2022 Summer School on the Langlands Program at IHES

THE LOCAL LANGLANDS CONJECTURE

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(SCIBED BY WENHAN DAI)

Abstract. We formulate the local Langlands conjecture for connected reductive groups over local fields, including the internal parametrization of L-packets.

Readme. This is a very preliminary version for the (closed-door) lecture series given by Oliver Taïbi at IHES Summer School 2022. *Please use with caution and do not disseminate.*

Due to the mistake and carelessness of the notetaker, it is missing parts and many references and is full of typos. Also, every sign has at least a 50% chance of being wrong.

Contents

Let F be a local field. We denoted by $\|\cdot\|$ the normalized absolute value of F. In the non-archimedean case it maps a uniformizer to q^{-1} where q is the cardinality of the residue field. If $F \simeq \mathbb{R}$ it is the usual absolute value, if $F \simeq \mathbb{C}$ it is given by $z \mapsto z\overline{z}$.

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1. Representations of reductive groups

1.1. **Setup.** In this section we focus on the case where F is non-archimedean and occasionally indicate the differences for the archimedean case.

Let G be a connected reductive group over F. We refer to [Bor91] [Spr98] [BT65] and [DGA+11] for fundamental results about reductive groups. Let C be an algebraically closed field of characteristic zero, for example $\mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$. We consider **smooth** representations of $G(F)$ with coefficients in C, i.e. pairs (V,π) where V is a vector space over C and $\pi : G(F) \to \mathrm{GL}(V)$ is a morphism of groups such that the map

$$\begin{aligned} G \times V &\longrightarrow V \\ (g,v) &\longmapsto \pi(g)v \end{aligned}$$

is continuous for the natural topology on G and the discrete topology on V. If π is implicit we will also denote $g \cdot v$ for $\pi(g)v$. Recall that such a representation is called **admissible** if for any compact open subgroup K of $G(F)$ the subspace

$$V^K = \{v \in V \mid \forall k \in K,\ \pi(k)v = v\}$$

of V has finite dimension. It is a non-trivial but well-known fact that *any irreducible representation is admissible*. Denote by $Z(G)$ the center of G. By a suitable generalization of Schur's lemma, any irreducible representation has a central character $Z(G)(F) \to C^\times$. For a smooth representation (V,π) of $G(F)$, its **contragredient** $(\widetilde{V},\tilde{\pi})$ is the space of K-finite linear forms on V.

Remark 1.1. In the case of an archimedean field F we only consider coefficients $C = \mathbb{C}$. The analogue of smooth representations are $(\mathfrak{g},K)$-modules where $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \operatorname{Lie} G(F)$ and K is a maximal compact subgroup of $G(F)$. For many notions it is necessary to relate $(\mathfrak{g},K)$-modules to continuous representations of $G(F)$ on topological vector spaces. See e.g. [Wal88, §3.4] for the relation between the two notions in the case of unitary irreducible representations.

1.2. **Parabolic induction and the Jacquet functor.** Let P be a parabolic subgroup of G. Let N be the unipotent radical of P and $M = P/N$ its reductive quotient. Recall that there exists a section $M \to P$, unique up to conjugation by $N(F)$. Let $\delta_P(p) = |\det(\mathrm{Ad}(p)|\operatorname{Lie}(N))|$ be the modulus character (of $M(F)$ acting on $N(F)$). We choose a square root $\sqrt{q}$ of q in C, allowing us to define $\delta_P^{1/2}$. If $C = \mathbb{C}$ we naturally choose $\sqrt{q} \in \mathbb{R}_{>0}$.

Let (V,σ) be a smooth representation of $M(F)$, which we can see as a representation of $P(F)$ trivial on $N(F)$. The normalized parabolically induced representation $i_P^G\sigma$ is the space of locally constant function $f : G(F) \to V$ such that for any $p \in P(F)$ and $g \in G(F)$ we have $f(pg) = \delta_P(p)^{1/2}\sigma(p)f(g)$, with left action by $(g \cdot f)(x) = f(xg)$. If σ is admissible (resp. has finite length) then $i_P^G\sigma$ is admissible (resp. has finite length). The introduction of $\delta_P^{1/2}$ in the definition are motivated by the fact that if $C = \mathbb{C}$ and (V,σ) is unitary, i.e. endowed with a $M(F)$-invariant Hermitian inner product, then $i_P^G\sigma$ has a natural $G(F)$-invariant Hermitian inner product. In particular if σ is admissible and unitarizable then $i_P^G\sigma$ is semi-simple.

For (π, V) a smooth representation of $G(F)$, denote by V_N the space of coinvariants for the action of $N(F)$, which is naturally a smooth representation π_N of $M(F)$. The **normalized Jacquet functor** applied to (π, V) is the smooth representation $r_P^G \pi = \delta_P^{1/2} \otimes \pi_N$ of $M(F)$ on the space V_N. It also preserves admissibility and the property of being of finite length.

Recall that an irreducible (hence admissible) smooth representation (V, π) of $G(F)$ is called **supercuspidal** if $V_N = 0$ for any parabolic $P = MN \subsetneq G$; or equivalently, if for every proper parabolic subgroup P, the Jacquet functor $r_P^G(\cdot)$ is zero. This is equivalent to all "matrix coefficients"

$$\begin{aligned} G(F) &\longrightarrow C \\ g &\longmapsto \langle \pi(g)v, \tilde{v} \rangle \end{aligned}$$

for $v \in V$ and $\tilde{v} \in \widetilde{V}$, being compactly supported modulo center. Note that if $\omega_\pi : Z(G(F)) \to C^\times$ is the central character of π then matrix coefficients of π are ω_π-equivariant.

