p-adic nonabelian Hodge theory via moduli stacks

Daxin Xu

(Joint w/ Ben Heuer)

§1 Introduction

X sm proj curve $/\mathbb{C}$.

Simpson corr:

$$
\begin{tikzcd}
(E,0) & \left\{\begin{array}{c}\text{stable Higgs bundles} \\ \text{of deg zero \& rk } n\end{array}\right\} \arrow[r, leftrightarrow, "1-1"', "\text{Simpson}"] & \left\{\begin{array}{c}\text{irreps of } \pi_1^{top}(X(\mathbb{C})) \\ \text{of dim } n\end{array}\right\} \\
E \arrow[u, mapsto] & \{\text{stable vec bundles}\} \arrow[u, hook] \arrow[r, leftrightarrow, "1-1", "NS"'] & \{\text{unitary reps}\} \arrow[u, hook]
\end{tikzcd}
$$

(by Narasimhan-Seshadri)

Notions
- Higgs bundle (E,θ), E/X vec bun,
  $\theta: E \to E \otimes \Omega^1_X$ $\mathcal{O}_X$-linear
- (E,θ) is semistable (resp. stable)
  if $(F,\varphi) \subsetneq (E,\theta)$ sub-Higgs bundle
  w/ $\mu(F) = \deg F / \mathrm{rk}\, F \leq \mu(E)$ (resp. $< \mu(E)$).

Moduli spaces

M_B moduli space of Betti loc sys.

M_{Dol} moduli space of semistable Higgs bdls of $\deg = 0$, $\mathrm{rk} = n$.

$\rightsquigarrow$ Simpson: $M_B \simeq M_{Dol}$ isom of diff manifold $/\mathbb{R}$.

§2 p-adic Simpson correspondence

$\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p} = K$. X sm proj curve $/\overline{\mathbb{Q}}_p$ + BC to K.

<u>Categorical</u> (Faltings, Abbes-Gross, Tsuji)

$$\begin{tikzcd}
\text{"Generalized reps"}/X \arrow[rr, leftrightarrow, "1-1"] & & \text{Higgs bundles}/X & \\
\mathrm{Rep}_K^{cont}(\pi_1^{et}(X)) \arrow[u, hook, "\text{fully faithful}"] \arrow[rr, "(*)"] & & HB^{ss,0}(X) \arrow[u, hook] & (E,0) \\
& & VB^{DW}(X) \arrow[u, hook] \arrow[llu, "\rho^{DW}"] & E \arrow[u, mapsto]
\end{tikzcd}$$

(1) <u>Conj</u> $(*)$ is an equivalence.

(2) Deninger-Werner $VB^{DW}(X)$:

E satisfies: $\exists$ semi-stable model $\mathcal{X}$ of X

& $\mathcal{E}$ model of $E/\mathcal{X}$ s.t. $\mathcal{E}_k$ / connected comp of $\mathcal{X}_k$.

is <u>strongly semistable</u> of deg zero.

(i.e. Y sm proper curve,

$F_Y^{n,*}(\mathcal{E}_k)$ is semistable, $\forall n$.

<u>Key</u> ρ^{DW} is a p-adic analogue of NS corr.

Thm (Xu, 2022) $(*)$ induces

$$\mathrm{Rep}_K^{cont}(\pi_1^{et}(X)) \simeq HB^{p,DW}(X):$$

Higgs bdl

$(E,\theta) \in HB^{p,DW}(X)$: $\exists f: Y \to X$ fin et s.t. $f^\circ(E,\theta) = (F,\varphi)$, $F \in VB^{DW}(Y)$

w/ φ is 'small' Higgs field,

f° "twisted pullback" of Higgs bdls.

Moduli stack

$X (= X^{an})$ viewed as adic space.

"Generalized rep": bdls for v-top on X.

v-top fpqc, proper proj curves are v-covers, proétale.

