Perfectoid spaces

Peter Scholze

(2011, at IAS)

(Related work of Kedlaya-Liu).

Fix a nonarch field K of char 0,

complete w.r.t. $|\cdot| : K \to \mathbb{R}_{\geq 0}$ non-discrete

s.t. residue field of char $p > 0$

$+ \ \Phi : K^+/p \longrightarrow K^+/p$, $x \mapsto x^p$ surjective.

(K deeply ramified field).

E.g. $K = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$, $\widehat{\mathbb{Q}_p(\mu_{p^\infty})}$, $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$.

Let $K' := \mathrm{Frac}(\varprojlim_\Phi K^+/p) = \varprojlim_\Phi K \longrightarrow K$.

$x' \in K' \longmapsto [x] \in K$ Teichmüller lift.

$|\cdot| : K' \to \mathbb{R}_{\geq 0}$ s.t. $|x'| = |[x']|$

Fix $\varpi' \in K'$ s.t. $1 > |\varpi'| \geq |p|$.

let $\varpi = [\varpi'] \in K \rightsquigarrow \varpi^{1/p^n} = [\varpi'^{1/p^n}] \in K$.

<u>Thm</u> (Fontaine-Winterbeger, Gabber-Ramero)

$$\mathrm{Gal}(\overline{K'}/K') \cong \mathrm{Gal}(\overline{K}/K).$$

<u>Idea</u> $\{\text{finite étale } K'\text{-alg}\} \cong \{\text{almost fin ét } K'^{+}\text{-alg}\}$

$\cong \{\text{almost fin ét } K'^{+}/\varpi'\text{-alg}\}$

$\cong \{\text{almost fin ét } K^{+}/\varpi\text{-alg}\}$

$\cong$ {almost fin ét K^+-alg}

$\cong$ {fin ét K-alg}.

"almost" = up to m_K-torsion.

(max ideal of K^+

($m_K^2 = m_K$ b/c $|\cdot|$ on K non-discrete).

Def'n A perfectoid K-alg is a top K-alg R s.t.

R^+ p-adically complete / separated

+ $\Phi : R^+/\varpi^{1/p} \xrightarrow{\sim} R^+/\varpi$ isom

($\Leftrightarrow$ R^+ perfect)

+ $R = R^+[p^{-1}]$ with induced top.

Same def'n works / K'. Then

Thm (Tilting) {perf'd K-alg} $\cong$ {perf'd K'-alg}

$R \longmapsto (\varprojlim R^+/\varpi)[\varpi^{-1}]$.

e.g. $K \mapsto K'$, any L/K finite $\mapsto L'$ as above.

$\rightsquigarrow$ $\hat{\bar{K}} \mapsto \hat{\bar{K}}'$.

$K\langle T^{1/p^\infty}\rangle \mapsto K'\langle T^{1/p^\infty}\rangle$.

In the following, need finiteness assumption (*).

To perfectoid R, associate "adic spectrum"

$X = \mathrm{Spa}\, R = \{$equiv class of cont val'n $v : R \to \Gamma \cup \{0\}\}$

top gen'd by rat'l subset $\mathcal{U} = \{v \in X \mid v(f_i) \le v(g),\ \forall i\}$
for $f_1, \dots, f_n, g \in R$ s.t. $(f_1, \dots, f_n) = R$.

$$\mathcal{O}_X(\mathcal{U}) = R\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \rangle.$$

$\rightsquigarrow$ Similarly, X' & $\mathcal{O}_{X'}$ constructed.

There is a map $R' \longrightarrow R$, $f' \longmapsto [f']$
$$R' = \varprojlim_{x \mapsto x^p} R$$

It induces $X \longrightarrow X'$ via $v \longmapsto v'$,
where v' s.t. $v'(f') = v([f'])$.

<u>Thm</u> $X \simeq X'$ homeomorphism, $\mathcal{O}_X, \mathcal{O}_{X'}$ sheaves.
If $\mathcal{U}' \subseteq X'$ rat'l subset, then
$\mathcal{O}_X(\mathcal{U}')$ & $\mathcal{O}_{X'}(\mathcal{U}')$ perfectoid algs
& are tilts of each other.

Can glue these affinoids to form general perfectoid spaces.

<u>Étale covers of perfectoid spaces</u>

<u>Prop</u> (Gabber-Ramero) If $\tilde{R}'/R'$ fin ét, R' perf'd K'-alg,
then $\tilde{R}'^{+}/R'^{+}$ almost fin étale.

