COMPLEX ANALYSIS

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Abstract. These notes are intended to be written for the PhD Entrance and Qualifying Examination at the Beijing International Center of Mathematical Research in 2022. The first previous chapters do follow [SS10] loosely so that I claim no originality. Another standard reference for these notes is [Lan03].

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Date: May 3, 2022.

1. Overview

The so-called "complex analysis" is the theory of complex numbers $\mathbb{C}$. Many modern mathematical subjects are based on the language of complex analysis. The fundamental notion here is called *holomorphicity*, which is regarded as analogous to the differentiability over $\mathbb{R}$. The holomorphic functions with a single variable strongly relate to Riemann surfaces.

The global version of complex analysis is applied in geometry and topology, i.e., the research on Riemann surfaces, particularly complex algebraic curves of dimension 1. More generally, the complex geometry and even algebraic geometry over $\mathbb{C}$ take care of those geometric objects of higher dimensions by considering holomorphic functions with several variables. The most basic tool we use in geometry is called *multi-variable complex analysis*.

Riemann zeta functions, as well as L-functions, are key objects in analytic number theory, whose properties are probed by complex analysis as well. As for (homogeneous) dynamic systems, analysts are interested in *Teichmuller spaces* as an advanced topic in modern complex analysis.

Outline

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2. Preliminaries

2.1. **Complex Numbers and Complex Plane.** The complex field $\mathbb{C} : \{z = x + iy \mid x, y \in \mathbb{R}\}$ with $i^2 = -1$ is canonically isomorphic to $\mathbb{R}^2$ as $\mathbb{R}$-vector spaces, where the isomorphism sends $x + iy$ to (x, y). The real part and the imaginary part of $z \in \mathbb{C}$ is defined by

$$\Re(z) := x, \quad \Im(z) := y.$$

Given this, the geometry of $\mathbb{C}$ is called the *complex plane*.

2.1.1. *Algebraic Properties of* $\mathbb{C}$. Say $\mathbb{C}$ can be endowed with two operations $+, \cdot$ via the following way. For any $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}$,

$$\begin{aligned} + : \mathbb{C} \times \mathbb{C} &\longrightarrow \mathbb{C} \\ (z_1, z_2) &\longmapsto z_1 + z_2, \end{aligned}$$

and

$$\begin{aligned} \cdot : \mathbb{C} \times \mathbb{C} &\longrightarrow \mathbb{C} \\ (z_1, z_2) &\longmapsto z_1 \cdot z_2, \end{aligned}$$

where $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$ and $z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + y_1y_2 + i(x_1y_2 + x_2y_1)$. These can be viewed as actions of $\mathbb{C}$ on $\mathbb{C}$ itself, and then $+$ is induced by $(\mathbb{R}^2, +)$ directly and $\cdot$ is induced by $\mathrm{GL}_2(\mathbb{R})$ (recall that $\mathbb{C} \cong \mathbb{R}^2$ canonically). It is easy to verify the following properties

- (Commutativity) $z_1 + z_2 = z_2 + z_1$, $z_1 \cdot z_2 = z_2 \cdot z_1$.
- (Associativity) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$.
- (Distributivity) $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$.
- (Additive & Multiplicative Identity) $z + 0 = z$, $z \cdot 1 = z$.
- (Additive Inverse) $z + (-z) = 0$.
- (Multiplicative Inverse) For all $z \in \mathbb{C}\backslash\{0\}$, there is $w \in \mathbb{C}$ such that $z \cdot w = 1$.

It turns out that $(\mathbb{C}, +, \cdot)$ is a field and is morally algebraically closed.

2.1.2. *Geometric Properties of* $\mathbb{C}$. Induced from the inner product on $\mathbb{R}^2$, the *absolute value* on $\mathbb{C}$ is defined by

$$|\cdot| : \mathbb{C} \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}$$

that sends $z = x+iy \in \mathbb{C}$ to $|z| = \sqrt{x^2+y^2} \in \mathbb{R}$. In fact, it satisfies the following norm properties:

- (Triangle Inequality) For all $z, w \in C$, $|z+w| \leqslant |z| + |w|$.
- (Homogeneity) For any $a \in C$ (as a scalar) and $z \in C$ (as a vector), $|a \cdot z| = |a||z|$.
- (Positivity) For all $z \in \mathbb{C}$, we have $|z| \geqslant 0$ with the equality holds if and only if $z = 0$.

This shows that the absolute value we have defined is a *norm* on $\mathbb{C}$, and then $(\mathbb{C}, |\cdot|)$ is a normed space.

Definition 2.1. Let $\{z_n\}_{n=1}^{\infty} = \{z_1, z_2, \ldots\}$ be a sequence in $\mathbb{C}$, we call $\{z_n\}_{n=1}^{\infty}$ is *convergent* if there exists $w \in C$ such that

$$\lim_{n\to\infty} |z_n - w| = 0.$$

This is denoted by $z_n \to w$.

Definition 2.2 (Cauchy Sequence)**.** A sequence $\{z_n\}_{n=1}^{\infty}$ is called *Cauchy* if for all $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that whenever $m, n \geqslant N$, $|z_m - z_n| < \varepsilon$ is valid.

Theorem 2.3. $(\mathbb{C}, |\cdot|)$ *is complete, i.e., any Cauchy sequence* $\{z_n\}_{n=1}^{\infty}$ *is convergent in* $\mathbb{C}$. *Hence* $(\mathbb{C}, |\cdot|)$ *is a Banach space, i.e., a complete normed space.*

Proof. Let $z_n = x_n + iy_n$ with $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ being two real sequences. Since $\{z_n\}$ is Cauchy, the completeness of $(\mathbb{R}, |\cdot|)$ shows that $\{x_n\}$ and $\{y_n\}$ are convergent in $\mathbb{R}$. Hence $\{z_n\}$ is convergent. The ingredient is that finite copies of Banach spaces is still Banach. ☐

Let's point out that the multiplication of complex numbers has a geometric interpretation. For any $z \neq 0$, it can be rewritten as the following polar coordinates:

$$z = re^{i\theta} = (r\cos\theta + ir\sin\theta),$$

where θ is called the *argument* of z, denoted by $\arg(z)$. Note that $\arg(z)$ is not unique for given z, but is unique modulo $2\pi\mathbb{Z}$. In the sense of $\mathrm{GL}_2(\mathbb{R})$, if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_1 e^{i\theta_1}$, then $z_1 z_2$ is nothing but the image of z_2 under the multiplication homomorphism by z_1, and mult_{z_1} is represented by

$$\begin{pmatrix} r_1 & \\ & r_1 \end{pmatrix} \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}).$$

2.1.3. *Topological Properties of* $\mathbb{C}$. Loosely speaking, the topological information on $\mathbb{C}$ is totally induced by that on $\mathbb{R}^2$. We begin with some notations. Given $z_0 \in \mathbb{C}$ and $r > 0$, one can define:

(1) the open disc of radius r centred at z_0:
$$D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\};$$
also, the unit disc is denoted by $\mathbb{D} = D_1(0)$;

(2) the closed disc of radius r centred at z_0:
$$\overline{D_r(z_0)} = \{z \in \mathbb{C} \mid |z - z_0| \leqslant r\};$$

(3) the circle of radius r centred at z_0:
$$C_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| = r\} = \partial D_r(z_0) = \partial\overline{D_r(z_0)}.$$

Jargon Watch: suppose a subset $\Omega \subset \mathbb{C}$ is given.

(1) A point $z \in \Omega$ is called an *interior point* if there is some $r > 0$ such that $D_r(z) \subset \Omega$. Denote the interior of Ω by
$$\mathrm{Int}(\Omega) = \{\text{interior points of } \Omega\}.$$
(2) The subset Ω is called *open* if $\Omega = \mathrm{Int}(\Omega)$. For example, $D_r(z_0)$ is open whereas $\overline{D_r(z_0)}$ is not.
(3) A point z is a *limit point* of Ω if there exists a sequence $\{z_n\}_{n=1}^{\infty}$ in Ω such that $z \notin \{z_n\}$ but $z_n \to z$ as $n \to \infty$. The *closure* of Ω is
$$\overline{\Omega} := \Omega \cup \{\text{limit points of } \Omega\}.$$
(4) The subset Ω is *closed* if $\Omega = \overline{\Omega}$. For example, $\overline{D_r(z_0)}$ is closed whereas $D_r(z_0)$ is not, and $\mathbb{C} = D_\infty(0)$ is open and closed.
(5) The *boundary* of Ω is defined as
$$\partial\Omega = \overline{\Omega} - \mathrm{Int}(\Omega).$$
For example, $\partial D_r(z_0) = \partial\overline{D_r(z_0)} = C_r(z_0)$.
(6) The subset Ω is *bounded* if there is a sufficiently large $r \gg 0$ such that $\Omega \subset D_r(z_0)$.

Exercise 2.4. Show that Ω is closed if and only if its complement $\Omega^{\mathrm{c}} = \mathbb{C} - \Omega$ is open.

In the upcoming context, we will discuss the notion of compactness, which is the most important topological property of the complex plane in the analysis theory.

Definition 2.5 (Compactness)**.** An *open covering* of Ω is a family of open sets $\{V_\alpha\}_{\alpha\in I}$ such that $\Omega \subset \bigcup_{\alpha\in I} V_\alpha$. A subset $\Omega \subset \mathbb{C}$ is called *compact* if every open covering of Ω has a finite subcovering.

Theorem 2.6. *Any subset in the vector space $\mathbb{R}^n$ with $n < \infty$ is compact if and only if it is closed and bounded. In particular, $\Omega \subset \mathbb{C} \cong \mathbb{R}^2$ is compact if and only if Ω is closed and bounded. Even equivalently, say every sequence $\{z_n\}$ in Ω has a convergent subsequence.*

Let's consider the decreasing chain property of non-empty compact subsets. Define the *diameter* of Ω as
$$\operatorname{diam}\Omega = \sup_{z,w\in\Omega} |z - w|.$$

Proposition 2.7. *Let $\Omega_1 \supset \Omega_2 \supset \cdots \Omega_n \supset \cdots$ be a sequence of non-empty compact subsets of $\mathbb{C}$ satisfying $\operatorname{diam}\Omega_n \to 0$ as $n \to \infty$. Then there is a unique $w \in \mathbb{C}$ such that $w \in \Omega_n$ for any n, or equivalently,*
$$\bigcap_{n\geqslant 1} \Omega_n = \{w\}.$$

Proof. Since $\Omega_n \neq \emptyset$, we can take $z_n \in \Omega_n$ to form a sequence $\{z_n\}_{n=1}^{\infty}$. Because of $\operatorname{diam}\Omega_n \to 0$, we see $\{z_n\}$ is a Cauchy sequence. Thus $z_n \to w$ for some $w \in \mathbb{C}$ by the completeness of $(\mathbb{C}, |\cdot|)$. Again by the definition of compactness, $w \in \Omega_n$ for any $n \geqslant 1$ since Ω_n is compact. $\square$

Definition 2.8 (Region)**.** The subset $\Omega \subset \mathbb{C}$ is called *connected* if Ω cannot be the union of two disjoint non-empty open sets. A connected open set is called a *region*.

Example 2.9. $\mathbb{C}$, $D_r(z_0)$, and $\overline{D_r(z_0)}$ are all connected as regions.

In summary, do remember the following:
(1) $(\mathbb{C}, +, \cdot)$ is an algebraically closed field.
(2) $(\mathbb{C}, |\cdot|)$ is a complete normed space, namely a Banach space.
(3) The topology on $\mathbb{C}$ is induced by that on $\mathbb{R}^2$.

2.2. **Holomorphic Functions.** Let $\Omega \subset \mathbb{C}$ be an open set and let $f : \Omega \to \mathbb{C}$ be a complex valued function.

Definition 2.10. Let $z_0 \in \Omega$, f is called *holomorphic* at z_0 if the following limit exists:

$$f'(z_0) := \lim_{h\to 0, h\in\mathbb{C}} \frac{f(z_0+h)-f(z_0)}{h}.$$

Note that z_0 can be attained from any direction by h here.

2.2.1. *The Ring of Holomorphic Functions.* In fact, f is holomorphic at z_0 if and only if

$$f(z_0+h) - f(z_0) = ah + h\varphi(h),$$

where $a \in \mathbb{C}$ and $\varphi(h) \to 0$ as $h \to 0$. The notation is

$$\mathscr{O}(\Omega) = \{\text{holomorphic functions on } \Omega\}.$$

Then $\mathscr{O}(\Omega)$ is non-empty, for example, all constant function and $f(z) = z$ are holomorphic. In the latter case, just note that $f'(z_0) = 1$ for all $z_0 \in \mathbb{C}$.

Proposition 2.11. *$\mathscr{O}(\Omega)$ has a structure of ring. In particular, if $f, g \in \mathscr{O}(\Omega)$, then*

- *$f + g \in \mathscr{O}(\Omega)$ and $(f+g)' = f' + g'$,*
- *$f \cdot g \in \mathscr{O}(\Omega)$ and $(f \cdot g)' = f'g + fg'$, and*
- *if $g(z_0) \neq 0$, then f/g is holomorphic at z_0, and $(f/g)' = (f'g - fg')/g^2$.*

Moreover, the $\mathscr{O}(\Omega)$ admits the chain rule, i.e., for any holomorphic f and g, we have

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z).$$

Remark 2.12. As rings of functions,

$$\{\text{polynomials in } z\} \subset \{\text{convergent series in } z\} \subset \mathscr{O}(\Omega).$$

Note that any $f : \Omega \to \mathbb{C}$ can always be factored through an embedding $\Omega \subset \mathbb{C}$ and then can be translated to another map

$$F : \Omega \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

such that $f(x+iy) = u(x,y) + iv(x,y)$. The keynote question is that *is there any property of F corresponding to holomorphicity of f?*

Exercise 2.13. Prove that f is holomorphic on Ω if and only if F is differentiable on Ω.

2.2.2. *Cauchy-Riemann Equation.* Let's suppose $f : \Omega \to \mathbb{C}$ is holomorphic, hence

$$f'(z) = \lim_{h\in\mathbb{C}, h\to 0} \frac{f(z+h)-f(z)}{h}$$

exists by definition. Consider taking different values of $h \to 0$ and say $h = h_1 + ih_2$.

(i) If $h = h_1$, then

$$f'(z) = f'(x,y) = \lim_{h_1\to 0} \frac{f(x+h_1,y)-f(x,y)}{h_1} = \frac{\partial f}{\partial x}(z).$$

(ii) If $h = ih_2$, then

$$f'(z) = f'(x,y) = \lim_{h_2\to 0} \frac{f(x,y+h_2)-f(x,y)}{ih_2} = -i\frac{\partial f}{\partial y}(z).$$

By the uniqueness of $f'(z)$ for fixed z, we obtain

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

By writing $f(z) = u(x,y) + iv(x,y)$, this equation is equivalent to

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

which is called the **Cauchy-Riemann equations**.

The claim is that by introducing the notations

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}), \quad \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}),$$

holomorphicity of f implies that

$$\frac{\partial f}{\partial \overline{z}} = 0, \quad \frac{\partial f}{\partial z} = f'(z).$$

Here the former equality is nothing but the Cauchy-Riemann equation.

On the other hand, it turns out that the Cauchy-Riemann equation implies holomorphicity as well.

Theorem 2.14. *Suppose $f = u + iv$ with u and v being differentiable. If f satisfies the Cauchy-Riemann equation, then f is holomorphic.*

Proof. Since u and v are differentiable, we get

$$u(x+h_1, y+h_2) - u(x,y) = \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + |h|\varphi_1(h),$$
$$v(x+h_1, y+h_2) - v(x,y) = \frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + |h|\varphi_2(h).$$

Using Cauchy-Riemann, it follows that

$$f(z+h) - f(z) = (\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y})h + |h|(\varphi_1(h) + i\varphi_2(h)).$$

Hence $f'(z)$ exists, and moreover

$$f'(z) = 2\frac{\partial u}{\partial z} = \frac{\partial f}{\partial z}.$$

This completes the proof. □

Remark 2.15. Some interpretation on derivatives towards z and $\overline{z}$.

(1) For $f : \Omega \subset \mathbb{C} \to \mathbb{C}$, we obtain

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Denote $dz = d(x+iy) = dx + idy$, and $d\overline{z} = d(x-iy) = dx - idy$, and then

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}$$

which is equivalent to the above equality. From this, we see given $f : \Omega \to \mathbb{C}$ differentiable, then f is holomorphic if and only if $df = (\partial f/\partial z)dz$.

(2) For $f : \Omega \to \mathbb{C}$, there is a bijection between (x,y) and $(z,\overline{z})$. Note that the chain rule with respect to $(z,\overline{z})$ yields to the second relation in (1).

(3) In the sense of probability over $\mathbb{R}^2$, one can regard $f : \Omega \to \mathbb{C}$ as a distribution function. Then $\partial f/\partial \overline{z} = 0$ leads to holomorphicity of f via the regularity of $\partial/\partial\overline{z}$ (as a functor). In particular, f is differentiable under this condition.

2.3. **Power Series.** In this section we use $\mathbb{C}$ as $(\mathbb{C}, |\cdot|)$. A *power series* is an expansion of the form $\sum_{n=0}^{\infty} a_n z^n$ for $a_n \in \mathbb{C}$. One can define the convergent (resp. divergent) series easily, and then the definition of *absolute convergent series* follows: say $\sum_{n=0}^{\infty} |a_n||z|^n$ converges as a real series.

2.3.1. *Radius of Convergence.*

Theorem 2.16. *Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists $0 \leqslant R \leqslant \infty$ such that*

(1) *if $|z| < R$, the series is absolutely convergent. The disc of convergence is given by*
$$\{z \in \mathbb{C} \mid |z| < R\};$$
(2) *if $|z| > R$, the series is divergent.*

Moreover, R has a explicit expression read as
$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}.$$

Proof. It's the same as the real case. The idea is to compare with the geometric series. □

Examples 2.17. Some calculation on radius of convergence.

(1) The power series $\sum_{n=0}^{\infty} z^n$ has a partial sum $\sum_{n=0}^{N} z^n = (z^{N+1}-1)/(z-1)$. When $N \to \infty$, the convergence condition is given by $|z| < 1$. On the other hand, Theorem 2.16 leads to $R = 1$ since $a_n = 1$ for all $n \geqslant 0$.
(2) Consider the exponential function
$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$
By Theorem 2.16, the radius is given by
$$\limsup_{n\to\infty} (\frac{1}{n!})^{1/n} = 0,$$
hence $R = \infty$, i.e., e^z convergent for every $z \in \mathbb{C}$. In this case we say e^z is *well-defined* on $\mathbb{C}$.
(3) Consider the trigonometric functions
$$\cos z := \frac{e^{iz} + e^{-iz}}{2} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!},$$
$$\sin z := \frac{e^{iz} - e^{-iz}}{2i} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$
Note that their sum is given by
$$e^{iz} = \cos z + i \sin z,$$
which is the same as the Euler formula on complex rotations.

Theorem 2.18. *The series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is always holomorphic in the disc of convergence. Moreover, $f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ in the disc of convergence, and $f'(z)$ has the same radius of convergence as that of $f(z)$.*

Proof. Note that $n^{1/n} \to 1$ as $n \to \infty$. Consequently,

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} |na_n|^{1/n},$$

which gives the same radius. □

2.3.2. *Complex Derivative of Power Series.* Let R be the radius of convergence of f. Taking $z_0 \in D_R(0) = \{z \in \mathbb{C} \mid |z| < R\}$, we aim to compute $f'(z_0)$. Let's first write

$$f(z) = S_N(z) + E_N(z) = \sum_{k\leqslant N} a_k z^k + \sum_{k>N} a_k z^k.$$

Assume there is some r such that $|z_0| < r < R$. By taking sufficiently small $h \in \mathbb{C}$ such that $|z_0 + h| < r$, one can rewrite the derivative as

$$\frac{f(z_0+h) - f(z_0)}{h} = \underbrace{\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0)}_{\mathrm{I}} + \underbrace{S_N'(z_0)}_{\mathrm{II}} + \underbrace{\frac{E_N(z_0+h) - E_N(z_0)}{h}}_{\mathrm{III}}.$$

Watch the following observations:

- For $N \gg 0$, since S_N' is the partial sum of f', $|S_N'(z_0) - f'(z_0)| < \varepsilon$ for any $\varepsilon > 0$.
- If $|h| \ll 1$, then $|\mathrm{I}| < \varepsilon$ for any $\varepsilon > 0$.
- In part III, using the equality $(z_0+h)^k - z_0^k = h((z_0+h)^{k-1} + (z_0+h)^{k-2}z_0 + \cdots + z_0^{k-1})$, we see
$$\begin{aligned}\mathrm{III} = \frac{1}{h}E_N(z_0+h) - E_N(z_0) &= \sum_{k>N} a_k((z_0+h)^k - z_0^k)\\ &= \sum_{k>N} a_k((z_0+h)^{k-1} + (z_0+h)^{k-2}z_0 + \cdots + z_0^{k-1})\\ &\leqslant \sum_{k>N} |a_k||(z_0+h)^{k-1} + (z_0+h)^{k-2}z_0 + \cdots + z_0^{k-1}|\\ &\leqslant \sum_{k>N} |a_k| \sum_{j=1}^{k} |(z_0+h)^{k-j} z_0^{j-1}|\\ &\leqslant \sum_{k>N} k|a_k|r^{k-1} \to 0\end{aligned}$$
as $N \to \infty$. The last inequality uses $|z_0| < r < R$ and $|z_0 + h| < r < R$. Again, note that $f'(z) = \sum_{k\geqslant 1} ka_k z^{k-1}$ is absolutely convergent in $D_R(0)$, and then for $N \gg 0$, $|\mathrm{III}| < \varepsilon$ for any $\varepsilon > 0$.

In summary, if f is a power series with radius of convergence R, whenever $h \to 0$,

$$\frac{f(z_0+h) - f(z_0)}{h} \to S_\infty'(z_0).$$

Namely, we have checked that the common real derivative algorithm of power series can be realized over $\mathbb{C}$ as expected.

Corollary 2.19. *A power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is ∞-complex differentiable in the disc of convergence $D_R(z_0)$.*

Definition 2.20 (Analyticity)**.** A function $f : \Omega \to \mathbb{C}$ is called *analytic* in Ω if for all $z_0 \in \Omega$, $f(z)$ can be realized as a power series expansion towards z_0, say

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with disc of convergence $D_R(z_0)$ for $R > 0$.

Note that analyticity implies holomorphicity. We will prove the converse implication by using the Cauchy integral formula later (see Subsection 3.6.2).

2.4. **Integration along Curves.**

Definition 2.21. A *parametrized curve* is a map $z : [a, b] \to \mathbb{C}$ defined over an real interval. It is called *smooth* if $z'(t)$ exists and is continuous on $[a, b]$ as a complex function (i.e., $[a, b] \subset \mathbb{R} \subset \mathbb{C} \to \mathbb{C}$), where

$$z'(a) = \lim_{h \to 0^+} \frac{z(a + h) - z(a)}{h}, \quad z'(b) = \lim_{h \to 0^-} \frac{z(b + h) - z(b)}{h}.$$

Definitions 2.22. A parametrized curve is called *piecewise-smooth* if $z : [a, b] \to \mathbb{C}$ is continuous and there are points $a_0, a_1, \ldots, a_N$ such that $a = a_0 < a_1 < \cdots < a_N = b$ and $z|_{[a_i, a_{i+1}]}$ is smooth for any $0 \leqslant i \leqslant N - 1$. Moreover, z is called *closed* if $z(a) = z(b)$; z is called *simple* if $z(t) \neq z(s)$ unless $t = s$ or $t = a, s = b$.

Example 2.23. For $t \in [0, 2\pi]$ and fixed z_0, $z(t) = z_0 + Re^{it}$ and $z(t) = z_0 + Re^{-it}$ are closed and simple parametrized curves. Whereas $z(t) = z_0 + Re^{2it}$ is closed but not simple, since it forms a 2-covering of a circle centred at z_0 with radius R.

Definition 2.24. Two parametrizations $z : [a, b] \to \mathbb{C}$ and $w : [c, d] \to \mathbb{C}$ are called *equivalent* if there is a continuous differentiable bijection $t : [a, b] \to [c, d]$ such that $t'(s) > 0$ (namely, t preserves the orientation) and $z = w \circ t$.

In the upcoming context, our convention dictates that a "curve" is always a piecewise-smooth curve. Let $\Gamma \subset \mathbb{C}$ be a curve with a parametrization $z : [a, b] \to \mathbb{C}$. Let f be a continuous function on Γ. We define the *integral along the curve* by the following complex-valued integral, say

$$\int_{\Gamma} f(z)dz := \int_a^b f(z(t))z'(t)dt.$$

Due to the chain rule, we point out that this integral is well-defined, i.e., it is independent of the choice of equivalent parametrization of Γ.

Proposition 2.25. *Given Γ parametrized by $z : [a, b] \to \mathbb{C}$.*

(1) *The integration along z has linearity, i.e.,*

$$\int_{\Gamma} (af(z) + bg(z))dz = a \int_{\Gamma} f(z)dz + b \int_{\Gamma} g(z)dz.$$

(2) *Suppose Γ^- is defined by $\widetilde{z} : [a, b] \to \mathbb{C}$ via $\widetilde{z}(t) = z(a + b - t)$. Then*

$$\int_{\Gamma} f(z)dz = -\int_{\Gamma^-} f(z)dz.$$

(3) *The integration is bounded from above as follows*

$$\left|\int_\Gamma f(z)dz\right| \leqslant \operatorname{length}(\Gamma)\cdot \sup_{z\in\Gamma}|f(z)|,$$

where $\operatorname{length}(\Gamma)=\int_a^b|z'(t)|dt$.

Proof. (3) Compute by definition

$$\left|\int_\Gamma f(z)dz\right| = \left|\int_a^b f(z(t))z'(t)dt\right| \leqslant \int_a^b |f(z(t))z'(t)|dt \leqslant \sup_{z\in\Gamma}|f(z)|\int_a^b|z'(t)|dt.$$

And (1) (2) are apparent. □

3. Cauchy Theorem and Its Applications

3.1. Motivation from Stokes Formula.

Theorem 3.1. *Let $f:\Omega\to\mathbb{C}$ be continuous. Assume there is a holomorphic function $F:\Omega\to\mathbb{C}$ such that $F'=f$ (here F is called a primitive of f). If $\Gamma\subset\mathbb{C}$ is a curve that begins at w_1 and ends at w_2, then*

$$\int_\Gamma f(z)dz = F(w_2)-F(w_1).$$

In particular, if Γ is closed, then $\int_\Gamma f(z)dz=0$.

Proof. First assume Γ is smooth with a parametrization $z:[a,b]\to\mathbb{C}$. By definition,

$$\int_\Gamma f(z)dz = \int_a^b f(z(t))z'(t)dt = \int_a^b F'(z(t))z'(t)dt.$$

In the sense of Cauchy-Riemann equation, we consider $F=F(z,\overline{z})$. Then

$$\frac{d}{dt}F(z(t)) = \frac{\partial F}{\partial z}\frac{dz(t)}{dt}+\frac{\partial F}{\partial \overline{z}}\frac{d\overline{z}(t)}{dt} = \frac{\partial F}{\partial z}z'(t) = F'(z(t))z'(t)$$

because of holomorphicity of F. Thus, the original integral becomes

$$\int_a^b F'(z(t))z'(t)dt = \int_a^b \frac{d}{dt}F(z(t))dt = F(z(b))-F(z(a)) = F(w_2)-F(w_1).$$

In the case where Γ is piecewise-smooth, the argument is similar. □

Example 3.2. The following function does not have any primitives, so that Theorem 3.1 fails to be true. Consider

$$\begin{aligned} f:\mathbb{C}\backslash\{0\} &\longrightarrow \mathbb{C}\\ z &\longmapsto 1/z. \end{aligned}$$

Define $\Gamma=\{z\mid |z|=1\}$ as the unit circle which is parametrized by $z(t)=e^{it}$ for $t\in[0,2\pi]$. Then

$$\int_\Gamma \frac{1}{z}dz = \int_0^{2\pi}\frac{1}{e^{it}}ie^{it}dt = 2\pi i\neq 0.$$

The emphasis lies on that $f(x)=1/x$ for $f:\mathbb{R}\backslash\{0\}\to\mathbb{R}$ has $\log|x|$ as a primitive on $\mathbb{R}\backslash\{0\}$. So the primitive condition is more subtle over $\mathbb{C}$.

Notice that Theorem 3.1 is a particular version of Stokes formula. Recall that in the real case, for $f : [a, b] \to \mathbb{R}$ that admits a permitive, if $dF/dt = f$ (i.e., $dF = fdt$), then

$$\int_a^b f(t)dt = \int_a^b dF = F(b) - F(a).$$

We can translate this Newton-Leibniz-type statement as

$$\int_a^b dF = \int_I dF = F(b) - F(a) = \int_{\partial I} F,$$

where $I = [a, b]$ and $\partial I = \{b\} - \{a\}$ (as a formal sum of points). Again, Theorem 3.1 can be interpreted as

$$\int_\Gamma dF = \int_{\partial\Gamma} F = F(w_2) - F(w_1)$$

for $\partial\Gamma = \{w_2\} - \{w_1\}$. More generally, the **Stokes formula** states that for a given manifold M and a differential form φ, one obtain

$$\int_M d\varphi = \int_{\partial M} \varphi.$$

3.2. **Local Cauchy Theorem.** The motivation of the Cauchy theorem is seeking the existence of primitives over $\mathbb{C}$. Say given a real continuous function $f : [a, b] \to \mathbb{R}$, then

$$F(x) := \int_a^x f(t)dt = \int_{\Gamma_x} f(t)dt$$

is naturally the primitive of f. The second equality is given by canonically defining $\Gamma_x = [a, x]$. For an analogy, if $f : \Omega \to \mathbb{C}$ is given, is there any complex primitive of f? Consider $z_0 \in \Omega$ and

$$F(z) := \int_{z_0}^z f(w)dw = \int_\Gamma f(w)dw$$

where Γ is a connected path from z_0 to z in Ω. The question is whether F is independent of the choice of Γ.

3.2.1. *Goursat's Theorem.* Let's say $f : \mathbb{D} \to \mathbb{C}$ where $\mathbb{D}$ is a unit disc. For $z \in \mathbb{D}$, we define Γ_z as the line segment from 0 to z. We need to verify that

$$F(z) := \int_{\Gamma_z} f(w)dw$$

is a primitive of f. Consider

$$F(z + h) - F(z) = \int_{\Gamma_{z+h}} f(w)dw - \int_{\Gamma_z} f(w)dw.$$

Recall that if F is a primitive, then for any closed curve Γ in $\mathbb{D}$, $\int_\Gamma f(w)dw = 0$. In particular, this can be divided as

$$\int_{\Gamma_{z+h}} f(w)dw + \int_\gamma f(w)dw + \int_{\Gamma_z^-} f(w)dw = 0$$

in which γ is defined as the oriented line segment from $z + h$ to z. We can rewrite the equality above as

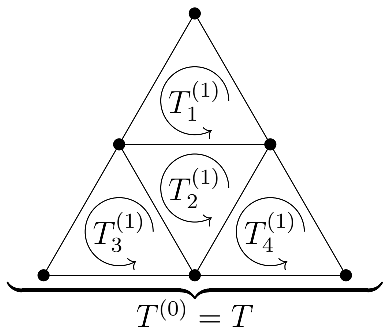
$$\int_{\Gamma_{z+h}} f(w)dw - \int_{\Gamma_z} f(w)dw = \int_{\widetilde{\gamma}} f(w)dw.$$

This observation is the so-called Goursat's theorem as follows.

Theorem 3.3 (Goursat)**.** *Suppose $f:\Omega\to\mathbb{C}$ is holomorphic and $T\subset\Omega$ is a triangle with* $\operatorname{Int}(T)\subset\Omega$*. Then*

$$\int_T f(z)dz = 0.$$

Proof. **Step 1: Triangulate Partition.**



In the picture above, the triangle $T = T^{(0)}$ is divided in to 4 parts, say $T_j^{(1)}$ with $1\leqslant j\leqslant 4$. These triangular bounds (as piecewise-smooth curves) are parametrized anticlockwise. It's easy to prove that

$$\int_T f(z)dz = \int_{T^{(0)}} f(z)dz = \sum_{j=1}^{4}\int_{T_j^{(1)}} f(z)dz,$$

and therefore

$$\left|\int_T f(z)dz\right| \leqslant 4\left|\int_{T_n^{(1)}} f(z)dz\right|$$

for some n. Let's denote $T^{(1)} = T_n^{(1)}$.

Step 2: Iteration. The construction of $T^{(1)}$ from $T^{(0)}$ can be reused to arise $T^{(i+1)}$ from $T^{(i)}$. Hence we get an inequality as in Step 1 of a higher order:

$$\left|\int_T f(z)dz\right| \leqslant 4^n\left|\int_{T^{(n)}} f(z)dz\right|.$$

Step 3: Estimates. Denote $\mathscr{T}^{(n)}$ for the solid triangle that is enclosed by $T^{(n)}$. In the remaining proof, we introduce the following notations, say

$$d^{(n)} := \operatorname{diam}\mathscr{T}^{(n)} = 2^{-n}d^{(0)},$$
$$p^{(n)} := \text{perimeter of } \mathscr{T}^{(n)} = 2^{-n}p^{(0)}.$$

Note that there is a sequence of compact sets

$$T^{(0)}\supset T^{(1)}\supset\cdots\supset T^{(n)}\supset\cdots$$

with $\operatorname{diam} T^{(n)}\to 0$ as $n\to\infty$. Thus there is a unique $w\in T^{(n)}$ for any n due to compactness and Proposition 2.7. Since f is holomorphic at w, by definition,

$$f(z) = \underbrace{f(w)+f'(w)(z-w)}_{f_0(z)} + \psi(z)(z-w),$$

where $\psi(z)\to 0$ as $z\to w$. One can observe that $f_0(z)$ has a primitive in Ω, and then

$$\int_{T^{(n)}} f(z)dz = \underbrace{\int_{T^{(n)}} f_0(z)dz}_{0} + \int_{T^{(n)}}\psi(z)(z-w)dz = \int_{T^{(n)}}\psi(z)(z-w)dz.$$

Then (3) of Proposition 2.25 is applied for

$$\left|\int_{T^{(n)}} f(z)dz\right| \leqslant p^{(n)} \sup_{z\in T^{(n)}} |\psi(z)| \sup_{z\in T^{(n)}} |z-w| \leqslant p^{(n)}d^{(n)} \sup_{z\in T^{(n)}} |\psi(z)|.$$

Accordingly, this implies that

$$\left|\int_T f(z)dz\right| \leqslant 4^n p^{(n)}d^{(n)} \sup_{z\in T^{(n)}} |\psi(z)| = p^{(0)}d^{(0)} \sup_{z\in T^{(n)}} |\psi(z)|.$$

However, $\sup_{z\in T^{(n)}} |\psi(z)| \to 0$ as $n\to\infty$ so that $\left|\int_T f(z)dz\right| = 0$. □

3.2.2. *Local Existence of Primitives.*

Theorem 3.4. *Let $D\subset\mathbb{C}$ be an open disc. Then any $f\in\mathscr{O}(D)$ has a primitive in D.*

Proof. Define $F(z) = \int_{\Gamma_z} f(w)dw$ with Γ_z being the oriented line segment from z_0 to z, where z_0 is the center of D. The claim followed is that $F'(z) = f(z)$ for all $z\in D$. This is valid because by Goursat's theorem,

$$\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\int_\gamma f(w)dw.$$

Again, γ is the oriented segment from z to $z+h$ here whose parametrization is, say,

$$\begin{aligned} w: [0,|h|] &\longrightarrow \mathbb{C} \\ t &\longmapsto z + ht/|h|. \end{aligned}$$

Now one can calculate the integral as

$$\int_\gamma f(w)dw = \int_0^{|h|} f(w(t))\frac{h}{|h|}dt.$$

Finally, it yields to that

$$\lim_{h\to 0, h\in\mathbb{C}} \frac{F(z+h)-F(z)}{h} = f(w(0)) = f(z).$$

The last step is not so obvious and is left as an exercise. □

Corollary 3.5 (Local Cauchy Theorem)**.** *Let $D\subset\mathbb{C}$ be an open disc and let $\Gamma\subset D$ be an arbitrary closed curve. Then for any $f\in\mathscr{O}(D)$,*

$$\int_\Gamma f(z)dz = 0.$$

Remark 3.6. Given a region $\Omega\subset\mathbb{C}$, recall that for $z\in\Omega$ and $f\in\mathscr{O}(\Omega)$, f has a primitive in disc D centred at z whenever $D\subset\Omega$. For another point $w\in\Omega$, we can still make a "parallel moving" of D at z to w (may need $\operatorname{diam} D$ to be sufficiently small).

3.3. **Global Cauchy Theorem.**

Definition 3.7. Let $\Gamma_0, \Gamma_1\subset\Omega$ be two curves with common endpoints, say α and β. Let $\gamma_0,\gamma_1 : [a,b]\to\Omega$ be parametrizations of them. We call Γ_0 and Γ_1 *homotopic* in Ω if for all $0\leqslant s\leqslant 1$, there is a curve Γ_s whose parametrization is given by $\gamma_s : [a,b]\to\Omega$ such that $\gamma_s(a)=\alpha$, $\gamma_s(b)=\beta$, and

$$\gamma_s|_{s=0} = \gamma_0, \quad \gamma_s|_{s=1} = \gamma_1,$$

and $\gamma_s(t)$ is jointly continuous with respect to $s\in[0,1]$ and $t\in[a,b]$.

One can quickly check that the homotopic relation is an equivalence relation. In a topological sense, homotopicity means a continuous deformation between two given curves.

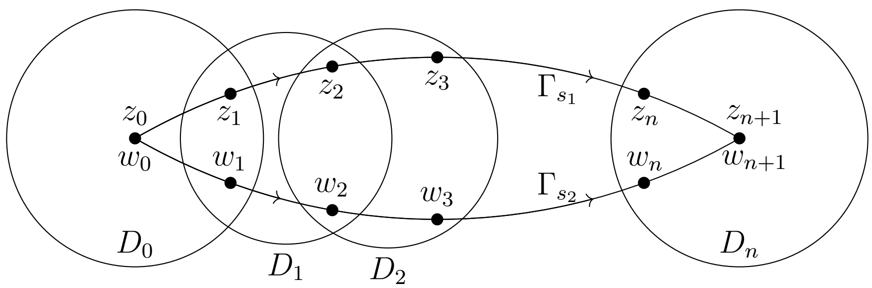
Theorem 3.8 (Homotopy Principle)**.** *Suppose $\Gamma_0, \Gamma_1 \subset \Omega$ with $\Gamma_0 \sim \Gamma_1$ homotopically. For any $f \in \mathscr{O}(\Omega)$,*

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz.$$

Proof. **Step 1: Local Equality.** The claim is that if $s_1, s_2 \in [0,1]$ are close enough, then

$$\int_{\Gamma_{s_1}} f(z)dz = \int_{\Gamma_{s_2}} f(z)dz.$$

To prove this, assume Γ_{s_1} and Γ_{s_2} are parametrized by $z, w : [a,b] \to \mathbb{C}$, respectively. Taking a partition on $[a,b]$ as $a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$ and denote $z_i = z(x_i) \in \Gamma_{s_1}$, $w_i = w(x_i) \in \Gamma_{s_2}$ for $0 \leqslant i \leqslant n+1$. There are a sequence of discs $\{D_i\}_{i=0}^n$ such that $\{z_i, w_i, z_{i+1}, w_{i+1}\} \in D_i$. See the picture below.



Now by local Cauchy theorem (see Corollary 3.5), for each D_i, there is a primitive F_i on D_i of f, i.e., $F_i'(z) = f(z)$ for all $z \in D_i$. On the intersection $D_i \cap D_{i+1}$, we see

$$(F_i - F_{i+1})' = 0,$$

which implies that $F_i - F_{i+1} \equiv \text{const}$, say C_i. Thus we obtain

$$F_{i+1}(z_{i+1}) - F_{i+1}(w_{i+1}) = F_i(z_{i+1}) - F_i(w_{i+1}).$$

Taking integrals, leads to

$$\begin{aligned}\int_{\Gamma_{s_1}} f(z)dz - \int_{\Gamma_{s_2}} f(z)dz &= \sum_{i=0}^{n} [F_i(z_{i+1}) - F_i(z_i)] - \sum_{i=0}^{n} [F_i(w_{i+1}) - F_i(w_i)] \\ &= (F_n(z_{n+1}) - F_n(w_{n+1})) - (F_0(z_0) - F_0(w_0)) \\ &= (F_n(\beta) - F_n(\beta)) - (F_0(\alpha) - F_0(\alpha)) = 0.\end{aligned}$$

Step 2: Iteration. Using the compactness of $[0,1]$, we can divide $[0,1]$ into subintervals $[s_i, s_{i+1}]$ with $|s_i - s_{i+1}| \ll 1$. Hence by Step 1, for all $t, s \in [s_i, s_{i+1}]$, $f(z)$ has the same integral along Γ_t and Γ_s. To sum up,

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz.$$

This completes the proof. □

Definition 3.9. A region $\Omega \subset \mathbb{C}$ is called *simply connected* if any two curves in Ω with common endpoints are homotopic.

Theorem 3.10 (Global Cauchy Theorem)**.** *If $\Omega \subset \mathbb{C}$ is simply connected, then all $f \in \mathscr{O}(\Omega)$ has a primitive.*

Proof. Fix a point $z_0 \in \Omega$. For any curve from z_0 to z, we define

$$F(z) := \int_{\Gamma_z} f(w)dw.$$

This is well-defined (i.e., independent of the choice of Γ_z) when Ω is simply connected. It's easy to check $F'(z) = f(z)$. $\square$

Alternative Proof. One may also fix Γ_z and define Γ_{z+h} as the combination of Γ_z and the segment from z to $z + h$. Thus,

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\int_\gamma f(w)dw.$$

Taking $h \to 0$, we set $F'(z) = f(z)$ again. $\square$

Remark 3.11. Do remember the Global Cauchy (Theorem 3.10) implies that all holomorphic integrals along closed curves in a simply connected region vanish.

The general philosophy of the Cauchy theorem lies in translating topological information (such as simply connected) on the complex plane into analytic information. Conversely, given a connected open subset $\Omega \subset \mathbb{C}$, the question is to determine whether Ω is simply connected or not whenever we assume for all $f \in \mathscr{O}(\Omega)$ and any closed curve $\Gamma \subset \Omega$, f has zero integral along Γ.

3.4. **The First Application: Evaluation of Some Integrals (I).** Our Cauchy theorem can be used to compute several types of real and complex integrals. Also, we will see more approaches to calculating, such as the residue formula (see Section 4.2), which is another corollary of the Cauchy theorem.

Example 3.12. Using the Cauchy theorem, we will calculate the Fourier transform of $e^{-\pi x^2}$. Note that in Fourier analysis, for any $f : \mathbb{R} \to \mathbb{R}$, one can define its Fourier transformation as

$$\widehat{f}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx.$$

The aim is to prove for any $\xi \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}dx = e^{-\pi\xi^2}.$$

Namely, the Fourier transformation of $f(x) = e^{-\pi x^2}$ is nothing but itself.

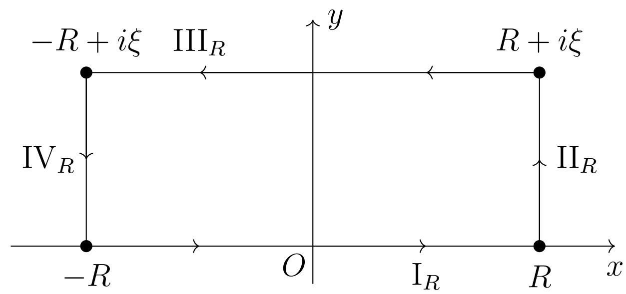
Proof. It is equivalent to prove

$$\int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2}dx = 1.$$

Notice that for $\xi = 0$, the formula is well-known as

$$\int_{-\infty}^{\infty} e^{-\pi x^2}dx = 1.$$

Let's prove for $\xi > 0$, and the remaining case $\xi < 0$ follows similarly. Consider the complex-variable function $f(z) = e^{-\pi z^2}$. Recall that in Example 2.17, we have seen that exponential functions are well-defined over $\mathbb{C}$, and hence $f \in \mathscr{O}(\mathbb{C})$. Then define the curve Γ_R for $R \in \mathbb{R}_{>0}$ as the clockwise oriented rectangle, which is shown in the following picture.



