Geometric Satake equivalence (II)

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(continued)

The Satake cat $S \longrightarrow \mathrm{Div}^d_{\mathcal{Y}}$ small v-stack,

$$\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}} = \mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}}} \times_{\mathrm{Div}^d_{\mathcal{Y}}} S,$$

e.g. $S = \mathrm{Spd}\,\mathbb{F}_q$, $\mathrm{Gr}_{G,\mathrm{Spd}\,\mathbb{F}_q/\mathrm{Div}^d_{\mathcal{Y}}} \cong \mathrm{Gr}_G^{\mathrm{Witt},\diamond}$.

$$\mathrm{Sat}(\mathrm{Hck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda) \subseteq D_{\acute{e}t}(\mathrm{Hck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$$

the full subcat of obj that are ULA and flat perverse

$A \otimes^{\mathbb{L}}_{\Lambda} M$ perverse, $\forall \Lambda$-mod M.

ULA: bounded, pullback to $\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$ is ULA $/S$.

$A \in D_{\acute{e}t}(\mathrm{Hck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{bd}$.

Prop (VI.6.2) A ULA $\Leftrightarrow$ $\mathrm{sw}^* A$ ULA,

where $\mathrm{sw}: \mathrm{Hck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}} \xrightarrow{\sim} \mathrm{Hck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$.

Prop (VI.6.4) $T \subset B \subset G$ max'l torus $\subseteq$ Borel.

A ULA $\Leftrightarrow$ $CT_B(A) \in D_{\acute{e}t}(\mathrm{Gr}_{T,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{bd}$ ULA.

$\Leftrightarrow$ $R\pi_{T,S,*} CT_B(A) \in D_{\acute{e}t}(S, \Lambda)$

locally const with perf'd fibres.

Proof 1st $\Leftrightarrow$:

$\Rightarrow$: Cor VI.3.5, $\pi_{T,S}$ ind-proper, Cor IV.2.12.

$\Leftarrow$: May assume S strictly tot discon and G split.

By Thm IV.2.23, to prove A ULA, it suffices to prove

$$p_1^* R\mathcal{H}om(A, R\pi_{G_S}^! \Lambda) \overset{\mathbb{L}}{\otimes}_\Lambda p_2^* A \xrightarrow{\sim} R\mathcal{H}om(p_1^* A, Rp_2^! A).$$

By Prop VI.4.2 applying to $G\times G$,

suffices to prove this after applying $CT_{B\times B}$

$\rightsquigarrow$ checking $CT_B(A)$ is ULA,

which is our given condition.

2st $(\Leftarrow)$: Prop IV.2.28 $\square$

Prop VI.6.5 G split, $d=1$, then

$$A \text{ ULA} \Longleftrightarrow \left(\begin{array}{c} \forall \mu \in X_*(T)^+,\ [\mu]^* A \text{ locally const} \\ \text{with perf'd fibres in } D_{\acute{e}t}(S,\Lambda), \\ [\mu]: S \longrightarrow \mathrm{Hck}_{G,S/\mathrm{Div}_X^1}. \end{array} \right)$$

pf uses affine flag var and Prop VI.5.7.

Similar results holds for $d\geq 1$ if $S \to (\mathrm{Div}_X^1)^d \longrightarrow \mathrm{Div}_X^d$

Cor VI.6.7 C/E alg closed perf'd, with res field k.

Then
$$\begin{array}{ccc} D_{\acute{e}t}^{ULA}(\mathrm{Hck}_{G,\mathrm{Spd}\,C/\mathrm{Div}_X^1}, \Lambda) & \xleftarrow{\sim} & D_{\acute{e}t}^{ULA}(\mathrm{Hck}_{G,\mathrm{Spd}\,\mathcal{O}_C/\mathrm{Div}_X^1}, \Lambda) \\ & & \downarrow s \\ & & D_{\acute{e}t}^{ULA}(\mathrm{Hck}_{G,\mathrm{Spd}\,k/\mathrm{Div}_X^1}, \Lambda). \end{array}$$

