Geometrization of the local Langlands correspondence

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Lecture 1: Introduction

Setup E nonarch local field
- either $E \cong \mathbb{F}_q((t))$
- or E finite extension of $\mathbb{Q}_p$ $(E = \mathbb{Q}_p)$.

Notation $\mathbb{F}_q$ residue field, $\pi \in \mathcal{O}_E$ uniformizer.
G/E reductive grp,
e.g. $G = GL_n, Sp_{2n}, SO_{2n}, U_n, \dots, E_8, G_2$
but $G \neq B \subset GL_n$.

$\rightsquigarrow$ Interested in rep theory of locally profinite grp $G(E)$.

Recall **Def'n** Let Γ locally profin grp. L field.
A smooth rep of Γ over L is an L-v.s.
+ map $\Gamma \to GL(V)$
s.t. $\forall v \in V$, $\mathrm{Stab}(v) \subseteq \Gamma$ is an open subgrp.

Examples (1) Let $K \subseteq \Gamma$ open cpt subgrp.
$K \twoheadrightarrow \bar{K}$ finite quotient,
$\rho: \bar{K} \to GL_n(L) = GL(V_0)$ rep of $\bar{K}$

$$\rightsquigarrow \quad \underset{\Gamma}{\circlearrowright}\ \text{c-}\mathrm{Ind}_K^\Gamma \rho = \left\{ f: \Gamma \to V_0 \;\middle|\; \begin{array}{l} f \text{ has compact support} \\ f(\gamma k) = f(\gamma)k,\ \forall \gamma \in \Gamma,\ k \in K \end{array} \right\}$$

This is a sm rep.

These rep'ns form compact projective generators
(at least if char $L=0$, or $\#K \in L^\times$).

e.g.
$$\begin{array}{c} K = GL_n(\mathcal{O}_E) \subseteq \Gamma = GL_n(E) \\ \downarrow \\ \bar{K} = GL_n(\mathbb{F}_q) \xrightarrow{\rho} GL(V_0) \end{array}$$

ρ supercusp rep of $GL_n(\mathbb{F}_q)$.

$\rightsquigarrow$ (up to center) $c\text{-}\mathrm{Ind}_K^\Gamma \rho$ is "irred supercusp repr".

(2) If $P \subseteq G$ parabolic with Levi L,
(V_0, φ_L) smooth rep of $L(E)$, then
$$\mathrm{Ind}_{P(E)}^{G(E)} V_0 = \left\{ f: G(E) \to V_0 \;\middle|\; \begin{array}{l} f(\gamma p) = f(\gamma)\cdot \bar{p}, \\ \forall \gamma \in G(E),\ p \in P(E) \end{array} \right\}$$
$\circlearrowleft$ $G(E)$ sm repr. $\qquad p \mapsto \bar{p} \in L(E)$

In some vague sense,
all irred sm reps of $G(E)$ are built via parabolic induction from supercuspidal reps of Levi subgrps.

(3) If $\mathbb{E}$ global field, $\mathbb{G}$ red grp $/\mathbb{E}$,
s.t. (G, E) arises as localization of $(\mathbb{G}, \mathbb{E})$,
then space of autom forms
$$\mathcal{A}(\mathbb{G}(\mathbb{E}) \backslash \mathbb{G}(\mathbb{A}_\mathbb{E}), \mathbb{C}) \circlearrowleft \mathbb{G}(\mathbb{A}_\mathbb{E}) \supseteq G(E)$$
is a smooth rep of $G(E)$.

In some sense, study of category of smooth rep'ns of $G(E)$ is local analogue of study of space of autom forms.

Conjecture (Langlands) $L=\mathbb{C}$.

(Assume for simplicity that G is split.)

There is a natural map

$$\mathrm{Irr}(G(E))/\cong \longrightarrow \mathrm{Hom}(W_E, \hat{G}(\mathbb{C}))/\hat{G}(\mathbb{C})\text{-conj}$$

$$\pi \longmapsto \varphi_\pi$$

"L-parameters"

where
- $\hat{G}$ = Langlands dual grp (see below)
- W_E = Weil grp of E

$$\begin{tikzcd} W_E \arrow[d, phantom, "\text{dense}\ \cap"] \arrow[r, two heads] & \mathbb{Z} \arrow[d, phantom, "\cap"] \\ \mathrm{Gal}(\bar{E}/E) \arrow[r] & \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) = \hat{\mathbb{Z}} \ni (\mathrm{Frob}: x \mapsto x^q). \end{tikzcd}$$

satisfying certain compatibilities

+ description of fibres of $\pi \longmapsto \varphi_\pi$.

Examples $\quad \widehat{GL_n} = GL_n$, $\quad \widehat{SL_n} = PGL_n$

$\widehat{Sp_{2n}} = SO_{2n+1}$, $\quad \widehat{SO_{2n}} = SO_{2n}$.

Examples of local Langlands

(1) $G = \mathbb{G}_m$.

$\rightsquigarrow$ $G(E) = E^\times$ abelian

$\mathrm{Irr}(E^\times) = \{\chi : E^\times \to \mathbb{C}^\times\}$ characters

$\hat{G} = \mathbb{G}_m$

$\rightsquigarrow$ $\mathrm{Hom}(W_E, \hat{G}(\mathbb{C})) = \mathrm{Hom}(W_E, \mathbb{C}^\times) = \mathrm{Hom}(W_E^{ab}, \mathbb{C}^\times)$.

Local class field theory:

$$W_E^{ab} \cong E^\times \cong \mathcal{O}_E^\times \times \mathbb{Z} \quad (\text{c.f. } \mathrm{Gal}(\bar{E}/E)^{ab} \cong \widehat{E^\times} \cong (\mathcal{O}_E^\times \times \hat{\mathbb{Z}}).$$

$\rightsquigarrow$ get desired bijection.

(2) $G = GL_n$, then $\hat{G} = GL_n$.

Thm (Laumon - Rapoport - Stuhler '90, $E = \mathbb{F}_q((t))$

Harris - Taylor; Henniart '01, $E/\mathbb{Q}_p$).

Let $L = \mathbb{C}$.

$$\begin{array}{ccccc}
\left\{\begin{array}{c}\text{supercusp irred repr} \\ \text{of } GL_n(E)\end{array}\right\}/\cong & \overset{\cong}{\longleftrightarrow} & \left\{\begin{array}{c}\text{irred } n\text{-dim'l repr} \\ W_E \to GL_n(\mathbb{C})\end{array}\right\}/\cong \\
\cap| & BZ \Downarrow & \cap| \\
\mathrm{Irr}(G(E))/\cong & \dashleftarrow\dashrightarrow & \mathrm{Hom}(W_E, \hat{G}(\mathbb{C}))/\cong
\end{array}$$

where BZ = Bernstein-Zelevinsky (70's).

<u>Example for GL_n</u> E'/E ext'n of deg n.

$\chi' : W_{E'} \to W_{E'}^{ab} \cong E'^\times \longrightarrow \mathbb{C}^\times$ character

"generic"

$\rightsquigarrow \varphi = \mathrm{Ind}_{W_{E'}}^{W_E} \chi'$ irred n-dim'l repr of W_E

$\Downarrow$

$\pi_\varphi = \pi_{\chi'}$ supercusp rep of $GL_n(E)$

"automorphic induction".

<u>Recent improvement</u> (Hellmann, Zhu, Ben-Zvi - Chen - Helm - Nadler)

Conjecturally describe whole category $\mathrm{Rep}(G(E))$

in terms of $\underbrace{\mathrm{Hom}(W_E, \hat{G}) / \hat{G}}_{\text{Artin stack.}}$

<u>Question</u> (1) How does W_E relate to $\mathrm{Rep}(G(E))$?

(2) Where does $\hat{G}$ come from ?

About $\hat{G}$: split grps over any field are classified by root data

$$(X, \Phi, X^*, \Phi^\vee) \quad \text{for } G$$

char lattice — X; roots — Φ; cochar lattice — X^*; coroots — $\Phi^\vee$

Also swap these: $(X^*, \Phi^\vee, X, \Phi)$ for $\hat{G}$

(This is again a root datum.)

<u>Goal of course</u>

- Give construction of map

$$\underset{\text{irrep}}{\pi} \longmapsto \underset{\text{L-param}}{\varphi_\pi}$$

that works uniformly for any reductive grp G and is <u>purely local</u>.

- In doing so, "explain" where W_E and $\hat{G}$ "come from".
- Formulate a form of local Langlands corresp. as equiv of cats, and (essentially) construct a functor in one direction.
- Extend everything from char 0 coefficients to Λ, $p \in \Lambda^\times$.

<u>Idea</u> Develop the geom Langlands program
on the Fargues - Fontaine curve
using geometry of perfectoid spaces / diamonds.

<u>Basic references</u>

(1) Berkeley lectures on p-adic geometry

(2) Lecture notes for a Montreal "Barbados" workshop.

<u>Big picture</u>

Contemplation of "space Spec E".

$$\begin{array}{ccc} & \mathrm{Spec}\,E & \\ \text{coherent} \swarrow & & \searrow \text{étale} \\ \text{vec bdls, } E\text{-vec spaces} & & \pi_1^{et}(\mathrm{Spec}\,E) = \mathrm{Gal}(\bar{E}/E) \end{array}$$

G-torsors

$$\underbrace{[*/G]}_{}(\mathrm{Spec}\,E) = \coprod_{\alpha\in H^1(E,G)} [*/G_\alpha(E)] \supseteq [*/G(E)]$$

G_α inner form of G.

quotient stack classifying all G-torsors

G-torsors up to isom

$\rightsquigarrow$ $\mathrm{Rep}(G(E)) = \mathrm{Shv}([*/G(E)])$
$\qquad\qquad\qquad\qquad \cap$
$\qquad\qquad\qquad \mathrm{Shv}([*/G](\mathrm{Spec}\,E))$

This is inspired by Bernstein, Vogan.

<u>But</u> Want to change $\mathrm{Gal}(\bar{E}/E)$ to W_E.

For scheme $X/\mathbb{F}_q$ replace X by $X_{\overline{\mathbb{F}}_q} \circlearrowleft \mathrm{Frob}$

$$\rightsquigarrow \quad \pi_1(X_{\overline{\mathbb{F}}_q}/\mathrm{Frob}) = \pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q}) \rtimes \mathbb{Z}$$
$$\cap \qquad\qquad \cap$$
$$\pi_1(X) = \pi_1^{\text{ét}}(X_{\overline{\mathbb{F}}_q}) \rtimes \hat{\mathbb{Z}}.$$

So this suggests replacing

Spec $\mathbb{F}_q((t))$ by Spec $\overline{\mathbb{F}}_q((t))$ / Frob

Spec E by Spec $\breve{E}$ / Frob

completion of max unram ext'n /E.

Then

$$\begin{tikzcd} & \operatorname{Spec} \breve{E}/\mathrm{Frob} \arrow[dl, squiggly, "\text{coherent}"'] \arrow[dr, squiggly, "\text{étale}"] & \\ \mathrm{Isoc}_E & & \pi_1^{\text{ét}} = W_E \end{tikzcd}$$

E-v.s. $\subseteq \mathrm{Isoc}_E$

$$\mathrm{Isoc}_E = \left\{ \begin{array}{l} \breve{E}\text{-vec spaces } V \\ + \text{Frob-lin isom } \phi : V \xrightarrow{\sim} V \end{array} \right\}$$

$\cong$ Dieudonné–Manin

$$\bigoplus_{\lambda \in \mathbb{Q}} \underbrace{\mathrm{Isoc}_E^\lambda}$$

isocrystal "pure of slope λ".

<u>note</u> $\mathrm{Isoc}_E^\lambda \cong D_\lambda$-vec spaces

where D_λ / E central division alg of invariant $\lambda \in \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$.

<u>G-torsors in Isoc_E</u> (Kottwitz)

$$G\text{-}\mathrm{Isoc} \cong \coprod_{b \in B(E,G)} [*/G_b(E)]$$

(G_b inner form of Levi subgrp of G).

$$H^1(E,G) \hookrightarrow B(E,G)$$

$$\alpha \longmapsto b \quad \rightsquigarrow \quad G_b = G_\alpha$$

Kottwitz, Kaletha: profitable to consider all G.

Need to be more geometric:

want a geometric stack of G-isocrystals.

<u>Stack of G-isocrystals</u>

Several ways to define this stack:

(0) Isoc_E is E-linear category,

so for any E-algebra A,

can consider G-torsors in $\mathrm{Isoc}_E \otimes_E A$.

$\Rightarrow$ Artin stack $/ E$

$$= \coprod_{b \in B(E,G)} [* / G_b]$$

as algebraic grp.

$\rightsquigarrow$ <u>alg</u> repr of <u>alg</u> grp G_b, not desired category.

(1) Replace $\overline{\mathbb{F}}_q$ by any (perfect) $\overline{\mathbb{F}}_q$-algebra R (discrete).

$\rightsquigarrow$ Replace $\operatorname{Spec} \overline{\mathbb{F}}_q((t)) / \mathrm{Frob}$ by $\operatorname{Spec} R((t)) / \mathrm{Frob}$

$\operatorname{Spec} \breve{E} / \mathrm{Frob}$ by $\operatorname{Spec} (W(R) \otimes_{W(\mathbb{F}_q)} E) / \mathrm{Frob}$

(<u>note</u> $\breve{E} = W(\overline{\mathbb{F}}_q) \otimes_{W(\mathbb{F}_q)} E$.)

Define stack on perfect $\overline{\mathbb{F}}_q$-alg

$$G\text{-}\mathrm{Isoc}: R \longmapsto \left\{ \begin{array}{l} G\text{-torsors on } \operatorname{Spec} R((t)) / \mathrm{Frob} \\ \text{resp. } \operatorname{Spec} (W(R) \otimes_{W(\mathbb{F}_q)} E) / \mathrm{Frob} \end{array} \right\}.$$

<u>Thm</u> (Rapoport-Kottwitz, Caraiani-Scholze, Ivanov, Anschütz)

$G\text{-}\mathrm{Isoc}$ is stack for v- (arc-) topology

($\rightsquigarrow$ fpqc ...)

