LECTURE NOTES ON HOMOLOGICAL METHOD IN ALGEBRAIC GEOMETRY

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1. Lecture 1: Introduction

In the spring semester of 2021, I will teach a course named 'homological algebra' at Peking University. The main reference for this course is Huybrechts's book 'Fourier-Mukai transforms in algebraic geometry' ([Huy06]). Besides this, we will also use parts of the following books: 'Methods of homological algebra', 'An introduction to homological algebra' and 'D-modules, perverse sheaves, and representation theory' (only the chapter of t-structures) i.e., [GM13], [Wei94], and [HT07].

A word of warning: Though the book 'Methods of homological algebra' is very intuitive and a great book, it contains enormous typos and mistakes, please read with care!!!

Before going on, let me give the contact information and the standard of evaluation. My email address: liuyucheng@bicmr.pku.edu.cn, my office is 30-2 Quan Zhai, you can visit my office in the regular office hours (Wed 3:30-5pm every odd week, Friday 1:30-3pm every week, Friday 5-6:30 pm every even week, the parity depends on the Campus calender) or by appointment.

There are 100 points for this course, 40 of them are based on your regular attendance, the other 60 depends on the grade of your exam. There will be no homework in this class. However, I will leave some exercises in class. If you have any questions about the exercises, please come to my office and talk to me. You might get some extra points for the visits.

Ok, now let us turn our attention to the contents of this course, homological algebra is a basic and important tool in almost every mathematical research areas, for example, algebraic geometry, algebraic topology, geometric representation theory, arithmetic geometry, model theory, mathematical physics etc. In this course, we will mostly focus on the applications of homological algebras on algebraic geometry. A student with basic knowledge in algebraic geometry (e.g. a student who have worked through the first three chapters of Hartshorne's book [Har13]) should be able to follow it. We will rarely touch the applications of homological algebra on other research areas because of my own ignorance in some of these areas.

The first several lectures will be about the general theory of homological algebra, for example, Abelian categories, triangulated categories, derived categories, spectral sequences etc. Then we will focus on the derived categories of coherent sheaves on algebraic varieties, we will study Fourier-Mukai transforms, D- equivalence, and Bondal-Orlov's reconstruction theorem etc. The last several lectures will be introduction to an advanced research topic, the topic depends on the interests of students, possible choices are motivic cohomology, perverse sheaf and decomposition theorems, Bridgeland stability conditions etc.

These lecture notes are used for teaching only, not for publication. I claim no originality of the results in these lecture notes. Some of the contents are directly taken from literature.

Let us warm up with some basic notion in category theory (we will ignore any set theoretical issues).

Definition 1.1. Let A and B be two categories. A functor $F : \mathcal{A} \to \mathcal{B}$ is fully faithful if for any two objects $A, B \in \mathcal{A}$ the induced map

$$F : Hom_{\mathcal{A}}(A, B) \longrightarrow Hom_{\mathcal{B}}(F(A), F(B))$$

is an isomorphism.

Remark 1.2. The functor is called full if the morphism is surjective, and faithful if the morphism is injective. We will show that a fully faithful functor between categories is 'like' an injective map between sets.

A morphism $F \to G$ between two functors $F, G : \mathcal{A} \to \mathcal{B}$ is given by morphisms $\phi_A \in Hom_{\mathcal{B}}(F(A), G(A))$ for any object $A \in \mathcal{A}$ and are functorial in A. i.e.,

$$G(f) \circ \phi_A = \phi_B \circ F(f)$$

for any morphism $f : A \to B$.

Then we can define when two functors are isomorphic.

Definition 1.3. We say two functors $F, G : \mathcal{A} \to \mathcal{B}$ are isomorphic if there exists a morphism of functors $\phi : F \to G$ such that the induced morphism $\phi_A : F(A) \to G(A)$ is an isomorphism for any object $A \in \mathcal{A}$.

Equivalently, F and G are isomorphic if there exist functor morphisms $\phi : F \to G$ and $\psi : G \to F$ with $\phi \circ \psi = id$ and $\psi \circ \phi = id$.

Definition 1.4. A functor $F : \mathcal{A} \to \mathcal{B}$ is called an equivalence if there exists a functor $F^{-1} : \mathcal{B} \to \mathcal{A}$ such that $F \circ F^{-1}$ is isomorphic to $id_{\mathcal{B}}$ and $F^{-1} \circ F$ is isomorphic to $id_{\mathcal{A}}$. One calls F^{-1} an inverse or, sometimes, quasi-inverse of F. Two categories $\mathcal{A}$ and $\mathcal{B}$ are called equivalent if there exists an equivalence $F : \mathcal{A} \to \mathcal{B}$.

Clearly, any equivalence is fully faithful. A partial converse is provided by

Proposition 1.5. *Let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful functor. Then F is an equivalence if and only if every object $B \in \mathcal{B}$ is isomorphic to an object of the form $F(A)$ for some $A \in \mathcal{A}$.*

Proof. In order to find the inverse functor F^{-1}, one chooses for any $B \in \mathcal{B}$ an object $A_B \in \mathcal{A}$ together with an isomorphism $\phi_B : F(A_B) \to B$. Then, let

$$F^{-1} : \mathcal{B} \longrightarrow \mathcal{A}$$

be the functor that associates to any object $B \in \mathcal{B}$ this distinguished object $A_B \in \mathcal{A}$ and for which $F^{-1} : Hom(B_1, B_2) \to Hom(F^{-1}(B_1), F^{-1}(B_2))$ is given by the composition of

$$Hom(B_1, B_2) \longrightarrow Hom(F(A_{B_1}), F(A_{B_2})),\ f \mapsto \phi_{B_2}^{-1} \circ f \circ \phi_{B_1}$$

and the inverse of the bijection

$$F : Hom(A_{B_1}, A_{B_2}) \longrightarrow Hom(F(A_{B_1}), F(A_{B_2}).$$

And it is easy to see F^{-1} is an inverse of F. □

This proposition immediately yields the following corollary.

Corollary 1.6. *Any fully faithful functor $F : \mathcal{A} \to \mathcal{B}$ defines an equivalence between $\mathcal{A}$ and the full subcategory of $\mathcal{B}$ consisting all objects $B \in \mathcal{B}$ isomorphic to $F(A)$ for some $A \in \mathcal{A}$.*

Remark 1.7. This corollary manifests the sentence "a fully faithful functor between categories is like an injective map between sets". And the full subcategory in corollary is called the essential image of F.

A historical remark: in ancient Rome, corollary is the gift to the guests given by the host after the banquet. So you should feel happy to see the word corollary.

In the following proposition we let $Fun(\mathcal{A}^{op}, \mathbf{Set})$ be the category of all contravariant functors, i.e., the objects are functors $F : \mathcal{A}^{op} \to \mathbf{Set}$ and the morphisms are functor morphisms. Consider the natural functor

$$\mathcal{A} \longrightarrow Fun(\mathcal{A}^{op}, \mathbf{Set}), A \mapsto Hom(\ , A)$$

Usually, we use h_A to denote the image od A in the category $Fun(\mathcal{A}^{op}, \mathbf{Set})$. And a functor $F \in Fun(\mathcal{A}^{op}, \mathbf{Set})$ is called representable if it is isomorphic to h_A for some $A \in \mathcal{A}$. Given morphism $f : A \to B$ in $\mathcal{A}$, there is an induced morphism of functors

$$h(f) : h_A \to h_B$$

sending $(g : C \to A) \in h_A(C)$ to the composition $(f \circ g : C \to B) \in h_B(C)$. Notice that we recover f as the image of id_A under $h(f)_A$.

If $F : \mathcal{A}^{op} \to \mathbf{Set}$ is a functor and $A \in \mathcal{A}$ is an object, then for a morphism of functors $\eta : h_A \to F$ we get an element $\tau_\eta \in F(A)$ as the image of $id_A \in h_A(A)$ under η_A.

Proposition 1.8 (Yoneda's Lemma)**.** *The collection of morphisms of functors $\eta : h_A \to F$ is a set and the map*

$$\{morphisms\ h_A \to F\} \to F(A), \eta \mapsto \tau_\eta$$

is a bijection. In particular, the natural functor is fully faithful.

Proof. We need to construct the inverse of map, i.e., given an object in $\tau \in F(A)$, we need to construct a morphism of functors $\eta_\tau : h_A \to F$. This is done very naturally.

Given any object $B \in \mathcal{A}$, we let

$$\eta_{\tau,B} : h_A(B) \to F(B)$$

be the morphism sending $f \in h_A(B)$ to $F(f)(\tau) \in F(B)$.

The functoriality of F implies the functoriality of η_τ. And this is inverse to the map in the Proposition.

We leave the routine verification as exercises.

$\square$

Remark 1.9. Notice that Yoneda's lemma is stronger than the statement that the natural functor is fully faithful. Philosophically, Yoneda's lemma means that if we know the relation from objects to an object A, we know A itself in the category. We also have a dual version of Yoneda's lemma for corepresentable functors.

Example 1.10. *If a category $\mathcal{C}$ has n objects, where $n \in \mathbb{Z}_{>0}$, and for any two objects, there exists a unique isomorphism between them. In particular, $\mathcal{C}$ is a setoid. Then we can not distinguish the objects. However, a representable functor is represented by a unique object up to a unique isomorphism by Yoneda's lemma.*

We will rarely work with completely arbitrary categories. All our categories and functors are assumed to be at least additive.

Definition 1.11. A category $\mathcal{A}$ is an additive category if for any two objects $A, B \in \mathcal{A}$ the set $Hom(A, B)$ is endowed with the structure of abelian group such that the following is satisfied:

- The compositions
$$Hom(A_1, A_2) \times Hom(A_2, A_3) \to Hom(A_1, A_3)$$
are bilinear.
- There exists a zero object $0 \in \mathcal{A}$, i.e., an object 0 such that $Hom(0, 0)$ is the trivial group with one element.
- For any two objects A_1, A_2 there exists an object $B \in \mathcal{A}$ with morphisms $j_i : A_i \to B$ and $p_i : B \to A_i$, $i = 1, 2$, which makes B the direct sum and the direct product of A_1 and A_2.

Exercise 1.12. *(a) Show that for any object $A \in \mathcal{A}$ in an additive category $\mathcal{A}$ there exist unique morphisms $0 \to A$ and $A \to 0$, i.e. the zero object is the initial and final object of $\mathcal{A}$.*

(b) Show that the zero object is unique up to a unique isomorphism, in particular, the full subcategory of zero objects in an additive category is a connected setoid (connected means that: for any two objects, there is a unique isomorphism between them).

Remark 1.13. The first item in the definition provides a ring structure on $Hom(A, A)$ for arbitrary object $A \in \mathcal{A}$. The last two items can be summarized by the statement: finite product and coproduct of the same objects are the same in an additive category (the second item is the special case when the index set is empty set).

A functor between additive categories $\mathcal{A}$ and $\mathcal{B}$ will usually be assumed to be additive, i.e. the induced maps
$$Hom(A, B) \to Hom(F(A), F(B))$$

are group homomorphisms.

Everything that has been said so far carries over to additive categories. In particular, an additive functor $F : \mathcal{A} \to \mathcal{B}$ which is an equivalence is in fact an additve equivalence, i.e. the inverse functor is also additive. The Yoneda'a lemma is modified as follows: For an additive category $\mathcal{A}$ we let $Fun(\mathcal{A}^{op}, \mathbf{Ab})$ be the category of contravariant additive functors $F : \mathcal{A} \to \mathbf{Ab}$, where $\mathbf{Ab}$ is the category of abelian groups. Then the Yoneda's lemma remains valid in that form.

We will go even one step further. Eventually, all the categories we are interested in have geometric origin, i.e. are defined in terms of certain varieties over some base field, we usually deal with the following special type of abelian categories. We denote by k an arbitrary field.

Definition 1.14. A k-linear category is an additive category $\mathcal{A}$ such that the groups $Hom(A, B)$ are k-linear vector spaces and such that all compositions are k-linear.

Additive functors between two k-linear additive categories over a common base field k will be assumed to be k-linear, i.e. for any two objects $A, B \in \mathcal{A}$, the map

$$F : Hom(A, B) \longrightarrow Hom(F(A), F(B))$$

is k-linear.

Once again, everything that has been mentioned before carries over literally to k-linear categories. Usually, we will state all abstract results for additive categories, but in application everything will be over a base field. In principle, though, it could happen that two k-linear categories are equivalent as ordinary categories without being equivalent as k-linear categories.

The Yoneda's lemma can again be adjusted by setting the target category to be $\mathbf{Vec}(k)$ of k-vector spaces.

Definition 1.15. An additive category $\mathcal{A}$ is called abelian if also the following condition holds true:

- Every morphism $f \in Hom(A, B)$ admits a kernal and a cokernal and the natural map $Coim(f) \to Im(f)$ is an isomorphism.

Recall that the image $Im(f)$ is a kernal for a cokernal $B \to Coker(f)$ and the coimage $Coim(f)$ is a cokernal for a kernal $Ker(f) \to A$.

I think this terminology is a little bit confusing at first glance, hence we provide some examples to end this section.

Example 1.16. *(i) Let R be a commutative ring. Then the category $\boldsymbol{Mod}(R)$ of R-modules is abelian. The full subcategory of finitely generated modules is abelian as well.*

(ii) Let X be a scheme. Then the categories $\boldsymbol{Coh}(X)$, and $\boldsymbol{Qcoh}(X)$ of all coherent respectively quasi-coheren sheaves on X are both abelian.

(iii) The category $\boldsymbol{Ab}$ of abelian groups is an abelian category, while the category FAb of free abelian groups is only additive, not abelian.

Exercise 1.17. *Prove the last statement in examples. Hint: Consider the following map*

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$$

*in **FAb**, show that the kernal and cokernal of this map in **FAb** are trivial, and the natural map between the image and coimage is not an isomorphism.*

2. Lecture 2: Serre functors and triangulated categories

In an abelian category $\mathcal{A}$, since we have kernal, cokernal, and image of a map, we can define what is a short exact sequence, and what is a (left, right) exact functor. I believe we all are familiar with these notions.

Definition 2.1. A functor $F : \mathcal{A} \to \mathcal{B}$ is left adjoint to a functor $G : \mathcal{B} \to \mathcal{A}$ if there exist isomorphisms

$$Hom(F(A), B) \to Hom(A, G(B))$$

for any two objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$ which are functorial in A and B.

Remark 2.2. (i) We also say that G is right adjoint to F in the definition, and denote it by $F \dashv H$. Suppose we have $F \dashv H$. Then $id_{F(A)} \in Hom(F(A), F(A))$ induces a morphism $A \to H(F(A))$. The naturality of isomorphisms in the definition ensures that these morphisms defines a functor morphism

$$h : id_{\mathcal{A}} \longrightarrow H \circ F.$$

In the same vein, we can get a functor morphism

$$g : F \circ H \longrightarrow id_{\mathcal{B}}.$$

(ii) If $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $F \dashv H$, then F is right exact and H is left exact. Note that F is an exact functor does not imply H is an exact functor (ask the students for examples, possible answer: the pull back functor is exact when the morphism is open immersion, the push forward functor is exact when the morphism is closed embedding, hence they are almost never both exact at the same time).

Exercise 2.3. *(i) Suppose $F \dashv H$. Show that*

$$f \mapsto (A \xrightarrow{h_A} H(F(A) \xrightarrow{H(f)} H(B))$$

describes the adjunction morphism $Hom(F(A), B) = Hom(A, H(B))$.

(ii) Prove the second remark.

(iii) Suppose $F \dashv H$. Show that for the induced morphisms $g : F \circ H \to id$ and $h : id \to H \circ F$ the composition

$$H \xrightarrow{h_{H()}} (H \circ F) \circ H = H \circ (F \circ H) \xrightarrow{H(g)} H$$

is the identity. We have similar statement for F.

(iv) Prove that h is a functor morphism.

Example 2.4. *Let $f : X \to Y$ be a morphism between two noetherian schemes X and Y. Then the pull back functor*

$$f^* : \boldsymbol{Qcoh}(Y) \to \boldsymbol{Qcoh}(X)$$

is left adjoint to

$$f_* \boldsymbol{Qcoh}(X) \to \boldsymbol{Qcoh}(Y).$$

The same holds for the categories of coherent sheaves if the map is proper.

Lemma 2.5. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor and $G \dashv F$. Then the induced functor morphism $g : G \circ F \to id_{\mathcal{A}}$ induces for any $A, B \in \mathcal{A}$ the following commutative diagram*

$$\begin{tikzcd}
Hom(A,B) \arrow[d, "\circ g_A"'] \arrow[dr, "F"] & \\
Hom(G(F(A)),B) \arrow[r, "\sim"] & Hom(F(A),F(B)).
\end{tikzcd}$$

We have the similar commutative diagram for G.

Proof. For any $f : A \to B$, we have the following commutative diagram by the naturality.

$$\begin{tikzcd}
Hom(G(F(A)),A) \arrow[r, "\sim"] \arrow[d, "f\circ"'] & Hom(F(A),F(A)) \arrow[d, "F(f)\circ"] \\
Hom(G(F(A)),B) \arrow[r, "\sim"] & Hom(F(A),F(B)).
\end{tikzcd}$$

Look at the image of $id_{F(A)}$, we get the commutative diagram. □

Corollary 2.6. *Suppose a fully faithful functor $F : \mathcal{A} \to \mathcal{B}$ admits a left adjoint $G \dashv F$. Then the natural functor morphism*

$$g : G \circ F \xrightarrow{\sim} id_{\mathcal{A}}$$

is an isomorphism.

We have a similar statement if $F \dashv G$.

Proof. Since $F : Hom(A,B) \to Hom(F(A, F(B)))$ is bijective, the commutative diagram in last lemma induces bijections

$$Hom(A,B) \xrightarrow{\sim} Hom((G \circ F)(A), B)$$

for all A and B. By the dual version of Yoneda's lemma, we get the corollary. □

The same argument also show the converse.

Corollary 2.7. *Suppose that we have $F \dashv G$. If the induced functor morphism $F \circ G \to id$ is an isomorphism, then G is fully faithful.*

Exercise 2.8. *Suppose $G \dashv F \dashv H$ and F fully faithful. Construct a canonical homomorphism $H \to G$*

2.1. Serre functors.

Definition 2.9. Let $\mathcal{A}$ be a k-linear category. A Serre functor ia a k-linear equivalence $S : \mathcal{A} \to \mathcal{A}$ such that for any two objects $A, B \in\in \mathcal{A}$ there is an isomorphism

$$\eta_{A,B} : Hom(A, B) \xrightarrow{\sim} Hom(B.S(A))^*$$

of k-vector spaces, which is functorial in A and B.

We write the induced pairing as

$$Hom(A, B) \times Hom(B, S(A)) \to k, \ (g, f) \mapsto \langle f|g\rangle.$$

In order to avoid any trouble with the dual, we usually assume that all Hom's in $\mathcal{A}$ are finite dimensional.

Exercise 2.10. *Suppose we have that $f \in Hom(B, S(A))$ and $g \in Hom(A, B)$. Prove that $\langle f|g\rangle = \langle f \circ g|id\rangle = \langle S(g) \circ f|id\rangle = \langle S(g)|f\rangle$.*

Lemma 2.11. *Let $\mathcal{A}$ and $\mathcal{B}$ be k-linear categories with finite dimensional Hom's. If they are endowed with Serre functors $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$ respectively. Then any k-linear equivalence commutes with Serre duality, i.e. there exists an isomorphism*

$$F \circ S_{\mathcal{A}} \simeq S_{\mathcal{B}} \circ F.$$

Proof. For any two objects $A, B \in \mathcal{A}$, we have

$$Hom(A, S(B)) \simeq Hom(F(A), F(S(B)) \text{ and } Hom(B, A) \simeq Hom(F(B), F(A)).$$

Together with

$$Hom(A, S(B)) \simeq Hom(B, A)^* \text{ and } Hom(F(B), F(A)) \simeq Hom(F(A), S(F(B))^*,$$

this yields a functorial isomorphism

$$Hom(F(A), F(S(B))) \simeq Hom(F(A), S(F(B)).$$

By the assumption F is an equivalence, hence Yoneda's lemma implies the statement. ∎

Exercise 2.12. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between k-linear categories $\mathcal{A}$ and $\mathcal{B}$ endowed with Serre functors $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$ respectively. Then*

$$G \dashv F \Rightarrow F \dashv S_{\mathcal{A}} \circ G \circ S_{\mathcal{B}}^{-1}$$

2.2. Triangulated categories.

Let us start with the definition of triangulated category.

Definition 2.13. Let $\mathcal{D}$ be an additive category. The structure of a triangulated category on $\mathcal{D}$ is given by an additive equivalence

$$T : \mathcal{D} \to \mathcal{D},$$

the shifted functor, and a set of distinguished triangles

$$A \to B \to C \to A[1]$$

subject to the axioms TR1-TR4 (here $A[1] \coloneqq T(A)$).

TR1 (i)Any triangle of the form

$$A \xrightarrow{id} A \to 0 \to A[1]$$

is distinguished.

(ii) Any triangle isomorphic to a distinguished triangle is distinguished. The morphism between triangles is defined as you imagine.

(iii) Any morphism $f : A \to B$ can be completed to a distinguished triangle

$$A \xrightarrow{f} B \to C \to A[1].$$

TR2 The triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is a distinguished triangle if and only if

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$

is a distinguished triangle.

TR3 Suppose there exists a commutative diagram of distinguished triangles with vertical arrows f and g:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, dotted, "h"'] & A[1] \arrow[d, "f[1]"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1].
\end{tikzcd}$$

Then the diagram can be completed to commutative diagram, i.e. to a morphism of triangles, by a (not necessarily unique) morphism $h : C \to C'$.

TR4 This one is called the octahedron axiom. Since Octahedron is difficult to type in latex, I will type it in a different form (We will draw octahedron in class, since it is helpful in simplicial method). If we have the following morphism between distinguished triangles

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "id"'] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & A[1] \arrow[d, "id"] \\
A \arrow[r] & B' \arrow[r] & C' \arrow[r] & A[1].
\end{tikzcd}$$

Then it can be completed by the following diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "id"'] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & A[1] \arrow[d, "id"] \\
A \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & A[1] \arrow[d] \\
0 \arrow[r] & E \arrow[r, "id"] & E \arrow[r] & 0
\end{tikzcd}$$

such that all the columns and rows are distinguished triangles.

Proposition 2.14. *Let*

$$A \xrightarrow{f} B \to C \to A[1]$$

be a distinguished triangle in a triangulated category $\mathcal{D}$. Then for any object $A_0 \in \mathcal{D}$ the following induced sequences are exact:

$$Hom(A_0, A) \to Hom(A_0, B) \to Hom(A_0, C)$$

$$Hom(C, A_0) \to Hom(B, A_0) \to Hom(A, A_0).$$

Proof. Suppose $f : A_0 \to B$ composed with $B \to C$ is the trivial morphism $A_0 \to B \to C$. Then we get

$$\begin{tikzcd} A_0 \arrow[r, "id"] \arrow[d, dotted] & A_0 \arrow[r] \arrow[d, "f"'] & 0 \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \end{tikzcd}$$

which allows us to lift f to a morphism $A_0 \to A$. $\square$

Remark 2.15. (i) Due to TR2, one obtains in fact a long exact sequence.

(ii) The object C in TR1 (iii) is sometimes called the cone of $f : A \to B$ and denoted by $Cone(f)$. By TR3 and Exercise 2.16 (iii), we know that $Cone(f)$ is unique but up to a nonunique isomorphism. This is very unfortunate, it means the cone construction may not be functorial. So we need more care to prove that a construction involving cone construction is in fact a functor, like what happens to vanishing cycle functor, projection functor in semi-orthogonal decomposition, and truncation functors etc (see [KS94]). In the following exercise (v), we provide a criterion when the mapping cone unique up to a unique isomorphism.

Exercise 2.16. *(i) Prove that in any distinguished triangle*

$$A \xrightarrow{f} B \xrightarrow{h} C \to A[1],$$

the composition $g \circ f$ is zero.

(ii) Suppose

$$A \to B \to C \to A[1]$$

is a distinguished triangle. Show that $A \to B$ is an isomorphism if and only if $C \simeq 0$.

(iii) Consider a morphism of distinguished triangles

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & A[1] \arrow[d, "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]. \end{tikzcd}$$

show that if two of them are isomorphisms, then so is the third.

(iv) Let

$$A \to B \to C \to A[1]$$

be a distinguished triangle in a triangulated category $\mathcal{D}$. Suppose that $C \to A[1]$ is trivial. Show that the triangle splits, i.e. $B \simeq A \oplus C$.

(v) Suppose two rows in the following diagram are distinguished triangles. If we have a morphism $f : B \to B'$, such that the composition $A \to B \xrightarrow{f} B' \to C'$ is 0, then the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, dotted] & B \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, dotted] & A[1] \arrow[d, dotted] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

can be completed by the dotted arrows to be a commutative diagram.

Moreover, if $Hom(A[1], C') = 0$, these dotted arrows are unique.

(vi) Prove that the direct sum of two distinguished triangles is still a distinguished triangle.

These exercises are easy, useful and important, please do it!!!

Definition 2.17. An additive functor

$$F : \mathcal{D} \to \mathcal{D}'$$

between triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ is called exact if the following two conditions are satisfied.

- There is a functor morphism
$$F \circ T_{\mathcal{D}} \xrightarrow{\sim} T_{\mathcal{D}'} \circ F,$$
- Any distinguished triangle
$$A \to B \to C \to A[1]$$
in $\mathcal{D}$ is mapped to a distinguished triangle
$$F(A) \to F(B) \to F(C) \to F(A)[1]$$
in $\mathcal{D}'$, where $F(A)[1]$ is identified with $F(A[1])$ via the functor morphism in (i).

The following proposition illustrates the difference of exact functor of triangulated functors and exact functor of abelian categories. The precise replacement of exact functor in abelian categories should be t-exact functor, which notion will be introduced in later lectures.

Proposition 2.18. *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor between triangulated categories. If $F \dashv H$, then H is also exact.*

We have similar statement for left adjoint functor.

Proof. Let T and T' be the shifted functors in $\mathcal{D}$ and $\mathcal{D}'$ respectively. Since F is an exact functor, one has isomorphisms $F \circ T \simeq T' \circ F$ and $F \circ T^{-1} \simeq T'^{-1} \circ F$.

$$\begin{aligned} Hom(A, H(T'(B))) &\simeq Hom(F(A), T'(B)) \simeq Hom(T'^{-1}(F(A)), B) \\ &\simeq Hom(F(T^{-1}(A)), B) \simeq Hom(T^{-1}(A), H(B)) \\ &\simeq Hom(A, T(H(B))) \end{aligned}$$

As everything is functorial, the Yoneda's lemma yields an isomorphism

$$H \circ T' \simeq T \circ H.$$

Next, we need to prove that H sends a distinguished triangle to a distinguished triangle. Let

$$A \to B \to C \to A[1]$$

be a distinguished triangle in $\mathcal{D}'$. The induced morphism $H(A) \to H(B)$ can be completed to a distinguished triangle

$$H(A) \to H(B) \to C_0 \to H(A)[1].$$

Hence, we get the following commutative diagram

$$\begin{tikzcd} F(H(A)) \arrow[r] \arrow[d] & F(H(B)) \arrow[r] \arrow[d] & F(C_0) \arrow[r] \arrow[d, dotted, "\xi"'] & F(H(A))[1] \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1] \end{tikzcd}$$

which can be completed by the dotted row according to axiom TR3.
Applying H to the whole diagram and using the adjunction $h : id \to H \circ F$ yields

$$\begin{tikzcd} H(A) \arrow[r] \arrow[d] & H(B) \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d, "h_{C_0}"'] & H(A)[1] \arrow[d] \\ H(F(H(A))) \arrow[r] \arrow[d] & H(F(H(B))) \arrow[r] \arrow[d] & H(F(C_0)) \arrow[r] \arrow[d, "H(\xi)"'] & H(F(H(A)))[1] \arrow[d] \\ H(A) \arrow[r] & H(B) \arrow[r] & H(C) \arrow[r] & H(A)[1] \end{tikzcd}$$

By Exercise 2.16, we get the following diagram

$$\begin{tikzcd} H(A)) \arrow[r] \arrow[d, "id"] & H(B)) \arrow[r] \arrow[d, "id"] & C_0 \arrow[r] \arrow[d, "H(\xi)\circ h_{C_0}"] & H(A)[1] \arrow[d] \\ H(A) \arrow[r] & H(B) \arrow[r] & H(C) \arrow[r] & H(A)[1] \end{tikzcd}$$

Here, we want to apply (iii) in Exercise 2.16, but we are not allowed to use that $H(A) \to H(B) \to H(C) \to H(A)[1]$ is distinguished. But by adjunction we know that for any A_0 the sequence

$$Hom(A_0, H(B)) \longrightarrow Hom(A_0, H(C)) \longrightarrow Hom(A_0, H(A)[1])$$

is exact. Hence, by five lemma, we obtain

$$Hom(A_0, C_0) \simeq Hom(A_0, H(C))$$

for all A_0. That proves the statement. □

A subcategory $\mathcal{D}' \subset \mathcal{D}$ of a triangulated category is a triangulated subcategory if $\mathcal{D}'$ admits the structure of a triangulated category such that the inclusion is exact, i.e. the shift functor is inherited from $\mathcal{D}$. If $\mathcal{D}' \subset \mathcal{D}$ is a full subcategory, then it is a triangulated category if and only if $\mathcal{D}'$ is invariant under the shifted functor and for any distinguished triangle $A \to B \to C \to A[1]$ in $\mathcal{D}$ with $A, B \in \mathcal{D}'$ the object C is isomorphic to an object in $\mathcal{D}'$.

Definition 2.19. A full triangulated subcategory $\mathcal{D}' \subset \mathcal{D}$ is called (left, or right) admissible if the inclusion functor admits (left, or right) both adjoints respectively.

For a subcategory $\mathcal{C} \subset \mathcal{D}$, we define its left and right orthogonals as

$$^{\perp}\mathcal{C} := \{G \in \mathcal{D} | Hom(G, F) = 0 \ for\ all\ F \in \mathcal{C}\}$$

$$\mathcal{C}^{\perp} := \{G \in \mathcal{D} | Hom(F, G) = 0 \ for\ all\ F \in \mathcal{C}\}$$

We conclude this section by including a Proposition without proof. This is because the proposition is obvious in geometric situations, but the abstract proof is not so easy. You can take it as an exercise of using Exercise 2.10.

Proposition 2.20. *Any Serre functor on a triangulated category over a field k is exact.*

3. More on triangulated categories

In the first half of this lecture, we aim at introducing a criterion that allow us to decide whether a given exact functor is fully faithful or even an equivalence. In the second half of this lecture, we will introduce some standard terminology in triangulated categories.

Let us begin with the definition of a spanning classes.

Definition 3.1. A collection Ω of objects in a triangulated category $\mathcal{D}$ ia a spanning class of $\mathcal{D}$ if for all $B \in \mathcal{D}$ the following two conditions hold:
(i) If $Hom(A, B[i] = 0)$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \simeq 0$.
(ii) If $Hom(B[i], A) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \simeq 0$.

Remark 3.2. In the definition, we see that a spanning class determines other objects in some sense, in the derived category of coherent sheaves on an algebraic variety, we have a natural choice of spanning class, the skyscraper sheaves of points. The next proposition implies that we only need to check the functor on spanning class to see whether the functor is fully faithful.

Proposition 3.3. *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor between triangulated categories with left and right adjoints : $G \dashv F \dashv H$.*

Suppose Ω is a spanning class of $\mathcal{D}$ such that for all objects $A.B \in \Omega$ and all $i \in \mathbb{Z}$ the natural homomorphisms

$$F : Hom(A, B[i]) \to Hom(F(A), F(B)[i]$$

are bijective. Then F is fully faithful.

Proof. First recall that H and G are both exact by Lecture 2. We have the following commutative diagram:

$$\begin{tikzcd} Hom(A,B) \arrow[r, "h_B\circ"] \arrow[d, "\circ g_A"] & Hom(A, H(F(B))) \arrow[d, "\sim"] \\ Hom(G(F(A)), B) \arrow[r, "\sim"] & Hom(F(A), F(B)) \end{tikzcd}$$

for arbitrary $A, B \in \mathcal{D}$.

We first show that for any $A \in \Omega$ the homomorphism $g_A : G(F(A)) \to A$ is an isomorphism, we complete the morphism by a distinguished triangle

$$G(F(A)) \xrightarrow{g_A} A \to C \to G(F(A))[1].$$

Applying $Hom(\ ,B)$ for arbitrary $B \in \mathcal{D}$ induces a long exact sequence which combined with the commutative lower triangle yields

$$\begin{tikzcd} Hom(C, B[i]) \arrow[r] & Hom(A, B[i]) \arrow[r, "\circ g_A"] \arrow[dr, "F"] & Hom(G(F(A)), B[i]) \arrow[d, "\sim"] \\ & & Hom(F(A), F(B)[i]). \end{tikzcd}$$

If $B \in \Omega$, then $F : Hom(A, B[i]) \to Hom(F(A), F(B)[i])$ is bijective by assumption. Hence, $Hom(C, B[i]) = 0$ for all $i \in \mathbb{Z}$ and all $B \in \Omega$. Hence $C \simeq 0$ and g_A is an isomorphism.

Note this immediately implies that for $A \in \Omega$ and $B \in \mathcal{D}$ in fact all homomorphisms in the diagram are bijections, in particular

$$h_B\circ : Hom(A, B) \simeq Hom(A, H(F(B))).$$

This applied to $B \in \mathcal{D}$ and using a distinguished triangle of the form

$$B \xrightarrow{h_B} H(F(B)) \to C \to B[1]$$

shows that $Hom(A, C[i]) = 0$ for all $i \in \mathbb{Z}$ and all $A \in \Omega$. Hence $C \simeq 0$ and, thus, $h_B : B \xrightarrow{\sim} H(F(B))$. In particular,

$$h_B\circ : Hom(A, B) \xrightarrow{\sim} Hom(A, H(F(B)))$$

fo any $A \in \mathcal{D}$. Using the commutative diagram, we get

$$F : Hom(A, B) \xrightarrow{\sim} Hom(F(A), F(B))$$

for all $A, B \in \mathcal{D}$.

$\square$

Suppose we already know that the functor is fully faithful. We only need to prove that the functor is essentially surjective to prove the functor is an equivalence. The following lemma provides a criterion, whose assumption is however difficult to check in practice. Building upon the arguments we shall, however construct Proposition 3.8, which turns out to be very useful.

Lemma 3.4. *Let $F : \mathcal{D} \to \mathcal{D}'$ be a fully faithful exact functor between triangulated categories and suppose that F has a right adjoint $F \dashv H$. Then F is an equivalence if and only if for any $C \in \mathcal{D}'$ the triviality of $H(C)$, i.e. $H(C) \simeq 0$, implies $C \simeq 0$.*

Proof. We know that $h_A : A \to HF(A)$ is an isomorphism. In order to prove the assertion, one has to verify that also the adjunction morphism $g_B : F(H(B)) \to B$ is an isomorphism for any $B \in \mathcal{D}'$. We complete the adjunction morphism by the following distinguished triangle

$$FH(B) \to B \to C \to FH(B)[1].$$

Since H is exact, we obtain a distinguished triangle in $\mathcal{D}$

$$HFH(B) \xrightarrow{H(g_B)} H(B) \to H(C) \to HFH(B)[1]$$

Since one knows that $H(g_B) \circ h_{H(B)} = id_{H(B)}$ and, therefore $H(g_B)$ is an isomorphism, this shows that $H(C) \simeq 0$. Hence, by assumption $C \simeq 0$, which implies that g_B is an isomorphism. □

Remark 3.5. As we have said a fully faithful functor between categories is like an injective map between sets. Hence the lemma means that if the 'inverse map' is also injective, then the functor is an equivalence.

Definition 3.6. A triangulated category $\mathcal{D}$ is decomposed into triangulated subcategories $\mathcal{D}_1$ and $\mathcal{D}_2 \subset \mathcal{D}$ if the following three conditions are satisfied:

- Both categories $\mathcal{D}_1$ and $\mathcal{D}_2$ contain objects non-isomorphic to 0.
- For all $A \in \mathcal{D}$ there exists a distinguished triangle
$$B_1 \to A \to B_2 \to B_1[1]$$
with $B_1 \in \mathcal{D}_i$, $i = 1, 2$.
- $Hom(B_1, B_2) = Hom(B_2, B_1) = 0$ for all $B_1 \in \mathcal{D}_1$ and $B_2 \in \mathcal{D}_2$.

A triangulated category can not be decomposed is called indecomposable.

Remark 3.7. Note that the last item is super strong, the notion is not very interesting, because the derived category of an integral scheme is indecomposable. We will be more interested in a weaker notion, the semi-orthogonal decomposition.

Proposition 3.8. *Let $F : \mathcal{D} \to \mathcal{D}'$ be a fully faithful exact functor between triangulated categories. Suppose that $\mathcal{D}$ contains objects not isomorphic to 0 and that $\mathcal{D}'$ is indecomposable.*

Then F is an equivalence of categories if and only if F has a left adjoint $G \dashv F$ and a right adjoint $F \dashv H$ such that for any object $B \in \mathcal{D}'$ one has: $H(B) \simeq 0$ implies $G(B) \simeq 0$.

Proof. In order to prove the proposition, one introduces two full triangulated subcategories $\mathcal{D}'_1, \mathcal{D}'_2 \subset \mathcal{D}$. The first one, $\mathcal{D}'_1$, is the image of F, i.e. the full subcategory of all objects B isomorphic to some $F(A)$. We have $F(H(B)) \simeq B$ for any $B \in \mathcal{D}'$. Indeed, if $B \simeq F(A)$, then

$$H(B) \simeq H(F(A)) \simeq A,$$

for F is fully faithful. Thus,

$$B \simeq F(A) \simeq F(H(B)).$$

The second category, $\mathcal{D}'_2$, consists of all objects $C \in \mathcal{D}'$ with $H(C) \simeq 0$. Clearly, both are triangulated subcategories of $\mathcal{D}'$.

The argument in the proof of the previous lemma show that any $B \in \mathcal{D}'$ can be decomposed by a distinguished triangle

$$B_1 \to B \to B_2 \to B_1[1]$$

with $B_i \in \mathcal{D}'_i$.

Furthermore, for all $B_1 \in \mathcal{D}'_1$ and $B_2 \in \mathcal{D}'_2$ we have

$$Hom(B_1, B_2) \simeq Hom(F(H(B_1)), B_2) \simeq Hom(H(B_1), H(B_2)) = 0$$

and

$$Hom(B_2, B_1) \simeq Hom(B_2, F(H(B_1))) \simeq Hom(G(B_2), H(B_1)) = 0.$$

Since, $\mathcal{D}'$ is indecomposable, by assumption, $\mathcal{D}'_1$ is nontrivial. Hence $\mathcal{D}'_2$ is trivial. This proves the equivalence. □

Remark 3.9. The proposition can be best applied when $G = H$. If F is fully faithful and $H \dashv F \dashv H$ then F is an equivalence whenever $\mathcal{D}'$ is indecomposable.

3.1. **Some terminology in triangulated categories.** In this subsection, we will discuss the notion of semi-orthogonal decomposition, full exceptional collection, and t-structures of derived categories.

Definition 3.10. An object $E \in \mathcal{D}$ in a k-linear triangulated category $\mathcal{D}$ is called exceptional if

$$Hom(E, E[l]) = \begin{cases} k & \text{if } l = 0 \\ 0 & \text{if } l \neq 0 \end{cases}$$

An exceptional sequence is a sequence $E_1, \cdots E_n$ of exceptional objects such that

$$Hom(E_i, E_j[l]) = 0$$

for all $i > j$ and all l.

An exceptional collection is full if $\mathcal{D}$ is generated by $\{E_i\}$, i.e. any full triangulated subcategory containing all objects E_i is equivalent to $\mathcal{D}$ (via the inclusion).

Lemma 3.11. *Let $\mathcal{D}$ be a k-linear triangulated category such that for any $A, B \in \mathcal{D}$ the vector space $\oplus_i Hom(A, B[i])$ is finite dimensional.*

If $E \in \mathcal{D}$ is exceptional, then the objects $\oplus E[i]^{\oplus j_i}$ form an admissible triangulated subcategory $\langle E \rangle$ of $\mathcal{D}$.

Proof. We only prove the right admissibility, the left admissibility can be proved similarly. Since E is exceptional, given an object C in $\mathcal{D}$, consider the evaluation morphism

$$\oplus_k Hom(E, C[k]) \otimes E[-k] \to C,$$

where the first object is in $\langle E \rangle$. Complete it to a distinguished triangle

$$\oplus_k Hom(E, C[k]) \otimes E[-k] \to C \to D.$$

Since E is exceptional, we get $Hom(E, D[p]) = 0$ for all p. Hence $D \in \langle E \rangle^{\perp}$. □

Remark 3.12. If $\mathcal{D}$ is the derived category of coherent sheaves on an algebraic variety, then the skyscraper sheaf is an exceptional object.

The following notion generalize the concept of a (full) exceptional collection.

Definition 3.13. A sequence of full admissible triangulated subcategories

$$\mathcal{D}_1, \cdots \mathcal{D}_n \subset \mathcal{D}$$

is semi-orthogonal if for all $i > j$

$$\mathcal{D}_j \subset \mathcal{D}_i^{\perp}.$$

Such a sequence defines a semi-orthogonal decomposition of $\mathcal{D}$ if $\mathcal{D}$ is generated by the $\mathcal{D}_i$, i.e. for any $F \in \mathcal{D}$, there is a sequence of morphisms

$$0 = F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 = F,$$

such that $cone(F_i \to F_{i-1}) \in \mathcal{D}_i$ for all $1 \leq i \leq n$.

Remark 3.14. The vanishing condition implies that the filtration and its "factors" are unique and factorial, see exercise 2.16 (v). The functor $\delta_i : \mathcal{D} \to \mathcal{D}_i$ given by the i-th "factor", i.e.

$$\delta_i(F) = cone(F_i \to F_{i-1})$$

is called the projection functor onto $\mathcal{D}_i$. In the special case $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$, the functor is the left adjoint of the inclusion $\mathcal{D}_1 \to \mathcal{D}$, while the functor δ_2 is the right adjoint of $\mathcal{D}_2 \to \mathcal{D}$.

Lemma 3.15. *Any semi-orthogonal sequence of full admissible triangulated subcategories $\mathcal{D}_1, \cdots, \mathcal{D}_n \subset \mathcal{D}$, i.e. defines a semi-orthogonal decomposition of $\mathcal{D}$, if and only if any object $A \in \mathcal{D}$ with $A \in \mathcal{D}_i^{\perp}$ for all $i = 1, \cdots, n$ is trivial.*

Proof. Suppose $\mathcal{D}_1, \cdots \mathcal{D}_n \subset \mathcal{D}$ is a semi-orthogonal decomposition. If $A_0 \in \cap \mathcal{D}_i^{\perp}$, then $\mathcal{D}_1, \cdots, \mathcal{D}_n \subset^{\perp} A_0$. Hence ${}^{\perp}A_0 \simeq \mathcal{D}$ and in particular, $Hom(A_0, A_0) = 0$ and thus $A_0 \simeq 0$.

Let us now assume that $\cap \mathcal{D}_i^{\perp} = \{0\}$. For simplicity, we assume $n = 2$, the general case can be done by induction. We have to show that for any object $A_0 \in \mathcal{D}$ is contained in the triangulated subcategory generated by $\mathcal{D}_1$ and $\mathcal{D}_2$. Since $\mathcal{D}_2$ is admissible, one finds a distinguished triangle

$$A \to A_0 \to B \to A[1]$$

with $A \in \mathcal{D}_2$ and $B \in \mathcal{D}_2^{\perp}$. The latter one can be decomposed further into a distinguished triangle

$$C \to B \to C' \to C[1]$$

with $C \in \mathcal{D}_1$ and $C' \in \mathcal{D}_1^{\perp}$ (use that $\mathcal{D}_1$ is admissible).

As $C \in \mathcal{D}_1 \subset \mathcal{D}_2^{\perp}$ and $B \in \mathcal{D}_2^{\perp}$, one finds that $C' \in \mathcal{D}_2^{\perp}$. Hence $C' \in \mathcal{D}_1^{\perp} \cap \mathcal{D}_2^{\perp}$, which implies $C' \simeq 0$ by assumption. This proves the lemma. □

We end this section by introducing the notion of t-structures and t-exactness.

Definition 3.16. Let $\mathcal{D}$ be an triangulated category. A t-structure on $\mathcal{D}$ is a pair of full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ satisfying the condition (i) and (ii) below. We denote $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$, $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$ for every $n \in \mathbb{Z}$. Then the conditions are:
(i) $Hom(X, Y) = 0$ for every $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$;
(ii) $\mathcal{D}^{\leq -1} \subset \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$.
(iii) every object $X \in \mathcal{D}$ fits into an exact triangle

$$\tau^{\leq 0}X \to X \to \tau^{\geq 1}X \to \cdots$$

with $\tau^{\leq 0}X \in \mathcal{D}^{\leq 0}$, $\tau^{\geq 1}X \in \mathcal{D}^{\geq 1}$, where $\tau^{\leq n} := \tau^{\leq 0}[-n]$ and $\tau^{\geq n} := \tau^{\geq 0}[-n]$ are called truncation functors (see Exercise 2.16 (v)).

The heart of the t-structure is $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$. It is an abelian category (see [HT07, Theorem 8.1.9]). The associated cohomology functors are defined by $H^0 = \tau^{\leq 0}\tau^{\geq 0}$, $H^i(X) = H^0(X[i])$.

We omit the proof of the fact that $\mathcal{A}$ is an abelian category, because it is formal and relatively lengthy.

Suppose $f : \mathcal{D} \to \mathcal{D}'$ is a functor between two triangulated categories with t-structures. We can define (left, right) t-exactness.

Definition 3.17. An exact functor f is called left t-exact if $f(\mathcal{D}^{\geq 0}) \subset \mathcal{D}'^{\geq 0}$.

An exact functor f is called right t-exact if $f(\mathcal{D}^{\leq 0}) \subset \mathcal{D}'^{\leq 0}$.

An exact functor f is called t-exact if it is both right and left t-exact.

Remark 3.18. Notice that an t-exact functor induces an exact functor between the hearts, which are abelian categories.

In the next lecture, we will introduce derived category of an abelian category. For any derived category, there is a standard t-structure.

4. Derived categories

In the following several lectures. We will focus on the construction of derived categories, derived functors, and the compatibilities of the functors, which involves spectral sequences.

Given an abelian category $\mathcal{A}$, we will construct a triangulated category $D(\mathcal{A})$, which is called derived category in the literature. This construction involves three steps: first one, we need the naive category $Kom(\mathcal{A})$, whose objects are complexes, and morphisms are chain maps between complexes. Then, we pass this to homotopy category $K(\mathcal{A})$ by quotient homotopy equivalence in morphisms. Lastly, we invert some quasi-isomorphisms. The last step is so called localization procedure.

Before going on, I want to remark that the second step is important. Otherwise, the class of quasi isomorphisms might not be a multiplicative class, and the addition and composition in the final product would possibly be ill-defined.

Let me explain the three steps more explicitly. Firstly, the the objects in naive category $Kom(\mathcal{A})$ are complexes, which is defined by the following diagram

$$\cdots \to A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \to \cdots$$

satisfying $d^i \circ d^{i-1} = 0$. This is usually denoted by $A^\bullet$.

A morphism $f : A^\bullet \to B^\bullet$ between two complexes $A^\bullet$ and $B^\bullet$ is given by a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[r, "d_A^{i+1}"] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] & B^i \arrow[r, "d_B^i"] & B^{i+1} \arrow[r, "d_B^{i+1}"] & \cdots
\end{tikzcd}$$

Definition 4.1. The category of complexes $Kom(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the category whose objects are complexes $A^\bullet$ in $\mathcal{A}$ and morphisms are morphisms between complexes.

Remark 4.2. You can easily verify that $Kom(\mathcal{A})$ is still an abelian category.

There is a natural shift functor on $Kom(\mathcal{A})$.

Definition 4.3. Let $A^\bullet$ be a given complex. Then $A^\bullet[1]$ is the complex with $(A^\bullet[1])^i := A^{i+1}$ and differential $d^i_{A[1]} := -d_A^{i+1}$.

The shift $f[1]$ of a morphism $f : A^\bullet \to B^\bullet$ is given by $f[1]^i := f^{i+1}$.

Recall that the cohomology $H^i(A^\bullet)$ of a complex $A^\bullet$ is the quotient

$$H^i(A^\bullet) := \frac{Ker(d^i)}{Im(d^{i-1})} \in \mathcal{A}.$$

Any complex morphism $f : A^\bullet \to B^\bullet$ induces natural homomorphisms

$$H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet).$$

Hence, $H^i : Kom(\mathcal{A}) \to \mathcal{A}$ can be viewed as a functor for any $i \in \mathbb{Z}$.

Before going further, we need to introduce two notions: homotopy equivalence; quasi-isomorphisms. The first one will be used in the second step, i.e. passage to homotopy category $H(\mathcal{A})$; the second one will be used in the third step, i.e. passage to derived category.

