Fall 2022 – Seminar on *Abelian Varieties* by David Mumford

Lecture 9-11

ALGEBRAIC THEORY VIA SCHEMES

9. Basic Theory of Group Schemes

9.1. **Categorical Perspective of Schemes.** Fix an algebraically closed field k. We use Sch_k to denote the category of schemes of finite type over k. For two objects X, S in Sch_k, we define an S-valued point of X to be a morphism $S \to X$ in Sch_k. Denote

$$\underline{X}(S) := \mathrm{Hom}_k(S, X).$$

The association $S \mapsto \underline{X}(S)$ defines a contravariant functor $\underline{X} : \mathsf{Sch}_k^{\mathrm{op}} \to \mathsf{Sets}$ from the opposite category of Sch_k. When X varies, we get a functor $\mathsf{Sch}_k \to \mathrm{Fun}(\mathsf{Sch}_k^{\mathrm{op}}, \mathsf{Sets})$, which is fully faithful. Granting this, we can view Sch_k as a full subcategory of $\mathrm{Fun}(\mathsf{Sch}_k^{\mathrm{op}}, \mathsf{Sets})$. Similarly, if we use Alg_k to denote the category of finitely generated k-algebras, then any $X \in \mathsf{Sch}_k$ defines a covariant functor

$$\underline{X} : \mathsf{Alg}_k \to \mathsf{Sets}, \quad \underline{X}(R) := \underline{X}(\mathrm{Spec}\, R) = \mathrm{Hom}_k(\mathrm{Spec}\, R, X).$$

The functor

$$\mathsf{Sch}_k \to \mathrm{Fun}(\mathsf{Alg}_k, \mathsf{Sets}), \quad X \mapsto \underline{X}$$

is fully faithful and we can view Sch_k as a full subcategory of $\mathrm{Fun}(\mathsf{Alg}_k, \mathsf{Sets})$.

Definition 9.1 (Group scheme)**.** A **group scheme** is a scheme G of finite type over k together with

- a multiplication morphism $m : G \times G \to G$,
- an identity point $e : \mathrm{Spec}\, k \to G$, and
- an inverse morphism $i : G \to G$,

such that the following axioms hold.

(1) (Associativity) The diagram is commutative:

$$\begin{tikzcd} G \times G \times G \arrow[r, "m \times 1_G"] \arrow[d, "1_G \times m"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

(2) (Axiom of the identity section) The diagram is commutative:

$$\begin{tikzcd} G \times \mathrm{Spec}\, k \arrow[r, "\mathrm{id}_G \times e"] \arrow[d, "\cong"'] & G \times G \arrow[d, "m"] \\ G \arrow[r, "\mathrm{id}_G"] \arrow[d, "\cong"'] & G \\ G \times \mathrm{Spec}\, k \arrow[r, "e \times \mathrm{id}_G"] & G \times G \arrow[u, "m"'] \end{tikzcd}$$

(3) The diagram is commutative:

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$$\begin{tikzcd} & G\times G \arrow[dr, "m"] & \\ G \arrow[ur, "(\mathrm{id}_G, i)"] \arrow[r] \arrow[dr, "(i,\mathrm{id}_G)"'] & \operatorname{Spec} k \arrow[r, "e"] & G \\ & G\times G \arrow[ur, "m"'] & \end{tikzcd}$$

Remark 9.2. (1) We use $\underline{G}$ to denote the functor $\mathsf{Sch}_k^{\mathrm{op}} \to \mathsf{Sets}$ associated to G. Then G is a group scheme if and only if $\underline{G}$ factors through the forgetful functor $\mathsf{Grp} \to \mathsf{Sets}$, i.e., $\underline{G}$ is the composite

$$\underline{G} : \mathsf{Sch}_k^{\mathrm{op}} \to \mathsf{Grp} \to \mathsf{Sets}.$$

(2) For a closed point $x \in G(k)$, we can define the right translation R_x to be the composite

$$G \cong G \times \operatorname{Spec} k \xrightarrow{(\mathrm{id}_G, x)} G \times G \xrightarrow{m} G$$

and we define the left translation L_x similarly. In general, for a k-scheme $S \in \mathsf{Sch}_k$ and an S-point $x \in \underline{G}(S)$, we define the right translation R_x to be the S-morphism

$$G \times S \xrightarrow{(m\circ(\mathrm{id}_G \times x), p_2)} G \times S.$$

One can check that $R_{xy} = R_y \circ R_x$ and define L_x in a similar way.

9.2. **Lie Algebras.** Let $X \in \mathsf{Sch}_k$ and $\Omega_X = \Omega^1_{X/k}$ be the sheaf of relative differentials in X/k.

Definition 9.3. (1) A **vector field** D on X is a k-linear map $D : \mathscr{O}_X \to \mathscr{O}_X$ such that for any open subset U of X,

$$(D(U) : \mathscr{O}_X(U) \to \mathscr{O}_X(U)) \in \operatorname{Der}_k(\mathscr{O}_X(U), \mathscr{O}_X(U)).$$

In other words, $D : \mathscr{O}_X \to \mathscr{O}_X$ is the composite

$$\mathscr{O}_X \xrightarrow{d} \Omega_X \xrightarrow{f} \mathscr{O}_X$$

where d is the canonical derivation and f is an $\mathscr{O}_X$-linear map.

(2) A **tangent vector** d of X at a closed point $x \in X$ (cf. [Har13, II, Exer 2.8]) is a k-derivation $d : \mathscr{O}_{X,x} \to k \in \operatorname{Der}_k(\mathscr{O}_{X,x}, k)$, for which

$$\begin{aligned} \operatorname{Der}_k(\mathscr{O}_{X,x}, k) &\cong \operatorname{Hom}_{\mathscr{O}_{X,x}}(\Omega^1_{\mathscr{O}_{X,x}/k}, k) \\ &\cong \operatorname{Hom}_{\mathscr{O}_{X,x}}(\Omega_{X,x}, k) \\ &\cong \operatorname{Hom}_k(\Omega_{X,x} \otimes_{\mathscr{O}_{X,x}} k, k) \\ &\cong \operatorname{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k), \end{aligned}$$

where $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$ is the maximal ideal. The last isomorphism is from [Har13, II, Prop 8.7]. Therefore, to give a tangent vector at x, say $d : \mathscr{O}_{X,x} \to k$, is equivalent to giving a k-linear map $t : \mathfrak{m}_x/\mathfrak{m}_x^2 \to k$.

(3) For a vector field $D : \mathscr{O}_X \to \mathscr{O}_X$ we define its **value** at $x \in X$ to be the tangent vector $\mathscr{O}_{X,x} \xrightarrow{D_x} \mathscr{O}_{X,x} \to k$.

For two schemes $X, Y \in \mathsf{Sch}_k$, we have a canonical isomorphism $\Omega_{X\times Y} \cong p_1^*\Omega_X \oplus p_2^*\Omega_Y$ where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are natural projections. Let

$$D : \mathscr{O}_X \xrightarrow{d} \Omega_X \xrightarrow{f} \mathscr{O}_X$$

be a vector field on X, we define a vector field $D \otimes 1$ on $X \times Y$ that corresponds to the $\mathscr{O}_{X\times Y}$-linear map

$$\Omega_{X\times Y} \xrightarrow{\quad\sim\quad} p_1^*\Omega_X \oplus p_2^*\Omega_Y \xrightarrow{(p_1^*(f),0)} \mathscr{O}_{X\times Y}$$

Definition 9.4. Let G be a group scheme over k. A vector field D on G is called **left invariant** if the following diagram commutes:

$$\begin{tikzcd}
\mathscr{O}_G \arrow[r, "D"] \arrow[d, "m^*"'] & \mathscr{O}_G \arrow[d, "m^*"] \\
\mathscr{O}_{G\times G} \arrow[r, "1\otimes D"] & \mathscr{O}_{G\times G}
\end{tikzcd}$$

Proposition 9.5. *For any tangent vector t at e_G to G, there is a unique left invariant vector field on G whose value at e_G is exactly t.*

Proof. First we give another expression of tangent vectors and vector fields. Let $\Lambda = k[\varepsilon]/(\varepsilon)^2$. Let A be a k-algebra and B be an A-algebra. Then we have a bijection between sets, say

$$\begin{tikzcd}
D \arrow[r, mapsto] & \varphi(a) = a \cdot 1_B + D(a)\varepsilon \\
\mathrm{Der}_k(A,B) \arrow[r, leftrightarrow] & \left\{ \varphi : A \to B \otimes_k \Lambda \;\middle|\; \begin{array}{c} \varphi \text{ is a } k\text{-algebra homomorphism} \\ \text{such that } \overline{\varphi} : A \to B \text{ is the structure map} \\ \text{read as } \overline{\varphi} : A \xrightarrow{\varphi} B \otimes_k \Lambda \xrightarrow{\operatorname{mod} \varepsilon} B \end{array} \right\} \arrow[d, leftrightarrow] \\
& \left\{ \varphi' : A \otimes_k \Lambda \to B \otimes_k \Lambda \;\middle|\; \begin{array}{c} \varphi' \text{ is a } \Lambda\text{-algebra homomorphism such that} \\ \varphi' \otimes_k \Lambda : A \to B \text{ is the structure map} \end{array} \right\}.
\end{tikzcd}$$

Under the above bijections:

(i) a tangent vector t to X at $x \in X$ corresponds to a morphism $\tilde{t} : \operatorname{Spec}\Lambda \to X$ such that the composite $\operatorname{Spec} k \to \operatorname{Spec}\Lambda \xrightarrow{\tilde{t}} X$ is basically the point $x \in X$.

(ii) a vector field D on X corresponds to a morphism over $\operatorname{Spec}\Lambda$:

$$\begin{tikzcd}
X \times \operatorname{Spec}\Lambda \arrow[rr, "\widetilde{D}"] \arrow[dr] & & X \times \operatorname{Spec}\Lambda \arrow[dl] \\
& \operatorname{Spec}\Lambda &
\end{tikzcd}$$

such that $\widetilde{D} \times_{\operatorname{Spec}\Lambda} \operatorname{Spec} k : X \to X$ is id_X.

