# The Hitchin map and the spectral base

Jie Liu

(Joint with Siqi He, Ngaiming Mok)

## Lecture 1

$X$ = proj mfd $/\mathbb{C}$. $n = \dim X$.

### § Def of Higgs bundles and Hitchin map

Def A Higgs bundle $/X$ is a pair $(E, \varphi)$ s.t.

(1) $E \to X$ holo vec bundle of rk $r$.

(2) $\varphi : E \to E \otimes \Omega_X^1$ (Higgs field) s.t. $\varphi \wedge \varphi = 0$. $(*)$

Examples (1) If $\dim X = 1$, then $(*)$ is automatic.

(2) $\omega \in H^0(X, \Omega_X^1)$, $E \xrightarrow{\times \omega} E \otimes \Omega_X^1$.

(3) Locally we can choose a trivialization of $E$.

May write $\varphi = \sum_{i=1}^{n} \underset{\substack{\uparrow \\ r\times r \text{ matrix.}}}{A_i} \cdot dz_i$

$(\varphi \wedge \varphi = 0 \iff [A_i, A_j] = 0,\ \forall i, j.)$

(4) As usual we can define (semi- / poly-) stability for Higgs bundles by considering $\varphi$-inv subsheaves of $E$.

$\mathcal{M}^{\text{stack}}_{\text{Higgs}}$ Moduli stack of Higgs buns

$\cup$

$\mathcal{M}_{\text{Higgs}}$ Moduli stack of polystab Higgs buns

$\cup$

$M_{\text{Dol}}$ Moduli space of top trivial polystab Higgs buns

- top trivial $\iff$ $\underset{\overset{\shortparallel}{0}}{c_i(E)} \in H^{2i}(X,\mathbb{Q})$

- $\mathcal{M}_{Dol} \xleftarrow[\text{Hodge}]{\text{non-abelian}} \dashrightarrow \mathcal{M}_B = \{\rho : \pi_1(X) \to GL_r(\mathbb{C}) \mid \text{reductive}\}/\sim$

  $\uparrow$ Betti moduli space.

  [Hitchin, Donaldson, Corlette, Simpson].

Def The Hitchin map is def'd as

$$h_X : \mathcal{M}^{\text{stack}}_{Higgs} \longrightarrow \underset{\text{Hitchin base.}}{A_X} = \bigoplus_{i=1}^{r} H^0(X, \mathrm{Sym}^i \Omega_X^i)$$

$$(E,\varphi) \longmapsto \text{Coeffs of char poly of } \varphi.$$

$$(\mathrm{tr}\,\varphi, \ldots, \underset{\overset{\shortparallel}{(-1)^r \det \varphi}}{\mathrm{tr}\,\varphi^r}).$$

Thm (Hitchin, Simpson)

(1) $h_X|_{\mathcal{M}_{Dol}} : \mathcal{M}_{Dol} \longrightarrow A_X$ is proper.

(2) If $\dim X = 1$, then $h_X|_{\mathcal{M}_{Dol}}$ is surj.

Q What happens if $\dim X \geq 2$ around $\mathcal{M}^{\text{stack}}_{Higgs} \xrightarrow{h_X} A_X$?

## § Spectral base

Def (Chen-Ngô)

The spectral base $\mathcal{S}_X \subseteq A_X$ is the subset def'd as

$(s_1, \ldots, s_r) \in \mathcal{S}_X \iff \forall x \in X, \exists w_1, \ldots, w_r \in T^*_x X$ s.t.

$$s_i(x) = (-1)^i \sigma_i(w_1, \ldots, w_r) \qquad (**)$$

where $\sigma_i$ = $i$-th symm poly in $r$ variables.

<u>Rmk</u> $w_i$'s may NOT be pairwise distinct.

<u>Facts</u> (1) $\mathcal{S}_X$ is closed & stable under natural $\mathbb{C}^*$-action

$$\mathbb{C}^* \curvearrowright \mathcal{A}_X = \underset{wt=1}{H^0(X, \Omega_X^1)} \oplus \underset{\cdots}{\cdots} \oplus \underset{wt=r}{H^0(X, \mathrm{Sym}^r \Omega_X^1)}.$$

(2) It is enough to check (**) for general pts of $X$:

· In particular, $\mathcal{S}_X$ is birat inv

$\hat{X} \xrightarrow{f} X$ birat proper.

$$\rightsquigarrow \quad f^*: \begin{array}{ccc} \mathcal{A}_X & \longrightarrow & \mathcal{A}_{\hat{X}} \\ \cup\! \mid & & \cup\! \mid \\ \mathcal{S}_X & \longrightarrow & \mathcal{S}_{\hat{X}}. \end{array}$$

<u>Prop</u> $\mathrm{Im}(h_X) \subseteq \mathcal{S}_X$.

