Spectral side of categorical Langlands

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$E/\mathbb{Q}_p$ fin, G/E red grp, <u>split</u>.

Fix a Whittaker datum (B, ψ) over L ($L/\mathbb{Q}_\ell$ algebraic).

$\ell \neq p$. $\sqrt{q} \in L$.

<u>CLLC (Fargues-Scholze)</u> $D_{lis}(\mathrm{Bun}_G, \bar{\mathbb{Q}}_\ell) \overset{\exists(!)}{\simeq} \mathrm{IndCoh}(\mathrm{Par}_G)$.

$\circlearrowleft$

T_V ($V \in \mathrm{Rep}\,\hat{G}$), T_V "=" $\underline{V} \otimes (-)$.

given by $i_{1!}\,c\mathrm{Ind}_U^G \psi \longmapsto \mathcal{O}_{Z^1/\hat{G}}$.

$(TU = B)$

* <u>Why do we care about CLLC?</u>

(1) D_{lis} & T_V know cohom of

local Sh vars & moduli of local shtukas.

"$i_b^* T_V i_{1!}\, c\mathrm{Ind}_K^G \rho$" $\in D(G_b(E))^\omega$.

Application of Fargues-Scholze + etc.

- vanishing thm (Koshikawa, Hamann-Lee).
- CLLC + spectral consideration

  $\rightsquigarrow$ new vanishing conjs (Hansen, Koshikawa).
- Some hidden str? beyond generic case.

  (related on Koshikawa-Shin).

(2) Fargues's Conj on eigenshvs. (More later).

Vaguely: $\exists$ relation w/ cohom of global Shimura var.

(automorphic).

$$\begin{tikzcd}
(\text{compact}) \subset Sh_{K^p} \arrow[r, "\pi_{HT}"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{F}\ell_{G_{\mathbb{Q}_p}} \arrow[d] \\
Igs_{K^p} \arrow[r, "\bar{\pi}_{HT}"] & \mathrm{Bun}_{G_{\mathbb{Q}_p}}
\end{tikzcd}$$

<u>Igusa stack</u>: Zhang: PEL case

Daniels - van Hoften - Kim - Zhang: Hodge case.

<u>Caraiani - Scholze</u> "$R\bar{\pi}_{HT*}\bar{\mathbb{Q}}_\ell$ is perverse"

<u>Fargues</u> Restriction of eigensheaves for discrete parameters should appear.

<u>Local-global compatibility</u> (at least)

Our insight: More is true on the spectral side.

(w/ Bertoloni-Meli).

<u>Main conj</u> (BM-K)

$\psi : W \times SL_2^D \times SL_2^A \to {}^LG \,/\bar{\mathbb{Q}}_\ell$ "generalized" A-param.

This contains all A-params for $\bar{\mathbb{Q}}_\ell \simeq \mathbb{C}$.

$\rightsquigarrow \exists\, \mathcal{F}_\psi \in D_{lis}(\mathrm{Bun}_G, \bar{\mathbb{Q}}_\ell)$

"sheared" eigensheaf, perverse.

$V \in \mathrm{Rep}(\hat{G})$, $\quad V \circ \psi = \bigoplus_i V_i$

wt i space w.r.t. $\mathbb{G}_m \subset SL_2^A$.

$T_V \mathcal{F}_\psi \cong \bigoplus_i V_i \otimes \mathcal{F}_\psi[-i]$ in $D(\mathrm{Bun}_G \times \mathrm{Div}^1, \bar{\mathbb{Q}}_\ell)$.

c.f. · Frenkel - Langlands, Ngo

· Ben-Zvi - Sakellaridis - Venkatesh.

Ex $\quad G$ semisimple, b basic.

$$\Rightarrow \quad i_b^* \mathcal{F}_\psi = \bigoplus_{\pi \in \Pi_\psi^{ABV}(G_b)} \pi^{\oplus \text{ mult} > 0}$$

- G_b pure inner form.
- Π_ψ^{ABV} = Adams - Barbasch - Vogan packet (c.f. Vogan, CFMMX).

Spectral side

Our project: (1) Construct eigensheaves

(2) Study str of $\mathrm{Coh}(\mathrm{Par}_G)$.

Related works: Zhu, Hansen-Mann.

Expected thing $\quad b$ basic, π_b supercusp rep of G_b.

Have $\quad i_{b!}\pi_b \longmapsto VB([V_\lambda / H_\lambda])$.

