Classical and categorical local Langlands correspondences

David Hansen

Lecture I

Notations
- E non-arch local field, uniformizer ϖ, res field $\mathbb{F}_q$, $q = p^r$.
- G/E conn red grp (often assumed to be quasi-split).
- Q minimal finite quotient of W_E which acts nontrivially on $\hat{G}$.

$\rightsquigarrow$ ${}^LG := \hat{G} \rtimes Q$.

Local Langlands, vague form

Expect a natural map

$$\Pi(G) \longrightarrow \Phi(G)$$

- $\Pi(G)$ = set of sm irreps of $G(E)$ / $\mathbb{C}$
- $\Phi(G)$ = set of L-parameters $W_E \times SL_2(\mathbb{C}) \to {}^LG$

with finite fibres and explicit image.

Categorical upgrade?

Hope: The classical LLC should be a set-theoretical shadow of some equiv of cats.

Fix $\ell \neq p$, $\bar{\mathbb{Q}}_\ell \simeq \mathbb{C}$.

Def Write $\mathrm{Par}_G = \left(\begin{array}{c}\text{stack of } \ell\text{-adically conts L-parameters} \\ \phi : W_E \to {}^L G(\bar{\mathbb{Q}}_\ell)\end{array}\right)$

$= Z^1(W_E, \hat{G})_{\bar{\mathbb{Q}}_\ell} / \hat{G}$ (Fargues-Scholze).

Thm (Helm, Zhu, Fargues-Scholze)

Par_G is a $\coprod$ of finite Artin stacks,

equi-dim'l of dim 0, locally closed $/ \bar{\mathbb{Q}}_\ell$.

Rmk (Can replace $\bar{\mathbb{Q}}_\ell$ with any $\mathbb{Z}_\ell[\sqrt{q}]$-alg Λ).

X_G^{spec} : coarse moduli Space

$\uparrow$ + canonical quotient $q : \mathrm{Par}_G \to X_G^{spec}$ semisimplification

Spectral Bernstein variety.

(closed pts of X_G^{spec}) = isom classes of semisimple

L-paras $\phi : W_E \to {}^L G(\bar{\mathbb{Q}}_\ell)$

If ϕ is ss, $x_\phi \in X_G^{spec}$ closed pt,

$$q^{-1}(x_\phi) = \bigcup_{\substack{\phi' \text{ with} \\ \phi'^{ss} \simeq \phi}} BS_{\phi'}.$$

$\phi \rightsquigarrow S_\phi = \mathrm{Cent}_{\hat{G}}(\mathrm{im}\,\phi)$.

Example $G = GL_2$, $\hat{G} = GL_2$.

ss L-parameters $\simeq \phi = \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix}$ or ϕ irred supercusp.

with $\chi_1, \chi_2 : W_E \to \bar{\mathbb{Q}}_\ell^\times$.

Note $q^{-1}(x_\phi) = BS_\phi$ "most of the time" (for all G).

For $G = GL_2$, $\varphi^{-1}\left(\begin{pmatrix} x_1 & \\ & x_2 \end{pmatrix}\right) = \mathcal{N}/\hat{G}$,

$\mathcal{N} \subset \mathrm{Lie}\,\hat{G}$ nilpotent cone.

$$\varphi^{-1}\Big(\underbrace{\begin{pmatrix} x_1 & \\ & x_1 \cdot |\cdot| \end{pmatrix}}_{\phi}\Big) \simeq \mathbb{A}^1 / \mathbb{G}_m^2 = \mathbb{G}_m / \mathbb{G}_m^2 = BS_{\phi'}$$

($\phi' = \chi$ Steinberg para).

$\rightsquigarrow pt/\mathbb{G}_m^2 = BS_\phi$.

<u>The automorphic side</u>

Main object:

$\mathrm{Bun}_G :=$ stack of G-bundles on the FF curve

Cheat sheaf:

$$|\mathrm{Bun}_G| \cong B(G) := \text{set of } F\text{-isocrystals w/ } G\text{-structure}$$
$$= G(\breve{E})/{}^{\sigma}G(\breve{E})$$

Fargues: as sets; Viehmann: as top spaces.

$$\begin{tikzcd} B(G) \arrow[r, "\nu"] \arrow[dr, "\kappa"] & \mathcal{N}(G) \\ B(G)_{bas} \arrow[u, hook] & \pi_1(G)_\Gamma \arrow[l, "\sim"'] \end{tikzcd}$$

Fargues-Scholze: $\pi_0(\mathrm{Bun}_G) = \pi_1(G)_\Gamma$

& $\exists$ a unique open stratum in each open component of Bun_G of the form $[*/\underline{G_b(E)}]$

where b is the unique basic elt with the correct κ.

