Irreducible components of affine Deligne-Lusztig varieties and orbital integrals

Rong Zhou

(Joint with Yihang Zhu).

<u>Motivation</u> p prime, $N \geq 3$, $p \nmid N$.

$X_0(N)$ level $\Gamma_0(N)$ modular curve $/\mathbb{Z}_{(p)}$.

classifies (E, C), E ell curve + $C \subseteq E[N]$ cyclic order N.

Supersingular locus: $X_0(N)^{ss}(\bar{\mathbb{F}}_p) = D^\times \backslash X_\mu(b) \times GL_2(\mathbb{A}_f^\infty)/K^p$

where
- D quat alg $/\mathbb{Q}$ ramfied at p, ∞.
- $K^p \subseteq GL_2(\mathbb{A}_f^p)$ compact open.
- $L = W(\bar{\mathbb{F}}_p)[\frac{1}{p}] \supset \mathcal{O}_L$ ring of int

  σ Frob.

$b = \begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix}$, $\mu(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$.

$\rightsquigarrow X_\mu(b) = \{ g \in GL_2(L)/GL_2(\mathcal{O}_L) \mid g^{-1} b \sigma(g) \in GL_2(\mathcal{O}_L)\mu(p)GL_2(\mathcal{O}_L) \}$

ADLV

$\updownarrow$ 1-1

Dieudonné mods in D-isocrystal $(L^2, b\sigma)$.

<u>Fact</u> $X_\mu(b) \cong D_{\mathbb{Q}_p}^\times / \mathcal{O}_{D_{\mathbb{Q}_p}}^\times$ (local description).

unique cpt open.

Then $X_0(N)^{ss} = D^\times \backslash (D \otimes \mathbb{A}_f)^\times / K$ (by Serre).

In general, $X_\mu(b)$ encodes the geometry of basic locus of Sh vars.

§ Chen–X. Zhu Conj

F local field $\supseteq \mathcal{O}_F \twoheadrightarrow \mathcal{O}_F/\varpi_F = \mathbb{F}_q$

$L \supseteq \mathcal{O}_L$, $L := \widehat{F^{ur}} = \breve{F}$, $\sigma \in \mathrm{Aut}(L/F)$ Frob.

Fix $G/\mathcal{O}_F$ reductive grp (split for simplicity)

$T \subseteq B$ max split torus, B Borel.

$\mu \in X_*(T)_+$, $b \in G(L)$.

$\rightsquigarrow$ $X_\mu(b) = \{ g \in G(L)/G(\mathcal{O}_L) \mid g^{-1} b \sigma(g) \in G(\mathcal{O}_L) \cdot \mu(\varpi_F) G(\mathcal{O}_L) \}$

ADLV in general.

Rmk (1) $X_\mu(b)$ is a (perfect) scheme $/\bar{\mathbb{F}}_q$ (X. Zhu, Bhatt–Scholze)

(in mixed char only
actual sch in equal char)

abs Fr is an isom

(2) Only depends on $[b] \in B(G) := G(L)/\sim$ $\leftarrow$ $(g \sim h^{-1} g \sigma(h))$

Fact $B(G)$ classified by $b \mapsto (\bar{\nu}_b \in X_*(T) \otimes \mathbb{Q}, \kappa_G(b) \in \pi_1(G))$

Newton pt Kottwitz pt

Dimension formula (well-known)

$$d := \dim X_\mu(b) = \langle \mu - \bar{\nu}_b, \rho \rangle - \tfrac{1}{2} \mathrm{def}_G(b)$$

"defect" (known).

Let $\Sigma := \{\text{irred comp in } X_\mu(b) \text{ of dim } d\}$

Usually $\# \Sigma = \infty$.

Def $J_b(F) := \{ g \in G(L) \mid g^{-1} b \sigma(g) = b \}$ called σ-centralizer

$\rightsquigarrow$ $J_b(F) \curvearrowright X_\mu(b)$ & Σ.

Conj (Chen-Zhu) $\#(J_b(F)\backslash\Sigma) = \dim \underbrace{V_\mu(\lambda_b)}_{\text{wt space}}$

$\#$ (irred comps of $X_\mu(b)$)

Explanation

Let $\hat{G}$ dual grp, $\hat{T}$ dual torus, $\mu \in X_*(T) = X^*(\hat{T})$

$\rightsquigarrow$ V_μ $\hat{G}$-rep of highest wt μ

λ_b = "best integral approx to $\bar{\nu}_b$"

$\lambda_b \in X_*(T)$

Also define defect $\quad \mathrm{def}_G(b) := \mathrm{rank}\, G - \mathrm{rank}\, J_b$.

Previous works:

Xiao-Zhu: case $\mathrm{def}_G(b) = 0$

($\Rightarrow$ Tate conj for Sh vars).

motivation of Chen-Zhu

Hamacher-Viehmann: μ minuscule, G split

Nie: μ = sum of dominant minuscule coweights

(in particular type A).

Thm (Zhou-Zhu) Conj holds for all (G, μ, b).

E.g. $E/\mathbb{Q}$ tot real, deg 2. p inert in E.

$\mathcal{Y}_0(N)/\mathbb{Z}_{(p)}$ Hilbert modular surface

$\mathcal{Y}_0(N)^{ss}_{\bar{\mathbb{F}}_p} = \mathcal{Y}_1 \sqcup \mathcal{Y}_2$ two families

$\mathcal{Y}_i = \mathbb{P}^1$-fibration over $D'\backslash(D'\otimes\mathbb{A}_f)^\times/K'$

where D' quat alg$/E$ ram at $\infty_1, \infty_2 | \infty$.

