Introduction (I)

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§1 Classical local Langlands conjecture

Setups E non-arch p-adic field.

G/E quasi-split reductive grp.

$\Gamma_E = \mathrm{Gal}(\bar{E}/E)$, $\hat{G}/\mathbb{C} \circlearrowleft \Gamma_E$.

$q = \# k_E$, fix $\sqrt{q}$.

Def'n An L-parameter for G is

$$\begin{tikzcd} \varphi : W_E \times SL_2(\mathbb{C}) \arrow[r] \arrow[dr] & \hat{G}(\mathbb{C}) \rtimes W_E \arrow[d] \\ & W_E = \text{Weil group} \end{tikzcd}$$

s.t. (1) $\varphi|_{W_E}$ is continuous, $\varphi|_{SL_2}$ alg,

(2) $\varphi(W_E)$ consists of semisimple elts.

LLC $\exists$ a map

$$\left\{ \begin{matrix} \text{smooth irred} \\ \text{rep'ns of } G(E) \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{L-parameters} \\ \text{for } G \end{matrix} \right\} \Big/ \hat{G}\text{-conj.}$$

$$\pi \longmapsto \varphi_\pi$$

satisfying certain compatibilities.

Fargues-Scholze : Constructed $\varphi_\pi^{ss} := \varphi_\pi \circ \iota_w$

$$\iota_w : W_E \longrightarrow W_E \times SL_2$$

$$w \longmapsto \left(w, \begin{pmatrix} \|w\|^{1/2} & 0 \\ 0 & \|w\|^{-1/2} \end{pmatrix}\right) \qquad (\|\mathrm{Frob}_q\| = q).$$

by doing geom. Langlands over FF curves.

· Representation side · Galois side

$$\underbrace{D_{lis}(\mathrm{Bun}_G, \mathbb{Z}_\ell)}_{\text{triangulated category}} \xleftrightarrow[\text{V. Lafforgues}]{\text{geom Satake}} D^b_{Coh,nilp}(\underbrace{Z^1(W_E, \widehat{G})}_{\text{certain scheme}/\mathbb{Z}_\ell}/\widehat{G})$$

§2 The Fargues-Fontaine curve

$\mathrm{Perf}_{\mathbb{F}_p}$:= category of perfectoid spaces in char p / $\mathbb{F}_p$.
equipped with pro-étale top, v-top.

Def'n A diamond is a pro-étale sheaf on $\mathrm{Perf}_{\mathbb{F}_p}$ that is a quotient of a perfectoid space in char p by a pro-étale equivalence relation.

Fact (1) A diamond is also a v-sheaf

(2) $\forall$ adic space $Z/\mathbb{Z}_p$, get a v-sheaf

$$Z^\diamond : \mathrm{Perf}_{\mathbb{F}_p} \longrightarrow \mathrm{Sets}$$
$$S \longmapsto \{S^\# : \text{untilting of } S \text{ with map } S^\# \to Z\}.$$

(3) $Z^\diamond$ is indeed a diamond if Z is analytic

(i.e. locally of the form $\mathrm{Spa}(A, A^+)$, A = Tate algebra).

In this case, $\underbrace{|Z^\diamond| = |Z|}_{\text{top spaces}}$, $Z^\diamond_{et} \cong Z_{et}$.

(In general, $|Z^\diamond|$ and $|Z|$ are different, but $|Z^\diamond| \twoheadrightarrow |Z|$).

Prop (Kedlaya-Liu) The functor

$$\left\{\begin{array}{c}\text{seminormal rigid} \\ \text{spaces} / \mathbb{Q}_p\end{array}\right\} \longrightarrow \{\text{diamonds over } \mathrm{Spa}(\mathbb{Q}_p)^\diamond\}$$

is fully faithful.

