A Quick Transcript on Proj Morphisms

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$$\mathrm{Mod}/\mathrm{Ring} \overset{?}{\longleftarrow\!-\!-\!\rightarrow} \mathrm{Qcoh\ Sh}/\mathrm{Space}.$$

<u>Def</u> $S = \bigoplus S_d$, $M = \bigoplus M_d \in \mathrm{Mod}_S$, $M_d \in \mathrm{Mod}_{S_d}$

$\rightsquigarrow \widetilde{M}/\mathrm{Proj}\,S$:

$$\widetilde{M}(U) := \left\{ \begin{array}{l} U \xrightarrow{s} \coprod_{\mathfrak{p}\in U} M_{(\mathfrak{p})} \\ \mathfrak{p} \longmapsto s(\mathfrak{p}) \in M_{(\mathfrak{p})} \end{array} \middle| \ \text{loc. free} \right\}$$

$$\text{where } M_{(\mathfrak{p})} := \left\{ \frac{g}{f} \in T^{-1}M \ \middle| \ \begin{array}{l} g\in M,\ f\in T \text{ homo}, \\ \deg g = \deg f \end{array} \right\}$$

$$T = \{ f \in M \mid \mathfrak{p} \text{ homo.} \} \ \underline{\text{multi}}.$$

"loc. free" $=$ $\forall \mathfrak{p} \in U$, $\exists\, \mathfrak{p} \in V \subseteq U$ nbhd & $m \in M$ homo.

s.t. $\forall \mathfrak{q} \in V$, $\exists f \in S$ w/ $\deg f = \deg m$

s.t. $s(\mathfrak{q}) = m/f \in M_{(\mathfrak{q})}$.

$\rightsquigarrow \widetilde{M} \in \mathrm{Sh}.$

<u>Prop</u> $S = \bigoplus S_d$, $M = \bigoplus M_d \in \mathrm{Mod}_S$. $X = \mathrm{Proj}\,S$.

(a) $\forall \mathfrak{p} \in S$, $(\widetilde{M})_{\mathfrak{p}} = M_{(\mathfrak{p})}$ stalk.

$= \{ s \in M_f \mid \deg s = 0 \}$

(b) $\forall f \in S_+$ homo, $\widetilde{M}|_{D_+(f)} \cong (M_{(f)})^\sim$

$\hat{\ }$

Gp.

via $D_+(f) \cong \operatorname{Proj} S_{(f)}$

$\updownarrow$

c.f. $D(f) \cong \operatorname{Spec} A_f$

(c) $\tilde{M} \in \operatorname{Qcoh}_{\mathcal{O}_X}$. if S nt, M f.g. $\Rightarrow \tilde{M} \in \operatorname{Coh}_{\mathcal{O}_X}$.

pf. note (b) $\Rightarrow$ (c). And (a) (b): follow (2.5). ✓

Def $S = \bigoplus S_d$, $X = \operatorname{Proj} S$. $\forall n \in \mathbb{Z}$,

sh : $\mathcal{O}_X(n) = \widetilde{S(n)}$

$\mathcal{O}_X(1)$ = the twisting sh of Serre.

Recall $S(n)_d = S_{d+n}$. lifting deg n.

$\rightsquigarrow \mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$. the twisting sh.

Prop $X = \operatorname{Proj} S = \operatorname{Proj} \bigoplus S_d$. $S = S_0 \langle S_1 \rangle$ S_0-alg.

(a) $\mathcal{O}_X(n) \in \operatorname{Pic}(X)$. i.e. invertible

i.e. loc. free w/ rank = 1.

Reminder: $\mathcal{F}$ free $\Leftrightarrow$ $\mathcal{F} \cong \mathcal{O}_X^{\otimes n}$.

$\mathcal{F}$ loc. free $\Leftrightarrow$ $\mathcal{F}|_U \cong (\mathcal{O}_X|_U)^{\otimes n}$.

$\mathcal{O}_X(n) \in \operatorname{Pic}(X) \Leftrightarrow \mathcal{O}_X(n)|_U \cong \mathcal{O}_X|_U$

(b) $M = \bigoplus M_d \Rightarrow \widetilde{M}(n) \cong \widetilde{M(n)}$.