We recall in the following theorem the notion of supercuspidal support.

Theorem 1.2. *Let π be an irreducible representation of $G(F)$.*

1. *There exists a parabolic subgroup $P = MN$ of G and a supercuspidal irreducible representation σ of $M(F)$ such that π embeds in $i_P^G \sigma$.*
2. *If $P' = M'N'$ is a parabolic subgroup of G and σ' is a supercuspidal irreducible representation of $M'(F)$ then π is isomorphic to a subquotient of $i_{P'}^G \sigma'$ if and only if there exists an element of $G(F)$ conjugating (M, σ) and (M', σ), where M and σ are given as in* (1)*.*

The conjugacy classes of (M, σ) may be called the **supercuspidal support of** π.

Proof. The first part is due to Jacquet: see [Cas, Theorem 5.1.2]. The second part seems to be due to Harish-Chandra: see [Sil79, Theorem 4.6.1, §5.3.1 and Theorem 5.4.4.1] for the "if" part. The "only if" part can be deduced from Bernstein center theory [Ber84a]. See also [BZ77]. □

The $G(F)$-conjugacy class of (M, σ) in the previous theorem is called the **supercuspidal support** of π.

1.3. **Asymptotic properties.** For the rest of this section we assume $C = \mathbb{C}$.

Definition 1.3. Let (V, π) be a smooth irreducible representation of $G(F)$. Let $\omega_\pi : Z(G(F)) \to \mathbb{C}^\times$ be its central character. If ω_π is unitary, then we say that π is **essentially square-integrable** if all of its matrix coefficients are square-integrable modulo center:

$$\forall v \in V,\ \forall \tilde{v} \in \widetilde{V}, \quad \int_{G(F)/Z(G(F))} |\langle \pi(g)v, \tilde{v} \rangle|^2 dg < \infty.$$

In general (without assuming that ω_π is unitary) there is a unique smooth character $\chi : G(F) \to \mathbb{R}_{>0}$ such that the central character of $\chi \otimes \pi$ is unitary [Cas, Lemma 5.2.5], and we say that π is essentially square-integrable if $\chi \otimes \pi$ is.

If π is an essentially square-integrable irreducible smooth representation of $G(F)$ and if ω_π is unitary then π is unitarizable.

Essential square-integrability can be checked on the Jacquet module of a representation, as recalled in Proposition 1.4 below. For a Levi subgroup M of G we denote by A_M the largest split torus in the centre of M. Denote $\mathfrak{a}_M^* := X^*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$. We have an isomorphism

$$\begin{aligned} \mathfrak{a}_M^* &\longrightarrow \mathrm{Hom}_{\mathrm{cont}}(A_M(F), \mathbb{R}_{>0}) \\ \chi \otimes s &\longmapsto (x \mapsto |\chi(x)|^s). \end{aligned} \tag{1.1}$$

Proposition 1.4 ([Wal03, Proposition III.1.1])**.** *Let (V, π) be an irreducible smooth representation of $G(F)$. Assume that the central character of π is unitary (we can reduce to this case by twisting). Then (V, π) is essentially square-integrable if and only if for every parabolic subgroup $P = MN$ of G, the absolute value of any character of $A_M(F)$ occurring in $r_P^G \pi$ is a linear combination with positive coefficients of the simple roots of A_M in N via the isomorphism* (1.1)*.*

Replacing "positive" by "non-negative" in this characterization we get the notion of **tempered representation**. This is also equivalent to a growth condition on coefficients [Wal03, Proposition III.2.2].

We have the following implications, for an irreducible smooth representation of $G(F)$ having unitary central character:

$$\text{supercuspidal} \implies \text{essentially square-integrable} \implies \text{tempered} \implies \text{unitarizable.}$$

For non-commutative G none of these implications is an equivalence.

Proposition 1.5 ([Wal03, Proposition III.4.1])**.**

(1) *Let $P = MN$ be a parabolic subgroup of F and σ an essentially square-integrable irreducible smooth representation of $M(F)$ having unitary central character. Then the induced representation $i_P^G \sigma$ is semi-simple, has finite length and any irreducible subrepresentation is tempered.*
(2) *Let (P, σ) and (P', σ') be two pairs as in* (1)*. Then $i_P^G \sigma$ and $i_{P'}^G \sigma'$ admit isomorphic irreducible subrepresentations if and only if the pairs (M, σ) and (M', σ') are conjugated by $G(F)$, and in this case the two induced representations are isomorphic.*
(3) *For any tempered irreducible smooth representation π of $G(F)$ there exists a pair (P, σ) as in* (1) *such that π is isomorphic to a subrepresentation of $i_P^G \sigma$.*

Remark 1.6. For $G = \mathrm{GL}_n$, parabolically induced representations as in Proposition 1.5 are always irreducible [Ber84b, §0.2] and so the proposition completely classifies tempered representations in terms of essentially square-integrable representations of smaller general liner groups.

For arbitrary G such induced representations are **generically irreducible** (see [Wal03, Proposition IV.2.2] for a precise statement), but decomposing such induced representations is a suitable problem in general.

The tempered representations are exactly the ones occurring in Harish-Chandra's Plancherel formula, expressing the values of any locally constant and compactly supported $f : G(F) \to \mathbb{C}$ in terms of the action of f in tempered representations (or expressing $f(1)$ in terms of the traces of f in tempered representations).

