# p-adic Riemann-Hilbert correspondence over the Robba ring

Ruochuan Liu

<u>Recall</u> $k/\mathbb{Q}_p$ fin, $X/k$ sm rig-ana var.

$$\begin{tikzcd} \left\{\begin{array}{c}\mathbb{Q}_p\text{-loc sys}\\ \text{on } X_{\text{ét}}\end{array}\right\} \arrow[r, "\text{Liu-Zhu}"] \arrow[d] & \left\{\begin{array}{c}\text{filtered flat connections on } X_{B_{dR}}\\ + \text{Gal}_k\text{-action} + \text{transversality}\end{array}\right\} . \arrow[d, "\text{Fil}^0", "\wr"'] \\ \left\{\begin{array}{c}B_{dR}^+\text{-loc sys}\\ \text{on } X_{\text{proét}}\end{array}\right\} \arrow[r, "\text{Gao-Min-Wang}"] & \left\{\begin{array}{c}\text{flat ``}t\text{-connection'' on } X_{B_{dR}^+}\\ + \text{Gal}_k\text{-action}\end{array}\right\} \end{tikzcd}$$

<u>Rmk</u>

- The functor $\mathbb{L} \longmapsto RH(\mathbb{L}) := R\nu_*(\hat{\mathbb{L}} \otimes \mathcal{O}B_{dR})$

$$= \nu_*(\hat{\mathbb{L}} \otimes \mathcal{O}B_{dR})$$

$$\nu : X_{\text{proét}}/X_{\mathbb{C}_p} \longrightarrow X_{\text{ét}}$$

- The connection is "mod $t$ nilp"

$$\Uparrow$$

Action of arith fundamental grp forces
the geom monodromy to be quasi-unipotent.

<u>Motivation</u>

- $B_{dR}^+ \simeq \widehat{\mathcal{O}}_{FF,\infty}$. Want to extend it to the FF curve.
- $(-)/k \rightsquigarrow (-)/\mathbb{C}_p$.

<u>Main result</u>

- $\mathcal{U} = \text{Spa}(R, R^+)$ affinoid perf'd $/\mathbb{C}_p$.

  $I = [r, s] \subset (0, \infty)$.

$\rightsquigarrow$ $\text{Spa}(\tilde{C}^I(\mathcal{U}), \tilde{C}^{I,+}(\mathcal{U}))$ rat'l localization of $\mathcal{U}$ w.r.t. $|[p^\flat]| \in p^I$.

$\rightsquigarrow$ period sheaf $\tilde{\mathcal{C}}^I$ ($\tilde{C}^I$ = Rabba ring).

Let $\mathfrak{X}$ p-adic sm formal sch / $\mathcal{O}_{\mathbb{C}_p}$.

Suppose it admits a flat lift $\tilde{\mathfrak{X}}$ to $A_{\inf}$.

$\rightsquigarrow$ $\tilde{\mathfrak{X}}^I$ = the base change of $\tilde{\mathfrak{X}}$ to $\tilde{\mathbb{C}}^I$ $(1 \in I)$.

$\tilde{X}$ = the base change of $\tilde{\mathfrak{X}}$ to $B_{dR}^+$.

<u>Thm A</u> (Chen-Liu-Wang-Zhu) $I \subset (\frac{1}{p-1}, p)$.

(1) $\exists$ period sheaf $\mathcal{O}\tilde{\mathbb{C}}^I$ on $X_{\text{proét}}$

$$\& \quad d : \mathcal{O}\tilde{\mathbb{C}}^I \longrightarrow \mathcal{O}\tilde{\mathbb{C}}^I \otimes \Omega^1_{\tilde{\mathfrak{X}}^I} \underbrace{\{-1\}}_{\substack{( \\ t\text{-connection.}}}$$

+ Poincaré lem.

