Derived Category

① Ind and Pro Categories Associated to a Category

Let $\mathcal{C}$ be any category.

$\rightsquigarrow$ $\mathcal{C}^{pro}$ = category of pro objects of $\mathcal{C}$ (replacement for limits).

$\mathcal{C}^{Ind}$ = category of ind objects of $\mathcal{C}$ (replacement for colimits).

<u>Def'n</u> $\mathcal{D}$ is a filtered category (resp. cofiltered category) if — opposite of the filtered.

1) $\mathcal{D} \neq \phi$,

2) $\forall X, Y \in \mathcal{D}$, $\exists Z$ & $X \to Z \leftarrow Y$ (resp. $X \leftarrow Z \to Y$)

3) $\forall f, g$ parallel, $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$\exists h : Y \to Z$ in $\mathcal{D}$ such that

$X \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{h} Z$ commutes $\Big\}$ $h \circ f = h \circ g$. (resp. $f \circ h = g \circ h$)

- $\mathcal{C}^{Ind}$: <u>objects</u> $F : \mathcal{D} \to \mathcal{C}$, $\mathcal{D}$ small filtered category

  <u>morphs</u> $\mathcal{C}^{Ind}(F, G)$ $F : \mathcal{D} \to \mathcal{C}$, $G : E \to \mathcal{C}$

  $= \lim_{d \in \mathcal{D}} \operatorname{colim}_{e \in E} \mathcal{C}(F(d), G(e))$.

  Morphs being this way follow from 3 properties:

  $$\begin{aligned} \mathcal{C} &\xrightarrow{\ i\ } \mathcal{C}^{Ind} \\ c &\longmapsto (1 \to c) \end{aligned}$$

1) Functor i is full and faithful.

2) Diagrams are colimits of themselves via the inclusion i.

3) object compact:

$$\mathrm{Hom}(X, \operatorname{colim}_i Y_i) = \operatorname{colim}_i \mathrm{Hom}(X, Y_i)$$

<u>Then</u>

$$\begin{aligned}
\mathcal{C}^{ind}(F,G) &\overset{2)}{=} \mathcal{C}^{ind}\left(\operatorname{colim}_{d\in D} F(d), \operatorname{colim}_{e\in E} G(e)\right) \\
&= \lim_{d\in D}\left[\mathcal{C}^{ind}\left(F(d), \operatorname{colim}_{e\in E} G(e)\right)\right] \\
&= \lim_{d\in D}\left[\operatorname{colim}_{e\in E} \mathcal{C}^{ind}(F(d), G(e))\right] \\
&= \lim_{d\in D} \operatorname{colim}_{e\in E} \mathcal{C}^{ind}(F(d), G(e))
\end{aligned}$$

contravariant Hom functor

compactness 3).

faithfulness 1).

- $\mathcal{C}^{pro}$: <u>objects</u> $F : D \longrightarrow \mathcal{C}$ D small filtered category.

<u>morphs</u> $\mathcal{C}^{pro}(F,G) = \lim_{e\in E} \operatorname{colim}_{d\in D} \mathcal{C}(F(d), G(e))$.

(cocompact).

② <u>Localization of Rings as Localizations of Categories</u>

$\boxed{\text{Localization}}$

RingCat = (category of categories w/ one object which are additive)

<u>morphs</u> additive morphs.

<u>Claim</u> RingCat $\simeq$ Ring

<u>Proof idea</u> $\mathcal{C} \in$ Ring Cat

$\ast$ = unique category

$\mathcal{C}(\ast, \ast) \in$ Ring

addition: $\mathcal{C}(*,*) \in Ab$

zero

unit: 1_* identity.

multiplication: $f, g \in \mathcal{C}(*,*)$, then $f\circ g$ defines mult.

these satisfy ring axioms.

$\boxed{\text{Decat}}$

$$\begin{aligned} \text{RingCat} &\longrightarrow \text{Ring} \\ \mathcal{C} &\longmapsto \mathcal{C}(*,*) \end{aligned}$$

Functorial, $F: \mathcal{C}_1 \to \mathcal{C}_2$

F will induce a ring hom.