Finally the "Langlands classification", that we recall below, classifies irreducible smooth representations of $G(F)$ in terms of tempered representations of Levi subgroups. For a connected reductive group M denote by $X^*(M)^\Gamma$ the abelian group of morphisms $M \to \mathrm{GL}_1$ (defined over F). The restriction morphism $X^*(M)^\Gamma \to X^*(A_M)$ is an isogeny (it is injective with finite cokernel) and so it induces an isomorphism $\mathrm{Res}^M_{A_M} : X^*(M)^\Gamma \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathfrak{a}^*_M$. We have an isomorphism

$$\begin{aligned} X^*(M)^\Gamma \otimes_{\mathbb{Z}} \mathbb{R} &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont}}(M(F), \mathbb{R}_{>0}) \\ \chi \otimes s &\longmapsto (x \mapsto |\chi(x)|^s). \end{aligned} \tag{1.2}$$

Fix a minimal parabolic subgroup P_0 of G and a Levi factor M_0 of P_0. Let $Y \subseteq X^*(A_{M_0})$ be the subgroup of characters which are trivial on $A_{M_0} \cap G_{\mathrm{der}}$. Recall from [BT65, Corollaire 5.8] that the set of roots of A_{M_0} in G is a root system in $(X^*(A_{M_0}, Y)$. Let $\Delta \subseteq X^*(A_{M_0})$ be the set of simple roots for the order corresponding to P_0. The rational Weyl group $N(A_{M_0}, G(F))/M_0(F)$ acts on $\mathfrak{a}^*_{M_0}$; fix an invariant inner product $(\cdot,\cdot)$ on $\mathfrak{a}^*_{M_0}$. For M a standard Levi subgroup of G the restriction map $X^*(A_{M_0}) \to X^*(A_M)$ induces a surjective map $\mathrm{Res}^{A_{M_0}}_{A_M} : \mathfrak{a}^*_{M_0} \to \mathfrak{a}^*_M$. We also have a composite map in the other direction

$$j^M_{M_0} : \mathfrak{a}^*_M \xrightarrow{(\mathrm{Res}^M_{A_M})^{-1}} X^*(M)^\Gamma \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\mathrm{Res}^M_{M_0}} X^*(M_0)^\Gamma \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\mathrm{Res}^{M_0}_{A_{M_0}}} \mathfrak{a}^*_{M_0}$$

and the composition $\mathrm{Res}^{A_{M_0}}_{A_M} \circ j^M_{M_0}$ is $\mathrm{id}_{\mathfrak{a}^*_M}$. In fact one can check that $j^M_{M_0} \circ \mathrm{Res}^{A_{M_0}}_{A_M}$ is the orthogonal projection $\mathfrak{a}^*_{M_0} \to j^M_{M_0}(\mathfrak{a}^*_M)$.

Theorem 1.7 ([Sil78, Theorem 4.1])**.**
(1) *Let P be a standard Levi subgroup of G (with respect to P_0) and M is Levi factor containing M_0. Let σ be a tempered irreducible smooth representation of $M(F)$ (in particular its central character is unitary). Let $\nu \in X^*(M)^\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ be such that for any $\alpha \in \Delta$ not occurring in M we have $(\mathrm{Res}^M_{A_{M_0}} \nu, \alpha) > 0$. Consider ν as a character of $M(F)$ via (1.2), and denote by σ_ν the twist of σ by this character. Then the induced representation $i^G_P(\sigma_\nu)$ admits a unique irreducible quotient $J(P, \sigma, \nu)$. Let $\overline{P}$ be a parabolic subgroup of G which is opposite to P. We have $\dim_{\mathbb{C}} \mathrm{Hom}_G(i^G_P(\sigma_\nu), i^G_{\overline{P}}(\sigma_\nu)) = 1$ and any nonzero element in this line identifies $J(P, \sigma, \nu)$ with the unique irreducible subrepresentation of $i^G_{\overline{P}}(\sigma_\nu)$.*
(2) *Let π be an irreducible smooth representation of $G(F)$. There exists a unique triple (P, σ, ν) as above such that π is isomorphic to the quotient $J(P, \sigma, \nu)$.*

Remark 1.8. It will be useful to reformulate the positivity condition on ν in terms of the absolute root system of G. First note that the condition does not depend on the choice of an admissible inner product on $\mathfrak{a}^*_{M_0}$. Let T be a maximal torus in $M_{0,F^{\mathrm{sep}}}$ and choose a Borel subgroup B of $G_{F^{\mathrm{sep}}}$ containing T and contained in $P_{0,F^{\mathrm{sep}}}$. Choose an admissible inner product $(\cdot,\cdot)_T$ on $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, i.e. one variant under the absolute Weyl group. Consider the restriction map $X^*(T) \to X^*(A_{M_0})$, inducing a surjective map $\mathrm{Res}^T_{A_{M_0}} : X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathfrak{a}^*_{M_0}$. It identifies $\mathfrak{a}^*_{M_0}$ with $\ker(\mathrm{Res}^T_{A_{M_0}})^\perp$, and we can endow $\mathfrak{a}^*_{M_0}$ with the restriction of $(\cdot,\cdot)_T$. It turns out that this restriction is also an admissible inner product on $\mathfrak{a}_{M_0}*$ for the relative Weyl group [BT65, §6.10]. The roots of A_{M_0} on $\mathrm{Lie}\, N$ are the restrictions of the roots of T on $\mathrm{Lie}\, N$. So the positivity condition in Theorem 1.7 is equivalent to $\langle \mathrm{Res}^M_T \nu, \alpha^\vee \rangle > 0$ for any simple root $\alpha \in X^*(T)$ which does not occur in M.

For analogous results in the case where F is archimedean see [Lan89] and [Wal88, Chapter 5].

1.4. **Harish-Chandra characters.** Denote by $C_c^\infty(G(F))$ the space of locally constant and compactly supported functions $G(F) \to \mathbb{C}$. Recall that any such function in bi-invariant under some compact open subgroup of $G(F)$.