Definition 4.4. Two morphisms of complexes

$$f, g : A^\bullet \to B^\bullet$$

are called homotopically equivalent, denoted by $f \sim g$, if there exists a collection of homomorphisms $h^i : A^i \to B^{i-1}$, $i \in \mathbb{Z}$, such that

$$f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i.$$

It is easy to see this is an equivalence relation. We can define the homotopy category $H(\mathcal{A})$ as the category whose objects are the objects of $Kom(\mathcal{A})$, and morphisms are

$$Hom_{H(\mathcal{A})}(A^\bullet, B^\bullet) := Hom_{Kom(\mathcal{A})}(A^\bullet, B^\bullet)/\sim .$$

Although the definition is natural, there is still something to check, we leave the easy verification as exercises.

Exercise 4.5. *(i) Homotopy equivalent class is stable under pre-compositions and post-compositions. This ensures that the composition is well defined in $H(\mathcal{A})$.*

(ii) If $f \sim g$, then $H^i(f) = H^i(g)$ for all i.

(iii) If $f : A^\bullet \to B^\bullet$ and $g : B^\bullet \to A^\bullet$ are given such that $f \circ g \sim id_B$ and $g \circ f \sim id_A$, then $H^i(f)^{-1} = H^i(g)$.

By previous exercise, the i-th homology functor $H^n : Kom(\mathcal{A}) \to \mathcal{A}$ factors through the homotopy category $H(\mathcal{A})$. Hence, the following definition makes sense in both $Kom(\mathcal{A})$ and $H(\mathcal{A})$. This definition plays an essential role in passage to derived category.

Definition 4.6. A morphism of complexes $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism (or qis, for short) if for all $i \in \mathbb{Z}$ the induced map $H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism.

Note that the projective resolution, injective resolution are quasi-isomorphisms. We will draw diagrams in class.

Notice that quasi-isomorphism are not equivalent relations. Then what should we do if we want to view a quasi-isomorphism as an isomorphism? We formally invert them.

4.1. **General localization procedure.** Let $\mathcal{C}$ be any additive category, S be a multiplicative class of morphisms (includes identity morphism, and closed under composition).

Our Goal is construct a morphism $\mathcal{C} \to \mathcal{C}[S^{-1}]$, elements of S becomes invertible in $\mathcal{C}[S^{-1}]$.

Definition 4.7. We call S a localizing class if we can complete the following diagrams by dotted arrows

(i) It satisfies the 2 out 3 property, i.e. for any two morphisms $f : A \to B$ and $g : B \to C$. If two of f, g, gf are in S, the nso is the third one.

(ii)

$$\begin{tikzcd} T \arrow[r, dotted] \arrow[d, dotted, "s' \in S"] & Z \arrow[d, "s \in S"] \\ X \arrow[r] & Y \end{tikzcd}$$

and

$$\begin{tikzcd} X \arrow[r] \arrow[d, "s\in S"] & Y \arrow[d, dotted, "s'\in S"] \\ Z \arrow[r, dotted] & T \end{tikzcd}$$

such that the corresponding dotted arrows are in S.

(iii) For any two morphisms $f, g : A \to B$.

The existence of $s \in S$ such that $f \circ s = g \circ s \Leftrightarrow$ The existence of $s' \in S$ such that $s' \circ f = s' \circ g$

Remark 4.8. We can think of these conditions by localizing a non-commutative ring (An additive category with essentially one object). The condition (ii) means that we can 'switch' the inverse element from S and normal morphisms in both ways.

We also have the notion of left localizing class and right localizing class, which essentially means that we can only switch the inverse element from S and normal morphisms in one particular way (left or right). We do not need this notion in this lecture.

Note that our definition of localizing set might be a little bit different from the definition in the literature. In the literature, usually, people do not require the first condition (2 out 3 property), they only need that S is closed under compositions. We add this condition to our definition because it simplifies most proofs, and most the examples (quasi isomorphisms, Serre subcategories) in this course satisfy this condition. We get the simplification with little cost.

Exercise 4.9. *Suppose we have that $\mathcal{A}$ an abelian category. A Serre subcategory $\mathcal{C}$ is a subcategory closed under subobject, quotients, and extensions. In particular, it is also an abelian category. Prove that the class of morphisms whose kernal and cokernal are in $\mathcal{C}$ forms a localizing class.*

Think about $\mathcal{A}$ as the category of abelian groups, and $\mathcal{C}$ as the subcategory of torsion groups. What is the the final localized category?

Definition 4.10. If we have $\mathcal{C}$ and S a localizing class. The localized category $\mathcal{C}[S^{-1}]$ is defined as follows: $Ob(\mathcal{C}[S^{-1}]) = Ob(\mathcal{C})$, morphisms between A and B are equivalence classes of the following roof diagrams.

$$\begin{tikzcd} & C \arrow[dl, "\in S"'] \arrow[dr] & \\ A & & B \end{tikzcd}$$

And two roofs are equivalent if we have the following commutative diagram

$$
\begin{tikzcd}
 & & D \arrow[dl, "\in S"'] \arrow[dr, "\in S"] & & \\
 & C \arrow[dl, "\in S"'] \arrow[drrr] & & C' \arrow[dlll, "\in S"] \arrow[dr] & \\
A & & & & B
\end{tikzcd}
$$

The composition law is defined by the following diagram

$$
\begin{tikzcd}
 & & F \arrow[dl, "\in S"'] \arrow[dr] & & \\
 & D \arrow[dl, "\in S"'] \arrow[dr] & & E \arrow[dl, "\in S"'] \arrow[dr] & \\
A & & B & & C
\end{tikzcd}
$$

The existence of this commutative diagram follows from the definition of localizing class.

Remark 4.11. There are several things to check in this definition. For example, the transitivity of equivalent relation, the composition is well defined etc, we leave the verification as exercises to readers. We will prove some in class.

There is a natural functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$.

One particular thing is that: morphisms in S becomes invertible.

$$
\begin{tikzcd}
 & & A \arrow[dl, "id"'] \arrow[dr, "id"] & & \\
 & A \arrow[dl, "id"'] \arrow[dr, "s"] & & A \arrow[dl, "s"'] \arrow[dr, "id"] & \\
A & & B & & A
\end{tikzcd}
$$

But the converse direction is not true.

Moreover, $\mathcal{C}[S^{-1}]$ is universal for such a property, i.e. if we have another functor

$$F : \mathcal{C} \to \mathcal{D}$$

such that any morphism in S becomes invertible after applying F, then this functor factors through $\mathcal{C}[S^{-1}]$.

Also, a roof can also be represented by a ditch

$$
\begin{tikzcd}
A \arrow[dr] & & B \arrow[dl, "\in S"] \\
 & C &
\end{tikzcd}
$$

by the definition of localizing set.

The following is the essential theorem of this section.

Theorem 4.12. *Quasi-isomorphisms form a localizing class in $H(\mathcal{A})$.*

The proof is postponed to the next subsection.

Remark 4.13. This is not true in $Kom(\mathcal{A})$. Consider the following diagram in the category of abelian groups

$$\begin{tikzcd} ? \arrow[r, dotted, "qis"] \arrow[d, dotted] & C^\bullet \arrow[d] \\ A^\bullet \arrow[r, "qis"] & B^\bullet \end{tikzcd}$$

where $A^\bullet = 0$, $B^\bullet = [\mathbb{Z} \xrightarrow{id} \mathbb{Z}]$, $C^\bullet = [0 \to \mathbb{Z}]$, and f, g are the obvious maps. Show that we can not fill in the diagram.

The derived category is denoted by $\mathcal{D}(\mathcal{A}) \coloneqq H(\mathcal{A})[qis^{-1}]$ is an additive category.

$$\begin{tikzcd} & & D \arrow[dl, "qis"'] \arrow[dr, "qis"] & & \\ & C \arrow[dl, "qis"'] \arrow[drrr] & & C' \arrow[dlll, "qis"] \arrow[dr] & \\ A & & & & B \end{tikzcd}$$

This procedure is like taking the common denominator in adding two fractals. Note that the diagram is not commutative.

We still have cohomology functors on $D(\mathcal{A})$, i.e. $H^n : H(\mathcal{A}) \to \mathcal{A}$ factors through $D(\mathcal{A})$ for $n \in \mathbb{Z}$. Since quasi-isomorphism becomes invertible in passage to cohomology functor.

4.2. **Cone construction and the proof of main theorem.**

Definition 4.14. Given a map of complexes $f : A^\bullet \to B^\bullet$ define the complex $Cone(f)^i = A^{i+1} \oplus B^i$, and the differentials are given by

$$\begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix}$$

We can check that this is a complex.

Remark 4.15. Reason for the name: in topology, given $f : X \to Y$ between two topological spaces. We have the topological space $Cone(f)$. If we look at the singular chain complexes, we get the construction of $Cone(f)$ as a complex.

Example 4.16. *If we view single objects A, B are complexes with the single object centered at degree 0, then $Cone(f : A \to B) = [A \xrightarrow{f} B]$, where B is in degree 0.*

The cone construction gives us a short exact sequence between complexes

$$0 \to B \to Cone(f) \to A[1] \to 0,$$

which gives us along exact sequence

$$\cdots \to H^i(B) \to H^i(Cone(f)) \to H^{i+1}(A) \to H^{i+1}(B) \to \cdots$$

Corollary 4.17. *f is a quasi-isomorphism if and only if $Cone(f)$ is acyclic, i.e. it has no cohomology.*

If we use the algebraic topological point of view, the following proposition is obvious. Indeed, $Cone(B \to Cone(f))$ is homotopy equivalent to $S(A)$. This is also the reason why people sometimes call the shift functor suspension functor.

Proposition 4.18.

$$Cone(B \to Cone(f)) \simeq A[1]$$

in homotopy category.

Proof. Consider the map

$$A[1] \to Cone(\tau) \to A[1],$$

where in the i-th degree, it is the following map

$$A^{i+1} \xrightarrow{\begin{pmatrix} -f \\ 1 \\ 0 \end{pmatrix}} B^{i+1} \oplus A^{i+1} \oplus B^i \xrightarrow{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^T} A^{i+1}$$

where the differential in the middle term is

$$\begin{pmatrix} -d_B & 0 & 0 \\ 0 & -d_A & 0 \\ 1 & f & d_B \end{pmatrix}$$

It is easy to check that these are chain maps.
The last thing to check is

$$Cone(\tau) \to A[1] \to Cone(\tau)$$

is homotopic to identity. The homotopy morphism is given by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

□

This is TR2 for $H(\mathcal{A})$. Moreover,

$$\begin{tikzcd} Cone(f) \arrow[r] \arrow[d, "id"] & A[1] \arrow[r, "-f"] \arrow[d] & B[1] \arrow[d, "id"] \\ Cone(f) \arrow[r] & Cone(\tau) \arrow[r] & B[1] \end{tikzcd}$$

commutes in the homotopy category.

Theorem 4.19. *A triangle*

$$A_1^\bullet \to A_2^\bullet \to A_3^\bullet \to A_1^\bullet[1]$$

in $H(\mathcal{A})$ is called distinguished if it is isomorphic in $H(\mathcal{A})$ to a triangle of the form

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau} Cone(f) \xrightarrow{\pi} A^\bullet[1].$$

Then toghether with the shift functor, $H(\mathcal{S})$ is a (pre)-triangulated category.

Proof. TR1 and TR2 are already proved, TR3 is obvious from the cone construction. TR4 is in [GM13, IV.2]. □

Now, we can prove the main theorem of this section. The condition (i) in the definition is obvious.

For condition (ii), we need the following diagram.

$$\begin{tikzcd}
Cone(\tau g)[-1] \arrow[r] \arrow[d] & Z \arrow[r, "\tau g"] \arrow[d, "g"] & Cone(f) \arrow[r] \arrow[d, "id"] & Cone(\tau g) \arrow[d, "\exists"] \\
Cone(\tau)[-1] \arrow[r] \arrow[d, "\sim"] & Y \arrow[r, "\tau"] \arrow[d, "id"] & Cone(f) \arrow[r] \arrow[d, "id"] & Cone(\tau) \arrow[d, "\sim"] \\
X \arrow[r, "f"] & Y \arrow[r, "\tau"] & Cone(f) \arrow[r] & X[1]
\end{tikzcd}$$

We need to check that $Cone(f)$ is homotopic to the cone of $Cone(\tau g)[-1] \to Z$, this follows from Proposition 4.18. Hence, we proved the condition (ii) for quasi-isomorphisms (the other direction can be proved similarly).

For the condition (iii) in the localizing set, assume $a : X \to Y$ be a morphism in $H(\mathcal{A})$ and $t : Z \to X$ such that $a \circ t = 0$. It suffices to find an $s \in S$ such that $s \circ t = 0$. Since $H(\mathcal{A})$ is a pretriangulated category, we can complete $t : Z \to X$ to a distinguished triangle

$$Z \xrightarrow{t} X \xrightarrow{g} Q \to Z[1].$$

Since $a \circ t = 0$, by Exercise 2.16 there exists a morphism $iQ \to Y$ such that $i \circ g = a$. Finally, we can complete i to a distinguished triangle. Here s is a quasi-isomorphism because Q is acyclic.

$$\begin{tikzcd}
Z \arrow[r, "t"] & X \arrow[r, "g"] \arrow[d, "id"] & Q \arrow[r] \arrow[d, "i"] & Z[1] \\
& X \arrow[r, "a"] & Y \arrow[d, "s"] & \\
& & W &
\end{tikzcd}$$

5. More on derived categories and derived functors

We will define the standard truncation functions in derived category $D(\mathcal{A})$. For any complex

$$\cdots \to A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots,$$

we can define $\tau^{\leq 0}(A^\bullet)$ as follows

$$\cdots \to A^{-1} \xrightarrow{d^{-1}} Ker(d^0) \to 0 \to \cdots,$$

and there is natural morphism $\tau^{\leq 0}(A^\bullet) \to A^\bullet$. And this morphism induces isomorphisms on cohomology objects in non-positive degree.

Similarly, one can define $\tau^{\geq 0}(A^\bullet)$ as follows

$$\cdots \to 0 \to coker(d^{-1}) \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots,$$

and there is a natural morphism $A^\bullet \to \tau^{\geq 0}(A^\bullet)$. And this morphism induces isomorphisms on cohomology objects in non-negative degree.

Hence, even though we can find a complex $A^\bullet$ with only one nontrivial cohomology object $H^0(A)$, such that there is no isomorphism between them in homotopy category. There always be one isomorphism between them in derived category. Indeed, we have the following roof,

$$\begin{tikzcd} & \tau^{\leq 0}(A^\bullet) \arrow[dl, "\in S"'] \arrow[dr] & \\ A^\bullet & & H^0(A^\bullet) \end{tikzcd}$$

Let us think about how does the localizing procedure interacts with the inclusion of a full subcategory.

Suppose we have a category $\mathcal{C}$ and a localizing class S. Consider a full subcategory $\mathcal{B} \subset \mathcal{C}$, and $S_\mathcal{B} = S \cap Mor_\mathcal{B}$. The following lemma provide a criterion when the natural functor $\mathcal{B}[S_\mathcal{B}^{-1}] \to \mathcal{C}[S^{-1}]$ is fully faithful.

Lemma 5.1. *Suppose the following conditions hold:*
(i) $S_\mathcal{B}$ is a localizing class in $\mathcal{B}$;
(ii) Either one of the following conditions is true:
(ii.a) $\forall X \xrightarrow{\in S} B \in \mathcal{B}$, there exists $B' \xrightarrow{\in S} X$ such that $B' \in \mathcal{B}$.
(ii.b) $\forall B \xrightarrow{\in S} X$ with $B \in \mathcal{B}$, there exists $X \xrightarrow{\in S} B' \in \mathcal{B}$.
Then the natural functor $\mathcal{B}[S_\mathcal{B}^{-1}] \to \mathcal{C}[S^{-1}]$ is fully faithful.

Proof. Firstly, the natural functor is surjective on morphisms. Because for any roof

$$\begin{tikzcd} & X \arrow[dl, "\in S"'] \arrow[dr] & \\ B_1 & & B_2 \end{tikzcd}$$

By condition (ii.a), we complete it by a morphism $B' \to X$. Hence it is equivalent to a morphism in $\mathcal{B}[S_{\mathcal{B}}^{-1}]$. For condition (ii.b), since we can switch a roof to a ditch, it is proved similarly.

Secondly, we need to prove that the natural functor is injective on morphisms. If any two morphism becomes equivalent in $\mathcal{C}[S^{-1}]$, we have the following diagram.

$$\begin{tikzcd}
& Y \arrow[ld, "\in S"] \arrow[rd, "\in S"] & \\
X \arrow[d, "\in S"'] \arrow[drr] & & X' \arrow[dll, "\in S"'] \arrow[d] \\
B_1 & & B_2
\end{tikzcd}$$

This diagram can be completed by $B \xrightarrow{\in S} Y$. Hence it is also equivalent in $\mathcal{B}[S_{\mathcal{B}}^{-1}]$. For condition (ii.b), we can prove it by switch to ditches. $\square$

Example 5.2. *There are several full subcategories in $H(\mathcal{A})$. For example, $H^+(\mathcal{A})$, $H^-(\mathcal{A})$, and $H^b(\mathcal{A})$. Let S be the class of quasi-isomorphisms.*

Since in the proof of 4.12, we mainly use the construction. You can check that quasi-isomorphisms and these subcategories satisfy our conditions (i). However, $H^+(\mathcal{A})$, $H^-(\mathcal{A})$ satisfy condition (ii) by using truncation functors. $H^b(\mathcal{A})$ may not satisfy condition (ii), but its inclusion to $H^+(\mathcal{A})$, or $H^-(\mathcal{A})$ satisfies condition (ii).

Hence, we have full subcategories $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, and $D^b(\mathcal{A})$ in $D(\mathcal{A})$.

Proposition 5.3. *Suppose $\mathcal{A}$ is an abelian category with enough injectives. For any $A^\bullet \in H^+(\mathcal{A})$ there exists a complex $I^\bullet \in H^+(\mathcal{A})$ with $I^i \in \mathcal{A}$ injective objects and a quasi-isomorphism $A^\bullet \to I^\bullet$.*

Proof. As $A^\bullet$ is a bounded below complex, we can proceed by induction as follows. Suppose we have constructed a morphism

$$f_i : A^\bullet \to (\cdots \to I^{i+1} \to I^i \to 0 \to \cdots)$$

such that $H^j(f_i)$ is bijective for $j < i$ and injective for $j = i$. Then one constructs a complex morphism

$$f_{i+1} : A^\bullet \to (\cdots \to I^{i+1} \to I^i \to I^{i+1} \to 0 \to \cdots)$$

which induces bijective maps $H^j(f_{i+1})$ for $j \leq i$ and an injective map $H^{i+1}(f_{i+1})$.

The construction of $I^0 \to I^1$ is easy. Suppose $A^\bullet$ is of the form

$$0 \to A^0 \to A^1 \to A^2 \to \cdots .$$

By assumption, there exists an injective objects I^0 and a monomorphism $A^0 \to I^0$. The induced morphism

$$f_0 : A^\bullet \to (I^0 \to 0 \to \cdots)$$

is our starting step.

Consider the object $(I^0 + A^1)/A^0$ and choose an injective object containing it. The morphism $I^0 \to I^1$ and $A^1 \to I^1$ are the obvious ones. The cohomological properties are easily verified.

For a general step, we have the following diagram

$$\begin{tikzcd}
 & C^i \arrow[rr, "d_C^i"] \arrow[dl, "f_i"'] \arrow[d, "pf_i"] & & C^{i+1} \arrow[d, "a"] & \\
I^i \arrow[r, "p"] & coker(d_I^{i-1}) \arrow[rr, "b"] & & coker(d_I^{i-1}) \amalg_{C^i} C^{i+1} \arrow[r, "c"] & I^{i+1}
\end{tikzcd}$$

Here we construct the cocartesian square with the node $coker(d_I^{i-1}) \amalg_{C^i} C^{i+1}$ and an injective hull of this node. Then we define $d_I^{i+1} = c \circ b \circ p$, $f^{i+1} = c \circ a$.

It is a complex, and we can show that this complex satisfies the cohomological property. □

Exercise 5.4. *Prove our last statement in the proof. Hint: use the fact that for a cocartesian diagram*

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{g} & D
\end{array}$$

we have a canonical isomorphism $coker(g) \to coker(f)$. If you do not know this fact, please take this as an exercise too.

Proposition 5.5. *If $A^\bullet \to B^\bullet$ ia a quasi-isomorphism in $H^+(\mathcal{A})$, and $I^\bullet$ is a complex of injective objects. Then*

$$Hom_{H(\mathcal{A})}(B^\bullet, I^\bullet) \to Hom_{H(\mathcal{A})}(A^\bullet, I^\bullet)$$

is an isomorphism.

Proof. Completing $B^\bullet \to A^\bullet$ to a distinguished triangle in $H^+(\mathcal{A})$. Then it suffices to prove that $Hom_{H(\mathcal{A})}(C^\bullet, I^\bullet) = 0$ for any acyclic complex $C^\bullet$.

This is done by induction, suppose we already have the h^j for $j \leq i$, then consider $g^i - d_I^{i-1} : C^i \to I^i$, which factor through $C^i/Im(d_C^{i-1})$, by the injectivity of I^i, this lifts to a morphism $h^{i+1} : C^{i+1} \to I^i$. □

Corollary 5.6. *If $A^\bullet$ is an arbitrary bounded below complex, and $I^\bullet$ is a bounded below complex of injective objects. Then*

$$Hom_{H(\mathcal{A})}(A^\bullet, I^\bullet) \simeq Hom_{D(\mathcal{A})}(A^\bullet, I^\bullet).$$

Proof. This is a natural map

$$Hom_{H(\mathcal{A})}(A^\bullet, I^\bullet) \to Hom_{D(\mathcal{A})}(A^\bullet, I^\bullet).$$

And we have to show that for any morphism

$$\begin{tikzcd} & B^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & I^\bullet \end{tikzcd}$$

in $D(\mathcal{A})$, there is a unique morphism $A^\bullet \to I^\bullet$ making the diagram commutative up to homotopy.

Moreover, we have to prove that for any two equivalent morphisms in $D(\mathcal{A})$, we get the same morphism $A^\bullet \to I^\bullet$. This can be easily seen by drawing a diagram. □

Remark 5.7. We have a dual version of projective resolutions. The precise statements are left to readers.

Theorem 5.8. *Let $\mathcal{A}$ be an abelian category with enough injective objects or projective objects, then*

$$Ext^i(A,B) \simeq Hom_{D(\mathcal{A})}(A, B[i])$$

for any $A, B \in \mathcal{A}$.

Proof. Recall the definition of Ext^i. We find a projective resolution of A, or a injective resolution of B. For instance, we assume that we have a injective resolution $B \to I^\bullet$, which is a quasi-isomorphism. Then we get a complex

$$Hom(A, I^0) \to Hom(A, I^1) \to Hom(A, I^2) \to \cdots .$$

$Ext^i(A,B)$ is the i-th homology group of this complex.

By the previous proposition and corollary, we have

$$Hom_{D(\mathcal{A})}(A, B[i]) \simeq Hom_{D(\mathcal{A})}(A, I^\bullet[i]) \simeq Hom_{H(\mathcal{A})}(A, I^\bullet[i]).$$

And it is easy to check that $Hom_{H(\mathcal{A})}(A, I^\bullet[i]) \simeq Ext^i(A,B)$. □

Remark 5.9. If we are interested in derived category of sheaves on algebraic varieties, then we do not have enough projective objects (unless the variety is affine). We always have enough injective objects.

Lemma 5.10. *Suppose we have a quasi-isomorphism $q : I^\bullet \to A^\bullet$, where $I^\bullet$ is a bounded below complex of injectives. Then there exists $f : I^\bullet \to A^\bullet$ such that $qf \sim id$.*

Proof. Consider $Cone(q)$, which is acyclic. Therefore the canonical map $Cone(q) \to I^\bullet[1]$ ia homotopic tp 0, this follows from 5.6. The homotopy map h provides a morphism $A^\bullet \to I^\bullet$ by restricting to $A^\bullet$ and a homotopy morphism between $qf \sim id$ by restricting to $I^\bullet$. □

Remark 5.11. Note that in the proof, we really need the target to be bounded below complex of injectives. We find find example of complex which is acyclic but not homotopic to 0. This implies that the quasi isomorphisms in the subcategory $H^+(I)$ (the bounded above complexes of injective objects) satisfies the condition of Lemma 5.1. And the quasi-isomorphisms in $H^+(I)$ is actually an isomorphism. Therefore, by lemma 5.1, we have a fully faithful functor $H^+(I) \simeq H^+(I)[qis^{-1}] \hookrightarrow D^+(\mathcal{A})$.

Proposition 5.3 implies that this functor is an equivalence if $\mathcal{A}$ has enough objects.

5.1. **Derived functors.** Suppose we have an additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories. We can define $F : H(\mathcal{A}) \to H(\mathcal{B})$ by applying it termwisely. If If F is exact, then F send quasi-isomorphisms to quasi-isomorphisms. So it induces a functor on derived categories.

However, in most cases, our functor F is only left or right exact. We can still define the derived functors by following method.

Now we assume F is left exact, $\mathcal{A}$ has enough injectives. Then we can define the right derived functor RF as the composition in the following diagram.

$$\begin{tikzcd} D^+(\mathcal{A}) \arrow[r, "RF"] \arrow[d, "\sim"] & D^+(\mathcal{B}) \\ H^+(\mathcal{I}_{\mathcal{A}}) \arrow[r, "F"] & H^+(\mathcal{B}) \arrow[u] \end{tikzcd}$$

More explicitly, suppose we have a bounded above complex $A^\bullet$, choose an injective resolution (which can be done by Proposition 5.3) $q : A^\bullet \to I^\bullet$, where q is a quasi-isomorphism. $RF(A^\bullet) \coloneqq F(I^\bullet)$.

Comparing the classical derived functors $R^iF : \mathcal{A} \to \mathcal{B}$. $R^iF \coloneqq H^i(RF(A))$.

And RF is an exact functors between triangulated category, since it the composition of three exact functors.

Remark 5.12. Note that $R^iF(A) = 0$ for $i < 0$ and $R^0F(A) \simeq F(A)$ for any $A \in \mathcal{A}$. Indeed, if

$$A \to I^0 \to I^1 \to \cdots$$

is an injective resolution, then

$$R^0F(A) = Ker(F(I^0) \to F(I^1)) = F(A)$$

as F is left exact.

An object $A \in \mathcal{A}$ is called F-acyclic if $R^iF(A) \simeq 0$ for $i \neq 0$.

Obviously, any injective objects are F-acyclic for any left exact functor F.

Proposition 5.13. *Under the above assumption any short exact sequence*

$$0 \to A \to B \to C \to 0$$

in the abelian category $\mathcal{A}$ gives rise to a long exact

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to \cdots$$

$$\cdots \to R^iF(B) \to R^iF(C) \to R^{i+1}F(A) \to \cdots$$

Proof. The short exact sequence is a distinguished triangle in $D(\mathcal{A})$, since $Cone(f) \to C$ is a quasi-isomorphism, it becomes invertible in $D(\mathcal{A})$. Hence we get a distinguished triangle

$$RF(A) \to RF(B) \to RF(C) \to RF(A)[1]$$

in $D(\mathcal{B})$.

Hence we get the long exact sequence. $\square$

We will review some derived functors in algebraic geometry. Suppose X is a Noetherian scheme, we will be in interested in $Coh(X)$ and $Qcoh(X)$.

Example 5.14.
- *The global section functor*
$$\Gamma : Qcoh(X) \to Ab$$
is a left exact functor. If X is a projective variety over a field k, then it becomes the left exact functor
$$\Gamma(X, -) : Coh(X) \to Vec_f(k),$$
where $Vec_f(k)$ is the category of finite dimensional vector space over k.
- *$Hom : Qcoh(X)^{op} \times Qcoh(X) \to Ab$. This is left exact on both positions.*
  For a projective variety over a field k, one has
$$Hom : Coh(X)^{op} \times Coh(X) \to Vec_f(k).$$
  Note that $\Gamma = Hom(\mathcal{O}_X, -)$.
- *Given a morphism $f : X \to Y$, we have $f_* Qcoh(X) \to Qcoh(Y)$, which is a left exact functor. Moreover, if f is proper, then it defines a left exact functor*
$$f_* : Coh(X) \to Coh(Y).$$
This in particular applies to any morphism of projective varieties.
  Note in general, $\Gamma(Y, -) \circ f_* = \Gamma(X, -)$ and if $Y = Spec(\mathbb{Z})$ (or $Spec(k)$), then $f_* = \Gamma$.
- *The sheaf Hom functor*
$$\mathcal{H}om : Qcoh(X)^{op} \times Qcoh(X) \to Qcoh(X)$$
, note that $\Gamma \circ \mathcal{H}om(\mathcal{F}, -) = Hom(\mathcal{F}, -)$.
- *Suppose X is endowed with a sheaf of commutative rings. Consider the abelian category of sheaves of R-modules $Sh_R(X)$. If $\mathcal{F} \in Sh_R(X)$, then*
$$\mathcal{F} \otimes_R (\ ) : Sh_R(X) \to Sh_R(X)$$
is a right exact functor.
- *Let $f : X \to Y$ be a morphism. Then the inverse image defines an exact functor*
$$f^{-1} : Sh(Y) \to Sh(X) \text{ and } f^{-1} : Sh_R(Y) \to Sh_{f^{-1}R}(X),$$
where R is any sheaf of rings on Y.
  Also recall that $f^* \dashv f_*$.

We know that $Qcoh(X)$ has enough injectives, hence we have derived functors $R\Gamma, RHom(), Rf_*$ and $R\mathcal{H}om$.

6. More on derived functors

In this section, we want to compose two derived functors, suppose we have

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

we know that

$$g_* f_* = (gf)_*,$$

we want to have

$$Rg_* Rf_* = R(gf)_*$$

But there is an issue: f_* does not take injective objects to injective objects, i.e. the following diagram may not be commutative.

$$\begin{tikzcd}
D^+(Qcoh(X)) \arrow[r, "Rf_*"] & D^+(QcohY) \arrow[r, "Rg_*"] & D^+(QcohZ) \\
H^+(InjX) \arrow[u, "\sim"] \arrow[ur, "f_*"] \arrow[urr, "(g\circ f)_*"] & H^+(InjY) \arrow[u, "\sim"] \arrow[ur, "g_*"] &
\end{tikzcd}$$

Definition 6.1. A sheaf F is flasque, or flabby. If $\forall V \subset U$, $F(U) \to F(V)$ is surjective.

Proposition 6.2. *(i) Injective sheaves are flasque.*
(ii) If F is flasque, then $H^{>0}(F) = 0$, i.e. F is Γ-acyclic.
(iii) The push-forwanrd functor f_* takes flasque sheaves to flasques sheaves.

Proof. See [Har13, III.2.4] for (i). For (ii), see [Har13, III.2.5], or take it as an exercise by using a Cěch proof.
(iii) is an easy exercise by looking at the definition. □

So, the idea is that instead by looking at an injective resolution, we will take the flasque resolution.

Definition 6.3. A class of objects I_F in $\mathcal{A}$ is called adapted to F if the following conditions are satisfied.
(i) It is closed under direct sum.
(ii) Every object in I_F is F-acyclic.
(iii) Every object in $\mathcal{A}$ can be embedded into an object in that class.

Remark 6.4. In some more general cases, if we do not have enough injectives, then we can not define R^iF at the first place. Hence we should change (ii) to: the functor F send acyclic complex in $H^+(I_F)$ to acyclic complex in $H^+(\mathcal{B})$.

The class of flasque sheaves are adapted to f_*. Given an adapted class I_F for $F : \mathcal{A} \to \mathcal{B}$, we can define an exact functor RF between triangulated categories $D^+(\mathcal{A}) \to D^+(\mathcal{B})$. This is given by the following diagram.

$$\begin{array}{ccc} D^+(I_F) & \xrightarrow{RF} & D^+(\mathcal{B}) \\ \uparrow & & \uparrow \\ H^+(I_F) & \xrightarrow{H(F)} & H^+(\mathcal{B}) \end{array}$$

The existence and exactness of RF in diagram follows from (ii) in the definition. And $D^+(I_F)$ is equivalent to $D^+(\mathcal{A})$ by lemma 5.1 and (iii) in the definition. Hence we get our derived functor.

Proposition 6.5. *Given left exact functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ between abelian categories. Suppose there exists an F-adapted class $R_{\mathcal{A}} \subset \mathcal{A}$ and a G- adapted class $R_{\mathcal{B}} \subset \mathcal{B}$, such that $F(R_{\mathcal{A}}) \subset R_{\mathcal{B}}$.*

Then

$$R(G \circ F) \simeq RG \circ RF$$

as functors from $D^+(\mathcal{A})$ to $D^+(\mathcal{C})$.

This proposition is intuitively easy to prove, but in fact it is not. The formalism makes the proof is cumbersome.

Notice that if you have an adapted class $R_{\mathcal{A}}$, you could throw in all F-acyclic objects to get a bigger one. But this is in tension with $F(R_{\mathcal{A}}) \subset R_{\mathcal{B}}$.

Next, we want to talk about derived functors of $\mathcal{H}om(E, -)$. Since $Qcoh(X)$ have enough injective objects, we get $R\mathcal{H}om$. Also, locally free sheaves are acyclic for the functor $\mathcal{H}om(-, F)$. So we can compute $R\mathcal{H}om(E^\bullet, F)$ by using a locally free resolution of $E^\bullet$.

Proposition 6.6. *If X is projective over a field k, then $Coh(X)$ has enough locally free objects.*

Proof. Given F, $\exists n \gg 0$ such that $F(n)$ is generated by global sections. Hence we have a surjection

$$\mathcal{O}_X^N(-n) \twoheadrightarrow F.$$

□

Therefore, every bounded above coherent sheaf is quasi-isomorphic to a bounded above complex of vector bundles.

Moreover, if X is regular (for example, smooth over an algebraically closed field k), then we can resolve a bounded complex by a bounded complex of locally free sheaves.

The reason is Hilbert syzygy theorem, let F be a sheaf, we can always find E_i vector bundles to resolve F.

$$0 \to E_n \to E_{n-1} \to \cdots \to E_0 \to F.$$

Exercise 6.7. *There is a counterexample when X is singular. On the cone $X = \{xy - z^2 = 0\} \subset \mathbb{A}^3$, try to resolve $\mathcal{O}_{x=z=0}$. Hint: the kernal of the map (x, z) is the image of*

$$M = \begin{pmatrix} z & y \\ -x & -z \end{pmatrix}$$

and the kernal of M is the image of M.

Remark 6.8. The categories of coherent sheaves rarely have enough projective objects (unless) X is affine.

For example, if $X = \mathbb{P}^1$. Then for $n \gg 0$, we know that $\Gamma(F(n)) \neq 0$.

This means

$$\begin{aligned} 0 &\neq Hom(\mathcal{O}_{\mathbb{P}^1}, F(n)) \\ &= Hom(\mathcal{O}_{\mathbb{P}^1}(-n), F) \\ &= Ext^1_{\mathbb{P}^1}(F, \mathcal{O}_{\mathbb{P}^1}(-n-2))^* \end{aligned}$$

Hence F is not projective. You can easily generalize this argument to any projective varieties.

Exercise 6.9. *$\mathcal{H}om(E, -)$ takes injective sheaves to flasque sheaves.*

By this exercise, we have

$$R\Gamma R\mathcal{H}om(E, -) = RHom(E, -).$$

What about $R\mathcal{H}om(E^\bullet, F^\bullet)$?

The lemma we need is the following. The proof uses the spectral sequence in the next subsection.

Lemma 6.10. *If $E^\bullet$ is acyclic, and $F^\bullet$ is a complex of injective objects, then $\mathcal{H}om(E^\bullet, F^\bullet)$ is acyclic. Or if $F^\bullet$ is acyclic, and $E^\bullet$ is a complex of locally free sheaves, then $\mathcal{H}om(E^\bullet, F^\bullet)$ is acyclic.*

6.1. **Spectral sequences associated with bi-complexes.** In this subsection, we will study Spectral sequences associated with double complexes (some people also call it bi-complex).

Definition 6.11. A double complex $K^{\bullet,\bullet}$ consists of objects $K^{i,j}$ for $i, j \in \mathbb{Z}$ and morphisms

$$d_I^{i,j} : K^{i,j} \to K^{i+1,j} \text{ and } d_{II}^{i,j} : K^{i,j} \to K^{i,j+1}.$$

satisfying

$$d_I^2 = d_{II}^2 = d_I d_{II} + d_{II} d_I = 0.$$

The total complex $K^\bullet := tot(K^{\bullet,\bullet})$ of a double complex $K^{\bullet,\bullet}$ is the complex $K^n = \oplus_{i+j=n} K^{i,j}$ with $d = d_I + d_{II}$.

(1) Pass it to cohomology in every column, and denote the objects at (p, q) by $E_1^{p,q}$. We have the horizontal maps.

$$H^{q+1}(K^{p,\bullet}) \xrightarrow{d_I} H^{q+1}(K^{p+1,\bullet}) \longrightarrow \cdots$$

$$H^{q}(K^{p,\bullet}) \xrightarrow{d_I} H^{q}(K^{p+1,\bullet}) \longrightarrow \cdots$$

And it is easy to see that each row is a complex. This is in fact the first page of a spectral sequence.

(2) Pass to cohomology again,denote the objects at (p,q) by $E_2^{p,q}$. At this time, the differentials are getting interesting. Let us think about it by using elements, whose correctness is ensured by embedding theorem.

If we have a cocycle $x \in K^{p,q}$ which represents an element in $H^q(K^{p,\bullet})$. Moreover, we assume it is zero under the map $d_I : H^q(K^{p,\bullet}) \to H^q(K^{p+1,\bullet})$. Hence $y := d_I(x) \in K^{p+1,q}$ is a coboundary of $K^{p+1,\bullet}$, i.e. there exists $z \in K^{p+1,q-1}$ such that $d_{II}(z) = y$, we apply d_I on z, we get an element t. If we trace the coboundary, we get a well defined map $E_2^{p,q} \to E^{p+2,q-1}$. This can be summarized by the following diagram

$$\begin{tikzcd}
0 & 0 & \\
x \arrow[u] \arrow[r] & y \arrow[u] \arrow[r] & 0 \\
& z \arrow[u] \arrow[r] & t \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

This is the second page of a spectral sequence.

The same argument provides us the differential of i-page of the spectral sequence from the previous pages. Very roughly, you can think $d_{r+1} = d_r \circ d_{r-1}^{-1} \circ d_r$. Continue doing this, we get $E_r^{p,q}$ for $r \geq 0$, and differentials $E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1}$ and $d_r^2 = 0$. We pass it to cohomology, we get the next page $E_{r+1}^{p,q}$ and its differentials.

Assume that for any (p,q) there exists an r_0 such that $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$ for $r \geq r_0$. In particular, $E_r^{p,q} \simeq E_{r_0}^{p,q}$ for all $r \geq r_0$. This object is called $E_\infty^{p,q}$.

Under some boundedness assumption on $K^{\bullet,\bullet}$, $E_\infty^{p,q}$ gives information about H^* of total complex $Tot(K)$.

$Tot(K)$ has a filtration

$$F^{p_0}(Tot(K))^n = \oplus_{p \geq p_0, p+q=n} K^{p,q}$$

by sub-complexes. This provides a filtration on $H^n(Tot(K))$. In fact the graded quotient $Gr^p H^n(Tot(K)) \simeq E_\infty^{p,n-p}$. By the boundedness assumption, we also have

$$\cap_p F^p(Tot(K)^n) = 0 \; and \; \cup_p F^p(Tot(K)^n) = Tot(K)^n.$$

This is written as

$$E_2^{p,q} \Longrightarrow H^{p+q}(Tot(K)),$$

where $\Longrightarrow$ means converge to.

One possible boundedness assumption is th following: $K^{p,q} = 0$ for $p \gg 0$. This is automatic if the double complex lies in left half plane, upper half plane, first or third quadrant etc.

One can also start by taking cohomology horizontally. These two procedures are symmetric under flip.

Remark 6.12. In fact, there is a natural generalization, a complex with filtrations. Assuming some boundedness condition, every filtered complex give rise to a spectral sequence. In most of the applications one does not go beyond E_2 or E_3. In the easiest situation the argument will go like this: Suppose we know that all differentials at the E_2-level are trivial, i.e. the spectral sequence degenerates at E_2 page. Then $Tot(K)^n$ admits a filtration whose subquotients are isomorphic to $E_2^{p,n-p}$.

Example 6.13. *Let R be a commutative ring, $A^\bullet$, $B^\bullet$ complexes of R-modules. We denote*

$$(A^\bullet \otimes B^\bullet) := \oplus_{p+q=n} A^p \otimes B^q,\ d(a_p \otimes b_q) = da_p \otimes b_q + (-1)^p a_p \otimes db_q$$

Proposition 6.14. *Assume $A^\bullet$ is a bounded above complex of flat R-modules. $B^\bullet$ is acyclic. Then $(A \otimes B)^\bullet$ is acyclic.*

This implies that if $B_1^\bullet \to B_2^\bullet$ is a quasi-isomorphism, then so is $A^\bullet \otimes B_1^\bullet \to A^\bullet \otimes B_2^\bullet$ (Only need to check that the construction commutes with tensor).

Proof. By assumption, we have $E_1^{p,q} = H^q(A^p \otimes B^\bullet) = 0$. Since $A^\bullet$ is bounded above, the boundedness assumption is satisfied. Hence the $H^n(A^\bullet \otimes B^\bullet) = 0$. □

Given $F : \mathcal{A} \to \mathcal{B}$ left exact functor, and $\mathcal{A}$ has enough injective objects. Hence we have $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$.

Question: Given $A^\bullet$ a bounded below complex of objects in $\mathcal{A}$. How to compute $H^n RF(A^\bullet) \in \mathcal{B}$?

Input: can compute $R^iF(A^p)$ or $R^iF(H^p(A^\bullet))$.

Answer: We have two spectral sequences.

We can construct double complex of injective resolutions of $A^\bullet$. This is Cartan-Eilenberg double complex. A Cartan Eilenberg resolution is a double complex $I^{\bullet,\bullet}$ together with the morphism $A^\bullet \to I^{\bullet,\bullet}$ such that:

- $I^{i,j} = 0$ for $j < 0$.
- The sequences
$$A^i \to I^{i,0} \to I^{i,1} \to \cdots$$
are injective resolutions of A^i inducing injective resolutions of $Ker(d_A^i)$, $Im(d_A^i)$ and $H^i(A^\bullet)$.
- The sequence $I^{\bullet,j}$ splits for all j, i.e. all short exact sequences
$$0 \to Ker d_I^{i,j} \to I^{i,j} \to Im(d_I^{i,j}) \to 0$$
split.

A Cartan-Eilenberg resolution exists whenever the abelian category contains enough injectives (see [GM13, III.7]). The Cartan-Eilenberg double complex lives in the first quadrant (up some translation). Therefore, the boundedness condition is satisfied. We can also check that $Tot(I^{\bullet,\bullet}) \leftarrow A^\bullet$ is a quasi-isomorphism.

This implies $RF(A^\bullet) \simeq F(Tot(I^{\bullet,\bullet})) = Tot(F(I^{\bullet,\bullet}))$. This means that we can approach $H^n(RF(A^\bullet))$ by two spectral sequences, one starts from horizontal map, the other one starts from vertical map.

If we start with the vertical map, we get

$$E_1^{p,q} = R^qF(A^p) \Longrightarrow H^nRF(A^\bullet).$$

If we start with the horizontal map, on the first page, every column becomes a resolution of $H^p(A^\bullet)$. Therefore, we get

$$E_2^{p,q} = R^qF(H^p(A^\bullet)) \Longrightarrow H^nRF(A^\bullet).$$

By the similar argument, we have the following generalization.

Proposition 6.15. *Let $F_1 : H^+(\mathcal{A}) \to H^+(\mathcal{B})$ and $F_2 : H^+(\mathcal{B}) \to H^+(\mathcal{C})$ be two exact functors. Suppose that $\mathcal{A}$ and $\mathcal{B}$ contains enough injectives and that the image under F_1 of injective I lie in an F_2 adapted class.*

Then for any complex

$$E_2^{p,q} = R^pF_2(R^qF_1(A^\bullet)) \Rightarrow E^n = R^n(F_2 \circ F_1)(A^\bullet).$$

Remark 6.16. Our second spectral sequence is the special case when F_1 is the identity map.

Since in the second spectral sequence, we start with the horizontal map. The differentials in each page is different from the usual definition. In many books, we just switch p, q so that it becomes coherent with the usual definition.

Example 6.17. *Leray spectral sequence: suppose we have $X \xrightarrow{g} Y \xrightarrow{f} Z$,*

$$R^n(f \circ g)_*F = H^nR(f \circ g)_*F = H^nRf_*(Rg_*F).$$

So by our second spectral sequence, we get

$$E_2^{p,q} = R^pf_*(R^qg_*F) \Longrightarrow R^n(f \circ g)_*(F).$$

Historical anecdote: Leray firstly stated it in a much less abstract way, constant sheaves on topological spaces. Before WWII, he was a great expert on PDEs. But he became afraid of working on PDEs, since Nazi may thought it as something applied. Therefore, he switched his area to topology (a safe subject) during WWII. He turned back to PDEs around 1950s.

Local-global extensions: suppose F, G are two coherent sheaves on X.

$$R\Gamma(X, R\mathcal{H}om(F, G)) = RHom(F, G).$$

Hence, we have the spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(F, G)) \Longrightarrow Ext^n(F, G).$$

Subexample: Assume X is a smooth variety over k, $p \in X$ is a closed point and $\mathcal{O}_p$ the skyscraper sheaf. We want to compute $Ext^n(\mathcal{O}_p, \mathcal{O}_p)$.

We have the spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{O}_p, \mathcal{O}_p)) \Longrightarrow Ext^n(\mathcal{O}_p, \mathcal{O}_p).$$

Notice that $\mathcal{E}xt^p(\mathcal{O}_p, \mathcal{O}_p)$ is supported on p, hence the higher cohomology vanishes. Therefore, we have $Ext^n(\mathcal{O}_p, \mathcal{O}_p) \simeq H^0(X, \mathcal{E}xt^n(\mathcal{O}_p, \mathcal{O}_p))$.

This result is useful. Indeed, it is much easier to find a local resolution rather than finding a global resolution. For example, in the smooth case, we can use Kozsul complex to resolve a skyscraper sheaf.

The following example is a spoiler of motivic cohomology.

A convieau spectral sequence: Suppose X is a smooth variety over a field k, and F a homotopy invariant Zariski sheaf with transfers. Then there is a canonical exact sequence of Zariski sheaves on X:

$$0 \to F \to \amalg_{codimz=0}(i_z)_*(F) \to \amalg_{codimz=1}(i_z)_*(F_{-1}) \to \cdots \to \amalg_{codimz=r}(i_z)_*(F_{-r}) \to \cdots .$$

Although there are several terms we do not introduce their definitions. At this time, we only need to know that it is a flasque resolution of F. Taking global sections yields a chain complex which computes the cohomology group $H^n(X, F)$. This gives the convieau spectral sequence

$$E_1^{p.q} = \oplus_{codimz=p} H_z^{p+q}(X, F) \Longrightarrow H^{p+q}(X, F)$$

degenerates, with $E_2^{p,0} = H^p(X, F)$ and $E_2^{p,q} = 0$ for $q \neq 0$.

Instead of giving a rigorous mathematical proof of this, I would rather say something about the intuition of this. F_{-1} is called the contraction of F, F_{-r} is the $r-th$ contraction of F. And we have $H^n_{Z\times\{0\}}(Z \times \mathbb{A}^r, F) \simeq H^{n-r}(Z, F_{-r})$.

7. Even more on derived functors

Given a morphism $f : X \to Y$ between algebraic varieties. We know the fact that locally free sheaves form an adapted class for the pull back functor f^*.

In this situation, since we do not have enough projectives, we have to change the condition (ii) in the definition to: the functor F send acyclic complex in $H^+(I_F)$ to acyclic complex in $H^+(\mathcal{B})$. This can be proved by looking at the affine charts.

We define $R\mathcal{H}om(F, -)$ by resolving the second term by injectives; $R\mathcal{H}om(-, G)$ by resolving the first term by vector bundles.

Same story for tensor product, we can resolve either term by locally free sheaves. Then we get the same result. This is the same by resolve both terms with locally free sheaves. Since it is proved in the last section that an acyclic complex tensor with a complex of locally free sheaves is again acyclic. This is true.

Furthermore, since tensor product of locally free sheaves are locally free sheaves. We have the associativity

$$(E^\bullet \otimes F^\bullet) \otimes G^\bullet \simeq E^\bullet \otimes (F^\bullet \otimes G^\bullet).$$

Pullback functor, let $f : X \to Y$ be a morphism between algebraic varieties. Then the inverse image defines an exact functor

$$f^{-1} : Sh(Y) \to Sh(X) \text{ and } f^{-1} : Sh_{\mathcal{O}_Y}(Y) \to Sh_{f^{-1}\mathcal{O}_Y}(X),$$

and we have $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ as part of the data of f.

Hence, we can define $f^* : Sh_{\mathcal{O}_Y}(Y) \to Sh_{\mathcal{O}_X}(X)$,

$$f^* : F \mapsto f^{-1}F \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X,$$

which is a right exact functor.

Since we have adapted class (locally free sheaves) for pull back functor. We get derived functor $Lf^* : D^-(QcohY) \to D^-(QcohX)$. And f^* sends locally free sheaves to locally free sheaves, hence it behaves well under composition.

