

Fujita Vanishing
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藤田消滅定理

B1 Thm (Fujita)

X proj. sch / k , H ample $\in \text{Coh} X$.

$\mathcal{F} \in \text{Coh } X$. $\exists M = m(\mathcal{F}, H)$ s.t.

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(nH + D)) = 0, \quad \forall i > 0, \quad n \geq m(\mathcal{F}, H)$$

↑
any $D \in \text{Coh} X$ (red).

proof. WLOG, $k = \bar{k}$. $X = \text{supp } \mathcal{F}$.

Rank $\mathcal{F} \in \text{Coh } X \Rightarrow \text{supp } \mathcal{F}$ w/ subsch str.

by $\ker(\mathcal{O}_X \rightarrow \text{End}_{\mathcal{O}_X}(\mathcal{F})) \in \text{Ideal}_{\mathcal{O}_X}$.

Induction on dim

Step 1 $\dim X = 0 \quad (\checkmark)$

Suppose validity for lower dim.

□

Step 2 Reduce to $X_{\text{red}} = X$.

p.f. Suppose ok for X_{red} . $M = m|_{\mathcal{O}_X}$ s.t. $M^r = 0$ ($r > 0$)

\Rightarrow filtration

$$\mathcal{F} \supset M \cdot \mathcal{F} \supset M^2 \cdot \mathcal{F} \supset \dots \supset M^r \cdot \mathcal{F} = 0,$$

s.t. $M^i \mathcal{F} / M^{i+1} \mathcal{F} \in \text{Coh}_{\mathcal{O}_X \text{red}}$

$$\Rightarrow \forall j > 0, H^j(X, (\mathcal{I}^{\otimes j})^r / \mathcal{I}^{r+j}) \otimes \mathcal{O}_X(mH+D) = 0.$$

$m \geq m(\mathcal{I}^r / \mathcal{I}^{r+1}, H).$

b/c $\mathcal{O}_{X_{\text{red}}}(H)$ ample.

Now $0 \rightarrow \mathcal{I}^{r+1} \rightarrow \mathcal{I}^r \rightarrow \mathcal{I}^r / \mathcal{I}^{r+1} \rightarrow 0$

$\mathcal{O}_X(mH+D) \xrightarrow{\text{twisting}} \cdots \xrightarrow{H^j(\cdot)} \cdots \rightarrow H^j(X, \mathcal{I}^{r+1} \otimes \mathcal{O}_X(mH+D))$
 $\rightarrow H^j(X, \mathcal{I}^r \otimes \mathcal{O}_X(mH+D))$
 $\rightarrow H^j(X, \mathcal{I}^r / (\mathcal{I}^{r+1} \otimes \mathcal{O}_X(mH+D))) \xrightarrow{\sim 0} \cdots$

long exact.

$$\Rightarrow H^j(X, \mathcal{I}^r \otimes \mathcal{O}_X(mH+D)) \quad i \geq 0,$$

$$H^j(X, \mathcal{I}^r \otimes \mathcal{O}_X(mH+D)) = 0, \quad r \gg 0.$$

$$\Rightarrow \mathcal{F} = \mathcal{I}^r \mathcal{F} \quad \text{OK}.$$

□

Step 3 Reduce to X irredu.

dévissement here!

p.f. Let $X = X_1 \cup \dots \cup X_k$.

$$\mathcal{I} = \mathcal{I}_{X_1} \quad \Rightarrow \quad 0 \rightarrow \mathcal{I} \cdot \mathcal{F} \rightarrow \mathcal{F} / \mathcal{I} \mathcal{F} \rightarrow 0.$$

\uparrow Supp on $X_2 \cup \dots \cup X_n$ \uparrow Supp on X_1 .

Assume $H^j(X, \mathcal{I} \mathcal{F} \otimes \mathcal{O}_X(mH+D)) = 0,$

wl $m \geq m(\mathcal{I} \mathcal{F}, H|_{X_2 \cup \dots \cup X_k}).$

$$Q \quad H^j(X, (\mathcal{F} / \mathcal{I} \mathcal{F}) \otimes \mathcal{O}_X(mH+D)) = 0$$

w/ $m \geq m(\mathcal{F}/\mathcal{G}, H|_{X_1})$.

$$\Rightarrow H^j(X, \mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0, \quad m > 0.$$

□

Step 4 Reduce to H very ample.

