

Divisors, Linear Systems, and Projective Embeddings

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Why we'll be interested in codim of Qcoh's?

§1 Weil Divisors

Hypothesis (x) (by Hartshorne):

X noe, int, sep, reg in codim 1.

i.e. $\forall x \in X, \mathcal{O}_{X,x}$ regular, Krull dim = 1.

Lemma A noe local ring of dim 1. TFAE:

- (a) A regular
 - (b) A normal
 - (c) A DVR.
- $\xrightarrow{\text{normalizing a 1-diml noe}}$ ring produces a regular ring.

Warning noe int domain A, A normal \Leftrightarrow A reg in codim 1
but no converse!

\Leftarrow : add Serre's S₂: $\text{codim}(\mathfrak{a}) = 0$, and

$\forall a \in A, \text{codim}(a) = 1$ when a is not a zero div.

Def'n prime (Weil) div on X = closed int subsch of codim 1.
irred + red

Weil div = \mathbb{Z} -formal comb of primes.

effective = coefficients ≥ 0 .

E.g. $k(x) = \text{func field of } X = \text{local ring of } X \text{ at } \eta \text{ gen pt.}$
 $= \text{Frac}(\mathcal{O}(U))$, $\forall \emptyset \neq U \subseteq X$ open affine sub

$\forall 0 \neq f \in k(x)$ no principal div can go to f :

$\forall z \subseteq X$ prime div, (\mathcal{O}_z, η_z) DVR with v_z valuation.

$$\text{so } (f) = \sum_z v_z(f) \cdot z.$$

only fin many $v_z(f) \neq 0$ (by noe).

\exists ϕ reg func for some $\phi \neq u \in k$

$v_z(\phi) = 0$ whenever $z \notin X - U$.

$\text{Div } X = \text{grp of Weil Divs on } X$.

subgrp: principal divs, $(f) + (g) = (fg)$. $\hookrightarrow \text{PD}_N(X)$.

$$\hookrightarrow \text{Div}(X)/\text{Div}^\circ(X) = \text{Cl}(X).$$

E.g. $X = \text{Spec } A$, A DDK domain (e.g. $A = G_k$ for k field)

$\Rightarrow \text{Cl}(X) = \text{ideal class group}$

$\text{Div}(X) = \text{grp of fractional ideals}$.

$$\sim (\text{lin equiv}): \mathfrak{a} \sim b \Leftrightarrow \mathfrak{a} = \lambda b$$

(differed by a principal div).

Example (Elliptic Curve) $k = \bar{k}$. $P(x, y, z) \in k[x, y, z]$ homo, $\deg = 3$.

$\hookrightarrow P$ defines a nonsing subvar $C \subseteq \mathbb{P}_k^2$.

Pick $o \in C(k)$, \exists surj $\deg: \text{Div } X \rightarrow \mathbb{Z}$

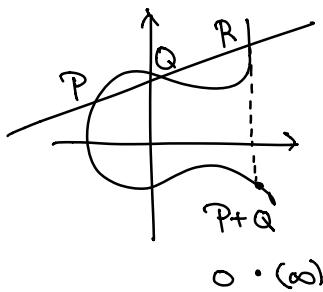
$$P \mapsto 1, P \text{ prime.}$$

note: $P \in \text{Div}^\circ(X) \Rightarrow \deg P = 0$

$\Rightarrow \deg$ factors through $\text{Div } X \rightarrow \text{Cl } X \rightarrow \mathbb{Z}$.

$$\ker(\text{Cl } X \rightarrow \mathbb{Z}) = ((P) - (o))_{P \in C(k)}$$

In fact: $(P) - (o)$ can be computed by the grp law.



$$\begin{aligned}
 (P) + (Q) + (R) &\sim (P) + (P+Q) + (0) \\
 \Rightarrow (P) - (0) + (Q) &\sim (P+Q) \\
 \Rightarrow ((P) - (0)) + ((Q) - (0)) &\sim (S) - (0).
 \end{aligned}$$

§2 Cartier Divisors

X not regular. A more restrictive notion.

X loc const sheaf on to $k(x)$.

Def'n A Cartier div on X : section of the sheaf $k(x)/G^*$.

$$\text{w.r.t. } \text{CaDiv} \xrightarrow{\quad\quad\quad} \text{WDiv}$$

$$D, \text{ repr'd by } f \in k(x) \text{ on } U \mapsto \bar{D}, \quad \text{with } \bar{D}|_U = \text{Div}(f).$$

components of primes that meet U .

Fact X normal \Rightarrow CaDiv \hookrightarrow WDiv inj.

b/c int closed noe domain $A = \bigcap_{\text{min prime}} A_p$

for minimal prime.

