Geometric Eisenstein series of the Fargues-Fontaine curve

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(Joint work in progress with Peter Scholze)

Goal of these talks

Formulate a strategy to prove the Harris-Viehmann (HV) Conj, by studying "Eisenstein series" in a certain geom setup.

Lecture 1 : Introduction

Fix a prime p.

Let $G / \mathbb{Q}_p$ be a conn'd red grp + assumed quasi-split.

Fix $T \subset B \subset G$ $\rightsquigarrow$ standard parabolics, standard Levis.

(T = max torus, B = Borel)

The HV conj relates the cohom of local shtuka spaces for G and for Levi subgrps $M \subset G$.

Local shtuka spaces

Axiomatically attached to certain triples (G, μ, b)

- G as before
- $b \in B(G) = G(\breve{\mathbb{Q}}_p) / (g \sim b g \sigma(b)^{-1})$. (Kottwitz set)

  ($\breve{\mathbb{Q}}_p = \widehat{\mathbb{Q}_p^{ur}} \circlearrowleft \sigma$)

$\rightsquigarrow$ $B(G)$ = isom classes of "Frob-isocrystals"
with G-str $/\overline{\mathbb{F}}_p$.

<u>note</u> Kottwitz: For all $b \in B(G)$, get

$\nu_b \in X_*(T)^{\Gamma_{\mathbb{Q}_p}}_{\mathbb{Q},\mathrm{dom}}$ Newton point

$+$ $\kappa_b \in \pi_1(G)_{\Gamma_{\mathbb{Q}_p}} = (X_*(T)/\mathbb{Z}\cdot\Phi^\vee)_{\Gamma_{\mathbb{Q}_p}}$ "Kottwitz point".

<u>Fact</u> (ν_b, κ_b) determine b uniquely.
$\llcorner$ Dieudonné–Manin theory: slope when $G = GL_n$.

Also get G_b = alg grp $/\mathbb{Q}_p$ w/ $G_b(\mathbb{Q}_p) = \{ g \in G(\mathbb{Q}_p) \mid b\,\sigma(g)\,b^{-1} = g \}$.

G quasi-split $\Rightarrow$ G_b = inner form of a Levi of G.

Now suppose $\mu \in X_*(T)_{\mathrm{dom}}$
$\rightsquigarrow$ μ determines a finite subset

$B(G,\mu) \subset \underbrace{B(G)}_{\text{infinite}}$ (Newton polygon lies below Hodge polygon.)

<u>Def'n</u> A <u>local shtuka datum</u> is a triple (G,μ,b)
where $b \in B(G,\mu)$ for G,μ as above.

Given such a datum, Scholze–Weinstein constructed
a moduli space $\mathrm{Sht}_{G,\mu,b}$

of mixed characteristic shtukas.

<u>Very rough idea</u>: $b \rightsquigarrow \mathcal{E}_b$ = a G-bdl on the FF curve

$$\rightsquigarrow \mathrm{Sht}_{G,\mu,b} = \left(\begin{array}{c}\text{moduli space of "}\mu\text{-bounded} \\ \text{mero maps } \underset{\substack{\circlearrowleft \\ G(\mathbb{Q}_p)}}{\mathcal{E}_1} \dashrightarrow \underset{\substack{\circlearrowleft \\ G_b(\mathbb{Q}_p)}}{\mathcal{E}_b}\text{"}\end{array}\right)$$

so $\mathrm{Sht}_{G,\mu,b}$ has natural commuting actions of $G(\mathbb{Q}_p)$ & $G_b(\mathbb{Q}_p)$.

Can make sense of the ℓ-adic intersection cohom of $\mathrm{Sht}_{G,\mu,b}$:

$R\Gamma(G,\mu,b)$ = Complex of smooth $G(\mathbb{Q}_p) \times G_b(\mathbb{Q}_p)$ $\overline{\mathbb{Q}}_\ell$-vec spaces

+ natural commuting action of W_E

($E \subset \overline{\mathbb{Q}}_p$ fixed field of $\mathrm{Stab}_{\Gamma_{\mathbb{Q}_p}}(\mu)$.)

Let Λ any $\mathbb{Z}_\ell$-alg

$\rightsquigarrow D(G(\mathbb{Q}_p),\Lambda)$ = derived cat of the abelian cat of smooth $G(\mathbb{Q}_p)$-reps on Λ-mods.

Get a functor

$$D(G_b(\mathbb{Q}_p),\overline{\mathbb{Q}}_\ell) \longrightarrow D(G(\mathbb{Q}_p),\overline{\mathbb{Q}}_\ell)^{BW_E}$$

$$\rho \longmapsto \underbrace{R\Gamma(G,\mu,b)}_{\text{full IC cohom of Sht}}[\rho].$$

Here $R\Gamma(G,\mu,b)[\rho] := R\Gamma(G,\mu,b) \overset{\mathbb{L}}{\otimes}_{C_c(G_b(\mathbb{Q}_p),\overline{\mathbb{Q}}_\ell)} \rho.$

<u>Thm</u> (Fargues - Scholze)

If ρ is a sm $\bar{\mathbb{Q}}_\ell$-rep of $G_b(\mathbb{Q}_p)$ of fin length (in particular, ρ irred)

then $H^i(R\Gamma(G,b,\mu)[\rho]) \neq 0$ for only finitely many i

& each H^i is of finite length as a $G(\mathbb{Q}_p)$-rep. + W_E-action.

$\rightsquigarrow$ Can make the following definition

<u>Def</u> $\mathrm{Mant}_{G,\mu,b} : \mathrm{Groth}(G_b(\mathbb{Q}_p)) \longrightarrow \mathrm{Groth}(G(\mathbb{Q}_p) \times W_E)$

$$\rho \longmapsto \sum_i (-1)^i H^i(R\Gamma(G,\mu,b)[\rho]).$$

When b is basic ($\Leftrightarrow G_b$ inner form of G)

$\mathrm{Mant}_{G,\mu,b}$ is conjecturally very well-understood (Kottwitz conjecture).

Partial results by Harris-Taylor, Shin, Fargues, Bertoloni-Meli - Nguyen, Hansen - Kaletha - Weinstein.

<u>Goal of these lectures</u> Understand the non-basic case.

Fix $M \subset G$ standard Levi, $b \in B(M)$.

