Lecture 3: Analytic Theory (III) - Algebraizability of Tori

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<u>Review</u> (1) $X = V/U$, $V = \mathbb{C}^g$, U = lattice of rank $2g$.

(2) H = a Hermitian form on V s.t. $E = \operatorname{im} H$ satisfies $E(U\times U) \subseteq \mathbb{Z}$.

(3) $\alpha : U \to \mathbb{C}_1^* = \{z \in \mathbb{C}^* \mid |z| = 1\}$ s.t.

$$\alpha(u_1 u_2) = \exp(i\pi E(u_1,u_2))\,\alpha(u_1)\alpha(u_2)$$

(4) $e_u(z) = \alpha(u)\exp(\pi H(z,u) + \frac{1}{2}\pi(u,u))$

(5) $L(H,\alpha)$ is the quotient of $\mathbb{C}\times V$ for the action of U

given by $\phi_u(\lambda, z) = (\alpha(u)\cdot\exp(\pi H(z,u) + \frac{1}{2}\pi(u,u))\cdot\lambda,\ z+u)$

The grp structure $L(H_1,\alpha_1)\otimes L(H_2,\alpha_2) = L(H_1+H_2, \alpha_1\alpha_2)$ (of $\operatorname{Pic}(X)$).

<u>Main Theorem</u>

$$\begin{array}{ccccccccc}
0 \to & \operatorname{Hom}(U, \mathbb{C}_1^*) & \to & \{\text{Group data of } (U,\alpha)\} & \to & \{\text{Group data of } H\} & \to 0 \\
& \downarrow\wr & & \downarrow\wr & & \downarrow\wr & \\
0 \to & \operatorname{Pic}^0(X) & \to & \operatorname{Pic}(X) & \to & \ker(H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)) & \to 0
\end{array}$$

This time: beginning with Chow's Theorem.

<u>Thm</u> X complete alg var, $Y \subseteq X_{Hol}$ closed analytic <u>subset</u>

Then (by GAGA) Y is Zariski closed in X.

<u>Def'n</u> A <u>theta-function</u> for (H,α) is a holomorphic func $\theta : V \to \mathbb{C}$

s.t. $\theta(z+u) = e_u(z)\theta(z) = \alpha(u)\exp(\pi H(z,u) + \frac{1}{2}H(u,u))\cdot\theta(z)$, $\forall z \in V$, $u \in U$.

<u>note</u> Elements in $H^0(X, L(H,\alpha)) \xleftrightarrow{1-1}$ theta funcs for (H,α).

<u>Prop</u> $L(H,\alpha)$ ample $\Rightarrow$ H positive definite.

Proof Step 1 Suppose H degenerate.

Let $N = \{x \in V : H(x,y) = 0, \forall y \in V\}$

$= \{x \in V : E(x,y) = 0, \forall y \in V\}$ b/c $H = E + iE$

Now $E|_{U\times U}$ is integral $\Rightarrow$ $N \cap U$ is a full lattice of N.

$\forall \theta : V \to \mathbb{C}$, we have

$$\theta(z+u) = \alpha(u)\theta(z), \quad \forall u \in N \cap U.$$

Take K to be a compact subset with $N = K + (N \cap U)$.

$\leadsto$ $|\theta(z_0 + z')| \leq \boxed{\sup_{\xi \in K} |\theta(z_0 + \xi)|} =: c(z_0)$, $\forall z \in N$.

Now by maximum principle on N:

$$\theta(z_0 + z') = \theta(z_0), \quad \forall z' \in N.$$

Denote $\eta : V \to V/N$, θ for (H, α) is of the form $\bar{\theta} \circ \eta$,

where $\bar{\theta} : \bar{V} = V/N \to \mathbb{C}$ theta-func for the lattice $\eta(U)$ & $(\bar{H}, \bar{\alpha})$

$\left(\begin{array}{l} \bar{H} : \text{nondegenerate}, \quad \theta(z) = 0 \ \forall z \in V \\ \bar{u} = \eta(u), \ \bar{\theta}(z + N) = 0, \ \forall z \in V \end{array}\right)$.

Step 2 When H is not positive definite:

$H(w,w) < 0$, $w \in V \setminus \{0\}$, $W = \mathrm{Span}\{w\}$, $V = U + K$, $z_0 \in V$

$\leadsto$ $w = d + u$, $d \in K$, $u \in U$.

