Divisors on Curves and Riemann-Roch

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§1 The Riemann-Roch Theorem

X = curve / $k = \bar{k}$ (proj, irred, 1-dim'l, nonsing).

⇝ canonical sheaf $\omega_{X/k} = \Omega_{X/k}$ since $\dim X = 1$.

Def'n canonical div = div def'd by any mero $f \in \omega_{X/k}$.

(Sorry: can div is not canonical in any sense).

$\mathrm{Div}\, X \longrightarrow \mathbb{Z}$, X ell curve. ⇝ $\mathrm{Div}\, X \to \mathrm{Cl}\, X \to \mathbb{Z}$.

$(P) \longmapsto 1$, $\forall P \in X(k)$ b/c any prin div has deg 0.

Shorthand $l(D) := \dim_k H^0(X, \mathcal{L}(D))$.

Thm (Riemann-Roch)

$\exists\, g = g(X) \geq 0$ s.t. $\forall D \in \mathrm{Div}(X)$ & $K \in \mathrm{CanDiv}(X)$,

$$\boxed{l(D) - l(K-D) = \deg D + 1 - g}.$$

Rmk Proved by Serre duality

But in application, it has no overt relationship to cohom.

Cor $g = l(K) = \dim_k \Gamma(X, \mathcal{L}(K)) = \dim_k \Gamma(X, \Omega_{X/k})$

Proof. $D = 0 \Rightarrow l(D) = 1$ b/c glob reg func on curve = const

(or indeed on any proj var)

$$\Rightarrow \underbrace{l(K) = g}_{\text{geometric genus of } \underline{K}}. \quad \square$$

$k = \mathbb{C} \Rightarrow$ this matches top genus of Riemann surface of X.

Cor $\deg K = 2g-2$.

Proof. $D = K \Rightarrow \deg K + 1 - g = \underset{g}{\underset{\parallel}{\ell(K)}} - \underset{1}{\underset{\parallel}{\ell(0)}}$

$$\Rightarrow \deg K = 2g-2. \qquad \square$$

Cor $\deg D > 2g-2$ or $(\deg D = 2g-2$ & $D \not\sim K)$, then
$$\ell(D) = \deg D + 1 - g \geq g-1.$$

Proof. $\deg D = 2g-2 \Rightarrow \deg(K-D) = 0$

If $f \in k(X)$, $f \neq 0$ s.t. $\underbrace{(f)+K-D}_{\deg = 0} \geq 0 \leftarrow$ i.e. $f \in \mathcal{L}(K-D)$

$\Rightarrow (f)+K-D \sim 0$

$\Rightarrow \ell(K-D) \neq 0$ only when $K \sim D$.

$\deg D > 2g-2 \Rightarrow \deg(K-D) < 0$

$\Rightarrow (f)+K-D < 0 \Rightarrow \ell(K-D) = 0. \qquad \square$

Cor $g \geq 2$. $\forall D \in \mathrm{Div}(X)$, $\deg D \geq 2g-1$, the complete linear system assoc to D defines a closed imm $D \hookrightarrow \mathbb{P}^r$.

§2 The Canonical (almost) Embedding

K canonical div $\rightsquigarrow$ complete lin system

$\rightsquigarrow K \hookrightarrow \mathbb{P}^r$ canonical (almost) emb

i.e. (almost) always a closed imm.

a few exceptions in low genus.

<u>Lemma</u> P any pt, $D \in \mathrm{Div}(X)$.

$$l(D) \le l(D+P) \le l(D)+1.$$

$(\Rightarrow l(D) \le \deg D + 1)$.

Proof. We have $0 \to \mathcal{L}(D) \to \mathcal{L}(D+(P)) \to \underset{\mathcal{O}_X/\mathcal{I}}{\overset{\mathcal{E}}{\shortparallel}} \to 0$

$\mathcal{I}$ = ideal sheaf defining $P \in X$.

$\Rightarrow l(D) \le l(D+P)$.

$\overset{\Gamma(X,-)}{\rightsquigarrow}$ $0 \to \Gamma(X, \mathcal{L}(D)) \to \Gamma(X, \mathcal{L}(D+(P))) \to \Gamma(X, \mathcal{E})$

1-dim'l / k

$\Rightarrow l(D+P) \le l(D)+1$. □

<u>Prop</u> The canonical emb is a closed imm

$\Leftrightarrow$ X not hyper-elliptic.

Proof. $D = (P)+(Q)$, $P, Q \in X(k)$. (not necessarily $P \neq Q$).