$\text{Decat}(F): \mathcal{C}_1(*,*) \to \mathcal{C}_2(*,*)$.

Localization of a Ring:

Let $R \in \text{Ring}$, $S \subseteq R$ mult closed.

$R \xrightarrow{f} R'$ localizes S iff $\forall s \in S$, $\exists r' \in R'$ s.t.

$f(s)r' = 1$.

$\leadsto$ category of localizers

objs (f, R')

Morphs $(f_1, R_1') \to (f_2, R_2')$,

$$\begin{tikzcd} & R_1' \arrow[d, "g"] \\ R \arrow[ur, "f_1"] \arrow[r, phantom, "\circlearrowleft"] \arrow[dr, bend right, "f_2"'] & R_2' \end{tikzcd}$$

Def'n The localization of R at S is the pair (u, R_S) which is initial in this category of localizations.

$\boxed{\text{Localization of a Category}}$

Given isom $\text{RingCat} \simeq \text{Ring}$

$S \longleftrightarrow$ morphs collection of closed under composition.

Def'n $\mathcal{C}$ category. $\mathcal{S}$ collection of morphs in $\mathcal{C}$.

$\mathcal{D}$ category. $F : \mathcal{C} \to \mathcal{D}$ localizes $\mathcal{S}$ if $\forall s \in \mathcal{S}$, $F(s)$ is an isom.

$\rightsquigarrow$ form category of localizers

$\rightsquigarrow$ localization of $\mathcal{C}$ at $\mathcal{S}$ is the initial object in the category if it exists.

③ Derived Categories (I)

Fix $\mathcal{A}$ = an abelian .

Def'n A cochain complex $(C^i, d^i) = C^\bullet$

$$\cdots \to C^{i-1} \xrightarrow{d} C^i \xrightarrow{d} C^{i+1} \to \cdots \qquad d^2 = 0$$

Def'n cohomology $H^i(C^\bullet) = \dfrac{Z^i(C^\bullet)}{B^i(C^\bullet)} = \dfrac{\ker(C^i \to C^{i+1})}{\mathrm{Im}(C^{i-1} \to C^i)}$.

Ques'n How do we make this functorial?

Vanilla Answer: cochain maps as morphs:

f

$$\begin{tikzcd}
\cdots \arrow[r] & C^{i-1} \arrow[r] \arrow[d, "f^{i-1}"'] & C^i \arrow[r] \arrow[d, "f^i"'] & C^{i+1} \arrow[r] \arrow[d, "f^{i+1}"'] & \cdots \\
\cdots \arrow[r] & D^{i-1} \arrow[r] & D^i \arrow[r] & D^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

$\rightsquigarrow$ Cochain($\mathcal{A}$) = cochain complexes + cochain maps.

exercise: H^i: Cochain($\mathcal{A}$) $\to \mathcal{A}$ well-def'd.

$$f : C^\bullet \to D^\bullet, \quad H^i(C^\bullet) \longrightarrow H^i(D^\bullet)$$
$$[c] \longmapsto [f(c)]$$

Def'n $f : C^\bullet \to D^\bullet$ a cochain map is quasi-isom

$\overset{\text{def}}{\Longleftrightarrow}$ $H^i(f)$ is an isom, $\forall i$.

$\boxed{\text{Slogan}}$ Derived Category = "Formally invert quasi-isoms in Cochain($\mathcal{A}$)".
Motto

<u>Def'n</u> $f, g \in \mathrm{Cochain}(\mathcal{A})(A^\cdot, B^\cdot)$

$f \sim g \Leftrightarrow f - g = d \circ h + h \circ d$
homotopic for some $h : A \to B[-1]$.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r] \arrow[dl, "h"'] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & A^i \arrow[r, "d"] \arrow[dl, "h"'] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & A^{i+1} \arrow[r] \arrow[dl, "h"'] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r] & B^i \arrow[r] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

$B^{i-1} = B[-1]^i$, $B^i = B[-1]^{i+1}$

<u>FACT</u> If $f \sim g$, then $H^i(f) = H^i(g)$, $\forall i$.

<u>GOAL</u>
- Form a category where morphs are only considered up to homotopy.
- Invert quasi-isom.

