On the de Rham flip-flopping in dual towers

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<u>Motivation</u> Coming from Colmez - Dospinescu - Niziol '20.

<u>Drinfeld LT towers</u>

$K/\mathbb{Q}_p$ fin, $C = \hat{\bar{\mathbb{Q}}}_p$, $d \geq 2$.

$G = GL_d(K)$, $G_0 = GL_d(\mathcal{O}_K)$, $G_n = 1 + \varpi^n M(\mathcal{O}_K)$, $n \geq 1$

$\check{G} = D^\times$, D/K central div alg, $\frac{1}{d}$ invertible.

$\check{G}_0 = \mathcal{O}_D^\times$, $\check{G}_n = 1 + \varpi_D^n \mathcal{O}_D$, $n \geq 1$.

$\rightsquigarrow$

$$\begin{tikzcd}
& \hat{\mathcal{M}}_\infty \simeq \hat{LT}_\infty \arrow[dl] \arrow[dr] & \\
\mathcal{M}_n \arrow[d, "{+\,G,\ \check{G},\ \mathcal{G}_K,\ \mathcal{O}_D^\times/\check{G}_n}"'] & & LT_n \circlearrowleft G_0,\ \check{G}_0,\ \mathcal{G}_K \arrow[d] \\
\mathcal{M}_0 \text{ "="} \mathbb{H}^{d-1}_K & & LT_0 \text{ "}\cong\text{" } \mathring{\mathbb{B}}^{d-1}_K .
\end{tikzcd}$$

$$\mathbb{H}^{d-1}_K = \mathbb{P}^d_C \setminus \bigcup_{H} H \text{ — rat'l hyperplane}$$

[CDN] $d = 2$. Interested in:

$$R\Gamma_{et}(\mathcal{M}_{n,C}, \mathbb{Q}_p) \leftsquigarrow R\Gamma_{proet}(\mathcal{M}_{n,C}, \mathbb{Q}_p)$$

$$\rightsquigarrow ?$$

Have $H^1_{dR,c}(\mathcal{M}_n) \circlearrowleft G$ admissible.

classical notion of adm:

$\forall H \subset G$ compact open, $|H^1_{dR,c}(\mathcal{M}_n)^H| < \infty$.

<u>Key lemmas</u> (Hodge flip-flopping)

(1) $H^1_{dR,c}(\mathcal{M}_\infty) \underset{\substack{| \\ G \times \check{G}\text{-equiv}}}{\simeq} H^1_{dR,c}(LT_\infty) := \operatorname{colim}_n H^1_{dR,c}(LT_n)$

$\Uparrow$

(2) $H^1_{dR,c}(\mathcal{M}_n)^{G_j} \overset{(*)}{\simeq} H^1_c(\widehat{\mathcal{M}}_\infty, \widehat{\mathcal{O}}) \underset{\substack{| \\ G_j \times \check{G}_n\text{-equiv.}}}{\simeq} H^1_c(\widehat{LT}_\infty, \widehat{\mathcal{O}})^{G_j \times \check{G}_n}$

$\simeq H^1_{dR,c}(LT_j)^{\check{G}_n}$.

<u>For (*)</u> Recall : X Stein $\Rightarrow H^i_c(X, \mathcal{F}) = 0$ for $i \neq d$

$\mathcal{F}$ loc free $\mathcal{O}_X$-mod

$\Rightarrow H^1_{dR,c}(X) \simeq H^0(H^1_c(X, \mathcal{O}) \xrightarrow{d} H^1_c(X, \Omega^1))$

$\simeq H^1_c(X, \mathcal{O})^{d=0}$.

<u>Q</u> Can we flip-flop Hodge cohom in dim $d>0$ using $\widehat{\mathcal{O}}$

or sheaf for perfectoid spaces?

only work in dim 1 but not in higher dim.

<u>Want</u> Generalize this lem to higher dims,

more general twin towers,

& other cohomologies (Hyodo-Kato).

