2022 Summer School on the Langlands Program at IHES

SHIMURA VARIETIES

SOPHIE MOREL

(SCRIBED BY WENHAN DAI)

Readme. This is a very preliminary version for the lecture series given by Sophie Morel at IHES Summer School 2022. *Please use with caution and do not disseminate.*

Due to the mistake and carelessness of the notetaker, it is missing parts and many references and is full of typos. Also, every sign has at least a 50% chance of being wrong.

If you know what this sentence means, the latest edited version by Sophie Morel of these notes will possibly be found at `Quatramaran` website (no guarantees though).

Contents

Date: July 12, 2022.

1. Locally symmetric spaces and Shimura varieties

1.1. **Locally symmetric spaces.** Let G be a semisimple algebraic group over $\mathbb{Q}$, for example SL_n, Sp_{2n}, or $\mathrm{SO}(p,q)$. We would like to present some "nice enough" whose cohomology is related to automorphic representations of G. A good reference for locally symmetric spaces is the introductory paper [Ji06] by Ji.

To simplify the presentation, we will assume here that $G(\mathbb{R})$ is connected. Let K_∞ be a maximal compact subgroup of $G(\mathbb{R})$, and let $X = G(\mathbb{R})/K_\infty$. If Γ is a discrete subgroup of $G(\mathbb{R})$ such that $\Gamma\backslash G(\mathbb{R})$ (or equivalently $\Gamma\backslash X$) is compact and Γ acts properly and freely on X,[1] then there is a classical connection between the cohomology of $\Gamma\backslash X$ and automorphic representations of $G(\mathbb{R})$, called **Matsushima's formula** (see Matsushima's paper [**?**]). We will state a modern reformulation in Lecture 3, but roughly it relates the Betti numbers of $\Gamma\backslash X$ and the multiplicities of representations of $G(\mathbb{R})$ in $L^2(\Gamma\backslash G(\mathbb{R}))$.

In fact, Matsushima's paper deals with semi-simple real Lie groups. Here, we have an algebraic group defined over $\mathbb{Q}$, so we have a particularly nice way to produce discrete subgroups of $G(\mathbb{R})$. Remember that a subgroup Γ of $G(\mathbb{Q})$ is called an **arithmetic subgroup** if there exists a closed embedding $G \subset \mathrm{GL}_N$ such that, setting $G(\mathbb{Z}) = G(\mathbb{Q}) \cap \mathrm{GL}_N(\mathbb{Z})$, we have that $\Gamma \cap G(\mathbb{Z})$ is of finite index in Γ and in $G(\mathbb{Z})$.[2] If Γ is small enough, then it acts properly and freely on X ([Ji06, Proposition 5.5]), so the quotient $\Gamma\backslash X$ is a real analytic manifold. Also, the quotient $\Gamma\backslash G(\mathbb{R})$ is compact if and only if G is anisotropic (over $\mathbb{Q}$), which means that G has no nontrivial parabolic subgroup defined over $\mathbb{Q}$ ([Ji06, Theorem 5.10]). (A subgroup of G is parabolic if it contains a Borel subgroup B of G.) If $\Gamma\backslash X$ is not compact but $G(\mathbb{R})$ has a discrete series, then there is an extension of Matsushima's formula, due to Borel and Casselman in [BC83], that involves L^2 cohomology of $\Gamma\backslash X$; see Lecture 3.

We actually would like to see automorphic representations of $G(\mathbb{A})$ (not just $G(\mathbb{R})$) in the cohomology of our spaces, so we will use adelic versions of $\Gamma\backslash X$. Let K be an open compact subgroup of $G(\mathbb{A}_f)$; for example, if we have chosen an embedding $G \subset \mathrm{GL}_N$, then we could take

$$K = G(\mathbb{A}_f) \cap \mathrm{Ker}(\mathrm{GL}_N(\widehat{\mathbb{Z}}) \to \mathrm{GL}_N(\mathbb{Z}/n\mathbb{Z})),$$

for some positive integer n (these are called **principal congruence subgroups**). Let

$$M_K = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K,$$

where the group K acts by right translations on the factor $G(\mathbb{A}_f)$, and the group $G(\mathbb{Q})$ acts by left translations on both factors simultaneously. Choose a system of representatives $(x_i)_{i\in I}$ of the (finite) quotient $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$, and set

$$\Gamma_i = G(\mathbb{Q}) \cap x_i K x_i^{-1}$$

for every $i \in I$. Then the Γ_i are arithmetic subgroups of $G(\mathbb{Q})$, and we have

$$M_K = \coprod_{i\in I} \Gamma_i\backslash X,$$

[1] This holds for example if Γ is torsion free, which happens when Γ is small enough.

[2] We can check that this definition does not depend on the embedding $G \subset \mathrm{GL}_N$, see [Ji06, Proposition 4.2].

so M_K is a real analytic manifold if K is small enough. But now we have an action of $G(\mathbb{A}_f)$ on the projective system $(M_K)_{K\subset G(\mathbb{A}_f)}$, so we get an action on $\varinjlim_K H^*(M_K)$, where H^* is any "reasonable" cohomology theory, for example Betti cohomology. If G is anisotropic over $\mathbb{Q}$, then Matsushima's result can be reformulated to give a description of this action in terms of irreducible representations of $G(\mathbb{A})$ appearing in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$, and there is also a version of the Borel-Casselman generalization.

There is another way to think about the action of $G(\mathbb{A}_f)$ on $(M_K)_{K\subset G(\mathbb{A}_f)}$, which does not involve a limit on K. Fix a Haar measure on $G(\mathbb{A}_f)$ such that open compact subgroups of $G(\mathbb{A}_f)$ have rational volume (this is possible because these groups are all commensurable); then every open subset of $G(\mathbb{A}_f)$ has rational volume. The **Hecke algebra** of G is the space $\mathcal{H}_G$ of locally constant functions with compact support from $G(\mathbb{A}_f)$ to $\mathbb{Q}$; if $f, g \in \mathcal{H}_G$, then the convolution product $f * g$ still has rational values by the choice of Haar measure, so convolution defines a multiplication on $\mathcal{H}_G$. For every open compact subgroup K of $G(\mathbb{A}_f)$, the **Hecke algebra at level** K is the subalgebra $\mathcal{H}_{G,K}$ of bi-K-invariant functions in $\mathcal{H}_G$; we have $\mathcal{H}_G = \bigcup_K \mathcal{H}_{G,K}$.