<u>Aux Constr</u>

$\mathrm{Comp}(\mathcal{A})^0$ <u>obj</u> : cochain comp.

<u>mor</u> :
$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r] \arrow[d, "f^{i-1}"] & A^i \arrow[r] \arrow[d, "f^i"] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r] & B^i \arrow[r] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

No Conditions (f^i).

$\mathrm{Comp}(\mathcal{A})^n$ <u>obj</u> : cochain maps

<u>mor</u> : $f^i : A^i \to B^{i+n} = B[n]^i$

a morph in $\mathrm{Comp}(\mathcal{A})^0$ btwn A & $B[n]$.

Special part: $A^\cdot, B^\cdot$ cochain comp.

$\mathrm{Comp}(\mathcal{A})(A^\cdot, B^\cdot)^* =$ cochain complex

$$d : \mathrm{Comp}(\mathcal{A})(A^\cdot, B^\cdot)^n \longrightarrow \mathrm{Comp}(\mathcal{A})(A^\cdot, B^\cdot)^{n+1}$$

$$df = d \circ f + (-1)^{n+1} f \circ d.$$

exercise: show $d^2 = 0$.

<u>Homotopy Category</u> $K(\mathrm{Comp}(\mathcal{A})) = K(\mathcal{A})$

<u>objects</u> cochain complexes

<u>morphs</u> $A^\cdot, B^\cdot$ cochain complexes

$$K(\mathcal{A})(A^\cdot, B^\cdot) := H^0(\mathrm{Comp}(\mathcal{A})(A^\cdot, B^\cdot)^*).$$

$$H^0(\mathrm{Comp}(\mathcal{A})(A^\cdot, B^\cdot)) = \frac{Z^0}{B^0} = \bigstar.$$

① $Z^0 = ?$. $df = d \circ f + (-1)^{n+1} f \circ d$ $\Big\}$ $n = 0$

$= d \circ f - f \circ d$

$df = 0 \Leftrightarrow d \circ f = f \circ d \Leftrightarrow f$ being a cochain map.

② $B^0 = ?$. $df = d \circ f + (-1)^{-1+1} f \circ d$

$= d \circ f + f \circ d =$ homotopy

$\rightsquigarrow \bigstar = \dfrac{\text{Cochain maps}}{\text{Homotopy}}.$

<u>Derived Category Def'n</u>

$K(\mathcal{A}) =$ homotopy cat, $\mathcal{S}(\mathcal{A}) =$ quasi-isoms.

$\rightsquigarrow D(\mathcal{A}) =$ Derived Category of $\mathcal{A}$

$$= \underbrace{K(\mathcal{A})_{\mathcal{S}(\mathcal{A})}}$$

localization of $K(\mathcal{A})$ at $\mathcal{S}(\mathcal{A})$.

⊕ Derived Categories (II)

Example $K^{\cdot} = (\cdots \xrightarrow{0} K^{i-1} \xrightarrow{0} K^{i} \xrightarrow{0} K^{i+1} \xrightarrow{0} \cdots)$

$H^{j} = Z^{j}/B^{j} = K^{j}/0 = K^{j}$.

"complex with zero maps has pieces isom. to own cohomology".

Def'n $K = K^{\cdot}$

$$(\cdots \to H^{i-1}K \xrightarrow{0} H^{i}K \xrightarrow{0} H^{i+1}K \to \cdots) = H^{\cdot}(K)$$

$$H^{i}(K^{\cdot}) = H^{i}(H^{\cdot}(K^{\cdot})) \qquad (*)$$

Question When $(*)$ induced by an isom. $K \to H^{\cdot}(K)$ in $D(\mathcal{A})$?

Def'n A complex concentrated at zero is

$$\begin{array}{ccccccc} (\cdots \to & 0 & \to & A & \to & 0 & \to \cdots) \\ & \uparrow & & \uparrow & & \uparrow & \\ & \deg -1 & & \deg 0 & & \deg 1 & \end{array}$$

Fact $\exists$ embedding $\mathcal{A} \to D(\mathcal{A})$ via

$$A \longmapsto \underbrace{(\cdots \to 0 \to A \to 0 \to \cdots)}_{\in\, D(\mathcal{A})}$$

Convention: From now on we will identify objects of A with their complex concentrated at zero.

