Stacks of p-divisible groups with additional structure

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(Joint with Z. Gardner & A. Mathew).

Abelian varieties ⇝ moduli : mod curve, Siegel modular vars,
$\Big\}$ local avatar some Shimura varieties.
p-div groups ⇝ local model : "Rapoport-Zink spaces".

<u>Questions</u> (1) What does it mean for a p-divisible grp to have trivial determinant?

(2) $SL_n \longrightarrow GL_n \longrightarrow PGL_n$
What happens when "push out" a p-div grp along this map?

<u>Dieudonné theory</u>

Over a perfect field k of char p.

$$\left\{\begin{array}{l}\text{p-div groups}\\ \text{up to isogenies}\end{array}\right\} \xleftrightarrow{\text{Dieudonné-Manin}} \left\{\begin{array}{l}\text{F-isocrystals of}\\ \text{slopes in } [0,1]\cap\mathbb{Q}\end{array}\right\}$$

$$\in \; (M \in \mathrm{Vect}(W(k)[\tfrac{1}{p}]) \;+\; F : \sigma^*M \xrightarrow{\sim} M).$$
$$\circlearrowleft \sigma$$

This is essentially the same as:

$$\left\{\begin{array}{l}\text{p-div groups}\\ \text{up to isoms}\end{array}\right\} \longleftrightarrow \left\{ (M,F,V) \;\middle|\; \begin{array}{l} M \in \mathrm{Vect}(W(k)), \\ p = (M \xrightarrow{V} \sigma^*M \xrightarrow{F} M) \end{array} \right\}$$

equivalently, $pM \subseteq F(\sigma^*M) \subseteq M$.

F-gauge interpretation of RHS

Roughly: - $N \in \mathrm{Vect}(W(k))$,
$\quad \mathrm{Fil}^0 N \subseteq N$, $\mathrm{Fil}^0 N \in \mathrm{Vect}(W(k))$ s.t. $N/\mathrm{Fil}^0 N$ is a k-vect space.
- $\xi : \sigma^* \mathrm{Fil}^0 N \xrightarrow{\simeq} N$.

Take $\mathrm{Fil}_p^\bullet W(k) = W(k)$ with the p-adic fil'n
$\mathrm{Fil}^i_{p,\pm} W(k) = p^i W(k)$.

$$\rightsquigarrow \quad \begin{tikzcd} \mathrm{Fil}^\bullet \sigma : \mathrm{Fil}_p^\bullet W(k) \arrow[r] \arrow[dr, "\mathrm{Fil}^\bullet_\pm \sigma"'] & \mathrm{Fil}_p^\bullet W(k) \arrow[d] \\ & \mathrm{Fil}^\bullet_{p,\pm} W(k). \end{tikzcd}$$

Def'n An <u>F-Gauge</u> over k is a pair $(\mathrm{Fil}^\bullet N, \xi)$
where (i) $\mathrm{Fil}^\bullet N$ is a filtered mod $/ \mathrm{Fil}_p^\bullet W(k)$
(ii) $\xi : \mathrm{Fil}^0\left((\mathrm{Fil}^\bullet_\pm \sigma)^* \mathrm{Fil}^\bullet N \right) \xrightarrow{\sim} N$.

Rmk If $\mathrm{Fil}^\bullet N$ is a filtered vect bdl,
then $\mathrm{Fil}^0\left((\mathrm{Fil}^\bullet_\pm \sigma)^* \mathrm{Fil}^\bullet N \right) = \sum_{i \in \mathbb{Z}} p^{-i} \sigma^* \mathrm{Fil}^i N$.

If $(\mathrm{Fil}^\bullet N \subseteq N, \xi)$ is as before,
then can prolong $\mathrm{Fil}^0 N \subseteq \mathrm{Fil}^{-1} N = N$
to a p-adic fil'n $\mathrm{Fil}^\bullet N$ to get an F-Gauge $/ k$.
This prolongation can be done using

$$N = P \oplus Q, \quad P, Q \in \mathrm{Vect}(W(k)).$$
$$\mathrm{Fil}^0 N = P \oplus p \cdot Q \quad (\subseteq N)$$

We now set $\mathrm{Fil}^i N = p^i P \oplus p^{i+1} Q$, $i \geq 0$.

