

§ 3.1 Affinoid Algebras.

Def 1: A K -alg A is called an **affinoid algebra** if there exists an epic morphism of K -algebras $\alpha: T_n \twoheadrightarrow A$ for some $n \in \mathbb{N}$.

Prop 2. (The category of affinoid admits (complete) Tensor product)

Prop 3. A is an affinoid K -alg.
(revisited. § 2.2).

(i) A is Noetherian

(ii) A is Jacobson

(iii). Noether Normalization

$\exists d \in \mathbb{Z}_{\geq 0} \quad T_d \hookrightarrow A$. finite.

Prop 4. A is affinoid K -alg.

$\Leftrightarrow A \subset A$ s.t. $\sqrt{g} = m$ is maximal.

$\Rightarrow A/g$ is finite dimensional over K .

Pf: A/g is finite dimensional

$\Leftarrow A/m^k$ is _____

$A \rightarrow A_m$.

$$\underbrace{A_m/m^k A_m}_{= A/m^k}$$

$$A_m/m A_m, \underbrace{m A_m/m^{i+1} A_m}_{\dots}$$

A/m^i - finite dimensional

$\dim_K A/m < \infty$.

Def: Residue norm.

A : affinoid K -alg.

$\alpha: T_n \rightarrow A$.

$|\cdot|_\alpha: A \rightarrow \mathbb{R}_{\geq 0}$.

$f \in T_n$. $\underline{\alpha(f)} \mapsto \inf_{\alpha \in \ker \alpha} |f - \alpha|$

Proposition 5. For an ideal $\alpha \subset T_n$, view the quotient $A = T_n/\alpha$ as an affinoid K -algebra via the projection map $\alpha: T_n \rightarrow T_n/\alpha$. The map $|\cdot|_\alpha: T_n/\alpha \rightarrow \mathbb{R}_{\geq 0}$ satisfies the following conditions:

- $|\cdot|_\alpha$ is a K -algebra norm, i.e. a ring norm and a K -vector space norm, and it induces the quotient topology of T_n on T_n/α . Furthermore, $\alpha: T_n \rightarrow T_n/\alpha$ is continuous and open.
- T_n/α is complete under $|\cdot|_\alpha$.
- For any $\bar{f} \in T_n/\alpha$, there is an inverse image $f \in T_n$ such that $|\bar{f}|_\alpha = |f|$. In particular, for any $\bar{f} \in T_n/\alpha$, there is an element $c \in K$ with $|\bar{f}|_\alpha = |c|$.

Pf: (i)
 $|\bar{f}|_\alpha = 0 \iff \bar{f} = 0 \iff f \in \ker \alpha$.
(α is closed).

$$\underline{|\bar{f}\bar{g}|} \leq |\bar{f}| \cdot |\bar{g}|$$

$$|\bar{s}|_2 = \inf \{ \cdot \mid \cdot \mid$$

$$= |f - a| \quad \text{for some } a \in \mathbb{R}.$$

$$|\bar{g}|_2 = |g - b|. \quad b \in \mathbb{R}$$

$$|\bar{f}\bar{g}|_2 \leq |(f-a)(g-b)|$$

$$= |f-a| \cdot |g-b|$$

$$= |\bar{f}|_2 \cdot \underline{|\bar{g}|_2}$$

$$|\bar{f} + \bar{g}|_2 \leq \max \{ |\bar{f}|_2, |\bar{g}|_2 \}.$$

$$|c\bar{f}|_2 = |c| \cdot |\bar{f}|. \quad \text{clear.}$$

Openness of α :

$$\alpha(B(\alpha, \varepsilon)) = B_\alpha(\alpha, \varepsilon).$$

\Leftarrow : Simple

\Rightarrow : α is strictly closed.

Continuity:

$$\|\alpha(f)\|_\alpha \leq \|f\|.$$

(ii) T_n/α is complete under $\|\cdot\|_\alpha$.

