Anabelian Geometry & ABC Conjecture
(A Review for Shinichi Mochizuki's work)

OUTLINE

- Anabelian Geometry.
- ABC Introduction
- Geometry of Frobenioids
- Big Hodge Theaters by Mochizuki.

① Anabelian Papers

1990 – Nakamura: 1-Crelle

1999 – Loc Pro-p – Inventiones

$\boxed{\text{2004 – AAGHC – Springer}}$ ☆ → AAG 1

2005 – Gal Sec – Nagoya

2006 – Semigraphs – RIMS

2007 – AACPHC – Kyoto

2007 – Combinatorial – Tohoku → AAG 2, AAG 3

② Kummer Classes of Functions

k field. $X \in Sch/k$ affine.

Construction $\mathcal{O}^\times(X) \xrightarrow{\kappa} H^1(X, \hat{\mathbb{Z}}(1)_X)$ baby kummer map.

$f \longmapsto \kappa_{X,f}$ ← kummer class

Recall: Baby Kummer Sequence

$$1 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 1 \qquad /k$$

We want the sequence on $Shv(X_{et}, Grp)$.

To do this, we base change to X:

$$1 \to \mu_{n,X} \to \mathbb{G}_{m,X} \xrightarrow{[n]} \mathbb{G}_{m,X} \to 1.$$

<u>Note</u> If $X' \to X$ is an étale cover,

$\rightsquigarrow \pi_1(X) \curvearrowright X'$

$\rightsquigarrow 1 \to \mu_{n,X}(X') \to \mathbb{G}_{m,X}(X') \to \mathbb{G}_{m,X}(X') \to 1.$

exact sequence of $\pi_1(X)$-modules.

We consider the associated long exact sequence

$$\begin{array}{ccc} H^0(X, \mathbb{G}_m) & \xrightarrow{\quad \mathfrak{K}_X^{(n)} \quad} & H^1(X, \mu_n). \\ \| & & \\ \mathcal{O}^\times(X) & & \end{array}$$

- <u>FACT</u> $\varprojlim H^1(X, \mu_{n,X}) = H^1(X, \varprojlim \mu_{n,X})$
$= H^1(X, \hat{\mathbb{Z}}(1)_X)$
- We get a map $\hat{\mathfrak{K}}_X : H^0(X, \mathbb{G}_m) \longrightarrow H^1(X, \hat{\mathbb{Z}}(1))$.
by taking limit. $\hat{\mathfrak{K}}_X = \varprojlim_n \mathfrak{K}_X^{(n)}$.

<u>Summary</u>

$$\begin{array}{ccc} \mathcal{O}^\times(X) & \longrightarrow & H^1(X, \hat{\mathbb{Z}}(1)) \\ \Psi & & \Psi \\ f & \longmapsto & \hat{\mathfrak{K}}_X(f) = \mathfrak{K}_{X,f} = \mathfrak{K}f \end{array}$$

Btw: $H^1(X, \hat{\mathbb{Z}}(1)) = H^1(\pi_1(X), \hat{\mathbb{Z}}(1))$.

$\boxed{\text{Functoriality}}$ There's a natural trans

$\hat{\mathfrak{K}} : H^0(-, \mathbb{G}_m) \longrightarrow H^1(-, \hat{\mathbb{Z}}(1))$

for functors $\mathrm{Sch}_K \longrightarrow \mathrm{Ab}$.

- If $\varphi : Y \to X$

we get

$$\begin{array}{ccc} H^0(X, \mathbb{G}_m) & \xrightarrow{\hat{\mathfrak{K}}_X} & H^1(X, \hat{\mathbb{Z}}(1)) \\ \varphi^* = H^0(\varphi, \mathbb{G}_m) \downarrow & & \downarrow \varphi^* = H^1(\varphi, \hat{\mathbb{Z}}(1)) \\ H^0(Y, \mathbb{G}_m) & \xrightarrow{\hat{\mathfrak{K}}_Y} & H^1(Y, \hat{\mathbb{Z}}(1)) \end{array}$$

Rmk This property is important b/c it will show us evaluation of functions at points factors through cohomology.

Group Theoretic Evaluation (cohomological).

- As in the previous subsec. : $Y = \operatorname{Spec} k$.

  & consider $\operatorname{Spec} k \xrightarrow{a} X$.

- This gives

$$\begin{tikzcd} f \in \mathcal{O}^\times(X) \arrow[r, "\hat{\kappa}_X"] \arrow[d, "a^*"] \arrow[rr, phantom] & H^1(X, \hat{\mathbb{Z}}(1)) \arrow[d, "a^*"] \\ f(a) \in k^\times = H^0(k, \mathbb{G}_m) \arrow[r, "\hat{\kappa}_k"] & H^1(k, \hat{\mathbb{Z}}(1)) \overset{\hat{\delta}^{-1}}{\simeq} \widehat{k^\times} \end{tikzcd}$$

$f \mapsto \kappa_f$, $\quad f \mapsto f(a)$, $\quad a^*\kappa_f \in H^1(k, \hat{\mathbb{Z}}(1))$

$\underbrace{f(a) \in k^\times = H^0(k, \mathbb{G}_m)}_{\text{usual eval. of func.}}$

$$\boxed{\hat{\kappa}_k(f(a)) = a^*(\hat{\kappa}_X f) \in H^1(k, \hat{\mathbb{Z}}(1))}$$

↑

relationship between evaluation of pts & kummer classes.

Let's recall

$$\delta_n = \kappa_k^{(n)} : k^\times/(k^\times)^n \xrightarrow{\sim} H^1(k, \mu_n).$$

$$\rightsquigarrow \boxed{\hat{\delta} : \widehat{k^\times} \longrightarrow H^1(k, \hat{\mathbb{Z}}(1))}$$

Putting these together:

$$\boxed{f(a) = \hat{\delta}^{-1}(a^* \hat{\kappa}_X f) \in \widehat{k^\times}}$$

In particular: $f(x) \in k^\times \hookrightarrow \widehat{k^\times}$.

$$k_f \in H^1(\pi_X, \hat{\mathbb{Z}}(1))$$

$$\begin{tikzcd} G_k \arrow[r, bend right, "s_x"'] & \pi_X \arrow[l] \end{tikzcd} \rightsquigarrow \boxed{\begin{array}{ccc} H^1(\pi_X, \hat{\mathbb{Z}}(1)) & \longrightarrow & H^1(G_k, \hat{\mathbb{Z}}(1)) \cong \widehat{k^\times} \\ k_f & \longmapsto & s^* k_f \end{array}}$$

- ⚠ only well-def'd up to conjugacy!
  1) We need to deal with this
  2) eventually, we want to just use
     $\mathrm{im}(s_x) = \underline{\underline{D_{\bar{x}/x}}} \leq \pi_X$, to evaluate the k_f.
     if it is a morph of groups (not outer).

can be given an anabelian interpretation!

③ Recovery of Δ_X from π_X

$\pi_X = \pi_1(X, \bar{x})$

$\bar{x} \in X(\bar{k})$

Recall $X \xrightarrow{f} \operatorname{Spec} k$.

$$\rightsquigarrow \boxed{1 \to \Delta_X \to \underbrace{\pi_X \to G_k}_{\text{"augmentation"}} \to 1} \quad ⊛$$

$\Delta_X = \pi_1(X_{\bar{k}}, \bar{x})$

Thm If k is a field or p-adic field then the sequence ⊛ can be constructed from π_X alone.

AAGHC, AAG1 §2.
↑
"Anabelian geometry of hyperbolic curves".

Number field case

Lemma (Fried-Jarden, Thm 5.10).
Let F be a # field. Every topologically finitely gen'd subgp of G_F which is normal & closed is trivial.

Lemma X curve $/K$.
$$\Delta_X \simeq \Delta_{g,r} = \langle a_i, b_i, c_j \mid \prod_{i=1}^{g} [a_i, b_i] \cdot \prod_{j=1}^{r} c_j = 1 \rangle$$

Lemma (Tamagawa AAGHC-Mochizuki)
Let X/F = variety over a # fld.
Suppose Δ_X is topologically finitely gen'd.
$\Rightarrow$ $\Delta_X = \left(\begin{array}{l} \text{unique max'l closed subgp of } \pi_X \\ \text{which is topologically finitely gen'd.} \end{array} \right)$
$= \bigcap \{ \Delta_0 : \Delta_0 \overset{cl}{\subseteq} \pi_X, \ \Delta \text{ top f.g.} \}$

proof From the exact sequence
$$1 \to \Delta_X \to \pi_X \xrightarrow{p_X} G_K \to 1$$
We get $\Delta_X := \ker(p_X)$.

Let Δ_0 be a max'l, closed, top f.g. subgp of π_X.

By the Lemma, we clearly have $\Delta_0 \subseteq \ker(p_X)$.

(Rmk We need to check the image is closed & normal).

since it's max'l $\Rightarrow \Delta_0 = \Delta_X$. $\square$

<u>Local Case</u>

<u>Lemma</u> Let $K/\mathbb{Q}_p$ be finite. Let $G = G_K$. Let ℓ be a prime.

$$\rightsquigarrow \dim_{\mathbb{Q}_\ell}(G^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_\ell) = \begin{cases} [K:\mathbb{Q}_p]+1, & \ell = p. \\ 1, & \ell \neq p. \end{cases}$$

<u>proof</u> By local class field theory,

$$G_K^{ab} \xrightarrow{\sim} \widehat{K^\times} = \varprojlim_n K^\times/(K^\times)^n.$$

We can now use description of $K^\times$:

- $K^\times = \mathcal{O}_K^\times \cdot \varpi^{\mathbb{Z}}$, ϖ = some uniformizer
- $\mathcal{O}_K^\times \simeq \mu(K)(1+\mathfrak{m})$, $\mathfrak{m} \triangleleft \mathcal{O}_K$ max'l ideal

  $\mu(K)$ = roots of $1 = \tau(k^\times) \simeq k^\times$,

  with k = residual fld $= \mathcal{O}_K/\mathfrak{m}$.
- $1+\mathfrak{m} \xrightarrow{\log} \mathfrak{m} = \varpi\mathcal{O}_K$.

  $$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad |x|<1,$$

  [btw, $\exp(x)$ is an inverse, $|x| < |p|^{1/(p-1)}$].
- $\mathfrak{m} = \varpi\mathcal{O}_K = \varpi(\omega_1\mathbb{Z}_p + \cdots + \omega_d\mathbb{Z}_p)$

  $\omega_1, \cdots, \omega_d$ basis of $\mathcal{O}_K$ as $\mathbb{Z}_p$-module,

  $d = [K:\mathbb{Q}_p]$.

Putting these together:

$$\mathcal{O}_K^\times = \mu(K)\exp(\varpi(\omega_1\mathbb{Z}_p + \cdots + \omega_d\mathbb{Z}_p)).$$
$$\simeq \left(\begin{smallmatrix}\text{some}\\ \text{torsion}\end{smallmatrix}\right) \times \mathbb{Z}_p^d.$$

- Also observe that $\widehat{\mathcal{O}_K^\times} \simeq \mathcal{O}_K^\times$ since
$$\widehat{\text{torsion}} = \text{torsion}, \quad \widehat{\mathbb{Z}_p} = \mathbb{Z}_p.$$
- The thm now follows from
$$\mathbb{Z}_p \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_\ell = \begin{cases} \mathbb{Q}_\ell, & \ell = p \\ 0, & \text{else}. \end{cases}$$
$$\widehat{K^\times} \simeq \hat{\mathbb{Z}} \times \widehat{\mathcal{O}_K^\times} = \underbrace{\hat{\mathbb{Z}}}_{+1} \times (\text{torsion}) \times \underbrace{\mathbb{Z}_p^d}_{d}. \qquad \square$$

<u>Lemma</u> $\mathbb{Z}_p \hat{\otimes} \mathbb{Z}_q = 0, \ p \neq q.$

<u>proof</u>. $a \otimes b = p^n a \otimes p^{-n} b$ i.e. every elt is arbitrarily small.

<u>Note</u> $\hat{\otimes}$ is essential !!

$\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ is funky (and nonzero).

<u>proof</u> $\mathbb{Z} \hookrightarrow \mathbb{Z}_\ell$. $\mathbb{Z}_p \otimes_{\mathbb{Z}} \cdot$ exact since $\mathbb{Z}_p/\mathbb{Z}$ flat.

$$\Rightarrow \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}_p \hookrightarrow \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \in \mathrm{CRing}_{\mathbb{Q}}.$$

<u>UPSHOT</u> Given a profinite group $G \simeq G_K$ one detect the residue characteristic of K.

<u>Def'n</u> Let G be a profinite group.

$$\dim_{\mathbb{Q}_\ell}(G) := \dim_{\mathbb{Q}_\ell}(G^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_\ell)$$
$$\text{"} \dim_\ell(G).$$

<u>Lemma</u> Given an exact sequence of prof groups

$$1 \to \Delta \to \pi \to G \to 1$$

We have $\dim_\ell(\pi) = \dim_\ell(G) + \dim_\ell(\Delta)$.

<u>proof</u> Check abelianazation presere that exact sequence & use dimension formula. $\square$

- Let $G \simeq G_K$, $K/\mathbb{Q}$ finite. We have seen:

$$\text{char } k = p \iff p \text{ is the unique prime where } \dim_p(G) \neq 1.$$
$$(\dim_p(G) = [K:\mathbb{Q}_p] + 1)$$

<u>Ques'n 1</u> How do get char k from π_X?

<u>Ques'n 2</u> How does this help us recover Δ_X?

<u>Theorem</u> (Recovering of Residue char from π_X).

Let $K/\mathbb{Q}_p$ be a f.d, X/K be a curve,

$$\dim_{\ell'}(\pi) - \dim_\ell(\pi) = \begin{cases} 0, & \ell, \ell' \text{ distinct} \neq p \\ [K:\mathbb{Q}_p] & \ell' = p,\ \ell \text{ distinct.} \\ \dots \end{cases}$$

proof of 1st thm

The point is that Δ^{ab} is profinite free

& hence $\dim_\ell(\Delta) = \dim_{\ell'}(\Delta)$ for all ℓ, ℓ'.

$$\dim_p(\pi) - \dim_\ell(\pi) = (\cancel{\dim_p(\Delta)} + \dim_p(G)) - (\cancel{\dim_\ell(\Delta)} + \dim_\ell(G))$$

$$= \dim_p(G) - \dim_\ell(G) = [K : \mathbb{Q}_p].$$

when we look at (ℓ', ℓ) the dimensions are constant because they are so on Δ & G. $\square$.

Some things that help me think about fundamental gps:

$\boxed{\Delta_X = \text{"openish subgroup of } \pi_X \text{ corresp to } X_{\bar{K}} \to X\text{"}.}$

$\boxed{1 \to \Delta_X \to \pi_X \to G_K \to 1}$

1) Can't do $X_{\bar{K}} \to X$.

Correspondence $\{\pi_0 \subseteq \pi_X \text{ open}\} \longleftrightarrow \{\text{fin ét cover } X_0 \to X\}$

2) Δ_X not open, it's closed.

Closedness of Δ_X

proof If it were open

$$\pi_X = \bigcup g\Delta_X = \bigcup_{i=1}^{n} g_i \Delta_X$$

$\Rightarrow \pi_X/\Delta_X$ finite. Contradiction!

this is the absolute Galois group

- $\Delta_X = \ker(\pi_X \xrightarrow{p} G_K) = p^{-1}(1)$

and $\{1\}$ is closed.

To see that $\{1\}$ is closed,

$$\pi : H \xrightarrow[\text{cont.}]{} H_i \qquad H = \varprojlim H_i$$

The kernels K_i of H_i are open.

$$\bigcup (H \setminus K_i) = H \setminus \{1\}$$

i.e. Every elt is eventually contained in a kernel.

$\square$

<u>Approx of $X_{\bar{K}} \to X$</u>

- Consider K_0/K finite ext'n. If we base change

$\operatorname{Spec} K_0 \to \operatorname{Spec} K$ via $X \to \operatorname{Spec} K$ then

étale map $\qquad X_{K_0} \longrightarrow X = X_K$.

- Étale maps corresp. to $\pi_0 := \pi_{X_{K_0}} \underset{op}{\subseteq} \pi_X$.

- Kunneth Formula:

$$\pi_0 = \underbrace{\pi_1(X \times_{\operatorname{Spec} K} \operatorname{Spec} K_0) \simeq \pi_X \times_{G_K} G_{K_0}}$$

- This is saying π_0 is the inverse image of G_{K_0} under p.

<u>Note</u>: $[\pi \times_G G_0 : \pi] = [G : G_0] = \deg(X_0/X)$.

$\pi = \pi_X$
$\pi_0 = \pi_{X_0}$
$G = G_K$

<u>Claim</u> If $[G : G_0] = d$, then $[\pi : p^{-1}(G_0)] = d$.

