Semi-linear operators on q-de Rham cohomology

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<u>q-de Rham</u> (Scholze, 2017).

$R/\mathbb{Z}$ sm alg. Choose an étale morphism

(can be $\mathbb{Z}[\frac{1}{N}]$) $\quad \square : \mathbb{Z}[x_1, \cdots, x_n] \to R$.

$$\rightsquigarrow \; q\Omega^{\square}_{R/\mathbb{Z}} := [R[\![q-1]\!] \xrightarrow{\nabla_q} \Omega^1_{R/\mathbb{Z}}[\![q-1]\!] \to \Omega^2_{R/\mathbb{Z}}[\![q-1]\!] \to \cdots]$$

Here
$$\nabla_q f = \sum_{i=1}^{n} \partial_{q,x_i} f \, dx_i = \sum_{i=1}^{n} \left(\frac{f(qx_i) - f(x_i)}{qx_i - x_i} \right) dx_i$$

(Think: q is the role of $1+\varepsilon$,
$qx_i - x_i = (q-1)x_i$ " $= \varepsilon x_i$ ").

<u>Thm</u> (Bhatt-Scholze, Wagner)

$q\Omega^{\square}_{R/\mathbb{Z}}$ is indep of the choice of $\square$ up to q-isom,
So we can glue to $q\Omega^{\bullet}_{X/\mathbb{Z}} \in \hat{D}(\mathbb{Z}[\![q-1]\!])$.

⚡ Warning: Still lack a conceptual understanding of $q\Omega^{\bullet}_{X/\mathbb{Z}}$.

<u>Note</u> $q\Omega^{\bullet}_{X/\mathbb{Z}} / (q-1) \simeq \Omega^{\bullet}_{X/\mathbb{Z}}$

q-de Rham gives a way to glue different p-adic Hodge Coh together (for varying p).

<u>Conj</u> (Scholze '17. §6)

(i) p prime. $\exists$ operator $\varphi_{qdR} \circlearrowleft (q\Omega^{\bullet}_{R/\mathbb{Z}})^{\wedge}_{p}$.
which on frames $q\Omega^{\square}_{R/\mathbb{Z}}$ are induced by $x_i \mapsto x_i^p$, $q \mapsto q^p$

(2) Moreover, $\exists\ \Gamma := \mathbb{Z}_p^\times \curvearrowright (q\text{-}\Omega^\bullet_{R/\mathbb{Z}})^\wedge_p$

semilinear w.r.t. $\Gamma \curvearrowright \mathbb{Z}_p[\![q-1]\!]$.

<u>Note</u> $\alpha \in \mathbb{Z}_p^\times$. $\gamma_\alpha : \mathbb{Z}_p[\![q-1]\!] \longrightarrow \mathbb{Z}_p[\![q-1]\!]$

$$q-1 \longmapsto q^\alpha - 1 = \sum_{n=1}^\infty \binom{a}{n}(q-1)^n$$

$$\binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!}.$$

(3) For any $n \in \mathbb{Z}$, $\exists\ \gamma_n \curvearrowright q\,\Omega^\bullet_{R[\frac{1}{n}]/\mathbb{Z}[\frac{1}{n}]}$

semilinear w.r.t. $\gamma_n : \mathbb{Z}[\frac{1}{n}][\![q-1]\!] \xrightarrow{\sim} \mathbb{Z}[\frac{1}{n}][\![q-1]\!]$

$$q \longmapsto q^n.$$

Moreover, if $n = p$ prime,

$$\varphi_{qdR} \curvearrowright \underbrace{\left((q\,\Omega^\bullet_{R/\mathbb{Z}})^\wedge_p[\tfrac{1}{p}]\right)^\wedge_{q-1}}_{\text{is}} \simeq (\Omega^\bullet_{R^\wedge_p/\mathbb{Z}_p}[\tfrac{1}{p}])[\![q-1]\!] \circlearrowleft \gamma_p$$

$$\simeq R\Gamma_{crys}((R/p)/\mathbb{Z}_p)[\tfrac{1}{p}][\![q-1]\!].$$

$\gamma_p \curvearrowright (q\,\Omega^\bullet_{R[\frac{1}{p}]/\mathbb{Z}[\frac{1}{p}]})^\wedge_p$ $\qquad \circlearrowleft \varphi_{cris}$

& $\varphi_{qdR} = \gamma_p \circ \varphi_{cris} = \varphi_{cris} \circ \gamma_p$.

<u>Local q-de Rham</u>

$X/\mathbb{Z}_p$ p-completely sm p-adic formal sch.

$(\mathbb{Z}_p[\![q-1]\!], [p]_q) \in \Delta$, $\quad \mathbb{Z}_p[\![q-1]\!]/[p]_q \cong \mathbb{Z}_p[\zeta_p]$.

$\circlearrowleft$ $\qquad [p]_q = \dfrac{q^p-1}{q-1}$

$\varphi : q \mapsto q^p$

Define $q\,\Omega^\bullet_{X/\mathbb{Z}_p} := R\Gamma_\Delta(X \times_{\mathbb{Z}_p} \mathbb{Z}_p[\zeta_p]/\mathbb{Z}_p[\![q-1]\!]) \in \hat{D}(\mathbb{Z}_p[\![q-1]\!])$.

