# Locally symmetric spaces : p-adic aspects

Laurent Fargues

(Joint with Miaofen Chen, Xu Shen)

## Keynote question

(By Grothendieck, ICM 70, at Nice).

Let $\mathcal{G}$ p-div grp $/\overline{\mathbb{F}}_p$.

Two aspects of data:

- $(D, \varphi)$ = rat'l Dieudonné mod
  - $D$ = fin-dim'l $\breve{\mathbb{Q}}_p$-v.s. , $\breve{\mathbb{Q}}_p = \widehat{\mathbb{Q}_p^{ur}} \circlearrowleft \sigma$
  - $\varphi$ = $\sigma$-linear automorph.
- $\mathcal{G} = A[p^\infty]$ for A ab var $/\overline{\mathbb{F}}_p$.

**Note** Have $D = H^1_{cris}(A/W(\overline{\mathbb{F}}_p))[\frac{1}{p}]$ $\circlearrowleft$ crystalline Frob.

This is a phenomenon of map

$$\left\{\begin{array}{l} p\text{-div grps } /\mathcal{O}_K \\ \text{isog to } \mathcal{G} \text{ mod } p \end{array}\right\} \xrightarrow{\text{Hodge fil'n}} \left\{\begin{array}{l} \text{Fil'ns of } D_K \\ \text{of dim} = \dim \mathcal{G} \end{array}\right\}.$$

where $K/\breve{\mathbb{Q}}_p$ Complete varying ext'n.

Q What is the str of image of this map in $Gr_d(D)$?

Is it an alg var? $\qquad d = \dim \mathcal{G}$

<u>Partial answers</u>

(I) Drinfeld:

$$\Omega := \mathbb{P}^{n-1}_{\mathbb{Q}_p} \setminus \bigcup_{H \in \check{\mathbb{P}}^{n-1}(\mathbb{Q}_p)} H \subset \mathbb{P}^{n-1}$$

rigid analytic open subset

remove profin set of alg var.

This is such a period space for p-div grps

+ particular type additive str.

(II) Serre-Tate-Katz: Ordinary ell curves.

Consider $\mathcal{G} = \mu_{p^\infty} \oplus \mathbb{Q}_p/\mathbb{Z}_p$.

$\rightsquigarrow \quad \mathfrak{X} = \mathrm{Def}(\mathcal{G}) \simeq \mathrm{Spf}(W(\bar{\mathbb{F}}_p)[\![x]\!])$

$\Rightarrow \quad \mathfrak{X}_\eta \simeq \mathring{\mathbb{B}}^1_{\breve{\mathbb{Q}}_p}$

generic fibre $/\breve{\mathbb{Q}}_p$ of dim 1 (open unit ball)

$$\rightsquigarrow \quad \begin{array}{ccc} \mathfrak{X}_\eta & \longrightarrow & \mathbb{A}^1 \subseteq \mathbb{P}^1 \\ x & \longmapsto & \log(1+x). \end{array} \quad \text{period map}$$

(III) Lubin-Tate - Lafaille - Gross - Hopkins

$\mathcal{G}$ = 1-dim'l formal p-div grp of ht $n$ $/\bar{\mathbb{F}}_p$.

(For $n=2$, this $\triangleq$ ss ell curve).

Then $\quad \mathfrak{X} = \mathrm{Def}(\mathcal{G}) \simeq \mathrm{Spf}(W(\bar{\mathbb{F}}_p)[\![x_1, \dots, x_{n-1}]\!])$

$\rightsquigarrow \quad \mathfrak{X}_\eta \simeq \mathring{\mathbb{B}}^{n-1}_{\breve{\mathbb{Q}}_p} \underset{\text{surj}}{\overset{\pi}{\twoheadrightarrow}} \mathbb{P}^{n-1}$ = period space

= entire flag mfd.

& $\pi$ is étale period map.

(IV) Fontaine:

Characterize the $K$-pts of the period space inside $Gr$ when $[K:\breve{\mathbb{Q}}_p]<\infty$.

$\rightsquigarrow$ "weak admissibility condition".

(V) More generally: Rapoport-Zink.

Let $\mathcal{G}/\bar{\mathbb{F}}_p$ + PEL additional str.

