HIGHER DIMENSIONAL GROSS–ZAGIER FORMULA

Abstract. These are the notes for the course given by Wei Zhang in 2022 Summer School on the Langlands Program at IHES. We focus on Gross–Zagier formula for higher dimensional Shimura varieties, with an emphasis on the arithmetic Gan–Gross–Prasad (AGGP) conjecture and the relative trace formula (RTF) approach.

Contents

1. Introduction

Let F be a global field and G a reductive group over F. Let π be a (tempered) cuspidal automorphic representation of G. Let $H \subset G$ be a subgroup. A fundamental question in the theory of automorphic forms is to study the H-period integral on π:

$$\int_{[H]} \varphi(h)dh, \quad \varphi \in \pi. \tag{1}$$

Here $[H]$ denotes the automorphic quotient $H(F)\backslash H(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of F. For many pairs (H, G), the period integrals are related to special values of certain L-functions attached to π. Examples include the Rankin–Selberg convolution L-function for $G = \mathrm{GL}_m \times \mathrm{GL}_n$, the standard L-function for GL_n. These examples involve *split* groups only; in particular, when F is a number field, the group G does not satisfy the following condition (unless G is commutative):

- $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ *is compact modulo its center.*

The compactness of $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ can often allow us to derive algebraicity of period integrals and, when combining other inputs, to relate L-values to interesting arithmetic invariants of an appropriate Galois representation associated to π. As an example in the non-split case, Waldspurger considered

$$H = \mathrm{Res}_{F'/F}\,\mathbb{G}_m, \quad G = B^\times, \tag{2}$$

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where $\mathrm{Res}_{F'/F}$ denotes the Weil restriction of scalar, F' is a quadratic field extension of F, with an embedding into a quaternion algebra B over F. In 1980s, Waldspurger [Wal85] proved a formula of the following form (suitably modifying the period integral to take account of the center of G)

$$\left|\int_{[H]} \varphi(h)dh\right|^2 = L(\pi_{F'}, 1/2) \times \text{"Local factors"},$$

where $\pi_{F'}$ denotes the base change of π to F'. The group $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ is compact modulo its center if and only if F is totally real and B is non-split at all archimedean places, in which case the formula of Waldspurger played a crucial role in the study of the Birch–Swinnerton-Dyer conjecture for (modular) elliptic curves over a totally real number field in the analytic rank zero case (see Bertolini–Darmon's variant of Kolyvagin system [BD05]). There has been a large number of (conjectural or proved) generalizations of Waldspurger's formula to more general pairs (H, G), see [Zha18, §2]. We refer to the lecture notes by Beuzart-Plessis, Chaudouard on this topic.

An analogous question to (1) is to upgrade the embedding of topological spaces $[H] \subset [G]$ to one with richer structure. When F is a (suitable) number field, we expect such upgrade when there are Shimura varieties associated to the groups H and G. The pioneering example is the Gross–Zagier formula [GZ86], proved about the same time as Waldspurger's formula, where one has, for $F = \mathbb{Q}$ and F' an imaginary quadratic field,

$$H = \mathrm{Res}_{F'/\mathbb{Q}}\, \mathbb{G}_m, \quad G = \mathrm{GL}_{2,\mathbb{Q}}.$$

Then the Gross–Zagier formula relates the Néron–Tate height pairing of π-isotypical component of a special cycle attached to H to the central derivative of L-function for π whose local components satisfies suitable conditions. After more than ten years, Shou-wu Zhang succeeded extending the Gross–Zagier formula from modular curves to a large class of Shimura curves (over $\mathbb{Q}$ and general totally real field) [Zha01a, Zha01b]; see also a different generalization to Shimura curves over $\mathbb{Q}$ due to Kudla–Rapoport–Yang [KRY06]. The most general form of Gross–Zagier formula for Shimura curves was obtained in [YZZ13] after another ten years. We will recall a special case in §2; we refer to [Zha13, §3] for a survey. The Gross–Zagier formula has a direct application to the Birch–Swinnerton-Dyer conjecture. Together with Kolyvagin's method of Euler system, it settles the rank part of the Birch–Swinnerton-Dyer conjecture for modular elliptic curves over $\mathbb{Q}$ when the analytic rank is at most one.

