Quotient by finite group schemes

1. Action of group scheme on scheme.

G. m: GxG-G

e: __

i: __

An action of _ G on _ X is a morphism:

 $\mu: G \times \chi \longrightarrow \chi$, s.t.

O the composite

X = Speck x X exid, axx 1 x

is id (e acts as id on X)

 \bigcirc the diagram is commute. $(g_1g_2 \cdot X = g_1(g_2 \cdot X))$

G×G×X (W)xid > G×X



 $\xrightarrow{G \times X} \xrightarrow{G} \times$

This is equivalent that G(S) acts on X(S) for every S. For every S-valued pt $\underline{x} \in G(S)$. gives an automorphism over S. XxS Lx, XxS \$ 10 P O x, y e G(S) = px · py = pxy XxS, hxs, (Cixt) Co cixt) XxS2 Mx XxSL Mx: = (pogo(idx x x), po) O: suttch worphism: XxG - GxX (xg) - (g.x) $f: X \rightarrow Y$ is called G-invariant, if diagram G×X X $\begin{array}{cccc} & & & & & & \downarrow f \\ & & & & & \downarrow f \\ & & & & & & & & \end{array}$

In particular, if we take $Y = A_K^{\dagger}$, f is called a G-invariant function.

An action is called free (a) the following morphism $\frac{(\mu,\,p_z):\;\;G\times\chi\;\;\longrightarrow\;\;\underset{=}{\times}\times\underset{=}{\times}}$

is a closed immersion.

$$X = X$$
, $Y = X$, $Y = X$, $Y = X$, $Y = X$

A lifting of the action p to f.

O play is functional in g.

A lifting of profis a isomorphism

\(\text{P.*(F)} \frac{\sigma}{\sigma}, \text{p*(F)}
\]

 $\frac{p_3^*(f)}{(m \times i d_x)^*(f)} \xrightarrow{(p_2, p_3)^*(\lambda)} \underbrace{\xi^*(f)}_{(id_a \times \mu)^*(\lambda)}$ sheaves on G×G×X $\xi : \mu \circ (\beta_2, \beta_3)$, $N = \mu \circ (m \times id_X)$ GxGxX - X Theorem: (A) Let G. finite group scheme, acts on X=SpecA. the orbit of ant pts in X is contained in an affine open subsets of X Then there exist a pair (Y, π) . Y scheme $\overline{h}: X \to Y$. $\longrightarrow S^2$. O As t70. sp. (Y, 17) = X/G O TI: X-1 Y is G-invariat, and To (Ox) G denotes the subsheef of T+(Ox) of G-inv. functions. Then

Of ma(Ox) of is an isomorphism.

Proof: (A) Assume: X = Spec A, G = Spec R. E^* : $R \rightarrow k$. $m^* \cdot R \rightarrow R \otimes_R R$ $M^* : A \rightarrow R \otimes_R A$. $Y = Spec A^G$ $A^G = B = Sae A | M^*(a) = 1 \otimes a |$

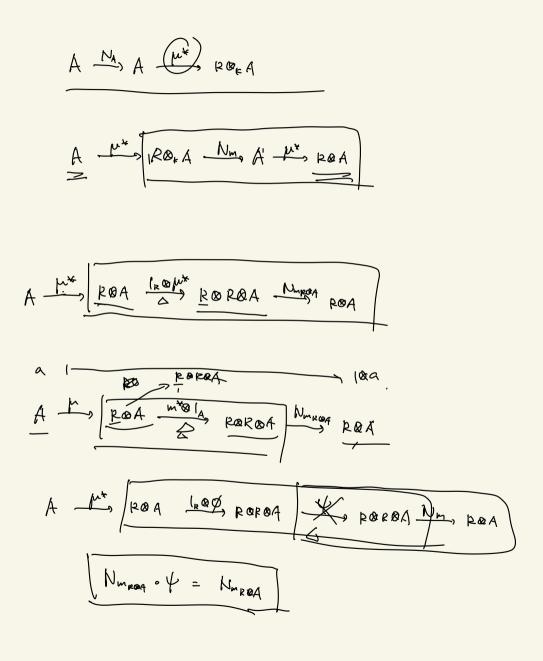
Nm_A: $R \otimes_k A \longrightarrow A$ be the norm map ($R \otimes_k A$ is free of finite rank over A). Nm_A is a homogenous poly. Function of degree n. ($n = dm_k R$).

 $N_A \cdot A \longrightarrow A$. $N_A(a) = N_{m_A} (\mu^*(a))$

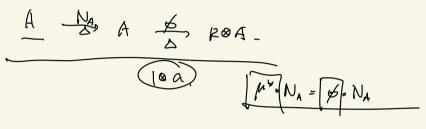
Note that if $f: B \to C$ homo, of k-alg. then. $R \otimes_{R} B \xrightarrow{l \otimes f} R \otimes_{R} C$ $R \otimes_{R} B \xrightarrow{l \otimes f} R \otimes_{R} C$ $R \otimes_{R} B \xrightarrow{l \otimes f} R \otimes_{R} C$ $R \otimes_{R} B \xrightarrow{l \otimes f} R \otimes_{R} C$ $R \otimes_{R} B \xrightarrow{l \otimes f} R \otimes_{R} C$ $R \otimes_{R} B \xrightarrow{l \otimes f} R \otimes_{R} C$ $R \otimes_{R} B \xrightarrow{l \otimes f} R \otimes_{R} C$ $R \otimes_{R} C \xrightarrow{l \otimes f} R \otimes_{R} C$ $R \otimes_{R} C \xrightarrow{l \otimes f} R \otimes_{R} C$ $R \otimes_{R} C \xrightarrow{l \otimes f} R \otimes_{R} C$ $R \otimes_{R} C \xrightarrow{l \otimes f} R \otimes_{R} C$ $R \otimes_{R} C \xrightarrow{l \otimes f} R \otimes_{R} C$ $R \otimes_{R} C \xrightarrow{l \otimes f} R \otimes_{R} C$

