An example-based introduction to Shimura varieties and their compactifications

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Lecture 1

Modular curves

$\Gamma \subseteq SL_2(\mathbb{Z})$ congruence subgrp

$\circlearrowright \mathfrak{H} = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$.

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \gamma . z = \dfrac{az+b}{cz+d}$.

& $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \underset{\cap \, \mathbb{C}^2}{\begin{pmatrix} z \\ 1 \end{pmatrix}} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} \sim \begin{pmatrix} \gamma z \\ 1 \end{pmatrix}$.

$\mathfrak{H} \hookrightarrow \mathbb{P}^1(\mathbb{C}) \rightsquigarrow \Gamma \backslash \mathfrak{H} \hookrightarrow \Gamma \backslash (\mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q}))$

cpt Riemann surface
& alg curve / $\mathbb{C}$.

Look at $\{$modular curves $\Gamma \backslash \mathfrak{H}\}_{\text{varying } \Gamma}$

- Hecke symmetry
- "Galois symmetry"
  - $\Gamma \backslash \mathfrak{H}$ a priori / $\mathbb{C}$
    but in fact / some number field.

More generally, $\Gamma \backslash \mathfrak{H}$ is a "double coset space".

Setup G = red alg grp $/\mathbb{Q}$

s.t. can talk about $G(\mathbb{A})$.

$$\mathbb{A} = \underbrace{\mathbb{A}_\infty}_{\mathbb{R}} \times \underbrace{\mathbb{A}^\infty}_{\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}} \quad \text{(away from } \infty)$$

$$\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}.$$

For simplicity, will define G as a grp functor $/\mathbb{Z}$.

$$R_{/\mathbb{Z}} \longmapsto G(R)_{\text{grp}}.$$

e.g. $GL_n(R)$ = invertible elts in $\mathrm{End}_R(R^{\oplus n})$ (any comm ring R)

$\cup$

$SL_n(R)$ elts of det 1.

e.g. $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

$$Sp_{2n}(R) = \{ g \in GL_{2n}(R) : {}^t g \cdot J_n \cdot g = J_n \}$$

$(\Rightarrow g \in SL_{2n}(R))$

w/ symplectic inner product $\langle x, y \rangle := {}^t x \cdot J_n \cdot y$

e.g. $p, q \in \mathbb{N}$, $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$

$$O_{p,q}(R) = \{ g \in GL_{p+q}(R) : {}^t g \, I_{p,q} \, g = I_{p,q} \}$$

$\cup$

$SO_{p,q}(R)$ det = 1.

$(O_n := O_{n,0},\ SO_n := SO_{n,0})$.

Suppose $G(\mathbb{R}) \curvearrowright$ some $\mathcal{D}$.

Consider for each cpt open subgrp $\mathcal{U} \subseteq G(\mathbb{A}^\infty)$

$$\rightsquigarrow X_{\mathcal{U}} := \underbrace{G(\mathbb{Q})}_{\text{acts diagonally}} \backslash (\mathcal{D} \times G(\mathbb{A}^\infty)) / \underbrace{\mathcal{U}}_{\text{acts on } G(\mathbb{A}^\infty).}$$

Now consider $\{X_{\mathcal{U}}\}_{\text{varying } \mathcal{U}}$.

<u>Note</u> $G(\mathbb{A}^\infty)$-action permutes $\mathcal{U}$'s by $\mathcal{U} \longmapsto g\mathcal{U}g^{-1}$.