A natural question is to generalize the Gross–Zagier formula for Shimura curves to higher dimensional Shimura varieties. The goal of this notes is to survey the developments this topic, with an emphasis on the arithmetic Gan–Gross–Prasad (AGGP) conjecture [GGP12] for $\mathrm{U}(n) \times \mathrm{U}(n+1)$ and the relative trace formula approach [Zha12, Rap20]. We will update the recent progress on some of the key local questions since Wei Zhang's previous survey [Zha18], namely the proof of the arithmetic fundamental lemma conjecture for all p-adic fields (with p odd) and of arithmetic transfer conjecture in certain parahoric cases, see §4.

In §5 we will briefly mention several other generalizations of the Gross–Zagier formula and some applications, including the AGGP conjecture in the $\mathrm{U}(n)\times\mathrm{U}(n)$-case formulated by Yifeng Liu, and the arithmetic Rallis inner product formula and Kudla's program on the arithmetic Siegel–Weil formula, as well as the higher derivative version of Gross–Zagier formula over a functional field.

2. Gross–Zagier formula for Shimura curves

We state a mild generalization of the Gross–Zagier formula. The restriction to the local ramification types here will have their generalization later in the high dimensional case.

Let $N > 1$ be an integer. Let $K = \mathbb{Q}[\sqrt{-D}]$ be an imaginary quadratic extension with fundamental discriminant $-D < 0$. For simplicity we assume $D > 4$ and $(D, N) = 1$. This determines a unique factorization

$$N = N^+ N^-$$

where the prime factors of N^+ (N^-, resp.) are all split (inert, resp.) in K. Also impose the *generalized Heegner hypothesis*

- *N^- is square-free with an even number of prime factors.*

We consider the indefinite quaternion algebra B over $\mathbb{Q}$ that is ramified precisely at all factors of N^-. Let $G = B^\times$ viewed as an algebraic group over $\mathbb{Q}$. We have the usual Shimura datum attached to G. Consider the (compactified) Shimura curve $X_{N^+,N^-} = X_U$ where the compact open $U \subset G(\mathbb{A}_f^\times) = B^\times(\mathbb{A}_f)$ is prescribed by

$$U_{N^+,N^-} = \prod_{\ell<\infty} U_\ell, \quad U_\ell = \begin{cases} \Gamma_0(N), & \ell \mid N^+, \\ \mathcal{O}_{B_\ell}^\times, & \ell \nmid N^+. \end{cases}$$

Equivalently, we may consider an Eichler order $\mathcal{O}_{B,N^+}$ in $\mathcal{O}_B$ with level N^+ and define

$$U_{N^+,N^-} = (\mathcal{O}_{B,N^+} \otimes \hat{\mathbb{Z}})^\times.$$

We have an isomorphism for the complex analytic space

$$X_{N^+,N^-}(\mathbb{C}) = B(\mathbb{Q})\backslash(\mathcal{H} \times B(\mathbb{A}_f))/(\mathcal{O}_{B,N^+} \otimes \hat{\mathbb{Z}})^\times \cup \{\text{cusps}\}.$$

In particular, if $N^- = 1$, the curve X_{N^+,N^-} is the (compactified) classical modular curve $X_0(N^+)$, whose complex points are $\Gamma_0(N^+)\backslash\mathcal{H}$ together with cusps.

