Lectures on Mod p Langlands Program for $GL_2(L)$ (1/4)

Yongquan Hu

<u>References</u>: Breuil-Herzig-Hu-Morra-Schraen (1), (2)
(1) Gelfand-Kirillov dim
(2) (φ,Γ)-mod
} $\bar{\rho}$ semi-simple.

Hu-Wang

§ Introduction

$L/\mathbb{Q}_p$ finite ext'n.

$$\bar{\rho}: G_L \longrightarrow GL_2(\bar{\mathbb{F}}_p) \overset{?}{\longleftrightarrow} \pi(\bar{\rho}): \text{sm adm rep'n of } GL_2(L).$$
$$\bar{\mathbb{F}}_p \supseteq \mathbb{F}$$

Known for $GL_2(\mathbb{Q}_p)$: Breuil, Colmez, Emerton

$\bar{\rho}$: irred. $\longleftrightarrow$ $\pi(\bar{\rho})$ supersingular (Breuil)

$\bar{\rho}$: reducible $\longleftrightarrow$ $\pi(\bar{\rho})$: $PS_1 \longrightarrow PS_2$.

When $L \neq \mathbb{Q}_p$, no classification for <u>s.s.</u>

- Breuil-Paskunas (2007) (infinite) family of s.s. rep.
- Hu, Schraen, Wu: s.s. are not of finite rep'n

$$0 \to \underset{\text{(not finite type)}}{\mathrm{Ker}} \longrightarrow \text{c-Ind}_{GL_2(\mathcal{O}_L)}^{GL_2(L)} \sigma \longrightarrow \underset{GL_2(L)\text{-rep'n}}{\pi} \to 0$$

- Le: $\exists$ non-adm smooth irred. s.s. rep.

<u>Candidate</u>: for $\pi(\bar{\rho})$, F tot real field.

D/F quaternion alg split above p;

at ∞: either non-split or split at only one place above ∞

$\bar{r} : G_F \to GL_2(\mathbb{F})$ cont. odd,

$\bar{r}|_{G_{F_v}} \simeq \bar{\rho}$, $U^v \subseteq (D \otimes \mathbb{A}_F^{\infty,v})^\times$ compact open

$(F_v \simeq L,\ v|p)$

$\rightsquigarrow \pi_v^D(\bar{r}) := \{ f : D^\times \backslash (D \otimes_F \mathbb{A}_F^{\infty})^\times / U^v \to \mathbb{F} \text{ cont} \} [m_{\bar{r}}]$.

$\circlearrowleft$ $GL_2(L)$ $\qquad ?\ \|$

eigenspace for $m_{\bar{r}}$ $\qquad \pi(\bar{\rho})^{\oplus d}$ (assume $d=1$)

(max'l ideal of Hecke alg).

<u>Goal</u> · property of $\pi(\bar{\rho})$? (finite length).

· "locality" of $\pi(\bar{\rho})$?

<u>Conjectural properties of $\pi(\bar{\rho})$</u>?

(1) As $K := GL_2(\mathcal{O}_L)$ - rep'n $\twoheadrightarrow GL_2(\mathbb{F}_q)$, $\mathcal{O}_L/(p) = \mathbb{F}_q$.

· Buzzard-Diamond-Jarvis (weight part for Serre conj).

$\mathrm{soc}_K \pi(\bar{\rho}) = \bigoplus_{\sigma \in W(\bar{\rho})} \sigma$. $W(\bar{\rho}) :=$ set of irred Γ-rep.

· [BP] Let $K_1 = \ker(K \twoheadrightarrow \Gamma)$,

unramified case:

$\not\Rightarrow$ $\pi(\bar{\rho})^{K_1} = D_0(\bar{\rho})$ fin. dim Γ-rep'n.

proved by Gee, Emerton-Gee-Savitt, Le

(ver. with multiplicities) all m_σ's are equal),

(by using a patching functor)

(2) As $GL_2(L)$-rep'n:

[BP] · $\pi(\bar{\rho})$ is generated by $\pi(\bar{\rho})^{K_1} = D_0(\bar{\rho})$ (i.e. finitely generated).

· $\pi(\bar{\rho})$ has finite length: $\begin{cases} 1 & \text{if } \bar{\rho} \text{ irred, ok!} \\ f+1 & \text{if } \bar{\rho} \text{ is reducible (generic)} \end{cases}$

$f=2$, ok ; $\underset{PS}{\pi_0} - \underbrace{\pi_1 - \cdots}_{S.S.} - \underset{PS}{\pi_f}$ (?)

