Bruhat-Tits buildings and p-adic period domains

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Motivation

p prime, $F/\mathbb{Q}_p$ fin, G/F red grp

$\mu: \mathbb{G}_{m,\bar{F}} \longrightarrow G_{\bar{F}}$ minuscule cochar.

$\{\mu^{-1}\}$ Conjugacy class

$\rightsquigarrow \mathcal{F}\ell(G,\mu^{-1})^{ad} / \breve{E}$ adic space assoc to the flag var.

$E = E(G,\{\mu^{-1}\})$ field of def, $\breve{E} = \widehat{E^{ur}}$.

Put $B(G) := G(\breve{F}) / \sigma\text{-conj} \underset{\text{Kottwitz}}{\cong} \{F\text{-isocrystals w/ } G\text{-str}\}$

$\cup\!\text{I}$ fin

$B(G,\mu^{-1}) \underset{\text{Fargues}}{\cong} \{G\text{-bdls on } / X_{C^\flat,F}\}$.

$\Downarrow$ $\qquad$ $X_{C^\flat,F}$: FF curve, $C = \hat{\bar{F}}$.

b

By Kedlaya-Liu, Scholze, etc. $\rightsquigarrow$ p-adic domain

$$\mathcal{F}\ell^a := \mathcal{F}\ell(G,\mu^{-1},b)^a \underset{\text{open}}{\subseteq} \mathcal{F}\ell(G,\mu^{-1})^{ad}.$$

<u>Know</u> $x \in \mathcal{F}\ell(G,\mu^{-1},b)^a(C) \iff \mathcal{E}_{b,x}$ triv G-bdl $/X_{C^\flat,F}$.

adm locus $\qquad$ $\mathcal{E}_{b,x}$: modification of $\mathcal{E}_b$ attached to

$x \in \mathcal{F}\ell(C) \cong Gr^{B_{dR}^+}_{G,\mu^{-1}}(C)$.

<u>In general</u> This is very mysterious.

[CFS]: b basic, $\mathcal{F}\ell^a = \mathcal{F}\ell(G,\mu^{-1},b)^{wa} \iff B(G,\mu^{-1})$ fully HN decomposable

$\mathcal{F}\ell(G,\mu^{-1},b)^{wa}$: RZ period domain

RZ period domain = moduli of wa filtered F-crystals w/ G-str
$\rightsquigarrow$ much easier to compute H_c^*.

<u>Examples</u> · GU: $\{\mu^{-1}\} \longleftrightarrow (1, n-1)$
SO: $\{\mu^{-1}\} \longleftrightarrow (2, n-2)$ } fully HN decomposable.

· GL_5: $\{\mu^{-1}\} \longleftrightarrow (1^2, 0^3)$ not fully HN decomposable.

<u>This talk</u> Find BT buildings of $G_b \subset \mathcal{F}\ell^a$.

<u>BT buildings v.s. p-adic period domains</u>

<u>Example</u> V/F v.s. of dim $n \geq 2$. $G = GL(V) \simeq GL_n/F$.

$\rightsquigarrow$ (reduced) BT building

$$\mathcal{B}(G,F) \cong \{\text{non-arch norms on } V\}/\underset{\text{homothety}}{\sim}.$$

$$\Omega := \underbrace{\mathbb{P}(V)^{an}}_{\text{Berkovich } F\text{-analytic space}} \setminus \bigcup_{\substack{H \text{ F-rat} \\ \text{hyperplane}}} H$$

<u>Berkovich</u> $\theta : \mathcal{B}(G,F) \hookrightarrow \underbrace{\mathbb{P}(V)^{an}}_{\text{seminorms on } \mathrm{Sym}\, V}$ embedding

factors through Ω s.t.

$\theta : \mathcal{B}(G,F) \hookrightarrow \Omega$ is a section of the Drinfeld map

$$\underbrace{r}_{\text{restriction of norms}} : \Omega \longrightarrow \mathcal{B}(G,F)$$

Now G/F red grp,
μ minuscule cochar $\rightsquigarrow \mathcal{F}\ell := \mathcal{F}\ell(G,\mu)^{Ber}/\ E = E(G,\{\mu\})$.

Rémy - Thuillier - Werner

$$\theta : \mathcal{B}(G,F) \longrightarrow \mathcal{F}\ell$$
$$x \longmapsto \theta(x)$$

$x \rightsquigarrow G_x \subset G_E^{an}$ affinoid subgrp $\rightsquigarrow$ $\theta(x)$ = the unique Shilov boundary $\in G_x$ s.t. $\forall\, K/E$, $G_x(K) = \mathrm{Stab}_{G(K)}(x_K)$.

"projection"

Such θ is conti, $G(F)$-equiv, injective if μ non-degenerate.

Caraiani - Scholze $\quad \mathcal{F}\ell = \coprod_{b \in B(G,\mu^{-1})} \mathcal{F}\ell^b \quad$ Newton stratification.

$\circlearrowleft G(F)$

C/E alg closed. $\quad x \in \mathcal{F}\ell^b(C) \iff b(\mathcal{E}_{1,x}) = b$.

$\exists!$ basic elt $b_0 \in B(G,\mu^{-1})$, $\mathcal{F}\ell^{b_0}$ open (Hodge-Tate) p-adic period domain.

E.g. $G = GL_n$, $\mu \leftrightarrow (1, 0^{n-1})$. $\quad \mathcal{F}\ell^{b_0} = \Omega$.

Question $\quad b(\mathcal{E}_{1,\theta(x)}) = ? \qquad x \in \mathcal{B}(G,F)$?

Thm1 $\quad \mathrm{Im}\,\theta \subset \mathcal{F}\ell^{b_0}$, i.e. $b(\mathcal{E}_{1,\theta(x)}) = b_0$, $\forall\, x \in \mathcal{B}(G,F)$.

