THE LOCAL LANGLANDS CONJECTURE

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Abstract. These are notes for the course given by Olivier Taïbi in 2022 Summer School on the Langlands Program at IHES. We formulate the local Langlands conjecture for connected reductive groups over local fields, including the internal parametrization of L-packets.

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Let F be a local field. We denoted by $\|\cdot\|$ the normalized absolute value of F. In the non-archimedean case it maps a uniformizer to q^{-1} where q is the cardinality of the residue field. If $F \simeq \mathbb{R}$ it is the usual absolute value, if $F \simeq \mathbb{C}$ it is given by $z \mapsto z\overline{z}$.

1. Representations of reductive groups

1.1. **Setup.** In this section we focus on the case where F is non-archimedean and occasionally indicate the differences for the archimedean case.

Let G be a connected reductive group over F. We refer to [Bor91] [Spr98] [BT65] and [DGA+11] for fundamental results about reductive groups. Let C be an algebraically closed field of characteristic zero, for example $\mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$. We consider **smooth** representations of $G(F)$ with coefficients in C, i.e.

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pairs (V, π) where V is a vector space over C and $\pi : G(F) \to \mathrm{GL}(V)$ is a morphism of groups such that the map

$$G \times V \longrightarrow V$$
$$(g, v) \longmapsto \pi(g)v$$

is continuous for the natural topology on G and the discrete topology on V. If π is implicit we will also denote $g \cdot v$ for $\pi(g)v$. Recall that such a representation is called **admissible** if for any compact open subgroup K of $G(F)$ the subspace

$$V^K = \{v \in V \mid \forall k \in K,\ \pi(k)v = v\}$$

of V has finite dimension. It is a non-trivial but well-known fact that *any irreducible representation is admissible*. Denote by $Z(G)$ the center of G. By a suitable generalization of Schur's lemma, any irreducible representation has a central character $Z(G)(F) \to C^\times$. For a smooth representation (V, π) of $G(F)$, its **contragredient** $(\widetilde{V}, \tilde{\pi})$ is the space of K-finite linear forms on V.

Remark 1.1. In the case of an archimedean field F we only consider coefficients $C = \mathbb{C}$. The analogue of smooth representations are $(\mathfrak{g}, K)$-modules where $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \operatorname{Lie} G(F)$ and K is a maximal compact subgroup of $G(F)$. For many notions it is necessary to relate $(\mathfrak{g}, K)$-modules to continuous representations of $G(F)$ on topological vector spaces. See e.g. [Wal88, §3.4] for the relation between the two notions in the case of unitary irreducible representations.

1.2. **Parabolic induction and the Jacquet functor.** Let P be a parabolic subgroup of G. Let N be the unipotent radical of P and $M = P/N$ its reductive quotient. Recall that there exists a section $M \to P$, unique up to conjugation by $N(F)$. Let $\delta_P(p) = |\det(\mathrm{Ad}(p)|\operatorname{Lie}(N))|$ be the modulus character (of $M(F)$ acting on $N(F)$). We choose a square root $\sqrt{q}$ of q in C, allowing us to define $\delta_P^{1/2}$. If $C = \mathbb{C}$ we naturally choose $\sqrt{q} \in \mathbb{R}_{>0}$.

Let (V, σ) be a smooth representation of $M(F)$, which we can see as a representation of $P(F)$ trivial on $N(F)$. The normalized parabolically induced representation $i_P^G \sigma$ is the space of locally constant function $f : G(F) \to V$ such that for any $p \in P(F)$ and $g \in G(F)$ we have $f(pg) = \delta_P(p)^{1/2}\sigma(p)f(g)$, with left action by $(g \cdot f)(x) = f(xg)$. If σ is admissible (resp. has finite length) then $i_P^G\sigma$ is admissible (resp. has finite length). The introduction of $\delta_P^{1/2}$ in the definition are motivated by the fact that if $C = \mathbb{C}$ and (V, σ) is unitary, i.e. endowed with a $M(F)$-invariant Hermitian inner product, then $i_P^G\sigma$ has a natural $G(F)$-invariant Hermitian inner product. In particular if σ is admissible and unitarizable then $i_P^G\sigma$ is semi-simple.

For (π, V) a smooth representation of $G(F)$, denote by V_N the space of coinvariants for the action of $N(F)$, which is naturally a smooth representation π_N of $M(F)$. The **normalized Jacquet functor** applied to (π, V) is the smooth representation $r_P^G\pi = \delta_P^{1/2} \otimes \pi_N$ of $M(F)$ on the space V_N. It also preserves admissibility and the property of being of finite length.

Recall that an irreducible (hence admissible) smooth representation (V, π) of $G(F)$ is called **supercuspidal** if $V_N = 0$ for any parabolic $P = MN \subsetneq G$; or equivalently, if for every proper parabolic subgroup P, the Jacquet functor $r_P^G(\cdot)$ is zero. This is equivalent to all "matrix coefficients"

$$G(F) \longrightarrow C$$
$$g \longmapsto \langle \pi(g)v, \tilde{v} \rangle$$

for $v \in V$ and $\tilde{v} \in \widetilde{V}$, being compactly supported modulo center. Note that if $\omega_\pi : Z(G(F)) \to C^\times$ is the central character of π then matrix coefficients of π are ω_π-equivariant.

We recall in the following theorem the notion of supercuspidal support.

Theorem 1.2. *Let π be an irreducible representation of $G(F)$.*

1. *There exists a parabolic subgroup $P = MN$ of G and a supercuspidal irreducible representation σ of $M(F)$ such that π embeds in $i_P^G\sigma$.*
2. *If $P' = M'N'$ is a parabolic subgroup of G and σ' is a supercuspidal irreducible representation of $M'(F)$ then π is isomorphic to a subquotient of $i_{P'}^G\sigma'$ if and only if there exists an element of $G(F)$ conjugating (M, σ) and (M', σ), where M and σ are given as in (1).*

The conjugacy classes of (M, σ) may be called the **supercuspidal support of** π.

Proof. The first part is due to Jacquet: see [Cas, Theorem 5.1.2]. The second part seems to be due to Harish-Chandra: see [Sil79, Theorem 4.6.1, §5.3.1 and Theorem 5.4.4.1] for the "if" part. The "only if" part can be deduced from Bernstein center theory [Ber84a]. See also [BZ77]. □

The $G(F)$-conjugacy class of (M,σ) in the previous theorem is called the **supercuspidal support** of π.

1.3. **Asymptotic properties.** For the rest of this section we assume $C=\mathbb{C}$.

Definition 1.3. Let (V,π) be a smooth irreducible representation of $G(F)$. Let $\omega_\pi : Z(G(F)) \to \mathbb{C}^\times$ be its central character. If ω_π is unitary, then we say that π is **essentially square-integrable** if all of its matrix coefficients are square-integrable modulo center:

$$\forall v \in V,\ \forall \tilde{v} \in \widetilde{V},\quad \int_{G(F)/Z(G(F))} |\langle \pi(g)v, \tilde{v}\rangle|^2 dg < \infty.$$

In general (without assuming that ω_π is unitary) there is a unique smooth character $\chi : G(F) \to \mathbb{R}_{>0}$ such that the central character of $\chi\otimes\pi$ is unitary [Cas, Lemma 5.2.5], and we say that π is essentially square-integrable if $\chi\otimes\pi$ is.

If π is an essentially square-integrable irreducible smooth representation of $G(F)$ and if ω_π is unitary then π is unitarizable.

Essential square-integrability can be checked on the Jacquet module of a representation, as recalled in Proposition 1.4 below. For a Levi subgroup M of G we denote by A_M the largest split torus in the centre of M. Denote $\mathfrak{a}_M^* := X^*(A_M)\otimes_{\mathbb{Z}}\mathbb{R}$. We have an isomorphism

$$\begin{aligned} \mathfrak{a}_M^* &\longrightarrow \mathrm{Hom}_{\mathrm{cont}}(A_M(F), \mathbb{R}_{>0}) \\ \chi\otimes s &\longmapsto (x \mapsto |\chi(x)|^s). \end{aligned} \tag{1.1}$$

Proposition 1.4 ([Wal03, Proposition III.1.1])**.** *Let (V,π) be an irreducible smooth representation of $G(F)$. Assume that the central character of π is unitary (we can reduce to this case by twisting). Then (V,π) is essentially square-integrable if and only if for every parabolic subgroup $P=MN$ of G, the absolute value of any character of $A_M(F)$ occurring in $r_P^G\pi$ is a linear combination with positive coefficients of the simple roots of A_M in N via the isomorphism (1.1).*

Replacing "positive" by "non-negative" in this characterization we get the notion of **tempered representation**. This is also equivalent to a growth condition on coefficients [Wal03, Proposition III.2.2].

We have the following implications, for an irreducible smooth representation of $G(F)$ having unitary central character:

$$\text{supercuspidal} \implies \text{essentially square-integrable} \implies \text{tempered} \implies \text{unitarizable}.$$

For non-commutative G none of these implications is an equivalence.

Proposition 1.5 ([Wal03, Proposition III.4.1])**.**
(1) *Let $P=MN$ be a parabolic subgroup of F and σ an essentially square-integrable irreducible smooth representation of $M(F)$ having unitary central character. Then the induced representation $i_P^G\sigma$ is semi-simple, has finite length and any irreducible subrepresentation is tempered.*
(2) *Let (P,σ) and (P',σ') be two pairs as in (1). Then $i_P^G\sigma$ and $i_{P'}^G\sigma'$ admit isomorphic irreducible subrepresentations if and only if the pairs (M,σ) and (M',σ') are conjugated by $G(F)$, and in this case the two induced representations are isomorphic.*
(3) *For any tempered irreducible smooth representation π of $G(F)$ there exists a pair (P,σ) as in (1) such that π is isomorphic to a subrepresentation of $i_P^G\sigma$.*

Remark 1.6. For $G=\mathrm{GL}_n$, parabolically induced representations as in Proposition 1.5 are always irreducible [Ber84b, §0.2] and so the proposition completely classifies tempered representations in terms of essentially square-integrable representations of smaller general liner groups.

For arbitrary G such induced representations are **generically irreducible** (see [Wal03, Proposition IV.2.2] for a precise statement), but decomposing such induced representations is a suitable problem in general.

The tempered representations are exactly the ones occurring in Harish-Chandra's Plancherel formula, expressing the values of any locally constant and compactly supported $f : G(F) \to \mathbb{C}$ in terms of the action of f in tempered representations (or expressing $f(1)$ in terms of the traces of f in tempered representations).

Finally the "Langlands classification", that we recall below, classifies irreducible smooth representations of $G(F)$ in terms of tempered representations of Levi subgroups. For a connected reductive group M denote by $X^*(M)^\Gamma$ the abelian group of morphisms $M \to \mathrm{GL}_1$ (defined over F). The restriction morphism $X^*(M)^\Gamma \to X^*(A_M)$ is an isogeny (it is injective with finite cokernel) and so it induces an isomorphism $\mathrm{Res}^M_{A_M} : X^*(M)^\Gamma \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathfrak{a}^*_M$. We have an isomorphism

$$\begin{aligned} X^*(M)^\Gamma \otimes_{\mathbb{Z}} \mathbb{R} &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont}}(M(F), \mathbb{R}_{>0}) \\ \chi \otimes s &\longmapsto (x \mapsto |\chi(x)|^s). \end{aligned} \tag{1.2}$$

Fix a minimal parabolic subgroup P_0 of G and a Levi factor M_0 of P_0. Let $Y \subseteq X^*(A_{M_0})$ be the subgroup of characters which are trivial on $A_{M_0} \cap G_{\mathrm{der}}$. Recall from [BT65, Corollaire 5.8] that the set of roots of A_{M_0} in G is a root system in $(X^*(A_{M_0}, Y)$. Let $\Delta \subseteq X^*(A_{M_0})$ be the set of simple roots for the order corresponding to P_0. The rational Weyl group $N(A_{M_0}, G(F))/M_0(F)$ acts on $\mathfrak{a}^*_{M_0}$; fix an invariant inner product $(\cdot,\cdot)$ on $\mathfrak{a}^*_{M_0}$. For M a standard Levi subgroup of G the restriction map $X^*(A_{M_0}) \to X^*(A_M)$ induces a surjective map $\mathrm{Res}^{A_{M_0}}_{A_M} : \mathfrak{a}^*_{M_0} \to \mathfrak{a}^*_M$. We also have a composite map in the other direction

$$j^M_{M_0} : \mathfrak{a}^*_M \xrightarrow{(\mathrm{Res}^M_{A_M})^{-1}} X^*(M)^\Gamma \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\mathrm{Res}^M_{M_0}} X^*(M_0)^\Gamma \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\mathrm{Res}^{M_0}_{A_{M_0}}} \mathfrak{a}^*_{M_0}$$

and the composition $\mathrm{Res}^{A_{M_0}}_{A_M} \circ j^M_{M_0}$ is $\mathrm{id}_{\mathfrak{a}^*_M}$. In fact one can check that $j^M_{M_0} \circ \mathrm{Res}^{A_{M_0}}_{A_M}$ is the orthogonal projection $\mathfrak{a}^*_{M_0} \to j^M_{M_0}(\mathfrak{a}^*_M)$.

Theorem 1.7 ([Sil78, Theorem 4.1])**.**
(1) *Let P be a standard Levi subgroup of G (with respect to P_0) and M is Levi factor containing M_0. Let σ be a tempered irreducible smooth representation of $M(F)$ (in particular its central character is unitary). Let $\nu \in X^*(M)^\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ be such that for any $\alpha \in \Delta$ not occurring in M we have $(\mathrm{Res}^M_{A_{M_0}} \nu, \alpha) > 0$. Consider ν as a character of $M(F)$ via (1.2), and denote by σ_ν the twist of σ by this character. Then the induced representation $i^G_P(\sigma_\nu)$ admits a unique irreducible quotient $J(P, \sigma, \nu)$. Let $\overline{P}$ be a parabolic subgroup of G which is opposite to P. We have $\dim_{\mathbb{C}} \mathrm{Hom}_G(i^G_P(\sigma_\nu), i^G_{\overline{P}}(\sigma_\nu)) = 1$ and any nonzero element in this line identifies $J(P, \sigma, \nu)$ with the unique irreducible subrepresentation of $i^G_{\overline{P}}(\sigma_\nu)$.*
(2) *Let π be an irreducible smooth representation of $G(F)$. There exists a unique triple (P, σ, ν) as above such that π is isomorphic to the quotient $J(P, \sigma, \nu)$.*

Remark 1.8. It will be useful to reformulate the positivity condition on ν in terms of the absolute root system of G. First note that the condition does not depend on the choice of an admissible inner product on $\mathfrak{a}^*_{M_0}$. Let T be a maximal torus in $M_{0,F^{\mathrm{sep}}}$ and choose a Borel subgroup B of $G_{F^{\mathrm{sep}}}$ containing T and contained in $P_{0,F^{\mathrm{sep}}}$. Choose an admissible inner product $(\cdot,\cdot)_T$ on $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, i.e. one variant under the absolute Weyl group. Consider the restriction map $X^*(T) \to X^*(A_{M_0})$, inducing a surjective map $\mathrm{Res}^T_{A_{M_0}} : X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathfrak{a}^*_{M_0}$. It identifies $\mathfrak{a}^*_{M_0}$ with $\ker(\mathrm{Res}^T_{A_{M_0}})^\perp$, and we can endow $\mathfrak{a}^*_{M_0}$ with the restriction of $(\cdot,\cdot)_T$. It turns out that this restriction is also an admissible inner product on $\mathfrak{a}_{M_0}*$ for the relative Weyl group [BT65, §6.10]. The roots of A_{M_0} on $\mathrm{Lie}\, N$ are the restrictions of the roots of T on $\mathrm{Lie}\, N$. So the positivity condition in Theorem 1.7 is equivalent to $\langle \mathrm{Res}^M_T \nu, \alpha^\vee \rangle > 0$ for any simple root $\alpha \in X^*(T)$ which does not occur in M.