(iii) for a vector field D on X, and t_x the value of D at $x \in X$, the morphism $\tilde{t}_x$ corresponds to the morphism

$$\operatorname{Spec}\Lambda \xrightarrow{\cong} \operatorname{Spec} k \times \operatorname{Spec}\Lambda \xrightarrow{(x,\mathrm{id}_X)} X \times \operatorname{Spec}\Lambda \xrightarrow{\widetilde{D}} X \times \operatorname{Spec}\Lambda \xrightarrow{p_1} X$$

Under the above expressions, we see that a vector field D on G is left invariant if and only if the following diagram is commutative:

$$\begin{tikzcd} G \times G \times \operatorname{Spec}\Lambda \arrow[r, "\mathrm{id}_G \times \widetilde{D}"] \arrow[d, "m \times \mathrm{id}_\Lambda"'] & G \times G \times \operatorname{Spec}\Lambda \arrow[d, "m \times \mathrm{id}_\Lambda"] \\ G \times \operatorname{Spec}\Lambda \arrow[r, "\widetilde{D}"] & G \times \operatorname{Spec}\Lambda \end{tikzcd}$$

Here all arrows are morphisms over $\operatorname{Spec}\Lambda$. We use $\widetilde{D}_1$ to denote the composite

$$\widetilde{D}_1 : G \times \operatorname{Spec}\Lambda \xrightarrow{\widetilde{D}} G \times \operatorname{Spec}\Lambda \xrightarrow{p_1} G.$$

Then D is left invariant if and only if for any $S \in \mathsf{Sch}_k$, $x, y \in \underline{G}(S)$, and $l \in \operatorname{Spec}\Lambda(S)$, $\widetilde{D}_1(xy, l) = x\widetilde{D}(y, l)$. (Caveat: one should be very careful about the order.) Alternatively, this is equivalent to say

$$\widetilde{D}_1(x, l) = x\widetilde{D}_1(e_G(S), l),$$

where $e_G(S) \in \underline{G}(S)$ is the identity element; note that $(e_G(S), l)$ is the value of $\widetilde{D}$ at e_G.

Now given a tangent vector t of G at e_G, we define a vector field D on X that corresponds to the following with $\tilde{t} : \operatorname{Spec}\Lambda \to G$,

$$\widetilde{D} : G \times \operatorname{Spec}\Lambda \xrightarrow{(\mathrm{id}_G, \tilde{t}, \mathrm{id}_\Lambda)} G \times G \times \operatorname{Spec}\Lambda \xrightarrow{(m, \mathrm{id}_\Lambda)} G \times \operatorname{Spec}\Lambda.$$

In other words, $\widetilde{D}_1$ satisfies $\widetilde{D}_1(x, l) = x\widetilde{D}_1(e_G(S), l)$ for each $S \in \mathsf{Sch}_k$, $x \in \underline{G}(S)$, and $l \in (\operatorname{Spec}\Lambda)(S)$. It renders that $\widetilde{D}$ is left invariant and has value t at e_G. Hence the uniqueness follows obviously. $\square$

Let D_1, D_2 be two vector fields on X. Their **Poisson bracket**

$$[D_1, D_2] = D_1 D_2 - D_2 D_1$$

is also a vector field on X. When $\operatorname{char}(k) = p > 0$, D_1^p is a vector field on X. When $X = G$ is a group scheme, the above two operators preserve left invariant vector fields.

Definition 9.6. The **Lie algebra of a group scheme** G is the k-vector space of left invariant vector fields, together with the operation of Poisson bracket (and the pth power operation if $\operatorname{char}(k) = p > 0$).

Proposition 9.7. *If G is a commutative group scheme, then its Lie algebra $\mathfrak{g}$ is abelian, i.e., $[D_1, D_2] = 0$ for all $D_1, D_2 \in \mathfrak{g}$.*

Proof. We first make the following observation. Let $X \in \mathsf{Sch}_k$, D_1, D_2 be vector fields on X, and $D_3 = [D_1, D_2]$. Let $\widetilde{D}_i : X \times \operatorname{Spec}\Lambda \to X \times \operatorname{Spec}\Lambda$ be the morphism corresponding to D_i for $i = 1, 2, 3$. I claim that $x_3 = x_1 x_2 x_1^{-1} x_2^{-1}$. The question is local on X so we can assume that $X = \operatorname{Spec} A$ is affine. The automorphism χ_i of $X \times \operatorname{Spec}\Lambda'$ over $\operatorname{Spec}\Lambda'$ corresponds to the Λ'-algebra automorphism

$$f_i : A[\varepsilon, \varepsilon']/(\varepsilon^2, \varepsilon'^2) \longrightarrow A[\varepsilon, \varepsilon']/(\varepsilon^2, \varepsilon'^2)$$

that is defined as follows:

$$\begin{aligned}
f_1 &: a_1 + a_2\varepsilon + a_3\varepsilon' + a_4\varepsilon\varepsilon' \longmapsto a_1 + (D_1(a_1)+a_2)\varepsilon + a_3\varepsilon' + (D_1(a_3)+a_4)\varepsilon\varepsilon',\\
f_2 &: a_1 + a_2\varepsilon + a_3\varepsilon' + a_4\varepsilon\varepsilon' \longmapsto a_1 + a_2\varepsilon + (D_2(a_1)+a_3)\varepsilon' + (D_2(a_2)+a_4)\varepsilon\varepsilon',\\
f_3 &: a_1 + a_2\varepsilon + a_3\varepsilon' + a_4\varepsilon\varepsilon' \longmapsto a_1 + a_2\varepsilon + a_3\varepsilon' + (D_3(a_1)+a_4)\varepsilon\varepsilon';
\end{aligned}$$

also, f_i^{-1} is given by replacing D_i by $-D_i$ in the above formulas. Hence

$$\begin{aligned}
& f_3^{-1} = f_2^{-1}\circ f_1^{-1}\circ f_2\circ f_1 : A\otimes_k \Lambda' \to A\otimes_k \Lambda',\\
\rightsquigarrow \qquad & \chi_3^{-1} = \chi_1\circ\chi_2\circ\chi_1^{-1}\circ\chi_2^{-1} : X\times \operatorname{Spec}\Lambda' \to X\times\operatorname{Spec}\Lambda'.
\end{aligned}$$

Now let D_1, D_2 be two left invariant vector fields on a commutative group scheme G that corresponds to the tangent vector $t_i : \operatorname{Spec}\Lambda \to G$ with $i = 1, 2$. Define $D_3 = [D_1, D_2]$ and $\tilde{t}_3$ that corresponds to D_3. For $i = 1, 2, 3$, define $T_i \in \underline{G}(\Lambda')$ to be the composite

$$T_i : \operatorname{Spec}\Lambda' \xrightarrow{\sigma_i} \operatorname{Spec}\Lambda \xrightarrow{\tilde{f}_i} G.$$

Thus, $\chi_i : G\times\operatorname{Spec}\Lambda' \to G\times\operatorname{Spec}\Lambda'$ be the right translation by $T_i \in \underline{G}(\Lambda')$. Then

$$\chi_3^{-1} = \chi_1\circ\chi_2\circ\chi_1^{-1}\circ\chi_2^{-1},$$

but $\chi_1\circ\chi_2\circ\chi_1^{-1}\circ\chi_2^{-1}$ is the right translation by $T_2^{-1}\cdot T_1^{-1}\cdot T_2\cdot T_2 \in \underline{G}(\Lambda')$, which is $e_G(\Lambda')$ as G is commutative. It follows that χ_3^{-1} (and hence χ_3) is the identity morphism so that $D_3 = [D_1, D_2] = 0$. $\square$

Theorem 9.8. *Any group scheme over a field k of characteristic 0 is automatically smooth (and in particular reduced).*

Proof. We can assume that k is algebraically closed and it suffices to prove the group scheme G is smooth at $e \in G$, the identity point. For simplicity, denote $\mathscr{O} = \mathscr{O}_{G,e}$, $\mathfrak{m} \subset \mathscr{O}$ the maximal ideal, and $\widehat{\mathscr{O}}$ the $\mathfrak{m}$-adic completion of $\mathscr{O}$. Denote $n = \dim_k \mathfrak{m}/\mathfrak{m}^2$. The multiplication map $m : G\times G \to G$ induces a continuous homomorphism

$$m^* : \widehat{\mathscr{O}} \longrightarrow \widehat{\mathscr{O}}\widehat{\otimes}_k\widehat{\mathscr{O}}.$$

Here $\widehat{\otimes}$ is the complete tensor product; that is, the $(1\otimes\widehat{\mathfrak{m}} + \widehat{\mathfrak{m}}\otimes 1)$-adic completion of $\widehat{\mathscr{O}}\otimes_k\widehat{\mathscr{O}}$. Since the two composites

$$\begin{tikzcd}
G \arrow[r, shift left, "{(\mathrm{id}_G, e)}"] \arrow[r, shift right, "{(e, \mathrm{id}_G)}"'] & G\times G \arrow[r, "m"] & G
\end{tikzcd}$$

are both identity, the composites

$$\begin{aligned}
&\widehat{\mathscr{O}} \xrightarrow{m^*} \widehat{\mathscr{O}}\widehat{\otimes}_k\widehat{\mathscr{O}} \longrightarrow \widehat{\mathscr{O}}\widehat{\otimes}_k k \cong \widehat{\mathscr{O}},\\
&\widehat{\mathscr{O}} \xrightarrow{m^*} \widehat{\mathscr{O}}\widehat{\otimes}_k\widehat{\mathscr{O}} \longrightarrow k\widehat{\otimes}_k\widehat{\mathscr{O}} \cong \widehat{\mathscr{O}}
\end{aligned}$$

are both identity as well. Thus, for any $a \in \widehat{\mathfrak{m}}$,

$$m^*(a) \in 1\otimes a + a\otimes 1 + \widehat{\mathfrak{m}}\widehat{\otimes}_k\widehat{\mathfrak{m}}.$$

Claim. *For any k-linear map $f : \widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2 \to k$, there is a k-derivation $D : \widehat{\mathscr{O}} \to \widehat{\mathscr{O}}$ such that $f = D|_{\widehat{\mathfrak{m}}} \bmod \widehat{\mathfrak{m}}$.*

Since we have a decomposition of k-vector spaces, $\widehat{\mathscr{O}} = k \oplus \widehat{\mathfrak{m}}$. A k-linear map $F : \widehat{\mathscr{O}} \to k$ could be found such that $F|_k = 0$ and $F|_{\widehat{\mathfrak{m}}}$ is the composite $\widehat{\mathfrak{m}} \to \widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2 \xrightarrow{f} k$. We define D to be the composite

$$\widehat{\mathscr{O}} \xrightarrow{m^*} \widehat{\mathscr{O}} \widehat{\otimes}_k \widehat{\mathscr{O}} \xrightarrow{1 \otimes F} \widehat{\mathscr{O}} \widehat{\otimes}_k k \xrightarrow{\cong} \widehat{\mathscr{O}}.$$

Clearly D is k-linear and $D(k) \equiv 0$ as $F(k) \equiv 0$. For $a \in \widehat{\mathfrak{m}}$,

$$D(a) = (1 \otimes F)(1 \otimes a + a \otimes 1 + b) = F(a) + (1 \otimes F)(b)$$

for some $b \in \widehat{\mathfrak{m}} \widehat{\otimes}_k \widehat{\mathfrak{m}}$. Consequently, $D(a) \bmod \widehat{\mathfrak{m}} = F(a) = f(a \bmod \mathfrak{m}^2)$. This proves the claim that $f = D|_{\widehat{\mathfrak{m}}} \bmod \widehat{\mathfrak{m}}$.

