$ is ≥ 2 if $N = 13$ or $N \geq 16$.

$\rightsquigarrow$ results of rational points on curves of genus ≥ 2.

Genus ≥ 2: Mordell Conjecture

Mordell made the following conjecture about 100 years ago (1922), known as the Mordell Conjecture. It became a theorem in 1983, proved by Faltings.

Theorem (Faltings '83; known as Mordell Conjecture)

If $g \geq 2$, then the set $C(K)$ is finite.

Feature of this theorem	When applied to Mazur's result on $X_1(N)$
➢ weak topological hypothesis, very strong arithmetic conclusion!	✎ $X_1(N)$ has only finitely many $\mathbb{Q}$-points if $N \geq 16$.
➢ no constructive yet.	✎ $X_1(N)(\mathbb{Q})$ cannot be determined by Faltings's Theorem.

Genus ≥ 2: Fermat's Last Theorem

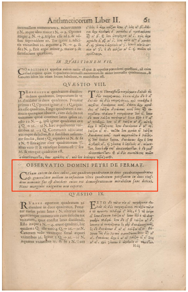
Fix $n \geq 4$ integer.

$$F_n : X^n + Y^n - 1 = 0.$$

Then $g(F_n) \geq 2$.

$$\Big\Downarrow \text{Faltings}$$

$\exists$ only finitely many $(x, y) \in \mathbb{Q}^2$ with $x^n + y^n = 1$.


Arithmeticorum Liber II. 61

QVÆSTIO VIII.

OBSERVATIO DOMINI PETRI DE FERMAT.

QVÆSTIO IX.

For this example, more is expected.

Theorem (Wiles, Taylor–Wiles, '95; known as Fermat's Last Theorem)

If x and y are rational numbers such that $x^n + y^n = 1$, then $(x, y) = (0, \pm 1)$ or $(x, y) = (\pm 1, 0)$.

Of course if n is furthermore assumed to be odd, then -1 cannot be attained.

Genus ≥ 2

From now on, we always assume that $g \geq 2$.
The example of Fermat's Last Theorem suggests that it can be extremely hard to compute $C(\mathbb{Q})$ for an arbitrary C!
Instead, here is a more achievable but still fundamental question.

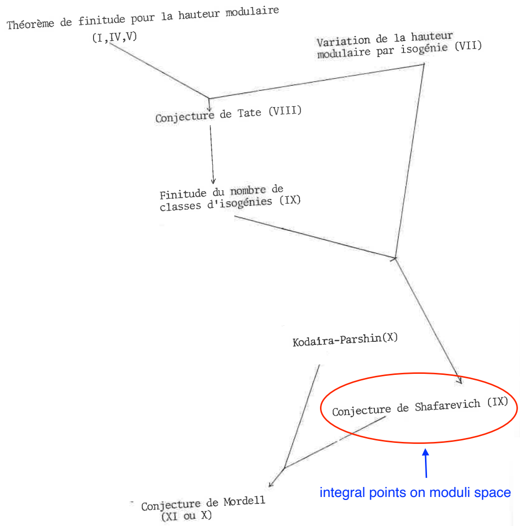
Question (Mordell, Weil, Manin, Mumford, Faltings, *etc.*)

Is there an "easy" upper bound for $\#C(K)$? How does $C(K)$ "distribute"?

Different grades of the question:

- Finiteness of $C(K)$
- Upper bound of $\#C(K)$
- Uniformity of bounds of $\#C(K)$
- Effective Mordell

Genus ≥ 2: Faltings's proof of the Mordell Conjecture



Extracted from « Séminaire sur les pinceaux arithmétiques, La conjecture de Mordell » (Astérisque 127), Lucien Szpiro.

Genus ≥ 2: a new proof by Vojta

In early 90s, Vojta gave a second proof to Faltings's Theorem with Diophantine method.

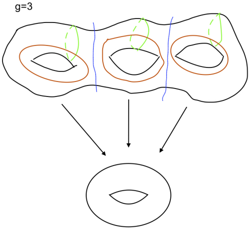
- In this proof, one sees some descriptions of distribution of algebraic points on C. They lead to an upper bound on $\#C(K)$.
- The proof was simplified by Bombieri. And generalized by Faltings to some high dimensional cases.

Abel-Jacobi embedding (since Abel 1823)

Fix any $P_0 \in C(K)$. There exists an embedding, called the *Abel-Jacobi embedding via P_0*,

$$\iota \colon C \to J$$

where $J \cong \mathbb{C}^g/\mathbb{Z}^{2g}$ is a complex torus of dimension g.



