A purity result for semi-stable local systems

Heng Du

<u>Motivation</u>

K p-adic field $\supset \mathcal{O}_K \ni \pi$

$$\mathrm{Rep}_{\mathbb{Z}_p}(G_K) \supseteq \mathrm{Rep}^{\mathrm{cR}}_{\mathbb{Z}_p}(G_K) \supseteq \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_p}(G_K) \supseteq \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Z}_p}(G_K).$$

semi-stable.

$\mathfrak{X} / \mathrm{Spf}\, \mathcal{O}_K$ semistable formal sch.

i.e. p-complete étale locally, $\mathfrak{X} = \mathrm{Spf}\, R$

s.t. $\exists$ p-complete étale map

$$R_0 = \mathcal{O}_K \langle T_1, \dots, T_m, T_{m+1}^{\pm 1}, \dots, T_n^{\pm 1} \rangle / (T_1 \cdots T_m - \pi) \to R.$$

& $X = \mathfrak{X}_\eta^{ad}$ / $\mathrm{Spa}\, K$ sm adic space.

$\rightsquigarrow \mathrm{Loc}_{\mathbb{Z}_p}(X_\eta, \bar{\eta})$.

Def (Faltings) $\mathbb{L} \in \mathrm{Loc}^{\mathrm{st}}_{\mathbb{Z}_p}(X, \mathfrak{X})$, $\mathfrak{X}_1 := \mathfrak{X} \otimes_{\mathcal{O}_K} (\mathcal{O}_K/\pi)$ mod π fibre.

$M_{\mathfrak{X}} : \mathcal{O}_{\mathfrak{X}} \cap (\mathcal{O}_{\mathfrak{X}}[\frac{1}{p}])^\times \longrightarrow \mathcal{O}_{\mathfrak{X}}$ log str on $\mathfrak{X}$

$\rightsquigarrow M_{\mathfrak{X}_1}$ log str on $\mathfrak{X}_1$.

$(\mathfrak{X}_1, M_{\mathfrak{X}_1})_{\mathrm{CRIS}} \ni (\mathcal{U}, T, \gamma)$

where $T \leftarrow \mathcal{U} \xrightarrow{f} \mathfrak{X}$

log PD-thickening

s.t. T has a log str given by M_T

s.t. $(\mathcal{U}, M_T|_{\mathcal{U}}) \xrightarrow{f} (\mathfrak{X}, M_{\mathfrak{X}})$ strict morph

Define cats of crystals & isocrystals

$CR((\mathfrak{X}_1, \mathcal{M}_{\mathfrak{X}_1}))$ in f.g. QCoh $\mathcal{O}$-mod

$\Downarrow$

obj $\mathcal{F}$ in $\mathcal{O}$-mod s.t.

- $\mathcal{F}_T$ is f.g. $\mathcal{O}$-mod
- $(U, T, \gamma) \xrightarrow{\varphi} (U', T', \gamma')$, $\varphi^* \mathcal{F}_{T'} \otimes \mathcal{O}_T \xrightarrow{\sim} \mathcal{F}_T$ isom.

$\rightsquigarrow$ $\mathrm{Vect}^{\varphi}_{\mathbb{Q}}((\mathfrak{X}_1, \mathcal{M}_{\mathfrak{X}_1}))$ fin loc free F-isocrystal

obj $= (\mathcal{F}_{\mathbb{Q}}, \varphi_{\mathcal{F}_{\mathbb{Q}}})$ w/ $\varphi_{\mathcal{F}_{\mathbb{Q}}} : F^*_{\mathfrak{X}_1} \mathcal{F}_{\mathbb{Q}} \xrightarrow{\sim} \mathcal{F}_{\mathbb{Q}}$.

where $F_{\mathfrak{X}_1} : \mathfrak{X}_1 \to \mathfrak{X}_1$.

fin loc free means:

$\forall (U, T = \mathrm{Spf}\, A, \gamma)$ s.t. A $\mathbb{Z}_p$-flat

$\mathcal{F}_{\mathbb{Q}}(U, T, \gamma)$ = loc free $A[\frac{1}{p}]$-mod.

Let $\mathbb{L} \in \mathrm{Loc}_{\mathbb{Z}_p}(X)$, $\mathbb{L}$ is $\mathfrak{X}$-semistable

$\exists$ $(\mathcal{F}_{\mathbb{Q}}, \varphi_{\mathcal{F}_{\mathbb{Q}}}) \in \mathrm{Vect}^{\varphi}_{\mathbb{Q}}(\mathfrak{X}_1, \mathcal{M}_{\mathfrak{X}_1})_{CRIS}$.

- $\mathcal{F}_{\mathbb{Q}}(\mathbb{B}_{cris})(U) := \mathcal{F}_{\mathbb{Q}}(\underbrace{A_{cris}(A^+), (p, \mathrm{Fil}^{\geq 1})}_{\in\, (\mathfrak{X}_1, \mathcal{M}_{\mathfrak{X}_1})_{CRIS}})$ $U = \mathrm{Spa}(A, A^+) \in X_{proet}$ affinoid

Input (Min-Wang) log str:

If $\mathrm{Spf}(A_{inf}(A^+)/\ker\theta) \to \mathfrak{X}$,

then $\exists!$ log str on $\mathrm{Spf}(A_{inf}(A^+))$ s.t. this is strict.

N.B. isocrystal + $\mathbb{B}_{cris}$ is a sheaf

$\Rightarrow$ $\mathcal{F}_{\mathbb{Q}}(\mathbb{B}_{cris})$ is a sheaf of $\mathbb{B}_{cris}$-mod on X_{proet}.

$\mathcal{F}_{\mathbb{Q}} \rightsquigarrow \varphi_{\mathcal{F}_{\mathbb{Q}}}$.

