2022 Summer School on the Langlands Program at IHES

SHIMURA VARIETIES

SOPHIE MOREL

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Readme. This is a very preliminary version for the (closed-door) lecture series given by Sophie Morel at IHES Summer School 2022. *Please use with caution and do not disseminate.*

Due to the mistake and carelessness of the notetaker, it is missing parts and many references and is full of typos. Also, every sign has at least a 50% chance of being wrong.

If you know what this sentence means, the latest edited version by Sophie Morel of these notes will possibly be found at `Quatramaran` website (no guarantees though).

Contents

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1. Locally symmetric spaces and Shimura varieties

1.1. **Locally symmetric spaces.** Let G be a semisimple algebraic group over $\mathbb{Q}$, for example SL_n, Sp_{2n}, or $\mathrm{SO}(p,q)$. We would like to present some "nice enough" whose cohomology is related to automorphic representations of G. A good reference for locally symmetric spaces is the introductory paper [Ji06] by Ji.

To simplify the presentation, we will assume here that $G(\mathbb{R})$ is connected. Let K_∞ be a maximal compact subgroup of $G(\mathbb{R})$, and let $X = G(\mathbb{R})/K_\infty$. If Γ is a discrete subgroup of $G(\mathbb{R})$ such that $\Gamma\backslash G(\mathbb{R})$ (or equivalently $\Gamma\backslash X$) is compact and Γ acts properly and freely on X,[1] then there is a classical connection between the cohomology of $\Gamma\backslash X$ and automorphic representations of $G(\mathbb{R})$, called **Matsushima's formula** (see Matsushima's paper [Mat67]). We will state a modern reformulation in Lecture 3, but roughly it relates the Betti numbers of $\Gamma\backslash X$ and the multiplicities of representations of $G(\mathbb{R})$ in $L^2(\Gamma\backslash G(\mathbb{R}))$.

In fact, Matsushima's paper deals with semi-simple real Lie groups. Here, we have an algebraic group defined over $\mathbb{Q}$, so we have a particularly nice way to produce discrete subgroups of $G(\mathbb{R})$. Remember that a subgroup Γ of $G(\mathbb{Q})$ is called an **arithmetic subgroup** if there exists a closed embedding $G \subset \mathrm{GL}_N$ such that, setting $G(\mathbb{Z}) = G(\mathbb{Q}) \cap \mathrm{GL}_N(\mathbb{Z})$, we have that $\Gamma \cap G(\mathbb{Z})$ is of finite index in Γ and in $G(\mathbb{Z})$.[2] If Γ is small enough, then it acts properly and freely on X ([Ji06, Proposition 5.5]), so the quotient $\Gamma\backslash X$ is a real analytic manifold. Also, the quotient $\Gamma\backslash G(\mathbb{R})$ is compact if and only if G is anisotropic (over $\mathbb{Q}$), which means that G has no nontrivial parabolic subgroup defined over $\mathbb{Q}$ ([Ji06, Theorem 5.10]). (A subgroup of G is parabolic if it contains a Borel subgroup B of G.) If $\Gamma\backslash X$ is not compact but $G(\mathbb{R})$ has a discrete series, then there is an extension of Matsushima's formula, due to Borel and Casselman in [BC83], that involves L^2 cohomology of $\Gamma\backslash X$; see Lecture 3.

We actually would like to see automorphic representations of $G(\mathbb{A})$ (not just $G(\mathbb{R})$) in the cohomology of our spaces, so we will use adelic versions of $\Gamma\backslash X$. Let K be an open compact subgroup of $G(\mathbb{A}_f)$; for example, if we have chosen an embedding $G \subset \mathrm{GL}_N$, then we could take

$$K = G(\mathbb{A}_f) \cap \mathrm{Ker}(\mathrm{GL}_N(\widehat{\mathbb{Z}}) \to \mathrm{GL}_N(\mathbb{Z}/n\mathbb{Z})),$$

for some positive integer n (these are called **principal congruence subgroups**). Let

$$M_K = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K,$$

where the group K acts by right translations on the factor $G(\mathbb{A}_f)$, and the group $G(\mathbb{Q})$ acts by left translations on both factors simultaneously. Choose a system of representatives $(x_i)_{i\in I}$ of the (finite) quotient $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$, and set

$$\Gamma_i = G(\mathbb{Q}) \cap x_i K x_i^{-1}$$

for every $i \in I$. Then the Γ_i are arithmetic subgroups of $G(\mathbb{Q})$, and we have

$$M_K = \coprod_{i\in I} \Gamma_i\backslash X,$$

[1] This holds for example if Γ is torsion free, which happens when Γ is small enough.

[2] We can check that this definition does not depend on the embedding $G \subset \mathrm{GL}_N$, see [Ji06, Proposition 4.2].

so M_K is a real analytic manifold if K is small enough. But now we have an action of $G(\mathbb{A}_f)$ on the projective system $(M_K)_{K\subset G(\mathbb{A}_f)}$, so we get an action on $\varinjlim_K \mathrm{H}^*(M_K)$, where H^* is any "reasonable" cohomology theory, for example Betti cohomology. If G is anisotropic over $\mathbb{Q}$, then Matsushima's result can be reformulated to give a description of this action in terms of irreducible representations of $G(\mathbb{A})$ appearing in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$, and there is also a version of the Borel-Casselman generalization.

There is another way to think about the action of $G(\mathbb{A}_f)$ on $(M_K)_{K\subset G(\mathbb{A}_f)}$, which does not involve a limit on K. Fix a Haar measure on $G(\mathbb{A}_f)$ such that open compact subgroups of $G(\mathbb{A}_f)$ have rational volume (this is possible because these groups are all commensurable); then every open subset of $G(\mathbb{A}_f)$ has rational volume. The **Hecke algebra** of G is the space $\mathcal{H}_G$ of locally constant functions with compact support from $G(\mathbb{A}_f)$ to $\mathbb{Q}$; if $f, g \in \mathcal{H}_G$, then the convolution product $f * g$ still has rational values by the choice of Haar measure, so convolution defines a multiplication on $\mathcal{H}_G$. For every open compact subgroup K of $G(\mathbb{A}_f)$, the **Hecke algebra at level** K is the subalgebra $\mathcal{H}_{G,K}$ of bi-K-invariant functions in $\mathcal{H}_G$; we have $\mathcal{H}_G = \bigcup_K \mathcal{H}_{G,K}$.

Fix K small enough. Then $\mathrm{H}^*(M_K)$ is basically the set of K-invariant vectors:

$$\mathrm{H}^*(M_K) = \varinjlim_{K'\subset G(\mathbb{A}_f)} \mathrm{H}^*(M_{K'})^K,$$

so it has an action of $\mathcal{H}_{G,K}$.[3] We can describe this action using Hecke correspondences: let $g \in G(\mathbb{A}_f)$, let K' be an open compact subgroup of $G(\mathbb{A}_f)$ such that $K' \subset K \cap gKg^{-1}$, then we have a **Hecke correspondence**

$$\begin{aligned} (T_1, T_g) : M_{K'} &\longrightarrow M_K \times M_K \\ (x,h) &\longmapsto ((x,h),(x,hg)), \end{aligned}$$

and T_1, T_g are both finite covering maps if K is small enough. Up to a scalar,[4] the action $\mathbb{1}_{KgK}$ on $\mathrm{H}^*(M_K)$ is given by pulling back cohomology classes along T_1, then pushing them forward along T_g.

We can also ask whether there is more structure on the spaces $\Gamma\backslash X$ (or M_K). For example, suppose that $G = \mathrm{SL}_2$ and $K_\infty = \mathrm{SO}(2)$. Then, for Γ an arithmetic subgroup of $\mathrm{SL}_2(\mathbb{Z})$, the space $\Gamma\backslash X$ is a modular curve, so it is (the set of complex points of) an algebraic variety defined over a function field F, and we can use the commuting actions of Hecke correspondences and of the absolute Galois group of F on its étale cohomology to construct some instance of the global Langlands correspondence for SL_2 or GL_2.

In order to generalize this picture, we first to know when the spaces $\Gamma\backslash X$ or M_K are the set of $\mathbb{C}$-points of an algebraic variety, and whether this algebraic variety is defined over a number field. As we will see later, another advantage over M_K over $\Gamma\backslash X$ is that, when the answer to the above question is "yes", then the M_K for K varying tend to all be defined over the same field, while this is not the case for the $\Gamma\backslash X$.

Remark 1.1. The first step is to check whether $\Gamma\backslash X$ has the structure of a complex manifold, and there are obvious obstructions to that. For example, if $G = \mathrm{SL}_3$ and $K_\infty = \mathrm{SO}(3)$,

[3] In fact, we can recover the action of $G(\mathbb{A}_f)$ on $\varinjlim_{K'\subset G(\mathbb{A}_f)} \mathrm{H}^*(M_{K'})$ from the action of $\mathcal{H}_{G,K}$ on M_K for every K small enough.

[4] Make scalar precise.

then $\Gamma\backslash X$ is 5-dimensional as a real manifold, so it cannot have the structure of a complex manifold. In fact, there is no structure of complex manifold on $\Gamma\backslash X$ for $G = \mathrm{GL}_d$ with $d \geqslant 3$, as we will now see.

Choose a $G(\mathbb{R})$-invariant Riemannian metric on $X = G(\mathbb{R})/K_\infty$ (such a metric is unique up to rescaling on each irreducible factor). Then X is a **symmetric space**, that is, a Riemannian manifold such that:

(a) The group of isometries of X acts transitively on X;
(b) For every $p \in X$, there exists a symmetry s_p of X (i.e. an involutive isometry) such that p is an isolated fixed point of s_p.

Moreover, the symmetric space X is of **noncompact type**, that is, it has negative curvature. For Γ a small enough arithmetic subgroup of $G(\mathbb{Q})$, the Riemannian manifold $\Gamma\backslash X$ is a **locally symmetric space**; in particular, it does not satisfy condition (a) anymore, and it satisfies a variant of condition (b) where we only ask for the symmetry to be defined in a neighborhood of the point. See Ji's notes [Ji06] for a review of locally symmetric spaces.

We say that X is a **Hermitian symmetric domain** if it admits a $G(\mathbb{R})$-invariant Hermitian metric. See Section 1 of Milne's notes [Mil05] for a review of Hermitian symmetric domains.

Example 1.2 (Siegel upper half space)**.** Let d be a positive integer. The **Siegel upper half space** $\mathfrak{h}_d^+$ is the set of symmetric $d \times d$ complex matrices in $Y \in M_d(\mathbb{C})$ such $\mathrm{Im}(Y)$ is positive definite; if $d = 1$, then this is just the usual upper half plane. Then the Siegel upper half space $\mathfrak{h}_d^+$ is a Hermitian symmetric domain. The proofs of the basic properties of $\mathfrak{h}_d^+$ can be found in Siegel's paper [Sie43].

We first need to see $\mathfrak{h}_d^+$ as a symmetric space. Let Sp_{2d} be the symplectic group of the symplectic form with matrix $\begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$, where $I_d \in \mathrm{GL}_d(\mathbb{Z})$ is the identity matrix. For every commutative ring R, we have

$$\mathrm{Sp}_{2d}(R) = \left\{ g \in \mathrm{GL}_{2d}(R) \,\middle|\, {}^t g \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \right\}.$$

Note that $\mathrm{Sp}_2 = \mathrm{SL}_2$. We make $\mathrm{Sp}_{2g}(\mathbb{R})$ act on $\mathfrak{h}_d^+$ by the following formula:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Y = (AY + B)(CY + D)^{-1},$$

where A, B, C, D are $d \times d$ matrices such that $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2d}(\mathbb{R})$ (see page 9 of [Sie43]). Then this action is transitive (see page 9 of [Sie43]). Let K_∞ be the stabilizer in $\mathrm{Sp}_{2d}(\mathbb{R})$ of $iI_d \in \mathfrak{h}_d^+$. Then $K_\infty = O(2d) \cap \mathrm{Sp}_{2d}(\mathbb{R})$ (this is easy to check directly), so it is a maximal compact subgroup of $\mathrm{Sp}_{2d}(\mathbb{R})$,[5] and we have

$$\mathfrak{h}_d^+ \simeq \mathrm{Sp}_{2d}(\mathbb{R})/K_\infty$$

as real analytic manifolds.

