

Serre Duality
Wenbin Dai

[§1] Preliminaries on Ext & Ext

State X ringed sp. $\mathcal{F} \in \text{Mod}_{\mathcal{O}_X}$.

$\Rightarrow \text{Hom}(\mathcal{F}, -)$, $\text{Hom}(-, \mathcal{F})$ left-exact.
 \uparrow \uparrow
 $\text{im} \in \text{Mod}_{\mathcal{F}(X, \mathcal{O}_X)}$ $\text{im} \in \text{Mod}_{\mathcal{O}_X}$, as sh.

$\text{Mod}_{\mathcal{O}_X}$ w/ enough inj. obj.

\Rightarrow right derived funs $\begin{cases} \text{Ext}^i (\text{glo ext'n}) \\ \text{Ext}^i (\text{sh ext'n}) \end{cases}$

compute via I^\bullet

Prop \mathcal{E} loc. free $\Rightarrow \text{Ext}^i(\mathcal{F} \otimes \mathcal{E}^\vee, \mathcal{G}) \cong \text{Ext}^i(\mathcal{F}, \mathcal{G} \otimes \mathcal{E})$

$$\text{Ext}^i(\mathcal{F}, \mathcal{G}) \otimes \mathcal{E}.$$

$$\& \text{Ext}^i(\mathcal{F} \otimes \mathcal{E}^\vee, \mathcal{G}) \cong \text{Ext}^i(\mathcal{F}, \mathcal{G} \otimes \mathcal{E})$$

can tensor as sh

Prop $U \subset X$ open, $\text{Ext}^i(\mathcal{F}, \mathcal{G})(U) = \text{Ext}^i(\mathcal{F}|_U, \mathcal{G}|_U)$

i.e. Ext^i is local.

Caution: \mathcal{E} Ext, \mathcal{E} genuinely diff.

e.g. $\text{Ext}^i(\mathcal{O}_X, \mathcal{F}) = H^i(X, \mathcal{F})$
 whereas $\text{Ext}^i(\mathcal{O}_X, \mathcal{F}) = \begin{cases} \mathcal{F}, & i=0, \\ 0, & \text{otherwise} \end{cases}$

Thm (Loc-glo spec'l seq'ce).

$$E_2^{P, \mathcal{F}} = \boxed{H^i(X, \underbrace{\text{Ext}^j(\mathcal{F}, \mathcal{G})}_{\text{loc.}}) \Rightarrow \text{Ext}^{i+j}(\mathcal{F}, \mathcal{G})}_{\text{glo.}}$$

Grothendieck spec'l seq'ce.

i.e. $\text{Hom} = \Gamma \circ \text{Hom.} \in \text{AbFun.}$

$$\text{e.g. } \text{Hom}_{\text{coh}}(\mathcal{F}, \mathcal{G}) = \Gamma(X, \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})).$$

In Abgp: enough proj.s $\Rightarrow \text{Ext} \neq 0$. (\checkmark)

In Sh: (X). not enough (e.g. \mathcal{E} free & not proj.).

But loc. free res $\mathcal{E} \Rightarrow \text{Ext} = 0$.

by acyc- \mathcal{O}_X (\Rightarrow acyc of \mathcal{E}) & spec'l seq'ce

(Cor X loc. nf, $\mathcal{F}, \mathcal{G} \in \text{Coh}_X \Rightarrow \text{Ext}^i(\mathcal{F}, \mathcal{G}) \in \text{Coh}_X$)

(X) for Qcoh, even for $i=0$.

§2 Thm (Sene duality). X CM + proj., $\text{pdim}_k X = n$

Yoneda $\Rightarrow \exists! \omega_X \in \text{Coh}_X$ s.t. (pure) $\omega_X \perp \mathcal{F}$

(a) \mathcal{F} vec. bundle $\rightsquigarrow H^2(X, \mathcal{F}) \times H^{n-2}(X, \mathcal{F}^\vee \otimes \omega_X) \rightarrow k$
perfect.

