ON THE MOD p JACQUET–LANGLANDS CORRESPONDENCE FOR GL_2

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Abstract. This lecture series by Yongquan Hu aims to cover necessary preliminaries to understand the mod p Jacquet–Langlands correspondence for GL_2 as well as some recent progresses around it. Most of the arguments are separated into the $\mathrm{GL}_2(\mathbb{Q}_p)$ case locally and the quaternionic case in a global sense. The main topics are as follows: (1) Serre weights and the construction by Breuil–Paškūnasvia deformations; (2) Gelfand–Kirillov dimension for some representations of GL_2 and its estimation; (3) Some finiteness and the local-global compatibility via Scholze's functor for étale p-adic cohomology of representations arising from mod p Jacquet–Langlands correspondence.

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These are the course notes of Yongquan Hu's lecture series at Beijing International Center of Mathematical Research in 2021. The note-taker claims no originality of the present contexts. However, all minor gaps and typos are due to the carelessness of the note-taker rather than the lecturer.

1. Introduction

Let L be a finite extension of $\mathbb{Q}_p$. We state the *classical local Langlands correspondence* as follows: there is a bijection between

(i) Frobenius-semisimple Weil–Deligne representations of $\mathrm{Gal}(\overline{L}/L)$ over $\mathbb{C}$;
(ii) Irreducible admissible smooth representations of $\mathrm{GL}_2(L)$ over $\mathbb{C}$.

We also propose another family of objects, read as

(iii) Irreducible admissible smooth representations of $D^\times$ over $\mathbb{C}$. Here D is a quaternion algebra with center L.

The *classical local Jacquet–Langlands correspondence* gives an injective map JL from (iii) to (ii) above, which is called the *Jacquet–Langlands transfer map*. Moreover, the image of JL consists of two parts:

$$\begin{aligned}\mathrm{Im}\,\mathrm{JL} &= \{\text{discrete series}\}\\ &= \{\text{supercuspidal representations}\} \cup \{\text{special series}\}.\end{aligned}$$

By fixing a field isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$, we may rewrite the objects above into a p-adic story. Furthermore, by modulo p and taking the residue field $\overline{\mathbb{F}}_p$, it becomes to a mod p story. In this lecture series, we focus on the mod p story and assume $L = \mathbb{Q}_p$.

1.1. The mod p local Langlands correspondence for GL_2.

Historically, the mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ is known by Breuil (together with Colmez). Let $\mathbb{F}$ be a sufficiently large finite extension of $\mathbb{F}_p$ in some fixed algebraic closure $\overline{\mathbb{F}}_p$. Then:

Theorem 1.1 (The mod p local Langlands correspondence for GL_2, Breuil)**.** *There is a bijection*

$$\left\{\begin{matrix}\text{Continuous representations of}\\ \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\text{ over }\mathbb{F}\text{ of dimension }2\end{matrix}\right\} \xrightarrow{\kappa} \left\{\begin{matrix}\text{Admissible smooth representation of}\\ \mathrm{GL}_2(\mathbb{Q}_p)\text{ over }\mathbb{F}\text{ with continuous character}\end{matrix}\right\}$$

$$(\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(\mathbb{F})) \longmapsto (\kappa(\overline{\rho}) : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathrm{GL}(V)).$$

Here V is an $\mathbb{F}$-vector space, which is possibly infinite-dimensional.

Note that due to the modulo p nature of Galois representations, the $\overline{\rho}$'s on the left-hand side are relatively easy to classify. However, since we do not require the representations on the right-hand side to be irreducible, there are four possible appearances of them, say

$$\mathrm{RHS} = \{\text{principal series}\} \cup \{\text{characters}\} \cup \{\text{special series}\} \cup \{\text{supersingular representations}\}.$$

Beware that this classification is not so refined that the four classes are possibly not disjoint. According to a big theorem of Breuil, the supersingular representations are exactly identified with the supercuspidal representations. Moreover, we have

- ◇ $\overline{\rho}$ is irreducible if and only if $\kappa(\overline{\rho})$ is supercuspidal (or equivalently, supersingular).
- ◇ $\overline{\rho}$ is reducible if and only if $\kappa(\overline{\rho})$ equals the direct sum of two principal series, i.e., $\kappa(\overline{\rho}) = \mathrm{PS}_1 \oplus \mathrm{PS}_2$.

Granting these, we are able to attain a complete understanding on the $\mathrm{GL}_2(\mathbb{Q}_p)$-side. Later on Breuil's work, Colmez has constructed a "functor" from $\mathrm{GL}_2(\mathbb{Q}_p)$-representations to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representations.

1.2. **The local-global compatibility.** We state the result proved by Emerton by beginning with the setups. Fix an integer $N \geqslant 5$. Let $Y(N)$ be the modular curve of level $\Gamma(N)$. Over $\mathbb{F}$, to propose more geometric information, it is natural to consider the étale cohomology of $Y(N)$ as well as the direct limit

$$H^1 := \varinjlim_{N} H^1_{\mathrm{et}}(Y(N) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{F}),$$

equipped with the action of $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$. Here for simplicity we denote $G_k := \mathrm{Gal}(\overline{k}/k)$ the absolute Galois group of a field k.

Let $\overline{r} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ be an irreducible absolute Galois representation, which is assumed to be odd under some mild conditions (for example, the Taylor–Wiles condition and some restricted condition on $\overline{r}|_{G_{\mathbb{Q}_p}}$).

Theorem 1.2 (The local-global compatibility, Emerton)**.** *Over the setups above, there exists a* $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$*-equivariant isomorphism*

$$(*) \qquad \mathrm{Hom}_{G_{\mathbb{Q}}}(\overline{r}, H^1) \simeq \widehat{\bigotimes_{\ell \text{ prime}}} \pi(\overline{r}|_{G_{\mathbb{Q}_\ell}}),$$

where the representation $\pi(\overline{r}|_{G_{\mathbb{Q}_\ell}})$ *is the image* $\kappa(\overline{r}|_{G_{\mathbb{Q}_\ell}})$ *under the mod* ℓ *local Langlands correspondence as in Theorem 1.1.*

Remark 1.3. We have to propose a modified version of Theorem 1.2 when

$$\overline{r}|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix} \otimes (\text{some twist}),$$

where ω is the mod p cyclotomic character.

1.3. **The mod p Jacquet–Langlands correspondence.** Let D be a quaternion algebra over $\mathbb{Q}_p$ with the ring of integers $\mathcal{O}_D$ (for the valuation v_D). Since $D^\times$ is compact modulo its center, with $[D^\times : \mathcal{O}_D^\times \mathbb{Q}_p^\times] = 2$, the representation theory of $D^\times$ mod p is transparent and easy to be caught. In fact, any irreducible representation of $D^\times$ over $\mathbb{F}$ has dimension $\leqslant 2$, which truly looks like the local Galois representation of $G_{\mathbb{Q}_p}$.

Unfortunately, even though one can write down a bijection between the $D^\times$-representations and $G_{\mathbb{Q}_p}$-representations due to the similitude above, this is not expected to be the correct one. The reason lies in that we are expecting the Jacquet–Langlands correspondence to be realized at the level of cohomology, i.e., in $(*)$, if we replaced the modular curve $Y(N)$ by some appropriate quaternionic Shimura curve $\mathscr{S}$, one would similarly expect a decomposition as $(*)$.[1] However, the "fake correspondence" above defines $\pi(\overline{r}|_{G_{\mathbb{Q}_p}})$ to be of infinite dimension (by Breuil, Diamond, and Scholze), and hence it is built by infinitely many pieces of irreducible representations in a highly semisimple way.

However, the main difficulty lies in how to describe this representation of infinite length, denoted as $\mathrm{JL}(\overline{\rho}) = \pi(\overline{r}|_{G_{\mathbb{Q}_p}})$, the p-component on the right-hand side of $(*)$. One way is looking at the filtration of $\mathrm{JL}(\overline{\rho})$ with respect to $\mathcal{O}_D^\times$.

Definition 1.4. Let V be an admissible smooth representation of $D^\times$. The $\mathcal{O}_D^\times$*-socle* of V is the maximal semisimple subrepresentation of V.

With this definition, we can get the socle filtration of V. Moreover, in the following theorem, the real number c is called the *Gelfand–Kirillov dimension* (or *GK-dimension*) of V, denoted by $\mathrm{GK}\dim(V)$, which measures the growth rate of the dimension of this socle filtration.

Theorem 1.5. *Let V be an admissible representation of $D^\times$. Let $H_n = 1 + p^n\mathcal{O}_D$. Then there exists a unique real number $c \in [0, 4]$ such that for all $n \in \mathbb{N}$,*

$$\dim V^{H_n} = \lambda \cdot p^{cn} + O(p^{(c-1)n}),$$

for some other real number $\lambda > 0$.

[1] Here, by an appropriate quaternionic Shimura curve $\mathscr{S}$, we mean $\mathscr{S}$ is defined for some quaternion algebra B over $\mathbb{Q}$, such that B is ramified at p (and hence there is an isomorphisms $B \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq D$.

Remark 1.6. In fact, we may view $V^\vee$ as a finitely generated module over the (non-commutative) Iwasawa algebra

$$\mathbb{F}[\![\mathcal{O}_D^\times]\!] := \varinjlim_n \mathbb{F}[\![\mathcal{O}_D^\times/H_n]\!]$$

and Theorem 1.5 is an analogue description of Hilbert–Kunz function.

The main theorem of our lecture is as follows.

Theorem 1.7 (Hu–Wang)**.** *Under Taylor–Wiles condition and a genericity condition,*

$$\mathrm{GK}\dim(\mathrm{JL}(\overline{\rho})) = 1.$$

Remark 1.8. Recently, Paškūnashas proved this result when $\overline{r}|_{G_{\mathbb{Q}_p}}$ is reducible, using Scholze's functor together with a result of Ludwig. However, Hu–Wang don't use Scholze's functor but rather deduce some consequences.

Proof Sketch of Theorem 1.7. The following context follows the case of $\mathrm{GL}_2(\mathbb{Q}_{p^f})$, proved by Breuil–Herzig–Hu–Morra–Schraen.

Step I. Let $\mathfrak{m}_D$ be the maximal ideal of $\mathbb{F}[\![\mathcal{O}_D^1/Z_D^1]\!]$. Let $\pi[\mathfrak{m}_D^3]$ be the subspace annihilated by $\mathfrak{m}_D^3$. Note that $\mathcal{O}_D^\times/\mathcal{O}_D^1$ is equipped with a $\pi[\mathfrak{m}_D^3]$-action, which is semisimple because $\mathcal{O}_D^\times/\mathcal{O}_D^1 \simeq \mathbb{F}_{p^2}^\times$. We introduce the following criterion:

- ⋄ If $\pi[\mathfrak{m}_D^3]$ is multiplicity-free for the $\mathbb{F}_{p^2}^\times$-action, then $\mathrm{GK}\dim(\pi) \leqslant 1$.

In commutative algebra, if M is a finitely generated module over a noetherian local ring $(A, \mathfrak{m})$, then the Krull dimension of M is always bounded by that of A. The upshot is that, to determine the size of M (or the Hilbert function on $\mathrm{Krull}\dim M/\mathfrak{m}^n M$, we observe

- the size of $M/\mathfrak{m}M$ corresponds to the number of generators of M;
- the size of $\mathfrak{m}M/\mathfrak{m}^2M$ corresponds to the first relation between the generators.

To understand this, one can consider the extreme example where M is cyclic, and then

$$\mathrm{Krull}\dim M \leqslant \dim_{A/\mathfrak{m}A} \mathfrak{m}M/\mathfrak{m}^2M,$$

which is bounded by the dimension of tangent space.

Now we suppose $\pi^\vee$ is finitely generated over the non-commutative Iwasawa algebra $\mathbb{F}[\![\mathcal{O}_D^1/Z_D^1]\!]$ and $\mathrm{gr}_{\mathfrak{m}_D}(\pi^\vee)$ is finitely generated over $\mathrm{gr}_{\mathfrak{m}_D}(\mathbb{F}[\![\mathcal{O}_D^1/Z_D^1]\!])$. The graded part is again non-commutative and admits an isomorphism to the universal enveloping algebra of the following graded Lie algebra:

$$\mathfrak{g} = \mathbb{F}e \oplus \mathbb{F}f \oplus \mathbb{F}h.$$

Here we relate the basis via $h = [e, f]$ and $[e, h] = [f, h] = 0$. More importantly,

$$\deg e = \deg f = 1, \quad \deg h = 2.$$

This explains why we need to proceed on the series of subspaces until $\pi[\mathfrak{m}_D^3]$.

Step II. We check the multiplicity-free condition in Step I above. We are require to computing the potentially crystalline Galois deformation ring of $\overline{\rho}$. In general, this process can be done by using Kisin's module in (integral) p-adic Hodge theory [BHH+20]. However, in our case, the issue lies in that a strong genericity condition is needed in addition, for example,

$$\overline{\rho} = \begin{pmatrix} \omega^{r_1} & * \\ 0 & 1 \end{pmatrix}, \quad 12 \leqslant r_1 \leqslant p - 13.$$

Here, as before, ω is the mod p cyclotomic character. Despite this, we choose to rather use Paškūnas' technique for the two purposes:

(i) to weaken the genericity condition, and
(ii) more conceptually, to avoid the heavy and explicit computation for the rings, just so the congruence relation is sufficient.

One should beware that Paškūnas' technique can only be applied to 2-dimensional representations of $G_{\mathbb{Q}_p}$. □

1.4. **Applications of Scholze's functor.** Let D be the central division algebra over the field L of invariant $1/n$. Scholze has constructed a cohomologically covariant δ-functor

$$\begin{aligned}\mathcal{S}^i : \mathsf{Mod}^{\mathrm{adm,sm}}_{\mathrm{GL}_n(L)}(\mathbb{F}) &\longrightarrow \mathsf{Mod}^{\mathrm{sm}}_{\mathrm{GL}\times D^\times}(\mathbb{F})\\ \pi &\longmapsto \mathcal{F}_\pi.\end{aligned}$$

Here π is any admissible smooth $\mathbb{F}$-representation of $\mathrm{GL}_n(L)$, and $\mathcal{F}_\pi$ is a (Weil-equivariant) sheaf on the étale site of the adic space $\mathbb{P}^{n-1}_{\breve{L}}$, and indeed, $\breve{L} = \mathbb{C}_p$. Scholze proved that this functor equals

$$\mathcal{S}^i : \pi \longmapsto H^1_{\mathrm{et}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi),$$

and moreover, $\mathcal{S}^i = 0$ if $i > 2(n-1)$ with $\mathcal{S}^0(\pi) = \mathcal{S}^0(\pi^{\mathrm{SL}_n(L)})$. In particular, when $n = 2$ in the GL_2-case, we get $\mathcal{S}^i = 0$ for all $i \geqslant 3$.

Due to Theorem 1.1, our expectation is that when $\overline{\rho}$ is generic, we can associate a $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $\kappa(\overline{\rho})$ to it, and

$$\begin{aligned}\mathcal{S}^0(\kappa(\overline{\rho})) &= \mathcal{S}^2(\kappa(\overline{\rho})) = 0,\\ \mathcal{S}^1(\kappa(\overline{\rho})) &= \overline{\rho}(-1) \otimes \mathrm{JL}(\overline{\rho}).\end{aligned}$$

Again, the core difficulty here is to understand the structure of $\mathrm{JL}(\overline{\rho})$, because the functor $\mathcal{S}^1(-)$ is extremely hard to compute. But besides, a result of Ludwig can be useful:

- ⋄ If $\pi = \mathrm{PS}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$, i.e., it is a principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$, then $\mathcal{S}^2(\pi) = 0$.

Theorem 1.9. *If π is supersingular and some minor genericity condition*[2] *is satisfied, then $\mathcal{S}^2(\pi) = 0$.*

Proof Sketch. Let $F/\mathbb{Q}$ be a totally real field with a fixed place $\mathfrak{p} \mid p$ such that $F_{\mathfrak{p}} \simeq \mathbb{Q}_p$. Let B be a definite quaternion algebra over F that splits at $\mathfrak{p}$. Let B' be another algebra over F that ramifies at $\mathfrak{p}$, splits at ∞, and has the same ramification behavior as B at other places. Thus,

$$B \otimes_F \mathbb{A}_F^{\mathfrak{p},\infty} \simeq B' \otimes_F \mathbb{A}_F^{\mathfrak{p},\infty}.$$

We then fix an open compact subgroup outside $\mathfrak{p}$, say

$$U^{\mathfrak{p}} \subset (B \otimes_F \mathbb{A}_F^{\mathfrak{p},\infty})^\times.$$

Consider for B that

$$S(U^{\mathfrak{p}}, \mathbb{F}) = \{f : B^\times \backslash (B \otimes \mathbb{A}_F^\infty)^\times / U^{\mathfrak{p}} \to \mathbb{F} \text{ continuous function}\},$$

equipped with an action of $(B \otimes F_{\mathfrak{p}})^\times = \mathrm{GL}_2(\mathbb{Q}_p)$ by rigid translator. And consider for B' that

$$H^1_{\mathrm{et}}(U^{\mathfrak{p}}, \mathbb{F}) := \varinjlim_{U_{\mathfrak{p}}} (X(U_{\mathfrak{p}} U^{\mathfrak{p}})_{\overline{F}}, \mathbb{F}),$$

equipped with an action of $G_F \times (B'_{\mathfrak{p}})^\times \simeq D^\times$, where $X(U_{\mathfrak{p}} U^{\mathfrak{p}})$ is the quaternionic modular curve associated to B'. On these setups, Scholze has discovered the following relation

$$\mathcal{S}^i(S(U^{\mathfrak{p}}, \mathbb{F})) \simeq H^1_{\mathrm{et}}(U^{\mathfrak{p}}, \mathbb{F}),$$

which can be understood as a kind of local-global compatibility. Moreover, taking $\mathbb{T}(U^{\mathfrak{p}})$ as the Hecke algebra of $U^{\mathfrak{p}}$, if $\overline{r} : G_F \to \mathrm{GL}_2(\mathbb{F})$ is a continuous absolutely irreducible representation corresponding to the maximal ideal $\mathfrak{m}_{\overline{r}} \subset \mathbb{T}(U^{\mathfrak{p}})$, then

$$\mathcal{S}^i(S(U^{\mathfrak{p}}, \mathbb{F})_{\mathfrak{m}_{\overline{r}}}) \simeq H^1_{\mathrm{et}}(U^{\mathfrak{p}}, \mathbb{F})_{\mathfrak{m}_{\overline{r}}},$$

and both sides enjoy the $\mathbb{T}(U^{\mathfrak{p}})_{\mathfrak{m}_{\overline{r}}}$-actions. So the isomorphism is $\mathbb{T}(U^{\mathfrak{p}})_{\mathfrak{m}_{\overline{r}}}$-equivariant.

Roughly, after taking dual module to work on finitely generated modules over the Iwasawa algebra, we are reduced to the situation with the following setups:

- A is a (commutative) local complete noetherian ring with Krull dimension d;
- M is an $A[\mathrm{GL}_2(\mathbb{Q}_p))]$-module that is flat over A and finitely generated as an $\mathbb{F}[\mathrm{GL}_2(\mathbb{Z}_p)]$-module of GK-dimension $d+1$;
- $\mathcal{S}^1(M) = M'$, which is an $A[D^\times]$-module and finitely generated over $\mathbb{F}[D^\times]$, is again of GK-dimension $d+1$;
- $\mathcal{S}^0(M) = 0$ by Ihara's lemma.

[2] It is read as $2 \leqslant r \leqslant p-3$, which will be explained later.

Our goal now is to show that $\mathcal{S}^1(M \otimes_A \mathbb{F})$ has GK-dimension 1. Easily, it can be seen that this is equivalent to M' being flat over A, or alternatively, $\mathcal{S}^2(M \otimes_A \mathbb{F}) = 0$. To prove this equivalence, note that A always contains a power series ring of Krull dimension d. So we may assume A is a regular noetherian ring. The following fact is at work:

- $\diamond$ (Miracle flatness, by Gee–Newton) *M' is flat over A if and only if* $\mathrm{GK}\dim(M' \otimes_A \mathbb{F}) = 1$.

In Paškūnas' theorem (c.f. Remark 1.8), i.e., when $\overline{r}|_{G_{\mathbb{Q}_p}}$ is reducible, the equivalent assertion $\mathcal{S}^2(M \otimes_A \mathbb{F}) = 0$ holds by Ludwig's theorem. But for us, we choose to prove $\mathrm{GK}\dim(\mathcal{S}^1(M \otimes_A \mathbb{F})) = 1$ first, and then it implies the vanishing result. □

Here comes one more result on $\mathcal{S}^1(\pi)$ for non-supersingular π. Assume $\overline{\rho}$ is irreducible and $\overline{\rho}^{\mathrm{ss}} = \chi_1 \oplus \chi_2$. We impulse another assumption that $\overline{\rho}$ is generic, i.e. $\chi_1\chi_2^{-1} \notin \{1, \omega^{\pm 1}\}$. Let

$$\overline{\rho}_1 = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}, \quad \overline{\rho}_2 = \begin{pmatrix} \chi_2 & * \\ 0 & \chi_2 \end{pmatrix}$$

be two non-split residual representations. Then

$$\mathcal{S}^1(\kappa(\overline{\rho}_i)) = \overline{\rho}_i(-1) \otimes \mathrm{JL}(\overline{\rho}_i)$$

for $D^\times$-representation $\mathrm{JL}(\overline{\rho}_i)$ with $i = 1, 2$. Also, we obtain the following result.

Theorem 1.10 (Hu–Wang)**.** *We have*

$$\mathrm{JL}(\overline{\rho}_1) = \mathrm{JL}(\overline{\rho}_2),$$

or namely, $\mathrm{JL}(\overline{\rho})$ *depends only on the supersingular part* $\overline{\rho}^{\mathrm{ss}}$ *of* $\overline{\rho}$*.*

Moreover, a similar result holds in the case where $\overline{\rho}^{\mathrm{ss}} = 1 \oplus \omega$.

2. Mod p representation theory

We state first a basic result in mod p representation theory. Let $\mathbb{F}$ be a sufficiently large finite extension of $\mathbb{F}_p$ (or simply $\mathbb{F} = \overline{\mathbb{F}}_p$).

Lemma 2.1. *Let H be a pro-p group and (ρ, V) be a smooth representation of H over $\mathbb{F}$. Then $V^H \neq 0$.*

Proof. Take $v \in V \backslash \{0\}$. Then v is fixed by some open subgroup $H' \subset H$, and hence $\langle H.v \rangle$ is finite-dimensional. Replacing V by $\langle H.v \rangle$, we can assume that V itself is finite-dimensional and that H is a finite p-group. The result is then classical, proved by counting the orbits: each orbit has cardinality 1 (being a fixed point) or a power of p, so the set of fixed points has cardinality $\geqslant p$; but it is nonempty as it always contains 0. □

In particular, we get the following consequence since V^H in Lemma 2.1 can be always viewed as a subrepresentation.

Corollary 2.2. *If H is a pro-p group, then the trivial representation is the unique (up to isomorphism) irreducible smooth $\mathbb{F}$-representation of H.*

2.1. The 2-dimensional $\mathbb{F}$-representation of $G_{\mathbb{Q}_p}$.

Denote $G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, the absolute Galois group. Let $I_{\mathbb{Q}_p}$ be the inertia subgroup of $G_{\mathbb{Q}_p}$. For $n \geqslant 1$, let $\omega_n : I_{\mathbb{Q}_p} \to \mathbb{F}^\times$ denote the fundamental character of Serre of level n, defined by

$$\omega_n(g) = \frac{g(p^{1/(p^n-1)})}{p^{1/(p^n-1)}} \in \mu_{p^n-1}(\overline{\mathbb{Z}}_p) \xrightarrow{\sim} \mu_{p^n-1}(\mathbb{F}_p), \quad \forall g \in I_{\mathbb{Q}_p}$$

for some choice of the (p^n-1)st root of p. If $n = 1$, then ω_1 is just the (restriction of) mod p cyclotomic character. It is obvious that $\omega_n^{p^n-1} = 1$; if $m \mid n$, then

$$\omega_n^{1+p^m+p^{2m}+\cdots+p^{(n/m-1)m}} = \omega_m.$$

Moreover, any continuous character of $I_{\mathbb{Q}_p}$ is a power of ω_n for some n (as the wild inertia $I_{\mathbb{Q}_p}^{\mathrm{wild}}$ is pro-p, it acts trivially).

