

A brief introduction to the trace formula and its stabilization (1/2)

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- Plan
- (1) Trace formula and its stab for G anisotropic red gp
 - (2) The same for G general.
 - (3) Application: Classical groups

G conn red / \mathbb{Q} , anisotropic

Assume (for simplicity)

- G_{der} derived subgp of G is simply conn
- G satisfies the Hasse principle.

$\nwarrow \text{O: doesn't satisfy; U: satisfies}$

§ Trace formula

interest

$$L^2(IG) = \bigoplus_{\pi} \pi^{\oplus m(\pi)}, \text{ finite } m(\pi)$$

$$f \in C_c^\infty(G(A)), R(f) \in L^2(IG)$$

\nwarrow action of right regular rep

$$\text{tr } R(f) = \sum_{\pi} m(\pi) \cdot \text{tr}(\pi(f)) \quad \text{spectral info}$$

Write $[Rf](x) = \int_{G(A)} f(g) f(gx) dg.$

$$= \int_{G(\mathbb{Q}) \backslash G(A)} \underbrace{\sum_{g \in G(\mathbb{Q})} f(x^{-1}gx) f(g)}_{K(x, g)} dg$$

kernel operator

geometric info

$$\begin{aligned} \rightarrow \text{tr } R(f) &= \int_{G(\mathbb{Q}) \backslash G(A)} K(x, x) dx \\ &= \sum_{x \in G(\mathbb{Q})} \text{vol}(G(\mathbb{Q}) \backslash G(A)) \int_{G_x(A) \backslash G(A)} f(x^{-1}\gamma x) dx. \end{aligned}$$

$G_x := \text{Cent}_G(x).$

$$\text{Thm (TF)} \quad \sum_{\pi} m(\pi) \operatorname{tr} \pi(f) = \sum_{\sigma \in I(G(\mathbb{Q}))} \tau(G_\sigma) \operatorname{Or}_\sigma(f)$$

spec side geom side ↗
↓ invariant distributions

w/ Tamagawa measure & $\tau(G_\sigma)$: Tamagawa number.

3 Stable Conjugacy

Def'n Let R be a \mathbb{Q} -alg ($R = \mathbb{Q}, \mathbb{Q}_v, A$).

(1) Two elts $\gamma_1, \gamma_2 \in G(R)$ are called stably conjugate

if $\gamma_1 \sim \gamma_2$ conj in $G(\bar{R})$, $\bar{R} = R \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$.

(2) The stable conj class of γ is $[\operatorname{Ad}(G(\bar{R})) \cdot \gamma] \cap G(R)$.

E.g. $G = \operatorname{SL}_2 / \mathbb{R}$.

$$\gamma_1 = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$$

$\rightsquigarrow \gamma_1 \sim \gamma_2$ in $\operatorname{SL}_2(\mathbb{C})$ but not in $\operatorname{SL}_2(\mathbb{R})$

$\Rightarrow \gamma_1, \gamma_2$ stably conj.

Fact Given $\gamma_1, \gamma_2 \in G(R)$ stably conj. Let $g \in G(\bar{R})$, $g\gamma_1 g^{-1} = \gamma_2$.

Then

(1) the class in $G(\bar{R})$ of

$\operatorname{Gal}_{\mathbb{Q}}, R = \mathbb{Q}, A \left\{ \begin{array}{l} \sigma \mapsto g^{-1} \sigma g \text{ in } H^1(T, G_\sigma(\bar{R})) \text{ is indep of } g, \\ \sigma \mapsto g^{-1} \sigma g \text{ in } H^1(T, G_\sigma(\bar{R})) \end{array} \right\} = \text{①}$ and will be denoted by $\operatorname{inv}(\gamma_1, \gamma_2)$.

\uparrow (2) $\gamma_2 \mapsto \operatorname{inv}(\gamma_1, \gamma_2)$ (fixing γ_1) is a bijection
 $\bar{\mathbb{Q}}_v = \mathbb{Q}_v \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$ $\{ \text{stab conj class of } \gamma_1 \} / G(R)$
 $\longleftrightarrow \ker(H^1(T, G_{\gamma_1}(\bar{R})) \rightarrow H^1(T, G(\bar{R})))$.