<u>Proof</u> $\varphi \wedge \varphi = 0 \iff [A_i, A_j] = 0, \ \forall i, j \quad (\varphi = \sum_i A_i dz_i)$.

Choose a suitable trivialization of $E$

s.t. $A_i$'s can be upper-triangulized simultaneously.

$$A_i = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \Rightarrow \varphi = \begin{pmatrix} w_1 & & * \\ & \ddots & \\ 0 & & w_r \end{pmatrix}, \quad w_i \in T_x^* X.$$

$\Rightarrow h_X(E, \varphi) = (s_1, \ldots, s_r) \in \mathcal{S}_X$. □

Consider

$$\begin{array}{ccccc} \mathcal{M}_{\mathrm{Higgs}}^{\mathrm{stack}} & \xrightarrow{h_X} & \mathcal{S}_X & \overset{\text{closed}}{\subseteq} & \\ \cup\! \mid & & \cup\! \mid & & \mathcal{A}_X \\ \mathcal{M}_{\mathrm{Dol}} & \xrightarrow{h_X|_{\mathcal{M}_{\mathrm{Dol}}}} & \mathcal{S}_{X,\mathrm{Dol}} & \underset{\text{closed}}{\subseteq} & \end{array}$$

(for $X$ = curve, $\mathcal{M}_{Dol} = \mathcal{M}^{Stack}_{Higgs}$.)

<u>Examples</u> (1) $\mathcal{Y}_{X,Dol} \subsetneq \mathcal{Y}_X$ [Bogomolov-de Oliveira]

Let $Y \subseteq A^3$ hypersurface of $\deg \gg 1$ (specially chosen)

$A^3$: 3-dim'l abelian var.

$\theta$ = involution of $A^3$.

$\rightsquigarrow$ $X \longrightarrow Y/\theta$ min resolution

s.t. $\pi_1(X) = \{1\}$ & $\mathcal{Y}_X \neq 0$ (2nd Symm power $0 \neq \mathcal{Y}_X^2 \subseteq \mathcal{A}_X^2$).

$\Downarrow$

$\mathcal{Y}_{X,Dol} = 0$,

(2) Shimura surface.

$\Gamma \subseteq \mathrm{Aut}(\mathbb{D} \times \mathbb{D})$, $\mathbb{D}$ = 1-dim'l disc.

irred torsion-free cocompact lattice.

$\Lambda = \mathbb{D} \times \mathbb{D} / \Gamma$ surface $\rightsquigarrow$ $\Omega_X^1 = \mathcal{L}_1 \oplus \mathcal{L}_2$.

Then for $m \geq 1$, have

$$H^0(X, \Omega_X^1) = H^0(X, \mathcal{L}_1^{\otimes m}) = H^0(X, \mathcal{L}_2^{\otimes m}) = 0$$

Let $\mathcal{A}_X^2 = H^0(X, \Omega_X^1) \oplus H^0(X, \mathrm{Sym}^2 \Omega_X^1)$

$(0 = s_1 \quad , \quad s_2) \in \mathcal{Y}_X$

$\Updownarrow$

$s_2(x) = w_x^2$ $(\forall x \in X, \exists w_x \in T_x^* X)$.

But $\mathrm{Sym}^2 \Omega_X^1 = \mathcal{L}_1^{\otimes 2} \oplus \mathcal{L}_1 \otimes \mathcal{L}_2 \oplus \mathcal{L}_2^{\otimes 2}$

$\Rightarrow$ $H^0(\mathrm{Sym}^2 \Omega_X^1) = H^0(X, \mathcal{L}_1 \otimes \mathcal{L}_2) = H^0(X, K_X)$

$\Rightarrow$ If $(s_1, s_2) \in \mathcal{Y}_X$, then $s_1 = s_2 = 0$.

$\exists \Gamma$ s.t. $H^0(X, K_X) = H^0(X, \mathrm{Sym}^2 \Omega_X^1) \neq 0 \Rightarrow \mathcal{A}_X^2 \neq 0$.

