Undergraduate Thesis

A Geometric Jacquet-Langlands Correspondence for Unitary Shimura Varieties mod Ramified p

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June 2023

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A GEOMETRIC JACQUET–LANGLANDS CORRESPONDENCE FOR MOD p UNITARY SHIMURA VARIETIES: THE RAMIFIED CASE

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Abstract. Let p be a prime that ramifies in a totally real field F^+ and splits in an imaginary quadratic field E. We propose a smooth Pappas–Rapoport splitting model over F^+E to describe a special fiber $\mathbf{Sh}(G)$ at p of $\mathrm{G}(\mathrm{U}(r,s)^d)$-Shimura variety. We then realize a geometric Jacquet–Langlands correspondence between unitary special fibers $\mathbf{Sh}(G)$ and $\mathbf{Sh}(G')$ with different signatures, which prompts a global description of Goren–Oort strata on $\mathbf{Sh}(G)$ as iterated $\mathbb{P}^1$-bundles over $\mathbf{Sh}(G')$.

Contents

Introduction

The Tate conjecture on algebraic cycles is one of the most important problems in arithmetic and algebraic geometry, which is closely related to other big problems such as the Hodge conjecture and the Birch–Swinnerton-Dyer conjecture. The general case of the Tate conjecture is far from being proved. The construction of cycles on modular varieties is regarded as the right track to approach the Tate conjecture and other central problems in arithmetic geometry. In a series of recent works [Hel10, Hel12, TX16, HTX17, TX19] by Helm, Tian, and Xiao, as well as [XZ17] by Xiao and Zhu, the authors aim to construct

Date: February 20, 2023.

cycles on certain Shimura varieties by using variant tools in which the geometric Satake equivalence and a geometrization of Jacquet–Langlands correspondence are involved. For example, [TX19] attacks the Tate conjecture for Hilbert modular varieties modulo an inert prime p, by proving that a Jacquet–Langlands transfer morphism is an isomorphism. These papers are deliberately based on a keynote idea of comparisons between different special fibers of Shimura varieties.

Following these, we introduce the *Pappas–Rapoport splitting model* for mod p special fibers of unitary Shimura varieties (Definition 1.1), which admits the prime p to be ramified in a totally real field. The goal of this presented paper is to propose a geometric Jacquet–Langlands correspondence over such a model, in order to describe the global geometry on ramified special fibers of unitary Shimura varieties (Section 5). More explicitly, by considering sections of modular bundles on abelian varieties, which are associated with any closed point of the unitary Shimura variety by the Pappas–Rapoport moduli problem, we construct a family of modular forms, called the *partial Hasse invariants* (Section 3), and can define their zero loci as the *Goren–Oort strata* (Section 4). The main theorem of this paper (Theorem 5.1) indicates that a Goren–Oort stratum is isomorphic to an iterated $\mathbb{P}^1$-bundle over another simpler unitary Shimura variety with a lower dimension. Moreover, for further purposes, our result can shed light on understanding the cycles defined by combinatorial information of Goren–Oort strata, and these cycles finally evolve into the Tate cycles of the middle dimension on the Shimura variety; also, figuring out the intersection relations between such cycles is supposedly the key to prove the Tate conjecture for special fibers under mild genericity conditions.

1. The Pappas–Rapoport splitting model. Let F^+ be a totally real field of degree $ef = [F^+ : \mathbb{Q}]$ and E be an imaginary quadratic field. Fix a prime $p > 2$ that ramifies in F^+ with inertia degree f and splits in E; put $F = F^+E$. It turns out to be exactly ef real embeddings of F^+, each of which is denoted by $\tau_j^{(i)} : F^+ \to \mathbb{R}$, where i and j run through $\{1, \dots, e\}$ and $\{1, \dots, f\}$, respectively. We can associate with each $\tau_j^{(i)}$ a pair of non-negative integers $(r_j^{(i)}, s_j^{(i)})$, called the signature of $\tau_j^{(i)}$, such that $r_j^{(i)} + s_j^{(i)} = 2$ (see Subsection 1.1 for more details). Let k_0 be a sufficiently large finite extension of $\mathbb{F}_p$ in a fixed algebraic closure $\overline{\mathbb{F}}_p$.

According to [RX17], the characteristic p fiber of the *Deligne–Pappas moduli space*, denoted by $\mathcal{M}_{k_0}^{\mathrm{DP}}$, parametrizes ef-dimensional abelian schemes A/S defined over a locally noetherian k_0-scheme S with classical additional structures [DP94]. The moduli problem of Deligne–Pappas is represented by a normal k_0-scheme, and its smooth locus $\mathcal{M}_{k_0}^{\mathrm{Ra}}$, called the Rapoport locus, parametrizes those abelian schemes A/S whose dualizing sheaf of invariant differentials $\omega_{A^\vee/S}$ is locally free of rank 1 as an $(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_S)$-module [Rap78]. After these, Andreatta and Goren have constructed some modular forms defined over $\mathcal{M}_{k_0}^{\mathrm{Ra}}$, called the *partial Hasse invariants* [Gor01, AG05], which factor the determinant of the Hasse–Witt matrix of the universal abelian scheme $\mathcal{A}_{k_0}^{\mathrm{Ra}}$.

When $e = 1$, or namely p is unramified in F^+, there are exactly f partial Hasse invariants that give rise to a good stratification of $\mathcal{M}_{k_0}^{\mathrm{Ra}} = \mathcal{M}_{k_0}^{\mathrm{DP}}$ (c.f. [GO00, AG05]). On the other hand, when p ramifies in F^+ with $e > 1$, the Rapoport locus is open and dense in $\mathcal{M}_{k_0}^{\mathrm{DP}}$ with the complement of codimension 2; and the partial Hasse invariants of [AG05] do not extend to $\mathcal{M}_{k_0}^{\mathrm{DP}}$ as one may expect. In addition, the number of such operators is strictly less than ef. For example, when p is totally ramified in F^+ with $f = 1$, only one partial Hasse invariant is defined on $\mathcal{M}_{k_0}^{\mathrm{Ra}}$: it is an eth root of the determinant of the Hasse–Witt matrix, up to sign.

The lack of partial Hasse invariants when p ramifies was in particular an obstruction in extending to the ramified setting the results proved in the unramified case. To remedy this, Reduzzi–Xiao [RX17] have worked with the characteristic p fiber of the *Pappas–Rapoport splitting model* constructed in [PR05] and made explicit by Sasaki in [Sas19]. This is a smooth k_0-scheme $\mathcal{M}_{k_0}^{\mathrm{PR}}$ endowed with a birational morphism $\mathcal{M}_{k_0}^{\mathrm{PR}} \to \mathcal{M}_{k_0}^{\mathrm{DP}}$ which is an isomorphism if and only if p is unramified in F^+, and which induces an isomorphism from a suitable open dense subscheme of $\mathcal{M}_{k_0}^{\mathrm{PR}}$ onto $\mathcal{M}_{k_0}^{\mathrm{Ra}}$. There is a natural notion of automorphic line bundles on $\mathcal{M}_{k_0}^{\mathrm{PR}}$. As a remark, the "splitting" in the name indicates that one can separate the geometry of the characteristic p fiber on this model, but it does not assume that p splits.

The Pappas–Rapoport splitting model $\mathcal{M}_{k_0}^{\mathrm{PR}}$ (Definition 1.1) parametrizes isomorphism classes of tuples $(A, \lambda, \rho, \underline{\mathscr{F}})$ where A is an abelian S-scheme endowed with a polarization λ and level structure ρ, and for each $j = 1, \ldots, f$ we are given a filtration

$$0 = \mathscr{F}_j^{(0)} \subseteq \mathscr{F}_j^{(1)} \subseteq \cdots \subseteq \mathscr{F}_j^{(e)} = \omega_{A^\vee/S,j}$$

of locally free sheaves of $\mathcal{O}_S$-modules. The signature condition further requires that each subquotient $\mathscr{F}_j^{(i)}/\mathscr{F}_j^{(i-1)}$ has rank $r_j^{(i)} \in \{0, 1, 2\}$, which equals to the signature associated to $\tau_j^{(i)}$. Also, fixing a choice of the uniformizer $\varpi \in \mathcal{O}_F$, we assume that the action of multiplication-by-ϖ on each subquotient is via $\tau_j^{(i)}(\varpi)$. It is necessary to point out that the labeling of $\tau_j^{(i)}$s determines the splitting model $\mathcal{M}_{k_0}^{\mathrm{PR}}$; this issue disappears when considering characteristic p fibers. For example, when $e = 2$ and $f = 1$, the splitting model is obtained by blowing up $\mathcal{M}_{k_0}^{\mathrm{DP}}$ in correspondence with its singularities, which are isolated points.

To construct sufficiently many correct partial Hasse invariants, one needs to separate the irreducible components of $\mathcal{M}_{k_0}^{\mathrm{PR}}$ and to consider the sections of $\mathcal{O}_S$-subbundles for two types of *partial Hasse invariants* (Subsection 3.2), which are respectively induced by

$$m_{\varpi,j}^{(i)} \colon \mathscr{F}_j^{(i)}/\mathscr{F}_j^{(i-1)} \longrightarrow \mathscr{F}_j^{(i-1)}/\mathscr{F}_j^{(i-2)}, \quad 1 < i \leqslant e,$$
$$\mathrm{Hasse}_{\varpi,j}^{(1)} \colon \mathscr{F}_j^{(1)}/\mathscr{F}_j^{(0)} \longrightarrow (\mathscr{F}_{j-1}^{(e)}/\mathscr{F}_{j-1}^{(e-1)})^{(p)}, \quad i = 1.$$

These leads to the desired ef partial Hasse invariants as follows:

$$h_j^{(i)} \in H^0(\mathcal{M}_{k_0}^{\mathrm{PR}}, (\mathscr{F}_j^{(i-1)}/\mathscr{F}_j^{(i-2)}) \otimes (\mathscr{F}_j^{(i)}/\mathscr{F}_j^{(i-1)})^{\otimes -1}), \quad 1 < i \leqslant e,$$
$$h_j^{(1)} \in H^0(\mathcal{M}_{k_0}^{\mathrm{PR}}, (\omega_{A^\vee/S,j-1}/\mathscr{F}_{j-1}^{(e-1)})^{\otimes p} \otimes (\mathscr{F}_j^{(1)})^{\otimes -1}), \qquad i = 1.$$

Using crystalline deformation theory of Grothendieck–Messing (Theorem 2.14), we prove the smoothness of $\mathcal{M}_{k_0}^{\mathrm{PR}}$, which further proposes the possibility for $h_j^{(i)}$s to exist. Moreover, we can regard the generalized partial Hasse invariants as functions on $\mathcal{M}_{k_0}^{\mathrm{PR}}$; Theorem 4.1 dictates that they cut out proper smooth divisors with simple normal crossings on $\mathcal{M}_{k_0}^{\mathrm{PR}}$.

To summarize the construction of the Pappas–Rapoport splitting model, note that objects as follows are mutually in bijections parametrized by the index $(i,j) \in \{1,\dots,e\} \times \{1,\dots,f\}$:

- a real embedding $\tau_j^{(i)}$ of F^+ inducing the embedding $\tau_j \colon \mathcal{O}_{F^+}/v \to k_0$;
- the subquotient $\mathscr{F}_j^{(i)}/\mathscr{F}_j^{(i-1)}$ as an $\mathcal{O}_S$-subbundle of $\omega_{A^\vee/S,j}$ for an abelian scheme A corresponding to an S-point on $\mathcal{M}_{k_0}^{\mathrm{PR}}$;
- a map of connection between subquotients, which equals $m_{\varpi,j}^{(i)}$ when $1 < i \leqslant e$ or $\mathrm{Hasse}_{\varpi,j}^{(i)}$ when $i = 1$;
- the generalized partial Hasse invariant $h_j^{(i)}$ as a mod p modular form on $\mathcal{M}_{k_0}^{\mathrm{PR}}$.

2. The geometric Jacquet–Langlands correspondence. Given an algebraic subgroup G over $\mathbb{Q}$ with some data in addition, the Pappas–Rapoport splitting model $\mathcal{M}_{k_0}^{\mathrm{PR}}$ defines the characteristic p special fiber of a Shimura variety for G, denoted by $\mathbf{Sh}(G)$. Acquired from Helm's original work [Hel10, Hel12], when G and G' are unitary groups whose real points are respectively isomorphic to $U(1,1) \times U(1,1)$ and $U(0,2) \times U(2,0)$ defined in terms of the CM field F, one can establish Jacquet–Langlands correspondences via the geometry of $\mathbf{Sh}(G)$ and $\mathbf{Sh}(G')$.

Historically, according to [Hel12], the Jacquet–Langlands correspondence predicts a bijection between automorphic representations of G and G' satisfying certain technical conditions; this holds at least when G and G' are quaternionic algebraic groups [JL70]. In general, while G and G' are varying beyond the quaternionic case and the GL_2 case, the theory of automorphic representations states the Jacquet–Langlands correspondence via the trace formula. But when $G = \mathrm{GL}_2$ and G is isomorphic to G' over all non-archimedean primes of $\mathbb{Q}$, Serre considers a modular curve together with sections of line bundles on it and relates the space of modular forms mod p arising from such sections to the space of functions on supersingular locus of characteristic p special fiber of the same modular curve [SL96]. The supersingular locus is a double coset for a quaternion algebra D and one can interpret the space of functions on such a space as a space of "mod p modular forms" for $D^\times$. In contrast to the traditional approach, which yields only a bijection between isomorphism classes of representations, Serre's result gives a canonical isomorphism between spaces that arise naturally from the geometry of the Shimura varieties attached to the groups GL_2 and $D^\times$.

In our case at work, the comparison in Theorem 5.1 can be viewed as a geometrization of Jacquet–Langlands correspondence. The geometric realization of Jacquet–Langlands correspondence was first studied by Ribet [Rib83, Rib89] who considers character groups of tori attached to the bad reduction of modular curves and Shimura curves at various primes; these character groups are free $\mathbb{Z}$-modules with actions of Hecke operators for GL_2 or D. After these, Helm constructed an auxiliary moduli problem to realize a certain type of Jacquet–Langlands correspondence between unitary Shimura varieties of different signatures [Hel10, Hel12]. They gave some examples of the cycles in the case of modular or Shimura curves and unitary Shimura varieties that realizes the Jacquet–Langlands correspondence geometrically. The geometric aspect of this technique is further developed by Tian and Xiao in [TX16]. This method helps to depict not only the global geometry of supersingular loci of special fibers but also the relationship with the Tate conjecture for Shimura varieties over finite fields.

The bulk of this paper (Section 5 and 6) is devoted to establishing the global result with its proof. Let T be a subset of in total ef real embeddings of F^+; so each element of T is of form $\tau_j^{(i)}$, regarded as a function on $\mathbf{Sh}(G)$. Define Z_T to be the intersection of vanishing loci on $\mathbf{Sh}(G)$ of all elements in T. Here comes a concise version of the main result of this paper.

Theorem 1. *Suppose G is an algebraic group over $\mathbb{Q}$ whose real points are isomorphic to $\prod_{j=1}^{f}\prod_{i=1}^{e} U(r_j^{(i)}, s_j^{(i)})$.*

(1) *The vanishing locus of any generalized partial Hasse invariant $h_j^{(i)}$ in $\mathcal{M}_{k_0}^{\mathrm{PR}}$ is a proper smooth divisor with simple normal crossings. In particular, Z_T has codimension $\#T$ in $\mathcal{M}_{k_0}^{\mathrm{PR}}$.*

(2) *There exists an integer N such that Z_T is isomorphic to a $(\mathbb{P}^1)^N$-bundle over the k_0-fiber $\mathbf{Sh}(G')$ of another unitary Shimura variety. More precisely, N depends only on T and the signatures $(r_j^{(i)})_{i,j}$, defined by Construction 5.5 via a combinatoric way.*

(3) *The following diagram commutes:*

$$\begin{tikzcd}
& Y_T \arrow[dl, "\cong"'] \arrow[dr, "\cong"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
Z_T \arrow[d] \arrow[ddr, phantom] \arrow[drr, dashed, "(\mathbb{P}^1)^N"] & & Z'_T \arrow[d, "(\mathbb{P}^1)^N"] \\
\mathbf{Sh}(G) & & \mathbf{Sh}(G').
\end{tikzcd}$$

Here Z'_T is the k_0-scheme representing the moduli problem that classifies another family of abelian schemes together with a collection of line subbundles on it; Y_T represents the data proposed by two isogenic families in Z_T and Z'_T. See Definitions 6.1 and 6.2 for details.

The proof of Theorem 1 depends on an elaborate analysis on Dieudonné modules of the abelian scheme corresponding to an S-point of $\mathbf{Sh}(G)$. In Subsection 6.1, we have defined a family of maps, written as $d_{\varpi,j}^{(i)}$ when $1 \leqslant i < e$ or $\mathrm{Hasse}_{\varpi,j}^{(e)}$ when $i = e$, which are essentially given by dividing by $[\varpi]$. It turns out that each $\mathscr{F}_j^{(i)}/\mathscr{F}_j^{(i-1)}$ is contained in the kernel of $d_{\varpi,j}^{(i)}$ or $\mathrm{Hasse}_{\varpi,j}^{(e)}$. This family of maps together with the family of

$m_{\varpi,j}^{(i)}$s and $\mathrm{Hasse}_{\varpi,j}^{(1)}$s emulate the characters of Frobenius and Verschiebung maps between direct factors of Dieudonné modules in the unramified case. Technically, to prove that $Y_T \to Z_T$ and $Y_T \to Z'_T$ are isomorphisms, we first check that both morphisms induce bijections on closed k-points for some perfect field k of characteristic p, and then apply Grothendieck–Messing theory on crystalline deformations to show that these bijections induce isomorphisms on tangent spaces of k-points.

3. Structure of the paper. In Section 1, we introduce the number-theoretic setups of the paper and establish the moduli problem for Pappas–Rapoport splitting model, based on [PR05] and [RX17]. In Section 2, we review some Dieudonné theory and Grothendieck–Messing deformation theory that will be frequently used in later sections. In Section 3, we construct the correct notion of transfer maps between subquotients, and hence the generalized partial Hasse invariants; the relations between these maps are summarized by Proposition 3.2. Section 4 is devoted to proving Theorem 4.1, the smoothness of Goren–Oort strata defined by vanishing loci of generalized partial Hasse invariants. In Section 5, we propose the main result Theorem 5.1; we also give an explicit combinatorial construction based on the defining subset of embeddings for the Goren–Oort strata, with calculation on some examples; the upshot lies in Construction 5.5 and 5.9 that defines two important subsets $I(T)$ and $\Delta(T)$ at work. In Section 6, we carefully complete the proof of the main theorem in several steps.

Acknowledgements

I am obtaining the School of Mathematical Sciences at Peking University and the Beijing International Center for Mathematical Research to thank for providing me with an exceptional chance to enrich myself with unforgettable experiences in life over the past five years. I express my gratitude to my advisor Professor Xiao Liang for his guidance during the writing process of this thesis. The insightful comments and suggestions by Yang Deding, Yang Xiangqian, Bai Ruiqi, Wang Zeyu, and Fu Hao are greatly appreciated.

Furthermore, I extend my heartfelt thank to Professor Yin Qizheng for every invaluable conversation we have had over the past five years and for his significant influence in shaping my values. I thank Professor Liu Ruochuan for his crucial assistance during my times of difficulty. I am grateful to Professors Wang Haoran, Hu Yongquan, Tian Yichao, and Tian Zhiyu for their wise counsel and impressive encouragement. I also thank Professor Yuan Xinyi, since his unassuming academics — plain, but more refined — have had a profound impact on me. Receiving their attentive instruction was an unforeseen delight at my first moment to enter Yan Yuan. I am very lucky to have such mentors to teach me what is the definition of ideal.

Again, the deepest gratitude goes to Professor Xiao Liang. With a phone call in the winter of 2019, he opened my eyes to the wonders of number theory and arithmetic geometry. He invested countless time and energy in me, taught with patience and generosity, and shared with me wisdom and expertise. He had built my essential groundwork to comprehend the hidden marvels behind the p-adic landscapes, with the Langlands program and p-adic Hodge theory involved, and so also inspired me softly and deftly, to pursue the Tate conjecture on mod p unitary Shimura variety, through working over ramified geometry thereof. He with kindness, moreover, cared for my health and well-being, strengthened my morale, smoothed my flaw, guided me through the hardships of life, and even helped me overcome irrational distress. Thus for life of my vision, in order to emulate his character, I have resolved to dedicate myself, thanks to his dedication in passion uttered, to the sense of truth and beauty in mathematics.

Moreover, I am deeply grateful to my family for the care and concern in both Kunming and Beijing. The final special thank goes to Bai Yuqin, Hu Rong, Min Rui, Ren Jinshan, Zhang Suyi, as well as Yang Yuhong together with his parents. During the past too many years, they have truly shared my every genuine laughter and every silent tear, at every very moment I had once skimmed. They always understand my mathematics with no idea. They always support and motivate me for no reason.

Many thanks. Merci à tous. 谢谢你们*.*

Part 1. Setups and Backgrounds

1. Pappas–Rapoport splitting model for unitary Shimura varieties

1.1. **Setups.** We begin with basic definitions and notations to define the unitary Shimura variety in Subsection 1.2 below.

Suppose E is an imaginary quadratic field, F^+ is a totally real field of degree $[F^+ : \mathbb{Q}] = ef > 1$, and the CM field $F = EF^+$ is their compositum. Denote $\mathcal{O}_F$ the ring of integers of the number field F. Throughout this paper, fix a rational prime p that splits in E_0 and ramifies in F^+ of index e, with inertia degree f. Let v be the unique prime ideal of F^+ such that $v \mid p$. Write the prime ideal factorization as $p\mathcal{O}_{F^+} = v^e$ and $p\mathcal{O}_F = \mathfrak{p}^e\overline{\mathfrak{p}}^e$, where $\overline{\mathfrak{p}}$ is the complex conjugate of $\mathfrak{p}$, and the complex conjugate $x \mapsto \overline{x}$ is identified with the nontrivial element in $\mathrm{Gal}(F/F^+)$.

Fix a finite extension k_0 over $\mathbb{F}_p$ such that any choice of the uniformizer $\varpi \in \mathcal{O}_{F^+}$ divides p. Denote $\sigma\colon k_0 \to k_0$ the arithmetic Frobenius automorphism $x \mapsto x^p$, for any $x \in k_0$. Then $k_0 = \mathbb{F}_{p^r} \subseteq \overline{\mathbb{F}}_p$ with some sufficiently divisible r.

Moreover, fix an isomorphism $\iota\colon \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ of fields and an embedding $W(k_0) \hookrightarrow \mathbb{C}$ that is compatible with ι, where $W(k_0)$ is the ring of Witt vectors in k_0 (i.e. the ring of integers with residue field k_0 which is unramified over $\mathbb{Q}_p$). Then for each embedding $\tau\colon F \hookrightarrow \mathbb{C}$, there is a natural map $\mathcal{O}_F \to W(k_0)$, at the level of rings of integers. It renders another map, which is in an abuse of notation still denoted by $\tau\colon \mathcal{O}_F/\mathfrak{p} \to k_0$. Accordingly, after τ running through all p-adic embeddings, we obtain

$$\tau_1, \ldots, \tau_f\colon \mathcal{O}_F/\mathfrak{p} \to k_0$$

that are related iteratively via the Frobenius automorphism, i.e., $\tau_{j+1} = \sigma \circ \tau_j$, where the subscripts are computed modulo f.

$$\begin{tikzcd}
\mathcal{O}_F/\mathfrak{p} \arrow[r, "\tau_1"] \arrow[rr, bend left, "\tau_2"] \arrow[rrrr, bend right, "\tau_f"'] & k_0 \arrow[r, "\sigma"] & k_0 \arrow[r, "\sigma"] & \cdots \arrow[r, "\sigma"] & k_0 \arrow[lll, bend right, "\sigma"']
\end{tikzcd}$$

The isomorphism $\iota\colon \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ induces a bijection $\iota^*\colon \mathrm{Hom}(F^+, \mathbb{C}) \xrightarrow{\sim} \mathrm{Hom}(F^+, \overline{\mathbb{Q}}_p)$. We define $\Sigma_\infty \coloneqq \mathrm{Hom}(F^+, \mathbb{R})$ to be the set of all archimedean embeddings of F^+. Since by assumption, there is only one place v above p in F^+, we henceforth confuse the set Σ_v of p-adic embeddings inducing the place v with Σ_∞. So for each $j \in \{1, \ldots, f\}$, there are exactly e different elements of Σ_v, denoted by

$$\tau_j^{(1)}, \ldots, \tau_j^{(e)},$$

inducing the embedding $\tau_j\colon \mathcal{O}_F/\mathfrak{p} \to k_0$. There is no canonical choice of such labeling, but we fix one for the rest of this paper.

Denote N the discriminant of E and choose a square root of N in E, written as $N^{1/2}$. Fix a choice of the square root $\sqrt{N}$ of N in $\mathbb{C}$, which uniquely determined an embedding

$E \hookrightarrow \mathbb{C}$. Given any real embedding $\tau\colon F^+ \to \mathbb{R}$ in Σ_∞, it would extend uniquely to a complex embedding $F = EF^+ \hookrightarrow \mathbb{C}$ via $E \hookrightarrow \mathbb{C}$ if the extension was required to preserve E. We again by an abuse of notation still denote it by $\tau\colon F \to \mathbb{C}$, satisfying $\tau|_E = \mathrm{id}_E$. Let else $\tau^\perp\colon F \to \mathbb{C}$ be the other extension of τ sending E to $\overline{E}$. Then each $\tau \in \Sigma_\infty$ induces $\tau, \tau^\perp \in \mathrm{Hom}(F, \mathbb{C})$ such that

$$\tau(x + yN^{1/2}) = \tau(x) + \tau(y)\sqrt{N}, \quad \tau^\perp(x + yN^{1/2}) = \tau(x) - \tau(y)\sqrt{N}.$$

Let D be a division algebra of dimension n^2 over its center F, equipped with a positive involution $*$ restricting to the complex conjugation on F. Fix a maximal order $\mathcal{O}_D$ of D that is stable under the involution $*$. We assume that D splits at $\mathfrak{p}$ and $\overline{\mathfrak{p}}$, and fix an isomorphism (which is a priori non-canonical)

$$D \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \mathrm{M}_n(F_{\mathfrak{p}}) \times \mathrm{M}_n(F_{\overline{\mathfrak{p}}}) \cong \mathrm{M}_n(\mathbb{Q}_{p^f}) \times \mathrm{M}_n(\mathbb{Q}_{p^f})$$

where $*$ switches the two direct factors. We denote $\mathfrak{e}$ the element of $D \otimes_{\mathbb{Q}} \mathbb{Q}_p$ that corresponds to the $(1,1)$-elementary matrix (i.e. an $n \times n$-matrix whose $(1,1)$-entry is 1 and whose other entries are 0) in the first factor via this isomorphism. Moreover, as F is purely imaginary, given a complex embedding $\tau = \tau_j^{(i)} \in \Sigma = \mathrm{Hom}(F, \mathbb{C})$, we obtain $D \otimes_{F,\tau} \mathbb{C} \simeq \mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(\mathbb{C})$.

Choose V to be a free left D-module of rank 1, equipped with a non-degenerate $\mathbb{Q}$-valued alternating pairing

$$\langle \cdot, \cdot \rangle\colon V \times V \longrightarrow \mathbb{Q}$$

satisfying that $\langle \alpha x, y \rangle = \langle x, \alpha^* y \rangle$ for each $\alpha \in D$. Accordingly, for each $\tau = \tau_j^{(i)} \in \Sigma$, the $(D \otimes_{F,\tau} \mathbb{C})$-module $V \otimes_{F,\tau} \mathbb{C}$ is free of rank 1. One can choose a suitable idempotent $\delta \in (D_{\mathbb{C},j}^{(i)})^{*=1}$ and note that $\delta(V \otimes_{F,\tau} \mathbb{C})$ is an n^2-dimensional Hermitian space over $\mathbb{C}$, equipped with a canonical non-degenerate Hermitian pairing $[\cdot,\cdot]_\tau = [\cdot,\cdot]_j^{(i)}$ on $V \otimes_{F,\tau} \mathbb{C}$, whose imaginary part is exactly induced from $\langle \cdot, \cdot \rangle$ on V via $\tau_j^{(i)}$. Whenever V is fixed, denote $r_\tau = r_j^{(i)}$ and $s_\tau = s_j^{(i)}$ the numbers of 1s and -1s in the signature of $[\cdot,\cdot]_j^{(i)}$, respectively. Then

$$r_j^{(i)} + s_j^{(i)} = n.$$

Fix a $\hat{\mathbb{Z}}$-lattice $\Lambda \subseteq V(\mathbb{A}_{\mathbb{Q},f})$ that is stable under the $\mathcal{O}_D$-action, such that $\langle \cdot, \cdot \rangle$ induces an integrally valued pairing $\Lambda \times \Lambda \to \hat{\mathbb{Z}}$; hence there is a dual map $\Lambda \to \mathrm{Hom}(T, \hat{\mathbb{Z}})$ with respect to it. Assume that the cokernel of this map has a prime-to-p order.

Given such V and D as above, define G to be an algebraic group over $\mathbb{Q}$ which satisfies for arbitrary $\mathbb{Q}$-algebra R that

$$G(R) \coloneqq \left\{ g \in \mathrm{Aut}_{D \otimes_{\mathbb{Q}} R}(V \otimes_{\mathbb{Q}} R) \;\middle|\; \begin{array}{c} \exists c(g) \in R^\times,\ \forall x, y \in V \otimes_{\mathbb{Q}} R, \\ \langle gx, gy \rangle = c(g)\langle x, y \rangle \end{array} \right\}.$$

This is a subgroup of $\mathrm{Aut}_{D\otimes_{\mathbb{Q}}R}(V\otimes_{\mathbb{Q}} R)$. It also induces a map $c\colon G(R)\to \mathrm{GL}_1(R)$, the similitude character. Consequently, $G(\mathbb{R})$ is isomorphic to a subgroup of

$$\prod_{j=1}^{f}\prod_{i=1}^{e}\mathrm{GU}_{\mathbb{R}}(r_j^{(i)},s_j^{(i)}),\quad r_j^{(i)}+s_j^{(i)}=n,$$

whose isomorphism image is given by $\{(g_j^{(i)})\colon c(g_j^{(i)})=c(g_{j+1}^{(i)}),\ c(g_j^{(i)})=c(g_j^{(i+1)})\}$. More explicitly, if we write the similitude character as $c\colon G\to \mathbb{G}_{m,\mathbb{Q}}$ and denote G^1 its kernel in G, then

$$G^1(\mathbb{R})\simeq \prod_{j=1}^{f}\prod_{i=1}^{e}\mathrm{U}(r_j^{(i)},s_j^{(i)}),$$

where each factor corresponds to an embedding $\tau_j^{(i)}\in\Sigma$.

Put $K_p=\mathbb{Z}_p^\times\times \mathrm{GL}_n(\mathcal{O}_{F_{\mathfrak{p}}})$ as a subgroup of $G(\mathbb{Q}_p)$, and fix another open compact subgroup $K^p\subseteq G(\mathbb{A}_{\mathbb{Q},f}^{(p)})$ such that $K=K^pK_p$ satisfies the neat condition, i.e. $G(\mathbb{Q})\cap xKx^{-1}$ is torsion-free for any $x\in G(\mathbb{A}_{\mathbb{Q},f})$. We also require that $g(\Lambda)=\Lambda$ for each $g\in K$, and that in the action

$$\prod_{\ell\nmid\infty}\Lambda_\ell \circlearrowleft \prod_{\ell\nmid\infty}K_\ell,$$

each K_ℓ is the largest subgroup of $G(\mathbb{Q}_\ell)$ preserving Λ_ℓ.

Morally, all choices above allow us to define the unitary Shimura variety $\mathbf{Sh}(G)$ associated with the algebraic group G. In this part of the paper, with an emphasis on the moduli problem, we usually write $\mathbf{Sh}(G)=\mathcal{M}_{K,\Lambda}^{\mathrm{PR},D}$ that is defined in the following context.

1.2. **The moduli problem.** If K is sufficiently small, there will be a scheme over $W(k_0)$, representing the Pappas–Rapoport functor defined in the upcoming context. This scheme *almost* turns out to be the unitary Shimura variety that we are interested in. We work on base schemes which are locally noetherian over $\mathcal{O}:=\mathcal{O}_{F^{\mathrm{Gal}},(p)}=\mathcal{O}_{F^{\mathrm{Gal}}}\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$, where F^{Gal} denotes the Galois closure of F.

Definition 1.1. The *Pappas–Rapoport functor*, denoted by $\underline{\mathcal{M}}_{K,\Lambda}^{\mathrm{PR},D}$ (or simply $\underline{\mathcal{M}}_{K,\Lambda}^{\mathrm{PR}}$), is the following functor from the category of locally noetherian $\mathcal{O}$-schemes $\mathsf{Sch}_{\mathcal{O}}^{\text{loc-noe}}$ that

$$\begin{aligned}\underline{\mathcal{M}}_{K,\Lambda}^{\mathrm{PR}}\colon\ \mathsf{Sch}_{\mathcal{O}}^{\text{loc-noe}} &\longrightarrow \mathsf{Sets}\\ S &\longmapsto \{(A,\lambda,\rho,\underline{\mathscr{F}})\}/\sim,\end{aligned}$$

where the quadruple satisfies the following conditions.

(1) A is an abelian scheme over S of relative dimension n^2ef, equipped with faithful $\mathcal{O}_D$-action satisfying the signature condition that we postpone to clarify.

(2) $\lambda\colon A\to A^\vee$ is a polarization on A, such that:

(2a) The Rosati involution associated to λ induces $*$ on $\mathrm{Im}(\mathcal{O}_D\to \mathrm{End}_S(A))$. Here $A^\vee$ is the Cartier dual scheme of A (see Subsection 2.1).

(2b) $(\operatorname{Ker}\lambda)[p^\infty] \cap A[\mathfrak{p}]$ is a finite flat closed subgroup scheme of $A[\mathfrak{p}]$ over S of order $(\# k_{\mathfrak{p}})^{n^2}$.

(3) $\rho = (\rho^p, \rho_p)$, the K-level structure, is a pair defined as follows.

(3a) ρ^p is a prime-to-p K-level structure on A, that is, a collection of K-level structures on each A_r/S_r, where the S_r is a connected component of S. With a geometric point $\overline{s}_r \in S_r$ fixed, the K-level structure on S_r is a $\pi_1^{\text{ét}}(S_r, \overline{s}_r)$-stable K^p-orbit of $\mathcal{O}_D$-linear isomorphism

$$\rho_r^p \colon \Lambda \otimes \hat{\mathbb{Z}}^{(p)} \xrightarrow{\sim} T_{\hat{\mathbb{Z}}^{(p)}}(A_{\overline{s}_r})$$

satisfying the compatible condition for Weil pairing; that is, there exists an isomorphism $\nu(\rho_r^p) \in \operatorname{Hom}(\hat{\mathbb{Z}}^{(p)}, \hat{\mathbb{Z}}^{(p)}(1))$ such that the following diagram commutes:

$$\begin{array}{ccc} \Lambda \otimes \hat{\mathbb{Z}}^{(p)} \times \Lambda \otimes \hat{\mathbb{Z}}^{(p)} & \xrightarrow{\langle \cdot,\cdot \rangle} & \hat{\mathbb{Z}}^{(p)} \\ \sim \big\downarrow \rho^p \qquad \sim \big\downarrow \rho^p & & \sim \big\downarrow \nu(\rho_r^p) \\ T_{\hat{\mathbb{Z}}^{(p)}}(A_{\overline{s}_r}) \times T_{\hat{\mathbb{Z}}^{(p)}}(A_{\overline{s}_r}) & \xrightarrow{\text{Weil pairing}} & \hat{\mathbb{Z}}^{(p)}(1). \end{array}$$

Here $T_{\hat{\mathbb{Z}}^{(p)}}(A_{\overline{s}_r})$ denotes the product of the ℓ-adic Tate modules of $A_{\overline{s}_r}$ for all $\ell \neq p$. For simplicity, it is customary to notate that

$$\rho^p \colon \Lambda \otimes \hat{\mathbb{Z}}^{(p)} \xrightarrow{\sim} T_{\hat{\mathbb{Z}}^{(p)}}(A).$$

(3b) ρ_p is a K-level structure at p on A, that is, a collection of $\mathcal{O}_D$-stable closed finite flat subgroup schemes $\rho_{\mathfrak{p}} = H_{\mathfrak{p}} \oplus H_{\overline{\mathfrak{p}}}$ in $A[\mathfrak{p}] \oplus A[\overline{\mathfrak{p}}]$ of order $(\# k_{\mathfrak{p}})^{n^2}$ such that we have $H_{\mathfrak{p}}^\vee \cong H_{\overline{\mathfrak{p}}}$ under the perfect pairing

$$A[\mathfrak{p}] \times A[\overline{\mathfrak{p}}] \longrightarrow \mu_p$$

induced by $\lambda \colon A \to A^\vee$.

