Igusa stacks and exotic Hecke correspondence (I)

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(Joint with Daniels, Kim, Zhang & with Sempliner).

§1 Introduction / Motivation

Fix g & a prime p.

Consider $\mathcal{A}_g$ moduli of ppavs $(A, \lambda : A \xrightarrow{\sim} A^t)$ / $\mathbb{Z}_p$.

Def (Caraiani-Scholze)

Let $(\mathbb{X}, \nu)$ be a polarized p-divisible grp / $\bar{\mathbb{F}}_p$

then $$\begin{array}{c} IG_{(\mathbb{X},\nu)} \\ \downarrow \\ \mathcal{A}_{g,\bar{\mathbb{F}}_p} \end{array}$$

is the moduli space of isoms

$$T \longmapsto \mathrm{Isom}((A_T[p^\infty], \lambda_T), (\mathbb{X}_T, \nu_T))$$

(isoms of p-div grps, respecting polarizations up to a scalar in $\mathbb{Z}_p^\times$.)

[CS] $IG_{(\mathbb{X},\nu)}$ is a perfect sch, and the image of $IG_{(\mathbb{X},\nu)} \to \mathcal{A}_{g,\bar{\mathbb{F}}_p}$ is a locally closed smooth equidim subset $C_{(\mathbb{X},\nu)} \subset \mathcal{A}_{g,\bar{\mathbb{F}}_p}$,

and $IG_{(\mathbb{X},\nu)} \to C^{\mathrm{perf}}_{(\mathbb{X},\nu)}$ is a proét torsor for the profin grp $\mathrm{Aut}_{\bar{\mathbb{F}}_p}(\mathbb{X}, \nu)$.

$IG_{(\mathbb{X},\nu)} \to C_{(\mathbb{X},\nu)}$ is an $\underline{\mathrm{Aut}}(\mathbb{X}, \nu)$-torsor

(*) If $(\mathbb{X}, \nu) \dashrightarrow (\mathbb{X}', \nu')$ is a quasi-isogeny, then there is a natural isom

$$\begin{tikzcd} IG(\mathbb{X},v) \arrow[rr, "\sim"] \arrow[dr] & & IG(\mathbb{X}',v') \arrow[dl] \\ & \mathcal{A}_{g,\bar{\mathbb{F}}_p} & \end{tikzcd}$$

Can reinterpret the def'n as follows:

$$\begin{tikzcd} IG(\mathbb{X},v) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \operatorname{Spec} \bar{\mathbb{F}}_p \arrow[d] \arrow[dd, bend left=60] \\ \mathcal{A}_{g,\bar{\mathbb{F}}_p} \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\ulcorner"] & \left\{ \begin{array}{c} \text{moduli stack of} \\ \text{polarized p-div grps} \end{array} \right\} \arrow[d] \\ \left\{ \begin{array}{l} (A,\lambda), \text{ up to} \\ \text{p-power isogeny} \end{array} \right\} \arrow[r] & \left\{ \begin{array}{c} \text{moduli stack of polarized p-div grps} \\ \text{up to quasi-isogeny} \end{array} \right\} \end{tikzcd}$$

[CS] The outer square is also a pullback.

(*) is proved by rewriting the def'n of $IG(\mathbb{X},v)$.

$$IG(\mathbb{X},v)(T) = \left\{ \begin{array}{l} (A,\lambda) \text{ over } T \text{ up to p-power isog} \\ + \text{ q-isog } (A[p^\infty],\lambda) \dashrightarrow (\mathbb{X}_T, v_T) \end{array} \right\}$$

<u>Note</u> Moduli $\mathcal{A}_{g,\bar{\mathbb{F}}_p}$ is the 1st information of Igusa stack.

It's fruitful to restrict to perfect test objects
(following Xiao-Zhu.)
and then can rewrite the diagram

$$\begin{tikzcd}
IG_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \operatorname{Spec} \overline{\mathbb{F}}_p \arrow[d, "b"] \\
\mathcal{A}_{g,\overline{\mathbb{F}}_p}^{perf} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Sht}^W_{GSp_{2g},\{\mu\}} \arrow[d] \\
{[\mathcal{A}^{perf}_{g,\overline{\mathbb{F}}_p}/p\text{-isog}]} & GSp_{2g}\text{-Isoc}
\end{tikzcd}$$

$\mathrm{Sht}^W_{GSp_{2g},\{\mu\}} = \{$Dieudonné mods of polarized p-div grps of height $2g\}$

<u>Def'n</u> A <u>GL_n-shtuka</u> over a perfectoid ring R is
a rank n proj $W(R)$-mod M
$+ \quad \varphi_M : \varphi^* M[\frac{1}{p}] \xrightarrow{\sim} M[\frac{1}{p}]$.

For this to be a (covariant) Dieudonné mod,
need $M \subset \varphi_M(\varphi^* M) \subset \frac{1}{p} M$.

<u>Exercise</u> The following is a pullback diagram

$$\begin{tikzcd}
\mathcal{A}_{g,\overline{\mathbb{F}}_p} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Sht}_{GSp_{2g},\{\mu\}} \arrow[d] \\
{[\mathcal{A}_{g,\overline{\mathbb{F}}_p}/p\text{-isog}]} \arrow[r] & GSp_{2g}\text{-Isoc}_{\le\mu}
\end{tikzcd}$$

$$[\mathcal{A}_{g,\overline{\mathbb{F}}_p}/p\text{-isog}] = IG_S^{perf}$$

Here $GSp_{2g}\text{-Isoc} = \bigcup_{[b]\in B(GSp_{2g})} [1/J_b(\mathbb{Q}_p)]$

$$\cup\!\!\text{I} \qquad\qquad \cup\!\!\text{I}$$

$GSp_{2g}\text{-Isoc}_{\le\mu} = \bigcup_{[b]\in B(GSp_{2g},-\mu)} [1/J_b(\mathbb{Q}_p)]$

$\rightsquigarrow$ Igusa stack

$$IGS^{perf} = \bigcup_{\substack{\text{symplectic} \\ \text{Newton polygons } b}} [IG^b / \underline{J_b(\mathbb{Q}_p)}]$$

<u>Slogan</u> The Igusa stack glues together all the perfect Igusa varieties.

Define $X(\mu, b)$ to fit in

$$\begin{tikzcd} X(\mu,b) \arrow[r] \arrow[d] & \operatorname{Spec} \overline{\mathbb{F}}_p \arrow[d, "b"] \\ Sht_{GSp_{2g},\mu} \arrow[r] & GSp_{2g}\text{-Isoc} \end{tikzcd}$$

<u>Exercise</u> Show that there is an isom

$$A_{g,\overline{\mathbb{F}}_p,[b]} = [IG_b \times X(\mu,b) / J_b(\mathbb{Q}_p)]$$

called Mantovan product formula.

<u>Imprecise thm</u> (DvHKZ)

For (G,X) of Hodge type we have perfect Igusa stacks (perfect special fibres of the Pappas-Rapoport integral models at parahoric level).