Sketch =

$$\overline{T_n}$$

 \Downarrow

$$T_n/\alpha.$$

$$x_1 - \dots - x_n$$

Cauchy?

$$\uparrow$$

$$y_1, \dots, y_n, \dots$$

Cauchy

$$\|x_{n+1} - x_n\| = \|y_{n+1} - y_n\|_\alpha.$$

Defn. If $|f|_a = |f|_{T_n} = |c|$ for some $c \in K$.

$\alpha: T_n \rightarrow A$. $\ker \alpha = \emptyset \subseteq T_n$

(1) $1 \cdot |\alpha|$.

(2) $f: V(\emptyset) \subset \underline{\mathbb{B}^n(K)} \rightarrow \overline{K}$.

$$|f|_{\sup} = \sup_{x \in \text{dom } f} |f(x)|.$$

$f(x) \in A/x$. \hookrightarrow

finite over K.



$|f|_{\sup} = 0 \iff f \in A$. is nilpotent.

Prop 6. Power multiplicative:

$$|f^n|_{\sup} = |f|^{\sup^n}, \forall f \in A, n \in \mathbb{Z}_{\geq 1}.$$

Prop 7. $\varphi: B \rightarrow A$ is a morphism between affinoid K -algebras.

Then $(\varphi(b))|_{\sup} \leq |b|_{\sup}, \forall b \in B$.

Pf: $|\varphi(b)(m)| \underset{m \in \text{Max } A}{\leq} |\varphi(b)(n)| \underset{n \in \text{Max } B}{\leq} |b|_{\sup}$.

Choose $n = \varphi^{-1}(m) \in B$.

n is maximal.

finite ext
over K .

$K \hookrightarrow B/n \hookrightarrow A/m$.
is a field

$$|\varphi(b)(m)| = |b(n)|, \quad \forall m \in \max A$$

$$\Rightarrow |\varphi(b)|_{\sup} \leq |b|_{\sup}. \quad \text{④}$$

Prop 8. $|\cdot|_{\sup} = |\cdot|$ (Grus norm)

on a Tche algebra T_n .

If: $f \in T_n$.

$$|f| = \max \{f(x) : x \in B^n(\bar{K})\}.$$

(maximum principle)

$$= |f|_{\sup}.$$

Recall: $\overline{B^n(\bar{K})} \rightarrow \max T_n$.

$$x = (x_1, \dots, x_n) \mapsto \max$$

$$= \ker (T_n \rightarrow \bar{K})$$

$$f \mapsto f(x_1, \dots, x_n)$$

Prop 9. A : affinoid K -alg.

$\alpha : T_n \rightarrow A$.

(•) α : Residue norm.

Then $|f|_{\text{sup}} \leq |f|_{\alpha}, \forall f \in A$.

$\Rightarrow |f|_{\text{sup}}$ is finite.

Pf: Choose $m \in \max A$. and $f \in A$.

$n = \alpha(m) \in \max T_n$.

$f \in A \xrightarrow{\text{preinf}} g \in T_n : |g| = |f|_{\alpha}$.

$|f(m)| = |g(n)| \leq |g| = |f|_{\alpha}$.

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Prop 10. $|f|_{\sup} = 0$

\Updownarrow

f is nilpotent.

If: A is Jacobson. \square .

• $| \cdot |_{\sup} \neq | \cdot |_2$.

Lem 11.

$$\begin{aligned} p(\xi) &= \xi^n + c_1 \xi^{n-1} + \dots + c_n \in K[\xi] \\ &= \prod_{j=1}^n (\xi - \alpha_j), \quad \alpha_j \in \overline{K}. \end{aligned}$$

Then $\max_{j=1, \dots, n} |\alpha_j| = \max_{i=1, \dots, n} |c_i|^{\frac{1}{i}}$.

