Exercise 2 Solutions

Problem 2.1. (Local Galois cohomology computation) Let K be a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ with residue field $\mathbb{F}_q$. Let V be a representation of G_K on an $\mathbb{F}_\ell$-vector space.

(1) Show that when $\ell \neq p$, $H^1(G_K, V) = 0$ unless $V^{G_K} \neq 0$ or $V^*(1)^{G_K} \neq 0$. When $\ell = p$ and $K = \mathbb{Q}_p$, what is $\dim H^1(G_{\mathbb{Q}_p}, V)$ "usually"?

(2) When $\ell \neq p$, compute without using Tate local duality and Euler charactersitic formula, in an explicit way, $\dim H^i(G_K, \mathbb{F}_\ell(n))$. Your answer will depend on congruences of q^n modulo ℓ. Observe that the dimenions coincide with the prediction by local Tate duality and Euler characteristic formula.

(3) When $\ell = p$ and K a finite extension of $\mathbb{Q}_p$, compute the dimension of $\dim H^i(G_K, \mathbb{F}_p(n))$. (Use Tate local duality and Euler characteristic formula.)

Solution. (1) Using Euler characteristic formula, we see that $\chi(G_K, V) = \dim H^0(G_K, V) - \dim H^1(G_K, V) + \dim H^2(G_K, V) = 0$. So $H^1(G_K, V) = 0$ unless either $H^0(G_K, V) = 0$ or $H^2(G_K, V) = 0$; the first condition is just $V^{G_K} = 0$, and the second condition (by local Tate duality) is equivalent to $H^0(G_K, V^*(1))^* = 0$ or equivalently $V^*(1)^{G_K} = 0$.

(2) (Let k denote the residue field of K.) We use Hoshchild–Serre spectral sequence,

$$E_2^{i,j} = H^i(G_k, H^j(I_K, \mathbb{F}_\ell(n)) \Rightarrow H^{i+j}(G_K, \mathbb{F}_\ell(n))$$

We know that $\mathbb{F}_\ell(n)$ is an unramified extension. As P_K is pro-p and $p \neq \ell$, we have

$$H^j(I_K, \mathbb{F}_\ell(n)) \cong H^j(I_K/P_K, \mathbb{F}_\ell(n)) \cong \begin{cases} \mathbb{F}_\ell(n) & j = 0 \\ \mathrm{Hom}(\widehat{\mathbb{Z}}(1), \mathbb{F}_\ell(n)) = \mathbb{F}_\ell(n-1) & j = 1 \\ 0 & j > 1. \end{cases}$$

From this, we deduce

$$H^0(G_K, \mathbb{F}_\ell(n)) = H^0(G_k, \mathbb{F}_\ell(n)) = \mathbb{F}_\ell(n)^{G_k} = \begin{cases} \mathbb{F}_\ell & \text{if } \ell \mid p^n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$H^2(G_K, \mathbb{F}_\ell(n)) = H^1(G_k, \mathbb{F}_\ell(n-1)) = \frac{\mathbb{F}_\ell(n-1)}{(\phi_k - 1)\mathbb{F}_\ell(n-1)} = \begin{cases} \mathbb{F}_\ell & \text{if } \ell \mid p^{n-1} - 1 \\ 0 & \text{otherwise.} \end{cases}$$

The first cohomology sits in an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(G_k, \mathbb{F}_\ell(n)) & \longrightarrow & H^1(G_K, \mathbb{F}_\ell(n)) & \longrightarrow & H^1(I_K, \mathbb{F}_\ell(n))^{G_k} & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ & & \begin{cases} \mathbb{F}_\ell & \text{if } \ell \mid p^n - 1 \\ 0 & \text{otherwise.} \end{cases} & & & & \mathbb{F}_\ell(n-1)^{G_k} = \begin{cases} \mathbb{F}_\ell & \text{if } \ell \mid p^{n-1} - 1 \\ 0 & \text{otherwise.} \end{cases} & & \end{array}$$

Combining these two, we obtain

$$\dim H^1(G_K, \mathbb{F}_\ell(n)) = \begin{cases} 2 & \text{if } p \equiv 1 \bmod \ell \\ 1 & \text{if exactly one of } p^n - 1 \text{ and } p^{n-1} - 1 \text{ is divisible by } \ell \\ 0 & \text{otherwise.} \end{cases}$$

We can check that the numerics agree with the prediction by local Tate duality and the Euler characteristic formula.

(3) $H^0(G_K, \mathbb{F}_p(n)) = 0$ unless $\mathbb{F}_p(n)$ is the trivial representation, which could either because $(p-1)|n$ or because K contains $\mathbb{Q}_p(\mu_p)$. In general, if we denote $d := [K \cap \mathbb{Q}_p(\mu_p) : \mathbb{Q}_p]$ (which is a divisor of $p-1$), then $\mathbb{F}_p(n)$ is the trivial representation if and only if $p-1$ divides dn.

By Tate local duality, $H^2(G_K, \mathbb{F}_p(n)) \cong H^0(G_K, \mathbb{F}_p(1-n))^*$ which is zero unless $p-1$ divides $d(n-1)$.

