Solid sheaves & Category $D_\square$ (II)

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§1 Embedding of Dét

Assume Λ discrete, so $n\Lambda = 0$ for pfn.

Naive embedding $\quad id : D_{ét}(X,\Lambda) \longrightarrow D_\square(X,\Lambda)$.

Dual embedding (Assume $\Lambda = \mathbb{Z}/m\mathbb{Z}$).

Recall $A \in D_{ét}(X,\Lambda)$, say A overcovergent if
$$R\Gamma(\mathrm{Spa}(C,C^+),A) \longrightarrow R\Gamma(\mathrm{Spa}\, C, A)$$
isom for all (C,C^+).

Let $D^{+}_{ét}(X,\Lambda) := \{\text{overcov complex}\} \subseteq D_{ét}(X,\Lambda)$.

Def'n Let $A \in D^{+}_{ét}(X,\Lambda)$. Define $A^\vee := R\mathcal{H}om_\square(A,\Lambda)$.

$\rightsquigarrow (-)^\vee : D^{+}_{ét}(X,\Lambda) \longrightarrow D_\square(X,\Lambda)$.

Thm $(-)^\vee$ fully faithful & t-exact.

Moreover, $A \xrightarrow{\sim} R\mathcal{H}om_\square(A^\vee,\Lambda)$.

Proof Assume X strictly tot disconn.

$\rightsquigarrow D^{+}_{ét}(X,\Lambda) = D(\pi_0|X|,\Lambda) = \mathrm{Ind}(D_{cons}(\pi_0|X|,\Lambda))$.

$(-)^\vee : D_{cons}(\pi_0|X|,\Lambda)^{op} \xrightarrow{\sim} D_{cons}(\pi_0|X|,\Lambda)$.

$$\begin{tikzcd}
\mathrm{Ind}(D_{cons}(\pi_0|X|,\Lambda))^{op} \arrow[r, "\sim"] \arrow[dd, equal] & \mathrm{Pro}(D_{cons}(\pi_0|X|,\Lambda)) \arrow[d] \\
 & \mathrm{Pro}(D_{ét,cons}(X,\Lambda)) \arrow[d] \\
D^{+}_{ét}(X,\Lambda) \arrow[r, "(-)^\vee"] & D_\square(X,\Lambda).
\end{tikzcd}$$

(this is indeed another def'n of the dual).

$$\Rightarrow \quad R\mathcal{H}om_{\square}(A^\vee, \Lambda), \qquad (A = \operatorname{colim} \mathcal{F}_i, \quad A^\vee = R\lim \mathcal{F}_i^\vee.)$$
$$\| $$
$$R\mathcal{H}om_{\square}(R\lim \mathcal{F}_i^\vee, \Lambda) = \operatorname{colim} R\mathcal{H}om_{\square}(\mathcal{F}_i^\vee, \Lambda) = A. \qquad \square$$

<u>Prop</u> (VII.4.2) $A \in D^+_{ét}(X,\Lambda)$ finite Tor-dim'l over $\Lambda = \mathbb{Z}/n\mathbb{Z}$.

Then $\forall B \in D^+_{ét}(X,\Lambda)$,

(1) $A^\vee \otimes^{\mathbb{L},\square} B^\vee \xrightarrow{\sim} (A \otimes^{\mathbb{L}} B)^\vee$

(2) $A \otimes^{\mathbb{L}} B \xrightarrow{\sim} R\mathcal{H}om_{\square}(A^\vee, B)$.

<u>Proof</u> (2) $R\mathcal{H}om_{\square}(A^\vee, B) = R\mathcal{H}om_{\square}(A^\vee, R\mathcal{H}om_{\square}(B^\vee, \Lambda))$

$= R\mathcal{H}om_{\square}(A^\vee \otimes^{\mathbb{L},\square} B^\vee, \Lambda)$.

So (1) $\Rightarrow$ (2).

(1) Assume $\Lambda = \mathbb{F}_\ell$, A, B sitting in deg 0, X strictly tot disconn.

