INTEGRAL MODELS OF SHIMURA VARIETIES

LECTURES BY MARK KISIN

These are notes expanded from three lectures by Mark Kisin at Clay Mathematical Institute. We mainly focus on Shimura varieties of abelian type. The note-taker claims the responsibility of all mistakes while disclaiming any originality.

Contents

1. Introduction to Shimura varieties

1.1. **Moduli space of elliptic curves.** The starting point lies in the case of modular curves with standard setups. Let X be the $\mathrm{GL}_2(\mathbb{R})$-conjugacy class whose representative is the map $h_0\colon \mathbb{C}^\times \to \mathrm{GL}_2(\mathbb{R})$ defined by $h_0(a+bi) = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$. Geometrically, one can check via taking the orbit of the point i that X consists of points in the upper and lower half planes. If we write $\mathbb{A}_f$ for the finite adeles over $\mathbb{Q}$, then, for any compact open subgroup $\mathsf{K} \subset \mathrm{GL}_2(\mathbb{A}_f)$,

$$\mathbb{X}_{\mathsf{K}} = \mathrm{GL}_2(\mathbb{Q})\backslash X \times \mathrm{GL}_2(\mathbb{A}_f)/\mathsf{K}$$

is a modular curve. If we identify X with $\mathbb{C} - \mathbb{R}$ as sets, then each point $h \in X$, representing a map $h\colon \mathbb{C}^\times \to \mathrm{GL}_2(\mathbb{R})$, is in correspondence with some $\tau_h \in \mathbb{C} - \mathbb{R}$. This procedure defines a lattice $\mathbb{Z} + \mathbb{Z}\tau_h \subset \mathbb{C} \simeq \mathbb{R}^2$ by τ_h, which is isomorphic to $\mathbb{Z}^2$. Then we obtain an elliptic curve $E_h = \mathbb{R}^2/(\mathbb{Z} + \mathbb{Z}\tau_h)$. It turns out that the modular curve $\mathbb{X}_{\mathsf{K}}$ parametrizes elliptic curves with extra level structures, and hence admits a moduli interpretation.

To describe the extra level structure in implication, we concern about elliptic curves. For any elliptic curve E over $\mathbb{C}$, define its *global Tate module* to be

$$\widehat{T}(E) \coloneqq \varprojlim_n E[n].$$

This is indeed a $\mathbb{Z}$-module. Also, we write the rational global Tate module as

$$\widehat{V}(E) \coloneqq \widehat{T}(E) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

As a remark, the name "global" is in the sense that the projective limit defining $\widehat{T}(E)$ is taken through all integer torsions. Rather, we may consider the (*rational*) *local Tate module at* p as a $\mathbb{Z}_p$-module in the following:

$$T_p(E) \coloneqq \varprojlim_m E[p^m], \qquad V_p(E) \coloneqq T_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

To each $(h,g) \in X \times \mathrm{GL}_2(\mathbb{A}_f)$ we attach E_h considered up to isogeny and

$$\epsilon_{h,g} \colon \widehat{V}(E_h) \xrightarrow{\sim} \mathbb{A}_f^2 \xrightarrow{g} \mathbb{A}_f^2 \pmod{\mathsf{K}}.$$

Note that (h,g) lies in a unique orbit up to the action of $\mathrm{GL}_2(\mathbb{Q})$ in $X \times \mathrm{GL}_2(\mathbb{A}_f)/\mathsf{K}$, and the pair $(E_h, \epsilon_{h,g})$ depends only on the image of (h,g) in $\mathbb{X}_{\mathsf{K}}$.

1.2. **Moduli space of polarized abelian varieties.**

1.2.1. *Siegel modular variety.* Let V be a vector space over $\mathbb{Q}$ equipped with an alternating pairing $\psi \colon V \times V \to \mathbb{Q}$, that is, a $\mathbb{Q}$-bilinear map such that $V \cong V^* = \mathrm{Hom}_{\mathbb{Q}}(V,\mathbb{Q})$ and $\psi(v,v) = 0$ for all $v \in V$. Take $G = \mathrm{GSp}(V,\psi)$ and let $X = S^{\pm}$ be the *Siegel double space*; the points of X are parametrized by maps $h \colon \mathbb{C}^\times \to \mathrm{GSp}(V_{\mathbb{R}},\psi) = G_{\mathbb{R}}\}$, satisfying that:

- (i) There is a Hodge decomposition $V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$, for which $V^{p,q}$ is the $\mathbb{C}$-vector space consisting of all $v \in V_{\mathbb{C}}$ such that $z \cdot v = z^{-p}\overline{z}^{-q}$ for each $z \in \mathbb{C}$.
- (ii) The symmetric pairing
$$V_{\mathbb{R}} \times V_{\mathbb{R}} \longrightarrow \mathbb{R}, \quad (x,y) \longmapsto \psi(x, h(i)y)$$
is definite.

The following fact would be useful:

- ◇ *If $V_{\mathbb{Z}} \subset V$ is a $\mathbb{Z}$-lattice and $h \in S^{\pm}$, then $V^{-1,0}/V_{\mathbb{Z}}$ is a polarized abelian variety.*

This leads to an interpretation of $\mathbf{Sh}_{\mathsf{K}}(\mathrm{GSp}, S^{\pm})$ as a moduli space for polarized abelian varieties with level structure.

1.2.2. *Shimura varieties and the canonical adelic interpretation.* To understand §1.2.1, we take a review on Shimura varieties. Let G be a connected reductive group over $\mathbb{Q}$ and X a conjugacy class of maps of algebraic groups over $\mathbb{R}$, whose elements are written as

$$h \colon \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \longrightarrow G_{\mathbb{R}}.$$

Here $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}$ denotes the Weil restriction. On real points, h appears to be

$$h \colon \mathbb{C}^\times \longrightarrow G(\mathbb{R}).$$

We assume that the pair (G,X) is a *Shimura datum* defined below.

Definition 1.1. The pair (G,X) is a *Shimura datum* if the following three axioms are satisfied.

- (SV1) The action on $\mathbb{C}^\times$ on $\mathrm{Lie}\, G_{\mathbb{C}}$ by conjugation is via the characters
$$z \longmapsto z\overline{z}^{-1}, 1, z^{-1}\overline{z}.$$
That is, $\mathrm{Lie}\, G_{\mathbb{C}}$ carries a Hodge structure of weights $\{(-1,1),(0,0),(1,-1)\}$; or equivalently, $\mathrm{Lie}\, G_{\mathbb{C}}$ admits the Hodge decomposition
$$\mathrm{Lie}\, G_{\mathbb{C}} = (\mathrm{Lie}\, G_{\mathbb{C}})^{-1,1} \oplus (\mathrm{Lie}\, G_{\mathbb{C}})^{0,0} \oplus (\mathrm{Lie}\, G_{\mathbb{C}})^{1,-1}$$
(see also the Hodge decomposition of condition (i) in §1.2.1).
- (SV2) The image $h(i)$ is a Cartan involution, meaning that twisting the real structure on G by $h(i)$ gives the compact form of $G_{\mathbb{R}}$. (Indeed, (SV1) implies $h(-1)$ is central, and hence $h(i)$ acts by an involution.)
- (SV3) The map $h \colon \mathbb{S} \to G_{\mathbb{R}}$ is nontrivial in each factor of $G_{\mathbb{Q}}^{\mathrm{ad}}$.

For us, the consequences of (SV1)–(SV3) in Definition 1.1 will be more important. To continue on, let $\mathsf{K} \subset G(\mathbb{A}_f)$ be a compact open subgroup. We obtain the following theorem.

Theorem 1.2 (Baily–Borel, complex uniformization)**.** *When* K *is small enough in certain sense,*

$$\mathbf{Sh}_{\mathsf{K}}(G,X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/\mathsf{K}$$

has a natural structure of an algebraic variety over $\mathbb{C}$*.*

In fact, $\mathbf{Sh}_{\mathsf{K}}(G,X)$ has a canonical model over a number field $E = E(G,X)$; here E is called the *reflex field*. In particular, the reflex field does not depend on K, by the work of Shimura, Deligne, etc.. This is a partial reason to why Shimura varieties are central objects in arithmetic geometry and number theory. In the present notes, we will again denote by $\mathbf{Sh}_{\mathsf{K}}(G,X)$ this canonical model as an algebraic variety over $E(G,X)$.