Thus the integral can be divided into

$$\int_{\Gamma_R} f(z)dz = \underbrace{\int_{-R}^{R} e^{-\pi x^2}dx}_{\mathrm{I}_R} + \underbrace{\int_{0}^{\xi} e^{-\pi(R+it)^2}dt}_{\mathrm{II}_R} + \underbrace{\int_{R}^{-R} e^{-\pi(x+i\xi)^2}dx}_{\mathrm{III}_R} + \underbrace{\int_{\xi}^{0} e^{-\pi(-R+it)^2}dt}_{\mathrm{IV}_R}.$$

As $R \to \infty$, we have

$$\mathrm{I}_R \to \int_{-\infty}^{\infty} e^{-\pi x^2}dx = 1.$$

As for part II, say

$$\mathrm{II}_R = \int_0^{\xi} e^{-\pi(R^2+2Rit-t^2)}dt = \int_0^{\xi} e^{-\pi R^2}e^{-2\pi Rit}e^{\pi t^2}dt.$$

Therefore, it is bounded as

$$|\mathrm{II}_R| \leqslant \int_0^{\xi} e^{-\pi R^2}e^{\pi t^2}dt \leqslant (e^{\pi\xi^2}e^{-\pi R^2}).$$

Since $e^{-\pi R^2} \to 0$ as $R \to \infty$, we get $\mathrm{II}_R \to 0$. Similarly, $\mathrm{IV}_R \to 0$ by symmetry (caution: III_R does not tend to 1). On the other hand, apply the Cauchy theorem to the piecewise-smooth closed curve Γ_R defined on $\mathbb{C}$, which is simply connected, the integral of f along with Γ_R vanishes. That is,

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)dz = 0 = \mathrm{I}_R + \mathrm{III}_R,$$

or equivalently,

$$\int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2}dx = \int_{-\infty}^{\infty} e^{-\pi x^2}dx = 1, \quad \xi > 0.$$

Again, the same argument for $\xi < 0$ completes the proof. □

3.5. **The Second Application: Cauchy Integral Formula.** It turns out that under some nice topological circumstances, the value of a holomorphic function at some point can be determined by an average of the boundary points of some neighbourhood.

Theorem 3.13 (Cauchy Integral Formula)**.** *Given an open subset $\Omega \subset \mathbb{C}$ and an open disc $D \subset \mathbb{C}$ with $\partial D \subset \Omega$, assume $f \in \mathscr{O}(\Omega)$. Then for all $z \in D$, we have*

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\xi)}{\xi - z}d\xi.$$

Proof. Consider the function $F(\xi) = f(\xi)/(\xi - z)$ for fixed $z \in D$. Define the curve

$$C_\varepsilon = \{w \mid |w - z| = \varepsilon\}.$$

Notice that F is holomorphic near z and ∂D is homotopic to C_ε. In the Homotopy Principle (Theorem 3.8), taking Ω to be the punctured disc centred at z with radius ε, and then

$$\int_{\partial D} F(\xi)d\xi = \int_{C_\varepsilon} F(\xi)d\xi.$$

Furthermore, one can compute

$$\int_{C_\varepsilon} F(\xi)d\xi = \int_{C_\varepsilon} \frac{f(\xi)}{\xi - z}d\xi = \int_{C_\varepsilon} \underbrace{\frac{f(\xi) - f(z)}{\xi - z}}_{\sim f'(z)} d\xi + \int_{C_\varepsilon} \frac{f(z)}{\xi - z}d\xi.$$

The punchline of this trick is read as follows. The first item has the same order as the integral of $f'(z)$. However, $f'(z)$ is bounded near z by holomorphicity of f. Hence the former term in the equality above tends to be 0 as $\varepsilon \to 0$.

Now, since C_ε has a parametrization $z(t) = z + \varepsilon e^{i\theta}$ for $\theta \in [0, frm{-}epi]$, we see

$$\int_{C_\varepsilon} \frac{f(z)}{\xi - z}d\xi = f(z)\int_0^{2\pi} \frac{1}{\varepsilon e^{i\theta}} ie^{i\theta}d\theta = 2\pi i f(z).$$

Therefore, after taking $\varepsilon \to 0$, we have

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\xi}{\xi - z}d\xi.$$

This is the Cauchy integral formula. □

3.5.1. *Applications of Cauchy Integral Formula.* The following more general version of Theorem 3.13 is given as an application. A quick observation of its proof shows that it contains no more information than the classical formula. However, it would be instrumental in proving the residue formula (see Theorem 4.8).

Theorem 3.14 (Higher Cauchy Integral Formula)**.** *With the same statement as in Theorem 3.13, f has infinitely many complex derivatives in Ω. Moreover,*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial D} \frac{f(\xi)}{(\xi - z)^{n+1}}d\xi.$$

Proof. It suffice to prove for $n \geqslant 1$. We first consider $n = 1$. Using the classical Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\xi)}{\xi - z}d\xi,$$

we have

$$\lim_{h \to 0, h \in \mathbb{C}} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{1}{2\pi i}\frac{1}{h}\int_{\partial D} \frac{f(\xi)}{\xi - z - h} - \frac{f(\xi)}{\xi - z}d\xi.$$

Note that

$$\frac{1}{\xi - z - h} - \frac{1}{\xi - z} = \frac{h}{(\xi - z - h)(\xi - z)},$$

and the equality becomes

$$f'(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\xi)}{(\xi - z)^2}d\xi.$$

Accordingly, one can apply this process inductively to tackle general $n \geqslant 2$. □

3.5.2. *Remarks on the Proof of Cauchy Integral Formula.* Recall that the homotopy principle (Theorem 3.8) tells us that if $\Gamma_0, \Gamma_1 \subset \Omega$ are two closed curves which are homotopic. Then for all $f \in \mathscr{O}(\Omega)$, we have $\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz$.

More generally, assume $\Gamma_0, \Gamma_1 : [0,1] \to C$ are two curves in Ω such that

$$\Gamma_0(0) = \alpha_0, \quad \Gamma_0(1) = \beta_0, \quad \Gamma_1(0) = \alpha_1, \quad \Gamma_1(1) = \beta_1$$

that are homotopic, i.e., for all $t \in [0,1]$ there exists $F(s,t) : [0,1] \times [0,1] \to \Omega$ such that

$$F(0,t) = \Gamma_0(t), \quad F(1,t) = \Gamma_1(t).$$

Claim: *for all $f \in \mathscr{O}(\Omega)$,*

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{I_0} f(z)dz + \int_{I_1} f(z)dz,$$

where I_0 (resp. I_1) is an arbitrary oriented curves from α_0 to α_1 (resp. from β_1 to β_0).

Proof of the Claim. It suffices to check that the curves Γ_0 and $I_0 + \Gamma_1 + I_1$ are homotopic. We define the following map

$$H(s,t) = \begin{cases} F((1+2s)t, 0), & 0 \leqslant t \leqslant s/(1+2s); \\ F(s, (1+2s)t - s), & s/(1+2s) \leqslant t \leqslant (s+1)/(1+2s); \\ F(-(1+2s)t + 1 + 2s, 1), & (s+1)/(1+2s) \leqslant t \leqslant 1. \end{cases}$$

In particular, for fixed s, $H(s,t)$ is the intermediate curve between $\Gamma_0(t)$ and $\Gamma_1(t)$. Moreover, $H(s,t)$ is continuous with respect to (s,t). The feature in need is the homotopy equivalence

$$H(0,t) = \Gamma_0(t) \quad \sim \quad H(1,t) = I_0 + \Gamma_1 + I_1.$$

From what we have proved for homotopy with common fixed endpoints,

$$\int_{\Gamma_0} f(z)dz = \int_{I_0+\Gamma_1+I_1} f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{I_0} f(z)dz + \int_{I_1} f(z)dz.$$

In particular, consider the special case where $\alpha_0 = \beta_0$ and $\alpha_1 = \beta_1$. Then Γ_0 and Γ_1 are closed curves such that

$$\int_{I_0} f(z)dz + \int_{I_1} f(z)dz = 0$$

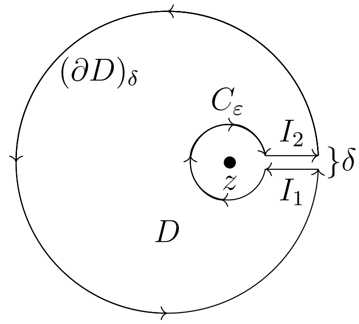
by taking $I_1 = I_0^-$. Hence for all $f \in \mathscr{O}(\Omega)$,

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz.$$

Note that this proof gives a slightly more robust relationship between the two integrals. □

Now we focus on another proof of the Cauchy integral formula (Theorem 3.13) using only the global Cauchy (Theorem 3.10) rather than the homotopy principle.

Alternative Proof. Keep the notation as in the original proof of Theorem 3.13. For any closed circle $D \subset \Omega$ that contains C_ε and any $\delta > 0$, define the piecewise-smooth closed curve Γ_δ as shown in the following.



For $\Gamma_\delta = (\partial D)_\delta + I_1 + C_\varepsilon + I_2$, by Cauchy theorem,

$$\int_{\Gamma_\delta} \frac{f(\xi)}{\xi - z} d\xi = \int_{(\partial D)_\delta} \frac{f(\xi)}{\xi - z} d\xi + \underbrace{\int_{I_1} \frac{f(\xi)}{\xi - z} d\xi + \int_{I_2} \frac{f(\xi)}{\xi - z} d\xi}_{0} + \int_{C_\varepsilon^-} \frac{f(\xi)}{\xi - z} d\xi = 0.$$

Letting $\delta \to 0$, we get

$$\int_{\partial D} \frac{f(\xi)}{\xi - z} d\xi + \int_{C_\varepsilon^-} \frac{f(\xi)}{\xi - z} d\xi = 0,$$

which implies that

$$\int_{\partial D} \frac{f(\xi)}{\xi - z} d\xi = \int_{C_\varepsilon} \frac{f(\xi)}{\xi - z} d\xi = 2\pi i f(z) + o(\varepsilon).$$

This proves the Cauchy integral formula whenever $\varepsilon \to 0$. $\square$

3.6. **More Corollaries of Cauchy Integral Formula.** The direct corollaries of the Cauchy theorem involve several outstanding results that we will introduce.

Proposition 3.15 (Cauchy Inequalities)**.** *Under the same statement as before, for all $f \in \mathscr{O}(\Omega)$ and $D_R(z_0) \subset \overline{D_R(z_0)} \subset \Omega$, we have*

$$|f^{(n)}(z_0)| \leqslant \frac{n!}{R^n} \|f\|_{\partial D_R}, \quad \forall n \geqslant 0,$$

where $\|f\|_{\partial D_R} = \max_{z \in \partial D_R} |f(z)|$.

Proof. Using Higher Cauchy integral formula (see Theorem 3.14),

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi = \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z(t))}{R^{n+1} e^{i(n+1)\theta}} i R e^{i\theta} d\theta$$

since we can parametrize ∂D_R via $z(t) = z_0 + Re^{i\theta}$ for $\theta \in [0, 2\pi]$. Hence

$$|f^{(n)}(z_0)| \leqslant \frac{n!}{2\pi} \int_0^{2\pi} \frac{|f(z(t))|}{R^n} d\theta \leqslant \frac{n!}{R^n} \max_{z \in \partial D_R} |f(z)|$$

as desired. This completes the proof. $\square$

3.6.1. *Liouville Theorem and Its Application.*

Corollary 3.16 (Liouville)**.** *Let $f \in \mathscr{O}(\mathbb{C})$ (say f is an entire function). If f is bounded, i.e., $|f| \leqslant M < \infty$ on $\mathbb{C}$, then f is a constant function.*

Proof. Using the first-order Cauchy inequality,

$$|f'(z_0)| \leqslant \frac{1}{R}\|f\|_{\partial D_R} \leqslant \frac{M}{R}$$

for any disc D_R with $R > 0$. The entire condition $f \in \mathscr{O}(\mathbb{C})$ guarantees that R can be any positive number. Making $R \to \infty$ to get $f'(z_0) = 0$ for arbitrary $z_0 \in \mathbb{C}$. Hence $f(z)$ is a constant function. □

Exercise 3.17. Show that for $f \in \mathscr{O}(\mathbb{C})$, if there is some constant $C < \infty$ such that $|f(z)| \leqslant CR^d$ for any $|z| \leqslant R$ together with some d (i.e., f has at most polynomial growth), then f must be a polynomial of degree at most d.

Here comes one of several approaches to prove the fundamental theorem of algebra using complex analysis.

Theorem 3.18 (Fundamental Theorem of Algebra)**.** *Every non-constant polynomial* $p(z) = \sum_{k=0}^{d} a_k z^k$ *with* $a_k \in \mathbb{C}$ *has a root in* $\mathbb{C}$.

Proof. If for all $z \in \mathbb{C}$ we have $p(z) \neq 0$, then the rational function $1/p(z) \in \mathscr{O}(\mathbb{C})$. In particular, $1/p(z)$ is a bounded entire function as $p(z)$ is a polynomial that is nowhere vanishing. Applying Liouville Theorem (Corollary 3.16), $1/p(z)$ is a constant, which yields to a contradiction. □

Corollary 3.19. *Every polynomial* $p(z) = a_d z^d + \cdots + a_1 z + a_0$ *with* $a_d \neq 0$ *has exactly* d *roots in* $\mathbb{C}$*, counted with multiplicity.*

Proof. Theorem 3.18 shows that there is $w_1 \in \mathbb{C}$ such that $p(w_1) = 0$. Making a change of variable, say $z = (z - w_1) + w_1$, we obtain

$$p(z) = \sum_{k=1}^{d} b_k (z - w_1)^k + b_0.$$

A simple comparison shows that $b_0 = 0$. Hence $(z - w_1) \mid p(z)$ and then $p(z) = (z - w_1)q(z)$ for another polynomial q with degree $d - 1$. Using induction on the degree of polynomials, one finally gets

$$p(z) = \prod_{k=1}^{d} (z - w_k)$$

for some $w_1, w_2, \ldots, w_k \in \mathbb{C}$. □

3.6.2. *Holomorphicity Implies Analyticity.* Given Definition 2.20 of analyticity, we have already seen it implies holomorphicity. The following context shows the converse via the Cauchy integral formula.

Theorem 3.20. *Given* $D_R(z_0) \subset \overline{D_R(z_0)} \subset \Omega$*, any* $f \in \mathscr{O}(\Omega)$ *has a power series expansion in* $D_R(z_0)$*. That is, for all* $z \in D_R(z_0)$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

Proof. Fix $z \in D_R(z_0)$. By Cauchy integral formula (Theorem 3.13),

$$f(z) = \frac{1}{2\pi i}\int_{\partial D_R} \frac{f(\xi)}{\xi - z}d\xi.$$

Let's write

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0) - (z - z_0)} = \frac{1}{\xi - z_0} \cdot \frac{1}{1 - \frac{z-z_0}{\xi - z_0}},$$

where $|\frac{z-z_0}{\xi-z_0}| \leqslant r < 1$ for some $r > 0$. Thus it admits a power expansion

$$\frac{1}{\xi - z} = \sum_{n=0}^{\infty} \frac{z - z_0}{\xi - z_0}^n = \sum_{n=0}^{\infty} \frac{1}{(\xi - z_0)^{n+1}} (z - z_0)^n.$$

Therefore,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial D_R} f(\xi) \sum_{n=0}^{\infty} \frac{1}{(\xi - z)^{n+1}} (z - z_0)^n d\xi \\ &= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{\partial D_R} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi (z - z_0)^n \\ &= \sum_{n=0}^{\infty} a_n (z - z_0)^n \end{aligned}$$

for any $z \in D_R(z_0)$. Note that $\sum_{n=0}^{\infty} \frac{1}{(\xi - z)^{n+1}}$ is uniformly convergent in $z \in D_R(z_0)$ just so the second equality holds. $\square$

Remark 3.21. From a topological aspect of view, $(\mathbb{Q}, |\cdot|)$, the rational numbers equipped with a usual absolute value, is not complete. There is a completion $(\mathbb{R}, |\cdot|)$ that is not algebraically closed. This phenomenon gives a motivation to consider field extensions

$$(\mathbb{Q}, |\cdot|) \subset (\mathbb{R}, |\cdot|) \subset (\mathbb{C} = \mathbb{R}^{\text{alg}}, |\cdot|).$$

The complex analysis theory primarily focuses on analytic functions (or, equivalently, holomorphic functions) on $\mathbb{C}$. However, the completion of $\mathbb{Q}$ is not unique whenever we replace the usual absolute value with other values. Given a prime p, the p-adic norm $|\cdot|_p$ is defined by

$$\forall x = \frac{p^a r}{s} \in \mathbb{Q}, \quad |x|_p := p^{-a}$$

where p neither divides r nor s for $r, s \in \mathbb{Z}$. Similarly, we obtain field extensions

$$(\mathbb{Q}, |\cdot|_p) \subset (\mathbb{Q}_p, |\cdot|_p) \subset (\mathbb{Q}_p^{\text{alg}}, |\cdot|_p) \subset (\mathbb{C}_p, |\cdot|_p).$$

Here the first completion $\mathbb{Q}_p$ is called the p-adic rational number field, which is not algebraically closed. Also, $\mathbb{Q}_p^{\text{alg}}$ is algebraically closed but not complete. To resolve this, taking $\mathbb{C}_p$ is enough. The theory to understand analytic functions defined on $\mathbb{C}_p$ is the so-called p-adic analysis.

3.6.3. *Analytic Continuation.* Thanks to Theorem 3.20, the holomorphic functions are analytic. From this, we wish to control all properties of an analytic function by a sequence of points. The following theorem makes the expectation morally valid. Note that the only subtlety here is the requirement that the limit point of this sequence must lie in the region.

Theorem 3.22 (Analytic Continuation)**.** *Let $\Omega \subset \mathbb{C}$ be an open connected region and $f \in \mathscr{O}(\Omega)$. Assume there is a sequence $\{z_n\}_{n=1}^{\infty} \subset \Omega$ with $z_n \neq w$ whereas $z_n \to w \in \Omega$, satisfying $f(z_n) = 0$ for any $n \in \mathbb{N}$. Then $f \equiv 0$ in Ω.*

Proof. Define the set of zeroes of f as follows (which is precisely open by definition):

$$S = \text{Int}\{z \in \Omega \mid f(z) = 0\}.$$

Claim: *as a non-empty open subset, S is also closed in Ω, i.e., $\overline{S} \cap \Omega = S$.*

Proof of the Claim. To prove this claim, we fix $w \in S$ and verify that there is a (non-empty) open set V around w such that $f = 0$ in V. Once this is valid, we are able to take any limit sequence $\{\xi_k\}_{k=0}^{\infty} \subset S$ that converges to some point $\xi \in \overline{S} \cap \Omega \subset \Omega$, and a similar argument shows that $\xi \in S$ as well. Using holomorphicity (hence analyticity) of f, we write

$$f(z) = \sum_{n=0}^{\infty} a_n (z-w)^n$$

for z lying near w. If f does not vanish constantly near w, then there exists $m \geqslant 0$ such that $a_m \neq 0$. This deduces to

$$\begin{aligned} f(z) &= a_m(z-w)^m + a_{m+1}(z-w)^{m+1} + \cdots \\ &= a_m(z-w)^m(1 + a_{m+1}(z-w) + \cdots) \\ &= a_m(z-w)^m(1+g(z)) \end{aligned}$$

for some g such that $g(w) = 0$, since $1 + a_{m+1}(z-w) + \cdots$ is convergent. Now consider to apply the condition $z_n \to w$ with $f(z_n) = 0$. We obtain

$$f(z_k) = a_m(z_k - w)(1 + g(z_k)) \neq 0,$$

which leads to a contradiction. So we have proved the claim.

Using the claim, we can easily get $\Omega \subset S$. Consequently, f vanishes everywhere in Ω. $\square$

Corollary 3.23. *Let $f, g \in \mathscr{O}(\Omega)$ for an open connected region Ω. Assume $f = g$ in some non-empty open set $V \subset \Omega$, then $f = g$ in Ω.*

3.7. **Further Applications.**

3.7.1. *Morera's Theorem.* The following theorem is the converse of Goursat's (Theorem 3.3).

Theorem 3.24 (Morera)**.** *Suppose f is a continuous function in some open disc D, and*

$$\int_T f(z)dz = 0$$

for any triangle T in D. Then $f \in \mathscr{O}(D)$.

Proof. Consider the primitive of f in Ω, say

$$F(z) := \int_{\Gamma_z} f(z)dz$$

for some fixed z_0 and a curve Γ_z from z_0 to z. The condition implies that F is independent of the choice of Γ_z since any curve can be approximated by a piecewise-linear curve (where the triangle division applies). In particular $F \in \mathscr{O}(\Omega)$ since $F'(z) = f(z)$, and thus F is analytic by Theorem 3.20. Again, $f = F'$ is also analytic, which is equivalent to holomorphic. $\square$

Exercise 3.25. Check that circles can replace the triangles in Theorem 3.24.

Theorem 3.26 (Holomorphic Approximation)**.** *Let $\{f_n\}_{n=0}^{\infty} \subset \mathscr{O}(\Omega)$. Assume $f_n \to f$ converges uniformly on every compact subset of Ω, denoted by $f_n \to f$ in $C^0_{\mathrm{loc}}(\Omega)$. Then $f \in \mathscr{O}(\Omega)$.*

Proof. For any triangle $T \subset \Omega$, since $f_n \to f$ in $C^0_{\mathrm{loc}}(\Omega)$,

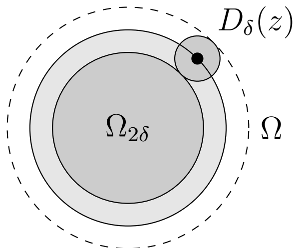
$$\int_T f(z)dz = \lim_{n\to\infty} \int_T f_n(z)dz = 0$$

because of $f_n \in \mathscr{O}(\Omega)$. By Morera Theorem, $f \in \mathscr{O}(\Omega)$. $\square$

Remark 3.27. Theorem 3.26 is not true in the real case.

Theorem 3.28 (Higher Local Convergence)**.** *Let* $\{f_n\}_{n=0}^{\infty} \subset \mathscr{O}(\Omega)$. *Assume* $f_n \to f$ *in* $C^0_{\mathrm{loc}}(\Omega)$. *Then* $f_n^{(k)} \to f^{(k)}$ *in* $C^0_{\mathrm{loc}}(\Omega)$ *for any* $k \geqslant 1$. *Or equivalently,* $f_n \to f$ *in* $C^\infty_{\mathrm{loc}}(\Omega)$.

Proof. We only need to verify $f_n' \to f'$ uniformly on every compact set of Ω. And by inductive arguments, this is equivalent to $f_n \to f$ in $C^\infty_{\mathrm{loc}}(\Omega)$. It suffices to verify that $f_n' \to f'$ uniformly on every $\Omega_\delta = \{z \in \Omega \mid \overline{D_\delta(z)} \subset \Omega\}$.



Using the Cauchy integral formula, for all $z \in \Omega_{2\delta}$, we have

$$(f_n' - f')(z) = \frac{1}{2\pi i}\int_{\partial D_\delta(z)} \frac{(f_n - f)(\xi)}{(\xi - z)^d} d\xi.$$

Therefore,

$$|f_n' - f'|(z) \leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{|f_n - f|(z)}{\delta^2}\delta d\theta \leqslant \frac{1}{\delta}\underbrace{\sup_{\xi\in\Omega_\delta}|f_n - f|(\xi)}_{\to 0}.$$

Hence $f_n' \to f'$ uniformly in $\Omega_{2\delta}$. ☐

Theorem 3.29. *Given an open subset* $\Omega \subset \mathbb{C}$ *(not necessarily connected), we define a function* $F(z,s)$ *on* $\Omega \times [0,1]$. *Assume that*

(1) $F(z,s)$ *is holomorphic with respect to* z *for any fixed* s, *and*
(2) F *is continuous with respect to* (z,s).

Then the function

$$f(z) = \int_0^1 F(z,s)ds$$

is holomorphic for z.

Proof. On condition (1), consider the Riemann sum

$$f_n(z) := \frac{1}{n}\sum_{k=1}^{n} F(z, \frac{k}{n}) \in \mathscr{O}(\Omega)$$

for any $n \geqslant 1$. We are going to prove that $f_n \to f$ uniformly on every compact set K of Ω, or namely, in $C^0_{\mathrm{loc}}(\Omega)$. Once this is valid, we obtain $f \in \mathscr{O}(\Omega)$ by holomorphic approximation (Theorem 3.26).

The condition (2) implies that F is uniformly continuous on $K \times [0,1]$. Hence

$$\sup_{z\in K}|F(z,s_1) - F(z,s_2)| < \varepsilon$$

whenever $|s_1 - s_2| < \delta \ll 1$. Furthermore,

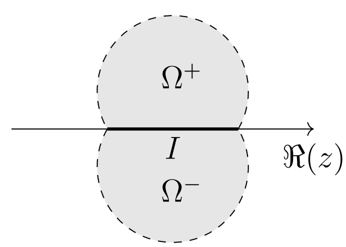
$$\begin{aligned}|f_n - f|(z) &= \left|\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}(F(z,\frac{k}{n}) - F(z,s))ds\right| \\ &\leqslant \sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}|F(z,\frac{k}{n}) - F(z,s)|ds \\ &\leqslant \sum_{k=1}^{n}\frac{1}{n}\varepsilon = \varepsilon\end{aligned}$$

whenever $1/n < \delta$. □

3.7.2. *Schwarz Reflection Principle.* Here comes another direct application of Morera's Theorem on symmetric regions.

Proposition 3.30 (Schwarz Reflection Principle)**.** *Let Ω be an open set that is symmetric with respect to the real axis, i.e., for any $z \in \Omega$, $\overline{z} \in \Omega$ as well. Denote*

$$\Omega^+ = \Omega \cap \{\Im(z) > 0\}, \quad \Omega^- = \Omega \cap \{\Im(z) < 0\}.$$



Suppose $f^\pm \in \mathscr{O}(\Omega^\pm)$ satisfy $f^+(x) = f^-(x)$ for all $x \in I = \Omega \cap \mathbb{R}$, and $f^\pm$ extend continuously to I. Then the function as follows is holomorphic in Ω.

$$f(z) := \begin{cases} f^+(z), & z \in \Omega^+; \\ f^+(z) = f^-(z) & z \in I; \\ f^-(z), & z \in \Omega^-. \end{cases}$$

Exercise 3.31. Prove Proposition 3.30. (Hint: Consider applying Morera theorem to verify that for any triangle $T \subset \Omega$, the integral of f along T is zero, whether T intersects with I or not.)

Corollary 3.32. *Let Ω be as above. Assume $f \in \mathscr{O}(\Omega^+)$ extends continuously on I and $f(x) \in \mathbb{R}$ for $x \in I$. Then there is $F \in \mathscr{O}(\Omega)$ such that $F|_{\Omega^+} = f$.*

Proof. Define F as follows: for all $z \in \Omega^-$, let $F(z) := \overline{f(\overline{z})}$. Then $F \in \mathscr{O}(\Omega^-)$ and $F(x) = f(x)$ for $x \in \mathbb{R}$. Applying Proposition 3.30 to F may complete the proof. □

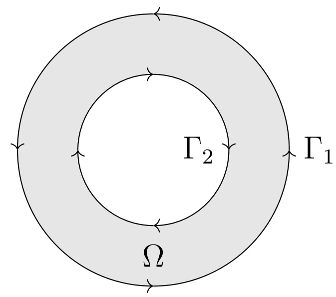
3.8. **A Geometric Point of View.** The final remark on a geometric point-view towards the Cauchy theorem and Cauchy integral formula comes. We introduce the *wedge product* of differential forms. Given two differential 1-forms dx, dy, we define their *linear wedge product* as $dx \wedge dy$ such that

$$dx \wedge dy = -dy \wedge dx, \quad dx \wedge dx = dy \wedge dy = 0.$$

Applying this to $dz = dx + idy$ and $d\overline{z} = dx - idy$, one deduce that

$$\frac{i}{2}dz \wedge d\overline{z} = \frac{i}{2}(dx + idy) \wedge (dx - idy) = dx \wedge dy.$$

3.8.1. *Remarks on Cauchy Theorem.* In the context of (local) Cauchy theorem (see Corollary 3.5), we consider $f \in \mathscr{O}(\Omega)$ and two closed curves in Ω, say Γ_1 and Γ_2, with two opposite orientations. In particular, by focusing only on the local case (recall that under the global situation, the region must be simply connected), assume Ω is an annulus as follows such that $\partial\Omega = \Gamma_1 + \Gamma_2$.



Punchline: using the language of wedge products, we can show that the local Cauchy is exactly implied by Stokes formula and Cauchy-Riemann equation.

Let's check this explicitly by hand. Now the statement of local Cauchy is

$$\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2^-} f(z)dz \quad \Longleftrightarrow \quad \int_{\partial\Omega} f(z)dz = 0,$$

which is also equivalent to

$$\int_{\Omega} d(f(z)dz) = \int_{\partial\Omega} f(z)dz = 0$$

by Stokes formula. Here the differential form on Ω is defined as

$$d(f(z)dz) := df(z) \wedge dz$$

via the wedge product. On the other hand, this can be computed explicitly via

$$df(z) \wedge dz = (\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}) \wedge dz = \frac{\partial f}{\partial z}\underbrace{dz \wedge dz}_{0} + \underbrace{\frac{\partial f}{\partial \overline{z}}}_{0} d\overline{z} \wedge dz = 0.$$

Recall that the second item vanishes because of the holomorphicity of f by the Cauchy-Riemann equation.

3.8.2. *Remarks on Cauchy Integral Formula.* One can apply a similar interpretation to prove Cauchy integral formula. Say $f \in \mathscr{O}(\Omega)$ and then

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\xi)}{\xi - z_0}d\xi = \frac{1}{2\pi i}\int_{\Omega} d(\frac{f(\xi)}{\xi - z_0}d\xi)$$

by Stokes formula. On the other hand, by definition again,

$$\begin{aligned} d(\frac{f(\xi)}{\xi - z_0}d\xi) &= \frac{\partial}{\partial \xi}(\frac{f(\xi)}{\xi - z_0})\underbrace{d\xi \wedge d\xi}_{0} + \frac{\partial}{\partial \overline{\xi}}(\frac{f(\xi)}{\xi - z_0})d\overline{\xi} \wedge d\xi \\ &= f(\xi)\frac{\partial}{\partial \overline{\xi}}(\frac{1}{\xi - z_0})d\overline{\xi} \wedge d\xi. \end{aligned}$$

Therefore, we get

$$f(z_0) = \frac{1}{2\pi i}\int_{\Omega} f(\xi)\frac{\partial}{\partial \overline{\xi}}(\frac{1}{\xi - z_0})d\overline{\xi} \wedge d\xi.$$

In a physical sense, the integral term

$$\frac{\partial}{\partial\bar{\xi}}(\frac{1}{\xi - z_0}) = c\delta_{z_0}$$

for some physical constant c. Here δ_{z_0} is called the *Dirac measure* at z_0 (corresponding to $f(\xi)/(\xi - z_0)$ as in the classical Cauchy integral formula).

4. Meromorphic Functions

In the previous chapter, based on holomorphicity, we begin with polynomials in a single complex variable, which yields definitions of rational and analytic functions. On the Cauchy integral formula, some local analysis induces the equivalence relation between holomorphicity and analyticity of complex functions defined on nice topological subspaces of the complex plane. Meromorphicity can be loosely understood as "weak holomorphicity with some singular points". We begin with discussions about special points defined by an arbitrary function in a single complex variable.

4.1. **Zeros and Poles.** Consider the following 3 examples that corresponds to some essential notions which will be defined later.

(1) (Removable Singularity) $f(z) = z$ at $z = 0$: f is well-defined (thus bounded) near $z = 0$;
(2) (Pole) $f(z) = 1/z$ at $z \to 0$: we have $|f(z)| \to \infty$.
(3) (Essential Singularity) $f(z) = e^{1/z}$ at $z \to 0$: there are many different cases, such as
  (i) whenever $z \to 0^+$ for $z \in \mathbb{R}$, $|f(z)| \to \infty$;
  (ii) whenever $z \to 0^-$ for $z \in \mathbb{R}$, $|f(z)| \to 0$;
  (iii) if $z = ix$ with $x \in R$, then $x \to 0$ leads to $z \to 0$ from the positive imaginary axis – in this case,
$$f(ix) = e^{-i/x} = \cos(-\frac{1}{x}) + i\sin(-\frac{1}{x})$$
  that oscillates rapidly.

Definition 4.1 (Zero)**.** For $f \in \mathscr{O}(\Omega)$, a point $z_0 \in \Omega$ is called a *zero* if $f(z_0) = 0$.

In fact, if $f(z_0) = 0$ then there exists a sufficiently small open neighborhood $V \subset \Omega$ of z_0 such that $f(z) \neq 0$ for any $z \in V\backslash\{z_0\}$ unless $f \equiv 0$ as a constant near z_0. In particular, the zeros of a non-constant holomorphic function are *isolated*.

Theorem 4.2 (Order of Zero)**.** *Given an open connected region Ω with $z_0 \in \Omega$, assume that $f(z_0) = 0$ and $f \not\equiv 0$ in Ω. Then there is a sufficiently small open neighborhood $V \subset \Omega$ of z_0 and a non-vanishing holomorphic function $g \in \mathscr{O}(V)$ together with a unique $m \in \mathbb{N}$ such that*

$$f(z) = (z - z_0)^m g(z), \quad \forall z \in V.$$

Proof. Using $f \in \mathscr{O}(\Omega)$, it is analytic in Ω by Theorem 3.20. In particular, it is analytic near $z_0 \in \Omega$. To be precise,

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

for z lying near z_0. Since f is not constantly vanishing, there is some $m < \infty$ such that $a_m \neq 0$. Finally, note that such smallest m does work. ☐

Notation 4.3. The unique integer in Theorem 4.2 is denoted by $m := \mathrm{ord}_{z_0}(f)$ and is called the order of f at z_0.

Definition 4.4 (Pole)**.** For $f \in \mathscr{O}(\Omega\backslash\{z_0\})$, call z_0 a *pole* of f if $1/f$ has a zero at z_0.

It turns out that if f has a pole at $z_0 \in \Omega$, then there is a sufficiently small open set V of z_0 such that

$$f(z) = (z - z_0)^{-m} g(z), \quad \forall z \in V,$$

where $g(z) \neq 0$ for any $z \in V$. Similarly, m here is called the order of the pole z_0, and it keeps the same notation. Furthermore, if the order of a zero (resp. pole) z_0 is exactly 1, we call z_0 a simple zero (resp. pole).

Theorem 4.5. *If f has a pole z_0 of order m, then near z_0, we have*

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-(m-1)}}{(z - z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z - z_0} + G(z)$$

where $a_{-m} \neq 0$ and $G(z)$ is holomorphic near z_0.

Proof. The condition forces f to be

$$f(z) = (z - z_0)^{-m} \sum_{k=0}^{\infty} b_k (z - z_0)^k = \frac{b_0}{(z - z_0)^m} + \frac{b_1}{(z - z_0)^{m-1}} + \cdots + \frac{b_{m-1}}{z - z_0} + b_m.$$

Letting $b_i = a_{i-m}$ gives the result in need. □

Definitions 4.6. The last coefficient a_{-1} in Theorem 4.5 is called the *residue* of f at the pole z_0 and is denoted by

$$\operatorname{res}_{z_0}(f) = a_{-1}.$$

Also, the function $f(z) - G(z)$ is called the *principal part* of f at the pole z_0.

Remark 4.7. Some unusual approaches to attain the order of zeros or poles.

(1) if z_0 is a simple pole, then

$$\operatorname{res}_{z_0}(f) = \lim_{z \to z_0} (z - z_0) f(z).$$

More generally, if z_0 is a pole of order m, then

$$\operatorname{res}_{z_0}(f) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} ((z - z_0)^m f(z)).$$

(2) If Ω is connected, then the poles and zeros of f in Ω are isolated whenever f is not a constant.

(3) If z_0 is a zero of f, then

$$\operatorname{ord}_{z_0}(f) = \max\{k \in \mathbb{N} \mid f^{(k)}(z_0) \neq 0\}.$$

4.2. **Residue Formula: Evaluation of Some Integrals (II).** In Section 3.4, we have given an approach to calculating some integral via the Cauchy theorem. Here comes another corollary of Theorem 3.8.

Theorem 4.8 (Single Residue Formula)**.** *For $f \in \mathscr{O}(\Omega \backslash \{z_0\})$ with z_0 being a pole of f, let $D \subset \Omega$ be an open disc containing z_0. Then*

$$\operatorname{res}_{z_0}(f) = \frac{1}{2\pi i} \int_{\partial D} f(z) dz.$$

Proof. Using the Cauchy theorem (or homotopy principle), for the circle with radius ε centred at z_0

$$\int_{\partial D} f(z) dz = \int_{C_\varepsilon} f(z) dz.$$

Since z_0 is a pole of f, we can write

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-(m-1)}}{(z-z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z-z_0} + G(z)$$

and hence

$$\int_{\partial D} f(z)dz = \underbrace{\int_{C_\varepsilon} \frac{a_{-m}}{(z-z_0)^m}dz + \cdots + \int_{C_\varepsilon} \frac{a_{-2}}{(z-z_0)^2}dz}_{0} + \underbrace{\int_{C_\varepsilon} \frac{a_{-1}}{z-z_0}dz}_{2\pi i a_{-1}} + \underbrace{\int_{C_\varepsilon} G(z)dz}_{0}.$$

The first part vanishes by applying the higher Cauchy integral formula (Theorem 3.14) to constant functions. The second part is valued by the classical Cauchy integral formula (Theorem 3.13), and the last part vanishes because $G(z)$ is holomorphic near z_0. As a result,

$$\operatorname{res}_{z_0}(f) = a_{-1} = \frac{1}{2\pi i}\int_{\partial D} f(z)dz$$

that gives the residue formula. □

The following corollary is given by applying Theorem 4.8 iteratively for generalizing it to more points.

Corollary 4.9 (Residue Formula)**.** *Suppose $f \in \mathscr{O}(\Omega\backslash\{z_1, z_2, \ldots, z_n\})$ with $z_1, z_2, \ldots, z_n$ being poles of f. Let $\Gamma \subset \Omega$ be the closed curve encompassing $\{z_1, z_2, \ldots, z_n\}$. Then*

$$\frac{1}{2\pi i}\int_\Gamma f(z)dz = \sum_{k=1}^{n} \operatorname{res}_{z_k}(f).$$

Now we move to apply the residue formula in the evaluation of integrals.

Example 4.10. For $0 < a < 1$, we are going to compute the real integral

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx.$$

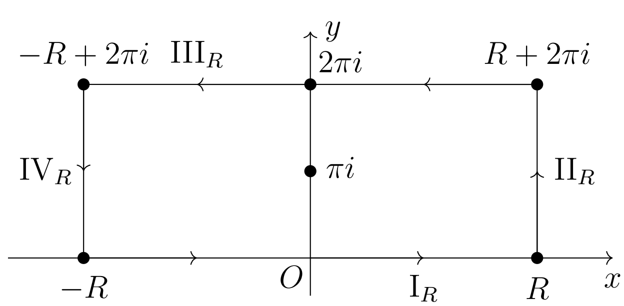
Set $f(z) = e^{az}/(1+e^z)$ as a complex function, then f has a pole at $z = \pi i$ with

$$\lim_{z\to\pi i}(z-\pi i)\frac{e^{az}}{1+e^z} = -e^{a\pi i}.$$

In particular, $z = \pi i$ is a simple pole of f. Hence

$$a_{-1} = \operatorname{res}_{\pi i}(f) = -e^{a\pi i}.$$

Now consider the following clockwise oriented triangle Γ_R whose intersection with $\Im(z)$-axis is $2\pi i$, the period of $1+e^z$.



Using residue formula to Γ_R:

$$\int_{\Gamma_R} f(z)dz = 2\pi i \operatorname{res}_{\pi i}(f) = -2\pi i e^{a\pi i}.$$

On the other hand, we do the calculation as

$$\int_{\Gamma_R} f(z)dz = \underbrace{\int_{-R}^{R} f(x)dx}_{\mathrm{I}_R} + \underbrace{\int_0^{2\pi} f(R+it)dt}_{\mathrm{II}_R} + \underbrace{\int_R^{-R} f(x+2\pi i)dx}_{\mathrm{III}_R} + \underbrace{\int_{2\pi}^{0} f(-R+it)dt}_{\mathrm{IV}_R}.$$

For part II, we get

$$|\mathrm{II}_R| = \left|\int_0^{2\pi} \frac{e^{a(R+it)}}{1+e^{R+it}}dt\right| \leqslant \int_0^{2\pi} \frac{e^{aR}}{|1+e^R\cdot e^{it}|}dt.$$

Since $e^R - 1 \leqslant |1+e^R\cdot e^{it}| \leqslant e^R+1$, the inequality further becomes

$$\int_0^{2\pi} \frac{e^{aR}}{|1+e^R\cdot e^{it}|}dt \leqslant \int_0^{2\pi} \frac{e^{aR}}{e^R-1}dt \leqslant Ce^{(a-1)R}$$

as $R\to\infty$ for $0<a<1$. Similarly, $|\mathrm{IV}_R| \to 0$ as well. Also,

$$\begin{aligned}\mathrm{III}_R &= -\int_{-R}^{R} \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}dx\\ &= -\int_{-R}^{R} \frac{e^{2a\pi i}\cdot e^{ax}}{1+e^x}dx\\ &= -e^{2a\pi i}\int_{-R}^{R}\frac{e^{ax}}{1+e^x}dx = -e^{2a\pi i}\mathrm{I}_R.\end{aligned}$$

Letting $R\to\infty$ and summing all pieces listed above, we see

$$-2\pi i e^{a\pi i} = \mathrm{I}_\infty - e^{2a\pi i}\mathrm{I}_\infty.$$

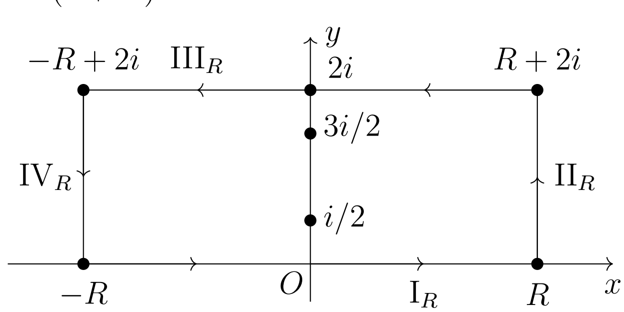
Therefore, the desired integral is

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx = \mathrm{I}_\infty = -\frac{2\pi i e^{a\pi i}}{1-e^{2a\pi i}} = \frac{\pi}{\sin \pi a}.$$

Example 4.11. In this example, we aim to calculate the following integral as a Fourier transformation (see Example 3.12) of $1/(\cosh \pi x)$, say

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{\cosh \pi x}dx,$$

where $\cosh z = (e^z + e^{-z})/2$. Consider the function $f(z) = e^{-2\pi i z\xi}/\cosh \pi z$, then $f(z)$ has poles at $e^{\pi z} = -e^{-\pi z}$. Equivalently, the poles are given by $ki + i/2$ with $k\in\mathbb{Z}$. Recall that $\cosh \pi z$ is a periodic function with $\cosh \pi(z+2i) = \cosh \pi z$.



We define Γ_R similarly as in Example 4.10. In a single period, $i/2$ and $3i/2$ are only poles. Note that

$$\operatorname{res}_{i/2}(f) = \lim_{z\to i/2}(z-\frac{i}{2})f(z) = \frac{e^{\pi\xi}}{\pi i},$$
$$\operatorname{res}_{3i/2}(f) = \lim_{z\to 3i/2}(z-\frac{3i}{2})f(z) = \frac{e^{3\pi\xi}}{\pi i}.$$

Hence by residue formula,

$$\int_{\Gamma_R} f(z)dz = 2\pi i(\operatorname{res}_{i/2}(f)+\operatorname{res}_{3i/2}(f)) = 2\pi i\cdot\frac{e^{\pi\xi}-e^{3\pi\xi}}{\pi i}.$$

On the other hand, we do the calculation as

$$\int_{\Gamma_R} f(z)dz = \underbrace{\int_{-R}^{R} f(x)dx}_{\mathrm{I}_R} + \underbrace{\int_0^2 f(R+it)dt}_{\mathrm{II}_R} + \underbrace{\int_R^{-R} f(x+2i)dx}_{\mathrm{III}_R} + \underbrace{\int_2^0 f(-R+it)dt}_{\mathrm{IV}_R}.$$

Again, we do similar operations to these parts. Firstly,

$$\begin{aligned}|\mathrm{II}_R| &= \int_0^2 \frac{e^{-2\pi i(R+it)\xi}}{(e^{\pi(R+it)}+e^{-\pi(R+it)})/2}dt\\ &\leqslant 2\int_0^2 \frac{e^{2\pi t\xi}}{e^{\pi R}\underbrace{|e^{\pi it}+e^{-2\pi R-\pi it}|}_{<\infty}}dt\\ &\leqslant C\frac{2e^{4\pi\xi}}{e^{\pi R}}\to 0\end{aligned}$$

as $R\to\infty$. Similarly, $|\mathrm{IV}_R|\to 0$ as well. As for another part,

$$\mathrm{III}_R = -\int_{-R}^{R}\frac{e^{-2\pi i(x+2i)\xi}}{\cosh\pi(x+2i)}dx = -e^{4\pi\xi}\int_{-R}^{R}\frac{e^{-2\pi ix\xi}}{\cosh\pi x}dx = -e^{4\pi\xi}\mathrm{I}_R.$$

Thus, letting $R\to\infty$, we get

$$(1-e^{4\pi\xi})\mathrm{I}_\infty = 2(e^{\pi\xi}-e^{3\pi\xi}).$$

Therefore, the result is given by

$$\int_{-\infty}^{\infty}\frac{e^{-2\pi ix\xi}}{\cosh\pi x}dx = \mathrm{I}_\infty = \frac{2(e^{\pi\xi}-e^{3\pi\xi})}{1-e^{4\pi\xi}} = \frac{1}{\cosh\pi\xi}.$$

This result dictates that the Fourier transform of $1/(\cosh\pi x)$ is itself.

