Fall 2022 – Seminar on *Abelian Varieties* by David Mumford

Lecture 5

ALGEBRAIC THEORY VIA VARIETIES

Cohomology and Base Change

The references for this section is [Har13, III, §12] and Conrad's lecture notes [Con00, §9].

Setups. Let $f : X \to Y$ be a proper morphism of noetherian schemes and $\mathscr{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Assume that $\mathscr{F}$ is flat over Y, i.e., for any $x \in X$, $\mathscr{F}_x$ is flat as an $\mathcal{O}_{Y,f(x)}$-module. For any $y \in Y$, we denote

$$X_y := X \times_Y \operatorname{Spec}(k(y))$$

and $\mathscr{F}_y$ the inverse image of $\mathscr{F}$ via the morphism $X_y \to X$.

Goal: For any $i \geqslant 0$, we want to understand the fiber cohomology $H^i(X_y, \mathscr{F}_y)$ as a function of $y \in Y$. And the idea is to find relations between the sheaf $R^i f_* \mathscr{F}$ and the cohomology groups $H^i(X_y, \mathscr{F}_y)$.

We assume the following result.

Theorem 5.1 (Proper base change)**.** *If $f : X \to Y$ is a proper morphism of locally noetherian schemes and $\mathscr{F}$ a coherent sheaf of $\mathcal{O}_X$-modules on X, then the direct image sheaves $R^p f_* \mathscr{F}$ are coherent sheaves of $\mathcal{O}_Y$-modules for all $p \geqslant 0$.*

When f is projective, this follows from [Har13, III, Thm 8.8]. As for the general case, it follows from EGA III, see [GD66, III, 3.2.1].

Theorem 5.2. *Let $f : X \to Y$ be a proper morphism of noetherian schemes with $Y = \operatorname{Spec} A$ affine, and $\mathscr{F}$ be a coherent sheaf of $\mathcal{O}_X$-module that is flat over Y. Then there exists a finite complex $K^\bullet$, say*

$$0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$$

of finitely generated projective A-modules and equivalences of functors

$$H^p(X \times_Y \operatorname{Spec}(\cdot), \mathscr{F} \otimes_A (\cdot)) = H^p(K^\bullet \otimes_A (\cdot)), \quad p \geqslant 0$$

on the category of A-algebras. Hence for any $B \in \mathsf{Alg}_A$,

$$H^p(X \times_Y \operatorname{Spec} B, \mathscr{F} \otimes_A B) \cong H^p(K^\bullet \otimes_A B), \quad p \geqslant 0.$$

Problem 5.3. Here the sheaf $\mathscr{F} \otimes_A B$ is the inverse image sheaf of $\mathscr{F}$ under the projection $X \times_Y \operatorname{Spec} B \to X$. How to give the association $B \mapsto H^p(X \times_Y \operatorname{Spec} B, \mathscr{F} \otimes_A B)$ rise to be a functor on the category of A-algebras? (To remedy this, one can use Čech cohomology, but how to make it formal?)

Remark 5.4.

1. Since $\mathscr{F}$ is flat over $Y = \operatorname{Spec} A$, for any affine open subset $U \subset X$, $\mathscr{F}(U)$ is flat as an A-module.
2. Since X is separated and noetherian, the coherent cohomology $H^*(X, \mathscr{F})$ can be computed by Čech cohomology with respect to finite affine open coverings, for any quasi-coherent sheaf $\mathscr{F}$ on X. The same is true for $X \times_Y \operatorname{Spec} B$.
3. As for $H^p(K^\bullet \otimes_A B)$, it is generally not a finitely generated algebra over A, and the cohomology does not commute with $(\cdot) \otimes_A B$ in most cases.

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Proof of Theorem 5.2. Let $\mathcal{U} = \{U_i\}_{i=0,\dots,n}$ be a finite affine open covering of X and $(C^\bullet(\mathcal{U},\mathscr{F}), d^\bullet)$ be the Čech cochain complex of alternating cochains with respect to the open covering $\mathcal{U}$ and the sheaf $\mathscr{F}$. In particular,

$$C^p(\mathcal{U},\mathscr{F}) = \bigoplus_{0\leqslant i_0<\cdots<i_p\leqslant n} \mathscr{F}(U_{i_0\cdots i_p})$$

is a free A-module for all p (being nonzero only when $0 \leqslant p \leqslant n$), and the Čech cohomology groups $H^\bullet(\mathcal{U},\mathscr{F})$ are isomorphic to $H^\bullet(X,\mathscr{F})$.