$\Rightarrow \mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$.

(c) $T = \bigoplus T_d \cong T_0\langle T_1\rangle$. $\varphi: S \to T$. preserv deg.

$U \subseteq Y = \mathrm{Proj}\, T$.

$$\begin{array}{ccc} \mathrm{Proj}\, T & \longrightarrow & \mathrm{Proj}\, S \\ \| & & \| \\ Y & & X \end{array}$$

$\rightsquigarrow f: U \hookrightarrow Y \to X$

$\Rightarrow f^*(\mathcal{O}_X(n)) \cong \mathcal{O}_Y(n)|_U$.

$f_*(\mathcal{O}_Y(n)|_U) \cong (f_*\mathcal{O}_U)(n)$.

⚠ $Y \to X$, $\left.\begin{array}{l} f_*\mathcal{O}_Y(n) \cong (f_*\mathcal{O}_Y)(n) \\ f^*(\mathcal{O}_X(n)) \cong \mathcal{O}_Y(n) \end{array}\right\}$ fail!

pf. (a). $f \in S_1 \rightsquigarrow \mathcal{O}_X(n)|_{D_+(f)} \cong (S(n)_{(f)})^\sim / \mathrm{Spec}\, S_{(f)}$,

free of rank 1 (?)

$S(n)_{(f)}$ free $S_{(f)}$-mod, rank 1.

b/c $S_{(f)} = \{g \in S_f,\ \deg g = 0\}$ gp.

$\rightsquigarrow S(n)_{(f)} = \{g \in S_f,\ \deg g = n\}$. $f^n S$.

f invertible in S_f.

note $S = S_0\langle S_1\rangle$ S_0-alg. $\Rightarrow$ $X \subseteq D_+(f)$, $f \in S_1$.

$\Rightarrow$ $\mathcal{O}(n) \in \mathrm{Pic}(X)$.

$S = S_0\langle S_1\rangle$ means the question is local on X.

(b) $(M \otimes_S N)^\sim \cong \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$. $\forall M, N \in \mathrm{Mod}_S$.

where $S = S_0\langle S_1\rangle$.

$\forall f \in S_1$, $(M \otimes_S N)_{(f)} = M_{(f)} \otimes_{S_{(f)}} N_{(f)}$.

$\tilde{M}(n) \cong \tilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \cong (M \otimes_S S(n))^\sim = (M(n))^\sim$.

(c) $\forall M \cong \bigoplus M_d \in \mathrm{Mod}_S$, $f^*\tilde{M} \cong (M \otimes_S T)^\sim|_U$.

$\forall T = \bigoplus T_n \in \mathrm{Mod}_N$, $f_*(\tilde{N}|_U) \cong ({}_S N)^\sim$.

& $\tilde{T} \cong f_* \mathcal{O}_U$ on X.

Philosophy twisting: relative rank (✓).

absolute rank (×)

Def $X = \mathrm{Proj} \bigoplus S_d$. $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$.

graded S-mod ass. to $\mathcal{F}$:

$$\boxed{\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n)) \in \mathrm{Gp}.}$$

$\mathcal{B}$

w/ str. graded S-mod

Str: $s \in S_d \rightsquigarrow s \in \Gamma(X, \mathcal{O}_X(d))$.

$\forall t \in \Gamma(X, \mathcal{F}(n)) \rightsquigarrow s \cdot t \in \Gamma(X, \mathcal{F}(n+d))$

by $\mathcal{F}(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(d) \cong \mathcal{F}(n+d)$.

$t \otimes s \longmapsto ts = st$

Prop $S = A[x_0, \dots, x_r]$, $r \geq 1$. $X = \operatorname{Proj} S = \mathbb{P}^r_A$.

$\Rightarrow \Gamma_*(\mathcal{O}_X) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(n)) \cong \bigoplus_{d \geq 0} S_d = S$.

pf. $X = \bigcup D_+(x_i)$. $t \in \Gamma(X, \mathcal{O}_X(n)) \longleftrightarrow \{t_i\}$, $t_i \in \Gamma(D_+(x_i), \mathcal{O}_X|_{D_+(x_i)}(n))$.

