On the category of qBC's

Pierre Colmez

Setup Y_K dagger var, $n \leq r$.

$\rightsquigarrow X^{r,n} := (H^n_{HK}(Y_C) \otimes B^+_{st})^{N=0, \varphi=p^r}$, $\quad C = \widehat{\overline{K}}$.

$F^{r,n} := H^n(\mathrm{Fil}^r(B^+_{dR} \otimes R\Gamma_{dR}(Y_K)))$ "sub"

$DR^{r,n} := H^n(B^+_{dR} \otimes R\Gamma_{dR}(Y_K) / \mathrm{Fil}^{(r)})$ "quot"

Conj (C_{st})

$$\begin{tikzcd}
\cdots \arrow[r] & DR^{r,n-1} \arrow[r] \arrow[d, equal] & H^n_{proét}(Y_C, \mathbb{Q}_p(r)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X^{r,n} \arrow[r] \arrow[d] & DR^{r,n} \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & DR^{r,n-1} \arrow[r] & F^{r,n} \arrow[r] & B^+_{dR} \otimes H^n_{dR}(Y_K) \arrow[r] & DR^{r,n} \arrow[r] & \cdots
\end{tikzcd}$$

The middle diagram is bicartesian.

$$\rightsquigarrow H^n_{proét}(Y_C, \mathbb{Q}_p(r)) = [\underbrace{B^+_{dR} \otimes H^{n-1}_{dR} / \mathrm{Fil}^r + X^{r,n-1}}_{①} - \underbrace{DR^{r,n-1} / (B^+_{dR} \otimes H^{n-1} / \mathrm{Fil}^r)}_{②}$$
$$- \underbrace{\ker(X^{r,n} \to B^+_{dR} \otimes H^n_{dR} / \mathrm{Fil}^{(r)})}_{③}]$$

① C-pts of a BC,

with no nonzero map to B^+_{dR}-mod (curvature >0)

② B^+_{dR}-mod, killed by t^2 (curvature $=0$)

③ C-pts of a BC, evaluated in a free B^+_{dR}-mod (curvature <0).

Def A sympathetic algebra S is a spectral C-Banach alg

w/ $x \longmapsto x^p$ is surjective on $(\{x : \|x-1\| < 1\})$

+ S separable ($\Leftrightarrow S^+/p$ countable $\Leftrightarrow \mathcal{O}_C/p$ countable).

S sympathetic $/C$ $\Rightarrow$ S perfectoid $/C$.

<u>TVS</u> A TVS W is a functor $S \longmapsto W(S)$ to $\mathbb{Q}_p$-top v.s.

S sympathetic

<u>Quotient</u> $W_2/W_1 \triangleq (W_1 \hookrightarrow W_2)$ if the map is strict.

<u>Note</u> $0 \to W_1 \to W \to W_2 \to 0$ is exact

$\Longleftrightarrow$ $0 \to W_1(S) \to W(S) \to W_2(S) \to 0$ is exact, $\forall S$.

<u>E.g.</u> $/\mathbb{Q}_p$, $S \longmapsto \mathcal{C}(\pi_0(S), \mathbb{Q}_p)$,

$\mathbb{G}_a : S \longmapsto S$, $\mathbb{B}_m : S \longmapsto B^+_{dR}(S)/t^m$.

<u>Def</u> A BC W is of the form

$$\begin{array}{ccccccc} & & 0 \to V_2 & & & & \\ & & & \searrow & & & \\ 0 & \longrightarrow & V_1 & \longrightarrow Y & \longrightarrow & \mathbb{G}_a^d & \to 0 \\ & & & & \searrow & & \\ & & & & & W \to 0 & \end{array}$$

W determined by d & difference of V_1 & V_2.

$\dim W = d$, $\operatorname{ht} W = \dim_{\mathbb{Q}_p} V_1 - \dim_{\mathbb{Q}_p} V_2$.

<u>Thm</u> BC is well-def'd & additive in exact seq.

<u>Def</u> A qBC W is a TVS + a filtration of curvature

$[W_{>0} - W_{=0} - W_{<0}]$.

- $W_{<0}$ is a BC with curvature <0
- $W_{>0}$ is a BC with curvature >0
- $W_{=0}$ is a B^+_{dR}-pair s.t.

$$W_{=0}(S) = \{ \mathbb{B}_m(S) \hat{\otimes}_{\mathbb{B}_m} W_1 \to \mathbb{B}_m(S) \hat{\otimes}_{\mathbb{B}_m} W_2 \}$$

where $(W_1 \to W_2)$ is a pair of $\mathbb{B}_m$-mods, $\mathbb{B}_m = \mathbb{B}_m(C)$.

Say W is a $\bar{q}BC$ if $(W_1 \to W_2) \triangleq W_2/W_1$,

i.e. $W_{=0}$ is a B_{dR}^+-mod.

