Representations of p-adic groups and Hecke algebras

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<u>Setup</u> $F/\mathbb{Q}_p$ finite or $F = \mathbb{F}_q((t))$, $F \supset \mathcal{O} \ni \varpi$, $\mathcal{O}/\varpi\mathcal{O} = \mathbb{F}_q$.

G conn reductive grp $/F$, e.g. $GL_n, SL_n, Sp_{2n}, SO_n$.

split over a tame ext

(e.g. $D^\times \cong GL_n$ over certain tame ext field.)

<u>Goal</u> We want to understand the category of all smooth $\underset{\underset{\bar{\mathbb{Q}}_\ell \text{ (or } \bar{\mathbb{F}}_\ell)}{\text{is}}}{\mathbb{C}}$-rep'ns of $G(F)$.

Building blocks = supercuspidal rep'ns.

<u>Fact</u> π irred rep of $G(F)$

$\Rightarrow \exists P \subseteq G$ para, $P = MN$, $\underbrace{M = \begin{pmatrix} \boxed{/} & & \\ & \boxed{///} & \\ & & \boxed{///} \end{pmatrix}, \; N = \begin{pmatrix} \boxed{\mathbb{1}} & * & * \\ & \boxed{\mathbb{1}} & * \\ & & \boxed{\mathbb{1}} \end{pmatrix}}_{\text{in } GL_n \text{ for example}}$

σ supercusp rep of $M(F)$,

s.t. $\pi \hookrightarrow \mathrm{Ind}_{P(F)}^{G(F)} \sigma \otimes \underset{\substack{\uparrow \\ \text{ext'n triv on } N}}{\mathbb{1}}$.

<u>Problem 1</u> Construct all supercusp rep'ns.

<u>Vague answer</u> Can do this under minor assumptions.

Bernstein decomp: $\mathrm{Rep}(G) \simeq \prod_{\{(M,\sigma)\}/\sim} \mathrm{Rep}(G)_{[M,\sigma]}$

$(M,\sigma) \sim (gMg^{-1}, \sigma(g^{-1}\_g) \otimes \underset{\substack{\uparrow \\ \text{unramified char.}}}{\chi})$, $g \in G(F)$

& $\mathrm{Rep}(G)_{[M,\sigma]}$ called Bernstein blocks which are reps whose irred subquotients are contained in $\mathrm{Ind}_{P'(F)}^{G'(F)} \sigma'$,

$$P' = M'N', \quad (M', \sigma') \sim (M, \sigma).$$

<u>Problem 2</u> Understand Bernstein blocks $\mathrm{Rep}(G)_{[M,\sigma]}$

<u>Example</u> $G = SL_2$.

(a) $M = G = SL_2$, σ = sc rep of $M(F)$ $(= G(F))$

$\rightsquigarrow \mathrm{Rep}(G)_{[G,\sigma]} = \{ \sigma, \sigma \oplus \sigma, \sigma \oplus \sigma \oplus \sigma, \cdots \}$

$(\mathrm{End}_{\mathbb{C}}\, \sigma = \mathbb{C}.)$

(b) $M = T = \left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} \right\}$ torus, σ = triv.

$\rightsquigarrow \mathrm{Rep}(G)_{[T,\mathrm{triv}]} \ni$ triv, St

$$\left(1 \longrightarrow \mathrm{triv} \longrightarrow \mathrm{Ind}_{B(F)}^{G(F)} \mathrm{triv} \longrightarrow \mathrm{St} \longrightarrow 1 \right)$$

Borel $\left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\}$

called principal block.

<u>Vague answer</u> (work in progress, Adler-Fintzen-Michra-Ohara)

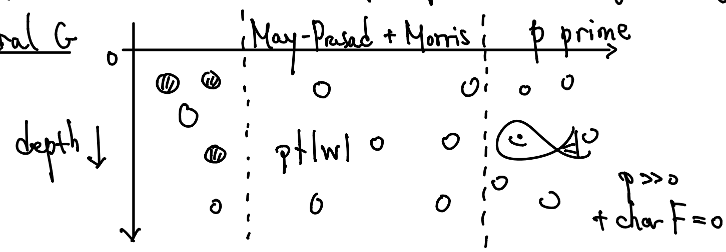
$$\mathrm{Rep}(G)_{[M,\sigma]} \simeq \mathrm{Rep}(G^{\circ})_{[M^{\circ}, \sigma_0]}$$

- σ_0 = depth zero rep'ns.
- RHS : rather well-understood.

<u>Construction of supercuspidal reps (and types)</u>

<u>Known</u> for GL_n, classical grps $(p \neq 2)$, inner forms of GL_n.

<u>For general G</u>



Moy – Prasad (1994/96), Morris (93/99) : depth - 0 rep'ns.
Adler, J.-K. Yu (1998/2001), Fintzen (2021).
J.-L. Kim (2007) : Yu's construction is exhaustive if $p \gg 0$ + char $F = 0$.
Fintzen (2021) : Yu's construction is exhaustive if $p \nmid |W|$, W Weyl grp.
Fintzen – Schwein (2024) : p small & depth small.

<u>Rmk</u> All those reps are of the form

$$\text{c-ind}_K^{G(F)} \rho = \left\{ f : G(F) \to V_\rho \;\middle|\; \begin{array}{l} f(kg) = \rho(k) f(g), \ \forall k \in K, g \in G(F) \\ f \text{ compactly supp} \end{array} \right\}$$

K compact mod center open subgrp of $G(F)$.

