Exercise 2 (due on October 26)

Choose 4 out of 8 problems to submit.

Problem 2.1. (Local Galois cohomology computation) Let K be a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ with residue field $\mathbb{F}_q$. Let V be a representation of G_K on an $\mathbb{F}_\ell$-vector space.

(1) Show that when $\ell \neq p$, $H^1(G_K, V) = 0$ unless $V^{G_K} \neq 0$ or $V^*(1)^{G_K} \neq 0$. When $\ell = p$ and $K = \mathbb{Q}_p$, what is $\dim H^1(G_{\mathbb{Q}_p}, V)$ "usually"?
(2) When $\ell \neq p$, compute without using Euler characteristic formula, in an explicit way, $\dim H^i(G_K, \mathbb{F}_\ell(n))$. Your answer will depend on congruences of q^n modulo ℓ. Observe that the dimensions coincidence with the prediction of Tate local duality and Euler characteristic formula.
(3) When $\ell = p$ and K a finite extension of $\mathbb{Q}_p$, compute the dimension of $\dim H^i(G_K, \mathbb{F}_p(n))$.

Problem 2.2. (Dimension of local Galois cohomology groups) Let K be a finite extension of $\mathbb{Q}_p$, and let V be a representation of G_K over a finite dimensional $\mathbb{F}_\ell$-vector space. Suppose that V is irreducible as a representation of G_K and $\dim V \geq 2$.

(1) When $\ell \neq p$, show that $H^1(G_K, V) = 0$. (Hint: compute H^2 using local Tate duality and then use Euler characteristic.)
(2) When $\ell = p$, what is $\dim H^1(G_K, V)$?

Problem 2.3. (An example of Poitou–Tate long exact sequence) Consider $F = \mathbb{Q}$, and let $S = \{p, \infty\}$ for an odd prime p. Determine each term in the Poitou–Tate exact sequence for the trivial representation $M = \mathbb{F}_p$. (Hint: usually $H^2(G_{F,S}, \mathbb{F}_p)$ is difficult to determine; but one can use Euler characteristic to help.)

Problem 2.4. (Cohomology of $\mathcal{O}_{F^S}[\frac{1}{S}]^\times$) Let F be a number field and S a finite set of places of F including all archimedean places and places above ℓ.

(1) Show that there is a natural exact sequence
$$1 \to \Big(\mathcal{O}_F[\tfrac{1}{S}]^\times \backslash \textstyle\prod_{v\in S} F_v^\times\Big) \times \prod_{v\notin S} \mathcal{O}_{F_v}^\times \to F^\times\backslash \mathbb{A}_F^\times \to \mathrm{Cl}(\mathcal{O}_F[\tfrac{1}{S}]) \to 1,$$
where $\mathrm{Cl}(\mathcal{O}_F[\frac{1}{S}])$ is the ideal class group of $\mathcal{O}_F[\frac{1}{S}]$, namely the quotient of the ideal class group $\mathrm{Cl}(\mathcal{O}_F)$ by the subgroup generated by ideals in S.
(2) By studying the exact sequence
$$1 \to \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \to \textstyle\prod_{v\in S}(F_v \otimes F^S)^\times \to \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times \backslash \prod_{v\in S}(F_v\otimes F^S)^\times \to 1,$$
show that
$$H^1\Big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times\Big) \cong \mathrm{Cl}(\mathcal{O}_F[\tfrac{1}{S}]),$$
and there is an exact sequence
$$0 \to H^2\Big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times\Big)\otimes \mathbb{Z}_\ell \to \bigoplus_{v\in S} \begin{cases} \mathbb{Q}_\ell/\mathbb{Z}_\ell & v \text{ non-arch} \\ \frac{1}{2}\mathbb{Z}/\mathbb{Z} & v = \mathbb{R} \text{ and } \ell = 2 \\ 0 & \text{otherwise} \end{cases} \to \mathbb{Q}_\ell/\mathbb{Z}_\ell \to 0.$$
For $i \geq 3$, we have
$$H^i\Big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times\Big)\otimes\mathbb{Z}_\ell \cong \bigoplus_{v \text{ real}} H^i(\mathbb{R}, \mathbb{C}^\times) \cong \bigoplus_{v\text{ real}} \begin{cases} \frac{1}{2}\mathbb{Z}/\mathbb{Z} & \ell = 2 \text{ and } i \text{ even,} \\ 0 & \text{otherwise.}\end{cases}$$

