The p-adic special series and p-adic upper-half plane

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Motivation

Setups

- p prime. $\mathbb{Z}_p \subset \mathbb{Q}_p \subset C := \mathbb{C}_p$.
- $G = GL_2(\mathbb{Q}_p)$, $\mathcal{G}_{\mathbb{Q}_p} := \mathrm{Gal}_{\mathbb{Q}_p}$.
- $L / \mathbb{Q}_p$ finite (big enough)
- $B_{dR}^+ \supset B_{cris}^+$ p-adic period rings.
- $P_+ := \{ \lambda = (\lambda_1, \lambda_2) \in \mathbb{N}^2 \mid \lambda_2 > \lambda_1 \geq 0 \}$.

  HT wts, e.g. $\rho = (0,1)$.

  $w(\lambda) = \lambda_2 - \lambda_1$, $|\lambda| = \lambda_2 + \lambda_1$.

p-adic Langlands

$$\left\{ \begin{matrix} \text{2-dim'l } L\text{-reps} \\ \text{of } \mathcal{G}_{\mathbb{Q}_p} \end{matrix} \right\} \underset{\mathbb{V}}{\overset{\Pi}{\rightleftarrows}} \left\{ \begin{matrix} \text{adm unitary Banach} \\ L\text{-reps of } G \end{matrix} \right\}.$$

s.t. $\mathbb{V} \circ \Pi = \mathrm{id}$.

$$V \text{ special of HT wt } \lambda \in P_+ \;\hat{=}\; \Pi \text{ s.t. } \Pi^{alg} \supset \underbrace{\underbrace{\mathrm{Sym}_L^{w(\lambda)-1} \mathrm{std} \otimes \det^{\lambda_1} \otimes |\det|}_{W_\lambda^*} \otimes \widetilde{\mathrm{St}}_L^{\infty}}_{\mathrm{St}_\lambda^{lalg}}$$

Goal Realize functor Π in the p-adic étale cohom of some space.

Generalize this to $E/\mathbb{Q}_p$ finite.

Def For V special,

$$D^\lambda_{\mathcal{L}} := \begin{cases} \cdot\ D_{st}(V) = Sp_2(|\lambda|) = Le_0 \oplus Le_1, \text{ slope } -\frac{|\lambda|}{2}. \\ \quad \| \qquad \begin{cases} \varphi(e_0) = p^{-\frac{|\lambda|-1}{2}} e_0, & Ne_0 = e_1 \\ \varphi(e_1) = p^{-\frac{|\lambda|+1}{2}} e_1, & Ne_1 = 0 \end{cases} \\ \cdot\ D_{dR}(V) \supset Fil^i_{\mathcal{L}}, \quad \mathcal{L} \in L, \\ \qquad Fil^i_{\mathcal{L}} = L(e_1 - \mathcal{L}e_0) \text{ for } -\lambda_2 + 1 \leq i \leq -\lambda_1. \end{cases}$$

Also, $D^\lambda_\infty :=$ same φ-mod w/ $N = 0$.

$\rightsquigarrow$ $V = V^\lambda_{\mathcal{L}} := V_{st}(D^\lambda_{\mathcal{L}})$, irred of wt $w(\lambda) > 1$.

$\Pi^\lambda_{\mathcal{L}} := \Pi(V^\lambda_{\mathcal{L}})$ adm unitary Banach L-rep.

encoded by $\mathcal{L} \in L$.

Get a bij b/w $\mathcal{L} \in \mathbb{P}^1(L) \longleftrightarrow$ unitary adm completion of St^{lalg}_λ.

<u>p-adic half-plane</u> $\mathbb{H}_{\mathbb{Q}_p} := \mathbb{P}^{1,an}_{\mathbb{Q}_p} \setminus \mathbb{P}^1(\mathbb{Q}_p)$ Stein curve.

$\circlearrowleft$

$\mathcal{G}_{\mathbb{Q}_p} \times G$

Drinfeld's modular description (1974)

$\pi_1(\mathbb{H}_{\mathbb{Q}_p}) \twoheadrightarrow \mathcal{O}_D^\times \subset D / \mathbb{Q}_p$ non-split quat alg.

& $\mathcal{O}_D^\times \hookrightarrow GL_2(L)$

$\rightsquigarrow$ $\mathbb{V}_p$ (resp. $\mathbb{V}_p^+$) L- (resp. $\mathcal{O}_L$) loc sys, G-equiv.

