Cohen-Macaulay Schemes and Serre Duality

<u>Goal</u> Extend Serre Duality to CM sch.

§1 Cohen-Macaulay Schemes and Duality

Choose ω_X° dualizing sheaf, $\dim X = n$

$\rightsquigarrow H^n(X, \omega_X^\circ) \xrightarrow{t} k$

$\rightsquigarrow \theta^i : \mathrm{Ext}_X^i(\mathcal{F}, \omega_X^\circ) \longrightarrow H^{n-i}(X, \mathcal{F})^\vee$.

both sides are δ-functors in $\mathcal{F} \in \mathrm{Coh}(X)^{op}$

<u>note</u> $\mathrm{Ext}_X^i(\bigoplus \mathcal{O}_X(m), \omega_X^\circ) = 0 \Rightarrow \mathrm{Ext}_X^i(-, \omega_X^\circ)$ effaceable.

By def'n, θ° isom.

local rings $\mathcal{O}_{X,x}$ are all CM, $\forall x \in X$

<u>Thm</u> TFAE: (a) X equidim & $\boxed{\text{CM}}$

i.e. irred comps have the same dim

(b) θ^i ($i \geq 0$) isom, $\forall \mathcal{F} \in \mathrm{Coh}(X)$.

<u>Punchline</u> (a) is a local condition whereas (b) seems not.

Indeed, a reg loc ring is always CM.

<u>Cor</u> X/k sm, then θ^i isom, $\forall i \geq 0$ & $\mathcal{F} \in \mathrm{Coh}(X)$.

§2 Proof of the Duality (I)

Start with (b) $\Leftrightarrow$ some loc condition.

<u>Lemma</u> TFAE to (b):

(c) $\forall \mathcal{F}$ loc free, $H^i(X, \mathcal{F}(-q)) = 0$, $\forall i < n$, $q \gg 0$.

(c') $H^i(X, \mathcal{O}_X(-q)) = 0$, $\forall i < n$, $q \gg 0$.

__Recall__ Serre vanishing: $H^i(X, \mathcal{F}(q)) = 0$, $\forall i > 0$, $q \gg 0$

(c) is some opposite sort of it.

Proof. (b) $\Rightarrow$ $\forall \mathcal{F} \in Coh(X)$ loc free, $\forall i < n$:

$$H^i(X, \mathcal{F}(-q)) = Ext_X^{n-i}(\mathcal{F}(-q), \omega_X^\circ)^\vee$$
$$= Ext_X^{n-i}(\mathcal{O}_X, \mathcal{F}^\vee \otimes \omega_X^\circ(q))^\vee \quad \text{by loc. free}$$
$$= H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X^\circ(q))^\vee$$

Serre vanishing $\Rightarrow$ it vanishes when $q \gg 0$, $n-i > 0$

$\Rightarrow$ (c).

(c) $\Rightarrow$ (c'): clear.

(c') $\Rightarrow$ $H^{n-i}(X, -)^\vee$ effaceable, $\forall i > 0$

(since $\mathcal{F}$ can be covered by $\bigoplus \mathcal{O}_X(n_i)$)

$\Rightarrow$ θ^i natural b/w two univ δ-functors

$\Rightarrow$ θ^i isom $\Rightarrow$ (b). $\square$

Next: Reformulate in local terms

__Lemma__ (b) $\Leftrightarrow$ (d) $\forall i < n$, $\mathcal{E}xt_P^{N-i}(j_*\mathcal{O}_X, \omega_P) = 0$. $j: X \hookrightarrow P = \mathbb{P}_k^N$ closed imm.

__Recall__ whatever X is, $\mathcal{E}xt_P^{N-i}(j_*\mathcal{O}_X, \omega_P) = 0$, $\forall i > n$.

(see notes for dualizing sheaf).

Proof. Serre duality on P (choosing $H^N(P, \omega_P) \cong k$):

$$H^i(X, \mathcal{O}_X(-q)) \cong H^i(P, j_*\mathcal{O}_X(-q))$$
$$\cong Ext_P^{N-i}(j_*\mathcal{O}_X(-q), \omega_P)^\vee$$
$$= Ext_P^{N-i}(j_*\mathcal{O}_X, \omega_P(q))^\vee$$

$\Rightarrow$ (c) $\Leftrightarrow$ "$Ext_P^{N-i}(j_*\mathcal{O}_X, \omega_P(q)) = 0$, $q \gg 0$, $i < n$".