Say b is well-chosen if $\nu_b \in X_*(T)^{\Gamma}_{\mathbb{Q}, M\text{-dom}}$ (a priori M-dominant) is actually G-dominant.

Intuition $(G = GL_n)$ $\begin{pmatrix} \boxed{} & & \\ & \boxed{} & \\ & & \boxed{} \end{pmatrix} = M$

blocks $\longleftrightarrow$ isocrystals

Then slopes of isocrystals are decreasing w.r.t. $(\searrow)$.

$\rightsquigarrow$ Get G_b, but also M_b = Levi of parab $P_b \subset G_b$

Def Fix (G, μ, b),

$W_G^M(\mu, b) :=$ multiset of $\nu \in X_*(T)_{M\text{-dom}}$ s.t. $b \in B(M, \nu)$

ν is counted with multiplicity $\dim \mathrm{Hom}_M(\underset{\text{highest wt}}{W_\nu}, \underset{\text{rep'ns}}{V_\mu|_M})$

Obs $\langle 2\rho_G - 2\rho_M, - \rangle : X_*(T) \longrightarrow \mathbb{Z}$

factors over the quotient $(X_*(T)/\mathbb{Z}\cdot\Phi^\vee(T, M))_\Gamma = \pi_1(M)_\Gamma$.

(target of Kottwitz map).

So can define $\varepsilon = \langle 2\rho_G - 2\rho_M, \kappa_b \rangle \in \mathbb{Z}$.

Conj (Generalized HV Conj)

Have an equality

$$\mathrm{Mant}_{G,\mu,b}(\mathrm{Ind}_{P_b(\mathbb{Q}_p)}^{G_b(\mathbb{Q}_p)} \rho) = (-1)^r \cdot \sum_{\nu \in W_G^M(\mu, b)} \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \mathrm{Mant}_{G,\nu,b}(\rho)\underbrace{\left(\tfrac{\varepsilon}{2}\right)}_{\text{Tate twist}}$$

in $\underbrace{\mathrm{Groth}(G(\mathbb{Q}_p) \times W_E)}$ for all $\rho \in \mathrm{Groth}(M_b(\mathbb{Q}_p))$.

($(\frac{\varepsilon}{2})$ only changes the Weil action.)

elts are finite sums $\sum_i n_i \cdot (\underset{G(\mathbb{Q}_p)}{\pi_i \circlearrowleft} \boxtimes \underset{W_E}{\sigma_i \circlearrowleft})$.

Rmks (1) This passes various sanity checks

- compatible with composition.
- when μ is miniscule (and $G_b = M_b$) all ν's are actually Weyl translates of μ and this reduces to the usual statement of the HV conj (cf. Bertoloni-Meli).

(2) Previous partial results:

Harris-Taylor, Boyer, Mantovan, Shin, Hansen, Gaisin-Imai

But no general strategy.

Goal of the remaining lectures:

Explain a strategy to prove this conj, in the "geometrization" framework of Fargues-Scholze.

Background from F-S

Main player Bun_G = stack of G-bdls on the FF curve.

↳ This is a reasonable geom obj:

As a top space, $|\mathrm{Bun}_G| \cong B(G)$ by theorem of Fargues.

⇝ Get a decomp. into locally closed substacks Bun_G^b, $b \in B(G)$.

note If $b \in B(G)$ is basic, then Bun_G^b is open in Bun_G & $\mathrm{Bun}_G^b \cong [*/G_b(\mathbb{Q}_p)]$.

Fix $\Lambda = \mathbb{Z}_\ell$-alg.

Then F-S define a natural cat

$$D(\mathrm{Bun}_G, \Lambda)$$

+ a semi-orthogonal decomp into pieces

$$D(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(\mathbb{Q}_p), \Lambda).$$

There are two natural finiteness conditions on objs of $D(\mathrm{Bun}_G, \Lambda)$.

- ULA sheaves $\leftarrow\dashrightarrow$ admissible rep'ns

  (Analogy in $G(\mathbb{Q}_p)$-reps:

  $A \in D(G(\mathbb{Q}_p), \Lambda)$ is adm if $\forall K \overset{\text{cos}}{\subseteq} G_b(\mathbb{Q}_p)$, $A^K \in \mathrm{Perf}(\Lambda)$.)
- Compact sheaves $\leftarrow\dashrightarrow$ fin gen'd rep'ns (if $\Lambda = \overline{\mathbb{Q}}_\ell$)

  (Analogy in $G(\mathbb{Q}_p)$-reps:

  rep's gen'd by cpt ind from open subgrps.)

<u>Fact</u> ULA sheaves $\circlearrowleft$ Verdier duality

Compact sheaves $\circlearrowleft$ Bernstein-Zelevinsky duality

Also have Hecke operators acting:

For any $V \in \mathrm{Rep}(\hat{G}_\Lambda)$, get a natural functor

$$T_V : D(\mathrm{Bun}_G, \Lambda) \longrightarrow D(\mathrm{Bun}_G, \Lambda)$$

generates compact & ULA sheaves separately.

<u>Fact</u> If $\Lambda = \overline{\mathbb{Q}}_\ell$, and (G, μ, b) is given.

then $R\Gamma(G,\mu,b)[\rho] = i_1^* T_{V_\mu^\vee} i_{b!} \rho$

$(i_b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$ open, $\mathbb{1} \in B(G))$.

<u>Idea</u> HV conj should follow from introducing some operation

$$\mathrm{Eis} : D(\mathrm{Bun}_M, \Lambda) \longrightarrow D(\mathrm{Bun}_G, \Lambda)$$

(which geometrizing parabolic inductions)

and studying how it interacts with Hecke operators on G & on M.

<u>Tentative plan</u>

<u>Lecture 2</u> Conjectures on Eis

+ <u>easy</u> proof of HV conditionally on those.

<u>Lecture 3</u> Unconditional results of Eis v.s. Hecke operators,

finiteness conditions.

<u>Lecture 4</u> Vanishing conjecture.

Lecture 2: The main Conjecture for Eis

Today
- Background from Lecture 1.
- Conjectures on Eis.
- Conditional reduction of HV conj from conjs on Eis.

Recall p prime, $G/\mathbb{Q}_p$ Conn red grp.

+ assume G quasi-split
$\cup$
$B \supset T$ fixed.

