Siegel modular schemes and their Compactifications over $\mathbb{C}$

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$g \geq 1$ fixed $\rightsquigarrow$ reductive grps $Sp_{2g} \subset GSp_{2g}$ $/\mathbb{Z}$

$\forall$ ring R, $\quad Sp_{2g}(R) = \left\{ \gamma \in M_{2g}(R) \,\middle|\, {}^t\gamma \begin{pmatrix} & I_g \\ -I_g & \end{pmatrix} \gamma = \begin{pmatrix} & I_g \\ -I_g & \end{pmatrix} \right\}$

$\cap\!\shortmid$

$GSp_{2g}(R) = \left\{ \gamma \in M_{2g}(R) \,\middle|\, {}^t\gamma \begin{pmatrix} & I_g \\ -I_g & \end{pmatrix} \gamma = c(\gamma) \cdot \begin{pmatrix} & I_g \\ -I_g & \end{pmatrix} \right\}$

for some character $c(\gamma) \in R^\times$.

note $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \gamma \in Sp_{2g}(R) \iff \begin{cases} A \cdot {}^tB = B \cdot {}^tA, \\ C \cdot {}^tD = D \cdot {}^tC, \\ A \cdot {}^tD - B \cdot {}^tC = I_g. \end{cases}$

$L = \mathbb{Z}^{2g}$, $\langle , \rangle : L \times L \longrightarrow \mathbb{Z}$

$$(x, y) \longmapsto {}^t x \begin{pmatrix} & I_g \\ -I_g & \end{pmatrix} y$$

then $Sp_{2g} = Sp(L, \langle \cdot, \cdot \rangle)$

$= \{ \gamma \in GL(L) \mid \langle \gamma x, \gamma y \rangle = \langle x, y \rangle, \ \forall x, y \in L \}$.

& Similar for GSp_{2g}.

$\rightsquigarrow$ exact seq $1 \to Sp_{2g} \to GSp_{2g} \xrightarrow{c} \mathbb{G}_m \to 1$

Siegel upper half space:

$\mathbb{H}_g = \{\Omega \in M_g(\mathbb{C}) \mid \Omega = {}^t\Omega,\ \mathrm{Im}\,\Omega > 0\}$

positive definite.

$\circlearrowleft$

$Sp_{2g}(\mathbb{R})$

$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \Omega \longmapsto (A\Omega + B)\cdot(C\Omega + D)^{-1}$.

We have K = max cpt of $Sp_{2g}(\mathbb{R})$

$\rightsquigarrow$ $Sp_{2g}(\mathbb{R})/K \xrightarrow{\sim} \mathbb{H}_g$

$\wr$ ← s.t. $c(\gamma) > 0$.

$GSp_{2g}(\mathbb{R})^+/K\mathbb{R}_{>0}$

$\rightsquigarrow$ $(GSp_{2g}/\mathbb{Q}, \pm\mathbb{H}_g)$ Siegel Shimura datum.

Schemes = cat of loc noe sch

S, X/S ab sch $\rightsquigarrow$ $X^\vee/S = \mathrm{Pic}^\circ(X/S)$
dual ab sch.

<u>Recall</u> A polarization of X/S is an S-morph

$\lambda : X/S \longrightarrow X^\vee/S$

s.t. $\forall$ geom pt $\bar{s} \to S$,

$\lambda_{\bar{s}} : X_{\bar{s}} \longrightarrow X^\vee_{\bar{s}}$ is of the form $\lambda_{\bar{s}} = \phi_{\mathcal{L}_{\bar{s}}}$

for some $\mathcal{L}_{\bar{s}}$ ample line bundle $/X_{\bar{s}}$.

$\rightsquigarrow$ $\lambda_* \mathcal{O}_X$ loc free $\mathcal{O}_{X^\vee}$-mod w/ const rank on each conn comp of S.

$\rightsquigarrow$ deg of $\lambda = d^2$, $d \geq 1$.

<u>principal polarization</u> if $\deg \lambda = 1$, i.e. λ isom.

(will assume X projective).

<u>Def</u> $\mathcal{A}_g$: Schemes $\longrightarrow$ Sets contravariant functor

$$S \longmapsto \{(X, \lambda)/S\}/\sim$$

- (X, λ) projective ppas of rel dim g /S.