4.3. **Meromorphicity on Singularities.**

4.3.1. *Removable Singularities.*

Theorem 4.12 (Riemann Extension Theorem)**.** *For all $f\in\mathscr{O}(\Omega\backslash\{z_0\})$, f is bounded near z_0 if and only if z_0 is a removable singularity, i.e., f extends to a holomorphic function in Ω. In particular, $f(z_0)$ is well-defined whenever f is bounded near z_0.*

Proof. It is clear that f is bounded near z_0 whenever z_0 is removable. Conversely, choose a sufficiently small open disc $D_R(z_0)\subset\Omega$ such that f is bounded in $D_R(z_0)$. For all $z\in D_R(z_0)$, we define

$$g(z) := \frac{1}{2\pi i}\int_{\partial D}\frac{f(\xi)}{\xi-z}d\xi.$$

Notice that $f(\xi)/(\xi - z)$ is continuous with respect to (ξ, z) and is holomorphic in z. Hence $g(z)$ is holomorphic on $D_R(z_0)$.

In the following context, we will verify that $g(z)$ is the desired extension of f, i.e., $g(z) = f(z)$ away from z_0. Applying Cauchy integral formula to f on $\Omega\backslash\{z_0\}$, then

$$g(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\xi)}{\xi - z}d\xi = \underbrace{\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{f(\xi)}{\xi - z}d\xi}_{I_\varepsilon} + \underbrace{\frac{1}{2\pi i}\int_{\widetilde{\Gamma}_\varepsilon}\frac{f(\xi)}{\xi - z}d\xi}_{f(z)},$$

where Γ_ε and $\widetilde{\Gamma}_\varepsilon$ are the circles centred at z_0 and z with a uniform radius ε, respectively. On right hand side of the equality above the second term is exactly $f(z)$ by Cauchy integral formula again. Also, $f(\xi)$ is bounded above by assumption, and $\xi - z$ is bounded below. Say $|f(\xi)/(\xi - z)| \leqslant M$ for some sufficiently large constant $M < \infty$. Accordingly,

$$|I_\varepsilon| \leqslant \frac{1}{2\pi}\int_0^{2\pi}\varepsilon M d\theta \leqslant C\varepsilon \to 0$$

as $\varepsilon \to 0$. Since $\varepsilon > 0$, we get $g(z) = f(z)$ for $z \neq z_0$. ☐

Corollary 4.13. *Let $f \in \mathscr{O}(\Omega\backslash\{z_0\})$. Then z_0 is a pole of f if and only if $|f(z)| \to \infty$ as $z \to z_0$ (i.e., f is unbounded near z_0).*

Proof. Suppose $|f(z)| \to \infty$ as $z \to z_0$. Then $1/f$ is bounded near z_0. By Theorem 4.12, it turns out that $1/f \in \mathscr{O}(\Omega)$ and $(1/f)(z_0) = 0$. Thus z_0 is a pole. The converse direction is clear. ☐

4.3.2. *Essential Singularities.*

Definition 4.14 (Essential Singularity)**.** Let $f \in \mathscr{O}(\Omega\backslash\{z_0\})$. The point z_0 is called an *essential singularity* of f if z_0 is neither a pole nor a removable singularity.

Example 4.15. As what we have seen in the beginning of Section 4.1, $f(z) = e^{1/z}$ has an essential singularity $z = 0$. We have seen the phenomenon that a function may have various values at an essential singularity attained from various directions.

Theorem 4.16 (Casorati-Weierstrass)**.** *Assume $f \in \mathscr{O}(D_r(z_0)\backslash\{z_0\})$ is defined over the punctured disc, where z_0 is an essential singularity. Then*

$$\overline{f(D_r(z_0)\backslash\{z_0\})} = \mathbb{C}.$$

Namely, the image of f is dense in $\mathbb{C}$.

Proof. Otherwise, there is some $w \in \mathbb{C} - \overline{D_r(z_0)\backslash\{z_0\}}$. Then there exists $\delta > 0$ such that $|f(z) - w| \geqslant \delta$ for any $z \in D_r(z_0)\backslash\{z_0\}$. In particular, we consider $g(z) := 1/(f(z) - w)$, then $|g| \leqslant 1/\delta$ on $D_r(z_0)\backslash\{z_0\}$. In other words, g is bounded near z_0 and is holomorphic on the small punctured region around z_0. From Riemann Extension Theorem 4.12, this implies that z_0 is a removable singularity of g. Here comes two cases:

- $g(z_0) = 0$, then $f(z) - w$ has a pole at z_0;
- $g(z_0) \neq 0$, then $f(z_0)$ is well-defined and z_0 is a removable singularity of f.

Neither the first nor the second case admits the assumption. So we get a contradiction. ☐

The following theorem shows the extreme complexity of essential singularities. Even the holomorphic ones have wild manifestations near this kind of singularity.

Theorem 4.17 (Big Picard Theorem)**.** *Assume $f \in \mathscr{O}(D_r(z_0)\backslash\{z_0\})$ has an essential singularity at z_0, then with at most one exception $w_0 \in \mathbb{C}$,*

$$\forall w \in \mathbb{C}\backslash\{w_0\}, \quad \#\{f^{-1}(w)\} = \infty.$$

4.3.3. *Meromorphicity Versus Rationality.*

Definition 4.18 (Meromorphicity)**.** A function $f : \Omega \to \mathbb{C}$ is called *meromorphic* if there is a sequence of points $\{z_n\}_{n=1}^{\infty} \subset \Omega$ without any limit points in Ω such that

(i) $f \in \mathscr{O}(\Omega \backslash \{z_n\}_{n=1}^{\infty})$, and
(ii) f has poles at $\{z_n\}_{n=1}^{\infty}$.

Recall from topology that the *extended complex plane* is the one-point compactification

$$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = \mathbb{S}^2 \subset \mathbb{R}^3$$

of $\mathbb{C}$ (as a Riemann surface). Note that the punctured 2-dimensional sphere is homeomorphic to $\mathbb{C}$ itself. If f is holomorphic in the set $\{z \in \mathbb{C} \mid |z| > R\}$ then we define

$$F(z) := f(\frac{1}{z}),$$

which is holomorphic in $D_{1/R}(0)\backslash\{0\}$. In convention, f is called to have a pole (resp. removable singularity or essential singularity) at ∞ if F has a pole (resp. removable singularity or essential singularity) at 0.

Definition 4.19. A meromorphic function on $\mathbb{C}$ which is either holomorphic at ∞ or has a pole at ∞ is called a *meromorphic function on* $\overline{\mathbb{C}}$.

Theorem 4.20 (Rationality)**.** *Every meromorphic function on $\overline{\mathbb{C}}$ is a rational function, i.e., the quotient of a polynomial by another polynomial (of any degree).*

Proof. Let f be a meromorphic function on $\overline{\mathbb{C}}$, then by definition $f(1/z)$ is either holomorphic or has a pole at $z = 0$. Thus f has only finite poles at $\overline{\mathbb{C}}$, denoted by $z_1, \dots, z_n \in \mathbb{C}$, since the zeros of $f(1/(z-w))$ must be isolated by Remark 4.7 for some fixed $w \in \overline{\mathbb{C}}$. Assume the principal parts of f at $z_1, \dots, z_n$ are $P_1, \dots, P_n$ for which P_k is a polynomial in $1/(z - z_k)$ (recall Definition 4.6). For z lying sufficiently close to z_k, one may write

$$f(z) = P_k(z) + h_k(z)$$

for some holomorphic function h_k defined near z_k. If ∞ is a pole of f then

$$f(\frac{1}{z}) = \widetilde{P}_\infty(z) + \widetilde{h}_\infty(z),$$

where $\widetilde{P}_\infty(z)$ is a polynomial in $1/z$ and $\widetilde{h}_\infty(z)$ is holomorphic near $z = 0$. Denote $P_\infty(z) = \widetilde{P}_\infty(1/z)$, which is a polynomial in z. Consider

$$H(z) = f(z) - P_\infty(z) - \sum_{k=1}^{n} P_k(z),$$

then $z_1, \dots, z_n$ are removable singularities of H. Thus $H \in \mathscr{O}(\mathbb{C})$ that is also bounded in $\mathbb{C}$. By Liouville (Corollary 3.16), H is a constant, say C. Therefore,

$$f(z) = C + P_\infty(z) + \sum_{k=1}^{n} P_k(z),$$

which is a rational function as required. ☐

In fact, Theorem 4.20 displays a phenomenon of "GAGA," which is the abbreviation of *Géométrie Algébrique et Géométrie Analytique* in French. The GAGA-type conclusions are about some hidden connection between analytic geometry and algebraic geometry.

Examples 4.21. Here comes some meromorphic functions on extended complex plane.

(1) $f(z) = z$ is holomorphic on $\mathbb{C}$ but with a pole at ∞ (since $F(z) := f(1/z) = 1/z$ has a pole at 0).
(2) $f(z) = 1/z$ has a pole at 0 and a zero at ∞.
(3) $f(z) = \sum_{k=0}^{\infty} a_k z^k$ has n zeros in $\mathbb{C}$ and a pole at ∞ of order n.

Corollary 4.22. *Any holomorphic function defined on $\overline{\mathbb{C}}$ (i.e., meromorphic function on $\overline{\mathbb{C}}$ without poles) is a constant.*

4.4. **The Argument Principle and Rouché Theorem.** Here are some observations:

- if f is holomorphic and has a zero z_0 of order n, then $f(z) = (z - z_0)^n g(z)$ with $g(z) \neq 0$ near z_0. Thus
$$\frac{f'}{f}(z) = \frac{n}{z - z_0} + G(z),$$
where $G = g'/g$ is also holomorphic near z_0. Here z_0 is the simple pole with residue n.
- if f is holomorphic in $\Omega\backslash\{z_0\}$ and has a pole z_0 of order n, then $f(z) = (z - z_0)^{-n} g(z)$ again. Thus
$$\frac{f'}{f}(z) = \frac{-n}{z - z_0} + G(z).$$
Here z_0 is the simple pole with residue $-n$.

From these, if f is meromorphic, then f'/f will have simple poles with residues given by the orders. These two extreme cases take care of the numbers of zeros and poles of a given meromorphic function. More generally, we obtain the following result.

Theorem 4.23 (Argument Principle)**.** *Assume f is meromorphic in some open set containing an open disc D, and f has no zeros and poles at ∂D. Then*
$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'}{f}(z)dz = \#\{\textit{zeros of } f \textit{ in } D\} - \#\{\textit{poles of } f \textit{ in } D\}.$$
Here the sizes on the right-hand side are counted with multiplicities.

Remark 4.24. Even without a suitably rigorous definition, one can formally write $f'/f = (\log f)'$, where
$$\log f(z) = \log(|f(z)|e^{i \arg f(z)}) = \log|f(z)| + i \arg f(z).$$
Consequently, we have
$$\frac{f'}{f} = \frac{d}{dz}\log|f(z)| + i\frac{d}{dz}\arg f(z).$$
And then
$$\int_{\partial D} \frac{f'}{f}(z)dz = \int_{f(\partial D)} \frac{1}{w}dw$$
by replacing $w = f(z)$.

As a typical application of the argument principle, the following result is widely used in counting zeros of holomorphic functions.

Corollary 4.25 (Rouché Theorem)**.** *Let $f, g \in \mathscr{O}(D)$ and $|f(z)| > |g(z)|$ on ∂D. Then $f + g$ and f have the same number of zeros in D.*

Proof. Consider the function
$$f_t(z) = f(z) + tg(z), \quad t \in [0, 1].$$
Then $f_0 = f$ and $f_1 = f + g$. Also note that $|f_t| \neq 0$ on ∂D since
$$|f_t(z)| \geqslant ||f(z)| - t|g(z)|| > 0$$
by assumption. Since f and g as well as f_t are holomorphic, the number of poles of f_t is forced to be 0. From the argument principle (Theorem 4.23), we see
$$\frac{1}{2\pi i}\int_{\partial D} \frac{f_t'}{f_t}(z)dz = n_t := \#\{\text{zeros of } f_t \text{ in } D\}.$$
Note that as a function in t, the left-hand side above is continuous with respect to t, whereas the right-hand side only takes values in $\mathbb{N}$ through a discontinuous way. This forces n_t to be a constant for any t. In particular, $n_0 = n_1$ just so f and $f + g$ have the same number of zeros in D. □

The following result is a further application of the Rouché theorem.

Proposition 4.26 (Open Mapping Theorem). *If $f : \Omega \to \mathbb{C}$ is a non-constant holomorphic function defined on an open connected region Ω, then f maps open sets to open sets. Namely, f is open as a map.*

Proof. Assume $w_0 = f(z_0)$ for any fixed $z_0 \in \Omega$. We need to verify if $w \in \mathbb{C}$ is close to w_0, then w also lands in the image of f. Denote
$$g(z) = f(z) - w = \underbrace{(f(z) - w_0)}_{F(z)} + \underbrace{(w_0 - w)}_{G(z)}.$$
Here $G(z)$ is a constant function in z. Choose $0 < \delta \ll 1$ such that $\{|z - z_0| \leqslant \delta\} \subset \Omega$ and $|f(z) - w_0| \geqslant \varepsilon$ for sufficiently small $\varepsilon > 0$ on the circle $C_\delta(z_0)$. The latter condition can be valid as z_0 is an isolated zero of the non-constant holomorphic function $f(z) - w_0$. Once we are given $|w - w_0| < \varepsilon$, by Rouché Theorem (Corollary 4.25), $F(z)$ and $(F + G)(z)$ have the same number of zeros in $C_\delta(z_0)$. As a result, there is $z \in \Omega$ such that $f(z) = w$ because $F(z)$ is already known to have a zero z_0. □

Proposition 4.27 (Maximum Principle). *Let Ω be an open connected region and let $f \in \mathscr{O}(\Omega)$. Then f cannot obtain a maximum in Ω unless f is a constant.*

Proof. Otherwise, there is $z_0 \in \Omega$ such that $|f(z_0)|$ is maximal. By the open mapping theorem (Proposition 4.26), f maps a small disc around z_0 to an open set of $f(z_0)$. This leads to a contradiction. □

Corollary 4.28. *Continuing on Proposition 4.27, assume moreover that Ω is bounded and f is continuous in $\overline{\Omega}$. Then*
$$\sup_{z\in\Omega} |f(z)| \leqslant \sup_{z\in\partial\Omega} |f(z)|.$$

Proof. Since Ω is bounded in $\mathbb{C} \cong \mathbb{R}^2$, we see $\overline{\Omega}$ is compact. The assumption on f deduces that f attains a maximum in $\overline{\Omega}$. Consequently, the maximum principle shows that if f is not a constant, this maximum must lie on $\partial\Omega$. □

Remark 4.29. The bounded requirement for Ω is essential in Corollary 4.28. For example, consider $F(z) = e^{-iz^2}$ defined on $\Omega = \{z = x + iy \mid x, y \geqslant 0\}$; here $|F(z)| = 1$ on $\partial\Omega$ but F is clearly unbounded in Ω.

As for unbounded sets, the Phragmén-Lindelöf theorem (Theorem 4.30) can be regarded as a various version of the maximum principle. We will use it to prove the Paley-Wiener theorem later (see Theorem 5.12).

Theorem 4.30 (Phragmén-Lindelöf)**.** *Suppose $D_\alpha \subset \mathbb{C}$ is an angular region of opening π/α with $\alpha > 1/2$, say,*

$$D_\alpha = \{z = re^{i\theta} \mid |\theta| < \frac{\pi}{2\alpha}, r > 0\}.$$

Let $f \in \mathscr{O}(D_\alpha)$ satisfy the following conditions:

(1) *$|f(z)| \leqslant M$ on ∂D_α;*
(2) *there is $0 < \beta < \alpha$ such that $|f(re^{i\theta})| \leqslant Ce^{r\beta}$ as $r \to \infty$.*

Then $|f(z)| \leqslant M$ for all $z \in D_\alpha$.

Proof. Fix $\gamma > 0$ such that $\beta < \gamma < \alpha$ and define

$$F_\varepsilon(z) = e^{-\varepsilon z^\gamma} f(z)$$

for $\varepsilon > 0$. We obtain

$$|F_\varepsilon(Re^{i\theta})| = e^{-\varepsilon R^\gamma \cos(\gamma\theta)}|f(Re^{i\theta})| \leqslant |f(Re^{i\theta})|$$

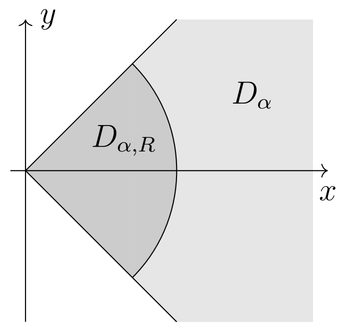
since $\gamma < \alpha$ implies $|\gamma\theta| < \pi\gamma/(2\alpha)$, and then $\cos\gamma\theta > 0$. Therefore,

$$|F_\varepsilon(z)| \leqslant M, \quad \forall z \in \partial D_\alpha.$$

Applying condition (2), since $\gamma > \beta$,

$$|F_\varepsilon(Re^{i\theta})| \leqslant e^{-\varepsilon R^\gamma \cos(\gamma\pi/(2\alpha))} \cdot Ce^{R^\beta} \to 0$$

as $R \to \infty$ for some constant $C < \infty$. By the maximum principle, $|F_\varepsilon(z)| \leqslant M$ for $z \in D_{\alpha,R}$ whenever $R \gg 1$. Here $D_{\alpha,R} = D_\alpha \cap \{|z| < R\}$ is defined as follows.



Letting $R \to \infty$ and we attain that $|F_\varepsilon| \leqslant M$ in D_α. Finally, letting $\varepsilon \to 0$ to get $|f(z)| \leqslant M$ in D_α. $\square$

Through the similar idea as in Theorem 4.30, we also have the result on a doubly infinite strip given as follows.

Theorem 4.31. *Let $S \subset \mathbb{C}$ be a doubly infinite strip, i.e., $S = \{z \in \mathbb{C} \mid -1 \leqslant \Re(z) \leqslant 1\}$. Let $f \in \mathscr{O}(S)$ with $|f(z)| \leqslant M$ for $z \in \partial S$. Assume f is bounded in S. Then $|f(z)| \leqslant M$ for all $z \in S$. Namely, the bound of f on ∂S extends to the interior region.*

Proof. Consider the function $F_\varepsilon(z) = e^{\varepsilon z^2} f(z)$ with $\varepsilon > 0$. Then

$$|F_\varepsilon(x + iy)| = e^{\varepsilon(x^2 - y^2)}|f(x + iy)|.$$

Since f is bounded on ∂S by M, for $T \gg 1$, one obtain

$$|F_\varepsilon(x \pm iT)| \leqslant e^{\varepsilon(x - T^2)}|f(x \pm iT)| \leqslant M.$$

Then $|F(z)| \leqslant M$ for $z \in \partial S_T$, where $S_T = z = x + iy \in \mathbb{C} \mid -1 \leqslant x \leqslant 1, -T \leqslant y \leqslant T$. Applying maximum principle (Proposition 4.27) to F_ε on S_T, we get $|F_\varepsilon(z)| \leqslant M$ in S_T. Finally, letting $T \to \infty$ and $\varepsilon \to 0$ gives $|f(z)| \leqslant M$ in S. □

4.5. **The Complex Logarithm.** The discussion on complex logarithms refers to a subtle phenomenon that the local complex geometry may not be compatible with that globally, even when we care only about the complex plane, which is the simplest geometric object over $\mathbb{C}$. Consider $z = re^{i\theta}$ with $r > 0$. Due to the experience in computing real logarithms, one can formally write

$$\log z := \log r + i\theta.$$

However, the first problem is that θ is not uniquely determined as different θ's can lead to the same value of z up to $2\pi\mathbb{Z}$. Let's make the following observations.

(1) If for some $z_0 \neq 0$, $\log z_0$ is defined, then $\log z$ is well-defined for z lying close to z_0 via the definition above.
(2) $\log z$ can be defined on $\mathbb{C}\backslash[0, \infty)$. Moreover,

$$z = re^{i\theta}, \quad 0 < \theta < 2\pi \quad \Longrightarrow \quad \log z = \log r + i\theta.$$

(3) $\log z$ can be defined on $\mathbb{C}\backslash i(-\infty, 0]$. Moreover,

$$z = re^{i\theta}, \quad -\frac{\pi}{2} < \theta < \frac{3\pi}{2} \quad \Longrightarrow \quad \log z = \log r + i\theta.$$

(4) $\log z$ cannot be well-defined on $\mathbb{C}\backslash\{0\}$.

To sum these observations up, the complex logarithm can be well-defined in some special (simply connected) regions that are not the whole complex plane. The admissible region must be truncated to a single period 2π in θ such that one cannot vary the argument continuously.

Theorem 4.32. *Assume $\Omega \subset \mathbb{C}$ is simply connected such that $0 \notin \Omega$ and $1 \in \Omega$. Then there exists a well-defined holomorphic function*

$$F(z) := \log_\Omega(z) \in \mathscr{O}(\Omega)$$

such that $e^{F(z)} = z$ for all $z \in \Omega$; and for $r > 0$ close to 1, we have $F(r) = \log r$.

Proof. The idea is naive: to construct $F(z)$ as a primitive of $1/z$. Since $0 \notin \Omega$, we see $f(z) = 1/z \in \mathscr{O}(\Omega)$. By the global Cauchy (Theorem 3.10), its primitive $F(z)$ is well-defined on the simply connected region, and moreover,

$$F(z) = \int_{\Gamma_z} f(w)dw$$

that is independent of the choice of the path Γ_z from 1 to z in Ω. Taking $z = 1$, we see $F(z) = F(1) = 0$ simply by definition of F. On the other hand, we note that

$$\frac{d}{dz}(e^{-F(z)}z) = -F'(z)e^{-F(z)}z + e^{-F(z)} = 0,$$

and then $e^{-F(z)}z$ is a constant function $e^{-F(1)} = 1$. Hence

$$e^{F(z)} = z, \quad F(r) = \int_1^r \frac{1}{x}dx = \log r.$$

□

Remark 4.33. If $\log z$ is well-defined in Ω, then for all $\alpha \in \mathbb{C}$, z^α is also well-defined via $z^\alpha := e^{\alpha \log z}$. Simultaneously, even if $\log z$ is well-defined in Ω,

$$\log(z_1 z_2) \neq \log z_1 + \log z_2$$

in general. A counterexample is easy to find: say

$$\log z = \log r + i\theta, \quad z \in \mathbb{C}\backslash(-\infty, 0], \quad -\pi < \theta < \pi$$

and take $z_1 = z_2 = e^{2\pi i/3}$. Then $z_1 z_2 = e^{4\pi i/3} = e^{i(-2\pi/3)}$. In this case,

$$\log(z_1 z_2) = -\frac{2\pi i}{3}, \quad \log z_1 + \log z_2 = \frac{4\pi i}{3}.$$

The following theorem generates Theorem 4.32 about the existence of complex logarithms. The log function is well-defined for an everywhere non-vanishing holomorphic function in a simply connected region.

Theorem 4.34. *Let $\Omega \subset \mathbb{C}$ be simply connected. Assume $f \in \mathscr{O}(\Omega)$ doesn't vanish anywhere in Ω. Then there is $g \in \mathscr{O}(\Omega)$ such that $f(z) = e^{g(z)}$ for all $z \in \Omega$. We denote $g(z) = \log_\Omega f(z)$.*

Proof. Through a similar idea, let's construct g as a primitive of f'/f. Fix $z_0 \in \Omega$ and define

$$g(z) := \int_{\Gamma_z} \frac{f'(w)}{f(w)} dw + C$$

where Γ_z is a path from z_0 to z and C is a constant such that $e^C = f(z_0)$ (or formally, $C = \log f(z_0)$). Again, $g(z)$ is well-defined and $g \in \mathscr{O}(\Omega)$ with $g'(z) = f'(z)/f(z)$. Then

$$\frac{d}{dz}(f(z)e^{-g(z)}) = 0$$

and $f(z)e^{-g(z)}$ is a constant function. Take $z = z_0$ to deduce that $e^{g(z)} = f(z)$. □

5. Fourier Analysis and Complex Analysis

In this chapter, we shall describe some interesting connections between complex function theory and Fourier analysis on the real line. The motivation for this study comes in part from the simple and direct relation between Fourier series on the circle and power series expansions of holomorphic functions in the disc, which we now investigate.

5.1. **Motivation: Mean-value Property.**

Recall Cauchy integral formula (Theorem 3.13 and 3.14) as follows. For $f \in \mathscr{O}(\Omega)$ and $\overline{D_R(z_0)} \subset \Omega$, we obtain

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial D_R(z_0)} \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

Now parametrize $\partial D_R(z_0)$ via $z_0 + Re^{i\theta}$ and we get

(1) (Mean-value Property) For $n = 0$,

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})}{Re^{i\theta}} Rie^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) d\theta. \end{aligned}$$

(2) More generally, for $n > 0$,

$$f^{(n)}(z_0) = \frac{n!}{2\pi R^n} \int_0^{2\pi} f(z_0 + Re^{i\theta}) e^{-in\theta} d\theta.$$

Remark 5.1. As for the case $n < 0$, note that $f(z)/(z-z_0)^{n+1}$ is holomorphic in Ω so its integral along $\partial D_R(z_0)$ vanishes. Hence

$$\int_0^{2\pi} f(z_0 + Re^{i\theta})e^{-in\theta}d\theta = 0, \quad n < 0.$$

Recall that holomorphicity is equivalent to analyticity. Then the (higher) mean-value equations above exactly reveal the coefficients of the analytic expansion at some z_0. Notice that these equations have to do with Fourier transformations on $\mathbb{R}$. To be more precise, the result above can be interpreted as a discrete version of the Fourier transform.

Here comes a quick review of Fourier transforms defined over $\mathbb{R}$. Let f be a nice function on $\mathbb{R}$ (with some decay condition or integrable condition satisfied, say). Its Fourier transform is defined as

$$\widehat{f}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}dx, \quad \xi \in \mathbb{R}.$$

Goal: in the present context, we aim to prove the following correspondence relation. Say the possibility of extending f to a holomorphic function is equivalent to some decay condition of $\widehat{f}$ at ∞. In other words, holomorphicity of a complex-valued function is determined (whereas not over-determined) by its restriction on $\mathbb{R}$ as well as the manifestation of its Fourier transformation at ∞.

Before the theoretical introduction, recall the following basic fact on the Fourier transform. It shows that the inversion of Fourier transformation only drops information on a zero-measure subset of $\mathbb{R}$.

Theorem 5.2 (Fourier Inversion on $\mathbb{R}$)**.** *If $f \in L^1(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$, then*

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi i\xi x}d\xi = f(x) \quad a.e.\ in\ \mathbb{R}.$$

5.2. **The Class $\mathfrak{F}$.** Now we introduce a class of functions that are particularly suited to our goal: proving theorems about the Fourier transform using complex analysis. Moreover, this class will be large enough to contain many essential applications.

Definition 5.3 (Moderate Decay)**.** Let f be a function on $\mathbb{R}$. We call f have moderate decay if

$$|f(x)| \leqslant \frac{A}{1+x^2}$$

for all $x \in \mathbb{R}$. In particular, for f continuous and of moderate decay,

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty.$$

Definition 5.4 (The Class $\mathfrak{F}$)**.** For $a > 0$, denote the class $\mathfrak{F}_a$ to the functions f satisfying the following conditions:

(i) $f \in \mathscr{O}(S_a)$, where $S_a = \{z \in \mathbb{C} \mid |\Im z| < a\}$;
(ii) there is a constant $A > 0$ such that $|f(x+iy)| \leqslant \frac{A}{1+x^2}$ for all $x \in \mathbb{R}$ and $|y| < a$.

And we define the class $\mathfrak{F} = \bigcup_{a>0} \mathfrak{F}_a$.

Examples 5.5. One must intuitively note that the class $\mathfrak{F}$ collects elements that behave well in a sufficiently narrow strip containing $\mathbb{R}$.

(1) $f(z) = e^{-\pi z^2} \in \mathfrak{F}_a$ for any $a > 0$.

(2) For any $0 < a < c$ with fixed constant $c > 0$,

$$f(z) = \frac{c}{c^2 + z^2} \in \mathfrak{F}_a.$$

(3) For any $0 < a < 1/2$,

$$f(z) = \frac{1}{\cosh \pi z} = \frac{2}{e^{\pi z} + e^{-\pi z}} \in \mathfrak{F}_a.$$

Exercise 5.6. Show that for all $f \in \mathfrak{F}_a$ and for any $0 < b < a$, we have $f^{(n)} \in \mathfrak{F}_b$. (Hint: using Cauchy integral formula.)

5.2.1. *Exponential Control for $\mathfrak{F}$.* The following theorem is in preparation for the result by Paley-Wiener in 1934 (Theorem 5.12), which will be useful in finding the support of Fourier transforms.

Theorem 5.7. *If $f \in \mathfrak{F}_a$ then $|\widehat{f}(\xi)| \leqslant Be^{-2\pi b|\xi|}$ for any $\xi \in \mathbb{R}$ and $0 \leqslant b < a$.*

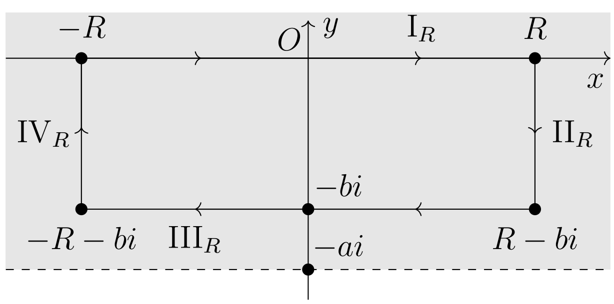
Proof. For any $\xi \in \mathbb{R}$ we obtain by definition that

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi} dx.$$

Consequently, it is bounded as

$$|\widehat{f}(\xi)| \leqslant \int_{-\infty}^{\infty} |f(x)| dx \leqslant \int_{-\infty}^{\infty} \frac{A}{1+x^2} dx \leqslant C$$

for all $\xi \in \mathbb{R}$ and some constants $A, C < \infty$. So the result is true for $b = 0$. Let $0 < b < a$ and denote $g(z) = f(z)e^{-2\pi i z\xi}$. It suffices to consider the case where $\xi > 0$, and the situation for $\xi \leqslant 0$ must be similar. The idea is the same as what we have used twice in Example 4.10 and 4.11 before. Suppose Γ_R is the piecewise-linear closed curve defined as follows.



Note that $g(z)$ is holomorphic in S_a by assumption. Now we obtain

$$0 = \int_{\Gamma_R} g(z) dz = \underbrace{\int_{-R}^{R} g(x) dx}_{\mathrm{I}_R} + \underbrace{\int_{0}^{-b} g(R + it) dt}_{\mathrm{II}_R} + \underbrace{\int_{R}^{-R} g(x - ib) dx}_{\mathrm{III}_R} + \underbrace{\int_{-b}^{0} g(-R + it) dt}_{\mathrm{IV}_R}$$

from the Cauchy integral formula (Theorem 3.13). Firstly, we obtain

$$|\mathrm{II}_R| \leqslant \int_0^{-b} \frac{A}{1+R^2} e^{-2\pi i (R+it)\xi} d\xi \sim \frac{C}{1+R^2} \to 0$$

for some constants $A, C < \infty$ as $R \to \infty$. Similarly, $|\mathrm{IV}_R| \to 0$ as well. Hence the equality above becomes

$$0 = \mathrm{I}_R + \mathrm{III}_R = \widehat{f}(\xi) + \int_{\infty}^{-\infty} f(x - ib) e^{-2\pi i (x - ib)\xi} dx.$$

And therefore,

$$\begin{aligned}|\widehat{f}(\xi)| &= \left|\int_{-\infty}^{\infty} f(x-ib)e^{-2\pi i(x-ib)\xi}dx\right| \\ &\leqslant \int_{-\infty}^{\infty} |f(x-ib)|e^{-2\pi b\xi}dx \\ &\leqslant e^{-2\pi b\xi}\int_{-\infty}^{\infty}\frac{A}{1+x^2}dx = Be^{-2\pi b\xi}.\end{aligned}$$

The remaining proof tackles the case $\xi \leqslant 0$. From this, we get

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x+ib)e^{-2\pi i(x+ib)\xi}dx.$$

Then the same inequality accomplishes the proof. □

The key ingredient in the proof of Theorem 5.7 above lies in the expression of Fourier transformation for $f \in \mathfrak{F}_a$ through a complex integral along some line $y = b$. This idea together with the Fubini theorem in real analysis deduce the following Fourier inversion. Proposition 5.8 is a modified version of Theorem 5.2 in complex analysis, which drops the "almost everywhere" condition.

5.2.2. *Fourier Inversion for $\mathfrak{F}$.*

Proposition 5.8 (Complex Fourier Inversion). *Given $f \in \mathfrak{F}_a$, then*

$$\forall x \in \mathbb{R}, \quad f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi i\xi x}d\xi.$$

Proof. Recall that we can rewrite the Fourier transform as

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x-ib)e^{-2\pi i(x-ib)\xi}dx.$$

Let's first consider the integral along the positive-half part. For $0 < b < a$, we obtain

$$\begin{aligned}\int_0^{\infty}\widehat{f}(\xi)e^{2\pi i\xi x}d\xi &= \int_0^{\infty}\left(\int_{-\infty}^{\infty} f(u)e^{-2\pi u\xi}du\right)e^{2\pi ix\xi}d\xi \\ &= \int_0^{\infty}\left(\int_{-\infty}^{\infty} f(u-ib)e^{-2\pi(u-ib)\xi}du\right)e^{2\pi ix\xi}d\xi \\ &= \int_{-\infty}^{\infty} f(u-ib)\int_0^{\infty} e^{-(2\pi b+2\pi(u-x)i)\xi}d\xi du \\ &= \int_{-\infty}^{\infty} f(u-ib)\cdot\frac{1}{2\pi b+2\pi i(u-x)}du \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(u-ib)}{(u-ib)-x}du \\ &= \frac{1}{2\pi i}\int_{L_1}\frac{f(\xi)}{\xi-x}d\xi.\end{aligned}$$

In the last row, the line $L_1 = \mathbb{R} - ib$. Similarly, for $L_2 = \mathbb{R} + ib$, we also have

$$\int_{-\infty}^{0}\widehat{f}(\xi)e^{2\pi i\xi x}d\xi = \frac{1}{2\pi i}\int_{L_2}\frac{f(\xi)}{\xi-x}d\xi.$$

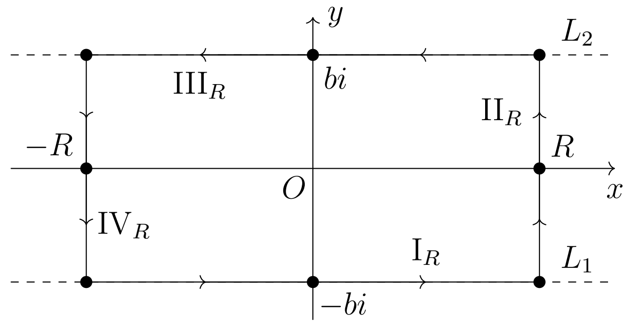
To sum these up, we get

$$\begin{aligned}\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi i\xi x}d\xi &= \int_{0}^{\infty} \widehat{f}(\xi)e^{2\pi i\xi x}d\xi + \int_{-\infty}^{0} \widehat{f}(\xi)e^{2\pi i\xi x}d\xi \\ &= \frac{1}{2\pi i}\int_{L_1+L_2} \frac{f(\xi)}{\xi - x}d\xi.\end{aligned}$$

On the other hand, from the Cauchy integral formula (Theorem 3.13),

$$f(x) = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{f(\xi)}{\xi - x}d\xi = \mathrm{I}_R + \mathrm{II}_R + \mathrm{III}_R + \mathrm{IV}_R,$$

where Γ_R is defined as in the picture. It suffices to show that $|\mathrm{II}_R|, |\mathrm{IV}_R| \to 0$ as $R \to \infty$.



The result in need is deduced from the typical argument. Say

$$|\mathrm{II}_R| = \left|\frac{1}{2\pi i}\int_{-b}^{b} \frac{f(R+it)}{R+it-x}dt\right| \leqslant \frac{1}{2\pi}\int_{-b}^{b} \frac{A}{1+R^2}\cdot\frac{1}{|R-x|}dt \leqslant \frac{C}{R^3} \to 0$$

for some constant $A, C < \infty$ as $R \to \infty$. Similarly, $|\mathrm{IV}_R| \to 0$ as well. Therefore,

$$f(x) = \lim_{R\to\infty} \frac{1}{2\pi i}\int_{\Gamma_R} \frac{f(\xi)}{\xi - x}d\xi = \frac{1}{2\pi i}\int_{L_1+L_2} \frac{f(\xi)}{\xi - x}d\xi = \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi i x\xi}d\xi.$$

This completes the proof. □

5.2.3. *Poisson Summation Formula for $\mathfrak{F}$.*

Theorem 5.9 (Complex Poisson Summation Formula)**.** *Given $f \in \mathfrak{F}_a$, we have*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n).$$

Proof. Consider the function $f(z)/(e^{2\pi iz} - 1)$, it has simple poles at every $n \in \mathbb{Z}$ with

$$\mathrm{res}_n \frac{f(z)}{e^{2\pi iz} - 1} = \lim_{z\in n}(z-n)\frac{f(z)}{e^{2\pi iz} - 1} = \frac{f(n)}{2\pi i}.$$

Now we fix $N \in \mathbb{N}$ and apply the residue formula on Γ_N. Here we keep the statement as in the proof of Proposition 5.8: Γ_N is defined by the picture above with $R = N + 1/2$. Hence

$$\int_{\Gamma_N} \frac{f(z)}{e^{2\pi iz} - 1}dz = 2\pi i \sum_{\text{pole } x\in\Gamma_N} \mathrm{res}_x \frac{f(z)}{e^{2\pi iz} - 1} = \sum_{|n|\leqslant N} f(n).$$

Claim: the integrals on the vertical segments tends to 0 as $N \to \infty$.

To show this claim, letting $N \to \infty$, we get

$$\lim_{N\to\infty} \int_{\Gamma_N} \frac{f(z)}{e^{2\pi i z}-1} dz = \lim_{N\to\infty} \sum_{|n|\leqslant N} f(n) = \sum_{n\in\mathbb{Z}} f(n).$$

Apply the same argument as in Proposition 5.8, this leads to

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{L_1} \frac{f(z)}{e^{2\pi i z}-1} dz + \int_{L_2} \frac{f(z)}{e^{2\pi i z}-1} dz.$$

We have some observations as follows.

- on $L_1 = \mathbb{R} - ib$, we see $e^{2\pi i z} = e^{2\pi i(x-ib)} = e^{2\pi b} e^{2\pi i x}$, thus $|e^{2\pi i z}| = e^{2\pi b} > 1$. Therefore,
$$\frac{1}{e^{2\pi i z}-1} = e^{-2\pi i z} \frac{1}{1-e^{-2\pi i z}} = e^{-2\pi i z} \sum_{n=0}^{\infty} e^{-2\pi i n z} = \sum_{n=0}^{\infty} e^{-2\pi i (n+1) z}.$$
- on $L_2 = \mathbb{R} + ib$, similarly, $|e^{2\pi i z}| = e^{-2\pi b} < 1$. Therefore,
$$\frac{1}{e^{2\pi i z}-1} = -\frac{1}{1-e^{2\pi i z}} = -\sum_{n=0}^{\infty} e^{2\pi i n z}.$$

So our calculation can be done:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} f(n) &= \int_{L_1} f(z) \sum_{n=0}^{\infty} e^{-2\pi i (n+1) z} dz - \int_{L_2} f(z) \sum_{n=0}^{\infty} e^{2\pi i n z} dz \\
&= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} f(x-ib) e^{-2\pi i (n+1)(x-ib)} dx + \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} f(x+ib) e^{-2\pi i (-n)(x+ib)} dx \\
&= \sum_{n=0}^{\infty} \widehat{f}(n+1) + \sum_{n=0}^{\infty} \widehat{f}(-n) = \sum_{n\in\mathbb{Z}} f(n).
\end{aligned}$$

This proves the Poisson summation formula. □

There are two precise applications of Theorem 5.9. It is used to deduce more formulas.

Example 5.10 (Functional Equation)**.** Recall that for $f(x) = e^{-\pi x^2}$, its Fourier transform is itself (Example 3.12):

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi x \xi} dx = e^{-\pi \xi^2}.$$

Thus, for $F(x) = e^{-\pi t (x+a)^2}$ with $t > 0$ and $a \in \mathbb{R}$, we have

$$\widehat{F}(\xi) = \int_{-\infty}^{\infty} F(x) e^{-2\pi i x \xi} dx = t^{-1/2} e^{2\pi i a \xi} e^{-\pi \xi^2/t}$$

The Poisson summation formula deduces that

$$\sum_{n\in\mathbb{Z}} e^{-\pi t (n+a)^2} = t^{-1/2} \sum_{n\in\mathbb{Z}} e^{-\pi n^2/t} \cdot e^{2\pi i a n}.$$

In particular, letting $a = 0$ and denoting $\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi t n^2}$ for $t > 0$, we get

$$\theta(t) = t^{-1/2} \theta(\frac{1}{t}).$$

This is an important functional equation in analytic number theory, which is relevant to the Riemann hypothesis.

Example 5.11. Recall Example 4.11 in which we have shown that $f(x) = 1/\cosh \pi x$ takes itself as its Fourier transform. One can also show that

$$F(x) = \frac{e^{-2\pi iax}}{\cosh(\pi x/t)}, \quad \widehat{F}(\xi) = \frac{t}{\cosh(\pi(\xi + a)t)}$$

for all $t > 0$ and $a \in \mathbb{R}$. Again, by Poisson summation formula, one deduces that

$$\sum_{n\in\mathbb{Z}} \frac{e^{-2\pi ian}}{\cosh(\pi n/t)} = \sum_{n\in\mathbb{Z}} \frac{t}{\cosh(\pi(n+a)t)}.$$

5.3. **Paley-Wiener Theorem.**

Theorem 5.12 (Paley-Wiener, 1934)**.** *Suppose $f : \mathbb{C} \to \mathbb{C}$ is continuous and of moderate decay on $\mathbb{R}$, i.e., for all $x \in \mathbb{R}$, $|f(x)| \leqslant A/(1 + x^2)$. Then the following are equivalent:*

(1) *f has an extension to a holomorphic function on $\mathbb{C}$ with $|f(z)| \leqslant Ae^{2\pi M|z|}$ for some constants $A, M > 0$, and for all $z \in \mathbb{C}$.*
(2) *$\widehat{f}$ is supported on $[-M, M]$, i.e., $\widehat{f}(\xi) = 0$ for any $|\xi| > M$.*

Proof. The converse direction (2) $\Rightarrow$ (1) is relatively easy. Suppose $\widehat{f}$ is supported on $[-M, M]$, then f is of moderate decay implies that the Fourier inversion for $\mathfrak{F}$ (Proposition 5.8) holds for f. In particular,

$$\forall x \in \mathbb{R}, \quad f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi ix\xi}d\xi = \int_{-M}^{M} \widehat{f}(\xi)e^{2\pi ix\xi}d\xi.$$

Let's define the complex-valued function

$$g(z) := \int_{-M}^{M} \widehat{f}(\xi)e^{2\pi iz\xi}d\xi.$$

Then $g \in \mathscr{O}(\mathbb{C})$ and $g(x) = f(x)$ over $\mathbb{R}$. Moreover, for any $z = x + iy$, we obtain

$$|g(z)| = \left|\int_{-M}^{M} \widehat{f}(\xi)e^{2\pi iz\xi}d\xi\right| \leqslant \int_{-M}^{M} |\widehat{f}(\xi)|e^{-2\pi y\xi}d\xi \leqslant Ae^{2\pi M|z|}$$

for some constant A. The last inequality above is given the exponential control (Theorem 5.7). Now we prove (1) $\Rightarrow$ (2) step by step.

