Locally analytic p-adic representations, coherent sheaves and patching

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Lecture 1

Setup E CM field, $F \subset E$ tot real subfield, $[E:F] = 2$.

p fixed prime, totally split in E.

$\underline{G}$ unitary grp split $/ E$,

$\rightsquigarrow$ fix $\mathfrak{p} | p$ of F, $\underline{G}(F_{\mathfrak{p}}) \simeq GL_n(\mathbb{Q}_p)$.

p-adic automorphic forms on G

Tame level $K^p \subseteq \underline{G}(\mathbb{A}_f^p)$ cpt open subgrp, small enough.

$\rightsquigarrow \mathcal{A}(K^p) := \{ f : \underbrace{\underline{G}(F) \backslash \underline{G}(\mathbb{A}_f) / K^p}_{\circlearrowleft\, \mathbb{T}_p} \longrightarrow \mathbb{Q}_p \text{ Conti} \}$

a p-adic Banach space.

$\rightsquigarrow \underset{\underline{G}(\mathbb{A}_f)}{\overset{\circlearrowleft}{\mathcal{A}}} := \varinjlim_{K^p} \mathcal{A}(K^p) \supset \mathcal{A}(K^p)$

$\overset{\circlearrowleft}{\mathbb{T}^S} = \bigotimes_{v \notin S} \mathcal{H}(\underline{G}(F_v), K_v)$

$(S = \{ v \text{ places of } F : K_v \text{ not hyperspecial} \})$

$K^p = \prod K_v$

Let $\pi = \pi_f \otimes \pi_\infty$ be an autom rep of $\underline{G}(\mathbb{A})$.

$\rightsquigarrow \rho : \mathrm{Gal}(\bar{E}/E) \longrightarrow GL_n(L)$, $L / \mathbb{Q}_p$ finite. (fix $\imath : \mathbb{C} \simeq \bar{\mathbb{Q}}_p$)

s.t. ρ compatible w/ LLC at places w of E, where $w \notin S$ & w split in E/F.

$\Rightarrow \quad \rho^\vee \simeq \rho^c \otimes \chi_{cyc}^{n-1} \qquad (\mathrm{Gal}(E/F) = \{1, c\})$.

<u>Rmk</u> ρ depends only on $\pi^S = \bigotimes'_{v \notin S} \pi_v$

So take
$$\begin{array}{ccc} \psi_\pi : \mathbb{T}^S & \longrightarrow & L \\ \downarrow & & \downarrow \\ \mathbb{C} & \overset{?}{\simeq} & \overline{\mathbb{Q}}_p \end{array}$$
character acting on $(\pi^S)^{K^S} \simeq \mathbb{C}$.

<u>Def</u> $\Pi(\rho) := \mathrm{Hom}_{L[\mathbb{T}^S]}(\psi_\pi, \mathcal{A}(K^p)_L)$
$\circlearrowleft \Pi(\rho)$: $G(\mathbb{F}_p) \simeq GL_n(\mathbb{Q}_p)$ $\qquad \circlearrowleft \mathcal{A}(K^p)$

This is a unitary p-adic Banach rep of $GL_n(\mathbb{Q}_p)$.

<u>Q</u> (p-adic local Langlands)

Is $\pi(\rho)$ determined by $\rho_p := \rho|_{\mathrm{Gal}(\overline{F}_{p'}/F_{p'})}$ $(p'|p)$.

$\rightsquigarrow$ A bit more precisely, do we have
$$\pi(\rho) \simeq \pi(\rho_p) \otimes (\pi_{S \setminus \{p\}})^{K^p}$$
$$\simeq \pi(e_p)^{\oplus n}$$
w/ $\pi(e_p)$ a rep depending only on ρ_p.