We still need to verify that D is a derivation, i.e., for all $a, b \in \widehat{\mathfrak{m}}$, we have $D(ab) = aD(b) + bD(a)$. By a direct computation,

$$\begin{aligned} m^*(ab) = &(a \otimes 1)m^*(b) + (b \otimes 1)m^*(a) \\ &+ (1 \otimes ab - ab \otimes 1 + S(1 \otimes a) + R(1 \otimes b) + RS). \end{aligned}$$

If we write $m^*(a) = 1 \otimes a + a \otimes 1 + R$, $m^*(b) = 1 \otimes b + b \otimes 1 + S$. In particular, $R, S \in \widehat{\mathfrak{m}} \widehat{\otimes}_k \widehat{\mathfrak{m}}$ and

$$T = 1 \otimes ab - ab \otimes 1 + S(1 \otimes a) + R(1 \otimes b) + RS \in \widehat{\mathscr{O}} \otimes 1 + \widehat{\mathscr{O}} \otimes \widehat{\mathfrak{m}}^2.$$

We infer that

$$\begin{aligned} D(ab) &= (1 \otimes F)(m^*(ab)) \\ &= a(1 \otimes F)(m^*(b)) + b(1 \otimes F)(m^*(a)) + \underbrace{(1 \otimes F)(T)}_{=0} \\ &= aD(b) + bD(a). \end{aligned}$$

Here $(1 \otimes F)(T) = 0$ because of $T \in \widehat{\mathscr{O}} \otimes 1 + \widehat{\mathscr{O}} \otimes \widehat{\mathfrak{m}}^2$. Choose $x_1, \ldots, x_n \in \widehat{\mathfrak{m}}$ such that $\{\overline{x}_1, \ldots, \overline{x}_n\}$ forms a k-basis of $\widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2$ and $\{f_i : \widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2 \to k \mid i = 1, \ldots, n\}$ be the dual basis. Let $D_i : \widehat{\mathscr{O}} \to \widehat{\mathscr{O}}$ be the k-derivation such that $D_i|_{\widehat{\mathfrak{m}}} \bmod \widehat{\mathfrak{m}} = f_i$ for each i. In particular, we have $D_i(x_j) \bmod \widehat{\mathfrak{m}} = \delta_{ij}$ for all $1 \leqslant i, j \leqslant n$.

Define a k-algebra homomorphism

$$\alpha : k[\![t_1, \ldots, t_n]\!] \to \widehat{\mathscr{O}}, \quad t_i \mapsto x_i, \quad i = 1, \ldots, n.$$

Since $\{x_1, \ldots, x_n\}$ generates $\widehat{\mathfrak{m}}$ by Nakayama's lemma, α is surjective. Define another k-algebra homomorphism

$$\beta : \widehat{\mathscr{O}} \to k[\![t_1, \ldots, t_n]\!], \quad f \mapsto \sum_{\alpha \in \mathbb{Z}^n_{\geqslant 0}} \overline{\left(\frac{D^\alpha f}{\alpha!}\right)} \cdot t^\alpha.$$

Here the operator $\overline{(\cdot)}$ means modulo $\widehat{\mathfrak{m}}$ (so that the coefficients are elements in k) the power series is defined through

$$\alpha! = \alpha_1! \cdots \alpha_n!, \quad D^\alpha f = D_1^{\alpha_1} \cdots D_n^{\alpha_n} f, \quad t^\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}.$$

By Leibniz's formula, one can check that β is a continuous homomorphism. Moreover,

$$\beta(x_i) \equiv t_i \mod (t_1, \ldots, t_n)^2, \quad i = 1, \ldots, n.$$

Hence β is surjective. The composite $\beta \circ \alpha : k[\![t_1, \ldots, t_n]\!] \to k[\![t_1, \ldots, t_n]\!]$ is onto and satisfies $\beta \circ \alpha \equiv \mathrm{id} \bmod (t_1, \ldots, t_n)^2$. Therefore, $\beta \circ \alpha$ is an isomorphism.[1] So α is injective and hence an isomorphism as well. This shows

$$\widehat{\mathscr{O}} \cong k[\![t_1, \ldots, t_n]\!],$$

which implies that $\widehat{\mathscr{O}}$ is regular, and so also is $\mathscr{O}$ itself. This proves G is smooth at e. $\square$

[1]For this implication, see [Eis13, §7].

10. Quotients by Finite Group Schemes

10.1. **The Group Scheme Action on Scheme.**

Definition 10.1 (Left action of schemes)**.** A **left action of a group scheme** G **on a scheme** X is a morphism $\mu : G \times X \to X$ such that

(1) the composite
$$X \xrightarrow{\cong} \operatorname{Spec} k \times X \xrightarrow{e_G \times 1_X} G \times X \xrightarrow{\mu} X$$
is the identity morphism;

(2) the diagram
$$\begin{tikzcd} G \times G \times X \arrow[r, "m \times 1_X"] \arrow[d, "1_G \times \mu"'] & G \times X \arrow[d, "\mu"] \\ G \times X \arrow[r, "\mu"] & X \end{tikzcd}$$
is commutative;

Remark 10.2. Indeed, we have the following equivalent characterization of a G-action on X.

(1) For any affine[2] scheme S we have a (left) $\underline{G}(S)$-action on $\underline{X}(S)$, which is functorial in S.

(2) More explicitly, for any $x \in \underline{G}(S)$, we have an automorphism over S; say the diagram
$$\begin{tikzcd} X \times S \arrow[rr, "T_x"] \arrow[dr, "p_2"'] & & X \times S \arrow[dl, "p_2"] \\ & S & \end{tikzcd}$$
commutes and is such that
  (i) $T_x \circ T_y = T_{xy}$ for all $x, y \in \underline{G}(S)$;
  (ii) for any morphism $f : S \to S'$ in Sch_k and $x \in \underline{G}(S')$, $x \circ f \in G(S)$.

We have another commutative diagram
$$\begin{tikzcd} X \times S \arrow[r, "T_{x \circ f}"] \arrow[d, "1_X \times f"'] & X \times S \arrow[d, "1_X \times f"] \\ X \times S' \arrow[r, "T_x"] & X \times S' \end{tikzcd}$$
For $x \in \underline{G}(S)$, the morphism $T_x : X \times S \to X \times S$ is given by (T'_x, p_2) where $T'_x : X \times S \to X$ is the composite
$$X \times S \cong S \times X \xrightarrow{f \times 1_X} G \times X \xrightarrow{\mu} X.$$
Conversely, the morphism μ can be recovered from the above datum. Take $S = G$ and $x = \mathrm{id}_G \in \underline{G}(G)$. Then μ is the composite
$$G \times X \xrightarrow{\cong} X \times G \xrightarrow{T_x} X \times G \xrightarrow{p_1} X.$$

Definition 10.3. A morphism $f : X \to Y$ is called G**-invariant** if the following diagram is commutative[3]

[2] Note that the problem is local.

[3] When $Y = \mathbb{A}^1 = \operatorname{Spec} k[T]$, one can talk about the G-invariant sections.

$$\begin{tikzcd} G \times X \arrow[r, "\mu"] \arrow[d, "p_2"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

More explicitly, for each $S \in \mathsf{Sch}_k$, $g \in \underline{G}(S)$, $x \in \underline{X}(S)$, we have $f(\mu(g,x)) = f(x)$.

The action of G on X is **free** if he morphism $(\mu, p_2) : G \times X \to X \times X$ is a closed immersion.

Definition 10.4. Let $\mathscr{F}$ be a coherent sheaf on X. A **lift of the action μ to $\mathscr{F}$** is an isomorphism

$$\lambda : p_2^* \mathscr{F} \xrightarrow{\sim} \mu^* \mathscr{F}$$

of sheaves on $G \times X$ such that the following diagram of sheaves on $G \times G \times X$ is commutative:

$$\begin{tikzcd} p_3^* \mathscr{F} \arrow[rr, "(p_2,p_3)^*(\lambda)"] \arrow[dr, "(m \times 1_X)^*(\lambda)"'] & & \xi^* \mathscr{F} \arrow[dl, "(1_G \times \mu)^*(\lambda)"] \\ & \eta^* \mathscr{F} & \end{tikzcd}$$

Here p_1, p_2, p_3 are natural projections from $G \times G \times X$ and

$$\xi : G \times G \times X \xrightarrow{(p_2,p_3)} G \times X \xrightarrow{\mu} X$$

$$\begin{tikzcd} \eta : G \times G \times X \arrow[r, "1_G \times \mu"] \arrow[dr, bend right, "m \times 1_X"'] & G \times X \arrow[r, "\mu"] & X \\ & G \times X \arrow[ur, "\mu"'] & \end{tikzcd}$$

10.2. **Classification of Quotients.**

Theorem 10.5 (Quotients by finite group schemes, somehow tedious)**.**

(A) *Let G be a finite group scheme acting on a scheme X such that the orbit of any point in contained in an affine open subset of X. Then there is a pair (Y, π) where Y is a scheme and $\pi : X \to Y$ a morphism, satisfying the following conditions:*

(1) *as a topological space, (Y, π) is the quotient of X for the action of the underlying finite group;*

