Exercise 3 (due on November 11)

Choose 4 out of 8 problems to submit.

Problem 3.1. (Examples of group schemes) We will consider several group schemes G over an affine base $S = \operatorname{Spec} k$, for k a general ring.

(1) $G = \mathbb{G}_m = \operatorname{Spec} k[x, x^{-1}]$. The comultiplication homomorphism is given by

$$m^* : k[x, x^{-1}] \longrightarrow k[y, y^{-1}] \otimes_k k[z, z^{-1}]$$

$$m^*(x) = y \otimes z.$$

Show that for every affine k-scheme $T = \operatorname{Spec} \ell$, $\mathbb{G}_m(T) = \ell^\times$. The induced group structure on $\mathbb{G}_m(T)$ is the usual multiplication. What are the inverse and identity maps of $\mathbb{G}_m$ in terms of rings?

(2) Let $n \in \mathbb{N}$. $G = \mu_n = \operatorname{Spec} k[x]/(x^n - 1)$. The comultiplication homomorphism is given by

$$m^* : k[x]/(x^n - 1) \longrightarrow k[y]/(y^n - 1) \otimes_k k[z]/(z^n - 1)$$

$$m^*(x) = y \otimes z.$$

Show that for every affine k-scheme $T = \operatorname{Spec} \ell$, $\mu_n(T) = \{a \in \ell^\times \mid a^n = 1\}$, and the induced group structure on $\mu_n(T)$ is the usual multiplication.

(3) Show that μ_n is a closed subgroup scheme of $\mathbb{G}_m$. More precisely, it is the kernel of the natural multiplication by n map in the following sense:

$$\begin{tikzcd} \mu_n \arrow[r] \arrow[d] & \mathbb{G}_m \arrow[d, "\mathrm{mult}_n"] \\ S \arrow[r, "e"] & \mathbb{G}_m \end{tikzcd}$$

where the vertical map is the multiplication by n on $\mathbb{G}_m$ and $S \to \mathbb{G}_m$ is the identity morphism. The diagram is Cartesian and μ_n is the Cartesian pullback of this diagram.

Remark: We often write

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{\mathrm{mult}_n} \mathbb{G}_m \to 1$$

as an exact sequence of group schemes.

(4) When k has characteristic $p > 0$, $G = \alpha_p = \operatorname{Spec} k[x]/(x^p)$ with comultiplication map given by

$$m^* : k[x]/(x^p) \longrightarrow k[y]/(y^p) \otimes_k k[z]/(z^p)$$

$$m^*(x) = y + z.$$

Show that this is a subgroup scheme, and show that when k is perfect, it is the kernel of the Frobenius morphism.

$$\mathrm{Frob} : \mathbb{G}_a \to \mathbb{G}_a; \quad \mathrm{Frob}^*(x) = x^p.$$

Problem 3.2. Let G be a profinite group. Show that the following two conditions are equivalent:

(1) for all finite length $\mathbb{Z}_\ell$-modules M with continuous G-action, $H^1(G, M)$ is finite;
(2) for all open compact subgroup $H \subseteq G$, $H^1(H, \mathbb{F}_\ell)$ is finite.

(Optional) Show that the same statement holds if we replace all H^1 by $H^{>0}$.

Problem 3.3. (Deformations with fixed determinant) In applications, it is often technically easier to consider deformations with a fixed determinant, which we discuss below. Let Γ, E, $\mathcal{O}$, ϖ, $\mathbb{F}$, $\bar{\rho}$ be as in the previous problem. Let $\chi : \Gamma \to \mathcal{O}^\times$ denote a lift of $\det \bar{\rho}$. Consider the functor

$$\begin{aligned} \mathrm{Def}_{\bar{\rho}}^{\square,\chi} : \mathsf{CNL}_{\mathcal{O}} &\longrightarrow \mathsf{Sets} \\ (R, \mathfrak{m}_R) &\longmapsto \left\{ \begin{array}{c} \text{continuous representations } \rho : \Gamma \to \mathrm{GL}_n(R) \\ \text{such that } \rho \bmod \mathfrak{m}_R = \bar{\rho}, \text{ and } \det \rho = \chi. \end{array} \right\} \end{aligned}$$

