Solid sheaves & Category $D_\square$ (I)

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§1 Solid sheaf

X spatial diamond.

$j: U \to X$ quasi-pro-étale.

$\rightsquigarrow U = \lim U_i$, $U_i \xrightarrow{j_i} U$ qcqs, étale.

Define $j_{\natural}\hat{\mathbb{Z}} := \lim_i j_{i,!}\hat{\mathbb{Z}}$,

where $\hat{\mathbb{Z}} := \lim \mathbb{Z}/n\mathbb{Z}$ as a sheaf of rings on $X_{\text{proét}}$.

<u>Def'n</u> $\mathcal{F} \in \mathrm{Mod}(X_{\text{proét}}, \hat{\mathbb{Z}})$,

$\mathcal{F}$ is called <u>solid</u> if for all $j: U \to X$,

$$\mathrm{Hom}(j_{\natural}\hat{\mathbb{Z}}, \mathcal{F}) = \mathcal{F}(U).$$

$\rightsquigarrow \mathrm{Solid}(X, \hat{\mathbb{Z}}) \subseteq \mathrm{Mod}(X, \hat{\mathbb{Z}})$.

<u>Thm</u> (1) $\mathrm{Solid}(X, \hat{\mathbb{Z}}) \subseteq \mathrm{Mod}(X, \hat{\mathbb{Z}})$ is an ω-admissible localization.

In particular, it is stable under limits, colimits, ext'ns;

it is compactly generated.

$$\mathrm{Solid}(X, \hat{\mathbb{Z}})^{\omega} = \mathrm{Mod}(X, \hat{\mathbb{Z}}) \cap \mathrm{Solid}(X, \hat{\mathbb{Z}}).$$

(2) $\mathrm{Solid}(X, \hat{\mathbb{Z}})^{\omega} = \mathrm{Pro}(\underbrace{\mathrm{Mod}_{\text{ét}, \text{cons}}(X, \hat{\mathbb{Z}})}_{\text{étale constructible torsion sheaves.}})$

(3) There's a left adjoint $(\cdot)^{\blacksquare}: \mathrm{Mod} \to \mathrm{Solid}$

of the fulgetful functor.

(Background with condensed math's).

Lem Let $\mathcal{F}$ be an étale cons sheaf / X spatial diamond.
Then $\mathcal{F}$ is rep'ble by a spatial diamond that's étale / X.

Lem Let $\mathcal{F}$ be a proét sheaf on X.
Assume $\mathcal{F}$ is rep'ble by a qcqs diamond.
Then $R\mathcal{H}om(\mathcal{F}, -)$ commutes with filtered colim.

Proof Have $\cdots \to \hat{\mathbb{Z}}[\mathcal{F}^3] \times \hat{\mathbb{Z}}[\mathcal{F}^2] \to \hat{\mathbb{Z}}[\mathcal{F}^2] \to \hat{\mathbb{Z}}[\mathcal{F}] \to \mathcal{F} \to 0$.
$\Rightarrow R\Gamma(\mathcal{F}, -) = R\mathcal{H}om(\hat{\mathbb{Z}}[\mathcal{F}], -)$ commutes with filtered colim since $\mathcal{F}$ qcqs. $\square$

Lem Let $\{\mathcal{F}_i\}$ be cofiltered inverse system of torsion étale cons sheaves.
Then $R\lim \mathcal{F}_i$ sits in deg 0.

Proof $\mathcal{F}_i = R\mathcal{H}om(R\mathcal{H}om(\mathcal{F}_i, \underline{S}'), \underline{S}')$.
Assume X is w-local.
$\rightsquigarrow R\lim \mathcal{F}_i(x) = R\mathcal{H}om(\operatorname{colim} R\mathcal{H}om(\mathcal{F}_i, \underline{S}'), \underline{S}')$
($\underline{S}'$ injective by VII.1.7)

Proof of the Thm

(1) Let $\{\mathcal{F}_i\}, \{\mathcal{G}_j\}$ be inverse systems of étale torsion cons sheaves.

$$\begin{aligned} & R\mathcal{H}om(R\lim \mathcal{F}_i, R\lim \mathcal{G}_j) \\ &= R\lim R\mathcal{H}om(R\lim \mathcal{F}_i, \mathcal{G}_j) \\ &= R\lim_j \operatorname{colim}_i R\mathcal{H}om(\mathcal{F}_i, \mathcal{G}_j) \quad (\text{resolve } \mathcal{F}_i \text{ by } \hat{\mathbb{Z}}[\mathcal{F}_i^m]) \end{aligned}$$

[Sch 17a, Prop 14.7] $\Rightarrow \operatorname{colim} R\Gamma(\mathcal{F}_i, \mathcal{G}_j) = R\Gamma(\mathcal{F}, \mathcal{G}_j)$.