Bun_G = stack of G-bdls on the FF curve.
(smooth Artin stack of dim 0)

& $|\mathrm{Bun}_G| \cong B(G)$ (by Fargues).

$$\begin{array}{ccc} |\mathrm{Bun}_G| & \cong & B(G) \\ \downarrow & & \downarrow \kappa \\ \pi_0(|\mathrm{Bun}_G|) & \cong & \pi_1(G)_\Gamma \end{array} \quad \text{(Fargues - Scholze)}$$

Strata $\mathrm{Bun}_G = \bigcup_{b \in B(G)} \underbrace{\mathrm{Bun}_G^b}$

$\quad (\simeq [*/G_b(\mathbb{Q}_p) \ltimes (\text{unipotent part})]$
smooth of dim $-\langle 2\rho, \nu_b \rangle$

& $i_b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$ locally closed immersion.

Note If b basic, then $i_b : [*/G_b(\mathbb{Q}_p)] \hookrightarrow \mathrm{Bun}_G$
is an open immersion.

Fix $\Lambda = \mathbb{Z}_\ell$-alg. F-S define cat $D(\mathrm{Bun}_G, \Lambda)$ & $D(\mathrm{Bun}_G^b, \Lambda)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cong D(G_b(\mathbb{Q}_p), \Lambda)$.

Have functors

$$\begin{tikzcd}
D(\mathrm{Bun}_G^b, \Lambda) \arrow[r, bend left, "i_{b\#}"] \arrow[r, bend right, "i_{b*}"'] \arrow[r, bend right=60, "i_{b!}"'] & D(\mathrm{Bun}_G, \Lambda) \arrow[l, "i_b^*"'] \arrow[l, bend left=80, "i_b^!"]
\end{tikzcd}$$

Here $i_{b\#} \vdash i_b^* \vdash i_{b*}$ $\left(\underline{\text{note}}\ \ b \text{ basic} \Rightarrow i_b \text{ open} \right.$

& $i_{b!} \vdash i_b^!$. $\left. \qquad \Rightarrow i_{b\#} = i_{b!}\ \ \&\ \ i_b^* = i_b^! . \right)$

Finiteness conditions

<u>Def'n</u> $A \in D(G_b(\mathbb{Q}_p), \Lambda)$ is <u>admissible</u> if

$A^K \in \mathrm{Perf}(\Lambda)$, $\forall$ open cpt subgrp $K \subset G(\mathbb{Q}_p)$.

There is a corresponding notion for sheaves

$\mathcal{F} \in D(\mathrm{Bun}_G, \Lambda)$, called ULA-ness.

<u>Fact</u> $\mathcal{F}$ is ULA $\iff$ $i_b^* \mathcal{F}$ admissible, $\forall b$.

Moreover, if $A \in D(G_b(\mathbb{Q}_p), \Lambda)$ adm,

then $i_{b!} A$ & $i_{b*} A$ are ULA.

Finally, have $R\mathcal{H}om(-, \Lambda) \curvearrowright D(G_b(\mathbb{Q}_p), \Lambda)$ "smooth duality"

$\curvearrowright D(\mathrm{Bun}_G, \Lambda)$ "Verdier duality".

These dualities intertwine $i_{b!}$ with i_{b*} (up to shifts).

& i_b^* with $i_b^!$

They induce self-equivalences satisfying dualities

on cpt & ULA objects.

Facts $A \in D(G_b(\mathbb{Q}_p), \Lambda)$ is compact
if it lies in the triangulated subcat gen'd by
objs of the form $\mathcal{C}_c(K\backslash G_b(\mathbb{Q}_p), \Lambda)$
$K \subset G_b(\mathbb{Q}_p)$ open pro-p cpt subgrp.

<u>On compact objs</u>: have a cohom duality
$$\mathbb{D}_{coh}(A) := R\mathrm{Hom}(A, \mathcal{C}_c(G_b(\mathbb{Q}_p), \Lambda))$$
preserving complexes and satisfying biduality.

- $A \in D(\mathrm{Bun}_G, \Lambda)$ is compact
$\iff i_{b!} i_b^* A$ is compact
and $=0$ for all but fin many b.
(Scholze: "support is quasi-cpt".)

Also on cpt objs, have a duality
$\mathbb{D}_{BZ}$ satisfying biduality.
Finally, for any $A \in D(G_b(\mathbb{Q}_p), \Lambda)$ cpt,
$i_{b*}A$ & $i_{b!}A$ are cpt
and $\mathbb{D}_{BZ}(i_{b!}A) \cong i_{b*}\mathbb{D}_{coh}(A)$.

If $\Lambda = \overline{\mathbb{Q}}_\ell$ and π = some adm irrep of $G_b(\mathbb{Q}_p)$
then $\mathbb{D}_{coh}(\pi) = \underbrace{\mathrm{Zel}(\pi)[-r]}$
"shift of some other irrep σ" ($r \in \mathbb{Z}$).

Fix $P = MN \subset G$ standard parabolic.
<u>Conjecture</u> There should exist a natural functor

$$\mathrm{Eis} = \mathrm{Eis}_P^G : D(\mathrm{Bun}_M, \Lambda) \to D(\mathrm{Bun}_G, \Lambda)$$

satisfying:

(1) (Compatible with parabolic induction)

Given $\pi \in D(M(\mathbb{Q}_p), \Lambda)$, then

$$\mathrm{Eis}_P^G \, i_{1!}\pi \cong i_{1!}\mathrm{Ind}_P^G \pi$$

& This is natural in π.

(2) Eis should be compatible with composition and extension of scalars $\Lambda \to \Lambda'$.

Moreover, if Λ is torsion, then should have

$$\mathrm{Eis} = p_! q^* \text{ for the maps } \begin{tikzcd} & \mathrm{Bun}_P \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathrm{Bun}_M & & \mathrm{Bun}_G \end{tikzcd}$$

(3) (Preservation of ULAness)

If $\mathcal{F} \in D(\mathrm{Bun}_M, \Lambda)$ is ULA with quasi-compact support, then so is $\mathrm{Eis}(\mathcal{F})$.

(Intuition: Ind_P^G preserves admissibility.)

(4) (Preservation of compactness)

If $\mathcal{F} \in D(\mathrm{Bun}_M, \Lambda)$ compact, then so is $\mathrm{Eis}(\mathcal{F})$.

