Final exam (due on January 22)

Let $\ell \geq 5$ be a prime number. The goal of this final exam is to prove the following level raising theorem, which is quite similar to Ribet's level lowering theorem: under some technical assumptions that we will specify in the text, suppose that $f = \sum_{n\geq 1} a_n q^n$ is a normalized cuspidal eigenform of level $\Gamma_1(N)$ and p is a prime relatively prime to ℓN such that $p \not\equiv 1 \bmod \ell$ and $\bar\rho_f(\mathrm{Frob}_p) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$, then there exists a normalized cuspidal eigenform of level $\Gamma_1(N) \cap \Gamma_0(p)$ that is new at p, such that $f \equiv g \bmod \ell$.

Each problem is worth 15 points. No more than 100 points will be given.

Problem 1. Let $p \neq \ell$ be another prime number.

(1) Show that $H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \mu_\ell) \cong \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^\ell$.
(2) Under this isomorphism, what does the canonical exact sequence $0 \to H^1(\mathrm{Gal}_{\mathbb{F}_p}, \mu_\ell) \to H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \mu_\ell) \to H^1(I_{\mathbb{Q}_p}, \mu_\ell)^{\mathrm{Frob}_p} \to 0$ look like?
(3) What is the dimension of $H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \mu_\ell)$? Using Tate duality to determine all $\dim H^i(\mathrm{Gal}_{\mathbb{Q}_p}, \mu_\ell)$. Verify that this agrees with the Euler-characteristic formula.

Problem 2. Let $p \not\equiv 1 \pmod{\ell}$ be another prime number. Let $\chi_{\mathrm{cycl}} : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathbb{Z}_\ell^\times$ denote the cyclotomic character.

(1) Show that $H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \chi_{\mathrm{cycl}})$ is free of rank one over $\mathbb{Z}_\ell$. (This is quite similar to Problem 1(1).)
(2) A generator of $H^1(\mathrm{Gal}_{\mathbb{Q}_p}, \chi_{\mathrm{cycl}})$ defines a extension

$$0 \to \chi_{\mathrm{cycl}} \to E \to \mathbb{Z}_\ell \to 0.$$

Describe this representation E as best as possible.
(3) What is the Weil–Deligne representation attached to $E \otimes \mathbb{Q}_\ell$?

Problem 3. Let $p \not\equiv \pm 1 \pmod{\ell}$ be another prime number. Consider the residual Galois representation $\bar\rho : \mathrm{Gal}_{\mathbb{F}_p} \to \mathrm{GL}_2(\mathbb{F}_\ell)$ given by $\mathbf{1} \oplus \bar\chi_{\mathrm{cycl}}^{-1}$ (or equivalently the unramified representation sending the Frobenius element to the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$).

Compute explicitly the framed deformation space $R_{\bar\rho}^{\square,\chi}$ (with $\chi = \chi_{\mathrm{cycl}}$) as a representation of $\mathrm{Gal}_{\mathbb{Q}_p}$, with fixed determinant being cyclotomic character. In fact show that $\mathrm{Spec}\, R_{\bar\rho}^{\square,\chi}$ is the union of two *smooth* irreducible components $\mathrm{Spec}\, R_{\bar\rho}^{\square,\chi,\mathrm{ur}}$ and $\mathrm{Spec}\, R_{\bar\rho}^{\square,\chi,\mathrm{St}}$, characterized by the following properties:

- $\mathrm{Spec}\, R_{\bar\rho}^{\square,\chi,\mathrm{ur}}$ is the subspace where the universal deformation is unramified, and
- $\mathrm{Spec}\, R_{\bar\rho}^{\square,\chi,\mathrm{St}}$ is the subspace where the trace of Frobenius element is $p+1$ (or equivalently closure of the subspace where $\rho(\tau)$ is nontrivial).

Problem 4. Continued with the previous problem. Taking advantage of the computations in Problem 1, find the dimension of $Z^1(\mathrm{Gal}_{\mathbb{Q}_p}, \mathrm{Ad}^\circ \bar\rho)$ and $H^2(\mathrm{Gal}_{\mathbb{Q}_p}, \mathrm{Ad}^\circ \bar\rho)$. Compare them with the tangent space of $\mathrm{Spec}\, R_{\bar\rho}^{\square,\chi}$ and the set of relations.