<u>Example</u> Bun_{GL_2}:

b = slope 0 $\quad$ b = slopes 1, −1 $\quad$ b = slopes 2, −2

$[*/\underline{GL_2(E)}] \rightsquigarrow [*/\,?] \rightsquigarrow [*/\,?] \rightsquigarrow \cdots$

b = slope $\frac{1}{2}$ $\quad$ b = slope 1, 0 $\quad$ b = slopes 2, −1

$[*/\underline{D}^\times] \rightsquigarrow [*/\,?] \rightsquigarrow [*/\,?] \rightsquigarrow \cdots$

where D/E quaternion alg

<u>General fact</u> $\mathrm{Bun}_G = \bigcup_{b \in B(G)} \mathrm{Bun}_G^b$

each $\mathrm{Bun}_G^b \simeq [*/\tilde{G}_b]$ with $\tilde{G}_b = \underset{\substack{\uparrow \\ \text{conn unipotent grp } v\text{-sheaf}}}{\tilde{G}_b^\circ} \rtimes \underline{G_b(E)}$.

& $\dim \mathrm{Bun}_G^b = -\langle 2\rho_G, \nu_b \rangle$.

Fix Λ a $\mathbb{Z}_\ell$-alg. FS define a cat of sheaves

$$D(\mathrm{Bun}_G, \Lambda)$$

glued semi-orthogonally from

$$D(\mathrm{Bun}_G^b, \Lambda) \underset{\substack{\uparrow \\ \text{Symm monoidal t-exact}}}{\cong} D(G_b(E), \Lambda).$$

For any b, have functors $(i_b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G)$

$$\dot{i}_b^* : D(\mathrm{Bun}_G, \Lambda) \to D(G_b(E), \Lambda)$$

$$\dot{i}_{b!} : D(G_b(E), \Lambda) \to D(\mathrm{Bun}_G, \Lambda)$$

Have adjoints

$$\dot{i}_{b!} \dashv \dot{i}_b^! , \quad \dot{i}_{b\natural} \dashv \dot{i}_b^* \dashv \dot{i}_{b*} .$$

Have natural transformations

$$\dot{i}_{b\natural} \to \dot{i}_{b!} \to \dot{i}_{b*}$$

with pullback to id along $\dot{i}_b^*$.

- $\dot{i}_{b\natural}$: supp at pts <u>less</u> special than b
- $\dot{i}_{b!}$: supp at b
- $\dot{i}_{b*}$: supp at pts <u>more</u> specified than b.

Renormalization For any b, $? \in \{\natural, !, *\}$, define

$$i_{b?}^{\text{ren}} A := i_{b?}(A \otimes \delta_b^{1/2})[-\langle 2\rho_G, \nu_b \rangle].$$

Likewise for $? \in \{!, *\}$ define

$$i_b^{?\,\text{ren}} A := (\delta_b^{-1/2} \otimes i_b^? A)[\langle 2\rho_G, \nu_b \rangle].$$

Here δ_b = modulus character of the parabolic assoc with ν_b.

- This changes nothing if b basic.

Finiteness and duality

The conditions for $A \in D(G_b(E), \Lambda)$:

- A is ULA, if $\forall K \subset G_b(E)$ pro-p open cpt, admissibility $A^K \in \mathrm{Perf}(\Lambda)$.
- A is Compact iff A lies in the thick triangulated subcat given by $\text{c-Ind}_K^{G_b(E)} \Lambda$, $K \subset G_b(E)$ pro-p open cpt.

Have $\mathbb{D}_{sm} \curvearrowright D(G_b(E), \Lambda)$ preserves $D(G_b(E), \Lambda)^{ULA}$

$\mathbb{D}_{sm} = R\mathcal{H}om(-, \Lambda)$

& $\mathbb{D}_{coh} \curvearrowright D(G_b(E), \Lambda)$ preserves $D(G_b(E), \Lambda)^{\omega}$

$\mathbb{D}_{coh} = R\mathrm{Hom}(-, C_c^\infty(G_b(E), \Lambda))$.

Def / Thm (1) $A \in D(\mathrm{Bun}_G)$ is ULA $\iff \forall b$, $i_b^* A$ is ULA

(2) $A \in D(\mathrm{Bun}_G)$ is compact

$\iff i_b^* A = 0$ and $i_b^* A$ cpt, $\forall\forall b$.

(3) $i_{b\natural}$, $i_{b!}$ preserve cptness.

