Stacks of global Langlands parameters

Xinwen Zhu

F = global / local field

G red grp / F.

F global:

$$\left\{\begin{matrix}\text{Autom forms} \\ \text{of } G\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\rho : \underset{\substack{\uparrow \\ \text{Langlands grp (e.g. Gal grp)}}}{\mathcal{L}_F} \to \hat{G} \\ \text{params}\end{matrix}\right\} / \sim .$$

F local:

$$\left\{\begin{matrix}\text{irred rep} \\ \text{of } G(F)\end{matrix}\right\} \longleftrightarrow \{L\text{-params}\}. \qquad (\text{Assume } G \text{ split}).$$

This may not be the perfect formulation for general G

(but it works well for GL_n).

<u>Idea</u> Plug in geometry to RHS.

$\rightsquigarrow$ Consider the moduli space of L-params $\mathfrak{X}_{\hat{G},F}$.

F global:

$$\text{Space of autom forms} \simeq \Gamma(\underbrace{\mathfrak{X}_{\hat{G},F}}_{\text{building blocks: } \coprod (\dots) \coprod_{\rho \text{ irred}} [*/Z_{\hat{G}}(\rho)]}, \overbrace{A}^{\text{some coh sheaf}})$$

F local:

$$\text{Cat of certain reps of } G(F) \simeq D^b_c(\mathrm{Coh}(\mathfrak{X}_{\hat{G},F}))$$

note Have local-global compatibility
as a map b/w two moduli stacks
(can pull / push coh sheaf).

- $F = \mathbb{F}_q(X)$, X sm proj curve / $\mathbb{F}_q$.
- F non-arch local field w/ res field $\mathbb{F}_q$.

$\rightsquigarrow$ $\mathcal{L}_F := W_F$,
$$\begin{tikzcd} W_F \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & \mathbb{Z} \arrow[d, hook] \\ \mathrm{Gal}(\bar{F}/F) \arrow[r] & \hat{\mathbb{Z}} \end{tikzcd}$$

Fix ℓ, $\ell \nmid q$.

~~Technically~~ For simplicity, $\hat{G} = GL_n$.

If F global, let $S \subset |X|$ be a finite (nonempty) set,
$U = |X| - S$.

$\rightsquigarrow$
$$\begin{tikzcd} W_{F,S} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & \mathbb{Z} \arrow[d, hook] \\ \pi_1(U) \arrow[r] & \hat{\mathbb{Z}} \end{tikzcd}$$

Def $\mathcal{X}^{\square}_{F,S} : \mathbb{Z}_\ell\text{-alg} \longrightarrow \text{Sets}$
$$A \longmapsto \left\{ \begin{array}{c} \text{strongly cont. homo} \\ \rho : W_{F,S} \to GL_n(A) \end{array} \right\}$$

Strong Continuity: ρ strongly cont if $\forall v \in A^n$, $K \subset W_{F,S}$ compact,
$\{\rho(K) v\}$ span a f.g. $\mathbb{Z}_\ell$-mod M
& $\rho : K \to \mathrm{Aut}_{\mathbb{Z}_\ell}(M)$ is cont.

Ex · A $\mathbb{Z}/\ell^n$-alg or $A = E/\mathbb{Q}_\ell$ finite:

Strong continuity $\Leftrightarrow$ continuity

(for A w/ discrete top).

· $A = \overline{\mathbb{Q}}_\ell$

Strong continuity $\Rightarrow$ $\rho : W_{F,S} \to GL_n(E) \subset GL_n(\overline{\mathbb{Q}}_\ell)$.

for some $E/\mathbb{Q}_\ell$ finite

Def $\mathfrak{X}_{F,S} := \mathfrak{X}^{\square}_{F,S}/GL_n$.

Thm $\mathfrak{X}^{\square}_{F,S}$ is represented by disjoint union of affine schemes:

$$\mathfrak{X}^{\square}_{F,S} = \bigcup_{\xi} \mathfrak{X}^{\square,\xi}_{F,S} \leftarrow \text{affine of fin type} / \mathbb{Z}_\ell.$$

$$+ \ \mathfrak{X}_{F,S} \longrightarrow \prod_{v \in S} \mathfrak{X}_{F_v}.$$

Key ingredients of pf:

(Check Artin-Lurie rep'bility axioms)

(1) Deformation theory (derived moduli problem)

(2) Suppose (A, m) complete noetherian local ring.

