Exercise 4 (due on November 30)

Choose 4 out of 8 problems to submit. (To be extended.)

For this exercise, we fix the coefficient field to be a finite extension E of $\mathbb{Q}_\ell$, with ring of integers $\mathcal{O}$, uniformizer ϖ, and residue field $\mathbb{F}$. The letter F is reserved to denote a number field, S a finite set of places including the ones dividing $\ell\infty$.

Problem 4.1. (Tangent space for relative deformation problem) Let T be a subset of S. For a continuous $G_{F,S}$-module M, define $\widetilde{H}^i_T(G_{F,S}, M)$ to be the cohomology of

$$\widetilde{\mathrm{R}\Gamma}_T\big(G_{F,S}, M\big) := \mathrm{Cone}\Big(\mathrm{R}\Gamma\big(G_{F,S}, M\big) \to \bigoplus_{v\in T} \mathrm{R}\Gamma\big(G_{F_v}, M\big)\Big)[-1].$$

Let $\bar\rho : G_{F,S} \to \mathrm{GL}_n(\mathbb{F})$ denote an absolutely irreducible representation and $\chi : G_{F,S} \to \mathcal{O}^\times$ a lift of $\det\bar\rho$. Assume that $\ell \nmid n$. Consider the following functor

$$\begin{array}{ccc}\mathrm{Def}^{\square_T,\chi}_{\bar\rho} : \mathsf{CNL}_{\mathcal{O}} & \longrightarrow & \mathsf{Sets} \\ A & \longmapsto & \left\{ (\rho, (h_v)_{v\in T}) \;\middle|\; \begin{array}{l} \bullet\ \text{for each } v,\ h_v \in \widehat{\mathrm{PGL}}_n(A), \\ \bullet\ \rho : G_{F,S} \to \mathrm{GL}_n(A) \text{ cont. repn. s.t.} \\ \bullet\ \rho \bmod \mathfrak{m}_A = \bar\rho \text{ and } \det\rho = \chi. \end{array} \right\} \Big/ \sim \end{array}$$

where $(\rho, (h_v)_{v\in T}) \sim (\rho', (h'_v)_{v\in T})$ if there exists $x \in \widehat{\mathrm{PGL}}_n(A)$ such that $\rho' = x\rho x^{-1}$ and $h'_v = h_v x^{-1}$ for every $v \in T$.

There is a natural morphism

$$\begin{array}{ccc}\mathrm{Def}^{\square_T,\chi}_{\bar\rho}(A) & \longrightarrow & \prod_{v\in T} \mathrm{Def}^{\square,\chi_v}_{\bar\rho_v}(A) \\ (\rho, (h_v)_{v\in T}) & \longmapsto & h_v\rho h_v^{-1}. \end{array}$$

This gives rise to a natural homomorphism

$$R^{\square_T}_{\mathrm{loc}} := \widehat{\bigotimes}_{v\in T} R^{\square,\chi_v}_{\bar\rho_v} \longrightarrow R^{\square_T,\chi}_{\bar\rho}$$

Let $\mathfrak{m}^{\square_T}_{\mathrm{loc}} := (\mathfrak{m}^{\square_v,\chi_v}_{\bar\rho_v}; v \in T)$ and let $\mathfrak{m}^{\square_T,\chi}_{\bar\rho}$ denote the maximal ideal of $R^{\square_T,\chi}_{\bar\rho}$.

