Compactification of Shimura varieties of abelian type

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Lecture 1: Char 0 theory

<u>Sh vars</u> G red grp / $\mathbb{Q}$,

$\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$, $\quad X = G(\mathbb{R})$-conj classes of $\{\mathbb{S} \to G_{\mathbb{R}}\}$.

$\rightsquigarrow$ (G, X) + axioms, called Shimura datum.

Assume
- no cpt type $\mathbb{Q}$-simple factors in G^{ad}.
- $h(i)$ induces Cartan involution of $G^{ad}_{\mathbb{R}}$.
- adjoint action on $\mathfrak{g}$ has type $(1,-1)$, $(0,0)$, $(-1,1)$.

<u>Ex</u> (V, ψ) s.t. $\dim_{\mathbb{Q}} V = 2g$ $(g>0)$

ψ alternative form on V.

- $h: x+iy \longmapsto \mathrm{diag}\{\underbrace{\begin{pmatrix} x & y \\ -y & x\end{pmatrix}, \cdots, \begin{pmatrix} x & y \\ -y & x\end{pmatrix}}_{g \text{ copies}}\} \in GL_{2g}(\mathbb{R})$
- $GSp(V,\psi)(R) := \{ g \in GL(V)(R) \mid \psi(gu, gv) = \nu(g)\psi(u,v),\ \nu(g) \in R^\times \}$.

  inducing similitude char

$$GSp(V,\psi) \to \mathbb{G}_m, \quad g \mapsto \nu(g).$$

This forms a Sh datum $(G, X) := (GSp(V,\psi), \underbrace{\{h\}}_{\text{Conj classes of } h})$

called Siegel Shimura datum.

<u>Note</u> Sh datum + cpt open subgrp (cos) $K \subset G(\mathbb{A}_f)$

$\rightsquigarrow \ Sh_K(G,X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K.$

When K sufficiently small,

$Sh_K(G,X)(\mathbb{C})$ algebrizes to $Sh_K(G,X)_{\mathbb{C}}$

& $Sh_K(G,X)_{\mathbb{C}}$ descends to $Sh_K(G,X)$ $/E$.

For $K_1 \subset K_2$, get $Sh_{K_1} \longrightarrow Sh_{K_2}$ $/E$

$\rightsquigarrow$ tower $(Sh_K)_K$.

<u>Toroidal cpt'n</u>

$$\begin{tikzcd} Sh_K \arrow[d, "\text{open dense}"] & \text{normal but not proper} & /E \\ Sh_K^{tor} & \text{normal \& proper} & /E \end{tikzcd}$$

Want a stratification $\underbrace{Sh_K^{tor} = \text{"}\coprod_Z \mathcal{Z}_Z\text{"}}_{\text{as top spaces}}$ + boundary data

Want each $Sh_{\mathcal{Z}_Z}^{tor,\wedge} \cong$ completion of some affine toric var.

Need torus-torsor of Pink:

$$\begin{tikzcd} S_Z^* \arrow[r, hook] \arrow[dr] & S_Z^*(\sigma) \arrow[d] \\ & \bar{S}_Z^* \end{tikzcd}$$

torsor under split torus E_Z.

S_Z^* is closely related to mixed Shimura varieties.

* From (G,X), how to construct mixed Sh vars?

<u>Step 1</u> Find admissible $\mathbb{Q}$-parabolic subgrps $Q \subset G$.

Meaning: $G^{ad} = G_1 \times G_2 \times \cdots \times G_n$,

G_i's = $\mathbb{Q}$-simple factors.

$\rightsquigarrow \quad Q^{ad} = Q_1 \times Q_2 \times \cdots \times Q_n,$

each $Q_i = G_i$ or max proper $\mathbb{Q}$-parabolic $\subset G_i$.

$\rightsquigarrow \quad Q :=$ preimage of Q^{ad} in G along $G \to G^{ad}$.

e.g. $\quad G = GSp(V, \psi)$,

$Q \;\hat{=}\;$ isotropic subspace $H \subset V$

$$\begin{array}{lccccccc} & 0 & \subset & H & \subset & H^{\perp} & \subset & V \quad \text{wt filtration} \\ \text{wt} & & & -2 & & -1 & & 0 \end{array}.$$

<u>Step 2</u> Find certain normal $\mathbb{Q}$-subgrp $P_Q \subset Q$

e.g. $\quad G = GSp(V, \psi)$,

$Q \rightsquigarrow P_Q$

s.t. $P_Q(R) := \{ g \in GSp(V,\psi)(R) \mid g|_{W_{-2}} = \nu(g)\cdot id,\ g|_{Gr_0^W} = id \}$.