$$= N_{\text{mred}} \circ (N_{\text{red}}) \cdot N_{\text{red}}$$

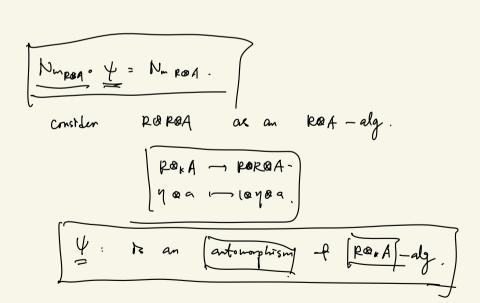
μ* , N.







Na(A) GB.



NA (A) & B. V. ACT) $G \cap X$, $G \cap X \times A'$ (G acts trivially on A') Similarly we can define: NAUT : ACT) - ACT) Y as A, Na(T) = NAUTO (T-a) $\chi_{\alpha}(T) = T^n + S_1 T^{n-1} - \cdots + S_n$ is G - invS1 - Sn ∈ B. Xa(T) is the characteristic poly of endomorphism of ROA defined by \mu^*(a) free A-mod NAMI (T-A) = N_AMI (T-ptai) 20 |: k & A → A is surj. (201) (p*a1) \$\frac{1}{4}\$ [\(\frac{\pu^*(a) - a}{a}\) defines zero map on A \(\((\cap\overline{a}\)) (\(\cap\overline{a}\)) μ*ca) - (@ a R&A/ p*(a) - a 1 on gotient A of ROA via

p*ca, - 180 in ROA det (1/201-6) = (2/2 (6) = 0) aeA a"+ s,a" --- + s, =0 5, - Sn &B => a is integral over B .=) A integral over B $O_{\Upsilon} \xrightarrow{\underline{C}} \pi_{\ast}(Q_{\Upsilon})^{G}$ Ti is G-im. 0y & The cox) & $O_{\gamma} \hookrightarrow (O_{\times})^{G}$ [TT : (0x) a coh. Oy-mod and is the kernel of $\pi_*(0_*)^{4} \xrightarrow{\Lambda} \pi_*(0_*) \otimes_{\mathbb{R}} \mathbb{R}$ f ~ p*cf, -fal $\langle er \rangle = \pi_*(Q_x)^{4}$. T* (10 A) = AA

 $\frac{\ker \lambda = \pi_*(O_X)^{a}}{\ker \lambda = O_Y = \beta = A^{\alpha}}.$ $\pi_*(\mathcal{O}_X)^{a} = \Lambda^{\alpha}$ β

The is
$$G = inv$$

The inverse of the inverse of $G = f$.

The invers

Gxx (cl.im) xxxx

Now since
$$\lambda$$
 is surj. Need to prove λ is inj.

View as A -mod

Now since λ is surj. $|R \otimes_{K} A|$ is generated by $|\mu^{*}(A)|$ (1802. These)

$$= \sum_{(0)} (100)^{k} (100)$$

 μ * a_1 form a basis \Rightarrow $l \otimes \mu$ * $\lambda_1 = l \otimes l \otimes \lambda_1$

 $=) \quad \mu^* \lambda_i = (\otimes \lambda_i =) \quad \lambda_i \in \beta_i$

A is a free B-mod with basts a, - an

λ: A⊗KA → R⊗KA

9181 - pr(a;)

Z pa; 20 => a;=0 ker λ >0 => (λ is injective. λ is smj.

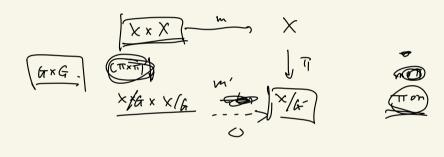
⇒
 \[
\lambda: is an isomphism
\]

(G 15 wormer of 48. G(S) is a wormal subgp of X(S) $\pi: X \to [x/a]$ is an expinorphism, $G = \ker f$

Corollary: X itself is a group schene, G would f. g.s

Converse by: $f: X \to Y$ weights in of group scheene, surj. Converse by: $f: X \to Y$ epi let $G = \ker f$

Conversely: $f: X \rightarrow Y$ epi let $G = \ker f$ $f: X \rightarrow X/G$.



Coro 2. $\left[\frac{\sqrt{-x/4}}{2}\right]$ g: cho o sheef on \times .

natural iso: $\pi^*(\pi_*g) \cong g \otimes_{k} R$ free

G $\Omega \times G$ G ΩF finite solver

U

G ~ × 1.F

$$U = (\underline{X} \times F)/G.$$

$$V = X/G.$$

U TSV is called the fibration with fibre F associated to the principal &-bundle.

Theorem: Ti: U-V. for ch. Fon V. we have

$$\chi(\pi^*(f)) = (\deg \pi) \cdot \chi(f)$$
degree of π - is defined as $|\pi(O_n)|$ rank over $|O_v|$