Perverse sheaves

Def / Prop VI.7.1 $\exists!$ t-structure $({}^pD^{\leq 0}, {}^pD^{\geq 0})$ on $D_{\acute{e}t}(\mathrm{Hck}_{G,S/\mathrm{Div}_X^1}, \Lambda)^{bd}$

s.t. $A \in {}^pD^{\leq 0} \Longleftrightarrow \forall$ geom. pt $\mathrm{Spa}(C,C^+) \to S$ &

open Schubert cell of $\mathrm{Hck}_{G,\mathrm{Spd}(C,C^+)/\mathrm{Div}_X^1}$

parametrized by $\mu_1, \dots, \mu_r \in X_*(T)^+$, the pullback of A to this cell sits in

coh deg $\leq -\sum_{i=1}^{r} \langle 2\rho, \mu_i \rangle$.

Define ${}^pD^{\geq 0} :=$ right ortho complement of ${}^pD^{\leq -1}$,

i.e. $B \in {}^pD^{\geq 0} \iff \forall A \in {}^pD^{\leq -1}$, $\mathrm{Hom}(A, B) = 0$.

And $({}^pD)^{\heartsuit} := \mathrm{Perv}(\mathrm{Hck}_{G, S/\mathrm{Div}_Y^d}, \Lambda) = {}^pD^{\leq 0} \cap {}^pD^{\geq 0}$.

<u>Prop VI.7.4</u> $\forall S' \to S \to \mathrm{Div}_Y^d$, pullback along $\mathrm{Hck}_{G, S'/\mathrm{Div}_Y^d} \to \mathrm{Hck}_{G, S/\mathrm{Div}_Y^d}$

is t-exact. Assume G splits. Consider

$$CT_B : D_{\acute{e}t}(\mathrm{Hck}_{G, S/\mathrm{Div}_Y^d}, \Lambda)^{bd} \to D_{\acute{e}t}(\mathrm{Gr}_{T, S/\mathrm{Div}_Y^d}, \Lambda)^{bd}.$$

In particular, $A \in {}^pD^{\leq 0} \iff CT_B(A)[\deg] \in D_{\acute{e}t}^{\leq 0}(\mathrm{Gr}_{T, S/\mathrm{Div}_Y^d}, \Lambda)$.

$\rightsquigarrow$ $A \in {}^pD^{\geq 0} \iff$ this holds true after pullback to all geom pt.

$A \in \mathrm{Perv}(\mathrm{Hck}_{G, S/\mathrm{Div}_Y^d}, \Lambda) \iff \forall$ geom pt $\mathrm{Spa}(C, C^+) \to S$,

the pullback of A to $\mathrm{Hck}_{G, \mathrm{Spa}(C, C^+)/\mathrm{Div}_Y^d}$

is perverse.

$$\mathrm{Sat}(\mathrm{Hck}_{G, S/\mathrm{Div}_Y^d}, \Lambda) = \mathrm{Perv}_{flat}^{ULA}(\mathrm{Hck}_{G, S/\mathrm{Div}_Y^d}, \Lambda).$$

<u>Prop VI.7.7</u> G split. $A \in \mathrm{Sat} \iff R\pi_* CT_B(A)[\deg] \in D_{\acute{e}t}(S, \Lambda)$

is étale loc. isom to a fin proj Λ-mod in deg 0.

<u>Example</u> $d = 1$, $\mu \in X_*(T)^+$, $j_\mu : \mathrm{Hck}_{G, \mathrm{Div}_Y^1, \mu} \hookrightarrow \mathrm{Hck}_{G, \mathrm{Div}_Y^1}$, $d_\mu = \langle 2\rho, \mu \rangle$.

<u>Prop VI.7.5 + VI.7.7</u> $\Rightarrow$ $\left.\begin{array}{l} {}^pj_{\mu !}\Lambda[d_\mu] := {}^p\mathcal{H}^0(j_{\mu !}\Lambda[d_\mu]) \\ {}^pRj_{\mu *}\Lambda[d_\mu] := {}^p\mathcal{H}^0(Rj_{\mu *}\Lambda[d_\mu]) \end{array}\right\} \in \mathrm{Sat}.$

$\mathrm{Sat} \hookrightarrow D_{\text{ét}}(\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$ fully faithful.