$\rightsquigarrow \operatorname{Pro}(\operatorname{Mod}_{\text{ét, cons}}) \hookrightarrow \operatorname{Mod}(X, \hat{\mathbb{Z}})$ fully faithful.

$R\lim \mathcal{F}_i = \lim \mathcal{F}_i$ rep'ble by a qcqs diamond.

$\Rightarrow \lim \mathcal{F}_i$ compact.

$\mathrm{Pro}(\mathrm{Mod}\,\text{ét},\mathrm{cons}) \subseteq \mathrm{Mod}(X,\hat{\mathbb{Z}})^{\omega}$, $j_!\hat{\mathbb{Z}} \in \mathrm{Pro}(\mathrm{Mod}\,\text{ét},\mathrm{cons})$.

$\mathcal{F} \longmapsto \mathrm{Hom}(j_!\hat{\mathbb{Z}},\mathcal{F})$, $\mathcal{F}(U)$ both commutes with filtered colimits.

$\Rightarrow \mathrm{Solid}(X,\hat{\mathbb{Z}})$ closed under limits, filtered colimits.

$\rightsquigarrow$ $\mathrm{Pro}(\mathrm{Mod}\,\text{ét},\mathrm{cons})$ generates $\mathrm{Solid}(X,\hat{\mathbb{Z}})$

$\Rightarrow \mathrm{Solid}(X,\hat{\mathbb{Z}}) = \mathrm{Ind}(\mathrm{Pro}(\mathrm{Mod}\,\text{ét}\,\mathrm{cons}))$. □

§2 Solid sheaves on small v-stack

Lem (VII.1.8) $f: X \to Y$ map of spatial diamonds,

(1) f^* commutes with $(-)^{\blacksquare}$,
in particular preserves the solid sheaves.

(2) If f is a v-covering (i.e. if $|f|$ injective)
then f^* reflects solid sheaves.
(if $f^*\mathcal{F}$ solid, so is $\mathcal{F}$).

Def'n X small v-stack. Define

$$\mathrm{Solid}(X,\hat{\mathbb{Z}}) \subseteq \mathrm{Mod}(X_v,\hat{\mathbb{Z}}), \quad \mathcal{F} \in \mathrm{Mod}(X_v,\hat{\mathbb{Z}}).$$

$\mathcal{F}$ is called solid if for all $Y \to X$, Y spatial diamond,
$\mathcal{F}|_{Y_{\text{proét}}}$ is a solid sheaf over Y.

Def'n X small v-stack.

$$D_{\blacksquare}(X,\hat{\mathbb{Z}}) := \{A \in D(X_v,\hat{\mathbb{Z}}) : \mathcal{H}^i(A) \in \mathrm{Solid}\}.$$

Rmk X spatial diamond.

Can define $D_\square(X, \hat{\mathbb{Z}}) := \{A \in D(X_{proét}, \hat{\mathbb{Z}}) : H^i(A) \in \mathrm{Solid}\}$

$\rightsquigarrow$ $D_\square(X_{proét}, \hat{\mathbb{Z}}) \xrightarrow[\sim]{\lambda_X^*} D_\square(X_v, \hat{\mathbb{Z}})$.

Prop $\mathrm{Hom}(j_! \hat{\mathbb{Z}}, A) = \mathrm{Hom}(\hat{\mathbb{Z}}[U], A)$, $\forall A \in D_\square(X_{proét}, \hat{\mathbb{Z}})$.

Proof A sitting in deg 0 & étale cons. [Sch 17a]. $\square$

§3 4-Functor formalism (VII. §2)

Let Λ be a condensed $\hat{\mathbb{Z}}^p$-alg. which is solid.