$\Gamma(D_+(x_i), \mathcal{O}_X|_{D_+(x_i)}(n)) = \mathcal{O}_X(n)(D_+(x_i))$.

w/ $t_i|_{D_+(x_ix_j)} = t_j|_{D_+(x_ix_j)}$.

t_i homo., deg $n \in S_{x_i}$, s.t. $S_{x_i} \longrightarrow S_{x_ix_j}$

$t_i \longmapsto t_i|_{D_+(x_ix_j)}$.

$\rightsquigarrow \Gamma_*(\mathcal{O}_X) = \left\{ (t_0, \dots, t_r) \;\middle|\; \begin{array}{l} \forall i, t_i \in S_{x_i} \ \& \ \forall i,j, \\ t_i|_{D_+(x_ix_j)} = t_j|_{D_+(x_ix_j)} \end{array} \right\}$

x_i not zero divisor $\in S$

$\Rightarrow S \hookrightarrow S_{x_i} \hookrightarrow S_{x_ix_j}$. inj.'s

$S_{x_ix_j} \subseteq S' = S_{x_0 \cdots x_r}$

$\Rightarrow \Gamma_*(\mathcal{O}_X) = \bigcap S_{x_i} \subseteq S'$

$\forall g \in S'$, $\exists!\ g = x_0^{i_0} \cdots x_r^{i_r} f(x_0, \dots, x_r)$. $x_i \nmid f$. homo.

$\rightsquigarrow g \in S_{x_i} \iff i_j \geq 0,\ \forall i \neq j.$

$\Rightarrow g \in \cap S_{x_i} \iff i_k \geq 0,\ \forall k. \Rightarrow \underbrace{\bigcap S_{x_i} = S}.$ $\square$.

<u>Caution</u> $\Gamma_*(\mathcal{O}_X) \neq S$ when S not poly ring.

<u>Lem</u> $\mathcal{L} \in \mathrm{Pic}(X)$. $f \in \Gamma(X, \mathcal{L})$. $X_f = \{x \in X : f_x \notin m_x \mathcal{L}_x\}$.

(c.f. 5.3). $\mathcal{F} \in \mathrm{Qcoh}/X$.

(a) X q-cpt, $s \in \Gamma(X, \mathcal{F})$ s.t. $s|_{X_f} = 0$.

$\Rightarrow \exists n > 0$ s.t. $f^n s = 0 \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$.

(b) $+ X \subseteq \overset{<\infty}{\bigcup} U_i$ s.t. $\mathcal{L}|_{U_i}$ free,

& $U_i \cap U_j$ q-cpt.

$t \in \Gamma(X_f, \mathcal{F}) \Rightarrow f^n t \in \Gamma(X_f, \mathcal{F} \otimes \mathcal{L}^{\otimes n}|_{X_f})$.

$\overset{\text{ext}}{\rightsquigarrow} s \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$.

pf. (a) $X = \overset{<\infty}{\bigcup} U$, $U = \mathrm{Spec} A$ s.t. $\mathcal{L}|_U$ free.

$\rightsquigarrow \psi : \mathcal{L}|_U \xrightarrow{\cong} \mathcal{O}_U$.

$\mathcal{F} \in \mathrm{Qcoh}. \Rightarrow \mathcal{F}|_U \cong \tilde{M}$. $s \in \Gamma(X, \mathcal{F}) \leftrightsquigarrow s \in M$.

$f \in \Gamma(X, \mathcal{L}) \rightsquigarrow f \in \Gamma(U, \mathcal{L}|_U)$

$\rightsquigarrow \psi(f) = g \in \Gamma(U, \mathcal{O}_U) = A$.

$X_f \cap U = D(g)$, X_f "im-non-vanishing part."

<u>Now</u> $s|_{X_f} = 0$, $g^n s = 0 \in M$.

$id \times \psi^{\otimes n} : \mathcal{F} \otimes \mathcal{L}|_U^{\otimes n} \cong \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_U^{\otimes n} \cong \mathcal{F}|_U$.

$\Rightarrow f^n s = 0 \in \Gamma(U, \mathcal{F} \otimes \mathcal{L}^{\otimes n}|_U)$.