Step 1: Stronger Growth Condition.

Assume $f \in \mathscr{O}(\mathbb{C})$ is controlled by a stronger growth condition, say

$$|f(x + iy)| \leqslant A'\frac{e^{2\pi M|y|}}{1 + x^2}$$

for some $A' > 0$. The claim is that $\widehat{f}(\xi) = 0$ whenever $|\xi| > M$.

(i) $\xi > M$: a similar computation through the Cauchy integral formula as if in the proof of Theorem 5.7 deduces that

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}dx = \int_{-\infty}^{\infty} f(x - iy)e^{-2\pi i(x-iy)\xi}dx$$

for all $y > 0$. Applying the stronger growth condition, we attain that

$$|\widehat{f}(\xi)| \leqslant \int_{-\infty}^{\infty} \frac{A'}{1 + x^2}e^{2\pi My - 2\pi y\xi}dx = \int_{-\infty}^{\infty} \frac{A'}{1 + x^2}dx e^{2\pi y(M-\xi)} \to 0$$

as $y \to \infty$, because of $y > 0$ and $M - \xi < 0$. Thus for $\xi > M$, $|\widehat{f}(\xi)| = 0$.

(ii) $\xi < -M$: same as in (i). One can compute

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}dx = \int_{-\infty}^{\infty} f(x+iy)e^{-2\pi i(x+iy)\xi}dx$$

for all $y > 0$ again. It can be verified that

$$|\widehat{f}(\xi)| \leqslant Ce^{2\pi y(\xi+M)} \to 0$$

for some constant C as $y \to \infty$. Thus for $\xi < -M$, $|\widehat{f}(\xi)| = 0$.

Step 2: Relaxing the Growth Condition.

Take $f \in \mathscr{O}(\mathbb{C})$ such that $|f(x+iy)| \leqslant Ae^{2\pi M|y|}$. The claim is that $\widehat{f}(\xi) = 0$ for all $|\xi| > M$ as well.

(i) $\xi > M$: consider for $\varepsilon > 0$ that

$$f_\varepsilon(z) := \frac{f(z)}{(1+i\varepsilon z)^2}.$$

It suffices to verify the following two facts. Firstly, the function f_ε satisfies the stronger growth condition in Step 1. That is,

$$|f_\varepsilon(x+iy)| \leqslant D\frac{e^{2\pi M|y|}}{1+x^2}$$

for another constant D. Applying the argument that we have used, we can immediately get $\widehat{f_\varepsilon}(\xi) = 0$ for $|\xi| > M$. Secondly, check that $\widehat{f_\varepsilon}(\xi) \to \widehat{f}(\xi)$ as $\varepsilon \to 0$. These are relatively easy to do (so we choose to omit the details).

(ii) $\xi < -M$: consider for $\varepsilon > 0$ that

$$f_\varepsilon(z) := \frac{f(z)}{(1-i\varepsilon z)^2}.$$

One can verify the conditions as in (i) again.

Step 3: Applying Phragmén-Lindelöf Maximum Principle.

We aim to prove that if $|f(x)| \leqslant 1$ for $x \in \mathbb{R}$ and $|f(z)| \leqslant e^{2\pi M|z|}$ for all $z \in \mathbb{C}$, then

$$|f(x+iy)| \leqslant e^{2\pi M|y|}.$$

For this, we consider the function

$$F(z) := f(z)e^{2\pi iMz}.$$

On $Q_1 = \{(x,y) \in \mathbb{R}^2 \mid x, y \geqslant 0\}$, we have

$$|F(x)| = |f(x)| \leqslant 1, \quad |F(iy)| = |f(iy)e^{-2\pi My}| \leqslant 1.$$

Hence $|F(z)| \leqslant 1$ for all $z \in \partial Q_1$. Also, the condition $|f(z)| \leqslant e^{2\pi M|z|}$ yields that

$$|F(z)| \leqslant e^{4\pi M|z|}$$

for all $z \in Q_1$. Now by the Phragmén-Lindelöf maximum principle (Theorem 4.30),

$$\forall z \in Q_1, \quad |F(z)| \leqslant 1.$$

Hence $|f(z)| \leqslant e^{-2\pi My}$ for all $z \in Q_1$. Applying the same argument to other quadrant closure Q_2, Q_3, and Q_4, we finally have $|f(z)| \leqslant e^{-2\pi My}$ for all $z \in \mathbb{C}$. Furthermore, note that the condition $|f(x)| \leqslant 1$ can be dropped without changing anything essentially. The result is naturally generated to

$$|f(z)| \leqslant e^{-2\pi M|z|}$$

for all $z \in \mathbb{C}$ as desired. □

Remark 5.13. The moderate decay condition for f in Theorem 5.12 can be replaced by some integrable property of f to attain a more general version.

6. Entire Function

Recall in Corollary 3.16 that f is called *entire* if $f \in \mathscr{O}(\mathbb{C})$. In Subsection 4.3.2 and 4.3.3, we have seen the complexity of the manifestation of the meromorphics at the infinity. In this chapter we are going to construct the hidden connection between the growth of f at ∞ and the zeros of f on $\mathbb{C}$. Morally, the intuition which will be proved is that the faster it growths at ∞, the more zeros it contains (Theorem 6.7). This result is compatible with the fundamental theorem of algebra (Theorem 3.18).

Also, it turns out that if an entire function has a finite (exponential) order of growth, then it can be specified by its zeros up to multiplication by a simple factor. The precise version of this assertion is the *Hadamard factorization theorem* (Theorem 6.13). It may be viewed as another instance of the general rule that was formulated before: under appropriate conditions, a holomorphic function is essentially determined by its zeros (Theorem 6.10).

6.1. Jensen's Formula.

Jensen's formula, central to much of the theory developed in this section, exhibits a deep connection between the number of zeros of a function in a disc and the (logarithmic) average of the function over the circle. In fact, Jensens formula not only constitutes a natural starting point for us, but also leads to the fruitful theory of value distributions, also called *Nevanlinna theory*.

The following result says that for a well-behaved holomorphic function on a disc, its central logarithmic value is and its logarithmic average along the boundary circle are almost mutually determined, where the difference is given by some information about zeros.

Theorem 6.1 (Jensen)**.** *Let $\Omega \subset \mathbb{C}$ be an open subset and let $D_R := D_R(0) \subset \Omega$. Suppose $f \in \mathscr{O}(\Omega)$ satisfies $f(0) \neq 0$ and is nonzero along ∂D_R. Assume $z_1, \dots, z_N$ are the zeros of f in D_R counted with multiplicities. Then*

$$\log|f(0)| = \sum_{k=1}^{N} \log|\frac{z_k}{R}| + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta.$$

Proof. Note that the formula is stable under additive. That is, if the result holds for f_1 and f_2 simultaneously, then it holds for $f_1 \cdot f_2$ as well. Denote

$$g(z) = \frac{f(z)}{\prod_{k=1}^{N}(z - z_k)},$$

then every z_k is a removable singularity of g. Thus $g \in OO(D_R)$ and $g \neq 0$ in D_R everywhere. Hence we may write

$$f(z) = g(z) \cdot \prod_{k=1}^{N}(z - z_k).$$

It suffices to verify the formula for g without zeros, that is, to show

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|d\theta.$$

If $g \neq 0$ in D_R, then $\log|g(z)|$ is harmonic in D_R and we can apply the mean-value property for harmonic functions. Furthermore, suppose D_R is simply connected and then $h = \log g$ is well-defined in D_R with $e^h = g$ by Theorem 4.32. Hence

$$\log|g(z)| = \Re|h(z)|.$$

This suggests us to apply the mean-value property to h and to take real parts. As a result, the Jensen's formula is valid for $g(z)$. On the other hand, let's check for the function $z-w$, where $w \in D_R$. One may need to prove

$$\log|w| = \log\frac{|w|}{R} + \frac{1}{2\pi}\int_0^{2\pi}\log|Re^{i\theta}-w|d\theta.$$

This is equivalent to say

$$\int_0^{2\pi}\log|e^{i\theta}-\frac{w}{R}|d\theta = 0.$$

Claim: for all $|a|<1$ we have

$$\int_0^{2\pi}\log|e^{i\theta}-a|d\theta = 0 \quad \Longleftrightarrow \quad \int_0^{2\pi}\log|1-ae^{i\theta}|d\theta = 0.$$

Proof of the Claim. For this, consider the function $F(z)=1-az$. Then $F\neq 0$ in the unit disc $\mathbb{D}$. Hence there is some G being holomorphic in $\mathbb{D}$ such that $e^G = F$ by Theorem 4.32 again. Thus,

$$\log|F| = \log|1-az| = \Re G(z).$$

Finally, applying the mean-value property to G, we get

$$0 = \log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log|F(e^{i\theta})|d\theta = \frac{1}{2\pi}\int_0^{2\pi}\log|1-ae^{i\theta}|d\theta.$$

This is enough to complete the proof by taking some sufficiently large R. □

In fact, the holomorphicity assumption in Theorem 6.1 can be dropped to deduce a general version of Jensen's formula.

Theorem 6.2 (General Jensen's Formula)**.** *Let $\Omega\subset\mathbb{C}$ be an open subset and $D_R := D_R(0)\subset\Omega$. Let f be a meromorphic function in Ω. Counting with multiplicities, assume $a_1,\dots,a_N$ are zeros and $b_1,\dots,b_N$ are poles of f in $\overline{D_R}$, respectively. Then for all $z\in D_R$ with $f(z)\neq 0$ and $f(z)\neq\infty$, we have*

$$\begin{aligned}\log|f(z)| =& \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|\cdot\frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}d\theta\\ &-\sum_{i=1}^N\log\left|\frac{R^2-\overline{a_i}z}{R(z-a_i)}\right| + \sum_{j=1}^M\log\left|\frac{R^2-\overline{b_j}z}{R(z-b_j)}\right|.\end{aligned}$$

Exercise 6.3. Prove Theorem 6.2 with a similar approach as in the proof of classical Jensen's formula. Consider the function

$$\psi_\alpha(z) := \frac{R^2-\overline{\alpha}z}{R(z-\alpha)}, \quad \alpha\in D_R.$$

First prove the result for $f(z)\cdot\prod_{i=1}^N\psi_{a_i}(z)\cdot(\prod_{j=1}^M\psi_{b_j}(z))^{-1}$.

6.2. **Zeros and the Order of Growth.** In the present context, we are doing some preparation works for the ultimate goal of this section: to construct the connection between zeros and the speed of growth at the infinity. Given $f\in\mathscr{O}(D_R)$, we denote

$$n_f(r) := \#\{z\in D_r \mid f(z)=0\} = \#(f^{-1}(0)\cap D_r),$$

counted with multiplicities.

Proposition 6.4. *Keep the same statement of Theorem 6.1 on f and Ω. Then*
$$\int_0^R n_f(r)\frac{dr}{r} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta - \log|f(0)|.$$

Proof. By Jensen's formula, we only need to verify
$$\int_0^R n_f(r)\frac{dr}{r} = \sum_{k=1}^N \log\frac{R}{|z_k|},$$
where $z_1, \dots, z_N$ are zeros of f in D_R. Let's define
$$\eta_k(r) = \begin{cases} 1, & |z_k| < r; \\ 0, & |z_k| \geqslant r. \end{cases}$$
Thus $n_f(r) = \sum_{k=1}^N \eta_k(r)$. On the other hand, we obtain
$$\int_0^R \eta_k(r)\frac{dr}{r} = \int_{|z_k|}^R \frac{dr}{r} = \log\frac{R}{|z_k|}.$$
Therefore,
$$\int_0^R n_f(r)\frac{dr}{r} = \int_0^R \sum_{k=1}^N \eta_k(r)\frac{dr}{r} = \sum_{k=1}^N \int_0^R \eta_k(r)\frac{dr}{r} = \sum_{k=1}^N \log\frac{R}{|z_k|}.$$
Note that the key point of this proof lies in the case of a single zero. □

Now we are defining the order of growth, which is to be applied at the infinity later.

Definition 6.5 (Order of Growth)**.** Given $f \in \mathscr{O}(\mathbb{C})$, if there exists some $\rho > 0$ and constants $A, B > 0$ such that $|f(z)| \leqslant Ae^{B|z|^\rho}$ for any $z \in \mathbb{C}$, i.e., $\log|f(z)| \leqslant B|z|^\rho + O(1)$ where $O(1)$ denotes a bounded term, then we call f has *order of growth* at most ρ. Then take $\rho_f := \inf\rho$ for all such ρ. And ρ_f is called *the order of growth* of f.

Examples 6.6. The subtlety in Definition 6.5 is that the order of growth is possibly not precise. For example, if f is a polynomial in z, then $\rho_f = 0$ whereas $|f(z)|$ cannot be bounded by $Ae^B < \infty$. Similarly, one can show that if $f(z) = \exp e^z$, then $\rho_f = \infty$. For a more prototypical example, consider $f(z) = e^z$ whose $\rho_f = 1$.

Theorem 6.7. *Let $f \in \mathscr{O}(\mathbb{C})$ with order of growth $\rho_f \leqslant \rho$. Then*

(1) *$n_f(r) \leqslant Cr^\rho$ for some $C > 0$ and $r \gg 1$;*
(2) *if $\{z_n\}_{n=1}^\infty$ are zeros of f with $z_k \neq 0$ for any k, then for all $s > \rho$ we have*
$$\sum_{k=1}^\infty \frac{1}{|z_k|^s} < \infty.$$

Proof. If $f(0) = 0$ then consider $f(z)/z^m$, where $m = \mathrm{ord}_0(f)$. So we may assume $f(0) \neq 0$ for convenience and by Proposition 6.4,
$$\int_0^R n_f(r)\frac{dr}{r} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta - \log|f(0)| \leqslant Ar^\rho$$
for some constant $A > 0$ by assumption. Let $R = 2r$ and note that
$$n_f(r)\log 2 = n_f(r)\int_r^{2r}\frac{ds}{s} \leqslant \int_r^{2r} n_f(s)\frac{ds}{s} \leqslant \int_0^R n_f(s)\frac{ds}{s}.$$

So there exists another constant $C > 0$ such that $n_f(r) \leqslant Cr^\rho$ for $r \gg 1$. This gives (1) as required. As for (2), we obtain

$$\sum_{|z_k|\geqslant 1} |z_k|^{-s} = \sum_{j=0}^{\infty} \sum_{2^j \leqslant |z_k| < 2^{j+1}} |z_k|^{-s} \leqslant \sum_{j=0}^{\infty} 2^{-js} n_f(2^{j+1})$$
$$\leqslant C \cdot \sum_{j=0}^{\infty} 2^{-js} 2^{(j+1)\rho} \leqslant C_1 \cdot \sum_{j=0}^{\infty} (2^{\rho-s})^j < \infty.$$

The last inequality in the first and the second row above respectively uses (1) and $\rho < s$. □

6.3. **Infinite Product.** A natural question is whether or not, given any sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$, there exists an entire function f with zeros precisely at the points of this sequence. A necessary condition is that $\{z_n\}_{n=1}^{\infty}$ do not accumulate, in other words we must have $\lim_{k\to\infty} |z_k| = \infty$, otherwise f would vanish identically by the analytic continuation (Theorem 3.22). Weierstrass proved that this condition is also sufficient by explicitly constructing a function with these prescribed zeros. A first guess is of course the product

$$\prod_{n=1}^{\infty} (z - z_n)$$

when the sequence of zeros is finite. In general, Weierstrass inserted factors in this product so that the convergence is guaranteed, yet no new zeros are introduced.

Before coming to the general construction, we review infinite products and study a basic example. Given $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$, say the product *converges* if the limit $\lim_{N\to\infty} \prod_{n=1}^{N}(1 + a_n)$ exists.

Proposition 6.8. *Whenever $\sum_{n=1}^{\infty} |a_n| < \infty$, the product $\prod_{n=1}^{N}(1 + a_n)$ converges and vanishes if and only if some factor $1 + a_k = 0$.*

Proof. For $|z| < 1$ we have the logarithmic expansion

$$\log(1 + z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}.$$

Suppose $\sum_{n=1}^{\infty} |a_n| < \infty$ and then $|a_n| < 1/2$ for $n \geqslant N_0 \gg 0$. Consequently, $\log(1 + a_n)$ is well-defined with $e^{\log(1+a_n)} = 1 + a_n$. Then we do calculation as

$$\prod_{n=1}^{\infty}(1 + a_n) = \prod_{n=1}^{N_0}(1 + a_n) \cdot \prod_{n=N_0+1}^{\infty} (1 + a_n)$$
$$= \prod_{n=1}^{N_0}(1 + a_n) \cdot \prod_{n=N_0+1}^{\infty} \exp(\log(1 + a_n))$$
$$= \prod_{n=1}^{N_0}(1 + a_n) \cdot \exp(\sum_{n=N_0+1}^{\infty} \log(1 + a_n)).$$

Note that $|\log(1 + z)| \leqslant 2|z|$ for $|z| < 1/2$. So there exists some constant B such that

$$\sum_{n=N_0+1}^{\infty} |\log(1 + a_n)| \leqslant \sum_{n=N_0+1}^{\infty} 2|a_n| \to B < \infty$$

by assumption. Hence the infinite product factors through a finite product as

$$\prod_{n=1}^{\infty}(1+a_n) = \prod_{n=1}^{N_0}(1+a_n)\cdot e^B$$

and the product is zero if and only if one of these factors is 0. $\square$

Proposition 6.9. *Let $\Omega \subset \mathbb{C}$ be an open subset and $F_n \in \mathscr{O}(\Omega)$. Assume there are $c_n > 0$ such that $\sum_{n=1}^{\infty} c_n < \infty$ and $|F_n(z) - 1| \leqslant c_n$ for all $z \in \Omega$. Then*

(1) *$\prod_{n=1}^{\infty} F_n(z) \to F(z)$ uniformly (with respect to z) on Ω for some $F \in \mathscr{O}(\Omega)$.*
(2) *If $F_n \neq 0$ for every $n \geqslant 1$ then for any $z \in \Omega$,*

$$\frac{F'}{F}(z) = \sum_{n=1}^{\infty}\frac{F_n'}{F_n}(z).$$

Proof. For (1), we can write $\prod_{n=1}^{\infty} F_n(z) = \prod_{n=1}^{\infty} 1 + (F_n(z) - 1)$. And for (2), just use the formula

$$\frac{(f\cdot g)'}{f\cdot g} = \frac{f'}{f} + \frac{g'}{g}.$$

The undisclosed details are left to readers. $\square$

Let's introduce the main theorem by Weierstrass, which dictates the existence of an entire function that vanishes at a given infinite sequence exactly. Moreover, such entire function is unique up to an exponential factor.

Theorem 6.10 (Weierstrass Infinite Product)**.** *Given $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ with $|a_n| \to \infty$ as $n \to \infty$. Then there exists some $f \in \mathscr{O}(\mathbb{C})$ with the zeros exactly at $z = a_n$. Any other such entire function is of the form $f(z)e^{g(z)}$, where $g \in \mathscr{O}(\mathbb{C})$.*

Before proving this, a lemma at work about *canonical factors* is in display.

Definition 6.11. For $k \geqslant 0$ we define the *canonical factors* as

$$E_0(z) = 1 - z, \quad E_k(z) = (1-z)\exp(\sum_{n=1}^{k}\frac{z^n}{n}).$$

Lemma 6.12. *If $|z| \leqslant 1/2$, then*

$$|1 - E_k(z)| \leqslant c|z|^{k+1}$$

for some $c > 0$ that is independent of k.

Proof. Whenever $|z| \leqslant 1/2$, we have $1 - z = \exp(\log(1-z))$ where

$$\log(1-z) = -\sum_{n=1}^{\infty}\frac{z^n}{n}.$$

Thus we can write for all $k \geqslant 1$ that

$$E_k(z) = \exp(\log(1-z) + \sum_{n=1}^{k}\frac{z^n}{n}) = \exp(-\sum_{n\geqslant k+1}\frac{z^n}{n}) = e^{w(z)}.$$

Here $|w(z)|$ is bounded as follows:

$$|w(z)| \leqslant |z|^{k+1}\sum_{n\geqslant k+1}\frac{|z|^{n-k+1}}{n} \leqslant |z|^{k+1}\sum_{j=0}^{\infty}(\frac{1}{2})^j = 2|z|^{k+1}$$

because of $|z| \leqslant 1/2$. In particular, $|w(z)| \leqslant 1$. Therefore,
$$|1 - E_k(z)| = |1 - e^{w(z)}| \leqslant e|w| \leqslant 2e|z|^{k+1}.$$
For the middle inequality above, recall that $e^w = \sum_{n=0}^{\infty} w^n/n!$, and then
$$|e^w - 1| \leqslant |w| \sum_{n=1}^{\infty} \frac{|w|^{n-1}}{n!} \leqslant |w| \sum_{n=1}^{\infty} \frac{1}{n!} = e|w|.$$
Taking $c = 2e$ does finish the proof. □

Now we move to the construction by Weierstrass.

Proof of Theorem 6.10. **Step 1: Existence.**

The most naive idea is to consider the infinite product
$$\prod_{n=1}^{\infty} (1 - \frac{z}{a_n}).$$
However, this product does not converge in general. Fortunately, the result is differed from this by another exponential factor. Obtaining the canonical factors (Definition 6.11), we move to
$$f(z) := z^m \prod_{n=1}^{\infty} E_n(\frac{z}{a_n}) = z^m \prod_{n=1}^{\infty} (1 - \frac{z}{a_n}) \exp(\sum_{k=1}^{n} \frac{(z/a_n)^k}{k}), \quad m \geqslant 0.$$
Claim: *$f \in \mathscr{O}(\mathbb{C})$ has a zero at $z = 0$ of order m and zeros at each a_n, but nowhere else.*
Proof of the Claim. For this, we first check that f is holomorphic in every disc $D_R(0)$ for $R > 0$. Write
$$\prod_{n=1}^{\infty} E_n(\frac{z}{a_n}) = \prod_{|a_n| \leqslant 2R} E_n(\frac{z}{a_n}) \prod_{|a_n| > 2R} E_n(\frac{z}{a_n}).$$
The motivation to consider this truncated product is that as $|a_n| \to \infty$, the finite part vanishes at $z = a_n$ for $|a_n| < R$ in $D_R(0)$, and the infinite part is convergent. Now for $z \in D_R$ and $|a_n| > 2R$, we have $|z/a_n| < 1/2$. Thus,
$$|1 - E_n(\frac{z}{a_n})| \leqslant c|\frac{z}{a_n}|^{n+1} \leqslant c(\frac{1}{2})^{n+1}$$
by Lemma 6.12. Now by Proposition 6.9, the infinite part $\prod_{|a_n| > 2R} E_n(z/a_n)$ converges uniformly to some holomorphic function in $D_R(0)$. Letting $R \to \infty$ finishes the proof of existence.
Step 2: Uniqueness.

This is relatively easy. If f_1 and f_2 are two such functions, then f_1/f_2 is holomorphic in $\mathbb{C}$ and $f_1/f_2 \neq 0$. Since $\mathbb{C}$ is simply connected, there exists some $g \in \mathscr{O}(\mathbb{C})$ such that $f_1/f_2 = e^g$ by Theorem 4.32. □

6.4. **Hadamard Factorization Theorem.** The main result: if an entire function has a finite (exponential) order of growth, then it can be specified by its zeros.

Recall Definition 6.5 that if $f \in \mathscr{O}(\mathbb{C})$ has finite order of growth, denoted by ρ_f, then for any $\varepsilon > 0$,
$$\log|f(z)| \leqslant A_\varepsilon |z|^{\rho_f + \varepsilon} + O(1)$$
as $|z| \to \infty$. In fact, if the sequence of zeros is given, say $\{a_n\}_{n=1}^{\infty} = f^{-1}(0)$, then
$$\sum_{a_n \neq 0} \frac{1}{|a_n|^{\rho_f + \varepsilon}} < \infty.$$

Theorem 6.13 (Hadamard Factorization). *Let $f \in \mathscr{O}(\mathbb{C})$ has a growth order $\rho_f < \infty$ and take $k = [\rho_f]$ as the integer part of ρ_f. If $\{a_n\}_{n=0}^{\infty}$ are the zeros of f that are away from 0, then*

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k(\frac{z}{a_n})$$

where P is a polynomial of degree at most k, and $m = \mathrm{ord}_0(f)$.

Proof. For $z \in D_R(0)$ we write

$$\prod_{n=1}^{\infty} E_k(\frac{z}{a_n}) = \prod_{|a_n| \leqslant 2R} E_k(\frac{z}{a_n}) \cdot \prod_{|a_n|>2R} E_k(\frac{z}{a_n}).$$

On the right hand side, the first term is a product of finite terms with zeros $z = a_n$ for $|a_n| < R$ in the disc D_R. Moreover, there is a constant C such that the second term satisfies

$$\prod_{|a_n|>2R} E_k(\frac{z}{a_n}) \leqslant C|\frac{z}{a_n}|^{k+1} \leqslant CR^{k+1}\frac{1}{|a_n|^{k+1}}$$

by Lemma 6.12 (because of $|z/a_n| < 1/2$ for $z \in D_R$). By Theorem 6.7 (2),

$$\sum_{n \geqslant 1} \frac{1}{|a_n|^{k+1}} < \infty,$$

which implies that $\prod_{n=1}^{\infty} E_k(z/a_n)$ is holomorphic in D_R. Letting $R \to \infty$, we define

$$E(z) = z^m \prod_{n=1}^{\infty} E_k(\frac{z}{a_n}) \in \mathscr{O}(\mathbb{C}).$$

Note that this function has the same zeros as $f(z)$. Therefore, the function $f(z)/E(z) \in \mathscr{O}(\mathbb{C})$ vanishes nowhere, that is, there is some $g(z) \in \mathscr{O}(\mathbb{C})$ such that $f(z)/E(z) = e^{g(z)}$. So

$$f(z) = e^{g(z)} z^m \prod_{n=1}^{\infty} E_k(\frac{z}{a_n}).$$

Now it suffices to control $g(z)$ by a polynomial of degree at most k.
Claim: *if f has the growth order ρ_f, then for all $s > \rho_f$, there is a constant C such that*

$$\prod_{n=1}^{\infty} E_k(\frac{z}{a_n}) \geqslant \exp(-C|z|^s)$$

on $|z| = r_m \to \infty$ as $m \to \infty$.
Assuming the claim, we get on $|z| = r_m \to \infty$ that

$$|e^{g(z)}| = e^{\Re(g)} = |\frac{f(z)}{E(z)}| \leqslant \frac{A\exp(B|z|^s)}{\exp(-C|z|^s)}$$

for some constants A, B. So $\Re(g)(z) \leqslant C|z|^s$ on $|z| = r_m \to \infty$. Using this condition, it can be shown that $g(z)$ is a polynomial of degree $\leqslant s$ (as an exercise). Let $s \to \rho_f$, we get $g(z)$ is a polynomial of degree $\leqslant k = [\rho_f]$. The proof of claim is omitted for convenience. □

Examples 6.14. There are some basic examples as applications of Theorem 6.13.

(1) $f(z) = e^z - 1$.
It is an entire function with $\rho_f = 1$ and $m = \mathrm{ord}_0(z) = 1$ with zeros at $z = 2\pi i n$ for all $n \in \mathbb{Z}$. Applying Hadamard factorization, we obtain

$$e^z - 1 = e^{az+b} z \prod_{n\in\mathbb{Z}\setminus\{0\}} (1 - \frac{z}{2\pi i n}) \exp(\frac{z}{2\pi i n}) = e^{az+b} z \prod_{n=1}^{\infty} (1 + \frac{z^2}{4\pi^2 n^2}$$

for some $a, b \in \mathbb{C}$. We use the following recipe to determine these constants:

$$\lim_{z\to 0} \frac{e^z - 1}{z} = 1 \quad \Longrightarrow \quad b = 0;$$

also, the infinite product is an even function with respect to z, which means that

$$\frac{e^z - 1}{e^{az} \cdot z} = \frac{e^{-z} - 1}{e^{-az} \cdot (-z)} \quad \Longrightarrow \quad a = \frac{1}{2}.$$

Therefore, the factorization of $f(z)$ is read as

$$e^z - 1 = e^{z/2} z \prod_{n=1}^{\infty} (1 + \frac{z^2}{4\pi^2 n^2}).$$

(2) $f(z) = \sin(\pi z)$.
It is apparent that $\rho_f = 0$ and $m = \mathrm{ord}_0(z) = 1$ with zeros at $z = n$ for all $n \in \mathbb{Z}$. Consequently,

$$\sin(\pi z) = e^a z \prod_{n\in\mathbb{Z}\setminus\{0\}} (1 - \frac{z}{n}) = e^a z \prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2})$$

for some constant $a \in \mathbb{C}$. Similarly, by considering

$$\lim_{z\to 0} \frac{\sin(\pi z)}{z} = \pi = e^a,$$

we get the desired factorization.

6.5. **Divisors.** Let $f \in \mathscr{O}(\mathbb{C})$ be a nonzero function. We define the divisor of f to describe its zeros and poles.

Definition 6.15 (Zero Divisor)**.** The following formal sum of points in $\mathbb{C}$ is called the *zero divisor* associated to f, say

$$Z(f) := \sum_{f(a)=0} \mathrm{ord}_a(f) \cdot a.$$

Here the sum runs through all points $a \in \mathbb{C}$ such that $f(a) = 0$.

Comparing with Theorem 6.10, we have the following neat result.

Theorem 6.16. *Given a discrete set $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$, there is an entire function $f \in \mathscr{O}(\mathbb{C})$ such that $\{a_n\}_{n=1}^{\infty}$ are exactly all the zeros of f (counted with multiplicity).*

Collecting the information in $\{a_n\}_{n=1}^{\infty}$ as a formal sum of points in $\mathbb{C}$, say

$$\sum_{k=1}^{\infty} m_k \cdot P_k, \quad m_k \in \mathbb{N},$$

then the theorem implies that this formal sum can be realized as $Z(f)$ for some $f \in \mathscr{O}(\mathbb{C})$.

Definitions 6.17 (Divisors)**.** A $\mathbb{Z}$-coefficient *divisor* in $\mathbb{C}$ is a formal sum

$$D = \sum_{k=1}^{\infty} m_k \cdot P_k$$

with $m_k \in \mathbb{Z}$, where the set $\{P_k\}_{k=1}^{\infty} \subset \mathbb{C}$ is discrete. A divisor is *effective* if all $m_k \geqslant 0$.

Let f be a meromorphic function in $\mathbb{C}$. Then the *divisor associated to* f is defined to be

$$(f) = Z(f) + P(f) = \sum_{f(a)=0} \operatorname{ord}_a(f) \cdot a + \sum_{f(a)=\infty} \operatorname{ord}_a(f) \cdot a.$$

The following theorem shows that divisors with $\mathbb{Z}$-coefficients in $\mathbb{C}$ are in a one-to-one correspondence with meromorphic functions on $\mathbb{C}$.

Theorem 6.18. *For any divisor D with $\mathbb{Z}$-coefficients in $\mathbb{C}$, there exists a meromorphic function f on $\mathbb{C}$ such that $D = (f)$.*

6.6. **Nevanlinna Theory.** Recall that Jensen's formula reveals a hidden connection between the number of zeros of a function in a disc and the (logarithmic) average of the function over the circle. Starting from the following *Poisson-Jensen formula*, which is a variant of Jensen's formula, we construct the theory developed by Nevanlinna in 1925.

Let f be a meromorphic function in $\overline{D_R}$, then for $z \in D_R$ with $f(z) \neq 0$ and $f(z) = \infty$, we have

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| \Re\left(\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\right) d\theta - \sum_{a \in D_R} \operatorname{ord}_a(f) \log|B_{R,a}(z)|,$$

where $B_{R,a}(z) = (R^2 - \overline{a}z)/R(z-a)$. In particular, if $z = 0$ is neither a zero nor a pole, i.e., it satisfies $f(0) \neq 0$ and $f(0) \neq \infty$, then

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| d\theta - \sum_{a \in D_R} \operatorname{ord}_a(f) \log\left|\frac{R}{a}\right|.$$

In general, if for those z landing near $z = 0$ we have an expansion $f(z) = c_f z^m + \cdots$ with $c_f \neq 0$, then by applying the equation above to $f(z)/z^m$ we get

$$(*) \qquad \log|c_f| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| d\theta - \sum_{0 \neq a \in D_R} \operatorname{ord}_a(f) \log\left|\frac{R}{a}\right| - m \log R.$$

We now introduce the number of poles of f in $\overline{D_r}$ (counted with multiplicity), say

$$n_f(r) = n_f(r, \infty) := \#(f^{-1}(\infty) \cap \overline{D_r}).$$

For $a \in \mathbb{C}$, we also define the number of solutions of $f(z) = a$ in $\overline{D_r}$ (counted with multiplicity) by

$$n_f(r, a) := n_{\frac{1}{f-a}}(r, \infty).$$

In particular, $n_f(0,0) - n_f(0,\infty) = m = \operatorname{ord}_0(f)$ is the difference of zeros and poles of f at $z = 0$. Using these sense, we are clear for the motivation of Nevanlinna's definition for the counting function.

Definition 6.19 (Nevanlinna Counting Function)**.** For $r > 0$ we define (for the second and the third terms in $(*)$) that

$$N_f(r) = N_f(r, \infty) := \sum_{0 \neq a \in D_r, f(a) = \infty} (-\operatorname{ord}_a(f)) \cdot \log\left|\frac{r}{a}\right| + n_f(0, \infty) \log r,$$

and

$$N_f(r,0) := \sum_{0\neq a\in D_r, f(a)=0} \mathrm{ord}_a(f)\cdot \log|\frac{r}{a}| + n_f(0,0)\log r.$$

Using the expressions of $N_f(r,\infty)$ and $N_f(r,0)$, Jensen's formula $(*)$ can be written as

$$(**)\qquad \log|c_f| + N_f(R,0) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta + N_f(R,\infty).$$

In fact, there is an explicit expression of $N_f(r)$ whose proof is leave as an exercise.

Proposition 6.20. *We have the equality*

$$N_f(r) = \int_0^r \frac{n_f(t)-n_f(0)}{t}dt + n_f(0)\log r.$$

Definition 6.21 (Proximity Function)**.** Let f be a meromorphic function in $\overline{D_R}$. Then for $0 < r < R$, we define

$$m_f(r) = m_f(r,\infty) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|d\theta.$$

Here $\log^+\alpha := \max(0,\log\alpha)$ for $\alpha > 0$. Also, for $a\in\mathbb{C}$, we define

$$m_f(r,a) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta})-a|}d\theta.$$

Remark 6.22. Note that $\log\alpha = \log^+\alpha - \log^+(1/\alpha)$ and $|\log\alpha| = \log^+\alpha + \log^+(1/\alpha)$.

Definition 6.23 (Nevanlinna Height Function)**.** For $r > 0$ we define

$$T_f(r) = T_f(r,\infty) := N_f(r,\infty) + m_f(r,\infty).$$

Note that the height function is the "pole part" of the right hand side of $(*)$. Again, by Jensen's formula $(**)$,

$$\begin{aligned}\log|c_f| + N_f(R,0) &= \frac{1}{2\pi}\int_0^{2\pi}\log^+|f(Re^{i\theta})|d\theta - \frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|f(Re^{i\theta})|}d\theta + N_f(R,\infty)\\ &= m_f(R,\infty) - m_{\frac{1}{f}}(R,\infty) + N_f(R,\infty).\end{aligned}$$

This is equivalent to

$$\log|c_f| + N_{\frac{1}{f}}(R,\infty) + m_{\frac{1}{f}}(R,\infty) = m_f(R,\infty) + N_f(R,\infty).$$

Therefore,

$$(1)\qquad \log|c_f| + T_{\frac{1}{f}}(R) = T_f(R).$$

Now let $a\in\mathbb{C}$. Applying Jensen's formula to $f(z)-a$, we get

$$\begin{aligned}\log|c_{f-a}| + N_{f-a}(R,0) &= \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})-a|d\theta + N_f(R,\infty)\\ &= N_f(R,\infty) + \frac{1}{2\pi}\int_0^{2\pi}\log^+|f(Re^{i\theta})-a|d\theta\\ &\quad - \frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|f(Re^{i\theta})-a|}d\theta.\end{aligned}$$

Consequently,

$$\log|c_{f-a}| + N_{\frac{1}{f-a}}(R) + m_{\frac{1}{f-a}}(R) = N_f(R) + \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(Re^{i\theta}) - a| d\theta.$$

Note that

$$\log^+(\alpha_1 + \cdots + \alpha_n) \leqslant \max_{1\leqslant i\leqslant n}(\log^+ \alpha_i) + \log n \leqslant \sum_{i=1}^n \log^+ \alpha_i + \log n.$$

In particular,

$$\begin{aligned} \log^+|f-a| &\leqslant \log^+|f| + \log^+|a| + \log 2, \\ \log^+|f| &\leqslant \log^+|f-a| + \log^+|a| + \log 2. \end{aligned}$$

So we get

$$T_{f-a}(R) = T_f(R) + O_a(1), \tag{2}$$

where $O_a(1)$ denotes a bounded term depending on a.

Theorem 6.24 (The First Main Theorem of Nevanlinna Theory)**.** *Let $R > 0$ and f be a meromorphic function defined on $\overline{D_R}$. Then*

(1) $\log|c_f| + T_{\frac{1}{f}}(R) = T_f(R)$;
(2) $T_{f-a}(R) = T_f(R) + O_a(1)$.

Theorem 6.25 (Cartan)**.** *Keep the same statement. We obtain*

$$\begin{aligned} T_f(r) &= \frac{1}{2\pi}\int_0^{2\pi} N_f(r, e^{i\theta}) d\theta + C \\ &= \frac{1}{2\pi}\int_0^{2\pi} N_f(r, e^{i\theta}) d\theta + \begin{cases} \log^+|f(0)|, & f(0) \neq \infty, \\ \log|c_f|, & f(0) = \infty. \end{cases} \end{aligned}$$

Proof Idea. For example, if $f(0) \neq \infty$, then apply Jensen's formula to $f(z) - e^{i\theta}$ and then integrate with respect to θ. □

Recall the definitions of n_f and N_f, we see they are increasing functions. Hence by Theorem 6.25, $T_f(r)$ is an increasing function with respect to r as well, and is convex with respect to $\log r$.

Theorem 6.26. *Let f be a meromorphic function on $\mathbb{C}$.*

(1) *If $T_f(R)$ is bounded as $R \to \infty$, then f is a constant.*
(2) *$T_f(R) \sim O(\log R)$ as $R \to \infty$ if and only if f is rational on $\mathbb{C}$.*

Let $f \in \mathscr{O}(\overline{D_r})$. Define $M_f(r) := \log\|f\|_r$, where $\|f\|_r = \sup_{|z|\leqslant r}|f(z)| = \sup_{|z|=r}|f(z)|$. Then

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})| d\theta \leqslant \max(M_f(r), 0).$$

Lemma 6.27. *Let $f \in \mathscr{O}(\overline{D_R})$, then for $0 < r < R$ we have*

$$M_f(r) \leqslant \frac{R+r}{R-r} m_f(R, \infty) - \frac{R-r}{R+r} m_f(R, 0) \leqslant \frac{R+r}{R-r} m_f(R).$$

Proof. Applying Jensen's formula to f (which is holomorphic), we get

$$\begin{aligned}\log|f(z)| &= \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|\Re(\frac{Re^{i\theta}+z}{Re^{i\theta}-z})d\theta - \sum_{a\in D_R,\mathrm{ord}_a(f)>0}\mathrm{ord}_a(f)\cdot\log|B_{R,a}(z)| \\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|\Re(\frac{Re^{i\theta}+z}{Re^{i\theta}-z})d\theta\end{aligned}$$

for z with $f(z)\neq 0$. Now for $z=re^{i\theta}$ and $r<R$, we have

$$\frac{R-r}{R+r}\leqslant\Re(\frac{Re^{i\theta}+z}{Re^{i\theta}-z})\leqslant\frac{R+r}{R-r}.$$

We write $\log|f(Re^{i\theta})| = \log^+|f(Re^{i\theta})| - \log^+\frac{1}{|f(Re^{i\theta})|}$. This completes the proof. □

Corollary 6.28. *For f that is holomorphic in $\overline{D_{2r}}$, we have*

$$M_f(r)\leqslant 3m_f(2r,\infty) = 3T_f(2r).$$

The latter equality holds because of the holomorphicity.

Proof Idea of Theorem 6.26. (1) Applying Liouville's Theorem (Corollary 3.16) is enough.

(2) The direction $\Rightarrow$ is easy by Cartan's Theorem 6.25. As for $\Leftarrow$, use the definitions of $T_f(R)$, $N_f(R)$ and so on. □

7. The Gamma and Zeta Functions

7.1. **The Gamma Function.** For $s>0$ we define

$$\Gamma(s) := \int_0^\infty e^{-t}t^{s-1}dt.$$

7.1.1. *Analytic Continuation.*

Proposition 7.1. *$\Gamma(\cdot)$ extends to a holomorphic function in the right-half plane $\Re(s)>0$ by replacing s by complex numbers.*

Proof. For $\varepsilon>0$ we define

$$F_\varepsilon(s) = \int_\varepsilon^{\frac{1}{\varepsilon}} e^{-t}t^{s-1}dt,$$

then $F_\varepsilon(\cdot)$ is holomorphic with respect to $s\in\mathbb{C}$.

Claim: *on every strip $S_{\delta,M}=\{s\in\mathbb{C}:\delta<\Re(s)<M\}$, the series of functions $F_\varepsilon\to\Gamma$ converges uniformly as $\varepsilon\to 0$.*

Proof of Claim. Note that for $t>0$ with any $s\in\mathbb{C}$, we have $t^s=\exp(s\log t)=\exp(\Re(s)\cdot\log t)\cdot\exp(i\Im(s)\log t)$. Hence $|t^s| = e^{\Re(s)\cdot\log t} = t^{\Re(s)}$. Now we denote $\sigma=\Re(s)$. Then

$$\begin{aligned}|F_\varepsilon(s)-\Gamma(s)| &= |\int_0^\varepsilon e^{-t}t^{s-1}dt + \int_{\frac{1}{\varepsilon}}^\infty e^{-t}t^{s-1}dt| \\ &\leqslant \int_0^\varepsilon e^{-t}t^{\sigma-1}dt + \int_{\frac{1}{\varepsilon}}^\infty e^{-t}t^{\sigma-1}dt \\ &\leqslant \int_0^\varepsilon t^{\sigma-1}dt + \int_{\frac{1}{\varepsilon}}^\infty e^{-t}t^{M-1}dt \\ &\leqslant \frac{\varepsilon^\delta}{\delta} + \int_{\frac{1}{\varepsilon}}^\infty e^{-t}t^{M-1}dt \to 0\end{aligned}$$

as $\varepsilon \to 0$. So we have proved the claim.

Now the claim implies that Γ is naturally a holomorphic function in $S_{\delta,M}$. This completes the proof. □

Proposition 7.2. *For $\Re(s) > 0$ we have $\Gamma(s+1) = s\Gamma(s)$. In particular, $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$.*

Proof. Using the formula

$$\int_{\varepsilon}^{\frac{1}{\varepsilon}} \frac{d}{dt}(e^{-t}t^{s-1})dt = -\int_{\varepsilon}^{\frac{1}{\varepsilon}} e^{-t}t^{s}dt + s\int_{\varepsilon}^{\frac{1}{\varepsilon}} e^{-t}t^{s-1}dt$$

and letting $\varepsilon \to 0$, we get

$$0 = -\Gamma(s+1) + s \cdot \Gamma(s),$$

that is, $\Gamma(s+1) = s\Gamma(s)$. In particular, since $\Gamma(1) = 1$, we get $\Gamma(n+1) = n!$. □

For $\Re(s) > 0$, we have $\Gamma(s) = \Gamma(s+1)/s$ by Proposition 7.1. And for $\Re(s) > -1$ we have $\Re(s+1) > 0$, which deduces that $\Gamma(s+1)/s$ is well-defined on $\Re(s) > -1$. So we define

$$F_1(s) := \frac{\Gamma(s+1)}{s}, \quad \Re(s) > -1.$$

Then $F_1(\cdot)$ is a meromorphic function on $\{s \in \mathbb{C} : \Re(s) > -1\}$ with a simple pole at $s = 0$, and

$$\operatorname{res}_0 F_1 = \lim_{s\to 0}(s-0)F_1(s) = \lim_{s\to 0}\Gamma(s+1) = \Gamma(1) = 1.$$

Also, we note that $F_1(s) = \Gamma(s)$ when $\Re(s) > 0$, i.e., F_1 is an analytic extension of Γ.