$\leadsto$ $|\theta(z_0 + w)| = |\theta(z_0 + d + u)| = |\theta(z+d)| \cdot \exp(\pi \mathrm{Re}\, H(z_0 + d, u) + \frac{1}{2} H(u,u))$

In which: $\mathrm{Re}\, H(z_0 + d, u) + \frac{1}{2} H(u,u)$

$= \mathrm{Re}\, H(z_0 + d, w) - \mathrm{Re}\, H(z_0 + d, d) + \frac{1}{2} H(w,w) + \frac{1}{2} H(d,d) - \mathrm{Re}\, H(w,d)$

$= \frac{1}{2} H(w,w) + \mathrm{Re}\, H(z_0, w) + c(d, z_0)$.

By the maximum principle, $\theta(z_0 + u) = 0 \Rightarrow \theta \equiv 0$. $\square$

Prop $\dim H^0(X, L(H,\alpha)) = \sqrt{\det E}$ $(= \mathrm{Pf}(E))$.

Lemma (Classification of alternating forms of free abelian grp)

$\exists$ a basis of U for which the matrix of E is of the form

$$[E] = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}, \quad D = \mathrm{diag}(d_1, \cdots, d_g)$$

with $\underbrace{d_1 | d_2 | \cdots | d_g}_{\text{invariant factors}}$ & $d_i > 0$.

$\Rightarrow \det D = \prod d_i = \sqrt{\det E} =: \mathrm{Pf}(E)$

is called the Pfaffian of E.

Now $U' = \mathrm{Span}\{e_1, \cdots, e_g\}$ satisfies

(1) $E|_{U' \times U'} = 0$

(2) $W = \mathbb{R} \cdot U'$, $V = W \oplus iW \simeq \mathbb{C} \otimes_{\mathbb{Z}} U' \simeq \mathbb{C} \otimes_{\mathbb{R}} W$

$\rightsquigarrow$ $H|_{W \times W}$ is real symmetric.

Then $\exists !$ symmetric complex bilinear form B on V s.t.

$$B|_{W \times W} = H|_{W \times W}, \quad H(z, u) = B(z, u), \ \forall z \in V, \ u \in W.$$

We can find $\lambda : V \to \mathbb{C}$, $\lambda|_U : W \to \mathbb{R}$

and $\alpha(u) = \exp(2\pi i \lambda(u))$ for $u \in U'$.

Why the Fourier analysis is in need:

Let $\theta^*(z) = \exp(-\frac{1}{2}\pi B(z,z)) \cdot \theta(z)$,

$$\theta^*(z+u) = \alpha(u) \cdot \exp\left(\pi(H-B)(z,u) + \frac{1}{2}\pi(H-B)(u,u)\right) \cdot \theta(z).$$

Let $\phi(z) = \exp(-2\pi i \lambda(z)) \cdot \theta^*(z)$,

$$\phi(z+u) = \phi(z), \quad \forall z \in V, \ u \in U'.$$

$\rightsquigarrow$ $\hat{U}' = \mathrm{Hom}_{\mathbb{Z}}(U, \mathbb{Z}) \subseteq \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$,

$$\phi(z) = \sum_{\chi \in \hat{U}'} c_\chi \exp(2\pi i \chi(z)). \quad \text{Periodic}$$

where $c_\chi = \alpha(u) \cdot \exp(i\pi \hat{u}(u) - 2\pi i \tau \chi(u) + \lambda(u)) \cdot c_{\chi - \hat{u}}$

Theorem of Lefschetz

Say TFAE: (1) Given any complex subtorus Y of X, there's an integer $N>0$, a section σ of $L^{\otimes N}$, and two pts $x_1, x_2 \in X$ with $x_1 - x_2 \in Y$ s.t. $\sigma(x_1)=0$ & $\sigma(x_2)\neq 0$.

(2) H is positive definite

(3) The space of holo sections of $L^{\otimes n}$ give an embedding of X as a closed complex submanifold in a proj space ($n \geq 3$).

Proof (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (1) obvious.

(2) $\Rightarrow$ (3): ① $n=3$, $L^{\otimes 3}$ very ample.

② Globally generated: $\theta \in H^0(X, L(H,\alpha))$, $a, b \in V$ with $\theta(z-a)\theta(z-b)\theta(z+a+b)$ is a section of $L(3H,\alpha)$.

$\rightsquigarrow$ $\alpha(u)^3 \cdot \exp\left(\pi H(z-a,u) + \pi H(z-b,u) + \pi H(z+a+b,u) + \frac{3\pi}{2} H(u,u)\right)$

$= \alpha(u)^3 \cdot \exp\left(\pi \cdot 3H(z,u) + \frac{1}{2}\pi \cdot 3H(u,u)\right)$.

