Fujita Vanishing (藤田消沒定理)

$\boxed{\S 1}$ <u>Thm</u> (Fujita)

X proj. sch $/k$, H <u>ample</u> $\in \mathrm{CaDiv}\, X$.

$\mathcal{F} \in \mathrm{Coh}_X$. $\exists\, M = m(\mathcal{F}, H)$ s.t.

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(nH + D)) = 0, \quad \forall i > 0, \quad n \geq m(\mathcal{F}, H)$$

any $D \in \mathrm{CaDiv}$ nef.

proof. WLOG, $k = \bar{k}$. $X = \mathrm{supp}\, \mathcal{F}$.

<u>Rmk</u> $\mathcal{F} \in \mathrm{Coh}_X \rightsquigarrow \mathrm{supp}\, \mathcal{F}$ w/ subsch str.

by $\ker(\mathcal{O}_X \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F})) \in \mathrm{Ideal}_{\mathcal{O}_X}$.

$\boxed{\text{Induction on dim}}$

<u>Step 1</u> $\dim X = 0$ ($\checkmark$)

Suppose validity for lower dim. $\square$

<u>Step 2</u> Reduce to $X_{red} = X$.

pf. Suppose ok for X_{red}. $\mathcal{N} = \mathrm{nil}\, \mathcal{O}_X$ s.t. $\mathcal{N}^r = 0$ $(r > 0)$

$\rightsquigarrow$ filtration

$$\mathcal{F} \supset \mathcal{N} \cdot \mathcal{F} \supset \mathcal{N}^2 \cdot \mathcal{F} \supset \cdots \supset \mathcal{N}^n \cdot \mathcal{F} = 0.$$

s.t. $\mathcal{N}^i \mathcal{F} / \mathcal{N}^{i+1} \mathcal{F} \in \mathrm{Coh}_{\mathcal{O}_{X_{red}}}$

$\Rightarrow \forall j>0,\ H^j(X, (\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}) \otimes \mathcal{O}_X(mH+D)) = 0.$

$m \geq m(\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}, H).$

b/c $\mathcal{O}_{X_{red}}(H)$ ample.

<u>Now</u> $0 \to \mathcal{N}^{i+1}\mathcal{F} \to \mathcal{N}^i\mathcal{F} \to \mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F} \to 0$

$\xrightarrow{\mathcal{O}_X(mH+D)}$ twisting $\cdots \xrightarrow{H^j(\cdot)} \cdots \to H^j(X, \mathcal{N}^{i+1}\mathcal{F} \otimes \mathcal{O}_X(mH+D))$

$\to H^j(X, \mathcal{N}^i\mathcal{F} \otimes \mathcal{O}_X(mH+D))$

$\to \underset{=0}{H^j(X, \mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F} \otimes \mathcal{O}_X(mH+D))} \to \cdots$

long exact.

$\Rightarrow H^j(X, \mathcal{N}^i\mathcal{F} \otimes \mathcal{O}_X(mH+D)) \quad i \geq 0.$

$\qquad \| $

$H^j(X, \mathcal{N}^r\mathcal{F} \otimes \mathcal{O}_X(mH+D)) = 0, \quad r \gg 0.$

$\rightsquigarrow \mathcal{F} = \mathcal{N}^0\mathcal{F}$ ok. $\square$

<u>Step 3</u> Reduce to X irred.

pf. Let $X = X_1 \cup \cdots \cup X_k$.

$\updownarrow$

$\mathcal{I} = \mathcal{I}_{X_1} \rightsquigarrow 0 \to \mathcal{I}\cdot\mathcal{F} \to \mathcal{F} \to \mathcal{F}/\mathcal{I}\mathcal{F} \to 0.$

dévissage here!

Supp on $X_2 \cup \cdots \cup X_n$ $\qquad$ Supp on X_1.

<u>Assume</u> $H^j(X, \mathcal{I}\mathcal{F} \otimes \mathcal{O}_X(mH+D)) = 0,$

w/ $m \geq m(\mathcal{I}\mathcal{F}, H|_{X_2 \cup \cdots \cup X_k}).$

& $H^j(X, (\mathcal{F}/\mathcal{I}\mathcal{F}) \otimes \mathcal{O}_X(mH+D)) = 0$

w/ $m \geq m(\mathcal{F}/\mathcal{I}\mathcal{F}, H|_{X_1})$.

$$\Rightarrow H^j(X, \mathcal{F} \otimes \mathcal{O}_X(mH+D)) = 0, \quad m \gg 0. \qquad \square$$

<u>Step 4</u> Reduce to H very ample.

pf. H ample $\rightsquigarrow$ lH very ample.