Fix K small enough. Then $H^*(M_K)$ is basically the set of K-invariant vectors:

$$H^*(M_K) = \varinjlim_{K'\subset G(\mathbb{A}_f)} H^*(M_{K'})^K,$$

so it has an action of $\mathcal{H}_{G,K}$.[3] We can describe this action using Hecke correspondences: let $g \in G(\mathbb{A}_f)$, let K' be an open compact subgroup of $G(\mathbb{A}_f)$ such that $K' \subset K \cap gKg^{-1}$, then we have a **Hecke correspondence**

$$\begin{aligned}(T_1, T_g) : M_{K'} &\longrightarrow M_K \times M_K\\ (x,h) &\longmapsto ((x,h),(x,hg)),\end{aligned}$$

and T_1, T_g are both finite covering maps if K is small enough. Up to a scalar,[4] the action $\mathbb{1}_{KgK}$ on $H^*(M_K)$ is given by pulling back cohomology classes along T_1, then pushing them forward along T_g.

We can also ask whether there is more structure on the spaces $\Gamma\backslash X$ (or M_K). For example, suppose that $G = \mathrm{SL}_2$ and $K_\infty = \mathrm{SO}(2)$. Then, for Γ an arithmetic subgroup of $\mathrm{SL}_2(\mathbb{Z})$, the space $\Gamma\backslash X$ is a modular curve, so it is (the set of complex points of) an algebraic variety defined over a function field F, and we can use the commuting actions of Hecke correspondences and of the absolute Galois group of F on its étale cohomology to construct some instance of the global Langlands correspondence for SL_2 or GL_2.

In order to generalize this picture, we first to know when the spaces $\Gamma\backslash X$ or M_K are the set of $\mathbb{C}$-points of an algebraic variety, and whether this algebraic variety is defined over a number field. As we will see later, another advantage over M_K over $\Gamma\backslash X$ is that, when the answer to the above question is "yes", then the M_K for K varying tend to all be defined over the same field, while this is not the case for the $\Gamma\backslash X$.

Remark 1.1. The first step is to check whether $\Gamma\backslash X$ has the structure of a complex manifold, and there are obvious obstructions to that. For example, if $G = \mathrm{SL}_3$ and $K_\infty = \mathrm{SO}(3)$,

[3] In fact, we can recover the action of $G(\mathbb{A}_f)$ on $\varinjlim_{K'\subset G(\mathbb{A}_f)} H^*(M_{K'})$ from the action of $\mathcal{H}_{G,K}$ on M_K for every K small enough.

[4] Make scalar precise.

then $\Gamma\backslash X$ is 5-dimensional as a real manifold, so it cannot have the structure of a complex manifold. In fact, there is no structure of complex manifold on $\Gamma\backslash X$ for $G = \mathrm{GL}_d$ with $d \geqslant 3$, as we will now see.

Choose a $G(\mathbb{R})$-invariant Riemannian metric on $X = G(\mathbb{R})/K_\infty$ (such a metric is unique up to rescaling on each irreducible factor). Then X is a **symmetric space**, that is, a Riemannian manifold such that:

(a) The group of isometries of X acts transitively on X;
(b) For every $p \in X$, there exists a symmetry s_p of X (i.e. an involutive isometry) such that p is an isolated fixed point of s_p.

Moreover, the symmetric space X is of **noncompact type**, that is, it has negative curvature. For Γ a small enough arithmetic subgroup of $G(\mathbb{Q})$, the Riemannian manifold $\Gamma\backslash X$ is a **locally symmetric space**; in particular, it does not satisfy condition (a) anymore, and it satisfies a variant of condition (b) where we only ask for the symmetry to be defined in a neighborhood of the point. See Ji's notes [Ji06] for a review of locally symmetric spaces.

We say that X is a **Hermitian symmetric domain** if it admits a $G(\mathbb{R})$-invariant Hermitian metric. See Section 1 of Milne's notes [**?**] for a review of Hermitian symmetric domains.

Example 1.2 (Siegel upper half space)**.** Let d be a positive integer. The **Siegel upper half space** $\mathfrak{h}_d^+$ is the set of symmetric $d \times d$ complex matrices in $Y \in M_d(\mathbb{C})$ such $\mathrm{Im}(Y)$ is positive definite; if $d = 1$, then this is just the usual upper half plane. Then the Siegel upper half space $\mathfrak{h}_d^+$ is a Hermitian symmetric domain. The proofs of the basic properties of $\mathfrak{h}_d^+$ can be found in Siegel's paper [**?**].

We first need to see $\mathfrak{h}_d^+$ as a symmetric space. Let Sp_{2d} be the symplectic group of the symplectic form with matrix $\begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$, where $I_d \in \mathrm{GL}_d(\mathbb{Z})$ is the identity matrix. For every commutative ring R, we have

$$\mathrm{Sp}_{2d}(R) = \left\{ g \in \mathrm{GL}_{2d}(R) \,\middle|\, {}^t g \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \right\}.$$

Note that $\mathrm{Sp}_2 = \mathrm{SL}_2$. We make $\mathrm{Sp}_{2g}(\mathbb{R})$ act on $\mathfrak{h}_d^+$ by the following formula:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Y = (AY + B)(CY + D)^{-1},$$

where A, B, C, D are $d \times d$ matrices such that $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2d}(\mathbb{R})$ (see page 9 of [**?**]). Then this action is transitive (see page 9 of [**?**]). Let K_∞ be the stabilizer in $\mathrm{Sp}_{2d}(\mathbb{R})$ of $iI_d \in \mathfrak{h}_d^+$. Then $K_\infty = O(2d) \cap \mathrm{Sp}_{2d}(\mathbb{R})$ (this is easy to check directly), so it is a maximal compact subgroup of $\mathrm{Sp}_{2d}(\mathbb{R})$,[5] and we have

$$\mathfrak{h}_d^+ \simeq \mathrm{Sp}_{2d}(\mathbb{R})/K_\infty$$

as real analytic manifolds.