Now let us accept one fact about modular forms, called Ihara's Lemma. (This is not quite the original Ihara's lemma, which is stated in terms of étale cohomology.) Let k denote the weight we consider; we assume k is even and $2 \leq k \leq \ell - 1$. Let N denote an integer that is not divisible by ℓ; let p be another prime that does not divide ℓN. Let S denote the set of all prime factors of N together with $\{p, \ell, \infty\}$.

Let $\bar\rho : \mathrm{Gal}_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}_\ell)$ denote an absolutely irreducible representation that is unramified at p. Let $\mathfrak{m}_{\bar\rho}$ be the maximal ideal of $\mathbb{T} := \mathbb{Z}_\ell[T_q; q \nmid pN]$ generated by ℓ and $T_q - \mathrm{tr}(\bar\rho(\mathrm{Frob}_q))^\sim$ (where $\mathrm{tr}(\bar\rho(\mathrm{Frob}_q))^\sim$ is any lift of $\mathrm{tr}(\bar\rho(\mathrm{Frob}_q)) \in \mathbb{F}_\ell$ to $\mathbb{Z}_\ell$. The ring $\mathbb{T}$ acts on the space of modular forms $S_k(\Gamma_1(N); \mathbb{Z}_\ell)$ and $S_k(\Gamma_1(N) \cap \Gamma_0(p); \mathbb{Z}_\ell)$. Let $S_k(\Gamma_1(N); \mathbb{Z}_\ell)_{\mathfrak{m}_{\bar\rho}}$ and $S_k(\Gamma_1(N) \cap \Gamma_0(p); \mathbb{Z}_\ell)_{\mathfrak{m}_{\bar\rho}}$ denote the localization of the corresponding $\mathbb{T}$-module at this maximal ideal $\mathfrak{m}_{\bar\rho}$. Let $\pi_1, \pi_2 : X(\Gamma_1(N) \cap \Gamma_0(p)) \to X_1(N)$ denote the two projections (here $X(\Gamma_1(N) \cap \Gamma_0(p))$ and $X_1(N)$ denote the corresponding modular curves over $\mathbb{Z}[\frac{1}{pN}]$ and $\mathbb{Z}[\frac{1}{N}]$, respectively.) Then take the following as a fact: the natural pullback

$$(\pi_1^*, \pi_2^*) : S_k(\Gamma_1(N); \mathbb{F}_\ell)^{\oplus 2}_{\mathfrak{m}_{\bar\rho}} \to S_k(\Gamma_1(N) \cap \Gamma_0(p); \mathbb{F}_\ell)_{\mathfrak{m}_{\bar\rho}} \tag{0.0.1}$$

is injective.[1] Somewhat in a dual situation, the following “trace map”

$$\pi_{1,*} \oplus \pi_{2,*} : S_k(\Gamma_1(N) \cap \Gamma_0(p); \mathbb{F}_\ell)_{\mathfrak{m}_{\bar\rho}} \to S_k(\Gamma_1(N); \mathbb{F}_\ell)^{\oplus 2}_{\mathfrak{m}_{\bar\rho}} \tag{0.0.2}$$

is surjective. (Recall that, in Exercise 6, we have proved this injectivity when the coefficients are $\mathbb{Q}_\ell$.)

Problem 5. (1) Let $\pi : X \to Y$ be a finite flat morphism of noetherian schemes and $\mathcal{L}$ an invertible sheaf on Y. Show that there is a natural trace map $\mathrm{Tr} : H^0(X, \pi^*\mathcal{L}) \to H^0(Y, \mathcal{L})$. (Hint: as $\pi_*\mathcal{O}_X$ is a finite locally free sheaf over Y, there is a well-defined trace map $\pi_*\mathcal{O}_X \to \mathcal{O}_Y$.)

(2) Show that the composition of the two maps (0.0.1) and (0.0.2) is given by

$$(\pi_{1,*} \oplus \pi_{2,*}) \circ (\pi_1^*, \pi_2^*) = \begin{pmatrix} p+1 & T_p \\ T_p \circ (\langle p \rangle^*)^{-1} & p+1 \end{pmatrix} : S_2(\Gamma_1(N); \mathbb{F}_\ell)^{\oplus 2}_{\mathfrak{m}_{\bar\rho}} \to S_2(\Gamma_1(N); \mathbb{F}_\ell)^{\oplus 2}_{\mathfrak{m}_{\bar\rho}}.$$

(after correctly chosen basis).