Def $R = \mathbb{Q}_v, A$. $f \in C_c^\infty(G(A))$, $\gamma \in G(R)$ semi-simple.

$$(1) O_\gamma(f) := \int_{G_\gamma(R) \backslash G(R)} f(x^{-1}\gamma x) dx$$

(2) When γ is regular,

$$SO_\gamma(f) := \int_{(G_\gamma \backslash G)(\mathbb{A})} f(x^{-1}\gamma x) dx = \sum_{\gamma' \not\sim \gamma} O_{\gamma'}(f).$$

(3) $d : C_c^\infty(G(R)) \rightarrow \mathbb{C}$ is stable

if $d(f) = 0$ whenever $O_\gamma(f) = 0, \forall \gamma$.

{Pre-stabilization}

$$V = V_{st} \oplus \Sigma, \quad \Sigma = \bigoplus_H \Sigma_H, \quad \text{with } \Sigma_H \rightarrow V_{st}^H.$$

$$\text{Write } TF_{\text{geom}}^G(f) = \sum_{\gamma \in \Gamma[G(\mathbb{Q})]} \tau(G_\gamma) O_\gamma(f)$$

$$\begin{aligned} \text{geom side of } TF &= \sum_{\gamma \in \Gamma[G(\mathbb{Q})]} \sum_{\gamma' \in \Gamma[G(\mathbb{A})]} \tau(G_{\gamma'}) O_{\gamma'}(f) \\ &\stackrel{[\dots]}{\sim} \text{stab classes} \quad \sum_{\gamma' \not\sim \gamma} O_{\gamma'}(f) \\ &\stackrel{[\dots]}{\sim} \text{conj classes} \end{aligned}$$

Kottwitz If γ, γ' are stab conj.

then the Tamagawa numbers $\tau(G_\gamma) = \tau(G_{\gamma'})$.

$$\therefore TF_{\text{geom}}^G(f) = \sum_{\gamma \in \Gamma[G(\mathbb{Q})]} \sum_{\gamma' \not\sim \gamma} O_{\gamma'}(f)$$

adelic orb int.

By the fact before,

param the sum $\sum_{\gamma' \not\sim \gamma} O_{\gamma'}(f)$

not groups \rightarrow $\text{ker} \rightarrow H^1(\Gamma, G_\gamma(\bar{\mathbb{Q}})) \xrightarrow{\beta_{\bar{\mathbb{Q}}}} H^1(\Gamma, G(\bar{\mathbb{Q}}))$ maps of sets

$\downarrow \alpha \qquad \downarrow \alpha \qquad \downarrow$
 $\text{ker} \rightarrow H^1(\Gamma, G_\gamma(\bar{\mathbb{A}})) \xrightarrow{\beta_{\bar{\mathbb{A}}}} H^1(\Gamma, G(\bar{\mathbb{A}}))$

param what would like to have
to make the orb int stable. $\downarrow \alpha$ \downarrow by Kottwitz
 $\pi_1(G_\gamma)_\Gamma, \text{tor}$ (later) \downarrow $\pi_1(G)_\Gamma, \text{tor}$

Assume γ regular ($\Rightarrow G_\gamma = T$ torus)

$$1 \rightarrow T(\bar{\mathbb{Q}}) \rightarrow T(\bar{A}) \rightarrow T(\bar{A})/T(\bar{\mathbb{Q}}) \rightarrow 1.$$

$$\hookrightarrow 1 \rightarrow \mathrm{H}^1(T, T(\bar{\mathbb{Q}})) \rightarrow \mathrm{H}^1(T, T(\bar{A})) \\ \rightarrow \mathrm{H}^1(T, T(\bar{A})/T(\bar{\mathbb{Q}})) \xrightarrow{\cong} X_{\ast}(T)_{T, \text{tor}}$$

↑
Tate-Nakayama

Borel alg. fundamental group $\pi_1(H)$

fin gen'd ab grp with F -action

for any conn red gp H over a field.