($\pi(e_p) = \mathrm{Hom}(\pi^p, \mathcal{A}(K^p)_L)$, $\pi^p = \bigotimes'_{v \neq p} \pi_v$)

Subspaces: loc an vectors & classical vectors:

$GL_n(\mathbb{Q}_p) \circlearrowright \mathcal{A}(K^p)^{la} := \{ \text{la fcts in } \mathcal{A}(K^p) \}$
$\cup\!|$
$GL_n(\mathbb{Q}_p) \circlearrowright \mathcal{A}(K^p)^{cl} := \{ \text{la polynomials in } \mathcal{A}(K^p) \}$

$\rightsquigarrow \quad \mathcal{A}(K^p)^* \cap \Pi(\rho) =: \Pi(\rho)^*, \quad * \in \{la, cl\}.$

<u>Known</u> (Caraiani, Emerton)

$$\Pi(\rho)^{cl} \simeq (\pi_p \otimes \underbrace{W_\lambda}_{\substack{\text{irred alg rep of } GL_n \\ \text{of highest wt } \lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \leftarrow\!\sim \text{ wt } \pi_\infty.}})^{\oplus n}$$

<u>But</u> Not enough to recover ρ_p!

<u>Note</u> $\pi_p \longleftrightarrow WD(\rho_p), \quad \lambda \longleftrightarrow HT(\rho_p).$

<u>Finite slope overconvergent forms</u>

$B = \begin{pmatrix} \nabla \end{pmatrix} \subset GL_n. \quad N = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}, \quad T = \begin{pmatrix} \diagdown \end{pmatrix}.$

$V \in \mathrm{Rep}_L B(\mathbb{Q}_p)$

$$\rightsquigarrow V^{N(\mathbb{Z}_p)} \circlearrowleft T(\mathbb{Q}_p)_+ = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in T(\mathbb{Q}_p) : v_p(a_1) \geq \cdots \geq v_p(a_n) \right\}.$$

$$t \cdot v = [N(\mathbb{Z}_p) : tN(\mathbb{Z}_p)t^{-1}]^{-1} \sum_{n \in N(\mathbb{Z}_p)/tN(\mathbb{Z}_p)t^{-1}} nt \cdot v$$

<u>Def</u> Let $\chi : T(\mathbb{Q}_p) \longrightarrow L^\times$ conti char (loc an).

$$\mathcal{A}(K^p)^{la}_\chi := \mathrm{Hom}_{T(\mathbb{Q}_p)}(\chi, \mathcal{A}(K^p)^{la, N(\mathbb{Z}_p)})$$

space of χ-eigenforms (of finite slope).

$$\Pi(\rho)_\chi := \Pi(\rho) \cap \mathcal{A}(K^p)^{la}_\chi.$$

<u>Rmk</u> If $\Pi(\rho)^{cl}_\chi = \Pi(\rho)^{cl} \cap \Pi(\rho)_\chi \neq 0,$

then $\chi = \chi_\lambda \cdot \chi_{sm} \hookrightarrow W_\lambda^{N(\mathbb{Z}_p)} \otimes J_N(\pi_p)$

$\qquad$) $\qquad$ a sm char

an alg char of wt λ

$$\pi_p \in JH\left(\left(\mathrm{Ind}_B^{GL_n} \chi_{sm}\right)^\infty\right) \simeq \mathrm{Ind}_B^{GL_n}(w \cdot \chi_{sm})^\infty, \quad w \in S_n.$$

(think: $\chi_{sm} \longleftrightarrow WD(\rho_p)$.)

Computation of $\Pi(\rho)_\chi^{la}$

Assume π_p is unramified & generic

$\Rightarrow \pi_p \simeq \left(\mathrm{Ind}_B^{GL_n} \chi_{sm}\right)^\infty$ & ρ_p is crystalline).