$\hat{\mathbb{Z}}^p := \varprojlim_{p \nmid n} \mathbb{Z}/n\mathbb{Z} \in \mathrm{Alg}_{cond} = \mathrm{Alg}(*_{proét})$.

the cat of condensed alg

$\rightsquigarrow$ $\Lambda \in \mathrm{Alg}(*_{proét}, \hat{\mathbb{Z}}^p)$ or $\Lambda \in \mathrm{Alg}(X_v, \hat{\mathbb{Z}}^p)$.

Def'n $D_\square(X, \Lambda) := \{$complex of v-Λ-mods that are solid $/\hat{\mathbb{Z}}\}$.

$\rightsquigarrow$ $D(X, (\Lambda, \hat{\mathbb{Z}}^p)_\square)$

From now on, X is a small v-stack.

Λ as above. $f : Y \to X$.

$\rightsquigarrow$ $f^* : D_\square(X, \Lambda) \longrightarrow D_\square(Y, \Lambda)$

$f_* : D_\square(Y, \Lambda) \longrightarrow D_\square(X, \Lambda)$

$(-) \otimes_\Lambda^{\mathbb{L}} (-) : D_\square(X, \Lambda)^2 \longrightarrow D_\square(X, \Lambda)$

$R\mathcal{H}om_{\Lambda, \square} : D_\square^{op} \times D_\square \longrightarrow D_\square(X, \Lambda)$.

(A) f^*

Lem $f : Y \to X$, $A \in D_\square(X, \Lambda)$.

Then $f^* A \in D(Y_v, \Lambda)$ lies in $D_\square(Y, \Lambda)$.

(B) f_*

<u>Prop</u> $f: Y \to X$, $A \in D_\square(Y, \Lambda)$.

Then $Rf_{v,*}A$ lies in $D_\square(X, \Lambda)$.

and we find a functor $Rf_*: D_\square(Y,\Lambda) \longrightarrow D_\square(X,\Lambda)$.

<u>Proof</u> Assume $\Lambda = \hat{\mathbb{Z}}^p$ & X spatial diamond

Y locally spatial diamond.

by taking a diamond hypercovering of Y.

Assume A splits in deg 0 & $A \in \mathrm{Mod}_{\text{ét,cons}}$.

[Sch 17a, Prop 17.6] $\Rightarrow$ $Rf_{v,*}A \in D_{\text{ét}}$. $\square$

(C) $(-)^{\mathbb{L},\square}: D(X_v, \Lambda) \longrightarrow D_\square(X, \Lambda)$ without proof.

(D) $(-) \otimes^{\mathbb{L}}_\Lambda (-): D_\square(X,\Lambda) \times D_\square(X,\Lambda) \longrightarrow D_\square(X,\Lambda)$.

$A \otimes^{\mathbb{L},\square} B := (A \otimes^{\mathbb{L}}_\Lambda B)^{\mathbb{L},\square}$.

$D_\square(X,\Lambda)$ is symm monoidal.

$(-)^{\mathbb{L},\square}: D(X,\Lambda) \longrightarrow D_\square(X,\Lambda)$ also symm monoidal.

<u>Property</u>: $(-)\otimes^{\mathbb{L},\square}(-)$ commutes with any pullback.

(E) $R\mathcal{H}om$.

<u>Lem</u> For any $A \in D(X_v, \Lambda)$, $B \in D_\square(X,\Lambda)$,

$R\mathcal{H}om_v(A, B) \in D_\square$.

In particular, it defines $R\mathcal{H}om: D_\square^{op} \times D_\square \to D_\square$.

(*) $R\mathcal{H}om$ is a <u>partial</u> right adjoint to $\otimes^{\mathbb{L},\square}$.

<u>Caveat</u> Projection formula

$$A \otimes^{\mathbb{L},\square} Rf_* B \xrightarrow{\sim} Rf_*(f^* A \otimes^{\mathbb{L},\square} B)$$

does NOT hold in general.

E.g. $f : \mathrm{Spa}\, C \to \mathbb{B}_C$, $A \in f_* \widehat{\mathbb{Z}}^p$, $B \in \widehat{\mathbb{Z}}^p$.

Lem Let $\Lambda = \widehat{\mathbb{Z}}^p$. Then $\otimes^{\mathbb{L},\square} : D_\square \times D_\square \to D_\square$ has homology dim 1.