Moreover, for any A-algebra B, $\{U_i \times_Y \operatorname{Spec} B\}_{i=0,\dots,n}$ is an affine open covering of $X \times_Y \operatorname{Spec} B$, and $C^\bullet(\mathcal{U},\mathscr{F}) \otimes_A B$ is the Čech cochain complex for this open covering and the sheaf $\mathscr{F} \otimes_A B$ on $X \times_Y \operatorname{Spec} B$. Therefore,

$$H^p(X \times_Y \operatorname{Spec} B, \mathscr{F} \otimes_A B) \cong H^p(C^\bullet(\mathcal{U},\mathscr{F}) \otimes_A B), \quad p \geqslant 0,$$

and this isomorphism is functorial for B. □

Lemma 5.5. *Let $C^\bullet$ be a cochain complex of A-modules (but each C^p may not be finitely generated over A) such that $H^i(C^\bullet)$ are finitely generated A-modules for all $i \geqslant 0$, and such that $C^\bullet$ is bounded on $[0,n]$.*[1] *Then there exists a complex $K^\bullet$ of finitely generated A-modules, bounded on $[0,n]$ and such that K^p is free for all $1 \leqslant p \leqslant n$, and a homomorphism of cochain complexes $\phi : K^\bullet \to C^\bullet$ such that ϕ induces isomorphisms $H^i(K^\bullet) \to H^i(C^\bullet)$ for all i; namely, ϕ is a quasi-isomorphism.*

Moreover, if all the C^p's are A-flat, then K^0 will be A-flat as well.

Proof. We will use descending induction on m to construct the following diagram

$$\begin{tikzcd}
 & K^m \arrow[r, dashed, "d_K^m"] \arrow[d, dashed, "\phi_m"] & K^{m+1} \arrow[r, "d_K^{m+1}"] \arrow[d, "\phi_{m+1}"] & K^{m+2} \arrow[r] \arrow[d, "\phi_{m+2}"] & \cdots \\
\cdots \arrow[r] & C^m \arrow[r, "d_C^m"] & C^{m+1} \arrow[r, "d_C^{m+1}"] & C^{m+2} \arrow[r] & \cdots
\end{tikzcd}$$

with the following properties:

(1) $d_K^{p+1} \circ d_K^p = 0$ for $p \geqslant m+1$;
(2) $\phi_{p+1} \circ d_K^p = d_C^p \circ \phi_p$ for $p \geqslant m+1$;
(3) ϕ_p induces an isomorphism of cohomology groups $H^p(K^\bullet) \to H^p(C^\bullet)$ for $p \geqslant m+2$ and a surjective homomorphism $\operatorname{Ker}(d_K^{m+1}) \to H^{m+1}(C^\bullet)$;
(4) K^p is a finite free A-module for $p \geqslant m+1$.

We are going to construct K^m, d_K^m, ϕ_m with the above properties. One can find finite free A-modules $(K')^m$ and $(K'')^m$, and surjective maps of A-modules:

$$(K')^m \twoheadrightarrow \operatorname{Ker}(\operatorname{Ker}(d_K^{m+1}) \to H^{m+1}(C^\bullet)),$$
$$(K'')^m \twoheadrightarrow H^m(C^\bullet).$$

Roughly speaking, the first surjection is to make ϕ_{m+1} into an isomorphism between cohomology groups; and the second surjection is to force ϕ_m to satisfy the desired property.

By construction, we have an inclusion $i'_m : (K')^m \to (K'')^{m+1}$ that factors through $\operatorname{Ker}(d_K^{m+1})$. Define

$$K^m := (K')^m \oplus (K'')^m, \quad d_K^m = (i'_m, 0) : K^m \to K^{m+1}.$$

[1]This is not a standard notation to say that $C^p \neq 0$ implies $0 \leqslant p \leqslant n$. Indeed, using the truncation functor, one may replace $C^\bullet$ with $\tau^{\geqslant 0}\tau^{\leqslant n}C^\bullet$.

Then property (1) and (4) hold for $p = m$, and ϕ_{m+1} induces an isomorphism $H^{m+1}(K^\bullet) \to H^{m+1}(C^\bullet)$. Since $(K'')^m$ is projective, we can lift the map $(K'')^m \to H^m(C^\bullet)$ to a map

$$\phi''_m : (K'')^m \to \mathrm{Ker}(d_C^m) \to C^m.$$

On the other hand, the composite

$$\begin{tikzcd}
(K')^m \arrow[r, "i'_m"] \arrow[dr, "i'_m"'] & K^{m+1} \arrow[r, "\phi_{m+1}"] & C^{m+1} \\
& \mathrm{Ker}(d_K^{m+1}) \arrow[u, hook] \arrow[r, "\phi_{m+1}"] & \mathrm{Ker}(d_C^{m+1}) \arrow[u, hook]
\end{tikzcd}$$

lies in $\mathrm{Ker}(d_C^{m+1})$ and is 0 in $H^{m+1}(C^\bullet)$. Then

$$(K')^m \xrightarrow{i'_m} \mathrm{Ker}(d_K^{m+1}) \xrightarrow{\phi_{m+1}} \mathrm{Ker}(d_C^{m+1})$$

factors through $\mathrm{im}(d_C^m)$. Since $(K')^m$ is projective, we can lift the map $(K')^m \to \mathrm{im}(d_C^m)$ to a map $\phi'_m : (K')^m \to C^m$ by the universal property. Finally we define

$$\phi_m = (\phi'_m, \phi''_m) : K^m \longrightarrow C^m.$$

It is straightforward to verify that $\phi_{m+1} \circ d_K^m = d_C^m \circ \phi_m$ and ϕ_m induces a surjective map

