Cohomology of p-adic local systems on the coverings of Drinfeld's half space

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<u>Notations</u> p prime, $\mathbb{Z}_p \subset \mathbb{Q}_p \subset \mathbb{Q}_p^{ur} \subset \breve{\mathbb{Q}}_p \subset C := \mathbb{C}_p$.

- $G := GL_2(\mathbb{Q}_p)$, $D/\mathbb{Q}_p$ non split quat alg
  $\check{G} := D^\times \supset \mathcal{O}_D$.
- $\mathcal{G}_{\mathbb{Q}_p} = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \supset W_{\mathbb{Q}_p}$ Weil grp.
- $L/\mathbb{Q}_p$ finite (big enough)
- $B_{dR}^+ \supset B_{st}^+ \supset B_{cr}^+$ p-adic period rings
- $P_+ := \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{N}^2 \mid \lambda_2 > \lambda_1 > 0\}$ wts,
  $w(\lambda) = \lambda_2 - \lambda_1, \quad |\lambda| = \lambda_2 + \lambda_1$.

<u>Introduction</u>

(I) The p-adic local Langlands (Colmez 2010)

$$\left\{\begin{array}{c}\text{2-dim'l } L\text{-rep of } \mathcal{G}_{\mathbb{Q}_p} \\ \text{de Rham of HT wt } \lambda \in P_+\end{array}\right\} \underset{\mathbb{V}}{\overset{\Pi}{\rightleftarrows}} \left\{\begin{array}{c}\text{unitary adm Banach} \\ L\text{-rep of } G \text{ s.t. } \Pi^{lalg} \neq 0\end{array}\right\}$$

$$V \qquad\qquad\qquad\qquad\qquad \Pi$$

s.t. $\mathbb{V} \circ \Pi = \mathrm{id}$.

Here $\Pi^{lalg} \simeq \underbrace{\mathrm{Sym}_L^{w(\lambda)-1} \otimes_L \det{}^{\lambda_1} \otimes |\det|}_{W_\lambda^*} \otimes \underbrace{\Pi^\infty}_{sm} = LL^{lalg}(V) \; (= LL_M^\lambda)$.

<u>Purpose</u> Realize Π in the cohom of some space.

$$\begin{tikzcd}
V \arrow[rr, mapsto] \arrow[dr] & & \Pi^{\infty} = LL_M \\
& M = D_{pst}(V) \arrow[ul, bend left, "V_{st}"] \arrow[ur, mapsto] & + \mathrm{Fil}
\end{tikzcd}$$

$\quad\quad\quad\quad\quad\quad\quad (\varphi, N, \mathcal{G}_{\mathbb{Q}_p})$-mod $/L$ of slope $1 - \frac{|\lambda|}{2}$.

$LL_M \rightsquigarrow JL_M$ sm L-rep'n of $\check{G}$ ($\check{G} \hookrightarrow GL_2(L)$).

s.t. $JL^{(alg)}(V) \; (= JL_M^{\lambda}) = JL_M \otimes \underbrace{\mathrm{Sym}_L^{w(\lambda)-1} \otimes \det^{\lambda_1} \otimes |\det|'}_{\check{W}_\lambda}$.

cuspidal: V — WD irred, $N=0$ $\sim$ Π^{∞} cuspidal — Π cuspidal

special: V — WD indecomp $N \neq 0$ $\sim$ $\Pi^{\infty} \simeq \mathrm{St}_L^{\infty} \otimes \chi$ — Π special

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ "L^2"

(II) The p-adic half plane

$$\mathbb{H}_{\mathbb{Q}_p} = \mathbb{P}^1_C \setminus \mathbb{P}^1(\mathbb{Q}_p) \circlearrowleft G.$$

$\quad\quad$ sm rigid Stein curve.

Drinfeld's modular description (1974)

- $\pi_1(\mathbb{H}_{\mathbb{Q}_p}) \twoheadrightarrow \check{G}^{+} = \mathbb{Q}_D^{\times}$,
- $\check{M}_C^{0} = \coprod_{m \in \mathbb{Z}} \mathbb{H}_C \circlearrowleft G$.