(4) $i_{b!}$, i_{b*} preserve ULAness.

Let $\pi : \mathrm{Bun}_G \to *$. Have

$$\mathbb{D}_{\mathrm{rel}} \circlearrowright D(\mathrm{Bun}_G, \Lambda)^{\mathrm{ULA}}$$

$$\mathbb{D}_{BZ} \circlearrowright D(\mathrm{Bun}_G, \Lambda)^{\omega}$$ characterized by

$$R\mathrm{Hom}(\mathbb{D}_{BZ}(A), B) \cong \pi_{\natural}(A \otimes B).$$

<u>Categorical local Langlands conj</u> (FS)

Assume G quasi-split. Then there should be a canonical equiv of cats:

$$D(\mathrm{Bun}_G, \bar{\mathbb{Q}}_\ell)^{\omega} \xrightarrow{\sim} \mathrm{Coh}(\mathrm{Par}_G) \overset{\shortparallel}{} D^{b,qc}_{coh}$$

depending only on a choice of Whittaker datum.

* <u>Where we are going?</u>

- Construct the functor which should realize this equiv
- Formulate a variant "with restricted variation".
- In this variant, formulate a conjectural matching of t-strs which on hearts is understoodable w/ classical LLC.

Lecture II

For simplicity, assume $\Lambda = \bar{\mathbb{Q}}_\ell$ in the rest lectures.

$$\begin{tikzcd} \Pi(G) \arrow[r, dashed, leftrightarrow] \arrow[d, dashed, leftrightarrow] & \Phi(G) \arrow[d, dashed, leftrightarrow] \\ D(\mathrm{Bun}_G) \arrow[r, dashed, leftrightarrow, "?"] & \mathrm{Coh}(\mathrm{Par}_G). \end{tikzcd}$$

Fargues-Scholze (first key constr'n)

There is a canonical map

$$\mathcal{O}(\mathrm{Par}_G) \longrightarrow \mathcal{Z}(D(\mathrm{Bun}_G))$$

$\mathcal{O}(\mathrm{Par}_G) \cong \mathcal{O}(X_G^{\mathrm{Spec}})$ global funcs; $\mathcal{Z}(D(\mathrm{Bun}_G))$ categorical center

(note $\mathcal{C}$ cat $\Rightarrow$ $\mathcal{Z}(\mathcal{C})$ = endos of the id functor.)

s.t.

$$\begin{tikzcd} \mathcal{O}(\mathrm{Par}_G) \arrow[r] \arrow[dr, "\Psi_G^b"'] & \mathcal{Z}(D(\mathrm{Bun}_G)) \arrow[d] \arrow[dl, phantom, "\circlearrowright"] \\ & \mathcal{Z}(D(G_b(E))) =: \mathcal{Z}(G_b(E)). \end{tikzcd}$$

$\rightsquigarrow$ Given $\pi \in \Pi(G)$, get $\mathcal{Z}(G) \longrightarrow \mathrm{End}(\pi) = \bar{\mathbb{Q}}_\ell$.

precomposing with Ψ_G^b, get

$$\mathcal{O}(X_G^{\mathrm{Spec}}) \longrightarrow \bar{\mathbb{Q}}_\ell.$$

$\rightsquigarrow$ $x_\pi \in X_G^{\mathrm{Spec}}$ $\iff$ φ_π a ss L-parameter.

<u>Thm</u> (Bernstein) $\exists$ a natural alg var X_{G_b} s.t.

$$\mathcal{O}(X_{G_b}) \cong \mathcal{Z}(G_b(E)).$$

- And each connected comp of X_{G_b} is the quotient of a torus by a finite grp.

- The closed pts on the Berstein var X_{G_b} are isom classes of pairs (M, σ), where $M \subset G_b$ a Levi and $\sigma \in \Pi(M)_{sc}$ irred supercusp rep.

Datum of $\Phi^b_G : \mathcal{O}(\mathrm{Par}_G) \to \mathfrak{Z}(G_b(E))$
$\Leftrightarrow$ datum of a map $\Phi'_G : X_{G_b} \to X^{\mathrm{Spec}}_G$.

<u>Thm</u> (FS) Given $b \in B(G)$, the diagram commutes:

$$\begin{tikzcd}
X_{G_b} \arrow[r, "\Phi'_{G_b}"] \arrow[dr, "\Phi^b_G"'] & X^{\mathrm{Spec}}_{G_b} \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \\
& X^{\mathrm{Spec}}_G
\end{tikzcd}$$

<u>Thm</u> (FS) There is a canonical $\bar{\mathbb{Q}}_\ell$-linear $\otimes$-action of $\mathrm{Perf}(\mathrm{Par}_G)$ on $D(\mathrm{Bun}_G)$,
- compatible with Hecke operators,
- preserving cpt $D(\mathrm{Bun}_G)^\omega$.