<u>§2 Mantovan II</u>

Consider (infin level) $A^{an}_{g,\infty,\mathbb{Q}_p}$ d'après Scholze

$$\begin{tikzcd} A^{an}_{g,\infty,\mathbb{Q}_p} \arrow[d, "GSp_{2g}(\mathbb{Z}_p)\text{-torsor}"] \\ A^{an}_{g,\mathbb{Q}_p} \end{tikzcd}$$

$$\rightsquigarrow \quad \begin{array}{ccc} \mathcal{A}^{an}_{g,\infty,\mathbb{Q}_p} & \xrightarrow{\pi_{HT}} & \mathrm{LagGr}^{ad} \subset \mathrm{Gr}(g,2g)^{ad} \\ \downarrow & & \\ \mathcal{A}^{an}_{g,\mathbb{Q}_p} & & (g=1,\ \mathrm{LagGr}^{ad} = \mathbb{P}^{1,ad}_{\mathbb{Q}_p}.) \end{array}$$

<u>$g=1$</u> $\mathcal{A}_{1,\mathbb{Q}_p}$ is the moduli of ell curves E

& $\mathcal{A}^{an}_{1,\mathbb{Q}_p}$ is parametrizing $(E, \eta_p : T_pE \xrightarrow{\sim} \mathbb{Z}_p^{\oplus 2})$.

<u>Scholze</u> For $E \xrightarrow{\pi} S$, S perf'd space,

$\exists$ a ses

$$1 \to R^1\pi_*\mathcal{O}_E \otimes_{\mathcal{O}_S} \hat{\mathcal{O}}_S \to R^1\pi_*\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \hat{\mathcal{O}}_S$$
$$\to \pi_*\Omega^1_{E/S} \otimes_{\mathcal{O}_S} \hat{\mathcal{O}}_S \to 1.$$

<u>Caraiani-Scholze</u> If we restrict to the good reduction locus $\mathcal{A}^{an}_{g,\infty,\mathbb{Q}_p}$, then the fibers of π_{HT} "are Igusa varieties".

To be precise, for $x \in \mathrm{LagGr}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$

there is a b s.t.

$$(\pi^\circ_{HT})^{-1}(x) \hookleftarrow (W(\mathrm{IG}_b) \otimes_{\mathrm{Spf}\,\mathbb{Z}_p^*} \mathrm{Spf}\,\mathcal{O}_{\mathbb{C}_p})^1_{ad}$$

open immersion keeping rk 1 pts.

<u>Which b?</u>

Well, there is a map
$$\begin{array}{c} (\mathrm{LagGr})^\diamond \\ \downarrow \mathrm{BL} \\ \mathrm{Bun}_{\mathrm{GSp}_{2g}} \end{array}$$

and $\mathrm{Bun}_{\mathrm{GSp}_{2g}}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p}) \cong B(\mathrm{GSp}_{2g})$.

($X^\diamond$ means thinking of X as a sheaf on perf'd spaces of char p.)

$g=1$. $\quad \mathbb{P}^1_{\mathbb{C}_p} = \underbrace{\mathbb{P}^1(\mathbb{Q}_p)}_{\text{ordinary}} \cup (\underbrace{\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{Q}_p)}_{\text{supersingular}})$

$b = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}.$

$\mathrm{Bun}_{GSp_{2g}}$ is the moduli stack of GSp_{2g} bundles on the Fargues–Fontaine curve.

<u>Conj</u> (Scholze)

$$\begin{tikzcd}
\mathcal{A}_{g,\infty,\mathbb{C}_p}^{\circ,\diamond} \arrow[r, "\pi^\circ_{HT}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{LanGr}^\diamond \arrow[d] \\
\mathrm{IGS} \arrow[r] & \mathrm{Bun}_{GSp_{2g}}
\end{tikzcd}$$

For GSp_{2g}, this is a theorem of Zhang in PEL + good red'n case).

<u>Fact</u> IGS is an Artin v-stack which is ℓ-coh sm of ℓ-dim 0 for all $\ell \neq p$.

Now can produce $R\pi^\circ_{HT,!}\mathbb{F}_\ell$, $R\pi^\circ_{HT,!}\overline{\mathbb{Q}}_\ell$.

<u>Thm</u> (DvHKZ) (s.t. admitting Pappas-Rapoport int models)

For (G,X) of Hodge type, this diagram exists

$$\begin{tikzcd}
Sh_{K^p}(G,X)^{\circ,\diamond} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Gr_{G,\mu^{-1}} \arrow[d, "BL"] \\
\mathrm{IGS}(G,X) \arrow[r] & \mathrm{Bun}_G
\end{tikzcd}$$

Relation $K_p \subset G(\mathbb{Q}_p)$

$$\begin{tikzcd} \mathrm{Sh}_{K^p}(G,X)^{\circ,\diamond} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Gr}_{G,\mu^{-1}} \arrow[d] \\ \mathrm{Sh}_K(G,X)^{\circ,\diamond} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & {[\mathrm{Gr}_{G,\mu^{-1}}/K_p]} \arrow[d] \\ \mathrm{IGS} \arrow[r] & \mathrm{Bun}_G \end{tikzcd}$$

There's a stack $\mathcal{G}(\mathbb{Z}_p) = K_p$

$$\mathrm{Sht}_{\mathcal{G},\{\mu\}} \longrightarrow \mathrm{Spd}\,\mathbb{Z}_p$$

s.t. $\mathrm{Sht}_{\mathcal{G},\{\mu\},\mathbb{Q}_p} = [\mathrm{Gr}_{G,\mu^{-1}}/K_p]$

and $\mathrm{Sht}^{\mathrm{red}}_{\mathcal{G},\{\mu\}} = \mathrm{Sht}^{W}_{\mathcal{G},\{\mu\}}$ ($= \mathrm{Sht}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$ in Xiao-Zhu).

[PR] $\exists$ an integral model $\mathcal{S}_K(G,X) \longrightarrow \mathrm{Spec}\,\mathbb{Z}_p$

s.t. $\mathcal{S}_K(G,X)^{\diamond} \longrightarrow \mathrm{Sht}_{\mathcal{G},\{\mu\}}$.

and $\mathcal{S}_K(G,X)^{\diamond}_{\mathbb{Q}_p} = \mathrm{Sh}_K(G,X)^{\circ,\diamond}$

Thm (DvHKZ)

$$\begin{tikzcd} \mathcal{S}_K(G,X)^{\diamond} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Sht}_{\mathcal{G},\{\mu\}} \arrow[d, "\mathrm{BL}^{\circ}"] \\ \mathrm{IGS}(G,X) \arrow[r] & \mathrm{Bun}_G \end{tikzcd}$$

The perfect version can then be recovered by applying "red" functor (Ian Gleason) (evaluating on $\mathrm{Spd}(R,R^+)$.)

