An introduction to kimberlites

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Let Perfd = {perfd spaces}

Perf = {char p perfd spaces}

Tilting perfectoid alg $R \rightsquigarrow R^\flat = \varprojlim_{x \mapsto x^p} R$.

Get a functor $\flat : \mathrm{Perfd} \longrightarrow \mathrm{Perf}$

with (a) $\mathrm{Spa}(R, R^+)^\flat = \mathrm{Spa}(R^\flat, R^{+\flat})$

(b) $|X| \simeq |X^\flat|$ homeomorphism

(c) $X_{\acute{e}t} \equiv X^\flat_{\acute{e}t}$ equiv of étale sites

(d) $\flat : \mathrm{Perfd}_{/X} \simeq \mathrm{Perf}_{/X^\flat}$.

Question What should $(\mathrm{Spa}\,\mathbb{Q}_p)^\flat$ be?

Want $\pi_1^{\acute{e}t}((\mathrm{Spa}\,\mathbb{Q}_p)^\flat) = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.

Naive attempt $\varprojlim_{x \mapsto x^p} \mathbb{Q}_p = \mathbb{F}_p$.

Second idea $\mathrm{Spa}(\mathbb{C}_p \hat{\otimes} \mathbb{C}_p) \rightrightarrows \mathrm{Spa}\,\mathbb{C}_p \longrightarrow \mathrm{Spa}\,\mathbb{Q}_p$.

$(\mathrm{Spa}\,\mathbb{Q}_p)^\flat = \mathrm{coeq}(X^\flat \rightrightarrows (\mathrm{Spa}\,\mathbb{C}_p)^\flat)$.

The v-top

A map $f : X \to Y$ in Perf is a v-cover if

$\forall$ map $g : \mathrm{Spa}(A, A^+) \to Y$, $\exists$ a commutative diagram

$$\begin{tikzcd} \mathrm{Spa}(R,R^+) \arrow[r] \arrow[d, "h"'] & X \arrow[d, "f"] \\ \mathrm{Spa}(A,A^+) \arrow[r, "g"] & Y \end{tikzcd} \quad \text{(not Cartesian)}$$

with h surjective.

<u>Key fact</u> Any map of affinoid perf'd

$$\mathrm{Spa}(R,R^+) \to \mathrm{Spa}(A,A^+)$$

is a spectral map of spectral top spaces.

<u>Non-example</u> $T = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$.

$R = C^0(T, \mathbb{C}_p)$, $R^+ = C^0(T, \mathcal{O}_{\mathbb{C}_p})$.

$\Rightarrow \quad |\mathrm{Spa}(R,R^+)| = T$.

$\forall t \in T = |\mathrm{Spa}(R,R^+)|$, have $ev_t : R \longrightarrow \mathbb{C}_p$, $f \longmapsto f(t)$

Then $\coprod_{t \in T} \mathrm{Spa}\, \mathbb{C}_p \longrightarrow \mathrm{Spa}(R,R^+)$ is a surjection but not a covering in v-top.

<u>Diamonds</u> A map of v-sheaves $\mathcal{F} \to \mathcal{G}$ is <u>quasi-proétale</u> if for all $Y = \mathrm{Spa}(R,R^+)$ (strictly, totally disconn space), have Cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ \mathcal{F} \arrow[r] & \mathcal{G} \end{tikzcd}$$

then X is rep'ble & pro-étale over Y.

<u>Def'n</u> A <u>diamond</u> $\mathcal{F}$ is a v-sheaf that is quasi-proét locally repble.

$$\begin{tikzcd}[column sep=small, row sep=small]
 & \{\text{diamonds}\} & \\
\{\text{v-sheaves}\} \arrow[ur, phantom, "\supseteq"] \arrow[dr, phantom, "\overset{\text{qcqs}}{\supseteq}"] & & \underbrace{\{\text{spatial diamonds}\}}_{\substack{\text{can control their} \\ \text{étale cohom.}}} \arrow[ul, phantom, "\subseteq"] \\
 & \{\text{spatial v-sheaves}\} \arrow[ur, phantom, "\supseteq"] &
\end{tikzcd}$$

$\diamond : \{\text{adic spaces} / \mathbb{Z}_p\} \longrightarrow \{\text{v-sheaves}\}$

$\mathrm{Spa}(A, A^+)^\diamond (R, R^+) = \{ (R^\#, u, f) \} / \sim$

where $\cdot$ $R^\# \in \mathrm{Perfd}$, $u : (R^\#)^\flat \xrightarrow{\sim} R$,

& $f : \mathrm{Spa}(R^\#, R^{\#+}) \longrightarrow \mathrm{Spa}(A, A^+)$.

for $(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_p}$.

<u>Facts</u> (a) $X^\diamond = X^\flat$ if $X \in \mathrm{Perfd}$

(b) $\mathrm{Spd}\, \mathbb{Q}_p = \mathrm{coeq}(\mathrm{Spd}\, \mathbb{C}_p \hat{\otimes} \mathbb{C}_p \rightrightarrows \mathrm{Spd}\, \mathbb{C}_p)$

(c) $\diamond : \mathrm{Perfd} \xrightarrow{\sim} \mathrm{Perf} / \mathrm{Spd}\, \mathbb{Z}_p$ is an equiv.

