The Langlands program and the moduli of bundles on the curve (2/3)

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§1 Local Langlands as geometric Langlands

$G/\mathbb{Q}_p$ conn red. $\rightsquigarrow$ $\mathrm{Bun}_G / \mathrm{Perf}_{\overline{\mathbb{F}}_p}$

$$S \longmapsto \{G\text{-bundles on } X_S\}.$$

Dual side $\ell \neq p$ "very good"

($G = GL_n$ at all ℓ, G classical for $\ell \neq 2$)

$\hat{G}/\mathbb{Z}_\ell$ $\quad \overset{\text{Dat's talk}}{\rightsquigarrow}$ moduli space of L-parameters

$\circlearrowleft \Gamma = \Gamma_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ $\qquad$ $\underbrace{Z^1(W_{\mathbb{Q}_p}, \hat{G})}_{\text{condensed 1-cocycle.}} / \hat{G}\text{-conj}.$

Spec R

Thm $Z^1(W_{\mathbb{Q}_p}, \hat{G})$ disjoint union of affine schs of fin type $/\mathbb{Z}_\ell$, flat, local complete intersection, of rel dim $/\mathbb{Z}_{\ell'} = \dim G$.

any prime ℓ'.

Pretend actually affine.

Thm $\exists$ reasonable triangulated category $D(\mathrm{Bun}_G, \mathbb{Z}_\ell)$ s.t.

or stable ∞-cat $\qquad$ (ℓ-adic sheaves on Bun_G)

(1) $D(\mathrm{Bun}_G, \mathbb{Z}_\ell)$ has semi-orthogonal decomp with graded pieces

$$D(\mathrm{Bun}_G^b, \mathbb{Z}_\ell) \simeq D(\text{smooth } G_b(\mathbb{Q}_p)\text{-rep'n} / \mathbb{Z}_\ell\text{-mod})$$
$$= D(G_b(\mathbb{Q}_p), \mathbb{Z}_\ell) \quad \text{for } b \in B(G).$$

(2) $D(\mathrm{Bun}_G, \mathbb{Z}_\ell)$ compactly generated,

with $A \in D(\mathrm{Bun}_G, \mathbb{Z}_\ell)$ compact

$\Leftrightarrow$ $\mathrm{supp}\, A \subset |\mathrm{Bun}_G| = B(G)$ finite and

$A|_{\mathrm{Bun}_G^b} \in D(G_b(\mathbb{Q}_p), \mathbb{Z}_\ell)$ compact for all b.

(generated by $c\text{-}\mathrm{Ind}_K^{G_b(\mathbb{Q}_p)}(\mathbb{1})$, K pro-p.)

(3) $D(\mathrm{Bun}_G, \mathbb{Z}_\ell)$ self dual $\longleftrightarrow$ Beinstein-Zelevinsky duality

(4) ULA sheaves $\longleftrightarrow$ adm repr

universally locally acyclic.

<u>Main Conjecture</u> (refined ver of categorical LLC)

Assume G quasi-split. Fix Whittaker datum $G \supset B \supset U$

& $\psi : U(\mathbb{Q}_p) \to W(\overline{\mathbb{F}}_\ell)^\times$.

Then $\exists$ canonical equivalence

$\curvearrowright D(G_b(\mathbb{Q}_p))$, b basic.

$$(*) \qquad D(\mathrm{Bun}_G, W(\overline{\mathbb{F}}_\ell))^{\omega} \simeq D^b_{\mathrm{coh}, \boxed{\mathrm{Nilp}}}(Z^1(W_{\mathbb{Q}_p}, \hat{G})_{W(\overline{\mathbb{F}}_\ell)} / \hat{G}).$$

means compact objects

Can ignore if invert ℓ.

Whittaker sheaf $W_\psi \longleftrightarrow \mathcal{O}$

$\qquad = j_{1,!}[c\text{-}\mathrm{Ind}_{U(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \psi]$, $j_b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$.

<u>Main Theorem</u> $\exists$ canonical action (all G)

$$\mathrm{Perf}(Z^1(W_{\mathbb{Q}_p}, \hat{G})/\hat{G}) \curvearrowright D(\mathrm{Bun}_G, \mathbb{Z}_\ell)^{(\omega)}$$

perfect complexes. "spectral action".

Moreover, $(*)$ is $\mathrm{Perf}(Z^1(W_{\mathbb{Q}_p}, \hat{G})/\hat{G})$-equivariant.

<u>Corollary</u> $\exists$ canonical map

$$\begin{array}{ccc}
\mathrm{End}(\mathcal{O}_{Z^1(W_{\mathbb{Q}_p}, \hat{G})/\hat{G}}) & \longrightarrow & \mathrm{End}(\mathrm{id}_{D(\mathrm{Bun}_G)}) \\
\parallel & & \parallel \\
R^{\hat{G}} & & \mathcal{Z}(\mathrm{Bun}_G) \text{ Beinstein center}
\end{array}$$

$$R^{\hat{G}} \searrow \; Z(G(\mathbb{Q}_p)) \; \swarrow \mathcal{Z}(\mathrm{Bun}_G)$$

Corollary (Construction of semisimple L-parameters)

$k/\mathbb{Z}_\ell$ alg closed field (e.g. $k = \bar{\mathbb{F}}_\ell, \bar{\mathbb{Q}}_\ell$).

π irr sm k-repr of $G(\mathbb{Q}_p)$.