(1) Show that

$$\Big(\mathfrak{m}^{\square_T,\chi}_{\bar\rho} \big/ \big((\mathfrak{m}^{\square_T,\chi}_{\bar\rho})^2, \mathfrak{m}^{\square_T}_{\mathrm{loc}}\big)\Big)^* \;\cong\; \mathrm{Ker}\,\Big(\mathrm{Def}^{\square_T,\chi}_{\bar\rho}(\mathbb{F}[\epsilon]) \to \prod_{v\in T} \mathrm{Def}^{\square,\chi_v}_{\bar\rho_v}(\mathbb{F}[\epsilon])\Big)$$

(2) Fill in details of the proof in class that

$$\mathrm{Ker}\,\Big(\mathrm{Def}^{\square_T,\chi}_{\bar\rho}(\mathbb{F}[\epsilon]) \to \prod_{v\in T} \mathrm{Def}^{\square,\chi_v}_{\bar\rho_v}(\mathbb{F}[\epsilon])\Big) \cong \widetilde{H}^1_T(G_{F,S}, \mathrm{Ad}^0\,\bar\rho).$$

Problem 4.2. (Relations for relative deformation problem) Continued with the previous problem and the notation therein, write J for the kernel of the map

$$R^{\square_T}_{\mathrm{loc}}[\![x_1, \ldots, x_t]\!] \twoheadrightarrow R^{\square_T,\chi}_{\bar\rho}.$$

Let $\mathfrak{m}$ denote the maximal ideal $(\mathfrak{m}^{\square_T}_{\mathrm{loc}}, x_1, \ldots, x_t)$ of $R^{\square_T}_{\mathrm{loc}}[\![x_1, \ldots, x_t]\!]$. Show that there is a natural injective map

$$\big(J/\mathfrak{m}J\big)^* \hookrightarrow \widetilde{H}^2_T(G_{F,S}, \mathrm{Ad}^0\,\bar\rho).$$

Problem 4.3. (More general Selmer duality) We start with local situation. For K a nonarchimedean local field, and let M a continuous $\mathcal{O}[G_K]$-module. Let

$$\mathcal{D} := \big[D^0 \to D^1 \to \dots\big] \quad \text{and} \quad \mathcal{D}^* := \big[(D^*)^0 \to (D^*)^1 \to \cdots\big]$$

be two complexes with morphisms $\mathcal{D} \to \mathrm{R\Gamma}(G_K, M)$ and $\mathcal{D}^* \to \mathrm{R\Gamma}(G_K, M^*(1))$ that induce *injective* maps on all cohomology groups. If under the natural cup product

$$\mathcal{D} \otimes \mathcal{D}^* \to \mathrm{R\Gamma}(G_K, M) \times \mathrm{R\Gamma}(G_K, M^*(1)) \xrightarrow{\cup} \mathrm{R\Gamma}(G_K, E/\mathcal{O}(1)) \to E/\mathcal{O}[-2],$$

we have

$$\begin{array}{ccccc} H^i(\mathcal{D}) & & H^{2-i}(\mathcal{D}^*) & & \\ \bigcap & & \bigcap & & \\ H^i(G_K, M) & \times & H^{2-i}(G_K, M^*(1)) & \xrightarrow{\cup} & H^2(G_K, E/\mathcal{O}(1)) \cong E/\mathcal{O}, \end{array}$$

$H^i(\mathcal{D})$ and $H^{2-i}(\mathcal{D}^*)$ are exact annihilators of each other, we say that $\mathcal{D}$ and $\mathcal{D}^*$ are *dual local conditions* for Galois cohomology.

A trivial example of dual local condition is $\mathcal{D} = 0$ and $\mathcal{D}^* = R\Gamma(G_K, M^*(1))$.

(1) Assume that ℓ does not divide the residual characteristic of K. Consider the continuous cochain complex $\mathrm{R\Gamma}(G_K, M)$:

$$C^0(G_K, M) \to C^1(G_K, M) \to C^2(G_K, M) \to \cdots$$

Set

$$\mathcal{D}^K(G_K, M) := \big[C^0(G_K, M) \to Z^1(G_K, M)\big]$$

$$\mathcal{D}_K(G_K, M^*(1)) := \big[C^0(G_K, M^*(1)) \to B^1(G_K, M^*(1))\big]$$

Show that $\mathcal{D}^K(G_K, M)$ and $\mathcal{D}_K(G_K, M^*(1))$ are dual local conditions.