The case $N^- = 1$ corresponds to the *classical Heegner hypothesis* in [GZ86]: *every prime factor* $\ell \mid N$ *is split in* K. Then there exists an ideal $\mathcal{N}$ of $\mathcal{O}_K$, the ring of integers of K, such that

$$\mathcal{O}_K/\mathcal{N} \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z}.$$

Then the elliptic curves $\mathbb{C}/\mathcal{O}_K$ and $\mathbb{C}/\mathcal{N}^{-1}$ are naturally related by a cyclic isogeny of degree N, therefore define a point, denoted by $x(1)$, on $X_0(N)$. By the theory of complex multiplication, the point $x(1)$ is defined over the Hilbert class field H of K. This depends on a (non-unique) choice of an ideal $\mathcal{N}$ of $\mathcal{O}_K$ above; but they all yield the same Gross–Zagier formula below. By class field theory, the Galois group $\mathrm{Gal}(H/K)$ is isomorphic to the class group $\mathrm{Pic}(\mathcal{O}_K)$. Similarly, under the general Heegner hypothesis, we can define a point $x(1) \in X_{N^+,N^-}(H)$.

Let $E/\mathbb{Q}$ be an elliptic curve with conductor N. Consider a modular parameterization

$$\phi : X_{N^+,N^-} \longrightarrow E.$$

If $N^- = 1$ we require this morphism to send ∞ to 0 of E; in general one needs to assume that it pushes forward the class of the Hodge bundle (in the Picard group) to a multiple of the class of 0 of E. Define

$$y(1) = \phi(x(1)) \in E(H), \quad y_K = \mathrm{tr}_{H/K}\, y(1) \in E(K).$$

Let f be the newform of weight two and $\Gamma_0(N)$-level associated to the elliptic curve E. Let $L(f/K,s) = L(E/K,s)$ be the L-function (without the archimedean factor) associated to the elliptic curve E base changed to K; here we take the classical normalization for $L(f/K,s)$, i.e., the center of functional equation is at $s = 1$. The generalized Heegner hypothesis ensures that the global root number

$$\epsilon(E/K) = -1.$$

Then the central value $L(f/K,1) = 0$ and the Gross–Zagier formula for X_{N^+,N^-} is as follows:

$$\frac{L'(f/K,1)}{(f,f)} = \frac{1}{\sqrt{|D|}} \frac{\langle y_K, y_K\rangle}{deg\phi}, \tag{3}$$

where $\langle y_K, y_K\rangle$ is the Néron–Tate height pairing over K and (f,f) is the Peterson inner product

$$(f,f) := 8\pi^2 \int_{\Gamma_0(N)\backslash\mathcal{H}} f(z)\overline{f(z)}dxdy = \int_{X_0(N)(\mathbb{C})} \omega_f \wedge \overline{i\omega}_f,$$

where $\omega_f := 2\pi i f(z)dz$. This is proved by Gross–Zagier in [GZ86] when $N^- = 1$, in general by S. Zhang in [Zha01a, Zha01b] and Yuan–Zhang–Zhang in [YZZ13].

Remark 2.1. To compare with the Birch–Swinnerton-Dyer conjectural formula for $L'(E/K,1)$, we consider a modular parameterization, still denoted by f,

$$f : X_0(N) \longrightarrow E,$$

which we assume maps the cusp ∞ to zero. The pull-back of the Néron differential ω on E is $f^*\omega = c{\cdot}\omega_f$ for a constant c. It is known that the constant c is an integer. If f is an optimal parameterization, it is called the Manin constant and conjecturally equal to 1. By a theorem of Mazur, we have $p \mid c \implies p \mid 2N$. Then an equivalent form of the formula (3) is

$$\frac{L'(E/K,1)}{|D|^{-1/2}\int_{E(\mathbb{C})}\omega\wedge\overline{\omega}} = \frac{1}{c^2}\frac{\deg f}{\deg\phi}\langle y_K, y_K\rangle.$$

3. Shimura varieties and the arithmetic Gan–Gross–Prasad conjecture

3.1. Special pairs of Shimura data and special cycles.

3.1.1. *Special pairs of Shimura data.* This subsection is largely from [Zha18]. We recall the notion of a *special pair of Shimura data*, which is an enhancement of a spherical pair (H,G) to a homomorphism of Shimura data $(H,X_H)\to(G,X_G)$.