Igusa stacks for Shimura varieties of Hodge type (II)

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(Joint with Daniels, Kim, Zhang)

<u>Recap</u> Fix a prime p. Let (G,X) be a Shimura datum of Hodge type with reflex field $\mathbb{E}$, and fix $v|p$ of $\mathbb{E}$ with completion $E = \mathbb{E}_v$.

Let μ be the (conj class over $\bar{\mathbb{Q}}_p$) of Hodge characters.

Fix $\mathcal{G}/\mathbb{Z}_p$ parahoric.

- $K_p = \mathcal{G}(\mathbb{Z}_p)$ & $K^p \subset G(\mathbb{A}_f^p)$. Let $K = K_p K^p$.

Write $Sh_K(G,X) \longrightarrow \operatorname{Spec} E$ for the base change $\mathbb{E} \to E$ of the Shimura var.

(*) <u>Technical ass'n</u> $(G,X,\mathcal{G})$ is "of global Hodge type", or $\mathcal{G}(\breve{\mathbb{Z}}_p)$ is the stabilizer of $x \in \mathcal{B}^e(G,\breve{\mathbb{Q}}_p)$.

e.g. can pretend $G/\mathbb{Z}_p$ reductive.

- <u>Pappas - Rapoport</u>

There is a system $\{S_K(G,X)\}_{K^p}$ of flat normal integral models over $\mathcal{O}_E$ of $\{Sh_K(G,X)\}_{K^p}$, uniquely characterized and functorial.

The most important part of the characterization is a map

$$S_K(G,X)^\diamond \longrightarrow Sht_{\mathcal{G},\mu} \qquad / \operatorname{Spd} \mathcal{O}_E$$

Here $\mathcal{S}_K(G,X)^\diamond$ is (the sheafification of)

$$\begin{array}{ccc} \mathrm{Perf}_{\mathbb{F}_p} & & p\text{-adic completion} \\ (S = \mathrm{Spd}(R,R^+)) \quad (R,R^+) & \longmapsto & (S^\#, \ \mathrm{Spf}\, R^{\#+} \to \widehat{\mathcal{S}}_K(G,X)). \end{array}$$

and $\mathrm{Sht}_{\mathcal{G}}$ is defined last time.

For technical conventions, we always work with

$$\mathcal{S}_{K_p}(G,X) = \varprojlim_{K^p} \mathcal{S}_{K^pK_p}(G,X).$$

<u>Conj</u> (Scholze)

$\exists$ v-sheaf $\mathrm{IGS}(G,X)$ on $\mathrm{Perf}_{\mathbb{F}_p}$ sitting in the pullback diagram

$$\begin{array}{ccc} \mathcal{S}_{K_p}(G,X)^\diamond & \xrightarrow{\pi_{crys}} & \mathrm{Sht}_{\mathcal{G},\mu} \\ {\scriptstyle q_{IGS}}\downarrow & & \downarrow {\scriptstyle BL^\circ}. \\ \mathrm{IGS}(G,X) & \xrightarrow{\overline{\pi}_{HT}} & \mathrm{Bun}_G \end{array}$$

$\leftarrow$ moduli stack of shtukas.

<u>Rmk</u> Scholze-Weinstein $\Rightarrow$ $\mathrm{Sht}_{\mathcal{G},\mu} \times_{\mathrm{Spd}\,\mathcal{O}_E} \mathrm{Spd}\,E = [\mathrm{Gr}_{G,\mu^{-1}} / \underline{\mathcal{G}(\mathbb{Z}_p)}]$.

(Seen: the diagram is different from that last time).

$$\mathrm{Sht}_{\mathcal{G}}(R,R^+) = \left\{ \begin{array}{l} S^\#, \ \mathcal{G}\text{-torsors } P \text{ on } S \times \mathbb{Z}_p := \mathrm{Spa}\, W(R^+) \setminus V(p[\varpi]) \\ \text{s.t. } \mathrm{Frob}_S^* P|_{S\times\mathbb{Z}_p \setminus S^\#} \xrightarrow{\sim} P|_{S\times \mathbb{Z}_p\setminus S^\#} \end{array} \right\}$$

$(S = \mathrm{Spd}(R,R^+).)$

Proved by Mingjia Zhang in her thesis for (G,X) of PEL type ($\mathbb{Q}$ $\mathcal{G}$ reductive).

<u>Thm</u> (DKvHZ) True if $(*)$ holds.

For the def'n we have to write down an equiv relation.

Choose $(G,X) \overset{i}{\hookrightarrow} (G_V, \mathcal{H}_V^{\pm})$

$$\begin{tikzcd} (G_V, \mathcal{H}_V^{\pm}) \\ GSp(V,\psi) \arrow[u] \end{tikzcd}$$ for symp space (V,ψ).

s.t. $\exists$ $\mathbb{Z}_p$-lattice $\Lambda \subset V \otimes \mathbb{Q}_p$ self-dual s.t.

$\mathcal{G}(\breve{\mathbb{Z}}_p)$ is the stabilizer of $\Lambda \otimes \breve{\mathbb{Z}}_p$ in $G(\breve{\mathbb{Q}}_p)$.

$\Rightarrow$ This gives an abelian var

$$\mathcal{A} \longrightarrow \mathcal{S}_{K_p}(G,X)$$

s.t. $\mathcal{S}_{K_p}(G,X)^{\diamond} \longrightarrow \mathrm{Sht}_{\mathcal{G},\mu} \longrightarrow \mathrm{Sht}_{GL(\Lambda)}$

is induced by the BKF module $\mathcal{A}[p^\infty]$.

+ tautological isom $\eta : V^p\mathcal{A} \xrightarrow{\sim} V \otimes \underline{\mathbb{A}_f^p}$

of pro-étale local systems.

<u>Equivalence relation</u>

Fix (R,R^+), $S = \mathrm{Spd}(R,R^+)$,

$$(S^{\#_1}, x_1 : \mathrm{Spf}\, R^{\#_1,+} \to \widehat{\mathcal{S}}_{K_p}(G,X)) \sim (S^{\#_2}, x_2 : \mathrm{Spf}\, R^{\#_2,+} \to \widehat{\mathcal{S}}_{K_p}(G,X))$$

if can choose $\varpi \in R^+$ s.t. $\exists$ map:

$$\begin{tikzcd} W(R^+) \arrow[r] \arrow[d] & R^{\#_i,+} \arrow[d, dashed] \\ R^+ \arrow[r] & R^+/\varpi \end{tikzcd}$$

$\exists$ $\mathcal{A}_{x_1} \otimes_{R^{\#_1,+}} R^+/\varpi \xrightarrow[q\text{-isog}]{f} \mathcal{A}_{x_2} \otimes_{R^{\#_2,+}} R^+/\varpi$

s.t.
$$\begin{tikzcd} V^p\mathcal{A}_{x_1} \arrow[r, "f"] \arrow[d, "\eta_{x_1}"'] \arrow[dr, phantom, "\circlearrowright"] & V^p\mathcal{A}_{x_2} \arrow[d, "\eta_{x_2}"] \\ V \otimes \underline{\mathbb{A}_f^p} \arrow[r, "="] & V \otimes \underline{\mathbb{A}_f^p} \end{tikzcd}$$

& s.t. the induced map

$\mathcal{E}(A_{x_1}) \longrightarrow \mathcal{E}(A_{x_2})$ of $GL(V)$ bundles on X_S is induced by a map of G-bundles.

$$\begin{array}{ccc} \mathcal{E}(A_{x_1}) & \xrightarrow{D(f)} & \mathcal{E}(A_{x_2}) \\ \| & & \| \\ (BL^\circ \circ \pi_{crys}(x_1)) \overset{G}{\times} GL(V) & & (BL^\circ \circ \pi_{crys}(x_2)) \overset{G}{\times} GL(V) \end{array}$$

Here $\mathbb{D}(f)$ is induced by Dieudonné theory over R^+/ϖ.

