

An introduction to kimberlites  
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Let  $\text{Perf}^d = \{\text{perfd spaces}\}$   
 $\text{Perf} = \{\text{char p perfd spaces}\}$

Tilting perfectoid alg  $R \rightsquigarrow R^\flat = \varprojlim_{x \mapsto x^\flat} R$ .

Get a functor  $b: \text{Perf}^d \rightarrow \text{Perf}$

with (a)  $\text{Spa}(R, R^\flat)^\flat = \text{Spa}(R^\flat, R^{+b})$

(b)  $|X| \simeq |X^\flat|$  homeomorphism

(c)  $X_{\text{et}} = X_{\text{et}}^\flat$  equiv of etale sites

(d)  $b: \text{Perf}^d /_X \simeq \text{Perf} /_{X^\flat}$ .

Question What should  $(\text{Spa } \mathbb{Q}_p)^\flat$  be?

Want  $\pi_i^{\text{et}}((\text{Spa } \mathbb{Q}_p)^\flat) = \text{Gal}(\bar{\mathbb{Q}}_p / \mathbb{Q}_p)$ .

Naive attempt  $\varprojlim_{x \mapsto x^\flat} \mathbb{Q}_p = \mathbb{F}_p$ .

Second idea  $\text{Spa}(\mathbb{Q}_p \hat{\otimes} \mathbb{Q}_p) \xrightarrow{\sim} \text{Spa } \mathbb{Q}_p \rightarrow \text{Spa } \mathbb{Q}_p$ .  
 $(\text{Spa } \mathbb{Q}_p)^\flat = \text{coeq}(x^\flat \Rightarrow (\text{Spa } \mathbb{Q}_p)^\flat)$ .

The v-top

A map  $f: X \rightarrow Y$  in  $\text{Perf}$  is a v-cover if

if map  $g: \text{Spa}(A, A^\flat) \rightarrow Y$ ,  $\exists$  a commutative diagram

$$\begin{array}{ccc} \mathrm{Spa}(R, R^+) & \longrightarrow & X \\ h \downarrow & & \downarrow f \\ \mathrm{Spa}(A, A^+) & \xrightarrow{\quad j \quad} & Y \end{array} \quad (\text{not Cartesian})$$

with  $h$  surjective.

Key fact Any map of affinoid perf'd  
 $\mathrm{Spa}(R, R^+) \rightarrow \mathrm{Spa}(A, A^+)$   
is a spectral map of spectral top spaces.

Non-example  $T = \{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \subseteq \mathbb{R}$ ,

$$R = C^\circ(T, G_p), \quad R^+ = C^\circ(T, \mathcal{O}_{G_p}).$$

$$\Rightarrow |\mathrm{Spa}(R, R^+)| = T.$$

$$\forall t \in T = |\mathrm{Spa}(R, R^+)|, \text{ have } \mathrm{ev}_t: R \longrightarrow G_p \\ f \longmapsto f(t)$$

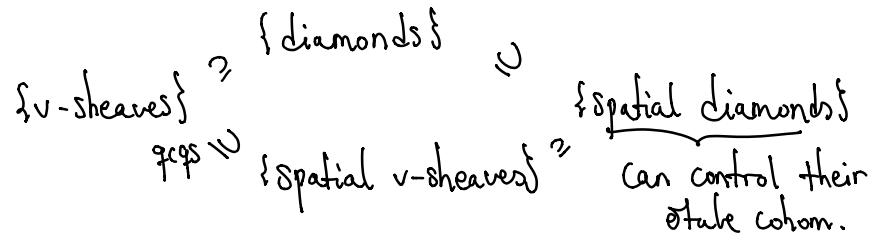
Then  $\coprod_{t \in T} \mathrm{Spa} G_p \longrightarrow \mathrm{Spa}(R, R^+)$  is a surjection  
but not a covering in  $v$ -top.

Diamonds A map of  $v$ -sheaves  $\mathcal{F} \rightarrow \mathcal{G}$  is quasi-proétale  
if for all  $Y = \mathrm{Spa}(R, R^+)$  (strictly, totally disconn space),

$$\text{have Cartesian diagram} \quad \begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{F} & \longrightarrow & \mathcal{G} \end{array}$$

then  $X$  is rep'ble & pro-étale over  $Y$ .

Def'n A diamond  $\mathcal{F}$  is a  $v$ -sheaf that is quasi-proét locally rep'ble.



$\diamond : \{ \text{adic spaces } / \mathbb{Z}_p \} \longrightarrow \{ v\text{-sheaves} \}$

$$\text{Spa}(A, A^+)^\diamond(R, R^+) = \{ (R^\#, u, f) \} / \sim$$

where  $R^\# \in \text{Perf}_d$ ,  $u : (R^\#)^b \xrightarrow{\sim} R$ ,

$$f : \text{Spa}(R^\#, R^{\#+}) \rightarrow \text{Spa}(A, A^+).$$

for  $(R, R^+) \in \text{Perf}_{\mathbb{Z}_p}$ .

Facts (a)  $X^\diamond = X^b$  if  $X \in \text{Perf}_d$

$$(b) \text{Spd } \mathbb{Q}_p = \text{cweq}(\text{Spd } \mathbb{Q}_p \widehat{\otimes} \mathbb{Q}_p \Rightarrow \text{Spd } \mathbb{Q}_p)$$

(c)  $\diamond : \text{Perf}_d \xrightarrow{\sim} \text{Perf} / \text{Spd } \mathbb{Z}_p$  is an equiv.

(d)  $X^\diamond$  is a diamond  $\Leftrightarrow X$  is analytic.

(d') If  $X$  analytic then  $X^\diamond_{\text{\'et}} \cong X_{\text{\'et}}$ .