If: \geq :

$$c_i = \pm \sum_{m_1 < \dots < m_i} \alpha_{m_1} \cdots \alpha_{m_i}$$

$$|c_i| \leq \max \{| \alpha_{m_1}|, \dots, | \alpha_{m_i}|\}.$$

$$\leq \left(\max_{\delta=1, \dots, r} |\alpha_\delta| \right)^t$$

$\Leftarrow :$

$I \subset \{1, \dots, r\}$. s.t.

$$|\alpha_i| = \max_{\delta=1, \dots, r} |\alpha_\delta| \Leftrightarrow i \in I.$$

$$t = |I|$$

$$|\alpha| = \left| \sum_{\delta_1, \dots, \delta_r} \alpha_{\delta_1} \cdots \alpha_{\delta_r} \right|.$$

$$= \left| \prod_{\delta \in I} \alpha_\delta \right| = \left(\max_{\delta=1, \dots, r} |\alpha_\delta| \right)^t.$$

$$\Rightarrow |\alpha|^{\frac{1}{t}} = \max |\alpha_\delta|. \quad \square.$$

$\sigma(p)$. $p \in A[\mathfrak{S}]$. A is normed ring.

$$\sigma(p) = \max_{j=1, \dots, r} |c_j|^{\frac{1}{r}} \quad (\text{Spectral value})$$

where $P = g^r + c_1 g^{r-1} + \dots + c_r$.

In Lem 11: $\max(|a_i|) = \sigma(p)$.

Lem 12:. A is a (semi-)normed ring.

$p, g \in A[\mathfrak{S}]$. are monic.

$$\sigma(pg) \leq \max(\sigma(p), \sigma(g)).$$

pf:. $P = \sum_{i=0}^m a_i g^{m-i}$ $a_m = b_m = 1$.

$$g = \sum_{j=0}^n b_j g^{n-j}$$

$$pg = \sum_{\lambda \geq 0}^{m+n} c_\lambda g^{m+n-\lambda} \quad c_\lambda = \sum_{i+j=\lambda} a_i b_j.$$

$$|c_n| \leq \max(\sigma(p), \sigma(g))^{-n}.$$

We know

$$|a_i| \leq \sigma(p)^i \leq \max(\sigma(p), \sigma(g))^i$$

$$|b_{\delta}| \leq \sigma(g)^{\delta} \leq \dots^{\delta}.$$

$$|c_n| = \left| \sum_{i+\delta=n} a_i b_{\delta} \right|$$

$$\leq \max_{i+\delta=n} |a_i| |b_{\delta}|$$

$$\leq \max_{i+\delta=n} \left(\max(\sigma(p), \sigma(g)) \right)^{\overbrace{i+\delta}^{itd.}}$$

(II).

Lemma 13. $T_d \hookrightarrow A$: finite norm of
 K -alg.

A is torsion-free as T_d -mod

$f \in A$.

(i) $\exists!$ monic $P_f = g^n + a_1 g^{n-1} + \dots + a_r$
 $\in T_d[\mathbb{S}]$ of minimal deg s.t. $P_f(f) = 0$.

And. $\ker \begin{bmatrix} T_d[\mathbb{S}] & \xrightarrow{\quad} A \\ g & \mapsto f \end{bmatrix}$ is

generated by P_f as a T_d -tors.

(ii) $y_1, \dots, y_s \xleftarrow{\text{max } A} A$

$$\begin{array}{ccc} | & & | \\ x & \xrightarrow{T_d} & f \end{array}$$

$\in \max T_d$.