Assume Thm for lH.

<u>note</u> $\mathcal{F} \in \mathrm{Coh}_X \Rightarrow \mathcal{F} \otimes \mathcal{O}_X(nH) \in \mathrm{Coh}_X$

$n = 0, \dots, l-1$.

$\rightsquigarrow m(\mathcal{F} \otimes \mathcal{O}_X(nH), lH) = m_n$.

$\rightsquigarrow m(\mathcal{F}, H) = l \cdot \max_n \{m_n + 1\}$

$$\Rightarrow H^j(X, \mathcal{F} \otimes \mathcal{O}_X(\underbrace{mH+D}_{\geq l(m_nH+H)+D})) = 0.$$

$$= (m_n+1)\underbrace{lH}_{\text{very ample}} + D \qquad \square$$

<u>Step 5</u> Reduce to show $H^1 = 0$.

pf. $H \in \mathrm{CaDiv}\, X$ nef $\rightsquigarrow$ $|H|$ lin system

$\cup$

A (general).

$$0 \to \mathcal{F} \otimes \mathcal{O}_X(-A) \to \mathcal{F} \to \mathcal{F}_A \to 0.$$

$\mathcal{F}_A$ = coker.

$\dim \operatorname{supp} \mathcal{F}_A < \dim \operatorname{supp} \mathcal{F} = \dim X$

$\rightsquigarrow \exists\, m(\mathcal{F}_A, H|_A)$ w/ $H^i(A, \mathcal{F}_A \otimes \mathcal{O}_A(mH+D)) = 0$.

by induction, $\forall i \geq 1$.

$\Rightarrow H^i(X, \mathcal{F} \otimes \mathcal{O}_X(\boxed{(m-1)A} + D)) = H^i(X, \mathcal{F} \otimes \mathcal{O}_X(\boxed{mA} + D))$

$\forall i \geq 2$.

Serre vanishing: $m \gg 0$ ✓

$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mA+D)) = 0$. $\forall i \geq 1$

$i \geq 2$ ok.

□

Step 6 Reduce to $\mathcal{F} = \mathcal{O}_X$.

pf. Assume Thm holds for $\mathcal{F} = \mathcal{O}_X$

$\rightsquigarrow 0 \to \mathcal{O}_X \xrightarrow{\alpha} \mathcal{F} \otimes \mathcal{O}_X(aH)$. inj. homo.

$a \gg 0$.

Consider $0 \to \mathcal{O}_X \to \mathcal{F} \otimes \mathcal{O}_X(aH) \to \operatorname{coker} \alpha \to 0$.

induct on rank $\mathcal{F}$. ✓ □

Step 7 char $k = 0$ (✓)

pf. $f: Y \to X$ resol.

$\rightsquigarrow 0 \to \underbrace{f_* \omega_Y} \to \mathcal{O}_X(bH) \to \overset{\text{coker}}{\overset{\shortparallel}{\mathcal{E}}} \to 0$

tor-free & rank $f_* \omega_Y = 1$

$\dim \operatorname{supp} \mathcal{E} < \dim X$.

And: $H^j(X, f_* \omega_Y \otimes \mathcal{O}_X(mH+D)) = 0$.

$\underline{\text{Kollár vanishing}}$ $\forall\, m>0,\ j>0$.

$\Rightarrow H^j(X, \mathcal{O}_X(bH) \otimes \mathcal{O}_X(mH+D))$

$= H^j(X, \mathcal{O}_X((b+m)H+D)) = 0, \quad j>0.$ □

<u>Step 8</u> Reduce to $\mathcal{F} = \omega_X$.

<u>Rmk</u> $\omega_X^\circ \simeq \mathcal{E}xt^{N-\dim X}_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N})$, $X \hookrightarrow \mathbb{P}^N$.

pf. Assume $\mathcal{F} = \omega_X$ ok.

$$0 \to \omega_X \xrightarrow{\beta} \mathcal{O}_X(cH) \to \operatorname{coker}\beta \to 0,$$

$\operatorname{rank} \omega_X = \operatorname{rank} \mathcal{O}_X(cH) = 1$

$\Rightarrow \dim(\operatorname{supp} \operatorname{coker}\beta) < \dim X$

induction on dim.:

ω_X ✓ $\operatorname{coker}\beta$ ✓ $\Rightarrow \mathcal{O}_X(cH)$ ✓

$\Rightarrow H^i(X, \mathcal{O}_X((c+m)H+D)) = 0, \quad i>0.$ □

<u>Now</u> Consider $\mathcal{F} = \omega_X$, $\operatorname{char} k = p > 0$.