<u>Prop VI.7.12</u> The Verdier dual $\mathbb{D}$ induces a functor

$$\mathbb{D} : \mathrm{Sat}^{op} \longrightarrow \mathrm{Sat}, \text{ with } \mathbb{D}^2 = \mathrm{Id}.$$

Moreover,

$$\begin{tikzcd} \mathrm{Sat}^{op} \arrow[r, "\mathbb{D}"] \arrow[d, "F_{G,S}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Sat} \arrow[d, "F_{G,S} = \bigoplus_i \mathcal{H}^i(R\pi_{G,S,*})"] \\ \mathrm{LocSys}(S,\Lambda) \arrow[r, "V \mapsto V^*"] & \mathrm{LocSys}(S,\Lambda) \end{tikzcd}$$

□

Convolution and fusion products

$D_{\text{ét}}(\mathcal{H}\mathrm{ck}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{bd}$ is a monoidal cat.

$$\begin{tikzcd} & L^+G \backslash LG \overset{L^+G}{\times} LG / L^+G \arrow[dl, "a"'] \arrow[dr, "b"] & \\ \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}}} \times_{\mathrm{Div}^d_{\mathcal{Y}}} \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}}} & & L^+G\backslash LG/L^+G = \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_{\mathcal{Y}}} \end{tikzcd}$$

$A_1 \star A_2 := Rb_{S*} a_S^* (A_1 \boxtimes A_2)$, associative.

<u>Prop VI.8.1</u> (1) If A_1, A_2 both ULA, then so is $A_1 \star A_2$.

(2) If $A_1, A_2 \in {}^pD^{\le 0}$, then $A_1 \star A_2 \in {}^pD^{\le 0}$.

(3) If $A_1, A_2 \in \mathrm{Sat}$, then $A_1 \star A_2 \in \mathrm{Sat}$.

note: (1) (2) $\Rightarrow$ (3).

I finite set, $d = |I|$. $\mathcal{H}\mathrm{ck}_G^I = \mathcal{H}\mathrm{ck}_{G,\mathrm{Div}^d_X} \times_{\mathrm{Div}^d_X} (\mathrm{Div}^1_X)^I$.

$\mathrm{Gr}_G^I = \mathrm{Gr}_{G,\mathrm{Div}^d_X} \times_{\mathrm{Div}^d_X} (\mathrm{Div}^1_X)^I$.

$\mathrm{Sat}_G^I(\Lambda) = \mathrm{Sat}(\mathcal{H}\mathrm{ck}_G^I, \Lambda)$.

- $F^I = \bigoplus_i \mathcal{H}^i(R\pi_{G,*}) : \mathrm{Sat}_G^I(\Lambda) \longrightarrow \mathrm{LocSys}((\mathrm{Div}^1_X)^I, \Lambda)$

  is $\mathrm{Rep}_{W_E^I}(\Lambda)$.

- If $I \to J$, then $W_E^J \to W_E^I$, and

$$\begin{tikzcd}
\mathrm{Sat}_G^I(\Lambda) \arrow[r] \arrow[d, "F^I"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Sat}_G^J(\Lambda) \arrow[d, "F^J"] \\
\mathrm{Rep}_{W_E^I}(\Lambda) \arrow[r] & \mathrm{Rep}_{W_E^J}(\Lambda)
\end{tikzcd}$$

- $I_1, \cdots, I_k$ finite sets. $I = \coprod_{j=1}^{k} I_j$.

Will construct a mon functor

$$* : \mathrm{Sat}_G^{I_1}(\Lambda) \times \cdots \times \mathrm{Sat}_G^{I_k}(\Lambda) \longrightarrow \mathrm{Sat}_G^I(\Lambda) \quad \text{fusion prod.}$$

$j : (\mathrm{Div}_X^1)^{I; I_1, \cdots, I_k} \subset (\mathrm{Div}_X^1)^I$ the open subset where $x_i \neq x_{i'}$ where $i, i' \in I$ lie in different I_j's.

$\rightsquigarrow$ can define similarly $\mathrm{Sat}_G^{I; I_1, \cdots, I_k}(\Lambda)$.