(b) $\mathcal{F} \in \text{Coh}_X$, $\text{Ext}^2(\mathcal{F}, \omega_X) \times H^{n-2}(X, \mathcal{F}) \rightarrow k$
 $\downarrow i=0$ perfect.

(b') $\mathcal{F} \in \text{Coh}_X$, $\text{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \rightarrow k$ perfect
(cM not needed).

(c) When $X \cong \mathbb{P}^n$, $\omega_X \cong k_X := X^n \mathcal{O}_X$.

Note (1): to construct $X \rightarrow \mathbb{P}^n$ flat.

\uparrow by Mireljeff's theorem.

counter-e.g. $X = \mathbb{P}^2 \coprod \mathbb{P}^2 \subseteq \mathbb{P}^4$. Serre fails.

Fact (b) $\Rightarrow X \in cM$.

$$(b') \rightsquigarrow t = \text{tr} : H^n(X, \omega_X) \rightarrow k \in H^n(X, \omega_X)^\vee$$

\curvearrowleft

(2) $\in H^n(\text{Hom}(\omega_X, \omega_X))$
by $\mathcal{F} = \omega_X$.

$$\rightsquigarrow \text{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \xrightarrow{\quad \quad \quad} k .$$

\hookrightarrow

$H^n(X, \omega_X) \xrightarrow{t}$

Fact $H^2(X, \mathcal{F}) \times H^{n-2}(X, \mathcal{F}^\vee \otimes \omega_X) \rightarrow H^n(X, \omega_X) \xrightarrow{\text{tr}} k$ (a)

$\text{Ext}^2(\mathcal{F}, \omega_X) \times H^{n-2}(X, \mathcal{F}) \rightarrow H^n(X, \omega_X) \xrightarrow{\text{tr}} k$ (b)

Strategy of pf $\oplus X = \mathbb{P}^n$ by cat.

② $\pi: X \rightarrow \mathbb{P}^n$ fin. flat \Rightarrow (b).

$$\omega_X \xrightarrow{\pi^* \omega_{\mathbb{P}^n}}$$

③ $\tilde{\iota}: X \hookrightarrow \mathbb{P}^N$ reg. emb. \Rightarrow (c)

$$\omega_X \simeq \mathcal{O}_X \simeq \mathbb{C}$$

⊕ Prop (b) \Rightarrow (a)

$$\text{pf. } H^2(X, \mathcal{F})^\vee \cong \text{Ext}^{n+1}(\mathcal{F}, \omega_X) \quad (\text{b})$$

$$\cong \text{Ext}^{n+1}(\mathcal{O}_X, \mathcal{F}^\vee \otimes \omega_X)$$

$$\cong H^{n+1}(X, \mathcal{F}^\vee \otimes \omega_X).$$

[§3] Serre duality holds for proj. sp.

Goal (b) w/ $\omega_{\mathbb{P}^n} = \mathbb{C}(-n-1) \Rightarrow$ (c). \Rightarrow (a).

Lem $X = \mathbb{P}^n$, $\omega_X = \mathbb{C}(-n-1) \Rightarrow$ (b').

pf. \exists canonical $\text{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \rightarrow H^n(X, \omega_X)$.
 choose $\mathcal{F} \cong \mathcal{O}(m)$ line bun.

$\Rightarrow \text{Hom}(\mathcal{O}(m), \mathbb{C}(-n-1)) \times H^n(\mathcal{O}(m)) \rightarrow H^n(\mathbb{C}(-n-1)) \cong \mathbb{C}$.

$$H^0(\mathbb{C}(-m-n-1)) \quad \left\langle \prod_{i=0}^{m+n} x_i^{n_i}, \sum n_i = m \right\rangle^{\text{exp} < 0}$$

$$< \prod_{i=0}^n x_i^{n_i}, \sum n_i = -m-n-1 \rangle^{\text{exp} > 0}$$

$$\langle \dots, \dots = -n-1 \rangle^{\text{exp} < 0}$$

\Rightarrow pairing map = multi. \Rightarrow perfect.

