Fargues-Fontaine curve and vector bundles

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§1 Fargues-Fontaine curve

E = non-arch local field with residue field $\mathbb{F}_q$ of char p

(either $E/\mathbb{Q}_p$ finite or $E = \mathbb{F}_q((t))$ of char p).

$\mathcal{O}_E \subseteq E$, $\pi \in \mathcal{O}_E$ uniformizer.

Recall R = perfect $\mathbb{F}_q$-algebra.

$W_{\mathcal{O}_E}(R)$:= the unique π-adically complete flat $\mathcal{O}_E$-algebra s.t. $W_{\mathcal{O}_E}(R)/\pi = R$.

Explicitly, $W_{\mathcal{O}_E}(R) = \begin{cases} W(R) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E, & \text{if } E/\mathbb{Q}_p \\ R \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_E, & \text{if char } E = p. \end{cases}$

In particular, $\exists!$ multiplicative lift

$$[\cdot] : R \to W_{\mathcal{O}_E}(R) \quad \text{(Teichmüller)}.$$

Universal property (cf. [FF] §1.1):

R = perfect $\mathbb{F}_q$-algebra, A = π-adically complete $\mathcal{O}_E$-algebra,

then
$$\mathrm{Hom}_{\mathcal{O}_E}(W_{\mathcal{O}_E}(R), A) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{F}_q}(R, A^\flat)$$
$$f \longmapsto f^\flat$$

where $A^\flat := \varprojlim_{x \mapsto x^q} A/\pi$.

§1.1 The adic space $\mathcal{Y}_S$

Setups
- $S = \mathrm{Spa}(R, R^+)$ affinoid perfectoid space over $\mathbb{F}_q$
- $\varpi \in R^+$ pseudo-uniformizer,

i.e. topologically nilpotent and invertible in R.

- $W_{\mathcal{O}_E}(R^+)$, endowed with $(\pi, [\varpi])$-adic topology.
- $\mathcal{Y}_S := (\mathrm{Spa}\, W_{\mathcal{O}_E}(R^+)) \setminus V([\varpi])$

$$= \{x \in \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) : |[\varpi]x| \neq 0\} \xhookrightarrow{\text{open}} \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+)$$

<u>Example</u> $\mathrm{char}\, E = p$, $\mathcal{O}_E = \mathbb{F}_q[\![t]\!]$, $E = \mathbb{F}_q(\!(t)\!)$.

$\rightsquigarrow W_{\mathcal{O}_E}(R^+) = R^+[\![t]\!]$

$\mathcal{Y}_S = \mathbb{D}_S$ = open unit disc over S with coordinate t.

Let $E_\infty = \mathbb{F}_q(\!(t^{1/p^\infty})\!) \Rightarrow \mathcal{Y}_S \hat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E_\infty} = \mathbb{D}_{S,\mathrm{perf}}$

In general, we have:

<u>Prop</u> (1) $\mathcal{Y}_S$ is an analytic adic space over $\mathcal{O}_E$.

(2) Let $E_\infty = \widehat{E(\pi^{1/p^\infty})}$, then $\mathcal{Y}_S \times_{\mathrm{Spa}\,\mathcal{O}_E} \mathrm{Spa}\,\mathcal{O}_{E_\infty}$ is a perfectoid space with tilt given by

$$S \times_{\mathbb{F}_q} \mathrm{Spa}\, \mathbb{F}_q[\![t^{1/p^\infty}]\!] =: \mathbb{D}_{S,\mathrm{perf}}.$$

<u>Prop</u> For any perfectoid space T over $\mathbb{F}_q$, with an untilt $T^\#$ over $\mathcal{O}_E$,

$$\underbrace{\mathcal{Y}_S(T^\#)}_{\mathrm{char}\, E} \simeq \underbrace{S(T)}_{\mathrm{char} = p}.$$

In particular, we find

$$\mathcal{Y}_S^\diamond(T) := \left\{ (T^\#, z) \,\middle|\, \begin{array}{l} T^\# \text{ untilt over } \mathcal{O}_E, \\ z \in \mathcal{Y}_S(T^\#) \end{array} \right\}$$

$$\simeq \left\{ (T^\#, \tilde{z}) \,\middle|\, \begin{array}{l} T^\# \text{ untilt over } \mathcal{O}_E \\ \tilde{z} \in S(T) \end{array} \right\}$$

$$= (\mathrm{Spd}\, \mathcal{O}_E \times S)(T)$$

Rmk $(\mathrm{Spd}\,\mathcal{O}_E)^\diamond$ = the v-sheaf on $\mathrm{Perf}_{\mathbb{F}_q}$ s.t.