For any $b \in B(E,G)$, get locally closed substack

$$\begin{array}{ccc} G\text{-}\mathrm{Isoc}^b & \subseteq & G\text{-}\mathrm{Isoc} \\ \wr\| & & \| \\ [*/G_b(E)] & & LG/_{\sigma\text{-conj}}LG \end{array}$$

locus where isom to b.

Gaitsgory, Genestier-V.Lafforgue, Zhu

Xiao-Zhu & Hemo-Zhu define

$$D(G\text{-}\mathrm{Isoc}, \overline{\mathbb{Q}}_\ell).$$

This has a semi-orthogonal decomp. into pieces

$$D(G\text{-}\mathrm{Isoc}^b, \overline{\mathbb{Q}}_\ell) \cong D(G_b(E), \overline{\mathbb{Q}}_\ell)$$

(fixing $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$). derived cat of smooth repr

<u>Q</u> How to get a relation to W_E & $\hat{G}$?

<u>Answer</u> Hecke operators.

These are related to <u>modifications</u> of G-torsors,

i.e. $\mathcal{E}_1, \mathcal{E}_2$ G-torsors / (Spec $R((t))$ / Frob),

+ isom $\mathcal{E}_1 \cong \mathcal{E}_2$ away from some divisor

$$\begin{tikzcd} D \arrow[r, phantom, "\subseteq"] \arrow[dr, "\wr"'] & \mathrm{Spec}\, R((t)) \arrow[d] \\ & \mathrm{Spec}\, R \arrow[u, dashed, bend right] \end{tikzcd}$$

(This require sections!)

$\Downarrow$

need to take R a Banach algebra.

(2) To any perfectoid affinoid alg $(R, R^+)/\mathbb{F}_q$,

Can associate Fargues-Fontaine curve

$$= \mathbb{D}^*_{\mathrm{Spa}(R,R^+)}/\mathrm{Frob}.$$

(resp. similar object in mixed char.)

$\rightsquigarrow$ moduli space of G-torsor on the FF curve Bun_G

$$\rightsquigarrow \quad D(\mathrm{Bun}_G, \bar{\mathbb{Q}}_\ell) \overset{\text{Conj}}{\simeq} D(G\text{-Isoc}, \bar{\mathbb{Q}}_\ell).$$

$$\begin{tikzcd}
& \mathrm{Hecke}_{G,\le\mu} = \{\mathcal{E}_1 \dashrightarrow_{\tilde{\mu}} \mathcal{E}_2 \text{ away from } D\} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathrm{Bun}_G & & \mathrm{Bun}_G \times \underbrace{\mathrm{Spa}\,\breve{E}}_{\text{parametrizing sections}}/\phi
\end{tikzcd}$$

$$T_\mu = p_{2!}\, p_1^* : \underset{\displaystyle \mathrm{Rep}(G(E))}{D(\mathrm{Bun}_G, \bar{\mathbb{Q}}_\ell)} \longrightarrow \underset{\displaystyle D(\mathrm{Bun}_G, \bar{\mathbb{Q}}_\ell)^{W_E}}{D(\mathrm{Bun}_G \times \mathrm{Spa}\,\breve{E}/\phi, \bar{\mathbb{Q}}_\ell)}$$

(by geom Satake, this is enumerated by $\mathrm{Rep}(\check{G})$.

This categorical str is <u>previously</u> what is needed to define L-parameters.

Lecture 2: The Fargues-Fontaine Curve (I)

Fix some nonarch local field E,

residue field $\mathbb{F}_q$ of char p, $\pi \in \mathcal{O}_E$ uniformizer.

- $E \cong \mathbb{F}_q((t))$ or
- $[E : \mathbb{Q}_p] < \infty$.

Goal "Make $\operatorname{Spec} E$ geometric".

Note
- $(\operatorname{Spec} E)_{Zar} = *$.
- $(\operatorname{Spec} E)_{\acute{e}t} = \{\text{finite separable } E\text{-algebras}\}^{op}$

$$= B\,\mathrm{Gal}(\bar{E}/E)$$

$$= \{\text{finite sets w/ cont } \mathrm{Gal}(\bar{E}/E)\text{-action}\}.$$

$\rightsquigarrow$ Look at $\mathrm{Gal}(\bar{E}/E)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_E & \longrightarrow & \mathrm{Gal}(\bar{E}/E) & \longrightarrow & \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) & \longrightarrow & 0 \\
 & & \text{inertia} & & & & \| & & \\
 & & & & & & \hat{\mathbb{Z}} \ni \mathrm{Frob} = \mathrm{Frob}_q & &
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_E & \longrightarrow & I_E & \longrightarrow & \prod_{\ell \neq p} \mathbb{Z}_\ell & \longrightarrow & 0 \\
 & & \text{wild inertia, pro-}p & & & & \| & & \\
 & & & & & & \hat{\mathbb{Z}}^p. & &
\end{array}$$

Local Tate duality

For all torsion $\mathrm{Gal}(\bar{E}/E)$-repr M (prime to p if $E \cong \mathbb{F}_q((t))$),

have
$$H^i_{\acute{e}t}(\operatorname{Spec} E, M) \otimes H^{2-i}_{\acute{e}t}(\operatorname{Spec} E, M^*(1))$$
$$\downarrow \text{perfect pairing} \qquad \text{away from } p \text{ if } E \cong \mathbb{F}_q((t)).$$
$$H^2_{\acute{e}t}(\operatorname{Spec} E, \mathbb{Q}/\mathbb{Z}(1)) \cong \mathbb{Q}/\mathbb{Z}.$$

Here $M^* = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$,

$\mathbb{Q}/\mathbb{Z}(1)$ = Tate twist = $\bigcup_n \mu_n$.

Obs This looks like Poincaré duality on a <u>compact Riemann surface</u>.

$\rightsquigarrow$ Want to turn Spec E into something like a compact Riemann surface.

<u>The equal char Fargues-Fontaine curve</u>

Let $E = \mathbb{F}_q((t))$, $\breve{E} = \overline{\mathbb{F}}_q((t)) \circlearrowleft \phi_{\overline{\mathbb{F}}_q}$

$\rightsquigarrow$ $\mathrm{Spec}\, \breve{E} = \mathrm{Spec}\, \overline{\mathbb{F}}_q((t))$

"formal punctured open unit disc $/ \overline{\mathbb{F}}_q$".

Make more space:

Fix $C / \overline{\mathbb{F}}_q$ complete alg closed nonarch field,

e.g. $C = \widehat{\overline{\mathbb{F}_q((u))}}$.

Consider $\mathrm{Spa}\, C \times_{\mathrm{Spa}\, \mathbb{F}_q} \mathrm{Spa}\, \mathbb{F}_q((t)) = \mathbb{D}_C^* \circlearrowleft \phi_C$

punctured open unit disc $\{x \mid 0 < |x| < 1\}$ over C.

<u>Def'n</u> The <u>Fargues-Fontaine curve</u> (for E, C) is

$$X_{C,E} := \mathbb{D}_C^* / \phi_C^{\mathbb{Z}}$$

This is an adic space over E.

<u>Thm</u> (Fargues-Fontaine)

(1) $H^0(X_{C,E}, \mathcal{O}_{X_{C,E}}) = E$.

(2) $\mathrm{F\acute{E}t}(X_{C,E}) = (\mathrm{Spec}\, E)_{\acute{e}t}$

  ↳ cat of f.ét covers over $X_{C,E}$.

(3) $H^i_{\acute{e}t}(X_{C,E}, M) = H^i_{\acute{e}t}(\mathrm{Spec}\, E, M)$.

<u>Some recollections on adic spaces</u>

Roughly, adic spaces are variants of schemes associated to certain topological rings (e.g. Banach algebras).

<u>Def'n</u> Let A topological ring.

(1) A is <u>adic</u> if there is some ideal $I \subseteq A$ s.t. $\{I^n \mid n \geq 0\}$ is a nbhd basis of 0

Such I is called an <u>ideal of definition</u>.

(↳ Not unique, but for any two I, J, $\exists n$: $I^n \subseteq J$, $J^n \subseteq I$.)

(2) A is <u>Huber</u> (= f-adic in Huber's papers) if $\exists$ open subring $A_0 \subseteq A$ that is adic w/ a f.g ideal of def'n.

Such A_0 is called a <u>ring of definition</u>.

<u>Rmk</u> Any such A admits completion $\hat{A}$, contains $\hat{A}_0 \subset \hat{A}$ as open subring, where $\hat{A}_0$ = I-adic completion of A_0.

Most important case

Def'n A Huber ring A is Tate if it contains a topologically nilpotent unit $\varpi \in A$ (pseudo-uniformizer).

Ex $\mathbb{F}_p((t)) \ni t$, $\mathbb{Q}_p \ni p$, $E \ni \pi$

any nonarch field, any Huber ring over a nonarch local field.

Rmk If K any nonarch field,

$\varpi \in K$ pseudo-uniformizer,

A/K complete Huber ring.

then A has natural str as Banach alg $/K$,

with $\{f \in A \mid |f_0| \leq 1\} = A_0$ "unit ball".

Any $A_0 \subseteq A$ ring of def'n has ϖ-adic top.

$\rightsquigarrow$ Define
$$\|\cdot\| : A \longrightarrow \mathbb{R}_{\geq 0}$$
$$a \longmapsto \inf_{\{n \mid \varpi^n a \in A_0\}} 2^n.$$

Fact $\exists$ Equiv of cats:
$$\left\{ \begin{array}{c} \text{Banach alg } /K \\ \text{w/ cont. maps} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Tate Huber} \\ \text{rings } /K \end{array} \right\}.$$

Def'n The valuation spectrum of Huber ring A is
$$\operatorname{Cont} A := \{ |\cdot| : A \to \Gamma \cup \{0\} \text{ cont val'n} \} / \sim$$
with top gen'd by opens

$\{ |f| \le |g| \ne 0 \} \subseteq \mathrm{Cont}\, A$ for $f, g \in A$.

Here: cont val'n:

- Γ totally ordered ab grp
  (e.g. $\mathbb{R}_{>0}$ or $\mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$ for $1 < \gamma < r$ for all $r \in \mathbb{R}_{>1}$).
- $|\cdot| : A \to \Gamma \cup \{0\}$ satisfies

  $|ab| = |a| \cdot |b|$,

  $|a+b| \le \max(|a|, |b|)$,

  $|0| = 0$, $|1| = 1$.

  $\forall \gamma \in \Gamma$, $\{a \mid |a| < \gamma\} \subset A$ open.
- Two cont val'ns $|\cdot|_1, |\cdot|_2$ are equivalent if

  $|a|_1 \ge |b|_1 \iff |a|_2 \ge |b|_2$.

  Equivalently, if Γ_1, Γ_2 are chosen minimal, then $\exists\, \Gamma_1 \xrightarrow{\sim} \Gamma_2$ s.t.

$$\begin{tikzcd} & \Gamma_1 \cup \{0\} \arrow[dd, "\wr"] \\ A \arrow[ur, "|\cdot|_1"] \arrow[dr, "|\cdot|_2"'] & \\ & \Gamma_2 \cup \{0\}. \end{tikzcd}$$

Def'n A <u>Huber pair</u> is a pair (A, A^+) where A = a Huber ring,

$A^+ \subseteq A^\circ$ open integrally closed subring of power-bdd elts $A^\circ \subseteq A$

$A^\circ = \bigcup_{A_0 \subseteq A} A_0$, A_0 ring of def'n.

Def'n (1) $\mathrm{Spa}(A, A^+) = \{ |\cdot| \mid |A^+| \le 1 \} \subseteq \mathrm{Cont}\, A$

(2) $\mathrm{Spa}\, A := \mathrm{Spa}(A, A^\circ)$.

Can endow $\mathrm{Spec}\, A$ with presheaf $\mathcal{O}_{\mathrm{Spec} A}$ ($\supseteq \mathcal{O}^+_{\mathrm{Spec} A}$)
of Huber rings on basis of rat'l subsets,
given by $\mathcal{U}\left(\frac{f_1, \dots, f_n}{g}\right) = \{ |f_i| \le |g| \ne 0 \}$
where $(f_1, \dots, f_n, g)$ generate an open ideal.

Note $\mathcal{O}_{\mathrm{Spa}(A,A^+)}\left(\mathcal{U}\left(\frac{f_1, \dots, f_n}{g}\right)\right) \supseteq \mathcal{O}^+_{\mathrm{Spa}(A,A^+)}\left(\mathcal{U}\left(\frac{f_1, \dots, f_n}{g}\right)\right)$

$\quad \| \qquad\qquad$ minimal choice that contains A^+ and all f_i/g's.

$A\left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle$

allows all conv series in f_i/g's.

Here $\mathcal{O}^+$ encodes elts with $|\cdot| \le 1$.

Thm (Huber, ...) In "all practical cases" $\mathcal{O}_{\mathrm{Spa}(A,A^+)}$ is a sheaf.
(But not always!)

Rmk Bambozzi-Kremnizer, Clausen-Scholze:
Non-sheafyness can be corrected by
allowing "derived Huber rings".
(But not relevant for this course.).

Def'n An adic space is a triple $(X, \mathcal{O}_X, \mathcal{O}_X^+)$

top space

sheaf of complete top rings

subsheaf of $\mathcal{O}_X$

that is locally of form $(\mathrm{Spa}(A, A^+), \mathcal{O}, \mathcal{O}^+)$
(formulate correct analogue of "locally ringed").

<u>Example</u> C nonarch field.

