Notes for *McGill lectures* by Patrick Allen

MODULARITY LIFTING

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Contents

1. Introduction

1.1. Galois representations attached to modular forms.

Setups. We make the most general statement of the course. Fix an integer $N \geqslant 1$ and define the following arithmetic subgroup of $\mathrm{SL}_2(\mathbb{Z})$ by

$$\Gamma_1(N) := \left\{ X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}) \,\middle|\, \det X = 1,\ X \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}.$$

For another integer $k \geqslant 1$, let $S_k(\Gamma_1(N), \mathbb{C})$ denote the space of cusp forms of weight k over $\mathbb{C}$ of level $\Gamma_1(N)$. Take a cusp form $f \in S_k(\Gamma_1(N), \mathbb{C})$ that is assumed to be a common eigenform for all Hecke operators T_ℓ and $\langle \ell \rangle$, where $\ell \nmid N$ is an arbitrary prime away from N. Write

$$T_\ell f = a_\ell f, \quad \langle \ell \rangle f = \chi(\ell) f;$$

here $a_\ell \in \mathbb{C}$ are eigenvalues with ℓ varying and $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}$ is a character. Fix an isomorphism $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$, and by abuse of notations, write $a_\ell = \iota(a_\ell)$ and $\chi = \iota \circ \chi$.

Remark 1.1. The choice of ι is not as brutal as it may seem. In fact, the theorem below only depends on the prime above p induced by ι in the field $\mathbb{Q}(a_\ell, \chi(\ell) : \ell \nmid N) \subset \mathbb{C}$, which is known to be finite over $\mathbb{Q}$. However, it is often convenient just to fix $\mathbb{C} \to \overline{\mathbb{Q}}_p$ once and for all and be done with it. For example, while dealing with multiple modular forms, this statement makes sense.

Denote $G_F := \mathrm{Gal}(\overline{F}/F)$ for the absolute Galois group of a number field F.

Theorem 1.2. *Fix the eigenform $f \in S_k(\Gamma_1(N), \mathbb{C})$ and the prime $\ell \nmid N$ as before. Choose* $\mathrm{Frob}_\ell : x \mapsto x^{-\ell}$ *for $x \in \overline{\mathbb{Q}}$ the geometric Frobenius element in the absolute Galois group $G_\mathbb{Q}$. Then there is a unique semisimple continuous Galois representation $\rho_f : G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ such that*

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(1) *for each prime $\ell \nmid Np$, ρ_f is unramified at ℓ and the characteristic polynomial of $\rho_f(\mathrm{Frob}_\ell)$ is $X^2 - a_\ell X + \chi(\ell)\ell^{k-1}$; and*
(2) *ρ_f is potentially semistable at p.*

Remark 1.3. (1) It is a consequence of Chebotarev density theorem that $\det \rho_f = \chi\epsilon_p^{1-k}$ with $\epsilon_p : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ the p-adic cyclotomic character.[1] In particular, if $c \in G_{\mathbb{Q}}$ is a choice of complex conjugacy, then since $\chi(-1) = (-1)^k$, we obtain $\det_{\rho_f}(c) = -1$; namely, ρ_f is an *odd Galois representation*.
(2) The notion of "potentially semistable" will be made precise later.

1.2. **Fontaine-Mazur theory and modularity lifting.**

Conjecture 1.4 (Fontaine-Mazur). *Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be a continuous irreducible representation that is unramified outside finitely many primes and is potentially semistable at p. Assume there is no $i \in \mathbb{Z}$ such that $\rho \otimes \epsilon_p^i$ is an even (i.e. not odd) representation factoring through a finite order quotient of $G_{\mathbb{Q}}$. Then there is $j \in \mathbb{Z}$ such that $\rho \otimes \epsilon_p^i \cong \rho_f$ for some cuspidal eigenform f.*

Remark 1.5. This conjecture is almost completely proven. In particular, it is known if ρ has regular Hodge–Tate weights and $p \geqslant 5$.

Conjecture 1.6 (Fontaine-Mazur-Langlands)**.** *Let F be a number field and denote $G_F := \mathrm{Gal}(\overline{F}/F)$. Suppose $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ is a continuous irreducible representation that is unramified at all but finitely many places of F and potentially semistable at all places above p. Then $\rho \cong \rho_{\pi,\iota}$ for some cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_f)$ and some chosen isomorphism $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$.*

We are particularly interested in the conjecture of Fontaine-Mazur-Langlands because of the following reasons:
(1) Philosophically, this is a non-abelian class field theory.
(2) Currently it is our only way to understand analytic properties (e.g. the analytic continuation) of arithmetic L-functions.

Example 1.7. Conjecture 1.4 (resp. Conjecture 1.6) morally implies the modularity, and hence analytic continuation, of the L-functions attached to elliptic curves over $\mathbb{Q}$ (resp. over a general number field F).

- If $F = \mathbb{Q}$ or F is real quadratic, this is known.
- If F is an imaginary quadratic field, then it is only known that a positive proportion of elliptic curves over F are of modularity.
- The mixed signature case where, for instance, $F = \mathbb{Q}(\sqrt[3]{2})$, is hopeless at present.

The known right track to prove Conjecture 1.4 (resp. Conjecture 1.6) is via two steps. Can first assume the following diagram is commutative; that is, to assume the continuous Galois representation ρ factors through $\mathrm{GL}_2(\overline{\mathbb{Z}}_p)$ and canonically becomes $\overline{\rho}$ by modulo $\mathfrak{m}_{\overline{\mathbb{Z}}_p}$.

$$\begin{tikzcd}
G_{\mathbb{Q}} \arrow[r, "\rho"] \arrow[dr, "\overline{\rho}"'] & \mathrm{GL}_2(\overline{\mathbb{Z}}_p) \subset \mathrm{GL}_2(\overline{\mathbb{Q}}_p) \arrow[d, "\mathrm{mod}\ \mathfrak{m}_{\overline{\mathbb{Z}}_p}"] \\
& \mathrm{GL}_2(\overline{\mathbb{F}}_p)
\end{tikzcd}$$

[1]Let F be a number field. Recall the definition for $\epsilon : G_F \to \mathbb{Z}_p^\times$ as follows. For $\sigma \in G_F$ and ζ_p the pth root of unity, it is such that $\sigma(\zeta_p) = \zeta_p^{\epsilon_p(\sigma)}$. In fact, ϵ_p is a continuous (and even algebraic) homomorphism that is unramified away from p. Furthermore, for any place $v \mid p$, we have $\epsilon_p(\mathrm{Frob}_v) = 1/\#k_v$.

Here $\overline{\mathbb{F}}_p$ is the residue field of the local ring of integers $\overline{\mathbb{Z}}_p$.

- ◇ (Residual modularity step) To prove that $\overline{\rho} \cong \overline{\rho}_g$ for some modular form g. This is due to the theory of residual modularity and is strongly relevant to Serre's conjecture.
- ◇ (Modularity lifting step) To prove that if $\overline{\rho} \cong \overline{\rho}_g$ for some modular form g, then $\rho \cong \rho_f$ for some modular form f. This is the so-called modularity lifting and it is also the crucial step in the proof of Fermat's Last Theorem by Andrew Wiles.

This course notes is dedicated to proving the second modularity lifting step. To be more precise, the second step can be divided into three sub-steps.

(a) Construct two $\mathbb{Z}_p$-algebras R and $\mathbb{T}$ to attain the correspondences

$$\left\{\begin{matrix}\mathbb{Z}_p\text{-algebra homomorphisms}\\ \mathbb{T} \longrightarrow \overline{\mathbb{Q}}_p\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Hecke eigensystems in a space of}\\ \text{modular forms that } \rho \text{ should arise from}\end{matrix}\right\},$$

$$\left\{\begin{matrix}\mathbb{Z}_p\text{-algebra homomorphisms}\\ R \longrightarrow \overline{\mathbb{Q}}_p\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Galois representations that}\\ \text{should conjecturally arise from}\\ \text{the above space of modular forms}\end{matrix}\right\}.$$

(b) Construct a $\mathbb{Z}_p$-algebraic map $R \to \mathbb{T}$.

(c) Show that the map in (b) is an isomorphism; or at least show that it induces an isomorphism $R^{\mathrm{red}} \cong \mathbb{T}^{\mathrm{red}}$.

Proposed Outline. For this course, to understand the whole proof of modularity lifting, we are to introduce the background preliminaries.

(1) Deformation theory and minimal modularity lifting for $\mathrm{GL}_2(\mathbb{Q})$.

(2) That of $\mathrm{GL}_2(F)$ for F totally real, and then for the non-minimal case. (Maybe a little higher-rank conjugate self-dual stuff included.)

(3) That of $\mathrm{GL}_2(F)$ for F a CM field.

2. Hecke-algebraically valued Galois representations for $\mathrm{GL}_2(\mathbb{Q})$

Setups. Fix a prime p and integers $k \geqslant 2$, $N \geqslant 4$.

- Fix a finite extension E of $\mathbb{Q}_p$ with $\mathcal{O}$ its ring of integers and ϖ the uniformizer. Denote the residue field $\mathbb{F} = \mathcal{O}/(\varpi)$ of size $q = p^f = |\mathbb{F}|$.
- Consider the arithmetic subgroup $\Gamma = \Gamma_1(N)$ of $\mathrm{SL}_2(\mathbb{Z})$.
- Let S be a finite set of "bad" primes that contains $\{p, \infty\} \cup \{q : q \mid N\}$. Denote $\mathbb{Q}_S$ the maximal extension of $\mathbb{Q}$ in some separable closure $\overline{\mathbb{Q}}$ that is unramified outside S. Let $G_{\mathbb{Q},S} = \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$ be the corresponding Galois group.
- Fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ and hence it leads to the isomorphism of the space of k-cusp forms $S_k(\Gamma, \overline{\mathbb{Q}}_p) \cong S_k(\Gamma, \mathbb{C})$; by abuse of notation, this isomorphism is again denoted by ι.

In this section, we work with local information, i.e. with Galois representations towards $\mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ instead of $\mathrm{GL}_2(\mathbb{Q})$ directly. However, the most important sort "Galois representation for $\mathrm{GL}_2(\mathbb{Q})$ arises from considering the double-coset actions $[\Gamma\alpha\Gamma]$ of Hecke operators induced from some $\alpha \in \mathrm{GL}_2(\mathbb{Q})$.

2.1. **The Hecke action and its eigensystems.**

Notation 2.1 (Universal Hecke algebras).

(1) The *universal Hecke algebra* is defined to be the following polynomial algebra

$$\mathbb{T}^{S,\mathrm{univ}} := \mathbb{Z}[T_\ell, S_\ell : \ell \notin S].$$

Equivalently, it is attained by adding T_ℓ and S_ℓ on $\mathbb{Z}$ for those $\ell \nmid pN$ and $\ell \neq \infty$.

(2) If A is a commutative ring, denote

$$\mathbb{T}_A^{S,\mathrm{univ}} := \mathbb{T}^{S,\mathrm{univ}} \otimes A.$$

(3) If M is a $\mathbb{T}_A^{S,\mathrm{univ}}$-module, define

$$\mathbb{T}_A^S(M) = \mathbb{T}^S(M) := \mathrm{im}(\mathbb{T}_A^{S,\mathrm{univ}} \to \mathrm{End}_A(M)).$$

The subscript A is omitted whenever there is no possible confusion.

Given the arithmetic level subgroup Γ, the space of complex cusp forms of weight k carries the action of universal Hecke algebras. More precisely, $\mathbb{T}_{\mathbb{C}}^{S,\mathrm{univ}}$ acts on $S_k(\Gamma, \mathbb{C})$ by

$$T_\ell = \left[\Gamma \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \Gamma\right], \quad S_\ell = \left[\Gamma \begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix} \Gamma\right] = \ell^{k-2}\langle \ell \rangle.$$

Moreover, $S_k(\Gamma, \mathbb{C})$ is a semisimple $\mathbb{T}_{\mathbb{C}}^{S,\mathrm{univ}}$-module whose valuation is given by

$$\mathbb{T}^S(S_k(\Gamma, \mathbb{C})) \cong \prod_{\text{eigensystems}} \mathbb{C}.$$

On the other hand, by fixing $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ we determine a p-adic embedding, and hence

$$\mathbb{T}^S(S_k(\Gamma, \overline{\mathbb{Q}}_p)) \cong \prod_{\text{eigensystems}} \overline{\mathbb{Q}}_p.$$

The two products above run over the eigensystems of the Hecke actions. The only upshot is that each eigensystem can be interpreted as a homomorphism of $\overline{\mathbb{Z}}_p$-algebras $\lambda : \mathbb{T}^S(S_k(\Gamma, \overline{\mathbb{Q}}_p)) \to \overline{\mathbb{Q}}_p$. So we virtually have a decomposition of the universal Hecke algebra action. It appears that, for each λ, there is a Galois representation

$$\rho_\lambda : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

such that for each "good" prime $\ell \notin S$, the characteristic polynomial of $\rho_\lambda(\mathrm{Frob}_\ell)$ equals $X^2 - \lambda(T_\ell)X + \ell\lambda(S_\ell)$, where $\lambda(T_\ell)$ and $\lambda(S_\ell)$ denote the $\overline{\mathbb{Q}}_p$-eigenvalues of T_ℓ and S_ℓ, respectively. So the Galois representation with avoidance S can be glued up via

$$\rho = \prod_\lambda \rho_\lambda : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{T}^S(S_k(\Gamma, \overline{\mathbb{Q}}_p))$$

such that for each $\ell \notin S$, the characteristic polynomial of $\rho(\mathrm{Frob}_\ell)$ truly equals $X^2 - T_\ell X + \ell S_\ell$. (Reminder: this is essentially a consequence of Theorem 1.6, which reveals the possibility to coarsely construct a correspondence between $\overline{\mathbb{Z}}_p$-algebra homomorphisms to eigensystems with values in $\overline{\mathbb{Q}}_p$.)

Goal. We are to construct such a Galois representation of an integral version, that is, a Galois representation arising with $\overline{\mathbb{Z}}_p$-coefficients instead.

2.2. **Eichler–Shimura relation: a geometric aspect of modular forms.**

Theorem 2.2 (Eichler–Shimura)**.** *For $k \geqslant 2$, there is an isomorphism of $\mathbb{T}_{\mathbb{C}}^{S,\mathrm{univ}}$-modules*

$$M_k(\Gamma, \mathbb{C}) \oplus \overline{S_k(\Gamma, \mathbb{C})} \cong H^1(\Gamma, \mathrm{Sym}^{k-2}\,\mathbb{C}^2).$$

We give some interpretation on this relation. Fix $\alpha \in \mathrm{GL}_2(\mathbb{Q})$. Note that the level subgroup Γ restricts to $\Gamma \cap \alpha^{-1}\Gamma\alpha$. The action of a double-coset operator $[\Gamma\alpha\Gamma]$ on $H^i(\Gamma, \mathrm{Sym}^{k-2}\,\mathbb{C}^2)$ is the composite

$$\begin{aligned} H^i(\Gamma, \mathrm{Sym}^{k-2}\,\mathbb{C}^2) &\xrightarrow{\mathrm{res}} H^i(\Gamma \cap \alpha^{-1}\Gamma\alpha, \mathrm{Sym}^{k-2}\,\mathbb{C}^2) \\ &\xrightarrow{\alpha_*} H^i(\alpha\Gamma\alpha^{-1} \cap \Gamma, \mathrm{Sym}^{k-2}\,\mathbb{C}^2) \\ &\xrightarrow{\mathrm{cores}} H^i(\Gamma, \mathrm{Sym}^{k-2}\,\mathbb{C}^2). \end{aligned}$$

The Eichler–Shimura isomorphism can be interpreted geometrically by considering the case where $k = 2$. We obtain for the modular curve $Y(\Gamma) = \Gamma\backslash\mathbb{H}$ with $N \geqslant 4$ that

$$H^1(\Gamma, \mathbb{C}) \cong H^1(Y(\Gamma), \mathbb{C}).$$

Consequently, the action of $[\Gamma\alpha\Gamma]$ inherits to the right hand side by considering

$$\begin{tikzcd} & Y(\Gamma \cap \alpha^{-1}\Gamma\alpha) \arrow[dl, "\pi_1"'] & Y(\alpha\Gamma\alpha^{-1} \cap \Gamma) \arrow[l, "\alpha"'] \arrow[dr, "\pi_2"] & \\ Y(\Gamma) & & & Y(\Gamma) \end{tikzcd}$$

where the action is given by the composite $\pi_{2,*} \circ \alpha^* \circ \pi_1^*$. Moreover, this geometric isomorphism holds for other coefficients:

$$H^1(\Gamma, \mathrm{Sym}^{k-2}\,\mathbb{C}^2) \cong H^1(\Gamma, \mathrm{Sym}^{k-2}\,\mathbb{Z}^2) \otimes_{\mathbb{Z}} \mathbb{C}.$$

In case the abelian group $H^1(\Gamma, \mathrm{Sym}^{k-2}\,\mathbb{Z}^2)$ is a finitely generated $\mathbb{Z}$-module. And then for the $\mathbb{Z}$-module $\mathcal{O}$ of finite rank,

$$H^1(\Gamma, \mathrm{Sym}^{k-2}\,\mathcal{O}^2) \cong H^1(\Gamma, \mathrm{Sym}^{k-2}\,\mathbb{Z}^2) \otimes_{\mathbb{Z}} \mathcal{O}$$

is a finitely generated $\mathcal{O}$-module. Recall that we have fixed $\iota : \overline{\mathbb{Q}}_p \simeq \mathbb{C}$ to determine a p-adic embedding. This together with the Eichler–Shimura isomorphism show that

$$\begin{aligned} S_k(\Gamma, \mathbb{C}) &\hookrightarrow H^1(\Gamma, \mathrm{Sym}^{k-2}\,\mathbb{C}^2) \\ &\simeq H^1(\Gamma, \mathrm{Sym}^{k-2}\,\overline{\mathbb{Q}}_p^2) \\ &\cong H^1(\Gamma, \mathrm{Sym}^{k-2}\,\mathcal{O}^2) \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p. \end{aligned}$$

All these maps are $\mathbb{T}^{S,\mathrm{univ}}$-equivariant.

2.3. **The scalar extension of $\overline{\mathbb{Q}}_p$-eigensystems.** Recall that we obtain the following natural map

$$\mathbb{T}^S(H^1(\Gamma, \mathrm{Sym}^{k-2}\,\overline{\mathbb{Q}}_p^2)) \to \mathbb{T}^S(S_k(\Gamma, \overline{\mathbb{Q}}_p)) \to \overline{\mathbb{Q}}_p$$

to determine a $\overline{\mathbb{Q}}_p$-eigensystem with respect to the action of H^1. Then choose a Hecke eigenform $g \in S_k(\Gamma, \mathbb{C}) \simeq S_k(\Gamma, \overline{\mathbb{Q}}_p)$ and consider

$$\lambda_g : \mathbb{T}^S(H^1(\Gamma, \mathrm{Sym}^{k-2}\,\mathcal{O}^2)) \to \mathcal{O}.$$

This on the level of the residue field gives

$$\overline{\lambda}_g : \mathbb{T}^S(H^1(\Gamma, \mathrm{Sym}^{k-2}\,\mathcal{O}^2)) \to \mathbb{F}.$$

In the upcoming context we fix a choice of $\mathcal{O}$ and denote $\mathbb{T}^S(\Gamma,k) := \mathbb{T}^S(H^1(\Gamma, \mathrm{Sym}^{k-2}\mathcal{O}^2))$. Then $\mathfrak{m} = \ker \overline{\lambda}_g$ is a maximal ideal of the local ring $\mathbb{T}^S(\Gamma,k)$. There is a continuous map of the compact Galois group $G_{\mathbb{Q},S}$ associated to $\mathfrak{m}$ and $\overline{\lambda}_g$, say

$$\overline{\rho}_{\mathfrak{m}} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$$

such that for all $\ell \notin S$, the characteristic polynomial of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_\ell)$ equals

$$X^2 - \overline{\lambda}_g(T_\ell)X + \ell\overline{\lambda}_g(S_\ell) \equiv X^2 - T_\ell X + \ell S_\ell \mod \mathfrak{m}.$$

Definition 2.3. The maximal ideal $\mathfrak{m}$ is *non-Eisenstein* if $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible.

Proposition 2.4. *If $\mathfrak{m}$ is non-Eisenstein, then the localized algebra $H^1(\Gamma, \mathrm{Sym}^{k-2}\mathcal{O}^2)_{\mathfrak{m}}$ at $\mathfrak{m}$ is a finite free $\mathcal{O}$-module.*

Recall that by definition, $\mathbb{T}(\Gamma,k)_{\mathfrak{m}} \subset \mathrm{End}_{\mathcal{O}}(H^1(\Gamma, \mathrm{Sym}^{k-2}\mathcal{O}^2)_{\mathfrak{m}})$. This deduces the consequence of the proposition as below.

Corollary 2.5. *Whenever $\mathfrak{m}$ is non-Eisenstein, the local algebra $\mathbb{T}(\Gamma,k)_{\mathfrak{m}}$ is $\mathcal{O}$-flat.*

Proof of Proposition 2.4 with $k=2$. We sketch the idea of the proof only. It can be first shown that $H^1(\Gamma,\mathcal{O})_{\mathfrak{m}}$ is p-torsion-free. After fixing a uniformizer ϖ of $\mathcal{O}$, we obtain a natural short exact sequence

$$0 \to \mathcal{O} \xrightarrow{\varpi} \mathcal{O} \to \mathbb{F} \to 0.$$

Taking the cohomology groups $H^i(\Gamma,-)$ and localizing them at $\mathfrak{m}$, we get

$$\cdots \to H^0(\Gamma,\mathbb{F})_{\mathfrak{m}} \to H^1(\Gamma,\mathcal{O})_{\mathfrak{m}} \xrightarrow{\varpi} H^1(\Gamma,\mathcal{O})_{\mathfrak{m}} \to \cdots.$$

To show that $H^1(\Gamma,\mathcal{O})_{\mathfrak{m}}$ is a finite free $\mathcal{O}$-module, it suffices to check whether $H^0(\Gamma,\mathbb{F})_{\mathfrak{m}} = 0$ or not. We consider the double-coset operator $[\Gamma\alpha\Gamma]$ for $\alpha \in \mathrm{GL}_2(\mathbb{Q})$ acting on $H^0(\Gamma,\mathbb{F})$. More precisely, the action is given by the composite

$$\begin{tikzcd}
H^0(\Gamma,\mathbb{F}) \arrow[r, "\mathrm{res}"] \arrow[d, equal] & H^0(\Gamma\cap\alpha^{-1}\Gamma\alpha,\mathbb{F}) \arrow[r, "\alpha"] \arrow[d, equal] & H^0(\alpha\Gamma\alpha^{-1}\cap\Gamma,\mathbb{F}) \arrow[r, "\mathrm{cores}"] \arrow[d, equal] & H^0(\Gamma,\mathbb{F}) \arrow[d, equal] \\
\mathbb{F} \arrow[r, "\mathrm{id}"] & \mathbb{F} \arrow[r, "\mathrm{id}"] & \mathbb{F} \arrow[r, "{[\Gamma:\alpha\Gamma\alpha^{-1}\cap\Gamma]}"] & \mathbb{F}
\end{tikzcd}$$

The right map on the bottom row denotes the scalar multiplication by the finite descent index of level subgroups $[\Gamma : \alpha\Gamma\alpha^{-1}\cap\Gamma]$. Therefore, by taking $\alpha = \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix}$ respectively, we see for any $\ell \notin S$,

- the operator T_ℓ acts on $H^0(\Gamma,\mathbb{F})$ by multiplicating $1+\ell$, and
- the operator S_ℓ has only trivial action on $H^0(\Gamma,\mathbb{F})$, i.e. by multiplicating 1.

So if $H^0(\Gamma,\mathbb{F})_{\mathfrak{m}} \neq 0$, we have

$$T_\ell = 1+\ell \bmod \mathfrak{m}, \quad S_\ell = 1 \bmod \mathfrak{m}.$$

On the other hand, this together with the Chebotarev density theorem dictates that

$$\overline{\rho}_{\mathfrak{m}} = 1 \oplus \overline{\epsilon}$$

where $\overline{\epsilon}$ is the cyclotomic character modulo p. This contradicts the assumption that $\mathfrak{m}$ is non-Eisenstein. ∎

Accordingly, we now obtain an approach

$$\mathbb{T}^S(\Gamma, k)_{\mathfrak{m}} \hookrightarrow \mathbb{T}^S(\Gamma, k)_{\mathfrak{m}} \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p = \prod_{\substack{\text{eigensystems}\\ \text{above } \mathfrak{m}}} \overline{\mathbb{Q}}_p$$

to define the (compact) Galois representation towards the universal Hecke algebra at $\mathfrak{m}$ through $\overline{\mathbb{Q}}_p$-eigensystems above $\mathfrak{m}$:

$$\rho = \prod \rho_i : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{T}^S(\Gamma, k)_{\mathfrak{m}} \otimes \overline{\mathbb{Q}}_p)$$

whose matrix algebra still carries $\overline{\mathbb{Q}}_p$-coefficients and for $\ell \notin S$, the characteristic polynomial for $\rho(\mathrm{Frob}_\ell)$ equals $X^2 - T_\ell X + \ell S_\ell \in \mathbb{T}(\Gamma, k)_{\mathfrak{m}}[X]$.

More fortunately, granting the following nontrivial result of Carayol, one would be able to descent the $\overline{\mathbb{Q}}_p$-coefficients into integers, i.e., $\overline{\mathbb{Z}}_p$-coefficients.

Theorem 2.6 (Carayol)**.** *Let A be a local ring with residue field F such that the Brauer group of F is trivial, R an A-algebra, and $A' = \prod_i A'_i \supset A$ a semi-local extension of A. That is, suppose each A'_i is a local ring with maximal ideal $\mathfrak{m}'_i$ and residue field F'_i. Assume there is an A-algebra representation*

$$\rho' = \prod_i \rho'_i : R \otimes_A A' \to M_n(A') = \prod_i M_n(A'_i)$$

such that

(1) $\mathrm{tr}\,\rho(r \otimes 1) \in A$ *for each* $r \in R$;
(2) $\overline{\rho}_i : R \otimes_A F'_i \to M_n(F'_i)$ *are all absolutely irreducible and such that* $\mathrm{tr}\,\overline{\rho}'_i(r \otimes 1) \in F$ *is independent of* i.

Then ρ' is conjugate to the scalar extension $\rho \otimes_A A'$ of some representation $\rho : R \to M_n(A)$.

In the situation of us, we take $A = \mathbb{T}^S(\Gamma, k)_{\mathfrak{m}}$, R to be the group algebra $\mathbb{T}^S(\Gamma, k)_{\mathfrak{m}}[G_{\mathbb{Q},S}]$, and F to be a finite field, with the extension $A' = \mathbb{T}^S(\Gamma, k)_{\mathfrak{m}} \otimes_{\mathcal{O}} \overline{\mathbb{Z}}_p$. Then the theorem dictates that our ρ can be extended to have $\overline{\mathbb{Z}}_p$-coefficients. Hence we attain

$$\rho = \prod \rho_i : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{T}^S(\Gamma, k)_{\mathfrak{m}} \otimes \overline{\mathbb{Z}}_p)$$

which further descends to

$$\rho_{\mathfrak{m}} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{T}^S(\Gamma, k)_{\mathfrak{m}}).$$

As desired, one still makes it valid for the characteristic polynomial for $\rho(\mathrm{Frob}_\ell)$ with $\ell \notin S$ to be equal to $X^2 - T_\ell X + \ell S_\ell \in \mathbb{T}(\Gamma, k)_{\mathfrak{m}}[X]$ again, with coefficients coming from the universal Hecke algebra arising from the cohomology modulo $\mathfrak{m}$.

3. An introduction to deformations of Galois representations

3.1. **Framed deformation for representations of profinite groups.** In this section, Γ will always denote a profinite group that we will usually assume to satisfy the following condition that is abbreviated as (Φ_p).

(Φ_p) For any open subgroup $H \subset \Gamma$, the maximal pro-p quotient of H is topologically finitely generated; or equivalently, the group $\mathrm{Hom}_{\mathrm{cont}}(H, \mathbb{F}_p)$ of continuous homomorphisms is finite.

Example 3.1. (1) If $K/\mathbb{Q}_\ell$ is a finite extension where ℓ is an arbitrary prime, then the absolute Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$ satisfies (Φ_p). In fact, any open subgroup H of G_K is topologically finitely generated.

(2) Let F be a number field and S be a finite set of places of F. Denote F_S the maximal unramified extension outside S of F in some fixed $\overline{F}$. Then $G_{F,S} = \mathrm{Gal}(F_S/F)$ satisfies condition (Φ_p). However, it is not known so far whether or not $G_{F,S}$ is topologically finitely generated.

Remark 3.2. The condition (Φ_p) is not actually necessary for what we do in the next, but it will be satisfied in the applications we care about (above the two statements in Example 3.1); omitting this condition leads to some non-noetherian rings, so we impose it for simplicity.

Notation 3.3. (1) Let CNL be the category whose objects are complete noetherian local rings A equipped with a fixed isomorphism $A/\mathfrak{m}_A \cong \mathbb{F}$, where $\mathfrak{m}_A$ is the maximal ideal of A. Any morphism in CNL is defined to be the local homomorphism $A \to B$ that is compatible with the identifications $A/\mathfrak{m}_A \cong \mathbb{F} \cong B/\mathfrak{m}_B$.
(2) Let Ar denote the full subcategory of CNL consisting of Artinian objects.
(3) Fix an object $\Lambda \in \mathsf{CNL}$. Let

$$\mathsf{CNL}_\Lambda := \text{the subcategory of } \mathsf{CNL} \text{ of } \Lambda\text{-algebras},$$
$$\mathsf{Ar}_\Lambda := \text{the subcategory of } \mathsf{Ar} \text{ of } \Lambda\text{-algebras}.$$

Remark 3.4. In fact, the categories CNL and Ar obtain the universal objects. Let $W(\mathbb{F})$ denote the ring of Witt vectors of $\mathbb{F}$. As the identification $\mathbb{F} \xrightarrow{\sim} A/\mathfrak{m}_A$ gives a CNL-morphism $W(\mathbb{F}) \to A$ for each A, we can write $\mathsf{CNL} = \mathsf{CNL}_{W(\mathbb{F})}$ and $\mathsf{Ar} = \mathsf{Ar}_{W(\mathbb{F})}$. Consequently, any object of CNL can be written as a quotient of $W(\mathbb{F})[\![x_1, \dots, x_n]\!]$ for some n. For a relative version, note that any object of CNL_Λ can be similarly written as a quotient of $\Lambda[\![x_1, \dots, x_n]\!]$ for some n.

Fix a continuous representation $\overline{\rho} : \Gamma \to \mathrm{GL}_n(\mathbb{F})$.

Definition 3.5. (1) For $A \in \mathsf{CNL}$, a *framed deformation* or a *lift* of $\overline{\rho}$ to A is a continuous homomorphism $\rho : \Gamma \to \mathrm{GL}_n(A)$ such that $\rho \bmod \mathfrak{m}_A = \overline{\rho}$, namely, the following diagram commutes:

$$\begin{tikzcd} \Gamma \arrow[r, "\rho"] \arrow[dr, "\overline{\rho}"'] & \mathrm{GL}_n(A) \arrow[d, "\bmod \mathfrak{m}_A"] \\ & \mathrm{GL}_n(\mathbb{F}) \end{tikzcd}$$

(2) Moreover, two lifts ρ and ρ' of $\overline{\rho}$ to A are *strictly equivalent* if there exists $g \in 1 + M_n(\mathfrak{m}_A) = \ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathbb{F}))$ such that $\rho' = g\rho g^{-1}$.
(3) A *deformation* of $\overline{\rho}$ to A is a strict equivalence class of lifts.

We will often abuse notations by denoting a deformation by a lift in its strict equivalence class.

Example 3.6. Recall the statement in the previous section. Fix $N \geqslant 4$. Let $\mathcal{O}$ be the ring of integers of some finite extension E of $\mathbb{Q}_p$ and $g \in S_k(\Gamma_1(N), \mathcal{O})$ be a normalized Hecke eigenform whose associated mod p Galois representation $\overline{\rho}_g : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ is absolutely irreducible (where p is a non-Eisenstein rational prime). Then if $f \in S_k(\Gamma_1(N), \mathcal{O})$ is a normalized Hecke eigenform congruent to g modulo $\mathfrak{m}_\mathcal{O}$, its p-adic Galois representation $\rho_f : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ yields a framed deformation of $\overline{\rho}_g$.[2]

Remark 3.7. In the most general case, one can replace GL_n's in the construction above with other (usually reductive) group schemes over $W(\mathbb{F})$, for which case some things become tricker.

[2]This ρ_f essentially arises from the same way as in Theorem 1.6.

Definition 3.8. (1) The *framed deformation functor* or *lifting functor* for $\overline{\rho}$ is defined to be

$$\begin{aligned} D^{\square}_{\overline{\rho}} : \mathsf{CNL} &\longrightarrow \mathsf{Sets} \\ A &\longrightarrow \{\text{framed deformations of } \overline{\rho} \text{ to } A\} \end{aligned}$$

(2) The *deformation functor* for $\overline{\rho}$ is defined to be

$$\begin{aligned} D_{\overline{\rho}} : \mathsf{CNL} &\longrightarrow \mathsf{Sets} \\ A &\longrightarrow \{\text{deformations of } \overline{\rho} \text{ to } A\} \end{aligned}$$

(3) Write $D^{\square}_{\overline{\rho},\Lambda}$ and $D_{\overline{\rho},\Lambda}$ for their respective restrictions to CNL_{Λ}. The subscript will be omitted if the notations $\overline{\rho}$ and Λ are clearly understood or fixed.

Exercise 3.9. Say a functor $F : \mathsf{CNL} \to \mathsf{Sets}$ is continuous if for every $A \in \mathsf{CNL}$, the natural map

$$F(A) \longrightarrow \varprojlim_{i} F(A/\mathfrak{m}_A^i)$$

is a bijection. Show that $D^{\square}_{\overline{\rho}}$ and $D_{\overline{\rho}}$ are continuous. As a consequence, $D^{\square}_{\overline{\rho}}$ and $D_{\overline{\rho}}$ are completely determined by their restrictions to Ar.

3.2. **Representability of the deformation functor.** Recall that a functor $F : \mathsf{CNL} \to \mathsf{Sets}$ is *representable* if there exists $R \in \mathsf{CNL}$ and an isomorphism of functors $F \cong \mathrm{Hom}_{\mathsf{CNL}}(R, -)$. And, if this is the case, then there exists a universal object $X^{\mathrm{univ}} \in F(R)$ corresponding to $\mathrm{id} \in \mathrm{Hom}_{\mathsf{CNL}}(R, R) \cong F(R)$ with the following universal property: for any $A \in \mathsf{CNL}$ and any $Y \in F(A)$, there is a unique CNL-morphism $\phi : R \to A$ such that $Y = F(\phi)(X^{\mathrm{univ}})$.

Proposition 3.10. *If Γ satisfies condition (Φ_p) then the functor $D^{\square}_{\overline{\rho}}$ is representable.*

Proof. By definition, we are to show that there exists an object $R^{\square} \in \mathsf{CNL}$ such that for each $A \in \mathsf{CNL}$, $D^{\square}_{\overline{\rho}}(A) \cong \mathrm{Hom}_{\mathsf{CNL}}(R^{\square}, A)$ is exactly the set of framed deformations of $\overline{\rho}$ to A. It suffices to show that each lifting of $\overline{\rho}$ to $\mathrm{GL}_n(A)$ is a descent of the universal framed deformation $\rho^{\square} : \Gamma \to \mathrm{GL}_n(R^{\square})$ along the unique ring homomorphism $R^{\square} \to A$. On the other hand, if $R^{\square}$ exists it must be some quotient of $W(\mathbb{F})[\![x_1, \ldots, x_t]\!]$ for some t. Recall that the Teichmüller lifting is a machine to lift a representation with $\mathbb{F}$-coefficients to that of $W(\mathbb{F})$-coefficients.