<u>Get</u>

$$\begin{array}{ccc} Sk_p(G,X)^\diamond & \longrightarrow & Sht_{\mathcal{G},\mu} \\ \downarrow & & \downarrow \\ IGS(G,X) & \longrightarrow & Bun_G \end{array}$$

<u>Rmk</u>

$$\begin{array}{ccc} Sk_p(G,X)^\diamond & \longrightarrow & Spd\ \mathcal{O}_E \\ \downarrow & \overset{\times}{\dashrightarrow} & \\ IGS(G,X) & & \end{array}$$

not over $\mathcal{O}_E$ (rather $\mathbb{F}_p$).

* <u>How do we prove the thm?</u>

(A) The analogous thm for $(G_V, \mathcal{H}_V, \mathcal{G}_\Lambda)$ [M. Zhang].
(complicated by the fact that BKF mods & shtukas are "not fully faithful" unless $R^+ = R^\circ$.)

(B) Rapoport-Zink uniformizations of isogeny classes
(Pappas-Rapoport & Gleason-Lim-Xu.)

Useful because, fibers of BL° over $Spd\ \bar{\mathbb{F}}_p$-points of Bun_G are integral local Shimura vars $\mathcal{M}^{int}_{\mathcal{G},b,\mu}$,
and uniformization says

$$\begin{tikzcd} & \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu} \arrow[dl, dashed, "\exists\ \text{RZ-unif}"'] \arrow[d] \\ \mathcal{S}_{K_p}(G,\mu) \arrow[r] \arrow[ur, phantom, "\circlearrowright"] & \mathrm{Sht}_{\mathcal{G},\mu} \end{tikzcd}$$

Step (B) deals with $(C, \mathcal{O}_C)$-points of the diagram.

(c) Combine (A) (B) with some complicated tech argument.

We also prove:

- $IGS(G,x)$ does not depend on K_p or $(G,X) \to (G_V, \mathcal{H}_V)$ & is functorial in $(G,X) \to (G',X')$.
  (unless $\pi_1(G)_{I_p}$ torsion-free.)
- Evaluating on perfect (discrete top) $\mathrm{Spd}(A,A)$ gives
$$\begin{tikzcd} \mathcal{S}_{K_p}(G,x)^{\mathrm{perf}}_{k_E} \arrow[r] \arrow[d] & \mathrm{Sht}^{W}_{\mathcal{G},\mu} \arrow[d] \\ IGS^{\mathrm{red}}(G,X) \arrow[r] & G\text{-}\mathrm{Isoc}. \end{tikzcd}$$

§ Consequences for cohomology

Assume $p>2$, $Sh_K(G,X) \to \mathrm{Spec}\, E$ proper, $G_{\mathbb{Q}_p}$ unramified.

$$\Rightarrow \quad \begin{tikzcd} Sh_{K^p}(G,X) \arrow[r, "\pi_{HT}"] \arrow[d] & Gr_{G,\mu^{-1}} \arrow[d] \\ IGS_{K^p}(G,X) \arrow[r, "\overline{\pi}_{HT}", "\text{proper}"'] & \mathrm{Bun}_G \end{tikzcd}$$

Thm (DKvHZ)

$\mathcal{F} = R\overline{\pi}_{HT*}\overline{\mathbb{F}}_\ell$ is a perverse Verdier self-dual sheaf on Bun_G.

pf. $\cdot$ $\mathcal{F}|_{\mathrm{Bun}_G^b} \in D(\mathrm{Bun}_G^b, \overline{\mathbb{F}}_\ell) \simeq D_{sm}(J_b(\mathbb{Q}_p), \overline{\mathbb{F}}_\ell)$.

It is $R\Gamma_{\acute{e}t}(IG_{b,\bar{\mathbb{F}}_p}, \bar{\mathbb{F}}_\ell)$.

Assumptions $\Rightarrow$ IG_b is affine.

- Artin vanishing gives half of the perversity.

Self-duality uses that

$IGS_{K^p}(G,X)$ & Bun_G have trivial dualizing sheaf.

If ℓ is very good for $\hat{G}$ in the sense of FS,

then $R\Gamma_{\acute{e}t}(Sh_{K^p}(G,X), \bar{\mathbb{F}}_\ell) = i_1^* T_\mu \mathcal{F}[-d](\frac{d}{2})$

where $d = \dim Sh_K(G,X)$,

$i_1 : \mathrm{Bun}_G^1 \hookrightarrow \mathrm{Bun}_G$.

This shows that the failure of $R\Gamma_{\acute{e}t}(Sh_{K^p}(G,X), \bar{\mathbb{F}}_\ell)$ to be in middle degree is caused by a failure of T_μ to be perverse t-exact.

<u>Hamann, Hamann-Lee</u>

Fix $\phi : W_{\mathbb{Q}_p} \longrightarrow {}^L G(\bar{\mathbb{F}}_\ell)$. Then it is reasonable to hope that for <u>generic</u> ϕ we have

$$T_\mu^\phi : D(\mathrm{Bun}_G, \bar{\mathbb{F}}_\ell)^\phi \longrightarrow D(\mathrm{Bun}_G, \bar{\mathbb{F}}_\ell)^\phi \text{ is exact.}$$

Proved it in some cases (GL_n, unramified & unitary, GSp_4).

We work out unconditional vanishing results for compact abelian type Shimura varieties of type C_2.

- F totally real, $G = \mathrm{Res}_{F/\mathbb{Q}} H$,
- H inner form of $GSp_{4,F}$.

compact mod center at at least one infinite place.

- $p > 2$, $p \neq \ell$ totally split in F s.t.

  $G \otimes \mathbb{Q}_p = \prod_{i=1}^{n} GSp_{4,\mathbb{Q}_p}$.

