Symmetric power functoriality for Hilbert modular forms

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F number field, $n \geq 1$.

$\mathbb{A}_F$ adèles, F_v completion at v.

Let $\pi \circlearrowleft GL_n(\mathbb{A}_F)$ autom rep.

Then $\pi = \bigotimes_v' \pi_v$, $\pi_v \circlearrowleft GL_n(F_v)$ irred adm.

$\forall v$, $\quad \mathrm{rec}_{F_v} : W_{F_v} \longrightarrow GL_n(\mathbb{C})$ associated Langlands para.

$\uparrow$ LLC for $GL_n(F_v)$

Conj (Langlands functoriality for GL_n)

Suppose that π is cuspidal, and

let $R : GL_n \to GL_m$ be an algebraic rep.

Then $\exists$ an automorphic rep $R_*(\pi) \circlearrowleft GL_m(\mathbb{A}_F)$

s.t. $\forall v$, $\quad \mathrm{rec}_{F_v}(R_*(\pi)_v) \cong R \circ \mathrm{rec}_{F_v}(\pi_v)$.

Examples (1) Can assume R irred.

$n = 1$, $R : GL_1 \to GL_1$, $\quad x \longmapsto x^N \quad (N \in \mathbb{Z})$.

If $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ Hecke char, $R_*(\chi) = \chi^N$.

(2) $n = 2$, $R = \mathrm{Sym}^m : GL_2 \to GL_{m+1}$ (up to twist).

e.g. $m = 2$, $\mathrm{Sym}^2 : GL_2 \to GL_3$

Gelbert-Jacquet (1978): $\mathrm{Sym}^2_* \pi$ ($= R_*(\pi)$) exists

& is cuspidal $\iff \pi \not\cong \pi \otimes \chi$ for any χ

$\iff \pi$ is NOT automorphically induced (AI).

<u>Applications</u> · Bounds towards Ramanujan conj.

· Langlands-Tunnell theorem.

· Kim-Shahidi. Kim established existence of $\mathrm{Sym}^3_*(\pi)$, $\mathrm{Sym}^4_*(\pi)$ for $\pi \circlearrowleft GL_2(\mathbb{A}_F)$.

If we restrict the class of allowable π, we can do more.

<u>Def'n</u> $\pi \circlearrowright GL_n(\mathbb{A}_F)$ cuspidal automorphic.

We say π is regular algebraic if it satisfies the following equivalent properties:

· $\forall v | \infty$, $\mathrm{rec}_{F_v}(\pi_v |\cdot|^{\frac{1-n}{2}})|_{\mathbb{C}^\times}$ is algebraic (identifying $\mathbb{C}^\times \leq W_{F_v}$ with $(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)(\mathbb{R})$.)

· There is an irred alg rep V of $(\mathrm{Res}_{F/\mathbb{Q}} GL_n)_{\mathbb{C}}$, a level subgrp $K \subseteq GL_n(\mathbb{A}_F^\infty)$, s.t. $(\pi^\infty)^K \neq 0$, and a Hecke-equivariant embedding

$$(\pi^\infty)^K \hookrightarrow H^*(\underset{\substack{\uparrow \\ \text{loc sym space of level } K.}}{X_K}, V).$$

e.g. If $F = \mathbb{Q}$, $n=2$, then a cuspidal $\pi \circlearrowright GL_2(\mathbb{A}_\mathbb{Q})$ corresponds canonically to one of

(i) holomorphic newform f of wt $k \geq 1$,

(ii) Maaß newform.

& π is regular alg $\iff$ π corr to (i).

e.g. If $F/\mathbb{Q}$ tot real, $n=2$, then

π reg alg $\iff$ π corr to a Hilbert modular newform
$f : \mathcal{H}^{[F:\mathbb{Q}]} \to \mathbb{C}$, $\mathcal{H}$ = upper-half plane,
of wt $(k_v)_{v|\infty}$, where $k_v \geq 2$ $\forall v$,
& (k_v mod 2) are indep of v.

<u>Thm</u> Let π 2 $GL_n(\mathbb{A}_F)$ be cuspidal regular algebraic (cAR).
Let F tot real $/\mathbb{Q}$.
Then $\forall p$ prime & $\forall \iota : \bar{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$,
$\exists$ a conti rep $r_\iota(\pi) : G_F \longrightarrow GL_n(\bar{\mathbb{Q}}_p)$
$$G_F = Gal(\bar{F}/F).$$
s.t. $\forall\forall v$, $WD(r_\iota(\pi)|_{G_{F_v}})^{F\text{-ss}} \cong \iota^{-1} rec_{F_v}(\pi_v |\cdot|^{\frac{1-n}{2}})$. $(*)$
Moreover, if π polarizable
(i.e. self-dual up to twist)
then $(*)$ holds for all finite places v.