<u>So</u> with reasonable $\mathcal{D}$,

$$\varinjlim_{\mathcal{U}} H^*(X_{\mathcal{U}}) \circlearrowleft G(\mathbb{A}^\infty) \quad \text{"Hecke symm"}$$

<u>Q</u> What $\mathcal{D}$'s should we consider?

e.g. $SL_n(\mathbb{R}) \curvearrowright \mathcal{Y}_n = \left\{ \begin{matrix} n\times n \text{ real mat} \\ \text{pos. def. } \det = 1 \end{matrix} \right\} \subseteq M_{n\times n}(\mathbb{R})$.

$$g \in SL_n(\mathbb{R}) \rightsquigarrow X \longmapsto {}^t g X g$$

<u>Obs</u> (1) This action is transitive

(2) (Stabilizer of I_n) $= SO_n(\mathbb{R})$

$$\Rightarrow \mathcal{Y}_n = SL_n(\mathbb{R}) / SO_n(\mathbb{R}).$$

For $n = 2$, get

$$\mathcal{Y}_2 = SL_2(\mathbb{R}) / SO_2(\mathbb{R}) \cong \mathfrak{h} \;/\mathbb{R}.$$

For general n (e.g. $n=3$),

might get odd dim'l space $/\mathbb{R}$.

Will give "nice" D
s.t. quotients of D by some Γ is "algebraic".

<u>Fact</u> $\#\left(G(\mathbb{Q})\backslash G(\mathbb{A}^{\infty})/\mathcal{U}\right) < \infty$

$$\Rightarrow X_{\mathcal{U}} = G(\mathbb{Q})\backslash \left(D \times G(\mathbb{A}^{\infty})\right)/\mathcal{U}$$
$$= G(\mathbb{Q})\backslash \Big(D \times \big(\underbrace{\textstyle\coprod_i}_{\text{finite}} G(\mathbb{Q})\, g_i\, \mathcal{U}\big)\Big)/\mathcal{U}$$
$$\overset{\text{(exercise)}}{=} \coprod_i \Gamma_i \backslash D,$$
with $\Gamma_i = G(\mathbb{Q}) \cap g_i \mathcal{U} g_i^{-1}$.

Want $\Gamma_i \backslash D$ to be an <u>alg var</u>.
($\Rightarrow$ Galois symm).

Will want D to be (finite unions of) Herm symm domains.
<u>Rmk</u> Will focus first on "connected components".

<u>Q</u> What D's should we consider?

e.g. (Siegel case) $G = Sp_{2n}(\mathbb{R}) \curvearrowright \mathfrak{H}_n$.

$\mathfrak{H}_n := \{ Z \in \mathrm{Sym}_n(\mathbb{C}) : \mathrm{Im}\, Z > 0 \}$
Siegel upper half space.

$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(\mathbb{R}) \rightsquigarrow gZ = (AZ+B)(CZ+D)^{-1}$

$A, B, C, D \in M_{n\times n}(\mathbb{R})$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} Z \\ 1 \end{pmatrix} = \begin{pmatrix} AZ+B \\ CZ+D \end{pmatrix} \underbrace{\sim}_{\text{(div by } CZ+D)} \begin{pmatrix} gZ \\ 1 \end{pmatrix}$$

More explicitly:

$$\mathfrak{h}_n = \left\{ Z \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \;\middle|\; \begin{array}{l} {}^t\begin{pmatrix} Z \\ 1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}}_{J_n} \begin{pmatrix} Z \\ 1 \end{pmatrix} = 0 \iff {}^tZ = Z \\ {}^t\begin{pmatrix} \bar{Z} \\ 1 \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} Z \\ 1 \end{pmatrix} < 0 \iff \mathrm{Im}\, Z > 0 \end{array} \right\}$$

(preserved by $Sp_{2n}(\mathbb{R})$.

$\rightsquigarrow Sp_{2n}(\mathbb{R}) \circlearrowright \mathfrak{h}_n$.

To show transversity (move all pts to iI_n)

$Z = X + iY$.

$$\begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Z \\ 1 \end{pmatrix} \sim \begin{pmatrix} iY \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \begin{pmatrix} iY \\ 1 \end{pmatrix} \sim \begin{pmatrix} i\underbrace{AY{}^tA} \\ 1 \end{pmatrix}$$

can choose A s.t. this is I_n.

Stabilizer at iI_n $= \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in Sp_{2n}(\mathbb{R}) \right\}$

$\cong U_n(\mathbb{R}) = \{ g \in GL_n(\mathbb{C}) : {}^t\bar{g} I_n g = I_n \}$.