7.1. **Compatibilities of derived functors.** We want to emphasis the relation between $RHom$ and $Hom_{\mathcal{D}}$.

Suppose $\mathcal{A}$ is an abelian category with enough injectives. $A^\bullet$ and $B^\bullet$ are complexes over $\mathcal{A}$.

We have underived $Hom^\bullet(A^\bullet, B^\bullet)$ complex of abelian groups.

$$Hom^n(A^\bullet, B^\bullet) = \oplus Hom_{\mathcal{A}}(A^i, B^{i+n}),$$

the differential $d : f \mapsto d_B f - (-1)^n f d_A$ from $Hom(A^\bullet, B^\bullet)^n \to Hom(A^\bullet, B^\bullet)^{n+1}$. This is the total complex of the natural double complex.

Easy observation: $H^0 Hom^\bullet(A^\bullet, B^\bullet) = Hom_{H(\mathcal{A})}(A^\bullet, B^\bullet)$.

Indeed, $f \in Hom^0(A^\bullet, B^\bullet)$ is a cocycle is equicalent to $f : A^\bullet \to B^\bullet$ is a chain map. If $f \in Hom^0(A^\bullet, B^\bullet)$ is a coboundary, then it means f is homotopic to 0.

This observation is consistent with the shift functor in the following sense. Consider $B^\bullet \mapsto B^\bullet[n]$, $Hom^\bullet(A^\bullet, B^\bullet) \simeq Hom(A^\bullet, B^\bullet)[n]$. Hence

$$Hom_{H(\mathcal{A})}(A^\bullet, B^\bullet[n]) = H^0 Hom^\bullet(A^\bullet, B^\bullet[n]) = H^n Hom^\bullet(A^\bullet, B^\bullet).$$

We pass it to derived category. Assume $B^\bullet$ bounded below, and $B^\bullet \to I^\bullet$ is a quasi isomorphism. Then $RHom(A^\bullet, B^\bullet) \simeq Hom^\bullet(A^\bullet, I^\bullet) \in D(\mathbf{Ab})$.

Corollary 7.1.

$$H^n RHom(A^\bullet, B^\bullet) \simeq Hom_{D(\mathcal{A})}(A^\bullet, B^\bullet[n]).$$

This immediately follows from the previous discussion and our lemma

$$Hom_{H(\mathcal{A})}(A^\bullet, I^\bullet) \simeq Hom_{D(\mathcal{A})}(A^\bullet, I^\bullet).$$

We more like focus on the left side, which is more computable by using local-to-global method.

7.2. **Serre duality revisited.** A simple version of Classical statement: Assume X is smooth projective variety over an algebraically closed field of dimension n. V a vector bundle on X. Then we have

$$H^i(X, V)^* \simeq H^{n-i}(X, V^\vee \otimes \omega_X),$$

where ω_X is the canonical bundle of X.

Example 7.2. *If $i = 0$, and $V = \mathcal{O}_X$. $k = H^0(X, \mathcal{O}_X)^* \simeq H^n(X, \omega_X)$. Hence, w have a canonical isomorphism $\tau : H^n(X, \omega_X) \to k$*

$$H^i(X, V) \otimes H^{n-i}(X, V^\vee \otimes \omega_X) \to H^n(X, V \otimes V^\vee \otimes \omega) \to H^n(\omega) \xrightarrow{\tau} k$$

Serre duality says this pairing is perfect.

Now we replace V by a bounded complex of vector bundles $V^\bullet$. $V^{\bullet\ \vee} = \mathcal{H}om(V^\bullet, \mathcal{O}_X)$, termwisely, it is defined as $\mathcal{H}om(V^\bullet, \mathcal{O})^n = \mathcal{H}om(V^{-n}, \mathcal{O})$.

We can also define $(V^\bullet)^\vee \otimes V^\bullet \to \mathcal{O}$. This consists of adding all the evaluation maps with correct signs, instead of giving the correct signs in general case, we provide an easy example.

Example 7.3. *Suppose $V^\bullet = [V^0 \to V^1]$, then $(V^\bullet)^\vee = [V^{1\ \vee} \to V^{0\ \vee}]$, where $V^{0\ \vee}$ sits in degree 0.*

$$(V^\bullet)^\vee \otimes V^\bullet : V^{1\ \vee} \otimes V^0 \to V^{0\ \vee} \otimes V^0 \oplus V^{1\ \vee} \otimes V^1 \to V^{0\ \vee} \otimes V^1.$$

There is natural chain map from this complex to the complex $\mathcal{O}_X$, with one the natural evaluation map, the other one the negative sign of evaluation map. And this example can be extended to arbitrary bounded complexes of vector bundles.

Therefore, we get a pairing between $H^i(X, V^\bullet) = H^i R\Gamma(X, V^\bullet)$ and $H^{n-i}(X, V^{\bullet\ \vee} \otimes \omega_X)$. These cohomology are usually called hypercohomology in literature.

The claim is that The pairing

$$H^i(X, V^\bullet) \otimes H^{n-i}(X, V^{\bullet\ \vee} \otimes \omega_X) \to k$$

is still perfect.

We will not prove this claim in a rigorous way. We sketch the idea of the proof. This is proved by induction on the length of $V^\bullet$, we use the following stupid truncation.

$$\begin{array}{ccccccccc}
 & & & & 0 & \longrightarrow & A^1 & \longrightarrow & \cdots \\
 & & & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{-1} & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{-1} & \longrightarrow & A^0 & \longrightarrow & 0 & &
\end{array}$$

This is short exact sequence of complexes, which gives long exact sequence of hypercohomology.

Proposition 7.4. *Serre duality says that we have a functorial isomorphism*

$$Hom(A^\bullet, B^\bullet)^* \simeq Hom(B^\bullet, A^\bullet \otimes \omega_X[n])$$

for any $A^\bullet, B^\bullet \in D^b(CohX)$.

Proof.

$$Hom(A^\bullet, B^\bullet) = H^0 RHom(A^\bullet, B^\bullet),$$

the right hand side is

$$H^0 R\Gamma(X, R\mathcal{H}om(A^\bullet, B^\bullet)) \simeq \mathbb{H}^0(X, A^{\bullet\vee} \otimes B^\bullet).$$

And analogously,

$$Hom(B^\bullet, A^\bullet \otimes \omega_X[n]) = \mathbb{H}^0(X, B^{\bullet\vee} \otimes A^\bullet \otimes \omega[n]) = \mathbb{H}^n(X, B^{\bullet\vee} \otimes A^\bullet \otimes \omega_X).$$

Hence Serre duality gives what we want.

□

Therefore $S = \otimes\omega_X[n] : D^b(CohX) \to D^b(CohX)$ is a Serre functor as we defined in Section 2.

The following compatibilities need some boundedness assumption. They follows easily from the classical compatibility of functors on sheaves and suitable resolution.

- $Lf^*(E^\bullet \otimes^L F^\bullet) = (Lf^*E^\bullet) \otimes^L (Lf^*F^\bullet)$ for any $E^\bullet$ and $F^\bullet$ in $D^-(QcohX)$. This is true because if we replace $E^\bullet$ and $F^\bullet$ by complexes of locally free sheaves. This is true in the nose.
- $R\mathcal{H}om(E^\bullet, R\mathcal{H}om(F^\bullet, G^\bullet)) \simeq R\mathcal{H}om(E^\bullet \otimes^L F^\bullet, G^\bullet)$.
- $R\mathcal{H}om(E^\bullet, F^\bullet) \otimes^L G^\bullet \simeq R\mathcal{H}om(E^\bullet, F^\bullet \otimes^L G^\bullet)$. One particular example is $F^\bullet = \mathcal{O}_X$, we get $E^\vee \otimes G^\bullet \simeq R\mathcal{H}om(E^\bullet, G^\bullet)$, where $E^\vee := R\mathcal{H}om(E^\bullet, \mathcal{O}_X)$. Here I think both $E^\bullet$ and $F^\bullet$ should br bounded complexes.
- Adjunction: let $f : X \to Y$ be a morphism between algebraic varieties. Then we have
$$Rf_* R\mathcal{H}om(Lf^*E^\bullet, F^\bullet) \simeq R\mathcal{H}om(E^\bullet, Rf_*F^\bullet)$$
by applying the derived functor $R\Gamma$, we get
$$RHom(Lf^*E^\bullet, F^\bullet) = RHom(E^\bullet, Rf_*F^\bullet).$$
- Projection formula:
$$Rf_*(E^\bullet \otimes Lf^*F^\bullet) \simeq (Rf_*E^\bullet) \otimes^L F^\bullet.$$
This can be compared with
$$\int e(x,y)f(x)dy = (\int e(x,y)dy) \cdot f(x).$$
- $R\mathcal{H}om(F^\bullet, E^\bullet \otimes G^\bullet) \simeq R\mathcal{H}om(r\mathcal{H}om(E^\bullet, F^\bullet), G^\bullet)$.
- Cohomology and base change: Suppose we have a fiber product diagram:
$$\begin{tikzcd} W \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{g}"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$
If g is flat, and f proper, then we have a functorial isomorphism
$$g^*Rf_* \xrightarrow{\sim} R\tilde{f}_*\tilde{g}^*.$$

Exercise 7.5. *In the last item, even without the flatness assumption one has the natural map*

$$Lg^*Rf_* \to R\tilde{f}_*L\tilde{g}^*.$$

Hint: use the functor morphism $g : id \to G \circ F$ for any adjoint pair $F \dashv G$.

Example 7.6. *Suppose $Z = Speck$, then we have the following diagram*

$$\begin{tikzcd} X \times Y \arrow[r, "p"] \arrow[d, "q"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Speck \end{tikzcd}$$

Then the isomorphism says that $\mathcal{O}_Y \otimes_k R\Gamma(E^\bullet)$ is isomorphic to $Rp_*q^*E^\bullet$.

Even more explicitly, suppose X is an elliptic curve, and $E^\bullet$ is the structure sheaf $\mathcal{O}_X$. Then $R\Gamma(\mathcal{O}_X) = k \xrightarrow{0} k$, and $g^*Rf_*(\mathcal{O}_X) \simeq \mathcal{O}_Y \xrightarrow{0} \mathcal{O}_Y$.

Example 7.7. *Suppose E is an elliptic curve, and $p_0 \in E$ a point. Then every degree 0 line bundle on E is isomorphic to $(O)_E(p - p_0)$ for a unique point $p \in E$. Therefore, the moduli space of line bundles of degree 0 on E is isomorphic to E itself. And we have the universal bundle $L := \mathcal{O}_{E\times E}(\Delta - E \times p_0)$ on $E \times E$.*

Here $L|_{p\times E} \simeq \mathcal{O}_E(p - p_0)$.

$$H^*(\mathcal{O}_E(p - p_0)) = \begin{cases} 0 \xrightarrow{0} 0, & \text{if } p \neq p_0 \\ k \xrightarrow{0} k, & \text{if } p = p_0 \end{cases}$$

We have the following diagram

$$\begin{tikzcd} p \times E \arrow[r, "j"] \arrow[d, "q"] & E \times E \arrow[d, "\pi"] \\ p \arrow[r, "i"] & E \end{tikzcd}$$

Hence we have

$$Li^*R\pi_*L \simeq Rq_*Lj^*L,$$

where the right hand is calculated before.

We want to understand the fact $R\pi_*L \simeq \mathcal{O}_{p_0}[-1]$ and we have $Li^*\mathcal{O}_{p_0} = 0$ if $p \neq p_0$ and $k \xrightarrow{0} k$ if $p = p_0$.

We can resolve $\mathcal{O}_{p_0}$ by the following sequence

$$0 \to \mathcal{O}_E(-p_0) \to \mathcal{O}_E \to \mathcal{O}_{p_0} \to 0.$$

Therefore, we have

$$Li^*\mathcal{O}_{p_0} = k \xrightarrow{x} k,$$

where the map x is zero if $p = p_0$ and nonzero, hence an isomorphism if $p \neq p_0$.

Notice that the previous example uses another version of base change theorem, this version is proved in [BO95].

Lemma 7.8. *Let $f : X \to Y$ be a smooth morphism of relative dimension r of smooth projective varties and $g : Y' \to Y$ a base change, with Y' being a smooth variety. Define X' as the Cartesian product $X' = X \times_Y Y'$.*

$$\begin{array}{ccc} X' & \xrightarrow{\tilde{f}} & Y' \\ \downarrow{\scriptstyle \tilde{g}} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

Then we have a functorial isomorphism

$$Lg^* Rf_* \xrightarrow{\sim} R\tilde{f}_* L\tilde{g}^*.$$

Proof. Notice that, since X and Y are smooth projective varieties. Then $D^b(Coh(X))$ and $D^b(Coh(Y))$ have Serre functors $\otimes\omega_X[dimX]$ and $\otimes\omega_Y[dimY]$ respectively. Hence by 2.12, we have that

$$f_* \dashv f^! = \omega_X[dimX] \otimes f^*(-) \otimes f^*(\omega_Y^{-1})[-dimY],$$

hence $f^! = f^*(-) \otimes \omega_{X/Y}[r]$.

Therefore, it suffices to prove that $f^! Rg_*$ is isomorphic to $R\tilde{g}_* \tilde{f}^!$. Indeed, we have

$$f^! g_*(-) = Lf^* Rg_*(-) \otimes \omega_{X/Y}[r].$$

Analogously,

$$R\tilde{g}_* \tilde{f}^!(-) \simeq R\tilde{g}_*(\tilde{f}^*(-) \otimes \omega_{X'/Y'}[r] \simeq \tilde{g}_*(\tilde{f}^*(-) \otimes \tilde{g}^* \omega_{X/Y}[r])$$

The latter isomorphism comes from the fact that for a smooth f differentials are compatible with base change.

Then by projection formula and flat base change theorem, we get what we want. $\square$

Remark 7.9. If you are familiar with *étale* cohomology, please compare these two base change theorems with the proper base change theorem and smooth base change theorem.

Note that in the proof, we use Exercise 2.12 to define a right adjoint functor of Rf_*. It is denoted by $f^!$. Hence we have

$$Hom(Rf_* A^\bullet, B^\bullet) \simeq Hom(A^\bullet, f^! B^\bullet).$$

Moreover, we have a sheafified version

$$R\mathcal{H}om(Rf_* A^\bullet, B^\bullet) \simeq Rf_* R\mathcal{H}om(A^\bullet, f^! B^\bullet),$$

this is also called Grothendieck-Verdier duality.

As a special case, if we take $A^\bullet = \mathcal{O}_X$, and $B^\bullet = \omega_Y[dimY]$, we have the following

$$Rf_* \omega_X[dimX] \simeq (Rf_* \mathcal{O}_X)^\vee \otimes \omega_Y[dimY].$$

Remark 7.10. In fact, Grothendieck-Verdier duality holds in much broader generality. What has to change in the definition is the Serre functors, where we change the canonical bundle to dualizing complex K_X. It turns out that K_X always exists when X is proper over a field k. Moreover, the variety X is Gorenstein if only if K_X is a line bundle in degree $-n$. Also, X is Cohen-Macaulay of pure dimension n if and only if K_X is a coherent sheaf in degree $-n$. And in the general case, we have to replace $D^b(CohX)$ by $D(Qcoh(X))$, while the latter category admits any small direct sums. This allows us to apply some general representability theorems.

Classical Serre duality and Serre functor in derived categories are special cases of Grothendieck-Verdier duality. Indeed, applied to $f : X \to Seck(k)$,

Exercise 7.11. *If we denote $\mathbb{D}_X$ is the dualizing functor*

$$F^\bullet \mapsto R\mathcal{H}om(F^\bullet, \omega_X[dimX]) = F^{\bullet\,\vee} \otimes \omega_X[dimX],$$

then prove that

$$f^! = \mathbb{D}_X \circ Lf^* \circ \mathbb{D}_Y^{-1}.$$

Moreover, show that Grothendieck-Verdier duality is equivalent to

$$Rf_* \circ \mathbb{D}_X \simeq \mathbb{D}_Y \circ Rf_*.$$

Corollary 7.12. *Let $i : X \hookrightarrow Y$ be a smooth subvariety of codimension c of a smooth projective variety. For any $F^\bullet \in D^b(X)$ and any $E^\bullet \in D^b(Y)$ there exists a functorial isomorphism*

$$Hom_X(F^\bullet, Li^*(E^\bullet) \otimes \omega_i[-c]) \simeq Hom_Y(i_*F^\bullet, E^\bullet),$$

where $\omega_i := \omega_X \otimes \omega_Y^*|X$.

In particular, this result allows one to compute the dual of structure sheaf of a smooth closed subvariety.

Corollary 7.13. *Let $i : X \hookrightarrow Y$ be a smooth closed subvariety of codimension c of a smooth projective variety. The derived dual of $i_*\mathcal{O}_X$ is given by*

$$(i_*\mathcal{O}_X)^\vee \simeq i_*\omega_X \otimes \omega_Y^*[-c].$$

Proof. This follows from

$$\begin{aligned} Hom_Y(G^\bullet, (i_*\mathcal{O}_X)^\vee) &\simeq Hom_Y(G^\bullet \otimes i_*\mathcal{O}_X, \mathcal{O}_Y) \\ &\simeq Hom_Y(i_*Li^*(G^\bullet), \mathcal{O}_Y) \\ &\simeq Hom_X(Li^*(G^\bullet), \omega_X \otimes i^*\omega_Y^*[-c]) \\ &\simeq Hom_Y(G^\bullet, i_*\omega_X \otimes \omega_Y^*[-c]) \end{aligned}$$

Hence, by Yoneda's lemma, we get the result. □

Exercise 7.14. *Let $i : X \hookrightarrow Y$ be a smooth closed subvariety of codimension $c > 1$ of a smooth projective variety. Show that the derived dual $I_X^\vee$ of its ideal sheaf satisfies*

$$\mathcal{H}^k(I_X^\vee) \simeq \begin{cases} \mathcal{O}_Y & \text{if } k=0 \\ i_*\omega_X \otimes \omega_Y^* & \text{if } k=c\text{-}1 \\ 0 & \text{otherwise.} \end{cases}$$

8. Untitled

8.1. **More on dualities.** In last section, we see that

$$R\mathcal{H}om(Rf_*A^\bullet, B^\bullet) \simeq Rf_*R\mathcal{H}om(A^\bullet, f^!B^\bullet),$$

which is called the Grothendieck-Verdier duality, if we take $B^\bullet$ to be $\mathcal{O}_Y$. We get

$$(Rf_*F^\bullet)^\vee \simeq Rf_*R\mathcal{H}om(F, \omega_f[dimX - dimY]).$$

This can be seen as a relative version of Serre duality, but one thing to notice is that this duality does not hold for individual higher direct images R^if_*. The following is the example illustrates this phenomenon.

Example 8.1. *The example is the same as the example in last section, assume E is an elliptic curve. Then we have a line bundle $L = \mathcal{O}_{E\times E}(\Delta - p_0 \times E)$ on $E \times E$. We look at the morphism $p : E \times E \to E$, which is the projection onto the second factor. Then the relative dualizing complex is $\mathcal{O}_{E\times E}[1]$. Hence Grothendieck-Verdier duality implies*

$$(Rf_*L)^\vee \simeq Rf_*R\mathcal{H}om(L, \mathcal{O}_{E\times E})[1] \simeq Rf_*(L^{-1}[1]).$$

Since Rf_*L can be resolved by a length 2 complex of locally free sheaves (this can be seen from a relative version of Cech resolution). Hence f_*L is a torsion free sheaf, on the other hand, it is supported on point p_0. Hence $f_*L = 0$. Similarly, f_*L^{-1} is also 0, and we have that $R^1f_*L = R^1f_*L^{-1} = \mathcal{O}_{p_0}[-1]$. Hence

$$(f_*L)^\vee \neq R^1f_*(L^{-1}).$$

In this case, Grothendieck-Verdier duality boils down to

$$\mathcal{E}xt^1(\mathcal{O}_{p_0}, \mathcal{O}_E) = \mathcal{O}_{p_0}.$$

8.2. **Hodge diamond of a K3 surface.** Let us review Grothendieck spectral sequence, i.e. the two spectral sequences we introduced in Section 6. If we start with the vertical map, we get

$$E_1^{p,q} = R^qF(A^p) \Longrightarrow H^nRF(A^\bullet).$$

If we start with the horizontal map, on the first page, every column becomes a resolution of $H^p(A^\bullet)$. Therefore, we get

$$E_2^{p,q} = R^qF(H^p(A^\bullet)) \Longrightarrow H^nRF(A^\bullet).$$

Cohomology of line bundles on $\mathbb{P}^4$.

$$\begin{pmatrix} & \mathcal{O}(-6) & \mathcal{O}(-5) & \mathcal{O}(-4) & \mathcal{O}(-3) & \mathcal{O}(-2) & \mathcal{O}(-1) & \mathcal{O} & \mathcal{O}(1) & \mathcal{O}(2) \\ h^4 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h^0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & 15 \end{pmatrix}$$

If X is a smooth proper intersection of a quadric and cubic inside $\mathbb{P}^4$ over complex numbers. We want to compute the hodge numbers $H^{p,q}(X) = H^q(\Omega_X^p)$.

Koszul resolution of $\mathcal{O}_X$.

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-5) \to \mathcal{O}_{\mathbb{P}^4}(-3) \oplus \mathcal{O}_{\mathbb{P}^4}(-2) \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_X \to 0$$

This means that $\mathcal{O}_X$ is quasi-isomorphic to

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-5) \to \mathcal{O}_{\mathbb{P}^4}(-3) \oplus \mathcal{O}_{\mathbb{P}^4}(-2) \to \mathcal{O}_{\mathbb{P}^4}.$$

Hence Grothendieck spectral sequence provides the following picture.

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -2 & -1 & 0 & \end{pmatrix}$$

where the bottom row and right column indicates the (p, q) position in the first page of spectral sequence. And it is easy to see that the spectral sequence degenerates starting from this page.

Hence we find that

$$h^i(\mathcal{O}_X) = \begin{cases} 1 & if\ i = 0, 2 \\ 0 & otherwise. \end{cases}$$

Similarly, we can compute $h^i(\mathcal{O}_X)(-1)$, $h^i(\mathcal{O}_X)(-2)$, and $h^i(\mathcal{O}_X)(-3)$.

While the spectral sequence of $h^i(\mathcal{O}_X)(-1)$ degenerates in the first page. For $h^i(\mathcal{O}_X)(-2)$ and $h^i(\mathcal{O}_X)(-3)$, we need to show that the boundary map is surjective. This canbe shown either by dimension reason or Serre duality.

Finally, we get the following results:

$$h^i(\mathcal{O}_X)(-1) = \begin{cases} 5 & if\ i = 2 \\ 0 & otherwise. \end{cases}$$

$$h^i(\mathcal{O}_X)(-2) = \begin{cases} 14 & if\ i = 2 \\ 0 & otherwise. \end{cases}$$

$$h^i(\mathcal{O}_X)(-3) = \begin{cases} 29 & if\ i = 2 \\ 0 & otherwise. \end{cases}$$

We can compute the canonical bundle of X by adjunction formula $\omega_X = \mathcal{O}(-5+2+3) = \mathcal{O}_X$.

Also, we have following normal sequence:

$$0 \to T_X \to T_{\mathbb{P}^4}|_X \to \mathcal{O}_X(2) \oplus \mathcal{O}_X(3) \to 0.$$

Hence, we have the exact sequence of cotangent bundle

$$0 \to \mathcal{O}_X(-3) \oplus \mathcal{O}_X(-2) \to \Omega^1_{\mathbb{P}^4}|_X \to \Omega^1_X \to 0.$$

To compute $\Omega_{\mathbb{P}^4}|_X$, we need Euler sequence

$$0 \to \Omega^1_{\mathbb{P}^4} \to \mathcal{O}_{\mathbb{P}^4}(-1)^5 \to \mathcal{O}_{\mathbb{P}^4} \to 0.$$

Therefore, Ω^1_X is quasi-isomorphic to

$$\mathcal{O}_X(-3) \oplus \mathcal{O}_X(-2) \to \mathcal{O}_X(-1)^5 \to \mathcal{O}_X,$$

where $\mathcal{O}_X(-1)^5$ sits in degree 0.

Then, we use Gothendieck spectral sequence. We get following picture.

$$\begin{pmatrix} 43 & 25 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & \end{pmatrix}$$

where the bottom row and right column indicates the (p, q) position in the first page of spectral sequence. There is some nontrivial boundary map in the first row. By dimension reason, the map from $25 \to 1$ must be surjective. By Serre duality, we know that $h^2(\Omega^1_X) = h^0(\Omega_X)$, the latter is zero by easy observation from the spectral sequence. This implies that $h^1(\Omega_X) = 43 + 1 - 25 + 1 = 20$. Therefore, we computed the Hodge diamond of a K3 surface.

8.3. **Fourier-Mukai transform.** From now on, since we will focus on the derived categories only. We will denote the derived functors as if they were not derived.

Assume X and Y are smooth projective variety over a field k. This assumption allows us to focus on $D(X) := D^b(CohX)$. Given an object $E \in D(X \times Y)$, define the Fourier-Mukai functor with kernal E

$$\Phi_E : D(X) \to D(Y).$$

defined by

$$\Phi_E(E) \to q_*(E \otimes p^* F),$$

where p, q are projections

$$\begin{tikzcd} & X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

Remark 8.2. Compare this with the classical Fourier transform

$$\hat{f}(y) = \int f(x) e^{-iixy} dy.$$

Pullback is like adding one extra variable, tensor with the kernal is like times the function e^{-ixy}, and pushforward is like taking integration.

And this kind of transform is ubiquitous in mathematics, we can find it in Chow groups, cohomology groups etc.

Our favorite functors are of Fourier-Mukai type.

Example 8.3. *(i) The identity*

$$id : D^b(X) \to D^b(X)$$

is naturally isomorphic to the Fourier-Mukai transform $\Phi_{\mathcal{O}_\Delta}$ *with the kernal being the structure sheaf of diagonal* $\Delta \subset X \times X$*. Indeed, with* $\iota : X \xrightarrow{\sim} \Delta \subset X \times X$ *denoting the diagonal embedding, one has*

$$\begin{aligned}\Phi_{\mathcal{O}_\Delta}(E) &= p_*(q^*E \otimes \mathcal{O}_\Delta)\\ &\simeq p_*(q^*E \otimes \iota_*\mathcal{O}_X)\\ &\simeq p_*(\iota_*(\iota^*q^*E \otimes \mathcal{O}_X))\\ &\simeq (p\circ\iota)_*(q\circ\iota)^*E \simeq E\end{aligned}$$

(ii) Let $f : X \to Y$ *be a morphism. Then*

$$f_* \simeq \Phi_{\mathcal{O}_{\Gamma_f}} : D^b(X) \to D^b(Y),$$

where Γ_f *is the graph of* f*.*

As a special instance, one may consider cohomology $H^*(X, -)$ *as the Fourier-Mukai transform* $\Phi_{\mathcal{O}_X} : D^b(X) \to D^b(Vec_f(k))$*, where* $X \subset X \times spec(k)$ *is considered as the graph of the structure morphism.*

Oppositely, we can use the same kernal for a Fourier-Mukai transform in the opposite direction which is nothing but the pullback functor $f^*D^b(Y) \to D^b(X)$*.*

In particular, the first example is a special case of the second example, and the proof are similar.

(iii) Let L *be a line bundle on* X*. Then* $E^\bullet \to E^\bullet \otimes L$ *defines an autoequivalence* $D^b(X) \to D^b(X)$ *which is isomorphic to the Fourier-Mukai transform with kernal* $\iota_*(L)$*, where* ι *is again the diagonal embedding of* X *in example (i).*

(iv) The shift functor $[1]$ *can be described as the Fourier-Mukai transform with kernal* $\mathcal{O}_\Delta[1]$*.*

(v) Consider once more the diagonal embedding $\iota : X \xrightarrow{\sim} \Delta \subset X \times X$*. Then*

$$\Phi_{\iota_*\omega_X^k} \simeq S^k[-nk],$$

where S *is the Serre functor* $F^\bullet \to f^\bullet \otimes \omega_X[n]$ *with* $n = dim(X)$*.*

(vi) This one is important in the theory of moduli spaces. Suppose we have P *a coherent sheaf on* $X \times Y$ *flat over* X*. Consider the Fourier-Mukai transform* $\Phi_P : D^b(X) \to D^b(Y)$*. If* $x \in X$ *is a closed point with* $k(x) \simeq k$*, then*

$$\Phi_P(k(x)) \simeq P_x,$$

where $P_x \coloneqq P|_{x\times Y}$ *is a sheaf on the fiber* $Y \times \{x\}$*.*

In the theory of moduli spaces, if we have a universal family P*. The look at the objects we want to parametrize, we can decide whether the Fourier-Mukai transform with kernal* P *is fully faithful or, even an equivalence. (vii) Suppose we have* P *a coherent sheaf on* $X \times Y$ *flat over* X*. This is commonly viewed as a family of coherent sheaves* P_x *or as a deformation of the sheaf* P_{x_0} *for a distinguished closed point* $x_0 \in X$*. For simplicity, we* *assume $k(x_0) = k$. A tangent vector v at x_0 is determined by a subscheme $Z_v \subset X$ of length two concentrated in $x_0 \in X$. Pulling-back*

$$0 \to k(x) \to \mathcal{O}_{Z_v} \to k(x) \to 0$$

and applying Φ_P yields

$$0 \to P_{x_0} \to P|_{Z_v \times Y} \to P_{x_0} \to 0$$

viewed as a sequence on Y this gives a class in $Ext^1_Y(P_{x_0}, P_{x_0})$. In this way, we obtain a linear map, the so called Kodaira-Spencer map,

$$\kappa(x_0) : T_{x_0}X \to Ext^1_Y(P_{x_0}, P_{x_0}).$$

In sum, we have following commutative diagram

$$\begin{tikzcd}
T_{x_0}X \arrow[r, "\kappa(x_0)"] \arrow[d, "\simeq"] & Ext^1_Y(P_{x_0}, P_{x_0}) \arrow[d, "\simeq"] \\
Hom(k(x_0), k(x_0)[1]) \arrow[r, "\Phi_P"] & Hom(P_{x_0}, P_{x_0}[1])
\end{tikzcd}$$

Remark 8.4. We have proved that any Serre functor is compatible with any equivalence. This is no longer true for arbitrary Fourier-Mukai transform.

For example, if $f : X \to Spec(k)$, then the Fourier-Mukai transform f_* maps a sheaf F to its cohomology $H^*(X, F)$ and in general

$$\begin{aligned}
S_{pt}H^0(X, F) &= H^0(X, F) \\
&\neq H^n(X, F \otimes \omega_X) \\
&\simeq H^0(X, F \otimes \omega_X[dimX]) \\
&= H^0(X, S_X(F))
\end{aligned}$$

A Fourier-Mukai transform functor has both left and right adjoint functors, and the adjoints are also of Fourier-Mukai type.

Definition 8.5. For any object $P \in D^b(X \times Y)$, we let

$$P_L := P^\vee \otimes p^*\omega_Y[dimY], \text{ and } P_R := P^\vee \otimes q^*\omega_X[dimX],$$

both objects in $D^b(X \times Y)$.

Remark 8.6. The induced Fourier-Mukai transforms $\Phi_{P_R} : D^b(Y) \to D^b(X)$ and $\Phi_{P_L} : D^b(Y) \to D^b(X)$ can equivalently be described as

$$\Phi_{P_L} \simeq \Phi_{P^\vee} \circ S_Y$$

and

$$\Phi_{P_R} \simeq S_X \circ \Phi_{P^\vee}$$

respectively.

Proposition 8.7. *Let $F = \Phi_P : D^b(X) \to D^b(Y)$ be the Fourier-Mukai transform with kernal P. Then*

$$H \coloneqq \Phi_{P_R} : D^b(Y) \to D^b(X) \text{ and } G \coloneqq \Phi_{P_L} : D^b(Y) \to D^b(X)$$

are right, respectively left adjoint to F.

Proof. This assertion is a direct consequence of Grothendieck-Verdier duality. Indeed, for any $E^\bullet \in D^b(X)$ and $\mathcal{F}^\bullet \in D^b(Y)$ one has a sequence of functorial isomorphisms:

$$\begin{aligned}
&Hom(G(\mathcal{F}^\bullet), E^\bullet) \\
&= Hom(q_*(P_L \otimes p^*\mathcal{F}^\bullet), E^\bullet) \\
&\simeq Hom(P_L \otimes p^*\mathcal{F}^\bullet, q^*E^\bullet \otimes p^*\omega_Y[dimY]) \\
&\simeq Hom(P^\vee \otimes p^*\mathcal{F}^\bullet, q^*E^\bullet) \\
&\simeq Hom(p^*\mathcal{F}^\bullet, P \otimes q^*E^\bullet) \\
&\simeq Hom(\mathcal{F}^\bullet, p_*(P \otimes q^*E^\bullet)) \\
&= Hom(\mathcal{F}^\bullet, F(E^\bullet))
\end{aligned}$$

Hence, we proved that $G \dashv F$. A similar argument proves $F \dashv H$. The reader can take it as an exercise. Or, you can take it as an easy consequence of Remark 8.6 and Exercise 2.10. □

Exercise 8.8. *Prove that this Proposition is consistent with our example (ii).*

This is certainly good news: the fully faithful criterion in Section 3 can be applied on Fourier-Mukai functor.

9. More on Fourier-Mukai transforms

In last section, we proved that any Fourier-Mukai functor has left and right adjoints. Therefore, we can apply our fully faithful criterion in Section 3.

Moreover, if we want to show a Fourier-Mukai functor is an equivalence, we need to consider compositions of Fourier-Mukai functors. And we will show that in general, any compositions of two Fourie-Mukai functors is still Fourier-Mukai functor.

Let X, Y, and Z be smooth projective varieties over a field k. Consider objects $P \in D^b(X \times Y)$ and $Q \in D^b(Y \times Z)$. Then define the object $R \in D^b(X \times Z)$ by the formula

$$R \coloneqq \pi_{XZ*}(\pi_{XY}^* P \otimes \pi_{YZ}^* Q),$$

where all the π's are projection from $X \times Y \times Z$ to their lower index respectively.

Proposition 9.1. *The composition*

$$D^b(X) \xrightarrow{\Phi_P} D^b(Y) \xrightarrow{\Phi_Q} D^b(Z)$$

is isomorphic to the Fourier-Mukai transform

$$\Phi_R : D^b(X) \to D^b(Z).$$

Proof. The proof is not difficult, but the notation may cause some troubles. The following diagram will lead us walking through the computation.

$$\begin{tikzcd}
 & X \times Y \times Z \arrow[dl, "\pi_{XY}"'] \arrow[d, "\pi_{XZ}"] \arrow[dr, "\pi_{YZ}"] & \\
X \times Y \arrow[d, "p"] \arrow[dr, "s"'] & X \times Z \arrow[dl, "p"] \arrow[dr, "u"'] & Y \times Z \arrow[dl, "r"] \arrow[d, "t"] \\
X & Y & Z
\end{tikzcd}$$

$$\begin{aligned}
\Phi_R(E) &= r_*(s^*E \otimes R) \\
&= r_*(s^*E \otimes \pi_{XZ*}(\pi_{XY}^*P \otimes \pi_{YZ}^*Q)) \\
&\simeq r_*(\pi_{XZ*}(\pi_X^*E \otimes \pi_{XY}^*P \otimes \pi_{YZ}^*Q)) \\
&\simeq \pi_{Z*}(\pi_{XY}^*(q^*E \otimes P) \otimes \pi_{YZ}^*Q) \\
&\simeq t_*\pi_{YZ*}(\pi_{XY}^*(q^*E \otimes P) \otimes \pi_{YZ}^*Q) \\
&\simeq t_*(\pi_{YZ*}\pi_{XY}^*(q^*E \otimes P) \otimes Q) \\
&\simeq t_*(u^*p_*(q^*E \otimes P) \otimes Q) = \Phi_Q(\Phi_P(E))
\end{aligned}$$

□

Remark 9.2. If the composition is not an equivalence, then the kernal R is in general not unique. The above choice of R is the natural one with respect to adjoint functors. More precisely, if

$$R \coloneqq \pi_{XZ*}(\pi_{XY}^*P \otimes \pi_{YZ}^*Q),$$

then

$$R_R \coloneqq \pi_{XZ*}(\pi_{XY}^*P_R \otimes \pi_{YZ}^*Q_R),$$

and similarly for R_L.

Indeed, applying Grothendieck-Verdier duality yields

$$\begin{aligned}
R_R &\simeq R^\vee \otimes s^*\omega_X[dim(X)] \\
&\simeq \mathcal{H}om(R, \mathcal{O}_{X\times Z}) \otimes s^*\omega_X[dim(X)] \\
&\simeq \pi_{XZ*}\mathcal{H}om(\pi_{XY}^*P \otimes \pi_{YZ}^*Q, \pi_Y^*\omega_Y[dim(Y)]) \otimes s^*\omega_X[dim(X)] \\
&\simeq \pi_{XZ*}(\pi_{XY}^*(P^\vee \otimes q^*\omega_X[dim(X)]) \otimes \pi_{YZ}^*(Q^\vee \otimes u^*\omega_Y[dim(Y)])) \\
&\simeq \pi_{XZ*}(\pi_{XY}^*P_R \otimes \pi_{YZ}^*Q_R)
\end{aligned}$$

Exercise 9.3. *(I) Let $P \in D^b(X \times Y)$ and $\Phi \coloneqq \Phi_P : D^b(X) \to D^b(Y)$ be the associated Fourier-Mukai transform. Verify the following assertions:*

(i) For $f : Y \to Z$ the composition $f_* \circ \Phi$ is isomorphic to the Fourier-Mukai transform with kernal $(id_X \times f)_*P \in D^b(X \times Z)$.

(ii) For $f : Z \to Y$ the composition $f^* \circ \Phi$ is isomorphic to the Fourier-Mukai transform with kernal $(id_X \times f)^*P \in D^b(X \times Z)$.

(iii) For $g : W \to X$ the composition $\Phi \circ g_*$ is isomorphic to the Fourier-Mukai transform with kernal $(g \times id_Y)^* P \in D^b(W \times Z)$.

(iv) For $g : X \to W$ the composition $\Phi \circ g^*$ is isomorphic to the Fourier-Mukai transform with kernal $(g \times id_Y)_* P \in D^b(X \times Z)$.

Exercise 9.4. *Consider two kernals $P_i \in D^b(X_i \times Y_i)$, $i = 1, 2$, and their exterior tensor product $P_1 \boxtimes P_2 \in D^b(X_1 \times X_2 \times Y_1 \times Y_2)$.*

(i) Consider the induced Fourier-Mukai transform $\Phi_{P_i} : D^b(X_i) \to D^b(Y_i)$, $i = 1, 2$, and $\Phi_{P_1 \boxtimes P_2} : D^b(X_1 \times X_2) \to D^b(Y_1 \times Y_2)$. Show that there exist isomorphisms

$$\Phi_{P_1 \boxtimes P_2}(F_1 \boxtimes F_2) \simeq \Phi_{P_1}(F_1) \boxtimes \Phi_{P_2}(F_2),$$

which are functorial in F_i, $i = 1, 2$.

(ii) Show for $R \in D^b(X_1 \times X_2)$ and its image $S := \Phi_{P_1 \boxtimes P_2}(R) \in D^b(Y_1 \times Y_2)$ the commutativity of the following diagram:

$$\begin{tikzcd}
D^b(X_1) \arrow[d, "\Phi_R"] & D^b(Y_1) \arrow[l, "\Phi_{P_1}"'] \arrow[d, "\Phi_S"] \\
D^b(X_2) \arrow[r, "\Phi_{P_2}"] & D^b(Y_2)
\end{tikzcd}$$

Note that P_1 in this time used to define a Fourier-Mukai transform in the opposite direction $D^b(Y_1) \to D^b(X_1)$.

The following nontrivial theorem clarify when a functor is of Fourier-Mukai type.

Theorem 9.5. *Let X and Y be two smooth projective varieties and let*

$$F : D^b(X) \to D^b(Y)$$

be a fully faithful functor. If F admits right (or left adjoint) functors, then there exists an object $P \in D^b(X \times Y)$ unique up to isomorphism such that F is isomorphic to Φ_P:

$$F \simeq \Phi_P.$$

Proof. We skip the proof of this highly nontrivial result. It is easier to prove the theorem if we assume that X is $\mathbb{P}^n$ the n-dimensional projective space. One can find the proof in Orlov's paper [Orl97]. And the existence of adjoints is also not necessary, because of the results in [BvdB03]. □

Remark 9.6. In particular, any equivalence is of Fourier-Mukai type. Later, this theorem was generalize to: if

$$Ext^i(F, F) = 0, \ \forall\ i < 0,$$

then

$$Ext^i(\Phi(F), \Phi(F)) = 0, \ \forall\ i < 0.$$

This implies Φ is of Fourier-Mukai type.

Also, we have a 2-category. The objects are smooth projective varieties, and 1 morphisms are kernals in the product, and the composition is given by Mukai's formula. 2-morphisms, if $E, F \in D^b(X \times Y)$, take $Hom^b_D(X \times Y)(E_1, E_2)$.

There is a functor from this 2-category to the 2-category of triangulated categories (1-morphism are functors, 2-morphism are natural transformations).

Orlov's result is very interesting, because it allows us to relate the abstract 2-category of triangulated categories with some geometric 2-categories.

There are many pathologies in the functor. As a warning, that one might lose information when we pass from objects in the derived category in the product to Fourier-Mukai functors.

Before giving the example, we need the following lemma.

Lemma 9.7. *Let C be a smooth projective curve. Then any object in $D^b(C)$ is isomorphic to a direct sum $\oplus E_i[i]$, where the E_i are coherent sheaves on C.*

Proof. We proceed by induction on the length of the complex. Suppose E is a complex of length k with $\mathcal{H}^i(E) = 0$ for $i < i_0$. Then by the standard t-structure, we have the following distinguished triangle

$$\tau^{\leq i_0} E \to E \to E_1 \to \tau^{\leq i_0} E[1]$$

with E_1 of length $k-1$ and $\mathcal{H}^i(E_1) = 0$ for $i \leq i_0$. Moreover, $\tau^{\leq i_0} E$ is quasi-isomorphic to $\mathcal{H}^{i_0}(E)[-i_0]$.

Therefore, we have following distinguished triangle

$$\mathcal{H}^{i_0}(E)[-i_0] \to E \to E_1 \to \mathcal{H}^{i_0}(E)[1 - i_0].$$

It suffices to prove that distinguished triangle splits, which is implied by

$$Hom(E_1, \mathcal{H}^{i_0}(E)[1 - i_0]) = 0.$$

By induction, we have $E^1 \simeq \oplus_{i > i_0} \mathcal{H}^i(E_1)[-i]$. Then

$$\begin{aligned} Hom(E_1, \mathcal{H}^{i_0}(E)[1 - i_0]) &= Hom(\oplus_{i > i_0} \mathcal{H}^i(E_1)[-i], \mathcal{H}^{i_0}(E)[1 - i_0]) \\ &= \oplus_{i > i_0} Ext^{i - i_0 + 1}(\mathcal{H}^i(E_1, \mathcal{H}^{i_0}(E))) \\ &= 0 \end{aligned}$$

by the dimension reason. □

Example 9.8. *Let E be an elliptic curve. Consider $\mathcal{O}_\Delta$ as an object in the derived category of $D^b(E \times E)$. Using Serre duality on the product, one finds that $Ext^2(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ is one-dimensional. Thus, there exists a nontrivial morphism*

$$\phi : \mathcal{O}_\Delta \to \mathcal{O}_\Delta[2]$$

in $D^b(E \times E)$.

As we have said in the remark, there induces a natural transformation between the corresponding Fourier-Mukai transforms, i.e. ϕ yields a morphism $\Phi_\phi : \Phi_{\mathcal{O}_\Delta} \to \Phi_{\mathcal{O}_\Delta[2]}$.

Note that both Fourier-Mukai transformations are equivalences. And in fact, we have

$$\Phi_\phi : id \to [2].$$

Now, one proves that Φ_ϕ is zero, although ϕ is not. Indeed, for a sheaf $\mathcal{F}$ on E. As $Ext^2(\mathcal{F},\mathcal{F})=0$ (since E is one-dimensional), the map $\Phi_\phi(\mathcal{F})$ must be zero. To conclude, one use the fact that any object in $D^b(E)$ is isomorphic to a direct sum of shifted sheaves. Hence, $\Phi_\phi(\mathcal{F}^\bullet)$ is zero for any complex $\mathcal{F}^\bullet \in D^b(E)$ by the naturality of Φ_ϕ.

Exercise 9.9. *(I) Show that Φ_P is an equivalence if and only if the following two conditions are satisfied:*
(i) $\pi_{13*}(\pi_{12}^*P\otimes\pi_{23}^*P_L)\simeq\mathcal{O}_{\Delta_X}$ and
(ii) $\pi_{13*}(\pi_{12}^*P_L\otimes\pi_{23}^*P)\simeq\mathcal{O}_{\Delta_Y}$.
A similar criterion works for P_L replaced by P_R.

(II) Use the uniqueness statement of Orlov's result and the description of the right and left adjoint functors of a Fourier-Mukai transform, in order to show the following description of the derived dual of $\mathcal{O}_\Delta$:

$$\mathcal{O}_\Delta^\vee\simeq\mathcal{O}_\Delta[-n]\otimes p^*\omega_X^*\simeq\mathcal{O}_\Delta[-n]\otimes q^*\omega_X^*.$$

(III) Let $P_i\in D^b(X_i\times Y_i)$, $i=1,2$, be objects such that

$$\Phi_{P_i}:D^b(X_i)\to D^b(Y_i)$$

are equivalences.

Show that the exterior product $P_1\boxtimes P_2\in D^b((X_1\times X_2)\times(Y_1\times Y_2))$ defines an equivalence

$$\Phi_{P_1\boxtimes P_2}:D^b(X_1\times X_2)\to D^b(Y_1\times Y_2).$$

Notice that the last exercise does not follow directly form Exercise 9.4 because in general $D^b(X_1)\boxtimes D^b(X_2)$ is just a subcategory of $D^b(D_1\times D_2)$.

Corollary 9.10. *Let X and Y be smooth projective varieties with equivalent derived categories $D^b(X)$ and $D^b(Y)$. Then $dim(X)=dim(Y)$.*

Proof. The proof is from [Kaw02]. By Orlov's result, we know that any equivalence $F: D^b(X)\simeq D^b(Y)$ is of the form Φ_P for some $P\in D^b(X\times Y)$. Moreover, F has a left adjoint given as the Fourier-Mukai functor $D^b(Y)\to D^b(X)$ with kernal $P_L=P^\vee\otimes p^*\omega_Y[dim(Y)]$ and a right adjoint given as the Fourier-Mukai functor $D^b(Y)\to D^b(X)$ with kernal $P_R=P^\vee\otimes q^*\omega_X[dim(X)]$.

Since F is an equivalence, its right and left adjoint are both quasi-inverse to F. Using the uniqueness of the Fourier-Mukai kernal, we conclude that P_L and P_R are isomorphic to each other in $D^b(X\times Y)$.

Hence

$$P^\vee=P^\vee\otimes(p^*\omega_X\otimes q^*\omega_Y^*[dim(X)-dim(Y)]).$$

With $P^\vee$ a nontrivial object in a bounded derived category, this immediately yield $dim(X)=dim(Y)$. □

Remark 9.11. In the proof, we have tacitly deduced one of the standard facts that is used over and over again, namely that the kernal $P\in D^b(X\times Y)$ of a Fourier-Mukai transform Φ_P which is an equivalence satisfies

$$P\otimes q^*\omega_X\simeq P\otimes p^*\omega_Y.$$

We will come back to this necessary condition in later lectures. There it will be turned into a sufficient criterion for a fully faithful functor to be an equivalence.

9.1. **Spanning classes in derived category.** In this subcategory, we will describe two most common spanning class in $D^b(X)$.

Proposition 9.12. *Let X be a smooth projective variety of dimension n. Then the objects of the form $k(x)$ with $x \in X$ a closed point span the derived category $D^b(X)$.*

Proof. It suffices to prove that for any non-trivial $\mathcal{F}^\bullet \in D^b(X)$ there exists closed points $x_1, x_2 \in X$ and integers i_1, i_2 such that

$$Hom(\mathcal{F}^\bullet, k(x_1)[i_1]) \neq 0 \text{ and } Hom(k(x_2), \mathcal{F}^\bullet[i_2]) \neq 0.$$

By Serre duality, it suffices to ensure the existence of x_2 and i_2.

We use the spectral sequence

$$E_2^{p,q} := Hom(k(x), \mathcal{H}^q(\mathcal{F}^\bullet)[p]) \Longrightarrow Hom(k(x), \mathcal{F}^\bullet[p+q]).$$

Since $\mathcal{F}^\bullet$ is nonzero, there exists a maximal m such that $\mathcal{H}^m(\mathcal{F}^\bullet) \neq 0$. With this choice of m all differentials with target $E_r^{n,m}$ are trivial. As Ext-groups above n between coherent sheaves are always trivial, one has $E_2^{p,q} = 0$ for $p > 0$. Hence all differentials with source $E_r^{n,m}$ are also trivial. Thus $E_\infty^{n,m} = E_2^{n,m}$.

