Lecture 13: Finite commutative group scheme

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Always assume $k=\bar{k}$, $\operatorname{char} k = p > 0$.

§0 Introduction

E.g. Discrete grps:

- $\mathbb{G}_m[p^n] = \operatorname{Spec} k[x,x^{-1}]/(x^{p^n}-1) = \mu_{p^n}$.
- $\mathbb{G}_a[p^n] = \operatorname{Spec} k[x]/(x^{p^n}) = \alpha_{p^n}$.
- $A[n]$ (or any kernel of isogeny).

Goal Classification of finite grp schemes $/k$.

(char 0 case: easy).

§1 Cartier duality

Setup $G = \operatorname{Spec} A / k \longleftrightarrow$ a Hopf alg structure on A:
$(m, \Delta, 1, \varepsilon, S)$

Brutal def'n $G^\vee = \operatorname{Spec} A^*$ alg grp (A^* Hopf alg).

A better interpretation:

recall Pontryagin duality $G^\vee := \operatorname{Hom}(G, \mathbb{Q}/\mathbb{Z}) = \operatorname{Hom}(G, S^1) = \operatorname{Hom}(G, \mathbb{C}^\times)$.

when $k = \operatorname{Spec} \mathbb{C}$.

Def'n/Prop (Cartier duality)

$\mathcal{H}om_{grp}(G, \mathbb{G}_m)$ is rep'd by $G^\vee = \operatorname{Spec} A^*$ as Zariski sheaf on Sch/k.

Proof $S = \operatorname{Spec} R$, $G^\vee(S) = \operatorname{Hom}_{grp}(A^*, R) = \{\alpha \in A_R \mid \Delta\alpha = \alpha \otimes \alpha,\ \varepsilon(\alpha) = 1\}$.

Take $\mathcal{H}om(G, \mathbb{G}_m)(S) = \operatorname{Hom}_{\text{Hopf Alg}}(R[x,x^{-1}], A_R)$

$$= \left\{ \alpha \in A_R \;\middle|\; \begin{array}{l} \alpha \text{ invertible}, \Delta\alpha = \alpha\otimes\alpha, \\ \text{and } \varepsilon(\alpha) = 1 \end{array} \right\}.$$

$\rightsquigarrow$ It suffices to check: the condition "α invertible" is free.

Check: $\underset{\alpha \cdot S\alpha}{\underset{\|}{m(1\otimes S)\alpha\otimes\alpha}} = \underset{\varepsilon(\alpha)=1}{\underset{\|}{m(1\otimes S)\cdot\Delta\alpha}}$

$\Rightarrow S\alpha = \alpha^{-1}$ automatically. $\square$

Cor $G^{\vee\vee} \cong G$, $G \times G^{\vee} \to \mathbb{G}_m$ perfect pairing.
and $(G\times H)^{\vee} = G^{\vee} \times H^{\vee}$ (functoriality).

E.g. (1) $(\mathbb{Z}/n\mathbb{Z})^{\vee} = \operatorname{Spec} k[\mathbb{Z}/n\mathbb{Z}]$. $(n,p)=1$
$= \operatorname{Spec} k[x]/(x^n-1) \simeq \mu_n$

(2) $(\mathbb{Z}/p^n\mathbb{Z})^{\vee} = \operatorname{Spec} k[\mathbb{Z}/p^n\mathbb{Z}]$
$= \operatorname{Spec} k[x]/(x-1)^{p^n} \simeq \mu_{p^n}$.

(3) $(\alpha_p)^{\vee} = \operatorname{Spec}(k[x]/(x^p))^{*} \cong \operatorname{Spec} k[y]/(y^p)$
$y: x \mapsto 1,\ x^i \mapsto 0\ (i\neq 1)$

(4) $(\alpha_{p^n})^{\vee} = ?$ (to appear later).

Recall: On $k[G]$,
alg str: $g\cdot h = gh$.
coalg str: $\Delta g = g\otimes g$

Cor $0 \to G^0 \to G \to \underbrace{\pi_0(G)}_{\text{"étale part"}} \to 0$. $\rightsquigarrow$ $G = \underset{G_e \times G_c}{\underset{\|}{G_{\text{ét}} \times G_{\text{conn}}}}$.

G^0: a conn component. "Connected part"

$\rightsquigarrow G = G_{ee} \times G_{ec} \times G_{ce} \times G_{cc}$.
$= G^p \times \{1\} \times G^p \times ?$

Next Goal: Classify finite grp sch of type cc.

§2 Higher one theory

Def'n A grp scheme G is of height 1 if $|G| = \{e\}$ and $\forall x \in \mathfrak{m}_e$, $x^p = 0$.

Lem Schematically, $G \simeq k[x_1,\dots,x_n]/(x_1^p,\dots,x_n^p)$.

Proof $\mathfrak{m}_e/\mathfrak{m}_e^2 = \mathrm{span}(\tilde{x}_1,\dots,\tilde{x}_n)$, $\tilde{x}_i$ = image of x_i.

$\rightsquigarrow (\mathfrak{m}_e/\mathfrak{m}_e^2)^\vee = \mathrm{Lie}\, G$. $D_i : A \to A$, $D_i(x_j) = 1 \pmod{\mathfrak{m}_e}$. $G = \mathrm{Spec} A$.

$\Rightarrow$ there's no relation. $\square$

For $G[F] \to G \xrightarrow{\mathrm{Frob}} G^{(1)}$, expect "$G[F] \approx \mathrm{Lie}\, G[F]$".

where $\mathrm{Lie}\, G[F] \simeq \mathrm{Lie}\, G \xrightarrow{0} \mathrm{Lie}\, G^{(1)}$.