- Why want this embedding?
  (i) $C(K) \subseteq J(K)$;
  (ii) $J(K)$ is simpler than $C(K)$ because $J(K)$ has the structure of abelian group (of finite rank by Mordell-Weil) while $C(K)$ is only a set!

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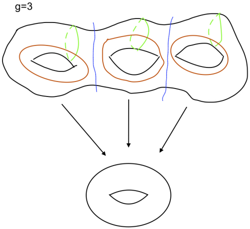

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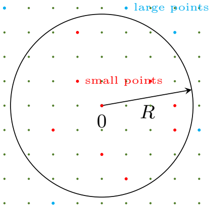
Vojta's proof of the Mordell Conjecture: Setup



Normalized height function $\hat{h}\colon J(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$ vanishing precisely on $J(\overline{\mathbb{Q}})_{\mathrm{tor}}$.

$\rightsquigarrow$ $\hat{h}\colon J(K) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}_{\geq 0}$ quadratic, positive definite.

$\rightsquigarrow$ Normed Euclidean space $(J(K) \otimes_{\mathbb{Z}} \mathbb{R}, |\cdot| := \hat{h}^{1/2})$, with $J(K)$ a lattice.

$\rightsquigarrow$ Inner product $\langle \cdot, \cdot \rangle$ on $J(K) \otimes_{\mathbb{Z}} \mathbb{R}$, and the angle of each two points in $J(K) \otimes_{\mathbb{Z}} \mathbb{R}$.

Vojta's proof of Mordell Conjecture: Mumford's work

A starting point is the following (consequence of) Mumford's Formula: For $P, Q \in C(\overline{\mathbb{Q}})$ with $P \neq Q$, we have

$$\frac{1}{g}\left(|P|^2 + |Q|^2 - 2g\langle P, Q\rangle\right) + O(|P| + |Q| + 1) \geq 0$$

As $g \geq 2$, the leading term is an indefinite quadratic form, which a priori could take any value. This gives a strong constraint on the pair (P, Q)!
$\rightsquigarrow$ Algebraic points are "sparse" in C!

Vojta's proof of Mordell Conjecture: Both inequalities

Theorem

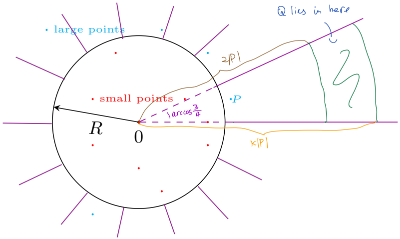
There exist $R = R(C)$ and $\kappa = \kappa(g)$ satisfying the following property. If two distinct points $P, Q \in C(\overline{\mathbb{Q}})$ satisfy $|Q| \geq |P| \geq R$ and

$$\langle P, Q\rangle \geq (3/4)|P||Q|,$$

then

- *(Mumford, '65)* $|Q| \geq 2|P|$;
- *(Vojta, '91)* $|Q| \leq \kappa|P|$.

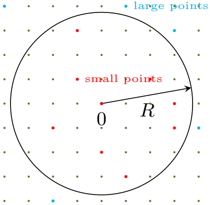
This finishes the proof of the Mordell Conjecture, with #large points$\leq (\log_2 \kappa + 1)7^{\mathrm{rk} J(K)}$.



If $P_1, \ldots, P_n$ are in the cone where P lies, then $\kappa|P| \geq |P_n| \geq 2|P_{n-1}| \geq \cdots \geq 2^n|P|$.
So in each cone there are $\leq \log_2 \kappa + 1$ large points!

$7^{\mathrm{rk} J(K)}$ such cones, according to the angle condition.

Genus ≥ 2: Classical bound



Attached to the curve C, there is a “canonically defined” number $h_{\mathrm{Fal}}(C)$ which measures the “complexity” of the coefficients of the equations defining the curve C.

Below, by abuse of notation use it for $\max\{1, h_{\mathrm{Fal}}(C)\}$.

Theorem (Bombieri ’91, de Diego ’97, Alpoge 2018)

- *One can take $R \sim h_{\mathrm{Fal}}(C)^{1/2}$.*
- *$\#$large points $\leq c(g)1.872^{\mathrm{rk}_{\mathbb{Z}} J(K)}$.* $\rightsquigarrow$ *A nice bound for #large points!*

For a bound of $\#C(K)$, we have:

Theorem (David–Philippon, Rémond 2000)

$$\#C(K) \leq c(g, [K:\mathbb{Q}], h_{\mathrm{Fal}}(C))^{1+\mathrm{rk}_{\mathbb{Z}} J(K)}.$$

Genus ≥ 2

Different grades of the question:

- ➢ Finiteness of $C(K)$ ✓
- ➢ Upper bound of $\#C(K)$ ✓
- ➢ Uniformity of bounds of $\#C(K)$
- ➢ Effective Mordell

Sparsity of algebraic points:

"sparsity" of large points

- ➢ Mumford's Inequality '65
- ➢ Vojta's Inequality '91
- ➢ ?✎
- ➢ ???