(2) Continued with the previous setup. Let $Z^1_{\mathrm{ur}}(G_K, M)$ denote the preimage of $H^1_{\mathrm{ur}}(G_K, M) \subseteq H^1(G_K, M)$ under the natural quotient $Z^1(G_K, M) \twoheadrightarrow H^1(G_K, M)$. Consider

$$\mathcal{D}_{\mathrm{ur}}(G_K, M) := \big[C^0(G_K, M) \to Z^1_{\mathrm{ur}}(G_K, M)\big]$$

which admits a natural morphism $\mathcal{D}_{\mathrm{ur}}(G_K, M) \to \mathrm{R\Gamma}(G_K, M)$.

Show that $\mathcal{D}_{\mathrm{ur}}(G_K, M)$ and $\mathcal{D}_{\mathrm{ur}}(G_K, M^*(1))$ are dual local conditions.

(3) Let ϕ_K denote a geometric Frobenius element of G_K. Let $\mathcal{D}'_{\mathrm{ur}}(G_K, M)$ denote the complex

$$\mathcal{D}'_{\mathrm{ur}}(G_K, M) := \big[M^{I_K} \xrightarrow{\phi_K - 1} M^{I_K}\big]$$

Construct a natural quasi-isomorphism $\mathcal{D}'_{\mathrm{ur}}(G_K, M) \xrightarrow{\sim} \mathcal{D}_{\mathrm{ur}}(G_K, M)$, i.e. there are two ways to represent this unramified local conditions.

<u>Caveat:</u> The unramified condition is not exact in M, i.e. for $0 \to M \to M' \to M'' \to 0$ a short exact sequence of continuous $\mathcal{O}[G_K]$-modules, $\mathcal{D}_{\mathrm{ur}}(G_K, M) \to \mathcal{D}_{\mathrm{ur}}(G_K, M') \to \mathcal{D}_{\mathrm{ur}}(G_K, M'') \to$ is NOT a distinguished triangle in general, because taking I_K-invariants is not exact.

(4) Now we switch to the global setup. Let F be a number field and let S be a finite set of places including those dividing $\ell\infty$. Let S_∞ denote the archimedean places of F. Let M be a continuous $\mathcal{O}[G_{F,S}]$-module. Assume that $\ell \geq 3$ to avoid archimedean troubles.

For each $v \in S \backslash S_\infty$, suppose that we are given a dual pair of local conditions $\mathcal{D}_v$ and $\mathcal{D}_v^*$ for $\mathrm{R}\Gamma(G_{F_v}, M)$ and $\mathrm{R}\Gamma(G_{F_v}, M^*(1))$. We define

$$\mathrm{R}\Gamma_{\mathcal{D}}(G_{F,S}, M) := \mathrm{Cone}\Big[\mathrm{R}\Gamma(G_{F,S}, M) \oplus \bigoplus_{v \in S\backslash S_\infty} \mathcal{D}_v \longrightarrow \bigoplus_{v \in S\backslash S_\infty} \mathrm{R}\Gamma(G_{F_v}, M)\Big][-1],$$

and $\mathrm{R}\Gamma_{\mathcal{D}^*}(G_{F,S}, M^*(1))$ similarly. We write $\widetilde{H}^i_{\mathcal{D}}(G_{F,S}, M)$ for the cohomology of $\mathrm{R}\Gamma_{\mathcal{D}}(G_{F,S}, M)$. Deduce from the global duality that there is a natural isomorphism

$$\widetilde{H}^i_{\mathcal{D}}(G_{F,S}, M)^* \cong \widetilde{H}^{3-i}_{\mathcal{D}^*}(G_{F,S}, M^*(1)).$$

Remark: In general, we do not need the injectivities on the cohomology groups of $\mathcal{D} \to \mathrm{R}\Gamma(G_K, M)$ and $\mathcal{D}^* \to \mathrm{R}\Gamma(G_K, M^*(1))$ to define dual local conditions. Instead, we need a certain derived version of duality.

