

# Cohomology of Projective Spaces

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## §1 Finitely Generated Sheaves

$(X, \mathcal{O}_X)$  loc. ringed space.  $\mathcal{F} \in \text{Sh}(\text{Mod}_{\mathcal{O}_X})$

Defn  $\mathcal{F}$  fin. gen'd if  $\forall x \in X, \exists x \in U \subseteq X, n \geq 0$

$$\text{s.t. } (\mathcal{O}_X|_U)^{\oplus n} \rightarrow \mathcal{F}|_U.$$

namely: loc. gen'd by fin. collections of sections.

Lemma  $M \in \text{Mod}_A$ .  $M$  f.g. as  $A$ -mod  $\Leftrightarrow M \in \text{Ob} \text{ f.g.}$

Caution Hartshorne: " $\text{f.g.} + \text{coh} = \text{coh}$ " (in general wrong)  
but valid when  $X = \text{loc. nt sch.}$

## §2 Odds and Ends

Lemma  $f: Z \rightarrow X$  closed immersion.  $\mathcal{F} \in \text{Sh}_{\text{Ab}}(Z)$

$$\Rightarrow \text{canonical } H^i(Z, \mathcal{F}) \xrightarrow{\cong} H^i(X, f_* \mathcal{F}) \quad (i \geq 0)$$

Proof.  $f_*$  preserves flasqueness.

$$Q (f_* \mathcal{F})_x = \begin{cases} \mathcal{F}_x, & x \in Z \\ 0, & x \notin Z \end{cases}$$

Note that  $H^0(Z, \mathcal{F}) \cong H^0(X, f_* \mathcal{F})$  (as  $\Gamma(\cdot, \cdot)$ ) canonically.

A flasque resolution of  $\mathcal{F}$

$\Gamma(Z, \cdot)$  same complexes on  $X \Rightarrow$  same  $H^i$ .

□

Lemma  $X$  noetherian top space.  $(\mathcal{F}_j)$  direct system of ab sh.

$$\Rightarrow \mathcal{F}_j \rightarrow \varinjlim_j \mathcal{F}_j \rightsquigarrow \varinjlim_j H^i(X, \mathcal{F}_j) \xrightarrow{\cong} H^i(X, \varinjlim_j \mathcal{F}_j).$$

### §3 Main Result

$\mathcal{F}, \mathcal{G} \in \text{Sh}(\text{Mod}_{\mathcal{O}_X})$ ,  $\mathcal{F}$  q-coh & flat (as  $\mathcal{O}_X$ -mod).

$\rightsquigarrow$  natural homomorphism

$$H^r(X, \mathcal{F}) \otimes_{\mathcal{O}_X} H^r(X, \mathcal{G}) \rightarrow H^r(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \quad (\forall r \geq 0)$$

of  $\Gamma(X, \mathcal{O}_X)$ -mods.

It comes from the facts:

can be computed  
using Čech complexes

- both sides are  $S$ -functors in  $\mathcal{G}$ 
  - (note:  $\mathcal{F}$  flat on  $\mathcal{F} \otimes_{\mathcal{O}_X} (\cdot)$ .
  - indeed,  $H^0(X, \mathcal{F})$  flat over  $\mathcal{O}_X$ )
- Left functor: effaceable ( $\Rightarrow$  univ).
- $\exists$  natural map for  $r=0$ .

Thm (Serre)  $\forall r \geq 1$ , put  $X = \mathbb{P}^r_A$ ,  $S = A[x_0, \dots, x_r]$ .

(a) The natural isom of graded  $S$ -mods:

$$S \xrightarrow{\cong} \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(n)).$$

(b)  $\forall 0 < i < r$ ,  $n \in \mathbb{Z}$ ,

$$H^i(X, \mathcal{O}_X(n)) = 0.$$

(c)  $H^r(X, \mathcal{O}_X(-r-i)) \cong A$ .

(d) Perfect pairing (as a natural  $A$ -linear map)

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-i)) \rightarrow H^r(X, \mathcal{O}_X(-r-i)) \cong A.$$

(of f.g.  $A$ -mods,  $\forall n \in \mathbb{Z}$ , via  $\text{Hom}(-, A)$ ).

(e)  $\forall i > r$ ,  $H^i(X, \mathcal{O}_X(n)) = 0$  ( $n \in \mathbb{Z}$ ).