$$\mathrm{Ker}(d_K^m) = (K'')^m \longrightarrow H^m(C^\bullet).$$

This finishes the construction for m. Now we have the following diagram

$$\begin{tikzcd}
& K^0 \arrow[r, "d_K^0"] \arrow[d, "\phi_0"] & K^1 \arrow[r, "d_K^1"] \arrow[d, "\phi_1"] & \cdots \\
0 \arrow[r] & C^0 \arrow[r, "d_C^0"] & C^1 \arrow[r, "d_C^1"] & \cdots
\end{tikzcd}$$

that satisfies (1)-(4) above. We replace K^0 by $K^0/(\mathrm{Ker}(d_K^0) \cap \mathrm{Ker}(\phi_0))$ and d_K^0, ϕ_0 by their induced maps. Then the new diagram satisfies all the properties (1)-(4) except that K^0 is no longer free.

We still need to prove that K^0 is A-flat. Let $C[-1]^\bullet$ be the complex shifted by -1 of the cochain complex $C^\bullet$, i.e.,

$$C[-1]^p := C^{p-1}, \quad d^p_{C[-1]} := -d_C^{p-1}.$$

Consider the mapping cone of the morphism $\phi : K^\bullet \to C^\bullet$, which is defined as follows:

$$\mathrm{Cone}(\phi)^p := K^p \oplus C^{p-1} = K^p \oplus C[-1]^p,$$

together with[2]

$$\begin{aligned} d^p_{\mathrm{Cone}(\phi)} : K^p \oplus C^{p-1} &\longrightarrow K^{p+1} \oplus C^p \\ (x, y) &\longmapsto (d_K^p(x), \phi_p(x) - d_C^{p-1}(y)). \end{aligned}$$

One can easily check that $(\mathrm{Cone}(\phi)^p, d^p_{\mathrm{Cone}(\phi)})_p$ is a cochain complex. Moreover, we have an exact sequence of cochain complexes for each p, say

[2]There is an alternative (and decorated) way to write the differential map as

$$d^p_{\mathrm{Cone}(\phi)} : K^p \oplus C[-1]^p \to K^{p+1} \oplus C[-1]^{p+1}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} d_K^p & 0 \\ \phi_p & d^p_{C[-1]} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C[-1]^p & \longrightarrow & K^p \oplus C[-1]^p & \longrightarrow & K^p & \longrightarrow & 0 \\
 & & y & \longmapsto & (0, y) & & & & \\
 & & & & (x, y) & \longmapsto & x & &
\end{array}$$

And we have a long exact sequence of cohomology groups

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^p(C[-1]^\bullet) & \longrightarrow & H^p(\mathrm{Cone}(\phi)^\bullet) & \longrightarrow & H^p(K^\bullet) & \xrightarrow{\delta^p} & H^{p+1}(C[-1]^\bullet) & \longrightarrow \cdots \\
 & \| & & & & & & \| & \\
 & H^{p-1}(C^\bullet) & & & & & & H^p(C^\bullet) &
\end{array}$$

Again, it is easy to verify that under the isomorphism $H^{p+1}(C[-1]^\bullet) \cong H^p(C^\bullet)$, the corresponding homomorphism δ^p is the one induced by the morphism ϕ_p^*, which is an isomorphism as well. Hence

$$H^p(\mathrm{Cone}(\phi)^\bullet) = 0, \quad \forall p.$$

So the cochain complex

$$\mathrm{Cone}(\phi)^\bullet : \quad 0 \to K^0 = \mathrm{Cone}(\phi)^0 \to \mathrm{Cone}(\phi)^1 \to \cdots \to \mathrm{Cone}(\phi)^{n+1} = C^n \to 0$$

is exact, in which $\mathrm{Cone}(\phi)^p$ is A-flat for all $p \geqslant 1$. Also, $\mathrm{Cone}(\phi)^\bullet$ breaks into n short exact sequences

$$0 \to \mathrm{Ker}(d^p_{\mathrm{Cone}(\phi)}) \to \mathrm{Cone}(\phi)^p \to \mathrm{Ker}(d^{p+1}_{\mathrm{Cone}(\phi)}) \to 0, \quad p = 1, \ldots, n.$$

Since $\mathrm{Ker}(d^{n+1}_{\mathrm{Cone}(\phi)}) = C^n$ is A-flat, so also is $\mathrm{Ker}(d^n_{\mathrm{Cone}(\phi)})$. We use descending induction and conclude that $\mathrm{Ker}(d^0_{\mathrm{Cone}(\phi)}) = K^0$ is A-flat. This proves the lemma. □

We apply Lemma 5.5 to the Čech cochain complex $C^\bullet = C^\bullet(\mathcal{U}, \mathscr{F})$ and obtain a cochain complex $K^\bullet$ and a cochain map $\phi : K^\bullet \to C^\bullet$ such that

(1) $K^\bullet$ is bounded on $[0, n]$;
(2) K^0 is finite and A-flat, and K^p are finite free A-modules for $p \geqslant 1$;
(3) ϕ is a quasi-isomorphism, i.e., for all p, $\phi_p : H^p(K^\bullet) \to H^p(C^\bullet)$ is an isomorphism.