Both sides sends colims to lims.

Assume $A, B \in D_{cons}(\pi_0|X|, \Lambda)$.

may not coh sm.

<u>Thm</u> (VII.4.3) $f: Y \to X$ proper, rep'ble in spatial diamonds, $\dim\operatorname{trg} < +\infty$

$\Lambda = \mathbb{Z}/n\mathbb{Z}$, $A \in D^+_{ét}(Y,\Lambda)$.

Then $(Rf_* A)^\vee = f_\natural A^\vee$.

(Recall: always $(f_\natural A)^\vee = Rf_* A^\vee$).

Resume on pf: by thm,

$$\begin{tikzcd}
D^+_{ét}(X,\Lambda) = \mathrm{Ind}(D_{cons}(\pi_0|X|,\Lambda)) \arrow[r, "(-)^\vee", "\sim"'] & \mathrm{Pro}(D_{cons}(\pi_0|X|,\Lambda)) \\
D^+_{ét}(Y,\Lambda) = \mathrm{Ind}(D_{cons}(\pi_0|Y|,\Lambda)) \arrow[r, "(-)^\vee", "\sim"'] \arrow[u, "Rf_*"] \arrow[ur, phantom, "\circlearrowright"] & \mathrm{Pro}(D_{cons}(\pi_0|Y|,\Lambda)). \arrow[u, "f_\natural"']
\end{tikzcd}$$

$\square$

§2 The duality (c.f. VII. §5)

ULA sheaves:

Def'n Assume Λ is quotient of $\widehat{\mathbb{Z}}^p$ of the form $\lim \mathbb{Z}/n\mathbb{Z}$ over some subset of $\mathbb{N}_+$. Assume $f: X \to S$ compactifiable, rep'ble in spatial diamonds, locally $\dim.\mathrm{trg} < +\infty$.

Define $D^{ULA}(X/S, \Lambda) := \lim_n D^{ULA}(X/S, \mathbb{Z}/n\mathbb{Z})$.

Equivalently, $D^{ULA}(X/S, \Lambda) \subseteq D_\square(X, \Lambda)$.

$$= \left\{ A \;\middle|\; \begin{array}{c} A_n := A \otimes^{\mathbb{L}}_\Lambda \mathbb{Z}/n\mathbb{Z} \text{ is ULA as } \mathbb{Z}/n\mathbb{Z}\text{-mod} \\ \text{and } A = R\lim A_n \end{array} \right\}.$$

$\rightsquigarrow$ Naive embedding:

$$\mathrm{id}: D^{ULA}(X/S, \Lambda) \longrightarrow D_\square(X, \Lambda).$$

Dual embedding: $A \in D^{ULA}(X/S, \Lambda)$, then $A_n \in D^{ULA}(X, \mathbb{Z}/n\mathbb{Z})$.

$\Rightarrow A_n^\vee \in D_\square(X, \mathbb{Z}/n\mathbb{Z})$,

$A^\vee := R\lim A_n^\vee \in D_\square(X, \Lambda)$.

Dual Verdier embedding:

$$\mathbb{D}^\vee: D^{ULA}(X/S, \Lambda) \longrightarrow D_\square(X, \Lambda).$$
$$A \longmapsto \mathbb{D}_{X/S}(A)^\vee.$$

To view algebraic sheaves as analytic sheaves.

Example C alg closed perf'd field.

$S = \mathrm{Spa}\, C$, $S^{alg} = \mathrm{Spec}\, C$, X^{alg}/S^{alg} ft sep scheme.

$X :=$ analytification of X^{alg}.

$\rightsquigarrow D^b_c(X^{alg}, \mathbb{Z}_\ell) \xrightarrow{can} D(X_{proet}, \mathbb{Z}_\ell)$ lands in $D^{ULA}(X/S, \mathbb{Z}_\ell)$.