Let $H = \ker \overline{\rho}$ as an open subgroup in Γ. Let $H^{(p)}$ be its maximal pro-p quotient and let $N = \ker(H \to H^{(p)})$. Then N is stable under any automorphism of H and is called a characteristic subgroup of Γ. In particular, N is normal in Γ and any lift $\rho : \Gamma \to \mathrm{GL}_n(A)$ of $\overline{\rho}$ to $A \in \mathsf{CNL}$ factors through Γ/N since $\ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathbb{F}))$ is a pro-p subgroup. Since Γ satisfies condition (Φ_p), Γ/N is topologically finitely generated with generators $\gamma_1, \ldots, \gamma_g$ for Γ/N say. For each $1 \leqslant s \leqslant g$, let $[\overline{\rho}(\gamma_s)] \in \mathrm{GL}_n(W(\mathbb{F}))$ be the Teichmüller lift of $\overline{\rho}(\gamma_s) \in \mathrm{GL}_n(\mathbb{F})$. Let F be the free profinite group on the set $\{\gamma_1, \ldots, \gamma_g\}$. Define the continuous homomorphism

$$\begin{aligned} r : F &\longrightarrow \mathrm{GL}_n(W(\mathbb{F})[\![\{X_s^{i,j}\}_{\substack{1\leqslant s\leqslant g \\ 1\leqslant i,j\leqslant n}}]\!]) \\ \gamma_s &\longmapsto [\rho(\gamma_s)](1 + (X_s^{i,j})). \end{aligned}$$

Define I to be the ideal of $W(\mathbb{F})[\![\{X_s^{i,j}\}]\!]$ generated by all matrix entries of $r(\sigma) - 1$ as σ runs through all elements in the kernel of the canonical surjection $F \to \Gamma/N$. Setting $R^{\square} = W(\mathbb{F})[\![\{X_s^{i,j}\}]\!]/I$, the representation r truly descends to

$$\rho^{\square} : \Gamma \longrightarrow \mathrm{GL}_n(R^{\square}).$$

This completes the proof as $R^{\square}$ represents $D^{\square}_{\overline{\rho}}$ with the universal object $\rho^{\square}$. $\square$

Theorem 3.11 (Mazur). *If Γ satisfies (Φ_p) and $\mathrm{End}_{\mathbb{F}[\Gamma]}(\overline{\rho}) = \mathbb{F}$, then the functor $D_{\overline{\rho}}$ is representable.*

Remark 3.12. One way to prove Mazur's representability theorem (à la Kisin) is to take the quotient of $D^\square$ by the free action of the smooth formal group $\widehat{\mathrm{PGL}_n}$, with using some results in SGA.

Let $\mathbb{F}[\varepsilon] := F[X]/(X^2)$ denote the ring of dual numbers.

Theorem 3.13 (Grothendieck). *Let $F : \mathsf{CNL} \to \mathsf{Sets}$ be a continuous functor such that $F(\mathbb{F})$ is a singleton. Then F is representable if and only if the following two conditions hold:*

(1) *for any morphisms $A \to C$ and $B \to C$ in Ar, the natural map $F(A \times_C B) \to F(A) \times_{F(C)} F(B)$ is a bijection, and*
(2) *as an $\mathbb{F}$-vector space, $F(\mathbb{F}[\varepsilon])$ has finite dimension.*

Here comes some explanation of condition (2). Using property (1), we can define with $A = B = \mathbb{F}[\varepsilon]$ and $C = \mathbb{F}$ that

$$F(\mathbb{F}[\varepsilon]) \times_{F(\mathbb{F})} F(\mathbb{F}[\varepsilon]) \cong F(\mathbb{F}[\varepsilon] \times_{\mathbb{F}} \mathbb{F}[\varepsilon]) \xrightarrow{F(\phi)} F(\mathbb{F}[\varepsilon])$$

where $\phi(a + b\varepsilon, a + c\varepsilon) = a + (b + c)\varepsilon$. Also define the scalar multiplication by $\alpha \in \mathbb{F}$ on $\mathbb{F}[\varepsilon]$ via $F(a + b\varepsilon \mapsto a + \alpha b\varepsilon)$.

We say that a homomorphism $A \to C$ in Ar is *small* if it is surjective, and its kernel is a principal ideal annihilated by $\mathfrak{m}_A$.

Theorem 3.14 (Schlessinger's representability criterion). *Let $F : \mathsf{CNL} \to \mathsf{Sets}$ be a continuous functor such that $F(\mathbb{F})$ is a singleton. For $\alpha : A \to C$ and $\beta : B \to C$ in Ar, consider the map*

$$\phi : F(A \times_C B) \to F(A) \times_{F(C)} F(B).$$

Then F is representable if and only if the following conditions are satisfied:

(H1) *if α is small, then ϕ is surjective;*
(H2) *if $A = \mathbb{F}[\varepsilon]$ and $C = \mathbb{F}$, then ϕ is bijective;*
(H3) $\dim_{\mathbb{F}} \mathbb{F}[\varepsilon] < \infty$*;*
(H4) *if $A = B$ and $\alpha = \beta$ is small, then ϕ is bijective.*

Now we are to prove Mazur's Theorem 3.11 on representability for $D_{\overline{\rho}}$. Recall that Γ is fixed as a profinite group and $\overline{\rho} : \Gamma \to \mathrm{GL}_n(\mathbb{F})$ is a continuous representation.

Proof of Theorem 3.11. Granting Schlessinger's representability criterion, it suffices to verify (H1)–(H4). We now only work for (H1) and (H4).

(H1) Take lifts ρ_A and ρ_B of $\overline{\rho}$ to A and B such that $\alpha \circ \rho_A$ and $\beta \circ \rho_B$ are $(1 + M_n(\mathfrak{m}_C))$-conjugate. Take $g \in 1 + M_n(\mathfrak{m}_C)$ such that $g(\alpha \circ \rho_A)g^{-1} = \rho_B$. Since α is surjective, we can lift g to $h \in 1 + M_n(\mathfrak{m}_A)$. Then the pair $(h\rho_A h^{-1}, \rho_B)$ defines a lift to $A \times_C B$, and $\phi(h\rho_A h^{-1}, \rho_B) = (\rho_A, \rho_B)$.

To do (H4) we first need a lemma.

Lemma. Assume that $\mathrm{End}_{\mathbb{F}[\Gamma]}(\overline{\rho}) = \mathbb{F}$. Then for any $C \in \mathsf{CNL}$ and any lift $\rho : \Gamma \to \mathrm{GL}_n(C)$ of $\overline{\rho}$, we obtain $\mathrm{End}_{C[\Gamma]}(\rho) = C$.

The proof of the lemma is left as an exercise. The recipe is to reduce to the Artinian case and use induction on the length of C.

(H4) Take $\alpha : A \to C$ a small map in Ar. We want to show

$$\phi : D_{\overline{\rho}}(A \times_C A) \to D_{\overline{\rho}}(A) \times_{D_{\overline{\rho}}(C)} D_{\overline{\rho}}(A)$$

is injective. Take $\rho, r \in D_{\overline{\rho}}^{\square}(A \times_C A)$ such that $\phi(\rho) = \phi(r)$ as deformations. On the level of framed deformations, we write

$$\begin{aligned} \rho &\mapsto (\rho_1, \rho_2) \in D_{\overline{\rho}}^{\square}(A) \times_{D_{\overline{\rho}}^{\square}(C)} D_{\overline{\rho}}^{\square}(A), \\ r &\mapsto (r_1, r_2) \in D_{\overline{\rho}}^{\square}(A) \times_{D_{\overline{\rho}}^{\square}(C)} D_{\overline{\rho}}^{\square}(A). \end{aligned}$$

By assumption, there is $g_i \in 1 + M_n(\mathfrak{m}_A)$ such that $\rho_i = g_i \Gamma_i g_i^{-1}$ for $i = 1, 2$. Note that, as lifts to C, $\alpha \circ \rho_1 = \alpha \circ \rho_2$ and $\alpha \circ r_1 = \alpha \circ r_2$. Also, compute that

$$\begin{aligned} \alpha \circ \rho_1 &= \alpha \circ (g_1 r_1 g_1^{-1}) \\ &= \alpha(g_1) \alpha \circ r_1 \alpha(g_1)^{-1} \\ &= \alpha(g_1) \alpha \circ r_2 \alpha(g_1)^{-1} \\ &= \alpha(g_1 g_2^{-1}) \alpha \circ \rho_2 \alpha(g_1 g_2^{-1})^{-1} \\ &= \alpha(g_1 g_2^{-1}) \alpha \circ \rho_1 \alpha(g_1 g_2^{-1})^{-1}. \end{aligned}$$

Therefore, $\alpha(g_1 g_2^{-1})$ commutes with $\alpha \circ \rho_1$. By the lemma, $\alpha(g_1 g_2^{-1}) \in 1 + \mathfrak{m}_C$, which can be further lifted to some element $a \in 1 + \mathfrak{m}_A$. Multiplying g_1 by a^{-1}, one can assume $\alpha(g_1) = \alpha(g_2)$. Then $g = (g_1, g_2) \in 1 + M_n(\mathfrak{m}_{A \times_C A})$ and this equals grg^{-1}.

The condition (H2) is skipped due to Theorem 3.13 and (H3) is planned to be postponed. $\square$

Exercise 3.15. Let $\Lambda \in \mathsf{CNL}$ be an object. Assume $\mathrm{End}_{\mathbb{F}[\Gamma]}(\overline{\rho}) = \mathbb{F}$ and let $R \in \mathsf{CNL}$ be the universal object that represents the functor $D_{\overline{\rho}} : \mathsf{CNL} \to \mathsf{Sets}$. Then the restriction of $D_{\overline{\rho}}$ to CNL_Λ is represented by $R \hat{\otimes}_{W(\mathbb{F})} \Lambda$. A similar result holds for $D_{\overline{\rho}}^{\square}$ without any additional condition on $\overline{\rho}$.

Remark 3.16. If $R^{\square}$ represents $D_{\overline{\rho}}^{\square}$, then we have a natural isomorphism of functors

$$D_{\overline{\rho}}^{\square}(-) \cong \mathrm{Hom}_{\mathsf{CNL}}(R^{\square}, -).$$

In particular, if $\rho^{\square} \in D_{\overline{\rho}}^{\square}(R^{\square})$ corresponds to $\mathrm{id} \in \mathrm{Hom}_{\mathsf{CNL}}(R^{\square}, R^{\square})$, then for any $A \in \mathsf{CNL}$ and $\rho \in D^{\square}(A)$, there is a unique $\phi : R^{\square} \to A$ in CNL, inducing (with abuse of notation) that $\phi : \mathrm{GL}_n(R^{\square}) \to \mathrm{GL}_n(A)$, such that $\rho = \phi \circ \rho^{\square}$, i.e. the diagram commutes

$$\begin{tikzcd} \Gamma \arrow[r, "\rho^{\square}"] \arrow[dr, "\rho"'] & \mathrm{GL}_n(R^{\square}) \arrow[d, "\phi"] \\ & \mathrm{GL}_n(A) \end{tikzcd}$$

3.3. **Tangent spaces.** For the representation $\overline{\rho}$ landing in $\mathrm{GL}_n(\mathbb{F})$, consider the adjoint representation space $\mathrm{ad}\,\overline{\rho} := \overline{\rho} \otimes \overline{\rho}^* = \mathrm{End}_{\mathbb{F}}(\overline{\rho})$, where $(-)^*$ denotes the dual of $\mathbb{F}$-vector spaces.[3] For simplicity, let $\mathrm{ad}\,\overline{\rho} = M_n(\mathbb{F})$ with adjoint Γ-action via the conjugation, i.e. for each $\sigma \in \Gamma$ and $X \in \mathrm{ad}\,\overline{\rho}$, define

$$\sigma \cdot X = \overline{\rho}(\sigma) X \overline{\rho}(\sigma)^{-1}.$$

For the $W(\mathbb{F})$-algebra $F[\varepsilon] \in \mathsf{CNL}$, we take a lift $\rho : \Gamma \to \mathrm{GL}_n(\mathbb{F}[\varepsilon])$ of $\overline{\rho}$. For every $\sigma \in \Gamma$, write

$$\rho(\sigma) = (1 + \varepsilon c(\sigma)) \overline{\rho}(\sigma), \quad \text{for some } c(\sigma) \in M_n(\mathbb{F}).$$

[3]In fact $\mathrm{ad}\,\overline{\rho} = \mathfrak{gl}_n$. Moreover, if we replace GL_n by some other group scheme, then $\mathrm{ad}\,\overline{\rho}$ would be the Lie algebra over $\mathbb{F}$.

Then for $\sigma, \tau \in \Gamma$, to make ρ to be a homomorphism,

$$\begin{aligned}
& \rho(\sigma\tau) = \rho(\sigma)\rho(\tau) \\
\iff\quad & (1+\varepsilon c(\sigma\tau))\overline{\rho}(\sigma\tau) = (1+\varepsilon c(\sigma))\overline{\rho}(\sigma) \cdot (1+\varepsilon c(\tau))\overline{\rho}(\tau) \\
\iff\quad & c(\sigma\tau)\overline{\rho}(\sigma\tau) = c(\sigma)\overline{\rho}(\sigma)\overline{\rho}(\tau) + \overline{\rho}(\sigma)c(\tau)\overline{\rho}(\tau) \\
\iff\quad & c(\sigma\tau) = c(\sigma) + \overline{\rho}(\sigma)c(\tau)\overline{\rho}(\sigma)^{-1} \\
\iff\quad & c \in Z^1(\Gamma, \operatorname{ad}\overline{\rho}).
\end{aligned}$$

Note that the second last row is the 1-cocycle condition, and $Z^1(\Gamma, \operatorname{ad}\overline{\rho})$ is defined to be the space of 1-cocycles on Γ with coefficients in $\operatorname{ad}\overline{\rho}$. In fact, we have proved the first part of Proposition 3.18 below.

Exercise 3.17. Check that the $\mathbb{F}$-vector space structures on $D^{\square}_{\overline{\rho}}(\mathbb{F}[\varepsilon])$ and $Z^1(\Gamma, \operatorname{ad}\overline{\rho})$ agree. If $R^{\square}$ represents $D^{\square}_{\overline{\rho}}$, show this also agrees with the "tangent space" $\operatorname{Hom}_{\mathbb{F}}(\mathfrak{m}_{R^{\square}}/\mathfrak{m}^2_{R^{\square},\rho}, \mathbb{F})$.

Moreover, we can consider the representative classes of two lifts $\rho_i = (1+\varepsilon c_i)\overline{\rho} \in D^{\square}_{\overline{\rho}}(\mathbb{F}[\varepsilon])$ with $i = 1, 2$. Define the equivalence relation $\rho_1 \sim \rho_2$ if they share the same deformation (up to some conjugation). Then

$$\begin{aligned}
\rho_1 \sim \rho_2 \quad \iff\quad & \exists X \in M_n(\mathbb{F}) \text{ such that } \rho_1 = (1+\varepsilon X)\rho_2(1-\varepsilon X) \\
\iff\quad & (1+\varepsilon c_1)\overline{\rho} = (1+\varepsilon X)(1+\varepsilon c_2)\overline{\rho}(1-\varepsilon X) \\
\iff\quad & c_1\overline{\rho} = X\overline{\rho} + c_2\overline{\rho} - \overline{\rho}X \\
\iff\quad & c_1(\sigma) = c_2(\sigma) + X - \overline{\rho}(\sigma)X\overline{\rho}(\sigma)^{-1} \text{ for all } \sigma \in \Gamma \\
\iff\quad & c_1 \text{ and } c_2 \text{ define the same cohomology class in } H^1(\Gamma, \operatorname{ad}\overline{\rho}).
\end{aligned}$$

Proposition 3.18. *As $\mathbb{F}$-vector spaces,*

$$D^{\square}_{\overline{\rho}}(\mathbb{F}[\varepsilon]) \cong Z^1(\Gamma, \operatorname{ad}\overline{\rho}), \quad D_{\overline{\rho}}(\mathbb{F}[\varepsilon]) \cong H^1(\Gamma, \operatorname{ad}\overline{\rho}).$$

Corollary 3.19. *If Γ satisfies the condition (Φ_p), i.e. for any open subgroup $H \leqslant \Gamma$, the set $\operatorname{Hom}_{\mathrm{cont}}(H, \mathbb{F}_p)$ is finite, then $D_{\overline{\rho}}(\mathbb{F}[\varepsilon])$ has finite dimension as an $\mathbb{F}$-vector space.*

Proof. Let $H = \ker\overline{\rho}$. By the inflation-restriction, we obtain the exact sequence

$$0 \to \underbrace{H^1(\Gamma/H, \operatorname{ad}\overline{\rho})}_{\text{finite since } \Gamma/H \text{ is finite}} \to H^1(\Gamma, \operatorname{ad}\overline{\rho}) \to H^1(H, \operatorname{ad}\overline{\rho}) \cong \underbrace{\operatorname{Hom}_{\mathrm{cont}}(H, \mathbb{F}^{n^2})}_{\text{finite by } (\Phi_p)} \to \cdots$$

So $\dim_{\mathbb{F}} H^1(\Gamma, \operatorname{ad}\overline{\rho}) < \infty$. $\square$

Remark 3.20. Assume $\operatorname{End}_{\mathbb{F}[\Gamma]}(\overline{\rho}) = \mathbb{F}$ and let R represent the functor $D_{\overline{\rho}}$. It can be shown that

$$R \cong W(\mathbb{F})[\![x_1, \ldots, x_g]\!]/(f_1, \ldots, f_r)$$

for some relations $f_1, \ldots, f_r$ on $x_1, \ldots, x_g$. Moreover,

$$g = \dim_{\mathbb{F}} H^1(\Gamma, \operatorname{ad}\overline{\rho}), \quad f = \dim_{\mathbb{F}} H^2(\Gamma, \operatorname{ad}\overline{\rho}).$$

Conjecture 3.21 (Mazur)**.** *Suppose F is a number field and S is a finite set of places of F containing all the primes above $p\infty$. Assume $\overline{\rho} : G_{F,S} \to \mathrm{GL}_n(\mathbb{F})$ is absolutely irreducible and let R represent the functor $D_{\overline{\rho}}$. Then*

$$\operatorname{Krull}\dim R = 1 + h_1 - h_2, \quad h_i = \dim_{\mathbb{F}} H^i(G_{F,S}, \operatorname{ad}\overline{\rho}).$$

Background in Leopoldt's conjecture. Let F be a number field and p be a rational prime. For $N \gg 0$, the p-adic logarithmic map with respect to a p-adic place v of F is read as $\log_v : (1+\varpi_v^N \mathcal{O}_{F_v})^\times \to (F_v, +)$, where ϖ_v is a choice of the uniformizer of $\mathcal{O}_{F_v}$. This naturally extends to

$$\log_v : \mathcal{O}_{F_v}^\times \to F_v, \quad x \mapsto \frac{1}{M}\log_p(x^M)$$

for M that is divisible by $N \cdot \#k_v$. Then there is a product logarithmic map

$$\mathcal{O}_F^\times \to \prod_{v|p} \mathcal{O}_{F_v}^\times \to \prod_{v|p} F_v \xrightarrow{\sim} \mathbb{Q}_p^{[F:\mathbb{Q}]}.$$

It is conjectured by Leopoldt that the image of $\mathcal{O}_F$ in $\mathbb{Q}_p^{[F:\mathbb{Q}]}$ spans a subspace of rank $r_1 + r_2 - 1$, which equals to $\mathrm{rank}_{\mathbb{Z}}\, \mathcal{O}_F^\times$ as a unit group. (This is only proved when F is an abelian extension of $\mathbb{Q}$ so far.)

To better understand the statement of Mazur's conjecture, a perfect exercise is to show that whenever $n = 1$ (or morally, $\overline{\rho}$ is the trivial representation), it is equivalent to Leopoldt's conjecture. The key observation is that $\dim R$ is exactly the $\mathbb{Z}_p$-rank of Γ.

4. Deformation conditions with examples

As before, we fix a complete noetherian local algebra $\Lambda \in \mathsf{CNL}$ and the representation $\overline{\rho} : \Gamma \to \mathrm{GL}_n(\mathbb{F})$. Usually, we take $\Lambda = \mathcal{O}$, the ring of integers in some totally ramified finite extension of $W(\mathbb{F})[\frac{1}{p}]$. We are interested in studying subfunctors $D \subset D_{\overline{\rho}}$ or $D \subset D_{\overline{\rho}}^{\square}$ consisting of the deformations subject to certain conditions (which correspond to explicit number-theoretic conditions at work).

4.1. **Deformations with a fixed determinant.** Let $\mathcal{O}$ be the ring of integers of some finite extension of $\mathbb{Q}_p$ and fix a continuous character $\psi : \Gamma \to \mathcal{O}^\times$ such that $\psi \bmod \mathfrak{m}_{\mathcal{O}} = \det \overline{\rho}$.

Define the subfunctor

$$D_{\overline{\rho}}^{\square,\psi} \subset D_{\overline{\rho}}^{\square} : \mathsf{CNL}_{\mathcal{O}} \longrightarrow \mathsf{Sets}$$

to be the framed deformation functor with fixed determinant ψ, that is, for each $A \in \mathsf{CNL}_{\mathcal{O}}$, the lifting $\rho \in D_{\overline{\rho}}^{\square}(A)$ lies in $D_{\overline{\rho}}^{\square,\psi}(A)$ if and only if $\det \rho = \alpha \circ \psi$ with $\alpha : \mathcal{O} \to A$ the structure map. By abuse of notation, we denote this by $\det \rho = \psi$ and omit the α. This condition is stable under conjugations by $1 + M_n(\mathfrak{m}_A)$. At the level of representative classes, we also get a subfunctor $D_{\overline{\rho}}^{\psi} \subset D_{\overline{\rho}}$.

Proposition 4.1. (1) *If $R^\square$ represents the functor $D_{\overline{\rho}}^{\square}$, then $D_{\overline{\rho}}^{\square,\psi}$ is represented by a quotient algebra of $R^\square$, denoted by $R^{\square,\psi}$.*

(2) *If further $\mathrm{End}_{\mathbb{F}[\Gamma]}(\overline{\rho}) = \mathbb{F}$ holds, then there is some quotient algebra R^ψ of R representing $D_{\overline{\rho}}^{\psi}$.*

Proof. Let $\rho^\square : \Gamma \to \mathrm{GL}_n(R^\square)$ be the universal deformation of $\overline{\rho}$ as in the proof of Proposition 3.10. The punchline lies in describing the determinant condition for an ideal. For this, let I be the ideal of $R^\square$ generated by

$$\{\det \rho^\square(\sigma) - \psi(\sigma) : \sigma \in \Gamma\}.$$

If $\rho \in D_{\overline{\rho}}^{\square}(A)$ corresponds to $\phi : R^\square \to A$, then

$$\begin{tikzcd} \Gamma \arrow[r, "\rho^\square"] \arrow[dr, "\rho"'] & \mathrm{GL}_n(R^\square) \arrow[d, "\phi"] \\ & \mathrm{GL}_n(A) \end{tikzcd}$$

From this, we see $\det\rho = \det(\phi\circ\rho^\square) = \psi$ if and only if ϕ factors through $R^{\square,\psi} = R^\square/I$. We point out the proof of $D^\psi_{\overline{\rho}}$ is by a similar argument since the condition $\det\rho = \psi$ does not depend on the conjugacy class of ρ. □

Let $\mathrm{ad}^0\,\overline{\rho} \subset \mathrm{ad}\,\overline{\rho}$ denote the subspace of those matrices with trace 0.

Proposition 4.2. *We obtain isomorphisms of $\mathbb{F}$-vector spaces*

$$D^{\square,\psi}_{\overline{\rho}}(\mathbb{F}[\varepsilon]) \cong Z^1(\Gamma, \mathrm{ad}^0\,\overline{\rho}), \quad D^\psi_{\overline{\rho}}(\mathbb{F}[\varepsilon]) \cong \mathrm{im}(Z^1(\Gamma,\mathrm{ad}^0\,\overline{\rho}) \to H^1(\Gamma,\mathrm{ad}\,\overline{\rho})).$$

Proof. Similarly, as in Subsection 3.3, there is some $c : \Gamma \to M_n(\mathbb{F})$ such that $\rho = (1 + +\varepsilon c)\overline{\rho} \in D^\square_{\overline{\rho}}(\mathbb{F}[\varepsilon])$. Once we assume $\det\overline{\rho} = \psi$ for some fixed determinant character,

$$\begin{aligned} \rho \in D^{\square,\psi}_{\overline{\rho}}(\mathbb{F}[\varepsilon]) \quad &\Longleftrightarrow \quad \det\rho = \det\overline{\rho} = \psi \\ &\Longleftrightarrow \quad \det(1+\varepsilon c) = 1 \\ &\Longleftrightarrow \quad 1 + \varepsilon\cdot\mathrm{tr}\,c = 1 \quad \text{(by some linear algebra)} \\ &\Longleftrightarrow \quad c \in Z^1(\Gamma,\mathrm{ad}^0\,\overline{\rho}). \end{aligned}$$

This proves the first statement, and the second one follows easily. One shall be careful that in general $\mathrm{im}(Z^1(\Gamma,\mathrm{ad}^0\,\overline{\rho}) \to H^1(\Gamma,\mathrm{ad}\,\overline{\rho})) \supset H^1(\Gamma,\mathrm{ad}^0\,\overline{\rho})$. □

4.2. **The deformation problem.** We then give a general statement of the deformation problem, which is crucially the same as that in the fixed-determinant case.

Definition 4.3. By a *deformation problem*, we mean a collection of data $\mathcal{D}$ of lifts (A,ρ) to objects $A \in \mathsf{CNL}_\Lambda$ for a fixed $W(\mathbb{F})$-algebra $\Lambda \in \mathsf{CNL}$, satisfying the following conditions

(1) $(\mathbb{F},\overline{\rho}) \in \mathcal{D}$;
(2) if $(A,\rho)\in\mathcal{D}$ and $\phi : A \to B$ is a morphism in CNL_Λ, then $(B,\phi\circ\rho)\in\mathcal{D}$ as well, where we denote any Λ-algebra morphism induced by the base change $A \to B$ by ϕ with an abuse of the notation;
(3) for two morphisms $A \to C$ and $B \to C$ in Ar_Λ, if $(A,\rho_A),(B,\rho_B)\in\mathcal{D}$, then $(A\times_C B, \rho_A\times\rho_B)\in\mathcal{D}$ as well;
(4) if $\{(A_i,\rho_i)\}_{i\in I}$ is an inverse system of elements of $\mathcal{D}$ with $\varprojlim_i A_i \in \mathsf{CNL}_\Lambda$, then $(\varprojlim_i A_i, \varprojlim_i \rho_i) \in \mathcal{D}$;
(5) as a set, $\mathcal{D}$ is closed under the strict equivalence, i.e., if $(A,\rho)\in\mathcal{D}$, then for any $(A,\rho')\sim(A,\rho)$, we have $(A,\rho')\in\mathcal{D}$;[4]
(6) if $A \hookrightarrow B$ is an injection in CNL_Λ and (A,ρ) is a lift such that $(B,\rho)\in\mathcal{D}$, then $(A,\rho)\in\mathcal{D}$.

Proposition 4.4. *Let $R^\square$ represent $D^\square_{\overline{\rho}}$ on CNL_Λ, and let $R^\square \to R$ be a quotient in CNL_Λ satisfying the following property:*

($*$) *for any lift $\rho : \Gamma \to \mathrm{GL}_n(A)$ and any $g \in 1 + M_n(\mathfrak{m}_A)$, the map $R^\square \to A$ induced by ρ factors through R if and only if the map $R^\square \to A$ induced by $g\rho g^{-1}$ factors through R.*

[4]Recall that in the prototypical case, $(A,\rho)\sim(A,\rho')$ if they are differed by conjugation and are the same while restricting to $\overline{\rho}$. That is, there exists some $g \in 1 + M_n(\mathfrak{m}_A) = \ker(\mathrm{GL}_n(A)\to\mathrm{GL}_n(\mathbb{F}))$ such that $\rho' = g\rho g^{-1}$.

Then the collection of lifts factoring through R forms a deformation problem. Moreover, every deformation problem arises in this way.

Proof. The first claim is easy. Now let $\mathcal{I}$ be the set of all ideals $I \leqslant R^\square$ such that $(R^\square/I, \pi\circ\rho^\square) \in \mathcal{D}$ where $\rho^\square$ is the universal lifting and $\pi : R \to R/I$ is the canonical projection of rings. We impose conditions (1)–(6) of Definition 4.3.

- By (1), $\mathcal{I} \neq \emptyset$.
- By (2)(6), $(A, \rho) \in \mathcal{D}$ if and only if $\ker(R^\square \to A) \in \mathcal{I}$.
- By (4), $\mathcal{I}$ is closed under nested intersections.
- By (3)(4), $\mathcal{I}$ is closed under finite intersections.

Then $\mathcal{I}$ contains a minimal element J such that $J \subseteq I$ for all element $I \in \mathcal{I}$, and $R = R^\square/J$ works. Note that condition $(*)$ is satisfied by Definition 4.3(5). $\square$

Let us resume the example before.

Example 4.5 (The fixed determinant condition)**.** Suppose the natural quotient map $R^\square \to R := R^\square/J$ corresponds to a deformation problem $\mathcal{D}$. We have a subspace

$$\mathcal{L}_{\mathcal{D}} \subseteq Z^1(\Gamma, \operatorname{ad}\overline{\rho}) \cong D^\square_{\overline{\rho}}(\mathbb{F}[\varepsilon])$$

corresponding to

$$\begin{aligned} \operatorname{Hom}_{\mathbb{F}}(\mathfrak{m}_R/(\mathfrak{m}_R^2, \mathfrak{m}_\Lambda), \mathbb{F}) &\xrightarrow{\sim} \operatorname{Hom}_{\mathbb{F}}(\mathfrak{m}_{R^\square}/(\mathfrak{m}_{R^\square}^2, \mathfrak{m}_\Lambda, I), \mathbb{F}) \\ &\hookrightarrow \operatorname{Hom}_{\mathbb{F}}(\mathfrak{m}_{R^\square}/(\mathfrak{m}_{R^\square}^2, \mathfrak{m}_\Lambda), \mathbb{F}) \\ &\xrightarrow{\sim} Z^1(\Gamma, \operatorname{ad}\overline{\rho}). \end{aligned}$$

And by condition $(*)$, the subspace $\mathcal{L}_{\mathcal{D}}$ contains all coboundaries, so by gluing up the strict equivalence classes, there is a corresponding quotient space $L_{\mathcal{D}}$ of $\mathcal{L}_D$ with

$$\mathcal{L}_{\mathcal{D}} \to L_{\mathcal{D}} \subset H^1(\Gamma, \operatorname{ad}\overline{\rho}),$$

and by using a similar exact sequence in the proof of Corollary 3.19,

$$\dim_{\mathbb{F}} \mathcal{L}_{\mathcal{D}} = \dim_{\mathbb{F}} L_{\mathcal{D}} + n^2 - \dim_{\mathbb{F}} H^0(\Gamma, \operatorname{ad}\overline{\rho}).$$

4.3. **More examples at work.** Recall our setups. We have a profinite arithmetic group Γ satisfying the finiteness condition (Φ_p) for some fixed prime p, and $I \trianglelefteq \Gamma$ is a normal subgroup.[5] Let $\mathcal{O}$ denote the ring of integers in some finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F} \cong \mathcal{O}/(\varpi)$ with a choice of the uniformizer ϖ. Consider the 2-dimensional representation $\overline{\rho} : \Gamma \to \mathrm{GL}_2(\mathbb{F})$. We will drop the subscript $\overline{\rho}$ in deformation functors since it is fixed from now on.

4.3.1. *The ordinary condition.* Fix a continuous character $\psi : I \to \mathcal{O}^\times$. Assume there are two characters $\overline{\chi}_1, \overline{\chi}_2 : \Gamma \to \mathbb{F}$ such that

$$\overline{\rho} = \begin{pmatrix} \overline{\chi}_1 & * \\ 0 & \overline{\chi}_2 \end{pmatrix}$$

such that $\overline{\rho}(I) \neq 1$ and $\overline{\chi}_1|_I = 1$. Consider the *ordinary deformation problem* $\mathcal{D}^{\mathrm{ord}}$ via the functor $D^{\mathrm{ord}} = D^{\mathrm{ord}}_{\overline{\rho}}$ defined as follows:

[5] This generalizes the situation where $I = I_K$ the inertia group of some p-adic field K, and $\Gamma = G_K$ the absolute Galois group. The philosophy is that we are to tackle those I, with some wild property, whose representation is hardly known by classical theory.

$$\begin{aligned} D^{\mathrm{ord}}: \mathsf{CNL}_{\mathcal{O}} &\longrightarrow \mathsf{Sets} \\ A &\longmapsto \left\{ \rho: \Gamma \to \mathrm{GL}_2(A) \,\middle|\, \begin{array}{l} \rho \text{ is a framed deformation of } \overline{\rho} \text{ that is strictly} \\ \text{equivalent to } \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix} \text{ with } \chi_1, \chi_2 \text{ respective} \\ \text{liftings of } \overline{\chi}_1, \overline{\chi}_2 \text{ such that } \chi_1|_I = 1,\ \chi_2|_I = \psi \end{array} \right\}. \end{aligned}$$

It can be checked that by Definition 4.3, $\mathcal{D}^{\mathrm{ord}}$ is actually a deformation problem.

Recall that one way to think about the deformation condition/problem is, to regard it, as a quotient R^{ord} of the universal lifting ring $R^{\square}$ that has a conjectural invariance property. If it is proved that the functor D^{ord} is represented by a quotient R^{ord} of $R^{\square}$, then it clearly satisfies the conjectural invariance condition $(*)$. Hence we are required to show the representability of D^{ord} on CNL.

The question can be converted into a relative version. Assume that D^{ord} is represented on $\mathsf{CNL}_{\mathcal{O}}$ by R^{ord}. Then

$$D^{\mathrm{ord}}(\mathbb{F}[\varepsilon]) \subset D^{\square}(\mathbb{F}[\varepsilon]).$$

Hence there is the embedding of tangent spaces

$$\mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}_{R^{\mathrm{ord}}}/(\mathfrak{m}_{R^{\mathrm{ord}}}^2, \mathfrak{m}_{\mathcal{O}}), \mathbb{F}) \hookrightarrow \mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}_{R^{\square}}/(\mathfrak{m}_{R^{\square}}^2, \mathfrak{m}_{\mathcal{O}}), \mathbb{F}).$$

By taking the $\mathbb{F}$-vector space duality this becomes a surjective map

$$\mathfrak{m}_{R^{\square}}/(\mathfrak{m}_{R^{\square}}^2, \mathfrak{m}_{\mathcal{O}}) \to \mathfrak{m}_{R^{\mathrm{ord}}}/(\mathfrak{m}_{R^{\mathrm{ord}}}^2, \mathfrak{m}_{\mathcal{O}}) \to 0.$$

It then follows from Nakayama's lemma that the map $R^{\square} \to R^{\mathrm{ord}}$ induced by the universal lifting to R^{ord} is surjective.

Upshot. We are reduced to showing that D^{ord} is representable on $\mathsf{CNL}_{\mathcal{O}}$.

4.3.2. *The Borel condition.* Define the *Borel deformation problem* $\mathcal{D}^{\mathrm{Bor}}$ via the functor

$$\begin{aligned} D^{\mathrm{Bor}}: \mathsf{CNL}_{\mathcal{O}} &\longrightarrow \mathsf{Sets} \\ A &\longmapsto \left\{ \rho: \Gamma \to \mathbf{B}_2(A) \,\middle|\, \begin{array}{l} \rho \text{ is a continuous homomorphism that is strictly} \\ \text{equivalent to } \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix} \text{ with } \chi_1, \chi_2 \text{ respective} \\ \text{liftings of } \overline{\chi}_1, \overline{\chi}_2 \text{ such that } \chi_1|_I = 1,\ \chi_2|_I = \psi \end{array} \right\}. \end{aligned}$$

Here $\mathbf{B}_n$ denotes the Borel subgroup of GL_n (in case it consists of upper-triangular matrices over A). The point lies in that we do not need the elements of $D^{\mathrm{Bor}}(A)$ to come from some $\overline{\rho}$. Also, define

$$\begin{aligned} L: \mathsf{CNL}_{\mathcal{O}} &\longrightarrow \mathsf{Sets} \\ A &\longmapsto \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \,\middle|\, x \in \mathfrak{m}_A \right\}. \end{aligned}$$

Then there is a natural transformation between functors

$$\phi: L \times D^{\mathrm{Bor}} \to D^{\mathrm{ord}}, \quad (u, \rho) \mapsto u\rho u^{-1} \text{ for } (u, \rho) \in L(A) \times D^{\mathrm{Bor}}(A)$$

for each $A \in \mathsf{CNL}_{\mathcal{O}}$.