For analogous results in the case where F is archimedean see [Lan89] and [Wal88, Chapter 5].

1.4. **Harish-Chandra characters.** Denote by $C^\infty_c(G(F))$ the space of locally constant and compactly supported functions $G(F) \to \mathbb{C}$. Recall that any such function in bi-invariant under some compact open subgroup of $G(F)$.

Let (V, π) be an admissible representation of $G(F)$. Any $f \in C^\infty_c(G(F))$ gives an endomorphism $\pi(f)$ of V via defining $\pi(f)v = \int_{G(F)} f(g)\pi(g)v dg$. By admissibility this integral is actually a finite sum. Moreover, the image of any $\pi(f)$ has finite range and we may consider $\Theta_\pi(f) = \mathrm{tr}\, \pi(f)$. The linear form $\Theta_\pi : C_c^\infty(G(F)) \to \mathbb{C}$ is called the **Harish-Chandra character** of π. A standard result in representation theory of finite-dimensional associative algebras implies that the Harish-Chandra characters Θ_π of the irreducible smooth representations of $G(F)$ (up to isomorphism) are linearly independent. In particular a smooth representation of finite length is characterized up to semi-simplification by its Harish-Chandra character.

Denote by G_{rs} the regular semi-simple locus in G, an open dense subscheme.

Theorem 1.9 ([HC99, Theorem 16.3])**.** *Assume that F is a non-archimedean local field of characteristic zero. Let (V, π) be an irreducible smooth representation of $G(F)$. Choose a Haar measure for $G(F)$. There exists a unique element of $L^1_{\mathrm{loc}}(G(F))$, also denoted Θ_π, such that for any $f \in C_c^\infty(G(F))$ we have*

$$\operatorname{tr} \pi(f) = \int_{G(F)} \Theta_\pi(g) f(g) dg.$$

Moreover, $\Theta_\pi(g)$ is represented by a unique locally constant function on $G_{\mathrm{rs}}(F)$.

Unfortunately this result does not seem to be known in full generality in positive characteristic, but see [CGH14]. Harish-Chandra characters behave well under induction [vD72].

See [Wal88, Chapter 8] for the archimedean case.

2. Langlands dual groups

We recall the definition of Langlands dual groups. We refer to [Bor79, §I.2] for details not recalled below. In this section F could be any field, $\overline{F}$ is a separable closure of F and we denote $\Gamma = \mathrm{Gal}(\overline{F}/F)$.

2.1. **Based root data.** Let G be a connected reductive group over F. There exists a finite separable extension E/F such that G_E admits a Killing pair (also called Borel pair) (B, T) [DGA$^+$11, Exposé XXII Corollaire 2.4 and Proposition 5.5.1]. We may do assume that E/F is a subextension of $\overline{F}/F$. Associated to (G_E, B, T) we have a based (reduced) root datum $(X, R, R^\vee, \Delta)$ where

- X is the group of characters of T,
- $R \subset X$ the set of roots of T in G_E,
- $R^\vee$ the set of coroots of T (a subset of $X^\vee = \mathrm{Hom}(X, \mathbb{Z})$, the group of cocharacters of T), and
- $\Delta \subset R$ the set of simple roots corresponding to B.[1]

The group $G(E)$ acts (by conjugation) transitively on the set of Killing pairs in G_E [DGA$^+$11, Exposé XXVI Corollaire 5.7 (ii) and Corollaire 1.8] and the (scheme-theoretic) stabilizer of (B, T) is T [DGA$^+$11, Exposé XXII Cor 5.3.12 and Proposition 5.6.1], which centralizes T. It follows that other choices of Killing pair in G_E yield based root data *canonically* isomorphic to $(X, R, R^\vee, \Delta)$, and so do other choices for E.

We also obtain a continuous action of Γ on this based root datum, that we now recall. The group $\mathrm{Gal}(E/F)$ acts on the set of closed subgroups of G_E: if $G = \operatorname{Spec} A$ for a Hopf algebra A over F and a closed subgroup H corresponds to an ideal I of $A \otimes_F E$, then for $\sigma \in \mathrm{Gal}(E/F)$ we let $\sigma(H)$ be the closed subgroup corresponding to $\sigma(I)$. If $K = \operatorname{Spec} B$ is a linear algebraic group over F and $\lambda : H \to K_E$ is a morphism, dual to a morphism of Hopf algebras $\lambda^\sharp : B \otimes_F E \to (A \otimes_F E)/I$, define $\sigma(\lambda) : \sigma(H) \to K_E$ as dual to

$$\sigma \circ \lambda^\sharp \circ \sigma^{-1} : B \otimes_F E \to (A \otimes_F E)/\sigma(I).$$

Now for $\sigma \in \mathrm{Gal}(E/F)$ there is a unique $T(E)g_\sigma \in T(E)\backslash G(E)$ such that we have $\sigma(B, T) = \mathrm{Ad}(g_\sigma^{-1})(B, T)$, and we get a well-defined isomorphism $\mathrm{Ad}(g_\sigma) : \sigma(T) \simeq T$. We obtain an action of Γ on $X = X^*(T)$ such that $\sigma \in \mathrm{Gal}(E/F)$ maps $\lambda : T \to \mathrm{GL}_{1,E}$ to $\sigma(\lambda) \circ \mathrm{Ad}(g_\sigma)^{-1}$. It is straightforward to check that this action preserves R and Δ and that the dual action on $X^\vee$ preserves $R^\vee$. We denote by brd_F the resulting functor from the groupoid of connected reductive groups over F to the groupoid of based root data with continuous action of Γ.

Definition 2.1. Let G be a connected reductive group over F. Define a *groupoid of inner twists* $\mathsf{IT}(G)$ as follows.

[1] Strictly speaking we should also include in the datum the bijection $R \to R^\vee$ as in [DGA$^+$11, Exposé XXI], or include the orthogonal of $R^\vee$ in X as in [BT65, §2.1].

- The objects of $\mathsf{IT}(G)$ are the inner twists of G, i.e. pairs (G', ψ) consisting of a connected reductive group G' over F and an isomorphism $\psi : G_{\overline{F}} \simeq G'_{\overline{F}}$ such that for any $\sigma \in \Gamma$ the automorphism $\psi^{-1}\sigma(\psi)$ of $G_{\overline{F}}$ is inner.
- A morphism between two inner twists (G_1, ψ_1) and (G_2, ψ_2) of G is an element $g \in G_{\mathrm{ad}}(\overline{F})$ such that for any $\sigma \in \Gamma$ we have

$$\psi_2^{-1}\sigma(\psi_2) = \mathrm{Ad}(\sigma(g))\psi_1^{-1}\sigma(\psi_1)\mathrm{Ad}(\sigma(g))^{-1}. \tag{2.1}$$

Remark 2.2. (1) One can check that any inner twist $\psi : G_{\overline{F}} \to G'_{\overline{F}}$ yields a canonical isomorphism $\mathrm{brd}_F(G) \simeq \mathrm{brd}_F(G')$.

(2) For an inner twist $\psi : G_{\overline{F}} \to G'_{\overline{F}}$ the map

$$\Gamma \to G_{\mathrm{ad}}(\overline{F}), \quad \sigma \mapsto \psi^{-1}\sigma(\psi)$$

is a 1-cocycle, i.e. an element of $Z^1_{\mathrm{cont}}(\Gamma, G_{\mathrm{ad}}) = Z^1(F, G_{\mathrm{ad}})$.

(3) The relation (2.1) imply that the isomorphism

$$\psi_2 \mathrm{Ad}(g)\psi_1^{-1} : G_{1,\overline{F}} \to G_{2,\overline{F}}$$

is defined over F, i.e. descends to an isomorphism $G_1 \simeq G_2$.

(4) For an inner twist (G', ψ) of G we have an isomorphism

$$\mathrm{Aut}(G', \psi) \to G'_{\mathrm{ad}}(F), \quad g \mapsto \psi(g).$$

Proposition 2.3. *Let b be a based root datum with continuous action of Γ. Let CRG_b be the groupoid of pairs (G, α) where G is a connected reductive group over F and $\alpha : b \simeq \mathrm{brd}_F(G)$ is an isomorphism of based root data with action of Γ, with obvious morphisms. In other words CRG_b is the groupoid fiber of b for brd_F.*

1. *There exists an object (G^*, α^*) of CRG_b such that G^* is quasi-split. Two such objects are isomorphic.*
2. *Any object (G, α) of CRG_b yields equivalences of groupoids*

$$Z^1(F, G_{\mathrm{ad}}) \xleftarrow{\sim} \mathsf{IT}(G) \xrightarrow{\sim} \mathsf{CRG}_b.$$

This gives in particular a bijection between $H^1(F, G_{\mathrm{ad}})$ and the set of isomorphism classes in CRG_b.

Proof. This is a reformulation of [DGA+11, Exposé XXIV Théorème 3.11] in the case where the base is the spectrum of a field. □

To sum up, we can "classify" connected reductive groups over F as follows:

- fix a representative in each isomorphism class of based root datum with continuous action of Γ;
- for each such representative b, fix a quasi-split connected reductive group G^* over F together with an isomorphism $\mathrm{brd}_F(G^*) \simeq b$;
- for each element of $H^1(F, G^*_{\mathrm{ad}})$ choose an inner twist (G, ψ) of G^* representing it.

Up to isomorphism each connected reductive group G over F arises in this way. It can happen that an isomorphism class of connected reductive groups arises more than once, because $H^1(F, G_{\mathrm{ad}}) \to H^1(F, \mathrm{Aut}(G))$ is not injective in general (equivalently, the functor brd_F is not full).

2.2. **Langlands dual groups.** Let C be an algebraically closed field of characteristic zero. Let G be a connected reductive group over F and let $\mathrm{brd}_F(G) = (X, R, R^\vee, \Delta)$ be its associated based root datum endowed with a continuous action of Γ. Let $(\widehat{G}, \mathcal{B}, \mathcal{T}, (X_\alpha)_{\alpha \in \Delta^\vee})$ be the *pinned connected reductive group* over C with the associated based root datum $(X^\vee, R^\vee, R, \Delta^\vee)$, i.e. the *dual* of $\mathrm{brd}_F(G)$ (ignoring the action of Γ from now). The choice of a pinning induces a splitting of the extension

$$1 \to \widehat{G}_{\mathrm{ad}} \to \mathrm{Aut}(\widehat{G}) \to \mathrm{Out}(\widehat{G}) \to 1$$

because the subgroup $\mathrm{Aut}(\widehat{G}, \mathcal{B}, \mathcal{T}, (X_\alpha)_{\alpha \in \Delta^\vee})$ of $\mathrm{Aut}(\widehat{G})$ maps bijectively onto $\mathrm{Out}(\widehat{G})$ [DGA+11, Exposé XXIV Théorème 1.3]. We also have an isomorphism

$$\mathrm{Out}(\widehat{G}) \simeq \mathrm{Aut}(X^\vee, R^\vee, R, \Delta^\vee) \simeq \mathrm{Aut}(X, R, R^\vee, \Delta)$$

and so we have an action of Γ on $\widehat{G}$ (preserving the pinning and factoring through a finite Galois group). Denote ${}^L G = \widehat{G} \rtimes \Gamma$ the Langlands dual group, also called L-group. It is sometimes useful (or just convenient) to replace Γ by a finite Galois group or by the Weil group in this semi-direct product.

One can give a more pedantic definition of Langlands dual group in order to avoid the inelegant choice of pinning. Namely, define an L-group for G as an extension ${}^L G$ of Γ by $\widehat{G}$, where $\widehat{G}$ is a split connected reductive group endowed with an isomorphism of its base root datum with the dual of that of G, such that the induced morphism $\Gamma \to \mathrm{Out}(\widehat{G})$ is as above, and endowed with a $\widehat{G}$-conjugacy class of splittings $\Gamma \to {}^L G$, called distinguished splittings, such that any (equivalently, one) of these splittings s preserves a pinning of $\widehat{G}$. It is not necessary to specify the pinning, since for a distinguished splitting s we have that $\widehat{G}^{s(\Gamma)}$ acts transitively on the set of such pinnings: see [Kot84, Corollary 1.7]. By the same argument, for any pinning of $\widehat{G}$ a distinguished splitting fixing it is unique up to

$$\ker(Z^1(\Gamma, Z(\widehat{G})) \to H^1(\Gamma, \widehat{G})) = B^1(\Gamma, Z(\widehat{G})).$$

Note that all distinguished splittings induce the same action of Γ on $Z(\widehat{G})$.

By Proposition 2.3 for two connected reductive groups G_1 and G_2 their Langlands dual groups ${}^L G_1$ and ${}^L G_2$ are isomorphic as extensions of Γ if and only if G_1 and G_2 are inner forms of each other, and in this case they are even isomorphic as extensions endowed with conjugacy classes of distinguished splittings.

The construction of the Langlands dual group is not functorial for arbitrary morphisms between connected reductive groups, however in the following cases functoriality is straightforward.