$\rightsquigarrow D^b_c(X^{alg}, \mathbb{Z}_\ell) \xrightarrow{can} D^{ULA}(X/S, \mathbb{Z}_\ell) \xrightarrow{\mathbb{D}^\vee} D_\square(X, \mathbb{Z}_\ell)^\omega$,

$\mathrm{Ind}\, D^b_c(X^{alg}, \mathbb{Z}_\ell) \xrightarrow{\mathbb{D}^\vee} D_\square(X, \mathbb{Z}_\ell)$.

For $f: X \to Y$,

$$\begin{tikzcd} \operatorname{Ind} D^b_c(X^{alg}, \mathbb{Z}_\ell) \arrow[r, "\mathbb{D}^\vee"] \arrow[d, "Rf^!"'] \arrow[dr, phantom, "\circlearrowright"] & D_\square(X, \mathbb{Z}_\ell) \arrow[d, "f^*"] \\ \operatorname{Ind} D^b_c(Y^{alg}, \mathbb{Z}_\ell) \arrow[r, "\mathbb{D}^\vee"] & D_\square(Y, \mathbb{Z}_\ell) \end{tikzcd}$$

for stack Y over schemes.

Define $\qquad \mathcal{D}_{\acute{e}t}(Y, \mathbb{Z}_\ell) := \varprojlim_{X \to Y} \operatorname{Ind} \mathcal{D}^b_c(X, \mathbb{Z}_\ell)$

with $Rf^!$ as a transition maps.

$$\mathbb{D}^\vee : D_{\acute{e}t}(Y, \mathbb{Z}_\ell) \longrightarrow D_\square(Y^\diamond, \mathbb{Z}_\ell).$$

Let $A \in D^{ULA}(X/S, \Lambda)$. Consider functor

$$f_\natural(\mathbb{D}(A)^\vee \otimes^{\mathbb{L}, \square}_\Lambda -) : D_\square(X, \Lambda) \longrightarrow D_\square(S, \Lambda).$$

<u>Lem</u> The functor above extends $Rf_!(A \otimes^{\mathbb{L}}_\Lambda (-))$.

More precisely, for $A \in D^{ULA}(X/S, \Lambda)$ finite Tor-dim'l, $B \in D_{\acute{e}t}(X, \mathbb{Z}/n\mathbb{Z})$,

we have $\qquad f_\natural(\mathbb{D}(A)^\vee \otimes^{\mathbb{L}, \square}_\Lambda B) = Rf_!(A_n \otimes^{\mathbb{L}}_{\mathbb{Z}/n\mathbb{Z}} B)$.

<u>Proof</u> LHS is pseudo-compact. RHS is an étale sheaf.

$\rightsquigarrow$ Suffices for $C \in D_{\acute{e}t}(X, \mathbb{Z}/n\mathbb{Z})$.

(isom after applying $R\mathrm{Hom}(-, C)$.).

Now $\qquad R\mathrm{Hom}(f_\natural(\mathbb{D}(A)^\vee \otimes B), C)$

$$= R\mathrm{Hom}(B, R\mathcal{H}om(\mathbb{D}(A)^\vee, f^*C))$$

$$= R\mathrm{Hom}(B, \mathbb{D}(A) \otimes f^*C)$$

$$= R\mathrm{Hom}(B, R\mathcal{H}om_{\acute{e}t}(A, Rf^!C))$$

$$= R\mathrm{Hom}(Rf_!(A \otimes B), C). \qquad \square$$

<u>Thm</u> (VII.5.3, Poincaré duality)

$f: X \to S$ proper, rep'ble in spatial diamonds, $\dim \operatorname{trg} < +\infty$.

$A \in D^{ULA}(X/S, \Lambda)$ with finite Tor-dim.

Then $\quad f_\natural(\mathbb{D}(A)^\vee \otimes^{\mathbb{L}} (-)) = Rf_* R\mathcal{H}om_\square(A^\vee, -)$.