Prop VI.9.3 The restriction functors

$$j^* : \mathrm{Sat}_G^I(\Lambda) \longrightarrow \mathrm{Sat}_G^{I; I_1, \cdots, I_k}(\Lambda)$$
$$j^* : \mathrm{LocSys}((\mathrm{Div}_X^1)^I, \Lambda) \longrightarrow \mathrm{LocSys}((\mathrm{Div}_X^1)^{I; I_1, \cdots, I_k}, \Lambda)$$

are fully faithful.

Since $\mathrm{Hck}_G^I \times_{(\mathrm{Div}_X^1)^I} (\mathrm{Div}_X^1)^{I; I_1, \cdots, I_k} \cong \prod_{j=1}^{k} \mathrm{Hck}_G^{I_j} \times_{\prod_j (\mathrm{Div}_X^1)^{I_j}} (\mathrm{Div}_X^1)^{I; I_1, \cdots, I_k}$

$\rightsquigarrow$ mon functor

$$\begin{tikzcd}
\prod_{j=1}^{k} \mathrm{Sat}_G^{I_j}(\Lambda) \arrow[r] \arrow[d] & \mathrm{Sat}_G^{I; I_1, \cdots, I_k}(\Lambda) \arrow[d] \\
\prod_{j=1}^{k} \mathrm{LocSys}((\mathrm{Div}_X^1)^{I_j}, \Lambda) \arrow[r, "\boxtimes"] & \mathrm{LocSys}((\mathrm{Div}_X^1)^{I; I_1, \cdots, I_k}, \Lambda)
\end{tikzcd}$$

Def'n/Prop VI.9.4 The above diagramm can be promoted as

$$\begin{tikzcd}
\prod_{j=1}^{k} \mathrm{Sat}_G^{I_j}(\Lambda) \arrow[r, "*"] \arrow[d, "\prod_{j=1}^{k} F^{I_j}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Sat}_G^I(\Lambda) \arrow[d, "F^I"] \\
\prod_{j=1}^{k} \mathrm{LocSys}((\mathrm{Div}_X^1)^{I_j}, \Lambda) \arrow[r, "\boxtimes"] & \mathrm{LocSys}((\mathrm{Div}_X^1)^I, \Lambda)
\end{tikzcd}$$

(functorially in $I_1, \dots, I_k$ and permutations of $I_1, \dots, I_k$.)

$\rightsquigarrow$ $\forall$ finite set I, the composite

$$\mathrm{Sat}_G^I(\Lambda) \times \cdots \times \mathrm{Sat}_G^I(\Lambda) \longrightarrow \mathrm{Sat}_G^{I \sqcup \cdots \sqcup I}(\Lambda) \longrightarrow \mathrm{Sat}_G^I(\Lambda).$$

defines a symm mon str on $\mathrm{Sat}_G^I(\Lambda)$, and

$$F^I : \mathrm{Sat}_G^I(\Lambda) \longrightarrow \mathrm{LocSys}((\mathrm{Div}_X^1)^I, \Lambda) \simeq \mathrm{Rep}_{W_E^I}(\Lambda)$$

is also a symm mon functor.

Tannakian dual and Langlands dual

General Tannakian reconstruction result (Prop VI.10.2)

$\Rightarrow$ $\exists$ Hopf alg $\mathcal{H}_G^I(\Lambda) \in \mathrm{Ind}(\mathrm{Rep}_{W_E^I}(\Lambda))$ s.t.

$$\mathrm{Sat}_G^I(\Lambda) \simeq \mathrm{CoMod}_{\mathcal{H}_G^I(\Lambda)}(\mathrm{Rep}_{W_E^I}(\Lambda))$$

$$\overset{\text{Prop VI.10.3}}{\Longrightarrow} \quad \mathcal{H}_G^I(\Lambda) \simeq \bigotimes_{i \in I} \mathcal{H}_G^{\{i\}}(\Lambda).$$

$\rightsquigarrow$ essential to study $\mathcal{H}_G(\Lambda) := \mathcal{H}_G^{\{*\}}(\Lambda) \in \mathrm{Ind}(\mathrm{Rep}_{W_E}(\Lambda))$.

