Zeta functions and Shimura varieties: Past, present, and the future

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(See also satellite conference in 2023 Summer at MCM.)

<u>Eichler - Shimura</u> (1950s)

$$\zeta(X_0(N), s) = \frac{\zeta(s)\cdot\zeta(1-s)}{\prod_{i=1}^{g} L(f_i, s)}$$

$\{f_i\}$ eigen-basis of $S_2(\Gamma_0(N))$

$\rightsquigarrow$ Generalize to more general Shimura vars:

Shimura datum (G, X), $\left(X = G(\mathbb{R})\text{-conj class of } h: \mathbb{S} \to G_{\mathbb{R}} \text{ + axioms.} \right)$

$\forall$ small $K \subset G(\mathbb{A}_f)$,

$\rightsquigarrow$ Smooth alg var $\begin{array}{c} Sh_K \\ \downarrow \\ \operatorname{Spec} E \end{array}$ $\quad E/\mathbb{Q}$ number field $\;=\; E(G,X)$.

$$Sh_K(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K.$$

Consider $IH^i := IH^i(\overline{Sh_K}) = H^i_{et}(\overline{Sh_K}, \bar{E}, j_{!*}\bar{\mathbb{Q}}_\ell)$ with $j: Sh_K \hookrightarrow \overline{Sh_K}$

($\overline{Sh_K}$: Baily-Borel cpt'n)

Can replace const sheaf $\bar{\mathbb{Q}}_\ell$ by LocSys $\mathcal{L}$ from $\operatorname{Rep}(G)$.

$IH^i \circlearrowleft Gal_E \times \mathcal{H}_K$.

<u>Known</u> As a Hecke mod,

$$\begin{array}{ccc} IH^i & = \bigoplus_{\pi_f} & \pi_f^K \otimes W^i(\pi_f) \\ \circlearrowleft & & \circlearrowleft \quad\quad \circlearrowleft \\ Gal_E \times \mathcal{H}_K & & \mathcal{H}_K \quad\quad Gal_E \end{array}$$

where π_f = fin part of $\pi \subset L^2_{disc}(G(\mathbb{Q})\backslash G(\mathbb{A}))$

$\uparrow$ disc autom rep of $G(\mathbb{A})$.

<u>Main question</u> To understand $W^i(\pi_f)$:

What is $W(\pi_f) = \sum_i (-1)^i W^i(\pi_f) \in Groth(Gal_E)$
$\circlearrowleft$
Gal_E (as a Gal_E-rep).

<u>Rough guess</u> $W(\pi_f)$ is given by composing

- the global L-para of π, $\rho_\pi : Gal_E \to {}^L G_E$, with
- $r_x : {}^L G_E \to GL_N(\bar{\mathbb{Q}}_\ell)$

highest wt rep of wt $-\mu_x$

(Not true b/c of endoscopy !)

Correct conjectural recipe given by Langlands-Kottwitz:

<u>LK</u> (1970s - 80s)

$W(\pi_f)$ has similar structure as <u>Arthur's multiplicity conj</u>.

Arthur's multiplicity conj

- $\pi \subset L^2_{disc}(G(\mathbb{Q})\backslash G(\mathbb{A}))$

  $\Rightarrow \pi \in \Pi_\psi(G(\mathbb{A}))$

  ψ global A-parameter,

  $$\mathcal{L}_{\mathbb{Q}} \times SL_2 \longrightarrow {}^LG.$$

- Conversely, $\forall\, \pi' \in \Pi_\psi(G(\mathbb{A}))$

  $$\pi' \rightsquigarrow \chi_{\pi'} : S_\psi \longrightarrow \mathbb{C}^\times, \quad \chi_{\pi'} = \prod_v \chi_{\pi'_v}$$

  Input more choices

  Here S_ψ = centralizer of ψ, assumed to be finite abelian

  multi of π' in $L^2_{disc}(G(\mathbb{Q})\backslash G(\mathbb{A}))$

  = multi of χ_{can} in $\chi_{\pi'}$.

  indep of π'.

$\boxed{\text{LK}}$ π_f s.t. $W(\pi_f) \neq 0 \Rightarrow \pi_f \in \Pi_\psi(G(\mathbb{A}_f))$

$$\prod_{v\neq\infty} \chi_{\pi_v} =: \chi_f : S_\psi \to \mathbb{C}^\times$$

Need to use Shimura datum X to get some modification $\tilde{\chi}_{\pi_f}$ of χ_{π_f}.