Finally, by Euler characteristic formula, we deduce that

$$\dim H^i(G_K, \mathbb{F}_p(n)) = \begin{cases} 2 & \text{if } p-1|d \text{ or equivalently } \mathbb{Q}_p(\mu_p) \subseteq K \\ 1 & \text{if exactly one of } dn \text{ and } d(n-1) \text{ is divisible by } p \\ 0 & \text{otherwise} \end{cases}$$

$\square$

Problem 2.2. (Dimension of local Galois cohomology groups) Let K be a finite extension of $\mathbb{Q}_p$, and let V be a representation of G_K over a finite dimensional $\mathbb{F}_\ell$-vector space. Suppose that V is irreducible as a representation of G_K and $\dim V \geq 2$.

(1) When $\ell \neq p$, show that $H^1(G_K, V) = 0$. (Hint: compute H^2 using local Tate duality and then use Euler characteristic.)
(2) When $\ell = p$, what is $\dim H^1(G_K, V)$?

Solution. (1) When $\dim V \geq 2$ and V is irreducible, V^{G_K} must be trivial; so $H^0(G_K, V) = 0$. Similarly, $V^*(1)$ is also irreducible; so we deduce similarly that $H^2(G_K, V) \cong H^0(G_K, V^*(1))^* = 0$. Finally, by Euler characteristic formula, we deduce that $H^1(G_K, V) = 0$.

(2) By the same argument as in (1), we deduce that $H^0(G_K, V) = H^2(G_K, V) = 0$. So Euler characteristic formula says that

$$H^1(G_K, V) = -\chi(G_K, V) = [K : \mathbb{Q}_p] \cdot \dim V.$$

$\square$

Problem 2.3. (An example of Poitou–Tate long exact sequence) Consider $F = \mathbb{Q}$, and let $S = \{p, \infty\}$ for an odd prime p. Determine each term in the Poitou–Tate exact sequence for the trivial representation $M = \mathbb{F}_p$. (Hint: usually $H^2(G_{F,S}, \mathbb{F}_p)$ is difficult to determine; but one can use Euler characteristic to help.)

Solution. As p is odd, the Tate cohomology $H^i_{\mathrm{Tate}}(G_{\mathbb{R}}, \mathbb{F}_p) = 0$ and the same for $\mathbb{F}_p(1)$. Write out the Poitou–Tate exact sequence for $\mathbb{F}_p$ as follows:

$$\begin{array}{l} 0 \longrightarrow H^0(G_{\mathbb{Q},S}, \mathbb{F}_p) \longrightarrow H^0(G_{\mathbb{Q}_p}, \mathbb{F}_p) \longrightarrow H^2(G_{\mathbb{Q},S}, \mathbb{F}_p(1))^* \\ \hookrightarrow H^1(G_{\mathbb{Q},S}, \mathbb{F}_p) \xrightarrow{\alpha} H^1(G_{\mathbb{Q}_p}, \mathbb{F}_p) \longrightarrow H^1(G_{\mathbb{Q},S}, \mathbb{F}_p(1))^* \\ \hookrightarrow H^2(G_{\mathbb{Q},S}, \mathbb{F}_p) \longrightarrow H^2(G_{\mathbb{Q}_p}, \mathbb{F}_p) \longrightarrow H^0(G_{\mathbb{Q},S}, \mathbb{F}_p(1))^* \longrightarrow 0 \end{array}$$

It is easy to see that $H^0(G_{\mathbb{Q},S}, \mathbb{F}_p) = \mathbb{F}_p$, $H^0(G_{\mathbb{Q}_p}, \mathbb{F}_p) = \mathbb{F}_p$, and

$$H^1(G_{\mathbb{Q},S}, \mathbb{F}_p) \cong \mathrm{Hom}(G_{\mathbb{Q},S}, \mathbb{F}_p) \cong \mathrm{Hom}\big(\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}_p), \mathbb{F}_p\big) \cong \mathbb{F}_p;$$

$$H^1(G_{\mathbb{Q}_p}, \mathbb{F}_p) \cong \mathrm{Hom}(G_{\mathbb{Q}_p}, \mathbb{F}_p) \cong \mathrm{Hom}\big(\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^\infty})/\mathbb{Q}_p), \mathbb{F}_p\big) \cong \mathbb{F}_p \oplus \mathbb{F}_p.$$

By Tate local duality, $H^2(G_{\mathbb{Q}_p}, \mathbb{F}_p) \cong H^0(G_{\mathbb{Q}_p}, \mathbb{F}_p(1))^* = 0$, and by global Euler characteristic formula,

$$\chi(G_{\mathbb{Q},S}, \mathbb{F}_p) = \chi(G_{\mathbb{Q}_p}, \mathbb{F}_p) + \chi(G_{\mathbb{R}}, \mathbb{F}_p) = -1 + 1 = 0.$$

So we deduce that $H^2(G_{\mathbb{Q},S}, \mathbb{F}_p) = 0$.