$\rightsquigarrow$ $\mathbb{D}_C^* = $ "$\{x \mid 0 < |x| < 1\}$" $\underset{\text{open}}{\subseteq} \mathbb{B}_C$.

Consider Tate algebra

$$C\langle T\rangle = \Big\{ \sum_{n\geq 0} a_n T^n \;\Big|\; a_n \in C,\ a_n \to 0 \Big\}.$$

For all $x \in C$, $|x| \leq 1$,

get map
$$C\langle T\rangle \xrightarrow{ev_x} C$$
$$\sum_{n\geq 0} a_n T^n \longmapsto \sum_{n\geq 0} a_n x^n.$$

So $\mathrm{Spa}\, C\langle T\rangle =: \mathbb{B}_C = $ "$\{x \mid 0 \leq |x| \leq 1\}$"

$\quad$ $\cup\!\!\mid$ $\qquad$ ↳ "closed unit disc".

$\mathbb{B}_C^* = \mathbb{B}_C \setminus \{0\} = \bigcup_{\varepsilon > 0} \underbrace{A(\varepsilon, 1)}$

where $A(\varepsilon, 1) \subseteq \mathbb{B}_C$ rat'l subset

↳ informally: $\{x \mid \varepsilon \leq |x| \leq 1\}$ annulus

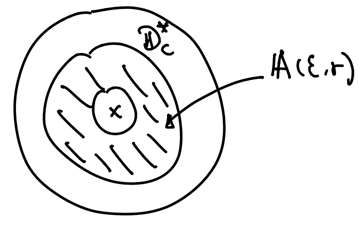
Formally, $A(\varepsilon, 1) := \mathrm{Spa}\, A_\varepsilon$,

$$A_\varepsilon := \Big\{ \sum_{n=-\infty}^{\infty} a_n T^n \;\Big|\; a_n \in C,\ |a_n| \xrightarrow{n \to \infty} 0,\ \varepsilon^n |a_n| \xrightarrow{n \to -\infty} 0 \Big\}.$$

Similarly, $\mathbb{D}_C^* = \bigcup_{\substack{\varepsilon > 0 \\ r < 1}} \underbrace{A(\varepsilon, r)}$

$\qquad$ " $\{\varepsilon \leq |T| \leq r\}$.

(punctured open unit disc).



<u>Beware</u> $\mathbb{D}_C^* \subseteq \mathbb{B}_C$ open, but not equal to $\underbrace{\{0 < |T| < 1\}}_{\text{not open}}$!

($|T| < 1$ defines a <u>closed</u> subset).

Problem: there is one pt $x \in \mathbb{B}_C = \mathrm{Spa}\, C\langle T\rangle$

s.t. $r < |T(x)| < 1$, $\forall r \in |C|$ with $r < 1$.

Namely, x is infinitesimally less than 1.

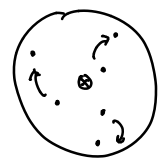
In fact, $\exists$ a natural radius map

$$\mathrm{rad}: \mathbb{B}_C \longrightarrow (0,1)$$
$$x \longmapsto |T(\tilde{x})|$$

This satisfies $\mathrm{rad} \circ \phi_C = \mathrm{rad}^{1/q}$.

$\Rightarrow$ action of ϕ_C on $\mathbb{D}_C^*$ free

& properly discontinuous.



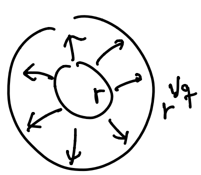
$\mathbb{D}_C^* \circlearrowleft \phi_C$

$\Rightarrow$ $X_{C,E} = \mathbb{D}_C^* / \phi_C^{\mathbb{Z}}$

$\|$ well-def'd adic space

$\mathbb{A}(r, r^{1/q})$ / identify boundary annuli.

So $\phi: \mathbb{A}(r,r) \xrightarrow{\sim} \mathbb{A}(r^{1/q}, r^{1/q})$



$\rightsquigarrow$

quotient $\mathbb{A}(r, r^{1/q})/\phi$

looks like complex torus.

Lecture 3: The Fargues-Fontaine curve (II)

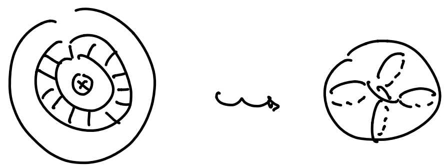
Setup for E, $\mathbb{F}_q$, π as before.

$C/\mathbb{F}_q$ complete alg closed nonarch field.

If $E = \mathbb{F}_q((t))$, consider

$$\operatorname{Spa} E \times_{\operatorname{Spa}\mathbb{F}_q} \operatorname{Spa} C = \mathbb{D}_C^* \circlearrowleft \phi_C \quad \text{punctured open unit disc.}$$

Def'n Fargues-Fontaine curve $X_{C,E} = \mathbb{D}_C^* / \phi_C^{\mathbb{Z}}$.



$\mathbb{D}_C^* \rightsquigarrow \mathbb{D}_C^*/\phi_C^{\mathbb{Z}}$

Classical pts

$$\left\{ \begin{matrix} \text{rigid-analytic} \\ \text{varieties } /C \end{matrix} \right\} \cong \left\{ \begin{matrix} \text{adic spaces "locally of} \\ \text{finite type" } / \operatorname{Spa} C \end{matrix} \right\}$$

(Tate '70s)

$$X(C) \longleftarrow\!\shortmid X$$

$\rightsquigarrow$ $X(C) \subseteq |X|$ "classical pts".

Locally, $X = \operatorname{Spa} A$, $A = C\langle T_1, \dots, T_n \rangle / I$

$\downarrow$

$\operatorname{Sp} A$, $|\operatorname{Sp} A| = X(C) = \operatorname{Spm} A$ (spec of max ideals)

$$= \{ (x_1, \dots, x_n) \in C^n \mid |x_i| \le 1,\ \forall f \in I,\ f(x_1, \dots, x_n) = 0 \}$$

Thm (Huber) $(\operatorname{Sp} A)_{\text{rig}} \cong \operatorname{Spa} A$

Grothendieck top on $\operatorname{Sp} A$ whose qc adm opens $=$ qc opens of $\operatorname{Spa} A$

and admissible covers of $(\mathrm{Sp} A)_{\mathrm{rig}}$ = covers of $\mathrm{Spa}\, A$.

<u>Facts</u> (1) For $\phi_C \circlearrowright |\mathbb{D}_C^*|$ classical pts are

$$(x \mapsto x^q) \circlearrowright \overset{\cup\!\shortmid}{\{} x \in C \mid 0 < |x| < 1 \}.$$

(2) For any connected affinoid $\mathrm{Spa}\, A \subseteq \mathbb{D}_C^*$,

A is a principal ideal domain.

$\Rightarrow$ $\mathbb{D}_C^*$ is "1-dim'l".

Check: max ideal corresp. to x is $(T-x)$.

By descent, can also define classical pts of $X_{C,E}$:

$$X_{C,E} \supseteq \{0 < |x| < 1\} / \phi =: X_{C,E}^{cl}.$$

Again, any conn affinoid subset of $X_{C,E}$

is the Spa(principal ideal domain).

<u>The mix char Fargues-Fontaine curve</u>

Now consider case $E/\mathbb{Q}_p$; still take $C/\mathbb{F}_q$.

<u>Question</u> What is "$\mathrm{Spa}\, E \times_{\mathrm{Spa}\, \mathbb{F}_q} \mathrm{Spa}\, C$" in this setup?

<u>Idea</u> In char p, deformed any $\mathbb{F}_q$-alg R to $\mathbb{F}_q[\![t]\!]$ by taking $R[\![t]\!]$.

<u>Note</u> If R perfect $\mathbb{F}_q$-alg, there exists a unique (up to isom) lift $\tilde{R}/\mathcal{O}_E$ that is flat, π-adically complete, with $\tilde{R}/\pi = R$.

One choice is $\tilde{R} = W(R) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E = W_{\mathcal{O}_E}(R)$.

"ramified Witt vectors".

using p-typical Witt vectors.

The idea behind this:

If $E \cong \mathbb{F}_q((t))$ with $\mathcal{O}_E \cong \mathbb{F}_q[[t]]$,

have $\operatorname{Spa} \mathcal{O}_E \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} \mathcal{O}_C = \operatorname{Spa} \mathcal{O}_C[[t]]$.

Need analog of $\mathcal{O}_C[[t]]$ in this case.

$\rightsquigarrow$ $\exists$ a Teichmüller map

$$[\cdot] : R \longrightarrow \tilde{R} = W_{\mathcal{O}_E}(R) \qquad \text{(multi, not additive)}$$

$$x \longmapsto \lim_{n \to \infty} \tilde{x}_n^{p^n}$$

where $\tilde{x}_n$ = any lift of $x^{1/p^n} \in R$ to $\tilde{R}$.

assumed to be perfect.

Then any elt of $\tilde{R}$ admits a unique expression

as $\sum_{n \geq 0} [r_n] \cdot \pi^n$, $r_n \in R$.

So $\tilde{R}$ looks like "$\mathcal{O}_E[[\pi]]$" in mixed char.

$\Rightarrow$ Analogue of

$$\operatorname{Spa} \mathbb{F}_q((t)) \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} C = \mathbb{D}_C^*$$

in mixed char is

$$\text{"}\operatorname{Spa} E \times_{\operatorname{Spa} \mathbb{F}_q} \operatorname{Spa} C\text{"} = Y_{C,E} \circlearrowleft \phi_C$$

$$\| \quad \{\pi \neq 0, [\varpi] \neq 0\} \subseteq \operatorname{Spa} W_{\mathcal{O}_E}(\mathcal{O}_C).$$

$(\pi \in \mathcal{O}_C, \varpi \in \mathcal{O}_E)$

<u>Def'n</u> The <u>Fargues-Fontaine curve</u> is

$$X_{C,E} = Y_{C,E} / \phi^{\mathbb{Z}} \text{ over } \operatorname{Spa} E.$$

<u>Thm</u> (Fargues–Fontaine, Kedlaya)

(1) There is a notion of classical pts

$$Y^{cl}_{C,E} \subseteq Y_{C,E}$$

s.t. for any connected affinoid $\mathrm{Spa}\, A \subseteq Y_{C,E}$,

- A is a principal ideal domain, and
- $\mathrm{Spm}\, A \xrightarrow{\cong} \mathrm{Spa}\, A \cap Y^{cl}_{C,E} \subseteq Y_{C,E}$.

(2) For any classical pt $y \in Y^{cl}_{C,E}$,

there is some $x \in C$, $0 < |x| < 1$

s.t. $\{y\} = V(\pi - [x])$.

<u>Caveat</u> This elt x is not unique.

If C not alg closed, $\pi - [x]$ could be of higher deg.

(3) For any classical pt $y \in Y^{cl}_{C,E}$,

the complete residue field at y is a complete alg closed field $C(y) \supseteq E$ with a distinguished isom

$$C(y)^\flat \cong C$$

$\rightsquigarrow$ This gives a bijection

$$Y^{cl}_{C,E} \cong \{\text{untilts } C^\# / E \text{ of } C\}.$$

<u>Aside</u> (Tilting) For complete alg closed field K s.t. $|p|_K < 1$,

can define a complete alg closed <u>nonarch</u> field

$$\begin{array}{ccc} K^\flat & := & \varprojlim_{x \mapsto x^p} K \quad \text{(as top multiplicative monoid)} \\ \cup\!\!\mid & & \cup\!\!\mid \\ \mathcal{O}_{K^\flat} & = & \varprojlim_{x \mapsto x^p} \mathcal{O}_K \xrightarrow{\cong} \varprojlim_{x \mapsto x^p} \mathcal{O}_K / p \end{array}$$

of char p

c.f. def'n of Teichmüller map.

Want to sketch proof of thm.

(a) First step. Construct injective map

$$\{C^\#/E \text{ untilt of } C\} \longrightarrow |\mathcal{Y}_{C,E}|$$

Say $C^\#$ untilt of C, so

$$\mathcal{O}_C \cong \varprojlim_{x \mapsto x^q} \mathcal{O}_{C^\#} \longrightarrow \mathcal{O}_{C^\#}$$

$$x \longmapsto x^\#$$

$\rightsquigarrow$ $\theta : W_{\mathcal{O}_E}(\mathcal{O}_C) \twoheadrightarrow \mathcal{O}_{C^\#}$ "Fontaine's map"

$$\sum_{n \geq 0} [x_n] \pi^n \longmapsto \sum_{n \geq 0} x_n^\# \pi^n.$$

$\rightsquigarrow$

$$\begin{array}{ccc} \operatorname{Spa} \mathcal{O}_{C^\#} & \hookrightarrow & \operatorname{Spa} W_{\mathcal{O}_E}(\mathcal{O}_C) \\ \cup\!\!| & & \cup\!\!| \\ \operatorname{Spa} C^\# & \hookrightarrow & \mathcal{Y}_{C,E} \end{array}$$

image $= y \in \mathcal{Y}_{C,E}$, Complete res field at y is $C^\#$.

$\rightsquigarrow$ $\{C^\#/E \text{ untilt of } C\} \hookrightarrow \mathcal{Y}_{C,E}$ injection

Define $\mathcal{Y}_{C,E}^{cl} :=$ image of all $C^\#/E$'s.

Aside (Residue field for pts in adic spaces)

If (A, A^+) Huber pair, $x \in \operatorname{Spa}(A, A^+)$,

$\rightsquigarrow$ $|\cdot|_x : A \longrightarrow \Gamma_x \cup \{0\}$

$\mathfrak{p}_x = \{f \in A \mid |f|_x = 0\}$ prime ideal

$\rightsquigarrow$ $\widehat{\operatorname{Frac}(A/\mathfrak{p}_x)} =: K(x)$ res field.

(b) Tilting for $\mathcal{Y}_{C,E}$.