(4) $\underline{\mathscr{F}}$ is a splitting filtration above $\mathfrak{p}$, i.e. a collection $(\mathscr{F}_j^{(i)})_{\substack{i=0,\dots,e \\ j=1,\dots,f}}$ such that:

(4a) Each direct summand $\omega_{A^\vee/S,j}$ of the sheaf $\omega_{A^\vee/S}$, on which $W(k_0)$ acts through τ_j, admits a filtration

$$0 = \mathscr{F}_j^{(0)} \subseteq \mathscr{F}_j^{(1)} \subseteq \cdots \subseteq \mathscr{F}_j^{(e)} = \omega_{A^\vee/S,j}$$

of locally free sheaves of $\mathcal{O}_S$-modules. We impulse a condition that for each $1 \leqslant i \leqslant e$ the subquotient $\mathscr{F}_j^{(i)}/\mathscr{F}_j^{(i-1)}$ is locally free of rank $r_j^{(i)} = n - s_j^{(i)} \in \{0, 1, \dots, n\}$ as an $\mathcal{O}_S$-module;

(4b) Each sheaf $\mathscr{F}_j^{(i)}$ is invariant under those S-endomorphisms on A that come from $\mathcal{O}_D$, i.e. $\alpha(\mathscr{F}_j^{(i)}) = \mathscr{F}_j^{(i)}$ for all $\alpha \in \operatorname{Im}(\mathcal{O}_D \to \operatorname{End}_S(A))$.

(4c) Denote $\varpi = \varpi_{\mathfrak{p}}$ a choice of the uniformizer for the completed local ring $\mathcal{O}_{D_{\mathfrak{p}}}$. Then the $\mathcal{O}_{D_{\mathfrak{p}}}$-action of multiplication of $[\varpi]$ on $\mathscr{F}_j^{(i)}/\mathscr{F}_j^{(i-1)}$ is given by $\tau_j^{(i)}(\varpi)$, so that

$$\begin{aligned}0 &= ([\varpi]-\tau_j^{(i)}(\varpi))\cdot \mathscr{F}_j^{(i)}/\mathscr{F}_j^{(i-1)}\\ &= (\varpi\otimes 1 - 1\otimes \tau_j^{(i)}(\varpi))\cdot \mathscr{F}_j^{(i)}/\mathscr{F}_j^{(i-1)}.\end{aligned}$$

Recall for Definition 1.1(2) the Rosati involution condition. We obtain a morphism $\eta\colon \mathcal{O}_D\to \mathrm{End}_S(A)$ of rings satisfying that, for each $b\in\mathcal{O}_D$, we have $\lambda\circ\eta(b^*)=\eta(b)^\vee\circ\lambda$. Here $\eta(b^*)$ is called the Rosati involution, which is a quasi-isogeny.

In Definition 1.1(3), while defining the prime-to-p K-level structure ρ^p, the compatibility condition between Hermitian pairing and Weil pairing is necessary. In fact, Weil pairing does not carry the "arithmetic symmetry" induced by Hermitian pairing. So we first turn the canonical Hermitian form $[\cdot,\cdot]$ into a $\mathbb{Q}$-valued symplectic form $\langle\cdot,\cdot\rangle$, and then compare it with Weil pairing.

The signature condition in Definition 1.1(1) can be explained via (4a) as follows. The relative de Rham homology $H_1^{\mathrm{dR}}(A/S)\coloneqq H^1_{\mathrm{dR}}(A^\vee/S)$ carries an action of $\mathcal{O}_F\otimes\mathcal{O}_S\simeq \bigoplus_{\tau\in\mathrm{Hom}(F,\mathbb{C})}\mathcal{O}_S$ just so there is a short exact sequence, induced by the Hodge filtration,

$$0\to\omega_{A^\vee/S}\to H_1^{\mathrm{dR}}(A/S)\to \mathrm{Lie}_{A/S}\to 0,$$

in which $\omega_{A^\vee/S}$ is the dualizing sheaf of invariant differentials. This Hodge filtration is compatible with the $\mathcal{O}_S$-action. Here we apply the homology theory instead of that of cohomology at work for the resulting covariant convenience. Alternatively, we can assume that A satisfies the Kottwitz determinant condition, that is, there is a natural ring homomorphism $\mathcal{O}_D\to\mathrm{End}_S A$ sending each $\alpha\in\mathcal{O}_D$ to its characteristic polynomial, given by

$$\mathrm{char\,poly}(\alpha)=\prod_{j=1}^f\prod_{i=1}^e(x-\tau_j^{(i)}(\alpha))^{s_j^{(i)}}(x-\tau_j^{(i),\perp}(\alpha))^{r_j^{(i)}}\in\mathrm{End}(\mathrm{Lie}_{A/S}).$$

Moreover, this condition of elements $\alpha\in\mathcal{O}_D$ acting on $\mathrm{Lie}_{A/S}$ can be reformulated in a geometric manner. Let M be a sheaf on S carrying a locally free $\mathcal{O}_D$-module structure. Then M further admits an action of $\mathcal{O}_D\otimes W(k_0)$, and hence breaks up as a direct sum $M=\bigoplus_\varphi M_\varphi$, where φ runs through all induced homomorphisms $\mathcal{O}_D\to W(k_0)$.

Hopefully, as each $\mathcal{O}_D$-action can be realized as an element of $\mathrm{End}_S A$, there exist natural decompositions

$$H_1^{\mathrm{dR}}(A/S)=\bigoplus_{j=1}^f H_1^{\mathrm{dR}}(A/S)_j\supseteq\bigoplus_{j=1}^f\omega_{A^\vee/S,j}=\omega_{A^\vee/S}.$$

It is known that $H_1^{\mathrm{dR}}(A/S)$ appears to be a locally free $(\mathcal{O}_D\otimes_{\mathbb{Z}}\mathcal{O}_S)$-module of rank n, and for each direct summand (cf. [Rap78, Lemme 1.3] and [Vol05, Proposition 2.11]), we have with $r_j^{(i)}\in\{0,1,\dots,n\}$ that

$$0\leqslant \mathrm{rank}_{\mathcal{O}_S}\,\omega_{A^\vee/S,j}=\sum_{i=1}^e r_j^{(i)}\leqslant ne=\mathrm{rank}_{\mathcal{O}_S} H_1^{\mathrm{dR}}(A/S)_j.$$

Also, each $\omega_{A^\vee/S,j}$ is stable under the $\mathcal{O}_{D_\mathfrak{p}}$-actions.

In fact, whenever K is sufficiently small, the Pappas–Rapoport functor is representable via a smooth object $\mathcal{M}^{\mathrm{PR}}_{K,\Lambda} \in \mathsf{Sch}^{\text{loc-noe}}_{\mathcal{O}}$ of finite type. For each base scheme S, the isomorphism classes of $(A, \lambda, \rho, \underline{\mathscr{F}})$ are parametrized by the morphisms $\mathcal{M}^{\mathrm{PR}}_{K,\Lambda} \to S$ of locally noetherian schemes over $W(k_0)$. The representability can be morally deduced from the following two facts:

(i) When $n = 2$, $\underline{\mathcal{M}}^{\mathrm{DP}}_{K,\Lambda}$ of Deligne–Pappas is representable by $\mathcal{M}^{\mathrm{DP}}_{K,\Lambda}$, see [DP94];

(ii) $\underline{\mathcal{M}}^{\mathrm{PR}}_{K,\Lambda}$ is relatively representable over $\mathcal{M}^{\mathrm{DP}}_{K,\Lambda}$ by a closed scheme of a Grassmannian, because for any S-point $(A, \lambda, \rho) \in \mathcal{M}^{\mathrm{DP}}_{K,\Lambda}(S)$, the dualizing sheaf $\omega_{A^\vee/S,j}$ is exactly locally free of rank e over $\mathcal{O}_S$ (cf. [RX17, Proposition 2.4]).

However, the scheme $\mathcal{M}^{\mathrm{PR}}_{K,\Lambda}$ is not connected in general [Hel12, p.43], but its connected components are pairwise isomorphic. A Shimura variety associated to G arises as a $W(k_0)$-scheme from each connected component of $\mathcal{M}^{\mathrm{PR}}_{K,\Lambda}$. Since we obtain the moduli problem at work, and a global description of $\mathcal{M}^{\mathrm{PR}}_{K,\Lambda}$ is in need, it would be more convenient to concern about $\mathcal{M}^{\mathrm{PR}}_{K,\Lambda}$ than about its connected components. Therefore, we will fraudulently call $\mathcal{M}^{\mathrm{PR}}_{K,\Lambda}$ a *unitary Shimura variety* (before or after a descent), which is indeed finite copies of the disjoint union of those (see discussions at the end of Subsection 1.4).

Remark 1.2. To make the splitting model better understood by the reader, we also comment on the comparison of different moduli spaces. In [DP94], the *Deligne–Pappas moduli problem* $\underline{\mathcal{M}}^{\mathrm{DP}}_{K,\Lambda}$ parametrizes isomorphism classes of (A, λ, ρ) and is representable by some locally noetherian $W(k_0)$-scheme $\mathcal{M}^{\mathrm{DP}}_{K,\Lambda}$. Moreover, the characteristic p fiber $\mathcal{M}^{\mathrm{DP}}_{K,\Lambda,k_0}$ is normal and contains the so-called *Rapoport smooth locus* $\mathcal{M}^{\mathrm{Ra}}_{K,\Lambda,k_0}$, which parametrizes (A, λ, ρ) such that each $\omega_{A^\vee/S,j}$ is locally free $(\mathcal{O}_D \otimes_{\mathbb{Z}} \mathcal{O}_S)$-modules of rank 1 for any $j = 1, \dots, f$.

As for a geometric interpretation, for example, when $n = 2$ and $e = 2$, the splitting model $\mathcal{M}^{\mathrm{PR}}_{K,\Lambda}$ is attained by blowing up $\mathcal{M}^{\mathrm{DP}}_{K,\Lambda}$ with respect to its isolated singularities, whose exceptional divisors are all (-2)-curves birationally equivalent to $\mathbb{P}^1$. Here the singularities of $\mathcal{M}^{\mathrm{DP}}_{K,\Lambda}$ correspond to the abelian schemes whose sheaves of invariant differentials are not free over $\mathcal{O}_D \otimes_{\mathbb{Z}} \mathcal{O}_S$, even if they have the same $\mathcal{O}_S$-ranks as on the ordinary locus. Definitely, there is a natural forgetful transformation by dropping the filtrations:

$$\pi\colon \mathcal{M}^{\mathrm{PR}}_{K,\Lambda} \longrightarrow \mathcal{M}^{\mathrm{DP}}_{K,\Lambda}, \quad (A, \lambda, \rho, \underline{\mathscr{F}}) \longmapsto (A, \lambda, \rho).$$

The original motivation in [PR05] to introduce $\underline{\mathscr{F}}$ is modeled on the twisted-product resolution of the affine Grassmannian for GL_2:

$$(\mathbb{P}^1)^{\tilde{\times} e} = \underbrace{\mathbb{P}^1 \tilde{\times} \mathbb{P}^1 \tilde{\times} \cdots \tilde{\times} \mathbb{P}^1}_{e \text{ copies}} \longrightarrow \mathbf{Gr}.$$

Also note that in case when p ramifies in F, $\mathcal{M}^{\mathrm{Ra}}_{K,\Lambda}$ is dense in $\mathcal{M}^{\mathrm{DP}}_{K,\Lambda}$, and the choices of the filtrations $\underline{\mathscr{F}}$ are always unique over $\mathcal{M}^{\mathrm{Ra}}_{K,\Lambda}$. For the trivial case with $e = 1$, we have $\mathcal{M}^{\mathrm{PR}}_{K,\Lambda} = \mathcal{M}^{\mathrm{DP}}_{K,\Lambda} = \mathcal{M}^{\mathrm{Ra}}_{K,\Lambda}$.

Notation 1.3. In the upcoming context, if no confusion arises, we will denote $\mathcal{M} = \mathcal{M}^{\mathrm{PR}}_{K,\Lambda}$, and $\mathcal{A} = \mathcal{A}^{\mathrm{PR}}_{K,\Lambda}$ for the universal abelian scheme over $\mathcal{M}$, arising from the Pappas–Rapoport functor. These are schemes over $\mathcal{O}_{F^{\mathrm{Gal}},(p)}$, and their special characteristic p fibers are defined to be the base changes to k_0, denoted by $\mathcal{M}_{k_0}$ and $\mathcal{A}_{k_0}$, respectively.

1.3. **Universal filtration over the splitting model.** Following the notations as before, consider the universal abelian scheme $\mathcal{A}$ over $\mathcal{M}$. By Definition 1.1, it is equipped with a universal polarization $\lambda\colon \mathcal{A} \to \mathcal{A}^\vee$, which further induces an isomorphism

$$\lambda^*\colon H_1^{\mathrm{dR}}(\mathcal{A}/\mathcal{M}) \xrightarrow{\sim} H_1^{\mathrm{dR}}(\mathcal{A}^\vee/\mathcal{M}) \xrightarrow{\sim} (H_1^{\mathrm{dR}}(\mathcal{A}/\mathcal{M}))^\vee.$$

Recall the paring $\langle\cdot,\cdot\rangle$ enjoys the property that for all $a \in \mathcal{O}_{D_\mathfrak{p}}$ and $x, y \in H_1^{\mathrm{dR}}(\mathcal{A}/\mathcal{M})_j$,

$$\langle ax, y\rangle = \langle x, ay\rangle, \quad \langle ax, x\rangle = 0.$$

Thus, taking the direct summand on which $W(k_0)$ acts through τ_j, we attain a non-degenerate symplectic pairing of Weil type, read as

$$\langle\cdot,\cdot\rangle\colon H_1^{\mathrm{dR}}(\mathcal{A}/\mathcal{M})_j \times H_1^{\mathrm{dR}}(\mathcal{A}/\mathcal{M})_j^\perp \longrightarrow \mathcal{O}_\mathcal{M},$$

where $H_1^{\mathrm{dR}}(\mathcal{A}/\mathcal{M})_j$ denotes the component admitting the action by $\tau_j^\perp$. For the geometric problems on special fibers, it suffices to explore the inner structure of $\mathcal{O}_{\mathcal{M}_{k_0}}$-bundle $\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j$.

Definition 1.4. Denote *the universal filtration*

$$\underline{\mathcal{F}} = (\mathcal{F}_j^{(i)})_{\substack{i=0,\dots,e\\ j=1,\dots,f}}$$

of the pullback $\omega_{\mathcal{A}_{k_0}^\vee/\mathcal{M}_{k_0}} := e^*\Omega^1_{\mathcal{A}^\vee/\mathcal{M}}$ via the zero section $e\colon \mathcal{M} \to \mathcal{A}^\vee$ of the sheaf of relative differentials of $\mathcal{A}_{k_0}^\vee/\mathcal{M}_{k_0}$. More precisely, this gives an $\mathcal{O}_{D_\mathfrak{p}}$-stable filtration

$$0 = \mathcal{F}_j^{(0)} \subseteq \mathcal{F}_j^{(1)} \subseteq \cdots \subseteq \mathcal{F}_j^{(e)} = \omega_{\mathcal{A}_{k_0}^\vee/\mathcal{M}_{k_0}} \subseteq \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j$$

for each $j = 1, \dots, f$, satisfying that

(1) for $i = 1, \dots, e$, each subquotient $\mathcal{F}_j^{(i)}/\mathcal{F}_j^{(i-1)}$ is a locally free sheaf of $\mathcal{O}_\mathcal{M}$-module of rank $\tilde{r}_j^{(i)} = n - \tilde{s}_j^{(i)} \in \{0, 1, \dots, n\}$;
(2) the action of $\mathcal{O}_{D_\mathfrak{p}}$ by $[\varpi]$ on the subquotient $\mathcal{F}_j^{(i)}/\mathcal{F}_j^{(i-1)}$ is given by $\tau_j^{(i)}(\varpi)$.

Unwinding the construction, $\mathcal{F}_j^{(i)}$ is a $\mathcal{O}_\mathcal{M}$-subbundle of $\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j$ and it is natural to consider the quotient bundle $\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j/\mathcal{F}_j^{(i)}$ for each $i = 0, \dots, e$. We are particularly interested in the part on which the action of $\mathcal{O}_{D_\mathfrak{p}}$ via $[\varpi]$ factors through $\tau_j^{(i)}\colon W(k_0) \to \overline{\mathbb{Z}}_p$. Define for $1 \leqslant i \leqslant e$ that

$$\mathcal{H}_j^{(i)} := \{z \in \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j/\mathcal{F}_j^{(i-1)} \colon ([\varpi] - \tau_j^{(i)}(\varpi)) \cdot z = 0\}.$$

Moreover, due to the assumptions on $\mathcal{A}$, the de Rham homology $\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})$ carries a faithful $\mathcal{O}_\mathcal{M}$-action, and thus can be regarded as a locally free sheaf of $\mathcal{O}_\mathcal{M}$-rank $2ef$ on $\mathcal{M}$. Depending on the context, we may write $\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}/\mathcal{M})$ (resp. $\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j$) instead of $H_1^{\mathrm{dR}}(\mathcal{A}/\mathcal{M})$ (resp. $H_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j$) to emphasize its sheaf position if necessary.

Proposition 1.5 (cf. [PR05, Proposition 5.2])**.** *For each $i = 0, \dots, e$ and $j = 1, \dots, f$, the sheaf $\mathcal{H}_j^{(i)}$ is a subbundle of $\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j/\mathcal{F}_j^{(i-1)}$ of rank 2 and contains $\mathcal{F}_j^{(i)}/\mathcal{F}_j^{(i-1)}$.*

Proof. First notice that the property is local and it suffices to prove the proposition for an arbitrary closed point $\operatorname{Spec} k_0 \to \mathcal{M}$. We can evaluate the de Rham cohomology over $\operatorname{Spec} k_0$ to obtain $\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_{k_0}$. To simplify the proof, we insert to choose that $k_0 = \overline{\mathbb{F}}_p$, which is algebraically closed. Recall that

$$\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0}) = \bigoplus_{j=1}^{f} \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j$$

is a locally free module of rank 2 over $\mathcal{O}_{\mathcal{M}} \otimes_{\mathbb{Z}} \mathcal{O}_D$, and each component corresponding to τ_j is locally free of rank 2 over

$$\mathcal{O}_{F_{\mathfrak{p}}} \otimes_{W(k_0),\tau_j} \mathcal{O}_{\mathcal{M}} \cong (k_0[x]/(x^e)) \otimes \mathcal{O}_{\mathcal{M}},$$

equipped with the action $[\varpi]$ by the uniformizer via the multiplication by x. Hence

$$(\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j)_{k_0} \cong (k_0[x]/(x^e))^{\oplus 2} \otimes \mathcal{O}_{\mathcal{M}}.$$

Since, by definition, $\mathcal{H}_j^{(i)}$ is the $[\varpi]$-torsion of $\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j/\mathcal{F}_j^{(i-1)}$, it boils down to show that for any nonzero isotropic $k_0[x]$-submodule $\mathcal{F}$ of $(\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j)_{k_0}$ with $\dim_{k_0} \mathcal{F} < e$, the quotient $\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})/\mathcal{F}$ has 2-dimensional $[\varpi]$-torsion. Without loss of generality, one may assume

$$\mathcal{F} \cong (x^a k_0[x]/(x^e)) \oplus (x^b k_0[x]/(x^e)), \quad 0 < a, b \leqslant e,\ a + b > e,$$

as $k_0[x]$-submodules after choosing an k_0-basis. As $a + b > e$,

$$\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})/\mathcal{F} \supseteq (x^{a-1} k_0[x]/(x^a)) \oplus (x^{b-1} k_0[x]/(x^b)).$$

It is also straightforward that $\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j/\mathcal{F}_j^{(i-1)} \supseteq \mathcal{F}_j^{(i)}/\mathcal{F}_j^{(i-1)}$. □

Recall that we have fixed a choice of labelings of $\tau_j^{(1)}, \dots, \tau_j^{(e)}$ of p-adic embeddings of F^+ into $\overline{\mathbb{Q}}_p$. Following the notation of [RX17], we set an automorphic line bundle

$$\omega_j^{(i)} := \mathcal{F}_j^{(i)}/\mathcal{F}_j^{(i-1)}.$$

This bundle will be used to describe the filtrations in constructing generalized partial Hasse invariants (see Section 3).

Remark 1.6. Prototypically, one can concern about the special fibers of certain $U(2)$-Shimura variety modulo an inert prime p. In [Hel12], the moduli problem is constructed as a particular case of Definition 1.1 where $D = \mathrm{M}_2(F)$, where the involution $*$ equal to complex conjugate transpose of matrices, and where the abelian scheme $(A, \lambda, \rho, \underline{\mathscr{F}})$ has relative dimension $\dim_{\mathbb{Q}}(V)/2 = 2ef$ over S, with the filtration subquotients $\mathscr{F}_j^{(i)}/\mathscr{F}_j^{(i-1)}$ being locally free $\mathcal{O}_{\mathcal{M}}$-modules of rank $r_j^{(i)} \in \{0, 1, 2\}$. In fact, given such a quadruple of the candidate $U(2)$-Shimura variety, by letting $\mathcal{O}_D \simeq \mathrm{M}_2(\mathcal{O}_F)$ acting faithfully on A^2 in the obvious way, we obtain a point $(A^2, \lambda^2, \rho^2, \underline{\mathscr{F}}^2) \in \mathcal{M}_{K,\Lambda}^{\mathrm{PR}}$. Conversely, fix an idempotent $\delta \in (\mathcal{O}_D)^{*=1}$; given $(A', \lambda', \rho', \underline{\mathscr{F}}') \in \mathcal{M}_{K,\Lambda}^{\mathrm{PR}}$, we have a prime-to-$p$ polarization

$\delta\lambda\colon \delta A \to (\delta A)^\vee$ induced from $\lambda\colon A \to A^\vee$, and thus $(\delta A, \delta\lambda, \delta\rho, \delta\underline{\mathscr{F}})$ is a closed point in the $U(2)$-Shimura variety.

1.4. **Representability and complex uniformization.** This subsection refers closely to [HTX17, §2.3]. Acquiescently, we take $\mathcal{M} = \mathcal{M}^{\mathrm{PR}}_{K,\Lambda}$ and $\mathcal{A} = \mathcal{A}^{\mathrm{PR}}_{K,\Lambda}$ across the upcoming discussions. Our goal now is to translate the $\mathbb{C}$-points of the unitary Shimura variety defined by the moduli problem above into a double coset form.

To construct the Shimura datum, on one hand we define a morphism between real algebraic groups

$$h\colon \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m \longrightarrow G_{\mathbb{R}}, \quad z \longmapsto \prod_{j=1}^{f}\prod_{i=1}^{e}\begin{pmatrix} zI_{r_j^{(i)}} & 0 \\ 0 & \overline{z}I_{s_j^{(i)}} \end{pmatrix}.$$

On the other hand, there is a map $\mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$ via $z \mapsto (z,1)$. Define the minuscule cocharacter μ_h to be the composition

$$\mu_h\colon \mathbb{G}_{m,\mathbb{C}} \longrightarrow \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \xrightarrow{\sim} \mathbb{S}_{\mathbb{C}} \xrightarrow{h_{\mathbb{C}}} G_{\mathbb{C}}.$$

Here the first copy of $\mathbb{G}_{m,\mathbb{C}}$ is the one indexed by $\mathrm{id} \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{C})$, and the second copy of $\mathbb{G}_{m,\mathbb{C}}$ is indexed by the complex conjugation.

It follows that (each connected components of) $\mathcal{M}$ is a smooth and projective Shimura variety over $\mathcal{O}$ of relative dimension

$$\dim_{\mathcal{O}} \mathcal{M} = \sum_{j=1}^{f}\sum_{i=1}^{e} r_j^{(i)} s_j^{(i)}.$$

In particular, when $n = 2$, one might happen to pick some abelian scheme $(A, \lambda, \rho, \underline{\mathscr{F}})$ in $\mathcal{M}$, so that all subquotients of $\underline{\mathscr{F}}$ are of non-splitting type (i.e. $r_j^{(i)}$ and $s_j^{(i)}$ are either 0 or 2 for each i and j); the Shimura variety is thus discrete of relative dimension 0.

Denote $\mathcal{M}(\mathbb{C})$ the complex points of $\mathcal{M}$ via this embedding. Let K_∞ be the stabilizer of h under the conjugation action, as a subgroup of $G(\mathbb{R})$. Then K_∞ is a maximal compact subgroup of $G(\mathbb{R})$, modulo the center. Let X_∞ be the $G(\mathbb{R})$-conjugacy class of h. More precisely, excluding the case where n is even and all $r_j^{(i)}$s and $s_j^{(i)}$s are equal to $n/2$, it can be realized as

$$X_\infty = \prod_{j=1}^{f}\prod_{i=1}^{e} U_{\mathbb{R}}(r_j^{(i)}, s_j^{(i)})/U_{\mathbb{R}}(r_j^{(i)}) \times U_{\mathbb{R}}(s_j^{(i)}).$$

According to [Kot92, pp. 398–400], we take

$$\mathrm{Ker}^1(\mathbb{Q}, G) = \begin{cases} \mathrm{Ker}(F^{+,\times}/\mathbb{Q}^\times N_{F/F^+}(F^\times) \to \mathbb{A}_{F^+}^\times/\mathbb{A}^\times N_{F/F^+}(\mathbb{A}_F^\times)), & 2 \nmid n; \\ (0), & 2 \mid n. \end{cases}$$

It intrinsically depends only on the parity of n and the extension $F/F^{\times}$. Then

$$\begin{aligned}\mathcal{M}(\mathbb{C}) &\simeq \coprod_{i\in\mathrm{Ker}^1(\mathbb{Q},G)} G(\mathbb{Q})\backslash(X_\infty \times (G(\mathbb{A}_f)/K)) \\ &\simeq \coprod_{i\in\mathrm{Ker}^1(\mathbb{Q},G)} G(\mathbb{Q})\backslash G(\mathbb{A})/K \times K_\infty,\end{aligned}$$

the disjoint union of $\#\,\mathrm{Ker}^1(\mathbb{Q},G)$ isomorphic copies.

1.5. **The $U(2)$-assumption.** For the majority of our purposes in this paper, we work over the Pappas–Rapoport splitting model for $n=2$ as in Definition 1.1.

Assumption 1.7. Unless otherwise stated, for all $i=1,\dots,e$ and $f=1,\dots,f$, the default possible value of $r_j^{(i)}$ (resp. $s_j^{(i)} = n - r_j^{(i)}$) lies in $\{0,1,2\}$.

2. Dieudonné modules and Grothendieck–Messing deformation

This section serves as a preliminary of the forthcoming works. The language of Dieudonné modules involves a more general description of de Rham homology theory and truly helps to construct the generalized partial Hasse invariants (see Section 3). Furthermore, the crystalline deformation theory of abelian varieties by Grothendieck–Messing, is the core technique and will be used over later. In particular, it essentially supports the smoothness for the intersection of the zero loci of the generalized partial Hasse invariants. We will carefully apply this deformation theory in proving Theorem 4.1 later.

2.1. **Notations and definitions.** Let k be a perfect field of characteristic $p>0$, equipped with the arithmetic Frobenius map $\sigma\colon k\to k$ via $x\mapsto x^p$. Denote $W(k)$ the ring of Witt vectors. When $k/\mathbb{F}_p$ is a finite extension, $W(k)$ is the ring of integers of the unique extension of $\mathbb{Q}_p$ with residue field k. We still write $\sigma\colon W(k)\to W(k)$ for the unique automorphism lifting σ on k. Let A be an abelian scheme over k. The pullback of A along σ is written as $A^{(p)} = A\times_{k,\sigma} k$, which is an abelian scheme as well.

- The *absolute Frobenius morphism* $\mathrm{Fr}\colon A\to A$ is a homeomorphism between the underlying topological spaces, sending each $f\in\mathcal{O}_A$ to f^p. The *relative Frobenius morphism* (or simply the *Frobenius morphism*) $F_{A/k}\colon A\to A^{(p)}$ is the unique fiber of F along the natural morphism $A^{(p)}\to A$, i.e. it is the morphism given by the diagram

$$\begin{tikzcd}
A \arrow[dr, dashed, "F_{A/k}"] \arrow[drr, bend left, "\mathrm{Fr}"] \arrow[ddr, bend right] & & \\
& A^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A \arrow[d] \\
& \operatorname{Spec} k \arrow[r, "\sigma^*"] & \operatorname{Spec} k.
\end{tikzcd}$$

  By definition, the Frobenius morphism commutes with fiber products. Consequently, $F_{A/k}$ is a homomorphism compatible with the group structures on A and $A^{(p)}$. When the meaning is clear, it is customary to simply notate $F_{A/k}$ by F.

- The *Cartier dual scheme of* A is a scheme $A^\vee$ defined by taking the connected component of $\mathrm{Pic}(A/k)$ that contains the zero section of $A \to \operatorname{Spec} k$. In the characteristic-$p$ case, consider the quotient group scheme $H = A^\vee / A^\vee_{\mathrm{red}}$, which is an infinitesimal group as it contains a single point only. Here $A^\vee_{\mathrm{red}}$ is clearly an abelian scheme and one can prove that $H = 0$. Thus, $A^\vee$, as an extension of an infinitesimal group by an abelian scheme, is abelian as well.
- The *Verschiebung morphism of* A is the Cartier dual of the Frobenius $F = F_{A^\vee/k} \colon A^\vee \to (A^\vee)^{(p)} = (A^{(p)})^\vee$, written as $V_A \colon A^{(p)} \to A$. Usually, we loosen the subscript and denote $V = V_A$ if no confusion arises.

Lemma 2.1 (Cartier duality)**.** *We obtain the following relations:*

$$V \circ F = p \cdot \mathrm{Id}_A, \quad F \circ V = p \cdot \mathrm{Id}_{A^{(p)}}.$$

Resuming our original setups, we consider the universal abelian scheme $\mathcal{A}/\mathcal{M}$. There are the relative Frobenius and the Verschiebung morphisms

$$F = F_{\mathcal{A}/\mathcal{M}} \colon \mathcal{A} \longrightarrow \mathcal{A}^{(p)}, \quad V \colon \mathcal{A}^{(p)} \longrightarrow \mathcal{A},$$

Simultaneously, the arithmetic Frobenius $\sigma \colon k_0 \to k_0$ induces morphism $\mathcal{M} \to \mathcal{M}$ of schemes. For any sheaf on $\mathcal{M}$, use the superscripted notation $\cdot^{(p)}$ to denote the pullback along this induced morphism. These further render

$$\begin{aligned} F \colon \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}^{(p)}/\mathcal{M}) &\longrightarrow \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}/\mathcal{M}), \\ V \colon \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}/\mathcal{M}) &\longrightarrow \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}^{(p)}/\mathcal{M}). \end{aligned}$$

Moreover, $\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}^{(p)}/\mathcal{M}) \cong \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}/\mathcal{M})^{(p)}$. Recall that by fixing an embedding of $W(k_0)$ into $\mathbb{C}$, each real embedding τ of F^+ corresponds to a unique map $\tau \colon \mathcal{O}_{F^+} \to k_0$ by abuse of notation. Consequently, there is a natural composition $\sigma \circ \tau \colon \mathcal{O}_{F^+} \to k_0 \to k_0$, which again corresponds to another real embedding $\sigma\tau$. Also, $\sigma^{-1}\tau$ can be defined in the same way. Then it turns out that, at the level of direct summands, the two induced morphisms can be reformulated more subtly, read as

$$\begin{aligned} F_\tau \colon \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}/\mathcal{M})_\tau^{(p)} &\longrightarrow \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}/\mathcal{M})_{\sigma\tau}, \\ V_\tau \colon \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}/\mathcal{M})_\tau &\longrightarrow \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}/\mathcal{M})_{\sigma^{-1}\tau}^{(p)}. \end{aligned}$$

In fact, we obtain

$$\begin{aligned} \operatorname{Ker} F_{\sigma^{-1}\tau} = \operatorname{Im} V_\tau = \omega^{(p)}_{\mathcal{A}^\vee/\mathcal{M},\sigma^{-1}\tau} &\subseteq \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}/\mathcal{M})^{(p)}_{\sigma^{-1}\tau}, \\ \operatorname{Ker} V_{\sigma\tau} = \operatorname{Im} F_\tau = \mathrm{Lie}_{\mathcal{A}/\mathcal{M},\sigma\tau} &\subseteq \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}/\mathcal{M})_{\sigma\tau}. \end{aligned}$$

Finally, notice that all these constructions are compatible with base changes to special fibers. We will check in Proposition 3.2 that the filtration $\underline{\mathcal{F}}$ (c.f. Definition 1.1) for defining the unitary Shimura variety is stable under V (and similarly, stable under F).

2.2. **The Dieudonné modules.** The ultimate goal of Dieudonné module theory is to understand the behaviors of abelian schemes over an arbitrary base, or at least over rings of integers (and, localizations and completions thereof). Obtaining the setups above, we are ready to approach Dieudonné module theory by first defining the base ring, in which the generators are simulations of the relative Frobenius and the Verschiebung.

Definition 2.2. The *Dieudonné ring* $\mathcal{D}_k$ over k is the associative $W(k)$-algebra, which is non-commutative when $k \neq \mathbb{F}_p$, generated by elements F and V subject to the relations

$$FV = VF = p, \quad Fc = \sigma(c)F, \quad cF = V\sigma(c)$$

for all $c \in W(k)$.

The following theorem serves as a mainstay to support the theory. It mainly describes the compatibility between the structures of $\mathcal{D}_k$-module and $W(k)$-module. The *contravariant* functor $\mathcal{D}$ below is called the *Dieudonné functor*. Philosophically, the motivation to define such a functor goes over pretending a p-torsion group scheme to be semilinear.

For this theorem, we define $\mathsf{FGS}(k)[p^\infty]^{\mathrm{op}}$ as the opposite category of finite p-torsion (i.e. the torsion of some power of p, or the torsion killed by some power of V) commutative group schemes over k, and define $\mathsf{Mod}(\mathcal{D}_k)$ for simplicity as the category of all $\mathcal{D}_k$-modules of finite $W(k)$-length.

Theorem 2.3 ([CCO14], Theorem 1.4.3.2)**.** *There is a categorical equivalence*

$$\begin{aligned} \mathcal{D}\colon \mathsf{FGS}(k)[p^\infty]^{\mathrm{op}} &\xrightarrow{\sim} \mathsf{Mod}(\mathcal{D}_k) \\ G_n &\longmapsto \mathcal{D}(G_n). \end{aligned}$$

In particular, letting A be an abelian scheme over k and $A[p^r]$ the p-torsion part of A annihilated by $[p^r]\colon A \to A$, we have the following for $\mathcal{D}(A[p^r])$ the associated $\mathcal{D}_k$-module to $A[p^r]$.

(1) *As a $W(k)$-module, $\mathcal{D}(A[p^r])$ has length r.*

(2) *The functor $\mathcal{D}$ is functorial in the base field, i.e., given another perfect field $k'/\mathbb{F}_p$ together with a map $k \to k'$, there is a natural isomorphism*

$$\mathcal{D}(A_{k'}[p^r]) \cong \mathcal{D}(A[p^r]) \otimes_{W(k)} W(k'),$$

where $A_{k'} = A \times_k k'$.

(3) *The relative Frobenius $F_{A[p^r]/k}\colon A[p^r] \to A[p^r]^{(p)}$ corresponds to the linearization*

$$\mathcal{D}(A[p^r]^{(p)}) \xrightarrow{\sim} \mathcal{D}(A[p^r])^{(p)} = \sigma^*\mathcal{D}(A[p^r]) \longrightarrow \mathcal{D}(A[p^r]).$$

(4) *The Verschiebung $V\colon A[p^r]^{(p)} \to A[p^r]$ corresponds to the linearization*

$$\mathcal{D}(A[p^r]) \longrightarrow \sigma^*\mathcal{D}(A[p^r]) = \mathcal{D}(A[p^r])^{(p)} \xrightarrow{\sim} \mathcal{D}(A[p^r]^{(p)}).$$

(5) *For the Cartier dual $A^\vee[p^r] = A[p^r]^\vee = \underline{\mathrm{Hom}}(A[p^r], \mathbb{G}_m)$, there is a natural isomorphism*

$$\mathcal{D}(A^\vee[p^r]) \cong \mathcal{D}(A[p^r])^*$$

of $\mathcal{D}_k$-modules. Here $\mathcal{D}(A[p^r])^*$ is the $W(k)[\frac{1}{p}]/W(k)$-dual of $\mathcal{D}(A[p^r])$; the F-operator and V-operator on $\mathcal{D}(A[p^r])^*$ are respectively semilinear duals to the V-operator and F-operator on $\mathcal{D}(A[p^r])$.