(1) Show that $\mathrm{Def}_{\bar{\rho}}^{\square,\chi}$ is representable by a ring $R_{\bar{\rho}}^{\square,\chi} \in \mathsf{CNL}_{\mathcal{O}}$. (One can use that $\mathrm{Def}_{\bar{\rho}}^{\square}$ is representable and realize $R_{\bar{\rho}}^{\square,\chi}$ as a quotient of $R_{\bar{\rho}}^{\square}$.
(2) Let $\mathrm{Ad}^0 \bar{\rho}$ denote the adjoint action on the "trace-zero" part of $\mathrm{Ad}(\bar{\rho})$. Namely, it is the kernel of the natural map $\mathrm{Ad}\,\bar{\rho} \cong \bar{\rho}^* \otimes \bar{\rho} \xrightarrow{\text{natural}} \mathbb{F}$; or explicitly, if we write $\bar{\rho}$ as $\Gamma \to \mathrm{GL}_n(\mathbb{F})$, then $\mathrm{Ad}\,\bar{\rho}$ is the Γ-representation on $\mathrm{M}_n(\mathbb{F})$, by $\gamma \in \Gamma$ sending $x \to \bar{\rho}(\gamma)x\bar{\rho}(x)^{-1}$, and $\mathrm{Ad}^0(\bar{\rho})$ is the subrepresentation acting on the trace zero subspace of $\mathrm{M}_n(\mathbb{F})$.
    Set $\mathbb{F}[\varepsilon] = \mathbb{F}[X]/(X^2)$. Construct and prove a natural isomorphism $\mathrm{Def}_{\bar{\rho}}^{\square,\chi} \cong H^1(\Gamma, \mathrm{Ad}^0 \bar{\rho})$

Remark: One way to see the advantage of working with $R_{\bar{\rho}}^{\square,\chi}$ is that, when $\bar{\rho}$ is absolutely irreducible, $H^0(\Gamma, \mathrm{Ad}^0 \bar{\rho}) = 0$. In other words, the representation has no automorphism. So the unframed deformation has no automorphism and we truly have a moduli space (as opposed to moduli stack.)

Problem 3.4. (Bounding relations in terms of second cohomology group) Keep the notation as in the previous problem. Let $\bar{\rho} : \Gamma \to \mathrm{GL}_n(\mathbb{F}) = \mathrm{GL}(V)$ be a continuous residual representation of Γ. Let $\chi : \Gamma \to \mathcal{O}^\times$ be a character that lifts $\det \bar{\rho}$. Let $R_{\bar{\rho}}^{\square,\chi}$ denote the universal framed deformation ring. Set $t = \dim H^1(\Gamma, \mathrm{Ad}^0 \bar{\rho})$. Then we have a surjection

$$\mathcal{O}[\![\underline{z}]\!] = \mathcal{O}[\![z_1, \dots, z_t]\!] \twoheadrightarrow R_{\bar{\rho}}^{\square,\chi}.$$

Let J denote its kernel. Let $\mathfrak{m} = (\varpi, z_1, \dots, z_t)$ denote the maximal ideals of $\mathcal{O}[\![\underline{z}]\!]$.

Let $\rho^{\mathrm{univ}} : \Gamma \to \mathrm{GL}_n(R_{\bar{\rho}}^{\square,\chi})$ denote the universal representation. We construct a map

$$\mathrm{Hom}(J/\mathfrak{m}J,\ \mathbb{F}) \to H^2(\Gamma, \mathrm{Ad}\,\bar{\rho}) \tag{3.4.1}$$

as follows: for each $\gamma \in \Gamma$, lift $\rho^{\mathrm{univ}}(\gamma) \in \mathrm{GL}_n(\mathcal{O}[\![\underline{z}]\!]/J) \cong \mathrm{GL}_n(R_{\bar{\rho}}^{\square,\chi})$ to an element $\tilde{\rho}(\gamma) \in \mathrm{GL}_n(\mathcal{O}[\![\underline{z}]\!]/\mathfrak{m}J)$. (We can choose this in a continuous way.) For $g_1, g_2 \in \Gamma$, define

$$c(g_1, g_2) := \tilde{\rho}(g_1 g_2)\tilde{\rho}(g_1)^{-1}\tilde{\rho}(g_2)^{-1} - 1 \in \mathrm{M}_n(\mathcal{O}[\![\underline{z}]\!]/\mathfrak{m}J).$$

(1) Show that the element $c(g_1, g_2)$ belongs to $\mathrm{M}_n(J/\mathfrak{m}J)$ and satisfies the cocycle condition: for $g_1, g_2, g_3 \in \Gamma$,

$$g_1 c(g_2, g_3) g_1^{-1} - c(g_1 g_2, g_3) + c(g_1, g_2 g_3) - c(g_1, g_2) = 0.$$

(Try to give a proof that "deduces" this cocycle condition, as opposed to plugging in the definition of $c(-,-)$ and check.) This shows that $c(-,-)$ defines a cocycle in $Z^2(\Gamma, J/\mathfrak{m}J \otimes \mathrm{Ad}\,\bar{\rho})$.

