The categorical local Langlands program

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(Joint with L. Mann)

Setup Fix E nonarch local field, res char p,
Let G quasi-split red grp $/E$.
Fix $\ell \neq p$.

Have $\mathrm{Bun}_G = \coprod_{b \in B(G)} [*/G_b(E)]$

$$\rightsquigarrow \quad D(\mathrm{Bun}_G, \bar{\mathbb{Q}}_\ell) \underset{i_{b!}}{\overset{i_b^*}{\rightleftarrows}} D(G_b(E), \bar{\mathbb{Q}}_\ell)$$

fully faithful

& $D(\mathrm{Bun}_G)$ is glued from these cats.

Take Par_G = stack of ℓ-adically conti maps $W_E \to {}^LG(\bar{\mathbb{Q}}_\ell)$.

$\downarrow$

$\coprod$ (reduced lci Artin stacks of pure dim 0).

+ a canonical map $\mathrm{Par}_G \xrightarrow{q} X_G^{\mathrm{Spec}}$.

Thm (Fargues - Scholze) $\exists$ canonical $\otimes$-action

$$\mathrm{QCoh}(\mathrm{Par}_G) \circlearrowright D(\mathrm{Bun}_G)$$
$$\mathcal{F} \in \mathrm{QCoh}(\mathrm{Par}_G), \quad A \in D(\mathrm{Bun}_G) \quad \rightsquigarrow \quad \mathcal{F} * A.$$

normalized by "$V * (-) = T_V$" (Hecke op).

Fix $(B = TU, \psi)$ Whittaker datum

$$\rightsquigarrow \quad W_\psi := \mathrm{ind}_{U(E)}^{G(E)} \psi.$$

$\rightsquigarrow$ Get $a_\psi : \mathrm{QCoh}(\mathrm{Par}_G) \longrightarrow D(\mathrm{Bun}_G)$

$$\mathcal{F} \longmapsto \mathcal{F} * i_{1!} W_\psi, \qquad i_1 : \mathrm{Bun}_G^1 \hookrightarrow \mathrm{Bun}_G \quad (b = 1)$$

Conj (LLC, updated ver) a_φ is fully faithful

& $\exists$ an equiv

$$\begin{tikzcd} D(\mathrm{Bun}_G) \arrow[rr, "\mathbb{L}_\varphi", "\sim"'] & & \mathrm{Ind\,Coh}(\mathrm{Par}_G) \\ & \mathrm{QCoh}(\mathrm{Par}_G) \arrow[ul, "a_\varphi"] \arrow[ur] & \end{tikzcd}$$

$(\mathrm{QCoh}(\mathrm{Par}_G) \simeq \mathrm{Ind\,Perf}(\mathrm{Par}_G))$.

Prop $\mathbb{L}_\varphi$ is unique if it exists.

If exists $\Leftrightarrow$ C_φ = the right adj of a_φ restricts to an equiv

$$C_\varphi : D(\mathrm{Bun}_G)^\omega \xrightarrow{\sim} \mathrm{Coh}(\mathrm{Par}_G).$$

Goal Prove the conj in some cases.

Key working hypothesis Assume C_φ is compatible w/ Eis series.

- Proved by Linus for GL_2.
- General case: Work in progress, Hamann-Hansen-Mann.

Thm CLLC is true for GL_2.

a specialization for much more general results.

Comment GLLC out of reach in general.

Def G is well-understood if $\underbrace{\text{FS L-param}}_{\text{ss action of a known LLC with good properties.}}$

Thm Assume G is well-understood.

(1) $\exists$ canonical adj functors

$$D(\mathrm{Bun}_G) \underset{R_\varphi}{\overset{\mathbb{L}_\varphi}{\rightleftarrows}} \mathrm{Ind\,Coh}(\mathrm{Par}_G).$$

$$\text{s.t.}\quad \begin{tikzcd} D(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}_\psi"] \arrow[dr, "a_\psi"'] & \mathrm{IndCoh}(\mathrm{Par}_G) \arrow[d] \arrow[dl, phantom, "\circlearrowright"] \\ & \mathrm{QCoh}(\mathrm{Par}_G) \end{tikzcd}$$

and both functors preserve cpt objs.