$\circlearrowleft \phi$ $\qquad$ Complete complex

<u>Thm</u> $\exists$ an action $\Gamma = \mathbb{Z}_p^\times$ on $q\,\Omega_{X/\mathbb{Z}_p}$ Semilinear w.r.t. $\Gamma \curvearrowright \mathbb{Z}_p[\![q-1]\!]$.

s.t. (1) Γ commutes with ϕ

(2) Γ is conti w.r.t. profinite top on Γ

and $(p, [p]_q)$-adic top on $q\Omega^\bullet_{X/\mathbb{Z}_p}$.

$$\underset{\cap}{} \quad D(\underbrace{\mathrm{Spf}(\mathbb{Z}_p[\![q-1]\!]) / \mathrm{Spf}(\mathcal{C}(\Gamma, \mathbb{Z}_p[\![q-1]\!]))}_{\text{quot stack}})$$

(3) Γ acts trivially on $q\Omega^\bullet_{X/\mathbb{Z}_p} / (q-1) \simeq \Omega^\bullet_{X/\mathbb{Z}_p}$

(4) If X proper sm $/ \mathbb{Z}_p$,

& we assume that $H^i(q\Omega^\bullet_{X/\mathbb{Z}_p})$ finite free,

then $H^i(q\Omega^\bullet_{X/\mathbb{Z}_p})$ is the Wach module

of the Gal rep $H^i(X_{\bar{\eta}}, \mathbb{Z}_p)$.

<u>Recall</u> $\{\text{Wach modules}\} := \left\{ \begin{array}{c} \text{fin free } \mathbb{Z}_p[\![q-1]\!]\text{-mods } M \\ + \varphi \curvearrowright M \curvearrowleft \Gamma \\ \text{s.t. } \Gamma \text{ is trivial on } M/(q-1) \\ \& \ \Gamma \text{ is conti} \end{array} \right\}$

$\parallel\!\wr$ Fact

$\left\{ \begin{array}{c} \text{crystalline reps of } \mathrm{Gal}_{\mathbb{Q}_p} \\ \text{on fin free } \mathbb{Z}_p\text{-mods} \end{array} \right\}$.

<u>Sketch</u> $R\Gamma(\mathrm{WCart}_{X \times_{\mathbb{Z}_p} \mathbb{Z}_p[\zeta_p] / \mathbb{Z}_p[\![q-1]\!]}, \mathcal{O})$

$\cong R\Gamma_{\Delta}((X \times_{\mathbb{Z}_p} \mathbb{Z}_p[\zeta_p]) / \mathbb{Z}_p[\![q-1]\!])$.

$\Gamma \curvearrowright \mathrm{WCart}_{X \times_{\mathbb{Z}_p} \mathbb{Z}_p[\zeta_p] / \mathbb{Z}_p[\![q-1]\!]} \circlearrowleft \mathrm{Spf}\, \mathcal{C}(\Gamma, \mathbb{Z}_p[\![q-1]\!])$.

<u>Global q-de Rham</u>

$X / \mathbb{Z}$ proper sm.

<u>Bhatt-Scholze, Wagner</u> Define

$$\begin{array}{ccc} q\Omega^\bullet_{X/\mathbb{Z}} & \longrightarrow & \prod_p q\Omega_{\hat{X_p}/\mathbb{Z}_p} \\ \downarrow & \ulcorner & \downarrow \\ q\Omega_{X/\mathbb{Z}} \hat{\otimes}_{\mathbb{Q}} \mathbb{Q}[\![q-1]\!] & \longrightarrow & \prod_p \Omega_{\hat{X_p}/\mathbb{Z}_p}[\tfrac{1}{p}] \hat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p[\![q-1]\!] \end{array}$$

(Sullivan arithmetic fracture).

$M \in D(\mathbb{Z})$,
$$\begin{tikzcd} M \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \prod_p M_p^\wedge \arrow[d] \\ M \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r] & \prod_p M_p^\wedge \otimes_{\mathbb{Z}} \mathbb{Q} \end{tikzcd}$$

$$\Rightarrow \quad q\Omega^\bullet_{X/\mathbb{Z}} / (q-1) \simeq \Omega^\bullet_{X/\mathbb{Z}}$$
$$(q\Omega^\bullet_{X/\mathbb{Z}})^\wedge_p \cong q\Omega^\bullet_{\hat{X}_p/\mathbb{Z}_p}.$$

Let $n \in \mathbb{Z}$, $\gamma_n \curvearrowright \Omega^\bullet_{X[\frac{1}{n}]/\mathbb{Z}[\frac{1}{n}]}$

$$\rightsquigarrow \quad \begin{tikzcd} \gamma_n \curvearrowright q\Omega^\bullet_{X[\frac{1}{n}]/\mathbb{Z}[\frac{1}{n}]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \prod_{p \nmid n} q\Omega_{\hat{X}_p/\mathbb{Z}_p} \arrow[d] \\ q\Omega^\bullet_{X[\frac{1}{n}]/\mathbb{Z}[\frac{1}{n}]} \hat{\otimes}_{\mathbb{Q}} \mathbb{Q}[\![q-1]\!] \arrow[r] & \prod_{p \nmid n} \Omega_{\hat{X}_p/\mathbb{Z}_p}[\tfrac{1}{p}] \hat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p[\![q-1]\!] \end{tikzcd}$$

$$\circlearrowleft$$
$$\gamma_n = \mathrm{id} \otimes (q \mapsto q^n)$$

Check: $\gamma_p \circ \phi_{cris} = \phi_{qdR} = \phi_{cris} \circ \gamma_p$.

<u>Remaining questions</u>

Q Does γ_p reflects some str of $(\hat{X}_p)_\eta$?

Ideally, $\left\{ \begin{matrix} \text{proper sm rigid-analytic} \\ \text{var } / \mathbb{Q}_p \end{matrix} \right\} \xrightarrow{?} \hat{D}(\mathbb{Q}_p[\![q-1]\!])$.