(2) Show that a different choice of $\tilde{\rho}(\gamma)$ defines a different cocycle $c'(-,-)$ that is differed by a coboundary. Therefore, we have a well-defined element in

$$H^2(\Gamma, J/\mathfrak{m}J \otimes \mathrm{Ad}\,\bar{\rho}).$$

(3) Given an $\mathbb{F}$-linear map $f : J/\mathfrak{m}J \to \mathbb{F}$, we obtain a class in $[c_f] \in H^2(\Gamma, \mathrm{Ad}\,\bar{\rho})$ represented by the cocycle

$$c_f(g_1, g_2) = f(c(g_1, g_2)) \in \mathrm{M}_n(\mathbb{F}).$$

This defines the needed map in (3.4.1). Given such a f, let J_f denote the kernel of $J \to J/\mathfrak{m}J \xrightarrow{f} \mathbb{F}$. Show that J_f is a ideal of $\mathcal{O}[\![\underline{z}]\!]$.

Show that the class $[c_f] = 0$ if and only if the representation $\rho^{\mathrm{univ}} : \Gamma \to \mathrm{GL}_n(R^{\square}_{\bar{\rho}})$ lifts to a representation $\Gamma \to \mathrm{GL}_n(\mathcal{O}[\![\underline{z}]\!]/J_f)$.

(4) Conclude that the natural map (3.4.1) is injective.

Optional: verify that the same argument above shows that, when $\mathrm{End}_{\mathbb{F}[\Gamma]}(\bar{\rho}) = \mathbb{F}$, if J denotes the kernel of $\mathcal{O}[\![t_1, \dots, t_{\dim H^1(\Gamma, \mathrm{Ad}\,\bar{\rho})}]\!] \to R_{\bar{\rho}}$ instead, then there is a natural injection

$$\mathrm{Hom}(J/\mathfrak{m}J,\ \mathbb{F}) \hookrightarrow H^2(\Gamma, \mathrm{ad}\bar{\rho}),$$

where $\mathfrak{m}$ is the maximal ideal of the corresponding power series ring.

Problem 3.5. (Explicitly computation of rank one deformations) Consider the cyclotomic character $\bar{\rho} = \bar{\chi}^n_{\mathrm{cycl}} : G_{\mathbb{Q}_p} \to \mathbb{F}_\ell^\times$. Compute its deformation ring (same for framed or unframed) explicitly as follows:

(1) Let $P^{(\ell)}$ denote the kernel of $I_{\mathbb{Q}_p} \twoheadrightarrow I_{\mathbb{Q}_p}/P_{\mathbb{Q}_p} \cong \widehat{\mathbb{Z}}^{(p)}(1) \twoheadrightarrow \mathbb{Z}_\ell(1)$. For $(R, \mathfrak{m}_R) \in \mathsf{CNL}_{\mathcal{O}}$ and $\rho_R : \Gamma \to R^\times$ be a lift of $\bar{\rho}$. Show that $\rho_R(P^{(\ell)}) = \{1\}$. Thus the deformation problem for $\bar{\rho}$ is the same as the deformation of representation of $\bar{\rho} : G_{\mathbb{Q}_p} \twoheadrightarrow G_{\mathbb{Q}_p}/P^{(\ell)} \to \mathbb{F}_\ell^\times$, as a representation of $G_{\mathbb{Q}_p}/P^{(\ell)}$.

(2) Compute explicit the deformation ring $R_{\bar{\chi}^n_{\mathrm{cycl}}}$. (The answer will depend on congruences $p \bmod \ell$.)

(3) Compare your result with the cohomology $H^i(G_{\mathbb{Q}_p}, \mathbb{F}_\ell)$ computed in the previous exercise, in terms of tangent spaces versus H^1 and relations versus H^2. Can you see the two tangent directions of $R_{\bar{\chi}^n_{\mathrm{cycl}}}$ versus the natural Res-Inf exact sequence on $H^1(G_{\mathbb{Q}_p}, \mathbb{F}_\ell)$?