Also, the space $\mathfrak{h}_d^+$ is an open subset of the complex vector space of symmetric matrices in $M_d(\mathbb{C})$, so it has an obvious structure of complex manifold. It remains to construct a $\mathrm{Sp}_{2d}(\mathbb{R})$-invariant Hermitian metric on $\mathfrak{h}_d^+$. Let $\mathcal{D}_d$ be the set of symmetric matrices $A \in M_d(\mathbb{C})$ such that $I_d - A^*A$ is positive definite; this is a bounded domain in the complex vector space of symmetric matrices in $M_d(\mathbb{C})$, hence is equipped with a canonical Hermitian metric called the **Bergman metric**, which has negative curvature (see for example, [**?**, Theorem 1.3]); in particular, this metric is invariant by all holomorphic automorphisms of $\mathcal{D}_d$. Now note that we have an isomorphism

$$h_d^+ \xrightarrow{\sim} \mathcal{D}_d, \quad X \longmapsto (iI_d - X)(iI_d + X)^{-1}$$

(whose inverse sends $A \in \mathcal{D}_d$ to $i(I_d - A)(I_d + A)^{-1}$), see [**?**, pp. 8-9]. We can give a formula for the resulting Hermitian metric on h_d^+: up to a positive scalar, it is given by

$$ds^2 = \mathrm{Tr}(\mathrm{Im}(Y)^{-2} dY\, \mathrm{Im}(Y)^{-1} d\overline{Y})$$

(see formula (28) on page 17 of [**?**]).

The isomorphism $h_d^+ \simeq \mathcal{D}_d$ is called a **bounded realization** of h_d^+.

We can give a complete classification of Hermitian symmetric domains (cf. [**?**, Theorem 1.21]), in terms of real algebraic groups:

Theorem 1.3. *Suppose that $G(\mathbb{R})$ is connected and adjoint. The locally symmetric space X is a Hermitian symmetric domain if and only if there exists a morphism of real Lie groups $u : \mathrm{U}(1) \to G(\mathbb{R})$ such that:*

(a) *The only characters of* $\mathrm{U}(1)$ *that appear in its representation* $\mathrm{Ad} \circ u$ *on* $\mathrm{Lie}(G(\mathbb{R}))$ *are* 1, z, *and* z^{-1};
(b) *Conjugation by $u(i)$ is a Cartan involution of $G(\mathbb{R})$, which means that $\{g \in G(\mathbb{C}) \mid g = u(i)\overline{g}u(i)^{-1}\}$ is compact;*
(c) *The projection of $u(i)$ to a simple factor of $G(\mathbb{R})$ is never equal to* 1.

Moreover, we can choose u such that K_∞ is the centralizer of u in $G(\mathbb{R})$, which means that X is isomorphic to set of conjugates of u by elements of $G(\mathbb{R})$.

We explain the construction of the morphism u. Suppose that X is a Hermitian symmetric domain, and let $p \in X$. For every $z \in \mathbb{C}$ with $|z| = 1$, multiplication by z on T_pX preserves the Hermitian metric and sectional curvatures, so there exists a unique isometry $u_p(z)$ of D fixing p and such that $T_p u_p(z)$ is multiplication by z. The uniqueness implies that $u_p(z)u_p(z') = u_p(zz')$ if $|z| = |z'| = 1$, so we get a morphism of groups from $\mathrm{U}(1)$ to the group of isometries of X, which is equal to $G(\mathbb{R})^0_{\mathrm{ad}}$.

[5]In fact, we have an isomorphism (with $X, Y \in \mathrm{GL}_d(\mathbb{R})$):

$$U(d) \xrightarrow{\sim} K_\infty, \quad X + iY \longmapsto \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}.$$

Example 1.4.
(1) If $G = \mathrm{Sp}_{2d}$, let $h : \mathbb{C}^\times \to G(\mathbb{R})$ be defined by
$$h(a+ib) = \begin{pmatrix} aI_d & -bI_d \\ bI_d & aI_d \end{pmatrix}.$$
Then we can take $u : \mathrm{U}(1) \to \mathrm{PSp}_{2d}(\mathbb{R})$ given by $u(z) = h(\sqrt{z})$. Note that u does not lift to a morphism from $U(1)$ into $G(\mathbb{R})$.
(2) If $G = \mathrm{PGL}_n$ with $n \geqslant 3$, then the centralizer of a character $u : \mathrm{U}(1) \to G(\mathbb{R})$ cannot be a maximal compact subgroup of $G(\mathbb{R})$ (exercise), so the locally symmetric space of maximal compact subgroups of $G(\mathbb{R})$ is not Hermitian.

Theorem 1.3 puts some pretty strong restrictions on the root systems of the simple factors of $G(\mathbb{R})$, see Theorem 1.25 of [**?**] and the table following it. In particular, the type A simple factors of $G(\mathbb{R})$ must be of the form $\mathrm{PSU}(p, q)$, and $G(\mathbb{R})$ can have no simple factor of type E_8, F_4 or G_2.

The natural next step would be to wonder for which Hermitian symmetric domains X the quotients $\Gamma\backslash X$ are algebraic varieties, but in fact it turns out that the answer is "for all of them", as was proved by Baily and Borel [BB66].

Theorem 1.5 (Baily-Borel)**.** *Suppose that $X = G(\mathbb{R})/K_\infty$ is a Hermitian symmetric domain. Then, for any torsion free arithmetic subgroup Γ of $G(\mathbb{Q})$, the quotient $\Gamma\backslash X$ has a canonical structure of algebraic variety over $\mathbb{C}$.*

The very rough idea is that the sheaf of automorphic forms on $\Gamma\backslash X$ of sufficiently high weight will define an embedding of $\Gamma\backslash X$ into a projective space.

Remember that we did not just want the locally symmetric spaces $\Gamma\backslash X$ to be algebraic varieties, we also wanted them to be defined over a number field, and we would ideally like the number field in question to only depend on G and K_∞. For this, it will actually be easier to work with reductive groups instead of semi-simple groups. As a motivation for this, and for the definition of Shimura varieties, we now spend some more time on the case of the symplectic group.

1.2. **The Siegel modular variety.** See the end of this subsection for some background on abelian schemes.

1.3. **The Siegel upper half space as a moduli space of abelian varieties over $\mathbb{C}$.** We use the notation of Example 1.2. It is well-known that $\mathfrak{h}_1^+$ parametrizes elliptic curves over $\mathbb{C}$: an element $\tau \in \mathfrak{h}_1^+$ is sent to the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, and $E_\tau \simeq E_{\tau'}$ if and only if $\tau, \tau' \in \mathfrak{h}_1^+$ they are conjugated under the action of $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{Sp}_2(\mathbb{Z})$; so we can recover τ from E_τ and the data of a symplectic isomorphism $H_1(E_\tau, \mathbb{Z}) \simeq \mathbb{Z}^2$ where $\mathbb{Z}^2$ is equipped with the standard symplectic form. We want to give a similar picture for higher-dimensional abelian varieties; in fact, the analogy works best if we consider abelian varieties with a principal polarization (Definition **??**).

