On stacks of p-adic local shtukas

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<u>Riemann's Thm</u> The functor

$$\left\{\begin{matrix}\text{abelian vars} \\ /\mathbb{C}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{integral Hodge} \\ \text{structures}\end{matrix}\right\}$$

$$A \longmapsto H_1(A, \mathbb{Z})$$

is fully faithful.

The essential image consists of polarizable integral Hodge strs of type $\{(-1,0),(0,-1)\}$.

<u>p-adic analogue</u> Fix $p \in \mathbb{Z}$ a prime. Let $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$.

<u>Thm</u> (Scholze-Weinstein)

$\exists$ equiv of cats b/w

cat of p-div grps / $\mathcal{O}_{\mathbb{C}_p}$

$\updownarrow$

cat of pairs (T, W), T a free $\mathbb{Z}_p$-mod

W a $\mathbb{C}_p$-subspace of $T \otimes \mathbb{C}_p(-1)$

$\quad$ Tate twist

<u>More geometric statement</u>

Given $n \geq m \geq 0$. Let $X^{n,m} \longrightarrow \operatorname{Spec} \mathbb{C}_p$

denote the Grassmannian parametrizing m-dim'l subspaces in n-dim'l v.s.

Let $X = \coprod_{m \leq n} \underline{GL_n(\mathbb{Z}_p)} \backslash X^{n,m}$.

This makes sense as a spatial diamond.

<u>Then</u>

$$\{p\text{-div grps} / \mathcal{O}_{\mathbb{C}_p}\}/\sim \; = X(\mathbb{C}_p).$$

* How does this work in families?

Let $Spa(R,R^+)$ an affinoid perfectoid $/\mathbb{Z}_p$ w/ p.u. $\varpi \in R$.

Have $\quad A_{inf} := W(R^{+,\flat})$

endowed w/ $(p,[\varpi])$-adic top.

Set surjective map $\theta : A_{inf} \twoheadrightarrow R^+$,

$\ker\theta = \langle \xi \rangle$.

Have period rings

$$B_{dR}^+ := \left(A_{inf}[\tfrac{1}{\varpi}]\right)^\wedge_\xi \quad (\xi\text{-adic completion})$$

$$B_{dR} := B_{dR}^+[\tfrac{1}{\xi}].$$

$$\rightsquigarrow \quad \mathcal{Y}_{(0,\infty)} \subseteq \mathcal{Y}_{[0,\infty)} \subseteq \mathcal{Y} \subseteq Spa(A_{inf}).$$

$$\begin{array}{ccc} | & | & | \\ p\cdot[\varpi]\neq 0 & [\varpi]\neq 0 & \text{analytic locus.} \end{array}$$

Define relative FF curve

$$X_{FF} := \mathcal{Y}_{(0,\infty)}/\varphi^{\mathbb{Z}}$$

- set closed imms

$$\infty : Spa(R,R^+) \hookrightarrow \mathcal{Y}_{[0,\infty)}.$$

- whenever $\mathbb{Q}_p \subseteq R$, also set

$$\infty : Spa(R,R^+) \hookrightarrow X_{FF}.$$

<u>Thm</u> (Fargues) Suppose $R = \mathbb{C}_p$, $R^+ = \mathcal{O}_{\mathbb{C}_p}$.

$\exists$ equiv of cats b/w

(a) cat of pairs (T, Ξ) where

T is a finite free $\mathbb{Z}_p$-mod,

$\Xi \subseteq T \otimes_{\mathbb{Z}_p} B_{dR}$ a B_{dR}^+-lattice.

(b) cat of quadruples $(\mathcal{F}, \mathcal{F}', \beta, T)$ where

$\mathcal{F}, \mathcal{F}'$ are vec bdls / X_{FF}, with $\mathcal{F}$ trivial

$\beta : \mathcal{F}|_{X_{FF} \setminus \infty} \overset{\sim}{\dashrightarrow} \mathcal{F}'|_{X_{FF} \setminus \infty}$ isom & β mero at ∞.

$T \subseteq H^0(X_{FF}, \mathcal{F})$ a $\mathbb{Z}_p$-lattice.

(c) cat of shtukas / $\mathbb{C}_p^\flat$ with legs at $\varphi^{-1}(\infty)$

i.e. a vec bdl $\mathcal{E}$ over $\mathcal{Y}_{[0,\infty)}$

$+$ isom $\varphi_{\mathcal{E}} : \varphi^* \mathcal{E}|_{\mathcal{Y}_{[0,\infty)} \setminus \varphi^{-1}(\infty)} \dashrightarrow \mathcal{E}|_{\mathcal{Y}_{[0,\infty)} \setminus \varphi^{-1}(\infty)}$

mero at $\varphi^{-1}(\infty)$.

(d) Same as in (c) but w/ $\mathcal{Y}$ instead of $\mathcal{Y}_{[0,\infty)}$.

(e) Breuil-Kisin-Fargues mods / A_{inf}.

i.e. finite free A_{inf}-mods M

$+$ isom $\varphi_M : \varphi^* M[\frac{1}{\varphi(\xi)}] \to M[\frac{1}{\varphi(\xi)}]$.

<u>Description</u> (a) in families gives $\underbrace{GL_n(\mathbb{Z}_p) \backslash Gr_{B_{dR}^+}}_{\text{spatial diamond}}$

(b) gives a Beauville-Laszlo morphism

$BL : \underline{GL_n(\mathbb{Z}_p)} \backslash Gr_{B_{dR}^+} \longrightarrow Bun_G$.