Let $E_\infty = E(\pi^{1/p^\infty})^\wedge = \left(\bigcup_n E(\pi^{1/p^n})\right)^\wedge.$

This is a "perfectoid field": $(x \mapsto x^p) \circlearrowleft \mathcal{O}_E/p$ is surjective.

$$\rightsquigarrow \quad \text{tilt} \quad \begin{array}{c} E_\infty^\flat \cong \mathbb{F}_q((t^{1/p^\infty})) \ni t \\ \| \\ \varprojlim_{x \mapsto x^p} E_\infty \ni (\pi, \pi^{1/p}, \pi^{1/p^2}, \dots) \end{array}.$$

$$\text{Claim} \quad (\underbrace{Y_{C,E} \times_{\mathrm{Spa}\, E} \mathrm{Spa}\, E_\infty}_{\text{This is a perf'd space.}})^\flat \cong \mathbb{D}_C^* \times_{\mathrm{Spa}\, \mathbb{F}_q((t))} \mathrm{Spa}\, \mathbb{F}_q((t^{1/p^\infty}))$$

$$\cap\!\mid \qquad\qquad\qquad \cap\!\mid$$

$$(W_{\mathcal{O}_E}(\mathcal{O}_C) \hat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E_\infty})^\flat \cong \mathrm{Spa}\, \mathcal{O}_C[[t^{1/p^\infty}]]$$

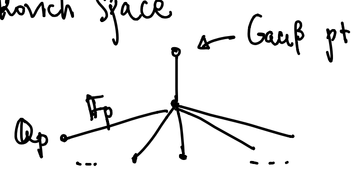
(c.f. $\mathcal{O}_C \otimes_{\mathbb{F}_q} \mathbb{F}_q[[\pi^{1/p^\infty}]]/\pi = \mathcal{O}_C \otimes_{\mathbb{F}_q} \mathbb{F}_q[[t^{1/p^\infty}]]/t$).

Moreover, classical pts biject under this corresp.

$$\rightsquigarrow \quad |\mathbb{D}_C^*| = |\mathbb{D}_C^* \times_{\mathrm{Spa}\, \mathbb{F}_q((t))} \mathrm{Spa}\, \mathbb{F}_q((t^{1/p^\infty}))|$$

invariant under perfection

tilting $\cong$ Berkovich space



$$\cong |Y_{C,E} \times_{\mathrm{Spa}\, E} \mathrm{Spa}\, E_\infty|$$

$$\downarrow$$

$$\rightsquigarrow \quad \begin{array}{ccc} |\mathbb{D}_C^*| & \longrightarrow & |Y_{C,E}| \\ \cup\!\mid & & \cup\!\mid \\ \mathbb{D}_C^{*,cl} & \longrightarrow & Y_{C,E}^{cl} \\ \| & & \psi \\ \{0 < |x| < 1,\ x \in C\} & & V(\pi - [x]) \\ \psi & \nearrow & \\ x & & \end{array}$$

This proves (1) (2) of thm.

& (a) + (b) proves (3) of thm.

Aside on perfectoid spaces

Def'n (1) A perfectoid Tate ring is a complete Tate ring A

$(\exists\, \varpi \in A$ top nilp unit, $\exists\, A_0 \subseteq A = A_0[\frac{1}{\varpi}]$ open, ϖ-adic$)$

if $\exists\, \pi$ s.t. $\pi^p \mid p$ in A° (A° ϖ-adic)

$(\Leftrightarrow A^\circ = A_0$ ring of def'n$)$.

$\Rightarrow$ p top nilp. $x \mapsto x^p \circlearrowright A^\circ/p$ is surjective.

(2) A perfectoid space is an adic space X covered by $\mathrm{Spa}(A, A^+)$ with A a perf'd Tate ring.

Example $A = E_\infty,\ C,\ \mathbb{F}_q((t^{1/p^\infty})),\ C\langle T^{1/p^\infty}\rangle$ are perfectoid rings.

If $A/\mathbb{F}_p$ Tate ring, then

A perfectoid $\Longleftrightarrow$ A perfect (i.e. $\phi : A \to A,\ x \mapsto x^p$ isom)

Tilting extends to perfectoid (Tate) rings:

$$\begin{tikzcd} A \arrow[r, mapsto] & A^\flat = \varprojlim\limits_{x \mapsto x^p} A \arrow[d] & \text{(with suitable def'n)} \\ & f^\# \in A & \end{tikzcd}$$

$f \mapsto f^\#$ (E.g. $(E_\infty\langle T^{1/p^\infty}\rangle)^\flat = E_\infty^\flat\langle T^{1/p^\infty}\rangle$)

and to perfectoid spaces

$$\begin{array}{ccc} X & \longmapsto & X^\flat \\ \cup\!\shortmid & & \cup\!\shortmid \\ \mathrm{Spa}(A, A^+) & \longmapsto & \mathrm{Spa}(A^\flat, A^{\flat +}) . \end{array}$$

Thm (1) "Tilting preserves top spaces"

i.e. $|X| \xrightarrow{\cong} |X^\flat|$

$x \longmapsto x^\flat$

$$\begin{array}{ccc} & X & \to \mathrm{Spec}\,\mathbb{Z}_p \\ X^\flat & \longrightarrow & \mathrm{Spec}\,\mathbb{F}_p \end{array}$$

If $X = \mathrm{Spa}(A, A^+)$, $X^\flat = \mathrm{Spa}(A^\flat, A^{\flat +})$,
then for $f \in A^\flat$, $|f(x^\flat)| = |f^\#(x)|$.

(2) Given perf'd space X,

$$\begin{array}{ccc} \{\text{perf'd spaces } Y/X\} & \xrightarrow{\sim} & \{\text{perf'd spaces } Y'/X^\flat\} \\ Y & \longmapsto & Y^\flat \end{array}$$

is an equiv of cats.

(3) (Bhatt - Scholze)
If $X = \mathrm{Spa}(A, A^+)$, $X^\flat = \mathrm{Spa}(A^\flat, A^{\flat +})$,
then the $\underbrace{\text{Zariski closed}}$ subsets of X & $X^\flat$ correspond.
vanishing locus of some ideal.

i.e. $(Z \subseteq |X|) \triangleq (Z^\flat \subseteq |X^\flat|)$

$(Z^\flat \rightsquigarrow Z$ is easy$)$

<u>Challenge</u> $X = \mathrm{Spa}\, C^\# \langle T^{1/p^\infty} \rangle \supseteq Z = V(T-1)$.
Show that $Z^\flat \subseteq \mathrm{Spa}\, C \langle T^{1/p^\infty} \rangle$ is Zariski closed.

Lecture 4: The Fargues-Fontaine curve (III)

As before, E nonarch local field, res field $\mathbb{F}_q$,

$\mathcal{O}_E \ni \pi$ uniformizer.

Let $C/\mathbb{F}_q$ complete nonarch alg closed field.

$\rightsquigarrow$ Fargues-Fontaine curve

$$X = X_C = X_{C,E} = Y_{C,E}/\phi_C^{\mathbb{Z}} \quad \text{adic space } / E$$

with $Y_{C,E} \underset{\text{open}}{\subseteq} \mathrm{Spa}\, \underbrace{W_{\mathcal{O}_E}(\mathcal{O}_C)}$ flat deformation of $\mathcal{O}_C$ to $\mathcal{O}_E$.

where $\pi \neq 0$ & $[\varpi] \neq 0$

($\varpi \in C$ pseudo-uniformizer).

<u>Recall</u> Classical pts $X_{C,E}^{cl} \subseteq |X_{C,E}|$

$\qquad\qquad\qquad\qquad \{$untilts $C^\#/E$ of $C\}/\phi_C^{\mathbb{Z}}$

- Any conn affinoid subset $\mathrm{Spa}\, A \subseteq X_{C,E}$ has A principal ideal domain ("1-dim'l") and $\mathrm{Spm}\, A = X_{C,E}^{cl} \cap |\mathrm{Spa}\, A| \subseteq |X_{C,E}|$.
- For any classical pt $y \in Y_{C,E}^{cl}$, $\exists\, t \in C$ s.t. $0 < |t| < 1$ & $\{y\} = V(\pi - [t])$.

Classification theorem for vector bundles

<u>Isocrystals</u> Recall:

<u>Def'n</u> An <u>isocrystal</u> is a pair (V, ϕ_V)

where V fin-dim'l $\breve{E}$-vect space

$+\ \phi_V : V \xrightarrow{\sim} V$ $\ \phi_{\breve{E}}$-linear automorphism.

(<u>Aside</u> $\breve{E} = W_{\mathcal{O}_E}(\overline{\mathbb{F}_q})[\frac{1}{\pi}]$ completion of max'l unramified ext'n $/E$.)

$\rightsquigarrow$ Isoc_E : an E-linear $\otimes$-category.

<u>Examples</u> (0) unit of $\otimes$: $(\breve{E}, \phi_{\breve{E}})$.

(1) 1-dim'l objects : $(\breve{E}, b\phi_{\breve{E}})$ for some $b \in \breve{E}^\times$.

(i) Any such is isom to $(\breve{E}, \pi^n \phi_{\breve{E}})$ for a unique $n \in \mathbb{Z}$.

Here n = slope of $(\breve{E}, \pi^n \phi_{\breve{E}})$.

(ii) Change of basis replaces b by

$$a^{-1} b\, \phi(a) = \underbrace{a^{-1}\phi(a)}_{\in\, \mathcal{O}_{\breve{E}}^\times}\, b \quad \text{for } a \in \breve{E}^\times$$

"ϕ-conjugation".

($a^{-1}\phi(a)$ can be any elt in $\mathcal{O}_{\breve{E}}^\times$).

(iii) $\mathrm{Hom}((\breve{E}, \pi^n\phi_{\breve{E}}), (\breve{E}, \pi^m\phi_{\breve{E}})) = \breve{E}^{\phi = \pi^{m-n}} = \begin{cases} E, & m = n \\ 0, & m \neq n. \end{cases}$

(2) If $\lambda = s/r \in \mathbb{Q}$, $(s,r) = 1$, $r > 0$,

get $(V_\lambda, \phi_{V_\lambda}) = \left(\breve{E}^r, \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & & \ddots & 1 \\ \pi^s & & & 0 \end{pmatrix} \phi_{\breve{E}} \right)$.

<u>Thm</u> (Dieudonné - Manin)

$$\mathrm{Isoc}_E \cong \bigoplus_{\lambda \in \mathbb{Q}} \mathrm{Isoc}_E^\lambda$$

"isoclinic of slope λ".

where $\mathrm{Isoc}_E^\lambda = (\text{f.d. } E\text{-v.s.}) \otimes V_\lambda$

(c.f. $\mathrm{End}(V_\lambda) = D_\lambda$,

central div alg of inv. λ.)

Sketch of proof

For any nonzero $V = (V, \phi_V) \in \mathrm{Isoc}_E$,

$\det V := \Lambda^{\mathrm{rk} V} V \cong (\breve{E}, \pi^n \phi_{\breve{E}}) \in \mathrm{Isoc}_E$ of rk 1.

let $\deg V :=$ n above,

$\mu(V) := \deg(V) / \underbrace{\mathrm{rk}(V)}_{\text{dim of underlying } \breve{E}\text{-v.s.}}$

$\rightsquigarrow$ "Harder-Narasimhan formalism of slopes":

$\rightsquigarrow$ (semi)stable objects of Isoc_E: (V, ϕ_V) is (semi)stable

if for all $0 \subsetneq (V', \phi_{V'}) \subsetneq (V, \phi_V)$,

one has $\mu(V') \leq \mu(V)$

(resp. $<$ means stable)

$\rightsquigarrow$ Any object $V = (V, \phi_V) \in \mathrm{Isoc}_E$ has a unique "Harder-Narasimhan fil'n"

decreasing separated exhaustive fil'n, $\mathbb{Q}$-indexed

$V^{\leq \lambda} \subseteq V$ in Isoc_E

s.t. each $V^{\lambda} := V^{\geq \lambda} / \bigcup_{\lambda' > \lambda} V^{\geq \lambda'}$

is semistable of slope λ.

Obs In this simple case, also

$\mu'(V) = -\mu(V) = -\deg(V)/\mathrm{rk}(V)$ gives HN fil'n

So this fil'n is canonically split,

and $\mathrm{Isoc}_E = \bigoplus_{\lambda \in \mathbb{Q}} \underbrace{\mathrm{Isoc}_E^{\lambda}}_{\text{semistable of slope } \lambda}$.

- For $\lambda = 0$: Want $\mathrm{Isoc}_{\breve{E}}^{0} \xleftrightarrow{\cong}$ f.d. E-v.s.

$$(W \otimes_E \breve{E},\ \mathrm{id} \otimes \phi_{\breve{E}}) \longleftarrow\!\shortmid\ W = (W, \mathrm{id})$$

or equivalently, for all $(V, \phi_V) \in \mathrm{Isoc}_{\breve{E}}^{0}$,

letting $W = V^{\phi_V = \mathrm{id}}$, one has $W \otimes_E \breve{E} \xrightarrow{\cong} V$.

<u>Idea</u> Show that V contains a ϕ_V-stable lattice $L \subseteq V$

i.e. $\phi_V(L) = L$.

finite free $\mathcal{O}_{\breve{E}}$-mod generating V (over $\breve{E}$)

Then $L^{\phi_L = \mathrm{id}}$ is fin. free over $\mathcal{O}_E$ of correct rank, by Artin–Schreier theory.