Lemma 2.3. (1) *Serre's fundamental character ω_n can be extended to $G_{\mathbb{Q}_{p^f}}$ if and only if $n \mid f$.*

(2) *Any character of $G_{\mathbb{Q}_{p^f}}$ has the form $\chi\omega_f^m$ for some integer $0 \leqslant m \leqslant p^f-2$ and some unramified character χ.*

Proof. To show (1), it suffices notice that the image of $\omega_n : I_{\mathbb{Q}_{p^f}} \to \mathbb{F}^\times$ lands in $\mu_{p^n-1}(\overline{\mathbb{F}}_p)$ via the isomorphism above, and factors through $\mathbb{F}_{p^f}^\times$. The assertion (2) follows from the fact that any representation of $G_{\mathbb{Q}_p}/I_{\mathbb{Q}_p}$ is unramified, and hence the corresponding character χ is unramified. Also, any continuous character of $I_{\mathbb{Q}_p}$ is of form ω_n^m, where $0 \leqslant m \leqslant p^f - 2$ as $\omega_n^{p^f-1} = 1$. □

In the upcoming context, let $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation. It possibly has two different appearances.

(1) If ρ is reducible, then, clearly,

$$\rho \cong \begin{pmatrix} \chi_1\omega^{m_1} & * \\ 0 & \chi_2\omega^{m_2} \end{pmatrix},$$

where χ_1, χ_2 are unramified characters, and m_is are two integers with $0 \leqslant m_i \leqslant p-2$.

(2) If ρ is irreducible, first look at the restriction of ρ to $I_{\mathbb{Q}_p}$. Since $I_{\mathbb{Q}_p}^{\mathrm{wild}}$ is a pro-p subgroup, it acts trivially on ρ. The quotient

$$I_{\mathbb{Q}_p}/I_{\mathbb{Q}_p}^{\mathrm{wild}} \simeq \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$$

is a prime-to-p abelian group (where the twist (1) means that the conjugation action of $G_{\mathbb{Q}_p}^{\mathrm{ur}}$ on $I_{\mathbb{Q}_p}/I_{\mathbb{Q}_p}^{\mathrm{wild}}$ is by cyclotomic character). So $\rho|_{I_{\mathbb{Q}_p}}$ is a direct sum of two characters $\chi_1 \oplus \chi_2$. Using the action of a well-chosen Frobenius element, i.e., $s \in G_{\mathbb{Q}_p}$ is a lifting of (arithmetic) Frobenius and acts trivially on $p^{1/(p^n-1)}$ (here n depends on χ_i), then one checks

$$\chi_i(sgs^{-1}) = \chi_i(g)^p$$

for $g \in I_{\mathbb{Q}_p}$, so $\{\chi_1, \chi_2\} = \{\chi_1^p, \chi_2^p\}$. If $\chi_1 = \chi_1^p$, then $\chi_2 = \chi_2^p$ and both χ_1 and χ_2 extend to $G_{\mathbb{Q}_p}$, a contradiction. Hence, $\chi_1 = \chi_2^p$ and $\chi_2 = \chi_1^p$, so $\chi_1^{p^2} = \chi_1$. This implies that χ_1 has the form ω_2^m with $0 \leqslant m \leqslant p^2 - 2$, and $p+1$ does not divide m as $\chi_1 \neq \chi_1^p$. Note that χ_i then extends to $G_{\mathbb{Q}_{p^2}}$, so there exists an unramified character $\eta : G_{\mathbb{Q}_{p^2}} \to \mathbb{F}^\times$ such that $\omega_2^m\eta \hookrightarrow \rho|_{G_{\mathbb{Q}_{p^2}}}$. By Frobenius reciprocity (Proposition 2.9), we obtain an isomorphism

$$\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}(\omega_2^m\eta) \cong \rho.$$

But the left-hand side is isomorphic to $(\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^m) \otimes \mu$ for some unramified character $\mu : G_{\mathbb{Q}_p} \to \mathbb{F}^\times$ which extends η.[3]

Lemma 2.4. (1) *Any irreducible 2-dimensional representation of $G_{\mathbb{Q}_p}$ over $\mathbb{F}$ is isomorphic to*

$$\rho(r,\chi) := (\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{r+1}) \otimes \chi$$

for some $0 \leqslant r \leqslant p-1$ and some smooth character $\chi : \mathbb{Q}_p^\times \to \mathbb{F}^\times$.

(2) *We have the following isomorphisms:*

$$\rho(r,\chi) \cong \rho(r, \chi\mu_{-1}) \cong \rho(p-1-r, \chi\omega^r) \cong \rho(p-1-r, \chi\omega^r\mu_{-1}).$$

Proof. For (1), write $m = m_0 + (p+1)m_1$ with $0 \leqslant m_0 \leqslant p$; as $p+1$ does not divide m, we have $m_0 \neq 0$. Then

$$\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^m \cong (\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{m_0}) \otimes \omega^{m_1}.$$

Also, (2) essentially follows from the definition. The only ambiguity lies in the technique to prove the second isomorphism. But this is a similar argument as in the proof of Theorem 2.15. □

[3]If η sends Frob^2, a generator of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$, to $x \in \mathbb{F}$, then we can take $\mu : G_{\mathbb{Q}_p} \to \mathbb{F}^\times$ to be the unramified character sending Frob to a fixed root $\sqrt{x} \in \mathbb{F}$ (this may require to enlarge $\mathbb{F}$).

2.2. **The $\mathrm{GL}_2(\mathbb{Q}_p)$-case.** We write $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and $B = B(\mathbb{Q}_p)$ the Borel subgroup. As in the classical case, irreducible smooth representations of G falls into four classes as follows.

(1) **Principal series:** $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$ for $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to \mathbb{F}^\times$. It is irreducible if and only if $\chi_1 = \chi_2$ (c.f. Theorem 2.6 below).
(2) **Characters:** $\chi \circ \det$, where $\chi : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ is a character.
(3) **Special series:** $\mathrm{Sp} \otimes (\chi \circ \det)$, where Sp is the quotient of $\mathrm{Ind}_B^G(\mathbb{1} \otimes \mathbb{1})$ by constant functions.
(4) **Supersingular representations:** other irreducible ones. (avatar of supercuspidal representations in classical situation).

Before giving the proof, we recall a useful criterion to prove irreducibility of a smooth modulo p representation (π, V) of G.

Lemma 2.5 (Irreducibility criterion)**.** *Let P be a pro-p subgroup of G. If for any $v \in V^P$ we have $V = \langle G \cdot v\rangle$, then V is irreducible.*

Proof. Let $V' \subset V$ be a sub-G-representation. Then V' contains some nonzero P-fixed vector by Lemma 2.1. By assumption, it generates V, so we must have $V' = V$. □

Theorem 2.6. (1) *$\mathrm{Ind}_B^G \chi_1 \otimes \chi_2$ is irreducible if and only if $\chi_1 \neq \chi_2$.*
(2) *If $\chi = \chi_1 = \chi_2$, then there is a non-split extension*

$$0 \longrightarrow \chi \circ \det \longrightarrow \mathrm{Ind}_B^G(\chi \otimes \chi) \longrightarrow \mathrm{Sp} \otimes (\chi \circ \det) \longrightarrow 0 \tag{$\dagger$}$$

and Sp *is irreducible.*

Proof. (1) **Step I.** Denote by I (resp. I_1) the so-called Iwahori (resp. pro-p-Iwahori) subgroup

$$I = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}, \quad I_1 = \begin{pmatrix} 1 + p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1 + p\mathbb{Z}_p \end{pmatrix}.$$

Then I_1 is a pro-p group, and hence $(\mathrm{Ind}_B^G \chi_1 \otimes \chi_2)^{I_1}$ is nonzero by Lemma 2.1. The Bruhat decomposition

$$G = BI_1 \sqcup B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B = BI_1 \sqcup B \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} I_1 = BI_1 \sqcup B\Pi I_1.$$

It tells us that $(\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2))^{I_1}$ is always 2-dimensional, spanned by the two functions f_1, f_2 characterized by the following properties:

$$\begin{aligned} \mathrm{Supp}\, f_1 &= BI_1, \quad & f_2(bg) &= 1, \\ \mathrm{Supp}\, f_2 &= B\Pi I_1, \quad & f_2(b\Pi g) &= 1, \end{aligned}$$

for all $b \in B$ and $g \in I_1$. By the criterion of Lemma 2.5, we need to prove that any vector of the form $af_1 + bf_2$ (with $a, b \in \mathbb{F}$) can generate $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$ as a G-representation.

Step II. It is easy to show that $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$ is generated by f_1 and so also by f_2 (as $\Pi \cdot f_1 = f_2$). If $h = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in I$, then

$$\begin{aligned} h \cdot f_1 &= \chi_1(a)\chi_2(d) f_1, \\ h \cdot f_2 &= \chi_1(d)\chi_2(a) f_1. \end{aligned}$$

So if $\chi_1|_{\mathbb{Z}_p^\times} \neq \chi_2|_{\mathbb{Z}_p^\times}$, we are done. Otherwise we need some finer analysis which we omit.

(2) Up to twist, we may assume χ is the trivial character $\mathbb{1}$. It is clear that $\mathrm{Ind}_B^G(\mathbb{1} \otimes \mathbb{1})$ contains the trivial representation $\mathbb{1}_G$. But the proof of the irreducibility of Sp is a little subtle. We need to show that when taking I_1-invariants, ($\dagger$) induces again a short exact sequence, namely, Sp^{I_1} is 1-dimensional, generated by the image of f_1 (note that $f_1 + f_2 \in \mathbb{1}_G$), so we have $\overline{f}_1 = -\overline{f}_2$ in the quotient Sp. This shows that Sp is irreducible by the criterion.

□

Remark 2.7. In term of Iwahori Hecke algebra, Theorem 2.6 means that $(\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2))^{I_1}$ is an irreducible module over the Iwahori–Hecke algebra $\mathcal{H}(I_1, \mathbb{1})$ (see Subsection 2.3 below). This algebra is generated by elements

$$\Pi, \quad h = \begin{pmatrix} [a] & 0 \\ 0 & d \end{pmatrix}, \quad \sum_{\lambda \in \mathbb{F}_p} \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix}.$$

In fact, the results of Theorem 2.6 hold for $\mathrm{GL}_2(L)$, where L is any non-archimedean local field of residual characteristic p. Next, we pass to supersingular representations, which are only classified for $\mathrm{GL}_2(\mathbb{Q}_p)$, up to present.

2.3. **Compact inductions.**

Definition 2.8. If H is an open subgroup of G, and (σ, W) is a smooth representation of H, define $\text{c-Ind}_H^G\sigma$ to be the space of all locally constant functions $f : G \to W$ such that

(i) $f(hg) = h \cdot f(g)$ for all $h \in H$ and $g \in G$;
(ii) $\mathrm{Supp}(f)$ is compact modulo H.

Let G act on $\text{c-Ind}_H^G\sigma$ by right translation:

$$(gf)(g') = f(g'g).$$

In this way, we get a smooth representation, but it is *not admissible*.

Proposition 2.9 (Frobenius reciprocity)**.** *Suppose H is an open subgroup of G. Let (π, V) be a smooth representation of G and (σ, W) be a smooth representation of H. Then*

$$\mathrm{Hom}_G(\text{c-Ind}_H^G\sigma, \pi) \cong \mathrm{Hom}_H(\sigma, \pi|_H).$$

We are mainly interested in the case when

$$H = KZ := \mathrm{GL}_2(\mathbb{Z}_p) \times \mathbb{Q}_p^\times$$

and σ is an irreducible representation of K, viewed as a representation of KZ by letting the p-scalar matrix acts trivially.

Lemma 2.10. *Irreducible $\mathbb{F}$-representations of $K = \mathrm{GL}_2(\mathbb{Z}_p)$ are of the form $\mathrm{Sym}^r\,\mathbb{F}^2 \otimes \det^m$ for $0 \leqslant r \leqslant p-1$ and $0 \leqslant m \leqslant p-2$. If σ is such a representation, then σ^{I_1} is 1-dimensional.*

Proof. First, the representation has a standard basis $\{X^iY^{r-i} : 0 \leqslant i \leqslant r\}$, with the action of $\mathrm{GL}_2(\mathbb{Z}_p)$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} X^iY^{r-i} = (\overline{a}X + \overline{c}Y)^i(\overline{b}X + \overline{d}Y)^{r-i};$$

so the action factors through $\mathrm{GL}_2(\mathbb{F}_p)$. Its I_1-invariant is 1-dimensional, spanned by X^r. Moreover, X^r generates the whole representation, so by the irreducibility criterion (Lemma 2.5), $\mathrm{Sym}^r\,\mathbb{F}^2 \otimes \det^m$ is irreducible.

Second, let K_1 denote the first principal congruence subgroup, the kernel of $K \twoheadrightarrow \mathrm{GL}_2(\mathbb{F}_p)$. Since K_1 is pro-p, it acts trivially on σ. Second, by Brauer's theory of modular characters, the numbers of isomorphism classes of $\mathbb{F}$-representation of $\mathrm{GL}_2(\mathbb{F}_p)$ is equal to the number of conjugacy classes whose order is prime-to-p; it is exactly $p(p-1)$. □

For simplicity, write the Iwahori–Hecke algebra as

$$\mathcal{H}(\sigma) := \mathcal{H}(KZ, \sigma),$$

which naturally acts on $\text{c-Ind}_{KZ}^G\sigma$. If π is a smooth representation of G, then $\mathcal{H}(\sigma)$ also acts on the vector space $\mathrm{Hom}_G(\text{c-Ind}_{KZ}^G\sigma, \pi)$, which by Frobenius reciprocity 2.9 is isomorphic to $\mathrm{Hom}_{KZ}(\sigma, \pi|_{KZ})$. Note that if π is admissible, then this space is finite-dimensional over $\mathbb{F}$.

Theorem 2.11 (Barthel–Livné)**.**

(1) *We have*

$$\mathcal{H}(\sigma) \cong \mathbb{F}[T],$$

and the action of $\mathcal{H}(\sigma)$ on $\text{c-Ind}_{KZ}^G\sigma$ is free.

(2) *Any irreducible smooth representation π of G with a central character is a quotient of*

$$\frac{\text{c-Ind}_{KZ}^G\sigma}{T-\lambda} \otimes (\chi \circ \det)$$

for some irreducible representation σ of K, $\lambda \in \mathbb{F}$, and some $\chi : \mathbb{Q}_p^\times \to \mathbb{F}^\times$.

(3) *If π in (2) is supersingular, then $\lambda = 0$.*

Proof. We only work on (2). Up to twist we may assume that $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ acts on π trivially. Let $v \in \pi$ be any nonzero vector. Since π is a smooth representation and K is compact, $\langle K, v\rangle$ is finite-dimensional (c.f. Lemma 2.1). By induction on dimension of $\langle K.v\rangle$, we see that $\langle K.v\rangle$ must contain some irreducible representation of K, say σ. We then get a KZ-equivariant injection $\sigma \hookrightarrow \pi$ by assumption on the action of Z. By Frobenius reciprocity, it induces a nonzero G-equivariant morphism

$$\text{c-Ind}_{KZ}^{G}\sigma \longrightarrow \pi$$

and this is further surjective because π is assumed to be irreducible.

We only prove the result under the extra assumption that π is *admissible*; the original proof of Barthel–Livné, without this restriction, is more complicated. Since π is admissible, the space $\mathrm{Hom}_G(\text{c-Ind}_{KZ}^{G}\sigma, \pi)$ is finite-dimensional and nonzero as seen above. As $\mathcal{H}(\sigma) \cong E[T]$ is commutative and $\mathbb{F}$ is sufficiently large (or simply you may assume $\mathbb{F} = \overline{\mathbb{F}}_p$ is algebraically closed), there exists an eigenvector in $\mathrm{Hom}_G(\text{c-Ind}_{KZ}^{G}\sigma, \pi)$, with eigenvalue $\lambda \in \mathbb{F}$ say. Then the result follows. ☐

We set, for $0 \leqslant r \leqslant p-1$, $\lambda \in \mathbb{F}$, and $\chi : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ a character, that

$$\pi(r, \lambda, \chi) := \frac{\text{c-Ind}_{KZ}^{G}\,\mathrm{Sym}^r\,\mathbb{F}^2}{T - \lambda} \otimes (\chi \circ \det).$$

Again by Barthel–Livné, $\pi(r, \lambda, \chi)$ is always “non-supersingular” if $\lambda \neq 0$.

Theorem 2.12 (Breuil, 2001)**.** *The representation $\pi(r, 0, \chi)$ is irreducible. As a consequence, any supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, with a central character, is isomorphic to some $\pi(r, 0, \chi)$.*

Proof Sketch. The key lies in the following first step.

Step I. Show that the subspace of I_1-invariants of $(\text{c-Ind}_{KZ}^{G}\sigma)/T$ is 2-dimensional. Precisely, there is a natural K-equivariant embedding $\sigma \hookrightarrow \text{c-Ind}_{KZ}^{G}\sigma$ (this can be obtained by Frobenius reciprocity). Fix a nonzero vector $v \in \sigma^{I_1}$ and set $w := \Pi(v)$. Then apply Lemma 2.13 below.

Step II. Each element of $((\text{c-Ind}_{KZ}^{G}\sigma)/T)^{I_1}$ generates $(\text{c-Ind}_{KZ}^{G}\sigma)/T$ as a G-representation. This is the easy part, as in the case of principal series.

Step III. Conclude by the irreducibility criterion 2.5. ☐

Lemma 2.13. *The images of $\{v, w\}$ in $(\text{c-Ind}_{KZ}^{G}\sigma)/T$ are linearly independent, and span the subspace $((\text{c-Ind}_{KZ}^{G}\sigma)/T)^{I_1}$.*

Proof. We may assume χ is trivial. Write $\sigma = \mathrm{Sym}^r\,\mathbb{F}^2 \otimes \mathbb{1} = \mathrm{Sym}^r\,\mathbb{F}^2$, then I acts on v via a character η which sends $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ to a^r. The vector w is also fixed by I_1, and I acts on w via η^s, so we obtain by Frobenius reciprocity

$$\mathrm{Ind}_I^K\,\eta^s \longrightarrow \text{c-Ind}_{KZ}^{G}\sigma.$$

There is a short exact sequence

$$0 \longrightarrow \mathrm{Sym}^r\,\mathbb{F}^2 \otimes \mathbb{1} \longrightarrow \mathrm{Ind}_I^K\,\eta^s \longrightarrow \mathrm{Sym}^{p-1-r}\,\mathbb{F}^2 \otimes \det{}^r \longrightarrow 0.$$

The point is that the subrepresentation $\mathrm{Sym}^r\,\mathbb{F}^2$ is exactly $T(\sigma)$, thus becomes 0 in $\text{c-Ind}_{KZ}^{G}\sigma$. Note that $\mathrm{Sym}^r\,\mathbb{F}^2$ and $\mathrm{Sym}^{p-1-r}\,\mathbb{F}^2 \otimes \det^r$ are not isomorphic. It proves that

$$(\mathrm{Sym}^r\,\mathbb{F}^2 \otimes \mathbb{1}) \oplus (\mathrm{Sym}^{p-1-r}\,\mathbb{F}^2 \otimes \det{}^r) \hookrightarrow \text{c-Ind}_{KZ}^{G}\sigma.$$

So we conclude the proof. ☐

Remark 2.14. The result of Theorem 2.12 is a feature of the base field $\mathbb{Q}_p$, which fails to be valid when $L \neq \mathbb{Q}_p$.

Theorem 2.15. *We have the following isomorphisms between supersingular representations of G, read as*

$$\pi(r, 0, \chi) \cong \pi(r, 0, \chi\mu_{-1}) \cong \pi(p-1-r, 0, \chi\omega^r) \cong \pi(p-1-r, 0, \chi\omega^r\mu_{-1}).$$

And these are all the descriptions of $\pi(r, 0, \chi)$.

Proof. We may assume $\chi = \mathbb{1}$. The first isomorphism is clear, coming from

$$\text{c-Ind}_{KZ}^G \sigma \cong (\mu_{-1} \circ \det) \otimes \text{c-Ind}_{KZ}^G \sigma.$$

Moreover, the third one follows from the first two, so it suffices to prove the second one. Lemma 2.13 implies that $\pi(r,0,1)$ contains $\mathrm{Sym}^{p-1-r}\,\mathbb{F}^2 \otimes \det^r$ as a sub-K-representation, which induces by Frobenius reciprocity a G-equivariant morphism

$$(\text{c-Ind}_{KZ}^G \mathrm{Sym}^{p-1-r}\,\mathbb{F}^2) \otimes \omega^r \cong \text{c-Ind}_{KZ}^G (\mathrm{Sym}^{p-1-r}\,\mathbb{F}^2 \otimes \det^r) \twoheadrightarrow \pi(r,0,1).$$

By the theorem 2.11 of Barthel-Livné, this surjection must factor through

$$\phi : \pi(p-1-r, \lambda, \mathbb{1}) \twoheadrightarrow \pi(r,0,1)$$

for some $\lambda \in \mathbb{F}$, and since $\pi(r,0,1)$ is supersingular, we have $\lambda = 0$ by Theorem 2.11(3). But $\pi(p-1-r,0,1)$ is irreducible by Breuil's theorem 2.12, so ϕ must be an isomorphism. □

2.4. **The mod p local Langlands correspondence.** Fix a finite extension E of $\mathbb{Q}_p$. We first normalize the local class field map $\iota : \mathbb{Q}_p^\times \hookrightarrow G_{\mathbb{Q}_p}^{\mathrm{ab}}$ so that uniformizers are sent to the geometric Frobenius.

Corollary 2.16. *There exists a (unique) bijection between the isomorphism classes of supersingular representations of* $\mathrm{GL}_2(\mathbb{Q}_p)$ *and the isomorphism classes of irreducible* 2*-dimensional representations of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ *over* E *such that*

$$\pi(r,0,\chi) \longleftrightarrow (\mathrm{Ind}\,\omega_2^{r+1}) \otimes \chi$$

for all $0 \leqslant r \leqslant p-1$ *and all* χ.

To take account of principal series, we define semisimple modulo p local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. For $0 \leqslant r \leqslant p-1$, we write $[p-3-r]$ for the unique integer in $\{0,\dots,p-2\}$ which is congruence to $p-3-r$ modulo $p-1$. For example, $[p-3-r] = p-2$ when $r = p-2$.

Construction 2.17 (Breuil)**.** The (semisimple) modulo p local Langlands correspondence when $\lambda \neq 0$ is given by the following rule:

$$\begin{pmatrix} \omega^{r+1}\mu_r & 0 \\ 0 & \mu_{\lambda^{-1}} \end{pmatrix} \longleftrightarrow \left(\frac{\text{c-Ind}_{KZ}^G \sigma_r}{T-\lambda}\right)^{\mathrm{ss}} \otimes \eta \oplus \left(\frac{\text{c-Ind}_{KZ}^G \sigma_{[p-3-r]}}{T-\lambda^{-1}} \otimes \omega^{r+1}\right)^{\mathrm{ss}} \otimes \eta.$$

Remark 2.18. In the language of principal series, the correspondence is written as

$$\begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix} \longleftrightarrow (\mathrm{Ind}_B^G\, \chi_1 \otimes \chi_2\omega^{-1}) \oplus (\mathrm{Ind}_B^G\, \chi_2 \otimes \chi_1\omega^{-1}).$$

We will also need the non-semisimple version of the correspondence, constructed by Colmez. Assume that ρ satisfies $\mathrm{End}_{G_{\mathbb{Q}_p}}(\rho) \cong \mathbb{F}$, and ρ is generic in the reducible case.

(1) If ρ is absolutely irreducible, then $\kappa(\rho)$ is the supersingular representation on the right-hand side of Corollary 2.16 above.
(2) If $\rho \sim \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ satisfies that $\chi_1\chi_2^{-1} \neq \omega^{\pm 1}, \mathbb{1}$, then there is an non-split exact sequence

$$0 \longrightarrow \mathrm{Ind}_{B(\mathbb{Q}_p)}^G(\chi_2 \otimes \chi_1\omega^{-1}) \longrightarrow \kappa(\rho) \longrightarrow \mathrm{Ind}_{B(\mathbb{Q}_p)}^G(\chi_1 \otimes \chi_2\omega^{-1}) \longrightarrow 0.$$

2.5. **Quaternionic case.** (Indeed, the results below hold true for more general D.) Let D be a finite dimensional central $\mathbb{Q}_p$-division algebra with $\dim_F D = 4$. The homomorphism $v : \mathbb{Q}_p^\times \to \mathbb{Z}$ extends to a surjective homomorphism $v_D : D^\times \to \mathbb{Z}$, which is indeed a valuation. Extend v_D to D by defining $v_D(0) = \infty$. Let $\mathcal{O}_D = \{x \in D : v_D(x) \geqslant 0\}$ and $\mathfrak{p}_D = \{x \in D : v_D(x) \geqslant 1\}$. Take the residue field $k_D = \mathcal{O}_D/\mathfrak{p}_D$. Then $\mathcal{O}_D$ is the unique maximal order in D, $\mathfrak{p}_D$ is the unique maximal ideal of $\mathcal{O}_D$, and k_D is isomorphic to $\mathbb{F}_{p^2}$. We have a chain of subgroups

$$\mathcal{O}_D^\times \supseteq 1 + \mathfrak{p}_D \supseteq 1 + \mathfrak{p}_D^2 \supseteq \cdots,$$

each of them is normal in $D^\times$. We have canonical isomorphisms

$$\mathcal{O}_D^\times/(1+\mathfrak{p}_D) \cong \mathbb{F}_{p^2}^\times, \quad (1+\mathfrak{p}_D^i)/(1+\mathfrak{p}_D^{i+1}) \cong \mathfrak{p}_D^i/\mathfrak{p}_D^{i+1}.$$

Fix an embedding $\xi : \mathbb{F}_{p^2} \to \mathbb{F}$. It serves the same role as ω_2 with order p^2-1.