(2) *the morphism $\pi : X \to Y$ is G-invariant, and if $\pi_*(\mathscr{O}_X)^G$ denotes the subsheaf of $\pi_* \mathscr{O}_X$ of G-invariant functions, the natural homomorphism $\mathscr{O}_Y \to \pi_*(\mathscr{O}_X)^G$ is an isomorphism. The pair (Y, π) is uniquely determined up to isomorphism by these conditions. The morphism π is finite and surjective; Y will be denoted X/G, and it has the functorial property that for any G-invariant morphism $f : X \to Z$, there is a unique morphism $g : Y \to Z$ such that $f = g \circ \pi$.*

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "f"'] & & Y \arrow[dl, dashed, "\exists! \; g"] \\ & Z & \end{tikzcd}$$

(B) *Suppose further that the action of G is free and $G = \operatorname{Spec} R$, $n = \dim_k R$. Then π is a flat morphism of degree n, i.e., $\pi_* \mathscr{O}_X$ is a locally free $\mathscr{O}_Y$-module of rank n, and the subscheme of $X \times X$ defined by the closed immersion*

$$(\mu, p_2) : G \times X \to X \times X$$

is equal to the subscheme $X \times_Y X \subset X \times X$. Finally, if $\mathscr{F}$ is a coherent $\mathscr{O}_Y$-module, $\pi_*\mathscr{F}$ has a natural defined G-action lifting that on X, and $\mathscr{F} \mapsto \pi^*\mathscr{F}$ is an equivalence between the category of coherent $\mathscr{O}_Y$-modules (resp. locally free $\mathscr{O}_Y$-modules of finite rank) and the category of coherent $\mathscr{O}_X$-modules with G-action (resp. locally free $\mathscr{O}_X$-modules of finite rank with G-action).

Remark 10.6. (1) The assumption that the orbit of any point in contained in an affine open subset of X can be expressed as follows: for any closed point $x \in X(k)$, the morphism

$$G \cong G \times \operatorname{Spec} k \xrightarrow{1_G \times x} G \times X \xrightarrow{\mu} X$$

factors through an open affine subset of X, i.e., we obtain $G \to U \subset X$. This holds for X quasi-projective over k.

(2) $G_{\text{red}} = \operatorname{Spec}(R_{\text{red}})$ is a closed subgroup scheme of G, and the action of G on X induces an action

$$\mu_{\text{red}} : G_{\text{red}} \times X \to X$$

of G_{red} on X. As a scheme over k,

$$G_{\text{red}} \cong \bigsqcup_{g \in G(k)} \operatorname{Spec} k,$$

and we are in the situation of varieties. Theorem 10.5 (A)(1) says that as a topological space, (Y, π) only depends on the action μ_{red} of G_{red} on X.

(3) Under the assumption of (B), we have an isomorphism

$$(\mu, p_2) : G \times X \xrightarrow{\sim} X \times_Y X,$$

and $\pi : X \to Y$ is faithfully flat. Let $\mathscr{F}$ be a coherent sheaf on X. Under the above isomorphism, a lift of the action μ to $\mathscr{F}$ becomes an isomorphism $\lambda : p_2^*\mathscr{F} \xrightarrow{\sim} p_1^*\mathscr{F}$ (sheaves on $X \times_Y X$) such that the diagram

$$\begin{tikzcd} p_3^*\mathscr{F} \arrow[rr, "p_{23}^*(\lambda)"] \arrow[dr, "p_{13}^*(\lambda)"'] & & p_2^*\mathscr{F} \arrow[dl, "p_{12}^*(\lambda)"] \\ & p_1^*\mathscr{F} & \end{tikzcd}$$

commutes.[4]

10.3. **Proof of Theorem (A).**

Proof. We can reduce to the case for $X = \operatorname{Spec} A$ affine. Recall that $G = \operatorname{Spec} R$ and $n = \dim_k R$. Consider the k-algebra homomorphisms in the following correspondences:

Algebraic Homomorphisms	Geometric Morphisms
$\varepsilon : R \to k$	$e : \operatorname{Spec} k \to G$
$m^* : R \to R \otimes_k R$	$m : G \times G \to G$
$\mu^* : A \to R \otimes_k A$	$\mu : G \times X \to X$

[4] This is the standard descent theory. See [MA67, Chap VII].

For any k-algebra S, $R \otimes_k S$ is a free S-module of rank n. We have a norm map $\mathrm{Nm}_S : R \otimes_k S \to S$, i.e., for any $x \in R \otimes_k S$ the multiplication by x defines an S-linear map $l_x : R \otimes_k S \to R \otimes_k S$, and $\mathrm{Nm}_S(x) = \det l_x$. Also,

$$\mathrm{Nm}_S(ax) = a^n \,\mathrm{Nm}_S(x), \quad \forall a \in S,\ x \in R \otimes_k S,$$

and Nm_S is multiplicative.

Also define $B = \{a \in A \mid \mu^*(a) = 1 \otimes a\} \subset A$ to be the k-subalgebra of A consisting of G-invariant sections; that is, for $a \in A$, the morphism $X \to \mathbb{A}^1$ corresponding to $k[T] \to A$, $T \mapsto G$ is G-invariant if and only if $a \in B$. Define the composite

$$N : A \xrightarrow{\mu^*} R \otimes_k A \xrightarrow{\mathrm{Nm}_A} A.$$

Note that N is multiplicative and k-homogeneous of degree n.

Claim. *$N(A) \subset B$, i.e., $\mu^*(N(a)) = 1 \otimes N(a)$ for each $a \in A$.*

Proof of Claim. We define two k-algebra homomorphisms (with their geometric correspondences) as follows:

$$\phi : A \to R \otimes_k A, \quad a \mapsto 1 \otimes a$$

corresponding to

$$p_2 : G \times X \to X, \quad (g, x) \mapsto x;$$

and

$$\psi : R \otimes_k R \otimes_k A \to R \otimes_k R \otimes_k A, \quad x \otimes y \otimes a \mapsto (m^*(x) \otimes 1) \cdot (1 \otimes y \otimes a)$$

corresponding to

$$\mathrm{Spec}\,\psi : G \times G \times X \to G \times G \times X, \quad (g, h, x) \mapsto (gh, h, x).$$

Firstly, we make a remark. If $f : S_1 \to S_2$ is a k-algebra homomorphism, then the diagram commutes, namely, $f \circ \mathrm{Nm}_{S_1} = \mathrm{Nm}_{S_2} \circ (1_R \otimes f)$.

$$\begin{tikzcd}
R \otimes_k S_1 \arrow[r, "\mathrm{Nm}_{S_1}"] \arrow[d, "1_R \otimes f"'] & S_1 \arrow[d, "f"] \\
R \otimes_k S_2 \arrow[r, "\mathrm{Nm}_{S_2}"] & S_2
\end{tikzcd}$$

So, by the above remark, we obtain that

$$\mu^* \circ N = \mu^* \circ \mathrm{Nm}_A \circ \mu^* = \mathrm{Nm}_{R \otimes_k A} \circ (1_R \otimes \mu^*) \circ \mu^*.$$

Moreover,

$$\mathrm{Nm}_{R \otimes_k A} \circ (1_R \otimes \mu^*) \circ \mu^* = \mathrm{Nm}_{R \otimes_k A} \circ (m^* \otimes 1_A) \circ \mu^* = \mathrm{Nm}_{R \otimes_k A} \circ \psi \circ (1_R \otimes \phi) \circ \mu^*$$

because of the two diagrams are commutative:

$$\begin{tikzcd}
G \times G \times X \arrow[r, "1_G \times \mu"] \arrow[d, "m \times 1_X"'] & G \times X \arrow[d, "\mu"] \\
G \times X \arrow[r, "\mu"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times G \times X \arrow[r, "1_G \times \mu"] \arrow[d, "m \times 1_X"'] & G \times G \times X \arrow[r, "\mathrm{id}_G \times p_2"] & G \times X \arrow[dl, "\mu"] \\
G \times X \arrow[r, "\mu"] & X &
\end{tikzcd}$$

Let us regard $R \otimes_k (R \otimes_k A)$ as an $R \otimes_k A$-algebra via the last two factors, i.e., via the k-algebra homomorphism

$$R \otimes_k A \to R \otimes_k R \otimes_k A, \quad r \otimes a \mapsto 1 \otimes r \otimes a.$$

Then $\psi : R \otimes_k R \otimes_k A \to R \otimes_k R \otimes_k A$ is an $R \otimes_k A$-algebra automorphism; equivalently, we need to verify that

$$\begin{tikzcd} G \times G \times X \arrow[rr, "\operatorname{Spec}\psi"] \arrow[dr, "p_{23}"'] & & G \times G \times X \arrow[dl, "p_{23}"] \\ & G \times X & \end{tikzcd}$$

is an automorphism. Thus,

$$\mathrm{Nm}_{R\otimes_k A} \circ \psi = \mathrm{Nm}_{R\otimes_k A}.$$

And therefore,

$$\mu^* \circ N = \mathrm{Nm}_{R\otimes_k A} \circ (1_R \otimes \phi) \circ \mu^* = \phi \circ \mathrm{Nm}_A \circ \mu^* = 1 \otimes N.$$

This proves our claim. $\square$

We extend the G-action on X to $X \times \mathbb{A}^1$ such that G acts trivially on $\mathbb{A}^1$ with $\mu \times \mathrm{id}_{\mathbb{A}^1} : G \times X \times \mathbb{A}^1 \to X \times \mathbb{A}^1$. Correspondingly, $\mu^* : A \to R \otimes_k A$ can be extended to a k-algebra homomorphism $A[T] \to R \otimes_k A[T]$. So we can extend the map $N : A \to A$ to $N : A[T] \to A[T]$. For $a \in A$, we set $\chi_a(T) = N(T - a)$ and we can extend χ_a to a k-algebra homomorphism $k[T] \to A[T]$ (and hence determines a morphism $X \times \mathbb{A}^1 \to \mathbb{A}^1$).

It is straightforward to verify that $\chi_a(T) \in A[T]$ is the characteristic polynomial of the A-linear map

$$l_{\mu^*(a)} : R \otimes_k A \to R \otimes_k A$$

that is induced by the multiplication by $\mu^*(a)$, and $\chi_a(T)$ is G-invariant, i.e., the morphism $X \times \mathbb{A}^1 \to \mathbb{A}^1$ determined by $\chi_a(T)$ is G-invariant. So

$$\chi_a(T) = T^n + s_1 T^{n-1} + \cdots + s_n \in A[T]$$

is monic of degree n, and $s_i \in B$ for all i; namely, $\chi_a(T) \in B[T]$.