Proof Construct map LHS $\to$ RHS

$$\Leftrightarrow \quad f^* f_\natural(\mathbb{D}(A)^\vee \otimes^{\mathbb{L}} B) \otimes^{\mathbb{L}} A^\vee \longrightarrow B.$$

$$\begin{tikzcd} & X \times_S X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & X \end{tikzcd}$$

$\Delta : X \to X \times_S X$.

$$\text{LHS} = \pi_{1,\natural}\pi_2^*(\mathbb{D}(A)^\vee \otimes^{\mathbb{L}} B) \otimes^{\mathbb{L}} A^\vee$$

$$= \pi_{1,\natural}\left(\pi_2^*(\mathbb{D}(A)^\vee \otimes^{\mathbb{L}} B) \otimes^{\mathbb{L}} \pi_1^* A^\vee\right).$$

$$\Leftrightarrow \quad \pi_2^* \mathbb{D}_{X/S}(A)^\vee \otimes \pi_1^* A^\vee \otimes \pi_2^* B) \longrightarrow \pi_1^* B.$$

Note: (1) $\exists$ a map $\Delta_\natural \Lambda \otimes \pi_2^* B \longrightarrow \pi_1^* B$.

(2) $\text{LHS} = \Delta_\natural(\Delta^* \pi_2^* B) = \Delta_\natural B$,

adjoint to $B \mapsto \Delta^* \pi_1^* B = B$.

$\rightsquigarrow$ Suffices to $\pi_2^* \mathbb{D}_{X/S}(A)^\vee \otimes \pi_1^* A^\vee \to \Delta_\natural \Lambda = (R\Delta_* \Lambda)^\vee$

$\Leftarrow \quad R\Delta_* \Lambda \longrightarrow \pi_2^* \mathbb{D}_{X/S}(A) \otimes \pi_1^* A$

$\Leftrightarrow \quad \Lambda \longrightarrow R\Delta^!(\pi_2^* \mathbb{D}(A) \otimes \pi_1^* A)$.

Also, $\text{RHS} = R\Delta^! R\mathcal{H}om(\pi_2^* A, R\pi_1^! A) = R\mathcal{H}om(A, A)$. ☺.

Assume S, X strictly tot discon.

Both sides commutes with fil'd colim in B.

Assume B compact $\in \mathrm{Solid}^{w}$, $B = R\lim \mathcal{F}_i$.

Both sides commutes with such limits.

$\Rightarrow$ $B \in D_{\mathrm{et}}(X, \Lambda)$ by Poincaré duality.

§3 Lisse sheaves

Want sm rep'ns of $G(E)$.

$D_\square$ "topological complete module".

D_{lis} "relatively discrete topological module".

For example, $D_\square(\mathrm{Spa}\, C, \Lambda) = D_\square(\Lambda, \hat{\mathbb{Z}}^p)$

$D_{lis}(\mathrm{Spa}\, C, \Lambda) = D(\Lambda) = \{\text{rel disc}\} \subseteq D_\square(\Lambda, \hat{\mathbb{Z}}^p)$.

Assume Λ relatively discrete $\mathbb{Z}_\ell$-alg

i.e. $\Lambda = \Lambda_{disc} \otimes_{\mathbb{Z}_\ell, disc} \mathbb{Z}_\ell$ as condensed rings.

e.g. $\Lambda = \mathbb{F}_\ell$, $\mathbb{Z}_\ell$, $\mathbb{Q}_\ell$, $\mathbb{C}_\ell$, $\mathcal{O}_{\mathbb{C}_\ell}$.

<u>Def'n</u> (VII.6.1) Let X = Artin v-stack.

Define $D_{lis}(X, \Lambda) \subset D_\square(X, \Lambda)$

to be the smallest triangular subcat containing $f_\natural \Lambda$

for $f: U \to X$ sep'ble rep'd by locally spatial diamonds,

coh sm, with U a locally spatial diamond,

and stable under $\bigoplus$.

<u>Lem</u> $\otimes^{\mathbb{L},\square}$ preserves D_{lis},

f^* preserves D_{lis} for any $f: Y \to X$ of Artin v-stacks.

