Exercise 1 (due on October 12)

The solutions to the following problems are to be submitted.

Problem 1.1. (Semisimplification of the reduction is well-defined) Let L be a finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}_L$, uniformizer ϖ_L, and residual field k_L. Let Γ be a compact topological group, and let $\rho : \Gamma \to \mathrm{GL}_n(L) = \mathrm{GL}(V)$ be a representation. So there exists an $\mathcal{O}_L$-lattice Λ that is stable under Γ-action, and define $\bar{\rho}_\Lambda$ to be the representation given by the Γ-action on $\Lambda/\varpi_L\Lambda$.

(1) Show that the semisimplification of $\bar{\rho}_\Lambda$ does not depend on the choice of the Γ-stable $\mathcal{O}_L$-lattice Λ. (Hint: consider two such lattices Λ and Λ'; first reduce to the case when $\varpi_L \cdot \Lambda \subseteq \Lambda' \subseteq \Lambda$.)
(2) When $\bar{\rho}^{\mathrm{ss}}$ is irreducible, show that for every two Γ-stable lattices Λ_1 and Λ_2, $\Lambda_1 = \varpi_L^n \cdot \Lambda_2$ for some $n \in \mathbb{Z}$.

Problem 1.2. ($\mathbb{Z}_p$-extensions of local fields and global fields) For a field k, let $k^{p\text{-ab}}$ denote its maximal pro-p-abelian extension, i.e. the union of all abelian Galois extensions of k whose Galois groups are pro-p-groups. Under the Galois theory, this corresponds to the maximal pro-p quotient of G_k^{ab}. Then $G_k^{p\text{-ab}} := \mathrm{Gal}(k^{p\text{-ab}}/k)$ is a $\mathbb{Z}_p$-module, and we write $r_{p\text{-ab}}(k)$ for its rank over $\mathbb{Z}_p$ (possibly infinite), or equivalently $r_{p\text{-ab}}(k) = \dim_{\mathbb{Q}_p} G_k^{p\text{-ab}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Answer the following questions on the computation of $r_{p\text{-ab}}$ for different fields.

(1) If $k = K$ is a finite extension of $\mathbb{Q}_p$, show that $r_{p\text{-ab}} = [K : \mathbb{Q}_p] + 1$.
(2) If $k = \mathbb{F}_q((t))$ is a function field, with q a power of p, show that $r_{p\text{-ab}} = \infty$. (Hint: the structure of $\mathbb{F}_q((t))^\times$ is discussed in, for example, Neukirch, Algebraic Number Theory, Page 140, Proposition II.5.7 (ii).)
<u>Remark:</u> Philosophically, one can understand this difference as: in the function field case, all fields look like $\mathbb{F}_{q'}((t'))$ for some uniformizer t'. So there is no "base field" like $\mathbb{Q}_p$. Because of this, we can always make $\mathbb{F}_q((t))$ an as large as possible extension of another $\mathbb{F}_{q'}((t'))$. So $r_{p\text{-ab}}(\mathbb{F}_q((t)))$ is infinite.
(3) If $k = F$ a global number field, then Dirichlet unit theorem says that the rank of the unit group $\mathrm{rank}(\mathcal{O}_F^\times) = r_1 + r_2 - 1$, where r_1 and r_2 are number of real embeddings and number of pairs of complex embeddings. Show that
$$r_{\mathrm{ab}}(F) \geq [F : \mathbb{Q}] - (r_1 + r_2 - 1) = r_2 + 1.$$
<u>Remark:</u> It is conjectured that this is an equality, so-called the Leopoldt Conjecture. This is known when $F/\mathbb{Q}$ is an abelian extension, but open in general.

Problem 1.3. (Restriction of a Galois representation) Let F be a number field and let $\bar{\rho} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be a continuous residual Galois representation. Let S be a finite set of places of F at which $\bar{\rho}$ is unramified. Show that there exists a finite <u>solvable</u> Galois extension E over F such that

(1) letting $\bar{\rho}|_{G_E}$ denote the restriction of $\bar{\rho}$ to G_E, then $\mathrm{Im}(\bar{\rho}) = \mathrm{Im}(\bar{\rho}|_{G_E})$,
(2) $\bar{\rho}|_{G_E}$ is everywhere unramified, and
(3) every $v \in S$ splits completely in E/F.

Problem 1.4. (Compatibility of corestriction map and Shapiro's lemma under cup product) Let G be a finite group and H a subgroup. Let A and B be two finite H-modules and C a finite G-module. Assume that we are given a natural H-module homomorphism
$$\psi : A \otimes B \to C,$$

i.e. $\psi(ha \otimes hb) = h\psi(a \otimes b)$ for $h \in H$, $a \in A$, and $b \in B$.