Fix $a \in A$. The map $\varepsilon : R \to k$ corresponding to the section $e : \operatorname{Spec} k \to G$ extends to an A-linear map $\varepsilon \otimes 1_A : R \otimes_k A \to A$ such that the composite $A \xrightarrow{\mu^*} R \otimes_k A \xrightarrow{\varepsilon\otimes 1_A} A$ is nothing but id_A. Thus the A-linear map $l_{\mu^*(a)-a} : R \otimes_k A \to R \otimes_k A$ induces the zero map on the quotient $\varepsilon \otimes 1_A : R \otimes_k A \to A$. It follows that

$$\chi_a(a) = \det(l_{a-\mu^*(a)}) = 0,$$

namely, a is integral over B. Hence A is integral over B. Since A is a finitely generated k-algebra, there exists a finitely generated k-subalgebra $B' \subset B$ such that A is integral and finite over B'. Then B is finite over B'. Hence B is a finitely generated k-algebra. If we use $\pi : X \to Y = \operatorname{Spec} B$ to denote the morphism corresponding to the inclusion $B \hookrightarrow A$, then π is definitely finite and surjective.

Now we prove that π separates orbits, i.e. for two closed points $x_1, x_2 \in X(k)$, if $G_{\mathrm{red}}(k) = G'$ and $G' \cdot x_1 \cap G' \cdot x_2 = \emptyset$, then $\pi(x_1) \neq \pi(x_2)$. Define

$$N_{\mathrm{red}} : A \xrightarrow{\mu^*_{\mathrm{red}}} R_{\mathrm{red}} \otimes_k A \xrightarrow{\mathrm{Nm}} A.$$

By the argument in the previous lectures for Chapter II, we can find $a \in A$ such that $a(g'x_1) = 1$, $a(g'x_2) = 0$ for all $g' \in G'$. Granting this, we obtain

$$N_{\text{red}}(a)(x_1) = 1, \quad N_{\text{red}}(a)(x_2) = 0.$$

From the commutative diagrams (with $\alpha \in R \otimes_k A$ arbitrarily fixed):

$$\begin{tikzcd}
A \arrow[r, "\mu^*"] \arrow[dr, "\mu^*_{\text{red}}"'] & R \otimes_k A \arrow[r, "l_\alpha"] \arrow[d] & R \otimes_k A \arrow[d] \\
& R_{\text{red}} \otimes_k A \arrow[r, "l_{\overline{\alpha}}"] & R_{\text{red}} \otimes_k A
\end{tikzcd}$$

We can verify that $N(a)(x_1) \neq 0$, $N(a)(x_2) = 0$. Their implications are that $l_{\overline{\alpha}}(x_1)$ is an isomorphism (hence is surjective), and $l_{\overline{\alpha}}(x_2)$ is not surjective, respectively. One can actually show

$$N_{\text{red}}(T-a)(x_1) = (T-1)^n, \quad N_{\text{red}}(T-a)(x_2) = T^n.$$

On the other hand, since $N(a) \in B$, it forces $\pi(x_1) \neq \pi(x_2)$.

By definition, $\pi_*(\mathscr{O}_X)^G$ is the kernel of the $\mathscr{O}_Y$-linear map

$$\pi_*\mathscr{O}_X \to \pi_*\mathscr{O}_X \otimes_k R, \quad f \mapsto \mu^*(f) - f \otimes 1.$$

Then $\pi_*(\mathscr{O}_X)^G$ is coherent on Y, and $\mathscr{O}_Y \cong \pi_*(\mathscr{O}_X)^G$. Finally, the universal property of (Y, π) naturally follows from the construction. This finishes the proof of (A). □

10.4. **Proof of Theorem (B).**

Proof. Given a coherent sheaf $\mathscr{F}$ on Y, we have a canonical isomorphism

$$\lambda : p_2^*(\pi^*\mathscr{F}) \to \mu^*(\pi^*\mathscr{F})$$

as the two composites

$$\begin{tikzcd}
G \times X \arrow[r, bend left, "\mu"] \arrow[r, bend right, "p_2"'] & X \arrow[r, "\pi"] & Y
\end{tikzcd}$$

are equal. One can verify that this defined a lift of μ to $\pi^*\mathscr{F}$, i.e., can check the diagram is commutative:

$$\begin{tikzcd}
p_3^*\pi^*\mathscr{F} \arrow[rr] \arrow[dr] & & \xi^*\pi^*\mathscr{F} \arrow[dl] \\
& \eta^*\pi^*\mathscr{F} &
\end{tikzcd}$$

Conversely, let $\mathscr{G}$ be a coherent sheaf on X and we have a lift of μ to $\mathscr{G}$. In case when $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$ are affine, $\mathscr{G} = \widetilde{N}$ for some A-module N. We define $\pi_*(\mathscr{G})^G$ to be the coherent $\mathscr{O}_Y$-module corresponding to the B-module

$$N^G = \{n \in N \mid \lambda(\underbrace{1 \otimes n}_{p_2^*(n)}) = \mu^*(n) = n \otimes_{A,\mu^*} 1 \in N \otimes_{A,\mu^*} (R \otimes_k A)\}.$$

We run this construction for all open affine G-stable subsets of X, and can define $\pi_*(\mathscr{G})^G$ in general.

Now we assume that the action of G on X is free. The requirement is to prove:

(1) π is flat; alternatively, $B \to A$ is flat;
(2) $G \times X \xrightarrow{\sim} X \times_Y X$ is an isomorphism;
(3) the functors

$$\mathsf{Mod}_{\mathscr{O}_Y} \to \mathsf{Mod}_{(G,\mathscr{O}_X)}, \quad \mathscr{F} \mapsto \pi^*\mathscr{F}$$

and

$$\mathsf{Mod}_{(G,\mathscr{O}_X)} \to \mathsf{Mod}_{\mathscr{O}_Y}, \quad \mathscr{G} \mapsto \pi_*(\mathscr{G})^G$$

are inverses to each other. For this, it suffices to show $T(\mathscr{G}) : \pi^*\pi_*(\mathscr{G})^G \to \mathscr{G}$ is an isomorphism for each $(G, \mathscr{O}_X)$-module $\mathscr{G}$.

Now we assume $X = \operatorname{Spec} A$ is affine. As the G-action is free, $(\mu, p_2) : G \times X \to X \times X$ is a closed immersion. Since it factors through $X \times_Y X$, we get a surjective k-algebra homomorphism

$$\varphi : A \otimes_B A \to R \otimes_k A, \quad a_1 \otimes a_2 \mapsto \mu^*(a_1)(1 \otimes a_2).$$

Then it boils down to prove that

(1') A is flat over $B = A^G$, and φ is injective;
(2') for each coherent (G, A)-module M, the natural map $M^G \otimes_B A \to M$ is an isomorphism;
(3') if M is a projective A-module, M^G is projective as a B-module.

We first explain that $(1')(2')$ imply $(3')$. It suffices to show that M^G is flat as a B-module, or equivalently, the functor

$$(\cdot) \otimes_B M^G : \mathsf{Mod}_B \to \mathsf{Mod}_B$$

is exact. Since $B \to A$ is faithfully flat by (1'), this is to prove that the functor

$$((\cdot) \otimes_B M^G) \otimes_B A : \mathsf{Mod}_B \to \mathsf{Mod}_A$$

is exact. For any B-module N, we have

$$\begin{aligned}(N \otimes_B M^G) \otimes_B A &\cong (N \otimes_B A) \otimes_A (A \otimes_B M^G)\\ &\cong (N \otimes_B A) \otimes_A M \qquad \text{by granting (2')}\\ &\cong N \otimes_B M.\end{aligned}$$

And since M is A-flat with A being B-flat, the functor is morally exact. Therefore, we are left to prove (1')(2').

(1') Replacing B by $B_\mathfrak{m}$ where $\mathfrak{m} \subset B$ is the maximal ideal and A by $A \otimes_B B_\mathfrak{m}$, we may assume, without loss of generality, that B is local and A is semi-local. Regard $A \otimes_B A$ and $R \otimes_k A$ as A-algebras via the second factor. The map

$$\varphi : A \otimes_B A \to R \otimes_k A, \quad a_1 \otimes a_2 \mapsto \mu^*(a_1)(1 \otimes a_2)$$

is a homomorphism of A-algebras. Since φ is onto, $R \otimes_k A$ is generated by $\mu^*(a)$ with $a \in A$ as an A-algebra. Since A is semi-local one can find $\{a_1, \ldots, a_n\}$ in A such that $\{\mu^*(a_i) \mid 1 \leqslant i \leqslant n\}$ form a basis of $R \otimes_k A$ as an A-module.[5]

[5] Here are more details about this step of argument. Since $R \otimes_k A$ is a free A-module of rank n, it suffices to show that $\{\mu^*(a_i) \mid 1 \leqslant i \leqslant n\}$ generates $R \otimes_k A$ as an A-module for some suitable $\{a_i\}_{1 \leqslant i \leqslant n}$. By Nakayama's lemma, it reduces to the case where $A = \prod_{i=1}^m k$. We are to prove the following: if M is a free A-module of rank n, and a k-subspace $\Sigma \subset M$ is a set of generators of M, then there exist $x_1, \ldots, x_n \in \Sigma$

Claim. For $a, \lambda_1, \dots, \lambda_n \in A$,

$$(*) \qquad \mu^*(a) = \sum_{i=1}^{n} (1 \otimes \lambda_i) \cdot \mu^*(a_i) \quad \Longleftrightarrow \quad a = \sum_{i=1}^{n} \lambda_i \cdot a_i \text{ with } \lambda_1, \dots, \lambda_n \in B.$$

For ($\Leftarrow$), apply μ^* to $a = \sum_{i=1}^n \lambda_i \cdot a_i$ and use the fact that $\mu^*(\lambda_i) = 1 \otimes \lambda_i$ as $\lambda_i \in B$. For ($\Rightarrow$), since $(1_R \otimes \mu^*)(\mu^* a) = (m^* \otimes 1_A)(\mu^* a)$ in $R \otimes_k R \otimes_k A$, we have

$$\begin{aligned} &\sum_{i=1}^{n} (1 \otimes \mu^*(\lambda_i))(1_R \otimes \mu^*)(\mu^*(a_i)) \\ =& \sum_{i=1}^{n} (1 \otimes 1 \otimes \lambda_i)(m^* \otimes 1_A)(\mu^*(a_i)) \\ =& \sum_{i=1}^{n} (1 \otimes 1 \otimes \lambda_i)(1_R \otimes \mu^*)(\mu^*(a_i)). \end{aligned}$$