(6) *For the F-operator of $\mathcal{D}_k$,*

$$\mathcal{D}(A[p^r])/F\mathcal{D}(A[p^r]) \cong (\mathrm{Tgt}_e\, A[p^r])^\vee,$$

where the right-hand side is the Cartier dual of the tangent space to $A[p^r]$ at its identity e.

Example 2.4. For further practice such as to describe Goren–Oort strata, the associated $\mathcal{D}_k$-modules of the basic finite p-torsion commutative group schemes over k are:

(a) $\mathcal{D}(\alpha_{p^n/k}) = \mathcal{D}_k/(F^n(\mathcal{D}_k) + V(\mathcal{D}_k))$, where $\alpha_{p^n/k} = \operatorname{Spec} k[T]/(T^{p^n})$ denotes the kernel of $(F_{\mathbb{G}_a/k})^n$);
(b) $\mathcal{D}(\underline{\mathbb{Z}/p\mathbb{Z}}_k) = \mathcal{D}_k/((F-1)(\mathcal{D}_k) + V(\mathcal{D}_k))$;
(c) $\mathcal{D}(\mu_{p^n/k}) = \mathcal{D}(\underline{\mathbb{Z}/p\mathbb{Z}}_k)^* = \mathcal{D}_k/(F^n(\mathcal{D}_k) + (V-1)(\mathcal{D}_k))$, where the group scheme $\mu_{p^n/k} = \operatorname{Spec} k[T]/(T^{p^n} - 1)$ is that of pth roots of unity over k.

Moreover, we are able to describe a Dieudonné module $\mathcal{D}$ as a $W(k)$-module equipped with a σ-linear map $F\colon M \to M$ and a σ^{-1}-linear map $V\colon M \to M$ with $\sigma\colon W(k) \to W(k)$. Thus,

(a') $\mathcal{D}(\alpha_{p^n/k}) \simeq k$ with $F = V = 0$;
(b') $\mathcal{D}(\underline{\mathbb{Z}/p\mathbb{Z}}_k) \simeq k$ with $F = \sigma$ and $V = 0$;
(c') $\mathcal{D}(\mu_{p^n/k}) \simeq k$ with $F = 0$ and $V = \sigma^{-1}$.

We are now able to use the language of Dieudonné functor, to define a p-adic module, which captures the connotative arithmetic information implied in p-torsions of A, by analogy with the Tate module away from p.

Definition 2.5. Let A be an abelian variety over a perfect field k of characteristic $p > 0$. The *Tate module* of A is defined as

$$T_p(A) := \varprojlim_n \mathcal{D}(A[p^n]).$$

Since $\mathcal{D}_k$ is defined over $W(k)$, we can tensor this $\mathcal{D}_k$-module to get a rational version

$$V_p(A) := T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

2.3. **p-divisible groups.**

Definition 2.6. A *p-divisible group* (or *Barsotti–Tate group*) over k of height h is an inductive system $G = (G_\nu, i_\nu)_{\nu \geqslant 0}$, where G_ν is a finite group scheme over k of rank $p^{h\nu}$, and there is an exact sequence

$$0 \longrightarrow G_\nu \xrightarrow{i_\nu} G_{\nu+1} \xrightarrow{[p^\nu]} G_{\nu+1}$$

for each ν, in which the last map is given by multiplication by p^ν.

For an abelian scheme A over k of dimension g, the result about counting points deduces that each $A[p^\nu]$ is of rank $p^{2g\nu}$. Let $i_\nu\colon A[p^\nu] \to A[p^{\nu+1}]$ be the natural p-torsion map, so that $A(p) \coloneqq (A[p^\nu], i_\nu)_{\nu \geqslant 0}$ admits a p-divisible group structure of height $2g$.

The upshot to introduce p-divisible groups here is that they are limits of finite p-torsion group schemes, and so the categorical equivalence functor $\mathcal{D}$ in Theorem 2.3 extends to an equivalence

$$\begin{aligned} \tilde{\mathcal{D}}\colon p\mathsf{Div}(k) &\xrightarrow{\ \sim\ } \mathcal{D}\mathsf{Mod}(\mathcal{D}_k) \\ \varinjlim G_n &\longmapsto \varprojlim \mathcal{D}(G_n) \end{aligned}$$

between

- the category $p\mathsf{Div}(k)$ of p-divisible groups over k, and
- the category $\mathcal{D}\mathsf{Mod}$ of Dieudonné modules (i.e. projective limits of $\mathcal{D}_k$-modules).

Explicitly, the target category is equivalent to another category whose objects are triples $(\mathcal{D}, F, V)$, where $\mathcal{D}$ is a free $W(k)$-module of finite length and $F\colon \mathcal{D} \to \mathcal{D}$ (resp. $V : \mathcal{D} \to \mathcal{D}$) is a σ-linear (resp. σ^{-1}-linear) operator, satisfying $VF = FV = p$. Moreover, the compatible conditions (1)–(6) in Theorem 2.3 stay valid after extension. In particular, $\tilde{\mathcal{D}}$ acts on p-divisible groups to abelian varieties by

$$A(p) = \varinjlim_n A[p^n] \longmapsto \varprojlim_n \mathcal{D}(A[p^n]).$$

Since $\tilde{\mathcal{D}}$ is the composition of projective limit and $\mathcal{D}$, we see $\tilde{\mathcal{D}}$ is a *covariant theory.*

In practice, an important result of Oda reveals the relationship between Dieudonné module theory and de Rham homology theory.

Theorem 2.7 ([Oda69], Corollary 5.11)**.** *Let A be an abelian scheme over a perfect field k of characteristic $p > 0$. Then there is a canonical isomorphism*

$$\psi\colon \mathcal{D}(A^\vee[p]) \xrightarrow{\sim} H_1^{\mathrm{dR}}(A/k)$$

of $\mathcal{D}_k$-modules with finite $W(k)$-lengths. Moreover, the submodule $V\mathcal{D}(A^\vee[p])$ is mapped onto the subspace $\omega_{A^\vee/k}$ along ψ, i.e. under ψ the following two exact sequences correspond to each other:

$$\begin{tikzcd} 0 \arrow[r] & V\mathcal{D}(A^\vee[p]) \arrow[r] \arrow[d, "\sim"] & \mathcal{D}(A^\vee[p]) \arrow[r] \arrow[d, "\psi"', "\sim"] & \mathcal{D}(A^\vee[p])/V\mathcal{D}(A^\vee[p]) \arrow[r] \arrow[d, "\sim"] & 0, \\ 0 \arrow[r] & \omega_{A^\vee/k} \arrow[r] & H_1^{\mathrm{dR}}(A/k) \arrow[r] & \mathrm{Lie}_{A/k} \arrow[r] & 0. \end{tikzcd}$$

Remark 2.8. When k is a perfect field of characteristic $\ell \neq p$ with A an abelian scheme over k of dimension g, the p-divisible group $A(p)$ appears to be étale, i.e. it is a constant group scheme after base change; equivalently, the F-operator is bijective on $\mathcal{D}(A(p))$. Denote $V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ the rational ℓ-adic Tate module of A. There is a Galois representation $\mathrm{Gal}(\overline{k}/k) \to \mathrm{GL}_{2g}(\mathbb{Q}_p)$ arising from

$$V_p(A) = A(p)(\overline{k}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, \quad V_p(A)^\vee = H^1_{\text{ét}}(A, \mathbb{Q}_p).$$

Moreover, there is one more exotic phenomenon:

- ⋄ (Tate's isogeny conjecture) *Suppose k is finitely generated over its prime field. Let A, B be two abelian varieties over k. Then there is a bijective*

$$\mathrm{Hom}(A,B)\otimes_{\mathbb{Z}}\mathbb{Q}_\ell \longrightarrow \mathrm{Hom}_{\mathrm{Gal}(\overline{k}/k)}(V_\ell(A),V_\ell(B))$$

is bijective.

This, being regarded as a machine that transfers geometric objects to linear-algebraic objects (and vice versa), is very convenient for studying abelian varieties via studying ℓ-adic Galois representations. Its name thus follows from a consequence that A, B are isogenous if and only $V_\ell(A) \simeq V_\ell(B)$ as $\mathrm{Gal}(\overline{k}/k)$-modules.

It is up to now already a theorem proved by Faltings [Fal83], which is as well as important evidence for the validness of the Tate conjecture. To rephrase, the isogeny conjecture dictates that taking the Tate module is a fully faithful functor, from an isogenous category of abelian varieties whose morphism sets are of form $\mathrm{Hom}(A,B)\otimes_{\mathbb{Z}}\mathbb{Q}_\ell$, to the category of ℓ-adic Galois representations. Simultaneously, in the same spirit, the Tate conjecture implies that the following functor is fully faithful: it is given by taking ℓ-adic étale cohomology, from the category of pure motives over k (defined using the numerical equivalence), to the category of ℓ-adic Galois representation.

Recall that k is of characteristic p. Designedly, our recipe for this paper is to stratify the special fiber $\mathcal{M}_{k_0}$ of moduli space defined in Definition 1.1 via isomorphism classes of p-torsions, to get Goren–Oort strata. One may also stratify via the isogeny classes of p-divisible groups to get various Newton polygons. These classifications lead to interesting cycles in Chow groups $\mathrm{Ch}^*(\mathcal{M})$.

2.4. **Crystalline deformation theory of Grothendieck–Messing via isocrystals.** This subsection firstly constructs crystals associated with p-divisible groups and then states the deformation theory of Grothendieck–Messing (Theorem 2.14). The primary references are [Mes72, Chapter IV] and [Wan09, §1.4]. For the first goal, our track is to define Dieudonné crystals for p-divisible groups, and then generalize the definition to those over some scheme S. As for the second step, we construct a functor that encodes the crystalline homology theory, by sending any p-divisible group to some crystal. Finally, the deformation theory of abelian schemes along a PD-thickening is established.

Throughout this part (and even in the proof of Theorem 4.1), let S be a locally noetherian scheme over a field k of characteristic p, together with $S_0 \hookrightarrow S$ a nilpotent immersion defined by an ideal I which is endowed with locally nilpotent divided powers. Assume p is locally nilpotent on S.

Definition 2.9. A *crystal* over k consists of the following data:

- ∘ a free $W(k)$-module M of finite rank;

- an injective σ-linear endomorphism F such that $pM \subseteq FM$ (where $\sigma\colon k \to k$ is the arithmetic Frobenius), i.e. $F\colon M \to M$ is injective, additive, and $F(\lambda x) = \sigma(\lambda)F(x)$ for any $\lambda \in W(k)$ with $x \in M$.

Also, an *isocrystal* over k is a finite-dimensional $W(k)[\frac{1}{p}]$-vector space N equipped with a bijective σ-linear automorphism $F\colon N \to N$.

By definition, one may loosely regard an isocrystal as a rational crystal. Granting this, the extended Dieudonné functor $\tilde{\mathcal{D}}\colon p\mathsf{Div}(k) \to \mathcal{D}\mathsf{Mod}(\mathcal{D}_k)$ (see page 21) can be rewritten as the categorical equivalence functor associating every p-divisible group G over k to a crystal $\tilde{\mathcal{D}}(G)$ (and thus an isocrystal $\tilde{\mathcal{D}}(G) \otimes_{W(k)} W(k)[\frac{1}{p}]$). However, the definition of $\tilde{\mathcal{D}}$ can be further generalized. Morally, the recipe is to replace $\tilde{\mathcal{D}}$ by $\mathbb{D}$ whose target objects are $\mathcal{F}$-crystals on the crystalline site $\mathsf{Crys}(G)$.

Definition 2.10. Fix an object X in the category of schemes Sch.

(1) The *crystalline site* $\mathsf{Crys}(X)$ on X is the following category.
  (i) The objects are pairs $(U \hookrightarrow T, \gamma)$, where
   - U is a Zariski open subscheme of X and $T \in \mathsf{Sch}$;
   - $U \hookrightarrow T$ is a locally nilpotent immersion of schemes;
   - $\gamma = (\gamma_n)$ are locally nilpotent divided powers on the defining ideal I of U in T.

  (ii) A morphism from $(U \hookrightarrow T, \gamma)$ to $(U' \hookrightarrow T', \gamma')$ is the commutative diagram
$$\begin{array}{ccc} U & \hookrightarrow & T \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ U' & \hookrightarrow & T', \end{array}$$
  where f is the inclusion and $\overline{f}$ is a divided power morphism, i.e. the morphism of sheaf of rings $\overline{f}^{-1}(\mathcal{O}_{T'}) \to \mathcal{O}_T$ is a divided power morphism.

(2) Let $\mathcal{F}$ be a fibered category on Sch which is a stack with respect to the Zariski topology. An *$\mathcal{F}$-crystal* on X is a Cartesian section of the fibered category $\mathcal{F} \times_{\mathsf{Sch}} \mathsf{Crys}(X)$, where
$$\mathsf{Crys}(X) \longrightarrow \mathsf{Sch}, \quad (U \hookrightarrow T, \gamma) \longmapsto T.$$
A morphism of $\mathcal{F}$-crystals is a morphism of Cartesian sections.

Construction 2.11. Note that an $\mathcal{F}$-crystal is in particular a sheaf on $\mathsf{Crys}(X)$. The construction of Definition 2.10 means that, given any $(U \hookrightarrow T, \gamma) \in \mathsf{Crys}(X)$, an object $Q_{(U \hookrightarrow T,\gamma)}$ in $\mathcal{F}_T$ naturally arises. Therefore, we can vaguely define the following covariant Dieudonné functor
$$\mathbb{D}\colon p\mathsf{Div}(S) \longrightarrow \mathsf{Crys}_{\mathcal{F}}(S)$$
to the category of $\mathcal{F}$-crystals over $\mathsf{Crys}(S)$. Moreover, for an arbitrary morphism in $\mathsf{Crys}(X)$, say $f\colon (U \hookrightarrow T, \gamma) \to (U' \hookrightarrow T', \gamma')$, we are given an isomorphism
$$u_{\overline{f}}\colon Q_{(U \hookrightarrow T,\gamma)} \xrightarrow{\sim} \overline{f}^{*} Q_{(U' \hookrightarrow T',\gamma')},$$

which is compatible with compositions, in the sense that for any morphism $g\colon (U' \hookrightarrow T', \gamma') \to (U'' \hookrightarrow T'', \gamma'')$, the following diagram commutes.

$$\begin{tikzcd} Q_{(U\hookrightarrow T,\gamma)} \arrow[r, "u_{\overline{f}}"] \arrow[rr, bend left, "u_{\overline{g}\circ\overline{f}}"] & \overline{f}^* Q_{(U'\hookrightarrow T',\gamma')} \arrow[r, "\overline{f}^*(u_g)"] & (\overline{g\circ f})^* Q_{(U''\hookrightarrow T'',\gamma'')} \end{tikzcd}$$

Obtaining this, for each $G_0 \in p\mathsf{Div}(S_0)$, we have an appropriate value of $\mathbb{D}(G_0)$ at $(S_0 \hookrightarrow S, \gamma) \in \mathsf{Crys}(S)$, written as

$$\mathbb{D}(G_0)_{(S_0\hookrightarrow S,\gamma)} \coloneqq \tilde{\mathcal{D}}(G_{0,S}),$$

where $G_{0,S}$ is any choice of a lift of $G_0|_{S_0}$ to S.

Remark 2.12. Fix a candidate scheme S. Suggestively, the punchline of Construction 2.11 can be understood as that the S-valued covariant $\mathbb{D}$-functor conducts like the covariant crystalline homology functor H_1^{cris}, and $\tilde{\mathcal{D}}$ serves as a projective limit of H_1^{dR}. After a descent from the level of p-divisible groups to that of p-torsion finite group schemes, the definition of $\mathbb{D}(G_0)_{(S_0\hookrightarrow S,\gamma)}$ leads to a canonical isomorphism $(*)$ below.

Now we are ready to state the crystalline deformation theory. For a p-divisible group G_0/S_0, we denote $\mathbb{D}(G_0/S_0)_S \coloneqq \mathbb{D}(G_0)_{(S_0\hookrightarrow S)}$, the value of crystal $\mathbb{D}(G_0)$ at $(S_0 \hookrightarrow S)$.

Definition 2.13. We define the category $p\mathsf{Div}^\dagger(S_0)$ as follows.

(1) The objects are the pairs $(G_0, \mathrm{Fil}^1 \hookrightarrow \mathbb{D}(G_0/S_0)_S)$, where G_0 is a p-divisible group on S_0 and Fil^1 is an admissible filtration of $\mathbb{D}(G_0/S_0)_S$, i.e. Fil^1 is locally a direct factor vector subbundle of $\mathbb{D}(G_0/S_0)_S$ which reduces $\omega_{G_0^\vee/S_0} \subseteq \tilde{\mathcal{D}}(G_0)$ on S_0.
(2) The morphisms are the pairs (u_0, ξ), where
   - $u_0\colon G_0 \to H_0$ is a morphism of S_0-groups;
   - $\xi\colon \mathrm{Fil}^1(G_0) \to \mathrm{Fil}^1(H_0)$ between vector subbundles matches the left commutative diagram below and reduces to the right one.

$$\begin{tikzcd} \mathrm{Fil}^1(G_0) \arrow[r, hook] \arrow[d, "\xi"'] & \mathbb{D}(G_0/S_0)_S \arrow[d] \\ \mathrm{Fil}^1(H_0) \arrow[r, hook] & \mathbb{D}(H_0/S_0)_S \end{tikzcd} \qquad \begin{tikzcd} \omega_{G_0^\vee/S_0} \arrow[r, hook] \arrow[d] & \tilde{\mathcal{D}}(G_0) \arrow[d] \\ \omega_{H_0^\vee/S_0} \arrow[r, hook] & \tilde{\mathcal{D}}(H_0) \end{tikzcd}$$

Theorem 2.14 (Grothendieck–Messing, [Mes72])**.** *The following functor*

$$\begin{aligned} p\mathsf{Div}(S) &\longrightarrow p\mathsf{Div}^\dagger(S_0) \\ G &\longrightarrow (G_0, \mathrm{Fil}^1 \hookrightarrow \mathbb{D}(G_0/S_0)_S) \end{aligned}$$

establishes an equivalence between the category of p-divisible groups over S and that of admissible pairs (G_0, Fil^1).

In particular, for our applications when G is the abelian scheme $A(p)$ over S, we obtain a useful consequence. Suppose $S_0 \hookrightarrow S$ is a locally nilpotent immersion between locally noetherian schemes, whose ideal of the definition has locally divided powers. Assume the prime p is locally nilpotent on S. Let A and A_0 be abelian schemes over S and S_0, respectively. Then the crystalline homology $H_1^{\mathrm{cris}}(A_0/S_0)_S$ is a locally free $\mathcal{O}_S$-module. Canonically, there exists a comparison isomorphism involving the de Rham homology:

$$H_1^{\mathrm{cris}}(A_0/S_0)_S \cong H_1^{\mathrm{dR}}(A/S).$$

Furthermore, $\omega_{A^\vee/S}$ is a subbundle of $H_1^{\mathrm{dR}}(A/S)$, and consequently its image ω along this isomorphism lands in $H_1^{\mathrm{cris}}(A_0/S_0)_S$, which lifts the subbundle $\omega_{A_0^\vee/S_0}$ of $H_1^{\mathrm{dR}}(A_0/S_0)$. Due to Theorem 2.14, the functor

$$A \longmapsto (A_0, \omega_{A^\vee/S} \hookrightarrow H_1^{\mathrm{cris}}(A_0/S_0)_S)$$

establishes an equivalence between the category of abelian schemes A/S and that of admissible pairs. Namely, knowing the lift ω, which is canonically isomorphic to $\omega_{A_0^\vee/S_0}$, allows us to recover A from A_0.

3. The generalized partial Hasse invariants

The partial Hasse invariant is some type of modular form that is defined over $\mathcal{M}$ and factors the determinant of the Hasse–Witt matrix of $\mathcal{A}_{k_0}$ with respect to the action of $\mathcal{O}_D$. Morally, the partial Hasse invariants describe the inner structures of the sheaf of differentials. Therefore, with the splitting model deprived, there are fewer partial Hasse invariants available when p ramifies (cf. [AG05]) unlike in the case when p is unramified in F. To remedy this, our goal in this section is to construct close analogs of the *partial Hasse invariants* given by Goren and Oort on the setting of Hilbert modular varieties [GO00, Gor01]. For this, we first begin with recalling the construction in the unramified case [TX16]. We also refer to the construction by Andreatta–Goren of some modular forms defined over the Rapoport locus [AG05].

From now on, we work on $U(2)$-Shimura varieties, just so each $r_j^{(i)} \in \{0,1,2\}$.

3.1. **The unramified case revisit.** Assume $e = 1$ and $f > 1$, so we can omit the superscript $\cdot^{(i)}$ in the filtration. In this particular case, $\mathcal{M} = \mathcal{M}^{\mathrm{PR}} = \mathcal{M}^{\mathrm{Ra}}$, and by definition,

$$\omega_j = \omega_{\mathcal{A}_{k_0}^\vee/\mathcal{M}_{k_0},j}.$$

Consider the σ^{-1}-linear Verschiebung map $V\colon \mathcal{A}_{k_0}^{(p)} \to \mathcal{A}_{k_0}$ that preserves the $\mathcal{O}_F$-action. It further induces a morphism

$$V_j\colon \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j \longrightarrow \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_{j-1}^{(p)}, \quad \operatorname{Ker} V_j = \operatorname{Lie}_{\mathcal{A}_{k_0}/\mathcal{M}_{k_0},j}.$$

So it is convenient to write its restriction instead, that is,

$$V_j|_{\omega_j}\colon \omega_j \longrightarrow (\omega_{j-1})^{(p)} := \omega_{(\mathcal{A}_{k_0}^{(p)})^\vee/\mathcal{M}_{k_0},j-1}$$

of sheaves of $(\mathcal{O}_D \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{M}})$-modules for $j = 1, \dots, f$.

While working with the unramified case, one can fix $\tau_j \in \Sigma$ such that $r_j = s_j = 1$. This condition is for guaranteeing ω_j to be a line bundle. A priori one obtains a natural isomorphism

$$\omega_{(\mathcal{A}_{k_0}^{(p)})^\vee/\mathcal{M}_{k_0}} \cong F^*\omega_{\mathcal{A}_{k_0}^\vee/\mathcal{M}_{k_0}},$$

where F is the absolute Frobenius morphism on $\mathcal{M}_{k_0}$. Thus, following the signature condition, we split the direct sum and get another natural isomorphism

$$(\omega_{j-1})^{(p)} \cong (F^*\omega_{\mathcal{A}_{k_0}^\vee/\mathcal{M}_{k_0}})_{j-1}.$$

For our fixed τ_j, we define $\tau_{j-1} = \sigma^{-1}\tau_j$. Then, with this convention,

$$(F^*\omega_{\mathcal{A}_{k_0}^\vee/\mathcal{M}_{k_0}})_{j-1} \cong F^*\omega_j.$$

To proceed on Goren's recipe, assume also that $r_{j-1} = s_{j-1} = 1$. So the right-hand side above is the pullback of a line bundle by Frobenius. Pullback of a line bundle by Frobenius sends transition functions to their pth powers and therefore sends a line bundle to pth tensor power of its image. It thus renders that $F^*\omega_j = \omega_{j-1}^{\otimes p}$. Combining all above, we attain that

$$\omega_j \longrightarrow (\omega_{j-1})^{(p)} \xrightarrow{\sim} \omega_{j-1}^{\otimes p}.$$

In particular, Verschiebung on $\mathcal{A}_{k_0}$ yields to a section of the line bundle $\mathrm{Hom}(\omega_j, \omega_{j-1}^{\otimes p})$. Note further that this recipe is canonical, and write the canonical section as

$$h_j \in H^0(\mathcal{M}_{k_0}, \omega_{j-1}^{\otimes p} \otimes \omega_j^{\otimes -1}).$$

This is the *partial Hasse invariant* at the place $\tau_j \in \Sigma$ when p is unramified.

3.2. **The ramified case.** We are now in a position to construct a generalized partial Hasse invariant. The main reference is [RX17, §3]. Fix an embedding $\tau = \tau_j^{(i)} \in \Sigma$. As a reminder, we choose ϖ in Subsection 1.1 to be a uniformizer for the ring of integers $\mathcal{O}_{F_\mathfrak{p}}$. Here comes an upshot spoiler:

- When $1 < i \leqslant e$, the desired section is essentially given by multiplication by $[\varpi]$;
- When $i = 1$, it is given by first dividing by $[\varpi^{e-1}]$, and then applying the Verschiebung map.

While dealing with the unramified case, the sheaf images have been computed to get the desired map. In contrast to this, more explicit and finer descriptions of the filtration structure are required over the splitting model.

To prepare for the construction, recall from Proposition 1.5 that, over $\mathcal{M}_{k_0}$, the bundle $\mathcal{H}_j^{(i)}$ is a rank 2 subbundle of $\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j/\mathcal{F}_j^{(i-1)}$ containing $\mathcal{F}_j^{(i)}/\mathcal{F}_j^{(i-1)}$; the sheaf $\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j$ is a locally free $\mathcal{O}_{\mathcal{M}_{k_0}}[x]/(x^e)$-module of rank 2, where the action of ϖ is given by multiplication by x. Analogously to the unramified case, the sheaf $\mathcal{H}_j^{(i)}$ serves the role of the $\tau_j^{(i)}$-component of $\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})$ at this moment. For simplicity of notation, we define

$$\omega_j^{(i)} := \mathcal{F}_j^{(i)}/\mathcal{F}_j^{(i-1)}.$$

Assume $1 < i \leqslant e$. Recall that we have defined $\mathcal{H}_j^{(i)}$ to be the sheaf that collects the elements in $\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j/\mathcal{F}_j^{(i-1)}$ annihilated by $[\varpi] - \tau_j^{(i)}(\varpi)$. For each local section $z \in \mathcal{H}_j^{(i)}$, we choose one of its lift $\tilde{z}$ to $\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j$. We vaguely denote $[\varpi](\tilde{z})$ the image of $\tilde{z}$ after multiplication by ϖ and construct a map

$$\begin{aligned} m_{\varpi,j}^{(i)} \colon \mathcal{H}_j^{(i)} &\longrightarrow \mathcal{F}_j^{(i-1)}/\mathcal{F}_j^{(i-2)} = \omega_j^{(i-1)} \\ z &\longmapsto [\varpi](\tilde{z}). \end{aligned}$$

Because $\mathcal{H}_j^{(i)}$ is annihilated by $[\varpi]$ from the definition, we observe that $m_{\varpi,j}^{(i)}$ is a well-defined homomorphism since different choices of the lift $\tilde{z}$ lead to a same image $[\varpi](\tilde{z})$. Restricting $m_{\varpi,j}^{(i)}$ to the subbundle induces the map

$$\begin{tikzcd} \mathcal{H}_j^{(i)} \supseteq \mathcal{F}_j^{(i)}/\mathcal{F}_j^{(i-1)} \arrow[r, dashed] \arrow[d, equal] & \mathcal{F}_j^{(i-1)}/\mathcal{F}_j^{(i-2)} \arrow[d, equal] \\ \omega_j^{(i)} \arrow[r, dashed, "\exists"] & \omega_j^{(i-1)} \end{tikzcd}$$

and thus a desired section

$$h_j^{(i)} \in H^0(\mathcal{M}_{k_0}, (\omega_j^{(i-1)}) \otimes (\omega_j^{(i)})^{\otimes -1}), \quad 1 < i \leqslant e.$$

This is the *generalized partial Hasse invariant* at $\tau_j^{(i)}$ with $1 < i \leqslant e$. Since we mainly concern about the relations between subquotients, by an abuse of the notation, we will always identify $m_{\varpi,j}^{(i)}|_{\mathcal{F}_j^{(i)}/\mathcal{F}_j^{(i-1)}}$ with $m_{\varpi,j}^{(i)}$. It actually depends on the choice of uniformizer ϖ, but the divisor it defines does not. So we have a well-defined stratification on the special fiber $\mathcal{M}_{k_0}$ defined by the vanishing locus of partial Hasse invariants.

Remark 3.1. The punchline of this construction is as follows. Beginning with the Verschiebung on the jth component, denoted by $V_j \colon \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j \to \omega^{(p)}_{\mathcal{A}_{k_0}^\vee/\mathcal{M}_{k_0},j-1}$, we get a restricted Verschiebung

$$V_j^{(i)} = V_j|_{\mathcal{F}_j^{(i)}/\mathcal{F}_j^{(i-1)}} \colon \mathcal{F}_j^{(i)}/\mathcal{F}_j^{(i-1)} \longrightarrow (\mathcal{F}_{j-1}^{(i)}/\mathcal{F}_{j-1}^{(i-1)})^{(p)}$$

and $V_j^{(i)}$ factors through $m_{\varpi,j}^{(i)}, m_{\varpi,j}^{(i-1)}, \ldots, m_{\varpi,j}^{(1)}$. Granting the fact $V_j(\mathcal{F}_j^{(i)}) \subseteq (\mathcal{F}_{j-1}^{(i)})^{(p)}$ that we will recount later in the proof of Proposition 3.2, $V_j^{(i)}$ is well-defined. Predictably, we may claim that $m_{\varpi,j}^{(i)}$ is surjective. This can be verified at each geometric closed point $\xi \colon \operatorname{Spec} k_0 \to \mathcal{M}_{k_0}$, on which the de Rham homology is of splitting type, i.e. $H_1^{\mathrm{dR}}(\mathcal{A}_\xi)_j \cong (\overline{\mathbb{F}}_p[x]/(x^e))^{\oplus 2}$. We may pick a basis so that

$$\begin{aligned} (\mathcal{F}_j^{(i-1)})_\xi &\simeq (x^a\overline{\mathbb{F}}_p[x]/x^e\overline{\mathbb{F}}_p[x]) \oplus (x^b\overline{\mathbb{F}}_p[x]/x^e\overline{\mathbb{F}}_p[x]), \\ (\mathcal{H}_j^{(i)})_\xi &\simeq (x^{a-1}\overline{\mathbb{F}}_p[x]/x^e\overline{\mathbb{F}}_p[x]) \oplus (x^{b-1}\overline{\mathbb{F}}_p[x]/x^e\overline{\mathbb{F}}_p[x]), \end{aligned}$$

with $0 < a, b \leqslant e$ and $a+b = 2e-(i-1)$.[1] Therefore, it is clear that $m_{\varpi,j}^{(i)}$ surjects $(\mathcal{H}_j^{(i)})_\xi$ onto $(\mathcal{F}_j^{(i-1)})_\xi/(\mathcal{F}_j^{(i-2)})_\xi$.

As the subquotient $\mathcal{F}_j^{(i-1)}/\mathcal{F}_j^{(i-2)}$ does not make sense when $i=1$, we need a variant of the usual construction of the partial Hasse invariants via the Verschiebung maps. Consider its source

$$\begin{aligned} \mathcal{H}_j^{(1)} &= \{z \in \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j/\{0\} \colon [\varpi]\cdot z = 0\} \\ &= (\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j)[\varpi], \end{aligned}$$

[1] Note that the condition $a+b = 2e-(i-1)$ forces $a, b \geqslant 1$.

the $[\varpi]$-torsion subbundle (i.e. the subbundle in $\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j$ annihilated by multiplication by ϖ), as an $\mathcal{O}_{\mathcal{M}_{k_0}}[x]/(x^e)$-module. So

$$\mathcal{H}_j^{(1)} \subseteq [\varpi]^{e-1}\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j.$$

It follows that each element in $\mathcal{H}_j^{(1)}$ is divisible by $[\varpi]^{e-1}$. We define that

$$\begin{aligned} V_j \circ [\varpi]^{1-e} \colon \mathcal{H}_j^{(1)} &\xrightarrow{(\cdot)/[\varpi]^{e-1}} \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j \xrightarrow{V_j} \omega^{(p)}_{\mathcal{A}^\vee_{k_0}/\mathcal{M}_{k_0},j-1} \\ z &\longmapsto z/[\varpi]^{e-1} \longmapsto V_j(z/[\varpi]^{e-1}). \end{aligned}$$

Here the choice of $z/[\varpi]^{e-1}$ is not unique, but its image along V_j does not depend on this choice; this is because any other choice should be $(z/[\varpi]^{e-1}) + [\varpi]z'$ for some $z' \in \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j$, yet we have

$$[\varpi]\cdot \frac{\omega^{(p)}_{\mathcal{A}^\vee_{k_0}/\mathcal{M}_{k_0},j-1}}{V(\mathcal{F}_j^{(e-1)})} = [\varpi]\cdot \frac{\omega^{(p)}_{\mathcal{A}^\vee_{k_0}/\mathcal{M}_{k_0},j-1}}{(\mathcal{F}_{j-1}^{(e-1)})^{(p)}} = 0.$$

For this reason, $V_j \circ [\varpi]^{1-e}$ factors through the quotient by $V(\mathcal{F}_j^{(e-1)})$), i.e.

$$\mathrm{Im}(V_j \circ [\varpi]^{1-e}) \subseteq \omega^{(p)}_{\mathcal{A}^\vee_{k_0}/\mathcal{M}_{k_0},j-1}/V(\mathcal{F}_j^{(e-1)}).$$

Finally, recall from Proposition 1.5 that $\mathcal{H}_j^{(1)}$ contains $\omega_j^{(1)} = \mathcal{F}_j^{(1)}/\mathcal{F}_j^{(0)} \simeq \mathcal{F}_j^{(1)}$, and

$$\omega^{(p)}_{\mathcal{A}^\vee_{k_0}/\mathcal{M}_{k_0},j-1}/(\mathcal{F}_{j-1}^{(e-1)})^{(p)} = (\mathcal{F}_{j-1}^{(e)}/\mathcal{F}_{j-1}^{(e-1)})^{(p)} = (\omega_{j-1}^{(e)})^{(p)} \cong (\omega_{j-1}^{(e)})^{\otimes p}.$$

Therefore, $V_j \circ [\varpi]^{1-e}$ induces the desired map

$$\begin{tikzcd} \mathcal{H}_j^{(1)} \arrow[r, "(\cdot)/{[\varpi]^{e-1}}"] & \omega_{\mathcal{A}^\vee_{k_0}/\mathcal{M}_{k_0},j}/\mathcal{F}_j^{(e-1)} \arrow[r, "V_j"] & \omega^{(p)}_{\mathcal{A}^\vee_{k_0}/\mathcal{M}_{k_0},j-1}/(\mathcal{F}_{j-1}^{(e-1)})^{(p)} \\ \omega_j^{(1)} \arrow[u, phantom, "\cup"] \arrow[rr, dashed, "\exists"] & & (\omega_{j-1}^{(e)})^{\otimes p} \arrow[u, phantom, "\wr"] \end{tikzcd}$$

and hence a section

$$h_j^{(1)} \in H^0(\mathcal{M}_{k_0}, (\omega_{j-1}^{(e)})^{\otimes p} \otimes (\omega_j^{(1)})^{\otimes -1}).$$

This is the *partial Hasse invariant* at $\tau_j^{(1)}$.

Besides, both the division map and Verschiebung are surjective, so also is $V_j \circ [\varpi]^{1-e}$. We can redefine the *circumvention Hasse map* for simplicity as

$$\begin{tikzcd} \mathrm{Hasse}^{(1)}_{\varpi,j} \colon \mathcal{H}_j^{(1)} \arrow[r] \arrow[d, phantom, "\parallel"] & (\omega_{\mathcal{A}^\vee_{k_0}/\mathcal{M}_{k_0},j-1}/\mathcal{F}_{j-1}^{(e-1)})^{(p)}. \arrow[d, phantom, "\parallel"] \\ (\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j)[\varpi] \arrow[r] & (\omega_{j-1}^{(e)})^{\otimes p} \end{tikzcd}$$

The relation between the circumvention Hasse map and the Verschiebung map is described as follows. This proposition also explains the motivation for constructing $\mathrm{Hasse}^{(1)}_{\varpi,j}$ that we wanted to define a "periodicity connection map".