This looks like $\begin{pmatrix} \nu(g) & & & * & * \\ & \ddots & & & \\ & & \nu(g) & & \\ & & & \boxed{\begin{array}{c}\text{smaller}\\ GSp\end{array}} & * \\ & & & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}$

but not $\begin{pmatrix} 1 & & & * & * \\ & \ddots & & & \\ & & 1 & & \\ & & & \boxed{GSp} & * \\ & & & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}$.

Now $\quad Unip(P_Q) = Unip(Q) = W_Q$.

$U_Q :=$ Center of W_Q.

$Q \rightsquigarrow (P_Q, W_Q, U_Q)$.

<u>Construction</u> $\quad h: \mathbb{S} \to G_{\mathbb{R}}$

$\rightsquigarrow \quad h_\infty : \mathbb{S}_{\mathbb{C}} \to P_{Q,\mathbb{C}}$

Key Hodge cochar remains the same along $h \rightsquigarrow h_\infty$ but the wt cochar are changed.

e.g.
$$h_\infty : x+iy \longmapsto \begin{pmatrix} \boxed{\begin{matrix} x^2+y^2 & & \\ & \ddots & \\ & & x^2+y^2 \end{matrix}} & & \\ & \boxed{\begin{matrix} x & -y & & & \\ y & x & & & \\ & & \ddots & & \\ & & & x & -y \\ & & & y & x \end{matrix}} & \\ & & \boxed{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}} \end{pmatrix}$$

$$\omega : x \longmapsto \begin{pmatrix} x^2 & & & & & & & \\ & \ddots & & & & & & \\ & & x^2 & & & & & \\ & & & x & & & & \\ & & & & \ddots & & & \\ & & & & & x & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix}.$$

In general, $P_Q :=$ smallest normal $\mathbb{Q}$-subgrp $\subset Q$ s.t. h_∞ factors through.

Step 3 (Pink) A $Q(\mathbb{R})\, U_Q(\mathbb{C})$-equiv map
$$\begin{aligned} X &\longrightarrow \pi_0(X) \times \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{Q,\mathbb{C}}) \\ x &\longmapsto ([x],\ h_{x,\infty}) \end{aligned}$$

$\rightsquigarrow$ For $X^+ \subset X$ conn comp,
$$D_{Q,X^+} := P_Q(\mathbb{R})\, U_Q(\mathbb{C})\text{-orbit of } X^+ \text{ in } \pi_0(X) \times \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{Q,\mathbb{C}}).$$

Now $Q \rightsquigarrow (P_Q, D_{Q,X^+})$ a mixed Shimura datum.

Step 4 Cusp label representative.
$$\Phi := (Q_\Phi,\ X^+_\Phi,\ g_\Phi),$$
where Q_Φ adm $\mathbb{Q}$-parabolic,
X^+_Φ conn comp $\subset X$, $g_\Phi \in G(\mathbb{A}_f)$.

Take $P_\Phi := P_{Q_\Phi}$, $D_\Phi := D_{Q_\Phi, X_\Phi^+}$.

$\rightsquigarrow$ $Sh_{K_\Phi}(P_\Phi, D_\Phi)(\mathbb{C}) = P_\Phi(\mathbb{Q}) \backslash D_\Phi \times P_\Phi(\mathbb{A}_f) / K_\Phi$,

$$K_\Phi = P_\Phi(\mathbb{A}_f) \cap g_\Phi K g_\Phi^{-1}.$$

$\rightsquigarrow$ algebrizes to $Sh_{K_\Phi, \mathbb{C}}$

$\rightsquigarrow$ descends to $Sh_{K_\Phi} / E = E(G, X)$.