Problem 6. Assume that $\det \bar\rho = \bar\chi_{\mathrm{cycl}}$. Assume that the prime p satisfies

- $p \not\equiv \pm 1 \pmod{\ell}$, and
- $\bar\rho(\mathrm{Frob}_p) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$.

Let $R^\chi_{\bar\rho}$ denote the universal deformation ring for $\bar\rho$, as a representation of $G_{\mathbb{Q},S}$ with fixed determinant $\chi = \chi_{\mathrm{cycl}}^{-1}$, and let $\rho^{\mathrm{univ}} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(R^\chi_{\bar\rho})$ denote a universal representation (unique up to conjugation by $(1 + \mathrm{M}_2(\mathfrak{m}_{R^\chi_{\bar\rho}}))^\times$). Show that $\rho^{\mathrm{univ}}(\mathrm{Frob}_p)$ has precisely two eigenvectors $v_+ = \binom{1}{X}$ and $v_- = \binom{Y}{1}$ with eigenvalues $\alpha, \beta \in R^\chi_{\bar\rho}$ such that $\alpha \equiv 1, \beta \equiv p \bmod \mathfrak{m}_{R^\chi_{\bar\rho}}$. Let τ_p denote a generator for the tame inertia subgroup at p. Show that $\tau_p(v_-) = 0$ and $\tau_p(v_+) = Vv_-$ for some $V \in \mathfrak{m}_{R^\chi_{\bar\rho}}$. (The argument is quite similar to Problem 3, except that we do not need a framing here.)

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[1] Here is a proof under the “fake assumption” that the modular curve does not have cusps (which is the case in some analogous situation of Shimura curves). If (f_1, f_2) belongs to the kernel, then $f_1(E) \otimes \omega_E^{\otimes k} = f(E/C) \otimes \omega_{E/C}^{\otimes k}$ for any subgroup C of E of order p. In particular, if f is zero at one elliptic curve E over $\mathbb{F}_p$, then it is zero at all other elliptic curves that are p-isogenous to E. If one such E is ordinary, one can show that the p-isogeny class has infinite many E which would imply $f_1 \equiv 0$; if one such E is supersinglar, use the fact that all supersingular elliptic curves over $\mathbb{F}_p$ are p-isogenous to deduce that f_1 must vanish at all supersingular points, so it is divisible by the Hasse invariant, which has weight $k-1$. This argument has the flaw that it cannot quite treat the case when f_1 vanishes only at cusps.

Let $R^{\square_S,\chi}_{\bar\rho,S}$ denote the deformation ring with frames at places in S. Show that the following diagram are commutative Cartesian diagrams:

$$\begin{tikzcd}
\operatorname{Spec} R^{\chi}_{\bar\rho}/(V) \arrow[d, hook] & \operatorname{Spec} R^{\square_S,\chi}_{\bar\rho}/(V) \arrow[l] \arrow[r] \arrow[d, hook] & \operatorname{Spec} R^{\square,\chi,\mathrm{ur}}_{\bar\rho_p} \arrow[d, hook] \\
\operatorname{Spec} R^{\chi}_{\bar\rho} & \operatorname{Spec} R^{\square_S,\chi}_{\bar\rho} \arrow[l] \arrow[r] & \operatorname{Spec} R^{\square,\chi}_{\bar\rho_p}
\end{tikzcd}$$

Here the leftwards arrows are adding frames at S, and the rightwards arrows corresponds to restricting the universal deformation of $\bar\rho$ as a $G_{\mathbb{Q},S}$-representation to a $G_{\mathbb{Q}_p}$-representation.

