Comments for *Algebraic Geometry* with EGA & SGA

RELATIONS BETWEEN FINITENESS CONDITIONS

(1) **Qcqsness $\iff$ quasi-compactness and quasi-separatedness**.
For a scheme X, qcqsness is equivalent to say X can be covered by finitely many affine open subsets, satisfying that the intersection any two of these open subsets can be covered by finitely many affine open subsets as well. In other words, we may describe the geometry on X by using finitely many rings together with finitely many homomorphisms between them.
A morphism $f : X \to Y$ is qcqs if and only if for any open affine subset $V \subseteq Y$, the open subscheme $f^{-1}(V) \subseteq X$ is qcqs.

(2) **Finite type $\iff$ local finite type and quasi-compactness** (c.f. [EGA I, §6.3]).
If $f : X \to Y$ is of finite type and Y is noetherian, then so also is X. Be careful that an open immersion is not necessarily of finite type.
Fix a scheme X and consider the category $\mathsf{Sch}_{/X}$ of all schemes over X of finite type. It is equivalent to a small category (i.e. the collections of objects and morphisms are both sets). When X is locally noetherian, this category contains all open subsets of X. When X is quasi-separated, this category at least contains all quasi-compact subsets of X (and hence all affine open subsets). One can introduce an appropriate Grothendieck topology[1] on $\mathsf{Sch}_{/X}$ to define abelian group sheaves. These abelian group sheaves lead to a nice abelian category, which admits a cohomology theory on it.[2]

(3) **Finite presentation $\iff$ qcqsness and local finite presentation**.
Usually, if a proposition only concerns about noetherian schemes and morphisms of finite type, then it can be generalized to a result about qcqs schemes and finitely presented morphisms, by taking the projective limits. This is due to the following theorem.

Theorem 1. *Let X be a qcqs scheme. Then there exists a filtered projective system $(X_i)_{i\in I}$, where each X_i is a $\mathbb{Z}$-scheme of finite type, the connected morphisms are affine and such that $X \simeq \varprojlim_i X_i$.*

The proof goes more complicated and we choose to omit. Anyway, granting the theorem, we may only concern about noetherian schemes (instead of qcqs schemes) and apply the noetherian condition more or less.

Remark 2. Affine morphisms are qcqs. Finite morphisms are of finite type.

School of Mathematical Sciences, Peking University, 100871, Beijing, China
Email address: `daiwenhan@pku.edu.cn`

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[1]A Grothendieck topology is not a classical topology. It is determined by two types of data called admissible opens and admissible open covers.

[2]Historically, an important issue here is that whenever X is not quasi-separated, one may need to seek for a small category with sufficiently many open subsets involved.