- Let G be a quasi-split connected reductive group and (B, T) a Borel pair defined over F. Choose a distinguished splitting $s_G : \Gamma \to {}^L G$ preserving a pinning $(\mathcal{B}, \mathcal{T}, (X_\alpha)_{\alpha \in \Delta^\vee})$ of $\widehat{G}$ and a distinguished splitting $s_T : \Gamma \to {}^L T$. Then the canonical isomorphism $\widehat{T} \simeq \mathcal{T}$ extends to an embedding ${}^L T \hookrightarrow {}^L G$ whose composition with s_T is s_G.
- For $G = G_1 \times_F G_2$ we can identify ${}^L G$ with ${}^L G_1 \times_\Gamma {}^L G_2$.
- A central isogeny (see [DGA+11, Exposé XXII Définition 4.2.9]) $G \to H$ induces a surjective morphism with finite kernel ${}^L H \to G$.
- There are weaker forms of functoriality. Let G be a connected reductive group and T a maximal torus of G defined over F. Choose a Borel subgroup B of $G_{\overline{F}}$ containing $T_{\overline{F}}$ and a splitting $s : \Gamma \to {}^L G$ preserving a pinning $(\mathcal{B}, \mathcal{T}, (X_\alpha)_\alpha)$ of $\widehat{G}$. We have a canonical isomorphism $\widehat{T} \simeq \mathcal{T}$, but the Galois actions differ by a 1-cocycle taking values in the Weyl group. In general we don't have a canonical embedding ${}^L T \hookrightarrow {}^L G$,[2] but note that the induced embedding $Z(\widehat{G}) \hookrightarrow {}^L T$ is Γ-equivariant.

In the next section we recall how the first case generalizes to parabolic subgroups in arbitrary connected reductive groups.

2.3. **Parabolic subgroups and L-embeddings.** A parabolic subgroup $\mathcal{P}$ of ${}^L G$ is a closed subgroup mapping onto Γ and such that $\mathcal{P}^0 := \mathcal{P} \cap \widehat{G}$ is a parabolic subgroup of $\widehat{G}$. The set of parabolic subgroups is clearly stable under conjugation by $\widehat{G}$. If $\mathcal{P}$ is a parabolic subgroup of ${}^L G$ then $\mathcal{P}$ is the normalizer of $\mathcal{P}^0$ in ${}^L G$.

Choose a Killing pair $(\mathcal{B}, \mathcal{T})$ of $\widehat{G}$. Recall that a parabolic subgroup of $\widehat{G}$ is conjugated to a unique one containing $\mathcal{B}$, and that parabolic subgroups of $\widehat{G}$ containing $\mathcal{B}$ correspond bijectively to subsets of $\Delta^\vee$ (or Δ, using the bijection $\alpha \mapsto \alpha^\vee$), by associating to $\mathcal{P}^0$ the set of $\alpha \in \Delta^\vee$ (seen as characters of $\mathcal{T}$) such that $-\alpha$ is a root of $\mathcal{T}$ in $\mathcal{P}^0$. Embed $\mathcal{B}$ in a pinning $(\mathcal{B}, \mathcal{T}, (X_\alpha)_{\alpha \in \Delta^\vee})$ of $\widehat{G}$, and let $s : \Gamma \to {}^L G$ be a distinguished section fixing this pinning. Then $\mathcal{B}s(\Gamma)$ is a (minimal) parabolic subgroup of ${}^L G$, and any parabolic subgroup of ${}^L G$ is conjugated under $\widehat{G}$ to one containing $\mathcal{B}s(\Gamma)$. A parabolic subgroup $\mathcal{P}^0$ of $\widehat{G}$ containing $\mathcal{B}$ is such that its normalizer $\mathcal{P}$ in ${}^L G$ maps onto Γ (i.e. $\mathcal{P}$ is a parabolic subgroup of ${}^L G$) if and only if the corresponding subset of $\Delta^\vee$ is stable under Γ. Therefore $\widehat{G}$-conjugacy classes of parabolic subgroups of ${}^L G$ also correspond bijectively to Γ-stable subsets of $\Delta^\vee$.

[2] See however [LS87, §2.6] and [Kal].

Using the bijection between Δ and $\Delta^\vee$ we obtain a bijection between the set of Γ-stable $G(\overline{F})$-conjugacy classes of parabolic subgroups of $G_{\overline{F}}$ and the set of $\widehat{G}$-conjugacy classes of parabolic subgroups of ${}^LG$. The obvious map from the set of $G(F)$-conjugacy classes of parabolic subgroups of G to the set of Γ-stable $G(\overline{F})$-conjugacy classes of parabolic subgroups of $G_{\overline{F}}$ is injective, and it is surjective if and only if G is quasi-split.

Recall from [Bor79, §3.4] that if $\mathcal{P}$ is a parabolic subgroup of ${}^LG$ and $\mathcal{M}^0$ is a Levi factor of $\mathcal{P}^0$ then the normalizer $\mathcal{M}$ of $\mathcal{M}^0$ in $\mathcal{P}$ maps onto Γ and $\mathcal{P}$ is the semi-direct product of its unipotent radical and $\mathcal{M}$. In this situation we say that $\mathcal{M}$ is a Levi factor of $\mathcal{P}$, and a Levi subgroup of ${}^LG$.

Let P be a parabolic subgroup of G. Choose a distinguished splitting $s : \Gamma \to {}^LG$ stabilizing a pinning $\mathcal{E} = (\mathcal{B}, \mathcal{T}, (X_\alpha)_{\alpha\in\Delta^\vee})$ of $\widehat{G}$, and let ${}^LP$ be the parabolic subgroup of ${}^LG$ corresponding to P and containing $\mathcal{B}$. Let $M = P/N$ be the reductive quotient of P. Taking Killing pairs inside P in the definition of brd_F we obtain an isomorphism between $\mathrm{brd}_F(M)$ and $(X, R_P, R_P^\vee, \Delta_P)$ where Δ_P is the set of simple roots $\alpha \in \Delta$ such that $-\alpha$ also occurs in P, $R_P = R \cap \mathrm{Span}(\Delta_P)$, $\Delta_P^\vee = \{\alpha^\vee \mid \alpha \in \Delta_P\}$, and $R_P^\vee = R^\vee \cap \mathrm{Span}(\Delta_P^\vee)$. Let $\mathcal{E}_M = (\mathcal{B}_M, \mathcal{T}_M, (Y_\alpha)_\alpha)$ be a pinning of $\widehat{M}$ and $s_M : \Gamma \to {}^LM$ a corresponding distinguished splitting. These choices determine an embedding

$$\iota[P, \mathcal{E}, s, \mathcal{E}_M, s_M] : {}^LM \longrightarrow {}^LG$$

characterized by the following properties.

(1) It maps $(\mathcal{B}_M, \mathcal{T}_M)$ to $(\mathcal{B}, \mathcal{T})$, and on $\mathcal{T}_M$ it is the isomorphism $\mathcal{T}_M \simeq \mathcal{T}$ induced by the above embedding $\mathrm{brd}_F(M) \hookrightarrow \mathrm{brd}_F(G)$,
(2) it maps $\mathcal{E}_M$ to $\mathcal{E}$, and
(3) we have $\iota[P, \mathcal{E}, s, \mathcal{E}_M, s_M] \circ s_M = s$.

The image of $\iota[P, \mathcal{E}, s, \mathcal{E}_M, s_m]$ is clearly a Levi subgroup of ${}^LG$. The formation of $\iota[P, \mathcal{E}, s, \mathcal{E}_M, s_M]$ satisfies obvious equivariance properties with respect to conjugation by $\widehat{M}$ and $\widehat{G}$. In particular we have an embedding $\iota_P : {}^LM \to {}^LG$ well-defined up to conjugation by $\widehat{G}$.

Lemma 2.4. *Let M be a Levi subgroup of G. Let P and P' be parabolic subgroups of G admitting M as a Levi factor. Then ι_P and $\iota_{P'}$ are conjugated by $\widehat{G}$.*

Proof. First we recall a general construction. Fix a pinning $\mathcal{E} = (\mathcal{B}, \mathcal{T}, (X_\alpha)_\alpha)$ in $\widehat{G}$ and a distinguished splitting $s : \Gamma \to {}^LG$ fixing it. For a Killing pair (B, T) in $G_{\overline{F}}$ we denote by $\gamma[(B, T), (\mathcal{B}, \mathcal{T})]$ the isomorphism $X^*(\mathcal{T}) \simeq X_*(T)$. Considering Weyl groups inside automorphism groups of tori this also induces an isomorphism

$$\omega[(B, T), (\mathcal{B}, \mathcal{T})] : W(T, G_{\overline{F}}) \simeq W(\mathcal{T}, \widehat{G})$$

We have an action of Γ on $W(T, G_{\overline{F}})$: for $\sigma \in \Gamma$ let $T(\overline{F})g_\sigma \in T(\overline{F})\backslash G(\overline{F})$ be the class for which $\sigma(B, T) = \mathrm{Ad}(g_\sigma^{-1})(B, T)$, then $x \mapsto \mathrm{Ad}(g_\sigma)(\sigma(x))$ induces an automorphism of $W(T, G_{\overline{F}})$. One can check that the isomorphism $\omega[(B, T), (\mathcal{B}, \mathcal{T})]$ is Γ-equivariant for this action on $W(T, G_{\overline{F}})$ and the action via s on $W(\mathcal{T}, \widehat{G})$.

Fix $\mathcal{E}$, s, $\mathcal{E}_M$ and s_M as above. Fix a Borel pair (B_M, T) in $M_{\overline{F}}$. This determines two Borel subgroups B and B' in $G_{\overline{F}}$ that are characterized by the properties $B \cap M_{\overline{F}} = B_M$ and $N_{\overline{F}} \subset B$ and similarly for B'. There is a unique $x \in W(T, G_{\overline{F}})$ for which $\mathrm{Ad}(x)(B, T) = (B', T)$. Let $n : W(\mathcal{T}, \widehat{G}) \to N(\mathcal{T}, \widehat{G})$ be the set-theoretic splitting determined by $\mathcal{E}$ [Spr98, §9.3.3]. Denote $w = n(\omega[(B, T), (\mathcal{B}, \mathcal{T})](x))$. We claim that we have

$$\mathrm{Ad}(w) \circ \iota[P', \mathcal{E}, s, \mathcal{E}_M, s_M] = \iota[P, \mathcal{E}, s, \mathcal{E}_M, s_M]. \tag{2.2}$$

To simplify notation in the rest of the proof we respectively abbreviate $\iota = \iota[P, \mathcal{E}, s, \mathcal{E}_M, s_M]$ as well as $\iota' = \iota[P', \mathcal{E}, s, \mathcal{E}_M, s_M]$.

First we check that ι and ι' coincide on $\mathcal{T}_M$. Denote $T' = T$ for clarity. We have $(B', T') = \mathrm{Ad}(x)(B, T)$ so if we also denote by $\mathrm{Ad}(x)$ the induced isomorphism $X_*(T) \simeq X_*(T')$ we have $\mathrm{Ad}(x)\gamma[(B, T), (\mathcal{B}, \mathcal{T})] = \gamma[(B', T'), (\mathcal{B}, \mathcal{T})]$. Here because $T' = T$ we obtain

$$\gamma[(B', T), (\mathcal{B}, \mathcal{T})] = \gamma[(B, T), (\mathcal{B}, \mathcal{T})] \circ \omega[(B, T), (\mathcal{B}, \mathcal{T})](x).$$

The isomorphism $\iota|_{\mathcal{T}_M} : \mathcal{T}_M \simeq \mathcal{T}$ is dual to the isomorphism

$$\gamma[(B_M, T), (\mathcal{B}_M, \mathcal{T}_M)]^{-1} \circ \gamma[(B, T), (\mathcal{B}, \mathcal{T})] : X^*(\mathcal{T}) \simeq X^*(\mathcal{T}_M).$$

Similarly $\iota'|_{\mathcal{T}_M} : \mathcal{T}_M \simeq \mathcal{T}$ is dual to the isomorphism

$$\begin{aligned}&\gamma[(B_M,T),(\mathcal{B}_M,\mathcal{T}_M)]^{-1}\circ\gamma[(B',T),(\mathcal{B},\mathcal{T})]\\=\ &\gamma[(B_M,T),(\mathcal{B}_M,\mathcal{T}_M)]^{-1}\circ\gamma[(B,T),(\mathcal{B},\mathcal{T})]\circ\omega[(B,T),(\mathcal{B},\mathcal{T})](x)\end{aligned}$$

and the equality

$$\iota'|_{\mathcal{T}_M} = \omega[(B,T),(\mathcal{B},\mathcal{T})](x)^{-1}\circ\iota|_{\mathcal{T}_M}$$

follows.

To check that the equality (2.2) holds on $\widehat{M}$ it is enough to check that we have $\mathrm{Ad}(w)\iota(Y_\alpha) = \iota'(Y_\alpha)$ for any $\alpha \in \Delta(\mathcal{T}_M, \mathcal{B}_M)$. We have

$$\iota(Y_\alpha) = X_\beta \text{ and } \iota'(Y_\alpha) = X_{\beta'}$$

where

$$\begin{aligned}\beta &= \gamma[(B,T),(\mathcal{B},\mathcal{T})]^{-1}\gamma[(B_M,T),(\mathcal{B}_M,\mathcal{T}_M)](\alpha),\\ \beta' &= \gamma[(B',T),(\mathcal{B},\mathcal{T})]^{-1}\gamma[(B_M,T),(\mathcal{B}_M,\mathcal{T}_M)](\alpha)\\ &= w^{-1}(\beta)\end{aligned}$$

both belong to $\Delta(\mathcal{T},\mathcal{B})$. By [Spr98, Proposition 9.3.5] we have $X_\beta = \mathrm{Ad}(w)(X_{\beta'})$.

Finally we need to check $\mathrm{Ad}(w)\circ s = s$, i.e. that w commutes with $s(\Gamma)$. For $\sigma\in\Gamma$ and $y \in W(\mathcal{T},\widehat{G})$ we have $s(\sigma)n(y)s(\sigma)^{-1} = n(\sigma(y))$ and so it is enough to check that $w\mathcal{T} \in W(\mathcal{T},\widehat{G})$ is fixed by Γ. For any $\sigma \in \Gamma$ there exists $g_\sigma \in M(\overline{F})$ such that $\sigma(B_M,T) = \mathrm{Ad}(g_\sigma^{-1})(B_M,T)$ and this implies $\sigma(B,T) = \mathrm{Ad}(g_\sigma^{-1})(B,T)$ and $\sigma(B',T) = \mathrm{Ad}(g_\sigma^{-1})(B',T)$ because N and N' are both defined over F. A simple computation shows that we have $\mathrm{Ad}(g_\sigma)(\sigma(x)) = x$ in $W(T,G_{\overline{F}})$, i.e. x is Γ-invariant. □

The lemma shows that for a Levi subgroup M of G we have an embedding $\iota_M : {}^LM \to {}^LG$, well-defined up to conjugation by $\widehat{G}$. We call the image of such an embedding a *relevant* Levi subgroup of ${}^LG$.

3. Langlands parameters

In this section F is a local field.

3.1. **Weil-Deligne groups.**

We briefly recall the definition of Weil-Deligne groups of local fields. We refer the reader to [Tat79] for more details.