Also, recall that for $q \gg 0$,

$$\mathrm{Ext}_P^{N-i}(j_*\mathcal{O}_X, \omega_P(q)) = \Gamma(P, \mathcal{E}xt_P^{N-i}(j_*\mathcal{O}_X, \omega_P(q)))$$
$$= \Gamma(P, \underbrace{\mathcal{E}xt_P^{N-i}(j_*\mathcal{O}_X, \omega_P)}_{\text{coherent on } P.}(q))$$

$$\Downarrow$$

$$\left(\begin{array}{c} \Gamma(P, \mathcal{E}xt_P^{N-i}(j_*\mathcal{O}_X, \omega_P)(q)) = 0, \quad q \gg 0 \\ \Leftrightarrow \mathcal{E}xt_P^{N-i}(j_*\mathcal{O}_X, \omega_P) = 0. \end{array} \right) \qquad \square$$

<u>Lemma</u> (b) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e):

$$\left(\begin{array}{l} \forall x \in X,\ A = \mathcal{O}_{P,x},\ I \subseteq A \text{ ideal defing } X \text{ at } x, \\ \Rightarrow \forall i < n,\ \mathrm{Ext}_A^{N-i}(A/I, A) = 0 \end{array} \right)$$

local condition, but still refers to the position of X in P (given by I here).

§3 The Cohen-Macaulay Condition

To get rid of the relative geom $X \subseteq P$.

<u>Prop</u> A reg loc ring, $M \in \mathrm{Mod}_A$ f.g. Then $\forall n \geq 0$, TFAE:

(a) $\mathrm{Ext}^i(M, A) = 0$, $\forall i > n$

(b) $\forall N \in \mathrm{Mod}_A$, $\mathrm{Ext}^i(M, N) = 0$, $\forall i > n$.

(c) $\exists$ proj resolution $0 \to L_n \to \cdots \to L_0 \to M \to 0$ of M at length $\leq n$.

Proof. Hartshorne Prop III.6.10A, Ex III.6.6.

Minimal length in (c) $= \mathrm{pd}_A(M)$, proj dim of M.

e.g. M proj $\Leftrightarrow \mathrm{pd}_A(M) = 0$.

Regular sequence: $x_1, \dots, x_n$, $x_i \in A$ s.t.

x_i not a zero div on $M/(x_1, \dots, x_{i-1})M$.

A loc ring $\rightsquigarrow$ depth M = max'l length of reg seq with $x_i \in \mathfrak{m}_A$.

Prop A reg local, $M \in \mathrm{Mod}_A$,

$$\boxed{\mathrm{pd}_A(M) + \mathrm{depth}_A(M) = \dim(A)}.$$

Proof. Hartshorne III.6.12A (& Matsumura).

Recall $\left(\begin{array}{l}\text{(e) } \forall x \in X,\ \mathcal{O}_{P,x} = A,\ I \subseteq A \text{ defining } X \text{ at } x \\ \Rightarrow \mathrm{Ext}_A^{N-i}(A/I, A) = 0,\quad i < n.\end{array}\right)$

$$\Leftrightarrow \mathrm{pd}_A(A/I) \le N - n \Leftrightarrow \mathrm{depth}_A(A/I) \ge n.$$

$$\dim A = \dim \mathbb{P}.$$

Trick: $M \in \mathrm{Mod}_{A/I} \Rightarrow \mathrm{depth}_A(M) = \mathrm{depth}_{A/I}(M)$

Lemma (b) $\Leftrightarrow$ (e) $\Leftrightarrow$ (f): $\forall x \in X$, $B = \mathcal{O}_{X,x}$, then $\mathrm{depth}_B(B) \ge n$.

On the other hand, always $\mathrm{depth}_B(B) \le \dim B \le n$

$\rightsquigarrow$ equiv to require $\underbrace{\mathrm{depth}_B(B) = \dim B}_{\text{"Cohen-Macaulay"}} = n$.

Fact Any regular loc ring is CM

(generators of cot space as a reg sequence).

But CM is more permissive:

e.g. local complete intersection $\Rightarrow$ CM.