Another essential fact is about the independence of conjugacy classes. More precisely, each h naturally induces a complex map $h_{\mathbb{C}}\colon \mathbb{S}(\mathbb{C}) \to G_{\mathbb{C}}$, and we define

$$\mu_h\colon \mathbb{C}^\times \xrightarrow{z\mapsto z\times 1} \mathbb{C}^\times \times \mathbb{C}^\times = \mathbb{S}(\mathbb{C}) \xrightarrow{h_{\mathbb{C}}} G_{\mathbb{C}}.$$

However, whereas μ_h is defined through h explicitly, the conjugacy class of μ_h does not depend on h. Further, $E(G,X)$ is exactly the field of definition of this conjugacy class.

1.2.3. *Abelian varieties with Hodge structure.* Heuristically, Shimura varieties can be regarded as moduli spaces of *abelian motives*. In particular, they carry variations of the Hodge structure.

Fix a $\mathbb{Q}$-vector space V. For any Shimura datum (G,X), if $G \to \mathrm{GL}(V)$ is a representation of G, we might get a morphism of E-varieties:

$$\mathcal{V} = G(\mathbb{Q})\backslash V \times X \times G(\mathbb{A}_f)/\mathsf{K} \longrightarrow \mathbf{Sh}_{\mathsf{K}}(G,X).$$

Moreover, if $s = (h,g) \in \mathbf{Sh}_{\mathsf{K}}(G,X)$, then

$$h\colon \mathbb{C}^\times \longrightarrow G(\mathbb{R}) \circlearrowright V_{\mathbb{R}}$$

gives a bigrading of the fiber $\mathcal{V}_s$. More explicitly,

$$\mathcal{V}_s \otimes_E \mathbb{C} = \bigoplus_{p,q} V^{p,q}$$

and for any $v \in V^{p,q}$, the action of $h(z)$ is given by

$$h(z)v = z^{-p}\overline{z}^{-q}v.$$

The condition that $\mathbb{C}^\times$ acts on $\operatorname{Lie} G_{\mathbb{C}}$ via the characters

$$z \longmapsto z\overline{z}^{-1}, 1, z^{-1}\overline{z}$$

implies *Griffiths transversality*. Recall that the point s corresponds to an abelian variety in the moduli space $\mathbf{Sh}_{\mathsf{K}}(G,X)$. We want to figure out the extra structure carried by the complex fiber $\mathcal{V}_s \otimes \mathbb{C}$.

Consider a morphism $G \to G'$ of reductive groups, inducing $X \to X'$, and hence a morphism $(G,X) \to (G,X')$ of Shimura data. When $G \hookrightarrow G'$, it further leads to

$$\mathbf{Sh}_{\mathsf{K}}(G,X) \longhookrightarrow \mathbf{Sh}_{\mathsf{K}'}(G',X')$$

for suitable K, K'. Resume on our prescribed constructions, if $(G,X) \hookrightarrow (\mathrm{GSp}(V,\psi), S^\pm)$ as before, then

$$\mathbf{Sh}_{\mathsf{K}}(G,X) \longhookrightarrow \mathbf{Sh}_{\mathsf{K}'}(\mathrm{GSp}, S^\pm)$$

and the LHS carries a family of abelian varieties, which are equipped with a collection of *Hodge cycles*: Let

$$V^{\otimes} := \bigoplus_{n,m} V^{\otimes n} \otimes V^{*\otimes m}.$$

Then $G \subset \mathrm{GSp}(V,\psi)$ is the stabilizer of a collection of tensors $\{s_\alpha\} \subset V^{\otimes}$. Each s_α is fixed by G and hence by $h(\mathbb{C}^\times)$ for $h \in X$, namely,

$$s_\alpha \in V^{\otimes} \cap (V_{\mathbb{C}}^{\otimes})^{0,0}.$$

So the family of abelian varieties on $\mathbf{Sh}_{\mathsf{K}}(G,X)$ carries a collection of Hodge cycles coming from the s_α.

Definition 1.3. A Shimura datum (G, X) is of *Hodge type* if the family of abelian varieties on $\mathbf{Sh}_{\mathsf{K}}(G, X)$ carries a collection of Hodge cycles. Further, if these extra structures can be taken to be endomorphisms of the abelian variety, then (G, X) is called of *PEL type*.

Example 1.4. Let W be a $\mathbb{Q}$-vector space with a quadratic form such that

$$\mathrm{SO}(W)_{\mathbb{R}} = \mathrm{SO}(n, 2).$$

There is a Shimura datum $(\mathrm{SO}(W), X)$, satisfying that for each $h \in X$,

$$\mathrm{Cent}(h) = \mathrm{SO}(n) \times \mathrm{SO}(2) \subset \mathrm{SO}(W)_{\mathbb{R}}.$$

One has an extension

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{CSpin}(W) \longrightarrow \mathrm{SO}(W) \longrightarrow 1$$

and

$$(\mathrm{SO}(W), X) \hookrightarrow (\mathrm{CSpin}(W), X') \hookrightarrow (\mathrm{GSp}(V), S^{\pm})$$

with $\dim V = 2^{n+1}$. For varying n,

- When $n = 19$, the Shimura variety $\mathbf{Sh}_{\mathsf{K}}(\mathrm{SO}(W), X)$ carries a family of K3 surfaces.
- When n is general, the motivic meaning is unclear.

Here, $(\mathrm{SO}(W), X)$ is an example of a Shimura datum of abelian type, which is a "quotient" of the one of Hodge type.

Definition 1.5. A Shimura datum (G, X) is called of *abelian type* if there exists a datum (G', X') of Hodge type and a morphism $G'^{\mathrm{der}} \to G^{\mathrm{der}}$ inducing

$$(G'^{\mathrm{ad}}, X'^{\mathrm{ad}}) \longrightarrow (G^{\mathrm{ad}}, X^{\mathrm{ad}}).$$

The Shimura data of abelian type include almost all (G, X) with G being a classical group. We have certain approaches to relate a Shimura datum of abelian type to the moduli space of abelian varieties with extra structure beyond Hodge structure.

1.3. **Arithmetic properties of Shimura varieties.** We are interested in $\mathbf{Sh}_{\mathsf{K}}(G, X)(\overline{\mathbb{F}}_p)$, the so-called *special fiber* carrying arithmetic information. Here are some applications and motivations to investigate it.

(a) Compute the Hasse–Weil zeta function of $\mathbf{Sh}_{\mathsf{K}}(G, X)$ in terms of automorphic L-functions (Langlands).
(b) Arakelov intersection theory à la Gross–Zagier and Kudla: This relates the intersection theory of arithmetic cycles on $\mathbf{Sh}_{\mathsf{K}}(G, X)$ to special values of derivatives of L-functions.
(c) Honda–Tate theory: Every abelian variety over $\overline{\mathbb{F}}_p$ is isogenous to the reduction of a CM abelian variety. Can we control $\mathbf{Sh}_{\mathsf{K}}(G, X)(\overline{\mathbb{F}}_p)$ in terms of CM points? (This turns out to be an input into (a)).
(d) Testing ideas about motives over $\overline{\mathbb{F}}_p$.

2. Integral models of abelian type

As before we assume G is a connected reductive group over $\mathbb{Q}$. Suppose now G has a reductive model $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$, called the *integral model of G*.

Let $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p$, where $\mathsf{K}_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ is the hyperspecial subgroup; choose $\mathsf{K}^p \subset G(\mathbb{A}_f^p)$ as a compact open subgroup.