$$(\mathrm{Spd}\,\mathcal{O}_E)(T) = \{\text{untilt of } T \text{ over } \mathcal{O}_E\}.$$

~~Proof~~ Use the universal property of $W_{\mathcal{O}_E}(-)$:

WLOG, $T = \mathrm{Spa}(R_1, R_1^+)$, $T^\# = \mathrm{Spa}(R_1^\#, R_1^{\#+})$

$$\Rightarrow \quad \mathcal{Y}_S(T^\#) = \{T^\# \to \mathcal{Y}_S / \mathcal{O}_E\}$$

$\mathcal{Y}_S \cong \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+)$.

$$= \{W_{\mathcal{O}_E}(R^+) \to R_1^{\#,+}, \text{ s.t. } [\varpi] \mapsto \text{sth invertible in } R_1^\#\}$$

univ property $\Rightarrow$ $= \{R^+ \to (R_1^\#)^{\flat,+} \cong R_1^+ \text{ s.t. } [\varpi] \mapsto \text{sth invertible in } R_1\}$

$$= S(T). \qquad \square$$

Remark (1) $\exists$ natural continuous maps

$$|\mathcal{Y}_S| \cong |\mathcal{Y}_S^\diamond| \cong |\mathrm{Spd}(\mathcal{O}_E) \times S| \to |S|$$

(whereas there's no morphism $\mathcal{Y}_S \not\to S$.)

(2) For $S' \hookrightarrow S$ affinoid perfectoid open subset,

$U \hookrightarrow \mathcal{Y}_S$ the corresponding open,

$\exists$ natural isomorphism $\mathcal{Y}_{S'} \simeq U$

$\Rightarrow$ can glue local construction to get $\mathcal{Y}_S$ for general perfectoid space S.

not necessarily affinoid.

$$\begin{tikzcd} {|U|} \arrow[r] \arrow[d] & {|S'|} \arrow[d] \\ {|\mathcal{Y}_S|} \arrow[r] & {|S|} \end{tikzcd}$$

(3) For general perfectoid space S, we have

$$\mathcal{Y}_S^\diamond \simeq \mathrm{Spd}(\mathcal{O}_E) \times S.$$

$\rightsquigarrow$ $\exists$ natural map as above

$$|\mathcal{Y}_S| = |\mathcal{Y}_S^\diamond| = |\mathrm{Spd}(\mathcal{O}_E) \times S| \to |S|.$$

Let S be an untilt over $\mathcal{O}_E$ of S $\longleftrightarrow$ a section of $\mathcal{Y}_S^\diamond \to S$

$\rightsquigarrow$ a morphism $S^\# \to \mathcal{Y}_S$ of adic spaces $/\mathcal{O}_E$.

<u>Prop</u> The map $S^\# \to \mathcal{Y}_S$ is a closed immersion of adic spaces that represents $S^\#$ as a closed Cartier divisor in $\mathcal{Y}_S$.

§1.2 Classical points of $\mathcal{Y}_C$

<u>Setups</u> C = some complete alg closed non-arch field $/\mathbb{F}_q$.

$S = \mathrm{Spa}\, C = \mathrm{Spa}(C, \mathcal{O}_C) = \{*\}$.

$\mathcal{Y}_C := \mathcal{Y}_{\mathrm{Spa}\, C}$.

<u>Example</u> (Classical points in positive characteristic)

Assume $\mathrm{char}\, E = p > 0 \Rightarrow \mathcal{O}_E = \mathbb{F}_q[\![t]\!]$, $E = \mathbb{F}_q((t))$.

$\Rightarrow \mathcal{Y}_C = \left(\mathrm{Spa}\left(\mathcal{O}_C \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_E\right)\right) \setminus V([\varpi])$

$= \mathbb{D}_C$, the open unit disc over C, with coordinate t.

$\rightsquigarrow$ Define $|\mathcal{Y}_C|^{cl} :=$ the set of classical pts

$= \{x \in C : |x| < 1\} \subseteq \mathcal{Y}_C$.

Trivially $|\mathcal{Y}_C|^{cl} \xleftrightarrow{1-1} \mathrm{Spd}(\mathcal{O}_E)(C) = \{\text{morphism } \mathbb{F}_q[\![t]\!] \to C\}$

$x \longmapsto (\mathbb{F}_q[\![t]\!] \to C,\ t \mapsto x)$.

