Fourier-Jacobi period on unitary group

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(Joint with Paul Boisseau & Hang Xue).

§ Rankin-Selberg integral

F number field, $\mathbb{A} = \mathbb{A}_F$, $G(\mathbb{A}) = G(\mathbb{A})^1 \times A_G^\infty$, $A_G^\infty \subset G(F_\infty)$

$G = GL_n \times GL_n$, $H = GL_n \overset{\Delta}{\hookrightarrow} G$,

π cusp autom rep of $G(\mathbb{A})$, central char triv on A_G^∞.

$\pi = \pi_1 \boxtimes \pi_2 \circlearrowleft G(\mathbb{A})$

$\rightsquigarrow L(s,\pi) = L(s, \pi_1 \times \pi_2)$.

$\Phi \in \mathcal{S}(\mathbb{A}_n)$, $GL_n(\mathbb{A}) \curvearrowright \mathcal{S}(\mathbb{A}_n)$ Weil rep

$$(g \cdot \Phi)(v) = \Phi(vg) \cdot |\det g|^{1/2}.$$

$$\Theta'(h, \Phi) = \sum_{v \in F_n \setminus 0} (h \cdot \Phi)(v), \quad h \in [GL_n].$$

<u>Input</u> $\varphi \in \pi$, $\Phi \in \mathcal{S}(\mathbb{A}_n)$

Define global RS period

$$P_H(\varphi, \Phi, \underset{\mathbb{C}}{\overset{\in}{s}}) = \int_{[H]} \varphi(h)\Theta'(h,\Phi) \cdot |\det h|^{s-\frac{1}{2}} \cdot dh.$$

<u>Thm</u> (JPSS) (1) $P_H(\varphi, \Phi, s)$ converges when $\mathrm{Re}\, s > 1$, + mero continuation.

(2) $\varphi = \otimes_v' \varphi_v$, $\Phi = \otimes_v \Phi_v$,

$$\Rightarrow \quad P_H(\varphi,\Phi,s) = \prod_v \underbrace{P_{H,v}(\varphi_v, \Phi_v, s)}_{= L(s,\pi_v) \text{ for almost } v}$$

Cor $L(\frac{1}{2},\pi)\neq 0 \iff \mathcal{P}_H(-,-,\frac{1}{2})\neq 0$ on π.

§ GGP conj for Fourier-Jacobi period

E/F quadratic, V skew-herm of dim n.

$\mathcal{U}=\mathcal{U}(V)\times\mathcal{U}(V)$, $\mathcal{U}'=\mathcal{U}(V)\xhookrightarrow{\Delta}\mathcal{U}$.

π cusp autom rep of $\mathcal{U}(V)(\mathbb{A})$,

$BC(\pi)$ generic $\Rightarrow BC(\pi)=\Pi_1\boxplus\Pi_2\boxplus\cdots\boxplus\Pi_k$ [KMSW]

isobaric sum

rep of $GL_n(\mathbb{A}_E)$

where Π_i cusp unitary, conj self-dual, pairwise distinct

$L(s,\Pi_i,As^{(-1)^{n+1}})$ has pole at 1.

Hermitian Arthur parameter

Fix $\mu: E^\times\backslash\mathbb{A}_E^\times\to\mathbb{C}^\times$, $\mu|_{\mathbb{A}^\times}=\eta_{E/F}$,

$\mathcal{U}(V)(\mathbb{A})\to Mp(V)(\mathbb{A})$, V symplectic space.

$\omega: \mathcal{U}(V)(\mathbb{A})\curvearrowright\mathcal{S}(L(\mathbb{A}))$

Define $\theta(h,\phi)=\sum_{v\in L}(\omega(h)\phi)(v)$.