- If X_0/X Galois, then

$$\pi_X \xrightarrow{q_0} G(X_0/X) \text{ w/ discrete top}$$

cont. & $q^{-1}(1) = \pi_0$ as we vary over the various π_0's:

$\bigcap \pi_0 = $ closed, $\bigcup \pi_0 = $ open.

Arith / Geom. Factorization of Curves

$Z \in \mathrm{Sch}_{\overline{K}}$.

$$\begin{tikzcd}
Z_0 \arrow[dr, "g"] \arrow[dd, "f"', "\text{fin. ét}"] & \boxed{\text{geom}} \\
& Z_{F_0} \arrow[dl, "a"] \\
Z & \boxed{\text{arith}}
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[r] & \Delta_Z \arrow[r] \arrow[d, equal] & \pi_Z \arrow[r, "p"] & G_F \arrow[r] & 1 \\
1 \arrow[r] & \Delta \arrow[r] & \pi_{Z_{F_0}} = p^{-1}(G_0) \arrow[r] & G_0 \arrow[r] \arrow[u, phantom, "\cup\ \mathrm{op}"] & 1 \\
1 \arrow[r] & \Delta_0 \arrow[r] \arrow[u, phantom, "\cup\ \mathrm{op}"] & \pi_{Z_0} \arrow[r] \arrow[u, phantom, "\cup"] & G_0 \arrow[r] \arrow[u, equal] & 1
\end{tikzcd}$$

$\Delta_Z = \Delta$, $\Delta_0 = \pi_{Z_0} \cap \Delta_{Z_0}$, $G_0 = p(\pi_{Z_0})$.

$$\deg f = \deg a \cdot \deg g$$

$$= \boxed{[G_F : G_0]\cdot[\Delta : \Delta_0] = [\pi : \pi_0]}$$

Lemma Let X be a curve $/K$ p-adic field.

Let $\pi_0 \subseteq \pi := \pi_X$.

$$\rightsquigarrow \frac{\dim_p(\pi_0) - \dim_\ell(\pi_0)}{[\pi : \pi_0]} = [K : \mathbb{Q}_p]$$

$$\Updownarrow$$

$\pi_0 = p^{-1}(G_0)$ for some $G_0 \subseteq G$ open.

$$1 \to \Delta_X \to \pi_X \xrightarrow{p} G_K \to 1. \qquad (G = G_K)$$

proof. Suppose $\pi_0 = p^{-1}(G_0)$.

$\Rightarrow \pi_0 = \pi_{X_{K_0}}$, where K/K_0 is a finite ext'n.

$$1 \to \Delta_0 \to \pi_0 \to G_{K_0} \to 1.$$

$\Delta_0 = \Delta_{X_{K_0}}$, $G_{K_0} = G_0$.

From our recovery of char k,

$$\dim_p(\pi_0) - \dim_\ell(\pi_0) = [K_0 : \mathbb{Q}_p].$$

From our claim: $[\pi : \pi_0] = [G : G_0] = [K_0 : K]$

$$\Rightarrow \frac{\dim_p(\pi_0) - \dim_\ell(\pi_0)}{[\pi : \pi_0]} = \frac{[K_0 : \mathbb{Q}_p]}{[K_0 : K]} = [K : \mathbb{Q}_p].$$

Conversely, suppose $\pi_0 \subseteq \pi$ open with

$$\frac{\dim_p(\pi_0) - \dim_\ell(\pi_0)}{[\pi : \pi_0]} = [K : \mathbb{Q}_p].$$

π_0 corresponds to some π_{X_0}. We can use our factorization of covers to get

$$[\pi : \pi_0] = [\Delta : \Delta_0] \cdot [G : G_0].$$

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_{F_0} \arrow[dl] \\ X & \end{tikzcd}$$

$$1 \longrightarrow \Delta_{X_0} \longrightarrow \pi_{X_0} \longrightarrow G_0 \longrightarrow 1.$$
$$\pi_{X_0} = \pi_0$$

$$\dim_p(\pi_0) - \dim_\ell(\pi_0) = [K_0 : \mathbb{Q}_p]$$
$$= [K_0 : K] \cdot [K : \mathbb{Q}_p]$$

putting this into our ass.,

$$\frac{[K_0 : K] \cdot \cancel{[K : \mathbb{Q}_p]}}{[\pi : \pi_0]} = \cancel{[K : \mathbb{Q}_p]} \Rightarrow [K_0 : K] = [\pi : \pi_0].$$

Our factorization $[\cancel{\pi : \pi_0}] = [G : G_0] \cdot [\Delta : \Delta_0]$
$= \cancel{[K : K_0]} \cdot [\Delta : \Delta_0]$

$$\Rightarrow [\Delta : \Delta_0] = 1$$

Hence the cover is completely arithmetic!

base change $\Delta_X \subseteq \pi_{X_0}$ / $\pi_{X_0} = \phi^{-1}(G_0)$. $\square$

We conclude the recovery of Δ_X by intersecting over all of these.

Theorem (Recovery of Δ_X from π_X) (Again).

Let $K/\mathbb{Q}_p$ be a finite ext'n.

Let X/K be curve.

$$\Delta_X = \bigcap \left\{ \pi_0 \overset{\text{clopen}}{\subseteq} \pi_X : \frac{\dim_p(\pi_0) - \dim_\ell(\pi_0)}{[\pi : \pi_0]} = [K:\mathbb{Q}_p] \right\}$$

④ **Section Injection**

SETUP K fld, $\widehat{K^\times} = \varprojlim K^\times/(K^\times)^n$ infinite.

(E.g. $K/\mathbb{Q}$ fin. $K/\mathbb{Q}_p$ fin, K RCF. K DVF).

"real closed field"

Recall X/K scheme, $X_{\bar{K}}$ connected, $\bar{K} = K^{alg}$.

- $\bar{x} \in X(\bar{K})$, $\boxed{1 \to \underset{\overset{\|}{\ker}}{\pi_1(X_{\bar{K}}, \bar{x})} \to \pi_1(X, \bar{x}) \to G_K \to 1}$.

Note

$$\begin{tikzcd} \operatorname{Spec} \bar{K} \arrow[r, "\bar{x}"] \arrow[dr, dashed, "f(\bar{x})"'] & X \arrow[d, "f"] \\ & \operatorname{Spec} K \end{tikzcd} \rightsquigarrow \sigma_{\bar{x}} : K \hookrightarrow \bar{K}$$

We need an embedding to define G_K

& $G_K = \pi_1(\mathrm{Spec}\, k, f(\bar{x}))$

(!)

<u>False statement</u>

Given $x \in X(k)$, $x : \mathrm{Spec}\, k \to X$.

$$\rightsquigarrow \begin{tikzcd} \pi(X) \arrow[r] & G_K \arrow[l, bend left, "S_x"] \end{tikzcd}$$

This doesn't parse as stated.

<u>Hope</u> These sections determine pts for <u>hyperbolic</u> curves. ?

<u>Recall</u> $X \subseteq \bar{X}$ = projective curve of genus g.

$X^c = \bar{X} \setminus X$, $\# X^c = r$.

Def'n: $2g-2+r>0 \iff X$ <u>hyperbolic</u>.

<u>Note</u>: $Y = \mathrm{Spec}\, k$. $\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$

$f \circ g = id_Y$

$\Updownarrow$

g is a section

<u>But</u> we don't get $g \circ f = id_X$!!!

· If $\pi_1(X, \bar{x}) \xrightarrow{f_*} \pi_1(Y, f(\bar{x}))$

$$\begin{tikzcd} \pi_1(X, \bar{x}) \arrow[r, "f_*"] \arrow[dr, phantom, "\not\cong"] & \pi_1(Y, f(\bar{x})) \arrow[d, "g_*"] \\ & \pi_1(X, g \circ f(\bar{x})) \end{tikzcd}$$

to get a map b/w these we need to choose an isom. & deal w/ indeterminacy.

Correct Form:

Given $X \underset{g}{\overset{f}{\rightleftarrows}} Y \ /k$,

$$\rightsquigarrow \pi_1(X, \bar{x}) \xrightarrow{f_*} \pi_1(Y, f(\bar{x}))$$

$[S_g] =$ (conj. class of sections)

$=$ ("polymorphism" of sections).

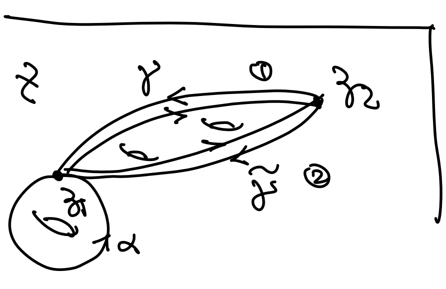
$$\boxed{\begin{array}{l} \underline{\text{Theorem}} \text{ (Section injection).} \\ \quad \text{Suppose } \hat{k}^{\times} \text{ infinite, } X/k,\ g \geq 2. \\ \qquad X(k) \hookrightarrow \dfrac{\Gamma(\pi_1(Y, f(\bar{x})), \pi_1(X, \bar{x}))}{\text{conjugation}} \\ \qquad \ \ x \longmapsto [S_x]. \end{array}}$$

Conjecture X/k hyperbolic, proper

The section map $x \mapsto [S_x]$ is a bijection.

⑤ Conjugacy Indeterminacy & Base points

$Z =$ topological space. $z_1, z_2 \in Z$.



$$\pi_1^{top}(Z, z_1) \xrightarrow{f_\gamma} \pi_1^{top}(Z, z_2).$$

① $f_\gamma(\alpha) = \gamma^{-1} \circ \alpha \circ \gamma$ ★

② $f_{\tilde{\gamma}}(\alpha) = \tilde{\gamma}^{-1} \circ \alpha \circ \tilde{\gamma}$.

Check: $f_{\gamma} = f \circ f_{\tilde{\gamma}} \circ f^{-1}$, $f = \gamma^{-1} \circ \tilde{\gamma}$.

<u>POINT</u> The isomorphism

$$\pi_1(Z, z_1) \longrightarrow \pi_1(Z, z_2)$$

requires a choose of path. This results in

$$\pi_1(Z, z_1) \longrightarrow \pi_1(Z, z_2)$$

only being well-defined up to conj. by an elt of the target.

This same idea works algebraically:

Z scheme, $\bar{z}_1$ closed pt.

$$\pi_1(Z, \bar{z}_1) = \mathrm{Aut}(F_{\bar{z}_1}), \quad \pi_1(Z, \bar{z}_2) = \mathrm{Aut}(F_{\bar{z}_2}).$$

- We relate these by a <u>choice</u> of

$$\gamma \in \pi_1(Z, \bar{z}_1, \bar{z}_2) := \mathrm{Isom}(F_{\bar{z}_1}, F_{\bar{z}_2}).$$

- <u>Note</u>: Any other $\tilde{\gamma}$ is of the form

$$\sigma_2 \circ \tilde{\gamma} \circ \sigma_1 \quad \text{where } \sigma_i \in \mathrm{Aut}(F_{\bar{z}_i}).$$

$$f_{\gamma} : \pi_1(Z, \bar{z}_1) \longrightarrow \pi_1(Z, \bar{z}_2)$$
$$\tau \longmapsto \gamma \circ \tau \circ \gamma^{-1}$$

Now look at $f_{\tilde{\gamma}}$:

$$f_{\tilde{\gamma}}(\tau) = \tilde{\gamma} \circ \tau \circ \tilde{\gamma}^{-1} = (\sigma_2 \gamma \sigma_1) \tau (\sigma_2 \gamma \sigma_1)^{-1}$$
$$= \sigma_2 \gamma \sigma_1 \tau \sigma_1^{-1} \gamma^{-1} \sigma_2^{-1}$$

$$\rightsquigarrow = \sigma_2 f_\gamma({}^{\sigma_1}\tau)\, \sigma_2^{-1} = {}^{\sigma_2}\left({}^{f_\gamma(\sigma_1)} f_\gamma(\tau)\right)$$
$$= {}^{\sigma_2 f_\gamma(\sigma_1)} f_\gamma(\tau).$$

This shows that

1) We need to make a choice when constructing
$$\pi_1(\bar{z}, \bar{z}_1) \longrightarrow \pi_1(\bar{z}, \bar{z}_2).$$

2) Regardless of the choice, this is a well-defined outer isom.

⑥ <u>LCFT on crack</u>

$K/\mathbb{Q}_p$ finite ext'n, k = residue field.

[<u>IDEA</u> $G_K \rightsquigarrow$ invariants of K.]

<u>Open Question</u>. What signature is used to describe all these reconstructions?

<u>Recall</u> $G_K^{ab} \xrightarrow[\sim]{\theta_K} \widehat{K^\times}$. artin map

$$\begin{array}{ccccccccc} 1 & \to & I_K & \to & G_K^{ab} & \to & G_K^{nr} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \widehat{\mathcal{O}_K^\times} & \to & \widehat{K^\times} & \xrightarrow{\text{ord}} & \widehat{\mathbb{Z}} & \to & 1. \end{array}$$

$G_K^{nr} \simeq G_k \simeq \widehat{\mathbb{Z}}$

From G_K we can construct:

$2 \nmid p(G)$

p = residue char $\leftarrow$ unique prime s.t. $\log_p(\#(G^{ab/tor}/\ell G^{ab/tor})) \geq 2$.

f = residue degree $\leftarrow \log_{p(G)}(1 + \#[(G^{ab})_{tors}]^{p(G)}) = f(G)$.

$d = [K : \mathbb{Q}_p]$, $\leftarrow \log_{p(G)}(\#(G^{ab/tor}/p(G).G^{ab/tor})) - 1 = d(G)$.

e = ramification index $\leftarrow d(G)/f(G) = e(G)$.

I = inertia group = $\ker(G_K \to G_k)$

$$J = \{N \triangleleft G \mid \text{normal open}, e(N) = e(G)\}, \quad \bigcap_{N \in J} N = I(G).$$

P = wild inertia. $\leftarrow J = \{N \triangleleft G \mid p(G) \nmid e(G)/e(N)\}$

$$\bigcap_{N \in J} N = P(G).$$

G_k = abs gal of residual. $\leftarrow G/I(G)$

$Fr \in G_k$. $\leftarrow$ Frob(G): unique elt of $G/I(G)$
whose image conj: $G/I(G) \to \mathrm{Aut}(I(G)/P(G))$
is mult. by $p(G)^{f(G)}$.

⑦ Kummer Theoretic Interpretation of $K^\times$

Lemma Let K be a fld.

$$\boxed{H^1(G_K, \hat{\mathbb{Z}}(1)(\bar{K})) \xrightarrow[\delta^{-1}]{\sim} \hat{K}^\times}$$

where $\hat{\mathbb{Z}}(1) = \varprojlim \mu_{n,K}$; $\hat{K}^\times = \varprojlim K^\times/(K^\times)^n$.

Baby version: (Kummer).

$$1 \to \mu_n(\bar{K}) \to \bar{K}^\times \xrightarrow{[n]} \bar{K}^\times \to 1$$

exact sequence of G_K-modules

taking group cohomology:

$$\cdots \to \underset{\overset{\|}{K^\times}}{H^0(G_K, \overline{K}^\times)} \xrightarrow{[n]} \underset{\overset{\|}{K^\times}}{H^0(G_K, \overline{K}^\times)} \xrightarrow{\delta_n} H^1(G_K, \mu_n(\overline{K}))$$

$$\to \underset{\text{Hilbert 90}}{\cancel{H^1(G_K, \overline{K}^\times)}} \to \cdots$$

$$\Rightarrow \underset{\overset{\|}{K^\times/(K^\times)^n}}{\operatorname{coker}([n] : K^\times \to K^\times)} \underset{\sim}{\xrightarrow{\delta_n}} H^1(G_K, \mu_n(\overline{K})).$$

taking limits: $\underset{\overset{\|}{\widehat{K}^\times}}{\varprojlim K^\times/(K^\times)^n} \xrightarrow{\overset{\hat{\delta}}{\|}\ \varprojlim \delta_n} \varprojlim H^1(G_K, \mu_n(\overline{K}))$

$$= H^1(G_K, \varprojlim \mu_n(\overline{K})) = H^1(G_K, \hat{\mathbb{Z}}(1)(\overline{K})).$$

$$\rightsquigarrow \boxed{\hat{\delta}^{-1} : H^1(G_K, \hat{\mathbb{Z}}(1)(\overline{K})) \to \widehat{K}^\times.}$$

8 A statement from Gal sec.

- K mixed char loc fld. or # fld.
- X/K hyperbolic curve of type (g, r).

$$X \xrightarrow{\text{str.}} \operatorname{Spec}(K)$$

$\overline{X}$ = compactification. $X^c = \overline{X} \setminus X$ = cusps. $\bar{x} \in X(\overline{K})$.

