Exercise 5 (due on December 23)

Choose 4 out of 8 problems to submit. (The problems are chronically ordered by the materials.) Let $\ell \geq 3$ be a prime number.

Problem 5.1. (Fontaine–Laffaille modules are weakly admissible) Let M be a $\mathbb{Z}_\ell$-free Fontaine–Laffaille module, that is a tuple $(M, \mathrm{Fil}^\bullet M, (\Phi^i)_i)$, consisting of

- a finite free $\mathbb{Z}_\ell$-module M,
- equipped with a decreasing filtration $\mathrm{Fil}^\bullet M$ by saturated $\mathbb{Z}_\ell$-submodules $\mathrm{Fil}^i M$ (i.e. $M/\mathrm{Fil}^i M$ is still a free $\mathbb{Z}_\ell$-module) such that $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{\ell-1} M = 0$; and
- $\Phi^i : \mathrm{Fil}^i M \to M$ are $\mathbb{Z}_\ell$-linear maps such that $\Phi^i|_{\mathrm{Fil}^{i+1}M} = \Phi^{i+1}$, and $M = \sum_i \Phi^i(\mathrm{Fil}^i M)$.

Prove the following statements:

(1) Choose a basis of M as $\{e_{0,1}, \dots, e_{0,r_0}, e_{1,1}, \dots, e_{1,r_1}, e_{2,1}, \dots, e_{\ell-2,r_{\ell-2}}\}$ so that

$$\begin{aligned}
\mathrm{Fil}^{\ell-2} M &= \mathbb{Z}_\ell e_{\ell-2,1} \oplus \cdots \oplus \mathbb{Z}_\ell e_{\ell-2,r_{\ell-2}}; \\
\mathrm{Fil}^{\ell-3} M &= \mathrm{Fil}^{\ell-2} M \oplus \mathbb{Z}_\ell e_{\ell-3,1} \oplus \cdots \oplus \mathbb{Z}_\ell e_{\ell-3,r_{\ell-3}}; \\
\cdots & \quad \cdots \\
\mathrm{Fil}^0 M &= \mathrm{Fil}^1 M \oplus \mathbb{Z}_\ell e_{0,1} \oplus \cdots \oplus \mathbb{Z}_\ell e_{0,r_0} = M.
\end{aligned}$$

Show that with respect to this basis, the matrix F for Φ^0 takes the form of

$$F = \mathrm{Diag}\big(\underbrace{1, \dots, 1}_{r_0}, \underbrace{\ell, \dots, \ell}_{r_1}, \underbrace{\ell^2, \dots, \ell^2}_{r_2}, \dots, \underbrace{\ell^{\ell-2}, \dots, \ell^{\ell-2}}_{r_{\ell-2}}\big) \cdot B$$

for a matrix $B \in \mathrm{GL}(\mathbb{Z}_\ell)$. (Note that it is not difficult to show that $B \in \mathrm{M}(\mathbb{Z}_p)$. To see that B is invertible, one needs to use the condition $M = \sum_i \Phi^i(\mathrm{Fil}^i M)$.)

(2) Deduce from (1) that viewing $M \otimes \mathbb{Q}_\ell$ as a filtered ϕ-module (with $\phi = \Phi^0$), the Newton degree of $M \otimes \mathbb{Q}_\ell$ is the same as the Hodge degree.

(3) Let N be a saturated submodule of M that is stable under the Φ^i-actions. Then the filtration of M induces a filtration on N. Show that N admits similar basis as in M so that Φ^0 takes a form similar to F above except that we cannot ensure that the corresponding B is an invertible integral basis.

Using this, prove that $M \otimes \mathbb{Q}_\ell$ is a weakly admissible filtered φ-module.

Problem 5.2. (Deformation of Fontaine–Laffaille modules) Let $\overline{M}$ denote the following Fontaine–Laffaille module: $\overline{M} = \mathbb{F}_\ell e_1 \oplus \mathbb{F}_\ell e_2$, with $\mathrm{Fil}^0 \overline{M} = \overline{M} \supset \mathrm{Fil}^1 \overline{M} = \mathbb{F}_\ell e_1 \supset \mathrm{Fil}^2 \overline{M} = 0$; $\Phi^1(e_1) = e_2$ and $\Phi^0(e_2) = e_1$.