Theorem 2.19. *Let $\pi : D^\times \to \mathrm{GL}_2(\mathbb{F})$ be a continuous $\mathbb{F}$-representation. Then π must have one of the following forms:*

(1) *π is irreducible and*
$$\pi \cong \mathrm{Ind}_{\mathbb{Q}_p^\times \mathcal{O}_D^\times}^{D^\times} \xi^m \otimes (\eta \circ \mathrm{Nrd}),$$
where $1 \leqslant m \leqslant p-1$. In this case,
$$\pi|_{\mathcal{O}_D^\times} \cong \xi^m \oplus \xi^{mp}.$$

(2) *π is reducible and*
$$\pi \cong \begin{pmatrix} \chi_1 \xi^{(p+1)m_1} & * \\ 0 & \chi_1 \xi^{(p+1)m_2} \end{pmatrix}$$
for some integers $0 \leqslant m_1, m_2 \leqslant p-2$ and characters χ_1, χ_2 which are trivial on $\mathcal{O}_D^\times$.

Proof. This is similar to the proof for Galois representations. Use the following facts:

- $1 + \mathfrak{p}_D$ is a pro-p group;
- $\mathcal{O}_D^\times / (1 + \mathfrak{p}_D) \cong \mathbb{F}_{p^2}^\times$ is abelian and prime-to-p;
- $\mathbb{Q}_p^\times \mathcal{O}_D^\times$ is of index 2 in $D^\times$;
- if ϖ_D is a uniformizer (with $\varpi_D^2 = p$), then $\varpi_D x \varpi_D^{-1} = x^p$ for $x \in \mathbb{F}_{p^2}^\times \hookrightarrow \mathcal{O}_D^\times$ via Teichmüller lifting.

These are sufficient to complete the proof. □

3. Serre weights

3.1. **Complement of mod p local Langlands: the non-semisimple case.** Recall from Theorem 1.1 that given $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$, the mod p local Langlands correspondence associates an admissible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ to it. Precisely,

- If $\overline{\rho} \sim (\mathrm{Ind}\, \omega_2^{r+1}) \otimes \chi$, then
$$\kappa(\overline{\rho}) = \pi(r, 0, \chi) := \frac{\text{c-Ind}_{KZ}^G \mathrm{Sym}^r \mathbb{F}^2}{T} \otimes (\chi \circ \det)$$
is a supersingular representation;
- If $\overline{\rho} \sim \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$, then there is a non-split short exact sequence
$$0 \longrightarrow \mathrm{Ind}_B^G(\chi_2 \otimes \chi_1 \omega^{-1}) \longrightarrow \kappa(\overline{\rho}) \longrightarrow \mathrm{Ind}_B^G(\chi_1 \otimes \chi_2 \omega^{-1}) \longrightarrow 0.$$

When $\overline{\rho}$ is semisimple, the classification is due to Breuil's work. Also, Colmez has given a more conceptual construction, which allows to treat also the non-semisimple case.

Definition 3.1. A (φ, Γ)-*module* over $\mathbb{F}(\!(T)\!)$ is a finite free $\mathbb{F}(\!(T)\!)$-module M, equipped with semi-linear actions of φ and Γ that commute to each other, where
$$\varphi : T \longmapsto T^p, \quad \gamma : T \longmapsto (1+T)^\gamma - 1$$
for all $\gamma \in \Gamma$. (Note that we have an isomorphism $\chi_{\mathrm{cycl}} : \Gamma \xrightarrow{\sim} \mathbb{Z}_p^\times$.) Moreover, a (φ, Γ)-module M is called *étale* if
$$M \otimes_{\mathbb{F}(\!(T)\!), \varphi} \mathbb{F}(\!(T)\!) \cong M.$$

Theorem 3.2 (Fontaine)**.** *There is an equivalence of categories between*
$$\{\textit{étale } (\varphi, \Gamma)\textit{-modules over } \mathbb{F}(\!(T)\!)\} \overset{V}{\longleftrightarrow} \{\mathbb{F}\textit{-representations of } G_{\mathbb{Q}_p}\}.$$
Moreover, this correspondence preserves module ranks over different bases.

Construction 3.3 (Colmez's observation)**.** If we have an $\mathbb{F}$-vector space M equipped with a continuous action of the mirabolic monoid $P^+ := \begin{pmatrix} \mathbb{Z}_p \backslash \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$, then we can define a structure of (φ, Γ)-module on M as follows:

(1) View it as an $\mathbb{F}[\![T]\!]$-module via the action of $\alpha = \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$, where the action is compatible with the isomorphism $\mathbb{F}[\![\alpha]\!] \simeq \mathbb{F}[\![T]\!]$, sending $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1$ to T.

(2) Identify $\begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & 1 \end{pmatrix}$ with Γ and $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ with φ; note that they do really commute.

The functor D of Colmez is defined as follows:

$$D(\pi) := \mathbb{F}(\!(T)\!)\,\hat{\otimes}_{\mathbb{F}[\![T]\!]}\,\pi^\vee,$$

where $\pi^\vee = \mathrm{Hom}_{\mathbb{F}}(\pi, \mathbb{F})$ denotes the Pontryagin dual equipped with the $\mathfrak{m}_{I_1}$-adic topology, with $\mathfrak{m}_{I_1}$ the maximal ideal of $\mathbb{F}[\![I_1]\!]$.[4]

Using the equivalence of Fontaine, we get a continuous representation of $G_{\mathbb{Q}_p}$ from a (φ, Γ)-module. In practice, we normalize the definition to get a covariant functor. Put

$$\mathbb{V}(\pi) := V(D(\pi))^\vee(1).$$

Based on the classification of irreducible smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representations at the beginning of Subsection 2.2, we have the following.

Theorem 3.4. *The functor $\mathbb{V} : \mathsf{Rep}_{\mathbb{F}}(G) \to \mathsf{Rep}_{\mathbb{F}}(G_{\mathbb{Q}_p})$ is an exact covariant functor. Moreover, on irreducible representations of form $\pi = \pi(r, 0, \chi)$, we have*

(1) *For principal series, $\mathbb{V}(\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2\omega^{-1})) = \chi_2$.*
(2) *For characters, $\mathbb{V}(\chi \circ \det) = 0$.*
(3) *For special series, $\mathbb{V}(\mathrm{Sp} \otimes (\chi \circ \det)) = \omega\chi$.*
(4) *For supersingular representations, $\mathbb{V}(\pi(r, 0, \chi)) = (\mathrm{Ind}\,\omega_2^{r+1}) \otimes \chi$.*

This allows to extend the mod p local Langlands correspondence (c.f. Corollary 2.16 and Construction 2.17) to the non-semisimple case.

Theorem 3.5. *There is a representation $\kappa(\overline{\rho})$ of finite length of G, unique up to isomorphism, such that*

(a) *$\mathbb{V}(\kappa(\overline{\rho})) \cong \overline{\rho}$;*
(b) *$\kappa(\overline{\rho})$ has central character $\det(\overline{\rho})\omega$;*
(c) *$\kappa(\overline{\rho})$ has no finite-dimensional G-subrepresentation;*
(d) *when $\overline{\rho}$ is reducible, say $\overline{\rho}^{\mathrm{ss}} \sim \chi_1 \oplus \chi_2$, then the semisimplication of $\kappa(\overline{\rho})$ is equal to*

$$\kappa(\overline{\rho})^{\mathrm{ss}} = (\mathrm{Ind}_B^G(\chi_2 \oplus \chi_1\omega^{-1}))^{\mathrm{ss}} \oplus (\mathrm{Ind}_B^G(\chi_1 \oplus \chi_2\omega^{-1}))^{\mathrm{ss}}.$$

Example 3.6 (Non-generic case)**.**

(1) If $\overline{\rho} \sim \begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix}$ and it is non-split, then $\kappa(\overline{\rho})$ has a filtration

$$\mathrm{Sp} \;—\; \mathbb{1}_G \;—\; \mathrm{Ind}_B^G(\omega \otimes \omega^{-1}).$$

Note that $\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\mathbb{1}, \omega)$ is 2-dimensional. Colmez proved that $\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\mathbb{1}, \mathrm{Sp})$ is also 2-dimensional and there is a natural bijection from it towards $\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\mathbb{1}, \omega)$.

(2) If $\overline{\rho} \sim \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix}$ and it is non-split, then $\kappa(\overline{\rho})$ has a filtration

$$\mathrm{Ind}_B^G(\omega \otimes \omega^{-1}) \;—\; \mathrm{Sp} \;—\; \mathbb{1}_G.$$

3.2. **Serre weights in $\mathrm{GL}_2(\mathbb{Q}_p)$-case.**

Definition 3.7. An irreducible smooth $\mathbb{F}$-representation σ of $K = \mathrm{GL}_2(\mathbb{Z}_p)$ is called a *Serre weight of $\overline{\rho}$* if

$$\mathrm{Hom}_K(\sigma, \kappa(\overline{\rho})) \neq 0.$$

Recall that the *socle* of a representation π is defined to be its maximal semisimple subrepresentation. Let $W(\overline{\rho})$ consist of irreducible representations which occur in $\mathrm{soc}_K(\kappa(\overline{\rho}))$, without counting the multiplicity.

Theorem 3.8. (1) *If $\overline{\rho} \sim (\mathrm{Ind}\,\omega_2^{r+1}) \otimes \chi$, then*

$$W(\overline{\rho}) = \{\mathrm{Sym}^r\,\mathbb{F}^2, \mathrm{Sym}^{p-1-r}\,\mathbb{F}^2 \otimes \det{}^r\} \otimes (\chi \circ \det).$$

[4] This is not the original definition of Colmez, because $\pi^\vee$ is not finitely generated as an $\mathbb{F}[\![T]\!]$-module, so the meaning of getting a (φ, Γ)-module of finite rank is not clear, which can be a potential obstruction to go further.

(2) *If* $\overline{\rho} \sim \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ *with* $\chi_1\chi_2^{-1} \neq \omega^{\pm 1}, \mathbb{1}$, *then*

$$W(\overline{\rho}) = \{\mathrm{Sym}^r \mathbb{F}^2\} \otimes (\chi_2 \circ \det),$$

where r *is determined by the rule* $\chi_1\chi_2^{-1}|_{I_{\mathbb{Q}_p}} = \omega^{r+1}$.

We take some preparation work before proving Theorem 3.8. For $0 \leqslant r \leqslant p-2$, let η_r denote the character $\mathbb{F}_p^\times \to \mathbb{F}^\times$ that sends a to a^r.

Lemma 3.9. *For* $0 \leqslant r \leqslant p-2$, *there is a short exact sequence*

$$0 \longrightarrow \mathrm{Sym}^r \mathbb{F}^2 \longrightarrow \mathrm{Ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)}(\mathbb{1} \otimes \eta_r) \longrightarrow \mathrm{Sym}^{p-1-r} \mathbb{F}^2 \otimes \det{}^r \longrightarrow 0.$$

Moreover, the sequence splits if and only if $r = 0$.

Proof. First, the coinvariant $(\mathrm{Sym}^r \mathbb{F}^2)_{U(\mathbb{F}_p)}$ is 1-dimensional spanned by Y^r, and $B(\mathbb{F}_p)$ acts via the character $\mathbb{1} \otimes \eta_r$. Thus, by Frobenius reciprocity, we deduce a nonzero (and hence injective) morphism $\mathrm{Sym}^r \mathbb{F}^2 \hookrightarrow \mathrm{Ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)}(\mathbb{1} \otimes \eta_r)$. On the other hand, the invariants of $\mathrm{Sym}^{p-1-r} \mathbb{F}^2 \otimes \det^r$ is also given by $\mathbb{1} \otimes \eta_r$ as

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \longmapsto a^{p-1-r}(ad)^r = d^r,$$

so we deduce an surjection

$$\mathrm{Ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)}(\mathbb{1} \otimes \eta_r) \twoheadrightarrow \mathrm{Sym}^{p-1-r} \mathbb{F}^2 \otimes \det{}^r.$$

Counting the dimension, the result follows. □

Remark 3.10. The two Jordan–Hölder factors of $\mathrm{Ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)}(\mathbb{1} \otimes \eta_r)$ are isomorphic, even when $r = (p-1)/2$, we also say $\mathrm{Ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)}(\mathbb{1} \otimes \eta_r)$ is multiplicity-free.

Now we are ready to tackle with the proof.

Proof of Theorem 3.8. (1) By definition, $\mathrm{Sym}^r \mathbb{F}^2$ embeds in $\pi(r, 0, \mathbb{1})$. Let $v \in (\mathrm{Sym}^r \mathbb{F}^2)^{I_1}$, and set $w := \Pi v$. Then we saw that w generates $\mathrm{Sym}^{p-1-r} \mathbb{F}^2 \otimes \det^r$ in $\pi(r, 0, \mathbb{1})$, so we already have

$$\mathrm{Sym}^r \mathbb{F}^2 \oplus (\mathrm{Sym}^{p-1-r} \mathbb{F}^2 \otimes \det{}^r) \hookrightarrow \mathrm{soc}_K(\pi(r, 0, \mathbb{1})).$$

Since $\dim_{\mathbb{F}} \pi(r, 0, \mathbb{1})^{I_1} = 2$ by Breuil's theorem 2.12, the above embedding is an equality.

(2) We always have $\mathrm{soc}_K(\pi) \subset \pi^{K_1}$, where K_1 denotes $\mathrm{Ker}(K \to \mathrm{GL}_2(\mathbb{F}_p))$ (a pro-p group). For principal series, the Iwasawa decomposition $G = B \cdot K$ implies

$$(\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2))^{K_1} = (\mathrm{Ind}_{B(\mathbb{Z}_p)}^K(\chi_1 \otimes \chi_2)|_{B(\mathbb{Z}_p)})^{K_1} = \mathrm{Ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)}(\overline{\chi_1} \otimes \overline{\chi_2}).$$

By definition and local class field theory, we have

$$\overline{\chi_1\chi_2}^{-1}\omega^{-1} = \eta_r,$$

where r is as in the statement, thus the socle of PS_1 is $\mathrm{Sym}^r \mathbb{F}^2 \otimes (\chi_2 \circ \det)$ by Lemma 3.9.

To conclude, we need to show that $0 \to \mathrm{PS}_1 \to \kappa(\overline{\rho}) \to \mathrm{PS}_2 \to 0$ induces an equality $\mathrm{soc}_K(\mathrm{PS}_1) = \mathrm{soc}_K(\kappa(\overline{\rho}))$. One checks that

$$\mathrm{soc}_K(\mathrm{PS}_2) = \sigma_2 := (\mathrm{Sym}^{p-3-r} \mathbb{F}^2 \otimes \det{}^{r+1}) \otimes (\chi_2 \circ \det).$$

We prove that the induced sequence

$$0 \longrightarrow \mathrm{PS}_1|_K \longrightarrow * \longrightarrow \sigma_2 \longrightarrow 0$$

is non-split. Otherwise, say $\sigma_2 \hookrightarrow \kappa(\overline{\rho})$, then there would be a morphism

$$\frac{\text{c-}\mathrm{Ind}_{KZ}^G \sigma_2}{T - \lambda} \longrightarrow \kappa(\overline{\rho})$$

for some $\lambda \in \mathbb{F}$. Looking at Jordan–Hölder factors of $\kappa(\overline{\rho})$, this λ is the unique element such that

$$\frac{\text{c-}\mathrm{Ind}_{KZ}^G \sigma_2}{T - \lambda} \cong \mathrm{PS}_2$$

by Theorem 2.11 of Barthel–Livné. However, this implies that $\kappa(\overline{\rho})$ splits, a contradiction. □

If $\overline{\rho} = \chi \oplus \chi_2$, then

$$W(\overline{\rho}) = \{\mathrm{Sym}^r \mathbb{F}^2, \mathrm{Sym}^{p-3-r} \mathbb{F}^2 \otimes \det^{r+1}\} \otimes (\chi_2 \circ \det).$$

It is a general principe that $W(\overline{\rho}) \subset W(\overline{\rho}^{\mathrm{ss}})$. Also, we see $\overline{\rho}$ is "more split" implies that $W(\overline{\rho})$ is larger via unwinding the definition.

Remark 3.11. In the argument above, one may notice that

$$\mathrm{soc}_K(\kappa(\overline{\rho})) = \bigoplus_{\sigma \in W(\overline{\rho})} \sigma.$$

This property is called *multiplicity one*.

Prototypically, the origin of the notion of Serre weight (Definition 3.7) comes from Serre's conjecture (and now a theorem of Khare–Wintenberger).

Conjecture 3.12 (Serre, 1987)**.** *Let $\overline{r} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation. Assume $\overline{r}$ is absolutely irreducible and odd, i.e.,* $\det \overline{r}(c) = -1$. *Then there exists a normalized cuspidal modular form $f = \sum_{n \geqslant 1} a_n q^n \in S_k(\Gamma_1(N))$ with $a_1 = 1$, of weight k and level N, such that for all prime $\ell \nmid pN$,*

$$\mathrm{Tr}(\overline{r}(\mathrm{Frob}_\ell)) = a_\ell.$$

Beyond this, Serre conjectured the minimal integer $k(\overline{\rho})$ associated to each residual Galois representation $\overline{\rho}$, such that $2 \leqslant k(\overline{r} \otimes \omega^i) \leqslant p+1$ for some $i \in \mathbb{Z}$. He also predicted the minimal level N for the normalized cuspidal modular form f, which is prime to p and equals the Artin conductor of $\overline{r}$. These are called *Serre's refined conjecture*.

Note that Conjecture 3.12 holds for Galois representations on $\mathbb{Q}$. When trying to generalize Serre's conjecture to totally real field case, Buzzard-Diamond-Jarvis observed that there is a bijection

$$\left\{(k,i) \;\middle|\; \begin{array}{c} 2 \leqslant k \leqslant p+1, \\ 0 \leqslant i \leqslant p-2 \end{array}\right\} \longleftrightarrow \left\{\begin{array}{c} \text{Irreducible representations} \\ \mathrm{Sym}^{k-2}\mathbb{F}^2 \otimes \det^i : \mathrm{GL}_2(\mathbb{Z}_p) \to \mathrm{GL}_2(\mathbb{F}) \end{array}\right\}.$$

3.3. **Serre weights in quaternion case.** First, we note that the representation $\mathrm{JL}(\overline{\rho})$ is defined on a global setup. So it is hard to obtain an explicit description, even for the Serre weights.

Setups. We introduce the following objects.

- Fix a number field F. Fix another finite extension E of $\mathbb{Q}_p$, with $\mathcal{O}$ the ring of integers and ϖ a choice of the uniformizer.
- Let B be a definite quaternion algebra with centre F; let Σ be the set of primes at which B ramifies.
- Assume B is ramified at p. Let $D := B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ be the quaternion algebra over $\mathbb{Q}_p$.
- Choose $U = \prod_v U_v \subset (B \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}}^\infty)^\times$ to be a compact open subgroup. Write $U^p = \prod_{v \nmid p} U_v$. Insert the assumption that $U_p = \mathcal{O}_D^\times$.
- Let A be a topological $\mathcal{O}$-algebra. Fix a continuous character $\psi : \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^{\infty,\times} \to A^\times$.
- For those characters ψ that are trivial on the intersection $U^p \cap \mathbb{A}_{\mathbb{Q}}^{\infty,\times}$, define $S_\psi(U, A)$ as the space of those continuous functions $f : B^\times \backslash (B \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}}^\infty)^\times / U \to A$ satisfying that
  - for any $z \in \mathbb{A}_{\mathbb{Q}}^{\infty,\times}$, the condition $f(gz) = \psi(z) f(g)$ holds.
- Excluding the p-component, we also define

$$S_\psi(U^p, A) := \varinjlim_{V_p \subset U_p} S_\psi(U^p V_p, A).$$

Remark 3.13. For any character ψ, we can consider $\hat{S}_\psi(U^p, A)$, the space of those continuous functions $f : B^\times \backslash (B \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}}^\infty)^\times / U \to A$ satisfying the same condition as above. Then

$$S_\psi(U^p, A) \subset \hat{S}_\psi(U^p, A).$$

The equality holds if the topology of A is discrete; for example, when $A \in \{\mathbb{F}, \mathcal{O}/\varpi^n, E/\mathcal{O}\}$. Moreover, whenever $A = \mathcal{O}$, the equality fails but

$$S_\psi(U^p, A) = \hat{S}_\psi(U^p, A)^{\mathrm{sm}},$$

the subspace consisting of smooth functions.

We may write

$$(B \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}}^{\infty})^{\times} = \coprod_{i \in I} B^{\times} t_i \mathbb{A}_{\mathbb{Q}}^{\infty,\times}$$

for some finite index set I and $t_i \in (B \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}}^{\infty})^{\times}$; here $\mathbb{A}_{\mathbb{Q}}^{\infty,\times}$ is identified with the center of $(B \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}}^{\infty})^{\times}$. We further assume for each $i \in I$ that

$$U(\mathbb{A}_{\mathbb{Q}}^{\infty})^{\times} \cap t_i^{-1} B^{\times} t_i = \mathbb{Q}^{\times}.$$

This is always possible when U^p is sufficiently small.

Lemma 3.14. (1) *$S_{\psi}(U^p, \mathbb{F})$ is an admissible representation of $D^{\times}$. Moreover, when restricted to $\mathcal{O}_D^{\times}$, it is injective in the category of smooth representations of $\mathcal{O}_D^{\times}$ over $\mathbb{F}$ with central character.*

(2) *$S_{\psi}(U^p, E/\mathcal{O})$ is injective as a smooth representation of $\mathcal{O}_D^{\times}$ over $\mathcal{O}$-module, with central character.*

Proof. Note that (2) follows from (1) together with the $\mathcal{O}$-divisibility. As for (1), it is clear that $S_{\psi}(U^p, \mathbb{F})$ is an admissible representation of $D^{\times}$, since for open subgroup $V_p \subset U_p = \mathcal{O}_D^{\times}$, the invariant subspace $S_{\psi}(U^p, \mathbb{F})^{V_p}$ consists of functions which factor through $B^{\times} \backslash (B \otimes \mathbb{A}_{\mathbb{Q}}^{\infty})^{\times} / U^p V_p \to \mathbb{F}$, and hence $S_{\psi}(U^p, \mathbb{F})^{V_p}$ is a finite set.

To show the injectivity, by the condition $U(\mathbb{A}_{\mathbb{Q}}^{\infty})^{\times} \cap t_i^{-1} B^{\times} t_i = \mathbb{Q}^{\times}$, the map sending $f \in S_{\psi}(U^p, \mathbb{F})$ to the function $(u \mapsto (f(t_i u))_{i \in I})$ induces an isomorphism

$$S_{\psi}(U^p, \mathbb{F}) \cong \bigoplus_{i \in I} \mathcal{C}_{\psi}(U_p, \mathbb{F}),$$

where $\mathcal{C}_{\psi}(U_p, \mathbb{F})$ denotes the space of continuous $\mathbb{F}$-valued functions on U_p with central character ψ. □

Now we introduce the Hecke algebra. Let S be the union of finite places

$$S = \Sigma \cup \{p\} \cup \{v \mid U_v \text{ is not maximal}\} \cup \{v \mid \psi \text{ ramifies}\}.$$

For each $v \notin S$, define Hecke operators as double-coset operators

$$T_v := U_v \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} U_v, \quad S_v := U_v \begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{pmatrix} U_v.$$

Let $\mathbb{T}(U^p V_p)$ be the endomorphism ring acting on $S_{\psi}(U^p V_p, \mathcal{O})$ that is generated by T_v and S_v. Put

$$\mathbb{T} := \varprojlim_{V_p \subset U_p} \mathbb{T}(U^p V_p).$$

Then $\mathbb{T}$ (also denoted by $\mathbb{T}(U^p)$) acts faithfully and semisimply on $S_{\psi}(U^p, \mathcal{O})$.

Let $\overline{r} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ be a continuous absolutely irreducible representation, unramified outside S. Given $\overline{r}$, define a maximal ideal of $\mathbb{T}$ to be the kernel of

$$\begin{aligned} \mathbb{T} &\longrightarrow \mathbb{F} \\ T_v &\longmapsto \mathrm{Tr}(\overline{r}(\mathrm{Frob}_v)) \\ S_v &\longmapsto \mathrm{N}(v)^{-1} \det(\overline{r}(\mathrm{Frob}_v)) \end{aligned}$$

where $v \notin S$. We may consider the localization $T_{\mathfrak{m}}$ as well as $S_{\psi}(U^p, A)_{\mathfrak{m}}$.