More generally, $\forall\, d \geq 1,\ n \geq 1$,

$\mathcal{A}_{g,d,n}$: Schemes $\longrightarrow$ Sets

$$S \longmapsto \{(X, \lambda, \eta)/S\}/\sim$$

- (X, λ) ab sch /S of rel dim d + $\deg \lambda = d^2$.
- $\eta : X[n] \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^{\oplus 2g}$, preserving pairings up to $(\mathbb{Z}/n\mathbb{Z})^*$.

<u>note</u> $\mathcal{A}_g = \mathcal{A}_{g,1,1}$.

$\mathcal{A}$: Schemes $\longrightarrow$ Sets contravariant

<u>Def</u> A coarse moduli sch of $\mathcal{A}$ is a sch A

& a morph $F : \mathcal{A} \longrightarrow h_A = \mathrm{Hom}_{sch}(-, A)$

s.t. (a) $\forall$ morph $G : \mathcal{A} \longrightarrow h_X$ for some sch X,

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[dr, "F"'] & h_X \\ & h_A \arrow[u] \end{tikzcd}$$

factors through F via a unique $A \to X$.

(b) $\forall$ alg closed field k,

$F(\mathrm{Spec}\, k) : \mathcal{A}(k) \xrightarrow{\sim} A(k)$ bijection.

<u>Thm 1</u> (Mumford, GIT)

$\forall\, g, d, n \in \mathbb{Z}_{\geq 1}$, the coarse moduli sch $A_{g,d,n}$ of $\mathcal{A}_{g,d,n}$ (abuse of notation) exists & it is faithfully flat / $\mathrm{Spec}\, \mathbb{Z}[\frac{1}{n}]$,

quasi-proj / $\mathrm{Spec}\, \mathbb{Z}[\frac{1}{np}]$ for <u>any</u> prime p

& in fact quasi-proj / $\mathrm{Spec}\, \mathbb{Z}[\frac{1}{n}]$.

Moreover, if $n \geq 3$, then $A_{g,d,n}$ actually smooth / $\mathrm{Spec}\, \mathbb{Z}[\frac{1}{nd}]$.

<u>pf ingredients</u>
- GIT
- ~~Artin's~~ method (alg stacks), c.f. [Faltings-Chai] Chap 1. §4.

In the following, mainly discuss A_g & $A_{g,1,n}$ / $\mathbb{C}$.

<u>Complex uniformization</u>

$$\{(X, \lambda, (\alpha_i)_{1 \leq i \leq 2g})\} / \sim \xrightarrow{\sim} \mathbb{H}_g$$

- $X / \mathbb{C}$ ab var. $\dim = g$, $\deg \lambda = 1$.
- $\alpha_i \in H_1(X(\mathbb{C}), \mathbb{Z})$ symplectic basis

i.e. s.t. $\langle \alpha_j, \alpha_k \rangle = \langle \alpha_{g+j}, \alpha_{g+k} \rangle = 0, \ \forall j, k$.

$\quad$ & $\langle \alpha_j, \alpha_{g+k} \rangle = -\langle \alpha_{g+j}, \alpha_k \rangle = \delta_{j,k}$.

<u>Fact</u> $\quad X(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}^g / \langle \alpha_1, \cdots, \alpha_g \rangle$. $\quad$ (Fix $\mathrm{Lie}\, X \simeq \mathbb{C}^g$.)

Let $\quad \Omega = (\alpha_1, \cdots, \alpha_g)$ period matrix

$$= \left(\underset{g\times g}{\Omega_1} \ \ \underset{g\times g}{\Omega_2} \right).$$

+ Riemannian relation $\Omega_2 {}^t\Omega_1 - \Omega_1 {}^t\Omega_2 = 0$.

$$2i\left(\Omega_2 \cdot {}^t\overline{\Omega_1} - \Omega_1 \cdot {}^t\overline{\Omega_2} \right) > 0.$$

$\rightsquigarrow$ The isom is by

$$(X, \lambda, (\alpha_i)) \longmapsto \Omega_2^{-1} \cdot \Omega_1.$$

Consider $\mathbb{Z}^{2g} \curvearrowright \mathbb{H}_g \times \mathbb{C}^g$

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} : (\Omega, z) \longmapsto (\Omega, z + \Omega n_1 + n_2).$$

$\rightsquigarrow \quad \mathcal{X}_g := \mathbb{Z}^{2g} \backslash (\mathbb{H}_g \times \mathbb{C}^g)$

$\quad\quad \downarrow \quad$ holo form of pp var

$\quad\quad \mathbb{H}_g \quad\quad$ w/ symp basis of $\mathbb{H}$.