Pf. H ample $\rightsquigarrow \mathcal{J}H$ very ample.

Assume Thm for $\mathcal{J}H$.

Note $\mathcal{F} \in \text{Coh}_X \Rightarrow \mathcal{F} \otimes \mathcal{O}_X(nH) \in \text{Coh}_X$

$$n = 0, \dots, l-1.$$

$$\Rightarrow m(\mathcal{F} \otimes \mathcal{O}_X(nH), \mathcal{J}H) = m_n.$$

$$\Rightarrow m(\mathcal{F}, H) = l \cdot \max_n \{m_n + 1\}$$

$$\Rightarrow H^j(X, \mathcal{F} \otimes \underbrace{\mathcal{O}_X(mH + D)}_{\geq l(m_n H + H) + D}) = 0.$$

$$\begin{aligned} &\geq l(m_n H + H) + D \\ &= (m_n + 1) \circledcirc H + D \end{aligned}$$

very ample

□

Step 5 Reduce to show $H' = 0$.

Pf. $H \in \text{Coh}_{\text{Div } X}$ nef \iff $|H|$ lin system
 \oplus A (general).

$$0 \rightarrow \mathcal{F} \otimes \mathcal{O}_X(-A) \rightarrow \mathcal{F} \rightarrow \mathcal{F}_A \rightarrow 0.$$

\parallel
coker.

$$\dim \text{supp } \mathcal{F}_A < \dim \text{supp } \mathcal{F} = \dim X$$

$$\Leftrightarrow \exists m(\mathcal{F}_A, H|_A) \text{ w/ } H^i(A, \mathcal{F}_A \otimes \mathcal{O}_A(mH+D)) = 0.$$

by induction, $\forall i \geq 1$.

$$\Rightarrow H^i(X, \mathcal{F} \otimes \mathcal{O}_X(\overline{\lfloor m \rfloor A} + D)) = H^i(X, \mathcal{F} \otimes \mathcal{O}_X(\overline{\lfloor m \rfloor A} + D))$$

Serre vanishing: $m \gg 0 \quad \checkmark$

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mA + D)) = 0, \quad \forall i \geq 1 \quad \left. \begin{array}{l} \forall i \geq 2 \\ i \geq 2 \\ \text{OK} \end{array} \right\}$$

□

Step 6 Reduce to $\mathcal{F} = \mathcal{O}_X$.

pf. Assume Thm holds for $\mathcal{F} = \mathcal{O}_X$

$$\Leftrightarrow 0 \rightarrow \mathcal{O}_X \xrightarrow{\alpha} \mathcal{F} \otimes \mathcal{O}_X(aH), \text{ inj. hom.}$$

$a \gg 0$.

Consider $0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{F} \otimes \mathcal{O}_X(aH) \rightarrow \text{coker } \alpha \rightarrow 0$.

induct on rank \mathcal{F} . $\quad \checkmark \quad \square$

Step 7 $\text{char } k = 0 \quad (\checkmark)$

pf. $f: Y \rightarrow X$ resol.

$$\Leftrightarrow 0 \rightarrow \underbrace{f_* \omega_Y}_{\text{for-free}} \rightarrow \mathcal{O}_X(bH) \xrightarrow{\text{coker }} \mathcal{E} \rightarrow 0$$

$\dim \text{supp } \mathcal{E} < \dim X$.

& $\text{rank } f_* \omega_Y = 1$

And: $H^j(X, f_* \omega_Y \otimes \mathcal{O}_X(mH+D)) = 0$.
Kollar vanishing $\forall m > 0, j \geq 0$.

$$\Rightarrow H^j(X, \mathcal{O}_X(bH) \otimes \mathcal{O}_X(mH+D)) \\ = H^j(X, \mathcal{O}_X(b+m)H+D) = 0, j > 0. \quad \square$$

Step 8 Reduce to $\mathcal{F} = \omega_X$.

Rmk $\omega_X^\circ \simeq \text{Ext}_{\mathbb{P}^N}^{N-\dim X}(\mathcal{O}_X, \omega_{\mathbb{P}^N})$, $X \hookrightarrow \mathbb{P}^N$.

