# p-adic geometry

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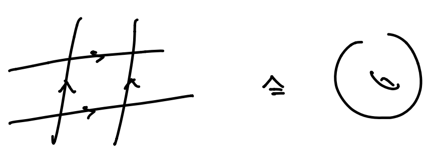
**Thm** $\left\{ \begin{array}{c} \text{elliptic curves} \\ E/\mathbb{Q} \end{array} \right\} / \text{isog} \underset{\text{Wiles}}{\overset{\text{Eichler, Shimura}}{\rightleftarrows}} \left\{ \begin{array}{c} \text{Hecke eigenforms of level } \Gamma_0(N) \text{ \& wt } 2 \\ f(q) = \sum_{n \geq 0} a_n q^n = q + a_2 q^2 + \cdots \in \mathbb{Q}[[q]] \end{array} \right\}$

$\rightsquigarrow \quad L(E, s) = L(f, s)$

$\Rightarrow \quad \forall p \nmid N, \quad \# E(\mathbb{F}_p) = p + 1 - a_p.$

- $E \setminus \{\infty\} : y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Q}.$
- $E(\mathbb{C}) = \mathbb{C}/\Lambda$
  - Riemann surface ($/\mathbb{R}$) of genus 1



$\mathbb{Z} \oplus \mathbb{Z} \simeq \Lambda \subseteq \mathbb{C} \qquad E(\mathbb{C}) = \mathbb{C}/\Lambda.$

<u>Note</u> <u>Thm</u> (Riemann)

$$\left\{ \begin{array}{c} \text{elliptic curves} \\ E/\mathbb{C} \end{array} \right\} / \text{isog} \;\cong\; \left\{ \begin{array}{l} \text{2-dim'l } \mathbb{Q}\text{-v.s. } V \\ + \; W \hookrightarrow V \otimes_{\mathbb{Q}} \mathbb{C} \quad \text{1-dim'l } \mathbb{C}\text{-v.s.} \\ \text{s.t. } W \cap \bar{W} = 0 \end{array} \right\}$$

$$E \longmapsto \left( \begin{array}{l} V = H^1(E, \mathbb{Q}) \\ W = H^0(E, \Omega_E^1) \to H^1(E, \mathbb{C}) \end{array} \right)$$

assoc to $0 \to \mathbb{C} \to \mathcal{O}_E \xrightarrow{d} \Omega_E^1 \to 0$)

$d$ = boundary map.

On side of mod forms:

$\Gamma = \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \mod N \right\}$

$\circlearrowright$

$\mathbb{H} = \{ \tau \in \mathbb{C} \mid \operatorname{Im} \tau > 0 \}$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := \frac{a\tau + b}{c\tau + d}$.

Def'n $f(q) = \sum a_n q^n$ is modular form of wt 2 & level $\Gamma_0(N)$.

if $\bar{f}(\tau) = f(e^{2\pi i \tau}) : \mathbb{H} \longrightarrow \mathbb{C}$ converges

$+ \quad \bar{f}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 \bar{f}(\tau), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

<u>Note</u> This is equiv to: holom 1-form $\bar{f}(\tau)\,d\tau$ on $\mathbb{H}$ being $\Gamma$-invariant.

$\Rightarrow$ Can view $\bar{f} \in H^0(\underbrace{\overline{\Gamma \backslash \mathbb{H}}}_{}, \Omega^1)$

cpt Riemann surface "modular curve".

<u>Hecke operators</u> $\gamma \in GL_2(\mathbb{Q}) \rightsquigarrow \Gamma' := \Gamma \cap \gamma^{-1} \Gamma \gamma$

$$\begin{tikzcd} & \Gamma' \arrow[dl] \arrow[dr, bend left, "\gamma"] & \\ \Gamma & & \Gamma \end{tikzcd}$$

$$\rightsquigarrow \quad \begin{tikzcd} & \overline{\Gamma' \backslash \mathbb{H}} \arrow[dl] \arrow[dr] & \\ \overline{\Gamma \backslash \mathbb{H}} & & \overline{\Gamma \backslash \mathbb{H}} \end{tikzcd}$$

This gives an endom of $H^0(\overline{\Gamma \backslash \mathbb{H}}, \Omega^1)$

<u>So</u> $p \nmid N$, $\gamma = \begin{pmatrix} p & \\ & 1 \end{pmatrix} \rightsquigarrow T_p \in \operatorname{End}(H^0(\overline{\Gamma \backslash \mathbb{H}}, \Omega^1))$

Def'n $\bar{f}$ Hecke eigenform if $f$ eigenvector of $T_p$ $(\forall p)$

Actually, $T_p(\bar{f}) = a_p \bar{f}$

"new form" : $N$ minimally chosen.