If we now choose a point x in the support of $\mathcal{H}^m(\mathcal{F}^\bullet)$, then

$$E_\infty^{n,m} = E_2^{n,m} = Hom(k(x), \mathcal{H}^m(\mathcal{F}^\bullet)[n]) \simeq Hom(\mathcal{H}^m(\mathcal{F}^\bullet), k(x))^* \neq 0$$

and hence $Hom(\mathcal{F}^\bullet, k(x)[n+m]) \neq 0$. □

Note that although the Grothendieck spectral sequence does not directly provide us the spectral sequence

$$E_2^{p,q} := Hom(\mathcal{H}^{-q}(\mathcal{F}^\bullet)[p], k(x)) \Longrightarrow Hom(\mathcal{F}^\bullet[p+q], k(x)).$$

This is also a spectral sequence, coming from resolving $k(x)$ by complex of injective objects. Then we get a double complex, hence a spectral sequence associated with it. You can check the spetral sequence is exactly the above one.

There is another choice for a spanning class of the derived category of coherent sheaves on a projective variety provided by the powers of an ample line bundle.

Let me introduce the abstract notion of ample sequence first.

Definition 9.13. A sequence of objects $L_i \in \mathcal{A}$, $i \in \mathbb{Z}$, in a k-linear abelian category $\mathcal{A}$ is called ample if for any object $A \in \mathcal{A}$ there exists an integer $i_0(A)$ such that for $i < i_0(A)$ the following conditions are satisfied:

(i) The natural morphism $Hom(L_i, A) \otimes L_i \to A$ is surjetive.
(ii) If $j \neq 0$, then $Hom(L_i, A[j]) = 0$.
(iii) $Hom(A, L_i) = 0$.

Remark 9.14. If X is a projective variety, L is an ample line bundle on it, and $\mathcal{A} = Coh(X)$. It is easy to check that L^i, $i \in \mathbb{Z}$ is an ample sequence. The routine check is left to the readers as an exercise.

Proposition 9.15. *Let L_i, $i \in \mathbb{Z}$ be an ample sequence in a k-linear abelian category $\mathcal{A}$ of finite homological dimension. Then, considered as objects in the derived category $D^b(\mathcal{A})$, the L_i span $D^b(A)$.*

Proof. The proof of this proposition consists of two steps proving the conditions (i) and (ii) in Definition 3.1.

Step one: Let $A^\bullet \in D^b(\mathcal{A})$ such that $Hom(L_i, A^\bullet[j]) = 0$ for all i and j. Suppose $A^\bullet$ is nontrivial, without lost of generality, we may assume that $A^\bullet$ is of the form

$$\cdots \to 0 \to A^0 \to A^1 \to \cdots$$

with $H^0(A^\bullet) \neq 0$.

Hence, $Hom(L_i, H^0(A^\bullet)) \hookrightarrow Hom(L_i, A^\bullet[n]) = 0$ for all i (comes from the long exact sequence by applying $Hom(L_i, -)$ on the distinguished triangle with respect to the standard t-structure). On the other hand, by condition (i) in Definition 9.13, the evaluation map

$$Hom(L_i, H^0(A^\bullet)) \otimes L_i \twoheadrightarrow H^0(A^\bullet)$$

is surjective for $i < i_0(H^0(A^\bullet))$. This yields a contradiction, hence $A^\bullet = 0$.

Step two: Let $A^\bullet \in D^b(\mathcal{A})$ such that $Hom(A^\bullet, L_i[j]) = 0$ for all i and j. If Serre duality is available, then this is equivalent to the condition in step one.

If not, the argument is slightly more involved and runs as follows. We may assume that $A^\bullet$ is of the form

$$\cdots \to A^{n-1} \to A^n \to 0 \to \cdots$$

with $H^n(A^\bullet) \neq 0$. The ampleness of $\{L_i\}$ allows us to construct a surjection

$$Hom(L_i, H^n(A^\bullet)) \otimes L_i \twoheadrightarrow H^n(A^\bullet)$$

for any $i < i_0(H^0(A^\bullet))$. Its kernal will be called B_1. Since $Hom(A^\bullet, L_i) = 0$, the ling exact sequence induced by

$$0 \to B_1 \to Hom(L_i, H^n(A^\bullet)) \otimes L_i \to H^n(A^\bullet) \to 0$$

yields an injection $Hom(A^\bullet, H^n(A^\bullet)) \hookrightarrow Hom(A^\bullet, B_1[1])$.

Then one continues to proceed with B_1 in the same way, i.e. one finds a surjection $Hom(L_i B_1) \otimes L_i \to B_1$ and denotes its kernal by B_2. As before, the long exact sequence yields an injection

$$Hom(A^\bullet, B_1[1]) \hookrightarrow Hom(A^\bullet, B_2[2]),$$

because $Hom(A^\bullet, L_i[j]) = 0$. Thus, recursively we obtain nested inclusions

$$Hom(A^\bullet, H^n(A^\bullet)) \hookrightarrow Hom(A^\bullet, B_1[1]) \hookrightarrow Hom(A^\bullet, B_2[2]) \hookrightarrow \cdots.$$

Since these exists a nontrivial morphism $A^\bullet \to H^n(A^\bullet)$, we obtain in this way for all $j > 0$ an object $B_j \in \mathcal{A}$ with $Hom(A^\bullet, B_j[j]) \neq 0$. This contradicts the assumption on the finiteness of homological dimension of $\mathcal{A}$ and $A^\bullet$ is bounded.

□

If you follow the proof of the theorem, we only used condition (i) in Definition 9.13. The other conditions are used in the proof of extending isomorphism of functors $j \xrightarrow{\sim} F|_{\mathcal{C}}$ (where $\mathcal{C}$ is the full subcategory whose objects are elements in the ample sequence) to an isomorphism on the whole derived category.

Corollary 9.16. *If X is smooth projective variety of dimension at least one, and L is an ample line bundle on X, then the powers L^i, $i \in \mathbb{Z}$, form a spanning class in $D^b(X)$.*

It follows directly from previous proposition. Note that the skyscraper sheaves covers the case when X is a point.

10. A digression to full exceptional collections

Firstly, let us review some basic results on the derived tensor product.

Recall that
$$Tor_i(\mathcal{F}^\bullet, \mathcal{E}^\bullet) := \mathcal{H}^{-i}(\mathcal{F}^\bullet \otimes \mathcal{E}^\bullet)$$
which can often computed via the spectral sequence
$$E_2^{p,q} = Tor_{-p}(\mathcal{H}^q(\mathcal{F}^\bullet), \mathcal{E}^\bullet) \Longrightarrow Tor_{-p-q}(\mathcal{F}^\bullet, \mathcal{E}^\bullet).$$

The argument that worked for right derived functors does not apply literally, as we do not know anything about Cartan-Eilenberg resolutions in this in this context (i.e. a Cartan-Eilenberg resolution of locally free sheaves). But an easy ad hoc argument goes as follows: We may resolve $\mathcal{E}^\bullet$ by locally free sheaves. Then $Tor_{-p}(\mathcal{H}^q(\mathcal{F}^\bullet), \mathcal{E}^\bullet)$ can be computed as the p-th cohomology of the complex $\mathcal{H}^q(\mathcal{F}^\bullet) \otimes \mathcal{E}^\bullet$. Similarly, $Tor_{-p-q}(\mathcal{F}^\bullet, \mathcal{E}^\bullet)$ can be computed as the $(p+q)$-th cohomology of the complex $\mathcal{F}^\bullet \otimes \mathcal{E}^\bullet$. The latter is the total complex of the natural double complex and the claimed spectral sequence corresponds to the standard spectral sequence for a double complex.

This can also be used on pullbacks. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then
$$f^* : Sh_{\mathcal{O}_Y}(Y) \to Sh_{\mathcal{O}_X}(X)$$
is by definition of the composition of the exact functor
$$f^{-1} : Sh_{\mathcal{O}_Y}(Y) \to Sh_{f^{-1}(\mathcal{O}_Y)}(X)$$
and the right exact functor
$$\mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} () : Sh_{f^{-1}(\mathcal{O}_Y)}(X) \to Sh_{\mathcal{O}_X}(X).$$
Thus, f^* is right exact and if $\mathcal{O}_X \otimes^L_{f^{-1}(\mathcal{O}_Y)} ()$ is the left derived functor of $\mathcal{O}_X \otimes f^{-1}(\mathcal{O}_Y)()$, then
$$Lf^* := (\mathcal{O}_X \otimes^L_{f^{-1}(\mathcal{O}_Y)} ()) \circ f^{-1} : D^-(Y) \to D^-(X).$$

To be precise, the arguments of the previous paragraph are not quite sufficient to derive the tensor product $\mathcal{O}_X \otimes^L_{f^{-1}(\mathcal{O}_Y)} ()$, as we only explained how to derive the tensor product of $\mathcal{O}_X$-modules. But the more general situation is handled in the similar way. And one obtains
$$E^2_{p,q} = L^p f^*(\mathcal{H}^q(\mathcal{E})^\bullet) \Longrightarrow L^{p+q} f^*(\mathcal{E}^\bullet),$$

where by definition $L^p f^*(\mathcal{F}^\bullet) = \mathcal{H}^p(Lf^*(\mathcal{F}^\bullet)))$.

Then we have following lemma, consider a flat morphism $S \to X$. If $x \in X$ is a closed point, we denote by $i_x : S_x \to S$ the closed embedding of the fiber over x.

Lemma 10.1. *Suppose $\mathcal{Q} \in D^b(S)$ and assume that for all closed points $x \in X$ the derived pullback $Li_x^*\mathcal{Q} \in D^b(S_x)$ is a complex concentrated in degree zero. i.e. a sheaf.*

Then $\mathcal{Q}$ is isomorphic to a sheaf which is flat over X.

Proof. In order to verify the claim we will apply spectral sequence to the inclusion i_x and obtain

$$E_2^{p,q} = \mathcal{H}^p(Li_x^*\mathcal{H}^q(\mathcal{Q})) \Longrightarrow \mathcal{H}^{p+q}(Li_x^*\mathcal{Q}).$$

By assumption, the right side is trivial except possibly for $p+q=0$. Choose m maximal with $\mathcal{H}^m(\mathcal{Q}) \neq 0$. Then there exists a closed point $x \in X$ with $E_2^{0,m} = \mathcal{H}^0(Li_x^*\mathcal{H}^m(\mathcal{Q})) \neq 0$ (this is just the ordinary pullback). But this non-triviality survives the passing to the limit in the spectral sequence and hence $m = 0$. For the same reason, $E_2^{0,-1} = \mathcal{H}^{-1}(Li_x^*\mathcal{H}^0(\mathcal{Q}))$ with $x \in X$ arbitrary also survives and must, therefore, be trivial. This shows that the sheaf $\mathcal{H}^0(\mathcal{Q})$ is actually flat over X.

Here we use the local criterion for flatness. In the following, we assume $A = \mathcal{O}_{X,x}$ is a local ring at a closed point $x \in X$, and $B = \mathcal{O}_{S,s}$, the local ring at the generic point s of a n irreducible component of S_x. Recall the simple fact from commutative algebra behind this: Let $A \to B$ be a local ring flat homomorphism and M a B module. In order to show that M is A flat, it suffices to show that M is flat over B. One has to show that for any finitely generated ideal $b \in B$ the map $b \otimes M \to M$ is injective. Of course, it suffices to show this for $n = m$, the maximal ideal of B. Suppose

$$0 \to N_1 \to N_2 \to M \to 0$$

is a short exact sequence of B-modules. The analogue of $\mathcal{H}^{-1}(Li_x^*\mathcal{H}^0(\mathcal{Q})) = 0$ for $\mathcal{H}^0(\mathcal{Q})$ replaced by M yields the injectivity of $N_1/mN_1 \to N_2/mN_2$. If N_2 is A-flat, then $N_2 \otimes m \to N_2$ is injective. Together with snake lemma, we get the injectivity of $m \otimes M \to M$.

Also note that the flatness of $\mathcal{H}^0(\mathcal{Q})$ over X implies the higher derived pullbacks $E_2^{p,0} = \mathcal{H}^p(Li_x^*\mathcal{H}^0(\mathcal{Q}))$ are trivial for $p < 0$.

The lst thing one has to check is that there is no non-trivial cohomology below, i.e. that $\mathcal{H}^q(\mathcal{Q}) = 0$ for $q < 0$. Suppose not; then choose m maximal among all $q < 0$ with $\mathcal{H}^q(\mathcal{Q}) \neq 0$ and x a closed point $x \in X$ in the support of $\mathcal{H}^m(\mathcal{Q})$.

Draw a picture of the second page od spectral sequence, we will see that this term will survive to the end, hence contradicts to our assumption. ☐

Notice that, we need the assumption $S \to X$ is flat, otherwise, one can easily find a counterexample on a closed embedding. Let us see some direct consequence of Orlov's existence theorem and this easy lemma.

Corollary 10.2. *Suppose $\Phi : D^b(X) \simeq D^b(Y)$ is an equivalence such that for any closed point $x \in X$ there exists a closed point $f(x) \in Y$ with*

$$\Phi(k(x)) \simeq k(f(x)).$$

Then $f : X \to Y$ defines an isomorphism and Φ is the composition of f_* with the twist by some line bundle $M \in Pic(Y)$, i.e.

$$\Phi \simeq (M \otimes (-)) \circ f_*.$$

Proof. In the first step one shows that there exists a morphism $X \to Y$ which on the set of closed points induces the given map f.

By Orlov's result, we know that Φ is a Fourier-Mukai transform Φ_P, then Lemma 2.12 implies that P is an X-flat sheaf on $X \times Y$. By assumption $P|_{\{x\}\times Y} \simeq k(f(x))$. Choosing local sections of P shows that it indeed defines a morphism $X \to Y$ inducing f on the closed points. By abuse of notation, the morphism will again be called f.

Next, one uses the assumption that Φ is an equivalence to prove that f is an isomorphism. Since the sheaves $k(x)$ span $D^b(X)$, their images span $D^b(Y)$. Thus, if $y \in Y$ is a closed point, then there exists a closed point $x \in X$ and an integer m with

$$Hom(\Phi(k(x)), k(y)[m]) \neq 0.$$

This implies that any $k(y)$ is of the form $k(f(x))$ for some closed point $x \in X$, i.e. f is surjective on the set of closed points.

Similarly, two different points $x_1 \neq x_2 \in X$ give rise to two different points $f(x_1) \neq f(x_2)$, i.e. f is injective on closed points. In characteristic 0, this already suffice to conclude that f as an isomorphism between two smooth projective varieties is an isomorphism.

Without the characteristic 0 assumption, one argues by using a quasi-inverse Φ^{-1} to produce an honest f^{-1}.

Eventually, P considered as sheaf on its support, which is the graph of f, is a sheaf of constant fiber dimension one and hence a line bundle. Using $Supp(P) \simeq Y$ given by the second projection allows us to view this line bundle as a line bundle M on Y. ☐

Orlov's result can also be used to give a somewhat round-about proof of the classical result of Gabriel saying that the abelian category of coherent sheaves on a scheme determines the scheme.

Corollary 10.3. *Suppose X and Y are smooth projective varieties. If there exists an equivalence $Coh(X) \simeq Coh(Y)$, then X and Y are isomorphic.*

Proof. Clearly, an equivalence

$$\Phi_0 : Coh(X) \xrightarrow{\sim} Coh(Y)$$

between the abelian categories can be extended to an equivalence

$$\Phi : D^b(X) \xrightarrow{\sim} D^b(Y)$$

of their derived categories.

A sheaf $\mathcal{F} \in Coh(X)$ is called indecomposable if for any non-trivial surjection

$$\mathcal{F} \twoheadrightarrow \mathcal{G}$$

with $\mathcal{G} \in Coh(X)$ is an isomorphism. It is easy to show that any indecomposable sheaf is of the form $k(x)$ with $x \in X$ a closed point.

The equivalence $\Phi_0 : Coh(X) \simeq Coh(Y)$ sends an indecomposable object to an indecomposable object. Hence, for any closed point $x \in X$ there exists a closed point $y \in Y$ with $\Phi_0(k(x)) \simeq k(y)$. This continues to hold for the extension $\Phi : D^b(X) \to D^b(Y)$. By previous corollary, we know that Φ is of the form $\mathcal{F}^\bullet \mapsto M \otimes \mathcal{F}^\bullet$ for some isomorphism $f : X \to Y$ and some line bundle M on Y.

Note that we have not only proved that X and Y are isomorphic, but that in fact any equivalence between their abelian categories is of the special form $\mathcal{F} \mapsto M \otimes \mathcal{F}$. $\square$

10.1. **Exceptional collections.** Before going into Bondal Orlov's reconstruction theorem, let us take a digression on exceptional collections in algebraic geometry.

Recall that we have defined what is exceptional collection in Section 3 in the following way.

Definition 10.4. An object $E \in \mathcal{D}$ in a k-linear triangulated category $\mathcal{D}$ is called exceptional if

$$Hom(E, E[l]) = \begin{cases} k & \text{if } l = 0 \\ 0 & \text{if } l \neq 0 \end{cases}$$

An exceptional sequence is a sequence $E_1, \cdots E_n$ of exceptional objects such that

$$Hom(E_i, E_j[l]) = 0$$

for all $i > j$ and all l.

An exceptional collection is full if $\mathcal{D}$ is generated by $\{E_i\}$, i.e. any full triangulated subcategory containing all objects E_i is equivalent to $\mathcal{D}$ (via the inclusion).

At this time, we know the following examples in $D^b(X)$.

Example 10.5.
(1) *Any line bundle on $\mathbb{P}^n$ is exceptional.*
Because

$$Ext^*(\mathcal{O}(t), \mathcal{O}(t)) = Ext^*(\mathcal{O}, \mathcal{O}) = H^*(\mathbb{P}^n, \mathcal{O})$$

satisfies the condition.

(2) *More generally, any line bundle on a Fano variety X ($\omega_X^\vee$ ample, for instance, any hypersurface of degree $\leq n$ in $\mathbb{P}^n$).*

$$Ext^*(L, L) = Ext^*(\mathcal{O}, \mathcal{O}) = H^*(X, \mathcal{O})$$

by Kodaira's vanishing theorem (because $\mathcal{O} - \omega_X$ is ample divisor), hence we need our variety defined over a field of characteristic zero.

(3) *If $\tilde{X}$ is the blow up of X at a smooth point x. Then we have the following diagram*

$$\begin{array}{ccc} E & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ x & \longrightarrow & X \end{array}$$

where $E \simeq \mathbb{P}^{n-1}$ is the exceptional divisor. Then any line bundle on E is exceptional in $D^b(\tilde{X})$.

Here is the proof of this statement: Consider the following sequence

$$0 \to \mathcal{O}_{\tilde{X}}(-E) \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_E \to 0.$$

Apply $Hom(-, \mathcal{O}_E)$ on this sequence. Since we have

$$Ext^i(\mathcal{O}_{\tilde{X}}, \mathcal{O}_E) = H^i(E, \mathcal{O}_E) = \begin{cases} k, & \textit{if } \ i=0 \\ 0 & \textit{otherwise.} \end{cases}$$

We get the following

$$0 \to Hom(\mathcal{O}_E, \mathcal{O}_E) \to k \to Hom(\mathcal{O}_{\tilde{X}}(-E), \mathcal{O}_E) \to Ext^1(\mathcal{O}_E, \mathcal{O}_E) \to 0,$$

and

$$Ext^i(\mathcal{O}_E, \mathcal{O}_E) \simeq Ext^{i-1}(\mathcal{O}_{\tilde{X}}(-E), \mathcal{O}_E)$$

for all $i \geq 2$. Hence it suffices to show that $Ext^*(\mathcal{O}_{\tilde{X}}(-E), \mathcal{O}_E) = 0$. This follows from

$$Ext^*(\mathcal{O}_{\tilde{X}}(-E), \mathcal{O}_E) = H^*(E, \mathcal{O}_E(E)) = H^*(E, \mathcal{O}_E(-1)) = 0.$$

(4) *There are many examples of exceptional objects in Grassmannians, flag varieties and so on.*
(5) *Non-examples: Skyscraper sheaf $\mathcal{O}_x$ on X with dimension ≥ 1. Because*

$$Ext^1(\mathcal{O}_x, \mathcal{O}_x) = T_x X.$$

(6) *Another non-example: Let C be a curve of genus bigger than zero, then any line bundle on C is not an exceptional object.*
This is because

$$Ext^*(L, L) = H^*(C, \mathcal{O}_C) = \begin{cases} 1 & \textit{if } \ {*}=0 \\ g & \textit{if } \ {*}=1 \end{cases}$$

(7) *Another non-example: if X is Calabi-Yau (ω_X is trivial, e.g. K3 surface, quintic 3fold etc.) Then*

$$Ext^n(E, E) = Ext^0(E, E)^* \neq 0$$

if $E \neq 0$, there is no exceptional objects in $D^b(X)$. The structure sheaf of a K3 surface is not an exceptional object, but a spherical object. Maybe we will talk about it in later lectures.

The following examples are exceptional collections.

Example 10.6. (1) *On $\mathbb{P}^n$, The sequence $\mathcal{O}$, $\mathcal{O}(1) \cdots \mathcal{O}(n)$ is an exceptional sequence. Notice that we can not go one step longer by*
(2) *More generally, if X is Fano. We can write $\omega_X = \mathcal{O}_X(-k)$ for some unique ample line bundle $\mathcal{O}(1)$. Then*

$$\mathcal{O}, \mathcal{O}(1), \cdots, \mathcal{O}(k-1)$$

is an exceptional collection by Kodaira vanishing. And similarly, we can not go one step further by adding $\mathcal{O}(k)$.

(3) *Let $\tilde{X} \coloneqq Bl_x\mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at a closed point x, and $f : \tilde{X} \to X$ be the projection, $H = f^*\mathcal{O}(1)$ is the pullback of a general hyperplane class. Then*

$$\mathcal{O}_E(-1), \mathcal{O}_{\tilde{X}}, \mathcal{O}_{\tilde{X}}(H), \mathcal{O}_{\tilde{X}}(2H)$$

is an exceptional collection.

Firstly, $R\Gamma(\mathcal{O}_{\tilde{X}}) = R\Gamma(Rf_*\mathcal{O}_{\tilde{X}}) = R\Gamma(\mathcal{O}_{\mathbb{P}^2})$, hence all the objects here are exceptional.

And

$$\begin{aligned}
&Ext^*(\mathcal{O}_{\tilde{X}}(H), \mathcal{O}_{\tilde{X}})\\
&= Ext^*(f^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\tilde{X}})\\
&= Ext^*(\mathcal{O}_{\mathbb{P}^2}(1), Rf_*\mathcal{O}_{\tilde{X}})\\
&= Ext^*(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}) = 0
\end{aligned}$$

Here $R^0f_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ by Zariski's main theorem (the base is normal, and the fibers are connected). For the vanishing of higher direct image, we prove it by induction in a general setting. Suppose we have smooth center Z of codimension c inside a smooth projective variety X, let $\tilde{X} \coloneqq Bl_ZX$, therefore we have the following diagram

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "\pi"] & \tilde{X} \arrow[d, "f"] \\
Z \arrow[r] & X
\end{tikzcd}$$

where π is a projection of $\mathbb{P}^{c-1}$ bundle. Then we have the short exact sequence

$$0 \to \mathcal{O}_{\tilde{X}}(-E) \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_E \to 0.$$

And $RF_*\mathcal{O}_E \to \mathcal{O}_Z$ by Grauert theorem in [Har13]. *Therefore, it suffices to show the vanishing of $Rf^{>0}\mathcal{O}_{\tilde{X}}(-E)$. We keep doing the induction*

$$0 \to \mathcal{O}_{\tilde{X}}(-2E) \to \mathcal{O}_{\tilde{X}}(-E) \to \mathcal{O}_E(-E) \to 0,$$

where $\mathcal{O}_E(-E) = \mathcal{O}_E(1)$ has no higher direct image. Therefore , it suffices to show the vanishing of $Rf^{>0}\mathcal{O}_{\tilde{X}}(-nE)$, for $n \gg 0$. This is the relative version of Serre's vanishing as $\mathcal{O}_{\tilde{X}}(-nE) = \mathcal{O}_f(n)$.

Similarly, we can put $2H$ instead of H, we get that there is non-trivial morphisms from right to left among the three line bundles.

$$\begin{aligned}
&Ext^*(\mathcal{O}_{\tilde{X}}(tH), \mathcal{O}_E(1))\\
&= Ext^*(f^*\mathcal{O}_{\mathbb{P}^2}(t), \mathcal{O}_E(-1))\\
&= Ext^*(\mathcal{O}_{\mathbb{P}^2}(t), Rf_*\mathcal{O}_E(-1)) = 0
\end{aligned}$$

The last inequality follows from $Rf_*\mathcal{O}_E(-1) = 0$. This is because

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"] & \tilde{X} \arrow[d] \\ x \arrow[r] & \mathbb{P}^2 \end{tikzcd}$$

If we go another direction in the diagram, we see that $Rf_*\mathcal{O}_E(-1) = 0$ follows from $H^*(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$.

(4) *Let $\tilde{X}$ denote the blow up of two points at $\mathbb{P}^2$, and $H = f^*\mathcal{O}(1)$ be the pullback of a general hyperplane class. Then*

$$\mathcal{O}_{E_1}(-1), \mathcal{O}_{E_2}(-1), \mathcal{O}_{\tilde{X}}, \mathcal{O}_{\tilde{X}}(H), \mathcal{O}_{\tilde{X}}(2H)$$

is an exceptional collection.

If $E, F \in D^b(X)$, we write $E \perp F$ to mean that $RHom(E, F) = 0$. It can be compared with the inner product, but this is not symmetric. Serre duality tells us $E \perp F$ is equivalent to $F \perp E \otimes \omega_X$. This is not symmetric unless X is Calabi-Yau.

You may wonder whether this exceptional collection is full or not. Here is the famous theorem of Beilinson.

Theorem 10.7. *On $\mathbb{P}^n$, the exceptional collection $\mathcal{O}$, $\mathcal{O}(1) \cdots \mathcal{O}(n)$ is full in $D^b(\mathbb{P}^n)$.*

So in particular, every $F \in D^b(\mathbb{P}^n)$ is isomorphic to a complex whose terms are sums of those $n+1$ line bundles in the derived category of $\mathbb{P}^n$.

Proof. First step: we generate all skyscraper sheaves by all . This is done by Koszul resolution. Suppose $x \in \mathbb{P}^n$ is cut out by n equations $l_1, \cdots, l_n$

$$0 \to \mathcal{O}(-n) \to \cdots \to \mathcal{O}(-2)^{\binom{n}{2}} \to \mathcal{O}(-1)^n \xrightarrow{l_1, \cdots, l_n} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_x \to 0$$

Twist this by n, we get a resolution of skyscraper sheaf by our exceptional collection.

Second step: let

$$\mathcal{C} := \langle \mathcal{O}, \cdots, \mathcal{O}(n) \rangle$$

be the triangulated category generated by the exceptional collection under cones and shifts.

Since $\mathcal{C}$ contains our spanning class, hence $\mathcal{C}^\perp =^\perp \mathcal{C} = \{0\}$.

The last step is show that this implies the theorem. This follows from Lemma 3.15. □

11. Exceptional collection and Semi-orthogonal decompositions

Let us recall some basic definitions and results we have already discussed (explicitly or implicitly) in Section 3.

Definition 11.1. A full triangulated subcategory $\mathcal{D}' \subset \mathcal{D}$ is called (left, or right) admissible if the inclusion functor admits (left, or right) both adjoints respectively.

Example 11.2. (1) *A fully faithful functor*

$$D^b(Y) \to D^b(X),$$

then its essential image is an admissible subcategory.

(2) *Non-example: Let $\mathcal{C}$ be the subcategory of $D^b(X)$ generated by skyscraper sheaves. This is because $\mathcal{C}^\perp = 0$, but $\mathcal{C} \neq D^b(X)$. All the objects in $\mathcal{C}$ has finite support. Hence it follows from Lemma 3.15.*

There is another equivalent definition of admissible categories.

Lemma 11.3. *Suppose we have a full triangulated subcategory $\mathcal{B} \subset \mathcal{D}$, then $\mathcal{B}$ is right admissible if and only if the following is true:*

For any object $X \in \mathcal{D}$, there exists distinguished triangle

$$B \to X \to C \to B[1]$$

where $B \in \mathcal{B}$ and $C \in \mathcal{B}^\perp$. Similarly, for left admissibility, it is equivalent to the existence of following distinguished triangle

$$C' \to X \to B \to C'[1]$$

where $C' \in^\perp \mathcal{B}$ and $B \in \mathcal{B}$.

Proof. We only prove it for the right admissible case. For any object $B_1 \in \mathcal{B}$, apply $Hom(B_1, -)$ on the distinguished triangle. We get long exact sequence

$$Hom(B_1, C[-1]) \to Hom(B_1, B) \to Hom(B_1, X) \to Hom(B_1, C).$$

Because $C \in \mathcal{B}^\perp$, we have isomorphism

$$Hom(B_1, B) \to Hom(B_1, X)$$

. Then we can set the right adjoint functor F by sending X to B, and this is a functor because of the diagram

$$\begin{tikzcd}
B \arrow[r] \arrow[d, dotted] & X \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dotted] & B[1] \arrow[d, dotted] \\
B' \arrow[r] & X' \arrow[r] & C' \arrow[r] & B'[1]
\end{tikzcd}$$

The dotted arrows are unique by Exercise 2.16.

Conversely, if we have right adjoint functor $F : \mathcal{D} \to \mathcal{B}$ of the inclusion functor. We have co-unit morphism $F(X) \to X$, which can be competed by a distinguished triangle

$$F(X) \to X \to C \to F(X)[1],$$

then for any object $B \in \mathcal{B}$, we apply $Hom(B, -)$, we get that $Hom(B, C) = 0$. We win. □

Lemma 11.4. *Suppose $\mathcal{B}$ is right admissible subcategory in $\mathcal{D}$, then $\mathcal{B}^\perp := \mathcal{C}$ is left admissible and $\mathcal{B} =^\perp \mathcal{C}$.*

Proof. Use the same triangle, we know that $\mathcal{C}$ is left admissible is equivalent to $\mathcal{B} = ^{\perp}\mathcal{C}$. Given $X \in ^{\perp}\mathcal{C}$, then we have the following exact triangle

$$B \to X \to C \to B[1].$$

Then for any $C' \in \mathcal{C}$, apply $Hom(-, C')$ on the triangle, we get that $Hom(C, C') = 0$ for any $C' \in \mathcal{C}$, hence $C \simeq 0$. Hence $X \simeq B$. □

Now recall that if E is an exceptional object, then $\langle E \rangle$ generated by E under direct sum and shifts is an admissible subcategory.

We know that $\langle E \rangle$ is isomorphic to the derived category of coherent sheaves on a point. Therefore, any exceptional object gives an fully faithful embedding

$$I : D^b(Spec(k)) \simeq D^b(Vec_f) \hookrightarrow D^b(X).$$

via $V^\bullet \mapsto E \otimes_k V^\bullet$. Hence by Example 11.2, we get another proof of this easy lemma.

Furthermore, this embedding can be viewed as a Fourier-Mukai transform with kernal $E \in D^b(Spec(k) \times X)$. And its right adjoint functor is given by

$$R : F \mapsto RHom(E, F).$$

If E is exceptional, and $F \in D^b(X)$ is an arbitrary object. Then the distinguished triangle we have seen before comes from the unit morphism $h : IR \to id$, i.e.

$$RHom(E, F) \otimes F \xrightarrow{h_F} F \to Cone(h_F) \to RHom(E, F) \otimes F[1],$$

and it is easy to see that $Cone(h_F) \in \langle E \rangle^{\perp}$. Usually, $Cone(h_F)$ is denoted by $L_E(F)[1]$, where $L_E(F)$ is the left mutation of F with respect to E. Similarly, we have right mutation of F with respect to E, denoted by $R_E(F)$, sitting in the following exact triangle

$$X \to Hom^\bullet(X, E)^* \otimes E \to R_E(X),$$

and it is easy to see that $R_E(F) \in ^{\perp}\langle E \rangle$.

More generally, $\mathcal{B} \subset \mathcal{D}$ is right admissible, then for any object X, we have the following exact triangle

$$B \to X \to L_{\mathcal{B}}(X)[1],$$

and $L_{\mathcal{B}} : \mathcal{D} \to \mathcal{B}^{\perp}$ is a functor, called the left mutation through $\mathcal{B}$.

Now recall the definition of semi-orthogonal decomposition.

Definition 11.5. A sequence of full admissible triangulated subcategories

$$\mathcal{D}_1, \cdots \mathcal{D}_n \subset \mathcal{D}$$

is semi-orthogonal if for all $i > j$

$$\mathcal{D}_j \subset \mathcal{D}_i^{\perp}.$$

Such a sequence defines a semi-orthogonal decomposition of $\mathcal{D}$ if $\mathcal{D}$ is generated by the $\mathcal{D}_i$, i.e. for any $F \in \mathcal{D}$, there is a sequence of morphisms

$$0 = F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 = F,$$

such that $cone(F_i \to F_{i-1}) \in \mathcal{D}_i$ for all $1 \leq i \leq n$.

Proposition 11.6. *Suppose we have a triangulated subcategory $\mathcal{A} \subset \mathcal{D}$ and $\mathcal{A} = \langle \mathcal{B}, \mathcal{C} \rangle$, where $\mathcal{B}$ and $\mathcal{C}$ are both (right, left) admissible triangulated subcategories in $\mathcal{D}$. Then $\mathcal{A}$ is (right, left) admissible and*

$$L_{\mathcal{A}}[1] = L_{\mathcal{B}}[1] \circ L_{\mathcal{C}}[1].$$

Proof. We only prove the right admissibility. The left admissibility is proved similarly. For any object $X \in \mathcal{D}$, we have exact triangle

$$C \to X \to L_{\mathcal{C}} X[1].$$

Then we apply $L_{\mathcal{B}}$ on $L_{\mathcal{C}} X[1]$, we get the following diagram

$$\begin{tikzcd}
 & L_{\mathcal{B}}(L_{\mathcal{C}}X)[2] \arrow[r, "id"] & L_{\mathcal{B}}(L_{\mathcal{C}}X)[2] \\
C \arrow[r] & X \arrow[r] \arrow[u] & L_{\mathcal{C}}X[1] \arrow[u] \\
C \arrow[r] \arrow[u, "id"] & A \arrow[r] \arrow[u] & B \arrow[u]
\end{tikzcd}$$

by Octahedron axiom. Each column and each row is an exact triangle. Here $A \in \mathcal{A}$ and

$$L_{\mathcal{B}}(L_{\mathcal{C}}X)[2] \in \mathcal{A}^{\perp} = \mathcal{B}^{\perp} \cap \mathcal{C}^{\perp}.$$

This is because $B \in \mathcal{C}^{\perp}$, and $L_{\mathcal{C}} X[1] \in \mathcal{C}^{\perp}$. □

Corollary 11.7. *If we have an exceptional collection $E_1, \cdots, E_n$, then the subcategory they generate*

$$\langle E_1, \cdots E_n \rangle$$

is an admissible category.

Proposition 11.8. *On the $\tilde{X} = Bl_x \mathbb{P}^2$, the exceptional collection*

$$\mathcal{O}_E(-1), \mathcal{O}_{\tilde{X}}, \mathcal{O}_{\tilde{X}}(H), \mathcal{O}_{\tilde{X}}(2H)$$

is a full exceptional collection.

Proof. We want to show that we can generated all skyscraper sheaves from the exceptional collection. If $p \notin E$, then $\mathcal{O}_p = Lf^* \mathcal{O}_{f(p)}$. Then Koszul complex guarantees that the three line bundles can generated $\mathcal{O}_p$.

If $p \in E$, then we have

$$L_i f^* \mathcal{O}_{f(p)} = \begin{cases} \mathcal{O}_E & \text{if } \text{ i=0}, \\ \mathcal{O}_E(-1) & \text{if } \text{ i=1} \end{cases}$$

This calculation is left to students as an exercise. Hence we have the following distinguished triangle

$$\mathcal{O}_E(-1)[1] \to Lf^* \mathcal{O}_{f(p)} \to \mathcal{O}_E.$$

Therefore, we can generated $\mathcal{O}_E$ and, hence the point $\mathcal{O}_p$ on E. □

Example 11.9. *Let $Q \subset \mathbb{P}^5$ is a smooth quadric fourfold. It is a Fano variety with $\omega_Q = \mathcal{O}(-4)$. Then the sequence*

$$\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3)$$

is an exceptional collection, but we will show that it is not full. More precisely, we will show that $\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3)\rangle^{\perp}$ is not trivial.

Let $P \subset Q$ be a plane (because quadric $2n$ folds contains two families of $\mathbb{P}^n$). Consider $\mathcal{O}_P(1)$. Then

$$RHom_Q(\mathcal{O}_Q(3), \mathcal{O}_P(1)) = R\Gamma(P, \mathcal{O}_P(-2)) = 0.$$

Similarly, we get $RHom(\mathcal{O}_Q(2), \mathcal{O}_P(1)) = 0$. But we have non-trivial morphism

$$\mathcal{O}_Q(1) \to \mathcal{O}_P(1) \to 0,$$

whose kernal is the ideal sheaf of P in Q twisted by one, denoted by $I(1)$.

Claim: $I(1) \in \langle \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3)\rangle^{\perp}$.

This is easy to check.

Then we calculate

$$Ext^i(\mathcal{O}_Q, I(1)) = \begin{cases} k^3, & \text{if i=0,} \\ 0 & \text{otherwise.} \end{cases}$$

Then we play the same game again.

$$0 \to S \to \mathcal{O}_Q^3 \to I(1) \to 0.$$

It is surjective because these sections are the linear forms on Q vanishing on P, the ideal of P is generated by these three 3 sections. Hence

$$S \in \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3)\rangle^{\perp}.$$

And S is nonzero, because its rank is 2, in fact, it is a vector bundle called Spinor bundle.

There is an interesting philosophy behind this example. Kuznetsov philosophy: On a Fano variety X with $\omega_X = \mathcal{O}_X(-k)$, exceptional collection

$$\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2), \cdots, \mathcal{O}_X(k-1)$$

is never full unless $X = \mathbb{P}^n$. The orthogonal

$$\langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2), \cdots, \mathcal{O}_X(k-1)\rangle^{\perp}$$

is always geometrically interesting.

For example, if X is the intersection two generic quadric hyper-surfaces in $\mathbb{P}^5$, then

$$\langle \mathcal{O}_X, \mathcal{O}_X(1)\rangle^{\perp} \simeq D^b(C),$$

where C is a genus 2 curve.

If X is cubic hyper-surface in $\mathbb{P}^5$, then

$$\langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\rangle^{\perp}$$

is a K3-category (i.e. its Serre functor is [2]), and sometimes it is the derived category of coherent sheaves on a K3 surface.

The following theorem can be viewed as an relative version of Beilinson decomposition.

Theorem 11.10. *Let E be a vector bundle of rank $n+1$ on X, we have $P = \mathbb{P}(E) \xrightarrow{f} X$. Then $f^* : D^b(X) \to D^b(P)$ is fully faithful, so is $f^*(-) \otimes \mathcal{O}_f(t)$ for $\forall t \in \mathbb{Z}$, where $\mathcal{O}_f(t)$ is the relative line bundle. Moreover, we have a semi-orthogonal decomposition.*

$$D^b(P) = \langle f^* D^b(X), f^* D(X) \otimes \mathcal{O}_f(1), \cdots, f^* D(X) \otimes \mathcal{O}_f(1) \rangle.$$

Proof. Given $F, G \in D^b(X)$, then

$$\begin{aligned} & Hom(f^*F, f^*G) \\ & = Hom(F, f_* f^* G) \\ & = Hom(F, G \otimes Rf_* \mathcal{O}_P) \\ & = Hom(F, G) \end{aligned}$$

Hence first half in the statement is proved.

Given $F, G \in D^b(X)$, then

$$\begin{aligned} & Hom(f^*F \otimes \mathcal{O}_f(i), f^*G \otimes \mathcal{O}_f(j)) \\ & = Hom(F, f_*(f^*G \otimes \mathcal{O}_f(j-i)) \\ & = Hom(F, G \otimes f_* \mathcal{O}_f(j-i)) \\ & = 0 \end{aligned}$$

The last equality follows from that $f_* \mathcal{O}_f(j-i) = 0$ for $-n-1 < j-i < 0$.

The last step is to show that they generate the points. Suppose that $p \in P$ is a closed point. This is obvious because we can generated any point in each fiber by Beilinson's decomposition.

More precisely, let $Y = f^{-1}(f(p)) \simeq \mathbb{P}^n$, and $\mathcal{O}_Y = f^* \mathcal{O}_{f(p)}$, then Koszul complex generates $\mathcal{O}_p$. □

11.1. **Semi-Orthogonal Decomposition in blow ups.** Next we will discuss blow-up of smooth centers in general, suppose we have $Z \subset X$ a smooth closed sub-variety in a smooth projective variety X of codimension c. Then we have the following diagram

$$\begin{array}{ccc} E & \xrightarrow{j} & \tilde{X} = Bl_Z X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X \end{array}$$

And $f_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X$, hence f^* is fully faithful by the proof in 11.10. Moreover, for each $t \in \mathbb{Z}$, we have functors $\Phi_t : D^b(Z) \to D^b(\tilde{X})$, $F \mapsto j_*(g^*F \otimes \mathcal{O}_E(tE))$. In fact, one can show that these functor are fully faithful.

Before we state the main theorem of this section, we prove a lemma which will be used in the proof of the following main theorem.

Lemma 11.11. *Let $j : D \to Z$ be an embedding of a smooth divisor in a smooth algebraic variety Z of dimension n. Consider for an object $A \in D^b(X)$, we have a distinguished* *triangle with the canonical second morphism:*

$$\bar{A} \to j^* j_* A \to A,$$

with $\bar{A} \simeq A \otimes \mathcal{O}_D(-D)[1]$.

Proof. The functor j_* coincides with Φ_E, where $E = \mathcal{O}_\gamma$ is the structure sheaf of the graph of j in $D \times Z$. The adjoint functor j^* is also a Fourier Mukai transform with the same kernal but in converse direction.

Hence, we know that $j^* j_*$ is the Fourier-Mukai transform with kernal

$$p_{13*}(p_{23}^* \mathcal{O}_\Gamma \otimes p_{12}^* \mathcal{O}_\Gamma)$$

in $D \times Z \times D$. Note that

$$p_{12}^* \mathcal{O}_\Gamma = \mathcal{O}_{\Gamma \times D}; \quad p_{23}^* \mathcal{O}_\Gamma = \mathcal{O}_{D \times \Gamma}.$$

Notice that although they do not meet transversely in $D \times Z \times D$, they meet transversely in the image of $\Delta_3 : D^3 \to D \times Z \times D$. This helps to calculate $\mathcal{H}^i(\mathcal{O}_{\Gamma \times D} \otimes \mathcal{O}_{D \times \Gamma})$. Indeed, one can consider the tensor product $\mathcal{O}_{\Gamma \times D} \otimes \mathcal{O}_{D \times \Gamma}$ as the restriction of $\mathcal{O}_{D \times \Gamma}$ to $\Gamma \times D$. Restricting first to the divisor D^3, we obtain :

$$\begin{aligned} &L^0 \Delta_3^* \mathcal{O}_{D \times \Gamma} = \mathcal{O}_{D \times \Gamma}, \\ &L^1 \Delta_3^* \mathcal{O}_{D \times \Gamma} = \mathcal{O}_{D \times \Gamma} \otimes \mathcal{O}_{D \times Z \times D}(-D^3), \\ &L^i \Delta_3^* \mathcal{O}_{D \times \Gamma} = 0, for\ i > 1. \end{aligned}$$

Then restriction to $G \times D \subset D \times Z \times D$ and projection along Z give that the complex $\mathcal{K} = p_{13*}(p_{23}^* \mathcal{O}_\Gamma \otimes p_{12}^* \mathcal{O}_\Gamma)$ has only two cohomology sheaves. Namely,

$$\mathcal{H}^0(\mathcal{K}) = \mathcal{O}_\Delta; \quad \mathcal{H}^{-1}(\mathcal{K}) = \mathcal{O}_\Delta \otimes i_* \mathcal{O}_D(-D),$$

where $i : D \to D \times D$ is the diagonal map.

Therefore, one has the distinguished triangle

$$\mathcal{O}_\Delta \otimes i_* \mathcal{O}_D(-D)[1] \to \mathcal{K} \to \mathcal{O}_\Delta.$$

Apply them as Fourier-Mukai functors on$A$, we get the proof. □

Note that the induced second morphism is the canonical unit morphism.

Theorem 11.12. *Assumed as the setting of this sub-section, then we have semi-orthogonal decomposition:*

$$D^b(\tilde{X}) = \langle \Phi_{c-1} D^b(Z), \cdots, \Phi_1 D^b(Z), f^* D^b(X) \rangle.$$

Proof. Firstly, let us prove that the functors Φ_i is fully faithful for each $i \in \mathbb{Z}$. Suppose we have $F, G \in D^b(Z)$, then we have

$$\begin{aligned} &Hom(j_*(g^* F \otimes \mathcal{O}_E(tE)), j_*(g^* G \otimes \mathcal{O}_E(tE))) \\ &= Hom(j^* j_*(g^* F \otimes \mathcal{O}_E(tE)), g^* G \otimes \mathcal{O}_E(tE)) \end{aligned}$$

By last lemma, we know that it suffices to show that

$$Hom(g^* F \otimes \mathcal{O}_E((t-1)E)[i], g^* G \otimes \mathcal{O}_E(tE)) = 0$$

for $i \in \mathbb{Z}$. This is because

$$\begin{aligned}
&Hom(g^*F \otimes \mathcal{O}_E((t-1)E)[i], g^*G \otimes \mathcal{O}_E(tE)) \\
&= Hom(g^*F, g^*G \otimes \mathcal{O}_E(E)[-i]) \\
&= Hom(F, g_*(g^*G \otimes \mathcal{O}_E(E))[-i]) \\
&= Hom(F, G \otimes g_*\mathcal{O}_E(E)[-i]) \\
&= 0
\end{aligned}$$

Because $g_*\mathcal{O}_E(tE) = 0$ for $0 < t < c$ by Grauert theorem. Similarly, for any $0 < j - i < c$ we have

$$\begin{aligned}
&Hom(g^*F \otimes \mathcal{O}_E(iE), g^*G \otimes \mathcal{O}_E(jE)) \\
&= Hom(g^*F, g^*G \otimes \mathcal{O}_E((j-i)E)) \\
&= Hom(F, g_*(g^*G \otimes \mathcal{O}_E((j-i)E))) \\
&= Hom(F, G \otimes g_*\mathcal{O}_E((j-i)E)) \\
&= 0
\end{aligned}$$

Suppose we have $E \in D^b(X)$ and $F \in D^b(Z)$. Then we have

$$\begin{aligned}
&Hom(f^*E, j_*(g^*F \otimes \mathcal{O}_E(tE)) \\
&= Hom(E, f_*j_*(g^*F \otimes \mathcal{O}_E(tE))) \\
&= Hom(E, i_*g_*(g^*F \otimes \mathcal{O}_E(tE))) \\
&= Hom(E, i_*(F \otimes g_*\mathcal{O}_E(tE))) \\
&= 0
\end{aligned}$$

by the same reason. Hence we proved the orthogonality of the decomposition. For the fullness, we only need to prove that that they can generated points. This is not hard, if $x \notin E$, then it is in $f^*D^b(X)$. If $x \in E$, then by Theorem 11.10, it can be generated by these $c-1$ admissible subcategories, similarly to case of blow up of $\mathbb{P}^2$. □

Remark 11.13. Note that although Φ_i is fully faithful. It is wrong to say that Φ_i is the composition of three fully faithful functors j_*, $\otimes\mathcal{O}_E(tE)$, and g^*. Indeed, the last two functors are fully faithful, and the underived version R^0f_* of a closed embedding is fully faithful in category of coherent sheaves. It is easy to see that the derived version of push-forward is no longer fully faithful since it involves the higher direct images.

Similarly, pullback of open immersions is fully faithful in abelian category of coherent sheaves. The derived version is no longer true.

12. Untilted

12.1. **Algebraic side of full exceptional collections.** In this subsection, we state the result without proof relating a strong exceptional collection with the category of representations of a quiver variety.

Definition 12.1. A full exceptional collection

$$\langle E_1, \cdots, E_n \rangle$$

is called strong if $Hom^n(E_i, E_j) = 0$ for any $i \le j$ and $n \ne 0$.

Definition 12.2. A quiver Q consists of a set of vertices V, and a set of arrows A. We have two maps $h, t : A \to V$. There is a natural algebra called path algebra, denoted by $k[Q]$, consisting of linear span of all paths (including the path of length 0, i.e. the vertices). The multiplication is given by concatenation. Indeed,

$$p_1 \cdot p_2 = \begin{cases} \text{combined path when it is poosible,} \\ 0, \ otherwise. \end{cases}.$$

If we have an ideal $I \subset k[Q]$, then it is called quiver with relations.

If we have a strong exceptional collection

$$E_1, \cdots, E_2,$$

then $A = Hom(\oplus E_i, \oplus E_i) = k[Q]/I$. The associated quiver is given by $V = \{1, \cdots n\}$, and arrows from i to j is given by the dimension of $Hom(E_i, E_j)$.

Example 12.3. (1) *On $D^b(\mathbb{P}^1)$, $\mathcal{O}, \mathcal{O}(1)$ is a strong exceptional collection. The associated quiver is Kronecker quiver.*
(2) *On $D^b(\mathbb{P}^2)$, $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$ is a strong exceptional collection. The associated quiver with relation $y_i x_j = y_j x_i$.*

The following theorem is due to Bondal.