<u>Note</u> (K. Zou) Take induced map on $\pi_0 \mathrm{End}_{id}$; recover the map $\mathcal{O}(\mathrm{Par}_G) \to \mathfrak{Z}(D(\mathrm{Bun}_G))$.

<u>Hecke operators</u>

$V \in \mathrm{Rep}(\hat{G}) \overset{[FS]}{\rightsquigarrow} T_V \in \mathrm{End}(D(\mathrm{Bun}_G))$.

On the other hand, have

$$\mathrm{Par}_G \longrightarrow B\hat{G}$$

$$V \in VB(\mathrm{Par}_G) \longleftarrow\!\shortmid V \in \mathrm{Rep}(\hat{G})$$

with compatibility

$$V * (-) \cong T_V(-).$$

Now assume G quasi-split.

Fix $B = T\mathcal{U} \subset G$, $\mathcal{U}$ = unip rad,

$\psi : \mathcal{U}(E) \to \bar{\mathbb{Q}}_\ell^\times$ generic char.

$\rightsquigarrow$ $W_\psi = \text{c-Ind}_{\mathcal{U}(E)}^{G(E)} \psi \curvearrowleft G(E)$, "large" but very reasonable.

It is projective, and $\forall\, \zeta \in \pi_0(X_{\hat{G}})$, $e_\zeta W_\psi$ is compact.

(Chan-Savin, Hansen) (Bushnell-Henniart)

Conj (FS) With the above choices, the functor

$$a_\psi : \mathrm{Perf}^{qc}(\mathrm{Par}_G) \longrightarrow D(\mathrm{Bun}_G)$$

$$\mathcal{F} \longmapsto \mathcal{F} * i_{1!} W_\psi$$

(where $i_1 : [*/\underline{G(E)}] \to \mathrm{Bun}_G$)

is fully faithful, and extends uniquely w/ image in $D(\mathrm{Bun}_G)^\omega$.

$\rightsquigarrow$ Get an equiv of cats

$$\mathbb{L}_\psi : \mathrm{Coh}(\mathrm{Par}_G) \xrightarrow{\sim} D(\mathrm{Bun}_G)^\omega.$$

Rmk (1) For some grps, can prove im $a_\psi \subset D(\mathrm{Bun}_G)^\omega$.

(2) full faithfulness (still totally open)

$$\Rightarrow \mathcal{O}(X_G^{\mathrm{Spec}}) \simeq \mathrm{End}(W_\psi).$$

<u>Next goal</u> Unconditional constr'n of a functor
$$C_\psi : D(\mathrm{Bun}_G) \longrightarrow \mathrm{QCoh}(\mathrm{Par}_G)$$
which is conjecturally $\mathbb{L}_\psi^{\dashv}$.

Par_G is very nice $\Rightarrow$ $\mathrm{Ind\,Perf} = \mathrm{QCoh}$.
Can formally upgrade spectral action to
$$a_\psi : \mathrm{QCoh}(\mathrm{Par}_G) \longrightarrow D(\mathrm{Bun}_G)$$
$$\mathcal{F} \longmapsto \mathcal{F} * i_{1!} W_\psi .$$

⚠ <u>Warning</u> $\mathbb{L}_\psi \neq a_\psi|_{\mathrm{Coh}}$.

Why? $\mathrm{Perf} \to \mathrm{Coh} \to \mathrm{QCoh}$

ind-complete $\rightsquigarrow$ $\mathrm{Ind\,Perf} = \mathrm{QCoh} \xrightarrow{\Xi} \mathrm{Ind\,Coh} \xrightarrow{\Psi} \mathrm{QCoh}$

fully faithful, $\Xi \dashv \Psi$.

$$\begin{tikzcd}
\mathrm{QCoh}(\mathrm{Par}_G) \arrow[r, "\Xi"] \arrow[dd, "a_\psi"'] & \mathrm{IndCoh}(\mathrm{Par}_G) \arrow[r] \arrow[ddl, dashed, "\cong ?"'] & \mathrm{QCoh}(\mathrm{Par}_G) \\
 & \mathrm{Coh}(\mathrm{Par}_G) \arrow[u, phantom, sloped, "\subset"] \arrow[dr, equal] & \\
D(\mathrm{Bun}_G) & D(\mathrm{Bun}_G)^\omega \arrow[l, hook'] \arrow[r, "C_\psi^\omega"'] & \mathrm{Coh}(\mathrm{Par}_G) \arrow[uu, hook]
\end{tikzcd}$$

<u>Def</u> $C_\psi : D(\mathrm{Bun}_G) \longrightarrow \mathrm{QCoh}(\mathrm{Par}_G)$
is the right adjoint of "enhanced Whittaker coefficient".