Let $\mathbb{S}$ be the torus $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m$ over $\mathbb{R}$ (i.e., we view $\mathbb{C}^\times$ as an $\mathbb{R}$-group). Recall that a Shimura datum (G,X_G) consists of a reductive group G over $\mathbb{Q}$, and a $G(\mathbb{R})$-conjugacy class $X_G=\{h_G\}$ of $\mathbb{R}$-group homomorphisms $h_G:\mathbb{S}\to G_{\mathbb{R}}$ (sometimes called Shimura homomorphisms) satisfying Deligne's list of axioms [Del71, 1.5]. In particular, X_G is a Hermitian symmetric domain.

Definition 3.1. A *special pair* of Shimura data is a homomorphism [Del71, 1.14] between two Shimura data

$$\delta:(H,X_H)\longrightarrow(G,X_G)$$

such that

(1) the homomorphism $\delta:H\to G$ is injective such that the pair (H,G) is spherical, and
(2) the dimensions of X_H and X_G (as complex manifolds) satisfy

$$\dim_{\mathbb{C}}X_H=\left\lfloor\frac{\dim_{\mathbb{C}}X_G}{2}\right\rfloor.$$

For a Shimura datum (G,X_G) we have a projective system of Shimura varieties $\{\mathrm{Sh}_K(G)\}$, indexed by compact open subgroups $K\subset G(\mathbb{A}_f)$, of smooth quasi-projective varieties (for neat K) defined over a number field E — the reflex field of (G,X_G), viewed as a subfield of $\mathbb{C}$.

For a special pair of Shimura data $(H,X_H)\to(G,X_G)$, compact open subgroups $K_H\subset H(\mathbb{A}_f)$ and $K_G\subset G(\mathbb{A}_f)$ such that $K_H\subset K_G$, we have a finite morphism (over the reflex field of (H,X_H))

$$\delta_{K_H,K_G}:\mathrm{Sh}_{K_H}(H)\longrightarrow\mathrm{Sh}_{K_G}(G).$$

The cycle $z_{K_H,K_G}:=\delta_{K_H,K_G,*}[\mathrm{Sh}_{K_H}(H)]$ on $\mathrm{Sh}_{K_G}(G)$ will be called the *special cycle* (for the level (K_H,K_G)). Very often we choose $K_H=K_G\cap H(\mathbb{A}_f)$ in which case we simply denote the special cycle by z_{K_G}.

When $\dim_{\mathbb{C}}X_G$ is even, the special cycles are in the middle dimension. When $\dim_{\mathbb{C}}X_G$ is odd, the special cycles are just below the middle dimension, and we will say that they are in the *arithmetic middle dimension* (in the sense that, once extending both Shimura varieties to suitable integral models, we obtain cycles in the middle dimension). The special cases in the middle dimension are very often related to the study of Tate cycles and automorphic period integrals, e.g., in the pioneering example of Harder, Langlands, and Rapoport [HLR86] and many of its generalizations. Below we focus on the case where the special cycles are in the arithmetic middle dimension.

3.1.2. *Gross–Zagier pair.* In the case of the Gross–Zagier formula [GZ86], one considers an embedding of an *imaginary* quadratic field F' into $\mathrm{Mat}_{2,\mathbb{Q}}$ (the algebra of 2×2-matrices), and the induced embedding

$$H=\mathrm{Res}_{F'/\mathbb{Q}}\,\mathbb{G}_m\hookrightarrow G=\mathrm{GL}_{2,\mathbb{Q}}.$$

Note that $H_{\mathbb{R}}\simeq\mathbb{C}^\times$ as $\mathbb{R}$-groups (upon a choice of embedding $F'\hookrightarrow\mathbb{C}$). This defines $h_H:\mathbb{S}\to H_{\mathbb{R}}$, and its composition with the embedding $H_{\mathbb{R}}\to G_{\mathbb{R}}$ defines $h_G:\mathbb{S}\to\mathbb{G}_{\mathbb{R}}$. We obtain a special pair $(H,X_H)\to(G,X_G)$, where

$$\dim X_G=1,\quad \dim X_H=0.$$

In the general case, we replace $F'/\mathbb{Q}$ by a CM extension F'/F of a totally real number field F, and replace $\mathrm{Mat}_{2,\mathbb{Q}}$ by a quaternion algebra B over F that is ramified at all but one archimedean places of F. Yuan–Zhang–Zhang proved Gross–Zagier formula in this generality in [YZZ13].

