Arithmetic Surfaces

A Dedekind dom of dim 1. $S = \operatorname{Spec} A$. (e.g. $A = \mathcal{O}_K$, K = number field.)

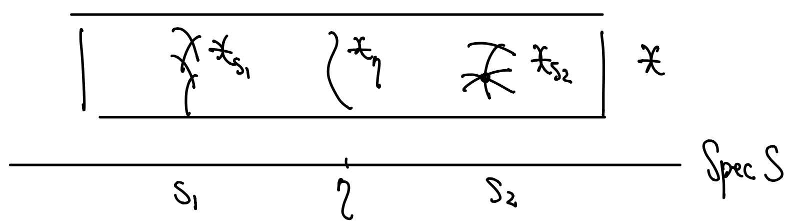
<u>Def:</u> An arith surface is a pair $(\mathfrak{X}, \pi)$

where $\cdot$ $\mathfrak{X}$ integral sch

$\cdot$ $\pi : \mathfrak{X} \to S = \operatorname{Spec} A$ flat proper. rel curve of fin type.

In particular, $\mathfrak{X}_\eta$ = generic fiber & $\dim \mathfrak{X} = 2$.

$\uparrow$ proj integral curve / $K = \operatorname{Frac} A$



<u>Note</u> $\mathfrak{X}$ regular ($\Rightarrow$ normal) $\Rightarrow$ $\mathfrak{X}_\eta$ smooth

However, $\mathfrak{X}_s$ can be singular or non-reduced or reducible.

<u>Q</u> Given X/K curve (proj sm, fin type, connected),

is there any "nice" $(\mathfrak{X}, \pi)$ s.t. $\mathfrak{X}_\eta \cong X$?

- If "nice" = normal, $\mathfrak{X}$ = spreading-out of X.
- If "nice" = regular, Lichtenbaum's thm.

<u>Idea</u> Spreading-out, blow-up, normalization, blow-up, ... (repeat) $\rightsquigarrow \mathfrak{X}$.

<u>Thm</u> (Lichtenbaum) $(\mathfrak{X}, \pi)$ regular $\Rightarrow$ $\pi : \mathfrak{X} \to S$ proj.

<u>pf sketch</u> <u>Step 1</u> Define the intersections.

$$\operatorname{Div}_S(\mathfrak{X}) \times \operatorname{Div}(\mathfrak{X}) \longrightarrow \mathbb{Z}$$

<u>idea</u> $\operatorname{Div}(X)$ = sheaf on X $\rightsquigarrow$ can pullback $\operatorname{Div}(X)$ along $\mathfrak{X} \to \mathfrak{X}_\eta \xrightarrow{\sim} X$

$\mathrm{Div}(X)$: sheaf on a nice proj var

$\Rightarrow$ Have well-def'd degrees.

<u>Step 3</u> Ampleness.

Construct $D \in \mathrm{Div}(\mathcal{X})$ s.t.

- $\mathrm{supp}(D)$ contains no fiber component.
- D meets every fiber component.

$\Rightarrow D|_X$ ample & D ample for π.

<u>Thm</u> (minimal model) Let X/K be a "nice" curve.

(1) $\exists$ regular int model $\mathcal{X}$ of X

(2) Can take $\mathcal{X}$ minimal (contains least info).

i.e. $\forall \mathcal{X}'$ of (1), $\pi' : \mathcal{X}' \dashrightarrow \mathcal{X} \xrightarrow{\pi} \mathrm{Spec}\, A$.

(3) $g(X) \geq 1 \Rightarrow \mathcal{X}^{min}$ in (2) is unique.

Call $(\mathcal{X}, \pi)$ in (2) rel minimal. Call $(\mathcal{X}, \pi)$ in (3) minimal.