(c) makes sense integrally & gives a v-stack

$Sht_{GL_n} \longrightarrow Spd\, \mathbb{Z}_p$.

(e) is important b/c :

Thm cat of p-div grps $/ R^+$

$$\updownarrow \sim$$

cat of BKF mods (M, φ_M) $/ A_{inf} = W(R^{+,\flat})$

s.t. $M \subseteq \varphi_M(M) \subseteq \frac{1}{\varphi(\xi)} \cdot M$.

Warning (c) is insensitive to choice of R^+;

But (d) & (e) are not !

- A naive fix is to replace R^+ by R° in (d) & (e).
  But this still fails for $R = \mathbb{Q}_{p^\infty}$.
- Second trail: replace R^+ by R°
  and then v-sheafify.

Setup Let $\mathrm{Spa}(R, R^+)$ be a product of pts

(i.e. $R^+ = \prod_{i \in I} C_i^+$, C_i^+ val'n ring

w/ $C_i = C_i^+[\varpi_i]$ alg closed non-arch field,

$R = R^+\left[\frac{1}{(\varpi_i)_{i \in I}}\right]$.)

Thm A (Gleason - Ivanov)

The following cats are equiv:

(c') cat of shtukas $/ R^\flat$ with legs at ∞.

(e') cat of BKF mods $/ A^\circ_{inf} = W(R^{\circ,\flat})$.

Cor If $\mathcal{G}$ is a parahoric grp sch $/ \mathbb{Z}_p$, Consider the functor

$$\mathcal{G}\text{-BKF} : \mathrm{Perf}_{\mathbb{F}_p} \longrightarrow \text{Groupoids}$$
$$(R, R^+) \longmapsto \{(R^\#, \mathcal{E})\}/\sim$$

- $R^\#$ an untilt.
- $\mathcal{E}$ GKF-mod $/ R^{\circ,\#}$

<u>Thm B</u> The v-stackification of $\mathcal{G}$-BKF agrees w/ $\mathrm{Sht}_{\mathcal{G}} \to \mathrm{Spd}\, \mathbb{Z}_p$ where $\mathrm{Sht}_{\mathcal{G}}$ = stack of p-adic shtukas.

Moreover, $\mathrm{Sht}_{\mathcal{G}, \mathbb{F}_p} = (\mathrm{Sht}_{\mathcal{G}}^{sch})^\dagger$.

<u>Aside</u> if X sch $/\mathbb{F}_p$. $X^\dagger(R, R^+) = X(\mathrm{Spec}\, R^\circ)$.

<u>Setup</u> Let $\mu \in X_*(G_{\bar{\mathbb{Q}}_p})$ conjugacy class of geometric cochars.

Then we have a stack of μ-bounded p-adic local shtukas

$$\mathrm{Sht}_{\mathcal{G}}^{\leq \mu} \longrightarrow \mathrm{Spd}\, \mathcal{O}_{\breve{E}(\mu)}$$

+ a map $\sigma : \mathrm{Sht}_{\mathcal{G}}^{\leq \mu} \longrightarrow \mathrm{Bun}_G$.

Given $\mathcal{G}$-shtuka $(\mathcal{E}, \varphi_{\mathcal{E}}) \rightsquigarrow \mathcal{E}|_{\mathcal{Y}_{[r,\infty)}}$ s.t. $\infty_R \notin \mathcal{Y}_{[0,\infty)}$.

$\mathcal{E}$ on $\mathcal{Y}_{[0,\infty)}$, $\varphi_{\mathcal{E}}$ on $\mathcal{Y}_{[0,\infty)} \setminus \infty_R$

<u>Thm</u> (Anschütz) $G = \mathcal{G} \times_{\mathrm{Spec}\, \mathbb{Z}_p} \mathrm{Spec}\, \mathbb{Q}_p$.

$$\mathrm{Bun}_G(\mathrm{Spd}\, \bar{\mathbb{F}}_p) \cong B(G) := G(\breve{\mathbb{Q}}_p) / \underset{\varphi\text{-conj}}{G(\breve{\mathbb{Q}}_p)}.$$

Kottwitz set

Given $b \in B(G)$, let

$$\mathrm{Sht}_{\mathcal{G}}^{\leq \mu}(b) := \mathrm{Sht}_{\mathcal{G}}^{\leq \mu} \times_{\mathrm{Bun}_G} \mathrm{Spd}\, \bar{\mathbb{F}}_p \longrightarrow \mathrm{Spd}\, \mathcal{O}_{\breve{E}}.$$

Then for some data,

$$\mathrm{Sht}_{\mathcal{G}}^{\le\mu}(b) = RZ^{\diamond} \quad \text{Rapoport-Zink space.}$$

<u>Thm C</u> (Gleason)

(a) $\mathrm{Sht}_{\mathcal{G}}^{\le\mu}$ is an Artin v-stack

(b) σ is rep'ble in $\underbrace{\text{locally spatial diamond}}$

↳ generalization of rigid spaces

(c) $\mathrm{Sht}_{\mathcal{G}}^{\le\mu}(b)$ is a locally $\underbrace{\text{spatial kimberlites}}$.

↳ behaving like formal schs.