\Rightarrow (b) w.r.t. $\mathcal{F} = \mathcal{E}^{\otimes m}$.

② Arbitrary \mathcal{F} .

$$\mapsto 0 \rightarrow \mathcal{G} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$$

ker " \oplus (mi) sum of line bds.

$$\begin{array}{ccccccc} & & & & \text{---} & & \\ \Rightarrow & 0 \rightarrow \text{Hom}(\mathcal{F}, \omega_X) & \rightarrow & \text{Hom}(\mathcal{E}, \omega_X) & \rightarrow & \text{Hom}(\mathcal{G}, \omega_X) & \\ \text{left-exact} & & & \downarrow \text{inj.} & \leftarrow \text{---} & \downarrow \cong & \downarrow \\ & 0 \rightarrow H^n(X, \mathcal{F})^\vee & \rightarrow & H^n(X, \mathcal{E})^\vee & \rightarrow & H^n(X, \mathcal{G})^\vee & \end{array}$$

note $\dim X = \dim \mathbb{P}^n = n \Rightarrow H^{n+1}(X, \mathcal{G}) = 0$.

nicer! $0 \rightarrow \text{Hom}(\mathcal{G}, \omega_X) \rightarrow H^n(X, \mathcal{G})^\vee$

By diag. chasing $\mapsto \text{Hom}(\mathcal{F}, \omega_X) \cong H^n(X, \mathcal{F})^\vee$

Prop $X = \mathbb{P}^n$, $\omega_X = \mathcal{O}(-n-1) \Rightarrow$ (b)

Pf. To show $\text{Ext}^i(-, \omega_X) \otimes H^{n-i}(X, -)^\vee$
univ. \mathcal{S} -functor.

(b'): $i=0 \rightsquigarrow$ isom.

• left-exact at $i=0$ ✓

• long exact seq'ce ✓

• coefficientability ①

check $\forall \mathcal{F} \in \text{Coh}_X$,

$$(\mathcal{O}(-m))^{\otimes N} \rightarrow \mathcal{F} \rightarrow 0, \quad m > 0.$$

$$\text{indeed: } H^{n-i}(\mathbb{P}^n, (\mathcal{O}(-m))^{\otimes N}) = 0, \quad \forall i > 0$$

$$\text{Ext}^i(\mathcal{O}(-m)^{\otimes N}, \omega) = H^i(\mathbb{P}^n, (\mathcal{O}(-m-n))^N) = 0 \\ \forall i > 0. \quad \square$$

[8] The dualizing sheaf exists in general

lem X equi-dim'l proj. k -sch.

$\Rightarrow \exists \pi: X \rightarrow \mathbb{P}^n$ fin.

$\mathcal{Q} X \circledcirc \mathcal{M} \Rightarrow \pi$ flat.

pf. $X \hookrightarrow \mathbb{P}^n$
 H codim 1, linear sub, $H \cap X = \emptyset$.

$\Rightarrow \mathbb{P}^n \setminus H \xrightarrow{\text{pr}} X \Rightarrow \pi$.

$X \subset M \Rightarrow$ apply miracle flatness. \square

! only works for $\#k = \infty$.

but can be modified to $\#k < \infty$.

Def $\pi: X \rightarrow Y$ fin. $\mathcal{Y} \in \text{Qcoh}_{\mathcal{O}_Y}$ $\xrightarrow{\text{?}} \pi^! \mathcal{Y} \in \text{Qcoh}_{\mathcal{O}_X}$.

Step 1 $Y = \text{Spec } \mathcal{B}$, $X = \text{Spec } \mathcal{A}$, $\mathcal{Y} = \tilde{N}$, $N \in \text{Mod}_{\mathcal{B}}$.
 $\Rightarrow M = \text{Hom}_{\mathcal{B}}(\mathcal{A}, N) \in \text{Mod}_{\mathcal{B}}$. $\Rightarrow M$.

$$\begin{array}{c} b \in \mathcal{B}, M_b \oplus \\ \downarrow \\ \tilde{M} \in \text{Qcoh } X. \end{array}$$

Step 2 Globally: $\boxed{\pi^! \mathcal{Y} := \text{Hom}_{\mathcal{O}_Y}(\pi_X^* \mathcal{O}_X, \mathcal{Y})}$. \mathcal{Y} coh.