(2) $$\left\{ \begin{matrix} \text{Small } \tilde{\mathbb{C}}^I\text{- loc Sys} \\ \text{on } X_{\text{proét}} \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{Small flat connections} \\ \text{on } \tilde{\mathfrak{X}}^I \end{matrix} \right\}.$$

$$\mathbb{L} \longmapsto (M, \nabla)$$

The functor: $\mathbb{L} \longmapsto R\nu_*(\mathbb{L} \otimes \mathcal{O}\tilde{\mathbb{C}}^I) = \nu_*(\mathbb{L} \otimes \mathcal{O}\tilde{\mathbb{C}}^I)$.

(3) $R\nu_*(\mathbb{L}) \simeq DR(M, \nabla)$

$\rightsquigarrow$ $R\Gamma(X_{\text{proét}}, \mathbb{L}) \simeq R\Gamma(\mathfrak{X}_{\text{ét}}, DR(M, \nabla))$.

<u>Rmk</u> (i) If $\mathbb{L}$ is Faltings-small, then $\hat{\mathbb{L}} \otimes \tilde{\mathbb{C}}^I$ is small in our setup.

(ii) Smallness is used for vanishing of higher cohom.

(iii) Where is the Frob on connections ?

<u>Thm B</u> (CLWZ) Same $\mathfrak{X}$.

(1) $\exists$ overconvergent dR period sheaf $\mathcal{O}\mathbb{B}_{dR}^{\dagger,+}$ & $d$

(2) $$\left\{ \begin{matrix} \text{Small } B_{dR}^+\text{-local} \\ \text{Systems on } X_{\text{proét}} \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{Small flat Connections} \\ \text{on } \tilde{X} \end{matrix} \right\}.$$

(3) Similar for coh.

(4) Compatible w/ Faltings's p-adic Simpson.

& RH for small $\tilde{\mathbb{C}}^{I}$-local system.

<u>Rmk</u>

- $\mathcal{O}\tilde{\mathbb{C}}^{I}$ cannot be mapped to $\mathcal{O}\mathbb{B}_{dR}$.
- Add more "analytic structure" to take smallness into account.

(In Yupeng Wang's thesis: $\underset{\substack{\text{locally polynomial}\\ \text{ring}}}{\mathcal{O}\mathbb{C}} \rightsquigarrow \underset{\substack{\text{locally + radius of conv}\\ \text{condition.}}}{\mathcal{O}\mathbb{C}^{\dagger}}$)

<u>Sketch of $\mathcal{O}\tilde{\mathbb{C}}^{I}$</u> $\quad \mathcal{U} = \mathrm{Spa}(S, S^{+}) \in X_{\mathrm{pro\acute{e}t}}$.

$$\rightsquigarrow \Sigma_{\mathcal{U}} := \left\{ \begin{array}{l} \mathcal{Y} = \mathrm{Spf}(\mathcal{R}) \to \mathfrak{X} \\ \text{s.t. } \mathcal{U} \to X \text{ factors over } Y \to X \end{array} \right\} \subset \mathfrak{X}_{\acute{e}t}.$$

$$\mathcal{Y} \in \Sigma_{\mathcal{U}} \rightsquigarrow \theta_{\mathcal{Y}} : \underset{\text{lift to Ainf}}{\tilde{\mathcal{R}}} \ \underset{(p,\xi)\text{-complete tensor product}}{\hat{\otimes}_{A_{\inf}}} A_{\inf}(\mathcal{U}) \longrightarrow S^{+}$$

<u>Fact</u> $\ker \theta_{\mathcal{Y}}$ is finitely gen'd.

Put $\tilde{S}_{\mathcal{Y}}^{I,(+)} := \tilde{\mathcal{R}}^{I,(+)} \hat{\otimes}_{\tilde{\mathbb{C}}^{I,(+)}} \tilde{\mathbb{C}}^{I,(+)}(\mathcal{U})$.