Let (V, π) be an admissible representation of $G(F)$. Any $f \in C_c^\infty(G(F))$ gives an endomorphism $\pi(f)$ of V via defining $\pi(f)v = \int_{G(F)} f(g)\pi(g)v dg$. By admissibility this integral is actually a finite sum. Moreover, the image of any $\pi(f)$ has finite range and we may consider $\Theta_\pi(f) = \operatorname{tr} \pi(f)$. The linear form $\Theta_\pi : C_c^\infty(G(F)) \to \mathbb{C}$ is called the **Harish-Chandra character** of π. A standard result in representation theory of finite-dimensional associative algebras implies that the Harish-Chandra characters Θ_π of the irreducible smooth representations of $G(F)$ (up to isomorphism) are linearly independent. In particular a smooth representation of finite length is characterized up to semi-simplification by its Harish-Chandra character.

Denote by G_{rs} the regular semi-simple locus in G, an open dense subscheme.

Theorem 1.9 ([HC99, Theorem 16.3])**.** *Assume that F is a non-archimedean local field of characteristic zero. Let (V, π) be an irreducible smooth representation of $G(F)$. Choose a Haar measure for $G(F)$. There exists a unique element of $L^1_{\mathrm{loc}}(G(F))$, also denoted Θ_π, such that for any $f \in C_c^\infty(G(F))$ we have*

$$\operatorname{tr} \pi(f) = \int_{G(F)} \Theta_\pi(g) f(g) dg.$$

Moreover, $\Theta_\pi(g)$ is represented by a unique locally constant function on $G_{\mathrm{rs}}(F)$.

Unfortunately this result does not seem to be known in full generality in positive characteristic, but see [CGH14]. Harish-Chandra characters behave well under induction [vD72].

See [Wal88, Chapter 8] for the archimedean case.

2. Langlands dual groups

We recall the definition of Langlands dual groups. We refer to [Bor79, §I.2] for details not recalled below. In this section F could be any field, $\overline{F}$ is a separable closure of F and we denote $\Gamma = \mathrm{Gal}(\overline{F}/F)$.

2.1. **Based root data.** Let G be a connected reductive group over F. There exists a finite separable extension E/F such that G_E admits a Killing pair (also called Borel pair) (B, T) [DGA+11, Exposé XXII Corollaire 2.4 and Proposition 5.5.1]. We may do assume that E/F is a subextension of $\overline{F}/F$. Associated to (G_E, B, T) we have a based (reduced) root datum $(X, R, R^\vee, \Delta)$ where

- X is the group of characters of T,
- $R \subset X$ the set of roots of T in G_E,
- $R^\vee$ the set of coroots of T (a subset of $X^\vee = \mathrm{Hom}(X, \mathbb{Z})$, the group of cocharacters of T), and
- $\Delta \subset R$ the set of simple roots corresponding to B.[1]

The group $G(E)$ acts (by conjugation) transitively on the set of Killing pairs in G_E [DGA+11, Exposé XXVI Corollaire 5.7 (ii) and Corollaire 1.8] and the (scheme-theoretic) stabilizer of (B, T) is T [DGA+11, Exposé XXII Cor 5.3.12 and Proposition 5.6.1], which centralizes T. It follows that other choices of Killing pair in G_E yield based root data *canonically* isomorphic to $(X, R, R^\vee, \Delta)$, and so do other choices for E.

We also obtain a continuous action of Γ on this based root datum, that we now recall. The group $\mathrm{Gal}(E/F)$ acts on the set of closed subgroups of G_E: if $G = \operatorname{Spec} A$ for a Hopf algebra A over F and a closed subgroup H corresponds to an ideal I of $A \otimes_F E$, then for $\sigma \in \mathrm{Gal}(E/F)$ we let $\sigma(H)$ be the closed subgroup corresponding to $\sigma(I)$. If $K = \operatorname{Spec} B$ is a linear algebraic group over F and $\lambda : H \to K_E$ is a morphism, dual to a morphism of Hopf algebras $\lambda^\sharp : B \otimes_F E \to (A \otimes_F E)/I$, define $\sigma(\lambda) : \sigma(H) \to K_E$ as dual to

$$\sigma \circ \lambda^\sharp \circ \sigma^{-1} : B \otimes_F E \to (A \otimes_F E)/\sigma(I).$$

Now for $\sigma \in \mathrm{Gal}(E/F)$ there is a unique $T(E)g_\sigma \in T(E)\backslash G(E)$ such that we have $\sigma(B, T) = \mathrm{Ad}(g_\sigma^{-1})(B, T)$, and we get a well-defined isomorphism $\mathrm{Ad}(g_\sigma) : \sigma(T) \simeq T$. We obtain an action of Γ on $X = X^*(T)$ such that $\sigma \in \mathrm{Gal}(E/F)$ maps $\lambda : T \to \mathrm{GL}_{1,E}$ to $\sigma(\lambda) \circ \mathrm{Ad}(g_\sigma)^{-1}$. It is straightforward to check that this action preserves R and Δ and that the dual action on $X^\vee$ preserves $R^\vee$. We denote by brd_F the resulting functor from the groupoid of connected reductive groups over F to the groupoid of based root data with continuous action of Γ.

Definition 2.1. Let G be a connected reductive group over F. Define a *groupoid of inner twists* $\mathsf{IT}(G)$ as follows.

- The objects of $\mathsf{IT}(G)$ are the inner twists of G, i.e. pairs (G', ψ) consisting of a connected reductive group G' over F and an isomorphism $\psi : G_{\overline{F}} \simeq G'_{\overline{F}}$ such that for any $\sigma \in \Gamma$ the automorphism $\psi^{-1}\sigma(\psi)$ of $G_{\overline{F}}$ is inner.