2.1. Existence of integral models of abelian type.

Conjecture 2.1 (Langlands–Milne)**.** *If K_p is hyperspecial and $\lambda \mid p$ is a prime of $E = E(G, X)$, then there is a smooth $\mathcal{O}_{E_\lambda}$-scheme $\mathscr{S}_{\mathsf{K}}(G, X)$ extending the E-scheme $\mathbf{Sh}_{\mathsf{K}}(G, X)$, such that the $G(\mathbb{A}_f^p)$-action on*

$$\mathbf{Sh}_{\mathsf{K}_p}(G, X) = \varprojlim_{\mathsf{K}^p} \mathbf{Sh}_{\mathsf{K}_p\mathsf{K}^p}(G, X)$$

extends to

$$\mathscr{S}_{\mathsf{K}_p}(G, X) = \varprojlim_{\mathsf{K}^p} \mathscr{S}_{\mathsf{K}_p\mathsf{K}^p}(G, X),$$

and $\mathscr{S}_{\mathsf{K}_p}(G,X)$ *satisfies the following extension property:*

- *If the ring* R *is regular and formally smooth over* $\mathcal{O}_{E_\lambda}$*, then*

$$\mathscr{S}_{\mathsf{K}_p}(G,X)(R) \xrightarrow{\sim} \mathscr{S}_{\mathsf{K}_p}(G,X)(R[1/p]).$$

Hyperspecial K_p subgroups exist if and only if G is quasi-split at p and split over an unramified extension. This implies K_p is maximal compact. Usually, one can only expect smooth models in this case.

Example 2.2 (Siegel case)**.** Take $(G,X) = (\mathrm{GSp}, S^\pm)$, defined by (V,ψ) as above. Let $V_\mathbb{Z} \subset V$ be a $\mathbb{Z}$-lattice, and K_p the stabilizer of

$$V_{\mathbb{Z}_p} = V_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}_p \subset V \otimes \mathbb{Q}_p.$$

In this case, K_p is hyperspecial if and only if a scalar multiple of ψ induces a perfect $\mathbb{Z}_p$-valued pairing on $V_{\mathbb{Z}_p}$. (Note that the existence of such a perfect pairing is equivalent to the splitting of G at p.) Then

$$\mathsf{K}_p = \mathrm{GSp}(V_{\mathbb{Z}_p}, \psi)(\mathbb{Z}_p),$$

the maximal compact subgroup of $G_{\mathbb{Q}_p}$. The choice of $V_\mathbb{Z}$ makes $\mathbf{Sh}_\mathsf{K}(\mathrm{GSp}, S^\pm)$ as a moduli space for polarized abelian varieties, which leads to a model $\mathscr{S}_\mathsf{K}(\mathrm{GSp}, S^\pm)$ over $\mathcal{O}_{(\lambda)}$.

The integral models $\mathscr{S}_\mathsf{K}(\mathrm{GSp}, S^\pm)$ are *smooth* over $\mathcal{O}_{(\lambda)}$ if and only if the degree of the polarization in the moduli problem is prime to p. This corresponds to the condition that ψ induces a perfect pairing on $V_{\mathbb{Z}_p}$.

The extension property for $\mathscr{S}_{\mathsf{K}_p}(\mathrm{GSp}, S^\pm)$ with hyperspecial K_p is motivated as follows. If R is an $\mathcal{O}_{E_\lambda}$-algebra, and $\mathcal{A}$ an abelian scheme over R, set

$$\widehat{V}^p(\mathcal{A}) = \left(\varprojlim_{p \nmid n} \mathcal{A}[n]\right) \otimes \mathbb{Q}.$$

This is a (pro-)étale group scheme. Attached to $x \in \mathscr{S}_{\mathsf{K}_p}(\mathrm{GSp}, S^\pm)(R)$ one has an abelian scheme $\mathcal{A}_x/R$ and an isomorphism

$$\epsilon_x \colon \widehat{V}^p(\mathcal{A}_x) \xrightarrow{\sim} V \otimes_\mathbb{Q} \mathbb{A}_f^p.$$

(a) Let R be a mixed characteristic discrete valuation ring, and $x \in \mathscr{S}_{\mathsf{K}_p}(\mathrm{GSp}, S^\pm)(R[1/p])$. Let $\overline{K}$ be an algebraic closure of $K = R[1/p]$. The existence of ϵ_x implies $\mathrm{Gal}(\overline{K}/K)$ acts trivially on $\widehat{V}^p(\mathcal{A}_x)$, and in particular, so does the inertia subgroup. This implies $\mathcal{A}_x$ has good reduction (Néron–Ogg–Shafarevich), so x extends to an R-point.

(b) If R is regular and formally smooth over $\mathcal{O}_{E_\lambda}$, one can use the same argument as in (a) to draw the same conclusion.

Theorem 2.3 (Existence of integral models)**.** *If* $p > 2$*,* K_p *is hyperspecial, and* (G,X) *is of abelian type, then* $\mathbf{Sh}_{\mathsf{K}_p}(G,X)$ *admits a smooth integral model*

$$\mathscr{S}_{\mathsf{K}_p}(G,X) = \varprojlim_{\mathsf{K}^p} \mathscr{S}_{\mathsf{K}_p\mathsf{K}^p}(G,X)$$

as in Conjecture 2.1.

Kottwitz's work contains many arguments about this result in PEL cases. In the case of Hodge type, $\mathscr{S}_{\mathsf{K}_p}(G,X)$ is given by taking the normalization of the closure of

$$\mathbf{Sh}_{\mathsf{K}_p}(G,X) \hookrightarrow \mathbf{Sh}_{\mathsf{K}'_p}(\mathrm{GSp}, S^\pm) \hookrightarrow \mathscr{S}_{\mathsf{K}'_p}(\mathrm{GSp}, S^\pm)$$

into a suitable moduli space of polarized abelian varieties. The idea of such a construction goes back to Milne, Faltings, Vasiu, etc.. The difficulty is to show that the resulting scheme is smooth. This uses deformation theory of p-adic divisible groups, which can be viewed as p-adic analogues of Hodge structures of weight 1.

2.2. **Compactification of integral models of abelian type.**

Theorem 2.4 (Madapusi Pera)**.** *Under the same assumptions as in Theorem 2.3, $\mathscr{S}_{\mathsf{K}_p}(G,X)$, which is of abelian type, has good toroidal and minimal compactifications.*

Remark 2.5. When (G,X) is of PEL type, another version of Theorem 2.4 is due to Kai-Wen Lan.

Theorem 2.6 (Kisin–Pappas)**.** *Good (but not smooth in general) models exist when $p>2$, K_p is parahoric, and $G_{\mathbb{Q}_p}$ splits over a tamely ramified extension of $\mathbb{Q}_p$.*

2.3. **The idea to prove Theorem 2.3.** We sketch out the ideas in proving the main theorem on existence of integral models when (G,X) is of Hodge type.

Constructing the integral model from the Siegel model. In this case, we have the embedding of Shimura data

$$(G,X)\longrightarrow(\mathrm{GSp}(V,\psi),S^{\pm}).$$

Then $\mathscr{S}_{\mathsf{K}}(G,X)$ is the normalization of the closure of the image of

$$\theta\colon \mathbf{Sh}_{\mathsf{K}}\longrightarrow\mathbf{Sh}_{\mathsf{K}'}(\mathrm{GSp},S^{\pm})\longrightarrow\mathscr{S}_{\mathsf{K}'}(\mathrm{GSp},S^{\pm}),$$

where $\mathscr{S}_{\mathsf{K}'}(\mathrm{GSp},S^{\pm})$ is an integral model of $\mathbf{Sh}_{\mathsf{K}'}(\mathrm{GSp},S^{\pm})$, called the *Siegel integral model*, obtained from its interpretation as a moduli space of polarized abelian varieties. Here we write $\mathsf{K}'=\mathsf{K}'_p\mathsf{K}'^p$, where $\mathsf{K}'_p\subset\mathrm{GSp}(V_{\mathbb{Q}_p})$ is the stabilizer of a lattice $V_{\mathbb{Z}_p}\subset V_{\mathbb{Q}_p}$. In fact, there is a "faithful representation"

$$G_{\mathbb{Z}_p}\longrightarrow\mathrm{GL}(V_{\mathbb{Z}_p})$$

due to the work of Prasad–Yu. Moreover, the limit process

$$\mathscr{S}_{\mathsf{K}_p}(G,X)\coloneqq\varprojlim_{\mathsf{K}^p}\mathscr{S}_{\mathsf{K}_p\mathsf{K}^p}(G,X)$$

is $G(\mathbb{A}_f^p)$-equivariant, and $\mathscr{S}_{\mathsf{K}_p}(G,X)$ satisfies the extension property as well as $\mathscr{S}_{\mathsf{K}'_p}(\mathrm{GSp},S^{\pm})$ does.