Pf. Assume $A, B \in \mathrm{Solid}$, X spatial diamond.

A, B compact,

$A = R\lim \mathcal{F}_i$, $B = R\lim \mathcal{G}_j$, $\mathcal{F}_i, \mathcal{G}_j$ étale torsion.

Claim $A \otimes^{\mathbb{L},\square} B = R\lim \underbrace{\mathcal{F}_i \otimes^{\mathbb{L},\square} \mathcal{G}_j}_{\text{étale cons.}}$ $\square$

Lem (VII.2.6, c.f. [Sch17a, 19.5])

Assume $X' = X \times_S S'$, $g' : X' \to X$.

Then $g'^* : D_\square(X, \Lambda) \to D_\square(X', \Lambda)$ is fully faithful

if (1) $S' = \mathrm{Spd}\, K' \to \mathrm{Spd}\, K = S$,

or (2) $S' = \mathrm{Spd}\, C \to \mathrm{Spd}\, K = S$,

or (3) $S' = \mathrm{Spd}\, C' \to \mathrm{Spd}\, C = S$.

K & K' alg closed fields

C & C' alg closed perf'd fields.

Pf. Exactly the same as [Sch17a, 19.5].

§4 Partially compactly supported cohomology $R\beta_{\pm,!}$

S, X spatial diamonds.

$X \to *$ proper finite dim. trg.

Let $\{U_{a,b}\} \subseteq X \times S$ as in IV.5.3, $U_a = \bigcup_b U_{a,b}$, $U_b = \bigcup_a U_{a,b}$.

$\rightsquigarrow$ $R\beta_{!+} C := \mathrm{colim}\, R\beta_* (j_{a!} C|_{U_a})$

$R\beta_{!-} C := \mathrm{colim}\, R\beta_* (j_{b!} C|_{U_b})$.

$\rightsquigarrow$ $R\beta_{!\pm} C : D_\square(X \times S, \Lambda) \to D_\square(S, \Lambda)$.

Thm (VII.2.10)

$\alpha : X \times S \to X$, $A \in D_\square(X)$, $C = \alpha^* A$.

Assume either (1) $A \in D^+$ or (2) X coh sm over $*$.

Then $R\beta_{!,\pm} C = 0$.

Proof Assume $\Lambda = \widehat{\mathbb{Z}}^p$, $S = \mathrm{Spa}\, \overline{k((t))}^\wedge$.

(2) reduces to (1) since can pass to Portnikov tower.

By hypercovering X, may assume X affinoid

and any finite étale cover of X splits.

Pick any map $X \xrightarrow{h} X' = \mathrm{Spa}\, \overline{k((t))}^\wedge$.

$$R\beta_{!\pm} \alpha^* A = R\beta'_{!\pm} \alpha'^* Rh_* A.$$

Replace A by $Rh_* A$. Assume $X = X' = \mathrm{Spa}\, \overline{k((t))}^\wedge$.

Exactly the same proof as in VI.5.3. □

§5 Relative cohomology $f_\natural$

Prop/Def'n (VII.3.1) $f : Y \to X$ any map of small v-stacks.

(1) $f^* : D_\square(X, \Lambda) \to D_\square(Y, \Lambda)$ admits a left adjoint

which satisfies the projection formula,

and the internal adjunction

$$R\mathcal{H}om(f_\natural A, B) = Rf_* R\mathcal{H}om(A, f^* B).$$

(2) $f_\natural$ is indep't of Λ, i.e. commutes with res'n of scalars.

(3) $f_\natural$ commutes with any pullback.

Proof (1)(2) formal arguments.

(3) $g^* f_\natural = f'_\natural g'^* \iff Rg'_* f'^* = f^* Rg_*$. □

Prop (VII.3.2)

$f : Y \to X$ proper, rep'ble in spatial diamonds. $\dim \mathrm{trg} < +\infty$, coh sm.

Then Rf_* has finite coh dim.

and commutes with direct sums.

Proof Assume X spatial diamond, $A = R\lim \mathcal{F}_i$.

$$Rf_* A = R\lim \boxed{Rf_* \mathcal{F}_i}. \quad \text{uniformly bounded.}$$

uniformly bounded, étale cons.