<u>Constr'n of $f \longmapsto E_f$</u> By Hodge theory:

$$\begin{array}{ccc} H^1(\overline{\Gamma\backslash \mathbb{H}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} & \hookleftarrow & H^0(\overline{\Gamma\backslash \mathbb{H}}, \Omega^1) \\ \circlearrowleft & & \circlearrowleft \\ T_p,\ \text{all } p \nmid N & & T_p,\ \text{all } p\nmid N \end{array}$$

Pass to $T_p = a_p$ - eigenspace

$$\rightsquigarrow \underbrace{V}_{\dim 2/\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \hookleftarrow \underbrace{W}_{\dim 1/\mathbb{C}}.$$

$\xRightarrow{\text{Riemann}}$ elliptic curve $E_f / \mathbb{C}$.

<u>Thm</u> $E_f$ is defined $/ \mathbb{Q}$ !

<u>Key idea</u> $\overline{\Gamma\backslash \mathbb{H}}$ is defined $/\mathbb{Q}$

(via moduli interpretation & canonical model).

+ $E_f$ factor of $\mathrm{Jac}(\overline{\Gamma\backslash \mathbb{H}})$.

<u>Variant 1</u> ell curve $/F$, $F \neq \mathbb{Q}$ number field.

<u>Assumption</u> There is <u>one</u> embedding $\iota : F \hookrightarrow \mathbb{R}$ ($\Leftarrow$ $\exists$ one real place)

e.g. $F$ CM $/\mathbb{Q}$ or totally real $/\mathbb{Q}$.

<u>Conj</u>

$$\left\{ \begin{array}{c} \text{ell curves} \\ E/F \end{array} \right\} / \text{isog} \xleftrightarrow{\text{bij}} \left\{ \begin{array}{c} \text{"weight 2 autom forms"} \\ \text{on } GL_2/F + \cdots \end{array} \right\}$$

$$E_f \longleftrightarrow f$$

$\forall$ places $v$ of $F$, res field $k(v) = \mathbb{F}_q$

$\# E_f(k(v)) = a_v \leftarrow$ Some Hecke eigenval.

There is a conj in the literature about how to construct $E_f$.

<u>For simplicity</u> Assume all other arch places of $F$ are complex.

Let $\Gamma \subseteq GL_2(\mathcal{O}_F)$ "new for $f$"

$\Gamma \xrightarrow{\iota} GL_2(\mathbb{R})^+ \curvearrowright \mathbb{H}$.

<u>Note</u> $F \neq \mathbb{Q}$, then $\Gamma \hookrightarrow GL_2(\mathbb{R})^+$ has dense image

so $\Gamma \backslash \mathbb{H}$ is very pathological.

<u>Nonetheless</u> Can extract Hodge structure!

$$H^i(\Gamma\backslash\mathbb{H}, \mathbb{Q}) := H^i(\Gamma, \mathbb{Q})$$
$$H^i(\Gamma\backslash\mathbb{H}, \Omega^1_{\mathbb{H}}) := H^i(\Gamma, \Omega^1(\mathbb{H}))$$

group cohom

coh cohom

Still have sequence

$$0 \to \mathbb{C} \to \mathcal{O}_{\mathbb{H}} \to \Omega^1_{\mathbb{H}} \to 0.$$

(can integrate all holom 1-forms w/ ambiguity = const).

$\rightsquigarrow$ boundary map $H^{i-1}(\Gamma\backslash\mathbb{H}, \Omega^1) \to H^i(\Gamma\backslash\mathbb{H}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$.

$\circlearrowleft$ Hecke operator $T_v$

$\rightsquigarrow$ pass to $T_v = a_v$ - eigenspace (say $i = [F:\mathbb{Q}]$).

$$W \hookrightarrow V \otimes_{\mathbb{Q}} \mathbb{C}, \quad W \cap \overline{W} = 0.$$

1-dim'l $/\mathbb{C}$ 2-dim'l $/\mathbb{Q}$

$\overset{\text{Riemann}}{\Longrightarrow}$ $E_f$ ell curve $/\mathbb{C}$ (up to isog).