We first introduce some notation about symplectic spaces and recall the definition of the (general) symplectic group as a group scheme over $\mathbb{Z}$. If R is a commutative ring, we denote by ψ_R the perfect symplectic pairing on R^{2d} with matrix $\begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$. So we have

$$\psi_R((x_1,\dots,x_d,y_1,\dots,y_d),(x'_1,\dots,x'_d,y'_1,\dots,y'_d)) = \sum_{i=1}^d x_i y'_i - \sum_{i=1}^g x'_i y_i.$$

The **general symplectic group** GSp_{2d} is the reductive group scheme over $\mathbb{Z}$ whose points in a commutative ring R are given by

$$\mathrm{GSp}_{2d}(R) = \{g \in \mathrm{GL}_{2d}(R) \mid \exists c(g) \in R^\times,\ \forall v,v' \in R^{2d},\ \psi_R(gv,gv') = c(g)\psi_R(v,v')\}.$$

The scalar $c(g)$ is called the **multiplier** of $g \in \mathrm{GSp}_{2d}(R)$. Sending g to $c(g)$ defines a morphism of group schemes $c : \mathrm{GSp}_{2d} \to \mathrm{GL}_1$, whose kernel Sp_{2d} is called the **symplectic group**.

Example 1.6. We have $\mathrm{GSp}_2 = \mathrm{GL}_2$ in which $c = \det$, and $\mathrm{Sp}_2 = \mathrm{SL}_2$.

1.3.1. *Complex abelian variety.* Let A be complex abelian variety of dimension d; we identify A and its set of complex points. Then A is a connected complex Lie group of dimension d, so we have $A \simeq \mathrm{Lie}(A)/\Lambda$, with $\mathrm{Lie}(A) \simeq \mathbb{C}^d$ the universal cover of A and $\Lambda = \pi_1(A) = H_1(A,\mathbb{Z}) \simeq \mathbb{Z}^{2d}$ a lattice in the underlying $\mathbb{R}$0vector space. Let $A^\vee$ be the dual abelian variety, i.e., the space of degree 0 line bundles on A (see Definition **??**). We can identify $\mathrm{Lie}(A^\vee)$ with the space of antilinear forms on $\mathrm{Lie}(A)$ and $H_1(A^\vee,\mathbb{Z})$ with the subspace $\Lambda^\vee$ of forms whose imaginary part takes integer values on Λ (see [**?**, §9]). For every positive integer n, we have

$$A[n] = \frac{1}{n}\Lambda/\Lambda, \quad A^\vee[n] = \frac{1}{n}\Lambda^\vee/\Lambda^\vee,$$

and the canonical pairing $A[n] \times A^\vee[n] \to \mu_n(\mathbb{C})$ is given by

$$(v,u) \mapsto e^{-2i\pi n\,\mathrm{Im}(u(v))}$$

(see [**?**, §24]). We then have a bijection between the set of polarizations on A and the set of positive definite Hermitian forms[6] H on $\mathbb{C}^{2d}$ such that the symplectic form $\mathrm{Im}(H)$ takes integer values on Λ; given such a form H, the corresponding isogeny λ_H from A to $A^\vee$ is given on $\mathbb{C}$-points by:

$$\begin{aligned} \lambda_H : \mathrm{Lie}(A)/\Lambda &\longrightarrow \mathrm{Lie}(A^\vee)/\Lambda^\vee \\ w &\longmapsto (v \mapsto H(v,w)). \end{aligned}$$

It follows that the Weil pairing (see Remark **??** (2)) corresponding to λ_H is the map

$$\begin{aligned} A[n] \times A[n] &\longrightarrow \mu_n(\mathbb{C}) \\ (v,w) &\longmapsto e^{-2i\pi n\,\mathrm{Im}(H(v,w))}. \end{aligned}$$

Note that we have $v,w \in \frac{1}{n}\Lambda$, so $\mathrm{Im}(H(v,w)) \in \frac{1}{n^2}\mathbb{Z}$.

In particular, the polarization λ_H is principal if and only if Λ is self-dual with respect to the symplectic form $\mathrm{Im}(H)$, that is,

$$\Lambda = \{w \in \mathrm{Lie}(A) \mid \forall v \in \Lambda,\ \mathrm{Im}(H(v,w)) \in \mathbb{Z}\}.$$

In that case, the symplectic $\mathbb{Z}$-module $(\Lambda, \mathrm{Im}(H))$ is isomorphic to $\mathbb{Z}^{2d}$ with the form $\psi_\mathbb{Z}$.

[6]We take Hermitian forms to be semi-linear in the first variable and linear in the second variable.

Let $\widetilde{\mathcal{M}}_d$ be the set of isomorphism classes of triples $(A, \lambda, \eta_{\mathbb{Z}})$, where A is a complex abelian variety of dimension d, λ is a principal polarization on A, and $\eta_{\mathbb{Z}}$ is an morphism of symplectic spaces from $H_1(A, \mathbb{Z})$ to $(\mathbb{Z}^{2d}, \psi_{\mathbb{Z}})$. We have an action of $\mathrm{Sp}_{2d}(\mathbb{Z})$ on $\widetilde{\mathcal{M}}_d$: if $c = (A, \lambda, \eta_{\mathbb{Z}}) \in \widetilde{\mathcal{M}}_d$ and $x \in \mathrm{Sp}_{2g}(\mathbb{Z})$, set $x \cdot c = (A, \lambda, x \circ \eta_{\mathbb{Z}})$.

If $(A, \lambda, \eta_{\mathbb{Z}}) \in \widetilde{\mathcal{M}}_d$, then $\Lambda = H_1(A, \mathbb{Z})$ is a lattice in the real vector space $\mathrm{Lie}(A)$, we have $A = \mathrm{Lie}(A)/\Lambda$ and we can recover the Hermitian form H_λ corresponding to λ from $\mathrm{Im}(H_\lambda)|_\Lambda$, which is sent to the form $\psi_{\mathbb{Z}}$ on $\mathbb{Z}^{2d}$ by the isomorphism $\eta_{\mathbb{Z}} : \Lambda \xrightarrow{\sim} \mathbb{Z}^{2d}$. If we see $\mathbb{R}^{2d}$ as a complex vector space via the isomorphisms (of real vector spaces)

$$\mathrm{Lie}(A) \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} \mathbb{R}^{2d},$$

then the Hermitian form on $\mathbb{R}^{2d}$ corresponding to H_λ is

$$(v, w) \longmapsto \psi_{\mathbb{R}}(iv, w) + i\psi_{\mathbb{R}}(v, w).$$

So $\eta_{\mathbb{Z}}$ determines all the data of the isomorphism class of $(A, \lambda, \eta_{\mathbb{Z}})$, except for the structure of complex vector space on $\mathbb{R}^{2d}$. This structure of complex vector space is equivalent to the data of an $\mathbb{R}$-linear endomorphism J of $\mathbb{R}^{2d}$ such that $J^2 = -1$ (the endomorphism J corresponds to multiplication by i). We also need the $\mathbb{R}$-bilinear map $\mathbb{R}^{2d} \times \mathbb{R}^{2d} \to \mathbb{C}$ defined by $(v, w) \longmapsto \psi_{\mathbb{R}}(J(v), w) + i\psi_{\mathbb{R}}(v, w)$ to be a positive definite Hermitian form on $\mathbb{R}^{2d}$. This is equivalent to the following conditions:

(a) $\psi_{\mathbb{R}}(J(v), J(w)) = \psi_{\mathbb{R}}(v, w)$ for all $v, w \in \mathbb{R}^{2d}$;
(b) the $\mathbb{R}$-bilinear form $(v, w) \mapsto \psi_{\mathbb{R}}(J(v), w)$ on $\mathbb{R}^{2d}$ (which is symmetric by (a)) is positive definite.

Conversely, if we have a complex structure J on $\mathbb{R}^{2d}$ satisfying (a) and (b), then we get a positive definite Hermitian form H on $\mathbb{R}^{2d}$ whose imaginary part takes integer values on the lattice $\mathbb{Z}^{2d}$, so the complex torus $\mathbb{R}^{2d}/\mathbb{Z}^{2d}$ has a polarization, hence is an abelian variety (for example by the Kodaira embedding theorem), and we gat an element of $\widetilde{\mathcal{M}}_d$.

So we get a bijection from $\widetilde{\mathcal{M}}_d$ to the set X' of endomorphisms J of $\mathbb{R}^{2d}$ such that $J^2 = -1$ and that J satisfies condition (a) and (b).

Now observe that, if W is a $\mathbb{R}$-vector space, then the data of an endomorphism J of W such that $J^2 = -1$ (i.e. of the structure of a $\mathbb{C}$-vector space on W) is equivalent to the data of a $\mathbb{C}$-linear endomorphism $J_{\mathbb{C}}$ of $W \otimes_{\mathbb{R}} \mathbb{C}$ such that

$$\mathrm{Ker}(J_{\mathbb{C}} - i \cdot \mathrm{id}) = \overline{\mathrm{Ker}(J_{\mathbb{C}} + i \cdot \mathrm{id})},$$

where $v \mapsto \overline{v}$ is the involution of $W \otimes_{\mathbb{C}} \mathbb{C}$ induced by complex conjugation on $\mathbb{C}$. This is equivalent to giving a $\mathbb{C}$-vector subspace E of $W \otimes_{\mathbb{R}} \mathbb{C}$ such that $W \otimes_{\mathbb{R}} \mathbb{C} = E \oplus \overline{E}$, i.e., a d-dimensional complex subspace E of $W \otimes_{\mathbb{R}} \mathbb{C}$ such that $E \cap \overline{E} = \{0\}$.[7]

[7] In fancy terms, we are saying that putting a structure of complex vector space on W is the same as putting a pure Hodge structure of type $\{(-1, 0), (0, -1)\}$ on it (or of type $\{(1, 0), (0, 1)\}$, depending on your normalization). When $W = \mathbb{R}^{2d}$ and the complex structure comes from an element $(A, \lambda, \eta_{\mathbb{Z}})$ of $\widetilde{\mathcal{M}}_d$, then this Hodge structure is the one induced by the isomorphism

$$H_1(A, \mathbb{R}) \xrightarrow{\eta_{\mathbb{Z}} \otimes \mathbb{R}} \mathbb{Z}^{2d} \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} \mathbb{R}^{2d}.$$

We apply this to $W = \mathbb{R}^{2d}$. Let J be a complex structure on $\mathbb{R}^{2d}$, and let E be the corresponding $\mathbb{C}$-vector subspace of $\mathbb{C}^{2d}$. Then condition (a) on J is equivalent to the fact that:

(a') $\psi_{\mathbb{C}}(v,w) = 0$ for all $v, w \in E$,

(i.e., to the fact that E is a Lagrangian subspace[8] of $\mathbb{C}^{2d}$), and condition (b) on J is equivalent to the fact that

(b') $-i\psi_{\mathbb{C}}(v,\overline{v}) \in \mathbb{R}_{>0}$ for all $v \in E \backslash \{0\}$.

Note that these two conditions on a $\mathbb{C}$-vector subspace E of $\mathbb{C}^{2d}$ imply that $E \cap \overline{E} = \{0\}$. So we get a bijection from X' to the set of Lagrangian subspaces E of $V_{\mathbb{C}}$ satisfying (b').

If we represent Lagrangian subspaces of $\mathbb{C}^{2d}$ by their bases, sees as complex matrices of size $d \times 2d$, then the action of $\mathrm{Sp}_{2d}(\mathbb{R})$ is just left multiplication. For example, the subspace E_0 corresponding to $J_d \in X'$ is the one with basis $\begin{pmatrix} iI_d \\ I_d \end{pmatrix}$.

More generally, if $Y \in \mathfrak{h}_d^+$, the subspace of $\mathbb{C}^{2d}$ with basis $\begin{pmatrix} Y \\ I_d \end{pmatrix}$ is a Lagrangian subspace satisfying condition (b'), and every such Lagrangian subspace is of that form. So we get bijections

$$\widetilde{\mathcal{M}}_d \simeq X' \simeq \mathfrak{h}_d^+,$$

and we can check that the second bijection is $\mathrm{Sp}_{2d}(\mathbb{R})$-equivariant. Unraveling the definitions, we see that $Y \in \mathfrak{h}_d^+$ corresponds to the element $(A_Y, \lambda_Y, \eta_{\mathbb{Z},Y})$ of $\widetilde{\mathcal{M}}_d$ such that $A_Y = \mathbb{C}^d/(\mathbb{Z}^d + Y\mathbb{Z}^d)$, λ_Y is the principal polarization given by the Hermitian form with matrix $\mathrm{Im}(Y)^{-1}$ on $\mathbb{C}^d$, and $\eta_{\mathbb{Z},Y} : \mathbb{Z}^d + Y\mathbb{Z}^d \xrightarrow{\sim} \mathbb{Z}^{2d}$ is the isomorphism sending $a \in \mathbb{Z}^d$ to $(a,0) \in \mathbb{Z}^d \times \mathbb{Z}^d = \mathbb{Z}^{2d}$ and $Ya \in Y\mathbb{Z}^d$ to $(0,a) \in \mathbb{Z}^d \times \mathbb{Z}^d = \mathbb{Z}^{2d}$.