$\Rightarrow$ tower of étale coverings $\check{M}_C^{\infty} = \{\check{M}_C^m\} \circlearrowleft G \times \check{G}$.

For $p \in \mathbb{Q}_p$ ($p \in G$), ${}^{p}M_C^m = \check{M}_C^m / p^{\mathbb{Z}}$ has a model $/\mathbb{Q}_p$.

$$\mathcal{G} = W_{\mathbb{Q}_p} \times G \times \check{G}.$$

$\rightsquigarrow$ L-local system $\mathbb{V}$, $\mathcal{G}$-equiv

$D \otimes |\cdot|$ (unitary central char, $\mathrm{rk}_L = 4$).

$$\mathrm{Sym}\, \mathbb{V} = \bigoplus_{k \geq 0} \mathrm{Sym}_L^k \mathbb{V}.$$

(III) $H^1_{ét} := \varinjlim_m \bigoplus_{k\geq 0} H^1_{ét}(\check{M}^m_C, \mathrm{Sym}^k_L \mathbb{V}(1))$ L-rep of G

Let V = $\mathcal{G}_{\mathbb{Q}_p}$-rep / L

$\rightsquigarrow$ $\Pi_{Dr}(V) := \mathrm{Hom}_{W_{\mathbb{Q}_p}\times\check{G}}(V \otimes JL^{(alg)}(V), H^1_{ét})^{\vee}$

stereotypical dual.

Thm (V, '23) (1) Let V abs irred L-rep of $\mathcal{G}_{\mathbb{Q}_p}$, $\dim \geq 2$.

$$\text{Then } \Pi_{Dr}(V) \simeq \begin{cases} \Pi(V), & V \text{ special or cuspidal } (\dim 2) \\ & \lambda = (0,1), \text{ cuspidal} \\ 0, & \text{else.} \end{cases}$$

(2) Let Π unitary adm + abs irred L-rep of G.

$$\text{Then } \mathrm{Hom}_{L[G]}(\Pi^{\vee}, H^1_{ét}) = \begin{cases} \mathbb{V}(\Pi) \otimes JL^{(alg)}(\mathbb{V}(\Pi)), & \Pi \text{ special or cuspidal} \\ 0, & \text{else.} \end{cases}$$

Proétale cohomology of strongly isotrivial p-adic opers

(I) **Def** Let X sm rigid space / $\mathbb{Q}_p$.

$\mathbb{V}$ a $\mathbb{Q}_p$-loc sys on X_{proet}.

Say $\mathbb{V}$ is <u>isotrivial</u> if $\exists\ D := D_{cr}(\mathbb{V})$ φ-mod s.t.

$$\mathbb{V} \otimes \mathbb{B}_{cr} \simeq D \otimes \mathbb{B}_{cr}$$

crystalline period sheaves

- $\mathbb{V} \longmapsto D_{cr}(\mathbb{V})$ fibre functor
- $\mathbb{V} \longmapsto D_{dR}(\mathbb{V}) := (\mathcal{E}, \nabla, \mathrm{Fil}^{\cdot})$ flat filtered bdl,

  $\mathbb{V}$ isotrivial, $\mathcal{E} = D \otimes \mathcal{O}_X$.

Say $\mathbb{V}$ is <u>strongly isotrivial</u> if $\nabla = \mathrm{id} \otimes d$.

<u>Point</u> · Universal loc sys on RZ spaces are strongly isotrivial

· Vst & a fundamental exact sequence.

<u>Def</u> Let $X/\mathbb{Q}_p$ sm rigid curve. $\mathbb{V}$ a $\mathbb{Q}_p$-loc sys on $X_{\text{proét}}$.

