Locally symmetric spaces and Galois representations (1/2)
"Higher reciprocity laws"
Peter Scholze

Want to describe relation between
(A) Certain hyperbolic 3 manifolds
"Bianchi manifolds" (Bianchi, 1892)
(B) Galois rep'ns. Let $\bar{\mathbb{Q}} \subseteq \mathbb{C}$ field of algebraic numbers
$$\bar{\mathbb{Q}} = \{x \in \mathbb{C} \mid \exists n, a_1, \dots, a_n \in \mathbb{Q} : x^n + a_1 x^{n-1} + \dots + a_n = 0\}.$$

Let $\mathrm{Gal}_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) = \mathrm{Aut}(\bar{\mathbb{Q}})$ large profinite group.
$i \mapsto -i$, $\sqrt{2} \mapsto -\sqrt{2}$, $\sqrt[3]{7} \mapsto \zeta_3 \sqrt[3]{7}$, $\cdots$.
Interested in (continuous) rep'ns
$$\rho : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow GL_n(k) \qquad (k = \mathbb{C}, \bar{\mathbb{Q}}_p, \bar{\mathbb{F}}_p)$$
Called Artin rep

e.g. given $\mathrm{Gal}_{\mathbb{Q}} \to GL_2(\mathbb{F}_3) \to PGL(\mathbb{F}_3) \cong S_4$
So $\mathrm{Gal}_{\mathbb{Q}}$ acts on $\{x_1, x_2, x_3, x_4\} = \{4 \text{ roots of } x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4 = 0\}$.

One standard question:
[Given $P(x) = x^n + \dots + a_n$, for which prime p does $P(x) \equiv 0 \mod p$ have a solution?]

Thm (Gauß' quadratic reciprocity)
Complete answer for all quad poly's.
e.g. $x^2 + 1 \equiv 0 \mod p \iff p \equiv 1 \mod 4$ (or $p = 2$).

In the early 20th century, this has been generalized to "class field theory", giving an answer for all P s.t. $\mathrm{Gal}_{\mathbb{Q}} \curvearrowright \{\text{zeros of } P\}$ acts through an abelian quotient.
(This works for any finite ext'n $K/\mathbb{Q}$ in place of $\mathbb{Q}$.)

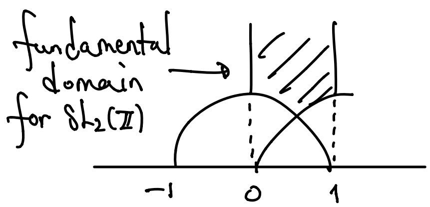
<u>Afterwards</u> Search for "nonabelian reciprocity laws".

<u>Eichler, Shimura</u> relation between

$$\text{elliptic curves } /\mathbb{Q} \longleftrightarrow \text{modular forms of wt } 2.$$

equations of the form $E: y^2 = x^3 + ax + b$, $a, b \in \mathbb{Q}$

diff'l forms on $\Gamma \backslash \mathbb{H}^2$
($\Gamma \leq SL_2(\mathbb{Z})$ cong subgrp.)



If $f(\tau)d\tau$ is such a mod form then $f(\tau) = f(\tau+1)$

$$\Rightarrow f(\tau) = \sum_{n \geq 0} a_n q^n, \quad q = e^{2\pi i \tau}.$$

This takes the following form:

<u>Thm</u> If $f = q + a_2 q^2 + \cdots + a_n q^n + \cdots$ modular eigenform of wt 2 s.t. $a_i \in \mathbb{Q}$ ($\Rightarrow a_i \in \mathbb{Z}$)

Then $\exists\, E_f / \mathbb{Q}$ ell curve s.t. for almost all p

$$\# E_f(\mathbb{F}_p) = (p+1) - a_p.$$

The map $f \mapsto E_f$ is explicit:

<u>Hodge theory</u> $H^1(\Gamma \backslash \mathbb{H}^2)$ related to <u>diff'l forms</u> = wt 2 mod forms.

But $\Gamma\backslash\mathbb{H}^2$ is an algebraic curve $/\mathbb{Q}$

Often, it is already the sought-for elliptic curve.