Now we introduce more local conditions:

- Assume that $\det\bar\rho = \bar\chi_{\mathrm{cycl}}^{-1}$ (in particular $\det\bar\rho(c) = -1$).
- Assume that N is a square-free integer, relatively prime to ℓ;
- Assume that for each prime q dividing N, $\bar\rho(I_{\mathbb{Q}_q})$ is conjugate to $\left(\begin{smallmatrix} 1 & \mathbb{F}_\ell \\ 0 & 1\end{smallmatrix}\right)$;
- For each prime q dividing N, we consider the local condition $\mathcal{D}_q$: parametrizing deformations $R^{\square,\mathcal{D}_q}_{\bar\rho_q}$ where the representation of ρ^{univ} has determinant $\chi_{\mathrm{cycl}}^{-1}$, factors through the quotient by the wild inertia subgroup, and a generator of the tame inertia subgroup acts via a nilpotent matrix;
- At ℓ, we consider the deformation ring $R^{\square,\mathcal{D}_\ell}_{\bar\rho_\ell}$ parametrizing crystalline representation of weight $(0,1)$, with determinant $\chi_{\mathrm{cycl}}^{-1}$.
- At the prime p, we assume that $\bar\rho(\mathrm{Frob}_p) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p\end{smallmatrix}\right)$, and we will consider the empty local condition, namely $R^{\mathcal{D}_p}_{\bar\rho_p} = R_{\bar\rho_p}$. But we will later consider its unramified variant $R^{\square,\mathrm{ur}}_{\bar\rho_p}$ and the Steinberg variant $R^{\square,\mathrm{St}}_{\bar\rho_p}$, in which case, we will note explicitly.

Recall that S denotes the set of all prime factors of $N\ell p$ together with $\{\infty\}$. Accept the fact that each $R^{\square,\mathcal{D}_v}_{\bar\rho_v}$ for $v \in S\backslash\{p\}$ is a power series ring, of dimension 3 if $v \neq \ell\infty$, of dimension 4 if $v = \ell$, and of dimension 2 if $v = \infty$.

These local conditions defines a deformation ring $R^{\mathcal{D}}_{\bar\rho}$ for deformations of $\bar\rho$ satisfying these local conditions and with determinant $\chi_{\mathrm{cycl}}^{-1}$. Consider the conditioned deformation ring $R^{\square_S,\mathcal{D}}_{\bar\rho}$ that fits in the following Cartesian square

$$\begin{tikzcd}
\operatorname{Spf} R^{\square_S,\mathcal{D}}_{\bar\rho} \arrow[r, hook] \arrow[d] & \operatorname{Spf} R^{\square_S,\chi}_{\bar\rho} \arrow[d] \\
\prod_{v\in S} \operatorname{Spf} R^{\square,\mathcal{D}_v}_{\bar\rho_v} \arrow[r, hook] & \prod_{v\in S} \operatorname{Spf} R^{\square,\chi}_{\bar\rho_v}.
\end{tikzcd}$$

Set $R^{\mathcal{D}}_{\mathrm{loc}} := \widehat{\bigotimes}_{v\in S} R^{\square,\mathcal{D}_v}_{\bar\rho_v}$. There exist r variables such that $R^{\mathcal{D}}_{\mathrm{loc}}[\![y_1,\dots,y_r]\!] \twoheadrightarrow R^{\square_S,\mathcal{D}}_{\bar\rho}$.

As we explained in class, $S_2(\Gamma_1(N);\mathbb{Z}_\ell)^\vee_{\mathfrak{m}_{\bar\rho}}$ and $S_2(\Gamma_1(N)\cap\Gamma_0(p);\mathbb{Z}_\ell)^\vee_{\mathfrak{m}_{\bar\rho}}$ are modules over $R^{\mathcal{D}}_{\bar\rho}/(V)$ and $R^{\mathcal{D}}_{\bar\rho}$ (where $(-)^\vee$ indicates taking dual $\mathrm{Hom}(-,\mathbb{Z}_\ell)$). Recall that $R^{\square_S,\mathcal{D}}_{\bar\rho} \simeq R^{\mathcal{D}}_{\bar\rho}[\![w_1,\dots,w_{3\#S}]\!]$ (we fix such an isomorphism). We define the patched version

$$\begin{aligned}
S_2(\Gamma_1(N);\mathbb{Z}_\ell)^{\vee,\square_S}_{\mathfrak{m}_{\bar\rho}} &\cong S_2(\Gamma_1(N);\mathbb{Z}_\ell)^\vee_{\mathfrak{m}_{\bar\rho}} \otimes_{R^{\mathcal{D}}_{\bar\rho}} R^{\square_S,\mathcal{D}}_{\bar\rho}; \\
S_2(\Gamma_1(N)\cap\Gamma_0(p);\mathbb{Z}_\ell)^{\vee,\square_S}_{\mathfrak{m}_{\bar\rho}} &\cong S_2(\Gamma_1(N)\cap\Gamma_0(p);\mathbb{Z}_\ell)^\vee_{\mathfrak{m}_{\bar\rho}} \otimes_{R^{\mathcal{D}}_{\bar\rho}} R^{\square_S,\mathcal{D}}_{\bar\rho}.
\end{aligned}$$