Proposition 3.2 ([RX17], Lemma 3.8)**.** *The Verschiebung map*

$$V_j \colon \omega_{\mathcal{A}^\vee_{k_0}/\mathcal{M}_{k_0},j} \longrightarrow \omega^{(p)}_{\mathcal{A}^\vee_{k_0}/\mathcal{M}_{k_0},j-1}$$

induces natural homomorphisms

$$V_j^{(i)} \colon \omega_j^{(i)} = \mathcal{F}_j^{(i)}/\mathcal{F}_j^{(i-1)} \longrightarrow (\mathcal{F}_{j-1}^{(i)}/\mathcal{F}_{j-1}^{(i-1)})^{(p)} = (\omega_{j-1}^{(i)})^{\otimes p}$$

for all $i = 1, \dots, e$. Moreover, we have the following commutative diagrams:

$$\begin{tikzcd}
\mathcal{F}_j^{(e)}/\mathcal{F}_j^{(e-1)} \arrow[r, "m_{\varpi,j}^{(e)}"] \arrow[d, "V_j^{(e)}"] & \mathcal{F}_j^{(e-1)}/\mathcal{F}_j^{(e-2)} \arrow[r, "m_{\varpi,j}^{(e-1)}"] \arrow[d, "V_j^{(e-1)}"] & \cdots \arrow[r, "m_{\varpi,j}^{(2)}"] & \mathcal{F}_j^{(1)} \arrow[d, "V_j^{(1)}"] \arrow[dlll, bend left, "\mathrm{Hasse}_{\varpi,j}^{(1)}"'] \\
(\mathcal{F}_{j-1}^{(e)}/\mathcal{F}_{j-1}^{(e-1)})^{(p)} \arrow[r, "(m_{\varpi,j-1}^{(e)})^{(p)}"] & (\mathcal{F}_{j-1}^{(e-1)}/\mathcal{F}_{j-1}^{(e-2)})^{(p)} \arrow[r, "(m_{\varpi,j-1}^{(e-1)})^{(p)}"] & \cdots \arrow[r, "(m_{\varpi,j-1}^{(2)})^{(p)}"] & (\mathcal{F}_{j-1}^{(1)})^{(p)}.
\end{tikzcd}$$

In other words, for $i = 1, \dots, e$,

$$V_j^{(i)} = (m_{\varpi,j-1}^{(i+1)})^{(p)} \circ \cdots \circ (m_{\varpi,j-1}^{(e)})^{(p)} \circ \mathrm{Hasse}_{\varpi,j}^{(1)} \circ m_{\varpi,j}^{(2)} \circ \cdots \circ m_{\varpi,j}^{(i)}.$$

In terms of functions, the section induced by $V_j^{(i)}$ is equal to

$$(h_{j-1}^{(i+1)})^p \cdots (h_{j-1}^{(e)})^p \cdot h_j^{(1)} \cdots h_j^{(i)} \in H^0(\mathcal{M}_{k_0}, (\omega_{j-1}^{(i)})^{\otimes p} \otimes (\omega_j^{(i)})^{\otimes -1}).$$

Proof. First verify that $V_j^{(i)}$ is well-defined. It boils down to prove $V_j(\mathcal{F}_j^{(i)}) \subseteq (\mathcal{F}_{j-1}^{(i)})^{(p)}$. (This property was used to check the well-definedness of the circumvention Hasse map before.) It is known that $\mathcal{F}_j^{(i)}$ is annihilated by $[\varpi]^i$. So we take a local section $z \in \mathcal{F}_j^{(i)}$ and consider

$$\begin{array}{ccc}
[\varpi]^{e-i}\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j & \xrightarrow{V_j} & [\varpi]^{e-i}\omega_{j-1}^{(p)} \\
\cup\!| & & \cup\!| \\
\mathcal{F}_j^{(i)} & \longrightarrow & [\varpi]^{e-i}(\mathcal{F}_{j-1}^{(i)})^{(p)}.
\end{array}$$

We can further write $z = [\varpi]^{e-i}z'$ for some $z' \in \mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})_j$. Thus, by the annihilating assumption,

$$[\varpi]^{e-i}(\omega_{\mathcal{A}_{k_0}^\vee/\mathcal{M}_{k_0},j-1}/\mathcal{F}_{j-1}^{(i)})^{(p)} = 0.$$

This proves the containment $V_j(z) = [\varpi]^{e-i}V_j(z') \in (\mathcal{F}_{j-1}^{(i)})^{(p)}$.

Now it remains with checking the commutativity of the above diagram. Recall that $m_{\varpi,j}^{(i)}$ is essentially given by multiplication by ϖ. So

$$m_{\varpi,j}^{(2)} \circ \cdots \circ m_{\varpi,j}^{(i)}(z) = [\varpi]^{i-1}(z) \in \mathcal{F}_j^{(1)},$$

and then

$$\begin{aligned}
\mathrm{Hasse}_{\varpi,j}^{(1)}([\varpi]^{i-1}(z)) &= V_j([\varpi]^{i-1}(z)/[\varpi]^{e-1}) \\
&= V_j([\varpi]^{i-1}([\varpi]^{e-i}z')/[\varpi]^{e-1}) = V_j(z').
\end{aligned}$$

As an immediate consequence of this, we get

$$((m_{\varpi,j-1}^{(i+1)})^{(p)} \circ \cdots \circ (m_{\varpi,j-1}^{(e)})^{(p)} \circ \mathrm{Hasse}_{\varpi,j}^{(1)} \circ m_{\varpi,j}^{(2)} \circ \cdots \circ m_{\varpi,j}^{(i)})(z) = [\varpi]^{e-i}V_j(z'),$$

which equals $V_j^{(i)}(z)$ as desired. $\square$

We remark that all these constructions can be dually done with $\tau_j^{(i),\perp}$ instead of with $\tau_j^{(i)}$. The existence of a prime-to-p polarization on $\mathcal{A}_{k_0}$ means that the $\tau_j^{(i)}$ and $\tau_j^{(i),\perp}$ components of $\mathcal{H}_1^{\mathrm{dR}}(\mathcal{A}_{k_0}/\mathcal{M}_{k_0})$, $\omega_{\mathcal{A}_{k_0}^\vee/\mathcal{M}_{k_0}}$, and $\omega_{\mathcal{A}_{k_0}^\vee/\mathcal{M}_{k_0}}$ are dual to each other, and hence that the bundle and section obtained from these dual components are mutually canonically isomorphic.

4. Smoothness of strata via crystalline deformation theory

This section is dedicated to establishing a global description of Goren–Oort stratification, and in particular, to prove Theorem 4.1 about the smoothness of strata on special fiber of the unitary Shimura variety. We resume all the statements and notations of the previous section. This is the desired Shimura variety mod p that interlaces crucially with the Jacquet–Langlands correspondence and carries algebro-geometric information via cycles.

We have pointed out before that the stratification induced on the special fiber $\mathcal{M}_{k_0}$ of the Pappas–Rapoport splitting model by the generalized partial Hasse invariants h_τ does not depend on the choice of uniformizer ϖ of $\mathcal{O}_{F_\mathfrak{p}}$, whereas h_τ does.

Throughout this section, for each p-adic embedding $\tau = \tau_j^{(i)}$ defined as usual, denote $h_\tau = h_j^{(i)}$ the corresponding partial Hasse invariant, with its zero locus $Z(h_\tau) = Z(h_j^{(i)})$ on $\mathcal{M}_{k_0}$. Given any subset $T \subseteq \Sigma_\infty$, define the strata

$$Z_T \coloneqq \bigcap_{\tau \in T} Z(h_\tau).$$

The stratification we obtain in this way is called *Goren–Oort stratification* for unitary Shimura varieties. When $T = \{\tau\}$, we also write $Z_{\{\tau\}} = Z_\tau$ for simplicity.

Theorem 4.1. *For each $\tau \in \Sigma$, the closed subscheme Z_τ is a proper and smooth divisor with simple normal crossings on $\mathcal{M}_{k_0}$. In particular, the stratum Z_T has a codimension equal to the cardinality of T in $\mathcal{M}_{k_0}$.*

Proof. The properness statement is relatively easy to tackle, but it requires a language of toroidal compactification and total Hasse invariant to run the argument [RX17, Theorem 3.10]. So we choose to omit the proof for it.

The smoothness statement is first proved in [Sas19] by following [PR05]. We mainly follow the combined proof in [RX17, Theorem 2.9] by applying Grothendieck–Messing deformation theory.

Let $S_0 \hookrightarrow S$ be a closed immersion of locally noetherian $W(k_0)$ schemes defined by an ideal I with locally nilpotent divided powers. Then S is a PD-thickening of S_0. Our goal is go lift any given S_0-point $(A_0, \lambda_0, \rho_0, \underline{\mathscr{F}}) \in Z_\tau(S_0)$ to some point in $Z_\tau(S)$ for each $\tau \in \Sigma$. For this, consider $\mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)$, the crystalline homology sheaf of A_0/S_0, which is locally free as an $(\mathcal{O}_D \otimes_{\mathbb{Z}} \mathcal{O}_{S_0}^{\mathrm{cris}})$-module of rank 2 by [Rap78, Lemme 1.3]. Similar to the de Rham case,

$$\mathcal{H}_1^{\mathrm{cris}}(A_0/S_0) = \bigoplus_{j=1}^{f} \mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_j$$

via the action of $\mathcal{O}_D$ on A_0, where each direct summand is separated from the $W(k_0)$ action via τ_j.

Correspondingly, each $\mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_j$ is a locally free $(\mathcal{O}_{S_0}^{\mathrm{cris}} \otimes_{W(k_0),\tau_j} \mathcal{O}_{F_\mathfrak{p}})$-module of rank 2. By evaluating the crystalline homology over S, we have

$$\mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_S = \bigoplus_{j=1}^{f} \mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_{S,j}.$$

Also, there is a Hodge filtration of crystalline homology

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_{A_0^\vee/S_0} & \longrightarrow & \mathcal{H}_1^{\mathrm{cris}}(A_0/S_0) & \longrightarrow & \mathrm{Lie}_{A_0/S_0} & \longrightarrow & 0, \\ & & \| & & \| & & \| & & \\ & & \bigoplus \omega_{A_0^\vee/S_0,j} & & \bigoplus \mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_j & & \bigoplus \mathrm{Lie}_{A_0/S_0,j} & & \end{array}$$

which further decomposes into a direct sum of short exact sequences at each $\mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_j$. Each $\omega_{A_0^\vee/S_0,j}$ is a locally free $\mathcal{O}_{S_0}$-module of rank $\sum_{i=1}^{e} r_j^{(i)}$, as a direct summand of $\mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_j$. One may note with caution that $\omega_{A_0^\vee/S_0,j}$ is not necessarily locally free as an $(\mathcal{O}_{S_0} \otimes_{W(k_0),\tau_j} \mathcal{O}_{F_\mathfrak{p}})$-module, but it is coherent.

By crystalline deformation theory (Theorem 2.14), there is a categorical equivalence functor, giving a correspondence

$$A \longleftrightarrow (A_0, \omega_{A^\vee/S} \hookrightarrow \mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_S),$$

where $\omega_{A^\vee/S}$ is image of the lift of $\omega_{A_0^\vee/S_0} \subseteq \mathcal{H}_1^{\mathrm{dR}}(A_0/S_0)$ to $\mathcal{H}_1^{\mathrm{dR}}(A/S)$ along the canonical comparison isomorphism $\mathcal{H}_1^{\mathrm{dR}}(A/S) \xrightarrow{\sim} \mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_S$. Therefore, to lift the abelian variety A_0 together with the $\mathcal{O}_D$-action $\mathcal{O}_D \to \mathrm{End}_{S_0}(A_0)$, it suffices to lift each $\omega_{A_0^\vee/S_0,j}$ to an $\mathcal{O}_D$-stable submodule $\tilde{\omega}_j \subseteq \mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_{S,j}$, which is as well as a locally free $\mathcal{O}_S$-subbundle. Moreover, once such lift A of A_0 is determined, the K-level structure ρ_0 lifts uniquely.

The polarization $\lambda_0 \colon A_0 \to A_0^\vee$ induces an isomorphism

$$\lambda_0^* \colon \mathcal{H}_1^{\mathrm{cris}}(A_0/S_0) \longrightarrow \mathcal{H}_1^{\mathrm{cris}}(A_0^\vee/S_0) \cong \mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)^\vee.$$

In particular, the isomorphism on jth component leads to a nondegenerate symplectic pairing

$$\langle \cdot, \cdot \rangle \colon \mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_j \times \mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_j \longrightarrow \mathcal{O}_{S_0}^{\mathrm{cris}}$$

satisfying

$$\langle ax, y \rangle = \langle x, ay \rangle,\ \langle ax, x \rangle = 0 \text{ for all } a \in \mathcal{O}_{F_\mathfrak{p}},\ x, y \in \mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_j.$$

The second equality is implied by $\langle ax, x \rangle = \langle x, ax \rangle = -\langle ax, x \rangle$ since the pairing is symplectic when 2 is not a zero divisor on S. In general case, we may assume that $S_0 = \mathrm{Spec}\, R_0$ is affine for some locally noetherian ring R_0, and $x \in \mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_{S',j}$ is a section over some PD-thickening $S_0 \hookrightarrow S' = \operatorname{Spec} R'$ where R' is noetherian. Write R' as a quotient of $\mathbb{Z}_p$-flat noetherian algebra $\tilde{R}$, and let $\tilde{R}_{\mathrm{PD}}$ denote the PD-envelop of the quotient map $\tilde{R} \twoheadrightarrow R' \twoheadrightarrow R_0$, so that we may evaluate the crystalline cohomology on the PD-thickening $\tilde{R}_{\mathrm{PD}} \to R_0$. Now $\tilde{R}_{\mathrm{PD}}$ is $\mathbb{Z}_p$-flat and so $\langle ax, x\rangle = 0$ holds over $\tilde{R}_{\mathrm{PD}}$; so it holds over R as R is a quotient of $\tilde{R}_{\mathrm{PD}}$ by the universal property of the PD-envelop.

The upshot to lift the polarization is as follows. A priori the submodule $\omega_{A_0^\vee/S_0,j}$ is a maximal isotropic submodule of $\mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)$ with respect to the pairing $\langle\cdot,\cdot\rangle$; the polarization λ_0 lifts if $\tilde{\omega}_j$ is also isotropic.

What we obtain now for each τ_j is the filtration

$$0 = \mathscr{F}_j^{(0)} \subseteq \mathscr{F}_j^{(1)} \subseteq \cdots \subseteq \mathscr{F}_j^{(e)} = \omega_{A_0^\vee/S_0,j} \subseteq \mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_j$$

valued over S_0 whose subquotient $\mathscr{F}_j^{(i)}/\mathscr{F}_j^{(i-1)}$ for $i \in \{1,\dots,e\}$ is a locally free sheaf of rank $r_j^{(i)} \in \{0,1,2\}$ over $\mathcal{O}_S$. We are to lift it to an $\mathcal{O}_D$-stable filtration (valued over S):

$$0 = \tilde{\mathscr{F}}_j^{(0)} \subseteq \tilde{\mathscr{F}}_j^{(1)} \subseteq \cdots \subseteq \tilde{\mathscr{F}}_j^{(e)} = \tilde{\omega}_j \subseteq \mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_{S,j}$$

such that for $i \in \{1,\dots,e\}$,

- each subquotient $\tilde{\mathscr{F}}_j^{(i)}/\tilde{\mathscr{F}}_j^{(i-1)}$ is a locally free sheaf of rank $r_j^{(i)}$ over $\mathcal{O}_S$ annihilated by $[\varpi]$,
- the $\mathcal{O}_D$-action on $\tilde{\mathscr{F}}_j^{(i)}/\tilde{\mathscr{F}}_j^{(i-1)}$ factors through $\tau_j^{(i)}\colon \mathcal{O}_D \to W(k_0)$.

We claim that all S-lifts are essentially determined by the lifts of the filtrations; more explicitly, a given lift $x = (A, \lambda, \rho, \underline{\tilde{\mathscr{F}}}) \in \mathcal{M}_{k_0}(S)$ of $x_0 = (A_0, \lambda_0, \rho_0, \underline{\mathscr{F}}) \in Z_\tau(S_0)$ further lies in $Z_\tau(S)$ if and only if for each $\tau_j^{(i)} \in \Sigma$ the sheaf $\tilde{\mathscr{F}}_j^{(i)}$ equals some fixed lift of $\mathscr{F}_j^{(i)}$. To prove the claim, we proceed inductively on $i \leqslant e$ and considering each $\tau = \tau_j^{(i)}$ independently.

(i) When $i = 1$, the lift x of x_0 belongs to Z_τ if and only if $\tilde{\mathscr{F}}_j^{(1)}$ is contained in the kernel of

$$\mathrm{Hasse}_{\varpi,j}^{(1)}\colon \mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_{S,j}[\varpi] \longrightarrow (\omega_{A^\vee/S,j-1}/\tilde{\mathscr{F}}_{j-1}^{(e-1)})^{(p)}.$$

Since the above map is surjective by the argument on page 28, its kernel is an $\mathcal{O}_S$-subbundle of rank 1, which coincides with $\tilde{\mathscr{F}}_j^{(1)}$ if and only if $x \in Z_\tau$. This shows that the $\tilde{\mathscr{F}}_j^{(1)}$ is uniquely determined by the condition $x \in Z_\tau$.

(ii) When $i > 1$, the lift x of x_0 belongs to Z_τ if and only if $\tilde{\mathscr{F}}_j^{(i)}$ is contained in the kernel of the surjective map

$$m_{\varpi,j}^{(i)}\colon \tilde{\mathcal{H}}_j^{(i)} \longrightarrow \tilde{\mathscr{F}}_j^{(i-1)}/\tilde{\mathscr{F}}_j^{(i-2)},$$

where $\tilde{\mathcal{H}}_j^{(i)}$ denotes the natural crystalline deformation of $\mathcal{H}_j^{(i)}$. So the kernel of $m_{\varpi_i,j}^{(i)}$ on $\tilde{\mathcal{H}}_j^{(i)}$ is a rank one $\mathcal{O}_S$-subbundle, and that it coincide with $\tilde{\mathscr{F}}_j^{(i)}$ if and only if $x \in Z_\tau$. This shows that the $\tilde{\mathscr{F}}_j^{(i)}$ is uniquely determined by the condition $x \in Z_\tau$.

Granting the claim, to finish the proof, it suffices to find the lift of $\underline{\mathscr{F}}$. For this, we define

$$\tilde{\mathcal{H}}_j^{(1)} := \{z \in \mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_{S,j} \colon ([\varpi] - \tau_j^{(1)}(\varpi))z = 0\}$$

to be a counterfeit de Rham homology. By condition $\langle ax, x\rangle = 0$ for λ_0, the isotropic condition on a lift $\tilde{\mathscr{F}}_j^{(1)}$ of $\mathscr{F}_j^{(1)}$ is automatically satisfied. Now suppose for induction that we have lifted $\mathscr{F}_j^{(i-1)}$ to $\tilde{\mathscr{F}}_j^{(i-1)}$ with the isotropic condition. Equivalently, we aim to find an $\mathcal{O}_D$-stable subsheaf $\tilde{\mathscr{F}}_j^{(i)}$ of $\mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_{S,j}$, which is a direct summand containing $\tilde{\mathscr{F}}_j^{(i-1)}$ and satisfies the two requirements listed before. Define

$$\tilde{\mathcal{H}}_j^{(i)} := \{z \in \mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_{S,j}/\tilde{\mathscr{F}}_j^{(i-1)} \colon ([\varpi] - \tau_j^{(i)}(\varpi))z = 0\}$$

which appears to be an $\mathcal{O}_S$-subbundle of rank 2 in $\mathcal{H}_1^{\mathrm{cris}}(A_0/S_0)_{S,j}/\tilde{\mathscr{F}}_j^{(i-1)}$; this can be verified by a similar argument as in the proof of Proposition 1.5. Thus, the lifting task can be rewritten as lifting the subquotient $\mathcal{F}_j^{(i)}/\mathcal{F}_j^{(i-1)} \subseteq (\tilde{\mathcal{H}}_j^{(i)})_{S_0}$ to an $\mathcal{O}_S$-subbundle of rank $r_j^{(i)}$. But this can be done due to the inductive hypothesis. $\square$

Part 2. A Geometrization of Jacquet–Langlands Correspondence

Depending on the choice of the base reductive group scheme, the special fibers of some Shimura varieties of PEL type can carry neat global geometry structures. The known results for this include but are not limited to those of $U(2)$-type or quaternionic GL_2-type, due to [HTX17] and [TX19], respectively. The global description comes from an isogenic comparison between different bundles over two Shimura varieties that are differed by signatures of base groups. In the upcoming sections, we specialize in the $U(2)$-Shimura varieties to give a global description of Goren–Oort stratification through Theorem 5.1.

5. Periodic cycles of embeddings and signature change

Given G as in Subsection 1.1 with signatures $(r_j^{(i)})$ for $1 \leqslant i \leqslant e$ and $1 \leqslant j \leqslant f$, we take $n = 2$ so that D has dimension 4 over F. In particular, $G(\mathbb{R})$ is isomorphic to a subgroup of

$$\prod_{j=1}^{f}\prod_{i=1}^{e} \mathrm{GU}_{\mathbb{R}}(r_j^{(i)}, s_j^{(i)}), \quad r_j^{(i)} + s_j^{(i)} = 2.$$

We point out that working on this $U(2)$ case is an essential assumption to establish the global geometric description, which deeply depends on the "flat nature" of GL_2; that is, the special fibers of Shimura varieties over GL_2 have simple and normal intersections, which fails to be valid in general cases.

This section describes the explicit machinery for constructing another algebraic group G' with different signatures, such that the unitary Shimura variety defined for G' records the essential information of that for G. Such a signature change trick was originally developed in [Hel12], and was successfully applied by [HTX17, TX19] to depict the global geometry of Goren–Oort strata on special fibers of unitary and quaternionic Shimura varieties at an inert prime p. Our argument is adapted from [TX16, §5].

To begin with, we denote $\mathbf{Sh}(G) = (\mathcal{M}_{K,\Lambda}^{\mathrm{PR}})_{\overline{\mathbb{F}}_p}$ the special fiber at $\overline{\mathbb{F}}_p$ of the unitary Shimura variety defined for G by the moduli problem $\underline{\mathcal{M}}_{K,\Lambda}^{\mathrm{PR}}$ in Definition 1.1. We will prove later that each closed Goren–Oort stratum of $\mathbf{Sh}(G)$ is a $(\mathbb{P}^1)^N$-bundle over $\mathbf{Sh}(G')$ for some appropriate integer N. Let $\mathcal{A}$ and $\mathcal{A}'$ be the universal abelian schemes over $\mathbf{Sh}(G)$ that we start with and over $\mathbf{Sh}(G')$ that we aim to, respectively.

5.1. Primary idea behind signature change.

Recall that Σ_∞ denotes the set of all real archimedean embeddings of F^+, identified with all p-adic embeddings by our assumption that there is only one place lying above p. We will consider the Goren–Oort stratification defined by some places in an arbitrary subset $T \subseteq \Sigma_\infty$. Also recall from Subsection 1.1 that the level structure of $\mathbf{Sh}(G)$ at p is given by an open compact subgroup

$$K_p = \mathbb{Z}_p^\times \times K_{\mathfrak{p}} = \mathbb{Z}_p^\times \times \mathrm{GL}_n(\mathcal{O}_{F_{\mathfrak{p}}}) \subseteq G(\mathbb{Q}_p).$$

This hypothesis, which essentially simplifies our argument, is indeed an avoidance of Iwahori level structures.

The moral idea is to establish a correspondence with the following two isomorphisms:

$$\begin{tikzcd}
& Y_T \arrow[dl, "\cong"'] \arrow[dr, "\cong"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
Z_T \arrow[d] \arrow[drr, dashed, "(\mathbb{P}^1)^N"] & & Z'_T \arrow[d, "(\mathbb{P}^1)^N"] \\
\mathbf{Sh}(G) & & \mathbf{Sh}(G').
\end{tikzcd}$$

Here Z'_T, arising with the universal abelian scheme $\mathcal{A}'$, is the $(\mathbb{P}^1)$-power bundle over the other special fiber $\mathbf{Sh}(G')$; also, Y'_T is the moduli space in Definition 6.1 that classifies both $\mathcal{A}$ and $\mathcal{A}'$ together with a quasi-isogeny $\mathcal{A} \to \mathcal{A}'$ with very small p-power degree. The two morphisms going downstairs are given by simply forgetting one of $\mathcal{A}$ and $\mathcal{A}'$. We postpone the proof to Section 6 that these are exactly isomorphisms, by checking that the natural forgetful morphisms are bijective on the closed points and that they induce isomorphisms on the tangent spaces. The key tool is the Grothendieck–Messing correspondence in Theorem 2.14. The main result is as follows.

Theorem 5.1. *For any subset $T \subseteq \Sigma_\infty$, the Goren–Oort stratum Z_T defined by T over the special fiber $\mathbf{Sh}(G)$ is isomorphic to a $(\mathbb{P}^1)^{\#I(T)}$-bundle over $\mathbf{Sh}(G')$, where*

$$I(T) = T' - T = \bigsqcup_{l=1}^{m} C'_i - \bigsqcup_{l=1}^{m} C_i$$

is the difference subset of embeddings in Σ_∞ that can be computed by Construction 5.5.

In this section, we will define the auxiliary geometric object Y_T and make Theorem 5.1 more transparent. Section 6 is devoted to proving this main theorem, and the proof will be separated into several parts on some propositions and lemmas.

5.2. **The unramified case revisited.** A prototypical description of global geometry in the unramified case can be found in [TX16, §1.5]. The main argument recipe is to figure out the relationship between chosen embeddings in T among a cycle of all embeddings. Precisely, the p-adic embeddings $\tau_j \colon F^+ \hookrightarrow \overline{\mathbb{Q}}_p$ of the totally real field are inductively contacted by Frobenius actions, i.e. $\sigma\tau_j = \sigma \circ \tau_j = \tau_{j+1}$ for all indices $j \pmod f$, which leads to a cycle structure for relative positions between any couple of embeddings. Let G be the $U(2)$-group of signatures $(r_1, \dots, r_f)$. Our construction works on the simple example that p is unramified and inert in F^+, with $[F^+ : \mathbb{Q}] = f$. Hopefully, by Theorem 5.1, for any subset $T \subseteq \Sigma_\infty$, the stratum Z_T is isomorphic to a $(\mathbb{P}^1)^N$-bundle over $\mathbf{Sh}(G')$ for some $N \in \mathbb{N}$.

Construction 5.2. Note that for each $\tau_j \in \Sigma_\infty$ with $j \in \{1, \dots, f\}$, the corresponding signature $r_j + s_j = 2$, and hence $r_j \in \{0, 1, 2\}$. We define the set of bad embeddings as

$$S_\infty := \{\tau_j \in \Sigma_\infty = \mathrm{Hom}(F^+, \overline{\mathbb{Q}}_p) \colon r_j \neq 1\},$$

and for each $\tau_j \in \Sigma_\infty - S_\infty$ that

$$n_j := \min\{n \in \mathbb{N}^* \colon \sigma^n \tau_j \in \Sigma_\infty - S_\infty\}.$$

An n-element subset C of $T - S_\infty \subseteq \Sigma_\infty - S_\infty$ is called a *proper chain* of length n if $C = \{\tau_{k_1}, \dots, \tau_{k_n}\} \neq \emptyset$ for some mutually distinct $k_1, \dots, k_n \in \{1, \dots, f\}$, satisfying $\tau_{k_{i+1}} = \sigma^{n_{k_i}} \tau_{k_i}$ for all $i = 1, \dots, n-1$ and $\sigma^{-1}\tau_{k_1}, \sigma^{n_{k_n}}\tau_{k_n} \notin T$.

(a) We first tackle the case where either the cardinality $\#(\Sigma_\infty - S_\infty)$ is even, or $\#(\Sigma_\infty - S_\infty)$ is odd and $T - S_\infty \subsetneq \Sigma_\infty - S_\infty$. Given an arbitrary proper chain $C = \{\tau_{k_1}, \dots, \tau_{k_n}\}$, we define it extended chain as

$$C' = \begin{cases} C, & \text{if } \#C \text{ is even}, \\ C \cup \{\sigma^{n_{k_n}} \tau_{k_n}\}, & \text{if } \#C \text{ is odd}. \end{cases}$$

Notice that in this case, whenever $\#C$ is odd, we always have $\sigma^{n_{k_n}}\tau_{k_n} \in \Sigma - (S_\infty \cup C)$, and the definition of C' is thus reasonable. On the other hand, there is apparently a unique decomposition to identify $T - S_\infty$ with a disjoint union of proper chains. Write $T - S_\infty = \bigsqcup_{i=1}^m C_i$ and define

$$I(T) = \bigsqcup_{i=1}^m C'_i - \bigsqcup_{i=1}^m C_i, \quad N = \#I(T) = \sum_{i=1}^m \#(C'_i - C_i).$$

We also construct the signature of G' as follows. Take

$$T' = (T - S_\infty) \cup I(T) = \bigsqcup_{i=1}^m C'_i = \{\tau_{l_1}, \dots, \tau_{l_k}\}$$

for some integer $k = k(T)$. Here the labeling is given by taking $l_1 < \cdots < l_k$ in $\mathbb{Z}/f\mathbb{Z}$. Note that all C'_i are even subsets, and hence k is even. Define G' to be the $U(2)$ group with the new signatures given by $s'_j = 2 - r'_j$ and

$$r'_j = \begin{cases} 0, & \text{if } j \in \{l_1, l_3, \dots, l_{k-1}\}, \\ 2, & \text{if } j \in \{l_2, l_4, \dots, l_k\}, \\ r_j, & \text{otherwise}. \end{cases}$$

In particular, when $T = \emptyset$, we have $I(T) = \emptyset$ with $N = 0$, and the theory can also be vacuous.

(b) Now it remains to consider the case where $\#(\Sigma_\infty - S_\infty)$ is odd, and $\Sigma_\infty - S_\infty = T - S_\infty$. Then C can be the whole set whereas there is no more place to add into. By contrast, we always let[2]

$$T' = T \cup \{v\}, \quad N = 0.$$

Moreover, when $\#T$ is even, we must put an Iwahori level structure at p to define $\mathbf{Sh}(G')$. We will omit any detailed discussion in this case.

In a sequel, the stratum Z_T over $\mathbf{Sh}(G)$ is isomorphic to a $(\mathbb{P}^1)^N$-bundle over $\mathbf{Sh}(G')$. The combinatorial nature of this construction will be revealed in the ramified case. So we postpone the explanation for why we do so to the next section.

[2]Recall that v is the unique place lying above p in F^+, with $p\mathcal{O}_{F^+} = v^e$. See Subsection 1.1.

Remark 5.3. It can be clear that there exists an equivalent algorithm to reconstruct T'. When $T = \Sigma_\infty - S_\infty$, if it has an odd cardinality, then we put $T' = T \cup \{v\}$; if else it has an even cardinality, then we put $T' = T$. Whenever $T \subsetneq \Sigma_\infty - S_\infty$, we can write $S_\infty \cup T = \bigsqcup C_i$ as the disjoint union of proper chains. Unlike before, by a proper chain C_i here we mean that there exists $\tau_j \in S_\infty \cup T$ together with some $m_j \geqslant 0$, such that $C_i = \{\tau_j, \sigma\tau_j, \dots, \sigma^{m_j}\tau_j\} \subseteq S_\infty \cup T$ and $\sigma^{-1}\tau_j, \sigma^{m_j+1}\tau_j \notin (S_\infty \cup T)$. Then we take $T' = \bigsqcup C_i'$ with C_i' defined by

$$C_i' = \begin{cases} C_i \cap T, & \text{if } \#(C_i \cap T) \text{ is even,} \\ (C_i \cap T) \cup \{\sigma^{n_{k_n}}\tau_{k_n}\}, & \text{if } \#(C_i \cap T) \text{ is odd.} \end{cases}$$

Finally, one can get $N = \#(T' - T)$ as desired.

Example 5.4. We propose two examples for the algorithm in Construction 5.2.

(a) Let $f = 12$ and $e = 1$, with

$$\Sigma_\infty = \{\tau_1, \dots, \tau_{12}\}, \quad S_\infty = \{\tau_3, \tau_4, \tau_6, \tau_8, \tau_9, \tau_{12}\}.$$

Let $\mathbf{Sh}(G)$ be the $U(2)$-Shimura variety of signatures $(r_1, \dots, r_{12})$. Consider the stratification Z_T. We choose T to be $\{\tau_2, \tau_7, \tau_{10}\}$. Then the decomposition is given by $\{\tau_2\} \sqcup \{\tau_7, \tau_{10}\}$, and by definition, it is modified to be

$$T' = (T - S_\infty) \cup I(T) = \{\tau_2, \tau_5\} \sqcup \{\tau_7, \tau_{10}\}, \quad I(T) = \{\tau_5\}.$$

Therefore, Z_T is isomorphic to a $\mathbb{P}^1$-bundle over $\mathbf{Sh}(G')$, where the signature of G' is given by

$$(r_1, 0, r_3, r_4, 2, r_6, 0, r_8, r_9, 2, r_{11}, r_{12}).$$

Indeed, by definition of S_∞ we know that exactly those elements lying outside S_∞ has signature 1, i.e. $r_1 = r_{11} = 1$ and $r_3, r_4, r_6, r_8, r_9, r_{12} \in \{0, 2\}$.

(b) Let $f = 5$ and $e = 1$, with

$$\Sigma_\infty = \{\tau_1, \dots, \tau_5\}, \quad S_\infty = \emptyset.$$

Suppose for simplicity that $T = \Sigma_\infty$. Then T contains a single cycle satisfying $\tau_{j+1} = \sigma\tau_j$ for $j \in \mathbb{Z}/5\mathbb{Z}$, which cannot be a proper chain. Then we take

$$T' = T \cup \{v\}, \quad N = 0,$$

and there is no Iwahori level structure involved. So Z_T is isomorphic to $\mathbf{Sh}(G')$, whose level structure is directly induced from that of $\mathbf{Sh}(G)$.

We give a summary of some features of Construction 5.2 in the unramified case. This may shed light on a similar argument over the Pappas–Rapoport splitting model. For our specialized purpose which is to construct Tate cycles, if we were without this signature change trick, the main obstruction would be an asymmetry caused by incorrect numbers of subquotients of rank 0 and rank 2. The general idea to remedy this is to make these subquotients have rather correct numbers, and hence to recover the hidden symmetry.

Concisely, the technique is to merge a couple of objects of rank 1 and to pretend that the compositum is of rank 2. So the algorithm appears to change the signatures $(1,1)$ into $(0,2)$ or $(2,0)$. Concerning the precise algorithm, the key steps are as follows.

(i) Attain a cycle of embeddings and decompose T into disjoint union of chains;

(ii) Modify all chains to make them have even cardinalities by adding the upcoming nearest embedding if necessary;

(iii) Transfer the signature parities in turn in the disjoint union of modified chains.

5.3. **The ramified case: on periodic cycles of $\mathfrak{p}$-adic embeddings.** By Definition 1.1(4c), the actions of $[\varpi]$ on subquotients $\mathcal{F}_j^{(i)}/\mathcal{F}_j^{(i-1)}$ are exactly determined by $\tau_j^{(i)}(\varpi)$. On the other hand, when $1 < i \leqslant e$, for each $j \in \{1, \dots, f\}$, the partial Hasse invariants

$$h_j^{(i)} \in H^0(\mathbf{Sh}(G), (\mathcal{F}_j^{(i-1)}/\mathcal{F}_j^{(i-2)}) \otimes (\mathcal{F}_j^{(i)}/\mathcal{F}_j^{(i-1)})^{\otimes -1})$$

are essentially given by multiplication-by-$[\varpi]$ maps $m_{\varpi,j}^{(i)}$. These indicate that the subquotients of the form $\mathcal{F}_j^{(i)}/\mathcal{F}_j^{(i-1)}$ are mutually contacted by partial Hasse maps when $1 < i \leqslant e$, and the case is similar when $i = 1$. Up to twists by p, the subquotients exactly correspond to p-adic embeddings $\{\tau_j^{(i)} \colon 1 \leqslant i \leqslant e, 1 \leqslant j \leqslant f\}$ in Σ_∞, inducing the place $\mathfrak{p}$. Then the following structure, called a *periodic cycle of* $\mathfrak{p}$*-adic embeddings*, corresponds to the commutative diagram in Proposition 3.2, in which elements are all of the $\tau_j^{(i)}$s in Σ_∞ and the connections are denoted by $\iota_j^{(i)}$.

$$\begin{tikzcd}
\tau_f^{(e)} \arrow[r, "\iota_f^{(e)}"] & \tau_f^{(e-1)} \arrow[r, "\iota_f^{(e-1)}"] & \cdots \arrow[r, "\iota_f^{(2)}"] & \tau_f^{(1)} \arrow[dlll, bend left, "\iota_f^{(1)}"'] \\
\tau_{f-1}^{(e)} \arrow[r, "\iota_{f-1}^{(e)}"'] & \tau_{f-1}^{(e-1)} \arrow[r, "\iota_{f-1}^{(e-1)}"'] & \cdots \arrow[r, "\iota_{f-1}^{(2)}"'] & \tau_{f-1}^{(1)} \arrow[dlll, bend left, "\iota_{f-1}^{(1)}"] \\
\cdots & \cdots & \cdots & \cdots \arrow[dlll, bend left, "\iota_2^{(1)}"'] \\
\tau_1^{(e)} \arrow[r, "\iota_1^{(e)}"] & \tau_1^{(e-1)} \arrow[r, "\iota_1^{(e-1)}"] & \cdots \arrow[r, "\iota_1^{(2)}"] & \tau_1^{(1)} \arrow[uuulll, bend right, "\iota_1^{(1)}"']
\end{tikzcd}$$

In contrast with the setups of τ_js that $\tau_{j+1} = \sigma \circ \tau_j$, for $\tau_j^{(i)}$s we only obtain the chain without commutative diagrams between any two embeddings, so the notion of connections is unable to be identified with that of maps between embeddings. In addition, notate for simplicity that

$$\iota(\tau_j^{(i)}) = \begin{cases} \tau_{(j-1) \bmod f}^{(e)}, & \text{if } i = 1, \\ \tau_j^{(i-1)}, & \text{if } 1 < i \leqslant e. \end{cases}$$

Here we mean by $(j-1) \bmod f$ the image of $j-1$ along $\mathbb{Z} \to \mathbb{Z}/f\mathbb{Z}$.