<u>p-divisible groups & F-Gauges in general</u>

<u>Thm</u> (Gardner - Madapusi - Mathew)

Let X formally of finite type $/ \operatorname{Spf} \mathbb{Z}_p$.

Then $\exists$ canonical equiv of cats

$$p\text{-div}_X \simeq \left\{ \begin{array}{l} \text{Vect bdl F-Gauges } / X \\ \text{of Hodge-Tate wts } \{0,1\} \end{array} \right\}$$

<u>Facts</u> (1) For $X = \operatorname{Spec} R$, $R / \mathbb{F}_p$ perfect alg,

this is due to Gabber & Lau.

(2) For X quasi-syntomic,

due to Anschütz - le Bras (2019)

& Mondal (2024).

<u>Rmk</u> These equivalences proceed by constructions of Dieudonné functor

$$p\text{-div}_X \longrightarrow \{ \text{F-Gauges over } X \}$$

The proof is <u>geometric</u>:

Will need to work with <u>n-truncated p-div grps</u>

also called BT_n 's.

<u>Defn</u> A BT_n is a fin flat p-power torsion comm grp sch G

s.t. (i) G is flat / $\mathbb{Z}/p^n\mathbb{Z}$ for $n \geq 2$.

(ii) If $n=1$, G lifts étale locally to a BT_2.

<u>Thm</u> (Grothendieck)

(1) BT_n is repr'ed by a smooth p-adic Artin stack of dim 0

(2) $BT_n \to BT_{n-1}$ are smooth for $n \geq 2$.

On the RHS, can work

$$\left\{\begin{matrix}\text{Vect bdl F-Gauges } / X \\ \text{of Hodge-Tate wts } \{0,1\}\end{matrix}\right\} \xrightarrow{\simeq} \varprojlim_n \left\{\begin{matrix}\text{Vect bdl F-Gauges } / X \\ \text{of HT wts } \{0,1\} \text{ and level } n\end{matrix}\right\}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad \| $$

$$\qquad\qquad\qquad\qquad\qquad\qquad \varprojlim_n \mathrm{Vect}^{syn}_{n,[0,1]}(X)$$

* What is $\mathrm{Vect}^{syn}_{n,[0,1]}(X)$?

<u>Example</u> (1) $X = \mathrm{Spec}\, K$, $n \geq 2$:

$\rightsquigarrow$ Elts in $\mathrm{Vect}^{syn}_{n,[0,1]}(X)$ are $(N_n, \mathrm{Fil}^0 N_n, \xi)$

Require N_n, $\mathrm{Fil}^0 N_n$ to be locally free / $W_n(K) := p^{-n}\cdot \mathrm{Fil}^n W(K)$

$\rightsquigarrow$ Have a map $\mathrm{Fil}^0 N_n \to N_n$ with coker killed by p.

(2) $X = \mathrm{Spf}\, R$,

$$\left\{\begin{matrix}\text{Vect bdl F-Gauges } / X \\ \text{of level } n\end{matrix}\right\} = \mathrm{Vect}\,(\underbrace{R^{syn} \otimes \mathbb{Z}/p^n\mathbb{Z}}_{})$$

syntomification of R

(everything is derived in general).

<u>Thm</u> (1) $\mathrm{Vect}^{syn}_{n,[0,1]}$ is rep'd by a smooth p-adic Artin stack / $\mathrm{Spf}\,\mathbb{Z}_p$

(2) $\mathrm{Vect}^{syn}_{n,[0,1]} \longrightarrow \mathrm{Vect}^{syn}_{n-1,[0,1]}$ is smooth for $n \geq 2$.