3.1.3. *Gan–Gross–Prasad pair.* Let F be a number field, and let $F' = F$ in the orthogonal case and F' a quadratic extension of F in the Hermitian case. Let W_n be a non-degenerate orthogonal space or Hermitian space with F'-dimension n. Let $W_{n-1} \subset W_n$ be a non-degenerate subspace of codimension one. Let G_i be $\mathrm{SO}(W_i)$ or $\mathrm{U}(W_i)$ for $i \in \{n-1, n\}$, and $\delta : G_{n-1} \hookrightarrow G_n$ the induced embedding. Let

$$G = G_{n-1} \times G_n, \quad H = G_{n-1}$$

with the "diagonal" embedding $\Delta : H \hookrightarrow G$ (i.e., the graph of δ). The pair (H, G) is spherical and we call it the *Gan–Gross–Prasad pair.*

Now we impose the following conditions.

(i) F is a totally real number field, and in the Hermitian case F'/F is a CM (i.e. totally imaginary quadratic) extension.

(ii) For an archimedean place $\varphi \in \mathrm{Hom}(F, \mathbb{R})$, denote by $\mathrm{sgn}_\varphi(W)$ the signature of $W \otimes_{F,\varphi} \mathbb{R}$ as an orthogonal or Hermitian space over $F' \otimes_{F,\varphi} \mathbb{R}$. Then there exists a distinguished real place $\varphi_0 \in \mathrm{Hom}(F, \mathbb{R})$ such that in the orthogonal case,

$$\mathrm{sgn}_\varphi(W_n) = \begin{cases} (2, n-2), & \varphi = \varphi_0, \\ (0, n), & \varphi \in \mathrm{Hom}(F, \mathbb{R}) \backslash \{\varphi_0\}; \end{cases}$$

and in the hermitian case,

$$\mathrm{sgn}_\varphi(W_n) = \begin{cases} (1, n-1), & \varphi = \varphi_0, \\ (0, n), & \varphi \in \mathrm{Hom}(F, \mathbb{R}) \backslash \{\varphi_0\}; \end{cases}$$

in addition, the quotient W_n/W_{n-1} is negative definite at every $\varphi \in \mathrm{Hom}(F, \mathbb{R})$ (so the signature of W_{n-1} is given by similar formulas).

Then Gan, Gross, and Prasad [GGP12, §27] prescribe Shimura data that enhance the embedding $H \hookrightarrow G$ to a homomorphism of Shimura data $(H, X_H) \to (G, X_G)$, where the dimensions are

$$\begin{cases} \dim X_G = 2n-5, & \dim X_H = n-3, \quad \text{in the orthogonal case,} \\ \dim X_G = 2n-3, & \dim X_H = n-2, \quad \text{in the Hermitian case.} \end{cases}$$

The reflex field is F' (which equals F in the orthogonal case) via the distinguished embedding φ_0.

3.2. **The arithmetic Gan–Gross–Prasad conjecture.**

3.2.1. *Height pairings.* Let X be a smooth proper variety over a number field E, and let $\mathrm{Ch}^i(X)$ be the group of codimension-i algebraic cycles on X modulo rational equivalence. We have a cycle class map

$$\mathrm{cl}_i : \mathrm{Ch}^i(X)_{\mathbb{Q}} \longrightarrow H^{2i}(X),$$

where $H^{2i}(X)$ is the Betti cohomology $H^*(X(\mathbb{C}), \mathbb{C})$. The kernel is the group of cohomologically trivial cycles, denoted by $\mathrm{Ch}^i(X)_0$.