<u>Step 9</u> $\operatorname{char} k = p$ (✓)

pf. $H \rightsquigarrow |H| \rightsquigarrow X \to \mathbb{P}^N$.

$F :=$ Frob morphism.

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d] \\ \mathbb{P}^N \arrow[r, "F"] & \mathbb{P}^N \end{tikzcd}$$

$\mathcal{O}_X \longrightarrow F_*\mathcal{O}_X$, apply $R\mathcal{H}om_{\mathbb{P}^N}(-, \omega^\bullet_{\mathbb{P}^N})$

$\rightsquigarrow R\mathcal{H}om_{\mathbb{P}^N}(F_*\mathcal{O}_X, \omega^\bullet_{\mathbb{P}^N}) \longrightarrow R\mathcal{H}om_{\mathbb{P}^N}(\mathcal{O}_X, \omega^\bullet_{\mathbb{P}^N})$

<u>Grothendieck duality</u>

$\Rightarrow R\mathcal{H}om_{\mathbb{P}^N}(F_*\mathcal{O}_X, \omega^\bullet_{\mathbb{P}^N}) \cong F_* R\mathcal{H}om_{\mathbb{P}^N}(\mathcal{O}_X, \omega^\bullet_{\mathbb{P}^N})$

$\rightsquigarrow \gamma : F_*\omega_X \longrightarrow \omega_X = \mathcal{E}xt^{N-\dim X}_{\mathbb{P}^N}(\mathcal{O}_X,$

(<u>Recall</u> Serre duality : $Rf_* \rightleftarrows f^!$, $f : X \to \operatorname{Spec} k$.

$\operatorname{Hom}_{D^b(\operatorname{Spec} k)}(Rf_*(\mathcal{F}), V) \cong \operatorname{Hom}_{D^b(X)}(\mathcal{F}, f^!(V))$.

vec. bundle on Spec k

= fin. dim'l v.s.)

$U \subseteq X$ <u>lissé</u> open. $\Rightarrow F_*\omega_X|_U \to \omega_X|_U \to 0$.

$\Rightarrow \mathcal{E}xt^k_{\mathbb{P}^N}(\mathcal{A}, \omega_{\mathbb{P}^N}) = 0, \quad \forall k > N - \dim X$

$:= \operatorname{coker}(\mathcal{O}_X \to F_*\mathcal{O}_X)$.

loc. free on U.

Consider $0 \to \overset{(v)}{\ker \gamma} \to \boxed{F_*\omega_X} \to \operatorname{im}\gamma \to 0$.

$\downarrow$

$0 \to \operatorname{im}\gamma \to \boxed{\omega_X} \to \underset{(v)}{\mathcal{C}} \longrightarrow 0$.

dévissage : $\dim X > \dim \operatorname{supp} \mathcal{C} \Rightarrow \mathcal{C}\ (v)$.

Step 5 $\Rightarrow \ker\gamma\ (v)$. i.e. $H^2(X, \ker\gamma \otimes \mathcal{O}_X(mH+D)) = 0$.

Induction on $\dim \mathcal{E}$

$\rightsquigarrow$ $H^1(X, F_*\omega_X \otimes \mathcal{O}_X(mH+D))$ surj.

$$\downarrow$$

$H^1(X, \omega_X \otimes \mathcal{O}_X(mH+D)) \to 0$

<u>note</u> By proj. formula:

$$H^1(X, \underbrace{F_*\omega_X \otimes \mathcal{O}_X(mH+D)}_{\text{proj. im.}}) \cong H^1(X, \underbrace{\omega_X \otimes \mathcal{O}_X(\overset{p = \operatorname{char} k.}{p}(mH+D))}_{\text{projector}}))$$

$\Rightarrow$ $H^1(X, \omega_X \otimes \mathcal{O}_X(mH+D)) \twoheadleftarrow H^1(X, \omega_X \otimes \mathcal{O}_X(p^e(mH+D)))$.