Claim. ϕ is an isomorphism between functors.

Granting the claim, one can show that D^{Bor} is representable by some $R^{\mathrm{Bor}} \in \mathsf{CNL}_{\mathcal{O}}$ by a similar proof for that $R^{\square}$ is representable. Moreover, we point out that L is representable by $\mathcal{O}[\![z]\!]$. As ϕ

is assume to be an isomorphism, D^{ord} is represented by

$$R^{\mathrm{ord}} = R^{\mathrm{Bor}} \hat{\otimes}_{\mathcal{O}} \mathcal{O}[\![z]\!] \cong R^{\mathrm{Bor}}[\![z]\!].$$

Proof of the Claim. It suffices to show that for each $A \in \mathsf{CNL}_{\mathcal{O}}$, the map

$$\phi : L(A) \times D^{\mathrm{Bor}}(A) \to D^{\mathrm{ord}}(A), \quad (u, \rho) \mapsto u\rho u^{-1}$$

is bijective. The surjectivity follows from the fact that any $g \in 1 + M_n(\mathfrak{m}_A)$ can be written as

$$g = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad x, b, 1-a, 1-d \in \mathfrak{m}_A.$$

To check the injectivity, if $u_1\rho_1 u_1^{-1} = u_2\rho_2 u_2^{-1}$ then $u\rho_1 u^{-1} = \rho_2$ with $u = u_2^{-1}u_1 \in L(A)$. We want that $u = 1$ or at least that u is upper-triangular. This follows from:

Subclaim. If $\rho \in D^{\mathrm{Bor}}(A)$ and $g \in 1 + M_2(\mathfrak{m}_A)$ are such that $g\rho g^{-1} \in D^{\mathrm{Bor}}(A)$, then g is upper-triangular.

One can reduce it to the case where A is Artinian, and then assume there is some i such that $\mathfrak{m}_A^i = 0$. Write

$$\rho(\sigma) = \begin{pmatrix} \chi_1(\sigma) & b(\sigma) \\ 0 & \chi_2(\sigma) \end{pmatrix}, \quad \sigma \in \Gamma.$$

Note that if $\sigma \in I$, then $\chi_1(\sigma) = 1$ and $\chi_2(\sigma) \in \psi(\sigma)$. By induction, can assume $\mathfrak{m}_A^{i+1} = 0$ and $g \in 1 + X$ with $X \in M_n(\mathfrak{m}_A^i)$. Then we take $X = \begin{pmatrix} u & v \\ x & y \end{pmatrix}$. Compute for $\sigma \in I$ that

$$\begin{aligned} g\rho(\sigma)g^{-1} &= \left(1 + \begin{pmatrix} u & v \\ x & y \end{pmatrix}\right) \begin{pmatrix} 1 & b(\sigma) \\ 0 & \psi(\sigma) \end{pmatrix} \left(1 - \begin{pmatrix} u & v \\ x & y \end{pmatrix}\right) \\ &= \begin{pmatrix} 1 - b(\sigma)x & * \\ (1-\psi(\sigma))x & * \end{pmatrix} \\ &= \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \quad \text{by assumption that } g\rho g^{-1} \in D^{\mathrm{Bor}}(A). \end{aligned}$$

Again by the assumption on $\overline{\rho}$, one can find $\sigma \in I$ such that

$$\text{either } \overline{\rho}(\sigma) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \overline{\rho}(\sigma) = \begin{pmatrix} 1 & * \\ 0 & \alpha \end{pmatrix} \text{ with some } \alpha \neq 1.$$

More or less this deduces that $b(\sigma) \in A^\times$ and $1 - \psi(\sigma) \in A^\times$. It proves that $x = 0$ and g are upper-triangular. ∎

4.3.3. *The case with $\ell = p$.* Let $\Gamma = G_K$ where K is a finite extension of $\mathbb{Q}_p$. Let $I = I_K$ be the inertia subgroup and $\psi = \epsilon_p^{1-k}$ for some $k \geqslant 2$, where ϵ_p denotes the p-adic cyclotomic character. For a variant version, consider $\mathcal{O}_K^{\times,(p)}$, the maximal pro-$p$ quotient of the unit group $\mathcal{O}_K^\times$. Take $\Lambda = \mathcal{O}[\![\mathcal{O}_K^{\times,(p)}]\!]$ with the ordinary deformation functor

$$D_\Lambda^{\mathrm{ord}} : \mathsf{CNL}_\Lambda \longrightarrow \mathsf{Sets}$$

as above. But at this time, we replace the fixed determinant $\psi : I_K \to \mathcal{O}^\times$ with $\Psi : I_K \to \Lambda^\times$, which is the universal enveloping character from the isomorphism $I_{K^{\mathrm{ab}}/K} \cong \mathcal{O}_K^\times$ of local class field theory (i.e. Ψ is the character that takes images with $\mathcal{O}$-coefficients in the maximal pro-p quotient of $I_{K^{\mathrm{ab}}/K}$.

Remark 4.6. Supposedly, one should be very careful about the setups. The subtlety here lies in that, even if we are in case $\ell = p$, the finitely generated $\mathbb{Z}_p$-algebra $\mathcal{O}$ is not necessarily the valuation ring of K. Keeping this in mind, the existence of Ψ could be particularly nontrivial.

4.3.4. *The case with* $\ell \neq p$. Let $\Gamma = G_K$ where K is a finite extension of $\mathbb{Q}_\ell$, with $\ell \neq p$. Let $I = I_K$ be the inertia group. We are to define the *minimally ramified deformation problem* $\mathcal{D}^{\min}$ of $\overline{\rho}$ in two cases, denoted by the functor $D^{\min}$.

(1) Suppose $\overline{\rho}(I)$ is non-split, i.e.

$$1 \neq \overline{\rho}(I) \subseteq \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}.$$

Taking $\psi = 1$, we get the deformation functor

$$\begin{aligned} D^{\min} : \mathsf{CNL}_{\mathcal{O}} &\longrightarrow \mathsf{Sets} \\ A &\longmapsto \left\{ \begin{array}{c} \text{framed deformations } \rho \text{ to } A \text{ such that } \rho(I) \text{ is} \\ \text{strictly equivalent to a subgroup of } \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \end{array} \right\}. \end{aligned}$$

(2) Suppose $\overline{\rho}(I)$ is split, i.e.

$$\overline{\rho} = \begin{pmatrix} \overline{\chi}_1 & 0 \\ 0 & \overline{\chi}_2 \end{pmatrix}, \quad \overline{\chi}_1|_I = 1, \quad \overline{\chi}_2|_I \neq 1.$$

We have the deformation functor

$$\begin{aligned} D^{\min} : \mathsf{CNL}_{\mathcal{O}} &\longrightarrow \mathsf{Sets} \\ A &\longmapsto \left\{ \begin{array}{c} \text{framed deformations } \rho \text{ to } A \text{ that is strictly equivalent to} \\ \left\{ \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix} \right\} \text{ with } \chi_1|_I = 1 \text{ and } \chi_2|_I \text{ given by the composite} \\ I_K \xrightarrow{\overline{\chi}_2} \mathbb{F}^\times \xrightarrow{\text{Teichmüller lifting}} \mathcal{O}^\times \to A^\times. \end{array} \right\}. \end{aligned}$$

Note that this case differs from the general example while using the determinant $\psi = \mathrm{Teich} \circ \overline{\chi}_2|_I$ as well as the structure of local Galois groups.

More generally, let $\overline{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ with K a finite extension of $\mathbb{Q}_\ell$ and $\ell \neq p$. Assume that $\overline{\rho}(I_K)$ has the a prime-to-p order as a subgroup in $\mathrm{GL}_2(\mathbb{F})$. Then there is a deformation condition

$$\begin{aligned} D^{\min} : \mathsf{CNL}_{\mathcal{O}} &\longrightarrow \mathsf{Sets} \\ A &\longmapsto \left\{ \begin{array}{c} \text{framed deformations } \rho \text{ to } A \text{ such that} \\ \rho(I_K) \xrightarrow[\text{mod } \mathfrak{m}_A]{\simeq} \overline{\rho}(I_K) \text{ is an isomorphism} \end{array} \right\}. \end{aligned}$$

This functor is also called the *minimally ramified deformation problem*.

Remark 4.7. In all the deformation problems above, one can also introduce the fixed determinants. That is, the functors $D^{\min,\psi}$ in both cases are representable by some $R^{\min,\psi}$.

Philosophy. The representations of wild inertia groups always appear to be very mysterious. There are several powerful theoretical approaches (e.g. the p-adic Hodge theory) to search for this. The modularity lifting takes less care of the explicit data about wild ramifications. A naive motivation to define $D^{\min}$ is to control that $\overline{\rho}(I_K)$ will not be more than wild along the lifting.

4.4. **Dimension of local deformation problems.** Let K be a finite extension of $\mathbb{Q}_p$ and let $\Gamma = G_K$, $I = I_K$. Consider the Galois representation

$$\overline{\rho} = \begin{pmatrix} \overline{\chi}_1 & * \\ 0 & \overline{\chi}_2 \end{pmatrix} : G_K \longrightarrow \mathrm{GL}_2(\mathbb{F})$$

satisfying that

(1) $\overline{\chi}_1|_I = 1$, $\overline{\chi}_2|_I \neq 1$;
(2) $\overline{\chi}_1\overline{\chi}_2^{-1} \neq \overline{\epsilon}_p$.

Here the second condition is some sort of non-degeneracy for a technical requirement. It shall be used to guarantee the formal smoothness of D^{ord} (cf. Proposition 4.11). Choose some determinant $\psi : I_{K^{\mathrm{ab}}/K} \to \mathcal{O}^\times$ as the lifting of $\overline{\chi}_2|_I$, and consider the conditional deformation functor $D^{\mathrm{ord}} : \mathsf{CNL}_{\mathcal{O}} \to \mathsf{Sets}$ that have been defined in Subsubsection 4.3.1. Then we obtain $\chi_1|_I = 1$ and $\chi_2|_I = \psi$. Recall from Subsubsection 4.3.2 that D^{ord} is represented by $R^{\mathrm{ord}} \in \mathsf{CNL}_{\mathcal{O}}$.

Goal. Under the above assumptions, when $\ell = p$,

$$R^{\mathrm{ord}} \cong \mathcal{O}[\![x_1, \dots, x_g]\!], \quad g = 4 + [K : \mathbb{Q}_p].$$

For this, we first list out some general facts at work for both $\ell = p$ and $\ell \neq p$ cases. Fix a prime p. Let ℓ be any prime with L a finite extension of $\mathbb{Q}_\ell$. Let V be a finite-dimensional $\mathbb{F}$-vector space with continuous G_L-action. As a reminder, $\mathbb{F}$ is a finite extension of $\mathbb{F}_p$.

Notation 4.8. Given an $\mathbb{F}$-vector space V, denote $V^* := \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ by its dual space. Let $V^*(1) = V^* \otimes \overline{\epsilon}_p$ be the dual vector space twisted by the p-adic cyclotomic character.

To run the dimension argument, it is useful to work with the Galois cohomology theory with duality.[6]

Theorem 4.9 (Local Tate duality). *For any $0 \leqslant i \leqslant 2$,*

$$H^i(G_L, V) \cong H^{2-i}(G_L, V^*(1))^*.$$

Theorem 4.10 (Local Euler characteristic formula).

$$\chi(G_L, V) := \sum_{i=0}^{2} (-1)^i \dim_{\mathbb{F}} H^i(G_L, V) = \begin{cases} 0, & \ell \neq p; \\ [L : \mathbb{Q}_p] \dim_{\mathbb{F}} V, & \ell = p. \end{cases}$$

To describe the (conditional) deformation problems, we consider the case when $V = \mathrm{ad}\,\overline{\rho}$. It turns out that there is a perfect pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathrm{ad}\,\overline{\rho} \times \mathrm{ad}\,\overline{\rho} &\longrightarrow \mathbb{F} \\ (X, Y) &\longmapsto \mathrm{tr}(XY). \end{aligned}$$

Therefore, $\mathrm{ad}\,\overline{\rho}$ is self-dual, and then $(\mathrm{ad}\,\overline{\rho})^*(1) = (\mathrm{ad}\,\overline{\rho})(1)$.

Proposition 4.11. *Under our assumption of $\overline{\rho}$, the functor D^{ord} is formally smooth. That is, for any $A \in \mathsf{Ar}$ and ideal $I \leqslant A$ such that $I^2 = 0$, the map*

$$D^{\mathrm{ord}}(A) \twoheadrightarrow D^{\mathrm{ord}}(A/I)$$

is surjective.

[6] For simplicity, we decline the language of Tate cohomology theory.

Proof. Use induction on the length of I. Firstly, one may assume that $I=(f)$ is principal and annihilated by $\mathfrak{m}_A$, so $I\cong\mathbb{F}$ as an $\mathcal{O}$-module. Without loss of generality, after taking some $\rho'\in D^{\mathrm{ord}}(A/I)$, can write

$$\rho'=\begin{pmatrix}\chi_1' & b'\\ 0 & \chi_2'\end{pmatrix},\quad \chi_1'|_I=1,\quad \chi_2'|_I=\psi,\quad b'\in Z^1(G_K,(A/I)(\chi_1'\chi_2'^{-1})).$$

We can lift $\chi_i':G_K\to(A/I)^\times$ to $\chi_i:G_K\to A^\times$ by simply lifting the Frobenius $\chi_i'(\mathrm{Frob}_K)\in(A/I)^\times$ to $A^\times$. Fix a choice of such a lift. Then it only remains to show that we can lift the cocycle to some $b\in Z^1(G_K,A(\chi_1\chi_2^{-1}))$. Since we can lift any coboundary, it suffices to show that

$$H^1(G_K,A(\chi_1\chi_2^{-1}))\to H^1(G_K,(A/I)(\chi_1\chi_2^{-1}))$$

is surjective. Notice that the map comes from the cohomology sequence

$$0\to I(\chi_1\chi_2^{-1})\to A(\chi_1\chi_2^{-1})\to (A/I)(\chi_1\chi_2^{-1})\to 0$$

of $\mathbb{F}$-vector spaces. Hence

$$\begin{aligned}&\operatorname{coker}(H^1(G_K,A(\chi_1\chi_2^{-1}))\to H^1(G_K,(A/I)(\chi_1\chi_2^{-1})))\\ &=H^2(G_K,I(\chi_1\chi_2^{-1}))\\ &\cong H^2(G_K,\mathbb{F}(\overline{\chi}_1\overline{\chi}_2^{-1}))\\ &\cong H^0(G_K,\mathbb{F}(\overline{\chi}_1^{-1}\overline{\chi}_2\overline{\epsilon}_p))^*\qquad\text{by local Tate duality.}\end{aligned}$$

However, we have assumed that $\overline{\chi}_1\overline{\chi}_2^{-1}\neq\overline{\epsilon}_p$, which implies $H^0(G_K,\mathbb{F}(\overline{\chi}_1^{-1}\overline{\chi}_2\overline{\epsilon}_p))=0$. ☐

One can further apply some commutative algebra to $\mathsf{CNL}_{\mathcal{O}}$ to get:

Corollary 4.12. *There is some integer $g\geqslant 1$ such that $R^{\mathrm{ord}}\cong\mathcal{O}[\![x_1,\dots,x_g]\!]$.*

We have already seen that g can be interpreted as "the dimension of the deformation problem", which is the dimension of the (co)tangent space over $\mathbb{F}$. Hence

$$\begin{aligned}g&=\dim_{\mathbb{F}}\mathfrak{m}_{R^{\mathrm{ord}}}/(\mathfrak{m}_{R^{\mathrm{ord}}}^2,\mathfrak{m}_{\mathcal{O}})\\ &=\dim_{\mathbb{F}}(\mathfrak{m}_{R^{\mathrm{ord}}}/(\mathfrak{m}_{R^{\mathrm{ord}}}^2,\mathfrak{m}_{\mathcal{O}}))^*\\ &=\dim_{\mathbb{F}}D^{\mathrm{ord}}(\mathbb{F}[\varepsilon]).\end{aligned}$$

Recall that $Z^1(G_K,\operatorname{ad}\overline{\rho})\cong D^{\square}_{\overline{\rho}}(\mathbb{F}[\varepsilon])\supset D^{\mathrm{ord}}(\mathbb{F}[\varepsilon])$ as vector spaces. Denote

$$H^1_{\mathrm{ord}}(G_K,\operatorname{ad}\overline{\rho})=\operatorname{im}(D^{\mathrm{ord}}(\mathbb{F}[\varepsilon])\to H^1(G_K,\operatorname{ad}\overline{\rho})).$$

Thus,

$$\begin{aligned}g&=\dim_{\mathbb{F}}H^1_{\mathrm{ord}}(G_K,\operatorname{ad}\overline{\rho})+2^2-\dim_{\mathbb{F}}H^0(G_K,\operatorname{ad}\overline{\rho})\\ &=\dim_{\mathbb{F}}H^1_{\mathrm{ord}}(G_K,\operatorname{ad}\overline{\rho})+\begin{cases}3, & \overline{\rho}\text{ non-split;}\\ 2, & \overline{\rho}\text{ split.}\end{cases}\end{aligned}$$

Define $\mathfrak{b}$ (resp. $\mathfrak{n}$) to be the subspace of upper-triangular matrices (resp. upper-triangular nilpotent matrices) in $\operatorname{ad}\overline{\rho}$. Note that both of these subspaces carry the Galois action of G_K, and that $\mathfrak{n}$ is also a subspace of $\mathfrak{b}$, satisfying $\dim_{\mathbb{F}}\mathfrak{b}=3$ and $\dim_{\mathbb{F}}\mathfrak{n}=1$. It is straightforward to construct the following composite of restrictions

$$\phi:H^1(G_K,\mathfrak{b})\to H^1(G_K,\mathfrak{b}/\mathfrak{n})\xrightarrow{\mathrm{Res}}H^1(I_K,\mathfrak{b}/\mathfrak{n}).$$

Moreover, the trace pairing $\langle\cdot,\cdot\rangle:(X,Y)\to\operatorname{tr}(XY)$ we have defined before induces the isomorphism $\mathfrak{n}^*\cong(\operatorname{ad}\overline{\rho})/\mathfrak{b}$. There is a natural exact sequence

$$0\to\mathfrak{b}\to\operatorname{ad}\overline{\rho}\to(\operatorname{ad}\overline{\rho})/\mathfrak{b}\to 0$$

of $\mathbb{F}$-vector spaces. After taking the cohomology, we obtain the map

$$\ker\phi \hookrightarrow H^1(G_K, \mathfrak{b}) \to H^1(G_K, \operatorname{ad}\overline{\rho}).$$

Proposition 4.13. *In fact,*

$$H^1_{\mathrm{ord}}(G_K, \operatorname{ad}\overline{\rho}) = \operatorname{im}(\ker\phi \to H^1(G_K, \operatorname{ad}\overline{\rho})).$$

Proof. Exercise. □

We then resume on computing $\dim_{\mathbb{F}} H^1_{\mathrm{ord}}(G_K, \operatorname{ad}\overline{\rho})$. The case $\ell \neq p$ is left as Exercise 4.14 at the end. We only tackle with $\ell = p$. Note that, as $\overline{\chi}_1 \neq \overline{\chi}_2$,

$$H^0(G_K, (\operatorname{ad}\overline{\rho})/\mathfrak{b}) = H^0(G_K, \mathbb{F}(\overline{\chi}_1^{-1}\overline{\chi}_2)) = 0.$$

So the map $H^1(G_K, \mathfrak{b}) \to H^1(G_K, \operatorname{ad}\overline{\rho})$ is injective and

$$H^1_{\mathrm{ord}}(G_K, \operatorname{ad}\overline{\rho}) \cong \ker\phi.$$

Again, taking the cohomology on $0 \to \mathfrak{n} \to \mathfrak{b} \to \mathfrak{b}/\mathfrak{n} \to 0$, we get

$$0 \to H^0(G_K, \mathfrak{n}) \to H^0(G_K, \mathfrak{b}) \to H^0(G_K, \mathfrak{b}/\mathfrak{n}) \to \cdots$$

in which

$$\dim_{\mathbb{F}} H^0(G_K, \mathfrak{b}/\mathfrak{n}) = 2, \quad H^0(G_K, \mathfrak{n}) = 0,$$

and

$$\dim_{\mathbb{F}} H^0(G_K, \mathfrak{b}) = \begin{cases} 1, & \overline{\rho} \text{ non-split;} \\ 2, & \overline{\rho} \text{ split.} \end{cases}$$

Also, on the other side,

$$\begin{array}{ccccccc} \cdots \longrightarrow & H^1(G_K, \mathfrak{b}) & \longrightarrow & H^1(G_K, \mathfrak{b}/\mathfrak{n}) & \longrightarrow & H^2(G_K, \mathfrak{n}). \\ & & & \| & & \| \\ & & & \operatorname{Hom}_{\mathrm{cont}}(G_K, \mathfrak{b}/\mathfrak{n}) & & H^0(G_K, (\operatorname{ad}\overline{\rho})/\mathfrak{b}) \\ & & & \| & & \| \\ & & & \operatorname{Hom}_{\mathrm{cont}}(K^\times, \mathbb{F})^{\oplus 2} & & 0 \end{array}$$

Therefore,

$$\begin{aligned} \dim_{\mathbb{F}} \ker\phi &= 2 + \dim_{\mathbb{F}} \ker(H^1(G_K, \mathfrak{b}) \to H^1(G_K, \mathfrak{b}/\mathfrak{n})) \\ &= 2 + \dim_{\mathbb{F}} \operatorname{im}(H^1(G_K, \mathfrak{n}) \to H^1(G_K, \mathfrak{b})) \\ &= 2 + \dim_{\mathbb{F}} H^1(G_K, \mathfrak{n}) - \dim_{\mathbb{F}} \ker(H^1(G_K, \mathfrak{n}) \to H^1(G_K, \mathfrak{b})) \\ &= 2 + \dim_{\mathbb{F}} H^1(G_K, \mathfrak{n}) - \dim_{\mathbb{F}} \operatorname{im}(H^0(G_K, \mathfrak{b}/\mathfrak{n}) \to H^1(G_K, \mathfrak{n})). \end{aligned}$$

Applying the local Euler characteristic formula (Theorem 4.10), as we are in case $\ell = p$,

$$\dim_{\mathbb{F}} H^1(G_K, \mathfrak{n}) = [K : \mathbb{Q}_p] - \dim_{\mathbb{F}} H^0(G_K, \mathfrak{n}) - \dim_{\mathbb{F}} H^2(G_K, \mathfrak{n}) = [K : \mathbb{Q}_p].$$

Finally, note that $H^0(G_K, \mathfrak{b}/\mathfrak{n})$ has trivial image in $H^1(G_K, \mathfrak{n})$ if and only if $\overline{\rho}$ splits. Hence

$$\dim_{\mathbb{F}} \operatorname{im}(H^0(G_K, \mathfrak{b}/\mathfrak{n}) \to H^1(G_K, \mathfrak{n})) = \begin{cases} 1, & \overline{\rho} \text{ non-split;} \\ 0, & \overline{\rho} \text{ split.} \end{cases}$$

To sum up all results above, we get

$$\begin{aligned} \dim_{\mathbb{F}} D^{\mathrm{ord}}(\mathbb{F}[\varepsilon]) &= 2 + [K : \mathbb{Q}_p] + \begin{cases} 3 - 1, & \overline{\rho} \text{ non-split;} \\ 2 - 0, & \overline{\rho} \text{ split.} \end{cases} \\ &= 4 + [K : \mathbb{Q}_p]. \end{aligned}$$

Exercise 4.14. Assume $\ell \neq p$ and say $L/\mathbb{Q}_\ell$ is finite. Consider the continuous homomorphism $\overline{\rho}: G_L \to \mathrm{GL}_2(\mathbb{F})$ that is either split or non-split:

(i) either $1 \neq \overline{\rho}(I_L) \subset \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$; or

(ii) $\overline{\rho} = \overline{\chi}_1 \oplus \overline{\chi}_2$ with $\overline{\chi}_1|_{I_L} = 1 \neq \overline{\chi}_2|_{I_L}$.

Let the functor $D^{\min}$ denote the minimally ramified deformation problem as we have defined in Subsubsection 4.3.4. Let $R^{\min}$ denote the corresponding representing object in $\mathsf{CNL}_{\mathcal{O}}$, where $\mathcal{O}$ is the ring of integers of some other finite extension of $\mathbb{Q}_p$. Show that

$$R^{\min} \cong \mathcal{O}[\![x_1, x_2, x_3, x_4]\!].$$

A final comment: one may find the crucial difference between this case and the $\ell = p$ case lies in the difference of the local Euler characteristic formula.

5. The global deformation problems

Setups. Fix a number field F and a prime number p. Let $S \supset \{v : v \mid p\}$ be a finite set of finite places of F containing the places above p. Denote F_S the maximal extension of F that is unramified outside $S \cup \{v : v \mid \infty\}$, with $G_{F,S} := \mathrm{Gal}(F_S/F)$. (Note that in particular, F_S is unramified away from $p\infty$.) Let $\mathcal{O}$ denote the ring of integers of another finite extension of $\mathbb{Q}_p$. Choose a maximal ideal $\mathfrak{m}_{\mathcal{O}}$ of $\mathcal{O}$ and take the residue field $\mathbb{F} = \mathcal{O}/\mathfrak{m}_{\mathcal{O}}$. Consider a continuous Galois representation $\overline{\rho}: G_{F,S} \to \mathrm{GL}_n(\mathbb{F})$. For a technical reason, we further assume that $p \nmid 2n$.

5.1. Global deformation problems with local conditions inserted.

Recall that we obtain a deformation functor

$$D_{\overline{\rho}} : \mathsf{CNL}_{\mathcal{O}} \longrightarrow \mathsf{Sets}$$

and it is representable by $R^{\mathrm{univ}}_{\overline{\rho}} \in \mathsf{CNL}_{\mathcal{O}}$ whenever $\mathrm{End}_{\mathbb{F}[G_{F,S}]}(\overline{\rho}) = \mathbb{F}$. Now we want to impose conditions. Note that for any place v of F, there is a natural transformation

$$D_{\overline{\rho}} \longrightarrow D_{\overline{\rho}_v}, \quad \overline{\rho}_v := \overline{\rho}|_{G_{F_v}},$$

going from the global to the local Galois group. More precisely, for $A \in \mathsf{CNL}_{\mathcal{O}}$, each $\rho \in D_{\overline{\rho}}(A)$ is sent to $\rho_v := \rho|_{G_{F_v}}$ by this transformation at A.

Definition 5.1 (Global deformation type)**.**

(1) A *global deformation problem* is the tuple of data $\mathcal{S} = (\overline{\rho}, S, \psi, \mathcal{O}, \{\mathcal{D}_v\}_{v\in S})$, in which
- $\psi : G_{F,S} \to \mathcal{O}^\times$ is a continuous homomorphism, and
- for each $v \in S$, $\mathcal{D}_v$ is a deformation problem for $\overline{\rho}_v$, i.e., $\mathcal{D}_v \subseteq \mathcal{D}^{\square,\psi}_{\overline{\rho}_v} \subseteq \mathcal{D}^{\square}_{\overline{\rho}_v}$ as deformation problems.

(2) A framed deformation ρ of $\overline{\rho}$ to $A \in \mathsf{CNL}_{\mathcal{O}}$ is *of type* $\mathcal{S}$ if
- ρ is unramified outside S,
- $\det \rho = \psi$, and
- $\rho_v \in \mathcal{D}_v(A)$ for each $v \in S$.

(3) A deformation is *of type* $\mathcal{S}$ if one of (and hence any of) the lift in its strict equivalence class is of type $\mathcal{S}$.

Note that a type $\mathcal{S}$ collects the information on local conditions over "bad" primes, together with a global determinant condition. It essentially provides a way to renovate some local-global compatibility, and different types provide different ways for the same subset S of places in F. For this, we get a functor

$$\begin{aligned} D_{\mathcal{S}} : \mathsf{CNL}_{\mathcal{O}} &\longrightarrow \mathsf{Sets} \\ A &\longmapsto \{\text{types of deformations of } \overline{\rho} \text{ to } A\}. \end{aligned}$$

Proposition 5.2. *If moreover the endomorphism condition* $\mathrm{End}_{\mathbb{F}[G_{F,S}]}(\overline{\rho}) = \mathbb{F}$ *holds, then the functor* $D_{\mathcal{S}}$ *is representable by a quotient* $R_{\mathcal{S}}$ *of* $R_{\overline{\rho}}^{\mathrm{univ}}$*.*

Proof. We have proved that fixing the determinant deformation problem is representable by a quotient $R_{\overline{\rho}}^{\psi}$ of $R_{\overline{\rho}}^{\mathrm{univ}}$. Choose any lift ρ in the class of the universal $R_{\overline{\rho}}^{\psi}$-valued deformation. Then for each $v \in S$, ρ_v induces a $\mathsf{CNL}_{\mathcal{O}}$-morphism

$$R_{\overline{\rho}_v}^{\square} \longrightarrow R_{\overline{\rho}}^{\psi},$$

where $R_{\overline{\rho}_v}^{\square}$ is the ring representing $D_{\overline{\rho}_v}^{\square}$. Recall that $\mathcal{D}_v \subseteq \mathcal{D}_{\overline{\rho}_v}^{\square}$ by definition, or namely, all local conditions appear from a condition on local Galois representation. Thus, as a subfunctor of $D_{\overline{\rho}_v}^{\square}$, the local functor D_v is representable by a the quotient of $R_{\overline{\rho}_v}^{\square}$, say R_v. Set

$$R_{\mathcal{S}}^{\square} = \hat{\bigotimes_{v\in S}} R_{\overline{\rho}_v}^{\square}, \quad R_{\mathcal{S}}^{\mathrm{loc}} = \hat{\bigotimes_{v\in S}} R_v.$$

Here both restricted tensor products are taken over $\mathcal{O}$-algebras. Then the functor $D_{\mathcal{S}}$ is represented by

$$R_{\mathcal{S}} := R_{\overline{\rho}}^{\psi} \otimes_{R_{\mathcal{S}}^{\square}} R_{\mathcal{S}}^{\mathrm{loc}}.$$

This quotient is independent of the choice of the lifting ρ of $\overline{\rho}$ in the class of the universal $R_{\overline{\rho}}^{\psi}$-deformations since the quotients $R_{\overline{\rho}_v}^{\square} \twoheadrightarrow R_v$ are all invariant under strict equivalence. $\square$

Technical Issue. We observe that $R_{\mathcal{S}}$ is an algebra over $R_{\mathcal{S}}^{\mathrm{loc}}$, yet in a non-canonical way. It is useful to have a variant construction of $R_{\mathcal{S}}$ that is canonically an algebra over $R_{\mathcal{S}}^{\mathrm{loc}}$.

5.2. **Relative deformation problems.** Fix a proper subset $T \subsetneq S$. Note that it may not contain the places above p.

Definition 5.3 (T-framed deformation)**.**

(1) A *T-framed lifting* of $\overline{\rho}$ to $A \in \mathsf{CNL}_{\mathcal{O}}$ is a tuple $(\rho, \{\beta_v\}_{v\in T})$ where ρ is a lift of $\overline{\rho}$ to A, and $\beta_v \in 1 + M_n(\mathfrak{m}_A) = \ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathbb{F}))$ for each $v \in T$.
(2) We say a T-framed lifting $(\rho, \{\beta_v\}_{v\in T})$ is *of type* $\mathcal{S}$ if ρ is of type $\mathcal{S}$.
(3) Two T-framed liftings $(\rho, \{\beta_v\}_{v\in T})$ and $(\rho', \{\beta_v'\}_{v\in T})$ are *strictly equivalent* if there exists some $g \in 1 + M_n(\mathfrak{m}_A)$ such that for all $v \in T$, $\rho' = g\rho g^{-1}$ and $\beta_v' = g\beta_v$.
(4) A *T-framed deformation* is a strict equivalence class of T-framed liftings.

Proposition 5.4. (1) *If* $\mathrm{End}_{\mathbb{F}[G_{F,S}]}(\overline{\rho}) = \mathbb{F}$ *or* $T \neq \emptyset$*, then the functor*

$$\begin{aligned} D_{\mathcal{S}}^{T} : \mathsf{CNL}_{\mathcal{O}} &\longrightarrow \mathsf{Sets} \\ A &\longmapsto \{T\text{-framed deformations of } \overline{\rho} \text{ to } A \text{ of type } \mathcal{S}\} \end{aligned}$$

is representable by some $R_{\mathcal{S}}^{T} \in \mathsf{CNL}_{\mathcal{O}}$*.*

(2) *If* $\mathrm{End}_{\mathbb{F}[G_{F,S}]}(\overline{\rho}) = \mathbb{F}$ *and* $T \neq \emptyset$*, then a choice of a lifting in the universal type-$\mathcal{S}$ deformation yields an isomorphism*

$$R_{\mathcal{S}}^{T} \cong R_{\mathcal{S}}[\![x_1, \dots, x_{n^2\#T-1}]\!].$$

Proof Sketch. The upshot is to observe that $n^2\#T$ is the dimension of the space of choices of $\{\beta_v\}_{v\in T}$. And the term -1 comes from simultaneously scaling each β_v by a chosen element in $1 + \mathfrak{m}_A$, which stabilizes ρ. $\square$

Notice that we have a well-defined map between deformation problems (as sets)

$$\mathcal{D}_{\mathcal{S}}^T \to \mathcal{D}_{\mathcal{S}}, \quad ((\rho, \{\beta_v\}_{v\in T}), A) \mapsto (\rho, A).$$

Also note that for any T-framed lifting $(\rho, \{\beta_v\}_{v\in T})$ in some T-framed deformation, we have a local lifting

$$\beta_v^{-1}\rho_v\beta_v : G_{F_v} \longrightarrow \mathrm{GL}_n(R_{\mathcal{S}}^T)$$

that is independent of the choice in the strict equivalence class. Therefore, by the universal property of the representability, we canonically get a morphism in $\mathsf{CNL}_{\mathcal{O}}$ that $R_v \to R_{\mathcal{S}}^T$. Gluing these local maps up over $\mathcal{O}$,

$$\hat{\bigotimes_{v\in T}} R_v = R_{\mathcal{S}}^{T,\mathrm{loc}} \longrightarrow R_{\mathcal{S}}^T.$$

5.3. **Relative tangent spaces.** The next goal is to describe (or explicitly compute) the relative cotangent space

$$\mathfrak{m}_{\mathcal{S}}/(\mathfrak{m}_{\mathcal{S}}^2, \mathfrak{m}^{T,\mathrm{loc}}).$$

Here $\mathfrak{m}_{\mathcal{S}}$ and $\mathfrak{m}^{T,\mathrm{loc}}$ are choices of the maximal ideals in $R_{\mathcal{S}}$ and $R_{\mathcal{S}}^{T,\mathrm{loc}}$, respectively.

Notation 5.5. Recall that F_S is the finite extension of F in some algebraic closure that is maximal unramified outside of S. Suppose M is an $\mathbb{F}[G_{F,S}]$-module of finite $\mathbb{F}$-dimension. Denote $C^\bullet(F_S/F, M)$ (resp. $C^\bullet(F_v, M)$) the complex of inhomogeneous cochain computing the $G_{F,S}$-cohomology (resp. G_{F_v}-cohomology) with coefficients in M.