Pf. Assume $\mathcal{F} = \omega_X$ ok.

$$0 \rightarrow \omega_X \xrightarrow{\beta} \mathcal{O}_X(cH) \rightarrow \text{Coker } \beta \rightarrow 0,$$

$$\text{rank } \omega_X = \text{rank } \mathcal{O}_X(cH) = 1 \\ \Rightarrow \dim(\text{supp Coker } \beta) < \dim X$$

induction on dim.:

$$\omega_X \vee \text{Coker } \beta \vee \Rightarrow \mathcal{O}_X(cH) \vee \\ \Rightarrow H^i(X, \mathcal{O}_X(c+m)H+D) = 0, i \geq 0. \quad \square$$

Now Consider $\mathcal{F} = \omega_X$, $\text{char } k = p > 0$.

Step 9 $\text{char } k = p$ (\vee)

p.f. $H \rightsquigarrow |H| \rightsquigarrow X \rightarrow \mathbb{P}^N$. $X \xrightarrow{F} X$
 $F := \text{Frob morphism}$, $\mathbb{P}^N \xrightarrow{F} \mathbb{P}^N$

$\mathcal{O}_X \rightarrow F_* \mathcal{O}_X$, apply $R\text{Hom}_{\mathbb{P}^N}(-, \omega_{\mathbb{P}^N})$

$$\Rightarrow R\text{Hom}_{\mathbb{P}^N}(F_* \mathcal{O}_X, \omega_{\mathbb{P}^N}) \rightarrow R\text{Hom}_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N})$$

Grothendieck duality

$$\Rightarrow R\text{Hom}_{\mathbb{P}^N}(F_* \mathcal{O}_X, \omega_{\mathbb{P}^N}) \cong F_* R\text{Hom}_{\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N})$$

$$\Rightarrow \gamma: F_* \omega_X \rightarrow \omega_X = \text{Ext}_{\mathbb{P}^N}^{N-\dim X}(\mathcal{O}_X,$$

(Recall Serre duality: $Rf_* \rightleftharpoons f^!, f: X \rightarrow \text{Spec } k$.)

$$\text{Hom}_{D^b(\text{Spec } k)}(Rf_*(\mathcal{F}), (\mathcal{V})) \cong \text{Hom}_{D^b(X)}(\mathcal{F}, f^!(\mathcal{V})).$$

vec. bundle on $\text{Spec } k$
 $= f^{-1}\text{loc. dim'l v.s.}$

$$U \subseteq X \text{ h̄issé open. } \Rightarrow F_* \omega_X|_U \rightarrow \omega_X|_U \rightarrow 0.$$

$$\Rightarrow \text{Ext}_{\mathbb{P}^N}^k(f), \omega_{\mathbb{P}^N}) = 0, \quad \forall k > N - \dim X$$

↑
 $\text{:= coker } (\mathcal{O}_X \rightarrow F_* \mathcal{O}_X).$

loc. free on U .

$$\begin{array}{c} \text{Consider } 0 \rightarrow \xrightarrow{(\gamma)} \ker \gamma \rightarrow \boxed{F_* \omega_X} \rightarrow \text{im } \gamma \rightarrow 0. \\ \quad \quad \quad \downarrow \\ 0 \rightarrow \text{im } \gamma \rightarrow \boxed{\omega_X} \xrightarrow{(\nu)} \mathcal{C} \rightarrow 0. \end{array}$$

Dévissage: $\dim X > \dim \text{supp } \mathcal{C} \Rightarrow \mathcal{C} \cong 0$.

Step 5 $\Rightarrow \ker \gamma \cong 0$, i.e. $H^2(X, \ker \gamma \otimes \mathcal{O}_X(mH+D)) = 0$.