(Emerton's Conjecture (?) :

f.g. + adm. $\Rightarrow$ fin length.)

· $\pi(\bar\rho)$ has Gelfand-Kirillov dim f.

<u>Recall</u> $K_n = 1+p^n M_2(\mathcal{O}_L)$, $\dim_{\mathbb{F}} \pi(\bar\rho)^{K_n}$.

$$\exists\ \underset{\substack{\uparrow\\ \text{integer}}}{0 \le c} \le \underset{\substack{\| \\ 4f}}{\dim K}, \quad a \ge b > 0$$

$$\rightsquigarrow\ b p^{nc} + O(p^{n(c-1)}) \le \dim_{\mathbb{F}} \pi(\bar\rho)^{K_n} \le a p^{nc} + O(p^{n(c-1)})$$

$$c := GK\text{-}(\pi(\bar\rho)).$$

<u>E.g.</u> · PS $\dim \left(\mathrm{Ind}_{B(L)}^{GL_2(L)} \chi\right)^{K_n} = p^{(n-1)f}(p^f+1) = p^{nf} + p^{(n-1)f}$. $GK = f$.

· $GL_2(\mathbb{Q}_p)$ (Morra):

$$\dim \pi^{K_n} = (p+1)(2p^{n-1}+1) + \begin{cases} p-3 \\ p-2 \end{cases} \leftarrow r \in \{0, p-1\}, \quad \pi \text{ s.s.}$$

$\Rightarrow GK(\pi) = 1$.

· $GK\text{-}\dim = 0$ iff $\dim_{\mathbb{F}} \pi < \infty$.

<u>Rmk</u> $\mathbb{F}[\![K_1]\!]$ local ring. $M = \pi(\bar\rho)^\vee$ is f.g. $\mathbb{F}[\![K_1]\!]$-mod

$$\pi(\bar\rho)^{K_1} \longleftrightarrow M/\mathfrak{m}_{\mathbb{F}[\![K_1]\!]} \subseteq M/\mathfrak{m}^2_{\mathbb{F}[\![K_1]\!]}$$

<u>Rmk</u> The importance of GK-dim: ($\Rightarrow M_\infty$ is flat over R_∞).

patching $\rightsquigarrow M_\infty$ / over $R_\infty = R_{\bar\rho}[\![x_1, \dots, x_g]\!]$ univ. def ring.

satisfies $M_\infty/\mathfrak{m}_{R_\infty} \simeq \pi(\bar\rho)^\vee$.

$\forall x : R_\infty \to \overline{\mathbb{Q}}_p$

$$\underset{?}{0 \ne} \left(\underbrace{M_\infty \otimes_{R_\infty, x} \mathcal{O}_E}_{p\text{-torsion}}\right)^d [\tfrac{1}{p}] \xleftrightarrow{p\text{-adic}} \Pi_x^{\mathrm{univ}}.$$

$(-)^d = \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}_E}(-, \mathcal{O}_E)$ duality.

§ Serre Weight

$$\begin{cases} \text{(i) Serre weight: } \pi(\bar\rho)^{K_1}, \cdots \\ \text{(ii) GK-dim}(\pi(\bar\rho)). \end{cases}$$

$\bar\rho \rightsquigarrow$ define the modular weight

$$W^?(\bar\rho) = \{\sigma \text{ irred } \Gamma\text{-rep or } K\text{-rep} \mid \mathrm{Hom}_K(\sigma, \pi(\bar\rho)) \neq 0\}$$

(all $\mathrm{soc}_K(\pi(\bar\rho))$.).

[BDJ] construct $W^{expl}(\bar\rho)$.

Notation: irred σ has the form $\quad (\Gamma = GL_2(\mathbb{F}_q))$

$$(r_0, \cdots, r_{f-1}) \otimes \det{}^a := \mathrm{Sym}^{r_0}\mathbb{F}^2 \otimes (\mathrm{Sym}^{r_1})^{\mathrm{Frob}} \otimes \cdots \otimes (\mathrm{Sym}^{r_{f-1}})^{\mathrm{Frob}^{f-1}} \otimes \det{}^a$$

$$0 \le r_i \le p-1, \quad 0 \le a \le p^f - 2.$$

(i) If $\bar\rho$ is reducible,

$$\bar\rho|_{I_L} \simeq \begin{pmatrix} \omega_f^{\sum_{i=0}^{f-1} p^i(r_i+1)} & * \\ 0 & 1 \end{pmatrix} \text{ up to twist.}$$

quotient

| generic condition: $0 \le r_i \le p-3$ but not all 0 or $p-3$

ω_f : Serre's fundamental character at level f.