Granting these conditions, we see K^p is projective as A-module for each $p \geqslant 0$. It remains to prove that for any A-algebra B,

$$\phi_B : H^p(K^\bullet \otimes_A B) \longrightarrow H^p(C^\bullet \otimes_A B)$$

is an isomorphism for each $p \geqslant 0$.

In fact, recall that the mapping cone $\mathrm{Cone}(\phi)^\bullet$ of ϕ breaks into short exact sequences

$$0 \to \mathrm{Ker}(d^p_{\mathrm{Cone}(\phi)}) \to \mathrm{Cone}(\phi)^p \to \mathrm{Ker}(d^{p+1}_{\mathrm{Cone}(\phi)}) \to 0, \quad p = 1, \ldots, n$$

and all the three terms are flat A-modules. Consequently, for each $p = 1, \ldots, n$,

$$0 \to \mathrm{Ker}(d^p_{\mathrm{Cone}(\phi)}) \otimes_A B \to \mathrm{Cone}(\phi)^p \otimes_A B \to \mathrm{Ker}(d^{p+1}_{\mathrm{Cone}(\phi)}) \otimes_A B \to 0$$

is also exact due to the flatness. In particular, the cochain complex $\mathrm{Cone}(\phi)^\bullet \otimes_A B$ is exact as well. On the other hand, $\mathrm{Cone}(\phi)^\bullet \otimes_A B$ is the mapping cone of $\phi_B = \phi \otimes_A B : K^\bullet \otimes_A B \to C^\bullet \otimes_A B$. So we have a long exact sequence of cohomology groups

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H^p(\mathrm{Cone}(\phi)^\bullet \otimes_A B) & \longrightarrow & H^p(K^\bullet \otimes_A B) & \xrightarrow{\phi_B} & H^{p+1}((C^\bullet \otimes_A B)[-1]) & \longrightarrow \cdots \\
 & & & & & \| & \\
 & & & & & H^p(C^\bullet \otimes_A B) &
\end{array}$$

Therefore, ϕ_B is an isomorphism for each p.

Now let $f : X \to Y$ be a proper morphism of noetherian schemes and $\mathscr{F}$ a coherent sheaf of $\mathcal{O}_X$-module on X that is flat over Y. Recall that for $y \in Y$, we define the fiber $X_y = X \times Y \operatorname{Spec}(k(y))$ and $\mathscr{F}_y$ the inverse image of $\mathscr{F}$ on X_y. (Caution: Y is not necessarily affine.)

Corollary 5.6. *Under the above notations, we have*

(1) *For every $p \geqslant 0$, the function*

$$Y \to \mathbb{Z}, \quad y \mapsto \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$$

is upper semicontinuous on Y. A function $h : Y \to \mathbb{Z}$ is, by definition, upper semicontinuous, if for all $n \in \mathbb{Z}$ the set $\{y \in Y \mid h(y) \geqslant n\}$ is a closed subset of Y.

(2) *The function*

$$Y \to \mathbb{Z}, \quad y \mapsto \chi(\mathscr{F}_y) = \sum_{p=0}^{\infty} (-1)^p \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$$

is locally constant on Y.

Proof. The question is local on Y so one may assume that $Y = \operatorname{Spec} A$ is affine. We apply the pervious Theorem 5.2 to the morphism $f : X \to Y$ and the sheaf $\mathscr{F}$, and obtain a cochain complex $K^\bullet$ such that

$$H^p(X_y, \mathscr{F}_y) \cong H^p(K^\bullet \otimes_A k(y)), \quad \forall p \geqslant 0, \ y \in Y.$$

Shrinking Y if necessary, we can assume that K^p is free for all p (the idea is to pretend K^p to be the pth Čech complex). For $p \geqslant 0$, we define

$$W^p := \operatorname{Coker}(d_K^{p-1} : K^{p-1} \to K^p).$$

So we have an exact sequence

$$W^p \xrightarrow{d_K^p} K^{p+1} \longrightarrow W^{p+1} \longrightarrow 0.$$

Applying the functor $(\cdot) \otimes_A k(y)$, we get

$$0 \to H^p(K^\bullet \otimes_A k(y)) \to W^p \otimes_A k(y) \to K^{p+1} \otimes_A k(y) \to W^{p+1} \otimes_A k(y) \to 0.$$

This is basically because the cokernel commutes with base changes, and so we have

$$W^p \otimes_A k(y) \cong \operatorname{Coker}(d_K^{p-1} \otimes_A k(y) : K^{p-1} \otimes_A k(y) \to K^p \otimes_A k(y)).$$

Therefore,

$$\begin{aligned} \dim_{k(y)} H^p(K^\bullet \otimes_A k(y)) = \dim_{k(y)} W^p \otimes_A k(y) - \dim_{k(y)} K^{p+1} \otimes_A k(y) \\ + \dim_{k(y)} W^{p+1} \otimes_A k(y). \end{aligned}$$

Since the function

$$y \mapsto \dim_{k(y)} K^{p+1} \otimes_A k(y)$$

is (locally) constant, it suffices to prove that the function

$$y \mapsto \dim_{k(y)} W^p \otimes_A k(y)$$

is upper semicontinuous.