Construction 5.5 (c.f. Construction 5.2)**.** We use the primary setup that p ramifies and $\mathfrak{p}$ is the only place above p in the CM field F, with $[F:\mathbb{Q}]=2ef$. Then Σ_∞ is identified with the collection of all $\tau_j^{(i)}$s where $j\in\{1,\dots,f\}$ and $i\in\{1,\dots,e\}$. Again, define the set of bad embeddings as $S_\infty:=\{\tau_j^{(i)}\in\Sigma_\infty\colon r_j^{(i)}\neq 1\}$, where $r_j^{(i)}=2-s_j^{(i)}\in\{0,1,2\}$. The signature of G is given by the $e\times f$ matrix $(r_j^{(i)})$. For any subset $T\subseteq\Sigma_\infty$, the difference set $I(T)$ can be established by using the following recipe.

An n-element subset C of $T-S_\infty\subseteq\Sigma_\infty-S_\infty$ is called a *proper chain* of length n if the following conditions are satisfied:

- There are mutually distinct pairs $(i_1,j_1),\dots,(i_n,j_n)\in\{1,\dots,e\}\times\{1,\dots,f\}$ such that $C=\{\tau_{j_1}^{(i_1)},\dots,\tau_{j_n}^{(i_n)}\}\neq\emptyset$.
- For $k=1,\dots,n-1$, the compositum of at most $ef-1$ connections in the whole periodic cycle of $\tau_j^{(i)}$s from $\tau_{j_k}^{(i_k)}$ to $\tau_{j_{k+1}}^{(i_{k+1})}$ does not pass through any other element of $T-S_\infty$; note in particular that it does not pass through $\tau_{j_l}^{(i_l)}$ for any $l\in\{1,\dots,n\}-\{k,k+1\}$. In other words, $\tau_{j_{k+1}}^{(i_{k+1})}=\iota^n(\tau_{j_k}^{(i_k)})$, where $n=\min\{1\leqslant\nu\leqslant ef-1\colon \iota^\nu(\tau_{j_k}^{(i_k)})\in T-S_\infty\}$.
- We have $\iota(\tau_{j_n}^{(i_n)})\notin T$, and $\tau_{j_1}^{(i_1+1)}\notin T$ when $1\leqslant i_1<e$ or $\tau_{(j_1+1)\bmod f}^{(1)}\notin T$ when $i_1=e$.

By definition, different proper chains are disjoint. Then we write $T-S_\infty=\bigsqcup_{i=1}^m C_i$ for some m.

(a) Assume either $\#(\Sigma_\infty-S_\infty)$ is even with T being arbitrary, or $\#(\Sigma_\infty-S_\infty)$ is odd with $T-S_\infty\subsetneq\Sigma_\infty-S_\infty$. For each proper chain $C=\{\tau_{j_1}^{(i_1)},\dots,\tau_{j_n}^{(i_n)}\}$ in the decomposition, we define

$$C'=\begin{cases}C, & \text{if } \#C \text{ is even},\\ C\cup\{\tau_{j_{n+1}}^{(i_{n+1})}\}, & \text{if } \#C \text{ is odd}.\end{cases}$$

Here $\tau_{j_{n+1}}^{(i_{n+1})}$ is the nearest embedding following from $\tau_{j_n}^{(i_n)}$ in the cycle, i.e. the compositum of at most $ef-1$ connections from $\tau_{j_n}^{(i_n)}$ to $\tau_{j_{n+1}}^{(i_{n+1})}$ does not pass through any other element of $T-S_\infty$. Note that $j_{n+1}\leqslant j_n$. Thus, we are able to take

$$I(T)=\bigsqcup_{i=1}^m C_i'-\bigsqcup_{i=1}^m C_i,\quad N=\#I(T)=\sum_{i=1}^m\#(C_i'-C_i).$$

Again, the signature of G' is as follows. Take the set (with an even cardinality)

$$T'=(T-S_\infty)\cup I(T)=\bigsqcup_{i=1}^m C_i'=\{\tau_{j_1}^{(i_1)},\dots,\tau_{j_k}^{(i_k)}\}$$

for some integer $k=k(T)$. Here, along the periodic cycle, the labeling is given by taking $\tau_{j_1}^{(i_1)}$ to be the nearest element following from $\tau_f^{(e)}$ and taking $\tau_{j_{\nu+1}}^{(i_{\nu+1})}$

to be the nearest element following from $\tau_{j_\nu}^{(i_\nu)}$ for $\nu \geqslant 2$, i.e.,

$$\tau_{j_1}^{(i_1)} = \iota^n(\tau_f^{(e)}), \qquad n = \min\{1 \leqslant \nu \leqslant ef-1 \colon \iota^\nu(\tau_f^{(e)}) \in T'\},$$
$$\tau_{j_{\nu+1}}^{(i_{\nu+1})} = \iota^m(\tau_{j_\nu}^{(i_\nu)}), \quad m = \min\{1 \leqslant \nu \leqslant ef-1 \colon \iota^\nu(\tau_{j_\nu}^{(i_\nu)}) \in T'\}.$$

Define G' to be the $U(2)$ group with the new signatures given by

$$r_j^{(i)\prime} = \begin{cases} 0, & \text{if } j \in \{j_1, j_3, \dots, j_{k-1}\}, \\ 2, & \text{if } j \in \{j_2, j_4, \dots, j_k\}, \\ r_j^{(i)}, & \text{otherwise,} \end{cases}$$

and $s_j^{(i)\prime} = 2 - r_j^{(i)\prime}$.

(b) Assume else $\#(\Sigma_\infty - S_\infty)$ is odd and $\Sigma_\infty - S_\infty = T - S_\infty$. Take $T' = T \cup \{v\}$ with $N = 0$. Input an Iwahori level structure at p for $\mathbf{Sh}(G')$ when $\#T$ is even.

We also construct the level structure of $\mathbf{Sh}(G')$. Recall that $K_p = \mathbb{Z}_p^\times \times \mathrm{GL}_n(\mathcal{O}_{F_\mathfrak{p}})$ is a subgroup of $G(\mathbb{Q}_p)$, and there is another fixed open compact subgroup $K^p \subseteq G(\mathbb{A}_{\mathbb{Q},f}^{(p)})$ such that $K = K^pK_p$ satisfies the neat condition. After establishing the signature of G' in (a) and in (b) when $\#T$ is odd, let V and V' be the defining vector spaces for G and G', respectively (see Subsection 1.1). Then there is an isomorphism $V \otimes_F \mathbb{A}_{\mathbb{Q},f} \simeq V' \otimes_F \mathbb{A}_{\mathbb{Q},f}$, which further identifies $G(\mathbb{A}_{\mathbb{Q},f})$ with $G'(\mathbb{A}_{\mathbb{Q},f})$ via an isomorphism. Let the open compact subgroup $K^{p\prime} \subseteq G'(\mathbb{A}_{\mathbb{Q},f}^{(p)})$ (resp. the lattice $\Lambda' \subseteq V'(\mathbb{A}_{\mathbb{Q},f})$) be the image of K^p (resp. $\Lambda \subseteq V(\mathbb{A}_{\mathbb{Q},f})$) along this isomorphism. We take $K' = K^{p\prime}K_p$ as the directly induced level structure defining $\mathbf{Sh}(G')$. One can easily check that $G'(\mathbb{Q}) \cap xK'x^{-1}$ is torsion-free for any $x \in G(\mathbb{A}_{\mathbb{Q},f})$. The neat condition for K' is thus valid.

Caution 5.6. In Construction 5.5(a), the new subset T' does not necessarily contain T. In Example 5.7(a1), one will notice that $T' = T - S_\infty \subsetneq T$.

Example 5.7. We propose two examples for the algorithm in Construction 5.5.

(a1) Let $f = 2$ and $e = 4$. Choose

$$S_\infty = \{\tau_2^{(1)}, \tau_2^{(3)}\}, \quad T = \Sigma_\infty - \{\tau_2^{(3)}, \tau_1^{(1)}, \tau_1^{(4)}\}.$$

The picture of the cycle is as follows, with the elements in $T - S_\infty$ being framed.

$$\begin{tikzcd}
\boxed{\tau_2^{(4)}} \arrow[r, "\iota_2^{(4)}"] & \tau_2^{(3)} \arrow[r, "\iota_2^{(3)}"] & \boxed{\tau_2^{(2)}} \arrow[r, "\iota_2^{(2)}"] & \tau_2^{(1)} \arrow[dlll, bend left, "\iota_2^{(1)}"] \\
\tau_1^{(4)} \arrow[r, "\iota_1^{(4)}"'] & \boxed{\tau_1^{(3)}} \arrow[r, "\iota_1^{(3)}"'] & \boxed{\tau_1^{(2)}} \arrow[r, "\iota_1^{(2)}"'] & \tau_1^{(1)} \arrow[ulll, bend right, "\iota_1^{(1)}"']
\end{tikzcd}$$

Note that $\#(\Sigma_\infty - S_\infty)$ is odd, but $T - S_\infty \subsetneq \Sigma_\infty - S_\infty$ holds. So we apply Construction 5.5(a) to write

$$T - S_\infty = \{\tau_2^{(4)}, \tau_2^{(2)}\} \sqcup \{\tau_1^{(3)}, \tau_1^{(2)}\}.$$

The two proper chains both have even cardinalities. So $T' = T - S_\infty$, $I(T) = \emptyset$, and $N = 0$. It follows that the signature change from G to G' is given by

$$\begin{aligned}\begin{pmatrix} r_2^{(4)} & r_2^{(3)} & r_2^{(2)} & r_2^{(1)} \\ r_1^{(4)} & r_1^{(3)} & r_1^{(2)} & r_1^{(1)} \end{pmatrix} &= \begin{pmatrix} 1 & r_2^{(3)} & 1 & r_2^{(1)} \\ 1 & 1 & 1 & 1 \end{pmatrix} \\ &\longmapsto \begin{pmatrix} 0 & r_2^{(3)} & 2 & r_2^{(1)} \\ 1 & 0 & 2 & 1 \end{pmatrix},\end{aligned}$$

where $r_2^{(1)}, r_2^{(3)} \in \{0, 2\}$. Also note that $r_j^{(i)} = 1$ unless $\tau_j^{(i)} \in S_\infty$. So we can transfer the matrices above into the form where the (i,j)th elements are given by $r_j^{(i)}$ or $r_j^{(i)\prime}$. Here comes an alternative way to write:

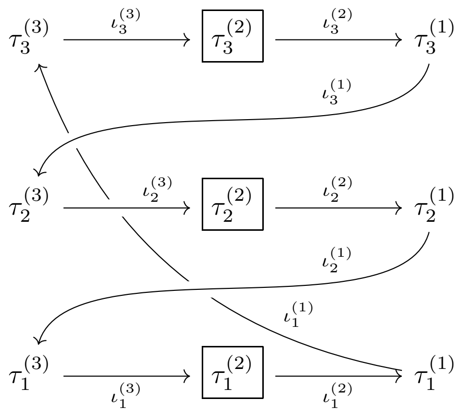
$$\begin{pmatrix} r_1^{(1)} & r_2^{(1)} \\ r_1^{(2)} & r_2^{(2)} \\ r_1^{(3)} & r_2^{(3)} \\ r_1^{(4)} & r_2^{(4)} \end{pmatrix} = \begin{pmatrix} 1 & r_2^{(1)} \\ 1 & 1 \\ 1 & r_2^{(3)} \\ 1 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & r_2^{(1)} \\ 2 & 2 \\ 0 & r_2^{(3)} \\ 1 & 0 \end{pmatrix}.$$

Therefore, the stratum Z_T is isomorphic to $\mathbf{Sh}(G')$, where the signature for G' is defined as above, and the level structure on $\mathbf{Sh}(G')$ is directly induced from that of $\mathbf{Sh}(G)$.

(a2) Let $f = e = 3$. Choose

$$S_\infty = \{\tau_3^{(1)}, \tau_2^{(1)}\}, \quad T = \{\tau_3^{(2)}, \tau_2^{(2)}, \tau_1^{(2)}\}.$$

The picture of the cycle is as follows, with the elements in $T - S_\infty$ being framed.



Again, we can apply Construction 5.5(a) to write

$$T - S_\infty = \{\tau_3^{(2)}\} \sqcup \{\tau_2^{(2)}\} \sqcup \{\tau_1^{(2)}\} = C_1 \sqcup C_2 \sqcup C_3.$$

These proper chains are all odd with single elements. Then we take

$$C_1' = C_1 \cup \{\tau_2^{(3)}\}, \quad C_2' = C_2 \cup \{\tau_1^{(3)}\}, \quad C_3' = C_3 \cup \{\tau_1^{(1)}\}.$$

Therefore, we have $I(T) = \{\tau_2^{(3)}, \tau_1^{(3)}, \tau_1^{(1)}\}$ and $N = 3$, with

$$T' = (T - S_\infty) \cup I(T) = \{\tau_3^{(2)}, \tau_2^{(3)}, \tau_2^{(2)}, \tau_1^{(3)}, \tau_1^{(2)}, \tau_1^{(1)}\}.$$

Furthermore, by noting that each $\tau_j^{(i)} \notin S_\infty$ has signature $r_j^{(i)} = 1$, we can write the signature change from G to G' as

$$\begin{pmatrix} r_3^{(3)} & r_3^{(2)} & r_3^{(1)} \\ r_2^{(3)} & r_2^{(2)} & r_2^{(1)} \\ r_1^{(3)} & r_1^{(2)} & r_1^{(1)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & r_3^{(1)} \\ 1 & 1 & r_2^{(1)} \\ 1 & 1 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & r_3^{(1)} \\ 2 & 0 & r_2^{(1)} \\ 2 & 0 & 2 \end{pmatrix},$$

or equivalently, with $r_3^{(1)}, r_2^{(1)} \in \{0, 2\}$,

$$\begin{pmatrix} r_1^{(1)} & r_2^{(1)} & r_3^{(1)} \\ r_1^{(2)} & r_2^{(2)} & r_3^{(2)} \\ r_1^{(3)} & r_2^{(3)} & r_3^{(3)} \end{pmatrix} = \begin{pmatrix} 1 & r_2^{(1)} & r_3^{(1)} \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 2 & r_2^{(1)} & r_3^{(1)} \\ 0 & 0 & 0 \\ 2 & 2 & 1 \end{pmatrix},$$

where the (i, j)th elements of the matrices are respectively given by $r_j^{(i)}$ and $r_j^{(i)\prime}$. As a result, the stratum Z_T on $\mathbf{Sh}(G)$ is isomorphic to a $(\mathbb{P}^1)^3$-bundle over $\mathbf{Sh}(G')$, where the level structure of $\mathbf{Sh}(G')$ is directly induced from that of $\mathbf{Sh}(G)$.

(b) Let $f = 2$ and $e = 4$ as in (a1). Choose

$$S_\infty = \{\tau_2^{(1)}\}, \quad T = \Sigma_\infty - \{\tau_2^{(1)}\}.$$

Then $T - S_\infty = \Sigma_\infty - S_\infty$ has an odd cardinality. So it happens to fit in Construction 5.5(b). Again, we take

$$T' = T \cup \{v\}, \quad N = 0.$$

In this case, there is no Iwahori level structure at p being involved.

Remark 5.8. Note that elements in $T' = \{\tau_{j_1}^{(i_1)}, \ldots, \tau_{j_k}^{(i_k)}\}$ actually form a subcycle of the whole periodic cycle for Σ_∞. For the signature change trick and the upcoming definition of exceptional embeddings, we have to begin with a choice of $\tau_{j_1}^{(i_1)}$ from the cycle. In Construction 5.5(a), we have given a non-canonical choice to let $\tau_{j_1}^{(i_1)}$ be the nearest element behind $\tau_f^{(e)}$ along connections. However, according to Remark 6.10, this choice does not affect the result of Theorem 5.1. Indeed, the algorithm dictates that a different choice $\tau_{j_1}^{(i_1)\prime}$ of $\tau_{j_1}^{(i_1)}$ either changes nothing or swaps the signatures 0 and 2 for G', depending on the parity of the number of connections between $\tau_{j_1}^{(i_1)}$ and $\tau_{j_1}^{(i_1)\prime}$.

5.4. **Exceptional embeddings in the cycle.** Fix a choice of $T \subseteq \Sigma_\infty$. Based on the algorithm above, we define an *exceptional subset* of embeddings in Σ_∞, denoted by $\Delta(T)$, to distinguish different geometric properties provided by Dieudonné modules at different places.

Construction 5.9. Resume on Construction 5.5(a) and keep the notations as before. Assume either $\#(\Sigma_\infty - S_\infty)$ is even with T being arbitrary, or $\#(\Sigma_\infty - S_\infty)$ is odd with $T - S_\infty \subsetneq \Sigma_\infty - S_\infty$. Since $T' = (T - S_\infty) \cup I(T)$ always has an even cardinality, write $T' = \{\tau_{j_1}^{(i_1)}, \ldots, \tau_{j_{2m}}^{(i_{2m})}\}$ for some integer k. We take the index $\epsilon \in \{1, \ldots, m\}$ so that $2\epsilon - 1$ runs through all odd integers in the interval $[1, 2m]$. For each $\tau_{j_{2\epsilon-1}}^{(i_{2\epsilon-1})}$, put elements in the periodic cycle of Σ_∞ that lie on the path of connections from $\tau_{j_{2\epsilon-1}}^{(i_{2\epsilon-1})}$ to $\tau_{j_{2\epsilon}}^{(i_{2\epsilon})}$

into a subset $\Delta_\epsilon(T) \subseteq \Sigma_\infty$, including $\tau_{j_{2\epsilon-1}}^{(i_{2\epsilon-1})}$ itself but not $\tau_{j_{2\epsilon}}^{(i_{2\epsilon})}$. Define the *exceptional embeddings* to be the elements in

$$\Delta(T) = \bigsqcup_{\epsilon=1}^{m} \Delta_\epsilon(T) \subseteq \Sigma_\infty.$$

Note that $\Delta(T)$ possibly intersects with S_∞.

Example 5.10. Suppose we are in the situations proposed by Example 5.7(a).

(a1) Write $T' = \{\tau_2^{(4)}, \tau_2^{(2)}, \tau_1^{(3)}, \tau_1^{(2)}\}$. Then we have $\epsilon \in \{1, 2\}$, and

$$\Delta_1(T) = \{\tau_2^{(4)}, \tau_2^{(3)}\}, \quad \Delta_2(T) = \{\tau_1^{(3)}\}.$$

So there are exactly 3 exceptional embeddings: $\Delta(T) = \{\tau_2^{(4)}, \tau_2^{(3)}, \tau_1^{(3)}\}$.

(a2) Write $T' = \{\tau_3^{(2)}, \tau_2^{(3)}, \tau_2^{(2)}, \tau_1^{(3)}, \tau_1^{(2)}, \tau_1^{(1)}\}$. Then we have $\epsilon \in \{1, 2, 3\}$, and

$$\Delta_1(T) = \{\tau_3^{(2)}, \tau_3^{(1)}\}, \quad \Delta_2(T) = \{\tau_2^{(2)}, \tau_2^{(1)}\}, \quad \Delta_3(T) = \{\tau_1^{(2)}\}.$$

It follows that $\Delta(T) = \{\tau_3^{(2)}, \tau_3^{(1)}, \tau_2^{(2)}, \tau_2^{(1)}, \tau_1^{(2)}\}$.

Here comes an important observation for later use.

Lemma 5.11. *For all $\tau_j^{(i)} \in \Delta(T)$, we always have $r_j^{(i)\prime} \in \{0, 2\}$.*

Proof. We know from Construction 6.1 that the signature of G' at each element of T' changes to be either 0 or 2. So it suffices to check the assertion for all ϵ and embeddings of $\Delta_\epsilon(T) - T'$. However, by definition, we have $\Delta_\epsilon(T) - T' \subseteq S_\infty$, which consists of $\tau_j^{(i)}$s such that $r_j^{(i)} = 1$. Since $S_\infty \cap T' = \emptyset$ and $r_j^{(i)\prime} = r_j^{(i)}$, we complete the proof. $\square$

Remark 5.12. As in Remark 5.8, depending on the parity, a different choice of $\tau_{j_1}^{(i_1)}$ can possibly switch $\Delta(T)$ into exactly one another set $\Delta^\dagger(T)$. More explicitly, we can compute in Example 5.10 that

(a1) $\Delta^\dagger(T) = \Delta_1^\dagger(T) \sqcup \Delta_2^\dagger(T) = \{\tau_2^{(2)}, \tau_2^{(1)}, \tau_1^{(4)}\} \sqcup \{\tau_1^{(2)}, \tau_1^{(1)}\}$.

(a2) $\Delta^\dagger(T) = \Delta_1^\dagger(T) \sqcup \Delta_2^\dagger(T) \sqcup \Delta_3^\dagger(T) = \{\tau_2^{(3)}\} \sqcup \{\tau_1^{(3)}\} \sqcup \{\tau_1^{(1)}, \tau_3^{(3)}\}$.

In particular, $\Delta^\dagger(T)$ and $\Delta(T)$ can have different cardinalities. But for our purposes, it suffices to fix the labeling and work with $\Delta(T)$ only.

6. Proofs of the two isomorphisms in moduli comparison

This section is devoted to completing the proof of Theorem 5.1 in the $U(2)$ case. We will first define two moduli problems Y_T and Z'_T, and then check that the morphisms $Y_T \to Z_T$ and $Y_T \to Z'_T$ are exactly isomorphisms. The main strategy is to prove by using Dieudonné theory that the closed points are in bijection, and the tangent spaces at each pair of corresponding closed points are isomorphic to each other. The essence of the proof involves an application of Grothendieck–Messing crystalline deformation theory (see Theorem 2.14).

To avoid the fussy and complicated discussion on Iwahori level structures, we always assume that $\mathbf{Sh}(G')$ is constructed by Construction 5.5(a) only. Equivalently, we assume that $T - S_\infty \subsetneq \Sigma_\infty - S_\infty$ whenever $\#(\Sigma_\infty - S_\infty)$ is odd.

6.1. **Algebraic configuration inside Dieudonné modules.** Before starting the proof, we first recall some notations regarding the theory of Dieudonné modules (c.f. Subsections 2.3 and 2.4). Let k be a perfect field and A be a polarized abelian k-scheme. We have canonical isomorphisms

$$H_1^{\mathrm{cris}}(A/k)_{W(k)} \cong \tilde{\mathcal{D}}(A), \quad H_1^{\mathrm{dR}}(A/k) \cong \mathcal{D}(A)$$

that are compatible with all the structures. For $j \in \{1, \dots, f\}$, the $W(k)$-action via τ_j gives a direct summand $\mathcal{D}(A)_j$ of $\mathcal{D}(A)$, as well as a corresponding Hodge filtration

$$0 \longrightarrow \omega_{A^\vee, j} \longrightarrow \mathcal{D}(A)_j \longrightarrow \mathrm{Lie}(A)_j \longrightarrow 0.$$

Also, by definition, $\mathcal{D}(A) = \tilde{\mathcal{D}}(A)/p\tilde{\mathcal{D}}(A)$ is the covariant Dieudonné module of $A[p]$. So under the reduction map $\tilde{\mathcal{D}}(A)_j \to \mathcal{D}(A)_j$, we can denote $\tilde{\omega}_{A^\vee, j}$ the preimage of $\omega_{A^\vee, j}$.

In a more general sense, we may replace $\operatorname{Spec} k$ with a locally noetherian k_0-scheme S and consider

$$\mathscr{H}_j^{A,(i)} := \{z \in H_1^{\mathrm{dR}}(A/S)/\mathscr{F}_j^{A,(i)} \colon ([\varpi] - \tau_j^{(i)}(\varpi)) \cdot z = 0\}.$$

If no confusion arises, then we can omit the superscript A and write $\mathscr{H}_j^{(i)}$ for simplicity. We have mentioned in Subsection 3.2 earlier that $\mathscr{H}_j^{(i)}$ is basically the $\tau_j^{(i)}$-component of $H_1^{\mathrm{dR}}(A/S)$, which is also an imitation of $H_1^{\mathrm{dR}}(A/S)_j$ in Deligne–Pappas module when p is unramified inert. Moreover, we respectively denote the Frobenius and Verschiebung maps on A as $F = F_A$ and $V = V_A$ and hence get the natural restrictions on subquotients in $H_1^{\mathrm{dR}}(A/S)$, say

$$F_j^{(i)} = F|_{(\mathscr{H}_j^{(i)})^{(p)}} \colon (\mathscr{H}_j^{(i)})^{(p)} \longrightarrow \mathscr{H}_{j+1}^{(i)},$$
$$V_j^{(i)} = V|_{\mathscr{H}_j^{(i)}} \colon \mathscr{H}_j^{(i)} \longrightarrow (\mathscr{H}_{j-1}^{(i)})^{(p)}.$$

We have already defined $\omega_j^{(i)} = \mathscr{F}_j^{A,(i)}/\mathscr{F}_j^{A,(i-1)}$ for shortness. Note that $\omega_j^{(i)}$ is a k-vector subspace of $\mathscr{H}_j^{(i)}$. The natural property of F and V leads to

$$\operatorname{Im} V_j^{(i)} = (\omega_{j-1}^{(i)})^{(p)} = \operatorname{Ker} F_{j-1}^{(i)}, \quad \operatorname{Im} F_{j-1}^{(i)} = \mathscr{H}_j^{(i)}/\omega_j^{(i)} = \operatorname{Ker} V_j^{(i)}$$

for all $j \in \{1, \dots, f\}$ and $i \in \{1, \dots, e\}$. Equivalently, we obtain the signature condition

$$r_j^{(i)} = \operatorname{rank}_{\mathcal{O}_S}(\operatorname{Im} V_{j+1}^{(i)}) = \operatorname{rank}_{\mathcal{O}_S} \omega_j^{(i)},$$
$$s_j^{(i)} = \operatorname{rank}_{\mathcal{O}_S}(\operatorname{Coker} V_{j+1}^{(i)}) = \operatorname{rank}_{\mathcal{O}_S} \mathscr{H}_j^{(i)}/\omega_j^{(i)}.$$

Also recall from Subsection 3.2 that $m_{\varpi,j}^{(i)}$ for all $1 \leqslant i \leqslant e$ are transfer maps between adjacent subquotients. More precisely,

$$m_{\varpi,j}^{(i)} \colon \mathscr{H}_j^{(i)} \longrightarrow \omega_j^{(i-1)} = \mathscr{F}_j^{(i-1)}/\mathscr{F}_j^{(i-2)} \subseteq \mathscr{H}_j^{(i-1)}.$$

For this, the signature condition is read as, for $1 \leqslant i < e$,

$$\begin{aligned} r_j^{(i)} &= \operatorname{rank}_{\mathcal{O}_S}(\operatorname{Im} m_{\varpi,j}^{(i+1)}) = \operatorname{rank}_{\mathcal{O}_S} \omega_j^{(i)}, \\ s_j^{(i)} &= \operatorname{rank}_{\mathcal{O}_S}(\operatorname{Coker} m_{\varpi,j}^{(i+1)}) = \operatorname{rank}_{\mathcal{O}_S} \mathscr{H}_j^{(i)}/\omega_j^{(i)}. \end{aligned}$$

Consider the perfect pairing $\langle\cdot,\cdot\rangle\colon H_1^{\mathrm{dR}}(A/S) \times H_1^{\mathrm{dR}}(A/S) \longrightarrow \mathcal{O}_S$ induce by the polarization $\lambda_A\colon A \to A^\vee$. Then the orthogonal complements of $\mathscr{H}_j^{(i)}$ and $\mathscr{H}_j^{(i-1)}$ lead us to another dual map, denoted by

$$d_{\varpi,j}^{(i)} \coloneqq m_{\varpi,j}^{(i+1),\vee}\colon \mathscr{H}_j^{(i)} \longrightarrow \mathscr{H}_j^{(i+1)}/\omega_j^{(i+1)} \simeq \mathscr{H}_j^{(i+1)}[\varpi] \subseteq \mathscr{H}_j^{(i+1)},$$

where $\mathscr{H}_j^{(i+1)}[\varpi]$ is the submodule of $\mathscr{H}_j^{(i+1)}$ annihilated by $[\varpi]$. This map is essentially the division-by-$[\varpi]$ (i.e. multiplication-by-$[\varpi^{-1}]$). As if in the previous case of $m_{\varpi,j}^{(i)}$, it is straightforward to check that $d_{\varpi,j}^{(i)}$ is a well-defined homomorphism between $\mathcal{O}_S$-modules. The signature condition can be thus rewritten as, for $1 < i \leqslant e$,

$$\begin{aligned} s_j^{(i)} &= \operatorname{rank}_{\mathcal{O}_S}(\operatorname{Im} d_{\varpi,j}^{(i-1)}) = \operatorname{rank}_{\mathcal{O}_S} \mathscr{H}_j^{(i)}/\omega_j^{(i)}, \\ r_j^{(i)} &= \operatorname{rank}_{\mathcal{O}_S}(\operatorname{Coker} d_{\varpi,j}^{(i-1)}) = \operatorname{rank}_{\mathcal{O}_S} \omega_j^{(i)}. \end{aligned}$$

Therefore, it follows that

$$\operatorname{Im} m_{\varpi,j}^{(i)} = \omega_j^{(i-1)} = \operatorname{Ker} d_{\varpi,j}^{(i-1)}, \quad \operatorname{Im} d_{\varpi,j}^{(i-1)} = \mathscr{H}_j^{(i)}/\omega_j^{(i)} = \operatorname{Ker} m_{\varpi,j}^{(i)}.$$

To understand the maps $m_{\varpi,j}^{(i)}$ and $d_{\varpi,j}^{(i)}$ above, we insert an example (c.f. Remark 3.1) as follows. When $k = \overline{\mathbb{F}}_p$, for $(A, \lambda, \rho, \underline{\mathscr{F}}) \in \mathbf{Sh}(G)(k)$, we have

$$H_1^{\mathrm{dR}}(A/k)_j \simeq (\overline{\mathbb{F}}_p[\varpi]/(\varpi^e)) \oplus (\overline{\mathbb{F}}_p[\varpi]/(\varpi^e)),$$

and after picking an appropriate basis, we have for some $0 < a, b \leqslant e$ and $a + b = 2e - i$ that

$$\begin{array}{ccccc} \mathscr{H}_j^{(i)} & = & (\varpi^{a-1}\overline{\mathbb{F}}_p[\varpi]/(\varpi^e)) & \oplus & (\varpi^{b-1}\overline{\mathbb{F}}_p[\varpi]/(\varpi^e)) \\ \cup\!\mid & & \cup\!\mid & & \cup\!\mid \\ \mathscr{F}_j^{(i)} & = & (\varpi^{a}\overline{\mathbb{F}}_p[\varpi]/(\varpi^e)) & \oplus & (\varpi^{b}\overline{\mathbb{F}}_p[\varpi]/(\varpi^e)). \end{array}$$

Moreover, there exists the circumvention Hasse map

$$\begin{array}{ccc} \mathrm{Hasse}_{\varpi,j}^{(1)}\colon \mathscr{H}_j^{(1)} & \xrightarrow{\qquad\qquad} & (\mathscr{F}_{j-1}^{(i)}/\mathscr{F}_{j-1}^{(e-1)})^{(p)} \\ \| & & \| \\ (H_1^{\mathrm{dR}}(A/S)_j)[\varpi] & \xrightarrow{\qquad\qquad} & (\omega_{j-1}^{(e)})^{(p)} \end{array}$$

in which $(H_1^{\mathrm{dR}}(A/S)_j)[\varpi]$ is the submodule of $H_1^{\mathrm{dR}}(A/S)_j$ annihilated by $[\varpi]$. By definition, $\mathrm{Hasse}_{\varpi,j}^{(1)}$ is essentially given by first dividing by $[\varpi]^{e-1}$ and then compositing with $V_j^{(e)}\colon \mathscr{H}_j^{(e)} \to (\mathscr{H}_{j-1}^{(e)})^{(p)}$. Define the *backward circumvention Hasse map* through

$$\mathrm{Hasse}_{\varpi,j}^{(e)} \coloneqq [\varpi]^{e-1} \circ F_j\colon (\mathscr{H}_j^{(e)})^{(p)} \longrightarrow \mathscr{H}_{j+1}^{(1)}.$$

Beware that the forward and the backward circumventions have superscripts (1) and (e), respectively. Accordingly, using the same argument as in the proof of Proposition 3.2, one can prove that each Frobenius map factors as

$$F_j^{(i)} = d_{\varpi,j+1}^{(i-1)} \circ \cdots \circ d_{\varpi,j+1}^{(1)} \circ \mathrm{Hasse}_{\varpi,j}^{(e)} \circ (d_{\varpi,j}^{(e-1)})^{(p)} \circ \cdots \circ (d_{\varpi,j}^{(i)})^{(p)}.$$

But supposedly, the reader is to concern with the subtlety behind the difference between the definition of $[\varpi]$ (resp. $[\varpi^{-1}]$) and $m_{\varpi,j}^{(i)}$ (resp. $d_{\varpi,j}^{(i)}$).

Summary. We obtain the following commutative diagrams to illustrate all the objects and the maps between them. Here fore each $\mathscr{H}_j^{(i)}$, we choose an isomorphism $\mathscr{H}_j^{(i)} \simeq \omega_j^{(i)} \oplus (\mathscr{H}_j^{(i)}/\omega_j^{(i)})$ to fix.

$$\begin{tikzcd}
\omega_j^{(e)} \arrow[d] \arrow[rr, shift left, Rightarrow] & \oplus & (\mathscr{H}_j^{(e)}/\omega_j^{(e)}) \arrow[r, bend left, "V_j^{(e)}"] & (\omega_{j-1}^{(e)})^{(p)} \arrow[l, bend left, "F_{j-1}^{(e)}"] \arrow[d] \arrow[rr, shift left, Rightarrow] \arrow[dddddlll, bend right, "\mathrm{Hasse}_{\varpi,j-1}^{(e)}"'] & \oplus & (\mathscr{H}_{j-1}^{(e)}/\omega_{j-1}^{(e)})^{(p)} \\
\vdots \arrow[d] \arrow[rr, shift left, Rightarrow] & & \vdots \arrow[u] & \vdots \arrow[d] \arrow[rr, shift left, Rightarrow] & & \vdots \arrow[u] \\
\omega_j^{(i)} \arrow[d, "m_{\varpi,j}^{(i)}"'] & \oplus & (\mathscr{H}_j^{(i)}/\omega_j^{(i)}) \arrow[u] \arrow[r, bend left, "V_j^{(i)}"] \arrow[dll] & (\omega_{j-1}^{(i)})^{(p)} \arrow[l, bend left, "F_{j-1}^{(i)}"] \arrow[d, "(m_{\varpi,j-1}^{(i)})^{(p)}"'] & \oplus & (\mathscr{H}_{j-1}^{(i)}/\omega_{j-1}^{(i)})^{(p)} \arrow[u] \arrow[dll] \\
\omega_j^{(i-1)} \arrow[d] \arrow[rr, shift left, Rightarrow] \arrow[urr] & \oplus & (\mathscr{H}_j^{(i-1)}/\omega_j^{(i-1)}) \arrow[u, "d_{\varpi,j}^{(i-1)}"'] \arrow[r, bend left, "V_j^{(i-1)}"] & (\omega_{j-1}^{(i-1)})^{(p)} \arrow[l, bend left, "F_{j-1}^{(i-1)}"] \arrow[d] \arrow[rr, shift left, Rightarrow] \arrow[urr] & \oplus & (\mathscr{H}_{j-1}^{(i-1)}/\omega_{j-1}^{(i-1)})^{(p)} \arrow[u, "(d_{\varpi,j-1}^{(i-1)})^{(p)}"'] \\
\vdots \arrow[d] \arrow[rr, shift left, Rightarrow] & & \vdots \arrow[u] & \vdots \arrow[d] \arrow[rr, shift left, Rightarrow] & & \vdots \arrow[u] \\
\omega_j^{(1)} & \oplus & (\mathscr{H}_j^{(1)}/\omega_j^{(1)}) \arrow[u] \arrow[r, bend left, "V_j^{(1)}"] \arrow[uuuuur, bend left, "\mathrm{Hasse}_{\varpi,j}^{(1)}"] & (\omega_{j-1}^{(1)})^{(p)} \arrow[l, bend left, "F_{j-1}^{(1)}"] & \oplus & (\mathscr{H}_{j-1}^{(1)}/\omega_{j-1}^{(1)})^{(p)} \arrow[u]
\end{tikzcd}$$

6.2. **Essential forward and backward Hasse maps.** To avoid a more complicated description for crystalline homology and Dieudonné modules arising from abelian schemes that are closed points of new moduli problems, we now define the *essential Hasse maps* for later use. This notion includes a forward map and a backward map for each pair (i,j).