Now the Poitou–Tate exact sequence has become

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{F}_p \arrow[r] & \mathbb{F}_p \arrow[r] & H^2(G_{\mathbb{Q},S}, \mathbb{F}_p(1))^* \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] & \\
& \mathbb{F}_p \arrow[r, "\alpha"] & \mathbb{F}_p \oplus \mathbb{F}_p \arrow[r] & H^1(G_{\mathbb{Q},S}, \mathbb{F}_p(1))^* \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] & \\
& 0 \arrow[r] & 0 \arrow[r] & H^0(G_{\mathbb{Q},S}, \mathbb{F}_p(1))^* \arrow[r] & 0
\end{tikzcd}$$

It is easy to see directly that $H^0(G_{\mathbb{Q},S}, \mathbb{F}_p(1)) = 0$. For the other two terms, we claim that the restriction map α is injective. This is because the non-trivial element in $H^1(G_{\mathbb{Q},S}, \mathbb{F}_p) \cong \operatorname{Hom}(G_{\mathbb{Q},S}, \mathbb{F}_p) \cong \mathbb{F}_p$ corresponds to the subextension $\mathbb{Q}(\zeta_{p^2})/\mathbb{Q}$ of degree p, which is totally ramified at p; so it induces a nontrivial element in $\operatorname{Hom}(G_{\mathbb{Q}_p}, \mathbb{F}_p)$. Thus α is injective.

From this, we deduce that $H^2(G_{\mathbb{Q},S}, \mathbb{F}_p(1))^* = 0$ and $H^1(G_{\mathbb{Q},S}, \mathbb{F}_p(1))^* = \mathbb{F}_p$. $\square$

Problem 2.4. (Cohomology of $\mathcal{O}_{F^S}[\frac{1}{S}]^\times$) Let F be a number field and S a finite set of places of F including all archimedean places and places above ℓ.

(1) Show that there is a natural exact sequence

$$1 \to \Big(\mathcal{O}_F[\tfrac{1}{S}]^\times \backslash \prod_{v \in S} F_v^\times\Big) \times \prod_{v \notin S} \mathcal{O}_{F_v}^\times \to F^\times \backslash \mathbb{A}_F^\times \to \mathrm{Cl}(\mathcal{O}_F[\tfrac{1}{S}]) \to 1, \tag{2.4.1}$$

where $\mathrm{Cl}(\mathcal{O}_F[\frac{1}{S}])$ is the ideal class group of $\mathcal{O}_F[\frac{1}{S}]$, namely the quotient of the ideal class group $\mathrm{Cl}(\mathcal{O}_F)$ by the subgroup generated by ideals in S.

(2) By studying the exact sequence

$$1 \to \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \to \prod_{v \in S}(F_v \otimes F^S)^\times \to \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \backslash \prod_{v \in S}(F_v \otimes F^S)^\times \to 1, \tag{2.4.2}$$

show that

$$H^1\Big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times\Big) \cong \mathrm{Cl}(\mathcal{O}_F[\tfrac{1}{S}]),$$

and there is an exact sequence

$$0 \to H^2\Big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times\Big) \otimes \mathbb{Z}_\ell \to \bigoplus_{v \in S} \begin{cases} \mathbb{Q}_\ell/\mathbb{Z}_\ell & v \text{ non-arch} \\ \frac{1}{2}\mathbb{Z}/\mathbb{Z} & v = \mathbb{R} \text{ and } \ell = 2 \\ 0 & \text{otherwise} \end{cases} \to \mathbb{Q}_\ell/\mathbb{Z}_\ell \to 0.$$

For $i \geq 3$, we have

$$H^i\Big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times\Big) \otimes \mathbb{Z}_\ell \cong \bigoplus_{v \text{ real}} H^i(\mathbb{R}, \mathbb{C}^\times) \cong \bigoplus_{v \text{ real}} \begin{cases} \frac{1}{2}\mathbb{Z}/\mathbb{Z} & \ell = 2 \text{ and } i \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

Remark: Using Kummer theory $1 \to \mu_\ell \to \mathcal{O}_{F^S}[\frac{1}{S}]^\times \to \mathcal{O}_{F^S}[\frac{1}{S}]^\times \to 1$, we can then use this to further compute $H^1(G_{F,S}, \mu_\ell)$.

Solution. (1) By definition, $\mathrm{Cl}(\mathcal{O}_F) \cong F^\times \backslash \mathbb{A}_{F,f}^\times / \prod_{v \text{ fin}} \mathcal{O}_{F_v}^\times$. So we have

$$\begin{aligned} \mathrm{Cl}(\mathcal{O}_F[\tfrac{1}{S}]) &\cong F^\times \backslash \mathbb{A}_{F,f}^\times \Big/ \Big(\prod_{v \notin S} \mathcal{O}_{F_v}^\times \prod_{v \in S \text{ fin}} F_v^\times \Big) \\ &\cong F^\times \backslash \mathbb{A}_{F}^\times \Big/ \Big(\prod_{v \notin S} \mathcal{O}_{F_v}^\times \prod_{v \in S} F_v^\times \Big) \end{aligned}$$

This implies an exact sequence

$$1 \to \mathrm{Ker} \to \prod_{v \notin S} \mathcal{O}_{F_v}^\times \times \prod_{v \in S} F_v^\times \to F^\times \backslash \mathbb{A}_{F,f}^\times \to \mathrm{Cl}(\mathcal{O}_F[\tfrac{1}{S}]) \to 0.$$

Here $\mathrm{Ker} = F^\times \cap \Big(\prod_{v \notin S} \mathcal{O}_{F_v}^\times \times \prod_{v \in S} F_v^\times \Big) \cong \mathcal{O}_F[\frac{1}{S}]^\times$. The exact sequence (2.4.1) follows.