- For general λ, use that if $(V, \phi_V) \in \mathrm{Isoc}_{\breve{E}}^{\lambda}$,

then $\underline{\mathrm{Hom}}((V_\lambda, \phi_{V_\lambda}), (V, \phi_V)) \in \mathrm{Isoc}_{\breve{E}}^{0}$

and use result for $\lambda = 0$ above. $\square$

<u>Back to Fargues–Fontaine curve</u>

<u>Note</u> $\overline{\mathbb{F}}_q \subseteq C \rightsquigarrow$

$$\begin{array}{ccc} Y_{C,E} & \longrightarrow & \mathrm{Spa}\,\breve{E} \\ \circlearrowleft & & \circlearrowleft \\ \phi_C & & \phi_{\breve{E}} \end{array}$$

$\rightsquigarrow$ pullback functor

$$\mathrm{Isoc}_{\breve{E}} \longrightarrow \{\phi_C\text{-equiv VB} / Y_{C,E}\} \underset{\text{descent}}{=\!=} \mathrm{VB}(X_{C,E}).$$

$$V \longmapsto \mathcal{E}(V)$$

Let $\mathcal{O}_{X_{C,E}}(\lambda) := \mathcal{E}(V_{-\lambda}) \in \mathrm{VB}(X_{C,E})$.

<u>Thm</u> (Fargues–Fontaine, Hartl–Pink, Kedlaya)

all E, '10 $\quad E = \mathbb{F}_q((t))$, '04 $\quad E$ p-adic, '04

Any vector bdl on $X_{C,E}$ is isom to a direct sum of $\mathcal{O}(\lambda)$'s.

Equiv'ly, the functor

$$\mathrm{Isoc}_E \longrightarrow \mathrm{VB}(X_{C,E})$$

induces a bijection on isom classes.

More precisely,

(1) Any $\mathcal{E} \in \mathrm{VB}(X_{C,E})$ admits a HN fil'n

$$\mathcal{E}^{\geq \lambda} \subseteq \mathcal{E}$$

s.t. each $\mathcal{E}^{\lambda} = \mathcal{E}^{\geq\lambda} / \bigcup_{\lambda' > \lambda} \mathcal{E}^{\geq \lambda'}$

is semistable of slope λ.

(2) $\mathrm{Isoc}_E^{\lambda} \xrightarrow{\sim} \mathrm{VB}(X_{C,E})^{\lambda}$ (semistable of slope λ).

(3) The HN fil'n for $\mathcal{E}$ splits (but not uniquely).

<u>Rmk</u> (1) Like for all sm proj curves.

(2) Similar to $\mathbb{P}^1$, but not other curves.

(3) Similar to $g=0$, but not to higher genus.

<u>Fact</u> Have cohom results:

$$H^0(X_{C,E}, \mathcal{O}(n)) = \begin{cases} \text{infin-dim'l } E\text{-v.s.}, & n>0 \\ E, & n=0 \\ 0, & n<0. \end{cases}$$

$$H^1(X_{C,E}, \mathcal{O}(n)) = \begin{cases} 0, & n>0 \\ 0, & n=0 \\ \text{infin-dim'l } E\text{-v.s.}, & n<0. \end{cases}$$

Let $C^\#/E$ some untilt of $C/\mathbb{F}_q$.

$\rightsquigarrow$ $\operatorname{Spa} C^\# \overset{i}{\hookrightarrow} X_{C,E}$ by Thm last time.

$$\rightsquigarrow \quad 0 \to \underset{\substack{\text{IS} \\ \mathcal{O}_{X_{C,E}}(-1)}}{\mathcal{I}} \to \mathcal{O}_{X_{C,E}} \to \underset{\substack{\| \\ i_* C^\#}}{i_* \mathcal{O}_{\operatorname{Spa} C^\#}} \to 0$$

$$\rightsquigarrow \quad \underbrace{H^1(X_{C,E}, \mathcal{O}_{X_{C,E}}(-1))} \cong C^\#/E$$

((recall $H^0(X_{C,E}, \mathcal{O}_{X_{C,E}}) \cong E$.)

"Banach-Colmez spaces":

built from f.d. $C^\#$-v.s. + f.d. E-v.s.

Similarly,

$$0 \to \mathcal{O}_{X_{C,E}} \to \mathcal{O}_{X_{C,E}}(1) \to i_* C^\# \to 0$$

$$\overset{H^0}{\rightsquigarrow} \quad 0 \to E \to H^0(X_{C,E}, \mathcal{O}(1)) \to C^\# \to 0.$$

<u>Next Goal</u> Sketch a new proof of VB classification theorem, using heavily perfectoid spaces, diamonds, v-descent, but no computations!

<u>Some reductions</u> (Same in all known proofs)

(a) <u>Classification of line bdls</u>

For this, first prove that "<u>$\mathcal{O}(1)$ is ample</u>".

<u>Thm</u> (Kedlaya-Liu, '15)

For any vect bdl $\mathcal{E}$ on $X_{C,E}$, all $n \gg 0$, the bdl $\mathcal{E}(n) = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ is globally gen'd, and $H^1(X_{C,E}, \mathcal{E}(n)) = 0$.

(<u>note</u>: $H^i(X_{C,E}, \mathcal{E}) = 0$ for $i \geq 2$ always.)

Cor (GAGA) Let $P = \bigoplus_{n \geq 0} H^0(X_{C,E}, \mathcal{O}(n))$, $X_{C,E}^{alg} = \mathrm{Proj}(P)$.

Then $|X_{C,E}^{alg}| = X_{C,E}^{cl} \cup \{\eta\}$.

$\rightsquigarrow$ natural map of locally ringed top spaces

$$X_{C,E} \xrightarrow{f} X_{C,E}^{alg}.$$

s.t. • $f^*: VB(X_{C,E}^{alg}) \xrightarrow{\sim} VB(X_{C,E})$ equiv of cats

• f preserves cohomology.

($X_{C,E}^{alg}$ regular, noetherian sch of Krull dim 1,

locally spectrum of a P.I.D.).

Cor Any line bdl $\mathcal{L} \in \mathrm{Pic}(X_{C,E})$ is isom to $\mathcal{O}(D)$ for some divisor $D \in \bigoplus_{x \in X_{C,E}^{cl}} \mathbb{Z}$.

Prop For any $x \in X_{C,E}^{cl}$, $\mathcal{O}(x) = \mathcal{I}_x^{-1} \cong \mathcal{O}_{X_{C,E}}(1)$.

(Next time: use Lubin-Tate theory.)

Cor $\mathcal{O}(D) \cong \mathcal{O}(\deg D)$, so $\mathrm{Pic}(X_{C,E}) \cong \mathbb{Z}$.

Can now define $\deg \mathcal{E} = \begin{pmatrix} \text{image of } \det \mathcal{E} = \wedge^{\mathrm{rk}\,\mathcal{E}} \mathcal{E} \\ \text{in } \mathbb{Z} \cong \mathrm{Pic}(X_{C,E}) \end{pmatrix}$

and then $\mu(\mathcal{E}) = \deg \mathcal{E} / \mathrm{rk}\, \mathcal{E}$ for $\mathcal{E} \neq 0$.

$\rightsquigarrow$ Harder-Narasimhan fil'n $\rightsquigarrow$ get (i) of main thm.

(b) Essential remaining step:

Classify bdls that are semistable of slope 0.

$$\begin{array}{ccc} \text{f.d. } E\text{-v.s.} & \xhookrightarrow{\sim} & VB(X_{C,E})^0 \\ W & \longmapsto & W \otimes_E \mathcal{O}_{X_{C,E}}. \end{array}$$

Key If $\mathcal{E} \neq 0 / X_{C,E}$ semistable of slope 0,
then $H^0(X_{C,E}, \mathcal{E}) \neq 0$.

Idea We will ourselves (a priori) to enlarge C.

(i) Consider the functor
$$C'/C \longmapsto H^0(X_{C',E}, \mathcal{E}|_{X_{C',E}}).$$
Want to think of this as a geometric object whose C'-valued pts are $H^0(X_{C',E}, \mathcal{E}|_{X_{C',E}})$.

(ii) Extend the functor on $\{C'/C\}$ to all perfectoid C-algebras.

$\rightsquigarrow$ get sheaf on category of (affinoid) perfectoid spaces.

Example Fix $C^\#/E$ untilt of C.
Let R perfectoid C-alg $\hat{=}$ untilt $R^\#$ perfectoid $C^\#$-alg
tilting equivalence.

$\rightsquigarrow$ $0 \to \mathcal{O}(-1) \to \mathcal{O} \to i_{R^\#,*}\mathcal{O}_{R^\#} \to 0$

$0 \to \mathcal{O}(-1) \to \mathcal{O} \to i_{C^\#,*}\mathcal{O}_{C^\#} \to 0$

$$\begin{tikzcd} \mathrm{Spa}\, R^\# \arrow[r, hook, "i_{R^\#}"] \arrow[d] & X_{R,E} \arrow[d] \\ \mathrm{Spa}\, C^\# \arrow[r, hook, "i_{C^\#}"] & X_{C,E} \end{tikzcd}$$

Then $H^1(X_{R,E}, \mathcal{O}(-1)) = R^\#/E$.
In equal char case $(E \cong \mathbb{F}_q((t)))$, $R^\# = R$.

So get $H^1(X_{C,E}, \mathcal{O}(-1)) = \mathbb{A}^1_C / E$

/ ($E \subseteq \mathbb{A}^1_C$ closed subset)

quotient of $\mathbb{A}^1_C$ by pro-étale equiv relation.

This will be the general picture!

$H^0(X, \mathcal{E})$, $H^1(X, \mathcal{E})$ = "quotients of perfectoid spaces under pro-étale equiv relations".

$\approx$ analogous to <u>diamonds</u>.

(cf. Artin's algebraic spaces
= quotients of schemes under étale equiv relations.)

It turns out: in <u>equal</u> char case

for $n>0$, $H^0(X_{C,E}, \mathcal{O}(n)) \cong \mathfrak{m}_{\mathcal{O}_C}^n$.

This $H^0(\mathcal{O}(n))$ is represented by $\mathbb{D}^n$
(n-dim'l open unit disc).

Lecture 5: Relative Fargues-Fontaine curve

Aim Classify vector bdls on "the" FF curve $X_{C,E}$.

Last time - Classification of line bdls.

The aim was then reduced to:

- Classification of VB semistable of slope 0.

(Proof relies on putting "geom structure" on

$H^i(X_{C,E}, \mathcal{E})$ $(i = 0, 1)$ for $\mathcal{E} \in VB(X_{C,E})$

"Banach-Colmez spaces".

Intermediate aim Explain these "geometric structures"

$\shortparallel$ diamonds

+ relative Fargues-Fontaine curve

$X_{S,E}$ for perfectoid space S.

"Étale cohomology of diamonds".

Recall (1) A perfectoid algebra $/\mathbb{F}_p$ is just

a perfect Tate algebra $R / \mathbb{F}_p$

(Huber's sense: $\exists$ top nilp unit ϖ.

$\underbrace{\hspace{3cm}}$

perfect Banach alg $/ \mathbb{F}_p((\varpi))$.

(2) A perfectoid space $/\mathbb{F}_p$ is an adic space $X / \mathbb{F}_p$

Covered by opens $U = \mathrm{Spa}(R, R^+) \subseteq X$

s.t. R perfect Tate, i.e. R perfectoid.

Let $E \supseteq \mathcal{O}_E \ni \pi$, $\mathcal{O}_E/(\pi) \simeq \mathbb{F}_q$ as before.

If $S = \mathrm{Spa}(R, R^+)$ $/\mathbb{F}_q$ is affinoid perfectoid.

can mimic construction of FF curve, replacing $\mathcal{O}_C$ by R^+:

Pick $\varpi \in R$ pseudo-unif.

$$\mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \supseteq \underset{\substack{\| \\ \{[\varpi] \neq 0\} \,/\, \mathcal{O}_E}}{\mathcal{Y}_{(R,R^+),E}} \supseteq \underset{\substack{\| \\ \{[\varpi] \neq 0,\ \pi \neq 0\} \,/\, E}}{Y_{(R,R^+),E}}$$

These $\mathcal{Y}$ & Y are <u>analytic</u> adic spaces.

(locally like $\mathrm{Spa}(T, T^+)$, for T = Tate alg.

Have continuous radius function

$$\begin{tikzcd} y \arrow[r, mapsto] & \log|[\varpi](\tilde{y})| / \log|\pi(\tilde{y})| \\ \mathrm{rad}: \mathcal{Y}_{(R,R^+),E} \arrow[r] & {[0,\infty)} \\ \cup\!| & \cup\!| \\ \phi_{R^+} \circlearrowright Y_{(R,R^+),E} \arrow[r] & (0,\infty) \circlearrowleft q \end{tikzcd}$$

free & totally disc.

so that $\mathrm{rad} \circ \phi_{R^+} = q \cdot \mathrm{rad}$.

<u>Def'n</u> $X_{(R,R^+),E} := Y_{(R,R^+),E} / \phi^{\mathbb{Z}}$ as an adic space $/E$.

"relative Fargues-Fontaine curve".

<u>Warning</u> $X_{(R,R^+),E} / E$ does NOT map to $S = \mathrm{Spa}(R, R^+)$!

<u>Example</u> $E = \mathbb{F}_q((t))$, $\mathrm{Spa}(R, R^+) = \{[\varpi] \neq 0\} \subseteq \mathrm{Spa}(R^+, R^+)$.

Then $\quad W_{\mathcal{O}_E}(R^+) = R^+[[t]]$.

$$\left(\text{``} W(R^+) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E \right)$$

$$\begin{aligned}
\rightsquigarrow \quad \mathcal{Y}_{(R,R^+),E} &= \mathrm{Spa}(R,R^+) \times_{\mathrm{Spa}\,\mathbb{F}_q} \mathrm{Spa}\,\mathbb{F}_q[[t]] \\
\cup| \quad &= \mathbb{D}_{\mathrm{Spa}(R,R^+)} \quad \text{open unit disc} \\
& \qquad \cup| \\
\phi \circlearrowright Y_{(R,R^+),E} &= \mathbb{D}^*_{\mathrm{Spa}(R,R^+)} \quad \text{punctured open unit disc}
\end{aligned}$$

<u>Note</u> In this case,

$$\begin{tikzcd}
Y_{(R,R^+),E} \arrow[r, phantom, "\subseteq"] \arrow[dr] & \mathcal{Y}_{(R,R^+),E} \arrow[d] \\
& \mathrm{Spa}(R,R^+)
\end{tikzcd}$$

but <u>not</u> after quotienting by ϕ !