Define (formally)

$$W^{expl}(\bar\rho) := \left\{ (s_0, \cdots, s_{f-1}) \otimes \theta \;\middle|\; \begin{array}{l} \exists J \subseteq \{0, 1, \cdots, f-1\} \text{ s.t.} \\ \bar\rho|_{I_L} \simeq \begin{pmatrix} \omega_f^{\sum_{j \in J} p^j(s_j+1)} & * \\ 0 & \omega_f^{\sum_{j \notin J} p^j (s_j+1)} \end{pmatrix} \otimes \theta. \\ \text{and comes from a Fontain-Laffaille mod.} \end{array} \right\}$$

E.g. $f = 2$:

$$\bar\rho|_{I_L} \simeq \underbrace{\begin{pmatrix} \omega_2^{(r_0+1)+p(r_1+1)} & * \\ 0 & 1 \end{pmatrix}}_{\text{split}} \simeq \begin{pmatrix} \omega_2^{r_0+2} & * \\ 0 & \omega_2^{p(p-1-r_1)} \end{pmatrix} \otimes \omega_2^{(p-1)+pr_1}$$

$J = \{1\}$

$$\simeq \begin{pmatrix} \omega_2^{p(r_1+2)} & * \\ 0 & \omega_2^{(p-1-r_0)} \end{pmatrix} \otimes \omega_2^{r_0+p(p-1)} \simeq \underbrace{\boxed{\begin{pmatrix} 1 & * \\ 0 & \omega_2^{(p-2-r_0)+p(p-2+r_1)} \end{pmatrix} \otimes \omega_2^{r_0+1+p(r_1+1)}}}_{\text{comes from FL iff } * = 0.}$$

$\exists$ crystalline lift with HT weights $((r_0+2,0),(0,p-1+r_1))$.

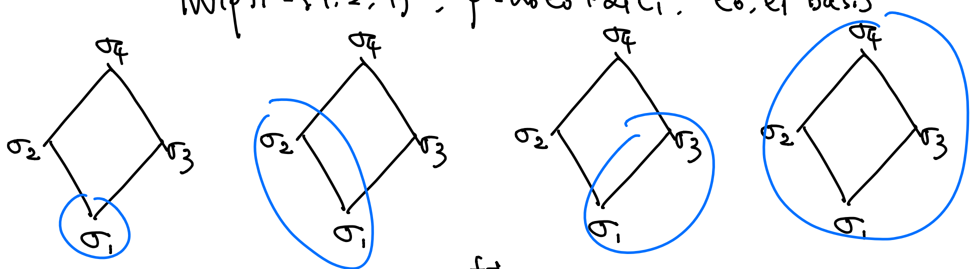
$$\omega_2^{p^2-1}=1.$$

$$W^{\exp}(\bar\rho)=\{\overset{\sigma_1}{(r_0,r_1)},\ \overset{\sigma_2}{(r_0+1,p-2-r_1)}\otimes\det{}^{p-1+pr_1},\ \underset{\sigma_3}{(p-2-r_0,r_1+1)}\otimes\det{}^{r_0+p(p-1)},\ \underset{\sigma_4}{(p-3-r_0,p-3-r_1)}\otimes\det{}^{(r_0+1)+p(r_1+1)}\}$$

split

In general, $W^{\exp}(\bar\rho)\subseteq W^{\exp}(\bar\rho^{ss})$, $\sigma_1\in W^{\exp}(\bar\rho)$, $\sigma_4\in W^{\exp}(\bar\rho)$ iff $\bar\rho$ splits

$|W(\bar\rho)|=\{1,2,4\}$, $\bar\rho=\alpha_0e_0+\alpha_1e_1$. e_0,e_1 basis



When $\bar\rho$ irred, $f=2$, $\bar\rho|_{I_L}\simeq\begin{pmatrix}\omega_{2f}^{\sum_{i=0}^{f-1}p^i(r_i+1)} & 0\\ 0 & \omega_{2f}^{p^f\cdot\sum_{i=0}^{f-1}p^i(r_i+1)}\end{pmatrix}$

with $J\subseteq\{0,1,\dots,2f-1\}\xrightarrow{\text{mod } f}\{0,\dots,f-1\}$ up to twist.

