Igusa stacks and p-adic Shimura varieties

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§1 Motivation

G red grp / F global field.

Global Langlands Conj:

$$\left\{\begin{array}{l}\text{cusp autom reps of } G\\ + \text{ extra conditions}\end{array}\right\}/\sim \quad \xleftrightarrow{1-1} \quad \left\{\begin{array}{l}\text{cont irred rep of}\\ \mathrm{Gal}(\bar{F}/F) \to {}^L G(\bar{\mathbb{Q}}_\ell)\end{array}\right\}/\underset{\text{weak}}{\sim}.$$

For F func field,

$X/\mathbb{F}_q$ sm proj curve,

- GL_2 (Drinfeld), GL_n (L. Lafforgue),

  G (Aut $\Rightarrow$ Galois, V. Lafforgue).

<u>Key</u> Cohom of moduli spaces of global/local shtukas.

For $F/\mathbb{Q}$ number field,

Scholze proposes $\mathbb{Q}$- or $\mathbb{Q}_p$-shtukas as analogues

$\rightsquigarrow$ $\mathbb{Z}$- or $\mathbb{Z}_p$-shtukas "integral models".

<u>Conj</u> (Scholze) $\exists\ \mathrm{Igs}_{K^p} / \mathbb{F}_p$ + maps sitting in a Cartesian diagram

$$\begin{tikzcd} \mathcal{S}_{K^p} \arrow[r, "\pi_{HT}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{F}\ell_{G,\mu^{-1}}^{\diamond} \arrow[d, "BL"] \\ \mathrm{Igs}_{K^p} \arrow[r] & \mathrm{Bun}_G \end{tikzcd}$$

§2 Setup

Perf_v site of perf'd spaces in char p + v-top.

w/ basis $\mathrm{Spa}(R, R^+)$. $R^+ = \prod_i C_i^+$
where $C_i = \bar{C}_i$ complete non-arch fields.
Objs of form $\mathrm{Spa}(R, R^+)$: product of geom pts.

<u>v-sheaf</u> e.g. E p-adic,

$$\mathrm{Spd}\, E : S \longmapsto \{S^\# \to \mathrm{Spa}\, E\}.$$

X/E scheme,

$$X^\diamond : \underset{\mathrm{Spa}(R,R^+)}{\overset{\|}{S}} \longmapsto \{ (\underset{\mathrm{Spa}(R^\#, R^{\#+})}{\overset{\|}{S^\#}}/E,\ \mathrm{Spec}\, R^\# \to X) \}.$$

Let $(G/\mathbb{Q}, X) \rightsquigarrow [\mu]$ conj class of minuscule cochar of G
called Hodge cochar
$[\mu] / E_0$ = reflex field. $v | p$, $E := E_{0,v}$.

$K \subset G(\mathbb{A}_{\mathbb{Q}f})$

$\rightsquigarrow (\mathrm{Sh}_K(G, X)_E)^\diamond =: \mathcal{S}_K$

$\mathcal{S}_{K^p} := \varprojlim_{K_p} \mathcal{S}_{K_pK^p}$ / $\mathrm{Spd}\, E$, $G := G_{\mathbb{Q}_p}$.

Define $(\mathcal{F}\ell_{G,\mu^{-1}})_E^\diamond := (G/\bar{P}_{\mu^{-1}})_E^\diamond$.

$\forall\ V \in \mathrm{Rep}_{\mathbb{Q}_p} G$,
$\mathbb{L} := \mathcal{S}_{K^p} \times^{K_p} V$ is Hodge-Tate w/ HT fil'n of type μ^{-1}.
$\rightsquigarrow$ get a map $\mathcal{S}_{K^p} \xrightarrow{\pi_{HT}} \mathcal{F}\ell_{G,\mu^{-1}}^\diamond$.