§ Application: Rigidity problem

**Def** $\rho : \pi_1(X) \to GL_r(\mathbb{C})$ a reductive repr.

(1) $\rho$ is rigid if $[\rho] \in \mathcal{M}_B$ is isolated

(2) $\rho$ is integral if it is conjugate to a map $\pi_1(X) \to GL_r(\mathcal{O}_K)$

($K$ = number field $\supseteq \mathcal{O}_K$).

**Thm** (Simpson, Arapura, Klingler, He-Liu-Mok).

$X$ proj mfd with $\mathcal{A}_X = 0$. Then

$\forall \rho : \pi_1(X) \to GL_r(\mathbb{C})$ reductive map is rigid & integral.

**Proof** (1) $h_X|_{\mathcal{M}_{Dol}} : \mathcal{M}_{Dol} \longrightarrow \mathcal{A}_{X,Dol}$ proper

$\simeq \mathcal{A}_X = 0$

$\Rightarrow \mathcal{M}_{Dol}$ compact

Have $\mathcal{M}_{Dol} \underset{\text{homeo}}{\cong} \mathcal{M}_B$ affine var $\Rightarrow \dim \mathcal{M}_B = 0$.

(2) Up to conjugate, $\rho$ is def'd / $K$ number field.

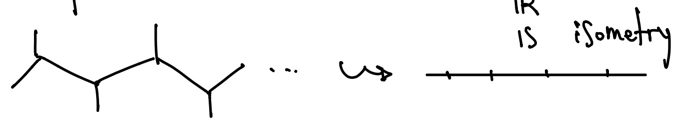
To prove integrality of $\rho$, it is enough to show that for $\forall v$ finite place of $K$,

$\rho : \pi_1(X) \to GL_r(K_v) =: G$ is bdd.

$$\begin{tikzcd} \tilde{X} \arrow[r, "\text{Lipschitz harmonic}", "b_\rho"'] \arrow[d, "\text{univ cover}"'] & \underbrace{\Delta(G)}_{\text{Bruhat-Tits building for } G.} \\ X & \end{tikzcd}$$

$\rho$-equiv [Gromov-Schoen]

[ E.g. $\Delta(G)$ for $G = GL_2(\mathbb{Q}_2)$ :




$\forall x \in \Delta(G)$, $\mathrm{Stab}_x \le G$ is bdd. ]

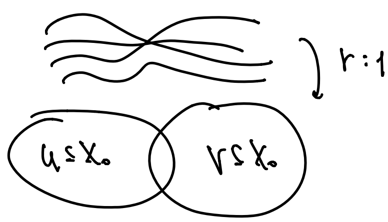
($b_\rho : \tilde{X} \to G$ exists when given any red rep $\rho$).

<u>Fact</u> $b_\rho$ constant $\Rightarrow$ $\rho$ is bdd.

Assume $b_\rho$ is not constant.

$\Rightarrow$ $\exists$ holo mult-sections of $\Omega_X^1 \to X$ over an open subset $X_0$ s.t. $X \setminus X_0$ has Hausdorff codim $\geq 2$.

$\uparrow$ by harmonicity of $b_\rho$.




$\rightsquigarrow$ $(s_1^\circ, \dots, s_r^\circ) \in \mathcal{S}_{X_0}$.

Now Lipschitz + Hausdorff codim$(X \setminus X_0) \geq 2$

$\Rightarrow$ $s_i^\circ \overset{\text{extends}}{\rightsquigarrow} s_i \in H^0(X, \mathrm{Sym}^i \Omega_X^1)$

$\Rightarrow$ $\underset{\neq 0}{(s_1, \dots, s_r)} \in \mathcal{S}_X$.

## Lecture 2

$$\begin{tikzcd}
\mathcal{M}^{\text{stack}}_{\text{Higgs}} \arrow[r, "h_X"] & \mathcal{S}_X \arrow[r, hook, "\text{closed}"] & \mathcal{A}_X = \bigoplus_{i=1}^r H^0(X, \mathrm{Sym}^i \Omega_X^1) \\
\mathcal{M}_{\text{Dol}} \arrow[u, hook] \arrow[r, "h_{X,\text{Dol}}"] & \mathcal{S}_{X,\text{Dol}} \arrow[u, hook] &
\end{tikzcd}$$

$s = (s_1, \dots, s_r) \in \mathcal{S}_X \iff \forall x, \exists w_1, \dots, w_r \in T_x^* X$

s.t. $s_i(x) = (-1)^i \sigma_i(w_1, \dots, w_r)$.

<u>Application</u> $\gamma_X = 0 \Rightarrow \forall [\rho] \in M_B$ is rigid & integral.