In our situation, we will apply this to $M = \mathrm{ad}\,\overline{\rho}$ and $\mathrm{ad}^0\,\overline{\rho}$. Recall that for every $v \in S$, there is a deformation problem $\mathcal{D}_v \subseteq \mathcal{D}_{\overline{\rho}_v}^{\square,\psi}$ which has a corresponding subspace $L_v \subseteq H^1(F_v, \mathrm{ad}^0\,\overline{\rho})$. The whole picture is as follows:

$$\begin{tikzcd}
D_v(\mathbb{F}[\varepsilon]) \cong \mathcal{L}_v \arrow[d, two heads] & \subseteq & Z^1(F_v, \mathrm{ad}^0\,\overline{\rho}) \arrow[d, two heads] \subseteq C^1(F_v, \mathrm{ad}^0\,\overline{\rho}) \\
L_v & \subseteq & H^1(F_v, \mathrm{ad}^0\,\overline{\rho}).
\end{tikzcd}$$

Definition 5.6. The complex $C_{\mathcal{S},T}^\bullet(\mathrm{ad}^0\,\overline{\rho})$ is defined as

$$C_{\mathcal{S},T}^i(\mathrm{ad}^0\,\overline{\rho}) = \begin{cases} C^0(F_S/F, \mathrm{ad}\,\overline{\rho}), & i = 0; \\ C^1(F_S/F, \mathrm{ad}^0\,\overline{\rho}) \oplus \bigoplus_{v\in T} C^0(F_v, \mathrm{ad}\,\overline{\rho}), & i = 1; \\ C^2(F_S/F, \mathrm{ad}^0\,\overline{\rho}) \oplus \bigoplus_{v\in T} C^1(F_v, \mathrm{ad}^0\,\overline{\rho}) \oplus \bigoplus_{v\in S\setminus T} \dfrac{C^1(F_v, \mathrm{ad}^0\,\overline{\rho})}{\mathcal{L}_v}, & i = 2; \\ C^i(F_S/F, \mathrm{ad}^0\,\overline{\rho}) \oplus \bigoplus_{v\in S} C^{i-1}(F_v, \mathrm{ad}^0\,\overline{\rho}), & i > 2. \end{cases}$$

with the boundary maps being

$$\partial : (\phi, \{\psi_v\}_v) \longmapsto (\partial\phi, \{\phi|_{G_{F_v}} - \partial\psi_v\}_v).$$

We denote the cohomology group of this complex by $H_{\mathcal{S},T}^i(\mathrm{ad}^0\,\overline{\rho})$, with its $\mathbb{F}$-dimension $h_{\mathcal{S},T}^i(\mathrm{ad}^0\,\overline{\rho})$. Similarly, we denote

$$h^i(F_S/F, \mathrm{ad}^0\,\overline{\rho}) = \dim_{\mathbb{F}} H^i(F_S/F, \mathrm{ad}^0\,\overline{\rho}), \quad h^i(F_v, \mathrm{ad}^0\,\overline{\rho}) = \dim_{\mathbb{F}} H^i(F_v, \mathrm{ad}^0\,\overline{\rho}).$$

As a reminder, we have assumed that $p \nmid 2n$. By definition, $\mathrm{ad}^0 \overline{\rho} \subseteq \mathrm{ad}\, \overline{\rho}$ is the subspace of those matrices with trace 0. We infer that whenever $p \nmid n$, the short exact sequence

$$0 \to \mathrm{ad}^0 \overline{\rho} \to \mathrm{ad}\, \overline{\rho} \to \mathbb{F} \to 0$$

splits, and this decomposition is $G_{F,S}$-equivariant. Also, the trace pairing $\langle \cdot, \cdot \rangle|_{\mathrm{ad}^0 \overline{\rho}} : \mathrm{ad}^0 \overline{\rho} \times \mathrm{ad}^0 \overline{\rho} \to \mathbb{F}$ that $(X, Y) \mapsto \mathrm{tr}(XY)$ is perfect on $\mathrm{ad}^0 \overline{\rho}$. Thus, we obtain an isomorphism

$$(\mathrm{ad}^0 \overline{\rho})^* \xrightarrow{\sim} \mathrm{ad}^0 \overline{\rho},$$

namely, $\mathrm{ad}^0 \overline{\rho}$ is self-dual as well as $\mathrm{ad}\, \overline{\rho}$. This will be useful in the upcoming computation of the relative tangent space.

Fix the global deformation problem $\mathcal{S} = (\overline{\rho}, S, \psi, \mathcal{O}, \{\mathcal{D}_v\}_{v \in S})$.

Proposition 5.7 (Cohomology interpretation of the relative tangent space)**.**

$$\mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}_{\mathcal{S}}/(\mathfrak{m}_{\mathcal{S}}^2, \mathfrak{m}^{\mathrm{loc}}), \mathbb{F}) \cong H^1_{\mathcal{S},T}(\mathrm{ad}^0 \overline{\rho}).$$

Proof. We are to work out the computation in a relative sense. Take a T-framed lifting $(\rho, \{\beta_v\}_{v \in T})$ of $\overline{\rho}$ to $\mathbb{F}[\varepsilon] \in \mathsf{CNL}_{\mathcal{O}}$. We naively want this lifting to be of type $\mathcal{S}$ and trivial when restricted to each $v \in T$. More precisely, the desired properties are as follows:

(1) $\det \rho = \psi = \det \overline{\rho}$,
(2) for all $v \in T$, $\rho_v = \beta_v \overline{\rho}_v \beta_v^{-1}$ (i.e. $\rho_v(\sigma) = \beta_v \overline{\rho}_v(\sigma) \beta_v^{-1}$ for all $\sigma \in G_{F_v}$), and
(3) for all $v \in S \backslash T$, $\rho_v \in D_v(\mathbb{F}[\varepsilon])$.

Write $\rho = (1 + \varepsilon \phi) \overline{\rho}$, for some $\phi \in Z^1(F_S/F, \mathrm{ad}\, \overline{\rho})$. As for $v \in T$, write $\beta_v = 1 + \varepsilon \alpha_v$, with some $\alpha_v \in \mathrm{ad}\, \overline{\rho} = C^0(F_v, \mathrm{ad}\, \overline{\rho})$. Then the conditions can be equivalently rewritten as

(1') $\phi \in Z^1(F_S/F, \mathrm{ad}^0 \overline{\rho})$,
(2') for all $v \in T$, $\phi|_{G_{F_v}} = (\sigma \mapsto \alpha_v - \sigma \alpha_v \sigma^{-1}) = \partial \alpha_v$, and
(3') for all $v \in S \backslash T$, $\phi|_{G_{F_v}} \in \mathcal{L}_v$.

Recall that $\partial(\phi, \{\alpha_v\}_{v \in T}) = (\partial \phi, \{\phi|_{G_{F_v}} - \partial \alpha_v\}_{v \in T})$. Then the conditions

$$(1)(2)(3) \iff (1')(2')(3') \iff \partial(\phi, \{\alpha_v\}_{v \in T}) = 0.$$

Two such cocycles $(\phi, \{\alpha_v\}_{v \in T})$ and $(\phi', \{\alpha'_v\}_{v \in T})$ induce strictly equivalent lifts if and only if there exists $g = 1 + \varepsilon \alpha$ such that $\phi' = \phi + \partial \alpha$, and that for all $v \in T$, $\alpha'_v = \alpha_v + \alpha$; or equivalently,

$$\phi' - \phi = \mathrm{im}(\partial : C^0_{\mathcal{S},T}(\mathrm{ad}^0 \overline{\rho}) \to C^1_{\mathcal{S},T}(\mathrm{ad}^0 \overline{\rho})).$$

This is the cocycle condition in $H^1_{\mathcal{S},T}(\mathrm{ad}^0 \overline{\rho})$, which completes the proof. □

5.4. **Greenberg–Wiles formula.** It turns out that we have an exact sequence

$$\begin{array}{l}
0 \longrightarrow H^0_{\mathcal{S},T}(\mathrm{ad}^0\,\overline{\rho}) \longrightarrow H^0(F_S/F, \mathrm{ad}\,\overline{\rho}) \longrightarrow \bigoplus_{v\in T} H^0(F_v, \mathrm{ad}\,\overline{\rho}) \\
\longrightarrow H^1_{\mathcal{S},T}(\mathrm{ad}^0\,\overline{\rho}) \longrightarrow H^1(F_S/F, \mathrm{ad}^0\,\overline{\rho}) \longrightarrow \bigoplus_{v\in T} H^1(F_v, \mathrm{ad}^0\,\overline{\rho}) \oplus \bigoplus_{v\in S\setminus T} (H^1(F_v, \mathrm{ad}^0\,\overline{\rho})/L_v) \\
\longrightarrow H^2_{\mathcal{S},T}(\mathrm{ad}^0\,\overline{\rho}) \longrightarrow H^2(F_S/F, \mathrm{ad}^0\,\overline{\rho}) \longrightarrow \bigoplus_{v\in T} H^2(F_v, \mathrm{ad}^0\,\overline{\rho}) \\
\longrightarrow H^2_{\mathcal{S},T}(\mathrm{ad}^0\,\overline{\rho}) \longrightarrow 0
\end{array}$$

where remember that $\mathcal{L}_v \twoheadrightarrow L_v \subseteq H^1(F_v, \mathrm{ad}^0\,\overline{\rho})$. Moreover, note that in the first row of H^0, the middle and the right term are taken with $\mathrm{ad}\,\overline{\rho}$ rather than $\mathrm{ad}^0\,\overline{\rho}$. This is not a big issue; as $\mathrm{ad}\,\overline{\rho} = \mathrm{ad}^0\,\overline{\rho} \oplus \mathbb{F}$, we see $H^0(F_S/F, \mathrm{ad}\,\overline{\rho}) = H^0(F_S/F, \mathrm{ad}^0\,\overline{\rho}) \oplus \mathbb{F}$, and $H^0(F_v, \mathrm{ad}\,\overline{\rho}) = H^0(F_v, \mathrm{ad}^0\,\overline{\rho}) \oplus \mathbb{F}$. Taking Euler characteristics on this sequence, we get

$$\begin{aligned}
\chi_{\mathcal{S},T}(\mathrm{ad}^0\,\overline{\rho}) = 1 - \#T + \chi(F_S/F, \mathrm{ad}^0\,\overline{\rho}) - \sum_{v\in S} \chi(F_v, \mathrm{ad}^0\,\overline{\rho}) \\
+ \sum_{v\in S\setminus T} (h^0(F_v, \mathrm{ad}^0\,\overline{\rho}) - \dim_{\mathbb{F}} L_v).
\end{aligned}$$

On the other hand, we obtain the following theorems on Galois cohomology.

Theorem 5.8 (Poitou-Tate)**.** *Let M be a finite-dimensional $\mathbb{F}$-vector space with continuous linear $G_{F,S}$-action. Set $M^* = \mathrm{Hom}_{\mathbb{F}}(M, \mathbb{F})$ the $\mathbb{F}$-linear dual module. Then all $H^i(G_{F,S}, M)$ are of finite length over $\mathcal{O}$, and there is an exact sequence:*

$$\begin{array}{l}
0 \longrightarrow H^0(F_S/F, M) \longrightarrow \bigoplus_{v\in S\cup\{v|\infty\}} H^0(F_v, M) \longrightarrow H^2(F_S/F, M^*(1))^* \\
\longrightarrow H^1(F_S/F, M) \longrightarrow \bigoplus_{v\in S} H^1(F_v, M) \longrightarrow H^1(F_S/F, M^*(1))^* \\
\longrightarrow H^2(F_S/F, M) \longrightarrow \bigoplus_{v\in S} H^2(F_v, M) \longrightarrow H^0(F_S/F, M^*(1))^* \longrightarrow 0
\end{array}$$

Theorem 5.9 (Global Euler characteristic formula)**.** *Let M be as before. Then*

$$\chi(F_S/F, M) = -[F:\mathbb{Q}]\dim_{\mathbb{F}} M + \sum_{v|\infty} h^0(F_v, M).$$

We apply these theorems to $M = \mathrm{ad}^0\,\overline{\rho} = (\mathrm{ad}^0\,\overline{\rho})^*$.

Notation 5.10. Let $L_v^{\perp} \subseteq H^1(F_v, \mathrm{ad}^0\,\overline{\rho}(1)) \cong H^1(F_v, \mathrm{ad}^0\,\overline{\rho})^*$ be the orthogonal complement of L_v under the local Tate duality. Define

$$H^1_{\mathcal{S}^{\perp},T}(\mathrm{ad}^0\,\overline{\rho}(1)) := \ker\left(H^1(F_S/F, \mathrm{ad}^0\,\overline{\rho}(1)) \to \bigoplus_{v\in S\setminus T} \frac{H^1(F_v, \mathrm{ad}^0\,\overline{\rho}(1))}{L_v^{\perp}} \right).$$

By the Poitou-Tate in Theorem 5.8 above, taking $M = \mathrm{ad}^0\,\overline{\rho}$ leads to

$$\cdots \longrightarrow H^1(F_S/F, \mathrm{ad}^0\,\overline{\rho}) \longrightarrow \begin{array}{c} \displaystyle\bigoplus_{v\in T} H^1(F_v, \mathrm{ad}^0\,\overline{\rho}) \\ \oplus \\ \displaystyle\bigoplus_{v\in S\setminus T} \frac{H^1(F_v, \mathrm{ad}^0\,\overline{\rho})}{L_v} \end{array} \longrightarrow H^1_{\mathcal{S}^{\perp},T}(\mathrm{ad}^0\,\overline{\rho}(1))^*$$
$$\longrightarrow H^2(F_S, F, \mathrm{ad}^0\,\overline{\rho}) \longrightarrow \bigoplus_{v\in S} H^1(F_v, \mathrm{ad}^0\,\overline{\rho}) \longrightarrow H^0(F_S/F, \mathrm{ad}^0\,\overline{\rho}(1))^* \longrightarrow 0$$

Comparing this with the first long exact sequence on page 25, we get

$$\begin{aligned} h^2_{\mathcal{S},T}(\mathrm{ad}^0\,\overline{\rho}) &= h^1_{\mathcal{S},T}(\mathrm{ad}^0\,\overline{\rho}(1)), \\ h^3_{\mathcal{S},T}(\mathrm{ad}^0\,\overline{\rho}) &= h^0(F_S/F, \mathrm{ad}^0\,\overline{\rho}(1)). \end{aligned}$$

The global and local Euler characteristic formulas (cf. Theorem 4.10, 5.9) dictate that (remember that $\{v : v \mid p\} \subseteq S$):

$$\chi_{\mathcal{S},T}(\mathrm{ad}^0\,\overline{\rho}) = 1 - \#T + \sum_{v\mid\infty} h^0(F_v, \mathrm{ad}^0\,\overline{\rho}) + \sum_{v\in S\setminus T} (h^0(F_v, \mathrm{ad}^0\,\overline{\rho}) - \dim_{\mathbb{F}} L_v).$$

To conclude, we obtain the following.

Theorem 5.11 (Greenberg–Wiles formula)**.**

$$\begin{aligned} h^1_{\mathcal{S},T}(\mathrm{ad}^0\,\overline{\rho}) = h^1_{\mathcal{S}^{\perp},T}(\mathrm{ad}^0\,\overline{\rho}(1)) &+ \sum_{v\in S\setminus T} (\dim_{\mathbb{F}} L_v - h^0(F_v, \mathrm{ad}^0\,\overline{\rho})) \\ &- \sum_{v\mid\infty} \big(h^0(F_v, \mathrm{ad}^0\,\overline{\rho}) - h^0(F_S/F, \mathrm{ad}^0\,\overline{\rho}(1))\big) + \begin{cases} \#T - 1, & T = \emptyset; \\ 0, & T = \emptyset. \end{cases} \end{aligned}$$

Comment. The punchline of Greenberg–Wiles formula is as follows. Let R be an abstract deformation ring of $\overline{\rho}$ subject to precise local deformation conditions at p and the places dividing ∞. For the prime p, the local conditions amount either to an "ordinary" or "finite-flat" restriction. One then interprets the dual of the reduced tangent space $\mathfrak{m}_{\mathcal{S}}/(\mathfrak{m}_{\mathcal{S}}^2, \mathfrak{m}^{\mathrm{loc}})$ of R in terms of Galois cohomology, in particular as a subgroup (i.e. Selmer group) of classes in $H^1(\mathbb{Q}, \mathrm{ad}^0\,\overline{\rho})$ satisfying local conditions. This can be thought of as analogous to a class group, and one does not have any a priori understanding of how large it can be although it has some finite dimension d. Using the Greenberg–Wiles formula, the obstructions in $H^2(\mathbb{Q}, \mathrm{ad}^0\,\overline{\rho})$ can be related to the relative tangent space, and allow one to realize R as a quotient of $W(\mathbb{F})[\![x_1, \ldots, x_d]\!]$ by d relations. In particular, if R was finite and free as a $W(\mathbb{F})$-module (as would be the case if $R = \mathbb{T}$ that we are to tackle with later) then R would be a complete intersection.

6. Taylor–Wiles primes

Using the same notations as before, fix again a global deformation problem

$$\mathcal{S} = (\overline{\rho}, S, \psi, \mathcal{O}, \{\mathcal{D}_v\}_{v\in S}),$$

where $\overline{\rho} : G_{F,S} \to \mathrm{GL}_2(\mathbb{F})$ is of rank 2.

Definition 6.1. A *Taylor–Wiles prime* (for $\mathcal{S}$) is a prime $v \notin S$ of F such that

(1) $q_v := \mathrm{Nm}(v) \equiv 1 \bmod p$, and
(2) $\overline{\rho}(\mathrm{Frob}_v)$ has distinct $\mathbb{F}$-rational eigenvalues.

We say a Taylor–Wiles prime v *has level* N with $N \geqslant 1$, if further

(3) $q_v \equiv 1 \bmod p^N$.

Remark 6.2. One can do assume that $\mathbb{F}$ is large enough such that all eigenvalues of all elements in $\overline{\rho}(G_{F,S})$ are defined over $\mathbb{F}$.

6.1. Diagonalizable local representation at Taylor–Wiles primes.

Proposition 6.3. *Let v be a Taylor–Wiles prime for $\mathcal{S}$. For any $A \in \mathsf{CNL}_{\mathcal{O}}$ and any lifting $\rho : G_{F_v} \to \mathrm{GL}_2(A)$ of $\overline{\rho}_v := \overline{\rho}|_{G_{F_v}}$, ρ is conjugate to a diagonal lifting $\begin{pmatrix}\chi_1 & 0\\ 0 & \chi_2\end{pmatrix}$.*

Proof. The proof can be reduced to the case where A is Artinian. Fix $\Phi \in G_{F_v}$ a lift of Frob_v. Since $\overline{\rho}(\mathrm{Frob}_v)$ has distinct $\mathbb{F}$-rational eigenvalues, there is a basis for ρ such that

$$\rho(\Phi) = \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}.$$

Since $\overline{\rho}(I_{F_v}) = 1$ and $\rho(I_{F_v}) \subseteq 1 + M_2(\mathfrak{m}_A)$, $\rho|_{I_{F_v}}$ factors through the tame inertia group $I_{F_v}^{\mathrm{tame}}$. Now fix a topological generator t of $I_{F_v}^{\mathrm{tame}}$. It suffices to prove that under our fixed basis $\rho(t)$ is diagonal.

We induct on the length of A. Can assume

$$\rho(t) = 1 + X \in 1 + M_n(\mathfrak{m}_A), \quad X = \begin{pmatrix}a & b\\ c & d\end{pmatrix}, \quad b, c \in \mathfrak{m}_A^n, \quad \mathfrak{m}_A^{n+1} = 0.$$

An easy check shows that X^k is diagonal if $k \geqslant 2$. We know that $\Phi^{-1} t \Phi = t^{q_v}$. Then

$$\begin{aligned}
0 &= \rho(\Phi^{-1})\rho(t)\rho(\Phi) - \rho(t)^{q_v}\\
&= 1 + \begin{pmatrix}a & \alpha^{-1}\beta b\\ \alpha\beta^{-1} c & d\end{pmatrix} - 1 - q_v \begin{pmatrix}a & b\\ c & d\end{pmatrix} + D\\
&= \begin{pmatrix}0 & (\alpha^{-1}\beta - 1)b\\ (\alpha\beta^{-1} - 1)c & 0\end{pmatrix} + D'
\end{aligned}$$

where D and D' are two diagonal matrices. The last equality is valid because $(q_v - 1)b = (q_v - 1)c = 0$, which is implied by Definition 6.1(1) and $b, c \in \mathfrak{m}_A^n$. On the other hand, both $\alpha^{-1}\beta - 1$ and $\alpha\beta^{-1} - 1$ are units in A since $\alpha \bmod \mathfrak{m}_A$ and $\beta \bmod \mathfrak{m}_A$ are the distinct eigenvalues of $\overline{\rho}$. Thus, $b = c = 0$. This completes the proof that ρ is diagonalizable. $\square$

6.2. **Unramified deformation ring as a quotient.** We now assume v is a Taylor–Wiles prime for $\mathcal{S}$. Let $R_v^{\square,\psi}$ be the universal framed deformation ring for $\overline{\rho}_v$ with a fixed determinant ψ, and let $\rho_v^{\psi} : G_{F_v} \to \mathrm{GL}_2(R_v^{\square,\psi})$ be the universal lifting.

Motivation. In the Galois tower from a local field F_v to its separable closure, the maximal unramified extension corresponds to a quotient group $G_{F_v}^{\mathrm{ur}}$ of G_{F_v}. We expect that the universal deformation ring of a given local deformation problem is also the quotient of the whole deformation ring.

By Proposition 6.3, the framed deformation ρ^{ψ} is conjugate to $\begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}$, where each $\chi_i : G_{F_v} \to \mathrm{GL}_1(R_v^{\square,\psi}) = (R_v^{\square,\psi})^{\times}$ and $\chi_1\chi_2 = \psi = \det \rho^{\psi}$. In particular, since ψ is unramified at v,

$$\chi_1|_{I_{F_v}} = \chi_2|_{I_{F_v}}.$$

Since $\overline{\rho}$ is unramified, $\chi_1|_{I_{F_v}}$ is a pro-p character of

$$I_{F_v^{\mathrm{ab}}/F_v} \cong k_v^{\times} \times \mathbb{Z}_q^d \times (\text{some finite } q\text{-group}),$$

where q is the residue character of v, and k_v is the residue field of F at v. Denote

$$\begin{aligned} \Delta_v &= \text{the maximal } p\text{-power quotient of } k_v^{\times}, \\ \mathcal{O}[\Delta_v] &= \text{the group algebra generated by } \Delta_v, \\ \mathfrak{a}_v &= \text{any ideal in the group algebra.} \end{aligned}$$

It follows that $\chi_1|_{I_{F_v}}$ determines an $\mathcal{O}[\Delta_v]$-algebra structure on $R_v^{\square,\psi}$. Moreover, note that there exists a natural ring homomorphism[7]

$$R_v^{\square,\psi} \longrightarrow R_v^{\mathrm{ur},\psi}$$

where $R_v^{\mathrm{ur},\psi}$ denotes the universal lifting ring for liftings ρ of $\overline{\rho}_v$ such that $\rho(I_{F_v}) = 1$ and $\det \rho = \psi$. It turns out that the kernel is an ideal of the form $\alpha_v R_v^{\square,\psi}$, since any unramified lifting to A with determinant ψ determines a map $\phi : R_v^{\square,\psi} \to A$ such that $\phi(\alpha_v) = 0$. Then

$$R_v^{\square,\psi}/\alpha_v R_v^{\square,\psi} \twoheadrightarrow R_v^{\mathrm{ur},\psi}.$$

And conversely, the universal unramified $R_v^{\square,\psi}/\alpha_v R_v^{\square,\psi}$-valued lifting is unramified. Hence the natural quotient $R_v^{\square,\psi} \twoheadrightarrow R_v^{\square,\psi}/\alpha_v R_v^{\square,\psi}$ factors through $R_v^{\mathrm{ur},\psi}$.

Proposition 6.4. *If v is a Taylor–Wiles prime for $\mathcal{S}$, then*

$$R_v^{\mathrm{ur},\psi} \cong R_v^{\square,\psi}/\alpha_v R_v^{\square,\psi}.$$

6.3. **A geometric dimension relation at Taylor–Wiles primes.** This subsection prepares some theoretical facts for introducing *Wiles' local-to-global approximation*, which can be seen as a miracle in various steps to prove Fermat's last theorem, in order to seek for the lifting of $\overline{\rho}$ that "arises from" a modular form, supposedly in an extremely vague sense. We plan to begin with Greenberg–Wiles formula (Theorem 5.11), which has geometric objects involved.

Now let Q be a finite set of Taylor–Wiles primes. Denote $\Delta_Q = \prod_{v \in Q} \Delta_v$, with corresponding group algebra $\mathcal{O}[\Delta_Q]$ and any ideal α_Q. We define the following global deformation problem for Q:

$$\mathcal{S}_Q = (\overline{\rho}, S \cup Q, \psi, \mathcal{O}, \{\mathcal{D}_v\}_{v \in S} \cup \{\mathcal{D}_v^{\psi}\}_{v \in Q}).$$

Here for each $v \in Q$, $\mathcal{D}_v^{\psi}$ is the deformation condition of all liftings of $\overline{\rho}_v$ with $\det \overline{\rho}_v = \psi|_{G_{F_v}}$. Then, assuming $\mathrm{End}_{\mathbb{F}[G_{F,S}]}(\overline{\rho}) = \mathbb{F}$, we have rings $R_{\mathcal{S}_Q}$ and $R_{\mathcal{S}}$ representing the deformation problems, and also $R_{\mathcal{S}_Q}^T$ and $R_{\mathcal{S}}^T$ for any $T \subsetneq S$.

[7] It is also natural to claim it is surjective.

Similarly as before, $R^T_{\mathcal{S}_Q}$ has the structure of an $\mathcal{O}[\Delta_Q]$-algebra, and the natural surjective ring homomorphism $R^T_{\mathcal{S}_Q} \twoheadrightarrow R^T_{\mathcal{S}}$ has kernel $\alpha_Q R^T_{\mathcal{S}_Q}$. Recall that for any (possibly empty) $T \subsetneq S$, the tangent space relative to $R^{T,\mathrm{loc}}_{\mathcal{S}}$ of $R^T_{\mathcal{S}}$ is given by a cohomology group $H^1_{\mathcal{S},T}(\mathrm{ad}^0\,\overline{\rho})$, and by Theorem 5.11, its dimension is

$$h^1_{\mathcal{S},T}(\mathrm{ad}^0\,\overline{\rho}) = h^1_{\mathcal{S}^\perp,T}(\mathrm{ad}^0\,\overline{\rho}(1)) + \sum_{v\in S\setminus T}(\dim_{\mathbb{F}} L_v - h^0(F_v, \mathrm{ad}^0\,\overline{\rho}))$$
$$- \sum_{v\mid\infty}\left(h^0(F_v, \mathrm{ad}^0\,\overline{\rho}) - h^0(F_S/F, \mathrm{ad}^0\,\overline{\rho}(1))\right) + \begin{cases} \#T - 1, & T = \emptyset; \\ 0, & T = \emptyset. \end{cases}$$

Here

- $h^1_{\mathcal{S}^\perp,T}(\mathrm{ad}^0\,\overline{\rho}(1)) := \ker\left(H^1(F_S/F, \mathrm{ad}^0\,\overline{\rho}(1)) \to \prod_{v\in S\setminus T} H^1(F_v, \mathrm{ad}^0\,\overline{\rho}(1))/L_v^\perp\right)$,
- $L_v \subseteq H^1(F_v, \mathrm{ad}^0\,\overline{\rho})$ is a subspace as the image of $D_v(\mathbb{F}[\varepsilon]) \cong \mathcal{L}_v \subseteq Z^1(F_v, \mathrm{ad}^0\,\overline{\rho})$ via the projection map, and
- $L_v^\perp \subseteq H^1(F_v, \mathrm{ad}^0\,\overline{\rho}(1))$ is the orthogonal complement of L_v under the local Tate duality.

Assumption 6.5. Assume the following conditions about local and global H^0 hold.

(1) Suppose $\overline{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible.
Consequence: there is no non-scalar $G_{F,S}$-equivariant homomorphism $\overline{\rho} \to \overline{\rho}(1)$ between $\mathbb{F}$-vector spaces (by abuse of notations); and hence,
$$H^0(F_S/F, \mathrm{ad}^0\,\overline{\rho}(1)) = 0.$$

(2) Suppose F is totally real and $\det\overline{\rho}(c_v) = -1$ (namely, $\overline{\rho}$ is "totally odd") for all $v \mid \infty$ in F, where c_v denotes the complex conjugation at v.
Consequence: then
$$h^0(F_v, \mathrm{ad}^0\,\overline{\rho}) = 1.$$

(3) Assume that for each $v \mid p$ (in particular $v \in S$), if $v \notin T$,
$$\dim_{\mathbb{F}} L_v - h^0(F_v, \mathrm{ad}^0\,\overline{\rho}) = [F_v : \mathbb{Q}_p].$$
(Recall that $L_v \subseteq H^1(F_v, \mathrm{ad}^0\,\overline{\rho})$ is the image of $\mathcal{L}_v \cong D_v(\mathbb{F}[\varepsilon]) \subseteq Z^1(F_v, \mathrm{ad}^0\,\overline{\rho})$.) And if $v \in T$, then R_v is $\mathcal{O}$-flat of relative Krull dimension $3 + [F_v : \mathbb{Q}_p]$ over $\mathcal{O}$.[8]
Comment: for example, this is true if
$$\overline{\rho}_v = \begin{pmatrix} \overline{\chi}_1 & 0 \\ 0 & \overline{\chi}_2 \end{pmatrix}, \quad \overline{\chi}_1|_{I_{F_v}} = 1,\ \overline{\chi}_2|_{I_{F_v}} \neq 1, \quad \mathcal{D}_v = \mathcal{D}_v^{\mathrm{ord},\psi},$$
where $\mathcal{D}_v^{\mathrm{ord},\psi}$ is the deformation problem corresponding to the local ordinary functor (with an additional fixed-determinant condition) we have defined in Subsubsection 4.3.1.

(4) Assume that for each $v \in S\setminus\{v : v \mid p\}$, if $v \notin T$, then
$$\dim_{\mathbb{F}} L_v - h^0(F_v, \mathrm{ad}^0\,\overline{\rho}) = 0.$$
And if $v \in T$, then R_v is $\mathcal{O}$-flat of relative Krull dimension 3 over $\mathcal{O}$.
Comment: for example, this is true if
$$\overline{\rho}|_{I_v} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \neq 1 \text{ or } \overline{\rho}_v = \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}$$
with $\chi_1(I_{F_v}) = 1$ and $\chi_2(I_{F_v}) \neq 1$, and $\mathcal{D}_v$ is the minimally ramified deformation problem with a fixed determinant in Subsubsection 4.3.4.

[8] In fact, one can compute by using the similar method in Subsection 4.4 that $\dim_{\mathbb{F}} D_v(\mathbb{F}[\varepsilon]) = 4 + [F_v : \mathbb{Q}_p]$.

Remark 6.6. In applications, the "if $v \in T$" part of (3)(4) above always hold, and the "if $v \notin T$" part holds if and only if D_v (or equivalently, R_v) is a formally smooth functor (resp. complete locally noetherian ring) over $\mathcal{O}$.

To conclude, we have the following result.

Theorem 6.7 (An important numerology by Wiles)**.** *Assume* (1)-(4) *in Assumption 6.5 hold.*

(1) *If $T = \emptyset$, then*

$$h^1_{\mathcal{S},\emptyset}(\mathrm{ad}^0\,\overline{\rho}) = h^1_{\mathcal{S}^\perp,\emptyset}(\mathrm{ad}^0\,\overline{\rho}(1)). \tag{$\diamondsuit$}$$

As a reminder,

$$H^1_{\mathcal{S}^\perp,T}(\mathrm{ad}^0\,\overline{\rho}(1)) := \ker\left(H^1(F_S/F,\mathrm{ad}^0\,\overline{\rho}(1)) \to \bigoplus_{v\in S\setminus T} H^1(F_v,\mathrm{ad}^0\,\overline{\rho}(1))/L_v^\perp\right).$$

(2) *If $T \supseteq \{v : v \mid p\}$, for example if $T = S$,*[9] *then*

$$h^1_{\mathcal{S},T}(\mathrm{ad}^0\,\overline{\rho}) = \#T - 1 - \underbrace{\sum_{v\mid\infty} h^0(F_v,\mathrm{ad}^0\,\overline{\rho})}_{=[F:\mathbb{Q}]} + h^1_{\mathcal{S}^\perp,T}(\mathrm{ad}^0\,\overline{\rho}(1)).$$

On the other hand, one can show that

$$\mathrm{Krull}\dim R^{T,\mathrm{loc}}_{\mathcal{S}} = 1 + 3\#T + [F:\mathbb{Q}].$$

Hence

$$\mathrm{Krull}\dim R^{T,\mathrm{loc}}_{\mathcal{S}} + h^1_{\mathcal{S},T}(\mathrm{ad}^0\,\overline{\rho}) = h^1_{\mathcal{S}^\perp,T}(\mathrm{ad}^0\,\overline{\rho}(1)) + 4\#T. \tag{$\heartsuit$}$$

We point out that in ($\heartsuit$), the last term equals $4\#T - 1 + 1$, where $4\#T - 1$ is the relative dimension of $R^{T,\mathrm{loc}}_{\mathcal{S}}$ (cf. Proposition 5.4(2)) and 1 is the dimension of $\mathcal{O}$ itself.

6.4. **Wiles' local-to-global approximation argument.** The punchline of Wiles' original idea is as follows. The local deformation rings are relatively easy to understand, and we want to "approximate" a global deformation problem by the local ones. However, there are enormous technical difficulties in implementing this idea. The most significant obstruction lands in some geometric dimension problem when the set of Taylor–Wiles primes becomes sufficiently large.

Again, let Q be a finite set of Taylor–Wiles primes in F. Recall that for $v \in Q$,

- $q_v = \mathrm{Nm}(v) \equiv 1 \bmod p$,
- $v \notin S$, and
- $\overline{\rho}(\mathrm{Frob}_v)$ has distinct $\mathbb{F}$-rational eigenvalues.

We further say v has level $N \geqslant 1$ if $q_v \equiv 1 \bmod p^N$. Denote $\psi_v := \psi|_{G_{F_v}}$ and $\overline{\rho}_v := \overline{\rho}|_{G_{F_v}}$. Define a global deformation datum

$$\mathcal{S}_Q = (\overline{\rho}, S\cup Q, \psi, \mathcal{O}, \{\mathcal{D}_v\}_{v\in S} \cup \{\mathcal{D}^{\square,\psi_v}_{\overline{\rho}_v}\}_{v\in Q}).$$

Question. How do ($\diamondsuit$) and ($\heartsuit$) change while replacing $\mathcal{S}$ by $\mathcal{S}_Q$?

Theorem 6.8 (Numerology for Taylor–Wiles primes)**.** *Assume* (1)-(4) *in Assumption 6.5 hold.*

(1) *If $T = \emptyset$, then*

$$h^1_{\mathcal{S}_Q,\emptyset}(\mathrm{ad}^0\,\overline{\rho}) = h^1_{\mathcal{S}_Q^\perp,\emptyset}(\mathrm{ad}^0\,\overline{\rho}(1)) + \#Q. \tag{$\clubsuit$}$$

[9] Recall that we have assumed that $T \subsetneq S$ for simplicity. But the case where $T = S$ is also admissible appears.