Also, the space $\mathfrak{h}_d^+$ is an open subset of the complex vector space of symmetric matrices in $M_d(\mathbb{C})$, so it has an obvious structure of complex manifold. It remains to construct a $\mathrm{Sp}_{2d}(\mathbb{R})$-invariant Hermitian metric on $\mathfrak{h}_d^+$. Let $\mathcal{D}_d$ be the set of symmetric matrices $A \in M_d(\mathbb{C})$ such that $I_d - A^*A$ is positive definite; this is a bounded domain in the complex vector space of symmetric matrices in $M_d(\mathbb{C})$, hence is equipped with a canonical Hermitian metric called the **Bergman metric**, which has negative curvature (see for example, [Mil05, Theorem 1.3]); in particular, this metric is invariant by all holomorphic automorphisms of $\mathcal{D}_d$. Now note that we have an isomorphism

$$h_d^+ \xrightarrow{\sim} \mathcal{D}_d, \quad X \longmapsto (iI_d - X)(iI_d + X)^{-1}$$

(whose inverse sends $A \in \mathcal{D}_d$ to $i(I_d - A)(I_d + A)^{-1}$), see [Sie43, pp. 8-9]. We can give a formula for the resulting Hermitian metric on h_d^+: up to a positive scalar, it is given by

$$ds^2 = \mathrm{Tr}(\mathrm{Im}(Y)^{-2} dY\, \mathrm{Im}(Y)^{-1} d\overline{Y})$$

(see formula (28) on page 17 of [Sie43]).

The isomorphism $h_d^+ \simeq \mathcal{D}_d$ is called a **bounded realization** of h_d^+.

We can give a complete classification of Hermitian symmetric domains (cf. [Mil05, Theorem 1.21]), in terms of real algebraic groups:

Theorem 1.3. *Suppose that $G(\mathbb{R})$ is connected and adjoint. The locally symmetric space X is a Hermitian symmetric domain if and only if there exists a morphism of real Lie groups $u : \mathrm{U}(1) \to G(\mathbb{R})$ such that:*

(a) *The only characters of $\mathrm{U}(1)$ that appear in its representation $\mathrm{Ad} \circ u$ on $\mathrm{Lie}(G(\mathbb{R}))$ are 1, z, and z^{-1};*
(b) *Conjugation by $u(i)$ is a Cartan involution of $G(\mathbb{R})$, which means that $\{g \in G(\mathbb{C}) \mid g = u(i)\overline{g}u(i)^{-1}\}$ is compact;*
(c) *The projection of $u(i)$ to a simple factor of $G(\mathbb{R})$ is never equal to 1.*

Moreover, we can choose u such that K_∞ is the centralizer of u in $G(\mathbb{R})$, which means that X is isomorphic to set of conjugates of u by elements of $G(\mathbb{R})$.

We explain the construction of the morphism u. Suppose that X is a Hermitian symmetric domain, and let $p \in X$. For every $z \in \mathbb{C}$ with $|z| = 1$, multiplication by z on T_pX preserves the Hermitian metric and sectional curvatures, so there exists a unique isometry $u_p(z)$ of D fixing p and such that $T_pu_p(z)$ is multiplication by z. The uniqueness implies that $u_p(z)u_p(z') = u_p(zz')$ if $|z| = |z'| = 1$, so we get a morphism of groups from $\mathrm{U}(1)$ to the group of isometries of X, which is equal to $G(\mathbb{R})^0_{\mathrm{ad}}$.

[5] In fact, we have an isomorphism (with $X, Y \in \mathrm{GL}_d(\mathbb{R})$):

$$U(d) \xrightarrow{\sim} K_\infty, \quad X + iY \longmapsto \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}.$$

Example 1.4. (1) If $G = \mathrm{Sp}_{2d}$, let $h : \mathbb{C}^\times \to G(\mathbb{R})$ be defined by

$$h(a+ib) = \begin{pmatrix} aI_d & -bI_d \\ bI_d & aI_d \end{pmatrix}.$$

Then we can take $u : \mathrm{U}(1) \to \mathrm{PSp}_{2d}(\mathbb{R})$ given by $u(z) = h(\sqrt{z})$. Note that u does not lift to a morphism from $U(1)$ into $G(\mathbb{R})$.

(2) If $G = \mathrm{PGL}_n$ with $n \geqslant 3$, then the centralizer of a character $u : \mathrm{U}(1) \to G(\mathbb{R})$ cannot be a maximal compact subgroup of $G(\mathbb{R})$ (exercise), so the locally symmetric space of maximal compact subgroups of $G(\mathbb{R})$ is not Hermitian.

Theorem 1.3 puts some pretty strong restrictions on the root systems of the simple factors of $G(\mathbb{R})$, see Theorem 1.25 of [Mil05] and the table following it. In particular, the type A simple factors of $G(\mathbb{R})$ must be of the form $\mathrm{PSU}(p,q)$, and $G(\mathbb{R})$ can have no simple factor of type E_8, F_4 or G_2.

The natural next step would be to wonder for which Hermitian symmetric domains X the quotients $\Gamma\backslash X$ are algebraic varieties, but in fact it turns out that the answer is "for all of them", as was proved by Baily and Borel [BB66].

Theorem 1.5 (Baily-Borel)**.** *Suppose that $X = G(\mathbb{R})/K_\infty$ is a Hermitian symmetric domain. Then, for any torsion free arithmetic subgroup Γ of $G(\mathbb{Q})$, the quotient $\Gamma\backslash X$ has a canonical structure of algebraic variety over $\mathbb{C}$.*

The very rough idea is that the sheaf of automorphic forms on $\Gamma\backslash X$ of sufficiently high weight will define an embedding of $\Gamma\backslash X$ into a projective space.

Remember that we did not just want the locally symmetric spaces $\Gamma\backslash X$ to be algebraic varieties, we also wanted them to be defined over a number field, and we would ideally like the number field in question to only depend on G and K_∞. For this, it will actually be easier to work with reductive groups instead of semi-simple groups. As a motivation for this, and for the definition of Shimura varieties, we now spend some more time on the case of the symplectic group.

1.2. **The Siegel modular variety.**

See the end of this subsection (1.3.5) for some background on abelian schemes.

1.3. **The Siegel upper half space as a moduli space of abelian varieties over $\mathbb{C}$.**

We use the notation of Example 1.2. It is well-known that $\mathfrak{h}_1^+$ parametrizes elliptic curves over $\mathbb{C}$: an element $\tau \in \mathfrak{h}_1^+$ is sent to the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, and $E_\tau \simeq E_{\tau'}$ if and only if $\tau, \tau' \in \mathfrak{h}_1^+$ they are conjugated under the action of $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{Sp}_2(\mathbb{Z})$; so we can recover τ from E_τ and the data of a symplectic isomorphism $\mathrm{H}_1(E_\tau, \mathbb{Z}) \simeq \mathbb{Z}^2$ where $\mathbb{Z}^2$ is equipped with the standard symplectic form. We want to give a similar picture for higher-dimensional abelian varieties; in fact, the analogy works best if we consider abelian varieties with a principal polarization (Definition 1.29).

We first introduce some notation about symplectic spaces and recall the definition of the (general) symplectic group as a group scheme over $\mathbb{Z}$. If R is a commutative ring, we denote by ψ_R the perfect symplectic pairing on R^{2d} with matrix $\begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$. So we have

$$\psi_R((x_1,\dots,x_d,y_1,\dots,y_d),(x'_1,\dots,x'_d,y'_1,\dots,y'_d)) = \sum_{i=1}^{d} x_i y'_i - \sum_{i=1}^{g} x'_i y_i.$$

The **general symplectic group** GSp_{2d} is the reductive group scheme over $\mathbb{Z}$ whose points in a commutative ring R are given by

$$\mathrm{GSp}_{2d}(R) = \{g \in \mathrm{GL}_{2d}(R) \mid \exists c(g) \in R^\times,\ \forall v, v' \in R^{2d},\ \psi_R(gv, gv') = c(g)\psi_R(v,v')\}.$$

The scalar $c(g)$ is called the **multiplier** of $g \in \mathrm{GSp}_{2d}(R)$. Sending g to $c(g)$ defines a morphism of group schemes $c : \mathrm{GSp}_{2d} \to \mathrm{GL}_1$, whose kernel Sp_{2d} is called the **symplectic group**.

Example 1.6. We have $\mathrm{GSp}_2 = \mathrm{GL}_2$ in which $c = \det$, and $\mathrm{Sp}_2 = \mathrm{SL}_2$.

1.3.1. *Complex abelian variety.* Let A be complex abelian variety of dimension d; we identify A and its set of complex points. Then A is a connected complex Lie group of dimension d, so we have $A \simeq \mathrm{Lie}(A)/\Lambda$, with $\mathrm{Lie}(A) \simeq \mathbb{C}^d$ the universal cover of A and $\Lambda = \pi_1(A) = \mathrm{H}_1(A,\mathbb{Z}) \simeq \mathbb{Z}^{2d}$ a lattice in the underlying $\mathbb{R}$0vector space. Let $A^\vee$ be the dual abelian variety, i.e., the space of degree 0 line bundles on A (see Definition 1.27). We can identify $\mathrm{Lie}(A^\vee)$ with the space of antilinear forms on $\mathrm{Lie}(A)$ and $\mathrm{H}_1(A^\vee,\mathbb{Z})$ with the subspace $\Lambda^\vee$ of forms whose imaginary part takes integer values on Λ (see [Mum08, §9]). For every positive integer n, we have

$$A[n] = \frac{1}{n}\Lambda/\Lambda, \quad A^\vee[n] = \frac{1}{n}\Lambda^\vee/\Lambda^\vee,$$

and the canonical pairing $A[n] \times A^\vee[n] \to \mu_n(\mathbb{C})$ is given by

$$(v,u) \mapsto e^{-2i\pi n \,\mathrm{Im}(u(v))}$$

(see [Mum08, §24]). We then have a bijection between the set of polarizations on A and the set of positive definite Hermitian forms[6] H on $\mathbb{C}^{2d}$ such that the symplectic form $\mathrm{Im}(H)$ takes integer values on Λ; given such a form H, the corresponding isogeny λ_H from A to $A^\vee$ is given on $\mathbb{C}$-points by:

$$\begin{aligned} \lambda_H : \mathrm{Lie}(A)/\Lambda &\longrightarrow \mathrm{Lie}(A^\vee)/\Lambda^\vee \\ w &\longmapsto (v \mapsto H(v,w)). \end{aligned}$$

It follows that the Weil pairing (see Remark 1.30 (2)) corresponding to λ_H is the map

$$\begin{aligned} A[n] \times A[n] &\longrightarrow \mu_n(\mathbb{C}) \\ (v,w) &\longmapsto e^{-2i\pi n\,\mathrm{Im}(H(v,w))}. \end{aligned}$$

Note that we have $v, w \in \frac{1}{n}\Lambda$, so $\mathrm{Im}(H(v,w)) \in \frac{1}{n^2}\mathbb{Z}$.

