

Canonical integral models of Shimura varieties
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Shimura datum (G, χ)

- $G = \text{red grp} / \mathbb{Q}$, connected
- $\chi = \text{conj class of } h: \text{Res}_{\mathbb{A}_F/\mathbb{Q}} \mathbb{G}_m \rightarrow G_{\mathbb{R}}$

Fact $X = \text{union of herm symm domains.}$

$K \subset G(\mathbb{A}_F)$ cpt open subgrp (+ neat).

$$\hookrightarrow \text{Sh}_K = \text{Sh}_X(G, \chi) = G(\mathbb{Q}) \backslash X^{\times} G(\mathbb{A}_F) / K$$

Note If $\exists g \in G$ s.t. $g^{-1} K' g \subset K$, $K, K' \subset G$ ws,
then get $\text{Sh}_K \rightarrow \text{Sh}_{K'}$.

$\cup \{ \text{Sh}_K \}_K$ tower.

Hodge cochar $g_c: \mathbb{G}_{m, \mathbb{C}} \rightarrow G_{\mathbb{C}}$ s.t. $g_c(z) = h_c(z, 1)$

\cup conj class of $\{g_c\}$ def'd over $E = \text{reflex field.}$

Then $\text{Sh}_K(G, \chi)$ carries

- can str of quasi-proj alg var / \mathbb{C}
(Baily-Borel, Borel).
- can str / $E = \text{number field}$
(Shimura, Deligne, Milne, Bruin).

$\underline{\mathbb{Q}}$ \ni canonical integral model? / $\mathbb{Q}_E = \text{ring of ints in } E$.

* Moduli of G -motives?

Siegel \subset PEL \subset Hodge \subset abelian \subset general type
 Can take Zariski closure of Some Siegel moduli sch / \mathbb{Z} $\xrightarrow{\quad \text{E}_4, E_7 \quad}$ (some twisted types of orthogonal grps)

v place of E , $v \nmid p$, $E_v \supset \mathbb{Q}_v = 0$
 \downarrow
 $k(v) = k = \text{res field.}$

$$K = K_p K^p, \quad K_p \subset G(\mathbb{Q}_p), \quad K^p \subset G(A_f^p).$$

"good places": $v \nmid p$, G extends to red grp / \mathbb{Z}_p
 $K_p = G(\mathbb{Z}_p)$ (hyperspecial)

Expect Canonical sm model / \mathbb{Q}_v .

Thm (Kisin, Vasiu, etc.) Ok if (G, x) of abelian type.

parahoric places

K_p = parahoric subgroup of $G(\mathbb{Q}_p)$

Aside $G(\mathbb{Q}_p) \subset \mathfrak{B}(G) = B-T$ building

$\xrightarrow[x]{\text{B-T}}$ Canonical sm grp sch $\mathfrak{g}_x / \mathbb{Z}_p$

w/ $\mathfrak{g}_x(\mathbb{Z}_p) = \text{stabilizer } G(\mathbb{Q}_p)_x$.

$\mathfrak{g}_x^\circ \subset \mathfrak{g}_x$ neutral comp.

Then

K_p parahoric $\overset{\text{Def}}{\iff} \exists x \in \mathfrak{B}(G) \text{ s.t. } K_p = \mathfrak{g}_x^\circ(\mathbb{Z}_p)$.

Denote $\mathcal{G} = \mathcal{G}_\infty^\circ$. So $K_p = \mathcal{G}(I_p)$.

In (1) G extends to red grp / I_p

$$K_p = \{g \in G(I_p) \mid g \text{ mod } p \in P(\mathbb{F}_p)\}$$

↑
parabolic of $G(\mathbb{F}_p)$.

(2) $G = \text{Res}_{F/\mathbb{Q}_p} G'$, G' red grp / \mathbb{Q}_F

$$K_p = G'(\mathbb{Q}_F).$$

Models $\{\mathcal{G}_K\}_K$, $K = \underbrace{K_p}_{\text{fixed}} \underbrace{K^p}_{\text{varying}}$ / $\mathcal{O} = \mathcal{O}_K$.

Properties (1) \mathcal{G}_K flat / \mathcal{O} , $\mathcal{G}_K \otimes_{\mathcal{O}} E_v = \text{Sh}_K \otimes_E E_v$

(2) (Hecke) $g^{-1} K^p g \subset K^p$

↪ $\mathcal{G}_{K_p K^p} \rightarrow \mathcal{G}_{K_p K^p}$ finite etale

(3) (Relative properness) Let $\mathcal{G}_{K_p} := \varprojlim_{K^p} \mathcal{G}_{K_p K^p}$.