- $\mathcal{O}_X (= \hat{\mathcal{O}}_X)$ structure sheaf on X_v (v-site on X)
- v-bdls: loc free $\mathcal{O}_X$-mods of fin type / X_v.

Scholze: $\mathrm{Perf}_{/K}$ perfectoid spaces $/K$,

+ v-top & v-stacks.

$\left[\begin{array}{l} G/K \text{ conn red grp } (G = GL_n). \\ \mathfrak{g} \text{ Lie alg of } G. \end{array}\right.$

Both are viewed as adic spaces via $(-)^{an}$.

(1) $\tau \in \{\acute{e}t, v\}$, $\mathrm{Bun}_{G,\tau} : \mathrm{Perf}_K \longrightarrow \mathrm{Grpoids}$

$$T \longmapsto \tau\ G\text{-bdls} / X_T := X \times_K T.$$

(2) $\nu : \tilde{X}_{T,v} \longrightarrow \tilde{X}_{T,\acute{e}t}$,

$$\Omega_{X_T} := R^1\nu_* \mathcal{O}_{X_T} \simeq \pi^* \underbrace{\nu^* \Omega^1_{X/K}}_{\text{as } v\text{-sheaf}}, \quad \pi : X_T \to X \text{ projection}.$$

$\rightsquigarrow$ $\mathrm{Hig}_{G,\tau} : \mathrm{Perf}_K \longrightarrow \mathrm{Grpoids}$

$$T \longmapsto \left\{ (E, \theta) \;\middle|\; \begin{array}{l} E \in \mathrm{Bun}_{G,\tau}(T) \\ \theta \in \Gamma(X_T, \mathrm{Ad}(E) \otimes \Omega_{X_T}) \end{array} \right\}$$

Goal "twisted isom" b/w $\mathrm{Bun}_{G,v}$ & $\mathrm{Hig}_{G,\acute{e}t}$ ($=: \mathrm{Hig}_G$)

over Hitchin base A.

Hitchin base $A = \left(\bigoplus_{i=1}^{n} \Gamma(X, \Omega_X^{\otimes i})\right) \otimes \mathbb{G}_a$.

$\exists\ h: \quad \mathrm{Hig}_{G,\tau} \longrightarrow A$

$(E, \theta: E \to E \otimes \Omega_X) \longmapsto$ (coeffs of char poly of θ)

$\Lambda^i\theta : \Lambda^i E \to \Lambda^i E \otimes \Omega_X^{\otimes i}$

$\mathrm{Tr}(\Lambda^i\theta) \in \Gamma(X, \Omega_X^{\otimes i})$

§3 $\tilde{h}$ for $\mathrm{Bun}_{G,v}$

<u>Thm</u> (Pan, Carmago, He, Heuer, ...)

Given a v-G-bundle $\mathcal{E} / X$,

$\exists$ a canonical Higgs field $\psi_{\mathcal{E}} \in \Gamma(X, \mathrm{Ad}(\mathcal{E}) \otimes \nu^*\Omega_X)$

s.t. (1) $\mathcal{E} \longmapsto (\mathcal{E}, \psi_{\mathcal{E}})$ fully faithful functor.

(2) $\nu^* :$ G-bdls $/ X_{\text{ét}} \xrightarrow{\sim}$ v-G-bdls w/ $\psi = 0$.

Rmk ψ = Sen operator for $\mathcal{E}$.

<u>pf idea</u> (1) $\tilde{h}: \mathrm{Bun}_{G,v} \longrightarrow \mathrm{Hig}_{G,v} \xrightarrow{h_v} A$.

(2) $\mathrm{Bun}_{G,\text{ét}}$ is a v-stack (satisfies v-descent)

$\begin{array}{c}\Downarrow\\ E\end{array} \longleftrightarrow (\mathcal{E}, \psi_{\mathcal{E}})$ by (1).

