Comments on the last bit of the proof of local Tate duality

Let K be a local field which is a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((\varpi_K))$ and let k_E be a finite extension of $\mathbb{F}_\ell$. We assume that $\ell \neq \mathrm{char} K$.

Towards the end of the proof, I was a little sketchy. Here is some more details. (I write $H^i(K, M)$ for $H^i(G_K, M)$ to save some space.) First of all, whenever we are in the situation that M is a continuous $k_E[G_K]$-module and L/K is a Galois extension such that G_L acts trivially on M, and moreover we have an exact sequence like $0 \to M \to \mathrm{Ind}_{G_L}^{G_K} M|_{G_L} \to Q \to 0$, then we have a commutative diagram

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & H^0(K,M) \arrow[r] \arrow[d] & H^0(K,\mathrm{Ind}\,M) \arrow[r] \arrow[d, "\cong"] & H^0(K,Q) \arrow[r] \arrow[d] & H^1(K,M) \arrow[r] \arrow[d] & H^1(K,\mathrm{Ind}\,M) \arrow[r] \arrow[d, "\cong"] & H^1(K,Q) \arrow[r] \arrow[d] & \cdots \\
\cdots \to H^3(K,Q^*(1))^* \arrow[r] & H^2(K,M^*(1))^* \arrow[r] & H^2(K,\mathrm{Ind}\,M^*(1))^* \arrow[r] & H^2(K,Q^*(1))^* \arrow[r] & H^1(K,M^*(1))^* \arrow[r] & H^1(K,\mathrm{Ind}\,M^*(1))^* \arrow[r] & H^1(K,Q^*(1))^* \arrow[r] & \cdots
\end{tikzcd}$$

The two red isomorphisms come from the Shapiro lemma, and that

$$H^i(L, M) \cong H^i(L, \mathbb{F}_\ell) \otimes_{\mathbb{F}_\ell} M \cong H^{2-i}(L, \mathbb{F}_\ell(1))^* \otimes_{\mathbb{F}_\ell} M \cong H^{2-i}(L, M^*(1))^*$$

when G_L acts trivially on M and $L = L(\mu_\ell)$.

Now, we prove that the natural map $H^i(G, M) \to H^{2-i}(G, M^*(1))^*$ is an isomorphism.

Step 1: $H^0(K, M) \to H^2(K, M^*(1))^*$ is injective for every continuous $k_E[G_K]$-module M. Make a diagram like above, then it is clear that $H^0(K, M) \to H^2(K, M^*(1))^*$ is injective.

Step 2: $H^1(K, M) \to H^1(K, M^*(1))^*$ is injective for every continuous $k_E[G_K]$-module M. Make a diagram like above, but note that Step 1 can be applied to Q as well (except that Q plays the role of M and we will get a different Q' to make the argument there), so we are in the situation:

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & H^0(K,M) \arrow[r] \arrow[d, hook] & H^0(K,\mathrm{Ind}\,M) \arrow[r] \arrow[d, "\cong"] & H^0(K,Q) \arrow[r] \arrow[d, hook] & H^1(K,M) \arrow[r] \arrow[d] & H^1(K,\mathrm{Ind}\,M) \arrow[r] \arrow[d, "\cong"] & H^1(K,Q) \\
H^3(K,Q^*(1))^* \arrow[r] & H^2(K,M^*(1))^* \arrow[r] & H^2(K,\mathrm{Ind}\,M^*(1))^* \arrow[r] & H^2(K,Q^*(1))^* \arrow[r] & H^1(K,M^*(1))^* \arrow[r] & H^1(K,\mathrm{Ind}\,M^*(1))^* \arrow[r] & H^1(K,Q^*(1))^*
\end{tikzcd}$$

By five lemma, the blue vertical map $H^1(K, M) \to H^1(K, M^*(1))^*$ is injective.