Consider the following deformation functor:

$$\begin{aligned}
\mathrm{Def}_{\overline{M}} : \mathsf{CNL}_{\mathbb{Z}_\ell} &\longrightarrow \mathsf{Sets} \\
A &\longmapsto \big\{\text{Fontaine–Laffaille module } M \text{ such that } M \otimes_A \mathbb{F}_\ell \cong \overline{M}\big\}/\sim
\end{aligned}$$

(Note that this an unframed deformation.) Show that $\mathrm{Def}_{\overline{M}}$ is represented by a two-variable formal power series over $\mathbb{Z}_\ell$. (Had we fixed the determinant, it would be only one-variable.)

Problem 5.3. (Some computation from Weil's conjecture) This is partially copied from Hartshorne GTM 52's Appendix C. Let X denote a projective smooth variety of dimension n over a finite field $\mathbb{F}_q$. Let $|X|$ denote the set of closed points of X; and for $x \in |X|$, write

$\deg(x) = [k_x : \mathbb{F}_q]$ for the degree at x. Define the zeta function of X to be

$$\zeta_X(t) := \prod_{x \in |X|} \frac{1}{1 - t^{\deg(x)}} \in \mathbb{Q}[\![t]\!].$$

Here t is a proxy of q^{-s} in the usual Riemann zeta function; but in function field, luckily, all residue fields at closed points are extensions of the *same* finite field $\mathbb{F}_q$.

(1) Write $N_r := \#X(\mathbb{F}_{q^r})$. Show that $\zeta_X(t) = \exp\Big(\sum_{r=1}^{\infty} N_r \frac{t^r}{r}\Big)$.
(2) Compute $\zeta_X(t)$ for $X = \mathbb{P}^n$ over $\mathbb{F}_q$.
(3) Consider the geometric q-Frobenius ϕ_q on each of the cohomology group $H^i_{\mathrm{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$. The Lefschetz trace formula says that

$$\#X(\mathbb{F}_{q^r}) = \sum_{i=0}^{2n} (-1)^i \mathrm{Tr}\big(\phi_q^r;\ H^i_{\mathrm{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)\big)$$

Show that plugging Lefschetz trace formula into the definition of $\zeta_X(t)$ gives

$$\zeta_X(t) = \frac{\prod_{i \text{ odd}} \det\big(1 - \phi_q t;\ H^i_{\mathrm{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)\big)}{\prod_{i \text{ even}} \det\big(1 - \phi_q t;\ H^i_{\mathrm{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)\big)}.$$

(In particular, $\zeta_X(t) \in \mathbb{Q}(t)$.
(4) (Optional) The Poincaré duality for étale cohomology says that there is a ϕ_q-equivariant isomorphism

$$H^i_{\mathrm{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \cong H^{2n-i}_{\mathrm{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)^*(-\dim X).$$

Show that this implies the following functional equation:

$$\zeta_X\Big(\frac{1}{q^n t}\Big) = \pm q^{nE/2} t^E \zeta_X(t),$$

where E is the Euler characteristic of X, namely, $E = \sum_i (-1)^i \dim H^i_{\mathrm{et}}\big(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell\big)$.

Problem 5.4. (Traces of differential forms) Consider the simplest case $h : \mathbb{C} \to \mathbb{C}$ with $h(z) = w^n$. Show that the trace map of differential forms is well-defined.

Problem 5.5. (Compatibility of Abel–Jacobi map with respect to functoriality maps) For X a projective smooth curve over $\mathbb{C}$, let $\mathrm{Jac}_X := H^0(X, \Omega^1_X)^\vee / H_1(X^{\mathrm{an}}, \mathbb{Z})$ denote the associated Jacobian, and let $\mathrm{Pic}^0(X)$ denote the Picard group of X, parametrizing line bundles of degree 0. Let $h : X \to Y$ be a finite (flat) morphism of projective smooth curves over $\mathbb{C}$. Show that the following diagrams of functorial maps commute:

$$\begin{tikzcd} \mathrm{Jac}_X \arrow[r, "(h^*)^\vee"] \arrow[d, "\mathrm{AJ}"', "\cong"] & \mathrm{Jac}_Y \arrow[d, "\mathrm{AJ}"', "\cong"] \\ \mathrm{Pic}^0(X) \arrow[r, "\mathcal{L} \mapsto \det(h_* \mathcal{L})"] & \mathrm{Pic}^0(Y) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Jac}_Y \arrow[r, "(\mathrm{Tr}_h)^\vee"] \arrow[d, "\mathrm{AJ}"', "\cong"] & \mathrm{Jac}_X \arrow[d, "\mathrm{AJ}"', "\cong"] \\ \mathrm{Pic}^0(Y) \arrow[r, "\mathcal{L} \mapsto h^* \mathcal{L}"] & \mathrm{Pic}^0(X). \end{tikzcd}$$

Problem 5.6. (L-function for modular forms and its functional equations) Let $f = \sum_{n \geq 1} a_n q^n$ denote a cuspidal modular forms of weight k, level $\Gamma_0(N)$. Suppose that f is normalized so that $a_1 = 1$, and that f is an eigenform for all Hecke operators.

