LECTURE 2 - FIBER SEQUENCES

The goal of this lecture is to introduce the fiber sequence generated by a map and relate it to the long exact sequence of the homotopy groups.

Definition 1. A surjective map $p : E \to B$ is a fibration if it satisfies the covering homotopy property. This means that if $h \circ i_0 = p \circ f$ in the diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & B \end{tikzcd}$$

then there exists $\tilde{h}$ that makes the diagram commute.

Proposition 2. *The pullback of a fibration is a fibration. This means that if $p : E \to B$ is a fibration and $g : A \to B$ is any map, then the induced map $A \times_g E \to A$ is a fibration.*

Proposition 3. *If $p : E \to B$ is a covering, the p is a fibration with a unique path lifting function s.*

Definition 4. For $f : X \to Y$, we define the *mapping path space* Nf to be

$$Nf = X \times_f Y^I = \{(e, \beta) | \beta(1) = p(e)\} \subset X \times Y^I.$$

Observe that f coincides with the composite

$$X \xrightarrow{\nu} NF \xrightarrow{\rho} Y,$$

where $\nu(x) = (x, c_{f(x)})$ and $\rho(x, \chi) = \chi(1)$. Let $\pi : Nf \to X$ be the projection. then $\pi \circ \nu = id$ and $id \simeq \nu \circ \pi$ since we can define a deformation $h : Nf \times I \to Nf$ of Nf onto $\nu(X)$ by setting

$$h(x, \chi)(t) = (x, \chi_t), \text{ where } \chi_t(s) = \chi((1 - t)s).$$

Definition 5. It can be checked that $\rho : Nf \to Y$ satisfies the covering homotopy property. We call it the *fibrant replacement* of f.

Translation of fibers along paths in the base space played a fundamental role in the study of covering spaces. Fibrations admit an up to homotopy version of that theory that well illustrates the use of the covering homotopy property (CHP).

Let $p : E \to B$ be a fibration with fiber F_b over $b \in B$ and let $i_b : F_b \to E$ be the inclusion. For a path $\beta : I \to B$ from b to b', the CHP gives a lift $\tilde{\beta}$ in the diagram

$$\begin{tikzcd} F_b \times \{0\} \arrow[rr, "i_b"] \arrow[d] & & E \arrow[d, "p"] \\ F_b \times I \arrow[r, "\pi_2"'] \arrow[urr, dashed, "\tilde{\beta}"] & I \arrow[r, "\beta"'] & B. \end{tikzcd}$$

Definition 6. Note that $\tilde{\beta}_1$ maps F_b to the fiber $F_{\beta(1)} = F_{b'}$. We call

$$[\tilde{\beta}_1] \in [F_b, F_{b'}]$$

the translation of fibers along the path class $[\beta]$.

Exercise 7. The definition claims that $[\tilde{\beta}_1]$ does not depend on the choice of β in its path class. Prove this fact by considering a diagram similar to the above one.

Definition 8. The dual notion of cones and suspensions are paths and loops. The *path space* of X is defined by $PX = F(I, X)$.

Definition 9. For a based map $f : X \to Y$, we define the *homotopy fiber* Ff to be

$$Ff = X \times_f PY = \{(x, \chi) | f(x) = \chi(1)\} \subset X \times PY.$$

Equivalently, Ff is the pullback displayed in the diagram

$$\begin{tikzcd} Ff \arrow[r] \arrow[d, "\pi"'] & PY \arrow[d, "p_1"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where $\pi(x, \chi) = x$. As a pullback of a fibration, π is a fibration.

Definition 10. Let $\iota : \Omega Y \to Ff$ be the inclusion specified by $\iota(\chi) = (*, \chi)$. The sequence

$$\cdots \to \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{-\Omega\iota} \Omega Ff \xrightarrow{-\Omega\pi} \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{\iota} Ff \xrightarrow{\pi} X \xrightarrow{f} Y$$

is called the fiber sequence generated by the map f.

Theorem 11. *For any based space Z, the induced sequence*

$$\cdots \to [Z, \Omega Ff] \to [Z, \Omega X] \to [Z, \Omega Y] \to [Z, Ff] \to [Z, X] \to [Z, Y]$$

is an exact sequence of pointed sets, or of groups to the left of $[Z, LoopY]$, or of Abelian groups to the left of $[Z, \Omega^2 Y]$.

Corollary 12. *If we let $Z = S^n$, then we get the long exact sequence*

$$\cdots \to \pi_n(Ff) \to \pi_n X \to \pi_n Y \to \pi_{n-1}(Ff) \to \cdots \to \pi_0(Ff) \to \pi_0 X \to \pi_0 Y.$$