Then localize at $\mathfrak{m}_p \subset \mathcal{H}(G, K_p)$, K_p hyperspecial

corresponding to $\pi_p = \prod_{i=1}^{n} \pi_{\mathfrak{p}_i}$,

s.t. $\pi_{\mathfrak{p}_i}$ has Satake parameters that are generic

$\phi_i : W_{\mathbb{Q}_p} \to GSp_4 \hookrightarrow GL_4$

Then ask $\phi_i(\mathrm{Frob}_p)$ to have eigenvals $\alpha_{1,i}, \cdots, \alpha_{4,i}$

with radius $\neq p^{\pm 1}$.

Igusa stacks for Shimura varieties of Hodge type (III)

Pol van Hoften

(Joint with Sempliner.)

Note (III) & (IV) : most about exotic correspondence.

Let E imag quad field, p inert in E.

Let V Herm E-v.s. with sign $(3,2)$.

W (the unique) herm E v.s. with $W \otimes_E \mathbb{A}_f \cong V \otimes \mathbb{A}_f$

and $W \otimes \mathbb{R}$ has sign $(4,1)$.

Choose self-dual $\mathcal{O}_{E_p}$-lattice Λ in $V \otimes \mathbb{Q}_p \cong W \otimes \mathbb{Q}_p$

$\rightsquigarrow (G,X) = (GU(V), X)$ & $(G', X') = (GU(W), X)$.

Fix $V \otimes \mathbb{A}_f \cong W \otimes \mathbb{A}_f$.

$\Rightarrow G(\mathbb{A}_f^p) = G'(\mathbb{A}_f^p) \supseteq K^p$ cpt open

& $K_p = \mathrm{Stab}_\Lambda \subset G(\mathbb{Q}_p) = G'(\mathbb{Q}_p)$.

Then by Kottwitz, $\exists$ integral models

$\mathscr{S}_K(G,X)$ & $\mathscr{S}_{K'}(G', X')$ over $\mathcal{O}_{E,p} = \mathbb{Z}_{p^2}$.

moduli spaces of ppavs $(A, \lambda, \iota, \eta^p)$ & (B, ψ, ι, η^p)

Here
- (A, λ) ppav of dim 5, up to prime-to-p isogeny
- $\iota : \mathcal{O}_{E,p} \hookrightarrow \mathrm{End}_{\mathbb{Z}_{(p)}}(A)$
- η^p K^p-level str s.t. Lie A is a free $\mathcal{O}_{E,(p)}$-mod of sign $(3,2)$.
- Similar for (B, ψ, ι, η^p) of sign $(4,1)$.

It turns out that $\mathcal{S}_K(G,X)_{\mathbb{F}_{p^2}}$ & $\mathcal{S}_K(G',X')_{\mathbb{F}_{p^2}}$ are closely related.
by

$$\begin{tikzcd} & \mathrm{Corr} \arrow[dl, "s"'] \arrow[dr, "t"] & \\ \mathcal{S}_K(G,X)_{\mathbb{F}_{p^2}} & & \mathcal{S}_K(G',X')_{\mathbb{F}_{p^2}} \end{tikzcd}$$

the moduli of (quasi-) isogenies

$$(A, \lambda, \iota, \eta^p) \dashrightarrow (B, \psi, \iota, \eta^p).$$

* Moreover, images of s & t are union of Newton strata.

<u>Lem</u> Corr is an (ind-) scheme.

pf Hilbert scheme argument.

<u>Obs</u> Corr is nonempty.

By moduli interpretation, enough to find a quasi-isog $(A[p^\infty], \lambda, \iota) \dashrightarrow (B[p^\infty], \psi, \iota)$.

Now it is a computation with $G_{\mathbb{Q}_p}$-isocrystals.
Concretely, we have $B(G_{\mathbb{Q}_p}) (= B(G))$ of isom classes of $G_{\mathbb{Q}_p}$-isocrystals / $\overline{\mathbb{F}}_p(\breve{\mathbb{Q}}_p)$.

$$\rightsquigarrow \underset{\substack{\| \\ B(G,X)}}{B(G,-\mu)}, \ \underset{\substack{\| \\ B(G,X')}}{B(G,-\mu')} \subset B(G).$$

<u>Need</u> $B(G,-\mu) \cap B(G,-\mu') \neq \phi$

For this, need to check that $\kappa_G(-\mu) = \kappa_G(-\mu')$ in

$$\pi_1(G)_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} = \pi_1(G^{ab})_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} = \pi_1(GU(1)) = \mathbb{Z}.$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| \ \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m$

Rmk This can be done by hand.

Recap Last time we know that given (G, X) of Hodge type,

$v | p$ in $\mathbb{E}$, $K^p \subset G(\mathbb{A}_f^p)$, $\mathcal{G}$-parahoric,

$$K_p = \mathcal{G}(\mathbb{Z}_p), \quad K = K^p K_p.$$

Then we get $E = \mathbb{E}_v$.

$\mathscr{S}_K(G, X) \;/\; \mathcal{O}_E$

$+ \quad \mathscr{S}_K(G, X)^\diamond \xrightarrow{\pi_{crys}} \mathrm{Sht}_{\mathcal{G}, \mu}$

"$\mathcal{G}$-bundles on the Cartier-Witt stack".

Thm (DKvHZ)

$$\begin{tikzcd} \mathscr{S}_K(G, X)^\diamond \arrow[r, "\pi_{crys}"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Sht}_{\mathcal{G}, \mu} \arrow[d] \\ \mathrm{IGS}_{K^p}(G, X) \arrow[r] & \mathrm{Bun}_G \end{tikzcd}$$

If we restrict π_{crys} to perfect $\mathbb{F}_p$-algebras A (with discrete top)

$$\mathscr{S}_K(G, X)^{perf}_{k_E} \xrightarrow{\pi^{red}_{crys}} \mathrm{Sht}^W_{\mathcal{G}, \mu}$$

Shen-Yu-Zhang + Xiao-Zhu.

Want to state, in this formalism, what "corr" should be.

Let Z be a G-torsor over $\mathrm{Spec}\, \mathbb{Q}$.

$$+ \text{ fix } Z \otimes \mathbb{A}_f^p = G \otimes \mathbb{A}_f^p.$$

Under some condition on $Z \otimes \mathbb{R}$,

$\exists$ Shimura datum X' for $G' = \mathrm{Aut}_G(Z)$. (previously $[Z] \leftrightarrow W$).

$[\, i : (G, X) \hookrightarrow (GSp_{2g}, \mathcal{H}^{\pm})\,]$ $\qquad GSp_{2g} \cong$

$\Rightarrow [\, \mathrm{Aut}_G(Z) \xrightarrow{i'} \mathrm{Aut}_{GSp_{2g}}(Z \overset{G}{\times} GSp_{2g})\,]$.

Want X' s.t. i' is a map of Shimura data.