Claim. For any finitely generated A-module M, the function

$$Y \to \mathbb{Z}, \quad y \mapsto \dim_{k(y)} M \otimes_A k(y)$$

is upper semicontinuous.

The proof of the claim is leave as an exercise. Granting the claim, (2) follows by taking alternating sum of the dimension equation above. □

Corollary 5.7. *Under the above notations, assume further that Y is reduced and connected. Then for all p, the following are equivalent.*

(1) *The function*
$$y \mapsto \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$$
is constant.

(2) *$R^p f_* \mathscr{F}$ is a locally free sheaf on Y, and for all $y \in Y$, the natural map*
$$R^p f_* \mathscr{F} \otimes_{\mathcal{O}_Y} k(y) \to H^p(X_y, \mathscr{F}_y)$$
is an isomorphism.

If any one of (1)(2) hold, we also have that
$$R^{p-1} f_* \mathscr{F} \otimes_{\mathcal{O}_Y} k(y) \cong H^{p-1}(X_y, \mathscr{F}_y)$$
for all $y \in Y$.

We can assume that $Y = \operatorname{Spec} A$ is affine and let $K^\bullet$ be the cochain complex in Theorem 5.2. Then (2) $\Longrightarrow$ (1) is obvious. So it boils down to prove (1) $\Longrightarrow$ (2).

Lemma 5.8. *Let Y be a reduced affine scheme and $\mathscr{F}$ be a coherent sheaf on Y. If*
$$\dim_{k(y)} \mathscr{F} \otimes_{\mathcal{O}_Y} k(y) = r$$
for all $y \in Y$ (as $k(y)$-vector spaces), then $\mathscr{F}$ is a locally free $\mathcal{O}_Y$-module of rank r.

Proof. Let $Y = \operatorname{Spec} A$ and $\mathscr{F} = \widetilde{M}$. Fix $y \in Y$ that correspond to $\mathfrak{p} \in \operatorname{Spec} A$. We choose $x_1, \ldots, x_r \in M_\mathfrak{p}$ such that the images of x_i's in $M \otimes_A k(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ form a basis of this $k(\mathfrak{p})$-vector space. By Nakayama's lemma, the $A_\mathfrak{p}$-linear homomorphism $\phi_\mathfrak{p} : A_\mathfrak{p}^r \to M_\mathfrak{p}$ determined by $x_1, \ldots, x_r$ is surjective. Then there exists $a \in A \backslash \mathfrak{p}$ such that $\phi_\mathfrak{p}$ extends to a surjective A_a-linear homomorphism $A_a^r \to M_a$. Replacing A by A_a, we can assume that there exists a surjective A-linear map
$$\phi : A^r \longrightarrow\!\!\!\!\!\rightarrow M.$$
For any $\mathfrak{q} \in \operatorname{Spec} A$, $\phi \otimes_A k(\mathfrak{q})$ is a surjective $k(\mathfrak{q})$-linear map of $k(\mathfrak{q})$-vector spaces of dimension r. Then $\phi \otimes_A k(\mathfrak{q})$ is an isomorphism. Let $K = \operatorname{Ker}(\phi)$, and hence
$$K_\mathfrak{q} \subset (\mathfrak{q}A_\mathfrak{q})^r, \quad \forall \mathfrak{q} \in \operatorname{Spec} A.$$
Since A is reduced, we have $K = 0$, and then ϕ is an isomorphism. So M is free. □

Lemma 5.9. *Let Y be a reduced noetherian affine scheme, and $\phi : \mathscr{F} \to \mathcal{G}$ be a morphism of finite and locally free $\mathcal{O}_Y$-modules. If*
$$\dim_{k(y)} \operatorname{im}(\phi \otimes_{\mathcal{O}_Y} k(y))$$
is locally constant, then we can find a decomposition of finite and locally free $\mathcal{O}_Y$-modules
$$\mathscr{F} = \mathscr{F}_1 \otimes \mathscr{F}_2, \quad \mathcal{G} = \mathcal{G}_1 \otimes \mathcal{G}_2$$
such that ϕ factors through $\mathcal{G}_1$, $\phi|_{\mathscr{F}_1} = 0$, and $\phi : \mathscr{F}_2 \to \mathcal{G}_1$ is an isomorphism.