$\forall e > 0$, $m \geq m_0$

$m_0(\omega_X, H)$.

indep. of D (nef)

Serre vanishing.

$\Rightarrow$ $H^1(X, \omega_X \otimes \mathcal{O}_X(mH+D)) = 0$, $\forall m \geq m_0$ $\square$

· Proof Finished!

<u>note</u> $F_*\omega_X \to \omega_X \to 0$ can be constructed by

<u>Lem</u> $f: V \twoheadrightarrow W$ proj. surj. b/w proj. var $/ k = \bar{k}$,

w/ $\dim V = \dim W = n$.

$\rightsquigarrow$ $\varphi: f_*\omega_V \twoheadrightarrow \omega_W$ generically surj.

cf. [T. Fujita, Semipositive line bundles, Cor 5.7].

pf. By def'n $H^n(V, \omega_V) \neq 0$.

<u>Léray</u> spec'l seq'ce:

$$E_2^{p,q} = H^p(W, R^q f_* \omega_V) \Rightarrow H^{p+q}(V, \omega_V).$$

Note: $\operatorname{Supp} R^q f_* \omega_V \subseteq W_q = \{ w \in W : \dim f^{-1}(w) \geq q \}$.

$\dim f^{-1}(W_q) < r \quad \forall q > 0 \Rightarrow \dim W_q < n - q, \ \forall q > 0.$

$\Rightarrow E_2^{n-q,q} = 0$ unless $q = 0$.

& $E_2^{n,0} = H^n(W, \underbrace{R^0 f_* \omega_V}_{= f_* \omega_V}) \neq 0$ by $H^n(V, \omega_V) \neq 0$.

By def of ω_W: $\operatorname{Hom}(f_* \omega_V, \omega_W) \neq 0$.

$\Downarrow$

$\varphi \neq 0$ w/ $\operatorname{im} \varphi \subseteq \omega_W$.

$\operatorname{Hom}(\operatorname{Im} \varphi, \omega_W) \neq 0 \Rightarrow H^n(W, \operatorname{Im} \varphi) \neq 0$.

$\uparrow$ GTM52, III, §7.

$\Downarrow$

$\dim \operatorname{supp} \operatorname{im} \varphi = n$.

$\Leftarrow$

$\varphi : f_* \omega_V \to \omega_W$ gen. surj.

$\uparrow$ $\operatorname{rk} = 1$. $\square$

Rmk Lemma w/ $R^q f_* \omega_V = 0, \ \forall q > 0 \Rightarrow H^n(W, f_* \omega_V) \simeq H^n(V, \omega_V) \cong k$.

$\forall \varphi : f_* \omega_V \to \omega_W$ nontriv. homo, $k = \bar{k}$. $\longrightarrow k$

$\exists\, a \in k \setminus \{0\}$ s.t. $\psi = a\varphi$. φ as in the Lem.

<u>note</u> f fin $\Rightarrow R^q f_* \omega_V = 0,\ \forall q > 0.$

$\Uparrow$

char $k = 0$ & V has only <u>rat. singularities</u>

$\uparrow$

by Grauert-Riemenschneider vanishing

or Kollár's tor-free thm.

<u>Lem</u> special case of Kollár's tor ...

cor of Kawamata-Viehweg vanishing.

$f : V \twoheadrightarrow W$ proj. surj.

V proj. sm, W proj. $\in Var_k$, $k = \bar{k}$, char $k = 0$

$\Rightarrow R^q f_* \omega_V = 0,\ \forall q > \dim V - \dim W.$

pf. $A \in CaDiv(W)$ suff. ample. s.t.

$$H^0(W, f_*\omega_V \otimes \mathcal{O}_W(A)) \simeq H^q(V, \omega_V \otimes \mathcal{O}_V(f^*A)).$$

& $R^q f_* \omega_V \otimes \mathcal{O}_W(A)$ glo. gen'd, $\forall q$.

<u>note</u> num. dim $\nu(V, f^*A) = \dim W$.

$\rightsquigarrow$ Can check $H^q(V, \omega_V \otimes \mathcal{O}_V(f^*A)) = 0.$

$\forall q > \dim V - \dim W = \dim V - \nu(V, f^*A).$

by Kawamata-Viehweg vanishing.

$\Rightarrow R^q f_* \omega_V = 0, \ \forall q > \dim V - \dim W.$ $\square$

<u>Thm</u> (weak generalization of Kodaira vanishing).