Generic condition: $1\le r_0\le p-2$, $0\le r_i\le p-3$, $i\neq 0$.

$$f=2\rightsquigarrow W^{\exp}(\bar\rho)=\left\{\begin{array}{l}(r_0,r_1),\ (r_0-1,p-2-r_1)\otimes\det{}^?,\\ (p-1-r_0,p-3-r_1)\otimes\det{}^?,\ (p-2-r_0,r_1+1)\otimes\det{}^?\end{array}\right\}.$$

§ Concerning about $D_0(\bar\rho)$

Thm (BP) $\exists$ unique f.dim Γ-rep $=GL_2(\mathbb{F}_q)$ $D_0(\bar\rho)$ s.t.

(i) $\operatorname{soc}_\Gamma D_0(\bar\rho)=\bigoplus_{\sigma\in W^{\exp}(\bar\rho)}\sigma$ (Fact: $W^?(\bar\rho)=W(\bar\rho)$.)

(ii) any weight of $W(\bar\rho)$ appears once in $D_0(\bar\rho)$

(iii) $D_0(\bar\rho)$ is max'l w.r.t. (i)(ii).

Moreover, $D_0(\bar\rho)$ is multip one, and $D_0(\bar\rho)=\bigoplus_{\sigma\in W(\bar\rho)}D_{0,\sigma}(\bar\rho)$.

Eg. $f=1$: $\bar\rho = \begin{pmatrix} \omega_1^{r+1} & * \\ 0 & 1 \end{pmatrix}$ non-split,

$$W(\bar\rho) = \{ \mathrm{Sym}^r \mathbb{F}^2 \} \qquad (r_0)$$

(i) $\Rightarrow$ $D_0(\bar\rho) \hookrightarrow \mathrm{Inj}_\Gamma \mathrm{Sym}^r \mathbb{F}^2$ (injective envelop)

$$\begin{tikzcd}[column sep=small]
\mathrm{Sym}^r \mathbb{F}^2 \arrow[r, dash] \arrow[dr, dash] & \mathrm{Sym}^{p-1-r} \otimes \det^a \arrow[dr, dash] & \\
& \mathrm{Sym}^{p-3-r} \otimes \det^{a+1} \arrow[r, dash] & \cancel{\mathrm{Sym}^r \mathbb{F}^2}
\end{tikzcd}$$

with $D_0(\bar\rho) = \mathrm{Sym}^r \mathbb{F}^2$ and $\mathrm{Sym}^r \mathbb{F}^2 \subseteq \mathrm{Inj}_\Gamma \mathrm{Sym}^r \mathbb{F}^2$.

Rmk $\pi(\bar\rho) \hookrightarrow \underbrace{\mathrm{Inj}_K \Big(\bigoplus_{\sigma \in W(\bar\rho)} \sigma \Big)[m_F]}_{\text{of } \infty\text{-dim'l with GK-dim} = 4f}$. with $K = GL_2(\mathcal{O}_L)$

If $\pi(\bar\rho)^{K_1} \subsetneq (\mathrm{Inj}_K(\cdots))^{K_1} = \mathrm{Inj}_\Gamma \mathrm{Sym}^r \mathbb{F}^2$ $\rightsquigarrow$ a control of $GK(\pi(\bar\rho))$.

Eg. When irred with $f=1$, $\bar\rho|_{I_L} = \begin{pmatrix} \omega_2^{r+1} & 0 \\ 0 & \omega_2^{p(r+1)} \end{pmatrix}$.

$\rightsquigarrow$ $W(\bar\rho) = \{ \mathrm{Sym}^r \mathbb{F}^2, \mathrm{Sym}^{p-1-r} \otimes \det^r \}$

$$D_0(\bar\rho) \hookrightarrow \begin{array}{c}
\begin{tikzcd}[column sep=small, row sep=tiny]
& \cancel{\mathrm{Sym}^{p-1-r}} \arrow[dr, dash] & \\
\mathrm{Sym}^r \arrow[ur, dash] \arrow[dr, dash] & & \cancel{\mathrm{Sym}^r} \\
& \mathrm{Sym}^{p-3-r} \arrow[ur, dash] &
\end{tikzcd} \\
\oplus \\
\begin{tikzcd}[column sep=small, row sep=tiny]
& \cancel{\mathrm{Sym}^r} \arrow[dr, dash] & \\
\mathrm{Sym}^{p-1-r} \arrow[ur, dash] \arrow[dr, dash] & & \cancel{\mathrm{Sym}^{p-1-r}} \\
& \mathrm{Sym}^{r-2} \arrow[ur, dash] &
\end{tikzcd}
\end{array}$$

§ Patching module / functor

$R_{\bar\rho}[[x_1, \cdots, x_g]] \to R_\infty$

Def A patching module M_∞ is f.g. $R_\infty[GL_2(L)]$-module s.t.