Conj (Oda, Darmon, Guitart - Masdeu - Sengün 2014)

$E_f$ defined $/F$ is the right ell curve.

Philosophy "$\Gamma \backslash \mathbb{H}$ is defined over $F$".

Concrete Conj $H^1(\Gamma, \Omega^1(\mathbb{H}^3))$ has $F$-str which is stable under Hecke operators.

Variant 2 $GL_2 / \mathbb{Q}$

Conj

$$\left\{ \begin{array}{c} \text{irred 2-dim'l repr} \\ \rho : Gal_{\mathbb{Q}} \to GL_2(\mathbb{C}) \\ \underline{\text{even}} \text{ (i.e. } \det \rho(\text{cplx conj}) = 1) \end{array} \right\} / \cong$$

$$\updownarrow \text{ bij}$$

$$\left\{ \begin{array}{c} \text{cuspidal Maaß eigenforms} \\ \text{of eigenval } 1/4 \end{array} \right\}$$

Maaß forms are very non-alg objects.

Def'n $\Gamma \subseteq SL_2(\mathbb{Z})$ congr subgrp.

$f : \mathbb{H} \to \mathbb{C}$ Maaß form of level $\Gamma$ ($f$ may not holomorphic !)

if (i) $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau)$

(ii) $\Delta(f) = \lambda f$

$\Delta(f) = -y^2 \left( \frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} \right)$ hyperbolic Laplacian on $\mathbb{H} = \{ x + iy \mid y > 0 \}$.

(iii) growth condition.

<u>Issue</u> Maaß form not holomorphic

$\Rightarrow$ cannot understand by $\Omega^1$-section on $\underbrace{\Gamma\backslash\mathbb{H}}_{\text{alg curve}}$

but need instead some real mfd for an analogue.

Conj (Selberg) $\lambda = 0$ or $\lambda \geq 1/4$ (spectral gap).

<u>Rmk</u> (1) odd $\rho \longleftrightarrow$ modular forms of wt 1.

Thm of Langlands - Tunnell, Deligne - Serre, $\cdots$, Khare - Wintenburger.

<u>But</u> Not known for Maaß forms!

(2) $\mathbb{Q} \rightsquigarrow K$ imag quad field

even/odd distinction disappears.

but lose all Shimura varieties!

Let $\pi = \bigotimes_{p\leq\infty} \pi_p$ cusp autom repr of $GL_2(\mathbb{A})$

Corresp. to Maaß form of eigenval $1/4$.

<u>Assumption</u> $\exists\, p<\infty$ s.t. $\pi_p$ disc series.

Choose $K/\mathbb{Q}$ imag quad & $p$ splits in $K$.

Let $\mathbb{H}^{\pm} = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$

$\circlearrowleft GL_2(\mathbb{R})$

$\rightsquigarrow$ p-adic analogue $\underbrace{\mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p)}_{\circlearrowleft GL_2(\mathbb{Q}_p)}$, $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$

Drinfeld's p-adic upper half plane

Have $\quad \mathcal{M}_{Dr,m} \longrightarrow \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p)$

$\circlearrowleft\ GL_2(\mathbb{Q}_p) \qquad$ ($m \geq$ "conductor of" $\pi_p$).

Called Drinfeld space.

$\rightsquigarrow \quad GL_2(\mathbb{Q}_p) \times GL_2(\mathbb{Q}_p) \circlearrowright \mathcal{M}_{Dr,m} \times \mathcal{M}_{Dr,m}$

$$\begin{array}{c} \uparrow \\ GL_2(\mathcal{O}_K[\tfrac{1}{p}]) \supseteq \Gamma \end{array}$$

Let $\omega / \mathcal{M}_{Dr,m} :=$ pullback of $\mathcal{O}(-1) / \mathbb{P}^1$.

Conj $\quad H^i(\Gamma \backslash \mathcal{M}_{Dr,m} \times \mathcal{M}_{Dr,m}, \omega \otimes \omega) \quad \circlearrowleft$ Hecke operators

$:= H^i(\Gamma, H^0(\mathcal{M}_{Dr,m} \times \mathcal{M}_{Dr,m}, \omega \otimes \omega))$

This sees Hecke eigenforms of (the base change of) $\pi$.