[1] Strictly speaking we should also include in the datum the bijection $R \to R^\vee$ as in [DGA+11, Exposé XXI], or include the orthogonal of $R^\vee$ in X as in [BT65, §2.1].

- A morphism between two inner twists (G_1, ψ_1) and (G_2, ψ_2) of G is an element $g \in G_{\mathrm{ad}}(\overline{F})$ such that for any $\sigma \in \Gamma$ we have

$$\psi_2^{-1}\sigma(\psi_2) = \mathrm{Ad}(\sigma(g))\psi_1^{-1}\sigma(\psi_1)\mathrm{Ad}(\sigma(g))^{-1}. \tag{2.1}$$

Remark 2.2.
(1) One can check that any inner twist $\psi : G_{\overline{F}} \to G'_{\overline{F}}$ yields a canonical isomorphism $\mathrm{brd}_F(G) \simeq \mathrm{brd}_F(G')$.
(2) For an inner twist $\psi : G_{\overline{F}} \to G'_{\overline{F}}$ the map

$$\Gamma \to G_{\mathrm{ad}}(\overline{F}), \quad \sigma \mapsto \psi^{-1}\sigma(\psi)$$

is a 1-cocycle, i.e. an element of $Z^1_{\mathrm{cont}}(\Gamma, G_{\mathrm{ad}}) = Z^1(F, G_{\mathrm{ad}})$.
(3) The relation (2.1) imply that the isomorphism

$$\psi_2 \mathrm{Ad}(g)\psi_1^{-1} : G_{1,\overline{F}} \to G_{2,\overline{F}}$$

is defined over F, i.e. descends to an isomorphism $G_1 \simeq G_2$.
(4) For an inner twist (G', ψ) of G we have an isomorphism

$$\mathrm{Aut}(G', \psi) \to G'_{\mathrm{ad}}(F), \quad g \mapsto \psi(g).$$

Proposition 2.3. *Let b be a based root datum with continuous action of Γ. Let CRG_b be the groupoid of pairs (G, α) where G is a connected reductive group over F and $\alpha : b \simeq \mathrm{brd}_F(G)$ is an isomorphism of based root data with action of Γ, with obvious morphisms. In other words CRG_b is the groupoid fiber of b for brd_F.*

(1) *There exists an object (G^*, α^*) of CRG_b such that G^* is quasi-split. Two such objects are isomorphic.*
(2) *Any object (G, α) of CRG_b yields equivalences of groupoids*

$$Z^1(F, G_{\mathrm{ad}}) \xleftarrow{\sim} \mathsf{IT}(G) \xrightarrow{\sim} \mathsf{CRG}_b.$$

This gives in particular a bijection between $H^1(F, G_{\mathrm{ad}})$ and the set of isomorphism classes in CRG_b.

Proof. This is a reformulation of [DGA$^+$11, Exposé XXIV Théorème 3.11] in the case where the base is the spectrum of a field. □

To sum up, we can "classify" connected reductive groups over F as follows:

- fix a representative in each isomorphism class of based root datum with continuous action of Γ;
- for each such representative b, fix a quasi-split connected reductive group G^* over F together with an isomorphism $\mathrm{brd}_F(G^*) \simeq b$;
- for each element of $H^1(F, G^*_{\mathrm{ad}})$ choose an inner twist (G, ψ) of G^* representing it.

Up to isomorphism each connected reductive group G over F arises in this way. It can happen that an isomorphism class of connected reductive groups arises more than once, because $H^1(F, G_{\mathrm{ad}}) \to H^1(F, \mathrm{Aut}(G))$ is not injective in general (equivalently, the functor brd_F is not full).

2.2. **Langlands dual groups.** Let C be an algebraically closed field of characteristic zero. Let G be a connected reductive group over F and let $\mathrm{brd}_F(G) = (X, R, R^\vee, \Delta)$ be its associated based root datum endowed with a continuous action of Γ. Let $(\widehat{G}, \mathcal{B}, \mathcal{T}, (X_\alpha)_{\alpha\in\Delta^\vee})$ be the *pinned connected reductive group* over C with the associated based root datum $(X^\vee, R^\vee, R, \Delta^\vee)$, i.e. the *dual* of $\mathrm{brd}_F(G)$ (ignoring the action of Γ from now). The choice of a pinning induces a splitting of the extension

$$1 \to \widehat{G}_{\mathrm{ad}} \to \mathrm{Aut}(\widehat{G}) \to \mathrm{Out}(\widehat{G}) \to 1$$

because the subgroup $\mathrm{Aut}(\widehat{G}, \mathcal{B}, \mathcal{T}, (X_\alpha)_{\alpha\in\Delta^\vee})$ of $\mathrm{Aut}(\widehat{G})$ maps bijectively onto $\mathrm{Out}(\widehat{G})$ [DGA+11, Exposé XXIV Théorème 1.3]. We also have an isomorphism

$$\mathrm{Out}(\widehat{G}) \simeq \mathrm{Aut}(X^\vee, R^\vee, R, \Delta^\vee) \simeq \mathrm{Aut}(X, R, R^\vee, \Delta)$$

and so we have an action of Γ on $\widehat{G}$ (preserving the pinning and factoring through a finite Galois group). Denote ${}^L G = \widehat{G} \rtimes \Gamma$ the Langlands dual group, also called L-group. It is sometimes useful (or just convenient) to replace Γ by a finite Galois group or by the Weil group in this semi-direct product.