<u>Lem</u> Forgetful $D_{lis} \hookrightarrow D_\square$ admit a right adjoint $(-)^{lis}$.

<u>Cor</u> f^* and $\otimes^{\mathbb{L},\square}$ as functors on D_{lis},

admit right adjoints $Rf_{lis,*}$, $R\mathcal{H}om_{lis}$,

given by applying $(-)^{lis}$ on Rf_* and $R\mathcal{H}om$.

Prop $X = \mathrm{Spa}\, C$ geom pt. Then

$$D_{lis}(X, \Lambda) = D(\Lambda).$$

Proof It suffices to show $f_\natural \Lambda \in D(\Lambda)$, $\forall f : Y \to X$ sep, coh sm, rep'ble in spatial diamonds.

Assume $\Lambda = \mathbb{Z}_\ell$,

$$\rightsquigarrow \quad f_\natural \Lambda = \lim f_\natural \mathbb{Z}/\ell^n\mathbb{Z} = \lim \underbrace{Rf_! Rf^! \mathbb{Z}/\ell^n\mathbb{Z}}_{\text{perfect constructible}}.$$

$\Rightarrow f_\natural \Lambda$ is perfect Λ-mod $\Rightarrow f_\natural \Lambda \in D(\Lambda)$. ok.

And $id_\natural \Lambda = \Lambda \in D_{lis} \Rightarrow D_{lis}(X, \Lambda) = D(\Lambda)$. $\square$

Prop X Artin v-stack. $\ell^m \Lambda = 0$.

Then $D_{lis}(X, \Lambda) \subseteq D_{et}(X, \Lambda)$. Equality holds if $(*)$.

$(*)$: $\exists\, U \to X$ sep. locally coh sm, surj,

U loc. spatial diamond,

s.t. U_{et} admits a collection of basis $\{U'\}$,

s.t. U' has bounded ℓ-coh dim.

Proof $f_\natural \Lambda = Rf_! Rf^! \Lambda \in D_{et} \Rightarrow D_{lis} \subseteq D_{et}$.

Assume $X = U$.

$(*) \Rightarrow D_{et}(X, \Lambda) = D(X_{et}, \Lambda)$.

It is generated by $j_! \Lambda = j_\natural \Lambda$ for $j : U \to X$ quasi-pro-étale.

$\Rightarrow D_{lis} = D_{et}$. $\square$

Prop (VII.6.7) X spatial diamond, $x \in |X|$ closed, $Z = \{x\}$.

Assume Z is rep'ble by $\mathrm{Spa}\, C$ generic pt, and Z is the cofil'd intersection of open nbhds V's

s.t. V qc, $R\Gamma(V, \mathbb{F}_\ell) = \mathbb{F}_\ell$.

Let $U = X \setminus Z$, $j : U \to X$, $i : \mathrm{Spa}\, C = Z \longrightarrow X$.

Then $D_{lis}(X, \Lambda)$ has semi-orthogonal decomposition

into $j_! D_{lis}(U, \Lambda)$ and $i_\natural D_{lis}(Z, \Lambda) = D(\Lambda)$.

<u>Proof</u> $R\mathrm{Hom}(j_! A, i_\natural B) = R\mathrm{Hom}(A, j^* i_\natural B) = 0$.

$\rightsquigarrow$ to show $D_{lis}(Z, \Lambda) \xrightarrow{i_\natural} D_{lis}(X, \Lambda) \longrightarrow (\mathrm{Coker}\, j_!)$

is an equivalence.

Here $\mathrm{Coker}\, j_!$ is generated by image of $f_\natural \Lambda$,

i.e. $\mathrm{cofib}(j_! j^* f_\natural \Lambda \longrightarrow f_\natural \Lambda)$.

<u>Claim</u> $f_\natural \Lambda$ is constant near Z, with value $i^* f_\natural \Lambda$.

May assume $\Lambda = \mathbb{Z}_\ell$.

$\rightsquigarrow$ $f_\natural \mathbb{Z}_\ell$ invertible, constant on an nbhd V.