Induction on $\dim \mathcal{C}$

$$\hookrightarrow H^1(X, F_* \omega_X \otimes \mathcal{O}_X(mH+D)) \xrightarrow{\text{surj.}} \\ H^1(X, \omega_X \otimes \mathcal{O}_X(mH+D)) \rightarrow 0$$

note By proj. formula:

$$H^1(X, F_* \omega_X \otimes \mathcal{O}_X(mH+D)) \cong H^1(X, \omega_X \otimes \underbrace{\mathcal{O}_X(p(mH+D))}_{\text{proj. im.}}) \xrightarrow{\text{proj.}} \\ p = \text{char } k.$$

$$\Rightarrow H^1(X, \omega_X \otimes \mathcal{O}_X(mH+D)) \leftarrow H^1(X, \omega_X \otimes \mathcal{O}_X(p^e(mH+D))),$$

$\forall e > 0, m \geq m_0$

$$\text{Serre vanishing.} \quad \begin{matrix} \uparrow \\ \text{mod}(\omega_X, H). \\ \text{indep. of } D \text{ (ref)} \end{matrix}$$

$$\Rightarrow H^1(X, \omega_X \otimes \mathcal{O}_X(mH+D)) = 0, \quad \forall m \geq m_0$$

□

• Proof Finished!

note $F_* \omega_X \rightarrow \omega_X \rightarrow 0$ can be constructed by

Lem $f: V \rightarrow W$ proj. surj. b/w proj. var / $k = \bar{k}$.

w/ $\dim V = \dim W = n$.

$\hookrightarrow \varphi: f_* \omega_V \rightarrow \omega_W$ generically surj.

cf. [T. Fujita, Semipositive line bundles, Cor 5.7].

Pf. By def'n $H^n(V, \omega_V) \neq 0$.

Leray spectral sequence:

$$E_2^{p,q} = H^p(W, R^q f_* \omega_V) \Rightarrow H^{p+q}(V, \omega_V).$$

Note: $\text{Supp } R^q f_* \omega_V \subseteq W_q$

$$\left\{ w \in W : \dim \varphi_f(w) \geq q \right\}.$$

$$\dim \varphi_f(W_q) < r \quad \forall q > 0 \Rightarrow \dim W_q < n - q, \quad \forall q > 0.$$

$$\Rightarrow E_2^{n-q, q} = 0 \text{ unless } q = 0.$$

$$\text{Q } E_2^{n,0} = H^n(W, R^0 f_* \omega_V) \neq 0 \text{ by } H^n(V, \omega_V) \neq 0.$$

By def of ω_W : $\text{Hom}(\varphi_f \omega_V, \omega_W) \neq 0$.

$$\downarrow \quad \varphi \neq 0 \Rightarrow \text{im } \varphi \subseteq \omega_W.$$

$$\text{Hom}(\text{Im } \varphi, \omega_W) \neq 0 \Rightarrow H^n(W, \text{Im } \varphi) \neq 0,$$

(GTM 122, III, §7). $\dim \text{Supp } \text{Im } \varphi = n$.

$$\varphi: \varphi_f \omega_V \rightarrow \omega_W \text{ gen. surj.}$$

$$rk = 1.$$

□

Rmk Lemma $\Rightarrow R^q f_* \omega_V = 0, \forall q > 0 \Rightarrow H^n(W, \varphi_f \omega_V) \simeq H^n(V, \omega_V)$.

$\forall \psi: \varphi_f \omega_V \rightarrow \omega_W$ nontriv. hom, $k = \bar{k} \rightarrow k$

$\exists a \in k \setminus \{0\}$ s.t. $\psi = a\varphi$. φ as in the Lem.

Note $f \circledcirc_{fin} \Rightarrow R^q f_* w_V = 0, \forall q > 0.$

$\text{char } k = 0 \& V \text{ has only } \underline{\text{rat. singularities}}$

by Grauert-Riemenschneider vanishing
or Kollar's tor-free thm.

Lem ↘ special case of Kollar's tor ...

cor of Kawamata-Viehweg vanishing.

$f: V \rightarrow W$ proj. surj.

V proj. sm, W proj. $\in \text{Var}_k$, $k = \bar{k}$, $\text{char } k = 0$

$\Rightarrow R^q f_* w_V = 0, \forall q > \dim V - \dim W.$

if. $A \in \text{Coh}_{\text{irr}}(W)$ suff. ample. s.t.

$$H^0(W, f_* w_V \otimes Q_W(A)) \cong H^0(V, w_V \otimes Q_V(f^* A)).$$

& $R^q f_* w_V \otimes Q_W(A)$ glo. gen'd, $\forall q$.