($T \subset H$ any max torus,

$$\uparrow \quad \pi_1(H) := X_{\ast}(T)/X_{\ast}(T_{\infty}).)$$

Canonical, up to isoms.

Kottwitz H/\mathbb{Q} ,

$$1 \rightarrow \mathrm{H}^1(H) \rightarrow \mathrm{H}^1(T, H(\bar{\mathbb{Q}})) \rightarrow \mathrm{H}^1(T, H(\bar{A})) \rightarrow \pi_1(H)_{T, \text{tor}} \text{ is exact}$$

Def $K(\gamma) := \ker(\pi_1(G_\gamma)_{T, \text{tor}} \rightarrow \pi_1(G)_{T, \text{tor}})^*$.

$$\hookrightarrow \mathrm{TF}_{\text{geom}}^G(f) = \sum_{\gamma \in \mathrm{I}(G(\mathbb{Q}))} \tau(G_\gamma) \cdot \# \mathrm{H}^1(G_\gamma) \\ \cdot |K(\gamma)|^{-1} \sum_{\gamma \in K(\gamma)} \sum_{\substack{\gamma' \in T(G(A)) \\ \gamma \cdot \gamma' = \gamma}} \chi(\mathrm{inv}(\gamma, \gamma')) \cdot O_{\gamma'}(f) \\ = \tau(G) \cdot \sum_{\gamma \in \mathrm{I}(G(\mathbb{Q}))} \sum_{\gamma \in K(\gamma)} \sum_{\substack{\gamma' \in T(G(A)) \\ \gamma \cdot \gamma' = \gamma}} \chi(\mathrm{inv}(\gamma, \gamma')) \cdot e(G_\gamma) O_{\gamma'}(f) \text{ of centralizer}$$

$e(G_\gamma)$ Kottwitz's sign
 $O_{\gamma'}(f)$ of centralizer

$$= O_{\gamma}^K(f).$$

Final result of pre-stabilization:

$$\mathrm{TF}_{\text{geom}}^G(f) = \tau(G) \cdot \sum_{\gamma \in \mathrm{I}(G(\mathbb{Q}))} \underbrace{\sum_{\gamma \in K(\gamma)} O_{\gamma}^K(f)}_{\text{"SO}_{\gamma}(f)" \text{ when } \gamma = 1.}$$

Remark Kottwitz proved that the global sign $e(G_\gamma) = 1$.
 & when $\kappa = 1$.

$$\tau(G) \sum_{\gamma \in [G(\mathbb{Q})]} SO_\gamma(f) = STF_{\text{geom}}^G(f).$$

$$O_\gamma(f).$$

§ Transfer

$$\text{Bij: } (\gamma, \kappa) \longleftrightarrow (H, K_H, \gamma_H)$$

where (H, K_H) = endoscopic elliptic grp data

$\gamma_H \in H(\mathbb{Q})$ (G, H) -reg elliptic.

H quasi-split \mathbb{Q} -grp

a very strict { w/ fixed isom $T^G \xrightarrow{\sim} T^H$.

Condition s.t. the Γ -str is twisted by Weyl grp.

κ clear of $\ker(\pi_{\mathcal{U}}(H) \rightarrow \pi_{\mathcal{U}}(G))_{\Gamma, \text{tor}}$.