Say $\mathbb{V}$ is a <u>$\mathbb{Q}_p$-oper of wt (a,b)</u>, $b \geq a$, if

- $\mathcal{E} = \mathrm{Fil}^{-b} \supsetneq \cdots \supsetneq \mathrm{Fil}^{-a} \supsetneq 0$
- $\mathrm{gr}_i \mathcal{E} := \mathrm{Fil}^{-i}/\mathrm{Fil}^{-i+1}$ is an invertible sheaf

  $(\mathrm{rk}\, \mathbb{V} = b-a+1)$.
- ∇ induces $\theta : \mathrm{gr}_i \mathcal{E} \xrightarrow{\sim} \mathrm{gr}_{i+1} \Omega_{\mathcal{E}}$ $\quad (a \leq i \leq b-1)$

  $\Omega_{\mathcal{E}} = \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$

$\Rightarrow$ $\mathbb{V}$ has HT wts $a, \cdots, b$

$\Rightarrow$ ∇ defines a diff operator $L_\nabla : \mathrm{gr}_a \mathcal{E} \to \mathrm{gr}_b \Omega_{\mathcal{E}}$ of order $b-a+1$.

<u>Oper complex</u> (for cohom)

$$R\Gamma_{\mathrm{op}}(X; \mathbb{V}) := (\mathrm{gr}_a \mathcal{E} \xrightarrow{L_\nabla} \mathrm{gr}_b \Omega_{\mathcal{E}}).$$

(II) Let X sm rigid Stein curve $/\mathbb{Q}_p$.

$\mathbb{V}$ a $\mathbb{Q}_p$-oper of wt (a,b),

$D_{dR}(\mathbb{V}) = (\mathcal{E}, \nabla, \mathrm{Fil}^\cdot)$.

<u>de Rham complex</u>

$$\begin{array}{ccc} R\Gamma_{dR}(X; \mathbb{V}) := & (\mathcal{E} \xrightarrow{\nabla} & \Omega_{\mathcal{E}}) \\ \mathrm{Fil}^k & \mathrm{Fil}^k & \mathrm{Fil}^{k-1} \end{array}$$

(automatic Griffith transversality).

Def - Prop Have a filtered map that is a strict q-isotrivial

$$R\Gamma_{op}(X;\mathbb{V}) \xrightarrow{\sim} R\Gamma_{dR}(X;\mathbb{V})$$

$\mathbb{V}$ is strongly isotrivial, $D := D_{cr}(\mathbb{V})$

$\rightsquigarrow$ define its isotrivial Hyodo - Kato cohom

$$R\Gamma_{HK}(X;\mathbb{V}) := R\Gamma_{HK}(X) \otimes \underset{(\varphi, N)\text{-mod.}}{D}$$

Then have a HK-map, strict quasi-isom

$$\iota_{HK} : R\Gamma_{HK}(X;\mathbb{V}) \xrightarrow{\sim} R\Gamma_{dR}(X;\mathbb{V}) \simeq R\Gamma_{dR}(X) \otimes D.$$

Define the isotrivial syntomic cohom (= derived Fontaine's recipe, from complex of Gal action)

$$R\Gamma_{syn}(X_C;\mathbb{V}(i)) := \left[\begin{array}{c} (R\Gamma_{HK}(X;\mathbb{V}) \hat{\otimes}^R B_{st}^+)^{N=0,\ \varphi=p^i} \\ \downarrow \\ (R\Gamma_{dR}(X;\mathbb{V}) \hat{\otimes}^R B_{dR}^+)/\mathrm{Fil}^i \end{array}\right]$$

Thm Have a period morphism

$$\alpha_i : R\Gamma_{syn}(X_C;\mathbb{V}(i)) \to R\Gamma_{proet}(X_C;\mathbb{V}(i))$$

becoming a strict quasi-isom after $\tau_{\leq i}$.

Pf FES + Bosco's thesis.

$\rightsquigarrow$ Get a diagram to compute H^1_{proet}.