(In general, take a piece of $\mathrm{Jac}(\Gamma\backslash\mathbb{H}^2)$.)

$$\begin{tikzcd} & \text{mod forms (analysis)} \\ \Gamma\backslash\mathbb{H}^2 \arrow[ur, bend left] \arrow[dr, bend right] & \\ & \text{elliptic curves (algebra)} \end{tikzcd}$$

Thm (Wiles et al) All elliptic curves $/\mathbb{Q}$ come from some f.

Langlands Conjectures vast generalization ($K/\mathbb{Q}$ finite)

$$\left\{\begin{matrix}\text{motives of rank } n \\ \text{over } K\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{automorphic forms} \\ \text{on } GL_n(\mathbb{A}_K)\end{matrix}\right\}$$

$$\overset{\curvearrowright}{\{\text{alg var } / K\}} \qquad \overset{\curvearrowright}{\{\mathrm{Gal}_K \to GL_n(\overline{\mathbb{Q}}_p)\}}$$

$GL_2/\mathbb{Q}$ is largely known

(except for "even Artin" $\longleftrightarrow$ "Maaß forms of eigenval 1/4").

Next case GL_2/K, K quad field, $K = \mathbb{Q}(\sqrt{d})$.

- $d>0$: Similar to $GL_2/\mathbb{Q}$

  (Hilbert-Blumenthal surface, Shimura curves, ...)
- $d<0$: (imag quad) Very different!

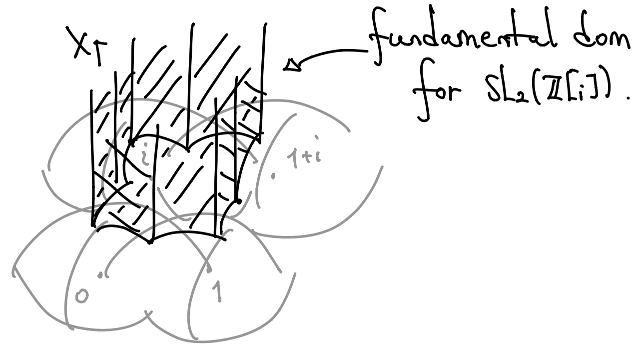
The relevant loc sym space is

$$X_\Gamma := \Gamma\backslash\mathbb{H}^3 \leftarrow \text{3-dim hyperbolic, } \mathbb{H}^3 = \mathbb{C}\times\mathbb{R}_{>0}$$

$$\Gamma \subseteq SL_2(\mathcal{O}_K) \text{ cong subgrp}$$

called "Bianchi manifolds".

Picture of X_Γ:



<u>Note</u> These are real 3-dim'l, hence has no complex str,
a fortiori no alg str.

sh var doesn't have big torsion!

(1) $H_1(\Gamma\backslash\mathbb{H}^3, \mathbb{Z}) = \underbrace{\mathbb{Z}^{b_1(X_\Gamma)}}_{\text{quite small}} \oplus \underbrace{(\text{torsion part})}_{\text{extremely big}}$ (exclusive feature of non-alg var)

<u>Thm</u> (Lück, 1994) $b_1(X_\Gamma) = o(\mathrm{vol}(X_\Gamma))$ when $\Gamma \to 0$ (suff small).

<u>Conj</u> (Bergeron-Venkatesh)
Torsion part grows exponentially in volume.

(2) $H_1(\Gamma\backslash\mathbb{H}^3, \mathbb{C})$ described by autom forms.

<u>Expect</u> (Langlands) $H_1(\Gamma\backslash\mathbb{H}^3, \mathbb{C}) \rightsquigarrow (\mathrm{Gal}_K \to GL_2(\overline{\mathbb{Q}}_p))$.

<u>Q</u> Also for torsion classes?

<u>Conj</u> (Gruneward 1970s, Ash 1990)
For any mod p torsion class
there is an assoc Galois repr $\mathrm{Gal}_K \to GL_2(\overline{\mathbb{F}}_p)$.