Say $\chi_{sm} = \chi_1 \otimes \cdots \otimes \chi_n : T(\mathbb{Q}_p) \simeq (\mathbb{Q}_p^\times)^n \longrightarrow L^\times$

Note
- $\rho_p \longleftrightarrow$ filtered φ-mod $(D_{cris}(\rho_p), \varphi, Fil_{dR})$
- $\varphi \sim \begin{pmatrix} \chi_1(p) & & & \\ & \chi_2(p)p^{-1} & & \\ & & \ddots & \\ & & & \chi_n(p)p^{1-n} \end{pmatrix}$.
- $(\mathrm{Ind}\, \chi_{sm})^\infty \simeq \mathrm{Ind}\, (\underbrace{w \cdot \chi_{sm}}_{= w \cdot (\chi_{sm} \delta_B^{-1/2}) \delta_B^{1/2}})^\infty, \quad w \in S_n.$

Def Let $w \in S_n$, Fil_w the unique complete flag of $D_{cris}(\rho_p)$,

$\qquad$ + stable under φ,

and the eigenvals of φ on successive subquotients of this flag

are $(\chi_{w^{-1}(1)}(p), \chi_{w^{-1}(2)}(p)p^{-1}, \cdots, \chi_{w^{-1}(n)}(p)p^{1-n}) = (w \cdot \chi_{sm})(p)$.

Fix $s_w \in S_n$ s.t. (Fil_w, Fil_{dR}) is in the relative position s_w in $GL_n/B\ (D_{cris}(\rho_p))$

$(\mathrm{Fl}_{rv}, \mathrm{Fl}_{dR}) \in \mathrm{GL}_n(1, S_w) \subset (\mathrm{GL}_n/B)^2.$

Thm (Coleman, Ding, Breuil-Hellmann-Schraen)

Under certain hypothesis: TW hyp, G quasi-split at finite places. etc.

$$\Pi(\rho)_\chi \neq 0 \iff \chi = \chi_{w'\cdot\lambda}(w \cdot \chi_{sm})$$

where $w, w' \in S_n$ s.t. $w' \leq S_w w_0$ for Bruhat order

(w_0 = longest elt in S_n).

<u>Question</u> · Compute $\dim \Pi(\rho)_\chi$

· $\chi = \chi_\lambda \chi_{sm}$ dominant, $0 \neq \Pi(\rho)_\chi^{cl} \subset \Pi(\rho)_\chi^{la}$ is this "=" ?

<u>Recall</u> $\dim \Pi(\rho)_\chi^{cl} = r = \dim \Pi_{S \setminus \{p\}}^{K^p}$.

Thm (1) (BHS) When $w'w_0$ or $S_w w_0$ is the product of distinct simple reflexions, have "mult one":

$$\dim \Pi(\rho)^{la}_{\chi_{w'\cdot\lambda}(w\cdot\chi_{sm})} = r.$$

(2) $n=3$. $S_w = 1$ ($S_w w_0 = s_1 s_2 s_1$) and $w' = 1$

$\Rightarrow \dim \Pi(\rho)_\chi^{la} = 2r = 2 \dim \Pi(\rho)_\chi^{cl}$ ($\chi = \chi_\lambda \cdot \chi_{sm}$, λ dominant)

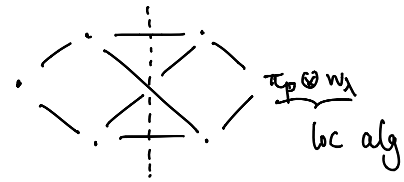
(Note: $\rho_p \simeq \eta_1 \oplus \eta_2 \oplus \eta_3$.)

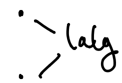
<u>Rmk</u> Breuil reciprocity law:

· Π la rep of $\mathrm{GL}_n(\mathbb{Q}_p)$, admissible

$\Pi_\chi \simeq \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{Q}_p)}(\mathrm{coind}_B^{\mathrm{GL}_n} \chi, \Pi)$ (coind = $\mathrm{Ind}^\vee$).

E.g. In GL_3 case, $\chi = \chi_\lambda \chi_{sm}$,

$\mathrm{Coind}_B^{GL_3} \chi =$ 

$\dim \mathrm{Hom}(\mathrm{coind}_B^{GL_3} \chi, \Pi(\rho_p)^{la}) = 2$ 

$\pi_p \otimes W_\lambda \hookrightarrow \Pi(\rho_p)^{al} \subset \Pi(\rho_p)^{la}$.

