p-adic Simpson correspondence

Mingjia Zhang

(Joint with B. Bhatt)

Corlette–Simpson nonabelian Hodge theory

$X/\mathbb{C}$ cpt Kähler.

Hodge theory $\quad H^1(X,\mathbb{C}) \simeq H^1(X,\mathcal{O}_X) \oplus H^0(X,\Omega^1_X)$.

$$\begin{array}{l} \quad\;\; \wr\wr \\ \mathrm{Hom}(\pi_1(X,x), \underbrace{\mathbb{C}}_{\substack{\| \\ \mathbb{G}_a(\mathbb{C})}}) \end{array} \quad (\mathbb{G}_a \text{ ab grp})$$

Q What if replacing $\mathbb{G}_a$ with general linear grp?

Def A Higgs bdl on X is a pair $(\mathcal{E},\theta)$

- $\mathcal{E}$ vect bdl $/X$
- $\theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ $\mathcal{O}_X$-linear s.t. $\theta \wedge \theta = 0$.

  ("Higgs field").

Thm (Colette, Simpson) $\exists$ equiv of tensor cats

$$\left\{ \begin{array}{l} \text{rep'ns of } \pi_1(X,x) \text{ on} \\ \text{f.d. } \mathbb{C}\text{-v.s.} \end{array} \right\} \simeq \left\{ \begin{array}{l} \text{semistable Higgs bdls } / X \\ \text{w/ vanishing rat'l Chern classes} \end{array} \right\}.$$

tensor: $(\mathcal{E}_1,\theta_1) \otimes (\mathcal{E}_2,\theta_2) = (\mathcal{E}_1 \otimes \mathcal{E}_2, \theta_1 + \theta_2)$

Q What if $\mathbb{C} \rightsquigarrow \mathbb{C}_p := \widehat{\overline{\mathbb{Q}}_p}$?

$(X/\mathbb{C}_p$ sm rigid space$)$.

- Faltings, Deninger - Werner, ...
- Heuer 2023 : for X sm proper (not only curve).

Prime characteristic ($p>0$)

$X/\mathbb{F}_p$ smooth, $\mathcal{D}_X :=$ sheaf of $\mathbb{F}_p$-alg gen'd by $\mathcal{O}_X, T_X / k$

w/ relation $\partial \cdot f - f \cdot \partial = \partial(f)$

$\partial\partial' - \partial'\partial = [\partial, \partial']$.

Thm (Bezrukavnikov - Mirković - Rumynin, Ogus - Vologodsky)

$\exists$ natural (in X) $\mathbb{G}_m$-gerbe (aka. $B\mathbb{G}_m$-torsor) $\mathcal{S}(X)$ on $(T^*X)_{\mathrm{et}}$

and an equiv of tensor cats

$$\mathcal{D}_X\text{-mod} \cong \mathrm{QCoh}(\mathcal{S}(X))_{\deg 1}$$

$B\mathbb{G}_m \times \mathcal{S}(X) \xrightarrow{a} \mathcal{S}(X)$

obj $= \mathcal{E}$ s.t. $a^*(\mathcal{E})$ deg 1.

Rmk (1) char 0 : Betti - de Rham - Dolbeault (Corlette - Simpson)

$\mathbb{C}$-locsys / vect bdl w/ flat conn $\subseteq \mathcal{D}_X$-mod / Higgs bdl

$\mathrm{QCoh}(\mathcal{S}(X))_{\deg 1} \simeq \mathrm{QCoh}(T^*X) \supseteq \mathrm{Higgs}(X)$

$\mathcal{S}(X)$ was split $\qquad (\mathcal{E}, \theta) \qquad \theta : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1$

$\mathcal{O}_{T^*X} = \mathrm{Sym}^\cdot T_X \longrightarrow \mathcal{E}nd(\mathcal{E})$.

$\theta =$ datum that provides $\mathcal{E}$ w/ an $\mathcal{O}_{T^*X}$-mod str.

But $\mathcal{P}(X)$ does not split if $\dim X > 0$.

(2) Azumaya property of D_X

$$Z(D_X) \longleftarrow \mathrm{Sym}^{\cdot}\, T_{X/k} = \mathcal{O}_{T^*X'}.$$

$$\partial^{p} - \partial^{[p]} \longleftarrow\!\shortmid\, \partial$$

p-th composite as vect field.

Fact D_X is an Azumaya algebra T^*X

$\mathcal{P}(X)$ gerbe of splittings of D_X

$\rightsquigarrow$ equiv follows formally from Morita equiv.