Smoothness of the integral model. To deduce Conjecture 2.1, we are going to show that $\mathscr{S}_{\mathsf{K}}(G,X)$ is a smooth $\mathcal{O}_{E_\lambda}$-scheme. The following is the main idea:

- ◇ Describe the local structure of $\mathscr{S}_{\mathsf{K}_p}(G,X)$ in terms of moduli of "p-adic Hodge structures", or more precisely, p-divisible groups.

2.4. **Hodge cycles.** Let $E(G,X)\subset K\subset\mathbb{C}$ be a field, and fix a point

$$x\in\mathbf{Sh}_{\mathsf{K}}(G,X)(K)\longrightarrow\mathbf{Sh}_{\mathsf{K}'}(\mathrm{GSp},S^{\pm})(K).$$

Then x gives rise to a polarized abelian variety $\mathcal{A}=\mathcal{A}_x$ over K. Here the embedding representation $G\hookrightarrow\mathrm{GL}(V)$ is defined by a collection of tensors $\{s_\alpha\}\in V^{\otimes}$. We have seen in §1.2.3 that these give rise to Hodge cycles

$$s_{\alpha,x}\in H_1(\mathcal{A}_{\mathbb{C}},\mathbb{Q})^{\otimes}=H^1(\mathcal{A}_{\mathbb{C}},\mathbb{Q})^{\otimes},$$

and hence, using the comparison between étale and singular cohomology, to

$$s_{\alpha,x,\ell}\in H^1_{\mathrm{et}}(\mathcal{A}_{\overline{K}},\mathbb{Q}_\ell)^{\otimes}\xrightarrow{\sim}H^1(\mathcal{A}_{\mathbb{C}},\mathbb{Q})^{\otimes}\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$$

for any prime ℓ. In this cohomology comparison, the $\mathbb{Q}_\ell$-vector space $H^1_{\mathrm{et}}(\mathcal{A}_{\overline{K}},\mathbb{Q}_\ell)$ is naturally equipped with an action of $\mathrm{Gal}(\overline{K}/K)$.

Theorem 2.7 (Deligne, *Hodge implies absolute Hodge*)**.** *The ℓ-adic Hodge cycles*

$$s_{\alpha,x,\ell}\in H^1_{\mathrm{et}}(\mathcal{A}_{\overline{K}},\mathbb{Q}_\ell)^{\otimes}$$

are fixed by $\mathrm{Gal}(\overline{K}/K)$.

This is motivated by the Hodge conjecture which predicts that the $s_{\alpha,x,\ell}$ are the classes of algebraic cycles, so would at least be fixed by an open subgroup $\mathrm{Gal}(\overline{K}/K)$. Now let $V_{\mathbb{Z}_{(p)}} = V_{\mathbb{Z}_p} \cap V \subset V_{\mathbb{Q}_p}$, a $\mathbb{Z}_{(p)}$-lattice in V. Choose $\{s_\alpha\} \subset V_{\mathbb{Z}_p}^{\otimes}$ such that $G_{\mathbb{Z}_p} \subset \mathrm{GL}(V_{\mathbb{Z}_p})$ is the stabilizer of $\{s_\alpha\}$. For $\ell = p$, the tensors

$$s_{\alpha,x,p} \in H^1_{\mathrm{et}}(\mathcal{A}_{\overline{K}}, \mathbb{Q}_p)^{\otimes} \circlearrowleft \mathrm{Gal}(\overline{K}/K)$$

are $\mathrm{Gal}(\overline{K}/K)$-invariant.

Now suppose K is a finite extension of E_λ with the residue field k. Assume $\mathcal{A}_x$ has good reduction, so that x reduces to a point $\overline{x} \in \mathbf{Sh}_{\mathsf{K}}(G,X)(k)$. Take

$$H^1_{\mathrm{et}}(\mathcal{A}_{x,\overline{K}}, \mathbb{Z}_p) \circlearrowleft \mathrm{Gal}(\overline{K}/K) \qquad \text{and} \qquad H^1_{\mathrm{cris}}(\mathcal{A}_{\overline{x}}/W(k)) \circlearrowleft \varphi$$

where φ denotes the absolute Frobenius action. In the crystalline cohomology, $W(k)$ is the ring of Witt vectors on k. For this, in classical p-adic Hodge theory, one has Fontaine's p-adic comparison isomorphism

$$\Phi\colon H^1_{\mathrm{et}}(\mathcal{A}_{x,\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{cris}} \xrightarrow{\sim} H^1_{\mathrm{cris}}(\mathcal{A}_{\overline{x}}/W(k)) \otimes_{W(k)} B_{\mathrm{cris}},$$

which is the analogue of the de Rham isomorphism. The coefficient ring B_{cris} is a $K_0 = W(k)[1/p]$-algebra with an action of $\mathrm{Gal}(\overline{K}/K)$ and φ. Along Φ,

$$H^1_{\mathrm{et}}(\mathcal{A}_{x,\overline{K}}, \mathbb{Z}_p)^{\otimes} \ni s_{\alpha,x,p} \longmapsto s_{\alpha,x,0} \in H^1_{\mathrm{cris}}(\mathcal{A}_{\overline{x}}/W(k)) \otimes \mathbb{Q}_p,$$

which is invariant by φ and lands in Fil^0 Attached to $\mathcal{A}_x$, we have a Néron model $\widetilde{\mathcal{A}}_x$, which is the "integral model" of $\mathcal{A}_x$, i.e., the abelian scheme over $\mathcal{O}_K$ extending $\mathcal{A}_x$. Define

$$\mathscr{G}_x := \varinjlim_n \widetilde{\mathcal{A}}_x[p^n].$$

This package (containing $H^1_{\mathrm{et}}(\mathcal{A}_{x,\overline{K}}, \mathbb{Z}_p)$, $H^1_{\mathrm{cris}}(\mathcal{A}_{\overline{x}}/W(k))$, and $\mathscr{G}_x$) is the p-adic analogue of a Hodge structure of weight 1. To equip our integral p-adic Hodge structure with a "G-structure" and consider its deformations, one needs the following.

Lemma 2.8 (Key lemma)**.** *We have*

$$\{s_{\alpha,x,0}\} \subset H^1_{\mathrm{cris}}(\mathcal{A}_{\overline{x}}/W(k))^{\otimes},$$

and the group fixing the $s_{\alpha,x,0}$ is a reductive subgroup

$$G_{W(k)} \subset \mathrm{GL}(H^1_{\mathrm{cris}}(\mathcal{A}_{\overline{x}}/W(k))).$$

This lemma 2.8 lets us do two things:

(1) The s_α extend to integral sections of the de Rham cohomology of the universal family
$$\mathcal{A} \longrightarrow \mathscr{S}_{\mathsf{K}}(G,X).$$
(2) One can define a deformation space of "p-divisible groups with G-structure", and show it is smooth (worked out by Faltings).

Then using (1), one can identify this deformation space with the completion of $\mathscr{S}_{\mathsf{K}}(G,X)$ at $\overline{x}$.

2.5. ***p*-adic Hodge theory.** For the proof of Lemma 2.8 we need some techniques from p-adic Hodge theory.

Definition 2.9. A *crystalline $\mathbb{Z}_p$-representation* is a finite free $\mathbb{Z}_p$-module L equipped with an action of $\mathrm{Gal}(\overline{K}/K)$, such that

$$\dim_{K_0}(L \otimes B_{\mathrm{cris}})^{\mathrm{Gal}(\overline{K}/K)} = \mathrm{rank}_{\mathbb{Z}_p} L.$$

Here one always has $\leqslant$ above. Denote $\mathsf{Rep}^{\mathrm{cris}}_{\mathbb{Z}_p}$ the category of crystalline $\mathbb{Z}_p$-representations.