To check $l(K-D) = l(K) - 2 = g-2$.

(P)+(Q).

Riemann-Roch $\Rightarrow l(K-D) = l(D) - \deg D - 1 + g$

$= l(D) + g - 3$

So $\exists$ embedding $X \hookrightarrow \mathbb{P}^1 \Leftrightarrow l(D) = 1, \forall D$ of deg 2.

<u>but</u> a failure defines 2-to-1 map to $\mathbb{P}^1$

when X hyperelliptic. □

<u>Strictly</u> also to check for D of deg 1.

but $l(D) > 0 \Rightarrow \exists f \in K(X)$ w/ single pole

$\rightsquigarrow X \to \mathbb{P}^1$ deg 1. i.e. $X \cong \mathbb{P}^1$.

Rmk canonical emb and variants of it (e.g. higher mult of K) | key tools to study moduli spaces M_g of curves with given g.

$\rightsquigarrow$ M_g reps $\mathrm{Fun}(\mathrm{Sch}, \underset{\text{family of curves of genus } g}{\mathrm{Curv}^g})$

it becomes rep'ble in $\underset{\text{cat of Deligne-Mumford stacks.}}{\underline{DM}}$

ext'n of Sch.

(the same as orbifolds extends manifolds).

§3 The Riemann-Hurwitz Formula

$f: X \to Y$ fin sep of curves

(i.e. induced field ext'n $k(X)/k(Y)$ sep).

Def'n The ramification div of f:

$$R = \sum_{P \in X(k)} \mathrm{length}(\underbrace{\Omega_{X/Y}}_{\text{mod of Kähler differentials}})_P (P).$$

Prop $K_X \sim f^* K_Y + R$

Proof. (c.f. Hartshorne Prop IV.2.3) Note that

$$0 \to \underbrace{f^*\Omega_{Y/k}}_{\text{check at } \eta} \to \underbrace{\Omega_{X/k} \to \Omega_{X/Y}}_{\text{by properties of Kähler diff.}} \to 0 \quad \text{exact.}$$

(b/c $k(X)/k(Y)$ sep)

$(\cdot) \otimes \Omega_{X/k}^\vee \rightsquigarrow 0 \to f^*\Omega_{Y/k} \otimes \Omega_{X/k}^\vee \to \mathcal{O}_X \to \Omega_{X/Y} \otimes \Omega_{X/k}^\vee \to 0.$

($k(X)/k(Y)$ fin) supp on fin many pts

($\Rightarrow \Omega_{X/Y} \cong \Omega_{X/Y} \otimes \Omega_{X/k}^\vee$)

$$\Rightarrow (f^*\Omega_{Y/k}) \otimes \Omega_{X/k}^{\vee} \cong \mathcal{O}_X / \Omega_{X/Y}.$$

$$\rightsquigarrow \quad f^*K_Y - K_X \sim -R \qquad \square$$

Riemann-Roch $\Rightarrow$ Riemann-Hurwitz formula

<u>Prop</u> $2g(X) - 2 = (\deg f)(2g(Y)-2) + \deg R$

$(K_X \sim f^*K_Y + R)$

where $\deg f = [k(X) : k(Y)]$.

Moreover, contribution of $P \in X(k)$ can be simply computed.

i.e. $Q = f(P)$, $t \in k(Y)$ s.t. $\mathfrak{m}_{Y,Q} = (t)$.

$\rightsquigarrow f^*(t)$ generates $\mathfrak{m}_{X,P}^{e}$, $e \geq 0$

Call $e = e_P$ ramif index of P.

$\Rightarrow$ length$(\Omega_{X/Y})_P \geq e_P - 1$,

"$=$" $\Leftrightarrow$ f tamely ram, i.e. char $k \nmid e_P$.

(<u>Recall</u> In alg number theory, L/K fin ext'n of complete DVF. $k = \mathcal{O}_K/\pi_K$.

L/K tamely ram $\Leftrightarrow$ char $k \nmid [L:K]$)

When $k = \mathbb{C}$: R-H formula has a top meaning.

Euler char χ additive,

- f unram $\Rightarrow \deg R = 0 \Rightarrow \chi(X) = \deg f \cdot \chi(Y)$.
- Otherwise $\Rightarrow \chi(X) = \deg f \cdot \chi(Y) - \sum_P (e_P - 1)$

those P s.t. $e_P > 1$

$(\chi(pt) = 1)$.