Note num. dim $H^0(V, f^* A) = \dim W$.

to (as check) $H^q(V, w_V \otimes Q_V(f^* A)) = 0$.

$$\forall q > \dim V - \dim W = \dim V - H^0(V, f^* A).$$

by Kawamata-Viehweg vanishing.

$$\Rightarrow R^q f_* \omega_V = 0, \forall q > \dim V - \dim W.$$

□

Thm (weak generalization of Kodaira vanishing).

X = n -dim'l cpt Kähler mfd.

L = line bd / X w/ curvature form $\sqrt{-1}\Theta(L)$
semi-positive.

L has $\geq k$ eigenvalues > 0 on $U \subseteq X$,
 U dense open.

$$\Rightarrow H^i(X, \omega_X \otimes L) = 0, \forall i > n-k.$$

Note $H^i(X, \omega_X \otimes L) \cong H^{n,i}(X, L)$

"
{ L -valued harmonic (n,i) -forms on X }.

Nakano's formula

we can check $H^{n,i}(X, L) = 0, \forall i+k \geq n+1$.

Prop $f: V \rightarrow W$ [proper] surj. b/w normal alg var.
w/ V_W conn. & $w \in W$.

$k = \tilde{k}$, $\text{char } k = 0$. V, W have only rat. sing.

$$\Rightarrow R^d f_* \omega_V \simeq \omega_W, d = \dim V - \dim W.$$

Pf. Comm. diagram s.t. (i) X, Y sm. alg. var

$$\begin{array}{ccc}
 X & \xrightarrow{\pi} & V \\
 g \downarrow & \lrcorner & \downarrow f \\
 Y & \xrightarrow{p} & W
 \end{array}
 \quad \begin{array}{l}
 \text{(ii) } \pi, p \text{ proj. birat.} \\
 \text{(iii) } g \text{ proj. \& sm. outside a} \\
 \text{SNC } \Sigma \in \text{Div } Y.
 \end{array}$$

Note $\forall i, R^d g_* \omega_X$ be free.

Grothendieck duality:

$$\rightsquigarrow Rg_* \mathcal{O}_X \simeq R\text{Hom}_Y(Rg_* \omega_X, \omega_Y).$$

$$\Rightarrow \mathcal{O}_Y \simeq \text{Hom}_Y(R^d g_* \omega_X, \omega_Y).$$

$$\text{Yoneda} \Rightarrow \omega_Y \simeq R^d g_* \omega_X. \quad p \text{ birat.}$$

$$\rightsquigarrow p_* R^d g_* \omega_X \simeq p_* \omega_Y \simeq \omega_W$$

$$\text{note } p_* R^d g_* \omega_X \simeq R^d (p \circ g)_* \omega_X$$

$$R^i p_* R^d g_* \omega_X = 0, \forall i > 0. \quad (\text{cf. sm base change})$$

$$\text{Also, } R^d (p \circ g)_* \omega_X \simeq R^d (f \circ \pi)_* \omega_X \simeq R^d f_* \omega_V.$$

π birat.

$$\text{since } R^i \pi_* \omega_X = 0, \forall i > 0 \quad \& \quad \omega_V \simeq \pi_* \omega_X.$$

$$\Rightarrow R^d f_* \omega_V \simeq \omega_W. \quad d = \dim V - \dim W.$$

□

[§3] Applications of Fujita vanishing

Thm $\mathcal{F} \in \text{Coh } X$, X proper / $k = \mathbb{K}$. $L \rightarrow X$ nef line bun.

$$\Rightarrow \dim H^q(X, \mathcal{F} \otimes L^{\otimes t}) \leq O(t^{m-q}), \quad m = \dim \text{supp } \mathcal{F}.$$

↑
a poly. $\in \mathbb{Z}[t]$.

Pf. Step 1 Assume X proj. Induct on q .