Moreover, should have

$$\mathrm{Eis}_P^G \, \mathbb{D}_{BZ} \mathcal{F} \cong \mathbb{D}_{BZ} \, \mathrm{Eis}_{P^-}^G \mathcal{F},$$

where P^- = the opposite parabolic.

(Intuition: If $\Lambda = \bar{\mathbb{Q}}_\ell$, $A \in D(M(\mathbb{Q}_p), \bar{\mathbb{Q}}_\ell)$ compact,
then so is $\mathrm{Ind}_P^G A$
and $\mathbb{D}_{coh} \mathrm{Ind}_P^G A \simeq \mathrm{Ind}_P^G \mathbb{D}_{coh} A$, by Bernstein.)
This is much harder than preserving admissibility.

(5) (Interaction with Hecke operators)
Eis commutes with Hecke operators in some filtered sense.
(see Lecture 3).

$\rightsquigarrow$ Write Eis' = the renormalized version of Eis defined as follows.
If $\mathcal{F} \in D(\mathrm{Bun}_M, \Lambda)$ lies in a single connected comp,
corresponding to elt $\nu \in X_*(M)_\Gamma$,
then $\quad \mathrm{Eis}'(\mathcal{F}) \cong \mathrm{Eis}(\mathcal{F}[-\langle 2\rho_G - 2\rho_M, \nu\rangle])$
(+ extend additively).
For this variant, have for $V \in \mathrm{Rep}(\hat{G}_\Lambda)$ that
$$[T_V \circ \mathrm{Eis}'] = [\mathrm{Eis}' \circ T_{V|_{\hat{M}}}]$$
as maps $\quad K_0(D(\mathrm{Bun}_M, \Lambda)) \longrightarrow K_0(D(\mathrm{Bun}_G, \Lambda))$
and compatible w/ finiteness conditions.

Final part of conjecture:

(6) (A simple formula in K_0)
Set $D_{fin}(\mathrm{Bun}_G, \Lambda)$ = full subcat of objs that are compact & ULA.
Take $\Lambda = \bar{\mathbb{Q}}_\ell$. Then
$$K_0 D_{fin}(\mathrm{Bun}_G, \Lambda) = \bigoplus_{b \in B(G)} K_0 \underbrace{\mathrm{Rep}(G_b(\mathbb{Q}_p))}_{\text{fin length adm reps } / \bar{\mathbb{Q}}_\ell}$$
$$\mathcal{F} \longmapsto \left(\sum_i (-1)^i H^i(i_b^* \mathcal{F})\right)_b$$

So by (3) (4), Eis induces a map

$$\begin{array}{ccc} [\mathrm{Eis}] : K_0 D_{fin}(\mathrm{Bun}_M, \bar{\mathbb{Q}}_\ell) & \longrightarrow & K_0 D_{fin}(\mathrm{Bun}_G, \bar{\mathbb{Q}}_\ell), \\ \wr\wr & & \wr\wr \\ \bigoplus_{b \in B(M)} K_0 \mathrm{Rep}(M_b(\mathbb{Q}_p)) & \longrightarrow & \bigoplus_{b \in B(G)} K_0 \mathrm{Rep}(G_b(\mathbb{Q}_p)) \end{array}$$

This [Eis] should be explicitly given by sending

$$A \in K_0 \mathrm{Rep}(M_b(\mathbb{Q}_p)) \longmapsto \mathrm{Ind}_{M_b}^{G_b} A \in K_0 \mathrm{Rep}(G_b(\mathbb{Q}_p))$$

and then extending additively.

i.e. $[\mathrm{Eis}] \simeq (\text{extension}) \circ \left(\bigoplus_{b \in B(M)} \mathrm{Ind}_{M_b}^{G_b}\right)$.

<u>Prop</u> This conjecture implies the HV conjecture !

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<u>Sketch</u> Fix $P = MU \subseteq G$,

(G, μ, b) with $b \in M(\mathbb{Q}_p)$ well-chosen.

Want to compute

$$\mathrm{Mant}_{G,\mu,b}(\mathrm{Ind}_{P_b}^{G_b} \rho) \in K_0 \mathrm{Rep}(G_b(\mathbb{Q}_p)) \quad \text{for } \rho \in K_0 \mathrm{Rep}(M_b(\mathbb{Q}_p)).$$

<u>Idea</u> $\mathrm{Mant}_{G,\mu,b} = [i_1^* T_{V_\mu^*} i_{b!}]$.

$\rightsquigarrow$ Want to compute $[T_{V_\mu^*} \circ \mathrm{Eis}(i_{b!}^M \rho)]$ in two different ways

($i_b^M : \mathrm{Bun}_M^b \hookrightarrow \mathrm{Bun}_M$.

Lecture 3: Unconditional commutativity & finiteness conditions

Last time Conjecture on Eis $\Rightarrow$ HV conjecture.

Fix (G,μ,b). $P = MU$ with $b \in M(\breve{\mathbb{Q}}_p)$
(as in statement of HV conj).

Recall Conjectures predict:

$$\exists\ \mathrm{Eis} : D(\mathrm{Bun}_M, \bar{\mathbb{Q}}_\ell) \longrightarrow D(\mathrm{Bun}_G, \bar{\mathbb{Q}}_\ell)$$

which preserve finiteness properties

+ satisfying $[T_V \circ \mathrm{Eis}] = [\mathrm{Eis} \circ T_V|_{\hat{M}}]$
("comm" w/ Hecke operator)

and s.t. the diagram

$$\begin{array}{ccc}
K_0 D_{\mathrm{fin}}(\mathrm{Bun}_M, \bar{\mathbb{Q}}_\ell) & \xrightarrow{[\mathrm{Eis}]} & K_0 D_{\mathrm{fin}}(\mathrm{Bun}_G, \bar{\mathbb{Q}}_\ell), \\
\wr\| & \circlearrowleft & \wr\| \\
\bigoplus_{b \in B(M)} K_0 \mathrm{Rep}(M_b(\mathbb{Q}_p)) & \xrightarrow{\mathrm{eis}} & \bigoplus_{b \in B(G)} K_0 \mathrm{Rep}(G_b(\mathbb{Q}_p)) \\
\mathcal{F}_b & \longmapsto & \mathrm{Ind}_{M_b}^{G_b} \mathcal{F}_b .
\end{array}$$

How to get HV from this?