$\rightsquigarrow$ Construct $\mathcal{M}/\operatorname{Spf}(W(\bar{\mathbb{F}}_p))$ formal sch.

locally like $\operatorname{Spf}(W(\bar{\mathbb{F}}_p)[[x_1,\dots,x_n]]\langle y_1,\dots,y_m\rangle/\text{Ideal})$

(deformation space by quasi-isog of $\mathcal{G}$).

Here $V(\text{Ideal}) \hookrightarrow \mathring{\mathbb{B}}^n \times \mathbb{B}^m$.

$\rightsquigarrow$ $\pi: \mathcal{M}_\eta \xrightarrow{\text{étale}} \mathcal{F}$ = flag var

period map

$\mathcal{F}^a := \operatorname{Im}(\pi) \subseteq \mathcal{F}$ open

<u>Fontaine's answer</u> $\mathcal{F}^a \subseteq \mathcal{F}^{wa} \subseteq \mathcal{F}$

weakly adm

$\mathcal{F}^{wa} = \mathcal{F} \setminus (\text{profin set of Schubert vars}) \underset{\text{open}}{\subseteq} \mathcal{F}$

s.t. $\mathcal{F}^a(K) = \mathcal{F}^{wa}(K)$ if $[K:\breve{\mathbb{Q}}_p]<\infty$.

<u>But</u> In general, $\mathcal{F}^a \neq \mathcal{F}^{wa}$, e.g. $\mathcal{F}^a(\mathbb{C}_p) \subsetneq \mathcal{F}^{wa}(\mathbb{C}_p)$.

For $X$ Berkovich space $/\mathbb{Q}_p$, $x \in X(\mathbb{C}_p)\setminus X(\bar{\mathbb{Q}}_p)$,

$X\setminus\{x\} \overset{\neq}{\hookrightarrow} X$, but same Tate "classical pts".

# Local Shimura varieties

Local Sh datum:

- $G/\mathbb{Q}_p$ red grp.
- $\mu : \mathbb{G}_{m,\bar{\mathbb{Q}}_p} \to G_{\bar{\mathbb{Q}}_p}$ minuscule cochar up to conjugation.
- $[b] \in B(G) = G(\breve{\mathbb{Q}}_p) / \sigma\text{-conj}$

  Kottwitz set $= \{G\text{-isocrystal}\}/\sim$

  role of "crystalline Frob".

  s.t. $[b] \in B(G,\mu) \subset B(G)$.

Can construct for level str $K$, local reflex field $\breve{E}$

$$\mathcal{M}(G,\mu,b)_K \;/\; \breve{E} \qquad \text{rigid analytic space}$$

$(K \subset G(\mathbb{Q}_p)$ compact open$)$

This is the local Shimura var

(c.f. Scholze's diamond, Kedlaya-Liu, FF curve).

$+ \quad \pi : \mathcal{M}(G,\mu,b)_K \xrightarrow{\text{étale}} \mathcal{F}$

twisted form of $G/P_\mu$.

Take $\mathcal{F}^a := \mathrm{Im}(\pi) \subset \mathcal{F}$ open,

$\mathcal{F}^a \subset \mathcal{F}^{wa} = \mathcal{F} \setminus$ profin set of Schubert vars.

Hartl: Classified all cases when $\mathcal{F}^a = \mathcal{F}^{wa}$ for $G = GL_n$.

<u>Thm</u> (Conjecture of Fargues & Rapoport)

For any $G$ (not necessarily quasi-split), Suppose $b$ is basic. Then

$$\mathcal{F}^a = \mathcal{F}^{wa} \iff B(G,\mu) \text{ is fully HN decomposable.}$$

<u>Explanation</u> $\cdot$ $b$ = generalization of $\underbrace{\text{isoclinic}}_{\text{one Dieudonné-Manin slope}}$

$\cdot$ $(G,\mu) \rightsquigarrow B(G,\mu) \subset B(G)$ finite subset

classifies the Newton strata of the reduction mod $p$ of the Shimura var.