And about the distribution / sparsity of points:

✎ Are there other descriptions of the "sparsity" of algebraic points on C? Or at least can we say something about "small" points?

Genus ≥ 2: Towards uniform bounds on $\#C(K)$

The cardinality $\#C(K)$ must depend on g.

Example

The hyperelliptic curve defined by

$$y^2 = x(x-1)\cdots(x-2023)$$

has genus 1012 *and has at least* 2025 *different rational points.*

The cardinality $\#C(K)$ must depend on $[K:\mathbb{Q}]$.

Example

The hyperelliptic curve

$$y^2 = x^6 - 1$$

has points $(1,0)$, $(2, \pm\sqrt{63})$, $(3, \pm\sqrt{728})$, *etc.*

Genus ≥ 2: Towards uniform bounds on $\#C(K)$

Here is a very ambitious bound.

Question

Is it possible to find a number $B(g, [K : \mathbb{Q}]) > 0$ such that

$$\#C(K) \leq B?$$

This question has an affirmative answer if one assumes a widely open conjecture of Bombieri–Lang on rational points on varieties of general type (Caporaso–Harris–Mazur, Pacelli, '97).

✎ Two divergent opinions towards this conditional result: either this ambitious bound is true, or one could use this to disprove this conjecture of Bombieri–Lang.

Genus ≥ 2: Mazur's Conjecture B

Theorem (Dimitrov-G'-Habegger, 2021)

If $g \geq 2$, then

$$\#C(K) \leq c(g, [K:\mathbb{Q}])^{1+\mathrm{rk}_{\mathbb{Z}} J(K)}$$

where J is the Jacobian of J. Moreover, $c(g, [K:\mathbb{Q}])$ grows at most polynomially in $[K:\mathbb{Q}]$.

- This proves Mazur's Conjecture B ('86, 2000).
- Compared to the classical result, the *height of C* is no longer involved in the bound.
- We showed that the constant c does not depend on $[K:\mathbb{Q}]$ assuming the relative Bogomolov conjecture. Kühne (2021) removed this dependence on $[K:\mathbb{Q}]$ unconditionally.

✎ If you believe that the Mordell–Weil rank $\mathrm{rk}_{\mathbb{Z}} J(K)$ is bounded for fixed g and K, then the ambitious bound on last page is true. If you do not believe the ambitious bound on the last page, then the Mordell–Weil rank is unbounded.

Previously known results on Mazur's Conjecture B

- ✎ By Diophantine method, based on Vojta's proof,
  - ➢ David–Philippon 2007: when $J \subseteq E^n$.
  - ➢ David–Nakamaye–Philippon 2007: for some particular families of curves.
  - ➢ Alpoge 2018: average number of $\#C(\mathbb{Q})$ when $g = 2$.
- ✎ By the Chabauty–Coleman method,
  - ➢ Stoll 2015: hyperelliptic curves when $\mathrm{rk}J(K) \leq g - 3$.
  - ➢ Katz–Rabinoff–Zureick-Brown 2016: when $\mathrm{rk}J(K) \leq g - 3$.

Example of a 1-parameter family

Example (DGH, 2019)

Let $s \geq 5$ be an integer and let C_s be the genus 2 hyperelliptic curve defined by

$$C_s : y^2 = x(x-1)(x-2)(x-3)(x-4)(x-s).$$

Then

$$\begin{aligned}\mathrm{rk}(J_s)(\mathbb{Q}) &\leq 2g\#\{p : \ p = 2 \text{ or } C_s \text{ has bad reduction at } p\} \\ &\leq 2g\#\{p : p \mid 2 \cdot 3 \cdot 5 \cdot s(s-1)(s-2)(s-3)(s-4)\} \\ &\ll_g \frac{\log s}{\log \log s}.\end{aligned}$$

This yields, for any $\epsilon > 0$,

$$\#C_s(\mathbb{Q}) \ll_\epsilon s^\epsilon.$$

Genus ≥ 2: New Gap Principle

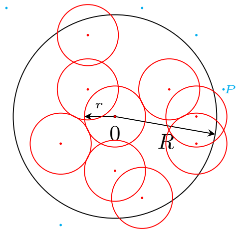
Our new contribution is a New Gap Principle.