E.g. $G = SL_2$,

$$\begin{array}{ccc} SL_2(\mathcal{O}) & \longrightarrow & SL_2(\mathcal{O}) / \begin{pmatrix} 1+\varpi\mathcal{O} & \varpi\mathcal{O} \\ \varpi\mathcal{O} & 1+\varpi\mathcal{O} \end{pmatrix}^{\det = 1} \\ & & \big\downarrow S \\ & & SL_2(\mathbb{F}_q) \xrightarrow[\text{cuspidal rep}]{\rho} V_\rho \end{array}$$

$\pi_\rho = \text{c-ind}_K^{G(F)} \rho$.

<u>Note</u> G general, G_x max cpt subgrp (G ss, simply conn)

$$\begin{array}{ccl} K = G_x & \longrightarrow & G_x / \underbrace{G_{x,0+}}_{\substack{\uparrow \\ \text{pro-}p \text{ unipotent radical}}} \\ & & S \big\downarrow \\ & & \mathbb{G}(\mathbb{F}_q) \text{ if } G \text{ ss simply conn.} \\ & & \text{cusp} \big\downarrow \rho \\ & & V_\rho \end{array}$$

$\pi_\rho = \text{c-ind}_K^{G(F)} \rho$ sc depth-zero rep of $G(F)$.

Types

Def A pair (K, ρ) with K cpt open subgrp of $G(F)$
ρ irrep of K
is called an <u>$[M,\sigma]$-type</u> if the following are equivalent
for all irreps π of $G(F)$:

(i) $\pi \in \mathrm{Rep}(G)_{[M,\sigma]}$

(ii) $\rho \hookrightarrow \pi|_K$, i.e. $\mathrm{Hom}_K(\rho, \pi) \neq \{0\}$.

Thm (Bushnell-Kutzko, 1998)

Suppose (K,ρ) is an $[M,\sigma]$-type

$$\mathrm{Rep}(G)_{[M,\sigma]} \simeq \mathcal{H}(K,\rho)\text{-mod} \quad (\text{Hecke})$$

where

$$\mathcal{H}(K,\rho) = \left\{ f : G(F) \to \mathrm{End}(V_\rho) \;\middle|\; \begin{array}{l} f(k_1 g k_2) = \rho(k_1) f(g) \rho(k_2) \\ \forall\, k_i \in K,\ g \in G(F) \\ + f \text{ compactly supp} \\ + \text{ convolution} \end{array} \right\}$$

Rmk $\mathcal{H}(K,\rho) \simeq \mathrm{End}_{G(F)}(\text{c-ind}_K^{G(F)} \rho)$

Slogan Can understand all irreps through compact open subgrps.

1993: Morris gave an explicit description of $\mathcal{H}(K,\rho)$
for the case of a union of depth-zero Bernstein blocks
$\mathrm{Rep}(G)_{[M,\sigma]}$, σ depth-zero.
([AFMO] can ignore "unions")
in terms of generators and relations

$$\mathcal{H}(K,\rho) \cong \underbrace{\mathbb{C}[\Omega(\sigma),\mu]}_{\text{complicated}} \ltimes \mathcal{H}_{aff}(W_{aff}(\sigma), q).$$

E.g. $G = SL_2$.

(a) $M = G$, $\sigma = \text{c-ind}_K^{G(F)} \rho$, (K,ρ) is a type of (G,σ)

$\mathcal{H}(K,\rho) \simeq \mathbb{C}$.

(b) $M = T$, $\sigma = \text{triv}$, $(K,\rho) = (I_w, \text{triv})$

$$I_w = \begin{pmatrix} \mathcal{O} & \varpi\mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}$$

$$\mathcal{H}(I_w, \text{triv}) = \mathbb{C}[I_w \backslash G(F) / I_w]$$

$$\simeq \mathbb{C}[T_w \mid w \in W_{aff} = \langle s_0, s_1 \mid s_i^2 = 1 \rangle]$$

$$T_{s_i} T_w = \begin{cases} T_{s_i w} & \text{if } \ell(s_i w) = \ell(s_i) + \ell(w) \\ q T_w + (q-1) T_{s_i w}, & \text{else.} \end{cases}$$

def'n of $\mathcal{H}_{aff}(W_{aff}, q)$.

<u>Dream</u> Let (K,ρ) be a Kim-Yu type for $\text{Rep}(G)_{[M,\sigma]}$.

Then $\exists\, G^0 \subset G$ twisted Levi subgrp,

$M^0 \subset G^0$ Levi, σ_0 of $M^0(F)$ that is depth 0

(K^0, ρ_0) a type of $\text{Rep}(G^0)_{[M^0,\sigma_0]}$

s.t. $\mathcal{H}(G, K, \rho) \simeq \underbrace{\mathcal{H}(G^0, K^0, \rho_0)}_{\text{Hecke alg for a depth 0 Bernstein block}}$ (isom of algs).

(+ Thm $\Rightarrow$ everything reduces to depth 0.)

Or $\text{Rep}(G)_{[M,\sigma]} \simeq \text{Rep}(G^0)_{[M^0,\sigma_0]}$.

<u>When is the dream true for general G?</u>

1998 Roche: $M = T$ a split max torus.

2021 Ohara: $M = G$.

<u>Thm</u> (in progress, Alder-Fintzen-Michra-Ohara)
The dream is always true!

<u>Example</u> $G = SL_2$, $G^\circ = SL_2$ (can choose)

or $G^\circ = T = \begin{pmatrix} * & \\ & * \end{pmatrix}$ or $G^\circ = T_{an}$

$\varphi_0 = $ char.