<u>Def'n / Prop</u> (cf. [FS] I.1.7)

The map $\mathrm{Spd}(\mathcal{O}_E)(C) \longrightarrow |\mathcal{Y}_C|$ $\rightsquigarrow$ $\mathrm{Spa}\, C^\# \to \mathcal{Y}_C$.

$C^\# \longmapsto$ (image of the natural map $\mathrm{Spa}(C^\#) \to \mathcal{Y}_C$).

is injective, and the image is contained in the set of closed points.

Write $|\mathcal{Y}_C|^{cl} \subseteq |\mathcal{Y}_C|$ for the image of the map above, called the set of classical pts of $\mathcal{Y}_C$.

Proof $\cdot$ $C^\# =$ complete residue field of $\mathcal{Y}_C$ at x,

$\cdot$ $W_{\mathcal{O}_E}(\mathcal{O}_C) \to \mathcal{O}_{C^\#} \to \mathcal{O}_C \simeq \mathcal{O}_{C^\#}^\flat \rightsquigarrow C \to C^{\#\flat}$. $\square$

<u>Properties of classical points</u>

(i) Pick $\pi \in E$ uniformizer, $E_\infty = E(\pi^{1/p^\infty})$.

<u>Recall</u> $\mathcal{Y}_C \hat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E_\infty}$ perfectoid, and $(\mathcal{Y}_C \hat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E_\infty})^\flat \simeq \mathbb{D}_{C,\mathrm{perf}}$.

$\Rightarrow |\mathbb{D}_C| \simeq |\mathbb{D}_{C,\mathrm{perf}}| \simeq |\mathcal{Y}_C \hat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E_\infty}| \twoheadrightarrow |\mathcal{Y}_C|$.

Prop (1) $|\mathbb{D}_C|^{cl} =$ the inverse image of $|\mathcal{Y}_C|^{cl} \subseteq |\mathcal{Y}_C|$.

(2) $x \in C$, $|x|<1$ is sent to the closed pt

def'd by the ideal $(\pi - [x]) \subseteq W_{\mathcal{O}_E}(\mathcal{O}_C)$.

(ii) Change of complete alg closed non-arch fields.

Prop Let C'/C be an ext'n of complete algebraically closed non-arch fields $/\mathbb{F}_q$.

$\rightsquigarrow$ inducing $\mathcal{Y}_{C'} \to \mathcal{Y}_C \rightsquigarrow |\mathcal{Y}_{C'}| \to |\mathcal{Y}_C|$.

(1) A point $x \in |\mathcal{Y}_C|$ is classical

$\Leftrightarrow$ the preimage in $|\mathcal{Y}_{C'}|$ of x is a classical pt.

(2) If $x \in |\mathcal{Y}_C|$ is a rk-1-pt that is not classical,

then $\exists\, C'/C$ s.t. the preimage in $|\mathcal{Y}_{C'}|$ of x

contains a nonempty open subset of $|\mathcal{Y}_{C'}|$.

(iii) Classical pts in terms of max'l ideals.

Prop Let $U = \mathrm{Spa}(B, B^+) \subseteq \mathcal{Y}_C$ be an affinoid subset.

(1) $\forall$ max'l ideal $\mathfrak{m} \subseteq B$, s.t. $B/\mathfrak{m}$ non-arch field

& $\rightsquigarrow \mathrm{Spm}\, B \hookrightarrow |U|$ injection.

This gives a bijection

$$\mathrm{Spm}\, B \longrightarrow |U|^{cl} := |U| \cap |\mathcal{Y}_C|^{cl} \subseteq |U|.$$

(2) U has finitely many connected components.

Assuming U conn, then B is a PID.

The pf of (1) depends on the following.

Key lemma (Zariski closed $\Rightarrow$ strongly Zariski closed.)

Let $S = \mathrm{Spa}(R, R^+)$ an affinoid perf'd with $S^\flat = \mathrm{Spa}(R^\flat, R^{\flat,+})$.

Then a closed subset $Z \subseteq |S|$, $Z = V(I)$, $I \subseteq R$ ideal

$\Leftrightarrow$ $Z \subseteq |S| = |S^\flat|$, $Z = V(J)$, $J \subseteq R^\flat$ ideal.

§1.3 The Fargues-Fontaine curve

Setups S = perf'd space / $\mathbb{F}_q$ $\rightsquigarrow$ $\mathcal{Y}_S \circlearrowleft \varphi$, the q-Frob.

$Y_S = \mathcal{Y}_S \setminus V(\pi)$ $\circlearrowleft \varphi$

$X_S := Y_S / \varphi^{\mathbb{Z}}$ the relative FF curve $/S$.

