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SOME PERSPECTIVES ON EISENSTEIN SERIES (1/2)

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(NOTES BY WENHAN DAI)

Abstract. This is a review of some developments in the theory of Eisenstein series since Corvallis. The talk closely follows [Lap22].

1. Introduction: A Story about SL_2

We begin with a picture that is familiar to everyone. Let $\mathbb{H}$ be the hyperbolic upper half plane with $\mathrm{SL}_2(\mathbb{R})$ action. Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ with $\mathcal{X} = \Gamma\backslash\mathbb{H}$, the automorphic space. Let

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbb{R} \right\} \subset \mathrm{SL}_2(\mathbb{R}), \quad \Gamma_\infty := N \cap \Gamma.$$

Consider their respective actions on $\mathbb{H}$, given by the horizontal translations. One gets

$$\mathcal{Y} = \Gamma_\infty\backslash\mathbb{H} \simeq \mathbb{R}_{>0} \times (\mathbb{Z}\backslash\mathbb{R}), \quad \mathcal{Z} = N\backslash\mathbb{H} \simeq \mathbb{R}_{>0}.$$

We have two natural projections

$$\begin{tikzcd} & \mathcal{Y} \arrow[dl] \arrow[dr] & \\ \mathcal{X} & & \mathcal{Z} \end{tikzcd}$$

The right map is proper because the fibers are just the circles in $\mathbb{Z}\backslash\mathbb{R}$. One can define a constant term from functions on $\mathcal{X}$ to functions on $\mathcal{Z}$, by first pulling back to $\mathcal{Y}$ and then pushing forward to $\mathcal{Z}$.

Upshot. Keep in mind that $\mathcal{X}$ is the target space we are to work with, and $\mathcal{Z}$ is basically an auxiliary space.

Consider the **constant term**

$$\mathcal{F}(\mathcal{X}) \to \mathcal{F}(\mathcal{Y}) \to \mathcal{F}(\mathcal{Z}), \quad f \mapsto \int_{\mathbb{Z}\backslash\mathbb{R}} f(x+iy)dx, \quad y > 0.$$

Here the function space $\mathcal{F}(\cdot)$ could be for instance L^1_{loc}, L^∞, or C^∞ (but not $\mathcal{C}_c$ or L^2)[1]. We list out some analytic properties for this.

(1) The adjoint to this operator is

$$(*) \qquad \mathcal{C}_c(\mathcal{Z}) \to \mathcal{C}_c(\mathcal{X}), \quad f \mapsto \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\gamma z), \quad z \in \mathbb{H}.$$

(2) The space $\mathcal{Z}$ (unlike $\mathcal{X}$) admits a left action by the torus

$$T = \left\{ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \middle| t \in \mathbb{R}_{>0} \right\} \subset \mathrm{SL}_2(\mathbb{R}).$$

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[1] The point here is that $\mathcal{F}(\cdot)$ cannot contain compactly supported functions. Otherwise it is not compatible with the adjoint operator which preserves the compactness of supports.

Note: this induces an action of T on $\mathcal{C}_c(\mathcal{Z})$.

(3) At first approximation, the theory of Eisenstein series aims to give a spectral decomposition/expansion of $(*)$ with respect to the action of T on $\mathcal{C}_c(\mathcal{Z})$.

The simplest Eisenstein series (introduced by Mass in 1949) is

$$E(z;s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^{s+\frac{1}{2}} = \sum_{\substack{(m,n)\in\mathbb{Z}^2\backslash\{(0,0)\} \\ \gcd(m,n)=1}} \frac{y^{s+\frac{1}{2}}}{|mz+n|^{2s+1}},$$

where $z = x+iy$. Roughly, this is a real analytic version of the modular forms considered by Eisenstein (with s the role of the weight). The series converges for $\mathrm{Re}(s) > 1/2$. It admits a meromorphic continuation to $\mathbb{C}$ and a functional equation $s \mapsto -s$.

Ultimately, the theory of Eisenstein series gives the non-discrete part of the spectral decomposition of $L^2(\mathcal{X})$ (equipped with the action of the Laplace operator ∇^2).[2]

Backgrounds. In the 1980s Joseph Bernstein came up with a new, simpler proof of the meromorphic continuation of the Eisenstein series. It is based on a general "soft" principle of meromorphic continuation. Its main feature is that it completely avoids the role of $L^2(\mathcal{X})$.[3] Indeed, no spectral theory is used beyond **rudimentary Fredholm theory for Banach spaces**. Lapid's first goal is to explain this proof, following [BL19].

Remark 1.1. The normalized Eisenstein series

$$E^*(z;s) = \sum_{(m,n)\in\mathbb{Z}^2\backslash\{(0,0)\}} \frac{y^{s+\frac{1}{2}}}{|mz+n|^{2s+1}} = \zeta(2s+1)E(z;s)$$

can be analytically continued exactly as Riemann did for $\zeta(s)$ (using Poisson summation formula). It has a simple pole at $s = 1$ and a functional equation

$$\pi^{-s}\Gamma(s)E^*(z;s) = \pi^s\Gamma(-s)E^*(z;-s).$$

This method is too special. Bernstein's proof is fundamentally different (and gives another proof of the meromorphic continuation of $\zeta(s)$).

2. Meromorphic Continuation of Eisenstein Series

In the upcoming context we introduce the proof of Bernstein for the analytic continuation for $E^*(z;s)$.

2.1. Basic Notions.

2.1.1. *Analytic Functions in HLCTVSs.* Let $\mathfrak{C}$ be a complex, Hausdorff, locally convex topological vector space (HLCTVS). Let $\mathfrak{C}'$ be the space of continuous linear forms on $\mathfrak{C}$.