<u>Local correspondence</u> $Z \otimes \mathbb{Q}_p$ induces $G\text{-Isoc} \xrightarrow[\alpha_Z]{\sim} G'\text{-Isoc}$

$\mathcal{E} \longmapsto \mathrm{Isom}_G(\mathcal{E}, Z)$

$(\mathrm{Bun}_G \xrightarrow[\alpha_Z]{\sim} \mathrm{Bun}_{G'}$ in [FS])

Define
$$\begin{tikzcd} \mathrm{Sht}^{W,Z}_{\mathcal{G},\mu \mid \mathcal{G}',\mu'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Sht}^{W}_{\mathcal{G}',\mu'} \arrow[d] \\ \mathrm{Sht}^{W}_{\mathcal{G},\mu} \arrow[r] & G'\text{-Isoc}. \end{tikzcd}$$

<u>Conj</u> (XZ with $Z \otimes \mathbb{Q}_p = \{1\}$, Sempliner in general)

$\exists$ perfect ind-scheme $\mathrm{Corr}^Z_{\mathcal{G},\mathcal{G}'}$

s.t.
$$\begin{tikzcd} & \mathrm{Corr}^Z_{\mathcal{G},\mathcal{G}'} \arrow[dl] \arrow[dr] \arrow[dd] & \\ \mathscr{S}_K(G,X)^{perf}_{k_E} \arrow[dd] \arrow[dr, phantom, "\square"] & & \mathscr{S}_{K'}(G',X')^{perf}_{k_E} \arrow[dd] \arrow[dl, phantom, "\square"] \\ & \mathrm{Sht}^{W,Z}_{\mathcal{G},\mu \mid \mathcal{G}',\mu'} \arrow[dl] \arrow[dr] & \\ \mathrm{Sht}^{W}_{\mathcal{G},\mu} & & \mathrm{Sht}^{W}_{\mathcal{G}',\mu'} \end{tikzcd}$$

with both squares Cartesian.

$H^1(\mathbb{Q}_p, G) \subset B(G)_{bas}$

Choose $a \in G(\breve{\mathbb{Q}}_p)$ s.t. $[a] = [Z]$.

<u>Kottwitz</u> $\exists\ \nu_a : G'_{\breve{\mathbb{Q}}_p} \xrightarrow{\sim} G_{\breve{\mathbb{Q}}_p}$

s.t. $\nu_a(\sigma'(g)) = a\,\sigma(\nu_a(g))\,a^{-1}$

then take $\beta : G'_{\breve{\mathbb{Q}}_p} \longrightarrow G_{\breve{\mathbb{Q}}_p}$, $g \longmapsto \nu_a(g) \cdot a$.

$\Rightarrow \quad B(G') \xrightarrow{\sim} B(G)$, $1 \longmapsto [a]$.

(Choose v'' in $E'' = E \cdot E'$ above v & v'.)

<u>Thm</u> (XZ) If $Z \otimes \mathbb{Q}_p$ is trivial, $\mathcal{G}, \mathcal{G}'$ hyperspecial, then the Conj holds.

<u>Thm</u> (vHS) If $i^{-1}(Sp_{2g}) \cap G$ is connected & $G \otimes \mathbb{Q}_p$ splits over an unr ext'n, then Conj holds.

<u>Examples</u> F tot real, $G = \mathrm{Res}_{F/\mathbb{Q}} GSp_{2g}$

$p = \mathfrak{p}_1 \cdots \mathfrak{p}_d$ splits completely in F.

Then get corr to (G', X') with (fixing $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$).

$$G' \otimes \mathbb{Q}_p = \prod_{i=1}^{r} GSp_{2g} \times \prod_{i=r+1}^{d} H_{\mathbb{Q}_p} \quad (H \hookrightarrow GSp_{2g})$$

$$G' \otimes \mathbb{R} = \prod_{i=1}^{r} GSp_{2g,\mathbb{R}} \times \prod_{i=r+1}^{d} H_{\mathbb{R}}$$

with $H_{\mathbb{R}}(\mathbb{R})$ cpt mod center.

In this case all Newton strata of (G', X') see this correspondence.

($g=1$ due to Tian – Xiao).

<u>But</u> Not covered by [XZ] since $G' \otimes \mathbb{Q}_p \neq G \otimes \mathbb{Q}_p$.

Caution: Thm does not strictly apply!

e.g. (G, X) abelian type, no Z exists.

Relation to Igusa stacks

Igusa stack $\mathrm{IGS}_{K^p}(G,X)$ gives

$$\begin{tikzcd} \mathscr{S}_K(G,X)^{\mathrm{perf}}_{k_E} \arrow[r] \arrow[d] & \mathrm{Sht}^{W}_{\mathcal{G},\mu} \arrow[d] \\ \mathrm{IGS}_{K^p}(G,X)^{\mathrm{red}} \arrow[r] & G\text{-}\mathrm{Isoc}\ (\xrightarrow{\alpha_Z} G'\text{-}\mathrm{Isoc}) \end{tikzcd}$$

(Note $\alpha_Z(B(G,-\mu)\cap B(G',-\mu')) \subset B(G') \rightsquigarrow G'\text{-}\mathrm{Isoc}_{\mu,\mu'}$)

* Slightly stronger conj

There is a $G(\mathbb{A}_f^p)$-equiv isom

$$\mathrm{IGS}_{K^p}(G,X)^{\mathrm{red}} \times_{G\text{-}\mathrm{Isoc}} G'\text{-}\mathrm{Isoc}_{\mu,\mu'} \xrightarrow[\sim]{\delta_Z} \mathrm{IGS}_{K^p}(G',X')^{\mathrm{red}} \times_{G'\text{-}\mathrm{Isoc}} G'\text{-}\mathrm{Isoc}_{\mu,\mu'}$$

Relation There is a pullback

$$\begin{tikzcd} \mathrm{Corr}^{Z}_{\mathcal{G},\mathcal{G}'} \arrow[r, "(s,t)"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathscr{S}_K(G,X)^{\mathrm{perf}}_{k_E} \times \mathscr{S}_{K'}(G',X')^{\mathrm{perf}}_{k_E'} \arrow[d] \\ \Gamma_{\delta_Z} \arrow[r] & \mathrm{IGS}_{K^p}(G,X)^{\mathrm{red}} \times \mathrm{IGS}_{K^p}(G',X')^{\mathrm{red}} \end{tikzcd}$$

Key More flexible than the conj before.

Can be easier to generalize to abelian type.

Could wonder

Maybe there is in fact

$$\mathrm{IGS}_{K^p}(G,X) \times_{\mathrm{Bun}_{G'}} \mathrm{Bun}_{G',\mu,\mu'} \xrightarrow{\sim} \mathrm{IGS}_{K^p}(G',X') \times_{\mathrm{Bun}_{G'}} \mathrm{Bun}_{G',\mu,\mu'} \quad ?$$

This would imply that we could glue together

$IGS_{K^p}(G, X)$ & $IGS_{K^p}(G', X')$

along their common open.

$\Rightarrow$ $R\bar{\pi}_{HT,(G',X')}\mathbb{Q}_\ell|_{Bun_{G',\mu,\mu'}} \cong \alpha_Z(R\bar{\pi}_{HT,(G,X)}\mathbb{Q}_\ell)|_{Bun_{G,\mu,\mu'}}$

<u>Thm</u>** (vHS) This expectation is true.

