Sen theory for locally analytic representations

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Classical Sen theory

$K/\mathbb{Q}_p$ finite, fix $K \hookrightarrow \overline{\mathbb{Q}}_p$.

$G_K = \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$, $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$.

V f. dim vec space $/\mathbb{Q}_p$, $V \circlearrowleft G_K$ cont rep.

<u>Sen</u>: $\Theta_V \in \mathrm{End}_{\mathbb{C}_p}(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)$ Sen operator

commutes with diagonal action of G_K

& eigenvals (up to ± 1) of Θ_V are HT-Sen wts of V.

Have $W := V \otimes_{\mathbb{Q}_p} \mathbb{C}_p = \bigoplus_{\lambda \text{ eigenval of } \Theta_V} W_\lambda$

generalized eigenspace decomp

If Θ_V is ss, eigenval $\in \mathbb{Z}$ $\Rightarrow$ This is HT decomp.

<u>Construction</u> (Sen, Berger-Colmez)

$$G_K\left(\begin{array}{l} \overline{\mathbb{Q}}_p \\ H_K \Big| \\ K_\infty = K(\mu_{p^\infty}) \\ \Gamma_K \Big| \\ K \end{array}\right.$$

- $\Gamma_K \cong$ an open subgrp of $\mathbb{Z}_p^\times$ (this is $\mathbb{Z}_p^\times$ when $K = \mathbb{Q}_p$). naturally a p-adic Lie grp
- $G_K \circlearrowright V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$
  $\Rightarrow G_K \circlearrowright (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{H_K}$ factors through Γ_K
  $\rightsquigarrow$ can take Γ_K-la part $(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{H_K, \Gamma_K\text{-la}}$
- $(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{H_K, \Gamma_K\text{-la}} \otimes_{K_\infty} \mathbb{C}_p \cong V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$.
  $\circlearrowleft$
  $\mathrm{Lie}\,\Gamma_K \cong \mathbb{Q}_p \ni 1$.

(<u>Rmk</u> K_∞ can be a fin ext $/\mathbb{Q}_p$)

The Lie Γ_K-action is K_∞-linear

+ isom $\Rightarrow$ it extends to $\mathbb{C}_p$-linear action

$\rightsquigarrow \Theta_V \in \mathrm{End}_{\mathbb{C}_p}(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)$.

<u>Question</u> Can we allow more general V (∞-dim'l) ?

Where do the la vectors come from ?

($\dim V < \infty \Rightarrow G_K \curvearrowright V$ "la").

<u>Def'n</u> G top grp (in practice $G = G_K$).

W $\mathbb{Q}_p$-Banach space as a G-rep.

Say W is a <u>locally analytic</u> rep'n of G if

$G \longrightarrow \mathcal{L}_s(W, W)$ is continuous.

strong norm topology

$(\|f\| = \sup_{x \in W^\circ} \|f(x)\|.)$

Or equivalently,

$\forall n \geq 0$, $G_n \leq G$ certain open subgrp. $G_n \triangleleft G_0$ normal,

s.t. · W° is G_0-stable,

· G_n acts <u>trivially</u> on $W^\circ / p^n W^\circ$

Here W° = unit ball in W.

<u>Rmk</u> G f.d. p-adic Lie grp

agree with the defn of Schneider-Teitelbaum.

<u>Main thm</u>

· $G_K \curvearrowright V$ = <u>la</u> rep of G_K

$$\begin{array}{ll} & \overline{\mathbb{Q}}_p \\ H_K & \quad | \\ & K_\infty = K(\mu_{p^\infty}) \\ \Gamma_n & \quad | \\ & K_n = K(\mu_{p^n}) \end{array}$$

- For $n \gg 0$, (completed $\hat{\otimes}$ b/c V not f.d.)

$$(V \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{H_K, \Gamma_n\text{-an}} \hat{\otimes}_{K_n} \mathbb{C}_p \cong V \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p$$

Lie $\Gamma_n \cong \mathbb{Q}_p \ni 1$

$\rightsquigarrow \Theta_V \in \mathrm{End}(V \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)$ Sen operator independent of n.

& Θ_V is functorial in V.

<u>Rmks</u> (1) Thm also obtained by Juan E. Rodriguez Camargo.

(2) $G_K(p)$ max'l pro-p quotient

topologically f.g.

(essential cor of class field theory)

V la $\iff$ $G_{K'}$ open subgrp of G_K

s.t. V^0 is $G_{K'}$-stable &

$\mathrm{im}(G_{K'} \to \mathrm{End}(V^0/pV^0))$ <u>finite</u>.