<u>Def'n</u> F tot real. Say $\rho : G_F \to GL_n(\bar{\mathbb{Q}}_p)$ is <u>automorphic</u>
if $\exists$ π CAR, s.t. $\rho \simeq r_\iota(\pi)$.

<u>Fact</u> If π 2 $GL_2(\mathbb{A}_F)$ + $Sym^m_*(\pi)$ 2 $GL_{m+1}(\mathbb{A}_F)$ exists,
then $Sym^m_*(\pi)$ is regular alg, and
$\exists$ isom $r_\iota(Sym^m_*(\pi)) \simeq Sym^m r_\iota(\pi)$
of reps $G_F \longrightarrow GL_{n+1}(\bar{\mathbb{Q}}_p)$.
Conversely, can try to show $Sym^m_*(\pi)$ exists
by showing $Sym^m r_\iota(\pi)$ is automorphic (for some π).

Thm (Newton-Thorne, 2019)

Let $\pi \wr GL_2(\mathbb{A}_\mathbb{Q})$ be CAR. Then $\forall m \geq 1$, $\mathrm{Sym}^m_*(\pi)$ exists.
It is cuspidal $\Leftrightarrow$ π is not automorphially induced.

Proof uses overconvergent modular forms
+ geometry of 2-adic tame level 1 eigenvar.

Thm (Newton-Thorne, 2022)

Let F tot real. $\pi \wr GL_2(\mathbb{A}_F)$ CAR.
Then $\forall m \geq 1$, $\mathrm{Sym}^m_*(\pi)$ exists,
and is cuspidal $\Leftrightarrow$ π is not AI.

Rmk (1) Proof is new even in case $F = \mathbb{Q}$.
(2) This implies the same result w/ "regular alg"
replaced by "π_∞ square-integrable"
$\leftrightarrow$ HMF of wt $(k_v)_{v|\infty}$, $k_v \geq 2$.

Strategy of [NT22]

Goes back to Clozel-Thorne (2014).
We argue by induction on $m \geq 1$,
using known cases of $m \leq 4$.
Want to use autom lifting thms,
which say that if $\exists$ isom of residual reps
$\overline{\mathrm{Sym}^m r_2(\pi)} \cong \overline{r_2(\Pi)} : G_F \to GL_{m+1}(\overline{\mathbb{F}}_p)$

for some Π & $GL_{m+1}(\mathbb{A}_F)$ CAR,
then $\mathrm{Sym}^m_*(\pi)$ is automorphic.

If p prime, $p < m < 2p$ (say $m = p + r - 1$), then
$\mathrm{Sym}^m : GL_{2,\mathbb{F}_p} \longrightarrow GL_{m+1,\mathbb{F}_p}$ is reducible:

so $\overline{\mathrm{Sym}^m r_2(\pi)}$ is isom to

$$(\underset{\substack{\uparrow \\ \det \overline{r_2(\pi)}.}}{\bar{\chi}^r} \otimes \mathrm{Sym}^{p-r-1}\overline{r_2(\pi)}) \oplus (\underset{\substack{\uparrow \\ \text{Frob twist}}}{{}^{\varphi_p}\overline{r_2(\pi)}} \otimes \mathrm{Sym}^{r-1}\overline{r_2(\pi)})$$

dim $\quad 1 \times (p-r) \quad + \quad 2 \times r$

Bertrand's postulate says that such a p always exists!
This is the basis of the induction.

<u>Caveat</u> We can prove ALT only under conditions
including
- necessary conditions, e.g. ρ de Rham,
- useful technical conditions,
  often including $\bar{\rho}$ irred, $p > \dim \bar{\rho}$.

