Partially de Rham family on HMFs

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Setup

$F/\mathbb{Q}$ tot real quad ext

$\bar{\mathbb{Q}}_p \simeq \mathbb{C}$, $\tau_1, \tau_2 : F \hookrightarrow \mathbb{C}$.

$G \simeq \mathrm{Res}_{F/\mathbb{Q}} GL_2 \supset B = \begin{pmatrix} * & * \\ & * \end{pmatrix}$.

$$\mathcal{F}l = (G/B)_{\mathbb{C}_p} = \underset{\substack{\circlearrowleft \\ B}}{\mathbb{P}^1 \times \mathbb{P}^1} = (\infty \times \infty) \amalg (\mathbb{A}^1 \times \infty) \amalg (\infty \times \mathbb{A}^1) \amalg (\mathbb{A}^1 \times \mathbb{A}^1)$$

X_K Hilb mod var.

Fix $K^p \subseteq G(\mathbb{A}^{p,\infty})$

$X_{K^p} := \varprojlim_{K_p} X_K$, $K = K_p K^p$ perfectoid.

$$\rightsquigarrow \quad \begin{tikzcd} & X_{K^p} \arrow[dl, "\pi_{K_p}"'] \arrow[dr, "\pi_{HT}"] & \\ X_{K^p K_p} & & \mathcal{F}l \end{tikzcd}$$

For $k_1, k_2 \in \mathbb{Z}$, $\omega^{k_1,k_2,sm} := R\pi_{HT,*}\Big(\underbrace{\varinjlim_{K_p} \pi_{K_p}^{-1} \omega^{k_1,k_2}_{X_{K_pK^p}}}_{\overset{\circ}{\mathcal{D}}(X_{K^p},\, an)}\Big)$

$$\rightsquigarrow \quad H^0(\mathcal{F}l, \omega^{k_1,k_2,sm}) = \varinjlim_{K_p} M_{k_1,k_2}(K^pK_p)$$

$$H^0(\infty \times \infty, \omega^{k_1,k_2,sm}) = \varinjlim_{K_p} M^{\dagger}_{k_1,k_2}(K^pK_p).$$

Conj Let $f \in H^0(\infty \times \infty, \omega^{k_1,k_2,sm})[\mathfrak{p}_f]$, $\mathfrak{p}_f \subseteq \mathbb{T}^S \xrightarrow{\chi_f} \mathbb{T}^S/\mathfrak{p}_f$.

$\mathbb{T}^S$-eigenform $\rightsquigarrow \rho_f : \mathrm{Gal}_F \to GL_2(\bar{\mathbb{Q}}_p)$.

Then $f \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \omega^{k_1,k_2,sm})$ iff ρ_f is de Rham at p.

Def $\tau : F \hookrightarrow \mathbb{Q}_p \rightsquigarrow \mathfrak{p}_\tau | \mathfrak{p}$ in F.

ρ_f is <u>τ-de Rham</u> if $\dim D_{dR}(\rho_f|_{\mathrm{Gal}_{F_{\mathfrak{p}_\tau}}})_\tau = 2$.

Lem ρ_f de Rham at $p \iff \rho_f$ τ-de Rham at all τ.

Conj f as above. Then

$f \in H^0(\mathbb{P}^1 \times \infty, \omega^{k_1,k_2,sm})$ iff ρ_f is $\underset{\substack{\uparrow \\ \text{at 1st component.}}}{\tau_1}$-de Rham.

Def $\omega^{k_1,k_2,sm}_{\infty\times\infty} := (\pi_{HT,*}\pi^*_{HT}\omega^{k_1,k_2}_{\infty\times\infty})$ for $k_1, k_2 \in \mathbb{Q}_p$.

$$\begin{array}{c} G/N \\ \downarrow T\text{-torsor} \\ \mathcal{F}l = G/B \ni \infty\times\infty. \end{array}$$

Thm (Partial result)

$k_1 \in \mathbb{Z}_{\geq 2}$, $k_2 \in \mathbb{Q}_p \setminus (\mathbb{Z} \cap [-k_1, k_1])$

(note In global Langlands, concern about $k_1 = k_2$ (parallel case)

But we do no cover it now.

Let $f \in H^0(\infty\times\infty, \omega^{1+k_1, 1+k_2, sm})[\rho_f] \rightsquigarrow \rho_f : G_F \to GL_2(\overline{\mathbb{Q}}_p)$.

Assume $(*)$, $\otimes\text{-}\mathrm{Ind}^{G_\mathbb{Q}}_{G_F}\rho_f$ is irred.

Then $f \in H^0(\mathbb{P}^1 \times \infty)$ iff ρ_f is τ_1-de Rham.

$$\left(\otimes\text{-}\mathrm{Ind}^{G_\mathbb{Q}}_{G_F}\rho_f = \underset{\substack{\circlearrowleft \\ G_F}}{\rho_f} \otimes \underset{\substack{\circlearrowleft \mathrm{Ad}_\sigma \\ G_F}}{\rho_f} \quad 4\text{-dim'l}\right).$$

Thm $\forall k_1, k_2$ w/ $k_1 \in \mathbb{Z}_{\geq 2}$,

$$f \in H^0(\mathbb{P}^1 \times \infty, \omega^{1+k_1, 1+k_2, sm})^{B(\mathbb{Q}_p), fs}[\mathfrak{P}_f].$$

Then $(*) \Rightarrow \rho_f$ τ_1-de Rham.

Rmk Ding's conj: $f \in M^{\dagger}_{1+k_1, 1+k_2}$ s.t. $U_{p_1} f = a_{p_1} f$

Then $v(a_{p_1}) < r_1 \overset{?}{\Longrightarrow} \rho_f$ τ_1-de Rham

$\Downarrow$ $\qquad \nearrow$

$f \in H^0(\mathbb{P}^1 \times \infty)$

known $H^0(\mathbb{P}^1 \times \infty, \omega^{1+k_1, 1+k_2, sm}) = 0$

but $H^1(\mathbb{P}^1 \times \infty, \omega^{1+k_1, 1+k_2, sm}) = ?$

Def $\widetilde{R\Gamma}(K^p, \mathbb{Q}_p) := \left(R\varprojlim_m \varinjlim_{K_p} R\Gamma(X_{K^pK_p}, \mathbb{Z}/p^m) \right) [\frac{1}{p}]$

$\circlearrowleft$

$Gal_{\mathbb{Q}} \times \mathbb{T}^S \times G(\mathbb{Q}_p)$.

$\widetilde{H}^i := H^i(\widetilde{R\Gamma}(K^p, \mathbb{Q}_p))$ complete cohom.

Thm $\widetilde{H}^2 \circlearrowleft \mathbb{T}^S \times Gal_{\mathbb{Q}}$ satisfies <u>Caley-Hamilton relation</u>

$\forall x \in \mathbb{Z}_p[Gal_{\mathbb{Q}}]$.

for $\otimes\text{-}\mathrm{Ind}_F^{\mathbb{Q}} \rho^{univ}$

$$\widetilde{H}^2[\mathfrak{P}_f] \overset{(*)}{\simeq} (\otimes\text{-}\mathrm{Ind}\, \rho_f)^{\oplus I}.$$

<u>Case of modular curve</u>

$\widetilde{H}^{1, la, b=(k-1,0)}$

(Incomplete.)