In particular, the polarization λ_H is principal if and only if Λ is self-dual with respect to the symplectic form $\mathrm{Im}(H)$, that is,

$$\Lambda = \{w \in \mathrm{Lie}(A) \mid \forall v \in \Lambda,\ \mathrm{Im}(H(v,w)) \in \mathbb{Z}\}.$$

In that case, the symplectic $\mathbb{Z}$-module $(\Lambda, \mathrm{Im}(H))$ is isomorphic to $\mathbb{Z}^{2d}$ with the form $\psi_{\mathbb{Z}}$.

[6] We take Hermitian forms to be semi-linear in the first variable and linear in the second variable.

Let $\widetilde{\mathcal{M}}_d$ be the set of isomorphism classes of triples $(A, \lambda, \eta_{\mathbb{Z}})$, where A is a complex abelian variety of dimension d, λ is a principal polarization on A, and $\eta_{\mathbb{Z}}$ is an morphism of symplectic spaces from $\mathrm{H}_1(A, \mathbb{Z})$ to $(\mathbb{Z}^{2d}, \psi_{\mathbb{Z}})$. We have an action of $\mathrm{Sp}_{2d}(\mathbb{Z})$ on $\widetilde{\mathcal{M}}_d$: if $c = (A, \lambda, \eta_{\mathbb{Z}}) \in \widetilde{\mathcal{M}}_d$ and $x \in \mathrm{Sp}_{2g}(\mathbb{Z})$, set $x \cdot c = (A, \lambda, x \circ \eta_{\mathbb{Z}})$.

If $(A, \lambda, \eta_{\mathbb{Z}}) \in \widetilde{\mathcal{M}}_d$, then $\Lambda = \mathrm{H}_1(A, \mathbb{Z})$ is a lattice in the real vector space $\mathrm{Lie}(A)$, we have $A = \mathrm{Lie}(A)/\Lambda$ and we can recover the Hermitian form H_λ corresponding to λ from $\mathrm{Im}(H_\lambda)|_\Lambda$, which is sent to the form $\psi_{\mathbb{Z}}$ on $\mathbb{Z}^{2d}$ by the isomorphism $\eta_{\mathbb{Z}} : \Lambda \xrightarrow{\sim} \mathbb{Z}^{2d}$. If we see $\mathbb{R}^{2d}$ as a complex vector space via the isomorphisms (of real vector spaces)

$$\mathrm{Lie}(A) \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} \mathbb{R}^{2d},$$

then the Hermitian form on $\mathbb{R}^{2d}$ corresponding to H_λ is

$$(v, w) \longmapsto \psi_{\mathbb{R}}(iv, w) + i\psi_{\mathbb{R}}(v, w).$$

So $\eta_{\mathbb{Z}}$ determines all the data of the isomorphism class of $(A, \lambda, \eta_{\mathbb{Z}})$, except for the structure of complex vector space on $\mathbb{R}^{2d}$. This structure of complex vector space is equivalent to the data of an $\mathbb{R}$-linear endomorphism J of $\mathbb{R}^{2d}$ such that $J^2 = -1$ (the endomorphism J corresponds to multiplication by i). We also need the $\mathbb{R}$-bilinear map $\mathbb{R}^{2d} \times \mathbb{R}^{2d} \to \mathbb{C}$ defined by $(v, w) \longmapsto \psi_{\mathbb{R}}(J(v), w) + i\psi_{\mathbb{R}}(v, w)$ to be a positive definite Hermitian form on $\mathbb{R}^{2d}$. This is equivalent to the following conditions:

(a) $\psi_{\mathbb{R}}(J(v), J(w)) = \psi_{\mathbb{R}}(v, w)$ for all $v, w \in \mathbb{R}^{2d}$;
(b) the $\mathbb{R}$-bilinear form $(v, w) \mapsto \psi_{\mathbb{R}}(J(v), w)$ on $\mathbb{R}^{2d}$ (which is symmetric by (a)) is positive definite.

Conversely, if we have a complex structure J on $\mathbb{R}^{2d}$ satisfying (a) and (b), then we get a positive definite Hermitian form H on $\mathbb{R}^{2d}$ whose imaginary part takes integer values on the lattice $\mathbb{Z}^{2d}$, so the complex torus $\mathbb{R}^{2d}/\mathbb{Z}^{2d}$ has a polarization, hence is an abelian variety (for example by the Kodaira embedding theorem), and we gat an element of $\widetilde{\mathcal{M}}_d$.

So we get a bijection from $\widetilde{\mathcal{M}}_d$ to the set X' of endomorphisms J of $\mathbb{R}^{2d}$ such that $J^2 = -1$ and that J satisfies condition (a) and (b).

Now observe that, if W is a $\mathbb{R}$-vector space, then the data of an endomorphism J of W such that $J^2 = -1$ (i.e. of the structure of a $\mathbb{C}$-vector space on W) is equivalent to the data of a $\mathbb{C}$-linear endomorphism $J_{\mathbb{C}}$ of $W \otimes_{\mathbb{R}} \mathbb{C}$ such that

$$\mathrm{Ker}(J_{\mathbb{C}} - i \cdot \mathrm{id}) = \overline{\mathrm{Ker}(J_{\mathbb{C}} + i \cdot \mathrm{id})},$$

where $v \mapsto \overline{v}$ is the involution of $W \otimes_{\mathbb{C}} \mathbb{C}$ induced by complex conjugation on $\mathbb{C}$. This is equivalent to giving a $\mathbb{C}$-vector subspace E of $W \otimes_{\mathbb{R}} \mathbb{C}$ such that $W \otimes_{\mathbb{R}} \mathbb{C} = E \oplus \overline{E}$, i.e., a d-dimensional complex subspace E of $W \otimes_{\mathbb{R}} \mathbb{C}$ such that $E \cap \overline{E} = \{0\}$.[7]

[7]In fancy terms, we are saying that putting a structure of complex vector space on W is the same as putting a pure Hodge structure of type $\{(-1, 0), (0, -1)\}$ on it (or of type $\{(1, 0), (0, 1)\}$, depending on your normalization). When $W = \mathbb{R}^{2d}$ and the complex structure comes from an element $(A, \lambda, \eta_{\mathbb{Z}})$ of $\widetilde{\mathcal{M}}_d$, then this Hodge structure is the one induced by the isomorphism

$$\mathrm{H}_1(A, \mathbb{R}) \xrightarrow{\eta_{\mathbb{Z}} \otimes \mathbb{R}} \mathbb{Z}^{2d} \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} \mathbb{R}^{2d}.$$

We apply this to $W = \mathbb{R}^{2d}$. Let J be a complex structure on $\mathbb{R}^{2d}$, and let E be the corresponding $\mathbb{C}$-vector subspace of $\mathbb{C}^{2d}$. Then condition (a) on J is equivalent to the fact that:

(a') $\psi_{\mathbb{C}}(v,w) = 0$ for all $v, w \in E$,

(i.e., to the fact that E is a Lagrangian subspace[8] of $\mathbb{C}^{2d}$), and condition (b) on J is equivalent to the fact that

(b') $-i\psi_{\mathbb{C}}(v,\overline{v}) \in \mathbb{R}_{>0}$ for all $v \in E\backslash\{0\}$.

Note that these two conditions on a $\mathbb{C}$-vector subspace E of $\mathbb{C}^{2d}$ imply that $E \cap \overline{E} = \{0\}$. So we get a bijection from X' to the set of Lagrangian subspaces E of $V_{\mathbb{C}}$ satisfying (b').

If we represent Lagrangian subspaces of $\mathbb{C}^{2d}$ by their bases, sees as complex matrices of size $d \times 2d$, then the action of $\mathrm{Sp}_{2d}(\mathbb{R})$ is just left multiplication. For example, the subspace E_0 corresponding to $J_d \in X'$ is the one with basis $\begin{pmatrix} iI_d \\ I_d \end{pmatrix}$.

More generally, if $Y \in \mathfrak{h}_d^+$, the subspace of $\mathbb{C}^{2d}$ with basis $\begin{pmatrix} Y \\ I_d \end{pmatrix}$ is a Lagrangian subspace satisfying condition (b'), and every such Lagrangian subspace is of that form. So we get bijections

$$\widetilde{\mathcal{M}}_d \simeq X' \simeq \mathfrak{h}_d^+,$$

and we can check that the second bijection is $\mathrm{Sp}_{2d}(\mathbb{R})$-equivariant. Unraveling the definitions, we see that $Y \in \mathfrak{h}_d^+$ corresponds to the element $(A_Y, \lambda_Y, \eta_{\mathbb{Z},Y})$ of $\widetilde{\mathcal{M}}_d$ such that $A_Y = \mathbb{C}^d/(\mathbb{Z}^d + Y\mathbb{Z}^d)$, λ_Y is the principal polarization given by the Hermitian form with matrix $\mathrm{Im}(Y)^{-1}$ on $\mathbb{C}^d$, and $\eta_{\mathbb{Z},Y} : \mathbb{Z}^d + Y\mathbb{Z}^d \xrightarrow{\sim} \mathbb{Z}^{2d}$ is the isomorphism sending $a \in \mathbb{Z}^d$ to $(a,0) \in \mathbb{Z}^d \times \mathbb{Z}^d = \mathbb{Z}^{2d}$ and $Ya \in Y\mathbb{Z}^d$ to $(0,a) \in \mathbb{Z}^d \times \mathbb{Z}^d = \mathbb{Z}^{2d}$.

Now we want an interpretation of the quotients $\Gamma\backslash\mathfrak{h}_d^+$, for Γ an arithmetic subgroup of $\mathrm{Sp}_{2d}(\mathbb{Q})$. We will do this for the groups $\Gamma(n) = \mathrm{Ker}(\mathrm{Sp}_{2d}(\mathbb{Z}) \to \mathrm{Sp}_{2d}(\mathbb{Z}/n\mathbb{Z}))$, where n is a positive integer (and $\Gamma(n)$ is called the **principal congruence subgroup** at level n). Note that any arithmetic group contains $\Gamma(n)$ for n divisible enough.

We will need the notion of a level structure; we give the general definition here.

Definition 1.7. Let S be a scheme, (A,λ) be a principally polarized abelian scheme of relative dimension d over S, and n be a positive integer. Than a **level n structure** on (A,λ) is a couple (η,φ), where

$$\eta : A[n] \xrightarrow{\sim} \underline{\mathbb{Z}/n\mathbb{Z}}_S^{2g}, \quad \varphi : \underline{\mathbb{Z}/n\mathbb{Z}}_S \xrightarrow{\sim} \mu_{n,S}$$

are isomorphisms of group schemes such that $\varphi \circ \psi_{\mathbb{Z}/n\mathbb{Z}} \circ \eta$ is the Weil pairing associated to λ on $A[n]$.

Remark 1.8. A level n structure on (A,λ) can only exist if n is invertible on S and $\mu_{n,S}$ is a constant group scheme.

Note that isomorphisms $\varphi : \underline{\mathbb{Z}/n\mathbb{Z}}_S \xrightarrow{\sim} \mu_{n,S}$ correspond to sections $\zeta \in \mu_n(S)$ generating $\mu_{n,S}$ (i.e. to primitive nth roots of 1 over S), by sending φ to $\zeta = \varphi(1)$. So we will also see level structures as couples (η,ζ), with $\zeta \in \mu_n(S)$ primitive.

[8]By definition, a maximal isotropic subspace.

Let $\zeta_n = e^{-2i\pi/n} \in \mu_n(\mathbb{C})$. If $Y \in \mathfrak{h}_d^+$, then $\frac{1}{n}\eta_{\mathbb{Z},Y}$ defines an isomorphism of groups $\eta_Y : A_Y[n] \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^{2d}$, and it follows from formula

$$A[n] \times A[n] \to \mu_n(\mathbb{C}), \quad (v, w) \mapsto e^{-2i\pi n \operatorname{Im}(H(v,w))}$$

that (η, ζ_n) is a level n structure on (A_Y, η_Y).