Also, $\quad Sp_{2g}(\mathbb{Z}) \curvearrowright \mathbb{H}_g \times \mathbb{C}^g$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : (\Omega, z) \longmapsto \left((A\Omega + B)\cdot(C\Omega + D)^{-1}, {}^t(C\Omega + D)^{-1} \cdot z \right)$$

$\rightsquigarrow \quad Sp_{2g}(\mathbb{Z}) \ltimes \mathbb{Z}^{2g} \curvearrowright \mathbb{H}_g \times \mathbb{C}^g$

$\rightsquigarrow \quad Sp_{2g}(\mathbb{Z}) \backslash \mathcal{H}_g \longrightarrow Sp_{2g}(\mathbb{Z}) \backslash \mathbb{H}_g .$

$\Rightarrow \quad \underset{\substack{\parallel \\ A^{an}_{g,\mathbb{C}} \text{ by def}}}{A_g(\mathbb{C})} \simeq Sp_{2g}(\mathbb{Z}) \backslash \mathbb{H}_g$ as cplx analytic spaces.

For any $\Gamma \subset Sp_{2g}(\mathbb{Z})$ subgrp fin index,

get $\Gamma \backslash \mathbb{H}_g \longrightarrow Sp_{2g}(\mathbb{Z}) \backslash \mathbb{H}_g$.

When $n \geq 3$,

$$\Gamma(n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z}) \;\middle|\; \begin{array}{l} A \equiv D \equiv I_g \bmod n, \\ B \equiv C \equiv 0 \bmod n \end{array} \right\}.$$

$$K(n) = \{ \gamma \in GSp_{2g}(\hat{\mathbb{Z}}) \mid \gamma \equiv I_{2g} \bmod n \}$$

<u>note</u> $\Gamma(n) \cong \underbrace{K(n) \cap Sp_{2g}(\mathbb{Q})}_{\text{in } GSp_{2g}(\mathbb{A}_f)}.$

Then $A_{g,1,n}(\mathbb{C}) \cong Sh_{K(n)}(GSp_{2g}, \pm\mathbb{H}_g)(\mathbb{C})$

$$\cong \coprod_{(\mathbb{Z}/n\mathbb{Z})^\times} \Gamma(n) \backslash \mathbb{H}_g .$$

<u>Siegel modular forms</u>

Recall $K = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in M_{2g}(\mathbb{R}) \;\middle|\; \begin{array}{l} A \cdot {}^tB = B \cdot {}^tA, \\ A\,{}^tA + B\,{}^tB = I_g \end{array} \right\} \xrightarrow{\sim} U_g(\mathbb{R})$

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \longmapsto A + iB .$$

$\rightsquigarrow K_{\mathbb{C}} \simeq GL_g(\mathbb{C})$.

<u>note</u> $Sp_{2g}(\mathbb{R})/K \xrightarrow{\sim} \mathbb{H}_g$, $K = \mathrm{Stab}(iI_g)$.

$\rho : GL_n(\mathbb{C}) = K_{\mathbb{C}} \longrightarrow GL(V_\rho)$ fin dim rep,

$\Gamma \subset Sp_{2g}(\mathbb{Z})$ fin index.

<u>Def</u> A Siegel mod form of wt ρ & level Γ is a holo fct $f : \mathbb{H}_g \to V_\rho$ s.t.

(1) $f(\gamma\Omega) = \rho(C\Omega + D) f(\Omega)$, $\forall \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$, $\Omega \in \mathbb{H}_g$

(2) f is holom at all cusps if $g = 1$.

If $\rho = \det^k$ for some $k \in \mathbb{N}$, wt k,

$R_k(\Gamma) = \{$ Siegel mod forms of wt k & level $\Gamma \}$.

<u>Thm 2</u> (1) The graded $\mathbb{C}$-alg $R(\Gamma) = \bigoplus_{k \in \mathbb{N}} R_k(\Gamma)$ is f.g. over $\mathbb{C}$.

(2) $\mathrm{trdeg}_{\mathbb{C}} R(\Gamma) = \binom{g+1}{2} + 1 = \frac{(g+1)g}{2} + 1$.