Conj (GGP) Suppose Π Herm A-para. Then (1) $\iff$ (2):

(1) $L(\frac{1}{2},\Pi\times\mu^{-1})\neq 0$

(2) $\exists V,\pi$ cusp rep, $BC(\pi)=\Pi$ and

$$\mathcal{P}_{\mathcal{U}'}(\varphi,\phi)=\int_{[\mathcal{U}']}\varphi(x)\overline{\theta(x,\phi)}\,dx\neq 0.$$

Rmk (a) Bessel period:

Jacquet-Rallis, W. Zhang, Yun, Xue, BPLZZ (isolation)

Zydor, BPCZ.

(b) FJ period: Xue, Liu.

§ RTF and proof of main conj

$\mathcal{P}_{\mathcal{U}'} \neq 0 \overset{\text{easy}}{\Longleftrightarrow} J_\pi \neq 0$, J_π distribution on $\mathcal{U}$.

$\Updownarrow$ Comparison of RTFs (most difficult).

$L(\frac{1}{2}, \pi \times \mu^{-1}) \neq 0 \Longleftrightarrow I_\pi \neq 0$. I_π distribution on $GL_n \times GL_n$.

π cuspidal: okay

π isobaric: harder.

$f \in \mathcal{S}(\mathcal{U}(\mathbb{A}))$, $K_f(x,y) := \sum_{\gamma \in \mathcal{U}(F)} f(x^{-1}\gamma y)$, $x, y \in [\mathcal{U}]$.

$R(f): L^2([\mathcal{U}]) \longrightarrow L^2([\mathcal{U}])$.

$\quad \| \quad \bigoplus_{\chi \in \mathfrak{X}(\mathcal{U})} L^2_\chi([\mathcal{U}])$, $\mathfrak{X}(\mathcal{U})$: cusp datum $= \{(\underset{\substack{\uparrow \\ \text{Levi}}}{M}, \pi)\}/\sim$.

$\rightsquigarrow K_f(x,y) = \sum_{\chi \in \mathfrak{X}(\mathcal{U})} K_{f,\chi}(x,y)$.

$$J_\chi(f \otimes \underbrace{\phi_1 \otimes \phi_2}_{\in \mathcal{S}(L(\mathbb{A}))}) = \int_{[\mathcal{U}'] \times [\mathcal{U}']} K_{f,\chi}(x,y) \cdot \overline{\theta(x,\phi_1)} \cdot \theta(y,\phi_2)\, dx\, dy.$$

<u>Claim</u> $J_\pi \neq 0 \Longleftrightarrow \mathcal{P}_{\mathcal{U}'} \neq 0$ on π.

Also, $K_{f,\pi}(x,y) = \sum_{\varphi \in \mathcal{B}(\pi)} (R(f)\varphi)(x) \cdot \overline{\varphi(y)}$.

$$\rightsquigarrow J_\pi(f \otimes \phi_1 \otimes \phi_2) = \int_{[\mathcal{U}'] \times [\mathcal{U}']} K_{f,\pi}(x,y) \overline{\theta(x,\phi_1)} \cdot \theta(y,\phi_2)$$

$$= \sum_{\varphi \in \mathcal{B}(\pi)} \mathcal{P}_{\mathcal{U}'}(R(f)\varphi, \phi_1) \overline{\mathcal{P}_{\mathcal{U}'}(\varphi, \phi_2)}$$

$G = GL_{n,E} \times GL_{n,E} \overset{\Delta}{\hookleftarrow} H = GL_{n,E}$.

$G' = GL_{n,F} \times GL_{n,F} \hookrightarrow G$.

$$I_\chi(f \otimes \Phi) = \int_{[H]\times[G]} K_{f,\chi}(h,g') \cdot \Theta_\mu(h, \Phi)\, \eta(g')\, dh\, dg'.$$

<u>Claim</u> $I_\pi \neq 0 \iff L(\tfrac{1}{2}, \Pi \times \mu^{-1}) \neq 0$.

<u>Note</u> The above integrals have convergence issues.

<u>Convergence issue</u>

Consider $\quad \int_{[H]} \varphi(h)\, \Theta(h, \Phi)\, dh$.