<u>Thm</u> (Scholze) The conj is true.

<u>Remarks</u> (1) Works for GL_n / K as well, K totally real.

(2) For torsion-free part, due to Harris-Lan-Taylor-Thorne.

(3) cf. forthcoming work of Boxer.

(4) $\exists$ version "mod p" $\Rightarrow$ p-adic Galois rep

for "many" autom rep'ns.

(5) <u>Example</u> (Figueiredo thesis, 98)

Consider $P(x) = x^4 - 7x^2 - 3x + 1$. $\quad$ ($\mathrm{disc} = 3^2 \times 61^2$.)

It generates A_4-ext of $\mathbb{Q}$, corr to "even Artin" repr

ramified at 3, 61.

($\Rightarrow$ does <u>not</u> come from $H^1(\Gamma\backslash\mathbb{H}^2, \mathbb{Z})$).

finds 3-torsion classes in $H_1(\Gamma\backslash\mathbb{H}^3, \mathbb{Z})$,

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}[i]) \;\middle|\; c \equiv 0 \bmod 3\cdot 61,\ d \equiv 1 \bmod 3 \right\}$$

whose Hecke eigenvalues $a_{\mathfrak{p}} \in \mathbb{F}_3$ ($\mathfrak{p}$ prime of $\mathbb{Z}[i]$)

seem to match.

<u>Cor</u> They do match, and thus

$\exists$ a solution to $X^4 - 7X^2 - 3X + 1 \equiv 0 \bmod \mathfrak{p} \iff a_{\mathfrak{p}} \neq 0$

Locally symmetric spaces and Galois representations (2/2)

Peter Scholze

Let F totally real or CM, $n \geq 1$.

$(\Gamma \rightsquigarrow)$ $K \subseteq GL_n(\mathbb{A}_{F,f})$ compact open

$(X_\Gamma \rightsquigarrow)$ $X_K = GL_n(F) \backslash (GL_n(\mathbb{A}_F) / \underset{\substack{\uparrow \\ \text{max'l cpt subgrp of } GL_n(F \otimes_{\mathbb{Q}} \mathbb{R})}}{K_\infty} K)$

$\left(= \coprod_{\text{finite}} (\text{congruence subgrp}) \backslash (\text{symm space})\right).$

<u>Thm</u> (Scholze) For any system of Hecke eigenvals $(a_{\mathfrak{p}})$ ($\mathfrak{p}$ prime of F) appearing in $H^i(X_K, \overline{\mathbb{F}}_p)$, there is cont $\rho : \mathrm{Gal}_F \longrightarrow GL_n(\overline{\mathbb{F}}_p)$ s.t. $\forall\forall\, \mathfrak{p}$, $\mathrm{tr}(\mathrm{Frob}_{\mathfrak{p}}) = a_{\mathfrak{p}}$.

<u>The big lines of the proof</u> (F im quad)

$$\left\{\begin{matrix} \overline{\mathbb{F}}_p\text{-cohom of loc symm} \\ \text{space for } GL_n/F \end{matrix}\right\} \xrightarrow[\substack{\text{Contribution} \\ \text{(Borel-Serre)}}]{\text{boundary}} \left\{\begin{matrix} \overline{\mathbb{F}}_p\text{-cohom of loc symm} \\ \text{space for } \underset{\substack{\uparrow \\ \text{quasi-split unitary}}}{U(n,n)}/\mathbb{Q} \end{matrix}\right\}$$

<u>Note</u> $U(n,n)(\mathbb{Q}) \supseteq \text{Siegel parabolic} \supseteq \text{Levi component} = \text{a subgrp of } GL_n(F)$

$$\text{Siegel parabolic} \updownarrow \text{boundary stratum of } Sh_{GL_n}$$

<u>Upshot</u> RHS: space is an alg var / $\mathbb{Q}$.