Define the *forward Hasse map*, denoted by $\mathrm{Hasse}^{\downarrow}$, as $m_{\varpi,j}^{(i)}$ when $1 < i \leqslant e$, and as $\mathrm{Hasse}_{\varpi,j}^{(1)}$ when $i = 1$. Similarly, the *backward Hasse map* $\mathrm{Hasse}^{\uparrow}$ is taken as $d_{\varpi,j}^{(i)}$ when $1 \leqslant i < e$, and as $\mathrm{Hasse}_{\varpi,j}^{(e)}$ when $i = e$. Thus,

$$\mathrm{Hasse}^{\downarrow} \colon \mathscr{H}_j^{(i)} \longrightarrow \mathscr{H}_j^{(i-1)}, \qquad \text{if } 1 < i \leqslant e,$$
$$\mathrm{Hasse}^{\downarrow} \colon \mathscr{H}_j^{(i)} \longrightarrow (\mathscr{H}_{j-1}^{(e)})^{(p)}, \quad \text{if } i = 1,$$

and

$$\begin{aligned} \mathrm{Hasse}^{\uparrow}\colon \mathscr{H}_j^{(i)} &\longrightarrow \mathscr{H}_j^{(i+1)}, && \text{if } 1 \leqslant i < e, \\ \mathrm{Hasse}^{\uparrow}\colon (\mathscr{H}_j^{(i)})^{(p)} &\longrightarrow \mathscr{H}_{j+1}^{(1)}, && \text{if } i = e. \end{aligned}$$

Notice, by the algebraic configuration in the previous subsection, that either of the following three claims exactly holds:

(i) $\mathrm{Hasse}^{\downarrow}$ is an isomorphism from $\mathscr{H}_j^{(i)}$; equivalently, $s_j^{(i)} = 0$.

(ii) $\mathrm{Hasse}^{\downarrow}$ from $\mathscr{H}_j^{(i)}$ and $\mathrm{Hasse}^{\uparrow}$ towards $\mathscr{H}_j^{(i)}$ both have cokernels of k-dimension one simultaneously; equivalently, $s_j^{(i)} = 1$.

(iii) $\mathrm{Hasse}^{\uparrow}$ is an isomorphism towards $\mathscr{H}_j^{(i)}$; equivalently, $s_j^{(i)} = 2$.

We then define the *essential forward Hasse map* through

$$\mathrm{Hasse}_{\mathrm{es}}^{\downarrow} = \begin{cases} \mathrm{Hasse}^{\downarrow}, & \text{if (i) or (ii)}, \\ (\mathrm{Hasse}^{\uparrow})^{-1}, & \text{if (iii)}. \end{cases}$$

Reversely, the *essential backward Hasse map* is read as

$$\mathrm{Hasse}_{\mathrm{es}}^{\uparrow} = \begin{cases} \mathrm{Hasse}^{\uparrow}, & \text{if (ii) or (iii)}, \\ (\mathrm{Hasse}^{\downarrow})^{-1}, & \text{if (i)}. \end{cases}$$

As a consequence of the definitions, we see each essential Hasse map always has a 1-dimensional cokernel unless it is an isomorphism. Moreover, as there is a linear correspondence between subquotients $\mathscr{F}_j^{(i)}/\mathscr{F}_j^{(i-1)}$ (as well as their twists by p-powers) and $\tau_j^{(i)}$, we see $\mathrm{Hasse}_{\mathrm{es}}^{\downarrow}(\mathscr{F}_j^{(i)}/\mathscr{F}_j^{(i-1)})$ is always contained the subquotient associated to the embedding $\iota(\tau_j^{(i)})$, and similarly for the reverse.

We also introduce notations $\iota^n(\mathscr{H}_j^{(i)})$ and $\iota^n(\omega_j^{(i)})$ to denote the $\iota^n(\tau_j^{(i)})$-components of $H_1^{\mathrm{dR}}(A/S)$ and $\omega_{A^\vee/S}$, respectively. The meaning can go similarly for $\iota^{-n}(\mathscr{H}_j^{(i)})$ and $\iota^{-n}(\omega_j^{(i)})$.

Given an embedding $\tau = \tau_j^{(i)} \in \Delta(T) = \bigsqcup_{\epsilon=1}^m \Delta_\epsilon(T) \subseteq \Sigma_\infty$, there is a unique index $\epsilon = \epsilon(i,j) \in \{1,\ldots,m\}$ so that $\tau \in \Delta_\epsilon(T)$. By Lemma 5.11, we see $\Delta_\epsilon(T) - T' \subseteq S_\infty$, whose elements are all with signature 0 or 2. This phenomenon motivates us to put

$$n(i,j) \coloneqq \min\{n \geqslant 1\colon \iota^n(\tau_j^{(i)}) \notin S_\infty\}.$$

Note that when $\tau_j^{(i)} \in \Delta(T) - T'$, the set S_∞ in definition can be replaced with $\Delta(T)$. The composite of essential Hasse maps for $n(i,j)$ times are given as follows:

$$\begin{aligned} \mathrm{Hasse}_{\mathrm{es}}^{\downarrow,n(i,j)}\colon \mathscr{H}_j^{(i)} &\xrightarrow{\sim} \iota(\mathscr{H}_j^{(i)}) \xrightarrow{\sim} \cdots \xrightarrow{\sim} \iota^{n(i,j)-1}(\mathscr{H}_j^{(i)}) \longrightarrow \iota^{n(i,j)}(\mathscr{H}_j^{(i)}), \\ \mathrm{Hasse}_{\mathrm{es}}^{\uparrow,n(i,j)}\colon \mathscr{H}_j^{(i)} &\longrightarrow \iota^{-1}(\mathscr{H}_j^{(i)}) \xrightarrow{\sim} \cdots \xrightarrow{\sim} \iota^{-n(i,j)+1}(\mathscr{H}_j^{(i)}) \xrightarrow{\sim} \iota^{-n(i,j)}(\mathscr{H}_j^{(i)}). \end{aligned}$$

In the definition of $\mathrm{Hasse}_{\mathrm{es}}^{\downarrow,n(i,j)}$, all morphisms are isomorphisms except the last one towards the subquotient corresponding to $\iota^{n(i,j)}(\tau_j^{(i)}) \notin S_\infty$. Similarly, in the definition of $\mathrm{Hasse}_{\mathrm{es}}^{\uparrow,n(i,j)}$, all morphisms are isomorphisms except the first one. The two excluded maps must have locally free cokernels of $\mathcal{O}_S$-rank 1.

6.3. **The two auxiliary moduli problems.** We define two moduli functors $\underline{Y}_T$ and $\underline{Z}'_T$ that are respectively representable by locally noetherian schemes Y_T and Z'_T over k_0.

Definition 6.1. The functor $\underline{Y}_T$ is defined as follows:

$$\begin{aligned} \underline{Y}_T\colon\ \mathsf{Sch}_{k_0}^{\text{loc-noe}} &\longrightarrow \mathsf{Sets} \\ S &\longmapsto \{(A,\lambda_A,\rho_A,\underline{\mathscr{F}}^A;B,\lambda_B,\rho_B,\underline{\mathscr{F}}^B;\phi)\}/\sim \end{aligned}$$

that sends a locally noetherian scheme S over k_0 to isomorphism classes of such tuples, satisfying the following conditions.

(1) $(A,\lambda_A,\rho_A,\underline{\mathscr{F}}^A)\in Z_T(S)$, where A is an abelian scheme over S of relative dimension $4ef$, equipped with faithful $\mathcal{O}_D$-action via the embedding homomorphism $\mathcal{O}_D\to\mathrm{End}_S(A)$.

(2) $(B,\lambda_B,\rho_B,\underline{\mathscr{F}}^B)\in\mathbf{Sh}(G')(S)$, where the signature and level structure of $\mathbf{Sh}(G')$ are given by Construction 5.5 (c.f. Definition 1.1).

(3) $\phi\colon A\to B$ is an $\mathcal{O}_D$-isogeny such that:

(3a) It is compatible with the polarizations, i.e., $p\lambda_A=\phi^\vee\circ\lambda_B\circ\phi$.

(3b) For $\tau=\tau_j^{(i)}\notin\Delta(T)$, the induced map

$$\phi_{*,j}^{(i)}\colon \mathscr{H}_j^{A,(i)}\xrightarrow{\sim}\mathscr{H}_j^{B,(i)}$$

is an isomorphism. Here we restrict the τ_j-stable induced homological map $\phi_{*,j}\colon H_1^{\mathrm{dR}}(A/S)_j\to H_1^{\mathrm{dR}}(B/S)_j$ to the ith subquotient in $\underline{\mathscr{F}}^A$ to define $\phi_{*,j}^{(i)}$ as $\phi_{*,\tau}=\phi_{*,j}|_{\mathscr{H}_j^{A,(i)}}$.

(3c) For $\tau=\tau_j^{(i)}\in\Delta(T)$,

$$\mathrm{Ker}\,\phi_{*,j}^{(i)}=\mathrm{Hasse}_{\mathrm{es}}^{\uparrow,n(i,j)}(\iota^{n(i,j)}(\mathscr{H}_j^{(i)})).$$

Here the Hasse maps and integer $n(i,j)$ are defined in Subsection 6.2, and $\mathrm{Hasse}_{\mathrm{es}}^{\uparrow,n(i,j)}$ denotes the composition of $\mathrm{Hasse}_{\mathrm{es}}^{\uparrow}$ for $n(i,j)$ times.

(4) The prime-to-p tame level structures are compatible, i.e. in the sense of modulo K', for the fixed $\hat{\mathbb{Z}}$-lattice $\Lambda'\subseteq V'(\mathbb{A}_{\mathbb{Q},f})$ with the isomorphism $\eta\colon\Lambda'\otimes_{\hat{\mathbb{Z}}}\hat{\mathbb{Z}}^{(p)}\xrightarrow{\sim}\Lambda\otimes_{\hat{\mathbb{Z}}}\hat{\mathbb{Z}}^{(p)}$, we have

$$T_{\hat{\mathbb{Z}}^{(p)}}(\phi)\circ\rho_A^p\circ\eta=\rho_B^p\colon\Lambda'\otimes_{\hat{\mathbb{Z}}}\hat{\mathbb{Z}}^{(p)}\longrightarrow T_{\hat{\mathbb{Z}}^{(p)}}(B)$$

as maps (see Subsection 1.1 and Definition 1.1(3a)). Here K' is defined at the end of Construction 5.5, $T_{\hat{\mathbb{Z}}^{(p)}}(B)$ (resp. $T_{\hat{\mathbb{Z}}^{(p)}}(A)$) denotes the product of the ℓ-adic Tate modules of B (resp. A) for all $\ell\neq p$, and $T_{\hat{\mathbb{Z}}^{(p)}}(\phi)\colon T_{\hat{\mathbb{Z}}^{(p)}}(A)\to T_{\hat{\mathbb{Z}}^{(p)}}(B)$ is the natural induced map.

Definition 6.2. The functor $\underline{Z}'_T$ is defined as follows:

$$\begin{aligned} \underline{Z}'_T\colon\ \mathsf{Sch}_{k_0}^{\text{loc-noe}} &\longrightarrow \mathsf{Sets} \\ S &\longmapsto \{(B,\lambda_B,\rho_B,\underline{\mathscr{F}}^B;\mathbb{J})\}/\sim \end{aligned}$$

that sends a locally noetherian scheme S over k_0 to isomorphism classes of such tuples, satisfying the following conditions.

(1) $(B, \lambda_B, \rho_B, \underline{\mathscr{F}}^B) \in \mathbf{Sh}(G')(S)$, the same as in Definition 6.1(2).
(2) $\mathbb{J}$ is (a choice of) a collection of subbundles $\mathscr{J}_\tau = \mathscr{J}_j^{(i)} \subseteq \mathscr{F}_j^{(i)}/\mathscr{F}_j^{(i-1)}$ that are locally free of rank 1 for each $\tau = \tau_j^{(i)} \in I(T)$ in the corresponding subquotient of τ, where $I(T)$ is given by Construction 5.5.

By the general theory of moduli spaces of abelian schemes due to Mumford, the moduli functors $\underline{Y}_T$ and $\underline{Z}'_T$ are representable by k_0-schemes Y_T and Z'_T of finite type.

Lemma 6.3. *The k_0-scheme Z'_T is a $(\mathbb{P}^1)^{\#I(T)}$-bundle over $\mathbf{Sh}(G')$.*

Proof. This is clear by the definition of Z'_T, in which we assign a collection of $\#I(T)$ line subbundles to each S-point $(B, \lambda_B, \rho_B, \underline{\mathscr{F}}^B)$. □

The following proposition is the key to proving our main Theorem 5.1, and the rest of this section contributes the proof of the proposition.

Proposition 6.4. *For any subset $T \subseteq \Sigma_\infty$, there exist two isomorphisms as follows:*

$$\begin{tikzcd} & Y_T \arrow[dl, "\cong"', "\eta_1"] \arrow[dr, "\cong", "\eta_2"'] & \\ Z_T & & Z'_T. \end{tikzcd}$$

6.4. **The first isomorphism $Y_T \to Z_T$.** The upshot of this step is that the information carried by $(B, \lambda_B, \rho_B, \underline{\mathscr{F}}^B)$ and ϕ in Definition 6.1 can be recovered from $(A, \lambda_A, \rho_A, \underline{\mathscr{F}}^A)$.

Proposition 6.5. *The natural forgetful functor*

$$\eta_1 \colon (A, \lambda_A, \rho_A, \underline{\mathscr{F}}^A; B, \lambda_B, \rho_B, \underline{\mathscr{F}}^B; \phi) \longmapsto (A, \lambda_A, \rho_A, \underline{\mathscr{F}}^A)$$

induces an isomorphism $\eta_1 \colon Y_T \xrightarrow{\sim} Z_T$.

Proof. By the representability of moduli schemes, it suffices to show that the natural map $Y_T \to Z_T$ induces a bijection on closed points and an isomorphism of the tangent spaces at each closed point. The proposition will follow thus from Lemmas 6.6 and 6.8 below. □

Lemma 6.6. *Suppose k is a perfect field of characteristic $p > 0$. Then for any k-valued point $(A, \lambda_A, \rho_A, \underline{\mathscr{F}}^A) \in Z_T(k)$, there exists a unique tuple of data $(B, \lambda_B, \rho_B, \underline{\mathscr{F}}^B; \phi)$ such that*

$$(A, \lambda_A, \rho_A, \underline{\mathscr{F}}^A; B, \lambda_B, \rho_B, \underline{\mathscr{F}}^B; \phi) \in Y_T(k).$$

Proof. The main task is to establish B from A and to check that B satisfies the desired conditions in Definition 1.1 and other compatibilities.

Step I. For this, we define first for each $\tau = \tau_j^{(i)} \in \Sigma_\infty$ a $W(k)$-module $M_\tau = M_j^{(i)}$ with the property

$$\mathscr{H}_j^{A,(i)} \subseteq M_j^{(i)} \subseteq [\varpi^{-1}] \cdot \mathscr{H}_j^{A,(i)}.$$

The construction is as follows.

- If $\tau_j^{(i)}$ is not exceptional, i.e. it lies outside $\Delta(T)$, then we take
$$M_j^{(i)} := \mathscr{H}_j^{A,(i)}, \quad \tau_j^{(i)} \notin \Delta(T).$$
- Otherwise, whenever $\tau_j^{(i)}$ is exceptional,
$$M_j^{(i)} := [\varpi^{-1}] \cdot \mathrm{Hasse}_{\mathrm{es}}^{\uparrow, n(i,j)}(\iota^{n(i,j)}(\mathscr{H}_j^{A,(i)})), \quad \tau_j^{(i)} \in \Delta(T).$$

By the definition of Z_T, the partial Hasse invariant $h_j^{(i)}$ for A vanishes at all $\tau_j^{(i)} \in T \subseteq \Sigma_\infty$. Alternatively, for each $\tau_j^{(i)} \in T$, we have $\mathrm{Hasse}^{\downarrow}(\mathscr{H}_j^{A,(i)}) = 0$, or even equivalently,
$$\mathrm{Hasse}_{\mathrm{es}}^{\uparrow, n(i,j)}(\iota^{n(i,j)}(\mathscr{H}_j^{A,(i)})) = \omega_j^{A,(i)}.$$

The key of the construction is that $M_j^{(i)}$ will play the role of $\mathscr{H}_j^{B,(i)}$ in the Dieudonné module of B. If so, then we can truly recover $\mathcal{D}(B)$ from $\mathcal{D}(A)$. Our goal now is to verify that such $M_j^{(i)}$s are stable under the essential Hasse maps. So we are to check for any $\tau_j^{(i)} \in \Sigma_\infty$ that,

$$(\dagger) \qquad \begin{cases} m_{\varpi,j}^{(i)}(M_j^{(i)}) \subseteq M_j^{(i-1)}, & \text{for } 1 < i \leqslant e, \\ \mathrm{Hasse}_{\varpi,j}^{(1)}(M_j^{(1)}) \subseteq (M_{j-1}^{(e)})^{(p)}, & \text{for } i = 1, \end{cases}$$

$$(\dagger\dagger) \qquad \begin{cases} d_{\varpi,j}^{(i)}(M_j^{(i)}) \subseteq M_j^{(i+1)}, & \text{for } 1 \leqslant i < e, \\ \mathrm{Hasse}_{\varpi,j}^{(e)}(M_j^{(e)}) \subseteq (M_{j+1}^{(1)})^{(p)}, & \text{for } i = e. \end{cases}$$

These can deduce $M_j^{(i)}$s are stable under $\mathrm{Hasse}_{\mathrm{es}}^{\downarrow}$ and $\mathrm{Hasse}_{\mathrm{es}}^{\uparrow}$. Note that it would not be sufficient if we checked ($\dagger$) and ($\dagger\dagger$) for elements in T only. In the upcoming context, we choose to check ($\dagger$), and ($\dagger\dagger$) can be deduced via the same argument by reversing all the directions of the maps involved. Importantly, the most essential observation lies in that

- ◇ *Key.* $\mathrm{Hasse}_{\mathrm{es}}^{\uparrow, n}$ never goes surjectively (and, in fact, has a 1-dimensional cokernel) as long as its source corresponds to an embedding with signature 1.

The first task is to consider $1 < i \leqslant e$. Assuming this, we distinguish several cases as follows.

(1a) If $\tau_j^{(i)}, \tau_j^{(i-1)} \notin \Delta(T)$, then
$$M_j^{(i)} = \mathscr{H}_j^{A,(i)}, \quad M_j^{(i-1)} = \mathscr{H}_j^{A,(i-1)}.$$
Hence $m_{\varpi,j}^{(i)}(M_j^{(i)}) \subseteq M_j^{(i-1)}$ by the nature of $m_{\varpi,j}^{(i)}$.

(1b) If $\tau_j^{(i)} \in \Delta(T)$ but $\tau_j^{(i-1)} \notin \Delta(T)$, then $M_j^{(i-1)} = \omega_j^{A,(i-1)}$ and $n(i,j) = 1$. So
$$\begin{aligned} m_{\varpi,j}^{(i)}(M_j^{(i)}) &= m_{\varpi,j}^{(i)}([\varpi^{-1}] \cdot \mathrm{Hasse}_{\mathrm{es}}^{\uparrow, n(i,j)}(\iota^{n(i,j)}(\mathscr{H}_j^{A,(i)}))) \\ &\subseteq [\varpi^{-1}] \cdot m_{\varpi,j}^{(i)}(\mathscr{H}_j^{A,(i)}) \\ &= [\varpi^{-1}] \cdot \omega_j^{A,(i-1)} = M_j^{(i-1)}. \end{aligned}$$

Hence the desired relation follows immediately.[3] Notice that the second row is a strict inclusion, because $\mathrm{Hasse}_{\mathrm{es}}^{\uparrow,n(i,j)}$ is a composition of $n(i,j)-1$ isomorphisms and a homomorphism whose cokernel is 1-dimensional.

(1c) If $\tau_j^{(i)},\tau_j^{(i-1)}\in\Delta(T)$, then $n(i,j)=n(i-1,j)+1$ by the definition of $n(i,j)$. In this case, we have

$$M_j^{(i-1)}=[\varpi^{-1}]\cdot\mathrm{Hasse}_{\mathrm{es}}^{\uparrow,n(i-1,j)}(\iota^{n(i-1,j)}(\mathscr{H}_j^{A,(i-1)}))$$

and

$$\begin{aligned}M_j^{(i)}&=[\varpi^{-1}]\cdot\mathrm{Hasse}_{\mathrm{es}}^{\uparrow,n(i,j)}(\iota^{n(i,j)}(\mathscr{H}_j^{A,(i)}))\\&=[\varpi^{-1}]\cdot\mathrm{Hasse}_{\mathrm{es}}^{\uparrow,n(i-1,j)}(\mathrm{Hasse}_{\mathrm{es}}^{\uparrow}(\iota(\iota^{n(i-1,j)}(\mathscr{H}_j^{A,(i)}))))\\&\subseteq[\varpi^{-1}]\cdot\mathrm{Hasse}_{\mathrm{es}}^{\uparrow,n(i-1,j)}(\iota^{n(i-1,j)}(\mathscr{H}_j^{A,(i)})).\end{aligned}$$

By the commutativity of $m_{\varpi,j}^{(i)}$ and $\mathrm{Hasse}_{\mathrm{es}}^{\uparrow}$, we thus get

$$\begin{aligned}m_{\varpi,j}^{(i)}(M_j^{(i)})&\subseteq[\varpi^{-1}]\cdot\mathrm{Hasse}_{\mathrm{es}}^{\uparrow,n(i-1,j)}(\iota^{n(i-1,j)}(m_{\varpi,j}^{(i)}(\mathscr{H}_j^{A,(i)})))\\&\subseteq[\varpi^{-1}]\cdot\mathrm{Hasse}_{\mathrm{es}}^{\uparrow,n(i-1,j)}(\iota^{n(i-1,j)}(\mathscr{H}_j^{A,(i-1)}))\\&=M_j^{(i-1)}.\end{aligned}$$

So we attain the inclusion as expected.

(1d) If $\tau_j^{(i)}\notin\Delta(T)$ but $\tau_j^{(i-1)}\in\Delta(T)$, then from Construction 5.9 we see $\tau_j^{(i-1)}\in T$, and hence the generalized partial Hasse invariant $h_j^{(i-1)}$ vanishes at $\tau_j^{(i-1)}$. Therefore, with $M_j^{(i)}=\mathscr{H}_j^{A,(i)}$,

$$\begin{aligned}M_j^{(i-1)}&=[\varpi^{-1}]\cdot\mathrm{Hasse}_{\mathrm{es}}^{\uparrow,n(i-1,j)}(\iota^{n(i-1,j)}(\mathscr{H}_j^{A,(i-1)}))\\&=[\varpi^{-1}]\cdot\omega_j^{A,(i-1)}\end{aligned}$$

by the equivalent expression of vanishing. Then

$$m_{\varpi,j}^{(i)}(M_j^{(i)})=m_{\varpi,j}^{(i)}(\mathscr{H}_j^{(i)})=\omega_j^{A,(i-1)}\subseteq M_j^{(i-1)},$$

which is as expected. Note that the vanishing condition is essentially used here since we are forced to derive a strict equality to describe $M_j^{(i-1)}$.

The next task is to complete the verification of (†) with $i=1$.

(2a) If $\tau_j^{(1)},\tau_{j-1}^{(e)}\notin\Delta(T)$, then

$$M_j^{(1)}=\mathscr{H}_j^{A,(1)},\quad M_{j-1}^{(e)}=\mathscr{H}_{j-1}^{A,(e)}.$$

So the inclusion is apparent like in (1a).

(2b) If $\tau_j^{(1)}\in\Delta(T)$ but $\tau_{j-1}^{(e)}\notin\Delta(T)$, then

$$M_j^{(1)}=[\varpi^{-1}]\cdot\mathrm{Hasse}_{\mathrm{es}}^{\uparrow,n(1,j)}(\iota^{n(1,j)}(\mathscr{H}_j^{A,(1)})),\quad M_{j-1}^{(e)}=\mathscr{H}_{j-1}^{A,(e)}.$$

[3] In fact, the statement implies $\tau_j^{(i-1)}\in T'$. But this property is hardly in need.

So we are able to compute with $n(1,j)=1$ that

$$\begin{aligned}\mathrm{Hasse}^{(1)}_{\varpi,j}(M^{(1)}_j) &= [\varpi^{-1}]\cdot \mathrm{Hasse}^{(1)}_{\varpi,j}(\mathrm{Hasse}^{\uparrow}_{\mathrm{es}}(\iota(\mathscr{H}^{A,(1)}_j)))\\ &\subseteq [\varpi^{-1}]\cdot \mathrm{Hasse}^{(1)}_{\varpi,j}(\mathscr{H}^{A,(1)}_j)\\ &= [\varpi^{-1}]\cdot (\omega^{(e)}_{j-1})^{(p)}\end{aligned}$$

where the last row equals $(\mathscr{H}^{A,(e)}_{j-1})^{(p)}=(M^{(e)}_{j-1})^{(p)}$. So we are done in this case.

(2c) If $\tau^{(1)}_j,\tau^{(e)}_{j-1}\in\Delta(T)$, then $n(1,j)=n(e,j-1)+1$, and we have

$$M^{(e)}_{j-1}=[\varpi^{-1}]\cdot \mathrm{Hasse}^{\uparrow,n(e,j-1)}_{\mathrm{es}}(\iota^{n(e,j-1)}(\mathscr{H}^{A,(e)}_{j-1})).$$

as well as

$$\begin{aligned}M^{(1)}_j &= [\varpi^{-1}]\cdot \mathrm{Hasse}^{\uparrow,n(1,j)}_{\mathrm{es}}(\iota^{n(1,j)}(\mathscr{H}^{A,(1)}_j))\\ &\subseteq [\varpi^{-1}]\cdot \mathrm{Hasse}^{\uparrow,n(e,j-1)}_{\mathrm{es}}(\iota^{n(e,j-1)}(\mathscr{H}^{A,(1)}_j)),\end{aligned}$$

through the computation similar to (1c). And hence

$$\begin{aligned}\mathrm{Hasse}^{(1)}_{\varpi,j}(M^{(1)}_j) &\subseteq [\varpi^{-1}]\cdot \mathrm{Hasse}^{(1)}_{\varpi,j}(\mathrm{Hasse}^{\uparrow,n(e,j-1)}_{\mathrm{es}}(\iota^{n(e,j-1)}(\mathscr{H}^{A,(1)}_j)))\\ &\subseteq [\varpi^{-1}]\cdot \mathrm{Hasse}^{\uparrow,n(e,j-1)}_{\mathrm{es}}(\iota^{n(e,j-1)}(\mathrm{Hasse}^{(1)}_{\varpi,j}(\mathscr{H}^{A,(1)}_j)))\\ &\subseteq [\varpi^{-1}]\cdot \mathrm{Hasse}^{\uparrow,n(e,j-1)}_{\mathrm{es}}(\iota^{n(e,j-1)}((\mathscr{H}^{A,(e)}_{j-1})^{(p)}))\\ &= (M^{(e)}_{j-1})^{(p)}.\end{aligned}$$

This is clearly contained in $(M^{(e)}_{j-1})^{(p)}$.

(2d) If $\tau^{(1)}_j\notin\Delta(T)$ but $\tau^{(e)}_{j-1}\in\Delta(T)$, then $\tau^{(e)}_{j-1}\in T$ and $M^{(1)}_j=\mathscr{H}^{A,(1)}_j$. Again, since the partial Hasse invariant vanishes,

$$\begin{aligned}M^{(e)}_{j-1} &= [\varpi^{-1}]\cdot \mathrm{Hasse}^{\uparrow,n(e,j-1)}_{\mathrm{es}}(\iota^{n(e,j-1)}(\mathscr{H}^{A,(e)}_{j-1}))\\ &= [\varpi^{-1}]\cdot \omega^{(e)}_{j-1}.\end{aligned}$$

Simultaneously,

$$\mathrm{Hasse}^{(1)}_{\varpi,j}(M^{(1)}_j)=\mathrm{Hasse}^{(1)}_{\varpi,j}(\mathscr{H}^{(1)}_j)=(\omega^{(e)}_{j-1})^{(p)}\subseteq (M^{(e)}_{j-1})^{(p)}$$

This is the same as (1d).

Step II. Now we have accomplished to verifying (†) and (††). Resume on the construction of B. Put $\tilde{M}^{(i)}_j=\tilde{\mathscr{H}}^{(i)}_j$ for $\tau^{(i)}_j\notin\Delta(T)$ and $\tilde{M}^{(i)}_j=[\varpi^{-1}]\mathrm{Hasse}^{\uparrow,n(i,j)}_{\mathrm{es}}(\tilde{\mathscr{H}}^{(i)}_j)$ for $\tau^{(i)}_j\in\Delta(T)$, where $\tilde{\mathscr{H}}^{(i)}_j$ is the preimage of $\mathscr{H}^{(i)}_j$ under the reduction map $\tilde{\mathcal{D}}(A)_j[\varpi^{-1}]\to\mathcal{D}(A)_j[\varpi^{-1}]$.[4] Then (†) and (††) are still valid while replacing each $M^{(i)}_j$ by $\tilde{M}^{(i)}_j$. Take

$$\tilde{M}=\bigoplus_{j=1}^{f}\bigoplus_{i=1}^{e}\tilde{M}^{(i)}_j.$$

[4]The notation $\tilde{\mathcal{D}}(A)_j[\varpi^{-1}]$ here (and also, similarly for $\tilde{\mathcal{D}}(B)_j[\varpi^{-1}]$ that will appear later) means the $\tilde{\mathcal{D}}(A)_j$-algebra by adding ϖ^{-1} as a monomial generator. This is in contrast to $\mathscr{H}^{(i)}_j[\varpi]$, which denotes the $[\varpi]$-torsion part of $\mathscr{H}^{(i)}_j$.

Then $\tilde{M}$ is a Dieudonné module with induced F and V on it, together with the induced $\mathrm{Hasse}_{\mathrm{es}}^{\uparrow}$ and $\mathrm{Hasse}_{\mathrm{es}}^{\downarrow}$ between its adjacent subquotients. Consider the quotient $\tilde{M}/\tilde{\mathcal{D}}(A)$. It corresponds to a finite subgroup scheme C of $A[p]$ that is stable under the action of $\mathcal{O}_D$ by the covariant Dieudonné theory. We put the quotient group scheme $B = A/C$ as well as the natural quotient map

$$\phi\colon A \longrightarrow A/C = B$$

to be the desired $\mathcal{O}_D$-isogeny, so that its induced map on Dieudonné modules is identified with the natural inclusion

$$\phi_*\colon \tilde{\mathcal{D}}(A) \longrightarrow \tilde{\mathcal{D}}(B) = M.$$

It is clear that ϕ_* satisfies Definition 6.1(3). Via this construction, we see the $\mathcal{O}_D$-action on A is inherited by B. So there is a ring homomorphism $\mathcal{O}_D \to \mathrm{End}_k(B)$. We also define the quasi-polarization of B via

$$\lambda_B\colon B \xleftarrow{\ \phi\ } A \xrightarrow{\ \lambda_A\ } A^\vee \xleftarrow{\ \phi^\vee\ } B^\vee.$$

Moreover, the algebraic configuration of $\mathcal{D}(B)$ is given by putting

$$\omega_j^{B,(i)} = \mathscr{F}_j^{B,(i)}/\mathscr{F}_j^{B,(i-1)} := \mathrm{Hasse}^{\downarrow}(\iota^{-1}(M_j^{(i)})) \subseteq \mathscr{H}_j^{B,(i)} := M_j^{(i)}.$$

As a consequence of this, the splitting filtration of B is

$$\mathscr{F}_j^{B,(i)} = \bigoplus_{l=1}^{i} \omega_j^{B,(l)} = \bigoplus_{l=1}^{i} \mathrm{Hasse}^{\downarrow}(\iota^{-1}(M_j^{(l)})).$$

We have to check that this abelian scheme B satisfies conditions (2)–(4) in Definition 1.1 for the Shimura variety $\mathbf{Sh}(G')$; after this, we also have to show that $\phi\colon A \to B$ is the $\mathcal{O}_D$-isogeny defined by Definition 6.1(4). For Definition 1.1(2b), it is equivalent to proving that, when viewing $\tilde{\mathcal{D}}(B)$ as a $W(k)$-lattice of $\tilde{\mathcal{D}}(A)[\varpi^{-1}]$ via the $\mathcal{O}_D$-isogeny $\phi\colon A \to B$, the perfect alternating pairing

$$\langle\cdot,\cdot\rangle_{\lambda_A,j}^{(i)}\colon \tilde{\mathscr{H}}_j^{A,(i)}[\varpi^{-1}] \times \tilde{\mathscr{H}}_j^{A,(i),\perp}[\varpi^{-1}] \longrightarrow W(k)[\varpi^{-1}]$$

for $j \in \{1,\dots,f\}$ and $i \in \{1,\dots,e\}$ induces a perfect pairing

$$\tilde{\mathscr{H}}_j^{B,(i)} \times \tilde{\mathscr{H}}_j^{B,(i),\perp} \longrightarrow W(k)[\varpi^{-1}].$$

By the construction of B, we have $\tilde{\mathscr{H}}_j^{B,(i)} = \tilde{\mathscr{H}}_j^{A,(i)}$ unless $\tau_j^{(i)} \in \Delta(T)$. So the statement is almost clear except for the case where $\tau_j^{(i)} \in \Delta(T)$. When $\Delta(T)$ is nonempty, in this situation, the perfect duality between $\tilde{\mathscr{H}}_j^{B,(i)}$ and $\tilde{\mathscr{H}}_j^{B,(i),\perp}$ follows from the equality

$$\langle \varpi^{-1}\mathrm{Hasse}_{\mathrm{es}}^{\uparrow,n(i,j)}(u), \mathrm{Hasse}_{\mathrm{es}}^{\uparrow,n(i,j)}(v)\rangle_{\lambda_A,j}^{(i)} = \langle u,v\rangle_{\lambda_A,j}^{(i)}$$

for all $u \in \iota^{n(i,j)}(\tilde{\mathscr{H}}_j^{A,(i)})$ and $v \in \iota^{n(i,j)}(\tilde{\mathscr{H}}_j^{A,(i),\perp})$. Finally, it is also clear that λ_B induces the Rosati involution $*$ on $\mathcal{O}_D$, so Definition 1.1(2a) is valid for B.

Step III. Within this step, our goal is to check that the abelian scheme B has the correct signature required by the moduli space $\mathbf{Sh}(G') = (\mathcal{M}_{K',\Lambda'}^{\mathrm{PR}})_{\mathbb{F}_p}$. This corresponds to Definition 1.1(4). It is equivalent to show that each subquotient $\omega_j^{B,(i)}$ is a k-vector space of dimension $r_j^{(i)\prime} \in \{0,1,2\}$. This assertion is written alternatively by Lemma 6.7, which we grant at this moment and choose to postpone the proof.