(III) Back to Drinfeld case $\mathbb{V} \simeq \underset{D\otimes L}{\mathbb{V}_e} \otimes \underset{G\times\check{G}\text{-equiv}}{\check{\mathbb{V}}_e}$, $e = (0,2)$

$\lambda \in P_+$, $\mathbb{V}_e$ corresp to the rep $\check{\mathbb{V}}_e$ of π_1

$$LW_\lambda := \mathrm{Sym}_L^{w(\lambda)-1}\, \mathbb{V}_e \otimes \det{}^{\lambda_1} \xrightarrow{D_{cr}} D_\lambda \simeq W_\lambda$$
$$\xrightarrow{D_{dR}} (\mathcal{E}_\lambda, \nabla_\lambda, \mathrm{Fil}_\lambda)$$

Prop (Schneider- Stuhler)

$\mathbb{V}_\lambda$ is a strongly isotrivial L-oper of wt $(\lambda_1, \lambda_2 - 1)$

and $L_\nabla = (u^+)^{w(\lambda)}$, $\quad u^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{gl}_2$.

Functor formula:

$$\mathrm{Sym}\, \mathbb{V} = \bigoplus_{\lambda \in P_+} V_\lambda \otimes \check{\mathbb{W}}_\lambda .$$

$$H^\lambda_* := H^1_*({}^pM_C^\infty, \mathbb{V}_\lambda(1))$$

$$* \in \{et, proet, dR, HK\}.$$

Get a G-equiv comm diagram of Fréchet spaces:

$B_\lambda = t^{\lambda_1} B^+_{dR}/t^{\lambda_2}$, $\quad t = \log[\varepsilon]$,

$\mathcal{O}^\infty_\lambda := \mathrm{gr}_{\lambda_2 - 1}\, \mathcal{E}_\lambda({}^pM_C^\infty) \qquad \omega^\infty_\lambda := \mathrm{gr}_{\lambda_1}\, \Omega\mathcal{E}_\lambda({}^pM_C^\infty)$.

$\pi_0^\infty = \varprojlim_m \pi_0({}^pM_C^m)$

$$\rightsquigarrow \begin{tikzcd} & U^0_\lambda \otimes_{\mathbb{Q}_{p^2}} W_\lambda^{\pi_0^\infty} \arrow[r] \arrow[d] & B_\lambda \hat{\otimes} \mathcal{O}^\infty_\lambda \arrow[r] \arrow[d, equal] & H^\lambda_{proet} \arrow[r] \arrow[d] & t^{\lambda_1} X^1_{st}(H^\lambda_{HK}[\lambda_1]) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B_\lambda \hat{\otimes} W_\lambda^{\pi_0^\infty} \arrow[r] & B_\lambda \hat{\otimes} \mathcal{O}^\infty_\lambda \arrow[r, "L_\nabla"] & B_\lambda \hat{\otimes} \omega^\infty_\lambda \arrow[r] & B_\lambda \hat{\otimes} H^\lambda_{dR} \arrow[r] & 0 \end{tikzcd}$$

where $\quad U^0_\lambda = t^{\lambda_1} (B^+_{cr})^{\varphi^2 = p^{w(\lambda)+1}}$

$\qquad X^1_{st}(H^\lambda_{HK}[\lambda_1]) = (H^\lambda_{HK} \hat{\otimes} B^+_{st})^{N=0, \varphi = p^{1-\lambda_1}}$.

Lem If $w(\lambda) > 1$, then

$$H^0_{proet}({}^pM_C^\infty, \mathbb{V}_\lambda(1)) = 0 \quad \& \quad \alpha \text{ is injective.}$$

Computing multiplicities

(I) Proét cohom vs étale cohom.

Prop $H^1_{ét} \hookrightarrow H^1_{proét}$ & $H^1_{ét} =$ G-bdd vectors of $H^1_{proét}$.

<u>Idea</u> $H^1_{proét}$ is dual to loc anal & dual to G-bdd vectors

= universal unitary completion.