Pick some $\rho \in \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(M_b(\mathbb{Q}_p))$, and evaluate the identity

$$[T_{V_\mu} \circ \mathrm{Eis}] = [\mathrm{Eis} \circ T_{V_\mu^*}|_{\hat{M}}] \quad \text{on } i_{b!}^M \rho .$$

$\rightsquigarrow$ LHS becomes $[T_{V_\mu} \circ \mathrm{Eis}(i_{b!}^M \rho)] = [T_{V_\mu^*}] \circ \mathrm{eis}[i_{b!}^M \rho]$

$$= [T_{V_\mu^*}] \circ i_b^G \mathrm{Ind}_{P_b}^{G_b} \rho .$$

& RHS becomes $\mathrm{eis}[T_{V_\mu^*}|_{\hat{M}}\, i_{b!}^M \rho] = \mathrm{eis} \sum_{\nu \in \mathcal{W}_G^M(\mu)} [T_{W_\nu} \circ i_{b!}^M \rho]$, where $\mathcal{W}_G^M(\mu)$ = multiset of $\nu \in X_*(T)_{M\text{-dom}}$'s

with multiplicity = $\dim \mathrm{Hom}(\mathcal{W}_\nu, T_\mu(\hat{M}))$.

In other words, $V_\mu|_{\hat{M}} = \bigoplus_{\nu \in \mathcal{W}_G^M(\mu)} \mathcal{W}_\nu$.

Now project everything to

$K_0 \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G(\mathbb{Q}_p))$ (summand corresp. to $b = 1$).

$\rightsquigarrow$ LHS becomes

$$[i_1^* T_{V_\mu^*} i_{b!}^G \mathrm{Ind}_{P_b}^{G_b} \rho] = \mathrm{Mant}_{G,\mu,b} \mathrm{Ind}_{P_b}^{G_b} \rho$$

& RHS becomes

$$\mathrm{Ind}_P^G \sum_{\nu \in \mathcal{W}_G^M(\mu)} [i_1^{M*} T_{\mathcal{W}_\nu} i_{b!}^M \rho] = \mathrm{Ind}_P^G \sum_{\nu \in \mathcal{W}_G^M(\mu)} \mathrm{Mant}_{M,\mu,b}(\rho)$$

$$= \mathrm{Ind}_P^G \sum_{\nu \in \mathcal{W}_G^M(\mu,b)} \mathrm{Mant}_{M,\mu,b}(\rho)$$

<u>Remaining lecture</u> Towards an <u>unconditional</u> proof of HV.

- Reduction to $\mathbb{Z}_\ell$-coeffs
  $\rightsquigarrow$ can define (a variant of) Eis unconditionally.
- Why should Eis preserve finiteness properties?
- "Combinatorial analysis" of Eis.

Eventual outcome of all this:

The HV conj is (unconditionally)

a consequence of the following Conj.

<u>Conj</u> (i) For any $b \in B(G)$, the function

$$i_{b\#} : D(G_b(\mathbb{Q}_p), \mathbb{Z}_\ell) \longrightarrow D(\mathrm{Bun}_G, \mathbb{Z}_\ell)$$

sends admissible objs to ULA objs.

($\Rightarrow$ $i_{b\#}$ sends adm & cpt objs to ULA & cpt objs)

(2) For any $b' \neq b \in B(G)$, the induced map

$$D_{cpt,adm}(G_b(\mathbb{Q}_p), \mathbb{Z}_\ell) \longrightarrow K_0 \operatorname{Rep}_{\overline{\mathbb{Q}}_\ell}(G_{b'}(\mathbb{Q}_p))$$

$$A \longmapsto [i_{b'}^* i_{b\#} A \otimes \mathbb{Q}_\ell]$$

is identically zero.

<u>Rmk</u> Hansen checked this conjecture by hand for $G = GL_n$ ($n \leq 4$)

$\Rightarrow$ HV is true in these cases.

(Method is very painful; does not geometrize)

<u>Reduction to integral coefficients</u>

<u>Def'n</u> Let π be an adm $\overline{\mathbb{Q}}_\ell$-rep of $G(\mathbb{Q}_p)$.

A <u>lattice</u> in π is a free $\overline{\mathbb{Z}}_\ell$-mod $\pi^\circ \subset \pi$

stable under $G(\mathbb{Q}_p)$-action

& containing a $\overline{\mathbb{Q}}_\ell$-basis of π.

This is a well-behaved notion b/c If π has fin length,

then
- any two π°'s are commeasurable,
- π° is fin gen'd,

and
- $(\pi^\circ \otimes \overline{\mathbb{Z}}_\ell / \underset{\text{max ideal}}{\mathfrak{m}})^{ss}$ depends only on π.

<u>Example</u> (1) Suppose π is irred & supercuspidal.

Then π admits a lattice

$\iff \omega_\pi : Z_G(\mathbb{Q}_p) \longrightarrow \bar{\mathbb{Q}}_\ell^\times$ has image in $\bar{\mathbb{Z}}_\ell^\times$.

(2) For any irred π, π admits a lattice

$\iff \varphi_\pi : W_{\mathbb{Q}_p} \longrightarrow {}^LG(\bar{\mathbb{Q}}_\ell)$ factors over (a conjugate of) ${}^LG(\bar{\mathbb{Z}}_\ell)$.

ss L-param (constructed by Fargues-Scholze)

Note $\Rightarrow$: Easy.

$\Leftarrow$: uses Dat's theory of v-tempered rep'ns.

Thm (Reduction) In proving the HV conj, it is enough to prove it for all $\rho \in \mathrm{Irr}_{\bar{\mathbb{Q}}_\ell}(M_b(\mathbb{Q}_p))$ which admit a lattice.

Intuition Both sides of the HV conj are "continuous" fcts of ρ, and the ρ's which admit lattices are "dense" in the set of all ρ's.

Notation Let $K_0 G := K_0 \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(G(\mathbb{Q}_p))$.

Recall For any $f \in \mathcal{C}_c(G(\mathbb{Q}_p), \bar{\mathbb{Q}}_\ell)$ and any $\pi \in \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(G(\mathbb{Q}_p))$, can form

$$\mathrm{tr}(f \mid \pi) = \int_{G(\mathbb{Q}_p)} f(g)\, \Theta_\pi(g)\, dg \in \bar{\mathbb{Q}}_\ell.$$

This extends to a linear form

$$\mathrm{tr}(f \mid -) : K_0 G \longrightarrow \bar{\mathbb{Q}}_\ell$$

called trace form.

