Theory of p-adic ordinary curves
(after S. Mochizuki)

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X sm proj curve $/K$, $g=2$

Def (1) GL_2-oper $/X$ is a triple (E, ∇, H)

- (E, ∇) rk 2 connection,
- $H \subseteq E$ subline bdl s.t.

$$g(\nabla): H \xrightarrow{\sim} E/H \otimes \Omega_X \quad \mathcal{O}_X\text{-linear}$$

is an isom.

(2) PGL_2-oper :

$(E, \nabla, H) \sim (E', \nabla', H')$ if $\exists$ rk 1 (L, ∇) line bdl connection

s.t. $(E, \nabla, H) \otimes (L, \nabla) \simeq (E', \nabla, H)$.

Such a class is called a PGL_2-oper.

Fact (1) A PGL_2-oper can be lifted to an SL_2-oper

$$\underset{(GL_2\text{-oper} + \det = (\mathcal{O}, d))}{\Vert}$$

An SL_2-oper (E, ∇, L)

$\rightsquigarrow$ $0 \to L \to E \to L^{-1} \to 0$. $\quad L^{\otimes 2} \simeq \Omega_X$

(K-S isom: $\kappa: L \xrightarrow{\sim} L^{-1} \otimes \Omega_X$)

depends on ∇.

$E =$ the unique nontriv ext'n in $\underset{\substack{\wr\mid \\ H^0(X, \mathcal{O}_X)}}{\mathrm{Ext}^1(L^{-1}, L)}$.

(2) Aut (a PGL_2-oper) $= \{1\}$.

(3) PGL_2-opers form a torsor under $\underbrace{\Gamma(X, \Omega_X^{\otimes 2})}_{\text{grp}}$

$\forall\, \theta \in \Gamma(X, \Omega_X^{\otimes 2})$,

$$\theta : L^{-1} \longrightarrow L \otimes \Omega_X^1 \quad \overset{L^{\otimes 2} = \Omega_X}{\Longleftrightarrow} \quad E \to L^{-1} \xrightarrow{\theta} L \otimes \Omega_X^1 \longrightarrow E \otimes \Omega_X^1$$

For fixed (E, L):

$$\nabla \longmapsto \nabla + \theta = \nabla' \quad \text{s.t. } \kappa = \kappa'$$

i.e. ∇ & ∇' have the same K-S map.

<u>Complex theory</u>

$/\mathbb{C}$, X Riemann surface, $g \geq 2$. $\mathbb{H}/\Gamma \cong X$.

$\rightsquigarrow$ $\rho_X : \underset{\pi_1^{top}(X)}{\overset{\|}{\Gamma}} \to PGL_2(\mathbb{R}) = \mathrm{Aut}_{Hol}(\mathbb{H})$ Canonical repn.

Have corresp.

$$L^{\otimes 2} = \Omega_X, \quad \kappa : L \xrightarrow{\sim} L^{-1} \otimes \Omega_X$$

$$\begin{tikzcd}
\rho_X \arrow[rr, leftrightarrow, "\text{Simpson}"] \arrow[dr, leftrightarrow] & & \left(L \oplus L^{-1}, \begin{pmatrix} 0 & \kappa \\ 0 & 0 \end{pmatrix}\right) \\
& (M, \nabla, L) \text{ oper} \arrow[ur, leftrightarrow, "gr"'] &
\end{tikzcd}$$

<u>Char p theory</u>

$/\mathbb{F}_p$ $\quad \mathcal{M} = \mathcal{M}_g$ moduli stack of genus $g \geq 2$ curve.

$\pi : \mathcal{C} \to \mathcal{M}$ universal curve.

Let $\mathcal{O}p$ = Moduli stack of $(X, \mathcal{E})$,

- X curve of genus g

- $\mathcal{E}$ PGL_2-oper $/X$.

$$\rightsquigarrow \quad \begin{array}{ccc} Op & \ni & (X,\mathcal{E}) \\ \downarrow & & \downarrow \\ M & \ni & X \end{array}$$

$Op \to M$ is an affine bdl assoc to

$\pi_*(\Omega^{\otimes 2}_{C/M}) \leftarrow$ loc free sheaf of rk $3g-3$

This is fibrewise a torsor under $\Gamma(X, \Omega_X^{\otimes 2})$.