(3) $\forall k$, $\dim_{\mathbb{C}} R_k(\Gamma) < +\infty$, and

$$\dim_{\mathbb{C}} R_k(\Gamma) = O\left(k^{\binom{g+1}{2}}\right).$$

(4) $R(\Gamma)$ embeds $\Gamma \backslash \mathbb{H}_g$ into $\mathrm{Proj}(R(\Gamma))(\mathbb{C})$ as an open dense subvar in Zar top.

Let $X = \mathcal{X}_{g,\Gamma}(\mathbb{C})$ univ cplx ab var.

$$\begin{array}{c} X = \mathcal{X}_{g,\Gamma}(\mathbb{C}) \\ \downarrow \pi \\ T = \Gamma \backslash \mathbb{H}_g \end{array}$$

$\mathcal{E} = \pi_* \Omega^1_{X/T}$ loc free of rk g

Set $\omega_{g,T} = \Lambda^g \mathcal{E}$. Then $R_k(\Gamma) = H^0(\Gamma \backslash \mathbb{H}_g, \omega_{g,T}^{\otimes k})$.

The minimal cpt'n

<u>Goal</u> Describe $\mathrm{proj}(R(\Gamma))(\mathbb{C})$ more explicitly.

Consider $\mathbb{H}_g^* = \{ (\gamma, \Omega) \mid \gamma \in Sp_{2g}(\mathbb{Q}),\ \Omega \in \mathbb{H}_r$ for some $0 \le r \le g \} / \sim$

$\circlearrowleft$
$Sp_{2g}(\mathbb{Q})$

$$(\gamma_1, \Omega_1) \sim (\gamma_2, \Omega_2)$$
$$\Updownarrow$$
$$\gamma_1 = \gamma_2 \ \&\ \Omega_2 = \underbrace{\mathrm{Pr}_{r(g)}(\gamma_2^{-1}\gamma_1)}_{\in Sp_{2r}(\mathbb{Q})} \cdot \Omega_1$$

$\forall\, 0 \le r \le g$, $\mathbb{H}_r \hookrightarrow \mathbb{H}_g^*$, let image $= \mathfrak{F}_r$.

$$\Omega \longmapsto [(1, \Omega)].$$

$\gamma \in Sp_{2g}(\mathbb{Q}) \rightsquigarrow \gamma \cdot \mathfrak{F}_r \subset \mathbb{H}_g^*$

rational boundary comp.

$$Sp_{2g}(\mathbb{Q}) \supset N_r = N(\mathfrak{F}_r)$$

$$\text{s.t. } N_r(\mathbb{Q}) = \left\{ \begin{pmatrix} A_1 & * & B_1 & * \\ * & A_2 & * & * \\ C_1 & 0 & D_1 & * \\ 0 & 0 & 0 & {}^tA_2^{-1} \end{pmatrix} \right\}.$$

Define a (satake) top on $\mathbb{H}_g^* \supset \mathfrak{F}_g^*$ as follows:

To be def'd.

$\Omega = X + iY \in \mathbb{H}_g$, $X, Y \in M_g(\mathbb{R})$, $X = {}^tX$, $Y = {}^tY > 0$.

Have Jacobi decomp $Y = {}^tBDB$ uniquely

w/ $B = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$ unip, $D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_g \end{pmatrix}$, $d_i > 0$.

$\forall u > 0$, define the Siegel subset $\mathfrak{F}_g(u) \subset \mathbb{H}_g$

$$\mathfrak{F}_g(u) := \left\{ X + iY \in \mathbb{H}_g \;\middle|\; \begin{array}{l} |x_{ij}| < u,\ \forall i,j,\ |b_{ij}| < u,\ \forall 1 \le i < j \le g, \\ 1 < u d_1,\ d_i < u d_{i+1},\ \forall 1 \le i \le g-1 \end{array} \right\}$$

$$\Rightarrow \bigcup_{u>0} \mathfrak{F}_g(u) = \mathbb{H}_g.$$

- $Sp_{2g}(\mathbb{Z}) \cdot \mathfrak{F}_g(u) = \mathbb{H}_g$ for $u \gg 0$. $\quad (*)$
- $\forall u > 0$, $\#\{\gamma \in Sp_{2g}(\mathbb{Z}) \mid \gamma\mathfrak{F}_g(u) \cap \mathfrak{F}_g(u) \neq \phi\} < \infty$.

Fix $u_0 \gg 0$ s.t. $(*)$ holds for $u \ge u_0$.