<u>Claim</u> For general perfectoid S,

$$\mathcal{Y}_{S,E} \supseteq Y_{S,E} \twoheadrightarrow X_{S,E} = Y_{S,E}/\phi^{\mathbb{Z}}$$

can be glued from affinoid case $(-)_{(R,R^+),E}$.

This gluing is easy for $E = \mathbb{F}_q((t))$:

$$\begin{aligned}
\mathcal{Y}_{S,E} &= S \times_{\mathrm{Spa}\,\mathbb{F}_q} \mathrm{Spa}\,\mathbb{F}_q[[t]] \qquad / \mathcal{O}_E \\
\cup| \quad & \qquad\qquad \cup| \\
Y_{S,E} &= S \times_{\mathrm{Spa}\,\mathbb{F}_q} \mathrm{Spa}\,\mathbb{F}_q((t)) \qquad / E \\
\big\downarrow \quad & \\
X_{S,E} &= \quad Y_{S,E}/\phi^{\mathbb{Z}}.
\end{aligned}$$

<u>Note</u> $\quad id = \phi \circlearrowright |S| \quad \rightsquigarrow \quad |X_{S,E}| \longrightarrow |S|$.

For $E/\mathbb{Q}_p$, would like to argue similarly in p-adic case.

"<u>Diamond equation</u>"

$$\begin{array}{ccc} \mathcal{Y}^\diamond_{S,E} & = & S \times (\mathrm{Spa}\, \mathcal{O}_E)^\diamond \\ \cup\!\!| & & \cup\!\!| \\ Y^\diamond_{S,E} & = & S \times (\mathrm{Spa}\, E)^\diamond \\ \downarrow & & \\ X^\diamond_{S,E} & = & Y^\diamond_{S,E} / \phi^{\mathbb{Z}}. \end{array}$$

<u>Diamonds</u>

<u>Idea</u> Diamonds = quotients of perfectoid spaces (of char p) by <u>pro-étale</u> equivalence relations.

$$\{\text{Analytic adic spaces} / \mathrm{Spa}\, \mathbb{Z}_p\} \longrightarrow \text{Diamonds}$$

$$\left(\begin{array}{c} \tilde{X} \\ {\scriptstyle \text{pro ét cover}} \downarrow \\ X \end{array} \right) \text{— perfectoid space} \longmapsto X^\diamond$$

"generalizing tilting".

Think of X "=" $\tilde{X}/R$, $R \subseteq \tilde{X} \times \tilde{X}$ perfectoid pro-étale equiv relation.

$\rightsquigarrow \quad X^\diamond := \tilde{X}^\flat / R^\flat.$

<u>(Pro-)Étale maps of perfectoid spaces</u>

<u>Def'n</u> Let $f: Y \to X$ map of perfectoid spaces (possibly of mixed char.)

(1) f is <u>finite étale</u> if for any open affinoid perfectoid

$$\mathcal{U} = \mathrm{Spa}(R, R^+) \subseteq X$$

(equivalently for a cover by such),

- the preimage $V = f^{-1}(\mathcal{U}) = \mathrm{Spa}(S, S^+) \subseteq Y$ is affinoid perfectoid, and
- S finite étale R-algebra, $S^+ \subseteq S$ integral closure of R^+.

$\rightsquigarrow \quad \mathrm{Spa}(R, R^+)_{\text{fét}} \simeq \mathrm{Spec}(R)_{\text{fét}}$.

<u>Implicit</u>: If S fét R-alg, R perfectoid, then S again perfectoid. c.f. $\sim$ Faltings's almost purity thm.

(2) f is <u>étale</u> if it is locally on Y of form

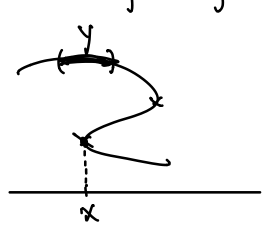
$$Y \hookleftarrow Y' \longrightarrow \mathcal{U} \hookrightarrow X.$$

open imm (for $Y' \hookrightarrow Y$), fin ét (for $Y' \to \mathcal{U}$), open imm (for $\mathcal{U} \hookrightarrow X$)

<u>Nontrivial fact</u> Composites of étale maps are étale.

└ Warning: Instead of perfectoid (adic) spaces, analogous assertions fails for étale maps of schemes.

$$Y \longrightarrow X$$



But this works for analytic adic spaces.

(3) f is <u>pro-étale</u> if locally on X and Y,

it is <u>affinoid pro-étale</u>, i.e.

$$\begin{array}{c} Y = \mathrm{Spa}(S, S^+) = \varprojlim_i \mathrm{Spa}(S_i, S_i^+) \\ \downarrow f = \varprojlim_i f_i \\ X = \mathrm{Spa}(R, R^+) \end{array}$$

In this cofiltered limit,

all $f_i : \mathrm{Spa}(S_i, S_i^+) \to \mathrm{Spa}(R, R^+)$ are étale,

& $S^+ = (\varinjlim S_i)^\wedge_\varpi$, $S = S^+[\frac{1}{\varpi}]$.

<u>Example</u> For $p \neq 2$,

$$\begin{array}{c} Y = \mathrm{Spa}(C\langle T^{1/2p^\infty}\rangle) \\ \downarrow \text{ extract } T^{1/2}. \\ X = \mathrm{Spa}(C\langle T^{1/p^\infty}\rangle) \end{array}$$

$\rightsquigarrow$ looks like it is ramified at origin.

<u>Claim</u> $\exists$ affinoid pro-étale cover $\tilde{X} \to X$

s.t.
$$\begin{tikzcd} \tilde{Y} \arrow[r, "\text{proét}"] \arrow[d, "\substack{\text{affinoid} \\ \text{proét}}"'] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \\ \tilde{X} \arrow[r, "\text{proét}"] & X \end{tikzcd}$$

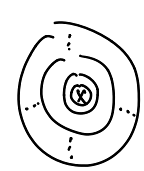
Let $U_n \subseteq X$ small disc $\{|T| \leq \frac{1}{p^n}\}$.

($U_{n,n+1} \subseteq U_n$ annulus $\{\frac{1}{p^{n+1}} \leq |T| \leq \frac{1}{p^n}\}$.

All are affinoid perfectoid.

For each n, let

$$\begin{array}{l} X_n = U_{0,1} \sqcup U_{1,2} \sqcup \cdots \sqcup U_{n-1,n} \sqcup U_n = \\ \downarrow \text{ étale cover} \\ \tilde{X} \end{array}$$



Then $\tilde{X} = \varprojlim_n X_n \longrightarrow X$ affinoid proét.

$$\begin{array}{c} \tilde{X} \\ \downarrow \\ \pi_0(\tilde{X}) = \mathbb{N} \cup \{\infty\}. \end{array}$$

fibre of $\tilde{X}$ over $n \in \mathbb{N}$ is $U_{n,n+1}$

fibre of $\tilde{X}$ over ∞ is $\mathrm{Spa}\, C$ = origin of X.

$\rightsquigarrow$ Easy to see:

$$\tilde{Y} = Y \times_X \tilde{X} \longrightarrow \tilde{X} \text{ affinoid pro-étale.}$$

<u>Note</u> About $\tilde{X} \to X$ pro-étale $\rightsquigarrow$ diamond.

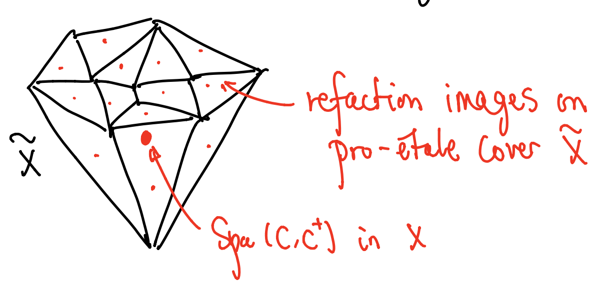
$$\begin{array}{ccc} \tilde{X}/R & & \\ \parallel & & \text{profinite} \\ X & \longrightarrow & \pi_0(X) \end{array}$$

each conn comp $= \mathrm{Spa}(C, C^+)$

C complete alg closed nonarch field

$C^+ \subseteq \mathcal{O}_C$ open valuation subring.

Diamond picture:



<u>Example</u> Any Zariski closed immersion is affinoid pro-étale.

$$X = \mathrm{Spa}(R, R^+) \supseteq V(f) = Z = \mathrm{Spa}(S, S^+)$$

Zariski closed.

$$S = R / \overline{(f^{1/p^\infty})} \supseteq S^+ = \text{integral closure of } R^+ \text{ in } S.$$

Let $U_n = \{ |f| \leq \frac{1}{p^n} \} \subseteq X$ rat'l open subset.

Then $V(f) = \bigcap_n U_n = \varprojlim_n U_n \subseteq X$

(Here $|f| = 0$ everywhere, so $f = 0$.

(perfectoid spaces are reduced, even uniform).

(c.f. Analog of henselization for schemes.)

Example $f : \mathrm{Spa}\, C\langle T^{1/p^\infty}\rangle \longrightarrow \mathrm{Spa}\, C$ not pro-étale !

as its fibre is positive-dim'l.

(Fibres of pro-ét maps are profin sets. like $\mathbb{N} \cup \{\infty\}$.)

Thm A map $f : Y \to X$ of affinoid perf'd spaces

is pro-étale locally on X affinoid pro-étale

iff for all geom (rk 1) pts

$$\mathrm{Spa}(C, \mathcal{O}_C) \hookrightarrow X,$$

the fibre product

$$Y \times_X \mathrm{Spa}(C, \mathcal{O}_C) \longrightarrow \mathrm{Spa}(C, \mathcal{O}_C)$$

is affinoid pro-étale ; equivalently, isom to

$$\underline{S} \times \mathrm{Spa}(C, \mathcal{O}_C) = \varprojlim_i (S_i \times \mathrm{Spa}(C, \mathcal{O}_C))$$

for a profinite set $S = \varprojlim_i S_i$.

Call such f quasi-pro-étale.

The v-topology

Def'n A map $f : Y \to X$ is a v-cover

if for any qc $U \subseteq X$, $\exists$ qc $V \subseteq Y$

s.t. $|V| \twoheadrightarrow |U|$ surjective.

<u>Def'n</u> Say f is a <u>pro-étale cover</u> if

- f is a v-cover, and
- f is quasi-pro-étale.

<u>Thm</u> (1) $X \longmapsto \mathcal{O}_X(X), \ \mathcal{O}_X^+(X)$

sheaves for v-topology on perfectoid spaces

(2) For any perf'd space X,

$\mathrm{Hom}(-, X)$ is a sheaf for v-top.

$\rightsquigarrow$ v-top is the canonical top for representability.

(3) $X \longmapsto VB(X)$ v-stack

(4) If $X = \mathrm{Spa}(R, R^+)$ affinoid, then

$$H^i_v(X, \mathcal{O}_X) = \begin{cases} R, & i=0 \\ 0, & i>0 \end{cases}$$

and $$H^i_v(X, \mathcal{O}_X^+) \overset{a}{=} \begin{cases} R^+, & i=0 \\ 0, & i>0. \end{cases}$$

almost equal, i.e. difference killed by ϖ^{1/p^n} for all n.

<u>Sketch of pf</u> First, prove that

$X \longmapsto \mathcal{O}_X^+(X)$ sheaf for <u>étale</u> top

$+$ $H^i_{\acute{e}t}(X, \mathcal{O}_X^+(X)) \overset{a}{=} 0$ for $i>0$. X affinoid.

<u>Note</u> By de Jong – van der Put combinatorics argument,

$$\text{étale covers} = \underbrace{\text{open covers}} + \underbrace{\text{finite étale covers}}_{\text{by Faltings's almost purity}}$$

"Canonical", i.e. finest top s.t. rep'ble objs are sheaves.

Then get similar assertions for

$\underbrace{\mathcal{O}_X^+/\varpi}$ (discrete).

This extends to affinoid pro-étale things by filt colims.

$\rightsquigarrow$ Get $\mathcal{O}_X^+/\varpi$ has good properties as pro-étale sheaf.

$\rightsquigarrow$ $\mathcal{O}_X^+ = \varprojlim_n \mathcal{O}_X^+/\varpi^n$ as well

$\mathcal{O}_X = \mathcal{O}_X^+[\frac{1}{\varpi}]$ as well.

Fact pro-étale locally, v-covers are faithfully flat on $\mathcal{O}_X^+/\varpi$-level.

Then use faithfully flat descent. "$\square$"

Upshot "Everything is flat in perfectoid world"!

e.g. Working pro-ét locally breaks the space so much

$\rightsquigarrow$ flatness is automatic.

Lecture 6: Diamonds and the relative Fargues-Fontaine curve

Recall E nonarch local field, $\mathbb{F}_q$ residue field.
$S / \mathbb{F}_q$ perfectoid space.

Aim Introduce rel FF curve:

$$X_{S,E} = Y_{S,E} / \phi^{\mathbb{Z}}$$
$$Y_{S,E} \text{ "} = S \times \operatorname{Spa} E \text{"}$$
$$+ \quad Y_{S,E}^{\diamond} = S \times (\operatorname{Spa} E)^{\diamond} \quad \text{"diamond formula".}$$

There is a functor extending tilting:

$$\begin{array}{ccc} X & \longmapsto & X^{\diamond} \\ \{\text{analytic adic spaces} / \mathbb{Z}_p\} & \longrightarrow & \{\text{diamonds}\} \\ \cup\!\!\mid & & \cup\!\!\mid \\ \{\text{perf'd spaces}\} & \longrightarrow & \{\text{perf'd spaces} / \mathbb{F}_p\} \\ X & \longmapsto & X^{\flat} \end{array}$$

Have defined pro-étale morphisms of perf'd spaces.