(1) Prove that
$$\int_0^\infty f(iy)y^s\frac{dy}{y} = (2\pi)^{-s}\Gamma(s)L(f,s)$$
where $L(f,s) = \sum_{n\geq 1} a_n/n^s$ is the L-function associated to f.

(2) Suppose for simplicity that f has level $\mathrm{SL}_2(\mathbb{Z})$. Let $L_\infty(f,s) := (2\pi)^{-s}\Gamma(s)$ denote the "L-factor at infinity", and write $\Lambda(f,s) := L(f,s)L_\infty(f,s)$ for the "complete L-function". Show that
$$\Lambda(f,s) = (-1)^{k/2}\Lambda(f,k-s).$$
(Hint: breaks up the integral above as $\int_0^1$ and $\int_1^\infty$, and use the relation $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.)

Problem 5.7. (Geometric realization of Hecke operator T_p) Recall that $Y_0(N)$ parameterizes elliptic curves together with a cyclic subgroup C_N or order N. Write $\mathcal{E}$ for the universal elliptic curve, and $\pi : \mathcal{E} \to Y_0(N)$ for the structure map and $e : Y_0(N) \to \mathcal{E}$ the zero section. Set $\omega_{E/Y_0(N)} := e^*\Omega^1_{E/Y_0(N)}$.

To a modular form $f \in S_k(\Gamma_0(N))$ and a prime $p \nmid N$, one can associate a section
$$f(z) \otimes d\tau^{\otimes k} \in H^0(Y_0(N), \omega).$$
We define the Hecke operator T_p on $S_k(\Gamma_0(N))$ using the following diagram:
$$\begin{tikzcd} & Y_0(pN) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Y_0(N) & & Y_0(N) \end{tikzcd} \qquad \begin{tikzcd} & (E, C_N, C_p) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ (E, C_N) & & (E/C_p, C_N + C_p/C_p). \end{tikzcd}$$
Here, we may alternatively view $Y_0(pN)$ as the moduli space of isogenies of elliptic curves $E \to E/C_p$ of degree p together with a cyclic subgroup C_N of E of order N. Let $\varphi : \pi_1^*\mathcal{E} \to \pi_2^*\mathcal{E}$ denote the universal isogeny on $Y_0(pN)$. Define
$$T_p : H^0(Y_0(N), \omega_2^{\otimes k}) \xrightarrow{\pi_2^*} H^0(Y_0(Np), \omega_2^{\otimes k}) \xrightarrow{\varphi^*} H^0(Y_0(Np), \omega_1^{\otimes k}) \xrightarrow{\frac{1}{p}\mathrm{Tr}} H^0(Y_0(N), \omega_1^{\otimes k})$$
Here we write ω_1 and ω_2 to distinguish sheaves on the left and right $Y_0(N)$. Show that this agrees with the usual definition of T_p.

Problem 5.8. (Monodromy of moduli space of elliptic curves at a cusp) Fix $N \geq 5$ and consider the quotient $Y_1(N) := \mathcal{H}/\Gamma_1(N)$. This space parametrizes elliptic curves E over $\mathbb{C}$ together with a point $P \in E$ of exact order N. Let $\pi : \mathcal{E} \to Y_1(N)$ be the universal elliptic curve; its fiber over $z \in Y_1(N)$ is denoted by $\mathcal{E}_z := \pi^{-1}(z)$. We study the pushforward of the sheaf $R^1\pi_*\underline{\mathbb{Z}}$, which is locally free sheaf of rank 2 over $Y_1(N)$; such that the stalk at the point z is precisely $H^1(E_z, \mathbb{Z})$. Consider the loop $\gamma : [0,1] \to Y_1(N)$ given by $\gamma(a) = a + 10i$. Looping around γ defines a homomorphism $H^1(E_{10i}, \mathbb{Z}) \to H^1(E_{10i}, \mathbb{Z})$, called the monodromy operator at $i\infty$. What is this operator in terms of 2-by-2 matrix?