A stacky p-adic Riemann-Hilbert Correspondence on Hitchin-small locus

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(Joint with Liu, Ma, Nie, Qu)

Introduction

1.1 Classical non-ab Hodge theory.

$X/\mathbb{C}$ sm proj var

Thm

$$\left\{ \begin{matrix} \mathbb{C}\text{-loc sys} \\ \text{on } X \end{matrix} \right\} \overset{RH}{\simeq} \left\{ \begin{matrix} \text{integrable conn} \\ \text{on } X \end{matrix} \right\}.$$

$\wr\!\mathrm{S}$ Simpson

$$\left\{ \begin{matrix} \text{Higgs bdls, rigid ss} \\ c_i = 0,\ \forall i \end{matrix} \right\}.$$

Moduli $\rightsquigarrow$

$$\begin{matrix} M_B & \simeq & M_{dR} \\ \wr\!\mathrm{S} & & \\ M_{Dol} & & \end{matrix}$$

homeomorphisms

1.2 p-adic non-ab Hodge theory

Setup κ perfect field of char p.

$W = W(\kappa)$, $K/W[\frac{1}{p}]$ fin tot ramified

$\mathcal{O}_K$, π, $E \in W(k)[\![u]\!]$.

$C = \hat{\bar{K}}$, $\mathcal{O}_C$, $\mathfrak{m}_C$.

- For Simpson Corresp. X/K or C sm rigid space.

  Faltings: generalized reps. (v-VB X_v).

- For RH Corresp. Scholze: $\mathbb{B}_{dR}^+$-loc sys on X_v.

- When X/K, Scholze, Liu-Zhu, Gao-Min-Wang

Let $\mathbb{B}_{dR,n}^+ := \mathbb{B}_{dR}^+/t^n$.

$$\begin{tikzcd}
\rightsquigarrow LS_r(X, \mathbb{B}^+_{dR,n}) \arrow[dr, "\tilde{h}"'] & & MIC(X_{B^+_{dR}}) \arrow[dl, "h"] \\
& \mathcal{A}_r & \\
& 0 \arrow[u, phantom, "\in" rotate] &
\end{tikzcd}$$

Consider objs $\in h^{-1}(0)$ (or $\in \tilde{h}^{-1}(0)$).

<u>Question</u> What happens if we work w/ X/C ?

Can we get a RH beyond the Hitchin fibre at 0?

<u>Equivalence on Hitchin-Small locus</u>

$\mathrm{Perf}_{\mathcal{C}}$:= v-site of all affinoid perf'd spaces $S = \mathrm{Spa}(A, A^+)$ / C.

<u>Assumption</u> X admits a flat lifting $\tilde{X}$ / B^+_{dR}

(e.g. X/K proper)

$n \geq 1$, $\tilde{X}_n := \tilde{X}/t^n$.

For $r \geq 1$,

$$LS_r(X, \mathbb{B}^+_{dR,n}) : \underset{\in}{S} \overset{\mathrm{Perf}_{\mathcal{C}}}{} \longrightarrow \text{Grpoid of } \mathbb{B}^+_{dR,n}\text{-local sys on } X \times S.$$

$$MIC_r(\tilde{X}_n) : S \in \mathrm{Perf}_{\mathcal{C}} \longrightarrow \text{Grpoid of integrable conn on } \tilde{X}_n \times_{B^+_{dR,n}} \mathbb{B}^+_{dR,n}(S).$$

<u>Connection</u> (D, ∇), $\nabla : D \longrightarrow D \otimes_{\mathcal{O}_{\tilde{X}_n \times \mathbb{B}^+_{dR}(S)}} \Omega^1_X(-1)$

D: $VB(\tilde{X}_n \times \mathbb{B}^+_{dR,n}(S))$

s.t. Leibniz rule w.r.t. $d : \mathcal{O}_{\tilde{X}_n \times B^+_{dR}} \longrightarrow \Omega^1_{\tilde{X} \times B^+_{dR}} \hookrightarrow \Omega^1_{\tilde{X} \times B^+_{dR}}(-1)$

behaves like t-conn.