$K \hookrightarrow \overline{K}$ (fnd by $\operatorname{str}(\bar{x})$).

$$\rightsquigarrow 1 \to \underset{\overset{\|}{\Delta_X}}{\pi_1(X_{\overline{K}}, \bar{x})} \to \underset{\overset{\|}{\Pi_X}}{\pi_1(X_K, \bar{x})} \to \underset{\overset{\|}{G_K}}{\operatorname{Gal}(\overline{K}/K)} \to 1$$

If we are sloppy about base pts,

$$\pi_1 : \mathrm{Sch}_k \longrightarrow (\text{Grp-SES})'$$

short exact sequence

$$\rightsquigarrow \begin{tikzcd} 1 \arrow[r] & \Delta_X \arrow[r] \arrow[d] & \pi_X \arrow[r] \arrow[d] & G_k \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \Delta_Y \arrow[r] & \pi_Y \arrow[r] & G_k \arrow[r] & 1 \end{tikzcd} \Bigg\} \text{ conj. class of morphism}$$

• (Grp-SES)' **obj** $1 \to G' \to G \to G'' \to 1$

morphs
$$\begin{tikzcd} 1 \arrow[r] & G_1' \arrow[r] \arrow[d, shift left] \arrow[d, shift right, "\alpha"'] & G_1 \arrow[r] \arrow[d, shift left] \arrow[d, shift right, "\beta"'] & G_1'' \arrow[r] \arrow[d, shift left] \arrow[d, shift right, "\gamma"'] & 1 \\ 1 \arrow[r] & G_2' \arrow[r] & G_2 \arrow[r] & G_2'' \arrow[r] & 1 \end{tikzcd}$$
polymorphisms.

(α, β, γ) are G_2'-orbits of these morphs under conj.

$\boxed{\text{Important instance}}$

$$\begin{tikzcd} \mathrm{Spec}\,\bar{k} \arrow[r, "a"] \arrow[dr] & X \arrow[d, "\text{str.}"] \\ & \mathrm{Spec}\,k \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \pi_1(X) \arrow[r] & G_k \arrow[l, bend right, "{[s_a]}"'] \end{tikzcd}$$

$[s_a] = \Delta_X$-(conj. classes of sections)

Note: $\cdot\ D_x \twoheadrightarrow G_{k(x)} \underset{\text{open}}{\subseteq} G_k$, D = decomposition grp

$\cdot\ I_x \simeq \hat{\mathbb{Z}}(1) = \mu_\infty(\bar{k})$ as a G_k-module.

<u>GalSec-Theorem</u> $X \subseteq \bar{X}$ over k # fld or mixed char.

i) $x, x' \in |X|$, $D_{x'} = D_x \Rightarrow x = x'$;

$x, x' \in |X^c|$, $I_{x'} = I_x \Rightarrow x = x'$.

ii) If $x \in |X|$, $\forall H \subseteq D_x$ open, $C_{\pi_X}(H) = D_x$.

If $x \in |X^c|$, $\forall H \subseteq I_x$ open, $C_{\Delta_X}(H) = I_x$. $\quad C \shortparallel$ cyclotome

iii) cuspidal inertia groups are precisely the cyclotomes in Δ_X.

<u>Application</u> This allows us to reconstruct decomposition groups of torsion points on punctured elliptic curves.

⇝ evaluate Kummer classes of torsion pts. (supposing we have them).

<u>proof of</u>: $I_{\tilde{y}_1/y_1} = I_{\tilde{y}_2/y_2} \Rightarrow \tilde{y}_1 = \tilde{y}_2$. $y_i \in |X^c|$.

Need a Lemma:

$$\begin{tikzcd}
\hat{\mathbb{Z}}(1) \arrow[r, "\text{diag}"] & \hat{\mathbb{Z}}(1) \otimes \mathbb{Z} \cdot X^c \arrow[r] & \Delta_X^{ab} \arrow[r] & \Delta_{\bar{X}}^{ab} \arrow[r] & 0 \\
\circlearrowleft G_K & \hat{\mathbb{Z}}(1) \cdot y \arrow[u, phantom, "\cup"] \arrow[r, mapsto] & I^{ab}_{\tilde{y}/y} = I^{ab}_{\tilde{y}'/y} \arrow[u, phantom, "\cup"] & &
\end{tikzcd}$$

from tame inertia character

Pf. If X is of type (g,r) then

$$\Delta_X = \Delta_{g,r} = \langle a_i, b_i, \cancel{c_j} \mid \prod_{i=1}^{g} [a_i, b_i] \cancel{\prod_{j=1}^{r} c_j} = 1 \rangle^\wedge$$

Back to proof: Tamagawa Prop-1.6.

$\downarrow$

Δ_X torsion free $\Rightarrow I_{\tilde{y}_1/y_1} \cap I_{\tilde{y}_2/y_2}$ either infinite or trivial.

Lemma Let $I = \langle g c_j g^{-1} \rangle \triangleleft \Delta_{g,r} = \Delta$.

$\Rightarrow N_\Delta(I) = I$.

proof. • Take $c \in I = \langle g c_j g^{-1} \rangle$

• Belyi $Z_\Delta(I) = I \quad (r \geq 2)$.

• $g \in N_\Delta(I)$, $g c g^{-1} = c^a$, $a \in \hat{\mathbb{Z}}^\times$.

• Taking abelianization $\Rightarrow a = 1 \Rightarrow g \in Z_\Delta(I)$.

• $r = 1$: • $I \subseteq [G, G]$; $g \in N_\Delta(I) = N$,

I, g are contained in an open subgroup of positive index.

[$\exists H$ open of index 2 where this works

$[\Delta : H] = 2 \Rightarrow H \simeq \underbrace{\Delta_{2g-1,2}}$

from Riemann - Hurwitz

then apply previous argument.] $\square$

Claim $N_{\Delta_X}(I_{\tilde{y}/y}) = I_{\tilde{y}/y}$ ← we showed this

Claim $N_{\pi_X}(I_{\tilde{y}/y}) = D_{\tilde{y}/y}$ } for y a cusp.

proof. • $D_{\tilde{y}/y} = \mathrm{stab}_{\pi_X}(\tilde{y})$.

$\gamma \in D_{\tilde{y}/y}$, $\gamma I_{\tilde{y}/y} \gamma^{-1} = I_{\gamma(\tilde{y})/\gamma(y)} = I_{\tilde{y}/y}$.

$\Rightarrow D_{\tilde{y}/y} \subseteq N_{\pi_X}(I_{\tilde{y}/y})$

• suppose $\gamma \in N_{\pi_X}(I_{\tilde{y}/y})$

$$I_{\gamma(\tilde{y})/\gamma(y)} = \gamma I_{\tilde{y}/y} \gamma^{-1} = I_{\tilde{y}/y}.$$

$$\Rightarrow I_{\gamma(\tilde{y})/\gamma(y)} \cap I_{\tilde{y}/y} = 1 \Rightarrow \gamma(\tilde{y}) = \tilde{y}.$$

$$\Rightarrow \gamma \in D_{\tilde{y}/y}. \quad \square$$

Let $\Delta = \Delta_{g,r} = \langle a_i, b_i, c_j \mid \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{r} c_j = 1 \rangle^{\wedge}$ profinite completion.

$I = \langle g c_j^{a} g^{-1} \mid 1 \le j \le r,\ g \in \Delta,\ a \in \hat{\mathbb{Z}} \rangle$

(compact subset)

<u>Recall</u> G profin. group.

$\Phi_p(G) = [G, G] \cdot G^p =$ smallest $N \triangleleft G$ such that G/N is isom. to $\bigoplus \mathbb{Z}/p$.

[<u>p-Frattin</u>: <u>Nakayama's Lemma</u>:

TFAE: $z \in I \iff \forall H \overset{op}{\le} \Delta$, $\forall \ell$ prime.

$\langle z \rangle \cap H \le [H, H] H^{\ell} \langle I \cap H \rangle$.]

$\Phi_\ell(H) \langle I \cap H \rangle =$ smallest normal subgroup w/ quot $\bigoplus \mathbb{Z}/\ell$ of killing inertia.

<u>proof of Nakayama</u>: $G = \Delta$.

<u>easy direction</u>: $z \in I \Rightarrow \forall H \overset{op}{\le} G$, $\forall \ell$ prime,

$\langle z \rangle \cap H \subseteq \Phi_\ell(H) \langle I \cap H \rangle$.

<u>converse</u>: Goal: $z \notin I \Rightarrow !(\forall H \overset{op}{\leq} G, \forall \ell \text{ primes}$

$\langle z \rangle \cap H \subseteq \Phi_\ell(H) \langle I \cap H \rangle)$.

$\Updownarrow$

$\exists H \overset{op}{\leq} G, \exists \ell \text{ prime}, \langle z \rangle \cap H \not\subseteq \Phi_\ell(H) \langle I \cap H \rangle$.

<u>Step 1</u> Reduce the problem to a cover,

i.e. $N \underset{op}{\subseteq} G$ where we know entire conj. classes

& hence cusps are not fixed

by a particular $z \notin I$.

<u>Step 2</u> Choices.

<u>Step 3</u> Show choices work.

<u>Step 1</u> :. Suppose $z \notin I$. Let C_i = conj cl of c_i

$C = \bigcup_{i=1}^{r} C_i$.

- By normal terminality,

  $g \langle c_i \rangle g^{-1}$ is normally terminal.

- $z \notin I$ has the property $\text{conj}_z \in \text{Aut}(C)$

  acts fixed pt freely on C.

- Compactness $\Rightarrow$ conj_z acts freely after

  passing to $C / N \cap C$ for some $N \triangleleft C$.

- Fix $Y \to X$ corresponding to $N \triangleleft G$. $G = \pi_1(X_{\bar{K}})$.

- Let Y^c = cusps of Y.

- $\forall y \in Y^c$, cover $x \in X^c$ the $I_{y/x} = I_{\tilde{y}/x}/N$ are finite cyclic. Also they are not fixed by conj $\hat{z}$.
- "Not fixed by" $\Leftrightarrow$ $z N \notin I_{y/x} = I_{\tilde{x}/x}/N$

  $\Leftrightarrow$ z does not fix any cusps.

<u>Notation</u>
- $w \in (\langle z \rangle \cap N)$ not identity.
- $I' = I \cap N$.
- ℓ rational prime s.t. $\langle w^\ell \rangle \subsetneq \langle w \rangle$.
- $B \leq N$ open such that $B \cap \langle w \rangle = \langle w^\ell \rangle$.
- $H = M\langle w \rangle$ (apply to $G = N$).

exists by a Lemma of Lubotzky

<u>Step 2</u> Choices.
- B s.t. $\langle w \rangle \cap B = \langle w^\ell \rangle$,
- $M \leq B$ s.t. $I' \cap B$, $\langle w \rangle \cap B$ disjoint mod M.
- $H = M\langle w \rangle$ (applied to G repl N).

<u>Step 3</u> Show $w \in I' \Rightarrow \langle w \rangle \cap H \nsubseteq \Phi_\ell(H)\langle I' \cap H \rangle$.
- $I' \cap H \leq M\langle w^\ell \rangle$.
- $\Phi_\ell(H) \leq M\langle w^\ell \rangle$.
- $w \notin M\langle w^\ell \rangle$ $(w \notin B)$. $\square$

<u>Nakayama characters</u>

$$1 \to \Delta_X \to \pi_X \xrightarrow{p} G_K \to 1$$

X/K curve. $J \lhd \Delta_X$ normally terminal, cyclic

$N = N_{\pi_X}(J)$, $N \cap \Delta_X = J$.

$$\chi_J : p(N) \simeq N/J \longrightarrow \mathrm{Aut}(J).$$

$$\boxed{\chi_J(g)(w) = \tilde{g}\, w\, \tilde{g}^{-1}}$$

$w \in J$, $g \in p(N)$, $\tilde{g} \in p^{-1}(g) \cap N$.

<u>Def</u> $J \triangleleft \Delta_X$ is a <u>cyclotome</u> (or max'l cyclic subgroup of cyclotomic type)

iff (1) $p(N_{\pi_X}(J)) \underset{op}{\subseteq} G_K$

(2) The associated χ_J is the cyclotomic character:

$\forall g \in p(N_{\pi_X}(J))$, $\forall \zeta$ root of 1.

$$\zeta^{\chi_J(g)} = g(\zeta) = \zeta^{\chi_{cyclo}(g)}.$$

<u>Easy Direction</u>

$y \in X^c$ cusp, I_y is a cyclotome.

(1) $\Leftarrow$ Normal Terminality

(2) $\Leftarrow$ Tame Inertia Character.

<u>Hard Direction</u>

Every cyclotome in Δ_X is I_y for some $y \in X^c$.

- Weight - Monodromy filtration + extra crap.

<u>Theorem</u> (Characterization of Cuspidal Inertia as Cyclotomes for # flds).

Let X/K be a curve over a number field.

Let $I \subseteq \Delta_X := \pi_1(X_{\bar{k}}, \bar{x})$.

I is a cyclotome $\iff$ $\exists x \in X^c = \bar{X} \setminus X$, $\tilde{x} \in \tilde{X}$ lifting x s.t. $I = I_{\tilde{x}/x}$.

proof. ($\Leftarrow$) If x is a cusp, $I_{\tilde{x}/x}$ is a cyclotome for G_k by the tame inertia char.

($\Rightarrow$) Suppose $\langle z \rangle$ is a cyclotome in Δ_X.
Suppose $\langle z \rangle$ is not a cuspidal inertia group.

$\underset{\Longrightarrow}{\text{Nakayama}}$ $\exists\, \ell$ prime, $\exists H \underset{op}{\subseteq} \Delta_X$, $w \in H \cap \langle z \rangle$.

such that $\boxed{w \notin [H, H] \cdot H^{\ell^\infty} \langle \underline{I} \cap H \rangle}$ $(*)$

$\underline{\text{Leak}}$ This will lead to a contradiction later.

$\underline{I}$: group gen'd by all conj. of all inertias.

Let $\tilde{H} = N_{\pi_X}(H)$ and $\tilde{k}$ be the field corresponding to the image in G_k.

Let $Y \to X$ be the cover corresponding to $\tilde{H}$ (i.e. $\tilde{H} = \pi_Y$).

From the theory of weight we have an exact seq'ce of $G_{\tilde{k}}$-modules

$$0 \to \mathbb{Z}_\ell(1) \to \mathbb{Z}_\ell(1) \otimes \bigoplus_{y \in Y^c} \mathbb{Z}_\ell \cdot y \to T_\ell(Y) := \Delta_Y^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Z}_\ell \to T_\ell(\bar{Y}) \to 0$$

$$T_\ell(\bar{Y}) = \Delta_{\bar{Y}}^{ab} \otimes_{\hat{\mathbb{Z}}} \mathbb{Z}_\ell$$

- This sequence is really related to the wt filtration.

More precisely if we restrict to a local field and tensor up to $\mathbb{Q}_\ell$ we get the following:

If $V_\ell(y) = T_\ell(y) \otimes \mathbb{Q}_\ell$

$V_\ell(\bar{y}) = T_\ell(\bar{y}) \otimes \mathbb{Q}_\ell$,

$I_\ell = \operatorname{coker} g_0$. $J_\ell = I_\ell \otimes \mathbb{Q}_\ell$

Then the sequence

$$0 \to J_\ell \to V_\ell(y) \to V_\ell(\bar{y}) \to 0$$

is really the weight filtration sequence.

- $V_\ell(y)^{(-2)} = J_\ell$.

$Gr^w_{-1}(V_\ell(y)) = V_\ell(y)^{(-1)}/V_\ell(y) \simeq V_\ell(\bar{y})$ (pure of wt -1).

- $Gr^w_{-2}(V_\ell(y)) = V_\ell(y)^{(-2)}/\cancel{V_\ell(y)^{(-3)}} = V_\ell(y)^{(-2)}$.

Using that $\langle z \rangle$ is a cyclotome implies

$$\rho_\ell(g)(\bar{\omega}) = \bar{\omega}^{\chi_\ell(g)}, \quad g \in G_{\tilde{K}}$$

we find that $\langle \bar{\omega} \rangle \simeq \mathbb{Z}_\ell(1)$.

So a $G_{\tilde{K}_v}$-module H has wt -2

$\Rightarrow \bar{\omega} \in I_\ell = J_\ell \cap T_\ell(y)$.

But this is a contradiction.

b/c condition $(*)$ is exactly the characterization of the integral wt -2 subspace of $V_\ell(y)$. $\square$

⑨ Torsion & The Fundamental Group

Let $\bar{X}/K$ be an elliptic curve.

Let X/K be the punctured curve. $\bar{X} - \{0\} = X$.

<u>Lemma</u> $\boxed{\bar{X}_{\bar{K}}(\bar{K})[\ell] \xrightarrow{\sim} \pi(X_{\bar{K}})^{ab}/\ell}$ $\qquad$ K char 0.

proof. $\bar{X}_{\bar{K}}[\ell](\bar{K}) \cong T_\ell(\bar{X}_{\bar{K}})/\ell \cong \pi_1(\bar{X}_{\bar{K}})^{ab}/\ell \cong \pi_1(X_{\bar{K}})^{ab}/\ell$.