Since $\{\mu^(a_i) \mid 1 \leqslant i \leqslant n\}$ is a basis of $R \otimes_k A$ as an A-module, $(1_R \otimes \mu^*)(\mu^*(a_i))$ is a basis of $R \otimes_k R \otimes_k A$ as an $R \otimes_k A$-module via the last two factors. (Here we have used $(R \otimes_k A) \otimes_{A,\mu^*} (R \otimes_k A) \cong R \otimes_k R \otimes_k A$.) Thus, in $R \otimes_k R \otimes_k A$,

$$1 \otimes \mu^*(\lambda_i) = 1 \otimes 1 \otimes \lambda_i.$$

So all λ_i's land in B. Apply $\varepsilon \otimes 1$ to $\mu^*(a) = \sum_{i=1}^n (1 \otimes \lambda_i)(\mu^*(a_i)$, we have $a = \sum_{i=1}^n \lambda_i \cdot a_i$. So we have proved $(*)$. However, $(*)$ implies that A is a free B-module with basis $\{a_1, \dots, a_n\}$. This shows A is flat over B. Moreover, the A-linear map $\varphi : A \otimes_B A \to R \otimes_k A$ is a map between free A-modules of rank n and takes a basis $\{a_i \otimes 1\}$ to a basis $\{\mu^*(a_i)\}$. So φ is an isomorphism.

(2') Morally, this follows from the general descent theory. We only list out a sketch. View $M \otimes_B A$ and $A \otimes_B M$ as $A \otimes_B A$-modules in the obvious way. Then a G-action on M is an isomorphism of $A \otimes_B A$-modules $\tau : A \otimes_B M \to M \otimes_B A$ such that

$$\begin{tikzcd} & A \otimes A \otimes M \arrow[dl, "1_A \otimes \tau"'] \arrow[dd, "(p_1 \circ \tau) \otimes 1_A \otimes (p_2 \circ \tau)"] \\ A \otimes M \otimes A \arrow[dr, "\tau \otimes 1_A"'] & \\ & M \otimes A \otimes A \end{tikzcd}$$

Note that the right vertical map is given by τ on the first and the third factors together with 1_A on the second factor.

such that $\{x_1, \dots, x_n\}$ is a basis of M as an A-module. To see this, one can use induction on $n = \operatorname{rank}_A M$. When $n = 1$, it suffices to find an element $x \in \Sigma$ such that if $x = (x^1, \dots, x^m)$ then $x^i \neq 0$ for all $i = 1, \dots, m$. We can prove this by induction on m and use the fact that $k = \overline{k}$ is algebraically closed. And hence k is infinite. In general, suppose the statement holds for n and M is a free A-module of rank $n+1$. Then one may find $x_1 \in M$ if we write $x_1 = (x_1^1, \dots, x_1^m)$ under the decomposition $A = \prod_{i=1}^m k$. Thus $x_1^i \neq 0$ for each $i = 1, \dots, m$, i.e., $Ax_1 \subset M$ is a free A-submodule of rank 1. Since A is isomorphic to m-copies of k, any (finitely generated) A-module is locally free and hence projective. Therefore, there exists $M_1 \subset M$ that is free of rank n such that $M = Ax_1 \oplus M_1$. Apply the inductive hypothesis to M_1 and get the desired $\{x_1, \dots, x_{n+1}\}$.

Define

$$N = \{m \in M \mid \tau(1 \otimes m) = m \otimes 1\}.$$

We need to show that $N \otimes_B A \to M$ is an isomorphism. Notice that

$$N = \mathrm{Ker}(\phi : M \to M \otimes_B A), \quad m \mapsto m \otimes 1 - \tau(1 \otimes m)$$

and $B \to A$ is flat, we have

$$N \otimes_B A = \left\{ \sum_i m_i \otimes a_i \in M \otimes_B A \,\middle|\, \sum_i m_i \otimes 1 \otimes a_i = \sum_i \tau(1 \otimes m_i) \otimes a_i \right\}.$$

Applying the commutative diagram above to $1 \otimes 1 \otimes m \in A \otimes A \otimes M$, we have

$$\tau(1 \otimes M) \subset N \otimes_B A$$

as subsets in $M \otimes_B A$. We view $M \otimes_B A$ as an A-module via the second factor, then $\tau(1 \otimes M)$ and $N \otimes_B A$ are A-submodules of $M \otimes_B A$; and $N \otimes_B A$ is generated by those $n \otimes 1$ with $n \in N$. Since $n \otimes 1 = \tau(1 \otimes n) \in \tau(1 \otimes M$, we have $M \otimes_B A = \tau(1 \otimes M)$. On the other hand, as $B \to A$ is faithfully flat, the map $M \to A \otimes_B M$ is injective and we have an isomorphism

$$\begin{tikzcd} M \arrow[r] \arrow[rr, bend left, "\sim"] & 1 \otimes M \arrow[r, "\tau"] & \tau(1 \otimes M) \\ m \arrow[r, mapsto] & 1 \otimes m & \end{tikzcd}$$

so we get a canonical isomorphism $N \otimes_B A \cong M$.

We have accomplished the proof of (B). $\square$

Interlude: On Seesaw's Theorem

This is a preliminary part of the upcoming lecture which recalls and generalizes the classical Seesaw's theorem we have mentioned in Chapter II.

Theorem 10.7 (Seesaw)**.** *Let X be a complete variety, T any variety, and $\mathscr{M}$ a line bundle on $X \times T$. Then the set*

$$T_1 = \{t \in T \mid \mathscr{M}|_{X\times\{t\}} \text{ is trivial on } X \times \{t\}\}$$

is closed in T, and if $p_2 : X \times T_1 \to T_1$ is the second projection, then $\mathscr{M}|_{X\times T_1} \cong p_2^*\mathscr{N}$ for some line bundle $\mathscr{N}$ on T_1. Also, T_1 has the reduced closed subscheme structure.

Proposition 10.8 (Generalized Seesaw)**.** *Let X be a complete variety, Y a scheme, and $\mathscr{M}$ a line bundle on $X \times Y$. Then there exists a unique and closed subscheme $Y_1 \hookrightarrow Y$ with the following properties.*

(1) *If $\mathscr{M}_1 = \mathscr{M}|_{X\times Y_1}$, there is a line bundle $\mathscr{N}_1$ on Y_1 and an isomorphism $p_2^*\mathscr{N}_1 \cong \mathscr{M}_1$ on $X\times Y_1$; or alternatively, if $i_1 : Y_1 \hookrightarrow Y$ denotes the first natural closed immersion, we obtain $p_2^*\mathscr{N}_1 \cong (1_X \times i_1)^*\mathscr{M}$.*

(2) *If $f : Z \to Y$ is a morphism such that there exists a line bundle $\mathscr{K}$ on Z and an isomorphism $p_2^*\mathscr{K} \cong (1_X \times f)^*\mathscr{M}$ on $X \times Z$, then f factors as*

$$f : Z \to Y_1 \hookrightarrow Y.$$

Remark 10.9. For any closed point $y_1 \in Y_1(k)$, we have

$$\mathscr{M}|_{X\times\{y_1\}} \cong \mathscr{M}_1|_{X\times\{y_1\}} \cong (p_2^*\mathscr{N}_1)|_{X\times\{y_1\}}$$

being trivial. So the closed subvarieties given by the above two Seesaw's are homeomorphic as topological spaces. But the closed subscheme Y_1 in the second proposition may have nonreduced closed subscheme structure so that the universal property (2) holds.

We refer the closed subscheme Y_1 of Y in Proposition 10.8 as the maximal closed subscheme of Y over which $\mathscr{M}$ is trivial. (Caveat: the notation is a little misleading as $\mathscr{M}|_{X\times Y_1}$ is NOT a trivial line bundle in general. Sorry for this!)

11. The Dual Abelian Variety in Any Characteristic

In Chapter II, we defined a reduced closed subscheme $K(\mathscr{L})$ of X, for every line bundle $\mathscr{L}$ on an abelian variety X, i.e.,

$$K(\mathscr{L}) = \{x \in X(k) \mid T_x^*\mathscr{L} \cong \mathscr{L}\}.$$

We want to make $K(\mathscr{L})$ as a (nonreduced) closed subgroup scheme of X.

Definition 11.1. Consider the line bundle $\mathscr{M} = m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^*\mathscr{L}^{-1}$ on $X \times X$. We define $K(\mathscr{L})$ to be the maximal closed subscheme of X such that $\mathscr{M}|_{X\times K(\mathscr{L})}$ is trivial.

Remark 11.2. We apply the generalized Seesaw theorem (Proposition 10.8) to $\mathscr{M} \in \mathrm{Pic}(X \times X)$ and get a line bundle $\mathscr{N}_1$ on $K(\mathscr{L})$ and an isomorphism $p_2^*\mathscr{N}_1 \cong \mathscr{M}|_{X\times K(\mathscr{L})}$. But $\mathscr{N}_1 \cong (p_2^*\mathscr{N}_1)|_{\{e_X\}\times K(\mathscr{L})} \cong \mathscr{M}|_{\{e_X\}\times K(\mathscr{L})}$ is trivial as $\mathscr{M}|_{\{e_X\}\times K(\mathscr{L})}$ is trivial. So that $\mathscr{M}|_{X\times K(\mathscr{L})}$ is trivial as well.

In the upcoming context we are to verify that $K(\mathscr{L})$ is a *closed subgroup scheme* of X. Recall we have defined the "translation by f" morphism, say T_f, as an automorphism over S as follows:

$$\begin{tikzcd} X \times S =: X_S \arrow[rr, "T_f"] \arrow[dr] & & X_S \arrow[dl] \\ & S & \end{tikzcd}$$

Also, $p_1 \circ T_f : X_S \to X$ is the composite

$$X \times S \xrightarrow{1_X \times f} X \times X \xrightarrow{m} X.$$

Here X is a commutative group scheme, so there is no difference between left and right translations.