Have a diagram

$$\begin{array}{ccc} LS_r(X, \mathbb{B}^+_{dR,n}) & & MIC_r(\tilde{X}_n) \\ \downarrow{\scriptstyle \bmod t} & & \downarrow{\scriptstyle \bmod t} \\ vBun_r(X) = LS_r(X, \mathbb{B}^+_{dR,1}) & & HIG_r(X) \\ & \searrow^{\tilde{h}_1} \quad \swarrow^{h} & \\ & \mathcal{A}_r & \end{array}$$

$\quad \llcorner\ S \in \mathrm{Perf}_C \longmapsto \bigoplus_{i=1}^r H^0(X \times S, \mathrm{Sym}^i \Omega^1_{X\times S}(-1))$

<u>Thm</u> (Heuer, $n=1$; J. Yu, $n \geq 2$)

$LS_r(X, \mathbb{B}^+_{dR,n})$ & $MIC_r(\tilde{X}_n)$ are small v-stacks.

<u>Hitchin-small locus</u> Assume X has a <u>semistable</u> model $\mathfrak{X}/\mathcal{O}_C$

$\mathfrak{X} = \mathrm{Spf}\, R$, $\quad$ Česnavičius-Koshikawa: $\mathcal{M}_{\mathfrak{X}} = \mathcal{O}_X^\times \cap \mathcal{O}_{\mathfrak{X}} \hookrightarrow \mathcal{O}_{\mathfrak{X}}$.

s.t. $\mathfrak{X}$ admits a flat lifting $\tilde{\mathfrak{X}}$ over A_{inf}.

$\square : \mathcal{O}_C \langle T_0, \cdots, T_r, T_{r+1}^{\pm 1}, \cdots, T_d^{\pm 1} \rangle / (T_0 \cdots T_r - p^a) \xrightarrow{\acute{e}t} R$.

$\qquad \llcorner\ p^a \in \mathfrak{m}_C$.

$\leadsto$ Lifting is compatible with log-str.

Def $\quad \mathcal{A}_r^\circ : S = \mathrm{Spa}(A, A^+) \longmapsto \bigoplus_{i=1}^r \underbrace{p^{> \frac{i}{p-1}}}_{\substack{\shortparallel \\ \{x \mid v_p(x) > \frac{i}{p-1}\} \subseteq \mathcal{O}_C}} H^0(\mathfrak{X}_{A^+}, \mathrm{Sym}^i \Omega^1_{\mathfrak{X}}\{-1\})$.

$\quad \mathcal{A}_r^\circ \subseteq \mathcal{A}_r$

$Z \in \{LS_r, MIC_r\} \leadsto Z^\circ := Z \times_{\mathcal{A}_r} \mathcal{A}_r^\circ$.

<u>Thm 1</u> (LMNQW) $\quad X = \mathfrak{X}_\eta^{ad}$,

$\mathfrak{X}/\mathcal{O}_C$ semistable, admitting a lift $\tilde{\mathfrak{X}}/A_{inf}$

$\qquad \downarrow$

Then $\exists$ equiv of stacks $\qquad \hat{X}/\mathbb{B}^+_{dR}$

$$\begin{array}{ccc} LS_r(X, \mathbb{B}^+_{dR,n})^\circ & \simeq & MIC_r(\tilde{X}_n)^\circ \\ & \searrow \quad \swarrow & \\ & \mathcal{A}_r^\circ & \end{array}$$

<u>Rmk</u> (1) When $n=1$,

- Anschutz - Heuer - Le Bras (Sm)

  based on prismatization of p-adic formal sch of BL

- Sheng - Wang (semistable)

  based on previous work w/ Min.

  $(\mathcal{O}\hat{\mathbb{C}}^{+}_{pd}, \theta)$, $\quad \mathcal{O}\mathbb{C} \subseteq \mathcal{O}\mathbb{C}^{\dagger} \subseteq \mathcal{O}\hat{\mathbb{C}}_{pd}$.

(2) When $n \geq 2$: know nothing before this work.

<u>Note</u> $\{\text{Hitchin - small } v\text{-VB}\} \simeq \{\text{HT crystals on } \mathfrak{X}_{\Delta}\}$

$$\begin{array}{c} \text{IS} \\ VB(\mathfrak{X}^{HT})[\tfrac{1}{p}]. \end{array}$$

"<u>Prof</u>"

<u>Thm 2</u> Let X, $\mathfrak{X}$, $\tilde{\mathfrak{X}}$ as before.