For $\Re(s) > -2$, we also define

$$F_2(s) := \frac{F_1(s+1)}{s} = \frac{\Gamma(s+2)}{(s+1)s},$$

then $F_2(\cdot)$ is meromorphic in $\{s \in \mathbb{C} : \Re(s) > -2\}$ and $F_2(s) = \Gamma(s)$ for $\Re(s) > 0$.

Now by induction, for $\Re(s) > -m$ where $m \in \mathbb{N}$, we define

$$F_m(s) := \frac{F_{m-1}(s)}{s} = \cdots = \frac{\Gamma(s+m)}{(s+m-1)(s+m-2)\cdots s}.$$

Then $F_m(\cdot)$ extends $\Gamma(\cdot)$ to a meromorphic function on $\Re(s) > -m$, with simple poles at $\delta = 0, -1, \cdots, -(m-1)$. Moreover,

$$\operatorname{res}_{s=-n} F_m = \lim_{s\to -n}(s+n)F_m(s) = \frac{(-1)^n}{n!}, \quad 0 \leqslant n \leqslant m-1.$$

Therefore, we have proved the following theorem.

Theorem 7.3 (Analytic Continuation)**.** *The Gamma function $\Gamma(\cdot)$ that is initially holomorphically defined on $\{s \in \mathbb{C} : \Re(s) > 0\}$ has an analytic continuation to a meromorphic function on $\mathbb{C}$ (which we denote by Γ as well), whose only singularities are simple poles at $s = 0, -1, \cdots, -m, \cdots$ with $\operatorname{res}_{-m}\Gamma = (-1)^m/m!$ for all $m \in \mathbb{N}$.*

Remark 7.4. The analytic continuation of Theorem 7.3 is unique, since $\mathbb{C}\backslash\{0, -1, \cdots, -m, \cdots\}$ is topologically connected.

Morally, the Gamma function $\Gamma(s)$ can be almost realized as a holomorphic function, and the only problem lies in the neighborhood of $s = 0$.

Proposition 7.5. *For $s \in \mathbb{C}$ such that $\Re(s) > 0$, we have*

$$\Gamma(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(s+k)} + \int_1^{\infty} e^{-t} t^{s-1} dt.$$

Proof. We do the computation directly. Fix some $\varepsilon > 0$,

$$\begin{aligned}
\Gamma(s) &= \int_0^{\infty} e^{-t} t^{s-1} dt \\
&= \int_0^{\varepsilon} e^{-t} t^{s-1} dt + \int_{\varepsilon}^{\infty} e^{-t} t^{s-1} dt \\
&= \int_0^{\varepsilon} t^{s-1} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} dt + \int_{\varepsilon}^{\infty} e^{-t} t^{s-1} dt \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^{\varepsilon} t^{k+s-1} dt + \int_{\varepsilon}^{\infty} e^{-t} t^{s-1} dt \\
&= \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k \varepsilon^{s+k}}{k!(s+k)}}_{\text{meromorphic}} + \underbrace{\int_{\varepsilon}^{\infty} e^{-t} t^{s-1} dt}_{\text{holomorphic}}.
\end{aligned}$$

In particular, by taking $\varepsilon = 1$, we get the desired result. ☐

7.1.2. *The Symmetry Property.*

Theorem 7.6 (Gamma Symmetry)**.** *For all $s \in \mathbb{C}$, we have*

$$\Gamma(s) \cdot \Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

In particular, for $s = 1/2$, we get $\Gamma(1/2) = \sqrt{\pi}$.

Proof. By analytic continuation (Theorem 7.3), we only need to check the formula on $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$. For $0 < \Re(s) < 1$, we have

$$\begin{aligned}
\Gamma(s) \cdot \Gamma(1-s) &= \int_0^{\infty} e^{-t} t^{s-1} dt \cdot \int_0^{\infty} e^{-u} u^{-s} du \\
&= \int_0^{\infty} e^{-t} t^{s-1} (\int_0^{\infty} e^{-u} u^{-s} du) dt \\
&= \int_0^{\infty} e^{-t} t^{s-1} (\int_0^{\infty} e^{-vt} (vt)^{-s} t du) dt \\
&= \int_0^{\infty} \int_0^{\infty} e^{-(1+v)t} v^{-s} dv dt \\
&= \int_0^{\infty} \frac{v^{-s}}{1+v} dv \\
&= \int_{-\infty}^{\infty} \frac{e^{(1-s)x}}{1+e^x} dx.
\end{aligned}$$

Here the change of variants are $u = vt$ and $v = e^x$ with $t > 0$. Recall that in Example 4.10, for $0 < a < 1$,

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin \pi a}.$$

Therefore, the desired integral is

$$\int_{-\infty}^{\infty} \frac{e^{(1-s)x}}{1+e^x} dx = \frac{\pi}{\sin \pi(1-s)} = \frac{\pi}{\sin \pi s}.$$

This completes the proof. $\square$

Remark 7.7. Note that for all $s \in \mathbb{C}$, we have $\Gamma(s) \neq 0$.

7.1.3. *The Growth of Gamma Functions.*

Theorem 7.8. *The function $1/\Gamma(\cdot)$ enjoys the following properties.*

(1) $1/\Gamma(\cdot) \in \mathscr{O}(\mathbb{C})$ *has simple zeros at $s = 0, -1, \ldots$, and it vanishes nowhere else.*
(2) *The order of growth of $1/\Gamma(\cdot)$ is 1, and for all $s \in \mathbb{C}$,*

$$\left|\frac{1}{\Gamma(s)}\right| \leqslant C_1 \exp(C_2 |s| \log |s|)$$

for some constants C_1 and C_2.

Proof. (1) By Theorem 7.6, the symmetry of Γ shows that

$$\frac{1}{\Gamma(s)} = \Gamma(1-s) \cdot \frac{\sin \pi s}{\pi},$$

where $\Gamma(1-s)$ has simple poles at $s = 1, 2, \ldots$ and $\sin \pi s / \pi$ has simple zeros at $s \in \mathbb{Z}$. Since $\Gamma(s) \neq 0$ for all $s \in \mathbb{C}$, we see $1/\Gamma(\cdot)$ is holomorphic in $\mathbb{C}$ with the only zeros at $s = 0, -1, \ldots$, which are all simple.

(2) Again, by Theorem 7.6,

$$\begin{aligned}
\frac{1}{\Gamma(s)} &= \Gamma(1-s) \cdot \frac{\sin \pi s}{\pi} \\
&= \left(\int_0^1 e^{-t} t^{-s} dt + \int_1^\infty e^{-t} t^{-s} dt\right) \cdot \frac{\sin \pi s}{\pi} \\
&= \underbrace{\frac{\sin \pi s}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1-s)}}_{\mathrm{I}} + \underbrace{\frac{\sin \pi s}{\pi} \int_1^\infty e^{-t} t^{-s} dt}_{\mathrm{II}}
\end{aligned}$$

For I, the trouble term is on

$$\frac{1}{n+1-s} = \frac{1}{n+1-\Re(s) - i\Im(s)}.$$

If $|\Im(s)| > 1$, then

$$\left|\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1-s)}\right| \leqslant C$$

for some constant C. Otherwise $|\Im(s)| \leqslant 1$, in this case $1/(n+1-s)$ can be infinite when $s = n+1$. For this, note that given any s, we have some k such that $k - 1/2 \leqslant \Re(s) < k + 1/2$. When $k \leqslant 0$,

$$|n+1-s| = |n+1-\Re(s) - i\Im(s)| \geqslant \frac{1}{2} \quad \Longrightarrow \quad \left|\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1-s)}\right| \leqslant C.$$

When $k > 0$, we have $n - k + 1/2 \leqslant n + 1 - \Re(s) \leqslant n - k + 3/2$. The case is valid for $n \neq k-1$ because of $|n+1-\Re(s)| \geqslant C$ for some C that is independent of k. It boils down to tackle to the case where $n = k - 1 > -1$. We obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1-s)} \frac{\sin \pi s}{\pi} = \underbrace{(-1)^{k-1} \frac{\sin \pi s}{(k-1)!(k-s)\pi}}_{\mathrm{A}} + \underbrace{\sum_{n \neq k-1} \frac{(-1)^n}{n!(n+1-s)} \frac{\sin \pi s}{\pi}}_{\mathrm{B}}.$$

In fact, the part A is bounded from above because of

$$\left|\frac{\sin \pi s}{s-k}\right| = \left|\frac{\sin \pi (s-k)}{s-k}\right| = \left|\frac{\sin \pi \xi}{\xi}\right|$$

for $\xi = s - k$. This is bounded on $\xi \in \{s \in \mathbb{C} : |\Re(s)| \leqslant 1, |\Im(s)| \leqslant 1\}$. On the other hand, by Euler's formula, we see

$$\sin \pi s = \frac{e^{i\pi s} - e^{-i\pi s}}{2i} \quad \Longrightarrow \quad |\sin \pi s| \leqslant e^{\pi |s|}.$$

Hence the part B is bounded by some $Ce^{\pi|s|}$. To sum up, $|\mathrm{I}| < \infty$.

As for II, since

$$\left|\int_1^{\infty} e^{-t} t^{-s} dt\right| \leqslant \int_1^{\infty} e^{-t} t^{\Re(s)} dt \leqslant \exp((|\Re(s)|+1) \cdot \log(|\Re(s)|+1)),$$

there is a constant C' such that

$$|\mathrm{II}| \leqslant \exp(C|s| \log|s|) \cdot \exp(C|s|) \leqslant \exp(C'|s| \log|s|).$$

Consequently,

$$|\mathrm{I} + \mathrm{II}| \leqslant C_1 \exp(C_2 |s| \log|s|).$$

for some constants C_1, C_2 that are independent of s. ☐

Starting with Theorem 7.8, the Hadamard factorization (Theorem 6.13) shows that

$$\frac{1}{\Gamma(s)} = e^{As+B} \cdot s \cdot \prod_{n=1}^{\infty} (1 + \frac{s}{n}) e^{-s/n}.$$

Here A, B are constants (to be determined). Note that

$$\lim_{s \to 0} \Gamma(s) \cdot s = \lim_{s \to 0} \Gamma(s+1) = \Gamma(1) = 1 \quad \Longrightarrow \quad 1 = \lim_{s \to 0} e^{As+B} \prod_{n=1}^{\infty} (1 + \frac{s}{n}) e^{-s/n} = e^B.$$

Hence $B = 0$. Letting $s = 1$, the equation becomes

$$1 = e^A \prod_{n=1}^{\infty} (1 + \frac{1}{n}) e^{-1/n} \quad \Longrightarrow \quad e^{-A} = \prod_{n=1}^{\infty} (1 + \frac{1}{n}) e^{-1/n}.$$

To compute A, we note that

$$\prod_{n=1}^{\infty} (1 + \frac{1}{n}) e^{-1/n} = \lim_{N \to \infty} \prod_{n=1}^{N} (1 + \frac{1}{n}) e^{-1/n} = \lim_{N \to \infty} \exp(\sum_{n=1}^{N} (\log(1 + \frac{1}{n}) - \frac{1}{n})).$$

Hence as $N \to \infty$,

$$\sum_{n=1}^{N}(\log(1+\frac{1}{n})-\frac{1}{n}) = -\sum_{n=1}^{N}\frac{1}{n}+\log\frac{2}{1}+\log\frac{3}{2}+\cdots+\log\frac{N}{N-1}+\log\frac{N+1}{N}$$
$$= \underbrace{-(\sum_{n=1}^{N}\frac{1}{n}-\log N)}_{\gamma}+\underbrace{\log\frac{N+1}{N}}_{\to 0} \to -\gamma,$$

where γ is the Euler constant. So we have $A = r$. We have proved the following theorem.

Theorem 7.9. *For all $s \in \mathbb{C}$, we obtain*

$$\frac{1}{\Gamma(s)} = e^{\gamma s}\cdot s\cdot\prod_{n=1}^{\infty}(1+\frac{s}{n})e^{-s/n}.$$

Here γ denotes the Euler constant.

7.2. **Riemann Zeta Function.** For $s \in \mathbb{R}$ satisfying $s > 1$, it is well-known that the series

$$\zeta(s) := \sum_{n=1}^{\infty}\frac{1}{n^s}$$

is convergent. By replacing the real number s by any $s \in \mathbb{C}$, we get the definition of *Riemann Zeta Function.*

Proposition 7.10. *The Riemann zeta function $\zeta(s)$ converges on $\{s \in \mathbb{C} : \Re(s) > 1\}$ and converges uniformly on $\{s \in \mathbb{C} : \Re(s) \geqslant 1+\delta\}$ for any $\delta > 0$. In particular, $\zeta(s)$ is holomorphic on $\{s \in \mathbb{C} : \Re(s) > 1\}$.*

Proof. Write $s = \sigma + it$, then $|1/n^s| = 1/n^\sigma$. For $\sigma \geqslant 1+\delta$ with $\delta > 0$, we have

$$|\sum_{n=1}^{\infty}\frac{1}{n^s}| \leqslant \sum_{n=1}^{\infty}\frac{1}{n^\sigma} \leqslant \sum_{n=1}^{\infty}\frac{1}{n^{1+\delta}}$$

in which the right item is called convergent. □

7.2.1. *The Zeta, Gamma and Theta Functions.*

Definition 7.11 (The Theta Function)**.** For a real number $t > 0$ we define

$$\Theta(t) := \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}.$$

We the list out some basic properties of $\Theta(t)$.

Proposition 7.12. *For the real variable $t > 0$, we have:*

1. *$\Theta(t) \leqslant Ct^{-1/2}$ for some constant $C > 0$ as $t \to 0^+$.*
2. *$|\Theta(t) - 1| \leqslant Ce^{-\pi t}$ for some constant $C > 0$ and any $t \geqslant 1$.*

Proof. (1) Recall the Poisson summation formula (Theorem 5.9) dictates that

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n),$$

where $\widehat{f}$ is the Fourier transform of f. Consider $f(x) = \exp(-\pi t(x+a)^2)$ with $t > 0$ and $a \in \mathbb{R}$. Then we get

$$\Theta(t) = t^{-1/2}\Theta(\frac{1}{t}), \quad t > 0.$$

From this formula, $\Theta(t) \leqslant Ct^{-1/2}$ is obvious.

(2) We write

$$\Theta(t) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 t},$$

in which for $t \geqslant 0$,

$$\sum_{n=1}^{\infty} e^{-\pi n^2 t} \leqslant \sum_{n=1}^{\infty} e^{-\pi n t} = \sum_{n=1}^{\infty} (e^{-\pi t})^n \leqslant Ce^{-\pi t}$$

if $\pi t \geqslant \delta > 0$ (in particular, this is valid for $t \geqslant 1$). Therefore, for $t \geqslant 1$ we have

$$0 \leqslant \Theta(t) - 1 \leqslant Ce^{-\pi t}.$$

This is exactly what we want. □

The following theorem reveals the hidden connection between the Zeta, Gamma and Theta functions.

Theorem 7.13 (The Xi Identity)**.** *If $\Re(s) > 1$ we have*

$$\pi^{-s/2} \cdot \Gamma(\frac{s}{2}) \cdot \zeta(s) = \frac{1}{2}\int_0^{\infty} u^{s/2-1}(\Theta(u) - 1)du.$$

Proof. Beginning with the definition of $\Theta(\cdot)$, we compute

$$\begin{aligned}\frac{1}{2}\int_0^{\infty} u^{s/2-1}(\Theta(u) - 1)du &= \int_0^{\infty} \sum_{n=1}^{\infty} u^{s/2-1} e^{-\pi n^2 u} du \\ &= \sum_{n=1}^{\infty} \int_0^{\infty} u^{s/2-1} e^{-\pi n^2 u} du.\end{aligned}$$

Here the second equality is because of Proposition 7.12. Letting $t = \pi n^2 u$, the right hand side becomes

$$\sum_{n=1}^{\infty} \pi^{-s/2}(\int_0^{\infty} e^{-t} \cdot t^{s/2-1} dt) n^{-s} = \pi^{-s/2} \cdot \Gamma(\frac{s}{2}) \cdot \zeta(s).$$

This completes the proof. □

People are truly interested in the LHS in Theorem 7.13.

Definition 7.14 (Xi Function)**.** For $\Re(s) > 1$, we define

$$\xi(s) := \pi^{-s/2} \cdot \Gamma(\frac{s}{2}) \cdot \zeta(s).$$

Theorem 7.15. *The Xi function enjoys the following properties.*

1. *$\xi(\cdot)$ is holomorphic in $\{s \in \mathbb{C} : \Re(s) > 1\}$.*
2. *$\xi(\cdot)$ has an analytic continuation to a meromorphic function on $\mathbb{C}$ with simple poles only at $s = 0, 1$.*
3. *$\xi(s) = \xi(1 - s)$ for any $s \in \mathbb{C}$.*

Proof. (1) is clear.

(2) By Theorem 7.13, for $\Re(s) > 1$ we have

$$\xi(s) = \frac{1}{2}\int_0^\infty u^{s/2-1}(\Theta(u)-1)du.$$

Denote $\psi(u) = (\Theta(u)-1)/2$, then by the identity $\Theta(t) = t^{-1/2}\Theta(1/t)$ we have

$$\psi(u) = u^{-1/2}\psi(\frac{1}{u}) + \frac{1}{2u^{1/2}} - \frac{1}{2}, \quad u > 0.$$

Consequently,

$$\begin{aligned}\xi(s) &= \int_0^\infty u^{s/2-1}\cdot\psi(u)du \\ &= \int_0^1 u^{s/2-1}\cdot\psi(u)du + \int_1^\infty u^{s/2-1}\cdot\psi(u)du \\ &= \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty (u^{s/2-1} + u^{-1/2-s/2})\psi(u)du,\end{aligned}$$

where the last equality is given by the variable exchanging $u \mapsto 1/u$ in the first integral. Now for $s \in \mathbb{C}$, we define

$$(*) \qquad \xi(s) := \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty (u^{s/2-1} + u^{-1/2-s/2})\psi(u)du.$$

Then $\xi(\cdot)$ is a meromorphic function on $\mathbb{C}$ with simple poles at $s = 0, 1$.

(3) Using $(*)$ above, we directly get the result. $\square$

Theorem 7.16 (Analytic Continuation)**.** *The Zeta function $\zeta(\cdot)$ that is initially holomorphically defined on $\{s \in \mathbb{C} : \Re(s) > 1\}$ has an analytic continuation to a meromorphic function on $\mathbb{C}$, whose singularity is a simple pole at $s = 1$.*

Proof. Note that $\zeta(s) = \pi^{s/2}\cdot\xi(s)/\Gamma(s/2)$ by Definition 7.14. Now Theorem 7.15 (2) shows that $\xi(s)$ has simple poles at $s = 0, 1$, and Theorem 7.8 (1) dictates that $\Gamma(s)$ has simple poles at $s = 0, -2, -4, \ldots$. $\square$

Remark 7.17. The continued definition of $\zeta(s)$ is given by

$$\zeta(s) = \begin{cases} \sum_{n=1}^\infty n^{-s}, & \Re(s) > 1; \\ \pi^{s/2}\cdot\xi(s)/\Gamma(s/2), & \Re(s) \leqslant 1. \end{cases}$$

Also note that $\zeta(s)$ has simple poles at $s = -2, -4, \ldots$.

7.2.2. *Zeros of Riemann Zeta Function.* We begin with the Euler identity without proof.

Proposition 7.18 (Euler Identity)**.** *For $\Re(s) > 1$ we have*

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}.$$

Proof. Note that on the connected region $\{s \in \mathbb{C} : \Re(s) > 1\}$, $\sum n^{-s}$ and $\prod(1-p^{-s})^{-1}$ are analytical functions with respect to s. Hence it suffices to check the equality for real numbers $s > 1$, and then the equality extends continuously.

The fundamental theorem of arithmetic shows that for all $n \in \mathbb{N}$, we have $n = p_1^{k_1} \cdots p_m^{k_m}$, where $p_1, \ldots, p_m$ are distinct primes. Then

$$\sum_{n=1}^{N} \frac{1}{n^s} \leqslant \prod_{p \leqslant N} (1 + \frac{1}{p^s} + \cdots + \frac{1}{p^{Ms}}) \leqslant \prod_{p \leqslant N} \frac{1}{1 - p^{-s}}, \quad M \gg 0.$$

The second inequality is because of

$$\frac{1}{1 - p^{-s}} = \sum_{k=0}^{\infty} p^{-ks} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots.$$

By taking $N \to \infty$, we have

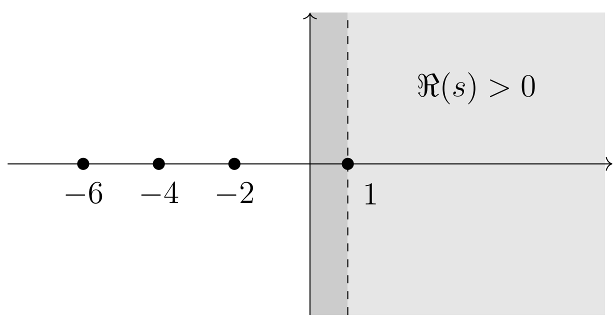
$$\sum_{n=1}^{\infty} \frac{1}{n^s} \leqslant \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

Similarly, one may deduce the converse inequality. This completes the proof. □

The immediate corollary of Proposition 7.18 is for $\Re(s) > 1$ we have $\zeta(s) \neq 0$. Recall that $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ and $\xi(s) = \xi(1-s)$. Thus,

$$\zeta(s) = \frac{\xi(1-s)}{\pi^{-s/2}\Gamma(s/2)} = \pi^{s-1/2} \cdot \frac{\Gamma((1-s)/2)}{\Gamma(s/2)} \cdot \zeta(1-s).$$

For $\{s \in \mathbb{C} : \Re(s) < 0\}$, we have $\zeta(1-s) \neq 0$ and $\Gamma((1-s)/2) \neq 0$ (since $\Gamma(\cdot) \neq 0$) on $\mathbb{C}$. Also, $1/\Gamma(s/2) = 0$ exactly at $s = -2, -4, \ldots$ Hence all zeros of $\zeta(s)$ in $\{s \in \mathbb{C} : \Re(s) < 0\}$ are $-2, -4, \ldots$



To sum up, we are to seek the zeros of $\zeta(\cdot)$ in the critical strip $\{s \in \mathbb{C} : 0 \leqslant \Re(s) \leqslant 1\}$.

8. Riemann Zeta Function and Prime Number Theory

Euler found, through his product formula for the zeta function, a deep connection between analytical methods and arithmetic properties of numbers, in particular primes. An easy consequence of Eulers formula is that the sum of the reciprocals of all primes, $\sum_p 1/p$, diverges, a result that quantifies the fact that there are infinitely many prime numbers. The natural problem then becomes that of understanding how these primes are distributed. With this in mind, we consider the following function:

$$\pi(x) := \#\{\text{primes} \leqslant x\} = \sum_{p \leqslant x} 1.$$

Then $\pi(x) = \pi([x])$ for any $x \geqslant 0$. A conjecture of Gauss made in 1792 (and independently, of Legendre in 1808) says that

$$(*) \qquad \lim_{x \to \infty} \frac{\pi(x)}{x/\log x} = 1.$$

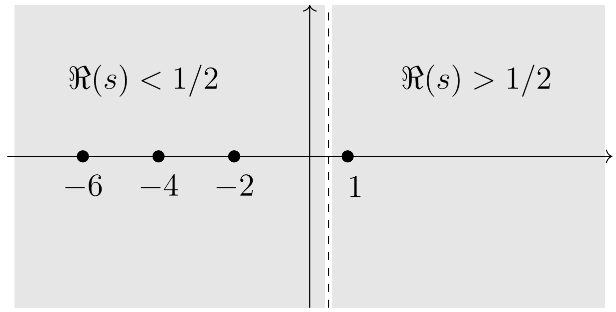
This is denoted as $\pi(x) \sim x/\log x$ as $x \to \infty$. On the work of Dirichlet (1837), Chebychev (1850s) and Riemann (1859), this conjecture is proved as the *Prime Number Theorem.*

Theorem 8.1 (Hadamard, de la Vallée Poussin, 1896)**.** *The conjecture* $(*)$ *is true.*

8.1. **The Riemann Memoir.** [1] In this subsection we list out some basic and important properties given by Riemann.

(A) The Zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ that is holomorphically defined in $\{s \in \mathbb{C} : \Re(s) > 1\}$ has an analytic continuation to a meromorphic function in $\mathbb{C}$ with a simple pole at $s = 1$.
(B) One can define $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, then $\xi(s) = \xi(1-s)$ for any $s \in \mathbb{C}$. Also, $\xi(\cdot)$ is meromorphic on $\mathbb{C}$ with simple poles at $s = 0, 1$.
(C) The Zeta function $\zeta(\cdot)$ has simple zeros at $s = -2, -4, \cdots$ (trivial zeros) and $\zeta(s) \neq 0$ for $\Re(s) > 1$. There are infinitely many nontrivial zeros of the form $\rho = \sigma + it$ for $0 \leqslant \sigma \leqslant 1$ and $t \in \mathbb{R}$ (i.e., living in the critical strip). Moreover, let $N(T) = \#\{\rho = \sigma + it : 0 \leqslant \sigma \leqslant 1, |t| \leqslant T\}$, then
$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$
as $T \to \infty$. This is proved by von-Mangoldt in 1895 and 1905.
(D) (The Product Formula) We have the following (to be proved as an exercise):
$$s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) = e^{-Bs} \prod_{\substack{\zeta(\rho)=0, \\ 0\leqslant\Re(\rho)\leqslant 1}} (1 - \frac{s}{\rho})e^{s/\rho}$$
where $B = 1 + \gamma/2 - \log 2\sqrt{\pi}$ and γ denotes the Euler constant. This is proved by Hadamard in 1893.
(E) (Riemann's Explicit Formula) Denote
$$\psi(x) := \sum_{\substack{p^m \leqslant x, \\ p \text{ prime}, \ n\in\mathbb{N}}} \log p = \sum_{n\leqslant x} \Lambda(n), \quad \psi^{\#}(x) := \sum_{n<x} \Lambda(n) + \frac{1}{2}\Lambda(x),$$
where
$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \text{ for some prime } p \text{ and } m \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$
Then the formula (proved by von-Mangoldt in 1895) is read as
$$\psi^{\#}(x) = x - \sum_{\substack{\zeta(\rho)=0, \\ 0\leqslant\Re(\rho)\leqslant 1}} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - x^{-2}), \quad x \geqslant 2.$$
This formula is powerful, whereas it is still very difficult to find out all zeros ρ lying in the critical strip.
(F) (Riemann Hypothesis) Any nontrivial zeros of $\zeta(s)$ is on the line $\Re(s) = 1/2$.

[1]Riemann (1859): *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* (English translation: *on the number of prime less than a given magnitude*).



Remark 8.2 (RH implies PNT). Suppose the Riemann Hypothesis is true. Then (without proof) as $x \to \infty$, we have

$$\psi(x) = \sum_{n \leqslant x} \Lambda(n) \sim x, \quad \psi^{\#}(x) = \sum_{n<x} \Lambda(n) + \frac{1}{2}\Lambda(x) \sim x.$$

This is equivalent to

$$\pi(x) = \sum_{p \leqslant x} 1 \sim \frac{x}{\log x},$$

which is nothing but the prime number theorem (PNT).

A Sketchy Proof for Riemann's Explicit Formula. Recall that for $\Re(s) > 1$ we have Euler identity

$$\zeta(s) = \prod_{p \text{ prime}} (1 - \frac{1}{p^s})^{-1},$$

and it implies that

$$\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= (-\log \zeta(s))' = \sum_{p \text{ prime}} (\log(1 - \frac{1}{p^s}))' \\
&= (-\sum_{p \text{ prime}} \sum_{m=1}^{\infty} \frac{p^{-ms}}{m})' = \sum_{p,m} (\log p) \cdot p^{-ms}.
\end{aligned}$$

We use the following sublemma as a fact (whose proof is leave as an exercise). For $y > 0$ and for any fixed $\alpha > 0$, we have

$$\lim_{T \to \infty} \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \frac{y^s}{s} ds = \begin{cases} 0, & \text{if } 0 < y < 1; \\ 1/2, & \text{if } y = 1; \\ 1, & \text{if } y > 1. \end{cases}$$

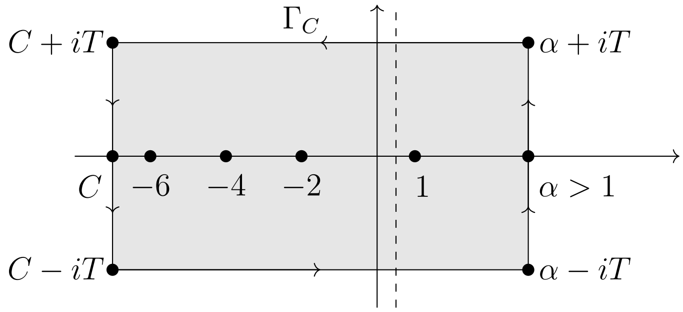
Obtaining this, we consider the following limit:

$$\begin{aligned}
\lim_{T \to \infty} \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \frac{y^s}{s} \cdot \frac{-\zeta'(s)}{\zeta(s)} ds &= \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \frac{y^s}{s} \cdot \sum_{p,m} (\log p) \cdot p^{-ms} ds \\
&= \sum_{p,m} (\log p) \cdot \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \frac{(yp^{-m})^s}{s} ds \\
&= \sum_{p^m < y} \log p + \sum_{p^m = y} \frac{1}{2} \log p = \psi^{\#}(y).
\end{aligned}$$

The last equality above takes the sublemma above at work. Next, we consider the function

$$F(s) = -\frac{y^s}{s} \cdot \frac{\zeta'(s)}{\zeta(s)}$$

and its integral along Γ_C defined as follows, where $\alpha > 1$ and $C \ll 0$.



Applying the residue formula (Theorem 4.8 and Corollary 4.9), we get

$$\frac{1}{2\pi i}\int_{\Gamma_C} F(s)ds = \sum_{F(z)=\infty} \operatorname{res}_{s=z} F(s).$$

To compute the right hand side, all poles of F are listed out below.

- $s = 1$ (simple pole):

$$\operatorname{res}_{s=1} F = \lim_{s\to 1}(s-1)\cdot F(s) = \lim_{s\to 1}\frac{y^s}{s}\cdot \lim_{s\to 1}(1-s)\frac{\zeta'(s)}{\zeta(s)} = y.$$

- $s = 0$ (simple pole):

$$\operatorname{res}_{s=0} F = -\lim_{s\to 0} s\cdot \frac{y^s}{s}\cdot\frac{\zeta'(s)}{\zeta(s)} = -\frac{\zeta'(0)}{\zeta(0)}.$$

- $s = \rho \neq 0$ with $0 \leqslant \Re(\rho) \leqslant 1$ and $|\Im(\rho)| \leqslant T$:

$$\operatorname{res}_{s=\rho} F = -\frac{y^\rho}{\rho}, \quad \zeta(\rho) = 0.$$

- $s = -2m$ with $m \in \mathbb{N}$ (simple poles):

$$\operatorname{res}_{s=-2m} F = -\frac{y^{-2m}}{2m}.$$

Finally, letting $T \to \infty$ and $C \to -\infty$, we get

$$\begin{aligned}\psi^{\#}(y) &= y - \lim_{T\to\infty}\sum_{\substack{\zeta(\rho)=0,\\ 0\leqslant\Re(\rho)\leqslant 1}}\frac{y^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \sum_{m=1}^{\infty}\frac{y^{-2m}}{2m}\\ &= y - \lim_{T\to\infty}\sum_{\substack{\zeta(\rho)=0,\\ 0\leqslant\Re(\rho)\leqslant 1}}\frac{y^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1-y^{-2}).\end{aligned}$$

This completes the proof. □

8.2. **The Prime Number Theorem.** We have claimed in Remark 8.2 that Riemann hypothesis implies the prime number theorem (PNT). This subsection is to make this explicit. The essential property at work is the following corollary of Riemann hypothesis.

Theorem 8.3 (Non-degeneracy)**.** $\zeta(s) \neq 0$ *when* $\Re(s) = 1$.

To prove Theorem 8.3, we first introduce several lemmas as follows.

Lemma 8.4. *If* $\Re(s) > 1$ *then*

$$\log \zeta(s) = \sum_{\substack{p \text{ prime}, \\ m \geqslant 1}} \frac{p^{-ms}}{m} := \sum_{n=1}^{\infty} c_n n^{-s},$$

with the coefficients given by

$$c_n = \begin{cases} 1/m, & \text{if } n = p^m; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By the Euler identity, for $\Re(s) > 1$, we have

$$\zeta(s) = \prod_{p \text{ prime}} (1 - \frac{1}{p^s})^{-1}.$$

So that for the real number $s > 1$, we have

$$\begin{aligned} \log \zeta(s) &= \log \prod_{p \text{ prime}} (1 - \frac{1}{p^s})^{-1} = - \sum_{p \text{ prime}} \log(1 - \frac{1}{p^s})^{-1} \\ &= \sum_{p \text{ prime}} \sum_{m=1}^{\infty} \frac{p^{-ms}}{m} = \sum_{p,m} \frac{p^{-ms}}{m}. \end{aligned}$$

On the other hand, $\zeta(s) \neq 0$ for $\Re(s) > 1$, which implies that $\log \zeta(s)$ is a well-defined holomorphic function. Accordingly, $\sum p^{-ms}/m$ is also a holomorphic function in $\Omega = \{s \in \mathbb{C} : \Re(s) > 1$. However, we know that Ω is a connected region, so

$$\log \zeta(s) = \sum_{p,m} \frac{p^{-ms}}{m}$$

for any s such that $\Re(s) > 1$. ☐

Lemma 8.5. *For any* $\theta \in \mathbb{R}$, *we have* $3 + 4\cos\theta + \cos 2\theta \geqslant 0$.

Proof. This follows from $3 + 4\cos\theta + \cos 2\theta = 2(\cos\theta + 1)^2$ at once. ☐

Lemma 8.6. *If* $\sigma > 1$ *and* $t \in \mathbb{R}$, *then* $\log |\zeta(\sigma)^3 \cdot \zeta(\sigma + it)^4 \cdot \zeta(\sigma + 2it)| \geqslant 0$.

Proof. We calculate directly, say

$$\begin{aligned} \text{LHS} &= 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma + it)| + \log|\zeta(\sigma + 2it)| \\ &= 3\Re \log \zeta(\sigma) + 4\Re \log \zeta(\sigma + it) + \Re \log \zeta(\sigma + 2it) \\ &= \Re \sum_{n=1}^{\infty} c_n \cdot 3n^{-\sigma} + \Re \sum_{n=1}^{\infty} c_n \cdot 4n^{-\sigma - it} + \Re \sum_{n=1}^{\infty} c_n \cdot n^{-\sigma - 2it}. \end{aligned}$$

Here the last equality is deduced from Lemma 8.4. On the other hand,

$$\begin{aligned} &\Re \sum_{n=1}^{\infty} c_n \cdot 3n^{-\sigma} + \Re \sum_{n=1}^{\infty} c_n \cdot 4n^{-\sigma - it} + \Re \sum_{n=1}^{\infty} c_n \cdot n^{-\sigma - 2it} \\ &= \sum_{n=1}^{\infty} c_n \cdot n^{-\sigma}(3 + 4\cos(t \log n) + \cos(2t \log n)) \geqslant 0. \end{aligned}$$

by Lemma 8.5. ☐

Proof of Theorem 8.3. Note that $\zeta(\cdot)$ is a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$. Then $\zeta(s) \neq 0$ for those s that are landing close to 1. We need to verify that $\zeta(1 + it) \neq 0$ for any $t \in \mathbb{R}$. Suppose not for the sake of contradiction. Then there is some $t_0 \neq 0$ such that $\zeta(1 + it_0) = 0$. Consequently,

$$|\zeta(\sigma + it_0)|^4 \leqslant C(\sigma - 1)^4$$

as $\sigma \to 1$ for some constant $C > 0$. For other terms, Since $s = 1$ is a simple pole of $\zeta(\cdot)$, we see $|\zeta(\sigma)|^3 \sim C|\sigma - 1|^{-3}$ as $\sigma \to 1$. Again, note that $\zeta(\cdot)$ is holomorphic for $s \neq 1$, we have $\zeta(\sigma + 2it_0)$ being bounded as $\sigma \to 1$. Therefore,

$$|\zeta(\sigma)^3 \cdot \zeta(\sigma + it)^4 \cdot \zeta(\sigma + 2it)| \leqslant C|\sigma - 1|, \quad \sigma \to 1.$$

This is contradicting with Lemma 8.6. □

Remark 8.7. By the symmetry of ξ, we have $\xi(s) = \pi^{-s/2} \cdot \Gamma(s/2) \cdot \zeta(s) = \xi(1 - s)$, hence $\zeta(s) \neq 0$ for $\Re(s) \neq 0$.

Theorem 8.8. *Theorem 8.3 implies the following prime number theorem: as $x \to \infty$,*

$$\pi(x) = \sum_{p \leqslant x} 1 \sim \frac{x}{\log x}.$$

The proof of Theorem 8.8 follows the proof by Zagier in 1997, which is based on the proof of Newman in 1980. It truly relies on the following result.

Theorem 8.9 (Tauberian Theorem)**.** *Let f be a bounded measurable function on $[0, \infty)$. Assume the Laplace transform*

$$g(z) = \int_0^\infty f(t) e^{-zt} dt$$

that is a holomorphic function for $\Re(z) > 0$ extends holomorphically in an open set containing $\{z \in \mathbb{C} : \Re(z) \geqslant 0\}$. Then the integral

$$\int_0^\infty f(t) dt = \lim_{T \to \infty} \int_0^T f(t) dt$$

converges and equals to $g(0)$, which is the value of the extended g at $z = 0$.

In the upcoming context we are to use the language of Φ function and φ function.

Definitions 8.10. We define

$$\Phi(s) := \sum_{p \text{ prime}} \frac{\log p}{p^s}, \quad \varphi(s) := \sum_{p \leqslant x} \log p.$$

Lemma 8.11. *$\Phi(s)$ is holomorphic for $\Re(s) > 1$.*

Lemma 8.12. *$\Phi(s) - (s-1)^{-1}$ extends holomorphically to an open set containing $\{s \in \mathbb{C} : \Re(s) \geqslant 1\}$.*

Proof. For $\Re(s) > 1$, the Euler identity $\zeta(s) = \prod (1 - p^{-s})^{-1}$ dictates that

$$(-\log \zeta(s))' = -\frac{\zeta'(s)}{\zeta(s)} = (-\sum_{p \text{ prime}} \log \frac{1}{1 - p^{-s}})' = \sum_{p \text{ prime}} \frac{\log p}{p^s - 1}.$$

Moreover, this can be written as

$$\sum_{p \text{ prime}} \frac{\log p}{p^s - 1} = \sum_{p \text{ prime}} (\frac{\log p}{p^s - 1} - \frac{\log p}{p^s}) + \Phi(s).$$

Therefore,

$$\Phi(s) = -\frac{\zeta'(s)}{\zeta(s)} - \sum_{p \text{ prime}} \frac{\log p}{p(p^s - 1)}.$$

Note that the second term is holomorphic for $\Re(s) > 1/2$. The first term is meromorphic with poles at the pole $s = 1$ of $\zeta(\cdot)$, as well as the zeros of $\zeta(\cdot)$. This together with Theorem 8.3 that $\zeta(s) \neq 0$ for $\Re(s) = 1$, we see $\zeta'(s)/\zeta(s)$ is holomorphic near $\{\Re(s) = 1\}$ (except a pole at $s = 1$).

Recall for $\Re(s) > 0$ that $\zeta(s) = (s-1)^{-1} +$ (a holomorphic function). As the derivation of analytic function is still analytic,

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + (\text{a holomorphic function})$$

near $s = 1$. To sum these up, the function

$$\Phi(s) - \frac{1}{s-1}$$

is holomorphically defined near $\{s \in \mathbb{C} : \Re(s) = 1\}$. □

Now we are ready to introduce the main theorem on PNT by using the function $\varphi(\cdot)$.

Theorem 8.13. *As $x \to \infty$, we have $\varphi(x) \sim x$, i.e., $\lim_{x\to\infty} \varphi(x)/x = 1$. Furthermore, this result implies PNT.*

Proof. The proof for $\varphi(x) \sim x$ is relatively easy. We are to do the second part. Note that

$$\varphi(x) = \sum_{p \leqslant x} \log p \leqslant \sum_{p \leqslant x} \log x = \pi(x) \cdot \log x,$$

which immediately implies that

$$\liminf_{x\to\infty} \frac{\pi(x)}{x/\log x} \geqslant \liminf_{x\to\infty} \frac{\varphi(x)}{x} = 1.$$

On the other hand, for all $\varepsilon > 0$,

$$\begin{aligned} \varphi(x) &\geqslant \sum_{x^{1-\varepsilon} < p \leqslant x} \log p \geqslant \sum_{x^{1-\varepsilon} < p \leqslant x} \log x \\ &= (1-\varepsilon) \cdot \log x \cdot (\pi(x) - \pi(x^{1-\varepsilon})) \\ &\geqslant (1-\varepsilon) \cdot \log x \cdot (\pi(x) - x^{1-\varepsilon}) \end{aligned}$$

Here the equality is deduced from the definition of $\pi(\cdot)$. Therefore,

$$\limsup_{x\to\infty} \frac{\pi(x)}{x/\log x} \leqslant \frac{1}{1-\varepsilon}.$$

By letting $\varepsilon \to 0^+$, we get

$$\limsup_{x\to\infty} \frac{\pi(x)}{x/\log x} \leqslant 1.$$

This finally proves Theorem 8.8. □

Lemma 8.14. *The following integral converges:*

$$\int_1^\infty \frac{\varphi(x) - x}{x^2} dx.$$

Proof. **Claim:** For $\Re(s) > 1$, by substituting $x = e^t$, we have

$$\Phi(s) = \sum_{p \text{ prime}} \frac{\log p}{p^s} = s \int_1^\infty \frac{\varphi(x)}{x^{s+1}} dx = s \int_0^\infty e^{-st} \varphi(e^t) dt.$$

Apply this claim without proof. Note that via $x = e^t$,

$$\int_1^\infty \frac{\varphi(x) - x}{x^2} dx = \int_0^\infty (\varphi(e^t) e^{-t} - 1) dt = \int_0^\infty f(t) dt,$$

where we denote $f(t) := \varphi(e^t)e^{-t} - 1$. Then consider the Laplace transform

$$\begin{aligned} g(s) = \int_0^\infty f(t) e^{-st} dt &= \int_0^\infty (\varphi(e^t) e^{-t} - 1) e^{-st} dt \\ &= \underbrace{\int_0^\infty e^{-(s+1)t} \cdot \varphi(e^t) dt}_{\Phi(s+1)/(s+1)} - \underbrace{\int_0^\infty e^{-st} dt}_{1/s} \\ &= \frac{1}{s+1} (\Phi(s+1) - \frac{1}{s} - 1). \end{aligned}$$

By Lemma 8.12, the function $g(\cdot)$ extends holomorphically to $\{s \in \mathbb{C} : \Re(s) \geqslant 0\}$. Now apply Tauberian theorem (Theorem 8.9), we see the integral

$$g(0) = \int_0^\infty f(t) dt$$

converges. This completes the proof. $\square$

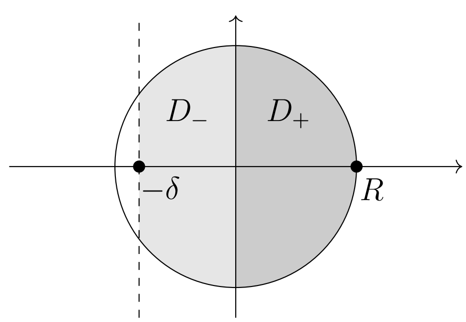
At the end of this section, we are going to prove Theorem 8.9.