<u>note</u> statement is intrinsic

i.e. indep. of $\psi : \mathcal{L}|_U \cong \mathcal{O}_U$.

$\Rightarrow n \gg 0$, $f^n s = 0 \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$

(b) $U_i \cap U_j$ q-cpt $\Rightarrow$ apply 5.3 (a).

keeping track of the twist. $\square$

<u>Rmk</u> hypothesis $\Leftarrow$ either X nt ($\Rightarrow U \subseteq X$ q-cpt).

or X qcs (q-cpt + sep)

($\Rightarrow$ Spec $A_i \cap$ Spec A_j = Spec A_{ij} q-cpt).

<u>Prop</u> $S = \bigoplus S_d = S_0 \langle S_1 \rangle$ S_0-alg. f.g. $\rightsquigarrow X = \mathrm{Proj}\, S$

$\mathcal{F} \in \underline{\underline{\mathrm{Qcoh}}}_X \rightsquigarrow \beta : (\Gamma_*(\mathcal{F}))^\sim \longrightarrow \mathcal{F}$.

"part $\mathcal{F}$ into graded".

$\updownarrow$ mod can be parted. (f.g. or not?).

pf. $f \in S_1$, $\Gamma_*(\mathcal{F})^\sim \in \mathrm{Qcoh}$.

$$s \in \Gamma_*(\mathcal{F})^\sim \longmapsto ?|_{D_+(f)} \rightsquigarrow \beta \text{ on } X. \qquad \underline{\text{Ex } 5.3}$$

$$\uparrow$$

$$m/f^d, \quad m \in \Gamma(X, \mathcal{F}(d)).$$

$$\text{s.t. } \deg m = \deg f^d = d$$

think $f^{-d} \in \Gamma(X, \mathcal{O}_X(-d)) / D_+(f)$.

$$\rightsquigarrow m \otimes f^{-d} \mapsto m f^{-d} \quad \text{no twist}$$

$$\cap \ \Gamma(D_+(f), \mathcal{F}).$$

$$\rightsquigarrow \beta|_{D_+(f)} : \Gamma_*(\mathcal{F}|_{D_+(f)})^\sim \longrightarrow \mathcal{F}|_{D_+(f)}$$

$$s \longmapsto m \otimes f^{-d}$$

$$\rightsquigarrow \beta : \Gamma_*(\mathcal{F})^\sim \longrightarrow \mathcal{F}$$

(5.14) $\Rightarrow f \in \Gamma(X, \mathcal{L})$, $\boxed{\mathcal{L} = \mathcal{O}(1)} \in \mathrm{Pic}(X)$.

$S = S_0\langle S_1 \rangle$ f.g. $\Rightarrow \exists f_1, \dots, f_r \in S_1$ s.t.

$$X \subseteq \bigcup D_+(f_i) \text{ cover}.$$

& $D_+(f_i) \cap D_+(f_j)$ affine.

$\mathcal{L}|_{D_+(f_i)}$ free.

$$\overset{(5.14)}{\Longrightarrow} \mathcal{F}(D_+(f)) \cong \Gamma_*(\mathcal{F})_{(f)}$$

$$\Rightarrow \beta_{(f)} : \Gamma_*(\mathcal{F})^\sim_{(f)} \xrightarrow{\cong} \mathcal{F}(D_+(f))$$

$$\beta|_{D_+(f)} : \Gamma_*(\mathcal{F}|_{D_+(f)})^\sim \xrightarrow{\cong} \mathcal{F}|_{D_+(f)} \rightsquigarrow \beta \ \checkmark \qquad \square$$

Cor (a) $Y \overset{i}{\hookrightarrow} \mathbb{P}^r_A = \mathbb{P}^r_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} A$, closed sub.

$\rightsquigarrow$ homo $I \subseteq S = A[x_0, \dots, x_r]$ s.t. $\tilde{I} = \mathcal{I}_Y$.

i.e. $0 \to \tilde{I} \to \mathcal{O}_{\mathbb{P}^r_A} \longrightarrow i^* \mathcal{O}_Y$. <u>Ex 2.3.12</u>

(b) $Y / \operatorname{Spec} A$ proj sch $\Leftrightarrow$ $Y \cong \operatorname{Proj} S$, $S = \oplus S_d$.

w/ $S_0 = A$. $S = S_0 \langle S_1 \rangle$ f.g.