<u>Examples</u> (1) (Berger-Colmez)

$$\begin{array}{l} L \\ | \\ \mathbb{Q}_p \end{array}$$

L alg Galois, $G = \mathrm{Gal}(L/\mathbb{Q}_p)$

f.d. p-adic Lie grp

$V = \mathcal{C}^{an}(G, \mathbb{Q}_p)$ (in general, $\mathcal{C}(G, \mathbb{Q}_p)^{G_0\text{-an}}$, $G_0 \triangleleft G$ pro-p uniform).

$\circlearrowleft G_{\mathbb{Q}_p}$ via left translation action.

Sen theory $\Rightarrow$ $\Theta_{\mathcal{C}^{an}} \in \mathrm{End}(\mathcal{C}^{an}(G, \mathbb{Q}_p) \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p) \cong \mathrm{End}(\mathcal{C}^{an}(G, \mathbb{C}_p))$

- $\Theta_{\mathcal{C}^{an}}$ is derivation
- $\Theta_{\mathcal{C}^{an}}$ commutes with right translation action (functoriality).

$\Rightarrow$ Thm (Sen) $\Theta_L := \Theta_{G\text{-an}} \in (\underbrace{\mathrm{Lie}\, G \otimes_{\mathbb{Q}_p} \mathbb{C}_p}_{\text{Comactified Lie alg}})^{G_{\mathbb{Q}_p}}$

(2) (Completed cohomology)

T ↻ $G_{\mathbb{Q}_p} \times G$ p-adically complete & p-torsion-free, cont

where G f.d. p-adic Lie grp (pro-p uniform)

e.g. $T = \mathbb{Z}_p$, $1 + p^2 M_n(\mathbb{Z}_p)$.

Let $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ admissible rep of G.

$\rightsquigarrow V^{G\text{-an}}$ is a la rep of $G_{\mathbb{Q}_p}$.

Have that

$$(V^{G\text{-an}})^\circ = (V^\circ \hat{\otimes}_{\mathbb{Z}_p} \mathcal{C}^{an}(G, \mathbb{Q}_p)^\circ)^G$$

$$(V^{G\text{-an}})^\circ/p \hookrightarrow (V^\circ/pV^\circ \otimes \mathcal{C}^{an}(G, \mathbb{Q}_p)^\circ/p)^G.$$

Fact $\mathcal{C}^{an}(G)^\circ/p$ is fixed by some open subgrp G' of G.

But this does not hold for cont funcs.

e.g. $G = \mathbb{Z}_p \Rightarrow \mathcal{C}^{an}(\mathbb{Z}_p)^\circ = \{ f(T) = \sum_{i=0}^{\infty} a_i T^i,\ a_i \in \mathbb{Z}_p,\ a_i \to 0 \}$

$f(T+p) - f(T) \in p\,\mathcal{C}^{an}(\mathbb{Z}_p)^\circ$.

$$\Rightarrow \underset{\circlearrowleft G_{\mathbb{Q}_p}}{(V^{G\text{-an}})^\circ/p} \hookrightarrow \underbrace{\underset{\circlearrowleft G_{\mathbb{Q}_p}}{(V^\circ/pV^\circ)^{G'}}}_{\text{f.d. by admissibility}} \otimes_{\mathbb{F}_p} \mathcal{C}^{an}(G)^\circ/p.$$

$\Rightarrow \mathrm{im}(G_{\mathbb{Q}_p} \to \mathrm{End}((V^{G\text{-an}})^\circ/p))$ is finite.

Sketch of proof of main thm

Can replace K by a finite Galois ext'n.

May assume G_K acts trivially in V°/pV°.

Rough idea: lift "mod p" analytic vectors.

$$(V \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^{H_K, \Gamma\text{-an}} \hat{\otimes}_K \mathbb{C}_p \cong V \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p$$

Essential difficulty Do not have enough analytic vecs

$\leadsto$ need to produce them on LHS.

Consider "integral Gal rep"

$$(V^\circ \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{C}_p})^{H_K} / p \overset{a}{=} V^\circ / p V^\circ \otimes_{\mathbb{F}_p} (\mathcal{O}_{K_\infty} / p)$$

$\cup$ almost $\cup$

"analytic vectors" $\approx V^\circ / p V^\circ \otimes (\mathcal{O}_K / p)$

lift this to $(V^\circ \hat{\otimes} \mathbb{C}_p)^{H_K} / p^n$, $n = 2, 3, \ldots$

(1) make sense of "mod p analytic vecs".

($\mathbb{B}^n_{\mathrm{sen}}$. Colmez's $\mathbb{B}_{\mathrm{sen}}$).

(2) obstruction is "zero" that is killed by p.