Definition 2.1. A function $f : \mathbb{C}^n \to \mathfrak{C}$ is **analytic** if for every $\mu \in \mathfrak{C}'$, the scalar function

$$\langle \mu, f(s)\rangle : \mathbb{C}^n \to \mathbb{C}$$

[2] It was developed by Selberg (1950s) and Langlands (1960s) by forming eigenfunctions with respect to ∇^2. Their proofs are a bit involved. Also, the theory of Eisenstein series is not limited to arithmetic lattices. At least the theory has an analytic aspect and the rough geometry of $\Gamma\backslash\mathbb{H}$ is in need.

[3] "The bed of Procrustes", as Langlands refers to it.

is analytic.[4]

If $\mathfrak{C}$ is Fréchet space, this definition is equivalent to strong analyticity, or to having a convergent Taylor series. In practice, analyticity is not very sensitive to the topology of $\mathfrak{C}$.

Let $U \neq \emptyset$ be the complement of a closed analytic set in $\mathbb{C}^n$. In particular, U is dense in $\mathbb{C}^n$.

Definition 2.2. Let $f : U \to \mathfrak{C}$ be an analytic function. We say that f is **meromorphic** on $\mathbb{C}^n$ if for every $s_0 \in \mathbb{C}^n$ there exists a polydisc D around s_0 and a holomorphic function $0 \not\equiv g : D \to \mathbb{C}$ such that the function

$$g(s)f(s) : U \cap D \to \mathfrak{C}$$

extends holomorphically to D.

Example 2.3. Let $\mathfrak{F}$ be another HLCTVS. Let $\mathfrak{L}(\mathfrak{C}, \mathfrak{F})$ be the space of continuous linear operators from $\mathfrak{C}$ to $\mathfrak{F}$ (with the topology of pointwise convergence).

An **analytic family of operators** is a function $A : \mathbb{C}^n \to \mathfrak{L}(\mathfrak{C}, \mathfrak{F})$ such that the scalar function

$$s \mapsto \langle \mu, A(s)v \rangle$$

is analytic for all $v \in \mathfrak{C}$ and $\mu \in \mathfrak{F}'$.

The composition of two analytic families of operators is analytic by Hartog's Theorem on separate holomorphicity.

2.1.2. *Analytic Systems of Linear Equations.* A system Ξ of linear equations (SLE) in $v \in V$ takes the form

$$\mu_i(v) = u_i, \quad i \in I,$$

where for every $i \in I$, u_i is a vector in a vector space U_i and $\mu_i : V \to U_i$ is a linear operator. We write $\mathrm{Sol}(\Xi)$ for the set of solutions of Ξ in V. Suppose that $\mathfrak{C}$ and $\mathfrak{F}_i$ are HLCTVSs. If for every $i \in I$,

$$u_i : \mathbb{C}^n \to \mathfrak{F}_i, \quad \mu_i : \mathbb{C}^n \to \mathfrak{L}(\mathfrak{C}, \mathfrak{F})$$

depend analytically on $s \in \mathbb{C}^n$, then we say that the system $\Xi(s)$, with $s \in \mathbb{C}^n$ on $v \in \mathfrak{C}$,

$$\mu_i(s)(v) = u_i(s), \quad i \in I$$

is an analytic system linear equations (ASLE). Without loss of generality, $\mathfrak{C}_i = \mathbb{C}$ for all i.

2.2. **Fredholm Theory.** A rough form of **principle of meromorphic continuation** is read as follows.

- ⋄ Suppose that an ASLE $\Xi = \Xi(s)$ has a unique solution $v(s)$ for an open set of s's. Then $v(s)$ admits meromorphic continuation to $\mathbb{C}^n$.

Unfortunately, this is not always the case. We have to impose some conditions.

Definition 2.4. We say that an ASLE Ξ is **of finite type** if there exists a finite-dimensional vector space L and an analytic family $\lambda(s)$ of injective operators $L \to \mathfrak{C}$ such that $\mathrm{Sol}(\Xi(s)) \subset \mathrm{Im}\,\lambda(s)$ for all s.

[4]Of course, we can replace $\mathbb{C}^n$ by any complex analytic manifold.

We can similarly speak about systems that are locally of finite type (LFT).

Theorem 2.5 (Principle of meromorphic continuation, PMC)**.** *Let $\Xi = (\Xi(s))_{s\in\mathbb{C}^n}$ be an ASLE on a HLCTVS $\mathfrak{C}$. Assume that Ξ is LFT. Let*

$$S = \{s \in \mathbb{C}^n \mid \textit{the system } \Xi(s) \textit{ admits a unique solution } v(s)\}.$$

Suppose that S has nonempty interior. Then

(1) *$\mathbb{C}^n \backslash S$ is a closed analytic subset of $\mathbb{C}^n$.*
(2) *v is holomorphic on S.*
(3) *v is meromorphic on $\mathbb{C}^n$.*

The proof is a simple application of Cramer's rule.

A basic tool for proving local finiteness is Fredholm theory.

Example 2.6 (A system of Fredholm type)**.** Suppose that $\mu_s, \nu_s : \mathfrak{H}_1 \to \mathfrak{H}_2$ are two analytic families of bounded operators between Hilbert spaces. Suppose that for all s, μ_s is a strict embedding and ν_s is a compact operator.

Then the homogeneous equation

$$\mu_s v = \nu_s v$$

on $v \in \mathfrak{H}_1$ is LFT.

In the example above, recall that **strict embedding** means that there exists constants $C_1, C_2 > 0$ such that

$$C_1\|v\| \leqslant \|\mu_s(v)\| \leqslant C_2\|v\|, \quad \forall v \in \mathfrak{H}_1.$$

A practical way to prove that an operator $A : \mathfrak{H}_1 \to \mathfrak{H}_2$ between Hilbert spaces is compact is to show that it is **Hilbert-Schmidt**. This means that

$$\sum \|Ae_i\|^2 < \infty$$

for any orthonormal basis e_i of $\mathfrak{H}_1$.