Now we want an interpretation of the quotients $\Gamma\backslash\mathfrak{h}_d^+$, for Γ an arithmetic subgroup of $\mathrm{Sp}_{2d}(\mathbb{Q})$. We will do this for the groups $\Gamma(n) = \mathrm{Ker}(\mathrm{Sp}_{2d}(\mathbb{Z}) \to \mathrm{Sp}_{2d}(\mathbb{Z}/n\mathbb{Z}))$, where n is a positive integer (and $\Gamma(n)$ is called the **principal congruence subgroup** at level n). Note that any arithmetic group contains $\Gamma(n)$ for n divisible enough.

We will need the notion of a level structure; we give the general definition here.

Definition 1.7. Let S be a scheme, (A,λ) be a principally polarized abelian scheme of relative dimension d over S, and n be a positive integer. Than a **level n structure** on (A,λ) is a couple (η,φ), where

$$\eta : A[n] \xrightarrow{\sim} \underline{\mathbb{Z}/n\mathbb{Z}}_S^{2g}, \quad \varphi : \underline{\mathbb{Z}/n\mathbb{Z}}_S \xrightarrow{\sim} \mu_{n,S}$$

are isomorphisms of group schemes such that $\varphi \circ \psi_{\mathbb{Z}/n\mathbb{Z}} \circ \eta$ is the Weil pairing associated to λ on $A[n]$.

Remark 1.8. A level n structure on (A,λ) can only exist if n is invertible on S and $\mu_{n,S}$ is a constant group scheme.

Note that isomorphisms $\varphi : \underline{\mathbb{Z}/n\mathbb{Z}}_S \xrightarrow{\sim} \mu_{n,S}$ correspond to sections $\zeta \in \mu_n(S)$ generating $\mu_{n,S}$ (i.e. to primitive nth roots of 1 over S), by sending φ to $\zeta = \varphi(1)$. So we will also see level structures as couples (η,ζ), with $\zeta \in \mu_n(S)$ primitive.

[8]By definition, a maximal isotropic subspace.

Let $\zeta_n = e^{-2i\pi/n} \in \mu_n(\mathbb{C})$. If $Y \in \mathfrak{h}_d^+$, then $\frac{1}{n}\eta_{\mathbb{Z},Y}$ defines an isomorphism of groups $\eta_Y : A_Y[n] \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^{2d}$, and it follows from formula

$$A[n] \times A[n] \to \mu_n(\mathbb{C}), \quad (v,w) \mapsto e^{-2i\pi n \operatorname{Im}(H(v,w))}$$

that (η, ζ_n) is a level n structure on (A_Y, η_Y).

Using the fact that $\mathrm{Sp}_{2d}(\mathbb{Z}) \to \mathrm{Sp}_{2d}(\mathbb{Z}/n\mathbb{Z})$ is surjective for every $n \in \mathbb{N}$, which follows from strong approximation for Sp_{2d},[9] we finally get:

Proposition 1.9. *Let n be a positive integer. The map $Y \mapsto (A_Y, \lambda_Y, \eta_Y)$ induces a bijection from $\Gamma(n)\backslash\mathfrak{h}_d^+$ to the set of isomorphism classes of triples (A, λ, η), where (A, λ) is a principally polarized complex abelian variety of dimension d and $\eta : A[n] \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^{2d}$ is an isomorphism of groups such that (η, ζ_n) is a level n structure on (A, λ).*

Now there is an obvious way to make $\Gamma(n)\backslash\mathfrak{h}_d^+$ into an algebraic variety.

1.3.2. *The connected Siegel modular variety.* Let $\mathcal{O}_n = \mathbb{Z}[1/n][T]/(T^n - 1)$. If S is a scheme over $\mathcal{O}_n$, we denote by $\varphi_0 : \underline{\mathbb{Z}/n\mathbb{Z}}_S \xrightarrow{\sim} \mu_{n,S}$ the isomorphism sending 1 to the class of T.

Definition 1.10. Let $\mathcal{M}'_{d,n}$ be the functor from the category of $\mathcal{O}_n$-schemes to the category of sets sending S to the set of isomorphisms classes of triples (A, λ, η), where (A, η) is a principally polarized abelian scheme of relative dimension d over S and $\eta : A[n] \xrightarrow{\sim} \underline{\mathbb{Z}/n\mathbb{Z}}_S^{2g}$ is an isomorphism of group schemes such that (η, φ_0) is a level n structure on (A, λ).

An isomorphism from (A, λ, η) to (A', λ', η') is an isomorphism of abelian varieties $u : A \xrightarrow{\sim} A'$ such that $\lambda' \circ u = u^\vee \circ \lambda$ and $\eta' = \eta \circ (u, u)$.

Theorem 1.11 (Mumford, cf. [FC90])**.** *Suppose that $n \geqslant 3$. Then the functor $\mathcal{M}'_{d,n}$ is representable by a smooth quasi-projective $\mathcal{O}_n$-scheme purely of dimension $d(d+1)/2$ and with connected geometric fibers, which we still denote by $\mathcal{M}'_{d,n}$ and call the* ***connected Siegel modular variety*** *of level n.*

Remark 1.12. If $n \in \{1, 2\}$, then triples (A, λ, η) as in Definition 1.10 may have automorphisms, so we should see $\mathcal{M}'_{d,n}$ as a stack. This stack will then be representable by a smooth Deligne-Mumford stack over $\mathcal{O}_n$ that is a finite étale quotient of the scheme $\mathcal{M}'_{d,3n}$.

We can now reformulate Proposition 1.9 in the following way.

Proposition 1.13. *Let $n \geqslant 3$ be an integer. Then the map $Y \mapsto (A_Y, \lambda_Y, \eta_Y)$ induces an isomorphism of complex manifolds from $\Gamma(n)\backslash\mathfrak{h}_d^+$ to $\mathcal{M}'_{d,n}(\mathbb{C})$.*

The fact that this is an isomorphism of complex manifolds is clear on the explicit formula for the bijection $\Gamma(n)\backslash\mathfrak{h}_d^+ \to \mathcal{M}'_{d,n}(\mathbb{C})$.