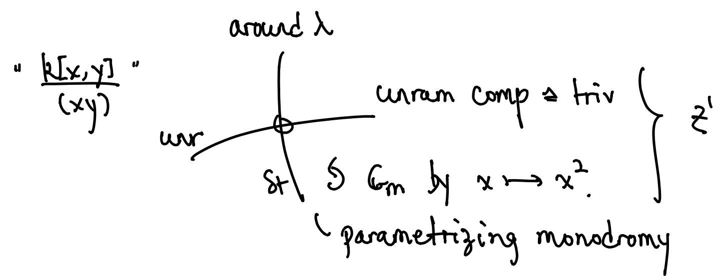
Here
- $\pi_b \rightsquigarrow \lambda$ ss L-param
  $\rightsquigarrow V_\lambda \circlearrowleft N$ monodromy
- $H_\lambda = \mathrm{Cent}(\lambda)$
- $[V_\lambda / H_\lambda] \subseteq \mathrm{Par}_G$.

Note $\quad$ If π_b unip cusp, $\pi_b \triangleq$ Lustig's cuspidal local system.

$$BS_{\phi_{disc}} \xhookrightarrow{\text{open}} [V_\lambda / H_\lambda].$$

Example $\quad G = PGL_2$, $\hat{G} = SL_2$, $\quad St \rightsquigarrow \lambda = $ ss L-param,

$\rightsquigarrow H_\lambda = \mathrm{Cent}(\lambda) = \mathbb{G}_m$.



Define $L_i :=$ $\mathcal{O}_{St}$ + equiv str twisted by $i \in \mathbb{Z}$.

(line bdl on St

$\rightsquigarrow$ (a) $0 \to L_2 \to \mathcal{O}_{Z^1} \to \mathcal{O}_{unr} \to 0$

$0 \to \mathcal{O}_{unr} \to \mathcal{O}_{Z^1} \to L_0 \to 0$

$$L_0 = \mathcal{O}_{St}$$

(b) $L_2 \hookrightarrow L_0 \hookrightarrow L_{-2} \hookrightarrow \cdots$

$L_0 \to L_2[2] \to L_4[4] \to \cdots$ (c.f. Hellmann)

Via CLLC: $i_{1!}\, St \triangleq L_0$

$i_{1!}\, \mathbb{1} \triangleq L_2[1]$

$i_{b_{1/2}}!\, \mathbb{1} \triangleq L_1$

$V \in \operatorname{Rep} SL_2 \rightsquigarrow \underbrace{L_i \oplus L_{i-2} \oplus \cdots \oplus L_{-i}}_{\text{wt decomp}}$ on St.

$\operatorname{Rep} SL_2 = \operatorname{Rep} \widehat{G}$

(Autom side) (Spectral side)

<u>Eigensheaf</u>

$$L^- := \begin{matrix} (L_0 \hookrightarrow L_{-2} \hookrightarrow L_{-4} \hookrightarrow \cdots) \\ \oplus \\ (L_1 \hookrightarrow L_{-1} \hookrightarrow L_{-3} \hookrightarrow \cdots) \end{matrix} \in \operatorname{IndCoh}$$

$$L^+ := \begin{matrix} (L_0[-1] \to L_2[1] \to L_4[3] \to \cdots) \\ \oplus \\ (L_1 \to L_3[2] \to L_5[4] \to \cdots) \end{matrix} \in \operatorname{IndCoh}$$

Fact: L^- is the eigensheaf for ϕ_{St}!

L^+ is the eigensheaf for $\phi_{\mathbb{1}}$!

* <u>How to construct an eigensheaf?</u>

Along $pt \to B\mu_2 \hookrightarrow St = [V_\lambda / H_\lambda]$,

$L^- =$ pushforward of $\mathcal{O}_{pt}$ reg rep of μ_2.

Easy to generalize L^-:

this works for "generic" params, N maximal.

But L^+ is hard to generalize.

<u>Rmk</u> Inspired by BZ-C-H-N, we use Koszul duality to construct eigensheaves:

$$\lambda \rightsquigarrow [V_\lambda / H_\lambda] \subseteq \mathrm{Par}_G.$$

$$(\mathrm{Par}_G)^\wedge_{[V_\lambda/H_\lambda]} \simeq \widehat{T}_*(V_\lambda/H_\lambda)[-1]$$

$$\Big\updownarrow \text{Koszul}$$

$$T^*(V_\lambda/H_\lambda).$$