Example 2.10. By definition, we immediately see that

$$L = H^1_{\mathrm{et}}(\mathcal{A}_{x,\overline{K}}, \mathbb{Z}_p)$$

is a crystalline $\mathbb{Z}_p$-representation. Actually, all the crystalline $\mathbb{Z}_p$-representations we work with will arise from this L by tensor operations.

Fix a uniformizer $\pi \in K$. Let $E(u) \in W(k)[u]$ be the Eisenstein polynomial for π. Set $\mathfrak{S} = W(k)[\![u]\!]$ equipped with a Frobenius φ acting as the usual Frobenius $u \mapsto u^p$ on $W(k)$.

Definition 2.11. Let $\mathsf{Mod}^{\varphi}_{/\mathfrak{S}}$ denote the category of finite free $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a Frobenius semi-linear isomorphism

$$1 \otimes \varphi \colon \varphi^*(\mathfrak{M})[1/E(u)] \xrightarrow{\sim} \mathfrak{M}[1/E(u)].$$

We have the following "Black (magic) Box" theorem.

Theorem 2.12. *There exists a fully faithful tensor functor*

$$\begin{aligned} \mathfrak{M} \colon \mathsf{Rep}^{\mathrm{cris}}_{\mathbb{Z}_p} &\longrightarrow \mathsf{Mod}^{\varphi}_{/\mathfrak{S}} \\ L &\longmapsto \mathfrak{M}. \end{aligned}$$

If we write $\mathfrak{M} = \mathfrak{M}(L)$, then

(1) *There is a canonical isomorphism*

$$D_{\mathrm{cris}}(L) := (L \otimes B_{\mathrm{cris}})^{\mathrm{Gal}(\overline{K}/K)} \xrightarrow{\sim} (\mathfrak{M}/u\mathfrak{M})[1/p],$$

compatible with Frobenius.

(2) *There is a canonical isomorphism*

$$\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathbb{Z}_p} L \xrightarrow{\sim} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{S}} \mathfrak{M},$$

where $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ is a faithfully flat and formally étale $\widehat{\mathfrak{S}}_{(p)}$-algebra.

(3) *If $L = H^1_{\mathrm{et}}(\mathcal{A}_{x,\overline{K}}, \mathbb{Z}_p)$, then*

$$\varphi^*(\mathfrak{M}/u\mathfrak{M}) \xrightarrow{\sim} H^1_{\mathrm{cris}}(\mathcal{A}_{\overline{x}}/W(k)).$$

Remark 2.13. The functor $\mathfrak{M}$ is still somewhat mysterious. It is constructed using p-adic Hodge theory. (But see Scholze's final talk.)

Proof of Key Lemma 2.8. We apply the above theory with

$$L = H^1_{\mathrm{et}}(\mathcal{A}_{x,\overline{K}}, \mathbb{Z}_p).$$

Recall that $G_{\mathbb{Z}_p} \subset \mathrm{GL}(L)$ is the reductive group defined by $\{s_{\alpha,x,p}\}$. We may view the $s_{\alpha,x,p}$ as morphisms $s_\alpha \colon \mathbb{1} \to L^{\otimes}$ in $\mathsf{Rep}^{\mathrm{cris}}_{\mathbb{Z}_p}$. Applying the functor $\mathfrak{M}$ we obtain morphisms $\widetilde{s}_\alpha \colon \mathbb{1} \to \mathfrak{M}^{\otimes}$ in $\mathsf{Mod}^{\varphi}_{/\mathfrak{S}}$. Specializing the $(\widetilde{s}_\alpha)$ at $u = 0$ gives

$$s_{\alpha,x,0} \in \varphi^*(\mathfrak{M}/u\mathfrak{M})^{\otimes} \xrightarrow{\sim} H^1_{\mathrm{cris}}(\mathcal{A}_{\overline{x}})^{\otimes},$$

which gives the first part of the lemma.

For the second part, we have to show $G_{\mathfrak{S}} \subset \mathrm{GL}(\mathfrak{M})$ defined by the $(\widetilde{s}_\alpha)$ is reductive. Let $\mathfrak{M}' = L \otimes \mathfrak{S}$ and

$$P \subset \underline{\mathrm{Hom}}_{\mathfrak{S}}(\mathfrak{M}', \mathfrak{M})$$

the subscheme of isomorphisms between $\mathfrak{M}'$ and $\mathfrak{M}$ taking $s_{\alpha,x,p}$ to $\widetilde{s}_\alpha$. The fibers of P are either empty or a torsor under $G_{\mathbb{Z}_p}$. We know the $s_{\alpha,x,p}$ define a reductive subgroup in $\mathrm{GL}(L)$ by assumption, so it suffices to show that P is a $G_{\mathbb{Z}_p}$-torsor; but all such torsors are trivial (Lang's lemma). So we have to show that P is flat over $\mathfrak{S}$ with non-empty fibers.

Claim. $P = \underline{\mathrm{Hom}}_{\mathfrak{S},s_\alpha}(\mathfrak{M}', \mathfrak{M}) \subset \underline{\mathrm{Hom}}_{\mathfrak{S}}(\mathfrak{M}', \mathfrak{M})$ is a $G_{\mathbb{Z}_p}$-torsor.

For R any $\mathfrak{S}$-algebra, we set $P_R = P \times_{\operatorname{Spec} \mathfrak{S}} \operatorname{Spec} R$.

Step I ($P_{\mathfrak{S}_{(p)}}$ is a $G_{\mathbb{Z}_p}$-torsor)**.** For this, using Theorem 2.12(2),

$$\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathbb{Z}_p} L \xrightarrow{\sim} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{S}} \mathfrak{M},$$

and thus $P_{\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}}$ is a trivial $G_{\mathbb{Z}_p}$-torsor. Since $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ is faithfully flat over $\widetilde{\mathfrak{S}}_{(p)}$ and $\mathfrak{S}_{(p)}$, we see $P_{\mathfrak{S}_{(p)}}$ is a $G_{\mathbb{Z}_p}$-torsor.

Step II ($P_{\mathfrak{S}[1/pu]}$ is a $G_{\mathbb{Z}_p}$-torsor)**.** Let $U \subset \operatorname{Spec}\mathfrak{S}[1/up]$ be the maximal open subset over which P is flat with nonempty fibers. By Step I, U contains the generic point, so the complement $\operatorname{Spec}\mathfrak{S}[1/p]\backslash U$ is finite. Since the $\widetilde{s}_\alpha$ are φ-invariant, and $\varphi^*(\mathfrak{M})[1/E(u)] \xrightarrow{\sim} \mathfrak{M}[1/E(u)]$, we have

$$P_{\mathfrak{S}[1/E(u)]} \xrightarrow{\sim} \varphi^*(P_{\mathfrak{S}[1/E(u)]}).$$

For each $y \in U$,

(i) If $\varphi^n(y) \neq (E(u))$ for all $n \geqslant 0$, then $\varphi^n(y) \notin U$ by descent.
(ii) If $(E(u)) \not\subset \varphi^{-n}(y)$ for all $n \geqslant 1$, then $\varphi^{-n}(y) \notin U$.

Note that one of these conditions always holds, which contradicts finiteness.

Step III (P_{K_0} is a $G_{\mathbb{Z}_p}$-torsor, where $\mathfrak{S} \to K_0 = W(k)[1/p]$ via $u \mapsto 0$)**.** Known that $D_{\text{cris}}(L[1/p]) = \mathfrak{M}/u\mathfrak{M}[1/p]$, we obtain the isomorphism

$$L \otimes B_{\text{cris}} \xrightarrow{\sim} B_{\text{cris}} \otimes \mathfrak{M}/u\mathfrak{M}.$$

Step IV ($P_{K_0[\![u]\!]}$ is a $G_{\mathbb{Z}_p}$-torsor)**.** There is a canonical φ-equivariant isomorphism

$$\mathfrak{M} \otimes_{\mathfrak{S}} K_0[\![u]\!] \xrightarrow{\sim} K_0[\![u]\!] \otimes_{K_0} (\mathfrak{M}/u\mathfrak{M})[1/p].$$

On the other hand, we can consider $P_{K_0[\![u]\!]} \xrightarrow{\sim} P_{K_0} \otimes_{K_0} K_0[\![u]\!]$, implying that $P_{K_0[\![u]\!]}$ is a $G_{\mathbb{Z}_p}$-torsor by Step III.