(2) *If $T \supseteq \{v : v \mid p\}$, for example if $T = S$, then*

$$(\spadesuit) \qquad \operatorname{Krull\,dim} R_{\mathcal{S}}^{T,\mathrm{loc}} + h^1_{\mathcal{S}_Q,T}(\mathrm{ad}^0\,\overline{\rho}) = h^1_{\mathcal{S}_Q^\perp,T}(\mathrm{ad}^0\,\overline{\rho}(1)) + 4\#T + \#Q.$$

Proof. Note that for $v \in Q$, $\mathcal{D}_v = \mathcal{D}^{\square,\psi_v}_{\overline{\rho}_v}$. Hence

$$D_v(\mathbb{F}[\varepsilon]) = D^{\square,\psi_v}_{\overline{\rho}_v}(\mathbb{F}[\varepsilon]) \cong \mathcal{L}_v \twoheadrightarrow L_v = H^1(F_v, \mathrm{ad}^0\,\overline{\rho}), \quad L_v^\perp = 0.$$

Therefore,

$$\begin{aligned} H^1_{\mathcal{S}_Q^\perp,T}(\mathrm{ad}^0\,\overline{\rho}) &= \ker\left(H^1(F_{S\cup Q}/F, \mathrm{ad}^0\,\overline{\rho}(1)) \to \begin{matrix} \displaystyle\bigoplus_{v\in S\setminus T} \frac{H^1(F_v,\mathrm{ad}^0\,\overline{\rho}(1))}{L_v^\perp} \\ \oplus \\ \displaystyle\bigoplus_{v\in Q} H^1(F_v,\mathrm{ad}^0\,\overline{\rho}(1)) \end{matrix} \right) \\ &= \ker\left(H^1_{\mathcal{S}^\perp,T}(\mathrm{ad}^0\,\overline{\rho}(1)) \to \bigoplus_{v\in Q} H^1(F_v,\mathrm{ad}^0\,\overline{\rho}(1)) \right). \end{aligned}$$

Comparing with the Greenberg–Wiles formula (Theorem 5.11), the right hand sides of ($\diamondsuit$) and ($\heartsuit$) should be added with

$$\begin{aligned} &\sum_{v\in Q} \dim_{\mathbb{F}} L_v - h^0(F_v, \mathrm{ad}^0\,\overline{\rho}) \\ =& \sum_{v\in Q} h^1(F_v,\mathrm{ad}^0\,\overline{\rho}) - h^0(F_v,\mathrm{ad}^0\,\overline{\rho}) && (\text{as } L_v \cong H^1(F_v,\mathrm{ad}^0\,\overline{\rho})) \\ =& \sum_{v\in Q} h^2(F_v,\mathrm{ad}^0\,\overline{\rho}) && (\text{by local Euler characteristic formula}) \\ =& \sum_{v\in Q} h^0(F_v,\mathrm{ad}^0\,\overline{\rho}(1)) && (\text{by local Tate duality}) \\ =& \sum_{v\in Q} h^0(F_v,\mathrm{ad}^0\,\overline{\rho}) && (\text{since } q_v \equiv 1 \bmod p) \\ =& \#Q. && (\text{since } \overline{\rho}(\mathrm{Frob}_v) \text{ has distinct eigenvalues}) \end{aligned}$$

This completes the proof. □

Next Goal. Note that on the right-hand sides of ($\diamondsuit$) and ($\heartsuit$), the h^1-terms are defined for $\mathcal{S}_Q^\perp$. Assume $h^1_{\mathcal{S}_Q^\perp,T}(\mathrm{ad}^0\,\overline{\rho}(1)) = 0$ and the number of Taylor–Wiles primes is exactly $h^1_{\mathcal{S}^\perp,T}(\mathrm{ad}^0\,\overline{\rho}(1))$. We are to prove that the left-hand sides of ($\diamondsuit$) and ($\heartsuit$) for $\mathcal{S}_Q$ depend only on $\mathcal{S}$.

A geometric interpretation of this phenomenon. On the scheme $\operatorname{Spec} R_{\mathcal{S}}^{T,\mathrm{loc}}$, each closed point is in bijection with the framed deformations of $\overline{\rho}$ with local conditions inserted. Consider a proper smooth $\mathbb{F}$-scheme X such that the cone $X \times_{\mathbb{F}} \operatorname{Spec} R_{\mathcal{S}}^{T,\mathrm{loc}} = \operatorname{Spec} R_{\mathcal{S}}^{T,\mathrm{loc}}[\![x_1,\dots,x_t]\!]$ for some integer t (which is actually computable using the stated theorems). This is the "ambient space" for $\operatorname{Spec} R_{\mathcal{S}}^T$. Then $\operatorname{Spec} R^T_{\mathcal{S}_Q}$ contains $\operatorname{Spec} R^T_{\mathcal{S}}$ as a proper subscheme when $Q \neq \emptyset$. Since Q is a finite set, they share the same $\mathbb{F}$-dimension and $\operatorname{Spec} R^T_{\mathcal{S}_Q}$ has finitely many more irreducible components than $\operatorname{Spec} R^T_{\mathcal{S}}$ does. Moreover, for different choices of Taylor–Wiles primes Q_1 and Q_2 say, $\operatorname{Spec} R^T_{\mathcal{S}_{Q_1}}$ and $\operatorname{Spec} R^T_{\mathcal{S}_{Q_2}}$ generally obtain different irreducible components outside $\operatorname{Spec} R^T_{\mathcal{S}}$. We will later define Q_N as a collection of Taylor–Wiles primes of level N with more conditions. (The existence of Q_N is highly nontrivial. See Proposition 6.12 below.) Consider

$$R_{\mathcal{S},\infty} := \varprojlim_N R_{\mathcal{S}_{Q_N}} \quad\rightsquigarrow\quad \operatorname{Spec} R_{\mathcal{S},\infty} := \varinjlim_N \operatorname{Spec} R_{\mathcal{S}_{Q_N}}.$$

It turns out that $\operatorname{Spec} R_{\mathcal{S},\infty}$, which depends only on $\mathcal{S}$, is smooth and fills up the whole ambient space. Even if the limit scheme $\operatorname{Spec} R_{\mathcal{S},\infty}$ is nice and contains $\operatorname{Spec} R_{\mathcal{S}}$, a dimension problem arises inevitably.

Definition 6.9. Let Γ be a subgroup of $\mathrm{GL}_2(\mathbb{F})$ acting absolutely irreducibly on $\mathbb{F}^2$ and such that the eigenvalues at any $\gamma \in \Gamma$ are $\mathbb{F}$-rational (but not necessarily distinct). Let ad^0 be the trace 0 subspace of $M_2(\mathbb{F})$ with an adjoint Γ-action. We say that Γ is *enormous* (or equivalently, say *adequate* or *big*) if it satisfies the following properties.

(E1) The group Γ has no quotient of order p,
(E2) $H^0(\Gamma, \mathrm{ad}^0) = H^1(\Gamma, \mathrm{ad}^0) = 0$, and
(E3) for any simple $\mathbb{F}[\Gamma]$-submodule W of ad^0, there is $\gamma \in \Gamma$ with distinct eigenvalues such that $W^\gamma \neq 0$.

Remark 6.10. In the case of rank 2, the definitions of adequateness, bigness, and enormousness are all equivalent. Generally, Definition 6.9 actually defines the *enormousness.* In the case where rank > 2, adequateness and bigness both have (E1) and (E2), but (E3) weakens

(B3) (Bigness) For any simple $\mathbb{F}[\Gamma]$-submodule W of ad^0, there is a semisimple $\gamma \in \Gamma$ with an eigenvalue of multiplicity 1, satisfying a more technical condition.
(A3) (Adequateness) For any simple $\mathbb{F}[\Gamma]$-submodule W of ad^0, there is a semisimple $\gamma \in \Gamma$ satisfying the same technical condition as in (B3).

Theorem 6.11 (Background in Wiles' 3-5 trick)**.** *Assume $\Gamma \subseteq \mathrm{GL}_2(\mathbb{F})$ acts absolutely irreducibly and $p > 2$ is an odd prime. Then Γ is enormous unless*

- *$p = 3$, and the image of Γ in $\mathrm{PGL}_2(\overline{\mathbb{F}}_3)$ is conjugate to $\mathrm{PSL}_2(\mathbb{F}_3)$, or*
- *$p = 5$, and the image of Γ in $\mathrm{PGL}_2(\overline{\mathbb{F}}_5)$ is conjugate to $\mathrm{PSL}_2(\mathbb{F}_5)$.*

Proposition 6.12. *Let $\mathcal{S}$ be as above and suppose $\Gamma = \overline{\rho}(G_{F(\zeta_p)})$ is enormous. Let $q = h^1_{\mathcal{S}^\perp,T}(\mathrm{ad}^0\,\overline{\rho}(1))$. Then for any $N \geqslant 1$, we can find a set of Taylor–Wiles primes Q_N of level N, i.e. $q_v \equiv 1 \bmod p^N$ for all $v \in Q_N$, such that*

(1) $\#Q_N = q$, and
(2) $H^1_{\mathcal{S}^\perp_{Q_N},T}(\mathrm{ad}^0\,\overline{\rho}(1)) = 0$.

Proof. Fix $N \geqslant 1$. Assuming we have a set of Taylor–Wiles primes $Q' = \{v_1, \ldots, v_{j-1}\}$ of level N with $1 \leqslant k \leqslant q$, and assuming

$$h^1_{\mathcal{S}^\perp_{Q'},T}(\mathrm{ad}^0\,\overline{\rho}(1)) = q - (j-1),$$

we show how to find an additional Taylor–Wiles prime v_j of level N such that

$$h^1_{\mathcal{S}^\perp_{Q' \cup \{v_j\}},T}(\mathrm{ad}^0\,\overline{\rho}(1)) = q - j.$$

Fix $0 \neq [\mathcal{R}] \in H^1_{\mathcal{S}^\perp_{Q'},T}(\mathrm{ad}^0\,\overline{\rho}(1))$ with $\mathcal{R}$ a cocycle representation of the chosen cohomology class $[\mathcal{R}]$. It suffices to show that there exist infinitely many Taylor–Wiles primes $v \notin S$ of F such that

(a) $q_v \equiv 1 \bmod p^N$,
(b) $\overline{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues, and

(c) the cohomology restriction

$$\mathbb{F}\cdot[\mathcal{R}] \xrightarrow[\sim]{\mathrm{Res}} H^1(F_v^{\mathrm{ur}}/F_v, \mathrm{ad}^0\,\overline{\rho}(1))$$

is an isomorphism.

If v satisfies (a) and (b), then

$$H^1(F_v^{\mathrm{ur}}/F_v, \mathrm{ad}^0\,\overline{\rho}(1)) \cong \mathrm{ad}^0\,\overline{\rho}/(\mathrm{Frob}_v - 1)\,\mathrm{ad}^0\,\overline{\rho}, \quad [\phi] \mapsto \phi(\mathrm{Frob}_v),$$

and the space $\mathrm{ad}^0\,\overline{\rho}/(\mathrm{Frob}_v - 1)\,\mathrm{ad}^0\,\overline{\rho}$ is 1-dimensional under (b). So we can replace (c) with

(c') $\mathrm{Res}_v(\mathcal{R})(\mathrm{Frob}_v) \notin (\mathrm{Frob}_v - 1)\,\mathrm{ad}^0\,\overline{\rho}$.

By the Chebotarev density theorem, the existence of infinitely many primes is equivalent to the positivity of their density, and hence the existence of the Galois conjugacy class. So it suffices to show that there exists $\sigma \in G_{F,S}$ such that

(a) $\sigma \in G_{F(\zeta_{p^N})}$,
(b) $\overline{\rho}(\sigma)$ has distinct eigenvalues, and
(c) $\mathcal{R}(\sigma) \notin (\sigma - 1)\,\mathrm{ad}^0\,\overline{\rho}$.

Let $L/F(\zeta_p)$ be the extension cut out by $\overline{\rho}|_{G_{F(\zeta_p)}}$.

$$\begin{tikzcd}
& L(\zeta_{p^N}) \arrow[dl, dash] \arrow[dr, dash] \arrow[dr, bend left, dash, "\Gamma"] & \\
L \arrow[dr, dash] \arrow[dr, bend right, dash, "\Gamma"'] & & F(\zeta_{p^N}) \arrow[dl, dash] \\
& F(\zeta_p) \arrow[d, dash] & \\
& F &
\end{tikzcd}$$

For simplicity, we denote

$$L_N := L(\zeta_{p^N}), \quad L_1 := L = L(\zeta_p), \quad F_N := F(\zeta_{p^N}), \quad F_1 := F(\zeta_p).$$

Note that by (E1) of enormousness, Γ has no quotient subgroup of order p, hence $L_N \cap F_N = F_1$.

Claim. $H^1(L_N/F, \mathrm{ad}^0\,\overline{\rho}(1)) = 0$.

To prove the claim, by inflation-restriction, we have

$$\begin{tikzcd}
& H^1(F_N/F, H^0(\Gamma, \mathrm{ad}^0\,\overline{\rho})) = 0 \arrow[d, phantom, "\parallel"] & & \\
0 \arrow[r] & H^1(F_N/F, (\mathrm{ad}^0\,\overline{\rho}(1))^{\mathrm{Gal}(L_N/F_N)}) \arrow[r, "\mathrm{Inf}"] & H^1(L_N/F, \mathrm{ad}^0\,\overline{\rho}(1)) \arrow[dl, "\mathrm{Res}"] & \\
& H^1(L_N/F_N, \mathrm{ad}^0\,\overline{\rho}(1)) \arrow[rr] \arrow[d, phantom, "\parallel"] & & 0 \\
& H^1(\Gamma, \mathrm{ad}^0\,\overline{\rho}) = 0 & &
\end{tikzcd}$$

Here in the first term, $(\mathrm{ad}^0\,\overline{\rho}(1))^{\mathrm{Gal}(L_N/F_N)} = H^0(\Gamma, \mathrm{ad}^0\,\overline{\rho})$. Hence the quotient space and the subspace both vanish by (E2). Then the claim follows.

Granting the claim, by inflation-restriction, the restriction

$$0 = H^1(L_N, F, \operatorname{ad}^0 \overline{\rho}(1)) \to H^1(F_S/F, \operatorname{ad}^0 \overline{\rho}(1)) \xrightarrow{\text{Res}} H^1(F_S/L_N, \operatorname{ad}^0 \overline{\rho}(1))^{\operatorname{Gal}(L_N/F)}$$

is injective (and hence an isomorphism). In fact,

$$\begin{aligned} 0 \neq \operatorname{Res}([\mathcal{R}]) &\in H^1(F_S/L_N, \operatorname{ad}^0 \overline{\rho}(1))^{\operatorname{Gal}(L_N/F)} \\ &\subseteq \operatorname{Hom}_\Gamma(\operatorname{Gal}(F_S/L_N), \operatorname{ad}^0 \overline{\rho}). \end{aligned}$$

Let W be a nonzero irreducible subrepresentation of the $\mathbb{F}$-span of $\mathcal{R}(\operatorname{Gal}(F_S/L_N)) \subseteq \operatorname{ad}^0 \overline{\rho}$. By (E3), we can find $\sigma_0 \in \operatorname{Gal}(L_N, F_N)$ such that $W^{\sigma_0} \neq 0$ and $\overline{\rho}(\sigma_0)$ has distinct eigenvalues. So if $\mathcal{R}(\sigma_0) \notin (\sigma_0 - 1) \operatorname{ad}^0 \overline{\rho}$, we take $\sigma = \sigma_0$ and we are done.

Now assume $\mathcal{R}(\sigma_0) \in (\sigma_0 - 1) \operatorname{ad}^0 \overline{\rho}$. Conjecturally, when it is necessary, we can assume that

$$\overline{\rho}(\sigma_0) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \alpha \neq \beta.$$

So $(\sigma_0 - 1) \operatorname{ad}^0 \overline{\rho} = \left\{ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\}$, which has no nonzero $\overline{\rho}(\sigma_0)$-invariant vectors. We infer that

$$\begin{aligned} & W \not\subseteq (\sigma_0 - 1) \operatorname{ad}^0 \overline{\rho} \\ \Longrightarrow \quad & \mathcal{R}(\operatorname{Gal}(F_S/L_N)) \not\subseteq (\sigma_0 - 1) \operatorname{ad}^0 \overline{\rho} \\ \Longrightarrow \quad & \text{there exists } \tau \in \operatorname{Gal}(F_S/L_N) \text{ such that } \mathcal{R}(\tau) \notin (\sigma_0 - 1) \operatorname{ad}^0 \overline{\rho}. \end{aligned}$$

Set $\sigma = \tau\sigma_0$. Then $\sigma \in G_{F_N}$, $\overline{\rho}(\sigma) = \overline{\rho}(\sigma_0)$, and

$$\mathcal{R}(\sigma) = \mathcal{R}(\tau\sigma_0) = \tau\mathcal{R}(\sigma_0) + \mathcal{R}(\tau) = \underbrace{\mathcal{R}(\sigma_0)}_{\in (\sigma_0 - 1) \operatorname{ad}^0 \overline{\rho}} + \underbrace{\mathcal{R}(\tau)}_{\notin (\sigma_0 - 1) \operatorname{ad}^0 \overline{\rho}}.$$

Here the second equality is due to the cocycle condition of $\mathcal{R}$. Then $\mathcal{R}(\sigma) \notin (\sigma_0 - 1) \operatorname{ad}^0 \overline{\rho} = (\sigma - 1) \operatorname{ad}^0 \overline{\rho}$. This concludes the proof. $\square$

If we further assume that D_v for $v \in S$ are nice, i.e. as in cases (1)(2) of Theorem 6.7 and 6.8, we get:

Corollary 6.13. *There is an integer $q \geqslant 0$ such that for all $N \geqslant 1$ there is a set Q_N of Taylor–Wiles primes of level N and a surjection*

$$R_{\mathcal{S}}^{T,\text{loc}}[\![x_1, \ldots, x_g]\!] \twoheadrightarrow R_{\mathcal{S}_{Q_N}}^T$$

for which

(1) *when $T = \emptyset$ and $R_{\mathcal{S}}^{T,\text{loc}} = \mathcal{O}$, we have $g = q$;*
(2) *when $T \supseteq \{v : v \mid p\}$ (for example $T = S$), we have*

$$\dim R_{\mathcal{S}}^{T,\text{loc}} + g = q + 4\#T.$$

Definition 6.14. A *Taylor–Wiles datum* $(Q, \{\alpha_v\}_{v\in Q})$ consists of a set Q of Taylor–Wiles primes and choices α_v of eigenvalue of $\overline{\rho}(\operatorname{Frob}_v)$ for each $v \in Q$.

We saw previously that if $\rho^{\text{univ}} : G_{F,S} \to \operatorname{GL}_2(R_{\mathcal{S}_Q})$ is the universal deformation of type $\mathcal{S}_Q$, then for any $v \in Q$,

$$\rho^{\text{univ}}|_{G_{F_v}} \cong \chi_{v,1} \oplus \chi_{v,2}$$

with $\chi_{v,i} \circ \operatorname{Art}_{F_v}|_{\mathcal{O}_F^\times} : \mathcal{O}_F^\times \to R_{\mathcal{S}_Q}^\times$ factoring through Δ_v, the maximal p-power ordinary quotient of $(\mathcal{O}_F/\mathfrak{m}_v)^\times$. A choice of eigenvalues $(\alpha_v)_{v\in Q}$ of $\overline{\rho}(\operatorname{Frob}_v)$ determines an ordering of $\chi_{v,1}$ and $\chi_{v,2}$ by $\chi_{v,1}(\operatorname{Frob}_v) = \alpha_v$. Thus, a Taylor–Wiles datum implies the information about an $\mathcal{O}$-algebra map

$\mathcal{O}[\Delta_Q] \to R_{\mathcal{S}_Q}$ (by sending $\delta \in \Delta_v$ to $\chi_{v,1}(\delta)$) and the surjection $R_{\mathcal{S}_Q} \twoheadrightarrow R_{\mathcal{S}}$, whose kernel is $\mathfrak{a}_Q$, an augmented ideal of $\mathcal{O}[\Delta_Q]$, where $\Delta_Q = \prod_{v\in Q} \Delta_v$.

By letting $q = \#Q$ with $Q = \{v_1, \ldots, v_q\}$, we have the following.

(1) Suppose $T = \emptyset$. Define

$$S_\infty := \mathcal{O}[\![y_1, \ldots, y_q]\!] \cong \mathcal{O}[\![\mathbb{Z}_p^q]\!] \supseteq (y_1, \ldots, y_q) = \mathfrak{a}_\infty$$

where $\mathfrak{a}_\infty$ is the augmented ideal. The diagram commutes

$$\begin{tikzcd} & \mathcal{O}[\![\mathbb{Z}_p^q]\!] \arrow[d, two heads] \arrow[ddl, dashed] \\ & \mathcal{O}[\Delta_Q] \arrow[d] \\ \mathcal{O}[\![x_1, \ldots, x_g]\!] \arrow[r, two heads] & R_{\mathcal{S}_Q}. \end{tikzcd}$$

in which the map $\mathcal{O}[\![y_1, \ldots, y_q]\!] \cong \mathcal{O}[\![\mathbb{Z}_p^q]\!] \twoheadrightarrow \mathcal{O}[\Delta_Q]$ sends $1 + y_i$ to the corresponding generator of Δ_{v_i}. Also,

$$R_{\mathcal{S}_Q}/\mathfrak{a}_\infty \cong R_{\mathcal{S}}.$$

And if Q is as in Corollary 6.13, then $g = q$.

(2) Suppose $T \supseteq \{v : v \mid p\}$. Fix an isomorphism

$$R^T_{\mathcal{S}_Q} \cong R_{\mathcal{S}_Q}[\![z_1, \ldots, z_{4\#T-1}]\!] \cong R_{\mathcal{S}_Q} \hat{\otimes}_{\mathcal{O}} \mathcal{T},$$

where $\mathcal{T} := \mathcal{O}[\![z_1, \ldots, z_{4\#T-1}]\!]$. Similarly, we have

$$\mathcal{O}[\![z_1, \ldots, z_{4\#T-1}, y_1, \ldots, y_q]\!] = \mathcal{T}[\![y_1, \ldots, y_q]\!] \cong \mathcal{T}[\![\mathbb{Z}_p^q]\!] \supseteq (y_1, \ldots, y_q) = \mathfrak{a}_\infty.$$

And the diagram commutes

$$\begin{tikzcd} & \mathcal{T}[\![\mathbb{Z}_p^q]\!] \arrow[d] \arrow[ddl, dashed] \\ & \mathcal{T}[\Delta_Q] \arrow[d] \\ R^{T,\mathrm{loc}}_{\mathcal{S}}[\![x_1, \ldots, x_g]\!] \arrow[r, two heads] & R^T_{\mathcal{S}_Q}. \end{tikzcd}$$

Also,

$$R^T_{\mathcal{S}_Q}/\mathfrak{a}_\infty \cong R^T_{\mathcal{S}}.$$

And if Q is as in Corollary 6.13, then

$$\dim R^{T,\mathrm{loc}}_{\mathcal{S}}[\![x_1, \ldots, x_g]\!] = \dim S_\infty.$$

7. Taylor–Wiles primes and modular forms

Setups. Fix $\mathcal{O}$ the ring of integers in some finite extension E of $\mathbb{Q}_p$. Choose a uniformizer $\varpi \in \mathcal{O}$ and let $\mathbb{F} = \mathcal{O}/(\varpi)$ be the residue field. Assume $\mathrm{char}(\mathbb{F}) = p > 2$. Let S be a finite set of primes with $p \in S$. Fix an absolute irreducible representation

$$\overline{\rho} : G_{\mathbb{Q},S} \longrightarrow \mathrm{GL}_2(\mathbb{F}).$$

Recall that a Taylor–Wiles datum of level $N \geqslant 1$ is a tuple $(Q, \{\alpha_v\}_{v\in Q})$ consisting of a finite set Q of primes such that $Q \cap S = \emptyset$ and for all $v \in Q$,

(TW1) $v \equiv 1 \bmod p^N$, and
(TW2) $\overline{\rho}(\mathrm{Frob}_v)$ has distinct $\mathbb{F}$-rational eigenvalues.

Also, for each $v \in Q$, $\alpha_v \in \mathbb{F}$ is a choice of eigenvalue.

Assume that $\overline{\rho} \cong \overline{\rho}_g$ with g a Hecke eigenform in $S_2(\Gamma, \mathcal{O})$ and $\Gamma_1(m) \subseteq \Gamma \subseteq \Gamma_0(M)$ for some integer $M \geqslant 1$ such that $\{\ell : \ell \mid M\} \subseteq S$ and that Γ is torsion-free. We define the arithmetic subgroups $\Gamma_1(Q) \subseteq \Gamma_Q \subseteq \Gamma_0(Q) \subseteq \Gamma$ as follows:

- $\Gamma_0(Q) = \Gamma \cap \Gamma_0(\prod_{v \in Q} v)$,
- $\Gamma_1(Q) = \Gamma \cap \Gamma_1(\prod_{v \in Q} v)$, and
- Γ_Q is the kernel of the map from $\Gamma_0(Q)$ to the maximal p-power quotient of $\Gamma_0(Q)/\Gamma_1(Q)$, for which the target is isomorphic to $\prod_{v \in Q} (\mathbb{Z}/v\mathbb{Z})^\times$.

For the third definition above we note that

$$\Gamma_0(Q)/\Gamma_Q \cong \Delta_Q = \prod_{v \in Q} \Delta_v.$$

For convenience, we write $S(\Gamma) := S_2(\Gamma, \mathcal{O})$ with fixed weight 2 in the upcoming context.

7.1. **Sketch to Taylor–Wiles geometrization method.**

Remark 7.1 (Analogy with Hida theory). First, recall the recipe to build a Hida family. For the space $S(\Gamma \cap \Gamma_1(p^{N+1}))^{\mathrm{ord}}$ with a deep level we take (co)invariant vectors under the group (say $p \nmid N$)

$$\Gamma \cap \Gamma_0(p^{N+1})/\Gamma \cap \Gamma_1(p^{N+1}) \cong (\mathbb{Z}/p^{N+1}\mathbb{Z})^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times \times \mathbb{Z}/p^N\mathbb{Z}.$$

Then we get a subspace $S(\Gamma \cap \Gamma_0(p^{N+1}))^{\mathrm{ord}}$. Applying Hida's idempotent, we obtain an isomorphism in a canonical sense:

$$S(\Gamma \cap \Gamma_0(p^{N+1}))^{\mathrm{ord}} \cong S(\Gamma \cap \Gamma_0(p))^{\mathrm{ord}}.$$

In this way, we get a canonical subspace of modular forms of weight 2 with a shallow level from the space with a deep level. Fixing a tame character, we can then build a module over the group algebra $\Lambda = \mathcal{O}[\![\mathbb{Z}_p]\!] \simeq \mathcal{O}[\![T]\!]$ such that modding out by the augmentation ideal recovers $S(\Gamma \cap \Gamma_0(p))^{\mathrm{ord}}$.

As for the construction by Taylor–Wiles, we take (co)invariants of $S(\Gamma_Q)_{\mathfrak{m}_Q}$ under

$$\Gamma_0(Q)/\Gamma_Q \cong \Delta_Q \approx (\mathbb{Z}/p^N\mathbb{Z})^q, \quad q = \#Q.$$

This outputs $S(\Gamma_0(Q))_{\mathfrak{m}_Q}$ by (TW1). Localizing at appropriate maximal ideals $\mathfrak{m}, \mathfrak{m}_Q$ of the Hecke algebras, there is a (non-canonical) isomorphism

$$S(\Gamma_0(Q))_{\mathfrak{m}_Q} \simeq S(\Gamma)_{\mathfrak{m}}.$$

We use this to build a module over $S_\infty = \mathcal{O}[\![\mathbb{Z}_p^q]\!] \simeq \mathcal{O}[\![y_1, \ldots, y_q]\!]$ such that modding out by the augmentation ideal recovers $S(\Gamma)_{\mathfrak{m}}$.

We have even more comments on Remark 7.1. Obviously, for any Taylor–Wiles prime v, there is $N \geqslant 1$ such that $v \equiv 1 \bmod p^N$ but $v \not\equiv 1 \bmod p^{N+1}$. So to pass from $\mathbb{Z}/p^N\mathbb{Z}$ to $\mathbb{Z}/p^{N+1}\mathbb{Z}$, and then in a limit to $\mathbb{Z}_p$, we will need to keep changing the Taylor–Wiles primes. The construction will thus be highly non-canonical, unlike Hida theory which is completely canonical.

Recall from Notation 2.1 that

$$\mathbb{T}^{S,\mathrm{univ}} = \mathcal{O}[\{T_\ell, S_\ell\}_{\ell \notin S}].$$

For a subset $\Sigma \subseteq S$ we also define

$$\mathbb{T}_\Sigma^{S,\mathrm{univ}} = \mathbb{T}^{S,\mathrm{univ}}[\{U_v\}_{v \in \Sigma}].$$

Here T_ℓ, S_ℓ, and U_v are just polynomial variables. But these universal Hecke algebras then act on spaces of modular forms, homology spaces, cohomology spaces, etc. by letting T_ℓ, S_ℓ, and U_v act as the operators with the same name. In particular, we take $\mathbb{T}^S(\Gamma)$ and $\mathbb{T}^S_\Sigma(\Gamma)$ to be the images of $\mathbb{T}^{S,\mathrm{univ}}$ and $\mathbb{T}^{S,\mathrm{univ}}_\Sigma$ in $\mathrm{End}_\mathcal{O} H^1(\Gamma,\mathcal{O})$, respectively.

By our assumption $\overline{\rho} \cong \overline{\rho}_g$ for $g \in S_2(\Gamma,\mathcal{O})$, we obtain a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^S(\Gamma)$ which we can also think of as a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{S,\mathrm{univ}}$ in the support of $H^1(\Gamma,\mathcal{O})$. We then consider the action

$$\mathbb{T}^S(\Gamma)_\mathfrak{m} \circlearrowright H^1(\Gamma,\mathcal{O})_\mathfrak{m} \cong H^1(Y,\mathcal{O})_\mathfrak{m}$$

for $Y = Y(\Gamma)$. Recall in the proof of Proposition 2.4 (with $k = 2$) on Page 5 that

$$H^i(\Gamma,\mathbb{F})_\mathfrak{m} = 0, \quad i \neq 1,$$

and as a consequence that $H^1(\Gamma,\mathcal{O})_\mathfrak{m} \cong H^1(Y,\mathcal{O})_\mathfrak{m}$ is torsion free. Hence

$$H^1(Y,\mathcal{O})_\mathfrak{m} \cong \mathrm{Hom}_\mathcal{O}(H_1(Y,\mathcal{O})_\mathfrak{m},\mathcal{O})$$

as $\mathbb{T}^{S,\mathrm{univ}}$-modules. Moreover, transpose identifies $\mathbb{T}^S(\Gamma)_\mathfrak{m}$ with the image of $\mathbb{T}^{S,\mathrm{univ}}_\mathfrak{m}$ in $\mathrm{End}_\mathcal{O}(H_1(Y,\mathcal{O})_\mathfrak{m})$.

Remark 7.2. Homology theory is more natural than cohomology theory for Taylor–Wiles patching as we wish to have a map from level Γ_Q to level Γ that is taking coinvariants by Δ_Q. (Although one can often use cohomology theory by using a trace map.)

7.2. **Universally-Hecke valued cohomology of modular curve.** Recall that we have a fixed Taylor–Wiles datum $(Q,\{\alpha_v\}_{v\in Q})$. We can pullback $\mathfrak{m} \subseteq \mathbb{T}^{S,\mathrm{univ}}$ to a maximal ideal of $\mathbb{T}^{S\cup Q,\mathrm{univ}}$ and we again denote it by $\mathfrak{m}$.

For each $v \in Q$, the characteristic polynomial

$$\mathbb{T}^{S,\mathrm{univ}}[X] \ni X^2 - T_vX + vS_v \equiv (X-\alpha_v)(X-\beta_v) \bmod \mathfrak{m},$$

which is also the Hecke polynomial of $\overline{g} := g \bmod \varpi \in S_2(\Gamma,\mathbb{F})$. By the theory of old forms, we know that there is $\overline{g}' \in S_2(\Gamma_0(\mathbb{Q}),\mathbb{F})$ that has the same T_ℓ, S_ℓ-eigenvalues as $\overline{g}$ for some $\ell \in S \cup Q$ and such that for any $v \in Q$,

$$U_v\overline{g} = \alpha_v\overline{g}.$$

Thus, choosing any lift $\widetilde{\alpha}_v \in \mathcal{O}$ of α_v and defining

$$\mathfrak{m}_Q = (\mathfrak{m},\{U_v - \widetilde{\alpha}_v\}_{v\in Q}) \subseteq \mathbb{T}^{S\cup Q,\mathrm{univ}}_Q$$

as a maximal ideal; both $\mathfrak{m} \subseteq \mathbb{T}^{S\cup Q,\mathrm{univ}}$ and $\mathfrak{m}_Q \subseteq \mathbb{T}^{S\cup Q,\mathrm{univ}}_Q$ are in the supports of

$$H^1(Y_0(Q),\mathcal{O}), \quad H_1(Y_0(Q),\mathcal{O}),$$

respectively, where

$$Y_0(Q) := \Gamma_0(Q)\backslash\mathbb{H}$$

is the modular curve defined by $\Gamma_0(Q)$. And we again have the duality between these spaces after respectively localizing at $\mathfrak{m}_Q$ or $\mathfrak{m}$.

Note also that since $\mathbb{T}^{S\cup Q}(\Gamma_0(Q))$ and $\mathbb{T}^{S\cup Q}_Q(\Gamma_0(Q))$ are finite $\mathbb{Z}_p$-algebras, the Hecke algebra $\mathbb{T}^{S\cup Q}(\Gamma_0(Q))_\mathfrak{m}$ is a complete local noetherian ring, and the localization of $\mathbb{T}^{S\cup Q}_Q(\Gamma_0(Q))$ at $\mathfrak{m} \subseteq \mathbb{T}^{S\cup Q}(\Gamma_0(Q))$ is thus a complete semilocal ring, and hence a product of its local rings of which $\mathbb{T}^{S\cup Q}_Q(\Gamma_0(Q))_{\mathfrak{m}_Q}$ is a complete semilocal ring as well. In particular, $H_1(Y_0(Q),\mathcal{O})_{\mathfrak{m}_Q}$ is a direct summand of $H_1(Y_0(Q),\mathcal{O})_\mathfrak{m}$.

Remark 7.3. The similar statements all hold with $\Gamma_0(Q)$ replaced by Γ_Q.

Proposition 7.4. *The natural map $H_1(Y_0(Q),\mathcal{O}) \to H_1(Y,\mathcal{O})$ induces an isomorphism*

$$H_1(Y_0(Q),\mathcal{O})_{\mathfrak{m}_Q} \cong H_1(Y,\mathcal{O})_\mathfrak{m}$$

of $\mathbb{T}^{S\cup Q,\mathrm{univ}}$-local modules.

This will be proved later by following Khare–Thorne (c.f. the context from page 42 to the end of this section).

Comment. Here is some way to think about Proposition 7.4. Say we just work with collections of eigenforms (instead of homology). If an eigenform f is new of level $\Gamma_0(v)$ at place v, then its Galois representation ρ_f at v has semisimplification $\chi_1 \oplus \chi_2$ with $\chi_1\chi_2^{-1} = \epsilon_p$, the p-adic cyclotomic character. Hence if $\overline{\rho}_f$ is unramified at v and $v \equiv 1 \bmod p$, then $\overline{\rho}_f(\mathrm{Frob}_v)$ does not has distinct eigenvalues.

Thus localization at $\mathfrak{m}$ kills all forms that are new at any $v \in Q$. The resulting Q-algebraic form in $\Gamma_0(Q)$ looks like 2 copies of the level Γ-forms and forcing $U_v \equiv \alpha_v$ acts on the first copy (again using that the eigenvalues are distinct).

The above argument uses characteristic 0 information, which is insufficient in "defect" cases (for example, over imaginary quadratic fields). Khare and Thorne's proof works purely over $\mathbb{F}$ and is applicable in those more general settings.

Recall that given $\overline{\rho} \cong \overline{\rho}_g$, we can get a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^S(\Gamma)$ that is also considered as a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{S,\mathrm{univ}}$. Here comes the key fact of the geometrization method.

Proposition 7.5 (Key fact, cohomological vanishing of modular curves)**.** *We have*

$$H^i(Y, \mathbb{F})_{\mathfrak{m}} = 0, \quad i \neq 1.$$

This has an immediate consequence. Apply the cohomology to the short exact sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{\varpi} \mathcal{O} \longrightarrow \mathbb{F} \longrightarrow 0$$

and then localize at $\mathfrak{m}$. Then

(a) $H^i(Y, \mathcal{O})_{\mathfrak{m}} = 0$ if $i \neq 1$;

(b) $H^1(Y, \mathcal{O})_{\mathfrak{m}}$ is p-torsion free, and

$$H^1(Y, \mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} \mathbb{F} \cong H^1(Y, \mathbb{F})_{\mathfrak{m}}.$$

Recall that for a fixed Taylor–Wiles datum $(Q, \{\alpha_v\}_{v \in Q})$, we can define $\Gamma_0(Q) := \Gamma \cap \Gamma_0(\prod_{v\in Q} v)$, which determines the modular curve $Y_0(Q) = \Gamma_0(Q)\backslash\mathbb{H}$. Also, there is a normal subgroup Γ_Q of $\Gamma_0(Q)$ such that $\Gamma(Q)/\Gamma_Q \simeq \Delta_Q$, the maximal p-power quotient of $\prod_{v\in Q}(\mathbb{Z}/v)^\times$. Hence we get another modular curve $Y_Q = \Gamma_Q\backslash\mathbb{H}$. Similarly, we have $\mathbb{T}^{S\cup Q}(\Gamma_0(Q))$ and $\mathbb{T}^{S\cup Q}(\Gamma_Q)$.