Example 2.7. Let $A : \mathfrak{H} \to L^2(X, \mu)$ be a bounded operator from a separable Hilbert space $\mathfrak{H}$ to an L^2-space. Then the following are equivalent.

- A is Hilbert-Schmidt;
- for almost all $x \in X$, the evaluation $\mathrm{ev}_x(u) := Au(x)$ is bounded on $\mathfrak{H}$ and $x \mapsto \|ev_x\| \in L^2(X, \mu)$.

2.3. The Equation that $E(z; s)$ Satisfies.

We take the setups as follows. Consider a real Lie group $G = \mathrm{SL}_2(\mathbb{R})$ together with its maximal compact subgroup $K = \mathrm{SO}(2)$. Then the upper-half plane $\mathbb{H} = G/K$. Also, for the modular subgroup $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ in G, we define

$$\mathcal{X} = \Gamma\backslash\mathbb{H} = \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2).$$

Also denote

$$\mathfrak{F}_{\mathrm{umg}} := \text{the space of functions of } \textbf{uniform moderate growth} \text{ on } \mathcal{X}.$$

This means that there exists $N > 0$ such that

$$|Xf(z)| \ll_X (\mathrm{Im}(z))^N, \quad \mathrm{Im}(z) > \frac{1}{2}$$

for every $X \in U(\mathfrak{g})^K$. The Eisenstein series

$$E(z;s) = \sum_{\gamma \in \pm\Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^{s+\frac{1}{2}} = \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2,\\ \gcd(m,n)=1}} \frac{y^{s+\frac{1}{2}}}{|mz+n|^{2s+1}}$$

converges for $\mathrm{Re}(s) > 1/2$ and defines a function in $\mathfrak{F}_{\mathrm{umg}}(\mathcal{X})$.

Consider the holomorphic system $\Xi(s)$ with $s \in \mathbb{C}$ on $\psi \in \mathfrak{F}_{\mathrm{umg}}(\mathcal{X})$ given by the following three sets of linear equations:

(a) $\delta(h)\psi = \hat{h}(s)\psi$ for all $h \in \mathcal{C}_c^\infty(G/\!\!/K)$;
(b) $(T_a - a^{-s})(\mathfrak{C}\psi(y) - y^{s+\frac{1}{2}}) \equiv 0$ for all $a > 0$;
(c) $(\psi, f)_{\mathcal{X}} = 0$ for every cusp form f on $\mathcal{X}$.

The list of notations appeared above is in the following.

- Denote by $\mathcal{C}_c^\infty(G/\!\!/K)$ the algebra of smooth, bi-K-invariant, compactly supported functions on G.
- This algebra acts on the right on $L^1_{\mathrm{loc}}(\mathbb{H})$. We denote this action by $f \mapsto \delta(h)f$.
- Let $\hat{h}(s)$ be the eigenvalue of the eigenfunction $(\mathrm{Im}(z))^{s+\frac{1}{2}}$ under $\delta(h)$. It can be computed explicitly.
- Denote $\mathfrak{C}f$ the constant term

$$\mathfrak{C}f(y) = \int_{\mathbb{Z}\backslash\mathbb{R}} f(x+iy)dx, \quad y > 0.$$

- For $a > 0$ denote by T_a the normalized shift operator on functions on $\mathbb{R}_{>0}$ given by

$$T_a f(y) = a^{-\frac{1}{2}} f(ay).$$

  These operators pairwise commute.
- $(\cdot,\cdot)_{\mathcal{X}}$ is the pairing with respect to the measure $\mu = \frac{dxdy}{y^2}$ on $\mathcal{X}$.

It is easy to check that the Eisenstein series satisfies $\Xi(s)$ for $\mathrm{Re}(s) > 1/2$. Note that the non-homogeneous equation (b) amounts to the equation

$$\mathfrak{C}E(y;s) = y^{s+\frac{1}{2}} + m(s)y^{-s+\frac{1}{2}}, \quad y > 0$$

for some function $m(s)$. (We do NOT need to know anything about $m(s)$.)[5] In order to apply PMC (Theorem 2.5), we show the following two statements.

(1) For $\mathrm{Re}(s) > 1/2$, the Eisenstein series $E(z;s)$ is the unique function of uniform moderate growth satisfying the equations (b) and (c).
(2) The ASLE (a) and (b) (already for a single $a > 0$) is LFT.

Proposition 2.8 (Uniqueness)**.** *Fix s with $\mathrm{Re}(s) > 1/2$. Suppose that $\psi \in \mathfrak{F}_{\mathrm{umg}}(\mathcal{X})$ satisfies*

$$(*) \qquad (T_a - a^{-s})(\mathfrak{C}\psi(y) - y^{s+\frac{1}{2}}) \equiv 0 \text{ for all } a > 0,$$

and

$$(**) \qquad (\psi, f)_{\mathcal{X}} = 0 \text{ for every cusp form } f \text{ on } \mathcal{X}.$$

[5] Ultimately, it can be computed as

$$m(s) = \sqrt{\pi}\frac{\Gamma(s)\zeta(2s)}{\Gamma(s+\frac{1}{2})\zeta(2s+1)}.$$

Then $\psi = E(z;s)$.

Proof. Let $\psi' = \psi - E(z;s) \in \mathfrak{F}_{\mathrm{umg}}(\mathcal{X})$. Then $\mathfrak{C}\psi'$ is proportional to $y^{\frac{1}{2}-s}$ by $(*)$, hence bounded on $y \geqslant 1/2$. By rapid decay of $\psi' - \mathfrak{C}\psi'$, for $y \geqslant 1/2$, ψ' is bounded. By Γ-invariance, ψ' is bounded on $\mathcal{X}$. Therefore, $\mathfrak{C}\psi'$ is bounded on $\mathbb{R}_{>0}$. Since $\mathrm{Re}(s) > 1/2$, we infer that $\mathfrak{C}\psi' \equiv 0$, so that ψ' is cuspidal. By $(**)$, $\psi' \equiv 0$ as required. □

2.4. **Local Finiteness.** We model the equations on an auxiliary Hilbert space, replacing the complicated space $\mathcal{X}$ by a simpler one which approximates it at the cusp.