Proof of Theorem 8.9. The bounded condition for f is essential. Assume $|f(t)| \leqslant M$ for $t \in [0, \infty)$. For $T > 0$ we define its truncated Laplace transform as

$$g_T(z) := \int_0^T f(t) \cdot e^{-zt} dt,$$

which is an entire function. We need to verify that

$$\lim_{T \to \infty} g_T(0) = g(0).$$



Apply the Cauchy integral formula (Theorem 3.13) to

$$G(z) = (g(z) - g_T(z)) \cdot e^{zT} \cdot (1 + \frac{z^2}{R^2}),$$

which is holomorphic in $D = \{|z| \leqslant R, \Re(z) \geqslant -\delta(R)\}$, we get

$$G(0) = \frac{1}{2\pi i} \int_{\partial D} \frac{G(z)}{z} dz.$$

Or equivalently,

$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_{\partial D} (g(z) - g_T(z)) \cdot e^{zT} \cdot (1 + \frac{z^2}{R^2}) \cdot \frac{1}{z} dz.$$

For convenience, we denote $\partial D_+ = \partial D \cap \{x > 0\}$ and $\partial D_- = \partial D \cap \{x \leqslant 0\}$. On ∂D_+, via $z = x + iy$,

$$|g(z) - g_T(z)| = |\int_T^\infty f(t) e^{-zt} dt| \leqslant M \cdot \int_T^\infty e^{-xt} dt = \frac{Me^{-xT}}{x}.$$

On the other hand, for $|z| = R$,

$$|e^{zT} \cdot (1 + \frac{z^2}{R^2}) \cdot \frac{1}{z}| = e^{xT} \cdot \frac{2|x|}{R^2}.$$

Combining these, we see

$$|\frac{1}{2\pi i} \int_{\partial D_+} (g(z) - g_T(z)) \cdot e^{zT} \cdot (1 + \frac{z^2}{R^2}) \cdot \frac{1}{z} dz| \leqslant \frac{M}{R}.$$

On ∂D_-, we choose to estimate $g_T(z)$ and $g(z)$ respectively. Say

$$|g_T(z)| = |\int_0^T f(t) e^{-zt} dy| \leqslant M \cdot \int_{-\infty}^T e^{-xt} dt = \frac{Me^{-xT}}{|x|}.$$

Note that $g_T(z)$ is entire, hence

$$\begin{aligned} &|\int_{\partial D} (g(z) - g_T(z)) \cdot e^{zT} \cdot (1 + \frac{z^2}{R^2}) \cdot \frac{1}{z} dz| \\ \leqslant &\int_{\Gamma_-} \frac{Me^{-xT}}{|x|} \cdot e^{xT} \cdot \frac{2|x|}{R^2} dz \leqslant \frac{M}{R} \end{aligned}$$

by local Cauchy theorem (Corollary 3.5). Here $\Gamma_- = \{|z| = R, \Re(z) \leqslant 0\}$ denotes the left semi-circle. For $g(z)$, $g(\cdot)$ is holomorphic on ∂D_-. Hence there exists some constant $K = K(R, \delta) > 0$ such that on ∂D_-,

$$|g(z) \cdot (1 + \frac{z^2}{R^2}) \cdot \frac{1}{z}| \leqslant K(R, \delta).$$

Note that e^{zT} is bounded on ∂D_- and $e^{zT} \to 0$ uniformly on every compact set of $\{z \in \mathbb{C} : \Re(z) < 0\}$. Then

$$\lim_{T \to \infty} |\frac{1}{2\pi i} \int_{\partial D_-} g(z) \cdot (1 + \frac{z^2}{R^2}) \cdot e^{zT} \cdot \frac{1}{z} dz| = 0.$$

Therefore,

$$\begin{aligned}\limsup_{T\to\infty}|g(0)-g_T(0)| &\leqslant \limsup_{T\to\infty}\Big|\frac{1}{2\pi i}\int_{\partial D_+}(g(z)-g_T(z))\cdot e^{zT}\cdot(1+\frac{z^2}{R^2})\cdot\frac{1}{z}dz\Big| \\ &\quad+\limsup_{T\to\infty}\Big|\frac{1}{2\pi i}\int_{\partial D_-}g_T(z)\cdot(1+\frac{z^2}{R^2})\cdot e^{zT}\cdot\frac{1}{z}dz\Big| \\ &\quad+\limsup_{T\to\infty}\Big|\frac{1}{2\pi i}\int_{\partial D_-}g(z)\cdot(1+\frac{z^2}{R^2})\cdot e^{zT}\cdot\frac{1}{z}dz\Big| \\ &\leqslant \frac{2M}{R}\to 0, \quad R\to\infty.\end{aligned}$$

This shows that $\lim_{T\to\infty}|g(0)-g_T(0)|=0$. □

Remarks 8.15 (Final Remarks on RH). Recall that $\zeta(\cdot)$ has two types of zeros: the nontrivial ones and the trivial ones. The trivial zeros are given by $\zeta(-2m)=0$ with $m\in\mathbb{N}$. The Riemann Hypothesis claims that all nontrivial zeros lie on the line $\Re(s)=1/2$, and in particular, the known fact is that $\zeta(s)\neq 0$ for $\Re(s)=1$. By symmetry, $\zeta(s)\neq 0$ on $\Re(s)=0$.

According to the known result by von-Mangoldt in 1905, as $T\to\infty$, we obtain an estimation for the number of zeros of $\zeta(\cdot)$ in the critical strip. Say

$$\begin{aligned}N(T) &:= \#\{s\in\mathbb{C}:\zeta(s)=0, 0<\Re(s)<1, |\Im(s)|\leqslant T\} \\ &= \frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+O(\log T), \quad T\to\infty.\end{aligned}$$

To study the hypothesis, we denote

$$M(T) := \#\{s\in\mathbb{C}:\zeta(s)=0, \Re(s)=1/2, |\Im(s)|\leqslant T\}.$$

In 1943, Selberg showed that $M(T)\geqslant A\cdot T\log T$ for some constant $A>0$ that is independent of T. In particular, this result implies

$$\frac{M(T)}{N(T)}\geqslant C>0.$$

Philosophically speaking, it shows that there are at least a certain proportion of zeros lie on $\Re(s)=1/2$. In 1974, Levinson had shown that $A\geqslant 1/3$; in 1991, Conrey had shown that $A\geqslant 2/5$.

9. Conformal Mappings: On Geometry of the Disc

We are to study the geometry of holomorphic functions. The problems and upshot ideas we present in this chapter are more geometric in nature than the ones we have seen so far. In fact, here we will be primarily interested in mapping properties of holomorphic functions. In particular, most of our results will be "global," as opposed to the more "local" analytical results proved in the first three chapters. The motivation behind much of our presentation lies in the following simple question:

- Given open sets $U, V\subset\mathbb{C}$, does there exist a holomorphic bijection between them?

By a holomorphic bijection we simply mean a function that is both holomorphic and bijective. (It will turn out that the inverse map is then automatically holomorphic.) A solution to this problem would permit a transfer of questions about analytic functions from one open set with little geometric structure to another with possibly more useful properties. The prime example consists in taking $V=\mathbb{D}$ the unit disc, where many ideas have been developed to study analytic functions. In fact, since the disc seems to be the most fruitful choice for V we are led to a variant of the above question:

- Given an open subset Ω of $\mathbb{C}$, what conditions on Ω guarantee that there exists a holomorphic bijection from Ω to $\mathbb{D}$?
- Given an open set $\Omega \subset \mathbb{C}$, what is the group of holomorphic automorphisms on Ω, i.e., how to find out $\mathrm{Aut}(\Omega) := \{f : \Omega \to \Omega \text{ conformal map}\}$?

9.1. **Conformal Equivalence and Examples.**

Definitions 9.1 (Conformality, Biholomorphicity)**.**

(1) Let $U, V \subset \mathbb{C}$ be open sets and $f : U \to V$ be holomorphic. Then f is called a *conformal map* or *biholomorphic map* if f is also bijective.

(2) If there exists a conformal map from U to V, then U, V are called *conformally equivalent* or *biholomorphically equivalent*.

Note that an equivalent definition of biholomorphicity is read as follows. There exists two holomorphic maps $F : U \to V$ and $G : V \to U$ such that $F \circ G = \mathrm{id}_V$ and $G \circ F = \mathrm{id}_U$. In some other materials, one may define *conformal mapping* as a map that preserves all local angles. We will no longer follow this definition.

Proposition 9.2. *If $f : U \to V$ is holomorphic and injective, then $f'(z) \neq 0$ for all $z \in U$. In particular, $f^{-1} : f(U) \to U$ is also holomorphic.*

Proof. If there is $z_0 \in U$ such that $f'(z_0) = 0$, then $f(z) - f(z_0) = a(z - z_0)^k + G(z)$ for those z lying near z_0, where $k \geq 2$, $a \neq 0$, and $\mathrm{ord}_{z_0} G \geqslant k + 1$. On the other hand, the condition that f is injective implies that f is not a constant. Then z_0 is an isolated zero of $f'(z)$, i.e., $f'(z) \neq 0$ for $z \neq z_0$ that are close to z_0. Hence the roots of $f(z) - f(z_0) - w$ are distinct near z_0 for some $w \neq 0$. Write

$$f(z) - f(z_0) - w = \underbrace{(a(z - z_0)^k - w)}_{F(z)} + G(z).$$

Note that $|F(z)| > |G(z)|$ on the circle $|z - z_0| = \delta$ for $0 < \delta \ll 1$. By Rouché Theorem (Corollary 4.25), $F(z) = a(z - z_0)^k - w$ and $F(z) + G(z)$ have the same number of zeros in $|z - z_0| < \delta$. Therefore, $f(z) - f(z_0) - w$ has $k \geq 2$ roots in $|z - z_0| < \delta$. This leads to a contradiction as $\delta \to 0$.

Let $g = f^{-1}$. For $w = f(z)$ that is close to $w_0 = f(z_0)$, we have

$$\frac{g(w) - g(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)}.$$

Consequently, $g'(w_0) = 1/f'(g(w_0)) \neq 0$. So f^{-1} is also holomorphic. □

Corollary 9.3. *The inverse of a conformal map is holomorphic.*

Remark 9.4. Here are some remarks for the sake of understanding Definitions 9.1.

(1) Suppose $f : U \to V$ is holomorphic and $f'(z) \neq 0$ for any $z \in U$. However, this does not imply the injectivity of f. For a counterexample, on $\mathbb{D}^* = \{0 < |z| < 1\}$,

$$f : \mathbb{D}^* \to \mathbb{D}^*; \quad z \mapsto z^2.$$

But $f'(z_0) \neq 0$ locally implies that f is (locally) biholomorphic near z_0.

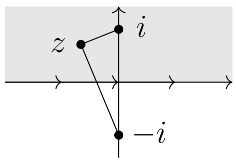
(2) On the terminology "conformal": let $f : U \to V$ be conformal. By Proposition 9.2, $f'(z) \neq 0$ for any $z \in U$. We claim that f preserves angles. To be more explicit, let Γ_1 and Γ_2 be two curves intersecting at $z \in \mathbb{C}$ with the intersection angle θ. Then $f(\Gamma_1)$ and $f(\Gamma_2)$ intersect at $f(z)$ with angle θ as well.

Examples 9.5. Here comes a series of examples on conformal maps. We are particularly interested to focus on the conformal equivalence class of $\mathbb{H}$.

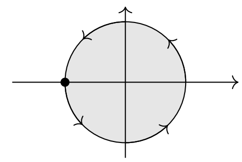
(1) The upper-half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ is conformally equivalent to the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, denoted as $\mathbb{H} \cong \mathbb{D}$. Note that for any $z \in \mathbb{H}$,

$$|F(z)| = \frac{|z-i|}{|z+i|} < 1.$$

Therefore, we get the holomorphic map $F : \mathbb{H} \to \mathbb{D}$. Its inverse is given by $G : \mathbb{D} \to \mathbb{H}$ with $G(w) = i(1-w)/(1+w)$.



$$\begin{tikzcd} \mathbb{H} \arrow[r, bend left, "F", "\sim"'] & \mathbb{D} \arrow[l, bend left, "G", "\sim"'] \end{tikzcd}$$

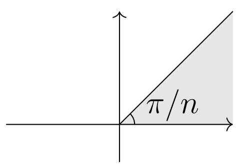


Then $F \circ G = \mathrm{id}_{\mathbb{D}}$ and $G \circ F = \mathrm{id}_{\mathbb{H}}$. Focusing on the image of the real line $\mathbb{R} \subset \mathbb{C}$ under F, for $z = x \in \mathbb{R}$,

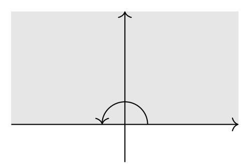
$$\frac{i-x}{i+x} = \frac{(i-x)^2}{1+x^2} = \frac{1-x^2}{1+x^2} + i\frac{2x}{1+x^2} = \cos(2t) + i\sin(2t).$$

by changing the variable $x = \tan t$ for $t \in (-\pi/2, \pi/2)$. In particular, $F(\infty) = F(-\infty) = -1$.

(2) We now define $S = \{z \in \mathbb{C} : 0 < \arg(z) < \pi/n\}$, and then $S \cong \mathbb{H}$ via $F : S \to \mathbb{H}$ and $G : \mathbb{H} \to S$ such that $F(z) = z^n$ and $G(w) = w^{1/n}$.



$$\begin{tikzcd} S \arrow[r, bend left, "F", "\sim"'] & \mathbb{H} \arrow[l, bend left, "G", "\sim"'] \end{tikzcd}$$



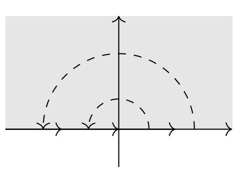
Note that the proportion $1/n$ can be replaced by any irrational number $\alpha \in \mathbb{R}$.

(3) Using the similar idea as in (2), $\mathbb{D}$ is conformally equivalent to the upper-half unit disc $\mathbb{D}_+ = \{z \in \mathbb{D} : \Im(z) > 0\}$, which is open as well. But the boundary behavior is not the same. In fact, there is a conformal map

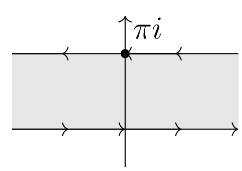
$$F : \mathbb{D}_+ \to \mathbb{H}; \quad z \mapsto -\frac{1}{2}(z + \frac{1}{z}).$$

To verify this, note that the equation $F(z) = w \in \mathbb{H}$ reduces to $z^2 + 2wz + 1 = 0$ that has two distinct roots whenever $w \neq \pm 1$.

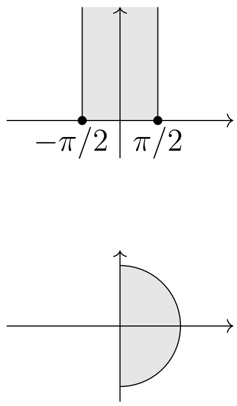
(4) Again, the upper-half plane can be conformally equivalent to a strip. Define $\Omega = \{z \in \mathbb{C} : 0 < \Im(z) < \pi\}$ and for $z = re^{i\theta} \in \mathbb{H}$ with $\theta \in [-\pi/2, 3\pi/2)$, we take $F : z \mapsto \log z = \log r + i\theta$ to see the result. Its inverse is given by $G : w \mapsto e^w$.



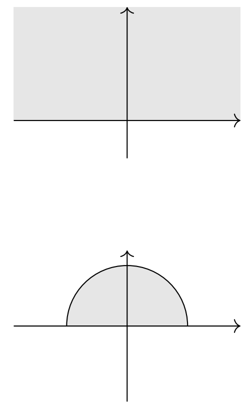
$$\begin{tikzcd} \mathbb{H} \arrow[r, bend left, "F", "\sim"'] & \Omega \arrow[l, bend left, "G", "\sim"'] \end{tikzcd}$$



(5) We define the half-strip $T = \{z \in \mathbb{C} : \Im(z) > 0, -\pi/2 < \Re(z) < \pi/2\}$. Note that the map $z \mapsto \exp(iz)$ takes T to the right half-disc $\mathbb{D}'_+ := \{|z| < 1, \Re(z) > 0\}$. This is immediate from the fact that if $z = x + iy$, then $e^{iz} = e^{ix}e^{-y}$. Also, we have the orientation given by multiplicating with i, say $\mathbb{D}'_+ \to \mathbb{D}_+$.



$$\begin{tikzcd}
T \arrow[r, bend right, "\sim" above, "z\mapsto \sin z" below] \arrow[d, bend left, "z\mapsto \exp(iz)" left, "\sim" right] & \mathbb{H} \\
\mathbb{D}'_+ \arrow[r, bend left, "z\mapsto iz" above, "\sim" below] & \mathbb{D}_+ \arrow[u, bend left, "\sim" left, "z\mapsto -\frac{z+1/z}{2}" right]
\end{tikzcd}$$



Combining these with the result of (3), we have a biholomorphic map $T \cong \mathbb{H}$. This is nothing but $\sin z$, because

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = -\frac{1}{2}(i\zeta + \frac{1}{i\zeta}), \quad \zeta = e^{iz}.$$

(6) For a non-example, we see $\mathbb{D} \ncong \mathbb{C}^* = \mathbb{C} - \{0\}$. Otherwise if there is some holomorphic map $f : \mathbb{C}^* \to \mathbb{D}$, it must be bounded, and equivalently, f has a removable singularity at 0 by the Riemann extension (Theorem 4.12). Moreover, it extends to some holomorphic and bounded map $g : \mathbb{C} \to \mathbb{D}$. From Liouville Theorem (Corollary 3.16), g must be a constant. This contradicts to the assumption.

(7) We claim $\mathbb{D} \ncong \mathbb{C}$. By Definitions 9.1, if $U \cong V$ then they have the same set of holomorphic functions. Moreover, there exists a group homomorphism $\mathscr{O}(U) \simeq \mathscr{O}(V)$ by Proposition 9.2. The result is given by the fact that there is some bounded non-constant holomorphic function on $\mathbb{D}$, whereas by Liouville Theorem (Corollary 3.16), there is no such bounded and non-constant entire function on $\mathbb{C}$.

9.2. **The Schwarz Lemma.** The statement and proof of the Schwarz lemma are both simple, but the applications of this result are far-reaching.

Lemma 9.6 (Schwarz)**.** *Suppose $f : \mathbb{D} \to \mathbb{D}$ is holomorphic with $f(0) = 0$. Then*

(1) *$|f(z)| \leqslant |z|$ for all $z \in \mathbb{D}$ with equality at some $z_0 \in \mathbb{D}$ if and only if $f(z) = e^{i\theta} \cdot z$ (i.e., f is a rotation);*

(2) *$|f'(0)| \leqslant 1$ with equality being valid if and only if f is a rotation.*

Proof. (1) Consider the function $g(z) = f(z)/z$, then $f(0) = 0$ implies that $z = 0$ is a removable singularity of g. If $|z| = r < 1$ then

$$\max_{|z|\leqslant r} |g(z)| = \max_{|z|=r} |g(z)| = \frac{1}{r} \max_{|z|=r} |f(z)| \leqslant \frac{1}{r}.$$

Letting $r \to 1$ from $r > 0$, we see for all $z \in \mathbb{D}$ that $|g(z)| \leqslant 1$. By the maximum principle (Proposition 4.27) applying to g, the equality holds if and only if $g(z) = C$ for some constant C such that $|C| = 1$, that is, $C = e^{i\theta}$ for some θ. Thus, $f(z) = e^{i\theta} \cdot z$.

(2) We still consider $g(z) = f(z)/z$. Note that

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} \frac{f(z)}{z} = \lim_{z\to 0} g(z) = g(0).$$

By (1), we get $|f'(0)| = |g(0)| \leqslant 1$ with equality if and only if $g(z) = e^{i\theta}$. This shows that $f(z) = e^{i\theta} \cdot z$. $\square$

9.2.1. $\mathrm{Aut}(\mathbb{D})$. The next goal is to apply Lemma 9.6 to understand the group $\mathrm{Aut}(\mathbb{D})$.

Examples 9.7. We list out some basic elements in $\mathrm{Aut}(\mathbb{D})$ as examples.

- The rotation: $z \mapsto e^{i\theta} \cdot z$.
- Given $\alpha \in \mathbb{D}$, we define
$$\psi_\alpha(z) := \frac{\alpha - z}{1 - \overline{\alpha} \cdot z},$$
then $\psi_\alpha \in \mathrm{Aut}(\mathbb{D})$. One can verify some properties such as $\psi_\alpha(0) = \alpha$, $\psi_\alpha(\alpha) = 0$, and $\psi_\alpha^2 = \psi_\alpha \circ \psi_\alpha = \mathrm{id}_{\mathbb{D}}$ (i.e. $\psi_\alpha^{-1} = \psi_\alpha$).

The following fundamental theorem dictates that the second example above almost represents all elements in $\mathrm{Aut}(\mathbb{D})$ (up to some rotation).

Theorem 9.8 (The Fundamental Theorem on $\mathrm{Aut}(\mathbb{D})$)**.** *For any $f \in \mathrm{Aut}(\mathbb{D})$, there are some $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$ such that*
$$f(z) = e^{i\theta} \cdot \frac{\alpha - z}{1 - \overline{\alpha} z}, \quad \forall z \in \mathbb{D}.$$

Proof. For $f \in \mathrm{Aut}(\mathbb{D})$, there is a unique $\alpha = f(0)$ such that $g(z) := \psi_\alpha \circ f(z) \in \mathrm{Aut}(\mathbb{D})$ and $g(0) = 0$. Note that $g : \mathbb{D} \to \mathbb{D}$ is holomorphic and satisfies the condition of the Schwarz Lemma (Lemma 9.6), for all $z \in \mathbb{D}$, $|g(z)| \leqslant |z|$. On the other hand, by Proposition 9.2, for each $g \in \mathrm{Aut}(\mathbb{D})$, we have $g^{-1} \in \mathrm{Aut}(\mathbb{D})$ and $g^{-1}(0) = 0$ as well. Again by Schwarz, $|g^{-1}(w)| \leqslant |w|$ for all $|w| \in \mathbb{D}$. Let $w = g(z)$ and then $|z| \leqslant |g(z)|$. Hence $|g(z)| = |z|$ for any $z \in \mathbb{D}$. This means that the equality in Lemma 9.6 holds, or equivalently, $g(z) = e^{i\theta} z$, denoted by $r_\theta(z)$, for some $\theta \in \mathbb{R}$. By definition, we get $\psi_\alpha \circ f = r_\theta$ and then $f = \psi_\alpha^{-1} \circ r_\theta = \psi_\alpha \circ r_\theta$ by Example 9.7. Finally, by replacing α by $\alpha \cdot e^{-i\theta}$, we finish the proof. ☐

9.2.2. $\mathrm{Aut}(\mathbb{H})$. Recall Example 9.5 (1) that $\mathbb{H}$ is conformally equivalent to $\mathbb{D}$ via
$$F : \mathbb{H} \to \mathbb{D}, \quad z \mapsto \frac{i - z}{i + z},$$
and hence we expect that $\mathrm{Aut}(\mathbb{H})$ can be expressed by $\mathrm{Aut}(\mathbb{D})$. Consider the composition
$$\begin{tikzcd} \mathbb{H} \arrow[r, "F"] \arrow[rrr, bend right, "F^{-1}\circ\varphi\circ F"'] & \mathbb{D} \arrow[r, "\varphi"] & \mathbb{D} \arrow[r, "F^{-1}"] & \mathbb{H} \end{tikzcd}$$

We know that for each $\varphi \in \mathrm{Aut}(\mathbb{D})$ (represented by Theorem 9.8), F induces an isomorphism
$$\begin{aligned} \Gamma_F : \mathrm{Aut}(\mathbb{D}) &\longrightarrow \mathrm{Aut}(\mathbb{H}) \\ \varphi &\longmapsto F^{-1} \circ \varphi \circ F \end{aligned}$$
whose image are given by conjugations of F.

Exercise 9.9. Fix $\varphi \in \mathrm{Aut}(\mathbb{D})$. Show that $\Gamma_F(\varphi)$ defined as above is of the form
$$z \mapsto \gamma.z := \frac{az + b}{cz + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
where $\gamma \in \mathrm{SL}_2(\mathbb{R})$, the special linear group over $\mathbb{R}$ (i.e. $ad - bc = 1$ and $a, b, c, d \in \mathbb{R}$).

Starting from this point of view, we define the *fractional linear transformation* as
$$f_M(z) = \frac{az + b}{cz + d}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

Theorem 9.10. *A map $g \in \mathrm{Aut}(\mathbb{H})$ if and only if $g = f_M$ for some $M \in \mathrm{SL}_2(\mathbb{R})$.*

Remark 9.11. Note that $f_M = f_{-M}$ for any $M \in \mathrm{SL}_2(\mathbb{R})$. By defining the equivalence relation $\sim$ by identifying M and $-M$, we see

$$\mathrm{Aut}(\mathbb{H}) \simeq \mathrm{PSL}_2(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R})/\sim,$$

which is the so-called *projective special linear group*.

At the end of this part, we will introduce a generalized version of the Schwarz Lemma 9.6 which drops the condition $f(0) = 0$.

Proposition 9.12 (Schwarz-Pick Lemma)**.** *Suppose $f : \mathbb{D} \to \mathbb{D}$ is holomorphic. Then for any $z \in \mathbb{D}$,*

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leqslant \frac{1}{1 - |z|^2}$$

with equality at some $z_0 \in \mathbb{D}$ if and only if $f \in \mathrm{Aut}(\mathbb{D})$.

Proof. Fix $z \in \mathbb{D}$ and consider the composition

$$F : 0 \mapsto z = g(0) \mapsto f(z) \mapsto 0 = h(f(z))$$

where $g(\xi) := (\xi + z)/(1 + \overline{z}\xi)$ and $h(\xi) = (\xi - f(z))/(1 - \overline{f(z)}\xi)$. Then $F = h \circ f \circ g : \mathbb{D} \to \mathbb{D}$ with $g, h \in \mathrm{Aut}(\mathbb{D})$ such that $F(0) = 0$. Now applying Lemma 9.6 to F, we get

$$|F'(0)| = |h'(f(z)) \cdot f'(z) \cdot g'(0)| = \frac{1 - |z|^2}{1 - |f(z)|^2}|f'(z)| \leqslant 1$$

with equality if and only if F is a rotation, i.e.,

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leqslant \frac{1}{1 - |z|^2}$$

with equality at some $z_0 \in \mathbb{D}$ if and only if $f \in \mathrm{Aut}(\mathbb{D})$. □

9.3. **Hyperbolic Geometry on $\mathbb{D}$.** Recall the Cauchy-Riemann equation is read as

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

for $dz = dx + idy$ and $d\overline{z} = dx - idy$. For a smooth function f, we see

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

If f is holomorphic, then $df(z) = f'(z)dz$ by definition.

Definition 9.13 (Kähler Metric)**.** Let $\Omega \subset \mathbb{C}$ be an open set. Suppose $g(z) > 0$ is a smooth function on $z \in \Omega$. A *Kähler Metric* on Ω is defined to be

$$ds^2(z) = g(z) \cdot |dz|^2,$$

where $|dz|^2 = (dx)^2 + (dy)^2$.

Definition 9.14 (Pull-back of Kähler Metric)**.** Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be open sets. Suppose $ds^2_{\Omega_2}(z) = g(z)|dz|^2$ is a metric on Ω_2 and $f : \Omega_1 \to \Omega_2$ is a holomorphic map. We define the *pull-back* of $ds^2_{\Omega_2}(z)$ along f as

$$f^*(ds^2_{\Omega_2}) := f^*(g|dz|^2) = (g \circ f) \cdot |df|^2 = (g \circ f) \cdot |f'(z)|^2|dz|^2.$$

Example 9.15 (Poincaré Metric on $\mathbb{D}$)**.** By taking $g(z) = 4/(1-|z|^2)^2$ in a Kähler metric, we get the Poincaré metric on $\mathbb{D}$:

$$ds^2_{\mathrm{P}}(z) := \frac{4|dz|^2}{(1-|z|^2)^2}, \quad ds_{\mathrm{P}} = \frac{2|dz|}{1-|z|^2}.$$

As for the pull-backs, we take $f \in \mathrm{Hol}(\mathbb{D},\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{D} \text{ holomorphic}\}$. Then

$$f^* ds^2_{\mathrm{P}}(z) = \frac{4}{(1-|f'(z)|^2)^2}|f'(z)|^2|dz|^2.$$

Through the holomorphic function f, $(\mathbb{D}, ds^2_{\mathrm{P}})$ is sent to $(\mathbb{D}, ds^2_{\mathrm{P}})$ as well. By cancelling the $|dz|^2$ term and applying the Schwarz-Pick lemma (Proposition 9.12), we see

$$f^* ds^2_{\mathrm{P}}(z) \leqslant ds^2_{\mathrm{P}}(z)$$

with equality holds at some $z_0 \in \mathbb{D}$ if and only if $f \in \mathrm{Aut}(\mathbb{D})$. Note that the inequality above is equivalent to the previous Schwarz lemma (Lemma 9.6).

Remark 9.16. One can check that the curvature of $(\mathbb{D}, ds^2_{\mathrm{P}})$ is a negative constant. The negativity here is often regarded as some "hyperbolic property" in complex geometry. Denote

$$\mathrm{Iso}(\mathbb{D}, ds^2_{\mathrm{P}}) = \{f : \mathbb{D} \to \mathbb{D} \text{ holomorphic with } f^* ds^2_{\mathrm{P}} = ds^2_{\mathrm{P}}\}$$

as the isometric group on $(\mathbb{D}, ds^2_{\mathrm{P}})$. Then the Schwarz-Pick shows that $\mathrm{Aut}(\mathbb{D}) = \mathrm{Iso}(\mathbb{D})$.

Example 9.17. Recall Example 9.5 (1) that for $\mathbb{H}$, we have the conformal equivalence

$$\varphi : \mathbb{H} \to \mathbb{D}; \quad z \mapsto \frac{z-i}{z+i}.$$

Then the pull-back of ds^2_{P} along φ is readily read as

$$\varphi^* ds^2_{\mathrm{P}}(z) = \frac{4|d\varphi(z)|^2}{(1-|\varphi(z)|^2)^2} = \frac{4}{(1-|\frac{z-i}{z+i}|^2)^2} \cdot \frac{4}{|z+i|^4} \cdot |dz|^2 = \frac{1}{y^2}|dz|^2$$

where $y = \Im(z)$. Then $(\mathbb{D}, ds^2_{\mathrm{P}}(z)) \simeq (\mathbb{H}, |dz|^2/|\Im(z)|^2)$. In this sense, we see the geometry on $\mathbb{D}$ is the same as that on $\mathbb{H}$.

9.3.1. *Poincaré Length.* Let $\Gamma \subset \mathbb{D}$ be a (piecewise smooth) curve in $\mathbb{D}$ joining two fixed points $a, b \in \mathbb{D}$. Assume Γ has a parametrization $z(t) = x(t) + iy(t) : [0,1] \to \mathbb{D}$ with $z(0) = a$ and $z(1) = b$. Then the *Poincaré length* of Γ with respect to $ds^2_{\mathrm{P}}(z)$ is given by

$$L(\Gamma) := \int_\Gamma ds_{\mathrm{P}}(z(t)) = \int_0^1 \frac{2}{1-|z(t)|^2}|dz(t)|$$

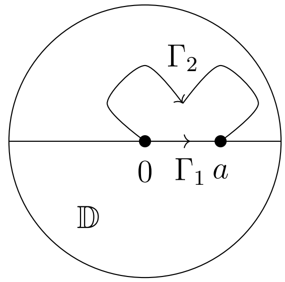
that is independent of the choice of the parametrization of Γ. Note that $dz(t) = (x'(t) + iy'(t))dt$ and then $|dz(t)| = (x'(t)^2 + y'(t)^2)^{1/2}dt$, we see

$$\int_\Gamma ds_{\mathrm{P}}(z(t)) = \int_0^1 \frac{2(x'(t)^2 + y'(t)^2)^{1/2}}{1-(x(t)^2 + y(t)^2)}dt.$$

Example 9.18. Consider $\Gamma_1 : z(t) = t$ with $0 \leqslant t \leqslant a$. Then

$$L(\Gamma_1) = \int_0^a \frac{2}{1-t^2}dt = \int_0^a (\frac{1}{1-t} + \frac{1}{1+t})dt = \log\frac{1+a}{1-a}.$$

Note that as $a \to 1$, we have $L(\Gamma_1) \to \infty$, which does not coincide with the intuition for classical Eulerian geometry.



Now consider $\Gamma_2 : z(t) = x(t) + iy(t)$ with $0 \leqslant t \leqslant 1$ such that $z(0) = 0$ and $z(1) = a$. Then

$$\begin{aligned} L(\Gamma_2) &= \int_0^1 \frac{2(x'(t)^2 + y'(t)^2)^{1/2}}{1 - x(t)^2 - y(t)^2} dt \\ &\geqslant \int_0^1 \frac{2|x'(t)|}{1 - x(t)^2} dt \geqslant \int_0^1 \frac{2dx(t)}{1 - x(t)^2} \\ &= \int_0^a \frac{2ds}{1 - s^2} = \log \frac{1+a}{1-a} = L(\Gamma_1). \end{aligned}$$

From this, we know that with respect to ds_{P}^2, the line segment Γ_1 from 0 to a is actually the shortest path.

Definition 9.19 (Poincaré Distance)**.** For $a, b \in \mathbb{D}$ we define the *Poincaré distance* from a to b as

$$\mathrm{dist}_{\mathrm{P}}(a, b) := \inf_{\Gamma} L(\Gamma),$$

where Γ runs through all curves joining a and b.

Then for $0 < a < 1$ we have $\mathrm{dist}_{\mathrm{P}}(0, a) = \log((1 + a)/(1 - a))$.

Exercise 9.20. Calculate the Poincaré distance on $\mathbb{D}$ as follow.

(1) For any $a \in \mathbb{D}$, show that

$$\mathrm{dist}_{\mathrm{P}}(0, a) = \log \frac{1 + |a|}{1 - |a|}.$$

(Hint: consider a rotation r_θ such that $r_\theta(a) = |a|$.)

(2) For any $a, b \in \mathbb{D}$, show that

$$\mathrm{dist}_{\mathrm{P}}(a, b) = \log \frac{|1 - a\overline{b}| + |a - b|}{|1 - a\overline{b}| - |a - b|}.$$

(Hint: consider $h(\xi) = (\xi - b)/(1 - \overline{b}\xi)$ such that $h(b) = 0$ and $h(a) = (a - b)/(1 - a\overline{b})$, and use the fact that the group action of $\mathrm{Aut}(\mathbb{D})$ preserves ds_{P}^2.)

Theorem 9.21. *For all $a, b \in \mathbb{D}$ and $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$, we have*

$$\mathrm{dist}_{\mathrm{P}}(f(a), f(b)) \leqslant \mathrm{dist}_{\mathrm{P}}(a, b).$$

Sketchy Idea for Proof. Use the definition of Poincaré distance and apply the Schwarz-Pick Lemma (Proposition 9.12). $\square$

9.3.2. *Kobayashi Pseudo-Distance.* Let $\Omega \subset \mathbb{C}$ be an open connected set with $x, y \in \Omega$. Consider a sequence of holomorphic maps $f_i : \mathbb{D} \to \Omega$ $(i = 1, 2, \ldots, m)$ and points $p_i, q_i \in \mathbb{D}$ satisfying the following Kobayashi condition:

$$(*) \qquad f_1(p_1) = x,\ f_m(q_m) = y; \quad f_i(q_i) = f_{i+1}(p_{i+1}).$$

Geometrically, this construction is nothing but a chain of discs connecting x and y.

Definition 9.22 (Kobayashi Hyperbolic)**.** We define the *Kobayashi pseudo-distance* as

$$d_{\mathrm{K}}(x, y) := \inf_{f_i, p_i, q_i} \sum_{i=1}^{m} \mathrm{dist}_{\mathrm{P}}(p_i, q_i),$$

where the index runs over all such f_i and p_i, q_i satisfying $(*)$. One can check that for any $x, y, z \in \Omega$,

$$d_{\mathrm{K}}(x, y) = d_{\mathrm{K}}(y, x) \geqslant 0, \quad d_{\mathrm{K}}(x, z) \leqslant d_{\mathrm{K}}(x, y) + d_{\mathrm{K}}(y, z).$$

Moreover, the region Ω is called *Kobayashi hyperbolic* if for all $x, y \in \Omega$ such that $x \neq y$, we always have $d_{\mathrm{K}}(x, y) > 0$.

We introduce the main result due to Kobayashi theory without proof (also leave as an exercise).

Theorem 9.23. *Consider $d_{\mathrm{K}}(\cdot, \cdot)$ on $\mathbb{D}$ and $\mathbb{C}$.*

(1) *For $\mathbb{D}$, we have $d_{\mathrm{K}} = \mathrm{dist}_{\mathrm{P}}$, thus $\mathbb{D}$ is Kobayashi hyperbolic.*
(2) *For $\mathbb{C}$, we have $d_{\mathrm{K}} \equiv 0$, thus $\mathbb{C}$ is not Kobayashi hyperbolic.*

9.4. **The Riemann Mapping Theorem.** The motivation of the Riemann mapping theorem comes from the following natural question.

- Given an open set $\Omega \subset \mathbb{C}$, is Ω conformally equivalent to $\mathbb{D}$?

For this, we make two trivial observations. Firstly, $\Omega \neq \mathbb{C}$ by Liouville (Corollary 3.16) because of the boundedness of $\mathbb{D}$. Secondly, Ω must be simply connected since the biholomorphic functions preserves all topology information.

Surprisingly, the Riemann mapping theorem dictates that these two necessary conditions are also sufficient to determine the conformal equivalence class of $\mathbb{D}$.

Theorem 9.24 (Riemann Mapping Theorem)**.** *Let Ω be a proper (i.e. $\Omega \neq \mathbb{C}$) and simply connected open set of $\mathbb{C}$. Fix $z_0 \in \Omega$. Then there exists a unique biholomorphic map $F : \Omega \to \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) > 0$.*

Proof of the Uniqueness. We prove the uniqueness first. If there are $F, G : \Omega \simeq \mathbb{D}$ satisfying that $F(z_0) = G(z_0) = 0$ and $F'(z_0), G'(z_0) > 0$, then $F \circ G^{-1} \in \mathrm{Aut}(\mathbb{D})$ satisfies $F \circ G^{-1}(0) = 0$. Thus,

$$F \circ G^{-1}(z) = e^{i\theta} \cdot z,$$

that is, $F \circ G^{-1}$ is a rotation on $\mathbb{D}$. However, the condition $F'(z_0), G'(z_0) > 0$ shows that

$$(F \circ G^{-1})'(0) = e^{i\theta} > 0$$

as a real number. Therefore, $e^{i\theta} = 1$ and $F = G$. ☐

Corollary 9.25. *Any two proper simply connected open sets of $\mathbb{C}$ are conformally equivalent.*

The proof for existence is hard. We will consider the function space

$$\mathfrak{F} = \{f : \Omega \to \mathbb{D} \mid f \text{ is holomorphic and injective such that } f(z_0) = 0\}.$$

Some preparation work for this is in need.

9.4.1. *Montel's Theorem.*

Theorem 9.26 (Montel)**.** *Let $\Omega \subset \mathbb{C}$ be an open set and $\mathfrak{F}$ be a family of holomorphic functions on Ω. Assume that $\mathfrak{F}$ is uniformly bounded on every compact set of Ω, i.e., for any compact subset $K \subset \Omega$, there is a constant $B(K) > 0$ such that for each $f \in \mathfrak{F}$, we have $\sup_{z\in K} |f(z)| \leqslant B(K)$. Then*

(1) *$\mathfrak{F}$ is* ***equicontinuous*** *on every compact subset of Ω, i.e., for any compact subset $K \subset \Omega$, for all $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that for each $f \in \mathfrak{F}$, $|f(z) - f(w)| < \varepsilon$ whenever $z, w \in K$ and $|z - w| < \delta(\varepsilon)$;*

(2) *$\mathfrak{F}$ is a* ***normal*** *family, i.e., each sequence in $\mathfrak{F}$ has a subsequence that converges uniformly on every compact subset of Ω.*

Sketchy Proof. (1) By Cauchy integral formula (Theorem 3.13), the condition that all $f \in \mathfrak{F}$ are uniformly bounded on compact sets implies that $\mathfrak{F}$ is equicontinuous on every compact set.

(2) By (1), $\mathfrak{F}$ is equicontinuous and uniformly bounded on every compact set. By Arzela-Ascoli theorem, $\mathfrak{F}$ is normal. □

Proposition 9.27. *Let $\Omega \subset \mathbb{C}$ be open and connected. Suppose $\{f_n\}_{n=1}^\infty$ is a series of injective and holomorphic functions on Ω. Assume $f_n \to f$ uniformly on every compact set of Ω, then f is also injective unless it is a constant.*

Proof. We argue by contradiction and suppose that f is not injective, so there exist distinct complex numbers z_1 and z_2 in Ω such that $f(z_1) = f(z_2)$. Define a new sequence by $g_n(z) = f_n(z) - f_n(z_1)$, so that g_n has no other zero besides z_1, and the sequence $\{g_n\}$ converges uniformly on compact subsets of Ω to $g(z) = f(z) - f(z_1)$. If g is not identically zero, then z_2 is an isolated zero for g (because Ω is connected); therefore, by the argument principle (Theorem 4.23),

$$1 = \frac{1}{2\pi i}\int_\gamma \frac{g'(\zeta)}{g(\zeta)}d\zeta,$$

where γ is a small circle centered at z_2 chosen so that g does not vanish on γ or at any point of its interior besides z_2. Therefore, $1/g_n$ converges uniformly to $1/g$ on γ, and since $g_n' \to g'$ uniformly on γ we have

$$\frac{1}{2\pi i}\int_\gamma \frac{g_n'(\zeta)}{g_n(\zeta)}d\zeta \to \frac{1}{2\pi i}\int_\gamma \frac{g'(\zeta)}{g(\zeta)}d\zeta$$

But this is a contradiction since g_n has no zeros inside γ, and hence

$$\frac{1}{2\pi i}\int_\gamma \frac{g_n'(\zeta)}{g_n(\zeta)}d\zeta = 0$$

for all n. This shows that $g \equiv 0$ and f must be a constant. □

9.4.2. *Proof of the Riemann Mapping Theorem.* The proof are listed in 3 steps.

- **Step 1.** Let $\Omega \subsetneq \mathbb{C}$ be a simply connected open set.

  Claim: Ω is biholomorphic to an open set of $\mathbb{D}$ containing 0.

  Proof of Claim. By translations and rescalings, it is enough to prove that Ω is conformally equivalent to a bounded open set of $\mathbb{C}$. Since Ω is proper and simply connected, there exists $\alpha \notin \Omega$ such that $z - \alpha \neq 0$ for any $z \in \Omega$. Consequently,

  $$f(z) := \log_\Omega(z - \alpha)$$

  is well-defined and holomorphic, and $e^{f(z)} = z - \alpha$. Pick $w \in \Omega$ then

  $$f(z) \neq f(w) + 2\pi i$$

for any $z \in om$. To see this, the case where $z = w$ is obvious. For $z \neq w$, if $f(z) = f(w) + 2\pi i$ then

$$z - \alpha = e^{f(z)} = e^{f(w)} = w - \alpha \quad \Rightarrow \quad z = w,$$

which leads to a contradiction. Moreover, there is an open disc D centered at $f(w) + 2\pi i$ such that $D \cap \overline{f(\Omega)} = \emptyset$. Otherwise there is a series $\{z_n\} \subset \Omega$ such that $f(z_n) \to f(w) + 2\pi i$, and hence $e^{f(z_n)} \to e^{f(w)+2\pi i}$ (i.e. $z_n \to w$). Thus we have $f(z_n)] \to f(w)$, which is a contradiction. We then consider

$$F(z) = \frac{1}{f(z) - (f(w) + 2\pi i)}.$$

As f is injective, for a fixed w, F is injective as well. Hence $F : \Omega \to F(\Omega)$ is biholomorphic. On the other hand, there is some $C > 0$ such that for all $z \in \Omega$, $|F(z)| \leqslant C$. These proves the claim.

- **Step 2.** By Step 1, we can assume $0 \in \Omega \subset \mathbb{D}$. Consider the following family

$$\mathfrak{F} := \{f : \Omega \to \mathbb{D} \mid f \text{ is holomorphic and injective such that } f(0) = 0\}.$$

Since $f(z) = z \in \mathfrak{F}$ we know at least $\mathfrak{F} \neq \emptyset$. Also, $\mathfrak{F}$ is uniformly bounded. Let $s = \sup_{f\in\mathfrak{F}} |f'(0)|$ then $s \geqslant 1$ since $f(z) = z \in \mathfrak{F}$. Moreover, one can prove that $s < \infty$. This is because

$$f'(0) = \frac{1}{2\pi i}\int_{|\xi|=r} \frac{f(\xi)}{\xi^2}d\xi \quad \Rightarrow \quad |f'(0)| \leqslant \frac{1}{r}$$

by higher Cauchy integral formula (Theorem 3.14). Choose a sequence $\{f_n\} \subset \mathfrak{F}$ such that $|f_n'(0)| \to s$. By applying Montel theorem (Theorem 9.26) to $\{f_n\}$, we see there exists a subsequence of $\{f_n\}$ that converges to f uniformly on every compact subset.