<u>Rmk</u> $Sht^{Z}_{G,\mu|G',\mu'}$ is nonempty

$\Leftrightarrow$ $\alpha_Z(B(G,-\mu) \cap B(G',-\mu')) \neq \phi$.

In the reinterpretation Corr has a simple def'n

but is yet unclear whether Corr $\neq \phi$.

Everything goes through the XZ result about $Corr(\bar{\mathbb{F}}_p)$.

Igusa stacks for Shimura varieties of Hodge type (IV)

Pol van Hoften

(Joint with Sempliner.)

<u>Today</u> Proof of results on exotic Hecke Corr.

<u>Notation</u> p fixed prime. (G, X) Hodge type.

Z is a G-torsor on Spec $\mathbb{Q}$

$+ \; Z \otimes \mathbb{A}_f^p \xrightarrow{\sim} G \otimes \mathbb{A}_f^p$ s.t. $Z \otimes \mathbb{R}$ satisfies the same condition.

Then $G' = \mathrm{Aut}_G(Z)$ admits a Shimura datum (G', X') of Hodge type.

(✗) <u>Technical ass'n</u>

$\exists \; i : (G, X) \hookrightarrow (G_V, \mathcal{H}_V)$

$G_V = GSp(V, \psi)$, (V, ψ) symplectic.

s.t. $i^{-1}(Sp_V) \cap G$ is connected.

We will use extended Shimura varieties and Igusa stacks.

If $\mathrm{Ker}^1(\mathbb{Q}, G) = Ш^1(\mathbb{Q}, G) = \mathrm{Ker}(H^1(\mathbb{Q}, G) \to \prod_v H^1(\mathbb{Q}_v, G))$

then $\mathrm{Sh}_K^{ext}(G, X) \xrightarrow[\text{non-can}]{\approx} \coprod_{\mathrm{Ker}^1(\mathbb{Q}, G)} \mathrm{Sh}_K(G, X)$.

<u>Punchline</u> In the PEL case, $\mathrm{Sh}_K^{ext}(G, X)$ has a moduli interpretation, but not for $\mathrm{Sh}_K(G, X)$.

Can define $\mathrm{Sh}^{ext}(G, X) := \mathrm{Sh}(G, X) \overset{A(G)}{\times} A^{ext}(G)$

with $\quad \mathcal{A}(G) \approx G(\mathbb{A}_f) \ltimes G^{ad}(\mathbb{Q})^+ / G(\mathbb{Q})_+$
$\quad\quad \circlearrowright$
$\quad\quad Sh(G,X)$

and $\exists$ canonical ext'n
$$1 \to \mathcal{A}(G) \to \mathcal{A}^{ext}(G) \to \ker^1(\mathbb{Q}, G) \to 1.$$
This $Sh^{ext}(G,X)$ is a better-behaved object.

(Fix v'' of $E'' = E \cdot E'$ above v & v'.)

<u>Thm</u> (vHS) Under ($\bowtie$), and if $G_{\mathbb{Q}_p}$ splits over an unram ext'n, then $\exists$ a $G(\mathbb{A}_f^p)$-equivariant isom
$$Igs^{ext}(G,X)^{red} \times_{G'\text{-Isoc}} G'\text{-Isoc}_{\mu,\mu'} \xrightarrow{\sim} Igs^{ext}(G',X')^{red} \times_{G\text{-Isoc}} G'\text{-Isoc}_{\mu,\mu'}.$$

<u>Recall</u> $G\text{-Isoc} \xrightarrow[\sim]{\alpha_Z} G'\text{-Isoc}$.

<u>Rmk</u> If $G_{\mathbb{Q}_p}$ quasi-split & $Z \otimes \mathbb{Q}_p = \{1\}$ (cf. XZ), then no longer need ($\bowtie$).

Choose $\mathcal{G}, \mathcal{G}'$ parahorics for G, G', and fix $i : (G,X) \hookrightarrow (G_V, \mathcal{H}_V)$, $i' : (G',X') \hookrightarrow (G_V, \mathcal{H}_V)$ as in ($\bowtie$).

Then get
$$\begin{array}{cc} \mathcal{A} & \mathcal{B} \\ \downarrow & \downarrow \\ \mathcal{S}^{ext}_{K_p}(G,X)^{perf} & \mathcal{S}^{ext}_{K_p}(G',X')^{perf} \end{array}$$
abelian schemes up to prime-to-p isogeny
+ tautological $V^p\mathcal{A} \xrightarrow[\sim]{\eta_{\mathcal{A}}} V \otimes \mathbb{A}_f^p$
$V^p\mathcal{B} \xrightarrow[\sim]{\eta_{\mathcal{B}}} V \otimes \mathbb{A}_f^p$.

Then $\mathrm{IGS}^{\mathrm{ext}}(G,X)^{\mathrm{red}}$ is the quotient of $\mathscr{S}^{\mathrm{ext}}_{K_p}(G,X)^{\mathrm{perf}}$

by $x \sim y$ if $\exists$ quasi-isog $A_x \overset{f}{\dashrightarrow} A_y$

s.t.
$$\begin{tikzcd}
V^p A_x \arrow[r, "f"] \arrow[d, "\eta_{A_x}"', "\wr"] \arrow[dr, phantom, "\circlearrowright"] & V^p A_y \arrow[d, "\eta_{A_y}", "\wr"'] \\
V \otimes \mathbb{A}_f^p \arrow[r, "="] & V \otimes \mathbb{A}_f^p
\end{tikzcd}$$

+ condition at p.

$$\mathcal{Z} \otimes \mathbb{A}_f^p = G \otimes \mathbb{A}_f^p \Rightarrow G \otimes \mathbb{A}_f^p = G' \otimes \mathbb{A}_f^p$$

$$\mathcal{Z} \overset{G}{\times} G_V \overset{\delta}{=} G_V \Rightarrow t^p \in G_V(\mathbb{A}_f^p) \text{ s.t. } i' = t^p \cdot i \cdot (t^p)^{-1}.$$

$$\rightsquigarrow \quad \begin{tikzcd}
G\text{-Isoc} \arrow[r, "\alpha_{\mathcal{Z}}"] \arrow[d, "i"'] \arrow[dr, phantom, "\Swarrow \delta_p"] & G'\text{-Isoc} \arrow[d, "i'"] \\
G_V\text{-Isoc} \arrow[r, "="] & G_V\text{-Isoc}
\end{tikzcd}$$

Define $\mathrm{IGSCorr}^{\mathcal{Z},\delta} \xhookrightarrow{(s,t)} \mathrm{IGS}^{\mathrm{ext}}(G,X)^{\mathrm{red}} \times \mathrm{IGS}^{\mathrm{ext}}(G',X')^{\mathrm{red}}$

to be (x,y) lies in IGSCorr

if $\exists\ A_x \overset{f}{\dashrightarrow} B_y$ s.t.
$$\begin{tikzcd}
V^p A_x \arrow[r, "f"] \arrow[d, "\eta_{A_x}"'] & V^p B_y \arrow[d, "\eta_{B_y}"] \\
V \otimes \mathbb{A}_f^p \arrow[r, "t^p"] & V \otimes \mathbb{A}_f^p
\end{tikzcd}$$

& s.t. $\mathbb{D}(A_x[p^\infty])[\frac{1}{p}] \xrightarrow[\sim]{f} \mathbb{D}(B_y[p^\infty])[\frac{1}{p}]$

is induced by an isom of G'-isocrystals.