Using the fact that $\mathrm{Sp}_{2d}(\mathbb{Z}) \to \mathrm{Sp}_{2d}(\mathbb{Z}/n\mathbb{Z})$ is surjective for every $n \in \mathbb{N}$, which follows from strong approximation for Sp_{2d},[9] we finally get:

Proposition 1.9. *Let n be a positive integer. The map $Y \mapsto (A_Y, \lambda_Y, \eta_Y)$ induces a bijection from $\Gamma(n)\backslash\mathfrak{h}_d^+$ to the set of isomorphism classes of triples (A, λ, η), where (A, λ) is a principally polarized complex abelian variety of dimension d and $\eta : A[n] \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^{2d}$ is an isomorphism of groups such that (η, ζ_n) is a level n structure on (A, λ).*

Now there is an obvious way to make $\Gamma(n)\backslash\mathfrak{h}_d^+$ into an algebraic variety.

1.3.2. *The connected Siegel modular variety.* Let $\mathcal{O}_n = \mathbb{Z}[1/n][T]/(T^n - 1)$. If S is a scheme over $\mathcal{O}_n$, we denote by $\varphi_0 : \underline{\mathbb{Z}/n\mathbb{Z}}_S \xrightarrow{\sim} \mu_{n,S}$ the isomorphism sending 1 to the class of T.

Definition 1.10. Let $\mathcal{M}'_{d,n}$ be the functor from the category of $\mathcal{O}_n$-schemes to the category of sets sending S to the set of isomorphisms classes of triples (A, λ, η), where (A, η) is a principally polarized abelian scheme of relative dimension d over S and $\eta : A[n] \xrightarrow{\sim} \underline{\mathbb{Z}/n\mathbb{Z}}_S^{2g}$ is an isomorphism of group schemes such that (η, φ_0) is a level n structure on (A, λ).

An isomorphism from (A, λ, η) to (A', λ', η') is an isomorphism of abelian varieties $u : A \xrightarrow{\sim} A'$ such that $\lambda' \circ u = u^\vee \circ \lambda$ and $\eta' = \eta \circ (u, u)$.

Theorem 1.11 (Mumford, cf. [FC90])**.** *Suppose that $n \geqslant 3$. Then the functor $\mathcal{M}'_{d,n}$ is representable by a smooth quasi-projective $\mathcal{O}_n$-scheme purely of dimension $d(d+1)/2$ and with connected geometric fibers, which we still denote by $\mathcal{M}'_{d,n}$ and call the **connected Siegel modular variety** of level n.*

Remark 1.12. If $n \in \{1, 2\}$, then triples (A, λ, η) as in Definition 1.10 may have automorphisms, so we should see $\mathcal{M}'_{d,n}$ as a stack. This stack will then be representable by a smooth Deligne-Mumford stack over $\mathcal{O}_n$ that is a finite étale quotient of the scheme $\mathcal{M}'_{d,3n}$.

We can now reformulate Proposition 1.9 in the following way.

Proposition 1.13. *Let $n \geqslant 3$ be an integer. Then the map $Y \mapsto (A_Y, \lambda_Y, \eta_Y)$ induces an isomorphism of complex manifolds from $\Gamma(n)\backslash\mathfrak{h}_d^+$ to $\mathcal{M}'_{d,n}(\mathbb{C})$.*

The fact that this is an isomorphism of complex manifolds is clear on the explicit formula for the bijection $\Gamma(n)\backslash\mathfrak{h}_d^+ \to \mathcal{M}'_{d,n}(\mathbb{C})$.

In particular, we showed that $\Gamma(n)\backslash\mathfrak{h}_d^+$ is the set of complex points of an algebraic variety defined over the number field $\mathbb{Q}(\zeta_n)$. Unfortunately, this number field depends on the level n. The issue is that we need a fixed primitive nth root of 1 in order to define the moduli problem $\mathcal{M}'_{d,n}$, so we need to be over a basis where such a primitive nth root exists. To fix this problem, we will allow the primitive nth root of 1 to vary.

[9]See [?] and [Pla69].

1.3.3. *The Siegel modular variety.*

Definition 1.14. Let n be a positive integer. The **Siegel modular variety** $\mathcal{M}_{d,n}$ is the functor from the category of $\mathbb{Z}/n\mathbb{Z}$-schemes to the category of sets sending a scheme S to the set of isomorphism classes of triples $(A, \lambda, \eta, \varphi)$, where (A, λ) is a principally polarized abelian scheme of relative dimension d over S and (η, φ) is a level n structure on (A, λ).

An isomorphism from (A, λ, η) to (A', λ', η') is an isomorphism of abelian varieties $u : A \xrightarrow{\sim} A'$ such that $\lambda' \circ u = u^\vee \circ \lambda$ and $\eta' = \eta \circ (u, u)$.

The group $\mathrm{GSp}_{2d}(\widehat{\mathbb{Z}})$ acts on $\mathcal{M}_{d,n}$: if $g \in \mathrm{GSp}_{2d}(\widehat{\mathbb{Z}})$ and $(A, \lambda, \eta, \varphi) \in \mathcal{M}_{d,n}(S)$, then

$$g \cdot (A, \lambda, \eta, \varphi) = (A, \lambda, g \circ \eta, c(g)^{-1}\varphi).$$

The kernel of this action is the group

$$K(n) = \mathrm{Ker}(\mathrm{GSp}_{2d}(\widehat{\mathbb{Z}}) \to \mathrm{GSp}_{2d}(\mathbb{Z}/n\mathbb{Z})).$$

If n divides m, then we have a morphism $\mathcal{M}_{d,m} \to \mathcal{M}_{d,n}$ that forgets part of the level m structure; this morphism is (representable) finite étale, and in fact it is a torsor under the finite group $K(n)/K(m)$.

We have the following variant of Theorem 1.11.

Theorem 1.15 (Mumford, cf. [FC90]). *Suppose that $n \geqslant 3$. Then the functor $\mathcal{M}_{d,n}$ is representable by a smooth quasi-projective $\mathcal{O}_n$-scheme purely of dimension $d(d+1)/2$, which we will denote by $\mathcal{M}_{d,n}$ and call the **Siegel modular variety** of level n.*

Remark 1.16. Let $K(n) = \mathrm{Ker}(\mathrm{GSp}_{2d}(\widehat{\mathbb{Z}}) \to \mathrm{GSp}_{2d}(\mathbb{Z}/n\mathbb{Z}))$. Then $\mathcal{M}_{d,n}$ is the Shimura variety for GSp_{2d} with level $K(n)$, or rather its integral model. If K is an open compact subgroup of $\mathrm{GSp}_{2d}(\mathbb{A}_f)$ that is small enough,[10] then we can also define the Shimura variety $\mathcal{M}_{d,K,\mathbb{Q}}$ with level K: choose n such that $K(n) \subset K$. Then $K(n)$ is a normal subgroup of K, so the group $K/K(n) \subset \mathrm{GSp}_{2d}(\mathbb{Z}/n\mathbb{Z})$ acts on $\mathcal{M}_{d,K,\mathbb{Q}}$, and we set $\mathcal{M}_{d,K,\mathbb{Q}} = \mathcal{M}_{d,n,\mathbb{Q}}/(K/K(n))$. It is easy to check that this does not depend on the choice of n.

In fact, for K an open compact subgroup of $\mathrm{GSp}_{2d}(\mathbb{A}_f)$, we have a direct definition of a level K structure on a principally polarized abelian scheme (see Section 5 of [Kot92]). For K small enough, the scheme $\mathcal{M}_{d,K,\mathbb{Q}}$ is the moduli space of a principally polarized abelian schemes with level K structure. In general, this moduli space is representable by a Deligne-Mumford stack. We can also define this moduli schemes over a localization of $\mathbb{Z}$, but the primes that we invert depend on K; see the discussion in Subsubsection **??**.

Let us explain the relationship between $\mathcal{M}_{d,n}$ and $\mathcal{M}'_{d,n}$. We define a map

$$s : (\mathbb{Z}/n\mathbb{Z})^\times \longrightarrow \mathrm{GSp}_{2d}(\mathbb{Z}/n\mathbb{Z}), \quad s(\alpha) = \begin{pmatrix} 0 & \alpha I_d \\ I_d & 0 \end{pmatrix};$$

note that s is a section of the multiplier $c : \mathrm{GSp}_{2d}(\mathbb{Z}/n\mathbb{Z}) \to (\mathbb{Z}/n\mathbb{Z})^\times$, and that it is not a morphism of groups.

Proposition 1.17. *The morphism*

[10] For example, $K \subset K(n)$ with $n \geqslant 3$.

$$\begin{aligned} \mathcal{M}'_{d,n} \times (\mathbb{Z}/n\mathbb{Z})^\times &\longrightarrow \mathcal{M}_{d,n,\mathcal{O}_n} \\ ((A,\lambda,\eta),\alpha) &\longmapsto (A,\lambda,s(\alpha)\circ\eta,\varphi_0\circ\alpha), \end{aligned}$$

where we see α as an automorphism of $\underline{\mathbb{Z}/n\mathbb{Z}}_S$ for any scheme S, is an isomorphism.

As a corollary, we get a description of the complex points of $\mathcal{M}_{d,n}$. Let $\mathfrak{h}_d = \mathfrak{h}_d^+ \cup (-\mathfrak{h}_d^+)$ be the set of symmetric matrices $Y \in M_d(\mathbb{C})$ such that $\mathrm{Im}(Y)$ is positive definite or negative definite. The action of $\mathrm{Sp}_{2d}(\mathbb{R})$ on $\mathfrak{h}_d$ extends to a transitive action of $\mathrm{GSp}_{2d}(\mathbb{R})$, given by the same formula. The stabilizer of $iI_d \in \mathfrak{h}_d$ in $\mathrm{GSp}_{2d}(\mathbb{R})$ is $\mathbb{R}_{>0}K_\infty$, so $\mathfrak{h}_d \simeq \mathrm{GSp}_{2d}(\mathbb{R})/\mathbb{R}_{>0}K_\infty$ as real analytic manifolds.

Corollary 1.18. *We have an isomorphism of complex manifolds*

$$\mathcal{M}_{d,n}(\mathbb{C}) \simeq \mathrm{GSp}_{2d}(\mathbb{Q})\backslash(\mathfrak{h}_d \times \mathrm{GSp}_{2d}(\mathbb{A}_f)/K(n))$$

extending the isomorphism of Proposition 1.9, where $K(n) = \mathrm{Ker}(\mathrm{GSp}_{2d}(\widehat{\mathbb{Z}}) \to \mathrm{GSp}_{2d}(\mathbb{Z}/n\mathbb{Z}))$ and $\mathrm{GSp}_{2d}(\mathbb{Q})$ acts diagonally on $\mathfrak{h}_d \times \mathrm{GSp}_{2d}(\mathbb{A}_f)$.