Then

$$\max_{j=1, \dots, s} |f(y_j)|$$

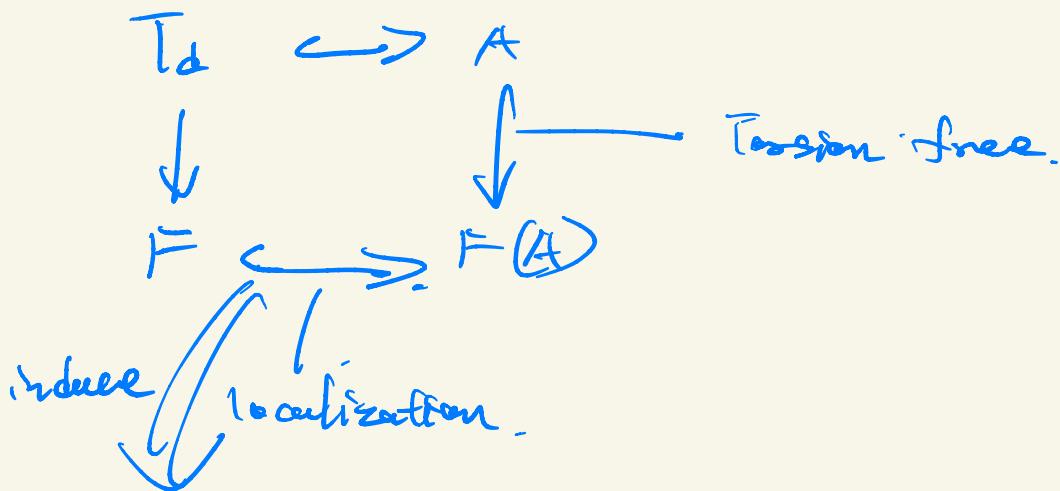
$$= \max_{i=1, \dots, r} |a_i(x)|$$

$(\forall i) (\Leftarrow c_i)$ by taking sup over Max T_d)

$$|f|_{\sup} = \max_{i=1, \dots, r} |a_i|_{\sup}^{\frac{1}{i}}$$

Pf. $F = Q(T_d)$. $F(A) = A \otimes_{\mathbb{Q}} F$

\exists , deg of inclusion.



$$\ker \left(\begin{matrix} F[\mathfrak{s}] & \xrightarrow{\quad} & F(A) \\ s & \mapsto & f \end{matrix} \right) = (P_f), \quad P_f \in F[\mathfrak{s}].$$

claim: $P_f \in T_d[\zeta]$.

$T_d \hookrightarrow A$ finite.

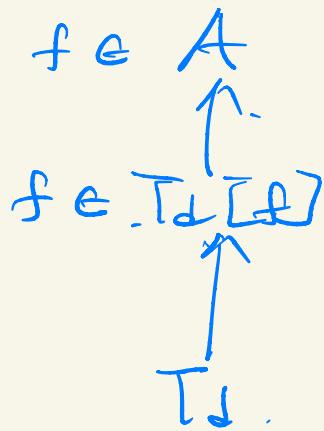
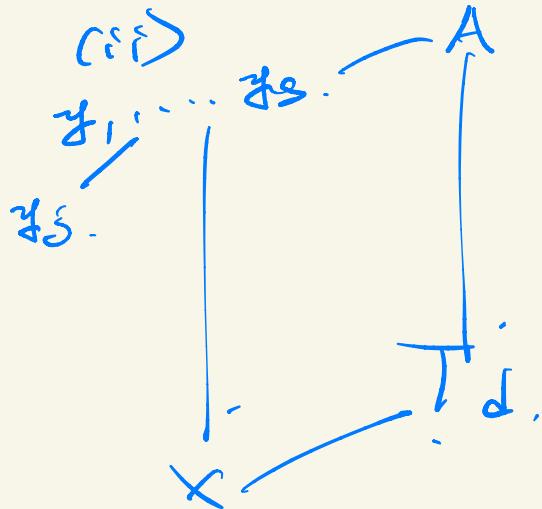
$f \in A \Rightarrow h \in T_d[\zeta] \subset F[\zeta].$
 $h(f) = 0.$ h is monic.

$\Rightarrow P_f | h$ over $F[\zeta]$.

$\Rightarrow P_f | h$ over $T_d[\zeta]$, as T_d is factorial.

($h \in T_d[\zeta]$, $h(f) = 0 \Rightarrow P_f | h$).