Fact $\pi^! \mathcal{Y} \in \text{Coh}_{\mathcal{O}_Y}$ or $\pi^! \mathcal{Y} \in \text{Coh}_{\mathcal{O}_X}$
when f fin. + $\mathcal{Y} \in \text{Coh}$.

$$\begin{array}{ccc} \text{Frob reciprocity} & \text{Hom}_X(M, \text{Hom}_{\mathcal{B}}(A, N)) & \cong \text{Hom}_{\mathcal{B}}(M_{\mathcal{B}}, N) \\ & N \rightarrow \mathcal{B} & \uparrow \\ & & M \rightarrow A. \end{array}$$

$$\stackrel{\text{glo.}}{\Rightarrow} \pi_X^* \text{Hom}_{\mathcal{O}_X}(F, \pi^! \mathcal{Y}) \cong \text{Hom}_{\mathcal{O}_Y}(\pi_X^* F, \mathcal{Y}).$$

$$\Rightarrow \underset{||}{\text{Hom}_{\mathcal{O}_X}(F, \pi^! \mathcal{Y})} \cong \underset{||}{\text{Hom}_{\mathcal{O}_Y}(\pi_X^* F, \mathcal{Y})}.$$

$$\Gamma(X, \dots) \qquad \qquad \Gamma(Y, \dots).$$

lem $\pi: X \rightarrow Y = \mathbb{P}^n$, $\omega_X = \pi^! \omega_Y$. $\Rightarrow (b)'.$

$$\begin{array}{ccc} \text{pf. Now } \text{Hom}_X(F, \pi^! \omega_Y) \times H^n(X, F) & & \\ \downarrow & \text{if} & \uparrow F \in \text{Coh}_X. \\ & & \end{array}$$

$$\text{Hom}_Y(\pi_X^* F, \omega_Y) \times H^n(Y, \pi_X^* F)$$

$$\begin{array}{ccccc}
 \rightarrow H_{\text{om}}(F, \pi^! \omega_Y) \times H^n(X, F) & \xrightarrow{\text{perfect}} & H^n(X, \pi^! \omega_Y) & \xrightarrow{f_X} & \\
 \cong \downarrow & & \uparrow H^n(Y, \pi_* \pi^! \omega_Y) & \downarrow & \\
 H_{\text{om}}(\pi_* F, \omega_Y) \times H^n(Y, \pi_* F) & \xrightarrow{\text{perfect}} & H^n(Y, \omega_Y) & \xrightarrow{f_Y} & \square
 \end{array}$$

Rmk "π flat" is not used $\rightarrow \boxed{(b) \nRightarrow X \text{ CM.}}$

LEM $\pi: X \rightarrow Y = \mathbb{P}^n$, $X \text{ CM}$ & $\omega_X = \pi^! \omega_Y \Rightarrow (b)$.

pf. check $\text{Ext}_X^i(-, \pi^! \omega_Y)$, $H^{n-i}(X, -)^\vee$ well-behaved.

(\Rightarrow univ. δ -fun's).

Again: $(\mathcal{O}_{X(-m)})^{\otimes N} \xrightarrow{\text{pr}} F \rightarrow 0$, $m \gg 0$.
 Coh.

$X, Y \xrightarrow[\text{emb}]{\text{pr}} \mathbb{P}^N \Rightarrow \pi^* \mathcal{O}_{Y(1)} \cong \mathcal{O}_{X(1)}$.