(3) $\mathcal{P}(X)$ multiplicative, i.e.

$$T^*X \longrightarrow B^2\mathbb{G}_m \in \mathcal{D}(\mathbb{Z})$$

$\rightsquigarrow$ Convolution str $(\mathrm{QCoh}(\mathcal{P}(X))_{\deg 1}, *)$.

(4) $\mathcal{P}(X)|_{\widehat{T^*X}^{\#}}$ can be splitted if X is lifted to $\mathbb{Z}/p^2$.

PD nbhd of 0 section on T^*X

$\rightsquigarrow$ Cartier transform:

$$D_X\text{-}\mathrm{Mod}_{\leq p-1} \simeq \mathrm{Higgs\ Mod}_{\leq p-1}.$$

(5) This equiv is of local nature.

<u>p-adic Simpson correspondence</u>

$C/\mathbb{Q}_p$ perf'd field. X/C sm rigid space.

Thm (Bhatt - Zhang)

$\exists$ nat'l (in X) multiplicative $\mathbb{G}_m$-gerbe $\mathcal{P}(X)$ on $(T^*X)(-1)$ of

called "Simpson gerbe". Tate twist

+ A canonical equiv of tensor cats

$$\mathrm{Vect}(X_{\text{proét}}) \cong \mathrm{Vect}^{\mathcal{H}}(X_{\text{ét}}, \mathcal{P}(X)) := \left\{ \begin{array}{l} \mathcal{E} \in \mathrm{QCoh}(\mathcal{P}(X))_{\deg 1} \\ \text{"loc free / } X\text{"} \end{array} \right\}.$$

Rmks (1) · C-locsys on $X_{\text{proét}} \hookrightarrow \mathrm{Vect}(X_{\text{proét}})$

$$\mathbb{L} \longmapsto \mathbb{L} \otimes_C \widehat{\mathcal{O}}_X$$

"generalized rep'n"

· $\mathcal{P}(X)$ nontrivial if $\dim X > 0$.

But if $C = \bar{C}$, choice of $\mathbb{X}$ = lift of X to $\mathbb{B}_2 := B_{dR}^+/\xi^2$

+ $\mathrm{Exp}: C \longrightarrow 1 + \mathfrak{m}_C$

one can recover split of $\mathcal{P}(X)$ simultaneously on all "spectral varieties", i.e.

$$Z := \bigcup_{(\mathcal{E}, \theta)} (\operatorname{supp} \theta \text{ on } \mathcal{E}) \subset T^*X(-1).$$

$\leadsto$ This recovers:

Thm (Faltings, ⋯, Heuer) Choices of $\mathbb{X}$ & Exp leads to a splitting of $\mathcal{P}(X)|_Z$, and hence an equiv of tensor cats

$$\mathrm{Vect}(X_{\text{proét}}) \cong \mathrm{Vect}^{\mathcal{H}}(X_{\text{ét}}, \mathcal{P}(X)) \underset{\mathbb{X}, \mathrm{Exp}}{\cong} \mathrm{Higgs}(X_{\text{ét}}).$$

(2) Upgrades to $\mathrm{Perf}(X_{\text{proet}}) \simeq \mathrm{Perf}^{\mathcal{H}}(X, \mathcal{P}(X))$.

(3) Compatible w/ char p story

& stacky approach (via HT stack).

If X/C has a sm formal model $\mathfrak{X}/\mathcal{O}_C$

$$\mathfrak{X}^{HT} \longrightarrow \mathfrak{X}$$

$BT^{\#}_{\mathfrak{X}}\{1\}$

PD-hull of 0-section Breuil-Kisin twist

Via Cartier duality, get $\mathbb{G}_m$-gerbe on $\widehat{T^*\mathfrak{X}\{-1\}}$

(say $\widehat{\mathcal{P}(\mathfrak{X})}$).

Then $\widehat{\mathcal{P}(X)}_\eta$ is $\mathbb{G}_m$-gerbe on $(\widehat{T^*\mathfrak{X}\{-1\}})_\eta \subset T^*X(-1)$.

- This canonically agrees w/ $\mathcal{P}(X)|_{(\widehat{T^*\mathfrak{X}\{-1\}})_\eta}$.
- On the other hand, can extend $\widehat{\mathcal{P}(\mathfrak{X})}$ to $\mathcal{P}(\mathfrak{X})$ on $T^*\mathfrak{X}\{-1\}$.

  & $\mathcal{P}(\mathfrak{X})_k$ (k = res field of C)

  agrees w/ $\mathbb{G}_m$-gerbe of splittings of $D_{\mathfrak{X}_k}$.