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \longleftrightarrow A + iB.$$

$\Rightarrow \mathfrak{h}_n \cong Sp_{2n}(\mathbb{R}) / U_n(\mathbb{R}) = G(\mathbb{R})$.

$\rightsquigarrow \mathfrak{h}_n \cong G(\mathbb{R}) \cdot h_0$, $\quad h_0 : U_1(\mathbb{R}) \longrightarrow Sp_{2n}(\mathbb{R})$

$$a + bi \longmapsto \begin{pmatrix} aI_n & -bI_n \\ bI_n & aI_n \end{pmatrix}$$

$U_n(\mathbb{R})$ = centralizer of h_0.

<u>note</u> $\mathfrak{h}_n$ stable under conj action of $Sp_{2n}(\mathbb{R})$ on $\mathfrak{h}_0$.

$J = h_0(i)$ defines a "complex str" on $\mathbb{R}^{2n}$.

Let $z = iI_n$. Consider

$$A_{h_0} = \underbrace{\mathbb{R}^{\oplus 2n}}_{\mathbb{C}\text{-v.s.}} / \underbrace{\mathbb{Z}^{\oplus 2n}}_{\text{lattice}}$$

$$\cong \mathbb{C}^{\oplus n} / (\mathbb{Z}^{\oplus n} \cdot z + \mathbb{Z}^{\oplus n}).$$

If $h = gh_0$, then for $z = g \cdot (iI_n)$,

$$A_h = \mathbb{R}^{\oplus 2n} / \mathbb{Z}^{\oplus 2n}$$

$\circlearrowleft$ cplx str $J = h(i)$

$$\cong \mathbb{C}^{\oplus n} / (\mathbb{Z}^{\oplus n} z \oplus \mathbb{Z}^{\oplus n}) \leftarrow \text{polarized.}$$

<u>Fact</u> $(A_h, \text{polar.}) \cong (A_{h'}, \text{polar.})$

$\iff h = \gamma h'$ for some $\gamma \in Sp_{2n}(\mathbb{Z}) = \Gamma$

<u>So</u> pts of $\underset{Sp_{2n}(\mathbb{Z})}{\overset{\shortparallel}{\Gamma}} \backslash \mathfrak{h}_n$ paramtrizing

(principal) polarized ab var / $\mathbb{C}$.

Replace Γ w/ congruence subgrp def'd by cond mod n

$\Rightarrow$ cond on "n-torsion pts" on ab vars

$\shortparallel$ "level strs".

<u>Next time</u> PEL / Hodge / abelian type, exceptional type.

Lecture 2

Last time

$Sp_{2n}(\mathbb{R}) \circlearrowright \mathfrak{h}_n = \{ Z \in \mathrm{Sym}_n(\mathbb{C}) \text{ s.t. } \mathrm{Im}\, Z > 0 \}$

$GSp_{2n}(\mathbb{R}) \circlearrowright \mathfrak{h}_n^{\pm} = \{ Z \in \mathrm{Sym}_n(\mathbb{C}) \text{ s.t. } \mathrm{Im}\, Z > 0 \text{ or } \mathrm{Im}\, Z < 0 \}.$

$\quad \parallel$

$\{ (g, r) \in GL_{2n}(\mathbb{R}) \times \mathbb{R}^{\times},\ {}^t g J_n g = r \cdot J_n \}$

r = similitude char.

$$h_0 : U_1(\mathbb{R}) \longrightarrow Sp_{2n}(\mathbb{R})$$
$$a + bi \longmapsto \begin{pmatrix} aI_n & -bI_n \\ bI_n & aI_n \end{pmatrix}$$

replaced w/

$$h_0 : \mathbb{C}^{\times} \longrightarrow GSp_{2n}(\mathbb{R})$$
$$a + bi \longmapsto \text{same}$$

note better for VHS (variation of Hodge str's).