The natural maps (π_1^*, π_2^*) and $\pi_{1,*} \oplus \pi_{2,*}$ give rise to

$$\begin{aligned} \text{injective map} \qquad & (\pi_{1,*} \oplus \pi_{2,*})^\vee : \big(S_2(\Gamma_1(N); \mathbb{Z}_\ell)^{\vee, \square_S}_{\mathfrak{m}_{\bar\rho}}\big)^{\oplus 2} \longrightarrow S_2(\Gamma_1(N) \cap \Gamma_0(p); \mathbb{Z}_\ell)^{\vee, \square_S}_{\mathfrak{m}_{\bar\rho}} \\ \text{surjective map} \qquad & (\pi_1^*, \pi_2^*)^\vee : S_2(\Gamma_1(N) \cap \Gamma_0(p); \mathbb{Z}_\ell)^{\vee, \square_S}_{\mathfrak{m}_{\bar\rho}} \longrightarrow \big(S_2(\Gamma_1(N); \mathbb{Z}_\ell)^{\vee, \square_S}_{\mathfrak{m}_{\bar\rho}}\big)^{\oplus 2} \end{aligned}$$

They remain injective/surjective after quotient by an ideal of $\mathbb{Z}_\ell[\![w_1, \dots, w_{3\#S}]\!]$.

In addition, we use as a fact that the kernel of $(\pi_{1,*} \oplus \pi_{2,*})^\vee$ and the cokernel of $(\pi_1^*, \pi_2^*)^\vee$ as $R_{\bar\rho}^{\square_S, \mathcal{D}}$-module are supported on $R_{\bar\rho}^{\square_S, \mathcal{D}, \mathrm{St}}$ (this is because when tensored with $\mathbb{Q}_\ell$, the kernel/cokernel corresponds to p-new forms, whose associated Galois representations have the corresponding properties.)

Problem 7. (Assume the needed existence of Taylor–Wiles prime.) Let the weight $k = 2$. Run the patching argument to get $\mathcal{M}_\infty(\Gamma_0(N))$ and $\mathcal{M}_\infty(\Gamma_0(Np))$, which are modules over $\mathbb{Z}_\ell[\![z_1, \dots, z_r, w_1, \dots, w_{3\#S}]\!] \to R^{\mathcal{D}}_{\mathrm{loc}}[\![y_1, \dots, y_r]\!]$. Show that the Krull dimension of $R^{\mathcal{D}}_{\mathrm{loc}}$ is $3\#S + 1$ (we already assumed that this is a power series ring and it remains to verify its dimension) and both $\mathcal{M}_\infty(\Gamma_0(N))$ and $\mathcal{M}_\infty(\Gamma_0(Np))$ are free modules over $\mathbb{Z}_\ell[\![z_1, \dots, z_r, w_1, \dots, w_{3\#S}]\!]$.

Problem 8. Prove that, as the corresponding maps above are injective/surjective, the following maps

$$\begin{gathered} (\pi_{1,*} \oplus \pi_{2,*})^\vee : \mathcal{M}_\infty(\Gamma_0(N))^{\oplus 2} \to \mathcal{M}_\infty(\Gamma_0(Np)) \\ (\pi_1^*, \pi_2^*)^\vee : \mathcal{M}_\infty(\Gamma_0(Np)) \to \mathcal{M}_\infty(\Gamma_0(N))^{\oplus 2} \end{gathered}$$

are injective/surjective. Moreover, the composition is given by

$$(\pi_1^*, \pi_2^*)^\vee \circ (\pi_{1,*} \oplus \pi_{2,*})^\vee = \begin{pmatrix} p+1 & T_p^\vee \\ T_p^\vee & p+1 \end{pmatrix} : \mathcal{M}_\infty(\Gamma_0(N))^{\oplus 2} \to \mathcal{M}_\infty(\Gamma_0(N))^{\oplus 2},$$

where $T_p^\vee$ is given by multiplication by $\mathrm{Tr}(\rho^{\mathrm{univ}}(\mathrm{Frob}_p))$. (It will be important later that this matrix is not an invertible matrix because $\mathrm{Tr}(\bar\rho(\mathrm{Frob}_p)) \equiv p + 1 \bmod \ell$.)