<u>Note</u>

$$Sh_{K_\Phi} \xrightarrow[\text{torus-torsor}]{(\ E_\Phi} \overline{Sh}_{K_\Phi} \xrightarrow{\text{proper}} Sh_{K_{\Phi,h}},$$

$$\triangleq \quad (P_\Phi, D_\Phi) \longrightarrow (P_\Phi / U_\Phi, \underbrace{D_\Phi / U_\Phi(\mathbb{C})}_{=: \overline{D}_\Phi}) \longrightarrow (\underbrace{P_\Phi / W_\Phi}_{=: P_{\Phi,h}}, \underbrace{\overline{D}_\Phi / W_\Phi(\mathbb{R})}_{=: D_{\Phi,h}}).$$

<u>Rmk</u> If (G, X) Hodge type, say

$$(G, X) \hookrightarrow (GSp(V, \psi), X'),$$

then $(P_{\Phi,h}, D_{\Phi,h})$ is a Shimura datum in usual sense.

<u>About E_Φ</u> $\quad E_\Phi := U_\Phi(\mathbb{C}) / \Lambda \quad$ (imagine $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^\times$).

where $\Lambda(-1) \subset U_\Phi(\mathbb{R}) \underbrace{(-1)}_{\text{twist by multi by } \frac{1}{2\pi i}}$ lattice

$$\Lambda = pr_2\left((Z_\Phi(\mathbb{Q})^\circ \times U_\Phi(\mathbb{Q})) \cap g_\Phi K g_\Phi^{-1} \right)$$

$$Z_\Phi = \mathrm{Cent}(P_\Phi).$$

<u>Warning</u> $U_\Phi \neq \mathrm{Cent}(G)$ (e.g. $G = GSp(V, \psi)$.)

Recall $\Phi \rightsquigarrow U_\Phi \rightsquigarrow \mathbb{P}_\Phi \subset U_\Phi(\mathbb{R})$ cone in $U_\Phi(\mathbb{R})$.

Let Σ cone decomp + axioms.

$\rightsquigarrow$ $\Sigma(\Phi)$ = cone decomp of $\mathbb{P}_\Phi$.

Thm (Pink) Given (Φ, σ), $\sigma \in \Sigma(\Phi)$ + good assumptions.

Then $\exists$ $Sh_K^{\Sigma} = \coprod Z_{[(\Phi,\sigma)]}$, $(\Phi,\sigma)/\sim = [(\Phi,\sigma)]$

s.t. $Sh^{\Sigma, \wedge}_{K, Z_{[(\Phi,\sigma)]}} \cong \underbrace{\Delta^{\circ}_{\Phi,K}}_{} \backslash \left(Sh_{K_\Phi}(\sigma)^{\wedge}_{Sh_{K(\Phi,\sigma)}} \right)$

acts through a finite quotient.

A partial order on (Φ,σ): $(\Phi_1,\sigma_1) \leq (\Phi_2,\sigma_2)$

$\iff \exists\, \gamma \in G(\mathbb{Q})$

s.t. • $\gamma P_{\Phi_1} \gamma^{-1} \subset P_{\Phi_2}$,

• $P_{\Phi_1}(\mathbb{Q})\gamma \cdot X^+_{\Phi_1} = P_{\Phi_2}(\mathbb{Q}) \cdot X^+_{\Phi_2}$.

• $\gamma g_{\Phi_1} = a\, g_{\Phi_2} \mod K$ for $a \in P_{\Phi_2}(\mathbb{A}_f)$.

• $U_{\Phi_2}(\mathbb{R})(-1) \xrightarrow{\gamma^{-1}(\cdot)\gamma} U_{\Phi_1}(\mathbb{R})(-1)$,

then $\gamma^{-1}\sigma_2\gamma \subset \sigma_1$ as a face.

Lecture 2: Abelian-type compactification

Last time (G, X) Shimura datum

Σ "admissible" cone decomp.

$$\begin{tikzcd} Sh_K \arrow[r, hook] \arrow[dr] & Sh_K^{\Sigma} = \coprod Z_{[(\Phi,\sigma)]} \arrow[d] \\ & \operatorname{Spec} E \end{tikzcd}$$

While constructing $Sh_K^{\Sigma}(\mathbb{C})$, it is not taking disjoint union of $Z_{[(\Phi,\sigma)]}$'s.