$\left(\begin{array}{l} \text{recall: } Y \xhookrightarrow{cl} \mathbb{P}^N_A \xrightarrow{proj.} \operatorname{Spec} A \Leftrightarrow Y \text{ proj.} \\ Y \overset{op}{\hookrightarrow} Y' \xhookrightarrow{cl} \mathbb{P}^N_A \longrightarrow \operatorname{Spec} A \Leftrightarrow Y \text{ q-proj.} \end{array} \right)$

pf. (a) $\mathcal{I}_Y \subseteq \mathcal{O}_X$ sub sh.

twisting functor $(\cdot)(n) = (\cdot) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ exact.

$\Gamma(X, \cdot)$: left-exact.

$\Rightarrow \Gamma_*(\mathcal{I}_Y) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{I}_Y(n)) \subseteq \Gamma_*(\mathcal{O}_X)$ submod.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{S}$ (5.13)

$\left(\begin{array}{l} \text{left} \\ \text{right} \end{array} \text{-exact preserves } \begin{array}{l} \text{sub} \\ \text{quot.} \end{array} \right)$

$\Rightarrow \Gamma_*(\mathcal{I}_Y)$ homo ideal $= I \subseteq S$.

$\mathcal{I}_Y \in \mathfrak{Qcoh}$ as ker $\Rightarrow \mathcal{I}_Y \overset{\beta}{\cong} \tilde{I}$ by partition.

$\Rightarrow Y \subseteq X$ sub. by I

<u>Fact</u> $\Gamma_*(\mathcal{I}_Y)$ = largest ideal $\subseteq S$ defining Y.

(b) "$\Rightarrow$" Y proj. $\Rightarrow$ $Y \hookrightarrow \mathbb{P}^r_A \longrightarrow \operatorname{Spec} A$

$\qquad\qquad\qquad\qquad \overset{\shortparallel}{\operatorname{Proj} S/I}$.

take $I \subseteq S_+ = \bigoplus_{d>0} S_d$ s.t. $(S/I)_0 = A$. ✓

"$\Leftarrow$": $\forall$ such S, $S = A[x_0, \dots, x_r]/I$ quot.

$\Rightarrow$ $\operatorname{Proj} S$ proj. $\square$

<u>Def</u> $Y \in \mathrm{Sch}$, $\mathcal{O}_{\mathbb{P}^r_Y}(1)$ = twisting sh $\in \mathrm{Sh}\, \mathbb{P}^r_Y$.

$\qquad \mathcal{O}_{\mathbb{P}^r_Y}(1) = g^*(\mathcal{O}(1))$. $\qquad \mathbb{P}^r_Y \longrightarrow \mathbb{P}^r_{\mathbb{Z}}$ $\quad g = pr_1$.

$\qquad\qquad \mathbb{P}^r_Y = \mathbb{P}^r_{\mathbb{Z}} \times_{\mathbb{Z}} Y$.

<u>note</u>: if $Y = \operatorname{Spec} A$, $\mathcal{O}_{\mathbb{P}^r_Y}(1) = \mathcal{O}_A(1)$ on $\operatorname{Proj} A[x_0, \dots, x_r]$

<u>Def</u> X/Y sch. $\mathcal{L} \in \operatorname{Pic}(X)$.

$\mathcal{L}$ $\underbrace{\text{very ample}}_{\text{vap.}}$ rel. to Y $\iff$ $\exists X \overset{i}{\hookrightarrow} \mathbb{P}^r_Y$ imm

s.t. $i^*\mathcal{O}_{\mathbb{P}^r_Y}(1) \cong \mathcal{L}$.

$i : X \hookrightarrow Z$ imm $\iff$ $X \overset{\text{open}}{\hookrightarrow} X' \overset{\text{closed}}{\hookrightarrow} Z$

[EGA II, 4.4.2] differs slightly.