Now $\forall i > 0$, $\text{Ext}_X^i(\mathcal{O}_{X(-m)}, \pi^! \omega_Y) \cong \text{Ext}_X^i(\mathcal{O}_X, \pi^! \omega_Y(m)) \stackrel{\text{if } m \gg 0}{=} 0$
 Serre vanishing $\rightarrow H^i(X, \pi^! \omega_Y(m))$

Also $H^{n-i}(X, \mathcal{O}_{X(-m)})^\vee \cong H^{n-i}(Y, \pi_*(\mathcal{O}_X \otimes \underbrace{\pi^* \mathcal{O}_Y(-m)}_{\pi^! \mathcal{O}_Y(-m)}))^\vee$

(a) of $Y = \mathbb{P}^n$ i.e. $\cong H^{n-i}(Y, \pi_*(\mathcal{O}_X \otimes \mathcal{O}_Y(-m)))^\vee$
 $H^i(Y, F)^\vee \cong H^{n-i}(Y, F^\vee \otimes \omega_Y) \rightarrow \cong H^i(Y, (\pi_*(\mathcal{O}_X)^\vee \otimes \mathcal{O}_Y(m)) \otimes \omega_Y)$.

Serre $\rightarrow = 0 \quad m \gg 0$.

□

Rmk (a): $\pi_* \mathcal{O}_X$ vec. bud. $\Leftarrow \pi$ fin. flat $\Leftarrow \text{CM}$.

[§5] The dualizing sheaf is the canonical bundle
for sm. var.

X sm. ($\Rightarrow \text{CM}$) proj. var. $\dim_k X = n$

$\Rightarrow X \rightarrow \mathbb{P}^N = Y$ reg. emb. w/ $\text{codim } N - n = r$.

[Step 1] Goal $\omega_X \cong \mathop{\mathrm{Ext}}\nolimits_Y^r(\pi_* \mathcal{O}_X, \omega_Y)$.
 $\mathop{\mathrm{Coh}}\nolimits_{\mathcal{O}_X}^r$.

Subtlety $X \xrightarrow{\pi} Y$, $y \in \mathcal{Q}_{\text{coh}}(Y)$.
 $\hookrightarrow \text{Hom}_Y(\pi_* \mathcal{O}_X, y) \in \mathcal{Q}_{\text{coh}}(Y)$ w/ \mathcal{O}_X -str.
 $\hookrightarrow \mathcal{O}_X \text{Hom}_Y(\pi_* \mathcal{O}_X, y)$ need.

b/c if $y \notin \mathcal{Q}_{\text{coh}}$, inj. sol. may not be quan.

• $X \rightarrow Y$ closed emb $\Rightarrow \exists \mathcal{O}_X$ -str. $\forall y$.

top reason: $\forall U \subseteq_{\text{open}} X$, $U = \pi^{-1}(V)$, $V \subseteq Y$ open

\Rightarrow data to define \mathcal{O}_X -action,

$\hookrightarrow \pi^! y$ ok. (recover $(\pi_*, \pi^!)$).

$\& \mathcal{O}_X \mathop{\mathrm{Ext}}\nolimits_Y^r(\pi_* \mathcal{O}_X, \omega_Y)$ makes sense.

Step 2 Prop $\text{Hom}_X(\mathcal{F}, \text{Ext}_Y^r(\pi_*\mathcal{O}_X, \omega_Y)) \cong \text{Ext}_Y^r(\pi_*\mathcal{F}, \omega_Y)$.

Goal \Leftarrow b/c By (b), (b'): $Y = P^N = P^{n+r}$

$$\begin{aligned} \text{Hom}_X(\mathcal{F}, \text{Ext}_Y^r(\pi_*\mathcal{O}_X, \omega_Y)) &\cong \text{Ext}_Y^r(\pi_*\mathcal{F}, \omega_Y) \\ &\stackrel{(b)}{\cong} H^n(Y, \pi_*\mathcal{F})^\vee \cong H^n(X, \mathcal{F})^\vee \\ &\stackrel{(b')}{\cong} \text{Hom}_X(\mathcal{F}, \omega_X). \end{aligned}$$

(b') $\Rightarrow \omega_X \cong \text{Ext}_Y^r(\pi_*\mathcal{O}_X, \omega_Y)$

Recall (b) $\text{Ext}_X^i(\mathcal{F}, \omega_X) \times H^{n-i}(X, \mathcal{F}) \rightarrow k$.

