Exercise 6 (due on January 6 or later by emails)

Choose 3 out of 6 problems to submit. (The problems are chronically ordered by the materials.) Let $\ell \geq 3$ be a prime number.

Problem 6.1. (Hecke operator for p-stabilization) Let p be a prime that does not divide N. We consider the natural maps:

$$\begin{aligned} \varphi : S_k(\Gamma_0(N))^{\oplus 2} &\longrightarrow S_k(\Gamma_0(pN)) \\ (f(z), g(z)) &\longmapsto f(z) - g(pz) \end{aligned}$$

(1) Recall from the class that if $f(z)$ is an eigen form for T_p-operator, then U_p-action on $f(z), f(pz)$ is given by $\left(\begin{smallmatrix} T_p & 1 \\ -p^{k-1} & 0 \end{smallmatrix}\right)$. Construct a natural action U' on $S_k(\Gamma_0(N))^{\oplus 2}$ given by $\left(\begin{smallmatrix} T_p & 1 \\ -p^{k-1} & 0 \end{smallmatrix}\right)$. Show that this is equivariant under the φ-action with the U_p-action on $S_k(\Gamma_0(pN))$. From this, deduce that φ must be injective.

(2) Let $h_1(N)$ denote the Hecke algebra on $S_k(\Gamma_0(N))$, that is the $\mathbb{Z}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(S_k(\Gamma_0(N))^{\oplus 2})$ generated by all Hecke operators T_ℓ with $\ell \nmid N$ and U_ℓ with $\ell | N$. We define $h_1(Np)$ similarly (note that we use U_p instead of T_p in this case). Write $h_1(Np)^{\mathrm{old}}$ for the quotient of $h_1(Np)$ given by restricting the endomorphisms of $h_1(Np)$ on the subspace $\varphi(S_k(\Gamma_0(N))^{\oplus 2})$. Write out $h_1(Np)^{\mathrm{old}}$ explicit in terms of $h_1(N)$. (This is an easy question.)

Problem 6.2. (Gauss–Manin connection and Kodaira–Spencer map for relative curves) Let S be a smooth variety over a field k of dimension n, and let $f : C \to S$ denote a proper smooth relative curve.

(1) Recall that the differential sheaf of C admits a filtration

$$0 \to f^*\Omega^1_{S/k} \to \Omega^1_{C/k} \to \Omega^1_{C/S} \to 0,$$

where $\Omega^1_{C/S}$ is locally free of rank 1. For $i \in \mathbb{N}$, we write $\Omega^i_{C/k} := \wedge^i \Omega^1_{C/k}$ for the sheaf of i-forms on C. Consider the de Rham complex $\Omega^\bullet_{C/k}$ given by $\big[\mathcal{O}_C \xrightarrow{d} \Omega^1_{C/k} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n+1}_{C/k}\big]$. Show that each term in this complex sits in a short exact sequence

$$0 \to f^*\Omega^i_{S/k} \to \Omega^i_{C/k} \to \Omega^1_{C/S} \otimes f^*\Omega^{i-1}_{S/k} \to 0$$

Moreover, putting them together, we get a short exact sequence of complexes of sheaves:

$$0 \to f^*\Omega^\bullet_{S/k} \to \Omega^\bullet_{C/k} \to \Omega^1_{C/S} \otimes f^*\Omega^\bullet_{S/k}. \tag{6.2.1}$$

Remark: 1. The above argument applies to more general setup where C is a proper smooth variety over S; or even the log-smooth case....

2. Applying the above discussion to the case when S is the modular curve and C the universal relative elliptic curve, we obtain the Kodaira–Spencer isomorphism for modular curves.

Problem 6.3. (Derivation on $\mathfrak{sl}_n$) Let $\mathbb{F}$ be a finite field and $\mathrm{char}\mathbb{F} \nmid n$. Let $\mathfrak{sl}_n := \mathrm{M}_n(\mathbb{F})^{\mathrm{tr}=0}$ denote the corresponding Lie algebra over $\mathbb{F}$. Show that every derivation of $\mathfrak{sl}_n$ are given by Lie bracket, i.e. if an $\mathbb{F}$-linear map $\theta : \mathfrak{sl}_n \to \mathfrak{sl}_n$ satisfies $\theta(xy) = \theta(x)y + x\theta(y)$, then there exists $a \in \mathfrak{sl}_n$ such that $\theta(x) = xa - ax$.

Problem 6.4. (Ultraproduct of different local Artinian rings) Let $(R_n, \mathfrak{m}_n)_{n\in\mathbb{N}}$ denote a collection of local Artinian rings, and set $R = \prod_{n\in\mathbb{N}} R_n$. For each element $(x_n) \in R$, define

$$Z((x_n)) := \{n \mid x_n \in \mathfrak{m}_n\} \subseteq \mathbb{N}$$

For a subset $A \subseteq \mathbb{N}$, let $e_A \in R$ denote the idempotent

$$(e_A)_n = \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{if } n \notin A. \end{cases}$$

It is clear that $Z(e_A) = A^c$.

