Comments for *Algebraic Geometry* with EGA & SGA

MORPHISMS OF SCHEMES

This sheet refers to [EGA I, §1.8, §2.2].

For two given schemes X, Y, a *morphism* $f : X \to Y$ is the following data:

(a) A continuous map $\psi : X \to Y$ of underlying topological spaces.
(b) A ring homomorphism of sheaves $\theta : \mathcal{O}_Y \to \psi_* \mathcal{O}_X$, such that for any $x \in X$, the induced map is a local homomorphism (i.e. $\mathcal{O}_{Y,\psi(x)} \to \mathcal{O}_{X,x}$ is a homomorphism of local rings).

It turns out that whenever the target Y is affine, say $Y = \operatorname{Spec} A$, then $X \to Y$ corresponds to a unique rig homomorphism $A \to \Gamma(X, \mathcal{O}_X)$. (Morally, the local condition in (b) guarantees that this is a one-to-one correspondence, see [EGA I, 1.8.1] for details.)

From this we know that for any morphism $f : X \to Y$ and any pair of affine open subsets $\operatorname{Spec} A$ of X with $\operatorname{Spec} B$ of Y, f is locally determined by the ring homomorphism $B \to A$ whenever $f(\operatorname{Spec} A) \subseteq \operatorname{Spec} B$.

Therefore, any local property of ring homomorphisms (and in particular any pointwise property) indicates a corresponding property of morphisms between schemes.

Definition 1 (Immersion)**.** A morphism $f : X \to Y$ is called an *immersion* if

(a) $\psi : X \to Y$ of underlying topological spaces is injective, and
(b) For any $x \in X$ there exists an affine open neighborhood $V = \operatorname{Spec} B$ of $\psi(x) \in Y$ such that
- $\psi^{-1}(V)$ is affine, i.e., $\psi^{-1}(V) = \operatorname{Spec} A \subseteq X$ for some ring A;
- $f|_V$ corresponds to a surjective ring homomorphism $\alpha : B \to A$.

Note by definition that, the second condition of (b) implies that $\psi(X) \cap V$ is a closed subset of V defined by an ideal $\ker \alpha$. Hence $\psi(X)$ is locally Zariski closed in Y.

Example 2. The following are two particular cases where f is an immersion.

(1) f is called an *open immersion* if, $\psi(X) \subseteq Y$ is open, V is sufficiently small, and $f|_V$ corresponds locally to an isomorphism $\alpha : B \to A$.
(2) f is called a *closed immersion* if, $\psi(X) \subseteq Y$ is closed.

Supposedly, see [EGA I, §4.2] to complete the argument.

Definition 3 (Affine morphism)**.** A morphism $f : X \to Y$ is called *affine* if, for any $y \in Y$, there exists an affine open neighborhood V of y such that $f^{-1}(V)$ (i.e. $\psi^{-1}(V)$) is an affine open in X.

If f is an affine morphism then for any affine open subset V of Y, $f^{-1}(V)$ is an affine open in X as well. In fact, may assume $Y = \operatorname{Spec} B$ is affine, to prove that so also is X.

For this, the useful idea is writing down the commutative diagram

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$$\begin{tikzcd} X \arrow[rr, "\alpha"] \arrow[dr] & & \operatorname{Spec} A \arrow[dl] \\ & Y & \end{tikzcd}$$

in which $A = \Gamma(X, \mathcal{O}_X)$. It suffices to prove that α is an isomorphism. For this, we are to find out a basic open subset V of Y such that $f^{-1}(V)$ is an affine open subset, and to show that $\alpha|_V$ is exactly an isomorphism. This argument needs necessarily a result in [EGA I, 9.3.3]. A more completed argument for this is in [EGA II, §1.3, §5.2].

Example 4. We use affine morphisms to interpret some geometric structures.

(1) The *vector bundle* is actually an affine morphism (c.f. [EGA II, §1.7]).
(2) The *finite morphism* $f : X \to Y$ is particularly an affine morphism. The following condition is further required. Given any affine open subset $V = \operatorname{Spec} B$ of Y, with $f^{-1}(V) = \operatorname{Spec} A \subseteq X$ correspondingly, the ring A is a finite B-algebra (i.e. A is of finite type as a B-module). It turns out that
   - All closed immersions are automatically finite morphisms.
   - A finite morphism is always closed, i.e. it maps Zariski closed subsets to Zariski closed subsets. Moreover, it can be open under appropriate conditions.[1]
(3) The *ramification cover* of Riemann surfaces can be read, in the language of schemes, as a finite morphism.

Remark 5. In the theory of manifolds it would be different to say a morphism is an immersion or an embedding. The scheme-theoretical immersion may look loosely like an embedding in general, however, this observation is not very correct, as the immersion implies information about structure sheaves.

For example, given an immersion $f : X \to Y$, it is not necessarily an isomorphism, even if f is a homeomorphism between underlying topological spaces.

In convention, an immersion can be written as an embedding if X and Y are both reduced schemes. Classical algebraic geometry only considers embeddings actually.

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[1]This phenomenon obtains a background about going-up and going-down theorems in commutative algebra. See [EGA II, §6.1].