Comments for *Algebraic Geometry* with EGA & SGA

FOUR ASPECTS FOR FINITENESS DESCRIPTION

Quasi-compactness

We refer to [EGA 0, 2.1.3] and [EGA I, §6.6].

Definition 1. A scheme X is *quasi-compact* if any one of its open cover admits a finite subcover.

For example, any affine scheme $X = \operatorname{Spec} A$ is always quasi-compact. Therefore, the quasi-compactness is equivalent to the ability of X to be covered by finitely many affine open subsets.

Furthermore, X is compact if it is quasi-compact and Hausdorff separated. However, a scheme is not necessarily Hausdorff separated under the generic conditions. So we use quasi-compactness as a substitution in scheme theory of compactness in manifold theory.

Definition 2. A morphism $f : X \to Y$ between schemes is called *quasi-compact* if for any quasi-compact open subset $V \subseteq Y$, the inverse image $f^{-1}(V) \subseteq X$ is quasi-compact.

It follows that all closed immersions are quasi-compact morphisms, whereas an open immersion is possibily not.

Quasi-separatedness

We refer to [EGA IV, §1.2].

Definition 3. A scheme X is *quasi-separated* if for any pair of two quasi-compact subsets of X, their intersection is still quasi-compact. A morphism $f : X \to Y$ is *quasi-separated* if for any quasi-separated open subset $V \subseteq Y$, the inverse image $f^{-1}(V) \subseteq X$ is quasi-separated.

Again, any affine scheme $X = \operatorname{Spec} A$ is always quasi-separated. And the quasi-separated property is equivalent to the condition that for any quasi-compact open subset U of X, the open immersion $U \hookrightarrow X$ is quasi-compact. One may alternatively consider affine open subsets $V \subseteq Y$ only for defining quasi-separated morphisms. (This equivalence requires a more complicated proof, so we choose to skip such a proof.)

Proposition 4. *Any locally noetherian scheme is quasi-separated. In fact, when X is a locally noetherian scheme, then for any open subset $U \subseteq X$, which is not necessarily quasi-compact, the open immersion $U \hookrightarrow X$ is quasi-compact.*

Note that the quasi-compact and quasi-separated conditions are essentially in the topological sense, without the structure sheaves involved.

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Local finite type

We refer to [EGA I, §6.6] and [EGA IV, §1.3].

Definition 5. A morphism $f : X \to Y$ is *locally of finite type* if

- for any $x \in X$ and $y = f(x)$, there exists an affine open neighborhood $\operatorname{Spec} A = U$ of x, as well as that $\operatorname{Spec} B = V$ of y, such that $f(U) \subseteq V$;
- the morphism $f|_U : U \to V$ corresponds to a ring homomorphism $B \to A$ so that A is a B-algebra of finite type.

In particular, when X and Y are both affine schemes, this is equivalent to say $\Gamma(X, \mathcal{O}_X)$ is a finite $\Gamma(Y, \mathcal{O}_Y)$-algebra of finite type. This is because both U, V can be taken to be basic open subsets, and X (resp. Y) can be covered by finitely many U's (resp. V's). It leads to finitely many sets of generators for $\Gamma(X, \mathcal{O}_X)$ over $\Gamma(Y, \mathcal{O}_Y)$. See [EGA I, 6.3.3] for more details.

Consequently, if $f : X \to Y$ is locally of finite type, then for any affine open subset $U = \operatorname{Spec} A \subseteq X$ and $V = \operatorname{Spec} B \subseteq Y$, whenever $f(U) \subseteq V$ holds, A must be a B-algebra of finite type.

Proposition 6. *Let $f : X \to Y$ be a morphism that is locally of finite type. If Y is locally noetherian, then so also is X.*

Local finite presentation

We refer to [EGA IV, §1.4].

Definition 7. A morphism $f : X \to Y$ is *locally of finite presentation* if

- for any $x \in X$ and $y = f(x)$, there exists an affine open neighborhood $\operatorname{Spec} A = U$ of x, as well as that $\operatorname{Spec} B = V$ of y, such that $f(U) \subseteq V$;
- the morphism $f|_U : U \to V$ corresponds to a ring homomorphism $B \to A$ so that A is a finite presented B-algebra.

The latter condition in Definition 7 above implies that for some $n, m \in \mathbb{N}$,

$$A \simeq B[z_1, \dots, z_n]/(h_1, \dots, h_m).$$

The proof of this assertion is complicated and subtle. One is supposed to refer to [EGA IV, 1.4.6]. Roughly, the idea is that when $B[z_1, \dots, z_n]/\mathfrak{b}$ is a finite presented B-algebra for an arbitrary ideal $\mathfrak{b}$ of $B[z_1, \dots, z_n]$, it turns out that $\mathfrak{b}$ must be an ideal finite type.

Accordingly, if X and Y are both affine schemes, the condition is equivalent to say $\Gamma(X, \mathcal{O}_X)$ is a finite presented $\Gamma(Y, \mathcal{O}_Y)$-algebra.

Remark 8. If we concerned about locally noetherian schemes only, then the notion of local finite presentation would be vacuous. In general situations, when this local finite presentation condition holds, we may reduce some problem to noetherian schemes by taking projective limits (c.f. [EGA IV, §8.9]).

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