Lecture 2

- G/F definite unitary grp / tot real field.
- E/F CM / $\mathbb{Q}$, $G_E \simeq GL_n$.
- $p | \mathfrak{p}$. $G(F_\mathfrak{p}) \simeq GL_n(\mathbb{Q}_p)$.
- $K^p \subseteq G(\mathbb{A}^{\infty,p})$, $\mathcal{A}(K^p)_L \circlearrowleft GL_n(\mathbb{Q}_p)$. $L/\mathbb{Q}_p$ finite.
- π autom rep of $G(\mathbb{A}) \rightsquigarrow \rho : \mathrm{Gal}(\bar{E}/E) \to GL_n(L)$

$\rho_p = \rho|_{\mathrm{Gal}(\bar{E}_p/E_p)}$

$\psi_\pi : \mathbb{T}^S \to L$.

- $\Pi(\rho)^{la} = \mathrm{Hom}_{\mathbb{T}^S}(\psi_\pi, \mathcal{A}(K^p)_L^{la})$.

<u>Hypothesis</u> ρ abs irred. π unram + generic at p.
+ T-W hypothesis.

Let $\chi : T(\mathbb{Q}_p) \to L^\times$ s.t. $\Pi(\rho)^{la}_\chi = \mathrm{Hom}_{T(\mathbb{Q}_p)^+}(\chi, \Pi(\rho)^{la, N(\mathbb{Z}_p)}) \neq 0$.

$\Rightarrow \chi = \underbrace{\chi_{w\cdot\lambda}}_{\text{alg}} \cdot \underbrace{\chi_R}_{\text{sm unram}}$

Here λ = dom wt $\leftrightarrow$ wt (π_∞)

$\wp$ = order on eigenvals of $\varphi \subset D_{cris}(\rho_p)$.

$w \in S_n$, $w \leq s_\wp w_0$ for some $s_\wp \in S_n$.

<u>Thm</u> (HHS) $n = 3$, $s_\wp = 1$. Then

$$\dim \Pi(\rho)^{la}_{\chi_\lambda \chi_\wp} = 2 \cdot \dim \Pi(\rho)^{la}_{\chi}.$$

<u>Patching method</u>

$\exists\ \Pi_\infty$ unitary adm L-Banach rep of $GL_n(\mathbb{Q}_p)$

$\circlearrowleft \quad \Pi_\infty^{\mathbb{Z}_p^s} \simeq \mathcal{A}(K^p)_{\bar\rho} \overset{\oplus}{\hookrightarrow} \mathcal{A}(K^p)_L$.

$$\begin{tikzcd}
R_\infty = R^{\square}_{\bar\rho_p}[[x_1, \ldots, x_g]] \arrow[r, two heads] \arrow[drr] & R_{\bar\rho} \arrow[r, two heads] & \widehat{\mathbb{T}}^S_{K^p} = \mathrm{Im}(\mathbb{T}^S \to \mathrm{End}\, \mathcal{A}(K^p)_{\bar\rho})^\wedge \arrow[d, "\psi_\pi"] \\
 & & L.
\end{tikzcd}$$

$\bar\rho = \rho \bmod p$.

$\Pi(\rho) \simeq \mathrm{Hom}_{R_\infty}(\psi_\pi, \Pi_\infty)$.

$\Rightarrow (\Pi(\rho)^{la}_{\chi})' \simeq (\Pi^{la}_{\infty,\chi})' \otimes_{R_\infty, \psi_\pi} L$.

Moreover, $(\Pi^{la}_{\infty,\chi})'$ is a f.g. $R_\infty[\frac{1}{p}]^\wedge$-mod ($\sim$ Emerton).

<u>Category $\mathcal{O}$</u> $\quad \mathfrak{g} = \mathfrak{gl}_n / \mathbb{Q}_p$.