Rmk Y nt sch. X/Y proj. $\Leftrightarrow$ proper + $\boxed{\exists \mathcal{L} \in \mathrm{Pic}(X) \text{ vap. rel. } Y.}$

weak by Chow's Lem.

"$\Rightarrow$" proj. $\Rightarrow$ proper

$\exists X \overset{i}{\hookrightarrow} \mathbb{P}^r_Y$ closed imm. $\Rightarrow \mathcal{O}_X(1)$ vap.

b/c $i^* \mathcal{O}_{\mathbb{P}^r_Y}(1) \cong \mathcal{O}_X(1)$

"$\Leftarrow$" $\mathcal{L}$ vap $\Rightarrow \mathcal{L} \cong i^* \mathcal{O}_{\mathbb{P}^r_Y}(1)$

but (Ex 4.4) $i(X)$ closed $\Rightarrow i$ closed imm

$\Rightarrow X$ proj $/Y$ through $\mathbb{P}^r_Y$.

Note $\mathcal{L}$ vap depends on $i : X \hookrightarrow \mathbb{P}^r_Y$

① $Y = \mathrm{Spec} A$, $X = \mathrm{Proj}\, S$, $S = S_0\langle S_1\rangle$ f.g.

$\Rightarrow \mathcal{O}_X(1)$ vap.

② However: $\exists T \not\cong S$ w/ $\mathrm{Proj}\, T \cong \mathrm{Proj}\, S$

$\rightsquigarrow$ same $\mathcal{O}_X(1)$.

Object Sh / proj. sch / nt ring.

Def X sch, $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$. $\mathcal{F}$ generated by global sections

$\Leftrightarrow \exists \{s_i\}_{i \in I} \subseteq \Gamma(X, \mathcal{F})$

s.t. $\forall x \in X$, stalk $\mathcal{F}_x = \langle s_{i,x} \rangle_{i \in I} \in \mathrm{Mod}_{\mathcal{O}_x}$.

<u>note</u> $\mathcal{F} = \langle s_i \rangle$, $s_i \in \Gamma(X, \mathcal{F}) \iff \mathcal{F} = \mathcal{O}_X^{\oplus n} / \mathcal{I}$. quot sh.

indeed: $\mathcal{O}_X^{\oplus |I|} \twoheadrightarrow \mathcal{F} \leftrightsquigarrow \{s_i\}_{i \in I}$.

E.g. $\mathcal{F} \in \mathrm{Qcoh}_{\mathcal{O}_{\mathrm{Spec} A}} \Rightarrow \mathcal{F} = \langle s_i \rangle$

b/c when $\mathcal{F}|_{\mathrm{Spec} B} \cong \tilde{M}$, $M = B \langle \bar{s}_i \rangle$.

$\rightsquigarrow \mathcal{F}_x \cong \tilde{M}_x = (B \langle \bar{s}_i \rangle_x)^\sim \Rightarrow \mathcal{F} = \langle s_i \rangle$.

E.g. $X = \mathrm{Proj}\, S$, $S = S_0 \langle S_1 \rangle \Rightarrow \mathcal{O}_X(1) = \langle \mathrm{im}\, S_1 \rangle$.

$\boxed{\text{BIG THM}}$ (Serre) "Higher twist becomes finite".

<u>Condition</u> "Best, compatible w/ intuitions":

$$\begin{array}{c} X \text{ proj sch} \\ \downarrow \\ \mathrm{Spec} A,\ A \text{ nt} \end{array} \rightsquigarrow \mathcal{F} \in \mathrm{Coh}_{\mathcal{O}_X}.$$

<u>Then</u> $\exists n_0$ s.t. $n \geq n_0 \Rightarrow \mathcal{F}(n) = \langle s_{ni} \rangle_{i \in I}$ w/ $\boxed{|I| < \infty}$

pf. $X \overset{i}{\hookrightarrow} \mathbb{P}^r_A$ closed imm. s.t. $i^* \mathcal{O}_{\mathbb{P}^r_A}(1) \cong \mathcal{O}_X(1)$.