X = n-dim'l cpt Kähler mfd.

$\mathcal{L}$ = line bd $/X$ w/ curvature form $\sqrt{-1}\,\Theta(\mathcal{L})$ semi-positive.

$\mathcal{L}$ has $\geq k$ eigenvalues > 0 on $U \subseteq X$, U dense open.

$\Rightarrow H^i(X, \omega_X \otimes \mathcal{L}) = 0, \ \forall i > n-k.$

<u>note</u> $H^i(X, \omega_X \otimes \mathcal{L}) \cong \mathcal{H}^{n,i}(X, \mathcal{L})$

$\quad = \{\mathcal{L}\text{-valued harmonic } (n,i)\text{-forms on } X\}.$

Nakano's formula

us can check $\mathcal{H}^{n,i}(X, \mathcal{L}) = 0, \ \forall i+k \geq n+1.$

<u>Prop</u> $f: V \to W$ $\boxed{\text{proper}}$ surj. b/w normal alg var. w/ V_w conn, $\forall w \in W$.

$k = \bar{k}$, char $k = 0$. V, W have only rat. sing.

$\Rightarrow R^d f_* \omega_V \simeq \omega_W$, $d = \dim V - \dim W$.

pf. Comm. diagram s.t. (i) X, Y sm. alg. var

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d, "f"] \\ Y \arrow[r, "p"'] & W \end{tikzcd}$$

(ii) π, p proj. birat.

(iii) g proj. & sm. outside a SNC $\Sigma \in \mathrm{Div}\, Y$.

<u>note</u> $\forall j$, $R^j g_* \omega_X$ loc. free.

Grothendieck duality:

$\rightsquigarrow Rg_* \mathcal{O}_X \simeq R\mathcal{H}om_Y(Rg_* \omega_X^\cdot, \omega_Y^\cdot)$.

$\Rightarrow \mathcal{O}_Y \simeq \mathcal{H}om_Y(R^d g_* \omega_X, \omega_Y)$.

Yoneda $\Rightarrow \omega_Y \simeq R^d g_* \omega_X$.

$\rightsquigarrow p_* R^d g_* \omega_X \simeq p_* \omega_Y \overset{p \text{ birat.}}{\simeq} \omega_W$

<u>note</u> $p_* R^d g_* \omega_X \simeq R^d (p \circ g)_* \omega_X$

$R^i p_* R^d g_* \omega_X = 0$, $\forall i > 0$. (cf. sm base change)

Also, $R^d (p \circ g)_* \omega_X \simeq R^d (f \circ \pi)_* \omega_X \underset{\pi \text{ birat.}}{\simeq} R^d f_* \omega_V$.

since $R^i \pi_* \omega_X = 0$, $\forall i > 0$ & $\omega_V \simeq \pi_* \omega_X$.

$\Rightarrow R^d f_* \omega_V \simeq \omega_W$. $d = \dim V - \dim W$. $\square$

$\boxed{\S 3}$ Applications of Fujita vanishing

<u>Thm</u> $\mathcal{F} \in \mathrm{Coh}\,X$, X proper / $k = \bar{k}$. $\mathcal{L} \to X$ nef line bun.

$$\Rightarrow \underbrace{\dim H^q(X, \mathcal{F} \otimes \mathcal{L}^{\otimes t})}_{\text{a poly.} \in \mathbb{Z}[t]} \leq O(t^{m-q}). \quad m = \dim \operatorname{supp} \mathcal{F}.$$

Pf. <u>Step 1</u> Assume X proj. Induct on q.

$H \geq 0 \in \mathrm{CaDiv}\,X$ ample. s.t. $\mathcal{L} \otimes \mathcal{O}_X(H)$ ample

$H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) \subset H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes t} \otimes \mathcal{O}_X(tH))$. $\forall t > 0$.