It suffices to check the condition that $\dim_k \operatorname{Coker} \phi_{*,\tau} = \dim_k \operatorname{Coker} \phi_{*,j}^{(i)} \in \{0,1\}$ and it equals 1 if and only if $\tau \in \Delta(T)$. By Definition 6.1(3b), $\phi_{*,\tau}$ is an isomorphism when $\tau \notin \Delta(T)$. Using the notations proposed by the proof of Lemma 6.7, we are just to consider two cases when $\tau = \tau_j^{(i)} \in \Delta(T)$:

(i) If $\iota^{-1}(\tau) \notin \Delta(T)$, then $\tau \in T'_{\mathrm{odd}}$ whose signature becomes 0 after the change. By the signature condition, $\phi_{*,\tau} = \phi_{*,j}^{(i)} \colon \mathscr{H}_j^{A,(i)} \to \mathscr{H}_j^{B,(i)}$ travels from a 1-dimensional k-vector space to zero in its target. Thus, $\dim_k \operatorname{Coker} \phi_{*,\tau} = 1$.

(ii) If $\iota(\tau) \in \Delta(T)$, then $\tau \notin T'$ and its signature does not change. Since the restrictions of ϕ_* commutes with essential Hasse maps on all subquotients and the cokernels of $\mathrm{Hasse}_{\mathrm{es}}$ in $\mathscr{H}_j^{A,(i)}$ and $\mathscr{H}_j^{B,(i)}$ are isomorphic, we have
$$\dim_k \operatorname{Coker} \phi_{*,\tau} = \dim_k \operatorname{Coker} \phi_{*,\iota^{-1}(\tau)}.$$
Obtaining this equality, we can shift the position of τ in $\Delta(T)$ without changing $\dim_k \operatorname{Coker} \phi_{*,\tau}$. So for our purpose, may assume $\iota^{-1}(\tau) \in T'_{\mathrm{odd}}$; namely, $\iota^{-1}(\tau)$ is the first element in $\Delta_\epsilon(T)$, where ϵ is the unique index such that $\tau \in \Delta_\epsilon(T) \subseteq \Delta(T)$. By the same argument as in (i) above, we have $\dim_k \operatorname{Coker} \phi_{*,\iota^{-1}(\tau)} = 1$, which implies $\dim_k \operatorname{Coker} \phi_{*,\tau} = 1$.

Step IV. It remains to check that $\phi \colon A \to B$ is the $\mathcal{O}_D$-isogeny in Definition 6.1(3)(4). The relation $p\lambda_A = \phi^\vee \circ \lambda_B \circ \phi$ in (3a) is an immediate consequence of the definition of λ_B. Since for $\tau_j^{(i)} \notin \Delta(T)$ we have $M_j^{(i)} = \mathscr{H}_j^{A,(i)}$, the induced map $\phi_{*,j}^{(i)}$ is thus the identity map between $\mathscr{H}_j^{A,(i)}$ and $\mathscr{H}_j^{B,(i)}$, which leads to (3b).

We are left with Definition 1.1(3) and 6.1(4). As for the level structure $(\rho_B^p, \rho_{B,p})$, we have constructed $K' = K^{p\prime} K_p$ in Construction 5.5, where $K_p = \mathbb{Z}_p^\times \times \mathrm{GL}_n(\mathcal{O}_{F_\mathfrak{p}})$ and $K^{p\prime}$ is the image of K^p in $G'(\mathbb{A}_{\mathbb{Q},f}^{(p)})$ along the isomorphism $V \otimes_F \mathbb{A}_{\mathbb{Q},f} \to V' \otimes_F \mathbb{A}_{\mathbb{Q},f}$. To fulfill Definition 6.1(4), the away-from-p tame level structure on B is chosen and determined as the composite
$$\rho_B^p \colon \Lambda' \otimes_{\hat{\mathbb{Z}}} \hat{\mathbb{Z}}^{(p)} \xrightarrow{\sim} \Lambda \otimes_{\hat{\mathbb{Z}}} \hat{\mathbb{Z}}^{(p)} \xrightarrow{\rho_A^p} T_{\hat{\mathbb{Z}}^{(p)}}(A) \xrightarrow{T_{\hat{\mathbb{Z}}^{(p)}}(\phi)} T_{\hat{\mathbb{Z}}^{(p)}}(B).$$
Note that $T_{\hat{\mathbb{Z}}^{(p)}}(\phi)$ is an isomorphism as ϕ is a p-isogeny. So the condition in Definition 1.1(3a) is automatically satisfied. We next aim to check (3b). When $T = \emptyset$, which forces $\Delta(T) = \emptyset$ by definition, the induced morphism $\phi_p \colon A[p^\infty] \to B[p^\infty]$ is an isomorphism. In this case, we only need to take the level structure $\rho_{B,p} = \phi_p(\rho_{A,p})$. Assume else that $T \neq \emptyset$. Using the polarization λ_B, it boils down to show that there exists a unique subgroup scheme $H_\mathfrak{p} \subseteq B[\mathfrak{p}]$ satisfying condition (3b). We are able to define a 1-dimensional k-vector subspace $\mathcal{D}(H_\mathfrak{p}) \subseteq \mathcal{D}(B)/p\mathcal{D}(B)$ by
$$\mathcal{D}(H_\mathfrak{p}) = \bigoplus_{j=1}^{f} \bigoplus_{i=1}^{e} \mathcal{D}(H_\mathfrak{p})_j^{(i)},$$
where each direct summand is given as follows.

- If $\tau_j^{(i)} \in \Delta(T)$, then after the modification by signature change, we have $r_j^{(i)\prime} \in \{0,1\}$. Moreover, in this case, $[\varpi] \cdot \mathscr{H}_j^{B,(i)} \subseteq \mathcal{D}(B)$ is isomorphic to a submodule of $\mathscr{H}_j^{A,(i)} \subseteq \mathcal{D}(A)$ with quotient isomorphic to k. We put

$$\mathcal{D}(H_{\mathfrak{p}})_j^{(i)} := \frac{p\mathscr{H}_j^{A,(i)}}{p([\varpi] \cdot \mathscr{H}_j^{B,(i)})}, \quad \tau_j^{(i)} \in \Delta(T).$$

  It turns out to be a line bundle annihilated by p.
- If $\tau_j^{(i)} \notin \Delta(T)$, then let n be the least positive integer such that $\iota^n(\tau_j^{(i)}) = \tau_{j^-}^{(i^-)} \in \Delta(T)$. Such n exists because $\Delta(T) \neq \emptyset$ by our assumption that $T \neq \emptyset$. Put

$$\mathcal{D}(H_{\mathfrak{p}})_j^{(i)} := \mathrm{Hasse}_{\mathrm{es}}^{\uparrow,n}(\mathcal{D}(H_{\mathfrak{p}})_{j^-}^{(i^-)}), \quad \tau_j^{(i)} \notin \Delta(T).$$

Notice that in the former case, the construction essentially depends on the previous conclusion that $\omega_j^{B,(i)}$ satisfies the signature condition of B, i.e. it has $\mathcal{O}_S$-rank $r_j^{(i)\prime} \in \{0,1,2\}$ defined in Construction 5.5. Using the vanishing of the generalized partial Hasse invariants at places in T, one may check easily that $\mathcal{D}(H_{\mathfrak{p}}) \subseteq \mathcal{D}(B[\mathfrak{p}])$ is a Dieudonné module. We define $H_{\mathfrak{p}} \subseteq B[\mathfrak{p}]$ as the finite subgroup scheme corresponding to $\mathcal{D}(H_{\mathfrak{p}})$ by covariant Dieudonné theory. Then $\mathcal{D}(H_{\mathfrak{p}})$ is canonically identified with the kernel of the induced map

$$\phi_{\mathfrak{p},*} \colon \mathcal{D}(B[\mathfrak{p}]) \xrightarrow{\sim} H_1^{\mathrm{dR}}(B[\mathfrak{p}]/k) \longrightarrow H_1^{\mathrm{dR}}((B[\mathfrak{p}]/H_{\mathfrak{p}})/k) \xrightarrow{\sim} \mathcal{D}(B[\mathfrak{p}]/H_{\mathfrak{p}}).$$

Therefore, $H_{\mathfrak{p}}$ satisfies condition (3b) in Definition 1.1. This shows the existence of such $\rho_{B,p}$. Now the proof of Lemma 6.6 is almost complete except for Lemma 6.7. $\square$

Lemma 6.7. *Fix the subset $T \subseteq \Sigma_\infty$ and the perfect field k of characteristic $p > 0$ as above. Define the indicator $\delta\colon \Sigma_\infty \to \{0,1\}$ by setting $\delta_X(\tau) = 1$ if and only if τ lies in the set $X \subseteq \Sigma_\infty$. Assume for each $\tau = \tau_j^{(i)}$,*

$$\dim_k \mathrm{Coker}\, \phi_{*,\tau} = \dim_k \mathrm{Coker}\, \phi_{*,j}^{(i)} = \delta_{\Delta(T)}(\tau).$$

Then $\dim_k \omega_j^{B,(i)} = r_j^{(i)\prime}$ for all $\tau_j^{(i)} \in \Sigma_\infty$ if and only if $\dim_k \omega_j^{A,(i)} = r_j^{(i)}$ for all $\tau_j^{(i)} \in \Sigma_\infty$.

Proof. We prove the sufficiency, and the necessity follows by reversing the argument. Using the signature condition about A for the Shimura variety Z_T, we have

$$s_j^{(i)} = 2 - r_j^{(i)} = \dim_k(\mathscr{H}_j^{A,(i)}[\varpi]) = \dim_k(\mathscr{H}_j^{A,(i)}/\mathrm{Hasse}_{\mathrm{es}}^{\downarrow}(\iota^{-1}(\mathscr{H}_j^{A,(i)}))).$$

Comparing the signature condition of A with that of B, we get

$$\begin{aligned}\dim_k(\mathscr{H}_j^{B,(i)}/\mathrm{Hasse}_{\mathrm{es}}^{\downarrow}(\iota^{-1}(\mathscr{H}_j^{B,(i)}))) + \dim_k(\iota^{-1}(\mathscr{H}_j^{B,(i)})/\iota^{-1}(\mathscr{H}_j^{A,(i)}))\\ = \dim_k(\mathscr{H}_j^{B,(i)}/\mathscr{H}_j^{A,(i)}) + \dim_k(\mathscr{H}_j^{A,(i)}/\mathrm{Hasse}_{\mathrm{es}}^{\downarrow}(\mathscr{H}_j^{A,(i)}))\end{aligned}$$

that is equivalent to

$$\dim_k(\mathscr{H}_j^{B,(i)}/\mathrm{Hasse}_{\mathrm{es}}(\iota^{-1}(\mathscr{H}_j^{B,(i)}))) = s_j^{(i)} + \delta_{\Delta(T)}(\tau_j^{(i)}) - \delta_{\Delta(T)}(\iota^{-1}(\tau_j^{(i)})),$$

where $\iota^{-1}(\tau)$ is the last previous element of τ such that $\iota(\iota^{-1}(\tau)) = \tau$. We aim to show that both sides of the expression above are equal to $2 - r_j^{(i)\prime}$. Recall from Construction 5.5 that by fixing a labeling that begins with $\tau_f^{(e)}$, we can write $T' = \{\tau_{j_1}^{(i_1)}, \ldots, \tau_{j_k}^{(i_k)}\}$ for some integer $k = k(T)$. This leads to a natural partition of T':

$$T' = T'_{\mathrm{odd}} \sqcup T'_{\mathrm{even}} \coloneqq \{\tau_{j_{2\epsilon-1}}^{(i_{2\epsilon-1})} \colon 1 \leqslant \epsilon \leqslant 1 + \lfloor k/2 \rfloor\} \sqcup \{\tau_{j_{2\epsilon}}^{(i_{2\epsilon})} \colon 1 \leqslant \epsilon \leqslant 1 + \lfloor k/2 \rfloor\}.$$

By the definition of $r_j^{(i)\prime}$, it follows that

$$r_j^{(i)\prime} - r_j^{(i)} = \delta_{T'_{\mathrm{even}}}(\tau_j^{(i)}) - \delta_{T'_{\mathrm{odd}}}(\tau_j^{(i)}) = \begin{cases} -1, & \text{if } \tau_j^{(i)} \in T'_{\mathrm{odd}}, \\ 1, & \text{if } \tau_j^{(i)} \in T'_{\mathrm{even}}, \\ 0, & \text{otherwise.} \end{cases}$$

The following case-by-case argument is the key to the proof.

(a) If $\iota^{-1}(\tau_j^{(i)}), \tau_j^{(i)} \notin \Delta(T)$, then $\tau_j^{(i)} \notin T'$ by the construction. So

$$\delta_{\Delta(T)}(\iota^{-1}(\tau_j^{(i)})) - \delta_{\Delta(T)}(\tau_j^{(i)}) = 0 - 0,$$
$$\delta_{T'_{\mathrm{even}}}(\tau_j^{(i)}) - \delta_{T'_{\mathrm{odd}}}(\tau_j^{(i)}) = 0.$$

(b) If $\iota^{-1}(\tau_j^{(i)}) \in \Delta(T)$ but $\tau_j^{(i)} \notin \Delta(T)$, then $\tau_j^{(i)} \in T'_{\mathrm{even}}$. Thus,

$$\delta_{\Delta(T)}(\iota^{-1}(\tau_j^{(i)})) - \delta_{\Delta(T)}(\tau_j^{(i)}) = 1 - 0,$$
$$\delta_{T'_{\mathrm{even}}}(\tau_j^{(i)}) - \delta_{T'_{\mathrm{odd}}}(\tau_j^{(i)}) = 1.$$

(c) If $\iota^{-1}(\tau_j^{(i)}), \tau_j^{(i)} \in \Delta(T)$, then $\tau_j^{(i)} \notin T'$. And consequently,

$$\delta_{\Delta(T)}(\iota^{-1}(\tau_j^{(i)})) - \delta_{\Delta(T)}(\tau_j^{(i)}) = 1 - 1,$$
$$\delta_{T'_{\mathrm{even}}}(\tau_j^{(i)}) - \delta_{T'_{\mathrm{odd}}}(\tau_j^{(i)}) = 0.$$

(d) If $\iota^{-1}(\tau_j^{(i)}) \notin \Delta(T)$ but $\tau_j^{(i)} \in \Delta(T)$, then $\tau_j^{(i)} \in T'_{\mathrm{odd}}$. This leads to

$$\delta_{\Delta(T)}(\iota^{-1}(\tau_j^{(i)})) - \delta_{\Delta(T)}(\tau_j^{(i)}) = 0 - 1,$$
$$\delta_{T'_{\mathrm{even}}}(\tau_j^{(i)}) - \delta_{T'_{\mathrm{odd}}}(\tau_j^{(i)}) = -1.$$

To conclude, we have

$$\begin{aligned} s_j^{(i)} + \delta_{\Delta(T)}(\tau_j^{(i)}) - \delta_{\Delta(T)}(\iota^{-1}(\tau_j^{(i)})) &= (2 - r_j^{(i)}) - (\delta_{T'_{\mathrm{even}}}(\tau_j^{(i)}) - \delta_{T'_{\mathrm{odd}}}(\tau_j^{(i)})) \\ &= 2 - r_j^{(i)} - (r_j^{(i)\prime} - r_j^{(i)}) \\ &= 2 - r_j^{(i)\prime}. \end{aligned}$$

This proves Lemma 6.7. □

Lemma 6.8. *The map $\eta_1 \colon Y_T \to Z_T$ induces an isomorphism of tangent spaces at every closed point.*

Proof. This proof is adapted from the proof of [TX16, Lemma 5.20]. Fix a perfect field k of characteristic $p > 0$. Let $y = (A, \lambda_A, \rho_A, \underline{\mathscr{F}}^A; B, \lambda_B, \rho_B, \underline{\mathscr{F}}^B; \phi) \in Y_T(k)$ be a k-point and $z = (A, \lambda_A, \rho_A, \underline{\mathscr{F}}^A) \in Z_T(k)$ be its image under the map η_1. We have to show that $Y_T \to Z_T$ induces an isomorphism of k-vector spaces between tangent spaces, denoted by

$$\eta_{1,y}^* \colon T_{Y_T,y} \xrightarrow{\sim} T_{Z_T,z} = T_{Z_T,\eta_1(y)}.$$

For the first order deformation, we set $\mathbb{I} = \mathrm{Spec}(k[\varepsilon]/\varepsilon^2)$. Then $\mathbb{I}$ is a nilpotent thickening of $\mathrm{Spec}\, k$, equipped with the divided power structure. By deformation theory, the tangent space $T_{Z_T,z}$ is identified with the $\mathbb{I}$-valued points $z_{\mathbb{I}} = (A_{\mathbb{I}}, \lambda_{A,\mathbb{I}}, \rho_{A,\mathbb{I}}, \underline{\mathscr{F}}^A_{\mathbb{I}}) \in Z_T(\mathbb{I})$ with reduction $z \in Z_T(k)$ modulo ε. Also, the crystal nature of $H_1^{\mathrm{cris}}(A/k)$ implies that there is a canonical isomorphism

$$H_1^{\mathrm{cris}}(A/k)_{\mathbb{I}} \cong H_1^{\mathrm{dR}}(A/k) \otimes_k (k[\varepsilon]/\varepsilon^2).$$

See Subsection 2.4 and Theorem 2.14 for more details. As an immediate consequence, the canonical Hodge filtration for $A_{\mathbb{I}}/\mathbb{I}$ is written as

$$0 \longrightarrow \omega_{A_{\mathbb{I}}^\vee/\mathbb{I}} \longrightarrow H_1^{\mathrm{cris}}(A/k)_{\mathbb{I}} \longrightarrow \mathrm{Lie}(A_{\mathbb{I}}/\mathbb{I}) \longrightarrow 0.$$

So $\omega_{A_{\mathbb{I}}^\vee/\mathbb{I}}$ is a subbundle of $H_1^{\mathrm{cris}}(A/k)_{\mathbb{I}}$ that lifts $\omega_{A^\vee/k}$. Denote $H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i)}$ the image of $\omega_j^{A,(i)} \otimes_k (k[\varepsilon]/\varepsilon^2)$ via the canonical isomorphism above. For lifting the abelian scheme A together with the $\mathcal{O}_D$-action $\mathcal{O}_D \to \mathrm{End}_k(A)$, it is equivalent to giving, for each $j = 1, \ldots, f$ and $i = 1, \ldots, e$, a direct factor $\omega_{\mathbb{I},j}^{A,(i)} \subseteq H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i)}$ that lifts $\omega_j^{A,(i)} \subseteq H_1^{\mathrm{dR}}(A/k)_j^{(i)}$ and satisfies:

(i) For each $\tau = \tau_j^{(i)} \in \Sigma_\infty$, the subbundles $\omega_{\mathbb{I},j}^{A,(i)}$ and $\omega_{\mathbb{I},j}^{A,(i),\perp}$ are orthogonal complements of each other under the perfect pairing

$$H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i)} \times H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i),\perp} \longrightarrow k[\varepsilon]/\varepsilon^2$$

induced by the polarization λ_A.

(ii) For each $\tau_j^{(i)} \in S_\infty$ we have $\omega_{\mathbb{I},j}^{A,(i)} = 0$ (and hence $\omega_{\mathbb{I},j}^{A,(i),\perp} = H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i),\perp}$).

(iii) For each $\tau_j^{(i)} \in T$, the lift $\omega_{\mathbb{I},j}^{A,(i)}$ coincides with $\mathrm{Hasse}_{\mathbb{I},\mathrm{es}}^{\downarrow,n(i,j)}(H_1^{\mathrm{cris}}(A^{(p^{d(i,j)})}/k)_{\mathbb{I},j}^{(i)})$, where $\mathrm{Hasse}_{\mathbb{I},\mathrm{es}}^{\downarrow}$ on the direct summands of crystalline homology are defined in the same way as $\mathrm{Hasse}_{\mathrm{es}}^{\downarrow}$ on the de Rham homology. The p-twisted power of A can be computed by taking $\iota^{n(i,j)}(\tau_j^{(i)}) = \tau_{j^-}^{(i^-)}$ and letting $d(i,j) = (j' - j) \bmod f$. Since we are on a special fiber with characteristic p, we write

$$\mathrm{Hasse}_{\mathbb{I},\mathrm{es}}^{\downarrow,n(i,j)}(H_1^{\mathrm{cris}}(A^{(p^{d(i,j)})}/k)_{\mathbb{I},j}^{(i)}) = \omega_{\mathbb{I},j}^{A,(i)} \otimes_k (k[\varepsilon]/\varepsilon^2).$$

Hopefully, we can make up an isomorphism between $\omega_{A_{\mathbb{I}}^\vee/\mathbb{I}}$ and $\omega_{A^\vee,\mathbb{I}} \otimes_k (k[\varepsilon]/\varepsilon^2)$. We have to show that given such $z_{\mathbb{I}} \in Z_T(\mathbb{I})$, or equivalently, given the lifts $\omega_{\mathbb{I},j}^{A,(i)}$ as above, there exists a unique $(B_{\mathbb{I}}, \lambda_{B,\mathbb{I}}, \rho_{B,\mathbb{I}}, \underline{\mathscr{F}}^B_{\mathbb{I}}; \phi_{\mathbb{I}})$ deforming $(B, \lambda_B, \rho_B, \underline{\mathscr{F}}^B; \phi)$ such that $y_{\mathbb{I}} = (A_{\mathbb{I}}, \lambda_{A,\mathbb{I}}, \rho_{A,\mathbb{I}}, \underline{\mathscr{F}}^A_{\mathbb{I}}; B_{\mathbb{I}}, \lambda_{B,\mathbb{I}}, \rho_{B,\mathbb{I}}, \underline{\mathscr{F}}^B_{\mathbb{I}}; \phi_{\mathbb{I}}) \in Y_T(\mathbb{I})$ (c.f. proof of Theorem 4.1). To show the existence of $B_{\mathbb{I}}$, it suffices to construct, for each $\tau_j^{(i)} \in \Sigma_\infty$, a direct factor $\omega_{\mathbb{I},j}^{B,(i)} \subseteq H_1^{\mathrm{cris}}(B/k)_{\mathbb{I},j}^{(i)}$ that lifts $\omega_j^{B,(i)} \subseteq \mathscr{H}_j^{B,(i)} \cong H_1^{\mathrm{dR}}(B/k)_j^{(i)}$.

- When $\tau = \tau_j^{(i)}$ and $\iota(\tau) \notin \Delta(T)$, to fulfill the condition in Definition 6.1(3b), both $\phi_{*,\tau}$ and $\phi_{*,\iota(\tau)}$ are isomorphisms. We take $\omega_{\mathbb{I},j}^{B,(i)} \subseteq H_1^{\mathrm{cris}}(B/k)_{\mathbb{I},j}^{(i)}$ to be the image of $\omega_{\mathbb{I},j}^{A,(i)} \subseteq H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i)}$ under the induced morphism $\phi_{*,\tau}^{\mathrm{cris}} \colon H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i)} \to H_1^{\mathrm{cris}}(B/k)_{\mathbb{I},j}^{(i)}$ of the crystalline homology.
- Otherwise, when either one of τ and $\iota(\tau)$ belongs to $\Delta(T)$, we can translate the statement of Lemma 6.7 into crystalline settings and apply it with Lemma 5.11 to show that $\omega_j^{B,(i)}$ is either zero of locally free of rank 2 as a $(k[\varepsilon]/\varepsilon^2)$-module. So there is a unique obvious lift $\omega_{\mathbb{I},j}^{B,(i)}$ without any other choice, which can be zero or the full corresponding subquotient.

From these, we get the deformation $B_{\mathbb{I}}$ of B together with the inherited natural $\mathcal{O}_D$-action. It is clear that the induced crystalline isogeny preserves each subquotient, i.e.,

$$\begin{array}{ccc} \phi_{*,j}^{\mathrm{cris},(i)} \colon H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i)} & \longrightarrow & H_1^{\mathrm{cris}}(B/k)_{\mathbb{I},j}^{(i)} \\ \cup\!\mid & & \cup\!\mid \\ \omega_{\mathbb{I},j}^{A,(i)} & \longrightarrow & \omega_{\mathbb{I},j}^{B,(i)}. \end{array}$$

Hence, ϕ deforms to an $\mathcal{O}_D$-equivariant isogeny of abelian schemes $\phi_{\mathbb{I}} \colon A_{\mathbb{I}} \to B_{\mathbb{I}}$ by [Lan13, 2.1.6.9].

We check now that $\phi_{\mathbb{I}}$ satisfies Definition 6.1(3). Note that the map $\phi_{\mathbb{I},*} \colon H_1^{\mathrm{dR}}(A_{\mathbb{I}}/\mathbb{I}) \to H_1^{\mathrm{dR}}(B_{\mathbb{I}}/\mathbb{I})$ is canonically identified with $\phi_*^{\mathrm{cris}} \colon H_1^{\mathrm{cris}}(A/k)_{\mathbb{I}} \to H_1^{\mathrm{cris}}(B/k)_{\mathbb{I}}$ by crystalline theory, which is, in turn, isomorphic to the base change of $\phi_* \colon H_1^{\mathrm{dR}}(A/k) \to H_1^{\mathrm{dR}}(B/k)$ via $k \hookrightarrow k[\varepsilon]/\varepsilon^2$. Let $\tau \in \Delta(T)$. Since the arithmetic Frobenius on $k[\varepsilon]/\varepsilon^2$ factors as

$$\begin{array}{ccccccc} k[\varepsilon]/\varepsilon^2 & \twoheadrightarrow & k & \longrightarrow & k & \hookrightarrow & k[\varepsilon]/\varepsilon^2, \\ x + y\varepsilon & \longmapsto & x & \longmapsto & x^p & \longmapsto & x^p = (x+y\varepsilon)^p \end{array}$$

we see that

$$\mathrm{Hasse}_{\mathbb{I},\mathrm{es}}^{\downarrow,n(i,j)}(\omega_{\mathbb{I},j}^{A,(i)}) = \mathrm{Hasse}_{\mathrm{es}}^{\downarrow,n(i,j)}(\omega_j^{A_{\mathbb{I}},(i)}) = \mathrm{Hasse}_{\mathrm{es}}^{\downarrow,n(i,j)}(\omega_j^{A,(i)}) \otimes_k (k[\varepsilon]/\varepsilon^2).$$

As it is the case after reduction modulo ε, for each $\tau_j^{(i)} \in \Delta(T)$ with $\iota^{n(i,j)}(\tau_j^{(i)}) = \tau_{j^-}^{(i^-)}$, we get

$$\mathrm{Ker}(\phi_{*,j^-}^{(i^-)} \colon \omega_{j^-}^{A_{\mathbb{I}},(i^-)} \to \omega_{j^-}^{B_{\mathbb{I}},(i^-)}) = \mathrm{Hasse}_{\mathrm{es}}^{\downarrow,n(i,j)}(\omega_j^{A_{\mathbb{I}},(i)}).$$

This shows that $\phi_{\mathbb{I}}$ satisfies the condition (3c). Conversely, it is clear that, if $B_{\mathbb{I}}$ and $\phi_{\mathbb{I}}$ satisfy Definition 6.1(3), then they have to be the form as above.

We now consider the polarization deformation. Let

$$\langle \cdot, \cdot \rangle_{\lambda_B}^{\mathrm{cris}} \colon H_1^{\mathrm{cris}}(B/k)_{\mathbb{I}} \times H_1^{\mathrm{cris}}(B/k)_{\mathbb{I}} \longrightarrow k[\varepsilon]/\varepsilon^2$$

be the crystalline pairing induced by the polarization λ_B. To prove that λ_B deforms (necessarily uniquely) to a polarization $\lambda_{B,\mathbb{I}} \coloneqq \lambda_{B_{\mathbb{I}}} \colon B_{\mathbb{I}} \to B_{\mathbb{I}}^\vee$ on $B_{\mathbb{I}}$, it suffices to check that $\langle \cdot, \cdot \rangle_{\lambda_B}^{\mathrm{cris}}$ vanishes on $\omega_{\mathbb{I},j}^{B,(i)} \times \omega_{\mathbb{I},j}^{B,(i),\perp}$ for all $\tau_j^{(i)} \in \Sigma_\infty$ (c.f. [Lan13, 2.1.6.9, 2.2.2.2, 2.2.2.6]).

- If $\tau_j^{(i)} \in S_\infty$, then $\tau_j^{(i)}$ has signature $r_j^{(i)} \in \{0, 2\}$. In such case the requirement is vacuous because one of $\omega_{\mathbb{I},j}^{B,(i)}$ and $\omega_{\mathbb{I},j}^{B,(i),\perp}$ is equal to 0 and the other one is equal to $H_1^{\mathrm{cris}}(B/k)_{\mathbb{I},j}^{(i)}$ by construction.
- If $\tau_j^{(i)} \notin S_\infty$, then $r_j^{(i)} = 1$ and the natural isomorphism

$$\begin{aligned}&H_1^{\mathrm{cris}}(B/k)_{\mathbb{I},j}^{(i)} \xrightarrow{\sim} H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i)}\\&(\text{resp. } H_1^{\mathrm{cris}}(B/k)_{\mathbb{I},j}^{(i),\perp} \xrightarrow{\sim} H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i),\perp})\end{aligned}$$

  sends $\omega_{\mathbb{I},j}^{B,(i)}$ to $\omega_{\mathbb{I},j}^{A,(i)}$ (resp. sends $\omega_{\mathbb{I},j}^{B,(i),\perp}$ to $\omega_{\mathbb{I},j}^{A,(i),\perp}$). The vanishing of $\langle\cdot,\cdot\rangle_{\lambda_B}^{\mathrm{cris}}$ on $\omega_{\mathbb{I},j}^{B,(i)} \times \omega_{\mathbb{I},j}^{B,(i),\perp}$ follows from the known statement with B replaced by A.

Therefore, we see that λ_B deforms to a polarization $\lambda_{B,\mathbb{I}}$ on $B_{\mathbb{I}}$. Also, the induced maps on the homology of both de Rham and crystalline versions are canonically identified, i.e., there is a commutative diagram

$$\begin{array}{ccc}H_1^{\mathrm{dR}}(B/\mathbb{I}) & \xrightarrow{\lambda_{B,\mathbb{I},*}^{\mathrm{dR}}} & H_1^{\mathrm{dR}}(B^\vee/\mathbb{I})\\ \cong\downarrow & & \downarrow\cong\\ H_1^{\mathrm{cris}}(B/k)_{\mathbb{I}} & \xrightarrow{\lambda_{B,\mathbb{I},*}^{\mathrm{cris}}} & H_1^{\mathrm{cris}}(B^\vee/k)_{\mathbb{I}}.\end{array}$$

Furthermore, the lower vertical map is in turn identified with the base change of $\lambda_{B,\mathbb{I},*}^{\mathrm{dR}}$ via $k \hookrightarrow k[\varepsilon]/\varepsilon^2$. So it is clear that the conditions regarding the polarization are preserved by the deformation $\lambda_{B,\mathbb{I}}$.

We next prove that the level structure ρ_B deforms to $\rho_{B,\mathbb{I}} = \rho_{B_{\mathbb{I}}}$. The deformation of the tame level structure is automatic because the prime-to-p isogeny $\phi_{\mathbb{I}}$ induces an isomorphism $T_{\hat{\mathbb{Z}}^{(p)}}(A_{\mathbb{I}}) \xrightarrow{\sim} T_{\hat{\mathbb{Z}}^{(p)}}(B_{\mathbb{I}})$. As for the deformation of the subgroup at p-adic places, let $H_{\mathfrak{p}}$ be the given $\mathcal{O}_D$-stable closed finite flat subgroup scheme of $A[\mathfrak{p}]$. Then we have a canonical quotient isogeny $f_{\mathfrak{p}}\colon A[\mathfrak{p}] \to A[\mathfrak{p}]/H_{\mathfrak{p}}$. To check Definition 1.1(3b), it suffices to show that $A[\mathfrak{p}]/H_{\mathfrak{p}}$ and $f_{\mathfrak{p}}$ deforms to be over $\mathbb{I}$. The abelian scheme $A[\mathfrak{p}]/H_{\mathfrak{p}}$ is equipped with an induced action of $\mathcal{O}_D$, a polarization $\lambda_{A[\mathfrak{p}]/H_{\mathfrak{p}}}$ satisfying conditions (i)(ii) on page 57. Using the same argument as above, the isogeny $f_{\mathfrak{p}}$ turns out to induce canonical isomorphisms

$$f_{\mathfrak{p},\mathbb{I},*,j}^{\mathrm{cris},(i)}\colon H_1^{\mathrm{cris}}(A[\mathfrak{p}]/k)_{\mathbb{I},j}^{(i)} \xrightarrow{\sim} H_1^{\mathrm{cris}}((A[\mathfrak{p}]/H_{\mathfrak{p}})/k)_{\mathbb{I},j}^{(i)}.$$

So each $\omega_j^{A[\mathfrak{p}]/H_{\mathfrak{p}},(i)}$ admits a unique lift to a direct summand of $H_1^{\mathrm{cris}}((A[\mathfrak{p}]/H_{\mathfrak{p}})/k)_{\mathbb{I},j}^{(i)}$. Such choices of liftings give rise to deformations $(A[\mathfrak{p}]/H_{\mathfrak{p}})_{\mathbb{I}}$ over $\mathbb{I}$ of $(A[\mathfrak{p}]/H_{\mathfrak{p}})$ over k. It is clear that $f_{\mathfrak{p}}$ also lifts to an isogeny $f_{\mathfrak{p},\mathbb{I}}\colon A[\mathfrak{p}]_{\mathbb{I}} \to (A[\mathfrak{p}]/H_{\mathfrak{p}})_{\mathbb{I}}$. Therefore, $\operatorname{Ker} f_{\mathfrak{p},\mathbb{I}}$ gives the required lift of $H_{\mathfrak{p}}$.

Finally, it remains to check that $\underline{\mathscr{F}}^B$ deforms to $\underline{\mathscr{F}}_{\mathbb{I}}^B$. From the same argument as in the proof of Theorem 4.1, since $\omega_{\mathbb{I},j}^{B,(i)}$ is given, one can naturally put

$$\mathscr{F}_{\mathbb{I},j}^{B,(i)} = \bigoplus_{k=1}^{i} \omega_{\mathbb{I},j}^{B,(k)}.$$

Note that each subquotient $\omega_{\mathbb{I},j}^{A,(i)}$ deforming from $\omega_j^{A,(i)}$ has the correct rank $r_j^{(i)}$, and $\mathscr{F}_{\mathbb{I},j}^{B,(i)}/\mathscr{F}_{\mathbb{I},j}^{B,(i-1)} = \omega_{\mathbb{I},j}^{B,(i)}$ is determined by $\omega_{\mathbb{I},j}^{A,(i)}$. Therefore, each $\omega_{\mathbb{I},j}^{B,(i)}$ is as well a locally free sheaf of rank $r_j^{(i)\prime} \in \{0,1,2\}$ over $\mathcal{O}_S$ annihilated by $[\varpi]$ in the sense of modulo p, and the $\mathcal{O}_D$-action on it factors through $\tau_j^{(i)}\colon \mathcal{O}_D \to W(k)$. This concludes the proof of Lemma 6.8. $\square$

6.5. **The second isomorphism $Y_T \to Z_T'$.** Let S be a locally noetherian k_0-scheme. For the S-valued point $y = (A, \lambda_A, \rho_A, \underline{\mathscr{F}}^A; B, \lambda_B, \rho_B, \underline{\mathscr{F}}^B; \phi)$ of Y_T, we define the morphism η_2 that sends y to $z' = (B, \lambda_B, \rho_B, \underline{\mathscr{F}}^B; \mathbb{J}) \in Z_T'(S)$, where $\mathbb{J}$ collects line bundles

$$\mathcal{J}_j^{(i)} = \phi_{*,j}^{(i)}(\omega_j^{A,(i)})$$

for the induced map $\phi_{*,j}^{(i)}\colon \omega_j^{A,(i)} \to \omega_j^{B,(i)}$ when $\tau_j^{(i)}$ runs through all elements in $I(T)$ defined in Construction 5.5. Also note that $I(T) \subseteq T'_{\mathrm{even}} \subseteq \Sigma_\infty - \Delta(T)$. So any $\tau_j^{(i)} \in I(T)$ does not lie in $\Delta(T)$, which implies that $\phi_{*,j}^{(i)}$ is an isomorphism by Definition 6.1(3b).

Proposition 6.9. *The morphism $\eta_2\colon Y_T \to Z_T'$ of schemes is an isomorphism.*

Proof. Fix a perfect field $k = \overline{\mathbb{F}}_p$ without loss of generality. We use the same strategy as in the proof of Lemma Proposition 6.5, by proving that η_2 induces a bijection on the closed points and this bijection induces isomorphisms on tangent spaces.