Observation • $K^{\cdot}$ cochain complex in $D^{+}(\mathcal{A})$ ← (Def: Complexes need to have $K^{i} = 0,\ \forall i < 0$).

$H^{0}(K^{\cdot})$ = stupid complex.

$$\begin{array}{ccccccc} \cdots \to & 0 & \to & H^{0}(K^{\cdot}) & \to & 0 & \to \cdots \\ & & & \downarrow & & \downarrow & \\ \cdots \to & 0 & \to & K^{0} & \to & K^{1} & \to \cdots \end{array}$$

so we have $H^0(K^\cdot) \hookrightarrow K^\cdot$ in $D^+(\mathcal{A})$.

<u>Def'n</u> A <u>simple complex</u> is

$$(\cdots \to 0 \to K^0 \to K^1 \to 0 \to \cdots)$$

i.e. concentrated in degree zero & one.

<u>Basic Computations</u>

$$H^0(0 \to A \xrightarrow{f} B \to 0) = \frac{Z^0}{B^0} = \ker(f).$$

$$H^1(0 \to A \xrightarrow{f} B \to 0) = \frac{Z^1}{B^1} = \frac{B}{\operatorname{im}(f)} = \operatorname{coker}(f).$$

$$B \twoheadrightarrow H^1(0 \to A \to B \to 0).$$

<u>Truncation</u> $K^\cdot = (\cdots \to K^{i-1} \to K^i \to K^{i+1} \to \cdots)$.

$$\tau_{\leq n}(K^\cdot) = (\cdots \to K^i \to \cdots \to K^{n-1} \to K^n \to 0 \to \cdots).$$

$$H^i(\tau_{\leq n} K^\cdot) = \begin{cases} H^i(K^\cdot), & i < n \\ K^n/\operatorname{im}(f^{i-1}), & i = n \\ 0, & i > n \end{cases}$$

<u>Shifting Operation</u> $K^\cdot$ = cochain complex

$K[j]$ = shifted to the left by j.

$K[j]^i = K^{i+j}$.

<u>Claim</u> $H^i(K[j]) = H^i(K)[j]$.

proof. $H^i(K[j]) = Z^i(K[j]) / B^i(K[j]) = \dfrac{\ker(K[j]^i \to K[j]^{i+1})}{\operatorname{im}(K[j]^{i-1} \to K[j]^i)}$

$$= \frac{\ker(K^{i+j} \to K^{i+j+1})}{\operatorname{im}(K^{i+j-1} \to K^{i+j})}$$

$$= Z^{i+j}(K) / B^{i+j}(K) = H^{i+j}(K)$$

which means what we wanted.

$\square$

Justifies: $H^i(K^j)[\ell]$ ← parentheses not needed.

$$H^{\cdot}(K[j]) = H^{\cdot}(K)[j].$$

↖ as complexes with zero d-maps.

⑤ <u>Derived Categories (III)</u>

| $\mathcal{A}$ = ab cat.

<u>Fun Example</u> Ext'n of H by V.

$$\cdots \to 0 \to V \to E \to H \to 0 \to \cdots \quad \text{exact.}$$

⇝ $\mathrm{Ext}^1(H,V) := (R^1\mathrm{Hom}(H,-))(V)$.

<u>Claim</u> $\mathrm{Hom}(H, V[1]) \simeq \mathrm{Ext}^1(H,V)$.

$\mathrm{Hom}(H, V[n]) \simeq \mathrm{Ext}^n(H,V)$.

↖ Hom in $D(\mathcal{A})$.

We will construct $\mathrm{Ext}^1(H,V) \longrightarrow \mathrm{Hom}(H,V[1])$.

Fix an ext'n $0 \longrightarrow V \to E \to H \longrightarrow 0$ assoc. to $\xi \in \mathrm{Ext}^1(H,V)$.

$$\begin{tikzcd}
 & & 0 & & \\
\cdots \arrow[r] & 0 \arrow[r] & H \arrow[u] \arrow[r] & 0 \arrow[r] & \cdots \\
0 \arrow[r] & V \arrow[u] \arrow[d] \arrow[r] & E \arrow[u] \arrow[d] \arrow[r] & 0 \arrow[u] \arrow[d] \arrow[r] & \cdots \\
V[1]: \cdots \arrow[r] & V \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

quasi isom.