- Let $\mathcal{Z} = \Gamma_\infty \backslash \mathbb{H}$.
- Take $\mathcal{S} = \{z \in \mathcal{Z} \mid y > c_0\}$, where $c_0 > 0$ is chosen so that the projection $p : \mathcal{S} \to \mathcal{X}$ is actually onto.
- For $N > 0$ consider the Hilbert space $\mathfrak{H}^N(\mathcal{S}) = L^2(\mathcal{S}; y^{-2N}\mu)$.
- The pullback $f \mapsto \tilde{f}$ by p gives rise to a Hilbert space $\mathfrak{H}^N(\mathcal{X})$ of functions on $\mathcal{X}$ with a strict embedding $\mathfrak{H}^N(\mathcal{X}) \to \mathfrak{H}^N(\mathcal{S})$.
- $\mathfrak{F}_{\mathrm{umg}}(\mathcal{X})$ is the union over N of the smooth part of $\mathfrak{H}^N(\mathcal{X})$.
- Any $f \in \mathfrak{H}^N(\mathcal{S})$ admits an orthogonal decomposition $f = \mathfrak{C}f + f_{\mathrm{cusp}}$ in which $\mathfrak{C}f_{\mathrm{cusp}} \equiv 0$.

Fix $a > 1$ and $h \in \mathcal{C}_c^\infty(G/\!\!/K)$.

Proposition 2.9 (Local Finiteness)**.** *The following ASLE on $f \in \mathfrak{H}^N(\mathcal{X})$ is of Fredholm type for $|\operatorname{Re} s| < N$ and $\hat{h}(s) \neq 0$.*

$$\begin{aligned} \hat{h}(s)\tilde{f}_{\mathrm{cusp}} &= \widetilde{\delta(h)}f, \\ \hat{h}(s)(\mathfrak{C}f)|_{[c_0,c_0a^2]} &= \mathfrak{C}(\delta(h)f)|_{[c_0,c_0a^2]}, \\ (T_a - a^s)(T_a - a^{-s})(\mathfrak{C}f) &= 0. \end{aligned}$$

More precisely, the operator

$$\begin{aligned} \mathfrak{H}^N(\mathcal{X}) &\to \mathfrak{H}^N(\mathcal{S}) \oplus L^2(\mathbb{R}_{>c_0}, y^{-2N}\frac{dy}{y^2}) \oplus L^2([c_0, c_0a^2]) \\ f &\mapsto (\tilde{f}_{\mathrm{cusp}}, (T_a - a^s)(T_a - a^{-s})(\mathfrak{C}f), (\mathfrak{C}f)|_{[c_0,c_0a^2]}) \\ \textit{resp.,}\quad f &\mapsto (\widetilde{\delta(h)}f_{\mathrm{cusp}}, 0, \mathfrak{C}(\delta(h)f)|_{[c_0,c_0a^2]}) \end{aligned}$$

is a strict embedding (resp., Hilbert-Schmidt).

The first statement boils down to the elementary fact that the operator

$$\begin{aligned} L^2(\mathbb{R}_+, e^{-2rx}dx) &\to L^2(\mathbb{R}_+, e^{-2rx}dx) \oplus L^2([0,1]), \\ f &\mapsto (f(x+1) - \lambda f(x), f|_{[0,1]}) \end{aligned}$$

is a strict embedding provided that $e^r > |\lambda|$. The second statement is standard.

3. A More General Picture over Number Fields

3.1. **The General Statement (in broad strokes).** These go on reductive groups over *number fields.*

Notation 3.1. Say G is a reductive group over a number field F. Fix a minimal parabolic subgroup P_0 of G defined over F. Let Ω be the Weyl group of G. Fix a "good" maximal compact subgroup K of $G(\mathbb{A})$. Define

$$\mathcal{P} := \text{the finite set of standard parabolic subgroups of } G \text{ defined over } F.$$

For any $P \in \mathcal{P}$ let $\mathcal{X}_P := P(F)U(\mathbb{A})\backslash G(\mathbb{A})$. Denote $m_P(g)$ the $M(\mathbb{A})$-part of $g \in G(\mathbb{A})$ in the Iwasawa decomposition. Denote $\Delta'_P \subset \Delta_0$ the set of simple roots whose restriction to $\mathrm{Lie}(U)$ is nontrivial.

Fix $P \in \mathcal{P}$ and an automorphic form φ on $\mathcal{X}_P$. For a quasi-character λ of $M(\mathbb{A})/M(\mathbb{A})^1$, the Eisenstein series

$$E(g, \varphi, \lambda) = \sum_{\gamma \in P(F)\backslash G(F)} \varphi(\gamma g) m_P(\gamma g)^\lambda$$

converges absolutely if $\mathrm{Re}(\lambda) \gg 0$ (that is, $\mathrm{Re}\langle \lambda, \alpha^\vee \rangle \gg 0$ for all $\alpha \in \Delta'_P$) and defines an automorphic form on $\mathcal{X}_G = G(F)\backslash G(\mathbb{A})$.

Proposition 3.2 (Cuspidal exponents of Eisenstein series)**.** *For any $Q = L \ltimes V \in \mathcal{P}$, we have*

$$\mathcal{E}_Q^{\mathrm{cusp}}(E(\varphi, \lambda)) \subset \{w(\lambda + \mu) \mid w \in \Omega^{\supset Q}(P),\ \mu \in \mathcal{E}_{P_w}^{\mathrm{cusp}}(\varphi)\}$$

where

- *$\mathcal{E}_R^{\mathrm{cusp}}(\phi)$ is the set (with multiplicities) of cuspidal exponents of ϕ along a parabolic subgroup $R \subset P$.*
- *$\Omega^{\supset Q}(P) = \{w \in \Omega$ right Ω_M-reduced $\mid wMw^{-1} \supset L\}$.*
- *For any $w \in \Omega \supset Q(P)$, P_w is the standard parabolic subgroup of P with Levi subgroup $w^{-1}Lw$.*

This follows from the computation of the constant term of $E(\varphi, \lambda)$ along Q using Bruhat decomposition (a global analogue to the geometric lemma of Bernstein-Zelevinsky).