We used: $\cdot\ \pi_1(\bar{X}_{\bar{K}}) \simeq T(\bar{X}_{\bar{K}}) = \prod_p T_p(\bar{X}_{\bar{K}})$,

$\cdot\ T(\bar{X}_K)/\ell = \bar{X}_{\bar{K}}(\bar{K})[\ell]$.

$\cdot\ \pi_1(\bar{X}_{\bar{K}})^{ab} = \pi_1(X_{\bar{K}})^{ab}$. $\qquad \square$

<u>Lemma</u> $\pi_1(\bar{X}_{\bar{K}}) \simeq T(X_{\bar{K}})$ as G_K-modules.

<u>Recall</u> $\cdot\ T(\bar{X}_{\bar{K}}) = \varprojlim \bar{X}_{\bar{K}}(\bar{K})[n]$
$\qquad\qquad (a_n)_{n\in\mathbb{N}},\ d a_n = a_{d|n},\ d|n$.

$\cdot$ The action of G_K on $\pi_1(\bar{X}_{\bar{K}})$:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\bar{X}_{\bar{K}}) \arrow[r] \arrow[d, "\text{conj.}"'] & \pi_1(\bar{X}_K) \arrow[r] \arrow[d, "\text{conj.}"'] & G_K \arrow[r] & 1 \\
1 \arrow[r] & \mathrm{Inn}(\pi_1(\bar{X}_{\bar{K}})) \arrow[r] & \mathrm{Aut}(\pi_1(\bar{X}_{\bar{K}})) & &
\end{tikzcd}$$

$\rightsquigarrow G_K \simeq \pi_1(\bar{X}_K)/\pi_1(\bar{X}_{\bar{K}}) \longrightarrow \mathrm{Out}(\pi_1(\bar{X}_{\bar{K}}))$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| $
$\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathrm{Aut}/\mathrm{Inn}$.

proof. Since every elliptic curve is p.p.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ principally polarized

$$E \simeq E^t \xrightarrow{f^t} (E')^t \simeq E' \xrightarrow{f} E$$
with the composite $E \to E$ equal to $[\deg(f)]$, $\Rightarrow$ $[n]$ $(n \in \mathbb{N})$ are cofinal system of étale cover.

- $\bar{x}$, $\pi_1(E, \bar{x}) := \mathrm{Aut}(F_{\bar{x}})$

  $F_{\bar{x}}([n] : E \to E) = [n]^{-1}(\bar{x})$.

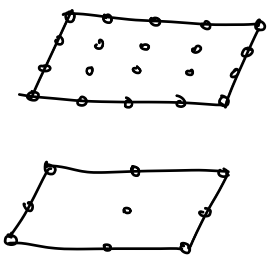
  and when $\bar{x} = \bar{0}$, $[n]^{-1}(\bar{x}) = E[n](\bar{k})$.

This gives a bijection of sets

$$E[n](\bar{k}) = [n]^{-1}(\bar{x}).$$

- Check transition maps perform as advertised.

$$\begin{tikzcd} E \arrow[d] \arrow[dd, bend left, "{[nm]}"] \\ E \arrow[d, "{[n]}"'] \\ E \end{tikzcd}$$



$[n]^{-1}(\bar{x}) \curvearrowleft E[n](\bar{k})$
$\shortparallel$
translate of $E[n](\bar{k})$.

$\mathrm{Aut}(F_{\bar{x}}) \ni \left(\begin{array}{c} \text{a system of torsion pts that} \\ \text{acts on each } [n]^{-1}(\bar{x}) \text{ as} \\ \text{the Tate residue.} \end{array} \right)$.

<u>Lemma.</u> $E = \bar{X}_{\bar{k}}$. $T(E)/\ell^n = E(\bar{k})[\ell^n]$.

proof. • $T(E) = \prod_p T_p(E)$

$T_p(E) = \varprojlim E(\bar{k})[p^n]$.

- Multiplication by ℓ on $E(\bar{k})[p^n]$ is surjective

  hence $T_p(E)/\ell^n T_p(E) = 0$.

- It remains to show $T_\ell(E)/\ell^n = E[\ell^n](\bar{K})$.
- General statement: M abelian grp.

  1) $M[\ell^{a+b}]/M[\ell^a] \xrightarrow[\sim]{[\ell^a]} M[\ell^b]$.

  2) $\ell^b M[\ell^{a+b}] = M[\ell^a]$.

We apply this to $M = E(\bar{K})$.

$\rightsquigarrow M[\ell^{a+b}]/\ell^b M[\ell^{a+b}]$

$= M[\ell^{a+b}]/M[\ell^a] \xrightarrow[\sim]{[\ell^a]} M[\ell^b]$.

$$\underbrace{E(\bar{K})[\ell^N]/\ell^b E(\bar{K})[\ell^N] \simeq E(\bar{K})[\ell^b]}. \qquad N = a+b$$

taking a limit of this gives the claim. □

$\boxed{\begin{array}{l} \underline{\text{General Idea}} \quad Z \subseteq \bar{Z} \text{ curves/ alg. closed fld.} \\ \Rightarrow \ 1 \to \underbrace{I}_{\substack{\text{grp generated by} \\ \text{inertia groups.}}} \to \pi_1(Z) \longrightarrow \pi_1(\bar{Z}) \longrightarrow 1. \end{array}}$

<u>Lemma</u> Every abelian cover of a punctured elliptic curve over an alg. closed fld is <u>unramified</u>,

i.e. $\pi_1(X_{\bar{K}})^{ab} = \pi_1(\bar{X}_{\bar{K}})^{ab}$.

Proof. We have an explicit description of the fundamental groups:

$$\pi_1(X_{\overline{K}}) \simeq \langle a, b, c \mid aba^{-1}b^{-1}c = 1 \rangle^{\wedge}$$

for single puncture

$$\pi_1(\overline{X}_{\overline{K}}) \simeq \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle^{\wedge}$$

Note that

$$(aba^{-1}b^{-1})^{ab} (c)^{ab} = 1 \Rightarrow c^{ab} = 1.$$

$$\Rightarrow \pi_1(X_{\overline{K}})^{ab} \simeq \pi_1(\overline{X}_{\overline{K}}) = \pi_1(\overline{X}_{\overline{K}})^{ab}. \qquad \square$$

(10) Mochizuki's Recovery of Inertia Groups of Cusps

Lemma • If $r \geq 1$, then

$$\Delta_{g,r} = \langle a_i, b_i, c_j \mid \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{r} c_j = 1 \rangle$$

is isomorphic to F_{2g+r-1}

$\qquad \| $

the free group on $2g+r-1$ generators

• If $r = 0$, then

$$\Delta_{g,r} \simeq (\text{some free group})$$

proof. $\Delta_{g,r} = F_{2g+r}/R$.

$$F_{2g+r} = \langle a_1, b_1, \cdots, a_g, b_g, c_1, \cdots, c_r \rangle$$

$$R = \langle \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{r} c_j = 1 \rangle_{F_{2g+r}}.$$

If we get $A = \langle a_1, b_1, \cdots, a_g, b_g \rangle$

$C = \langle c_1, \cdots, c_{r-1} \rangle$.

We claim $F_{2g+r-1} \cong \langle A, C \rangle \longrightarrow \Delta_{g,r}$.

is an isom.

- The map is surjective since we can solve for c_r using the relation.
- Remains to show the map is injective.

Suppose $\exists\, w \in \langle A, C\rangle$ such that $w \in R$.

$$\leadsto \quad w = \prod_{i=1}^{N} w_i r_0 w_i^{-1} \in R.$$

$$r_0 = \prod_{i=1}^{g} [a_i, b_i] \cdot \prod_{j=1}^{r} c_j$$

consider the map $f: F_{2g+r} \longrightarrow \langle c_r \rangle$

which maps $a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_{r-1}$ to 1.

- $f(w) = 1$, since $w \in \langle A, C\rangle$.
- $f(w) = f\left(\prod_{i=1}^{N} w_i r_0 w_i^{-1}\right) = \prod_{i=1}^{N} \cancel{f(w_i)} \cdot c_r \cdot \cancel{f(w_i)^{-1}} = c_r^N.$

$\Rightarrow N = 0$ and hence $w = 1$.

This proves injectivity. $\square$

$\boxed{\text{Abelian Recover of \# of punctures}}$ AAGHC

Statement (1) If X/k curve of type (g, r),

$k/\mathbb{Q}$ finite ext'n,

$$\boxed{r = \dim_{\mathbb{Q}_\ell}(V_\ell^{(-2)}) - \dim_{\mathbb{Q}_\ell}(V_\ell^{(0)})}$$

where $V_\ell := \Delta_X^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$ and

$V_\ell^{(w)} :=$ (pure component of wt w),

(2) The expression on the RHS of $(*)$ is anabelian.

proof. (1) Let $\overline{X}$ be the compactification of X,

$$\overline{V}_\ell := \Delta_{\overline{X}}^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell,$$

$$I_\ell = \mathbb{Q}_\ell(1) \otimes \Big[\bigoplus_{x \in X^c} \mathbb{Z}_x \Big] \qquad \boxed{\dim_{\mathbb{Q}_\ell}(I_\ell) = r}$$

Nakayama

$$0 \to I_\ell \to V_\ell \to \overline{V}_\ell \to 0$$

is an exact sequence of G_π-modules

$$\rightsquigarrow \quad 0 \to I_\ell^{(w)} \to V_\ell^{(w)} \to \overline{V}_\ell^{(w)} \to 0$$

$$\begin{aligned}
\rightsquigarrow \quad \dim_{\mathbb{Q}_\ell}(I_\ell) &= \dim_{\mathbb{Q}_\ell}(I_\ell^{(-2)}) \\
&= \dim_{\mathbb{Q}_\ell}(V_\ell^{(-2)}) - \dim_{\mathbb{Q}_\ell}(\overline{V}_\ell^{(-2)}) \\
&= \dim_{\mathbb{Q}_\ell}(V_\ell^{(-2)}) - \dim_{\mathbb{Q}_\ell}(\overline{V}_\ell^{(0)}) \\
&= \dim_{\mathbb{Q}_\ell}(V_\ell^{(-2)}) - \dim_{\mathbb{Q}_\ell}(V_\ell^{(0)}).
\end{aligned}$$

principally polarized abelian variety

X PPAV, $T_\ell(X^t) \simeq T_\ell(X)^\vee(1) \implies T_\ell(X)(-1) \simeq T_\ell(X)$

$\quad\quad\quad\;\; \| $

$\quad\quad T_\ell(X) \quad X^t = $ dual AV.

---

(2) $\circ$ $\Delta_X \rightsquigarrow 1 \to \underset{=\Delta}{\Delta_X} \to \underset{=\pi}{\pi_X} \to G \to 1$, $\quad G = \pi/\Delta$

- $\mathrm{Frob}(G)$, + representation on Δ_X.
  - + $q = p^f$.

Makes sense to talk about wt decomposition of V_ℓ. $\square$

$\boxed{\text{Recovery of inertia groups at cusp (local case)}}$

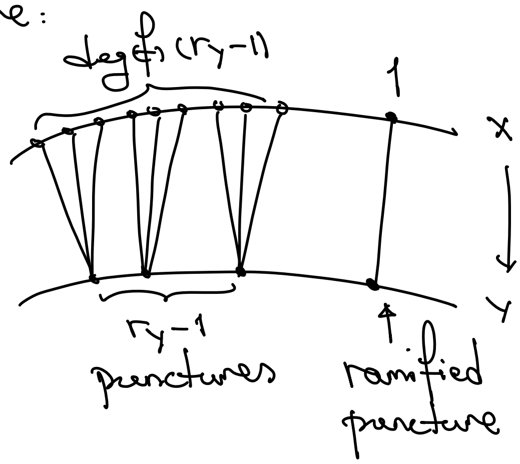
Lemma Let $k/\mathbb{Q}_p$ be a fld.

Let $f: X \to Y$ be a morphism of curves of type (g_X, r_X) & (g_Y, r_Y).

The map $\bar{f}: \overline{X} \to \overline{Y}$ is totally ramified at a single pt. & unramified elsewhere iff

$$\boxed{r_X = \deg(f)(r_Y - 1) + 1.}$$

proof by picture:



$\square$

⑪ What is the ABC Conjecture

Conjecture (Oesterlé - Masser)

$\forall \varepsilon > 0$, $\exists C_\varepsilon$ such that for all triples $(a,b,c) \in \mathbb{Z}^3$, a,b,c coprimes such that

$$a + b = c.$$

We have

$$\max\{|a|, |b|, |c|\} \leq C_\varepsilon \cdot \mathrm{rad}(abc)^{1+\varepsilon}.$$

Def $N \in \mathbb{Z}$, $\mathrm{rad}(N) = \prod_{p \mid N} p$. E.g. $\mathrm{rad}(2^3 \cdot 5^4) = 2 \cdot 5$

$\boxed{\text{Variants}}$ "Weak Form":

for a single fixed ε_0 (i.e. $\exists \varepsilon_0, \exists C_{\varepsilon_0}$).

"Explicit weak Form":

for a single fixed ε_0 & C_{ε_0} is specified.

example (explicit form $\varepsilon_0 = 1$)

$C_{\varepsilon_0} = 1$. $\max(|a|, |b|, |c|) \le \mathrm{rad}(abc)^2$.

If $a, b > 0$, $\boxed{a+b \le \mathrm{rad}(ab(a+b))^2}$. ← rigidity.

$\boxed{\text{Consequences of ABC}}$

prime factorization $a = p^\alpha$, $b = q^\beta$.

where p, q are two distinct primes, $\alpha, \beta \in \mathbb{N}$.

$$p^\alpha + q^\beta = \mathrm{rad}(p^\alpha q^\beta (p^\alpha + q^\beta))^2$$
$$\le p^2 q^2 \underbrace{\mathrm{rad}(p^\alpha + q^\beta)}$$

needs to have large prime factors

or no prime power factors.

sub example $p = 2$, $q = 3$, $\alpha = \beta = 10$.

$\Rightarrow 2^{10} + 3^{10} = 60073 = 13 \cdot 4621$ ← This is the large prime factor of $2^{10} + 3^{10}$ that ABC predicts.

Fermat's Last Thm | ABC weak explicit form $\varepsilon_0 = 1 \Rightarrow$ FERMAT.

Let (x, y, z) be coprime & satisfy $x^n + y^n = z^n$.

By the weak form of ABC w/ $\varepsilon_0 = 1$ & $C_{\varepsilon_0} = 1$.

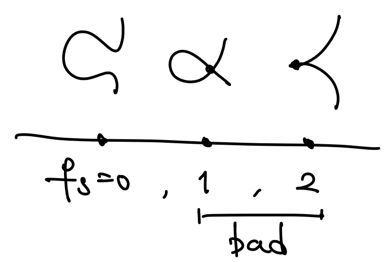
we have $z^n \leq \mathrm{rad}(x^n y^n z^n)^2 = (xyz)^2 \leq z^6$

$\Rightarrow n \leq 6$. proof for small n, show no such solution exists for $n \geq 2$.

⑬ Szpiro's Conjecture

SETUP: S 1-dim'l scheme, normal, irred.

$K = K(S)$, E elliptic over K.



Def (Divisor form of conductor).

$$\mathrm{Cond}^*(E/K) = \sum_{s \in |S|} f_s \cdot [s] \in \mathrm{Div}(S)$$

$$f_s = \begin{matrix}\text{conductor}\\ \text{exponent}\end{matrix} = \begin{cases} 0, & E \text{ has good red. at } s \\ 1, & E \text{ has semistable red. at } s. \\ \geq 2, & \text{else} \end{cases}$$

Remarks on Conductors

- S geometric (S = curve over $k = k^a$)

  $f_s \leq 2$ (at worst additive red.)

- S arith Lots of ways to define f_s

  ★ Ogg Formula:

  $$\mathrm{ord}(\Delta^{\min}) = f_s + \begin{pmatrix}\#\text{ irred comps}\\ \text{of reduction}\end{pmatrix} - 1.$$

  related to Néron models

  ★ Artin conductor of Gal on the p-torsion (at $\mathfrak{p}$)

General Idea of Szpiro Inequalities:

" Δ_E^{min} divisor $\leq$ (ε, k) + $c(6+\varepsilon)$ cond divisor".

sth. bdd

<u>Arithmetic Form of Szpiro</u>

- $K = \kappa(S) = \#$ fld, $S = \operatorname{Spec} \mathcal{O}_K$.

  $\operatorname{Cond}(E/K) := \prod_{\mathfrak{p}} \mathfrak{p}^{f_{\mathfrak{p}}} \triangle \mathcal{O}_K$

- $K = \mathbb{Q}$,

  $\operatorname{Cond}(E/\mathbb{Q}) = \prod_p p^{f_p} \in \mathbb{Z}$.

$$\boxed{\begin{array}{l}\underline{\text{Szpiro's Conjecture}} \quad \forall \varepsilon, \exists C_\varepsilon, \forall E/\mathbb{Q}, \\ \qquad \max(|\Delta_E^{min}|, |c_4|^3) \leq C_\varepsilon \cdot \operatorname{cond}(E/\mathbb{Q})^{1+\varepsilon}.\end{array}}$$

What do I want to show?