(b') $\text{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \rightarrow k$.

The pf of Prop: first need

lem $\forall i < r, \text{Ext}_Y^i(\pi_*\mathcal{O}_X, \omega_Y) = 0$.

pf. $\text{Ext}_Y^i(\pi_*\mathcal{O}_X, \omega_Y) \in \text{Coh}_Y$

To show $m \gg 0$:

$$0 = H^0(Y, \text{Ext}_Y^i(\pi_*\mathcal{O}_X, \omega_Y)(m)).$$

$$H^0(Y, \text{Ext}_Y^i(\pi_*\mathcal{O}_X, \omega_Y(m)))$$

Serre vanishing: $H^j(Y, \dots) = 0, \forall j > 0$.

\Rightarrow loc-glo. spec'l seq'ce legen. at $E_2^{p,q} (m \gg 0)$.

$$\begin{array}{ccc}
\Rightarrow H^i(Y, \text{Ext}_Y^i(\pi_*\mathcal{O}_X, \omega_Y(m))) & H^j(X, \text{Ext}_X^j(\mathcal{F}, g)) = E_2^{i,j} \\
& \xleftarrow{\text{if}} & \downarrow \\
\text{Ext}_Y^i(\pi_*\mathcal{O}_X, \omega_Y(m)) . & & \text{Ext}_X^{i+j}(\mathcal{F}, g) . \\
& \text{if} & \\
\text{Ext}_Y^i(\pi_*\mathcal{O}_X(-m), \omega_Y) \stackrel{(b)}{\cong} H^{N-i}(Y, \pi_*\mathcal{O}_X(-m))^{\vee} & & \\
\uparrow & & \\
\text{Ext}_X^i(\mathcal{F}, \omega_X) \cong H^{n-i}(X, \mathcal{F})^{\vee} & & \\
i < r \Rightarrow N-i > N-r = n \Rightarrow H^{N-i}(\dots) = 0 . & & \square \\
\uparrow & & \\
\dim \text{supp } \pi_*\mathcal{O}_X(-m) . & &
\end{array}$$

Pf of Prop Grothendieck spec'l seq'ce
on $\text{Hom}_Y(\pi_*\mathcal{O}_X, -)$ & $\text{Hom}_X(\mathcal{F}, -)$.

\mathcal{G} inj. $\Rightarrow \text{Hom}_Y(\pi_*\mathcal{O}_X, \mathcal{G}) = \pi^*\mathcal{G}$ acyc. for $\text{Hom}_X(\mathcal{F}, -)$.

$$\Rightarrow \text{Ext}_X^i(\mathcal{F}, \text{Ext}_Y^j(\pi_*\mathcal{O}_X, \omega_Y)) = 0, \forall j < r.$$

$\Rightarrow E_2^{i,j}$ degen.

$$\Rightarrow \text{Ext}_Y^r(\pi_*\mathcal{F}, \omega_Y) \cong \text{Hom}_X(\mathcal{F}, \text{Ext}_Y^i(\pi_*\mathcal{O}_X, \omega_Y)).$$

Step 3 Lem $\text{Ext}_Y^r(\pi_*\mathcal{O}_X, \omega_Y) \cong \det \mathcal{N}_{X/Y} \otimes_{\mathcal{O}_X} \omega_Y|_X$.
natural.

$\Rightarrow (c) \vee b/c K_X$ satisfies same adjunction.

note up to now, haven't used X sm.

pf: (GTM52, Thm III.7.11).

Ext : compute locally.

use Koszul cplx: $y \xrightarrow{\text{cut out}} x$.

by $\underbrace{x_1, \dots, x_r}$

reg. seq'ce

\rightsquigarrow free resol. of $\pi_* \mathcal{O}_X$.

$\mathrm{Hom}(-, \omega_Y) \mapsto \omega_Y / (x_1, \dots, x_r) \omega_Y = \omega_Y|_X$

(looking (oe.) has dim r cohom.
& 0 otherwise.)

pink A feature of the whole proof:

I generally push the hardest ingredients to the end.