Assume $R\Gamma(V, \mathbb{F}_\ell) = \mathbb{F}_\ell$

$\Rightarrow \exists!$ isoms $(\underline{M/^L \ell^m M})|_V \xrightarrow{\sim} (f_\natural \Lambda)|_V \otimes \mathbb{Z}/\ell^m\mathbb{Z}$

lifting the given $(\underline{M/^L \ell M})|_V \xrightarrow{\sim} (f_\natural \Lambda)|_V$.

Take $R\lim \Rightarrow$ $\underline{M}|_V \xrightarrow{\sim} (f_\natural \Lambda)|_V$.

$\Rightarrow$ $\mathrm{Cofib}(j_! j^! f_\natural \Lambda \to f_\natural \Lambda) = \mathrm{Cofib}(j_! M \to M)$

is gen'd by $\mathrm{Cofib}(j_! \Lambda \to \Lambda)$.

$\Rightarrow$ $i_\natural : \underset{\substack{\parallel \\ D(\Lambda)}}{D_{lis}(Z, \Lambda)} \longrightarrow \mathrm{Coker}\, j_!$ generates the target

$\Lambda \longmapsto \mathrm{Cofib}(j_! \Lambda \to \Lambda)$.

It remains to show

$$R\mathrm{Hom}(\mathrm{Cofib}(j_! \Lambda \to \Lambda), \mathrm{Cofib}(j_! \Lambda \to \Lambda)) = \Lambda.$$

But $\mathrm{LHS} = \varinjlim_{V \supseteq Z} R\Gamma(V, \mathbb{F}_\ell) = \mathbb{F}_\ell$. $\square$

§4 $\mathrm{D_{lis}}(\mathrm{Bun}_G)$

E local field with res field $\mathbb{F}_q$. $k = \overline{\mathbb{F}}_q$ alg closed.

G/E reductive grp. Work over $* = \mathrm{Spd}\, k$.

C any alg closed perf'd field $/k$.

$\rightsquigarrow$ $\mathrm{Spa}\, C = \mathrm{Spa}(C, \mathcal{O}_C)$ generic pt.

__Recall__ $\exists$ natural map $\underline{G_b(E)} \hookrightarrow \widetilde{G}_b(E)$

inducing $s : [*/\underline{G_b(E)}] \longrightarrow [*/\widetilde{G}_b(E)] = \mathrm{Bun}_G^b$.

A commutative diagram

$$\begin{tikzcd}
D(G_b(E), \Lambda) \arrow[dr, bend right] & & \\
& \mathrm{D_{lis}}(\mathrm{Bun}_G^b, \Lambda) \arrow[r, "s^*"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{D_{lis}}([*/\underline{G_b(E)}], \Lambda) \arrow[d] \\
& \mathrm{D_{lis}}(\mathrm{Bun}_{G,C}^b, \Lambda) \arrow[r, "s_C^*"] & \mathrm{D_{lis}}([\mathrm{Spa}\, C/\underline{G_b(E)}], \Lambda)
\end{tikzcd}$$

VII.7.1 : all functors above are equivalences.

__Prop__ (VII.7.2) $b \in B(G)$, $i^b : \mathrm{Bun}_G^b \to \mathrm{Bun}_G$.

$\rightsquigarrow$

$$\begin{tikzcd}
{[*/G_b(E)]} \arrow[r, hook, "q'"] \arrow[d, "s"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{M}_b \arrow[d, "\pi_b"] & {[*/G_b(E)]} \arrow[l, hook', "q_b"'] \\
\mathrm{Bun}_G^b \arrow[r, "i^b"] & \mathrm{Bun}_G . &
\end{tikzcd}$$

$\Rightarrow$ $\pi_{b,\natural} q_b^*$ is a left adjoint of $\boxed{s^*} i^{b*}$ $\leftarrow$ equivalence.

pf : c.f. Prop V.4.2.