Basic example $Z = \mathrm{Spa}\,\mathbb{Q}_p = (\mathrm{Spa}\,\mathbb{C}_p)/\Gamma_{\mathbb{Q}_p} = \mathrm{Spa}(\widehat{\mathbb{Q}_p(\zeta_{p^\infty})})/\mathbb{Z}_p^\times$.

$$\begin{aligned}\rightsquigarrow Z^\diamond &= \mathrm{Spa}(\mathbb{Q}_p)^\diamond/\Gamma_{\mathbb{Q}_p} \cong \mathrm{Spa}(\mathbb{C}_p^\flat)/\Gamma_{\mathbb{Q}_p}\\ &= \mathrm{Spa}(\widehat{\mathbb{Q}_p(\zeta_{p^\infty})})^\diamond/\mathbb{Z}_p^\times\\ &\cong \mathrm{Spa}(\mathbb{F}_p((t^{1/p^\infty})))/\mathbb{Z}_p^\times\end{aligned}$$

$t = \varepsilon - 1$, $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathbb{C}_p^\flat$.

Def'n For any $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_p}$, define

$$Y_S := \mathrm{Spa}(W(R^+)) \setminus V(p[\varpi])$$

$\leftarrow$ adic space $\circlearrowleft$ Frobenius ϕ_S.

the Teichmüller lift of ϖ.

$\varpi \in R^+$ pseudo-uniformizer.

Define the Fargues-Fontaine curve over S

$$X_S := Y_S/\phi_S^{\mathbb{Z}}$$

But No morphism $Y_S \dashrightarrow S$ of adic spaces

Prop (1) $Y_S^\diamond = S \times \mathrm{Spa}(\mathbb{Q}_p)^\diamond$

(2) $X_S^\diamond = S/\phi_S^{\mathbb{Z}} \times \mathrm{Spa}(\mathbb{Q}_p)^\diamond$.

§3 Vector bundles on X_S

Fix an alg closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$, and work with $\mathrm{Perf}_{\overline{\mathbb{F}}_p}$.

Define $\mathrm{Isoc}_{\overline{\mathbb{F}}_p} := \left\{ (D, \phi_D) : \begin{array}{l} D \text{ is a fin-dim'l } \breve{\mathbb{Q}}_p := W(\overline{\mathbb{F}}_p)[\tfrac{1}{p}]\text{-space} \\ \phi_D : D \to D \text{ is a Frobenius-linear bijection} \end{array} \right\}$

$\exists$ $(-) \overset{\mathbb{L}}{\otimes} (-)$ functor

$$\begin{aligned}\mathrm{Isoc}_{\overline{\mathbb{F}}_p} &\longrightarrow \mathrm{Bun}(X_S)\\ (D, \phi_D) &\longmapsto \underbrace{(D \otimes_{\breve{\mathbb{Q}}_p} \mathcal{O}_{Y_S})}_{}/(\phi_D \otimes \phi_S)^{\mathbb{Z}}\end{aligned}$$

$=: \mathcal{E}(D)$.

Notation Let $(D_\lambda, \phi_{D_\lambda})$ be the simple isocrystal of slope $\lambda \in \mathbb{Q}$.

Define $\quad \mathcal{O}_{X_S}(\lambda) := \mathcal{E}(D_{-\lambda})$, $\lambda = \frac{r}{s}$, $s>0$, $(r,s)=1$

(to guarantee that $\mathcal{O}_{X_S}(1)$ ample)

$\rightsquigarrow$ $\mathcal{O}_{X_S}(\lambda)$ is a v.b. on X_S of rank r & deg s.

Thm (Fargues-Fontaine) $C(\supseteq \overline{\mathbb{F}}_p)$ alg closed non-arch field.

Then any vector bundle on X_C is a direct sum of $\mathcal{O}_{X_C}(\lambda)$'s.

Moreover, · If $\lambda \geq 0$, $H^1(X_C, \mathcal{O}_{X_C}(\lambda)) = 0$

· If $\lambda < 0$, $H^0(X_C, \mathcal{O}_{X_C}(\lambda)) = 0$.

Remark For $\lambda > 0$, $H^0(X_C, \mathcal{O}_{X_C}(\lambda))$ is a large space, $\dim_{\mathbb{Q}_p} = \infty$.