Theorem 12.4. *Suppose we have*

$$\mathcal{D} = \langle E_1, \cdots, E_n \rangle$$

has a strong exceptional collection. Then its homo algebra A is a path algebra with relations. Moreover, enhanced triangulated category, is equivalent to $D^b(mod - A)$.

Remark 12.5. We will not give the proof. The theorem is not surprising, because $D^b(mod - A)$ admits a natural strong exceptional collection $P_i = e_i A$ (they all are projective objects), the ideal of span of paths staring from vertex i, then the converse direction . And the quiver is directed without loops.

12.2. **Koszul complex and Eagon-Northcott complex.** In this subsection, we will review the construction of Koszul complex and Eagon-Northcott complex.

Firstly, let us quickly review what is Koszul complex associated with a morphism $\phi : E \to R$, where R is a commutative ring, E is a finitely generated R-module, ϕ is a homomorphism. Then for each $q \in \mathbb{Z}_{\ge 0}$, we have the map

$$\phi_q : \wedge^q E \to \wedge^{q-1} E,$$

which is given by the following formula

$$e_1 \wedge \cdots \wedge e_q \mapsto \Sigma_i (-1)^{i-1} \phi(e_i) e_1 \wedge \cdots \hat{e_1} \cdots \wedge e_q.$$

It is easy to check that $\phi_{q-1}\phi_q = 0$, so we get a complex, denoted by $K^\bullet(\phi) := \{(K_q, d_q) := (\wedge^q E, \phi_q)\}$. It is called the Koszul complex of ϕ. More generally, if M is a finitely generated R-module, $K(\phi, M)$ is the tensor product $K^\bullet(\phi) \otimes M$.

An important example is the following. Let us start with an ideal I generated by $(x_1, \cdots, x_n)$ in R. Let $E = R^n$, and $\phi : E \to R$ be the map determined by the sequence $(x_1, \cdots, x_n)$. The first two terms form the start of a free resolution of R/I. The next term consists of the "trvial" relations among the generators. In general, there may be other relations as well. It is indeed a resolution when the sequence $(x_1, \cdots, x_n)$ is a regular sequence.

Exercise 12.6. *The Koszul complex, $K(x_1, \cdots, x_n)$, associated with $x_1, \cdots, x_n \in R$ can be viewed as the cone of the multiplication by x_n morphism:*

$$K(x_1, \cdots, x_n) \simeq Cone(a_n : K(x_1, \cdots, x_{n-1})) \to K(x_1, \cdots, x_{n-1})).$$

Use this to prove that the Koszul complex of a regular sequence is exact (except the last place). Hint: in the induction step, we can replace $K(x_1, \cdots, x_{n-1})$ by $R/(x_1, \cdots, x_{n-1})$.

Also, $K(x_1, \cdots, x_n)$ can be view as the total complex of $K(x_1, \cdots, x_{n-1}) \otimes K(x_n)$.

Remark 12.7. This previous exercise also illustrates that the derived tensor product of structure sheaves of two closed sub-variety which intersect transversally is a sheaf.

More generally, we have the Eagon Northcott complex associated with a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \cdot & \cdot & \cdots & \cdot \\ a_{s1} & a_{s2} & \cdots & a_{sr} \end{pmatrix}$$

where $s \leq r$, and a_{ij} are elements in the commutative ring R. The sub-determinants of A of order s will generate an ideal. This ideal will be denoted by (A).

Let K be the exterior algebra generated by $X_1, X_2 \cdots, X_r$, then the k-th row of A determines a differentiation Φ_k on K; in fact

$$\Phi_k(X_{i_1} X_{i_2} \cdots X_{i_n}) = \Sigma_{p=1}^n (-1)^{p+1} a_{k i_p} X_{i_1} \cdots \hat{X}_{i_p} \cdots X_{i_n}.$$

Furthermore, one easily verifies that

$$\Phi_k \Phi_h + \Phi_h \Phi_k = 0,$$

for $k \neq h$.

Next let $Y_1, Y_2, \cdots, Y_s$ be s new symbols and, in the polynomial ring $R[Y_1, Y_2, \cdots, Y_s]$, denote by T_n the R-module consisting of all forms of degree n. We are now ready to describe Eagon-Northcott complex denoted by R^A. It has component modules $R_0^A, R_1^A, \cdots, R_{r-s+1}^A$, where

$$R_{q+1}^A = K_{s+q} \otimes_R T_q$$

for $q = 0, 1, \cdots, r - s$, and $R_0^A = R$.

Now, we explain how does the differentiation homomorphism d work. Note that if $q > 0$, then R^A_{q+1} has an R-base consisting of the elements

$$X_{i_1} \cdots X_{i_{s+q}} \otimes Y_1^{v_1} \cdots Y_s^{v_s} \ (1 < i_1 \leq \cdots < i_{s+q} \leq r;\ v_1 + \cdots + v_s = q).$$

When $q > 0$ the result of operating with d on a member of such a basis is given by the formula

$$d(X_{i_1} \cdots X_{i_{s+q}} \otimes Y_1^{v_1} \cdots Y_s^{v_s}) = \Sigma_j^* \Phi_j(X_{i_1} \cdots X_{i_{s+q}}) \otimes Y_1^{v_1} \cdots Y_j^{v_j - 1} \cdots Y_s^{v_s},$$

where the star attached to the summation sign means that we sum only over those values of j which $v_j > 0$. For $q = 0$, we assign the formula

$$d(X_{i_1} \cdots X_{i_s} \otimes 1) = det \begin{pmatrix} a_{1i_1} & a_{1i_2} & \cdots & a_{1i_s} \\ a_{2i_1} & a_{2i_2} & \cdots & a_{2i_s} \\ \cdot & \cdot & \cdots & \cdot \\ a_{si_1} & a_{si_2} & \cdots & a_{si_s} \end{pmatrix}$$

It is not hard to show that $d^2 = 0$. We leave this verification to the readers.

In the special case when $s = 1$, that is when the matrix consists of a single row, and it is easy to see that the complex is essentially the Koszul complex.

We list some basic properties of the complex R^A without the proofs, since it is not the purpose of this lecture notes.

Proposition 12.8. *Assume that E is a finitely generated R-module, we can define $E^A = R^A \otimes_R E$. If (A) annihilates a non-zero element of E, then $H_{s-r+1}(E^A) \neq 0$.*

R^A has an intimate connexion with similar complexes determined by certain submatrices of A. We will assume that $1 < s < r$. This is in order that the appropriate submatrices shall exist and have the right kind of shape.

Denote by L the matrix which results when last column of A is deleted, and by M the matrix obtained when both the last column and the first row are removed.

Thus

$$L = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,r-1} \\ a_{21} & a_{22} & \cdots & a_{2,r-1} \\ \cdot & \cdot & \cdots & \cdot \\ a_{s1} & a_{s2} & \cdots & a_{s,r-1} \end{pmatrix}$$

and

$$M == \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2,r-1} \\ a_{31} & a_{32} & \cdots & a_{3,r-1} \\ \cdot & \cdot & \cdots & \cdot \\ a_{s1} & a_{s2} & \cdots & a_{s,r-1} \end{pmatrix}$$

It is clear that R^L can be regarded as a subcomplex R^A and, indeed, R^L_q is a direct summand of R^A_q. To describe the connexion with R^M, we introduce certain R homomorphisms

$$\mu_q : R^L_q \to R^M_q,\ (q = 1, 2, \cdots, r - s + 1)$$

Since $R^L_{r-s+1} = 0$, it will suffice if we describe the mappings

$$\mu_{q+1} : R^L_{q+1} \to R^M_{q+1},\ (q = 0, \cdots, r-s-1).$$

To this end we recall that R^L_{q+1} has an R base consisting of all products

$$X_{i_1} \cdots X_{i_{s+q}} \otimes Y_1^{v_1} \cdots Y_s^{v_s}\ (1 < i_1 \leq \cdots < i_{s+q} \leq r-1;\ v_1 + \cdots + v_s = q).$$

It will be convenient to describe the corresponding base for R^M_{q+1} by

$$U_{j_1} U_{j_2} \cdots U_{j_{s+q-1}} \otimes V_2^{n_2} V_3^{n_3} \cdots V_s^{n_s},$$

where now $1 \leq j_1 < j_2 < \cdots < j_{s+q-1} \leq r-1$ and $n_2 + \cdots n_s = q$. With this notation, μ_{q+1} $(q = 0, 1, \cdots, r-s-1)$ is obtained by putting

$$\mu_{q+1}(X_{i_1} \cdots X_{i_{s+q}} \otimes Y_1^{v_1} \cdots Y_s^{v_s}) = \Sigma_{p=1}^{s+q} (-1)^{p+1} a_{1 i_p} U_{i_1} \cdots \hat{U_{i_p}} \cdots U_{j_{s+q}} \otimes V_1^{n_1} V_2^{n_2} \cdots V_s^{n_s},$$

where it is understood that the right hand side is zero if $v_1 \geq 1$. The special property of these mappings is that

$$d_{q+1}\mu_{q+1} + \mu_q d_{q+1=0}\ (q = 1, 2, \cdots, r-s).$$

Once again, the verification is straightforward computation left to the reader.

Hence $(-1)^{q+1}\mu_{q+1}$ defines a morphism of complex, then we denote by $R^{L,M}$ the cone of this morphism.

Lemma 12.9. *There exists a mapping $R^A \to R^{L,M}$ of complexes in such a way that*

$$0 \to R^L \to R^A \to R^{L,M} \to 0$$

an exact sequence of complexes. Moreover, each of the exact sequences

$$0 \to R^L_q \to R^A_q \to R^{L,M}_q \to 0$$

splits.

As a consequence of this lemma, there exists a morphism $R^{L,M}[-1] \to R^L$, such that R^A is quasi-isomorphic to the cone of this morphism.

Proof. We only need to construct the morphism $\phi : R^A \to R^{L,M}$ satisfying the result. We will describe the action of ϕ_{q+1} on the individual generators of R^A_{q+1}. Assume that $0 \leq q \leq r-s$. Since ϕ_{q+1} is to vanish on R^L_{q+1}, it is only necessary to show how generators of the form

$$X_{i_1} \cdots X_{i_{s+q-1}} X_r \otimes Y_1^{v_1} \cdots Y_s^{v_s}\ (1 \leq i_1 < \cdots < i_{s+q-1} \leq r-1;\ v_1 + \cdots v_s = q)$$

are to mapped.

Note that $R^{L,M}_{q+1}$ is a free module with a base consisting of all products

$$U_{j_1} U_{j_2} \cdots U_{j_{s+q-1}} \otimes V_2^{n_2} V_3^{n_3} \cdots V_s^{n_s},$$

where now $1 \leq j_1 < j_2 < \cdots < j_{s+q-1} \leq r-1$ and $n_2 + \cdots n_s = q$, together with the products

$$X_{i_1} \cdots X_{i_{s+q-1}} \otimes Y_1^{v_1} \cdots Y_s^{v_s}\ (1 < i_1 \leq \cdots < i_{s+q-1} \leq r-1;\ v_1 + \cdots + v_s = q-1)$$

it being understood that the second set is empty when $q = 0$. We now define ϕ_{q+1} by putting

$$\phi_{q+1}(X_{i_1} \cdots X_{i_{s+q-1}} X_r \otimes Y_1^{v_1} \cdots Y_s^{v_s}) = U_{i_1} \cdots U_{i+s+q-1} \otimes V_1^{v_1} \cdots V_s^{v_s}$$

if $v_1 = 0$, while if $v_1 > 0$ we put

$$\phi_{q+1}(X_{i_1} \cdots X_{i_{s+q-1}} X_r \otimes Y_1^{v_1} \cdots Y_s^{v_s}) = X_{i_1} \cdots X_{i_{s+q-1}} \otimes Y_1^{v_1 - 1} \cdots Y_s^{v_s}.$$

There might be a change of signs. Then straightforward computation show that they consists a morphism ϕ satisfying the required properties.

□

Remark 12.10. In general, the complex R^A may have nontrivial higher homology groups, and these groups are annihilated by some power of (A) (i.e. $(A)^h H(E^A) = 0$ for some h, and $h = 1$ for Koszul complex, see [EN62, Proposition 2] for a proof). This lemma is a generalization of Exercise 12.6.

Similar to the Koszul complex, Eagon-Northcott complex is a free resolution of $R/(A)$ under certain good conditions.

Suppose that R is noetherian, let E be a finitely generated R-module and I an ideal such that $IE \neq E$. A sequence $u_1, u_2 \cdots, u_p$ of elements in I is said to be an M-regular sequence on E contained in I if for each $1 \leq i \leq p$, u_i is not a zero-divisor in $E/(u_1, \cdots, u_{i-1})E$. Such a sequence is said to be maximal if every element of I is a zero-divisor in $E/(u_1, \cdots, u_{i-1})E$. The length of a maximal regular sequence is well defined (independent of the regular sequence) by [AB58]. And it is denoted by $depth_R(I, E)$, the depth of I on E. Then we have the following theorem.

Theorem 12.11. *Let R be a Noetherian ring and E a finitely generated R-module. Further, let (A) be an ideal defined by an $s \times r$ matrix ($s \leq r$) and assume that $(A)E \neq E$. If now q is the largest value of m such that $H_m(E^A) \neq 0$, then we have*

$$depth((A), E) + q = r - s + 1$$

and, in particular $depth((A), E) \leq r - s + 1$.

Proof. See [EN62, Thereom 1]. □

As a corollary, we have the following result.

Corollary 12.12. *Let R be a Noetherian ring and E a finitely generated R-module. Further, let (A) be an ideal defined by an $s \times r$ matrix ($s \leq r$) and assume that $(A)E \neq E$. If now $depth((A), E) = r - s + 1$. Then E^A is a free resolution of $E/(A)E$.*

12.3. **In the geometric context.** Given a vector bundle E of rank r on X and a section $\sigma \in H^0(X, E^\vee)$, the construction of Koszul complex provide us a complex of vector bundles

$$K(\sigma) : \wedge^r E \to \cdots \to \wedge^p E \xrightarrow{i_\sigma} \wedge^{p-1} E \to \cdots \to \mathcal{O}_X.$$

If σ is a regular section, i.e. the rank of E coincides with the codimension of the zero locus $Z = V(\sigma)$, this complex provides a resolution of the ideal sheaf I_Z of Z. More generally, given a line bundle L and a homomorphism of vector bundles $\sigma : E \to L$, we obtain a complex of vector bundles

$$K(\sigma) : \cdots \wedge^p E \otimes L^q \xrightarrow{\delta} \wedge^{p-1} E \otimes L^{q+1} \to \cdots$$

where δ is defined as the composition

$$\wedge^p E \otimes L^q \xrightarrow{\iota \times id} \wedge^{p-1} E \otimes E \otimes L^q \xrightarrow{id \otimes (\mu \circ \sigma)} \wedge^{p-1} E \otimes L^{q+1}.$$

Locally, we should think

$$\iota(e_1 \wedge \cdots \wedge e_p) = \Sigma_{i=1}^p (-1)^i e_1 \wedge \cdots \hat{e_i} \wedge \cdots \wedge e_p \otimes e_i.$$

Given a line bundle L, put $V = H^0(X, L)$. Applying the above construction to the evaluation map $ev : V \otimes \mathcal{O}_X \to L$, we obtain a Koszul complex of vector bundles

$$K(X, L) : \cdots \wedge^p V \otimes L^q \xrightarrow{\delta} \wedge^{p-1} V \otimes L^{q+1} \to \cdots .$$

If we apply the global section functor and take the cohomology group, we get something called the Koszul group $K_{p,q}(X, L)$, which was introduced by M.Green in his papers [Gre84a] and [Gre84b]. It became a relatively important tool in algebraic geometry in last several decades. Unfortunately, we will not talk about it in this lecture notes.

Example 12.13. *(1) Let V be a k-vector space of dimension $r + 1$. Applying the above construction to the line bundle $\mathcal{O}_{\mathbb{P}}(1)$ on $\mathbb{P}(V^\vee)$, we obtain an exact complex of locally free sheaves*

$$0 \to \wedge^{r+1} V \otimes \mathcal{O}_{\mathbb{P}}(-r-1) \to \cdots \to V \otimes \mathcal{O}_{\mathbb{P}}(-1) \to \mathcal{O}_{\mathbb{P}} \to 0.$$

If we take r-dimensional subspace $W := k\{x_0, x_1, \cdots, x_{r-1}\} \subset V$, then applying the construction, we get

$$0 \to \wedge^r W \otimes \mathcal{O}_{\mathbb{P}}(-r) \to \cdots \to W \otimes \mathcal{O}_{\mathbb{P}}(-1) \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_x \to 0$$

where x is point $x_0 = \cdots = x_{r-1} = 0$.

These two exact sequences can also be seen as taking $E = V \otimes \mathcal{O}(-1)$ ($W \otimes \mathcal{O}(-1)$), $\sigma = (x_0, \cdots, x_r)$ ($\sigma = (x_0, \cdots, x_{r-1})$) respectively.

Hence, the exceptional collection $\langle \mathcal{O}, \cdots, \mathcal{O}(r) \rangle$ generates $\mathcal{O}(r + 1)$ and $\mathcal{O}_x$.

(2) Another example used by Beilinson to construct his decomposition of $D^b(\mathbb{P}^n)$ is the following lemma.

Lemma 12.14. *There exists a natural locally free resolution of $\mathcal{O}_\Delta$, the structure sheaf of diagonal in $\mathbb{P}^n \times \mathbb{P}^n$, of the form*

$$0 \to \wedge^n (\mathcal{O}(-1) \boxtimes \Omega(1)) \to \cdots \to \mathcal{O}(-1) \boxtimes \Omega(1) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_\Delta \to 0.$$

Proof. Let us write $\mathbb{P}^n$ as $\mathbb{P}(V)$. The Euler sequence sequence can the be written as

$$0 \to \Omega(1) \to V^* \otimes \mathcal{O} \to \mathcal{O}(1) \to 0.$$

Recall that the fiber of $\mathcal{O}(-1)$ at a point $l \in \mathbb{P}(V)$ is by definition identified with the line $l \subset V$. Also, the fiber of $\Omega(1)$ on l is the subspace of those linear maps $\phi : V \to k$ that are trivial on $l \subset V$ (we should think V as the tangent space, hence at the point l, the direction along l should vanish in the tangent space of $\mathbb{P}(V)$). Thus the Euler sequence at a point $l \in \mathbb{P}(V)$ is

$$0 \to l^{\perp} \to V^* \to l^* \to 0.$$

The homomorphism

$$\mathcal{O}(-1) \boxtimes \Omega(1) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}$$

at a point $(l, l') \in \mathbb{P}(V) \times \mathbb{P}(V)$ is by definition given by $(x, \phi) \mapsto \phi(x)$, where $x \in l$ and $\phi|_{l'} = 0$. Clearly, the image of this map is the ideal sheaf of the diagonal $\Delta \subset \mathbb{P}(V) \times \mathbb{P}(V)$.

Then we apply the standard Koszul complex construction, we get the exact sequence, since the rank equals the codimension. $\square$

Note that the structure sheaf of the diagonal $\Delta \subset X \times X$ for more complicated varieties (e.g. elliptic curve, K3 surface) can not be resolved by sheaves of the form $\mathcal{F} \boxtimes \mathcal{G}$.

(3) Eagon-Northcott complex can be used in Grassmannians. The following example is an explicit calculation in [BC09]. *Let V be a 7 dimensional vector space, and denote $G(2, V)$ by G the Grassmannian of planes in V. Let $K \subset V$ be a 3 dimensional subspace in V. We assume that in a basis $e_1, \cdots e_7$ of V, K is spanned by e_5, e_6, e_7. Let $x_1, \cdots, x_7$ be the dual basis of V^*. The subspace $AnnK \subset V^*$, is spanned by $x_1, \cdots, x_4$, and thus $\wedge^2 AnnK$ has basis*

$$x_1 \wedge x_2, \cdots, x_3 \wedge x_4.$$

We define $S \subset G$ to be the locus of two planes $T \subset V$ which intersect K non-trivially. Then we have the following proposition.

Proposition 12.15. *Denote by T the rank two tautological bundle on G, and let $A = AnnK$. Then there exists a resolution of $\mathcal{O}_S$ on G of the form:*

$$0 \to \wedge^4 A \otimes Sym^2(T)(-1) \to \wedge^3 A \otimes T(-1) \to \wedge^2 A \otimes \mathcal{O}_G(-1) \to \mathcal{O}_G \to \mathcal{O}_S \to 0.$$

Proof. In the Zariski open subspace $U \simeq \mathbb{C}^{10} \subset G$ in which points T are given by

$$T = Span(a_{11}e_1 + \cdots + a_{15}e_5 + e_6, a_{21}e_1 + \cdots + a_{25}e_5 + e_7),$$

the cycle S is characterized by the condition that the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

has rank one. Because S is cut out by $\wedge^2 AnnK$ (as if $T \cap K \neq 0$, then the image $\bar{T}$ of T in V/K has dimension at most one, hence $\wedge^2 \bar{T} = 0$ in $\wedge^2 V/K$). This complex is exact follows from the fact that S is Cohen-Macaulay. For a proof of this fact, please see [BC09]. $\square$

13. The classification of vector bundles on $\mathbb{P}^1$ and elliptic curve

In this section, we will talk about the classification of vector bundles on $\mathbb{P}^1$ and elliptic curves.

The case of $\mathbb{P}^1$ is very simple, it is Grothendieck theorem (it is generally attributed to Grothendieck because he proved it in modern language).

Theorem 13.1. *Let E be a vector bundle on $\mathbb{P}^1$. Then there exists unique integers $a_1, \cdots, a_n \in \mathbb{Z}$ satisfying $a_1 > a_2 > \cdots a_n$ and unique nonzero vector spaces $V_1, \cdots, V_n$ such that E is isomorphic to*

$$E \simeq (\mathcal{O}(a_1) \otimes V_1) \oplus \cdots \oplus (\mathcal{O}(a_n) \otimes V_n).$$

Proof. For the existence of the decomposition we proceed by induction on the rank $r(E)$. If $r(E) = 1$, there is nothing to prove. Hence, we assume $r(E) > 1$. By Serre duality and Serre vanishing, for $a \gg 0$ we have

$$Hom(\mathcal{O}(a), E) = H^1(\mathbb{P}^1, E^\vee(a-2))^* = 0.$$

We pick the largest integer $a \in \mathbb{Z}$ (the existence of such a line bundle is given by Serre's globally generation theorem) such that $Hom(\mathcal{O}(a), E) \neq 0$ and a nonzero morphism $\phi \in Hom(\mathcal{O}(a), E)$. Since $\mathcal{O}(a)$ is torsion free of rank 1, ϕ is injective. Consider the exact sequence

$$0 \to \mathcal{O}(a) \xrightarrow{\phi} E \to F = coker(\phi) \to 0.$$

We claim that F is a vector bundle of rank $r(E) - 1$. Indeed, if it is not the case, there is a torsion sub-sheaf $T_F \hookrightarrow F$ supported at dimension 0. In particular, we have a morphism from the sky scraper sheaf $k(x) \to F$. But this gives a nonzero map $\mathcal{O}(a+1) \to E$, contradicting the maximality of a.

By induction F splits as a direct sum

$$F \simeq \oplus \mathcal{O}(b_j)^{\oplus r_j}.$$

The second claim if that $b_j \leq a$ for all j. If not, there is a nonzero morphism $\mathcal{O}(a+1) \hookrightarrow F$. Since $Ext^1(\mathcal{O}(a+1), \mathcal{O}(a)) = H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$, this morphism lifts to a nonzero map $\mathcal{O}(a+1) \hookrightarrow E$, contradicting again the maximality of a. Finally, we obtain a decomposition of E as a direct sum of line bundles since

$$Ext^1(\oplus_j \mathcal{O}(b_j)^{\oplus r_j}, \mathcal{O}(a)) = \oplus_j H^1(\mathbb{P}^1, \mathcal{O}(a - b_j))^{\oplus r_j} = 0.$$

□

Before starting to classify the vector bundles on elliptic curves, we need to introduce the notion of (semi)stable vector bundles on curves and HN filtration.

Definition 13.2. Let C be a smooth projective curve over an algebraically closed field k. We call a vector bundle E on C (semi-)stable if for any sub-coherent sheaf $F \subset E$, we have that

$$\mu(F)(\leq) < \mu(E),$$

where

$$\mu(E) := \begin{cases} \frac{deg(E)}{rk(E)} & \text{if rk(E) is positive,} \\ +\infty & \text{if rk(E)=0, i.e. } E \text{ is torsion.} \end{cases}$$

Exercise 13.3. *Prove the following statements: (1) If $E \in Coh(C)$ is semi-stable, then it is either a vector bundle or a torsion sheaf.*

(2) A vector bundle E on C is (semi)stable if and only if for all non-trivial subbundles $F \subset E$ with $r(F) < r(E)$ the inequality $\mu(F) < (\leq)\mu(E)$ holds.

(3) Prove the see-saw principle in the curve case, i.e. if we have the following short exact sequence of nontrivial objects

$$0 \to K \to E \to Q \to 0.$$

Then $\mu(K) < \mu(E)$ is equivalent to $\mu(E) < \mu(Q)$.

(4) If E and F are (semi)stable vector bundles, and $\mu(E) > \mu(F)$, then $Hom(E, F) = 0$.

(5) The category of semi-stable sheaves with fixed slope is an Abelian category.

We recall the following basic fact.

Theorem 13.4. *Let E be a nonzero coherent sheaf on C. Then there is a unique filtration (called Harder-Narasimhan filtration)*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

of coherent sheaves such that $A_i = E_i/E_{i-1}$ is semi-stable for all $i = 1, \cdots n$ and

$$\mu(A_1) > \mu(A_2) > \cdots > \mu(A_n).$$

Proof. We will skip the details of the proof since the students do not choose Bridgeland stability conditions as our final subject. Instead, we will draw a nice and cute picture in the class and indicate the HN-polygon. For the details of the proof, see [HN75] and [MS17]. □

Now, we follow the path in [HP05] to recover the classical result of Atiyah in [Ati57] by applying the Fourier-Mukai transform.

Let E be an elliptic curve with fixed base-point $p_0 \in E$ over an algebraically closed field of characteristic 0 (we view p_0 as the identity point in the group structure of E). Then we can identify E with its Jacobian of degree 0 line bundle $\hat{E} := Pic^0(E)$ via $E \to \hat{E}$, $x \mapsto \mathcal{O}_E(x - p_0)$. We will write $t_a : E \to E$, $p \mapsto p + a$ for the group law on E in order to distinguish between addition of points and of divisors. The choice of p_0 also allows defining the normalized Poincaré line bundle on $E \times E$ by $\mathcal{P} := \mathcal{O}_{E\times E}(\Delta - E \times p_0 - p_0 \times E)$, i.e. $\mathcal{P}|_{E\times\{x\}} \simeq \mathcal{O}_E(x - p_0)$ and $\mathcal{P}|_{\{p_0\}\times E} \simeq \mathcal{O}_E$.

Then the Poincaré bundle $\mathcal{P}$ defines a functor as follows

$$\Phi_{\mathcal{P}} : D(E) \to D(E)$$

.

Mukai showed in [Muk81] that Poincaré bundles actually give equivalences on all Abelian varieties. He also proved an involution property valid for principle polarized Abelian varieties which in our case reads as

$$\Phi_{\mathcal{P}} \circ \Phi_{\mathcal{P}} = (-1)^*[-1].$$

Indeed, the composition $\Phi_{\mathcal{P}} \circ \Phi_{\mathcal{P}}$ has a kernal

$$p_{13*}(p_{12}^*\mathcal{P} \otimes p_{23}^*\mathcal{P}) =: p_{13*}K.$$

Using cohomology and base change together with the definition of $\mathcal{P}$, we see $R^1p_{13*}K \otimes k(a,b) = H^1(E, \mathcal{O}_E(a+b-2p_0))$. This already shows that $R^1p_{13*}K$ is a line bundle supported on $\Delta' := \{(x,-x) : x \in E\}$ because $H^1(E, \mathcal{O}_E(a+b-2p_0)) \neq 0 \iff \mathcal{O}_E(a+b-2p_0)$ is trivial, which is equivalent to $a = -b$ in the group structure of E. Furthermore, $R^0p_{13*}K = 0$, and hence $p_{13*}K = R^1p_{13*}K[-1]$ is concentrated in degree 1. Finally notice that $\phi_{\mathcal{P}}(\mathcal{O}_E) = k(p_0)[-1]$ and $\Phi_{\mathcal{P}}(k(p_0)) = \mathcal{O}_E$ and so the line bundle on Δ' mentioned above is trivial. Altogether we obtain $\phi_{\mathcal{P}}^2 = (-1)^*[-1]$. This also proves that $\Phi_{\mathcal{P}}$ is an equivalence.

The classical version of the transform defined above is the ring endomorphism of the even cohomology ring

$$\Phi_{ch(\mathcal{P})} : H^0(E) \oplus H^2(E) \to H^0(E) \oplus H^2(E),\ \alpha \mapsto p_{2*}(ch(\mathcal{P}) \cdot p_1^*(\alpha)),$$

A basic fact from calculation of Grothendieck Riemann-Roch: Any choice of kernal G in $D^b(E \times E)$ gives us a commutative diagram

$$\begin{tikzcd}
D(E) \arrow[r, "ch"] \arrow[d, "\Phi_G"] & H^{2*}(E) \arrow[d, "\Phi_{ch(G)}"] \\
D(E) \arrow[r, "ch"] & H^{2*}(E)
\end{tikzcd}$$

This essentially comes from the triviality of Todd class of elliptic curves. The Chern character of the Pincaré bundle in $H^*(E \times E)$ is readily read off from the definition as

$$ch(\mathcal{P}) = 1 + [\Delta] - [E \times pt] - [pt \times E] - [pt \times pt]$$

(using $N_{\Delta/E\times E} = \mathcal{O}_E$ for $[\Delta]^2 = deg(c_1(N_{\Delta/E\times E})) = 0$) and hence

$$\Phi_{ch(\mathcal{P})}(r[E]+d[pt]) = p_{2*}(r[E\times E]+r[\Delta]-r[E\times pt]-r[pt\times E]-[pt\times pt]+d[pt\times E]) = d[E]-r[pt]$$

Therefore, we can write $\Phi_{ch(\mathcal{P})}$ in the following way

$$\Phi_{ch(\mathcal{P})} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : H^{2*}(E) \to H^{2*}(E).$$

Now, we can start the classification.

Lemma 13.5. *Let F be a locally free sheaf of rank r and degree d. Then we have the implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) with*
(i) F is stable, (ii) F is simple,
(iii) F is indecomposable,
(iv) F is semi-stable.
If moreover r and d are co-prime, then we also have (iv) $\Longrightarrow$ (i), so that all four properties are equivalent.

Recall the definition simple means $Hom(F, F) = k$, or equivalently, all its endomorphisms are either trivial or isomorphism. Indecomposable means that if $F \simeq G \oplus H$, then either G or H is zero object.

Proof. The implications $(i) \Longrightarrow (ii) \Longrightarrow (iii)$ are valid for arbitrary varieties and sheaves. For any smooth projective curves, $(iv) \Longrightarrow (i)$ is also always true. So we only need to prove that $(iii) \Longrightarrow (iv)$, which is a special property on Elliptic curves.

So assume that F is decomposable, take the first HN factor F_1 (it is also called the maximal destabilizing object of F in the literature). This would lead to an exact sequence

$$0 \to F_1 \to F \to F_2 \to 0$$

with $Hom(F_1, F_2) = 0$ because the quotient F_2 can be filtered by semi-stable bundles with slope smaller that $\mu(F_1)$. But from Serre duality we infer

$$Ext^1(F_2, F_1) = Hom(F_1, F_2) = 0$$

. Since F is indecomposable we finally get $F_2 = 0$ and $F = F_1$ is indeed semi-stable. □

Lemma 13.6. *Let $r > 0$ and d be integers and L a line bundle of degree d. A stable vector bundle on E with rank r and degree d exists $\Longleftrightarrow$ $(r, d) = 1$.*

Proof. Fix integers r and d with $(r, d) = 1$. Remember that we have chosen an origin p_0 on E. There is another elliptic curve $\tilde{E}$ together with a morphism $\pi_r : \tilde{E} \to E$ such that $E = \tilde{E}/G$ is a finite quotient of order r, and $G \simeq \mathbb{Z}/(r)$ acts without fixed points on $\tilde{E}$. This can be done by taking the unramified covering of E given by a subgroup in $\pi_1(E) = \mathbb{Z}^2$ of index r. The fiber $\pi_r^{-1}(p_0)$ consists of r points, among which we choose a base point $\tilde{p}_0$ for $\tilde{E}$. After that, we can also chose a generator $\tilde{g}$ of $\pi_r^{-1}(p_0)$ (considered as a subgroup of $\tilde{E}$).

Now take a line bundle $\tilde{L}$ on $\tilde{E}$ od degree d, e.g. $\mathcal{O}_{\tilde{E}}(dp_0)$. The projection $\pi_r : \tilde{E} \to E$ is a finite, unramified morphism, and thus $V := \pi_{r*}\tilde{L}$ is a sheaf locally free of rank r and degree d (since π_{r*} is exact, we have $\chi(E, V) = \chi(\tilde{E}, \tilde{L})$). It is simple because

$$\begin{aligned} Hom_E(V, V) &= Hom_{\tilde{E}}(\pi_{r*}\tilde{L}, \pi_{r*}\tilde{L}) = Hom_{\tilde{E}}(\pi_r^*\pi_{r*}\tilde{L}, \tilde{L}) \\ &= Hom_{\tilde{E}}(\oplus_{g\in G} g^*\tilde{L}, \tilde{L}) = \oplus_{g\in G} H^0(\tilde{E}, \tilde{L} \otimes g^*\tilde{L}^\vee) = k \end{aligned}$$

using that only $\mathcal{O}_{\tilde{E}}$ has nontrivial sections among line bundles of degree 0. The third equality follows from the base change of following Cartesian diagram

$$\begin{tikzcd} \tilde{E} \times G \arrow[r, "\rho"] \arrow[d, "pr"] & \tilde{E} \arrow[d, "\pi_r"] \\ \tilde{E} \arrow[r, "\pi_r"] & E, \end{tikzcd}$$

where ρ is the map of group action, pr is the projection. By previous lemma, V is also stable.

We will prove the other direction by using Fourier-Mukai transform in the second half of this section.

□

Remark 13.7. The assertions of the lemma can be rephrased using the moduli space $\mathcal{M}(r,d)$ of stable vector bundles of rank r and degree d:

$\mathcal{M}(r,d)$ is not empty $\iff (r,d)=1$,

13.1. **Universal bundles.** The following Proposition is well known, I think there is a gap in the proof in the paper [HP05].

Proposition 13.8. *Given co-primes integers r and d, there is a universal bundle $\mathcal{G}$ on $E\times E$ parametrizing stable vector bundles of rank r and degree d, i.e. $\Phi_{\mathcal{G}}: D(E)\to D(E)$ is an equivalence such that all $\Phi_{\mathcal{G}}(k(p))$ are stable of rank r and degree d.*

Proof. The above construction of stable vector bundles can also be described in terms of Fourier-Mukai transforms. Consider the graph $\Gamma\subset\tilde{E}\times E$ of π_r and its structure sheaf $\mathcal{O}_\Gamma\in D(\tilde{E}\times E)$ as a kernal. Then we have $\pi_{r*}=\Phi_{\mathcal{O}_\Gamma}$. Furthermore, consider next the Poincaré bundle $\tilde{\mathcal{P}}^d$ of degree d line bundles on $\tilde{E}$. We will also assume that $\tilde{\mathcal{P}}^d$ is normalized by requiring it to be symmetric. Then the composition $\Phi_{\mathcal{O}_\Gamma}\circ\Phi_{\tilde{\mathcal{P}}^d}: D(\tilde{E})\to D(E)$ takes the sky scraper sheaf to stable bundles with correct rank and degree. However, this map is over-parametrized (hence the composite kernal is not universal bundle): two points $\tilde{x}$ and $\tilde{y}$ lead to the same bundle if they are in the same π_r-fiber.

Thus, it is necessary to divide out the G-action. This is possible if and only if the composite kernal $\mathcal{K}\in D(\tilde{E}\times E)$ descends. This in turn means that there is a $\mathcal{G}\in D(E\times E)$ such that $\mathcal{K}=(\pi_r\times id_E)^*\mathcal{G}$. A necessary and sufficient condition for this is the existence of a G-linearization on $\mathcal{K}$.

Note that the generator $\tilde{g}$ of $G\simeq\mathbb{Z}/(r)$ acts on $\tilde{E}\times E$ by translation $\tilde{g}$ on the first factor and trivially on the second. We write $t:=t_{\tilde{g},p_0}$ for this translation. A G linearization is a set of isomorphisms $\lambda_g: g^*\mathcal{K}\xrightarrow{\sim}\mathcal{K}$ satisfying the obvious compatibility. Because G is acyclic, it is sufficient to consider only for the generator. Now

$$\begin{aligned}
&t^*\mathcal{K}\simeq\mathcal{K}\\
\iff&\Phi_{t^*\mathcal{K}}\simeq\Phi_{\mathcal{K}}\\
\iff&\pi_{r*}\circ\Phi_{t^*\tilde{\mathcal{P}}^d}\simeq\pi_{r*}\circ\Phi_{\tilde{\mathcal{P}}^d}\\
\iff&\pi_{r*}\circ t_{\tilde{g}}^*\circ\Phi_{\tilde{\mathcal{P}}^d}\circ\Phi_{\tilde{\mathcal{P}}^d}^{-1}\simeq\pi_{r*}\\
\iff&\pi_{r*}\circ(t_{\tilde{g}}^{-1})_*\simeq\pi_{r*}\\
\Longleftarrow&\pi_r\circ t_{\tilde{g}}^{-1}=\pi_r
\end{aligned}$$

Note that the first equivalence is only true because $\mathcal{K}$ is a sheaf, not a complex, see [CS12] for the proof of sheaf case and counterexample in the complex case. Let me sketch the proof for the sheaf case: By [Gro61, Theorem 3.4.4], we know that there is a isomorphism between $\mathcal{K}$ and the sheaf associated to $M:=\oplus_{m\in\mathbb{Z}}p_{2*}(\mathcal{K}\otimes p_1^*\mathcal{O}_{\tilde{E}}(m))$ where p_{2*} is underived. Since for $m \gg 0$ there are functorial isomorphisms

$$p_{2*}(\mathcal{K} \otimes p_1^* \mathcal{O}_{\tilde{E}}(m)) = \Phi_{\mathcal{K}}(\mathcal{O}_{\tilde{E}}(m)),$$

so the graded module M and $N := \oplus_{m\in\mathbb{Z}} p_{2*}(\sqcup^*\mathcal{K} \otimes p_1^*\mathcal{O}_{\tilde{E}}(m))$ are isomorphic to in sufficient high degrees. Hence, taking the associated sheaf, we get a isomorphism $t^*\mathcal{K} \simeq \mathcal{K}$.

But this is not enough to say that $\mathcal{K}$ is G linearized, since we do not check the compatibility, which is essential in the descent theory.

We also have the following equations

$$\begin{aligned}\Phi_{\mathcal{K}}(k(\tilde{x})) &= p_{2*}((\pi_4 \times id_E)^*\mathcal{G} \otimes p_1^*(k(\tilde{x}))) \\ &= p_{2*}(\iota_{\tilde{x}}^*(\pi_r \times id_E)^*\mathcal{G}) \\ &= p_{2*}(\mathcal{G}|_{\{\pi_r(\tilde{x})\}\times E}) \\ &= \Phi_{\mathcal{G}}(k(\pi_r(\tilde{x})))\end{aligned}$$

where $\iota_{\tilde{x}} : \{\tilde{x}\} \times E \hookrightarrow \tilde{E} \times E$.

And this induces an equivalence of $D(E)$ will be discussed in later lectures. $\square$

Remark 13.9. Note that the existence of universal family is equivalent to the assumption $(r, d) = 1$, indeed, it is no longer true when $(r, d) > 1$. For the construction of universal bundles, see [HL10, Section 4.6].

Here we consider vector bundles of arbitrary rank r and degree d. We denote $\tilde{r} := \frac{r}{(r,d)}$ and $\tilde{d} := \frac{d}{(r,d)}$. From Proposition 13.8 we have a universal bundle for stable vector bundles of rank $\tilde{r}$ and degree $\tilde{d}$. Our aim is the following description of semi-stable sheaves on E.

Proposition 13.10. *Let $\mathcal{S}(r, d)$ be the set of all isomorphism classes of semi-stable bundles of rank r and degree d. There is an isomorphism between $\mathcal{S}(r, d)$ and the set $Torsion_{length=(r,d)}$ of torsion sheaves of length (r, d)*

$$\Phi_{\mathcal{G}} : Torsion_{length=(r,d)} \xrightarrow{\sim} \mathcal{S}(r, d).$$

Proof. Remember that $\mathcal{G}$ was the universal bundle on $E\times E$ constructed in last proposition. The Fourier-Mukai transform $\Phi_{\mathcal{G}}$ here is meant in the same direction as there, i.e., taking points to stable vector bundles.

First, take an arbitrary torsion sheaf T on E of length (r, d). Then, it is obvious that $\Phi_{\mathcal{G}}(T)$ is a locally free sheaf of rank $\tilde{r}(r, d) = r$. It has degree d because of $ch(\Phi_{\mathcal{G}(T)}) = \Phi_{ch(\mathcal{G})}(ch(T)) = (r, d)(\tilde{r}[E] + \tilde{d}[pt])$. Finally, it is semi-stable because all T can be filtered in a composition series of sky scraper sheaves, hence F is a successive extension of stable vector bundles of rank $\tilde{r}$ and degree $\tilde{d}$.

On the other hand, let F be semi-stable with rank r and degree d. We are looking for a T with $\Phi_{\mathcal{G}}(T) = F$. In order to do this, we will utilize the transform $\Phi_{\mathcal{G}^\vee}$ with the dual of the universal bundle as kernal. It can be easily seen that $\Phi_{\mathcal{G}^\vee} = \Phi_{\mathcal{G}}^{-1}[-1]$ from the fact that $\Phi_{\mathcal{G}}$ is an equivalence, hence we have $\Phi_{ch(\mathcal{G}^\vee)}(\tilde{r}[E] + \tilde{d}[pt]) = -[pt]$ and assume that $\Phi_{ch(\mathcal{G}^\vee)}([pt]) = \tilde{r}[E] + e[pt]$ follow from that $\mathcal{G}$ is locally free vector bundle of rank $\tilde{r}$. This allows us to write $\Phi_{ch(\mathcal{G}^\vee)}$ and $\Phi_{ch(\mathcal{G})}$ as matrices

$$\Phi_{ch(\mathcal{G}^\vee)} = \begin{pmatrix} -\tilde{d} & \tilde{r} \\ -\frac{1+\tilde{d}e}{\tilde{r}} & e \end{pmatrix}$$

and

$$\Phi_{ch(\mathcal{G}))} = \begin{pmatrix} -e & \tilde{r} \\ -\frac{1+\tilde{d}e}{\tilde{r}} & \tilde{d} \end{pmatrix}$$

Now by Serre's theorem, we have a two-term resolution $0 \to A^{-1} \to A^0 \to F \to 0$ such that $R^0 p_{2*}(\mathcal{G}^\vee \otimes p_1^* A^i) = 0$ for both $i = 0, -1$ using sufficiently anti-ample twists. Applying $\Phi_{\mathcal{G}^\vee}$, we get a long exact sequence

$$0 \to H^0(\Phi_{\mathcal{G}^\vee}(F)) \to B^{-1} \xrightarrow{\beta} B^0 \to H^1(\Phi_{\mathcal{G}^\vee}(F)) \to 0$$

where $B^i = \Phi_{\mathcal{G}^\vee}(A^i)$ are locally free sheaves. We want to show that β is injective, hence $\Phi_{\mathcal{G}^\vee}(F)$ will be a sheaf concentrated in degree 1. Assume that β is not injective, then there is an injection $\mathcal{O}_E(-M) \hookrightarrow ker(\beta)$ for some $M \gg 0$.

Applying $\Phi_{\mathcal{G}}$ on the following chain map between complexes

$$\begin{array}{ccc} \mathcal{O}_E(-M) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ B^{-1} & \xrightarrow{\beta} & B^0, \end{array}$$

we get a map $\gamma : \Phi_{\mathcal{G}}(\mathcal{O}_E(-M)) \to \Phi_{\mathcal{G}}(\Phi_{\mathcal{G}^\vee}(F)) = F[-1]$.

By increasing M some more, we may assume that $\Phi_{\mathcal{G}}(\mathcal{O}_E(-M))$ is concentrated in degree 1. Hence, the morphism γ is one morphism between bundles sitting in degree 1 and the homological consideration above shows that

$$ch(\Phi_{\mathcal{G}}(\mathcal{O}_E(-M))) = \begin{pmatrix} -e & \tilde{r} \\ \frac{1+\tilde{d}e}{-\tilde{r}} & d \end{pmatrix} \begin{pmatrix} 1 \\ -M \end{pmatrix} = \begin{pmatrix} -e - \tilde{r}M \\ \frac{1+\tilde{d}e}{-\tilde{r}} - \tilde{d}M \end{pmatrix}.$$

Note that $\Phi_{\mathcal{G}}(\mathcal{O}_E(-M))$ is simple because $\mathcal{O}_E(-M)$ is simple and that twp numbers are coprime to each other, hence stable by Lemma 13.5. Thus, the morphism γ is one between stable vector bundles with slopes

$$\mu(\Phi_{\mathcal{G}}(\mathcal{O}_E(-M))) = \frac{\frac{1+\tilde{d}e}{\tilde{r}} + \tilde{d}M}{e + \tilde{r}M} = \frac{1}{\tilde{r}(e+\tilde{r}M)} + \frac{\tilde{d}}{\tilde{r}} > \frac{\tilde{d}}{\tilde{r}} = \mu(F)$$

which is a contraction for $M \gg 0$.

By now we know that β is injective, hence $\Phi_{\mathcal{G}^\vee}(F)$ is concentrated in degree 1. The numerical invariants can be computed as follows

$$ch(\Phi_{\mathcal{G}^\vee}(F)) = \begin{pmatrix} -\tilde{d} & \tilde{r} \\ -\frac{1+\tilde{d}e}{\tilde{r}} & e \end{pmatrix} \begin{pmatrix} r \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ -(r,d) \end{pmatrix}$$

and this shows that $T = coker(\beta)$ is a torsion sheaf of length (r,d) with $\Phi_{\mathcal{G}}(T) = F$, as claimed.

□

Remark 13.11. The bijection between torsion sheaves and semi-stable bundles given by the proposition also allows the identification of indecomposable objects on both sides. Explicitly, torsion sheaves of the form $k[\epsilon]/\epsilon^l$ gives rise to indecomposable bundles and vice versa. Especially, we obtain an equivalence

$$\Phi_{\mathcal{G}} : E \xrightarrow{\sim} \{F \in \mathcal{S}(r,d) \text{ } indecomposable\}, \ p \mapsto \Phi_{\mathcal{G}}(k(p)[\epsilon]/\epsilon^{(r,d)}).$$

This in Atiyah's main theorem.

Note that this implies that the space $\mathcal{S}(r,d)$ is neither reduced nor separable if $(r,d) > 1$. There is natural equivalence relation to solve this issue. The so-called S-equivalence. Two semi-stable bundles V_1 and V_2 are S-equivalent if the graded objects of their Jordan-Hölder filtrations are isomorphic $gr_{JH}(V_1) \simeq gr_{JH}(V_2)$.

The quotient $\mathcal{M}(r,d) := \mathcal{S}(r,d)/\sim$, where $\sim$ denote the S-equivalence, is the moduli space of semi-stable vector bundles of rank r and degree d. Then $\mathcal{M}(r,d) \simeq Sym^r(E)$ is separated and reduced.

14. Bondal-Orlov reconstruction theorem

Starting from this section, we will gradually introduce Bondal-Orlov reconstruction theorem. Let us start with the basic notation of support of a complex.

Definition 14.1. The support of a complex $F \in D^b(X)$ is the union of the supports of all its cohomology sheaves, i.e. it is the closed subset

$$supp(F) := \cup supp(\mathcal{H}^i(F)).$$

Lemma 14.2. *Suppose $F \in D^b(X)$ and $supp(F) = Z_1 \amalg Z_2$, where $Z_1, Z_2 \subset X$ are disjoint closed subsets. Then $F \simeq F_1 \oplus F_2$ with $supp(F_i) \subset Z_i$ for $i = 1, 2$.*

Proof. We prove it by induction on the length of the complex. The assertion is clear for complex of length zero. Let F be a complex of length at least two. Suppose m is minimal with $0 \neq \mathcal{H}^m(F) =: H$. The sheaf H may be decomposed as $H \simeq H_1 \oplus H_2$ with $supp(H_i) \subset Z_j$. Consider the natural distinguished triangle given the standard t-structure

$$H[-m] \to F \to G \to H[1-m],$$

where $\mathcal{H}^q(G) = \mathcal{H}^q(F)$ for $q > m$ and $\mathcal{H}^q(G) = 0$ for $q \leq m$. Thus the induction hypothesis applies to G and we may write $G = G_1 \oplus G_2$ with $supp(G_i) \subset Z_i$. Next, we use spectral sequence

$$E_2^{p,q} = Hom(\mathcal{H}^{-q}(G_1), H_2[p]) \Longrightarrow Hom(G_1, H_2[p+q])$$

to prove $Hom(G_1, H_2) = 0$. This follows from the assumption that their support are disjoint (using the local-to-global spectral sequence, this reduces to a local statement, which is then obvious). Similarly, one fined $Hom(G_2, H_1[1-m]) = 0$. This proves that $F \simeq F_1 \oplus F_2$ where F_i are chosen to complete $G_j \to H_j[1-m]$ to a distinguished triangle

$$F_j \to G_j \to H_j \to F_j[1].$$

Hence, we win. □

Let us show how to characterize certain geometric structures, like points or line bundles, intrinsically as objects in derived category.