(2) Taking Galois cohomology of (2.4.2), we deduce

$$\begin{aligned} &1 \longrightarrow \mathcal{O}_F[\tfrac{1}{S}]^\times \longrightarrow \textstyle\prod_{v \in S} F_v^\times \longrightarrow H^0\big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \backslash \prod_{v \in S} (F_v \otimes F^S)^\times\big) \\ &\hookrightarrow H^1\big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times\big) \longrightarrow {\color{red}0} \longrightarrow H^1\big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \backslash \textstyle\prod_{v \in S} (F_v \otimes F^S)^\times\big) \\ &\hookrightarrow H^2\big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times\big) \longrightarrow \textstyle\bigoplus_{v \in S} H^2(G_{F_v}, F_v^{S,\times}) \longrightarrow H^2\big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \backslash \prod_{v \in S} (F_v \otimes F^S)^\times\big) \longrightarrow 0 \end{aligned}$$

The red zero follows from Hilbert 90, and F_v^S denote the completion of F^S along any place of F^S above v.

Plugging in the input of cohomology of $H^i\big(G_{F,S}, \mathcal{O}_{F^S}[\frac{1}{S}]^\times \backslash \prod_{v \in S} (F_v \otimes F^S)^\times\big)$ from class field theory, we deduce that

$$\begin{aligned} &1 \longrightarrow \mathcal{O}_F[\tfrac{1}{S}]^\times \longrightarrow \textstyle\prod_{v \in S} F_v^\times \longrightarrow F^\times \backslash \mathbb{A}_F^\times / \prod_{v \notin S} \mathcal{O}_{F_v}^\times \\ &\hookrightarrow H^1\big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times\big) \longrightarrow {\color{red}0} \longrightarrow 0 \\ &\hookrightarrow H^2\big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times\big) \otimes \mathbb{Z}_\ell \longrightarrow \textstyle\bigoplus_{v \in S} H^2(G_{F_v}, F_v^{S,\times}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow \cdots \end{aligned}$$

Here, for the last row, we tensored with $\mathbb{Z}_\ell$ to take only the ℓ-part.

Comparing this exact sequence with the one in (1), we deduce that $H^1(G_{F,S}, \mathcal{O}_{F^S}[\frac{1}{S}]^\times) \cong \mathrm{Cl}(\mathcal{O}_F[\frac{1}{S}])$. The description of H^2 and H^3 follow from this as well.

$\square$

Problem 2.5. (A step in the proof of local Euler characteristic formula) Consider the following situation. Let K be a finite extension of $\mathbb{Q}_p$ such that $K = K(\mu_p)$. Let L/K be a finite cyclic extension with Galois group H of order relatively prime to p. Let N be a finite $\mathbb{F}_p[H]$-module. Our goal is to compute

$$\dim \Big(\big(\mathcal{O}_L^\times / (\mathcal{O}_L^\times)^p \big) \otimes N(-1) \Big)^H$$

(1) Consider the logarithmic map

$$\begin{aligned} \log_p : \mathcal{O}_L^\times/\mu(L) &\longrightarrow L \\ a &\longmapsto \tfrac{1}{n}\log_p(a^n) \end{aligned}$$

where n is taken sufficiently divisible so that $a^{p^n} \in 1 + p^2\mathcal{O}_L$ so that $\log_p(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ makes sense. Show that $\log_p$ is well-defined homomorphism (and independent of the choice of n), and that it induces an isomorphism

$$\log_p : \big(\mathcal{O}_L^\times/\mu(L)\big) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong L.$$

(2) Show that for any two $\mathcal{O}_L$-lattices $\Lambda_1, \Lambda_2 \in L$, we have

$$\dim(\Lambda_1/p\Lambda_1 \otimes N)^H = \dim(\Lambda_2/p\Lambda_2 \otimes N)^H.$$

(3) Recall that $L \cong K[H]$ as H-modules by Hilbert 90. From this and (2), deduce that

$$\dim\Big(\big(\mathcal{O}_L^\times/(\mathcal{O}_L^\times)^p\big) \otimes N(-1)\Big)^H = \dim N^H + \dim N \cdot [K : \mathbb{Q}_p].$$

Solution. (1) Take n to be divisible by $\#k_L^\times$, so that for every $a \in \mathcal{O}_L^\times$, $a^n \equiv 1 \bmod \varpi_L$. Then $\log_p(a^n) := (a^n-1) - \frac{1}{2}(a^n-1)^2 + \frac{1}{3}(a^n-1)^3 - \cdots$ converges to an element in L, and the sequence convergences uniformly. By algebraic relations, it is easy to see that $\log_p(a^n) + \log_p(b^n) = \log_p((ab)^n)$ for $a, b \in \mathcal{O}_L^\times$. From this, it is immediate to see that defining $\log_p(a)$ as $\frac{1}{n}\log_p((a^n))$ is a well-defined homomorphism.