Szpiro $\Rightarrow$ Weak Form of ABC.

Szpiro is connected to ABC through Frey.

<u>Lemma</u> (Frey) Let (a, b, c) be coprime integers with

$$a + b = c.$$

Define the Weierstrass form

$$\mathcal{E}_{a,b,c} : \underbrace{y^2 = x(x+a)(x-b)}_{\text{FREY CURVE}}$$

We can explicitly compute the minimal Weierstrass model of $\mathcal{E}_{a,b,c}$ ...

I) $16 \mid abc$ $\quad a \equiv 1 \pmod 4$ $\quad \rightsquigarrow \quad (y')^2 + x'y' = (x')^3 + \frac{a-b-1}{4}(x')^2 - \frac{ab}{16}x'$

$\underbrace{b \equiv 0 \pmod{16}}_{\text{WLOG}}$ $\quad c_4' = a^2 + ab + b^2$

$\Delta' = (abc)^2/2^8.$

II) $16 \nmid abc$ $\quad y^2 = x^3 + (a-b)x^2 - abx$

$c_4 = 16(a^2 + b^2 + ab)$

$\Delta = 16(abc)^2.$

<u>proof idea</u>: In the case $16 \mid abc$ we transform

$$y^2 = x(x+a)(x-b)$$

via, $\begin{cases} x = 4x', \\ y = 8y' + 4x' \end{cases}$, then use the formulas for disc:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

$c_4' = b_2^2 - 24b_4$

$\Delta = \dots$

$b_2 = a_1^2 + 4a_2, \quad b_4 = a_1a_3 + 2a_4,$

$b_6 = a_3^2 + 4a_6, \quad b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 - a_2a_3^2 - a_4^2.$

evaluate what happens after an admissible change of variables.

<u>proof that Szpiro $\Rightarrow$ Weak ABC:</u>

<u>$16 \nmid abc$</u>: $c_4 = 16(a^2 + ab + b^2)$, $\Delta = 16(abc)^2$

$$\begin{aligned} \max(|a|,|b|,|c|) &\le \max(|c_4|^3, |\Delta|) \\ &= \max((16(a^2+ab+b^2))^3, 16(abc)^2) \\ &\underset{\text{Szp}}{\le} C_\varepsilon \cdot \mathrm{Cond}(E)^{6+\varepsilon} \\ &\le C_\varepsilon 2^{f_2(6+\varepsilon)} \mathrm{rad}(abc)^{6+\varepsilon} \end{aligned}$$

the last step is b/c

$$\mathrm{Cond}(E) = 2^{f_2} \prod_{p \mid \Delta, p \neq 2} p \leq 2^{f_2} \cdot \prod_{p \mid abc} p = 2^{f_2} \mathrm{rad}(abc).$$

from eq'n $f_p = 1,\ p \neq 2$

$\rightarrow f_2 \leq 8 + 7 \cdot 2 = 22$

- $\mathrm{Cond}(E/\mathbb{Q}) \mid (\Delta_E = 16(abc)^2)$,

$16 \nmid abc$

This tells us:

$$\max(|a|, |b|, |c|) \leq C_\varepsilon \cdot 2^{22(6+\varepsilon)} \cdot \mathrm{rad}(abc)^{6+\varepsilon}$$

If we take $\varepsilon = 1$, we get a form of ABC for $\varepsilon_0 = 6$.

<u>$16 \mid abc$</u>: $c_4' = a^2 + ab + b^2$, $\Delta' = 2^{-8}(abc)^2$.

$$\max(|c_4'|^3, |\Delta|) = \max\left(|a^2+ab+b^2|^3, \left|\frac{abc}{2^8}\right|\right).$$

$$\geq \max(|a|, |b|, |c|).$$

From the form of the eqn (+ work) the Weierstrass form has multiplicative red.

$\forall p,\ f_p = 1 \Rightarrow \mathrm{Cond}(E) = \mathrm{rad}(\Delta_E)$

$$= \prod_{p \mid \frac{(abc)^2}{2^8}} p \leq \mathrm{rad}(abc).$$

$$\max(|a|, |b|, |c|) \leq \max\left(|a^2+ab+b^2|^3, \left|\frac{abc}{2^8}\right|\right).$$

$$\leq C_\varepsilon\, \mathrm{Cond}(E/\mathbb{Q})^{6+\varepsilon}$$

$$\overset{\text{Szp}}{\leq} C_\varepsilon \cdot \mathrm{rad}(abc)^{6+\varepsilon}.$$

So plugging in $\varepsilon = \varepsilon_0$ gives our weak ABC. □

(14) Weierstrass Models of Elliptic Curves

<u>Def</u> An elliptic curve is a marked curve of genus 1.

$$\begin{tikzcd} E \arrow[d, "\pi"', "\text{proper}"] \\ S \arrow[u, bend right, "e = \text{``zero section''}"'] \end{tikzcd}$$

<u>Theorem</u> Every elliptic curve E/K (K fld) has a "Werstrass Model"

- Some embedding $E \subseteq \mathbb{P}^2_K$

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

[projective model $[X, Y, Z]$, $x = X/Z$, $y = Y/Z$]

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

[so that the marked point becomes $[0,1,0]$ in this model.

moreover, such an equation is unique up to

$$x = u^2x' + r$$
$$y = u^3y' + u^3sx' + t, \qquad u \in K^\times,\ s, t, r \in K. \ ]$$

<u>proof idea</u>

Riemann-Roch (using $\mathcal{O}_E(6(\text{pt at } \infty))$).

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

<u>INVARIANTS</u>

$$\left[\begin{array}{ll} b_2 = a_1^2 + 4a_2, & b_6 = a_3^2 + 4a_6 \\ b_4 = 2a_4 + a_1a_3, & b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 \end{array}\right.$$

$$\rightsquigarrow \begin{cases} c_4 = b_2^2 - 24 b_4 \\ c_6 = -b_2^3 + 36 b_2 b_4 - 216 b_6 \end{cases}$$

Δ = discriminant $= -b_2^2 b_8 - 8 b_4^3 - 27 b_6^2 + 9 b_2 b_4 b_6$

$j = c_4^3 / \Delta$ = j-invariant.

ω = invariant diff $= \dfrac{dx}{2y + a_1 x + a_3} = \dfrac{dy}{3x^2 + 2a_2 x + a_4 - a_1 y}$

If char $k \neq 2$: $y = \frac{1}{2}(\tilde{y} - a_1 \tilde{x} - a_3)$,

$x = \tilde{x}$

$\rightsquigarrow \tilde{y} = 4\tilde{x}^3 + b_2 \tilde{x}^2 + 2b_4 \tilde{x} + b_6$.

If char $k \neq 2 \& 3$: $y' = \tilde{y}/108$

$x' = (\tilde{x} - 3b_2)/36$

$\rightsquigarrow (y')^2 = (x')^3 - 27 c_4 x - 54 c_6$.

<u>Invariants have relations</u>

$$\begin{cases} 4 b_8 = b_2 b_6 - b_4^2 \\ 1728 \Delta = c_4^3 - c_6^2 \end{cases}$$

<u>Basic Case</u>: char $k \neq 2, 3$. $y^2 = x^3 + Ax + B$.

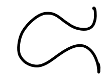
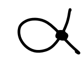
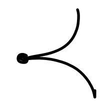
$\Rightarrow \Delta = -16(4A^3 + 27B^2)$

$j = -1728 (4A)^3 / \Delta$.

<u>Thm</u> A The curve

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

is classified as follows

Δ	c_4	name	picture
$\neq 0$	$\cdots$	nonsingular	
$= 0$	$\neq 0$	node	
	$= 0$	cusp	

B) two elliptic curves w/ same j-invariant are isom. over $\bar{K}$. $(\Leftrightarrow)$.

$\forall E/K$, K global fld. $\forall v \in \mathrm{Val}(K)_{fin} \simeq |\mathrm{Spec}\, \mathcal{O}_K| \ni \mathfrak{p}_v$

$\exists$ a Weierstrass model

$$y_v^2 + a_{1,v} x_v y_v + a_{3,v} y_v = x_v^3 + a_{2,v} x_v^2 + a_{4,v} x_v + a_{6,v}.$$

$b_2 = a_1^2 + 4a_2$, $b_4 = 2a_4^2 + 4a_2$,

$b_6 = a_3^2 + 4a_6$, $b_8 = a_2^2 a_6 + 4a_3 a_6 - a_1 a_3 a_4$

We can arrange so this at last a cusp. ($p \neq 2$).

$$\boxed{\Delta_v = -b_2 b_8 - 8b_4^3 - 27b_6 + 9 b_2 b_4 b_6}$$

This eq'n is <u>minimal</u> if $\mathrm{ord}_v(\Delta_v)$ is minimal among all possible Weierstrass models.

<u>Def'n</u> E/K be an elliptic curve over a global fld K.

$$\Delta_{E/K}^{min} := \underline{\text{minimal discriminant.}}$$
$$= \prod_{v \in \mathrm{Val}(K)_{fin}} \mathfrak{p}_v^{\mathrm{ord}(\Delta v)} \triangleleft \mathcal{O}_K.$$

<u>Qstn</u> Can we find a single Weierstrass eqn over K

which is minimal for all $v \in Val(K)$?

<u>Ans</u>. No. - But some interesting things come out if...

Start with some rando model.

$E_{rando} : y^2 + a_1 xy + a_3 y = x^3 + \underline{a_2} x^2 + a_4 x + a_6 \quad / K.$

$\forall v \in Val(K)$, can get to a minimal model

by $x = u_v^2 x_v + r_v$,

$y = u_v^3 y_v + s_v u_v^2 x_v + t_v$

The discriminants are related by

$$\Delta_{rando} = u_v^{12} \Delta_v \quad \leftarrow \star$$

Define the crazy ideal:

$$\mathfrak{A}_{rando} := \prod_{v \nmid \infty} I_v^{-ord_v(u_v)}$$

"alpha"

$$\Delta_{E/K}^{min} = (\Delta_{rando}) \cdot \underbrace{\mathfrak{A}_{rando}^{12}}_{\text{improvement on } \Delta_{rando}}$$

<u>Theorem</u> 1) $[\mathfrak{A}_{rando}] \in Cl(K)$

independent of the "random" model.

2) There exists a global Weierstrass model

$\Downarrow$ iff $[\mathfrak{A}_{rando}] = 0 \in Cl(K)$.

3) If $\# Cl(K) = 1$, then there exist global Weierstrass models [E.g. $K = \mathbb{Q}$].

proof. (1) Let E, E' be two models. $\leadsto$ $\Delta = u^{12} \cdot \Delta'$.

$$\Delta'(\mathfrak{a}')^{12} = \Delta^{min} = \Delta(\mathfrak{a})^{12} = u^{12}\Delta'(\mathfrak{a})^{12}$$
$$= \Delta'(u\mathfrak{a})^{12}.$$

$\Rightarrow \mathfrak{a}' = u\cdot\mathfrak{a}.$ principal.

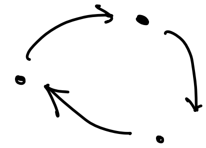
so $[\mathfrak{a}'] = [\mathfrak{a}] \in Cl(K)$. $\square$

⑮ Étale-like v.s. Crystal-Like / Frob-Like

(according to Mochizuki).

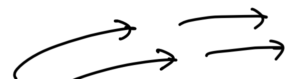
étale-like $\longleftrightarrow$ don't care about orderings

Literally "orderings".



"GROUPS"

Frob-Like $\longleftrightarrow$ affected by orderings.



"POSETS"
(with rigidity).

__Def'n__ A morphism α in a category $\mathcal{C}$ is irreducible if
$\alpha = \beta \circ \gamma \Rightarrow$ either β or γ is an isomorphism.

__Def'n__ A morphism $\beta : B \to A$ has pre-pullbacks
(also called FSM: fiberwise surjective morphism).

iff $\forall \gamma : C \to A$, $\exists$
$$\begin{tikzcd} D \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "\beta"] \\ C \arrow[r, "\gamma"'] & A \end{tikzcd}$$

__Def'n__ A category $\mathcal{D}$ is étale-like iff

a) $\forall \beta$ morphism in $\mathcal{D}$,

β has pre-pullback & β is not an isom.

$\Rightarrow$ β factors into a finite number of morphs with pre-pullbacks which are irred.

b) $\forall A \in \mathcal{D}, \exists N \in \mathbb{N}$ s.t.

every chain of pre-pullbackable irred. morphs starting at A has length $< N$.

$A^{\infty} \to \times \to \times \to \times \cdots$: ∞-length.

⑯ Monoid (Set)

$(M, +) = M$ monoid.

Frobenius-like

$M \to M/M^{I} = M^{\#} \in$ Sharp Monoids $\xrightarrow{I}$ 0

Sharp Monoids $\xrightarrow{gp}$ (???)

$M \to M^{I}$ = invertible elts $\in$ Grps

$M \to M^{gp}$ = groupification $\in$ Grps $\xrightarrow{\#}$ 0

étale-like

Name	Def'n	Example	Nonexample
M sharp	$M^{I} = 0$	$(\mathbb{N}, +)$	Any group
M integral	$\cdot M \hookrightarrow M^{gp}$ injective $\cdot$ Cancellation law	$\mathbb{N}$	1_1 · 2 3 4 … 1_2
M saturated	$\forall a \in M^{gp}, \forall n \in \mathbb{N}$, $na \in M \Rightarrow a \in M$.	$\mathbb{N}$	$11\mathbb{N} + 23\mathbb{N}$
M divisorial	sharp + integral + saturated	$\mathbb{N}$	any one of those guys.

basic notions

Another property:

- M <u>perfect</u> iff for all $n \in \mathbb{N}$ the map

$$[n] : M \longrightarrow M \quad \text{is a bijection.}$$

<u>example</u> $(\mathbb{Q}_{\geq 0}, +)$. <u>nonexample</u>: $\mathbb{N}$.

with this property comes the operation of perfection...

<u>Construct a "Division Category"</u>

$$\left.\begin{tikzcd} 1 & 2 \arrow[rr, bend right, "2|4"'] \arrow[rrrr, bend right, "2|6"'] & 3 \arrow[rrr, bend left, "3|6"] & 4 & 5 & 6 & \cdots \end{tikzcd}\right\}$$

Call this category <u>Division Cat</u>.

Perfection Constr:

$$\mathcal{D} : \text{Division Cat} \longrightarrow \text{Mon}$$

$$i \longmapsto I_i = M \text{ (some copy of } M\text{)}.$$

$$M = I_i \xrightarrow{[j]} I_{ij} = M \in \text{Mor}.$$

<u>Def</u> M monoid. $M^{pf} := \operatorname{colim}(\mathcal{D})$.

17 <u>What is Frobenioid?</u>

"Abstraction of Divisors on Schemes".

<u>Def</u> Let $\mathcal{D}$ be a category. Let $\Phi : \mathcal{D} \to \text{Mon}$ contravariant.

("pullback maps")

An <u>elementary Frobenioid</u> is a category $\mathbb{F}_\Phi$

<u>objects</u>: objects of $\mathcal{D}$.

<u>Morphs</u>: $S \xrightarrow{(f,\alpha,n)} T \xrightarrow{(g,\beta,m)} R$

$f \in D(S,T)$ base morph

$\alpha \in \Phi(S)$ divisor

$n \in \mathbb{N}$ degree.

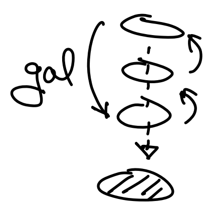
$$(g,\beta,m)\circ(f,\alpha,n) = (g\circ f,\ f^*\beta + m\alpha,\ mn).$$

<u>IDEAS</u> D = category of coverings of some scheme

Φ = monoid of divisors for the cover.

like Galois categories:

[<u>Def</u> A category $\mathcal{C}$ is <u>Galois</u> iff every endomorphism is an automorphism.]



<u>Baby Most Example</u> $D = \bullet$, $\Phi = \underline{M}$ constant monoid.

$\rightsquigarrow$ $\underline{\mathbb{F}_\Phi} = \mathbb{F}_M$; $(a,n)\cdot(b,m) = (a+nb, mn)$. <u>like $M \rtimes \mathbb{N}$</u>

cat w/ one obj End($*$)

- The standard Frbd is $\mathbb{F}_M$ for $M = \mathbb{Z}_{\geq 0}$.

<u>Def</u> Let $\mathbb{F}_\Phi$ be an elementary Frbd.

A <u>Frbd</u> is a pair $(\mathcal{X}, F)$ where $\mathcal{X} \in$ Cat,

& $F: \mathcal{X} \xrightarrow[\text{functor}]{} \mathbb{F}_\Phi$ [2-cat].