## § Quotient of bounded symmetric domains (BSDs)

<u>Def</u> $\Omega \subseteq \mathbb{C}^n$ is called a BSD if $\forall x \in \Omega$,
$\exists\ \sigma : \Omega \to \Omega$ a biholo involusion s.t.
$x$ is an isolated fixed pt of $\sigma$.

<u>Examples</u> (1) $\mathbb{B}^n \subseteq \mathbb{C}^n$ ball

(2) $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$ disk product

(3) BSD $= \prod$ irred BSDs

$\exists$ a classification of irred BSDs:

4 classical types + 2 excep types.

(4) $\Omega \subseteq M =$ cpt dual of $\Omega$

$\llcorner$ Herm symm space of cpt type (Fano & homogeneous).

e.g. $\Omega = \mathbb{B}^n \subseteq \mathbb{P}^n = M$

$\Omega_{I(m,n)} \subseteq Gr(m, m+n)$.

<u>Rmk</u> $\Omega$ BSD, $\Gamma \subseteq \mathrm{Aut}(\Omega)$ an irred torsion-free cocompact lattice

$\rightsquigarrow X = \Omega/\Gamma$ proj mfd.

<u>Note</u> (1) $\Omega_X^1$ big $\Rightarrow \limsup h^0(X, \mathrm{Sym}^m \Omega_X^1) \sim O(m^{2\dim X - 1})$ as $m \to +\infty$

$\Rightarrow \mathcal{A}_X$ is very large for $r \gg 1$.

(2) Margulis' super-rigidity thm

$\Rightarrow \Gamma$ is arith if rank $\Omega \geq 2$

$\Rightarrow [\rho] \in \mathcal{M}_B$ is rigid & integral

$\Rightarrow \mathcal{G}_{X, \mathrm{Dol}} = 0$.

(3) $\operatorname{rank} \Omega = 1 \Leftrightarrow \Omega = \mathbb{B}^n$ ball.

<u>Thm A</u> (He-Liu-Mok)

$X = \Omega/\Gamma$, $\operatorname{rank} \Omega \geq 2$, $\Gamma$ = irred torsion-free cocpt lattice.

Then $\mathcal{G}_X = 0$.

<u>Idea of the proof of Thm A</u> ($r=2$, $\Omega$ irred)

(1) minimal characteristic bundle

$$\begin{tikzcd}
\mathcal{C}_x \arrow[r] \arrow[d, hook] & x \arrow[d, hook] & \tilde{x} \arrow[l] \arrow[d, hook] & \\
\mathcal{C} \arrow[r] & X & \Omega \arrow[l] \arrow[r, phantom, "\overset{\text{dual}}{\subseteq}"] & M \\
\mathbb{P}(T_X) \arrow[u, phantom, "\cong", sloped] \arrow[ur, "\text{geometric}"'] & & &
\end{tikzcd}$$

( Fano mfd of $\rho(M) = 1$

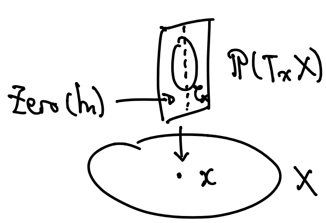
$\mathcal{C}_x \subseteq \mathbb{P}(T_x X) \cong \mathbb{P}(T_{\tilde{x}} \Omega) = \mathbb{P}(T_{\tilde{x}} M)$

$\parallel$ "tangent directions of lines in $M$ passing through $\tilde{x}$"

$\parallel$ "VMRT of $M$ at $\tilde{x}$".

e.g. $\mathbb{B}^n \subset \mathbb{P}^n$. $\mathcal{C}_x = \mathbb{P}(T_x X)$

e.g. $\Omega_{I(m,n)} \subseteq Gr(m, m+n)$, $\mathcal{C}_x = \mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \xrightarrow{\mathcal{O}(1,1)} \mathbb{P}(T_x X)$




<u>Fact</u> $\mathcal{C}_x \subseteq \mathbb{P}(T_x X)$ is linearly normal.

(2) Finsler metric rigid thm

Thm (Mok's rigidity thm)

rank $\Omega \geq 2$, $X = \Omega/\Gamma$.

$\Gamma$ = irred tor-free cocpt lattice.

$h$ = Finsler pseudo-metric on $\mathcal{O}_{\mathbb{P}(TX)}(-1) \to \mathbb{P}(TX)$

$g$ = Herm metric on $\mathcal{O}_{\mathbb{P}(TX)}(-1) \to \mathbb{P}(TX)$

induced by canonical metric on $TX$.