For $z_0 \in V$, take $a, b \in V$ s.t.

$$\theta(z_0-a)\cdot\theta(z_0-b)\cdot\theta(z_0+a+b) \neq 0.$$

If $H^0(X, L(3H,\alpha^3))$ has a basis $\theta_0, \cdots, \theta_d$,

$$\Theta: X \longrightarrow \mathbb{P}^d.$$

$$z \longmapsto (\theta_0(z), \cdots, \theta_d(z))$$

③ Separable points: if not, $\exists z_1, z_2 \in V$, $z_1 - z_2 \notin U$, $\gamma \in \mathbb{C}^*$ s.t. for any $\phi \in H^0(X, L(3H,\alpha^3))$, $\phi(z_1) = \gamma\phi(z_2)$.

$\rightsquigarrow$ $\theta(z_1-a)\theta(z_1-b)\theta(z_1+a+b) = \gamma\theta(z_2-a)\theta(z_2-b)\cdot\theta(z_2+a+b)$

viewed as func's of a.

Denote $\omega = \frac{d\theta}{\theta}$, then $-\omega(z_1-a) + \omega(z_1+a+b) = -\omega(z_2-a) + \omega(z_2+a+b)$

which means $\omega(z_2+z) - \omega(z_1+z)$ is invariant in z

$\rightsquigarrow \quad \omega(z_2+z) - \omega(z_1+z) = d\,l(z)$ (l linear)

$\Theta(z_1+z_2) = A\cdot\exp(l(z))\cdot\Theta(z+z_1)$.

$\Rightarrow \quad e^{\pi H(\sigma,u)} = e^{l(u)}, \quad \forall u\in U. \quad \sigma = z_2 - z_1.$

$\Rightarrow \quad \pi H(\sigma,u) - l(u) \in 2\pi i\mathbb{Z},$

$$\underbrace{\pi H(u,\sigma) - l(u)}_{=0} + 2\pi i\, E(\sigma,u)$$

$\pi H(u,\sigma) = l(u), \ \forall u\in U. \quad E(\sigma,u)\in\mathbb{Z}, \ \forall u\in U.$

$\rightsquigarrow \quad \sigma\in U^{\perp} := \{x\in V : E(x,w)\in\mathbb{Z}, \forall w\in U\}.$

Let $u' = u + \mathbb{Z}\sigma$, then $\det_u E > \det_{u'} E$

<u>but</u> $\Theta(z+\sigma) = A\, e^{l(z)}\,\Theta(z) = A\cdot\exp(\pi H(z,\sigma) + \frac{1}{2}\pi H(\sigma,\sigma))\cdot\Theta(z)$.

④ Separate tangent vectors

If $z_0\in V$, tangent vector $\sum_{i=1}^{g}\alpha_i\cdot\frac{\partial}{\partial z_i} \mapsto 0$

Then $\exists\,\alpha_0\in\mathbb{C}$ s.t. for all $\phi\in H^0(X, L(3H,\alpha^3))$.

$$\alpha_0\phi(z_0) + \sum_{i=1}^{g}\alpha_i\frac{\partial\phi}{\partial z_i}(z_0) = 0,$$

i.e. $D(\log\phi)(z_0) = -\alpha_0$, where $D = \sum_{i=1}^{g}\alpha_i\frac{\partial}{\partial z_i}$.

Recall $\phi(z) = \Theta(z-a)\,\Theta(z-b)\,\Theta(z+a+b)$.

Denote $f(z) = D(\log\Theta)(z)$,

$$f(z_0-a) + f(z_0-b) + f(z_0+a+b) = -\alpha,$$

and f is linear

$\Rightarrow \quad \exists\,\alpha\in V\setminus\{0\}$ s.t. $\Theta(z+\lambda\alpha) = \exp(\lambda^2+\lambda f(z))\cdot\Theta(z), \ \forall\lambda\in\mathbb{C}$.

$\Rightarrow \quad \lambda\alpha\in U^{\perp}, \ \forall\lambda\in\mathbb{C}$ (contradiction). $\square$

<u>Remarks</u> (1) The alg dim of X, say $a(X)$, is the trdim of $\mathcal{M}(X)$ (the field of mero funcs). then X alg $\Leftrightarrow a(X) = g$.

(2) $g \geq 2$, almost all torus aren't algebraic.

U has a basis $e_1, \cdots, e_{2g}$

$\rightsquigarrow \Pi = (e_1, \cdots, e_{2g}) \in M_{g \times 2g}(\mathbb{C})$

$\exists$ nondegenerate alternative matrix $A \in M_{2g \times 2g}(\mathbb{Z})$ s.t.

$\Pi A^{-1} \Pi^T = 0$, $i \Pi A^{-1} \bar{\Pi} > 0$.

e.g. When $g=2$, take the basis as $(1,0)$, $(0,1)$, (ai, bi), (ci, di)

with $ad - bc \notin \mathbb{Q}$.