Step V (P is a $G_{\mathbb{Z}_p}$-torsor)**.** So far we have proved that P is a torsor over

$$\operatorname{Spec}\mathfrak{S}_{(p)}, \qquad \operatorname{Spec}\mathfrak{S}[1/pu], \qquad \operatorname{Spec}K_0[\![u]\!].$$

This covers $U = \operatorname{Spec}\mathfrak{S} - \{\text{closed points}\}$. So $P|_U$ is a $G_{\mathbb{Z}_p}$-torsor. Consider

$$(\mathrm{GL}(V_{\mathbb{Z}_p})/G_{\mathbb{Z}_p})(U) \longrightarrow H^1(G_{\mathbb{Z}_p}, U) \longrightarrow H^1(\mathrm{GL}(V_{\mathbb{Z}_p}), U) = 0.$$

As $G_{\mathbb{Z}_p}$ is reductive, $\mathrm{GL}(V_{\mathbb{Z}_p})/G_{\mathbb{Z}_p}$ is affine, and a U-point of $\mathrm{GL}(V_{\mathbb{Z}_p})/G_{\mathbb{Z}_p}$ extends to $\operatorname{Spec}\mathfrak{S}$. Hence any $G_{\mathbb{Z}_p}$-torsor over U extends to $\operatorname{Spec}\mathfrak{S}$, and is thus trivial. Therefore, $P|_U$ is trivial and there is a $\mathfrak{M}|_U \xrightarrow{\sim} \mathfrak{M}'|_U$ taking $\widetilde{s}_\alpha$ to s_α. Since any vector bundle over U has a canonical extension to $\mathfrak{S}$, this implies that P is the trivial $G_{\mathbb{Z}_p}$-torsor. □

3. Mod p special fibers of integral models

Suppose (G, X) is a Shimura datum of abelian type. Fix a hyperspecial subgroup $\mathsf{K}_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$ and choose a compact open subgroup $\mathsf{K}^p \subset G(\mathbb{A}_f^p)$. Then for $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p \subset G(\mathbb{A}_f)$, the Shimura variety $\mathbf{Sh}_{\mathsf{K}}(G,X)$ over $E = E(G,X)$ arises with the complex uniformization

$$\mathbf{Sh}_{\mathsf{K}}(G,X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/\mathsf{K}.$$

By Theorem 2.3 we have the $G(\mathbb{A}_f^p)$-equivariant integral model

$$\mathscr{S}_{\mathsf{K}_p}(G,X) = \varprojlim_{\mathsf{K}^p} \mathscr{S}_{\mathsf{K}_p\mathsf{K}^p}(G,X)$$

of $\mathbf{Sh}_{\mathsf{K}_p}(G,X)$. This gives rise to a notion of when two points in $\mathscr{S}_{\mathsf{K}_p}(G,X)(\overline{\mathbb{F}}_p)$ are *isogenous*. Fixing a prime $\lambda \mid p$ in E, we want to do the following.

> *Goal.* We want to describe $\mathscr{S}_{\mathsf{K}_p}(G,X)(\overline{\mathbb{F}}_p)$ as a set equipped with the action of $G(\mathbb{A}_f^p)$ and the Frobenius Φ of $\kappa(\lambda)$.

To motivate the conjecture and results, suppose

$$(G,X) \hookrightarrow (\mathrm{GSp}, S^\pm)$$

is of Hodge type, so that $\mathscr{S}_{\mathsf{K}_p}(G,X)$ carries a family of abelian varieties equipped with certain Hodge cycles.

If $x \in \mathscr{S}_{\mathsf{K}_p}(G,X)(\overline{\mathbb{F}}_q)$ for some $q = p^r$, then attach to $\mathcal{A}_x$ we have

$$s_{\alpha,x,\ell} \in H^1_{\text{et}}(\mathcal{A}_{x,\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell)^\otimes, \quad \ell \neq p,$$

and

$$s_{\alpha,x,0} \in H^1_{\mathrm{cris}}(\mathcal{A}_x/W(\mathbb{F}_q))^{\otimes}.$$

Here the $s_{\alpha,x,\ell}$ comes from the definition and the $s_{\alpha,x,0}$ is given by part of Key Lemma 2.8. For $\ell \neq p$, the group $H^1_{\mathrm{et}}(\mathcal{A}_{x,\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell)$ is equipped with an action of Frob_q; Frob_q fixes the cycles $s_{\alpha,x,\ell}$ and hence gives rise to an element $\gamma_\ell \in G(\mathbb{Q}_\ell)$. The crystalline cohomology

$$\begin{array}{ccc} H^1_{\mathrm{cris}}(\mathcal{A}_x/W(\mathbb{F}_q)) & \xrightarrow{\sim} & V^*_{\mathbb{Z}_p} \otimes W(\mathbb{F}_q) \\ s_{\alpha,x,0} & \longmapsto & s_\alpha \end{array}$$

is equipped with a semi-linear action of the absolute Frobenius which fixes the tensors $s_{\alpha,x,0}$ and hence gives rise to an element $\delta \in G(W(\mathbb{F}_q)[1/p])$.

Denote by σ the absolute Frobenius on $G(W(\mathbb{F}_q))[1/p]$. Then δ is defined up to σ-conjugacy by elements of $G(W(\mathbb{F}_q))$, say

$$\delta \longmapsto g^{-1}\delta\sigma(g),$$

and satisfies

$$\delta \in G(W(\mathbb{F}_q)) \cdot \mu^\sigma(p^{-1}) \cdot G(W(\mathbb{F}_q)).$$

One knows that the characteristic polynomial of γ_ℓ acting on $H^1_{\mathrm{et}}(\mathcal{A}_{x,\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell)$ does not depend on ℓ, and there is a similar compatibility with

$$\gamma_p = \delta\sigma(\delta)\cdots\sigma^{r-1}(\delta).$$

That is, the γ's are conjugate in $\mathrm{GL}(V)$. One might further aks whether they are conjugate in G; this turns out to be true, but it is not immediately obvious.

3.1. **Isogeny classes and special locus.** Let $x, x' \in \mathscr{S}_{\mathsf{K}_p}(G,X)(\overline{\mathbb{F}}_p)$.

Definition 3.1. We say $\mathcal{A}_x, \mathcal{A}_{x'}$ are *isogeneous* if there exists an isomorphism in the isogeny category

$$\begin{array}{ccc} \mathcal{A}_x & \xrightarrow{\sim} & \mathcal{A}_{x'} \\ s_{\alpha,x',\ell} & \longmapsto & s_{\alpha,x,\ell} \end{array}$$

for both $\ell \neq 0$ and $\ell = 0$.

More generally, if R is a $\mathbb{Q}$-algebra, one can consider

$$\mathbf{Isog}_{s_\alpha}(\mathcal{A}_x, \mathcal{A}'_x)(R) := \{\iota \in \mathrm{Hom}_{\mathbb{Q}}(\mathcal{A}_x, \mathcal{A}_{x'})^\times \colon \iota \text{ respects the } s_\alpha\}.$$

Then $\mathbf{Isog}_{s_\alpha}(\mathcal{A}_x, \mathcal{A}_{x'})$ is a $\mathbb{Q}$-scheme for any s_α; glue up all these to have the $\mathbb{Q}$-scheme $\mathbf{Isog}(\mathcal{A}_x, \mathcal{A}_{x'})$. When $x = x'$ we get a $\mathbb{Q}$-group

$$I_x := \mathbf{Isog}(\mathcal{A}_x, \mathcal{A}_x).$$

The Tate conjecture suggests (which will proved later) that

$$I_x \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} G_{\gamma_\ell}$$

for each ℓ (and $\ell = p$ in particular). However, this looks like a little bit of a miracle.