$\mathbb{V}_p$ rk 2, HT wt $(0,1)$.

$\forall\ \lambda \in P_+$,

$$\mathbb{V}^{(+)}_\lambda := Sym_L^{w(\lambda)-1} \mathbb{V}_p^{(+)} \otimes (\wedge^2 \mathbb{V}_p^{(+)})^{\lambda_1} \quad G\text{-equiv.}$$

Let $* \in \{\text{ét}, \text{proét}\}$.

$$\underset{\mathcal{G}_{\mathbb{Q}_p} \times G.}{\overset{}{H^i_* \circlearrowleft}} = \bigoplus_{\lambda \in P_+} H^i_*[\lambda] = \bigoplus_{\lambda \in P_+} H^i_*(\mathbb{H}_C; \mathbb{V}_\lambda(1)).$$

If V L-rep of $\mathcal{G}_{\mathbb{Q}_p}$, $\Pi_*(V) := \mathrm{Hom}_{\mathcal{G}_{\mathbb{Q}_p}}(V, H^1_*)'$ dual

Thm (Vanhaecke '24)

- Let V abs irred L-rep of $\mathcal{G}_{\mathbb{Q}_p}$ of dim ≥ 2.

$$\Pi_{\text{ét}}(V) = \begin{cases} \Pi(V), & V \text{ special} \\ \quad = \Pi^\lambda_{\mathcal{L}} \text{ if } V = V^\lambda_{\mathcal{L}} & \\ 0, & \text{else.} \end{cases}$$

- Let Π adm unitary, abs irred, Banach L-rep.

$$\mathrm{Hom}_G(\Pi', H^1_{\text{ét}}) \simeq \begin{cases} \mathbb{V}(\Pi), & \Pi \text{ special} \\ \quad = V^\lambda_{\mathcal{L}} \text{ if } \Pi = \Pi^\lambda_{\mathcal{L}} & \\ 0, & \text{else.} \end{cases}$$

Proétale cohom of $\mathbb{V}_\lambda$

By Scholze, $\mathbb{V}_\lambda \rightsquigarrow (\mathcal{E}_\lambda, \nabla_\lambda, \mathrm{Fil}^\bullet_\lambda)$.

Prop $\mathbb{V}_\lambda$ is a $\underbrace{\text{Strongly}}_{②}$ $\underbrace{\text{isotrivial}}_{①}$ $\underbrace{L\text{-open}}_{③}$ of wt $(\lambda_1, \lambda_2 - 1)$.

① $\exists D_\lambda$ φ-mod s.t. $\mathcal{E}_\lambda \simeq D_\lambda \otimes \mathcal{O}$, $\varphi^2 = p^{1-|\lambda|}$.

② $\nabla_\lambda = \mathrm{id} \otimes d$

③ $R\Gamma_{dR}(\mathbb{H}_{\mathbb{Q}_p}, \mathcal{E}_\lambda) \simeq R\Gamma_{\text{op}}[\lambda] \simeq [\underbrace{\mathrm{Gr}_{1-\lambda_2}\mathcal{E}_\lambda(\mathbb{H}_{\mathbb{Q}_p}) \xrightarrow{\partial^{w(\lambda)}} \mathrm{Gr}_{\lambda_1}\Omega^1_{\mathcal{E}_\lambda}(\mathbb{H}_{\mathbb{Q}_p})}_{\text{line bdls.}}]$

<u>Syntomic cohom</u> $R\Gamma_{syn}[\lambda] := V_{st}(R\Gamma_{Op}[\lambda])$.

Comparison : $R\Gamma_{syn}[\lambda] \simeq R\Gamma_{pro\acute{e}t}[\lambda]$.

<u>Thm</u> $w = w(\lambda) > 1$. Get an exact seq

$$0 \longrightarrow \mathbb{Q}_\lambda^0 \otimes_L W_\lambda \to H^1_{pro\acute{e}t}[\lambda] \longrightarrow B_\lambda \hat{\otimes} (St_\lambda^{lan})'$$

$$\longrightarrow \mathbb{Q}_\lambda^1 \hat{\otimes}_L (St_\lambda^{lalg})' \longrightarrow 0.$$