$\Rightarrow i_* \mathcal{F} \in \mathrm{Coh} / \mathbb{P}^r_A$ (Ex 3.5.5). & $i_*(\mathcal{F}(n)) = (i_* \mathcal{F})(n)$

5.12 or Ex 5.1(d).

$\mathcal{F}(n) = (S_{ni}) \Longleftrightarrow$ so does $i_*\mathcal{F}(n)$

$\uparrow$ In fact, $i_*(\mathcal{F}(n)) = (S_{ni})$.

$\leadsto$ reduce to $X = \mathbb{P}^r_A = \mathrm{proj}\, A[x_0, \cdots, x_r]$.

<u>Now</u> $X \subseteq \bigcup_{i=0}^{r} D_+(x_i) = \bigcup U_i$ $\quad \mathcal{F} \in \mathrm{Coh}$.

$\Rightarrow \forall i, \exists M_i$ f.g. $B_i = A[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$ - mod.

s.t. $\widetilde{M_i} \cong \mathcal{F}|_{U_i}$

$\leadsto \forall i, M_i = B_i \langle s_{ij} \rangle$.

$\exists n$ s.t. $x_i^n s_{ij} \xrightarrow{\text{ext}} t_{ij} \in \Gamma(X, \mathcal{F}(n))$.

$\cap$

$\Gamma(U_i, \mathcal{F}(n)|_{U_i})$.

$n \gg 0$, $\mathcal{F}(n)|_{U_i} \cong \widetilde{M_i'}$, $M_i' \in \mathrm{Mod}_{B_i}$.

w/ $M_i' = B_i \langle x_i^n s_{ij} \rangle \Rightarrow \mathcal{F}(n) = (t_{ij})$. $\square$

<u>Cor</u> X proj / A nt. $\mathcal{F} \in \mathrm{Coh}_{\mathcal{O}_X}$

$\Rightarrow \mathcal{F} \cong \mathcal{E}/\mathcal{I} = \bigoplus_{i=1}^{N} \mathcal{O}_X(n_i)/\mathcal{I}$.

n_i various. by Serre.

Pf. Let $\mathcal{F}(n) = (s_i)_{i=1}^{N}$, $n \gg 0$ $\leadsto \bigoplus_{i=1}^{N} \mathcal{O}_X \longrightarrow \mathcal{F}(n) \longrightarrow 0$.

$\xrightarrow{(\cdot) \otimes \mathcal{O}_X(-n)} \bigoplus_{i=1}^{N} \mathcal{O}_X(-n) \longrightarrow \mathcal{F} \rightarrow 0$. $\square$

Thm k field, A f.g. k-alg. X proj. $/A$. $\mathcal{F} \in \mathrm{Coh}\,\mathcal{O}_X$.

$\Rightarrow \Gamma(X, \mathcal{F})$ f.g. A-mod.

In particular, $A = k \Rightarrow \Gamma(X, \mathcal{F})$ v.s. w/ $\dim_k < \infty$.

Pf. $X = \mathrm{Proj}\, S$, $S = S_0\langle S_1\rangle$ f.g.

$\rightsquigarrow M = \Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))$. $\Rightarrow \widetilde{M} \overset{\beta}{\cong} \mathcal{F}$.

- $n \gg 0$, $\mathcal{F}(n) = (s_{ni})^{<\infty}$, $s_{ni} \in \Gamma(X, \mathcal{F}(n))$.

$\rightsquigarrow M' = \langle s_{ni}\rangle_i \subseteq M$. submod.

$\Rightarrow M'$ f.g. S-mod. & $M' \hookrightarrow M \rightsquigarrow \widetilde{M}' \hookrightarrow \widetilde{M} = \mathcal{F}$.

$\rightsquigarrow \widetilde{M}'(n) \hookrightarrow \mathcal{F}(n)$. isom

b/c $\mathcal{F}(n) = (s_{ni})$, $s_{ni} \in M'$.

$\xrightarrow{(\cdot)(-n)} \widetilde{M}' \cong \mathcal{F}$. M' f.g. S-mod.

<u>To show</u> M f.g. S-mod $\Rightarrow \Gamma(X, \widetilde{M})$ f.g. A-mod.

finite fil: $0 = M^0 \subseteq M^1 \subseteq \cdots \subseteq M^r = M$.

graded mods.