Example $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$

$U_{p,q}(\mathbb{R}) = \{ g \in GL_{p+q}(\mathbb{C}) : {}^t\bar{g}\, I_{p,q}\, g = I_{p,q} \}$

$(p \geq q)$

$$\mathcal{D}_{p,q} = \left\{ u \in M_{p,q}(\mathbb{C}) : {}^t\begin{pmatrix} \bar{u} \\ 1 \end{pmatrix} \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix} \begin{pmatrix} u \\ 1 \end{pmatrix} = {}^t\bar{u} \cdot u - I_q < 0 \right\}$$

bounded realization.

e.g. $p = q = 1$, $\mathcal{D}_{1,1} = \{ |u| < 1 \} \subseteq \mathbb{C}$.

Alternative realization:

$$U'_{p,q}(\mathbb{R}) = \{ g \in GL_{p+q}(\mathbb{C}) : {}^t\bar{g}\, J_{p,q}\, g = J_{p,q} \}$$

(different def'n

$$\text{w/ } J_{p,q} = \begin{pmatrix} & & \boxed{I_q} \\ & \boxed{S} & \\ \boxed{-I_q} & & \end{pmatrix} \qquad S \text{ size } p-q.$$

$$\begin{array}{ccc} q & p-q & q \end{array}$$

(skew-herm s.t. $-iS > 0$

i.e. $J_{p,q}^* = {}^t\bar{J}_{p,q} = -J_{p,q}$.

$$U'_{p,q}(\mathbb{R}) \curvearrowright \mathfrak{h}_{p,q} = \left\{ \begin{array}{l} \begin{array}{c} \\ q \\ p-q \end{array} \begin{array}{c} q \\ \begin{pmatrix} Z \\ W \end{pmatrix} \end{array} \in M_{q\times q}(\mathbb{C}) \times M_{(p-q)\times q}(\mathbb{C}) \text{ s.t.} \\ -i\, {}^t\begin{pmatrix} \bar{Z} \\ \bar{W} \\ 1 \end{pmatrix} J_{p,q} \begin{pmatrix} Z \\ W \\ 1 \end{pmatrix} = -i\left({}^t\bar{Z} - Z + {}^t\bar{W} S W\right) < 0 \end{array} \right\}$$

Rmk Pairing Q on complexified v.s.

$$Q(v,v) = 0 \quad -iQ(\bar{v},v) < 0.$$

Now assume $p = q$.

$$\Rightarrow \mathfrak{h}_{p,q} = \{ Z \in M_{q,q}(\mathbb{C}) : -i({}^t\bar{Z} - Z) < 0 \iff \text{"}\underbrace{\mathrm{Im}}_{\text{skew-herm part}} Z > 0\text{"} \}$$

$$M_{q,q}(\mathbb{C}) \simeq \mathrm{Herm}_q(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$$

<u>Example</u> $SO^*_{2n}(\mathbb{R}) :=$ subgrp of $SO_{2n}(\mathbb{C})$ preserving skew-herm pairing def'd by $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

$$\begin{array}{c} SO^*_{2n}(\mathbb{R}) \\ \| \\ SO_{2n}(\mathbb{R}) \cap U'_{n,n} \end{array}$$

$$\mathfrak{h}_{SO^*_{2n}} = \left\{ \begin{array}{l} Z \in M_n(\mathbb{C}) : {}^t\begin{pmatrix} Z \\ 1 \end{pmatrix}\begin{pmatrix} Z \\ 1 \end{pmatrix} = {}^tZZ + 1 = 0 \\ \& \; -i\,{}^t\begin{pmatrix} \bar{Z} \\ 1 \end{pmatrix}\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\begin{pmatrix} Z \\ 1 \end{pmatrix} = -i({}^t\bar{Z} - Z) < 0 \end{array} \right\}$$

When $n = 2k$, then $SO^*_{4k}(\mathbb{R})$ can be realized as the subgrp of $Sp_{2k}(\mathbb{H})$ of $2k \times 2k$ mats / $\mathbb{H} \simeq \mathbb{C}^2$ (Hamiltonians) preserving the skew-herm pairing def'd by $\begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$.