(1) Check that, for every $(x_n), (y_n) \in R$, $Z((x_n) + (y_n)) \supseteq Z((x_n)) \cap Z((y_n))$ and $Z((x_n) \cdot (y_n)) = Z((x_n)) \cup Z((y_n))$.
(2) For a prime ideal $\mathfrak{p} \subset R$, we may define

$$Z(\mathfrak{p}) := \{Z((x_n)) \mid (x_n) \in R\}$$

Prove that $Z(\mathfrak{p}) = \{A^c \mid e_A \in \mathfrak{p}\}$.
(3) Further, prove that $Z(\mathfrak{p})$ is an ultrafilter.
(4) Conversely, if $\mathfrak{F}$ is an ultrafilter, show that

$$\mathfrak{p}(\mathfrak{F}) := \{(x_n) \in R \mid Z((x_n)) \in \mathfrak{F}\}$$

is a prime ideal.

Therefore, there is a one-to-one correspondence between ultrafilters of $\mathbb{N}$ and prime ideals of R.

Problem 6.5. (Pseudo-representations) We have seen that the traces somehow determine the semisimplification of a representation (under mild characteristic constraints). The following notion grow from this, and was first introduced by Wiles.

Let Γ denote a profinite group and R a topological ring. A continuous R-valued pseudo-representation of dimension d, for some $d \in \mathbb{N}$ is a continuous function $T : G \to R$ with the following properties:

(i) $T(\mathrm{id}) = d$ and $d!$ is a non-zero-divisor of R,
(ii) For all $g_1, g_2 \in \Gamma$, one has $T(g_1g_2) = T(g_2g_1)$,
(iii) $d \geq 0$ is minimal such that the following condition holds: for all $g_1, \dots, g_{d+1} \in \Gamma$,

$$\sum_{\sigma\in S_{d+1}} \mathrm{sgn}(\sigma) T_\sigma(g_1, \dots, g_{d+1}) = 0,$$

where $T_\sigma : \Gamma^{d+1} \to R$ is defined as follows: suppose that $\sigma \in S_{d+1}$ has cycle decomposition

$$\sigma = (i_1^{(1)}, \dots, i_{r_1}^{(1)}) \cdots (i_1^{(s)}, \dots, i_{r_s}^{(s)}) = \sigma_1 \cdots \sigma_s;$$

then $T_\sigma(g_1, \dots, g_{d+1}) := T\big(g_{i_1^{(1)}} \cdots g_{i_{r_1}^{(1)}}\big) \cdots T\big(g_{i_1^{(s)}} \cdots g_{i_{r_s}^{(s)}}\big)$

(1) Write out what condition (iii) means when $d = 2$.
(2) Show that when $\rho : \Gamma \to \mathrm{GL}_d(R)$ is a representation, then $T(g) := \mathrm{tr}(\rho(g))$ defines a pseudo-representation. (If this is too difficult, prove this for $d = 2$.)

Remark: (a) Richard Taylor showed that if R is an algebraically closed field of characteristic $> d$ or characteristic zero, then any pseudo-presentation is associated to a unique semisimple representation $\rho : G \to \mathrm{GL}_d(R)$.

(b) The reason one introduces pseudo-representations is that, when $\bar{\rho}$ is not irreducible, the usual deformation ring does not exist, yet we may deform the associated pseudo-representation. Plus, the information of traces is readily available from modular forms.

Problem 6.6. (Ultraproduct $\prod \mathbb{F}_p$) Let $\mathfrak{F}$ be a non-principal ultrafilter of $\mathbb{N}$ (which we identify with the set of primes). Consider the product $\prod_{p \text{ prime}} \mathbb{F}_p$ and the ultraproduct $U_{\mathfrak{F}}((\mathbb{F}_p))$.

(1) Show that $U_{\mathfrak{F}}((\mathbb{F}_p))$ contains $\mathbb{Q}$ and is a field of characteristic zero. (Show that $\mathbb{Z} \to U_{\mathfrak{F}}((\mathbb{F}_p))$ is injective.)
(2) Give a condition for $\mathfrak{F}$ for $U_{\mathfrak{F}}((\mathbb{F}_p))$ to contain the quadratic extension $\mathbb{Q}(\sqrt{5})$ say.
(3) Show that $U_{\mathfrak{F}}((\mathbb{F}_p))$ is NOT countable; in particular, it must be transcendental over $\mathbb{Q}$.