Definition 14.3. Let $\mathcal{D}$ be a k-linear triangulated category with a Serre functor S. An object $P \in \mathcal{D}$ is called point like of codimension d if

(1) $S(P) \simeq P[d]$,
(2) $Hom(P, P[i]) = 0$ for $i < 0$, and
(3) $k(P) := Hom(P, P)$ is a field, i.e. it is a simple object.

As we continue to assume that all Hom's are finite-dimensional, the field $k(P)$ in (3) is automatically a finite field extension of k. So, if k is algebraically closed, it is just k.

Lemma 14.4. *Suppose F is a simple object in $D^b(X)$ with zero-dimensional support. If $Hom(F, F[i]) = 0$ for $i < 0$, then*

$$F \simeq k(x)[m]$$

for some closed point $x \in X$ and some integer m.

Proof. Let us show that F is concentrated in exactly one closed point. Otherwise, F could be written as a direct sum $F \simeq F_1 \oplus F_2$ with F_i nonzero, $j = 1, 2$. However, simple implies indecomposable, hence a contradiction.

Thus we may assume that the support of all cohomology sheaves $\mathcal{H}^i$ of F consists of the same closed point $x \in X$. Set

$$m_0 := max\{i|\mathcal{H}^i \neq 0\} \text{ and } m_1 := min\{i|\mathcal{H}^i \neq 0\}.$$

Since both $\mathcal{H}^{m_0}$ and $\mathcal{H}^{m_1}$ are concentrated in $x \in X$, there exists a non-trivial morphism $\mathcal{H}^{m_0} \to \mathcal{H}^{m_1}$.

The composition

$$F[m_0] \to \mathcal{H}^{m_0} \to \mathcal{H}^{m_1} \to F[m_1],$$

is nontrivial by looking at the cohomology group. By the condition (ii) in the definition, we conclude with $m_0 = m_1$. Hence, $F \simeq T[m]$ with T a coherent sheaf with support in x and $m = m_0 = m_1$.

The only such sheaf which is also simple is $k(x)$. Hence we win. □

The following propositions are taken from [BO01].

Proposition 14.5. *Let X be a smooth projective variety. Suppose that ω_X or ω_X^* is ample. Then the point like objects in $D^b(X)$ are the objects which are isomorphic to $k(x)[m]$, where $x \in X$ is a closed point and $m \in \mathbb{Z}$.*

Proof. It is easy to verify that, the sky-scraper sheaf $k(x)$ of a closed point $x \in X$, as well as any shift $k(x)[m]$, does satisfy all three conditions in the definition. Let us now assume that $P \in D^b(X)$ satisfies these three conditions. We denote by $\mathcal{H}^i$ the coholomogy sheaves of P, which are all zero. Then condition (1), which ensures $\mathcal{H}^i \otimes \omega_X[n] \simeq \mathcal{H}^i[d]$, combining with the boundedness of F yield $d = n$ and $\mathcal{H}^i \simeq \mathcal{H}^i \otimes \omega_X$.

Since ω_X and ω_X^* is ample, the latter condition shows that $\mathcal{H}^i$ is supported in dimension zero. Indeed, the Hilbert polynomial $P_{\mathcal{F}}(k) = \chi(\mathcal{F} \otimes \omega_X^k)$ of any coherent sheaf is of degree $dimsupp(\mathcal{F})$. Hence the assertion follows from Lemma 14.4. □

Remark 14.6. The condition on the canonical bundle is necessary. E.g. if ω_X is trivial, then $\mathcal{O}_X$ is a point like object.

One may also try to characterize line bundles as objects in the derived category. Let us first give the abstract definition.

Definition 14.7. Let $\mathcal{D}$ be a k-linear triangulated category with a Serre functor S. An object $L \in \mathcal{D}$ is called invertible if for any point line object $P \in \mathcal{D}$ there exists $n_P \in \mathbb{Z}$ (depending also on L) such that

$$Hom(L, P[i]) = \begin{cases} k(P) & if\ i = n_P \\ 0 & otherwise. \end{cases}$$

Proposition 14.8. *Let X be a smooth projective variety. Any invertible object in $D^b(X)$ is of the form $L[m]$ with L a line bundle on X and $m \in \mathbb{Z}$. Conversely, if ω_X or ω_X^* is ample, then for any line bundle L and any $m \in \mathbb{Z}$ the object $L[m] \in D^b(X)$ is invertible.*

Proof. Let us suppose that L is invertible object in $D^b(X)$ and let m be maximal with $\mathcal{H}^m := \mathcal{H}^m(L) \neq 0$. In particular, there exists the natural morphism

$$L \to \mathcal{H}^m[-m]$$

inducing the identity on the $m-$th cohomology.

Pick a point $x_0 \in supp(\mathcal{H}^m)$. Then there exists a non-trivial homomorphism $\mathcal{H}^m \to k(x_0)$. Hence,

$$0 \neq Hom(\mathcal{H}^m, k(x_0)) = Hom(L, k(x_0)[-m])$$

and, therefore, $n_{k(x_0)} = -m$.

From the spectral sequence

$$E_2^{p,q} = Hom(\mathcal{H}^{-q}(L), k(x_0)[p]) \Longrightarrow Hom(L, k(x_0)[p+q]),$$

we see that

$$E_2^{1,-m} = Hom(\mathcal{H}^m, k(x_0)[1]) = Hom(L, k(x_0)[1+m]) = 0.$$

Thus as soon as $x_0 \in X$ is in the support of $\mathcal{H}^m$, we obtain $Ext^1(\mathcal{H}^m, k(x_0) = 0$.

Next, we shall apply the following standard result in commutative algebra: any finite module M over an arbitrary noetherian local ring (A, m) with $Ext^1(M, A/m) = 0$ is free.

The local-to-global spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{H}^m, k(x_0))) \Longrightarrow Ext^{p+q}(\mathcal{H}^m, k(x_0))$$

allows us to pass from the global vanishing $Ext^1(\mathcal{H}^m, k(x_0)) = 0$ to the local one

$$\mathcal{E}xt^1(\mathcal{H}^m, k(x_0)) = 0.$$

More precisely, as $\mathcal{E}xt^0(\mathcal{H}^m, k(x_0))$ is concentrated in $x_0 \in X$, one has

$$E_2^{2,0} = H^2(X, \mathcal{E}xt^0(\mathcal{H}^m, k(x_0))) = 0.$$

Therefore, $E_2^{0,1} = E_\infty^{0,1}$. Since $\mathcal{E}xt^1(\mathcal{H}^m, k(x_0))$ is also concentrated in $x_0 \in X$, it is a globally generated sheaf. Hence

$$H^0(X, \mathcal{E}xt^1(\mathcal{H}^m, k(x_0))) = E_2^{0,1} = 0$$

implies $\mathcal{E}xt^1(\mathcal{H}^m, k(x_0)) = 0$. Then the aforementioned result from commutative algebra shows that $\mathcal{H}^m$ is free in $x_0 \in X$.

Since X is irreducible, we have in particular $supp(\mathcal{H}^m) = X$. Thereby, there exists for any point $x \in X$ a surjection $\mathcal{H}^m \to k(x_0)$. Hence,

$$Hom(L, k(x)[-m]) = Hom(\mathcal{H}^m, k(x)) \neq 0.$$

In particular, $n_{k(x)}$ does not depend on x. As by assumption, the sheaf $\mathcal{H}^m$ has constant dimension one. Hence, $\mathcal{H}^m$ is a line bundle.

It remains to show that $\mathcal{H}^i = 0$ for $i < m$. We use again the spectral sequence. Since $\mathcal{H}^m$ is locally free, he row $E_2^{q,-m}$ is trivial except for $q = 0$. Indeed,

$$E_2^{q,-m} = Ext^q(\mathcal{H}^m, k(x)) = H^q(X, \mathcal{H}^{m\vee} \otimes k(x)) = 0$$

for $q > 0$.

The rest of the argument is by induction. Assume we have shown that $\mathcal{H}^i = 0$ for $i_0 < i$. Then $E_2^{0,-i_0} = E_\infty^{0,-i_0}$. Since

$$E^{-i_0} = Hom(L, k(x)[-i_0]) = 0,$$

this implies that $Hom(\mathcal{H}^{i_0}, k(x)) = 0$ for any $x \in X$, i.e. $\mathcal{H}^{i_0} = 0$.

Let us show that conversely any line bundle L on X, and hence any shift $L[m]$, defines an invertible object in $D^b(X)$ whenever the (anti-)canonical bundle is ample.

By Proposition 14.5 the assumption on the canonical bundle implies that point like object in $D^b(X)$ are of the form $k(x)[m]$ for some closed point $x \in X$ and some $m \in \mathbb{Z}$. Hence,

$$\begin{aligned} Hom(L[m], P[i]) &\simeq Hom(L[m], k(x)[i+n]) \\ &= H^0(X, L^*(x)[i+n-m]) \\ &= H^{i+n-m}(X, L^*(x)) = 0 \end{aligned}$$

except for $i = m - n$ when it is $k(x)$. We set $n_p := m - n$. □

Proposition 14.9. *Let X and Y be smooth projective varieties and assume that the (anti-)canonical bundle of X is ample. If there exists an exact equivalence $D^b(X) \simeq D^b(Y)$, then X and Y are isomorphic. In particular, the (anti-)canonical bundle of Y is also ample.*

Proof. Let us first sketch the original idea of the proof which is strikingly simple. Assume that under an equivalence $F : D^b(X) \to D^b(Y)$ the structure sheaf $\mathcal{O}_X$ is mapped to $\mathcal{O}_Y$. Since any equivalence is compatible with Serre functors and $dim(X) = dim(Y) =: n$, this proves

$$\begin{aligned} F(\omega_X^k) &= F(S_X^k(\mathcal{O}_X))[-kn] \\ &\simeq S_Y^k(F(\mathcal{O}_X))[-kn] \\ &\simeq S_Y^k(\mathcal{O}_Y)[-kn] = \omega_Y^k \end{aligned}$$

Using that F is fully faithful, we conclude from this that

$$\begin{aligned} H^0(X, \omega_X^k) &= Hom(\mathcal{O}_X, \omega_X^k) \\ &\simeq Hom(F(\mathcal{O}_X), F(\omega_X^k)) \\ &= Hom(\mathcal{O}_Y, \omega_Y^k) = H^0(Y, \omega_Y^k) \end{aligned}$$

for all k.

The product in $\oplus H^0(X, \omega_X^k)$ can be expressed in terms of compositions: namely, for $s_i \in H^0(X, \omega_X^{k_i}) = Hom(\mathcal{O}_X, \omega_X^{k_i})$ one has

$$s_1 \cdot s_2 = S_X^{k_1}(s_2)[-k_1 n] \circ s_1$$

and similarly for sections on Y. Hence, the induced bijection

$$\oplus H^0(X, \omega_X^k) \simeq \oplus H^0(Y, \omega_Y^k)$$

is a ring isomorphism. If the canonical bundle of Y is also ample, then this shows

$$X \simeq Proj(\oplus H^0(X, \omega_X^k)) \simeq Proj(\oplus H^0(Y, \omega_Y^k) \simeq Y.$$

Thus, under the two assumptions that $F(\mathcal{O}_X) \simeq \mathcal{O}_Y$ and that ω_Y (or ω_Y^*) is ample we have proved the assertion. Let us explain how to reduce to this situation $F(\mathcal{O}_X) = \mathcal{O}_Y$.

As we already have proved Corollary 10.2, we will take a short cut in the proof.

As the notion of point like and invertible objects in D^b are intrinsic, an exact equivalence $F : D^b(X) \to D^b(Y)$ induces bijections

$$\begin{tikzcd} \{\text{Point like objects in } D^b(X)\} \arrow[r, "\sim"] \arrow[d, "\simeq"] & \{\text{point like objects in } D^b(Y)\} \\ \{k(x)[m] | x \in X,\ m \in \mathbb{Z}\} & \{k(x)[m] | x \in X,\ m \in \mathbb{Z}\} \arrow[u, hook] \end{tikzcd}$$

and

$$\begin{tikzcd} \{\text{Invertible objects in } D^b(X)\} \arrow[r, "\sim"] \arrow[d, "\simeq"] & \{\text{Invertible objects in } D^b(Y)\} \\ \{L[m] | L \in Pic(X),\ m \in \mathbb{Z}\} & \{M[m] | M \in Pic(X),\ m \in \mathbb{Z}\} \arrow[u, hookleftarrow] \end{tikzcd}$$

As we have seen before, the point like object in $D^b(X)$ are all of the form $k(x)[m]$ for $x \in X$ a closed point and $m \in \mathbb{Z}$. By Proposition 14.8, any line bundle L, in particular $L = \mathcal{O}_X$ is an invertible object in $D^b(X)$. Thus $F(\mathcal{O}_X)$ is also an invertible object in $D^b(Y)$ and hence, of the form $M[m]$ for some line bundle M on Y. Compose F with the two equivalence given by $M^* \otimes (-)$ and the shift $[-m]$. The new equivalence, which we continue to call F,satisfies $F(\mathcal{O}_X) \simeq \mathcal{O}_Y$.

We first prove that point like objects in $D^b(Y)$ are of the form $k(y)[m]$.

Due to the first diagram, one finds for any closed point $y \in Y$ a closed point $x_y \in X$ and an integer m_y such that $k(y) \simeq F(k(x)[m_y])$. Suppose there exists a point like object $P \in D^b(Y)$ which is not of the form $k(y)[m]$ and denote by $x_p \in X$ the closed point with $F(k(x_P)[m_P]) \simeq P$ for a certain $m_P \in \mathbb{Z}$.

Note that $x_P \neq x_y$ for all $y \in Y$. Hence we have for all $y \in Y$ and all $m \in \mathbb{Z}$

$$\begin{aligned} Hom(P, k(y)[m]) &= Hom(F(k(x_P))[m_P], F(k(x_y))[m_y + m]) \\ &= Hom(k(x_P), k(x_y)[m_y + m - m_P]) \\ &= 0 \end{aligned}$$

Since the objects $k(y)[m]$ form a spanning class in $D^b(Y)$, this implies $P \simeq 0$, which is absurd. Hence, point like objects in $D^b(Y)$ are exactly the objects of the form $k(y)[m]$.

Together with $F(\mathcal{O}_X) \simeq \mathcal{O}_Y$ this also implies that for any closed point $x \in X$ there exists a closed point $y \in Y$ such that $F(k(x)) \simeq k(y)$ (no shifts). This is due to the easy observation that $m = 0$ if and only $Hom(\mathcal{O}_Y, k(y)[m]) \neq 0$.

Hence, by Corollary 10.2, we see that there exists an isomorphism $f : X \to Y$. Moreover, we can conclude that F is of the form $f_* \circ (\otimes M[m])$, where $M \in Pic(Y)$ and $m \in \mathbb{Z}$.

□

Remark 14.10. In the previous proof, we use Corollary 10.2 to take a short cut without proving that ω_Y is ample. For the original proof, please see [BO01] and [Huy06, Proposition 4.11].

Corollary 14.11. *Let X be a smooth projective variety with ample (anti-)canonical bundle. The group of auto-equivalences is generated by: i) automorphisms of X, ii) the shift functor T, and iii) twists by line bundles.*

In other words, one has

$$Aut(D^b(X)) \simeq \mathbb{Z} \times (Aut(X) \ltimes Pic(X)).$$

Remark 14.12. Much more interesting is the group $Aut(D^b(X))$ for a smooth projective variety X with trivial canonical bundle. A complete description in the case of abelian varieties, due to Mukai and Orlov, see [Orl02]. The case of K3 surfaces of Picard rank 1 is given by Bayer and Bridgeland in [Kaw02].

Corollary 14.13. *Let C be a curve of genus $g(C) \neq 1$ and let Y be a smooth projective variety. Then there exists an exact equivalence $D^b(C) \simeq D^b(Y)$ if and only if Y is a curve isomorphic to C.*

In fact, the statement is also true for elliptic curves. It can been by passage to cohomology and $E \simeq H^{0,1}(E)/H^1(E, \mathbb{Z})$. For details, please consult [Huy06, Section 5.2].

15. Derived category and canonical bundle

In the smooth projective curve case, the canonical bundle is either ample, anti-ample, or trivial. In the higher dimensional case, it becomes more complicated. We will investigate how much positivity of the canonical bundle of a variety is preserved under derived equivalence.

For the discussion in higher dimensional case, we need to recall the definitions of Kodaira dimension.

Definition 15.1. Let X be a smooth projective variety and $L \in PIc(X)$. The Kodaira dimension $kod(X, L)$ of L on X is the integer m such that

$$h^0(X, L^l) \coloneqq dim H^0(X, L^l)$$

grows like a polynomial of degree m for $l \gg 0$. By definition, $kod(X, L) = -\infty$ if $h^0(X, L^l) = 0$ for all $l > 0$.

There are equivalent descriptions of the Kodaira dimension: E.g. under the assumption that $kod(X, L) \geq 0$, one has

$$\begin{aligned} kod(X, L) &= max\{dim(Im(\phi_{L^l}))|l \geq 0\} \\ &= trdeg_k Q(R(X, L)) - 1 \end{aligned}$$

Here, $\phi_{L^l} : X \dashrightarrow \mathbb{P}^{h^0(L^l)-1}$ is the rational map defined by the linear system $|L^l|$, $R(X, L)$ is the canonical ring of L,i.e.

$$R(X, L) \coloneqq \oplus_{l \geq 0} H^0(X, L^l)$$

and $Q(R(X, L))$ is the field of fractions. Note that $kod(X, L) \leq dim(X)$ for any line bundle L.

The case that interests us most here is when $L \simeq \omega_X$. Then one calls

$$kod(X) \coloneqq kod(X, \omega_X)$$

the Kodaira dimension of X and

$$R(X) \coloneqq R(X, \omega_X)$$

the canonical ring of X. A standard fact in higher dimensional algebraic geometry says that the Kodaira dimension is a birational invariant.

Then we have the following Proposition.

Proposition 15.2. *Suppose X and Y are smooth projective varieties with equivalent derived categories $D^b(X) \simeq D^b(Y)$. Then there exists a ring isomorphism $R(X) \simeq R(Y)$ and in particular, $kod(X) = kod(Y)$.*

Note that we can not argue as in Proposition 14.9 to reduce to the case when $F(\mathcal{O}_X) = \mathcal{O}_Y$, since we do not assume ω_X is (anti-)ample, hence $\mathcal{O}_X$ may not be an invertible object.

Proof. By Orlov's existence result. We can assume that there exists a complex $P \in D^b(X \times Y)$ such that $\Phi_P : D^b(X) \to D^b(Y)$ is isomorphic to the given equivalence. And its inverse functor is a Fourier-Mukai functor with kernal $Q := P^\vee \otimes q^*\omega_X[n] \simeq P^\vee \otimes p^*\omega_Y[n]$, where $n = dim(X) = dim(Y)$.

One can also show that $\Phi_Q : D^b(X) \to D^b(Y)$ is also an equivalence (note the change of direction). One first show that Φ_Q is fully faithful. Indeed,

$$\begin{aligned} Hom(\Phi_Q(k(x)), \Phi_Q(k(y))[i]) &\simeq Hom(i_x^*P^\vee, i_y^*P^\vee[i]) \\ &\simeq Hom((i_x^*P)^\vee, (i_y^*P)^\vee[i]) \\ &\simeq Hom(i_y^*P, i_x^*P[i]) \\ &\simeq Hom(\Phi_P(k(y)), \Phi_P(k(x))[i]) \\ &\simeq Hom(k(y), k(x)[i]) \\ &\simeq Hom(k(x), k(y)[i]). \end{aligned}$$

The second isomorphism use that dualizing and pull-back commute. The last isomorphism is because both sides are zero when $x \neq y$. Hence, we prove that Φ_Q is full-faithful. It is an equivalence because $Q \otimes q^*\omega_X \simeq Q \otimes p^*\omega_Y \simeq P^\vee \otimes p^*\omega_Y \otimes q^*\omega_X[n]$.

Next, use the kernal $Q \boxtimes P \in D^b(()X \times X) \times (Y \times Y))$ to define the Fourier-Mukai equivalence

$$\Phi_{Q\boxtimes P} : D^b(X \times X) \to D^b(Y \times Y).$$

Denote $\Phi_{Q\boxtimes P}(\iota_*\omega_X^k)$ by $S \in D^b(Y \times Y)$ (we use ι for both diagonal inclusions $X \hookrightarrow X \times X$ and $Y \hookrightarrow Y \times Y$).

Then $\Phi_S : D^b(Y) \to D^b(Y)$ is an equivalence that can be computed as the composition

$$D^b(Y) \xrightarrow{\Phi_Q} D^b(X) \xrightarrow{\Phi_{\iota_*\omega_X^k}} D^b(X) \xrightarrow{\Phi_P} D^b(Y).$$

Since $\Phi_{\iota_*\omega_X^k}$ is isomorphic to $S_X^k[-kn]$ and since any equivalence commutes with the Serre functor S_X and S_Y, we obtain that $\Phi_S = S_Y^k[-kn]$. Hence, due to the uniqueness of the Fourier-Mukai kernal $S \simeq \iota_*\omega_Y^k$. Thus, for all $k \in \mathbb{Z}$ we have $\Phi_{Q\boxtimes P}(\iota_*\omega_X^k) \simeq \iota_*\omega_Y^k$. Since $\Phi_{Q\boxtimes P}$ is an equivalence, we obtain

$$Hom_{X\times X}(\iota_*\omega_X^k, \iota_*\omega_X^l) \simeq Hom_{Y\times Y}(\iota_*\omega_Y^k, \iota_*\omega_Y^k)$$

for all $k, l \in \mathbb{Z}$. The case $k = 0$ and $l \geq 0$ induces the claimed bijection

$$\begin{aligned} H^0(X, \omega_X^l) =& Hom_{X\times X}(\iota_*\mathcal{O}_X, \iota_*\omega_X^l) \\ &\simeq Hom_{Y\times Y}(\iota_*\mathcal{O}_Y, \iota_*\omega_Y^l) \\ &= H^0(Y, \omega_Y^l) \end{aligned}$$

As in the proof of Proposition 14.9, this is in fact a ring isomorphism.

□

Exercise 15.3. *Show that the same arguments also provide a ring isomorphism of the anti-canonical rings, i.e.* $R(X, \omega_X^*) \simeq R(Y, \omega_Y^*)$ *and hence* $kod(X, \omega_X^*) = kod(Y, \omega_Y^*)$.

15.1. **geometric aspects of the Fourier-Mukai kernal.** In this subsection, we prove a series of technical but useful facts that shed light on the geometry of the support of the Fourier-Mukai kernal P of an equivalence

$$\Phi_P : D^b(X) \to D^b(Y).$$

Sometimes, P is a locally free sheaf on $X \times Y$, e.g. the Poincaré sheaf on the product of abelian variety and its dual, and the nothing interesting can be said about $supp(P)$, which is just $X \times Y$. However, often the kernal is concentrated on a smaller subvariety, e.g. on the graph of a morphism or a correspondence, then encodes information about the geometric relation between X and Y. This usually happens if the canonical bundle of the varieties enjoy some kind of positivity. Most of the material is taken from [Kaw02].

Throughout this subsection we shall consider a Fourier-Mukai equivalence

$$\Phi_P : D^b(X) \to D^b(Y)$$

between the derived categories on smooth projective varieties X and Y.

We have the following lemma on support.

Lemma 15.4. *For any $F \in D^b(X)$ one has*

$$supp(F) = supp(F^\vee).$$

Proof. Consider the spectral sequence

$$E_2^{p,q} := \mathcal{E}xt^p(\mathcal{H}^{-q}(F), \mathcal{O}) \Longrightarrow \mathcal{E}xt^{p+q}(F, \mathcal{O}) \simeq \mathcal{H}^{p+q}(F^\vee)$$

From this, one immediately concludes $supp(F^\vee) \subset supp(F)$. Similarly, using $F^{\vee\vee} \simeq F$ one show the other inclusion and thus obtains equality. □

Hence, together with the easy fact that tensor product with a line bundle does not change the support of a complex one has

$$supp(P) = supp(P^\vee) = supp(P_R) = supp(P_L).$$

This is in fact true without Φ_P being an equivalence. In this case when Φ_P is an equivalence, one has $P \otimes q^*\omega_X \simeq P \otimes p^*\omega_Y$.

In the sequel we will often abbreviate

$$\mathcal{H}^i := \mathcal{H}^i(P)$$

and use $\mathcal{H}^i \otimes q^*\omega_X \simeq \mathcal{H}^i \otimes p^*\omega_Y$.

Lemma 15.5. *The natural projection $supp(P) \to X$ is surjective.*

Proof. We shall use the spectral sequence

$$E_2^{r,s} = Tor_{-r}(\mathcal{H}^s, q^*k(x)) \Longrightarrow Tor_{-(r+s)}(P, q^*k(x)).$$

Thus, for a closed point $x \in X$ in the complement of $q(supp(P))$, the derived tensor product $P \otimes q^*k(x)$ is trivial. Therefore, $\Phi_P(k(x)) = 0$, which is absurd for an equivalence. □

As the situation is symmetric and $supp(P) = supp(P_R)$, one immediately derives from the lemma also the surjectivity of the other projection $supp(P) \to Y$.

Corollary 15.6. *There exists an integer $i \in \mathbb{Z}$ and an irreducible component Z of $supp(\mathcal{H}^i)$ that projects onto X.*

Lemma 15.7. *Let C be a complete reduced curve and let $\phi : C \to X \times Y$ be a morphism with image in $supp(P)$. Then*

$$deg(\phi^* q^* \omega_X) = deg(\phi^* p^* \omega_Y).$$

In other words, the pull-backs $q^*\omega_X|_{supp(P)}$ and $p^*\omega_Y|_{supp(P)}$ are numerically equivalent.

Proof. We may assume that the curve is irreducible and smooth. Then there exists an integer i with $\phi(C) \subset supp(\mathcal{H}^i)$, i.e. the underived pull-back $\phi^*\mathcal{H}^i$ is a sheaf with a possibly nontrivial torsion part $T(\phi^*\mathcal{H}^i)$, but such that its locally free part $F := \phi^*\mathcal{H}^i/T(\phi^*\mathcal{H}^i)$ is nontrivial. In other words, F is locally free of positive rank, say r.

On the other hand, as Φ_P is an equivalence, one has $\mathcal{H}^i \otimes q^*\omega_X \simeq \mathcal{H}^i \otimes p^*\omega_Y$. Pulled-back to C it yields $F \otimes \phi^*q^*\omega_X \simeq F \otimes \phi^*p^*\omega_Y$ and after taking determinants $\phi^*q^*\omega_X^r \simeq \phi^*p^*\omega_Y^r$. We win. □

We recall definition and simple fact for nef line bundle.

Definition 15.8. A line bundle L on a proper scheme X over a field k is called nef if for any morphism $\phi : C \to X$ from a complete reduced curve C one has

$$deg(\phi^* L) \geq 0.$$

Of course, it suffices to test curves that are embedded into X, as one might replace $\phi : C \to X$ by the image $C' = \phi(C)$ (use $deg(\phi^*L) = deg(\phi) \cdot deg(L|_{C'})$). In another direction, it suffices to test $\phi : C \to X$ with C smooth and irreducible, as we always pass to the normalization of C.

Lemma 15.9. *Let $\pi : Z \to X$ be a projective morphism of proper schemes and $L \in Pic(X)$.*
(i) If L is a nef line bundle on X then $\pi^*(L)$ is nef.
(ii) If π is surjective, then L is nef if and only if π^*L is nef.

Proof. (i) follows easily from the definition. Let $\phi : C \to X$ be a given curve, the nthe composition with π yields $\pi \circ \phi : C \to X$.

To see (ii), one constructs for any irreducible curve $\phi : C \to X$ a ramified cover $\psi : \tilde{C} \to C$ by an irreducible curve $\tilde{C}$ such that $\phi \circ \psi : \tilde{C} \to X$ factorizes over $Z \to X$. Using $deg(\psi^*\phi^*L) = deg(\psi) \cdot deg(\phi^*L)$ this finishes the proof.

The construction of $\psi : \tilde{C} \to C$ is standard algebraic geometry: By working with the fiber product $C \times_X Z$, we may reduce to the claim that any dominant projective morphism $Z \to C$ onto a curve admits a multi-section. By embedding Z into some $\mathbb{P}^n \times C$ this may be achieved by intersecting with a generic linear subspace in $\mathbb{P}^n$ of the appropriate dimension. □

Definition 15.10. A line bundle L is called numerically trivial if for any curve $\phi : C \to X$ one has $deg(\phi^* L) = 0$. Clearly, L is numerically trivial if and only if L and L^* are both nef.

Corollary 15.11. *The canonical bundle ω_X is numerically trivial if and only if the canonical bundle ω_Y is.*

Proof. Suppose ω_X is numerically trivial. Then in particular $deg(\phi^* q^* \omega_X) = 0$ for any curve $\phi : C \to X \times Y$. Thus, the lemma shows that $p^* \omega_Y|_{supp(P)}$ is numerically trivial.

By lemma 15.9 and 15.5, it is easy to see that both ω_Y and ω_Y^* are nef, hence ω_Y is numerically trivial. □

Corollary 15.12. *Suppose $Z \subset supp(P)$ is a closed subvariety such that the restriction of ω_X (or its dual ω_X^*) to its image of $q : Z \to X$ is ample. Then $p : Z \to Y$ is a finite morphism.*

Proof. Suppose $p : Z \to Y$ is not finite. Since p is a proper morphism, there exists an irreducible curve $\phi : C \hookrightarrow Z$ such that $p \circ \phi : C \to Y$ is constant. Thus $\phi^* p^* \omega_Y$ is a trivial line bundle on C. Lemma 15.7 shows that $\phi^* q^* \omega_X$ is also numerically trivial. As $p \circ \phi$ is constant, the composition $q \circ \phi$ is necessarily non-trivial. Since ω_X is ample on $q(Z)$ and hence $q(\phi(C))$, this yields a contradiction. □

Before giving a refined version of the same principle, we state the following easy fact:

Lemma 15.13. *Let Z be a normal variety over a field k and let F be a coherent sheaf on Z generically of rank r. If $L_1, L_2 \in Pic(Z)$ are two line bundles such that $F \otimes L_1 \simeq F \otimes L_2$, then $L_1^r \simeq L_2^r$.*

Proof. Without loss of generality, we can assume that F is torsion free. Since Z is normal, this means that F is locally free on the open complement U of a codimension two subset.

As $det(F \otimes L_i|_U) \simeq (det(F) \otimes L_i^r)|_U$, one has $L_1^r|_U \simeq L_2^r|_U$. The induced trivialization section $s \in H^0(U, L_1^r \otimes L_2^{-r})$ extends to a section $\tilde{s} \in H^0(Z, L_1^r \otimes L_2^r)$, which automatically is trivializing and, hence, induces an isomorphism $L_1^r \simeq L_2^r$. The last two statements follow from $codim(Z \backslash U) \geq 2$ and the normality of Z. □

Lemma 15.14. *Let $Z \subset supp(P)$ be a closed irreducible subvariety with normalization $\mu : \tilde{Z} \to Z$. Then there exists an integer $r > 0$ such that*

$$\pi_X^* \omega_X^r \simeq \pi_Y^* \omega_Y^r,$$

where $\pi_X := q \circ \mu$ and $\pi_Y := p \circ \mu$.

Proof. By the assumption, there exists an integer i with $Z \subset supp(\mathcal{H}^i)$, i.e. $\mu^* \mathcal{H}^i$ is a coherent sheaf on $\tilde{Z}$ of generically positive rank, say $r > 0$. Pulling-back $\mathcal{H}^i \otimes q^* \omega_X \simeq \mathcal{H}^i \otimes p^* \omega_Y$ via μ to the normal variety $\tilde{Z}$ allows one to conclude by Lemma 15.13. □

Exercise 15.15. *Suppose $\Phi_P : D^b(X) \to D^b(Y)$ is an equivalence with kernal $P \in D^b(X \times Y)$ such that $supp(P) = X \times Y$. Show that ω_X and ω_Y are both of same finite order, i.e. there exists $r \geq 0$, such that $\omega_X^r \simeq \mathcal{O}_X$ and $\omega_Y^r \simeq \mathcal{O}_Y$.*

The following lemma will be needed.

Lemma 15.16. *Let $i : T \hookrightarrow X$ be a closed subscheme, Then for any $F \in D^b(X)$ one has*

$$supp(F) \cap T = supp(i^*F).$$

Proof. One direction is easy. Indeed, if $x \notin supp(F)$, then the restriction $F|_U$ of F to an open neighbourhood $x \in U \subset X$ is trivial. Hence, also $i_U^*(F|_U) = i^*F|_{U\cap T}$ is trivial, where $i_U : U \cap T \hookrightarrow U$. Thus, $x \notin supp(i^*F)$. This proves $supp(i^*F) \subset supp(F) \cap T$.

Conversely, let $x \in supp(F)$. If i_0 is maximal with $x \in supp(\mathcal{H}^{i_0}(F))$, then

$$Tor_0(\mathcal{H}^{i_0}(F), k(x)) \neq 0.$$

As $Tor_{-p}(\mathcal{H}^q(F), k(x)) = 0$ for $p > 0$. The spectral sequence

$$E_2^{p,q} = Tor_{-p}(\mathcal{H}^q(F), k(x)) \Longrightarrow Tor_{-(p+q)}(F, k(x)) = \mathcal{H}^{p+q}(F(x))$$

shows that $\mathcal{H}^{i_0}(F(x)) \neq 0$.

Hence, $x \in supp(i^*(F))$. □

Lemma 15.17. *The fibers of the projection $supp(P) \to X$ are connected.*

Proof. Suppose there exists a point $x \in X$ over which the fiber is not connected. Write $supp(P)\cap(\{x\}\times Y) = Y_1 \amalg Y_2$ as disjoint union of two non-empty closed subsets $Y_1, Y_2 \subset Y$.

By Lemma 15.16, we have $supp(P) \cap (\{x\} \times Y) = supp(P|_{\{x\}\times Y})$. Hence, $\Phi_P(k(x))$ is decomposable, a contradiction to the assumption Φ_P is an equivalence.

□

Corollary 15.18. *Let $Z \subset supp(P)$ be an irreducible component that surjects onto X. If $dim(Z) = dim(X)$, then $q : Z \to X$ is a birational morphism. Moreover, if such a component exists, then no other component of $supp(P)$ dominants X.*

Proof. Let us prove the last assertion first. Recall that due to Lemma 15.17 every fiber of $supp(P) \to X$ is connected. Consider the generic fiber of $\cup Z_i \to X$, where the Z_i are the irreducible components of $supp(P)$ different form Z. It is either empty or contains the corresponding (zero-dimensional!) fiber of $Z \to X$. The latter would imply $Z \subset \cup Z_i$ which is absurd.

In order ro prove that $g : Z \to X$ is birational, we pick a generic point $x \in X$. The intersection

$$\{y_1 \cdots, y_l\} \coloneqq Z \cap (\{x\} \times Y)$$

is finite and disjoint from any other irreducible component of $supp(P)$. Applying previous lemma, we get $l = 1$, i.e. $Z \to X$ is birational. □

Corollary 15.19. *Suppose there exists a closed point $x_0 \in X$ such that*

$$\Phi_P(k(x_0)) \simeq k(y_0)$$

for a closed point $y_0 \in Y$. Then one finds an open neighborhood $x_0 \in U \subset X$ and a morphism $f : U \to Y_0$ with $f(x_0) = y_0$ and such that

$$\Phi_P(k(x)) = k(f(x))$$

for all closed point $x \in U$.

Proof. See [Huy06, Corollary 6.14]. The basic idea is that: the assumption implies that the fiber of $supp(P) \to X$ is zero-dimensional. This clearly holds true for all points in an open neighborhood $U \subset X$ of x_0. Hence $\Phi_P(k(x)) \simeq k(y)[m]$, and m is locally constant by semi-continuity.

Argue as in Corollary 10.2, we get the morphism f. □

16. More on the geometry of derived equivalence

Recall the definition of numerical Kodaira dimension.

Definition 16.1. The numerical Kodaira dimension $v(X, L)$ of a line bundle L on a projective scheme X is the maximal integer m such that there exists a proper morphism $\phi : W \to X$ with W of dimension m with

$$([\phi^*(L)]^m \cdot W) \neq 0,$$

the intersection number $([M]^m \cdot W)$ of a line bundle M on a proper scheme W of dimension m is the degree m coefficient of the polynomial $\chi(W, M^l)$.

In fact, we can take W to be normal in the definition, since we can take the normalization.

After the technical preparations in the last section, the following result is proven easily.

Proposition 16.2. *Let X and Y be smooth projective varieties with euqivalent derived categories $D^b(X)$ and $D^b(Y)$. Then the equality of numerical Kodaira dimension holds: $v(X) = v(Y)$.*

Proof. There exists $i \in \mathbb{Z}$ and an irreducible component Z of $supp(\mathcal{H}^i)$ such that $p : Z \to Y$ is surjective.

Denote the normalization of Z by $\mu : \tilde{Z} \to Z$ and two projections to X and Y by $\pi_X = q \circ \mu$ and $\pi_Y = p \circ \mu$. Due to lemma 15.14, one finds an integer $r > 0$ with $\pi_X^* \omega_X^r \simeq \pi_Y^* \omega_Y^r$.

Hence, by definition of numberical Kodaira dimension, we get $v(X) = v(Y)$. □

Proposition 16.3. *Let X and Y be two smooth projective varieties over an algebraically closed field. Suppose there exists an exact equivalence*

$$D^b(X) \to D^b(Y).$$

If $kod(X, \omega_X) = dim(X)$ or $kod(X, \omega_X^*) = dim(X)$, then X and Y are birational and, morre precisely, there exists a birational correspondence.

$$\begin{tikzcd} & Z \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

with $\pi_X^* \omega_X \simeq \pi_Y^* \omega_Y$.

Proof. We shall only treat the case $kod(X,\omega_X) = dim(X)$, the other being completely analogous. Let $H \subset X$ be a smooth ample hyper-surface. The exact sequence

$$0 \to \mathcal{O}(-H) \to \mathcal{O} \to \mathcal{O}_H \to 0$$

induces an exact sequence

$$0 \to H^0(\omega_X^l(-H)) \to H^0(\omega_X^l) \to H^0(\omega_X^l|_H)$$

fr any l. If $kod(X,\omega_X) = dim(X)$, then $dimH^0(X,\omega_X^l)$ grows like $l^{dim(X)}$. On the other hand, as $dim(H) < dim(X)$, the dimension of $H^0(H,\omega_X^l|_H)$ has smaller growth. Thus, for $l \gg 0$ the line bundle $\omega_X^l(-H)$ has a section. In other words, We have

$$\omega_X^l \simeq \mathcal{O}(H) \otimes \mathcal{O}(D)$$

with H ample and D effective.

We know that there exists an irreducible component Z of $supp(P)$ that surjects onto X. Moreover, the pull-backs of (some power of) ω_X and ω_Y under $\pi_X : \tilde{Z} \to X$ and $\pi_Y : \tilde{Z} \to Y$ coincide, where $\tilde{Z} \to Z$ is the normalization.

Let us show that

$$\pi_Y : \tilde{Z} \backslash \pi_X^{-1}(D) \to Y$$

is quasi-finite, i.e. has finite fibers. In other words, at most curves completely mapped into D via π_X are contracted by the projection to Y.

Suppose that there exists an irreducible curve $C \subset \tilde{Z}$ contracted by π_Y and such that $C \nsubseteq \pi_X^{-1}(D)$. Then $deg\pi_Y^*(\omega_Y)|_C = 0$. On the other hand,

$$deg\pi_X^*(\omega_X)|_C \geq (1/l)deg\pi_X^*\mathcal{O}(H)|_C,$$

as the intersection of $\pi_X(C)$ and D consists of at most finitely many points. Moreover, since C is contracted by π_Y, the projection $\pi_X : C \to X$ must be finite. As H is ample this implies $deg\pi_X^*\mathcal{O}_H|_C > 0$. Hence, we get a contradiction.

Hence, $Z \to Y$ is generically finite and thus $dim(Z) \leq dim(Y)$. On the other hand, Z dominates X and hence $dim(X) \leq dim(Z)$. As $dim(X) = dim(Y)$, this shows that the correspondence $X \xleftarrow{\pi_X} \tilde{Z} \xrightarrow{\pi_Y} Y$ are generically finitely onto X and onto Y.

Now apply Corollary 15.18 to conclude that we have in fact constructed a birational correspondence

$$\begin{tikzcd} & \tilde{Z} \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

Moreover, by construction $\pi_X^*\omega_X^r \simeq \pi_Y^*\omega_Y^r$ for some $r > 0$.

On the other hand,

$$\pi_X^*\omega_X + \mathcal{O}(\Sigma a_i E_i) \simeq \pi_Y^*\omega_Y + \mathcal{O}(\Sigma a_i' E_i),$$

where E_i are exceptional with respect to π_X and π_Y. (Well, this can only be ensured if, e.g. $\tilde{Z}$ is smooth, but we may actually replace $\tilde{Z}$ by a desingularization. If fact, if the isomorphism exists on a desingularization, it also exists on the normal variety $\tilde{Z}$.)

Passing to the r-th power shows $\mathcal{O}(\Sigma r(a_i - a_i')E_i) \simeq \mathcal{O}$. Thus it suffices to show that whenever a linear combination $\Sigma\alpha_i E_i$ is linearly equivalent to zero, then all α_i are trivial.

Here is the sketch of the argument. Away from the pairwise intersection of the different exceptional divisors, they can all be contracted at once. So we suppose for simplicity that there is a single contraction $\tilde{Z} \to X$ contracting all E_i. Suppose $\Sigma\alpha_i E_i$ is linearly equivalent to zero with $\alpha_i < 0$ for $i \leq k$ and $\alpha_i \geq 0$ for $i > k$. We may assume $k > 0$, otherwise change the global sign.

Now, let $s \in H^0(\mathcal{O}(-\Sigma_1^k \alpha_i E_i)$ be the unique section vanishing to order $-\alpha_i$ along the divisors E_i, $i = 1, \cdots k$. A trivializing section of $\mathcal{O}(\Sigma\alpha_i E_i)$ multiplied by s would yield a section of $\mathcal{O}(\Sigma_{i \geq k+1}\alpha_i E_i)$ vanishing along the divisors E_i with $i \leq k$. However, $\mathcal{O}(\Sigma_{i\geq k+1}(\alpha_i E_i))$ admits only one gloval section up to scaling, namely the one vanishing along E_i, $i > k$ (of order α_i).

Indeed, by contracting the exceptional divisors E_i, $i \geq k+1$, two sections of $\mathcal{O}(\Sigma_{i\geq k+1}\alpha_i E_i)$ give rise to two functions on the complement of a closed subset of X of codimension ≥ 2 which by Hartogs differ by a scalar factor.

This yields contradiction.

□

Definition 16.4. Two varieties X and Y are called K-equivalent if there exists a birational correspondence

$$\begin{tikzcd} & \tilde{Z} \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

with $\pi_X^*\omega_X \simeq \pi_Y^*\omega_Y$.

Corollary 16.5. *Two $D-$equivalent smooth projective varieties X and Y with X of maximal Kodaira dimension are K-equivalent.*

Remark 16.6. Without the assumption of Kodaira dimension, the statement is false. We can find counter-examples on Calabi-Yau threefolds, and abelian varieties.

There are well known examples of smooth projective varieties X and Y with isomorphic derived categories but is not birational to each other, for example, see [BC09].

16.1. **Fully faithful criterion.** Consider the Fourier-Mukai transform $\Phi_P : D^b(X) \to D^b(Y)$ between the derived categories of two smooth projective varieties X and Y given by an object $P \in D^b(X \times Y)$.

Proposition 16.7. *The functor Φ_P is fully faithful if and only if for any two closed points $x, y \in X$ one has*

$$Hom(\Phi_P(k(x)), \Phi_P(k(y)[i]) = \begin{cases} k \ if \ x = y \ and \ i = 0 \\ 0, \ if \ x \neq y \ or \ i < 0 \ or \ i > dim(X) \end{cases}$$

Note that this is much stronger than our general method to test fully faithfulness to the case of spanning class given by closed points. As it asserts that the difficult cohomology groups $Ext^i(k(x, k(y)))$ with $0 < i \leq dim(X)$ need not to be tested.

Proof. The proof is an application of Proposition 3.3. The verification is relatively long, we divide it into several steps. Since the Fourier-Mukai transform $F := \Phi_P$ admits a left adjoint $G := \Phi_{P_L}$ and a right adjoint $H := \Phi_{P_R}$. We may check the fully faithfulness on sky-scraper sheaves (plus some shifts). i.e., we only need

$$Hom(k(x), k(y)[i] \to Hom(F(k(x)), F(k(y))[i])$$

are bijective for any closed points x and y and any integer i. The assumption in the proposition as a priori yields the bijectivity only for $i \notin [1, dim(X)]$.

Step 1. Reduction to $G(F(k(x))) \simeq k(x)$

By lemma 2.5 we know that the bijectivity of

$$Hom(k(x), k(x)[i]) \to Hom(F(k(x)), F(k(x))[i])$$

is equivalent to the bijectivity of

$$Hom(k(x), k(x)[i]) \xrightarrow{\circ g_k(x)} Hom(G(F(k(x))), k(x)[i]),$$

which is induced by the adjunction morphism $g : G \circ F \to id_{D^b(X)}$.

If we can shoe that $G(F(k(x))) \simeq k(x)$, then the adjunction morphism

$$g_{k(x)} : G(F(k(x))) \to k(x)$$

is either an isomorphism, which immediately yields the bijectivity we need, or $g_{k(x)}$ is zero.

We can actually exclude that $g_{k(x)}$ is zero, since the composition of

$$F(g_{k(x)}) : F(G(F(k(x)))) \to F(k(x))$$

with the adjunction morphism

$$h_{F(k(x))} : F(k(x)) \to F(G(F(k(x))))$$

yields the identity and $F(k(x)) \neq 0$ due to the assumption $End(F(k(x))) = k$.

Step 2: Proof of $G(F(k(x))) \simeq k(x)$ under additional hypothesis

Let us fix a closed point $x \in X$. We shall first show $G(F(k(x))) \simeq k(x)$ under two additional assumptions:

(i) $G(F(k(x)))$ is a sheaf and

(ii) The homomorphism

$$Hom(k(x), k(x)[i]) \to Hom(F(k(x)), F(k(x))[i])$$

is at least injective for $i = 1$. Let us denote $G(F(k(x)))$ by F, which is a sheaf by additional assumption. Then we have

$$Hom(F, k(y)) = Hom(F(k(x)), F(k(y))) = 0$$

for any closed point $y \neq x$. Hence, F is concentrated in x. As explained earlier, the adjunction morphism $\delta := g_{k(x)} : F \to k(x)$ is not trivial and hence surjective. We have to show that δ is in fact bijective. Consider the short exact sequence

$$0 \to ker(\delta) \to F \xrightarrow{\delta} k(x) \to 0.$$

Clearly, $ker(\delta)$ is also concentrated on X and in order to show that $Ker(\delta) = 0$ it suffices to prove that $Hom(Ker(\delta), k(x)) = 0$. Applying $Hom(-, k(x))$ to the short exact sequence and using $Hom(F, k(x)) = k$, yields the exact sequence

$$0 \to Hom(ker(\delta), k(x)) \to Hom(k(x), k(x)[1]) \xrightarrow{\circ\delta} Hom(F, k(x)[1]).$$

The last map is injective due to our additional assumption (ii) and hence $Ker(\delta) = 0$.

Step 3: Verification of the additional hypothesis (i) We shall use the following general lemma.

Lemma 16.8. *Let X be a smooth projective variety, $x \in X$ a closed point, and $F \in D^b(X)$. Suppose $Hom(F, k(y)[i]) = 0$ for any closed point $y \neq x$ and any $i \in \mathbb{Z}$ and $Hom(F, k(x)[i]) = 0$ for $i < 0$ and $i > dim(X)$.*

Then F is isomorphic to a sheaf concentrated in $x \in X$.

Proof. We will write $\mathcal{H}^q$ for the cohomology sheaves of F. For a fixed point $y \in X$ we consider the spectral sequence

$$E_2^{p,q} := Hom(\mathcal{H}^{-q}, k(y)[p]) \Longrightarrow Hom(F, k(y)[p+q]).$$

Let m_0 be maximal with $y \in supp(\mathcal{H}^{m_0})$. Then $E_2^{0,m_0} \neq 0$ implies

$$E_\infty^{0,m_0} = Hom(F, k(y)[-m_0]) \neq 0.$$

This implies that $y = x$ and $-d \leq m_0 \leq 0$, where $d = dim(X)$. In other words, all cohomology sheaves of F are concentrated in $x \in X$ and in degree $-d \leq i \leq 0$.