Cor Π unitary adm Banach L-rep
$$\mathrm{Hom}_G(\Pi, H^1_{ét}) \simeq \mathrm{Hom}_G((\Pi^{\mathrm{lalg}})^\vee, H^1_{proét})$$

<u>Subtle pt</u> A lattice $W^+ \subset W$ doesn't define a G-stable lattice in $H^1_{ét}$.

To remedy: $\Gamma \subset G$ cocompact,

$Y \subset {}^P M_C^\infty$ big enough affinoid, $m \geq 1$.
$$H^{1,\dagger}_{ét} := \left\{ v \in H^+_{ét} \;\middle|\; \begin{array}{c} p^m \mathrm{Res}_Y(\gamma \cdot v) \in H^1_{ét}(Y, \gamma^{-1}\mathrm{Sym}\, W^+(1)), \\ \forall \gamma \in \Gamma \end{array} \right\}.$$

(II) de Rham L-rep = pst L-rep.

- $\lambda \in P_+$
- M L-$(\varphi, N, \mathcal{G}_{\mathbb{Q}_p})$-mod of slope $1 - \frac{|\lambda|}{2}$
  special or cuspidal.
- $0 \subsetneq \mathcal{L} \subsetneq M_{dR} := (M \otimes_{\mathbb{Q}_p^{nr}} C)^{\mathcal{G}_{\mathbb{Q}_p}}$, L-v.s. of dim 2.
$$V^\lambda_{M,\mathcal{L}} := \mathrm{Ker}\left((M \otimes_{\mathbb{Q}_p^{nr}} B^+_{st})^{\substack{N=0 \\ \varphi=p}} \to M_{dR} \otimes B^+_{dR} / (\mathcal{L} \otimes t^{\lambda_1} B^+_{dR} + M_{dR} \otimes t^{\lambda_2} B^+_{dR})\right)$$

Then $\Pi^\lambda_{M,\mathcal{L}} := \Pi(V^\lambda_{M,\mathcal{L}})$ ($\mathcal{L}$ is inexplicit).

But $\Pi^\lambda_{M,\mathcal{L}} \supseteq LL^\lambda_M = {}^{\mathrm{lalg}}\Pi^\lambda_{M,\mathcal{L}}$ ($\mathcal{L}$ explicit)

reducible

${}^{la}\Pi^{\lambda}_{M,\mathcal{L}}$ = adm completions of $\mathbb{L}\mathbb{L}_M^\lambda$ in terms of $\mathcal{L} \subset M_{dR}$.

(II) <u>Cuspidal case</u>.

For X a $\mathbb{Q}_p[\check{G}]$-mod, define

$$X[M] := \mathrm{Hom}_{\check{G}}(JL_M, X).$$

Let M be cuspidal.

Prop

$$\begin{array}{ccccc} H^\lambda_{HK}[M] & \simeq & M \otimes_L \mathbb{L}\mathbb{L}_M^\lambda & & \\ \downarrow & \circlearrowright & \downarrow & & (\mathrm{CDN} + \mathrm{wts}) \\ H^\lambda_{dR}[M] & \simeq & M_{dR} \otimes_L \mathbb{L}\mathbb{L}_M^\lambda & & \end{array}$$

Indeed,

$$\begin{array}{ccc} \tilde{H}^\lambda_{HK}[M] & \simeq & \check{M} \hat{\otimes}_L (\mathbb{L}\mathbb{L}_M^\lambda)^\vee \\ \downarrow & \circlearrowright & \\ \tilde{H}^\lambda_{dR}[M] & \simeq & (M_{dR} \otimes C) \hat{\otimes}_L (\mathbb{L}\mathbb{L}_M^\lambda)^\vee \end{array}$$

where $\tilde{H}^\lambda_{HK} = H^1_{HK}({}^pM^\infty_C, \mathbb{V}_\lambda(1))$

$\tilde{H}^\lambda_{dR} = H^1_{dR}({}^pM^\infty_C, \mathbb{V}_\lambda(1))$ $\quad \mathbb{Q}_p$-v.s.