$\rightsquigarrow \mathcal{O}^{(b)}_\lambda \subset \mathcal{U}(\mathfrak{g})\text{-Mod}^{fg}$ full subcat of M's s.t.

- M is semisimple as $\mathcal{U}(\mathfrak{t})$-mod
- M is locally $\mathcal{U}(\mathfrak{b})$-finite, $\mathfrak{b} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.
- $\forall\ z \in Z(\mathfrak{g}) = Z(\mathcal{U}(\mathfrak{g}))$, $\exists\ r \geq 1$ s.t. $(z - \chi_\lambda(z))^r M = 0$.

χ_λ char of $Z(\mathfrak{g})$ on M_λ, $z \in Z(\mathfrak{g})$.

Example $M(w\cdot\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} w\cdot\lambda$. $w \in S_n$.

$L(w\cdot\lambda) = \mathrm{CoSoc}(M(w\cdot\lambda))$ simple

$P(w\cdot\lambda) =$ proj envelop of $L(w\cdot\lambda)$.

$w\cdot\lambda = w(\lambda+\rho) - \rho$.

Let $(-)^\vee$ = internal duality on $\mathcal{O}_\lambda$,

w/ $L(w\cdot\lambda)^\vee \simeq L(w\cdot\lambda)$.

Note $\exists$ exact functor (Orlik - Strauch)

$$\mathcal{O}_\lambda \longrightarrow \mathrm{Rep}_L^{la} GL_n(\mathbb{Q}_p)$$

Write $\mathcal{F} := \mathcal{F}_B^{GL_n}(-, \chi_{\rho})$. Then

$\mathcal{F}(M(w\cdot\lambda)^\vee) \simeq \mathrm{Ind}_B^{GL_n}(\chi_{w\cdot\lambda}\chi_\rho)^{la}$

$\mathcal{F}(M(w\cdot\lambda)) \simeq \mathrm{coInd}_B^{GL_n}(\chi_{w\cdot\lambda}\chi_\rho)$.

Have $\Pi^{la}_{\infty, \chi_{w\lambda}\chi_\rho} \simeq \mathrm{Hom}_{GL_n(\mathbb{Q}_p)}(\mathcal{F}(M(w\cdot\lambda)), \Pi^{la}_\infty)$.

Def $M \in \mathcal{O}_\lambda$. $\mathcal{M}_\infty(M) := \mathrm{Hom}_{GL_n(\mathbb{Q}_p)}(\mathcal{F}(M), \Pi^{la}_\infty)' \otimes_{R_\infty[\frac{1}{p}]^{rig}} R$.

Here $\gamma_\pi : R_\infty \to L$ w/ $\forall\, z \in (\mathrm{Spf}\, R_\infty)^{rig}$, $R := \hat{R}_{\infty, z}$.

& $R_\infty[\frac{1}{p}]^\wedge = \mathcal{O}((\mathrm{Spf}\, R_\infty)^{rig})$.

Then $\mathcal{M}_\infty : \mathcal{O}_\lambda \longrightarrow \mathrm{Coh}(\mathrm{Spec}\, R)$ exact functor.

Let $\mathcal{X} := \mathrm{Spec}\, R \simeq \mathcal{X}_{\rho_p}[\![x_1, \ldots, x_g]\!]$, $\mathcal{X}_{\rho_p} = \mathrm{Spec}\, R^{\square}_{\rho_p}$.

* Support of $\mathcal{M}_\infty(M)$?

$\exists\, R^{\square}_{\rho_p} \twoheadrightarrow R^{qtri, k}_{\rho_p}$ quotient parametrizing quasi-trianguline deformation of ρ_p.

This holds if $D_{rig}(\rho_A)[\frac{1}{t}] \sim \begin{pmatrix} R_A[\frac{1}{t}] & & * \\ & \ddots & \\ 0 & & R_A[\frac{1}{t}] \end{pmatrix}$

for some $\rho_A : \mathrm{Gal}_{F_p} \to GL_n(A)$ (but not all).