Roughly In Pink, Faltings-Chai, Lan, Madapusi $\frown$ Hodge

analytically — Siegel — PEL + integral model

Define Sh_K^{Σ} via its étale covering ("pro-étale way").

Recall · $Sh^{\Sigma,\wedge}_{K,Z([\Phi,\sigma])} \cong \Delta^{\circ}_{\Phi,K} \backslash Sh_{K_\Phi}(\sigma)^{\wedge}_{Sh_{K_\Phi,\sigma}}$.

where $\Delta_{\Phi,K} = (\mathrm{Stab}_{Q_\Phi(\mathbb{Q})}(D_\Phi) \cap P_\Phi(\mathbb{A}_f) g_\Phi K g_\Phi^{-1}) / P_\Phi(\mathbb{Q})$

$\Delta^{\circ}_{\Phi,K} \subset \Delta_{\Phi,K}$ subgrp stabilizing $\sigma \in \Sigma(\Phi)$.

indep of σ when K neat.

· $Q_\Phi \to \mathrm{Aut}(U_\Phi)$, $U_\Phi(\mathbb{R})(-1) \supset \mathbb{P}_\Phi$ cone / $\mathbb{R}$

$\cup$

Λ_Φ lattice / $\mathbb{Z}$.

Have $\Delta_{\Phi,K} \circlearrowright U_\Phi(\mathbb{R})(-1)$ preserving $\mathbb{P}_\Phi$.

$\Sigma(\Phi)$ cone decomp of $\mathbb{P}_\Phi$.

· On generic fibre,

$\Delta_{\Phi,K} \circlearrowright Sh_{K_{\Phi,h}}(P_\Phi, D_\Phi)(\mathbb{C})$ by left multiplication

Note Σ sm $\Rightarrow$ Sh^{Σ}_K sm alg space

Σ proj $\Rightarrow$ Sh^{Σ}_K proj.

* Toroidal cpt'n of integral models of ab-type Sh vars

(G,X) ab type if $\exists$ Hodge-type (G_0, X_0)

$+$ $G_0^{der} \to G^{der}$ central iso

s.t. $(G^{ad}, X^{ad}) \cong (G_0^{ad}, X_0^{ad})$.

Fix p prime. (G_0, X_0) Hodge type

$\downarrow$

$(G^\ddagger, X^\ddagger)$ Siegel $(G^\ddagger = GSp(V,\psi))$.

$\rightsquigarrow$ integral model / $\mathcal{O}_{E_0,(v)}$, $v | p$. $E_0 = E(G_0, X_0)$.

Let $\mathscr{S}_{K_0}$ = normalization in $Sh_{K_0}(G_0, X_0)$

$K = K_p K^p$, $K^p \subset G(\mathbb{A}_f^p)$ neat open cpt,

$K_p \subset G(\mathbb{Q}_p)$ open cpt.

Choose $K_p^\ddagger \subset G^\ddagger(\mathbb{Q}_p)$ (by enlarging (V,ψ), may assume $K_p^\ddagger = \mathrm{Stab}(V_{\mathbb{Z}_p})$)

self-dual.

$\rightsquigarrow$ $\mathcal{S}_{K^\ddagger}$ is a moduli space of p.p.a.v.s.

$\rightsquigarrow$ $Sh_{K_0} \xrightarrow{K_0^p \subset K^{\ddagger,p}} Sh_{K^\ddagger} \longrightarrow \mathcal{S}_{K^\ddagger}$.

<u>Thm</u> (Madapusi-Pera, 2019, Hodge type)

$\Sigma^\ddagger$ adm sm & proj cone decomp of $(G^\ddagger, X^\ddagger, K^\ddagger)$

Σ induced by $\Sigma^\ddagger$ (with (G,X) given).

Note $\forall\ \Phi_0 \hookrightarrow \Phi^\ddagger$, have

$$\mathcal{U}_{\Phi_0}(\mathbb{R})(-1) \longrightarrow \mathcal{U}_{\Phi^\ddagger}(\mathbb{R})(-1).$$

Let $\mathcal{S}_{K_0}^{\Sigma} :=$ normalization in Sh_{K_0} of $\mathcal{S}_{K^\ddagger}^{\Sigma^\ddagger}$

generic fibre of $Sh_{K_0}^{\Sigma}$, proper.