<u>Lifting thm</u> (Scholze)

$$\left\{\begin{matrix} \overline{\mathbb{F}}_p\text{-cohom of loc sym} \\ \text{space for } U(n,n)/\mathbb{Q} \end{matrix}\right\} \overset{\text{lifting}}{\rightsquigarrow} \left\{\begin{matrix} \text{cusp form} \\ \text{on } U(n,n) \end{matrix}\right\}$$

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{cusp form} \\ \text{on } U(n,n)\end{array}\right\} \arrow[r, "\text{Laibesse, Shin, Morel, Arthur, Mok}", "\text{relying Ngô, Waldspurger}"'] & \left\{\begin{array}{c}\text{conjugate self-dual} \\ \text{rep'ns of } GL_{2n}/F\end{array}\right\} \arrow[d, "\text{by all these people}"] \\
\{Gal_F \to GL_{2n}(\bar{\mathbb{Q}}_p)\} \arrow[d] & \left\{\begin{array}{c}\text{cusp form on} \\ U(2n-1,1)\end{array}\right\} \arrow[l, "\text{Kottwitz, Clozel}"', "\text{Harris-Taylor, Shin, Cheinevier-Harris}"] \\
\{Gal_F \to GL_{2n}(\bar{\mathbb{F}}_p)\} &
\end{tikzcd}$$

(case of Langlands functoriality "twisted endoscopy")

use étale cohom of these Shimura vars

<u>Lifting thm</u> (Scholze)

Let Sh_K be a Shimura var of Hodge type.

(i.e. $\exists$ embedding $Sh_K \hookrightarrow \mathcal{A}_g$)

For any system $(a_\wp)$ of eigenvals appearing in $H^i(Sh_K, \bar{\mathbb{F}}_p)$ there exists a classical cuspidal eigenform f on $Sh_{K'}$

(same grp but $K' \subseteq K$ larger level at p)

s.t. Hecke eigenval of $f \equiv a_\wp \mod \wp$.

<u>Rmk</u> (1) In particular this provides congruences between Eisenstein series & cusp forms.

(2) Also interesting in compact case.

In that case, recent work with Caraiani, same result is proved, but increasing level at $l \neq p$ instead.

(3) Have to go from singular cohomology to functions.

Over $\mathbb{C}$, this is done using Hodge theory.

Here, use (integral) p-adic Hodge theory.

(∗) Proof makes heavy use of p-adic geometry of Shimura varieties.

<u>Rest of the talk</u> Explain this geometry in simplest case.

<u>The modular curve</u> $\Gamma \subseteq SL_2(\mathbb{Z})$, $M_\Gamma/\mathbb{Q}$ with $M_\Gamma(\mathbb{C}) = \Gamma\backslash \mathbb{H}^2$.

moduli space of elliptic curves with level-Γ-structure.

(First work $/\mathbb{C}$)

$$\varprojlim_\Gamma M_\Gamma(\mathbb{C}) \approx \mathbb{H}^2 = \left\{ (E,\alpha) \;\middle|\; \begin{array}{l} E/\mathbb{C} \text{ elliptic curve,} \\ \alpha : H_1(E,\mathbb{Z}) \cong \mathbb{Z}^2 \text{ orientation-preserving} \end{array} \right\}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\mathbb{P}^1(\mathbb{C}) \quad \text{Hodge fil'n}: \mathbb{C}^2 \overset{\alpha}{\cong} H_1(E,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H_1^{dR}(E) \twoheadrightarrow \operatorname{Lie} E.$$

Then want a p-adic analogue of this.

$\mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$. need analogue of Hodge fil'n $/\mathbb{C}_p$.

<u>Hodge fil'n $/\mathbb{C}_p$</u> Let $X/\mathbb{C}_p$ proper smooth rigid-analytic var

(e.g. $X = E$ elliptic curve)

<u>Thm</u> (Hodge, Deligne if X algebraic, Scholze, Conrad-Gabber)

The Hodge-de Rham spectral sequence

$$E_1^{ij} = H^j(X, \Omega_X^i) \Rightarrow H^{i+j}_{dR}(X)$$

& degenerates at E_1.

$\Rightarrow$ Hodge-de Rham filtration on $H^i_{dR}(X)$.