<u>Prop</u> C_ψ is characterized by the formula:
$\forall\, A \in D(\mathrm{Bun}_G)$, $\mathcal{F} \in \mathrm{QCoh}(\mathrm{Par}_G)$,
$$R\mathrm{Hom}(i_{1!} W_\psi, \mathcal{F} * A) \simeq R\Gamma(\mathrm{Par}_G, \mathcal{F} \otimes C_\psi(A)).$$

Obs Take $\mathcal{F} = \mathcal{O}$, get

$$\mathrm{RHom}(W_\psi, i_1^* A) \cong \mathrm{RHom}(i_{1!} W_\psi, A)$$
$$\cong R\Gamma(\mathrm{Par}_G, C_\psi(A)).$$

space of Whittaker models of $i_1^* A$.

Note C_ψ is "linear over the spectral action",

i.e. $C_\psi(\mathcal{G} * A) \cong \mathcal{G} \otimes C_\psi(A)$,

for $A \in D(\mathrm{Bun}_G)$ & $\mathcal{G} \in \mathrm{QCoh}(\mathrm{Par}_G)$.

Conj The functor C_ψ induces an equiv of cats

$$C_\psi : D(\mathrm{Bun}_G)^\omega \xrightarrow{\sim} \mathrm{Coh}(\mathrm{Par}_G).$$

Restricted variation version:

Def $A \in D(\mathrm{Bun}_G)$ is <u>finite</u> if it is compact & ULA.

$\Leftrightarrow$ $\forall b$, $\bigoplus H^n(i_b^* A)$ has finite length

and $\forall \forall b$, $\bigoplus H^n(i_b^* A) = 0$.

Denote by $D(\mathrm{Bun}_G)_{\mathrm{fin}} = D(\mathrm{Bun}_G)^{\mathrm{ULA}} \cap D(\mathrm{Bun}_G)^\omega$.

Prop $D(\mathrm{Bun}_G)_{\mathrm{fin}}$ is a thick triangulated subcat,

preserved by Hecke operators & $\mathbb{D}_{BZ}$.

If $\pi \in \Pi(G_b)$, $i_{b!}^{\mathrm{ren}} \pi$ and $i_{b*}^{\mathrm{ren}} \pi$ are finite.

Let $\mathrm{Coh}(\mathrm{Par}_G)_{\mathrm{fin}} \subset \mathrm{Coh}(\mathrm{Par}_G)$ full subcat of coh complexes

supported set-theoretically on <u>finitely</u> many fibres of

$q : \mathrm{Par}_G \longrightarrow X_G^{\mathrm{Spec}}$ the canonical quotient.

Conj $C_\psi : D(\mathrm{Bun}_G) \to \mathrm{QCoh}(\mathrm{Par}_G)$ restricts to an equiv of cats

$$C_\psi : D(\mathrm{Bun}_G)_{\mathrm{fin}} \xrightarrow{\sim} \mathrm{Coh}(\mathrm{Par}_G)_{\mathrm{fin}}.$$

<u>Obvious question</u> What does this have to do with the actual local Langlands?

<u>Best hope</u> The categorical equiv is t-exact w.r.t. the perv t-str on $D(\mathrm{Bun}_G)$ and some exotic perv coh t-str on $\mathrm{Coh}(\mathrm{Par}_G)$.

- Irred objs in $\heartsuit$ in Bun_G

  $\xleftrightarrow{\sim} \{(b, \pi)\}$, where $b \in B(G)$, $\pi \in \Pi(G_b)$.
- Irred objs in $\heartsuit$ in Par_G

  $\xleftrightarrow{\sim} \{(\phi, \rho)\}$, where ϕ Frob-ss L-para

  ρ irrep of centralizer S_ϕ.

Def Let ϕ be a ss L-parameter.

(1) ϕ is $\underbrace{\text{generous}}_{\substack{\shortparallel \\ \text{generic + nice to you}}}$ if $q^{-1}(X_\phi) \simeq BS_\phi$.

(2) ϕ is semisimple generic if $L(s, \mathrm{ad}\,\phi)$ is holo at $s=1$.

$\Leftrightarrow$ Par_G is smooth in a Zariski nbhd of $q^{-1}(X_\phi)$.