Then \mathcal{G}_{K_p} DVR flat / \mathcal{O} ,

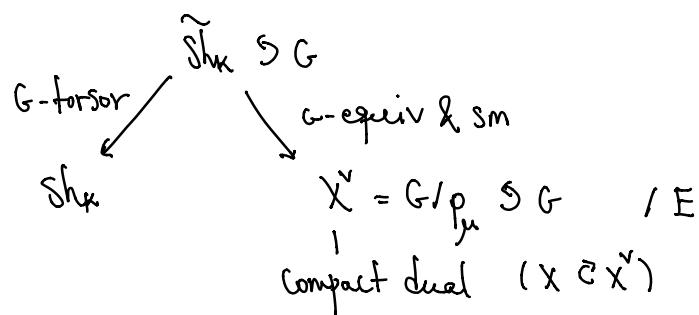
$$\mathcal{G}_{K_p}(\mathbb{Q}) = \text{Sh}_{K_p}(A[\frac{1}{p}]).$$

- Closed pts: $\mathcal{G}_{K_p}(k) \xrightarrow[\text{Frob}_v]{} \mathcal{G}_{K_p}$ Hecke (Langlands - Rapoport conj)

- (formal) local structure:

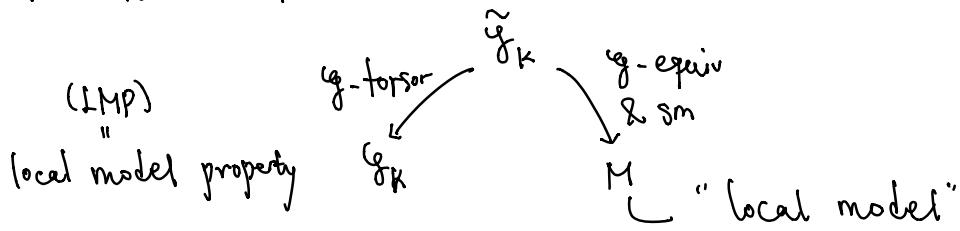
Assume max \mathbb{R} -split torus of $Z(G)$ is \mathbb{Q} -isotropic

↪ principal autom bundle:



Think of : $\tilde{\text{Sh}}_K$ gives variation of Hodge fil'n
 values in X^\vee gives position of Hodge fil'n.

This should extend to



where M = a canonical model / G of X^\vee

N.B. $M = M^{\text{loc}} \underset{\text{g}}{\otimes} \text{g}$ depends only on K_p & $\{\mu\}$
 local stuff.

satisfying (1) M flat / G

$$(2) M \otimes_G E_v = X^\vee \underset{E}{\otimes} E_v \quad \text{g-equiv}$$

(3) $M \otimes_G k$ reduced (difficult!)
 special fibre

$$M(k) = "g\text{-adm set"} \subset \frac{G(W(k)[\frac{1}{p}])}{g(W(k))}$$

$W(k)$ = Witt vectors

(4) functorial, invariant under central ext'n.

Scholze constructs v-sheaf M^*

try to characterize M (if this local model exists)

Constr'n of M (Pappas-Zhu, He-Pappas-Rapoport, Levin; Hodge type)

(exclude some grp's for $p=2, 3$)

Thm 1 (Kisin-Pappas)

Assume (i) (G, x) of Hodge type

(ii) G splits over a tame ext'n $/ \mathbb{Q}_p$,

$$p > 2, p \nmid |\Gamma_{\bar{\pi}}(G_{\text{der}})|, \quad \mathfrak{g}_x = \mathfrak{g}_x^\circ$$

Then $\exists \mathfrak{g}_k$ with (i) (ii) + (LMP)

* Canonicity & characterization of M :

• (\mathfrak{g}, g) - displays:

Zink displays, $p > 2$,

R = complete weth local ring w/ $R/\mathfrak{m} = k$.

$$W(R) = \{(r_0, r_1, \dots) \mid r_i \in R\}$$

$$\begin{matrix} \varphi, V \\ \text{Frob} \end{matrix} \quad \begin{matrix} \text{s.t.} \\ \text{def} \end{matrix} \quad V(r_0, r_1, \dots) = (0, r_0, r_1, \dots) \quad \& \quad \varphi \circ V = p.$$

$$\text{Turns out: } W(R) = \bigcup_{\mathfrak{f}} W(\mathfrak{f}) \oplus W(m) \quad (\text{Zink's constr'n})$$

$$\hat{W}(R) = W(k) \oplus \hat{W}(m)$$

$$\begin{matrix} \varphi, V \\ \text{def} \end{matrix} \quad \begin{matrix} \text{ii} \\ \{r_i \rightarrow 0 \text{ in } p\text{-adic top}\} \end{matrix}$$