Fix $1 < \lambda < \frac{1}{|\xi|_{I}}$.

$$\rightsquigarrow \mathrm{Spa}(\tilde{S}_{\mathcal{Y}}^{\lambda,I}, \tilde{S}_{\mathcal{Y}}^{\lambda,I,+}) := \text{rat'l localization of}$$
$$\mathrm{Spa}\left(\tilde{S}_{\mathcal{Y}}^{I}\left[\frac{\ker \theta_{\mathcal{Y}}}{\xi}\right], \tilde{S}_{\mathcal{Y}}^{I,+}\left[\frac{\ker \theta_{\mathcal{Y}}}{\xi}\right]\right) \text{ w.r.t. } \left|\frac{\ker \theta_{\mathcal{Y}}}{\xi}\right| \le \lambda$$

Take limit for $\lambda \to 1^{+}$ & sheafification.

<u>Locally</u> $\quad \mathfrak{X} = \mathrm{Spf}\, \mathcal{R}, \quad \psi : \mathcal{O}_{\mathbb{C}_p}\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle \longrightarrow \mathcal{R}^{\text{étale}}$

$$\mathcal{O}\tilde{\mathbb{C}}^{I}\big|_{X_{\infty}} = \bigcup_{\lambda \to 1^{+}} \tilde{\mathbb{C}}^{I}\left\langle \frac{Y_1}{\lambda}, \dots, \frac{Y_d}{\lambda} \right\rangle, \quad Y_i := \frac{T_i - [T_i^{\flat}]}{\xi T_i}.$$

<u>Applications</u> e.g. (Le Bras) $X = \mathbb{A}^n_{\mathbb{C}_p}$. $\mathbb{L} = \mathbb{B}^+_{dR}$.

$$R\Gamma(\mathbb{A}^n_{\mathbb{C}_p}, \mathbb{B}^+_{dR}) \simeq R\Gamma(\mathcal{O}_{\tilde{X}} \to \Omega^1_{\tilde{X}}\{-1\} \to \Omega^2_{\tilde{X}}\{-2\} \to \cdots).$$

$$\Rightarrow \quad H^i(X_{proet}, \mathbb{B}^+_{dR}) \simeq H^i(\mathcal{O}_{\tilde{X}}(\tilde{X}) \to \Omega^1_{\tilde{X}}(\tilde{X})\{-1\} \to \cdots)$$

$$\simeq \ker(\Omega^i(X) \xrightarrow{d} \Omega^{i+1}(X))$$

$$(\mathcal{O}_{\tilde{X}}(\tilde{X}) \to \Omega^1_{\tilde{X}}(\tilde{X}) \to \Omega^2_{\tilde{X}}(\tilde{X}) \to \cdots \text{ is exact}).$$

Work in progress w/ Yu, Wang, Zhu:

$\mathfrak{X} = \mathfrak{X}_0 \otimes_{\mathcal{O}_k} \mathcal{O}_{\mathbb{C}_p}$, $k/\mathbb{Q}_p$ finite unramified.

$$H^i(X_{et}, L\eta_t R\nu_* \tilde{\mathbb{C}}^I) \simeq H^i(\mathcal{O}_{\mathfrak{X}^I} \to \Omega^1 \to \Omega^2 \to \cdots)$$

$$\simeq \tilde{\mathbb{C}}^I \otimes H^i_{dR}(X_0)$$

$$\simeq \tilde{\mathbb{C}}^I \otimes H^i_{rig}((\mathfrak{X}_0)_s/\mathcal{O}_k).$$

Grosse-Klönne.

For certain $\mathfrak{X}_0$, $H^i(X_{et}, L\eta_* R\nu_* \tilde{\mathbb{C}}^I)$ gives rise to

$\mathcal{E}(H^i_{rig}((\mathfrak{X}_0)_s/\mathcal{O}_k))$ vect bdl on FF curve.

- $\mathbb{A}^1$-invariance of $H^i(X_{et}, L\eta_t R\nu_* \tilde{\mathbb{C}}^I)$.