Thus $T_d[\zeta] \rightarrow A$ has kernel
 $\zeta \mapsto f.$
(P_f).



$$\text{Max } A \rightarrow \text{Max } T_d[f] \rightarrow \text{Max } T_d.$$

$$y_0, \underline{\hspace{2cm}}, z_0, \underline{\hspace{2cm}}, x.$$

$$|f(y_0)| = |f(z_0)|.$$

$$T_d/x \hookrightarrow T_d[f]/\overline{z_0} \hookrightarrow A/\overline{y_0}$$

$$\overline{f}$$

$$\max |f(y_0)| = \max |f(z_0)|.$$

Thus WLOG, $A = T_d[f]$.

$$A = T_d[A] \stackrel{\text{by (i)}}{\equiv} T_d[S]/(P_S).$$

$$y_1, \dots, y_s \rightarrow T_d[\bar{x}] = T_d[S]/(P_S).$$

$y_1 \downarrow$ $\downarrow A$
 $x \longrightarrow T_d$

Set $L = T_d/x$ (is finite over k)

$$A/x = L[S]/(\bar{P}_S).$$

\bar{P}_S has roots in \bar{L} , denoted by

$$\alpha_1, \dots, \alpha_r$$

$$\max_{\bar{S}} |f(y_i)| = \max_{i=1, \dots, r} |\alpha_i|.$$

\uparrow
 \bar{P}_S

$$= \max_i |\alpha_i(x)|^{\frac{1}{r}} \quad \textcircled{D}$$

Lem 14. Let $\varphi: \mathcal{B} \rightarrow \mathcal{A}$ is finite hom between affineoid K -algebras.

Then for any $f \in \mathcal{A}$. \exists equation.

$$f = f(b_1) f^{(n-1)} + \dots + b_n = 0. \quad b_i \in \mathcal{B}.$$

$$\|f\|_{\sup} = \max_{i=1, \dots, n} \|b_i\|_{\sup}^{\frac{1}{i}}$$

Thm 15. A is an affine K-algebra.

$f \in A$.

Then \exists $\text{relex } A$ s.t.

$$|f(x)| = |f|_{\sup}.$$

If: A. maximal probes p_1, \dots, p_s .

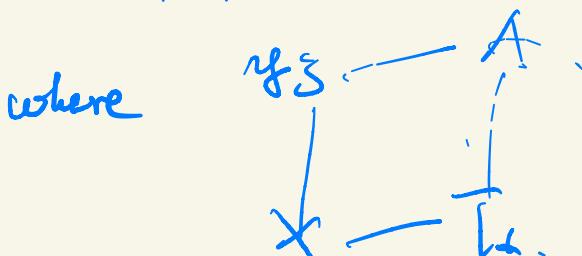
WLOG. A is an int domain.

\Rightarrow A finite monic $T_d \subset A$.

Choose $f \in A$.

$f^n + a_1 f^{n-1} + \dots + a_n = 0$ over T_d .

$$\Rightarrow \max_{j=1, \dots, s} |f(y_j)| = \max_{i=1, \dots, s} |a_i(x)|^{\frac{1}{n}}$$



$$|\alpha_1 \dots \alpha_r(x)| = |\alpha_1 \dots \alpha_r|$$

|| ||

$$\prod |\alpha_i(x)| = |\alpha_1| \dots |\alpha_r|$$

for some $x \in \max T_d$.

$$\Rightarrow |\alpha_i(x)| = (\alpha_i|, \vartheta_i)$$

$$\max_{y \in S} |f(y)| = \max_{i \in \{1, \dots, r\}} |\alpha_i(x)|^{\frac{1}{r}}$$

$$= \max_{i \in \{1, \dots, r\}} |\alpha_i|^{\frac{1}{r}}$$

$$= |f|_{\sup} \text{ (Lem 4).}$$

 (4).

Prop 16.: A: Affine K-alg

fGA.

$\Rightarrow \exists n \in \mathbb{Z}_{\geq 0}$ s.t. $|f|_{\sup}^n \in |K|$

Pf: Lem 4: $T_A \xrightarrow{\text{fixee}} A$.

$$|f|_{\sup} = \max_{i=1, \dots, r} |b_i|_{\sup}$$

$$= |b_{\delta}|_{\sup}$$

$$|f|^{\frac{r}{\delta}}_{\sup} = |b_{\delta}|_{\sup} \in |K|.$$

Theorem 7. f GA, TFAE.