Remark: Using Kummer theory $1 \to \mu_\ell \to \mathcal{O}_{F^S}[\frac{1}{S}]^\times \to \mathcal{O}_{F^S}[\frac{1}{S}]^\times \to 1$, we can then use this to further compute $H^1(G_{F,S}, \mu_\ell)$.

Problem 2.5. (A step in the proof of local Euler characteristic formula) Consider the following situation. Let K be a finite extension of $\mathbb{Q}_p$ such that $K = K(\mu_p)$. Let L/K be a finite cyclic extension with Galois group H of order relatively prime to p. Let N be a finite $\mathbb{F}_p[H]$-module. Our goal is to compute

$$\dim \Big(\big(\mathcal{O}_L^\times/(\mathcal{O}_L^\times)^p\big) \otimes N(-1)\Big)^H$$

(1) Consider the logarithmic map

$$\begin{aligned} \log_p : \mathcal{O}_L^\times/\mu(L) &\longrightarrow L \\ a &\longmapsto \tfrac{1}{p^n}\log_p(a^{p^n}) \end{aligned}$$

where n is taken sufficiently divisible so that $a^{p^n} \in 1 + p^2\mathcal{O}_L$ so that $\log_p(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ makes sense. Show that $\log_p$ is well-defined homomorphism (and independent of the choice of n), and that it induces an isomorphism

$$\log_p : \big(\mathcal{O}_L^\times/\mu(L)\big) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong L.$$

(2) Show that for any two $\mathcal{O}_L$-lattices $\Lambda_1, \Lambda_2 \in L$, we have

$$\dim(\Lambda_1/p\Lambda_1 \otimes N)^H = \dim(\Lambda_2/p\Lambda_2 \otimes N)^H.$$

(In terms of writing, it might be better to compare $(\Lambda_i/\varpi_L\Lambda_i \otimes N)^H$ first.)

(3) Recall that $L \cong K[H]$ as H-modules by Hilbert 90. From this and (2), deduce that

$$\dim \Big(\big(\mathcal{O}_L^\times/(\mathcal{O}_L^\times)^p\big) \otimes N(-1)\Big)^H = \dim N^H + \dim N \cdot [K : \mathbb{Q}_p].$$

Problem 2.6. (Comparing first Galois cohomology classes and extensions of Galois representations) If you are unfamiliar with the background of this problem, one can consult the short note on this topic, available on the webpage.

Let k be a field with discrete topology, and let G be a finite group acting k-linearly on a finite dimensional k-vector space M. Let $\rho : G \to \mathrm{GL}_k(M)$ be the representation.

(1) Given a cohomology class $[c] \in H^1(G, M)$, represented by cocycle $g \mapsto c_g \in M$, show that the following map defines a representation of G on $E_c := M \oplus k$:

$$g \mapsto \begin{pmatrix} \rho(g) & c_g \\ 0 & 1 \end{pmatrix}.$$

(2) Show that if $(c_g)_{g\in G}$ and $(c'_g)_{g\in G}$ define the same cohomology class, the representations E_c and $E_{c'}$ defined in (1) are isomorphic.

(3) By definition, there exists an exact sequence $0 \to M \to E_c \to k \to 0$. Taking the G-cohomology gives a connecting homomorphism

$$k^G = k \xrightarrow{\delta} H^1(G, M)$$

Show that $\delta(1) = [c]$.

(4) (Optional) Given an exact sequence of $k[G]$-modules

$$0 \to M \to E_1 \to E_2 \to k \to 0,$$

we may write F as the image of $E_1 \to E_2$ and thus get two short exact sequences

$$0 \to M \to E_1 \to F \to 0, \quad \text{and} \quad 0 \to F \to E_2 \to k \to 0$$

This way, the boundary maps of the group cohomology defines two maps

$$\delta : k = H^0(G, k) \longrightarrow H^1(G, F) \longrightarrow H^2(G, M)$$

and thus the image $\delta(1)$ defines a cohomology class $[c]$ in $H^2(G, M)$.