Then $\mathfrak{h}_{SO^*_{4k}} \cong \{ Z \in \mathrm{Herm}_k(\mathbb{H}) : \mathrm{Im}\, Z > 0 \}$.

<u>Example</u> $SO_{p,q}(\mathbb{R}) = \{ g \in SL_{p+q}(\mathbb{R}) : {}^t g\, I_{p,q}\, g = I_{p,q} \}$

$\circlearrowleft$

$V = \mathbb{R}^{p+q}$ $\qquad I_{p,q} = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$.

$$\mathfrak{h}_{SO_{p,q}} = \{ v \in V_{\mathbb{C}} : {}^t v\, I_{p,q}\, v = 0,\ {}^t \bar{v}\, I_{p,q}\, v < 0 \} / \mathbb{C}^\times$$

(working over $\mathbb{P}^1(V_{\mathbb{C}})$).

note: $v \in \mathfrak{h}_{SO_{p,q}}$ herm if $q = 2$.

Assume $q = 2$:

Decompose $V = \mathbb{R}^{\oplus p} \oplus \mathbb{R}^{\oplus 2}$ w/ pairings $\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$, $\begin{pmatrix} I_{p-1} & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$

(signature $(1,1)$)

Denote coordinates of $V_{\mathbb{C}}$ by $z_1, \dots, z_p, w_1, w_2$.

$$\text{Then } \mathfrak{h}_{SO_{p,q}} = \left\{ \begin{array}{l} z_1^2 + \dots + z_{p-1}^2 - z_p^2 + w_1 w_2 = 0 \\ |z_1|^2 + \dots + |z_{p-1}|^2 - |z_p|^2 + \bar{w}_1 w_2 + w_1 \bar{w}_2 < 0 \end{array} \right\}.$$

If w_2 were 0 then

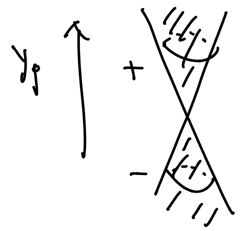
$|z_1|^2 + \dots + |z_{p-1}|^2 < |z_p|^2 = |z_1^2 + \dots + z_{p-1}^2|$ can't be true.

May assume $w_2 = 1$ (by div. nonzero values)

$\Rightarrow w_1 = z_p^2 - z_1^2 - \cdots - z_{p-1}^2$

$\Rightarrow \mathfrak{h}_{SO_{p,q}} \cong \{ Z \in \mathbb{C}^{\oplus p} : Y = \operatorname{Im} Z > 0 \}$
$(q=2)$

in the sense that if $y_i = \operatorname{Im} z_i$,
$Y = (y_1, \cdots, y_p)$, then $y_p^2 > y_1^2 + \cdots + y_{p-1}^2$.



light cone (2 conn components)
$\circlearrowleft$
the subgrp of $SO_{p,q}(\mathbb{R})$ of elts
w/ spinor norm $+1$
& preserving the $+$ component

$Spin_{p,q}$: can be embeded in a
(very large) symplectic grp.
of Hodge type
(i.e. parametrized by ab vars w/ Hodge tensors)

But $SO_{p,q}$ cannot!

of abelian type
- same conn comp as Hodge-type
- not the same rat'l str

Sp_{2n}	$U_{p,q}$ (over $\mathbb{C}$, $\cong GL_{p+q}$)	SO_{2n}^*	$SO_{p,2}$
Type C	Type A	Type D (type $D_{\mathbb{H}}$)	Type B, p odd or D, p even ($D_{\mathbb{R}}$)

__Rmk__ When there's mixture of types:

- $D_{\mathbb{R}}$ & $D_{\mathbb{H}}$ in the same $\mathbb{Q}$-simple factor of G
- __Not__ ab type.