The computation of the constant term gives more information. While most of the terms involve nontrivial intertwining operators and should be treated as unknown (apart from their exponents), the term corresponding to $w = e$ is simply the constant term $\mathfrak{C}_Q\varphi$ of φ itself (interpreted as 0 if $Q \not\subset P$).

3.2. **Uniqueness Property for Automorphic Forms.** Denote by $\mathfrak{C}_Q^{\mathrm{cusp}}\phi$ the cuspidal projection of $\mathfrak{C}_Q\phi$.

Proposition 3.3 (Uniqueness property)**.** *The exponents of the difference $\mathfrak{C}_Q^{\mathrm{cusp}}E(\varphi, \lambda) - \mathfrak{C}_Q^{\mathrm{cusp}}\varphi$ are contained (as a multiset) in*

$$\{w(\lambda + \mu) \mid w \in \Omega^{\supset Q}(P)\backslash\{e\},\ \mu \in \mathcal{E}_{P_w}^{\mathrm{cusp}}(\varphi)\}.$$

Moreover, this property determines $E(\varphi, \lambda)$ uniquely, at least if $\mathrm{Re}(\lambda) \gg 0$.

This uniqueness property follows from a general result on automorphic forms which is a strengthening of a basic result of Langlands.

Proposition 3.4. *Let ϕ be an automorphic form on $\mathcal{X}_G$, not identically zero. Then, there exists $Q \in \mathcal{P}$ and $\lambda \in \mathcal{E}_Q^{\mathrm{cusp}}(\phi)$ such that $\mathrm{Re}\,\lambda + \rho_Q$ lies in the closure of the positive Weyl chamber.*

This is proved by reducing it to a corank one statement, where the argument is very similar to the SL_2 case explained above.

The deduction of the uniqueness statement for $E(\varphi, \lambda)$ follows by observing that if $\operatorname{Re}\lambda \gg 0$ and $w \in \Omega^{\supset Q}(P) \backslash \{e\}$, then $w \operatorname{Re}\lambda$ is far from the positive Weyl chamber of Q.

For the local finiteness we need an additional set of equations. They are of the form

$$\delta(h_i(\lambda))\psi = c_i(\lambda)\psi, \quad i \in I,$$

where for each $i \in 1$, $h_i(\lambda)$ is a holomorphic family of functions in $\mathcal{C}_c^\infty(G(\mathbb{A}))$ and $c_i(\lambda)$, $i \in I$ are holomorphic functions with no common zeros. By a basic result of Harish-Chandra, such equations are satisfied by any holomorphic family of automorphic forms.

Assume for simplicity that the center of G is F-anisotropic.

Proposition 3.5 (The shape of the equations). *The following ASLE on $f \in \mathfrak{F}_{\mathrm{umg}}(\mathcal{X})$ is LFT:*

$$\delta(h(s))f = f, \quad D_\alpha(s)(T_{a_\alpha})(\mathfrak{C}_{P_\alpha} f) = 0 \textit{ for every } \alpha \in \Delta_0.$$

Here, $h(s)$ is a holomorphic family of functions in $\mathcal{C}_c^\infty(G(\mathbb{A}))$ (for s in a complex analytic manifold) and for every $\alpha \in \Delta_0$,

- *$D_\alpha(s)$ is a holomorphic family of monic polynomials in one variable of degree m_α;*
- *P_α is the maximal parabolic subgroup of G corresponding to α;*
- *T_{a_α} is the normalized left translation by a fixed element a_α of the center of the Levi part of P_α such that $|\alpha(a_\alpha)| > 1$.*

As in the SL_2 case, to prove this we pass to an auxiliary ASLE on a weighted Hilbert space $\mathfrak{H}^\lambda(\mathcal{X})$ for a certain parameter $\lambda \in \mathfrak{a}_0^*$.

3.3. **Weighted Hilbert Spaces.** *Setups.* Here are some sorts of the weighted L^2-space.

(a) Fix a Siegel domain $\mathcal{S}$ of $P_0(F)\backslash G(\mathbb{A})$ such that the projection $p : \mathcal{S} \to \mathcal{X}$ is onto.
(b) Define a weighted L^2-space

$$\mathfrak{H}^\lambda(\mathcal{S}) = L^2(\mathcal{S}, m_{P_0}(x)^{-2\lambda} dx).$$

(c) The pullback $f \mapsto f^{\mathcal{S}}$ by p gives rise to a Hilbert space $\mathfrak{H}^\lambda(\mathcal{X})$ of functions on $\mathcal{X}$ with a strict embedding in $\mathfrak{H}^\lambda(\mathcal{S})$.
(d) We have $\mathfrak{F}_{\mathrm{umg}}(\mathcal{X}) = \bigcup_\lambda \mathfrak{H}^\lambda_\infty(\mathcal{X})$.
(e) The space $\mathfrak{H}^\lambda(\mathcal{S})$ admits a Harish-Chandra decomposition

$$\mathfrak{H}^\lambda(\mathcal{S}) = \oplus P \in \mathcal{P} \mathfrak{H}^\lambda_{\mathrm{cusp}}(\mathcal{S}_P)$$

where $\mathcal{S}_P$ is the image of $\mathcal{S}$ under the projection to $U(\mathbb{A})P_0(F)\backslash G(\mathbb{A})$.
(f) For any $f \in \mathfrak{H}^\lambda(\mathcal{X})$ denote by $f_P^{\mathcal{S}}, P \in \mathcal{P}$ the components of $f^{\mathcal{S}}$ with respect to this decomposition.

