Pointwise criteria for p-adic local systems

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<u>Motivation</u> Let K be a p-adic field. $V \circlearrowleft \mathrm{Gal}_K$.

<u>Q</u> When does V come from geom ?

$$\mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{Gal}_K) \supseteq \mathrm{Rep}^{dR}_{\mathbb{Q}_p}(\mathrm{Gal}_K) \supseteq \mathrm{Rep}^{st}_{\mathbb{Q}_p}(\mathrm{Gal}_K) \supseteq \mathrm{Rep}^{crys}_{\mathbb{Q}_p}(\mathrm{Gal}_K).$$

$$\begin{array}{c} \cup \\ V \end{array}$$

if V comes from $H^i_{et}(X_{\bar{K}}, \mathbb{Q}_p) \Rightarrow V \in \mathrm{Rep}^{dR}_{\mathbb{Q}_p}(\mathrm{Gal}_K)$.

if X $\exists$ semistable red'n $\Rightarrow V \in \mathrm{Rep}^{st}_{\mathbb{Q}_p}(\mathrm{Gal}_K)$

if X $\exists$ good red'n $\Rightarrow V \in \mathrm{Rep}^{cris}_{\mathbb{Q}_p}(\mathrm{Gal}_K)$

<u>Today</u> Relative setup:

Assume X_η is a rigid space $/K$.

$$V \in \mathrm{Loc}_{\mathbb{Z}_p}(X_\eta)_{\mathbb{Q}_p} \supseteq \mathrm{Loc}^{*}_{\mathbb{Z}_p}(X_\eta)_{\mathbb{Q}_p}. \quad * \in \{dR, st, crys\}.$$

<u>Q</u> When does V come from geom ?

i.e. $f_\eta : Y_\eta \to X_\eta$ proper sm, V subquot of $R^i f_{\eta *}\mathbb{Q}_p$.

<u>Known</u> (i) Scholze: $R^i f_{\eta *}\mathbb{Q}_p \in \mathrm{Loc}^{dR}_{\mathbb{Z}_p}(X_\eta)_{\mathbb{Q}_p}$.

Liu-Zhu: $R^i f_{\eta *} V'$ is dR if V' is dR.

Guo-Reinecke: if $\exists f: X \to Y$ proper sm map of sm p-adic formal nbhds.

s.t. f_η = generic fibre of f,

then $R^i f_{\eta *} V'$ is crys if V' is crys

Q What about semistable loc sys ?

(2) Thm (Liu-Zhu)

X_η connected. $V \in \mathrm{Loc}(X_\eta)$.

V is dR $\Leftrightarrow$ $\exists$ classical pt $z \in X_\eta$

s.t. $V|_z$ is dR.

Q Is there an analogue of st / crys loc sys ?

(specificially for classical pts ?)

Pointwise criteria

Thm A (Guo-Yang)

If $\exists f: Y \to X$ proper semistable integrally b/w p-adic formal schs,

then $R^i f_{\eta *} \mathbb{Q}_p$ is st loc sys / X_η.

Thm PC (Guo-Yang)

Assume X is a semistable (resp. sm) p-adic formal sch,

$V \in \mathrm{Loc}(X_\eta)$.

Then V is st (resp. crys) $\Leftrightarrow$ so is $V|_z$ for many

classical pts $z \in X_\eta$.

Rmk (i) To prove Thm A:

ingredients: Thm PC + C_{st}-conj

$H^i_{et}(Z_{\bar{k}}, \mathbb{Q}_p)$ is st rep if Z has proper semistable red'n

(2) "Many" includes "every":

Let $\mathcal{C}$ a set of classical pts.

Introduce "effectiveness" of $\mathcal{C}$ (meaning of "many")

Roughly,
$$\begin{array}{ccc} \bigcup_{z \in \mathcal{C}} \mathrm{Gal}_{K(z)} & \longrightarrow & \pi_1^{alg}(X_\eta) \\ \cup\!\! \mid & & \cup\!\! \mid \\ \bigcup_{z \in \mathcal{C}} I_{K(z)} & \xrightarrow[\uparrow]{\alpha} & I_{X_\eta} \end{array}$$

if $\mathrm{Im}\,\alpha$ is top generating étale ext'n of X_η.

Q Let $\mathcal{C}$ = CM pts. Is $\mathcal{C}$ effective?

If $X'_\eta \xrightarrow{g_\eta} X_\eta$ fét, not extended to an étale map over X,

Then $\exists z \in X_\eta$, s.t. $g_\eta^{-1}(z)$ has a component z'

s.t. $K(z')/K(z)$ is not unr.

(3) Thm (Guo-Yang) (ℓ any prime)

Assume X is sm p-adic formal sys, $\mathcal{L} \in \mathrm{Loc}_{\mathbb{Z}_\ell}(X_\eta)$

Then $\mathcal{L}$ is unr (on X) $\leadsto$ can extend to integral model.

$\Updownarrow$

$\mathcal{L}|_z$ is unr. $\forall z \in \mathcal{C}$.

Crystalline RH enhancement

Recall If $X = \mathrm{Spec}\,K$, $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. To prove V is semistable,

Can do: (1) prove: V is dR

$\leadsto \exists$ fin Gal tot ram ext'n K'/K s.t. $V|_{K'}$ is st.

(2) $D_{st}(V|_{K'}) \circlearrowleft \mathrm{Gal}(K'/K)$ is a trivial action.

This relies on the constr'n $D_{st}(-)$ on $\mathrm{Rep}(\mathrm{Gal}_{K'})$.

<u>Thm RH</u> (Guo-Yang) X sm $/ \mathcal{O}_K$, $\exists$ comm diagram

$$\begin{tikzcd} & \mathrm{Isoc}^{\varphi,N}(X_s) \arrow[dd, "N=0"] & & & \mathcal{E}_{st,T} \arrow[dd] \\ \mathrm{Loc}(X_\eta) \arrow[ur] \arrow[dr] \arrow[r, phantom, "\circlearrowleft"] & & T \arrow[urr, mapsto] \arrow[drr, mapsto] & & \\ & \mathrm{Isoc}^{\varphi}(X_s) & & & \mathcal{E}_{cris,T} \end{tikzcd}$$

loc free

$X_s := X \times_{\mathcal{O}_K} \mathrm{Spec}\, \mathcal{O}_K/\pi$.

(1) $\mathrm{rk}\, \mathcal{E}_{crys,T} \le \mathrm{rk}\, \mathcal{E}_{st,T} \le \mathrm{rk}\, T$

T is crys (resp. st) $\Longleftrightarrow$ $\mathrm{rk}\, \mathcal{E}_{crys,T} = \mathrm{rk}\, T$

(resp. $\mathrm{rk}\, \mathcal{E}_{st,T} = \mathrm{rk}\, T$).

(2) (Pullback compatibility)

$f : X' \to X$ of 2 sm p-adic formal schs.

$\exists$ natural inj $f_s^* \mathcal{E}_{*,T} \overset{\alpha_f}{\hookrightarrow} \mathcal{E}_{*, f_\eta^{-1}T}$, $* \in \{st, cris\}$.

(3) (Sm base change)

α_f is an isom if f is p-adically sm.

(4) (Compatible with D_{dR})

$\exists$ inj $\mathcal{E}_{cris,T}(X, X_{proet}) \hookrightarrow \mathcal{E}_{st,T}(X, X_{proet}) \hookrightarrow D_{dR}(T)$

as flat conn $/ X_\eta$.

also recover the pt case.

<u>Rmk</u> Tan-Tong: crys case of D_{crys}