Set $\mathfrak{F}_g^* := \coprod_{r=0}^{g} \overline{\mathfrak{F}_r(u_0)}$,

where $\overline{\mathfrak{F}_r(u_0)}$ = closure of $\mathfrak{F}_r(u_0)$ in $\mathbb{H}_r$.

$\overline{\mathfrak{F}_0(u_0)}$ = pt.

<u>Define</u> a top on $\mathfrak{F}_g^*$: $\forall \Omega \in \overline{\mathfrak{F}_r(u_0)} \subset \mathfrak{F}_g^*$

a basis of nbhds of Ω is given by

$$\left\{ \bigcup_{r \le s \le g} W_{r,s}(U, c) \right\}_{U \ni \Omega,\ c \in \mathbb{R}_{>0}}$$

where $\forall$ open subset $U \subset \overline{\mathfrak{F}_r(u_0)}$, $c > 0$, $0 \le r, s \le g$,

$$W_{r,s}(\mathcal{U}, c) := \left\{ \begin{array}{l} \Omega' = \begin{array}{c} \begin{pmatrix} \Omega_1 & * \\ * & * \end{pmatrix} \begin{array}{l} r \\ s-r \end{array} \\ \begin{array}{cc} r & s-r \end{array} \end{array} = X + iY \in \mathfrak{F}_s(u_0) \\ \text{s.t. } \Omega_1 \in \mathcal{U}_1,\ c < d_{r+1} \text{ in the Jacobi} \\ \quad \text{decomp } Y = {}^t B D B,\ D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_g \end{pmatrix} \end{array} \right\}$$

Def Satake top on $\mathbb{H}_g^*$:

$\forall x \in \mathbb{H}_g^*$, a basis of nbhds of x is given by $\mathcal{U} \subseteq \mathbb{H}_g^*$

s.t. (a) $\forall \gamma \in Sp_{2g}(\mathbb{Q})$, $\gamma\mathcal{U} \cap \mathfrak{F}_g^*$ is an open nbhd of γx in $\mathfrak{F}_g^*$ whenever $\gamma \cdot x \in \mathfrak{F}_g^*$.

(b) $\forall \gamma \in Sp_{2g}(\mathbb{Z})$ s.t. $\gamma x = x$, we have $\gamma\mathcal{U} = \mathcal{U}$.

This is indep of u_0 chosen, with nice properties.

<u>Thm 3</u> $\forall \Gamma \subset Sp_{2g}(\mathbb{Z})$ fin index.

(1) $\Gamma \backslash \mathbb{H}_g^*$ has the str of normal cplx analytic space.

It has a natural fin stratification w/ strata loc closed analytic subspaces of the form $\Gamma' \backslash \mathbb{H}_r$, $0 \le r \le g$.

(2) $\Gamma \backslash \mathbb{H}_g^* \simeq \mathrm{Proj}(R(\Gamma))(\mathbb{C})$ isom of analytic spaces

Thus $\Gamma \backslash \mathbb{H}_g^*$ is proj.

<u>Rmk</u> (i) $\Gamma_g = Sp_{2g}(\mathbb{Z})$. $\Gamma_g \backslash \mathbb{H}_g^* = \coprod_{r=0}^{g} \Gamma_r \backslash \mathbb{H}_r$, $\Gamma_r = Sp_{2r}(\mathbb{Z})$.

(ii) For some $k>0$, $\omega_{g,\Gamma}^{\otimes k}$ extends to $\Gamma\backslash\mathbb{H}_g^*$.

<u>Toroidal cpt'n</u> Γ fixed. $X = X_\Gamma \hookrightarrow X^{\min}$ $/\mathbb{C}$

X: quasi-proj alg var; $X^{\min}$: normal proj.

[AMRT]

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, bend right] & X_\Sigma^{tor} \arrow[d, "\text{blow-down}"] \\ & X^{\min} \end{tikzcd}$$

X_Σ^{tor}: sm; $X^{\min}$: $/\mathbb{C}$

$\rightsquigarrow$

$$\begin{tikzcd} \Gamma\backslash\mathbb{H}_g \arrow[r, hook] \arrow[dr, bend right] & X_\Sigma^{tor}(\mathbb{C}) \arrow[d, "\text{blow-down}"] \\ & \Gamma\backslash\mathbb{H}_g^* \end{tikzcd}$$