Problem 9. Complete the proof of the level raising problem following the steps below:

(1) As f is already given, $\mathcal{M}_\infty(\Gamma_0(N))$ defines a free module over $R^{\mathcal{D}, \mathrm{ur}}_{\mathrm{loc}}[\![y_1, \dots, y_r]\!]$.

(2) Consider the kernel of

$$(\pi_1^*, \pi_2^*)^\vee : \mathcal{M}_\infty(\Gamma_0(Np)) \to \mathcal{M}_\infty(\Gamma_0(N))^{\oplus 2}$$

As at each finite level, this is supported on $R_{\bar\rho}^{\square_S, \mathcal{D}, \mathrm{St}}$. Show that this implies that this kernel is supported on $R^{\mathcal{D}, \mathrm{St}}_{\mathrm{loc}}[\![y_1, \dots, y_r]\!]$, and it is a finite free $R^{\mathcal{D}, \mathrm{St}}_{\mathrm{loc}}[\![y_1, \dots, y_r]\!]$-module.

The existence of g is equivalent to that this kernel is non-trivial.

(3) Show that when $\bar\rho(\mathrm{Frob}_p) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$, the matrix $\begin{pmatrix} p+1 & T_p^\vee \\ T_p^\vee & p+1 \end{pmatrix}$ is not invertible and therefore cannot induce an isomorphism.

Yet if $\ker(\pi_1^*, \pi_2^*)^\vee$ is trivial, or equivalently $(\pi_1^*, \pi_2^*)^\vee$ is an isomorphism, we may recover the map

$$S_2(\Gamma_0(N))^{\vee, \oplus 2}_{\mathfrak{m}_{\bar\rho}} \xrightarrow{(\pi_{1,*} \oplus \pi_{2,*})^\vee} S_2(\Gamma_0(Np))^\vee_{\mathfrak{m}_{\bar\rho}} \xrightarrow[\cong]{(\pi_1^*, \pi_2^*)^\vee} S_2(\Gamma_0(N))^{\vee, \oplus 2}_{\mathfrak{m}_{\bar\rho}}$$

Show that in this case, the former map cannot be injective after tensoring with $\mathbb{F}_\ell$. This gives a contradiction, and hence $\ker(\pi_1^*, \pi_2^*)^\vee$ cannot be trivial.

Final remark: If we write $R_{\mathrm{loc}}^{\mathcal{D}}$ as $\mathbb{Z}_\ell[\![Z_1,\dots,Z_r,X,Y]\!]/(XY)$, with $R_{\mathrm{loc}}^{\mathcal{D}}/(X) \cong R_{\mathrm{loc}}^{\mathcal{D},\mathrm{unr}}$ and $R_{\mathrm{loc}}^{\mathcal{D}}/(Y) \cong R_{\mathrm{loc}}^{\mathcal{D},\mathrm{St}}$, then it is expected that

$$\mathcal{M}_\infty(\Gamma_0(N)) \cong (R_{\mathrm{loc}}^{\mathcal{D}})^{\oplus 2m} \quad \text{and} \quad \mathcal{M}_\infty(\Gamma_0(Np)) \cong \left(R_{\mathrm{loc}}^{\mathcal{D}} \oplus R_{\mathrm{loc}}^{\mathcal{D},\mathrm{unr}}\right)^{\oplus m}$$

for some multiplicity $m \in \mathbb{N}$. The fact that the composition $(\pi_1^*,\pi_2^*)^\vee \circ (\pi_{1,*} \oplus \pi_{2,*})^\vee$ is not invertible is reflected by the fact that the composition of the following $\mathbb{Z}_\ell[\![X,Y]\!]/(XY)$-module maps is not an isomorphism

$$\mathbb{Z}_\ell[\![X,Y]\!]/(X) \to \mathbb{Z}_\ell[\![X,Y]\!]/(XY) \to \mathbb{Z}_\ell[\![X,Y]\!]/(X)$$

(A natural way for the first map to work is to multiply by Y, and a natural way to write down a second map is the natural quotient map.)

This is a philosophy in modularity lifting techniques: the singularities of the local deformation space and the patched modules over them reflects congruence relations of modular forms.