For $S = \mathrm{Spa}(R, R^+)$, $X_S := \left(\mathrm{Spa}\, W(R^+) \setminus V(p[\varpi])\right) / \varphi_S^{\mathbb{Z}}$ rel FF curve
ϖ p.u. $\circlearrowleft \varphi$

$$\mathrm{Bun}_G : \mathrm{Perf} \longrightarrow \text{Groupoids}$$
$$S \longmapsto G\text{-torsors }(X_S).$$

$$\mathcal{F}\ell_{G,\mu^{-1}}^{\diamond} \longleftrightarrow \left(S \mapsto \{(S^\#/E,\ \mathcal{E}_{triv} \overset{\mu^{-1}}{\dashrightarrow} \mathcal{E})\}\right.$$

Define $\quad BL : (S^\#, \mathcal{E}_{triv} \dashrightarrow \mathcal{E}) \longmapsto \mathcal{E}.$

e.g. $G = GL_2,\ X = \mathcal{H}^{\pm},\ \mu = (1,0),\ E_0 = \mathbb{Q}.$

$$\mathcal{S}_{K^p}/\mathrm{Spd}\,\mathbb{Q}_p : \left(S \longmapsto \{(S^\#, \mathcal{E}/\mathrm{Spec}\,R^\#, T_p\mathcal{E} \xrightarrow[\sim]{\alpha} \mathbb{Z}_p^{\oplus 2})\}/\sim\right)^{ad}$$

$$\mathcal{F}\ell_{GL_2,\mu^{-1}} = \mathbb{P}^1 = GL_2/B^{std}$$

Then

$$\mathcal{S}_{K^p} \xrightarrow{\quad \pi_{HT} \quad} \mathbb{P}^{1,\diamond}_{\mathbb{Q}_p}$$
$$(S^\#, \mathcal{E}, \alpha) \longmapsto (S^\#, \mathrm{Lie}\,\mathcal{E} \hookrightarrow T_p\mathcal{E} \otimes_{\mathbb{Z}_p} \mathcal{O}_{S^\#} \xrightarrow[\sim]{\alpha} \mathcal{O}_{S^\#}^{\oplus 2}).$$

In this case BL is as follows:

$$S^\# \overset{i}{\hookrightarrow} X_S,$$

$$\mathcal{E} := \lim\left(\begin{tikzcd} & \mathcal{O}_{S^\#}^{\oplus 2} \arrow[d] \\ i_* \mathrm{Lie}\,\mathcal{E} \arrow[r] & i_* \mathcal{O}_{S^\#}^{\oplus 2} \end{tikzcd}\right)$$

§3 Partial results

(1) (Zhang, 23) If (G, X) is PEL of type AC in Kottwitz classification, K_p hyperspecial, G unram, then conj holds on the good red'n locus.

$$\begin{tikzcd} \mathcal{S}^{\circ}_{K^p} \subset \{\mathcal{S}_{K^p}\}_{K^p} \arrow[r, "\pi_{HT}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F}\ell^{\diamond}_{G,\mu^{-1}} \circlearrowleft G(\mathbb{Q}_p) \arrow[d, "BL"] \\ G(\mathbb{A}_f^p) \circlearrowright \{\mathrm{Igs}_{K^p}\}_{K^p} \arrow[r] & \mathrm{Bun}_G. \end{tikzcd}$$

+ minimal cpt'n

+ integral model

+ Hecke equiv.

<u>Rmk</u> (i)

$$\begin{tikzcd} \mathscr{S}^{\diamond}_{K} / \mathrm{Spd}\, \mathcal{O}_E \arrow[r] \arrow[d] & \mathrm{Sht}_{\mathcal{G},\mu} \arrow[d] \\ \mathrm{Igs}_{K^p} \arrow[r] & \mathrm{Bun}_G \end{tikzcd}$$

where $\mathrm{Sht}_{\mathcal{G},\mu}$ = moduli stack of $\mathcal{G}$-shtukas with 1 leg bounded by μ.

(Berkeley lect of p-adic geometry)

$\mathcal{G}/\mathbb{Z}_p$ reductive model of G.

$\rightsquigarrow \quad \mathrm{Sht}_{\mathcal{G},\mu,E} := [\mathcal{F}\ell^{\diamond}_{G,\mu^{-1}} / \underline{K_p = \mathcal{G}(\mathbb{Z}_p)}]$.

(ii) Restricted to a basic Newton stratum $\in B(G,\mu^{-1})$

$\rightsquigarrow$ Rapoport–Zink p-adic uniformization.

(2) (Daniels – van Hoften – Kim – Zhang, in progress)

If (G,X) of Hodge type, K_p parahoric, then Conj holds for good red'n locus.