Straightforward that s & t are monomorphisms.

but not clear at all that $\mathrm{IGSCorr}^{(-)} \neq \emptyset$.

* Want Image of s (or t) is a certain union of Newton strata.

Enough to deal with $\overline{\mathbb{F}}_p$-pts.

Sketch of XZ proof

- Enough to check one isog class at a time.
- Those all have CM lifts,

  and then reduce to the case of $G = T$.
- Then do hard work.

But cannot deal with some one situation.

What do we do?

$$\begin{tikzcd} G \arrow[r, "i"] \arrow[rr, bend right, "c"'] & G_V \arrow[r] & \mathbb{G}_m \end{tikzcd}$$

For F totally real, we consider $H = H_F \subset \mathrm{Res}_{F/\mathbb{Q}}\, G_F$ to be the (identity component of) inverse image of $\mathbb{G}_m \subset \mathrm{Res}_{F/\mathbb{Q}}\, \mathbb{G}_{m,F}$ under c_F.

If $F/\mathbb{Q}$ is Galois, then

$\Gamma = \mathrm{Gal}(F/\mathbb{Q})$ acts on $(H, Y) \hookleftarrow (G, X)$.

Lem $p > 2$. Can choose F s.t.

(1) $Z \times^G H$ is trivial at p,

(2) H is quasi-split at p,

(3) F is tame,

(4) F unram at p.

pf (i) Uses (⊠) and that may arrange F_p for $\mathfrak{p} | p$ to contain $\mathbb{Q}_{p^s}$.

(Can arrange F cyclic & inert at p since $p \neq 2$.) □

Then $XZ+\varepsilon$ gives us (with $H' = \mathrm{Aut}_H(Z \overset{G}{\times} H)$.)

$$\mathrm{IGS}^{\mathrm{ext}}(H, y)^{\mathrm{red}} \times_{H'\text{-Isoc}} H'\text{-Isoc}_{\mu, \mu'} \xrightarrow{\sim} \mathrm{IGS}^{\mathrm{ext}}(H', y')^{\mathrm{red}} \times_{H'\text{-Isoc}} H'\text{-Isoc}_{\mu, \mu'}.$$

Can check it is Γ-equivariant.

Now, would like to take Γ-fixed pts.

Thm (vHS) The natural map

$$\mathrm{IGS}^{\mathrm{ext}}(G, x) \xrightarrow{\sim} \mathrm{IGS}^{\mathrm{ext}}(H, y)^{\Gamma} \times_{\mathrm{Bun}_H^{\Gamma}} \mathrm{Bun}_{G,\mu}$$ is an isom.

Consider $G(\mathbb{Q}) \backslash G(\mathbb{A}) \to (H_1(\mathbb{Q}) \backslash H_1(\mathbb{A}))^{\Gamma}$

Serre's non-abelian Galois coh tells that this is a bijection iff

$$\ker(H^1(\Gamma, H_1(\mathbb{Q})) \to H^1(\Gamma, H_1(\mathbb{A})))$$ is trivial.

A computation using inflation-restriction shows this is $\ker(Ш^1(\mathbb{Q}, G) \to Ш^1(\mathbb{Q}, H_1))$.

Goes away for extended Shimura varieties.

To deal with H, squeeze $H(\mathbb{Q}) \backslash H(\mathbb{A})$ between G & H_1.

Again, general theory says

$K_{\infty,1}, K_1$ Γ-stable,

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty^\Gamma K^\Gamma \to (H_1(\mathbb{Q})\backslash H_1(\mathbb{A})/K_{\infty,1}K_1)^{h\Gamma}$$

is an isom if $H^1(\Gamma, K_{\infty,1}) = 0$ (automatic)

& $H^1(\Gamma, K_1) = 0$.

If F is tame, $\exists$ cofinal $K_1 \subset H_1(\mathbb{A}_f)$ s.t. $H^1(\Gamma, K_1) = 0$.

Let $K = K_1 \cap H(\mathbb{A}_f)$. Then

$(H(\mathbb{Q})\backslash H(\mathbb{A}_f) \times Y / K)^\Gamma$ can again be squeezed

(for cofinal collection of K_1).

<u>Upshot</u> $\exists$ cofinal collection of $K_1 \subset H_1(\mathbb{A}_f)$ s.t.

$$Sh^{ext}_{K_1^\Gamma}(G,X) \xrightarrow{\sim} Sh^{ext}_K(H,Y)^\Gamma$$

$\Rightarrow$ similar statement on good reduction loci holds.

<u>§ Proof thm on Igusa stacks</u>

$$\begin{tikzcd} & (\ldots) \arrow[r] \arrow[d, "\Gamma"] & \mathrm{Bun}_{G,\mu} \arrow[d] \\ \mathrm{IGS}^{ext}(G,X) \arrow[ur, "\simeq"] \arrow[r] & \mathrm{IGS}^{ext}(H,Y)^\Gamma \arrow[r] & \mathrm{Bun}_H^{h\Gamma} \end{tikzcd}$$

Can check the isom after base change via the v-cover

$$BL : \mathrm{Gr}_{G,\mu^{-1}} \longrightarrow \mathrm{Bun}_{G,\mu}.$$

Then we get $Sh^{ext}(G,X)^{\circ,\diamond} \longrightarrow Sh^{ext}(H,Y)^{\circ,\diamond}$

which is an isom by the above. $\square$

<u>Cor</u> For $K = K_p K^p$, with K_p Γ-stable Iwahori,

K^p Γ-stable s.t. $H^1(\Gamma, K^p) = 0$,

get $\quad \mathcal{J}_{K^p}(G,X)^\diamond \xrightarrow{\sim} \mathcal{J}_K(H,Y)^{\diamond,\Gamma} \times_{\mathrm{Sht}^{h\Gamma}_{H,\mu}} \mathrm{Sht}_{\mathcal{G},\mu}$

$\mathrm{Sht}_{\mathcal{G},\mu} \xrightarrow{\sim} \mathrm{Sht}^{h\Gamma}_{H_1,\mu}$.

<u>Cor</u> Exotic corr of Igusa varieties for

$b \in B(G,-\mu) \cap B(G',-\mu')$

<u>Hope</u> Would like to prove new instances of Jacquet-Langlands functoriality.

So far. we can do something for $\mathrm{Res}_{F/\mathbb{Q}}\, GSp_4$

at a totally split p.