If $\pi = \pi_1 \times \pi_2$, $\varphi = \varphi_1 \otimes \varphi_2$,

$\Rightarrow \int_{[H]} \varphi(h)\, \Theta(h, \Phi)\, dh = \int_{[H]} \varphi_1(h) \cdot \varphi_2(h) \cdot \Theta(h, \Phi)\, dh$.

Similarly, define for $GL_n \times GL_{n+1}$ that

$$\int_{[GL_n]} \varphi_1(h) \cdot \varphi_2(h)\, dh.$$

Define Jacobi grp $\quad J_n = GL_n \ltimes (F_n \times F^n \times F)$

F_n row vec, F^n column vec.

$\rightsquigarrow J_n$ mimic of GL_{n+1}.

$GL_n(\mathbb{A}) \curvearrowright \mathcal{S}(\mathbb{A}_n)$ extends to Weil-Schrödinger rep of J_n.

$\Theta(j, \Phi) = \sum_{v \in F_n} (j \cdot \Phi)(v)$.

$\varphi_2(h) \cdot \Theta(j, \Phi)$ automorphic form on $J_n(\mathbb{A})$.

Need a truncation.

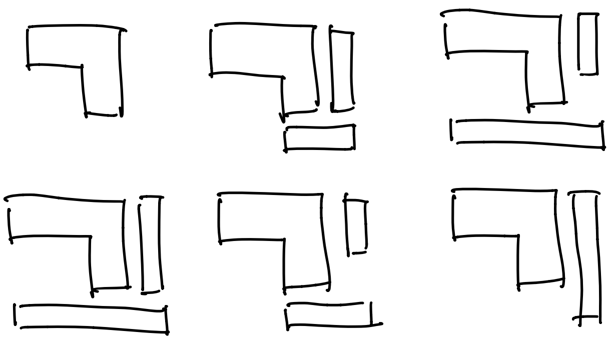
<u>Def</u> A D-parabolic of J_n is a subgrp of form $P(\lambda)$

with $\quad \lambda : \mathbb{G}_m \longrightarrow J_n$ cochar.

$P = P(x)$ std para if $P \cap GL_n$ is standard.

$\mathcal{F}$: set of std D-parabolics.

<u>Example</u> $P \cap GL_n$ can be



Consider the truncated period

$$\int_{[H]} \varphi_1(h)\, \Lambda^T(\varphi_2(h)\, \Theta(h, \Phi))\, dh$$

where $\Lambda^T \varphi = \sum_{P \in \mathcal{F}} \varepsilon_P \sum_{\gamma \in P_n(F) \backslash GL_n(F)} \hat{\tau}_{P_{n+1}} (H_{P_{n+1}}(\gamma h) - T_{P_{n+1}})\, \varphi_P(\gamma h)$.

Consider $\int_{[H] \times [G']} K_f(h, g') \cdot \Theta(h, \Phi)\, \eta(g')\, dh\, dg'$.

Define

$$I_\chi^T(f \otimes \Phi) = \int_{[H] \times [G']} \Big(\sum_{P \in \mathcal{F}} \varepsilon_P \sum_{\substack{\gamma \in P_H(F) \backslash H(F) \\ \delta \in P'(F) \backslash G'(F)}} \hat{\tau}_{P_{n+1}} (H_{P_{n+1}}(\delta, g_i') - T_{P_{n+1}})$$

$$K_{f, P, \chi}(\gamma h, \delta g')\, {}_P\Theta(\gamma h, \Phi) \Big)\, \eta(g')\, dh\, dg'.$$

<u>Thm</u> (1) I_χ^T is absolute convergent,

$T \longmapsto I_\chi^T(f)$ is an exponential of T

pure polynomial w/ const term $I_\chi(f \otimes \Phi)$.

(2) $\sum_\chi I_\chi(f)$ is abs convergent, $I(f) := \sum_\chi I_\chi(f)$.