§4 Symmetry of $h, \tilde{h}$ (à la Ngô)

<u>Def</u> (Spectral curve)

$\forall\ b = (b_i) \in A(K) = \bigoplus_{i=1}^{n} \Gamma(\Omega_X^{\otimes i})$, define

$X_b = \mathrm{Spec}_{\mathcal{O}_X}\left(\mathrm{Sym}_{\mathcal{O}_X} \Omega_X^{\vee} / (T^n - b_1 T^{n-1} + \cdots + (-1)^n b_n)\right)$

$\pi_b : X_b \to X$ finite of deg n.

If $(E, \theta) \in \mathrm{HB}(X)$, $h(E, \theta) = b$, then $\theta : \Omega_X^\vee \to \mathcal{E}nd(E) \rightsquigarrow \mathrm{Sym}_{\mathcal{O}_X} \Omega_X^\vee$ - action on E

$\rightsquigarrow \mathcal{O}_{X_b}$ - Mod structure on E.

LHS ↻ RHS

$$(*) \quad \pi_{b,*} : \{\text{line bdls} / X_b\} \longrightarrow \{HB(X) \text{ w/ } h(E,\theta) = b\}$$
$$(L, (E,\theta)) \longmapsto \pi_{b,*}(L \otimes_{\mathcal{O}_{X_b}} E).$$

Def $\mathrm{Perf}_{/A} :=$ v-site of perfectoid spaces $/A$.

Define $\mathcal{P}$ Picard v-stack $/ \mathrm{Perf}_A$.

$$\mathcal{P} : (b : T \to A) \longmapsto \text{grpoids of étale line bdls} / X_b.$$

$(*) \rightsquigarrow \mathcal{P} \times \mathrm{Hig}_G \longrightarrow \mathrm{Hig}_G$ over $\mathrm{Perf}_{/A}$.

Over $A^\circ \hookrightarrow A$ where X_b is sm,

$(*)$ is simply transitive & Hig_G is a triv $\mathcal{P}$-torsor.

note Over $A^\circ \hookrightarrow A$, $\mathcal{P}$-action $\rightsquigarrow \mathrm{Bun}_{G,v}$ is a $\mathcal{P}$-torsor.

$\tilde{h} : \mathrm{Bun}_{G,v} \to A$

$$\rightsquigarrow \mathcal{P} \times \mathrm{Bun}_{G,v} \longrightarrow \mathrm{Bun}_{G,v} \qquad \text{over } A$$
$$(L, \mathcal{E}) \longmapsto \pi_{b,*}(\nu^*(L) \otimes_{\mathcal{O}_{X_b}} \mathcal{E})$$

where $\mathcal{E} \in \mathrm{Bun}_{G,v}$, $\tilde{h}(\mathcal{E}) = b$, $L \in \mathcal{P}(b)$

$\psi_{\mathcal{E}} \rightsquigarrow \mathcal{O}_{X_b}$-mod str on $\mathcal{E} / X_v$.

Thm (Heuer-Xu)

$\exists$ a $\mathcal{P}$-torsor $\mathcal{H}$ on $\mathrm{Perf}_{/A}$.

and a twisted isom

$$\mathcal{S} : \mathcal{H} \overset{\beta}{\times} \mathrm{Hig}_G \xrightarrow{\sim} \mathrm{Bun}_{G,v} \quad \text{over } \mathrm{Perf}\,\mathbb{A}$$

Hig_G, $\mathrm{Bun}_{G,v}$ small v-stacks

+ Scholze top on $|\mathrm{Hig}_G(K)|$ & $|\mathrm{Bun}_{G,v}(K)|$.

Thm Fix $\tilde{X}$ flat lift of X to B_{dR}^+/ξ^2.

$\mathrm{Exp} : K \longrightarrow 1 + \mathfrak{m}_K$ a section of $\log : 1 + \mathfrak{m}_K \to K$.

Then $\forall\, b \in A(K)$ gives a splitting of $\mathcal{H}(b) / \mathcal{S}(b)$

$\rightsquigarrow |\mathrm{Hig}(K)| \simeq |\mathrm{Bun}_{G,v}(K)|$.