May assume $\mathcal{L}$ ample by replacing $\mathcal{L} \mapsto \mathcal{L} \otimes \mathcal{O}_X(H)$.

$$\rightsquigarrow \dim H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) \underset{\text{Serre vanishing.}}{=} \chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes t}), \quad t \gg 0.$$

$$\Rightarrow \dim H^0(\mathcal{F} \otimes \mathcal{L}^{\otimes t}) \leq O(t^m). \quad \deg \leq m = \dim \operatorname{supp} \mathcal{F}.$$

When $q > 0$, $\exists A \in \mathrm{CaDiv}\,X$ very ample.

s.t. $H^q(X, \mathcal{F} \otimes \underbrace{\mathcal{O}_X(A) \otimes \mathcal{L}^{\otimes t}}_{\text{“}\approx \mathcal{O}_X(A + H)\text{''}}) = 0$ by Fujita. $\forall t \geq 0$.

$$\underset{mH}{\underset{\uparrow}{A}} \qquad \underset{D}{\underset{\uparrow}{H}}$$

$A \rightsquigarrow \mathfrak{d}_A = |A| \ni D$ gen'l elt. $\rightsquigarrow \alpha : \mathcal{F} \otimes \mathcal{O}_X(-D) \xrightarrow{\text{inj.}} \mathcal{F}$

$$\rightsquigarrow 0 \to \mathcal{F} \otimes \mathcal{O}_X(-D) \xrightarrow{\alpha} \mathcal{F} \longrightarrow \underset{\text{coker}}{\underset{\parallel}{\mathcal{F}_D}} \to 0.$$

$$\Rightarrow \dim H^q(X, \mathcal{F}\otimes\mathcal{L}^{\otimes t}) \le \dim H^{q-1}(D, \mathcal{F}_D \otimes \mathcal{O}_D(A)\otimes \mathcal{L}^{\otimes t})$$
$$\le O(t^{m-q}).$$

induction hyp. ok.

Step 2 General X. nt induction on $\operatorname{supp}\mathcal{F}$.

May assume $X = \operatorname{supp}\mathcal{F}$ var. (i.e. int)

X proper, Chow's Lemma

$$\begin{array}{ccc} \exists f : V & \xrightarrow[\dashleftarrow]{} & X \\ | & & | \\ \mathcal{G} = f^*\mathcal{F} & & \mathcal{F} \end{array}$$
birat., V proj. var.

$\rightsquigarrow \beta : \mathcal{F} \to f_*\mathcal{G}$.

$\rightsquigarrow \exists \phi \ne U \subseteq X$ Zar open s.t. $\beta|_U$ isom.

Consider

$$0 \to \ker\beta \to \mathcal{F} \to \operatorname{im}\beta \to 0.$$
$$0 \to \operatorname{im}\beta \to f_*\mathcal{G} \to \operatorname{coker}\beta \to 0.$$

By induction: $\dim H^q(X, \ker\beta\otimes\mathcal{L}^{\otimes t}) \le O(t^{m-q})$

$\dim H^{q-1}(X, \operatorname{coker}\beta\otimes\mathcal{L}^{\otimes t}) \le O(t^{m-q})$.

$\Rightarrow$ suff. to show $\dim H^q(X, f_*\mathcal{G}\otimes\mathcal{L}^{\otimes t}) \le O(t^{m-q})$.

Léray: $E_2^{p,q} = H^p(X, R^q f_*\mathcal{G}\otimes\mathcal{L}^{\otimes t}) \Rightarrow H^{p+q}(V, \mathcal{G}\otimes(f^*\mathcal{L})^{\otimes t})$

$$\Rightarrow \dim H^q(X, f_*\mathcal{G}\otimes\mathcal{L}^{\otimes t})$$

$$\leq \sum_{j\geq 1} \underbrace{\dim H^{q-j-1}(X, R^j f_*\mathcal{G}\otimes\mathcal{L}^{\otimes t})}_{\leq O(t^{m-q})} + \underbrace{\dim H^q(V, \mathcal{G}\otimes(f^*\mathcal{L})^{\otimes t})}_{\leq O(t^{m-q})}.$$

$\Uparrow$ $\qquad\qquad$ (V proj.)

$\dim \operatorname{Supp} R^j f_*\mathcal{G} \leq \dim X - j - 1.\ \forall j \geq 1.$

by ind. hyp.

$$\Rightarrow \dim H^q(X, \mathcal{F}\otimes\mathcal{L}^{\otimes t}) \leq O(t^{m-q}).$$

$\square$

<u>Cor</u> (Fujita's numerical characterization of nef & big line bundles).

$\mathcal{L}$ nef line bun. / V <u>proper</u> alg. var / $k = \bar{k}$ w/ $\dim_k V = n$.

any char k !

$\rightsquigarrow \kappa(X, \mathcal{L}) = 0 \iff \mathcal{L}^n > 0$

self-intersection #.