(d) $X^\diamond$ is a diamond $\Leftrightarrow$ X is analytic.

(d+) If X analytic then $X^\diamond_{\text{ét}} \cong X_{\text{ét}}$.

(e) $\left\{ \begin{array}{c} \text{semi-normal} \\ \text{rigid spaces} / \mathbb{Q}_p \end{array} \right\} \overset{\diamond}{\hookrightarrow} \{\text{diamonds} / \mathrm{Spd}\, \mathbb{Q}_p\}$.

<u>Highlights of diamonds</u>

(a) Existence of local Shimura varieties

(b) New diamonds : $\cdot$ $\mathrm{Spd}\, \mathbb{Q}_p \times_{\mathbb{F}_p} \mathrm{Spd}\, \mathbb{Q}_p$

$\cdot$ B^+_{dR} - Grassmanian

$\cdot$ moduli spaces of p-adic shtukas

understand local Langlands correspondence.

(c) full 6-functor formalism for spatial diamonds.

<u>Kimberlites</u>

$$
\begin{tikzcd}
 & \{\text{formal schs} / \mathbb{Z}_p\} \arrow[dl, "\text{special fibre}"'] \arrow[d, "\diamond"] \arrow[dr, "\text{generic fibre}"] & \\
\{\text{schemes} / \mathbb{F}_p\} \arrow[d, "\text{perfection}"'] & \boxed{\{\text{Kimberlites}\}} \arrow[dl, "\text{special}"'] \arrow[d] \arrow[dr, "\text{generic}"] & \{\text{rigid spaces} / \mathbb{Q}_p\} \arrow[d, "\diamond"] \\
\{\text{perfect schs} / \mathbb{F}_p\} & \{v\text{-sheaves}\} . & \{\text{spatial diamonds}\}
\end{tikzcd}
$$

<u>Examples</u> (a) $\operatorname{Spd} \mathbb{Z}_p \times_{\mathbb{F}_p} \operatorname{Spd} \mathbb{Z}_p$

(b) Beilinson–Drinfeld Grassmanian

$$\text{generic} \left\{ \downarrow B_{dR}^{+}\text{-Grass} \right. \qquad \left. \downarrow \text{Witt vector Grass} \right\} \text{special}$$

(c) Integral models of moduli space of shtukas.

<u>Properties of a spatial kimberlite K</u>:

(a) attatched a reduced special fiber $K^{red} \in \{\text{perfect sch}\}$

(b) analytic locus $K^{an} \in \{\text{spatial diamond}\}$

(c) equipped with a specialization map $sp : |K^{an}| \to |K^{red}|$.

(d) notion of completion along locally closed imm $S \hookrightarrow K^{red}$.

(e) well-behaved formal étale site $(K_{f\text{-}et})$:

if $K = \mathfrak{X}^{\diamond}$, $\mathfrak{X}$ formal sch,

then $(\mathfrak{X}^{\diamond})_{f\text{-}et} = \mathfrak{X}_{et}$. $\rightsquigarrow$ formal nearby cycles.

Here (a) & (b) part of axioms.

<u>Example</u> Given $\lambda \in \mathbb{Q}^{\geq 0}$ one considers

absolute Banach-Colmez space $BC(\mathcal{O}(\lambda))$

$\rightsquigarrow$ $BC(\mathcal{O}(\lambda))(R, R^+) = H^0(X_{FF,R}, \mathcal{O}(\lambda))$

with $BC(\mathcal{O}(\lambda)) = \begin{cases} \underline{\mathbb{Q}_p} / \mathbb{B}^n_{\mathbb{F}_p}, & 0 \leq \lambda < 1, \\ \text{spatial kimberlite}, & \lambda > 1. \end{cases}$

where $\mathbb{B}^n_{\mathbb{F}_p} = \operatorname{Spd} \mathbb{F}_p[[t_1, \dots, t_n]]$.

<u>Highlights</u>

(a) Rep'bility & normality of local models of integral Shimura varieties [AGLR].

(b) Computation of connected components of ADLV [GLX].

Let $\mathcal{G}/\mathbb{Z}_p$ be a parahoric grp sch, $[\mu : \mathbb{G}_m \longrightarrow G_{\overline{\mathbb{Q}}_p}]$.

$\rightsquigarrow$ $\mathrm{Sht}_{\mathcal{G},\mu}$ = stack of p-adic $\mathcal{G}$-shtukas bounded by μ

Bun_G = stack of G-bundles over FF curve.

$\rightsquigarrow$ $\exists$ special polygon map

$$\sigma : \mathrm{Sht}_{\mathcal{G},\mu} \longrightarrow \mathrm{Bun}_G.$$

<u>Theorem</u> (a) $\mathrm{Sht}_{\mathcal{G},\mu}$ is an Artin v-stack.

(b) The map σ is rep'ble in locally spatial diamonds.

(c) The $\operatorname{Spd} \overline{\mathbb{F}}_p$-fibres of σ are locally spatial kimberlites.

Logic : (c) $\Rightarrow$ (b) $\not\Rightarrow$ (a).