(2) The proof is similar to that of Exercise 1.1. By multiplication by a power of p, one may assume that $\Lambda_1 \subseteq \Lambda_2 \subseteq p^{-n}\Lambda_1$ for some $n \in \mathbb{N}$. By further consider a sequence of modules $\Lambda_1 + p^{n-1}\Lambda_2$, $\Lambda_1 + p^{n-2}\Lambda_2$, $\dots$, $\Lambda_1 + \Lambda_2 = \Lambda_2$, in turn, we may assume (at each step), $\Lambda_1 \subseteq \Lambda_2 \subseteq p^{-1}\Lambda_1$. Now consider the tautological exact sequences

$$0 \to p\Lambda_2/p\Lambda_1 \to \Lambda_1/p\Lambda_1 \to \Lambda_1/p\Lambda_2 \to 0 \tag{2.5.1}$$

$$0 \to \Lambda_1/p\Lambda_2 \to \Lambda_2/p\Lambda_2 \to \Lambda_2/\Lambda_1 \to 0 \tag{2.5.2}$$

As the order of H is relative prime to p, taking H invariants is exact. So we have

$$\begin{aligned} \dim(\Lambda_1/p\Lambda_1 \otimes N)^H &\overset{(2.5.1)}{=} \dim(p\Lambda_2/p\Lambda_1 \otimes N)^H + \dim(\Lambda_1/p\Lambda_2 \otimes N)^H \\ &= \dim(\Lambda_2/p\Lambda_1 \otimes N)^H + \dim(\Lambda_1/p\Lambda_2 \otimes N)^H \\ &\overset{(2.5.2)}{=} \dim(\Lambda_2/p\Lambda_2 \otimes N)^H. \end{aligned}$$

(3) From the exact sequence

$$1 \to \mu(L) \to \mathcal{O}_L^\times \to \mathcal{O}_L^\times/\mu(L) \to 1$$

we deduce that

$$1 \to \mu(L)/\mu(L)^p \to \mathcal{O}_L^\times/(\mathcal{O}_L^\times)^p \to (\mathcal{O}_L^\times/\mu(L))/(\mathcal{O}_L^p/\mu(L))^p \to 1$$

Thus,

$$\begin{aligned} &\dim\Big(\big(\mathcal{O}_L^\times/(\mathcal{O}_L^\times)^p\big) \otimes N(-1)\Big)^H \\ = \; &\dim\Big(\big(\mathcal{O}_L^\times/(\mu(L))/(\mathcal{O}_L^\times/(\mu(L)))^p\big) \otimes N(-1)\Big)^H + \dim\Big(\mu(L)/\mu(L)^p \otimes N(-1)\Big)^H. \end{aligned}$$

It is not hard to see that $\mu(L)/\mu(L)^p \cong \mu_p$ by sending ζ to $\zeta^{\#\mu(L)/p}$. So the last term is equal to $\big(\mu_p \otimes N(-1)\big)^H = \dim N^H$.

To compute $\dim\Big(\big(\mathcal{O}_L^\times/(\mu(L))/(\mathcal{O}_L^\times/(\mu(L)))^p\big) \otimes N(-1)\Big)^H$, we may use (1) and (2) to choose a different lattice in $L \simeq K[H]$ for the convenience of the computation. From this, we deduce

$$\dim\Big(\big(\mathcal{O}_L^\times/(\mu(L))/(\mathcal{O}_L^\times/(\mu(L)))^p\big)\otimes N(-1)\Big)^H = \dim\Big(\mathcal{O}_K/p[H]\otimes N(-1)\Big)^H = [K:\mathbb{Q}_p]\cdot \dim N.$$

Here the in the last step, we used the fact that $\mathbb{F}_p[H]\otimes N(-1) \cong \mathbb{F}_p[H]^{\dim N}$ as $\mathbb{F}_p[H]$-module. Combining this with the above computation, we deduce (3). □

Problem 2.6. (Comparing first Galois cohomology classes and extensions of Galois representations) If you are unfamiliar with the background of this problem, one can consult the short note on this topic, available on the webpage.

Let k be a field with discrete topology, and let G be a finite group acting k-linearly on a finite dimensional k-vector space M. Let $\rho : G \to \mathrm{GL}_k(M)$ be the representation.

(1) Given a cohomology class $[c] \in H^1(G, M)$, represented by cocycle $g \mapsto c_g \in M$, show that the following map defines a representation of G on $E_c := M \oplus k$:

$$g \mapsto \begin{pmatrix} \rho(g) & c_g \\ 0 & 1 \end{pmatrix}.$$

(2) Show that if $(c_g)_{g\in G}$ and $(c'_g)_{g\in G}$ define the same cohomology class, the representations E_c and $E_{c'}$ defined in (1) are isomorphic.

(3) By definition, there exists an exact sequence $0 \to M \to E_c \to k \to 0$. Taking the G-cohomology gives a connecting homomorphism

$$k^G = k \xrightarrow{\ \delta\ } H^1(G, M)$$

Show that $\delta(1) = [c]$.

(4) (Optional) Given an exact sequence of $k[G]$-modules

$$0 \to M \to E_1 \to E_2 \to k \to 0,$$

we may write F as the image of $E_1 \to E_2$ and thus get two short exact sequences

$$0 \to M \to E_1 \to F \to 0, \quad \text{and} \quad 0 \to F \to E_2 \to k \to 0$$

This way, the boundary maps of the group cohomology defines two maps

$$\delta : k = H^0(G, k) \longrightarrow H^1(G, F) \longrightarrow H^2(G, M)$$

and thus the image $\delta(1)$ defines a cohomology class $[c]$ in $H^2(G, M)$.

Now, suppose that we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & k \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & E_1' \arrow[r] & E_2' \arrow[r] & k \arrow[r] & 0. \end{tikzcd}$$

Show that the second cohomology class defined by these two exact sequences are the same.