Ideas $\quad \forall\, K/E$, non arch,

$$\begin{tikzcd} x_K \in \mathcal{B}(G,K) \arrow[r, "\theta_K"] & \mathcal{F}\ell_K \arrow[d, "pr_{K/E}"] \\ x \in \mathcal{B}(G,F) \arrow[u, hook] \arrow[r] & \mathcal{F}\ell \end{tikzcd}$$

Can compute $y \in \theta_K(x_K)$ explicitly $\Rightarrow \mathrm{trdeg}(\widetilde{K(y)} \mid \widetilde{K}) = \dim \mathcal{F}\ell$

$\mathcal{F}\ell_K^{ad} \twoheadrightarrow \mathcal{F}\ell_K \qquad \Rightarrow \dim \overline{\{y\}} = \dim \mathcal{F}\ell$

$\Rightarrow \overline{\{y\}} \subset \mathcal{F}\ell_K^{ad, b_0}$ by dim formula of Newton strata.

Put $\mathcal{B}_\mu(G,F) := \mathrm{Im}\,\theta \subset \overline{\mathcal{B}_\mu(G,F)}$

$\overline{\mathcal{B}_\mu(G,F)}$ = the closure inside $\mathcal{F}\ell$.

[RTW] $\overline{\mathcal{B}_\mu(G,F)} = \coprod_{\tau\text{-relevant } Q} \mathcal{B}_\tau(Q_{ss}, F)$

$\{\mu\} \rightsquigarrow t_\mu$ F-type $\rightsquigarrow \tau$ F-rational type.

- Q F-parab of G:

  $\rightsquigarrow$ $OSC_\tau(Q) := \{P$ parab of type τ s.t. $P \cap Q$ parab$\}$.

- Q τ-relevant: the max parab rep $OSC_\tau(Q')$

Thm 2 Q proper τ-rel $\Rightarrow \mathcal{B}_\tau(Q_{ss}, F) \subset \mathcal{F}\ell^b$.

With b = image of the basic $b_{L_Q,0} \in B(L_Q, \mu^{-1})$ in $B(G, \mu^{-1})$.

$L_Q \subset Q \twoheadrightarrow Q_{ss}$.

Retraction maps for GL_n

Thm 3 $G = GL_n$, $\mu \leftrightarrow (1^d, 0^{n-d})$ s.t. $(d,n) = 1$.

Then $\exists$ a conti map $r : \mathcal{F}\ell^{b_0} \longrightarrow \mathcal{B}(G,F)$

s.t. $r \circ \theta = id$.

Moreover, for $d=1$, r = Drinfeld map.

Sketch of pf

Step 1 $\mathcal{F}\ell^{b_0} \subset \mathcal{F}\ell^{HN = b_0} = \mathcal{F}\ell^{ss}$ = semi-stable locus of filtered F-v.s.

GIT description.

<u>van der Put - Voskuil</u> $(d, n) = 1 \Rightarrow \mathcal{Fl}^{ss} = \mathcal{Fl}^{s}$ (Semi-Stable = Stable).

<u>Step 2</u> $A \subset \mathcal{B}(G, F)$ apartment $\triangleq T$ max torus

$\cup$

o Special vertex

$\rightsquigarrow A \simeq \mathrm{Hom}(X^*(T), \mathbb{R})$

$\mathcal{Fl}, T, \mathcal{L} / \mathcal{O}_F, \quad \mathcal{Fl}(T, \mathcal{L})^s \underset{\text{open}}{\subset} \mathcal{Fl} / \mathcal{O}_F$

$\rightsquigarrow \mathcal{Fl}(T, \mathcal{L})^s \underset{\text{open}}{\subset} \mathcal{Fl}^{\mathrm{Ber}}$

$\rightsquigarrow \gamma_{T,o} : \mathcal{Fl}(T, \mathcal{L})^s \longrightarrow \mathrm{Hom}(X^*(T), \mathbb{R})$

s.t. $\forall K/F$ non-arch,

- $\gamma_{T,o}(\mathcal{Fl}(T, \mathcal{L})^s(\mathcal{O}_K)) = 0$
- $\gamma_{T,o}(tx) = -v_T(t) + \gamma_{T,o}(x), \quad \forall x \in \mathcal{Fl}(T, \mathcal{L})^s(K)$.

$v_T : T(K) \cong \mathrm{Hom}(X^*(T), K^\times) \longrightarrow \mathrm{Hom}(X^*(T), \mathbb{R})$

$\rightsquigarrow \gamma_{T,o}$ Conti & indep of choices of o.

<u>Step 3</u> $A_1, A_2 \subset \mathcal{B}(G, F)$ apartments,

$z \in A_1 \cap A_2$ interior pt of $A_1 \cup A_2$

$\Rightarrow \gamma_{A_1}^{-1}(z) = \gamma_{A_2}^{-1}(z)$.

$\rightsquigarrow \forall z \in \mathcal{B}(G, F), \quad Y_z := \bigcap_{z \in A} \gamma_A^{-1}(z)$ fin intersection.

Have: $Y_z \subset \mathcal{Fl}^s = \bigcap_T \mathcal{Fl}(T, \mathcal{L})^s$

$\mathcal{Fl}^s = \coprod_{z \in \mathcal{B}(G,F)} Y_z \quad \Rightarrow \gamma : \mathcal{Fl}^s \longrightarrow \mathcal{B}(G, F)$ s.t. $\gamma(Y_z) = z$.

<u>Step 4</u> γ is Conti, & $\gamma \circ \theta = \mathrm{Id}$.

<u>Step 5</u> For $d = 1$, γ = Drinfeld map.