If $F\simeq\mathbb{C}$ define $\mathrm{W}_F = F^\times$. If $F\simeq\mathbb{R}$ define W_F as the unique non-split central extension

$$1\to\overline{F}^\times\to\mathrm{W}_F\to\mathrm{Gal}(\overline{F}/F)\to 1$$

where $\mathrm{Gal}(\overline{F}/F)$ acts on $\overline{F}^\times$ in the natural way. Explicitly, $\mathrm{W}_F = \overline{F}^\times \sqcup j\overline{F}^\times$ with $j^2=-1$.

If F is a non-archimedean local field, we have a short exact sequence of topological groups

$$1\to I_F\to\mathrm{Gal}(\overline{F}/F)\to\mathrm{Gal}(\overline{k}/k)\to 1$$

where k is the residue field of F and I_F is called the inertia subgroup of $\mathrm{Gal}(\overline{F}/F)$. Since k is finite, say of cardinality q, $\mathrm{Gal}(\overline{k}/k)$ is isomorphic to $\widehat{\mathbb{Z}}$ and topologically generated by the (arithmetic) Frobenius automorphism $x\mapsto x^q$. This automorphism generates a natural subgroup $\mathbb{Z}$ of $\mathrm{Gal}(\overline{k}/k)$, and the *Weil group* W_F is defined as its preimage, a dense subgroup of $\mathrm{Gal}(\overline{F}/F)$. Instead of the induced topology, we endow W_F with the topology making I_F an open subgroup, with its topology induced from that of $\mathrm{Gal}(\overline{F}/F)$.

Recall that the Artin reciprocity map is an isomorphism $\mathrm{W}_F^{\mathrm{ab}}\simeq F^\times$. Composing with the norm $\|\cdot\| : F^\times\to\mathbb{R}_{>0}$ we get a continuous morphism still denoted $\|\cdot\| : \mathrm{W}_F\to\mathbb{R}_{>0}$.

For non-archimedean F, we now recall three possible definitions for the Weil-Deligne group.

(1) $\mathrm{W}'_F := \mathbb{G}_a\rtimes\mathrm{W}_F$, where the action of W_F on $\mathbb{G}_a$ is by $w(x) = \|w\|x$.
(2) $\mathrm{WD}_F := \mathrm{W}_F\times\mathrm{SL}_2$, where the second factor is the algebraic group over $\mathbb{Q}$.
(3) The (unnamed) locally compact topological group $\mathrm{W}_F\times\mathrm{SU}(2)$.

For Archimedean F it will be convenient to denote $\mathrm{WD}_F = \mathrm{W}_F$.

3.2. **Langlands parameters.** First assume that F is non-archimedean.

For the first version of the Weil-Deligne group, a *Weil-Deligne Langlands parameter*[3] is a pair (ρ, N) such that

- $\rho : \mathrm{W}_F \to {}^LG$ is a continuous representation, i.e. there exists an open subgroup U of I_F which acts trivially on $\widehat{G}$ and is mapped to $1 \times U \subset \widehat{G} \rtimes \Gamma$, such that the composition with the projection ${}^LG \to \Gamma$ is the usual map,
- $N \in \operatorname{Lie} \widehat{G}$ satisfies $\rho(w)N\rho(w)^{-1} = \|w\|N$ for all $w \in \mathrm{W}_F$ (this forces N to be nilpotent), and
- for any $w \in \mathrm{W}_F$ (equivalently, for some $w \in \mathrm{W}_F \backslash I_F$) we have that $\rho(w)$ is semi-simple.

One of the motivations for using the first version of the Weil-Deligne group, rather than the other two, is the ℓ-adic monodromy theorem [Tat79, Theorem 4.2.1]. This roughly says that for a prime ℓ not equal to the residual characteristic of F and for $C = \overline{\mathbb{Q}}_\ell$,[4] any continuous morphism $\mathrm{W}_F \to {}^LG$ *for the natural topology on* $\widehat{G}$ compatible with ${}^LG \to \Gamma$ is given by a pair (ρ, N) satisfying the first two conditions above. Continuous ℓ-adic Galois representations occur naturally in algebraic geometry (Tate modules of elliptic curves over F, or more generally in the étale cohomology of varieties defined over F). Another reason for preferring W'_F is that this version requires fewer "choices of a square root of q" in the local Langlands correspondence, and is more obviously compatible with parabolic induction (property (10) in Conjecture 4.1 below).

For the second version WD_F, over any field C of characteristic zero, *Langlands parameters* are defined as morphisms $\phi : \mathrm{W}_F \times \mathrm{SL}_2(C) \to {}^LG$ which are compatible with ${}^LG \to \Gamma$, continuous and semi-simple on the first factor and algebraic on the second factor.

For the third version, we need to assume $C = \mathbb{C}$ and we consider continuous (for the natural topology on $\widehat{G}$) semi-simple morphisms $\phi : \mathrm{W}_F \times \mathrm{SU}(2) \to {}^LG$ which are compatible with ${}^LG \to \Gamma$. By restriction via $\mathrm{SU}(2) \subset \mathrm{SL}_2(\mathbb{C})$ we obtain exactly the same morphisms as in the second version, essentially because $\mathrm{SL}_2(\mathbb{C})$ is the complexification of the compact Lie group $\mathrm{SU}(2)$.

Recall that we have already chosen a square root of q in C in order to normalize parabolic induction. We have a natural map from Langlands parameters to Weil-Deligne Langlands parameters:

$$\phi \mapsto \left(\phi \circ \iota_W, d\phi|_{\mathrm{SL}_2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right)$$

where $\iota_W(w) = (w, \operatorname{diag}(\|w\|^{1/2}, \|w\|^{1/2}))$. By a refinement of the Jacobson-Morozov theorem (see [GR10, Lemma 2.1]) this induces a bijection between sets of $\widehat{G}$-conjugacy classes of parameters.

If F is archimedean we assume $C = \mathbb{C}$ and define Langlands parameters as semi-simple continuous morphisms $\phi : \mathrm{W}_F \to {}^LG$ which are compatible with ${}^LG \to \Gamma$.

We will denote by $\Phi(G)$ the set of $\widehat{G}$-conjugacy classes of Langlands parameters taking values in ${}^LG$. As explained above all versions of the Weil-Deligne group give equivalent sets of $\widehat{G}$-conjugacy classes.

3.3. **Reductions.** We briefly recall from [SZ] the Langlands classification for parameters. Assume $C = \mathbb{C}$ and let $\mathrm{cl}(\phi) \in \Phi(G)$. Applying the polar decomposition to $\phi(w)$ for any $w \in \mathrm{W}_F$ with positive valuation, we find a canonical tuple $({}^LP, {}^LM, \phi_0, \chi)$ satisfying the following conditions.

- ${}^LP$ is a parabolic subgroup of ${}^LG$ and ${}^LM$ is a Levi subgroup of ${}^LP$. We denote by $\widehat{N}$ the unipotent radical of ${}^LP$.
- ϕ_0 is a Langlands parameter taking values in ${}^LM$ and bounded on W_F.
- $\chi \in Z^1(\mathrm{W}_F, X_*(Z(\widehat{M})^{\Gamma,0}) \otimes_{\mathbb{Z}} \mathbb{R}_{>0})$ where $X_*(Z(\widehat{M})^{\Gamma,0}) \otimes_{\mathbb{Z}} \mathbb{R}_{>0}$ is seen as a subgroup of $X_*(Z(\widehat{M})^{\Gamma,0}) \otimes_{\mathbb{Z}} \mathbb{C} = Z(\widehat{M})^0$.
- The eigenvalues of $\chi(\mathrm{Frob})$ (resp. $\chi(x)$ for any $x > 1$) on $\operatorname{Lie} \widehat{N}$ are all greater than 1 if F is non-archimedean (resp. if F is archimedean).
- $\phi = \phi_0 \chi$.

This corresponds to the Langlands classification (Theorem 1.7). This reduction explains why we are mainly interested in bounded parameters ϕ. We will also call such parameters *tempered*.

[3]This terminology is not standard.

[4]One could work with a finite extension of $\mathbb{Q}_\ell$ instead.

The following proposition does not assume $C = \mathbb{C}$.

Proposition 3.1 ([Bor79, Proposition 3.6])**.** *Let $\phi : \mathrm{WD}_F \to {}^LG$ be a Langlands parameter. The Levi subgroups of ${}^LG$ which are minimal among those containing $\phi(\mathrm{WD}_F)$ are all conjugated under the centralizer of ϕ in $\widehat{G}$.*

This proposition may be seen as a generalization of the isotypical decomposition of a semi-simple linear group representation. A Langlands parameter ϕ is called essentially discrete if this Levi subgroup is ${}^LG$, i.e. if ϕ is "${}^LG$-irreducible". This condition is equivalent to $\mathrm{Cent}(\phi, Gh)/Z(\widehat{G})^\Gamma$ being finite. A Langlands parameter ϕ is called *relevant* if this Levi subgroup is relevant (see Subsection 2.3).

3.4. **Weil restriction.** Let E be a finite subextension E of $\overline{F}/F$ and let $\Gamma_E = \mathrm{Gal}(\overline{F}/E)$ be the corresponding open subgroup of Γ. Let G_0 be a connected reductive group over E. Let $G = \mathrm{Res}_{E/F}\, G_0$ be the *Weil restriction*, a connected reductive group over F such that the topological groups $G(F)$ and $G_0(E)$ are isomorphic. Recall from [Bor79, §5] that we may identify $\widehat{G}$ endowed with its action of Γ with the induction from Γ_E to Γ of $\widehat{G_0}$. By Shapiro's lemma we have a bijection $\Phi(G) \simeq \Phi(G_0)$.

4. The local Langlands conjecture

4.1. **Crude local Langlands correspondence.** Denote by $\Pi(G)$ the set of isomorphism classes of irreducible admissible representations of $G(F)$ (in the archimedean case, $(\mathfrak{g}, K)$-modules).

Conjecture 4.1. *There should exist maps $\mathrm{LL} : \Pi(G) \to \Phi(G)$ for all connected reductive groups G over F, satisfying the following properties. Denote $\Pi_\phi(G) = \mathrm{LL}^{-1}(\phi)$.*

(1) *If G is a torus then LL should be the bijection that Langlands deduced from class field theory* [Bor79, §9].
(2) *For any G all fibers of LL should be finite and the image of LL should contain all essentially discrete parameters.*
(3) *If $G = G_1 \times G_2$ then, using the identification of ${}^LG$ with ${}^LG_1 \times_\Gamma {}^LG_2$, for any irreducible admissible representation $\pi \simeq \pi_1 \otimes \pi_2$ of $G(F)$ we should have $\mathrm{LL}(\pi) = (\mathrm{LL}(\pi_1), \mathrm{LL}(\pi_2))$.*
(4) *If $\theta : G \to H$ is a central isogeny with dual $\widehat{\theta} : {}^LH \to {}^LG$ then for $\pi \in \Pi(H)$ and any constituent π' of the restriction $\pi|_{G(F)}$, which is semi-simple of finite length, we should have $\mathrm{LL}(\pi') = \widehat{\theta} \circ \mathrm{LL}(\pi)$.*
(5) *In the setup of Subsection 3.4 we should have a commutative diagram*
$$\begin{array}{ccc} \Pi(G) & \xrightarrow{\mathrm{LL}} & \Phi(G) \\ \downarrow{\sim} & & \downarrow{\sim} \\ \Pi(G_0) & \xrightarrow{\mathrm{LL}} & \Phi(G_0) \end{array}$$
where the left vertical map is induced by the isomorphism $G(F) \simeq G_0(E)$ and the right vertical map comes from Shapiro's lemma.
(6) *For an irreducible smooth representation π of $G(F)$ we should have that π is essentially square-integrable if and only if $\mathrm{LL}(\pi)$ is discrete.*
(7) *Let M be a Levi subgroup of G. Recall the embedding $\iota_M : {}^LM \hookrightarrow {}^LG$, well-defined up to $\widehat{G}$-conjugacy by Lemma 2.4. If σ is an irreducible smooth representation of $M(F)$ which is essentially square-integrable and has unitary central character then for any constituent π of $i_P^G\sigma$ we should have $\mathrm{LL}(\pi) = \iota_M \circ \mathrm{LL}(\sigma)$.*
(8) *In the situation of Theorem 1.7 we should have*
$$\mathrm{LL}(J(P, \sigma, \mu)) = \iota_P \circ \mathrm{LL}(\sigma \otimes \nu)$$
(9) *Assume $F \simeq \mathbb{R}$ and choose $\overline{F} \simeq \mathbb{C}$. We may reduce to this case if $F \simeq \mathbb{C}$ by (5) above. Then LL should be compatible with infinitesimal characters in the following sense. Let π be an irreducible $(\mathfrak{g}, K)$-module. The restriction of $\mathrm{LL}(\pi)$ to $\mathbb{C}^\times$ is conjugated to a morphism of the form $z \mapsto z^\lambda \overline{z}^\mu$ where $\lambda, \mu \in X_*(\mathcal{T}) \otimes_{\mathbb{Z}} \mathbb{C}$ satisfy $\lambda - \mu \in X_*(\mathcal{T})$ and $z^\lambda \overline{z}^\mu$ is a suggestive notation for $(z\overline{z})^{(\lambda+\mu)/2}(z/|z|)^{\lambda-\mu}$. The infinitesimal character of π should be identified to λ by the Harish-Chandra isomorphism.*

(10) *Assume that F is non-archimedean. If $P = MN$ is a parabolic subgroup of G and σ is an irreducible smooth representation of $M(F)$, then for any irreducible subquotient π of $i_P^G \sigma$ we should have $\mathrm{LL}(\pi) \circ \iota_W = \iota_M \circ \mathrm{LL}(\sigma) \circ \iota_W$. Equivalently, the same but just for supercuspidal σ.*

(11) *Assume that F is non-archimedean. If ϕ is essentially discrete and trivial on the factor SL_2 of WD_F, then every element of $\Pi_\phi(G)$ should be supercuspidal.*

We warn the reader that there are actually two versions of the conjecture, corresponding to the two possible normalizations of the Artin reciprocity map in local class field theory. According to [KS, §4] these should be related by a certain automorphism of ${}^L G$, which according to [AV16] and [Kal13] is itself related to taking contragredient representations. Thus another way to state the relation between the two normalizations is to say that we should obtain one from the other by composing with the involution $\pi \mapsto \tilde{\pi}$.

Cases for which the conjecture is known include the archimedean case [Lan89], general linear groups over non-archimedean fields [LRS93, Hen00, HT01, Sch13], GSp_4 over finite extensions of $\mathbb{Q}_p$ [GT11], quasi-split classical groups [Art13, Mok15]. More cases will be discussed later.

The rest of this section is devoted to remarks on the properties in the conjecture.

4.1.1. *Compatibility with the case of tori.* The functoriality assumptions (3) and (4) imply the following compatibilities with the case of tori.