Then

(1) $\mathcal{S}_{K_0}^{\Sigma} \cong \coprod \mathcal{Z}_{[(\Phi,\sigma)]}$

generic fibre of $Z_{[(\Phi,\sigma)]}$

(2) $\mathcal{S}_{K_0, \mathcal{Z}_{[(\Phi_0,\sigma)]}}^{\Sigma, \wedge} \cong \underbrace{\Delta^\circ_{\Phi,K}}\backslash \mathcal{S}_{K_{\Phi_0}}(\sigma)^{\wedge}_{\mathcal{S}_{K_{\Phi_0},\sigma}}$

trivial when (G_0, X_0) Hodge type + K_0 neat.

(3)
$$\begin{tikzcd}
\mathcal{S}_{K_{\Phi_0}} \arrow[r, "E_{\Phi_0}\text{-torsor}"] & \bar{\mathcal{S}}_{K_{\Phi_0}} \arrow[r, "\text{proper}"] & \mathcal{S}_{K_{\Phi_0},h} \\
Sh_{K_{\Phi_0}} \arrow[r] \arrow[u, hook] & \overline{Sh}_{K_{\Phi_0}} \arrow[r] \arrow[u, hook] & Sh_{K_{\Phi},h} \arrow[u, hook]
\end{tikzcd}$$

& $\bar{\mathcal{S}}_{K_{\Phi_0}} \to \mathcal{S}_{K_{\Phi_0},h}$ is an ab sch-torsor in some cases.

So this is a "factorization" thm.

Let $K_{0,p}$ quasi-parahoric.

(i.e. stabilizes a parahoric, $K^\circ_{0,p} \subset K_{0,p} \subset \mathrm{Stab}$).

If choose Hodge embedding $(G_0, X_0, K_{0,p}) \hookrightarrow (G^\ddagger, X^\ddagger, K^\ddagger_p)$

exactly as Daniels - van Hoften - Kim - Zhang '24.

$\rightsquigarrow$ there is certain functoriality for $\mathscr{S}_{K_{\Phi_0}}$.

$\forall \gamma \in G_0^{ad}(\mathbb{Q}) \ \exists \ \mathscr{S}_{K_{\Phi_0}} \longrightarrow \mathscr{S}_{K_{\gamma\Phi_0}}$, ${}^\gamma\Phi_0 = \gamma \cdot \Phi_0$ by char 0 theory.

extending $Sh_{K_{\Phi_0}} \longrightarrow Sh_{K_{\gamma\Phi_0}}$.

<u>Thm</u> (Wu, abelian type) (G_2, X_2) abelian type.

$K_2 = K_{2,p} K_2^p$ open cpt in $G_2(\mathbb{Q}_p)$.

Choose n BT stabilizers $\mathscr{G}_{x_i}(\mathbb{Z}_p)$

s.t. $K_{2,p} \subset \bigcap_{i=1}^{n} \mathscr{G}_{x_i}(\mathbb{Z}_p)$.

Let $E_2 = E(G_2, X_2)$. Choose (G_0, X_0) $(E_0 = E(G_0, X_0))$ Hodge type

$\rightsquigarrow E' = E_2 \cdot E_0$.

s.t. $\forall v_2 | p$ in E_2, v_2 splits completely in E'.

<u>Then</u> Can construct $\mathscr{S}^{tor}_{K_2} / \mathcal{O}_{E_2, (v_2)}$.

<u>Rough plan</u> Write $G' := (G_2 \times G_0/\mathbb{K}) / G_2^{der}$

$(\mathbb{K} := \ker(G_0^{der} \to G_2^{der})$

$\rightsquigarrow$ $G' \hookrightarrow G$ for some G with quasi-split Z_G.

$(G')^{der} = G^{der}$.

Then
$$\begin{matrix} (G_0, X_0) \xrightarrow{\pi^b} \\ (G_2, X_2) \xrightarrow{\pi^a} \end{matrix} (G, X_a) \approx (G, X_b) \longrightarrow (G^{ad}, X_a^{ad}) \approx (G^{ad}, X_b^{ad}).$$

Note · X_a, X_b are differed by a homo $c: \mathbb{S} \to Z_{G,\mathbb{R}}$.