This follows from the fact that

$$\mathrm{GSp}_{2d}(\mathbb{Q})\backslash(\mathfrak{h}_d \times \mathrm{GSp}_{2d}(\mathbb{A}_f)/K(n)) \simeq \mathrm{GSp}_{2d}(\mathbb{Q})^+\backslash(\mathfrak{h}_d^+ \times \mathrm{GSp}_{2d}(\mathbb{A}_f)/K(n)),$$

where $\mathrm{GSp}_{2d}(\mathbb{Q})^+ = \{g \in \mathrm{GSp}_{2d}(\mathbb{Q}) \mid c(g) > 0\}$, and from strong approximation for Sp_{2d},[11] which implies that c induces a bijection

$$\mathrm{GSp}_{2d}(\mathbb{Q})^+\backslash\mathrm{GSp}_{2d}(\mathbb{A}_f)/K(n) \xrightarrow{\sim} \mathbb{Q}_{>0}\backslash\mathbb{A}_f^\times/c(K(n)) \simeq \widehat{\mathbb{Z}}^\times(1+n\widehat{\mathbb{Z}}) \simeq (\mathbb{Z}/n\mathbb{Z})^\times.$$

For every $i \in (\mathbb{Z}/n\mathbb{Z})^\times$, we choose $x_i \in \mathrm{GSp}_{2d}(\mathbb{A}_f)$ lifting i and we set

$$\Gamma(n)_i = \mathrm{GSp}_{2d}(\mathbb{Q})^+ \cap x_iK(n)x_i^{-1} = \mathrm{Sp}_{2d}(\mathbb{Q}) \cap x_iK(n)x_i^{-1}.$$

Then the $\Gamma(n)_i$ are arithmetic subgroups of $\mathrm{Sp}_{2d}(\mathbb{Q})$, and we have

$$\mathrm{GSp}_{2d}(\mathbb{Q})\backslash(\mathfrak{h}_d \times \mathrm{GSp}_{2d}(\mathbb{A}_f)/K(n)) \simeq \coprod_{i\in(\mathbb{Z}/n\mathbb{Z})^\times} \Gamma(n)_i\backslash\mathfrak{h}_d^+$$

as complex manifolds.

In fact, for $i \in (\mathbb{Z}/n\mathbb{Z})^\times$, we can take $x_i = \begin{pmatrix} 0 & a_iI_d \\ I_d & 0 \end{pmatrix}$ with $a_i \in \widehat{\mathbb{Z}}^\times$ lifting i. In particular, we have $x_i \in \mathrm{GSp}_{2d}(\widehat{\mathbb{Z}})$; as $K(n)$ is a normal subgroup of $\mathrm{GSp}_{2d}(\widehat{\mathbb{Z}})$, we get $x_iK(n)x_i^{-1} = K(n)$, hence $\Gamma(n)_i = \Gamma(n)$, and finally

$$\mathrm{GSp}_{2d}(\mathbb{Q})\backslash(\mathfrak{h}_d \times \mathrm{GSp}_{2d}(\mathbb{A}_f)/K(n)) \simeq \coprod_{i\in(\mathbb{Z}/n\mathbb{Z})^\times} \Gamma(n)_i\backslash\mathfrak{h}_d^+.$$

Remark 1.19. If K is a small enough open compact subgroup of $\mathrm{GSp}_{2d}(\mathbb{A}_f)$, then we get an isomorphism of complex manifolds:

$$\mathcal{M}_{d,K}(\mathbb{C}) \simeq \mathrm{GSp}_{2d}(\mathbb{Q})\backslash(\mathfrak{h}_d \times \mathrm{GSp}_{2d}(\mathbb{A}_f)/K).$$

[11] See [?] and [Pla69].

1.3.4. *Heche correspondence.* We can also descend the Hecke correspondences before to morphisms of schemes over $\mathbb{Z}[1/n]$.

We proceed as in Section 3 of [Lau05]. Let $g \in \mathrm{GSp}_{2d}(\mathbb{A}_f)$, and let K, K' be small enough open compact subgroups of $\mathrm{GSp}_{2d}(\mathbb{A}_f)$ such that $K' \subset K \cap gKg^{-1}$. We want to define finite étale morphisms $T_1, T_g : \mathcal{M}_{d,K'} \to \mathcal{M}_{d,K}$, and the Hecke correspondence associated to (g, K, K') is the couple (T_1, T_g).

Choose $n \geqslant 3$ such that $K(n) \subset K'$; then $\mathcal{M}_{d,K'} = \mathcal{M}_{d,n}/(K'/K(n))$ and $\mathcal{M}_{d,K} = \mathcal{M}_{d,n}/(K/K(n))$. The morphism T_1 just forgets part of the level structure: as $K'/K(n) \to K/K(n)$, we have an obvious morphism $T_1 : \mathcal{M}_{d,K'} \to \mathcal{M}_{d,K}$.

To define T_g, we first consider the following special case: if $g \in \mathrm{GL}_{2d}(\widehat{\mathbb{Z}}) \cap \mathrm{GSp}_{2d}(\mathbb{A}_f)$, let $x = (A, \lambda, \eta, \varphi) \in \mathcal{M}_{d,n}(S)$. Let u be the endomorphism of $\underline{\mathbb{Z}/n\mathbb{Z}}^{2d}$ with matrix g. Then T_g sends the class of x in $\mathcal{M}_{d,K'}(S)$ to the class of $(A', \lambda', \eta', \varphi) \in \mathcal{M}_{d,n}(S)$, where A' is the quotient of A by the finite flat subgroup scheme $\eta^{-1}(\operatorname{Ker} u)$ of $A[n]$ and λ', η' are the morphisms deduced from λ, $g \circ \eta$.

Note that, if $g = aI_{2d}$ with $a \in \widehat{\mathbb{Z}} \cap \mathbb{A}_f^\times$ and $K' = K$, then the morphisms $T_g : \mathcal{M}_{d,K} \to \mathcal{M}_{d,K}$ is an isomorphism.

Finally, for a general $g \in \mathrm{GSp}_{2d}(\mathbb{A}_f)$, we write $g = a^{-1}g_0$ with $a \in (\widehat{\mathbb{Z}} \cap \mathbb{A}_f^\times)I_{2d}$ and $g_0 \in \mathrm{GL}_{2d}(\widehat{\mathbb{Z}}) \cap \mathrm{GSp}_{2d}(\mathbb{A}_f)$, and we set $T_g = T_{g_0} \circ T_a^{-1}$.

Remark 1.20. If we use instead the general definition of a level K structure from Section 5 of [Kot92], then it becomes much easier to define the Hecke correspondences; see Section 6 ibid..

Remark 1.21. We have two ways to think of $\mathcal{M}_{d,n}(\mathbb{C})$: as an adelic double quotient or as finite disjoint union of spaces $\Gamma(n)\backslash\mathfrak{h}_d^+$, which are locally symmetric spaces associated to the semi-simple group Sp_{2d}. The first description is more convenient to see the action of adelic Hecke operators, and the second description is a bit more concrete and has simpler combinatorics. Note also that the complex manifold $\Gamma(n)\backslash\mathfrak{h}_d^+$ is isomorphic to the set of $\mathbb{C}$-points of the algebraic variety $\mathcal{M}'_{d,n}$, but this algebraic variety is defined over the field $\mathbb{Q}[T]/(T^n - 1)$, which depends on n. On the other hand, the adelic double quotient $\mathrm{GSp}_{2d}(\mathbb{Q})\backslash(\mathfrak{h}_d \times \mathrm{GSp}_{2d}(\mathbb{A}_f)/K(n)$ is isomorphic to the set of $\mathbb{C}$-points of the algebraic variety $\mathcal{M}_{d,n}$, which is defined over $\mathbb{Q}$. So if we want to consider Shimura varieties as a projective system of algebraic varieties over a number field, then it makes sense to use the adelic double quotients, because they are all defined on the same field.

1.3.5. *Background on abelian schemes.* Let S be a scheme. We denote by Sch/S the category of S-schemes.

Definition 1.22. (1) An **abelian scheme** over S is an S-group scheme $A \to S$ which is smooth and proper with geometrically connected fibers. If S is the spectrum of a field k, an abelian scheme over S is also called an **abelian variety** over k.

(2) A **morphism of abelian schemes** over S is a morphism of S-group schemes between abelian schemes over S.

Proposition 1.23. *Let A be an abelian scheme over S. Then*

(1) *The S-group scheme $A \to S$ is commutative;*

(2) *The morphism $A \to S$ has connected fibers.*

Definition 1.24. Let A be an abelian scheme over S, and let $e : S \to A$ be its zero section. We consider the following two functors from $(\mathsf{Sch}/S)^{\mathrm{op}}$ to the category of sets:

(a) The functor $\mathrm{Pic}_{A/S,e}$ sending an S-scheme $T \to S$ to the set of isomorphism classes of couples $(\mathcal{L}, \varphi)$, where $\mathcal{L}$ is an invertible sheaf on $A \times_S T$, $e_T = e \times_S T : T \to A \times_S T$, and $\varphi : \mathcal{O}_T \xrightarrow{\sim} e_T^* \mathcal{L}$ is an isomorphism. An isomorphism from $(\mathcal{L}, \varphi)$ to $(\mathcal{L}', \varphi')$ is an isomorphism of $\mathcal{O}_T$-modules $\alpha : \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ such that $(e_T^* \alpha) \circ \varphi = \varphi'$;

(b) The subfunctor $\mathrm{Pic}^0_{A/S,e}$ of $\mathrm{Pic}_{A/S,e}$ sending sending an S-scheme $T \to S$ to the set of isomorphism classes of couples $(\mathcal{L}, \varphi)$ as in (a) such that, for every point t of T, every smooth projective curve C over the residue field $\kappa(t)$ of t, and every morphism of $\kappa(t)$-schemes $f : C \to A \times_S t$, the line bundle $f^*(\mathcal{L}|_{A \times_S t})$ is of degree 0 on C.

Remark 1.25. (1) The functor $\mathrm{Pic}_{A/S,e}$ can be made into a functor into the category of abelian groups: if T is an S-scheme and $(\mathcal{L}, \varphi)$, $(\mathcal{L}', \varphi')$ represent elements of $\mathrm{Pic}_{A/S,e}(T)$, their product is represented by $(\mathcal{L} \otimes_{\mathcal{O}_T} \mathcal{L}', \varphi \otimes \varphi')$, where $\varphi \otimes \varphi'$ is the isomorphism

$$\mathcal{O}_T = \mathcal{O}_T \otimes_{\mathcal{O}_T} \mathcal{O}_T \xrightarrow[\varphi \otimes \varphi']{\sim} (e_T^* \mathcal{L}) \otimes_{\mathcal{O}_T} (e_T^* \mathcal{L}') = e_T^*(\mathcal{L} \otimes_{\mathcal{O}_T} \mathcal{L}').$$

Moreover, for every S-scheme T, the set $\mathrm{Pic}^0_{A/S,e}(T)$ is a subgroup of $\mathrm{Pic}_{A/S,e}(T)$.

(2) If $X \to S$ is a scheme over S, then the relative Picard functor on Sch/S is the fppf sheafification of the functor $T \mapsto \mathrm{Pic}(X \times_S T)$ (where, for Y a scheme, we denote by $\mathrm{Pic}(Y)$ the set of isomorphism classes of line bundles on Y, that is an abelian group for the tensor product); see [Aut, Situation 0D25]. We can also define a subfunctor $\mathrm{Pic}^0_{X/S}$ of $\mathrm{Pic}_{X/S}$ as in Definition 1.24. By [Aut, Lemma 0D28], if A is an abelian scheme over S, then there is an isomorphism of functors in abelian groups $\mathrm{Pic}_{A/S} \xrightarrow{\sim} \mathrm{Pic}_{A/S,e}$, inducing an isomorphism $\mathrm{Pic}^0_{A/S} \xrightarrow{\sim} \mathrm{Pic}^0_{A/S,e}$.

(3) We can upgrade $A \mapsto \mathrm{Pic}_{A/S,e}$ and $A \mapsto \mathrm{Pic}^0_{A/S,e}$ to contravariant functors in A: if $f : A \to B$ is a morphism of abelian schemes over S, then it induces a natural transformation $f^* : \mathrm{Pic}_{B/S,e} \to \mathrm{Pic}_{A/S,e}$ sending $(\mathcal{L}, \varphi)$ to $(f^*(\mathcal{L}), f^*(\varphi))$, and f^* sends $\mathrm{Pic}^0_{B/S,e}$ to $\mathrm{Pic}^0_{A/S,e}$.