(so-called Banach-Colmez space)

· There's a family version of this vanishing result.

From now on, we work with $E = \mathbb{Q}_p$. $G/\mathbb{Q}_p$ reductive grp.

(More general E can be reduced to this.)

Def'n Bun_G is the v-stack

$$\begin{aligned} \mathrm{Perf}_{\overline{\mathbb{F}}_p} &\longrightarrow \text{Groupoids} \\ S &\longmapsto \{G\text{-bundles on } X_S\} \end{aligned}$$

Thm $\big(\underbrace{\text{Fargues-Fontaine}}_{GL_n}, \underbrace{\text{Fargues}}_{G}, \underbrace{\text{Anschütz}}_{\mathbb{Q}_p \rightsquigarrow \mathbb{F}_p((t))}\big)$

$$\exists \text{ bijection} \quad \begin{aligned} B(G) &\xrightarrow{\sim} \mathrm{Bun}_G(C)/\cong \\ [b] &\longmapsto \mathcal{E}_b := (G \times_{\breve{\mathbb{Q}}_p} Y_S)/(b \times \phi_S)^{\mathbb{Z}}. \end{aligned}$$

$B(G) := G(\breve{\mathbb{Q}}_p)/\sigma\text{-conj}$. $\quad g_1 \overset{\sigma}{\sim} g_2$ iff $\exists h$ s.t. $g_1 = h^{-1} g_2 \sigma(h)$.

__Recall__ $\exists$ injection $\underset{\substack{\text{IS} \\ \mathrm{Isoc}_{\overline{\mathbb{F}}_p}/\cong}}{B(G)} \xrightarrow{(\nu,\kappa)} (X_*(T)^+_{\mathbb{Q}})^{\Gamma_{\mathbb{Q}_p}} \times \pi(G)_{\Gamma_{\mathbb{Q}_p}}$

κ: Kottwitz map (generalization of degree map)

$\pi_1(G) = X_*(T)/(\text{coroot lattice})$

$T \subseteq G$ max'l torus, $X_*(T) := X_*(T_{\overline{\mathbb{Q}}_p})$.

E.g. $G = GL_n$: $X_*(T) = \bigoplus_{i=1}^{n} \mathbb{Z}e_i$,

root lattice $\bigoplus_{i=1}^{n-1} \mathbb{Z}(e_i - e_{i+1})$.

$\rightsquigarrow \pi_1(GL_n) \cong \mathbb{Z}$.

E.g. $G = PGL_2$: $\pi_1(G) \cong \mathbb{Z}/2\mathbb{Z}$.

__Thm__ (Kedlaya-Liu, Fargues-Scholze, Hansen, Viehmann)

(1) $\nu: |\mathrm{Bun}_G| = B(G) \longrightarrow (X_*(T)^+_{\mathbb{Q}})^{\Gamma_{\mathbb{Q}_p}}$ is semicontinuous

(2) $\kappa: |\mathrm{Bun}_G| \longrightarrow \pi_1(G)_{\Gamma_{\mathbb{Q}_p}}$ is locally constant.

Actually, $\pi_0(|\mathrm{Bun}_G|) \cong \pi_1(G)_{\Gamma_{\mathbb{Q}_p}}$.

(Equivalently, $|\mathrm{Bun}_G| \cong \boxed{B(G)}$ order topology

$(\nu,\kappa) \le (\nu',\kappa')$ if $\nu \le \nu'$ in dominant order and $\kappa = \kappa'$

(3) Each connected component of Bun_G has a __unique__ semi-simple pt $\mathcal{E}_b$ with $b \in B(G)_{\text{basic}}$.