Lemma 7.6. *Denote*

$$\mathfrak{m}^Q := \mathfrak{m} \cap \mathbb{T}^{S\cup Q,\mathrm{univ}}.$$

Then the following natural inclusion is an isomorphism:

$$\mathbb{T}^{S\cup Q}(\Gamma)_{\mathfrak{m}^Q} \xrightarrow{\sim} \mathbb{T}^S(\Gamma)_{\mathfrak{m}}.$$

Proof. By Nakayama's lemma, it suffices to prove

$$\mathbb{T}^S(\Gamma)_{\mathfrak{m}}/\mathfrak{m}^Q = \mathbb{F}.$$

This further reduces to proving that for all $v \in Q$,

$$T_v,\ S_v \bmod \mathfrak{m}^Q \in \mathbb{F}.$$

Since $\overline{\rho}$ is absolutely irreducible, we have Galois representations

- $\rho_{\mathfrak{m}} : G_{\mathbb{Q},S} \longrightarrow \mathrm{GL}_2(\mathbb{T}(\Gamma)_{\mathfrak{m}})$ such that for any $\ell \in S$, the characteristic polynomial $\rho_{\mathfrak{m}}(\mathrm{Frob}_\ell) = X^2 - T_\ell X + \ell S_\ell$.
- $\rho_{\mathfrak{m}^Q} : G_{\mathbb{Q},S\cup Q} \longrightarrow \mathrm{GL}_2(\mathbb{T}(\Gamma)_{\mathfrak{m}^Q})$ such that for any $\ell \notin S \cup Q$, the characteristic polynomial $\rho_{\mathfrak{m}^Q}(\mathrm{Frob}_\ell) = X^2 - T_\ell X + S_\ell$.

Consider the Galois representation $\rho := \rho_{\mathfrak{m}} \bmod \mathfrak{m}^Q$. Then for all $\ell \in S \cup Q$ we have

$$\begin{aligned}\operatorname{tr}\rho(\mathrm{Frob}_\ell) &\equiv \operatorname{tr}\rho_{\mathfrak{m}}(\mathrm{Frob}_\ell) \equiv T_\ell \equiv \operatorname{tr}\rho_{\mathfrak{m}^Q}(\mathrm{Frob}_\ell) \bmod \mathfrak{m}^Q \\ &= \operatorname{tr}\overline{\rho}(\mathrm{Frob}_\ell) \in \mathbb{F}.\end{aligned}$$

By the continuity of $\operatorname{tr}\rho$, we deduce that $\operatorname{tr}\rho$ is $\mathbb{F}$-valued. In particular for $v \in Q$,

$$T_v \bmod \mathfrak{m}^Q = \operatorname{tr}\rho(\mathrm{Frob}_v) \in \mathbb{F}.$$

Similarly, one can use det in place of tr to show that $S_v \bmod \mathfrak{m}^Q \in \mathbb{F}$ for all $v \in Q$ as well. $\square$

In particular, we see that

$$H^i(Y,\mathcal{O})_{\mathfrak{m}^Q} = H^i(Y,\mathcal{O})_{\mathfrak{m}}$$

by Lemma 7.6 and similarly,

$$H^i(Y,\mathbb{F})_{\mathfrak{m}^Q} = H^i(Y,\mathbb{F})_{\mathfrak{m}}.$$

Because of this, we will just write $\mathfrak{m} = \mathfrak{m}^Q$ from now on. Define

$$\mathbb{T}_Q^{S\cup Q,\mathrm{univ}} = \mathbb{T}^{S\cup Q,\mathrm{univ}}[\{U_v\}_{v\in Q}].$$

For each $v \in Q$, choose $\widetilde{\alpha}_v \in \mathcal{O}$ a preimage of $\alpha_v \in \mathbb{F}$ and define

$$\mathfrak{m}_Q = \mathfrak{m}_\alpha = (\mathfrak{m}, \{U_v - \widetilde{\alpha}_v\}_{v\in Q}) \subseteq \mathbb{T}_Q^{S\cup Q,\mathrm{univ}}.$$

From the theory of old forms, we see $\mathfrak{m}_Q$ is in the support of $H^1(Y_0(Q),\mathbb{F})$ (this will be in display in the proof of Proposition 7.7 below). Assuming this fact, note that

$$\mathbb{T}^{S\cup Q}(\Gamma_0(Q)) \longrightarrow \mathbb{T}_Q^{S\cup Q}(\Gamma_0(Q))$$

is a map of finite $\mathcal{O}$-algebras. Then $\mathbb{T}^{S\cup Q}(\Gamma_0(Q))_{\mathfrak{m}}$ is a complete local ring, and $\mathbb{T}_Q^{S\cup Q}(\Gamma_0(Q))_{\mathfrak{m}}$ is a complete semilocal ring. So $\mathbb{T}_Q^{S\cup Q}(\Gamma_0(Q))_{\mathfrak{m}}$ is isomorphic to a product of its local rings, for which $\mathbb{T}_Q^{S\cup Q}(\Gamma_0(Q))_{\mathfrak{m}_Q}$ is one of these copies.

Proposition 7.7. *The natural map $H^1(Y,\mathcal{O}) \to H^1(Y_0(Q),\mathcal{O})$ defines an isomorphism of $\mathbb{T}^{S\cup Q,\mathrm{univ}}$-modules*

$$H^1(Y,\mathcal{O})_{\mathfrak{m}} \xrightarrow{\sim} H^1(Y_0(Q),\mathcal{O})_{\mathfrak{m}_Q}$$

with inverse induced by the trace map, up to $\mathcal{O}^\times$.

Exercise 7.8. Prove Proposition 7.7. For this, first note that since both sides are free $\mathcal{O}$-modules, and

$$\begin{aligned}H^1(Y,\mathcal{O})\otimes_{\mathcal{O}}\mathbb{F} &\cong H^1(Y,\mathbb{F})_{\mathfrak{m}},\\ H^1(Y_0(Q),\mathcal{O})\otimes_{\mathfrak{m}_Q}\mathbb{F} &\cong H^1(Y_0(Q),\mathbb{F})_{\mathfrak{m}_Q}.\end{aligned}$$

Thus, it suffices to show that

$$H^1(Y,\mathbb{F})_{\mathfrak{m}} \simeq H^1(Y_0(Q),\mathbb{F})_{\mathfrak{m}_Q}.$$

7.3. **Geometric level raising via Satake isomorphism.**

Setups. Denote $K = \mathrm{GL}_2(\mathbb{Z}_v)$ and

$$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \,\middle|\, c \equiv 0 \bmod v \right\}.$$

For an open compact subgroup U of K, define $\mathcal{H}_U$ as the convolution algebra of compactly supported bi-U-invariant functions $f : \mathrm{GL}_2(\mathbb{Q}_v) \to \mathbb{F}$. Indeed, $\mathcal{H}_U$ is generated by double coset operators $[UgU]$ when g runs through all elements of $\mathrm{GL}_2(\mathbb{Q}_v)$. Set

$$M := H^1(Y_0(Q),\mathbb{F})_{\mathfrak{m}}, \quad N := H^1(Y,\mathbb{F})_{\mathfrak{m}}.$$

Then M is an $\mathcal{H}_I$-module and N is an $\mathcal{H}_K$-module. Some isomorphisms below will depend on a choice of a square root $v^{1/2}$ of v in our coefficient field $\mathbb{F}$. But $v \equiv 1 \bmod p$, so we can choose

$$v^{1/2} := 1.$$

The keynote question above these constructions is to find out the relation between M and N. This will be called the "geometric level raising" or "cohomological level raising".

We have the Hecke operators

$$T_v = \left[K \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} K\right], \quad S_v = \left[K \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} K\right]$$

and $\mathcal{H}_K = \mathbb{F}[T_v, S_v]$. Let T be the diagonal torus in GL_2 and $X_*(T)$ be the group of cocharacters of form $\mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ with

$$\lambda_1(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}, \quad \lambda_2(t) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}.$$

Let $W = \{1, w_0\}$ be the Weyl group.

The Satake isomorphism is stated as

$$\begin{aligned} \mathcal{H}_K &\xrightarrow{\quad\cong\quad} \mathbb{F}[X_*(T)]^W \\ T_v = v^{1/2} T_v &\longmapsto \lambda_1 + \lambda_2 \\ S_v = v S_v &\longmapsto \lambda_1 \lambda_2. \end{aligned}$$

We have an analog description of $\mathcal{H}_I$ as follows.

Lemma 7.9. *Using the fact that $v \equiv 1 \bmod p$, we have*

$$\mathbb{F}[X_*(T) \rtimes W] \xrightarrow{\cong} \mathcal{H}_I.$$

This sends $\lambda \in X_*(T)_+ = \{a\lambda_1 + b\lambda_2 \mid a > b\}$ to $[I\lambda(v)I]$, and sends $w \in W$ to $[I\widetilde{w}I]$, where $\widetilde{w} \in N(T)$ is a lift of w. Under this isomorphism, the centre $Z(\mathcal{H}_I)$ of $\mathcal{H}_I$ corresponds to $\mathbb{F}[X_*(T)]^W$ and the composite

$$\mathbb{F}[X_*(T)]^W \cong Z(\mathcal{H}_I) \longrightarrow \mathcal{H}_K, \quad f \longmapsto [K]f$$

is the Satake isomorphism.

Proof. This lemma follows from Berstein presentation, or Iwahori–Matsumura presentation of $\mathcal{H}_I$. See §5 of Chandrashekhar B. Khare and Jack A. Thorne, *Potential automorphy and the Leopoldt conjecture* (Amer. J. Math. **139** (2017), no. 5, 1205–1273) for more details. ☐

Note for later that under this isomorphism,

$$\lambda_1 \longmapsto [I\lambda_1(v)I] = \left[I \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} I\right] = U_v.$$

Lemma 7.10. *The inclusion $N \subseteq M$ is split by $M \to N$, $x \mapsto [K]x$.*

Proof. Geometrically, the map

$$M = H^1(Y_0(Q), \mathbb{F})_{\mathfrak{m}} \longrightarrow N = H^1(Y, \mathbb{F})_{\mathfrak{m}}, \quad x \longmapsto [K]x$$

is the trace map (in terms of group cohomology, it is the corestriction map). Thus the composite

$$N \longrightarrow M \xrightarrow{[K]} N$$

is multiplication by $[K : I] = v + 1 \in \mathbb{F}^\times$ as $v = 1$ in $\mathbb{F}$ and $p > 2$. ☐

Note also that in $\mathcal{H}_I$,

$$[K] = [I] + [Iw_0I] = 1 + w_0 = \sum_{w \in W} w.$$

Since $|W|$ is invertible in $\mathbb{F}$, we have $M^W = \left(\sum_{w \in W} w\right) \cdot M$. It is also known that $N \subseteq M^W$. Then Lemma 7.10 gives

$$N = M^W = [K]M = \left(\sum_{w \in W} w\right) \cdot M.$$

On the other hand, T_v and S_v act on N by $\alpha_v + \beta_v$ and $\alpha_v\beta_v$, respectively. Define the maximal ideal

$$\mathfrak{n} = (\lambda_1 + \lambda_2 - \alpha_v - \beta_v, \lambda_1\lambda_2 - \alpha_v\beta_v) \subset \mathbb{F}[X_*(T)]^W \cong \mathbb{F}[T_v, S_v].$$

then $N_{\mathfrak{n}} = N$ as a local module. Since $\alpha_v \neq \beta_v$, these are precisely two maximal ideals $\mathfrak{m}_\alpha, \mathfrak{m}_\beta \subset \mathbb{F}[X_*(T)] = \mathbb{F}[\lambda_1, \lambda_2]$ above $\mathfrak{n}$, given by

$$\mathfrak{m}_\alpha = (\lambda_1 - \alpha_v, \lambda_2 - \beta_v), \quad \mathfrak{m}_\beta = (\lambda_1 - \beta_v, \lambda_2 - \alpha_v).$$

Note that since λ_1 corresponds to U_v under the isomorphism of Lemma 7.9, we have

$$M_{\mathfrak{m}_\alpha} = H^1(Y_0(Q), \mathbb{F})_{\mathfrak{m}_Q}.$$

So we want to show that the composite

$$N \longrightarrow M \longrightarrow M_{\mathfrak{m}_\alpha}$$

is an isomorphism. Also note that $N = N_{\mathfrak{n}}$ and $\mathfrak{n} = \mathfrak{m}_\alpha \cap \mathbb{F}[X_*(T)]$, so it suffices to show that, on the level of local modules,

$$N \longrightarrow M_{\mathfrak{n}} \longrightarrow M_{\mathfrak{m}_\alpha}$$

is an isomorphism. For this, the following lemma is in need.

Lemma 7.11. *We have $M_{\mathfrak{n}} \cong M_{\mathfrak{m}_\alpha} \oplus M_{\mathfrak{m}_\beta}$, and the nontrivial element $w_0 \in W$ of the Weyl group maps $M_{\mathfrak{m}_\alpha}$ isomorphically to $M_{\mathfrak{m}_\beta}$.*

Proof. Since M is finite-dimensional over $\mathbb{F}$ as a vector space, the action of $\mathbb{F}[X_*(T)]_{\mathfrak{n}}$ on $M_{\mathfrak{n}}$ factors through an Artinian quotient algebra A of $\mathbb{F}[X_*(T)]$. Since $\mathfrak{m}_\alpha$ and $\mathfrak{m}_\beta$ are the two distinct maximal ideals of the Artinian ring A, we have $A \cong A_{\mathfrak{m}_\alpha} \times A_{\mathfrak{m}_\beta}$, which induces the decomposition

$$M \cong M_{\mathfrak{m}_\alpha} \oplus M_{\mathfrak{m}_\beta}.$$

It is straightforward from the definitions that W permutes $\mathfrak{m}_\alpha$ and $\mathfrak{m}_\beta$, and the second assertion follows. $\square$

Now we can summarize. Since $N = M^W = [K]M = M_{\mathfrak{n}}^W = [K]M_{\mathfrak{n}}$, it follows that the composites

$$N \longrightarrow M_{\mathfrak{n}} \cong M_{\mathfrak{m}_\alpha} \oplus M_{\mathfrak{m}_\beta} \longrightarrow M_{\mathfrak{m}_\alpha}$$

and

$$M_{\mathfrak{m}_\alpha} \longrightarrow M_{\mathfrak{m}_\alpha} \oplus M_{\mathfrak{m}_\beta} \cong M_{\mathfrak{n}} \xrightarrow{[K]} N$$

are inverse isomorphisms, up to $\mathbb{F}^\times$.

7.4. **A work review of Khare–Thorne.**

Proposition 7.12. *$H_1(Y_Q,\mathcal{O})_{\mathfrak{m}_Q}$ is a free $\mathcal{O}[\Delta_Q]$-module and the natural map*

$$H_1(Y_Q,\mathcal{O})_{\mathfrak{m}_Q} \longrightarrow H_1(Y_0(Q),\mathcal{O})_{\mathfrak{m}_Q}$$

induces an isomorphism from the Δ_Q-coinvariants of $H_1(Y_Q,\mathcal{O})_{\mathfrak{m}_Q}$ to $H_1(Y_0(Q),\mathcal{O})_{\mathfrak{m}_Q}$.

Combining Proposition 7.4 and Proposition 7.12 above, we get the following variant version, which is the main result of this subsection.

Proposition 7.13. *$H_1(Y_Q,\mathcal{O})_{\mathfrak{m}_Q}$ is a free $\mathcal{O}[\Delta_Q]$-module and the natural map*

$$H_1(Y_Q,\mathcal{O})_{\mathfrak{m}_Q} \longrightarrow H_1(Y,\mathcal{O})_{\mathfrak{m}}$$

induces an isomorphism from the Δ_Q-coinvariants of $H_1(Y_Q,\mathcal{O})_{\mathfrak{m}_Q}$ to $H_1(Y,\mathcal{O})_{\mathfrak{m}}$.

Our main proposition is true while localizing at $\mathfrak{m}=\mathfrak{m}\cap\mathbb{T}^{S\cup Q,\mathrm{univ}}$ and in fact, the localization at $\mathfrak{m}_Q$ is a direct summand of it (which will be discussed below).

To prove Proposition 7.13, we first recall a key fact that if $i\neq 1$,

$$H_i(Y_Q,\mathbb{F})_{\mathfrak{m}_Q}=\mathrm{Hom}(H^i(Y_Q,\mathcal{O})_{\mathfrak{m}_Q},\mathbb{F})=0$$

and, as a consequence,

$$H_i(Y_Q,\mathcal{O})_{\mathfrak{m}_Q}=\begin{cases}\text{vanishes} & \text{if } i\neq 1,\\ \text{is } \mathcal{O}\text{-free} & \text{if } i=1.\end{cases}$$

Proof. We switch to group homology for the proof. The Hochschild-Serre spectral sequence gives

$$H_i(\Delta_Q,H_j(\Gamma_Q,\mathcal{O}))\Longrightarrow H_{i+j}(\Gamma_0(Q),\mathcal{O}).$$

Localizing at $\mathfrak{m}$ and using the above key fact, we get

$$H_0(\Delta_Q,H_1(\Gamma_Q,\mathcal{O})_{\mathfrak{m}})\cong H_1(\Gamma_0(Q),\mathcal{O})_{\mathfrak{m}}.$$

Now it remains to prove that $H_1(\Gamma_Q,\mathcal{O})_{\mathfrak{m}_Q}$ is free over $\mathcal{O}[\Delta_Q]$. We morally insert a fact from commutative algebra: an $\mathcal{O}[\Delta_Q]$-module M is free if and only if M is flat; if and only if $\mathrm{Tor}_1^{\mathcal{O}[\Delta_Q]}(M,\mathbb{F})=0$.

First, again using Hochschild-Serre,

$$0=H_2(\Gamma_0(Q),\mathcal{O})_{\mathfrak{m}_Q}=H_1(\Delta_Q,H_1(\Gamma_Q,\mathcal{O})_{\mathfrak{m}_Q})=\mathrm{Tor}_1^{\mathcal{O}[\Delta_Q]}(H_1(\Gamma_Q,\mathcal{O})_{\mathfrak{m}_Q},\mathcal{O}).$$

Then tensoring

$$0\longrightarrow\mathcal{O}\xrightarrow{\varpi}\mathcal{O}\longrightarrow\mathbb{F}\longrightarrow 0$$

over $\mathcal{O}[\Delta_Q]$ with $H_1(\Gamma_Q,\mathcal{O})_{\mathfrak{m}_Q}$ and using the above equality, we have an exact sequence

$$\begin{tikzcd}
0=\mathrm{Tor}_1^{\mathcal{O}[\Delta_Q]}(H_1(\Gamma_Q,\mathcal{O})_{\mathfrak{m}_Q},\mathcal{O}) \arrow[r] & \mathrm{Tor}_1^{\mathcal{O}[\Delta_Q]}(H_1(\Gamma_Q,\mathcal{O})_{\mathfrak{m}_Q},\mathbb{F}) \arrow[dl] & & \\
H_1(\Gamma_Q,\mathcal{O})_{\mathfrak{m}_Q}\otimes_{\mathcal{O}[\Delta_Q]}\mathcal{O} \arrow[r, "\varpi"] \arrow[d, equal] & H_1(\Gamma_Q,\mathcal{O})_{\mathfrak{m}_Q}\otimes_{\mathcal{O}[\Delta_Q]}\mathcal{O} \arrow[r, "\varpi"] \arrow[d, equal] & H_1(\Gamma,\mathbb{F})_{\mathfrak{m}_Q} \arrow[r] & 0. \\
H_1(\Gamma_0(Q),\mathcal{O})_{\mathfrak{m}_Q} \arrow[r, "\varpi"] & H_1(\Gamma_0(Q),\mathcal{O})_{\mathfrak{m}_Q} \arrow[ur, bend right] & &
\end{tikzcd}$$

But the lower horizontal map $H_1(\Gamma_0(Q),\mathcal{O})_{\mathfrak{m}_Q}\longrightarrow H_1(\Gamma_0(Q),\mathcal{O})_{\mathfrak{m}_Q}$ is injective, so

$$\mathrm{Tor}_1^{\mathcal{O}[\Delta_Q]}(H_1(\Gamma_Q,\mathcal{O})_{\mathfrak{m}_Q},\mathbb{F})=0.$$

This completes the proof due to the commutative algebra fact. □

Recall that if we have a global deformation datum

$$\mathcal{S}=(\overline{\rho},S,\psi,\mathcal{O},\{\mathcal{D}_v\}_{v\in S}).$$

Then we have an augmented deformation datum

$$\mathcal{S}_Q=(\overline{\rho},S,\psi,\mathcal{O},\{\mathcal{D}_v\}_{v\in S}\cup\{\mathcal{D}_v^{\square,\psi}\}_{v\in Q}),$$

where $R_{\mathcal{S}_Q}$ is an $\mathcal{O}[\Delta_Q]$-algebra such that

$$R_{\mathcal{S}_Q}/\mathfrak{a}_Q\cong R_{\mathcal{S}}$$

with $\mathfrak{a}_Q$ the augmented ideal. On the other hand, we also have Galois representations

$$\rho_{\mathfrak{m}}:G_{\mathbb{Q},S}\longrightarrow \mathrm{GL}_2(\mathbb{T}^S(\Gamma)_{\mathfrak{m}}),$$

and

$$\rho_{\mathfrak{m}_Q}:G_{\mathbb{Q},S}\longrightarrow \mathrm{GL}_2(\mathbb{T}_Q^{S\cup Q}(\Gamma)_{\mathfrak{m}_Q}).$$

If they are respectively of type $\mathcal{S}$ and $\mathcal{S}_Q$, then we have

$$\begin{tikzcd}
R_{\mathcal{S}_Q} \arrow[r, hook] \arrow[d, two heads, "\bmod \mathfrak{a}_Q"'] & H_1(Y_Q,\mathcal{O})_{\mathfrak{m}_Q} \arrow[d, two heads, "\bmod \mathfrak{a}_Q"] \\
R_{\mathcal{S}} \arrow[r, hook] & H_1(Y,\mathcal{O})_{\mathfrak{m}}.
\end{tikzcd}$$

8. Interlude: Local-global compatibility

In the upcoming context, we briefly introduce the local-global compatibility theorem together with its concrete consequences.

Theorem 8.1 (Local-global compatibility)**.** *Let f be a cuspidal Hecke eigenform with associated cuspidal automorphic representation π_f of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. Let p be a prime and let $\iota:\mathbb{C}\xrightarrow{\sim}\overline{\mathbb{Q}}_p$ be an isomorphism, and*

$$\rho_{f,\iota}:G_{\mathbb{Q}}\longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

be the associated Galois representation. Then

(1) *For any prime ℓ,*

$$\mathrm{WD}(\rho_{f,\iota}|_{G_{\mathbb{Q}_\ell}})^{\mathrm{Frob-ss}}\cong \mathrm{LL}(\pi_\ell\otimes|\det|_\ell^{-1/2})\otimes_{\mathbb{C},\iota}\overline{\mathbb{Q}}_p.$$

Here the LHS is the Frobenius-semisimplification of Weil-Deligne representation associated with $\rho_{f,\iota}|_{G_{\mathbb{Q}_\ell}}$; the RHS is the (suitably normalized, e.g. irreducible admissible smooth representations) local Langlands correspondence.

(2) *$\rho_{f,\iota}|_{G_{\mathbb{Q}_p}}$ is de Rham with Hodge–Tate weights 0 and $k-1$, where k is the wight of f.*

The reader who is not familiar with the background of Langlands correspondence may regard the theorem as a vague description. However, we will make the theorem concrete in particular cases later. We also remark that (1) is the classical local-global compatibility, and to access (1), assertion (2) is in need, which implies more information.

Concrete consequences. Assume $\ell\neq p$. Take $f\in S_2(\Gamma_1(N),\overline{\mathbb{Q}}_p)$ a newform that is not coming from $\Gamma_1(M)$ for any $M<N$. Let η be Nebentypus of f and C be the conductor of ι (so $C\mid N$ follows). By abuse of the notation, denote again by η the Galois character $\eta:G_{\mathbb{Q}}\to\overline{\mathbb{Q}}_p$ corresponding to η via class field theory. Let ϵ be the p-adic cyclotomic character.

- If $\ell \nmid N$, ρ_f is unramified at ℓ and the characteristic polynomial of $\rho_f(\mathrm{Frob}_\ell)$ is
$$X^2 - a_\ell X + \eta(\ell)\ell,$$
where a_ℓ equals to T_ℓ-eigenvalues of f.
- If $\ell \parallel N$ and $\ell \nmid C$ (at ℓ, f is a newform of level $\Gamma_0(\ell)$), then
$$\rho_f|_{G_{\mathbb{Q}_\ell}} \cong \begin{pmatrix} \gamma & * \\ 0 & \gamma\epsilon^{-1} \end{pmatrix}$$
where Γ is the unramified character with $\gamma(\mathrm{Frob}_\ell)$ equal to U_ℓ eigenvalue of f, and ρ_f is the associated global Galois representation such that $1 \neq \rho_f(I_\ell) \subseteq \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$.
- If $\ell \parallel N$ and $\ell \parallel C$, then
$$\rho_f|_{G_{\mathbb{Q}_\ell}} \cong \gamma \oplus \gamma^{-1}\epsilon^{-1}\eta$$
where γ is the unramified character with $\gamma(\mathrm{Frob}_\ell)$ equal to U_ℓ eigenvalue of f.
- If $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is irreducible, then $\ell^2 \mid N$.
- If $p \nmid N$ and the T_p-eigenvalue a_p of f is a unit, then
$$\rho_f|_{G_{\mathbb{Q}_p}} \cong \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}.$$
Moreover, χ_1 is unramified and $\chi_1(\mathrm{Frob}_p)$ is a unit root of $X^2 - a_p X + \eta(p)p$; and $\chi_2|_{I_p} = \epsilon^{-1}$.

9. The Galois patching

9.1. Running assumptions for patching.

Setups. Fix

- $g \in S_2(\Gamma_1(N), \overline{\mathbb{Q}}_p)$ a newform with η its Nebentypus;
- p a prime number;
- $E/\mathbb{Q}_p$ a finite extension with ring of integers $\mathcal{O}$, and a choice of the uniformizer ϖ with the corresponding residue field $\mathbb{F} = \mathcal{O}/(\varpi)$.

Let $\overline{\rho} := \overline{\rho}_g : G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be the associated mod p representation. Assume $\mathbb{F}$ is sufficiently large so that the eigenvalues of all $\overline{\rho}(\sigma)$ for $\sigma \in G_\mathbb{Q}$ are lying in $\mathbb{F}$.

We assume

- ◇ $p > 2$ and $p \nmid N$;
- ◇ $\overline{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is absolutely irreducible with enormous (a.k.a. adequate) image (valid for $p \geqslant 7$);
- ◇ N is a square-free integer and $\overline{\rho}$ is ramified at all $\ell \mid N$ (restrictive) and η has prime-to-p order. Equivalently, we assume that $\overline{\rho}$ is modular of weight 2 and level $N(\overline{\rho})$, the Artin conductor, and $N(\overline{\rho})$ is square-free.
- ◇ (*unnecessary, for convenience only*)
$$\overline{\rho}|_{G_{\mathbb{Q}_p}} \cong \begin{pmatrix} \overline{\chi}_1 & * \\ 0 & \overline{\chi}_2 \end{pmatrix}$$
with $\overline{\chi}_1|_{I_p} = 1$ and $\overline{\chi}_2|_{I_p} = \overline{\epsilon}^{-1}$.

We then define a global deformation problem
$$(\overline{\rho}, S, \psi, \mathcal{O}, \{\mathcal{D}_v\}_{v \in S})$$

by $S = \{\ell : \ell \mid N\} \cup \{p\}$, $\psi = \eta\epsilon^{-1}$, and

$$\mathcal{D}_v = \begin{cases} \mathcal{D}_v^{\min} & \text{if } v \mid N, \\ \mathcal{D}_v^{\mathrm{ord}} & \text{if } v = p. \end{cases}$$

The definition for $\mathcal{D}_v^{\min}$ is restrictive for modularity lifting purposes but still has interesting consequences.

Let $\Gamma_1(N) \leqslant \Gamma \leqslant \Gamma_0(N)$ be

$$\Gamma = \ker(\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\eta} \overline{\mathbb{Q}}_p^\times).$$

Assume that

- $\diamond$ Γ is torsion-free. (One can get something similar around this, but we assume it for simplicity.)

Let $\mathfrak{m} \subseteq \mathbb{T}^{S,\mathrm{univ}}$ be the maximal ideal corresponding to $\overline{\rho}$.

Theorem 9.1. *The Galois representation $\rho_{\mathfrak{m}} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{T}^S(\Gamma)_{\mathfrak{m}})$ lifting $\overline{\rho}$ is of type $\mathcal{S}$. Consequently, there is a map in $\mathsf{CNL}_{\mathcal{O}}$*

$$R_{\mathcal{S}} \longrightarrow \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$$

and it is surjective.

Goal. We are to show this map $R_{\mathcal{S}} \twoheadrightarrow \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$ is an isomorphism in $\mathsf{CNL}_{\mathcal{O}}$.

Proof. Loosely, the theorem is essentially a consequence of the following:

1. (1) $\mathbb{T}^S(\Gamma)_{\mathfrak{m}} \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p \cong \prod \overline{\mathbb{Q}}_p$, with the product running over Hecke eigensystems in $S_2(\Gamma, \overline{\mathbb{Q}}_p)$ that are congruent to that of g mod p, and $\mathbb{T}^S(\Gamma)_{\mathfrak{m}}$ is p-torsion free.
2. (2) The local-global compatibility of these eigensystems.

Take a Hecke eigenform $f \in S_2(\Gamma, \mathcal{O})$ that is congruent to g mod ϖ. For any such f,

- ρ_f is unramified away from pN since $\Gamma_1(N) \leqslant \Gamma$;
- Nebentypus χ for f factors through $(\mathbb{Z}/N\mathbb{Z})^\times / \ker\eta$ by definition of Γ. But $f \equiv g$ implies $\chi \equiv \eta \bmod \varpi$. Since η has prime-to-p conductor, $\ker\eta = \ker\overline{\eta}$. Hence $\chi = \eta$ and $\det\rho_f = \eta\epsilon^{-1}$.

Now take $\ell \mid N$ and let C be conductor of η. Consider two different cases.

(a) If $\ell \nmid C$, we know that

$$1 \neq \rho_f(I_\ell) \subseteq \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$$

and $\overline{\rho}(I_\ell) \neq 1$. So ρ_f does yield a minimal deformation problem.

(b) If $\ell \mid C$, we know that

$$\rho_f|_{I_\ell} = 1 \oplus \eta$$

and η has prime-to-p order, so $\eta(I_\ell) = \overline{\eta}(I_\ell)$, which is the minimal conductor in this case.

Due to some facts from Fontaine–Laffaille theory,

$$\rho_f|_{G_{\mathbb{Q}_p}} \cong \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

with $\chi_1|_{I_p} = 1$ and $\chi_2|_{I_p} = \epsilon^{-1}$. So $\rho_f|_{G_{\mathbb{Q}_p}}$ satisfies the ordinary global deformation problem $\mathcal{D}^{\mathrm{ord}}$.

On the other hand, we obtain a Galois representation

$$\rho_{\mathfrak{m}} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{T}^S(\Gamma)_{\mathfrak{m}})$$

lifting ρ, and is unramified outside S, so we have an induced map in $\mathsf{CNL}_{\mathcal{O}}$, say

$$R_{\overline{\rho}}^{\text{univ}} \longrightarrow \mathbb{T}^S(\Gamma)_{\mathfrak{m}},$$

which is surjective since $\mathbb{T}^S(\Gamma)_{\mathfrak{m}}$ is generated by T_ℓ, S_ℓ for $\ell \notin S$, and the characteristic polynomial of $\rho_{\mathfrak{m}}(\mathrm{Frob}_\ell)$ is

$$X^2 - T_\ell X + \ell S_\ell.$$

Thus, T_ℓ, S_ℓ are in image of $R_{\overline{\rho}}^{\text{univ}} \to \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$. We want to show this map factors through $R_{\mathcal{S}}$. Since $\mathbb{T}^S(\Gamma)_{\mathfrak{m}} \hookrightarrow \mathbb{T}^S(\Gamma)_{\mathfrak{m}} \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p$, it suffices to prove that

$$R_{\overline{\rho}}^{\text{univ}} \longrightarrow \mathbb{T}^S(\Gamma)_{\mathfrak{m}} \longrightarrow \mathbb{T}^S(\Gamma)_{\mathfrak{m}} \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p$$

factors through $R_{\mathcal{S}}$. But

$$\rho_{\mathfrak{m}} \otimes \overline{\mathbb{Q}}_p = \prod_f \rho_f : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2\left(\prod_p \overline{\mathbb{Q}}_p\right)$$

and each ρ_f is of type $\mathcal{S}$. This completes the proof. $\square$

9.2. **Patching argument for minimal local deformation.** We assume the running assumptions as before. So $g \in S_2(\Gamma, \mathcal{O})$ and

$$\mathcal{S} = (\overline{\rho} = \overline{\rho}_g, S = \{\ell \mid N\} \cup \{p\}, \psi = \eta\epsilon^{-1}, \mathcal{O}, \{\mathcal{D}_v^{\min}\}_{v|N} \cup \{\mathcal{D}_p^{\text{ord}}\}).$$

Given $\overline{\rho}_g$, a maximal ideal $\mathfrak{m} \subset \mathbb{T}^{S,\text{univ}}$ arises, and we have a surjective $\mathsf{CNL}_{\mathcal{O}}$-algebra map $R_{\mathcal{S}} \twoheadrightarrow \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$. The goal is to show this map is an isomorphism.

Let $(Q, \{\alpha_v\}_{v\in Q})$ be a Taylor–Wiles datum. So we have Δ_Q. Let $\mathbb{T}^{S\cup Q}(\Gamma_Q)_{\mathfrak{m}_Q}$ be the subalebra of $\mathrm{End}_{\mathcal{O}}(H_1(Y, \mathcal{O})_{\mathfrak{m}_Q})$ generated by T_ℓ, S_ℓ for any $\ell \notin S \cup Q$ and $\langle\sigma\rangle$ for all $\sigma \in \Delta_Q$.

Theorem 9.2. *We have a continuous Galois representation*

$$\rho_Q : G_{\mathbb{Q},S\cup Q} \longrightarrow \mathrm{GL}_2(\mathbb{T}^{S\cup Q}(\Gamma)_{\mathfrak{m}_Q})$$

such that the following conditions are satisfied:

(a) *For any $\ell \in S \cup Q$, the characteristic polynomial of $\rho_Q(\mathrm{Frob}_\ell)$ equals $X^2 - T_\ell X + \ell S_\ell$;*
(b) *For any $v \in S$, $\rho_Q|_{G_{\mathbb{Q}_v}} \in \mathcal{D}_v$.*
(c) *For any $v \in Q$,*

$$\rho_Q|_{I_v} \cong \begin{pmatrix} 1 & 0 \\ 0 & \chi_v \end{pmatrix},$$

where the post-composition with Artin reciprocity map $\chi_v \circ \mathrm{Art}_{Q_v}(\delta) = \langle\delta\rangle$.

Proof. Leave it as an exercise. The argument is to use the same recipe as that to deduce concrete consequences of local-global compatibility (Theorem 8.1). $\square$

Note that $\eta_Q = (\det \rho_Q)\psi^{-1}$ is a finite p-power-order character, hence it admits a square root $\eta_Q^{1/2}$. Thus, $\rho_Q \otimes \eta_Q^{1/2}$ is of type $\mathcal{S}_Q$. So there exists a map

$$R_{\mathcal{S}_Q} \longrightarrow \mathbb{T}^{S\cup Q}(\Gamma)_{\mathfrak{m}_Q}$$

and $H_1(Y_Q, \mathcal{O})_{\mathfrak{m}_\alpha}$ is an $R_{\mathcal{S}_Q}$-module that is compatible with the $\mathcal{O}[\Delta_Q]$-structure.