"Category w/ Dangly bits".

<u>Rmk</u> • Technically is a pre-Frbd according (need rigidity).

• A morphism of Frbds is a functor.

• We will be able to "forget about F" via rigidity thms.

(18) Model Frobenioids

Def'n Model Frbd data on a category D will be a triple (Φ, B, d).

- $\Phi : D \longrightarrow$ Mon divisoral
- $B : D \longrightarrow$ Mon grp.
- $d : B \longrightarrow \Phi^{gp}$ natural trans

Philosophy Model Frbd Data $\rightsquigarrow$ Frbd.

Example X/K variety over a fld. (normal, integral). $D = \mathrm{Cov}(X)$ (finite étale).
covers

- $\Phi(Y) = \mathrm{Eff}(Y)$; $B(Y) = k(Y)^\times$;

$d = \mathrm{div} : k(Y)^\times \longrightarrow \mathrm{Eff}^{gp}(Y) = \mathrm{Div}(Y)$.

Construction of Model Frbds

Let D be a base category.

Let (Φ, B, d) be model Frbd data.

Let $\mathbb{F}_\Phi$ be the elt Frbd assoc to Φ.

We will construct a category

$$\begin{array}{c} \mathrm{Frd}(\Phi, B, d) \\ \downarrow \\ \mathbb{F}_\Phi . \end{array}$$

<u>objects</u> $X = (V_X, \alpha_X)$.

$V_X \in D$, $\alpha_X \in \Phi(V_X)$.

<u>Morphs</u> $X \xrightarrow{\varphi} Y$

$((f_\varphi, \eta_\varphi, n_\varphi), u_\varphi)$ $\quad$ $\mathrm{div}(\varphi)$ $\qquad$ "Riemann-Hurwitz like condition".

$f_\varphi : V_X \longrightarrow V_Y$,
$\eta_\varphi \in \Phi(V_X)$,
$n_\varphi \in \mathbb{N}$.
$\Big]$ eff Frbd.

- $u_\varphi \in B(X)$.

$f_\varphi^* \alpha_Y + d(u_\varphi)$
$= n_\varphi \alpha_X + \underbrace{\mathrm{div}(\varphi)}_{\eta_\varphi}$

$\deg(\varphi)$

$X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$

$\rightsquigarrow ((f_\psi, \mathrm{div}(\psi), n_\psi), u_\psi) \circ ((f_\varphi, \mathrm{div}(\varphi), n_\varphi), u_\varphi).$

$= ((f_\psi \circ f_\varphi, f_\varphi^* \mathrm{div}(\psi) + n_\psi \mathrm{div}(\varphi), n_\psi n_\varphi), n_\varphi u_\psi + u_\varphi).$

How do we see this is a Frbd?

$\mathrm{Ob}: (V_X, \alpha_X) \longmapsto V_X \qquad \mathbb{F}_\Phi$

$\mathrm{Mor}: ((f_\varphi, \mathrm{div}(\varphi), \deg(\varphi)), u_\varphi) \longmapsto (f_\varphi, \mathrm{div}(\varphi), \deg(\varphi)).$

This gives the functor

$$\begin{tikzcd} \mathrm{Frd}(\Phi, B, d) \arrow[rr] \arrow[dr] & & \mathbb{F}_\Phi \arrow[dl] \\ & D & \end{tikzcd}$$

1-commutative

& shows we constructed a Frbd

Two main examples come from

$\boxed{\text{Arithmetic Frbd Data}}$

$F = $ fld, $\overline{F} = $ large closure. $D = \mathrm{Cov}(\mathrm{Spec}\, F)$.

F'/F finite. $\Phi(\mathrm{Spec}(F')) = \widehat{\mathrm{Eff}}(F')$

$$B(\mathrm{Spec}(F')) = (F^\times)^\times$$
$$d : (F')^\times \longrightarrow \widehat{\mathrm{Eff}}(F')^{gp} = \underbrace{\widehat{\mathrm{Div}}(F')}_{\text{Arakelov divisors}}$$

Def ("Frbd of arith origin") $\mathrm{Frd}^{ar}(\overline{F}/F) = \mathrm{Frd}(\Phi, B, d)$.

$\boxed{\text{Geometric Frbd Data}}$

X = normal variety over a fld, integral.

$\mathcal{D} = \mathrm{Cov}(X)$ (connected, finite, étale).

$\Phi(Y) = \mathrm{Eff}(Y)$, $B(Y) = k(Y)^\times$, $d(u) = \mathrm{div}(u) \in \mathrm{Div}(Y)$, $u \in k(Y)^\times$.

Def (Frbd of geom. origin). $\mathrm{Frd}^{geom}(X) = \mathrm{Frd}(\Phi, B, d)$.

(19) Adjectives Associated with Frbds

Frbd elts, Morph of Frbd elts, etc...

- Let $\mathcal{X} = \mathrm{Frd}(\Phi_{\mathcal{X}}, B_{\mathcal{X}}, d_{\mathcal{X}})$. $\xrightarrow{\varphi} \cdots$

$$\begin{array}{c} \mathcal{X} \\ \text{elt.}\big\downarrow \\ \mathcal{A} = \mathbb{F}_{\Phi_{\mathcal{X}}} \end{array} \qquad \underset{\varphi_{elt}}{\longrightarrow} \cdots$$

all over a base $\mathcal{D}_{\mathcal{X}}$.

- For $x = (U_x, \alpha_x)$, $y = (V_y, \alpha_y) \in \mathcal{X}$,

$$x \xrightarrow{\varphi} y, \quad \varphi = (\underbrace{f_\varphi, \mathrm{div}(\varphi)}_{\varphi_{elt}}, \mathrm{deg}(\varphi), u_\varphi).$$

A morphism must satisfy
$$f_\varphi^* \alpha_y + \mathrm{div}(u_\varphi) = \mathrm{deg}(\varphi)\, \alpha_x + \mathrm{div}(\varphi) \quad \text{as elts of } \Phi_{\mathcal{X}}(V_x)^{\circ}.$$

ONE MORPHISM PREDICATES

$$\varphi = (\underbrace{f_\varphi, \operatorname{div}(\varphi), \deg(\varphi)}_{\varphi_{eft}}, u_\varphi)$$

Condition	Meaning	Name (given by Mochizuki)
$(*, *, 1)$	degree 1	"linear"
$(*, 0, *)$	no assoc divisor	"isometric"
$(\text{isom}, *, *)$	base map isom	base isom base identity

<u>Recall</u>

$$\begin{tikzcd} \mathcal{X} \arrow[d, "F"'] & X \arrow[r, "\varphi"] \arrow[d] & Y \arrow[d] \\ \mathcal{S} & S \arrow[r] & T \end{tikzcd}$$

identity X, S, Y, T
with h_X, h_Y, $\underbrace{h_S, h_T}$

Take $G : \mathcal{S} \to$ whatever
$\rightsquigarrow G \circ F : \mathcal{X} \to$ whatever
$\quad\quad = F^* G$

$$Y \times_{F^*T} F^*S \xleftarrow{\tilde{\varphi}} X$$

natural map from φ.

<u>Def'n</u> If $\tilde{\varphi}$ isom. then say "<u>φ is induced by pullbacks</u>".

Condition	Meaning	Name
$(\text{isom}, *, 1)$	...	prestep
$(\text{isom}, *, 1)$ & not an isom.	...	step

If $\circledast$ is primary, then this is a primary step / prestep.

<u>example</u> "isometric prestep".

isometric $(*, 0, *)$ + prestep $(\text{isom}, *, 1)$ = isometric prestep $(\text{isom}, *, 1)$.

<u>example</u> "linear base isom".

linear $(*, *, 1)$ + base isom. $(\text{isom}, *, *)$

$= \text{linear base isom.} \ (\text{isom}, *, 1)$.

<u>note</u> linear base isom = isometric prestep.

Composition: $\varphi = (f, \text{div}(\varphi).\ d, u\varphi)$.

Suppose $\varphi = \varphi_1 \circ \varphi_2 \circ \varphi_3$. If $(\text{isom}, *, 1) \circ (\text{isom}, 0, 1) \circ (*, *, *)$

or

$(*, *, 1) \circ (\text{isom}, 0, 1) \circ (\text{isom}, *, *)$

Then φ_2 is an isom.

$\Leftrightarrow$ " φ is coangular".

$\boxed{\text{TWO MORPH PREDICATES}}$

$$(\varphi_1)_{\text{eH}} = (f_1, \alpha_1, d_1) \qquad (\varphi_2)_{\text{eH}} = (f_2, \alpha_2, d_2)$$

<u>Conditions</u>	<u>Names</u>
$\text{dom}(f_1) = \text{dom}(f_2)$ & $\alpha_1 = \alpha_2$ $\text{rng}(f_1) = \text{rng}(f_2)$	"metrically equivalent".
$f_1 = f_2$ (f_i morphs in $D\mathfrak{X}$)	φ_1, φ_2 are "base equivalent"
$\Phi(f_1) = \Phi(f_2)$	"div equivalent".

$\boxed{\text{TWO OBJECT PREDICATES}}$

- $X = (V_X, \alpha_X)$, $Y = (V_Y, \alpha_Y)$.

  $V_X \simeq V_Y \Leftrightarrow X$ & Y are base isomorphic.

ONE OBJECT PREDICATES

- $X = (V_X, \alpha_X) \in \mathcal{X}$.

$$[\forall X \xrightarrow{\varphi} Y,\ \varphi_{elt} = (\text{isom}, 0, 1) \Rightarrow \varphi \text{ isom}]$$

$$\Updownarrow$$

X <u>isotropic</u>.

(20) <u>Archimedean Flds</u>

Def'n A <u>local archimedean fld</u> will be a topological fld isomorphic to $\mathbb{R}$ or $\mathbb{C}$.

Note if $i : K \hookrightarrow L$ is a morphism of local archimedean flds which is not an isom, then

$$\exists \alpha : K \xrightarrow{\sim} \mathbb{R},\ \exists \beta : L \xrightarrow{\sim} \mathbb{C}$$

such that $\beta \circ i \circ \alpha^{-1} : \mathbb{R} \longrightarrow \mathbb{C}$ is the standard inclusion.

<u>Angle / Magnitude Decomposition</u>

K local arch fld.

$\mathcal{O}_K^\times$ = group of units. $\quad \text{ord}(K^\times) = K^\times / \mathcal{O}_K^\times$.

<u>Note</u> 1) $|\cdot| : \text{ord}(K^\times) \longrightarrow \mathbb{R}_{>0}$ an isom.

2) $K^\times \simeq \underbrace{\mathcal{O}_K^\times}_{\text{ang}} \times \underbrace{\text{ord}(K^\times)}_{\text{mag}}$.

<u>example</u> 1) $K = \mathbb{C}$, $\mathbb{C}^\times = U(1) \times \mathbb{R}_{>0}$.

2) $K = \mathbb{R}$, $\underbrace{\mathcal{O}_K^\triangleright = [-1, 1]}_{\text{cool.}}$, $\leftarrow$ we don't have the ring of integers. But we do have the <u>monoid</u> of it.

Def Let K be an archimedean fld.

An *angular region* is a subset

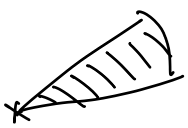
$$B \times C = A \subseteq K^\times$$

such that

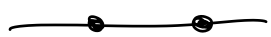
1) $B \subseteq \mathcal{O}_K^\times$, and it intersects each conn. comp. in a conn. comp.

2) $C \subseteq \operatorname{ord}(K^\times) \simeq \mathbb{R}_{>0}$ is an interval of the form $(0, \lambda]$.

1) Angles

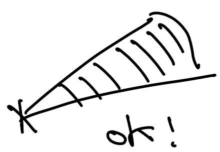


or

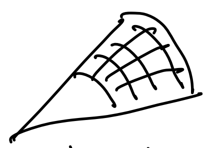


regions

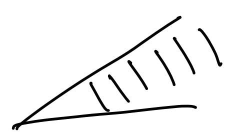
2) Mags



ok!



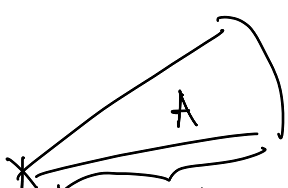
NOT ok



inf

NOT ok.

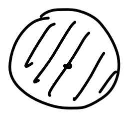
Def



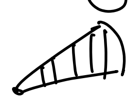
∂A = boundary

λ = tip

Def An angular region $A = B \times C$ isotropic if $B = \mathcal{O}_K^\times$ (full angle).



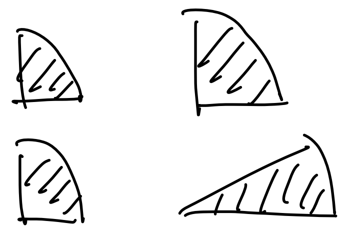
isotropic



not isotropic.

Def Let $A = B \times C$ & $A' = B' \times C'$ be angular regions

A & A' are coangular $\iff$ $B = B'$.

Yes  coangular.

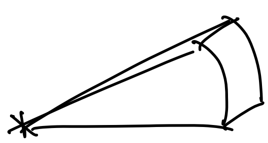
No not coangular.

Rmk If $V \in \mathrm{Vect}_K$, then we can talk about angular regions in K

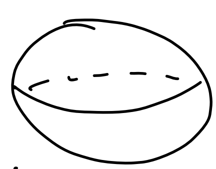
$$\mathrm{ord}(V) := V^*/\mathcal{O}_K^\times = R(V),$$

where $V^* = V \setminus \{0\}$.

(E.g. $V = \mathbb{R}^3$, $\mathrm{ord}(V) = \mathbb{R}^3/\mathbb{R}^\times$ = directions).



angular region



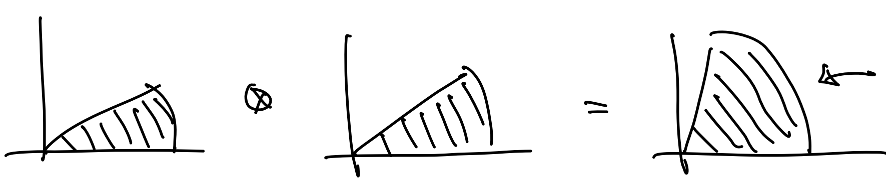
isotropic region

(coangularity makes sense too).

Tensor products of angular regions

$V_1, V_2 \in \mathrm{Vect}_K$. $A_1 \subseteq V_1$, $A_2 \subseteq V_2$ angular regions

$$A_1 \otimes A_2 := \{a_1 \otimes a_2 : a_1 \in A_1 \ \& \ a_2 \in A_2\} \subseteq V_1 \otimes_K V_2.$$



twice.

Under morphs of flds

Let $f : \mathrm{Spec}\, K' \to \mathrm{Spec}\, K$ be a fin étale cover.

$A \subseteq V \in \mathrm{Vect}_K$ angular region $f^*(V, A) := (V', A')$

where $V' = f^* V = V \otimes_K K'$

$$A' = \begin{cases} \text{image of } A. & f \text{ isom} \\ \text{isotropic orb of } A, & f \text{ not isom.} \end{cases}$$

5

$\mathrm{Cov}(\mathbb{R})^\circ$ = stupid category of covers of $\mathbb{R}$.

<u>Note</u> $\dfrac{\mathrm{Cov}(\mathbb{R})^\circ}{\text{isom.}} = \{\mathbb{R}, \mathbb{C}\}$

<u>Def'n</u> (c.f. pg 28, Frbd II)

The <u>elementary archimedean Frbd</u> is defined to be

$$\mathbb{F}_\infty = \mathbb{F}_{\Gamma_\infty} / \mathrm{Cov}(\mathbb{R})^\circ$$

where $\Gamma_\infty : \mathrm{Cov}(\mathbb{R})^\circ \longrightarrow \mathrm{Mon}$ $\quad \overset{\Gamma_K}{\shortparallel}$.

$$K \longmapsto \mathrm{ord}(K) = K^\times / \mathcal{O}_K^\times.$$

<u>Def'n</u> A_∞ = (The basic archimedean Frbd)/ $\mathrm{Cov}(\mathbb{R})^\circ$

<u>objects</u> $(\mathrm{Spec}\, K, V, A)$

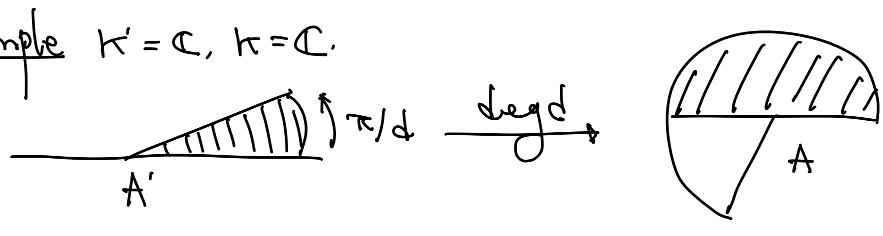
- $\mathrm{Spec}\, K$ cover of $\mathrm{Spec}\, \mathbb{R}$
- $V \in \mathrm{Vect}_K$
- $A \subset V$ angular region

<u>Morphs</u> $(\mathrm{Spec}\, K', V', A') \xrightarrow[\varphi = (f,\, \sigma_\varphi,\, \underset{d}{\underset{\shortparallel}{\deg(\varphi)}})]{\varphi} (\mathrm{Spec}\, K, V, A)$

$f_\varphi : \mathrm{Spec}\, K' \to \mathrm{Spec}\, K$

$\sigma_\varphi : (V')^{\otimes d} \longrightarrow f^* V = V \otimes_K K'$ such that $\sigma_\varphi((A')^{\otimes d}) \subseteq A \otimes_K K'$.