<u>p-curvature map</u>

$$\begin{tikzcd} X \arrow[dr, "F_{X/k}"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ & k \arrow[r, "F_k"] & k \end{tikzcd}$$

(E,∇) rk r connection on X/k

$\rightsquigarrow \psi_\nabla : F^*_{X/k}(T_{X'/k}) \longrightarrow \mathcal{E}nd(E)$ horizontal map

$\partial \longmapsto (\nabla_\partial)^p - \nabla_{\partial^{(p)}}$ $\quad \mathcal{O}_X$-linear

$\psi_\nabla : E \longrightarrow E \otimes F^*_{X/k}(\Omega_{X'})$

Hitchin map $\rightsquigarrow$ $h(\psi_\nabla) \in \bigoplus_{i=1}^{r} \Gamma(X, F^*_{X/k}(\Omega_{X'}^{\otimes i}))$

$\rightsquigarrow h_p(\psi_\nabla) \in \bigoplus_{i=1}^{r} \Gamma(X', \Omega_{X'}^{\otimes i})$ $\quad$ (Laszlo-Pauly).

$\rightsquigarrow$ p-Hitchin $h_p : \mathcal{M}^r_{dR}(X/k) \to$ Hitchin base of X'

Apply this to PGL_2-oper (E,∇,H):

$\psi : E \to E \otimes F^*_{X/k}(\Omega_{X'})$ $\quad ((E,\nabla) : SL_2\text{-lift})$

$\psi : F^*_{X/k}(T_{X'}) \to Ad(E) \leftarrow$ rk 3

PGL_2's Hitchin base of X' $= \Gamma(X', \Omega_{X'}^{\otimes 2})$.

Def $\mathcal{B} \to \mathcal{M}$ Hitchin base: affine bdl / $\mathcal{M}$
assoc to $\pi_*(\Omega^{\otimes 2}_{\mathcal{C}^F/\mathcal{M}})$.

$$\begin{tikzcd} \mathcal{C}^F \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d] \\ \mathcal{M} \arrow[r, "F_{\mathcal{M}}"] & \mathcal{M} \end{tikzcd}$$

Look at

$$\begin{tikzcd} \Gamma(X, \Omega_X^{\otimes 2}) - \mathrm{Op} \arrow[r, "\text{p-Hitchin}", "h_p"'] \arrow[dr] & \mathcal{B} - \Gamma(X', \Omega_{X'}^{\otimes 2}). \arrow[d] \\ & \mathcal{M} \arrow[u, bend right, "0"'] \end{tikzcd}$$

sm / $\mathbb{F}_p$.

All dim $= 3g-3$.

Thm (S. Mochizuki; PGL_2: Berzukavikov – Chen – Zhu; G)
h_p is finite flat of deg p^{3g-3}.

Rmk Interested in opers of p-cover $= 0$ ($\mathcal{M} \xrightarrow{0} \mathcal{B}$)
Thm $\Rightarrow$ these are fin many & no more than $3g-3$.

Def $\mathcal{E}$ PGL_2-oper / X/k

(1) $\mathcal{E}$ is nilp if $h_p(\mathcal{E}) = 0$

$$\begin{tikzcd} \mathcal{N} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Op} \arrow[d] \\ \mathcal{M} \arrow[r] & \mathcal{B} \end{tikzcd}$$

$\mathcal{N}$ classifies nilp PGL_2-opers
finite / $\mathcal{M}$. deg $= p^{3g-3}$.
(a priori not sm)

(2) $\mathcal{E}$ adm if $\psi_{\mathcal{E}} : F^*_{X/k}(T_{X'}) \to \mathrm{Ad}(\mathcal{E})$ nowhere vanishes.

(**Prop** Given $\mathcal{E} \in \mathcal{N}$, $\mathcal{E}$ adm $\Leftrightarrow$ $\mathcal{N}$ sm at $\mathcal{E}$.)

(3) $H_{\mathcal{E}} : F_{X/k}^*(T_{X'}) \xrightarrow{\psi_{\mathcal{E}}} Ad(\mathcal{E}) \longrightarrow \mathcal{H}om(L, L^{-1}) \simeq T_X$.

$$\begin{array}{c} \wr\wr \\ T_X^{\otimes p} \end{array} \iff H_{\mathcal{E}} \in \Gamma(X, \Omega_X^{\otimes p-1})$$

square Hasse invariant.

(4) $\Phi_{\mathcal{E}} : H^1(X', T_{X'}) \xrightarrow{F_{X/k}^*} H^1(X, F_{X/k}^*(T_{X'})) \xrightarrow{H_{\mathcal{E}}} H^1(X, T_X)$

$\mathcal{E}$ is <u>ordinary</u> if $\Phi_{\mathcal{E}}$ is an isom.