- The map LL should be compatible with central characters in the following sense. Let Z be the maximal central torus in G so that we have a surjective morphism ${}^L G \to {}^L Z$. Then all elements of $\Pi_\phi(G)$ should have (isomorphism class of) central character of $Z(F)$ determined by composing ϕ with this surjection and applying LL^{-1}.
- Langlands defined (see [Bor79, §10.2]) a morphism

$$H^1_{\mathrm{cont}}(\mathrm{W}_F, Z(\widehat{G})) \to \mathrm{Hom}_{\mathrm{cont}}(G(F), \mathbb{C}^\times).$$

  For a continuous 1-cocycle $c : \mathrm{W}_F \to Z(\widehat{G})$ with corresponding character $\chi : G(F) \to \mathbb{C}^\times$ we should have $\mathrm{LL}(\pi \otimes \chi) = c\mathrm{LL}(\pi)$ for $\pi \in \Pi(G)$.

4.1.2. *Reduction to the discrete case.* Using Proposition 1.5, the Langlands classification (Theorem 1.7, including Remark 1.8) and the "Langlands classification for parameters" (see Subsection 3.3), properties (7) and (8) imply that π is tempered if and only if $\mathrm{LL}(\pi)$ is tempered. In fact we see that these parallel results for smooth representations of reductive groups and Langlands parameters reduce the construction of LL to the essentially square-integrable case, and with property (2) we see that the image of LL should be the set of relevant Langlands parameters.

4.1.3. *The unramified case.* From properties (1), (7), and (8) it follows that if G is unramified and K is a hyperspecial compact open subgroup of $G(F)$ then on K-unramified irreducible representations of $G(F)$ (i.e. representations having non-zero K-invariants) the map LL is given by the *Satake isomorphism*. More precisely in this case the minimal Levi subgroup M_0 is an unramified torus and unramified representations of $G(F)$ are parametrized by orbits under the rational Weyl group of continuous characters $\chi : M_0(F) \to \mathbb{C}^\times$. The unramified representation π corresponding to the orbit of χ is the unique unramified constituent of $i_B^G \chi$, for any Borel subgroup B of G containing M_0. We have $\mathrm{LL}(\pi) = \iota_{M_0} \circ \mathrm{LL}(\chi)$. In the tempered case, that is when χ is unitary, this follows immediately from property (7). The general case is more suitable, and can be deduced from the Gindikin-Karpelevich formula [Cas80, Theorem 3.1] (see [CS80, p. 219] for the values of the constants in the case of an unramified group).[5]

[5]To be honest the arguments in [Cas80] assume that χ is regular but similar arguments work using only partial regularity.

4.1.4. *The semi-simplified correspondence and algebraicity.* For non-archimedean F property (10) says that the map $\mathrm{LL}^{\mathrm{ss}} : \pi \mapsto \mathrm{LL}(\pi) \circ \iota_W$ is compatible with the notion of supercuspidal support (Theorem 1.2). This suggests the following conjecture.

Conjecture 4.2. *Assume that F is non-archimedean. Let C be any algebraically closed field of characteristic zero and choose a square root $\sqrt{q} \in C$. There should exist for each connected reductive group G over F a map $\mathrm{LL}^{\mathrm{ss}}$ from the set of isomorphism classes of smooth irreducible representations of $G(F)$ over C to the set of $\widehat{G}$-conjugacy classes of continuous semi-simple morphisms $\mathrm{W}_F \to {}^LG$ which are compatible with ${}^LG \to \Gamma$, satisfying the obvious analogue of* (1), (3), (4) *in Conjecture 4.1, as well as the following analogues of properties* (10) *and* (11)*:*

(1) *If $P = MN$ is a parabolic subgroup of G and σ is an irreducible smooth representation of $M(F)$ then for any irreducible subquotient π of $i_P^G\sigma$ we should have $\mathrm{LL}^{\mathrm{ss}}(\pi) = \iota_M \circ \mathrm{LL}^{\mathrm{ss}}(\sigma)$.*
(2) *If $\mathrm{LL}^{\mathrm{ss}}(\pi)$ is essentially discrete then π should be supercuspidal.*

These maps should be functorial in $(C, \sqrt{q})$.[6]

Conjecture 4.1 implies the case $C = \mathbb{C}$ of Conjecture 4.2. Note that properties (6), (7), and (8) in Conjecture 4.1 make essential use of the topology on the coefficient field $\mathbb{C}$. The notion of essentially discrete Langlands parameter is purely algebraic (it does not rely on the topology of the coefficient field) so there ought to be a purely algebraic characterization of essentially square-integrable representations.[7] Assuming Conjecture 4.1 one can show that the map $\mathrm{LL} \circ \iota_W$ determines the map LL, by considering first the case of tempered representations and using the decomposition in Subsubsection 4.1.3 and the fact that an $\mathfrak{sl}_2$ triple is determined by its semi-simple element up to conjugation. In general Conjecture 4.2 does not immediately imply Conjecture 4.1: this would require proving a non-trivial integrality property for the Jacquet module of essentially square-integrable representations. In the case of general linear groups however the construction of the map LL was reduced to the supercuspidal case by Zelevinsky [Zel80].

For $C = \overline{\mathbb{Q}}_\ell$ where ℓ does not equal the residue characteristic of F, Genestier-Lafforgue [GL] (in positive characteristic) and Fargues-Scholze [FS] have constructed maps $\mathrm{LL}^{\mathrm{ss}}$ satisfying all properties in Conjecture 4.2 except for functoriality with respect to the coefficient field, which seems to remain open.

4.1.5. *Cuspidal parameters.* Note that property (10) implies that property (11) should be an equivalence, i.e. if all elements of Π_ϕ are supercuspidal then ϕ should be essentially discrete and trivial on SL_2. Contrary to the case of GL_n, in general supercuspidals do not correspond to discrete parameters which are trivial on SL_2, more precisely there are discrete parameters ϕ which are non-trivial on SL_2 and such that Π_ϕ contains supercuspidal representations. A related matter is that the classification of essentially discrete representations in terms of supercuspidal representations (of Levi subgroups) is more complicated in general than in the case of GL_n.

4.1.6. *Characterizations of the correspondence.* The list of properties in Conjecture 4.1 is not exhaustive, and these properties are certainly not enough to characterize the map LL. In particular we did not discuss the relation with L-functions and ϵ-factors. We refer the interested reader to [Har] for a survey of the possible characterizations.

4.2. **Refined local Langlands for quasi-split groups.** In some applications having just the map LL is too crude, e.g. to formulate the global multiplicity formula for the automorphic spectrum of a connected reductive group over a global field, and so we would like to understand the fibers $\Pi_\phi(G)$.

In this section we assume that G is quasi-split. For a Langlands parameter $\phi : \mathrm{WD}_F \to {}^LG$ denote $S_\phi = \mathrm{Cent}(\phi, \widehat{G})$ (a reductive subgroup of $\widehat{G}$), and define $\overline{S}_\phi = S_\phi / Z(\widehat{G})^\Gamma$. Recall that a parameter ϕ

[6]One could certainly avoid the choice of a square root of q by modifying Langlands dual groups. We do not attempt to explain this here, see [BG14, §5.3].

[7]More precisely Conjecture 4.1 implies that the characterization in Proposition 1.4 can be reformulated as follows. Up to twisting by a character we may assume that the central character of π has finite order. Then (V, π) should be essentially square-integrable if and only if for any parabolic subgroup $P = MN$ of G and for any character χ of $A_M(F)$ occurring in $r_P^G\pi$ there exists an integer $N \geqslant 1$ such that χ^N is equal to $\prod_\alpha \|\alpha\|^{n_\alpha}$ for some integers $n_\alpha > 0$, where the product ranges over the simple roots of A_M in $\mathrm{Lie}\, N$.

is *discrete* if and only if $\overline{S}_\phi$ is finite. It can happen that $\pi_0(\overline{S}_\phi)$ is non-abelian (even in the principal series case, that is if ϕ factors through $\iota_T : {}^LT \to {}^LG$ where T is part of a Borel pair (B,T) defined over F). For $F = \mathbb{R}$ however, it is always abelian, in fact there is a maximal torus $\mathcal{T}$ of $\widehat{G}$ such that $S_\phi \cap \mathcal{T}$ meets every connected component of S_ϕ. For a finite group A denoted by $\mathrm{Irr}(A)$ the set of isomorphism classes of irreducible representations of A over $\mathbb{C}$.

Conjecture 4.3. *For each Langlands parameter ϕ there should exist a bijection*

$$\mathrm{Irr}(\pi_0(\overline{S}_\phi)) \longrightarrow \Pi_\phi(G).$$

Langlands's classification (Theorem 1.7) again reduces the construction of this bijection to the tempered case. So we assume from now on that ϕ is tempered. The bijection in Conjecture 4.3 is not canonical in general: it depends on the choice of a Whittaker datum (up to conjugation by $G(F)$).

We briefly recall the notions of Whittaker datum and generic representation for a quasi-split connected reductive group G. Choose a Borel subgroup B with unipotent radical U. For a Galois orbit $\mathcal{O}$ on the set of simple roots, the group $U_{\mathcal{O}} = (\prod_{\alpha\in\mathcal{O}} U_\alpha(\overline{F}))^{\mathrm{Gal}_F}$ is isomorphic to a finite separable extension $F_{\mathcal{O}}$ of F. We have a natural surjective morphism from $U(F)$ to $\prod_{\mathcal{O}} U_{\mathcal{O}}$. Choosing a nontrivial morphism $U_{\mathcal{O}} \to \mathbb{C}^\times$ for each orbit $\mathcal{O}$ yields a morphism $\theta : U(F) \to \mathbb{C}^\times$, called a generic character. A *Whittaker datum* $\mathfrak{w}$ for G is such a pair (U,θ). The adjoint group $G_{\mathrm{ad}}(F)$ acts transitively on the set of such pairs, and so there are only finitely many $G(F)$-conjugacy classes of Whittaker data. An irreducible smooth representation (π, V) of $G(F)$ is called $\mathfrak{w}$-generic if there is a non-zero linear functional $V \to \mathbb{C}$ such that $\lambda(\pi(u)v) = \theta(u)\lambda(v)$ for all $u \in U(F)$ and $v \in V$.

Conjecture 4.4 (Shahidi)**.** *There should be a unique $\mathfrak{w}$-generic representation in each $\Pi_\phi(G)$. The conjectural bijection $\iota_{\mathfrak{w}} : \Pi_\phi(G) \to \mathrm{Irr}(\pi_0(\overline{S}_\phi))$, which depends on $\mathfrak{w}$, should map this $\mathfrak{w}$-generic representation to the trivial representation of $\overline{S}_\phi$.*

In order to characterize the bijections $\iota_{\mathfrak{w}}$ we have to introduce endoscopic data. Let $s \in S_\phi$ be a semi-simple element. From the pair (s, ϕ) one can construct the following objects. For $\pi \in \Pi_\phi(G)$ denote $\langle s, \pi\rangle_{\mathfrak{w}} = \mathrm{tr}(\iota_{\mathfrak{w}}(\pi))(s)$. On the one hand we have

$$\Theta^{\mathfrak{w}}_{\phi,s} = \sum_{\pi\in\Pi_\phi(G)} \langle s, \pi\rangle_{\mathfrak{w}} \Theta_\pi.$$

This is a virtual character on $G(F)$. In the case $s = 1$ we introduce the special notation

$$S\Theta_\phi = \Theta^{\mathfrak{w}}_{\phi,1}.$$

The reason for not recording w in the notation in this case will be explained below.

On the other hand we consider the complex connected reductive subgroup $\mathcal{H}^0 = \mathrm{Cent}(s, \widehat{G})^0$ of $\widehat{G}$. It contains $\phi(1 \times \mathrm{SL}_2)$ and is normalized by $\phi(\mathrm{W}_F)$. Thus $\mathcal{H} = \mathcal{H}^0 \cdot \phi(\mathrm{W}_F)$ is a subgroup of ${}^LG$, which is an extension $1 \to \mathcal{H}^0 \to \mathcal{H} \to \mathrm{W}_F \to 1$. The resulting morphism $\mathrm{W}_F \to \mathrm{Out}(\mathcal{H}^0)$ factors through the Galois group of a finite extension of F. By Proposition 2.3 there exists a quasi-split connected reductive group H over F together with an inner class of isomorphisms $\iota : \mathcal{H}^0 \simeq \widehat{H}$ such that the above morphism $\mathrm{W}_F \to \mathrm{Out}(\mathcal{H}^0)$ and the morphism $\mathrm{W}_F \to \mathrm{Out}(\widehat{H})$ used to define ${}^LH = \widehat{H} \rtimes \mathrm{W}_F$ correspond to each other via η, and for any two such groups H_1 and H_2 we have an isomorphism $H_1 \simeq H_2$, well-defined up to $H_{1,\mathrm{ad}}(F)$. It may unfortunately happen that the two extensions $\mathcal{H}$ and ${}^LH$ of W_F are not isomorphic. We shall ignore this difficulty, as its resolution is not terribly exciting (see [KS99, Lemma 2.2.A]). So let's assume there exists an isomorphism of extensions ${}^L\eta : \mathcal{H} \to {}^LH$. Then $\mathfrak{e} = (H, s, {}^L\eta)$ is called an extended endoscopic triple. By construction we have a unique Langlands character $S\Theta_{\phi_H}$ on $H(F)$.

The two virtual characters $\Theta^{\mathfrak{w}}_{\phi,s}$ and $S\Theta_{\phi_H}$ are expected to be related by a certain kernel function. This function, called the *Langlands-Shelstad transfer factor*, is itself non-conjectural and explicit. It is a function

$$\Delta[\mathfrak{w}, \mathfrak{e}] : H(F)_{G,\mathrm{sr}} \times G(F)_{\mathrm{sr}} \to \mathbb{C}$$

whose construction depends on the Whittaker datum and the extended endoscopic triple. We will not recall the definition of $\Delta[\mathfrak{w}, \mathfrak{e}]$ (which is rather technical, see [LS87, KS99, KS]), but let us recall what its support is (a correspondence between strongly regular semisimple conjugacy classes in $G(F)$

and G-strongly regular semisimple stable conjugacy classes in $H(F)$), and recall a meaningful variance property.

Definition 4.5. Recall that an element of $G(\overline{F})$ is called *strongly regular* if its centralizer is a torus. Two semisimple strongly regular elements $\delta, \delta' \in G(F)$ are called stably conjugate if there exists $g \in G(\overline{F})$ such that $g\delta g^{-1} = \delta'$.

Using maximal tori and identifications of Weyl groups one can define [KS99, Theorem 3.3.A] a canonical map m from semisimple conjugacy classes in $H(\overline{F})$ to from semisimple conjugacy classes in $G(\overline{F})$. A conjugacy class in $H(\overline{F})$ is called G-strongly regular elements of $H(F)$. The map m enjoys the following properties.