Now, suppose that we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & k \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & E_1' \arrow[r] & E_2' \arrow[r] & k \arrow[r] & 0. \end{tikzcd}$$

Show that the second cohomology class defined by these two exact sequences are the same.

Problem 2.7. (An explicit computation of local Galois cohomology when $\ell \neq p$) Let K be a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, with ring of integers $\mathcal{O}_K$ and residue field k_K. Let ℓ be a prime different from p. Let M be a finite G_K-module that is ℓ^∞-torsion. Following the instruction below to give another proof of the Euler characteristic for local Galois cohomology when $\ell \neq p$:

$$\chi(G_K, M) := \sum_{i=0}^{2} (-1)^i \cdot \mathrm{length}_{\mathbb{Z}_\ell} H^i(G_K, M) = 0. \tag{2.7.1}$$

(1) Let I_K and P_K denote the inertia subgroup and the wild inertia subgroup of G_K. Show that $H^{>0}(P_K, M) = 0$ for any G_K-module M that is ℓ^∞-torsion. Using the Hoshchild–Serre spectral sequence to deduce that, for every $i \geq 0$,

$$H^i(I_K, M) \cong H^i(I_K/P_K, M^{I_K}).$$

(2) Let $P_{K,\ell}$ denote the kernel of $I_K \to I_K/P_K \xrightarrow{t_{\xi,\ell}} \mathbb{Z}_\ell(1)$. Show that we have $H^i(I_K, M) \cong H^i(\mathbb{Z}_\ell(1), M^{P_{K,\ell}})$.

(3) Put $N := M^{P_{K,\ell}}$, and write τ for a generator of I_K/P_K, then we have

$$H^0(I_K, M) \cong N^{\tau=1}, \quad H^1(I_K, M) \cong N/(\tau - 1)N.$$

Note also that the second isomorphism is given by evaluating the cochain at τ; so the Frobenius action on $N/(\tau - 1)N$ is twisted by the inverse of cyclotomic character. Thus, we should have wrote $N(-1)/(\tau - 1)N(-1)$ instead.

(4) Let ϕ_K denote a Frobenius element. Show that we have isomorphisms:

$$H^0(G_K, M) \cong (N^{\tau=1})^{\phi_K=1}, \quad H^2(G_K, M) \cong \frac{N(-1)}{(\tau-1)N(-1)} \Big/ (\phi_K - 1).$$

For $H^1(G_K, M)$, we may describe the unramified part and singular part of it as follows:

$$\begin{tikzcd} 0 \arrow[r] & H^1(G_{k_K}, M^{I_K}) \arrow[r] \arrow[d, "\cong"] & H^1(G_K, M) \arrow[r] & H^1(I_K, M)^{G_{k_K}} \arrow[r] \arrow[d, "\cong"] & 0. \\ & \left(\frac{N(-1)}{(\tau-1)N(-1)}\right)^{\phi_K=1} & & N^{\tau=1}/(\phi_K - 1)N^{\tau=1} & \end{tikzcd}$$

(5) From this, deduce Euler characteristic formula (2.7.1) directly.

(6) Using the discussion above to prove the following isomorphism of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(G_{k_K}, M^{I_K}) & \longrightarrow & H^1(G_K, M) & \longrightarrow & H^1(I_K, M)^{G_{k_K}} & \longrightarrow & 0 \\ & & \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} & & \\ 0 & \longrightarrow & \big(H^1(I_K, M^*(1))^{G_{k_K}}\big)^* & \longrightarrow & H^1(G_K, M)^* & \longrightarrow & \big(H^1(G_{k_K}, (M^*(1))^{I_K})\big)^* & \longrightarrow & 0. \end{array}$$

(Hint: first show that the subgroup $H^1(G_{k_K}, M^{I_K})$ and $H^1(G_{k_K}, (M^*(1))^{I_K})$ annihilate each other. This is because such pairing factors through the cup product.

$$H^1(G_{k_K}, M^{I_K}) \times H^1(G_{k_K}, (M^*(1))^{I_K}) \to H^2(G_{k_K}, \mu_{\ell^\infty}) = 0.$$

After this, it is enough to show that $\#H^1(G_{k_K}, M^{I_K}) = \#H^1(I_K, M^*(1))^{G_{k_K}}$, which makes use of the discussion above.)