- $B_\lambda := t^{\lambda_1} B_{dR}^+ / t^{\lambda_2}$,
- $U_\lambda^i := t^{\lambda_1}(B_{cris}^+)^{\varphi^2 = w+(-1)^i} \rightsquigarrow \mathbb{Q}_\lambda^i := \dfrac{B_\lambda \otimes L}{U_\lambda^i \otimes_{\mathbb{Q}_{p^2}} L}$

  v.s. $/\mathbb{Q}_{p^2}$
- $St_\lambda^{lan} := \mathcal{C}^{lan}(\mathbb{P}^1(\mathbb{Q}_p), W_\lambda^*)/W_\lambda^* \simeq Gr_{\lambda_1}\Omega^1_{\mathcal{E}_\lambda}(\mathbb{H}_{\mathbb{Q}_p})$.

$$W_\lambda^* = Sym_L^{w(\lambda)-1} \otimes \det^{\lambda_1} \otimes |\det|^i$$

<u>Computation of multiplicities</u>

<u>Proét $\Rightarrow$ ét</u>

<u>Prop</u> $H^1_{\acute{e}t} \hookrightarrow H^1_{pro\acute{e}t}$. $H^1_{\acute{e}t}$ = G-bdd vectors of $H^1_{pro\acute{e}t}$.

<u>Cor</u>
- $\widehat{\Pi_{pro\acute{e}t}(V)} = \Pi_{\acute{e}t}(V)$
- π fin length, $Hom_G(\pi', H^1_{\acute{e}t}) \simeq Hom_G((\pi^{lan})', H^1_{pro\acute{e}t})$.

<u>Prop</u>
- $w(\lambda) > 1 \Rightarrow H^i_{pro\acute{e}t}[\lambda] = 0$, $i \neq 1$.
- $(H^{1,+}_{\acute{e}t}[\lambda]/p^k)^{\mathcal{G}_K}$, $K/\mathbb{Q}_p$ finite.

  This is dual of a sm rep of fin length.

<u>Prop</u> (Schraen '11 + ε)

In the desired cat of mods / la L-rep w/ fixed central char,

$$\mathcal{H}om_G\big(\underbrace{({}^{\mathrm{lan}}\Pi_{\mathcal{L}}^{\lambda})'}_{P_{\mathcal{L}}^{\lambda}}[-1],\ R\Gamma_{\mathrm{op}}[\lambda]\big) \simeq D_{\mathcal{L}}^{\lambda}.$$

Note · $\mathrm{Hom}((\Pi_{\mathcal{L}}^{\lambda})', H^1_{\acute{e}t}) = \mathrm{Hom}(P_{\mathcal{L}}^{\lambda}, H^1_{\mathrm{pro\acute{e}t}}[\lambda])$

$\simeq \mathrm{Hom}_G(P_{\mathcal{L}}^{\lambda}, R\Gamma_{\mathrm{pro\acute{e}t}}[\lambda])$

$\simeq V_{\mathrm{st}}(\mathcal{H}om_G(P_{\mathcal{L}}^{\lambda}[-1], R\Gamma_{\mathrm{op}}[\lambda]))$

$\simeq V_{\mathcal{L}}^{\lambda}$.

· $\mathrm{Hom}(\pi', H^1_{\acute{e}t}) \simeq \mathrm{Hom}((\pi^{\mathrm{lan}})', H^1_{\mathrm{pro\acute{e}t}})$

$\hookrightarrow \mathrm{Hom}(\mathrm{St}_{\lambda}^{\mathrm{lan}}, \pi^{\mathrm{lan}}) =: H$.

Also, · $\pi^{\mathrm{lalg}} = 0$

$\Rightarrow H \simeq \mathrm{Hom}(B^{\lambda}, \pi^{\mathrm{lan}})$, $\widehat{B}^{\lambda} = \widehat{\mathrm{St}_{\lambda}^{\mathrm{lan}}/\mathrm{St}_{\lambda}^{\mathrm{lalg}}} = 0$.

· $\pi^{\mathrm{lalg}} \neq 0$

$\Rightarrow \mathrm{St}_{\lambda}^{\mathrm{lalg}} \hookrightarrow \pi = \Pi_{\mathcal{L}}^{\lambda}$.

Thm $0 \to W_{\lambda}^{\oplus 2} \to \Pi_{\mathrm{pro\acute{e}t}}(V_{\mathcal{L}}^{\lambda})' \to (\mathrm{St}_{\lambda}^{\mathrm{lan}})' \to 0$.