Problem 3.6. (Galois deformation version of Leopoldt conjecture) In the famous paper by Mazur, he explained (at least when $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}(V)$ is absolutely irreducible) why the inequality

$$\text{(3.6.1)} \qquad \mathrm{Krull}\dim R_{\bar{\rho}} \geq \dim H^1(G_{\mathbb{Q}}, \mathrm{Ad}\,V) - \dim H^2(G_{\mathbb{Q}}, \mathrm{Ad}\,V).$$

achieving the equality should be considered as a generalization of the Leopoldt conjecture. We reproduce his discussion here. (For historic reasons, we use p-adic coefficients.)

Background on Leopoldt conjecture. This was briefly touched in Problem 1.2(3). Let F be a number field. For a p-adic place v of F, for $N \gg 0$, the p-adic logarithmic map $\log_v : (1 + \varpi_v^N \mathcal{O}_{F_v})^\times \to (F_v, +)$ extends naturally to a map $\log_v : \mathcal{O}_{F_v}^\times \to F_v$ by $\log_p(x) := \frac{1}{M}\log_p(x^M)$ for any M divisible by $N \cdot \# k_v$. Then consider the natural map

$$\mathcal{O}_F^\times \to \prod_{v|p} \mathcal{O}_{F_v}^\times \xrightarrow{\prod \log_v} \prod_{v|p} F_v \cong \mathbb{Q}_p^{[F:\mathbb{Q}_p]}.$$

Leopoldt conjectures that the image of $\mathcal{O}_F^\times$ in $\mathbb{Q}_p^{[F:\mathbb{Q}_p]}$ spans a rank $\mathcal{O}_F^\times = r_1+r_2-1$ dimensional subspace. (This is only known when F is an abelian extension of $\mathbb{Q}$.)

(1) Let $\Gamma = \mathbb{Z}_\ell$ or $\mathbb{Z}/\ell^n$. Compute the universal deformation ring of the trivial representation $\mathrm{tr} : \Gamma \to \mathbb{F}_\ell^\times$.

From this, infer the deformation ring of the trivial representation of a pro-ℓ-abelian group. What is the Krull dimension of such deformation ring?

Remark: In generally, for the trivial representation of a profinite group $\bar{\rho} = \mathrm{tr} : \Gamma \to \mathbb{F}_\ell^\times$ with Γ satisfying the finiteness condition in Problem 3.2, the universal deformation ring R_{tr} is precisely $\mathcal{O}[\![\Gamma^{\mathrm{ab},\ell}]\!]$, where $\Gamma^{\mathrm{ab},\ell}$ is the maximal pro-ℓ abelian quotient of Γ, and

$$\mathcal{O}[\![\Gamma^{\mathrm{ab},\ell}]\!] = \varprojlim_{\substack{H<\Gamma^{\mathrm{ab},\ell}\\ \text{open}}} \mathcal{O}[\Gamma^{\mathrm{ab},\ell}/H].$$

In fact, the computation did in (1) essentially proved this.

(2) Now consider a number field F and ℓ a prime. Let $G_{F,\ell}$ denote the Galois group of the maximal extension of F that is unramified outside ℓ. Consider the deformation of the trivial representation $\mathrm{tr} : G_{F,\ell} \to \mathbb{F}_\ell^\times$. Show that, on the one hand, the Krull dimension of R_{tr} is the $\mathbb{Z}_\ell$-rank of $G_{F,\ell}^{\mathrm{ab}}$ (which was computed in Problem 1.2(3)). On the other hand, using Euler characteristic formula, compute $\dim H^1(G_{F,\ell}, \mathbb{F}_\ell) - \dim H^2(G_{F,\ell}, \mathbb{F}_\ell)$. Explain why, in this case, the equality of (3.6.1) is equivalent to the Leopoldt conjecture.