<u>pf idea</u> $\exists N \gg 0$, $\tilde{R}'^{+}/R'^{+}$ fét up to ϖ'^{N}-torsion
Φ bij $\Rightarrow$ Same up to $\varpi'^{N/p}$-torsion

$\Rightarrow \cdots \Rightarrow$ Same eq to $m_{K'}$-torsion.

Prop $\Rightarrow$ $\tilde{R}'$ perf'd, tilt $\tilde{R}/R$ fet,
then $\tilde{R}^+/R^+$ almost fet.

To any perf'd space X/K with tilt X': get
fully faithful
$$\{Y'/X' \text{ fet}\} \hookrightarrow \{Y/X \text{ fet}\}.$$
Thm of F-W / G-R implies that it is locally an equiv.
$\Rightarrow$ equivalence always.

<u>Thm</u> Let R perf'd K-alg + $(*)$.
Let $\tilde{R}/R$ fin ét. Then $\tilde{R}^+/R^+$ almost fin ét.

Can introduce étale site $X_{ét}$.

<u>Cor</u> $X_{ét} \cong X'_{ét}$

<u>Cor</u> $\varprojlim_{\varphi} (\mathbb{P}^{n}_{K})^{rig}_{ét} \cong (\mathbb{P}^{n}_{K'})^{rig}_{ét}$
$\varphi(x_0 : \cdots : x_n) = (x_0^p : \cdots : x_n^p)$.

Same for toric varieties.

<u>Expect</u> · Also works for (ordinary) abelian vars
$\varprojlim_{\varphi} (X^{ord}_{\mathbb{Q}_p(y_{p^\infty})})_{ét} \cong (X^{ord}_{\mathbb{F}_q((t^{1/p^\infty}))})_{ét}$

denote by
Canonical subgrp

tubular nbhd of ord locus on mod curve

<u>1st application</u> de Rham comparison

(related work of Andreatta - Iovita in crystalline case,

+ Brinon.)

<u>Pro-étale site</u> Assume everything is qcqs.

Let X adic space of fin type / K p-adic field

($\approx$ X rigid space)

(recall: {rigid spaces} $\cong$ {fin-type qs adic spaces}.)

$X_{pro\acute{e}t}$:

- objects : formal filtered inverse lim $\mathcal{U} = \varprojlim_i \mathcal{U}_i \to X$
  of $\mathcal{U}_i \to X$ étale, s.t.
  $\mathcal{U}_i \to \mathcal{U}_j$ fin ét surj for $i,j \gg 0$.
- $\mathcal{U} \to V$ étale morph if $\exists$ $\mathcal{U}_0 \to V_0$ of $\mathcal{U}_0, V_0$ étale / X
  s.t. $\mathcal{U} = \mathcal{U}_0 \times_{V_0} V$.
- $\mathcal{U} \to V$ proétale if $\mathcal{U} = \varprojlim \mathcal{U}_i \to V$
  of $\mathcal{U}_i \to V$ étale,
  s.t. $\mathcal{U}_i \to \mathcal{U}_j$ fin ét surj for $i,j \gg 0$.
- Cover: $\{\mathcal{U}_i\} \to \mathcal{U}$ of $\mathcal{U}_i \to \mathcal{U}$ proét
  s.t. they cover in top sense.

Have sheaves such as $\hat{\mathbb{Z}}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$, $\hat{\mathbb{Q}}_p = \hat{\mathbb{Z}}_p \otimes \mathbb{Q}$.

<u>Prop</u> $X = \mathrm{Spa}\, K$, then

$$X_{pro\acute{e}t} \cong \{\text{profinite subset + cont } G_K\text{-action}\}$$

$$\mathcal{U} \longmapsto \mathcal{U}(\hat{\bar{K}}).$$

The pro-ét maps = $\underline{\underline{\text{open}}}$ & cont G_K-equiv maps.

pro-ét covers = top covers by open cont G_K-equiv maps.

If M top G_K-mod, let

$$\mathcal{F}_M(\mathcal{U}) := \mathrm{Hom}_{\mathrm{cont}, G_K\text{-equiv}}(\mathcal{U}(\hat{\bar{K}}), M)$$

<u>Fact</u> $\mathcal{F}_M$ sheaf on $X_{\mathrm{pro\acute{e}t}}$.

& $H^i_{\mathrm{cont}}(G_K, M) = H^i(X_{\mathrm{pro\acute{e}t}}, \mathcal{F}_M)$

abstract sheaf.

<u>note</u> Before, $\hat{\mathbb{Z}}_p = \mathcal{F}_{(\mathbb{Z}_p, \text{can top})}$.