<u>Illustrative example</u> $[W_{<0} - W_{=0} - W_{>0}]$

$$\begin{array}{c} \| \end{array}$$

$$[H^1(X_{FF}, \mathcal{E}) - H^0(X_{FF}, \mathcal{F}) - H^0(X_{FF}, \mathcal{E})]$$

Def A morphism $W \to W'$ of qBCs is a morphism of TVSs respecting the filtration, s.t. $W_{=0} \to W'_{=0}$ is B_{dR}^+-linear.

Thm (1) If W is a reasonable qBC, then

$W_{>0}$ is the biggest sub-BC of curvature >0

$W_{<0}$ is the biggest quotient-BC of curvature <0.

i.e. the filtration is unique.

(2) If W, W' are reasonable of the same type, then $\mathrm{Hom}_{qBC}(W, W') = \mathrm{Hom}_{TVS}(W, W')$.

Thm qBC's form an abelian cat & the ht fct is additive,

i.e. $\mathrm{ht}\, W = \mathrm{ht}\, W_{>0} + \mathrm{ht}\, W_{<0}$ $(\mathrm{ht}\, W_{=0} = 0)$.

Thm W $\bar{q}BC$ $\Rightarrow$ non-canonically,

$$W \simeq W_{>0} \oplus W_{=0} \oplus W_{<0}$$

W qBC $\Rightarrow$ non-canonically,

$$W \simeq W_{\geq 0} \oplus W_{<0},$$

but $0 \to W_{>0} \to W_{\geq 0} \to W_{=0} \to 0$

is not necessarily split.

<u>Anschütz - Le Bras</u> $\mathrm{Ext}^n_{BC}(W, W') = 0$ for $n \geq 2$, $\forall\, W, W' \in BC$.

or VS (need more work than BC)

<u>Thm</u> (1) If $W \in \bar{q}BC$, $W' \in BC$, then

$$\mathrm{Ext}^n_{qBC}(W, W') = 0, \quad n \geq 2.$$

Application: for Poincaré duality, need $W' = \mathbb{Q}_p$.

(2) If $W \in \bar{q}BC$, $W' \in BC$, then $\mathrm{Ext}^2_{qBC}(W', W) = 0$.

<u>Rmk</u> (i) Not true that $\mathrm{Ext}^2_{qBC}(W, W') = 0$ for any $W, W' \in qBC$.

b/c $\mathrm{Ext}^1 \neq 0$ for TVS.

e.g. $0 \to \underset{\text{loc const}}{LC(\mathbb{Z}_p, \mathbb{Q}_p)} \to \underset{\text{loc an}}{LA(\mathbb{Z}_p, \mathbb{Q}_p)} \xrightarrow{d} LA(\mathbb{Z}_p, \mathbb{Q}_p) \to 0$ not split.

e.g. $0 \to W_1 \to (\mathcal{U}_1 \hat{\otimes} W_1) + W \to \mathcal{U}_1 \hat{\otimes} W \to \mathbb{B}_1 \otimes W_2 \to 0$

w/ $0 \to W_1 \to W \to W_2 \to 0$

$0 \to \mathbb{Q}_p \to \mathcal{U}_1 \to \mathbb{B}_1 \to 0$.

(ii) If $W \in qBC$, can happen that $\mathrm{Ext}^2(W, \mathbb{Q}_p) \neq 0$.

(iii) Possibly, can happen $\mathrm{Ext}^n = 0$ if $n \geq 3$

e.g. $W = (W_1 \to W_2)$.

<u>Slogan</u> Most of the work is to deal with top issues of $\mathbb{B}^+_{dR}$-mods.

<u>Thm</u> W_1, W_2 top $\mathbb{B}_m$-mods of the same type.

$$\mathbb{W}_i := (S \mapsto \mathbb{B}_m(S) \hat{\otimes}_{\mathbb{B}_m} W_i).$$

Then for $n = 0$ or 1, $\mathrm{Ext}^n_{B^+_{dR}}(W_2, W_1) \to \mathrm{Ext}^n_{\mathbb{B}^+_{dR}}(\mathbb{W}_2, \mathbb{W}_1)$

$$\xrightarrow{\sim} \mathrm{Ext}^n_{TVS}(\mathbb{W}_2, \mathbb{W}_1).$$

Ex $n=1$, $W_1 = W_2 = C$,

$$\mathrm{Hom}(\mathbb{G}_a, \mathbb{G}_a) \simeq \mathrm{Hom}(C, C) = C$$

$$\mathrm{Ext}^1(\mathbb{G}_a, \mathbb{G}_a) \simeq \mathrm{Ext}^1_{B_{dR}}(C, C) \simeq C \quad \left(0 \to C \to \underset{\underset{B_{dR}^+/t^2}{\|}}{B_2} \to C \to 0\right).$$

Works / TVS or VS.

__Note__ $S, S' \in qBC$, then $\mathrm{Hom}(S, S') \times W(S) \longrightarrow W(S')$

bounded $\longmapsto$ bounded.