Lemma 3.15. *Lemma 3.14 holds true after everything being localized at $\mathfrak{m}$.*

Proof. This is because $T_{\mathfrak{m}}$ isa complete noetherian semi-local ring and $S_{\psi}(U^p, A)_{\mathfrak{m}}$ is a direct summand of $S_{\psi}(U^p, A)$. □

Lemma 3.16. *Let W be a continuous representation of $\mathcal{O}_D^{\times}$ on a finite $\mathcal{O}$-torsion-free module. Then the following statements are equivalent:*

(i) $\mathrm{Hom}_{\mathcal{O}_D^{\times}}(\sigma, S_{\psi}(U^p, \mathbb{F})_{\mathfrak{m}}) \neq 0$ *for some Jordan–Hölder factor σ of $W/\varpi W$;*

(ii) $\mathrm{Hom}_{\mathcal{O}_D^{\times}}(W/\varpi W, S_{\psi}(U^p, \mathbb{F})_{\mathfrak{m}}) \neq 0$*;*

(iii) $\mathrm{Hom}_{\mathcal{O}_D^{\times}}(W, \hat{S}_{\psi}(U^p, \mathcal{O})_{\mathfrak{m}}) \neq 0.$

Moreover, if W is smooth, then the above are also equivalent to

(iv) $\mathrm{Hom}_{\mathcal{O}_D^{\times}}(W, S_{\psi}(U^p, \mathcal{O})_{\mathfrak{m}}) \neq 0.$

Proof. The equivalence between (i) and (ii) is because $S_\psi(U^p,\mathbb{F})_\mathfrak{m}$ is an injective object, by Lemma 3.15. Also, (ii) and (iii) are equivalent because $\hat{S}_\psi(U^p,\mathcal{O})_\mathfrak{m}$ is dually projective, i.e.,

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(\hat{S}_\psi(U^p,\mathcal{O})_\mathfrak{m},\mathcal{O}) \cong \mathrm{Hom}_{\mathbb{F}}(S_\psi(U^p,E/\mathcal{O})_\mathfrak{m},\mathbb{F}).$$

After this, using that $W^d/\varpi W^d = (W/\varpi W)^\vee$, it is easy to see that

$$\text{(ii)} \iff \mathrm{Hom}_{\mathcal{O}_D^\times}(\hat{S}_\psi(U^p,\mathcal{O})^d, W^d) \neq 0 \iff \mathrm{Hom}_{\mathcal{O}_D^\times}(\hat{S}_\psi(U^p,\mathcal{O})^d/\varpi, W^d/\varpi) \neq 0 \iff \text{(iii)}.$$

Moreover, in case that W is smooth, (iii) is clearly equivalent to (iv) as $S_\psi(U^p,\mathcal{O})_\mathfrak{m}$ identifies with $\hat{S}_\psi(U^p,\mathcal{O})^{\mathrm{sm}}_\mathfrak{m}$ (see Remark 3.13). □

Recall that $\mathcal{O}_D^\times/U_D^1 \cong \mathbb{F}_{p^2}^\times$, so that any irreducible representation of $\mathcal{O}_D^\times$ is a character.

Definition 3.17. A character χ of $\mathcal{O}_D^\times$ is called a *Serre weight* of $\overline{r}: G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{F})$ if

$$\mathrm{Hom}_{\mathcal{O}_D^\times}(\chi, S_\psi(U^p,\mathbb{F})_\mathfrak{m}) \neq 0.$$

The set of Serre weights of $\overline{r}$ is denoted by $W_B(\overline{r})$.

Let ϖ_D denote a fixed uniformizer of $D^\times$ with $\varpi_D^2 = p$. Then

$$\varpi_D^{-1}[\lambda]\varpi_D = [\lambda^p], \quad \forall \lambda \in \mathbb{F}_{p^2}^\times.$$

Lemma 3.18. *The elements in the set $W_B(\overline{r})$ always appear with their couple, i.e., if $\chi \in W_B(\overline{r})$, then so also $\chi^p \in W_B(\overline{r})$.*

Theorem 3.19. *Consider a character $\chi: \mathcal{O}_D^\times \to \mathbb{F}_p^\times$ with the norm map $\mathbb{F}_{p^2}^\times \to \mathbb{F}_p^\times$.*

1. *Assume χ does not factor through the norm. Then $\chi \in W_B(\overline{r})$ if and only if $\overline{r}$ lifts to a modular Galois representation $G_\mathbb{Q} \to \mathrm{GL}_2(E)$, which at p is potentially crystalline of type $[\chi]\oplus[\chi^p]$ with Hodge–Tate weights $(0,1)$.*
2. *Assume χ factors through the norm. Then $\chi \in W_B(\overline{r})$ if and only if $\overline{r}$ lifts to a modular Galois representation $G_\mathbb{Q} \to \mathrm{GL}_2(E)$, which at p is potentially semistable, but not potentially crystalline, of type $[\chi]\oplus[\chi]$ with Hodge–Tate weights $(0,1)$.*

Our main object to study is the eigenspace of $\mathfrak{m}$, say

$$\pi_B(\overline{r}) := S_\psi(U^p,\mathbb{F})[\mathfrak{m}].$$

Since the action of $\mathbb{T}$ commutes with that of $D^\times$, it is easy to see that $\chi \in W_B(\overline{r})$ if and only if $\mathrm{Hom}_{\mathcal{O}_D^\times}(\chi, \pi_B(\overline{r})) \neq 0$. We have the following useful lemma, which gives more information of $\pi_B(\overline{r})$.

Lemma 3.20. *Let $\sigma_1 \subset \sigma$ be two $\mathcal{O}_D^\times$-representations such that $\iota: \sigma_1 \hookrightarrow \pi_B(\overline{r})$. Assume also*

$$\mathrm{JH}(\sigma/\sigma_1) \cap W_B(\overline{r}) = \emptyset,$$

where $\mathrm{JH}(-)$ denotes the (spectrum of) Jordan–Hölder factors. Then ι extends to an embedding $\sigma \hookrightarrow \pi_B(\overline{r})$.

Proof. By assumption, following Lemma 3.16, we have

$$\mathrm{Hom}_{\mathcal{O}_D^\times}(\sigma/\sigma_1, S_\psi(U^p,\mathbb{F})_\mathfrak{m}) = 0.$$

Applying Lemma 3.15, there exists a (unique) embedding $\sigma \hookrightarrow S_\psi(U^p,\mathbb{F})_\mathfrak{m}$ extending ι. So it suffices to show that the image of σ along this extended embedding is contained in $\pi_B(\overline{r})$, or namely, is annihilated by $\mathfrak{m}$.

For this, let $f \in \mathfrak{m}$ be viewed as an endomorphism of $S_\psi(U^p,\mathbb{F})_\mathfrak{m}$; we need to show the composition

$$\sigma \hookrightarrow S_\psi(U^p,\mathbb{F})_\mathfrak{m} \xrightarrow{f} S_\psi(U^p,\mathbb{F})_\mathfrak{m}$$

is zero. By assumption, $f|_{\sigma_1} = 0$, so we obtain an induced morphism $f: \sigma/\sigma_1 \to S_\psi(U^p,\mathbb{F})_\mathfrak{m}$ (by an abuse of the notation). However, this f must be zero due to the assumption. □

In practice, we suppose that $\sigma_1 = \mathrm{soc}_K(\pi_B(\overline{r}))$ and σ is the maximal representation of $\mathrm{Inj}_{\mathcal{O}_D^\times}\sigma_1$ (the injective envelope of σ_1 as a representation of $\mathcal{O}_D^\times$, which is isomorphic to $S_\psi(U^p,\mathbb{F})_\mathfrak{m}$), such that $\mathrm{JH}(\sigma/\sigma_1) \cap W_B(\overline{r}) = \emptyset$ is satisfied, then $\sigma \hookrightarrow \pi_B(\overline{r})$.

3.4. **Tame types and inertial local Langlands correspondence.** Let $W_{\mathbb{Q}_p}$ denote the Weil group of the Galois group of $\mathbb{Q}_p$.

Definition 3.21. An *inertial type* is a 2-dimensional representation $\tau : I_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ with an open kernel, which can be extended to a representation of $W_{\mathbb{Q}_p}$. An inertial type τ is said to be *tame* if it is trivial on the wild inertia subgroup, i.e. $\tau|_{I^{\mathrm{wild}}_{\mathbb{Q}_p}} = 0$.

Lemma 3.22. *An inertial type always has one of the following forms:*

(i) *τ is reducible and isomorphic to the sum $\chi_1 \oplus \chi_2$ of two characters which extend to $W_{\mathbb{Q}_p}$;*
(ii) *τ is reducible and isomorphic to $\eta_1 \oplus \eta_2$, where η_1, η_2 don't extend to $W_{\mathbb{Q}_p}$; in this case, η_i extends to $W_{\mathbb{Q}_{p^2}}$ and $\eta_2 = \eta_1^{\mathrm{conj}}$.*
(iii) *τ is irreducible.*

An inertial type τ in case (i) above is called a *principal series*. In case (ii) and (iii), τ is called a *supercuspidal type*. Also, τ is called a *discrete series* if it is either supercuspidal, or is of case (i) with $\tau \cong \chi \oplus \chi$.

Theorem 3.23 (Henniart)**.** *Let τ be an inertial type.*

(1) *There exists a unique smooth irreducible representation $\sigma(\tau)$ of $K = \mathrm{GL}_2(\mathbb{Z}_p)$, such that for any infinite-dimensional smooth irreducible representation π of G,*
$$\mathrm{Hom}_K(\sigma(\tau), \pi) \neq 0 \iff \mathrm{WD}(\pi)|_{I_{\mathbb{Q}_p}} \cong \tau.$$
Here $\mathrm{WD}(\pi)$ denotes the Weil–Deligne representation associated to π.
(2) *Similarly, there exists a unique smooth irreducible representation $\sigma^{\mathrm{cris}}(\tau)$ of $K = \mathrm{GL}_2(\mathbb{Z}_p)$, such that*
$$\mathrm{Hom}_K(\sigma^{\mathrm{cris}}(\tau), \pi) \neq 0 \iff \mathrm{WD}(\pi)|_{I_{\mathbb{Q}_p}} \cong \tau \text{ and } N = 0.$$

In both cases, the Hom *space is 1-dimensional.*

We always have $\sigma(\tau) = \sigma^{\mathrm{cris}}(\tau)$ except when $\tau = \chi \oplus \chi$ is a scalar, for which $\sigma(\chi \oplus \chi) = \mathrm{St} \otimes (\chi \circ \det)$ and $\sigma^{\mathrm{cris}}(\chi \oplus \chi) = \chi \circ \det$. Here St denotes the Steinberg representation of $\mathrm{GL}_2(\mathbb{F}_p)$ over $\overline{\mathbb{Q}}_p$.

Example 3.24. We will be only interested in Lemma 3.22(ii). There is a unique irreducible $(p-1)$-representation $\Theta(\eta)$ characterized by the isomorphism
$$\Theta(\eta) \otimes \mathrm{St} \cong \mathrm{Ind}_{\mathbb{F}_{p^2}^\times}^{\mathrm{GL}_2(F_p)} \eta.$$
Then, when τ is of case (ii), we have $\sigma(\tau) = \Theta(\eta)$.

Gee–Geraghty [GG15] developed an analogous theory for $D^\times$. Let τ be a discrete series of inertial type. Then there exists $\sigma_D(\tau)$, a finite-dimensional representation of $\mathcal{O}_D^\times$ such that
$$\mathrm{Hom}_{\mathcal{O}_D^\times}(\sigma_D(\tau), \pi_D) \neq 0 \iff \mathrm{LL}^{-1}(\mathrm{JL}(\pi_D))|_{I_{\mathbb{Q}_p}} \cong \tau.$$
Here LL denotes the functor of local Langlands correspondence.

Example 3.25. Let $\eta : \mathbb{F}_{p^2}^\times \to E^\times$ with $\eta \neq \eta^p$. Let $\tau := \eta \oplus \eta^p$ be the supercuspidal inertial type associated to η, where it is customary to denote by η the composition $I_{\mathbb{Q}_p} \to \mathbb{F}_{p^2}^\times \xrightarrow{\eta} E^\times$. Then $\sigma(\tau) = \Theta(\eta)$ and $\sigma_{D,\tau}|_{\mathcal{O}_D^\times} = \eta \oplus \eta^p$.

Proof of Theorem 3.19. Let $\chi \in W_B(\overline{r})$ and $\eta := [\chi]$. By Lemma 3.16, we have
$$\mathrm{Hom}_{\mathcal{O}_D^\times}(\eta, S_\psi(U^p, \mathcal{O})_{\mathfrak{m}}) \neq 0.$$
This implies that there is a cuspidal automorphic form on $B^\times$ that, locally at p, has type $\eta \oplus \eta^p$. In particular, by the Jacquet–Langlands correspondence, it corresponds to a cuspidal automorphic form on $\mathrm{GL}_2(\mathbb{Q})$; moreover, by the local-global compatibility (the version of Saito, c.f. Theorem 1.2) the associated Galois representation r is potentially crystalline of type $\sigma(\tau)$ with Hodge–Tate weights $(0, 1)$. $\square$

This motivates the following definition.

Definition 3.26. Let $W_D^?(\overline{\rho})$ denote the set of χ such that $\overline{\rho}$ has a potentially crystalline lift of type $[\chi] \oplus [\chi^p]$ with Hodge–Tate weights $(0, 1)$ (c.f. Theorem 3.19(1)).

It follows from Theorem 3.19 that $W_B(\overline{r}) \subset W_D^?(\overline{\rho})$. Indeed, we have the following. (See [Gee11] for the proof.)

Theorem 3.27. *We have $W_B(\overline{r}) = W_D^?(\overline{\rho})$.*

3.5. **Explicit form of $W_D^?(\overline{r})$.** Fix an embedding $\xi : \mathbb{F}_{p^2} \to \mathbb{F}^\times$; let $\alpha = \xi^{p-1}$ so that $\alpha^p = \alpha^{-1}$.

Theorem 3.28. *Let ζ denote the character ξ^{p+1}, and let α denote the character ξ^{p-1}.*

(1) *Assume $\overline{\rho}$ is irreducible and*

$$\overline{\rho} \sim (\mathrm{Ind}\, \omega_2^{r+1}) \otimes (\textit{some twist}).$$

(1a) *If $r \neq 0$ or $p-1$, then $\chi \in W_D^?(\overline{\rho})$ if and only if $\chi \in \{\xi^r, \xi^{pr}, \xi^r\alpha^{-1}, \xi^{pr}\alpha\}$.*
(1b) *If $r = 0$ or $p-1$, then $\chi \in W_D^?(\overline{\rho})$ if and only if $\chi \in \{\alpha^{-1}, \alpha\}$.*

(2) *Assume $\overline{\rho}$ is reducible and*

$$\overline{\rho}|_{I_{\mathbb{Q}_p}} \sim \begin{pmatrix} \omega^{r+1} & * \\ 0 & 1 \end{pmatrix}.$$

Then $\chi \in W_D(\overline{\rho})$ if and only if $\chi \in \{\xi^r\alpha^{-1}, \xi^{pr}\alpha\}$.

Proof. Equivalently, we need to show that there exists a lift of $\overline{\rho}$ of type $\Theta([\chi])$ and with Hodge–Tate weights $(0,1)$. See [GS11] for an explicit construction of such deformations.

There is a way to reduce the statement to GL_2-case. Using an analog of Lemma 3.16 in GL_2-case, we have the following equivalent statements: $\sigma \in W(\overline{\rho})$ and $\dim \sigma \leqslant p-2$, if and only if $\overline{\rho}$ has a potentially crystalline lift with Hodge–Tate weights $(0,1)$ and tame cuspidal type τ; moreover, we have $\sigma \in \mathrm{JH}(\overline{\sigma(\tau)})$ if and only if $\overline{\eta} \in W_D^?(\overline{\rho})$. We shall need the following fact:

- $\diamond$ Let $\eta : \mathbb{F}_{p^2}^\times \to E^\times$ with $\eta \neq \eta^p$. Write $\eta = [\xi]^{a+1+(p+1)b}$ with $0 \leqslant a \leqslant p-1$ and $0 \leqslant b \leqslant p-2$. Then
$$\overline{\Theta(\eta)}^{\mathrm{ss}} \cong (\mathrm{Sym}^{a-1}\mathbb{F}^2 \otimes \det^{b+1}) \oplus (\mathrm{Sym}^{p-2-a}\mathbb{F}^2 \otimes \det^{a+1+b}),$$
with the convention that $\mathrm{Sym}^{-1}\mathbb{F}^2 = 0$.

For example, in (1a) above, take $\sigma_1 := \mathrm{Sym}^r \mathbb{F}^2 \in W(\overline{\rho})$ and $a = r+1$, $b = p-2$, so that $\sigma_1 \in \mathrm{JH}(\overline{\Theta(\eta)})$, then we obtain

$$\xi^{r+1+(p+1)(p-2)} = \xi^r\alpha^{-1} \in W_D^?(\overline{\rho}).$$

We then deduce $\xi^{pr}\alpha \in W_D^?(\overline{\rho})$ by Lemma 3.18. Similarly, take $\mathrm{Sym}^{p-1-r}\mathbb{F}^2 \otimes \det^r \in W(\overline{\rho})$, we will get the other pair $\xi^r, \xi^{pr} \in W_D^?(\overline{\rho})$. ∎

4. Deformation theory

4.1. **Classical theory.** As before, let E be a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}$ its ring of integers. Let $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation such that $\mathrm{End}_{G_{\mathbb{Q}_p}}(\overline{\rho}) = \mathbb{F}$. Consider $\mathcal{C}$ the category of (commutative) local Artinian $\mathcal{O}$-algebras with residue field $\mathbb{F}$, in which the morphisms are local $\mathcal{O}$-algebra homomorphisms. Consider the functor

$$\mathrm{Def}_{\overline{\rho}} : \mathcal{C} \longrightarrow \mathsf{Sets},$$

sending $(A, \mathfrak{m}_A)$ to the set of deformations of $\overline{\rho}$ over A modulo the strict equivalence.[5] Fix $\psi : \mathbb{Q}_p^\times \to \mathcal{O}^\times$ which lifts to $\det\overline{\rho}$. Let $\mathrm{Def}_{\overline{\rho}}^\psi$ be the sub-functor by requiring $\det\overline{\rho}_A = \psi$.

Theorem 4.1 (Mazur)**.** (1) *$\mathrm{Def}_{\overline{\rho}}$ is (pro-)represented by $(R_{\overline{\rho}}, \rho^{\mathrm{un}})$, which is universal in the sense that, for any deformation ρ_A, there exists a unique local homomorphism $\phi : R_{\overline{\rho}} \to A$ such that $\phi \circ \rho^{\mathrm{un}} = \rho_A$. Similarly, $\mathrm{Def}_{\overline{\rho}}^\psi$ is (pro-)represented by $R_{\overline{\rho}}^\psi$.*

(2) *$R_{\overline{\rho}}$ (resp. $R_{\overline{\rho}}^\psi$) is a local complete noetherian flat $\mathcal{O}$-algebra of relative dimension 5 (resp. 3) over $\mathcal{O}$, with residue field $\mathbb{F}$.*

[5] Recall that, given a continuous representation $\rho_A : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(A)$ such that $\rho_A \pmod{\mathfrak{m}_A} \simeq \overline{\rho}$, we have defined $\overline{\rho}_A \sim \rho'_A$ if and only if $\rho_A = M\rho'_A M^{-1}$ for some matrix $M \equiv \mathbb{1} \pmod{\mathfrak{m}_A}$.

Mazur showed that $R_{\overline{\rho}}$ is isomorphic to a quotient of an $\mathcal{O}$-power series ring with $\dim_{\mathbb{F}} H^1(G_{\mathbb{Q}_p}, \mathrm{ad}(\overline{\rho}))$ generators through $\dim_{\mathbb{F}} H^2(G_{\mathbb{Q}_p}, \mathrm{ad}(\overline{\rho}))$ relations. Here, we write $\mathrm{ad}(\overline{\rho})$ for the representation $\mathrm{End}_{\mathbb{F}}(\overline{\rho})$ on which $G_{\mathbb{Q}_p}$ acts by conjugation.

When ρ is irreducible, or $\rho \sim \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ with $\chi_1^{-1}\chi_2 \neq \omega$, we have, by local Euler characteristic formula and Tate duality, that

$$\dim_{\mathbb{F}} H^1(G_{\mathbb{Q}_p}, \mathrm{ad}(\overline{\rho})) = 5, \quad \dim_{\mathbb{F}} H^2(G_{\mathbb{Q}_p}, \mathrm{ad}(\overline{\rho})) = 0,$$

so that

$$R_{\overline{\rho}} \cong \mathcal{O}[\![X_1, X_2, X_3, X_4, X_5]\!].$$

Similarly, one has

$$R_{\overline{\rho}}^{\psi} \cong \mathcal{O}[\![X_1, X_2, X_3]\!].$$

If $x : R_{\overline{\rho}}^{\mathrm{un}}[1/p] \to \overline{\mathbb{Q}}_p$ is a closed point, with E' a finite extension of E, we get a genuine p-adic representation

$$\rho_x^{\mathrm{un}} : G_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

via the specialization. Fix a p-adic Hodge type $(\mathbf{w}, \tau, \psi)$, where

- $\mathbf{w} = (a, b)$ is a pair of integers with $a < b$,
- $\tau : I_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ is an inertial type, and
- $\psi : G_{\mathbb{Q}_p} \to \mathcal{O}^{\times}$ is a continuous character such that $\psi \equiv \det\overline{\rho} \pmod{\varpi}$.

Definition 4.2. We say ρ_x^{un} is of type $(\mathbf{w}, \tau, \psi)$ if

(a) it is potentially semi-stable of Hodge–Tate weights $\mathbf{w}$;
(b) $\mathrm{WD}(\rho_x^{\mathrm{un}})|_{I_{\mathbb{Q}_p}} \cong \tau$, where $\mathrm{WD}(\rho_x^{\mathrm{un}})$ denotes the Weil–Deligne representation associated to ρ_x^{un} by Fontaine;
(c) its determinant $\det\rho_x^{\mathrm{un}} = \psi$.

Theorem 4.3. *There exists a unique reduced $\mathcal{O}$-flat quotient of $R_{\overline{\rho}}^{\psi}$, denoted by $R_{\overline{\rho}}^{\psi}(k, \tau)$ (resp. $R_{\overline{\rho}}^{\psi,\mathrm{cris}}(k, \tau)$), parametrizing all potentially semi-stable (resp. potentially crystalline) deformations of type (k, τ, ψ) of $\overline{\rho}$.*

Whenever nonzero, $R_{\overline{\rho}}^{\psi}(k, \tau)$ (resp. $R_{\overline{\rho}}^{\psi,\mathrm{cris}}(k, \tau)$) is a local complete reduced flat $\mathcal{O}$-algebra, equidimensional of relative $\mathcal{O}$-dimension 1.

In the following, we introduce the Breuil–Mézard conjecture. For a complete noetherian local ring A, let $e(A)$ denotes the Hilbert–Samuel multiplicity. Precisely, if $\dim A = d$, then $e(A)$ is equal to $d!$ times the leading coefficient of the Hilbert polynomial of A. Given the pair $(\mathbf{w}, \tau)$, we define

$$\sigma(\mathbf{w}, \tau) := \mathrm{Sym}^{b-a-1} E^2 \otimes \det{}^a \otimes \sigma(\tau),$$
$$\sigma^{\mathrm{cris}}(\mathbf{w}, \tau) := \mathrm{Sym}^{b-a-1} E^2 \otimes \det{}^a \otimes \sigma^{\mathrm{cris}}(\tau).$$

Conjecture 4.4 (Breuil–Mézard)**.** *There exists integers $\mu(\sigma)$ for all irreducible $\mathbb{F}$-representations σ of K, such that for any type $(\mathbf{w}, \tau)$,*

$$e(R_{\overline{\rho}}^{\psi}(\mathbf{w}, \tau)/\varpi) = \sum_{\sigma} \mu(\sigma)[\overline{\sigma(\mathbf{w}, \tau)} : \sigma].$$

Here $[\overline{V} : \sigma]$ denotes the multiplicity of σ in $\overline{V}$. Also, the similar statement holds for $e(R_{\overline{\rho}}^{\psi,\mathrm{cris}}(\mathbf{w}, \tau)/\varpi)$.

We actually have $\mu(\sigma) = 0$ if and only if $\sigma \in W(\overline{\rho})$. This gives a simple way to predict the size of the deformation ring of a given type and Hodge–Tate weights.