One can give a more pedantic definition of Langlands dual group in order to avoid the inelegant choice of pinning. Namely, define an L-group for G as an extension ${}^L G$ of Γ by $\widehat{G}$, where $\widehat{G}$ is a split connected reductive group endowed with an isomorphism of its base root datum with the dual of that of G, such that the induced morphism $\Gamma \to \mathrm{Out}(\widehat{G})$ is as above, and endowed with a $\widehat{G}$-conjugacy class of splittings $\Gamma \to {}^L G$, called distinguished splittings, such that any (equivalently, one) of these splittings s preserves a pinning of $\widehat{G}$. It is not necessary to specify the pinning, since for a distinguished splitting s we have that $\widehat{G}^{s(\Gamma)}$ acts transitively on the set of such pinnings: see [Kot84, Corollary 1.7]. By the same argument, for any pinning of $\widehat{G}$ a distinguished splitting fixing it is unique up to

$$\ker(Z^1(\Gamma, Z(\widehat{G})) \to H^1(\Gamma, \widehat{G})) = B^1(\Gamma, Z(\widehat{G})).$$

Note that all distinguished splittings induce the same action of Γ on $Z(\widehat{G})$.

By Proposition 2.3 for two connected reductive groups G_1 and G_2 their Langlands dual groups ${}^L G_1$ and ${}^L G_2$ are isomorphic as extensions of Γ if and only if G_1 and G_2 are inner forms of each other, and in this case they are even isomorphic as extensions endowed with conjugacy classes of distinguished splittings.

The construction of the Langlands dual group is not functorial for arbitrary morphisms between connected reductive groups, however in the following cases functoriality is straightforward.

- Let G be a quasi-split connected reductive group and (B, T) a Borel pair defined over F. Choose a distinguished splitting $s_G : \Gamma \to {}^L G$ preserving a pinning $(\mathcal{B}, \mathcal{T}, (X_\alpha)_{\alpha\in\Delta^\vee})$ of $\widehat{G}$ and a distinguished splitting $s_T : \Gamma \to {}^L T$. Then the canonical isomorphism $\widehat{T} \simeq \mathcal{T}$ extends to an embedding ${}^L T \hookrightarrow {}^L G$ whose composition with s_T is s_G.
- For $G = G_1 \times_F G_2$ we can identify ${}^L G$ with ${}^L G_1 \times_\Gamma {}^L G_2$.
- A central isogeny (see [DGA+11, Exposé XXII Définition 4.2.9]) $G \to H$ induces a surjective morphism with finite kernel ${}^L H \to G$.

- There are weaker forms of functoriality. Let G be a connected reductive group and T a maximal torus of G defined over F. Choose a Borel subgroup B of $G_{\overline{F}}$ containing $T_{\overline{F}}$ and a splitting $s : \Gamma \to {}^L G$ preserving a pinning $(\mathcal{B}, \mathcal{T}, (X_\alpha)_\alpha)$ of $\widehat{G}$. We have a canonical isomorphism $\widehat{T} \simeq \mathcal{T}$, but the Galois actions differ by a 1-cocycle taking values in the Weyl group. In general we don't have a canonical embedding ${}^L T \hookrightarrow {}^L G$,[2] but note that the induced embedding $Z(\widehat{G}) \hookrightarrow {}^L T$ is Γ-equivariant.

In the next section we recall how the first case generalizes to parabolic subgroups in arbitrary connected reductive groups.

2.3. **Parabolic subgroups and L-embeddings.** A parabolic subgroup $\mathcal{P}$ of ${}^L G$ is a closed subgroup mapping onto Γ and such that $\mathcal{P}^0 := \mathcal{P} \cap \widehat{G}$ is a parabolic subgroup of $\widehat{G}$. The set of parabolic subgroups is clearly stable under conjugation by $\widehat{G}$. If $\mathcal{P}$ is a parabolic subgroup of ${}^L G$ then $\mathcal{P}$ is the normalizer of $\mathcal{P}^0$ in ${}^L G$.

Choose a Killing pair $(\mathcal{B}, \mathcal{T})$ of $\widehat{G}$. Recall that a parabolic subgroup of $\widehat{G}$ is conjugated to a unique one containing $\mathcal{B}$, and that parabolic subgroups of $\widehat{G}$ containing $\mathcal{B}$ correspond bijectively to subsets of $\Delta^\vee$ (or Δ, using the bijection $\alpha \mapsto \alpha^\vee$), by associating to $\mathcal{P}^0$ the set of $\alpha \in \Delta^\vee$ (seen as characters of $\mathcal{T}$) such that $-\alpha$ is a root of $\mathcal{T}$ in $\mathcal{P}^0$. Embed $\mathcal{B}$ in a pinning $(\mathcal{B}, \mathcal{T}, (X_\alpha)_{\alpha \in \Delta^\vee})$ of $\widehat{G}$, and let $s : \Gamma \to {}^L G$ be a distinguished section fixing this pinning. Then $\mathcal{B}s(\Gamma)$ is a (minimal) parabolic subgroup of ${}^L G$, and any parabolic subgroup of ${}^L G$ is conjugated under $\widehat{G}$ to one containing $\mathcal{B}s(\Gamma)$. A parabolic subgroup $\mathcal{P}^0$ of $\widehat{G}$ containing $\mathcal{B}$ is such that its normalizer $\mathcal{P}$ in ${}^L G$ maps onto Γ (i.e. $\mathcal{P}$ is a parabolic subgroup of ${}^L G$) if and only if the corresponding subset of $\Delta^\vee$ is stable under Γ. Therefore $\widehat{G}$-conjugacy classes of parabolic subgroups of ${}^L G$ also correspond bijectively to Γ-stable subsets of $\Delta^\vee$.