(2) If a_ψ is fully faithful, then $\mathbb{R}_\psi$ fully faithful.

(3) If $G = GL_n$, $\mathbb{L}_\psi \circ i_{1,!}$ = Ben-Zvi-Chen-Helm-Nadler functor

& a_ψ is fully faithful.

<u>Key new Constr'n</u> An "explicit" partial right adj to $\mathbb{L}_\psi$

+ a "formula" for $\mathbb{R}_\psi$ on some subcat.

Let X a reasonable derived Stack / $\overline{\mathbb{Q}}_\ell$.

<u>Def</u> $\mathrm{IndCoh}(X) \ni \mathcal{F}$ is admissible if

$\cup$

$\mathrm{Adm}(X)$ $\quad$ $R\mathrm{Hom}(\mathcal{G}, \mathcal{F}) \in \mathrm{Perf}(X)$, $\forall\, \mathcal{G} \in \mathrm{Coh}(X)$.

<u>Fact</u> If CLLC is true, $\mathrm{Adm}(\mathrm{Par}_G) \cong D(\mathrm{Bun}_G)^{ULA}$.

$\mathbb{D}_{adm} \circlearrowright \mathrm{IndCoh}(X)$ by $R\mathrm{Hom}(\mathcal{G}, \mathbb{D}_{adm}\mathcal{F})$

$$\cong$$

$$R\mathrm{Hom}(\underbrace{\mathbb{D}_{GS}}_{\text{Grothendieck-Serre duality}}\mathcal{G}, \mathcal{F})^\vee.$$

$\mathbb{D}_{adm}$ is a perfect duality on $\mathrm{Adm}(X)$.

<u>Thm</u> $\mathrm{Adm}(\mathrm{Par}_G) \cap \mathrm{Coh}(\mathrm{Par}_G) = \underbrace{\mathrm{Coh}(\mathrm{Par}_G)_{fin}}$

Supp on finite fibres of $q : \mathrm{Par}_G \to X_G^{Spec}$.

& $\forall\, \mathcal{F} \in \mathrm{Adm}(\mathrm{Par}_G)$, $\mathbb{D}_{adm}\mathcal{F} \in \mathrm{QCoh}(\mathrm{Par}_G) \subset \mathrm{IndCoh}(\mathrm{Par}_G)$.

<u>Def</u> $t_\psi : \mathrm{Coh}(\mathrm{Par}_G)_{fin} \longrightarrow D(\mathrm{Bun}_G)$

$$\mathcal{F} \longmapsto \mathbb{D}_{Verd}\, a_\psi^{-1} \overbrace{\mathbb{D}_{tw,adm}\mathcal{F}}^{\in \mathrm{QCoh}}.$$

Thm If G well-understood, then

- t_ψ has image $\in D(\mathrm{Bun}_G)_{\mathrm{fin}}$.
- $\mathbb{R}_\psi|_{\mathrm{Coh}(\mathrm{Par}_G)_{\mathrm{fin}}} = t_\psi$.

Thm Assume G is well-understood. Then

$$\text{CLLC for } G \iff t_\psi : \mathrm{Coh}(\mathrm{Par}_G)_{\mathrm{fin}} \xrightarrow{\sim} D(\mathrm{Bun}_G)_{\mathrm{fin}} \text{ equiv.}$$

<u>Endgame for G_2</u> Check that t_ψ is essentially surj onto $D(\mathrm{Bun}_{G_2})_{\mathrm{fin}}$.

This is only hard at $\phi \sim \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ or $\phi \sim \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$.

use a good local model
and then use tables computed by Bertoloni-Meli – Koshikawa.