<u>Fact</u> $\exists$ enough trace forms in the sense that

if $\ell(\pi) = 0$ for all trace forms ℓ, then $\pi = 0$.

<u>Def'n</u> Say functor $\alpha : K_0 G \to K_0 H$ is <u>continuous</u>

if $\forall$ trace form $\ell : K_0 H \to \overline{\mathbb{Q}}_\ell$, $\ell \circ \alpha$ is a trace form on $K_0 G$.

<u>Example</u> If $P = MU \subset G$, then

$\mathrm{Ind}_P^G : K_0 M \to K_0 G$, $\mathrm{Jac}_G^P : K_0 G \to K_0 M$

(Jacquet module

are continuous in this sense.

<u>Prop A</u> (Hansen-Kaletha-Weinstein)

For any (G, μ, b), any $f \in \mathcal{C}_c(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ and any $w \in W_E$,

the linear form $K_0 G_b \longrightarrow \overline{\mathbb{Q}}_\ell$

$$\rho \longmapsto \mathrm{tr}(f \circ w \mid \mathrm{Mant}_{G,\mu,b}(\rho))$$

for some $f \in \mathcal{C}_c(G_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ is a trace form.

Taking $w = 1 \in W_E$, get:

$\mathrm{Mant}_{G,\mu,b} : K_0 G_b \longrightarrow K_0 G$ is continuous.

<u>Prop B</u> (Hansen-Kaletha-Weinstein)

If $\ell : K_0 G \longrightarrow \overline{\mathbb{Q}}_\ell$ is a trace form

s.t. $\ell(\pi) = 0$, $\forall \pi \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G(\mathbb{Q}_p))$ admitting a lattice,

then $\ell = 0$ ("density").

<u>Pf of Prop A</u> Uses trace Paley-Wiener theorem + the FS machinery.

<u>Pf of Prop B</u> Uses an ℓ-adic analogue of Langlands classification due to Dat.

<u>Proof of Reduction thm</u>

Consider the map

$$h : K_0 M_b \longrightarrow K_0 G$$

$$\rho \longmapsto \mathrm{Mant}_{G,\mu,b} \, \mathrm{Ind}_{P_b}^{G_b} \rho - (-1)^* \sum_{\nu \in W_G^M(\mu, b)} \mathrm{Ind}_P^G \mathrm{Mant}_{M,\mu,b}(\rho)$$

By assumption,

$h(\rho) = 0$ if ρ admits a lattice.

<u>Want</u> $h = 0$.

ETS: $\forall$ trace form $\ell : K_0 G \longrightarrow \bar{\mathbb{Q}}_\ell$, $\ell \circ h = 0$.

For any such ℓ, $\ell \circ h : K_0 M_b \to \bar{\mathbb{Q}}_\ell$ is a trace form by Prop A.

Since $(\ell \circ h)(\rho) = 0$ for all ρ admitting a lattice.

$\Rightarrow$ $\ell \circ h = 0$ by Prop B.

$\Rightarrow$ $(\ell \circ h)(\rho) = 0$ for any ρ & any ℓ.

$\Rightarrow$ $h(\rho) = 0$ for any ρ. $\square$

By further small reductions, can work for $\mathbb{Q}_E$-coeffs (not only $\bar{\mathbb{Z}}_\ell$-coeffs, $E/\mathbb{Q}_\ell$ finite).

Why helpful?

After all, def'n of $\mathrm{Eis}: D(\mathrm{Bun}_M, \mathcal{O}_E) \to D(\mathrm{Bun}_G, \mathcal{O}_E)$ is still missing.

<u>Point</u> After passing to ℓ-adically complete objects, can define Eis unconditionally.

<u>Def'n</u> Fix ℓ prime.

$$\hat{D}(\mathrm{Bun}_G, \mathcal{O}_E) := \varprojlim_n D(\mathrm{Bun}_G, \mathcal{O}_E/\ell^n).$$

($D_{\acute{e}t}(\mathrm{Bun}_G, \mathcal{O}_E)$ in Fargues–Scholze

Then $\exists$ a natural functor

$$\gamma: D(\mathrm{Bun}_G, \mathcal{O}_E) \longrightarrow \hat{D}(\mathrm{Bun}_G, \mathcal{O}_E)$$
$$\mathcal{F} \longmapsto \varprojlim_n (\mathcal{F} \otimes^{\mathbb{L}} \mathcal{O}_E/\ell^n)$$

which commutes w/ colims + admits a right adjoint δ (by Lurie's derived functor thm).

In general, γ and δ can throw away information.
However, for ULA objs, everything behaves as well as possible.

<u>Prop</u> γ, δ are mutually inverse equivalences of cats

$$\begin{tikzcd} D(\mathrm{Bun}_G, \mathcal{O}_E)_{\mathrm{ULA}} \arrow[r, bend left, "\gamma"] & \hat{D}(\mathrm{Bun}_G, \mathcal{O}_E)_{\mathrm{ULA}} \arrow[l, bend left, "\delta"] \end{tikzcd}.$$

<u>Idea of pf</u> This can be checked on strata, using
<u>Fact</u> (A nontrivial result in FS) $\forall$ open $j: U \hookrightarrow \mathrm{Bun}_G$,
j_* preserves ULA objs in both settings and are compatible. $\square$

In the $\hat{D}$ ($= D_{\acute{e}t}$) setting, have a <u>full</u> 6-functor formalism.

$\rightsquigarrow$ Can define

$$\widehat{\mathrm{Eis}} : \hat{D}(\mathrm{Bun}_M, \mathbb{Q}_E) \longrightarrow \hat{D}(\mathrm{Bun}_G, \mathbb{Q}_E)$$
$$\mathcal{F} \longmapsto p_! q^* \mathcal{F},$$

$$\begin{tikzcd} & \mathrm{Bun}_P \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathrm{Bun}_M & & \mathrm{Bun}_G . \end{tikzcd}$$

Conj $\widehat{\mathrm{Eis}}$ preserves ULA objs, and

the "true" Eis functor on ULA objs agrees with $\delta_G \circ \widehat{\mathrm{Eis}} \circ \gamma_M$.