Problem 3.7. (Schlessinger's criterion) There is a different approach to the representability using Schlessinger's criterion. We outline the key steps below. We first establishing the general Schlessinger criterion. Let E be a finite extension of $\mathbb{Q}_\ell$, with ring of integers $\mathcal{O}$, uniformizer ϖ, and residue field k. Let $\mathsf{CNL}_\mathcal{O}$ denote the category of complete noetherian local $\mathcal{O}$-algebras $(A, \mathfrak{m}_A)$ such that the structure map $\mathcal{O} \to A$ induces an isomorphism $k \cong A/\mathfrak{m}_A$. We will often write $k[\varepsilon] := k[X]/(X^2)$ for a typical element in $\mathsf{CNL}_\mathcal{O}$. In $\mathsf{CNL}_\mathcal{O}$, a morphism $A \to B$ (necessarily sends $\mathfrak{m}_A$ to $\mathfrak{m}_B$) is called *small* if it is surjective and the kernel is a principal ideal annihilated by $\mathfrak{m}_A$.

For Schlessinger criterion, one often considers the following (not very commonly used product): if $A \to C$ and $B \to C$ are two morphisms in $\mathsf{CNL}_\mathcal{O}$, then

$$A \times_C B := \{(a, b) \in (A, B) \mid a \text{ and } b \text{ have the same image in } C\}$$

is an object in $\mathsf{CNL}_\mathcal{O}$. Let $F : \mathsf{CNL}_\mathcal{O} \to \mathsf{Sets}$ be a functor, we then have a natural map

$$F(A \times_C B) \to F(A) \times_{F(C)} F(B). \tag{3.7.1}$$

(1) Prove that if F is representable, then $F(k)$ is a single point, $F(k[\varepsilon])$ is a finite dimensional vector space, and for every pair of morphisms $A \to C$ and $B \to C$ in $\mathsf{CNL}_\mathcal{O}$, (3.7.1) is a bijection.

The Schlessinger's criterion is somehow the converse of (1): suppose that $F : \mathsf{CNL}_\mathcal{O} \to \mathsf{Sets}$ is a functor satisfying the following conditions

(H1) (3.7.1) is a surjection whenever $B \to C$ is small;
(H2) (3.7.1) is a bijection when $C = k$ and $B = k[\varepsilon]$;
(H3) $F(k[\varepsilon])$ is finite dimensional;
(H4) (3.7.1) is a bijection when $A \to C$ and $B \to C$ are equal and small;

then F is representable.

(2) Show that for a representation $\bar{\rho} : \Gamma \to \mathrm{GL}_n(\mathbb{F})$ with Γ satisfying the finiteness condition of Problem 3.2 and $\mathrm{End}_{\mathbb{F}[\Gamma]}(\bar{\rho}) = \mathbb{F}$, the (unframed) deformation functor $\mathrm{Def}_{\bar{\rho}}$ satisfies the Schlessinger criterion. (Hint: it is useful to note that $\mathrm{GL}_n(A \times_C B) = \mathrm{GL}_n(A) \times_{\mathrm{GL}_n(C)} \mathrm{GL}_n(B)$.)

Problem 3.8 (Level raising/lowering deformation)**.** Let K be a finite extension of $\mathbb{Q}_p$ and and let $\ell \neq p$ be another prime. Suppose that $p \not\equiv \pm 1 \pmod{\ell}$. Fix a geometric Frobenius element ϕ and a tame generator τ in $G_K/P_K \cong \widehat{\mathbb{Z}}^{(p)}(1) \rtimes \widehat{\mathbb{Z}}$ (the quotient of the Galois group by the wild inertia subgroup). Consider the following residual representation

$$\begin{aligned} \bar{\rho} : G_K/P_K &\longrightarrow \mathrm{GL}_2(\mathbb{F}_\ell) \\ \phi &\longmapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \\ \tau &\longmapsto \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \end{aligned}$$

Check that $\bar{\rho}(\phi)^{-1}\bar{\rho}(\tau)\bar{\rho}(\phi) = \bar{\rho}(\tau)^p$.

In two cases $a = 0$ and $a \neq 0$ (or essentially equivalently $a = 1$), compute the framed deformation ring $R^{\square}_{\bar{\rho}}$. (Hint: via the action of ϕ, any lift $\rho : G_K \to \mathrm{GL}_2(A)$ admits two eigenvectors v_1 and v_2 with eigenvalues $\lambda_1, \lambda_2 \in A$ such that $\lambda_1 \equiv 1 \pmod{\ell}$ and $\lambda_2 \equiv p \pmod{\ell}$.)

Remark: When $a = 0$, this is the deformation that is crucially related to the level raising/lowering for modular forms. In this case, there are representations ρ_{unr} (unramified) and ρ_{st} (where τ acts nontrivially or Steinberg) that lift ρ.