Laumon sheaf and mod p Langlands Correspondence

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Introduction

$E/\mathbb{Q}_p$ finite, $\ell \neq p$. G/E red grp.

FS: $\pi \longmapsto \varphi_\pi^{ss}$ — ss L-param $W_E \longrightarrow {}^L G(\bar{\mathbb{Q}}_\ell$ or $\bar{\mathbb{F}}_\ell)$.

sm irrep of $\underbrace{G(E)}_{\text{loc cpt}}$ w/ coeffs in $\bar{\mathbb{Q}}_\ell$ or $\bar{\mathbb{F}}_\ell$.

This is a ss LLC.

Standard geom Langlands methods

(Drinfeld, Laumon, Frenkel-Gaitsgory-Vilonen).

To construct $\varphi \longmapsto \pi(\varphi)$.

Cusp L-param

But Does not directly work in our context.

b/c "descent issue" in mod ℓ case.

However, everything should work well for the p-adic Langlands.

↝ use this p-adic case + motivic argument

motivic spectral action

to treat any $\ell \neq p$ or $\ell = p$.

Starting point Breuil, Colmez, Emerton, Paskunas, ...

$\exists$ good p-adic Langlands corresp. for $GL_2(\mathbb{Q}_p)$.

ζ but not for $GL_2(E)$ for $E/\mathbb{Q}_p$, $E \neq \mathbb{Q}_p$.

<u>Today</u> A functor $\mathrm{Rep}_{\bar{\mathbb{F}}_q} \Gamma_E \longrightarrow \mathrm{Rep}_{\bar{\mathbb{F}}_q} \begin{pmatrix} E^\times & E \\ 0 & 1 \end{pmatrix}$

$\Gamma_E = \mathrm{Gal}(\bar{E}/E)$.

$$\rho \longmapsto \pi(\rho)\big|_{\begin{pmatrix} E^\times & E \\ 0 & 1 \end{pmatrix}}.$$

<u>Later</u> * Extend this to a rep of $GL_2(E)$ ("descent")
when ρ is irred & $\dim \rho = 2$.

* By mod p^n for all $n \geq 1$, get Banach LLC :

$L/\mathbb{Q}_p$, $\mathrm{Rep}_L \Gamma_E \longrightarrow \{\text{Banach reps of } \begin{pmatrix} E^\times & E \\ 0 & 1 \end{pmatrix}\}$.

* Locally analytic version $\rho \longmapsto \pi(\rho)^{la}$.

Using the analytic prismatization of $\mathrm{Spa}\, E$:

$\mathrm{Rep}_{\mathbb{Q}_p} \Gamma_E \rightsquigarrow$ VBs on the analytic pairs of $\mathrm{Spa}\, E$.

<u>Key word</u>: Holonomy.

<u>Warm up</u> Recall (Laumon): $X/k = \bar{k}$ sm proj curve.

$\mathcal{E} = \bar{\mathbb{Q}}_\ell$ - irred rk n loc sys $/X$.

$d\ (= [E:\mathbb{Q}_p]) \geq 1$.

$$\rightsquigarrow \quad \pi_d : X^d \longrightarrow \mathrm{Div}^d_X = X^d/\mathfrak{S}_d$$

$$(x_1, \cdots, x_d) \longmapsto \sum_{i=1}^d [x_i]$$

Cartier div

Let $S_d \mathcal{E} := (\pi_{d*} \mathcal{E}^{\boxtimes d})^{\mathfrak{S}_d}$

<u>Fact</u> $S_d \mathcal{E} \in \mathrm{Perv}(\mathrm{Div}^d_X, \bar{\mathbb{Q}}_\ell)$.