Step I. We show first that η_2 induces a bijection on closed points. Consider the closed k-point $z' = (B, \lambda_B, \rho_B, \underline{\mathscr{F}}^B; \mathbb{J}) \in Z_T'(k)$. We have to show that there exists a unique point $y = (A, \lambda_A, \rho_A, \underline{\mathscr{F}}^A; B, \lambda_B, \rho_B, \underline{\mathscr{F}}^B; \phi) \in Y_T(k)$ with $\eta_2(y) = z'$. To prove this, we basically reverse the construction in the proof of Lemma 6.6.

We start by reconstructing A from B and $\mathbb{J}$. By covariant Dieudonné theory, our goal is to construct a Dieudonné module $M \subseteq \mathcal{D}(B)[\varpi^{-1}]$ with $\mathcal{D}(B) \subseteq M \subseteq p^{-1}\mathcal{D}(B)$ as follows. Before separating the cases, we write $T \cup S_\infty$ as a disjoint union of proper chains C_ks. Beware of that in Construction 5.5 we rather decompose $T - S_\infty$ into proper chains. In the following, for each choice of C_k, we omit the index k and write $C_k = \{\tau_{j_1}^{(i_1)}, \dots, \tau_{j_m}^{(i_m)}\}$ for some n. Note that $C_k \cap T$ is a proper chain in T. If $\#(C_k \cap T)$ is odd, then $C_k' - C_k$ has exactly a single element, denoted by $\tau_{j_{m+1}}^{(i_{m+1})}$. Whenever m is odd (resp. even), let $n_1, \dots, n_m$ (resp. $n_1, \dots, n_{m-1}$) be the integers between 1 and $ef - 1$ such that $\iota^{n_\nu}(\tau_{j_\nu}^{(i_\nu)}) = \tau_{j_{\nu+1}}^{(i_{\nu+1})}$ for all $1 \leqslant \nu \leqslant m$ (resp. $1 \leqslant \nu \leqslant m-1$).

- Assume $T \subsetneq \Sigma_\infty$. If $\tau_j^{(i)} \notin \Delta(T)$, then we take
$$M_j^{(i)} := \mathscr{H}_j^{B,(i)}.$$
Suppose else in the following $\tau_j^{(i)} \in \Delta(T)$ and it lies between two elements of some extended proper chain $C_k' = \{\tau_{j_1}^{(i_1)}, \dots, \tau_{j_{2m}}^{(i_{2m})}\}$ of $C_k \subseteq T \cup S_\infty$, i.e., there exists integers $1 \leqslant l \leqslant 2m-1$ and $0 \leqslant u(i,j) \leqslant n_l - 1$ so that $\tau_j^{(i)} = \iota^{u(i,j)}(\tau_{j_l}^{(i_l)})$.
  - When $\#(C_k \cap T)$ is odd, we have $\tau_{j_{2m}}^{(i_{2m})} \in C_k' - C_k \subseteq I(T)$. Accordingly, the line bundle $\mathcal{J}_{j_{2m}}^{(i_{2m})}$ can be defined. Let $m(i,j)$ be the integer between 0 and

$ef-1$ such that $\iota^{m(i,j)}(\tau_j^{(i)}) = \tau_{j_{2m}}^{(i_{2m})}$. Put

$$M_j^{(i)} := [\varpi^{-1}] \cdot \mathrm{Hasse}_{\mathrm{es}}^{\uparrow, m(i,j)}(\mathcal{J}_{j_{2m}}^{(i_{2m})}).$$

- ○ When $\#(C_k \cap T)$ is even, the issue is that there is no $\mathcal{J}$ involved in this construction because $I(T) \supseteq C'_k - C_k = \emptyset$. We have $n(i,j) = m(i,j) + n_{2m}$, where n_{2m} is the minimal integer such that $\tau_{j_{2m+1}}^{(i_{2m+1})} := \iota^{n_{2m}}(\tau_{j_{2m}}^{(i_{2m})}) \notin S_\infty$,[5] which is of signature 1. It follows that $\iota^{n(i,j)}(\tau_j^{(i)}) = \tau_{j_{2m+1}}^{(i_{2m+1})}$. Now we set

$$M_j^{(i)} := [\varpi^{-1}] \cdot \mathrm{Hasse}_{\mathrm{es}}^{\uparrow, n(i,j)}(\mathscr{H}_{j_{2m+1}}^{B,(i_{2m+1})}).$$

- Assume $T = \Sigma_\infty$. Let $H_\mathfrak{p} \subseteq B[\mathfrak{p}]$ be the finite flat closed subgroup scheme given by $\rho_{B,p}$. Then we obtain a Dieudonné submodule $\mathcal{D}(H_\mathfrak{p}) \subseteq \mathcal{D}(B[\mathfrak{p}])$, which by definition necessarily has 1-dimensional direct factors $\mathscr{H}_j^{H_\mathfrak{p},(i)}$ for all $\tau_j^{(i)} \in \Sigma_\infty$ as k-vector spaces. If $\tau_j^{(i)} \notin \Delta(T)$, we take

$$M_j^{(i)} := \mathscr{H}_j^{B,(i)}.$$

Otherwise, for $\tau_j^{(i)} \in \Delta(T)$, define

$$M_j^{(i)} := [\varpi^{-1}] \cdot \mathscr{H}_j^{H_\mathfrak{p},(i)},$$

where $\mathscr{H}_j^{H_\mathfrak{p},(i)}$ denotes the obvious subspace of $\mathscr{H}_j^{B,(i)}$ defined earlier.

Now we check that each $M_j^{(i)}$ is stable under the essential Hasse maps. So we are to check, for any $\tau_j^{(i)} \in \Sigma_\infty$, that

$$(\star) \qquad \begin{cases} m_{\varpi,j}^{(i)} M_j^{(i)} \subseteq M_j^{(i-1)}, & \text{for } 1 < i \leqslant e, \\ \mathrm{Hasse}_{\varpi,j}^{(i)}(M_j^{(1)}) \subseteq (M_{j-1}^{(e)})^{(p)}, & \text{for } i = 1, \end{cases}$$

$$(\star\star) \qquad \begin{cases} d_{\varpi,j}^{(i)} M_j^{(i)} \subseteq M_j^{(i+1)}, & \text{for } 1 \leqslant i < e, \\ \mathrm{Hasse}_{\varpi,j}^{(e)}(M_j^{(e)}) \subseteq (M_{j+1}^{(1)})^{(p)}, & \text{for } i = e. \end{cases}$$

Similar to the proof of Lemma 6.6, we do check ($\star$) only and separate the verification into cases. After that, a similar argument can show ($\star\star$) directly. Moreover, from the proof of Lemma 6.6 we see the essence idea keeps invariant while dealing with $1 < i \leqslant e$ and $i = 1$. So it boils down to assuming $1 < i \leqslant e$ without loss of generality.

(a) Let $\tau_j^{(i)}, \tau_j^{(i-1)} \notin \Delta(T)$. Then whatever T is, we have

$$M_j^{(i)} = \mathscr{H}_j^{B,(i)}, \quad M_j^{(i-1)} = \mathscr{H}_j^{B,(i-1)}.$$

So ($\star$) follows immediately.

[5] Using the notation before, we see $n_{2m} = n(i_{2m}, j_{2m})$ and $\tau_{j_{2m+1}}^{(i_{2m+1})} = \tau_{j_{2m}^+}^{(i_{2m}^+)}$.

(b) Let $\tau_j^{(i)} \in \Delta(T)$ with $\tau_j^{(i-1)} \notin \Delta(T)$. We always obtain $M_j^{(i-1)} = \mathscr{H}_j^{B,(i-1)}$. If $T \neq \Sigma_\infty$, then $\mathrm{Hasse}_{\mathrm{es}}^{\uparrow,m(i,j)}(\mathcal{J}_{j_{2m}}^{(i_{2m})})$ is the backward image of a line bundle and $\mathrm{Hasse}_{\mathrm{es}}^{\uparrow,n(i,j)}(\mathscr{H}_{j_{2m+1}}^{(i_{2m+1})})$ equals a 1-dimensional k-vector subspace. Thus, whether or not $\#(C_k \cap T)$ is odd, we always have

$$\dim_k m_{\varpi,j}^{(i)}([\varpi] \cdot M_j^{(i)}) \leqslant 1 < 2 = \dim_k \mathscr{H}_j^{B,(i-1)}.$$

This dimension estimate immediately indicates that

$$m_{\varpi,j}^{(i)}(M_j^{(i)}) = [\varpi^{-1}] \cdot m_{\varpi,j}^{(i)}([\varpi] \cdot M_j^{(i)}) \subseteq \mathscr{H}_j^{B,(i-1)} = M_j^{(i-1)}.$$

If $T = \Sigma_\infty$, recall that $\mathscr{H}_j^{H_{\mathfrak{p}},(i)}$ is already defined to be 1-dimensional. So the inclusion holds for the same reason.

(c) Let $\tau_j^{(i)}, \tau_j^{(i-1)} \in \Delta(T)$. We first assume $T \neq \Sigma_\infty$. The construction definitely makes two places in the same proper chain C_k. Also, observe that $m(i,j) = m(i-1,j)+1$ with $\#(C_k \cap T)$ being odd, or $n(i,j) = n(i-1,j)+1$ with $\#(C_k \cap T)$ being even. In a sequel,

$$\begin{aligned} \text{either} \qquad & M_j^{(i)} = [\varpi^{-1}] \cdot \mathrm{Hasse}_{\mathrm{es}}^{\uparrow,m(i,j)}(\mathcal{J}_{j_{2m}}^{(i_{2m})}), \\ & M_j^{(i-1)} = [\varpi^{-1}] \cdot \mathrm{Hasse}_{\mathrm{es}}^{\uparrow,m(i-1,j)}(\mathcal{J}_{j_{2m}}^{(i_{2m})}), \\ \text{or} \qquad & M_j^{(i)} = [\varpi^{-1}] \cdot \mathrm{Hasse}_{\mathrm{es}}^{\uparrow,n(i,j)}(\mathscr{H}_{j_{2m+1}}^{B,(i_{2m+1})}), \\ & M_j^{(i-1)} = [\varpi^{-1}] \cdot \mathrm{Hasse}_{\mathrm{es}}^{\uparrow,n(i-1,j)}(\mathscr{H}_{j_{2m+1}}^{B,(i_{2m+1})}). \end{aligned}$$

Similar to the argument of (b), in both cases,

$$\dim_k m_{\varpi,j}^{(i)}([\varpi] \cdot M_j^{(i)}) \leqslant 1 \leqslant \dim_k([\varpi] \cdot M_j^{(i-1)}).$$

On one hand, it remains to consider the situation where both sides are of dimension 1. On the other hand, note that the statement implies $m(i-1,j) \geqslant 1$ or $n(i-1,j) \geqslant 1$, and the following two are isomorphisms:

$$\mathrm{Hasse}_{\mathrm{es}}^{\uparrow} \colon \mathrm{Hasse}_{\mathrm{es}}^{\uparrow,m(i-1,j)}(\mathcal{J}_{j_{2m}}^{(i_{2m})}) \xrightarrow{\sim} \mathrm{Hasse}_{\mathrm{es}}^{\uparrow,m(i,j)}(\mathcal{J}_{j_{2m}}^{(i_{2m})}),$$
$$\mathrm{Hasse}_{\mathrm{es}}^{\uparrow} \colon \iota^{-n(i-1,j)}(\mathscr{H}_{j_{2m+1}}^{B,(i_{2m+1})}) \xrightarrow{\sim} \iota^{-n(i,j)}(\mathscr{H}_{j_{2m+1}}^{B,(i_{2m+1})}).$$

Note that the second map above is nothing but $\mathscr{H}_j^{B,(i-1)} \xrightarrow{\sim} \mathscr{H}_j^{B,(i)}$. Therefore, in this case, $m_{\varpi,j}^{(i)}([\varpi] \cdot M_j^{(i)})$ and $[\varpi] \cdot M_j^{(i-1)}$ must be identified. As for the case where $T = \Sigma_\infty$, the dimension estimate goes as

$$\begin{aligned} \dim_k m_{\varpi,j}^{(i)}([\varpi] \cdot M_j^{(i)}) &= \dim_k m_{\varpi,j}^{(i)}(\mathscr{H}_j^{H_{\mathfrak{p}},(i)}) \leqslant 1, \\ \dim_k([\varpi] \cdot M_j^{(i-1)}) &= \dim_k \mathscr{H}_j^{H_{\mathfrak{p}},(i-1)} \geqslant 1. \end{aligned}$$

Since $\mathscr{H}_j^{H_{\mathfrak{p}},(i-1)}$ arose from subquotients of B in a unique way, thanks to the isomorphism reason of $\mathrm{Hasse}_{\mathrm{es}}^{\uparrow}$ on B above, one may check easily that if both sides are simultaneously of dimension 1, then they must equal. So we have finished.

(d) Let $\tau_j^{(i)} \notin \Delta(T)$ with $\tau_j^{(i-1)} \in \Delta(T)$. Then $M_j^{(i)} = \mathscr{H}_j^{B,(i)}$ is always valid. Note by the proof of Lemma 6.7 that $\tau_j^{(i-1)} \in T'_{\mathrm{odd}}$, and hence

$$\dim_k m_{\varpi,j}^{(i)}(\mathscr{H}_j^{B,(i)}) = \dim_k \omega_j^{B,(i-1)} = r_j^{(i)\prime} = 0.$$

So the inclusion relation is trivial. The approach for (d) here differs from that for (1d) and (2d) to the proof of Lemma 6.6, because before the signature change, we only have $r_j^{(i-1)} = 1$ on A.

To finish the definition of $M \subseteq p^{-1}\mathcal{D}(B)$, we put

$$\tilde{M} = \bigoplus_{j=1}^{f} \bigoplus_{i=1}^{e} \tilde{M}_j^{(i)}.$$

Here $\tilde{M}_j^{(i)}$ is the preimage of $M_j^{(i)}$ under the reduction map $\tilde{\mathcal{D}}(B)_j[\varpi^{-1}] \to \mathcal{D}(B)_j[\varpi^{-1}]$ (resp. $\tilde{\mathcal{D}}(H_{\mathfrak{p}})_j \to \mathcal{D}(H_{\mathfrak{p}})_j$) when $T \subsetneq \Sigma_\infty$ (resp. $T = \Sigma_\infty$).

Step II. Now we are to construct the data $y = (A, \lambda_A, \rho_A, \underline{\mathscr{F}}^A; B, \lambda_B, \rho_B, \underline{\mathscr{F}}^B; \phi) \in Y_T(k)$. Consider the quotient Dieudonné module

$$\tilde{M}/\tilde{\mathcal{D}}(B) \subseteq p^{-1}\tilde{\mathcal{D}}(B)/\tilde{\mathcal{D}}(B) = \mathcal{D}(B).$$

Then by covariant Dieudonné theory, $\tilde{M}/\tilde{\mathcal{D}}(B)$ corresponds to a unique closed finite group scheme $G \subseteq B[p]$ that is $\mathcal{O}_D$-stable. We put $A = B/G$, which is automatically equipped with an $\mathcal{O}_D$-action inherited from that of B. Also, there is a natural projection $\psi\colon B \to A$. Take the $\mathcal{O}_D$-isogeny

$$\phi\colon A = B/G \longrightarrow B$$

such that $\phi \circ \psi = [p]_B$, the multiplication-by-p on B. We then define the prime-to-p polarization of A as the quasi-isogeny

$$\lambda_A\colon A \xrightarrow{\phi} B \xrightarrow{\lambda_B} B^\vee \xrightarrow{\phi^\vee} A^\vee.$$

Moreover, take the splitting filtration $\underline{\mathscr{F}}^A$ as

$$\omega_j^{A,(i)} = \mathscr{F}_j^{A,(i)}/\mathscr{F}_j^{A,(i-1)} := \mathrm{Hasse}^{\downarrow}(\iota^{-1}(M_j^{(i)})) \subseteq \mathscr{H}_j^{A,(i)} := M_j^{(i)}.$$

We aim to verify that λ_A is a genuine isogeny satisfying condition of Definition 1.1(2). If so, since λ_B is a prime-to-p polarization, so also is λ_A. Note that $\operatorname{Ker}\phi = G$ is a maximal isotropic subgroup of $A[p]$, so $p\lambda_A = \phi^\vee \circ \lambda_B \circ \phi$, where $p \nmid \deg \lambda_A$. This deduces (2a), and (2b) follows for the same reason. It suffices to show that we have a natural inclusion

$$\mathscr{H}_j^{A,(i)} \subseteq (\mathscr{H}_j^{A,(i),\perp})^\vee = \{v \in \mathscr{H}_j^{B,(i)}[\varpi^{-1}]\colon \langle v, w\rangle_{\lambda_B} \in W(k) \text{ for all } w \in \mathscr{H}_j^{A,(i),\perp}\}$$

which is indeed an isomorphism. However, by the construction of A, this is clear for all $\tau_j^{(i)} \in \Delta(T)$ (and hence for all their complex conjugates as the duality is reciprocal). Whenever $\tau_j^{(i)} \notin \Delta(T)$, it follows from the same statement for B.

We now define the level structure ρ_A on A. For the prime-to-p level structure ρ_A^p, we define it to be the K'-orbit of the isomorphism class

$$\rho_A^p\colon \Lambda'\otimes_{\hat{\mathbb{Z}}}\hat{\mathbb{Z}}^{(p)} \xrightarrow[\sim]{\rho_B^p} T_{\hat{\mathbb{Z}}^{(p)}}(B) \xleftarrow[\sim]{\phi_*} T_{\hat{\mathbb{Z}}^{(p)}}(A),$$

where K' and Λ' are defined in Construction 5.5. Via the obvious isomorphism $B[\mathfrak{p}^\infty]\cong A[\mathfrak{p}^\infty]$ between p-divisible groups, the closed subgroup schemes correspond to each other. So we naturally get $\rho_{A,p}$ from $\rho_{B,p}$. This finishes the construction of all the data $y=(A,\lambda_A,\rho_A,\underline{\mathscr{F}}^A;B,\lambda_B,\rho_B,\underline{\mathscr{F}}^B;\phi)$.

To see that y is indeed a k-point of Y_T, we have to check that y satisfies all the conditions of Definition 6.1, among which (2) and (4) are obvious from the construction. Moreover, the Kottwitz signature condition on $\underline{\mathscr{F}}^A$ follows from Lemma 6.7 immediately. So for 6.1(1), it remains to show that the generalized partial Hasse invariants at τ for A vanishes if and only if $\tau\in T$; but this is implied by the known fact that $M_j^{(i)}$s are stable under the actions of $\mathrm{Hasse}_{\mathrm{es}}^{\uparrow}$ and $\mathrm{Hasse}_{\mathrm{es}}^{\downarrow}$. As for 6.1(3c), whenever $\tau_j^{(i)}\in\Delta(T)$,

$$[\varpi]\cdot\mathscr{H}_j^{A,(i)}=\begin{cases}\mathrm{Hasse}_{\mathrm{es}}^{\uparrow,m(i,j)}(\mathcal{J}_{j_{2m}}^{(i_{2m})}), & \text{if } T\neq\Sigma_\infty \text{ and } 2\nmid\#(C_k\cap T),\\ \mathrm{Hasse}_{\mathrm{es}}^{\uparrow,n(i,j)}(\mathscr{H}_{j_{2m+1}}^{B,(i_{2m+1})}), & \text{if } T\neq\Sigma_\infty \text{ and } 2\mid\#(C_k\cap T),\\ \mathscr{H}_j^{H_{\mathfrak{p}},(i)}, & \text{otherwise.}\end{cases}$$

Via the definition of ϕ_*, with the fact that $n(i,j)=m(i,j)+n(i_{2m},j_{2m})>m(i,j)$, the relations above conclude the proof because $\phi_{*,j}^{(i)}([\varpi]\cdot\mathscr{H}_j^{A,(i)})=0$.

Step III. Let $y\in Y_T(k)$ and $z'=\eta_2(y)\in Z_T'(k)$ as before. We have shown that they are coupled by the bijection. Now we prove, using the same method à la Grothendieck–Messing of Lemma 6.8, that $\eta_2\colon Y_T\to Z_T'$ induces an isomorphism of tangent spaces

$$\eta_{2,y}^*\colon T_{Y_T,y}\xrightarrow{\sim}T_{Z_T',z}=T_{Z_T',\eta_2(y)}.$$

Set $\mathbb{I}=\mathrm{Spec}(k[\varepsilon]/\varepsilon^2)$. The tangent space $T_{Z_T',z}$ is identified with the set of first order deformations $z_{\mathbb{I}}=(B_{\mathbb{I}},\lambda_{B,\mathbb{I}},\rho_{B,\mathbb{I}},\underline{\mathscr{F}}_{\mathbb{I}}^B,\mathbb{J}_{\mathbb{I}})\in Z_T'(\mathbb{I})$ of $z\in Z_T'(k)$, where $\mathbb{J}_{\mathbb{I}}$ is the collection of subbundles $\mathcal{J}_{\mathbb{I},j}^{(i)}$ of $\omega_j^{B_{\mathbb{I}},(i)}=H_1^{\mathrm{cris}}(B/k)_{\mathbb{I},j}^{(i)}$. We have to show that every point $z_{\mathbb{I}}$ lifts uniquely to a deformation $y_{\mathbb{I}}=(A_{\mathbb{I}},\lambda_{A,\mathbb{I}},\rho_{A,\mathbb{I}},\underline{\mathscr{F}}_{\mathbb{I}}^A;B_{\mathbb{I}},\lambda_{B,\mathbb{I}},\rho_{B,\mathbb{I}},\underline{\mathscr{F}}_{\mathbb{I}}^B;\phi_{\mathbb{I}})\in Y_T(\mathbb{I})$ with $\eta_2(y_{\mathbb{I}})=z_{\mathbb{I}}$. Recall from the proof of Lemma 6.8 that, a priori we have two induced maps for each $\tau_j^{(i)}\in\Sigma_\infty$, read as

$$\phi_{*,j}^{\mathrm{cris},(i)}\colon H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i)}\longrightarrow H_1^{\mathrm{cris}}(B/k)_{\mathbb{I},j}^{(i)},$$
$$\phi_{*,j}^{\mathrm{dR},(i)}\colon \mathscr{H}_j^{A,(i)}\longrightarrow\mathscr{H}_j^{B,(i)}.$$

The crystalline nature of H_1^{cris} implies that $\phi_{*,j}^{\mathrm{cris},(i)}=\phi_{*,j}^{\mathrm{dR},(i)}\otimes_k(k[\varepsilon]/\varepsilon^2)$.

We now construct $A_{\mathbb{I}}$ and the isogeny $\phi_{\mathbb{I}}\colon A_{\mathbb{I}}\to B_{\mathbb{I}}$ from the data about $B_{\mathbb{I}}$. It suffices to specify, for each $\tau_j^{(i)}\in\Sigma_\infty$, a subbundle $\omega_{\mathbb{I},j}^{A,(i)}\subseteq H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i)}$ that lifts $\omega_j^{A,(i)}$ and satisfies $\phi_{*,j}^{\mathrm{cris},(i)}(\omega_{\mathbb{I},j}^{A,(i)})\subseteq\omega_{\mathbb{I},j}^{B,(i)}=\omega_j^{B_{\mathbb{I}},(i)}$.

- If $\iota^{-1}(\tau_j^{(i)}),\tau_j^{(i)}\notin\Delta(T)$, then by Definition 6.1(3b), $\phi_{*,j}^{\mathrm{dR},(i)}$ is an isomorphism, and so also is $\phi_{*,j}^{\mathrm{cris},(i)}$. Hence we are forced to take

$$\omega_{\mathbb{I},j}^{A,(i)}:=(\phi_{*,j}^{\mathrm{cris},(i)})^{-1}(\omega_{\mathbb{I},j}^{B,(i)})=(\phi_{*,j}^{\mathrm{cris},(i)})^{-1}(\omega_j^{B_{\mathbb{I}},(i)}).$$

- Whenever $\tau_j^{(i)} \in I(T)$, we have the isomorphism $\phi_{*,j}^{\mathrm{cris},(i)}$ and are able to define the element in $\mathbb{J}_{\mathbb{I}}$. Put
$$\omega_{\mathbb{I},j}^{A,(i)} := (\phi_{*,j}^{\mathrm{cris},(i)})^{-1}(\mathcal{J}_{\mathbb{I},j}^{(i)}).$$
Note that this partially covers the first case above.
- If $\iota^{-1}(\tau_j^{(i)}), \tau_j^{(i)} \in \Delta(T)$, then due to the signature condition and Lemma 5.11, $r_j^{(i)}$ keeps invariant during the signature change process. Also, we note that $\tau_j^{(i)} \in S_\infty$, implying that $\omega_j^{A,(i)}$ is of dimension 0 or 2 as a k-vector subspace. Then the choice of $\omega_{\mathbb{I},j}^{A,(i)}$ is unique, i.e. it is either zero or the whole $H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i)}$.
- In all other cases, we always have $\tau_j^{(i)} \in T$ whose partial Hasse invariants vanishes. Take
$$\omega_{\mathbb{I},j}^{A,(i)} := \begin{cases} \mathrm{Hasse}_{\mathbb{I},\mathrm{es}}^{\uparrow,n(i,j)}(\iota^{n(i,j)}(H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i)})), & \tau_j^{(i)} \in \Delta(T)\cap T, \\ H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i)}, & \tau_j^{(i)} \notin \Delta(T)\cap T. \end{cases}$$
Here the first line equals nothing but $\mathrm{Ker}\,\mathrm{Hasse}_{\mathbb{I},\mathrm{es}}^{\downarrow,n(i,j)}$. Indeed, it is not a forced choice at this moment, but it will become the unique choice when we have constructed the lift $A_{\mathbb{I}}$ and require $A_{\mathbb{I}}$ to have vanishing partial Hasse invariants.

Now the construction of $\omega_{\mathbb{I},j}^{A,(i)}$ has been finished. We are to verify the condition
$$\phi_{*,j}^{\mathrm{cris},(i)}(\omega_{\mathbb{I},j}^{A,(i)}) \subseteq \omega_{\mathbb{I},j}^{B,(i)} = \omega_j^{B_{\mathbb{I}},(i)}.$$
For the first and the second cases, this is obviously valid. Note that $r_j^{(i)} \neq 1$ always implies $r_j^{(i)\prime} \neq 1$. It follows that if $\omega_{\mathbb{I},j}^{A,(i)} = H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i)}$, then $\omega_{\mathbb{I},j}^{B,(i)}$ must equal $H_1^{\mathrm{cris}}(B/k)_{\mathbb{I},j}^{(i)}$, which shows the desired inclusion via $\phi_{*,j}^{\mathrm{cris},(i)}(H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i)}) \subseteq H_1^{\mathrm{cris}}(B/k)_{\mathbb{I},j}^{(i)}$. Since $\phi_{*,j}^{\mathrm{cris},(i)}$ sends the trivial bundle to the trivial bundle, we are on one hand left with the last case where $\omega_{\mathbb{I},j}^{A,(i)} = \mathrm{Hasse}_{\mathbb{I},\mathrm{es}}^{\downarrow,n(i,j)}(H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i)})$; on the other hand, by the argument of Step II, A satisfies Definition 6.1(3c), which dictates that the desired condition is trivial over k at the level of de Rham homology. So we are done by lifting the trivial bundle to the crystalline setting over $k[\varepsilon]/\varepsilon^2$.

In the following, we must check that the partial Hasse invariant $h_j^{(i)}$ of $A_{\mathbb{I}}$ vanishes if $\tau_j^{(i)} \in T$; or equivalently,
$$\mathrm{Hasse}_{\mathbb{I},\mathrm{es}}^{\uparrow,n(i,j)}(\iota^{n(i,j)}(H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i)})) = \omega_{\mathbb{I},j}^{A,(i)} \iff \tau_j^{(i)} \in T.$$
Unwinding our construction, this is clear for $\tau_j^{(i)} \in \Delta(T)\cap T$. As for when $\tau_j^{(i)} \notin \Delta(T)\cap T$, we essentially have $n(i,j) = 0$, so the left-hand side also equals $H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i)}$.

By duality, we have to take $\omega_{\mathbb{I},j}^{(i),\perp}$ to be the orthogonal complement of $\omega_{\mathbb{I},j}^{(i)}$ under the perfect pairing
$$\langle\cdot,\cdot\rangle_{\lambda_A}^{\mathrm{cris}} \colon H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i)} \times H_1^{\mathrm{cris}}(A/k)_{\mathbb{I},j}^{(i),\perp} \longrightarrow k[\varepsilon]/\varepsilon^2$$
induced by the polarization λ_A. This guarantees the vanishing of partial Hasse invariants at the conjugate places of those in T. This condition conversely forces the uniqueness of

our choice of $A_{\mathbb{I}}$, as indicated earlier. From the construction, $\omega_{A^\vee,\mathbb{I}} = \bigoplus_{j=1}^{f}\bigoplus_{i=1}^{e}\omega_{\mathbb{I},j}^{A,(i)}$ is isotropic under the pairing $\langle\cdot,\cdot\rangle_{\lambda_A}^{\mathrm{cris}}$. This concludes checking $(A,\lambda_A,\rho_A,\underline{\mathscr{F}}^A)\in Z_T(S)$ of Definition 6.1(1).

The lift of the level structure $\rho_{A,\mathbb{I}}$ is automatic for the tame part $\rho_{A,\mathbb{I}}^p$, and can be done in a unique way as at the end of the proof to Lemma 6.8. The condition 6.1(4) also follows, and it then remains to check the condition 6.1(3). The relation $p\lambda_A = \phi^\vee\circ\lambda_B\circ\phi$ naturally leads to $p\lambda_{A,\mathbb{I}} = \phi_{\mathbb{I}}^\vee\circ\lambda_{B,\mathbb{I}}\circ\phi_{\mathbb{I}}$ by base change. To check that $\phi_{\mathbb{I},*,j}^{\mathrm{cris},(i)}$ is an isomorphism for $\tau_j^{(i)}\notin\Delta(T)$, it suffices to show that its image has the same rank as $r_j^{(i)}$. However, this is implied by the equality

$$\phi_{\mathbb{I},*,j}^{\mathrm{cris},(i)}(\omega_{\mathbb{I},j}^{A,(i)}) = \phi_{*,j}^{\mathrm{dR},(i)}(\omega_j^{A,(i)})\otimes_k (k[\varepsilon]/\varepsilon^2).$$

Hence (3b) follows. We besides have

$$\begin{aligned}\phi_{\mathbb{I},*,j^-}^{\mathrm{cris},(i^-)}(\mathrm{Hasse}_{\mathbb{I},\mathrm{es}}^{\downarrow,n(i,j)}(\omega_{\mathbb{I},j}^{A,(i)})) &= \phi_{\mathbb{I},*,j^-}^{\mathrm{cris},(i^-)}(\mathrm{Hasse}_{\mathrm{es}}^{n(i,j)}(\omega_j^{A_{\mathbb{I}},(i)}))\\ &= \phi_{*,j^-}^{\mathrm{dR},(i^-)}(\mathrm{Hasse}_{\mathrm{es}}^{n(i,j)}(\omega_j^{A,(i)}))\otimes_k (k[\varepsilon]/\varepsilon^2),\end{aligned}$$

where $\tau_{j^-}^{(i^-)} = \iota^{n(i,j)}(\tau_j^{(i)})$. Therefore, (3c) for $A_{\mathbb{I}}$ follows from (3c) for A. □

Remark 6.10. Previously, in Remarks 5.8 and 5.12, we have seen there are two different modifications for the signature change, and so also there would be another choice $\Delta^\dagger(T)$ to replace $\Delta(T)$. Technically, the most significant difference between them lies in swapping the signatures in T' between 0 and 2. However, unwinding the proofs to Lemmas 6.6, 6.8, and Proposition 6.9, we see the combinatorics are essentially the same for both modifications. The main reason is that the essential Hasse maps concerns about separating the embeddings into two classes only: one is of signature 1, and the other is of signature 0 or 2.

To summarize, the proof of Theorem 5.1 follows from Lemma 6.3 and Proposition 6.4, and the latter is proved by Propositions 6.5 and 6.9. Recall that Proposition 6.5 is implied by Lemmas 6.6 and 6.8; for the sake of simplicity, 6.9 shrinks all the lemmas into its 3-step proof, based on the same framework as of 6.5.

References

[AG05] F. Andreatta and E.Z. Goren. *Hilbert Modular Forms: mod p and p-adic Aspects*. Number 819. American Mathematical Society, 2005. URL, arXiv:0308040.

[CCO14] C.-L. Chai, B. Conrad, and F. Oort. *Complex Multiplication and Lifting Problems*, volume 195 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2014. URL.

[DP94] P. Deligne and G. Pappas. Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant. *Compositio Mathematica*, 90(1):59–79, 1994. URL.

[Fal83] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent Math*, 73:349–366, 1983. URL.

[GO00] E.Z. Goren and F. Oort. Stratifications of Hilbert modular varieties. *Journal of Algebraic Geometry*, 9:111–154, 2000. URL.

[Gor01] E.Z. Goren. Hasse invariants for Hilbert modular varieties. *Israel Journal of Mathematics*, 122:157–174, 2001. URL.

[Hel10] D. Helm. Towards a geometric Jacquet–Langlands correspondence for unitary Shimura varieties. *Duke Mathematical Journal*, 155(3):483–518, 2010. URL, arXiv:0511146.

[Hel12] D. Helm. A geometric Jacquet–Langlands correspondence for $U(2)$ Shimura varieties. *Israel Journal of Mathematics*, 187:37–80, 2012. URL, arXiv:0409458.

[HTX17] D. Helm, Y. Tian, and L. Xiao. Tate cycles on some unitary Shimura varieties mod p. *Algebraic Number Theory*, 11(10):2213–2288, 2017. URL, arXiv:1410.2343.

[JL70] H. Jacquet and R.P. Langlands. *Automorphic forms on* $\mathrm{GL}(2)$, volume 114 of *Lecture Notes in Mathematics*. Springer Berlin, Heidelberg, 1970. URL.

[Kot92] R. Kottwitz. Points on some Shimura varieties over finite fields. *Journal of the American Mathematical Society*, 5(2):373–444, 1992. URL.

[Lan13] K.-W. Lan. *Arithmetic compactifications of PEL-type Shimura varieties*, volume 36 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2013. URL.

[Mes72] W. Messing. *The crystals associated to Barsotti–Tate groups: with applications to abelian schemes*. Number 264 in Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1972. URL.

[Oda69] T. Oda. The first de Rham cohomology group and Dieudonné modules. *Annales scientifiques de l'École Normale Supérieure*, Ser.4, 2(1):63–135, 1969. URL.

[PR05] G. Pappas and M. Rapoport. Local models in the ramified case, II: Splitting models. *Duke Mathematical Journal*, 127(2):193–250, 2005. URL, arXiv:0205021.

[Rap78] M. Rapoport. Compactifications de l'espace de modules de Hilbert–Blumenthal. *Compositio Mathematica*, 36(3):255–335, 1978. URL.

[Rib83] K. Ribet. Mod p Hecke operators and congruences between modular forms. *Inventiones mathematicae*, 71(1):193–205, 1983. URL.

[Rib89] K. Ribet. Bimodules and abelian surfaces. In J. Coates, R. Greenberg, B. Mazur, and I. Satake, editors, *Algebraic Number Theory — in honor of K. Iwasawa*, volume 17 of *Advanced Studies in Pure Mathematics*, pages 359–407. Academic Press, New York, 1989. URL.

[RX17] D. Reduzzi and L. Xiao. Partial Hasse invariants on splitting models of Hilbert modular varieties. *Annales scientifiques de l'École Normale Supérieure*, 50:579–607, 2017. URL, arXiv:1405.6349.

[Sas19] S. Sasaki. Integral models of Hilbert modular varieties in the ramified case, deformations of modular Galois representations, and weight one forms. *Inventiones Mathematicae*, 215:171–264, 2019. URL.

[SL96] J.P. Serre and R. Livné. Two letters on quaternions and modular forms (mod p). *Israel Journal of Mathematics*, 95:281–299, 1996. URL.

[TX16] Y. Tian and L. Xiao. On Goren–Oort stratification for quaternionic Shimura varieties. *Compositio Mathematica*, 152:2134–2220, 2016. URL, arXiv:1308.0790.

[TX19] Y. Tian and L. Xiao. Tate cycles on quaternionic Shimura varieties over finite fields. *Duke Mathematical Journal*, 168:1551–1639, 2019. URL, arXiv:1410.2321.

[Vol05] I. Vollaard. On the Hilbert–Blumenthal moduli problem. *Journal of the Institute of Mathematics of Jussieu*, 4(4):653–683, 2005. URL, arXiv:0307325.

[Wan09] H. Wang. Moduli spaces of p-divisible groups and period morphisms. Master's thesis, Universiteit Leiden, 2009. URL.

[XZ17] L. Xiao and X. Zhu. Cycles on Shimura varieties via geometric Satake. Preprint, 2017. arXiv:1707.05700.

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