Two more $\mathbb{R}$-simple cases (of exceptional type) E_6, E_7:

(1) $\underbrace{E_{7(-25)}}_{\text{Cartan index}} \curvearrowright \mathfrak{h}_{E_7} = \{ Z \in \mathrm{Herm}_3(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C} : \mathrm{Im}\, Z > 0 \}$

$$\mathbb{O} = \mathbb{R} \oplus \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_7,$$

s.t. $e_i^2 = 1$ & $(e_i e_{i+1}) e_{i+3} = e_i (e_{i+1} e_{i+3}) = 1$.

__Fact__ $\dim_{\mathbb{C}} \mathfrak{h}_{E_7} = 27$.

(2) $E_{6(-14)} \curvearrowright \mathfrak{h}_{E_6} \leftarrow \dim_{/\mathbb{C}} = 16$.

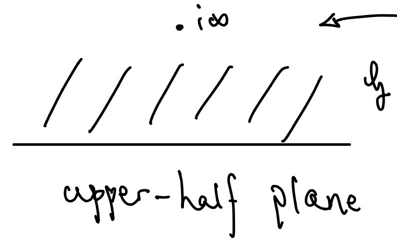
We have compactifications of quotients like $\Gamma \backslash D$.

$$\begin{tikzcd}
X_\Gamma := \Gamma\backslash D \arrow[r, hook] & (\Gamma\backslash D)^{tor}_\Sigma \arrow[dr] \arrow[rr, dashed, leftrightarrow, "\times"', "\text{no map}"'] & & (\Gamma\backslash D)^{BS} \arrow[dl] \\
& & (\Gamma\backslash D)^{min} &
\end{tikzcd}$$

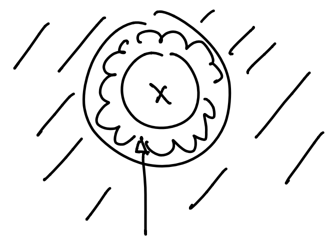
__note__ Sh vars are always quasi-proj vars

- $(\Gamma\backslash D)^{min}$: Satake-Baily-Borel, normal proj var.
- $(\Gamma\backslash D)^{BS}$: Borel-Serre, manifold w/ "corners".
- $(\Gamma\backslash D)^{tor}_\Sigma$: can be sm proj for good Σ.

Why not $(\Gamma\backslash D)^{\min}$?



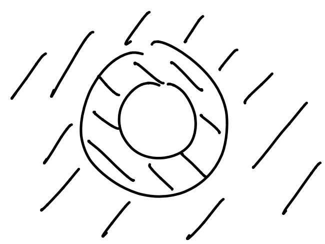
near $i\infty$: $\Gamma\backslash D$ looks like



$\exists$ info along boundary

<u>But</u> Add one pt $\leadsto$ will lose cohom classes if compactify into.

<u>Remedy</u> BS cpt'n looks like



$(\Gamma\backslash D)^{BS}$.

Lecture 3

Begin with $\Gamma \backslash D \hookrightarrow (\Gamma\backslash D)^{\min}$ Satake - Baily - Borel

proj normal var.

<u>Idea</u> Just like in the modular curve case.

$$\Gamma \backslash \mathfrak{h} \hookrightarrow \Gamma \backslash (\mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q}))$$

(w/ suitable top).

Here $\mathbb{P}^1(\mathbb{Q}) = \underbrace{SL_2(\mathbb{Q})}_{\text{acts in proj Coord.}} \cdot \infty$, $\infty = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

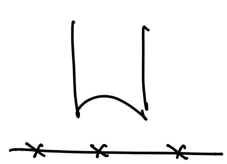
<u>note</u> elts of $SL_2(\mathbb{Q})$ preserving $\begin{pmatrix} * \\ 0 \end{pmatrix}$

are of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.

What we need in general are

the <u>max</u> rat'l parabolic subgrps of G.

(parabolic P : s.t. G/P proj var).



$\mathbb{P}^1(\mathbb{Q})$

"reduction theory":

only rat'l boundary comp's matter.

<u>Example 1</u> $\underbrace{Sp_4(\mathbb{Q})}_{\text{std parabolics}} \curvearrowright \mathfrak{h}_2$

(<u>Recall</u> Sp_{2n} respects the pairing def'd by $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.)