Namely, the information in  $S$  is completely determined by

$$\left\{ \begin{array}{l} H^0 \rightarrow T_*(X) = \bigoplus H^0(X, \mathcal{O}_X(n)) \\ H^i (0 < i < r) : \text{vanishing} \\ H^r \rightarrow \text{"dual" to } H^0 \\ H^i (i > r) : \text{vanishing} \end{array} \right.$$

Proof. To compute  $\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n)$ .

(a) Checked earlier (see e.g. Hartshorne Prop II.5.13).

$$\left\{ \begin{array}{l} \text{Čech : } \mathcal{D} = \{ D_i(x_i) \mid i=0, \dots, r \}, \sum^2(D_i, \mathcal{O}_X(n)) = 0, \forall i > r \\ \Rightarrow H^i(X, \mathcal{O}_X(n)) = 0, \forall i > r. \end{array} \right.$$

c.f. Hartshorne Thm II.2.7.

(c) To compute  $H^i(X, \mathcal{F})$ , we need

$$\text{coker}(d^M : \prod_{k=0}^r S_{x_0 \dots x_k \dots x_{k+1} \dots x_r} \rightarrow S_{x_0 \dots x_r}).$$

View  $S_{x_0 \dots x_r} = A[x_0^{\pm}, \dots, x_r^{\pm}]$  as a free  $A$ -mod.

(gen'd by  $x_0^{e_0} \dots x_r^{e_r}$  monomials).

$$\Rightarrow \text{im } d^M = \text{Span}_A \{ x_0^{e_0} \dots x_r^{e_r} \mid e_i \geq 0 \}.$$

$\Rightarrow H^r(X, \mathcal{F})$  free  $A$ -mod gen'd by  $x_0^{e_0} \dots x_r^{e_r}$ ,  $e_i \leq 0$ .

$\Rightarrow$  In  $\deg -r-1$ , exactly one monomial

$$(x_0^{-1} \dots x_r^{-1}) = H^r(X, \mathcal{O}_X(-r-1)) \cong A,$$

$$(d) n < 0: H^0(X, \mathcal{O}_X(n)) = S_{-n} = 0$$

$$\Rightarrow H^0(X, \mathcal{O}_X(-n-r-1)) = 0$$

b/c  $\nexists$  monomials of  $\deg > -r-1$  w.r.t.  $x_0^{e_0} \dots x_r^{e_r}$ ,  $e_i < 0$ .

$\Rightarrow$  nothing to check.

May assume  $n \geq 0$  thereafter.

$$H^0(X, \mathcal{O}_X(n)) = \text{Span}_A \{ x_0^{e_0} \cdots x_r^{e_r} : e_0 + \cdots + e_r = n, e_i \geq 0 \}$$

$$\Rightarrow H^r(X, \mathcal{O}_X(-n-r-1)) = \text{Span}_A \{ x_0^{e_0} \cdots x_r^{e_r} : e_0 + \cdots + e_r = -n-r-1, e_i < 0 \}.$$

↪ Description of the pairing:

multiplying together and throw away  $x_0^{-1} \cdots x_r^{-1}$ .  
note this also implies (c).

(b) Induction on  $r$ .

①  $r=1$ : nothing to check (never  $0 < i < r$ )

② Localize:  $\check{C}^i(X, \mathcal{O}_X(n))_{x_r} = \check{C}^i(D_r(x_r), \mathcal{O}_{X(r)}|_{D_r(x_r)})$   
 affine.  $\mathcal{Q}\text{coh}$  on affine

⇒  $\check{C}^i(X, \mathcal{O}_X(n))_{x_r}$  acyclic.

On the other hand,  $(\cdot)_{x_r}$  is exact.

↪  $(\cdot)_{x_r}$  commutes ( $\cong H^i(\cdot)$ ).

i.e.  $\forall i > 0$ ,  $H^i(X, \mathcal{O}_X(n))_{x_r} = 0$ .

(in other words,  $\forall s \in H^i(X, \mathcal{O}_X(n))$ ,  $x_r^d \cdot s = 0$ ,  $d \in \mathbb{N}$ ).

It suffices to show  $\forall 0 < i < r$ ,

$(\cdot) \times x_r : H^i(X, \mathcal{O}_X(n)) \rightarrow H^i(X, \mathcal{O}_X(n))$  injective.