<u>Note</u> $S_d \mathcal{E} = j_{!*} \mathcal{F}$ wh

ere if $\mathcal{U} = \{(x_i)_i \mid \forall i \neq j, x_i \neq x_j\} \subset X^d$,

$\mathcal{F}$ = loc sys assoc to the S_d-equiv loc sys $(\mathcal{E}^{\boxtimes d})|_{\mathcal{U}}$.

$$\begin{tikzcd} \mathcal{U} \arrow[d, "S_d"] & \\ \pi_d(\mathcal{U}) \arrow[r, hook, "j"] \arrow[d, equal] & \mathrm{Div}^d_X . \\ \text{multi-free divisors.} & \end{tikzcd}$$

<u>Generally</u> For $\mathcal{E}$ VB on X, $S_d \mathcal{E} := (\pi_{d*} \mathcal{E}^{\boxtimes d})^{\mathfrak{S}_d}$.

This is a coh sheaf $/ \mathrm{Div}^d_X$.

<u>Lem</u> $S_d \mathcal{E}$ is a vect bdl on Div^d_X.

<u>Compatibility with RH</u>

<u>Recall</u> Katz, Emerton-Kisin, Bhatt-Lurie:

$X = \mathbb{F}_q$-sch.

$\mathrm{RH} : D_{et}(X, \mathbb{F}_q) \hookrightarrow D_{coh}(\mathcal{O}_{X^{1/p^\infty}})^{\varphi = id}$ fully faithful.

$X^{1/p^\infty} := \varprojlim_{\mathrm{Fr}} X$

adjoint to Sol (sol'n functor)

$\mathrm{Sol} : (\mathcal{E}, \varphi) \longmapsto \tilde{\mathcal{E}}^{\varphi = id}$

qcoh on X^{1/p^∞} $\quad$ $\varphi : \mathcal{E} \xrightarrow{\sim} \mathcal{E}$ $\quad$ $\tilde{(-)}$ = étaleness.

Rmk Katz corresp.

$$\{\mathbb{F}_q\text{-étale loc sys } / X\} \simeq \left\{ (\mathcal{E}, \varphi) \;\middle|\; \begin{array}{l} \mathcal{E} \text{ VB } / X^{1/p^\infty} \\ \varphi : \mathcal{E} \xrightarrow{\sim} \mathcal{E} \end{array} \right\}.$$

Prop X curve / $\mathbb{F}_q$ (sm proj), $\mathcal{F} = \mathbb{F}_q$-étale loc sys / X

$\rightsquigarrow$ $(\mathcal{E}, \varphi) = RH(\mathcal{F})$

$$\Rightarrow \quad RH(S_d \mathcal{F}) = \varinjlim_{\varphi} (\underbrace{S_d \mathcal{E}}_{\text{VB by lem}}, \varphi).$$

Prop $\Rightarrow$ $RH(S_d\mathcal{F})$ is holonomy (i.e. a perfection of some VB).

<u>The real thing</u> $E/\mathbb{Q}_p$. $d = [E : \mathbb{Q}_p]$, $\mathbb{F}_q = \mathcal{O}_E/\pi$.

$$\begin{array}{c} \mathbb{B}^{\varphi=\pi} \\ \downarrow \\ * = \mathrm{Spd}\, \overline{\mathbb{F}}_q \end{array} \qquad \text{absolute BC}, \simeq \mathrm{Spd}(\overline{\mathbb{F}}_q[[T^{1/p^\infty}]]).$$

$$\rightsquigarrow \quad \mathrm{Rep}_{\overline{\mathbb{F}}_q}(\Gamma_E) = \{ \overline{\mathbb{F}}_q\text{-étale loc sys on } \underbrace{(\mathbb{B}^{\varphi=\pi} \setminus \{0\}) / \underline{E}^\times}_{\text{"Div}^1\text{" curve.}} \}$$

<u>Then</u> $RH : \mathrm{Rep}_{\overline{\mathbb{F}}_q}(\Gamma_E) \xrightarrow{\sim} \text{VB on } \mathrm{Div}^1$

$$\rho \longmapsto \underbrace{\mathcal{F}_\rho}_{\text{étale loc sys } ((\varphi,\Gamma)\text{-mod}).} \otimes_{\overline{\mathbb{F}}_q} \mathcal{O}$$

$\mathcal{E}_\rho = \mathcal{F}_\rho \otimes \mathcal{O}$ seen as an $\underline{E}^\times$-equiv VB on $\mathbb{B}^{\varphi=\pi} \setminus \{0\}$.