If $\Theta_h \leq 0$, then $h|_{\mathcal{C}} = c\, g|_{\mathcal{C}}$.

(3) $s = (s_1, s_2) \in \mathcal{G}_X$.

Want $s_1 = s_2 = 0$.

- If $s_1 \neq 0$ then $0 \neq s_1 \in H^0(X, \Omega_X^1) = H^0(\mathbb{P}TX, \mathcal{O}_{\mathbb{P}TX}(1))$

  $\rightsquigarrow$ $h_1(v) = \| s_1(v) \|$ with $\Theta_h = -\operatorname{div} s_1 \leq 0$

  By rigidity thm, get $h_1 = c\, g|_{\mathcal{C}}$

  $\Rightarrow$ For $x \in X$ general pt,

  $0 \neq h_1|_{\mathcal{C}_x} = c\, g|_{\mathcal{C}_x} \Rightarrow c_1 \neq 0$ $(*)$

  as $\mathrm{Zero}(h_1|_{\mathbb{P}T_xX}) = \mathrm{Zero}(s_1(x)) =$ hyperplane in $\mathbb{P}T_xX$.

  (e.g. $s_2(x) = z^2 + y^2$, $\mathcal{C}_x = \{ z^2 + y^2 = 0 \}$.)

- $s_1 = 0$ + $s = (s_1, s_2) \in \mathcal{G}_X \Rightarrow \forall x \in X$, $s_2(x) = w_x^2$, $w_x \in T^*X$,

  $\rightsquigarrow$ $h_2(v) = \| s_2(v) \|^{1/2}$.

  Apply a same argument yields a contradiction.

## § Chen-Ngô's conjecture & spectral correspondence

Conj (Chen-Ngô) $h_X : \mathcal{M}_{Higgs}^{stack} \longrightarrow \mathcal{G}_X$ is always surjective.

Rmk (1) Cannot replace $\mathcal{M}_{Higgs}^{stack}$ by $\mathcal{M}_{Dol}$.

(2) Conj holds if $\dim X = 2$ (Chen-Ngô, Song-Sun).

$\rightsquigarrow$ spectral correspondence

(Beauville-Narasimhan-Ramanan corr if $\dim X = 1$).

Def Given $s = (s_1, \ldots, s_r) \in \mathcal{S}_X$, the <u>spectral var</u> $X_s$ is def'd by

$$\{\lambda^r - s_1\lambda^{r-1} + \cdots + (-1)^r s_r = 0\} \subseteq T^*X \xrightarrow{\pi} X$$

$$\lambda^r - s_1\lambda^{r-1} + \cdots + (-1)^r s_r \in H^0(T^*X, \pi^* \mathrm{Sym}^r \Omega_X^1).$$

Thm (Chen-Ngô) Assume that $X_s \to X$ is $r:1$.

i.e. $\lambda^r - s_1\lambda^{r-1} + \cdots + (-1)^r s_r = 0$ has distinct roots at a generic pt.

$$\{(E,\varphi) \mid h_X(E,\varphi) = s\}/\sim \;\xleftrightarrow{1-1}\; \left\{\begin{array}{l}\text{maximal Cohen-Macaulay} \\ \text{sheaves of rk 1 on } X_s\end{array}\right\}/\sim$$

$$(\pi_*\mathcal{F}, \pi_*\lambda) \longleftarrow\!\shortmid\; \mathcal{F}$$

where $\lambda$ is char'd by $\mathcal{F} \xrightarrow{\times\lambda} \mathcal{F} \otimes \pi^*\Omega_X^1$

$\rightsquigarrow \pi_*\mathcal{F} \xrightarrow{\pi_*\lambda} \pi_*\mathcal{F} \otimes \Omega_X^1$.

- maximal $\Rightarrow$ $\mathrm{supp}\,\mathcal{F} = X_s$
- Cohen-Macaulay $\Leftrightarrow$ $\pi_*\mathcal{F}$ locally free.

<u>Thm (*)</u> (Chen-Ngô, Song-Sun)

CN conj holds if $\dim X = 2$.

<u>Thm B</u> (He-Liu-Mok)

Given $s \in \mathcal{S}_X$, $\exists\, (\mathcal{E}, \varphi)$ w/ $\mathcal{E}$ reflexive s.t. $h_X(\mathcal{E}, \varphi) = s$.

Thm B $\Rightarrow$ Thm $(*)$ b/c nef sheaves are loc free in codim 2.