Put $\tilde{\mathfrak{g}} := \{ (gB, \theta) \in GL_n/B \times \mathfrak{g} \mid g^{-1}\theta g \in \mathfrak{b} \}$.

$$X = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} = \{ (g_1B, \theta, g_2B) \in GL_n/B \times \mathfrak{g} \times GL_n/B \mid g_i^{-1}\theta g_i \in \mathfrak{b} \}$$

$$\downarrow \pi$$

$$GL_n/B \times GL_n/B.$$

$\rightsquigarrow$ $X = \bigcup_{w \in S_n} X_w$, $X_w = \overline{\pi^{-1}(GL_n(1,w))}$ irred comp of X.

<u>Thm</u> (BHS) $\exists\, x = (g_1B, \theta, g_2B) \in X$,

s.t.

$$\begin{tikzcd} & \mathfrak{X}_{\rho_p}^{\square, \mathrm{qtri}} \arrow[dl, "GL_n"'] \arrow[dr, "\text{formally sm.}"] & \\ \mathfrak{X}_{\rho_p}^{\mathrm{qtri}} = \mathrm{Spec}\, R_{\rho_p}^{\mathrm{qtri}} & & X_x^{\wedge} \end{tikzcd}$$

<u>Prop</u> (BHS, HHS) $\forall\, M \in \mathcal{O}_\lambda$, $\mathrm{Supp}\, M_\infty(M) \subset \mathfrak{X}_{\rho_p}^{\mathrm{qtri}, R}[\![x_1, \dots, x_g]\!]$.

Have map $X = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} \xrightarrow{\kappa} \mathfrak{t} \times_{\mathfrak{t}/W} \mathfrak{t}$ $(W = S_n)$

$$(g_1B, \theta, g_2B) \longmapsto (g_1^{-1}\theta g_1 \bmod \mathfrak{n},\ g_2^{-1}\theta g_2 \bmod \mathfrak{n}).$$

$\rightsquigarrow$ $\bar{X} = \kappa^{-1}(\{0\} \times \mathfrak{t})$.

<u>Thm</u> (Bezrukavnikov) $\exists$ exact functor

$$\mathcal{O}_\lambda \xrightarrow{\mathcal{B}} \mathrm{Coh}^{GL_n}(\bar{X})$$

s.t. (1) $\forall\, w \in S_n$, $\mathcal{B}(M(w\cdot\lambda)^\vee) \simeq \mathcal{O}_{\bar{X}_w}$, $\bar{X}_w = \bar{X} \times_X X_w$.

(2) $\forall\, w \in S_n$, $\mathcal{B}(M(w\cdot\lambda)) \simeq \omega_{\bar{X}_w}$ ($\bar{X}_w$ Cohen-Macaulay).

Pull-push $\mathcal{B}$ into $\mathcal{B}_\infty : \mathcal{C}_\lambda \to \mathrm{Coh}(\mathfrak{X}_{\bar{\rho}_p}^{\mathrm{qtri}} [\![x_1, \cdots, x_g]\!])$.

<u>Thm</u> (HHS) For $n = 3$ (also for $n=2$),

$\exists$ an isom of functors $\mathcal{M}_\infty \simeq \mathcal{B}_\infty^{\oplus r}$ $\quad$ ($r = \dim \pi_{S\mathcal{U}(p)}^{K^p}$).

<u>Consequence</u> $\quad n = 3$, $S_{\mathfrak{p}} = 1$.

$$\mathcal{M}_\infty(M(x)) \simeq (\omega^{\wedge}_{\bar{X}_{w_0}, x} [\![x_1, \cdots, x_g]\!])^{\oplus r}.$$

$\bar{X}_{w_0}$ is Cohen-Macaulay but not Gorenstein at x.

$\dim \omega_{\bar{X}_w} \otimes k(x) = 2$.