(\Leftarrow) $|f|_{\sup} \leq 1$.

(\Rightarrow) $\exists f^n + a_1 f^{n-1} + \dots + a_n = 0$ with
coefficients $a_i \in A$ s.t $|a_i|_A \leq 1$.

(\Leftarrow). If $f^n|_A$ is bounded.

Sketch: $T_d \xrightarrow{\text{fixee}} A$

$\Leftarrow \Rightarrow$ By lem 4:

\exists equation with $|a_i|_{\sup} = |a_i| \leq 1$.

where $a_i \in T_d$.

$T_d \hookrightarrow T_n \xrightarrow{\text{d}} A$.

Contractive $\Rightarrow |a_i|_A \leq 1$.

(iii) \Rightarrow (iii).

$$A^\circ = \{g \in A : \|g\|_a \leq 1\}.$$

(ii) \Rightarrow $A^{\circ}[f]$ is finite over A° .
norms are bounded
 $\Rightarrow \|f^n\|_a$ is bounded.

$$(iv) \Rightarrow (i). \|f\|_{\sup} = \underline{\|f\|_{\sup}^n} \leq \underline{\|f^n\|_a} \cdot \overline{\|f^n\|_a}^{1/n} \geq 1$$
$$\Rightarrow \|f\|_{\sup} \leq 1.$$

Cor (B). TFAE . $\alpha: \mathbb{T}_n \rightarrow A$.

(i) If $|f|_{\sup} < 1$.

(ii). $\{f^n\}_{\alpha}$ is a zero seq.

Sketch: (i) \Rightarrow (ii).

$$|f|_{\sup} = |\alpha|. \quad \alpha \in K.$$

$$\Rightarrow |c^{-1} f^n|_{\sup} = 1.$$

Thm 17. $\{\|c^{-n} f^n\|_2\}_n$ is bounded.

$\Rightarrow \{\|f^n\|_2\}_n$ is a zero seq.

$\Rightarrow \{\|f^n\|_2\}_n$ is _____.

Lem 19. A . f. . . - the & A .

(i) $\varphi : K \subset S_1, \dots, S_n \rightarrow A$.

$$s_i \mapsto f_i.$$

$$\Rightarrow |f_i|_{\sup} \leq 1, \forall i.$$

[Contractive under $| \cdot |_{\sup}$].

(ii) $|f_i|_{\sup} \leq 1$.

$\Rightarrow \exists !$ K-mon. (No demand of continuity).

$\varphi : K \subset S_1, \dots, S_n \rightarrow A$ s.t.

$$s_i \mapsto f_i.$$

If: (i). Q.

(ii).

Prop 20. If α and $\beta: B \rightarrow A$ between
affined K -algs is continuous w.r.t
any res norm on A & B .

And all res norms on affined
 K -alg are equivalent.

If:

$\alpha: T_B \rightarrow A$
 $\beta: T_B \rightarrow A$

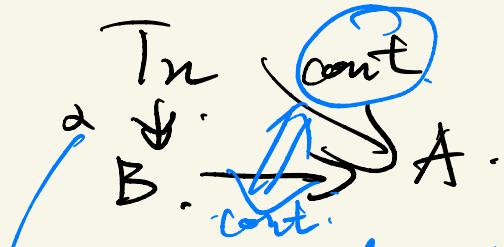
$$\begin{array}{ccc}
 A & \xrightarrow{\text{id}} & A \\
 \downarrow & \swarrow \text{id} & \downarrow \\
 1 - | \alpha & & 1 - | \beta
 \end{array}$$

is continuous.

\Rightarrow Equivalent,

First cover:

$$B \rightarrow A. \quad B: 1. l_2. \quad \alpha = T_n \rightarrow B.$$



B - is endowed with root topology.

④.

Example.

$$A < (\mathbb{S}).$$

$$\left| \sum_{v \in V} \alpha_v \delta^v \right| \leq \max_{v \in V^n} |\alpha_v|_2.$$