Problem 2.8. Fix a prime number ℓ. Let F be a number field and S a finite set of places that includes all archimedean places and places above ℓ. For an extension L of F, we write S_L for the set of places of L that lies over places in S. Let F^S denote the maximal Galois extension of F that is unramified outside S. We compare the cohomology groups (for $i = 1, 2$)

$$H^i\Big(G_{F,S}, \Big(F^{S,\times}\backslash\mathbb{A}_{F^S}^\times\Big)\Big) \otimes \mathbb{Z}_\ell \quad \text{with} \quad H^i\Big(G_{F,S}, \Big(\mathcal{O}_{F^S}[\tfrac{1}{S}]^\times\backslash \textstyle\prod_{v\in S}(F_v \otimes F^S)^\times\Big)\Big) \otimes \mathbb{Z}_\ell.$$

(1) For a finite extension $L \subset F^S$, show that we have an exact sequence

$$1 \to \Big(\mathcal{O}_L[\tfrac{1}{S}]^\times\backslash \textstyle\prod_{w\in S_L} L_w^\times\Big) \times \prod_{w\notin S_L} \mathcal{O}_{L_w}^\times \to L^\times\backslash\mathbb{A}_L^\times \to \mathrm{Cl}(\mathcal{O}_L[\tfrac{1}{S}]) \to 1,$$

where $\mathrm{Cl}(\mathcal{O}_L[\frac{1}{S}])$ is the ideal class group of $\mathcal{O}_L[\frac{1}{S}]$. (This is Problem 2.4(1) earlier.)

(2) Show that the limit $\varprojlim_{L\subset F^S} \mathrm{Cl}(\mathcal{O}_L[\frac{1}{S}])$ is trivial. (Hint: A property of Hibert class field theory is that, if L^{Hilb}/L is the Hilbert class field of L, namely the maximal unramified abelian extension of L, every ideal of L becomes principal in L^{Hilb}. Using the commutative diagram for compatibility of Artin maps with ideal class groups

$$\begin{array}{ccc} L'^\times\backslash\mathbb{A}_{L'}^\times & \xrightarrow{\mathrm{Art}_{L'}} & G_{L'}^{\mathrm{ab}} \\ {\scriptstyle\text{natural}}\uparrow & & \uparrow{\scriptstyle\mathrm{Ver}} \\ L^\times\backslash\mathbb{A}_L^\times & \xrightarrow{\mathrm{Art}_L} & G_L^{\mathrm{ab}}, \end{array}$$

to show that this boils down to the following group theoretic statement: Let G be a pro-finite group and H its commutator group, then the transfer map $G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ is the zero map. This is known as the Artin's principal ideal theorem. The class field theory by Artin–Tate has a proof of this, on their page)

(3) Deduce that

$$H^0\Big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times\backslash \textstyle\prod_{v\in S}(F_v \otimes F^S)^\times\Big) \cong F^\times\backslash\mathbb{A}_F^\times/\prod_{v\notin S}\mathcal{O}_{F_v}^\times,$$

$$H^1\Big(G_{F,S}, \mathcal{O}_{F^S}[\tfrac{1}{S}]^\times\backslash \textstyle\prod_{v\in S}(F_v \otimes F^S)^\times\Big) = 0 \quad \text{and}$$

$$H^2\Big(G_{F,S}, \Big(\mathcal{O}_{F^S}[\tfrac{1}{S}]^\times\backslash \textstyle\prod_{v\in S}(F_v \otimes F^S)^\times\Big)\Big) \otimes \mathbb{Z}_\ell \cong \mathbb{Q}_\ell/\mathbb{Z}_\ell.$$