Example 4.5. Let $\overline{\rho} = \mathrm{Ind}\,\omega_2^{r+1}$, so that

$$W(\overline{\rho}) = \{\sigma_1, \sigma_2\} = \{\mathrm{Sym}^r \mathbb{F}^2, \mathrm{Sym}^{p-1-r} \mathbb{F}^2 \otimes \det{}^r\}.$$

(1) If $\tau = \mathbb{1} \oplus [\eta_\tau]$ (recall that $\eta_r : \mathbb{F}_p^{\times} \to \mathbb{F}^{\times}$ via $a \mapsto a^r$), then

$$\sigma(\tau) = \mathrm{Ind}_I^K(\mathbb{1} \otimes \eta_r), \quad \mathrm{JH}(\overline{\sigma(\tau)}) = \{\sigma_1, \sigma_2\}.$$

So, assuming Breuil–Mézard conjecture 4.4, we have

$$R_{\overline{\rho}}^{\psi,\mathrm{cris}}((0,1), \tau) \neq 0, \quad e(R_{\overline{\rho}}^{\psi,\mathrm{cris}}((0,1), \tau)/\varpi) = 2.$$

In fact, an explicit computation shows that

$$R_{\overline{\rho}}^{\psi,\mathrm{cris}}((0,1),\tau) \simeq \mathcal{O}[\![X,Y]\!]/(XY-p).$$

(2) If $\tau = [\eta] \oplus [\eta_\tau]$ for some $\eta : \mathbb{F}_{p^2}^\times \to \mathbb{F}^\times$ which does not factor through $\mathbb{F}_p^\times$, then $\sigma(\tau)$ is a cuspidal representation of $\mathrm{GL}_2(\mathbb{F}_p)$, and at most one of σ_1 and σ_2 lies in $\mathrm{JH}(\overline{\sigma(\tau)})$. If this is the case, then

$$R_{\overline{\rho}}^{\psi,\mathrm{cris}}((0,1),\tau) \simeq \mathcal{O}[\![X]\!].$$

4.2. **Deformations of $\mathrm{GL}_2(\mathbb{Q}_p)$-representations.** Colmez has defined an exact covariant functor $\mathbb{V} : \mathsf{Rep}_{\mathbb{F}}(G) \to \mathsf{Rep}_{\mathbb{F}}(G_{\mathbb{Q}_p})$ (see Theorem 3.4). It can be extended to unitary Banach space representations (i.e., a Banach space Π equipped with a continuous action of G, such that $\|gv\| = \|v\|$ for all $g \in G$ and $v \in \Pi$) of G as follows. Fix a Banach space $(\Pi, \|\cdot\|)$. Let $\Pi^0 := \{v \in \Pi : \|v\| \leqslant 1\}$ be the unit ball and assume that $\Pi^0/\varpi\Pi^0$ has finite length. Define

$$\mathbb{V}(\Pi^0) := \varinjlim_n \mathbb{V}(\Pi^0/\varpi^n\Pi^0), \quad \mathbb{V}(\Pi) := \mathbb{V}(\Pi^0)[1/p].$$

Kisin observed that $\mathbb{V}$ should be viewed as a transformation between deformation functors (assuming $\mathrm{End}_{G_{\mathbb{Q}_p}}(\overline{\rho}) = \mathbb{F}$ and so $\mathrm{End}_G(\kappa(\overline{\rho})) = \mathbb{F}$):

$$\mathbb{V} : \mathrm{Def}_{\kappa(\overline{\rho})} \longrightarrow \mathrm{Def}_{\overline{\rho}},$$

hence a map between the deformation space $\mathrm{Spec}\, R_{\kappa(\overline{\rho})}^{\psi\chi_{\mathrm{cycl}}} \longrightarrow \mathrm{Spec}\, R_{\overline{\rho}}^{\psi}$. Later on, Colmez checked that the natural map tangent spaces

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p),\psi\chi_{\mathrm{cycl}}}(\kappa(\overline{\rho}),\kappa(\overline{\rho})) \longrightarrow \mathrm{Ext}^1_{G_{\mathbb{Q}_p},\psi}(\overline{\rho},\overline{\rho})$$

is injective, so $\mathbb{V}$ induces a closed immersion via $\mathrm{Def}_{\kappa(\overline{\rho})} \to \mathrm{Def}_{\overline{\rho}}$.

Remark 4.6. It is in fact nontrivial to see that if

$$0 \longrightarrow \kappa(\overline{\rho}) \longrightarrow \mathcal{E} \longrightarrow \kappa(\overline{\rho}) \longrightarrow 0$$

is an extension with central character $\psi\chi_{\mathrm{cycl}}$, then $\mathbb{V}(\mathcal{E})$ automatically has determinant ψ. But we ignore this issue here.

Colmez had a reverse construction of $\mathbb{V} : \rho \mapsto \Pi(\rho)$, and he showed that $\mathbb{V}(\Pi(\rho)) \cong \rho$ for "trianguline" points. Since such points form a dense subset of $\mathrm{Spec}\, R_{\overline{\rho}}^{\psi}[1/p]$, the transformation $\mathbb{V}$ is actually an isomorphism.

As a consequence, $\mathbb{V}$ sends the universal deformation of $\kappa(\overline{\rho})$ to the universal deformation of $\overline{\rho}$. In practice, we shall work on the dual side. Precisely,

Definition 4.7. Assume $\overline{\rho}$ is generic and $\mathrm{End}_{G_{\mathbb{Q}_p}}(\overline{\rho}) \cong \mathbb{F}$. Define N to be the universal deformation of $\kappa(\overline{\rho})^\vee$.

By definition, N simultaneously carries actions of G and $R_{\overline{\rho}}^{\psi}$, commuting with each other. We list some important properties of N.

Theorem 4.8 (Colmez, Paškūnas)**.**

($N1$) *N is a flat $R_{\overline{\rho}}^{\psi}$-module, and*

$$\mathbb{F} \otimes_{R_{\overline{\rho}}^{\psi}} N \cong \kappa(\overline{\rho})^\vee.$$

($N2$) $\mathrm{End}_G(N) \cong R_{\overline{\rho}}^{\psi}$ *and $\mathbb{V}(N)$ is isomorphic to ρ^{un} as $R_{\overline{\rho}}^{\psi}[\![G_{\mathbb{Q}_p}]\!]$-module*[6]*, and for any closed $\mathbb{Q}_p$-point $x : R_{\overline{\rho}}^{\psi}[1/p] \to \overline{\mathbb{Q}}_p$,*

$$\mathbb{V}(N \otimes_{R_{\overline{\rho}}^{\psi},x} \overline{\mathbb{Q}}_p) \cong \rho_x^{\mathrm{un}}.$$

($N3$) *N is projective in the category $\mathfrak{C}(\mathcal{O})$ defined below.*

($N4$) *There exists $x \in R_{\overline{\rho}}^{\psi}$ such that N/xN is isomorphic to a projective envelope of $\bigoplus_{\sigma\in W(\overline{\rho})} \sigma^\vee$ as a K-representation.*

The properties ($N1$) and ($N2$) follow from the construction. Below we explain how to prove ($N3$) and ($N4$). We first explain the category $\mathfrak{C}(\mathcal{O})$.

[6] More rigorously, the functor $\mathbb{V}$ here should be dualized.

Definition 4.9. A smooth $\mathbb{F}$-representation π of G is called *locally admissible* if $\langle G.v\rangle$ is admissible for any $v \in \pi$.

Remark 4.10.
(1) The compact induction $\text{c-Ind}_{KZ}^{G}\sigma$ is *not* locally admissible.
(2) For each $n \geqslant 1$, $\text{c-Ind}_{KZ}^{G}\sigma/T^n$ is admissible (because $\text{c-Ind}_{KZ}^{G}\sigma/T$ is admissible by Breuil's theorem 2.12), so

$$\varinjlim_{n\geqslant 1} \frac{\text{c-Ind}_{KZ}^{G}\sigma}{T^n}$$

is locally admissible.

Let $\mathfrak{C}(\mathbb{F})$ be the *dual* category of locally admissible $\mathbb{F}$-representations of G (with central character $\psi\chi_{\text{cycl}}$); let $\mathfrak{C}(\mathcal{O})$ be the *dual* category of locally admissible $\mathcal{O}$-torsion representations of G (with central character $\psi\chi_{\text{cycl}}$).

Lemma 4.11 (Emerton)**.** *Injective objects exist in the category of locally admissible $\mathbb{F}$-representations of G or that of $\mathcal{O}$-torsion representations of G. Dually, projective objects exist in $\mathfrak{C}(\mathbb{F})$ or $\mathfrak{C}(\mathcal{O})$.*

4.2.1. *Proof of* 4.8($N3$). We only indicate the proof when $\overline{\rho}$ is irreducible, so $\pi := \kappa(\overline{\rho})$ is irreducible and supersingular.

Definition 4.12. Let $\tilde{P}$ be a projective envelope of $\kappa(\overline{\rho})^\vee$ in $\mathfrak{C}(\mathcal{O})$. Let

$$\tilde{E} := \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}).$$

It is a general fact that $\tilde{E}$ is a local ring with the maximal ideal

$$\tilde{\mathfrak{m}} := \{\phi \in \tilde{E} : \mathrm{pr}\circ\phi = 0\},$$

where $\mathrm{pr} : P \to \kappa(\overline{\rho})^\vee$.

Remark 4.13. As can be seen from Remark 4.10,

$$\varinjlim_{n\geqslant 1} \frac{\text{c-Ind}_{KZ}^{G}\sigma_1}{T^n} \hookrightarrow \tilde{P}^\vee,$$

so $\tilde{P}^\vee$ is not admissible; equivalently, $\tilde{P}$ is not finitely generated as $\mathcal{O}[\![K]\!]$-module. Actually, this is the only obstruction for $\tilde{P}$ to be admissible (c.f. ($N4$)).

Proposition 4.14. *$\tilde{P}$ is a deformation of $\pi^\vee$ to $\tilde{E}$, i.e., $\tilde{P}$ is a flat $\tilde{E}$-module and $\mathbb{F}\otimes_{\tilde{E}}\tilde{P} \cong \pi^\vee$.*

Proof Upshot. The key is to prove that the block containing π is just $\{\pi\}$. This means that if π' is another irreducible smooth representation of G, and if

$$\mathrm{Ext}^1_G(\pi,\pi') \neq 0 \text{ or } \mathrm{Ext}^1_G(\pi',\pi) \neq 0,$$

then $\pi' \cong \pi$. As a consequence, any irreducible subquotient of $\tilde{P}$ is isomorphic to $\pi^\vee$; this is a necessary condition for $\tilde{P}$ to be flat over $\tilde{E}$. The rest of the proof is more or less formal. □

Proposition 4.15. *$\tilde{P}$ is isomorphic to the universal deformation of $\pi^\vee$ to the category of not-necessarily-commutative artinian $\mathcal{O}$-algebras.*

Proof. Since $\tilde{P}$ is projective, there exists $h : \tilde{P} \to M$ making the diagram

$$\begin{tikzcd} \tilde{P} \arrow[r, two heads] \arrow[d, "h"'] & \pi^\vee \arrow[d, equal] \\ M \arrow[r, two heads, "\alpha"] & \mathbb{F}\otimes_A M \end{tikzcd}$$

commute. We claim that the map $A \mapsto \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, M)$ sending a to $a\circ h$ is an isomorphism. A dévissage, using the fact that $\tilde{P}$ is projective and M is flat over A, reduces to the case $\ell_{\mathcal{O}}(A) = 1$ and the result is clear. □

Let (M,α) be a deformation of S to a local artinian $\mathcal{O}$-algebra $(A,\mathfrak{m}_A)$, which is not necessarily commutative. Then Proposition 4.15 means that there exists $\phi:\tilde{E}\to A$ such that

$$M\cong A\otimes_{\varphi,\tilde{E}}\tilde{P}.$$

To show $(N3)$, it is equivalent to show $N=P$. Comparing the definition of N and $\tilde{P}$, we need to show $\tilde{E}$ is a *commutative* ring. In general, a ring of endomorphisms is not commutative; so this turns out to be a serious issue. However, Paškūnasproved that it was indeed a commutative ring, using the following criterion.

By the above discussion, $\tilde{E}$ is a complete noetherian local $\mathcal{O}$-algebra, and can be generated by 3 elements over $\mathcal{O}$, with

$$\dim_{\mathbb{F}}\tilde{\mathfrak{m}}/(\varpi,\tilde{\mathfrak{m}}^2)=\dim_{\mathbb{F}}\mathrm{Ext}^1_{G,\psi\chi_{\mathrm{cycl}}}(\pi^\vee,\pi^\vee)=3.$$

Moreover, there exists a surjection $\tilde{E}\twoheadrightarrow R^{\psi}_{\overline{\rho}}\cong\mathcal{O}[\![x_1,x_2,x_3]\!]$; to show it is an isomorphism, by Nakayama's lemma, it suffices to show it induces an isomorphism after modulo ϖ:

$$E:=\tilde{E}/\varpi\tilde{E}\xrightarrow{\sim}\mathbb{F}[\![x_1,x_2,x_3]\!].$$

Theorem 4.16 (Criterion for commutativity)**.** *Assume that for every exact sequence*

$$0\longrightarrow\pi^\vee\longrightarrow\mathcal{E}\longrightarrow\pi^\vee\longrightarrow 0$$

such that $\dim_{\mathbb{F}}\mathrm{Hom}_G(\mathcal{E},\pi^\vee)=1$ *and* $\dim_{\mathbb{F}}\mathrm{Ext}^1(\mathcal{E},\pi^\vee)\leqslant 3$*. Then* $\mathcal{E}$ *is commutative and isomorphic to* $\mathbb{F}[\![x_1,x_2,x_3]\!]$*.*

Proof. Note that the condition in Theorem 4.16 is equivalent to that: the image of $\mathrm{Ext}^1(\mathcal{E},\pi^\vee)\to\mathrm{Ext}^1(\pi^\vee,\pi^\vee)$ is one-dimensional over $\mathbb{F}$.

It suffices to show that the graded ring $\mathrm{gr}_{\mathfrak{m}}(E)$ is commutative. In fact, using the equality $\dim_{\mathbb{F}}\tilde{\mathfrak{m}}/(\varpi,\tilde{\mathfrak{m}}^2)=\dim_{\mathbb{F}}\mathrm{Ext}^1_{G,\psi\chi_{\mathrm{cycl}}}(\pi^\vee,\pi^\vee)=3$ above, this will imply that $\mathrm{gr}_{\mathfrak{m}}(E)$ is a quotient of a polynomial ring with 3 variables, thus the induced map $\mathrm{gr}_{\mathfrak{m}}(E)\to\mathrm{gr}(\mathbb{F}[\![x_1,x_2,x_3]\!])$ must be an isomorphism. Consequently,

$$\mathfrak{m}^i/\mathfrak{m}^{i+1}\cong(x_1,x_2,x_3)^i/(x_1,x_2,x_3)^{i+1},\quad\forall i\geqslant 0.$$

Taking limits, we obtain $E\cong\mathbb{F}[\![x_1,x_2,x_3]\!]$.

Therefore, it further suffices to prove that

$$E/\mathfrak{m}^3\longrightarrow\mathbb{F}[\![x_1,x_2,x_3]\!]/(x_1,x_2,x_3)^3$$

is an isomorphism. In fact, this will imply that $E/\mathfrak{m}^3$ is commutative, so the commutator of any two elements in $\mathrm{gr}^1_{\mathfrak{m}}(E)$ is zero, so $\mathrm{gr}_{\mathfrak{m}}(E)$ is commutative, and we can conclude by the argument above.

So we may assume that $\mathfrak{m}^3=0$ in E. Using the natural isomorphism

$$\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P},M)\hat{\otimes}_{\tilde{E}}\tilde{P}\xrightarrow{\sim}M,$$

the equivalent assumption essentially implies the following: for any $t\in\mathfrak{m}\backslash\mathfrak{m}^2$, there exists a unique quotient ring of E which has dimension 3 over $\mathbb{F}$, such that $\mathfrak{m}_A/\mathfrak{m}_A^2\cong\mathbb{F}t$; precisely, $A=\mathbb{F}\oplus\mathbb{F}t\oplus\mathbb{F}t^2$. One then shows using this description that $E\cong\mathbb{F}[\![x_1,x_2,x_3]\!]/(x_1,x_2,x_3)^2$. □

Remark 4.17. Indeed, Paškūnashas verified the condition in Theorem 4.16 by complicated computation on Ext^1 groups.

4.2.2. *Proof of* 4.8$(N4)$. To proceed with, we need a result of Breuil and Paškūnas [BP12].

Theorem 4.18. *There exists* $\Omega\in\mathsf{Rep}^{\mathrm{sm}}_{\mathbb{F}}(G)$ *such that*

$$\pi\hookrightarrow\Omega,\quad\Omega|_K\cong\mathrm{Inj}_K(\sigma_1\oplus\sigma_2).$$

Lemma 4.19. *We have* $\dim_{\mathbb{F}}\mathrm{Ext}^1_G(\pi,\Omega)=1$ *and* $\mathrm{Ext}^i_G(\pi,\Omega)=0$ *for* $i\geqslant 2$*.*

Proof. Write $I(\sigma):=\text{c-Ind}^G_{KZ}\sigma$ for simplicity. Recall that

$$\mathrm{soc}_K\kappa(\overline{\rho})=\sigma_1\oplus\sigma_2,\quad\kappa(\overline{\rho})\cong I(\sigma_1)/T$$

by Breuil's theorem 2.12. Using the exact sequence

$$0\longrightarrow I(\sigma_1)\xrightarrow{T}I(\sigma_1)\longrightarrow\pi\longrightarrow 0,$$

we obtain a long exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_G(\pi,\Omega) \arrow[r] & \mathrm{Hom}_G(\sigma_1,\Omega) \arrow[r] & \mathrm{Hom}_G(\sigma_1,\Omega) \arrow[dll, out=-10, in=170] & \\
& \mathrm{Ext}^1(\pi,\Omega) \arrow[r] & \mathrm{Ext}^1_K(\sigma_1,\Omega) \arrow[r] & \mathrm{Ext}^1_K(\sigma_1,\Omega) \arrow[r] & \cdots
\end{tikzcd}
$$

where we have used the isomorphism

$$\mathrm{Ext}^i_G(I(\sigma_1),\Omega) \cong \mathrm{Ext}^i_K(\sigma_1,\Omega)$$

by Shapiro's lemma. Since $\Omega|_K$ is injective as a K-representation by Theorem 4.18, we have $\mathrm{Ext}^i_K(\sigma_1,\Omega) = 0$ for $i \geqslant 1$. This proves the vanishing assertion that $\mathrm{Ext}^i_G(\pi,\Omega) = 0$ for $i \geqslant 2$. The first assertion $\dim_{\mathbb{F}} \mathrm{Ext}^1_G(\pi,\Omega) = 1$ is also clear because $\dim_{\mathbb{F}} \mathrm{Hom}_G(\pi,\Omega) = 1$ by construction. □

By Lemma 4.19 and the fact that the block containing π is just $\{\pi\}$, we obtain an injective resolution of Ω, read as

$$0 \longrightarrow \Omega \longrightarrow \mathrm{Inj}_G\,\pi \longrightarrow \mathrm{Inj}_G\,\pi \longrightarrow 0.$$

Dually, it gives

$$0 \longrightarrow P \longrightarrow P \longrightarrow \Omega^\vee \longrightarrow 0.$$

The first map $P \to P$ gives rise to an element in $\mathrm{End}_G(P) \cong R^\psi_{\overline{\rho}}/\varpi$, say $\overline{x}$. Pick any lift x in $R^\psi_{\overline{\rho}}$, then $\tilde{P}/x\tilde{P}$ is isomorphic to a projective envelope of $\sigma_1^\vee \oplus \sigma_2^\vee$.

4.3. **Reconstruction of deformation rings.** If σ is a finite $\mathcal{O}$-module (resp. $\mathbb{F}$-module) with a continuous action of K, we define

$$M(\sigma) := N \hat{\otimes}_{\mathcal{O}[\![K]\!]} \sigma.$$

This defines an exact functor $M(-)$ as N is projective (and hence flat). An equivalent definition can be

$$M(\sigma) := \mathrm{Hom}_{\mathcal{O}[\![K]\!]}(N, \sigma^\vee)^\vee.$$

Using 4.8($N0$), this implies that $M(\sigma)$ is a finitely generated $R^\psi_{\overline{\rho}}$-module.

Theorem 4.20 (Paškūnas)**.** *Let $\mathbf{w},\tau$ be as above. Let Θ be any K-stable lattice in $\sigma(\mathbf{w},\tau)$ (resp. $\sigma^{\mathrm{cris}}(\mathbf{w},\tau)$). Then $R^\psi_{\overline{\rho}}/\operatorname{Ann}_{R^\psi_{\overline{\rho}}}(M(\Theta))$ is equal to $R^\psi_{\overline{\rho}}(\mathbf{w},\tau)$ (resp. $R^{\psi,\mathrm{cris}}_{\overline{\rho}}(\mathbf{w},\tau)$).*

Proof. For any $x : R^\psi_{\overline{\rho}} \to \overline{\mathbb{Q}}_p$, we have

$$M(\Theta) \otimes_{R^\psi_{\overline{\rho}}} \overline{\mathbb{Q}}_p \cong \mathrm{Hom}_K(N \otimes_{R^\psi_{\overline{\rho}},x} \overline{\mathbb{Q}}_p, \Theta^\vee)^\vee.$$

If ρ_x denotes the corresponding Galois representation, which is a deformation of $\overline{\rho}$, then $(N \otimes_{R^\psi_{\overline{\rho}}} \overline{\mathbb{Q}}_p)^\vee$ is just the unitary Banach space representation $\Pi_x := \mathrm{LL}(\rho_x)$; for this, see in Theorem 4.8($N2$) that

$$\mathbb{V}(N \otimes_{R^\psi_{\overline{\rho}}} \overline{\mathbb{Q}}_p) \cong \rho_x^{\mathrm{un}}.$$

By Colmez's theorem on locally algebraic vectors of Π_x, $\mathrm{Hom}_K(\Theta,\Pi_x) \neq 0$ if and only if x is a deformation of type $(\mathbf{w},\tau)$ (see Definition 4.2), thus x lies in $\operatorname{Spec} R^\psi_{\overline{\rho}}(\mathbf{w},\tau)$. In other words,

$$\operatorname{Supp} M(\Theta) = \operatorname{Spec} R^\psi_{\overline{\rho}}(\mathbf{w},\tau).$$

Since by definition $R^\psi_{\overline{\rho}}(\mathbf{w},\tau)$ is reduced, we get

$$R^\psi_{\overline{\rho}}/\sqrt{\operatorname{Ann}(M(\Theta))} = R^\psi_{\overline{\rho}}(\mathbf{w},\tau).$$

To really conclude, we need to show that $\operatorname{Ann}(M(\Theta))$ is a radical ideal. This is a hard theorem whose proof uses a result of Dospinescu [Dos15]; we omit the details. □

5. Applications of deformation theory

5.1. **Cyclicity.** Sometimes (for simple types $(\mathbf{w},\tau)$), it is possible to choose $\Theta \subset \sigma(\mathbf{w},\tau)$ such that $M(\Theta)$ is cyclic as an $R_{\overline{\rho}}^{\psi}$-module. When this is the case, we deduce from Theorem 4.20 that

$$M(\Theta) \cong R_{\overline{\rho}}^{\psi}(\mathbf{w},\tau).$$

Note that by Nakayama's lemma, the cyclicity is equivalent to saying that $M(\Theta)\otimes_{R_{\overline{\rho}}^{\psi}}\mathbb{F}$ is 1-dimensional as an $\mathbb{F}$-vector space; also, this is equivalent to

$$\dim_{\mathbb{F}} \mathrm{Hom}_K(\Theta/p\Theta, \kappa(\overline{\rho})) = 1,$$

by the definition $M(\Theta)$ above.

Example 5.1. Assume $\overline{\rho} \sim \mathrm{Ind}\,\omega_2^{r+1}$, so $W(\overline{\rho}) = \{\sigma_1,\sigma_2\}$. Consider the principal series type $\tau = \mathbb{1}\oplus[\eta_r]$, such that $\mathrm{JH}(\sigma(\tau)) = \{\sigma_1,\sigma_2\}$. In this case, Proposition 5.2 below does not apply, but we can find a lattice inside $\sigma(\tau)$, say Θ, such that

$$0 \longrightarrow \sigma_1 \longrightarrow \Theta/p\Theta \longrightarrow \sigma_2 \longrightarrow 0.$$

Indeed, we have

$$\Theta = \mathrm{Ind}_I^K(\mathbb{1}\otimes[\eta_r]), \quad \Theta/p\Theta \cong \mathrm{Ind}_I^K(\mathbb{1}\otimes\eta_r).$$

By Frobenius reciprocity,

$$\mathrm{Hom}_K(\Theta/p\Theta,\kappa(\overline{\rho})) \cong \mathrm{Hom}_I(\mathbb{1}\otimes\eta_r,\kappa(\overline{\rho}))$$

is 1-dimensional (recall Breuil's theorem on $\kappa(\overline{\rho})^{I_1}$). By Proposition 5.2 and Theorem 4.20,

$$M(\Theta) \cong R_{\overline{\rho}}^{\psi,\mathrm{cris}}((0,1),\tau).$$

Similarly, there is another lattice $\Theta' \subset \sigma(\tau)$ such that

$$0 \longrightarrow \sigma_2 \longrightarrow \Theta'/p\Theta' \longrightarrow \sigma_1 \longrightarrow 0$$

and we have $M(\Theta') \cong R_{\overline{\rho}}^{\psi,\mathrm{cris}}((0,1),\tau)$.