Proposition 3.6. *The following ASLE on $f \in \mathfrak{H}^\lambda(\mathcal{X})$ is of Fredholm type (and in particular, LFT) provided that λ is sufficiently positive.*

$$D_\alpha(s)(T_{a_\alpha})(f_P^{\mathcal{S}}) = 0 \textit{ for all } \alpha \in \Delta'_P, \quad f_P^{\mathcal{S}}|_{\mathcal{S}'_P} = (\delta(h(s))f)_P^{\mathcal{S}}|_{\mathcal{S}'_P}$$

for every $P \in \mathcal{P}$, where $\mathcal{S}'_P = \mathcal{S}_P \backslash \bigcup_{\alpha \in \Delta'_P} a_\alpha^{m_\alpha} \mathcal{S}_P$. More precisely, for every P,

(1) *The operator* $\mathfrak{H}^\lambda(\mathcal{S}_P) \to \mathfrak{H}^\lambda(\mathcal{S}_P)^{\Delta'_P} \oplus \mathfrak{H}^\lambda(\mathcal{S}'_P)$

$$f \mapsto ((D_\alpha(s)(T_{\partial_\alpha})(f))_{\alpha\in\Delta'_p}, f|_{\mathcal{S}'_p})$$

is a strict embedding, if $e^{\langle\lambda, H_0(a_\alpha)\rangle} > |r|$ *for an arbitrary root* r *of* $D_\alpha(s)$.

(2) *The operator*

$$\mathfrak{H}^\lambda(\mathcal{X}) \to \mathfrak{H}^\lambda(\mathcal{S}'_P), \quad f \mapsto (\delta(h(s))f)_P^{\mathcal{S}}|_{\mathcal{S}'_P}$$

is Hilbert-Schmidt[6].

Remark 3.7. (1) The proof gives meromorphic continuation for intertwining operators and functional equations

$$E(M(w,\lambda)\varphi, w\lambda) = E(\varphi,\lambda),$$
$$M(w_2w_1,\lambda) = M(w_2, w_1\lambda)M(w_1,\lambda).$$

(2) The case of a global function field (i.e., positive characteristic) is easier. (Use an algebraic version of the PMC. The uniqueness result suffices. No further analysis is needed.)

Questions.

(1) It would be desirable to have a more robust statement for Eisenstein series induced from smooth automorphic forms (i.e., $\mathfrak{z}$-finite, but not necessarily K-finite, function of uniform moderate growth).

With the approach described, this would entail a more flexible form of the PMC (relaxing the LFT condition). There are other approaches to extend the meromorphic continuation from the K-finite case to the smooth case.

(2) Is there a version of the PMC that guarantees that the solution is a function of finite order?

4. Spectral Decomposition of $L^2(G(F)\backslash G(\mathbb{A}))$

For any $P \in \mathcal{P}$ denote by $\mathcal{A}_P$ the space of automorphic forms on $\mathcal{X}_P$,

$$\mathcal{A}_P^2 = \{\varphi \in \mathcal{A}_P \mid \delta_P^{-\frac{1}{2}}\varphi(\cdot g) \in L^2(A_M M(F)\backslash M(\mathbb{A})),\ \forall g \in G(\mathbb{A})\}.$$

This is an inner product space with respect to integration over $A_M\backslash\mathcal{X}_P$. Its completion is

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A})).$$

By the functional equations, for any $w \in \Omega(P,Q)$ the intertwining operators

$$M(w,\lambda) : \mathcal{A}_P^2 \longrightarrow \mathcal{A}_Q^2, \quad \lambda \in \mathfrak{a}^*_{P,C}$$

are unitary (and in particular, holomorphic) on $i\mathfrak{a}_P^*$ and extend to isometries

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A})) \longrightarrow \mathrm{Ind}_{Q(\mathbb{A})}^{G(\mathbb{A})} L^2_{\mathrm{disc}}(A_L L(F)\backslash L(\mathbb{A})).$$

[6]Due to Gelfand-Piatetski-Shapiro.

Theorem 4.1 (Langlands)**.** *The bilinear map*

$$f, \varphi \in \mathcal{C}_c^\infty(\mathrm{i}\mathfrak{a}_p^*) \times \mathcal{A}_P^2 \longmapsto E_{f\otimes\varphi} = \int_{\mathrm{i}\mathfrak{a}_p^*} f(\lambda)E(\varphi,\lambda)d\lambda,$$

which is suitably normalized, induces an isometry of Hilbert spaces

$$\Theta : \left(\bigoplus_P L^2(\mathrm{i}\mathfrak{a}_P^*; \mathrm{Ind}_{P(\mathbb{A})}^{G(A)} L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A})))\right)_\Omega \longrightarrow L^2(\mathcal{X}_G)$$

where the Ω-coinvariants space on the left-hand side is isomorphic to the closed subspace defined by the relations

$$f^Q(w\lambda) = M(w,\lambda)f^P(\lambda), \quad \lambda \in \mathrm{i}_P^*, \quad w \in \Omega(P,Q).$$

Langlands proved this theorem in his treatise on Eisenstein series. His argument is a *tour de force*. Lapid will explain a simpler approach, due to Delorme [Del20] (combined with a technical simplification). It uses as input

- ◇ the meromorphic continuation of Eisenstein series, and
- ◇ another key idea of Bernstein from the 1980s ("on the support of the Plancherel measure").

The method is analogous to the Plancherel formula in the local case (Harish-Chandra, Waldspurger) extended to symmetric spaces by Delorme around 2000. This was revisited and extended by Sakellaridis-Venkatesh, but we will not discuss it here.