<u>Local coordinates</u>

For $0 \le r \le g-1$, $\Omega \in \mathbb{H}_g$, $\Omega = \begin{array}{cc} \begin{pmatrix} t & w \\ {}^t w & \tau \end{pmatrix} & \begin{array}{c} r \\ g-r \end{array} \\ \begin{array}{cc} r & g-r \end{array} & \end{array}$,

"Siegel domain of 3rd kind".

$$D_r = \left\{ \begin{pmatrix} t & w \\ {}^t w & \tau \end{pmatrix} \in M_g(\mathbb{C}) \;\middle|\; \begin{array}{l} t \in \mathbb{H}_r, \\ \tau \in M_{g-r}(\mathbb{C}) \\ {}^t\tau = \tau \end{array} \right\}$$

$D_r \supset \mathbb{H}_g$

$U_r \subset Sp_{2g}/\mathbb{Q}$ s.t. $U_r(\mathbb{Q}) = \left\{ \begin{pmatrix} I_r & & & \\ & I_{g-r} & & b \\ & & I_r & \\ & & & I_{g-r} \end{pmatrix} \;\middle|\; \begin{array}{c} {}^t b = b \\ \cap \\ M_{g-r}(\mathbb{Q}) \end{array} \right\}$

as a vec grp

$\rightsquigarrow$ as a $\mathbb{Z}$-grp s.t. $U_r(\mathbb{Z}) = \Gamma \cap U_r(\mathbb{Q})$

$U_r(\mathbb{Z})^* = \mathrm{Hom}(U_r(\mathbb{Z}), \mathbb{Z})$

$U_r(\mathbb{R})$
$\cup$
$\bar{C}_r$ closure
$\cup$
C_r positive cone

Let $\sigma \subset \bar{C}_r$ be a top-dim cone generalized by a $\mathbb{Z}$-basis $\{\xi_1, \dots, \xi_n\}$ of $U_r(\mathbb{Z})$,

$n = \binom{g-r+1}{2}$.

Let $\{e_1, \dots, e_n\} \subseteq U_r(\mathbb{Z})^*$ dual basis

s.t. $(U_r(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C})^\times \cong (\mathbb{C}^\times)^n$ torus

w/ $\exp: U_r(\mathbb{C}) \longrightarrow (U_r(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C})^\times$.

Fact $D_r \cong U_r(\mathbb{C}) \times \mathbb{C}^k \times \mathfrak{F}_r$, $k = r(g-r)$.

$$\begin{tikzcd}
\mathbb{H}_g \arrow[d] & \subset & U_r(\mathbb{C}) \times \mathbb{C}^k \times \mathfrak{F}_r \arrow[d, "\exp \times id \times id"] & (\tau, w, t) \arrow[d, mapsto] \\
U_r(\mathbb{Z}) \backslash \mathbb{H}_g \arrow[d] \arrow[dr, hook] & \subset & \underbrace{(U_r(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C})^\times \times \mathbb{C}^k}_{\star} \times \mathfrak{F}_r & (\exp(2\pi i (e_j(\tau)), w, t) \\
\Gamma \backslash \mathbb{H}_g \arrow[dr, hook] & (U_r(\mathbb{Z}) \backslash \mathbb{H}_g)_\sigma \arrow[d] & & \\
& \widetilde{(\Gamma \backslash \mathbb{H}_g)} & &
\end{tikzcd}$$

$(U_r(\mathbb{Z}) \backslash \mathbb{H}_g)_\sigma$ = interior of closure of $U_r(\mathbb{Z}) \backslash \mathbb{H}_g$ in $(\star)_\sigma$.

by theory of torus embed.

Let $\Sigma_{\mathfrak{F}_r} = \{\sigma_\alpha\}$ rat'l polyhydral cone decomp of $\bar{C}_r$.

$\sigma \in \Sigma_{\mathfrak{F}_r}$ $\rightsquigarrow$ $X_{\mathfrak{F}_r, \sigma} = (U_r(\mathbb{Z}) \backslash \mathbb{H}_g)_\sigma$

+ action by $G_\ell(\mathcal{F}_r) \subset Sp_{2g}/\mathbb{Q}$
$\cong$
GL_{g-r}

similar as above

s.t. $G_\ell(\mathcal{F}_r)(\mathbb{Q}) = \left\{ \begin{pmatrix} I_r & & & \\ & u & & \\ & & I_r & \\ & & & {}^t u^{-1} \end{pmatrix} \middle| \, u \in GL_{g-r}(\mathbb{Q}) \right\}$

acts trivially U_r, $\bar{C}_r$ by conjugation.