Next, we look at another special case.

Proposition 5.2. *Let $\sigma \in W(\overline{\rho})$ be a Serre weight. Then $M(\sigma)$ is a cyclic $R_{\overline{\rho}}^{\psi}$-module and is isomorphic to $\mathbb{F}[\![x]\!]$ where $x \in R_{\overline{\rho}}^{\psi}$ is as in 4.8(N4).*

Proof. Consider the short exact sequence

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0.$$

By $(N4)$, N/xN is projective as representation of K. Hence, by applying $\mathrm{Hom}_K(-,\sigma^\vee)^\vee$, we obtain

$$0 \longrightarrow M(\sigma) \xrightarrow{x} M(\sigma) \longrightarrow \mathrm{Hom}_K(N,\sigma^\vee)^\vee \longrightarrow 0.$$

Again by $(N4)$, the last term is 1-dimensional over $\mathbb{F}$, from which the result follows. □

5.2. **Regularity.** Recall that $\mathcal{O}$ is ramified over $\mathbb{Z}_p$.

Proposition 5.3. *Let $\mathbf{w},\tau$ be as above. Assume that there exist two K-stable lattices Θ,Θ' in $\sigma(\mathbf{w},\tau)$ (resp. $\sigma^{\mathrm{cris}}(\mathbf{w},\tau)$) such that the following conditions hold:*

(i) *$p\Theta \subset \Theta' \subset \Theta$ and $\dim_{\mathbb{F}}\mathrm{Hom}_K(\Theta/p\Theta,\kappa(\overline{\rho})) = \dim_{\mathbb{F}}\mathrm{Hom}_K(\Theta',\kappa(\overline{\rho})) = 1$;*
(ii) *taking into account multiplicities, $\mathrm{JH}(\Theta/\Theta')$ contains exactly one element in $W(\overline{\rho})$.*

Then $R_{\overline{\rho}}^{\psi}(\mathbf{w},\tau)$ (resp. $R_{\overline{\rho}}^{\psi,\mathrm{cris}}(\mathbf{w},\tau)$) is a regular local ring.

Proof. We only treat the case for $R_{\overline{\rho}}^{\psi}(\mathbf{w},\tau)$. The condition (i) implies that $M(\Theta)$ and $M(\Theta')$ are both cyclic modules over $R_{\overline{\rho}}^{\psi}$, hence are isomorphic to $R_{\overline{\rho}}^{\psi}(\mathbf{w},\tau)$ by Theorem 4.20. The short exact sequence

$$0 \longrightarrow \Theta' \longrightarrow \Theta \longrightarrow \Theta/\Theta' \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow M(\Theta') \xrightarrow{f} M(\Theta) \longrightarrow M(\Theta/\Theta') \longrightarrow 0.$$

Since both $M(\Theta)$ and $M(\Theta')$ are isomorphic to $R_{\overline{\rho}}^{\psi}(\mathbf{w},\tau)$, the morphism f is equal to multiplication by some element $y \in R_{\overline{\rho}}^{\psi}(\mathbf{w},\tau)$. On the other hand, by Proposition 5.2, condition (ii) implies that

$M(\Theta/\Theta')$ is isomorphic to $\mathbb{F}[\![x]\!]$. This means that $R_{\overline{\rho}}^{\psi}(\mathbf{w},\tau)/(y)$ is a regular local ring of Krull dimension 1. Since $R_{\overline{\rho}}^{\psi}(\mathbf{w},\tau)$ has Krull dimension 2, it is also regular. □

Example 5.4. Emerton–Gee–Savitt computed explicitly the ring in Example 5.1 and attained the result that

$$R_{\overline{\rho}}^{\psi,\mathrm{cris}}((0,1),\tau)\simeq \mathcal{O}[\![X,Y]\!]/(XY-p),$$

which is a regular ring (as $\mathcal{O}$ is unramified).

One may use Proposition 5.3 to give another proof of regularity of $R_{\overline{\rho}}^{\psi}((0,1),\tau)$ without explicit computation. Indeed, one may check that (with the notion of Example 5.1): if Θ' is chosen to satisfy $\Theta'\subset\Theta$ but $\Theta'\not\subset p\Theta$, then there exists a short exact sequence[7]

$$0\longrightarrow\Theta'\longrightarrow\Theta\longrightarrow\sigma_2\longrightarrow 0.$$

Then we can conclude by Proposition 5.3.

5.3. **Gluing lattices.** Consider the following situation:

- V_1, V_2 are two non-isomorphic irreducible locally algebraic representations of K,
- $\Theta_i\subset V_i$ is an $\mathcal{O}$-lattice for $i=1,2$, and
- for each $\sigma\in W(\overline{\rho})$, there exists K-equivariant surjections $r_i:\Theta_i\twoheadrightarrow\sigma$.

Let L be the fibered product of r_1 and r_2, i.e.

$$0\longrightarrow\Theta\longrightarrow\Theta_1\oplus\Theta_2\xrightarrow{r_1-r_2}\sigma\longrightarrow 0.$$

We also call Θ the *gluing* of Θ_1 and Θ_2 along σ.

Proposition 5.5. *Fix $\sigma\in W(\overline{\rho})$. Assume $M(\Theta_1)$ and $M(\Theta_2)$ are both cyclic $R_{\overline{\rho}}^{\psi}$-modules. Let Θ be the gluing of Θ_1 and Θ_2 along σ. Then $M(\Theta)$ is cyclic if and only if*

$$\mathrm{Ann}(M(\Theta_1))+\mathrm{Ann}(M(\Theta_2))=\mathrm{Ann}(M(\sigma)).$$

Proof. It follows from the following fact of commutative algebra. Let $(R,\mathfrak{m}_R)$ be a commutative noetherian local ring. Let $\mathcal{I}_0,\mathcal{I}_1,\mathcal{I}_2$ be ideals of R such that $\mathcal{I}_1,\mathcal{I}_2\subset\mathcal{I}_0\subset\mathfrak{m}_R$. Consider the natural surjective homomorphism $R/\mathcal{I}_1\oplus R/\mathcal{I}_2\twoheadrightarrow R/\mathcal{I}_0$. Then $\mathrm{Ker}(R/\mathcal{I}_1\oplus R/\mathcal{I}_2\twoheadrightarrow R/\mathcal{I}_0)$ is a cyclic R-module if and only if $\mathcal{I}_1+\mathcal{I}_2=\mathcal{I}_0$. □

Note that $p\in\mathcal{I}_0$, so to deduce the cyclicity of $M(L)$, it requires (at least) $p\in\mathcal{I}_1+\mathcal{I}_2$.

Example 5.6. Assume $\overline{\rho}\sim\mathrm{Ind}\,\omega^{r+1}$ as in previous examples. Take τ_1 to be the principal series type and τ_2 to be the cuspidal type. Since $\sigma_1:=\mathrm{Sym}^r\,\mathbb{F}^2$ occurs in both $\sigma(\tau_1)$ and $\sigma(\tau_2)$, we can find $\Theta_i\subset\sigma(\tau_i)$ such that $\Theta_i\twoheadrightarrow\sigma$ and σ is cosocle. Precisely,

$$0\longrightarrow\mathrm{Sym}^{p-1-r}\,\mathbb{F}^2\otimes\det{}^r\longrightarrow\Theta_1/p\Theta_1\longrightarrow\sigma_1\longrightarrow 0,$$

$$0\longrightarrow\mathrm{Sym}^{p-3-r}\,\mathbb{F}^2\otimes\det{}^{r+1}\longrightarrow\Theta_2/p\Theta_2\longrightarrow\sigma_1\longrightarrow 0.$$

One checks that the gluing lattice $\Theta\subset\sigma(\tau_1)\oplus\sigma(\tau_2)$ has mod p reduction isomorphic to $\mathrm{Proj}_{\mathrm{GL}_2(\mathbb{F}_p)}(\sigma_1)$, i.e.,

$$\Theta/p\Theta\cong(\sigma_1\;-\!\!-\;\mathbb{F}^2\otimes\det{}^r\oplus\mathrm{Sym}^{p-3-r}\,\mathbb{F}^2\otimes\det{}^{r+1}\;-\!\!-\;\sigma_1).$$

By explicit description of $\kappa(\overline{\rho})^{K_1}$ (which is multiplicity free), we see that

$$\mathrm{Hom}_K(\Theta/p\Theta,\kappa(\overline{\rho}))$$

is 1-dimensional, so $M(\Theta)$ is a cyclic $R_{\overline{\rho}}^{\psi}$-module by Proposition 5.2, and by Proposition 5.5,

$$\mathcal{I}_1+\mathcal{I}_2=\mathcal{I}_0:=\mathrm{Ann}_{R_{\overline{\rho}}^{\psi}}(M(\sigma_1)).$$

Note that $M(\Theta_2/p\Theta_2)\cong M(\sigma_1)\cong\mathbb{F}[\![x]\!]$ follows from the second short exact sequence above. Thus,

$$\mathcal{I}_2+(p)=\mathrm{Ann}_{R_{\overline{\rho}}^{\psi}}(M(\Theta_2))+(p)=\mathrm{Ann}_{R_{\overline{\rho}}^{\psi}}(M(\Theta_2/p\Theta_2))=\mathcal{I}_0$$

and so $\mathcal{I}_1+\mathcal{I}_2=(p,\mathcal{I}_2)$.

[7]If $\mathcal{O}$ is not unramified, then we would rather get

$$0\longrightarrow\Theta'\longrightarrow\Theta\longrightarrow(\mathcal{O}/p)\otimes_{\mathbb{F}}\sigma_2\longrightarrow 0.$$

Then $M((\mathcal{O}/p)\otimes_{\mathbb{F}}\sigma_2)\simeq(\mathcal{O}/p)[\![x]\!]$, which is not a regular ring.

6. Gelfand–Kirillov dimension

6.1. **Definition of Gelfand–Kirillov dimension.** Let G be a compact p-adic analytic group, or equivalently a closed subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$ for some $n \geqslant 1$. Define $G_1 := G$ and inductively $G_{i+1} := \overline{G_i^p[G_i, G]}$ for $i \geqslant 1$. Then $\{G_i\}$ forms a decreasing chain, called the *lower p-series* of G.

Definition 6.1. The group G is *uniform* if it satisfies:

- $G/\overline{G^p}$ is abelian for odd p, or $G/\overline{G^4}$ is abelian for $p = 2$,
- G is topologically finitely generated, and
- $[G : G_2] = [G_i : G_{i+1}]$ for all $i \geqslant 1$.

When G is uniform and pro-p, if $[G : G_2] = p^d$ with non-negative integer d, then we say G has dimension d. Also, d is equal to the cardinality of a minimal set of (topological) generators of G.

Example 6.2. Consider subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$.

(1) K_1/Z_1 is a uniform group of dimension 3.
(2) I_1/Z_1 is not uniform. First, we have the Iwahori decomposition
$$I_1 = (I_1 \cap U^-)(I_1 \cap T)(I_1 \cap U).$$
Second, using the following identity
$$\begin{pmatrix} 1 & 0 \\ -p(1+p)^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \begin{pmatrix} 1 & -(1+p)^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+p & 0 \\ 0 & (1+p)^{-1} \end{pmatrix},$$
we deduce that the group $I_1 \cap \mathrm{SL}_2(\mathbb{Q}_p)$ has the following two topological generators:
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}.$$
So I_1/Z_1 is generated by two elements.

Theorem 6.3. *Assume G is uniform and pro-p of dimension d. Let π be an admissible smooth $\mathbb{F}$-representation of G. There exist $0 \leqslant c \leqslant d$ and real numbers $a \geqslant b > 0$ such that*[8]
$$bp^{nc} + O(p^{n(c-1)}) \leqslant \dim_{\mathbb{F}}(\pi^{G_n}) \leqslant ap^{nc} + O(p^{n(c-1)}).$$
The integer c (or $-\infty$ if $\pi = 0$) is called the Gelfand–Kirillov dimension of π.

Remark 6.4. If G' is a p-adic analytic group, say $\mathrm{GL}_2(\mathbb{Q}_p)$ or $D^\times$, and π is an admissible smooth $\mathbb{F}$-representation of G', we define its GK-dimension to be the dimension when restricted to a uniform pro-p subgroup. This clearly does not depend on the choice of the subgroup.

Now we give another definition of $\dim_G(\pi)$ in the theory of Iwasawa algebra. Let Λ be the *Iwasawa algebra* of G over $\mathbb{F}$, i.e.,
$$\Lambda := \mathbb{F}[\![G]\!] = \varprojlim \mathbb{F}[G/N],$$
where the inverse limit is taken over the open normal subgroups N of G. It is always a complete (left and right) noetherian ring (by Lazard), and is local if G is pro-p, an integral domain if G has no (nontrivial) p-torsion elements.

A finitely generated (left) Λ-module is said to be of *grade* (or *codimension*) c if
$$\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0, \quad \forall i < c$$
and is nonzero for $i = c$; the grade of the zero module is defined to be ∞. We denote the grade by $j_\Lambda(M)$. If M is nonzero, then $j_\Lambda(M) \leqslant d$.

Let π be an admissible smooth $\mathbb{F}$-representation of G. Let d be the dimension of G. The dual $\pi^\vee := \mathrm{Hom}_{\mathbb{F}}(\pi, \mathbb{F})$ is a finitely generated Λ-module.

Theorem 6.5. *The Gelfand–Kirillov dimension of π is equal to*
$$d - j_\Lambda(\pi^\vee).$$

[8] In fact, one can prove that $a \geqslant b \geqslant 1/c!$.

6.2. **p-valuation.** Let G be a compact p-adic analytic group.

Definition 6.6. A *p-valuation* ω on G is a real-valued function

$$\omega : G\backslash\{1\} \longrightarrow (0,\infty)$$

which, with the convention $\omega(1)=\infty$, satisfies for any $g,h\in G$ that

(a) $\omega(g) > 1/(p-1)$;
(b) $\omega(g^{-1}h) \geqslant \min(\omega(g),\omega(h))$;
(c) $\omega([g,h]) \geqslant \omega(g)+\omega(h)$;
(d) $\omega(g^p) = \omega(g)+1$.

The p-valuation ω is called *saturated* if any $g\in G$ satisfying $\omega(g) > p/(p-1)$ lies in a pth power of G.

Take $h=1$ in (b) we get $\omega(g^{-1}) \geqslant \omega(g)$, and by symmetry,

$$\omega(g^{-1}) = \omega(g).$$

One also checks using (b) and (c) that

$$\omega(ghg^{-1}) = \omega(h),$$

and if $\omega(g)\neq\omega(h)$, then

$$\omega(gh) = \min(\omega(g),\omega(h)).$$

Whenever G has nontrivial p-torsion elements, there exists no p-valuation on G because $\omega(\cdot)=\infty$ on such elements by (d).

Example 6.7. The followings are examples of p-valuations in practice.

(1) Let L be a finite extension of $\mathbb{Q}_p$, with valuation v normalized such that $v(p)=1$. As for the first principal congruence subgroup $K_1\subset \mathrm{GL}_2(\mathcal{O}_L)$, a p-valuation on K_1 is given by $\omega(g) := v(g-1)$, where for $(m_{ij})\in M_2(L)$ we set

$$v(m_{ij}) := \min\{v_p(m_{ij})\}.$$

(2) Consider the pro-p-Iwahori subgroup

$$I_1 = \begin{pmatrix} 1+p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}.$$

We cannot define ω in the same way as in (1), because its value sends $\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}\in I_1$ to 0. But one can embed I_1 into $\mathrm{GL}_2(\mathbb{Q}_p(\sqrt{p}))$ as follows. Let D be the diagonal matrix

$$D = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p}\end{pmatrix}$$

and define for $x\in I_1$ that

$$\omega(x) := v(D^{-1}xD - I_2).$$

Then ω is a saturated p-valuation on I_1. We may identify I_1/Z_1 with $I_1\cap \mathrm{SL}_2(\mathbb{Z}_p)$. For

$$g = \begin{pmatrix} 1+pa & b \\ pc & 1+pd\end{pmatrix} \in I_1\cap \mathrm{SL}_2(\mathbb{Z}_p),$$

the definition of $\omega(\cdot)$ gives

$$\omega(g) = \min\left\{1+v_p(a), \frac{1}{2}+v_p(b), \frac{1}{2}+v_p(c), 1+v_p(d)\right\}.$$

(3) For $(1+\mathfrak{p}_D)/Z_1$, we set $v_D(a) := v_p(\mathrm{Nrd}_D(a))$ and

$$\omega(g) := \frac{1}{2}v_D(g-1).$$

Again, this is a saturated p-valuation.

For any $\nu > 0$, we put

$$G_\nu := \{g \in G : \omega(g) \geqslant \nu\}, \quad G_{\nu+} := \{g \in G : \omega(g) > \nu\}$$

and form the graded abelian group

$$\mathrm{gr}\, G := \bigoplus_{\nu>0} \mathrm{gr}_\nu G = \bigoplus_{\nu>0} G_\nu/G_{\nu+}.$$

This is an $\mathbb{F}_p$-vector space by 6.6(d). It becomes a Lie algebra via

$$\begin{aligned} \mathrm{gr}_\nu G \times \mathrm{gr}_{\nu'} G &\longrightarrow \mathrm{gr}_{\nu+\nu'} G \\ (\xi, \eta) &\longmapsto [\xi, \eta] = [g, h] G_{(\nu+\nu')+} \end{aligned}$$

where g, h are representatives of ξ, η, respectively, and $[g, h] := ghg^{-1}h^{-1}$. It also carries an action of $\mathbb{F}[\varepsilon]$ (where ε is a formal variable) via

$$\varepsilon : gG_{\nu+} \longmapsto g^p G_{(\nu+1)+}.$$

We have the following facts:

- $\mathrm{gr}\, G$ is torsion-free as an $\mathbb{F}_p[\varepsilon]$-module.
- (G, ω) is saturated if and only if

$$\varepsilon(\mathrm{gr}\, G) = \bigoplus_{\nu > p/(p-1)} \mathrm{gr}_\nu G.$$

As a consequence, when (G, ω) is saturated, $\mathrm{gr}\, G$ is a finite free $\mathbb{F}_p[\varepsilon]$-module with a basis given by an $\mathbb{F}_p$-basis of $\bigoplus_{0<\nu\leqslant p/(p-1)} \mathrm{gr}_\nu G$.

Example 6.8. Below we assume $p \geqslant 5$.

(1) Let $G = K_1/Z_1$ which is isomorphic to $K_1 \cap \mathrm{SL}_2(\mathbb{Z}_p)$. We only need to determine an $\mathbb{F}_p$-basis of $\mathrm{gr}_1 G$, which is given by the images of

$$\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \quad \begin{pmatrix} 1+p & 0 \\ 0 & (1+p)^{-1} \end{pmatrix}.$$

Denote them by e, f, h, respectively. One checks that

$$[e, f] = \varepsilon h, \quad [h, e] = 2\varepsilon e, \quad [h, f] = -2\varepsilon f.$$

(2) Let $G = I_1/Z_1$ which is isomorphic to $I_1 \cap \mathrm{SL}_2(\mathbb{Z}_p)$. We only need to determine an $\mathbb{F}_p$-basis of $\mathrm{gr}_{1/2} G \oplus \mathrm{gr}_1 G$ which is given by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \quad \begin{pmatrix} 1+p & 0 \\ 0 & (1+p)^{-1} \end{pmatrix}.$$

Denote them by e, f, h, respectively. Then one checks that

$$[e, f] = h, \quad [h, e] = 2\varepsilon e, \quad [h, f] = -2\varepsilon f.$$

We will use later the base of $\mathrm{gr}\, G$ to $\mathbb{F}_p$, i.e.,

$$\mathfrak{g}_{\mathbb{F}_p} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\varepsilon], \varepsilon \mapsto 0} \mathrm{gr}\, G = \mathbb{F}_p e \oplus \mathbb{F}_p f \oplus \mathbb{F}_p h,$$

satisfying

$$[e, f] = h, \quad [h, e] = [h, f] = 0.$$

(3) Let $G = (1 + \mathfrak{p}_D)/Z_1$. Again, we only need to determine an $\mathbb{F}_p$-basis of $\mathrm{gr}_{1/2} G \oplus \mathrm{gr}_1 G$ which is given by

$$1 + \varpi_D, \quad 1 + \varpi_D[t], \quad 1 + p(t - t^p)$$

where $t \in \mu_{p^2-1} \backslash \mu_{p-1} \hookrightarrow \mathcal{O}_D^\times$. Moreover, this gives the same Lie algebra as in (2).

6.3. **Iwasawa algebras.** Now we consider the algebra $\mathbb{Z}_p[\![G]\!]$. The p-valuation ω on G induces a filtration on $\mathbb{Z}_p[\![G]\!]$ as follows. For $\nu \geqslant 0$, let J_ν denote the smallest closed $\mathbb{Z}_p$-submodule of $\mathbb{Z}_p[\![G]\!]$ which contains all elements of the form

$$p^\ell (g_1 - 1)\cdots(g_s - 1)$$

with $s \geqslant 0$, $g_i \in G$, and

$$\ell + \omega(g_1) + \cdots + \omega(g_s) \geqslant \nu.$$

Let $J_{\nu+} := \bigcup_{\nu' > \nu} J_{\nu'}$ and

$$\mathrm{gr}_J \Lambda := \bigoplus_{\nu \geqslant 0} J_\nu / J_{\nu+}.$$

The homomorphism of abelian groups

$$\begin{aligned} \mathcal{L}_\nu : \mathrm{gr}_\nu G &\longrightarrow J_\nu / J_{\nu+} \\ gG_{\nu+} &\longmapsto (g-1) + J_{\nu+} \end{aligned}$$

extends to a homomorphism of graded $\mathbb{F}_p[\varepsilon]$-Lie algebra

$$\mathcal{L} : \mathrm{gr}\, G \longrightarrow \mathrm{gr}_J \mathbb{Z}_p[\![G]\!],$$

where the $\mathbb{F}_p[\varepsilon]$-algebra structure on $\mathrm{gr}_J \mathbb{Z}_p[\![G]\!]$ is given through the isomorphism $\mathbb{F}_p[\varepsilon] \xrightarrow{\sim} \mathrm{gr}\, \mathbb{Z}_p$ (sending ε to the principal part of $p + p^2\mathbb{Z}_p$).

Let $U_{\mathbb{F}_p[\varepsilon]}(\mathrm{gr}\, G)$ be the universal enveloping algebra of $\mathrm{gr}\, G$ as an $\mathbb{F}_p[\varepsilon]$-Lie algebra. Then by the universal property we have a homomorphism of associative $\mathbb{F}_p[\varepsilon]$-algebras

$$\tilde{\mathcal{L}} : U_{\mathbb{F}_p[\varepsilon]}(\mathrm{gr}\, G) \longrightarrow \mathrm{gr}_J\, \mathbb{Z}_p[\![G]\!].$$

Theorem 6.9 ([Laz65, Chap III, 2.1.2])**.** *The map $\tilde{\mathcal{L}}$ is an isomorphism.*

The filtration J_ν on $\mathbb{Z}_p[\![G]\!]$ induces a filtration on $\mathbb{F}_p[\![G]\!]$, which we denote by $\overline{J}_\nu$.

Corollary 6.10. *As a consequence, there is an isomorphism*

$$U_{\mathbb{F}_p}(\mathbb{F}_p \otimes_{\mathbb{F}_p[\varepsilon]} \mathrm{gr}\, G) \cong \mathrm{gr}_{\overline{J}}\, \mathbb{F}_p[\![G]\!].$$

Example 6.11.
(1) If G is uniform of dimension d, then $\mathrm{gr}_{\overline{J}}\, \mathbb{F}_p[\![G]\!]$ is a commutative polynomial ring of d variables.
(2) For $G = I_1/Z_1$ (and similarly $G = (1+\mathfrak{p}_D)/Z_1$), we have

$$\mathrm{gr}_{\overline{J}}\, \mathbb{F}_p[\![G]\!] \cong U_{\mathbb{F}_p}(\mathfrak{g}_{\mathbb{F}_p})$$

where $\mathfrak{g}_{\mathbb{F}_p}$ is defined in Example 6.8(2).

6.4. **The $\mathfrak{m}$-adic filtration.** Let $\mathfrak{m}$ denote the maximal ideal of $\mathbb{F}_p[\![G]\!]$. In practice, we need to consider the $\mathfrak{m}$-adic filtration on $\mathbb{F}_p[\![G]\!]$ (as a finitely generated module over $\mathbb{F}_p[\![G]\!]$ that carries naturally a compatible $\mathfrak{m}$-adic filtration).