<u>Strategy for HV conj</u>

(1) Prove that $\widehat{\mathrm{Eis}}$ preserves ULA objs.

(2) Prove that $\widehat{\mathrm{Eis}}$ interacts with Hecke operators in the expected "comm" way.

(3) "Combinatorial analysis":

Reduce the vanishing conj on Eis to the conj on i_{b*} stated earlier.

Lecture 4: More computation for $\widehat{\mathrm{Eis}}$

Plan
- Interaction of $\widehat{\mathrm{Eis}}$ with Hecke operators.
- Why $\widehat{\mathrm{Eis}}$ should preserve ULA objs.
- No Combinatorics!

  Rather, discuss relationship between $\widehat{\mathrm{Eis}}$ & $i_{b\#}$ when $G = GL_2$.

Recall G quasi-split $\supset P = MU$.

$$\begin{tikzcd} & \mathrm{Bun}_P \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathrm{Bun}_M & & \mathrm{Bun}_G \end{tikzcd} \quad \rightsquigarrow \quad \widehat{\mathrm{Eis}} : \widehat{D}(\mathrm{Bun}_M, \mathcal{O}_E) \longrightarrow \widehat{D}(\mathrm{Bun}_G, \mathcal{O}_E).$$
$$\mathcal{F} \longmapsto p_! q^* \mathcal{F}.$$

Interaction of $\widehat{\mathrm{Eis}}$ with Hecke operators

For simplicity, assume G split.

Let $V \in \mathrm{Rep}_{\mathcal{O}_E}(\hat{G})$ be some alg rep of $\hat{G} / \mathcal{O}_E$.

$\rightsquigarrow$ Get Hecke operator
$$T_V : \widehat{D}(\mathrm{Bun}_G, \mathcal{O}_E) \longrightarrow \widehat{D}(\mathrm{Bun}_G, \mathcal{O}_E)^{BW_{\mathbb{Q}_p}}.$$

For any such V, $\exists$ (many) filtrations
$$0 = V_0 \subset V_1 \subset \cdots \subset V_k = V|_{\hat{M}}$$
with property that each $W_i = V_i / V_{i-1} \in \mathrm{Rep}_{\mathcal{O}_E}(\hat{M})$

is α_i-isotypic for some $\alpha \in X^*(Z(\hat{M})^\Gamma) \cong \pi_1(M)_\Gamma$.

Thm Notations as above, the fil'n can be chosen s.t. $T_V \circ \widehat{\mathrm{Eis}}$ admits a fil'n

with graded pieces $\widehat{\mathrm{Eis}} \circ T_{W_i}[-d_i](-\frac{d_i}{2})$

where $d_i = \langle \alpha_i, 2\rho_G - 2\rho_M \rangle$.

$$\Rightarrow \quad [T_V \circ \widehat{\mathrm{Eis}}] = \sum_{i=0}^{k} [\widehat{\mathrm{Eis}} \circ T_{W_i}[-d_i](-\tfrac{d_i}{2})].$$

$\widehat{\mathrm{Eis}}$ v.s. ULA sheaves

ULA sheaves : setups

"smooth-locally nice"

Let $f: X \to Y$ be a "reasonable" map of Artin stacks.

$\rightsquigarrow$ Have $f_!$, $f^!$ with usual properties.

Fix a prime ℓ.

Def'n $\mathcal{F} \in \hat{D}(X, \mathcal{O}_E)$ is called f-ULA (universally locally acyclic)

if (i) $\mathcal{F}$ is "overconvergent",

i.e. $\forall\, x : \mathrm{Spa}(C, C^+) \hookrightarrow X$,

$x^* \mathcal{F} \otimes^{\mathbb{L}} \mathcal{O}_E/\ell \in D(\mathrm{Spa}(C, C^+), \mathcal{O}_E/\ell)$ is constant.

(ii) $R\mathcal{H}om(\mathcal{F}, f^!(-)) : D(Y, \mathcal{O}_E) \to D(X, \mathcal{O}_E)$

commutes with all direct sums,

and still after any base change on Y.

This is some kind of finiteness condition:

Example (i) If $Y = \mathrm{Spa}(C, \mathcal{O}_C)$ geom pt,

$X = \mathfrak{X}^{an}$, $\mathfrak{X}$ some alg var $/C$,

then for any $\mathcal{F} \in D^b_c(\mathcal{X}, \mathcal{O}_E)$, $\mathcal{F}^{an} \in D(X, \mathcal{O}_E)$ is f-ULA
(for $f: X \to Y$ str map).

(2) If $f: X = \mathrm{Bun}_G \longrightarrow Y = \mathrm{Spd}\, \bar{\mathbb{F}}_p$, then
$\mathcal{F} \in \hat{D}(\mathrm{Bun}_G, \mathcal{O}_E)$ is f-ULA
$\iff$ all stalks $\otimes^{\mathbb{L}} \mathcal{O}_E/\ell$ are admissible.

Note (1) f-ULAness is sm-local on the source
& v-locally on the target.

(2) f-ULAness is stable under sm pullback and proper pushforward
i.e. for $X \xrightarrow{g} Y \xrightarrow{f} Z$,

- if $\mathcal{F}$ f-ULA on Y, g sm, then $g^*\mathcal{F}$ is $g \circ f$-ULA
- if $\mathcal{F}$ $f \circ g$-ULA on X, g proper, then $g_*\mathcal{F}$ is f-ULA.

(2') For $X \xrightarrow{g} Y \xrightarrow{f} Z$,
given $\mathcal{F}$ on Y f-ULA & $\mathcal{G}$ on X g-ULA,
then $\mathcal{G} \otimes g^*\mathcal{F}$ is $f \circ g$-ULA.

Back to Eis

$$\begin{tikzcd} & \mathrm{Bun}_P \arrow[dl, bend right, "q"'] \arrow[dr, bend left, "p"] & \\ \mathrm{Bun}_M \arrow[dr, dashed, "\pi_M"'] & & \mathrm{Bun}_G \arrow[dl, dashed, "\pi_G"] \\ & \mathrm{Spd}\, \bar{\mathbb{F}}_p & \end{tikzcd} \quad \rightsquigarrow \quad \widehat{\mathrm{Eis}} = p_! q^*.$$

Q Does $\widehat{\mathrm{Eis}}$ preserve ULAness ?