(1) The map m is Γ-equivariant.
(2) If $\gamma \in H(F)$ is semisimple G-strongly regular then $m([\gamma]) \cap G(F)$ is a non-empty[8] finite union of $G(F)$-conjugacy classes. In this situation we say that (the stable conjugacy class of) γ and (the conjugacy class) of $\delta \in m([\gamma]) \cap G(F)$ *match*. Given a strongly regular stable conjugacy class for G, there are finitely many stable conjugacy classes for H in its preimage by m.
(3) For any matching pair $(\gamma, \delta) \in H(F)_{G,\mathrm{sr}} \times G(F)_{\mathrm{sr}}$, denoting $T_H = \mathrm{Cent}(\gamma, H)$ and $T = \mathrm{Cent}(\delta, g)$ (maximal tori of H and G), there is a canonical isomorphism $T_H \simeq T$ identifying γ and δ.

Let δ be a strongly regular element of $G(F)$, and denote $T = \mathrm{Cent}(\delta, G)$. The set of $G(F)$-conjugacy classes $[\delta']$ which are stably conjugate to δ is parametrized by $\ker(H^1(F,T) \to H^1(F,G))$, by mapping δ' to $\mathrm{inv}(\delta, \delta') := (\sigma \mapsto \sigma(g)^{-1}g)$ where as above $g\delta^{-1}g = \delta'$. Recall from [Tat66] that the Tate-Nakayama isomorphism for tori over F identifies $H^1(F,T)$ with

$$\widehat{H}^{-1}(E/F, X_*(T)) = X_*(T)^{N_{E/F}=0}/I_{E/F}X_*(T) \tag{4.1}$$

where E/F is any finite Galois subextension of $\overline{F}/F$ splitting T, $N_{E/F}$ is the norm map, and for a $\mathbb{Z}[\mathrm{Gal}(E/F)]$-module Y we denote by $I_{E/F}Y$ the submodule $\sum_{\sigma \in \mathrm{Gal}(E/F)} (\sigma - 1)Y$. Note that the right-hand side of (4.1) can also be described as the torsion subgroup of the coinvariants $X_*(T)_\Gamma$. Kottwitz interpreted this isomorphism in terms of Langlands dual groups and generalized it to connected reductive groups in [Kot86]. Recall that $\widehat{T}$ is a torus over $\mathbb{C}$ endowed with an isomorphism $X^*(\widehat{T}) \simeq X_*(T)$. Using the exactness of the functor mapping a finitely generated abelian group A to the diagonalizable group scheme Z with character group A (considered as a sheaf on the étale site of $\mathbb{C}$, say) we see that $X^*(\widehat{T})_\Gamma$ is identified with $X^*(\widehat{T}^\Gamma)$. It follows that the Tate-Nakayama isomorphism may be written as

$$\alpha_T : H^1(F,T) \simeq \mathrm{Irr}(\pi_0(\widehat{T}^\Gamma)). \tag{4.2}$$

It is formal to check that this definition is functorial in T. As for the Artin reciprocity map it would be just as natural to consider the same isomorphism composed with $x \mapsto x^{-1}$.

Theorem 4.6 ([Kot86, Theorem 1.2])**.** *There is a unique extension of the above family of isomorphisms to a family of maps of pointed sets*

$$\alpha_G : H^1(F,G) \to \mathrm{Irr}(\pi_0(Z(\widehat{G})^\Gamma))$$

for connected reductive group G, "functorial" in the following sense. For any morphism $H \to G$ which is either the embedding of a maximal torus in a connected reductive group G or a central isogeny between connected reductive groups we have a commutative diagram

$$\begin{tikzcd}
H^1(F,H) \arrow[r] \arrow[d, "\alpha_H"'] & H^1(F,G) \arrow[d, "\alpha_G"] \\
\mathrm{Irr}(\pi_0(Z(\widehat{H})^\Gamma)) \arrow[r] & \mathrm{Irr}(\pi_0(Z(\widehat{G})^\Gamma))
\end{tikzcd}$$

[8] For non-emptiness the fact that G is quasi-split is essential.

where the bottom horizontal map is the one induced by the Γ*-equivariant map* $Z(\widehat{G}) \to Z(\widehat{H})$ *recalled (in both cases) at the end of Subsection 2.2.*

For two connected reductive groups G_1 *and* G_2 *we have* $\alpha_{G_1 \times G_2} = \alpha_{G_1} \times \alpha_{G_2}$.

In [Kot86] this is proved in the case where F has characteristic zero but the same proof works for all local fields, using Bruhat and Tit's generalization of Kneser's theorem [BT87]. If F is non-archimedean then each α_G is a bijection, in particular $H^1(F, G)$ has a commutative group structure. In the archimedean case the kernel and image of α_G are described. We will also denote $\alpha_G(c)(s) = \langle c, s\rangle$.

We resume the above notation: $(H, s, {}^L\eta)$ is an extended endoscopic triple, $(\gamma, \delta) \in H(F)_{G,\mathrm{sr}} \times G(F)_{\mathrm{sr}}$ is a matching pair, $T_H = \mathrm{Cent}(\gamma, H)$ and $T = \mathrm{Cent}(\delta, G)$ and we have a canonical isomorphism $T_H \simeq T$. By Theorem 4.6 the kernel of $H^1(F, T) \to H^1(F, G)$ is identified with the group of characters of $\pi_0(\widehat{T}^{\mathrm{Gal}_F})$ which are trivial on $Z(\widehat{G})^{\mathrm{Gal}_F}$. The element ${}^L\eta(s) \in Z(\widehat{H})^{\mathrm{Gal}_F}$ defines an element $s_{\gamma,\delta}$ of $\widehat{T}_H^{\mathrm{Gal}_F} \simeq \widehat{T}^{\mathrm{Gal}_F}$. We can finally state the variance property of transfer factors: we have

$$\Delta[\mathfrak{w}, \mathfrak{e}](\gamma, \delta') = \Delta[\mathfrak{w}, \mathfrak{e}](\gamma, \delta)\langle \inf(\delta, \delta'), s_{\gamma,\delta}\rangle^{-1}. \tag{4.3}$$

As for the Artin reciprocity map and the pairing (4.2) there are several natural normalizations for the transfer factors [KS, §4], and for half of these normalizations the exponent -1 on the right-hand side should be removed. The relation (4.3) is far from characterizing $\Delta[\mathfrak{w}, \mathfrak{e}]$ because it does not compare the values at unrelated matching pairs.

Conjecture 4.7. *Let* G *be a quasi-split connected reductive group over* F. *Let* $\phi : \mathrm{WD}_F \to {}^L G$ *be a tempered Langlands parameter.*

1. *The map* $S\Theta_\phi : G_{\mathrm{rs}}(F) \to \mathbb{C}$ *should be invariant under* ***stable*** *conjugacy.*
2. *For any semisimple* $s \in S_\phi$ *and any strongly regular semisimple* $G(F)$*-conjugacy class* $[\delta]$ *we should have*
$$\Theta^{\mathfrak{w}}_{\phi,s}(\delta) = \sum_{\gamma \in H(F)/\mathrm{st}} \Delta[\mathfrak{w}, \mathfrak{e}](\gamma, \delta) S\Theta_{\phi_H}(\gamma).$$

Remark 4.8.
1. The equation uniquely determined $\iota_{\mathfrak{w}}$ when provided it exists, due to the linear independence of characters. In particular, one can deduce how $\iota_{\mathfrak{w}}$ should depend on $\mathfrak{w}$. Namely, to each pair $\mathfrak{w}$ and $\mathfrak{w}'$ one can associate unconditionally a character $(\mathfrak{w}, \mathfrak{w}')$ of S_ϕ and then $\iota_{\mathfrak{w}'}(\pi) = \iota_{\mathfrak{w}}(\pi) \otimes (\mathfrak{w}, \mathfrak{w}')$. See [Kal13, §3] for details. In particular, $\dim(\iota_{\mathfrak{w}}(\pi))$ is independent of the choice of $\mathfrak{w}$, and hence $S\Theta_\phi$ is also independent.
2. While Conjecture 4.1 readily induces to the discrete case using Harish-Chandra's work, the putative analogous reductions for Conjectures 4.3 and 4.7 appear to be more subtle, involving the study of intertwining operators. See [KS88] for character formulas in the case of principal series representations.
3. Implicity in the conjecture is the fact that the choice of a semisimple s in its connected component in $\pi_0(\overline{S}_\phi)$ is irrelevant. One can reduce to the case where s is "generic" (implying that ϕ_H is essentially discrete) by parabolic induction (which behaves well with respect to $S\Theta$).
4. This conjecture reduces the characterization of the local Langlands correspondence to a characterization of the stable functions $S\Theta_\phi$.

4.3. **Refined Langlands correspondence for non-quasi-split groups.** Recall from Proposition 2.3 that two connected reductive groups that are inner forms of each other have isomorphic Langlands dual groups, and thus the "same" Langlands parameters. Vogan's idea is to consider the L-packets $\Pi_\phi(G)$, for a given ϕ and G varying in an inner class, as one big L-packet Π_ϕ. It is natural to take the quasi-split group given in Proposition 2.3 as "base point" in the inner class because we already have a satisfying conjecture in this case, and for reasons explained below. So we fix a quasi-split group G^*. Recall that isomorphism classes of inner twists of G^* are parametrized by $H^1(F, G^*_{\mathrm{ad}})$. We may consider the groupoid of triples (G, ψ, π) where (G, ψ) is an inner twist of G^* and π is an irreducible smooth representation of $G(F)$, with the obvious notion of isomorphism. The problem with this definition is that for an inner twist (G, ψ) of G^* its automorphism group in $\mathsf{IT}(G^*)$ is $G_{\mathrm{ad}}(F)$, which acts non-trivially on the set of isomorphism classes of irreducible smooth representations of $G(F)$.

This motivates the introduction of *pure inner twists*: augment the datum (G,ψ) with a 1-cocycle $z:\Gamma\to G^*(\overline{F})$ lifting

$$\Gamma \longrightarrow G_{\mathrm{ad}}(\overline{F})$$
$$\sigma \longmapsto \psi^{-1}\sigma(\psi).$$

This effectively solves the above problem but creates a new one because the map $H^1(F,G^*)\to H^1(F,G^*_{\mathrm{ad}})$ is not surjective in general. For $F=\mathbb{R}$, Adams, Barbasch and Vogan [ABV92] found an ad-hoc generalization of $Z^1(\mathbb{R},G^*)$, called strong real forms, that surjects onto $H^1(\mathbb{R},G^*_{\mathrm{ad}})$. Kottwitz suggested using his theory of isocrystals with additional structure [Kot85, Kot97] in the case of non-archimedean fields of characteristic zero as a generalization of $H^1(F,G^*)$. This suggestion was implemented completely by Kaletha and will be recalled below, but unfortunately it does not capture all inner forms of a given quasi-split group in general. Kaletha later introduced another generalization of inner forms, called *rigid* inner forms, for any local field F of characteristic zero and which captures all inner forms. Specializing to $F=\mathbb{R}$ recovers strong real forms. It turns out that all of these generalizations can be understood as replacing the Galois group Γ (or the étale site of $\operatorname{Spec} F$) by an appropriate *Galois gerb*. We summarize the three theories (pure, isocrystal and rigid) for a local field F of characteristic zero below and refer the interested reader to Dillery's paper [Dil] for the generalization to functions fields, which uses Čech cohomology instead of Galois cohomology and also provides a more conceptual point of view using actual gerbs.

In characteristic zero and for a commutative band, following [LR87] the above mentioned Galois gerbs may prosaically be defined as group extensions

$$1\to u(\overline{F})\to\mathcal{E}\to\Gamma\to 1$$

where u is a commutative group scheme over F and the action by conjugation of Γ on $u(\overline{F})$ coincides with the usual one. In practice u is a projective limit of groups $(u_i)_{i\geqslant 0}$ of multiplicative type and finite type over F with surjective morphisms between them, and the extension $\mathcal{E}$ is built from a class in $H^2_{\mathrm{cont}}(\Gamma,u(\overline{F}))$ where $u(\overline{F})$ is endowed with the topology induced by the discrete topology on each $u_i(\overline{F})$. Note that we have set-theoretic sections $\Gamma\to\mathcal{E}$, endowing $\mathcal{E}$ with a natural topology. Define $H^1_{\mathrm{alg}}(\mathcal{E},G)\subset H^1_{\mathrm{cont}}(\mathcal{E},G(\overline{F}))$ as the subset of classes of 1-cocycles $\mathcal{E}\to G(\overline{F})$ whose restriction to $u(\overline{F})$ is given by an algebraic morphism from $u_{\overline{F}}$ to $G_{\overline{F}}$. Define $H^1_{\mathrm{bas}}(\mathcal{E},G)\subset H^1_{\mathrm{alg}}(\mathcal{E},G)$ as the set of classes of cocycles for which the algebraic morphism $u_{\overline{F}}\to G_{\overline{F}}$ takes values in the center $Z(G)(\overline{F})$. By the cocycle condition it descends in the case to a morphism $u\to Z(G)$ defined over F. Note that such a morphism is induced from a morphism $u_i\to Z(G)$ for some index i because the center of G has finite type over F. We will also consider, for Z a subgroup scheme of $Z(G)$, the subset $H^1(u\to\mathcal{E},Z\to G)$ of $H^1_{\mathrm{bas}}(\mathcal{E},G)$ consisting of classes of cocycles whose associated map $u\to Z(G)$ factors through $\mathbb{Z}$.

We consider three cases in parallel.

(1) If we take $u=1$ we obtain the trivial extension $\mathcal{E}^{\mathrm{pur}}=\Gamma$, recovering the usual Galois cohomology group $H^1(F,G)$.

(2) Consider the pro-torus u over F with character group

$$X^*(u)=\begin{cases}\mathbb{Q} & \text{if } F \text{ is non-archimedean,}\\ \frac{1}{2}\mathbb{Z} & \text{if } F\simeq\mathbb{R}.\end{cases}$$

(Exclude the case $F\simeq\mathbb{C}$ here because it is essentially trivial.) We have

$$H^2_{\mathrm{cont}}(\Gamma,u(\overline{F}))\simeq\begin{cases}\widehat{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{Q} & \text{if } F \text{ is non-archimedean,}\\ \mathbb{Z}/2\mathbb{Z} & \text{if } F\simeq\mathbb{R}.\end{cases}$$

Let $\mathcal{E}^{\mathrm{iso}}$ be the extension of Γ by $u(\overline{F})$ corresponding to the class of 1.