On the other hand, $Hom(\mathcal{H}^{-q}, k(x)[p]) = 0$ for $p \notin [0, dim(X)]$ and, therefore, $E_2^{p,q} = 0$ for $p \notin [0, dim(X)]$ in the spectral sequence.

Let now m_1 be the minimal with $\mathcal{H}^{m_1} \neq 0$. By what has been shown, we know $m_1 \leq m_0 \leq 0$. Since the sheaf $\mathcal{H}^{m_1}$ is concentrated in x, one finds, by applying Serre duality, that $Hom(\mathcal{H}^{m_1}, k(x)[d]) \simeq Hom(k(x), \mathcal{H}^{m_1})^* \neq 0$.

A quick look at the spectral sequence reveals that $E_2^{d,-m_1}$ survives to the end, hence $E^{d-m_1} = Hom(F, k(x)[d-m_1])$. By assumption, the latter group is zero id $d - m_1 > d$, which thus only leaves the possibility $m_1 = m_0 = 0$. Hence, we win.

□

Thus, we have proved the first of our two additional assumptions in Step 2, namely that $G(F(k(x)))$ is a sheaf for any $x \in X$. Indeed, $F := G(F(k(x)))$ satisfies the assumption of the lemma, because

$$\begin{aligned} Hom(F, k(y)[i]) &\simeq Hom(G(F(k(x))), k(y)[i]) \\ &\sim Hom(F(k(x)), F(k(y))[i]) = 0 \end{aligned}$$

for $i \notin [0, dim(X)]$ or $x \neq y$ by assumption.

Stap 4: Verification of the additional hypothesis (ii) for generic x The composition $G \circ F$ is a Fourier-Mukai transform. We denote its kernal by Q. As we have just seen $i_X^* Q \simeq G(F(k(x)))$ is a sheaf, for any closed point $x \in X$. Here i_x^* is the derived pull-back of the inclusion $i_x : \{x\} \times X \hookrightarrow X \times X$. By lemma 10.1, we know that Q is a sheaf on $X \times X$ flat over the first factor.

Let us prove that $Hom(k(x), k(x)[1] \to Hom(F(k(x)), F(k(x)))$ is injective for generic $x \in X$. The composition with the functor G yields the map

$$\kappa(x) : Hom(k(x), k(x)[1]) \to Hom(G(F(k(x))), G(F(k(x))[1])$$

and we will show the injectivity of this map. This is clearly sufficient to ensure our additional assumption (ii).

Using the flatness of Q, we know that $\kappa(x)$ is the Kodaira-Spencer map of the flat family Q over $X \times X$ defining $G \circ F$.

On the other hand, the map $f : x \mapsto Q_x$ is injective, since for any x the sheaf $Q_x = G(F(k(x)))$ is concentrated in x. Hence the tangent map $\kappa(x) := df(x)$ is injective for $x \in X$ generic (Note here, we definitely use the assumption that characteristic is zero!).

Step 5. End of proof By Step 2, we know that for a generic point $x \in X$, $Q_x \simeq k(x)$. On the other hand, Q is flat over X and hence the Hilbert polynomial of Q_x is independent of $x \in X$. As we know in addition that Q_x is concentrated in x for any $x \in X$, we know that $Q_x \simeq k(x)$ for any $x \in X$.

□

17. Matrix factorization

The story of this section goes back to [BE77].

We have already seen the matrix factorization in Exercise 6.7. Let us quickly recall the example, let $X = xy = z^2$ be a quadratic cone in $\mathbb{A}^3$. Consider the line $x = z = 0$ and its minimal resolution.

We can also view this as the conic curve C in $\mathbb{P}^2$, and $z = x = 0$ is a closed point in C. Then we get a periodic complex

$$\cdots \to \mathcal{O}_C(-3)^{\oplus 2} \xrightarrow{M} \mathcal{O}_C(-2)^{\oplus 2} \xrightarrow{M} \mathcal{O}_C(-1)^{\oplus 2} \xrightarrow{(x,z)} \mathcal{O}_C \to \mathcal{O}_{pt} \to 0,$$

where $M = \begin{pmatrix} z & y \\ -x & -z \end{pmatrix}$. Notice that $M^2 = (z^2 - xy)id$.

Definition 17.1. A graded matrix factorization of a homogeneous polynomial of degree d is a pair of vector bundles $E = \oplus \mathcal{O}_{\mathbb{P}^n}(m_i)$, $F = \oplus \mathcal{O}_{\mathbb{P}^n}(n_i)$, and a pair of maps

$$E \xrightarrow{M} F \xrightarrow{N} E(d)$$

such that $NM = f \cdot 1_E$ and $MN = f \cdot 1_F$.

Note that we have $rank(E) = rank(F)$ from the definition.

We can continue

$$\cdots \to E \xrightarrow{M} F \xrightarrow{N} E(d) \xrightarrow{M} F(d) \xrightarrow{N} E(2d) \to \cdots$$

It is like a chain complex complex, but with $d^2 = f$, not 0.

Eventually, we will consider the category of matrix factorizations with chain maps module chain homotopy.

Example 17.2. *We have seen this in Example 11.9. Consider a minimal resolution of $\mathcal{O}_P$, the structure sheaf of a plane P in quadric hyper-surface $Q \subset \mathbb{P}^5$, where Q is cut out by $q = x_1y_1 + x_2y_2 + x_3y_3 = 0$. Assume that the plane P is cut out by $x_1 = x_2 = x_3 = 0$.*

$$0 \leftarrow \mathcal{O}_P \leftarrow \mathcal{O}_Q \leftarrow \mathcal{O}_Q(-1)^3 \leftarrow \mathcal{O}(-2)^4 \leftarrow \mathcal{O}(-3)^4 \leftarrow \mathcal{O}_Q(-4)^4 \leftarrow \mathcal{O}(-5)^4 \leftarrow \cdots$$

This becomes 2-periodic after $\mathcal{O}_Q(-2)^4$, which is a resolution of the Spinor bundle S. To get the matrices, we use a trick. We consider the matrices M and N with $MN = NM = (x_1y_1 + x_2y_2) \cdot 1_{2\times 2}$ first. For example,

$$M = \begin{pmatrix} x_1 & -x_2 \\ y_2 & y_1 \end{pmatrix} \text{ and } N = \begin{pmatrix} y_1 & x_2 \\ -y_2 & x_1 \end{pmatrix}$$

Then the matrices

$$A = \begin{pmatrix} M & -y_3 \cdot 1_{2\times 2} \\ x_3 \cdot 1_{2\times 2} & N \end{pmatrix} \quad B = \begin{pmatrix} N & y_3 \cdot 1_{2\times 2} \\ -x_3 \cdot 1_{2\times 2} & M \end{pmatrix}$$

satisfies that $AB = BA = q \cdot 1_{4\times 4}$.

Example 17.3. *Let us resolve $\mathcal{O}_{pt}$ on a cubic curve in $\mathbb{P}^2$. Let $C = f = 0 \subset \mathbb{P}^2$, $deg(f) = 3$, $p = \{x = y = 0\} \in C$, so $f \in (x, y)$. Therefore, we can write $f = xq_1 + yq_2$ with $deg(q_i) = 2$.*

(Of course, the choice of q_1 and q_2 is non-unique.)

Let $M = \begin{pmatrix} x & q_2 \\ -y & q_1 \end{pmatrix}$ and $N = \begin{pmatrix} q_1 & -q_2 \\ y & x \end{pmatrix}$ then $NM = MN = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$.

Then we have following resolution

$$0 \leftarrow \mathcal{O}_{pt} \leftarrow \mathcal{O}_C \xleftarrow{(x,y)} \mathcal{O}(-1)^2 \xleftarrow{M} \mathcal{O}(-2) \oplus \mathcal{O}(-3) \xleftarrow{N} \mathcal{O}(-4)^2 \leftarrow \cdots.$$

Claim: $ker(M) = im(N)$.

In general, denote $R = k[x, y, z]$ and $S = R/(f)$. Suppose we have two morphisms $S^2 \xrightarrow{M} S^2$, $S^2 \xrightarrow{N} S^2$ such that $MN = NM = f \cdot id$, and if $M \begin{pmatrix} g \\ h \end{pmatrix} = 0$ for $g, h \in S$.

Then $M \begin{pmatrix} \tilde{g} \\ \tilde{h} \end{pmatrix} = f \begin{pmatrix} k \\ l \end{pmatrix}$, where $\tilde{g}, \tilde{h} \in R$ are representatives of g, h, $k, l \in R$.

Therefore, we have $M \begin{pmatrix} \tilde{g} \\ \tilde{h} \end{pmatrix} = MN \begin{pmatrix} k \\ l \end{pmatrix}$, since R is a polynomial ring, and the determinant of M is nontrivial, hence M is not a zero-divisor. Therefore, we get

$$\begin{pmatrix} \tilde{g} \\ \tilde{h} \end{pmatrix} = N \begin{pmatrix} k \\ l \end{pmatrix}$$

and descends to S

$$\begin{pmatrix} g \\ h \end{pmatrix} = N \begin{pmatrix} \bar{k} \\ \bar{l} \end{pmatrix}.$$

This example has moduli, unlike the quadric examples.

Example 17.4. *Same as in last example, we assume that C is a cubic curve in $\mathbb{P}^2$. Let M be a 3×3 matrix of linear forms, and N its adjugate matrix (a 3×3 matrix of quadratic forms). Then $NM = MN = f1_{3\times 3}$, where $f = det(M)$ is a plane cubic equation.*

(The rank of a matrix drop by 1 in codimension 1, and the rank drop by 2 in codimension 4, since we are in $\mathbb{P}^2$, for a generic choice, it is a smooth cubic curve).

Therefore, we get a matrix factorization

$$\cdots \xleftarrow{N} \mathcal{O}(-1)^3 \xleftarrow{M} \mathcal{O}(-2)^3 \xleftarrow{N} \mathcal{O}(-3)^3 \xleftarrow{M} \cdots$$

is a matrix factorization of f.

Can we fix f, deform M, N to get something that is not homotopy equivalent?

Yes, consider on $\mathbb{P}^2$, we have

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^3 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1)^3 \to coker(M) \to 0.$$

The cokernal is a line bundle L on C.

We can compute the cohomology group of L, we get $H^*(C, L) = 0$. Hence L is a degree 0 line bundle on the elliptic curve C by Riemann-Roch.

This definitely have moduli. In fact, get every degree 0 line bundle except $\mathcal{O}_C$ in this way.

We will not prove it, instead we do the easy thing: the parameter count. We have $9\times 3 = 27$ choices for M. And the map $M \mapsto AMB$, for $A, B \in GL_3(k)$ gives the excessive dimension count, which is $9 + 9 - 1 = 17$.

On the other hand, we have $10-1$ dimensional cubic curves, and plus one extra dimension for the degree 0 line bundle on C. Hence, at least the dimension match up. But this moduli is not compact; it contains all the generic degree 0 line bundle without cohomology, but miss the structure sheaf $\mathcal{O}_C$.

For $\mathcal{O}_C$, this comes from a silly matrix factorization.

$$\mathcal{O}_C \xleftarrow{1} \mathcal{O}_C \xleftarrow{f} \mathcal{O}_C(-3) \xleftarrow{1} \mathcal{O}_C(-3) \leftarrow \cdots .$$

The idea is to put this silly matrix factorization and the general matrix factorization together. This will lead us to the general notion of matrix factorization.

Let L be any degree 0 line bundle on C. We have the following long exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-4)^3 \to \mathcal{O}_{\mathbb{P}^2}(-3)^9 \to \mathcal{O}_{\mathbb{P}^2}(-2)^6 \to L \to 0.$$

(Also think as free modules over $R = k[x, y, z]$.) We want to make a matrix factorization from this long exact sequence. We twist this long exact by $\mathcal{O}_{\mathbb{P}^2}(-3)$, we get another long exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-7)^3 \to \mathcal{O}_{\mathbb{P}^2}(-6)^9 \to \mathcal{O}_{\mathbb{P}^2}(-5)^6 \to L(-3) \to 0.$$

And f induces a chain map between these two long exact sequences, while restricting on $L(-3)$ is zero. Since f and 0 are two lifts of the same map $f : L(-3) \to L$. They are homotopic to each other, hence there exists maps h, such that $f = dh + hd$, where d are differentials for each long exact sequence (here we abuse of notation by omitting the lower index). Hence, we have the following diagram

$$\mathcal{O}(-8)^6 \oplus \mathcal{O}(-7)^3 \xrightarrow{(h,d)^T} \mathcal{O}(-6)^9 \xrightarrow{(d,h)} \mathcal{O}(-5)^6 \oplus \mathcal{O}(-4)^3 \xrightarrow{(h,d)^T} \mathcal{O}(-3)^9 \to \mathcal{O}(-2)^6 \to L.$$

We will draw a more vertical digram in class. This is a matrix factorization if the composition $h_0 \circ h_1 = 0$. This is indeed the case, we will prove it in class.

18. Landau-Ginzburg models and matrix factorizations

There are several definitions of Landau-Ginsburg models in mathematical literature. By now the author is not capable of seeing the equivalences between them. For example, the definition of LG model in [Orl09] looks different from the definition in [ASS14].

We will mainly follow the definition in [Orl09], and list the definition in [ASS14] in the subsection as auxiliary contents (it is all because of the author's ignorance, it does not mean one is more important than the other one to any extend).

Before introduce the notion of Landau-Ginzburg models, we need the notion of triangulated categories of singularities.

Let X be a scheme over a field k. We will say that it satisfies condition (ELF) if it is separated, noetherian, of finite Krull dimension, and the category of coherent sheaves coh(X) has enough locally free sheaves.

The last condition means that for any coherent sheaf $\mathcal{F}$ there is a vector bundle $\mathcal{E}$ and an epimorphism $\mathcal{E} \twoheadrightarrow \mathcal{F}$. For example any quasi-projective scheme satisfies these conditions.

Recall that as in the proof of injective case: for any bounded above complex of coherent sheaves C on X there is a bounded above complex of locally free sheaves P and a quasi-isomorphism of the complexes $P \to C$.

Also recall that we have a brutal truncation $\delta^{\geq k}$, for any complex C. We have

$$\delta^{\geq k} C = \cdots \to 0 \to C^k \to C^{k-1} \to C^{k+2} \to \cdots .$$

This is a subcomplex of C. The quotient $C/\delta^{\geq k}C$ is another brutal truncation $\delta^{\leq k-1}C$.

Note that this is very different from the standard truncation functors of a derived category, which id denoted by $\tau^{\leq k}, \tau^{\geq k}$.

Definition 18.1. A bounded complex of coherent sheaves will be called a perfect complex if it is quasi-isomorphic to a bounded complex of locally free sheaves of finite type.

Lemma 18.2. *Any complex C, which is isomorphic to a bounded complex of locally free sheaves in $D^b(X)$, is perfect.*

Note that this is not a trivial statement, since there exists examples of complexes which are isomorphic in derived category, but not quasi-isomorphic.

Proof. We can represent the isomorphism in the derived categories as a roof

$$\begin{tikzcd} & E \arrow[dl, "s"'] \arrow[dr, "t"] & \\ P & & C \end{tikzcd},$$

where P is a bounded complex of locally free sheaves and s, t are quasi-isomorphisms.

There is a bounded above complex Q of locally free sheaves and quasi-isomorphism $Q \to E$. Consider the standard truncation $\tau^{\geq -k}Q$ for sufficiently large k. As E is bounded there is a morphism $r : \tau^{\geq -k}Q \to E$ that is also a quasi-isomorphism.

To prove the lemma, it suffices to show that $\tau^{\leq -k}Q$ is a complex of locally free sheaves. Consider the composition $sr : \tau^{\geq -k}Q \to P$ which is a quasi-isomorphism. The cone of sr is a bounded acyclic complex all terms of which, except maybe the leftmost term, are locally free. This implies that the leftmost term is locally free as well, because the kernal of an epimorphism of a locally free sheaves is locally free. Thus we win. □

The perfect complexes form a full triangulated subcategory $perf(X) \subset D^b(X)$, which is thick. Recall a thick triangulated category is a strict full triangulated subcategory and if whenever $X \oplus Y$ is isomorphic to an object of $perf(X)$ then both X and Y are isomorphic to objects of $perf(X)$. Hence the kernal of the quotient functor is equivalent to $perf(X)$.

Definition 18.3. We define a triangulated category $D_{sg}(X)$ as the quotient of the triangulated category $D^b(X)$ by the full triangulated category $perf(X)$ and call it as a triangulated category of singularities of X.

Remark 18.4. It is known that if a scheme X is as above and is regular in addition then the subcategory of perfect complexes coincides with the whole bounded derived category of coherent sheaves. Hence $D_{sg}(X)$ is trivial in this case.

Lemma 18.5. *Let X be as above and let $\mathcal{F}$ be a coherent sheaf on X such that for any point $x \in Sing(X)$ it is locally free in some neighborhood of x. Then it is a perfect complex.*

Proof. There is a bounded above complex Q of locally free sheaves and quasi-isomorphism $Q \to \mathcal{F}$. Consider $\tau^{\geq -k}Q$ for sufficiently large k. There is a morphism $r : \tau^{\geq -k}Q \to \mathcal{F}$ that is also a quasi-isomorphism. To prove the lemma it is sufficient to show that $\tau^{\geq -k}Q$ is a complex of locally free sheaves. All terms of this complex, except maybe the leftmost term, are locally free.

But for any point $x \in Sing(X)$ the leftmost term is also locally free in some neighborhood of x, because $\mathcal{F}$ is locally free there. If now $x \notin Sing(X)$ then there is a neighborhood U of X which is regular. Hence the leftmost term is locally free on U under the assumption $k > dimX$ by Hilbert syzygy theorem. □

Lemma 18.6. *Let X be as above. Then any object $A \in D_{sg}(X)$ is isomorphic to an object $\mathcal{F}[k]$ where $\mathcal{F}$ is a coherent sheaf.*

Lemma 18.7. *Consider the locally free bounded above resolution $P \simeq A$. Consider a brutal truncation $\delta^{\geq -k}P$ for sufficiently large $k \gg 0$. Denote by $\mathcal{F}$ the cohomology $H^{-k}(\delta^{\geq -k}P)$. It is clear that $A \simeq \mathcal{F}[k+1]$ in $D_{sg}(X)$.*

Here is the local property for triangulated categories of singularities.

Proposition 18.8. *Let X be as above and let $j : U \hookleftarrow X$ be an embedding of an open subscheme such that $Sing(X) \subset U$. Then the functor $\bar{j}^* : D_{sg}(X) \to D_{sg}(U)$ is an equivalence of triangulated categories.*

Proof. Since X is noetherian and j is an open embedding. We have that the composition j^*Rj_* is isomorphic to the identity functor (since j^* is exact and Rj_* is fully faithful). Take an object $B \in perf(U)$ and consider $Rj_*(B)$. It is easy to see that the object $Rj_*(B)$ belongs to $perf(X)$. Actually, this condition is local. For U it is fulfilled and for $X\backslash sing(X)$ as for smooth scheme it is evident. Thus the functor Rj_* induces the functor

$$R\bar{j}_* : D_{sg}(U) \to D_{sg}(X).$$

Moreover, the functor Rj_* is right adjoint to $\bar{j}^*$.

For any object $A \in D^b(X)$ we have a canonical map $\mu_A : A \to Rj_*j^*A$. A cone $C(\mu_A)$ of this map is an object whose support belongs to $X\backslash U$ and does not intersect $Sing(X)$. Hence $C(\mu_A$ belongs to the subcategory $perf(X)$. Thus the morphism μ_A becomes an isomorphism in $D_{sg}(X)$. Therefore, the functor

$$\bar{j}^* : D_{sg}(X) \to D_{sg}(U)$$

is fully faithful. On the other hand, we know that $j^*Rj_*(B) \simeq B$ for any $B \in D^b(Qcoh(U))$. Hence $\bar{j}^*$ is an equivalence. □

Remark 18.9. Note that there is a mistake in the proof. We only prove that $\bar{j}^*$ is an equivalence in the triangulated category of singularities of quasi-coherent sheaves. Since j is not proper. To conclude it is an equivalence in $D_{sg}(X)$, we need [Har13, Ex 5.15] saying that any coherent sheaf on U can be obtained as the restriction of a coherent sheaf on X. Combing this with lemma 18.7. And also we use the following result in the proof.

Exercise 18.10. *Let $\mathcal{M}$ and $\mathcal{N}$ be full triangulated subcategories in triangulated categories $\mathcal{C}$ and $\mathcal{D}$ respectively. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be adjoint pair of exact functors such that $F(\mathcal{M}) \subset \mathcal{V}$ and $G(\mathcal{N}) \subset \mathcal{M}$. Then they induce functors*

$$\bar{F} : \mathcal{C}/\mathcal{M} \to \mathcal{D}/\mathcal{N},\ \ \bar{N} : \mathcal{D}/\mathcal{N} \to \mathcal{C}/\mathcal{M}$$

which are adjoint.

The Landau-Ginzburg model in [Orl09] consists of the following data: a smooth variety (or a regular scheme) X and a regular function X such that the morphism $W : X \to \mathbb{A}^1$ is flat.

With any closed point $w_0 \in \mathbb{A}^1$ we can associated a differential $\mathbb{Z}/2\mathbb{Z}$-graded category $DG_{w_0}(W)$, an exact category $Pair_{w_0}(W)$ and a triangulated category $DB_{w_0}(W)$. We give construction of these categories under the condition $X = Spec(A)$ is affine. The general case is more involved.

Since the category of coherent sheaves on an affine scheme $X = Spec(A)$ is the same as the category of finitely generated A-module we will frequently go from sheaves to modules and back. Note that under this equivalence locally free sheaves are the same as projective modules.

Objects of these categories are ordered pairs

$$\bar{P} := (P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} P_1)$$

where P_0, P_1 are finitely generated projective A-modules and the composition p_0p_1 and p_1p_0 are the multiplication by the element $(W - w_0) \in A$.

Morphisms from $\bar{P}$ to $\bar{Q}$ in the category $DG_{w_0}(W)$ form $\mathbb{Z}/2\mathbb{Z}$-graded complex

$$\mathbb{H}om(\bar{P}, \bar{Q}) = \oplus_{i,j} Hom(P_i, Q_j)$$

with a natural grading on the parity of $(i - j)$, and with a differential D acting on homogeneous elements of degree k as

$$Df = q \circ f - (-1)^k f \circ p.$$

The space of morphisms $Hom(\bar{P}, \bar{Q})$ in the category $DB_{w_0}(W)$ $(Pair_{w_0}(W)$ $)$ is the 0-th cohomology group of the complex (the kernal of 0-th differential).

$$Hom_{Pair_{w_0}(W)} = Z^0(\mathbb{H}om(\bar{P}, \bar{Q})),\ Hom_{DB_{w_0}(W)} = H^0(\mathbb{H}om(\bar{P}, \bar{Q})).$$

Thus a morphism $f : \bar{P} \to \bar{Q}$ in the category $DB_{w_0}(W)$ is a pair $f_1 : P_1 \to Q_1$ and $f_0 : P_0 \to Q_0$ such that $f_1p_0 = q_0f_0$ and $q_1f_1 = f_0p_1$. The morphism is zero if there exists two morphism $s : P_0 \to Q_1$ and $t : P_1 \to Q_0$ such that $f_1 = q_0t + sp_1$ and $f_0 = tp_0 + q_1s$.

Remark 18.11. Such construction appeared many years ago in the paper [Eis80] and is known for specialist in singular theory as a matrix factorization. Then people found its application in physics in 21st century.

The category $DB_{w_0}(W)$ can be endowed with a natural structure of a triangulated category. To determine it we have to define a translation functor [1] and a class of exact triangles.

The translation functor can be defined as a functor that take an object $\bar{P}$ to he object

$$\bar{P}[1] = (P_0 \xrightarrow{-p_0} P_1 \xrightarrow{-p_1} P_1).$$

We see that the functor [2] is the identity functor.

For any morphism $f : \bar{P} \to \bar{Q}$, we define a mapping cone $C(f)$ as an object

$$C(f) = (Q_1 \oplus P_0 \xrightarrow{c_1} Q_0 \oplus P_1 \xrightarrow{c_0})$$

such that

$$c_0 = \begin{pmatrix} q_0 & f_1 \\ 0 & -p_1 \end{pmatrix}, c_1 = \begin{pmatrix} q_1 & f_0 \\ 0 & -p_0 \end{pmatrix}.$$

There are maps $g : \bar{Q} \to C(f)$, $g = (id, 0)$ and $h : C(f) \to \bar{P}[1]$, $h = (0, -id)$.

Then we can define the class of exact triangles and prove it is a triangulated category as in the homotopy category case.

Definition 18.12. We define a category of D-branes of type B (B-branes) on X with the superpotential W as the product

$$DB(W) = \Pi_{w \in \mathbb{A}^1} DB_w(W).$$

Note that since X is regular, the set of points on $\mathbb{A}^1$ with singular fibers is finite (see [Har13, III,Cor.10.7], we suppose here that we work over a field of characteristic 0). It will be shown that $DB_w(W)$ is trivial if the fiber over point w is smooth. Hence, it is in fact a finite product.

18.1. **Categories of pairs and categories of singularities.** In this subsection, we will sketch the the equivalence between the category of B-branes in a Landau-Ginsburg model and triangulated categories of singular fibers, introduced above. The readers should also consult [Eis80, Section 6].

We further suppose $w_0 = 0$. Denote by X_0 the fiber of $f : X \to \mathbb{A}^1$ over the point 0. With any pair $\bar{P}$ we can associate a short exact sequence

$$0 \to P_1 \xrightarrow{p_1} P_0 \to coker(p_1) \to 0.$$

We can attach to an object $\bar{P}$ the sheaf $coker(p_1)$. This is a sheaf on X_0, since W annihilates it. In fact $cok : Pair_0(W) \to coh(X_0)$ is a functor.

Lemma 18.13. *The functor $cok : Pair_0(W) \to coh(X_0)$ is full.*

Proof. The proof is very easy. Any map $g : coker(p_1) \to coker(q_1)$ can be extended to a map of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r, "p_1"] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & coker(p_1) \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & Q_1 \arrow[r, "q_1"] & Q_0 \arrow[r] & coker(q_1) \arrow[r] & 0 \end{tikzcd}$$

because P_0, P_1 are projective. To prove the lemma, it suffices to show that $f_1 p_0 = q_0 f_0$. But this follows easily from composing with the injection q_1. □

Then this functor induces a functor $F : DB_0(W) \to D_{sg(X_0)}$. By some standard arguments (like full implies faithful if the functor has no kernal) and the results on $D_{sg}(X)$ when X is Gorenstein scheme. We can prove it is an equivalence. For details, see [Orl09, Section 2, Section 3].

For the general case, we state a non-trivial theorem of Orlov in [Orl09] without proof.

Theorem 18.14. *Let X be the affine space $\mathbb{A}^N$ and let W be a homogeneous polynomial of degree d. Let $Y \subset \mathbb{P}^{N-1}$ be the hyper-surface of degree d that is given by the equation $W = 0$. Then, there is the following between the triangulated category of graded B-branes $DGrB(W)$ and the derived category of coherent sheaves $D^b(Y)$:*

(i) if $d < N$, i.e., Y is a Fano variety, there is a semi-orthogonal decomposition

$$D^b(Y) \simeq \langle \mathcal{O}_Y(d - N + 1), \cdots, \mathcal{O}_Y, DGrB(W)\rangle;$$

(ii) if $d > N$, i.e., Y is a variety of general type, there is a semi-orthogonal decomposition

$$DGrB(W) \simeq \langle F^{-1}q(k(r + 1)), \cdots, F^{-1}q(k), D^b(Y)\rangle,$$

where $q : D^b(gr - A) \to D^{gr}_{Sg}(A)$ is the natural projection, and $F : DGrB \xrightarrow{\sim} D^{gr}_{Sg}(A)$ is the equivalence in [Orl09, Proposition 3.5].

(iii) if $d = N$, i.e., Y is a Calabi-Yau variety, there is an equivalence

$$DGrB(W) \xrightarrow{\sim} D^b(coh(Y)).$$

18.2. Another definition of Landau-Ginsburg models.

Definition 18.15. A Landau-Ginzburg model consists of the following data: X is a scheme or a (stack) over $\mathbb{C}$; a $\mathbb{C}^*$ action on X called R-charge with -1 acting trivially; an element $W \in \Gamma(\mathcal{O}_X)$ is an equivariant of weight 2 (it is called super-potential).

Example 18.16. *(0) Let X be a regular scheme, $\mathbb{C}^*$ acts trivially. $W = 0$. Then we will recover $D^b(X)$.*

(1) Let $X = \mathbb{C}_{x,p}$, and $\mathbb{C}^*$ acts on the first coordinate by weight 0, and weight 2 on the second coordinate. Let $W = xp$.

(2) Let $E \to B$ a vector bundle over a smooth scheme B (e.g. total space of $\mathcal{O}(-3) \to \mathbb{P}^2$), then $s \in \Gamma(E^*)$ determines a function W on $tot(E)$. Let $\mathbb{C}^*$ acts on $X = tot(E)$ with weight 2 on fibers. Eventually, $D(X, W) \simeq D(\{s = 0\} \subset B)$ if $codim\{s = 0\} = rank(E)$.

A matrix factorization consists of: a $\mathbb{C}^*$-equivariant vector bundle E on X, and a map $d : E \to E[1]$, note $E[1]$ is change of the $\mathbb{C}^*$ action, not the shift, such that $d^2 = W \cdot id_E$.

Example 18.17. *(0) Let X be a regular scheme over $\mathbb{C}$, $\mathbb{C}^*$ acts trivially on X. Hence an equivariant bundle F can be split into direct sum of F_i, with $\mathbb{C}^*$ acts on each F_i with weight i. Then $d^2 = W = 0$, this gives us a complex of vector bundles.*

(1) We consider the matrix factorization of Landau-Ginsburg model in Example 18.16.(1). Take $E = \mathcal{O}_X \oplus \mathcal{O}_X[-1]$, then $E[1] = \mathcal{O}[1] \oplus \mathcal{O}_X$. The matrix $d = \begin{pmatrix} 0 & p \\ x & 0 \end{pmatrix}$, $d^2 = w \cdot id$.

19. Flips and flops

In this section, we go back to investigate the relation between $D-$equivalences and $K-$equivalences.

Firstly, we need to do some technical preparations. We want to compute various Tor- and Ext-groups arising naturally in situations like closed embeddings and blow-ups. E.g. one needs to know exactly how to compute $\mathcal{E}xt^i_X(\mathcal{O}_Y, \mathcal{O}_Y)$ for a closed subvariety $j : Y \hookleftarrow X$ or the pull-back $q^*\mathcal{O}_Y$ under the blow-up $q : \tilde{X} \to X$ of X along Y. The strategy in both cases is to treat a linearized version and then to glue the local information.

We shall first discuss in detail the situation of a closed embedding

$$j : Y \hookrightarrow X$$

of codimension c.

In a first step, we will suppose that Y is given as the zero locus of a regular section $s \in H^0(X, E)$ of a locally free sheaf E of rank c. In this case, its structure sheaf can be resolved by Koszul complex

$$0 \to \wedge^c E^* \to \cdots \to E^* \to \mathcal{O}_X \to j_*\mathcal{O}_Y \to 0,$$

with morphisms given by contraction with s. Note that in this case the normal bundle of $Y \subset X$ is given by

$$N := N_{Y/X} \simeq E|_Y.$$

More precisely, differentiating the section $s \in H^0(X, E)$ induces a canonical morphism $T_X|_Y \to E|_Y$ which provide the canonical isomorphism.

Proposition 19.1. *Suppose $j : Y \hookrightarrow X$ with normal bundle N is the zero locus of a regular section of a locally free sheaf of rank $c = codim(Y \subset X)$. Then there exists a canonical isomorphism*

$$j^* j_* \mathcal{O}_Y \simeq \oplus \wedge^k N^*[k]$$

and for any $F \in D^b(Y)$ one has:

$$j_* j^* j_* F \simeq j_* \mathcal{O}_Y \otimes j_* F \simeq j_*(\oplus \wedge^k N^*[k] \otimes F)$$

$$\mathcal{H}om(j_* \mathcal{O}_Y, j_* F) \simeq j_*(\oplus \wedge^k N[-k] \otimes F).$$

Proof. As before, we denote by $s \in H^0(X, E)$ the section defining Y. Thus the Koszul complex allows us to compute $j^* j_* \mathcal{O}_Y$ as pulling back the complex to Y. As the differentials in the Koszul complex is given by contraction with the defining section s, they become trivial on Y. In other words, $j^* j_* \mathcal{O}_Y \simeq \oplus \wedge^k E^*|_Y[k]$. Hence, we get the first isomorphism.

To prove the second isomorphism, split the resolution into short exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \wedge^{i+1} E^* \arrow[rr] \arrow[dr] & & \wedge^i E^* \arrow[rr] \arrow[dr] & & \wedge^{i-1} E^* \arrow[r] & \cdots \\
& & M_{i+1} \arrow[ur] & & M_i \arrow[ur] & &
\end{tikzcd}$$

As $j_* F$ is concentrated in Y and the morphisms $M_{i+1} \to \wedge^i E^*$ vanish along Y, each short sequence

$$0 \to M_i \to \wedge^i E^* \to M_i \to 0,$$

considered as a distinguished triangle yields isomorphism

$$M_i \otimes j_* F \simeq (\wedge^i E^* \otimes j_* F) \oplus (M_{i+1}[1] \otimes j_* F).$$

Putting things together and using

$$\wedge^i E^* \otimes j_* F \simeq j_*(j^* \wedge^i E^* \otimes F) \simeq j_*(\wedge^i N^* \otimes F)$$

proves the second isomorphism.

To prove the last isomorphism, we first write

$$\mathcal{H}om(j_* \mathcal{O}_Y, j_* F) \simeq \mathcal{H}om(\wedge^\bullet E^*, j_* F) \simeq (\wedge^\bullet E^*)^\vee \otimes j_* F.$$

Argue as before, split into short exact sequence, this yields split distinguished triangles. $\square$

By abuse of notation, we often write simply $\mathcal{O}_Y$ instead of $j_* \mathcal{O}_Y$.

Corollary 19.2. *Under the same assumption of the proposition one has for any $F \in D^b(Y)$:*

$$\mathcal{H}^l(j^*j_*F) \simeq \oplus_{s-r=l} \wedge^r N^* \otimes \mathcal{H}^s(F)$$

and

$$\mathcal{E}xt^l_X(j_*\mathcal{O}_Y, j_*F) \simeq j_*(\oplus_{r+s=l} \wedge^r N \otimes \mathcal{H}^s(F)).$$

Proof. As j_* is exact, one has $\mathcal{H}^L \circ j_* \simeq j_* \circ \mathcal{H}^l$. Thus the proposition implies $j_*\mathcal{H}^l(j^*j_*F) \simeq \mathcal{H}^l(j_* \oplus \wedge^k N^*[k] \otimes F) \simeq j_*(\oplus \wedge^k N^* \otimes \mathcal{H}^{k+l}(F))$, which yields the first assertion. The second assertion is proved similarly. Note that in both cases we use that tensor product with the locally free sheaf N commutes with taking cohomology. □

Remark 19.3. There is a different approach for the construction of the isomorphisms

$$\mathcal{E}xt^l_X(j_*\mathcal{O}_Y, j_*\mathcal{O}_Y) \simeq \wedge^l N.$$

(Here, we set $F = \mathcal{O}_Y$ in the second isomorphism in the corollary). This one works without assuming that $Y \subset X$ is the zero set of a regular section.

By definition $N \coloneqq N_{Y/X} \simeq \mathcal{E}xt^0(\mathcal{I}_Y, \mathcal{O}_Y)$. Applying $\mathcal{H}om(-, \mathcal{O}_Y)$ to the short exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

yields a canonical homomorphism $N \to \mathcal{E}xt^1(\mathcal{O}_Y, \mathcal{O}_Y)$, which is in fact an isomorphism as $\mathcal{E}xt^1_X(\mathcal{O}_X, \mathcal{O}_Y) = 0$. This proves the claim for $l = 1$.

For $l > 1$ one considers the induces homomorphism

$$\wedge^l N \simeq \wedge^l \mathcal{E}xt^1_X(\mathcal{O}_Y, \mathcal{O}_Y) \to \mathcal{E}xt^l_X(\mathcal{O}_Y, \mathcal{O}_Y),$$

where the latter is given by the cup product. That this is indeed an isomorphism can be checked by a local calculation.

Corollary 19.4. *Let $j : Y \hookrightarrow X$ be a smooth hyper-surface. Then*

$$j^*j_*\mathcal{O}_Y \simeq \mathcal{O}_Y \oplus \mathcal{O}_Y(-Y)[1]$$

and for any $F \in D^b(Y)$ there exists a distinguished triangle and an isomorphism:

$$F \otimes \mathcal{O}_Y(-Y)[1] \to j^*j_*F \to F$$

and

$$j_*j^*j_*F \simeq j_*F \oplus j_*(F \otimes \mathcal{O}_Y(-Y))[1].$$

Proof. The isomorphisms comes from our proposition. And distinguished triangle was proved in Lemma 11.11. □

As it turns out, the computation of cohomology sheaves in Corollary 19.2 remains valid even if the subvariety Y is not given as the zero section of a locally free sheaf.

Proposition 19.5. *Let $j : Y \hookrightarrow X$ be an arbitrary closed embedding of sooth varieties. Then there exist isomorphisms*

$$\mathcal{H}^i(j^*j_*\mathcal{O}_Y) \simeq \wedge^{-i} N*_{Y/X}$$

and

$$\mathcal{E}xt^i_X(j_*\mathcal{O}_Y, j_*\mathcal{O}_Y) \simeq \wedge^i N_{Y/X}.$$

Proof. Choose a global locally free resolution $G^\bullet \to \mathcal{O}_Y$ and consider the induced free resolution $G^\bullet_y \to \mathcal{O}_{Y,y}$ of $\mathcal{O}_{X,y}$-modules for any point $y \in Y$. Locally around $y \in Y \subset X$, i.e. on an open neighborhood $y \in U \subset X$, we may find a locally free sheaf E of rank c together with a regular section $s \in H^0(U, E)$ defining $Y \cap U$. This yields the second free resolution $\wedge^\bullet E^*_y \to \mathcal{O}_{Y,y}$ (this one is a minimal resolution, since the differential restricting on y becomes trivial).

Using the projectivity of free modules, we obtain a morphism of complexes $\phi : G^\bullet_y \to \wedge^\bullet E^*_y$. Pulling back via $j : Y \hookrightarrow X$ and taking cohomology yields isomorphisms $\mathcal{H}^i(j^*\phi)$: $\mathcal{H}^i(j^*G^\bullet)_y \simeq \mathcal{H}^i(j^*G^\bullet_y) \xrightarrow{\sim} \mathcal{H}^i(j^* \wedge^\bullet E^*_y) \simeq \wedge^{-i} N^*_y$.

Another choice of the isomorphism ϕ is homotopic to the original one and thus induces the same map on cohomology. For another choice of the minimal resolution, say defined by a section $\tilde{s}$ of $\tilde{E}$, any isomorphism $E \simeq \tilde{E}$ that send s to $\tilde{s}$ induces the identity on N. Hence the identification $\mathcal{H}^i(j^*G)_y \simeq \wedge^{-i} N^*_y$ is independent of any choice and thus leads to a global isomorphism $\mathcal{H}^i(j^*j_*\mathcal{O}_Y) \simeq \wedge^{-i} N^*$.

The proof for the second assertion is similar.

□

Recall the following examples we have done before.

Example 19.6. *(i) Consider the diagonal embedding $\iota : X \xrightarrow{\sim} \Delta \subset X \times X$, the conormal bundle of which is by definition the cotangent bundle ω^1_X. Equivalently, $N_{\Delta/X\times X} \simeq \iota_* T_X$. Thus the above isomorphism can in this situation be written as*

$$\mathcal{E}xt^i_{X\times X}(\iota_*\mathcal{O}_X, \iota_*\mathcal{O}_X) \simeq \wedge^i T_X.$$

(ii) This can also be applied to a closed point $x \in X$. It leads to the following description

$$Ext^i_X(k(x), k(x)) \simeq \wedge^i Ext^1_X(k(x), k(x)) \simeq \wedge^i T_x.$$

19.1. **The standard flip.** Suppose X contains a smooth subvariety Y isomorphic to $\mathbb{P}^k$ and such that the normal bundle $N := N_{\mathbb{P}^n/X}$ is isomorphic to $\mathcal{O}(-1)^{\oplus l+1}$. So, $l+1 = codim(Y \subset X)$. Using the adjunction formula for $Y \subset X$, we obtain

$$\omega_X|_{\mathbb{P}^k} \simeq \mathcal{O}(l-k).$$

The blow-up $q : \tilde{X} \to X$ of X along $\mathbb{P}^k$ produces an exceptional divisor $E \simeq \mathbb{P}(N)$ which is isomorphic to $\mathbb{P}^k \times \mathbb{P}^l$. The two projections will be denoted by $\pi : \mathbb{P}^k \times \mathbb{P}^l \to \mathbb{P}^k$ and $\pi' : \mathbb{P}^k \times \mathbb{P}^l \to \mathbb{P}^l$. In particular, $q|_E = \pi$.

Let us first of all compute the relevant line bundles on $\tilde{X}$ and their restriction to E. As we know

$$\omega_{\tilde{X}} \simeq q^*\omega_X \otimes \mathcal{O}(lE)$$

and

$$\omega_E \simeq (\omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(E))|_E.$$

If we write $\mathcal{O}(a,b)$ for the line bundle $\pi^*\mathcal{O}(a) \otimes \pi'^*\mathcal{O}(b)$ on $\mathbb{P}^k \times \mathbb{P}^l$ this yields

$$\mathcal{O}(-k-1,-l-1) \simeq \pi^*(\omega_X|_{\mathbb{P}^k}) \otimes \mathcal{O}_E((l+1)E).$$

Altogether, we have

$$\mathcal{O}_E(E) \simeq \mathcal{O}(-1,-1)$$

and

$$\omega_{\tilde{X}}|_E \simeq \mathcal{O}(-k,-l).$$

In particular, the Fujiki-Nakano criterion applies and yields a contraction $p : \tilde{X} \to X'$, which is a blow-up of $\mathbb{P}^l \subset X'$ with exceptional divisor E such that the restriction of p to E equals π'. One furthermore shows that $N' := N_{\mathbb{P}^l/X'} \simeq \mathbb{O}(-1)^{\oplus k+1}$. We will tacitly assume that X' is projective as well.

Some words about Fujiki-Nakano criterion, it is about the converse direction of blow-ups which allows us to contract negative divisors. More precisely, if a divisor $E \subset \tilde{X}$ of a complex manifold has a structure of being projective bundle over a projective variety Y, i.e. there exists $\pi : E = \mathbb{P}(N) \to Y$. And the line bundle $\mathcal{O}(E)$ on the fibers of π is of degree -1, then the projection π extends to a morphism $q : \tilde{X} \to X$ onto a complex manifold. A word of warning: X does not have to be projective even all all $\tilde{X}$, E and Y are projective.

The situation is not quite symmetric, at least not when $k \neq l$. Indeed, for $l < k$ the restriction ω_X to $\mathbb{P}^k$ is the negative line bundle $\mathcal{O}(l-k)$, whereas the restriction $\omega_{X'}$ to $\mathbb{P}^l$ is the positive line bundle $\mathcal{O}(k-l)$. Changing the sign of the canonical bundle in this way is usually called a flip. The diagram

$$\begin{tikzcd}
 & & E = \mathbb{P}^k \times \mathbb{P}^l \arrow[d, "i"] \arrow[dll, "\pi"'] \arrow[drr, "\pi'"] & & \\
 & & \tilde{X} \arrow[dl, "q"'] \arrow[dr, "p"] & & \\
\mathbb{P}^k \arrow[r] & X & & X' & \mathbb{P}^l \arrow[l]
\end{tikzcd}$$

is called a standard flip. In the special case $k = l$ the restriction of the canonical bundle on both sides is trivial and, in particular, does not change when passing from X to X'. This is called the standard flop.

In general, flips are birational transformations which are performed in order to increase the positivity of the canonical bundle (and eventually to reach a minimal model). General flops are birational transformations that describe the passage from one minimal model to another one.

Now, we are able to compare the derived categories of X and X' inside the bigger derived category $D^b(\tilde{X})$. Roughly speaking, the derived category of the variety whose canonical bundle is more negative tends to be bigger.

Exercise 19.7. *Show that $q_* \circ p^*$ is the Fourier-Mukai transform $\Phi_{\mathcal{O}_{\tilde{X}}}$, where the structure sheaf $\mathcal{O}_{\tilde{X}}$ is viewed as the structure sheaf of the image $(p, q) : \tilde{X} \to X' \times X$.*

Proposition 19.8. *Let $X \leftarrow \tilde{X} \to X'$ be the standard flip with $l \leq k$ as constructed above. Then*

$$q_* \circ p^* : D^b(X') \to D^b(X)$$

is fully faithful. If $k = l$, it defines an equivalence.

Proof. The proof uses the two semi-orthogonal decomposition of $D^b(\tilde{X})$ induced by the two blow-up maps q and p. We use the notation

$$D^b(\tilde{X}) = \langle D_{-l}, \cdots, D_{-1}, D^b(X)\rangle$$

and

$$D^b(\tilde{X}) = \langle D'_{-k}, \cdots, D'_{-1}, D^b(X')\rangle,$$

with $D_b = i_*(\pi^* D^b(\mathbb{P}^k) \otimes \mathcal{O}(0, b))$ and $D_b = i_*(\pi'^* D^b(\mathbb{P}^l) \otimes \mathcal{O}(a, 0))$.

In order to prove the assertion, we have to show that for arbitrary $E, F \in D^b(X')$, we have

$$Hom(E, F) \simeq Hom(q_*p^*E, q_*p^*F).$$

As we have shown in this case, the functor p^* is fully faithful and hence

$$Hom(E, F) \simeq Hom(p^*E, p^*F).$$

On the other hand, we have

$$Hom(q_*p^*E, q_*p^*F) \simeq Hom(q^*q_*p^*E, p^*F).$$

Thus, it suffices to prove that the adjunction morphism $q^*q_*p^*E \to p^*$ induces a bijection

$$Hom(p^*E, P^*F) \xrightarrow{\sim} Hom(q^*q_*p^*E, p^*F)$$

for any $F \in D^b(X')$. As we have already shown in earlier lectures, these functorial morphisms are indeed compatible.

To this end, we first complete the adjunction morphism $q^*q_*p^*E \to p^*$ to a distinguished triangle

$$q^*q_*p^*E \to p^*E \to H \to q^*q_*p^*E[1].$$

It suffices to prove that $Hom(H, p^*F) = 0$ for any $F \in D^b(X')$. It follows from the following two things:

(i)$Hom(H, i_*\mathcal{O}(a, b)) = 0$ if $-k \leq a \leq -1$ and $-l \leq b \leq -1$.

(ii) $H \in D^b(X)^{\perp}$.

(i) is easy to prove, note that under the numerical condition, we have $i_*\mathcal{O}(a, b) \in D'_a \cap D_b$ and hence $i_*\mathcal{O}(a, b) \in D^b(X)^{\perp} \cap D^b(X')^{\perp}$. In particular,

$$Hom(q^*q_*p^*E, i_*\mathcal{O}(a, b)) = 0 = Hom(p^*E, i_*\mathcal{O}(a, b)).$$

Hence (i) follows from the long exact sequence.

For (ii), apply q_* on the triangle and use that $q_*q^*q_*p^*E \simeq q_*p^*E$, as q^* is fully faithful. This yields that $q_*H = 0$. Therefore, by adjunction $Hom(q^*G, H) \simeq Hom(G, q_*H) = 0$ for any $G \in D^b(X)$.

We have a full exception collection of $D^b(X)^{\perp}$:

$$\begin{aligned} D^b(X)^{\perp} = \langle & \mathcal{O}(-k,-l), \cdots, \mathcal{O}(0,-l), \\ & \mathcal{O}(-k+1,-l+1), \cdots, \mathcal{O}(1,-l+1), \\ & \cdots \\ & \mathcal{O}(-k+l-1,-1), \cdots, \mathcal{O}(l-1,-1)\rangle. \end{aligned}$$

This can be written as

$$D^b(X)^{\perp} = \langle D^1, D^2 \rangle$$

with $D^1 \coloneqq \{\mathcal{O}(a,b)\}_{-k \le a < 0}$ and $D^2 \coloneqq \{\mathcal{O}(a,b)\}_{0 \le a \le l-1}$. To see this, we need

$$Hom(i_*\mathcal{O}(a,b), i_*\mathcal{O}(a',b')[m]) = 0$$

for $\mathcal{O}(a,b) \in D^2$ and $\mathcal{O}(a',b') \in D^1$.

Using Grothendieck-Verdier duality and Corollary 19.4. It suffices to show that

$$H^*(\mathbb{P}^k \times \mathbb{P}^l, \mathcal{O}(a'-a, b'-b)) = 0 = H^*(\mathbb{P}^k \times \mathbb{P}^l, \mathcal{O}(a'-a-1, b'-b-1))$$

and, by using the Künneth formula, further to $H^*(\mathbb{P}^l, \mathcal{O}(b'-b)) = 0$ for $a'-a \le -k-1$ and $H^*(\mathbb{P}^l, \mathcal{O}(b'-b-1)) = 0$ for $a'-a-1 \le -k-1$. This vanishing follows from numerical consideration and left as an exercise.

