Overconvergence of étale (φ,Γ)-modules and prismatic F-crystals

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__Recall__ Scholze: $\mathrm{Spa}\,\mathbb{Z}_p[[T]]$ (p,T)-adic
$$\ni x_0 = V(p,T)$$

$\rightsquigarrow$ $X = \mathrm{Spa}\,\mathbb{Z}_p[[T]] \setminus \{x_0\}$.

$\kappa : |X| \longrightarrow [0,\infty]$

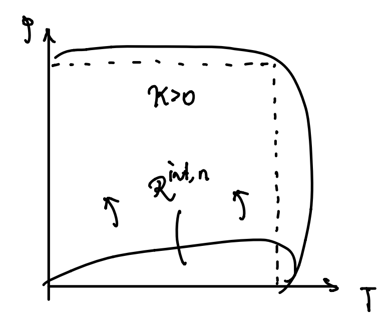
$x \longmapsto |\log T(\tilde{x})| / |\log p(\tilde{x})|$, $\quad \tilde{x} \rightsquigarrow x$ (rk 1 pt).

__Fact__ κ is conti.

__Obs__ $\{x \in |X| \mid \kappa(x) > \frac{1}{n}$ (resp. $< \frac{1}{n}$)$\}$

$$= \{x \in |X| \mid |T(\tilde{x})|^n \geq |p(\tilde{x})| \neq 0 \text{ (resp. } 0 \neq |T(\tilde{x})|^n \leq |p(\tilde{x})|)\}.$$

$$= \{x \in |X| \mid \left|\frac{T(\tilde{x})^n}{p(\tilde{x})}\right| \geq 1 \text{ resp. } \leq 1\}.$$



So $\mathbb{Z}_p\langle T, \frac{T^n}{p}\rangle[\frac{1}{p}] \;\hat{=}\; \{\kappa \geq \frac{1}{n}\}$

$\mathbb{Z}_p\langle T, \frac{p}{T^n}\rangle[\frac{1}{p}] \;\cong\; \{\kappa \leq \frac{1}{n}\}$

$=: \mathcal{R}^{int,n}$

__Def__ $\mathcal{R}^{int} := \bigcup_n \mathcal{R}^{int,n}$ integral Robba ring.

$\kappa > 0 \;\cong\;$ adic open unit disc.

__Setup__ $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p \supseteq \mathcal{O}_{\mathbb{C}_p}$, $A_{inf} := A_{inf}(\mathcal{O}_{\mathbb{C}_p})$

$\tilde{X} := \mathrm{Spa}(A_{inf}) \setminus V(p[p^\flat])$, $\quad p^\flat = (p, p^{1/p}, p^{1/p^2}, \dots) \in \mathcal{O}_{\mathbb{C}_p}^\flat$.

$[p^\flat] \in A_{inf}$.

$$\tilde{\kappa} : |\tilde{X}| \longrightarrow [0,\infty]$$
$$x \longmapsto \log|[p^b](\tilde{x})| / \log|p(\tilde{x})| .$$

$$\tilde{\kappa} \le \frac{1}{n} \quad \triangleq \quad \tilde{\mathcal{R}}^{int,n} := A_{inf}\langle \frac{p}{[p^b]^n} \rangle [\frac{1}{[p^b]}]$$

$$\rightsquigarrow \quad \tilde{\mathcal{R}}^{int} := \bigcup_n \tilde{\mathcal{R}}^{int,n}.$$

Have $\varphi \circlearrowright A_{inf}$, $\tilde{\kappa} \circ \varphi = p\tilde{\kappa} \rightsquigarrow \varphi \circlearrowright \tilde{\mathcal{R}}^{int}$.

$$\varphi \circlearrowright \mathbb{Z}_p[[T]] \hookrightarrow A_{inf}$$
$$T \longmapsto [\varepsilon]-1 . \quad \varepsilon = (1, \zeta_p, \zeta_{p^2}, \cdots) \in \mathcal{O}_{\mathbb{C}_p}^b .$$
$$\varphi(T) = (1+T)^p - 1$$
$$\rightsquigarrow \mathcal{R}^{int} \hookrightarrow \tilde{\mathcal{R}}^{int} . \qquad (*)$$

This $(*)$ is essentially due to:

<u>Lem</u> (Cherbonnier – Colmez)

$$\underbrace{\log|T(\tilde{x})| - \frac{p}{p-1}\log|[p^b](\tilde{x})|}_{\log|\frac{T}{[p^b]^{\frac{p}{p-1}}}(\tilde{x})|} = 0 \quad \text{when } \tilde{\kappa}(x) \text{ is close to } 0.$$

<u>Rmk</u> Fact: $\log|(p-[p^b])(\tilde{x})| - \log|[p^b](\tilde{x})| = 0$ when $\tilde{\kappa}(x)$ is close to 0.