Proof. Write $Y = \operatorname{Spec} A$ and $\mathscr{F} = \widetilde{M}$, $\mathcal{G} = \widetilde{N}$ for locally free A-modules M, N of finite rank; $\phi : M \to N$ is an A-linear map. For any $\mathfrak{p} \in \operatorname{Spec} A$,
$$\dim_{k(y)} \operatorname{Coker}(\phi \otimes_A k(y)) = \dim_{k(y)} N \otimes_A k(y) - \dim_{k(y)} \operatorname{im}(\phi \otimes_A k(y))$$
is locally constant. By Lemma 5.8, $\operatorname{Coker} \phi$ is a locally free A-module of finite rank. Define
$$N_1 := \operatorname{Ker}(N \to \operatorname{Coker} \phi) = \operatorname{im} \phi.$$
So we have an exact sequence
$$0 \to N_1 \to N \to \operatorname{Coker} \phi \to 0.$$

We see that N_1 is locally free of finite rank, and there is a decomposition

$$N = N_1 \oplus N_2$$

such that $N_2 \cong \operatorname{Coker}\phi$ under the natural map $N \to \operatorname{Coker}\phi$. Also define $M_1 = \operatorname{Ker}\phi$. We have an exact sequence

$$0 \to M_1 \to M \xrightarrow{\phi} N_1 \to 0.$$

This shows that M_1 is locally free of finite rank. Moreover, notice that the exact sequence splits at M. So there is a decomposition $M = M_1 \oplus M_2$ such that $\phi|_{M_2} : M_2 \to N_1$ is an isomorphism. □

Now we are ready to prove the corollary.

Proof of Corollary 5.7. Applying Theorem 5.2 to $f : X \to Y$ and $\mathscr{F}$, we attain a cochain complex $K^\bullet$ such that for each $p \geqslant 0$,

$$H^p(X_y, \mathscr{F}_y) = H^p(K^\bullet \otimes_A k(y)).$$

Therefore,

$$\begin{aligned}
&\dim_{k(y)} H^p(X_y, \mathscr{F}_y)\\
=&\dim_{k(y)} \operatorname{Ker}(d_K^p \otimes_A k(y)) - \dim_{k(y)} \operatorname{im}(d_K^{p-1} \otimes_A k(y))\\
=&\dim_{k(y)} K^p \otimes_A k(y) - \dim_{k(y)} \operatorname{im}(d_K^p \otimes_A k(y)) - \dim_{k(y)} \operatorname{im}(d_K^{p-1} \otimes_A k(y))
\end{aligned}$$

is constant. Hence

$$\underbrace{\dim_{k(y)} \operatorname{im}(d_K^p \otimes_A k(y))}_{=\phi_1(y)} - \underbrace{\dim_{k(y)} \operatorname{im}(d_K^{p-1} \otimes_A k(y))}_{=\phi_2(y)}$$

is locally constant. Shrinking Y if necessary, we can assume that $\phi_1(y) + \phi_2(y) = C$ (constant) on Y. Since $\phi_1(y)$ and $\phi_2(y)$ are lower semicontinuous, there is a natural stratification on Y, read as

$$\begin{aligned}
Y &= \bigsqcup_{n=0}^{c} \{y \in Y \mid \phi_1(y) = n,\ \phi_2(y) = c - n\}\\
&= \bigsqcup_{n=0}^{c} \{y \in Y \mid \phi_1(y) \leqslant n,\ \phi_2(y) \leqslant c - n\}.
\end{aligned}$$

Since Y is connected, ϕ_1 and ϕ_2 are constant on Y. Now we can apply Lemma 5.9 to $d_K^p : K^p \to K^{p+1}$ and $d_K^{p-1} : K^{p-1} \to \operatorname{Ker}(d_K^p)$, to see there is a decomposition of locally free A-modules of finite rank:

$$\begin{array}{ccccccccc}
 & & Z^{p-1} \oplus (K')^{p-1} & & B^p \oplus H^p \oplus (K')^p & & B^{p+1} \oplus (K')^{p+1} & & \\
 & & \| & & \| & & \| & & \\
\cdots & \longrightarrow & K^{p-1} & \xrightarrow{\quad d_K^{p-1}\quad} & K^p & \xrightarrow{\quad d_K^p\quad} & K^{p+1} & \longrightarrow & \cdots
\end{array}$$

such that

$$\begin{aligned}
Z^{p-1} &= \operatorname{Ker}(d_K^{p-1}), & d_K^{p-1} &: (K')^{p-1} \xrightarrow{\cong} B^p = \operatorname{im}(d_K^{p-1});\\
B^p \oplus H^p &= \operatorname{Ker}(d_K^p), & d_K^p &: (K')^p \xrightarrow{\cong} B^{p+1} = \operatorname{im}(d_K^p).
\end{aligned}$$

Therefore, for any A-algebra B,

$$H^p(K^\bullet \otimes_A B) \cong H^p \otimes_A B \cong H^p(K^\bullet) \otimes_A B.$$

Since $R^p f_* \mathscr{F}$ corresponds to the A-module

$$H^p(X, \mathscr{F}) \cong H^p(K^\bullet) \cong H^p,$$

we have that $R^p f_*\mathscr{F}$ is a locally free A-module of finite rank, and

$$(R^p f_*\mathscr{F}) \otimes_A B \cong H^p \otimes_A B \cong H^p(K^\bullet \otimes_A B) \cong H^p(X_y, \mathscr{F}_y).$$