Conditional on certain standard conjectures on algebraic cycles, there is a height pairing defined by Beilinson and Bloch,

$$(\cdot,\cdot)_{\mathrm{BB}} : \mathrm{Ch}^i(X)_{\mathbb{Q},0} \times \mathrm{Ch}^{d+1-i}(X)_{\mathbb{Q},0} \longrightarrow \mathbb{R}, \quad d = \dim X, \tag{4}$$

see the paper of Künnemann [Kün98] for the exact statements. This is unconditionally defined when $i = 1$ (the Néron–Tate height), or when X is an abelian variety [Kün01]. In some situations, cf. [Rap20, §6.1], one can define the height pairing unconditionally in terms of the arithmetic intersection theory of Arakelov and Gillet–Soulé [GS90, §4.2.10]. This is the case when there exists a smooth proper model $\mathcal{X}$ of X over $\mathcal{O}_E$ (this is also true for Deligne–Mumford (DM) stacks X and $\mathcal{X}$).

3.2.2. *The arithmetic Gan–Gross–Prasad conjecture.* We consider the special cycle in the Gan–Gross–Prasad setting §3.1.3, which we also call the *arithmetic diagonal cycle* [Rap20]. We will state a version of the arithmetic Gan–Gross–Prasad conjecture assuming some standard conjectures on algebraic cycles (cf. [Rap20, §6]), in particular, that we have the height pairing (4).

For each $K \subset G(\mathbb{A}_f)$, one can construct "Hecke–Künneth" projectors that project the total cohomology of Shimura variety $\mathrm{Sh}_K(G)$ (or its toroidal compactification) to the odd-degree part (cf. [Rap20, §6.2] in the Hermitian case; the same proof works in the orthogonal case). Then we apply this projector to define a cohomologically trivial cycle $z_{K,0} \in \mathrm{Ch}^{n-1}(\mathrm{Sh}_K(G))_0$ (with $\mathbb{C}$-coefficient). The classes $\{z_{K,0}\}_{K\subset G(\mathbb{A}_f)}$ are independent of the choice of our projectors (cf. [Rap20, Remark 6.11]), and they form a projective system (with respect to push-forward).

We form the colimit

$$\mathrm{Ch}^{n-1}(\mathrm{Sh}(G))_0 := \varinjlim_{K\subset G(\mathbb{A}_f)} \mathrm{Ch}^{n-1}(\mathrm{Sh}_K(G))_0.$$

The (conditionally defined) height pairing (4) with $\{z_{K,0}\}_{K\subset G(\mathbb{A}_f)}$ defines a linear functional

$$\mathscr{P}_{\mathrm{Sh}(H)} : \mathrm{Ch}^{n-1}(\mathrm{Sh}(G))_0 \longrightarrow \mathbb{C}.$$

This is the arithmetic version of the automorphic period integral in §1. The group $G(\mathbb{A}_f)$ acts on the space $\mathrm{Ch}^{n-1}(\mathrm{Sh}(G))_0$. For any representation π_f of $G(\mathbb{A}_f)$, let $\mathrm{Ch}^{n-1}(\mathrm{Sh}(G))_0[\pi_f]$ denote the π_f-isotypic component of the Chow group $\mathrm{Ch}^{n-1}(\mathrm{Sh}(G))_0$. Conjecturally the subspace $\mathrm{Ch}^{n-1}(\mathrm{Sh}(G))_0[\pi_f]$ should be an admissible representation of $G(\mathbb{A}_f)$ and $\mathrm{Ch}^{n-1}(\mathrm{Sh}(G))_0$ should be a direct sum of $\mathrm{Ch}^{n-1}(\mathrm{Sh}(G))_0[\pi_f]$'s for π_f appearing in the middle degree cohomology $H^{2n-3}(\mathrm{Sh}(G),\mathbb{C})$; see [Rap20, §6, Remark 6.9].