Now let Λ vary, $\mathcal{H}_G(\Lambda)$ compatible.

$$\begin{array}{c} \mathrm{Sat}_G(\hat{\mathbb{Z}}^p) = \varprojlim_{p \nmid n} \mathrm{Sat}_G(\mathbb{Z}/n\mathbb{Z}) \\ F \big\downarrow \\ \mathrm{Rep}^c_{W_E}(\hat{\mathbb{Z}}^p) = \varprojlim_{p \nmid n} \mathrm{Rep}_{W_E}(\mathbb{Z}/n\mathbb{Z}) \end{array}$$

$\rightsquigarrow$ $\mathcal{H}_G \in \mathrm{Ind}(\mathrm{Rep}^c_{W_E}(\hat{\mathbb{Z}}^p)) \longleftrightarrow$ affine grp scheme $\check{G}/\hat{\mathbb{Z}}^p \circlearrowleft W_E$.

Thm VI.11.1 $\exists$ canonical W_E-equiv isom. $\check{G} \simeq \hat{G}$,

where we make an explicit cyclotomic twist

on the usual W_E-action on $\hat{G}$.

$$\overset{\Lambda = \hat{\mathbb{Z}}^p}{\rightsquigarrow} \quad \mathrm{Sat}_G^I(\Lambda) \simeq \mathrm{Rep}(\hat{G}^I, \mathrm{Rep}_{W_E^I}(\Lambda)) = \mathrm{Rep}((\hat{G} \rtimes W_E)^I, \Lambda).$$

G/E reductive grp $\rightsquigarrow$ root datum $(X^*, \Phi, X_*, \Phi^\vee, X^*_+, X^+_*)$
$\circlearrowleft \mathrm{Gal}(\bar{E}/E)$. $\quad \cup\!\!\!\cup\ \Delta \quad \cup\!\!\!\cup\ \Delta^\vee$

$\rightsquigarrow$ Chevalley grp scheme $\hat{G}/\mathbb{Z} \circlearrowleft \mathrm{Gal}(\bar{E}/E)$
coming from the Langlands dual of G
$(X_*, \Phi^\vee, X^*, \Phi, X^+_*, X^*_+)$.
$\quad \cup\!\!\!\cup\ \Delta^\vee \quad \cup\!\!\!\cup\ \Delta$

- $\hat{T} \subset \hat{B} \subset \hat{G}$, $\forall a \in \Delta^\vee$,

$\psi_a : \mathrm{Lie}\, \hat{U}_a \xrightarrow{\sim} \mathbb{Z}$ fixed pinnings.
$\circlearrowleft \mathrm{Gal}(\bar{E}/E)$-equiv.

Need to work over $\mathbb{Z}_\ell$ or $\mathbb{Z}/\ell^n\mathbb{Z}$, then come with a Tate twist

$$\psi'_a : \mathrm{Lie}\, \hat{U}_a \xrightarrow{\sim} \mathbb{Z}_\ell\boxed{(1)}$$ $\leftarrow$ cyclotomic twist.

To prove Thm VI.11.1, one proceeds as

- if $G = T$ torus, $\mathrm{Gr}_{T, \mathrm{Div}^1_X} \cong X_*(T) \times \mathrm{Div}^1_X$,

$$\mathrm{Sat}_T = \bigoplus_{X_*(T)} \mathrm{Rep}^c_{W_E}(\Lambda).$$

$\Rightarrow \hat{T} = \check{T}$.

- for general G, $CT_B[\deg] : \mathrm{Sat}_G \to \mathrm{Sat}_T$,
commuting with fibre functors $\Rightarrow \hat{T} = \check{T} \longrightarrow \check{G}$,
using $A_\mu := {}^p j_{\mu,!}\mathbb{Z}_\ell$ to show $\check{T} \to \check{G}$ is a closed immersion.
- analyze $\check{G}_{\mathbb{Q}_\ell}$ and $\mathrm{Sat}_G(\mathbb{Q}_\ell)$ by A_μ and Prop VI.7.5
to find $\check{B} \subset \check{G}$ s.t. $\check{B}_{\mathbb{Q}_\ell} \subset \check{G}_{\mathbb{Q}_\ell}$ Borel.
    - $G = PGL_2$, and rk 1 grps
    - general case uses again hyperbolic localization.
    - arguments on grp schemes, e.g. [PY06, Cor 5.2].