Pro-étale local structure of perfectoid spaces

Def'n A perfectoid space X is (strictly) totally disconnected if it is qcqs (in fact, affinoid), and every
· étale cover splits (for strictly tot disc)

resp. • open cover splits (for tot disc).

(c.f. étale covers = open covers + finite étale covers
(by de Jong – van der Put).)

Prop X is (strictly) totally disconnected
iff X qcqs and all fibres of

$$X \longrightarrow \pi_0(X)$$

profinite set

are of form $\mathrm{Spa}(K, K^+)$,
where K is a perfectoid field.

(= complete nonarch + nondiscontinuously valued field
s.t. Φ surj on $\mathcal{O}_K/p$)

and $\mathfrak{m}_{\mathcal{O}_K} \subseteq K^+ \subseteq \mathcal{O}_K$ open val'n subring.

(resp. K is further alg closed for strictly tot disc case.)

Rmk (Understanding $\mathrm{Spa}(K, K^+)$.) K^+ open $\Rightarrow$ $K^+ \supseteq \mathfrak{m}_{\mathcal{O}_K}$

$\rightsquigarrow$ $\underbrace{K^+/\mathfrak{m}_{\mathcal{O}_K}}_{\text{valuation ring.}} \subseteq \mathcal{O}_K/\mathfrak{m}_{\mathcal{O}_K} \simeq k$ field

$\rightsquigarrow$ $|\mathrm{Spa}(K, K^+)| \cong |\mathrm{Spec}(K^+/\mathfrak{m}_{\mathcal{O}_K})|$

Totally ordered chain of specializations,

w/ generic pt $\hat{=}$ $\mathrm{Spa}(K, \mathcal{O}_K) \hookrightarrow \mathrm{Spa}(K, K^+)$.

$\rightsquigarrow$ $\mathrm{Spa}(K, K^+)$ = (unique) rk 1 generalization of any pt

Cor Assume $X = \mathrm{Spa}(R, R^+)$ totally disc,

$f : Y = \mathrm{Spa}(S, S^+) \to X$ any affinoid adic space $/X$.

Then $R^+/\varpi \to S^+/\varpi$ flat

for any pseudo-unif $\varpi \in R^+$.

(and faithfully flat if $|f|$ is surjective).

<u>Proof</u> Can be checked on connected components.

(flat ~ local property)

Then $(R, R^+) = (K, K^+)$, so K^+ val'n ring.

[<u>Note</u>: $S^+ \subseteq S = S^+[\frac{1}{\varpi}]$ is ϖ-torsion free,
so S^+ is flat $/K^+$.]

Hence S^+/π flat over K^+/ϖ.

For faithful flatness, use

$$|\mathrm{Spa}(K, K^+)| \cong |\mathrm{Spec}(K^+/\varpi)|. \qquad \square$$

This allows us to deduce <u>v-descent results</u>

(i.e all maps $f: Y \to X$ s.t. X, Y qc & $|f|$ surj)

from pro-étale descent and faithfully flat descent.

<u>Def'n</u> A <u>diamond</u> is a pro-étale sheaf Y on

$\mathrm{Perf} := \{\text{perfectoid spaces} / \mathbb{F}_p\}$

that can be written in form

$$Y = X / R$$

where - X perfectoid space

- $R \subseteq X \times X$ equiv. relation repr'd by a perf'd space s.t. $s, t : R \to X$ are pro-étale.

Here, use Yoneda embedding

$$\begin{array}{ccc} \mathrm{Perf} & \longrightarrow & \{\text{pro-étale sheaves on } \mathrm{Perf}\} \\ X & \longmapsto & \mathrm{Hom}(-, X). \end{array}$$

<u>Facts</u> (1) Category of diamonds has all fibre products, products, cofilt inv lims (all nonempty lims)

But <u>no</u> final object!

The final obj (if exists) should be $\mathrm{Spa}\,\mathbb{F}_p$.

but not perf'd space as <u>not analytic</u>

(no top-nilp unit),

and cannot adjoin one through pro-étale covers.

(2) If $f: Y \to X$ quasi-pro-étale map, then

$$Y \text{ diamond } \underset{\Rightarrow}{\overset{\text{automatic}}{\Leftarrow}} X \text{ diamond}$$

if f surj as map of pro-ét sheaves.

<u>Def'n</u> A map $f: Y \to X$ of pro-étale sheaves on Perf is <u>quasi-pro-étale</u> if for all strictly tot disc perf'd spaces X', $X' \to X$, the fibre product $f': Y' = Y \times_X X' \longrightarrow X'$ is repr'd in perf'd spaces & is pro-étale.

<u>Facts</u> (3) Y is a diamond $\Leftrightarrow$ $\exists\, f: X \to Y$ surj + quasi-pro-ét & X perfectoid space.

(4) Can introduce underlying top space

$$|Y| = |X| / |R|$$
(is independent of presentation).

<u>Example</u> Fix geom base pt $S = \mathrm{Spa}(C, \mathcal{O}_C)$.

$$\mathrm{ProFin} \hookrightarrow \mathrm{Perf}_{/S}$$
$$T = \varprojlim_i T_i \longmapsto \varprojlim_i (T_i \times S) = \underline{T} \times S$$
$$= \mathrm{Spa}(\mathrm{Cont}(T, C), \mathrm{Cont}(T, \mathcal{O}_C)).$$

<u>Recall</u> Any $T \in \mathrm{CHaus}$ (cpt Hausdorff spaces) satisfies $T \cong \tilde{T} / R$,
- $\tilde{T}$ profin set = tot disc cpt Haus space
- $R \subseteq \tilde{T} \times \tilde{T}$ closed equiv relation.

$\rightsquigarrow$ $\mathrm{CHaus} \hookrightarrow \{\text{diamond} / S\}$

$$T \longmapsto (X \in \mathrm{Perf}_{/S} \mapsto \mathrm{Cont}(|X|, T))$$

This sheaf is rep'd by $(\underline{\tilde{T}} \times \mathrm{Spa}(C, \mathcal{O}_C)) / (\underline{R} \times \mathrm{Spa}(C, \mathcal{O}_C))$.

<u>Def'n</u> (1) A diamond $Y = X / R$ is <u>spatial</u> if
- it is qcqs ($\Rightarrow$ can choose X, R qcqs)
- $|Y|$ is <u>spectral</u>.
  (i.e. $|Y| \simeq \varprojlim$ (finite T_0 spaces),
  $|Y| \simeq |\mathrm{Spec}\, A|$ for some ring A,
  & has a good basis of qc open subsets)

& - $|X| \to |Y|$ is spectral
  (i.e. preimage of qc open is qc open).

(2) Y is <u>locally spatial</u> if it has an open cover by spatial $\mathcal{U} \subset Y$.
($\rightsquigarrow$ $|Y|$ is locally spatial).

<u>Note</u> (1) Y spatial $\Leftrightarrow$ Y locally spatial + $|Y|$ qcqs.
(2) In practice, all relevant diamonds are locally spatial.

<u>Rmk</u> {locally spatial diamonds}
has all
- (fibre) products
- all cofilt lims w/ qcqs transition maps.

<u>Structure of locally spatial diamonds</u>

Let $|Y|$ = underlying locally spectral of Y.
For each $y \in |Y|$, has localization
$$\varprojlim_{\mathcal{U} \ni y} \mathcal{U} = Y_y \subseteq Y \quad \text{of } Y \text{ at } y,$$
$\rightsquigarrow$ $Y_y = \mathrm{Spa}(C, C^+) / \underline{G}$
w/ C = complete alg closed nonarch field
$\mathfrak{m}_{\mathcal{O}_C} \subseteq C^+ \subseteq \mathcal{O}_C$ valuation subring.
G profin grp acting continuously & faithfully on C.

<u>Diamond functor</u>

To construct $\{\text{analytic adic spaces} / \mathbb{Z}_p\} \longrightarrow \{\text{diamonds}\}$
$$X \longmapsto X^{\diamond}.$$

Def'n / Prop For analytic adic space $X/\mathbb{Z}_p$, the functor

$$X^\diamond : S \in \mathrm{Perf} \longmapsto \left\{ \begin{array}{c} S^\#/\mathbb{Z}_p \text{ untilt of } S \\ + \text{ map } S^\# \to X \end{array} \right\}$$

defines a locally spatial diamond.

Moreover, there are canonical equiv.

$$|X| \cong |X^\diamond|, \quad X_{\text{ét}} \cong X^\diamond_{\text{ét}}.$$

If X is perfectoid, then $X^\diamond \cong X^\flat$.

Slogan $X^\diamond$ remembers top info about X,

but forgets structure map to $\mathrm{Spa}\,\mathbb{Z}_p$.

Sketch If X perfectoid,

$$\{S^\# + S^\# \to X\} \xrightarrow{\sim} \{S \to X^\flat\}$$

by tilting equivalence.

$\rightsquigarrow$ $X^\diamond$ is rep'd by $X^\flat$,

$$\left. \begin{array}{c} |X^\diamond| \cong |X^\flat| \cong |X| \\ X^\diamond_{\text{ét}} \cong X^\flat_{\text{ét}} \cong X_{\text{ét}} \end{array} \right\} \text{ by tilting equiv for top spaces \& étale sites.}$$

In general, use that any $\underset{\text{analytic adic space } (/\mathbb{Z}_p)}{\underset{\cap}{X}}$ admits pro-étale surjection

$\tilde{X} \to X$ from $\tilde{X}$ perfectoid.

General construction of $\tilde{X}$ (by Colmez, Faltings):

Locally $X = \mathrm{Spa}(A, A^+)$. If $A/\mathbb{Q}_p$ for simplicity,

adjoining x^{1/p^∞} is pro-étale whenever $x \in A^\times$,

(for example $x \in 1 + pA^+$).

This defines affinoid pro-étale perfectoid cover. $\square$

<u>Rmk</u> <u>Prop</u> (Kedlaya-Liu)

$$\left\{\begin{matrix}\text{seminormal rigid-analytic} \\ \text{spaces } / \mathbb{Q}_p\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{diamonds} \\ /(\mathrm{Spa}\,\mathbb{Q}_p)^\diamond\end{matrix}\right\}$$

$$X \longmapsto X^\diamond$$

is a fully faithful functor.

<u>Note</u> $(\mathrm{Spa}\,\mathbb{Q}_p)^\diamond(S) = \{S^\# \text{ untilt } / \mathbb{Q}_p \text{ of } S\}$
parametrizes untilts of S.

<u>Back to relative Fargues-Fontaine curve</u>

If $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}/\mathbb{F}_q$,

$$\rightsquigarrow \quad \mathcal{Y}_{S,E} = \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \setminus \{[\varpi] = 0\}.$$

$$\cup\!|$$

$$Y_{S,E} = \{\pi \neq 0\}.$$

<u>Thm</u> ("Diamond equation")

$$Y_{S,E}^\diamond = S \times (\mathrm{Spa}\, E)^\diamond.$$

Equivalently, given perf'd space $T / \mathbb{F}_q$,

untilts $T^\# / Y_{S,E}$ $\triangleq$ untilts $T^\# / E$ + maps $T \to S$.

<u>Sketch</u>. Given $T^\# / E$, need to see that

maps $T^\# \longrightarrow Y_{S,E} \,/\, E \;\hat{=}\;$ maps $T \longrightarrow S \,/\, \mathbb{F}_q$.

Let $T^\# = \mathrm{Spa}(A, A^+)$. Then for $S = \mathrm{Spa}(R, R^+)$,

$$T^\# \to Y_{S,E} \subseteq \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \text{ are given by}$$

maps $W_{\mathcal{O}_E}(R^+) \longrightarrow A^+$

s.t. $[\varpi], \pi \longmapsto$ units of A.

(automatic for $\varpi \in R^+$, as $T^\# / E$.

Adjunction between $W_{\mathcal{O}_E}$(perf rings)

& tilting: $W_{\mathcal{O}_E}(R^+) \longrightarrow A^+$ s.t. $[\varpi] \mapsto$ unit of A

$$\hat{=} \text{ maps } \begin{pmatrix} R^+ \longrightarrow (A^+)^\flat = \varprojlim_{x \mapsto x^p} A/\pi \\ \varpi \longmapsto \text{unit of } A^\flat. \end{pmatrix}$$

$$\hat{=} \text{ maps } \quad \underset{\mathrm{Spa}(A^\flat, A^{\flat +})}{T} \longrightarrow \underset{\mathrm{Spa}(R, R^+)}{S}$$

$\square$

Obtain a canonical map

$$\begin{tikzcd} {|Y_{S,E}|} \arrow[d] & \cong {|Y^\diamond_{S,E}|} \cong & {|S \times (\mathrm{Spa}\, E)^\diamond|} \arrow[d] \\ {|S|} \arrow[u, bend right] & & {|S|} \end{tikzcd}$$

of top spaces (but not for adic spaces $Y_{S,E} \not\to S$)

<u>Prop</u> For $S' \subseteq S$ open affinoid subset,

$$Y_{S',E} \hookrightarrow Y_{S,E} \quad \text{open immersion}$$

with $|Y_{S',E}| = |Y_{S,E}| \times_{|S|} |S'|$.

Can prove this by diamond equations. $\square$

$\rightsquigarrow$ Can glue $Y_{S,E} \circlearrowleft \phi_S$

for general perfectoid spaces $S / \mathbb{F}_q$

s.t. $Y_{S,E}^\diamond \cong S \times (\mathrm{Spa}\, E)^\diamond$.