Lemma 11.3. *Set $\mathscr{L}_S = p_1^*\mathscr{L} \in \mathrm{Pic}(X_S)$. Then $f \in K(\mathscr{L})(S)$ if and only if $T_f^*\mathscr{L}_S \cong \mathscr{L}_S \otimes p_2^*\mathscr{N}$ for some $\mathscr{N} \in \mathrm{Pic}(S)$.*

Proof. By direct computation, we have

$$T_f^*\mathscr{L}_S = T_f^*p_1^*\mathscr{L} \cong (1_X \times f)^*(m^*\mathscr{L}),$$

and hence $T_f^*\mathscr{L}_S|_{\{e_X\}\times S} \cong f^*\mathscr{L}$; the restriction $\mathscr{L}_S|_{\{e_X\}\times S}$ is trivial. So if $T_f^*\mathscr{L}_S \cong \mathscr{L}_S \otimes p_2^*\mathscr{N}$ for some $\mathscr{N} \in \mathrm{Pic}(S)$, by restricting to $\{e_X\} \times S$, we have $\mathscr{N} \cong f^*\mathscr{L}$. Hence

$$\begin{aligned} & T_f^*\mathscr{L}_S \cong \mathscr{L}_S \otimes p_2^*\mathscr{N} \\ \Longleftrightarrow\ & (1_X \times f)^*m^*\mathscr{L} \cong p_1^*\mathscr{L} \otimes p_2^*f^*\mathscr{L} \\ \Longleftrightarrow\ & (1_X \times f)^*m^*\mathscr{L} \otimes p_1^*\mathscr{L}^{-1} \otimes p_2^*(f^*\mathscr{L})^{-1} \cong (1_X \times f)^*\mathscr{M} \text{ is trivial on } X \times S \\ \Longleftrightarrow\ & f \text{ factors through } K(\mathscr{L}). \end{aligned}$$

This is equivalent to say $f \in K(\mathscr{L})(S)$. $\square$

It follows from Lemma 11.3 that $K(\mathscr{L})(S)$ is a subgroup of $X(S)$.

Hence $K(\mathscr{L})$ is a subgroup scheme of X. Now we are ready to construct the dual abelian variety over any characteristic. Fix an ample line bundle $\mathscr{L}$ on X. Then $K(\mathscr{L})$ is a closed finite subgroup scheme of X. Define $\widehat{X} = X/K(\mathscr{L})$ where $K(\mathscr{L})$ acts on X via translation and $\pi : X \to \widehat{X}$ is the natural morphism. One can verify that $\widehat{X}$ is also an abelian variety and π is an isogeny of abelian varieties, i.e., a finite surjective homomorphism. Consequently,

$$\widehat{X}(k) \xrightarrow{\sim} X(k)/K(\mathscr{L})(k) \xrightarrow[\phi_{\mathscr{L}}]{\sim} \mathrm{Pic}^0(X).$$

There is another isomorphism of abelian groups $\widehat{X}(k) \cong \mathrm{Pic}^0(X)$. As before, we want to define the Poincaré bundle $P \in \mathrm{Pic}(X \times \widehat{X})$ such that $(1_X \times \pi)^*(P) = \mathscr{M}$, via the isogeny $1_X \times \pi : X \times X \to X \times \widehat{X}$ with its kernel $K = 1 \times K(\mathscr{L})$. So it suffices to define a lift of the action of K on $X \times X$ to $\mathscr{M}$. More precisely, we need to find an isomorphism

$$\lambda : p_2^* \mathscr{M} \to \mu^* \mathscr{M}$$

where $p_2 : K \times (X \times X) \to X \times X$ is the natural projection. (Also recall that $\mu : K \times (X \times X) \to X \times X$ is the translation morphism.

In general, for a scheme S and an S-valued point $(e, x) : S \to K = 1 \times K(\mathscr{L})$ of K (so $x \in K(\mathscr{L})(S)$), let

$$T_{(e,x)} : X_S \times_S X_S \to X_S \times_S X_S$$

be the translation by $(e, x) \in K(S) \subset (X \times X)(S)$ and $T_x : X_S \to X_S$ be the translation by $x \in X(S)$. Let $\mathscr{M}_S$ be the inverse image of $\mathscr{M}$ under the projection $X_S \times_S X_S \cong S \times X \times X \to X \times X$ and $\mathscr{L}_S$ the image of $\mathscr{L}$ under $X_S = X \times S \to X$. Then we have $T_{(e,x)}^* \mathscr{M}_S \cong m_S^* T_x^* \mathscr{L}_S \otimes p_{1,S}^* \mathscr{L}_S^{-1}$. Since $x \in K(\mathscr{L})(S)$ we have an isomorphism

$$T_x^* \mathscr{L}_S \cong \mathscr{L}_S \otimes p_S^* \mathscr{N}, \qquad \text{for some } \mathscr{N} \in \mathrm{Pic}(S).$$

Here $p_S : X_S = X \times S \to S$ is the natural projection. Fix such an isomorphism and we obtain an isomorphism on $X_S \times_S X_S$:

$$\lambda_S : \mathscr{M}_S \xrightarrow{\sim} T_{(e,x)}^* \mathscr{M}_S.$$

In particular we take $S = 1 \times K(\mathscr{L}) = K$ and $(e, x) \in K(S)$ to be the identity morphism. We get an isomorphism

$$\lambda : p_2^* \mathscr{M} \to \mu^* \mathscr{M}$$

as before. Here λ cannot be chosen arbitrarily as it must satisfy some extra condition. In general, we want to have a "canonical" isomorphism $\lambda_S : \mathscr{M}_S \to T_{(e,x)}^* \mathscr{M}_S$ on $X_S \times_S X_S$ for all S. Fortunately, this can be done by restricting to $e_S \times_S S \hookrightarrow X_S \times_S X_S$. (Check this; as an exercise).

As a consequence, we obtain a Poincaré bundle P on $X \times \widehat{X}$ such that $P|_{\{e_X\} \times \widehat{X}}$ is trivial and for all $\alpha \in \widehat{X}(k)$, $P|_{X \times \{\alpha\}}$ corresponds to the element in $\mathrm{Pic}^0(X)$ under the isomorphism $\widehat{X}(k) \cong \mathrm{Pic}^0(X)$, i.e., $(\widehat{X}, P)$ satisfies the first property of Theorem 8.3, in Chapter II, that characterizes $\widehat{X}$. But some modification towards the second property is required. It generalizes as follows.

Theorem 11.4. *Let S be any scheme. Let $\mathscr{L} \in \mathrm{Pic}(X \times S)$ be such that $\mathscr{L}|_{\{e_X\} \times S}$ is trivial and $\mathscr{L}|_{X \times \{s\}} \in \mathrm{Pic}^0(X)$ for each closed point $s \in S(k)$. Then there exists a unique morphism $\phi : S \to \widehat{X}$ such that $\mathscr{L} \cong (1_X \times \phi)^* P$.*

Proof. As before, we consider the line bundle $\mathscr{M} = p_{13}^*P \otimes p_{12}^*\mathscr{L}^{-1}$ on $X \times S \times \widehat{X}$ and let Γ_S be the maximal closed subscheme of $S \times \widehat{X}$ over which $\mathscr{M}$ is trivial.[6] The goal is to show

$$f : \Gamma_S \hookrightarrow S \times \widehat{X} \xrightarrow{p_1} S$$

is an isomorphism. We know f is a homeomorphism on the underlying topological spaces. It suffices to show that for any closed subscheme S_0 of S such that $|S_0|$ is a single point of S. Then the morphism

$$f \times_S S_0 : \Gamma_S \times_S S_0 \to S_0$$

is an isomorphism. In fact, if this is valid, then f is bijective on closed points, and hence f is quasi-finite. Since f is proper, we have f being finite by the Zariski Main Theorem.

The statement follows from the fact. let $(A, \mathfrak{m})$ be a local ring and B a finite A-algebra. If $A/\mathfrak{m}^n \to B/\mathfrak{m}^n B$ is an isomorphism for any m, then $A \to B$ is an isomorphism. So we may assume $S = \operatorname{Spec} B$ where B is a finite local k-algebra and $S = \{s\}$ a single point. Moreover, we can assume that $\mathscr{L}|_{X \times \{s\}}$ is trivial. Consider the line bundle $\mathscr{M} = p_{13}^*P \otimes p_{12}^*\mathscr{L}^{-1}$ on $X \times S \times \widehat{X}$. Since $\mathscr{M}|_{\{e_X\} \times \{s\} \times \widehat{X}} \cong P|_{\{e_X\} \times \widehat{X}}$ is trivial (and hence belongs to $\operatorname{Pic}^0(\widehat{X})$, we have $\mathscr{M}|_{\{x\} \times \{s\} \times \widehat{X}} \in \operatorname{Pic}^0(\widehat{X})$ for all $x \in X(k)$. On the other hand,

$$\pi^*(\mathscr{M}|_{\{x\} \times \{s\} \times \widehat{X}}) \cong (T_x^*\mathscr{L}_a) \otimes \mathscr{L}_a^{-1},$$

where $\mathscr{L}_a$ is the ample line bundle on X we have chosen before to construct $\widehat{X}$. So there are only finitely many $x \in X(k)$ such that $\mathscr{M}|_{\{x\} \times \{s\} \times \widehat{X}}$ is trivial.

Since $H^i(\widehat{X}, \mathscr{L}_{\widehat{X}}) = 0$ for all $i \geqslant 0$ and $0 \neq \mathscr{L}_{\widehat{X}} \in \operatorname{Pic}^0(\widehat{X})$, the support of $R^i p_{12,*}\mathscr{M}$ on $X \times S$ is the disjoint union of finitely many closed points. So

$$H^n(X \times S, R^i p_{12,*}\mathscr{M}) = 0, \quad n \geqslant 1.$$

By the Leray spectral sequence

$$H^i(X \times S, R^j p_{12,*}\mathscr{M}) \Rightarrow H^{i+j}(X \times S \times \widehat{X}, \mathscr{M})$$

we have the canonical isomorphisms

$$H^n(X \times S \times \widehat{X}, \mathscr{M}) \cong H^0(X \times S, R^n p_{12,*}\mathscr{M}).$$

Now apply the projection formula (cf. [Har13, III, Exer 8.3]),

$$\begin{aligned} R^n p_{12,*}\mathscr{M} &= R^n p_{12,*}(p_{13}^*P \otimes p_{12}^*\mathscr{L}^{-1}) \\ &\cong R^n p_{12,*}p_{13}^*P \otimes \mathscr{L}^{-1} \\ &\cong R^n p_{12,*}p_{13}^*P. \end{aligned}$$

The last step above uses the fact that $R^n p_{12,*}p_{13}^*P$ has support on finitely many closed points. Therefore, in summary,

$$H^n(X \times S \times \widehat{X}, \mathscr{M}) \cong H^n(X \times S \times \widehat{X}, p_{13}^*P) \underset{\substack{\uparrow \\ \text{by flat base change theorem}}}{\boxed{\cong}} H^n(X \times \widehat{X}, P) \otimes_k B.$$

[6]Here recall the fact at work that $\mathscr{M}|_{\{e_X\} \times S \times \widehat{X}}$ is trivial.