<u>Thm</u> (1) $\mathcal{E}$ ordinary $\iff h_p : Op \to B$ is étale at $\mathcal{E}$

(2) ordinary $\Rightarrow$ adm $\Rightarrow H_{\mathcal{E}} \neq 0$.

<u>Def</u> X is <u>(hyperbolic) ordinary</u> if $\exists$ <u>no</u> PGL_2-oper $/X$.

$$\begin{array}{ccc} \mathcal{N}^{ord} & \hookrightarrow & \mathcal{N} \quad \text{open locus} \\ \downarrow & & \downarrow \deg p^{3g-3} \\ \mathcal{M}^{ord} & \hookrightarrow & \mathcal{M} \end{array}$$

<u>Thm</u> (S. Mochizuki) $\mathcal{M}^{ord} \neq \phi$

<u>Conj</u> $\mathcal{M}^{ord} = \mathcal{M}$.

<u>Prop</u> $\mathcal{E}$ PGL_2-oper. Take an SL_2-lift.

$\mathcal{E}$ is nilp adm $\iff \exists!$ Frob str on $\mathcal{E}$:

i.e. $\exists!$ lift $\mathfrak{X}_2'$ of $\mathfrak{X}'$ to $W(k)$ and an isom

$\varphi_{\mathcal{E}} : C_{\mathfrak{X}_2'}^{-1}(gr(E), \kappa) \xrightarrow{\sim} (E, \nabla)$ as PGL_2-bdl + Connection

Pf ($\Leftarrow$) ψ_D locally $= F^*_{X/k}(\kappa)$ is nilp & nowhere vanishing

($\Rightarrow$) $\mathcal{E}$ is nilp ord $\Rightarrow$ $(\mathcal{E}, \varphi_{\mathcal{E}})$, $\mathfrak{X}_2'$ can be uniquely lifted to $W(k)$. $\square$

$(\mathbb{P}^1, S)$ works for X (genus g) w/ r marked pts D

(1) s.t. $2g-2+r>0$.

GL_2-opers : $(E, \nabla : E \to E \otimes \Omega^{\log}, H)$

$gr(\nabla)$ is an isom

(2) $\forall\, x \in D$, residue $M_x \neq 0$

rk 2 Connection $\longleftrightarrow$ diff operators of order 2

ψ $\qquad\qquad\qquad\qquad$ L.

- ψ nilp $\Leftrightarrow$ L has a polynomial sol'n
- $\psi = 0$ $\Leftrightarrow$ L is full set of sol'n

e.g. (1) $\mathcal{M}_{0,3}$ $(\mathbb{P}^1, \{0,1,\infty\})$

equation : $t(t-1)\left(\frac{\partial}{\partial t}\right)^2 + (2t-1)\frac{\partial}{\partial t} + \frac{1}{4} = 0$

has sol'n $u(t) = \sum_{i=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{i}^2 \cdot t^i$.

(ψ nilp but $\psi \neq 0$).

$y^2 = x(x-1)(x-\lambda)$

$$\begin{array}{ccc} E & \longrightarrow & \bar{E} \\ \pi \downarrow & & \downarrow \\ \mathbb{P}^1 - \{0,1,\infty\} & \longrightarrow & \mathbb{P}^1 \end{array}$$

$\varphi \circlearrowright (R^1\pi^{\log}_{dR}(\bar{E}/\mathbb{P}^1), \nabla_{GM}, H = e^*\Omega^{\log}_{E/\mathbb{P}^1})$

zero of $u(t)$ = supersing locus of E_λ

= zeros of $H_{\mathcal{E}}$.

(2) $\mathcal{M}_{0,4}$, $(\mathbb{P}^1, \{0,1,\lambda,\infty\})$

1-dim'l family of rigid connections

$$L_\beta = \left(\frac{\partial}{\partial t}\right)^2 + \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{t-\lambda}\right)\frac{\partial}{\partial t} + \frac{t-\beta}{t(t-1)(t-\lambda)} \qquad (\beta = \lambda+1)$$

has a polynomial sol'n deg $= p-1$.

$$\Downarrow$$

$$\begin{tikzcd} \mathcal{N} \arrow[r, hook] \arrow[dr, "\deg = 1"'] & \mathcal{N}_{0,4} \arrow[d] \\ & \mathcal{M}_{0,4} \ \text{sm} \end{tikzcd}$$

$\mathcal{N}_1 \to \mathcal{M}_{0,4}$ is an isom

$$\Downarrow$$

$\mathcal{N}_1 \subseteq \mathcal{N}^{ord}$

(In particular, $(\mathbb{P}^1, \{0,1,\lambda,\infty\})$ is ordinary).