We focus now on checking the residual automorphy of $\mathrm{Sym}^m \overline{r_2(\pi)}$.
By a level-raising result (Thorne 2022)
it suffices to check the residual automorphy of the 2 factors

$$\bar{\chi}^r \otimes \mathrm{Sym}^{p-r-1}\overline{r_2(\pi)} = \bar{\chi}^r \otimes \mathrm{Sym}^{m-2r}\overline{r_2(\pi)}$$

residual automorphy follows by induction.

$${}^{\varphi_p}\overline{r_2(\pi)} \otimes \mathrm{Sym}^{r-1}\overline{r_2(\pi)} = {}^{\varphi_p}\overline{r_2(\pi)} \otimes \mathrm{Sym}^{m-p}\overline{r_2(\pi)}.$$

residual automorphy of $\otimes$-factors follows by induction.

<u>But</u> Do not know $\otimes$-product functoriality!

__Strategy__ Reduce to case π has wt 0, and

choose a congruence $\pi \equiv \pi'$ (mod p)

s.t. $\quad {}^{\varphi_p} r_2(\pi) \otimes \mathrm{Sym}^{r-1} r_2(\pi')$ is (Hodge-Tate) regular.

(lift φ_p to $G_{\mathbb{Q}_p}$)

__Note__ $\mathrm{Aut}(\mathbb{C})$ acts on the set of CAR of $GL_2(\mathbb{A}_F)$

(by e.g. conjugation of Hecke eigenvalues).

We have ${}^{\varphi_p} r_2(\pi) = r_2({}^{\iota\varphi_p\iota^{-1}}\pi)$.

We will in fact show $r_2({}^{\sigma}\pi) \otimes \mathrm{Sym}^{r-1} r_2(\pi')$

is automorphic for any $\sigma \in \mathrm{Aut}(\mathbb{C})$.

__First case__ $\sigma = 1$. Then $\exists$ an isom

$$\overline{r_2(\pi)} \otimes \mathrm{Sym}^{r-1} \overline{r_2(\pi')} \cong \overline{r_2(\pi)} \otimes \mathrm{Sym}^{r-1} \overline{r_2(\pi)}$$
$$\cong (\bar{\chi} \otimes \mathrm{Sym}^{r-2} \overline{r_2(\pi)}) \oplus \mathrm{Sym}^{r} \overline{r_2(\pi)}.$$

(Clebsch-Gordan formula).

Residual automorphy follows by induction.

__Second case__

Note ${}^{\sigma}\pi$ only depends on image of σ in $\mathrm{Gal}(K_\pi/\mathbb{Q})$

where K_π is the normal closure of definition of π.

Suppose $\exists$ a prime l, place $v | l$ of K_π,

$\delta_v \in I_{v|l}$ (inertia grp).

such that $\sigma = \delta_v$.

Then $r_2({}^{\sigma}\pi) \otimes \mathrm{Sym}^{r-1} r_2(\pi)$ is automorphic.

Why? True $\iff$ some member of the compatible system

$$(r_\jmath({}^\sigma\pi) \otimes \mathrm{Sym}^{r-1} r_\jmath(\pi))_\jmath$$

is automorphic.

We choose $\jmath : \bar{\mathbb{Q}}_\ell \to \mathbb{C}$ so that $\jmath^{-1}|_{K_\pi}$ induces the place v.

Then $\overline{r_\jmath({}^{\delta_v}\pi)} \simeq^{\jmath^{-1}\delta_v \jmath} \overline{r_\jmath(\pi)} \cong \overline{r_\jmath(\pi)}$,

so $\overline{r_\jmath({}^{\delta_v}\pi)} \otimes \mathrm{Sym}^{r-1} \overline{r_\jmath(\pi')} \cong \overline{r_\jmath(\pi)} \otimes \mathrm{Sym}^{r-1} \overline{r_\jmath(\pi')}$

and the residual rep is autom.

<u>General case</u>

Spec $\mathbb{Z}$ simply conn $\Rightarrow$ $\mathrm{Gal}(K_\pi/\mathbb{Q})$ generated by inertia groups.

$\Rightarrow$ can write $\sigma|_{K_\pi} = \delta_{v_1} \cdots \delta_{v_s}$ for primes $\ell_1, \dots, \ell_s$,

$\delta_{v_i} \in I_{v_i/\ell_i} \subseteq \mathrm{Gal}(K_\pi/\mathbb{Q})$.

The compatible systems $(r_\jmath(\pi) \otimes \mathrm{Sym}^{r-1} r_\jmath(\pi))$

$(r_\jmath({}^\sigma\pi) \otimes \mathrm{Sym}^{r-1} r_\jmath(\pi))$

are then connected by a chain of congruences

in residue char $\ell_1, \dots, \ell_s$.

$\rightsquigarrow$ This requires a new "functoriality lifting thm"!