(1) Show that ψ induces a natural well-defined G-module homomorphism

$$\tilde{\psi} : \operatorname{Ind}_H^G A \otimes \operatorname{Ind}_H^G B \to C$$

given by, for $f_A \in \operatorname{Ind}_H^G A$, $f_B \in \operatorname{Ind}_H^G B$:

$$\tilde{\psi}(f_A \otimes f_B) := \sum_{g \in H\backslash G} g^{-1}\psi(f_A(g) \otimes f_B(g)).$$

(Here $\operatorname{Ind}_H^G A := \{f : G \to V;\ \mid f(hg) = h(f(g)) \text{ for } h \in H, g \in G\}$ is the standard induced representation; G acts on it by $(g \star f)(x) := f(xg^{-1})$ for $g, x \in G$.)

(2) Show that, for any $i, j \geq 0$, the following diagram of cup products commutes:

$$\begin{tikzcd}
H^i(H,A) \arrow[d, "\text{Shapiro Lemma}"', "\cong"] & \times & H^j(H,B) \arrow[d, "\text{Shapiro Lemma}"', "\cong"] \arrow[r, "\cup_\psi"] & H^{i+j}(H,C) \arrow[d, "\text{corestriction}"] \\
H^i(G,\operatorname{Ind}_H^G A) & \times & H^j(G,\operatorname{Ind}_H^G B) \arrow[r, "\cup_{\tilde{\psi}}"] & H^{i+j}(G,C).
\end{tikzcd}$$

(Hint: use dimension shifting to reduce to $i = j = 0$, and then make an explicit computation.)

Additional problems.

(*No credit. No need to hand in.*)

Problem 1.5. (Lattices in a representation by example) Let $\ell \geq 3$ be a prime number Let Γ be a compact topological group. Let $\chi_1, \chi_2 : \Gamma \to \mathbb{Z}_\ell^\times$ be two continuous characters, whose reductions modulo ℓ are different. Let $\rho : \Gamma \to \mathrm{GL}_2(\mathbb{Q}_\ell) = \mathrm{GL}(V)$ be an *irreducible* representation.

(1) Prove (by abstract nonsense) that for every $g \in \Gamma$, $\mathrm{Tr}(\rho(g)) \in \mathbb{Z}_\ell$, and that the associated semisimple residual Galois representation $\bar{\rho}^{\mathrm{ss}} \cong \bar{\chi}_1 \oplus \bar{\chi}_2$ if and only if, for every $g \in \Gamma$,
$$\mathrm{Tr}(\rho(g)) \equiv \chi_1(g) + \chi_2(g) \mod \ell.$$

(2) Assume the equivalent conditions in (1) hold. Show that there exists a lattice Λ_1 admitting a basis for which the associated residual Galois representation $\bar{\rho}_{\Lambda_1}$ takes the form of
$$\bar{\rho}_{\Lambda_1}(g) = \left(\begin{smallmatrix} \bar{\chi}_1(g) & \bar{b}(g) \\ 0 & \bar{\chi}_2(g) \end{smallmatrix}\right)$$
for some nonzero map $\bar{b} : \Gamma \to \mathbb{F}_p$. Show that such Λ_1 is unique up scalar. (This is not hard; but I believe that such an argument first appeared in Ribet's famous converse to Herbrand theorem, in K. Ribet, A modular construction of unramified p-extensions of $\mathbb{Q}(\mu_p)$, *Invent. Math.* **34** (1976), 151–162.

(3) We may reverse the role of $\bar{\chi}_1$ and $\bar{\chi}_2$ in (2) to get a lattice Λ_2 (canonical up to a unique scalar), so that $\bar{\rho}_{\Lambda_2}$ is a non-trivial extension of $\bar{\chi}_1$ by $\bar{\chi}_2$. By possibly rescaling Λ_2, we may assume that $\Lambda_1 \subseteq \Lambda_2 \subseteq \ell^{-n}\Lambda_1$, with subquotients for each inclusion is isomorphic to $\mathbb{Z}_\ell/\ell^n$. This n can be viewed as an invariant that describes how similar this ρ is to a direct sum of two characters. Show that there exists two characters $\chi_i^{(n)} : \Gamma \to (\mathbb{Z}_\ell/\ell^n\mathbb{Z}_\ell)^\times$ with $i = 1, 2$, such that for every $g \in \Gamma$,
$$\mathrm{Tr}(\rho(g)) \equiv \chi_1^{(n)}(g) + \chi_2^{(n)}(g) \mod \ell^n.$$