<u>example</u> $K' = \mathbb{C}$, $K = \mathbb{C}$.



Def $x \in F_\infty$ is <u>naively isotropic</u> iff
angle $(x) = \odot$ whole thing.

Def $f: X \to Y$ morph in F_∞ is <u>naively coangular</u> iff
angle $(Y) = f(\text{angle}(X))$.

<u>Question/Exercise</u>

- Recall that we had a notion of isotropic and coangular for abstract Frobenioids.
  Show that a Frbd elt in F_∞ is naively isotropic
  $\Leftrightarrow$ it's isotropic.
- Similar question for coangularity.

<u>Def'n</u> Let D be totally epimorphic, connected category.
Let $F: D \to \mathrm{Cov}(\mathbb{R})^\circ$.
Let A_∞ be the basic archimedean Frbd.
$$F^* A_\infty \overset{\text{def}}{=} \underline{\text{Archimedean Frbd}}.$$

$\rightsquigarrow$ <u>Note</u> $\begin{pmatrix}\text{underlying cat.} \\ \text{of } A_\infty\end{pmatrix} = A_\infty \times_{\mathrm{Cov}(\mathbb{R})^\circ} D$

(underlying monoid) $= F^* \Gamma_\alpha = \Gamma_\infty \circ F$.

<u>Lemma/Qstn/Exercise</u>

Let $F: D \to \mathrm{Cov}(\mathbb{R})^\circ$. $D =$ conn, locally epi. Then
$$\mathbb{F}_{F^*\Phi} = F^* \mathbb{F}_\Phi .$$

proof of objects

$$F^* \mathbb{F}_{\Gamma_\infty} := \mathbb{F}_{\Gamma_\infty} \times_{\mathrm{Cov}(\mathbb{R})^\circ} D$$

$$\mathrm{obj}(\mathbb{F}_{\Gamma_\infty} \times_{\mathrm{Cov}(\mathbb{R})^\circ} D) = \cancel{\mathrm{obj}(\mathbb{F}_{\Gamma_\infty}) \times_{\mathrm{Cov}(\mathbb{R})^\circ}} \mathrm{Obj}(D)$$
$$= \mathrm{Obj}(D).$$

(21) Basic p-adic Frobenioid (Frbd II, Ex 1.1).

Def Let $D_p = \mathrm{Cov}(\mathbb{Q}_p)^\circ$.

Basic p-adic Frbd $= \mathcal{F}_p = \mathrm{Frd}(\Phi, \mathcal{B}, \mathrm{ord})$.

$$\Phi(K) = \mathrm{ord}(\mathcal{O}_K^\triangleright)_{\mathbb{R}} = \underline{\mathcal{O}_K^\triangleright / \mathcal{O}_K^\times \simeq \mathbb{N}}$$

$\mathcal{B}(K) = K^\times$

$$\left[\begin{array}{l} \mathrm{div}: \mathcal{B}(K) \longrightarrow \Phi(K)^{gp} \\ \qquad a \longmapsto \mathrm{ord}(a) \end{array}\right.$$

$\mathrm{ord}(\mathbb{Z}_p^\triangleright) = \mathbb{Z}_p^\triangleright / \mathbb{Z}_p^\times$
$= p^{\mathbb{N}} \simeq \mathbb{N}.$
$\mathbb{Z}_p^\triangleright = p^{\mathbb{N}} \cdot \mathbb{Z}_p^\times$

comes from taking the natural surjection

$$K^\times \longrightarrow \mathrm{ord}(K^\times) := K^\times / \mathcal{O}_K^\times.$$

Mochizuki does this.

Def $\mathrm{MLB}(\mathbb{Q}_p)$:= covers w/ "metrized" line bundles

$$\downarrow$$

$\mathrm{Cov}(\mathbb{Q}_p)^\circ$

objects (K, V, μ), $K \in \mathrm{Cov}(\mathbb{Q}_p)^\circ$, $V \in \mathrm{Vect}_K$ 1-dim'l

$\mu \in \mathrm{ord}(V)_{\mathbb{R}}$, $\mathrm{ord}(V) = V / \mathcal{O}_K^\times$.

morphs $(\mathrm{Spec}\, K, V, \mu) \xrightarrow[(f, \sigma, d)]{} (\mathrm{Spec}\, L, W, \nu)$ L/K.

$f: \mathrm{Spec}\, L \to \mathrm{Spec}\, K$

$\sigma: V^{\otimes d} \to W_K$ s.t. $\sigma(\mu^{\otimes d}) = [c] \cdot \nu_K$.

for some $c \in \mathrm{ord}(\mathcal{O}_K^\triangleright)_\mathbb{R}$, $\mathrm{ord}(\mathcal{O}_K^\triangleright) = \mathcal{O}_K^\triangleright/\mathcal{O}_K^\times$.

<u>Lemma</u> (Frbd II, pg 8) $\boxed{\mathcal{MLB}(\mathbb{Q}_p) \twoheadrightarrow \mathcal{F}_p^{\mathbb{R}}}$.

(2) <u>Localization</u> (Example 1.4. Frbd II)

Let F = # fld. $\tilde{F}$ = alg closure.

$v \in \mathrm{Spec}\,\mathcal{O}_F = \mathrm{Val}(F) \setminus \mathrm{Val}(F)_\sigma$

F_v = completion, $\tilde{F}_v$ = completion of $\tilde{F}$.

$\varphi_v : \mathrm{Cov}(F)^\circ \longrightarrow (\mathrm{Cov}(F_v)^\circ)^\perp$

closure under product by Mochizuki.

"pull back a cover" $= \varphi_v : \mathrm{Cov}(K)^\circ \longrightarrow \underline{(\mathrm{Cov}(K_v)^\circ)}^\perp$.

covers are no longer connected.

<u>1st Descr</u> $\mathrm{Cov}(\mathbb{Q})^\circ \simeq \mathcal{B}(G_\mathbb{Q})^\circ$, $\mathrm{Cov}(\mathbb{Q}_p) \simeq \mathcal{B}(G_{\mathbb{Q}_p})^\circ$ } category of connected G-sets.

Then the inclusion $G_{\mathbb{Q}_p} \hookrightarrow G_\mathbb{Q}$ shows how a $G_\mathbb{Q}$-set can become a $G_{\mathbb{Q}_p}$-set.

<u>2nd Descr</u>

$$\begin{tikzcd}
\mathrm{PULLBACK} \arrow[r] \arrow[d] & \overline{X} \arrow[d] \\
\mathrm{Spec}\,\mathbb{Q}_p \arrow[r] & \mathrm{Spec}\,\mathbb{Q}
\end{tikzcd}$$

<u>Def'n</u> (Frbd II, pg 12) $\mathrm{Cov}(F_v, F) = \left(\begin{array}{c}\text{connected finite étale} \\ \text{coverings of } \mathrm{Spec}\,F_v \text{ w/} \\ \text{localization map}\end{array}\right)$

<u>objects</u> $(\mathcal{P}, \mathcal{Q}, i : \mathcal{P} \to \varphi_p(\mathcal{Q}))$.

$P \in \mathrm{Cov}(F_v)$, $Q \in \mathrm{Cov}(F)$.

<u>Morphisms</u> $P \xrightarrow{\alpha} P'$, $Q \xrightarrow{\beta} Q'$ s.t.

$$\begin{tikzcd} P \arrow[r, "p(\alpha)"] \arrow[d, "i"'] \arrow[dr, phantom, "\circlearrowright"] & P' \arrow[d, "i'"] \\ P_P(Q) \arrow[r, "\varphi_P(\beta)"] & P_P(Q') \end{tikzcd}$$

<u>Example 1.4(iii) FRBD II</u>

The main results from Frbd II can be applied to p-adic Frobenioids.

<u>Qstn</u> Does this include the "degree/radification trick for arithmetic Frbds"?

⑳ Tempered Coverings & The Tempered Fundamental Group

<u>1st way</u> (work with Berkovich spaces).

- $\exists$ a notion of finite étale morphs for Berk. spaces.

<u>Def'n</u> $f: S \to S'$ will be <u>étale</u> provided $\exists$ covering $(U_i \to S)$ such that

$$f^{-1}(U_i) = \coprod V_{ij}$$

where $f|_{V_{ij}}$ is finite étale.

<u>Def'n</u> $f: S \to S'$ étale is <u>tempered</u> if $\exists\, T \to S$ finite étale such that

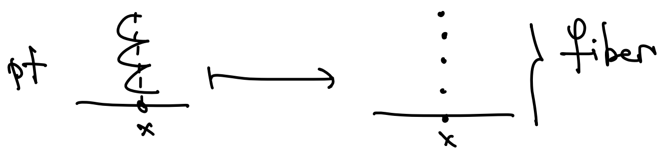
$$f\times_S T : S'\times_S T \longrightarrow S\times_S T$$

is a topological covering.

* verify this is a site.

* Define (for $x\in X$) a functor

$$F_x : \mathrm{Cov} \longrightarrow \mathrm{Set}$$



$$* \quad \pi_1^{temp}(X,x) = \mathrm{Aut}(F_{\bar{x}})$$

<u>2nd way</u> Take a pro-system of finite étale galois pt. covers

$$(S_i, s_i) \longrightarrow (X,x)$$

then take $\qquad$ limit over filtered category

$$\boxed{\pi_1^{temp}(X,x) = \varprojlim \mathrm{Gal}(S_i^\infty/X)}$$

of finite étale Galois covers.

- Suppose $(S,s)\longrightarrow(X,x)$ is a finite étale gal cover.

$$(S^\infty, s^\infty) \longrightarrow (S,s) \quad \text{universal topological cover.}$$

$$\downarrow$$

$$(X,x)$$

$G(S^\infty/X) = \mathrm{Gal}(S^\infty/X)$

$= \{(\hat{g}, g) \in \mathrm{Aut}(S^\infty/X)\times G(S/X) : p\circ\hat{g} = g\circ p\}.$

- Suppose

$$\begin{array}{ccc} (S_1, s_1) & \xleftarrow{\ \psi\ } & (S_2, s_2) \\ & {}_{\pi_1}\searrow \quad \swarrow_{\pi_2} & \\ & (X,x) & \end{array}$$

$\rightsquigarrow \psi^\infty : (S_2^\infty, s_2^\infty) \longrightarrow (S_1^\infty, s_1^\infty)$

$$\mathrm{Gal}(S_2^\infty/X) \longrightarrow \mathrm{Gal}(S_1^\infty/X)$$

$$(\hat{g}_2, g_2) \longmapsto (\hat{g}_1, g_1)$$

$\exists\, \hat{g}_1 \,\&\, g_1$ doing this

$$\begin{cases} \psi^\infty\circ\hat{g}_2 = \hat{g}_1\circ\tilde{\psi}, \\ \psi\circ g_2 = g_1\circ\psi \end{cases}$$

24 Basic Tempered Frobenioids

X = Berkovich space

Π = Tempered Fundamental Group of X

$= \pi_1^{temp}(X, x)$.

$\mathcal{D} = \mathcal{B}(\pi)^\circ = \mathrm{Cov}^{temp}(X)$.

Baby Version of Divisor Monoid for Tempered Frobenioids:

1. $Y \to X$ a tempered cover.
2. $S \to Y$ finite étale galois covering. (*)
3. $S^\infty \to S$ universal topological cover.
4. $\mathcal{S}^\infty \to \mathcal{S}$ formal model of universal topological cover. $/ \mathrm{Spf}(\mathcal{O}_K)$.
5. Take effective divisor $\mathrm{Eff}(\mathcal{S}^\infty)$.
6. Finally we take

$$\mathrm{Eff}(\mathcal{S}^\infty)^{G(\mathcal{S}^\infty/Y)}$$

6 steps $\longrightarrow$ baby version of the divisor monoid.

Full Version:

$$(\mathcal{S}_i) \longrightarrow Y$$

pro-system of finite étale Gal coverings

$$\boxed{\Phi_X^{temp}(Y) = \varinjlim \mathrm{Eff}(\mathcal{S}_i^\infty)^{G(\mathcal{S}_i^\infty/Y)}}$$ monoid for tempered Frobenioids

- To get the principal divisors:

$$\mathcal{P}_X^{temp}(Y) = \varinjlim_i \mathrm{Mer}(\mathcal{S}_i^\infty)^{G(\mathcal{S}_i^\infty/Y)}$$

meromorphic function.

• To get the divisor map

$$\mathrm{div}_X^{temp} : \Phi_X^{temp} \longrightarrow \mathcal{D}_X^{temp}$$

we just take the divisor of a mero. func.

<u>Def'n</u> The <u>basic tempered Frobenioid associated</u> to a Berkovich space X over a complete local fld K

$$\mathrm{Frd}^{temp}(X/K) = \mathrm{Frd}(\Phi_X^{temp}, \mathcal{D}_X^{temp}, \mathrm{div}_X^{temp}).$$

- "Base changing" $\mathcal{D} \longrightarrow \mathrm{Cov}^{temp}(X)$.
- "Base theoretic hull".

(25) Distortion Phenomena in IUT by Mochizuki.

$$\mathcal{F} = \mathrm{Frd}(\Phi_{\mathcal{F}}, \mathcal{D}_{\mathcal{F}}, \mathrm{div}_{\mathcal{F}}).$$

$$\begin{array}{l} \mathcal{F} \\ \downarrow \mathrm{elt} \\ \mathbb{F}_{\Phi_{\mathcal{F}}} \\ \downarrow \\ \mathcal{D} = \text{base category}. \end{array}$$

<u>Def</u> For $A \in \mathcal{F}$:

- $O^{\times}(A) = \{\sigma \in \mathrm{Aut}(A) : \sigma_{\mathrm{elt}} = \mathrm{id}\}$.
- $O^{\triangleright}(A) = \{\sigma \in \mathrm{End}(A) : \sigma_{\mathrm{elt}} = \mathrm{id}\}$.

<u>Def</u> For $A \in \mathcal{F}$ there is a notion of a <u>Picard group</u>

$$\mathrm{Pic}_{\mathcal{F}}(A) = \Phi(A)^{gr}/\mathcal{P}(A).$$

$$\Phi_{\mathcal{F}} = \Phi, \quad \mathcal{P}_{\mathcal{F}} = \mathcal{P}.$$

<u>Def</u> An element $A \in \mathcal{F}$ is <u>Frobenius trivial</u> if there exists some

$$\phi : (\mathbb{N}, +) \longrightarrow \mathrm{End}(A).$$

(having bunch of commuting lifts of the Frb - like a $\underline{\Lambda\text{-scheme}}$).

Go Arithmetic

- $F = \#$ fld
- $\bar{F}$ = alg. closure.

$$\mathcal{X} = \mathrm{Frd}^a(\tilde{\bar{F}}/F) \quad [\text{Built from Arakelov divisors}].$$

$$\boxed{\begin{array}{l}\underline{\text{Lemma}} \text{ For all } X \in \mathcal{X}, \\ \qquad \mathcal{O}^{\Delta}(X) = \mathcal{O}^{\times}(X) \xrightarrow{\sim} \mu(F).\end{array}}$$

proof. $A \in \mathrm{Frd}^a(\hat{\bar{F}}/F)$ w/ base $\mathrm{Spec}(K)$.

$$\mathcal{O}^{\Delta}(A) = \mathcal{O}^{\times}(A) \simeq \mu(K).$$

- $A = (K, D)$, $D \in \widehat{\mathrm{Div}}(K)$.
- $\varphi \in \mathcal{O}^{\Delta}(A)$ endomorphism w/ $\varphi_{ett} = id$.

$$\varphi = (\varphi_{ett}, u_\varphi) = (\underbrace{f_\varphi, z_\varphi, d_\varphi}_{\varphi_{ett}}, u_\varphi)$$

$f_\varphi = id$, $z_\varphi = 0$, $d_\varphi = 1$.

condition on ways,

$$\cancel{f_\varphi^* D} + \mathrm{div}(u_\varphi) = \cancel{d_\varphi D} + \underbrace{\mathrm{div}(\varphi)}_{} \overset{}{=} 0 .$$

$\Rightarrow \mathrm{div}(u_\varphi) = 0$.