Solution. (1) It is enough to check that

$$\begin{pmatrix}\rho(g) & c_g\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}\rho(h) & c_h\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\rho(gh) & c_{gh}\\ 0 & 1\end{pmatrix}$$

The only non-trivial equality spells $c_g+\rho(g)c_h=c_{gh}$. This is precisely the cocycle condition.

(2) Let $m\in M$ be taken so that $c_g=c'_g+gm-m$ for every $g\in G$. Then we can check immediately that

$$\begin{pmatrix}\rho(g) & c_g\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & -m\\ 0 & 1\end{pmatrix}\begin{pmatrix}\rho(g) & c'_g\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}$$

This means that the representation on $M\oplus k$ given by c_g and c'_g are conjugate and hence isomorphic.

(3) We following the definition of the connecting homomorphism: given $1\in k$, we lift it to column vector $\binom{0}{1}\in E_c$. Then apply the differential map in group cohomology, we deduce a cycle

$$c'_g:=g\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}c_g\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}=c_g\in M.$$

This verifies that $\delta(1)=[c]$.

(4) The commutative diagram of long exact sequences splits into two commuting short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & M \arrow[r] & E'_1 \arrow[r] & F' \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & F \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & k \arrow[r] \arrow[d, equal] & 0\\
0 \arrow[r] & F' \arrow[r] & E'_2 \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

Then we have a commutative diagram

$$\begin{tikzcd}
k=H^0(G,k) \arrow[r] \arrow[d, equal] & H^1(G,F) \arrow[r] \arrow[d] & H^2(G,k) \arrow[d, equal]\\
k=H^0(G,k) \arrow[r] & H^1(G,F') \arrow[r] & H^2(G,k)
\end{tikzcd}$$

So the image of 1 along the top row is the same as the image of 1 along the bottom row, i.e. the cohomology class defined by "equivalent" extensions are the same. □

Problem 2.7. (An explicit computation of local Galois cohomology when $\ell\neq p$) Let K be a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, with ring of integers $\mathcal{O}_K$ and residue field k_K. Let ℓ be a prime different from p. Let M be a finite G_K-module that is ℓ^∞-torsion. Following the instruction below to give another proof of the Euler characteristic for local Galois cohomology when $\ell\neq p$:

$$\chi(G_K,M):=\sum_{i=0}^{2}(-1)^i\cdot\mathrm{length}_{\mathbb{Z}_\ell}H^i(G_K,M)=0. \tag{2.7.1}$$

(1) Let I_K and P_K denote the inertia subgroup and the wild inertia subgroup of G_K. Show that $H^{>0}(P_K,M)=0$ for any G_K-module M that is ℓ^∞-torsion. Using the Hoshchild–Serre spectral sequence to deduce that, for every $i\geq 0$,

$$H^i(I_K,M)\cong H^i(I_K/P_K,M^{P_K}).$$

(2) Let $P_{K,\ell}$ denote the kernel of $I_K \to I_K/P_K \xrightarrow{t_{\xi,\ell}} \mathbb{Z}_\ell(1)$. Show that we have $H^i(I_K, M) \cong H^i(\mathbb{Z}_\ell(1), M^{P_{K,\ell}})$.

(3) Put $N := M^{P_{K,\ell}}$, and write τ for a generator of I_K/P_K, then we have

$$H^0(I_K, M) \cong N^{\tau=1}, \quad H^1(I_K, M) \cong N/(\tau-1)N.$$

Note also that the second isomorphism is given by evaluating the cochain at τ; so the Frobenius action on $N/(\tau-1)N$ is twisted by the inverse of cyclotomic character. Thus, we should have wrote $N(-1)/(\tau-1)N(-1)$ instead.

(4) Let ϕ_K denote a Frobenius element. Show that we have isomorphisms:

$$H^0(G_K, M) \cong (N^{\tau=1})^{\phi_K=1}, \quad H^2(G_K, M) \cong \frac{N(-1)}{(\tau-1)N(-1)}\Big/(\phi_K - 1).$$

For $H^1(G_K, M)$, we may describe the unramified part and singular part of it as follows:

$$\begin{tikzcd} 0 \arrow[r] & H^1(G_{k_K}, M^{I_K}) \arrow[r] \arrow[d, "\cong"] & H^1(G_K, M) \arrow[r] & H^1(I_K, M)^{G_{k_K}} \arrow[r] \arrow[d, "\cong"] & 0. \\ & \left(\frac{N(-1)}{(\tau-1)N(-1)}\right)^{\phi_K=1} & & N^{\tau=1}/(\phi_K-1)N^{\tau=1} & \end{tikzcd}$$

(5) From this, deduce Euler characteristic formula (2.7.1) directly.

(6) Using the discussion above to prove the following isomorphism of exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & H^1(G_{k_K}, M^{I_K}) \arrow[r] \arrow[d, "\cong"] & H^1(G_K, M) \arrow[r] \arrow[d, "\cong"] & H^1(I_K, M)^{G_{k_K}} \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \left(H^1(I_K, M^*(1))^{G_{k_K}}\right)^* \arrow[r] & H^1(G_K, M)^* \arrow[r] & \left(H^1(G_{k_K}, (M^*(1))^{I_K})\right)^* \arrow[r] & 0. \end{tikzcd}$$

(Hint: first show that the subgroup $H^1(G_{k_K}, M^{I_K})$ and $H^1(G_{k_K}, (M^*(1))^{I_K})$ annihilate each other. This is because such pairing factors through the cup product.

$$H^1(G_{k_K}, M^{I_K}) \times H^1(G_{k_K}, (M^*(1))^{I_K}) \to H^2(G_{k_K}, \mu_{\ell^\infty}) = 0.$$

After this, it is enough to show that $\#H^1(G_{k_K}, M^{I_K}) = \#H^1(I_K, M^*(1))^{G_{k_K}}$, which makes use of the discussion above.)