In particular, we showed that $\Gamma(n)\backslash\mathfrak{h}_d^+$ is the set of complex points of an algebraic variety defined over the number field $\mathbb{Q}(\zeta_n)$. Unfortunately, this number field depends on the level n. The issue is that we need a fixed primitive nth root of 1 in order to define the moduli problem $\mathcal{M}'_{d,n}$, so we need to be over a basis where such a primitive nth root exists. To fix this problem, we will allow the primitive nth root of 1 to vary.

[9]See [?] and [?].

1.3.3. *The Siegel modular variety.*

Definition 1.14. Let n be a positive integer. The **Siegel modular variety** $\mathcal{M}_{d,n}$ is the functor from the category of $\mathbb{Z}/n\mathbb{Z}$-schemes to the category of sets sending a scheme S to the set of isomorphism classes of triples $(A, \lambda, \eta, \varphi)$, where (A, λ) is a principally polarized abelian scheme of relative dimension d over S and (η, φ) is a level n structure on (A, λ).

An isomorphism from (A, λ, η) to (A', λ', η') is an isomorphism of abelian varieties $u : A \xrightarrow{\sim} A'$ such that $\lambda' \circ u = u^\vee \circ \lambda$ and $\eta' = \eta \circ (u, u)$.

The group $\mathrm{GSp}_{2d}(\widehat{\mathbb{Z}})$ acts on $\mathcal{M}_{d,n}$: if $g \in \mathrm{GSp}_{2d}(\widehat{\mathbb{Z}})$ and $(A, \lambda, \eta, \varphi) \in \mathcal{M}_{d,n}(S)$, then

$$g \cdot (A, \lambda, \eta, \varphi) = (A, \lambda, g \circ \eta, c(g)^{-1}\varphi).$$

The kernel of this action is the group

$$K(n) = \mathrm{Ker}(\mathrm{GSp}_{2d}(\widehat{\mathbb{Z}}) \to \mathrm{GSp}_{2d}(\mathbb{Z}/n\mathbb{Z})).$$

If n divides m, then we have a morphism $\mathcal{M}_{d,m} \to \mathcal{M}_{d,n}$ that forgets part of the level m structure; this morphism is (representable) finite étale, and in fact it is a torsor under the finite group $K(n)/K(m)$.

We have the following variant of Theorem 1.11.

Theorem 1.15 (Mumford, cf. [FC90])**.** *Suppose that $n \geqslant 3$. Then the functor $\mathcal{M}_{d,n}$ is representable by a smooth quasi-projective $\mathcal{O}_n$-scheme purely of dimension $d(d+1)/2$, which we will denote by $\mathcal{M}_{d,n}$ and call the* ***Siegel modular variety*** *of level n.*

Remark 1.16. Let $K(n) = \mathrm{Ker}(\mathrm{GSp}_{2d}(\widehat{\mathbb{Z}}) \to \mathrm{GSp}_{2d}(\mathbb{Z}/n\mathbb{Z}))$. Then $\mathcal{M}_{d,n}$ is the Shimura variety for GSp_{2d} with level $K(n)$, or rather its integral model. If K is an open compact subgroup of $\mathrm{GSp}_{2d}(\mathbb{A}_f)$ that is small enough,[10] then we can also define the Shimura variety $\mathcal{M}_{d,K,\mathbb{Q}}$ with level K: choose n such that $K(n) \subset K$. Then $K(n)$ is a normal subgroup of K, so the group $K/K(n) \subset \mathrm{GSp}_{2d}(\mathbb{Z}/n\mathbb{Z})$ acts on $\mathcal{M}_{d,K,\mathbb{Q}}$, and we set $\mathcal{M}_{d,K,\mathbb{Q}} = \mathcal{M}_{d,n,\mathbb{Q}}/(K/K(n))$. It is easy to check that this does not depend on the choice of n.

In fact, for K an open compact subgroup of $\mathrm{GSp}_{2d}(\mathbb{A}_f)$, we have a direct definition of a level K structure on a principally polarized abelian scheme (see Section 5 of [**?**]). For K small enough, the scheme $\mathcal{M}_{d,K,\mathbb{Q}}$ is the moduli space of a principally polarized abelian schemes with level K structure. In general, this moduli space is representable by a Deligne-Mumford stack. We can also define this moduli schemes over a localization of $\mathbb{Z}$, but the primes that we invert depend on K; see the discussion in Subsubsection **??**.

Let us explain the relationship between $\mathcal{M}_{d,n}$ and $\mathcal{M}'_{d,n}$. We define a map

$$s : (\mathbb{Z}/n\mathbb{Z})^\times \longrightarrow \mathrm{GSp}_{2d}(\mathbb{Z}/n\mathbb{Z}), \quad s(\alpha) = \begin{pmatrix} 0 & \alpha I_d \\ I_d & 0 \end{pmatrix};$$

note that s is a section of the multiplier $c : \mathrm{GSp}_{2d}(\mathbb{Z}/n\mathbb{Z}) \to (\mathbb{Z}/n\mathbb{Z})^\times$, and that it is not a morphism of groups.

Proposition 1.17. *The morphism*

[10] For example, $K \subset K(n)$ with $n \geqslant 3$.

$$\begin{aligned} \mathcal{M}'_{d,n} \times (\mathbb{Z}/n\mathbb{Z})^\times &\longrightarrow \mathcal{M}_{d,n,\mathcal{O}_n} \\ ((A,\lambda,\eta),\alpha) &\longmapsto (A,\lambda,s(\alpha)\circ\eta,\varphi_0\circ\alpha), \end{aligned}$$

where we see α as an automorphism of $\underline{\mathbb{Z}/n\mathbb{Z}}_S$ for any scheme S, is an isomorphism.

As a corollary, we get a description of the complex points of $\mathcal{M}_{d,n}$. Let $\mathfrak{h}_d = \mathfrak{h}_d^+ \cup (-\mathfrak{h}_d^+)$ be the set of symmetric matrices $Y \in M_d(\mathbb{C})$ such that $\mathrm{Im}(Y)$ is positive definite or negative definite. The action of $\mathrm{Sp}_{2d}(\mathbb{R})$ on $\mathfrak{h}_d$ extends to a transitive action of $\mathrm{GSp}_{2d}(\mathbb{R})$, given by the same formula. The stabilizer of $iI_d \in \mathfrak{h}_d$ in $\mathrm{GSp}_{2d}(\mathbb{R})$ is $\mathbb{R}_{>0}K_\infty$, so $\mathfrak{h}_d \simeq \mathrm{GSp}_{2d}(\mathbb{R})/\mathbb{R}_{>0}K_\infty$ as real analytic manifolds.