$u_\varphi = \Gamma(A) = K^\times$, (natural image of u_φ).

$$\underbrace{\Phi(A)^{gp} = \bigoplus_y K_y^\times / \mathcal{O}_{K_y}^\times}_{\text{multiplicatively}} \ni \mathrm{div}(u_\varphi)$$

the statement $\mathrm{div}(u_\varphi) = 0$ is really saying $u_\varphi \equiv 1 \bmod \mathcal{O}_{K_y}^\times$, $\forall y$.

This is the same as saying $u_\varphi \in \mu(K)$.

<u>Rmk</u> $\check{H}^\circ(M) := \{ s \in M : \|s\|_y \leq 1 \}$; $\check{H}^\circ(\mathcal{O}_K) = \mu(K) \cup \{1\}$.

($M = \mathcal{O}_K$ w/ abs values)

<u>Lemma</u> (Mochizuki's Baby Distortion Lemma). [Frbd I, Thm 6.4]

1) Let $\mathcal{X} = \mathrm{Frd}^a(\tilde{F}/F)$

If $X \in \mathcal{X}$ is Frobenius trivial then

$$\mathrm{Pic}_{\mathcal{X}^{\mathbb{R}}}(X^{\mathbb{R}}) \xrightarrow[\widehat{\deg}_{X^{\mathbb{R}}}]{\sim} \mathbb{R}.$$

2) Let $\mathcal{X} = \mathrm{Frd}^a(\tilde{F}/F)$, $\mathcal{Y} = \mathrm{Frd}^a(\tilde{K}/K)$.

Suppose $\psi : \mathcal{X} \longrightarrow \mathcal{Y}$ such that

$\psi^{\mathbb{R}} : \mathcal{X}^{\mathbb{R}} \xrightarrow{\sim} \mathcal{Y}^{\mathbb{R}}$. $\Big)$ <u>symmetry breaking</u>.

<u>Then</u> $\exists \deg(\psi) \in \mathbb{R}$ such that for all $X \in \mathcal{X}^{\mathbb{R}}$ that are Frobenius trivial,

distorsion $\leftarrow$ $\deg(\psi) \, \widehat{\deg}_X \geq \widehat{\deg}_Y \cdot \mathrm{Pic}_{\mathcal{Y}^{\mathbb{R}}}(\psi^{\mathbb{R}})$

(Here $Y^{\mathbb{R}} = \psi^{\mathbb{R}}(X^{\mathbb{R}})$, which is automatically Frobenius trivial.)

<u>NOTE</u>

$$\begin{tikzcd} \mathrm{Pic}_{\mathcal{X}}(X) \arrow[dr, "\widehat{\deg}_X"] \arrow[d, "\mathrm{Pic}(\psi)"'] & \\ \mathrm{Pic}_{\mathcal{Y}}(Y) \arrow[r, "\widehat{\deg}_Y"'] & \mathbb{R} \end{tikzcd}$$

$\mathbb{R} \leftarrow \begin{cases} X = X^{\mathbb{R}}, \\ Y = Y^{\mathbb{R}}, \\ \psi = \psi^{\mathbb{R}} \end{cases}$

3) If ψ is an equivalence of categories then $\deg(\psi) = 1$.

<u>Proof Ideas</u> 1) Dirichlet Unit Theorem.
2) Ordering Properties.
3) Chebotarev.

(26) Hodge Theaters (I)

Confused Groups and Torsors.

<u>IUT-Def'n-6.1.i</u> : ($\mathbb{F}_\ell$-groups) / ± 1
($\mathbb{F}_\ell$-torsors) / $\mathbb{X} \pm$.

<u>Def</u> The category $\{\mathbb{F}_\ell\text{-group}\}/\pm 1$ is defined by:

<u>objects</u> $f^{\pm}: E \Longrightarrow \mathbb{F}_\ell$ polymorphs from a set E to the $\mathbb{Z}$-module $\mathbb{F}_\ell$.

<u>morphs</u> $f^{\pm} \longrightarrow g^{\pm}$ are equivalence classes of morphs of sets

$$E_f \xrightarrow{\alpha} E_g$$

such that $f^{\pm} = g^{\pm} \circ \alpha$.

$$\boxed{\begin{aligned} f^{\pm} &= (\pm 1) \cdot f \\ &= \{f, -f\} \end{aligned}}$$

<u>Composition</u>
$$\begin{tikzcd} E_f \arrow[r, "\alpha"] \arrow[rr, bend right, "\beta\circ\alpha"'] & E_g \arrow[r, "\beta"] & E_h \end{tikzcd} \quad ☆$$

$$f^{\pm} = g^{\pm} \circ \alpha, \; g^{\pm} = h^{\pm} \circ \beta \;\Rightarrow\; f^{\pm} = h^{\pm} \circ \beta \circ \alpha \qquad \text{ok!}$$

<u>Notation</u> If $f^{\pm}: E \Rightarrow \mathbb{F}_\ell$ is a $\mathbb{F}_\ell$-group$/\{\pm 1\}$, then we may just denote it by E or f.

In this case,

$$E = E_f \quad \& \quad f = f_E.$$

There is a functor

$$\{\mathbb{F}_\ell\text{-groups}\}/(\pm 1) \xrightarrow{M} \mathrm{Mod}_{\mathbb{F}_\ell}$$

because $\{f,-f\}: E \longrightarrow \mathbb{F}_\ell$ has well-defined module str.

If $e \in E$ then $f^{\pm}(n \cdot e) = n f^{\pm}(e)$;

the maps are (± 1)-equivariant.

Def'n We define the category the additive action

$$\{\mathbb{F}_\ell\text{-torsors}\}/(\mathcal{A}^{\pm}).$$

objects $f^{\mathcal{A}\pm}: T \Rrightarrow \mathbb{F}_\ell$ polymorphs of bijections from a set T to $\mathbb{F}_\ell$ viewed as $\mathbb{F}_\ell$-torsor.

The polymorph is an $\mathbb{F}_\ell^{\mathcal{A}\pm}$-orbit of a single bijection $f: T \longrightarrow \mathbb{F}_\ell$

i.e. $f^{\mathcal{A}\pm} = \mathbb{F}_\ell^{\mathcal{A}\pm} \circ f$,

morphs $f \xrightarrow{\alpha} g$ are given by $\alpha: T_f \to T_g$ such that $f^{\mathcal{A}\pm} = g^{\mathcal{A}\pm} \circ \alpha$.

Composition similar to the $(\mathbb{F}_\ell\text{-group})/(\pm 1)$.

Notation We simply denote an $(\mathbb{F}_\ell\text{-torsor})/(\mathcal{A}\pm)$ by simply $f^{\mathcal{A}\pm}$ or T if $f^{\mathcal{A}\pm}: T \Rrightarrow \mathbb{F}_\ell$ is such an object.

Def'n Let $T \in (\mathbb{F}_\ell\text{-torsor})/(\mathcal{A}\pm)$.

We define $\mathrm{Aut}_+(T) = \mathrm{Aut}_+(f_T^{\mathcal{A}\pm})$

$$\begin{aligned}&\| \\ &\{\alpha : T \to T \text{ bijection of sets} : \\ &\quad \exists \lambda \in \mathbb{F}_\ell,\ \forall e \in E,\ (f_T \circ \alpha)(e) = f_T(e) + \lambda_\alpha \}\end{aligned}$$

depends on a choice of representative.

Let's show that $\mathrm{Aut}_+(T)$ is independent of the representative f_T.

$$f_T^{\mathcal{A}\pm} = \mathbb{F}_\ell^{\mathcal{A}\pm} \circ f_T.$$

Let $\gamma : \mathbb{F}_\ell \longrightarrow \mathbb{F}_\ell$ be in $\mathbb{F}_\ell^{\mathcal{A}\pm}$ given by

$$\gamma(z) = \mu_\gamma z + \lambda_\gamma$$

$$\mu_\gamma \in \{\pm 1\},\ \lambda_\gamma \in \mathbb{F}_\ell.$$

$$\begin{aligned}((\gamma \circ f_T) \circ \alpha)(e) &= \gamma((f_T \circ \alpha)(e)) \\ &= \gamma(f_T(e) + \lambda_\gamma) \\ &= \mu_\gamma(f_T(e) + \lambda_\alpha) + \lambda_\gamma \\ &= \mu_\gamma f_T(e) + \lambda_\gamma + \mu_\gamma \lambda_\alpha \\ &= \underbrace{(\gamma \circ f_T)(e) + \mu_\gamma \lambda_\alpha}_{\text{still a translation}}\end{aligned}$$

$\Rightarrow$ $\mathrm{Aut}_+(T)$ is independent of representative of $f^{\mathcal{A}\pm}$.

<u>Notation</u> $(\gamma \circ f_T) \circ \alpha = \gamma^\alpha \circ f_T$

$$\mu_{\gamma^\alpha} = \mu_\gamma$$

$$\lambda_{\gamma^\alpha} = \mu_\gamma \lambda_\alpha + \lambda_\gamma$$

Check that this action doesn't collapse the polymorphism:

$$\gamma_1 \circ f_T \circ \alpha = \gamma_2 \circ f_T \circ \alpha \qquad ⊛$$

$$\mu_{\gamma_1^\alpha} = \mu_{\gamma_1} \qquad \Big| \qquad \mu_{\gamma_2^\alpha} = \mu_{\gamma_2}$$

$$\lambda_{\gamma_1^\alpha} = \mu_{\gamma_1}\lambda_\alpha + \lambda_{\gamma_2} \qquad \Big| \qquad \lambda_{\gamma_2^\alpha} = \mu_{\gamma_2}\lambda_\alpha + \lambda_{\gamma_2}.$$

Supposing ⊛, $\mu_{\gamma_1} = \mu_{\gamma_2}$.

$$\cancel{\mu_{\gamma_1}\lambda_\alpha} + \lambda_{\gamma_1} = \cancel{\mu_{\gamma_2}\lambda_\alpha} + \lambda_{\gamma_2} \Rightarrow \lambda_{\gamma_1} = \lambda_{\gamma_2}$$
$$\Rightarrow \gamma_1 = \gamma_2.$$

So the action of translation of polymorphism is well-defined.

<u>Def'n</u> $\mathrm{Aut}_\pm(T) := \{\alpha : T \to T \text{ bijection} : f_T \circ \alpha = \gamma_\alpha \circ f_T$ where $\gamma_\alpha \in \mathbb{F}_\ell^{\rtimes\pm}\}$

Checking that this is well-def'd is similar to $\mathrm{Aut}_+(T)$.

$\rightsquigarrow$ $\mathrm{Aut}_+(T) \le \mathrm{Aut}_\pm(T)$ in particular.

- Let $T \in (\mathbb{F}_\ell\text{-torsor})/(\rtimes\pm)$.

  obvious fact: $\mathrm{Aut}_\pm(T) \in \mathrm{Ab}$.

- Less obvious:

  $\mathrm{Aut}_\pm(T)$ can be given a $(\mathbb{F}_\ell\text{-group})/(\pm 1)$ structure.

<u>Construction</u> Pick any representative f_T of $f_T^{\rtimes\pm}$.

Then define $f^{\pm}_{\mathrm{Aut}_+(T)} := (\pm 1)\cdot(\alpha \overset{g}{\mapsto} \lambda_\alpha)$

where $f_T \circ \alpha = f_T(e) + \lambda_\alpha$.

$$\rightsquigarrow \mathrm{Aut}_+(T) \underset{-g}{\overset{g}{\rightrightarrows}} \mathbb{F}_\ell$$
$$\alpha \longmapsto \lambda_\alpha.$$

To see that this polymorphism is well-defined, pick another representative $\gamma \circ f_T$, $\gamma \in \mathbb{F}_\ell^{\rtimes \pm}$.

$$
\begin{aligned}
(\gamma \circ f_T \circ \alpha)(e) &= \gamma((f_T \circ \alpha)(e)) \\
&= g_\gamma (f_T(e) + \lambda_\alpha) + \lambda_\gamma \\
&= g_\gamma f_T(e) + \lambda_\gamma + g_\gamma \lambda_\alpha \\
&= \gamma \circ f_T(e) + g_\gamma \lambda_\alpha
\end{aligned}
$$

and we use that

$$g_\gamma \lambda_\alpha \in \{\lambda_\alpha, -\lambda_\alpha\}. \qquad \square$$

(27) Hodge Theater (II)

A First Look of the Big Hodge Theater.

Fix:
- X/F_0 punctured ell curve over a # fld.
- K/F_0 set hypoth.
- $\underline{\epsilon} \in \underline{C}_K^c$ nonzero
- $V = \underline{\mathbb{V}}$ the image of a section of

$$S(\mathrm{val}(F_0)) \subseteq \mathrm{val}(K) \underset{S}{\rightleftarrows} \mathrm{val}(F)$$

Def'n A prime strip of Frobenioids is tuples

$$\mathfrak{F} = (\mathcal{F}_v)_{v \in V}$$

$$\text{where } \mathcal{F}_v = \begin{cases} \mathrm{TFrd}(\underline{\underline{X}}_v) & v \nmid \infty \text{ bad} \\ \mathrm{GFrd}(\underline{X}_v) & v \nmid \infty \text{ good} \\ \text{aut hol spaces}, & v \mid \infty. \end{cases}$$

- A morphism of prime strips is just a collection of morphisms.

<u>Notations</u> $Z/K \rightsquigarrow Z_v := Z \times_{\operatorname{Spec} K} K_v$.

<u>Def'n</u> • A <u>capsule</u> of prime strips for a set T

is $\mathfrak{F}_T = (\mathfrak{F}_t)_{t\in T}$.

• A morphism of capsules is just a collection of morphisms.

$\boxed{\text{Big Hodge Theater}}$

$$\begin{tikzcd}
\text{add/geom} & \mathfrak{F}_{\Delta\pm} & & \mathfrak{F}_{\circledast} & \text{mult/arith} \\
\Theta^{ell}\text{-bridge} & & & & \Theta\text{-bridge} \\
& (\mathfrak{F}_t)_{t\in T} = \mathfrak{F}_T \arrow[uu, shift left, "\psi^{ev}_{\Delta\pm}"'] \arrow[uu, shift right] \arrow[rr, bend left, "\text{``glue''}"] \arrow[dd, shift left, "\psi^{sync}_{\Delta\pm}"] \arrow[dd, shift right] & & \mathfrak{F}_J = (\mathfrak{F}_j)_{j\in J} \arrow[uu, shift left, "\psi^{ev}_{\circledast}"] \arrow[uu, shift right] \arrow[dd, shift left, "\psi^{sync}_{\circledast}"] \arrow[dd, shift right] & (\text{LOCAL}) \\
\Theta^{\pm}\text{-bridge} & & & & \text{NF-bridge} \\
& \mathrm{GFrd}(\underline{X}_K) \arrow[d, equal] & & \mathrm{GFrd}(\underline{C}_K) \arrow[d, equal] & \\
& \mathcal{D}^{\circledcirc\pm} & & \mathcal{D}^{\circledcirc} & (\text{GLOBAL})
\end{tikzcd}$$

$T \simeq \mathbb{F}_\ell \simeq \underline{X}_K^{c}$ as $\mathbb{F}_\ell^{\Delta\pm}$-sets,

$J \simeq \mathbb{F}_\ell/\{\pm1\} = |\mathbb{F}_\ell| \simeq \underline{C}_K^{c}$ as $\mathbb{F}_\ell^{*}$-set

<u>Base Bridges</u>

- $(\psi^{ev}_{\circledast})_{j,v} = \mathrm{Isom}(\mathcal{B}(\underline{X}_v), \mathcal{D}_{\circledast,v}) \circ (\text{eval sec}) \circ (\text{base map}) \circ \mathrm{Isom}(\mathcal{D}_{j,v}, \mathcal{B}(\underline{X}_v))$.
- $(\psi^{sync}_{\circledast})_{j,v} = [j]\circ(\psi^{sync}_{\circledast})_{1,v}$, $[j]\in\mathbb{F}_\ell$.

  $(\psi^{sync}_{\circledast})_{1,v} = \mathrm{Aut}_{\underline{\varepsilon}}(\mathcal{D}^{\circledcirc}) \circ \alpha_v \circ \mathrm{Aut}(\mathcal{D}_{1,v})$
- $(\psi^{sync}_{\Delta\pm})_{t,v} = [t]\circ(\psi^{sync}_{\Delta\pm})_{0,v}$, $[t]\in\mathbb{F}_\ell^{\Delta\pm}$

  $(\psi^{sync}_{\Delta\pm})_{0,v} = \mathrm{Aut}^{c}(\mathcal{D}^{\pm\circledcirc}) \circ \alpha_{0,v} \circ \mathrm{Aut}^{+}(\mathcal{D}_{0,v})$.
- $(\psi^{ev}_{\Delta\pm})_{t,v} = \alpha_{t,v} \circ \mathrm{Aut}^{+}(\mathcal{D}_{0,v})$