Generalized Euler characteristic for Selmer groups of non-CM elliptic curves

Yukako Kezuka

§1 Setups & Introduction

G a compact p-adic Lie grp.

$\Lambda(G) = \varprojlim_{\mathcal{U}} \mathbb{Z}_p[G/\mathcal{U}]$, $\Gamma \simeq \mathbb{Z}_p$.

We have a structure theory for

f.g. $\Lambda = \Lambda(\Gamma) \simeq \mathbb{Z}_p[[T]]$ - modules :

M f.g. Λ-module, then

$$M \sim \Lambda^r \oplus \left(\bigoplus_i \Lambda/(p^{\mu_i})\right) \oplus \left(\bigoplus_j \Lambda/(f_j(T)^{n_j})\right).$$

(c.f. structure theory for f.g. ab grp $G \simeq \mathbb{Z}^r \oplus (\bigoplus_i \mathbb{Z}/(n_i))$.)

If G is torsion then $\# G = \prod_i n_i$.

The corresponding invariant for a f.g. torsion $(r=0)$ Λ-mod is called a <u>characteristic element</u>,

which is $(f_M) = (p^{\mu} \cdot f)$, $\mu = \sum \mu_i$, $f = \prod f_j^{n_j}$.

F number field or global func field.

F_∞/F p-adic Lie ext'n, $\Sigma = \mathrm{Gal}(F_\infty/F)$.

Call F_∞ <u>admissible</u> if

(1) F_∞/F unram outside a finite set of places

(2) $F^{cyc} \subset F_\infty$, F^{cyc} = cycl $\mathbb{Z}_p$-ext'n of F.

(3) Σ has no elt of order p.

Def 1 Given a discrete p-primary Σ-module Y,

if $H^i(\Sigma, Y)$ are finite, say Y has a finite Σ-Euler characteristic $\chi(\Sigma, Y) = \prod_{i \geq 0} \#(H^i(\Sigma, Y))^{(-1)^i}$.

If M is a f.g. torsion Λ-mod, the char elt

$$f_M \longleftrightarrow \text{Pontryagin dual } M^\vee = \mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$$

via $$|f_M(0)|_p^{-1} = \chi(\Gamma, M^\vee).$$

Set $\Gamma = \mathrm{Gal}(F^{cyc}/F)$, $H = \mathrm{Gal}(F_\infty / F^{cyc})$.

We restrict to the cat of mods

$$\mathfrak{M}_H(\Sigma) := \text{f.g. } \Lambda(\Sigma)\text{-mods } M \text{ s.t. } M/M(p) \text{ is a f.g. } \Lambda(H)\text{-mod.}$$

Given $M \in \mathfrak{M}_H(G)$, Coates - Schneider - Sujatha introduced "Akashi series" which coincides with the char elt when M is a f.g. torsion Λ-mod.

§ Elliptic curves over global func fields

F global func field of char $\ell \geq 5$.

E/F ell curve.

Def'n Say E has CM if $\dim(\mathrm{End}\, E \otimes \mathbb{Q}) > 1$.

Thm (Deuring) E ell curve $/F$ of char >0.

Then E has CM $\Leftrightarrow$ F is a finite field.

Now $p > 5$ prime, $p \neq l$.

F^{cyc}/F cycl $\mathbb{Z}_p$-ext'n.

F_∞/F admissible p-adic Lie ext'n.

$\Gamma = \mathrm{Gal}(F^{cyc}/F)$, $\Sigma = \mathrm{Gal}(F_\infty/F)$.

$\mathrm{Sel}(E/F_\infty)$ p-primary Selmer grp of E/F_∞.

<u>Prop</u> (Sechi, 2006)

Let E/F ell curve with no CM.

S set of places in F where E has bad red'n or no potentially good red'n.

Assume $F_\infty = F(E_{p^\infty})$ and assume $\mathrm{Sel}(E/F)$ is finite.

Then $\chi(\Sigma, \mathrm{Sel}(E/F_\infty)) = \chi(\Gamma, \mathrm{Sel}(E/F^{cyc})) \cdot \prod_{v \in S} |L_v(E,1)|_p$

where $L_v(E,s)$ = Euler factor of $L(E,s)$ at v.

We extend Sechi's result in 2 directions:

(1) Let F_∞ = more general adm p-adic Lie ext'n.

(2) Let $\mathrm{Sel}(E/F)$ be infinite, by allowing $E(F)$ to be infinite & assuming $Ш(E/F)(p)$ is finite.

<u>Rmk</u> When $\mathrm{Sel}(E/F)$ is infinite, $\mathrm{Sel}(E/F_\infty)$ does not have finite Σ-Euler char.

<u>Strategy</u> Assume an additional finiteness condition (Fin). just enough so we can show the Pontryagin dual

$$X(E/F_\infty) = \mathrm{Sel}(E/F_\infty)^\vee \in \mathfrak{M}_H(\Sigma).$$

<u>Theorem</u> (Deng - Kezuka - Li)

Let F_∞/F = an adm p-adic Lie ext'n.

E/F ell curve with no CM.

Assume Ш$(E/F)(p)$ is finite & (Fin).

Then $\mathrm{Sel}(E/F_\infty)$ has finite "generalized Σ-Euler char"

$\iff$ $\mathrm{Sel}(E/F^{cyc})$ has finite "generalized Γ-Euler char".

In this case we have

$$\chi(\Sigma, \mathrm{Sel}(E/F_\infty)) = \chi(\Sigma, \mathrm{Sel}(E/F^{cyc})) \cdot \prod_{v\in S} |L_v(E,1)|_p$$

where S is the set of primes of F

whose inertia subgrps in Σ are finite.

<u>Generalized Euler char</u>

Given a discrete p-primary Γ-mod Y, define

$$d_0 : H^0(\Sigma, Y) = \underset{\substack{\| \\ (Y^H)^\Gamma}}{H^0(\Gamma, \overset{\substack{H^0(H,Y) \\ \|}}{Y^H})} \longrightarrow \underset{\substack{\| \\ (Y^H)_\Gamma}}{H^1(\Gamma, \overset{\substack{H^0(H,Y) \\ \|}}{Y^H})} \hookrightarrow H^1(\Sigma, Y)$$

Similarly define d_1.

Then $(H^i(\Sigma, Y), d_i)$ form a complex w/ cohom $\mathcal{H}_i$.

<u>Def 2</u> If all $\mathcal{H}_i$ are finite, define the generalized Σ-Euler char of Y to be

$$\chi(\Sigma, Y) := \prod_{i\geq 0} \# \mathcal{H}_i^{(-1)^i}.$$

<u>Rmks</u> (1) If all $H^i(\Sigma, Y)$ are finite, then Def 2 = Def 1.

(2) If $H^0(\Sigma, Y)$ & $H^1(\Sigma, Y)$ are infinite

but $H^i(\Sigma, Y)$ are finite for $i > 1$,

then Y has finite gen Σ-Euler char

iff $\ker d_0$ & $\operatorname{coker} d_0$ are finite,

in which case

$$\chi(\Sigma, Y) = \frac{\# \ker d_0}{\# \operatorname{coker} d_0} \cdot \prod_{i > 1} \# (H^i(\Sigma, Y))^{(-1)^i}$$

(c.f. Zerbes 2008, CSS 2003 in number field case.)