Problem 4.4. (Another interpretation of the conditions for Taylor–Wiles primes) Assume that $\ell \geq 3$. (This problem requires some input from the previous problem.)

Let M be a finite dimensional $\mathbb{F}$-vector spaces with continuous $G_{F,S}$-actions. Let Q be a finite set of places disjoint from S (in particular, each place in Q is relatively prime to ℓ).

(1) For each $v \in Q$, consider the unramified local condition $\mathcal{D}_{\mathrm{ur}}(G_{F_v}, M)$. Collectively, let $\mathcal{D}_{Q\text{-ur}}$ denote the local condition which is trivial at places in S and $\mathcal{D}_{\mathrm{ur}}(G_{F_v}, M)$ at each $v \in Q$. Show that we have an isomorphism

$$H^i(G_{F,S}, M) \cong \widetilde{H}^i_{\mathcal{D}_{Q\text{-ur}}}(G_{F,S\cup Q}, M).$$

Show that there is a natural injective morphism

$$\widetilde{H}^1_S(G_{F,S}, \mathrm{Ad}^\circ \bar{\rho}) \to \widetilde{H}^1_Q(G_{F,S}, \mathrm{Ad}^\circ \bar{\rho}). \tag{4.4.1}$$

Hint: Here is one way to prove this. In fact, we prove a statement that is more general than this. Let $\mathcal{D}_S$ and $\mathcal{D}_S^*$ denote a dual pair of local conditions for $\mathrm{R}\Gamma(G_{F,S}, M)$ and $\mathrm{R}\Gamma(G_{F,S}, M^*(1))$. Then we may extend these dual pair to a dual pair of local conditions $\mathcal{D}_S \oplus \mathcal{D}_{Q\text{-ur}}$ and $\mathcal{D}_S^* \oplus \mathcal{D}_{Q\text{-ur}}^*$ for $\mathrm{R}\Gamma(G_{F,S\cup Q}, M)$ and $\mathrm{R}\Gamma(G_{F,S\cup Q}, M^*(1))$, by taking the unramified local conditions at places at Q. Then, we have natural isomorphisms

$$\widetilde{H}^i_{\mathcal{D}_S \oplus \mathcal{D}_{Q\text{-ur}}}(G_{F,S}, M) \cong \widetilde{H}^i_{\mathcal{D}_S^* \oplus \mathcal{D}_{Q\text{-ur}}^*}(G_{F,S\cup Q}, M). \tag{4.4.2}$$

The reason for this generalization is that one can prove (4.4.1) and (4.4.2) relatively directly for $\widetilde{H}^0$ and $\widetilde{H}^1$. Then one can invoke the duality from the previous for $\widetilde{H}^2$ and $\widetilde{H}^3$. For the empty local condition, one needs to consider full local condition as its dual.

(2) Now assume that $M^{G_{F,S}} = 0$. Let $\mathcal{D}_{S\text{-full}}$ denote the full local condition at S, that is Using the natural isomorphism (4.4.2), we deduce an (injective) natural map

$$\widetilde{H}^1_{\mathcal{D}_{S\text{-full}}}(G_{F,S}, M) \cong \widetilde{H}^1_{\mathcal{D}_{S\text{-full}} \oplus \mathcal{D}_{Q\text{-ur}}}(G_{F,S\cup Q}, M) \to \widetilde{H}^1_{\mathcal{D}_{S\text{-full}}}(G_{F,S\cup Q}, M)$$

Show that this is surjective (and thus an isomorphism) if and only if the natural map

$$H^1(G_{F,S}, M^*(1)) \to \bigoplus_{v \in Q} H^1(G_{F_v}, M^*(1))$$

is surjective.

Applying this problem to the case when $M = \mathrm{Ad}^\circ \bar{\rho}$ for an absolutely irreducible representation of $G_{F,S}$, we give an alternative proof of a step in the search of Taylor–Wiles primes, without relying on numerical computations (and under less additional conditions).