Theorem 1.26. *Let A be an abelian scheme over S. Then $\mathrm{Pic}^0_{A/S,e}$ is representable by an abelian scheme over S.*

Proof. We know that $\mathrm{Pic}^0_{A/S,e}$ is representable by an algebraic space by a result of M. Artin (see [Art69] or [Aut, Lemma 0D2C]). We can check on the moduli problem that this algebraic space is proper and smooth, and its fibers over points of S are abelian varieties by the classical theory of the dual abelian variety (see sections II.8 and III.13 of Mumford's book [Mum08]). It remains to prove that the algebraic space representing $\mathrm{Pic}^0_{A/S,e}$ is a scheme; this is due to Raynaud, and a proof is given in [FC90, Theorem 1.9]. □

Definition 1.27. Let A be an abelian scheme over S. The abelian scheme over S representing $\mathrm{Pic}^0_{A/S,e}$ is called the **dual abelian scheme** of A and denoted by $A^\vee$. In particular, we get a couple $(\mathcal{P}_A, \varphi_A)$ representing the element of $\mathrm{Pic}^0_{A/S,e}(A^\vee)$ corresponding to $\mathrm{id}_{A^\vee}$, with $\mathcal{P}_A$ a line bundle on $A \times_S A^\vee$, called the **Poincaré line bundle**.

If $f : A \to B$ is a morphism of abelian schemes over S, we denote by $f^\vee : B^\vee \to A^\vee$ the morphism corresponding to the natural transformation $f^* : \mathrm{Pic}^0_{B/S,e} \to \mathrm{Pic}^0_{A/S,e}$ of Remark 1.25.

Remark 1.28. Let $e : S \to A^\vee$ be the unit section. Then the pullback of $\mathcal{P}_A$ by $A \times_S e : A = A \times_S A \to A \times_S A^\vee$ is the line bundle on A corresponding to the element e of $A^\vee(S) = \mathrm{Pic}^0_{A/S,e}(S)$; in other words, it is isomorphic to the trivial line bundle $\mathcal{O}_A$. So $\mathcal{P}_A$ defines an element of $\mathrm{Pic}^0_{A/S,e}(A)$, that is, a morphism of S-schemes $A \to A^{\vee\vee}$, called the **biduality morphism**. The **biduality theorem** says that the biduality morphism is an isomorphism. For S the spectrum of a field, this is proved in Section III.13 of [Mum08], and the general case reduces to this by looking at the fibers of points of S.

Let A be an abelian scheme over S and let $\mathcal{L}$ be a line bundle on A. We denote by $\mu, p_1, p_2, \varepsilon : A \times_S A \to A$ the addition morphism, the first projection, the second projection and the zero morphism respectively. Then the line bundle $(\mu^*\mathcal{L}) \otimes (p_1^*\mathcal{L}^{\otimes -1}) \otimes (p_2^*\mathcal{L}^{\otimes -1}) \otimes (\varepsilon^*\mathcal{L})$ on $A \times_S A$ is trivial when restricted to $S \times_S A$ via the zero section of A, hence it defines an element of $\mathrm{Pic}^0_{A/S,e}(A)$, corresponding to a morphism of S-schemes $\lambda(\mathcal{L}) : A \to A^\vee$. Moreover, the theorem of the cube (see for example Section III.10 of [Mum08]).

Definition 1.29. Let A be an abelian scheme over S. A **polarization** on A is a morphism of abelian schemes $\lambda : A \to A^\vee$ such that, for every algebraically closed field k and every morphism $\mathrm{Spec}\, k \to S$, the morphism $\lambda \times_S \mathrm{Spec}\, k : A \times_S \mathrm{Spec}\, k \to A^\vee \times_S \mathrm{Spec}\, k = (A \times_S \mathrm{Spec}\, k)^\vee$ is of the form $\lambda(\mathcal{L})$, for $\mathcal{L}$ an ample line bundle on $A \times_S \mathrm{Spec}\, k$. We say that a polarization is **principal** if it is an isomorphism.

A **principally polarized abelian scheme** over S is a pair $(A,$
lambda), where A is an abelian scheme over S and λ is a principal polarization on A.

Remark 1.30.
(1) A polarization on A is always an isogeny, i.e. finite and faithfully flat.
(2) Let λ be a polarization on A, and let n be a positive integer. Then, composing $\lambda : A[n] \to A^\vee[n]$ with the canonical pairing $A[n] \times A^\vee[n] \to \mu_{n,S}$, we get a pairing $A[n] \times A[n] \to \mu_{n,S}$, called the **Weil pairing** associated to λ. If λ is principal, this is a perfect pairing.

1.4. **Shimura varieties over $\mathbb{C}$.** Remember the upshot of Subsection 1.2: if we want algebraic varieties that are all defined over the same number field, and Hecke correspondences that are also defined on this number field, it is better to work with adelic double quotient for a reductive group such as GSp_{2d} rather than with locally symmetric spaces for a semi-simple group such as Sp_{2d}. This (and Theorem 1.3) motivates the definition of Shimura data, due to Deligne in [Del71].

1.4.1. *The Serre torus.* Let $\mathbb{S}$ be $\mathbb{C}^\times$ seen as an algebraic group over $\mathbb{R}$; this is called the **Serre torus**. In other words, the group $\mathbb{S}$ is the Weil restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$ of GL_1, so that $\mathbb{S}(R) = (R \otimes_\mathbb{R} \mathbb{C})^\times$ for every $\mathbb{R}$-algebra R. We denote by w the injective morphism $\mathrm{GL}_{1,\mathbb{R}} \to \mathbb{S}$ corresponding to the inclusion $\mathbb{R}^\times \subset \mathbb{C}^\times$.

We have $\mathbb{S}(\mathbb{C}) = (\mathbb{C} \otimes_\mathbb{R} \mathbb{C})^\times \xrightarrow{\sim} \mathbb{C}^\times \times \mathbb{C}^\times$, where the isomorphism sends $a \otimes 1 + b \otimes i$ to $(a+ib, a-ib)$. So the abelian group $\mathrm{Hom}(\mathbb{S}_\mathbb{C}, \mathrm{GL}_{1,\mathbb{C}})$ of characters of $\mathbb{S}$ is free of rank 2 and generated by the characters z and $\overline{z}$ corresponding to the two projections of $\mathbb{C}^\times \times \mathbb{C}^\times$ on

$\mathbb{C}^\times$. We denote by $r : \mathrm{GL}_{1,\mathbb{C}} \to \mathbb{S}_\mathbb{C}$ the injective morphism corresponding to the injection of the first factor in $\mathbb{C}^\times \times \mathbb{C}^\times$.

If V is a real vector space and $\rho : \mathbb{S} \to \mathrm{GL}(V)$ is a morphism of algebraic groups (i.e. a representation of $\mathbb{S}$ on V), then we have $V_\mathbb{C} := V \otimes_\mathbb{R} \mathbb{C} = \bigoplus_{p,q\in\mathbb{Z}} V^{p,q}$, where $V^{p,q}$ is the subspace of $V_\mathbb{C}$ on which $\mathbb{S}_\mathbb{C}$ acts by the character $z^{-p}\overline{z}^{-q}$; moreover, as ρ is defined over $\mathbb{R}$, we have $\overline{V^{p,q}} = V^{q,p}$ for all $p, q \in \mathbb{Z}$. Let $m \in Z$. We say that ρ is **of weight** m if $\rho \circ r : \mathrm{GL}_1 \to \mathrm{GL}(V)$ is equal to $x \mapsto x^m \mathrm{id}_V$.

Remark 1.31. If $\rho : \mathbb{S} \to \mathrm{GL}(V)$ is of weight m, we have $V^{p,q} = 0$ unless $p + q = m$, so the decomposition $V_\mathbb{C} = \bigoplus p, qV^{p,q}$ is a pure Hodge structure of weight m on V. In fact, representations of weight m of $\mathbb{S}$ on V are in bijection with pure Hodge structures of weight m on V.

1.4.2. *Shimura data.*

Definition 1.32. A **Shimura datum** is a couple (G, h), where G is a connected reductive algebraic group over $\mathbb{Q}$ and $h : \mathbb{S} \to G_\mathbb{R}$ is a morphism of real algebraic groups such that:

(a) The image of $h \circ w : \mathrm{GL}_{1,\mathbb{R}} \to G_\mathbb{R}$ is central;

(b) If $\mathfrak{g} = \mathrm{Lie}(G_\mathbb{R})$ and $\mathfrak{g}_\mathbb{C} = \bigoplus_{p,q\in\mathbb{Z}} \mathfrak{g}^{p,q}$ is the decomposition induced by the representation $\mathrm{Ad}\circ h : \mathbb{S} \to \mathrm{GL}(\mathfrak{g})$, then we have $\mathfrak{g}^{p,q} = 0$ unless $(p, q) \in \{(-1, 1), (0, 0), (1, -1)\}$;

(c) Conjugation by $h(i)$ induces a Cartan involution of $G_{\mathrm{der}}(\mathbb{R})$ (see Theorem 1.3);

(d) G_{der} has no normal subgroup (defined over $\mathbb{Q}$) whose group of R-points is compact.[12]

Let (G, h) be a Shimura datum. We denote by K_∞ the centralizer of h in $G(\mathbb{R})$ and by X the set of $G(\mathbb{R})$-conjugates of h. Then K_∞ contains the center of $G(\mathbb{R})$, and $K_\infty \cap G_{\mathrm{der}}(\mathbb{R})^0$ is equal to the centralizer of $h_0(i)$ in $G_{\mathrm{der}}(\mathbb{R})^0$, hence is a maximal compact subgroup of $G_{\mathrm{der}}(\mathbb{R})^0$ by condition (c). We have $X \simeq G(\mathbb{R})/K_\infty$, and Theorem 1.3 implies that there is a $G(\mathbb{R})$-invariant complex structure on X such that the connected components of X are Hermitian symmetric domains.

Example 1.33. Take $G = \mathrm{GSp}_{2d}$. Up to conjugation, there exists a unique morphism $h : \mathbb{S} \to \mathrm{GSp}_{2d}$ satisfying conditions (a)–(c) of Definition 1.32 and such that $h \circ w = xI_{2d}$ for every $x \in \mathbb{R}^\times$. An element of that class is given by

$$h(a + ib) = \begin{pmatrix} aI_d & -bI_d \\ bI_d & aI_d \end{pmatrix}.$$

For this h, we have $K_\infty = \mathrm{GSp}_{2d}(\mathbb{R}) \cap \mathrm{GO}(2d)$, and we can check that the map $\mathrm{GU}(d) \to K_\infty$ sending $X + iY \in \mathrm{GU}(d)$ (with $X, Y \in M_d(\mathbb{R})$) to $\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}$ is an isomorphism of Lie groups. So $K_\infty = \mathbb{R}_{>0}K'_\infty$, where $K_\infty = \mathrm{Sp}_{2d}(\mathbb{R}) \cap \mathrm{O}(d)$ is the maximal compact subgroup of $\mathrm{Sp}_{2d}(\mathbb{R})$ that was called K_∞ in Subsections 1.1 and 1.2. This implies that $X \simeq \mathfrak{h}_d$.

The couple (GSp_{2d}, h) is called a **Siegel Shimura datum**.

Let K be an open compact subgroup of $G(\mathbb{A}_f)$. We set

$$M_K(G, h)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K,$$

[12] Note that $G_{\mathrm{der}}(\mathbb{R})$ could still have compact normal algebraic subgroups, as long as they are not defined over $\mathbb{Q}$.

where the group K acts by right translations on the factor $G(\mathbb{A}_f)$, and the group $G(\mathbb{Q})$ acts by left translations on both factors simultaneously. This is the **Shimura variety at level** K associated to the Shimura datum K.