Solution. (1) Since P_K is pro-p, by definition $H^j(P_K, M) = \varinjlim_{H \lhd P_K} H^j(P_K/H, M^H) = 0$ when $j > 0$ as M is ℓ^∞-torsion. Thus, the Hoshchild–Serre spectral sequence degenerates:

$$H^i(I_K/P_K, H^j(P_K, M)) = \begin{cases} 0 & \text{if } j > 0 \\ H^i(I_K/P_K, M^{P_K}) & \text{if } j = 0. \end{cases}$$

So we deduce that for every $i \geq 0$,

$$H^i(I_K, M) \cong H^i(I_K/P_K, M^{P_K}).$$

(2) By exactly the same argument as in (1) with P_K replaced by $P_{K,\ell}$, we deduce that for every $i \geq 0$,

$$H^i(I_K, M) \cong H^i(I_K/P_{K,\ell}, M^{P_{K,\ell}}) \cong H^i(\mathbb{Z}_\ell(1), M^{P_{K,\ell}}).$$

(3) By cohomology of procyclic group, we deduce that

$$H^0(I_K, M) \cong H^0(\mathbb{Z}_\ell(1), N) = N^{\tau=1},$$

$$H^1(I_K, M) \cong H^1(\mathbb{Z}_\ell(1), N) = N(-1)/(\tau-1)N(-1).$$

(4) This follows immediately from the description of $H^i(I_K, M)$ above and Hoshchild spectral sequence.

(5) We note the following two exact sequences

$$0 \to (N^{\tau=1})^{\phi_K=1} \to N^{\tau=1} \xrightarrow{\phi_K-1} N^{\tau=1} \to N^{\tau=1}/(\phi_K-1)N^{\tau=1} \to 0,$$

$$0 \to \Big(\frac{N(-1)}{(\tau-1)N(-1)}\Big)^{\phi_K=1} \to \frac{N(-1)}{(\tau-1)N(-1)} \xrightarrow{\phi_K-1} \frac{N(-1)}{(\tau-1)N(-1)} \to \frac{N(-1)}{(\tau-1)N(-1)}\Big/(\phi_K-1) \to 0.$$

From these, we deduce immediately that

$$\mathrm{length}_{\mathbb{Z}_\ell} H^0(G_K, M) = \mathrm{length}_{\mathbb{Z}_\ell} (N^{\tau=1})^{\phi_K=1} = \mathrm{length}_{\mathbb{Z}_\ell} \big(N^{\tau=1}/(\phi_K-1)N^{\tau=1}\big) = \mathrm{length}_{\mathbb{Z}_\ell} H^1(I_K, M)^{G_{k_K}}.$$

$$\begin{aligned}\mathrm{length}_{\mathbb{Z}_\ell} H^2(G_K, M) &= \mathrm{length}_{\mathbb{Z}_\ell} \frac{N(-1)}{(\tau-1)N(-1)}\Big/(\phi_K-1)\\ &= \mathrm{length}_{\mathbb{Z}_\ell} \Big(\frac{N(-1)}{(\tau-1)N(-1)}\Big)^{\phi_K=1} = \mathrm{length}_{\mathbb{Z}_\ell} H^1(G_{k_K}, M^{I_K}).\end{aligned}$$

Euler characteristic follows from this immediately.

(6) First note that we have the following commutative diagram

$$\begin{tikzcd} H^1(G_{k_K}, M^{I_K}) \arrow[d] & \times & H^1(G_{k_K}, (M^*(1))^{I_K}) \arrow[r] \arrow[d] & H^2(G_{k_K}, \mu_{\ell^\infty}) = 0 \arrow[d] \\ H^1(G_K, M) & \times & H^1(G_K, M^*(1)) \arrow[r] & H^2(G_K, \mu_{\ell^\infty}) \end{tikzcd}$$

So under the local Tate duality, $H^1(G_{k_K}, M^{I_K})$ annihilates $H^1(G_{k_K}, (M^*(1))^{I_K})$. As the local Tate duality is a perfect pairing, it is enough to verify that

$$\#H^1(G_{k_K}, M^{I_K}) = \#H^1(I_K, M^*(1))^{G_{k_K}}.$$

Using the above description of the terms, this is equivalent to prove that

$$\#\Big(\frac{N(-1)}{(\tau-1)N(-1)}\Big)^{\phi_K=1} = \#(N^*(1))^{\tau=1}/(\phi_K-1)(N^*(1))^{\tau=1}$$

This follows from the fact that the following two exact sequences are dual to each other

$$0 \to \Big(\frac{N(-1)}{(\tau-1)N(-1)}\Big)^{\phi_K=1} \to \frac{N(-1)}{(\tau-1)N(-1)} \xrightarrow{\phi_K-1} \frac{N(-1)}{(\tau-1)N(-1)}$$

$$(N^*(1))^{\tau=1} \xrightarrow{\phi_K-1} (N^*(1))^{\tau=1} \to (N^*(1))^{\tau=1}/(\phi_K-1)(N^*(1))^{\tau=1} \to 0$$