This proves (2). Moreover, in this case,

$$\begin{aligned}(R^{p-1} f_*\mathscr{F}) \otimes_A k(y) &\cong H^{p-1}(X, \mathscr{F}) \otimes_A k(y)\\ &\cong \operatorname{Ker}(d_K^{p-1}) \otimes_A k(y)/\operatorname{im}(d_K^{p-1}) \otimes_A k(y)\\ &\cong Z^{p-1} \otimes_A k(y)/\operatorname{im}(d_K^{p-1} \otimes_A k(y))\\ &\cong H^{p-1}(K^\bullet \otimes_A k(y)).\end{aligned}$$

Therefore,

$$(R^{p-1} f_*\mathscr{F}) \otimes_A k(y) \cong H^{p-1}(X_y, \mathscr{F}_y)$$

for all $y \in Y$. □

Corollary 5.10. *Under the above notations (Y may not be reduced or connected), assume that $H^p(X_y, \mathscr{F}_y) = 0$ for some p and all $y \in Y$. Then the rational map*

$$R^{p-1} f_*\mathscr{F} \otimes_{\mathcal{O}_Y} k(y) \xrightarrow{\cong} H^{p-1}(X_y, \mathscr{F}_y)$$

is an isomorphism for all $y \in Y$.

Proof. Let $K^\bullet$ be the cochain complex by Theorem 5.2. Fix $y \in Y$ such that

$$H^p(X_y, \mathscr{F}_y) \cong H^p(K^\bullet \otimes_A k(y)) = 0.$$

Then the sequence

$$K^{p-1} \otimes_A k(y) \xrightarrow{d_K^{p-1} \otimes_A k(y)} K^p \otimes_A k(y) \xrightarrow{d_K^p \otimes_A k(y)} K^{p+1} \otimes_A k(y)$$

is exact. We can decompose the $k(y)$-vector space $K^p \otimes_A k(y)$ as $\overline{W}_1 \oplus \overline{W}_2$ such that

$$\overline{W}_1 = \operatorname{im}(d_K^{p-1} \otimes_A k(y))$$

and $d_K^p \otimes_A k(y)|_{\overline{W}_2}$ is injective. Let $\{\overline{x}_1, \dots, \overline{x}_r\}$ be a basis of $\overline{W}_1$ and $\{\overline{y}_1, \dots, \overline{y}_s\}$ be a basis of $\overline{W}_2$. For $i = 1, \dots, s$, denote

$$\overline{z}_i = d_K^p \otimes_A k(y)(\overline{y}_i) \in K^{p+1} \otimes_A k(y),$$

and extend $\{\overline{z}_1, \dots, \overline{z}_s\}$ to a basis $\{\overline{z}_1, \dots, \overline{z}_n\}$ of $K^{p+1} \otimes_A k(y)$. We choose lifting $x_i \in \operatorname{im}(d_K^{p-1})$ of $\overline{x}_i$ for $i = 1, \dots, r$, $y_i \in K^p$ of $\overline{y}_j$ for $j = 1, \dots, s$, and $z_i \in K^{p+1}$ of $\overline{z}_l$ for $l = 1, \dots, s$. Shrinking A by a localization A_a at a such that $a(y) \neq 0$, one may assume that $\{x_1, \dots, x_r, y_1, \dots, y_r\}$ is a basis of K^p, and $\{z_1, \dots, z_n\}$ is a basis of K^{p+1}. Let W_1, W_2 be the free modules generated by $x_1, \dots, x_r$ and $y_1, \dots, y_s$, respectively. Then $K^p = W_1 \oplus W_2$, where $W_1 \subset \operatorname{im}(d_K^{p-1})$ and $d_K^p|_{W_2}$ is injective. Hence $W_1 = \operatorname{im}(d_K^{p-1})$. As W_1 is free, it is projective. So there is a decomposition $K^{p-1} = W_1 \oplus \operatorname{Ker}(d_K^{p-1})$. Now we have two exact sequences

$$K^{p-2} \xrightarrow{d_K^{p-2}} \operatorname{Ker}(d_K^{p-1}) \longrightarrow H^{p-1}(K^\bullet) \cong H^{p-1}(X, \mathscr{F}) \longrightarrow 0,$$

and

$$\begin{tikzcd} K^{p-2} \otimes_A k(y) \arrow[r, "d_K^{p-2} \otimes_A k(y)"] & \operatorname{Ker}(d_K^{p-1} \otimes_A k(y)) \arrow[r] \arrow[d, equal] & H^{p-1}(K^\bullet \otimes_A k(y)) \arrow[r] \arrow[d, equal] & 0. \\ & \operatorname{Ker}(d_K^{p-1}) \otimes_A k(y) & H^{p-1}(X_y, \mathscr{F}_y) & \end{tikzcd}$$

$$(\text{by } K^{p-1} = W_1 \oplus \operatorname{Ker}(d_K^{p-1}))$$

Since the cokernel is stable under base changes, we have an isomorphism

$$\begin{array}{c} \boxed{H^{p-1}(X,\mathscr{F})} \otimes_A k(y) \xrightarrow{\cong} H^{p-1}(X_y,\mathscr{F}_y). \\ \| \\ R^{p-1}f_*\mathscr{F} \end{array}$$