⇐  $0 \rightarrow S(-i) \xrightarrow{x_r} S \rightarrow S/(x_r) \rightarrow 0$  exact

(of graded  $S$ -mod's)

Consider  $H \cong \mathbb{P}^{r+1}_A$  hyperplane  $x_r=0$ ,  $j: H \hookrightarrow X$

↪  $0 \rightarrow \mathcal{O}_X(-i) \rightarrow \mathcal{O}_X \rightarrow \bigoplus_{n \in \mathbb{Z}} (j_* \mathcal{O}_H)(n) \rightarrow 0$ .

↪  $\delta^\circ: H^i(X, \bigoplus \cdots) \rightarrow H^i(X, \mathcal{O}_X(-i))$  zero.

(ind hyp)  $\Rightarrow H^i(X, \bigoplus (j_* \mathcal{O}_H)(n)) \cong H^i(H, \bigoplus \mathcal{O}_H(n))$ ,  $\forall 0 < i < r-1$

$\Rightarrow (\cdot) \times x_r$  bijective,  $\forall 0 \leq i \leq r-2$ . }  $\Rightarrow H^i(X, \mathcal{O}_X(n)) = 0$ .

&  $(\cdot) \times x_r$  inj. ( $i=r-1$ )  $(0 < i < r)$  □

## §4 Finiteness of Cohomology on Projective Schemes

(The truly powerful vanishing thm by Serre)

Little Thm (Hartshorne II.5.17, without nf hyp).

$\varphi: X \rightarrow \mathbb{P}_A^r$  closed imm,  $r \geq 1$ .

Let  $\mathcal{O}_X(1) = \varphi^{-1}(\mathcal{O}_{\mathbb{P}_A^r}(1))$ .  $\mathcal{F} \in \text{Coh}(X)$  f.g.

$\Rightarrow \mathcal{F}(n)$  ger'd by finitely many global sections,  $n \gg 0$ .

Cor  $\exists n \ll 0$  s.t.  $\bigoplus_{i=1}^m \mathcal{O}_X(n) \longrightarrow \mathcal{F}$ .

Big Thm (Serre vanishing) A nf.  $X \xrightarrow{f} \mathbb{P}_A^r$  closed imm ( $r \geq 1$ ).

$\mathcal{O}_X(1) := f^*\mathcal{O}_{\mathbb{P}}(1)$ .  $\mathcal{F} \in \text{Coh}(X)$  f.g.

Then (a)  $H^i(X, \mathcal{F})$  (as  $A$ -mod) f.g.  $\forall i \geq 0$ .

(b)  $H^i(X, \mathcal{F}(n)) = 0$ ,  $n \gg 0$ ,  $\forall i \geq 0$ .

Proof. (Descending induction on  $i$ ).

$X$  admits a good cover by  $\leq r+1$  open affines

$\Rightarrow H^i(X, \mathcal{F}) = 0$ ,  $\forall i > r$ .

Cor  $\Rightarrow \mathcal{F} = \mathcal{E}/\mathcal{G}$ ,  $\mathcal{E} = \bigoplus_{i=1}^m \mathcal{O}_X(n_i)$ .

Nt hyp  $\Rightarrow \mathcal{G}$  f.g.

$$0 \rightarrow \mathcal{G} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$$

$$\hookrightarrow \dots \rightarrow H^i(X, \mathcal{E}) \rightarrow H^i(X, \mathcal{F}) \rightarrow H^{i+1}(X, \mathcal{G}) \rightarrow \dots$$

$$\bigoplus_{i=1}^m H^i(X, \mathcal{O}_X(n_i)) \quad \begin{matrix} \text{f.g. as } A\text{-mod.} \\ (\text{ind hyp}) \end{matrix}$$

$$\Rightarrow H^i(X, \mathcal{F}) \text{ f.g. by nt. } \Rightarrow (a).$$

For (b): twisting by  $n \gg 0$

$$\hookrightarrow \cdots \rightarrow H^i(X, \xi_{(n)}) \rightarrow H^i(X, \mathcal{F}_{(n)}) \xrightarrow{\text{ind hyp}} H^{i+1}(X, \mathcal{G}_{(n)}) \rightarrow \cdots$$

(by explicit calculation)

(ind hyp)

$$\Rightarrow H^i(X, \mathcal{F}_{(n)}) = 0.$$

□