(i) By definition, f_n are all injective and $s \geqslant 1$. Hence f is injective as well.

(ii) Since $|f_n| \leqslant 1$ in Ω, we have $|f| \leqslant 1$ in Ω. Applying the maximum principle (Proposition 4.27) to f on the open connected region Ω, we have $|f| < 1$ in Ω. Therefore, the image of f is strictly contained in $\mathbb{D}$.

(iii) Note that $f(0) = 0$.

From (i)-(iii) above, f readily lies in $\mathfrak{F}$ and $|f'(0)| = s$.

- **Step 3.** Let $f : \Omega \to \mathbb{D}$ be the map constructed in Step 2.

Claim: f is surjective, that is, $f(\Omega) = \mathbb{D}$; and therefore f is biholomorphic.

Proof of Claim. Otherwise there is some $\alpha \in \mathbb{D}$ such that for all $z \in \Omega$, $f(z) \neq \alpha$. Consider

$$\widetilde{f} = \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ f$$

where $\psi_\alpha(z) = (\alpha - z)/(1 - \overline{\alpha}z)$ as in Example 9.7. Also, $g : \psi_\alpha \circ f(\Omega) \to \mathbb{C}$ is defined on a simply connected region $\psi_\alpha \circ f(\Omega)$ by $g(w) = \sqrt{w} = \exp((\log w)/2)$, which is injective. Then $\widetilde{f} \in \mathfrak{F}$ where $\mathfrak{F}$ is the same as in Step 2. By the definition of $\widetilde{f}$,

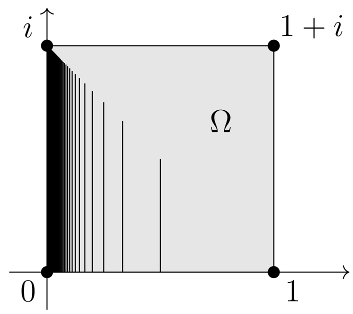
$$f = \Phi \circ \widetilde{f} := \underbrace{\psi_\alpha^{-1} \circ g^{-1} \circ \psi_{g(\alpha)}^{-1}}_{\Phi} \circ \widetilde{f}$$

where $g^{-1}(w) = w^2$. Thus, $f'(0) = \Phi'(\widetilde{f}(0)) \cdot \widetilde{f}'(0) = \Phi'(0) \cdot \widetilde{f}'(0)$. Note that $\Phi : \mathbb{D} \to \mathbb{D}$ satisfies $\Phi(0) = 0$ and Φ is not injective. By the Schwarz lemma (Lemma 9.6), $|\Phi'(0)| < 1$ and $|\widetilde{f}'(0)| > |f'(0)|$, which contradicts with the definition $|f'(0)| = \sup_{h\in\mathfrak{F}} |h'(0)|$.

The whole proof for Theorem 9.24 is accomplished.

Example 9.28 (Topological Comb)**.** We consider a classical example in topology that is (globally) path-connected but not locally path-connected. Say

$$\Omega = \{s \in \mathbb{C} : 0 < \Re(s) < 1, 0 < \Im(s) < 1\} - \bigcup_{n=1}^{\infty} [\frac{1}{n}, \frac{1}{n} + \frac{n-1}{n} i].$$



That is the open set by removing a series of the "comb intervals" from the interior of the square with vertices $0, 1, i, 1+i$. Note that Ω is simply connected and open with $\partial\Omega = \overline{\Omega} \backslash \Omega$ being the union of the comb space and the edges of the square. By Riemann mapping theorem (Theorem 9.24), there is a conformal map $F : \Omega \to \mathbb{D}$.

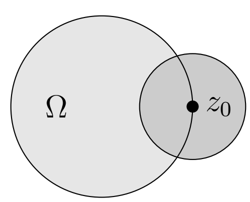
Remark 9.29. The conformal equivalence relation implies the topological homeomorphism. For (the most important) example, suppose $\Omega_1, \Omega_2 \subsetneq \mathbb{C}$ are two simply connected open sets that are conformally equivalent, then $\Omega_1 \simeq \Omega_2$ as a topological homeomorphism. However, given a map that preserves all local angles, it need not be a homeomorphism unless it is bijective.

9.5. **Correspondence of Boundaries.**

Definition 9.30 (Regularity)**.** Let $\Omega \subset \mathbb{C}$ be a bounded region. A point $z_0 \in \partial\Omega$ is called *regular* if there exists $r(z_0) > 0$ such that for all $0 < r < r(z_0)$ we have

$$\Omega \cap \{z \in \mathbb{C} : |z - z_0| = r\} = \{z_0 + re^{i\theta} : \theta_1(r) < \theta < \theta_2(r)\}$$

for some constants with $\theta_1(r) < \theta_2(r)$, which are continuous with respect to r.



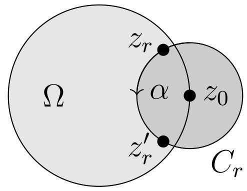
If every point of $\partial\Omega$ is regular, then we say Ω is *regular*.

Example 9.31. If $\partial\Omega$ is C^1 with some corners, then Ω is regular. In particular, if $\partial\Omega$ is piecewise smooth, then Ω is regular.

Theorem 9.32 (Boundary Correspondence)**.** *Let $\Omega \subset \mathbb{C}$ be an open set that is bounded, simply connected, and regular. Then any conformal map $F : \Omega \to \mathbb{D}$ extends to a continuous bijection $F : \overline{\Omega} \to \overline{\mathbb{D}}$. In particular, F induces a homeomorphism from $\partial\Omega$ to $\partial\mathbb{D}$.*

Upshot for Proof: we need to verify $\lim_{z \to z_0, z \in \Omega} F(z)$ exists for any $z_0 \in \partial\Omega$.

We take a lemma as the preparation work. For each $0 < r < r(z_0)$ we denote $C_r = \{z \in \mathbb{C} : |z - z_0| = r\}$. For any given two points $z_r, z_r' \in \Omega \cap C_r$, let $\rho(r) := |F(z_r) - F(z_r')|$. This statement essentially uses the regularity assumption.



Lemma 9.33. *We have* $\liminf_{r\to 0}\rho(r) = 0$.

Proof. We take α as the arc on C_r from z_r to z_r'. Note that F is holomorphic, and hence

$$F(z_r') - F(z_r) = \int_\alpha F'(\xi)d\xi.$$

If $\limsup_{r\to 0}\rho(r) > 0$, i.e., there is some $C > 0$ together with $0 < R \ll 1$ such that $\rho(r) \geqslant C$ for any $0 < r < R$. On the other hand,

$$\begin{aligned}\rho(r) = |\int_\alpha F'(\xi)d\xi| &\leqslant \int_{\theta_1(r)}^{\theta_2(r)} |F'(\xi)|rd\theta \\ &\leqslant (\int_{\theta_1(r)}^{\theta_2(r)} |F'(\xi)|^2 \cdot rd\theta)^{1/2} \cdot (\int_{\theta_1(r)}^{\theta_2(r)} rd\theta)^{1/2} \\ &\leqslant (2\pi r)^{1/2}(\int_{\theta_1(r)}^{\theta_2(r)} |F'(\xi)|^2 \cdot rd\theta)^{1/2}.\end{aligned}$$

Here the second inequality is the Cauchy-Schwarz. This is equivalent to

$$\frac{\rho(r)^2}{r} \leqslant 2\pi \int_{\theta_1(r)}^{\theta_2(r)} |F'(\xi)|^2 \cdot rd\theta.$$

After taking the integral for $0 < r < R$, we have

$$\begin{aligned}\int_0^R \frac{\rho(r)^2}{r}dr &\leqslant 2\pi \int_0^R \int_{\theta_1(r)}^{\theta_2(r)} |F'(\xi)|^2 \cdot rd\theta dr \\ &\leqslant 2\pi \int_\Omega |F'(\xi)|^2 dxdy \\ &= 2\pi \int_{F(\Omega)} dxdy = 2\pi \int_{\mathbb{D}} dxdy = 2\pi^2.\end{aligned}$$

However, as $\rho(r) \geqslant C$ from the assumption,

$$\int_0^R \frac{\rho(r)^2}{r}dr \geqslant C^2 \int_0^R \frac{dr}{r} = \infty.$$

This leads to a contradiction. Thus, $\liminf_{r\to 0}\rho(r) = 0$. $\square$

Exercise 9.34. In the proof of Lemma 9.33 above, we have used the following fact. Let $f : U \to f(U)$ be a conformal map. Prove that

$$\int_U |f'(z)|^2 dxdy = \int_{f(U)} dxdy.$$

Proof for Theorem 9.32. We first prove $\lim_{z\to z_0, z\in\Omega} F(z)$ exists. Otherwise, there are two sequences $\{z_1, z_2, \ldots\}$ and $\{z_1', z_2', \ldots\}$ in Ω with $z_k \to z_0$ and $z_k' \to z_0$ but $F(z_k) \to \xi$, $F(z_k') \to \xi'$ such that $\xi \neq \xi'$. Note that $\xi, \xi' \in \partial\mathbb{D}$ as $F : \Omega \to \mathbb{D}$ is a conformal equivalence. This contradicts with Lemma 9.33.

Now define $F(z_0) = \lim_{z\to z_0, z\in\Omega} F(z)$ for $z_0 \in \partial\Omega$. Then $F : \overline{\Omega} \to \overline{\mathbb{D}}$ is continuous by Lemma 9.33 again. Applying similar argument to $F^{-1} : \mathbb{D} \to \Omega$, we get a continuous extension $F^{-1} : \overline{\mathbb{D}} \to \overline{\Omega}$. It can be verified that

$$F \circ F^{-1} = \mathrm{id}_{\overline{\mathbb{D}}}, \quad F^{-1} \circ F = \mathrm{id}_{\overline{\Omega}}.$$

Therefore, F is a continuous bijection. □

Remarks 9.35. We have some comments on the boundary correspondence and the Riemann mapping theorem.

(1) For the uniqueness in the Riemann mapping theorem (Theorem 9.24), we have the following. Let $\Omega \subset \mathbb{C}$ be a proper and simply connected region with $\partial\Omega$ being a closed piecewise smooth curve. Take three distinct points $z_1, z_2, z_3 \in \partial\Omega$. For arbitrary and distinct $a, b, c \in \partial\mathbb{D}$, there exists a unique conformal map $F : \Omega \to \mathbb{D}$ such that the homeomorphic extension $F : \overline{\Omega} \to \overline{\mathbb{D}}$ maps z_1, z_2, z_3 to a, b, c, respectively.

(2) The boundary correspondence (Theorem 9.32) also holds for domains in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Here $\{\infty\}$ (as a point or a region) is called regular if $\{0\}$ is regular in $\partial\Omega^{-1} := \partial\{z^{-1} : z \in \Omega\}$.

Theorem 9.36 (Extended Riemann Mapping Theorem and Boundary Correspondence)**.** *Suppose $\Omega \subset \overline{\mathbb{C}}$ is a simply connected open subset that is proper (i.e., $\Omega \neq \mathbb{C}$ or $\overline{\mathbb{C}}$). Then*

(1) *there is a conformal map $F : \Omega \to \mathbb{D}$;*

(2) *furthermore, if Ω is also regular, we have the correspondences of boundaries as in Theorem 9.32.*

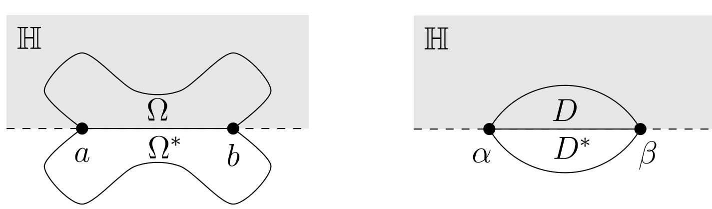
9.6. **Applications of Riemann Mapping Theorem.** This part refers to [Kod07, pp. 224-241]. We will introduce two types of applications about reflections and modular functions respectively.

9.6.1. *The Principle of Reflection.* Recall the Schwarz reflection principle (Proposition 3.30) which deduces the following theorem.

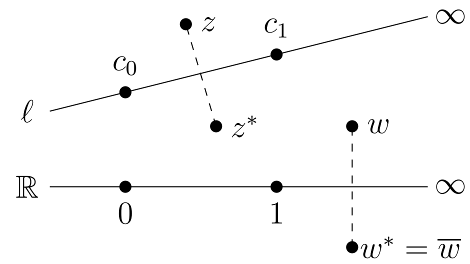
Theorem 9.37. *Suppose Ω, D are open subsets of the upper-half plane $\mathbb{H}$ whose boundaries intersect $\mathbb{R}$ with at least an interval. Assume (a, b) is part of boundary of Ω and (α, β) is part of boundary of D on $\mathbb{R}$. If $F : \Omega \to D$ is a conformal map extending homeomorphically such that $F : \Omega \cup (a, b) \to D \cup (\alpha, \beta)$, then we can extend F to a conformal map as follows:*

$$\widetilde{F} : \Omega \cup (a, b) \cup \Omega^* \longrightarrow D \cup (\alpha, \beta) \cup D^*$$
$$z \longmapsto \begin{cases} F(z), & z \in \Omega \cup (a, b); \\ \overline{F(\overline{z})}, & z \in \Omega^*. \end{cases}$$

Here Ω^* and D^* denote the reflection image of Ω and D with respect to the real axis, respectively. Or equivalently, $\Omega^* := \{z \in \mathbb{C} : \overline{z} \in \Omega\}$ and $D^* := \{z \in \mathbb{C} : \overline{z} \in D\}$.



Now we are to study reflections with respect to a line or a circle. Consider the equation $\lambda(w) := (c_1 - c_0)w + c_0$. If $w \in \mathbb{R}$, then $\lambda(w) = 0$ is the equation of ℓ that passes through $c_0, c_1 \in \mathbb{C}$. If $w \in \mathbb{C}$, then $\lambda(\cdot) : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a conformal equivalence and maps $\mathbb{R} \cup \{\infty\}$ onto $\ell \cup \{\infty\}$; and such that $\lambda(0) = c_0$, $\lambda(1) = c_1$, and $\lambda(\infty) = \infty$.

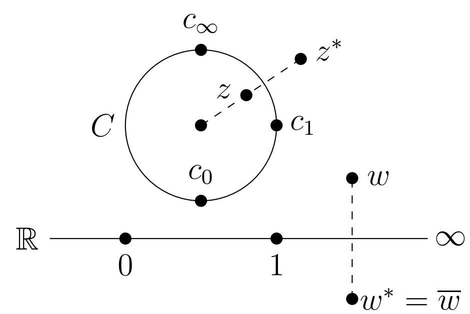


Exercise 9.38. Denote z^* the image of $z \in \mathbb{C}$ under the reflection by ℓ.

(1) Let $z = \lambda(w)$. Show that $z^* = \lambda(\overline{w})$.
(2) If μ is another linear fractional transform, i.e.,
$$\mu(z) = \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}).$$
which maps $\mathbb{R} \cup \{\infty\}$ to $\ell \cup \{\infty\}$. Prove that $z = \mu(w)$ implies $z^* = \mu(\overline{w})$.

Remark 9.39. Suppose $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. Then
$$\varphi(z) = \frac{az+b}{cz+d} \in \mathrm{Aut}(\overline{C}).$$
Note that $\varphi(\infty) = a/c$ and $\varphi(-d/c) = \infty$. So
$$\varphi : \overline{\mathbb{C}} \backslash \{\infty, -\frac{d}{c}\} = \mathbb{C} \backslash \{-\frac{d}{c}\} \simeq \overline{\mathbb{C}} \backslash \{\infty, \frac{a}{c}\} = \mathbb{C} \backslash \{\frac{a}{c}\}.$$

Now let C be a circle on $\mathbb{C}$ and c_0, c_1, c_∞ be three distinct points on C. Set $\lambda(\cdot) : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ by
$$\lambda(w) = \frac{(c_0 - c_\infty)(c_\infty - c_1)}{(c_1 - c_0)w + (c_\infty - c_1)} + c_\infty.$$
It can be verified that $\lambda(0) = c_0$, $\lambda(1) = c_1$, and $\lambda(\infty) = c_\infty$.

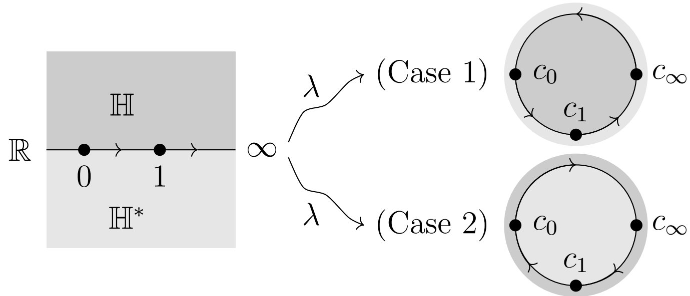


The idea to deal with the reflections with respect to a circle is to regard a line ℓ as a circle with infinite radius, say $\{z \in \ell : |z + \infty| = \infty\}$. Then the linear fractional transforms always map a circle to another circle. In particular, λ maps $\mathbb{R} \cup \{\infty\}$ onto C.

Definition 9.40. For $z = \lambda(w)$ with λ defined as above, the *reflection of z with respect to the circle C* is defined by
$$z^* = \lambda(\overline{w}) = \frac{(c_0 - c_\infty)(c_\infty - c_1)}{(c_1 - c_0)\overline{w} + (c_\infty - c_1)} + c_\infty.$$

Remark 9.41. It can be checked that the definition of $z \mapsto z^*$ is independent of the choice of λ, or equivalently, independent of the choice of points c_0, c_1, c_∞. For any linear fractional transformation $\mu \in \mathrm{GL}_2(\mathbb{C})$ which maps $\mathbb{R} \cup \{\infty\}$ onto C, we have $z = \mu(w)$ implying $z^* = \mu(\overline{w}) = \lambda(\overline{w})$.

Note that whether λ preserves the direction of $\mathbb{R}$ from $-\infty$ to ∞ or not leads to two different cases.



In Case 1 above, λ maps $\mathbb{H}$ conformally to the interior of C; but in Case 2, it maps $\mathbb{H}^*$ to the interior of $\mathbb{C}$. One can prove that in any case, $\lambda(\cdot)$ can be written as

$$z = \lambda(w) = c + Re^{i\theta} \cdot \frac{w - w_0}{w - \overline{w_0}}$$

for some $\theta \in \mathbb{R}$, where $c \in \mathbb{C}$ and $R > 0$ denotes the center and the radius of C, respectively. The point w_0 is chosen to be on $\mathbb{H}$ in Case 1, and on $\mathbb{H}^*$ in Case 2. Therefore, $z = \lambda(w)$ has its reflection with respect to C given by

$$z^* = \lambda(\overline{w}) = c + Re^{i\theta} \cdot \frac{\overline{w} - w_0}{\overline{w} - \overline{w_0}}.$$

By an easy comparison on expressions of $z = \lambda(w)$ and $z^* = \lambda(\overline{w})$, we get the **Circle-Power Theorem** in Euclid geometry:

$$(\overline{z - c}) \cdot (z^* - c) = R^2.$$

There is a consequence of this result at once, say for $r > 0$ and $\varphi \in \mathbb{R}$,

$$z - c = re^{i\varphi} \quad \Longleftrightarrow \quad z^* - c = \frac{R^2}{r} e^{i\varphi}.$$

As expected, note that the reflection map with respect to C is an involution (that is, $(z^*)^* = z$) and preserves the circle. According to the convention, write $c^* = \infty$ and $\infty^* = c$ for the center c of C. Finally, proof of the following proposition is left as an exercise.

Proposition 9.42. *Reflections with respect to a line or a circle are invariant under linear fractional transforms. In other words, the reflections commute with the action of* $\mathrm{GL}_2(\mathbb{C})$*. To be more precise, if* C *is a line or a circle and* $\mu \in \mathrm{GL}_2(\mathbb{C})$*, then*

$$\mu(z^*) = \mu(z)^*.$$

On the right hand side, $\mu(z) \mapsto \mu(z)^*$ *is a reflection with respect to* $\mu(C)$*.*

Theorem 9.43 (The Principle of Reflection)**.** *We make the following statements.*

- C *is a circle in* $\mathbb{C}$ *with center* c*.*
- Ω *is a connected open subset contained in the interior of* C *or the exterior of* C*, satisfying* $c \notin \Omega$*.*
- $\gamma \subsetneq C$ *is a part of the boundary of* Ω*, i.e.,* $\gamma \subset \partial\Omega = \overline{\Omega} \backslash \Omega$*.*
- $D \subset \mathbb{H}$ *is a connected open subset with a real interval* (α, β) *as a part of* ∂D*.*

If the conformal map $f : \Omega \to D$ *extends to a homeomorphism* $f : \Omega \cup \gamma \to D \cup (\alpha, \beta)$*, then* f *extends to a conformal map*

$$g : \Omega \cup \gamma \cup \Omega^* \to D \cup (\alpha, \beta) \cup D^*,$$

which is defined by

$$g(z) = \begin{cases} f(z), & z \in \Omega \cup \gamma; \\ \overline{f(z^*)}, & z \in \Omega^*. \end{cases}$$

Note that Theorem 9.43 generalizes Theorem 9.37 but preserves all essential ingredients. In short, the result dictates that if a conformal map extends to some boundary of the reflection axis (which is a circle or a line), then it extends to the reflection image as well.

Proof. Assume Ω is in the interior of C with $c \notin \Omega$. Recall that for some $\theta \in \mathbb{R}$,

$$\lambda(w) = c + Re^{i\theta} \cdot \frac{w - w_0}{w - \overline{w_0}}.$$

We can choose some θ such that $\lambda(\infty) \in C \backslash \gamma$. Set $w_0 = \lambda^{-1}(c)$. Note that $\lambda : \mathbb{C}\backslash\{\overline{w_0}\} \simeq \mathbb{C}\backslash\{\lambda(\infty)\}$ is conformal. Conversely,

$$\lambda^{-1} : \mathbb{C}\backslash\{\lambda(\infty)\} \to \mathbb{C}\backslash\{\overline{w_0}\}$$

is conformal. Since $c \notin \Omega$, we have $\infty \notin \Omega^*$. Hence $\Omega \cup \gamma \cup \Omega^* \subset \mathbb{C}\backslash\{\lambda(\infty)\}$. Applying the Schwarz reflection principle (Proposition 3.30) to $g \circ \lambda^{-1}$, we get the result. The case where Ω lies in the exterior of C is similar. $\square$

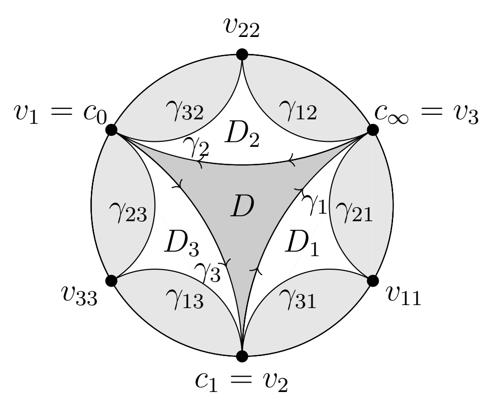
Remark 9.44. The same result as in Theorem 9.43 also holds when C is a line in $\mathbb{C}$ if Ω is on one side of C and $\gamma \subset C$ is a segment such that $\gamma \subset \overline{\Omega}\backslash\Omega$.

Moreover, in case Ω is a simply connected open set of $\mathbb{C}$ such that $\Omega \cup C = \emptyset$, $\gamma \subset \partial\Omega$, and $\gamma \subsetneq C$, where C is a circle or a line. By the Riemann mapping theorem (Theorem 9.24), we have the following result.

Theorem 9.45. *A conformal map $f : \Omega \to \mathbb{H}$ whose extension maps γ to a segment $(\alpha, \beta) \subset \partial\mathbb{H}$ can extend to a conformal map*

$$g : \Omega \cup \gamma \cup \Omega^* \xrightarrow{\sim} \mathbb{H} \cup (\alpha, \beta) \cup \mathbb{H}^*.$$

9.6.2. *Construction of a Modular Function.* Let's recall Theorem 9.23 that $\mathbb{D}$ is (Kobayashi) hyperbolic. Let $C = \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ with distinct points $(c_0, c_1, c_\infty) \in C$. The partition of $\mathbb{D}$ is given as follows.



Step 0. As D is simply connected, by Riemann mapping theorem and boundary correspondence (Theorem 9.36), there exists a unique conformal map $f : D \to \mathbb{H}$ whose extension

$$f : \overline{D} \to \overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$$

is a homeomorphism satisfying $f|_{\partial D} : \partial D \to R \cup \{\infty\}$ with $f(c_0) = 0$, $f(c_1) = 1$, and $f(c_\infty) = \infty$. Then the correspondence is read as

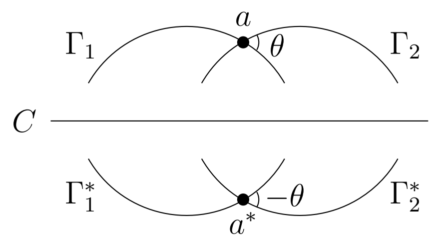
$$f(\gamma_1) = (1, \infty), \quad f(\gamma_2) = (-\infty, 0), \quad f(\gamma_3) = (0, 1).$$

Step 1. Reflections with respect to γ_j $(j = 1, 2, 3)$.

Proposition 9.46. *Let C be a line or a circle and let c be the center of C if C is a circle.*

(1) *Let Γ_1, Γ_2 be smooth curves in $\mathbb{C}$ intersecting at $a \neq c$, let θ be the angle between Γ_1, Γ_2 at a. If $\Gamma_1^*, \Gamma_2^*, a^*$ are the reflection images with respect to C, then the angle between Γ_1^*, Γ_2^* is $-\theta$.*

(2) *The reflection $z \mapsto z^*$ with respect to C maps circles (resp. lines) onto circles (resp. lines).*



Proof. Note that (1)(2) hold for $C = \mathbb{R} \cup \{\infty\}$. For the general case, apply a linear fractional transform $\lambda : \mathbb{R} \cup \{\infty\} \to C$ to complete the proof. $\square$

Applying Proposition 9.46 to our setting, we get the following result.

- The reflections with respect to γ_j map $\partial\mathbb{D}$ to $\partial\mathbb{D}$.
- The reflections with respect to γ_j map the intersection angle (with $\partial\mathbb{D}$) $\pi/2$ to $\pi/2$.
- The reflections with respect to γ_j map D to the interior of $\mathbb{D}$.

By the principle of reflections (Theorem 9.43) applying to the reflection with respect to γ_1, we extend f to a conformal map

$$f_1 : D \cup \gamma_1 \cup D^* \xrightarrow{\sim} \mathbb{H} \cup (1, \infty) \cup \mathbb{H}^*.$$

Here $f_1(D^*) = \mathbb{H}^*$ and $f_1(\gamma_1) = (1, \infty)$. It is also such that $f_1(\gamma_{21}) = (-\infty, 0)$ and $f_1(\gamma_{31}) = (0, 1)$. Similarly, we have maps f_2 and f_3. Denote

$$S^{(1)} = D \cup \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup D_1 \cup D_2 \cup D_3.$$

Define $g_1 : S^{(1)} \to \mathbb{C}$ as

$$g_1(z) = \begin{cases} f(z), & z \in D \cup \gamma_j \ (j = 1, 2, 3); \\ f_j(z), & z \in D_j \ (j = 1, 2, 3). \end{cases}$$

Then $g_1 : S^{(1)} \to \mathbb{C}\backslash\{0, 1\}$ is a holomorphic map.

Step 2. Reflections with respect to γ_{ij} $(i, j = 1, 2, 3)$.

By the similar construction as in Step 1, we get another holomorphic map

$$g_2 : S^{(2)} \to \mathbb{C}\backslash\{0, 1\}.$$

Again, using the induction, we have for all $n \in \mathbb{N}$ that

$$g_n : S^{(n)} \to \mathbb{C}\backslash\{0, 1\}.$$

Here $D \subset S^{(1)} \subset S^{(2)} \subset \cdots \subset S^{(n)}$ and $\bigcup_{n=1}^{\infty} S^{(n)} = \mathbb{D}$. Gluing these up, we get a holomorphic map

$$g : \mathbb{D} \to \mathbb{C}\backslash\{0,1\},$$

which is the so-called **modular function on** $\mathbb{D}$.

As for some application, the following result is a corollary for the existence of g.

Proposition 9.47 (Little Picard Theorem)**.** *If $f : \mathbb{C} \to \mathbb{C}\backslash\{0,1\}$ is a holomorphic function, then f is a constant.*

Proof. From the existence of the modular function g, we obtain a commutative diagram

$$\begin{tikzcd} & \mathbb{D} \arrow[d, "g"] \\ \mathbb{C} \arrow[ur, "\widehat{f}"] \arrow[r, "f"] & \mathbb{C}\backslash\{0,1\} \end{tikzcd}$$

such that f has a holomorphic lifting $\widehat{f}$ such that $g \circ \widehat{f} = f$. But by Liouville (Corollary 3.16), $\widehat{f}$ must be a constant as it is bounded and holomorphic on $\mathbb{C}$. Then f is a constant as well. ☐

Note that Proposition 9.47 also holds for those functions to the punctured complex plane with exactly two points missed, i.e., for $f : \mathbb{C} \to \mathbb{C}\backslash\{c_0, c_1\}$ with $c_0 \neq c_1 \in \mathbb{C}$.

10. An Introduction to Elliptic Functions

In short, elliptic functions are meromorphic functions defined on $\mathbb{C}/L$, where L is a lattice of $\mathbb{C}$. These functions are called "elliptic" because the domain $\mathbb{C}/L \simeq \mathbb{C}/\mathbb{Z}^2$ can be not only interpreted as a torus but also an elliptic curve.

10.1. Basics on Elliptic Functions.

Definition 10.1 (Lattice)**.** A *lattice* L of $\mathbb{C}$ is a subgroup of $(\mathbb{C}, +)$ which is generated by $\omega_1, \omega_2 \in \mathbb{C}$ over $\mathbb{Z}$ such that ω_1, ω_2 generates $\mathbb{C}$ over $\mathbb{R}$. That is,

$$L = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}.$$

Without loss of generality, we assume $\Im(\omega_1/\omega_2) > 0$.

Definition 10.2 (Elliptic Function)**.** An *elliptic function* f with respect to a lattice L of $\mathbb{C}$ is a non-constant meromorphic function on $\mathbb{C}$ which is L-periodic, i.e.,

$$f(z + \omega) = f(z), \quad \forall z \in \mathbb{C},\ \omega \in L.$$

Or equivalently, for all $z \in \mathbb{C}$,

$$f(z) = f(z + \omega_1) = f(z + \omega_2).$$

Remark 10.3. For the second condition on L-periodicity above, let $\omega_1, \omega_2 \in \mathbb{C}$ be arbitrary. If $\omega_1/\omega_2 \in \mathbb{Q}$ then f is periodic with a single period. If $\omega_1/\omega_2 \in \mathbb{R}\backslash\mathbb{Q}$, then f must be a constant.

Proposition 10.4. *An elliptic function which is entire is a constant function.*

Proof. An elliptic function descends to a function on the torus $\mathbb{C}/L$, which is compact. The equivalence relation in $\mathbb{C}/L$ is given by

$$z_1 \equiv z_2 \bmod L \quad \Longleftrightarrow \quad z_1 - z_2 = m\omega_1 + n\omega_2 \text{ for some } (m,n) \in \mathbb{Z}^2.$$

If the function is entire, then it is bounded. By Liouville (Corollary 3.16), it must be a constant. ☐

Definition 10.5 (Fundamental Parallelogram)**.** Let $L = [\omega_1, \omega_2]$ be a lattice of $\mathbb{C}$ and suppose $\alpha \in \mathbb{C}$. Then the set

$$P = \{\alpha + t_1\omega_1 + t_2\omega_2 : 0 \leqslant t_1 < 1, 0 \leqslant t_2 < 1\}$$

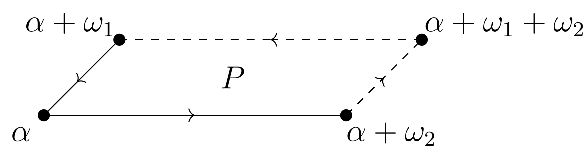
is called a *fundamental parallelogram* of L.

It's not hard to see that if f is elliptic, then f is determined by its behavior in P.

Theorem 10.6. *Let f be elliptic with respect to L and P be a fundamental parallelogram for L. Assume f has no poles on ∂P. Then*

$$\sum_{z \in P} \operatorname{res}_z f = 0.$$

Proof. Suppose P has a vertex, say α as follows.



By the residue formula (Theorem 4.8), we have

$$\begin{aligned}
2\pi i \sum_{z\in P} \operatorname{res}_z f &= \int_{\partial P} f(z)dz \\
&= \int_{\alpha}^{\alpha+\omega_2} f(z)dz + \int_{\alpha+\omega_2}^{\alpha+\omega_1+\omega_2} f(z)dz + \int_{\alpha+\omega_1+\omega_2}^{\alpha+\omega_1} f(z)dz + \int_{\alpha+\omega_1}^{\alpha} f(z)dz \\
&= \int_{\alpha}^{\alpha+\omega_2} f(z)dz + \int_{\alpha+\omega_2}^{\alpha+\omega_1+\omega_2} f(z)dz - \int_{\alpha}^{\alpha+\omega_2} f(z)dz - \int_{\alpha+\omega_2}^{\alpha+\omega_1+\omega_2} f(z)dz \\
&= 0
\end{aligned}$$

by the double-periodicity. $\square$

Corollary 10.7. *The total number of poles (counted with multiplicities) of an elliptic function f in P is not less than 2.*

Proof. If f has no poles on ∂P, then the result follows from Theorem 10.6. Otherwise f has poles on ∂P, then consider a slight perturbation of P to $P + h$ with $|h| \ll 1$. By applying Theorem 10.6 to $P + h$ again, we get the result. $\square$

Theorem 10.8. *Let P be a fundamental parallelogram and f be an elliptic function. Let $\{a_n\}_{n=1}^N$ be the collection of all zeros and poles of f in P with order $\operatorname{ord}_{a_i} f = m_i$, respectively. (Recall that $m_i > 0$ if a_i is a zero and that $m_i < 0$ if a_i is a pole.) Then*

$$\sum_{i=1}^N m_i = 0.$$

Proof. Note that f'/f is elliptic as well as f since it is meromorphic. Therefore, by the argument principle (Theorem 4.23), if f has no zeros or poles along ∂P, then

$$\sum_{i=1}^N m_i = \frac{1}{2\pi i}\int_{\partial P} \frac{f'(z)}{f(z)}dz = 0$$

as f'/f is elliptic (c.f. the proof of Theorem 10.6 above). Again, if f has zeros or poles on ∂P, then consider a slight perturbation $P + h$ and apply the same argument. $\square$

Exercise 10.9. Keeping the same hypothesis as in Theorem 10.8, prove that

$$\sum_{i=1}^{N} m_i a_i \equiv 0 \bmod L,$$

i.e. $\sum_{i=1}^{N} m_i a_i = k\omega_1 + \ell\omega_2$ for some $(k, \ell) \in \mathbb{Z}^2$. (Hint: consider the integral $\int_{\partial P}(zf'(z)/f(z))dz$ and apply the residue formlua (Theorem 4.8).)

10.2. **Weierstrass $\wp$ Function.** Suppose $[\omega_1, \omega_2]$ is a lattice of $\mathbb{C}$ and

$$L^* := \{m\omega_1 + n\omega_2 : (m, n) \in \mathbb{Z}^2\backslash(0, 0)\} = L\backslash\{(0, 0)\}.$$

The Weierstrass $\wp$ Function is defined over $\mathbb{C}$ but essentially depends on the choice of L.

Definition 10.10. The *Weierstrass $\wp$ function* for L is defined as

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega \in L^*} (\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2})$$

for all $z \in \mathbb{C}$.

Theorem 10.11. *$\wp$ is elliptic with respect to L.*

Proof. The first step is to verify that the sum in $\wp$ converges uniformly on all compact sets that include no lattice points. For $|z| < \infty$ staying away from L,

$$\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = \frac{-z^2 + 2z\omega}{\omega^4 - 2z\omega^3 + \omega^2 z^2} \sim O(\frac{1}{|\omega|^3}).$$

The following fact will be at work for this.
Fact: for $\lambda > 2$, the infinite sum converges:

$$\sum_{\omega \in L^*} \frac{1}{|\omega|^\lambda} = \sum_{(m,n)\in\mathbb{Z}^2\backslash\{(0,0)\}} \frac{1}{|m\omega_1 + n\omega_2|^\lambda} < \infty.$$

Coming back to the proof, note that $|\omega| = |m\omega_1 + n\omega_2| \sim |m| + |n|$ and hence

$$\frac{1}{|\omega|^\lambda} \sim \frac{1}{(|m| + |n|)^\lambda}.$$

To estimate the right hand side, for fixed n,

$$\begin{aligned}
\frac{1}{|n|^\lambda} + 2\sum_{m\geqslant 1} \frac{1}{(|m| + |n|)^\lambda} &= \frac{1}{|n|^\lambda} + 2\sum_{k\geqslant |n|+1} \frac{1}{k^\lambda} \\
&\leqslant \frac{1}{|n|^\lambda} + 2\int_{|n|}^{\infty} \frac{1}{x^\lambda}dx \\
&\leqslant \frac{1}{|n|^\lambda} + \frac{C}{|n|^{\lambda-1}}.
\end{aligned}$$

Using this property, we see

$$\begin{aligned}
\sum_{(m,n)\in\mathbb{Z}^2\backslash\{(0,0)\}} \frac{1}{(|m| + |n|)^\lambda} &= \sum_{m\neq 0} \frac{1}{|m|^\lambda} + \sum_{m\in\mathbb{Z}, n\neq 0} \frac{1}{(|m| + |n|)^\lambda} \\
&\leqslant \sum_{m\neq 0} \frac{1}{|m|^\lambda} + \sum_{n\neq 0}(\frac{1}{|n|^\lambda} + \frac{C}{|n|^{\lambda-1}}) \\
&< \infty
\end{aligned}$$

as $\lambda > 2$ due to the fact above. Therefore, $\wp$ is a meromorphic function on $\mathbb{C}$ with a double pole at each $\omega \in L$. Furthermore, $\wp$ is even, i.e. $\wp(z) = \wp(-z)$. Also note that

$$\wp'(z) = -2\sum_{\omega\in L}\frac{1}{(z-\omega)^3}$$

and thus $\wp'$ is L-periodic and odd. It suffices to check whether $\wp$ is L-periodic or not. As for any z,

$$\wp'(z+\omega_1) = \wp'(z) \quad \Rightarrow \quad \wp(z+\omega_1) - \wp(z) = c$$

for some constant c. The claim is that $c = 0$. To see this, let $z = -\omega_1/2$ and get

$$\wp(\frac{\omega_1}{2}) = \wp(-\frac{\omega_1}{2}) + c = \wp(\frac{\omega_1}{2}) + c \quad \Rightarrow \quad c = 0.$$

Here the second equality holds because $\wp(\cdot)$ is even. Thus $\wp(z+\omega_1) = \wp(z)$ and similarly, $\wp(z+\omega_2) = \wp(z)$. $\square$

Note that the set of all elliptic functions (with respect to a fixed lattice L) forms a field, denoted by $\mathfrak{m}(\mathbb{C}/L)$, which contains $\mathbb{C}$ as the constant field. Here $\mathfrak{m}(\mathbb{C}/L)$ is called the function field of the torus $\mathbb{C}/L$.

Theorem 10.12. *The field $\mathfrak{m}(\mathbb{C}/L)$ is generated by $\wp$ and $\wp'$, i.e.,*

$$\mathfrak{m}(\mathbb{C}/L) = \mathbb{C}(\wp, \wp').$$

Or equivalently, any elliptic function on $\mathbb{C}/L$ is a rational function of $\wp$ and $\wp'$.

Proof. If f is elliptic, we write

$$f(z) = \underbrace{\frac{f(z)+f(-z)}{2}}_{\text{even, elliptic}} + \underbrace{\frac{f(z)-f(-z)}{2}}_{\text{odd, elliptic}}.$$

If f is odd, then $f\cdot\wp'$ is even. We only need to prove that the field of even elliptic functions is equal to $\mathbb{C}(\wp)$.

Fact. Let f be an even elliptic function, then:

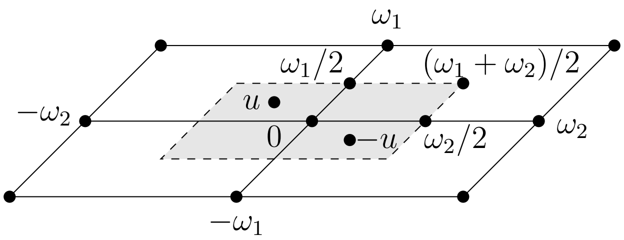
- if f has a zero (resp. pole) of order m at some point u, then f has also a zero (resp. pole) of order m at $-u$;
- if $u \equiv -u \bmod L$ (or $2u \equiv 0 \bmod L$), then f has either a zero or a pole of even order at u.

Using the fact in particular, f has a zero or a pole of even order at $z = 0$. Hence there exists some $m \in \mathbb{Z}$ such that $f\cdot\wp^m$ has no poles or zeros at $z = 0$ (thus at all points of L).

We now assume $u \not\equiv 0 \bmod L$ and let $g(z) := \wp(z) - \wp(u)$. The result above shows that $g(z)$ has a zero of even order at u if $2u \equiv 0 \bmod L$, i.e., $u \equiv \omega_1/2, \omega_2/2, (\omega_1+\omega_2)/2 \bmod L$. By Theorem 10.8,

$$\sum_{z\in P}\mathrm{ord}_z(g) = 0$$

so $\mathrm{ord}_u(g) = 2$ if $u \equiv \omega_1/2, \omega_2/2, (\omega_1+\omega_2)/2 \bmod L$. Under the same assumption, g has zeros at u and $-u$ of order 1.



Without loss of generality, we can assume f has no zeros or poles at points of L. Let $u_1, u_2, \ldots, u_r$ be points in P where f has a zero or pole. Let

$$m_i = \begin{cases} \operatorname{ord}_{u_i} f, & \text{if } 2u_i \not\equiv 0 \bmod L; \\ (\operatorname{ord}_{u_i} f)/2, & \text{if } 2u_i \equiv 0 \bmod L. \end{cases}$$

Define $G(z) := \prod_{i=1}^{r}(\wp(z)-\wp(u_i))^{m_i}$, then G has the same order at u_i as f does. Then $f(z)/G(z)$ is entire (and elliptic) so that $f(z)/G(z) = C$ for some constant C by Liouville (Corollary 3.16). $\square$

10.2.1. *The Canonical Elliptic Curve.* For the half-periods $\omega_1/2$, $\omega_2/2$, and $(\omega_1+\omega_2)/2$, denote

$$e_1 = \wp(\frac{\omega_1}{2}), \quad e_2 = \wp(\frac{\omega_2}{2}), \quad e_3 = \wp(\frac{\omega_1+\omega_2}{2}).$$

Then the equation $\wp(z) = e_1$ (resp. e_2, e_3) has a double root at ω_1 (resp. $\omega_2/2$, $(\omega_1+\omega_2)/2$) because of the fact in the proof of Theorem 10.12. Also, e_1, e_2, e_3 are distinct. Moreover,

$$\wp'(\frac{\omega_1}{2}) = \wp'(\frac{\omega_2}{2}) = \wp'(\frac{\omega_1+\omega_2}{2}) = 0$$

and the order equals to 1 at every point. Then $\wp'(z)^2$ and $(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)$ have the same zeros and poles in P. So there is a constant C such that

$$\frac{\wp'(z)^2}{(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)} = C.$$

A natural question for this is to ask for the value of C. Consider the power series of $\wp$ and $\wp'$ near $z = 0$:

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \sum_{\omega\in L^*}(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}) \\
&= \frac{1}{z^2} + \sum_{\omega\in L^*}(\frac{1}{\omega^2}\cdot\frac{1}{(1-z/\omega)^2} - \frac{1}{\omega^2}) \\
&= \frac{1}{z^2} + \sum_{\omega\in L^*}(\frac{1}{\omega^2}\cdot(1+\frac{z}{\omega}+(\frac{z}{\omega})^2+\cdots)^2 - \frac{1}{\omega^2}) \\
&= \frac{1}{z^2} + \sum_{\omega\in L^*}\sum_{m=1}^{\infty}(2m+1)\cdot(\frac{z}{\omega})^{2m}\cdot\frac{1}{\omega^2} \\
&= \frac{1}{z^2} + \sum_{m=1}^{\infty} c_m\cdot z^m,
\end{aligned}$$

where $c_m = \sum_{\omega\in L^*}(m+1)/\omega^{m+2}$. Denote

$$E_m(L) = E_m := \sum_{\omega\in L^*}\frac{1}{\omega^m},$$

which is the *Eisenstein series of order* m. By this,

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{n=1}^{\infty}(2n+1)E_{2n+2}(L)\cdot z^{2n}\\ &= \frac{1}{z^2} + 3E_4\cdot z^2 + 5E_6\cdot z^4 + 7E_8\cdot z^6 + \cdots ,\end{aligned}$$

and

$$\begin{aligned}\wp'(z) &= -\frac{2}{z^3} + \sum_{n=1}^{\infty}2n(2n+1)E_{2n+2}(L)\cdot z^{2n-1}\\ &= -\frac{2}{z^3} + 6E_4\cdot z + 20E_6\cdot z^3 + 42E_8\cdot z^5 + \cdots ,\end{aligned}$$

Therefore, by a comparison on leading terms of these two equations, we have $C = 4$. In other (geometric) words, the point $(\wp(z), \wp'(z))$ that is parametrized by $z \in \mathbb{C}\backslash L$ lies on the cubic curve

$$\{(x,y)\in\mathbb{C}^2 : y^2 = 4(x-e_1)(x-e_2)(x-e_3)\}\subset\mathbb{C}^2.$$

Again, from the two equations above,

$$\begin{aligned}\wp'(z)^2 &= \frac{4}{z^6} - \frac{24E_4}{z^2} - 80E_6 + \cdots ,\\ \wp(z)^3 &= \frac{1}{z^6} + \frac{9E_4}{z^2} + 15E_6 + \cdots ,\\ 60E_4\wp(z) &= \frac{60E_4}{z^2} + 180E_4^2z^2 + \cdots .\end{aligned}$$

By comparison, we see the function $\wp'(z)^2 - 4\wp(z)^3 + 60E_4\wp(z) + 140E_6$ is holomorphic near $z = 0$ and vanishes at $z = 0$. Then

$$\wp'(z)^2 = 4\wp(z)^3 - 60E_4\wp(z) - 140E_6.$$

Denote $g_2 = 60E_4$ and $g_3 = 140E_6$. Then $\wp'(z) = 4\wp(z)^2 - g_2\wp(z) - g_3$.