Lemma 6.12. *Assume $G = I_1/Z_1$ or $(1+\mathfrak{p}_D)/Z_1$. Then, up to rescalar, the $\overline{J}$-filtration on $\mathbb{F}_p[\![G]\!]$ coincides with the $\mathfrak{m}$-adic filtration. Precisely,*

$$\overline{J}_{i/2} = \mathfrak{m}^i, \quad i \geqslant 0.$$

We finally obtain the following result. Note that $\mathbb{F}[\![G]\!] = \mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}_p[\![G]\!]$.

Theorem 6.13. *Assume $G = I_1/Z_1$ or $(1+\mathfrak{p}_D)/Z_1$. Then,*

(1) $\mathrm{gr}_{\mathfrak{m}}\mathbb{F}[\![G]\!]$ *is isomorphic to* $\mathbb{F} \otimes_{\mathbb{F}_p} U(\mathfrak{g}_{\mathbb{F}_p})$, *with* $\deg e = \deg f = 1$ *and* $\deg h = 2$.
(2) h *is a regular sequence of central elements of* $\mathrm{gr}_{\mathfrak{m}}\mathbb{F}[\![G]\!]$, *and* $\mathrm{gr}_{\mathfrak{m}}\mathbb{F}[\![G]\!]/(h)$ *is isomorphic to* $\mathbb{F}[e,f]$, *a commutative polynomial ring in two variables.*

6.5. **H-action.** We only treat the GL_2 case. Let
$$H := \left\{ \begin{pmatrix} [a] & 0 \\ 0 & [d] \end{pmatrix} : a, d \in \mathbb{F}_p^\times \right\},$$
viewed as a subgroup of I. It acts naturally on $\mathbb{F}[\![I_1/Z_1]\!]$ via conjugation, and also on $\mathrm{gr}^i_{\mathfrak{m}}(\mathbb{F}[\![I_1/Z_1]\!])$. One checks that for $1 \leqslant i \leqslant p-1$,
$$\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} - 1 \equiv i \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1 \right) \bmod \mathfrak{m}^2$$
which implies that in $\mathfrak{m}/\mathfrak{m}^2$,
$$\begin{pmatrix} [a] & 0 \\ 0 & [d] \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} [a] & 0 \\ 0 & [d] \end{pmatrix}^{-1} - 1 = (ad^{-1}) \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1 \right).$$
This means that H acts on e via the character α, where $\alpha : H \to \mathbb{F}^\times$ sends $\begin{pmatrix} [a] & 0 \\ 0 & [d] \end{pmatrix}$ to ad^{-1}. Similarly, one checks that H acts on f (resp. h) via α^{-1} (resp. $\mathbb{1}$).

Remark 6.14. In the case of $(1+\mathfrak{p}_D)/Z_1$, the group $[\mathbb{F}_{p^2}^\times]$ acts naturally on $\mathbb{F}[\![(1+\mathfrak{p}_D)/Z_1]\!]$, and also on $\mathrm{gr}_{\mathfrak{m}}\mathbb{F}[\![(1+\mathfrak{p}_D)/Z_1]\!]$. But the elements e, f constructed in Example 6.8(3) are not eigenvectors of $\mathbb{F}_{p^2}^\times$. However, one can construct $X, Y \in \mathbb{F}[\![(1+\mathfrak{p}_D)/Z_1]\!]$ such that for all $t \in \mathbb{F}_{p^2}^\times$,
$$[t] \cdot X = \alpha(t) X, \quad [t] \cdot Y = \alpha^{-1}(t) Y,$$
where $\alpha : \mathbb{F}_{p^2}^\times \to \mathbb{F}^\times$ is the character sending t to t^{p-1}.

6.6. **A control theorem.** Now we are ready to prove the following criterion for controlling the GK-dimension.

Theorem 6.15. *Let π be an admissible smooth representation of $G := \mathcal{O}_D^\times/Z_1$ over $\mathbb{F}$. Assume that for each character χ such that $\mathrm{Hom}_G(\chi, \pi) \neq 0$, we have the equality*
$$[\pi[\mathfrak{m}^3] : \chi] = [\pi[\mathfrak{m}] : \chi]$$
where $\mathfrak{m}$ denotes the maximal ideal of $\mathbb{F}[\![(1+\mathfrak{p}_D)/Z_1]\!]$. Then
$$\mathrm{GK}\dim(\pi) \leqslant 1.$$

Proof. The dual $\pi^\vee$ is a finitely generated $\mathbb{F}[\![G]\!]$-module, so $\mathrm{gr}_{\mathfrak{m}}\pi^\vee$ is finitely generated over $\mathrm{gr}_{\mathfrak{m}}\mathbb{F}[\![G]\!]$. Moreover the graded $\mathrm{gr}_{\mathfrak{m}}\mathbb{F}[\![G]\!]$-module $\mathrm{gr}_{\mathfrak{m}}\pi^\vee$ is generated by its homogeneous elements of degree 0.

Let $\mathfrak{a}_G$ be the left ideal of $\mathrm{gr}_{\mathfrak{m}}\mathbb{F}[\![G]\!]$ generated by xy, yx (of degree 2). We easily see that $\mathfrak{a}_G$ is in fact a two-sided ideal of $\mathrm{gr}_{\mathfrak{m}}\mathbb{F}[\![G]\!]$. The assumption implies that xy, yx act trivially on $\mathrm{gr}_{\mathfrak{m}}\pi^\vee$, hence so does $\mathfrak{a}_G$. Since $\mathrm{gr}_{\mathfrak{m}}\mathbb{F}[\![G]\!]/\mathfrak{a}_G$ is isomorphic to $\mathbb{F}[x,y]/(xy)$, which is a commutative ring of Krull dimension 1, the result follows. $\square$

It is natural to make the following definition.

Definition 6.16. Let $\mathcal{C}$ denote the category of admissible smooth $\mathbb{F}$-representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ or $D^\times$, such that $\mathrm{gr}_{\mathfrak{m}}\pi^\vee$ is annihilated by some power of $\mathfrak{a}_G$.

Any $\pi \in \mathcal{C}$ has Gelfand–Kirillov dimension 1. Our main result says that the representation $\pi_B(\overline{r})$ lies in $\mathcal{C}$ (under some genericity condition). It is clear that $\mathcal{C}$ is an abelian category, and is stable under extensions (using Artin–Rees lemma).

7. Estimating Gelfand–Kirillov dimension in quaternion case

7.1. **An upper bound of GK-dimension.** Recall that we have defined an admissible smooth representation of $D^\times$:
$$\pi_B(\overline{r}) := S_\psi(U^p, \mathbb{F})[\mathfrak{m}]$$
where B is a definite quaternion algebra over $\mathbb{Q}$, $\overline{r} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ an absolutely irreducible continuous representation, assumed to be modular, and $D := B \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

Our aim is to understand the structure of $\pi_B(\overline{r})$. It can be proven that $\pi_B(\overline{r})$ is infinite dimensional (as $\mathbb{F}$-vector space), so it is natural to look at its $\mathcal{O}_D^\times$-socle filtration. We have determined the structure of the socle $\pi_B(\overline{r})[\mathfrak{m}]$ without counting the multiplicity. For example, by Theorem 3.28,

- if $\overline{\rho} := \overline{r}|_{G_{\mathbb{Q}_p}} \sim \mathrm{Ind}(\omega_2^{r+1})$ and $r \neq 0, p-1$, then
$$W_B(\overline{r}) = \{\xi^r, \xi^{pr}, \xi^r\alpha^{-1}, \xi^{pr}\alpha\};$$
- if $\overline{\rho} \sim \begin{pmatrix} \omega^{r+1} & * \\ 0 & 1 \end{pmatrix}$ and is generic, then
$$W_B(\overline{r}) = \{\xi^r\alpha^{-1}, \xi^{pr}\alpha\}.$$

The main result of the lecture is the following.

Theorem 7.1. *Under some genericity condition on $\overline{\rho}$, we have*
$$\mathrm{GK}\dim(\pi_B(\overline{r})) \leqslant 1.$$

For simplicity, let's assume that each Serre weight occurs with multiplicity one in $\mathrm{soc}_{\mathcal{O}_D^\times} \pi_B(\overline{r})$, i.e.,
$$\pi_B(\overline{r})[\mathfrak{m}] = \bigoplus_{\chi \in W_B(\overline{\rho})} \chi.$$
In other words, $[\pi_B(\overline{r})[\mathfrak{m}] : \chi] = 1$ for any $\chi \in W_B(\overline{r})$. Then, as a direct consequence of Theorem 6.15, it is sufficient to show that for any $\chi \in W_B(\overline{r})$,
$$[\pi_B(\overline{r})[\mathfrak{m}^3] : \chi] = 1,$$
or equivalently,
$$\mathrm{Hom}_{\mathcal{O}_D^\times}(V_3 \otimes \chi, \pi_B(\overline{r})) = 1$$
where by definition $V_3 := \mathrm{Proj}_{\mathcal{O}_D^\times/\mathfrak{m}^3} \mathbb{1}$.

7.2. **The patched module.** In the setting of quaternion algebra (or GL_2-case beyond $\mathrm{GL}_2(\mathbb{Q}_p)$), one has a construction of patched modules (due to Caraiani–Emerton–Gee–Geraghty–Paškūnas–Shin) which serves a replacement of Paškūnas' $\tilde{P}$. Roughly, M_∞ is a finitely generated $R_\infty[D^\times]$-module, where
$$R_\infty := R_{\overline{\rho}}^{\psi}[x_1, \ldots, x_g]$$
for some $n \geqslant 1$ (the x_i are called patched variables), satisfying the following conditions.

(a) $M_\infty/\mathfrak{m}_{R_\infty} \cong \pi_B(\overline{r})^\vee$.

(b) For any type $(\mathbf{w}, \tau)$, the action of R_∞ on $M_\infty(\sigma_D(\mathbf{w}, \tau))$ factors through
$$R_\infty(\mathbf{w}, \tau) := R_\infty \otimes_{R_{\overline{\rho}}^{\psi}} R_{\overline{\rho}}^{\psi}(\mathbf{w}, \tau).$$
Here, $M_\infty(-)$ is defined as $\mathrm{Hom}_{\mathcal{O}_D^\times}(M_\infty, (-)^\vee)^\vee$ and $\sigma_D(\mathbf{w}, \tau)$ is the representation of $\mathcal{O}_D^\times$ defined by
$$\mathrm{Sym}^{b-a-1} E^2 \otimes \det{}^a \otimes \sigma_D(\tau).$$

(c) There exists another regular local ring S_∞, and a local map $S_\infty \hookrightarrow R_\infty$ such that $M_\infty/\mathfrak{m}_{S_\infty}$ is projective as $\mathcal{O}_D^\times$-representation. Moreover,
$$\dim S_\infty = \dim R_\infty(\mathbf{w}, \tau). \tag{$\diamondsuit$}$$
Equivalently, this requires $S_\infty \cong \mathcal{O}[\![y_1, \ldots, y_{g+1}]\!]$ in our situation.

Recall that $\mathcal{O}_D^\times$ embeds into $\mathrm{GL}_2(\mathbb{Z}_{p^2})$ and then embeds into $\mathrm{GL}_2(\mathcal{O})$ via the embedding $\mathrm{GL}_2(\mathbb{Z}_{p^2}) \subset \mathrm{GL}_2(\mathcal{O})$. An explicit embedding is given by
$$\varpi_D \longmapsto \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}, \quad a \longmapsto \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix}$$
for all $a \in \mathbb{Q}_{p^2}$. We are particularly interested in the case $(a, b) = (-1, 2)$. Equipped with this $\mathcal{O}_D^\times$-action, $\mathrm{Sym}^2 E^2 \otimes \det^{-1}$ is an irreducible representation of $\mathcal{O}_D^\times$. Note that the action of $\mathcal{O}_D^\times$ stabilizes the lattice $\mathrm{Sym}^2 \mathcal{O}^2 \otimes \det^{-1}$.

Lemma 7.2. *As a representation of $\mathcal{O}_D^\times$,*
$$(\mathrm{Sym}^2 \mathcal{O}^2 \otimes \det{}^{-1})/p \cong \mathrm{Sym}^2 \mathbb{F}^2 \otimes \det{}^{-1},$$
and the semisimplification is given by
$$(\mathrm{Sym}^2 \mathcal{O}^2 \otimes \det{}^{-1})^{\mathrm{ss}} = \alpha \oplus \mathbb{1} \oplus \alpha^{-1}.$$

Proof. It is a direct verification. For each $a \in \mu_{p^2-1}$,

$$[a]X^2 = \begin{pmatrix} [a] & 0 \\ 0 & [a^p] \end{pmatrix} X^2 = a^2(aa^p)^{-1}X^2 = a^{-(p-1)}X^2 = \alpha^{-1}(a)X^2.$$

Similarly, XY has eigencharacter $\mathbb{1}$ and Y^2 has eigencharacter α. $\square$

7.3. **The Gluing argument.** To prove the main result Theorem 7.1, it suffices to verify the isomorphism

$$\mathrm{Hom}_{\mathcal{O}_D^\times}(V_3 \otimes \chi, \pi_B(\overline{r})) = 1.$$

It is further equivalent to show that $M_\infty(V_3 \otimes \chi)$ is a cyclic R_∞-module.

Step I. Fix $\chi \in W_B(\overline{r})$. We do this by gluing the following three types:

- $\mathbf{w}_0 = (0,1)$, $\tau_0 = [\chi] \oplus [\chi^p]$, and take $\sigma_D(\mathbf{w}_0, \tau_0) = [\chi]$;
- $\mathbf{w}_1 = (-1,2)$, $\tau_1 = [\chi\alpha] \oplus [(\chi\alpha)^p]$, and take $\sigma_D(\mathbf{w}_1, \tau_1) = (\mathrm{Sym}^2 E^2 \otimes \det^{-1}) \otimes [\chi]$;
- $\mathbf{w}_2 = (-1,2)$, $\tau_2 = [\chi\alpha^{-1}] \oplus [(\chi\alpha^{-1})^p]$, and take $\sigma_D(\mathbf{w}_2, \tau_2) = (\mathrm{Sym}^2 E^2 \otimes \det^{-1}) \otimes [\chi\alpha^{-1}]$.

The Jordan–Holder factors of reduction mod p of $\bigoplus_{i=0}^2 \sigma_D(\mathbf{w}_i, \tau_i)$ exactly gives $\mathrm{JH}(V_3 \otimes \chi)$.

Step II. Choose an integral lattice $\Theta_i \subset \sigma_D(\mathbf{w}_i, \tau_i)$ such that $\Theta_i/p\Theta_i$ has cosocle χ; this is always possible.

Proposition 7.3. *For $i = 0, 1, 2$, $M_\infty(\Theta_i)$ is a cyclic R_∞-module.*

Proof. This uses that $R_\infty(\mathbf{w}_i, \tau_i)$ is a regular local ring, as $R_{\overline{\rho}}^{\psi}(\mathbf{w}_i, \tau_i)$ is (see Proposition 5.3).

Recall Auslander–Buchsbaum theorem: if A is a noetherian local ring and $M \neq 0$ is a finitely generated A-module such that $\mathrm{proj}\dim M < \infty$, then

$$\mathrm{depth}\, M + \mathrm{proj}\dim M = \mathrm{depth}\, A.$$

Note that, when A is regular, the condition $\mathrm{proj}\dim M < \infty$ is automatic.

Apply it to $M_\infty(\Theta_i)$, which is a Cohen–Macauley module of Krull dimension equal to $\dim S_\infty$ by condition (c) above, thus

$$\mathrm{depth}\, M_\infty(\Theta_i) = \dim S_\infty = \dim R_\infty(\mathbf{w}_i, \tau_i)$$

by ($\diamondsuit$). The formula of Auslander–Buchsbaum for $M_\infty(\Theta_i)$ implies that $M_\infty(\Theta_i)$ is projective, and hence free. $\square$

Step III. We glue the lattices Θ_i for $i = 0, 1, 2$. For example, let

$$0 \longrightarrow \Theta' \longrightarrow \Theta_0 \oplus \Theta_1 \longrightarrow \chi \longrightarrow 0$$

be the gluing of Θ_0 and Θ_1 along the common quotient χ. To show that $M_\infty(\Theta')$ is cyclic, it is equivalent to show

$$\mathrm{Ann}_{R_\infty} M_\infty(\Theta_0) + \mathrm{Ann}_{R_\infty} M(\Theta_1) = \mathrm{Ann}_{R_\infty} M_\infty(\chi).$$

The latter condition can be translated to relations between local deformation rings $R_{\overline{\rho}}^{\psi}(-)$ (i.e. forgetting patching variables), which can be proved via $\mathrm{GL}_2(\mathbb{Q}_p)$-side, because by work of Barthel–Livné, Breuil, Paškūnas, Morra, we have a complete understanding about $\kappa(\overline{\rho})$.

7.4. **A lower bound of GK-dimension.** First recall the miracle flatness from commutative algebra.

Theorem 7.4 (Miracle flatness)**.** *Let R be a noetherian local ring and M be a finitely generated R-module. Assume that*

- *R is regular,*
- *M is Cohen–Macaulay, and*
- *$\dim R + \dim M \otimes_R \mathbb{F} = \dim M$.*

Then M is a flat R-module.

In fact, one always have

$$\dim R + \dim M \otimes_R \mathbb{F} \geqslant \dim M$$

and the equality holds if and only if M is flat.

Gee–Newton generalized this to finitely generated modules over non-commutative Iwasawa algebras. Let G be compact p-adic analytic group, which we assume to be pro-p. Let R be a noetherian local

$\mathcal{O}$-algebra, M be a finitely generated $R[\![G]\!] := R \hat{\otimes}_{\mathcal{O}} \mathcal{O}[\![G]\!]$-module. For simplicity, assume R is a power series ring over $\mathcal{O}$, say

$$R \cong \mathcal{O}[\![x_1, \dots, x_n]\!].$$

By identifying R with $\mathcal{O}[\![\mathbb{Z}_p^n]\!]$, we may identify $R[\![G]\!]$ with $\mathcal{O}[\![G \times \mathbb{Z}_p^d]\!]$. Thus we can define the grade of M, and define its δ-dimension[9] to be

$$\delta_{R[\![G]\!]}(M) := (n + 1 + \dim G) - j_{R[\![G]\!]}(M).$$

The module M is said to be *Cohen–Macaulay* if

$$\mathrm{Ext}^j_{R[\![G]\!]}(M, R[\![G]\!]) \neq 0 \text{ only for } j = j_{R[\![G]\!]}(M).$$

Theorem 7.5 (Gee–Newton, [GN22, Appendix A])**.** *Let R, M be as above. Assume that M is Cohen–Macaulay. Then*

$$\dim R + \delta(M \otimes_R \mathbb{F}) \geqslant \delta(M)$$

and the equality holds if and only if M is flat over R. Moreover, if this is the case, then $M \otimes_R \mathbb{F}$ is also Cohen–Macaulay.

Proof. For the first statement, by induction it suffices to show $\delta(M/xM) \geqslant \delta(M) - 1$ for $x \in \mathfrak{m}_R$. Setting $M[x^\infty] := \bigcup_{n \geqslant 1} M[x^n]$, we have a short exact sequence

$$0 \longrightarrow M[x^\infty] \longrightarrow M \longrightarrow M/M[x^\infty] \longrightarrow 0$$

and $\delta(M) = \max(\delta(M[x^\infty]), \delta(M/M[x^\infty]))$. Since M is finitely generated, we have $M[x^\infty] = M[x^n]$ for some $n \gg 0$. Also note that the induced map

$$x : M/M[x^\infty] \longrightarrow M/M[x^\infty]$$

is injective. We are thus reduced to treat separately the two cases: (a) x is nilpotent on M; (b) x is not a zero-divisor. In case (a), it is easy to see that $\delta(M) = \delta(M/xM)$. In case (b), we have a short exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0;$$

if $\delta(M/xM) \leqslant \delta(M) - 2$, then the map

$$x : \mathrm{Ext}^{j(M)}(M, R[\![G]\!]) \longrightarrow \mathrm{Ext}^{j(M)}(M, R[\![G]\!])$$

would be an isomorphism, which is not possible by Nakayama's lemma (for $x \in \mathfrak{m}_R$).

As for the second statement, we need to show $\delta(M/xM) \leqslant \delta(M) - 1$ if and only if x is injective on M.

(i) If x is injective, then we have a short exact sequence about $x : M \to M$ as above. A Hilbert-polynomial-type argument then shows that M/xM must have a smaller size than M.
(ii) The converse use the Cohen–Macaulayness of M: we see M is pure, i.e. any nonzero submodule of M has the same grade as M (hence the same size as M). Thus, if $M[x] \neq 0$, again a Hilbert-polynomial-type argument shows that M/xM has the same size as M, which means that $\delta(M/xM) = \delta(M)$.

□

Corollary 7.6. *Assume that $\overline{\rho} := \overline{r}|_{G_{\mathbb{Q}_p}}$ is generic (and hence R_∞ is formally smooth).*

(1) $\mathrm{GK}\dim \pi_B(\overline{r}) = 1$ *and* $\pi_B(\overline{r})^\vee$ *is a Cohen–Macaulay module.*
(2) M_∞ *is flat over* R_∞*.*
(3) $\mathbb{T}_{\mathfrak{m}}$ *is a complete intersection ring.*

Proof. By the construction of M_∞ one has

$$\delta(M_\infty) = \dim R_\infty + 1,$$

so implies $\delta(\pi_B(\overline{r})^\vee) \geqslant 1$ by Gee–Newton 7.5. Combined with the main result in last lecture, we obtain (1); the others assertions except (3) follow from Theorem 7.5 again.

For (3), we use that fact that there exists a surjection

$$R_{\overline{r}} := R_\infty/\mathfrak{m}_{S_\infty} \longrightarrow\!\!\!\!\!\rightarrow \mathbb{T}_{\mathfrak{m}_{\overline{r}}}.$$

[9] If π is an admissible representation of G, then δ-dimension of $\pi^\vee$ is just the GK-dimension of π. We should think of δ-dimension as Krull dimension in commutative algebras.

By choosing a regular sequence of S_∞ which generates $\mathfrak{m}_{S_\infty}$, we show inductively that $R_{\overline{r}}$ acts faithfully on $S^1(U^p, \mathcal{O})_{\mathfrak{m}_{\overline{r}}}$. Since $\mathbb{T}_{\mathfrak{m}_{\overline{r}}}$ also acts faithfully, the surjection is an isomorphism. □

Corollary 7.7. *Assume that $\overline{\rho}$ is generic. Then $\pi_B(\overline{r})$ does not admit nonzero quotients which are finite-dimensional as $\mathbb{F}$-vector spaces.*

Proof. Otherwise, we obtain dually a nonzero submodule $M \hookrightarrow \pi_B(\overline{r})^\vee$, which is finite-dimensional. Let $\Lambda := \mathbb{F}[\![(1+\mathfrak{p}_D)/Z_1]\!]$. For finite-dimensional M, its grade is 3, so $\mathrm{Ext}^3_\Lambda(M, \Lambda) \neq 0$. But the inclusion $M \hookrightarrow \pi_B(\overline{r})^\vee$ thus implies that

$$\mathrm{Ext}^3_\Lambda(\pi_B(\overline{r})^\vee, \Lambda) \neq 0$$

as $\mathrm{Ext}^4_\Lambda(-, \Lambda) = 0$. This leads to a contradiction to the Cohen–Macaulayness of Corollary 7.6. □

We explain why we are interested in the flatness of M_∞. Given any point $x : R^\psi_{\overline{\rho}} \to \mathcal{O}'$, where $\mathcal{O}'$ is a finite extension of $\mathbb{Z}_p$, we are able to extend it to a point $x : R_\infty \to \mathcal{O}'$ by, for example, a modularity lifting argument. Define

$$\Pi(x) := (M_\infty \otimes_{R_\infty} \mathcal{O}')^d \otimes_{\mathcal{O}'} E'$$

which is a unitary Banach space representation of $D^\times$; it is expected to be the Jacquet–Langlands correspondence to ρ_x. But it is not clear from the definition that $\Pi(x)$ is nonzero: indeed, $M_\infty \otimes_{R_\infty} \mathcal{O}'$ could be an $\mathcal{O}$-torsion module, so that $(-)^d = 0$. Gee–Newton observed that this non-nullity is implied by the stronger statement that M_∞ is flat, which turns out to be equivalent to $\mathrm{GK}\dim(\pi_B(\overline{r})) \leqslant 1$.

Note that one can also talk about the δ-dimension of $\Pi(x)$, viewing $\Pi(x)^d$ as a finitely generated module over $\mathcal{O}[\![(1+\mathfrak{p}_D)/Z_1]\!][1/p]$.

Theorem 7.8. *For any point $x : R^\psi_{\overline{\rho}} \to \mathcal{O}'$, $\Pi(x)$ is nonzero and has δ-dimension 1. Moreover, its mod p reduction is isomorphic to $\pi_B(\overline{r})$.*

Remark 7.9. This non-nullity at all points differs from the classical Jacquet–Langlands correspondence, as only discrete series of $\mathrm{GL}_n(L)$-representations show up in the correspondence. More precisely, for ρ_x non-discrete series, it is expected that $\Pi(x)^{\mathrm{alg}} = 0$.