(3) Consider the pro-finite algebraic group u over F with character group $X^*(u)$ the set of locally constant functions $f:\Gamma\to\mathbb{Q}/\mathbb{Z}$ satisfying $\sum_{\sigma\in\Gamma}f(\sigma)=0$ in F is archimedean. We have

$$H^2_{\mathrm{cont}}(\Gamma,u(\overline{F}))\simeq\begin{cases}\widehat{\mathbb{Z}} & \text{if } F \text{ is non-archimedean,}\\ \mathbb{Z}/2\mathbb{Z} & \text{if } F\simeq\mathbb{R}.\end{cases}$$

(As above we exclude the case $F \simeq \mathbb{C}$.) Let $\mathcal{E}^{\mathrm{rig}}$ be the extension of Γ by $u(\overline{F})$ corresponding to the class of -1.

We have the following generalizations of the Tate-Nakayama isomorphisms.

Theorem 4.9. *We have natural maps*

$$\kappa_G : H^1_{\mathrm{bas}}(\mathcal{E}^{\mathrm{iso}}, G) \longrightarrow X^*(Z(\widehat{G})^\Gamma)$$

extending the maps α_G of Theorem 4.6, i.e. sitting in commutative diagrams

$$\begin{tikzcd} H^1(F,G) \arrow[r, "\alpha_G"] \arrow[d, hook] & \mathrm{Irr}(\pi_0(Z(\widehat{G})^\Gamma)) \arrow[d, hook] \\ H^1_{\mathrm{bas}}(\mathcal{E}^{\mathrm{iso}}, G) \arrow[r, "\kappa_G"] & X^*(Z(\widehat{G})^\Gamma) \end{tikzcd}$$

and functorial in G similarly to Theorem 4.6 (in the case of an inclusion of a maximal torus $T \subset G$ we have to restrict to elements of $H^1_{\mathrm{bas}}(\mathcal{E}^{\mathrm{iso}}, T)$ for which the induce map $u \to T$ factors through $Z(G)$).

The map κ_G is bijective if F is non-archimedean.

Proof. See [Kot, Proposition 13.1 and Proposition 13.4] and [Kal18, §3.1]. □

For a connected reductive group G over F and a finite central subgroup scheme Z denote $\overline{G} = G/Z$. We have a dual map $\widehat{\overline{G}} \to \widehat{G}$; denote by $Z(\widehat{\overline{G}})^+$ the preimage of $Z(\widehat{G})^\Gamma$ in $Z(\widehat{\overline{G}})$.

Theorem 4.10 ([Kal16, Corollary 5.4])**.** *We have natural maps*

$$H^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G) \longrightarrow X^*(Z(\widehat{\overline{G}})^+)$$

extending the maps α_G and functorial in $Z \to G$ as in Theorem 4.6.

These maps are bijective in the non-archimedean case.

We also have natural maps $H^1(u \to \mathcal{E}^?, Z \to G) \to H^1(F, G/Z)$, and the above generalizations of the Tate-Nakayama morphism are also compatible with $\alpha_{G/Z}$. One can deduce that the maps

$$H^1_{\mathrm{bas}}(\mathcal{E}^{\mathrm{iso}}, G) \to H^1(F, G/Z(G)^0)$$

and

$$H^1(u \to \mathcal{E}^{\mathrm{rig}}, Z(G_{\mathrm{der}}) \to G) \longrightarrow H^1(F, G_{\mathrm{ad}})$$

are both surjective. In particular all inner forms can be realized as rigid inner twists, or as isocrystal inner twists if the center of G is connected. In general not all inner forms can be realized as isocrystal inner twists, e.g. when G is split semisimple but not adjoint.

There is [Kal18, §3.3] a natural map of extensions $\mathcal{E}^{\mathrm{rig}} \to \mathcal{E}^{\mathrm{iso}}$, inducing $H^1_{\mathrm{bas}}(\mathcal{E}^{\mathrm{iso}}, G) \to H^1_{\mathrm{bas}}(\mathcal{E}^{\mathrm{rig}}, G)$ for any group G. The relation with Theorem 4.9 and 4.10 is not so obvious.

Conjecture 4.11. *Let G^* be a quasi-split connected reductive group over F. Let $\mathfrak{w}$ be a Whittaker datum for G^*. Let $\phi : \mathrm{WD}_F \to {}^LG^*$ be a tempered Langlands parameter. Let $? \in \{\mathrm{pur}, \mathrm{iso}, \mathrm{rig}\}$. Define $\Pi^?_\phi$ as the set of isomorphism classes of pairs (z, π) where $z \in H^1_{\mathrm{bas}}(\mathcal{E}^?, G^*)$ and $\pi \in \Pi_\phi(G^*_z)$. Define*

1. $Z^{\mathrm{pur}} = 1$, $\mathcal{S}^{\mathrm{pur}}_\phi = \pi_0(S_\phi)$, *and* $\mathcal{Z}^{\mathrm{pur}} = \pi_0(Z(\widehat{G})^\Gamma)$;
2. $Z^{\mathrm{iso}} = Z(G)^0$, $\mathcal{S}^{\mathrm{iso}}_\phi = S_\phi/(S_\phi \cap \widehat{G}_{\mathrm{der}})^0$, *and* $\mathcal{Z}^{\mathrm{iso}} = Z(\widehat{G})^\Gamma$;
3. Z^{rig} *is any finite subgroup scheme of* $Z(G)$, $\mathcal{S}^{\mathrm{rig}}_\phi = \pi_0(S^+_\phi)$ *where* S^+_ϕ *is the preimage of* S_ϕ *in* $\widehat{\overline{G}}$ *and* $\mathcal{Z}^{\mathrm{rig}} = \pi_0(Z(\widehat{\overline{G}})^+)$.

There should exist a bijection $\iota_\mathfrak{w}$ making the following diagram commutative.

$$\begin{tikzcd} \Pi^?_\phi \arrow[r, "\iota_\mathfrak{w}", "\sim"'] \arrow[d] & \mathrm{Irr}(\mathcal{S}^?_\phi) \arrow[d] \\ H^1(u \to \mathcal{E}^?, Z^? \to G^*) \arrow[r] & X^*(\mathcal{Z}^?). \end{tikzcd}$$

Here the left vertical map is induced by the forgetful map $(z, \pi) \mapsto z$, the right vertical map is induced by the obvious map $\mathcal{Z}^? \to \mathcal{S}_\phi^?$ and the bottom horizontal map is given by Theorem 4.6 (resp. 4.9, resp. 4.10) in the pure (resp. isocrystal, resp. rigid) case.

The relation with Conjecture 4.1 is that for any $z \in Z^1(u \to \mathcal{E}^?, Z^? \to G^*)$ we should have $\Pi_\phi(G_z^*) = \{\pi \mid (z,\pi) \in \Pi_\phi^?\}$.

As for Conjectures 4.3 and 4.7, the map $\iota_{\mathfrak{w}}$ in Conjecture 4.11 should be characterized by endoscopic character relations. In order to state these relations we need normalized transfer factors. Their definition was suggested by Kottwitz and established by Kaletha in the case of pure inner forms [Kal11, §2.2] and extended to the isocrystal and rigid case by Kaletha [Kal14, Kal16].

Let (G, ψ, z) be a pure/isocrystal/rigid inner twist of G^* and $\phi : \mathrm{WD}_F \to {}^LG$ a tempered Langlands parameter. Consider a semi-simple $s \in S_\phi$ if $? \in \{\mathrm{pur}, \mathrm{iso}\}$ or $s \in S_\phi^+$ if $? = \mathrm{rig}$. As in Subsection 4.2 we obtain an extended endoscopic triple[9] $\mathfrak{e} = (H, s, {}^L\eta)$ and a tempered Langlands parameter $\phi_H : \mathrm{WD}_F \to {}^LH$. Consider matching strongly regular $\gamma \in H(F)$ and $\delta \in G(F)$. Using Steinberg's theorem [Ste65, Theorem I.7] we see that for any strongly regular $\delta \in G(F)$ there exists $\delta^* \in G^*(F)$ stably conjugate to δ, i.e. for which there exists $g \in G^*(\overline{F})$ satisfying $\psi(g^{-1}\delta^* g) = \delta$. Clearly δ^* is also strongly regular; denote its centralizer in G^* by T^*. In this situation let $\mathrm{inv}[\psi, z](\delta^*, \delta) \in H^1(u \to \mathcal{E}^?, Z^? \to T^*)$ be the class of $w \mapsto g z_w w(g)^{-1}$. This class does not depend on the choice of g. Similarly to the quasi-split case we can associate $s_{\gamma,\delta^*} \in \widehat{T^*}^\Gamma$ (resp. $\widehat{T^*}^\Gamma$, resp. $\widehat{\widehat{T^*}}^+$) to s and the matching pair (γ, δ^*), and pair it with $\inf(\delta^*, \delta)$ using Theorem 4.6 (resp 4.9, resp. 4.10). In analogy with (4.3) define

$$\Delta[\mathfrak{w}, \mathfrak{e}, \psi, z](\gamma, \delta) = \Delta[\mathfrak{w}, \mathfrak{e}](\gamma, \delta^*)\langle \inf(\delta^*, \delta), s_{\gamma,\delta^*}\rangle^{-1}.$$

In turns out that this is well-defined, i.e. the right-hand side does not depend on the choice of δ^*, and this defines a normalization of transfer factors for $(H, s, {}^L\eta)$. Again there are several natural normalizations and in half of these normalizations the exponent -1 should be removed.

We can now formulate the generalization of Conjecture 4.7. As in Subsection 4.2 we abbreviate $\langle s, \pi\rangle_{\mathfrak{w},z} = \mathrm{tr}\, \iota_{\mathfrak{w}}(z,\pi)(s)$ and define

$$\Theta_{\phi,s}^{\mathfrak{w},z} = e(G_z) \sum_{\pi \in \Pi_\phi(G_z)} \langle s, \pi\rangle_{\mathfrak{w},z} \Theta_\pi$$

where $e(G_z)$ is the sign defined by Kottwitz [Kot83].

Conjecture 4.12. *In the setting of Conjecture 4.11, for any $z \in Z^1(u \to \mathcal{E}^?, Z^? \to G^*)$, any strongly regular $G_z(F)$-conjugacy class $[\delta]$ and any semisimple $s \in S_\phi$ (resp. S_ϕ, resp. S_ϕ^+) we should have*

$$\Theta_{\phi,s}^{\mathfrak{w},z}(\delta) = \sum_{\gamma \in H(F)/\mathrm{st}} \Delta[\mathfrak{w}, \mathfrak{e}, \psi, z](\gamma, \delta) S\Theta_{\phi_H}(\gamma).$$

By linear independence of characters the conjecture implies that packets $\Pi_{\phi_H}(H)$ for all endoscopic groups of G^* — all quasi-split groups — should determine the refined Langlands correspondence for all pure/isocrystal/rigid inner forms of G^*.

If we fix an inner twist (G, ψ) of G^* then it may be realized as a rigid inner twists in more than one way: one can multiply $z \in Z^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G^*)$ by any $c \in Z^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to Z)$. By [Kal18, §6], Conjectures 4.11 and 4.12 for z imply the same conjectures for cz. This implies the same invariance property for pure inner twists. Presumably a similar invariance property should be valid in the isocrystal case.

4.4. **Reduction to the isocrystal case.** Let G^* be a quasi-split connected reductive group over a p-adic field F. As explained above all inner forms of G^* can be reached using the rigid theory, and one might be tempted to simply forget the pure and isocrystal versions. They are simpler however, and the relative complexity of the rigid version is exacerbated in the global setting. Another reason to favor the isocrystal version is that it seems more naturally related to geometric incarnations of the correspondence, as in [FS]. It is thus useful to relate the isocrystal and rigid versions (the relation between the pure and isocrystal versions being rather obvious.)

[9]A refined one in the rigid case, i.e. s belongs to the cover $\widehat{\widehat{G}}$ of $\widehat{G}$.

As explained in [Kal18, §4], for $z^{\mathrm{iso}} \in Z^1_{\mathrm{bas}}(\mathcal{E}^{\mathrm{iso}}, G^*)$ and $z^{\mathrm{rig}} \in Z^1_{\mathrm{bas}}(\mathcal{E}^{\mathrm{rig}}, G^*)$ is pullback via $\mathcal{E}^{\mathrm{rig}} \to \mathcal{E}^{\mathrm{iso}}$, the relevant (i.e. given restriction to center) representations of centralizers are the same and the endoscopic character relations are also the same. In [Kal18, §5] Kaletha construct an embedding $G^* \to \widetilde{G}^*$ with normal image and abelian cokernel such that the center of $\widetilde{G}^*$ is connected and such that Conjecture 4.11 and 4.12 for G^* and $\widetilde{G}^*$ are equivalent. Since these conjectures for $\widetilde{G}^*$ can be reduced to the isocrystal case, it would be enough to prove Conjecture 4.11 and 4.12 for all quasi-split groups in the isocrystal setting to deduce them for all quasi-split groups in the rigid setting, yielding "the" refined Langlands correspondence for all connected reductive groups.

4.5. **Relation with the crude version.** By [Kal16, Lemma 5.7] Conjecture 4.11 recovers the relevance condition on parameters discussed in 4.1.2.

One can formulate a more precise version of property (4) in Conjecture 4.1. Let $f : G_1^* \to G_2^*$ be a central isogeny between quasi-split connected reductive groups over F, inducing a dual map $\hat{f} : {}^LG_2 \to {}^LG_1$. Let $\phi_2 : \mathrm{WD}_F \to {}^LG_2$ be a tempered Langlands parameter and denote $\phi_1 = \hat{f} \circ \phi_2$. Let $? \in \{\mathrm{pur}, \mathrm{rig}, \mathrm{iso}\}$. We use the same notation as in Conjecture 4.11, choosing finite central subgroups Z_i^{rig} in the rigid case. Up to enlarging these groups we may assume that Z_1^{rig} contains the kernel of f and that its image is Z_2^{rig}. Let $z_1 \in Z^1(u \to \mathcal{E}^?, Z^? \to G_1^*)$ and let z_2 be its image in $Z^1(u \to \mathcal{E}^?, Z^? \to G_2^*)$. Denote $G_1 = G^*_{1,z_1}$ and $G_2 = G^*_{2,z_2}$. In all three cases $\hat{f}$ induces a morphism $S^?_{\phi_2} \to S^?_{\phi_1}$. Let $\mathfrak{w}$ be a Whittaker datum for G_1^* and G_2^*.

Conjecture 4.13. *For any $\pi_2 \in \Pi_{\phi_2}(G_2)$ we should have*

$$\pi_2|_{G_1(F)} \simeq \bigoplus_{\pi_1 \in \Pi_{\phi_1}(G_1)} m(\pi_1, \pi_2)\pi_1$$

where $m(\pi_1, \pi_2)$ is the multiplicity of $\iota_{\mathfrak{w}}(z_2, \pi_2)$ in the restriction of $\iota_{\mathfrak{w}}(z_1, \pi_1)$ to $S^?_{\phi_2}$.