<u>Idea</u>	<u>Rigid analytic</u>		<u>Perfectoid</u>
	(filtered) de Rham cohom	$\Longleftrightarrow$	$\mathbb{B}_{dR}$ — relative period ring.
	$(\varphi, N, \mathscr{G}_K)$ HK cohom	$\Longleftrightarrow$	$\mathbb{B}_{\log}$ $(\sim \mathbb{B}^+_{st})$

<u>Abstract towers</u>

T adic space $/K$, $X, \check{X}$ sm rigid-analytic $/K$.

$$\begin{tikzcd} & T \circlearrowleft G\times\check{G} \arrow[dl, "\pi"', "G"] \arrow[dr, "\check{\pi}", "\check{G}"'] & \\ X & & \check{X} \\ \circlearrowleft & & \circlearrowleft \\ \check{G} & & G \end{tikzcd}$$

Assume
- $G, \check{G}$ profinite, p-adic Lie grps
- $\pi, \check{\pi}$ equivariant torsors
- All actions are conti.

<u>Thm 1</u> (de Rham flip-flopping)

(1) $X, \check{X}$ Stein or qc $\Rightarrow$ in $D(K_\square)$,

$r \geq 0$, $R\Gamma(\check{G}, F^r R\Gamma_{dR,*}(X)) \simeq R\Gamma(G, F^r R_{dR,*}(\check{X}))$.

$* \in \{\phi, c\}$

pass to gradings $\Downarrow$

(2) (Hodge flip-flopping)

$$R\Gamma(\check{G}, \Omega^j(X)) \simeq R\Gamma(G, \Omega^j(\check{X})).$$

$$R\Gamma(\check{G}, H^d_c(X, \Omega^j)) \simeq R\Gamma(G, H^d_c(\check{X}, \Omega^j)).$$

(3) (Lie alg flip-flopping)

$$R\Gamma(\check{\mathfrak{g}}, \underbrace{R\Gamma_{dR,c}(X_\infty)}) \simeq R\Gamma(\mathfrak{g}, R\Gamma_{dR,c}(\check{X}_\infty)).$$

Sm rep of $\check{G}$ $\Rightarrow$ $\check{\mathfrak{g}}$-action is triv.

Explanation:

$$R\Gamma(\check{\mathfrak{g}}, K) \otimes^{L\square}_K \underbrace{R\Gamma_{dR,c}(X_\infty)} \simeq R\Gamma(\mathfrak{g}, K) \otimes^{L\square}_K \underbrace{R\Gamma_{dR,c}(\check{X}_\infty)}.$$

$$R\Gamma_{dR,c}(X_\infty) := \operatorname*{colim}_{\substack{H\subset G \\ \text{cpt open}}} R\Gamma_{dR,c}(X_H) \qquad R\Gamma_{dR,c}(\check{X}_\infty) := \operatorname*{colim}_{\substack{H\subset G \\ \text{cpt open}}} R\Gamma_{dR,c}(\check{X}_{\check{H}})$$

<u>Cor</u> (1) Let $X, \check{X}$ Stein, $G, \check{G}$ p-adic Lie grps

s.t. $\dim_K H^i(\mathfrak{g}, K) = \dim_K H^i(\check{\mathfrak{g}}, K)$ $\quad(*)$

Then $\exists$ cpt open $H \subset G$, $\check{H} \subset \check{G}$, s.t.

$\exists\, \alpha_i : H^i_{dR,c}(X_\infty) \simeq H^i_{dR,c}(\check{X}_\infty)$ in $[d, 2d]$.

$H \times \check{H}$-equiv

Rmk $H \times \check{H}$-equiv (not $G \times \check{G}$) is enough if they both act trivially on Lie alg cohom.

(2) $X = \mathcal{M}_0$, $\check{X} = LT_0$

$\Rightarrow$ (a) α_i is $G \times \check{G}$-equiv

(b) $H^i_{dR,c}(X_\infty)$ is adm as a G-rep.