Definition 3.2 (Special locus)**.** A point of $(h,g) \in \mathbf{Sh}_{\mathsf{K}_p}(G,X)(\mathbb{C})$ is called *special* if $h\colon \mathbb{C}^\times \to G(\mathbb{R})$ factors through $T(\mathbb{R})$, where $T \subset G$ is a subtorus *defined over* $\mathbb{Q}$.

$$\begin{tikzcd} h\colon \mathbb{C}^\times \arrow[r] \arrow[dr] & G(\mathbb{R}) \\ & T(\mathbb{R}) \arrow[u] \end{tikzcd}$$

Note that the special points on $\mathbf{Sh}_{\mathsf{K}_p}(G,X)(\mathbb{C})$ correspond to CM abelian varieties. The following is the structure theorem of special fibers, dictating that the isogeny class of each mod p point contains the reduction of a special point.

Theorem 3.3 (Kisin, Madapusi Pera, Shin)**.** *Suppose $G_{\mathbb{Q}_p}$ is quasi-split. Then any $x \in \mathscr{S}_{\mathsf{K}}(G,X)(\overline{\mathbb{F}}_p)$ is isogenous to the reduction of a special point.*

Remark 3.4. When (G, X) is of PEL type, another version of Theorem 3.3 is due to Kottwitz and Zink.

3.2. **The Langlands–Rapoport conjecture.** The idea is to describe the set of isogeny classes, and then the points in each isogeny class. One has the following result, conjectured by Kottwitz and Langlands–Rapoport in greater generality.

Heuristically, in the following the ϕ parameterize "G-isogeny classes", while the $S(\phi)$ parameterize the points in a given isogeny class. We will indicate the definition of the ϕ and then explain the definition of $S(\phi)$. The definition of the ϕ involves the *fundamental groupoid* P of the category of motives over $\overline{\mathbb{F}}_p$. Then $P_{\mathbb{Q}}$ is a pro-torus. The ϕ run over representations $\phi\colon P \to G$ satisfying certain conditions.

It is easier to explain some invariants which can be attached to each ϕ. The idea is that one can attach to each isogeny class the conjugacy class of Frobenius, as for abelian varieties above.

Construction 3.5. Fix an integer $r \gg 0$. Then what attached to ϕ is a triple

$$(\gamma_0, (\gamma_\ell)_{\ell\neq p}, \delta),$$

where

- $\gamma_0 \in G(\mathbb{Q})$ is semi-simple, defined up to conjugacy in $G(\overline{\mathbb{Q}})$ (namely, up to stable conjugacy).
- $\gamma_\ell \in G(\mathbb{Q}_\ell)$ for each $\ell \neq p$ is a semi-simple conjugacy class, stably conjugate to $\gamma_0 \in G(\overline{\mathbb{Q}}_\ell)$.
- $\delta \in G(\operatorname{Fr} W(\mathbb{F}_{p^r}))$ is an element defined up to *Frobenius* conjugacy $\delta \mapsto g^{-1}\delta\sigma(g)$ (where σ is the absolute Frobenius) such that $N\delta = \delta\sigma(\delta)\cdots\sigma^{r-1}(\delta)$ is stably conjugate to γ_0.

The data is required to satisfy certain conditions (corresponding to those on the ϕ). Notice that the compatibility between the γ_ℓ for abelian varieties is built in here. Using these, we can define $S(\phi)$.

Definition 3.6.

$$S(\phi) \coloneqq \varprojlim_{\mathsf{K}^p} I_\phi(\mathbb{Q})\backslash X_p(\phi) \times X^p(\phi)/\mathsf{K}^p.$$

The expression on the right is purely group theoretic, in the sense that it does not involve any algebraic geometry. As a set, each coproduct component $S(\phi) \coloneqq I_\phi(\mathbb{Q})\backslash X_p(\phi) \times X^p(\phi)$ is intended to correspond to the points in a fixed isogeny class in the left side. Here

- $X_p(\phi)$ is the set of p-power isogenies, which can be identified with a subset of $G(\mathbb{Q}_p^{\mathrm{ur}})/G_{\mathbb{Z}_p}(\mathbb{Z}_p^{\mathrm{ur}})$.
- $X^p(\phi) \xrightarrow{\sim} G(\mathbb{A}_f^p)$ is the $G(\mathbb{A}_f^p)$-torsor of prime-to-p isogenies.
- $I_\phi(\mathbb{Q})$ is an algebraic group over $\mathbb{Q}$; when $\mathscr{S}_{\mathsf{K}_p}(G, X)$ is a moduli space for abelian varieties, I_ϕ corresponds to automorphisms of the abelian varieties (up to isogeny) with extra structure; modulo the center, I_ϕ is a compact form (in the sense of modulo Z_G) of the centralizer G_{γ_0}, where $\gamma_0 \in G(\mathbb{Q})$ corresponds to the conjugacy class of Frobenius.

Moreover, we can make $X_p(\phi)$ more explicit. Recall that $S(\phi)$ is meant to parametrize points in a fixed isogeny class. Let $\mathcal{O}_L = W(\overline{\mathbb{F}}_p)$ and $L = \operatorname{Fr}\mathcal{O}_L$. Then we have

$$X_p(\phi) = \{g \in G(L)/G(\mathcal{O}_L)\colon g^{-1}\delta\sigma(g) \in G(\mathcal{O}_L)\mu^\sigma(p^{-1})G(\mathcal{O}_L)\},$$

where $\mu\colon \mathbb{G}_m \to G$ is a cocharacter attached to $h \in X$. The defining condition above corresponds to a group theoretic version of the usual condition on the shape of Frobenius on the Dieudonné module of a p-divisible group.

Theorem 3.7. *Suppose $p > 2$, K_p is hyperspecial, and (G, X) is of abelian type. There is a bijection*

$$\mathscr{S}_{\mathsf{K}_p}(G, X)(\overline{\mathbb{F}}_p) \xrightarrow{\sim} \coprod_\phi I_\phi(\mathbb{Q})\backslash X_p(\phi) \times X^p(\phi) = \coprod_\phi S(\phi),$$

compatible with the Frobenius action and the $G(\mathbb{A}_f^p)$-action. The isomorphism is up to conjugating the action of $I_\phi(\mathbb{Q})$ by an element of $I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$.

Remark 3.8. The coproduct $\coprod_\phi S(\phi)$ can be an infinite disjoint union, not the stratification in the sense of algebraic geometry.

3.3. **p-isogenies.** Consider the Shimura datum (G, X) of Hodge type. We want to recognize the difficulties compared to the PEL type (at least in understanding Theorem 3.7).

(1) Suppose $x \in \mathscr{S}_{\mathsf{K}_p}(G,X)(\overline{\mathbb{F}}_p)$ so that we obtain the associated abelian variety $\mathcal{A}_x$. Note that if $g \in X_p(\delta)$ then gx is followed by $\mathcal{A}_{gx}$. However, it is not clear that whether $gx \in \mathscr{S}_{\mathsf{K}_p}(G,X)(\overline{\mathbb{F}}_p)$ or not, because this is defined as a closure and has no easy moduli description. So it is not clear if there is a map

$$\begin{array}{ccc} X_p & \dashrightarrow & \mathscr{S}_{\mathsf{K}_p}(G,X) \\ g & \mapsto & \mathcal{A}_{gx}. \end{array}$$

(2) Given $x \in \mathscr{S}_{\mathsf{K}_p}(G,X)(\overline{\mathbb{F}}_p)$, we may fetch the tuple $(\delta, (\gamma_\ell)_{\ell\neq p})$. However, the existence of γ_0 in Construction 3.5 is not clear: it is *not* known that the γ_ℓ's are stably conjugate.

(3) Even once one has a (conjectural) map

$$\begin{array}{ccc} X_p \times G(\mathbb{A}_f^p) & \dashrightarrow & \mathscr{S}_{\mathsf{K}_p}(G,X) \\ g & \mapsto & \mathcal{A}_{gx}, \end{array}$$

it is not clear that the stabilizer of a point is a compact form of G_{γ_0} modulo Z_G.