We are ready to state the arithmetic Gan–Gross–Prasad conjecture [GGP12, §27], parallel to the global GGP Conjecture in [Zha14b] for the central value.

Conjecture 3.2. *Let π be a tempered cuspidal automorphic representation of $G(\mathbb{A})$, appearing in the cohomology $H^*(\mathrm{Sh}(G))$. The following statements are equivalent.*

(1) *The linear functional $\mathscr{P}_{\mathrm{Sh}(H)}$ does not vanish on the π_f-isotypic component $\mathrm{Ch}^{n-1}(\mathrm{Sh}(G))_0[\pi_f]$.*
(2) *The space $\mathrm{Hom}_{H(\mathbb{A}_f)}(\pi_f,\mathbb{C})\neq 0$ and the first order derivative $L'(1/2,\pi,R)\neq 0$.*

In the orthogonal case with $n\leqslant 4$, and when the ambient Shimura variety is a curve ($n=3$), or a product of three curves ($n=4$), the conjecture is unconditionally formulated. The case $n=3$ is proved by Yuan–Zhang–Zhang in [YZZ]; in fact they proved a refined version. When $n=4$ and in the triple product case (i.e., the Shimura variety $\mathrm{Sh}_K(G)$ is a product of three curves), Yuan–Zhang–Zhang formulated a refined version of the above conjecture and proved it in some special cases, cf. [YZZ13].

Since the structure of $\mathrm{Ch}^{n-1}(\mathrm{Sh}(G))_0$ as a representation of $G(\mathbb{A}_f)$ depends on widely open conjectures on algebraic cycles over number fields, there seems no hope to prove the above conjecture. Therefore it is desirable to formulate more accessible ones. Fix $K=\prod_v K_v\subset G(\mathbb{A}_f)$ and consider the Hecke algebra $\mathscr{H}(G,K)$ of bi-K-invariant functions (valued in $\mathbb{C}$). Let S be a large finite set such that K_v is hyperspecial for all $v\notin S$ and let $\mathscr{H}(G,K^S)=\bigotimes_{v\notin S}\mathscr{H}(G,K_v)$ denote the away-from-S spherical Hecke algebra. Let E denote the reflex field and let $R(f)\in\mathrm{Ch}^{2n-3}(\mathrm{Sh}_K(G)\times_{\mathrm{Spec}\,E}\mathrm{Sh}_K(G))$ denote the associated Hecke correspondence. Define

$$\mathrm{Int}_{\mathrm{BB}}(f) := (R(f)*z_{K,0},z_{K,0})_{\mathrm{BB}} \tag{5}$$

whenever the right hand side is defined, for example, by verifying the conditions in [Kün98].

Conjecture 3.3. *Assume that the height pairing (5) is well-defined. There is a decomposition*

$$\mathrm{Int}_{\mathrm{BB}}(f)=\sum_\pi \mathrm{Int}_\pi(f)$$

for all $f\in\mathscr{H}(G,K)$, such that

(1) *the sum runs over all cohomological (with respect to the trivial coefficient system) automorphic representations of $G(\mathbb{A})$ with non-trivial K-invariants, that appear in the middle degree cohomology $H^{2n-3}(\mathrm{Sh}_K(G))$;*
(2) Int_π *is an eigen-distribution for the spherical Hecke algebra $\mathscr{H}(G,K^S)$ with eigen-character λ_π in the sense that for all $f_0\in\mathscr{H}(G,K)$ and $f\in\mathscr{H}(G,K^S)$,*

$$\mathrm{Int}_\pi(f)=\lambda_\pi(f)\cdot\mathrm{Int}_\pi(f_0),$$

where λ_π is the "eigen-character" of $\mathscr{H}(G,K)$ associated to π.