In particular, $H^n(X \times S \times \widehat{X}, \mathscr{M})$ are free B-modules for all $n \geqslant 0$. Similarly, we consider the projection $p_{23} : X \times S \times \widehat{X} \to S \times \widehat{X}$. As $\mathscr{L}|_{X\times\{s\}}$ is trivial by our assumption, we get

$$\mathscr{M}|_{X\times\{s\}\times\{\alpha\}} \cong P|_{X\times\{\alpha\}} \otimes \mathscr{L}^{-1}|_{X\times\{s\}} \cong P|_{X\times\{\alpha\}},$$

which further implies that $M|_{X\times\{s\}\times\{\alpha\}} \in \mathrm{Pic}^0(X)$ for all $\alpha \in \widehat{X}(k)$ and it is trivial if and only if $\alpha = e_{\widehat{X}}$. We infer that $R^i p_{23,*}\mathscr{M}$ is supported at the point $(s, e_{\widehat{X}})$ of $S \times \widehat{X}$. Then

$$H^n(X \times S \times \widehat{X}, \mathscr{M}) \cong H^0(S \times \widehat{X}, R^n p_{23,*}\mathscr{M}) = (R^n p_{23,*}\mathscr{M})_{(s,e_{\widehat{X}})},$$

the stalk of the sheaf at the closed point $(s, e_{\widehat{X}})$. For simplicity, we use $\mathscr{O}$ to denote the stalk $\mathscr{O}_{\widehat{X},e_{\widehat{X}}}$ of $\widehat{X}$ at $e_{\widehat{X}}$. Then the stalk A of $S \times \widehat{X}$ at $(s, e_{\widehat{X}})$ is given by $B \otimes_k \mathscr{O}$. Consider the following Cartesian diagram

$$\begin{tikzcd}
\mathscr{M}_A \arrow[d, dash] & \mathscr{M} \arrow[l] \arrow[d, dash] \\
X \times \mathrm{Spec}\, A \arrow[r] \arrow[d, "p"'] & X \times S \times \widehat{X} \arrow[d, "p_{23}"] \\
\mathrm{Spec}\, A \arrow[r] & S \times \widehat{X}
\end{tikzcd}$$

and we have $R^i p_{23,*}\mathscr{M}|_{(s,e_{\widehat{X}})} \cong R^i p_* \mathscr{M}_A$. Since $p : X \times \mathrm{Spec}\, A \to \mathrm{Spec}\, A$ is proper and flat, and $\mathscr{M}_A$ is a line bundle on $X \times \mathrm{Spec}\, A$, by the main theorem (Theorem 5.2) in *Cohomology and base change*, there is a finite complex

$$K^\bullet : \quad 0 \to K^0 \to K^1 \to \cdots \to K^g \to 0$$

of finitely generated free A-modules such that

$$H^i(K^\bullet) \cong R^i p_{23,*}\mathscr{M}|_{(s,e_{\widehat{X}})} \cong R^i p_* \mathscr{M}_A = H^i(X \times \mathrm{Spec}\, A, \mathscr{M}_A),$$

where $g = \dim X = \dim \mathscr{O}$. This is crucial in the computation of cohomology groups of P and $\mathscr{O}_X$.[7]

Lemma 11.5. *Let $\mathscr{O}$ be a regular local ring of $\dim g$, and*

$$0 \to K^0 \to \cdots \to K^g \to 0$$

be a complex of finitely generated free $\mathscr{O}$-modules. If $H^i(K^\bullet)$ are artinian $\mathscr{O}$-modules, we have $H^i(K^\bullet) = 0$ for each $0 \leqslant i < g$.

Resume on. By this lemma, we see that $R^i p_{23,*}\mathscr{M} = 0$ for each $0 \leqslant i < g$, and we get an exact sequence of A-modules:

$$0 \to K^0 \to \cdots \to K^g \to N \to 0$$

such that $N = (R^g p_{23,*}\mathscr{M})_{(s,e_{\widehat{X}})} \cong H^g(X \times S \times \widehat{X}, \mathscr{M})$ which is a free B-module.

Now we apply $\mathrm{Hom}_A(\cdot, A)$ to the complex $K^\bullet$, and get another complex

$$\widehat{K}^\bullet : \quad 0 \to \widehat{K}^g \to \cdots \to \widehat{K}^0 \to 0$$

and by the lemma above, we get an exact sequence

$$0 \to \widehat{K}^g \to \cdots \to \widehat{K}^0 \to K \to 0$$

[7] Exercise: we only know $K^\bullet$ should be bounded by the theorem. Why is it bounded on $[0, g]$?

of A-modules. Since

$$\begin{aligned} H^0(K^\bullet \otimes_A k) &\cong H^0(X \times \{s\} \times \{e_{\widehat{X}}\}, \mathscr{M}|_{X\times\{s\}\times\{e_{\widehat{X}}\}}) \\ &\cong k = \operatorname{Ker}(K^0 \otimes_A k \to K^1 \otimes_A k), \end{aligned}$$

we see $K \otimes_A k = \operatorname{Ker}(\widehat{K}^1 \otimes_A k \to \widehat{K}^0 \otimes_A k)$ is 1-dimensional over k. Thus, for some ideal I, there is an isomorphism of A-modules $K \cong A/I$. Then one can show the closed subscheme Γ_S of $S \times \widehat{X}$ is the closed subscheme of $\operatorname{Spec} A$ defined by the ideal I and the map $B \to B \otimes_k \mathscr{O} = A \to A/I$ is an isomorphism. In other words, the composite $\Gamma_S \hookrightarrow S \times \widehat{X} \xrightarrow{p_1} S$ is an isomorphism. So we get a morphism

$$\phi : S \xrightarrow{\sim} \Gamma_S \xrightarrow{p_2} \widehat{X}$$

which is the unique morphism we need. $\square$

The importance of the proof is that it helps us to compute the cohomology groups of P and $\mathscr{O}_X$.

Corollary 11.6. *As for the cohomology groups of P, we have*

$$H^i(X \times \widehat{X}, P) = \begin{cases} 0, & i \neq g = \dim X; \\ k, & i = g = \dim X. \end{cases}$$

Proof. In the previous proof, we take $S = \operatorname{Spec} k$ and $\mathscr{L}$ is trivial. So that

$$H^n(X \times \widehat{X}, P) \cong H^n(K^\bullet), \quad n \geqslant 0.$$

In this case $\Gamma_S = \operatorname{Spec} k$ and $\phi : S \to \widehat{X}$ is given by $e_{\widehat{X}}$. Thus, $K \cong k$ and we have an exact sequence of A-modules:

$$0 \to \widehat{K}^g \to \widehat{K}^{g-1} \to \cdots \to \widehat{K}^0 \to k \to 0,$$

i.e., $\widehat{K}^\bullet$ is a free resolution of k. Since $\mathscr{O} = \mathscr{O}_{\widehat{X}, e_{\widehat{X}}}$ is a regular local ring of dimension g, the $\mathscr{O}$-module k has a standard resolution by free $\mathscr{O}$-modules that is called the Koszul complex $L_\bullet$ and is defined as follows.

Let $(x_1, \ldots, x_g)$ be a system of generators of $\mathscr{O}$. Take

$$L_k := \text{free } \mathscr{O}\text{-modules with basis } \{e_{i_1\cdots i_k} \mid 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant g\},$$

and the differentials

$$d_k : L_k \to L_{k-1}, \quad e_{i_1\cdots i_k} \mapsto \sum_{l=1}^{k} (-1)^l \chi_{i_l} e_{i_1\cdots \widehat{i_l} \cdots i_g}.$$

Then we have a resolution

$$0 \to L_g \to L_{g-1} \to \cdots \to L_0 \to k \to 0$$

of k. Hence $L_\bullet$ is homotopic to $\widehat{K}^\bullet$ as chain complexes. Therefore,

$$H^i(X \times \widehat{X}, P) \cong H^i(K^\bullet) \cong H_{g-i}(L_\bullet) = \begin{cases} 0, & i \neq g; \\ k, & i = g. \end{cases}$$

For more details, see [Mat80, §18]. $\square$

Corollary 11.7. *Let $g = \dim X$. Then*

$$\dim_k H^p(X, \mathscr{O}_X) = \binom{g}{p}.$$

Proof. Using the same notation as in the proof of Corollary 11.6 above. We have

$$H^p(X, \mathscr{O}_X) \cong H^p(K^\bullet \otimes_A k) \cong H^p(L_\bullet \otimes_A k) = L_{g-p}.$$

Hence

$$\dim_k H^p(X, \mathscr{O}_X) = \binom{g}{g-p} = \binom{g}{p}.$$

□

Corollary 11.8. *There is a canonical isomorphism between the tangent space at $e_{\widehat{X}}$ on $\widehat{X}$ and $H^1(X, \mathscr{O}_X)$. That is,*

$$\operatorname{Lie} \widehat{X} \cong H^1(X, \mathscr{O}_X).$$

Proof. Trivial. □

References

[Eis13] David Eisenbud. *Commutative algebra: with a view toward algebraic geometry*, volume 150 of *Graduate Text in Mathematics*. Springer Science & Business Media, 2013.

[Har13] Robin Hartshorne. *Algebraic geometry*, volume 52. World Publishing Co., Beijing, China, 2013.

[MA67] Jacob P. Murre and Siva Anantharaman. Lectures on an introduction to Grothendieck′s theory of the fundamental group. 1967.

[Mat80] Hideyuki Matsumura. *Commutative algebra*. Benjamin/Cummings Pub. Co, Reading, Mass, 2nd edition, 1980.

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