□

Problem 2.8. Fix a prime number ℓ. Let F be a number field and S a finite set of places that includes all archimedean places and places above ℓ. For an extension L of F, we write S_L for the set of places of L that lies over places in S. Let F^S denote the maximal Galois extension of F that is unramified outside S. We compare the cohomology groups (for $i = 1, 2$)

$$H^i\Big(G_{F,S}, \Big(F^{S,\times}\backslash\mathbb{A}^\times_{F^S}\Big)\Big) \otimes \mathbb{Z}_\ell \quad \text{with} \quad H^i\Big(G_{F,S}, \Big(\mathcal{O}_{F^S}[\tfrac{1}{S}]^\times\backslash \textstyle\prod_{v\in S}(F_v \otimes F^S)^\times\Big)\Big) \otimes \mathbb{Z}_\ell.$$

(1) For a finite extension $L \subset F^S$, show that we have an exact sequence

$$1 \to \Big(\mathcal{O}_L[\tfrac{1}{S}]^\times \backslash \prod_{w \in S_L} L_w^\times\Big) \times \prod_{w \notin S_L} \mathcal{O}_{L_w}^\times \to L^\times \backslash \mathbb{A}_L^\times \to \mathrm{Cl}(\mathcal{O}_L[\tfrac{1}{S}]) \to 1, \tag{2.8.1}$$

where $\mathrm{Cl}(\mathcal{O}_L[\frac{1}{S}])$ is the ideal class group of $\mathcal{O}_L[\frac{1}{S}]$. (This is Problem 2.4(1) earlier.)

(2) Show that the limit $\varprojlim_{L \subset F^S} \mathrm{Cl}(\mathcal{O}_L[\frac{1}{S}])$ is trivial. (Hint: A property of Hibert class field theory is that, if L^{Hilb}/L is the Hilbert class field of L, namely the maximal unramified abelian extension of L, every ideal of L becomes principal in L^{Hilb}. Using the commutative diagram for compatibility of Artin maps with ideal class groups

$$\begin{tikzcd} L'^\times \backslash \mathbb{A}_{L'}^\times \arrow[r, "\mathrm{Art}_{L'}"] & G_{L'}^{\mathrm{ab}} \\ L^\times \backslash \mathbb{A}_L^\times \arrow[r, "\mathrm{Art}_L"] \arrow[u, "\text{natural}"] & G_L^{\mathrm{ab}}, \arrow[u, "\mathrm{Ver}"'] \end{tikzcd}$$

to show that this boils down to the following group theoretic statement: Let G be a pro-finite group and H its commutator group, then the transfer map $G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ is the zero map. This is known as the Artin's principal ideal theorem. The class field theory by Artin–Tate has a proof of this, on their page)

(3) Deduce that

$$H^0\Big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \backslash \prod_{v \in S} (F_v \otimes F^S)^\times\Big) \cong F^\times \backslash \mathbb{A}_F^\times / \prod_{v \notin S} \mathcal{O}_{F_v}^\times,$$

$$H^1\Big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \backslash \prod_{v \in S} (F_v \otimes F^S)^\times\Big) = 0 \quad \text{and}$$

$$H^2\Big(G_{F,S}, \Big(\mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \backslash \prod_{v \in S} (F_v \otimes F^S)^\times\Big)\Big) \otimes \mathbb{Z}_\ell \cong \mathbb{Q}_\ell/\mathbb{Z}_\ell.$$

Solution. (1) This is problem 2.4 earlier.

(2) As L^{Hilb} is a subfield of F^S (as it is everywhere unramified), by the cited Artin's principal ideal theorem, the natural map $\mathrm{Cl}(\mathcal{O}_L[\frac{1}{S}]) \to \mathrm{Cl}(\mathcal{O}_{L^{\mathrm{Hilb}}}[\frac{1}{S}])$ is the zero map. So the limit is zero.

(3) Taking limit of (2.8.1), we deduce that

$$\mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \backslash \prod_{v \in S} (F_v \otimes F^S)^\times \times \prod_{v \notin S} \mathcal{O}_{F^S \otimes F_v}^\times \;\cong\; (F^S)^\times \backslash (\mathbb{A}_F \otimes F^S)^\times.$$

Note that $H^{>0}(G_{F,S}, \mathcal{O}_{F^S \otimes F_v}^\times)$, so we immediately deduce that for $i \geq 1$,

$$H^i\big(G_{F,S}, \big(\mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \backslash \prod_{v \in S} (F_v \otimes F^S)^\times\big)\big) \otimes \mathbb{Z}_\ell \cong H^i\big(G_{F,S}, (F^S)^\times \backslash (\mathbb{A}_F \otimes F^S)^\times\big) \otimes \mathbb{Z}_\ell \cong \begin{cases} 0 & \text{if } i = 1 \\ \mathbb{Q}_\ell/\mathbb{Z}_\ell & \text{if } i = 2. \end{cases}$$

For H^0, we have instead an isomorphism

$$H^0\big(G_{F,S}, \big(\mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \backslash \prod_{v \in S} (F_v \otimes F^S)^\times\big)\big) \times \prod_{v \notin S} F_v^\times \xrightarrow{\cong} F^\times \backslash \mathbb{A}_F^\times.$$

We deduce the description of H^0 immediately from this. $\square$