<u>Note</u> Over $\mathbb{C}_p$, $H^i_{et}(X,\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{C}_p$ <u>not</u> canonically isom to $H^i_{dR}(X)$

(Need Fontaine's B_{dR}.)

<u>Thm</u> (Tate (1967) for abelian vars, Faltings (1990) if X algebraic, Scholze, Conrad-Gabber)

There is a Hodge-Tate spectral sequence

$$E_2^{i,j} = H^i(X, \Omega_X^j)(-j) \Rightarrow H^{i+j}_{\acute{e}t}(X, \mathbb{Z}_p) \otimes \mathbb{C}_p.$$

It degenerates at E_2

$\Rightarrow$ Hodge-Tate fil'n on $H^i_{\acute{e}t}(X, \mathbb{Z}_p) \otimes \mathbb{C}_p$.

<u>Example</u> $E / \mathbb{C}_p$ elliptic curve.

$$0 \to (\mathrm{Lie}\, E^*)^* \to H_1^{dR}(E) \to \mathrm{Lie}\, E \to 0.$$

$$0 \to (\mathrm{Lie}\, E)(1) \to T_pE \otimes_{\mathbb{Z}_p} \mathbb{C}_p \to (\mathrm{Lie}\, E^*)^* \to 0$$

Back to modular curve:

After fixing isom $\alpha : T_pE \cong \mathbb{Z}_p^2$, get "Hodge-Tate period map".

<u>Thm</u> (Scholze)

(1) There exists a perfectoid space $\mathcal{M}_{\Gamma(p^\infty)} / \mathbb{C}_p$ which parametrizes (E, α), where

- $E / \mathbb{C}_p$ elliptic curve
- $\alpha : T_pE \cong \mathbb{Z}_p^2$.

(2) There is a Hodge-Tate period map

$$\pi_{HT} : \mathcal{M}_{\Gamma(p^\infty)} \to \underset{\substack{\uparrow \\ \text{rigid-analytic var}}}{\mathbb{P}^1_{\mathbb{C}_p}}$$

via $(E, \alpha) \mapsto (\mathbb{C}_p^2 \overset{\alpha}{\cong} T_pE \otimes \mathbb{C}_p \twoheadrightarrow (\mathrm{Lie}\, E^*)^*)$.

It contracts $GL_2(\mathbb{A}_f^p)$-orbits.

$$\begin{array}{ccccccc}
\mathcal{M}_{\Gamma(p^\infty)} & = & \mathcal{M}^{ss}_{\Gamma(p^\infty)} & \sqcup & \mathcal{M}^{ord}_{\Gamma(p^\infty)} & \sqcup & \mathcal{M}^{bad}_{\Gamma(p^\infty)} \\
\downarrow & & \downarrow \text{interesting} & & \searrow & & \swarrow \\
\mathbb{P}^1_{\mathbb{C}_p} & = & \Omega^2 & \sqcup & & \mathbb{P}^1(\mathbb{Q}_p) &
\end{array}$$

$\Omega^2 = \mathbb{P}^1_{\mathbb{C}_p} - \mathbb{P}^1(\mathbb{Q}_p)$ Drinfeld's p-adic upper-half plane

(cf. $\mathcal{H}^{\pm} = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$.)

conn comps of $\mathcal{M}^{ord} \sqcup \mathcal{M}^{bad}$ are contracted to pts!

$$\mathcal{M}^{ss}_{\Gamma(p^\infty)} = \coprod_{\text{finite}} \mathcal{M}_{LT,\infty} \quad \text{(Lubin-Tate space w/ infinite level).}$$

$$\cong \coprod_{\text{finite}} \mathcal{M}_{Dr,\infty} \quad \text{(by Faltings, Fargues, Scholze-Weinstein)}$$

$$\downarrow \text{profinite étale cover}$$

$$\mathcal{M}^{ss}_{\Gamma(p^\infty)} \xrightarrow{\pi_{HT}} \Omega^2$$

<u>Strategy of proof of Lifting thm</u>

Understand cohom of $\mathcal{M}_{\Gamma(p^\infty)}$ using Leray spectral sequence corr to π_{HT}.