$\Rightarrow$ $Sh_K(G, X_a)(\mathbb{C}) \cong Sh_K(G, X_b)(\mathbb{C})$

c causes the difference of descent data of (G, X_a) & (G, X_b).

e.g. RSZ Shimura var : $Z \times GU^{\otimes} / \mathcal{K}$.

Rmk $X \to Y$ / $\mathcal{O}_{E_2, V_2}$ a torus-torsor.

It is not well-behaved under normalization,

but is well-behaved under pulling back open & closed embeddings.

<u>Construction</u> $\mathcal{G}_{x_i}(\mathbb{Z}_p)$ stabilizer $\subset G(\mathbb{Q}_p)$.

$K_{2,p} \rightsquigarrow \bigcap_i \mathcal{G}_{x_i}(\mathbb{Z}_p)$.

$K_{0,p} \subset \bigcap \pi^{b,-1}\mathcal{G}_{x_i}(\mathbb{Z}_p)$.

from $\{\pi^{b,-1}\mathcal{G}_{x_i}(\mathbb{Z}_p)\}$ + Construction by DvHKZ & PR.

· Arbitrary $\underset{(G_2, X_2)}{\Sigma_2} \leq \Sigma_2'$ refinement, sm & proj

$\Rightarrow$ $\pi^a: \Sigma_2 \to \underset{(G, X_a) \approx (G, X_b)}{\Sigma}$ $\rightsquigarrow$ $\pi^a: \Sigma_2' \to \Sigma$

· $\underset{\alpha}{I_{G/G_0}} := \mathrm{Stab}_{G(\mathbb{Q})} X_0^+ \cdot \pi^b G_0(\mathbb{A}_f) \backslash G(\mathbb{A}_f) / K$.

$\rightsquigarrow$ $\pi^b: \underset{(G_0, X_0)}{\Sigma_0^\alpha} \to \Sigma$ + $\Sigma_0^\alpha \to \underset{(G^\ddagger, X^\ddagger)}{\Sigma^{\ddagger,\alpha}}$.

$$\begin{array}{ccc} (\alpha; G_0, X_0; \Sigma_0^\alpha) & \longrightarrow & (G^\ddagger, X^\ddagger, \Sigma^\ddagger)^\alpha \\ \downarrow & & \text{by Madapusi.} \\ (G_2, X_2; \Sigma_2') & \hookrightarrow & (G, X_a, \Sigma) \approx (G, X_b, \Sigma) \end{array}$$

So this constr'n generalizes Madapusi's work.

$$\mathscr{S}_{K_2}^{\Sigma_2'} \xrightarrow[\text{in } Sh_{K_2}^{\Sigma_2}]{\text{normalization}} \mathscr{S}_K^{\Sigma} := \coprod_{\alpha \in I_{G/G_0}} \mathscr{S}_{K_0^\alpha}^{\Sigma_0^\alpha} / \Delta_{K^\alpha}(G_0, G)$$

acts through fin grp

<u>Resulting Thm</u> (Wu)

$$\mathscr{S}_{K_2}^{\Sigma_2'} = \coprod Z_{[(\Phi_2, \sigma_2)]}$$

$$\exists \quad \mathscr{S}_{K_{\Phi_2}} \xrightarrow{\text{torus torsor}} \bar{\mathscr{S}}_{K_{\Phi_2}} \xrightarrow{\text{proper}} \mathscr{S}_{K_{\Phi},h}$$

s.t. $\underbrace{\Delta^\circ_{\Phi_2, K_2}}_{} \backslash \mathscr{S}_{K_{\Phi_2}, \sigma_2} \cong Z_{[(\Phi_2, \sigma_2)]}$

non-trivial in ab type (but triv in Hodge type).

$$\rightsquigarrow \quad \mathscr{S}^*_{K_{\Phi_2}} \xrightarrow{\text{torus torsor}} \bar{\mathscr{S}}^*_{K_{\Phi_2}} \xrightarrow{\text{proper}} \mathscr{S}_{K_{\Phi},h}$$

$$\mathscr{S}^*_{K_{\Phi_2}} = \Delta^\circ_{\Phi_2, K_2} \backslash \mathscr{S}_{K_{\Phi_2}}$$