Then $\exists$ period sheaf w/ flat conn

$$(\mathcal{O}\mathbb{B}^{+}_{dR,pd},\ d : \mathcal{O}\mathbb{B}^{+}_{dR,pd} \to \mathcal{O}\mathbb{B}^{+}_{dR,pd} \otimes \Omega^{1}_{\tilde{\mathfrak{X}}}(-1))$$

s.t. $0 \to \mathbb{B}^{+}_{dR} \to DR(\mathcal{O}\mathbb{B}^{+}_{dR,pd}, d)$ exact

& (1) $\forall$ Hit-small $\mathbb{B}^{+}_{dR}$-loc sys $\mathbb{L}$,

$$R\nu_{*}(\mathbb{L} \otimes \mathbb{B}^{+}_{dR,pd}) = \nu_{*}(\mathbb{L} \otimes \mathcal{O}\mathbb{B}^{+}_{dR,pd}) = D(\mathbb{L})$$

$(D(\mathbb{L}), \nabla_{\mathbb{L}})$ is Hit-small flat conn on $\tilde{X}_n$.

$$\nabla_{\mathbb{L}} = \nu_{*}(\mathrm{id}_{\mathbb{L}} \otimes d)$$

(2) $\forall$ Hit-small int conn (D, ∇) on $\tilde{X}_n$,

$$(D \otimes \mathcal{O}\mathbb{B}^{+}_{dR,pd})^{\nabla = 0} =: \mathbb{L}(D, \nabla)$$

is a Hit-small $\mathbb{B}^{+}_{dR,n}$-loc sys on X_v.

(3) (1) & (2) $\leadsto$ an equiv of cats

$$\{\text{Hit-small } \mathbb{B}^+_{dR,n}\text{-loc sys}\} \cong \{\text{Hit-small conn}\}$$

& compatible w/ cohom.

<u>Rmk</u> (1) $\{\text{Small } \mathbb{B}^+_{dR}\text{-loc sys}\} \subseteq \{\text{Hit-small } \mathbb{B}^+_{dR}\text{-loc sys}\}$.

See R. Liu's talk (Small = Faltings-Small).

e.g. A generalized rep where geom Sen is nilp is Hit-small but NOT necessarily small.

(2) $\mathcal{O}\mathbb{B}_{dR} \hookrightarrow \mathcal{O}\mathbb{B}^{\dagger,+}_{dR}[\frac{1}{t}] \hookrightarrow \mathcal{O}\mathbb{B}^+_{dR,pd}[\frac{1}{t}]$.

This work is compatible w/ RH of Scholze, Liu-Zhu, Gao-Min-Wang & p-adic RH in Liu's talk.

(3) $\exists$ rel ver of $\mathcal{O}\mathbb{B}^+_{dR,pd}$, $\mathcal{O}\mathbb{B}^{\dagger,+}_{dR}$, $\mathcal{O}\tilde{\mathbb{C}}^I$ in the sense that:

if $f: \mathcal{Y} \to \mathfrak{X}$ + lift $\tilde{f}: \tilde{\mathcal{Y}} \to \tilde{\mathfrak{X}}$ / Ainf,

then $\tilde{f}^+ \mathcal{O}\mathbb{P}_{\tilde{\mathfrak{X}}} \to \mathcal{O}\mathbb{P}_{\tilde{\mathcal{Y}}} \xrightarrow{d} \mathcal{O}\mathbb{P}_{\tilde{\mathcal{Y}}} \otimes \Omega^1_{\tilde{\mathcal{Y}}/\tilde{\mathfrak{X}}}(-1) \to \cdots$ exact.

(4) $\exists$ analogue when $\mathfrak{X}$ is a semistable / A^+ admits a lifting / Ainf(S)

$S = \mathrm{Spa}(A = A^+[\frac{1}{p}], A^+) \in \mathrm{Perf}_C$.