$\rightsquigarrow$ Semisimple L-parameter $\varphi_\pi : W_{\mathbb{Q}_p} \longrightarrow \hat{G}(k)$

1-cocycle up to $\hat{G}$-conj.

$$R^{\hat{G}} \longrightarrow \mathcal{Z}(G(\mathbb{Q}_p)) \longrightarrow \mathrm{End}_k(\pi) = k.$$

$\rightsquigarrow$ k-point of $\underbrace{\mathrm{Spec}\, R^{\hat{G}}}_{\text{Coarse moduli}} = Z^1(W_{\mathbb{Q}_p}, \hat{G}) /\!\!/ \hat{G}$.

Coarse quotient

i.e. semisimple L-parameter.

Q Does this agree with other LLC?

A Yes, for tori, for GL_n, for inner forms of GL_n.

(Hansen-Kaletha-Weinstein)

§2 Construction of spectral action

For simplicity, G split.

usual curve / $\mathbb{F}_q$

Raskin's talk $\{ F_I : \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G}^I) \longrightarrow D(\pi_1^{\mathrm{Weil}}(X)^I) \}$

functorial in a fin set I

exactly the data produced in Cong Xue's talk.

$\|$

$D_{qc}(\mathrm{Hom}(\pi_1^{\mathrm{Weil}}, \hat{G}) / \hat{G})$

$\in$

Drinf, $F_I(V) = R\Gamma(\underbrace{\mathrm{Drinf}}_{= (\omega_{Z^1(-)/\hat{G}})} \otimes V \circ \rho^{\mathrm{univ}})$.

Thm (Fargues-Scholze)

$\mathcal{C}$ $\mathbb{Z}_\ell$-linear idempotent-complete stable ∞-category

$(= \mathcal{D}(\mathrm{Bun}_G, \mathbb{Z}_\ell)^\omega.)$

Giving an action of $\mathrm{Perf}(\mathrm{Hom}(W_{\mathbb{Q}_p}, \hat{G})/\hat{G})$ on $\mathcal{C}$

is equiv to giving

$\forall$ finite sets I, exact, $\mathbb{Z}_\ell$-linear monoidal

$$\mathrm{Rep}^{f.p.}_{\mathbb{Z}_\ell}(\hat{G}^I) \longrightarrow \underbrace{\mathrm{End}(\mathcal{C})^{W^I_{\mathbb{Q}_p}}}_{W^I_{\mathbb{Q}_p}\text{-equiv objects}}$$

fin proj mod

$$V \longmapsto (T_V : \mathcal{C} \to \mathcal{C}^{W^I_{\mathbb{Q}_p}})$$

functorial in I $\qquad$ $\{x \in \mathcal{C} + \text{action of } W^I_{\mathbb{Q}_p} \to \mathrm{Aut}(x)\}$

<u>Construction</u> $T_V : D(\mathrm{Bun}_G) \longrightarrow D(\mathrm{Bun}_G \times (\mathrm{Div}^1)^I) \supseteq D(\mathrm{Bun}_G)^{W^I_{\mathbb{Q}_p}}$, $\quad \pi_1(\mathrm{Div}^1) = W_{\mathbb{Q}_p}$.

use Hecke operators

<u>Example</u> $|I| = 1$, $G = GL_2$, $\hat{G} = GL_2$, $V = \mathrm{Std}$ repr.

$$\begin{tikzcd}
 & H \arrow[dl] \arrow[dr] & & \\
* \arrow[dr, "\text{given by } \mathcal{O}^2"'] & & \mathrm{Hecke} \arrow[dl] \arrow[dr] & \\
 & \mathrm{Bun}_2 & & \mathrm{Bun}_2 \times \mathrm{Div}^1
\end{tikzcd}$$

$\mathrm{Hecke} = \{(\mathcal{E}, \mathcal{E}', x, f)\}$

where $\mathcal{E}, \mathcal{E}' \in \mathrm{Bun}_2$, $x \in X$, $f : \mathcal{E} \hookrightarrow \mathcal{E}'$

with coker f deg 1 skyscraper sheaf at x.

H: parametrizes untilts (up to Frob) $S^\sharp$

+ points of $\mathbb{P}^1_{S^\sharp}$.

$H \longrightarrow (\mathrm{Spa}\, \breve{\mathbb{Q}}_p)^\diamond/\phi = \mathrm{Div}^1$. fibres are $\mathbb{P}^1$'s.

Also, $\Omega^2 \subseteq (\mathbb{P}^1_C)^\diamond \supseteq \mathbb{P}^1(\mathbb{Q}_p)$

$$\begin{tikzcd}
 & \Omega^2 \arrow[dl] \arrow[d] \arrow[dr] & \mathbb{P}^1(\mathbb{Q}_p) \\
* & \mathrm{Bun}_2 \times \mathrm{Div}^1 & \mathrm{Hecke} \arrow[l]
\end{tikzcd}$$

$\mathcal{O}(1/2)$ $\qquad$ $\mathcal{O} \oplus \mathcal{O}(1)$

Rmk (About perverse sheaves)

moduli space of modifications of trivial G-bundle

$\overset{\shortparallel}{\mathrm{Gr}_G}$ "affine Grassmannian".

$\rightsquigarrow \mathrm{Gr}_G(\mathbb{C}_p)$

$\overset{\shortparallel}{}$

$G(B_{dR})/G(B_{dR}^+)$ with $B_{dR}^+ = \widehat{\mathcal{O}}_{X_{\mathbb{C}_p},\mathbb{C}_p}$ complete dvr.

$\Vert$ $\qquad B_{dR} = \mathrm{Frac}\, B_{dR}^+$.

$LG(\mathbb{Q}_p)/L^+G(\mathbb{Q}_p)$.

Thm $\mathrm{Perv}_{L^+G}(\mathrm{Gr}_G) \simeq \mathrm{Rep}\, \widehat{G}$.

$\circlearrowleft_{\Gamma} \qquad \circlearrowleft_{\Gamma}$.