Then $\quad \# J_b(F)\backslash\Sigma = 2$.

§ Idea of proof

Previous works: reduce to "superbasic" case
essentially: $G = GL_n$

But reduction can encounter complicated combinatorics.

<u>New idea</u> X quasi-proj var $/\mathbb{F}_q$, dim d,
w/ C irred components of dim d.

<u>Lang–Weil bound</u> ("weak Weil conj")

$$X(\mathbb{F}_{q^s}) = C \cdot q^{sd} + o(q^{sd}).$$

For $s \geq 1$, F_s/F unram ext of deg s.

May assume: G simple adjoint, <u>not</u> of type A
(to avoid comb issue).

- b basic ($\bar{\nu}_b = 0$, J_b inner form of G).
- b descent ($b \in G(F_{s_0})$, $\gamma_s := \mathrm{Nm}(b) = b \cdot \sigma(b) \cdots \sigma^{s-1}(b) = 1$
$X_\mu(b)$ is def'd over $\mathbb{F}_{q^s}$.)

<u>Prop</u>
$$TO_b(f_{\mu,s}) = \sum_{Z \in J_b(F) \backslash \Sigma} \frac{q^{sd}}{\mathrm{vol}(\mathrm{Stab}_Z(J_b(F))} + o(q^{sd}).$$

where $f_{\mu,s} \in \mathcal{H}_s := \mathcal{H}(G(F_s) /\!/ G(\mathcal{O}_{F_s}), \mathbb{C})$ $\left(\begin{array}{c}\text{indicator functions of} \\ G(\mathcal{O}_{F_s})\, \mu(\varpi_F)\, G(\mathcal{O}_{F_s})\end{array}\right)$

spherical Hecke alg

$$TO_b(f_{\mu,s}) := \int_{J_b(F) \backslash G(F_s)} f(g^{-1} b \sigma(g))\, dg$$

Haar measure $\mathrm{vol}(G(\mathcal{O}_{F_s})) = 1$, $\mathrm{vol}(G(\mathcal{O}_F)) = 1$.

Need to understand:

(a) Asymptotics of $TO_b(f_{\mu,s})$

(b) $\mathrm{vol}(\mathrm{Stab}_Z(J_b(F)))$.

(A) Base change fundamental lemma:

$$TO_b(f_{\mu,s}) = O_{\gamma_s}(BC(f_{\mu,s})) = BC(f_{\mu,s})(1).$$

($BC: \mathcal{H}_s \longrightarrow \mathcal{H}_1 \leftarrow$ to understand this).

Rmk (1) stability is ok.

(2) BCFL is only known for p-adic fields.

But can show truth of Conj is indep of char F.

<u>Satake transform</u>

$$\begin{tikzcd} \mathrm{Sat}_s : \mathcal{H}_s \arrow[r, "\sim"] \arrow[d, "BC"'] & \mathbb{C}[X_*(T)]^{W_0} \arrow[d, "BC"] \\ \mathrm{Sat}_1 : \mathcal{H}_1 \arrow[r, "\sim"] & \mathbb{C}[X_*(T)]^{W_0} \end{tikzcd}$$

W_0 Weyl.

easy to compute BC on RHS

2 bases of $\mathbb{C}[X_*(T)]^{W_0}$ indexed by $X_*(T)_+$

$$\mathrm{Sat}_s(f_{\mu,s}) := f_{\mu,s}, \quad m_\mu = \sum_{\substack{\lambda \in \text{Weyl orbit} \\ \text{of } \mu}} e^\lambda$$

<u>Upshot</u> $BC(m_\mu) = m_{s\mu}$.

We show $TO_b(f_{\mu,s}) + o(q^{sd}) = TO_b(\chi_\mu)\cdot q^{s\langle\mu,\rho\rangle}$ (χ_μ = char of V_μ)

$$= q^{s\langle\mu,\rho\rangle} BC(\chi_\mu)(1)$$

$$= q^{s\langle\mu,\rho\rangle} \sum_{\lambda\le\mu} \dim V_\mu(\lambda)\, m_{s\lambda}(1).$$

$m_{s\lambda}(1) = M^\circ_{s\lambda}(q^{-1}) \leftarrow$ related to Kazhdan-Lusztig polynomials.

<u>Key computation</u> $\lambda \neq \lambda_b^+$, $q^{\langle\mu,\rho\rangle} M^\circ_{s\lambda}(q^{-1}) = o(q^{sd})$.

$$(*) \sum_{Z \in J_b(F)\backslash\Sigma} \frac{q^{sd}}{\mathrm{vol}(\mathrm{Stab}_Z(J_b(F)))} = \dim V_\mu(\lambda_b) \cdot q^{s\langle\mu,\rho\rangle} M^\circ_{s\lambda_b^+}(q^{-1}).$$

(B) We show $\exists\, R(T) \in \mathbb{Z}(T)$ which only depends on affine root system of G

s.t. (i) $R(q) = $ LHS of $(*)$,

(ii) $R(0) = \# J_b(F)\backslash\Sigma$ number of irred comps in $X_\mu(b)$

- $R(q) = (\dim V_\mu(\lambda_b)) \cdot \underbrace{\lim_{s\to\infty} q^{\frac{1}{2}\mathrm{def}_G(b)} M^\circ_{s\lambda_b^+}(q^{-1})}_{= \, S(q)}$

$S(q)$ with $S(T) \in \mathbb{Z}(T)$ s.t. $S(0) = 1$ (use $\mu = \lambda_b^+$).

- $R(0) = \dim V_\mu(\lambda_b)$