<u>Fact</u> Can take $\Gamma \cap G_\ell(\mathcal{F}_r)(\mathbb{Q})$ - inv cone decomp $\Sigma_{\mathcal{F}_r}$ modulo $\Gamma \cap G_\ell(\mathcal{F}_r)(\mathbb{Q})$ w/ only finitely many orbits.

Called "$\bar{\Gamma}_{\mathcal{F}_r}$ - admissible".

Then $(X_{\mathcal{F}_r,\sigma})_\sigma$ can be glued into a space
$$X'_{\Sigma_{\mathcal{F}_r}} \longrightarrow \Gamma \backslash \mathbb{H}_g^*$$

$$U_r \subset \overset{G_\ell(\mathcal{F}_r)}{\overset{\cap}{N_r}} \subset Sp_{2g}/\mathbb{Q}, \quad \text{then}$$

$$\Gamma_{\mathcal{F}_r} = \frac{\Gamma \cap N_r(\mathbb{Q})}{U_r(\mathbb{Z})} \text{ acts properly discontinuously on } X'_{\Sigma_{\mathcal{F}_r}}.$$

$X_{\Sigma_{\mathcal{F}_r}}$ = the quotient, then

$$\begin{array}{ccc} \Gamma_{\mathcal{F}_r} \backslash \mathbb{H}_g & \longrightarrow & \Gamma \backslash \mathbb{H}_g \\ \cap & & \cap \\ X_{\Sigma_{\mathcal{F}_r}} & \longrightarrow & \Gamma \backslash \mathbb{H}_g^* . \end{array}$$

$\mathcal{F}$ any rat'l boundary comp.

$\rightsquigarrow \Sigma_{\mathcal{F}} = \{\sigma_\alpha\}$, $\bar{\Gamma}_{\mathcal{F}}$ - adm cone decomp of $\overline{C(\mathcal{F})}$.

$\rightsquigarrow X_{\Sigma_{\mathfrak{F}}}$ $\qquad W := \coprod_{\mathfrak{F}} X_{\Sigma_{\mathfrak{F}}}$ $\circlearrowleft \Gamma$

if $\mathfrak{F} \subset \mathfrak{F}'$, $X_{\Sigma_{\mathfrak{F}'}} \to X_{\Sigma_{\mathfrak{F}}}$ étale

$$\begin{tikzcd} \Sigma = (\Sigma_{\mathfrak{F}})_{\mathfrak{F}} \text{ adm} \arrow[r] & \text{quotient} \arrow[d] \\ & X^{\text{tor}}_{\Sigma}(\mathbb{C}) = \tilde{X}_{\Sigma} \end{tikzcd}$$

<u>Thm 4.</u> (1) $\tilde{X}_{\Sigma}$ is the unique Hausdorff analytic space containing $\Gamma \backslash \mathbb{H}_g$ as an open dense subset

s.t. (a) $\forall$ rat boundary $\mathfrak{F}$, $\exists$ open morph $\pi_{\mathfrak{F}}$

s.t.
$$\begin{tikzcd} U_{\mathfrak{F}}(\mathbb{Z}) \backslash \mathbb{H}_g \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & X'_{\Sigma_{\mathfrak{F}}} \arrow[d, "\pi_{\mathfrak{F}}"] \\ \Gamma \backslash \mathbb{H}_g \arrow[r, hook] & \tilde{X}_{\Sigma} \end{tikzcd}$$

(b) $\forall$ pt of $\tilde{X}_{\Sigma}$, it is the image of $\pi_{\mathfrak{F}}$ for some $\mathfrak{F}$ and $\tilde{X}_{\Sigma}$ proj.

(2) $\tilde{X}_{\Sigma}$ cpt normal alg space, and $\exists \Sigma$ s.t. $\tilde{X}_{\Sigma}$ is smooth & projective.

(3)
$$\begin{tikzcd} \Gamma \backslash \mathbb{H}_g \arrow[r, hook] \arrow[dr, hook] \arrow[r, phantom, shift right=4ex, "\circlearrowleft"] & \tilde{X}_{\Sigma} \arrow[d] \\ & \Gamma \backslash \mathbb{H}_g^* \end{tikzcd}$$

<u>Rmk</u>
- Can describe the boundary of $\tilde{X}_{\Sigma}$ using theory of semi-ab var.
- arith cpt'n also exists. c.f. [Faltings - Chai].