Step 3: $H^2(K, M) \to H^0(K, M^*(1))^*$ is injective for every continuous $k_E[G_K]$-module M. For this, we already know from Step 2 that $H^1(K, Q) \hookrightarrow H^1(K, Q^*(1))^*$ is injective (except that Q plays the role of M and we will get a different Q' to make the argument in Step 2, which further relies on a different Q'' in Step 1). We argue same way as above using five lemmas.

Step 4: $H^3(K, M) \cong 0$. We look at the end of the above diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H^2(K,M) \arrow[r] \arrow[d] & H^2(K,\mathrm{Ind}\,M) \arrow[r] \arrow[d, "\cong"] & H^2(K,Q) \arrow[r] \arrow[d, hook] & H^3(K,M) \arrow[r] \arrow[d] & H^3(K,\mathrm{Ind}\,M) = 0 \\
\cdots \to H^0(K,Q^*(1))^* \arrow[r, phantom] & H^0(K,M^*(1))^* \arrow[r] & H^0(K,\mathrm{Ind}\,M^*(1))^* \arrow[r] & 0 & &
\end{tikzcd}$$

Here $H^3(K, \mathrm{Ind}\, M) = H^3(L, M) = 0$ by our earlier calculation. Easy diagram chasing implies that $H^3(K, M) = 0$ and $H^2(K, Q) \cong H^0(K, \mathrm{Ind}\, M^*(1))^*$.

After this, we may apply dimension shifting techniques to show that $H^4(K, M) \cong H^3(K, Q) = 0$ and so on so forth.

Step 5: Now we reverse the direction of the argument but using the exact sequence $0 \to R \to \operatorname{ind}_{G_L}^{G_K} M|_{G_L} \to M \to 0$. (Yet note that G_K/G_L is finite, so $\operatorname{ind}_{G_L}^{G_K} M|_{G_L} \cong \operatorname{Ind}_{G_L}^{G_K} M|_{G_L}$.) Then we start from the right end of the exact sequence.

$$\begin{tikzcd}
\cdots \arrow[r] & H^2(K,R) \arrow[r] \arrow[d] & H^2(K,\operatorname{ind} M) \arrow[r] \arrow[d, "\cong"] & H^2(K,M) \arrow[r] \arrow[d] & H^3(K,R)=0 \\
\cdots \arrow[r] & H^0(K,R^*(1))^* \arrow[r] & H^0(K,\operatorname{ind} M^*(1))^* \arrow[r] & H^0(K,M^*(1))^* \arrow[r] & 0
\end{tikzcd}$$

It follows that $H^2(K,M) \twoheadrightarrow H^0(K,M^*(1))^*$ is surjective. This is for all continuous $k_E[G_K]$-module M.

Step 6: Prove that $H^1(K,M) \twoheadrightarrow H^1(K,M^*(1))^*$ is surjective for all continuous $k_E[G_K]$-module M. Make a diagram like above, but note that Step 1 can be applied to R as well (except that R plays the role of M and we will get a different R' to make the argument there), so we are in the situation:

$$\begin{tikzcd}
\cdots \arrow[r] & H^1(K,R) \arrow[r] \arrow[d] & H^1(K,\operatorname{ind} M) \arrow[r] \arrow[d, "\cong"] & H^1(K,M) \arrow[r] \arrow[d] & H^2(K,R) \arrow[r] \arrow[d, two heads] & H^2(K,\operatorname{ind} M) \arrow[r] \arrow[d, "\cong"] & H^2(K,M) \arrow[r] \arrow[d, two heads] & 0 \\
\cdots \arrow[r] & H^1(K,R^*(1))^* \arrow[r] & H^1(K,\operatorname{ind} M^*(1))^* \arrow[r] & H^1(K,M^*(1))^* \arrow[r] & H^0(K,R^*(1))^* \arrow[r] & H^0(K,\operatorname{ind} M^*(1))^* \arrow[r] & H^0(K,M^*(1))^* \arrow[r] & 0
\end{tikzcd}$$

It follows that the blue downward arrow is surjective. Then repeat this process.....