As in Subsection 1.1, if $(x_i)_{i\in I}$ is a system of representatives of the (finite) quotient $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$, and if $\Gamma_i = G(\mathbb{Q}) \cap x_i K x_i^{-1}$ for every $i \in I$, then the Γ_i are arithmetic subgroups of $G(\mathbb{Q})$, and we have

$$M_K(G,h)(\mathbb{C}) = \coprod_{i\in I} \Gamma_i\backslash X.$$

Hence it follows from Theorem 1.5 that $M_K(G,h)(\mathbb{C})$ is (the set of complex points of) a quasi-projective algebraic variety over $\mathbb{C}$, smooth if K is small enough.

Again as in Subsection 1.1, we have Hecke correspondences between the $M_K(G,h)(\mathbb{C})$, which are finite maps, hence morphisms of algebraic varieties. This defines an action of $G(\mathbb{A}_f)$ on the projective system $(M_K(G,h)(\mathbb{C}))_{K\subset G(\mathbb{A}_f)}$, or on its limit

$$M(G,h)(\mathbb{C}) = \varprojlim M_K(G,h)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f),$$

and we have $M_K(G,h)(\mathbb{C}) = M(G,h)(\mathbb{C})/K$ for every open compact subgroup K of $G(\mathbb{A}_f)$.

1.4.3. *Morphisms of Shimura varieties.* Let (G_1,h_1) and (G_2,h_2) be Shimura data, and let $u : G_1 \to G_2$ be a morphism of algebraic groups such that $u \circ h_1$ and h_2 are conjugated under $G_2(\mathbb{R})$; we say that u is a **morphism of Shimura data**. Then u induces a morphism of complex manifolds $X_1 \to X_2$, so, for all $K_1 \subset G_1(\mathbb{A}_f)$, $K_2 \subset G_2(\mathbb{A}_f)$ open compact subgroups such that $u(K_1) \subset K_2$, we get a morphism of quasi-projective varieties $u(K_1,K_2) : M_{K_1}(G_1,h_1)(\mathbb{C}) \to M_{K_2}(G_2,h_2)(\mathbb{C})$. We can also think of this as a morphism of $\mathbb{C}$-schemes $u : M(G_1,h_1)(\mathbb{C}) \to M(G_2,h_2)(\mathbb{C})$.

Proposition 1.34. [Del71, Proposition 1.15] *If G_1 is an algebraic subgroup of G_2 and u is the inclusion, then, for every open compact subgroup K_1 of $G_1(\mathbb{A}_f)$, there exists an open compact subgroup $K_2 \supset K_1$ of $G_2(\mathbb{A}_f)$ such that $u(K_1,K_2) : M_{K_1}(G_1,h_1)(\mathbb{C}) \to M_{K_2}(G_2,h_2)(\mathbb{C})$ is a closed immersion.*

1.4.4. *Connected components.* For (G,h) equal to the Shimura datum of Example 1.33 and $K = \mathrm{Ker}(\mathrm{GSp}_{2d}(\widehat{\mathbb{Z}}) \to \mathrm{GSp}(\mathbb{Z}/n\mathbb{Z}))$, we have seen that the multiplier $c : \mathrm{GSp}_{2d} \to \mathrm{GL}_1$ induces a bijection

$$\pi_0(M_K(G,h)(\mathbb{C})) \xrightarrow{\sim} \mathbb{Q}_{>0}\backslash\mathbb{A}_f^\times/c(K) = \mathbb{Q}^\times\backslash\mathbb{A}^\times/c(K_\infty K).$$

In fact, it follows from real approximation and the Hasse principle that this works for many Shimura varieties:

Theorem 1.35 (Deligne, see [Del71, 2.7])**.** *Let $\nu : G \to T := G/G_{\mathrm{der}}$ be the quotient morphism, and suppose that G_{der} is simply connected. Then, for every open compact subgroup K of $G(\mathbb{A}_f)$, the map ν induces an isomorphism of groups*

$$\pi_0(M_K(G,h)(\mathbb{C})) \xrightarrow{\sim} T(\mathbb{Q})\backslash T(\mathbb{A})/\nu(K_\infty K).$$

In other words, ν induces an isomorphism

$$\pi_0(M(G,h)(\mathbb{C})) \xrightarrow{\sim} \pi_0(T(\mathbb{A})/T(\mathbb{Q}))/\pi_0(K_\infty).$$

1.5. **Canonical models.** In the situation of Example 1.33, we have seen that the algebraic varieties $M_K(G,h)(\mathbb{C})$ and all the Hecke correspondences are defined over $\mathbb{Q}$. We would like to generalize this kind of result to other Shimura varieties.

1.5.1. *Model of a Shimura variety.* First we need to say what we mean by a model.

Definition 1.36. Let (G,h) be a Shimura datum, and F be a subfield of $\mathbb{C}$. A **model** of the projective system $(M_K(G,h)(\mathbb{C}))_K$ over F is the data:

- for every open compact subgroup K of $G(\mathbb{A}_f)$, of a quasi-projective variety M_K over F and isomorphism $\iota_K : M_K \otimes_F \mathbb{C} \xrightarrow{\sim} M_K(G,h)(\mathbb{C})$;
- for every $g \in G(\mathbb{A}_f)$ and all open compact subgroups K, K' of $G(\mathbb{A}_f)$ such that $gK'g^{-1} \subset K$, of a morphism of F-varieties $T_{g,K,K'} : M_{K'} \to M_K$,

such that:

(i) For all g, K, K' as above, the morphism $\iota_K \circ T_{g,K,K',\mathbb{C}} \circ \iota_{K'}^{-1} : M_{K'}(G,h)(\mathbb{C}) \to M_K(G,h)(\mathbb{C})$ sends the class of (x,h) in $M_{K'}(G,h)(\mathbb{C})$ to the class of (x,hg) in $M_K(G,h)(\mathbb{C})$;
(ii) If K is an open compact subgroup of $G(\mathbb{A}_f)$ and $g \in K$, then $T_{g,K,K} = \mathrm{id}_{M_K}$;
(iii) If K, K', K'' are open compact subgroups of $G(\mathbb{A}_f)$ and $g, h \in G(\mathbb{A}_f)$ are such that $gK'g^{-1} \subset k$ and $hK''h^{-1} \subset K'$, then $T_{g,K,K'} \circ T_{h,K',K''} = T_{gh,K,K''}$;
(iv) If K, K' are open compact subgroups of $G(\mathbb{A}_f)$ such that K' is a normal subgroup of K, then the morphism $T_{g,K',K'}$ for $g \in K$ define an action of K/K' on $M_{K'}$ (this follows from (ii) and (iii)), and $T_{1,K,K'} : M_{K'} \to M_K$ induces an isomorphism $M_{K'}/(K/K') \to M_K$.

If we have a model $(M_K)_K$ of $(M_K(G,h)(\mathbb{C}))_K$ over F, we write $M = \varprojlim_K M_K$ (where the transition morphisms are given by the $T_{1,K',K}$). This is an F-scheme with an action of $G(\mathbb{A}_f)$, and we have a $G(\mathbb{A}_f)$-equivariant isomorphism $M \otimes_F \mathbb{C} \xrightarrow{\sim} M(G,h)(\mathbb{C})$.

In particular we get an action of $\mathrm{Gal}(\overline{F}/F)$ on $\pi_0(M(G,h)(\mathbb{C})) \xrightarrow{\sim} \pi_0(M \otimes_F \overline{F})$, which must commute with the action of $G(\mathbb{A}_f)$. Under the hypothesis of Theorem 1.35, we have

$$\pi_0(M \otimes_F \overline{F}) \xrightarrow{\sim} \pi_0(T(\mathbb{A})/T(\mathbb{Q}))/\pi_0(K_\infty)$$

with $T = G/G^{\mathrm{der}}$, and $G(\mathbb{A}_f)$ acts transitively on this set of connected components (Proposition 2.2 of [Del71]). So every element of $\mathrm{Gal}(\overline{F}/F)$ acts by translation by an element of $\pi_0(T(\mathbb{A})/T(\mathbb{Q}))/\pi_0(K_\infty)$, and the action of $\mathrm{Gal}(\overline{F}/F)$ comes from a morphism of groups $\mathrm{Gal}(\overline{F}/F) \to \pi_0(T(\mathbb{A})/T(\mathbb{Q}))/\pi_0(K_\infty)$, that necessarily factors through the maximal abelian quotient $\mathrm{Gal}(\overline{F}/F)^{\mathrm{ab}}$.

Suppose that F is a number field. Then global class field theory[13] gives an isomorphism

$$\mathrm{Gal}(\overline{F}/F)^{\mathrm{ab}} \xrightarrow{\sim} \pi_0(F^\times \backslash \mathbb{A}_F^\times)$$

where $\mathbb{A}_F$ is the ring of adeles of F, so the action of $\mathrm{Gal}(\overline{F}/F)$ on $\pi_0(M \otimes_F \overline{F})$ comes from a morphism of groups

$$\lambda_M : \pi_0(F^\times \backslash \mathbb{A}_F^\times) \to \pi_0(T(\mathbb{Q}) \backslash T(\mathbb{A}))/\pi_0(K_\infty),$$

called the **reciprocity law of the model**.

[13] Normalized so that local uniformizers correspond to geometric Frobenius elements.

1.5.2. *The case of tori.* We consider the case where $G = T$ is a torus. Let $h : \mathbb{S} \to T_{\mathbb{R}}$ be any morphism of real algebraic groups. Then h trivially satisfies the conditions of Definition 1.32, so we get a Shimura datum (T, h), and $X = T(\mathbb{R})/\operatorname{Cent}_{T(\mathbb{R})}(h)$ is a singleton. For every open compact subgroup K of $T(\mathbb{A}_f)$,

$$M_K(T, h)(\mathbb{C}) = T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K$$

is a finite set, and we have

$$M(T, h)(\mathbb{C}) = T(\mathbb{Q})\backslash T(\mathbb{A}_f),$$

which is a profinite set. Giving a model of the Shimura variety of (T, h) over a subfield F of $\mathbb{C}$ is the same as giving an action of $\operatorname{Gal}(\overline{F}/F)$ over $T(\mathbb{Q})\backslash T(\mathbb{A}_f)$ (commuting with the action of $T(\mathbb{A}_f)$ by translation), i.e. a morphism of groups $\operatorname{Gal}(\overline{F}/F) \to T(\mathbb{Q})\backslash T(\mathbb{A}_f)$. If F is a number field, this is equivalent to giving a morphism of groups

$$\lambda : \pi_0(F^\times\backslash\mathbb{A}_F^\times) \to \pi_0(T(\mathbb{Q})\backslash T(\mathbb{A})).$$

It is natural to construct such a morphism from a morphism of algebraic groups $F^\times \to T$, where $F^\times$ is seen as an algebraic group over $\mathbb{Q}$ (so that, for example, we have $F^\times(\mathbb{A}) = (\mathbb{A} \otimes_{\mathbb{Q}} F)^\times = \mathbb{A}_F^\times$). We already have a morphism $h : \mathbb{S} \to T_{\mathbb{R}}$, which gives a morphism of complex algebraic groups $h_{\mathbb{C}} : \mathbb{S}_{\mathbb{C}} \to T_{\mathbb{C}}$. Remember that $\mathbb{S}_{\mathbb{C}} \simeq \mathrm{GL}_{1,\mathbb{C}} \times \mathrm{GL}_{1,\mathbb{C}}$, and that we denoted by $r : \mathrm{GL}_{1,\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$ the embedding of the first factor (see Subsubsection 1.4.1). We get a morphism $h_{\mathbb{C}} \circ r : \mathrm{GL}_{1,\mathbb{C}} \to T_{\mathbb{C}}$. As T is an algebraic group over $\mathbb{Q}$, this morphism is defined over a finite extension of $\mathbb{Q}$ in $\mathbb{C}$, and we call this extension F. We get a morphism of F-algebraic groups $\mathrm{GL}_{1,F} \to T_F$, hence a morphism of $\mathbb{Q}$-algebraic groups $F^\times \to \operatorname{Res}_{F/\mathbb{Q}} T_F$, where $\operatorname{Res}_{F/\mathbb{Q}} T_F$ is the algebraic group that sends a $\mathbb{Q}$-algebra R to $T(R \otimes_{\mathbb{Q}} F)$. Composing this with the norm $N_{F/\mathbb{Q}} : \operatorname{Res}_{F/\mathbb{Q}} T_F \to T$, we finally get a morphism $r(h) : F^\times \to T$, called the **reciprocity morphism for** (T, h). We take λ_M to be induced by $r(h)$.