<u>Realizing Galois actions</u>

$G_{\mathbb{Q}_p} := \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \circlearrowright A_{inf} \rightsquigarrow \tilde{\mathcal{R}}^{int,(n)}$

$G_{\mathbb{Q}_p}$ stabilizes each $\tilde{\mathcal{R}}^{int,(n)}$

& The action factors through $G_{\mathbb{Q}_p} \to \underset{\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)}{\overset{\shortparallel}{\Gamma}} \xrightarrow[\sim]{\chi} \mathbb{Z}_p^\times .$

<u>Def</u> $\mathrm{Mod}^{et}_{\mathcal{R}^{int}}(\varphi,\Gamma)$ (resp. $\mathrm{Mod}^{et}_{\tilde{\mathcal{R}}^{int}}(\varphi, G_{\mathbb{Q}_p})$) consists of:

objs (M, φ_M, ρ)

- M finite free mod $/ \mathcal{R}^{int}$ (resp. $\tilde{\mathcal{R}}^{int}$)
- $\varphi_M : \varphi^* M \xrightarrow{\sim} M$.
- ρ conti semilin action $\Gamma(G_{\mathbb{Q}_p}) \curvearrowright M$ commuting w/ φ_M.

Thm (Cherbonnier - Colmez)

$$\mathrm{Mod}^{\text{ét}}_{\mathcal{R}^{int}}(\varphi, \Gamma) \xrightarrow{\sim} \mathrm{Mod}^{\text{ét}}_{\tilde{\mathcal{R}}^{int}}(\varphi, G_{\mathbb{Q}_p}).$$

Rmk (1) C-C used Tate-Sen method to prove this.

(2) Berger - Colmez : Tate - Sen formulation
find imperfect Robba ring by taking la vectors.

(3) Kedlaya-Liu : A different pf,
using θ-map & defining imperfect period rings.

(4) Gao - Loyeton, Gao - Liu, etc :
similar result for (φ, τ)-mods

(5) Andreatta - Brinon :
Such result for $\mathbb{Z}_p$-loc sys,
(assuming the base admits a chart).

F - crystals

$R = \mathbb{Z}_p$. $R^{\circ}_{\mathbb{Z}_p}$:= absolute transversal prism site

$(A, I) \rightsquigarrow A/I$ is p-torsion free.

e.g. $\cdot$ $(\mathbb{Z}_p[\![q-1]\!], ([p]_q = \frac{q^p - 1}{q-1}))$, $\varphi(q) = q^p$

$\cdot$ $(A_{inf}, \ker \theta)$.

Def

$$\mathcal{O}_{\Delta} : (A, I) \longmapsto A$$

$$\mathcal{I}_{\Delta} : (A, I) \longmapsto I$$

$$\tilde{\mathcal{O}}_{\Delta} : (A, I) \longmapsto \big(\varinjlim_{\varphi} A\big)^{\wedge}_{(p,I)} =: A_{perf}$$

$$\mathcal{O}_{\mathcal{E},\Delta} : (A, I) \longmapsto A[\tfrac{1}{I}]^{\wedge}_{p}$$

$$\tilde{\mathcal{O}}_{\mathcal{E},\Delta} : (A, I) \longmapsto A_{perf}[\tfrac{1}{I}]^{\wedge}_{p} .$$

$$\tilde{\mathcal{O}}^{\dagger}_{\mathcal{E},\Delta} : (A, I) \longmapsto \bigcup_{n} A_{perf}\langle \tfrac{p}{\varphi(I)^n} \rangle [\tfrac{1}{\varphi(I)}] .$$

$$\rightsquigarrow \quad \begin{tikzcd} \mathcal{O}^{\dagger}_{\mathcal{E},\Delta} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{O}_{\mathcal{E},\Delta} \arrow[d] \\ \tilde{\mathcal{O}}^{\dagger}_{\mathcal{E},\Delta} \arrow[r] & \tilde{\mathcal{O}}_{\mathcal{E},\Delta} . \end{tikzcd}$$

Def $\mathrm{Vect}(R_{\Delta}, *)^{\varphi=1}$, $* \in \{\mathcal{O}_{\mathcal{E},\Delta}, \tilde{\mathcal{O}}_{\mathcal{E},\Delta}, \mathcal{O}^{\dagger}_{\mathcal{E},\Delta}, \tilde{\mathcal{O}}^{\dagger}_{\mathcal{E},\Delta}\}$

$\rotatebox[origin=c]{90}{$\in$}$

$(\mathcal{M}_{\Delta}, \varphi_{\mathcal{M}_{\Delta}})$ as objs

- $\mathcal{M}_{\Delta}$ is a crystal in finite perfectives* module
- $\varphi_{\mathcal{M}_{\Delta}} : \varphi^{*}\mathcal{M}_{\Delta} \xrightarrow{\sim} \mathcal{M}_{\Delta}$.