A Yes, if q smooth!
No always, if p not proper!

Remedy strategy Enlarge the diagram above by "compactifying" the morphism p.

$$\begin{tikzcd} & \widetilde{Bun_P} \arrow[dl, bend right, "\tilde{q}"'] \arrow[dr, bend left, "\tilde{p}"] & \\ & Bun_P \arrow[u, hook, "j"] \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Bun_M & & Bun_G \end{tikzcd}$$

Here $\widetilde{Bun_P}$ = the so-called Drinfeld cpt'n of Bun_P.

<u>Conj</u> (1) $\tilde{p}$ is proper (on individual conn comps of $\widetilde{Bun_P}$)

(2) $j_! \mathcal{O}_E$ is $\tilde{q}$-ULA.

<u>Prop</u> This conj $\Rightarrow$ $\widehat{Eis}$ preserves ULA objs with qc supports.

<u>Proof</u> Have $\widehat{Eis}(\mathcal{F}) = p_! q^* \mathcal{F} \cong \tilde{p}_! j_! q^* \mathcal{F}$

(by commutativity)

$\cong \tilde{p}_! (\tilde{q}^* \mathcal{F} \otimes j_! \mathcal{O}_E)$.

(b/c both vanish on boundary of $\widetilde{Bun_P}$)

$\cong \tilde{p}_* (\tilde{q}^* \mathcal{F} \otimes j_! \mathcal{O}_E)$

(by Conj (1), $\tilde{p}$ proper).

But now, if $\mathcal{F}$ is π_M-ULA (& has qc support),

so then $\tilde{q}^* \mathcal{F} \otimes j_! \mathcal{O}_E$ is $\pi_M \circ \tilde{q}$ - ULA

(by Conj (2) + property (2') before).

$\Rightarrow$ $\tilde{q}^* \mathcal{F} \otimes j_! \mathcal{O}_E$ is $\pi_G \circ \tilde{p}$ - ULA

(by commutativity).

Then $\tilde{p}_* (\tilde{q}^* \mathcal{F} \otimes j_! \mathcal{O}_E)$ is π_G - ULA

(proper pushforward). $\square$

Rmks · Conj (2) is true in the classical setup (Braverman - Gaitsgory).

Should be within reach, by studying "local models" for $\tilde{q}$.

· Conj (1) is true for $P=B$.

The case of $G = GL_2$

B upper triangular, $T \subset B$ diagonal.

$$\rightsquigarrow \quad \begin{tikzcd} & \mathrm{Bun}_B \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathrm{Bun}_T & & \mathrm{Bun}_G \end{tikzcd}$$

$$\mathrm{Bun}_T = \coprod_{b \in B(T) \cong \mathbb{Z}^2} [*/T(\mathbb{Q}_p)].$$

$$b \in B(T) \Leftrightarrow (m,n) \in \mathbb{Z}^2 \Leftrightarrow \mathcal{O}(m) \oplus \mathcal{O}(n).$$

Q Suppose $\mathcal{F} \in D(\mathrm{Bun}_T, \overline{\mathbb{Q}}_\ell)$ is concentrated on one stratum, say corresp. to $(m,n) \in \mathbb{Z}^2$.

What is $\mathrm{Eis}(\mathcal{F})$ in this case?

Three cases

(i) $m > n$. Then any B-bdl with a reduction to $\mathcal{O}(m) \oplus \mathcal{O}(n)$ is actually isom to $\mathcal{O}(m) \oplus \mathcal{O}(n)$.

The fibre $\mathrm{Bun}_B^{(m,n)}$ of q over (m,n)-stratum is

$$[*/\mathrm{Aut}(\mathcal{O}(m) \oplus \mathcal{O}(n))] = [*/T(\mathbb{Q}_p) \ltimes (\text{unipotent part})].$$

and $\mathrm{Bun}_B^{(m,n)} \xhookrightarrow{p} \mathrm{Bun}_G$ is the inclusion of the HN stratum of Bun_G labelled by (m,n).

$\Rightarrow \quad \mathrm{Eis}(\mathcal{F}) = i_{(m,n),!}\,\mathcal{F}$ for $i_{(m,n)} = p$.

(ii) $m=n$. Then the fibre of q is

$$[*/B(\mathbb{Q}_p)].$$

and p maps this fibre to $[*/GL_2(\mathbb{Q}_p)] \underset{\text{open}}{\subset} \mathrm{Bun}_{GL_2}$.

$\Rightarrow \quad \mathrm{Eis}(\mathcal{F}) = i_{(m,m),!}\,\mathrm{Ind}_B^G\,\mathcal{F}$

(iii) $m<n$. Then the fibre of q is

the moduli space $\mathcal{M}^{(m,n)}$ of diagrams

$$\left(\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{L}' & \longrightarrow & 0 \\ & & \mathrm{rk}\,1 & & \mathrm{rk}\,2 & & \mathrm{rk}\,1 & & \\ & & \deg m & & & & \deg n & & \end{array}\right).$$

$\rightsquigarrow$
$$\begin{tikzcd} & \mathcal{M}^{(m,n)} \arrow[dl, "\text{forget } \mathcal{E}"'] \arrow[dr, "p"] & \\ {[*/T(\mathbb{Q}_p)]} & & \mathrm{Bun}_G \end{tikzcd}$$

forget $\mathcal{E}$ $\qquad$ forget $\mathcal{L}$ & $\mathcal{L}'$.

The image in $|\mathrm{Bun}_G|$ contains the HN (n,m)-stratum

but it also contains all pts

which are "more semistable" than this one.

<u>Obs</u> These spaces $\mathcal{M}^{(m,n)}$ are exactly "local charts" used in FS.

and in this case $p_! q^* \mathcal{F} = i_{(n,m),*}\mathcal{F}$ (up to shift).

<u>Note</u> For general G,

$$\begin{tikzcd} & \mathcal{M}_b \arrow[dl, "q_b"'] \arrow[dr, "p_b"] & \\ {[*/G_b(\mathbb{Q}_p)]} & & \mathrm{Bun}_G \end{tikzcd}$$

$\rightsquigarrow \quad i_{b*} = p_{b,!}\, q_b^*$ up to shift.