Sato–Tate for Bianchi modular forms

James Newton

(Joint with Boxer, Calegari, Gee, Thorne).

§1 Compatible systems of Galois reps

F number field, M coeff number field.

$d > 0$. S = finite set of ramified places.

Def'n $(r_\lambda : G_F = \mathrm{Gal}(\bar{F}/F) \longrightarrow GL_d(\bar{M}_\lambda) \mid \text{cont ss})_\lambda$ finite places of M

is a compatible system, unram outside S

if (1) r_λ unram outside $S \cup \{v \mid Nm(\lambda)\}$

& for $v \notin S$, $v \nmid Nm(\lambda)$:

$r_\lambda(\mathrm{Frob}_v)$ has char poly $P_v(t) \in M[t]$ indep of λ.

(2) For $\underbrace{\text{almost all}}$ λ,

($\lambda \mid \ell \in \mathbb{Q}$, ℓ in Dirichlet density 1 set).

r_λ is crystalline, i.e. HT wts indep of λ.

$\rightsquigarrow$ $\mathcal{R} = (r_\lambda)_\lambda$.

$\mathcal{R}$ called irred if r_λ are irred for almost all λ.

$\mathcal{R}$ called autom if $\forall \iota : M \hookrightarrow \mathbb{C}$,

$\exists \pi$ cuspidal autom rep of $GL_d(\mathbb{A}_F)$ unram outside S,

for $v \notin S$, $P_v(t) \longleftrightarrow$ Satake parameter of π_v.

$t(\pi_v) \in GL_d(\mathbb{C})$.

Expect $\mathcal{R}$ irred $\Rightarrow$ $\mathcal{R}$ autom & further motivic.

§2 Main theorem

Thm $d=2$, F CM field, $\mathcal{R}$ irred 2-dim'l compatible system.

(labelled) HT weights are all $(0,m)$ for $m \geq 1$ fixed.

Fix $\tau : F_v \hookrightarrow \bar{M}_\lambda$.

Then (1) $\mathcal{R}$ is pure of wt m, i.e.

roots of $P_v(t)$ are wt m Weil numbers.

(2) $\exists F'/F$ CM extension with $\mathcal{R}|_{G_{F'}}$ autom.

(3) Assume $\mathcal{R}$ is not induced

(induced = come from 1-dim irreps / quadratic field which are well-understood.)

then $\forall n \geq 1$, $\exists F_n'/F$ CM with $\mathrm{Sym}^n \mathcal{R}|_{G_{F_n'}}$ automorphic.

Remark (a) The autom compatible systems are not shown to be motivic.

(in contrast to the case where $F = \mathbb{Q}$ or tot real field.)

Instead we use (3) to deduce (1) (d'après Langlands).

$|\alpha^n|$. $\forall n \geq 1$ $\rightsquigarrow$ $|\alpha|$.

(b) (3) $\xRightarrow{\text{Tate \& Serre}}$ Sato-Tate equidistribution

(goes back to Taylor "mero continuation of rk 2 L-fcns".)

Corollary F imag quad field.

π reg alg cuspidal autom rep of $GL_2(\mathbb{A}_F)$.

(Contributes to cohom of loc symm space)

Then (1) π satisfies Ramanujan Conj.

i.e. π_v essentially tempered for all v.

(3) ($\Leftarrow$ Thm (2)) Sato-Tate conj for π.

<u>Proof</u> Use compatible system R_π attached to π (+ crystallinity)

Harris-Soudry-Taylor, ACCGHLNSTT.

<u>History</u> (a) $F = \mathbb{Q}$. (without potential)

- Khare-Wintenberger
- Calegari
- Newton-Thorne

(b) F tot real

- (2) Taylor
- (3) Taylor, Clozel, Harris, Shepherd-Barron. Barnet-Lamb-Gee-Geraghty

("more motivic").

(c) 10 author paper: $m=1$. F CM field.

<u>Strategy</u> Find chain of congruences connecting $R|_{G_{F'}}$ to an autom compatible system

$\rightsquigarrow$ apply automorphy lifting thms to progagate automorphy through the chain.

Need better automorphy lifting thms to carry out this strategy.

(1) Caraiani-Newton: local-global compatibility.

(2) Purely local results on Gal deformation rings.

$K/\mathbb{Q}_p$, $\bar{\rho}: G_K \to GL_d(k)$ $(k/\mathbb{F}_p)$.

$R^{crys,0} = $ CNL-$Alg_{W(k)}$.

$\uparrow$ classifies lifts of $\bar{\rho}$ which are crys with consecutive HT weights.

<u>Thm</u> (Ihara avoidance)

$p > d$, $R^{crys,0}/p$ is generically reduced.

Pf Emerton-Gee $\rightsquigarrow$ prove for "generic" $\bar{\rho}$.

$\bar{\rho}$ fully irred, non-split.

<u>Rmk</u> "$p > d$": See T. Liu "on a Conj of Breuil".