This completes the proof. $\square$

Corollary 5.11. *If $R^k f_*\mathscr{F} = 0$ for $k \geqslant k_0$, then*

$$H^k(X_y,\mathscr{F}_y) = 0, \quad \forall y \in Y,\ k \geqslant k_0.$$

Corollary 5.12 (Flat base change)**.** *If B is a flat A-algebra, then*

$$H^p(X \times_Y \operatorname{Spec} B, \mathscr{F} \otimes_A B) \cong H^p(X,\mathscr{F}) \otimes_A B.$$

Corollary 5.13 (Seesaw's theorem)**.** *Let X be a complete[3] variety and T be any variety. Choose a line bundle $\mathscr{L} \in \operatorname{Pic}(X \times T)$. Then the set*

$$T_1 := \{t \in T \mid \mathscr{L}|_{X\times\{t\}} \text{ is trivial on } X \times \{t\}\}$$

is closed in T, and $\mathscr{L}|_{X\times T_1} \cong p_2^*\mathcal{M}$ for some $\mathcal{M} \in \operatorname{Pic}(T_1)$, where $p_2 : X \times T_1 \to T_1$ is the second projection.

Lemma 5.14. *A line bundle (i.e., an invertible sheaf) $\mathcal{M}$ on a complete variety X is trivial if and only if*

$$\dim H^0(X,\mathcal{M}) > 0, \quad \dim H^0(X,\mathcal{M}^{-1}) > 0.$$

Proof. Exercise. $\square$

Proof of Seesaw's Theorem. It follows from Lemma 5.14 that

$$\begin{aligned} T_1 &= \{t \in T \mid \mathscr{L}|_{X\times\{t\}} \text{ is trivial on } X \times \{t\}\} \\ &= \left\{ t \in T \;\middle|\; \begin{array}{c} \dim_{k(t)} H^0((X\times T)\times_T \operatorname{Spec}(k(t)), \mathscr{L} \otimes_{\mathcal{O}_T} k(t)) > 0, \text{ and} \\ \dim_{k(t)} H^0((X\times T)\times_T \operatorname{Spec}(k(t)), \mathscr{L}^{-1} \otimes_{\mathcal{O}_T} k(t)) > 0 \end{array} \right\}. \end{aligned}$$

By the semicontinuity theorem (Corollary 5.6), T_1 is closed in T. We regard T_1 as a reduced closed subscheme of T, and $p_2 : X \times T_1 \to T_1$ is a proper morphism of noetherian schemes. Denote for simplicity that $\mathscr{L}_1 = \mathscr{L}|_{X\times T_1}$. By definition of T_1, for any $t \in T_1$,

$$\dim_{k(t)} H^0((X\times T_1)\times_{T_1} \operatorname{Spec}(k(t)), \mathscr{L}_1 \otimes_{\mathcal{O}_{T_1}} k(t)) > 0$$

By Corollary 5.7, $\mathcal{M} := p_{2,*}\mathscr{L}_1$ is an invertible sheaf on T_1 and the natural map

$$p_{2,*}\mathscr{L}_1 \otimes_{\mathcal{O}_{T_1}} k(t) \longrightarrow H^0(X\times\{t\}, \mathscr{L}_1|_{X\times\{t\}})$$

is an isomorphism for any $t \in T_1$.

We prove that the natural morphism $p_2^*\mathcal{M} \to \mathscr{L}_1$ is an isomorphism. In fact, for any $t \in T_1$, the sheaf $p_2^*\mathcal{M}|_{X\times\{t\}}$ is the inverse image of $\mathcal{M}$ under

$$X \times \{t\} \hookrightarrow X \times T_2 \xrightarrow{p_2} T_2.$$

It is the trivial invertible sheaf on $X \times \{t\}$ and is the pullback of the $k(t)$-vector space $p_{2,*}\mathscr{L}_1 \otimes_{\mathcal{O}_{T_1}} k(t)$ under $X \times \{t\} \to \{t\} = \operatorname{Spec}(k(t))$. On the other hand, $\mathscr{L}_1|_{X\times\{t_1\}}$ is also trivial and the restriction of $p_2^*\mathcal{M} \to \mathscr{L}_1$ on $X \times \{t\}$ corresponds to the morphism

$$p_{2,*}\mathscr{L}_1 \otimes_{\mathcal{O}_{T_1}} k(t) \longrightarrow H^0(X\times\{t\}, \mathscr{L}_1|_{X\times\{t\}})$$

of global sections. Therefore, the restriction of $p_2^*\mathcal{M} \to \mathscr{L}_1$ on $X \times \{t\}$ is an isomorphism for each $t \in T_1$. This is enough to show that $p_2^*\mathcal{M} \to \mathscr{L}_1$ is itself an isomorphism. $\square$

[3]Can be replaced with properness.

Remark 5.15. We can assume that T is a (reduced) scheme of finite type over an algebraically closed field k.

References

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School of Mathematical Sciences, Peking University, 100871, Beijing, China
Email address: daiwenhan@pku.edu.cn