8. Applications of Scholze's functor and Ludwig's result

8.1. **Scholze's functor.** Let L be a finite extension of $\mathbb{Q}_p$. Let $\breve{L}$ denote the completion of the maximal unramified extension of L in a fixed $\overline{L}$. Let D be the central division algebra over L of invariant $1/n$. To any admissible smooth $\mathbb{F}$-representation π of G, Scholze associates a Weil-equivariant sheaf $\mathcal{F}_\pi$ on the étale site of the adic space $\mathbb{P}^{n-1}_{\breve{L}}$.

More precisely, define the sheaf $\mathcal{F}_\pi$ on $(\mathbb{P}^{n-1}_{\breve{L}})_{\mathrm{et}}$ by setting

$$\mathcal{F}_\pi(U) := \mathbf{Map}_{\mathrm{cont}, \mathrm{GL}_n(L)\times D^\times}(|U \times_{\mathbb{P}^{n-1}_{\breve{L}}} \mathcal{M}_{\mathrm{LT},\infty}|, \pi)$$

for $U \in (\mathbb{P}^{n-1}_{\breve{L}})_{\mathrm{et}}$. Here $\mathcal{M}_{\mathrm{LT},\infty}$ is the perfectoid space of Lubin–Tate tower at infinite level, which admits the Gross–Hopkins period map the association

$$\pi_{\mathrm{GH}} : \mathcal{M}_{\mathrm{LT},\infty} \longrightarrow \mathbb{P}^{n-1}_{\breve{L}}.$$

Scholze proved that

$$\mathcal{S}^i : \pi \longmapsto H^i_{\mathrm{et}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi)$$

defines a covariant cohomological δ-functor from $\mathsf{Rep}^{\mathrm{adm}}_{\mathbb{F}}(\mathrm{GL}_n(L))$ to the category of admissible smooth representations of $D^\times$ which carries a continuous commuting action of G_L.

We denote the functors by $\mathcal{S}^i$. The following result comes from [Sch18].

Theorem 8.1 (Scholze)**.** *Let π be an admissible smooth $\mathbb{F}$-representation of $\mathrm{GL}_n(L)$.*

(1) $\mathcal{S}^i(\pi) = 0$ *for* $i > 2(n-1)$.
(2) *If π is an injective $\mathrm{GL}_n(\mathcal{O}_L)$-representation then $\mathcal{S}^i(\pi) = 0$ for $i > n-1$.*

(3) *The natural map*

$$\mathcal{S}^0(\pi^{\mathrm{SL}_n(L)}) \hookrightarrow \mathcal{S}^0(\pi)$$

is an isomorphism. In particular, if $\pi^{\mathrm{SL}_n(L)} = 0$ *then* $\mathcal{S}^0(\pi) = 0$. *Moreover, if* π *carries a central character, then* $\mathcal{S}^0(\pi)$ *is always finite-dimensional (as* $\mathbb{F}$*-vector space).*[10]

Originally, Scholze defined the functors for admissible representations on $\mathcal{O}$-torsion modules, but it directly extends to locally admissible representations.

We will also need to work on the modules via the Pontryagin duality. Namely, we will consider the covariant homological δ-functor $\{\check{\mathcal{S}}^i\}_{i\geqslant 0}$ defined by

$$\check{\mathcal{S}}^i : M \longmapsto (\mathcal{S}^i(M^\vee))^\vee.$$

8.2. **Local-global compatibility à la Scholze.** From now on, we assume $G = \mathrm{GL}_2(\mathbb{Q}_p)$.

- Suppose $B/\mathbb{Q}$ is an indefinite quaternion algebra (i.e. split at ∞) that is ramified at p; so $B \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong D$.
- Suppose $B'/\mathbb{Q}$ is a definite quaternion algebra that splits at p, so $B' \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong M_2(\mathbb{Q}_p)$; suppose B' has the same ramification behavior as B at all the other places.
- Fix an isomorphism $B^\times(\mathbb{A}^p_{\mathbb{Q},f}) \cong B'^\times(\mathbb{A}^p_{\mathbb{Q},f})$ and an open compact subgroup $U^p \subset B^\times(\mathbb{A}^p_{\mathbb{Q},f}) \cong B'^\times(\mathbb{A}^p_{\mathbb{Q},f})$.
- On B', define $S^1(U^p,\mathbb{F})$ as before; on B, we define $H^i(U^p,\mathbb{F})$ for $i = 0,1,2$ as

  $$H^i(U^p,\mathbb{F}) := \varinjlim_{U_p \subset D^\times} H^i_{\mathrm{et}}(\mathrm{Sh}_{U^pU_p,\overline{\mathbb{Q}}},\mathbb{F}).$$

  Both of them carry an action of the Hecke algebra $\mathbb{T}$, and $H^i(U^p,\mathbb{F})$ carries an extra action of $G_{\mathbb{Q}}$.
- Let $\overline{r} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ and $\mathfrak{m}_{\overline{r}} \subset \mathbb{T}$ be as before. We may consider

  $$S(U^p,\mathbb{F})_{\mathfrak{m}_{\overline{r}}}, \quad S(U^p,\mathbb{F})[\mathfrak{m}_{\overline{r}}]$$

  as well as

  $$H^1(U^p,\mathbb{F})_{\mathfrak{m}_{\overline{r}}}, \quad H^1(U^p,\mathbb{F})[\mathfrak{m}_{\overline{r}}].$$
- Let M'_∞ be the patched module for B', and M_∞ be the one for B, both over R_∞. Then

  $$M'_\infty/\mathfrak{m}_{R_\infty} \cong (S(U^p,\mathbb{F})[\mathfrak{m}_{\overline{r}}])^\vee, \quad M_\infty/\mathfrak{m}_{R_\infty} \cong (H^1(U^p,\mathbb{F})[\mathfrak{m}_{\overline{r}}])^\vee.$$

Remark 8.2. In previous sections we have worked with definite quaternion algebra which is ramified at p, for example $\pi_B(\overline{r})$. But the results remain true with $\pi_B(\overline{r})$ replaced by $H^1(U^p,\mathbb{F})[\mathfrak{m}_{\overline{r}}]$.

We collect some known results about $\mathcal{S}^i$ in the case of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Theorem 8.3. *Assume* $G = \mathrm{GL}_2(\mathbb{Q}_p)$.

(1) *There is a canonical* $\mathbb{T}_{\mathfrak{m}_{\overline{r}}}[G_{\mathbb{Q}_p} \times D^\times]$*-equivariant isomorphism*

$$\mathcal{S}^1(S(U^p,\mathbb{F}))_{\mathfrak{m}_{\overline{r}}} \cong H^1(U^p,\mathbb{F})_{\mathfrak{m}_{\overline{r}}}.$$

More generally, there is a canonical $R_\infty[G_{\mathbb{Q}_p} \times D^\times]$*-equivariant isomorphism*

$$\check{\mathcal{S}}^1(M'_\infty) = M_\infty.$$

(2) *We have* $\mathcal{S}^i(S(U^p,\mathbb{F})_{\mathfrak{m}_{\overline{r}}}) = 0$ *for* $i = 0,2$. *Similarly,*

$$\check{\mathcal{S}}^0(M'_\infty) = \check{\mathcal{S}}^2(M'_\infty) = 0.$$

(3) *There is an inclusion*

$$\mathcal{S}^1(S(U^p,\mathbb{F})[\mathfrak{m}_{\overline{r}}]) \subset H^1_{\mathrm{et}}(U^p,\mathbb{F})[\mathfrak{m}_{\overline{r}}]$$

whose cokernel is finite-dimensional.

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[10] This is because $\pi^{\mathrm{SL}_n(L)}$ is finite-dimensional (which uses that there is no nontrivial extension between characters of $\mathrm{GL}_n(L)$ admitting central character), and that $\mathcal{S}^0(\chi \circ \det)$ is finite-dimensional.

Proof. (1) Use p-adic uniformization theorem of Čerednik.

(2) For $i=2$ it is Theorem 8.1(2); for $i=0$ it is Ihara's lemma and Theorem 8.1(3). The statement for M'_∞ is a consequence of the construction.

(3) Let $(f_1,\dots,f_n)$ be a set of generators of $\mathfrak{m}_{\overline{r}}$. Write $V = S(U^p,\mathbb{F})_{\mathfrak{m}_{\overline{r}}}$ for simplicity. Then there is a left-exact sequence

$$0 \longrightarrow V[\mathfrak{m}_{\overline{r}}] \longrightarrow V \xrightarrow{(f_1,\dots,f_n)} \bigoplus_{i=1}^n V.$$

Let Q and C be the image and the cokernel of $(f_1,\dots,f_n)$, respectively. Then

- We have $\mathcal{S}^0(Q)=0$ as $\mathcal{S}^0(Q)\hookrightarrow \mathcal{S}^0(\bigoplus_{i=1}^n V)$; also, $\mathcal{S}^2(Q)=0$ as $\mathcal{S}^2(V)\twoheadrightarrow \mathcal{S}^2(Q)$.
- Thus,
$$0 \longrightarrow \mathcal{S}^1(V[\mathfrak{m}_{\overline{r}}]) \longrightarrow \mathcal{S}^1(V) \xrightarrow{\alpha} \mathcal{S}^1(Q) \longrightarrow \mathcal{S}^2(V[\mathfrak{m}_{\overline{r}}]) \longrightarrow 0.$$
- Moreover, the sequence
$$0 \longrightarrow Q \longrightarrow \bigoplus_{i=1}^n V \longrightarrow C \longrightarrow 0$$
induces the cohomological sequence
$$0 \longrightarrow \mathcal{S}^0(C) \longrightarrow \mathcal{S}^1(Q) \xrightarrow{\beta} \mathcal{S}^1\left(\bigoplus_{i=1}^n V\right).$$

On the other hand, $H^1_{\mathrm{et}}(U^p,\mathbb{F})[\mathfrak{m}_{\overline{r}}]$ is identified with the kernel of

$$\beta\circ\alpha : \mathcal{S}^1(V) \longrightarrow \mathcal{S}^1\left(\bigoplus_{i=1}^n V\right) = \bigoplus_{i=1}^n \mathcal{S}^1(V)$$

and we want to control the cokernel of the following map

$$\gamma : \mathcal{S}^1(V[\mathfrak{m}_{\overline{r}}]) \longrightarrow \ker(\beta\circ\alpha);$$

but it embeds in $\ker(\beta)=\mathcal{S}^0(C)$ and we conclude by Theorem 8.1(3). □

Corollary 8.4. *When $R^\psi_{\overline{\rho}}$ is formally smooth, the inclusion in Theorem 8.3(3) is an equality, i.e.*

$$\mathcal{S}^1(S(U^p,\mathbb{F})[\mathfrak{m}_{\overline{r}}]) = H^1_{\mathrm{et}}(U^p,\mathbb{F})[\mathfrak{m}_{\overline{r}}].$$

Proof. This follows from Corollary 7.7. □

Finally, we recall the following theorem [Lud17].

Theorem 8.5 (Ludwig)**.** *If π is a principal series, then $\mathcal{S}^2(\pi)=0$.*

8.3. **Local-global compatibility à la Emerton.** Recall that $\kappa(\overline{\rho})$ denotes the representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ corresponding to $\overline{\rho}$ by mod p LLC.

Theorem 8.6 ([Eme06])**.** *There is an isomorphism*

$$S(U^p,\mathbb{F})[\mathfrak{m}_{\overline{r}}] \cong \kappa(\overline{\rho})^{\oplus s}$$

of $\mathrm{GL}_2(\mathbb{Q}_p)$-representations, for some $s\geqslant 1$.

Proof. In fact, Emerton treated the modular curve case; the generalization in the case of definite quaternion algebras was done by Dospinescu–Le Bras [DB17]. □

Scholze also showed that $H^1(U^p,\mathbb{F})[\mathfrak{m}_{\overline{r}}]$ is $\overline{r}$-typic, meaning that

$$\overline{r}\otimes \mathrm{Hom}_{G_{\mathbb{Q}}}(\overline{r}, H^1(U^p,\mathbb{F})[\mathfrak{m}_{\overline{r}}]) \xrightarrow{\sim} H^1(U^p,\mathbb{F})[\mathfrak{m}_{\overline{r}}].$$

In summary, restricted to $G_{\mathbb{Q}_p}$ we get an $G_{\mathbb{Q}_p}\times D^\times$-equivariant isomorphism[11]

$$\mathcal{S}^1(\kappa(\overline{\rho})) \cong \overline{\rho}\otimes \mathrm{JL}(\overline{\rho})$$

for some admissible smooth representation $\mathrm{JL}(\overline{\rho})$ of $D^\times$.

[11]In the formula we should indeed take $\overline{\rho}(-1)$ instead of $\overline{\rho}$.

8.4. Paškūnas' argument.

Proposition 8.7. *Assume that $R^{\psi}_{\overline{\rho}}$ is formally smooth. Then M'_∞ is a flat R_∞-module. Moreover, the following statements are equivalent:*

(1) $\mathrm{GK}\dim \mathcal{S}^1(S(U^p,\mathbb{F})[\mathfrak{m}_{\overline{r}}]) = 1$;
(2) $\mathcal{S}^2(S(U^p,\mathbb{F})[\mathfrak{m}_{\overline{r}}]) = 0$.

Remark 8.8. (1) Paškūnashas proved the following result: If $\overline{\rho}$ is reducible, then

$$\mathrm{GK}\dim \mathcal{S}^1(S(U^p,\mathbb{F})[\mathfrak{m}_{\overline{r}}]) = 1.$$

He used Ludwig's theorem 8.5, together with the equivalence in Proposition 8.7.

(2) Since we have shown that $\mathrm{GK}\dim \mathcal{S}^1(S(U^p,\mathbb{F})[\mathfrak{m}_{\overline{r}}]) = 1$ under some genericity conditions, we can deduce

$$\mathcal{S}^2(S(U^p,\mathbb{F})[\mathfrak{m}_{\overline{r}}]) = 0.$$

Using the local-global compatibility result of Theorem 8.6, it implies $\mathcal{S}^2(\kappa(\overline{\rho})) = 0$. In particular, when $\overline{\rho}$ is irreducible, we deduce the vanishing of $\mathcal{S}^2(\pi)$ for π supersingular.

Proof. The flatness of M'_∞ follows from Gee–Newton and the fact that $\kappa(\overline{\rho})$ has GK-dimension 1 (and so also is $S(U^p,\mathbb{F})[\mathfrak{m}_{\overline{r}}]$). When π is non-supersingular, it can be directly checked[12]; when π is supersingular, it is proved by Paškūnas(and Morra).

Now we prove the equivalence. It is proved by an induction. Choose $x \in R_\infty$ and consider

$$0 \longrightarrow M'_\infty \xrightarrow{x} M'_\infty \longrightarrow M'_\infty/xM'_\infty \longrightarrow 0.$$

Applying $\breve{\mathcal{S}}^1$ gives the long cohomological exact sequence. Note that $\breve{\mathcal{S}}^0(-) = 0$ in our situation and $\breve{\mathcal{S}}^2(M'_\infty) = 0$ by the projectivity of M'_∞. Hence there is a short exact sequence

$$0 \longrightarrow \breve{\mathcal{S}}^2(M'_\infty/xM'_\infty) \longrightarrow \breve{\mathcal{S}}^1(M'_\infty) \xrightarrow{x} \breve{\mathcal{S}}^1(M'_\infty) \longrightarrow \breve{\mathcal{S}}^1(M'_\infty/xM'_\infty) \longrightarrow 0.$$

As seen in the proof of Theorem 8.3, $\breve{\mathcal{S}}^2(M'_\infty/xM'_\infty) = 0$ if and only if x is injective, if and only if $\breve{\mathcal{S}}^1(M'_\infty/xM'_\infty)$ has δ-dimension smaller by 1 than $\breve{\mathcal{S}}^1(M'_\infty)$. Then the induction finishes the proof. □

8.5. **Generic case.** Recall the representation $\mathrm{JL}(\overline{\rho})$ introduced after Theorem 8.6:

$$\mathcal{S}^1(\kappa(\overline{\rho})) \cong \overline{\rho} \otimes \mathrm{JL}(\overline{\rho}).$$

Theorem 8.9. *Assume that $\overline{\rho}$ is reducible and generic. Then $\mathrm{JL}(\overline{\rho})$ depends only on $\overline{\rho}^{\mathrm{ss}}$.*

Proof. Write $\overline{\rho}_1$ (resp. $\overline{\rho}_2$) for the non-split extension

$$\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix} \chi_2 & * \\ 0 & \chi_1 \end{pmatrix}.$$

Recall that there exist exact sequences

$$0 \longrightarrow \pi_1 \longrightarrow \kappa(\overline{\rho}_1) \longrightarrow \pi_2 \longrightarrow 0,$$
$$0 \longrightarrow \pi_2 \longrightarrow \kappa(\overline{\rho}_2) \longrightarrow \pi_1 \longrightarrow 0,$$

where $\pi_1 := \mathrm{Ind}_{B(\mathbb{Q}_p)}^G \chi_1 \otimes \chi_2\omega^{-1}$ and $\pi_2 := \mathrm{Ind}_{B(\mathbb{Q}_p)}^G \chi_2 \otimes \chi_1\omega^{-1}$. We have for $i \in \{1,2\}$ that

$$\mathcal{S}^1(\kappa(\overline{\rho}_i)) = \overline{\rho}_i \otimes \mathrm{JL}(\overline{\rho}_i).$$

Note that $\mathcal{S}^0(\pi_i) = \mathcal{S}^2(\pi_i) = 0$ for $i \in \{1,2\}$ by Ludwig's theorem. Hence, applying the functor $\mathcal{S}^i$ and using the equality above, we obtain

$$0 \longrightarrow \mathcal{S}^1(\pi_1) \xrightarrow{\iota_1} \overline{\rho}_1 \otimes \mathrm{JL}(\overline{\rho}_1) \longrightarrow \mathcal{S}^1(\pi_2) \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{S}^1(\pi_2) \xrightarrow{\iota_2} \overline{\rho}_2 \otimes \mathrm{JL}(\overline{\rho}_2) \longrightarrow \mathcal{S}^1(\pi_1) \longrightarrow 0.$$

We state the following lemma at work.

Lemma 8.10. *When restricted to $G_{\mathbb{Q}_p}$, $\mathcal{S}^1(\pi_1)$ (resp. $\mathcal{S}^1(\pi_2)$) is semisimple and any irreducible subquotient of it is isomorphic to χ_1 (resp. χ_2).*

[12] In fact, we have

$$\dim_{\mathbb{F}}(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2)^{K_n} = |B(\mathbb{Q}_p)\backslash \mathrm{GL}_2(\mathbb{Q}_p)/K_n| = p^n(p+1),$$

where $K_n := 1 + p^n M_2(\mathbb{Z}_p)$.

Proof of Lemma. Since $\overline{\rho}_1$ is non-split, we have

$$\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi_2, \overline{\rho}_1 \otimes \mathrm{JL}(\overline{\rho}_1)) = 0.$$

As a consequence,

$$\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi_2, \mathcal{S}^1(\pi_1)) = 0$$

by the first short exact sequence above. Also, applying $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi_2, -)$ to the second short exact sequence above leads to an isomorphism

$$\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi_2, \mathcal{S}^1(\pi_2)) \cong \mathrm{Hom}_{G_{\mathbb{Q}_p}}(\chi_2, \overline{\rho}_2 \otimes \mathrm{JL}(\overline{\rho}_2)) \cong \mathrm{JL}(\overline{\rho}_2)$$

where the last isomorphism follows from the definition of $\overline{\rho}_2$. This implies a $G_{\mathbb{Q}_p} \otimes D^\times$-equivariant embedding

$$\chi_2 \otimes \mathrm{JL}(\overline{\rho}_2) \hookrightarrow \mathcal{S}^1(\pi_2)$$

and one checks that its composition with ι_2 coincides with the morphism obtained by tensoring the inclusion $\chi_2 \hookrightarrow \overline{\rho}_2$ by $\mathrm{JL}(\overline{\rho}_2)$. Using these, a diagram chasing gives a surjection

$$\chi_1 \otimes \mathrm{JL}(\overline{\rho}_2) \twoheadrightarrow \mathcal{S}^1(\pi_1).$$

This proves the lemma. ☐

Now resume on the proof of Theorem 8.9. Referring to the proof of Lemma 8.10, we similarly have an embedding

$$\chi_1 \otimes \mathrm{JL}(\overline{\rho}_1) \hookrightarrow \mathcal{S}^1(\pi_1).$$

We claim that this embedding is an isomorphism. Indeed, the injection $\iota_1 : \mathcal{S}^1(\pi_1) \to \overline{\rho}_1 \otimes \mathrm{JL}(\overline{\rho}_1)$ induces a $G_{\mathbb{Q}_p} \times D^\times$-equivariant embedding

$$\mathcal{S}^1(\pi_1)/(\chi_1 \otimes \mathrm{JL}(\overline{\rho}_1)) \hookrightarrow (\overline{\rho}_1 \otimes \mathrm{JL}(\overline{\rho}_1))/(\chi_1 \otimes \mathrm{JL}(\overline{\rho}_1)) \cong \chi_2 \otimes \mathrm{JL}(\overline{\rho}_1),$$

which forces $\mathcal{S}^1(\pi_1)/(\chi_1 \otimes \mathrm{JL}(\overline{\rho}_1)) = 0$ by Lemma 8.10, proving the claim. In a similar way, the embedding $\chi_2 \otimes \mathrm{JL}(\overline{\rho}_2) \hookrightarrow \mathcal{S}^1(\pi_2)$ is also an isomorphism and consequently so is it for $\overline{\rho}_1$. Therefore, the result follows. ☐

8.6. **Non-generic case.** We only state the theorem without proof. Let

$$\overline{\rho}_1 \sim \begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix}, \quad \overline{\rho}_2 \sim \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix}.$$

Recall that

$$\kappa(\overline{\rho}_1) \cong (\mathrm{Sp} \;\text{—}\; \mathbb{1}_G \;\text{—}\; \mathrm{Ind}_B^G\, \omega \otimes \omega^{-1}),$$

$$\kappa(\overline{\rho}_2) \cong (\mathrm{Ind}_B^G\, \omega \otimes \omega^{-1} \;\text{—}\; \mathrm{Sp} \;\text{—}\; \mathbb{1}_G^{\oplus 2}).$$

Theorem 8.11. *There exist exact sequences*

$$0 \longrightarrow \mathbb{1}_{D^\times} \longrightarrow \mathrm{JL}(\overline{\rho}_1) \longrightarrow V \longrightarrow 0$$

and

$$0 \longrightarrow V \longrightarrow \mathrm{JL}(\overline{\rho}_2) \longrightarrow (\mathbb{1}_{\mathcal{D}^\times})^{\oplus 2} \longrightarrow 0.$$

Moreover, $\mathrm{JL}(\overline{\rho}_2)$ *is isomorphic to the universal extension of* $(\mathbb{1}_{\mathcal{D}^\times})^{\oplus 2}$ *by* V.

Note that $R^{\psi}_{\overline{\rho}_1}$ is formally smooth, but $R^{\psi}_{\overline{\rho}_2}$ is not. So we have

$$\mathcal{S}^1(\kappa(\overline{\rho}_1)) \cong \overline{\rho}_1 \otimes \mathrm{JL}(\overline{\rho}_1)$$

but the inclusion

$$\mathcal{S}^1(\kappa(\overline{\rho}_2)) \subset \overline{\rho}_2 \otimes \mathrm{JL}(\overline{\rho}_2)$$

is strict.

One key ingredient in the proof is the following precision of Scholze's result. We have claimed in Theorem 8.3(3) that the inclusion

$$\mathcal{S}^1(S(U^p, \mathbb{F})[\mathfrak{m}_{\overline{r}}] \subset H^1_{\mathrm{et}}(U^p, \mathbb{F})[\mathfrak{m}_{\overline{r}}]$$

has a finite-dimensional cokernel. Moreover, we have the following result.

Lemma 8.12. *With the above notations, the cokernel of the inclusion in Theorem 8.3(3) for ρ_2, i.e., that of*

$$\mathcal{S}^1(\kappa(\overline{\rho}_2)) \subset \overline{\rho}_2 \otimes \mathrm{JL}(\overline{\rho}_2),$$

is equal to

$$\breve{\mathcal{S}}^0(\mathrm{Tor}_1^{R_\infty}(\mathbb{F}, M'_\infty)).$$

The above lemma reduces the cokernel to the information about $\mathrm{Tor}_1^{R_\infty}(\mathbb{F}, M'_\infty)$, which turns to be more transparent via the isomorphism

$$\mathrm{Tor}_1^{R_\infty}(\mathbb{F}, M'_\infty) \cong \mathrm{Tor}_1^{R^\psi_{\overline{\rho}_2}}(\mathbb{F}, \tilde{P}),$$

where $\tilde{P}$ is the projective envelope of $\kappa(\overline{\rho}_2)^\vee$. This can be computed using Paškūnas's results.

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