Theorem $\{\text{finite grp schs of ht } 1/k\} \xrightarrow{\sim} \{\text{fin-dim'l } p\text{-Lie alg}/k\}$

$$G \longmapsto \mathrm{Lie}\, G$$

Resp. $\{\text{finite commutative grp schs of ht } 1/k\}$ $\quad G \quad G_{ce} \quad G_{cc}$

$$\downarrow \qquad\qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\left\{ (V,\varphi) \,\middle|\, \begin{array}{l} V = \text{fin-dim'l v.s. } /k, \\ \varphi : V \to V \text{ semilinear}, \\ \text{s.t. } \varphi(\lambda v) = \lambda^p \varphi(v) \end{array} \right\} \qquad \mathrm{Lie}\, G \quad (V,\varphi) \quad (V,\varphi)$$

φ isom $\quad \varphi$ nilpotent.

Proof. $G(V,\varphi) = \mathrm{Spec}(\underbrace{\mathrm{Sym}(V)}_{\varphi}/(\alpha^p - \varphi(\alpha), \alpha \in V))^*$.

$\Delta\alpha = \alpha \otimes 1 + 1 \otimes \alpha$.

Prop $\mathrm{Lie}\, G = \{\alpha \in \Gamma(G, \mathcal{O}_G)^* \mid \Delta\alpha = \alpha \otimes 1 + 1 \otimes \alpha, \varepsilon(\alpha) = 1\}$ = primitive elements.

Proof α primitive $\Leftrightarrow \alpha(f \cdot g) = \alpha(f) \cdot \varepsilon(g) + \varepsilon(f) \cdot \alpha(g)$

$$\Rightarrow \begin{cases} \alpha(1^p) = p \cdot \alpha(1) = 0 \\ \varepsilon(f) = \varepsilon(s) = 0 \end{cases} \Rightarrow \alpha(\mathfrak{m}_e^2) = 0 \Rightarrow \alpha \in \mathrm{Lie}\, G.$$

$\Downarrow$

$\alpha(fg) = 0$

Conversely, $\alpha \in \mathrm{Lie}\, G \Rightarrow \alpha$ primitive. $\square$

A priori we obtain $V \hookrightarrow \mathrm{Lie}\, G(V,\varphi)$

and $G(V,\varphi)$ of ht 1: $\alpha(e^i) = 1$, $\alpha(e^j) = 0$ $(j \neq i)$.

$$\Rightarrow \alpha^p\left(\prod_i e_i^{a_i}\right) = (\alpha \otimes \cdots \otimes \alpha)\left(\prod_{i=1}^{n} (e_i \otimes \cdots \otimes 1 + 1 \otimes e_i \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes e_i)^{a_i}\right).$$

Lemma & dimension comparison $\Rightarrow V \xrightarrow{\sim} \mathrm{Lie}\, G(V,\varphi)$.

$\mathrm{Lie}\, G \hookrightarrow \Gamma(G, \mathcal{O}_G)^* \to \Gamma(G, \mathcal{O}_G) \to \mathrm{Sym}(\mathrm{Lie}(G)/(\alpha^p - \varphi(\alpha))$

$G(\mathrm{Lie}\, G) \hookrightarrow G$ & dim argument $\Rightarrow G(\mathrm{Lie}\, G) \simeq G$.

E.g. $\alpha_{p^n} = k[x]/x^{p^n}$.

{primitive elements} $= \mathrm{Span}\{x, x^p, x^{p^2}, \ldots, x^{p^{n-1}}\}$.

$\rightsquigarrow (\alpha_{p^n})^\vee$ is of height 1.

§3 General theory

Ref: Pink, Finite Group Schemes

Some clues ① Prop above $\Rightarrow \mathrm{Lie}\, G \simeq \mathrm{Hom}(G^\vee, \mathbb{G}_a)$.

② Can write $(\alpha_{p^n})^\vee$ more explicitly.

W = Witt vector ring, scheme $/k$,

$\wr$ $\mathrm{Spec}\, k[x_0, x_1, \ldots]$ with actions $\begin{cases} F \text{ (Frobenius)} & (x_0, x_1, \ldots) \mapsto (x_0^p, x_1^p, \ldots) \\ V \text{ (Verschiebung)} & (x_0, x_1, \ldots) \mapsto (0, x_0, x_1, \ldots). \end{cases}$

Define $W_n = W/V^n$, $W_n^m = W_n[F^m]$.

(e.g. $W_1 = W/V = \mathbb{G}_a$, $W_1^n = \alpha_{p^n}$).

$$\rightsquigarrow D(k) := \underset{\circlearrowleft\, \sigma}{W(k)}\{F, V\} \Big/ \begin{pmatrix} VF = FV = p \\ F(\lambda x) = \sigma(\lambda) F(x) \\ V(\sigma(\lambda) x) = \lambda V(x) \end{pmatrix}.$$

Big Theorem (1) {finite commutative p-group scheme $/k$}^{op} $\quad G$

$\sim\downarrow \qquad \downarrow$

{left $D(k)$-modules of fin length} $\quad \varinjlim_n \mathrm{Hom}_{gp}(G, W_n^m) =: M(G)$.

with cc $\longleftrightarrow$ F, V nilpotent

ec $\longleftrightarrow$ F isom, V nilpotent.

ce $\longleftrightarrow$ F nilpotent, V isom.

(2) $M^\vee = \mathrm{Hom}_{W(k)}(M, W(k)[\frac{1}{p}]/W(k)) \circlearrowleft F, V \; (FV = VF)$

$\Rightarrow \quad M(G^\vee) = M(G)^\vee$