<u>Symmetrization</u>

$$\begin{array}{rcl} \pi_d : (\mathbb{B}^{\varphi=\pi} \setminus \{0\})^d & \longrightarrow & \mathbb{B}^{\varphi=\pi^d} \setminus \{0\} \\ (t_1, \cdots, t_d) & \longmapsto & t_1 \cdots t_d. \end{array}$$

<u>Thm</u> (Fargues) $A_d := \{ (\mu_1, \cdots, \mu_d) \in (\underline{E}^\times)^d \mid \prod_{i=1}^d \mu_i = 1 \}$.

Then π_d is quasi-proét surjective,

& $(\mathbb{B}^{\varphi=\pi} \setminus \{0\})^d / \underbrace{\Delta_d \rtimes \mathfrak{S}_d}_{\text{pro-ét quotient}} \xrightarrow{\sim} \mathbb{B}^{\varphi=\pi^d} \setminus \{0\}$.

From now on, $\rho \in \mathrm{Rep}_{\overline{\mathbb{F}}_q}(\Gamma_E) \rightsquigarrow \mathcal{F}_\rho$ étale

$\rightsquigarrow \mathcal{E}_\rho = \mathcal{F}_\rho \otimes \mathcal{O}$.

Def $S_d \mathcal{F}_\rho := (\pi_{d*} \mathcal{F}_\rho^{\boxtimes d})^{\Delta_d \rtimes \mathfrak{S}_d}$ an étale sheaf.

$S_d \mathcal{E}_\rho := (\pi_{d*} \mathcal{E}_\rho^{\boxtimes d})^{\Delta_d \rtimes \mathfrak{S}_d}$ v-sheaf of $\mathcal{O}$-mods.

Can prove: $\underbrace{S_d \mathcal{E}_\rho}_{(\varphi,\Gamma)\text{-mod}} = \underbrace{RH}_{\text{Mann's RH corresp.}}(S_d \mathcal{F}_\rho)$

$\Rightarrow$ "$S_d \mathcal{E}_\rho$ is quasi-coherent".

Thm (Holonomicity) If $d < p$, then

$\exists\ \underbrace{\mathcal{M}}_{\text{sub-}\mathcal{O}\text{-mod}} \subset S_d \mathcal{E}_\rho$ stable under φ (i.e. $\varphi(\mathcal{M}) \subset \mathcal{M}$)

s.t. $\mathcal{M}$ is a perfect complex

& "$\bigcup_{n \geq 0} \varphi^{-n}(\mathcal{M})$ is dense in $S_d \mathcal{E}_\rho$".

Thm (Very difficult) If $d<p$

then $S_d \mathcal{E}_\rho$ is generated by its global sections.

Fact $H^0(S_d \mathcal{E}_\rho) = \Gamma\Big(\underbrace{(\mathbb{B}^{\varphi=\pi} \setminus \{0\})^d}_{\text{Stein \& perfectoid}}, \underbrace{\mathcal{E}_\rho^{\boxtimes d}}_{\text{VB}}\Big)^{\Delta_d \rtimes \mathfrak{S}_d} \neq 0$ if $\rho \neq 0$

This can also be very difficult even if $\rho = \chi_1 \oplus \chi_2$, $\chi_1 \neq \chi_2$.

<u>Lem</u> $d<p$, k char p field. Then

$$\begin{array}{c} k[x_1,\dots,x_d]/(\sigma_0,\dots,\sigma_{d-1}) \\ \circlearrowleft \\ \mathfrak{S}_d \end{array} \simeq k[\mathfrak{S}_d]$$

e.g. $d=[E:\mathbb{Q}_p]$,

$$\begin{array}{c} \bar{\mathbb{F}}_q[\![E]\!] \\ \circlearrowleft \\ E^\times \end{array} = \begin{array}{c} \mathcal{O}(\mathbb{B}^{\varphi=\pi^d}) \\ \circlearrowleft \\ E^\times \end{array}$$