<u>note</u> $\mathcal{L}$ big $\iff \kappa(V, \mathcal{L}) = n$.

pf. Well-known fact: $\chi(V, \mathcal{L}^{\otimes t}) - \frac{\mathcal{L}^n}{n!} t^n \leq O(t^{n-1})$.

Thm $\Rightarrow \underbrace{\dim H^0(V, \mathcal{L}^{\otimes t})}_{\leq O(t^n)} - \chi(V, \mathcal{L}^{\otimes t}) \leq O(t^{n-1})$.

$\dim \operatorname{Supp} \mathcal{O}_V = n$

$\Rightarrow \kappa(V, \mathcal{L}) = n \iff \mathcal{L}^n > 0.$

<u>note</u> $\mathcal{L}$ nef $\Rightarrow \mathcal{L}^n \geq 0.$ $\square$

Cor $\mathcal{L}$ nef & big / V proj. var / $k = \bar{k}$. $\dim V = n$.

$$\Rightarrow \forall \mathcal{F} \in \mathrm{Coh}_V,\ \dim H^q(V, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) \le O(t^{n-q-1}).$$

$\rightsquigarrow$ In particular: $H^n(V, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) = 0$, $t \gg 0$.

pf. $A \in \mathrm{CaDiv}(V)$ ample s.t.

$$H^i(V, \mathcal{F} \otimes \mathcal{O}_V(A) \otimes \mathcal{L}^{\otimes t}) = 0,\quad i > 0,\ t \ge 0.$$

$\mathcal{L}$ big $\Rightarrow |\mathcal{L}^{\otimes m} \otimes \mathcal{O}_V(-A)| \ne \emptyset$ by def.

$$\underset{D}{\underset{\cup\!\!\!\shortmid}{\ }}$$

$\rightsquigarrow \gamma : \mathcal{F} \otimes \mathcal{O}_V(-D) \longrightarrow \mathcal{F}$ homo.

$$\rightsquigarrow \dim H^q(V, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) \le \dim H^q(V, \mathrm{coker}\,\gamma \otimes \mathcal{L}^{\otimes t}) + \dim H^q(V, \mathrm{im}\,\gamma \otimes \mathcal{L}^{\otimes t})$$

$$\begin{tikzcd}
0 \arrow[r] & \ker\gamma \arrow[r] & \mathcal{F} \otimes \mathcal{O}_V(-D) \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] & \mathrm{coker}\,\gamma \arrow[r] & 0 \\
& & \mathrm{im}\,\gamma \arrow[ur] \arrow[d, shift left] & & & \\
& & 0 \arrow[u, shift left] & & &
\end{tikzcd}$$

$$\& \dim H^q(V, \mathrm{im}\,\gamma \otimes \mathcal{L}^{\otimes t}) \le \overset{0}{\overset{\parallel}{\underbrace{\dim H^q(V, \mathcal{F} \otimes \mathcal{O}_V(-D) \otimes \mathcal{L}^{\otimes t})}}} + \dim H^{q+1}(V, \ker\gamma \otimes \mathcal{L}^{\otimes t})$$

$$= \dim H^{q+1}(V, \ker\gamma \otimes \mathcal{L}^{\otimes t}),\quad t \ge m$$

by Fujita

$$\forall t \ge m,\ H^q(V, \mathcal{F} \otimes \mathcal{O}_V(-D) \otimes \mathcal{L}^{\otimes t}) \simeq H^q(V, \mathcal{F} \otimes \mathcal{O}_V(A) \otimes \mathcal{L}^{\otimes (t-m)}) = 0$$

b/c $D \in |\mathcal{O}_V(-A) \otimes \mathcal{L}^{\otimes m}|$ $\rightsquigarrow$ $-D \in |\mathcal{O}_V(-A) \otimes \mathcal{L}^{\otimes -m}|$.

<u>note</u> $\dim H^q(V, \operatorname{coker} \gamma \otimes \mathcal{L}^{\otimes t}) \leq O(t^{n-1-q})$.

b/c $\operatorname{supp} \operatorname{coker} \gamma \subseteq D$.

& $\dim H^{q+1}(V, \ker \gamma \otimes \mathcal{L}^{\otimes t}) \leq O(t^{n-q-1})$

$\Rightarrow \dim H^q(V, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) \leq O(t^{n-q-1})$. $\square$