(e)  $\begin{cases} \text{semi-normal} \\ \text{rigid spaces } / \mathbb{Q}_p \end{cases} \xrightarrow{\diamond} \{ \text{diamonds } / \text{Spd } \mathbb{Q}_p \}.$

### Highlights of diamonds

(a) Existence of local Shimura varieties

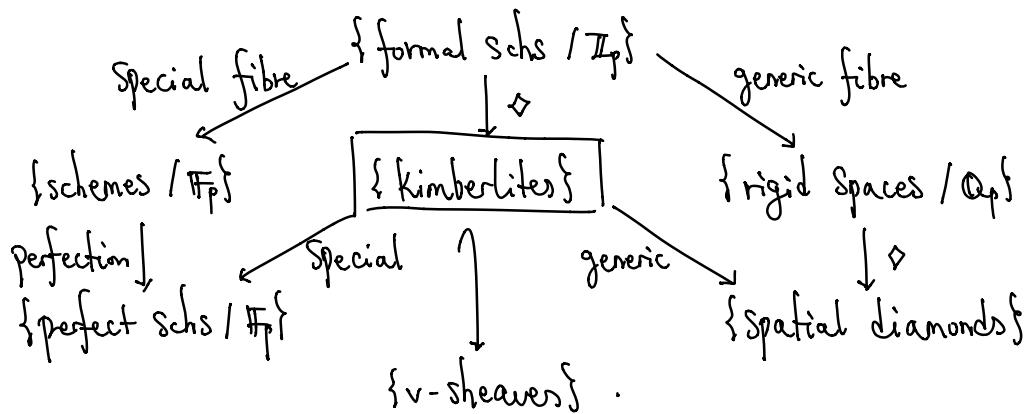
(b) New diamonds : •  $\text{Spd } \mathbb{Q}_p \times_{\mathbb{F}_p} \text{Spd } \mathbb{Q}_p$

•  $B_{\text{dp}}^+$  - Grassmannian

• moduli spaces of  $p$ -adic sheaves  
understand local Langlands correspondence.

(c) full 6-functor formalism for spatial diamonds.

### Kimberlites



Examples (a)  $\mathrm{Spd} \mathbb{I}_p \times_{\mathbb{F}_p} \mathrm{Spd} \mathbb{I}_p$

(b) Beilinson-Drinfeld Grassmannian

$$\begin{array}{ccc} \text{generic} & & \text{special} \\ \downarrow & & \downarrow \\ \mathbb{B}_{\mathbb{R}}^+ \text{-Grass} & & \text{Witt vector Grass} \end{array}$$

(c) Integral models of moduli space of shtukas.

### Properties of a spatial kimberlite $K$ :

- { (a) attached a reduced special fiber  $K^{\mathrm{red}} \in \{\text{perfect sch}\}$
- { (b) analytic locus  $K^{\mathrm{an}} \in \{\text{spatial diamond}\}$
- { (c) equipped with a specialization map  $\mathrm{Sp}: |K^{\mathrm{an}}| \rightarrow |K^{\mathrm{red}}|$ .
- { (d) notion of completion along locally closed imm  $S \hookrightarrow K^{\mathrm{red}}$ .
- { (e) well-behaved formal étale site  $(K_{\mathrm{f},\mathrm{et}})$  :
  - if  $K = \mathbb{X}^\diamond$ ,  $\mathbb{X}$  formal sch,
  - then  $(\mathbb{X}^\diamond)_{\mathrm{f},\mathrm{et}} = \mathbb{X}_{\mathrm{et}}$ .  $\rightsquigarrow$  formal nearby cycles.

Here (a) & (b) part of axioms.

Example Given  $\lambda \in \mathbb{Q}^\times$  one considers

absolute Banach-Colmez space  $BC(\mathcal{O}(\lambda))$

$$\hookrightarrow BC(\mathcal{O}(\lambda))(\mathbb{R}, \mathbb{R}^+) = H^0(X_{\mathbb{F}_p, \mathbb{R}}, \mathcal{O}(\lambda))$$

with  $BC(\mathcal{O}(\lambda)) = \begin{cases} \mathbb{Q}_p / \mathbb{B}_{\mathbb{F}_p}^\circ, & 0 \leq \lambda < 1, \\ \text{spatial kimberlite}, & \lambda > 1. \end{cases}$

where  $\mathbb{B}_{\mathbb{F}_p}^\circ = \text{Spd } \mathbb{F}_p[[t_1, \dots, t_n]].$

### Highlights

(a) Rep'ability & normality of local models of integral Shimura varieties [AGLR].

(b) Computation of connected components of ADLV [GLX].

Let  $\mathfrak{g}/\mathbb{Z}_p$  be a parahoric grp sch,  $[\mu: G_m \rightarrow G_{\mathbb{A}_f^\times}]$ .

$\hookrightarrow Sht_{\mathfrak{g}, \mu} = \text{stack of } p\text{-adic } \mathfrak{g}\text{-shtukas}$   
bounded by  $\mu$

$Bun_G = \text{stack of } G\text{-bundles over FF curve.}$

$\hookrightarrow \exists$  special polygon map

$$\sigma: Sht_{\mathfrak{g}, \mu} \longrightarrow Bun_G.$$

Theorem (a)  $Sht_{\mathfrak{g}, \mu}$  is an Artin  $v$ -stack.

(b) The map  $\sigma$  is rep'ble in locally spatial diamonds.

(c) The  $\text{Spd } \mathbb{F}_p$ -fibres of  $\sigma$  are locally spatial kimberlites.

Logic : (c)  $\Rightarrow$  (b)  $\Rightarrow$  (a).