Theorem (New Gap Principle, Dimitrov–G'–Habegger + Kühne, 2021)

Assume $g \geq 2$. Each $P \in C(\overline{\mathbb{Q}})$ satisfies

$$\#\{Q \in C(\overline{\mathbb{Q}}) : \hat{h}_L(Q-P) \leq c_1 h_{\mathrm{Fal}}(C)\} \leq c_2$$

for some positive constants c_1 and c_2 depending only on g.



$R^2 = c_0(g) h_{\mathrm{Fal}}(C)$
$r^2 = c_1(g) h_{\mathrm{Fal}}(C)$
$\#$ small balls to cover all small points $\leq (R/r)^{\mathrm{rk} J(K)}$
$\#$ of points in each ball $\leq c_2$

- The Bogomolov Conjecture, proved by Ullmo and S.Zhang ('98), gives this result with c_1 and c_2 depending on C (but don't know how).
- The New Gap Principle is another phenomenon of the "sparsity" of algebraic points in C of genus ≥ 2. It says that algebraic points in $C(\overline{\mathbb{Q}})$ are in general far from each other in a quantitative way.
- It implies that #small rational points $\leq c'(g)^{1+\mathrm{rk} J(K)}$ by a simple packing argument.

Genus ≥ 2

Different grades of the question:

- Finiteness of $C(K)$ ✓
- Upper bound of $\#C(K)$ ✓
- Uniformity of bounds of $\#C(K)$
  ✓"subject" to the Mordell–Weil rank
- Effective Mordell

Sparsity of algebraic points:

- Mumford's Inequality -'65
- Vojta's Inequality -'91
- New Gap Principle -2021 (Dimitrov–G'–Habegger + Kühne)
- ???✎

And:

- ✎ Mumford's and Vojta's Inequalities to describe that large algebraic points are "sparse" in C.
- ✎ New Gap Principle gives another description on how all algebraic points are "sparse" in C.
- ✎ Effective Mordell is a conjectural statement which describes where to find the rational points ("no large rational points").

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✎ Effective Mordell is a conjectural statement which describes where to find the rational points ("no large rational points").

Genus ≥ 2: Effective Mordell

Conjecture (Effective Mordell, made by Szpiro)

There exists an effectively computable $c = c(g, [K:\mathbb{Q}], \mathrm{disc}(K/\mathbb{Q})) > 0$ such that $\hat{h}(P) \leq c h_{\mathrm{Fal}}(C)$ for all C/K and $P \in C(K)$.

- Effective Mordell tells us where to find all the rational points on C ("no large rational points")!
- Little is known about Effective Mordell.
- Checcoli, Veneziano, and Viada proved results in this direction when $C \subsetneq E^n$ for some elliptic curve E with $\mathrm{rk}E(K) < n$ (modification if E has CM) and C is *transverse*, following the method of Manin–Demjanenko.

✎ Another approach to compute $C(K)$ is the Chabauty–Coleman–Kim method, by obtaining sharp bounds on $\#C(K)$. Currently:

- Chabauty–Coleman: $K = \mathbb{Q}$ and $\mathrm{rk}J(\mathbb{Q}) < g$.
- Quadratic Chabauty: $K = \mathbb{Q}$ and $\mathrm{rk}J(\mathbb{Q}) = g$, in various publications of Jennifer Balakrishnan in collaboration with Besser, Müller, Dogra *et al*.

Genus ≥ 2: Algebraic torsion points, finiteness

$\exists$ a natural way to divide $C(\overline{\mathbb{Q}})$ into equivalence classes, called torsion packets.

For any $P \in C(\overline{\mathbb{Q}})$, the torsion packet $\mathrm{TP}(C, P)$ containing P is the set of $Q \in C(\overline{\mathbb{Q}})$ such that the divisor $m[P] - m[Q]$ is a principle divisor on C for some positive $m \in \mathbb{Z}$.

It is known that for the Abel–Jacobi map $j_P \colon C \to J$ based at P,

$$\mathrm{TP}(C, P) = j_P(C)(\overline{\mathbb{Q}}) \cap J(\overline{\mathbb{Q}})_{\mathrm{tor}}.$$

Theorem (Raynaud '83, known as the Manin-Mumford Conjecture)

If $g \geq 2$, then each torsion packet is a finite set.