This uses the fact $p > 2$.

$$\mathcal{I}_R := V\widehat{W}(R) \subset \widehat{W}(R), \quad \widehat{W}(R)/\mathcal{I}_R \simeq R.$$

Def Dieudonné display over R is (M, M_1, F) (with $n, d \in \mathbb{N}$)

- $M \simeq \widehat{W}(R)^n \supset M_1 \supset \mathcal{I}_R M$
- $F: M_1 \rightarrow M$

s.t. (1) $M/M_1 \simeq R^d$

(2) $F(a m_1) = \varphi(a) F(m_1), \quad \forall a \in \widehat{W}(R), m_1 \in M_1$

(3) $F(V(a)m) = a F(V(a)m), \quad \forall a \in \widehat{W}(R), m \in M$

(4) $\langle \text{Im } F \rangle = M_1$.

Thm (Zink) \exists equiv of cts

$$\left\{ \begin{array}{l} \text{p-div grps / R} \\ \text{of ht n & $\dim d$} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Dieudonné displays of} \\ \text{type (n, d)} \end{array} \right\}$$

Def Let R be as above + normal flat / \mathcal{I}_p .

A (\mathfrak{g}, μ) -display is triple $\mathcal{D} = (P, \varphi, F)$

- P \mathfrak{g} -torsor on $\widehat{W}(R)$
- $\varphi: P \otimes_{\widehat{W}(R)} R \rightarrow M$
 - ↳ depends only on (\mathfrak{g}, μ)
- $\hookrightarrow \mathfrak{g}$ -torsor Q
- $F: Q \xrightarrow{\sim} P$.

Thm 2 Same assns as in Thm 1. Then

(a) $\forall x \in \mathfrak{g}_k(k)$, $\exists \mathcal{D}_x$ (\mathfrak{g}, μ) -display over $\widehat{\mathcal{O}}_{\mathfrak{g}_k, x}$ which is

- "versal"
- "associated" to the pro-étale K_p -cover

$$\varprojlim_{K_p} \mathrm{Sh}_{K_p K^p} \rightarrow \mathrm{Sh}_{K_p K^p}.$$

(b) If $\mathfrak{Y}_K, \mathfrak{Y}'_K$ models of Sh_K satisfying (1)(2)(3),
and support (\mathfrak{g}, μ) -data as in (a), then

$$\mathfrak{Y}_K \simeq \mathfrak{Y}'_K.$$

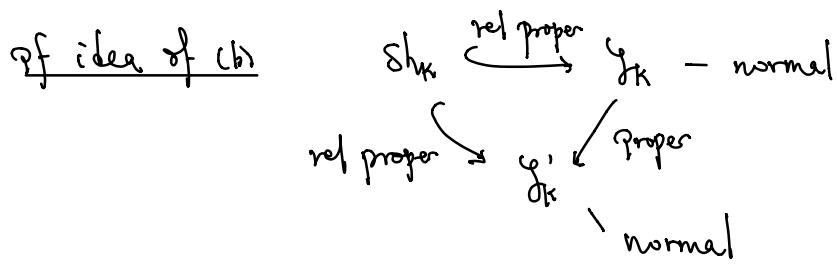
Cor \mathfrak{Y}_K of Thm 1 (by Kisin-Pappas) indep of choices of (\mathfrak{g}, μ)
 |
 namely canonical.

Prob (Meaning of Thm 2(a))

"associated" : $\mathfrak{g} \hookrightarrow G_{\mathbb{Q}_p}$
 $\mu \mapsto \mu_n = \text{minuscule char of } G_{\mathbb{Q}_p}$
 can be done for Hodge type.

Get $\{(\mathfrak{g}, \mu) \text{-display}\} \rightarrow \{(G_{\mathbb{Q}_p}, \mu_n) \text{-display}\}$
 ||
 $\{$ Dieudonné display of type (n, d) $\}$
 is Zink's thm
 $\{$ p-div groups / R of ht n & dim d $\}$

- ↳ Tate modns / $R[\frac{1}{p}]$
- ↳ \mathbb{Z}_p^n étale local systems
- ↳ $G_{\mathbb{Q}_p}(\mathbb{Z}_p)$ -torsor.



Same generic fibre Sh_K

- \Rightarrow same (g, μ) -display on g_K & g'_K
- \Rightarrow same p-Liv grp.

Apply Furihi main thm + formal local isom \Rightarrow pf. \square