Moreover, Constr of $\mathrm{Igs}^{\circ}(G,X)$ is functorial in (G,X).

<u>Rmk</u> In both cases,

$$\mathrm{Igs}^{\circ}_{K^p}(G,X) : \underset{\substack{\mathrm{Spa}(R,R^+) \\ \rotatebox{90}{$\in$} \\ \varpi}}{\overset{}{S}} \longrightarrow \left\{ \begin{array}{l} \text{Obj: } \mathcal{E}/(R^+/\varpi), \\ \text{Mor: quasi-isog} \end{array} \right\}$$

(e.g. GL_2)

§4 Application to cohomology

(1) Torsion vanishing

(CS, Koshikawa, Hamann-Lee).

If (G, X) Hodge type, K_p hyperspecial,

$Sh_K(G, X) / E_0$ proper.

Assume $G = G_{\mathbb{Q}_p}$ has a "nice" local Langlands corr (Hamann).

that is compatible with FS's constr,

then $\forall\, \mathfrak{m} \subset \mathcal{H}_p := \bar{\mathbb{F}}_\ell(K_p \backslash G(\mathbb{Q}_p)/K_p)$, $\ell \neq p$

s.t. FS parameter $\varphi_{\mathfrak{m}}$ is "generic" (Hamann),

have $\quad H^i(Sh_K(G, X), \bar{\mathbb{F}}_\ell)_{\mathfrak{m}} \neq 0$

$\Rightarrow i = \dim_{E_0} Sh_K(G, X) =: d$.

<u>Pf sketch</u> ① Igs_{K^p} is ℓ-cohomologically smooth of ℓ-dim 0

$\rightsquigarrow \quad \mathbb{D}_{Igs_{K^p}} = \bar{\mathbb{F}}_\ell[0] \in D_{\acute{e}t}(Igs_{K^p}, \bar{\mathbb{F}}_\ell)$.

$\Rightarrow$ For $Igs_{K^p} / \bar{\mathbb{F}}_p \xrightarrow{\bar{\pi}_{HT}} Bun_G$,

$R\bar{\pi}_{HT,*}\bar{\mathbb{F}}_\ell$ is perverse for t-str on $D_{\acute{e}t}(Bun_G, \bar{\mathbb{F}}_\ell)$.

② $R\Gamma(Sh_K(G, X), \bar{\mathbb{F}}_\ell)_{\mathfrak{m}} = i_1^* T_\mu (R\bar{\pi}_{HT,*}\bar{\mathbb{F}}_\ell)_{\varphi_{\mathfrak{m}}}[-d](\frac{-d}{2})$.

$[*/G(\mathbb{Q}_p)] \cong Bun_G^1 \xhookrightarrow{i_1} Bun_G$.

Assumption $\Rightarrow$ t-exactness of $i_1^* T_\mu$.

$\Rightarrow$ desired result.

(2) Global-local compatibility (constr of Gal rep).

In the setting of [CS, non-compact, 2019],

$$Igs_{K^p} \xrightarrow{\bar{\pi}} Bun_G.$$

$\mathcal{F}_! := R\bar{\pi}_!\bar{\mathbb{F}}_\ell$, $\quad \mathcal{F}_* := R\bar{\pi}_*\bar{\mathbb{F}}_\ell$.

$\circlearrowleft \mathbb{T}^S$

For $\mathfrak{m} \subset \mathbb{T}^S$ in the support of $\mathcal{F}_!$ or $\mathcal{F}_*$,

$\rightsquigarrow$ Galois rep $\bar{\rho}_{\mathfrak{m}}$ w/ $\mathbb{F}_\ell$-coeff

If $\mathfrak{m}$ non-Eisenstein

$$\Rightarrow (\mathcal{F}_!)_{\mathfrak{m}} = (\mathcal{F}_*)_{\mathfrak{m}} \text{ \& both are perverse.}$$

<u>Expect</u> $(\bar{\rho}_{\mathfrak{m}}|_{G_{F,v}})^{ss} \rightsquigarrow \varphi_{\mathfrak{m}}$

And if $\varphi_{\mathfrak{m}}$ generic, then

$$(\mathcal{F}_!)_{\mathfrak{m},\varphi_{\mathfrak{m}}} = (\mathcal{F}_!)_{\mathfrak{m}}.$$