Corollary 1.18. *We have an isomorphism of complex manifolds*

$$\mathcal{M}_{d,n}(\mathbb{C}) \simeq \mathrm{GSp}_{2d}(\mathbb{Q})\backslash(\mathfrak{h}_d \times \mathrm{GSp}_{2d}(\mathbb{A}_f)/K(n))$$

extending the isomorphism of Proposition 1.9, where $K(n) = \mathrm{Ker}(\mathrm{GSp}_{2d}(\widehat{\mathbb{Z}}) \to \mathrm{GSp}_{2d}(\mathbb{Z}/n\mathbb{Z}))$ and $\mathrm{GSp}_{2d}(\mathbb{Q})$ acts diagonally on $\mathfrak{h}_d \times \mathrm{GSp}_{2d}(\mathbb{A}_f)$.

This follows from the fact that

$$\mathrm{GSp}_{2d}(\mathbb{Q})\backslash(\mathfrak{h}_d \times \mathrm{GSp}_{2d}(\mathbb{A}_f)/K(n)) \simeq \mathrm{GSp}_{2d}(\mathbb{Q})^+\backslash(\mathfrak{h}_d^+ \times \mathrm{GSp}_{2d}(\mathbb{A}_f)/K(n)),$$

where $\mathrm{GSp}_{2d}(\mathbb{Q})^+ = \{g \in \mathrm{GSp}_{2d}(\mathbb{Q}) \mid c(g) > 0\}$, and from strong approximation for Sp_{2d},[11] which implies that c induces a bijection

$$\mathrm{GSp}_{2d}(\mathbb{Q})^+\backslash\mathrm{GSp}_{2d}(\mathbb{A}_f)/K(n) \xrightarrow{\sim} \mathbb{Q}_{>0}\backslash\mathbb{A}_f^\times/c(K(n)) \simeq \widehat{\mathbb{Z}}^\times(1+n\widehat{\mathbb{Z}}) \simeq (\mathbb{Z}/n\mathbb{Z})^\times.$$

For every $i \in (\mathbb{Z}/n\mathbb{Z})^\times$, we choose $x_i \in \mathrm{GSp}_{2d}(\mathbb{A}_f)$ lifting i and we set

$$\Gamma(n)_i = \mathrm{GSp}_{2d}(\mathbb{Q})^+ \cap x_iK(n)x_i^{-1} = \mathrm{Sp}_{2d}(\mathbb{Q}) \cap x_iK(n)x_i^{-1}.$$

Then the $\Gamma(n)_i$ are arithmetic subgroups of $\mathrm{Sp}_{2d}(\mathbb{Q})$, and we have

$$\mathrm{GSp}_{2d}(\mathbb{Q})\backslash(\mathfrak{h}_d \times \mathrm{GSp}_{2d}(\mathbb{A}_f)/K(n)) \simeq \coprod_{i\in(\mathbb{Z}/n\mathbb{Z})^\times} \Gamma(n)_i\backslash\mathfrak{h}_d^+$$

as complex manifolds.

In fact, for $i \in (\mathbb{Z}/n\mathbb{Z})^\times$, we can take $x_i = \begin{pmatrix} 0 & a_iI_d \\ I_d & 0 \end{pmatrix}$ with $a_i \in \widehat{\mathbb{Z}}^\times$ lifting i. In particular, we have $x_i \in \mathrm{GSp}_{2d}(\widehat{\mathbb{Z}})$; as $K(n)$ is a normal subgroup of $\mathrm{GSp}_{2d}(\widehat{\mathbb{Z}})$, we get $x_iK(n)x_i^{-1} = K(n)$, hence $\Gamma(n)_i = \Gamma(n)$, and finally

$$\mathrm{GSp}_{2d}(\mathbb{Q})\backslash(\mathfrak{h}_d \times \mathrm{GSp}_{2d}(\mathbb{A}_f)/K(n)) \simeq \coprod_{i\in(\mathbb{Z}/n\mathbb{Z})^\times} \Gamma(n)_i\backslash\mathfrak{h}_d^+.$$

Remark 1.19. If K is a small enough open compact subgroup of $\mathrm{GSp}_{2d}(\mathbb{A}_f)$, then we get an isomorphism of complex manifolds:

$$\mathcal{M}_{d,K}(\mathbb{C}) \simeq \mathrm{GSp}_{2d}(\mathbb{Q})\backslash(\mathfrak{h}_d \times \mathrm{GSp}_{2d}(\mathbb{A}_f)/K).$$

[11] See [?] and [?].

1.3.4. *Heche correspondence.* We can also descend the Hecke correspondences before to morphisms of schemes over $\mathbb{Z}[1/n]$.

We proceed as in Section 3 of [**?**]. Let $g \in \mathrm{GSp}_{2d}(\mathbb{A}_f)$, and let K, K' be small enough open compact subgroups of $\mathrm{GSp}_{2d}(\mathbb{A}_f)$ such that $K' \subset K \cap gKg^{-1}$. We want to define finite étale morphisms $T_1, T_g : \mathcal{M}_{d,K'} \to \mathcal{M}_{d,K}$, and the Hecke correspondence associated to (g, K, K') is the couple (T_1, T_g).

Choose $n \geqslant 3$ such that $K(n) \subset K'$; then $\mathcal{M}_{d,K'} = \mathcal{M}_{d,n}/(K'/K(n))$ and $\mathcal{M}_{d,K} = \mathcal{M}_{d,n}/(K/K(n))$. The morphism T_1 just forgets part of the level structure: as $K'/K(n) \to K/K(n)$, we have an obvious morphism $T_1 : \mathcal{M}_{d,K'} \to \mathcal{M}_{d,K}$.

2. Arithmetic Shimura varieties

References

[BB66] L. Baily and A. Borel. Compactification of arithmetic quotients of bounded symmetric domains. *Ann. of Math*, 84(2):442–528, 1966.

[BC83] Armand Borel and William Casselman. L^2-cohomology of locally symmetric manifolds of finite volume. *Duke Math. J.*, 50(3):625–647, 1983.

[FC90] Gerd Faltings and Ching-Li Chai. Degeneration of abelian varieties. *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*, 22, 1990. Springer-Verlag, Berlin. With an appendix by David Mumford.

[Ji06] Lizhen Ji. Lectures on locally symmetric spaces and arithmetic groups, lie groups and automorphic forms. *AMS/IP Stud. Adv. Math*, 37:87–146, 2006. Amer. Math. Soc., Providence, RI.

School of Mathematical Sciences, Peking University, 100871, Beijing, China

Email address: `daiwenhan@pku.edu.cn`