Theorem (Baker–Poonen, 2001)

There are at most finitely many $P \in C(\overline{\mathbb{Q}})$ such that $\#\mathrm{TP}(C, P) > 2$.
In other words, there exists a constant $B(C) > 0$ such that: the cardinality of $\{P \in C(\overline{\mathbb{Q}}) : \#\mathrm{TP}(C, P) > 2\}$ is at most $B(C)$.

Genus ≥ 2: Algebraic torsion points, uniformity

A step of proving the New Gap Principle introduced before is the following Uniform Manin–Mumford Conjecture.

Theorem (Kühne 2021, Uniform Manin–Mumford)

There exists a constant $c'(g)$ such that

$$\#\mathrm{TP}(C, P) \leq c'(g).$$

- DeMarco–Krieger–Ye 2020: $g = 2$ bielliptic, using Arithmetic Dynamics.
- Kühne's proof uses his equidistribution theorem, in addition to techniques & results from Dimitrov–G'–Habegger 2021.
- Yuan 2022: New proof using adelic line bundles over quasi-projective varieties of Yuan–Zhang 2021.
- G'–Habegger 2023: New proof which does not use equidistribution (stronger result called Relative Manin–Mumford). Pila–Zannier method, and more careful study of degeneracy loci compared to DGH.

Genus ≥ 2: Algebraic torsion points, questions

- Is it possible to give an explicit formula for $c'(g)$?
  - Over function fields, explicit $c'(g)$ by Looper–Silverman–Wilms (2021).
  - The proofs of Kühne and Yuan are currently not effective.
  - The proof of G'–Habegger is in principle effective (subject to Binyamini's effective Pila–Wilkie counting result). However, the extracted bound is too large.
- Is it possible to make Baker–Poonen's result uniform? For example, does there exist a constant $B = B(g) > 0$ such that the cardinality of

$$\{P \in C(\overline{\mathbb{Q}}) : \#\mathrm{TP}(C, P) > 6\}$$

  is at most $B(g)$?
  - The number 6 here is suggested by the proof of G'–Habegger.
- There are many recent exciting analogous / related results for Arithmetic Dynamics by DeMarco–Mavraki–Schmidt, Gauthier–Vigny, *etc.*

High Dimension: Mordell–Lang

The setup of Mordell–Lang: Over $\overline{\mathbb{Q}}$

- A = abelian variety of dimension $g \geq 1$;
- $X \subseteq A$ geometrically irreducible subvariety;
- $\Gamma < A(\overline{\mathbb{Q}})$ a finite rank subgroup.

Theorem (Mordell–Lang Conjecture, Falting '91 + Hindry '88)

$$X(\overline{\mathbb{Q}}) \cap \Gamma = \bigcup_{i=1}^{n} (x_i + B_i(\overline{\mathbb{Q}})) \cap \Gamma.$$

Again, we have results concerning the distribution of algebraic/rational points in X.

- (Rémond 2000) generalized Mumford (for points in Γ!) and Vojta Inequalities;
- (G'–Ge–Kühne 2021) generalized New Gap Principle.

From these we obtain:

Theorem (G'–Ge–Kühne 2021; known as Uniform Mordell–Lang)

There exists a partition such that $n \leq c(g, \deg X)^{1+\mathrm{rk}\Gamma}$.

A tool (Degeneracy loci) and a bigness result

- $\mathcal{A} \to S$ an abelian scheme.
- $X \subseteq \mathcal{A}$ irreducible subvariety, dominant to S.

✎ (G' 2020) For each $t \in \mathbb{Z}$, one can define the t-th degeneracy locus $X^{\deg}(t)$ of X. ⇝Important tool to study these uniformity results.

Theorem (G' 2020, example of application of $X^{\deg}(0)$)

Let $\mathcal{L}$ be a relatively ample line bundle on $\mathcal{A}/S$ which is trivial along the 0-section, let $\widetilde{\mathcal{L}}$ be the associated adelic line bundle (as defined by Yuan–Zhang).
Then $\widetilde{\mathcal{L}}|_X$ is NOT a big line bundle if and only if there exists an abelian subscheme $\mathcal{B}$ of $\mathcal{A} \to S$ such that $\dim X - \dim(\iota \circ p)(X) > \dim \mathcal{B} - \dim S$, under

$$\begin{tikzcd} \mathcal{A} \arrow[r, "p"] \arrow[d] & \mathcal{A}/\mathcal{B} \arrow[r, "\iota"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{A}_{g'} \arrow[d] \\ S \arrow[r, "="] & S \arrow[r] & \mathbb{A}_{g'}. \end{tikzcd}$$

Thanks!