## LECTURE 2 - FIBER SEQUENCES

The goal of this lecture is to introduce the fiber sequence generated by a map and relate it to the long exact sequence of the homotopy groups.

**Definition 1.** A surjective map  $p: E \to B$  is a fibration if it satisfies the covering homotopy property. This means that if  $h \circ i_0 = p \circ f$  in the diagram

$$Y \xrightarrow{f} E$$

$$\downarrow i_0 \qquad \downarrow \tilde{h} \qquad \uparrow \qquad \downarrow p$$

$$Y \times I \xrightarrow{h} B$$

then there exists  $\tilde{h}$  that makes the diagram commute.

**Proposition 2.** The pullback of a fibration is a fibration. This means that if  $p: E \to B$  is a fibration and  $g: A \to B$  is any map, then the induced map  $A \times_q E \to A$  is a fibration.

**Proposition 3.** If  $p: E \to B$  is a covering, the p is a fibration with a unique path lifting function s.

**Definition 4.** For  $f: X \to Y$ , we define the mapping path space Nf to be

$$Nf = X \times_f Y^I = \{(e,\beta) | \beta(1) = p(e)\} \subset X \times Y^I.$$

Observe that f coincides with the composite

$$X \xrightarrow{\nu} NF \xrightarrow{\rho} Y$$
,

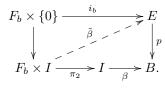
where  $\nu(x)=(x,c_{f(x)})$  and  $\rho(x,\chi)=\chi(1)$ . Let  $\pi:Nf\to X$  be the projection. then  $\pi\circ\nu=id$  and  $id\simeq\nu\circ\pi$  since we can define a deformation  $h:Nf\times I\to Nf$  of Nf onto  $\nu(X)$  by setting

$$h(x, \chi)(t) = (x, \chi_t)$$
, where  $\chi_t(s) = \chi((1 - t)s)$ .

**Definition 5.** It can be checked that  $\rho: Nf \to Y$  satisfies the covering homotopy property. We call it the *fibrant replacement* of f.

Translation of fibers along paths in the base space played a fundamental role in the study of covering spaces. Fibrations admit an up to homotopy version of that theory that well illustrates the use of the covering homotopy property (CHP).

Let  $p: E \to B$  be a fibration with fiber  $F_b$  over  $b \in B$  and let  $i_b: F_b \to E$  be the inclusion. For a path  $\beta: I \to B$  from b to b', the CHP gives a lift  $\tilde{\beta}$  in the diagram



**Definition 6.** Note that  $\tilde{\beta}_1$  maps  $F_b$  to the fiber  $F_{\beta(1)} = F_{b'}$ . We call

$$[\tilde{\beta}_1] \in [F_b, F_{b'}]$$

the translation of fibers along the path class  $[\beta]$ .

**Exercise 7.** The definition claims that  $[\tilde{\beta}_1]$  does not depend on the choice of  $\beta$  in its path class. Prove this fact by considering a diagram similar to the above one.

**Definition 8.** The dual notion of cones and suspensions are paths and loops. The path space of X is defined by PX = F(I, X).

**Definition 9.** For a based map  $f: X \to Y$ , we define the homotopy fiber Ff to be

$$Ff = X \times_f PY = \{(x, \chi) | f(x) = \chi(1)\} \subset X \times PY.$$

Equivalently, Ff is the pullback displayed in the diagram

$$Ff \longrightarrow PY$$

$$\downarrow p_1$$

$$X \longrightarrow Y$$

where  $\pi(x,\chi) = x$ . As a pullback of a fibration,  $\pi$  is a fibration.

**Definition 10.** Let  $\iota: \Omega Y \to Ff$  be the inclusion specified by  $\iota(\chi) = (*, \chi)$ . The sequence

$$\cdots \to \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{-\Omega \iota} \Omega F f \xrightarrow{-\Omega \pi} \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{\iota} F f \xrightarrow{\pi} X \xrightarrow{f} Y$$

is called the fiber sequence generated by the map f.

**Theorem 11.** For any based space Z, the induced sequence

$$\cdots \to [Z,\Omega Ff] \to [Z,\Omega X] \to [Z,\Omega Y] \to [Z,Ff] \to [Z,X] \to [Z,Y]$$

is an exact sequence of pointed sets, or of groups to the left of [Z, LoopY], or of Abelian groups to the left of  $[Z, \Omega^2Y]$ .

Corollary 12. If we let  $Z = S^n$ , then we get the long exact sequence

$$\cdots \to \pi_n(Ff) \to \pi_n X \to \pi_n Y \to \pi_{n-1}(Ff) \to \cdots \to \pi_0(Ff) \to \pi_0 X \to \pi_0 Y.$$