$\cdot$ $[b] \in B(G,\mu)$ basic $\hat{=}$ "supersingular" closed stratum

$\cdot$ Fully HN decomposable = the Newton polygon of non-basic elts in $B(G,\mu)$ touches the Hodge polygon def'd by $\mu$ outside its extremities.

(polygon $\in$ Weyl chamber corresp. to $G$).

<u>Example</u> $G = SO(n)/\mathbb{Q}_p$ quasi-split

$+ \begin{pmatrix} & & 1 \\ & \cdots & \\ 1 & & \end{pmatrix} \rightsquigarrow$ Sh var assoc to $SO(2,n-2)$ of abelian type

$\mu(z) = \mathrm{diag}(z, \underbrace{1, \cdots, 1}_{n-2}, z^{-1})$.

Then $B(G,\mu)$ is fully HN decomposable

$\Rightarrow \mathcal{F}^a = \mathcal{F}^{wa} \subseteq \mathcal{F}$.

$\rightsquigarrow$ For $n = 21$, allows us to compute the $p$-adic period space of (polarized) K3 surfaces with supersingular reductions.

$\underline{Q}$ How to prove this thm?

$\underline{A}$ Use the FF curve.

$C/\breve{\mathbb{Q}}_p$ alg closed complete $\rightsquigarrow$ $C^\flat/\mathbb{F}_p$

$\rightsquigarrow$ $X_{C^\flat}$ the curve = Dedekind sch / $\mathbb{Q}_p$.

$+ \ \infty \in |X_{C^\flat}|$, $k(\infty) = C$.

<u>Construction</u> $\exists$ $\{\text{Isocrystals}\} \longrightarrow \{\text{Vect bdls} / X_{C^\flat}\}$

It can be enhanced to

$$\begin{array}{ccc} G(\breve{\mathbb{Q}}_p) & \longrightarrow & \{G\text{-bdls} / X_{C^\flat}\} \\ b & \longmapsto & \mathcal{E}_b. \end{array}$$

<u>Thm</u>

$$\begin{array}{ccc} B(G) & \xrightarrow{\sim} & H^1_{et}(X_C, G) \quad \text{isom} \\ [b] & \longmapsto & [\mathcal{E}_b] \end{array}$$

Moreover, $b$ basic $\Longleftrightarrow$ $\mathcal{E}_b$ semi-stable (à la Atiyah-Bott).

uses $\mu$ minuscule

Now for $x \in \mathcal{F}(G,\mu)(C)$ $\rightsquigarrow$ $\mathcal{E}_{b,x}$

$\mathcal{E}_{b,x}$ = modification at $\infty$ of $\mathcal{E}_b$ given by $x$.

Then get

$$\mathcal{F}^a(G,\mu)(C) = \{x \in \mathcal{F}(G,\mu)(C) \mid \mathcal{E}_{b,x} \text{ is semi-stable}\}.$$

<u>Pf sketch</u> Let $G$ quasi-split. Fix a Borel subgrp.

Let $\mathcal{E} / X := X_{C^\flat}$ $G$-bdl.

$\mathcal{E}$ semi-stable $\Longleftrightarrow$ $\forall$ $P$ std parabolic, $\mathcal{E}_P$ reduction of $\mathcal{E}$ to $P$,

$\forall \chi \in X^*(P/Z_G)^+$, $\deg(\chi_* \mathcal{E}_P) \leq 0$.

$x \in \mathcal{F}(G,\mu)(C)$ weakly admissible

$\iff \forall P,\ \forall b_P$ $\underbrace{\text{red'n of } b \text{ to } P}_{\text{sub-isocrystal}}$,

$\rightsquigarrow$ $\mathcal{E}_{b_P}$ red'n of $\mathcal{E}_b$ to $P$

$\rightsquigarrow$ $(\mathcal{E}_{b,x})_P$ red'n of $\mathcal{E}_{b,x}$ to $P$

( red'ns of parabolics transfer via modifications)

$\forall \chi \in X^*(P/Z_G)^+,\quad \deg(\chi_*(\mathcal{E}_{b,x})_P) \leq 0.$

Admissible : test on all red'ns of $\mathcal{E}_{b,x}$ to $P$'s.

Weakly adm : test on all red'ns coming from a red'n of $b$.