WLoG $A = \overline{\mathbb{F}}_\ell[\![t]\!]$.

$$\mathfrak{X}_{F,S}(A) \supseteq \left\{ \begin{array}{l} \rho : W_{F,S} \longrightarrow GL_n(A) \\ \quad \cup \\ \pi_1(\overline{U}) \nearrow \quad \text{Im is finite} \end{array} \right\}$$

$$\updownarrow$$

$$\{\rho : W_{F,S} \longrightarrow GL_n(\overline{\mathbb{F}}_\ell[t]/t^i)\}$$

$$\updownarrow$$

$$\{\rho : W_{F,S} \longrightarrow GL_n(\overline{\mathbb{F}}_\ell[\![t]\!])\}$$

$\llcorner$ Cont. w.r.t. t-adic top.

This is de Jong's Conj (proved by Gaitsgory using Langlands corr. over fct fields).

<u>Q</u> What if F = number field?

(in this case $\mathcal{L}_F = ?$). $\quad S \supset \{\infty, \ell\}$

Conjecturally:

$$\begin{tikzcd} \mathfrak{X}_{F,S} \arrow[r] & \prod_{v \in S_f} \mathfrak{X}_{F_v} \times \mathfrak{X}_{F_\infty} \\ \mathfrak{X}^{odd}_{F,S} \arrow[u] \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \mathfrak{X}^{odd}_{F_\infty} = \{\rho : Gal(\mathbb{C}/\mathbb{R}) \xrightarrow[odd]{} \widehat{G}\} \arrow[u] \end{tikzcd}$$

$G = Res_{F/\mathbb{Q}} GL_n$

Will be moduli of cont ℓ-adic reps.

<u>Rmk</u> <u>c.f.</u> In previous fct field case:

$$\begin{tikzcd} \mathfrak{X}_{F,S\cup\{w\}} \arrow[r] & \prod_{v \in S} \mathfrak{X}_{F_v} \times \mathfrak{X}_{F_w} \\ \mathfrak{X}_{F,S} \arrow[u] \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \mathfrak{X}^{unr}_{F_w} = GL_n/\text{conj} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

<u>Easy</u>: This is Cartesian
$\rightsquigarrow$ local unr $\Rightarrow$ global unr.

<u>Def</u> $\mathfrak{X}^{odd,\square}_{F,S} : Nilp_{\mathbb{Z}_\ell} \longrightarrow$ Sets cont rep'n.

$\mathfrak{X}^{odd}_{F,S} := \mathfrak{X}^{odd,\square}_{F,S} / GL_n$.

<u>Conj 1</u> $\mathfrak{X}^{odd}_{F,S} \longrightarrow \prod_{v \in S} \mathfrak{X}_v$ is relatively rep'ble by a <u>scheme</u>.

($v | \ell$, $\mathfrak{X}_v$ = EG stack).

Equivalently,

<u>Conj 1'</u> $\mathfrak{X}_{F,S}^{\mathrm{odd,\,cris,\,(k_v)}} := \mathfrak{X}_{F,S}^{\mathrm{odd}} \times_{\prod_{v|\ell} \mathfrak{X}_v} \prod_{v|\ell} \mathfrak{X}_v^{\mathrm{cris,\,(k_v)}}$

ℓ-adic formal over $\mathrm{Spf}\, \mathbb{Z}_\ell$.

<u>Thm</u> (Emerton-Gee-Pan-Zhu)

$F = \mathbb{Q}$, $\ell \geq 5$, $n = 2$, Conj 1 holds.

(need autom input, not purely geometric).

<u>Conj 2</u> $\mathfrak{X}_{F,S}^{\mathrm{odd,\,cris,\,(k_v)}}$ can be algebraized over $\mathbb{Z}_\ell$.

<u>Ex</u> $F = \mathbb{Q}$, $S = \{5, 11, \infty\}$.

$\bar{\rho} = \mathbb{1} \oplus \overline{\mathrm{cycl}}^{-1} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_5)$.

$\mathfrak{X}_{\mathbb{Q},S}^{(!)}$ (!) = fin flat at 5, unip at 11.

$\downarrow$

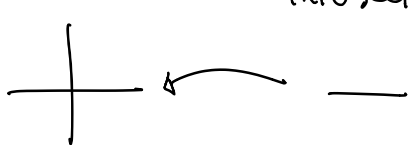
$\mathbb{T}_0(11)$

$\parallel$

$\mathrm{Spec}(\mathbb{Z}_5 \times_{\mathbb{F}_5} \mathbb{Z}_5)$ $\quad \mathbb{F}_5$ — cusp form, Eis series

$\uparrow$

intersection at special fibre.

 $= \mathbb{A}^1/\mathbb{G}_m = \mathrm{Spec}\, \mathbb{Z}_5[u]/\mathbb{G}_m$, wt $u = 2$

$(\mathrm{Spec}\, \mathbb{Z}_5[x,y]/(xy-25))/\mathbb{G}_m$

wt x = wt y = 2.