<u>Proof of Thm B</u>

(1) Reduces to $X_s \to X$ is $r:1$.

Indeed, $[X_s] = \sum m_i [X_s^i]$

multiplicity of general pts — irred comp

$$P(\lambda) = \lambda^r - s_1 \lambda^{r-1} + \cdots + (-1)^r s_r = \prod P_i(\lambda)^{m_i}.$$

Set $(\mathcal{E}, \varphi) = \bigoplus_{i=1}^{l} (\mathcal{E}_i, \varphi_i)^{\oplus m_i} \Rightarrow h_X(\mathcal{E}, \varphi) = s.$

(2) $(\pi_* \mathcal{O}_{X_s}, \pi_* \lambda) \xrightarrow{\text{taking nef hull}} ((\pi_* \mathcal{O}_{X_s})^{**}, (\pi_* \lambda)^{**})$, with $(\pi_* \mathcal{O}_{X_s})^{**} = \mathcal{F}$

can be very bad

Rmk (i) If $X_s \to X$ is $r:1$, $\mathcal{F}$ carries a natural $\mathcal{O}_X$-alg str.

$$\rightsquigarrow \quad \begin{tikzcd} \mathrm{Spec}_{\mathcal{O}_X} \mathcal{F} =: \widetilde{X}_s \arrow[rr] \arrow[dr] & & (X_s)_{red} \arrow[dl] \\ & X & \end{tikzcd}$$

finite, isom in codim 1.

Can replace $X_s$ by $\widetilde{X}_s$ in the spectral corr.

(ii) If $\dim X = 2$ then $\widetilde{X}_s$ is Cohen-Macauley.

<u>§ Chen-Ngô conj in rank 2 case</u>

<u>Thm C</u> (He-Liu) Assume $r = 2$.

For $s \in \mathcal{S}_X$ s.t. $X_s \to X$ is $2:1$,

$\exists\ \widetilde{X}_S \longrightarrow (X_S)_{red}$ finite, isom in codim 1 s.t.

(1) $\widetilde{X}_S$ is Cohen-Macauley

(2) spectral corr holds for $\widetilde{X}_S$.

Rmk If $r=2$, then $h_X|_{\mathcal{M}_{Higgs}} \twoheadrightarrow \mathcal{S}_X$.

(Thm C $\Rightarrow$ CN conj for $r=2$).

<u>Proof of Thm C</u>

(1) $s=(s_1,s_2)\in\mathcal{S}_X$, $\varphi\longmapsto\varphi-\frac{1}{2}s_1=:\varphi'$

Reduce to $s_1=0$.

(2) $s_1=0$ + $s=(s_1,s_2)\in\mathcal{S}_X \Leftrightarrow s_2(x)=w_x^2$ $(\forall x\in X,\ \exists w_x\in T^*X)$

$\Leftrightarrow s_2$ rank 1 symm diff

(3) $s\in\mathcal{B}_X \Rightarrow s=\alpha^2\bar{L}$ with $\alpha: L\to\Omega_X^1$ inj in codim 1

$\bar{L}\in H^0(X,L^{\otimes 2})$.

Indeed, $s\in H^0(\mathbb{P}TX,\mathcal{O}_{\mathbb{P}TX}(2))$.

Then $s\in\mathcal{B}_X \Leftrightarrow \operatorname{div} s = \underset{\text{horizontal}}{2\Delta} + \underset{\text{vertical}}{D}$.

$\Delta \rightsquigarrow \alpha: L\to\Omega_X^1$

$D \rightsquigarrow \tau\in H^0(X,L^{\otimes 2})$.

(4) $\widetilde{X}_S$ = double cover defined by $\tau$

$$\begin{array}{ccc} \widetilde{X}_S & & \\ \downarrow & & \\ X & \quad \widetilde{X}_S \subseteq L \xrightarrow[\alpha]{\text{embedding in codim 1}} & \Omega_X^1 \\ & & \cup \\ & \quad \searrow & X_S \end{array}$$

here $\widetilde{X}_S\subseteq L$ hypersurface $\Rightarrow \widetilde{X}_S$ Cohen-Macauley.

$\square$

<u>Cor</u> If $X$ = proj mfd of dim $\geq 2$ s.t. $h^{1,1}(X) = 1$, then $\delta_X^2 = 0$.

e.g. $X$ = fake proj space (ball quotient)

i.e. $b_i(X) = b_i(\mathbb{P}^n)$ + $X \neq \mathbb{P}^n$.