Have $\alpha : \mathbb{L} \otimes \mathbb{B}_{cris} \xrightarrow[\sim]{\varphi} \mathcal{F}_{\mathbb{Q}}(\mathbb{B}_{cris})$ of $\mathbb{B}_{cris}$-mod.

Rmk (1) Def depends on $\mathfrak{X}_1$ (Dwork's trick) $\Leftarrow$ $\mathfrak{X}_S := \mathfrak{X} \otimes_{\mathcal{O}_K} \mathcal{O}_K/\pi$

(2) semistable $\Rightarrow$ de Rham

$\mathfrak{X} \rightsquigarrow Sp_{\mathfrak{X}} : |X| \longrightarrow |\mathfrak{X}| = |\mathfrak{X}_1|$

Def $x \in |X|$ rk 1 pt is called $\mathfrak{X}$-Shilov pt if $Sp_{\mathfrak{X}}(x)$ is a generic pt in $|\mathfrak{X}|$.

This is inspired by

Aside def (Bhatt-Hansen) $X \supset \mathcal{U} = Spa(A, A^+) \ni x$ rk 1.

$x \in |X|$ weak Shilov pt if $\exists$ $\mathcal{U}$ affinoid

$Sp_{A^\circ} : \mathcal{U} \to |Spa(A^\circ/A^{\circ\circ})|$

$x \longmapsto$ generic pt.

Thm 1 (Purity, Du-Liu-Moon-Shimizu)

For $\mathbb{L} \in Loc_{\mathbb{Z}_p}(X)$, TFAE:

(1) $\mathbb{L}$ is $\mathfrak{X}$-semistable

(Morita: $\mathcal{O}\mathbb{B}_{st}$-adm rep)

(2) $\mathbb{L}|_{\overline{K(x)}}$ is semistable as Gal rep of $Gal(\overline{K(x)}/K(x))$ for x weakly Shilov pt.

(3) $\mathbb{L}|_{\overline{K(\eta)}}$ is semistable as Gal rep of $Gal(\overline{K(x)}/K(x))$ for $\mathfrak{X}$-Shilov pt.

Note (3) $\Rightarrow$ purity for semistable loc sys.

Rmk · $\mathfrak{X}$-semistable is indep of choice of $\mathfrak{X}$.

· Being shilov pt ($\Leftrightarrow$ generic pt) is a top property

· Can check this property étale locally

$$\begin{tikzcd}
\mathrm{Loc}_{\mathbb{Z}_p}(X) \arrow[r, "\sim", "\text{Min-Wang}" above=8pt, "+\text{Bhatt-Scholze}"'] \arrow[d, "\mathrm{res}"] & \mathrm{Vect}((\mathfrak{X}, M_{\mathfrak{X}})_{\Delta}, \mathcal{O}_{\Delta}[\tfrac{1}{I}]^{\wedge}_{p})^{\varphi=1} \arrow[d] & \mathrm{Vect}^{an,\varphi}((\mathfrak{X}, M_{\mathfrak{X}})_{\Delta}, \mathcal{O}_{\Delta}) \arrow[l, "T_{\mathfrak{X}}"'] \arrow[d] \\
\mathrm{Loc}_{\mathbb{Z}_p}(K(\eta)) \arrow[r, "\sim"] & \mathrm{Vect}((\mathcal{O}_{K(\eta)}, \mathbb{N})^{\wedge}_{\Delta}, \mathcal{O}_{\Delta}[\tfrac{1}{I}]^{\wedge}_{p})^{\varphi=1} & \mathrm{Vect}^{\varphi}((\mathcal{O}_{K(\eta)}, \mathbb{N})^{\wedge}_{\Delta}, \mathcal{O}_{\Delta}) \arrow[l, "T_{\eta}"']
\end{tikzcd}$$

$$\ni \mathcal{M}$$

<u>Thm 1'</u> (Prismatic purity theory)

$\mathcal{M}$ in ess image of $T_{\mathfrak{X}}$

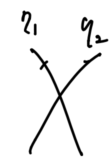
$\Leftrightarrow$ $\mathcal{M}|_{\mathcal{O}_{K(\eta)}}$ in ess image of T_{η}.

<u>Thm 2</u> T_η indeed an equiv of

$$\mathrm{Vect}^{\varphi}((\mathcal{O}_{K(\eta)}, \mathbb{N})^{\wedge}_{\Delta}, \mathcal{O}_{\Delta}) \xrightarrow{\sim} \mathrm{Rep}^{st}_{\mathbb{Z}_p}(G_{K(\eta)}).$$

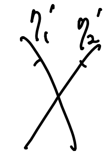
$\ulcorner$ OB_{st}-adm

<u>Example</u> $R = \mathbb{Z}_p\langle x, y\rangle / (xy - p^2) \rightsquigarrow X = \mathrm{Spa}\ \mathbb{Q}_p\langle \frac{p^2}{x}, x\rangle$



$\downarrow$ $x \mapsto (x')^2,\ y \mapsto (y')^2$ $\qquad f \uparrow 2:1$

$R' = \mathbb{Z}_p\langle x', y'\rangle / (x'y' - p) \rightsquigarrow X' = \mathrm{Spa}\ \mathbb{Q}_p\langle \frac{p}{x'}, x'\rangle$



$\mathfrak{X} = \mathrm{Spf}\ R.$

For $\mathbb{L} = f_* \mathbb{Q}_p$,

· $\mathbb{L}_{\bar{\eta}_i}$ (when $p>2$) semistable $K(\eta_i')/K(\eta_i)$ unram.

- $$\begin{tikzcd}[row sep=small] x : X = Y = p \\ x' : X' = Y' = \sqrt{p} \arrow[u, squiggly] \end{tikzcd}$$

H_x is not semistable.