# Canonical integral models of Shimura varieties

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Shimura datum $(G, X)$

- $G$ = red grp / $\mathbb{Q}$, connected
- $X$ = conj class of $h: \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$

<u>Fact</u> $X$ = union of herm symm domains.

$K \subset G(\mathbb{A}_f)$ cpt open subgrp (+ neat).

$\rightsquigarrow$ $Sh_K = Sh_K(G, X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K$

<u>Note</u> If $\exists g \in G$ s.t. $g^{-1} K' g \hookrightarrow K$, $K, K' \subset G$ cos,

then get $Sh_{K'} \longrightarrow Sh_K$.

$\rightsquigarrow$ $\{Sh_K\}_K$ tower.

Hodge cochar $\mu: \mathbb{G}_{m,\mathbb{C}} \longrightarrow G_{\mathbb{C}}$ s.t. $\mu(z) = h_{\mathbb{C}}(z, 1)$

$\rightsquigarrow$ conj class $\{\mu\}$ def'd over $E$ = reflex field.

Then $Sh_K(G, X)$ carries

- can str of quasi-proj alg var / $\mathbb{C}$ (Baily-Borel, Borel).
- can str / $E$ = number field (Shimura, Deligne, Milne, Borovoi).

<u>Q</u> $\exists$ canonical integral model? / $\mathcal{O}_E$ = ring of ints in $E$.

* Moduli of G-motives?

$$\underbrace{\text{Siegel} \subset \text{PEL} \subset \text{Hodge}} \subset \text{abelian} \subset \text{general type}$$

Can take Zariski closure of some Siegel moduli sch $/\mathbb{Z}$

$E_6, E_7$ (some twisted types of orthogonal grps)

$v$ place of $E$, $v|p$. $E_v \supset \mathcal{O}_v = \mathcal{O}$

$$\downarrow$$

$k(v) = k$ = res field.

$K = K_p K^p$, $K_p \subset G(\mathbb{Q}_p)$, $K^p \subset G(\mathbb{A}_f^p)$.

"<u>good places</u>": $v|p$, $G$ extends to red grp $/\mathbb{Z}_p$

$K_p = G(\mathbb{Z}_p)$ (hyperspecial)

<u>Expect</u> Canonical sm model $/\mathcal{O}_v$.

<u>Thm</u> (Kisin, Vasiu, etc.) OK if $(G, X)$ of abelian type.

<u>parahoric places</u>

$K_p$ = parahoric subgrp of $G(\mathbb{Q}_p)$

<u>Aside</u> $G(\mathbb{Q}_p) \curvearrowright \mathcal{B}(G)$ = B-T building

$$\begin{array}{l} \rotatebox{90}{$\in$} \\ x \underset{\text{theory}}{\overset{\text{B-T}}{\rightsquigarrow}} \text{canonical sm grp sch } \mathcal{G}_x / \mathbb{Z}_p \end{array}$$

w/ $\mathcal{G}_x(\mathbb{Z}_p)$ = Stabilizer $G(\mathbb{Q}_p)_x$.

$\mathcal{G}_x^\circ \subset \mathcal{G}_x$ neutral comp.

Then

$$K_p \text{ parahoric} \overset{\text{Def}}{\iff} \exists\, x \in \mathcal{B}(G) \text{ s.t. } K_p = \mathcal{G}_x^\circ(\mathbb{Z}_p).$$

Denote $\mathcal{G} = \mathcal{G}_x^\circ$. So $K_p = \mathcal{G}(\mathbb{Z}_p)$.

<u>Ex</u> (1) $G$ extends to red grp $/ \mathbb{Z}_p$

$$K_p = \{ g \in G(\mathbb{Z}_p) \mid g \bmod p \in \underset{\substack{\uparrow \\ \text{parabolic of } G(\mathbb{F}_p).}}{P(\mathbb{F}_p)} \}$$

(2) $G = \mathrm{Res}_{F/\mathbb{Q}_p} G'$, $G'$ red grp $/ \mathcal{O}_F$

$$K_p = G'(\mathcal{O}_F).$$

<u>Models</u> $\{\mathcal{S}_K\}_K$, $K = \underbrace{K_p}_{\text{fixed}} \underbrace{K^p}_{\text{varying}}$ $/ \mathcal{O} = \mathcal{O}_{K}$.