$H > 0 \in \text{Coh } X$ ample. s.t. $L \otimes \mathcal{O}_X(H)$ ample

$$H^0(X, \mathcal{F} \otimes L^{\otimes t}) \subset H^0(X, \mathcal{F} \otimes L^{\otimes t} \otimes \mathcal{O}_X(-H)). \quad \forall t > 0.$$

May assume L ample by replacing $L \mapsto L \otimes \mathcal{O}_X(H)$.

$$\Rightarrow \dim H^0(X, \mathcal{F} \otimes L^{\otimes t}) = \chi(X, \mathcal{F} \otimes L^{\otimes t}), \quad t > 0.$$

↑
Serre vanishing.

$$\Rightarrow \dim H^0(\mathcal{F} \otimes L^{\otimes t}) \leq O(t^m). \quad \deg \leq m = \dim \text{supp } \mathcal{F}.$$

When $q > 0$, $\exists A \in \text{Coh } X$ very ample.

$$\text{s.t. } H^q(X, \underbrace{\mathcal{F} \otimes \mathcal{O}_X(A) \otimes L^{\otimes t}}_0) = 0 \quad \text{by Fujita.}$$

$$\begin{matrix} & \approx \mathcal{O}_X(A + H) \\ & \uparrow \quad \uparrow \\ & mH \quad D. \end{matrix}$$

$$\lambda \mapsto b_A = |\lambda| \mapsto D \text{ gen'l ext.} \Rightarrow \alpha: \mathcal{F} \otimes \mathcal{O}_X(-D) \xrightarrow{\text{ing.}} \mathcal{F}$$

$$\Rightarrow 0 \rightarrow \mathcal{F} \otimes \mathcal{O}_X(-D) \xrightarrow{\alpha} \mathcal{F} \rightarrow \mathcal{F}_D \xrightarrow{\text{coker}} 0.$$

$$\Rightarrow \dim H^q(X, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) \leq \dim H^{q-1}(D, \mathcal{F}_D \otimes \mathcal{O}_D(A) \otimes \mathcal{L}^{\otimes t})$$

$\curvearrowleft \leq O(t^{m-q}).$

induction hyp. ok.

Step 2 General X. nt induction on $\text{supp } \mathcal{F}$.

May assume $X = \text{supp } \mathcal{F}$ var. (i.e. int)

X proper, Chow's lemma

$$\exists f: V \xrightarrow{\sim} X \text{ birat. , } V \text{ proj. var.}$$

$$g = f^* \mathcal{F} \quad f$$

$$\Rightarrow \beta: \mathcal{F} \rightarrow f_* g.$$

$\Rightarrow \exists \phi \neq 0 \subseteq X$ Zar open s.t. $\beta|_U$ isom.

Consider

$$0 \rightarrow \ker \beta \rightarrow \mathcal{F} \rightarrow \text{im } \beta \rightarrow 0.$$

$$0 \rightarrow \text{im } \beta \rightarrow f_* g \rightarrow \text{coker } \beta \rightarrow 0.$$

By induction: $\dim H^q(X, \ker \beta \otimes \mathcal{L}^{\otimes t}) \leq O(t^{m-q})$

$$\dim H^{q-1}(X, \text{coker } \beta \otimes \mathcal{L}^{\otimes t}) \leq O(t^{m-q}).$$

\Rightarrow suff. to show $\dim H^q(X, f_* g \otimes \mathcal{L}^{\otimes t}) \leq O(t^{m-q}).$

$$\text{L\'eray: } E_2^{p,q} = H^p(X, R^q f_* g \otimes \mathcal{L}^{\otimes t}) \Rightarrow H^{p+q}(V, g \otimes (f^* \mathcal{L})^{\otimes t})$$

$$\begin{aligned}
&\Rightarrow \dim H^q(X, f_* g \otimes L^{\otimes t}) \\
&= \sum_{j \geq 1} \underbrace{\dim H^{f+j-1}(X, R^j f_* g \otimes L^{\otimes t})}_{\leq O(t^{m-q})} + \underbrace{\dim H^f(V, g \otimes (f^* L)^{\otimes t})}_{\leq O(t^{m-f})}. \\
&\quad \uparrow \quad (\text{V proj.}) \\
&\dim \text{Supp } R^j f_* g \leq \dim X - j - 1. \quad \forall j \geq 1. \\
&\quad \text{by ind. hyp.} \\
&\Rightarrow \dim H^f(X, g \otimes L^{\otimes t}) \leq O(t^{m-f}). \quad \square
\end{aligned}$$