Proposition 10.13 (Weierstrass Canonical Form). *For any $z \in \mathbb{C}\backslash L$, the point $(\wp(z), \wp'(z))$ is on the cubic curve defined by*

$$A_{\mathbb{C}} : y^2 = 4(x-e_1)(x-e_2)(x-e_3) = 4x^3 - g_2x - g_3 \subset \mathbb{C}^2 \subset \mathbb{P}^2_{\mathbb{C}}.$$

This curve is called an elliptic curve of Weierstrass canonical form. Moreover, as e_1, e_2, e_3 are distinct, the discriminant of equation $4x^3 - g_2x - g_3 = 0$ is nonzero, say

$$\Delta = g_2^3 - 27g_3^2 \neq 0.$$

Remark 10.14 (j-invariant). Continuing with Proposition 10.13, define the j-invariant by

$$J = \frac{g_2^3}{\Delta},\quad j = 1728\cdot\frac{g_2^3}{\Delta} = 2^6\cdot 3^3\cdot\frac{g_2^3}{\Delta}.$$

Then J and $j = 1728J$ are invariants of L. Also note that there is a (non-canonical) isomorphism

$$\begin{aligned}(\mathbb{C}/L)\backslash\{(0,0)\} &\xrightarrow{\sim} \overline{A_{\mathbb{C}}}\backslash\{\infty\}\\ z &\longmapsto (1, \wp(z), \wp'(z)).\end{aligned}$$

This isomorphism interprets why *a complex torus can be regarded as an elliptic curve.*

10.2.2. *Fourier Expansion and q-Expansion.* Keep the same assumption as before. By considering $F(z) := f(\omega_2 z)$, we see F is elliptic with respect to a new lattice $[\tau = \omega_1/\omega_2, 1]$ with $\tau \in \mathbb{H}$ (recall that we have assumed $\Im(\omega_1/\omega_2) > 0$.

Definition 10.15. For $\tau \in \mathbb{H}$, we call $L = [\tau, 1]$ a *normalized lattice* of $\mathbb{C}$.

As for the Eisenstein series

$$E_m(\tau) = \sum_{(k,\ell)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(k\tau+\ell)^m}, \quad m \in 2\mathbb{N}, \quad \tau \in \mathbb{H}.$$

For $m > 2$, $E_m(\tau)$ is absolutely convergent. However, for $m = 2$, it is not absolutely convergent but $\sum_k \sum_\ell (k\tau+\ell)^{-2}$ is convergent, i.e., we can define

$$E_2(\tau) := \sum_{k=0,\ell\in\mathbb{Z}\setminus\{0\}} \frac{1}{\ell^2} + \sum_{k\neq 0} \sum_{\ell\in\mathbb{Z}} \frac{1}{(k\tau+\ell)^2}.$$

The remaining task of this part is to expand $E_{2k}(\tau)$. By Hadamard factorization theorem (Theorem 6.13),

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2}).$$

Using this, we have

$$\pi \frac{\cos \pi z}{\sin \pi z} = (\log(\sin \pi z))' = (\log(\pi z \prod_{n=1}^{\infty}(1 - \frac{z^2}{n^2})))' = \frac{1}{z} + \sum_{n=1}^{\infty}(\frac{1}{z-n} + \frac{1}{z+n}).$$

Applying the Euler identity (Proposition 7.18), it turns out to be

$$\pi \frac{\cos \pi z}{\sin \pi z} = \pi \frac{(e^{i\pi z} + e^{-i\pi z})/2}{(e^{i\pi z} - e^{-i\pi z})/2i} = \pi i \frac{q+1}{q-1} = \pi i + \frac{2\pi i}{q-1},$$

where $q = q_z = e^{2\pi i z}$ (for $z \in \mathbb{H}$ we have $|q| < 1$). Thus, whenever $z \in \mathbb{H}$,

$$\pi i + \frac{2\pi i}{q-1} = \pi i - 2\pi i \sum_{\nu=0}^{\infty} q^\nu.$$

In conclusion, the formula we need is read as

$$\frac{1}{z} + \sum_{n=1}^{\infty}(\frac{1}{z-n} + \frac{1}{z+n}) = \pi i - 2\pi i \sum_{\nu=0}^{\infty} q^\nu.$$

Taking derivatives with respect to z, it becomes

$$\begin{aligned} -\frac{1}{z^2} - \sum_{n=1}^{\infty}(\frac{1}{(z-n)^2} + \frac{1}{(z+n)^2}) &= -\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} \\ &= -2\pi i \sum_{\nu=0}^{\infty} \nu q^{\nu-1}(2\pi i)q \\ &= -(2\pi i)^2 \sum_{\nu=0}^{\infty} \nu q^\nu. \end{aligned}$$

One may repeat the same operation recursively, and by induction,

$$(*) \qquad (-1)^{k-1} \cdot (k-1)! \cdot \sum_{n\in\mathbb{Z}} \frac{1}{(\tau-n)^k} = -\sum_{\nu=1}^{\infty} (2\pi i)^k \nu^{k-1} q^\nu, \quad k \in \mathbb{Z}_{>0}.$$

Remark 10.16. The same result in $(*)$ can be obtained by applying the Poisson summation formula (Theorem 5.9) to $f(z) = 1/(z+\tau)^k$ for $\tau \in \mathbb{H}$ (see [SS10, Chapter 4, Exercise 7]).

Now we are ready to get the expansion for $E_{2k}(\tau)$:

$$\begin{aligned}
E_{2k}(\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^{2k}} &= \sum_{m=0,n\neq 0} \frac{1}{n^{2k}} + \sum_{m\neq 0}\sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^{2k}} \\
&= 2\zeta(2k) + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^{2k}} \\
&= 2\zeta(2k) + 2\sum_{m=1}^{\infty}\sum_{\nu=1}^{\infty} \frac{(2\pi i)^{2k}\cdot \nu^{2k+1}}{(2k-1)!} q_{m\tau}^{\nu} \\
&= 2\zeta(2k) + 2\sum_{m=1}^{\infty}\sum_{\nu=1}^{\infty} \frac{(2\pi i)^{2k}\cdot \nu^{2k+1}}{(2k-1)!} q_{\tau}^{m\nu},
\end{aligned}$$

where $q_z = e^{2\pi i z}$ and the second last equality is deduced from $(*)$. This is the **q-expansion for Eisenstein series**. Denote $\sigma_k(n) = \sum_{d\mid n} d^k$ in which the sum runs through all positive divisors for n. Then the expansion formula can be rewritten as

$$E_{2k}(\tau) = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi i n\tau}.$$

10.3. **Arithmetic Properties of Elliptic Curves.**

10.3.1. *The Modular Function.* Let's begin with the setups. The *modular group* is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ defined by

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z},\ ad-bc = 1 \right\}.$$

Recall that $\mathrm{SL}_2(\mathbb{R})$ has an action on $\mathbb{H}$. More explicitly, for $z \in \mathbb{H}$ and $\alpha \in \mathrm{SL}_2(\mathbb{R})$, we define

$$\alpha(z) := \frac{az+b}{cz+d}, \quad \Im\alpha(z) = \frac{(ad-bc)\Im(z)}{|cz+d|^2} = \frac{\Im(z)}{|cz+d|^2}.$$

Also recall that the automorphism group for the upper-half plane is

$$\mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}.$$

Definitions 10.17 (Fundamental Domain)**.**

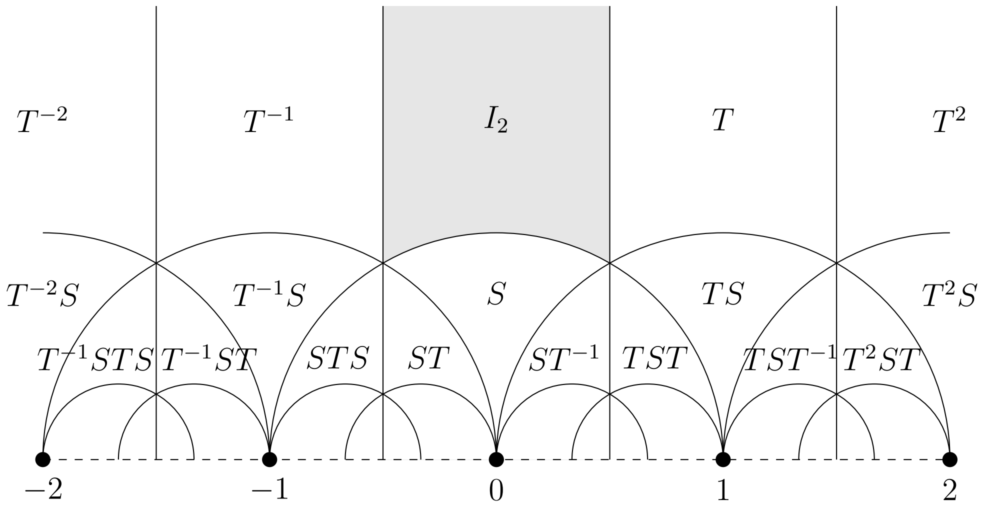
(1) An *orbit* of Γ is defined as $\Gamma.z = \{\alpha(z) : \alpha \in \Gamma\}$ for a fixed $z \in \mathbb{H}$.
(2) A subset $D \subset \mathbb{H}$ is called a *fundamental domain* for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ if every orbit of Γ has at least one element in D, and any two elements of D are in the same orbit if and only if they lie on the boundary of D.

In short, a fundamental domain of $\mathbb{H}$ for Γ can be regarded as a domain that generates $\mathbb{H}$ via the action of Γ.

Proposition 10.18. *The discrete modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is generated by*

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In other words, every $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ can be written as $T^m S^n$ or $S^m T^n$ for some $(m,n) \in \mathbb{Z}^2$. Furthermore, the following picture describes the action of T and S on $\mathbb{H}$ from the fundamental domain $D = I_2$.



Theorem 10.19. *The subset*

$$D = \{z \in \mathbb{H} : |\Re(z)| \leqslant \frac{1}{2}, |z| \geqslant 1\}$$

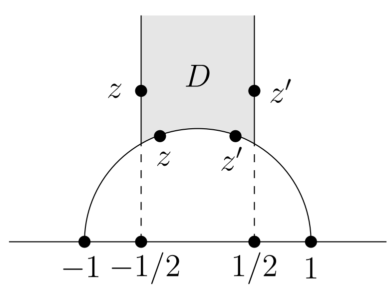
is a fundamental domain for Γ. Moreover, if $z, z' \in D$ are in the same orbit of Γ (i.e., $z' = \alpha(z)$ for some $\alpha \in \Gamma$), then either $\alpha = T^{\pm 1}$ or $\alpha = S^{\pm 1}$.

- *For the case $\alpha = T^{\pm 1}$, the points z and z' are on the vertical lines of ∂D. The action is given by the horizontal translation*

$$T(z) = \frac{z+1}{0+1} = z + 1.$$

- *For the case $\alpha = S^{\pm 1}$, the points z and z' are on the base arc of ∂D. The action is given by the reflection with respect to the vertical axis*

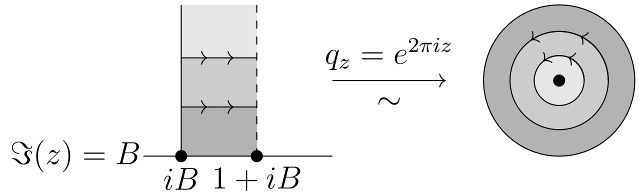
$$S(z) = \frac{0 + (-1)}{z + 0} = -\frac{1}{z}.$$



10.3.2. *Automorphic Functions of Degree $2k$.* For a real number $B > 0$, we define *the truncated upper-half plane* by

$$\mathbb{H}_B = \{z \in \mathbb{C} : \Im(z) > B\} \subset \mathbb{H}.$$

The map $z \mapsto e^{2\pi i z} = q_z$ gives a holomorphic mapping from $\mathbb{H}_B$ to $D^*(0, e^{-2\pi B}) := \{z \in \mathbb{C} : 0 < |z| < e^{-2\pi B}\}$. Consider $\mathbb{H}_B/\langle T \rangle$, the quotient space of $\mathbb{H}_B$ modulo translations by integers, i.e., $z_1 \sim z_2$ if $z_1 = z_2 + m$ for some $m \in \mathbb{Z}$ in $\mathbb{H}_B$.



Remark 10.20. If a meromorphic function f on $\mathbb{H}_B$ has period 1, i.e., $f(z+1) = f(z)$ for all $z \in \mathbb{H}_B$, then f descends to a function f^* on $D^*(0, e^{-2\pi B})$, where $f^*(q_z) := f(z)$.

Definition 10.21. The function f is called meromorphic (resp. holomorphic) at ∞ if f^* defined as in Remark 10.20 above is meromorphic (resp. holomorphic) at 0.

Definition 10.22 (The $\mathrm{SL}_2(\mathbb{Z})$-action)**.** Let f be a meromorphic function on $\mathbb{H}$ and $\alpha \in \Gamma = \mathrm{SL}_2(\mathbb{Z})$. For fixed integer $k \geqslant 0$, define

$$(T_k(\alpha)f)(z) := f(\alpha(z)) \cdot (cz+d)^{-2k} = f(\frac{az+b}{cz+d})(cz+d)^{-2k}, \quad \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Definition 10.23 (Automorphic Forms)**.** A function $f \in \mathfrak{m}(\mathbb{H})$ is called an *automorphic form of weight* $2k$ *with respect to* Γ if

(1) for any $\alpha \in \Gamma$, we have $T_k(\alpha)f = f$;
(2) f is meromorphic at ∞.

Example 10.24. There exists an one-to-one correspondence:

$$\left\{\begin{array}{c} \text{Functions } G : L \to \mathbb{C} \text{ of lattices which} \\ \text{are homogeneous with } \deg = -2k, \\ \text{i.e. } G(\lambda L) = \lambda^{-2k} G(L),\ \lambda \in \mathbb{C}\backslash\{0\} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{c} \text{Functions } g : \mathbb{H} \to \mathbb{C} \text{ satisfying that} \\ g(\alpha(z)) = (cz+d)^{2k} g(z) \text{ for all } \alpha \in \Gamma \end{array}\right\}$$

$$G(L) \longmapsto g(z) := G([z,1])$$

$$G([\tau,1]) = g(\tau) \longleftarrow\!\shortmid\ g(z)$$

In particular, the Eisenstein series

$$E_{2k}(L) = \sum_{\omega \in L^*} \frac{1}{\omega^{2k}}$$

gives an automorphic function.

11. Jacobi's Theta Functions

This section is devoted to a closer look at the theory of theta functions and some of its applications to combinatorics and number theory. The theta function of Jacobi is given by the series

$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}$$

which converges for all $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$.

A remarkable feature of the theta function is its dual nature. When viewed as a function of z, we see it in the arena of elliptic functions, since Θ is periodic with period 1 and "quasi-period" τ. When considered as a function of τ, Θ reveals its modular nature and close connection with the partition function and the problem of representation of integers as sums of squares.

The two main tools allowing us to exploit these links are the *triple-product* for Θ and its *transformation law*. Once we have proved these theorems, we give a brief introduction to the connection with partitions, and then pass to proofs of the celebrated theorems about representation of integers as sums of two or four squares.

11.1. **The Triple-Product Formula.** We begin our closer look at Θ as a function of z, with τ fixed, by recording its basic structural properties, which to a large extent characterize it.

11.1.1. *Basic Statements.*

Proposition 11.1. *The theta function $\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}$ enjoys the following properties.*

1. *For $\tau \in \mathbb{H}$ fixed, $\Theta(z|\tau)$ is entire with respect to z; for $z \in \mathbb{C}$ fixed, $\Theta(z|\tau)$ is holomorphic with respect to τ.*
2. *$\Theta(\cdot|\tau)$ is periodic with period 1, that is,*
$$\Theta(z+1|\tau) = \Theta(z|\tau).$$
3. *$\Theta(\cdot|\tau)$ is quasi-periodic with period τ, that is,*
$$\Theta(z+\tau|\tau) = \Theta(z|\tau)\cdot e^{-\pi i\tau}\cdot e^{-2\pi i z}.$$
4. *$\Theta(z|\tau) = 0$ whenever $z = (1+\tau)/2 + n + m\tau$ for $m, n \in \mathbb{Z}$.*

Proof. (1) Assume $\Im(\tau) = t \geqslant t_0 > 0$ and $|z| \leqslant M$. Then
$$|\Theta(z|\tau)| \leqslant \sum_{n\in\mathbb{Z}} |e^{\pi i n^2\tau} e^{2\pi i n z}| \leqslant 2\sum_{n\geqslant 0} e^{-\pi n^2 t_0} e^{2\pi n M} < \infty.$$

(2) This is obvious since $e^{2\pi i n(z+1)} = e^{2\pi i n z}$.

(3) We compute
$$\begin{aligned}\Theta(z+\tau|\tau) &= \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau} e^{2\pi i n(z+\tau)}\\ &= \sum_{n\in\mathbb{Z}} e^{\pi i(n^2+2n)\tau} e^{2\pi i n z}\\ &= \sum_{n\in\mathbb{Z}} e^{\pi i(n+1)^2\tau} e^{2\pi i(n+1)z} e^{-\pi i\tau} e^{-2\pi i z}\\ &= \Theta(z|\tau)\cdot e^{-\pi i\tau}\cdot e^{-2\pi i z}.\end{aligned}$$

(4) By (2)(3), it boils down to verify $\Theta(\frac{1+\tau}{2}|\tau) = 0$. This is given by
$$\Theta(\frac{1+\tau}{2}|\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau} e^{\pi i n(1+\tau)} = \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i(n^2+n)\tau}.$$

Note that for $n \geqslant 0$, $n^2+n = (-n-1)^2 + (-n-1)$ and $-n-1$ has the different parity from that of n. Thus
$$\sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i(n^2+n)} = \sum_{n\in\mathbb{Z}} (-1)^{-n-1} e^{\pi i((-n-1)^2+(-n-1))} = 0.$$

Thus, $\Theta(z|\tau) = 0$ whenever $z = (1+\tau)/2 + n + m\tau$ for $m, n \in \mathbb{Z}$. □

Theorem 11.2 (Jacobi's Triple-Product Formula, 1829)**.** *For $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$,*
$$\sum_{n\in\mathbb{Z}} q^{n^2} e^{2\pi i n z} = \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n-1}e^{2\pi i z})(1+q^{2n-1}e^{-2\pi i z}),$$
where $q = e^{\pi i\tau}$. By defining the right hand side as $\Pi(z|\tau)$, we write $\Theta(z|\tau) = \Pi(z|\tau)$.

Corollary 11.3. *Set $z = 0$ in the triple-product formula, we get*

$$\Theta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau} = \Theta(0|\tau) = \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1})^2.$$

In particular, $\Theta(\tau) \neq 0$ for any $\tau \in \mathbb{H}$ (c.f. Definition 7.11).

Proposition 11.4. *For any fixed $\tau \in \mathbb{H}$, the function*

$$(\log \Theta(z|\tau))'' = \frac{\Theta(z|\tau)\Theta''(z|\tau) - \Theta'(z|\tau)^2}{\Theta(z|\tau)}$$

is an elliptic function with periods 1 and τ and has double poles at $z = (1+\tau)/2 + m + n\tau$ for $m, n \in \mathbb{Z}$.

Remark 11.5. There is indeed some constant c_τ such that

$$(\log \Theta(z|\tau))'' = \wp(z - (1+\tau)/2; \tau) + c_\tau.$$

Here $\wp(z;\tau)$ denotes the Weierstrass $\wp$-function defined by the lattice $[\tau, 1]$.

11.1.2. *Proof of the Triple-Product Formula.* The proof of Theorem 11.2 ramifies into the following 3 steps.

- **Step 1.** We prove $\Pi(z|\tau)$ also satisfies properties (1)-(4) in Proposition 11.1.
  (1) For $\tau \in \mathbb{H}$ with $\Im(\tau) \geqslant t_0 > 0$, we have

  $$|q| = e^{-\pi\Im(\tau)} \leqslant e^{-\pi t_0} < 1,$$

  and

  $$|(1-q^{2n})(1+q^{2n-1}e^{2\pi i z})(1+q^{2n-1}e^{-2\pi i z})| = 1 + O(|q|^{2n-1}e^{2\pi|z|}).$$

  On the other hand, the series $\sum_{n\in\mathbb{Z}} |q|^{2n-1}$ converges and hence $\Pi(z|\tau)$ satisfies (1).
  (2) This is again obvious.
  (3) We compute

$$\begin{aligned}
\Pi(z+\tau|\tau) &= \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2\pi i(z+\tau)})(1+q^{2n-1}e^{-2\pi i(z+\tau)}) \\
&= \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n+1}e^{2\pi i z})(1+q^{2n-3}e^{-2\pi i z}) \\
&= \prod_{n=1}^{\infty}(1-q^{2n}) \cdot \prod_{n=1}^{\infty}(1+q^{2n+1}e^{2\pi i z}) \cdot \prod_{n=1}^{\infty}(1+q^{2n-3}e^{-2\pi i z}) \\
&= \Pi(z|\tau) \cdot \frac{1+q^{-1}e^{-2\pi i z}}{1+qe^{2\pi i z}} \\
&= \Pi(z|\tau) \cdot q^{-1} \cdot e^{-2\pi i z} \\
&= \Pi(z|\tau) \cdot e^{-\pi i \tau} \cdot e^{-2\pi i z}.
\end{aligned}$$

  (4) Note that for $\tau \in \mathbb{H}$, $|q|^{2n} \neq 1$. Therefore, $\Pi(z|\tau) = 0$ if and only if $(1+q^{2n-1}e^{2\pi i z})(1+q^{2n-1}e^{-2\pi i z}) = 0$ for some $n \in \mathbb{Z}$. This is also equivalent to $z = (1+\tau)/2 + n + m\tau$ for $m, n \in \mathbb{Z}$.

- **Step 2.** For $\tau \in \mathbb{H}$ fixed, consider
$$F(z) = \frac{\Theta(z|\tau)}{\Pi(z|\tau)}.$$
Then $F(z)$ is holomorphic and doubly-periodic with periods 1 and τ by Step 1 (1)-(3). By Liouville's Theorem (Corollary 3.16),
$$F = c(\tau)$$
for some constant $c(\tau)$ which is depending on τ.
- **Step 3.** We are to prove the claim that $c(\tau) = 1$ for any $\tau \in \mathbb{H}$. From Step 2, $\Theta(z|\tau) = c(\tau) \cdot \Pi(z|\tau)$.

Sublemma. $c(\tau) = c(4\tau)$.

Proof. Set $z = 1/2$ in $\Theta(z|\tau) = c(\tau) \cdot \Pi(z|\tau)$ to get
$$\begin{aligned}\sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} &= c(\tau) \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n-1})(1 - q^{2n-1}) \\ &= c(\tau) \prod_{n=1}^{\infty} (1 - q^n) \prod_{n=1}^{\infty} (1 - q^{2n-1}),\end{aligned}$$
This shows that
$$c(\tau) = \frac{\sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}}{\prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n-1})}.$$
Again, by setting $z = 1/4$, we can a similar process renders that
$$c(\tau) = \frac{\sum_{m\in\mathbb{Z}} (-1)^m q^{4m^2}}{\prod_{m=1}^{\infty} (1 - q^{4m})(1 - q^{8m-4})}.$$
A comparison is enough to show $c(\tau) = c(4\tau)$. ∎

By induction applying on the sublemma, we see for any $k \geqslant 1$, $c(\tau) = c(4^k\tau)$ for any $\tau \in \mathbb{H}$. On the other hand, as $k \to \infty$,
$$q_{4^k\tau} = e^{\pi i \cdot 4^k \tau} \to 0 \quad \Rightarrow \quad c(\tau) = 1.$$

The proof for Theorem 11.2 is accomplished.

11.2. **Modular Character of Θ.** We still work on the modular group $\mathrm{SL}_2(\mathbb{Z})$. Recall the definition for theta functions that
$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}.$$
From this, the immediate consequence is
$$\Theta(z|\tau + 2) = \Theta(z|T^2(\tau)) = \Theta(z|\tau), \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$
where $T \in \mathrm{SL}_2(\mathbb{Z})$ is the generator for horizontal translation. Again, by Proposition 10.18, $\mathrm{SL}_2(\mathbb{Z})$ is generated by
$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
The natural question is that under the action of S towards $\tau \in \mathbb{H}$, what property does the theta function obtain.

Theorem 11.6. *For $\tau \in \mathbb{H}$, we have*

$$\Theta(z|S(\tau)) = \Theta(z|-\frac{1}{\tau}) = \sqrt{\frac{\tau}{i}}e^{\pi i\tau z^2}\Theta(z\tau|\tau)$$

for all $z \in \mathbb{C}$, where $\sqrt{\alpha} = |\alpha|^{1/2}\exp(i(\arg\alpha)/2)$ with $0 < \arg\alpha < \pi$.

Proof. By the analytic continuation (Theorem 3.22), it suffices to check the identity for $z = x \in \mathbb{R}$ and $\tau = it$ with $t > 0$. For this, we obtain

$$\text{LHS} = \Theta(x|-\frac{1}{it}) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2/t}e^{2\pi inx},$$

$$\text{RHS} = t^{1/2}e^{-\pi tx^2}\sum_{n\in\mathbb{Z}} e^{-\pi n^2/t}e^{-2\pi tnx} = t^{1/2}\sum_{n\in\mathbb{Z}} e^{\pi tht(x+n)^2}.$$

By applying the Poisson summation formula (Theorem 5.9) to $f(y) = e^{-\pi t(y+x)^2}$ we get the identity. □

Recall Definition 7.11 that by letting $z = 0$,

$$\Theta(\tau) := \Theta(0|\tau) = \sum_{n\in\mathbb{Z}} e^{\pi in^2\tau}.$$

Corollary 11.7. *For all $\tau \in \mathbb{H}$,*

$$\Theta(-\frac{1}{\tau}) = \sqrt{\frac{\tau}{i}}\Theta(\tau).$$

On the other hand, by definition again, note that

$$\Theta(1+\tau) = \sum_{n\in\mathbb{Z}} e^{\pi in^2(1+\tau)} = \sum_{n\in\mathbb{Z}}(-1)^n e^{\pi in^2\tau} = \Theta(\frac{1}{2}|\tau).$$

Corollary 11.8. *For all $\tau \in \mathbb{H}$, as $\Im(\tau) \to \infty$,*

$$\Theta(1-\frac{1}{\tau}) = \sqrt{\frac{\tau}{i}}(2e^{\pi i\tau/4} + \cdots) \sim \sqrt{\frac{\tau}{i}}\cdot 2e^{\pi i\tau/4}.$$

Proof. By the equality above, plugging in $-1/\tau$, we have

$$\Theta(1-\frac{1}{\tau}) = \Theta(\frac{1}{2}|-\frac{1}{\tau}).$$

Using Theorem 11.6, we can compute

$$\begin{aligned}
\Theta(\frac{1}{2}|-\frac{1}{\tau}) &= \sqrt{\frac{\tau}{i}}e^{\pi i\tau/4}\Theta(\frac{\tau}{2}|\tau) \\
&= \sqrt{\frac{\tau}{i}}e^{\pi i\tau/4}\sum_{n\in\mathbb{Z}} e^{\pi in^2\tau}e^{\pi in\tau} \\
&= \sqrt{\frac{\tau}{i}}\sum_{n\in\mathbb{Z}} e^{\pi i(n+1/2)^2\tau} \\
&= \sqrt{\frac{\tau}{i}}(2e^{\pi i\tau/4} + \sum_{n\neq 0,-1} e^{\pi i(n+1/2)^2\tau}).
\end{aligned}$$

So it remains to estimate the second term. We obtain

$$|\sum_{n\neq 0,-1} e^{\pi i(n+1/2)^2\tau}| \leqslant 2\sum_{k\geqslant 1} e^{-\pi(k+1/2)^2 t} \sim e^{-O(t)}.$$

Thus, the higher terms can be sufficiently small. □

11.3. **Combinatoric Applications: Generating Functions.** Given a sequence $\{F_n\}_{n=0}^{\infty}$, we have a *generating function*

$$F(x) = \sum_{n=0}^{\infty} F_n x^n.$$

The properties of this function correspond to the properties of the sequence $\{F_n\}_{n=0}^{\infty}$, and particularly, the generating function usually has combinatoric interpretations for various sequences.

11.3.1. *Partition Function.* Given $n \in \mathbb{N}$, a *partition* of n is defined as a unordered series of non-negative integers whose sum is exactly n. For example, by defining

$$P(n) := \#\{\text{Partitions of } n\},$$

we have the following basic counting results.

n	Partitions of n	$P(n)$
1	1+0	1
2	1+1, 2+0	2
3	1+1+1, 2+1, 3+0	3
$\vdots$	$\vdots$	$\vdots$

(1) (Euler Identity) The generating function for $\{P(n)\}$ can be explicitly computed by the *Euler identity*: for $|x| < 1$,

$$\sum_{n=0}^{\infty} P(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}.$$

To prove this, note first that

$$\frac{1}{1-x^k} = \sum_{m=0}^{\infty} x^{km} = 1 + O(x^k),$$

thus the product $\prod_{k=1}^{\infty} 1/(1-x^k)$ converges. Moreover,

$$\begin{aligned}\prod_{k=1}^{\infty} \frac{1}{1-x^k} &= \prod_{k=1}^{\infty}\sum_{m=0}^{\infty} x^{km} \\ &= (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)\cdots\end{aligned}$$

Focusing on the right hand side, we are to find out the coefficients of x^n for all $n \geqslant 0$. Since each monomial in each fact has coefficient 1, the coefficient of x^n is nothing but the number of partitions. Then

$$\text{RHS} = \sum_{n=0}^{\infty} P(n)x^n.$$

(2) (Odd Partitions Correspond to Unequal Partitions) Denote

$$P_{\text{odd}}(n) = \#\{\text{Partitions of } n \text{ into odd integers}\},$$
$$P_{\text{un}}(n) = \#\{\text{Partitions of } n \text{ into unequal integers}\}.$$

The the claim is $P_{\text{odd}}(n) = P_{\text{un}}(n)$ for each $n \geqslant 0$. The observation based on understanding the Euler identity in (1) is useful:

$$\sum_{n=0}^{\infty} P_{\text{odd}}(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^{2k-1}},$$
$$\sum_{n=0}^{\infty} P_{\text{un}}(n)x^n = \prod_{k=1}^{\infty}(1+x^k).$$

To show these two products are the same, we say

$$\begin{aligned}\prod_{k=1}^{\infty} \frac{1}{1-x^{2k-1}} &= \frac{\prod_{k=1}^{\infty}(1-x^k)^{-1}}{\prod_{k=1}^{\infty}(1-x^{2k})^{-1}} \\ &= \frac{\prod_{k=1}^{\infty}(1-x^{2k})}{\prod_{k=1}^{\infty}(1-x^k)} \\ &= \frac{\prod_{k=1}^{\infty}(1-x^k)\prod_{k=1}^{\infty}(1+x^k)}{\prod_{k=1}^{\infty}(1-x^k)} \\ &= \prod_{k=1}^{\infty}(1+x^k).\end{aligned}$$

(3) (Euler's Pentagonal Counting) Denote

$$P_{\text{un}}^{\text{even}}(n) = \#\{\text{Partitions of } n \text{ into an even number of unequal integers}\},$$
$$P_{\text{un}}^{\text{odd}}(n) = \#\{\text{Partitions of } n \text{ into an odd number of unequal integers}\}.$$

The result for Euler concerns about their difference.

$$P_{\text{un}}^{\text{even}}(n) - P_{\text{un}}^{\text{odd}}(n) = \begin{cases}(-1)^k, & \text{if } n = k(3k+1)/2 \text{ for some } k \in \mathbb{Z}; \\ 0, & \text{otherwise.}\end{cases}$$

Here $n = k(3k+1)/2$ is called a pentagonal number, which can be interpreted as the number of small stones piled into a Pentagon with k as the side length. From the construction, we have

$$\begin{aligned}\sum_{n=0}^{\infty} P_{\text{un}}^{\text{even}}(n) - P_{\text{un}}^{\text{odd}}(n)x^n &= \sum_{n=0}^{\infty} \sum_{\substack{n=n_1+\cdots+n_r \\ n_1,\ldots,n_r \text{ distinct}}} (-1)^r \cdot x^n \\ &= \prod_{n=1}^{\infty}(1-x^n).\end{aligned}$$

To prove Euler's result, we only need to verify the following:

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{k\in\mathbb{Z}}(-1)^k x^{k(3k+1)/2}.$$

Let's recall the triple-product formula (Theorem 11.2), say

$$\sum_{n\in\mathbb{Z}} q^{n^2} e^{2\pi i n z} = \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2\pi i z})(1+q^{2n-1}e^{-2\pi i z}).$$

For convenience we set $\tau = 3u$ and $z = (1+u)/2$. It turns out to be

$$\begin{aligned}\sum_{n\in\mathbb{Z}} e^{3\pi i n^2 u}\cdot(-1)^n e^{\pi i n u} &= \sum_{n\in\mathbb{Z}}(-1)^n e^{\pi i n(3n+1)u}\\ &= \prod_{n=1}^{\infty}(1-e^{\pi i(6n)u})(1-e^{\pi i(6n-2)u})(1-e^{\pi i(6n-4)u})\\ &= \prod_{n=1}^{\infty}(1-e^{2\pi i n u}).\end{aligned}$$

The required result follows from simply replacing $e^{2\pi i u}$ by x.

11.3.2. *Sums of Squares.* The second example is about the famous problem on how to decompose an integer into the sum of two squares. This is an application of the theta function on analytic number theory.

Given $n \in \mathbb{N}$, denote

$$r_k(n) := \#\{n : \text{there exist } x_1, \ldots, x_k \in \mathbb{N} \text{ such that } n = x_1^2 + \cdots + x_k^2\}.$$

The most impressive result on counting the number of two-squares is as follows. Denote

$$\begin{aligned} d_1(n) &= \#\{\text{Divisors of } n \text{ of the form } 4k+1\},\\ d_3(n) &= \#\{\text{Divisors of } n \text{ of the form } 4k+3\}.\end{aligned}$$

Theorem 11.9 (Two-Square Theorem)**.**

$$r_2(n) = 4(d_1(n) - d_3(n)).$$

Note that $\Theta(\tau) = \Theta(0|\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau} = \sum_{n\in\mathbb{Z}} q^{n^2}$. Then

$$\Theta(\tau)^2 = \sum_{n_1\in\mathbb{Z}} q^{n_1^2}\sum_{n_2\in\mathbb{Z}} q^{n_2^2} = \sum_{(n_1,n_2)\in\mathbb{Z}^2} q^{n_1^2+n_2^2} = \sum_{n=0}^{\infty} r_2(n)q^n.$$

Lemma 11.10. *Theorem 11.9 is equivalent to the identities for $\tau \in \mathbb{H}$:*

$$\Theta(\tau)^2 = 2\sum_{n\in\mathbb{Z}}\frac{1}{q^n+q^{-n}} = 1 + 4\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}},$$

where $q = q_\tau = e^{\pi i\tau}$.

Proof. The following equalities hold without any assumption:

$$\begin{aligned}1+4\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}} &= 1+4\sum_{n=1}^{\infty}\frac{q^n(1-q^{2n})}{1-q^{4n}}\\ &= 1+4\sum_{n=1}^{\infty}\Big(\frac{q^n}{1-q^{4n}}-\frac{q^{3n}}{1-q^{4n}}\Big)\\ &= 1+4\sum_{n=1}^{\infty}\Big(\sum_{m=0}^{\infty}q^{n(4m+1)}-\sum_{m=0}^{\infty}q^{n(4m+3)}\Big)\\ &= 1+4\sum_{k=1}^{\infty}(d_1(k)-d_3(k))q^k.\end{aligned}$$

The last equality is valid because $d_1(k)$ and $d_3(k)$ count the number of divisors of k that are of the forms $4m+1$ and $4m+3$, respectively; and therefore, $\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} q^{n(4m+1)} = \sum_{k=1}^{\infty} d_1(k)q^k$ and similarly for $d_3(k)$. Assuming Theorem 11.9, the right hand side above is exactly $\Theta(\tau)^2$. $\square$

To prove the two-square theorem (Theorem 11.9), denote

$$C(\tau) := 2\sum_{n\in\mathbb{Z}} \frac{1}{q^n + q^{-n}} = \sum_{n\in\mathbb{Z}} \frac{1}{\cos(n\pi\tau)},$$

where $q = e^{\pi i\tau}$ again, and the second equality is deduced from $e^{\pi in\tau} + e^{-\pi in\tau} = 2\cos(n\pi\tau)$. We need to verify for $\tau \in \mathbb{H}$ that $\Theta(\tau)^2 = C(\tau)$.

Proposition 11.11. *Denote $G(\tau) := \Theta(\tau)^2$ (or equivalently, $G(\tau) := C(\tau)$). Then*

1. *$G(\tau+2) = G(\tau)$;*
2. *$G(\tau) = (i/\tau)\cdot G(-1/\tau)$;*
3. *$G(\tau) \to 1$ as $\Im(\tau)\to\infty$;*
4. *$G(1-1/\tau) \sim 4(\tau/i)\cdot e^{\pi i\tau/2}$ as $\Im(\tau)\to\infty$.*

Proof. Note that (1)(3) follow from the definition of $C(\tau)$ at once. For (2)(4), note that

$$\cosh(iz) = \cos z = \frac{e^{iz}+e^{-iz}}{2}.$$

Recall that in Example **??** (c.f. Example 4.11), we have used the Poisson summation formula (Theorem 5.9) to $f(x) = e^{-2\pi ixa}/\cosh(\pi x/t)$ with $a \in \mathbb{R}$ and $t > 0$, in order to get

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \frac{e^{-2\pi ian}}{\cosh(\pi n/t)} = \sum_{n\in\mathbb{Z}} \frac{t}{\cosh(\pi(n+a)t)} = \sum_{n\in\mathbb{Z}} \widehat{f}(n).$$

In particular, if we set $a = 0$, then

$$\sum_{n\in\mathbb{Z}} \frac{1}{\cosh(\pi n/t)} = \sum_{n\in\mathbb{Z}} \frac{t}{\cosh(\pi nt)}.$$

Therefore, via the variable change $\tau = it$ with $t > 0$,

$$\begin{aligned} C(\tau) &= \sum_{n\in\mathbb{Z}} \frac{1}{\cos(\pi nit)} = \sum_{n\in\mathbb{Z}} \frac{1}{\cosh(\pi nt)} \\ &= t^{-1}\sum_{n\in\mathbb{Z}} \frac{1}{\cosh(\pi n/t)} \\ &= \frac{i}{it}C(-\frac{1}{it}) = \frac{i}{\tau}C(-\frac{1}{\tau}). \end{aligned}$$

By the analytic continuation (Theorem 3.22), the formula in (2) holds for all $\tau \in \mathbb{H}$. Again, by setting $a = 1/2$, we get (4) through the same argument. $\square$

Proposition 11.12. *Let $f \in \mathscr{O}(\mathbb{H})$ and assume that*

- *$f(\tau+2) = f(\tau)$ for all $\tau \in \mathbb{H}$;*
- *$f(-1/\tau) = f(\tau)$ for all $\tau \in \mathbb{H}$;*
- *$f(\tau)$ is bounded.*

Then f is a constant.

Proof. Assume f is not a constant for the sake of contradiction. Denote the basic actions from $\mathrm{SL}_2(\mathbb{Z})$ by

$$T_2 : \tau \mapsto \tau + 2, \quad S : \tau \mapsto -\frac{1}{\tau}.$$

Consider $G = \langle T_2, S\rangle$, the group generated by T_2 and S.

Claim 1: when G acts on $\mathbb{H}$,

$$\mathcal{F} = \{\tau \in \overline{\mathbb{H}} : |\Re(\tau)| \leqslant 1, |\tau| \geqslant 1\}$$

is a fundamental domain.

Setting $z = e^{\pi i \tau}$, the function $f_1(z) := f(\tau)$ is a well-defined holomorphic function on $\mathbb{D}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ as $f(\tau + 2) = f(\tau)$. Moreover, f_1 is bounded as well as f is. Consequently, f_1 extends holomorphically from $\mathbb{D}^*$ to $\mathbb{D}$, i.e. the limit

$$f_1(0) = \lim_{z\to 0} f_1(z) = \lim_{\Im(\tau)\to\infty} f(\tau)$$

is well-defined. Applying the maximum principle (Proposition 4.27) to f_1, on the open connected region $\Omega = \{z \in \mathbb{C} : |\Re(z)| < 1, |z| > 1\}$, we attain

$$\lim_{\Im(\tau)\to\infty} |f(\tau)| < \sup_{\tau\in\mathcal{F}} |f(\tau)|.$$

Claim 2: the limit $\lim_{\Im(\tau)\to\infty} f(1 - 1/\tau)$ exists and

$$\lim_{\Im(\tau)\to\infty} |f(1 - \frac{1}{\tau})| < \sup_{\tau\in\mathcal{F}} |f(\tau)|.$$

For the second claim, set $F(\tau) = f(1 - \frac{1}{\tau})$, then F is periodic of period 1. Let

$$\mu(\tau) = \frac{1}{1-\tau}, \quad \mu^{-1}(\tau) = 1 - \frac{1}{\tau}, \quad T(\tau) = \tau + 1.$$

Then for any $n \in \mathbb{Z}$,

$$f(\mu^{-1} \circ T^n \circ \mu(\tau)) = f(\tau).$$

Therefore, the function $F(\tau) = f(\mu^{-1}(\tau))$ satisfies

$$F(T^n\tau) = F(\tau)$$

for all $n \in \mathbb{Z}$. In particular, $F(\tau + 1) = F(\tau)$ and therefore

$$f_2(z) := F(\tau), \quad z = e^{2\pi i\tau}$$

is a well-defined holomorphic function on $\mathbb{D}^*$, which is bounded since f is bounded. Thus, f_2 extends to $\mathbb{D}$ by Riemann extension (Theorem 4.12). Again, apply the maximum principle (Proposition 4.27) to f_2,

$$\lim_{\Im(\tau)\to\infty} |f(1 - \frac{1}{\tau})| < \sup_{\tau\in\mathcal{F}} |f(\tau)|.$$

Then f attains its maximum at some point $z_0 \in \mathbb{H}$, contradicting with the assumption that f is not a constant. Therefore, f must be a constant. □

Now the proof of the two-square theorem is easy to catch.

Proof of Theorem 11.9. By applying Proposition 11.12 to $C(\tau)/\Theta(\tau)^2$, it must be a constant. The conditions of the proposition are satisfied by Proposition 11.11. Again, it can be shown that the constant value is 1. □

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