Thm (Du-Liu)

$$\begin{tikzcd} \mathrm{Vect}(R^{\circ}_{\Delta}, \mathcal{O}^{\dagger}_{\mathcal{E},\Delta})^{\varphi=1} \arrow[r, "\sim"] \arrow[d, "\wr"'] & \mathrm{Vect}(R^{\circ}_{\Delta}, \mathcal{O}_{\mathcal{E},\Delta})^{\varphi=1} \arrow[d, "\wr"] \\ \mathrm{Vect}(R^{\circ}_{\Delta}, \tilde{\mathcal{O}}^{\dagger}_{\mathcal{E},\Delta}) \arrow[r, "\sim"] & \mathrm{Vect}(R^{\circ}_{\Delta}, \tilde{\mathcal{O}}_{\mathcal{E},\Delta})^{\varphi=1} \end{tikzcd}$$

<u>pf sketch</u> R_{qrsp} := cat of quasi-regular semifectoid rings.

$\subset$ $\mathbb{Z}_p$-flat $\qquad\qquad$ $\shortparallel$ Semi-perf'd

$$\rightsquigarrow \quad \mathrm{Vect}(R^{\circ}_{\Delta}, \mathcal{O}^{\dagger}_{\mathcal{E},\Delta})^{\varphi=1} = \lim_{\substack{(\Delta_S,(d))\\ S \in R^{\circ}_{qrsp}}} \mathrm{Vect}(S^{\circ}_{\Delta}, \mathcal{O}^{\dagger}_{\mathcal{E},\Delta})^{\varphi=1}$$

S_{Δ} admits a final obj $(\Delta_S, (d))$.

$$(\mathcal{M}_{\Delta}, \varphi_{\mathcal{M}_{\Delta}}) \rightsquigarrow (\mathcal{M}_{\hat{R}}^{int}, \varphi_{\mathcal{M}_{\hat{R}}^{int}}, f^{\dagger})$$

- $\mathcal{O}^{\dagger}_{\mathcal{E}_{\Delta}}(A_{\inf}, \ker\theta) = \tilde{R}^{int}$
  $\quad\quad\quad\quad\quad\quad \overset{\parallel}{(p-[p^{\flat}])}$
- $f^{\dagger}: M_{\tilde{R}^{int}} \otimes_{\tilde{R}^{int}, p_1} B^{\dagger} \xrightarrow{\sim} M_{\tilde{R}^{int}} \otimes_{\tilde{R}^{int}, p_2} B^{\dagger}$
  cocycle condition of $B^{\dagger}$-mods
  $B^{\dagger} := \mathcal{O}^{\dagger}_{\mathcal{E},\Delta}(\Delta_{\mathcal{O}_{\mathbb{C}_p}} \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{C}_p}, (d))$.

$\Rightarrow$ $\quad \mathrm{Vect}(R^{\circ}_{\Delta}, \mathcal{O}_{\mathcal{E}_{\Delta}})^{\varphi=1} = \{(M_{W(\mathcal{O}^{\flat}_{\mathbb{C}_p})}, \varphi, f)\}$

$(M_{W(\mathcal{O}^{\flat}_{\mathbb{C}_p})}, \varphi)$ always descends to $\tilde{R}^{int}$

$\rightsquigarrow (T \otimes_{\mathbb{Z}_p} \tilde{R}^{int}, \varphi = \mathrm{id} \otimes \varphi)$

So reduces to $\mathcal{O}^{\dagger}_{\mathcal{E}_{\Delta}}(\Delta_{\mathcal{O}_{\mathbb{C}_p}} \hat{\otimes} \mathcal{O}_{\mathbb{C}_p}, d)^{\varphi=1} = \mathcal{O}_{\mathcal{E},\Delta}(\Delta_{\mathcal{O}_{\mathbb{C}_p}} \hat{\otimes} \mathcal{O}_{\mathbb{C}_p}, d)^{\varphi=1}$

$= C^{0}(G_{\mathbb{Q}_p}, \mathbb{Z}_p)$. $\quad\quad \square$

<u>Rmk</u> The pf works for $\mathfrak{X}$ (locally $= \mathrm{Spf}\, R$).

R s.t. R admits a quasi-syntomic covering

by a perfectoid ring S

s.t. all local systems are trivial on $\mathrm{Spf}(S)_{\eta}$.

(R noeth "$\approx$" R regular domain).