Rmk [BMO 23] Best hope restricted on irred objs is true!

Lecture III

Classical Langlands

Conj ($B(G)_{bas}$ LLC) Fix any L-parameter $\phi \in \Phi(G)$.

(1) $\forall\, b \in B(G)_{bas}$, $\exists$ a natural finite-to-one map

$$\Pi(G_b) \longrightarrow \Phi(G).$$

Let $\Pi_\phi(G_b)$ be the fiber at $\phi \in \Phi(G)$.

(2) $\exists$ natural bijection

$$\iota_\phi : \coprod_{b \in B(G)_{bas}} \Pi_\phi(G_b) \xrightarrow{\sim} \mathrm{Irr}(S_\phi^\natural)$$

$$S_\phi^\natural = S_\phi / (\hat{G}_{der} \cap S_\phi)^\circ, \quad S_\phi = \mathrm{Cent}_{\hat{G}}(\phi).$$

$$(\rightsquigarrow Z(\hat{G})^\Gamma \to S_\phi^\natural).$$

s.t.

$$\begin{tikzcd}
\coprod_{b \in B(G)_{bas}} \Pi_\phi(G_b) \arrow[r, "\sim"', "\iota_\phi"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Irr}(S_\phi^\natural) \arrow[d] \\
B(G)_{bas} \arrow[r, "\kappa_G", "\sim"'] & X^*(Z(\hat{G})^\Gamma).
\end{tikzcd}$$

obvious projection   Kottwitz isom   induced by restr along $Z(\hat{G})^\Gamma \to S_\phi^\natural$.

(recall $\pi_0(\mathrm{Bun}_G) = \pi_1(G)_\Gamma \cong X^*(Z(\hat{G})^\Gamma)$.)

Example $G = GL_n$, $S_\phi = S_\phi^\natural = \mathbb{G}_m$, $B(G)_{bas} = \mathbb{Z}$.

Each $\Pi_\phi(G_b)$ is a single pt.

*Upgrade (Bertoloni-Meli – Oi)

Thm Fix G quasi-split & pinned.

Assume the $B(G)_{bas}$ LLC is true for G

and for all its standard Levi subgrps.

Then $\exists$ a canonical bij

$$\coprod_{b\in B(G)} \Pi_\phi(G_b) \xrightarrow{\sim} \mathrm{Irr}(S_\phi)$$

where $\Pi_\phi(G_b) =$ fiber of $\Pi(G_b) \to \Phi(G_b) \to \Phi(G)$ over ϕ

So $\Pi_\phi(G_b)$ is a fin union of L-packets.

This is known for GL_n, GSp_4, other classical grps, etc.

<u>Example</u> (1) If ϕ is discrete, then

$\Pi_\phi(G_b)$ is empty for non-basic b & $S_\phi = S_\phi^\natural$.

(2) $G = GL_2$, $\phi = \chi_1 \oplus \chi_2$ (random chars)

LCFT $\Rightarrow$ χ_1, χ_2 chars of $E^\times$ $\Rightarrow$ $S_\phi = \mathbb{G}_m \times \mathbb{G}_m$

$\Rightarrow$ $\mathrm{Irr}(S_\phi) = \mathbb{Z}^2$.

- $\{b \text{ basic}\} = \mathbb{Z}$, $b \in 2\mathbb{Z} \Leftrightarrow G_b = GL_2$, $\Pi_\phi(G_b) = \{i_B^G(\chi_1 \oplus \chi_2)\}$

  $b \in 2\mathbb{Z}+1 \Leftrightarrow G_b = D^\times$, $\Pi_\phi(G_b) = \phi$.

- $\{b \text{ non-basic}\} = \{(m,n) \in \mathbb{Z}^2 \text{ with } m > n\}$.

  $b = (m,n) \Leftrightarrow G_b = E^\times \times E^\times$,

  $\Pi_\phi(G_b) = \{\chi_1 \boxtimes \chi_2, \chi_2 \boxtimes \chi_1\}$.

Varying ϕ in the previous thm, see that should exist a can bij of pairs

$$\begin{tikzcd}
(b\in B(G), \pi \in \Pi(G_b)) & (\phi \in \Phi(G), \rho \in \mathrm{Irr}(S_\phi)) \\
(b,\pi) \arrow[r, leftrightarrow, "\text{BM-O bij}"] \arrow[d, leftrightarrow, dashed, "?"'] & (\phi,\rho) \arrow[d, leftrightarrow, dashed, "?"] \\
D(\mathrm{Bun}_G)^\omega \arrow[r, leftrightarrow, "\sim"] & \mathrm{Coh}(\mathrm{Par}_G)
\end{tikzcd}$$

Note When ϕ is generous, things seem as simple as possible.

$$q^{-1}(x_\phi) = BS_\phi$$

Prop (1) ϕ discrete L-para, ϕ generous $\Leftrightarrow$ ϕ supercusp.