  ("sheared prismatization",

  Bhatt - Kaledin - Mathew - Vologodsky - Zhang)

(4) "Analytic approach"

(Anschütz - Le Bras - Rodriguez Camargo - Scholze)

$$X^{HT} \longrightarrow X$$

$BT^{\dagger}_X(1)$ - gerbe

analytic HT stack.

Construction & Examples

(<u>Heuer</u>) $\nu : X_{\text{proét}} \longrightarrow X_{\text{ét}}$

$$R^i\nu_*\hat{\mathcal{O}}^\times = \begin{cases} \mathcal{O}^\times, & i=0 \\ \Omega^i(-i), & i\geq 1. \end{cases}$$

$$\rightsquigarrow \quad \mathcal{O}^\times \longrightarrow \tau_{\leq 1} R\nu_*\hat{\mathcal{O}}^\times \longrightarrow \underbrace{\Omega^1(-1)[-1] \xrightarrow{\partial} \mathcal{O}^\times[1]}$$

undo the shift, this is $T^*X(-1) \longrightarrow B^2\mathbb{G}_m$

$\rightsquigarrow$ $\mathcal{P}(X)$ multiplicative

Simpson functor: $X_{\text{proét}}$ basis affinoid perfectoid over S.

$$\begin{tikzcd}
 & & \mathcal{P}(X) \arrow[d] \\
(T^*X(-1))_S \arrow[urr, dashed, "\exists f_S"] \arrow[rr] \arrow[d, "\pi_S"'] \arrow[drr, phantom, "\ulcorner", very near start] & & T^*X(-1) \arrow[d] \\
S \arrow[rr] & & X
\end{tikzcd}$$

Canonical splitting $\frown f_S$

$$\mathcal{S}_X : \mathrm{Perf}^{\mathcal{H}}(X, \mathcal{P}(X)) \longrightarrow \mathrm{Perf}(X_{\text{proét}}) = \varprojlim_{S\to X} \mathrm{Perf}(S)$$

$$\mathcal{E} \longmapsto (R\pi_{S*}{}^{\text{ll}}f_S^*\mathcal{E})_S$$

<u>Example</u> (Drinfeld) $X = \mathbb{G}_m$, $T^*X(-1) \cong \mathbb{G}_m \times \mathbb{G}_a(-1)$

$$\alpha = [0 \to \mathbb{Z}_p(1) \to \widetilde{\mathbb{G}}_m \to \mathbb{G}_m \to 1]$$

$$\beta = [1 \to \mu_{p^\infty} \to 1 + \mathfrak{m}_C\mathcal{O}_X^+ \xrightarrow{\log} \mathcal{O}_X \to 0]$$

Then $\mathrm{Ext}^1(\mathbb{G}_m, \mathbb{Z}_p(1)) \otimes \mathrm{Ext}^1(\mathbb{G}_a(-1), \mu_{p^\infty}(-1)) \xrightarrow{\cup} \mathrm{Ext}^2(\mathbb{G}_m \times \mathbb{G}_a(-1), \mu_{p^\infty})$

$$\alpha \otimes \beta(-1) \longmapsto [\mathcal{P}(X)].$$

<u>Example</u> (Heuer) Rank 1 case, X sm proper, $C = \bar{C}$.

Fix $\mathbb{X}/B_2$ & Exp. $\nu : X_{proet} \to X_{et}$.

Then by Leray spectral seq,

$$\begin{array}{l} 0 \to \mathrm{Pic}(X_{et}) \to \mathrm{Pic}(X_{proet}) = H^1(X_{proet}, \hat{\mathcal{O}}^{\times}) \xrightarrow{\text{HT log}} H^0(X, \Omega^1_X(-1)) \to 0. \\ \qquad\quad \mathrm{Exp}\uparrow \qquad\qquad \mathrm{Exp}\uparrow \\ \& \quad 0 \to H^1(X, \mathcal{O}_X) \to H^1(X_{proet}, \hat{\mathcal{O}}) \xrightarrow{\text{HT}} H^0(X, \Omega^1_X(-1)) \to 0. \quad \curvearrowleft s_{\mathbb{X}} \end{array}$$

<u>Concluding rmk</u> $\mathrm{Pic}(X_{proet}) \cong \mathrm{Pic}(X_{et}) \oplus H^0(X, \Omega^1_X)$

$$\mathrm{Hom}_{cts}(\pi_1(X), C^{\times}) \cong \begin{pmatrix} \text{line bdls w/ vanishing} \\ \text{rat'l Chern classes} \end{pmatrix} \oplus H^0(X, \Omega^1_X)$$

very similar to Corlette-Simpson corresp $/\mathbb{C}$.