<u>Properties</u> (1) $\mathcal{S}_K$ flat $/ \mathcal{O}$, $\mathcal{S}_K \otimes_{\mathcal{O}} E_v = Sh_K \otimes_E E_v$

(2) (Hecke) $g^{-1} K'^p g \subset K^p$

$$\rightsquigarrow \mathcal{S}_{K_p K'^p} \longrightarrow \mathcal{S}_{K_p K^p} \text{ finite étale}$$

(3) (Relative properness) Let $\mathcal{S}_{K_p} := \varprojlim_{K^p} \mathcal{S}_{K_p K^p}$.

Then $\forall R$ DVR flat $/ \mathcal{O}$,

$$\mathcal{S}_{K_p}(R) = Sh_{K_p}(R[\tfrac{1}{p}]).$$

- Closed pts: $\mathcal{S}_{K_p}(k) \overset{\curvearrowleft \text{ Hecke}}{\underset{\circlearrowleft \text{ Frob}_v}{}}$ (Langlands–Rapoport Conj)

- (formal) local structure:

  Assume max $\mathbb{R}$-split torus of $Z(G)$ is $\mathbb{Q}$-isotropic

  $\rightsquigarrow$ principal autom bundle:

$$\begin{tikzcd}
 & \widetilde{Sh}_K \circlearrowleft G \arrow[dl, "G\text{-torsor}"'] \arrow[dr, "G\text{-equiv \& sm}"] & \\
Sh_K & & X^\vee = G/P_\mu \circlearrowleft G \quad / E
\end{tikzcd}$$

compact dual $(X \subset X^\vee)$

Think of: $Sh_K$ gives variation of Hodge fil'n

values in $X^\vee$ gives position of Hodge fil'n.

This should extend to

(LMP) = local model property

$$\begin{tikzcd}
 & \widetilde{\mathcal{S}}_K \arrow[dl, "\mathcal{G}\text{-torsor}"'] \arrow[dr, "\mathcal{G}\text{-equiv \& sm}"] & \\
\mathcal{S}_K & & M
\end{tikzcd}$$

"local model"

where $M$ = a canonical model $/ \mathcal{O}$ of $X^\vee$

<u>N.B.</u> $M = M^{loc}$ $\circlearrowleft$ $\mathcal{G}$ depends only on $\underbrace{K_p \ \& \ \{\mu\}}_{\text{local stuff.}}$

satisfying (1) $M$ flat $/ \mathcal{O}$

(2) $M \otimes_{\mathcal{O}} E_v = X^\vee \otimes_E E_v$ $\mathcal{G}$-equiv

(3) $M \otimes_{\mathcal{O}} k$ <u>reduced</u> (difficult!)

special fibre

$M(k) =$ "$\mu$-adm set" $\subset \dfrac{G(W(k)[\frac{1}{p}])}{\mathcal{G}(W(k))}$

$W(k) =$ Witt vectors

(4) functorial, invariant under central ext'n.

Scholze constructs v-sheaf $\mathbb{M}^{\diamond}$

$\rightsquigarrow$ to characterize $\mathbb{M}$ (if this local model exists)

<u>Constr'n of $\mathbb{M}$</u> (Pappas-Zhu, He-Pappas-Rapoport, Levin; Hodge type)

(exclude some grps for $p=2,3$)

<u>Thm 1</u> (Kisin-Pappas)

Assume (1) $(G, X)$ of Hodge type

(2) $G$ splits over a tame ext'n $/\mathbb{Q}_p$,

$p>2$, $p \nmid |\pi_1(G_{der})|$, $\mathcal{G}_x = \mathcal{G}_x^{\circ}$

Then $\exists\ \mathcal{G}_K$ with (1)(2)(3) + (LMP)

* Canonicality & characterization of $\mathbb{M}$:

· <u>$(\mathcal{G}, \mu)$-displays</u>:

Zink displays, $p>2$,

$R$ = complete noeth local ring w/ $R/\mathfrak{m} = k$.

$$W(R) = \{(r_0, r_1, \ldots) \mid r_i \in R\}$$

$\circlearrowleft$

$\varphi, V$ s.t. $V(r_0, r_1, \cdots) = (0, r_0, r_1, \cdots)$

$\parallel$ & $\varphi \circ V = p$.