(works after applying $\mathrm{Hom}_{\mathcal{G}_{\mathbb{Q}_p}}(V_{\mathcal{L}}^{\lambda}, -)$.).

Have $\mathrm{Ext}^1((\mathrm{St}_{\lambda}^{\mathrm{lan}})', W_{\lambda}) \simeq \mathrm{Hom}(\mathbb{Q}_p^{\times}, L)$

$({}^{\mathrm{lan}}\Pi_{\mathcal{L}}^{\lambda})' \twoheadrightarrow (\Sigma_{\mathcal{L}}^{\lambda})' \longleftarrow v_p - \mathcal{L}\cdot\log$

Prop

$$\begin{array}{ccc} \Pi_{\mathrm{pro\acute{e}t}}(V_{\mathcal{L}}^{\lambda}) & \simeq & \Sigma_{\mathcal{L}}^{\lambda} \oplus W_{\lambda}^{*} \\ \downarrow & & \downarrow \\ \Pi_{\acute{e}t}(V_{\mathcal{L}}^{\lambda}) & \simeq & \Pi_{\mathcal{L}}^{\lambda} \end{array}$$

Note $\pi(V) = \mathrm{Hom}(V, H^1_{\acute{e}t})'$, $V \hookrightarrow \mathrm{Hom}(\pi(V)', H^1_{\acute{e}t})$.

The conjecture

Setup $F/\mathbb{Q}_p$ finite.

$\mathbb{H}_F := \mathbb{P}^{1,an}_F \setminus \mathbb{P}^1(F)$, W_λ^*, $St_\lambda^{F\text{-}lan}$, St_λ^{lalg} similarly.

$G = GL_2(F)$, $\mathcal{G}_F = Gal_F \rightsquigarrow \pi_*$.

$\tau : F \hookrightarrow \bar{\mathbb{Q}}_p$ fixed.

Can describe certain completions of St_λ^{lalg} in terms of $\mathcal{L} \in L$.

$$\mathrm{Ext}^1(W_\lambda^*, St_\lambda^{F\text{-}lan}) \simeq \mathrm{Hom}^{F\text{-}lan}(F^\times, L)$$

$$\Sigma_{\mathcal{L}}^\lambda \longleftarrow\!\!\shortmid\; v_F - \mathcal{L}\cdot\log \qquad \hat{=}\; V_{\mathcal{L}}^\lambda$$

with HT wt $\begin{cases} (0,1), & \tau' \neq \tau \\ \lambda, & \text{at } \tau. \end{cases}$

Thm $\mathcal{L} \neq \infty$, $w(\lambda) > 1$.

$\hat{\Sigma}_{\mathcal{L}}^\lambda$ is a nonzero Banach L-rep of G,

with a natural adm quotient $\pi_{\mathcal{L}}^\lambda$, s.t. ${}^{lalg}\pi_{\mathcal{L}}^\lambda \supset St_\lambda^{lalg}$.

Note

- $\pi_{proét}(V_{\mathcal{L}}^\lambda) \simeq \Sigma_{\mathcal{L}}^\lambda \oplus W_\lambda^* \Rightarrow \pi_{ét}(V_{\mathcal{L}}^\lambda) = \hat{\Sigma}_{\mathcal{L}}^\lambda$.
- $\Gamma \subset G$ Schottky subgrp s.t. $\mathbb{H}_F/\Gamma \simeq E_q$, $\log q = \mathcal{L}$.

$$H^1_{ét}(E_{q,\mathbb{C}}, W_\lambda(1)) \simeq V_{\mathcal{L}}^\lambda \otimes (\mathbb{1} \oplus \chi_2).$$
$$\text{IS}$$
$$H^1_{ét}[\lambda]^\Gamma.$$

- $L = \mathrm{Hom}_\Gamma(\mathbb{1}, \pi_{ét}(V_{\mathcal{L}}^\lambda)') = \mathrm{Hom}(\hat{\Sigma}_{\mathcal{L}}^\lambda, \underbrace{\mathcal{C}(G/\Gamma, L)}_{\text{adm}})$

$$\downarrow$$
$$\pi_{\mathcal{L}}^\lambda \qquad \text{(i.e. } \pi^{F\text{-}lan} \subset \pi \text{ dense)}$$

"**Conj**" Every adm unitary completion of St_λ^{lalg} of F-lan type w/ lalg vectors $= St_\lambda^{lalg}$ is of the form $\pi_{\mathcal{L}}^\lambda$.