Prop (VII.3.3) $f : Y \to X$ proper & rep'ble in spatial diamonds.

$\dim \mathrm{trg} < +\infty$, coh sm.

Then Rf_* has projection formula

$$Rf_* A \otimes^{\mathbb{L},\blacksquare} B = Rf_*(A \otimes^{\mathbb{L},\blacksquare} f^* B).$$

Proof Assume $B \in D^-$, B sitting in deg 0.

X spatial diamond. $A, B \in \mathrm{Mod}_{\text{ét, cons}}$.

Following from proj formula of étale sheaves. $\square$

Prop (VII.3.4)

$f : Y \to X$ proper & rep'ble in spatial diamonds, $\dim \mathrm{trg} < +\infty$, coh sm.

$g : X' \to X$ with base change $f' : Y' \to X'$, $g' : Y' \to Y$.

Then $$g_\natural Rf'_* = Rf_* g'_\natural.$$

Proof Assume $\Lambda = \widehat{\mathbb{Z}}^p$, A sitting in deg 0, X spatial diamond.

By hypercovering X', assume X' spatial diamond.

$A = R\lim \mathcal{F}_i$ compact.

$\rightsquigarrow$ both sides $\underbrace{\text{pseudo-compact}}$

bounded above with cpt cohomology sheaf.

Suffices for all $B \in D_{\text{ét},\text{cons}}(X, \lambda)$.

$$R\mathcal{H}om(g_\natural f'_* A, B) = R\mathcal{H}om(Rf_* g'_\natural A, B)$$

$$\iff R\mathcal{H}om(A, g'^* Rf^! B) = R\mathcal{H}om(A, Rf'^! g^* B).$$

<u>Construction</u>

$$\begin{tikzcd} Y \arrow[dr, "\Delta"] & & \\ & Y \times_X Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] \\ & Y \arrow[r, "f"] & X \end{tikzcd}$$

f proper, rep'ble in spatial diamond, $\dim.\mathrm{trg} < +\infty$, coh sm.

$$\rightsquigarrow Rf_* A = Rf_* \pi_{2,\natural} \Delta_\natural \Delta^* \pi_1^* A$$

$$= f_\natural R\pi_{1,*} \Delta_\natural \Delta^* \pi_1^* A$$

$$= f_\natural R\pi_{1,*}(\Delta_\natural \Lambda \otimes^{\mathbb{L},\square} \pi_1^* A)$$

$$= f_\natural (R\pi_{1,*} \Delta_\natural \Lambda \otimes^{\mathbb{L},\square} A).$$

<u>Prop</u> (VII.3.5) $R\pi_{1,*}\Delta_\natural \Lambda$ is invertible,

whose inverse is canonically isomorphic to

$$Rf^! \Lambda := (R\lim(Rf^! \mathbb{Z}/n\mathbb{Z})) \otimes^{\mathbb{L}}_{\hat{\mathbb{Z}}^p} \Lambda.$$

In particular, $f_\natural A = Rf_*(A \otimes^{\mathbb{L},\square} Rf^! \Lambda)$.

<u>Proof</u> Assume $\Lambda = \hat{\mathbb{Z}}^p$.

Rf_* admits a right adjoint $A \mapsto R\mathcal{H}om(R\pi_{1,*}\Delta_\natural \Lambda, f^* A)$.

This preserves $D_{\text{ét}}$, and identifies with $Rf^!$ on $D_{\text{ét}}$.

$\rightsquigarrow$ a map $R\pi_{1,*}\Delta_\natural \Lambda \longrightarrow (Rf^! \Lambda)^{-1}$

suffices to prove the isomorphism.

Pick any $B \in D_{\text{ét}}(X, \Lambda)$.

$\rightsquigarrow$ Suffices to show

$$R\mathrm{Hom}(R\pi_{1,*}\Delta_\natural\Lambda, B) = (Rf^!\Lambda)\otimes^{\mathbb{L}}_{\Lambda} B.$$

But $$\mathrm{LHS} = R\pi_{1,*}R\mathcal{H}om(\Delta_\natural\Lambda, R\pi_1^! B)$$
$$= \Delta^* R\pi_1^! B = B\otimes_{\mathbb{Z}/n\mathbb{Z}} Rf^!\mathbb{Z}/n\mathbb{Z}. \qquad \square$$