So if (G, h) is a Shimura datum with G a torus, we get a canonically defined model of the associated Shimura variety over the field of definition of $h_{\mathbb{C}} \circ r$.

1.5.3. *The reflex field.* Let (G, h) be a Shimura datum satisfying the hypothesis of Theorem 1.35, and let $\nu : G \to T := G/G_{\mathrm{der}}$ be the quotient morphism. We have an isomorphism, induced by ν:

$$\pi_0(M(G, h))(\mathbb{C}) \xrightarrow{\sim} \pi_0(T(\mathbb{A})/T(\mathbb{Q}))/\pi_0(K_\infty).$$

Suppose that we have a model $(M_K)_K$ of the Shimura variety of (G, h) over a number field $F \subset \mathbb{C}$. We would expect the Shimura variety of $(T, \nu \circ h)$ to also have a model over F, and the isomorphism above to be $\operatorname{Gal}(\overline{F}/F)$-equivariant, where the action on the right hand side is given by the morphism $r(\nu \circ h) : F^\times \to T$ constructed in Subsubsection 1.5.2.

Remember that $r : \mathrm{GL}_{1,\mathbb{C}} \to \mathbb{S}_{\mathbb{C}} \simeq \mathrm{GL}_{1,\mathbb{C}} \times \mathrm{GL}_{1,\mathbb{C}}$ is the embedding of the first factor (see 1.4.1 again). By the previous paragraph, we would expect F to contain the field of definition of $\nu \circ h \circ r$. In fact it would make sense to take F to be the field of definition of $h_{\mathbb{C}} \circ r : \mathrm{GL}_{1,\mathbb{C}} \to G_{\mathbb{C}}$, except that h is only significant up to conjugation. This motivates the following definition.

Definition 1.37. Let (G, h) be a Shimura datum. The **reflex field** $F(G, h)$ of (G, h) is the field of definition of the conjugacy class of $h_{\mathbb{C}} \circ r : \mathrm{GL}_{1,\mathbb{C}} \to G_{\mathbb{C}}$.

Let $F = F(G,h)$. Then F is a finite extension of $\mathbb{Q}$ in $\mathbb{C}$, and, for every morphism ρ of G into a commutative algebraic group, the morphism $\rho_{\mathbb{C}} \circ h_{\mathbb{C}} \circ r$ is defined over F. Note that $h_{\mathbb{C}} \circ r$ itself is not necessarily defined over F.

Example 1.38. Let $E = \mathbb{Q}[\sqrt{-d}]$ be an imaginary quadratic extension of $\mathbb{Q}$, let $p \geqslant q \geqslant 1$ be integers, and set $n = p + q$. Let

$$J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \in \mathrm{GL}_n(\mathbb{Z}).$$

For every commutative ring R, we denote by $x \mapsto \overline{x}$ the involution of $R \otimes_{\mathbb{Z}} \mathcal{O}_E$ induced by the nontrivial element of $\mathrm{Gal}(E/\mathbb{Q})$, and, for every $Y \in M_n(R \otimes_{\mathbb{Q}} E)$, we write $Y^* = {}^t\overline{Y}$.

The general unitary group $\mathrm{GU}(p,q)$ is the $\mathbb{Z}$-group scheme defined by

$$\mathrm{GU}(p,q)(R) = \{g \in \mathrm{GL}_n(R \otimes_{\mathbb{Z}} \mathcal{O}_E) \mid \exists c(g) \in R^\times,\ g^* J g = c(g) J\}$$

for every commutative ring R. Then $\mathrm{GU}(p,q)_{\mathbb{Q}}$ is a connected reductive algebraic group, and we have a morphism of group schemes $c : \mathrm{GU}(p,q) \to \mathrm{GL}_1$, whose kernel is the unitary group $\mathrm{U}(p,q)$.

Let $h : \mathbb{S} \to \mathrm{GU}(p,q)_{\mathbb{R}}$ be the morphism defined by

$$h(z) = \begin{pmatrix} zI_p & 0 \\ 0 & \overline{z}I_q \end{pmatrix} \in \mathrm{GU}(p,q)(\mathbb{R}).$$

Then (G,h) is a Shimura datum, and K_∞ is the set of matrices $\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$ such that $g_1 \in \mathrm{GL}_p(\mathbb{C})$, $g_2 \in \mathrm{GL}_q(\mathbb{C})$ and there exists $c \in \mathbb{R}$ with $g_1^* g_1 = cI_p$ and $g_2^* g_2 = cI_q$. We use the isomorphism $\mathbb{C} \otimes_{\mathbb{Q}} E \xrightarrow{\sim} \mathbb{C} \times \mathbb{C}$ sending $x \otimes 1 + y \otimes \sqrt{-d}$ to $(x + \sqrt{-d}y, x - \sqrt{-d}y)$ to identify $\mathrm{GU}(p,q)(\mathbb{C})$ to a subgroup of $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$; note that the involution $g \mapsto \overline{g}$ of $\mathrm{GU}(p,q)(\mathbb{C})$ corresponds to switching of the two factors. With this convention, we have for every $z \in \mathbb{C}^\times$ that

$$h_{\mathbb{C}} \circ r(z) = \left(\begin{pmatrix} zI_p & 0 \\ 0 & I_q \end{pmatrix}, \begin{pmatrix} I_p & 0 \\ 0 & zI_q \end{pmatrix} \right).$$

It is easy to check that $h_{\mathbb{C}} \circ r$ is defined over E but not over $\mathbb{Q}$. On the other hand, the reflex field of (G,h) is E if $p > q$ and $\mathbb{Q}$ if $p = q$.

Example 1.39. Let (G,h) be the Shimura datum of Example 1.33 (so that $G = \mathrm{GSp}_{2d}$). For every $z \in \mathbb{C}^\times$, we have

$$h_{\mathbb{C}} \circ r(z) = \begin{pmatrix} \frac{1}{2}(z+1)I_d & -\frac{1}{2i}(z-1)I_d \\ \frac{1}{2i}(z-1)I_d & \frac{1}{2}(z+1)I_d \end{pmatrix} = P \begin{pmatrix} zI_d & 0 \\ 0 & I_d \end{pmatrix} P^{-1},$$

where

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}}I_d & \frac{i}{\sqrt{2}}I_d \\ \frac{i}{\sqrt{2}}I_d & \frac{1}{\sqrt{2}}I_d \end{pmatrix}.$$

So the reflex field of (G,h) is $\mathbb{Q}$.

1.5.4. *Canonical models.* We are now ready to define canonical models.

Definition 1.40. Let (G,h) be a Shimura datum and let $F = F(G,h)$. A **canonical model** of $M(G,h)(\mathbb{C})$ is a model $(M_K)_K$ over F such that, for every torus $u : H \subset G$ and every $h' : \mathbb{S} \to H_{\mathbb{R}}$ such that $u \circ h'$ and h are $G(\mathbb{R})$-conjugated (i.e. such that u induces a morphism of Shimura data from (H,h') to (G,h)), the morphism

$$u : M(H,h')(\mathbb{C}) \to M(G,h)(\mathbb{C})$$

is defined over the compositum $F \cdot F(H,h') \subset \mathbb{C}$, where we use as model of $M(H,h')(\mathbb{C})$ over $F(H,h')$ the one defined in Subsubsection 1.5.2.

Example 1.41. (1) If G is a torus, then the model of Subsubsection 1.5.2 is a canonical model of $M(G,h)(\mathbb{C})$.
(2) If (G,h) is the Shimura datum of Example 1.33 (so that $G = \mathrm{GSp}_{2d}$), then the schemes $(\mathcal{M}_{d,K,\mathbb{Q}})_{K \subset \mathrm{GSp}_{2d}(\mathbb{A}_f)}$ of Subsubsection 1.3.3 form a canonical model of $M(G,h)(\mathbb{C})$. This is not obvious but follows from the main theorem of complex multiplication; see Section 4 of [Del71].

At the time of Deligne's paper [Del71], it was not known whether all Shimura varieties has canonical models (spoiler: this is now known to be true, see Theorem **??**), but it was possible to prove their uniqueness. If $u : (H,h') \to (G,h)$ is a morphism of Shimura varieties as in Definition 1.40 (so that H is a subtorus of G), the image in $M(G,h)(\mathbb{C})$ of the points of $M(H,h)(\mathbb{C})$ are called **special points**. The fact that canonical models are uniquely characterized relies on the following two points:

(i) Special points are dense in $M(G,h)(\mathbb{C})$;
(ii) For every finite extension $F' \subset \mathbb{C}$ of $F(G,h)$, there exists $u : (H,h') \to (G,h)$ as above such that $F(H,h')$ and F' are linearly disjoint over $F(G,h)$ (Théorème 5.1 of [Del71]).

From this, we can deduce:

Theorem 1.42 (Corollaire 5.4 of [Del71])**.** *Let $u : (G_1,h_1) \to (G_2,h_2)$ be a morphism of Shimura data. Then the corresponding morphism $M(G_1,h_1)(\mathbb{C}) \to M(G_2,h_2)(\mathbb{C})$ of Shimura is defined over any common extension F of $F(G_1,h_1)$ and $F(G_2,h_2)$ in $\mathbb{C}$.*

Corollary 1.43. *Let (G,h) be a Shimura datum. Then a canonical model of $M(G,h)(\mathbb{C})$ is unique up to unique isomorphism if it exists.*

Corollary 1.44. *Let (G,h) be a Shimura datum satisfying the hypothesis of Theorem 1.35, let $\nu : G \to T := G/G_{\mathrm{der}}$ be the quotient morphism, and let $F = F(G,h)$. Then the action of $\mathrm{Gal}(\overline{F}/F)$ on $\pi_0(M(G,h)(\mathbb{C})) \xrightarrow{\sim} \pi_0(T(\mathbb{Q})\backslash T(\mathbb{A}))/\pi_0(K_\infty)$ is given by the inverse of the reciprocity morphism for $(T, \nu \circ h)$.*

Apply Theorem 1.42 to $\nu : (G,h) \to (T, \nu \circ h)$. Using the same techniques as for Theorem 1.42, we also get the following very useful result.

Proposition 1.45 (Corollarie 5.7 of [Del71])**.** *Let $u : (G_1,h_1) \to (G_2,h_2)$ be a morphism of Shimura data such that the underlying morphism of algebraic groups is a closed immersion and that $F(G_1,h_1) \subset F(G_2,h_2)$. If $M(G_2,h_2)(\mathbb{C})$ has a canonical model, then so does $M(G_1,h_1)(\mathbb{C})$.*

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School of Mathematical Sciences, Peking University, 100871, Beijing, China
Email address: daiwenhan@pku.edu.cn