For simplicity we will *assume* that the Eisenstein series for $\varphi \in \mathcal{A}_P^2$ is holomorphic on $\mathrm{i}\mathfrak{a}_P^*$. In fact, it is not necessary to assume this – it eventually follows from the proof.

Definition 4.2. An automorphic form $\varphi \in \mathcal{A}_P$ is **tempered** if there exists k such that

$$|\varphi(g)| \ll e^{\langle\rho,H(g)\rangle}(1 + \|H(g)\|)^k, \quad \forall g \in A_M\mathcal{S}.$$

Equivalently, every exponent $\lambda \in \mathcal{E}_Q(\varphi)$, $Q \subset P$ is subunitary (i.e., $\mathrm{Re}\,\lambda$ is a sum of simple roots Δ_Q^P with non-positive coefficients.)

We denote by $\mathcal{A}_P^{\mathrm{temp}}$ the space of tempered automorphic forms. Then $\mathcal{A}_P^2 \subset \mathcal{A}_P^{\mathrm{temp}}$. And if $\varphi \in \mathcal{A}_P^2$, then $E(\varphi,\lambda) \in \mathcal{A}_G^{\mathrm{temp}}$ for all $\lambda \in \mathrm{i}\mathfrak{a}_P^*$. Moreover, for any $\phi \in \mathcal{A}_P^2$ and $\psi \in \mathcal{A}_P^{\mathrm{temp}}$ the tempered distribution $p(\phi,\psi)$ on $\mathfrak{a}_p$

$$f \in \mathcal{S}(\mathfrak{a}_P) \mapsto \langle \phi\cdot(f\circ H_P), \psi\rangle_{\mathcal{X}_P}$$

is a polynomial exponential whose exponents are unitary and contained in the set $\mathcal{E}_P(\varphi) + \overline{\mathcal{E}_P(\psi)}$.

Definition 4.3. Let $\varphi \in \mathcal{A}_G^{\mathrm{temp}}$. The **weak constant term** $\varphi_P^{\mathrm{weak}}$ is the part of φ_P corresponding to the exponents λ with $\mathrm{Re}\,\lambda = 0$.

We have $\varphi_P^{\mathrm{weak}} \in \mathcal{A}_P^{\mathrm{temp}}$. If $\varphi \in \mathcal{A}_G^2$, then $\varphi_P^{\mathrm{weak}} \equiv 0$ for all proper P. For any $\psi \in \mathcal{A}_G^{\mathrm{temp}}$,

$$\psi \equiv 0 \quad \Longleftrightarrow \quad p(\varphi,\psi_P^{\mathrm{weak}}) \equiv 0 \text{ for every } P \in \mathcal{P} \text{ and } \varphi \in \mathcal{A}_P^2.$$

Key Statement. For any $p \in \mathcal{PE}^{\mathrm{unit}}(\mathfrak{a}_P)$ (i.e., a polynomial exponential with unitary exponents) denote by $\partial(p)$ the convolution (on $\mathcal{C}_c^\infty(\mathrm{i}\mathfrak{a}_P^*)$) by the Fourier transform of p, viewed as a finitely supported distribution on $\mathrm{i}\mathfrak{a}_P^*$.

Proposition 4.4. *Let $\varphi \in \mathcal{A}_P^2$. Then, for any $\psi \in \mathcal{A}_G^{\mathrm{temp}}$ and $f \in \mathcal{C}_c^\infty(\mathrm{i}\mathfrak{a}_P^*)$ we have*

$$\left\langle \int_{\mathrm{i}\mathfrak{a}_P^*} f(\lambda) E(\varphi, \lambda) d\lambda, \psi \right\rangle_{\mathcal{X}_G} = \partial(p(\varphi, \psi_P^{\mathrm{weak}})) f(0).$$

In particular, if $\psi \in \mathcal{A}_G^2$ and $P \subsetneq G$, then $\langle E(\varphi, \lambda), \psi \rangle_{\mathcal{X}_G} = 0$.

The left-hand side is well defined since the "wave packet"

$$\int_{\mathrm{i}\mathfrak{a}_p^*} f(\lambda) E(\varphi, \lambda) d\lambda$$

is in the Harish-Chandra Schwartz space of $\mathcal{X}_G$.

Why is this enough?

(1) First, by Bernstein, only tempered automorphic forms may contribute to $L^2(\mathcal{X})$. The proposition therefore implies that the wave packets span a dense subspace of $L^2(\mathcal{X})$.

(2) Second, if $\psi = E(\varphi', \mu)$ with $\varphi' \in \mathcal{A}_Q^2$ and $\mu \in \mathrm{i}\mathfrak{a}_{Q^*}^*$, then,

$$\psi_P^{\mathrm{weak}} = \sum_{w \in \Omega \supset Q(P)} E^P(M(w^{-1}, \mu)\varphi', w^{-1}\mu),$$

Hence,

$$p(\varphi, \psi_P^{\mathrm{weak}}) = \sum_{w \in \Omega(Q,P)} p(\varphi, (M(w, \mu)\varphi')_{w\mu}).$$

Thus,

$$\left\langle \int_{\mathrm{i}\mathfrak{a}_p^*} f(\lambda) E(\varphi, \lambda) d\lambda, \psi \right\rangle_{\mathcal{X}_G} = \sum_{w \in \Omega(Q,P)} f(w\mu) \langle \varphi, M(w, \mu)\varphi' \rangle_{A_M \backslash \mathcal{X}_P}.$$

One way to prove the key statement (Proposition 4.4) is using Arthur's truncation operator Λ^T. Assume for simplicity that the center of G is F-anisotropic.