Using the bijection between Δ and $\Delta^\vee$ we obtain a bijection between the set of Γ-stable $G(\overline{F})$-conjugacy classes of parabolic subgroups of $G_{\overline{F}}$ and the set of $\widehat{G}$-conjugacy classes of parabolic subgroups of ${}^L G$. The obvious map from the set of $G(F)$-conjugacy classes of parabolic subgroups of G to the set of Γ-stable $G(\overline{F})$-conjugacy classes of parabolic subgroups of $G_{\overline{F}}$ is injective, and it is surjective if and only if G is quasi-split.

Recall from [Bor79, §3.4] that if $\mathcal{P}$ is a parabolic subgroup of ${}^L G$ and $\mathcal{M}^0$ is a Levi factor of $\mathcal{P}^0$ then the normalizer $\mathcal{M}$ of $\mathcal{M}^0$ in $\mathcal{P}$ maps onto Γ and $\mathcal{P}$ is the semi-direct product of its unipotent radical and $\mathcal{M}$. In this situation we say that $\mathcal{M}$ is a Levi factor of $\mathcal{P}$, and a Levi subgroup of ${}^L G$.

Let P be a parabolic subgroup of G. Choose a distinguished splitting $s : \Gamma \to {}^L G$ stabilizing a pinning $\mathcal{E} = (\mathcal{B}, \mathcal{T}, (X_\alpha)_{\alpha \in \Delta^\vee})$ of $\widehat{G}$, and let ${}^L P$ be the parabolic subgroup of ${}^L G$ corresponding to P and containing $\mathcal{B}$. Let $M = P/N$ be the reductive quotient of P. Taking Killing pairs inside P in the definition of brd_F we obtain an isomorphism between $\mathrm{brd}_F(M)$ and $(X, R_P, R_P^\vee, \Delta_P)$ where Δ_P is the set of simple roots $\alpha \in \Delta$ such that $-\alpha$ also occurs in P, $R_P = R \cap \mathrm{Span}(\Delta_P)$, $\Delta_P^\vee = \{\alpha^\vee \mid \alpha \in \Delta_P\}$, and $R_P^\vee = R^\vee \cap \mathrm{Span}(\Delta_P^\vee)$. Let $\mathcal{E}_M = (\mathcal{B}_M, \mathcal{T}_M, (Y_\alpha)_\alpha)$ be a pinning of $\widehat{M}$ and $s_M : \Gamma \to {}^L M$ a corresponding distinguished splitting. These choices determine an embedding

$$\iota[P, \mathcal{E}, s, \mathcal{E}_M, s_M] : {}^L M \longrightarrow {}^L G$$

characterized by the following properties.

[2] See however [LS87, §2.6] and [Kal].

(1) It maps $(\mathcal{B}_M, \mathcal{T}_M)$ to $(\mathcal{B}, \mathcal{T})$, and on $\mathcal{T}_M$ it is the isomorphism $\mathcal{T}_M \simeq \mathcal{T}$ induced by the above embedding $\mathrm{brd}_F(M) \hookrightarrow \mathrm{brd}_F(G)$,
(2) it maps $\mathcal{E}_M$ to $\mathcal{E}$, and
(3) we have $\iota[P, \mathcal{E}, s, \mathcal{E}_M, s_M] \circ s_M = s$.

The image of $\iota[P, \mathcal{E}, s, \mathcal{E}_M, s_m]$ is clearly a Levi subgroup of ${}^L G$. The formation of $\iota[P, \mathcal{E}, s, \mathcal{E}_M, s_M]$ satisfies obvious equivariance properties with respect to conjugation by $\widehat{M}$ and $\widehat{G}$. In particular we have an embedding $\iota_P : {}^L M \to {}^L G$ well-defined up to conjugation by $\widehat{G}$.

Lemma 2.4. *Let M be a Levi subgroup of G. Let P and P' be parabolic subgroups of G admitting M as a Levi factor. Then ι_P and $\iota_{P'}$ are conjugated by $\widehat{G}$.*

Proof. First we recall a general construction. Fix a pinning $\mathcal{E} = (\mathcal{B}, \mathcal{T}, (X_\alpha)_\alpha)$ in $\widehat{G}$ and a distinguished splitting $s : \Gamma \to {}^L G$ fixing it. For a Killing pair (B, T) in $G_{\overline{F}}$ we denote by $\gamma[(B,T),(\mathcal{B},\mathcal{T})]$ the isomorphism $X^*(\mathcal{T}) \simeq X_*(T)$. Considering Weyl groups inside automorphism groups of tori this also induces an isomorphism

$$\omega[(B,T),(\mathcal{B},\mathcal{T})] : W(T, G_{\overline{F}}) \simeq W(\mathcal{T}, \widehat{G})$$

We have an action of Γ on $W(T, G_{\overline{F}})$: for $\sigma \in \Gamma$ let $T(\overline{F}) g_\sigma \in T(\overline{F}) \backslash G(\overline{F})$ be the class for which $\sigma(B,T) = \mathrm{Ad}(g_\sigma^{-1})(B,T)$, then $x \mapsto \mathrm{Ad}(g_\sigma)(\sigma(x))$ induces an automorphism of $W(T, G_{\overline{F}})$. One can check that the isomorphism $\omega[(B,T),(\mathcal{B},\mathcal{T})]$ is Γ-equivariant for this action on $W(T, G_{\overline{F}})$ and the action via s on $W(\mathcal{T}, \widehat{G})$.