__Prop__ (VII.4.3) For any qc open $U \subseteq \mathrm{Bun}_G$,

$\mathrm{D_{lis}}(U, \Lambda)$ has semi-ortho decomp into $\mathrm{D_{lis}}(\mathrm{Bun}_G^b, \Lambda)$.

Same is valid after base change to $\mathrm{Spa}\, C$

and $\mathrm{D_{lis}}(U, \Lambda) \xrightarrow{\pi^*} \mathrm{D_{lis}}(U_C, \Lambda)$ is an equivalence.

Prop (VI.7.4) $D_{lis}(\mathrm{Bun}_G, \Lambda)$ is compactly generated.

Then $A \in D_{lis}(\mathrm{Bun}_G, \Lambda)$ compact $\iff$ $\forall b \in B(G)$, $i^{b*}A$ compact & A has finite support.

Moreover, $b \in B(G)$, $K \subseteq G_b(E)$ open pro-p,

$$f_K : \mathcal{M}_{b,K} := \widetilde{\mathcal{M}}_b / \underline{K} \longrightarrow \mathrm{Bun}_G,$$

$A_K^b = f_{K\natural}\Lambda \in D_{lis}(\mathrm{Bun}_G, \Lambda)$ compact & generate $D_{lis}(\mathrm{Bun}_G, \Lambda)$.

Berstein-Zelevinsky duality (for lisse sheaves)

$\pi : \mathrm{Bun}_G \longrightarrow *$,

BZ-pairing
$$D_{lis}(\mathrm{Bun}_G, \Lambda)^2 \longrightarrow D(\Lambda)$$
$$(A, B) \longmapsto \pi_\natural(A \otimes^{\mathbb{L}} B).$$

Prop (VI.7.6) (cf. V.5.1)

$\exists$ $\mathbb{D}_{BZ} : (D_{lis}(\mathrm{Bun}_G, \Lambda)^{\omega})^{op} \longrightarrow D_{lis}(\mathrm{Bun}_G, \Lambda)^{\omega}$

s.t. $\mathbb{D}_{BZ}^2 = \mathrm{id}$, $R\mathrm{Hom}(\mathbb{D}_{BZ}(A), B) = \pi_\natural(A \otimes^{\mathbb{L}} B)$,

$\forall B \in D_{lis}(\mathrm{Bun}_G, \Lambda)$.

Moreover, if A is supported on an open U, so is $\mathbb{D}_{BZ}(A)$, and

$$\mathbb{D} : (D_{lis}(U, \Lambda)^{\omega})^{op} \longrightarrow D_{lis}(U, \Lambda)^{\omega}.$$

And in particular it recovers the classical BZ duality on $D(G_b(E), \Lambda)$ when $U = \{b\}$.

Cor (VI.7.7) c.f. V.6.1

$V \subseteq U \subseteq \mathrm{Bun}_G$, $j : V \to U$, $A \in D_{lis}(V)$.

Then $j_! R\mathcal{H}om_{lis}(A, \Lambda) = R\mathcal{H}om_{lis}(Rj_* A, \Lambda)$.

<u>Def'n</u> (VII.7.8) $A \in D_{lis}(Bun_G, \Lambda)$, $\pi : Bun_G \longrightarrow *$.

Say A is <u>π-ULA</u> if

$$pr_1^* R\mathcal{H}om_{lis}(A, \Lambda) \otimes^{L} pr_2^* A = R\mathcal{H}om_{lis}(pr_1^* A, pr_2^* A)$$

where $pr_i : Bun_G^2 \longrightarrow Bun_G$.

<u>Prop</u> (VII.7.9) c.f. V.7.1

$A \in D_{lis}(Bun_G, \Lambda)$ is ULA $\Leftrightarrow$ $\forall\, b \in B(G)$, $i^{b*} A \in D_{lis}(Bun_G^b, \Lambda)$ corresponds to $M \in D(G_b(E), \Lambda)$, s.t. $\forall\, K \subseteq G_b(E)$ open & pro-p, M^K is a perfect complex.