Cor (Fujita's numerical characterization
of nef & big line bundles).

$$\begin{aligned}
L \text{ nef line bun. / V proper alg. var / } k = k \\
\text{w/ } \dim_k V = n. \\
\text{asy char } k! \\
\text{as } K(X, L) = 0 \Leftrightarrow L^n > 0 \\
\text{t self-intersection \#}.
\end{aligned}$$

Note L big $\Leftrightarrow K(V, L) = n$.

pf. Well-known fact: $\chi(V, L^{\otimes t}) - \frac{L^n}{n!} t^n \leq O(t^{n-1})$,

$$\text{Thm } \Rightarrow \underbrace{\dim H^0(V, L^{\otimes t})}_{\leq O(t^n)} - \chi(V, L^{\otimes t}) \leq O(t^{n-1}).$$

$$\Rightarrow K(V, L) = n \Leftrightarrow L^n > 0.$$

$$\dim \text{Supp } G_V = n \quad \text{note } L \text{ nef } \Rightarrow L^n > 0. \quad \square$$

Cor L nef & big / V proj. var / $k = \bar{k}$. $\dim V = n$.

$\Rightarrow \forall F \in \text{Coh}_V, \dim H^q(V, F \otimes L^{\otimes t}) \leq O(t^{n-q-1})$,

\Rightarrow In particular: $H^n(V, F \otimes L^{\otimes t}) = 0, t \gg 0$.

Pf. $A \in \text{Coh}(V)$ ample s.t.

$$H^2(V, F \otimes \mathcal{O}_V(A) \otimes L^{\otimes t}) = 0, t \geq 0, t \geq 0.$$

L big $\Rightarrow |L^{\otimes m} \otimes \mathcal{O}_V(-A)| \neq \emptyset$ by def.
 \Downarrow
 D

$\Rightarrow \gamma: F \otimes \mathcal{O}_V(-D) \rightarrow F$ homo.

$\Rightarrow \dim H^q(V, F \otimes L^{\otimes t}) \leq \dim H^q(V, \text{coker } \gamma \otimes L^{\otimes t})$
 $\quad \quad \quad + \dim H^q(V, \text{im } \gamma \otimes L^{\otimes t})$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker \gamma & \rightarrow & F \otimes \mathcal{O}_V(-D) & \xrightarrow{F} & \text{coker } \gamma \rightarrow 0 \\ & & \downarrow & & \downarrow \text{im } \gamma & & \\ & & 0 & & 0 & & \end{array}$$

$$\begin{aligned} \dim H^q(V, \text{im } \gamma \otimes L^{\otimes t}) &\leq \underbrace{\dim H^q(V, F \otimes \mathcal{O}_V(-D) \otimes L^{\otimes t})}_{\text{by Fujita}} \\ &\quad + \dim H^{q+1}(V, \ker \gamma \otimes L^{\otimes t}) \\ &= \dim H^{q+1}(V, \ker \gamma \otimes L^{\otimes t}), t \geq m \end{aligned}$$

$$\forall t \geq m, H^q(V, F \otimes \mathcal{O}_V(-D) \otimes L^{\otimes t}) \simeq H^q(V, F \otimes \mathcal{O}_V(A) \otimes L^{\otimes (t-m)}) = 0$$

$$\text{b/c } D \in |Q_V(-A) \otimes \mathcal{L}^{\otimes m}| \Rightarrow -D \in |Q_V(-A) \otimes \mathcal{L}^{\otimes m}|.$$

$$\text{note } \dim H^q(V, \text{coker } \gamma \otimes \mathcal{L}^{\otimes t}) \leq O(t^{m-q}). \quad \left. \begin{array}{l} \\ \text{b/c } \text{supp coker } \gamma \subseteq D. \end{array} \right\}$$

$$\& \dim H^{q+1}(V, \ker \nu \otimes \mathcal{L}^{\otimes t}) \leq O(t^{n-q-1})$$

$$\Rightarrow \dim H^q(V, \mathcal{F} \otimes \mathcal{L}^{\otimes t}) \leq O(t^{n-q}). \quad \square$$