Lemma 4.5. *For any $\phi, \psi \in \mathcal{A}_G$ the function*

$$\langle \Lambda^T \phi, \psi \rangle_{\mathcal{X}_G} = \langle \phi, \Lambda^T \psi \rangle_{\mathcal{X}_G}$$

is a polynomial exponential in T whose exponents are contained in the set

$$\bigcup_P (\mathcal{E}_P(\varphi) + \overline{\mathcal{E}_P(\psi)}).$$

In particular, if $\phi, \psi \in \mathcal{A}_G^{\mathrm{temp}}$, then the exponents are subunitary. In this case, we denote by $\langle \Lambda^T \phi, \psi \rangle_{\mathcal{X}_G}^{\mathrm{unit}}$ the unitary part of this polynomial exponential.

Proposition 4.6. *Let $\psi \in \mathcal{A}_G^{\mathrm{temp}}$ and $\varphi \in \mathcal{A}_P^2$. Fix $T \in \mathfrak{a}_0$. Then, as a distribution on $\mathrm{i}\mathfrak{a}_P^*$, $\langle \Lambda^T E(\varphi, \cdot), \psi \rangle_{\mathcal{X}_G}^{\mathrm{unit}}$ is equal to*

$$\sum_Q \sum_{w \in \Omega(P,Q)} \partial(p(M(w, w^{-1} \cdot)\varphi, \psi_Q^{\mathrm{weak}}))(\widehat{\chi_w^T}) \circ w$$

where $\chi_w^T = (-1)^{\#\{\alpha\in\Delta_Q \mid w^{-1}\alpha<0\}}$ *times the characteristic function of the projection to* $\mathfrak{a}_Q$ *of the set*

$$T + \left\{ \sum_{\alpha\in\Delta_0} c_\alpha \alpha^\vee \mid c_\alpha > 0 \Longleftrightarrow w^{-1}\alpha < 0 \right\}.$$

In particular, if $\psi \in \mathcal{A}_G^2$ *and* $P \neq G$*, then*

$$\langle \Lambda^T E(\varphi,\lambda), \psi\rangle_{\mathcal{X}_G}^{\mathrm{unit}} \equiv 0.$$

Here,

$$\partial : C^\infty(\mathrm{i}\mathfrak{a}_Q^*) \otimes \mathcal{PE}^{\mathrm{unit}}(\mathfrak{a}_Q) \to \mathrm{End}(\mathcal{D}(\mathrm{i}\mathfrak{a}_Q^*))$$

is the linear map (rather than homomorphism) with $C^\infty(\mathrm{i}\mathfrak{a}_Q^*)$ acting by multiplication and $p \in \mathcal{PE}^{\mathrm{unit}}(\mathfrak{a}_Q)$ acting by convolution by $\hat{p}$.

Remark 4.7. Taking ψ to be an Eisenstein series, we obtain Arthur's asymptotic inner product of truncated Eisenstein series. (Arthur's original proof used Langlands's work.)

The key statement (Proposition 4.4) follows from the last proposition by taking the limit in T as $\min_{\alpha\in\Delta_0}\langle\alpha, T\rangle \to \infty$, since the limit (as tempered distributions) of χ_w^T is the constant function 1 if $w = e$ (and $Q = P$) and 0 otherwise.

Proposition 4.8. *For any* $\varphi \in \mathcal{A}_P$ *and* $\operatorname{Re}\lambda \gg 0$, $\Lambda^T E(\varphi,\lambda)$ *is the sum over* $Q \in \mathcal{P}$ *and* $w \in \Omega^{\supset Q}(P)$ *of*

$$\sum_{\gamma\in Q(F)\backslash G(F)} \Lambda^{T,Q}(M(w,\lambda)\varphi_{P_w})_{w\lambda}(\gamma g)\chi_w(H_Q(\gamma g) - T).$$

Here, $\Lambda^{T,Q}$ *is a relative truncation operator and* χ_w *is the function on* $\mathfrak{a}_Q$ *given by*

$$\chi_w(X) = \begin{cases} (-1)^{|D_{Q,+}(X)|} & \textit{if } (*) \textit{ holds },\\ 0 & \textit{otherwise}\end{cases}$$

where $D_{Q,+}(\sum_{\alpha\in\Delta_Q} x_\alpha\alpha^\vee) = \{\alpha\in\Delta_Q \mid x_\alpha > 0\}$ *and the condition* $(*)$ *is*

$$D_{Q,+}(X) = \{\alpha \in \Delta_Q \mid w^{-1}\alpha < 0 \textit{ or } w^{-1}\alpha \in \Delta_{P_w}^P \textit{ and } \langle\alpha, X\rangle \leqslant 0\}.$$

In particular, $\chi_w(\cdot - T) = \chi_w^T$ *if* $w \in \Omega(P,Q)$*.*

Remark 4.9. If φ is cuspidal, this was Langlands's original ad hoc definition of truncation of Eisenstein series. Subsequently, Arthur realized (in the late 1970s) that one can define a truncation operator on the space of locally L^1 functions on $\mathcal{X}_G$.

How to conclude the Proposition 4.6 from Proposition 4.8? χ_w can be expressed as a linear combination of characteristic functions of simplicial cones. Thus, its Laplace transform is a rational function with hyperplane singularities. Using the last proposition and simple geometric properties of χ_w, we can compute the inner product $\langle \Lambda^T E(\varphi,\lambda), \psi\rangle_{\mathcal{X}_G}$ as a sum of contributions over $Q \in \mathcal{P}$ and $w \in \Omega^{\supset Q}(P)$. One sees that only $w \in \Omega(P,Q)$ may contribute to $\langle \Lambda^T E(\varphi,\lambda), \psi\rangle_{\mathcal{X}_G}^{\mathrm{unit}}$. This contribution can be computed explicitly, and yields the penultimate proposition.

To be continued in Lecture 2/2.

References

[BL19] Joseph Bernstein and Erez Lapid. On the meromorphic continuation of eisenstein series. 2019. Available at arXiv:1911.02342.

[Del20] Patrick Delorme. On the spectral theorem of langlands. 2020. Available at arXiv:2006.12893.

[Lap22] Erez Lapid. Some perspectives on eisenstein series. 2022. Available at arXiv:2204.02914.

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