(2) ϕ generous $\Rightarrow$ S_ϕ° is a torus.

(3) ϕ generous $\Rightarrow$ q flat in a nbhd of $q^{-1}(x_\phi)$

& $\exists$ natural regular closed imm

$$i_\phi : BS_\phi = \mathrm{Par}_G \times_{X_G^{spec}} \{x_\phi\} \hookrightarrow \mathrm{Par}_G$$

and Par_G sm in a nbhd of im i_ϕ.

Conj 1 Suppose ϕ is generous and $(b,\pi) \leftrightarrow (\phi,\rho)$ matches under the BM-O bij.

Then (1) $a_\psi(i_{\phi *}\rho) := i_{\phi *}\rho * i_{1!} W_\psi \simeq i_{b!}^{ren}\pi$ in $D(\mathrm{Bun}_G)$

(2) $i_{bh}^{ren}\pi \xrightarrow{\sim} i_{b!}^{ren}\pi \xrightarrow{\sim} i_{b*}^{ren}\pi$.

Rmk (1) + Compatibility of classical LLC with duality $\Rightarrow$ (2) (pf in notes).

Conj 2 Let ϕ be generous. Then the object

$$\mathcal{F}_\phi = \bigoplus_{\substack{b \in B(G) \\ \pi \in \Pi_\phi(G_b)}} i_{b!}^{ren} \pi^{\oplus \dim \tau_\psi(b,\pi)}$$

$$(= a_\psi(i_{\psi *}\mathcal{O}(S_\phi)).)$$

is a perverse Hecke eigensheaf w/ eigenval ϕ.

[FS] $V \in \mathrm{Rep}(^L G)$

$\rightsquigarrow T_V : D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_G)^{BW_E}$.

should have $\mathcal{F}_\phi \xrightarrow{T_V} T_V(\mathcal{F}_\phi) = \mathcal{F}_\phi \boxtimes \underbrace{V\circ\phi}_{\circlearrowleft W_E}$

$$\begin{array}{c} \mathcal{F}_\phi \xrightarrow{T_V} T_V(\mathcal{F}_\phi) \\ \big\downarrow \text{forget} \\ T_V(\mathcal{F}_\phi) = \mathcal{F}_\phi^{\otimes \dim V}. \end{array}$$

Resulting formula: in $D(\mathrm{Bun}_G)^{BW_E}$

$$T_V\, i_{b!}^{\mathrm{ren}} \pi \simeq \bigoplus_{(b',\pi')} i_{b'!}^{\mathrm{ren}} \pi' \boxtimes \underbrace{\mathrm{Hom}_{S_\phi}(\iota_\psi(b,\pi)^\vee \otimes \iota_\psi(b',\pi'), V\circ\phi)}_{\circlearrowleft W_E}.$$

This is implied by Conj 1.

When ϕ is supercusp, this recovers the strongest form of Kottwitz's conj.

<u>What is known when ϕ supercusp?</u>

- Conj 2 already formulated by Fargues in 2014.
- Conj 1 : stated in FS
- Conj 1(2) & Conj 2 : known for GL_n, GSp_4, unram $U_{2n+1}/\mathbb{Q}_p$, SO_{2n+1}.
- Conj 1 is known for GL_n & unram $U_{2n+1}/\mathbb{Q}_p$:

  FS + HKW + { Hansen (SO_{2n+1}), Hamann (GSp_4), Bertoloni Meli–Hamann–Nguyen ($U_{2n+1}/\mathbb{Q}_p$) }

<u>Beyond ϕ supercusp?</u>

Hamann: If G split, $B = TU \subset G$, $\phi: W_E \to {}^LT \subset {}^LG$ is generous (& ℓ-integral + generous mod ℓ).

and enough is known about FS LLC for G.

Then Conj 2 & Conj 1(2) are true.

t-structures on $D(\mathrm{Bun}_G)$

It turns out there are 2 natural t-strs on $D(\mathrm{Bun}_G)$.

(I) <u>Perverse t-str</u> ${}^pD^{\leq 0}$: $A \in D(\mathrm{Bun}_G)$ s.t. $i_b^{*\mathrm{ren}} A \in D^{\leq 0}(G_b)$

${}^pD^{\geq 0}$: $A \in D(\mathrm{Bun}_G)$ s.t. $i_b^{!\mathrm{ren}} A \in D^{\geq 0}(G_b)$.