Apply $[M]$ to the diagram before:

$M = L \otimes \mathbb{Q}_p^{w}$ (rk 2).

$$\begin{array}{ccccccccc} 0 & \to & B_\lambda \hat{\otimes} \mathcal{O}^\infty_\lambda[M] & \to & H^\lambda_{proet}[M] & \to & t^{k_1} X^1_{st}(M[\lambda_1]) \hat{\otimes}_L (\mathbb{L}\mathbb{L}_M^\lambda)^\vee & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & B_\lambda \hat{\otimes} \mathcal{O}^\infty_\lambda[M] & \to & B_\lambda \hat{\otimes} \omega^\infty_\lambda[M] & \to & B_\lambda \otimes M_{dR} \otimes (\mathbb{L}\mathbb{L}_M^\lambda)^\vee & \to & 0 \end{array}$$

By Dospinescu-Le Bras (2017) + Colmez's wt change (2019)

The inverse image of $\mathcal{L} \otimes_L \mathbb{L}\mathbb{L}_M^\lambda \subset M_{dR} \otimes_L \mathbb{L}\mathbb{L}_M^\lambda$ in the lower line:

$$0 \longrightarrow (\widetilde{\Pi}_M^{\lambda})^{\vee} \longrightarrow ({}^{la}\Pi_{M,\mathcal{L}}^{\lambda})^{\vee} \longrightarrow (\amalg_M^{\lambda})^{\vee} \longrightarrow 0 \qquad (*)$$

$$(\widetilde{\Pi}_M^{\lambda})^{\vee} = \mathcal{O}_\lambda^{\infty}[M]$$

- Apply $\mathrm{Hom}_{\mathfrak{g}_{\mathcal{O}_p}}(V_{M,\mathcal{L}}^{\lambda}, -)$:

$\rightsquigarrow$ $\mathrm{Hom}_{\mathfrak{g}_{\mathcal{O}_p}}(V_{M,\mathcal{L}}^{\lambda}, B_\lambda) = 1\text{-dim'l}$

$\mathrm{Hom}_{\mathfrak{g}_{\mathcal{O}_p}}(V_{M,\mathcal{L}}^{\lambda}, t^{k_1} X_{st}^{1}(M[\lambda_1])) = \mathcal{L}$

$\mathrm{Hom}_{\mathfrak{g}_{\mathcal{O}_p}}(V_{M,\mathcal{L}}^{\lambda}, H_{et}^{1}) \simeq \mathrm{Hom}_{\mathfrak{g}_{\mathcal{O}_p}}(V_{M,\mathcal{L}}^{\lambda}, H_{proet}^{\lambda}[M])^{G\text{-bdd}}$

<u>Aside</u> $({}^{la}\Pi_{M,\mathcal{L}}^{\lambda})^{\vee, G\text{-bdd}} \simeq (\Pi_{M,\mathcal{L}}^{\lambda})^{\vee}$,

For π $(= \Pi_{M,\mathcal{L}}^{\lambda})$ any L-rep of G,

$$\mathrm{Hom}_G(\pi^{\vee}, \omega_\lambda^{\infty}[M]) = \begin{cases} \mathcal{L}, \\ 0. \end{cases}$$

(IV) <u>Special case</u>

$M = \mathrm{Sp}_2(|\lambda|-2)$, slope $1 - \frac{|\lambda|}{2}$, $N \neq 0$, $w(\lambda) > 1$.

Remove M from the index

$\rightsquigarrow$ $\Pi_{\mathcal{L}}^{\lambda} := \Pi_{M,\mathcal{L}}^{\lambda}$, $V_{\mathcal{L}}^{\lambda} := V_{M,\mathcal{L}}^{\lambda}$.