__Picture of GL_2__ $\kappa: |\mathrm{Bun}_{GL_2}| \to \pi_1(GL_2) \cong \mathbb{Z}$ essentially the deg map.

deg 0: $\mathcal{O} \oplus \mathcal{O} \rightsquigarrow \mathcal{O}(1) \oplus \mathcal{O}(-1) \rightsquigarrow \mathcal{O}(-2) \oplus \mathcal{O}(2) \rightsquigarrow \cdots$

deg 1: $\mathcal{O}(\tfrac{1}{2}) \rightsquigarrow \mathcal{O} \oplus \mathcal{O}(1) \rightsquigarrow \mathcal{O}(-1) \oplus \mathcal{O}(2) \rightsquigarrow \mathcal{O}(-2) \oplus \mathcal{O}(3) \rightsquigarrow \cdots$

deg 2: $\mathcal{O}(1) \oplus \mathcal{O}(1) \rightsquigarrow \mathcal{O} \oplus \mathcal{O}(2) \rightsquigarrow \mathcal{O}(-1) \oplus \mathcal{O}(3) \rightsquigarrow \cdots$

codim 0 strata

codim 1

codim 2

More generally:

Thm (1) $\mathrm{Bun}_G^{ss} = \coprod_{\underbrace{b\in B(G)_{basic}}_{\xrightarrow[\kappa]{\simeq} \pi_1(G)_{\Gamma_{\mathbb{Q}_p}}}} \mathrm{Bun}_G^b \underset{open}{\subseteq} \mathrm{Bun}_G$.

$\mathrm{Bun}_G^b \cong [*/G_b(\mathbb{Q}_p)]$ classifying stack, where $G_b := \mathrm{Aut}(\mathcal{E}_b)$.

$\forall S \in \mathrm{Perf}_{\overline{\mathbb{F}}_p}$, $G_b \times_{\mathbb{Q}_p} X_S = \mathrm{Aut}_{X_S}(\mathcal{E}_b)$.

Actually, G_b is an inner form of G.

$G_b(\mathbb{Q}_p) = \{g \in G(\breve{\mathbb{Q}}_p) \mid g^{-1} b\sigma(g) = b\}$.

In particular,

$$\mathrm{Rep}(G_b(\mathbb{Q}_p)) \cong \mathrm{Shv}([*/G_b(\mathbb{Q}_p)])$$
$$= \mathrm{Shv}(\mathrm{Bun}_G^b) \xrightarrow{i_{b,!}} \mathrm{Shv}(\mathrm{Bun}_G).$$

note $b=1 \Rightarrow G_b = G$.

(2) For $b \in B(G)$, $\mathrm{Bun}_G^b \subseteq \mathrm{Bun}_G$ is a locally closed substack.

$\mathrm{Bun}_G^b = [*/\mathcal{G}_b]$, $\mathcal{G}_b := \mathrm{Aut}(\mathcal{E}_b)$

$\exists$ an exact sequence of sheaves of groups:

$$1 \to \mathcal{G}_b^\circ \to \mathcal{G}_b \to G_b(\mathbb{Q}_p) \to 1$$

connected comp. "unipotent perfectoid grp".

$\rightsquigarrow \mathrm{Rep}(G_b(\mathbb{Q}_p)) \cong \mathrm{Shv}[*/G_b(\mathbb{Q}_p)] \cong \mathrm{Shv}[*/\mathcal{G}_b] = \mathrm{Shv}(\mathrm{Bun}_G^b)$

E.g. (1) $b = \begin{pmatrix} 0 & p^{-1} \\ 1 & 0 \end{pmatrix}$, the isocrystal D_b has slope $-\frac{1}{2}$.

$\mathcal{E}_b = \mathcal{O}(\frac{1}{2})$, $G_b = D^\times$

($D/\mathbb{Q}_p$ quaternion division algebra.)

(2) $b = \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}$, $\mathcal{E}_b = \mathcal{O}(1) \oplus \mathcal{O}$

$$\mathrm{Aut}(\mathcal{E}_b) = \mathrm{Aut}(\mathcal{O}(1)\oplus\mathcal{O}) = \begin{pmatrix} \mathbb{Q}_p^\times & \boxed{H^0(X_C, \mathcal{O}(-1))} \\ 0 & \mathbb{Q}_p^\times \end{pmatrix} \quad \boxed{H^0(X_C, \mathcal{O}(-1))} = \mathcal{G}_b^\circ.$$

$\cdot\ 0 = H^0(X_C, \mathcal{O}(-1)) = \mathrm{Hom}_{X_C}(\mathcal{O}(1), \mathcal{O})$

· $\mathcal{G}_b^\circ = H^0(\mathcal{O}(-1))$ is rep'd (as grp) by the perf'd open unit disc $\{x \in R \mid |x-1| < 1\}$.