← $H^0 = H$, $H^i = 0$ else.

← $H^0 = \frac{Z^0}{B^0} = \frac{E}{V} \simeq H$, $H^{-1} = 0$, $H^i = 0$ else.

⇝ $\boxed{\mathrm{Ext}^1(H,V) \longrightarrow \mathrm{Hom}(H, V[1])}$. This is an isom.

building towards descrp of morphs in $D^+(\mathcal{A})$ we need to talk about injectives.

I inj
⇕
$\mathrm{Hom}(-, I)$ exact

<u>Def'n</u> $\mathcal{A}$ <u>has enough injectives</u> if $\forall A \in \mathcal{A}$,

$\exists\, \mathrm{magic}(A) \in \mathrm{inj}(A)$,

$$A \hookrightarrow \mathrm{magic}(A).$$

Suppose $\mathcal{A}$ has enough injectives.

<u>Claim 1</u> We can find an injective resolution of $A \in \mathcal{A}$:

$$A \hookrightarrow \underbrace{I^0 \to I^1 \to I^2}_{\text{an exact sequence}} \to \cdots$$

basic strategy: $A \to \underset{\substack{\| \\ \mathrm{magic}(A).}}{I} \to I/A \to \mathrm{magic}(I/A)$

gives the first step:

$$A \to \underset{\substack{\| \\ I^0}}{I} \to \underset{\substack{\| \\ I^1}}{\mathrm{magic}(I/A)}$$

$\leadsto$ then continue like this & check properties.

<u>Claim 2</u> Suppose $g: A^\cdot \to B^\cdot$ is a quasi-isom.
If $B^\cdot$ is built from injectives then
$\exists f: B^\cdot \to A^\cdot$ such that

$$f \circ g \sim \mathrm{id}_{A^\cdot} \leftarrow \text{homotopy to the identity.}$$

necessary to work in
$D^+(\mathcal{A}) \ni K^\cdot$, i.e. $K^i = 0$, $\forall i < 0$.

<u>Corollary of 1</u> Suppose $A \in \mathcal{A}$, & let $(0 \to I^0 \to I^1 \to \cdots)$ be an injective resolution. Then

$$A \simeq I^\cdot \text{ in } D^+(\mathcal{A}).$$

proof.

$$\begin{array}{llllllll} A & 0 \to & \overset{\deg 0}{A} & \to & 0 & \to & 0 & \to \cdots \\ & & \downarrow & & \downarrow & & \downarrow & \\ I^\cdot & 0 \to & I^0 & \to & I^1 & \to & I^2 & \to \cdots \end{array}$$

$\rightsquigarrow$ $H^0(0 \to A \to 0) = A$

$H^i(0 \to A \to 0) = 0, \quad i \neq 0.$

$H^0(I^\cdot) = Z^0/B^0 = A/0 = A$

$H^i(I^\cdot) = 0$ by exactness.

"A is quasi-isom. to its injective res. $A \to I^\cdot$".

Description of morphs in $D^+(\mathcal{A})$.

notation: $inj(\mathcal{A})$ injective objects of $\mathcal{A}$.

<u>Theorem</u> The composition

$$\begin{tikzcd} K^+(inj(\mathcal{A})) \arrow[r, two heads] \arrow[rr, bend right] & K^+(\mathcal{A}) \arrow[r] & D^+(\mathcal{A}) \end{tikzcd}$$

is an equivalence of categories.

proof. <u>Lem</u> $\forall K^\cdot \in K^+(\mathcal{A})$, $\exists I^\cdot \in K^+(inj(\mathcal{A}))$,

$K^\cdot \simeq I^\cdot$.

"every complex is quasi-isom to a complex of injectives".

Other part of proof uses claim 2;

that $A^\cdot \xrightarrow{f} I^\cdot$ quasi-isom, $\exists g : I^\cdot \to A^\cdot$ s.t.

$g \circ f \sim id_{A^\cdot}$.