Then $${}^{la}\Pi_{\mathcal{L}}^{\lambda} : \underbrace{\underbrace{\underbrace{\mathrm{St}_\lambda^{lalg}}_{= \amalg_M^{\vee}} - B_\lambda}_{\mathrm{St}_\lambda^{la}} - W_\lambda^{*}}_{\Sigma_{\mathcal{L}}^{\lambda}} - B^{\lambda}$$

B_λ, B^{λ}: irred principle series.

$$H_*^{\lambda}[M] = H_*^{1}(\mathbb{H}_C, \mathbb{V}_\lambda(1)) =: H_*^{\lambda}.$$

<u>Prop</u> (Schrean + E) In the derived cat of modules over the la L-vectors with fixed central char.

here $\quad \mathbb{H}\mathrm{om}(\underbrace{({}^{la}\Pi_{\mathscr{L}}^{\lambda})'}_{P_{\mathscr{L}}^{\lambda}}[-1],\ R\Gamma_{dR}^{\lambda}) \simeq D_{pst}(V_{\mathscr{L}}^{\lambda}) + \mathrm{Fil}$

$R\Gamma_{dR}^{\lambda} = R\Gamma_{dR}(\mathbb{H}_{\mathbb{Q}_p}, \mathbb{V}_{\lambda}) \in \mathscr{D}(C_{\mathbb{Q}_p})$ (∞-cat)

$$\left(\begin{array}{l} N: R\Gamma_{HK} \longrightarrow R\Gamma_{HK} \\ \quad H^0 \longrightarrow H^1[-1]. \end{array} \right)$$

Then $\quad \mathrm{Hom}_G((\Pi_{\mathscr{L}}^{\lambda})', H^1_{\acute{e}t}) \simeq \mathrm{Hom}_G(P_{\mathscr{L}}^{\lambda}, H^{\lambda}_{pro\acute{e}t})$

$\simeq \mathrm{Hom}_G(P_{\mathscr{L}}^{\lambda}[-1],\ R\Gamma^{\lambda}_{pro\acute{e}t})$, $\quad R\Gamma^{\lambda}_{pro\acute{e}t} \simeq R\Gamma^{\lambda}_{syn}$

$\simeq V_{st}(\mathbb{H}\mathrm{om}_G(P_{\mathscr{L}}[-1],\ R\Gamma^{\lambda}_{dR})) \simeq V_{\mathscr{L}}^{\lambda}.$

Apply $\quad \mathrm{Hom}(V_{\mathscr{L}}^{\lambda}, -)$

$$0 \longrightarrow W_{\lambda}^{\oplus 2} \longrightarrow \mathrm{Hom}(V_{\mathscr{L}}^{\lambda}, H^{\lambda}_{pro\acute{e}t}) \longrightarrow (\mathrm{St}_{\lambda}^{la}) \longrightarrow 0$$

$\rightsquigarrow \quad \mathrm{Ext}^1(({\mathrm{St}_{\lambda}^{la}})^{\vee}, W_{\lambda}) \simeq \mathrm{Hom}(\mathbb{Q}_p^{\times}, L) \simeq \text{“} L\, v_p \oplus L \log \text{”}$

$$(\Sigma_{\mathscr{L}}^{\lambda}) \longleftarrow\!\!\longmapsto \log - \mathscr{L} v_p.$$

if contains $({}^{la}\Pi_{\mathscr{L}}^{\lambda})^{\vee} \twoheadrightarrow (\Sigma_{\mathscr{L}}^{\lambda})^{\vee}.$

Thm

$$\begin{array}{ccc} \mathrm{Hom}(V_{\mathscr{L}}^{\lambda}, H^{\lambda}_{pro\acute{e}t}) & \simeq & W_{\lambda} \oplus (\Sigma_{\mathscr{L}}^{\lambda})^{\vee} \\ \cup & & \cup \ \ G\text{-bdd} \\ \mathrm{Hom}(V_{\mathscr{L}}^{\lambda}, H^1_{\acute{e}t}) & \simeq & (\Pi_{\mathscr{L}}^{\lambda})^{\vee} \end{array}$$