Fix $\mathcal{E}$, s, $\mathcal{E}_M$ and s_M as above. Fix a Borel pair (B_M, T) in $M_{\overline{F}}$. This determines two Borel subgroups B and B' in $G_{\overline{F}}$ that are characterized by the properties $B \cap M_{\overline{F}} = B_M$ and $N_{\overline{F}} \subset B$ and similarly for B'. There is a unique $x \in W(T, G_{\overline{F}})$ for which $\mathrm{Ad}(x)(B,T) = (B', T)$. Let $n : W(\mathcal{T}, \widehat{G}) \to N(\mathcal{T}, \widehat{G})$ be the set-theoretic splitting determined by $\mathcal{E}$ [Spr98, §9.3.3]. Denote $w = n(\omega[(B,T),(\mathcal{B},\mathcal{T})](x))$. We claim that we have

$$\mathrm{Ad}(w) \circ \iota[P', \mathcal{E}, s, \mathcal{E}_M, s_M] = \iota[P, \mathcal{E}, s, \mathcal{E}_M, s_M]. \tag{2.2}$$

To simplify notation in the rest of the proof we abbreviate $\iota = \iota[P, \mathcal{E}, s, \mathcal{E}_M, s_M]$ and $\iota' = \iota[P', \mathcal{E}, s, \mathcal{E}_M, s_M]$.

First we check that ι and ι' coincide on $\mathcal{T}_M$. Denote $T' = T$ for clarity. We have $(B', T') = \mathrm{Ad}(x)(B,T)$ so if we also denote by $\mathrm{Ad}(x)$ the induced isomorphism $X_*(T) \simeq X_*(T')$ we have $\mathrm{Ad}(x)\gamma[(B,T),(\mathcal{B},\mathcal{T})] = \gamma[(B',T'),(\mathcal{B},\mathcal{T})]$. Here because $T' = T$ we obtain

$$\gamma[(B',T),(\mathcal{B},\mathcal{T})] = \gamma[(B,T),(\mathcal{B},\mathcal{T})] \circ \omega[(B,T),(\mathcal{B},\mathcal{T})](x).$$

The isomorphism $\iota|_{\mathcal{T}_M} : \mathcal{T}_M \simeq \mathcal{T}$ is dual to the isomorphism

$$\gamma[(B_M,T),(\mathcal{B}_M,\mathcal{T}_M)]^{-1} \circ \gamma[(B,T),(\mathcal{B},\mathcal{T})] : X^*(\mathcal{T}) \simeq X^*(\mathcal{T}_M).$$

Similarly $\iota'|_{\mathcal{T}_M} : \mathcal{T}_M \simeq \mathcal{T}$ is dual to the isomorphism

$$\begin{aligned} &\gamma[(B_M,T),(\mathcal{B}_M,\mathcal{T}_M)]^{-1} \circ \gamma[(B',T),(\mathcal{B},\mathcal{T})] \\ = &\gamma[(B_M,T),(\mathcal{B}_M,\mathcal{T}_M)]^{-1} \circ \gamma[(B,T),(\mathcal{B},\mathcal{T})] \circ \omega[(B,T),(\mathcal{B},\mathcal{T})](x) \end{aligned}$$

and the equality

$$\iota'|_{\mathcal{T}_M} = \omega[(B,T),(\mathcal{B},\mathcal{T})](x)^{-1} \circ \iota|_{\mathcal{T}_M}$$

follows.

To check that the equality (2.2) holds on $\widehat{M}$ it is enough to check that we have $\mathrm{Ad}(w)\iota(Y_\alpha) = \iota'(Y_\alpha)$ for any $\alpha \in \Delta(\mathcal{T}_M, \mathcal{B}_M)$. We have

$$\iota(Y_\alpha) = X_\beta \text{ and } \iota'(Y_\alpha) = X_{\beta'}$$

where

$$\begin{aligned}\beta &= \gamma[(B,T),(\mathcal{B},\mathcal{T})]^{-1}\gamma[(B_M,T),(\mathcal{B}_M,\mathcal{T}_M)](\alpha),\\ \beta' &= \gamma[(B',T),(\mathcal{B},\mathcal{T})]^{-1}\gamma[(B_M,T),(\mathcal{B}_M,\mathcal{T}_M)](\alpha)\\ &= w^{-1}(\beta)\end{aligned}$$

both belong to $\Delta(\mathcal{T},\mathcal{B})$. By [Spr98, Proposition 9.3.5] we have $X_\beta = \mathrm{Ad}(w)(X_{\beta'})$.

Finally we need to check $\mathrm{Ad}(w) \circ s = s$, i.e. that w commutes with $s(\Gamma)$. For $\sigma \in \Gamma$ and $y \in W(\mathcal{T}, \widehat{G})$ we have $s(\sigma)n(y)s(\sigma)^{-1} = n(\sigma(y))$ and so it is enough to check that $w\mathcal{T} \in W(\mathcal{T}, \widehat{G})$ is fixed by Γ. For any $\sigma \in \Gamma$ there exists $g_\sigma \in M(\overline{F})$ such that $\sigma(B_M,T) = \mathrm{Ad}(g_\sigma^{-1})(B_M,T)$ and this implies $\sigma(B,T) = \mathrm{Ad}(g_\sigma^{-1})(B,T)$ and $\sigma(B',T) = \mathrm{Ad}(g_\sigma^{-1})(B',T)$ because N and N' are both defined over F. A simple computation shows that we have $\mathrm{Ad}(g_\sigma)(\sigma(x)) = x$ in $W(T, G_{\overline{F}})$, i.e. x is Γ-invariant. $\square$

The lemma shows that for a Levi subgroup M of G we have an embedding $\iota_M : {}^LM \to {}^LG$, well-defined up to conjugation by $\widehat{G}$. We call the image of such an embedding a *relevant* Levi subgroup of ${}^LG$.

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