Thm There exists $f \in \mathcal{C}_c(H(\mathbb{A}))$ st.

$$O_\gamma^K(f) = SO_{\gamma H}^H(f^K).$$

pf: contains fund lemma

→ passing to Lie algs and use an observation of Waldspurger

$$\begin{aligned} \hookrightarrow TF_{\text{geom}}^G(f) &= \tau(G) \cdot \sum_{\substack{(H, K_H) \\ \text{ell}}} |\text{Out}(H)|^{-1} \cdot \sum_{\substack{\gamma^H \in [H(\mathbb{Q})] \\ \text{ell } (G, H)-\text{reg}}} SO_{\gamma^H}^H(f^H). \\ &\quad \text{ell endoscopic data} \\ &= \sum_{(H, K_H) \in \text{Ell}(G)} Z(G, H) \cdot \underbrace{STF_{\text{geom}}^H(f^H)}_{\in V_{\text{st}}^H}. \end{aligned}$$

§ Stabilization of the spectral side

$$TF_{\text{Spec}}^G(f) = \sum_{\pi} m(\pi) \cdot \text{tr}(\pi(f)).$$

Remember G is still anisotropic.

(conj (Arthur)) $L^2([IG]) = \bigoplus_{\pi} \bigoplus_{\pi} \pi^{m(t, \pi)}$.
where $\gamma : \mathcal{L}_Q \times \text{Sl}_2(\mathbb{C}) \rightarrow {}^t G$

$$\begin{aligned} \pi \in \text{IT}_{\gamma} &\leftarrow L\text{-packet } \prod_{\pi_w} \langle \pi_w, - \rangle, \\ m(t, \pi) &= \text{mult}(\mathcal{E}_t, (\pi, -)) \\ &= |\mathcal{S}_t|^{-1} \sum_{x \in \mathcal{S}_t} \mathcal{E}_t(x) \cdot \langle \pi, x \rangle. \end{aligned}$$

When both S_t, S_{t_w} are abelian,

$$m(t, \pi) = \begin{cases} 0, & \langle \pi, - \rangle \neq \mathcal{E}_t, \\ 1, & \langle \pi, - \rangle = \mathcal{E}_t. \end{cases}$$

$$L^2([IG]) = \bigoplus_{\pi} \bigoplus_{\pi : \langle \pi, - \rangle = \mathcal{E}_t} \pi.$$

$$\hookrightarrow TF_{\text{Spec}}^G(f) = \sum_t \sum_{\pi} |\mathcal{S}_t|^{-1} \sum_{x \in \mathcal{S}_t} \mathcal{E}_t(x) \langle \pi, x \rangle \text{tr}(\pi(f))$$

$$\text{Take } S_t \ni s_t = t(1, \begin{pmatrix} -1 & \\ & -1 \end{pmatrix})$$

and consider the summand on RHS with $x = s_t$.

$$= \sum_t |\mathcal{S}_t|^{-1} \sum_{x \in \mathcal{S}_t} \mathcal{E}_t(x, S_t) \prod_{\pi_w \in \text{IT}_{\gamma_{t_w}}} \langle \pi_w, x \cdot S_t \rangle \text{tr}(\pi_w(f_w)).$$

Using $\text{Bij} : (\psi, S) \longleftrightarrow (H, K, \gamma^H)$,

$$\sum_{\pi_w \in \text{IT}_{\gamma_{t_w}}} \langle \pi_w, x \cdot S_t \rangle \text{tr}(\pi_w(f_w)) = \sum_{\pi_w^H \in \text{IT}_{\gamma_{t_w}^H}} \langle \pi_w^H, S_t^H \rangle \cdot \text{tr}(\pi_w^H(f_w^H)).$$

$\stackrel{\text{defn}}{=} S \Theta_{\gamma^H}^H(f_w^H)$ Stable character of the parameter γ^H .

$$\hookrightarrow TF_{\text{Spec}}^G(f) = \sum_{(H, K) \in E_{\text{ell}}(G)} \zeta(G, H) \cdot \sum_{\gamma^H} \mathcal{E}_{\gamma^H}(S_t^H) S \Theta_{\gamma^H}(f^H).$$

\Rightarrow get the stabilization of spectral side.

Conj (Stable multiplicity formula, Arthur).

$$\text{STF}_{\text{geom}}^G(f) = \text{STF}_{\text{spec}}^G(f) := \sum_p \epsilon_p(S_p) S \otimes_{\mathbb{Z}_p} (f).$$