Borel: $\left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & \\ & & * & * \end{pmatrix} \right\} \longleftrightarrow \begin{pmatrix} * \\ 0 \\ 0 \\ 0 \end{pmatrix} \subset \begin{pmatrix} * \\ * \\ 0 \\ 0 \end{pmatrix}.$

Max ones: are $Sp_4(\mathbb{Q})$-conjs of the following:

Siegel: $\left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix} \right\} \subset \begin{pmatrix} * \\ * \\ 0 \\ 0 \end{pmatrix}$

Klingen: $\left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & \\ & * & * & * \end{pmatrix} \right\} \subset \begin{pmatrix} * \\ 0 \\ 0 \\ 0 \end{pmatrix}$

<u>Aside</u> General principle (for "classical grps" def'd by pairings): each rat'l parabolic preserves some flag of "isotypic" (rat'l) subspaces.

<u>Note</u> $Sp_{2n}(\mathbb{Q})$ has max rat'l parabolics.

$$\text{Conjugate to} \quad \begin{array}{c} \begin{array}{cccc} r & n-r & r & n-r \end{array} \leftarrow \text{unip radical} \\ \begin{pmatrix} \boxed{*} & * & * & * \\ & * & * & * \\ & & \boxed{*} & \\ & * & \boxed{*} & * \end{pmatrix} \begin{array}{l} r \\ n-r \\ r \\ n-r \end{array} \end{array}$$

$\rightsquigarrow G_{n-r} \cong Sp_{2(n-r)}.$

Levi: $\underbrace{GL_r}_{\tilde{G}_\ell} \times \underbrace{Sp_{2(n-r)}}_{G_h}$ Herm part.

The boundary component of coord: $\begin{pmatrix} r\,\boxed{\infty_r}^{\,r} & \\ & \boxed{Z_r}^{\,n-r}\, n-r \end{pmatrix} \sim \begin{pmatrix} 1 & Z_r \\ 0 & 1 \end{pmatrix}$ (in proj coord).

$z_r \in \mathfrak{h}_{n-r} \rightsquigarrow$ This boundary comp $\cong \mathfrak{h}_{n-r}$.

$$
\begin{aligned}
\mathfrak{h}_n^* = \mathfrak{h} &\cup Sp_{2n}(\mathbb{Q})\,\mathfrak{h}_{n-1} \leftarrow \dim \tfrac{n}{2}(n-1). \\
&\cup Sp_{2n}(\mathbb{Q})\,\mathfrak{h}_{n-2} \leftarrow \dim \tfrac{n-1}{2}(n-2) \\
&\cup \cdots \\
&\cup Sp_{2n}(\mathbb{Q}) \cdot \mathfrak{h}_0 \leftarrow \dim 0.
\end{aligned}
$$

<u>note</u> $\mathfrak{h}_n^*$ has dim $\frac{n}{2}(n+1)$ w/ Satake top.

$$
\begin{array}{ccc}
(\Gamma \backslash \mathfrak{h}_n)^{\min} & = & \Gamma \backslash \mathfrak{h}_n^* \\
\cup & & \\
\Gamma \backslash \mathfrak{h} & &
\end{array}
$$

stratified by images of "smaller $\Gamma \backslash \mathfrak{h}_n$".

<u>Example 2</u> $U_{p,q}(\mathbb{Q}) \curvearrowright \mathfrak{h}_{p,q}$.

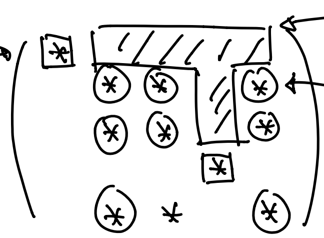
def'd by some Herm pairing over imag quad K w/ sgn (p,q) at ∞.

$$
\begin{pmatrix} & & I_q \\ & S & \\ -I_q & & \end{pmatrix} \curvearrowright \left\{ \begin{array}{c} \begin{pmatrix} * \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ r \end{array} \begin{array}{c} r \\ q-r \\ p-q \\ r \\ q-r \\ \ \end{array} \right\} \text{ isotropic rat'l}
$$

S size $p-q$.