· $\dim \mathcal{G}_b = \dim \mathcal{G}_b^\circ = \langle 2\rho, \nu_b \rangle$.

$\dim(\mathrm{Bun}_G^b) = -\langle 2\rho, \nu_b \rangle$.

(3) Bun_G is a cohomologically smooth Artin stack of dim 0 and each Bun_G^b is a smooth locally closed substack of dim $-\langle 2\rho, \nu_b \rangle$.

§3 ℓ-adic sheaves on Bun_G

<u>Thm</u> For any $\mathbb{Z}_\ell$-alg Λ, there's a reasonable category

$$D(\mathrm{Bun}_G, \Lambda) \ (= D_{lis}(\mathrm{Bun}_G, \Lambda))$$

s.t. (1) $D(\mathrm{Bun}_G, \Lambda)$ has a semi-orthogonal decomposition with graded pieces

$$D(\mathrm{Bun}_G^b, \Lambda) \simeq D(\text{smooth } G_b(\mathbb{Q}_p)\text{-rep'n in } \Lambda\text{-mod}) =: D(G_b(\mathbb{Q}_p), \Lambda).$$

(2) $D(\mathrm{Bun}_G, \Lambda)$ is compactly generated and an object $A \in D(\mathrm{Bun}_G, \Lambda)$ is compact iff

$$\left(\begin{array}{l} i_b^* A = 0 \text{ for almost all } b \in B(G) \\ \text{and } \forall b,\ i_b^* A \in D(G_b(\mathbb{Q}_p), \Lambda) \text{ is generated by } c\text{-}\mathrm{Ind}_K^{G_b(\mathbb{Q}_p)}(\Lambda) \\ \quad \text{with } K \subseteq G_b(\mathbb{Q}_p) \text{ open compact pro-}p \text{ subgrp.} \end{array} \right)$$

§4 ULA-sheaves

<u>Def'n</u> $f: X \to S$ a compactifiable map of loc. spatial diamonds with $\dim = \mathrm{trdeg}\, f < \infty$. $A \in D_{\acute{e}t}(X, \Lambda)$.

(i) A is <u>f-loc acyclic</u> if

(a) $\forall\, \bar{x} \to X$ geom pt with image $\bar{s} \in S$, and $\bar{t} \in S_{\bar{s}}$,

the map $R\Gamma(X_{\bar{x}}, A) \xrightarrow{\sim} R\Gamma(X_{\bar{x}} \times_{S_{\bar{s}}} S_{\bar{t}}, A)$ isom.

Here $X_{\bar{x}} := \varprojlim_{\substack{x : \mathrm{Spec}(C) \to X \\ \searrow U \nearrow \text{étale}}} U$.

(b) $\forall$ separated étale map $j : U \to X$ s.t. $f \circ j$ quasi-compact,

$$R(f \circ j)_! (A|_U) \in D_{\acute{e}t}(S, \Lambda)$$

is perfect-constructible.

(ii) A is <u>f-ULA</u> if it is locally acyclic after arbitrary base change.

<u>Miracle</u> <u>Thm</u> An object $A \in D(\mathrm{Bun}_G, \Lambda)$ is ULA (for $\mathrm{Bun}_G \to *$)

iff $\forall\, b \in B(G)$, $i_b^* A \in D(G_b(\mathbb{Q}_p), \Lambda)$ is <u>admissible</u>

i.e. $\forall\, K \subseteq G_b(\mathbb{Q}_p)$ pro-p subgroup, open compact,

$(i_b^* A)^K$ is perfect.