This gives a t-str on $D(\mathrm{Bun}_G)$

+ truncations preserve $D(\mathrm{Bun}_G)^{\mathrm{ULA}}$.

Verdier duality exchanges ${}^pD^{\leq 0} \cap D^{\mathrm{ULA}} \longleftrightarrow {}^pD^{\geq 0} \cap D^{\mathrm{ULA}}$.

Get $\mathrm{Perv}(\mathrm{Bun}_G)^{\mathrm{ULA}} \subset \mathrm{Perv}(\mathrm{Bun}_G)$

Irred objs in $\mathrm{Perv}(\mathrm{Bun}_G)^{\mathrm{ULA}}$ are parametrized by pairs (b, π).

Define $i_{b!*}^{\mathrm{ren}} \pi := \mathrm{im}({}^pH^0(i_{b!}^{\mathrm{ren}} \pi) \to {}^pH^0(i_{b*}^{\mathrm{ren}} \pi))$.

$(\Rightarrow \mathbb{D}_{\mathrm{Verd}}(i_{b!*}^{\mathrm{ren}} \pi) \simeq i_{b!*}^{\mathrm{ren}} \pi^\vee.)$

(II) <u>hadal t-str</u> a t-str on $D(\mathrm{Bun}_G)^\omega$ with

${}^hD^{\leq 0}$ = gen'd under finite ext'ns by $i_{b\natural}^{\mathrm{ren}} D^{\leq 0}(G_b)^\omega$.

${}^hD^{\geq 0}$ = gen'd under finite ext'ns by $i_{b!}^{\mathrm{ren}} D^{\geq 0}(G_b)^\omega$.

Pf in notes that this defines a bounded & nondeg t-str on $D(\mathrm{Bun}_G)^\omega$.

Let $\mathrm{Had}(\mathrm{Bun}_G)$ = abelian heart.

Truncation functors preserve $D(\mathrm{Bun}_G)_{\mathrm{fin}}$!

$\rightsquigarrow$ $\mathrm{Had}(\mathrm{Bun}_G)_{\mathrm{fin}} \subset \mathrm{Had}(\mathrm{Bun}_G)$ a Serre subcat and all $A \in \mathrm{Had}(\mathrm{Bun}_G)_{\mathrm{fin}}$ have finite length.

$\{(b, \pi)\} \cong$ irred objs in $\mathrm{Had}(\mathrm{Bun}_G)_{\mathrm{fin}}$

& (b, π)

$\rightsquigarrow \mathcal{G}_{(b,\pi)} := \mathrm{im}({}^hH^0(i_{b\natural}^{\mathrm{ren}} \pi) \to {}^hH^0(i_{b!}^{\mathrm{ren}} \pi))$.

If ϕ ss, $D(\mathrm{Bun}_G)_\phi^{\mathrm{ULA}}$ (ϕ-locally) satisfies

$\forall b, \forall n$, each irred subquot of $H^n(i_b^{*\mathrm{ren}} A)$ has FS param ϕ.

Turns out that $\forall n$, $\exists$ a functorial decomp

$$A \cong A_\phi \times A^\phi, \quad A_\phi \in D(\mathrm{Bun}_G)_\phi^{\mathrm{ULA}}$$

$\rightsquigarrow \mathrm{Perv}(\mathrm{Bun}_G)_\phi^{\mathrm{ULA}} \subset \mathrm{Perv}(\mathrm{Bun}_G)^{\mathrm{ULA}}$ Serre subcat

dual functor.

A finite $\Rightarrow$ $A \cong \bigoplus_\phi A_\phi$, only fin many summands needed.

$$\begin{array}{ccc} D(\mathrm{Bun}_G)_{\mathrm{fin}} & \cong \bigoplus_\phi & D(\mathrm{Bun}_G)_{\mathrm{fin},\phi} \\ \uparrow & & \uparrow \\ \mathrm{Had}(\mathrm{Bun}_G)_{\mathrm{fin}} & \cong \bigoplus_\phi & \mathrm{Had}(\mathrm{Bun}_G)_{\mathrm{fin},\phi} \end{array}$$

<u>Next time</u> Will explain the following picture:

If ϕ ss generic, have a matching of t-strs

$$D(\mathrm{Bun}_G)_{\mathrm{fin},\phi} \xrightarrow{C_\phi} \mathrm{Coh}(\mathrm{Par})_{\mathrm{fin},\phi}$$

perverse $\longleftrightarrow$ standart

hadal $\longleftrightarrow$ perverse coh

Lecture IV