(4) Assume $\ell \geq 3$ and that Γ contains an element $\mathbf{c}$ of order 2 (e.g. a complex conjugation in $G_\mathbb{Q}$) for which $\bar{\chi}_1(\mathbf{c}) = 1$ and $\bar{\chi}_2(\mathbf{c}) = -1$. Prove that the converse to (3) holds, namely, if there exists characters $\chi_1, \chi_2 : \Gamma \to \mathbb{Z}_\ell^\times$ such that
$$\chi_1(\mathbf{c}) = 1, \quad \chi_2(\mathbf{c}) = -1, \quad \text{and} \quad \mathrm{Tr}(\rho(g)) \equiv \chi_1(g) + \chi_2(g) \bmod \ell^m,$$
then the invariant n from (3) satisfies $n \geq m$.

(I do not know if (4) holds without the additional assumption on the existence of the element $\mathbf{c}$.)

<u>Remark:</u> In the aforementioned paper of Ribet, he considered the representation ρ associated to a cuspidal eigenform that is congruent to an Eisenstein series modulo p.

Problem 1.6. (Extensions of groups)

(1) If H is a normal subgroup of a finite group G and H is abelian, show that the quotient group $\Gamma := G/H$ acts naturally on H by conjugation, i.e. for $\gamma \in \Gamma$, pick a lift $\tilde{\gamma}$ of γ in G, then we let γ acts on H by
$$\gamma \star h := \tilde{\gamma} h \tilde{\gamma}^{-1}.$$
Show that this action is well-defined.

(In this situation, we call G an extension of Γ by H.)

(2) Given a finite group Γ acting on a finite abelian group H, then the set of isomorphism classes of extensions of Γ by H can be identified with $H^2(\Gamma, H)$. The identification is given as follows: for G an extension of Γ by H as in (1), for each $\gamma \in \Gamma$, we fix a lift $g(\gamma) \in G$, and set

$$f_{\gamma,\gamma'} := g(\gamma) \cdot g(\gamma') \cdot g(\gamma\gamma')^{-1} \in H.$$

Show that this defines a 2-cocycle in $Z^2(\Gamma, H)$, and a different choice of $g(\gamma)$ amounts to chancing the above 2-cocycle by a 2-coboundary.

Show that G is a semi-direct product $H \rtimes \Gamma$ if and only if the corresponding class of G in $H^2(\Gamma, H)$ is trivial.

Problem 1.7. (Ramification filtration for Lubin–Tate tower) This is a slight generalization of the cyclotomic case we did in class. Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$, uniformizer ϖ and residue field $k_K = \mathbb{F}_q$. Consider the Lubin–Tate formal group $\mathcal{F}_\varpi$ associated to the polynomial $f(x) = x^q - \varpi_K$. Then adjoining the ϖ^∞-torsion of the formal group defines a tower of extension $K_n = K(\pi_n)$ with π_n a generator of $\mathcal{F}_\varpi[\varpi^n]$, i.e. $\pi_1 \in K^{\mathrm{alg}}$ is a nonzero root of $f(x) = 0$ and $f(\pi_i) = \pi_{i-1}$ for $i \geq 2$. Set $K_\infty := \bigcup_{n\geq 1} K_n$. Write $K_n^{\mathrm{unr}} := K_n K^{\mathrm{unr}}$ for $n \geq 1$. Then under the Artin map, we have a canonical isomorphism

$$\mathrm{Art}_K : \mathcal{O}_K^\times/(1 + \varpi^n\mathcal{O}_K)^\times \cong \mathrm{Gal}(K_n^{\mathrm{unr}}/K^{\mathrm{unr}}), \quad \text{and} \quad \mathrm{Art}_K : \mathcal{O}_K^\times \cong \mathrm{Gal}(K_\infty^{\mathrm{unr}}/K^{\mathrm{unr}})$$

For each K_n/K, compute the lower numbering ramification filtration and the upper numbering ones, and check that when $m \geq n$,

$$\mathrm{Gal}(K_m/K)^v\mathrm{Gal}(K_m/K_n)/\mathrm{Gal}(K_m/K_n) = \mathrm{Gal}(K_n/K)^v, \qquad \text{for every } v > 0.$$

Show that when taking the inverse limit, we get when $v > 0$

$$\mathrm{Gal}(K_\infty^{\mathrm{unr}}/K^{\mathrm{unr}})^v \cong (1 + \varpi^{\lceil v\rceil}\mathcal{O}_K)^\times.$$

Problem 1.8. Decide whether there exists an abelian extension of $K = \mathbb{Q}(\sqrt{3})$ of degree 3, ramified only at the primes of $\mathcal{O}_K$ above 5.

Remark: This is just a very explicit example. It is important to be able to apply abstract heavy theory to a concrete example.