<u>Come to p-adic Hodge theory</u>

Have map $\nu : X_{\mathrm{pro\acute{e}t}} \longrightarrow X_{\mathrm{\acute{e}t}}$ projection.

Have sheaves of relative periods on $X_{\mathrm{pro\acute{e}t}}$,

e.g. $\mathbb{B}_{dR}^{\nabla +} = \varprojlim_n \left(W(\varprojlim_{\Phi} \nu^* \mathcal{O}_X^+/p)[p^{-1}] \right) / (\ker\theta)^n$

where θ-map given by

$$\theta : W(\varprojlim_{\Phi} \nu^*(\mathcal{O}_X^+/p)) \longrightarrow \hat{\mathcal{O}}_X^+$$
$$\hat{\mathcal{O}}_X^+ = \varprojlim \mathcal{O}_X^+/p^n.$$

& $\mathcal{O}_X^+(\mathcal{U}) := \{ x \in \mathcal{O}_X(\mathcal{U}) \mid v(x) \le 1 \text{ for any } v \in \mathcal{U} \}$.

e.g. $\mathcal{O}\mathbb{B}_{dR}^+ = \varprojlim_n \left(\nu^*\mathcal{O}_X \otimes_{\mathbb{Z}_p} W(\varprojlim_{\Phi} \nu^* \mathcal{O}_X^+/p) \right) / (\ker\theta)^n$.

(integral version).

<u>Prop</u> X sm of dim n.

(1) On affinoid perf'd spaces covering X,

can explicitly describe these sheaves.

(2) Analogue of Poincaré lem:

$$0 \to \mathbb{B}_{dR}^{\nabla+} \to \mathbb{B}_{dR}^{+} \xrightarrow{\nabla} \mathbb{B}_{dR}^{+} \otimes_{\mathcal{O}_X} \Omega_X^1$$
$$\xrightarrow{\nabla} \cdots \to \cdots \xrightarrow{\nabla} \mathbb{B}_{dR}^{+} \otimes_{\mathcal{O}_X} \Omega_X^n \to 0$$

(3) $\nu_* \mathbb{B}_{dR}^+ = \mathcal{O}_X$.

So $H^i(X_{\hat{\bar{K}}, \text{proet}}, \mathbb{B}_{dR}^{\nabla+}) \cong H^i_{dR}(X_{\hat{\bar{K}}, \text{proet}}, \mathbb{B}_{dR}^+)$.

by taking H^i on (2).

For $\nu_{\hat{\bar{K}}} : X_{\hat{\bar{K}}, \text{proet}} \to X_{\hat{\bar{K}}, \text{ét}}$,

Prop $R\nu_{\hat{\bar{K}}, *} \mathbb{B}_{dR, X} = \mathcal{O}_X \hat{\otimes}_{\mathcal{O}_K} \mathbb{B}_{dR, K}$.

Cor X algebraic $\Rightarrow H^i_{dR}(X_{\hat{\bar{K}}, \text{proet}}, \mathbb{B}_{dR}) = H^i_{dR}(X, \mathcal{O}_X) \otimes_K B_{dR}$.

Get a map $H^i_{\text{ét}}(X_{\hat{\bar{K}}}, \mathbb{Q}_p) \to H^i(X_{\hat{\bar{K}}, \text{proet}}, \mathbb{B}_{dR}^{\nabla+}) \cong H^i_{dR}(X) \otimes_{\bar{K}} B_{dR}$.

Usual trick shows this is an isom.

Thm Let A_0 smooth affinoid alg. $S_0 = \operatorname{Spec} A_0$.

$f_0 : X_0 \to S_0$ sm proper.

Let $\mathbb{L}$ de Rham sm $\mathbb{Q}_p$-sheaf on X_0,

with assoc filtered $\mathcal{O}(X_0)$-mod $\mathcal{E}$ + integral connection.

Put $\mathcal{E} = \nu_*(\mathbb{L} \otimes_{\mathbb{Q}_p} \mathbb{B}_{dR})$.

Then $R^i f_* \mathbb{L}$ de Rham assoc to $R^i f_*^{dR} \mathcal{E}$.

2nd application Prove the wt-monodromy conj:

<u>Thm</u> Let $X \subset T$ / K p-adic field
be a sm complete intersection in a proper toric var T.
Then $H^i(X_{\bar{K}}, \mathbb{Q}_\ell)$ $(\ell \neq p)$
satisfies weight monodromy conj
(i.e. pure of wt i).