(a) $M_\infty / m_{R_\infty} \simeq \pi(\bar\rho)^\vee$ $\longleftarrow$ (minimal). M_∞ is projective $S_\infty[[K]]$-mod.

(b) $\forall$ type (w, τ) $\quad w$: of HT wts $(a_i, b_i)_{0 \le i \le f-1}$ $\rightsquigarrow$ $R_{\bar\rho}(w, \tau)$ Kisin.

$\updownarrow$ $\quad \tau : I_L \to GL_2(E)$

$\sigma(w, \tau) = K$-rep fin-dim $\supseteq \mathcal{O}$ lattice

Define $M_\infty(\Theta) := \mathrm{Hom}_K(M_\infty, \Theta^\vee)^\vee$ is max'l Cohen-Macaulay (f.g.) $R_\infty(\omega,\tau)$-module.

$$\begin{array}{c} \circlearrowleft \\ R_\infty(\omega,\tau) \\ \parallel \\ R_\infty \otimes_{R_{\bar\rho}} R_{\bar\rho}(\omega,\tau). \end{array}$$

<u>Fact</u> If $R_{\bar\rho}(\omega,\tau)=0$, then $M_\infty(\Theta)=0$.

- $M_\infty(-) : \mathcal{O}_E[\![K]\!]\text{-mod}$ (finite generated over $\mathcal{O}$) $\longrightarrow$ f.g. R_∞-mod. is an exact functor.

<u>Cor</u> For $\Theta \subseteq \sigma(\omega,\tau)$ lattice,

$$M_\infty(\Theta) \neq 0 \iff M_\infty(\Theta/p\Theta) \neq 0,$$

$$\iff \exists\, \sigma \in \underset{\substack{\uparrow \\ \text{Jordan-Hodge factors}}}{JH}(\Theta/p\Theta),\ M_\infty(\sigma)\neq 0$$

<u>Thm</u> $W^?(\bar\rho) = W(\bar\rho)$.

Pf. ① $W^?(\bar\rho) \subseteq W(\bar\rho)$.

<u>Fact</u> (a) in def'n $\Rightarrow M_\infty(\Theta)/m_{R_\infty} \simeq \mathrm{Hom}_K(\underbrace{\Theta, \pi(\bar\rho)}_{})^\vee$

typically, as $(\pi(\bar\rho)^\vee, \Theta^\vee)$.

$\rightsquigarrow \sigma \in W^?(\bar\rho) \iff \mathrm{Hom}_K(\sigma, \pi(\bar\rho)) \neq 0 \iff M_\infty(\sigma)\neq 0$.

Assume $\exists\, \omega = (0,1)$, τ = tame type.

$\sigma \in JH(\overline{\sigma(\tau)})$, $R_{\bar\rho}((0,1),\tau) = 0 \Rightarrow M_\infty(\sigma) = 0$.

<u>Lemma</u> If $\sigma \notin W(\bar\rho)$ then such a τ always exists.

tame type
- PS type $\rightarrow \sigma(\tau) = \mathrm{Ind}_{B(\mathbb{F}_q)} \tilde\chi_1 \otimes \tilde\chi_2$ (char 0, $(q+1)$-dim)
- cusp $\longrightarrow \sigma(\tau)$ = cusp rep'n $(q-1)$-dim.

$\overline{\sigma(\tau)} = \left(\sigma(\tau)^\circ / p\right)^{ss}$:

$f=1$: $\overline{\sigma(\tau)} = \mathrm{Ind}_B^\Gamma(\omega^a \otimes 1) \rightsquigarrow \overbrace{\mathrm{Sym}^a \mathbb{F}^2 \otimes \mathrm{Sym}^{p-1-a} \otimes \det^a}^{W(\bar{\rho}) \text{ for } \bar{\rho} \text{ irred.}}$

$\mathrm{Sym}^b \notin W(\bar{\rho})$.