Frob

Turns out: 
$$\begin{array}{ccccc} W(R) & = & W(k) & \oplus & W(\mathfrak{m}) \\ \cup & & & & \cup \\ \hat{W}(R) & = & W(k) & \oplus & \hat{W}(\mathfrak{m}) \\ \circlearrowleft & & & & \parallel \\ \varphi, V & & & & \{r_i \to 0 \text{ in } p\text{-adic top}\} \end{array}$$
(Zink's constr'n)

This uses the fact $p>2$.

$$I_R := V\hat{W}(R) \subset \hat{W}(R), \quad \hat{W}(R)/I_R \simeq R.$$

Def <u>Dieudonné display</u> over $R$ is $(M, M_1, F)$ (with $n, d \in \mathbb{N}$)

- $M \simeq \hat{W}(R)^n \supset M_1 \supset I_R M$
- $F : M_1 \to M$

s.t. (1) $M/M_1 \simeq R^d$

(2) $F(a m_1) = \varphi(a) F(m_1)$, $\forall a \in \hat{W}(R)$, $m_1 \in M_1$

(3) $F(V(a) m) = a F(V(1) m)$, $\forall a \in \hat{W}(R)$, $m \in M$

(4) $\langle \mathrm{Im}\, F \rangle = M$.

<u>Thm</u> (Zink) $\exists$ equiv of cats

$$\left\{ \begin{array}{c} p\text{-div grps } /R \\ \text{of ht } n \text{ \& dim } d \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Dieudonné displays of} \\ \text{type } (n, d) \end{array} \right\}$$

Def Let $R$ be as above + normal flat $/ \mathbb{Z}_p$.

A <u>$(\mathcal{G}, \mu)$-display</u> is triple $\mathcal{D} = (P, q, F)$

- $P$ $\mathcal{G}$-torsor on $\hat{W}(R)$
- $q : P \otimes_{\hat{W}(R)} R \to M$

($M$ depends only on $(\mathcal{G}, \mu)$)

$\rightsquigarrow$ $\mathcal{G}$-torsor $Q$

- $F : Q \xrightarrow{\sim} P$.

<u>Thm 2</u> Same assns as in Thm 1. Then

(a) $\forall x \in \mathcal{S}_K(k)$, $\exists \mathcal{D}_x$ $(\mathcal{G}, \mu)$-display over $\hat{\mathcal{O}}_{\mathcal{S}_K, x}$ which is

- "versal"
- "associated" to the pro-étale $K_p$-cover

$$\varprojlim_{K_p'} Sh_{K_p'K^p} \longrightarrow Sh_{K_pK^p}.$$

(b) If $\mathscr{S}_K$, $\mathscr{S}_K'$ models of $Sh_K$ satisfying (1)(2)(3), and support $(\mathcal{G},\mu)$-data as in (a), then

$$\mathscr{S}_K \simeq \mathscr{S}_K'.$$

<u>Cor</u> $\mathscr{S}_K$ of Thm 1 (by Kisin-Pappas) indep of choices (of $(\mathcal{G},\mu)$)
namely canonical.

<u>Rmk</u> (Meaning of Thm 2 (a))

<u>"associated"</u> : $\mathcal{G} \hookrightarrow GL_n$

$\mu \longmapsto \mu_d$ = miniscule char of $GL_n$

can be done for Hodge type.

Get $\{(\mathcal{G},\mu)\text{-display}\} \to \{(GL_n,\mu_d)\text{-display}\}$

$\quad \parallel$

$\{$Dieudonné display of type $(n,d)\}$

$\quad \wr\wr$ Zink's thm

$\{$p-div grps $/R$ of ht $n$ & dim $d\}$

$\rightsquigarrow$ Tate mods $/ R[\frac{1}{p}]$

$\rightsquigarrow$ $\mathbb{Z}_p^n$ étale local systems

$\rightsquigarrow$ $GL_n(\mathbb{Z}_p)$-torsor.

<u>pf idea of (b)</u>

$$\begin{tikzcd} Sh_K \arrow[r, hook, "\text{rel proper}"] \arrow[dr, hook, "\text{rel proper}"'] & \mathscr{S}_K \arrow[d, "\text{proper}"] & \text{– normal} \\ & \mathscr{S}'_K & \text{normal} \end{tikzcd}$$

Same generic fibre $Sh_K$

$\Rightarrow$ same $(\mathcal{G}, \mu)$-display on $\mathscr{S}_K$ & $\mathscr{S}'_K$

$\Rightarrow$ same p-div grp.

Apply Zariski main thm + formal local isom $\Rightarrow$ pf. $\square$