Proposition 9.3. *There is an integer $q \geqslant 0$, a $\mathsf{CNL}_{\mathcal{O}}$-algebra R_∞, and a finitely generated R_∞-module M_∞ satisfying the following*

$$\begin{tikzcd}
S_\infty := \mathcal{O}[\![y_1, \ldots, y_q]\!] \arrow[d] & \\
R_\infty := \mathcal{O}[\![x_1, \ldots, x_q]\!] \arrow[r, two heads] & R := R_{\mathcal{S}} \\
\circlearrowright & \circlearrowright \\
M_\infty \arrow[r, two heads] & M := H_1(Y, \mathcal{O})_{\mathfrak{m}}.
\end{tikzcd}$$

(1) *M_∞ is a finite free S_∞-module.*
(2) *We have surjective maps $R_\infty \twoheadrightarrow R$ and $M_\infty \twoheadrightarrow M$ such that* $\ker(R_\infty \twoheadrightarrow R) \subseteq \mathfrak{a}R_\infty$ *and* $\ker(M_\infty \twoheadrightarrow M) = \mathfrak{a}M_\infty$*, where $\mathfrak{a} = (y_1, \ldots, y_q) \subset S_\infty$.*

Assuming this proposition from now on, we have

Theorem 9.4. *The surjection*

$$R_{\mathcal{S}} \longrightarrow \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$$

is an isomorphism of local complete intersections.

Proof. Since M_∞ is free, S_∞ and the S_∞-module structures factors through R_∞, we have

$$1 + q \geqslant \dim_{R_\infty}(M_\infty) \geqslant \operatorname{depth}_{R_\infty}(M_\infty) \geqslant \operatorname{depth}_{S_\infty}(M_\infty) = 1 + q.$$

So all these inequalities are equalities. Since R_∞ is regular, M_∞ has finite length projective resolution (by Serre). The Auslander–Buchsbaum formula then gives

$$\operatorname{proj}\dim_{R_\infty}(M_\infty) = \operatorname{depth}(R_\infty) - \operatorname{depth} R_\infty(M_\infty) = (1 + q) - (1 + q) = 0.$$

So M_∞ is a projective R_∞-module, hence is free as R_∞ is local.

Therefore, $M \cong M_\infty/\mathfrak{a}M_\infty$ is a free $R_\infty/\mathfrak{a}R_\infty$-module. As this action factors through surjections,

$$R_\infty/\mathfrak{a}R_\infty \longrightarrow R_{\mathcal{S}} \longrightarrow \mathbb{T}^S(\Gamma)_{\mathfrak{m}}.$$

These two maps are isomorphisms. Finally, $R_{\mathcal{S}}$ is a complete intersection, since we have found a presentation

$$R_{\mathcal{S}} \cong \mathcal{O}[\![x_1, \ldots, x_q]\!]/(y_1, \ldots, y_q)$$

and $\dim R_{\mathcal{S}} = \dim \mathbb{T}^S(\Gamma)_{\mathfrak{m}} = 1$. $\square$

Proof of Proposition 9.3. Set $q = h^1_{\mathcal{S}^\perp}(\mathbb{Q}, \operatorname{ad}^0_{\bar\rho}(1))$, $d = \operatorname{rank}_{\mathcal{O}} M$, and $S_\infty = \mathcal{O}[\![\mathbb{Z}_p^q]\!]$. For any $N \geqslant 1$, set

$$\begin{aligned}
\mathfrak{a}_N &:= \ker(S_\infty \twoheadrightarrow \mathcal{O}[(\mathbb{Z}/p^N\mathbb{Z})^q]) \subset S_\infty, \\
S_N &:= S_\infty/(\varpi^N, \mathfrak{a}_N), \\
\partial_N &:= (\varpi^N, \operatorname{Ann}_R(M)^N) \subset R.
\end{aligned}$$

We define a patching datum of level N to be a triple

$$(f, X, g)$$

where

- $f : R_\infty \to R/\partial_N$ is a surjection in $\mathsf{CNL}_{\mathcal{O}}$;
- X is an $R_\infty \otimes_{\mathcal{O}} S_N$-module, finite free over S_N and such that

$$\begin{aligned}
\operatorname{im}(S_N \to \operatorname{End}_{\mathcal{O}} X) &\subseteq \operatorname{im}(R_\infty \to \operatorname{End}_{\mathcal{O}} X), \\
\operatorname{im}(\mathfrak{a} \to \operatorname{End}_{\mathcal{O}} X) &\subseteq \operatorname{im}(\ker f \to \operatorname{End}_{\mathcal{O}} X).
\end{aligned}$$

- $g : X/\mathfrak{a} \xrightarrow{\sim} M/(\varpi^N)$ is an isomorphism of R_∞-modules.

Two patching data (f, X, g) and (f', X', g') of level N are isomorphic if $f = f'$, and there is an isomorphism $X \to X'$ of $R_\infty \otimes_{\mathcal{O}} S_N$-modules compatible with g and g'.

Notice that there are only finitely many isomorphism classes of patching data of a fixed level N. Note also that if $M \geqslant N \geqslant 1$ are two integers, and $D = (f, X, g)$ is a patching datum of level M, then

$$D \bmod N := (f \bmod \partial_N, X \otimes_{S_M} S_N, g \otimes_{S_M} S_N)$$

is a patching datum of level N. For each $N \geqslant 1$ we can choose a Taylor–Wiles datum $(Q_N, \{\alpha_v\}_{v \in Q_N})$ of level $N \geqslant 1$ such that

$$|Q_N| = q, \quad h^1_{\mathcal{S}^\perp_{Q_N}}(\mathbb{Q}, \mathrm{ad}^0_{\overline{\rho}}(1)) = 0.$$

By all our work so far, for any $N \geqslant 1$, we can then define a patching datum of level N by

$$D_N := (f_N, X_N, g_N)$$

with $f_N : R_\infty \twoheadrightarrow R_{\mathcal{S}_{Q_N}} \twoheadrightarrow R \twoheadrightarrow R/\partial_N$, where the first map $\mathcal{O}[\![x_1, \dots, x_g]\!] \twoheadrightarrow R_{\mathcal{S}_{Q_N}}$ comes from the fact that the relative to $\mathcal{O}$-tangent spaces of $R_{\mathcal{S}_{Q_N}}$ has dimension equal to $h^1_{\mathcal{S}_{Q_N}}(Q, \mathrm{ad}^0\,\overline{\rho}) = q$ under our running assumptions. Also, g_N is induced from the isomorphism from the Δ_{Q_N}-coinvariants of $H_1(Y_{Q_N}, \mathcal{O})_{\mathfrak{m}_{Q_N}}$ to $H = H_1(Y, \mathcal{O})_{\mathfrak{m}}$. We denote

$$H_N := H_1(Y_{Q_N}, \mathcal{O})_{\mathfrak{m}_{Q_N}} \otimes_{S_\infty} S_N.$$

Then for any $M \geqslant N \geqslant 1$ we have a patching datum of level N, say

$$D_{M,N} := D_M \bmod N = (f_{M,N}, X_{M,N}, g_{M,N}).$$

Since for any fixed $N \geqslant 1$ there are infinitely many $M \geqslant 1$ and only finitely many isomorphism classes of level N, we can find a subsequence $(M_N)_{N \geqslant 1}$ of $(M)_{M \geqslant 1}$ such that

$$D_{M_{N+1}, N+1} \bmod N \cong D_{M_N, N}.$$

We then define

$$\begin{aligned} M_\infty &:= \varprojlim_N X_{M_N}, \\ (R_\infty \twoheadrightarrow R) &:= \varprojlim_N f_{M_N, N}, \\ (M_\infty \twoheadrightarrow M) &:= \varprojlim_N g_{M_N, N}. \end{aligned}$$

Since

$$\begin{aligned} \mathrm{im}(S_N \to \mathrm{End}_{\mathcal{O}} X_{M_N,N}) &\subseteq \mathrm{im}(R_\infty \to \mathrm{End}_{\mathcal{O}} X_{M_N,N}), \\ \mathrm{im}(\mathfrak{a} \to \mathrm{End}_{\mathcal{O}} X_{M_N,N}) &\subseteq \mathrm{im}(\ker f_{M_N,N} \to \mathrm{End}_{\mathcal{O}} X_{M_N,N}), \end{aligned}$$

we have that

$$\begin{aligned} \mathrm{im}(S_\infty \to \mathrm{End}_{\mathcal{O}} M_\infty) &\subseteq \mathrm{im}(R_\infty \to \mathrm{End}_{\mathcal{O}} M_\infty), \\ \mathrm{im}(\mathfrak{a} \to \mathrm{End}_{\mathcal{O}} M_\infty) &\subseteq \mathrm{im}(\ker(R_\infty \twoheadrightarrow R) \to \mathrm{End}_{\mathcal{O}} M_\infty). \end{aligned}$$

And since S_∞ is a power series ring, we can choose a map $S_\infty \to R_\infty$ lifting $S_\infty \to \mathrm{End}_{\mathcal{O}} M_\infty$. The resulting diagram

$$\begin{tikzcd} S_\infty \arrow[r] & R_\infty \arrow[r, two heads] \arrow[d, phantom, "\circlearrowright"] & R \arrow[d, phantom, "\circlearrowright"] \\ & M_\infty \arrow[r, two heads] & M \end{tikzcd}$$

satisfies the statement of the proposition. $\square$

9.3. **Patching argument in the non-minimal case, à la Kisin.** The following is one of the phenomena (or corollaries) of our $R = \mathbb{T}$ big theorem.

Theorem 9.5. *Let $p > 2$ be a prime and let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be a continuous irreducible representation satisfying the following conditions.*

(1) *ρ is unramified outside finitely many primes;*
(2) *We have*
$$\rho|_{G_{\mathbb{Q}_p}} \cong \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$
with $\chi_1|_{I_p} = 1$ and $\chi_2|_{I_p} = \epsilon^{-1}$.
(3) *$\overline{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is absolutely irreducible with adequate image;*
(4) *For all $\ell \neq p$ at which ρ is ramified,*
   - *either $\rho|_{I_\ell} \cong \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}$ with $\rho(I_\ell) \xrightarrow{\sim} \overline{\rho}(I_\ell)$ an isomorphism,*
   - *or $\rho|_{I_\ell}$ is isomorphic to the image of $\overline{\rho}(I_\ell)$ in $\left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$ and $\overline{\rho}(I_\ell) \neq 1$;*

   And that
$$\overline{\rho}|_{G_{\mathbb{Q}_p}} \cong \begin{pmatrix} \overline{\chi}_1 & * \\ 0 & \overline{\chi}_2 \end{pmatrix}$$
   with $\overline{\chi}_1\overline{\chi}_2^{-1} \neq 1, \overline{\epsilon}$.
(5) *$\overline{\rho} \cong \overline{\rho}_g$ for $g \in S_2(\Gamma_1(N), \overline{\mathbb{Q}}_p)$ with $N = \prod'_{\ell \neq p} \ell$, where the restricted product runs through those ℓ such that ρ is ramified at ℓ.*

Then
$$\rho \cong \rho_f$$
for some Hecke eigenform $f \in S_2(\Gamma_1(N), \overline{\mathbb{Q}}_p)$.

Proof. One can check those assumptions (1)–(5) of the theorem implies the following. After fixing a model for $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}')$ for $\mathcal{O}'$ ring of integers of a finite extension of $\mathbb{Q}_p$, ρ defines a homomorphism $R_{\mathcal{S}} \to \mathcal{O}'$ of $\mathcal{O}$-algebras with $\mathcal{S}$ defined as in the Taylor–Wiles deformation datum.

On the other hand, we obtain the isomorphism
$$R_{\mathcal{S}} \cong \mathbb{T}^S(\Gamma)_{\mathfrak{m}}, \quad S = \{\ell : \ell \mid N\} \cup \{p\},$$
and this implies there is an $\mathcal{O}$-algebra homomorphism
$$\lambda : \mathbb{T}^S(\Gamma)_{\mathfrak{m}} \longrightarrow \mathcal{O}'$$
such that for any $\ell \notin S$, the characteristic polynomial of $\rho(\mathrm{Frob}_\ell)$ happens to equal to
$$X^2 - \lambda(T_\ell)X + \ell\lambda(S_\ell).$$
Moreover, such a λ is the eigensystem of some $f \in S_2(\Gamma_1(N), \overline{\mathbb{Q}}_p)$. $\square$

Remark 9.6. In fact, condition (4) in Theorem 9.5 is restrictive. There are some techniques to get rid of it.

- ◇ Wiles: inserting a numerical condition to replace (4). This is hard to generalize and is experiencing a revival.
- ◇ Kisin: presenting global deformation rings as algebras over local framed deformation rings. We will explain this.

Let us continue to assume that $\overline{\rho} \cong \overline{\rho}_g$ is modular and $\overline{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is absolutely irreducible with adequate image. But let us drop the "minimal hypothesis". If so, the level $\Gamma = \Gamma_1(N)$ possibly has a non-square-free N, and maybe we want to allow lifts ramified at ℓ for some ℓ at which $\overline{\rho}$ is unramified.

Say we have a deformation datum

$$\mathcal{S} = (\overline{\rho}, S, \mathcal{O}, \psi, \{\mathcal{D}_v\}_{v\in S}), \quad \mathcal{D}_v \subset \mathcal{D}^{\square,\psi}_{\overline{\rho}|_{G_{\mathbb{Q}_\ell}}}$$

such that we can prove

$$\rho_{\mathfrak{m}} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{T}^S(\Gamma)_{\mathfrak{m}})$$

is of type $\mathcal{S}$ and such that we expect all type $\mathcal{S}$ deformations of $\overline{\rho}$ to come from $\mathbb{T}^S(\Gamma)_{\mathfrak{m}}$. Also assume that for any $v \in S$ the ring R_v representing $\mathcal{D}_v$ is $\mathcal{O}$-flat and pure of dimension

$$\begin{cases} -1+3 & \text{if } v \neq p, \\ -1+3+1 & \text{if } v = p. \end{cases}$$

If $F = \mathbb{Q}$ in particular, then this is exactly $1 + 3 + [F_v : \mathbb{Q}_p]$, where $1 = \dim \mathcal{O}$, and 3 is the dimension of the space of values of Frob.

We consider places at $T = S$ and let

$$R^{\mathrm{loc}}_{\mathcal{S}} := \hat{\bigotimes_{v\in S}} R_v,$$

which is $\mathcal{O}$-flat of dimension $2 + 3\#S$. Recall also that

$$R^S_{\mathcal{S}} \cong R_{\mathcal{S}} \mathbin{\hat{\otimes}_{\mathcal{O}}} \mathcal{T}, \quad \mathcal{T} = \mathcal{O}[\![z_1, \ldots, z_{4\#S-1}]\!].$$

Proposition 9.7. *There is $q \geqslant 0$ and a diagram*

$$\begin{tikzcd}
S_\infty := \mathcal{T}[\![y_1,\ldots,y_q]\!] \arrow[d] & \\
R_\infty := R^{\mathrm{loc}}_{\mathcal{S}}[\![x_1,\ldots,x_g]\!] \arrow[r, two heads] \arrow[d, phantom, "\circlearrowright"] & R := R_{\mathcal{S}} \arrow[d, phantom, "\circlearrowright"] \\
M_\infty \arrow[r, two heads] & M := H_1(Y,\mathcal{O})_{\mathfrak{m}}.
\end{tikzcd}$$

in which M_∞ is an R_∞-module. This diagram satisfies:

(1) *M_∞ is a finite free S_∞-module;*

(2) *We have surjective maps $R_\infty \twoheadrightarrow R$ and $M_\infty \twoheadrightarrow M$ such that*

$$\ker(R_\infty \twoheadrightarrow R) \subset \mathfrak{a}R_\infty, \quad \ker(M_\infty \twoheadrightarrow M) = \mathfrak{a}M_\infty,$$

where $\mathfrak{a} = (z_1, \ldots, z_{4\#S-1}, y_1, \ldots, y_q)$;

(3) *$\dim S_\infty = \dim R_\infty$, i.e., $4\#S + q = g + 2 + 3\#S$.*

Proof Sketch. This patching argument is similar to that in the proof of Proposition 9.3, by using

- computations of Galois cohomology, see Case (2) of Theorem 6.7 (Wiles' numerology);
- framed deformation rings $R^S_{\mathcal{S}_{Q_N}}$ to define the maps

$$\begin{tikzcd}
& R^S_{\mathcal{S}_{Q_N}} \arrow[dr, two heads] & \\
R_\infty \arrow[ur, two heads] \arrow[rr, two heads, "f_N"] & & R/\partial_N.
\end{tikzcd}$$

- modules X_N defined using a free $\mathcal{T}[\Delta_{Q_N}]$-module
$$H_1(Y_{Q_N}, \mathcal{O})_{\mathfrak{m}_{Q_N}} \hat{\otimes}_{R_{\mathcal{S}_{Q_N}}} R^S_{\mathcal{S}_{Q_N}} \cong H_1(Y_{Q_N}, \mathcal{O})_{\mathfrak{m}_{Q_N}} \hat{\otimes}_{\mathcal{O}} \mathcal{T}.$$

□

Let us proceed as before:
$$\dim R_\infty \geqslant \dim_{R_\infty}(M_\infty) \geqslant \operatorname{depth}_{R_\infty}(M_\infty) \geqslant \operatorname{depth}_{S_\infty}(M_\infty) = \dim S_\infty,$$
and $\dim R_\infty = \dim S_\infty$ so all these inequalities are equalities. Further M_∞ is a Cohen-Macaulay R_∞-module and $\operatorname{Supp}_{R_\infty}(M_\infty)$ is a union of irreducible components of $\operatorname{Spec} R_\infty$.

Proposition 9.8. *If* $\operatorname{Supp}_{R_\infty}(M_\infty) = \operatorname{Spec} R_\infty^{\mathrm{red}}$ *then* $\operatorname{Supp}_R(M) = \operatorname{Spec} R^{\mathrm{red}}$, *and the surjection*
$$R = R_{\mathcal{S}} \longrightarrow \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$$
has a nilpotent kernel.

Proof. Take any $\mathfrak{p} \in \operatorname{Spec} R$, and let $\mathfrak{p}_\infty$ be its pullback image to R_∞. Then $(M_\infty)_{\mathfrak{p}_\infty} \neq 0$ by assumption. Since M_∞ is finitely generated over R_∞, Nakayama's lemma renders that
$$M_{\mathfrak{p}} \cong (M_\infty/\mathfrak{a}M_\infty)_{\mathfrak{p}} = (M_\infty)_{p_\infty}/\mathfrak{a}(M_\infty)_{p_\infty} \neq 0.$$
So $\mathfrak{p} \in \operatorname{Supp}_R(M)$. This implies that $\operatorname{Ann}_R(M)$ is nilpotent and since R-action on M factors through $R = R_{\mathcal{S}} \twoheadrightarrow \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$, and $\mathbb{T}^S(\Gamma)_{\mathfrak{m}}$ acts faithfully on M, this map has nilpotent kernel. □

Remark 9.9. $R_{\mathcal{S}}^{\mathrm{red}} \cong \mathbb{T}^S(\Gamma)_{\mathfrak{m}}$ is good enough for modularity lifting. (But this is not enough for adjoint Bloch–Kato conjectures).

So we want to know that M_∞ has full support in $\operatorname{Spec} R_\infty$; here
$$\operatorname{Spec} R_\infty = \operatorname{Spec} R_{\mathcal{S}}^{\mathrm{loc}}[\![x_1, \ldots, x_g]\!] \longrightarrow \operatorname{Spec} R_{\mathcal{S}}^{\mathrm{loc}}$$
is a bijection on irreducible components, and any irreducible component X of $\operatorname{Spec} R_{\mathcal{S}}^{\mathrm{loc}}$ is of the form
$$X = \prod_{v \in S} X_v$$
with X_v an irreducible component of $\operatorname{Spec} R_v$ since $R_{\mathcal{S}}^{\mathrm{loc}} = \hat{\bigotimes}_{v \in S} R_v$. So for each $v \in S$, we want to

(1) understand irreducible components of $\operatorname{Spec} R_v$, and
(2) produce congruence from g (as $\overline{\rho} \cong \overline{\rho}_g$), which lies on one component, to other modular forms lying on other components.

For this, here come two cases at work:

- Suppose $v \nmid p$. Use level raising/lowering using Ihara's lemma (but we don't know how to generalize this to higher rank), as Taylor's Ihara avoidance trick.
- Suppose $v \mid p$. This is more difficult; it is related to the Breuil-Mézard conjecture and the weight part of Serre's conjecture.

Short Summary. Previously we have discussed two ingredients of the patching argument:

(1) Minimal modularity lifting as a consequence of our $R \cong \mathbb{T}$ big theorem, and
(2) Non-minimal modularity lifting as a consequence of an $R^{\mathrm{red}} \cong \mathbb{T}$ theorem provided we can show M_∞ has full support over
$$\operatorname{Spec} R_\infty = \operatorname{Spec} \left(\hat{\bigotimes_{v \in S}} R_v \right) [\![x_1, \ldots, x_g]\!],$$
in which each R_v is called a local lifting ring.

9.4. **Patching over totally real fields and cyclic base change.** In what follows we assume that there is a fixed isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$.

Theorem 9.10. *Let F be a totally real field and let $p \geqslant 5$ be a prime that is unramified in F. Let*

$$\rho : G_F \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

be a continuous irreducible Galois representation satisfying the following:

(1) *ρ is unramified outside finitely many primes;*
(2) *for each $v \mid p$, $\rho|_{F_v}$ is crystalline with all labelled Hodge–Tate weights $\{0,1\}$;*
(3) *$\overline{\rho}|_{G_{F(\zeta_p)}}$ is (absolutely) irreducible with adequate image;*
(4) *$\overline{\rho} \cong \overline{\rho}_g$ for g a Hilbert modular cuspform of parallel weight 2 and level prime-to-p.*

Then

$$\rho \cong \rho_f$$

for f a Hilbert modular cuspform of parallel weight 2.

Remark 9.11. Note that in Theorem 9.10, we need no assumptions on the ramification of ρ or level of g at $v \nmid p$.

Using cyclic base change (due to Saito, Shintani), we have

Theorem 9.12 (Base change)**.** *Let L/F be a totally real solvable Galois extension. Let ρ and g be as in Theorem 9.10 above.*

(1) (Going-up) *If $\rho_g|_{G_L}$ is irreducible, then there is a Hilbert cuspform h over L such that h is the base change of g. In particular,*
$$\rho_h \cong \rho_g|_{G_L}.$$
(2) (Going-down) *If $\rho|_{G_L} \cong \rho_h$ for a Hilbert modular cuspform h over L, then $\rho \cong \rho_f$ for a Hilbert modular cuspform f over F.*

Lemma 9.13. *Let K be a number field and let S be a finite set of places of K. For each $v \in S$, let K'_v/K_v be a finite extension. Then there exists a finite solvable Galois extension L/K such that for any w of L above $v \in S$, $L_w \cong K'_v$ as K_v-algebras.*

Proof. It suffices to prove the lemma with L given by a sequence of cyclic extensions, replacing it with its Galois closure if necessary. By induction, we are then reduced to the cyclic case, which is an application of the Grunwald–Wang theorem. $\square$

9.5. **Jacquet–Langlands correspondence.** Let $S_p = \{v \in F : v \mid p\}$ and $S_\infty = \{v \in F : v \mid \infty\}$. Let Σ be the set of finite places of F containing all at which ρ or g is ramified. Note that $\Sigma \cap S_p = \emptyset$ by assumption in Theorem 9.10. Let $M/F(\zeta_p)$ be the extension cut out by $\overline{\rho}|_{G_{F(\zeta_p)}}$. Then M/F is finite Galois, so we can find a finite set V of finite places of F such that any nontrivial conjugacy class in $\mathrm{Gal}(M/F)$ is of form $\langle \mathrm{Frob}_v \rangle$ for some $v \in V$ and such that $V \cap (\Sigma \cup S_p) = \emptyset$.

We apply Lemma 9.13 with

$$K = F, \quad S = S_p \cup S_\infty \cup \Sigma \cup V,$$

and conditions

- for $v \in S_p$, $K'_v = F_v$;
- for $v \in S_\infty$, $K'_v = F_v \cong \mathbb{R}$;
- for $v \in \Sigma$, K'_v/F_v is such that $\rho|_{G_{K'_v}}$ is either unramified or unipotently ramified and similarly for $\overline{\rho}_g$, and $\overline{\rho}|_{G'_{K_v}}$ is trivial; we assume moreover that the residue field of K'_v has cardinality $\equiv 1 \bmod p$ (will be explained later);

- for $v \in V$, $K'_v = F_v$.

Then we obtain a solvable finite Galois extension L/F such that

(a) Each $v \mid p$ in F splits completely in L; in particular, p is unramified in L.
(b) L/F is totally real.
(c) If $\rho|_{G_L}$ is ramified at w, the ramification is unipotent. And if g is ramified at w, then g has Iwahori level. The residue field at any such w has cardinality $q_w \equiv 1 \bmod p$. Moreover, $[L : F]$ is even.
(d) $L \cap M = F$, so $\overline{\rho}|_{G_{L(\zeta_p)}}$ is also irreducible with adequate image.

Applying the base change theorem (Theorem 9.12) and replacing F with L, we can assume that for any $v \in \Sigma$,

- $\rho(I_v)$ is unipotent (possibly trivial);
- g has Iwahori or full level at v;
- $\mathrm{Nm}(v) \equiv 1 \bmod p$;
- $\overline{\rho}|_{G_{F_v}} = 1$.

In particular, $\det \rho$ and $\det \rho_g$ are both finite unramified characters twisted by ϵ^{-1}. After twisting, we can assume that

$$\det \rho = \det \rho_g = \eta \epsilon^{-1}$$

with η finite order and unramified.

Finally, replacing F by a quadratic extension, disjoint from $M(\zeta_p)/F$, and in which p is unramified, we can assume that $[F : \mathbb{Q}]$ is even. Now let D be the (unique up to isomorphisms) quaternion algebra over F such that

$$D \otimes_F F_v \cong \begin{cases} \mathbb{H} & \text{if } v \mid \infty, \\ M_2(F_v) & \text{if } v \nmid \infty. \end{cases}$$

We fix a maximal order $\mathcal{O}_D$ of D and an isomorphism

$$\mathcal{O}_D \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \cong M_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}) \cong \prod_{v \nmid \infty} M_2(\mathcal{O}_{F_v}),$$

and hence an isomorphism

$$(D \otimes_F \mathbb{A}_f^\infty)^\times \cong \mathrm{GL}_2(\mathbb{A}_F^\infty)$$

taking $(\mathcal{O}_D \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})^\times$ to $\mathrm{GL}_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}) \cong \prod_{v \nmid \infty} \mathrm{GL}_2(\mathcal{O}_{F_v})$. Fix an open compact subgroup U of $(\mathcal{O}_D \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})^\times$, which we identify with one of $\mathrm{GL}_2(\mathcal{O}_{F_v})$. We will make a precise choice of U later.

Now choose $E/\mathbb{Q}_p$ finite with the ring of integers $\mathcal{O}$ such that ρ takes values in $\mathrm{GL}_2(\mathcal{O})$, conjugating if necessary. For any $\mathcal{O}$-algebra A, define

$$S_{2,\eta}(U, A) := \left\{ f : D^\times \backslash (D \otimes \mathbb{A}_F^\infty)^\times \to A \;\middle|\; \begin{array}{c} f \text{ continuous, } f(guz) = \eta(z) f(g) \text{ for all} \\ g \in (D \otimes_F \mathbb{A}_F^\infty)^\times,\ u \in U,\ z \in (\mathbb{A}_F^\infty)^\times \end{array} \right\}.$$

Abusing notation, we again write η for the (finite order) character

$$\eta \circ \mathrm{Art}_F : F^\times \backslash \mathbb{A}_F^\times \longrightarrow \mathcal{O}^\times.$$

For any finite place v of F such that $U_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$, the double coset operators

$$T_v = \left[\mathrm{GL}_2(\mathcal{O}_{F_v}) \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{F_v}) \right],$$
$$S_v = \left[\mathrm{GL}_2(\mathcal{O}_{F_v}) \begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{F_v}) \right].$$

acting on $S_{2,\eta}(U,A)$. (Note that S_v simply acts on $\eta(\varpi_v)$, so we could have omitted these operators.)

Theorem 9.14 (Jacquet–Langlands)**.** *Recall we have a fixed isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$. There is an equality*

$$\left\{ \begin{matrix} \mathcal{O}\text{-algebra homomorphism} \\ \lambda : \mathbb{T}^{S,\mathrm{univ}} \longrightarrow \overline{\mathbb{Q}}_p \end{matrix} \;\middle|\; \begin{matrix} \lambda \text{ is the eigensystem for a Hilbert modular cuspform} \\ \text{of parallel weight 2, level } U, \text{ and Nebentypus } \eta \end{matrix} \right\}$$
$$\|$$
$$\left\{ \begin{matrix} \mathcal{O}\text{-algebra homomorphism} \\ \lambda : \mathbb{T}^{S,\mathrm{univ}} \longrightarrow \overline{\mathbb{Q}}_p \end{matrix} \;\middle|\; \begin{matrix} \lambda \text{ is the eigensystem for an eigenform } f \in S_{2,\eta}(U, \overline{\mathbb{Q}}_p) \\ \text{not factoring through the reduced norm of } D \end{matrix} \right\}.$$

Comment. The Hecke eigensystems that factor through the reduced norm of D are Eisenstein, i.e., have associated Galois representations that are reducible. It thus suffices to prove that

$$\rho \cong \rho_f,$$

for some $f \in S_{2,\eta}(U,\mathcal{O})$; and we can assume that $\overline{\rho} \cong \overline{\rho}_g$ for some $g \in S_{2,\eta}(U,\mathcal{O})$ (by enlarging $\mathcal{O}$ if necessary).

9.6. **Taylor's Ihara avoidance.** Again, let $\rho : G_F \to \mathrm{GL}_2(\mathcal{O})$ be as in last subsection with $\overline{\rho} \cong \overline{\rho}_g$, where, by Jacquet–Langlands, $g \in S_{2,\eta}(U,\mathcal{O})$, for

$$U \subseteq (\mathcal{O}_D \hat{\otimes}_{\mathbb{Z}} \hat{\mathbb{Z}})^\times \cong \prod_{v \nmid \infty} \mathrm{GL}_2(\mathcal{O}_{F_v})$$

as follows. Denote Σ the set of finite places at which either ρ or g is ramified.

- For $v \notin \Sigma$,
$$U_v = \mathrm{GL}_2(\mathcal{O}_{F_v});$$
- For $v \in \Sigma$,
$$U_v = \mathrm{Iw}_v := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F_v}) : c \equiv 0 \bmod \varpi_v \right\},$$
  where ϖ_v is the chosen uniformization for F_v.

Recall that if A is a topological $\mathcal{O}$-module, then we have defined $S_{2,\eta}(U,A)$ in page 54. Writing

$$(D \otimes_F \mathbb{A}_F^\infty)^\times = \bigsqcup_{i \in I} D^\times t_i U (\mathbb{A}_F^\infty)^\times$$

for some index set I, we obtain an isomorphism

$$\begin{aligned} S_{2,\eta}(U,A) &\xrightarrow{\cong} \bigoplus_{i \in I} A(\eta^{-1})^{(U(\mathbb{A}_F^\infty)^\times \cap t_i^{-1} D t_i)/F^\times} \\ f &\longmapsto (f(t_i))_{i \in I}. \end{aligned}$$

Lemma 9.15. *Each $(U(\mathbb{A}_F^\infty)^\times \cap t_i^{-1} D t_i)/F^\times$ is finite and (since $p \geqslant 5$ and unramified in F), has order prime-to-p.*

Corollary 9.16. *The functor $A \longmapsto S_{2,\eta}(U,A)$ is exact for fixed U. In particular, $S_{2,\eta}(U,\mathcal{O})$ is a free $\mathcal{O}$-module and*

$$S_{2,\eta}(U,\mathcal{O})/(\varpi) \cong S_{2,\eta}(U,\mathbb{F}).$$

Here ϖ is a choice of the uniformizer of $\mathcal{O}$.

Given a Taylor–Wiles datum $(Q, \{\alpha_v\}_{v\in Q})$ for $\overline{\rho}$, we can proceed as before and define levels $U_0(Q)$ and U_Q (as analogues of $\Gamma_0(Q)$ and Γ_Q) by

(i) if $v \notin Q$, $U_0(Q)_v = U_{Q,v} := U_v$, and
(ii) if $v \in Q$,

$$U_0(Q)_v := \mathrm{Iw}_v, \qquad U_{Q,v} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Iw}_v : ad^{-1} \in \ker(\mathcal{O}_{F_v}^\times \to \Delta_v) \right\}.$$

Here Δ_v is the maximal p-power ordinary quotient of $(\mathcal{O}_{F_v}/\varpi_v)^\times$ and $U_0(Q)/U_Q \cong \Delta_Q$. One can again define maximal ideals

$$\mathfrak{m} \in \mathbb{T}^{S,\mathrm{univ}}, \quad \mathfrak{m}_Q \in \mathbb{T}_Q^{S\cup Q,\mathrm{univ}}.$$

It turns out that $S_{2,\eta}(U_Q, \mathcal{O})_{\mathfrak{m}_Q}$ is a free $\mathcal{O}[\Delta_Q]$-algebra with Δ_Q-coinvariants, and is isomorphic to $S_{2,\eta}(U, \mathcal{O})_{\mathfrak{m}}$ as $\mathbb{T}^{S\cup Q,\mathrm{univ}}$-modules.

Recall that for $v \in \Sigma$, $\mathrm{Nm}(v) \equiv 1 \bmod p$. Fix a nontrivial character of p-power order

$$\chi_v : \mathcal{O}_{F_v}^\times \twoheadrightarrow (\mathcal{O}_{F_v}/\varpi_v)^\times \to \mathcal{O}^\times.$$

We then have

$$\chi = \prod_{v\in\Sigma} \chi_v : U = \prod_{v\nmid\infty} U_v \longrightarrow \mathcal{O}^\times, \quad \left(\begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix} \right)_{v\nmid\infty} \longmapsto \prod_{v\nmid\infty} \chi_v(a_v d_v^{-1}).$$

Then for a topological $\mathcal{O}$-module A, define

$$S_{2,\eta}^\chi(U, A) := \left\{ f : D^\times \backslash (D \otimes \mathbb{A}_F^\infty)^\times \to A \,\middle|\, \begin{array}{c} f \text{ continuous, } f(guz) = \eta(z)\chi(u)^{-1} f(g) \text{ for} \\ \text{all } g \in (D \otimes_F \mathbb{A}_F^\infty)^\times,\ u \in U,\ z \in (\mathbb{A}_F^\infty)^\times \end{array} \right\}.$$

Note that

$$S_{2,\eta}^\chi(U, \mathcal{O})/(\varpi) \cong S_{2,\eta}^\chi(U, \mathbb{F}) = S_{2,\eta}(U, \mathbb{F}) \cong S_{2,\eta}(U, \mathcal{O})/(\varpi).$$

We again do the same things and have $S_{2,\eta}^\chi(U_Q, \mathcal{O})_{\mathfrak{m}_Q}$ a free $\mathcal{O}[\Delta_Q]$-module with Δ_Q-coinvariants isomorphic to $S_{2,\eta}^\chi(U, \mathcal{O})_{\mathfrak{m}}$. Also, as $\mathcal{O}[\Delta_Q]$-modules,

$$S_{2,\eta}^\chi(U, \mathcal{O})_{\mathfrak{m}_Q}/(\varpi) \cong S_{2,\eta}(U_Q, \mathcal{O})_{\mathfrak{m}_Q}.$$

Assume that for any $v \in \Sigma$, $\overline{\rho}|_{G_{F_v}} = 1$. (Recall that Σ is the set of finite places of F containing all at which ρ or g is ramified.)

Theorem 9.17 (Taylor)**.** *Let $v \in \Sigma$.*

(1) *There is a local deformation problem $\mathcal{D}_v^1$ corresponding to possible lifts ρ of $\overline{\rho}|_{G_{F_v}}$ such that*
- $\det \rho = \eta\epsilon^{-1}$, *and*
- *the characteristic polynomial of $\rho(\sigma)$ is $(X-1)^2$ for all $\sigma \in I_{F_v}$.*

Moreover, the deformation ring R_v^1 representing the functor $\mathcal{D}_v^1$ satisfies
- *all irreducible components of* $\operatorname{Spec} R_v^1$ *have dimension* 3 *and characteristic* 0 *generic points, and*
- *any irreducible component of* $\operatorname{Spec} R_v^1/(\varpi)$ *is contained in a unique irreducible component of* $\operatorname{Spec} R_v^1$.