Prop (Hartshorne Prop II.6.11)

X loc factorial (i.e. $\forall x \in X$, $\mathcal{O}_{X,x}$ UFD)

$$\Rightarrow \text{CaDiv}(x) \xrightarrow{\cong} \text{WDiv}(x).$$

(In particular, it holds for X reg.)

b/c reg loc \Rightarrow factorial by comm alg.

E.g. $X = \text{Spec } k[x,y,z]/(xy-z^2)$, $I = (x,z)$ defines a WDiv
but it's not a CaDiv.

Principal CaDiv: def'd by single $f \in k(x)$.

$$\text{w.r.t. } \text{CaDiv}(x) := \text{CaDiv}(x)/\text{CaDiv}^0(x).$$

§3 The Picard Group

$\text{Pic}(X) := (\mathcal{L} \text{ invertible; } \otimes) \in \text{Grp}$.

usually the same as $\text{Coh}(X)$.

Namely, $D \in \text{Coh}(X)$, $\mathcal{L}(D) \subseteq \mathcal{K}$ subsheaf s.t.

$$\mathcal{L}(D)(U) = \left\{ f \in \mathcal{K}(X) : (f) + (D) \right\}_{U \geq 0}.$$

Assume X normal $\Rightarrow \mathcal{L}(D)$ be free of rank 1

$\Rightarrow \mathcal{L}(D)$ inv

$\hookrightarrow \text{Coh}(X) \rightarrow \text{Pic}(X)$ homo, kills $\text{Coh}^0(X)$.

$\hookrightarrow \text{Coh}(X) \rightarrow \text{Pic}(X)$ always injective

• X integral \Rightarrow surj. (Hartshorne Prop II.6.15).

Note D effective $\Rightarrow 1 \in \mathcal{K}(X)$ defines global sec on $\mathcal{L}(D)$.

\mathcal{L} loc principal $\Rightarrow \mathcal{L}|_U \cong \mathcal{O}_U$ locally

When we do so,

subsh of $\mathcal{L}(D)$ gen'd by $1 \longleftrightarrow I \subseteq \mathcal{O}_X$ ideal sheaf
defining $D \subseteq X$ closed.

i.e. $D = \text{zero locus of certain section of } \mathcal{L}(D)$.

More generally, (even if $D \geq 0$),

can view $D = \text{Zero}(f)$, f mero sec of $\mathcal{L}(D)$.

(i.e. the zero locus of $\mathcal{L}(D) \otimes_{\mathcal{O}_X} \mathcal{K}(X)$).

Indeed: If $f \in \mathcal{L}(D)$ mero, $\text{Zero}(f) \sim D$ linearly.

§4 Linear Systems

X int sep sch of fin type / k any field.
 L inv sheaf on X .

Def'n A linear system def'd by L

$$= \{ \underset{\substack{\uparrow \\ f \in H}}{\text{Zero}(f)}, f \in H : H \text{ } k\text{-subspace of } H^0(X, L) \}.$$

if $H = H^0(X, L)$: complete lin system.

$$h \in H \text{ w.s. } X \rightarrow \mathbb{P}_k^n, n = \dim_k H - 1$$

- H may have a base pt w.s. this constr fails
 i.e. $P \in \bigcap_{f \in H} \text{Zero}(f)$

In fact, we get $X \subseteq \mathbb{P}_k^n \Leftrightarrow H$ has no base pt.

Suppose now $k = \bar{k}$. X proj, 1-dim'l, irredu, nonsing (i.e. curve)

Consider $\exists D \in \text{Div}(X)$, $L(D)$ w.s. complete lin system.

$$(a) \text{ Can get } X \rightarrow \mathbb{P}_k^n \Leftrightarrow \forall x \in X(k), \dim_k H^0(X, L(D-x)) = \dim_k H^0(X, L(D)) - 1,$$

namely, $\exists f \in L(D)$ s.t. $f(x) \neq 0$.

$$(b) X \xrightarrow{i} \mathbb{P}_k^n \text{ injective} \Leftrightarrow \forall x \neq y \in X(k),$$

$$(\text{as sets}) \quad \dim_k H^0(X, L(D-x-y)) = \dim_k H^0(X, L(D)) - 2$$

namely, $\exists f \in L(D)$ s.t. $f(x) \neq 0$, $f(y) = 0$ and vice versa.

$$(c) X \xrightarrow{i} \mathbb{P}_k^n \text{ in (b) closed imm} \Leftrightarrow \forall x \in X(k),$$

$$\dim_k H^0(X, L(D-2x)) = \dim_k H^0(X, L(D)) - 2$$

namely, $\exists f, g \in L(D)$ s.t. $f(x) \neq 0$, $g(x) = 0$ of order 1.

Rmk (c) ensures $T_x X \hookrightarrow T_{i(x)} \mathbb{P}_k^n$.

when X embeds into \mathbb{P}^n_k
we compute $\dim_k H^0(X, L(D))$, $\forall D \in \text{Div } X$
(abettet by Riemann-Roch).