Hence our condition implies that $H \in D^2$. It suffices to show that $p^*F \in D^{2\perp}$ for any $F \in D^b(X')$, i.e. $Hom(i_*\mathcal{O}(a,b), p^*F[n]) = 0$ for all $0 \le a \le l-1$ and $n \in \mathbb{Z}$.

This follows from the following equation

$$\begin{aligned} Hom(i_*\mathcal{O}(a,b), p^*F[*]) &\simeq Hom(\mathcal{O}(a,b), i^!p^*F[*]) \\ &\simeq Hom(\mathcal{O}(a,b), i^*p^*F \otimes \omega_E \otimes \omega_{\tilde{X}}^*|_E[*]) \\ &\simeq Hom(\mathcal{O}, \pi'^*(F|_{\mathbb{P}^l}(-b)) \otimes \pi^*\mathcal{O}(l-k-1-a)[*]) \\ &\simeq H^*(\mathbb{P}^l, F(-b)) \otimes H^*(\mathbb{P}^k, \mathcal{O}(l-k-1-a)) \\ &= 0 \end{aligned}$$

for $-k \le l-k-1-a < 0$ if $0 \le a \le l-1$ and $l \le k$. □

20. The graded restriction rule and variation of GIT

Suppose we have a variety X as a geometric quotient $X = Y/\mathbb{C}^*$. How do we understand $D^b(X)$ in terms of $D^b(Y)$? For example, $\mathbb{P}^n = \mathbb{C}^{n+1}/\mathbb{C}^*$ (deleting 0). So geometry on $\mathbb{P}^n$ is approximated by $\mathbb{C}^*$-equivariant geometry on $\mathbb{C}^{n+1}$, but not quite.

We will try to make this sentence mathematically precise in terms of derived categories. We will compare $D^b(X)$ and $D^b_{\mathbb{C}^*}(Y)$.

The point is sometimes we have different choices X_1 and X_2 from same quotient $Y/\mathcal{C}^*$ depending on he choice of $\mathbb{C}^*$ linearized-line bundle (if you are familiar with geometric invariant theory, this will makes sense to you). So the comparison provides a way of relating $D^b(X_1)$ and $D^b(X_2)$ (sometimes they are equivalent to each other).

We will focus on the simplest case $Y = \mathbb{C}^n$. The techniques and ideas we use, like many other techniques, has repeated appeared in history. In particular, as far as the author can tell, it is due to Beilinson, Kawamata, Van der Bergh etc. The idea also appeared in string theory.

As a warm-up, we will do Beilinson's theorem again in a slightly different perspective.

Let $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1}$ in the usual way. Let $R = \mathbb{C}[x_0, \cdots, x_n]$ be its coordinate ring, and $\lambda : x_i \mapsto \lambda x_i$. Then weight decomposition gives the usual grading on R.

And we know that $Coh(\mathbb{C}^{n+1})$ is equivalent to R-modules in this case, and the $\mathbb{C}^*$-equivariant sheaf is equivalent to R-module with a $\mathbb{C}^*$ action and some compatibilities. This is equivalent to graded R-module.

Therefore, $Coh_{\mathbb{C}^*}(\mathbb{C}^{n+1})$ is equivalent to the abelian category of graded R modules. Denote its derived category by $D^b_{\mathbb{C}^*}(\mathbb{C}^{n+1}) \simeq D^b(\text{gr-R-mod})$.

Example 20.1. *R can be viewed as a graded R-module, it corresponds to the structure sheaf on $\mathbb{C}^{n+1}$. If we twist the grading on R by d, we get a module $R(d)$. This corresponds to $\mathcal{O}(d)$ the trivial line bundle with $\mathbb{C}^*$ acts on the fiber with weight d.*

Then

$$\begin{aligned} Hom(R, R(d)) &= \{\textit{equivariant homomorphisms } R \to R(d)\} \\ &= R_d = \{\textit{homogenous polynomials of degree } d\} \end{aligned}$$

But $\mathbb{P}^n = (\mathbb{C}^{n+1} \backslash 0)/\mathbb{C}^*$. Basic fact: $Coh(\mathbb{P}^n) \simeq Coh_{\mathbb{C}^*}(\mathbb{C}^{n+1}\backslash 0)$. This follows from the descent theory we mentioned before.

Have a restriction functor

$$Coh_{\mathbb{C}^*}(\mathbb{C}^{n+1}) \xrightarrow{\rho^*} Coh_{\mathbb{C}^*}(\mathbb{C}^{n+1}\backslash 0).$$

This operation is called proj, and is related to Serre's equivalence. It sends $R(d)$ to $\mathcal{O}(d)$ on $\mathbb{P}^n$.

It is an exact functor, since we restrict to an open subset. Hence we get $\rho^* : D^b_{\mathbb{C}^*}(\mathbb{C}^{n+1}) \to D^b(\mathbb{P}^n)$.

Note that on the left side, there is no higher cohomology groups. And it takes the graded R module M to M_0, its invariant piece. Since we are in characteristic 0, taking invariants is an exact functor. Hence on the left hand side, it has no higher cohomology.

But on the right side, we do have higher cohomology. Therefore, the functor ρ^* is not fully faithful. Another point of view, ρ^* has a kernal, e.g. $\mathcal{O}_0$ corresponds to the module $R/(x_0, \cdots, x_n)$. And this sheaf $\mathcal{O}_0$ has a Koszul complex

$$0 \to \mathcal{O}(-n-1) \to \mathcal{O}(-n)^{\oplus n+1} \to \cdots \to \mathcal{O}(1)^{\oplus n+1} \to \mathcal{O} \to \mathcal{O}_0 \to 0.$$

Restricting it on $\mathbb{P}^n$, we get an exact sequence of vector bundles, where the last term goes to 0. And this gives $H^n(\mathbb{P}^n, \mathcal{O}(-n-1)) = \mathbb{C}$.

There is one solution: consider the triangulated subcategory

$$\mathcal{W} = \langle \mathcal{O}, \mathcal{O}(1), \cdots, \mathcal{O}(n) \rangle \subset D^b_{\mathbb{C}^*}(\mathbb{C}^{n+1})$$

generated by $\mathcal{O}, \mathcal{O}(1), \cdots, \mathcal{O}(n)$. This is called window by Ed Segal by abuse of terminology, since the term window has a different meaning in string theory.

The claim is that $\rho^* : \mathcal{W} \to D^b(\mathbb{P}^n)$ is an equivalence. In some sense, we can lift $D^b(\mathbb{P}^n)$ into $D^b_{\mathbb{C}^*}(\mathbb{C}^{n+1})$. This is equivalent to Beilinson's decomposition.

The proof of the claim is very easy, we only to check the fully faithfulness of the pair of generators, then it follows from the classical Serre's calculation. The essential surjectivity follows from what we have done before.

Of course, we can choose different objects, for example $\mathcal{W} = \langle \mathcal{O}(k), \mathcal{O}(1+k), \cdots, \mathcal{O}(n+k)\rangle$ for any $k \in \mathbb{Z}$.

Let me introduce a new example.

Another example, let $\mathcal{C}^*$ acts on $\mathbb{C}^{n+2}$ with weights $1, \cdots, 1, -d$ for $d \in \mathbb{N}$, i.e.

$$\lambda(x_0, \cdots, x_n, y) = (\lambda x_0, \cdots, \lambda x_n, \lambda^{-d} y).$$

It gives a grading on the ring $R = \mathbb{C}[x_0, \cdots, x_n, y]$ different from the usual grading.

We will delete $\{x_0 = \cdot = x_n = 0\} = \mathbb{A}^1_y$, and let $X = (\mathbb{C}^{n+2} \backslash \mathbb{A}^1_y)/\mathbb{C}^*$. Consider the map $\pi : X \to \mathbb{P}^n_{x_0;\cdots;x_n}$ and the fiber of this map are lines. It is the total space of $\mathcal{O}(-d)$ on $\mathbb{P}^n$. It is not a proper scheme.

Same ss before, we have $\rho^* : D_{\mathbb{C}^*}(\mathbb{C}^{n+2}) \to D^b(X)$. We need to check how homomorphisms change under this functor. On $\mathbb{C}^{n+2}$, we have

$$Ext^k(\mathcal{O}(i), \mathcal{O}(j)) = \begin{cases} 0 & if\ k \neq 0 \\ R_{j-i} & if\ k = 0 \end{cases}$$

Note that each R_{j-i} is infinite dimensional. For example, R_0 is the span of monomials like $p(x)y^t, deg(p) = dt$.

On X, $Ext^k(\mathcal{O}(i), \mathcal{O}(j)) = H^k(X, \mathcal{O}(j-i)) = H^k(\mathbb{P}^n, \mathcal{O}(j-i) \oplus \mathcal{O}(j-i+d) \oplus \cdots)$.

In particular, we get that for $k = 0$, it is also the span of monomials like $p(x)y^t, deg(p) = dt$. For $k > 0$, if $j - i \geq -n$, then $H^k(X, \mathcal{O}(j-i)) = 0$ for $k > 0$. This indicates that we should define $\mathcal{W} = \langle \mathcal{O}, \cdots, \mathcal{O}(n)\rangle \subset D^b_{\mathbb{C}^*}(\mathbb{C}^{n+2})$. Then $\rho^*\mathcal{W} \to D^b(X)$ is fully faithful. Also, a slightly delicate thing is that ρ^* is essentially surjective. Hence, it is an equivalence.

We have a structure morphism $X \to SpecH^0(X, \mathcal{O}_X) = SpecR_0$. Note that $R_0 = \mathbb{C}[x_0, \cdots, x_n]^{\mathbb{Z}_d}$, so $SpecR_0 = \mathbb{C}^{n+1}/\mathbb{Z}_d$. It has the quotient singularity, and X resolve the singularity.

20.1. **Atiyah flop/stndard flop.** Let $\mathbb{C}^*$ acts on $\mathbb{C}^4$ of weights $1, 1, -1, -1$ on the coordinates x_1, x_2, y_1, y_2 respectively. Then $X_+ = (\mathbb{C}^4 \backslash \{x_1 = x_2 = 0\})/\mathbb{C}^*$.

We have a morphism $\pi_+ : X_+ \to \mathbb{P}^1_{x_1;x_2}$, and X_+ is the total space of $\mathcal{O}(-1)^{\oplus 2}$ on $\mathbb{P}^1$.

Proceeds as in the previous example, we end up considering $\mathcal{W} = \langle \mathcal{O}, \mathcal{O}(1)\rangle \subset D^b_{\mathbb{C}^*}(\mathbb{C}^4)$. We have $\rho_+^* : D^b_{\mathbb{C}^*}(\mathbb{C}^4) \to D^b(X_+)$ and $\rho_+^* : \mathcal{W} \xrightarrow{\sim} D^b(X_+)$ is an equivalence.

On the other hand, we have another choice: $X_- = (\mathbb{C}^4 \backslash \{y_1 = y_2 = 0\})/\mathbb{C}^*$. By symmetry, it is the total space of $\mathcal{O}(-1)^{\oplus 2}$ on $\mathbb{P}^1_{y_1,y_2}$.

X_+ and X_- are birational 3-folds.

Remark 20.2. X_+ and X_- are isomorphic, but this is misleading. Because this is a local model for flops in 3-folds.

By the same argument $\rho_-^* : \mathcal{W} \xrightarrow{\sim} D^b(X_-)$ is an equivalence. So we construct a derived equivalence between $D^b(X_+)$ and $D^b(X_-)$. And we have the diagram

$$\begin{tikzcd} X_+ \arrow[dr] & & X_- \arrow[dl] \\ & Spec\mathbb{C}[a,b,c,d]/(ad-bc) & \end{tikzcd}$$

The bottom variety has singularity od 3-fold ordinary double point. And X_+, X_- are resolution of it. There is a roof $\tilde{X} = Bl_{\mathbb{P}^1}(X_+) = Bl_{\mathbb{P}^1}(X_-)$, this is the total space of the line bundle $\mathcal{O}(-1,-1) \to \mathbb{P}^1 \times \mathbb{P}^1$.

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\pi_+"'] \arrow[dr, "\pi_-"] & \\ X_+ & & X_- \end{tikzcd}$$

Theorem 20.3. *The functor $(\pi_-)_*(\pi_+)^* : D^b(X_+) \to D^b(X_-)$ is an equivalence.*

This equivalence is the same as the one we get $\mathcal{W} = \langle \mathcal{O}, \mathcal{O}(1) \rangle$, and restrict it to X_+ and X_- respectively.

More general example, let $\mathbb{C}^*$ acts on $\mathbb{C}^{n+k}$ by weights $1, \cdots, 1, -1, \cdots, -1$ and the coordinates are denoted by $x_1, \cdots, x_n, y_1 \cdots, y_k$ respectively. By symmetry, we can assume that $n \leq k$.

Similarly, we have $X_+ = (\mathbb{C}^{n+k} \backslash \{x_i = 0 | \forall 1 \leq i \leq n\})/\mathbb{C}^*$, which is the total space of vector bundle $\mathcal{O}(-1)^{\oplus k}$ over $\mathbb{P}^{n-1}_{x_1,\cdots,x_n}$, and $X_- = (\mathbb{C}^{n+k} \backslash \{y_i = 0 | \forall 1 \leq i \leq k\})/\mathbb{C}^*$, which is total space of the vector bundle $\mathcal{O}(-1)^{\oplus n}$ over $\mathbb{P}^{k-1}_{y_1,\cdots,y_k}$.

For X_+ define $\mathcal{W} = \langle \mathcal{O}, \cdots, \mathcal{O}(n-1) \rangle \subset D^b_{\mathbb{C}^*}(\mathbb{C}^{n+k})$, then $\rho_+^* : \mathcal{W} \xrightarrow{\sim} D^b(X_+)$, or $\mathcal{W}' = \langle \mathcal{O}, \cdots, \mathcal{O}(k-1) \rangle \subset D^b_{\mathbb{C}^*}(\mathbb{C}^{n+k})$, and $\rho_-^* : \mathcal{W}' \xrightarrow{\sim} D^b(X_-)$. Hence we get an embedding $D^b(X_-) \subset D^b(X_+)$. This is the local version of Orlov's embedding result (Proposition 19.8) we have discussed in last section.

Note that, it is in fact an admissible subcategory and $D^b(X_+) \xrightarrow{\sim} \mathcal{W} \xrightarrow{\rho_-^*} D^b(X_-)$ is the adjoint functor.

In the special case when $k = 1$, then X_+ is the total space of $\mathcal{O}(-1)$ over $\mathbb{P}^{n-1}$, and X_- is affine space $\mathbb{A}^n$. Hence we get an embedding $D^b(\mathbb{A}^n) \hookrightarrow D^b(X_+)$, this also follows from that X_+ is the blow up of $\mathbb{A}^n$ at the point 0. Hence the section of semi-orthogonal decompositions tell us it is an embedding.

Another example, $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1}$ with weights $1, \cdots, 1, -d$ ($d \geq 1$ is a positive integer) and the coordinates functions are $x_1, \cdots, x_n, p$ respectively. Then X_+ is the total space $\mathcal{O}(-d)$ over $\mathbb{P}^{n-1}$. On the other hand, $X_- = \{p \neq 0\}/\mathbb{C}^* = [\mathbb{A}^n/\mathbb{Z}_d]$ is an orbifold (algebraic space, Deligne-Mumford stacks).

We can think it as setting $p = 1$, and $\mathbb{A}^n$ with a residual action of $\mathbb{Z}_d \subset \mathbb{C}^*$. Hence we have the following diagram

$$\begin{tikzcd} X_+ \arrow[rr, dotted] \arrow[dr] & & X_- \arrow[dl] \\ & \mathbb{A}^n/\mathbb{Z}_d & \end{tikzcd}$$

We know that $\mathcal{W} = \langle \mathcal{O}, \cdots, \mathcal{O}(n-1)\rangle \subset D^b_{\mathbb{C}^*}(\mathcal{C}^{n+1})$ is equivalent to $D^b(X_+)$.

On the orbifold side, which window should we choose? Line bundles on $[\mathbb{A}^n/\mathbb{Z}_d]$ are indexed by irreducible representations of $\mathbb{Z}_d$. Hence we have non-isomorphic line bundles $\mathcal{O}, \mathcal{O}(1), \cdots, \mathcal{O}(d-1)$ and $\mathcal{O}(d) \simeq \mathcal{O}$.

Therefore, if we choose $\mathcal{W}' = \langle \mathcal{O}, \mathcal{O}(1), \cdots, \mathcal{O}(d-1)\rangle \subset D^b_{\mathbb{C}^*}(\mathbb{C}^{n+1})$. Then the restriction functor $\rho^*_- : \mathcal{W}' \to D^b([\mathbb{C}^n/\mathbb{Z}_d]) := D^b_{\mathbb{Z}_d}(\mathbb{C}^n)$. This functor is essentially surjective.

On X_-, there is no higher cohomology since taking invariant with respect to $\mathbb{Z}_d$ is an exact functor. But there is a different problem, since we cut out a divisor (unlike in previous case, we cut out a higher co-dimensional thing), this may change our Ext^0. In fact, $Ext^0(\mathcal{O}, \mathcal{O}(-d))$ changes, because $\mathcal{O}(-d) \simeq \mathcal{O}$ on X_- and it is not the case in $\mathbb{C}^{n+1}$. But in the window, it is OK. Hence, ρ^*_- is an equivalence at the end.

We compare $D^b(X_+)$ and $D^b(X_-)$, there are 3 cases:

(1) If $n > d$, $\mathcal{W}' \subset \mathcal{W}$, so $D^b(X_-) \hookrightarrow D^b(X_+)$.
(2) If $n < d$, $\mathcal{W} \subset \mathcal{W}'$, so $D^b(X_+) \hookrightarrow D^b(X_-)$.
(3) If $n = d$, then we have equivalence $D^b(Tot(K_{\mathbb{P}^{n-1}})) \simeq D^b([\mathbb{C}^n/\mathbb{Z}_d])$. This is 'McKay correspondence'. This is the Calabi-Yau case, the left hand side is a non compact Calabi-Yau manifold. On the right hand side, it is a Calabi-Yau orbifold, i.e. $\mathbb{Z}_n$ acts trivially on $det(\mathbb{C}^n)$.

We focus on the simplest example of a very general story, which is developed in [HL15] and [BFK19].

The general setting: we have a reductive group G acting on a smooth projective Y. Given a G linearized line bundle L on Y, we have an open subset $Y^{ss} \subset Y$ (the semi-stable locus), such that $X = Y^{ss}/G$ is a nice variety or an orbifold.

Then there is a window $\mathcal{W} \subset D^b_G(Y)$, such that the restriction functor $\rho : \mathcal{W} \xrightarrow{\sim} D^b(X)$ is an equivalence.

And different choice of linearized line bundles gives us different quotients. Then we can lift the derived category in the bigger derived category $D^b_G(Y)$. In general, the windows are so different that we cannot compare them. But in special cases, we ca compare them.

Example 20.4. *Suppose $G = \mathbb{C}^*$ acts on a smooth projective variety Y with single fixed point $p \in Y$. There are two kinds of flows, one flow out from p, and one flow into p (the flows comes from $\mathbb{C}^*$ action). These gives us two possible semi-stable locus, one excludes the flowing in manifolds, the other one excludes the flowing out manifold.*

How do we choose the window $\mathcal{W}$? We cannot choose finite number of line bundles anymore.

We define $\mathcal{W}$ as follows: $\mathcal{W} = \{E \in D^b_G(Y), such\ that\ E|_p only\ have\ weights\ in\ [o,n\text{-}1]\}$, where n depends on the $\mathbb{C}^*$ action on the tangent space of Y at p.

One more remark, this can be generalized to the category of matrix factorizations if we define the R-charge to be $\mathbb{C}^*$ invariant.

21. An introduction to motives

21.1. **The Grothendieck ring of varieties.** The last two lectures will be an brief introduction to motivic cohomology. As a brief introduction it is, I will be sketchy on the proofs. We will take a bird's eye view on the land-scape of this area.

To set terms, let $\mathcal{V}_k$ be the set of varieties over k.

We define the Grothendieck ring of variaties $R = R(\mathcal{V}_k)$ (this notation is not standard) as follows.

(1) As an additive group, R is generated by symbols of the form $[X]$, where X is a variety up to isomorphism.
(2) The additive structure of R is generated by the following. If $Z \subset X$ is a closed embedding, with complement U, then we say $[X] = [U] + [Z]$.
(3) Multiplication in R is defined by the relation if W and X are varieties, then $[W][X] = [W \times X]$.

This turn R into a commutative ring.

Example 21.1. $0 = [\emptyset]$, $1 = [pt]$. *For convenience, define $\mathbb{L} = [\mathbb{A}^1]$. The $[\mathbb{P}^2] = \mathbb{L}^2 + \mathbb{L} + 1$. And if $W \to X$ is a $\mathbb{P}^n$ bundle (in the Zariski topology), show that $[W] = [X][\mathbb{P}^n]$.*

This ring is sometimes called the ring of baby motives. There are some maps from $\mathcal{V}_k$ to a ring A that descends to a map from R to that ring A. Such a map is called motivic measure.

Example 21.2. *Let $k = \mathbb{F}_p$ be the finite field of cardinality q (where q is a prime number). Then each variety has a finite number of rational points. Notice that this respects the addition and multiplication relations in R. Hence it is a motivic measure.*

In characteristic 0, by the virtue of resolution of singularities. We know that R is generated as an abelian group by classes of smooth projective varieties.

If $Z \subset X$ is a closed embedding of a smooth projective variety in another smooth projective variety, then let $Bl_Z X$ denote the blow up of X along Z with exceptional divisor $E_Z X$, then clearly

$$[Bl_Z X] - [E_Z X] = [X] - [Z].$$

Bittner's beautiful results says that these relations are enough (see [Bit04]).

Theorem 21.3. *R is isomorphic to the abelian group on classes of smooth irreducible projective varieties, modulo such relations.*

Remark 21.4. The proof is not hard, it uses the Weak Factorization Theorem of Abramovich, Karu, Matsuki, and Wlodarcyzk (see [AKMWo02]), which states that any birational map between smooth varieties can be factored into a sequences of blow ups and blow downs along smooth center. This theorem is only proved in characteristic 0.

Bittner's result is very useful, since it only involves a simple fashion of relations. It allows us to determine new motivic measures.

If you know a little about the properties of Hodge structures, and how they change by blowing up, show that the Hodge structure on h^i is also a motivic measure.

There is also an unexpected application.

Theorem 21.5. *There is a duality on $R_{\mathbb{L}}$ sending X to $\mathbb{L}^{-dimX}X$ for smooth irreducible projective varieties.*

Proof. Suppose Z is codimension c in X, and both are smooth projective varieties. Then we have

$$\begin{aligned} [E_Z X] &= [\mathbb{P}^{c-1}][Z] \\ (\mathbb{L}-1)[E_Z X] &= (\mathbb{L}^c - 1)[Z] \\ [BL_Z X] - \mathbb{L}[E_Z X] &= [X] - \mathbb{L}^c[Z] \end{aligned}$$

Dividing by $\mathbb{L}^{dimX}$, we get

$$[Bl_Z X]/\mathbb{L}^{dimX} - [E_Z X]/\mathbb{L}^{dim(X)-1} = [X]/\mathbb{L}^{dimX} - [Z]/\mathbb{L}^{dimZ}$$

as desired. □

In particular, $\mathbb{L} \mapsto 1/\mathbb{L}$, as $[\mathbb{P}^1] \mapsto 1 + 1/\mathbb{L}$.

Remember that $R_{\mathbb{L}} = R[1/\mathbb{L}]$, so this is one of many ways telling us to invert $\mathbb{L}$ (other reasons includes applying homotopy method in the theory). Note that Localization is not always injective; $R \to R_{\mathbb{L}}$ will kill any class α such that $\alpha\mathbb{L} = 0$.

For this reason there has long been a question/conjecture: the class $\mathbb{L}$ is not a zero divisor. In other words, $R \to R_{\mathbb{L}}$ is an injection.

But unfortunately, this conjecture is false, by the work of Borisov see [Bor18]. We will briefly describe the example later.

There is also an application on stable birational geometry (see [LL03]). Recall that two irreducible varieties X and Y are birational if they have 'isomorphic open subsets'. They are said to be stably birational if $X \times \mathbb{A}^m$ is birational to $Y \times \mathbb{A}^n$ for some m and n. Even taking into account the difference in dimension, stable birationality is a strictly weaker equivalence relation than birationality (for example, there exists examples of nonrational variety but is stable rational).

Theorem 21.6. *Denote $\mathcal{M}$ the multiplicative monoid of smooth projective variety. Let $\Phi : \mathcal{M} \to G$ be a homomorphism of monoids such that*

(i) $\Phi([X]) = \Phi([Y])$ if X and Y are birational to each other.

(ii) $\Phi([\mathbb{P}^n]) = 1$ for all $n \geq 0$.

Then there exists a unique ring isomorphism

$$\Psi : R \to \mathbb{Z}[G]$$

such that $\Phi([X]) = \Psi(X)$.

Proof. It suffices to how that Φ respect Bittner's relation. Suppose Z is codimension c in X, and both are smooth projective varieties. Then we have

$$\begin{aligned}\Phi([Bl_Z X]) - \Phi([E_Z X]) &= \Phi([Bl_Z X]) - \Phi([Z] \times \mathbb{P}^{c-1}) \\ &= \Phi([Bl_Z X]) - \Phi([Z]) \\ &= \Phi([X]) - \Phi([Z])\end{aligned}$$

The last two equalities follows from the assumption. □

There is a universal homomorphism of monoids $\Phi_{SB} : \mathcal{M} \to SB$, where SB denote the group of equivalent classes under the stable birational equivalence.

Clearly, this homomorphism Φ_{SB} satisfies the assumption. Hence there exists a unique ring homomorphism $\Psi : R \to \mathbb{Z}[SB]$. Moreover, we can determine the kernal of this map.

Proposition 21.7. *The kernal of Ψ_{SB} is generated by $\mathbb{L}$.*

Proof. Since $\Psi_{SB}(\mathbb{P}^1) = \Psi_{SB}([pt] + [\mathbb{A}^1]) = 1$, we get $(\mathbb{L}) \subset ker(\Psi_{SB})$.

Suppose that $a \in ker(\Psi_{SB})$, we can write $a = [X_1] + \cdots [X_k] - [Y_1] - \cdots - [Y_l]$, where X_i and Y_j are smooth projective varieties.

Apply Ψ_{SB} on a, we get that $k = l$ and X_i is stably birational to Y_i after renumbering.

Since $[X_i \times \mathbb{P}^n] - [X_i] = [X_i] \cdot [\mathbb{A}^1 + \cdots + \mathbb{A}^{r-1}]$, we can further assume that X_i is birational to Y_i. By weak factorization theorem, it suffices to check that $[BL_Z Y] - [Y] \in (\mathbb{L})$. This is obvious. □

21.2. **The class of Grassmannians.** We have a simple form for Grassmannians.

Proposition 21.8. *For $2 \leq k \leq n-1$, we have*

$$[G(k, n)] = [G(k, n-1)] + \mathbb{L}^{n-k} \cdot [G(k-1, n-1)].$$

Proof. Choose $e_1, \cdots, e_n$ to be a basis of $\mathbb{C}^n$, and F is the hyperplane orthogonal to e_n, define an open subset $U \subset G(k, n)$ in the following way

$$U = \{T \in G(k, n) | dim(T \cap F) = k - 1\}.$$

i.e., U contains the k-plane that intersects F properly. There is a natural projection $\pi : U \to G(k-1, F)$ by sending T to $T \cap F$.

It is easy to see what is the fiber of π. Let $S \in G(k-1, F)$, then

$$\pi^{-1}(S) \simeq \mathbb{P}(\mathbb{C}^n/S) \backslash \mathbb{P}(F/S) \simeq \mathbb{A}^{n-k}.$$

Let H be a complementary subspace of S in F. There exists an open subset $V = \{S' \in G(k-1, F) | S' \oplus H = F\}$. For any $S' \in V$, we have that $\mathbb{C}^n/S' = H \oplus \mathbb{C}e_n$. Hence π is a Zariski locally trivial fibration.

Therefore $[G(k, n)] = [G(k, n-1)] + [U] = [G(k, n-1)] + \mathbb{L}^{n-k} \cdot [G(k-1, n-1)]$. □

Exercise 21.9. *(1) Use the previous proposition to prove that*

$$(\mathbb{L} - 1)(\mathbb{L}^2 - 1) \cdot (\mathbb{L}^k - 1)[G(k, n)] = (\mathbb{L}^{n-k+1} - 1) \cdots (\mathbb{L}^n - 1).$$

If you think the calculation is too boring, you can try the following exercise.

(1)' Use flag varieties to show that

$$[\mathbb{P}^1][\mathbb{P}^2]\cdots[\mathbb{P}^{k-1}][G(k,n)] = [\mathbb{P}^{n-k}]\cdots[\mathbb{P}^{n-1}].$$

By some combinatorics, we can show that $[G(k,n)]$ can be written as a polynomial of $\mathbb{L}$. It is actually the sum of Schubert cells. And by previous results, we know that $G(k,n)$ is stable rational (in fact, it is rational since the coefficient of top cell is 1). You can also use this exercise to calculate the numbers of Schubert cells of a fixed dimension in $G(k,n)$.

Unfortunately, the most interesting part of Schubert calculus, the intersection theory of Schubert classes, is not encoded in this equation.

21.3. **Pfaffian Grassmannian mirror varieties.** Let V be a 7 dimensional vector space, W a generic 7 dimensional space of skew forms on V. We define $X_W = \{T \in G(2,7)|w|_T = 0 \; for \; all \; w \in W\}$, $Y_W = \{w \in \mathbb{P}W | rkw < 6\}$.

There are some statements about X_w and Y_w which we will not prove.

Theorem 21.10. *(1) The varieties X_W and Y_W are smooth Calabi-Yau threefolds for a generic choice of W.*

(2) The varieties X_W and Y_W are not isomorphic, or even birational to each other.

Proof. This is not a proof, just some words of explaining. For (1), the triviality of canonical bundle of Y_w follows from

$$0 \to \mathcal{O}_{\mathbb{P}^6}(-7) \xrightarrow{P^T} \mathcal{O}_{\mathbb{P}^6}(-4)^{\oplus 7} \xrightarrow{N} \mathbb{O}_{\mathbb{P}^6}(-3)^{\oplus 7} \xrightarrow{P} \mathcal{O}_{\mathbb{P}^6} \to \mathcal{O}_{Y_w} \to 0.$$

Here $P = \begin{pmatrix} P_1 \\ \cdots \\ P_7 \end{pmatrix}$ is the matrix of Pfaffians of N. And N is the skew-symmetric 7×7 matrix, we know that $ad(N) = P \cdot P^T$. Hence, the sequence is exact.

The triviality of canonical bundle of X_W follows form Koszul complex.

For (2), it is because $Pic(X_w) = Pic(Y_w) = 1$, $H_X^3 = 42$, and $H_Y^3 = 14$. Hence they are not isomorphic. They are also not birational. In deed, if they are birational to each other, since they are minimal models (the canonical divisors are nef), they will by connected by flops. But $Pic(X_W) = Pic(Y_W) = 1$, hence there is no flops, they are isomorphic. This contradicts the facts we know. □

We consider $H \subset G(2,V) \times \mathbb{P}W$:

$$H = \{(T_2, \mathbb{C}w)|w|_{T_2} = 0\}.$$

Then there are projections

$$\begin{tikzcd} & H \arrow[dl, "p"'] \arrow[dr, "q"] & \\ G(2,7) & & \mathbb{P}W \end{tikzcd}$$

This provides two ways of calculating the motive $[H]$, after some calculation, we get $([X_w]-[Y_W])\mathbb{L}^6=0$ in the Grothendieck ring.

Therefore, if we can show that $[X_w]-[Y_W]$ is nonzero, then $\mathbb{L}$ is a zero divisor. Assume that $[X_w]-[Y_W]=0$, then $X_w\times\mathbb{P}^n$ is birational to $Y_W\times\mathbb{P}^n$, after taking the maximal rational connected quotient, we get X_W is birational to Y_W, which is impossible.

21.4. **Motivic zeta function.** Motivic zeta function is a central player in the theory of baby motives. This is a notion which was formally defined remarkably late- in an unpublished manuscript of Kapranov in 2000-and should have been defined far earlier in many people's opinion (maybe Grothendieck had discussed the idea in a letter to Serre, I am not sure about it since I am not a historian).

Definition 21.11. The motivic zea function of X is defined as

$$Z_X(t):=\Sigma[Syn^nX]t^n\in R[[t]].$$

Exercise 21.12. *Prove that if $X=U\amalg Z$ where Z is closed and U is open, then $Z_X(t)=Z_U(t)Z_Y(t)$.*

Thus we can really define $Z_X(t)$ for any X in the Grothendieck ring R. We have a map of groups

$$Z:R\to(1+tR[[t]])\subset R[[t]]^\times,$$

turning $+$ into $\times$.

Example 21.13. *Let $k=\mathbb{F}_q$ be the finite field with q elements. Here we have the point counting function $\sharp:R\to\mathbb{Z}$. Applying this to motivic zeta function, we get the Weil zeta function.*

If you know the original definition of Weil zeta function, please do the exercise. If you do not know the original definition (the original one can be seen as the generating series of effective 0-cycles), you can take this as the definition.

Exercise 21.14. *Show that $\sharp(Z_X(t))=\eta_X(t)$.*

Weil introduced his zeta function by his thoughts around the Weil conjectures connecting number theory and topology and algebraic geometry. In particular, the first part of the Weil conjecture is that the Weil zeta function is always rational. This is proved by Dwork (see [Dwo60]).

This leads to a natural question: Is $Z_X(t)$ always rational? There are several evidence for this conjecture.

Here k can be any field. If X is a curve, the answer is yes.

If $k=\mathbb{C}$, then it is true for 'Hodge structures'.

If $k=\mathbb{F}_q$, then it is true for point counting by the rationality of the Weil zeta function $\eta_X(t)$..

This would even have strong consequences in characteristic 0. If we know a power series is actual rational, say $f(t)=g(t)/h(t)$, then from $f(t)h(t)=g(t)$ will provide a cursive way to compute the coefficients of f.

So if w knew the first few Sym^nX, we would know them all.

But this conjecture is disproved in [LL03].

Theorem 21.15. *The conjecture is not true if $k = \mathbb{C}$. In fact, there is a motivic measure to a field, such that for all smooth projective surfaces with $h^{2,0} = h^{0,2} \geq 2$, the motivic zeta function is not rational.*

However, the motivic measure they come up turns out to vanish on $\mathbb{L}$, so their argument does not show that the zeta function is not rational in $R_{\mathbb{L}}$.

Later, Larsen and Lunts gave a new motivic measure in [LL20], which showed that the motivic zeta function is not rational even after inverting $\mathbb{L}$.

22. Motivic cohomology

The theory of motivic cohomology was introduced by Voevodsky in [Voe00]. Let k be a field. We denote by Sm/k the category of smooth schemes over k.

For a pair X, Y of smooth schemes over k denote by $c(X, Y)$ the free abelian group generated by integral closed sub-schemes W in $X \times Y$ which are finite over X and surjective over a connected component of X. An element of $c(X, Y)$ is called a finite correspondence from X to Y. The following are directly taken from [Voe00, Section 2], and is very different from what I covered in the class.

Let now X_1, X_2, X_3 be a triple of smooth schemes over k, $\phi \in c(X_1, X_2)$ and $\psi \in c(X_2, X_3)$. Consider the product $X_1 \times X_2 \times X_3$ and let $\pi_i : X_1 \times X_2 \times X_3 \to X_i$ be the corresponding projections. One can verify easily that the cycles $(\pi \times \pi_2)^*\phi$ and $(\pi_2 \times \pi_3)^*\psi$ are in general position. Let $\psi * \phi$ be their intersection. We set $\psi \circ \phi = (\pi_1 \times \pi_3)_*(\psi * \phi)$. Note that the pushforward is well defined since ϕ (resp. ψ) is finite over X_1 (resp. X_2).

For any composable triple of finite correspondence α, β, γ one has

$$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$$

and therefore one can define a category $SmCor(k)$ such that the objects of $SmCor(k)$ are smooth schemes of finite type over k, morphisms are finite correspondences and compositions of morphisms are defined above. We write the objects of $SmCor(k)$ which corresponds to a smooth scheme by $[X]$.

For any morphism $f : X \to Y$ its graph Γ_f is a finite correspondence from X to Y. It gives us a functor $[-] : Sm/k \to SmCor(k)$.

One can easily see that the category $SmCor(k)$ is additive and one has $[X \amalg Y] = [X] \oplus [Y]$.

Consider the homotopy category $\mathcal{H}^b(SmCor(k))$ of bounded complexes over $SmCor(k)$. We are going to define the triangulated category of effective geometrical motives over k as a localization of $\mathcal{H}^b(SmCor(k))$. Let T be the class of complexes of the following two forms:

1. For any smooth scheme X over k the complex

$$X \times \mathbb{A}^1 \xrightarrow{\pi_1} [X]$$

belongs to T.

2. For any smooth scheme X over k and an open covering $X = U \cup V$ of X the complex

$$[U \cap V] \xrightarrow{[j_u]\oplus[j_V]} [U] \oplus [V] \xrightarrow{[i_U]\oplus(-[i_V])} [X]$$

belongs to T (all the maps are obvious open embeddings).

Let $\bar{T}$ denote the minimal thick subcategory of $\mathcal{H}^b(SmCor(k))$ which contains T.

Definition 22.1. The triangulated category $DM^{eff}_{gm}(k)$ of effective geometric motives over k is the pseudo-abelian envelop of the localization of $\mathcal{H}^b(SmCor(k))$ with respect to the thick subcategory $\bar{T}$. We denote the obvious functor $Sm/k \to DM^{eff}_{gm}(k)$ by M_{gm}.

You can ignore the term pseudo-abelian envelop (which essentially means formally adding kernals and cokernals of projectors), Voevodsky use it only for comparison with the 'classical motives'.

For a pair of smooth schemes X and Y over k we set

$$[X] \otimes [Y] = [X \times Y].$$

Furthermore, we can define the tensor product

$$c(X_1, Y_1) \otimes c(X_2, Y_2) \to c(X_1 \times X_2, Y_1 \times Y_2)$$

which gives us a tensor category structure on $SmCor(k)$. This structure defines in the usual way a tensor triangulated category structure on $\mathcal{H}^b(SmCor(k))$, which can be descended to the category $DM^{eff}_{gm}(k)$.

Proposition 22.2. *The category $DM^{eff}_{gm}(k)$ has a tensor triangulated category structure such that for any pair X, Y of smooth schemes over k there is a canonical isomorphism $M_{gm}(X \times Y) \simeq M_{gm}(X) \otimes M_{gm}(Y)$.*

Note that the unit object is $M_{gm}(Spec(k))$. We will denote it by $\mathbb{Z}$. For any smooth scheme X over k the structure morphism $X \to Spec(k)$ gives us a morphism $M_{gm}(X) \to \mathbb{Z}$. There is a canonical distinguished triangle $\tilde{M}_{gm}(X) \to M_{gm}(X) \to \mathbb{Z} \to \tilde{M}_{gm}(X)[1]$ where $\tilde{M}_{gm}(X)$ is the reduced motive of X.

We define the Tate object $\mathbb{Z}(1)$ of $DM^{eff}_{gm}(k)$ as $\tilde{M}_{gm}(\mathbb{P}^1)[-2]$. We further define $\mathbb{Z}(n)$ to be the $n-$th tensor product of $\mathbb{Z}(1)$. For any object A in $DM^{eff}_{gm}(k)$ we denote by $A(n)$ the object $A \otimes \mathbb{Z}(n)$.

Finally we define the triangulated category $DM_{gm}(k)$ of geometric motives over k as the category obtained from $DM^{eff}_{gm}(k)$ by inverting $\mathbb{Z}(1)$.

More precisely, the objects of $DM_{gm}(k)$ are pairs of the form (A, n) where A is an effective geometric motive and $n \in \mathbb{Z}$. The morphisms are defined by the following formula

$$Hom_{DM_{gm}}((A, n), (B, m)) = \lim_{k \geq -n, -m} Hom_{DM^{eff}_{gm}}(A(k + n), B(k + m)).$$

The category $DM_{gm}(k)$ is again a tensor triangulated category (the tensor structure is a little bit subtle, we need to show that $\mathbb{Z}(1)$ is a reflexive object, i.e., the permutation involution on $\mathbb{Z}(1) \otimes \mathbb{Z}(1)$ is identity in DM^{eff}_{gm}).

22.1. **Summary of results.** For a field k which admits resolution of singularities we can extend the functor $M_{gm} : Sm/k \to DM_{gm}^{eff}$ to a functor $Sch/k \to DM_{gm}^{eff}$ from the category of all schemes of finite type over k. The extended functor has the following main properties:

Kunneth formula: For schemes of finite type X, Y over k one has a canonical isomorphism $M_{gm}(X \times Y) = M_{gm}(X) \otimes M_{gm}(Y)$.

Homotopy invariance: For a scheme of finite type X over k the morphism $M_{gm}(X \times \mathbb{A}^1) \to M_{gm}(X)$ is an isomorphism.

Meyer-Vietoris axiom: For a scheme X of finite type over k and an open covering $X = U \cap V$ of X one has a canonical distinguished triangle of the form

$$M_{gm}(U \cap V) \to M_{gm}(U) \oplus M_{gm}(V) \to M_{gm}(X) \to M_{gm}(U \cap V)[1].$$

Blow-up distinguished triangle: For a scheme X of finite type over k and a closed subscheme Z in X denote by $p_Z : X_Z \to X$ the blow up of X along Z. Then there is a canonical distinguished triangle of the form

$$M_{gm}(p_Z^{-1}(Z)) \to M_{gm}(X_Z) \oplus M_{gm}(Z) \to M_{gm}(X) \to M_{gm}(p_Z^{-1}(Z))[1].$$

Projective bundle theorem: Let X be a scheme of finite type over k and E be a vector bundle on X. Denote by $p : \mathbb{P}(E) \to X$ the projective bundle over X associated with E. Then one has a canonical isomorphism:

$$M_{gm}(\mathbb{P}(E)) = \oplus_{n=0}^{rkE-1} M_{gm}(X)(n)[2n].$$

Note that this extension is not delicate enough to extend motivic cohomology theory to singular varieties. For instance the Picard group for a smooth scheme X is canonically isomorphic to the motivic cohomology group $Hom_{DM_{gm}^{eff}}(M_{gm}(X), Z(1)[2])$. We do not have 'motivic' description for the Picard groups of arbitrary varieties though, since the functor $X \mapsto Pic(X)$ considered on the category of all schemes is not homotopy invariant and does not have the descent property for general blow-ups.

It is Voevodsky's hope that there is another more subtle approach to 'motives' of singular varieties which makes use of some version of 'reciprocity functors' introduced by Bruno Kahn instead of homotopy invariant functors considered in this paper which gives 'right' answers for all schemes.

22.2. **Motives with compact support.** For any field k which admits resolution of singularities Voevodsky constructed a functor M_{gm}^c from the category of schemes of finite type over k and proper morphisms to the category DM_{gm}^{eff} which has the following properties:

1. For a proper scheme X over k one has a canonical isomorphism

$$M_{gm}^c(X) = M_{gm}(X).$$

2. For a scheme X of finite type over k and a closed sub-scheme Z in X one has a canonical distinguished triangle

$$M_{gm}^c(Z) \to M_{gm}^c(X) \to M_{gm}^c(X - Z) \to M_{gm}^c(Z)[1].$$

3. For a flat equi-dimensional morphism $f : X \to Y$ of schemes of finite type over k there is a canonical morphism

$$M^c_{gm}(Y)(n)[2n] \to M^c_{gm}(X)$$

where $n = dim(X/Y)$.

4. For any scheme of finite type X over k one has a canonical isomorphism $M^c_{gm}(X \times \mathbb{A}^1) = M^c_{gm}(X)(1)[2]$.

22.3. **Blow-ups of smooth varieties and Gysin distinguished triangles.** Let k be a perfect field, X be a smooth scheme over k and Z be a smooth closed sub-scheme in X everywhere of codimension c. Then one has:

Motives of blow-ups. There is a canonical isomorphism

$$M_{gm}(X_Z) = M_{gm}(X) \oplus (\oplus_{n=1}^{c-1} M_{gm}(Z)(n)[2n]).$$

Gysin distinguished triangle. There is a canonical distinguished triangle

$$M_{gm}(X - Z) \to M_{gm}(X) \to M_{gm}(Z)(c)[2c] \to M_{gm}(X - Z)[1].$$

The quasi-invertibility of the Tate object. Let k be a field which admits resolution of singularities. Then for any object $A, B \in DM^{eff}_{gm}(k)$ the obvious morphism

$$Hom(A.B) \to Hom(A(1), B(1))$$

is an isomorphism. In particular the functor

$$DM^{eff}_{gm} \to DM_{gm}$$

is a full embedding.

22.4. **Duality.** For any field k which admits resolution of singularities that category $DM_{gm}(k)$ is a 'rigid tensor triangulated category'. More precisely one has:

1. For any pair of objects $A, B \in DM_{gm}$ there exists the internal Hom-object

$$\underline{\mathrm{Hom}}_{DM}(A, B).$$

We set A^* to be $\underline{\mathrm{Hom}}_{DM}(A, Z)$.

2. For any object A in $DM_{gm}(k)$ the canonical morphism $A \to (A^*)^*$ is an isomorphism.

3. For any pair of objects A, B in DM_{gm} there are canonical isomorphisms

$$\underline{\mathrm{Hom}}_{DM}(A, B) = A^* \otimes B$$

and

$$(A \otimes B)^* = A^* \otimes B^*.$$

For any smooth equi-dimensional scheme X of dimension n over k there is a canonical isomorphism

$$M_{gm}(X)^* = M^c_{gm}(X)(-n)[-2n].$$

Let X be a smooth equi-dimensional scheme of dimension n over k and Z be a closed sub-scheme of X. Applying duality to the localization sequence (22.2.2) for M^c_{gm} we get the following generalized Gysin distinguished triangle

$$M_{gm}(X - Z) \to M_{gm}(X) \to M^c_{gm}(Z)^*(n)[2n] \to M_{gm}(X - Z)[1].$$

22.5. **Relation to the algebraic cycle homology theories.** Let again k be a field which admits resolution of singularities and X be a scheme of finite type over k.

Higher Chow group. If X is quasi-projective and equidimensional of dimension n the groups $Hom(\mathbb{Z}(i)[j], M^c_{gm}(X))$ (the Borel-Moore homology in our theory) are canonically isomorphic to the higher Chow group $CH^{n-i}(X, j-2i)$.

Suslin homology. For any X of finite type over a perfect field k the groups (homology in our theory) $Hom(\mathbb{Z}[j], M_{gm}(X))$ are isomorphic to the Suslin homology groups $h_j(X)$.

Bivariant cycle cohomlogy. For any X, Y of finite type over k any $i \geq 0$ and any j the group $Hom(M_{gm}(X)(i)[j], M^c_{gm}(Y))$ is isomorphic to the bivariant cycle cohomolgy group $A_{j,j-2i}(X, Y)$.

22.6. **Relations to Chow motives.** For any pair of smooth projective varieties X, Y over a field which admits resolution of singularities the group $Hom_{DM_{gm}(k)}(M_{gm}(X), M_{gm}(Y))$ is canonically isomorphism to the group of cycles of dimension $dim(X)$ on $X \times Y$ modulo rational equivalence. In particular the full additive subcategory by objects of the form $M_{gm}(n)[2n]$ for smooth projective X over k and $n \in \mathbb{Z}$ is canonically equivalent as a tensor additive category to the category of Chow motives over k. Moreover, any distinguished triangle with all three vertices being in the subcategory splits.

22.7. **Motivic complex and the homotopy t-structures.** Voevodsky constructed for any perfect field k an embedding of the category $DM^{eff}_{gm}(k)$ to a bigger tensor triangulated category $DM^{eff}_{-}(k)$ of motivic complexes over k. The image of $DM^{eff}_{gm}(k)$ is 'dense' in $DM^{eff}_{-}(k)$ in the sense that the smallest triangulated subcategory in $DM^{eff}_{-}(k)$ which is closed with respect to direct sums and contains the image of $DM^{eff}_{gm}(k)$ coincides with $DM^{eff}_{-}(k)$.

Almost by definition the category $DM^{eff}_{-}(k)$ has a (non-degenerate) t-structure whose heart is the abelian category $HI(k)$ of homotopy invariant Nisnevich sheaves with transfers on Sm/k. Note that this t-structure is not the desired 'motivic t-structure' on $DM^{eff}_{-}(k)$ whose heart is the abelian category of (effective) mixed motives over k.

Voevodsky also constructed the following canonical functors

$$DM^{eff}_{-}(k) \to DM^{eff}_{-,et}(k) \to DM_h(k)$$

and the second functor is an equivalence if k admits resolution of singularities. The first functor becomes an equivalence after tensoring with $\mathbb{Q}$. Voevodsky also show that the category $DM^{eff}_{-,et}(k, \mathbb{Z}/n\mathbb{Z})$ of the etale motivic complexes with $\mathbb{Z}/n\mathbb{Z}$ coefficients is equivalent to the derived category of xomplexes bounded from the above over the abelian category of sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules on the small etale site $Spec(k)_{et}$ for n prime to $char(k)$ and $DM^{eff}_{-,et}(k, \mathbb{Z}/p\mathbb{Z}) = 0$ for $p = char(k)$.

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