$\forall i$. $\boxed{M^i/M^{i-1} \cong (S/\mathfrak{p}_i)(n_i)}$.

$\mathfrak{p}_i$ homo $\subseteq S$ prime.

$\rightsquigarrow$ fil on $\tilde{M}$: $0 \to \tilde{M}^{i-1} \to \tilde{M}^i \to \tilde{M}^i/\tilde{M}^{i-1} \to 0$

$\Gamma(X, \cdot)$ left-exact :

$$0 \to \Gamma(X, \tilde{M}^{i-1}) \to \Gamma(X, \tilde{M}^i) \to \Gamma(X, \tilde{M}^i/\tilde{M}^{i-1}).$$

$\rightsquigarrow$ $\Gamma(X, \tilde{M})$ f.g. $/A$ $\Leftarrow$ $\Gamma(X, (\widetilde{S/\mathfrak{p}})(n))$ f.g. $\forall \mathfrak{p}, n$.

Reduce to : S graded int.

"$S_0\langle S_1\rangle$ f.g. & $S_0 = A$ f.g. int. $/k$.

$\Rightarrow$ $\Gamma(X, \mathcal{O}_X(n))$ f.g. A-mod, $\forall n$.

Let $S_1 = A\langle x_0, \dots, x_r\rangle \in \mathrm{Mod}_A$, $x_i \in S_1$.

S int $\Rightarrow$ $\mathrm{mult}_{x_0}: S(n) \xhookrightarrow{\cdot x_0} S(n+1)$, $\forall n$.

$\rightsquigarrow$ $\Gamma(X, \widetilde{S(n)}) \hookrightarrow \Gamma(X, \widetilde{S(n+1)})$

$\rightsquigarrow$ $\Gamma(X, \mathcal{O}_X(n)) \hookrightarrow \Gamma(X, \mathcal{O}_X(n+1))$. $\forall n$.

$\rightsquigarrow$ <u>to prove</u> : $\Gamma(X, \mathcal{O}_X(n))$ f.g. for $n \gg 0$.

Let $S' = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{O}_X(n))$ $\rightsquigarrow$ $S \subseteq S' \subseteq \bigcap_{i=0}^{r} S_{x_i}$

<u>Need</u> S' int $/S$.

$s' \in S'$ homo, deg $d \geq 0$ $\rightsquigarrow$ $s' \in S_{x_i}$, $\forall i$

$\Rightarrow$ $\exists$ $n \gg 0$ s.t. $x_i^n s' \in S$.

$S_1 = \langle x_i \rangle \Rightarrow a_{im} t^m$ mono. in x_i

s.t. $\langle a_{im} t^m \rangle = S_m, \forall m$.

$\rightsquigarrow$ May assume $ys' \in S$, $\forall y \in S_n$.

$\rightsquigarrow$ $\deg s' > 0 \Rightarrow \forall y \in S_{\geq n} = \bigoplus_{e \geq n} S_e$. $ys' \in S_{\geq n}$.

induction $\Rightarrow \forall q \geq 1$, $y \cdot (s')^q \in S_{\geq n}$, $\forall y \in S_{\geq n}$.

take $y = x_0^n \rightsquigarrow \forall q \geq 1$, $(s')^q \in \frac{S}{x_0^n}$.

$\uparrow$

f.g. sub-S-mod. / Frac S'.

$\Rightarrow$ s' int / S. $\Rightarrow S' \subseteq \bar{S} \subseteq \operatorname{Frac} S$.

[Atiyah-MacDonald 1. p.59]

Apply: (I.3.9A) (thm for fin. of int closure).

S f.g. k-alg. $\Rightarrow$ S' f.g. S-mod.

$\Rightarrow \forall n$, S_n' f.g. S_0-mod. $\leftarrow$ ☺. $\square$

<u>Indeed</u> pf $\Rightarrow S_n' = S_n$ when $n \gg 0$.

<u>Rmk</u> (I.3.4a) $\xrightarrow{\text{generalizes}}$ this pf.

Also: cohom. $\Rightarrow$ another pf. (III, 5.2.1).