<u>Note</u> Max parabolic are conj to

$\boxed{*} = GL_r(K) \rightarrow$



unip radical

$\circledast = U_{p-r,q-r}(\mathbb{Q})$ def'd by $\begin{pmatrix} & & I_{q-r} \\ & S & \\ -I_{q-r} & & \end{pmatrix}$.

$$\mathfrak{h}^*_{p,q} = \mathfrak{h}_{p,q} \cup \bigcup_{0\le r\le q} \left(U_{p,q}(\mathbb{Q})\, \mathfrak{h}_{p-r,q+r} \right)$$

$$\rightsquigarrow \quad (\Gamma \backslash \mathfrak{h}_{p,q})^* = \Gamma \backslash \mathfrak{h}^*_{p,q}.$$

<u>Example revisited</u>

For $G(\mathbb{R}) = SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$,

if $G(\mathbb{Q}) = SL_2(\mathbb{Q}) \times SL_2(\mathbb{Q})$,

then $G(\mathbb{Q}) \curvearrowright \mathfrak{h} \times \mathfrak{h}$.

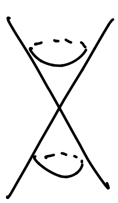
$$\rightsquigarrow (\mathfrak{h} \times \mathfrak{h})^* = \mathfrak{h}^* \times \mathfrak{h}^* = \mathfrak{h} \times \mathfrak{h} \cup G(\mathbb{Q})\,(\infty \times \mathfrak{h})$$
$$\cup\ G(\mathbb{Q})\,(\mathfrak{h} \times \infty)$$
$$\cup\ G(\mathbb{Q})\,(\infty \times \infty).$$

If F = tot real quad / $\mathbb{Q}$, then

$$G(\mathbb{Q}) = SL_2(F) \curvearrowright \mathfrak{h} \times \mathfrak{h}$$

$$(\mathfrak{h} \times \mathfrak{h})^* = \underset{\dim 2}{\mathfrak{h} \times \mathfrak{h}} \cup \underbrace{G(\mathbb{Q})\text{"}\infty\text{"}}_{\dim 0,\ \text{only one orbit.}}$$

$\rightsquigarrow (\Gamma \backslash (\mathfrak{h} \times \mathfrak{h}))^{\min}$ = Hilb mod surface w/ 0-dim cusps.



normal but not regular / smooth.

<u>Examples</u> (3) $\mathfrak{h}^*_{SO^*_{2n}} = \mathfrak{h}_{SO^*_{2n}} \cup \bigcup_{0<r\le\lfloor \frac{n}{2}\rfloor} G(\mathbb{Q}) \cdot \mathfrak{h}_{SO^*_{2n-4r}}$.

$\dim = \frac{n}{2}(n-1)$.

(4) $\mathfrak{h}^*_{SO_{p,2}} = \mathfrak{h}_{SO_{p,2}} \cup G(\mathbb{Q})\cdot \mathfrak{h}_1 \cup G(\mathbb{Q})\cdot \mathfrak{h}_0 .$

(5) $\mathfrak{h}^*_{E_7} = \mathfrak{h}_{E_7} \cup \underbrace{G(\mathbb{Q})\cdot \mathfrak{h}^{(+)}_{SO_{10,2}}}_{\dim = 10} \cup \underbrace{G(\mathbb{Q})\cdot \mathfrak{h}_1}_{\dim = 1} \cup \underbrace{G(\mathbb{Q})\cdot \mathfrak{h}_0}_{\dim = 0} .$

(5') $\underbrace{\mathfrak{h}^*_{E_6}}_{\dim = 16} = \mathfrak{h}_{E_6} \cup \underbrace{G(\mathbb{Q})\cdot \mathfrak{h}_{5,1}}_{\dim = 5} \cup G(\mathbb{Q})\cdot \mathfrak{h}_0$