4.6. **A few known cases.** In the case of real groups Conjectures 4.11 and 4.12 were proved by Shelstad in many papers, see [She08a, She10, She08b] and [Kal16, S5.6].

Arthur [Art13] proved Conjectures 4.3 and 4.7 for quasi-split special orthogonal[10] and symplectic groups over non-archimedean fields of characteristic zero using the stabilization of the twisted trace formula. In this case the stable characters $S\Theta$ are characterized by twisted endoscopy for the group GL_N with its automorphism $\theta : g \mapsto {}^tg^{-1}$ and the correspondence for general linear groups. Note that endoscopic groups of special orthogonal or symplectic groups are product of similar groups and general linear groups. Mok [Mok15] followed the same strategy to prove the conjectures for quasi-split unitary groups over non-archimedean local fields of characteristic zero.

Gan-Takeda [GT11] and Chan-Gan [CG15] proved Conjectures 4.11 and 4.12 for the groups GSp_4 over non-archimedean local fields of characteristic zero, although the normalization of transfer factors in the case of the non-quasi-split inner form was ad hoc.

5. Gerbs and Tannakian categories

We briefly mention the more conceptual point of view on gerbs and Tannakian categories.

We first recall the equivalence between certain gerbs and Tannakian categories [Riv72, Théorème 3] as corrected by [Del90]. We consider fpqc stacks over F. Recall that a *gerb* is a stack in groupoids admitting local sections and such that any two objects are locally isomorphic. A gerb $\mathcal{C}$ is said to *have affine band* if for any scheme S over F and any two objects x, y of $\mathcal{C}_S$ the sheaf $\underline{\mathrm{Isom}}_S(x, y)$ is representable by an affine scheme over S. If this holds for one non-empty S and one pair (x, y) then $\mathcal{C}$ has affine band [Del90, p. 131].

A representation R of a gerb $\mathcal{C}$ is a morphism from $\mathcal{C}$ to the stack of quasi-coherent sheaves (over varying schemes over F). For a scheme S over F and an object x of $\mathcal{C}_S$ the quasi-coherent sheaf $R(x)$ over S is automatically flat, and if it has finite rank n for some pair (S, x) then $R(y)$ has the same rank for any object y of $\mathcal{C}$ [Del90, §3.5]. In that case we may see R as a morphism from $\mathcal{C}$ to the stack of vector bundles of rank n (equivalently, GL_n-torsors). Finite-dimensional representations of $\mathcal{C}$ form a category $\mathsf{Rep}(\mathcal{C})$, that can be endowed with a tensor product (taking tensor products of vector

[10] In the even orthogonal case Arthur proved these conjectures "up to outer automorphism".

bundles). In fact $\mathsf{Rep}(\mathcal{C})$ is a tensor category over F (in the sense of [Del90, §2.1]). Because $\mathcal{C}$ has local sections the tensor category $\mathsf{Rep}(\mathcal{C})$ is even Tannakian, i.e. it admits a fibre functor [Del90, §1.9] over some non-empty scheme over F.

To any tensor category $\mathcal{T}$ over F we can associate the fibered category (over schemes over F) of fibre functors of $\mathcal{T}$, denoted by $\mathsf{Fib}(\mathcal{T})$. If $\mathcal{T}$ is Tannakian then $\mathsf{Fib}(\mathcal{T})$ is a gerb having affine band and the natural tensor functor $\mathcal{T} \to \mathsf{Rep}(\mathsf{Fib}(\mathcal{T}))$ is an equivalence. Conversely for a gerb $\mathcal{C}$ we also have a natural morphism of stacks $\mathcal{C} \to \mathsf{Fib}(\mathsf{Rep}(\mathcal{C}))$ which is an equivalence if and only if $\mathcal{C}$ has affine band.

For a gerb $\mathcal{C}$ having affine band and a linear algebraic group G over F we can consider morphisms of stacks from $\mathcal{C}$ to the gerb BG of G-torsors, generalizing the notion of representation of $\mathcal{C}$. Such a morphism may also be interpreted as a G-torsor on $\mathcal{C}$ (see [Dil, §2.4]). By the correspondence recalled above such a morphism amounts to a morphism of tensor categories $\mathsf{Rep}(G) \to \mathsf{Rep}(\mathcal{C})$. Here we have identified $\mathsf{Rep}(BG)$ with the category of finite-dimensional representations of G over F. The set[11] of isomorphism classes of morphisms $\mathcal{C} \to BG$ will be denoted by $H^1(\mathcal{C}, G)$.

We now assume that F has characteristic zero and specialize to the case of a gerb $\mathcal{C}$ whose band u is commutative, and so is an fpqc sheaf of commutative groups over F, and is representable by an affine (commutative group) scheme. Any group scheme over F is isomorphic to a projective limit, over a directed poset I, of group schemes of finite type $(u_i)_{i\in I}$. We assume further that I may be chosen to be countable. Recall that this implies that any projective limit over I of non-empty sets is itself not empty. We may identify $\mathcal{C}$ with a projective limit of gerbs $\mathcal{C}_i$ bounded by u_i (equivalently, we may identify the Tannakian category $\mathsf{Rep}(\mathcal{C})$ with a union of tensor subcategories admitting a tensor generator). Recall from [Riv72, Chapitre III Théorème 3.1.3.3] or [Del90, Corollaire 6.20] that $\mathcal{C}_i$ admits a section over a finite extension of F. It follows that $\mathcal{C}$ admits a section over $\overline{F}$. By the same projective limit argument we have that $\mathcal{C}_{\overline{F}}$ have only one isomorphism class (i.e. every $u_{\overline{F}}$-torsor is trivial). This implies that the group $\mathrm{Aut}_{\mathcal{C}}(x)$ is an extension $\mathcal{E}$ of Γ by $\mathrm{Aut}_{\mathcal{C}_{\overline{F}}}(x) = u(\overline{F})$. Moreover for any $x \in \mathcal{C}_{\overline{F}}$ the two pullbacks $p_1^* x$ and $p_2^* x$ in $\mathcal{C}_{\overline{F}\otimes_F\overline{F}}$ are isomorphic [Dil, Lemma 2.61]. Fix such an isomorphism $\varphi : p_1^* x \simeq p_2^* x$. Pulling back φ via the morphisms

$$\begin{aligned} \overline{F} \otimes_F \overline{F} &\longrightarrow \overline{F} \\ x \otimes y &\longmapsto x\sigma(y) \end{aligned}$$

for $\sigma \in \Gamma$ gives us a (set-theoretic) splitting $\Gamma \to \mathrm{Aut}_{\mathcal{C}}(x)$. Taking the "coboundary" $d\varphi = (p_{13}^*\varphi)^{-1} \circ (p_{23}^*\varphi) \circ (p_{12}^*\varphi)$ yields an automorphism of the pullback of x via the first projection $\overline{F} \to \overline{F}^{\otimes 3}$, i.e. an element of $u(\overline{F}^{\otimes 3})$, and one can check that it is a Čech 2-cocycle [Dil, Fact 2.31]. Conversely any such 2-cocycle induces a gerb bounded by u [Dil, Proposition 2.36], and two gerbs bound by u are isomorphic if and only if their associated class in $\check{H}^2(\overline{F}/F, u)$ are equal. Yet another projective limit argument shows that we have a natural isomorphism $\check{H}^2(\overline{F}/F, u) \simeq H^2_{\mathrm{cont}}(\Gamma, u(\overline{F}))$. For a linear algebraic group G over F and a morphism of stacks $R : \mathcal{C} \to BG$, R factors through $\mathcal{C}_i$ for some $i \in I$ (equivalently, $\mathsf{Rep}(G)$ has a tensor generator [DM82, Proposition 2.20(b)] and so the tensor functor $\mathsf{Rep}(G) \to \mathsf{Rep}(\mathcal{C})$ factor through a sub-tensor category of $\mathsf{Rep}(\mathcal{C})$ generated by a single object). Choosing a trivialization of the $G_{\overline{F}}$-torsor $R(x)$ we obtain a morphism $u_{\overline{F}} \to G_{\overline{F}}$. One can check that restricting R to $\mathcal{E} = \mathrm{Aut}_{\mathcal{C}}(x)$ gives a continuous 1-cocycle $\mathcal{E} \to G(\overline{F})$ whose restriction to $u(\overline{F})$ is (induced by) the above morphism of group schemes over $\overline{F}$. We obtain a map $H^1(\mathcal{C}, G) \to H^1_{\mathrm{alg}}(\mathcal{E}, G(\overline{F}))$ and one can check that it is bijective. (The least obvious part is perhaps the fact that a morphism $\mathcal{C} \to BG$ can be constructed from a cocycle in $Z^1_{\mathrm{alg}}(\mathcal{E}, G)$: one can reduce to constructing a morphism $\mathcal{C}_i \to \mathsf{Rep}(G)$ for some $i \in I$ and this may be done using *finite* Galois descent.)

For F a non-archimedean local field of characteristic zero the gerb corresponding to $\mathcal{E}^{\mathrm{iso}}$ was historically first reduced via its corresponding Tannakian category, the category of isocrystals. We briefly recall this notion. Let L be the completion of the maximal unramified extension of F. Denote by σ the Frobenius automorphism of L. An isocrystal is a finite-dimensional vector space V over L endowed with a σ-linear bijection $\Phi : V \to V$. They form a tensor category Isoc_F for the obvious notion of tensor product. (Among other axioms, we indeed have $\mathrm{End}_{\mathsf{Isoc}_F}(1) = L^\sigma = F$.) We have an obvious

[11] We ignore set-theoretic issues here.

fibre functor for Isoc_F over L, namely $(V,\Phi) \mapsto V$, and so Isoc_F is Tannakian. By the Dieudonné-Manin classification theorem the tensor category Isoc_F has a simple structure: it is semi-simple and its simple objects are parametrized by $\mathbb{Q}$. We briefly recall this classification and refer the reader to [Riv72, Chapitre VI §3.3] for more details and references. Fix a uniformizer ϖ of F. For $r/s \in \mathbb{Q}$ for coprime $r, s \in \mathbb{Z}$ with $s > 0$ we may construct the corresponding simple object of Isoc_F as follows. Let $S(r/s)$ be L^s and define a σ-linear automorphism of $S(r/s)$ as σ (on coordinates) post-composed with the linear automorphism of L^s with matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & \cdots & 0 & 1 \\ \varpi^r & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

This defines a simple object $S(r/s)$ in Isoc_F. The isomorphism class of $S(r/s)$ does not depend on the choice of uniformizer ϖ, and any simple object is isomorphic to $S(q)$ for a uniquely determined $q \in \mathbb{Q}$. Denote by F_s the unramified extension of degree s of F in L. The F-algebra $\mathrm{End}_{\mathsf{Isoc}_F}(S(r/s))$ embeds in the matrix algebra $M_s(L)$, in fact it embeds in $M_s(F_s)$ and it is a central simple algebra over F which is a division ring and is split by F_s. Its invariant in $H^2(F, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}$ is simply the image of r/s. Any isocrystal (V, Φ) decomposes canonically as $\bigoplus_{r/s \in \mathbb{Q}} V_{r/s}$ where

$$V_{r/s} = L \otimes_{F_s} V^{\varpi^{-r}\Phi^s}$$

is the isotypic component isomorphic to a finite sum of copies of $S(r/s)$. The rational numbers q for which $V_q \neq 0$ are called the *slope* of (V, f) and the above decomposition is called the *slope decomposition*. An isocrystal (V, Φ) is said to be pure of slope $q \in \mathbb{Q}$ if $V_{q'} = 0$ for all $q' \neq q$. The tensor product of two isocrystals which are pure of slopes q_1 and q_2 is also pure, of slope $q_1 + q_2$. The tensor category Isoc_F is the union of its tensor subcategories $\mathsf{Isoc}_{F,s}$ consisting of all isocrystals (V, f) whose slopes q all satisfy $qs \in \mathbb{Z}$. the Tannakian category $\mathsf{Isoc}_{F,s}$ admits a fibre functor over F_s, namely

$$\omega_s : (V, \Phi) \mapsto \bigoplus_{r \in \mathbb{Z}} V^{\varpi^{-r}\Phi^s}.$$

If s divides s' then we have an obvious identification between $F_{s'} \otimes_{F_s} \omega_s$ and $\omega_{s'}$. We obtain a fibre functor ω for Isoc_F over the maximal unramified extension of F. Thanks to the description of $\mathrm{End}_{\mathsf{Isoc}_F}(S(r/s))$ recalled above we can compute the band u_s of (the gerb of fibre functors of) $\mathsf{Isoc}_{F,s}$ as the (commutative!) multiplicative group $\mathbb{G}_m$ over F. For an F_s-algebra A and $x \in A^\times$, x acts on the slope r/s part $A \otimes_{F_s} V^{\varpi^{-r}\Phi^s}$ by multiplication by x^r. For s dividing s' the natural morphism $u_{s'} \to u_s$ can be checked to be $x \mapsto x^{s'/s}$, and so the band u of (the gerb of fibre functors of) Isoc_F is the split protorus with character group $\mathbb{Q}$. The class of the gerb in

$$H^2(F, u) \simeq H^2_{\mathrm{cont}}(\Gamma, u(\overline{F})) \simeq \varprojlim_s H^2(F, u_s) \simeq \varprojlim_s \mathbb{Q}/\mathbb{Z} \simeq \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$$

(the second isomorphism is because each $H^1(F, u_s)$ vanishes and so $\varprojlim_s H^1(F, u_s)$ also vanishes) can be computed from the above description of endomorphisms of simple isocrystals and is simply equal to 1.

For a connected linear algebraic group G over F we can identify the set of isomorphism classes of tensor functors $\mathsf{Rep}(G) \to \mathsf{Isoc}_F$ with $B(G) := G(L)/\sim$ where $g_1 \sim g_2$ if and only if there exists $x \in G(L)$ for which $g_2 = x g_1 \sigma(x)^{-1}$ (i.e. $\sim$ is the σ-conjugacy relation). This is because $H^1(L, G_L)$ is trivial and so there is up to isomorphism only one fibre functor for $\mathsf{Rep}(G)$ over L, namely $\omega_{G,L} : (V, \rho) \mapsto L \otimes_F V$. It follows that any tensor functor $\mathsf{Rep}(G) \to \mathsf{Isoc}_F$ is isomorphic to one of the form $(V, \rho) \mapsto (L \otimes_F V, \Phi_{V,\rho})$. It is clear that setting $\Phi_{V,\rho} = \sigma \otimes \mathrm{id}_V$ gives a tensor functor, and any other tensor functor differs from this one by an automorphism of the fibre functor $\omega_{G,L}$, i.e. by an element of $G(L)$. A similar argument shows that two elements of $G(L)$ induce isomorphic tensor functors if and only if they are σ-conjugated.

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