<u>Pf of Thm 1</u> In Thm 2, every functor is functorial in S. Apply Rmks. □

<u>Construction of $\mathcal{O}\mathbb{B}^+_{dR,pd}$</u>

For simplicity, $\mathfrak{X} = \mathrm{Spf}(\mathcal{O}_C\langle T^{\pm1}\rangle)$, $R = \mathcal{O}_C\langle T^{\pm1}\rangle$

$\tilde{\mathfrak{X}} = \mathrm{Spf}(\mathrm{Ainf}\langle T^{\pm1}\rangle)$, $\tilde{R} = \mathrm{Ainf}\langle T^{\pm1}\rangle$

$$X_\infty = \mathrm{Spa}(\underset{\overset{\parallel}{A}}{C\langle T^{\pm 1/p^\infty}\rangle}, \underset{\overset{\parallel}{A^+}}{\mathcal{O}_C\langle T^{\pm 1/p^\infty}\rangle}) \in \mathrm{Perfd}.$$

$\rightsquigarrow$ $\tilde{R}\,\hat{\otimes}_{A_{\inf}} \mathbb{A}_{\inf}(X_\infty) \xrightarrow{\theta} A^+$, $\ker\theta = (T - [T^\flat], \xi)$

$$\Big\downarrow (p, \xi, \ker\theta)\text{-cplt}$$

$A_{\mathfrak{X}} = \mathbb{A}_{\inf}(X_\infty)[\![V]\!]$, $V = T - [T^\flat]$.

$$\Big\updownarrow \text{blow-up alg}$$

$A_{\mathfrak{X}}[\frac{u \cdot \ker\theta}{\xi}] \simeq \mathbb{A}_{\inf}(X_\infty)[\frac{uV}{\xi}]$, $\quad u = [\varepsilon^{1/p}] - 1$,

$\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}_{C^\flat}$.

$$\Big\downarrow (p,\xi)\text{-cplt, pd-envelope}$$

$B_{\mathfrak{X}} = \mathbb{A}_{\inf}(X_\infty)[\frac{uV}{\xi}]^\wedge_{pd,(p,\xi)}$ over $\mathbb{A}_{\inf}(X_\infty)[u]^\wedge_{pd\,(p,\xi)}$.

$$\Big\downarrow (-)\otimes_{\mathbb{A}_{\inf}(X_\infty)[u]^\wedge_{pd}} \tilde{\mathbb{C}}^{1,+}(X_\infty)$$

$S_{\mathfrak{X}}$ $\qquad \tilde{\mathbb{C}}^{1,+} := \mathbb{A}_{\inf}[(\frac{[p^\flat]}{p})^{\pm}]^\wedge_p$.

Fact u admits pd-powers in $\tilde{\mathbb{C}}^{1,+}$ & u invertible after inverting p.

$$\Big\downarrow (p^{-1}\xi)\text{-adic cplt}$$

$\mathbb{A}_{dR}(X_\infty)\langle W\rangle_{pd}$, $\quad W = \frac{uV}{\xi}$.

__Aside__ $\tilde{\mathbb{C}}^{1,+} \to \hat{\mathcal{O}}_X^+$. $\quad 1 - \frac{[p^\flat]}{p} = \frac{\xi}{p}$.

Here $\mathbb{A}_{dR} := (p^{-1}\xi)$-adic cplt of $\tilde{\mathbb{C}}^{1,+}$

$\mathbb{A}_{dR}\langle W\rangle_{pd} := \lim_n \mathbb{A}_{dR}/(p^{-1}\xi)^n \langle W\rangle_{pd}$ — p-adic cplt

$+\ d : \mathbb{A}_{dR}\langle W\rangle \longrightarrow \mathbb{A}_{dR}\langle W\rangle \otimes \Omega^1_{\mathfrak{X}}\{-1\}$ $\quad \rightsquigarrow \quad u\frac{\partial}{\partial W} \otimes \frac{dT}{\xi}$.

Now put $\quad \mathcal{O}\mathbb{B}^+_{dR,pd}(X_\infty) := (\mathbb{A}_{dR}(X_\infty)\langle W\rangle_{pd}[\frac{1}{p}])^\wedge_\xi$

$+\quad d = u\frac{\partial}{\partial W} \otimes \frac{dT}{\xi}$.

$\Rightarrow\ \mathrm{DR}(\mathcal{O}\mathbb{B}^+_{dR,pd}, d) \simeq \mathbb{B}^+_{dR}$.