Take $X = \mathrm{Spf}\, R$ and let $\mathcal{F} \in \mathrm{Vect}^{syn}_{n,[0,1]}(\mathrm{Spf}\, R)$.
Consider at level n that

$$\Gamma_{syn}(\mathcal{F}) : \mathrm{CAlg}^{p\text{-nilp}}_{R/} \longrightarrow \mathcal{D}^{\leq 0}(\mathbb{Z})$$
$$C \longmapsto \tau^{\leq 0} R\Gamma(C^{syn} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{F}|_{C^{syn} \otimes \mathbb{Z}/p^n\mathbb{Z}}).$$

(p-nilp comm alg below R

Then $\Gamma_{syn}(\mathcal{F})$ is rep'd by a BT_n.

<u>Thm</u> Have an isom of sm p-adic Artin stacks

$$\mathrm{Vect}^{syn}_{n,[0,1]} \longrightarrow BT_n$$
$$\mathcal{F} \longmapsto \Gamma_{syn}(\mathcal{F}).$$

<u>Idea</u> Suffices to show

$$\mathrm{Vect}^{syn}_{n,[0,1]}(X) \xrightarrow{\cong} BT_n(X) \qquad \text{is an equiv of groupoids}$$

when $X = \mathrm{Spec}\, R$, $R = $ sm $\mathbb{F}_p$-alg.

Here, can get an inverse

$$BT_n(X) \longrightarrow \mathrm{Vect}^{syn}_{n,[0,1]}(X)$$

using
- crystalline Dieudonné theory
- prismatic cohom of classifying stacks.

<u>Pf of isom</u> <u>Step 1</u> (Drinfeld, for smooth inputs)

$$\mathrm{Vect}^{syn}_{n,[0,1]} \otimes \mathbb{F}_p$$
$$\downarrow \quad \text{gerbe bounded by a fin flat étale grp sch}$$
alg stack — $\mathrm{FZip}_{[0,1]} := \{\text{F-Zips of HT wts } 0,1\}$ (Pink-Wedhorn-Ziegler)
$\quad\quad\quad\quad // $
$\quad\quad\quad \mathrm{Disp}_1$ (by Zink)

Step 2

$$\begin{array}{c} \mathrm{Vect}^{\mathrm{Syn}}_{n,[0,1]} \otimes \mathbb{F}_p \\ \downarrow \\ \mathrm{Vect}^{\mathrm{Syn}}_{n-1,[0,1]} \otimes \mathbb{F}_p \\ \Downarrow \\ \mathcal{F}_{n-1} \text{ as an } R\text{-pt} \end{array}$$

is a torsor for the syntomic cohom of $\mathrm{End}(\mathcal{F}_{n-1}) \otimes \mathbb{Z}/p\mathbb{Z}$.

Step 3 (Grothendieck-Messing theory)

Let $R' \twoheadrightarrow R$ (pro-) nilpotent divided power thickening

e.g. $\mathbb{Z}_p \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$ if $p>2$

$\mathbb{Z}_2 \twoheadrightarrow \mathbb{Z}/4\mathbb{Z}$ if $p=2$.

$\rightsquigarrow$ For $P^- \subseteq GL_n$ parabolic subgrp corresp to a 1-step fil'n,

$$\begin{tikzcd} \mathrm{Vect}^{\mathrm{Syn}}_{n,[0,1]}(R') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & BP^-(R'/^{\mathbb{L}}p^n) \arrow[d] \\ \mathrm{Vect}^{\mathrm{Syn}}_{n,[0,1]}(R) \arrow[r] & BGL_h(R'/^{\mathbb{L}}p^n) \times_{BGL_h(R'/^{\mathbb{L}}p)} BP^-(R'/^{\mathbb{L}}p) \end{tikzcd}$$

$h = \text{rank}$

Obs

$$R \longmapsto BGL_h(R/^{\mathbb{L}}p^n)$$
$$R \longmapsto BP^-(R/^{\mathbb{L}}p^n)$$

are smooth p-adic Artin stacks.

Step 4 Apply this Cartesian diagram to

- $R \twoheadrightarrow R/^{\mathbb{L}}p$ (if $p>2$)
- $R \twoheadrightarrow R/^{\mathbb{L}}4 \longrightarrow R/^{\mathbb{L}}2$ (if $p=2$). $\square$