Example $\mathrm{char}\, E = p > 0 \Rightarrow Y_S = \underbrace{\mathbb{D}_S \setminus \{0\}}$.

punctured open unit disc with coordinate t

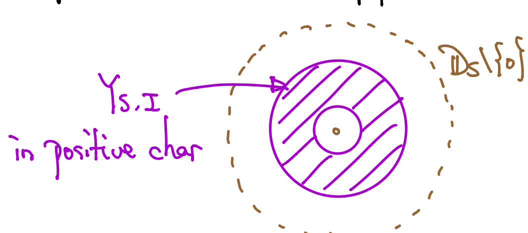
(1) The action φ on Y_S is free and totally discontinuous.

(2) Assume $S = \mathrm{Spa}(R, R^+)$, with pseudo-uniformizer ϖ.

For $0 < a \leq b < \infty$, $a, b \in \mathbb{Q}$,

$$Y_{S,[a,b]} := \{ |\varpi|^{\frac{1}{a}} \leq |t| \leq |\varpi|^{\frac{1}{b}} \neq 0 \}$$
$$\subseteq \mathbb{D}_S^* \; (:= \mathbb{D}_S \setminus \{0\}).$$

Then $\varphi : Y_{S,[1,1]} \longrightarrow Y_{S,[q,q]}$. (isom of affinoids).



Rmk (1) Similar statements hold when $E/\mathbb{Q}_p$.

(2) In particular, when S = affinoid perf'd

$\Rightarrow X$ qcqs.

Rmk (1) Can think X_S as a family of absolute Fargues-Fontaine curves

$$\left(X_{\mathrm{Spa}(k(x),k(x)^+)}\right)_{x\in S}.$$

(2) $\mathcal{Y}_S^\diamond \cong S\times \mathrm{Spd}\, \mathcal{O}_E \Rightarrow Y_S^\diamond \cong S\times \mathrm{Spd}\, E$

$\Rightarrow X_S^\diamond \cong S\times \mathrm{Spd}\, E/(\varphi\times id)^{\mathbb{Z}}$.

(3) Given an untilt $S^\#$ of S over E

$\rightsquigarrow$ a natural closed immersion

$S^\# \hookrightarrow Y_S$ (closed Cartier divisor.)

For $S^\#$ an untilt of S over E, the composite map

$$S^\# \hookrightarrow Y_S \longrightarrow X_S$$

is still a closed Cartier divisor, and depends only on

$$S \longrightarrow \mathrm{Spd}\, E \longrightarrow \mathrm{Spd}\, E/\varphi^{\mathbb{Z}}.$$

$\rightsquigarrow$ any map $S\to \mathrm{Spd}\, E/\varphi^{\mathbb{Z}}$ defines a closed Cardiv $D\subseteq X_S$.

$\rightsquigarrow$ get an injection

$$\overline{|(\mathrm{Spd}(E)/\varphi^{\mathbb{Z}})(S)|} \longleftrightarrow \{\text{closed Car div of } X_S\}.$$

parametrizes Cardiv of deg 1 over X_S.

Def'n $\mathrm{Div}^1 := \mathrm{Spd}(E)/\varphi^{\mathbb{Z}}$ $(\subseteq \mathrm{Div}(X))$.

Fact $\mathrm{Div}^1 \to *$ proper, rep'ble in spatial diamonds, and coh smooth.

Def'n/Prop (classical pts for the FF curve X_C)

Let $|X_C|^{cl} := |Y_C|^{cl}/\varphi^{\mathbb{Z}} \subseteq |X_C| = |Y_C|/\varphi^{\mathbb{Z}}$ cl-pts of X_C.

(1) Then $|X_C|^{cl} \xleftrightarrow{\text{bijection}} \underbrace{(\mathrm{Spd}\, E/\varphi^{\mathbb{Z}})(C) = \mathrm{Div}^1(C)}_{\text{the set of untilts of } C \text{ up to Frob.}}$,

$(C^\#, C \xrightarrow{\sim} C^{\#\flat}) \circlearrowright^{\varphi} (C^\#, C \xrightarrow[\sim]{\varphi} C \xrightarrow{\sim} C^{\#\flat})$.

(2) For any affinoid open subset $U = \mathrm{Spa}(B, B^+) \subseteq X_C$ — $Y_C \to X_C$ locally split.

$\exists$ a bijection $\mathrm{Spm}\, B \longrightarrow |U|^{cl} = |U| \cap |X_C|^{cl}$.

Any such U has only finitely many conn comps and U conn $\Rightarrow$ B PID.