Rmk · Cor HK version, compatible w/ $(\check{\varphi}, N, \check{\mathfrak{g}}_K)$

· Thm 1 HK version, twisted by B_{log}

Related work In progress w/ Rodriguez-Camargo on dR analytic stacks

$$\left.\begin{array}{ll} H_{dR}(X_n) \rightsquigarrow H(\hat{X}_\infty, B_{dR}) \\ X_{dR}(X_n) \qquad X_{dR}(\hat{X}_\infty) \end{array}\right\} \text{HK analytic stacks (analytic prismatization).}$$

Pf of Cor Induction.

· $i=0$, for α_i. both grps are trivial.

(Recall: $R\Gamma_{dR,c}(X) = [H^d_c(X, \mathcal{O}) \xrightarrow{d} H^d_c(X, \Omega^1) \longrightarrow \cdots \longrightarrow H^d_c(X, \Omega^d)][-d]$.)

Inductive step: OK if $s \leq i$. Want $i+1$.

Idea Thm 1 $\rightsquigarrow$ $H^{i+1}(R\Gamma(\check{\mathfrak{g}}, K) \otimes R\Gamma_{dR,c}(X_\infty))$

is

$H^{i+1}(R\Gamma(\mathfrak{g}, K) \otimes R\Gamma_{dR,c}(\check{X}_\infty))$.

Künneth $\rightsquigarrow$ $\underbrace{H^0(\check{\mathfrak{g}}, K) \otimes^{\square}_K H^{i+1}_{dR,c}(X_\infty)}_{\pi_1} \oplus \underbrace{H^1(\check{\mathfrak{g}}, K) \otimes^{\square}_K H^i_{dR,c}(X_\infty) \oplus \cdots \oplus H^{i+1}(\check{\mathfrak{g}}, K) \otimes H^0_{dR,c}(X_\infty)}_{\tilde{\pi}}$

$\simeq \underbrace{H^0(\mathfrak{g}, K) \otimes^{\square}_K H^{i+1}_{dR,c}(\check{X}_\infty)}_{\pi_2} \oplus \underbrace{H^1(\mathfrak{g}, K) \otimes^{\square}_K H^i_{dR,c}(\check{X}_\infty) \oplus \cdots \oplus H^{i+1}(\mathfrak{g}, K) \otimes H^0_{dR,c}(\check{X}_\infty)}_{\tilde{\pi}}$

Now need: $(\pi_1 \oplus \tilde{\pi} \simeq \pi_2 \oplus \tilde{\pi} \overset{?}{\Rightarrow} \pi_1 \oplus \pi_2 \ (**))$.

To prove (**): Need "complete reducibility".

① Can reduce to adm rep of cpt grps.

② Prove admissibility $\rightsquigarrow$ Krull-Schmidt thm

$\rightsquigarrow$ Pass to mods / $\mathcal{H}(G, K)$.

③ HK : $(\varphi, N, \mathcal{G}_K)$, $\mathcal{G}_K$ cpt grp

φ: Dieudonné - Manin

④ $H^i_{dR,c}(\mathcal{M}_n)$ adm $\overset{\text{Thm 1}}{\rightsquigarrow}$ $H^i_{dR,c}(LT_j)$ finite rk.

b/c $\updownarrow$ Poincaré duality

$H^*_{dR}(LT_j)$ $\leftarrow$ finite rk by Grosse-Khöne. □

<u>pf of Thm 1</u> (1) $\Rightarrow$ (3) (R-R)

$$R\Gamma(\check{\mathcal{G}}, R\Gamma_{dR,c}(X_\infty)) = \operatorname*{colim}_{H} \operatorname*{colim}_{\check{H}} R\Gamma(\check{H}, R\Gamma_{dR,c}(X_H))$$

<u>Main lem</u> (B_{dR} flip-flop)

$$R\Gamma(\check{G}, F^r R\Gamma_{dR}(X_C/B_{dR})) \simeq R\Gamma(G, F^r R\Gamma_{dR}(\check{X}_C/B_{dR}))$$

Pf Use Bosco's result:

$$F^r R\Gamma_{dR}(X_C/B_{dR}) \simeq R\Gamma_{proét}(X_C, F^r B_{dR})$$

$$\underset{\text{Galois descent}}{\simeq} R\Gamma_{proét}(\check{X}_{\infty,C}, B_{dR}).$$

This does not work for HK cohom.