Compose the global L-para ψ w/ r_X

$$\rightsquigarrow V_\psi = \overline{\mathbb{Q}}_\ell^{\oplus d} \circlearrowleft \mathrm{Gal}_E \times S_\psi.$$

Can decompose $V_\psi = \bigoplus_{\chi : S_\psi \to \mathbb{C}^\times} V_{\psi,\chi}$ ← endoscopy: with diff multiplicity for each $V_{\psi,\chi}$.

χ: eigenchar; $V_{\psi,\chi} \circlearrowleft \mathrm{Gal}_E$.

$$\underline{\text{Conj}}\ (\text{LK})\quad W(\pi_f) = \sum_{\substack{\psi \text{ s.t.} \\ \pi_f \in \Pi_\psi(G(\mathbb{A}_f))}} \sum_{\chi : S_\psi \to \mathbb{C}^\times} (\text{multi of } \chi \text{ in } \tilde{\chi}_{\pi_f}) \cdot (\pm 1) \cdot [V_{\psi,\chi}]$$

(± 1): depends only on ψ & χ.

+ cohom condition of ψ_∞.

$\underline{\text{Thm}}$ (In progress, KSZ)

(A) Define $W_c(\pi_f)$ using $H_c^*(Sh_K)$.

For unitary Sh var of form $G = \mathrm{Res}_{F/\mathbb{Q}}\, \mathcal{U}$

where · $F/\mathbb{Q}$ tot real

· $\mathcal{U}$ honest unitary grp w/ CM $\tilde{F}/F$.

$\boxed{F \neq \mathbb{Q}}$

But allow arbitrary sgn.

$\underline{\text{Conj}}$ (LK) is true w/ $W(\pi_f)$ replaced by $W_c(\pi_f)$

(B) Same Sh var, $W(\pi_f) = W_c(\pi_f)$ in $\mathrm{Groth}(\mathrm{Gal}_E)$.

↑ not true if $F = \mathbb{Q}$.

$\underline{\text{Explanations}}$ (B) By purity $W(\pi_f) = \sum_i (-1)^i W^i(\pi_f)$ has no cancellation!

(In principle $W(\pi_f) \rightsquigarrow W^i(\pi_f)$, $\forall i$).

Not true for $W_c(\pi_f)$.

(Not agree at each degree).

[Brylinski-Labesse, 1984] in Hilbert-Blumental case.

(A) (1) For G honest unitary,

A-paras, A-packets, etc.

are constructed in Mok, KMSW.

Very sensitive to the center of grp.

e.g. $U \not\rightsquigarrow GU$ by changing the center.

PEL-unitary : above results do not apply!

For honest unitary G, the Sh var is of ab type.

not PEL!

(2) Global L-para of ψ?

In the context of KMSW & Mok,

ψ essentially built from conjugate self-dual cuspidal autom rep'ns π_i of $GL_{n_i, \tilde{F}}$.

It turns out: if ψ contributes to IH,

then each π_i is regular algebraic

(up to twist).

Thm (Chenevier - Clozel - Harris - Kottwitz - Labesse - Shin - Taylor)

π_i has an assoc Gal rep of $\mathrm{Gal}_{\tilde{F}}$.

One builds from these Gal reps

the "global L-para of ψ" (Bellaïche - Chenevier).

Str of proof of Part (A)

(1) $K = K^p K_p$. $K_p \subset G(\mathbb{Q}_p)$ hyperspecial ($\forall\forall p$)

Count mod p pts of a "good" p-adic integral model of Sh_K.

(2) Compare result in (1) with

Arthur - Selberg stable trace formula

for all $\underbrace{\text{endoscopic grps}}$ H's

as opposed to G itself.

(3) Apply spectral expansion of S^H

and go back to G

(Need: Input endoscopic classification).

Thm (KSZ, 2021)

(1) & (2) ok for all Sh vars of abelian type.

<u>Thm</u> (Haines - Zhou - Zhu, in progress)

(1) & (2) ok for parahoric K_p.

<u>Ass</u> · ab type, $p \neq 2$, $G_{\mathbb{Q}_p}$ quasi-split

(G^{ad}, X^{ad}) no factor D^H.