On Beilinson-Bloch-Kato conjecture for products of elliptic curves

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(at MCM)

- High dim'l B-SD
- Personal motivation.

Solving Diophantine equations / $\mathbb{Q}$

$\rightsquigarrow$ Consider algebraic curves, classified by genus g:

- $g=0$: quadratic equation
- $g=1$: elliptic curves

  e.g. $y^2 = f(x)$, $\deg f = 3$ or $Q_1 \cap Q_2$, $Q_i \subseteq \mathbb{P}^3$ quadratics
- $g \geq 2$: (Mordell, Faltings) finitely many sol'ns / $\mathbb{Q}$

  e.g. $y^2 = f(x)$, $\deg f \geq 5$.

  $x^n + y^n = z^n$ $(n \geq 4)$ Fermat type.

More interesting example:

$Q_1 \cap Q_2 \cap Q_3$, $Q_i \subseteq \mathbb{P}^4$

Humbert curve $\sum_{i=1}^{5} a_i^j X_i^2 = 0$ $\leftarrow$ genus $=5$
$(j=1,2,3)$

<u>But</u> $\mathrm{Aut} \supseteq \mu_2^5 / \Delta(\mu_2)$

$\rightsquigarrow$ $Q_1 \cap Q_2 \cap Q_3$ can quotient in 5 different ways to get a genus 1 ell curve.

Its Jacobian $\simeq \prod_{i=1}^{5} E_i$, $E_i \simeq C/\mu_2$ (i-th coordinate).

$g=1$ E elliptic curve ($\rightsquigarrow E(\mathbb{Q})$ ab grp).

Mordell (-Weil) conj : $E(\mathbb{Q})$ is fin gen'd.

& weak MW : $E(\mathbb{Q})/2E(\mathbb{Q}) \subset \mathrm{Sel}_2(E/\mathbb{Q})$ finite grp.

(Hermite - Minkowski).

Aside Roughly, $\mathrm{Sel}_2(E/\mathbb{Q}) \longleftrightarrow \left\{ \begin{array}{l} C = Q_1 \cap Q_2, \\ \mathrm{Jac}(C) \simeq E \end{array} \right\}$

locally trivial.

Uniformity (D-H-G, K, Yuan)

Let X curve with $g_X = g \geq 2$. $K/\mathbb{Q}$ finite.

$$\# X(K) \leq C(g)^{\mathrm{rank}\, \mathrm{Jac}_X(K) + 1}$$

$C(g)$ = absolute const.

Abel-Jacobi case: Assume $X \hookrightarrow \mathrm{Jac}_X$.

(Chabauty-Kim find all sol'ns if applicaple.

(conditionally).

B-SD conjecture $\mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) = \mathrm{ord}_{s=1} L(E,s)$

(known if $\mathrm{ord}\, L \leq 1$).

Def L-fct $\longleftrightarrow \{ \# E(\mathbb{F}_p) \mid p \text{ prime} \}$

$$\prod_p \frac{1}{1 - a_p \cdot p^{-s} + p^{1-2s}}, \qquad \# E(\mathbb{F}_p) = p+1-a_p$$

($\mathrm{Re}(s) > \frac{3}{2}$, absolutely convergent.

Taniyama - Shimura conj: $L(E,s) \triangleq L(f,s)$
(pf by Wiles) f mod form of wt 2.

<u>B-SD Conj</u> (1960s, original version)

$\Downarrow$

Riemann hypothesis $\displaystyle\prod_{p<N} \frac{\# E(\mathbb{F}_p)}{p} \sim (\log N)^{\text{rank}}$.

<u>"Thm"</u> (As a consequence of DGH + K/Y, Kato, Rohrlich)

$X/\mathbb{Q}$ s.t. $J_X = \prod_i E_i$.

$$\mathbb{Q}(\mu_{p^\infty}) = \bigcup_{n\geq 1} \mathbb{Q}(e^{2\pi i/p^n})$$

Then $\# X(\mathbb{Q}(\mu_{p^\infty})) < \infty$.

<u>Rmk</u> This thm needs L-fcts to prove boundedness (whereas Mordell-Weil doesn't).

Rohrlich's work: L-fct order $<\infty \overset{\text{Kato}}{\Longrightarrow}$ rk $J_X(\mathbb{Q}(\mu_{p^\infty})) < \infty$

$\Longrightarrow \# X(\mathbb{Q}(\mu_{p^\infty})) < \infty$.

<u>Higher dim'l B-SD: BBK conj</u>

Chow grp: $X/\mathbb{Q} \rightsquigarrow Ch^i(X) = \dfrac{\{\text{codim } i \text{ alg cycles}\}}{\sim \text{rat'l equiv}}$

$$\begin{array}{c} Ch^i(X) \\ \cup\!\shortmid \\ Ch^i(X)_0 \quad \text{homologically trivial cycles} \\ \Big\downarrow{\scriptstyle \text{cycle class map}} \\ H^{2i}(X) \end{array}$$

$$\begin{array}{ccccc}
\text{Then} \quad \text{MW grp} & \longrightarrow & \text{Weak MW grp} & \rightsquigarrow & \text{L-fct} \\
\downarrow & & \downarrow \text{ fix a prime } p. & & \downarrow \\
Ch^i(X)_0 & \longrightarrow & \text{Bloch-Kato Selmer grp} & \rightsquigarrow & L(s, H^{2i-1}_{et}). \\
& & H^1_f(\mathbb{Q}, \underbrace{H^{2i-1}_{et}(\bar{X})(i)}_{wt=-1}) & &
\end{array}$$

<u>Tate's obs</u> $\operatorname{rk} Ch^i(X)/\text{hom} = -\operatorname{ord}_{s=i} L(s, H^{2i}_{et}(\bar{X}))$

<u>Bloch</u> (recurring fantasy) not known to be finite

$$\begin{array}{ccc}
\operatorname{rk} Ch^i(X)_0 & = & \operatorname{ord}_{s=i} L(s, H^{2i-1}_{et}(\bar{X})) \\
\| \quad \downarrow AJ & & \\
\operatorname{rk} H^1_f(\mathbb{Q}, H^{2i-1}_{et}(\bar{X})(i)) & \text{// B-K conj.} &
\end{array}$$

<u>Thm</u> (LTXZZ, ~2019)

E_1, E_2 ell curves $/\mathbb{Q}$, non-CM.

Let $X = E_1^n \times E_2^{n+1}$

$V = \operatorname{Sym}^n H^1(E_1) \otimes \operatorname{Sym}^{n+1} H^1(E_2)$ $\mathbb{Q}_p$-repr of $\operatorname{Gal}_\mathbb{Q}$.

Let $K/\mathbb{Q}$ imag quad. Then

$$\operatorname{ord}_{s=n+1} L(s, V) = 0 \implies H^1_f(s, V(n+1)) = 0.$$

(B-SD for rank 0).

entire mero conti by Newton-Thorne.

<u>Thm</u> (LTXZZ + Disegni-Zhang)

For p-adic L-fct case, AFL & p-adic GZ formula.

$$\operatorname{ord}_{s=n+1} L_p(s, V) = 1 \Rightarrow \operatorname{rank} H^1_f = 1. \quad (p \gg 0).$$

DZ $\Downarrow$ GGP cycle $\neq 0$ $\Nearrow$ LTXZZ, Euler system