So we need to know that the group

$$I_x = \mathrm{Aut}(\mathcal{A}_x, (s_\alpha))$$

defined earlier is big enough. In the PEL case one can deduce this from Tate's theorem.

The following are some resolutions of (1)–(3).

Step I (The existence of $X_p = X_p(\phi) \to \mathscr{S}_{\mathsf{K}_p}(G,X)$)**.**

We are to solve (1). Let $x \in \mathscr{S}_{\mathsf{K}_p}(G,X)(\overline{\mathbb{F}}_p)$ and choose $\tilde{x} \in \mathscr{S}_{\mathsf{K}_p}(G,X)(\overline{\mathbb{Q}}_p)$ lifting x. Then

$$\begin{array}{ccc} G(\mathbb{Q}_p) & \longrightarrow & \mathscr{S}_{\mathsf{K}_p}(G,X)(\overline{\mathbb{Q}}_p) \\ \tilde{x} & \longmapsto & g\tilde{x}. \end{array}$$

Reduction of isogenies mod p gives a map

$$\begin{array}{ccc} G(\mathbb{Q}_p) & \longrightarrow & X_p \\ g & \longmapsto & g_0. \end{array}$$

For $g \in G(\mathbb{Q}_p)$, define the map $X_p \to \mathscr{S}_{\mathsf{K}_p}(G,X)(\overline{\mathbb{F}}_p)$ at g_0 by sending g_0 to the reduction of $g\tilde{x}$.

On the other hand, elements of $X_p(\phi)$ are not usually of the form g_0. But $X_p(\phi)$ has a "geometric structure" and that $\tilde{x}$ can be chose so that the composite map

$$G(\mathbb{Q}_p) \longrightarrow X_p(\phi) \longrightarrow \pi_0(X_p(\phi))$$

is surjective. (This uses the joint work by Mark Kisin, Miaofen Chen, and Emma Viehmann, describing $\pi_0(X_p(\phi))$ explicitly.) So

$$X_p \longrightarrow \mathscr{S}_{\mathsf{K}_p}(G,X)$$

is well-defined at some point in every connected component. (The map is well-defined on a connected component once it is defined at a point. This is a deformation-theoretic argument.)

Step II (Tate's theorem redux)**.**

To solve (3) that I_x is big enough, one uses a geometric argument. For $\ell \neq p$ define a group I_ℓ over $\mathbb{Q}_\ell$ by

$$I_\ell := \mathrm{Aut}_{\mathrm{Frob}_q}(H^1_{\mathrm{et}}(\mathcal{A}_{x,\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell), s_{\alpha,x,\ell}).$$

Recall that

$$I = I_x = \mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x, s_{\alpha,x,?}), \quad ? \in \{\ell, 0\}$$

where $\ell \neq p$.

Theorem 3.9 (Tate)**.** *For $\ell \neq p$ we have*

$$I \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} I_\ell.$$

Proof. Fix a compact open $\mathsf{K}_\ell \subset G(\mathbb{Q}_\ell)$. Then we have the following:

$$\begin{array}{ccc} I(\mathbb{Q})\backslash I_\ell(\mathbb{Q}_\ell)/(I(\mathbb{Q}_\ell)\cap \mathsf{K}_\ell) & \subset & I(\mathbb{Q})\backslash G(\mathbb{Q}_\ell)/\mathsf{K}_\ell \\ \downarrow & & \downarrow \\ \mathscr{S}_{\mathsf{K}}(G,X)(\mathbb{F}_q) & \subset & \mathscr{S}_{\mathsf{K}}(G,X)(\overline{\mathbb{F}}_p). \end{array}$$

The finiteness of the target of the left vertical map implies that

$$|I(\mathbb{Q})\backslash I_\ell(\mathbb{Q}_\ell)/(I(\mathbb{Q}_\ell)\cap \mathsf{K}_\ell)| < \infty,$$

which is the first ingredient used by Tate. It follows that

$$I(\mathbb{Q}_\ell)\backslash I_\ell(\mathbb{Q}_\ell) \text{ is compact.}$$

Now we choose ℓ so that I_ℓ is a split group (use the compatible system — this choice of ℓ is also made by Tate). The theorem follows from the following.

> *Fact* (Bruhat–Tits). Let I' be a connected algebraic group over $\mathbb{Q}_\ell$ whose reductive quotient is split. If $I \subset I'$ is a closed subgroup such that $I(\mathbb{Q}_\ell)\backslash I'(\mathbb{Q}_\ell)$ is compact, then I contains a Borel subgroup of I'.

For $q = p^r$ with $r \gg 0$, I_ℓ is connected. By the fact (with $I' = I_\ell$) we see $I_{\mathbb{Q}_\ell}\backslash I_\ell$ is projective. As I is reductive, $I_{\mathbb{Q}_\ell}\backslash I_\ell$ is also affine and connected, hence a point. ☐

From the proof above we easily get the result for all q and the theorem is proved for a set of primes ℓ of positive density. Also,

$$\operatorname{rank} I = \operatorname{rank} G_{\gamma_\ell} = \operatorname{rank} G.$$

Using this, one can show the following.

Theorem 3.10. *Every isogeny class in $\mathscr{S}_{\mathsf{K}}(G,X)(\overline{\mathbb{F}}_p)$ contains a point which admits a special lifting.*

Note that Theorem 3.10 implies the existence of γ_0 stably conjugate to all γ_ℓ. This solves problem (2). Finally, this implies that $\dim I_\ell$ does not depend on ℓ, which gives for all ℓ that

$$I \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} I_\ell.$$

3.4. **From Hodge type to abelian type.** The process going from the Hodge case to the abelian case involves the action of $G^{\mathrm{ad}}(\mathbb{Q})^+$ on the tower

$$\mathbf{Sh}(G,X) = \varprojlim_{\mathsf{K}} \mathbf{Sh}_{\mathsf{K}}(G,X) = \varprojlim_{\mathsf{K}} G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/\mathsf{K}.$$

Deligne showed that it is defined over $E(G,X)$. For hyperspecial level $\mathsf{K}_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$, one shows that the action of $G^{\mathrm{ad}}_{\mathbb{Z}_p}(\mathbb{Z}_p)$ extends to $\mathscr{S}_{\mathsf{K}_p}(G,X)$ by giving a moduli-theoretic description.

3.5. **Twisting abelian varieties.** Let $\mathcal{A}/S$ be an abelian scheme with an action of an affine group scheme Z in the isogeny category. Let $\mathbf{P}$ be a Z-torsor, with affine ring $\mathcal{O}_{\mathbf{P}}$. For an S-scheme T we set

$$\mathcal{A}^{\mathbf{P}}(T) := (\mathcal{A}(T) \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbf{P}})^Z.$$

This turns out to be a sheaf represented by an abelian scheme, up to isogeny. There are two ways to view this construction.

- If one thinks of $\mathcal{A}$ as some motivic fundamental group G_{Mot} corresponding to a representation V, then Z acts on V and commutes with G_{Mot}, and $\mathcal{A}^{\mathbf{P}}$ corresponds to the G_{Mot}-representation $(V \otimes_{\mathbb{Q}} \mathcal{O}_{\mathbf{P}})^Z$.

- This construction is related to the familiar one twisting a CM-elliptic curve by an ideal in he ring of integers of its CM field. However, this last construction is trivial, up to isogeny — it gives an isogenous elliptic curve.

We now describe the action of $G^{\mathrm{ad}}(\mathbb{Q})^+$ when (G, X) is of Hodge type. Let $x \in \mathbf{Sh}_{\mathsf{K}}(G, X)$ and $\mathcal{A}_x$ the corresponding abelian variety. Then $\mathcal{A}_x$ is equipped with an action of Z_G. If $\gamma \in G^{\mathrm{ad}}(\mathbb{Q})^+$ then the fiber of $G \to G^{\mathrm{ad}}$ over γ is a Z_G-torsor $\mathbf{P}_\gamma$. On underlying abelian varieties, the action is given by $\mathcal{A}_{\gamma(x)} = \mathcal{A}_x^{\mathbf{P}_\gamma}$.

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