(2) *There is a local deformation problem $\mathcal{D}_v^{\chi_v}$ corresponding to possible lifts ρ of $\overline{\rho}|_{G_{F_v}}$ such that*
- $\det \rho = \eta\epsilon^{-1}$, *and*
- *the characteristic polynomial of $\rho(\sigma)$ is $(X - \chi_v(\sigma))(X - \chi_v^{-1}(\sigma))$ for all $\sigma \in I_{F_v}$.*

Moreover, the deformation ring $R_v^{\chi_v}$ representing the functor $\mathcal{D}_v^{\chi_v}$ satisfies
- $\operatorname{Spec} R_v^{\chi_v}$ *is irreducible of dimension* 3 *with a characteristic* 0 *generic point.*

As a remark, note that $R_v^1/(\varpi) = R_v^{\chi_v}/(\varpi)$. We define a pair of global deformation problem for $? \in \{1, \chi\}$, say

$$\mathcal{S}^? = (\overline{\rho}, S - (\Sigma \cup \{v : v \mid p\}), \eta\epsilon^{-1}, \mathcal{O}, \{\mathcal{D}_v\}_{v|p} \cup \{\mathcal{D}_v^?\}_{v\in\Sigma}).$$

Here for $v \mid p$, the local problem $\mathcal{D}_v$ corresponds to crystalline liftings with all labelled Hodge-Tate weights $\{0, 1\}$.

Fact. For $v \mid p$, this $\mathcal{D}_v$ is represented by $R_v \cong \mathcal{O}[\![z_1, \dots, z_{3+[F_v:\mathbb{Q}_p]}]\!]$. Also, one can show all Galois representations that are valued in $\mathrm{im}(\mathbb{T}^{S,\mathrm{univ}} \to \mathrm{End}_{\mathcal{O}} S_{2,\eta}(U, \mathcal{O})_{\mathfrak{m}})$ (resp. $\mathrm{im}(\mathbb{T}^{S,\mathrm{univ}} \to \mathrm{End}_{\mathcal{O}} S_{2,\eta}^{\chi}(U, \mathcal{O})_{\mathfrak{m}})$) are of type $\mathcal{S}^1$ (resp. $\mathcal{S}^{\chi}$).

On the other hand, our fixed Galois representation $\rho : G_F \to \mathrm{GL}_2(\mathcal{O})$ is of type $\mathcal{S}^1$. As before we can augment it with a Taylor-Wiles datum $(Q, \{\alpha_v\}_{v\in Q})$ to get deformation data $\mathcal{S}_Q^1$ and $\mathcal{S}_Q^{\chi}$. Then we patch both data simultaneously, in cooperating with an isomorphism mod ϖ between the two patching data, and get a pair of diagrams:

$$\begin{tikzcd} S_\infty \arrow[r] & R_\infty^1 \arrow[r, two heads] \arrow[d, phantom, "\circlearrowright"] & R_{\mathcal{S}^1} \arrow[d, phantom, "\circlearrowright"] \\ & M_\infty^1 \arrow[r, two heads] & S_{2,\eta}(U, \mathcal{O})_{\mathfrak{m}} \end{tikzcd} \qquad \begin{tikzcd} S_\infty \arrow[r] & R_\infty^{\chi} \arrow[r, two heads] \arrow[d, phantom, "\circlearrowright"] & R_{\mathcal{S}^{\chi}} \arrow[d, phantom, "\circlearrowright"] \\ & M_\infty^{\chi} \arrow[r, two heads] & S_{2,\eta}^{\chi}(U, \mathcal{O})_{\mathfrak{m}} \end{tikzcd}$$

that are identified mod ϖ. What are known is for $? \in \{1, \chi\}$, $M_\infty^?$ is supported on a nonempty union of irreducible components of $\mathrm{Spec}\, R_\infty^?$. However, what is required is that M_∞^1 should have full support in $\mathrm{Spec}\, R_\infty^1$.

Note that for $? \in \{1, \chi\}$,

$$R_\infty^? = \left(\hat{\bigotimes_{v\in\Sigma}} R_v^?\right) \hat{\otimes} \left(\hat{\bigotimes_{v|p}} R_v\right) [\![x_1, \dots, x_g]\!].$$

(Here $\hat{\bigotimes}_{v|p} R_v$ is smooth by assumptions.) We obtain a bijection on irreducible components

$$\mathrm{Spec}\, R_\infty^? \longrightarrow \prod_{v\in\Sigma} \mathrm{Spec}\, R_v^?.$$

Taylor's theorem 9.17(2) renders that $\mathrm{Spec}\, R_\infty^{\chi}$ is irreducible, so M_∞^{χ} has full support. Hence $M_\infty^1/(\varpi) \cong M_\infty^{\chi}/(\varpi)$ has full support over $\mathrm{Spec}\, R_\infty^1/(\varpi) = \mathrm{Spec}\, R_\infty^{\chi}/(\varpi)$. Then

- $\mathrm{Supp}_{R_\infty} M_\infty^1$ is a union of irreducible components,
- $M_\infty^1/(\varpi)$ has full support in $\mathrm{Spec}\, R_\infty^1/(\varpi)$, and
- by Theorem 9.17(1), each irreducible component of $\mathrm{Spec}\, R_\infty^1/(\varpi)$ is contained in a *unique* irreducible component of $\mathrm{Spec}\, R_\infty^1$.

Therefore, M_∞^1 has full support over R_∞^1. Then from the argument before, we get that the action of $R_{\mathcal{S}^1}$ on $S_{2,\eta}(U, \mathcal{O})_{\mathfrak{m}}$ has a nilpotent kernel, hence ρ arises from an eigenform $f \in S_{2,\eta}(U, \mathcal{O})_{\mathfrak{m}}$, namely

$$\rho \cong \rho_f.$$

This finishes the Ihara avoidance argument of Taylor.

9.7. **Patching à la Calegari–Geraghty.**

9.7.1. *On CM fields.* Let F be a CM field, i.e. a quadratic imaginary extension of some totally real field F^+ over $\mathbb{Q}$. The fact that F contains a totally real field was used in the following two situations.

- ◇ **Galois side:**
  In the minimal case, recall that
  $$h^1_{\mathcal{S}}(\mathrm{ad}^0\overline{\rho}) = h^1_{\mathcal{S}^\perp}(\mathrm{ad}^0\overline{\rho}(1)).$$
  If we kill the dual Selmer group with $q = h^1_{\mathcal{S}^\perp}(\mathrm{ad}^0\overline{\rho}(1))$ many Taylor–Wiles primes, which form a set Q with $|Q| = q$, then we can write $R_{\mathcal{S}_Q}$ as a quotient of $\mathcal{O}[\![x_1,\dots,x_q]\!]$.
- ◇ **Automorphic side:**
  After localizing at the non-Eisenstein maximal ideal $\mathfrak{m}$, cohomology is constructed in a single degree, i.e.
  $$H^*(Y,\mathbb{F})_{\mathfrak{m}} = H^d(Y,\mathbb{F})_{\mathfrak{m}}$$
  for some "middle degree" d. Therefore, at Taylor–Wiles level, $H_d(Y_Q,\mathcal{O})_{\mathfrak{m}}$ is a free $\mathcal{O}[\Delta_Q]$-module.

Combining these two sides, we have

$$\begin{tikzcd}
S_\infty \arrow[r] & R_\infty \arrow[r, two heads] \arrow[d, phantom, "\circlearrowright"] & R := R_{\mathcal{S}} \arrow[d, phantom, "\circlearrowright"] \\
& M_\infty \arrow[r, two heads] & M := H_d(Y,\mathcal{O})_{\mathfrak{m}}
\end{tikzcd}$$

with M_∞ a free S_∞-module and $\dim R_\infty = \dim S_\infty$.

9.7.2. *On general number fields.* Now say F is any number field with $[F:\mathbb{Q}] = r+2s$, where r and s denote the numbers of real places and complex places, and
$$\overline{\rho}: G_F \longrightarrow \mathrm{GL}_2(\mathbb{F})$$
is continuous and $\overline{\rho}|_{G_{F(\zeta_p)}}$ is also irreducible. Assume further that if v is a real place and c_v is a choice of complex conjugation of v, then $\det\overline{\rho}(c_v) = -1$.

- ◇ **Galois side:**
  Assume we are in a minimal regular setting, i.e.,
  - $v \mid p$, we have regular-weight crystalline deformations;
  - all local deformations are formally smooth with
  $$\dim L_v - h^0(F_v,\mathrm{ad}^0\overline{\rho}) = \begin{cases} [F_v:\mathbb{Q}_p] & \text{if } v \mid p, \\ 0 & \text{if } v \nmid p. \end{cases}$$

  Then
  $$\begin{aligned} & h^1_{\mathcal{S}}(\mathrm{ad}^0\overline{\rho}) \\ &= h^1_{\mathcal{S}^\perp}(\mathrm{ad}^0\overline{\rho}(1)) + \sum_{v\in S}(\dim L_v - h^0(F_v,\mathrm{ad}^0\overline{\rho})) - \sum_{v\mid\infty} h^0(F_v,\mathrm{ad}^0\overline{\rho}) \\ &= h^1_{\mathcal{S}^\perp}(\mathrm{ad}^0\overline{\rho}(1)) + [F:\mathbb{Q}] - r - 3s \\ &= h^1_{\mathcal{S}^\perp}(\mathrm{ad}^0\overline{\rho}(1)) - s. \end{aligned}$$
- ◇ **Automorphic side:**
  Let $X = \prod_{v\mid\infty}\mathrm{PGL}_2(F_v)/U_\infty$, where U_∞ is the maximum component in $\mathrm{PGL}_2(F_v)$. Then
  $$\begin{aligned} X &\cong (\mathrm{PGL}_2(\mathbb{R})/\mathrm{PO}(2))^r \times (\mathrm{PGL}_2(\mathbb{C})/\mathrm{PU}(2))^s \\ &\cong \mathbb{H}_2^r \times \mathbb{H}_3^s, \end{aligned}$$

where $\mathbb{H}_d$ denotes the hyperbolic d-space.

If $U = \prod_{v\nmid\infty} U_v \leqslant \prod_{v\nmid\infty} \mathrm{PGL}_2(\mathcal{O}_{F_v})$ is open compact and sufficiently small, then

$$Y(U) := \mathrm{PGL}_2(\mathbb{Q})\backslash X \times \mathrm{PGL}_2(\mathbb{A}_F^\infty)/U$$

is a smooth manifold and for any $\mathcal{O}$-algebra A, the cohomology $H^*(Y(U), A)$ has an action of $\mathbb{T}^{S,\mathrm{univ}} := \mathcal{O}[\{T_v\}_{v\notin S}]$ with S a finite set of finite places containing $\{v : v \mid p\} \cup \{v : U_v \neq \mathrm{PGL}_2(\mathcal{O}_{F_v})\}$.

Fix an isomorphism of fields $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$.

Theorem 9.18 (Harder)**.** *We obtain a Hecke stable decomposition*

$$H^*(Y(U), \mathbb{C}) = H^*_{\mathrm{cusp}}(Y(U), \mathbb{C}) \oplus H^*_{\mathrm{Eis}}(Y(U), \mathbb{C})$$

with

(*a*) *the cusp part*

$$H^*_{\mathrm{cusp}}(Y(U), \mathbb{C}) = \bigoplus_\pi ((\pi^\infty)^U)^{m_i(\pi_\infty)}$$

being $\mathbb{T}^{S,\mathrm{univ}}$-equivariant with the sum ranging over cuspidal continuous representations of $\mathrm{PGL}_2(\mathbb{A}_F)$, where the multiplicity $m_i(\pi_\infty) = 0$ for all but finitely many π_∞;

(*b*) *$\mathbb{T}^{S,\mathrm{univ}}$ action on $H^*_{\mathrm{Eis}}(Y(U), \mathbb{C})$ being "Eisenstein".*

Theorem 9.19 (Borel–Wallach)**.** *Set $q_0 = r + s$ (so $\dim Y(U) = 2r + 3s = 2q_0 + s$). Let $\lambda : \mathbb{T}^{S,\mathrm{univ}} \to \mathbb{C}$ be an eigensystem corresponding to a cuspidal automorphic representation π of $\mathrm{PGL}_2(\mathbb{A}_F)$ such that π_∞ is tempered. If $H^*_{\mathrm{cusp}}(Y(U), \mathbb{C})[\lambda] \neq 0$ then*

$$H^i_{\mathrm{cusp}}(Y(U), \mathbb{C})[\lambda] \neq 0 \iff i \in [q_0, q_0 + s].$$

What is important is that it is suggestive of the following.

Key Philosophy (For more general rank Selmer group). In nice situations,

$$\dim(\text{Selmer group}) - \dim(\text{Dual Selmer group}) = -\delta$$

if and only if its cohomology groups are supported in an interval of length $\delta + 1$.

For $\mathrm{PGL}_{2,F}$, we denote s by δ.

Conjecture 9.20 (Ash, Calegari–Geraghty)**.** *Let $\mathfrak{m} \in \mathrm{Max}\,\mathbb{T}^{S,\mathrm{univ}}$ such that $H^*(Y(U), \mathbb{F})_\mathfrak{m} \neq 0$. Then there exists a continuous semisimple representation*

$$\overline{\rho}_\mathfrak{m} : G_{F,S} \longrightarrow \mathrm{GL}_2(\mathbb{F})$$

such that for all $v \notin S$,

$$\mathrm{char\,poly}\,\overline{\rho}_\mathfrak{m}(\mathrm{Frob}_v) = X^2 - T_v X + \mathrm{Nm}(N) \bmod \mathfrak{m}.$$

Assuming this conjecture from now on, we call $\mathfrak{m}$ *non-Eisenstein* if $\overline{\rho}_\mathfrak{m}$ is absolutely irreducible.

Conjecture 9.21 (Calegari–Geraghty)**.** *If $\mathfrak{m}$ is non-Eisenstein, then*

$$H^i(Y(U), \mathbb{F})_\mathfrak{m} = 0, \quad \textit{for } i \notin [q_0, q_0 + \delta].$$

Now say $\overline{\rho} = \overline{\rho}_\mathfrak{m}$ with $\mathfrak{m}$ non-Eisenstein and $H^*(Y(U), \mathbb{F})_\mathfrak{m} \neq 0$.

Goal. We are to construct a diagram

$$\begin{tikzcd} S_\infty \arrow[r] & R_\infty \arrow[rr, two heads] \arrow[d, phantom, "\circlearrowright"] & & R := R_\mathcal{S} \arrow[d, phantom, "\circlearrowright"] \\ & H_\infty(C_\infty) \arrow[rr, two heads] & & H_*(C) := H_*(Y(U), \mathcal{O})_\mathfrak{m} \end{tikzcd}$$

where

- S_∞ is the power series ring over $\mathcal{O}$ with augmented ideal $\mathfrak{a}$;
- $\dim R_\infty - \dim S_\infty = -\delta$;
- C_∞ is a cocomplex of finite free S_∞-modules constructed in degrees $[q_0, q_0+\delta]$ and $C \cong C_\infty \bmod \mathfrak{a}$ is a complex of finite free $\mathcal{O}$-modules with $H_*(C) = H_*(Y(U), \mathcal{O})_\mathfrak{m}$;
- $H_\infty(C_\infty)$ is a finite R_∞-module.

Assuming this, we have the following.

Theorem 9.22. (1) *$\operatorname{Supp}_{R_\infty} H_{q_0}(C_\infty)$ is a nonempty union of irreducible components.*

(2) *If every irreducible component of $\operatorname{Spec} R_\infty$ is in $\operatorname{Supp}_{R_\infty} H_{q_0}(C_\infty)$, then the kernel of $R_\mathcal{S} \to \operatorname{End}_\mathcal{O} H_{q_0}(Y(U), \mathcal{O})_\mathfrak{m}$ is nilpotent.*

(3) *If $R_\infty \cong \mathcal{O}[\![x_1, \dots, x_g]\!]$ with $1+g = \dim S_\infty - \delta$, then $H_{q_0}(Y(U), \mathcal{O})_\mathfrak{m}$ is a free $R_\mathcal{S}$-module.*

Proof. (1) The S_∞-action on $H_*(C_\infty)$ factors through R_∞ and $H_*(C_\infty)$ is a finite R_∞-module, so for any $i \in \mathbb{Z}$,

$$(*) \qquad \operatorname{depth}_{S_\infty} H_i(C_\infty) \leqslant \operatorname{depth}_{R_\infty} H_i(C_\infty) \leqslant \dim_{R_\infty} H_i(C_\infty) \leqslant \dim R_\infty = \dim S_\infty - \delta.$$

We claim that

$$\operatorname{depth}_{S_\infty} H_{q_0}(C_\infty) = \dim S_\infty - \delta.$$

Assuming this, all inequalities above are equalities. This proves (1).

(2) For any $\mathfrak{p} \in \operatorname{Spec} R_\mathcal{S}$, let $\mathfrak{p}_\infty$ be its pullback to R_∞, then $H_{q_0}(C_\infty)_{\mathfrak{p}_\infty} \neq 0$ by assumption. Hence

$$\begin{aligned} H_{q_0}(Y(U), \mathcal{O})_\mathfrak{p} &= H_{q_0}(C_\infty \otimes_{S_\infty} \mathcal{O})_\mathfrak{p} \\ &= (C_{\infty, q_0}/(\mathfrak{a}, \operatorname{im} d_{q_0-1}))_\mathfrak{p} \\ &= (H_{q_0}(C_\infty)/\mathfrak{a})_\mathfrak{p} \\ &= H_{q_0}(C_\infty)_{\mathfrak{p}_\infty}/\mathfrak{a} \neq 0 \end{aligned}$$

by Nakayama's lemma. Here d_* is the differential map of complex C_∞.

(3) Since R is regular and $\dim_{R_\infty} H_{q_0}(C_\infty) = \operatorname{depth}_{R_\infty} H_{q_0}(C_\infty)$ by $(*)$ above, Auslander–Buchsbaum formula then implies that $H_{q_0}(C_\infty)$ is a projective, and hence free R_∞-module. So $H_{q_0}(Y(U), \mathcal{O})_\mathfrak{m} = H_{q_0}(Y(U), \mathcal{O})_\mathfrak{m}$ factors through $R_\mathcal{S}$, so $R_\infty/\mathfrak{a} \cong R_\mathcal{S}$.

After shifting, we can replace q_0 by 0 and the claim in (1) follows from the following lemma. This completes the proof. ☐

Lemma 9.23. *Let S be a local regular noetherian ring and $n = \dim S$. Let $P = P_\bullet$ be a (homological) complex of finite free S-modules constructed in degrees $[0, \delta]$. Then*

$$\dim H_*(P) \geqslant n - \delta;$$

moreover, if equality holds, then

(1) *P is a projective restriction of $H_0(P)$;*
(2) *$H_0(P)$ has depth $n - \delta$.*

Proof. Let $d_n : P_n \to P_{n-1}$ be the differential and let $m \geqslant 0$ be the largest integer such that $H_m(P) \neq 0$. Then

$$0 \to P_\delta \to P_{\delta-1} \to \cdots \to P_m$$

is exact until the final term, so is a projective restriction of $M : M_m = P_m/\operatorname{im} d_{m+1}$. Thus $\operatorname{proj\,dim} M \leqslant \delta - m$. On the other hand,

$$H_m(P) = \ker d_m/\operatorname{im} d_{m+1} \subseteq M$$

so $\dim H_m(P) \geqslant \operatorname{depth} M$ (which is a canonical algebraic fact). Then we obtain

$$\dim H_m(P) \geqslant \operatorname{depth} M = n - \operatorname{proj}\dim M \geqslant n - \delta + m$$

by Auslander–Buchsbaum formula. Now if $\dim H_*(P) \leqslant n - \delta$, one must have $m = 0$, P is a projective restriction of $H_0(P)$, and all $\geqslant$ above are equalities. So

$$\operatorname{depth} H_0(P) = n - \delta.$$

□

Keynote question. How do we (conjecturally, at least) create our patched diagram?

- **Galois side:**
  Just as before, if $\overline{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible with enormous image, then for any $N \geqslant 1$ we can find a Taylor–Wiles datum Q_N of level N such that $h^1_{\mathcal{S}^{\perp}_{Q_N}}(\operatorname{ad}^0 \overline{\rho}(1)) = 0$ and $|Q_N| = q$ is independent of N. Consequently, $R^S_{\mathcal{S}_{Q_N}}$ is a quotient of $R_\infty := R^{\mathrm{loc}}_{\mathcal{S}}[\![x_1, \ldots, x_g]\!]$ with

  $$g = h^1_{\mathcal{S}_{Q_N},T}(\operatorname{ad}^0 \overline{\rho}) = q + \#S - 1 - \delta.$$

  So $\dim R_\infty = \dim S_\infty - \delta$.
- **Automorphic side:**
  Let $G = \mathrm{PGL}_2$ and $X = G(F \otimes_{\mathbb{Q}} \mathbb{R})/K$, where K is a maximal compact subgroup. Let $U \leqslant G(\mathbb{A}_F^\infty)$ be a sufficiently small subgroup. Take

  $$Y(U) = G(F)\backslash X \times G(\mathbb{A}_F^\infty)/U.$$

  Given a Taylor–Wiles datum $(Q, \{\alpha_v\}_{v\in Q})$, one can still define $U_Q \leqslant U_0(Q) \leqslant U$, where $U_0(Q)$ is Iwahori at each $v \in Q$, and $U_0(Q)/U_Q \cong \Delta_Q$. One can still define $\mathfrak{m}_Q \in \mathbb{T}_Q^{S\cup Q,\mathrm{univ}}$.

However, on automorphic side there is a technical issue

$$\begin{aligned} H_*(Y(U_Q), \mathcal{O})_{\mathfrak{m}_Q} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} &\cong H_*(Y(U_0(Q)), \mathcal{O}[\Delta_Q])_{\mathfrak{m}_Q} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} \\ &\ncong H_*(Y_0(Q), \mathcal{O})_{\mathfrak{m}_Q} \end{aligned}$$

because H_* and $\otimes$ don't commute unless $H_* = H_d$.

To remedy this, we use instead a complex C_Q of free $\mathcal{O}[\Delta_Q]$-modules that compute $H_*(Y(U_Q), \mathcal{O})_{\mathfrak{m}_Q}$. Then $C_Q \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O}$ computes $H_*(Y(U_0(Q)), \mathcal{O})_{\mathfrak{m}_Q}$.

Here comes another problem. We really want a Hecke action on C_Q, and an action of $R_{\mathcal{S}_Q}$ via a map $R_{\mathcal{S}_Q} \to \mathbb{T}^{S\cup Q}(-)$. For a Hecke action, one can use singular chains for C_Q, i.e. the chains that compute singular homology. But for patching, we need C_Q to be a bounded complex of free $\mathcal{O}[\Delta_Q]$-modules (want finitely many isomorphism classes of "patching data of level N"). But this won't be preserved by the functor $\mathbb{T}^{S,\mathrm{univ}}(-)$.

To resolve this, it is most natural to work in the derived categories $D(\mathcal{O})$ and $D(\mathcal{O}[\Delta_Q])$ of $\mathcal{O}$-modules and $\mathcal{O}[\Delta_Q]$-modules, respectively.

Say R is a ring. Roughly $D(R)$ is constructed as follows.

- Let $\mathbf{Ch}(R)$ be the category of complexes of R-modules.
- Let $K(R)$ be another category whose objects are the same as $\mathbf{Ch}(R)$, and

  $$\operatorname{Hom}_{K(R)}(X, Y)\operatorname{Hom}_{\mathbf{Ch}(R)}(X, Y)/\sim$$

  where $\sim$ is the chain homotopy equivalence relation.
- Then $D(R)$, the derived category over R, is the category obtained from $K(R)$ by formally inverting all quasi-isomorphisms (i.e. chain maps that induce an isomorphism on H_*'s).

So each $f \in \operatorname{Hom}_{D(R)}(X, Y)$ is represented by

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y, \end{tikzcd}$$

where the left morphism is a quasi-isomorphism of complexes, and the right is just a map of complexes. There are subcategories $D(R)^-$ and $D(R)^+$ of objects that are bounded above and below, respectively. Similarly, we defined $K(R)^-$ and $K(R)^+$. Let $K(R)^{-,\mathrm{proj}}$ be the subcategory of $K(R)^-$ consisting of bounded above complexes of projective R-modules. Then

$$K(R)^{-,\mathrm{proj}} \xrightarrow{\sim} D(R)^-$$

is an equivalence of categories.

Now let A be a ring. We discuss some facts and operations in the derived category $D(A)$. Let $C \in D(A)^-$ (we identify chain and cochain comoplexes by $C_i = C^{-i}$). Choose a complex P of projective A-modules isomorphic to C in $D(A)$. If M is an A-module, then

$$C \otimes_A^{\mathbb{L}} M = P \otimes_A M,$$

i.e., $(C \otimes_A^{\mathbb{L}} M)_i = P_i \otimes_A M$. And

$$R\operatorname{Hom}_A(C, M) := \operatorname{Hom}_A(P_{-i}, M);$$

if $f \in R\operatorname{Hom}_A(C, M)$, then $d(f) = (-1)^{i+1} f \circ d_P$. It turns out that these functors

$$R\operatorname{Hom}_A(C, -), \quad C \otimes_A^{\mathbb{L}} (-)$$

are well-defined, i.e. independent of the choice of P up to unique isomorphisms in $D(A)$. Also, if $M = B$ is an A-algebra, then

$$(-) \otimes_A^{\mathbb{L}} B : D(A)^- \longrightarrow D(B)^-.$$

There is a spectral sequence

$$(E_2)_{i,j} = \operatorname{Tor}_j^A(H_i(C), M) \Longrightarrow H_{i+j}(C \otimes_A^{\mathbb{L}} M).$$

Fact. $D(A)$ is idempotent complete, i.e. if $e \in \operatorname{Hom}_{D(A)}(C)$ satisfies $e^2 = e$ then we have a direct sum decomposition $C = eC \oplus (1-e)C$.

Definition 9.24. (1) An object $C \in D(A)$ is *perfect* if there is an isomorphism in $D(A)$ from C to a bounded complex of finite projective A-modules.
(2) Suppose A is a local noetherian ring. We call a complex *minimal* if it is a bounded complex of finite projective (equivalently, free) A-modules and the differentials mod $\mathfrak{m}_A$ are all 0.

We insert a fact as follows without proof.

Proposition 9.25. *Let A be a local noetherian ring. Then any perfect complex in $D(A)$ is isomorphic to a minimal complex.*

Corollary 9.26. *Let A be a local noetherian ring and C a perfect complex in $D(A)$. If $H_*(C \otimes_A^{\mathbb{L}} A/\mathfrak{m}_A)$ is concentrated in degrees $[a,b]$, then C is isomorphic in $D(A)$ to a complex concentrated in degrees $[a,b]$.*

The consequence above is useful for the following reason. Recall that we want to build a diagram

$$\begin{tikzcd} S_\infty \arrow[r] & R_\infty \arrow[r, two heads] \arrow[d, phantom, "\circlearrowright"] & R := R_{\mathcal{S}} \arrow[d, phantom, "\circlearrowright"] \\ & H_\infty(C_\infty) \arrow[r, two heads] & H_0(C) \end{tikzcd}$$

with C_∞ concentrated in degrees $[q_0, q_0+\delta]$. Note that there exists a natural map

$$\mathrm{End}_{D(A)}(C) \longrightarrow \mathrm{End}_A(H_*(A)).$$

Also, if C is perfect and concentrated in $[0,d]$, and $f \in \mathrm{End}_{D(A)}(C)$ such that f acts as 0 on $H_*(A)$, then $f^{d+1}=0$ in $\mathrm{End}_{D(A)}(C)$. Consequently, let C be a perfect complex, then the kernel of $\mathrm{End}_{D(A)}(C) \to \mathrm{End}_A H_*(C)$ is nilpotent.

Resume on the case where $G = \mathrm{PGL}_{2,F}$. Again, let $U \leqslant G(\mathbb{A}_F^\infty)$ be a sufficiently small open subgroup. Then there is a perfect complex $C(U) \in D(\mathcal{O})$ such that

$$H_*(C(U)) = H_*(Y(U),\mathcal{O})$$

and $H^*(Y(U),\mathcal{O})$ is computed by $R\,\mathrm{Hom}(C(U),\mathcal{O})$. Also, there exists an $\mathcal{O}$-algebra map

$$\mathbb{T}^{S,\mathrm{univ}} \longrightarrow \mathrm{End}_{D(\mathcal{O})}(C(U)),$$

where the target is a finite rank $\mathcal{O}$-algebra. Let $\mathbb{T}^S(U)$ be the image of this map. It is a finite rank $\mathcal{O}$-module. Then $\mathbb{T}^S(U)$ is semilocal and equals the product of its local rings. Hence for our given $\mathfrak{m} \in \mathrm{Max}\,\mathbb{T}^{S,\mathrm{univ}}$, the local module $C(U)_\mathfrak{m}$ exists in $D(\mathcal{O})$ and $H_*(C(U))_\mathfrak{m} = H_*(Y(U),\mathcal{O})_\mathfrak{m}$. Note that the kernel

$$\ker(\mathbb{T}^S(U)_\mathfrak{m} \longrightarrow \mathrm{End}_\mathcal{O}(H_*(Y(U),\mathcal{O})_\mathfrak{m}))$$

is nilpotent.

Conjecture 9.27 (Calegari–Geraghty)**.** *There is a continuous Galois representation*

$$\rho_\mathfrak{m} : G_{F,S} \longrightarrow \mathrm{GL}_2(\mathbb{T}(U)_\mathfrak{m})$$

such that for all $v \notin S$,

$$\mathrm{char\,poly}\,\rho_\mathfrak{m}(\mathrm{Frob}_v) = X^2 - T_vX + \mathrm{Nm}(v).$$

Moreover,

1. *If $U_v = G(\mathcal{O}_{F_v})$ and p is unramified in F, then $\rho_\mathfrak{m}|_{G_{F_v}}$ is Fontaine–Laffaille with all labelled Hodge–Tate weights $\{0,1\}$ (recall that $p \geqslant 3$ stated).*
2. *If $v \in S$ with $v \nmid p$ and U_v contains a pro-ℓ Iwahori subgroup with ℓ the residue character of v, then for any $\sigma \in I_{F_v}$,*

$$\mathrm{char\,poly}\,\rho_\mathfrak{m}(\sigma) = (X - \langle \mathrm{Art}_{F_v}^{-1}(\sigma)\rangle)(X - \langle \mathrm{Art}_{F_v}^{-1}(\sigma)^{-1}\rangle)$$

and also a description of the characteristic polynomial of $\rho_\mathfrak{m}(\mathrm{Frob}_v)$.

By Conjecture 9.27, we get a map $R_\mathcal{S} \longrightarrow \mathbb{T}(U)_\mathfrak{m}$ for appropriate $\mathcal{S}$. In particular, $R_\mathcal{S}$ acts on $H_*(C(U)_\mathfrak{m}) = H_*(Y(U),\mathcal{O})_\mathfrak{m}$. Therefore, by adding Taylor–Wiles data Q, we can construct a perfect complex $C(U_Q)_{\mathfrak{m}_Q} \in D(\mathcal{O}[\Delta_Q])$ such that

$$C(U_Q)_{\mathfrak{m}_Q} \otimes^{\mathbb{L}}_{\mathcal{O}[\Delta_Q]} \mathcal{O} \cong C(U)_\mathfrak{m}.$$

In particular, note that

$$C(U_Q)_{\mathfrak{m}_Q} \otimes^{\mathbb{L}}_{\mathcal{O}[\Delta_Q]} \mathbb{F} \cong C(U)_\mathfrak{m} \otimes^{\mathbb{L}}_{\mathcal{O}[\Delta_Q]} \mathbb{F}$$

which computes $H_*(Y(U),\mathbb{F})_\mathfrak{m}$.

Conjecture 9.28 (Calegari–Geraghty)**.** *Recall that $\mathfrak{m}$ is supposedly non-Eisenstein. Then*

$$H_*(Y(U),\mathbb{F})_\mathfrak{m} = 0$$

for $i \in [q_0, q_0+\delta]$. Then $C(U_Q)_{\mathfrak{m}_Q}$ is concentrated in $[q_0, q_0+\delta]$ (note $2q_0+\delta = \dim Y(U)$).

With these two conjectures above, we can run a patching argument to get our desired diagram. Whenever F is an imaginary quadratic field, Conjecture 9.28 can be proved. This is because $\dim Y(U) = 3$. We only need to understand H^0 (and dually, H^3). However, this is very hard in general.

Workaround by Khare–Thorne. Say we only care about $R_{\mathcal{S}}^{\mathrm{red}} \cong \mathbb{T}^S(U)_{\mathfrak{m}}^{\mathrm{red}}$. Alternatively, we consider

$$R_{\mathcal{S}}\left[\frac{1}{p}\right] \cong \mathbb{T}^S(U)_{\mathfrak{m}}\left[\frac{1}{p}\right].$$

We at least know that

$$H_i(Y(U), \mathbb{F}) = 0, \quad i \notin [0, \dim Y(U)].$$

Denote $d := \dim Y(U)$. We can still patch to get a diagram

$$\begin{tikzcd} S_\infty \arrow[r] & R_\infty \arrow[r, two heads] \arrow[d, phantom, "\circlearrowleft"] & R \arrow[d, phantom, "\circlearrowleft"] \\ & H_\infty(C_\infty)/I_\infty \arrow[r, two heads] & H_*(Y(U), \mathcal{O})_{\mathfrak{m}}/I. \end{tikzcd}$$

But we only know C_∞ is constructed in degree $[0, d]$. We want it to concentrate in $[q_0, q_0 + \delta]$. Say we know $H_{q_0}(Y(U), \mathcal{O})_{\mathfrak{m}}\left[\frac{1}{p}\right] \neq 0$. Then localize the diagram above at an augmented ideal $\mathfrak{a}$ of $S_\infty \to \mathcal{O}$. Then it becomes

$$\begin{tikzcd} S_{\infty,\mathfrak{a}} \arrow[r] & R_{\infty,\mathfrak{a}} \arrow[r, two heads] \arrow[d, phantom, "\circlearrowleft"] & R_{\mathcal{S}}[1/p] \arrow[d, phantom, "\circlearrowleft"] \\ & H_*(C_{\infty,\mathfrak{a}}) \arrow[r, two heads] & H_*(Y(U), \mathcal{O})_{\mathfrak{m}}[1/p] \end{tikzcd}$$

and

$$C_{\infty,\mathfrak{a}} \otimes^{\mathbb{L}}_{S_{\infty,\mathfrak{a}}} E \cong C \otimes^{\mathbb{L}}_{\mathcal{O}} E, \quad E = \mathcal{O}\left[\frac{1}{p}\right].$$

Thanks to a result of Franks and Borel–Wallach,

$$H_*(C \otimes^{\mathbb{L}}_{\mathcal{O}} E) = H_*(Y(U), \mathcal{O})_{\mathfrak{m}}\left[\frac{1}{p}\right]$$

Apply Calegari–Geraghty argument to the localized diagram, we see $R_{\mathcal{S}}[\frac{1}{p}] \to \mathbb{T}(U)_{\mathfrak{m}}[\frac{1}{p}]$ has nilpotent kernel, assuming $H_{q_0}(C_{\infty,\mathfrak{a}})$ has full support in $\operatorname{Spec} R_{\infty,\mathfrak{a}}$.

Remark 9.29. We want there to be a nice Galois representation for this argument to work: if $\mathfrak{m}$ is non-Eisenstein, then $H^*(Y(U), \mathcal{O})_{\mathfrak{m}}[1/p]$ is all cuspidal.

For Conjecture 9.27, it can be proved if F is a CM field together with many technical conditions up to replacing $\mathbb{T}^S(U)_{\mathfrak{a}}$ by $\mathbb{T}^S(U)_{\mathfrak{m}}/I$ for a nilpotent ideal I with nilpotent degree depending only on F and $n = 2$ (with $G = \mathrm{PGL}_n$). One can build I into patching and still get

$$R_{\mathcal{S}}^{\mathrm{red}} \cong \mathbb{T}^S(U)^{\mathrm{red}}.$$

This crucially relies on viewing $\mathrm{Res}_{F/F^t}\mathrm{GL}_n$ as a Levi subgroup of a $2n$-dimensional unitary algebraic group over F^t. Via Borel–Serre compactification, we can find the cohomology groups of the locally symmetric space associated with GL_n in the cohomology of the unitary Shimura variety. This is the initial motivation for the restriction to CM fields.

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