Slogan $Y_{S,E}$ is the analytic adic space $/ E$

with $Y_{S,E}^\diamond = S \times (\mathrm{Spa}\, E)^\diamond$ over $(\mathrm{Spa}\, E)^\diamond$.

$\rightsquigarrow$ Varing in "families" w.r.t. S.

Def'n $X_{S,E} := Y_{S,E} / \phi^{\mathbb{Z}}$ "relative Fargues-Fontaine curve".

Apply diamond formula to get

$$X_{S,E}^\diamond \cong S / \phi_S^{\mathbb{Z}} \times (\mathrm{Spa}\, E)^\diamond.$$

Note $(\mathrm{Spa}\, E)^\diamond = \mathrm{Div}_Y^1$

$((\mathrm{Spa}\, E)^\diamond \times (\mathrm{Spa}\, E)^\diamond) / \Sigma_2 = \mathrm{Div}_Y^2$.

Prop All diamonds are v-sheaves.

Rmk For any adic space $X / \mathbb{Z}_p$, can define $X^\diamond$ as v-sheaf.

$$S \longmapsto \{ S^\sharp \text{ untilt} + S^\sharp \longrightarrow X \}$$

But $|X^\diamond| \longrightarrow |X|$ usually far from an isom

for X not perf'd (c.f. works of Ian Gleason).

Back to BC spaces : $S \longmapsto H^1(X_{S,E}, \mathcal{O}(-1)) = (\mathbb{A}_E^1)^\diamond / E$.

Lecture 7: Untilts, $\mathcal{O}(1)$, and Lubin-Tate theory

<u>Setup</u> E nonarch local field, $\mathcal{O}_E \ni \pi$, $\mathbb{F}_q \subset \overline{\mathbb{F}}_q$.

$\breve{E} = W_{\mathcal{O}_E}(\overline{\mathbb{F}}_q)[\frac{1}{\pi}]$ completion of max unram ext'n.

$S \in \mathrm{Perf}_{\mathbb{F}_q}$ perf'd space / $\mathbb{F}_q$.

$$\rightsquigarrow \quad \begin{array}{c} Y_{S,E} \\ \downarrow \\ X_{S,E} = Y_{S,E}/\phi^{\mathbb{Z}} \end{array} \qquad Y_{S,E}^{\diamond} = S \times (\mathrm{Spa}\, E)^{\diamond}.$$

Given an untilt $S^\#$ of S,

locally $S = \mathrm{Spa}(R, R^+)$, $S^\# = \mathrm{Spa}(R^\#, R^{\#+})$

and there is a canonical surjection

$$\theta: W_{\mathcal{O}_E}(R^+) \twoheadrightarrow R^{\#+}$$

$$\begin{aligned} (\hat{=} \quad & R^+ \longrightarrow R^{\#+}/\pi \\ \hat{=} \quad & R^+ = \varprojlim_{\phi} R^{\#+}/\pi \longrightarrow R^{\#+}/\pi .) \end{aligned}$$

with $\ker \theta = (\xi)$, ξ = nonzero-divisor in $W_{\mathcal{O}_E}(R^+)$.

(This is a general result on $\underbrace{\text{integral perfectoid rings}}_{\text{"perfect prisms.}}$)

$$\rightsquigarrow \quad \begin{array}{c} S^\# \\ \parallel \\ V(\xi) \end{array} \hookrightarrow \begin{array}{c} \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \setminus \{\pi = 0 \text{ or } [\varpi] = 0\} \\ \parallel \\ Y_{S,E} = \{\pi \neq 0,\ [\varpi] \neq 0\}. \end{array}$$

presents $S^\#$ as a Cartier divisor in $Y_{S,E}$.

<u>Def'n</u> Let X uniform analytic adic space.

$\left(\begin{array}{l} \text{i.e. analytic} = \text{locally } \mathrm{Spa}(R, R^+),\ R \text{ Tate.} \\ \quad \text{uniform} = R \text{ uniform + spectral norm is a norm} \\ \qquad\qquad \text{i.e. } R^\circ \subseteq R \text{ bounded } (\Leftrightarrow R^+ \subseteq R \text{ bounded}). \end{array}\right)$

A <u>closed Cartier divisor</u> on X is an ideal sheaf

$$\mathcal{I} \subseteq \mathcal{O}_X, \quad \text{locally free of rk } 1,$$

s.t. $\forall$ affinoid $U \subseteq X$,

$$\mathcal{I}(U) \hookrightarrow \mathcal{O}_X(U) \text{ has closed image.}$$

$\rightsquigarrow$ $Z = (V(\mathcal{I}), \mathcal{O}_X/\mathcal{I} + \text{valuations}) \subseteq X$

defines an adic space itself.

<u>Prop</u> (1) $V(\xi) = S^\# \hookrightarrow Y_{S,E}$ is a closed Cartier divisor.

(2) Also $S^\# \hookrightarrow Y_{S,E} \twoheadrightarrow X_{S,E}$

defines a closed Cartier divisor on $X_{S,E}$.

(c.f. Prop 11.3.1 in Berkeley lectures.)

<u>Def'n</u> Let $\mathrm{Div}^1_Y, \mathrm{Div}^1_X : \mathrm{Perf}_{\overline{\mathbb{F}}_q} \longrightarrow \mathrm{Sets}$ (to work geometrically)

be the functors taking S to the set of closed Cartier divisors on $Y_{S,E}$ (resp. $X_{S,E}$) that locally on S arise as

$$S^\# \hookrightarrow Y_{S,E} \quad (\text{resp. } S^\# \hookrightarrow X_{S,E})$$

for untilt $S^\# / E$.

$\rightsquigarrow$ $\mathrm{Div}^1_Y, \mathrm{Div}^1_X$ = "moduli space of deg 1 Cartier divisor".

Prop (1) $\mathrm{Div}^1_Y = (\mathrm{Spa}\,\breve{E})^\diamond$

(2) $\mathrm{Div}^1_X = \mathrm{Div}^1_Y / \phi_E^{\mathbb{Z}} = (\mathrm{Spa}\,\breve{E})^\diamond / \phi_E^{\mathbb{Z}}$.

In (2), note that $(\mathrm{Spa}\,\breve{E})^\diamond = \overbrace{(\mathrm{Spa}\,E)^\diamond}^{\circlearrowleft \phi_E} \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$.

b/c $(\mathrm{Spa}\,E)^\diamond$ is a functor on $\underset{\circlearrowleft \phi_S}{S} \in \mathrm{Perf}_{\mathbb{F}_q}$.

Proof (1) By def'n, $(\mathrm{Spa}\,\breve{E})^\diamond \twoheadrightarrow \mathrm{Div}^1_Y$

as any Cartier divisor param by Div^1_Y

locally comes from an untilt $S^\# / \breve{E}$.

But conversely,

a closed Cartier divisor on $Y_{S,E}$ determines $Z \subseteq Y_{S,E}$,

locally on S, this gives an untilt $S^\#$.

$\rightsquigarrow$ $Z/\breve{E}$ is an untilt $S^\#$ of S

$\rightsquigarrow$ $(\mathrm{Spa}\,\breve{E})^\diamond \xrightarrow{\sim} \mathrm{Div}^1_Y$.

(2) Take quotient by Frob. $\square$

Warning $X^\diamond_{S,E} = (S/\phi_S^{\mathbb{Z}}) \times (\mathrm{Spa}\,\breve{E})^\diamond$,

But $\underbrace{\mathrm{Div}^1_X}$ $= (\mathrm{Spa}\,\breve{E})^\diamond / \phi_S^{\mathbb{Z}}$.

("mirror FF curve", only a diamond,

(i.e. not from an adic space).

not quasi-separated, not locally spatial.

underlying top space $= |\mathrm{Spa}\,\breve{E}| = pt$.

So moduli of "pts" on the curve $\neq$ the curve.

$\mathcal{O}(1)$ + Lubin-Tate theory

<u>Recall</u> $\mathcal{O}_{X_{S,E}}(1)$ is the line bdl on $X_{S,E}$ corresp. to $(\breve{E}, \pi^{-1}\sigma)$.

In particular, by descent

$$H^0(X_{S,E}, \mathcal{O}_{X_{S,E}}(1)) = H^0(Y_{S,E}, \mathcal{O}_{Y_{S,E}})^{\phi_S = \pi}.$$

<u>Goal</u> If $S^\#$ any untilt $/E$ of S, then

$$I_{S^\#} \subseteq \mathcal{O}_{X_{S,E}}$$

is (at least after pro-ét localization on S) isom to $\mathcal{O}_{X_{S,E}}(-1)$.

$\rightsquigarrow$ (In this sense, $S^\# \hookrightarrow X_{S,E}$ is of deg 1).

To do this, need to construct

$$\text{maps} \quad \mathcal{O}(-1) \cong I_{S^\#} \hookrightarrow \mathcal{O}$$

$$\triangleq \quad \text{maps} \quad \mathcal{O} \longrightarrow \mathcal{O}(1)$$

$$\triangleq \quad \text{elts} \in H^0(X_{S,E}, \mathcal{O}(1)) \text{ vanishing along } S^\#.$$

<u>So</u> will give a formula for $H^0(X_{S,E}, \mathcal{O}(1))$ in terms of a Lubin-Tate formal grp.

<u>Recall</u> A <u>Lubin-Tate formal grp</u> is a 1-dim'l formal grp $G / \mathcal{O}_{\breve{E}}$ + an $\mathcal{O}_E$-action s.t. the two induced actions on Lie G agree, and "of ht 1".

i.e. $[\pi]_G(x) = \pi x + \underbrace{a_2 x^2 + \cdots}_{O(x^2)}$

(meaning of ht: mod π, first nonzero coeff is necessarily $a_{q^h} X^{q^h}$, $h \in \{1, 2, \cdots, \infty\}$ ht of G.)

Thm Such G is unique up to isom.

Example $E = \mathbb{Q}_p$,

$\rightsquigarrow$ G = formal multiplication grp

$= \mathrm{Spf}\, \breve{\mathbb{Z}}_p [\![X]\!] \quad (\breve{\mathbb{Z}}_p = W(\bar{\mathbb{F}}_p))$.

$X +_G Y = (1+X)(1+Y) - 1$

Check: $\pi = p \in \mathbb{Q}_p$, $[\pi]_G(X) = pX + O(X^2)$.

In general, $G \times_{\mathcal{O}_{\breve{E}}} \breve{E} \underset{\log_G}{\cong} \mathbb{G}_{a, \breve{E}}$ additive grp.

Rmk One can choose $G \cong \mathrm{Spf}\, \mathcal{O}_{\breve{E}} [\![X]\!]$

so that $\log_G(X) = X + \frac{1}{\pi} X^q + \frac{1}{\pi^2} X^{q^2} + \cdots$.

$\rightsquigarrow$ $X +_G Y = \exp_G(\underbrace{\log_G(X) + \log_G(Y)}) \in \mathcal{O}_{\breve{E}} [\![X, Y]\!]$.

where $\exp_G := \log_G^{-1}$.

Connection to local class field theory

For any $n \geq 1$, $G[\pi^n] \subseteq G$ kernel of multi-by-π^n on G.

$G[\pi^n] = \mathrm{Spf}\, \mathcal{O}_{\breve{E}} [\![X]\!] / ([\pi^n]_G(X))$.

(e.g. $E = \mathbb{Q}_p$, get $G = \mathrm{Spf}\, \breve{\mathbb{Z}}_p [\![X]\!]$

$\rightsquigarrow$ $G[\pi^n] = \mu_{p^n}$ = grp of p^n-th roots of unity.)

$$\rightsquigarrow \quad \underbrace{G[\pi^n]}_{\substack{\circlearrowleft \\ (\mathcal{O}_E/\pi^n)^\times}} \times_{\mathcal{O}_{\breve{E}}} \breve{E} = \coprod_{i=0}^{n} \operatorname{Spec} \breve{E}_i,$$

with $\breve{E} = \breve{E}_0 \subset \breve{E}_1 \subset \cdots \subset \breve{E}_n \subset \cdots$

adjoin a primitive π^n-torsion pt.

<u>Thm</u> The maximal abelian ext'n of E is

$$\bigcup_{i=0}^{\infty} \breve{E}_i \quad \text{(up to completion issues).}$$

And have a canonical isom

$$\operatorname{Gal}(\breve{E}_n/\breve{E}) = (\mathcal{O}_E/\pi^n)^\times.$$

Also, $\breve{E}_\infty$ = completion of $\bigcup_{i=0}^{\infty} \breve{E}_n$ is perfectoid.

<u>"Universal cover" $\tilde{G}$ of G</u>

<u>Def'n</u> $\tilde{G} := \varprojlim_{[\pi]_G} G$ (in cat of formal schs).

$$\rightsquigarrow \quad \begin{tikzcd} & \tilde{G} \arrow[dl, "f_n"'] \arrow[d, "f_1"] \arrow[dr, "f_0"] & \\ G \quad \cdots & G & G \end{tikzcd}$$

<u>Prop</u> $\tilde{G} \cong \operatorname{Spf} \mathcal{O}_{\breve{E}}[[x^{1/p^\infty}]]$.

<u>Proof</u> Enough to show: $\tilde{G} \times_{\mathcal{O}_{\breve{E}}} \bar{\mathbb{F}}_q \cong \operatorname{Spf} \bar{\mathbb{F}}_q[[x^{1/p^\infty}]]$

But $[\pi]_G(x) \equiv x^q \mod \pi$. $\square$

Have proj maps $\tilde{G} \xrightarrow{f_n} G$

$$\triangleq \left(\mathcal{O}_{\breve{E}}[[\tilde{x}^{1/p^\infty}]] \longleftarrow \mathcal{O}_{\breve{E}}[[x_n]] \right).$$

Due to the restriction on the maximal size of this file, I have to truncate the notes at this point.