Moreover, if such a representation π is tempered, then

$$\mathrm{Int}_\pi(f)=2^{-\beta_\pi}\mathscr{L}'(1/2,\pi)\prod_{v<\infty}\mathbb{I}_{\pi_v}(f_v). \tag{6}$$

We explain the undefined terms above. Here $\mathscr{L}(s,\pi)$ is the L-functions appearing in the GGP conjecture:

$$\mathscr{L}(s,\pi)=\Delta_n\frac{L(s,\pi,R)}{L(1,\pi,\mathrm{Ad})}, \tag{7}$$

where $L(s,\pi,R)$ is the Rankin–Selberg convolution L-function of the base change of the two factors of $\pi=\pi_{n-1}\boxtimes\pi_n$. We also write $\mathscr{L}(s,\pi_v)$ for the corresponding local factor at a place v. The constant β_π is a power of 2 depending on the number of cuspidal factors in the isobaric sum of the base change of π. The local spherical characters $\mathbb{I}_{\pi_v}$ to the triple (G,H,H) is defined as

$$\mathbb{I}_{\pi_v}(f_v):=\sum_{\phi_v\in\mathrm{OB}(\pi_v)}\alpha_v(\pi_v(f_v)\phi_v,\phi_v),\quad f_v\in\mathscr{C}_c^\infty(G(F_v)),$$

where the sum runs over an orthonormal basis $\mathrm{OB}(\pi_v)$ of π_v and α_v is the normalized local canonical invariant form of Ichino–Ikeda [II10], an integration of matrix coefficient of unitary tempered representation π_v with an invariant inner product $\langle -,-\rangle_{\pi_v}$:

$$\alpha_v(\phi_v,\varphi_v)=\frac{1}{\mathscr{L}(1/2,\pi_v)}\int_{H(F_v)}\langle\pi_v(h)\phi_v,\varphi_v\rangle_{\pi_v}dh.$$

The formula (6) is a generalization of the Gross–Zagier formula, and should be compared with the refinement due to Ichino–Ikeda of the global Gan–Gross–Prasad conjecture, reformulated in terms of local spherical characters (see [Zha14a, Conj. 1.6]).

Now the hope is to find as many K and functions $f\in\mathscr{H}(G,K)$ as we can such that the height pairing (5) is defined. Along this line, we formulate unconditional conjectures below for some special levels $K\subset G(\mathbb{A}_f)$ for unitary groups.

3.3. **The case of smooth integral models.** In fact in [Rap20] Rapoport, Smithling, and W. Zhang work with a variant of the Shimura data. They modify the groups $\mathrm{Res}_{F/\mathbb{Q}}G$ and $\mathrm{Res}_{F/\mathbb{Q}}H$ defined previously to be

$$\begin{aligned}
Z^{\mathbb{Q}}&:=G\mathrm{U}_1=\{z\in\mathrm{Res}_{F'/\mathbb{Q}}\mathbb{G}_m\mid\mathrm{Nm}_{F'/F}(z)\in\mathbb{G}_m\},\\
\widetilde{H}&:=G(\mathrm{U}_1\times\mathrm{U}(W_{n-1}))=\{(z,h)\in Z^{\mathbb{Q}}\times G\mathrm{U}(W_{n-1})\mid\mathrm{Nm}_{F'/F}(z)=c(h)\},\\
\widetilde{G}&:=G(\mathrm{U}_1\times\mathrm{U}(W_{n-1})\times\mathrm{U}(W_n))\\
&=\{(z,h,g)\in Z^{\mathbb{Q}}\times G\mathrm{U}(W_{n-1})\times G\mathrm{U}(W_n)\mid\mathrm{Nm}_{F'/F}(z)=c(h)=c(g)\},
\end{aligned}$$

where the symbol c denotes the unitary similitude factor. Then we have

$$\widetilde{H}\xrightarrow{\sim}Z^{\mathbb{Q}}\times\mathrm{Res}_{F/\mathbb{Q}}H,\qquad\widetilde{G}\xrightarrow{\sim}Z^{\mathbb{Q}}\times\mathrm{Res}_{F/\mathbb{Q}}G.$$

We then define natural Shimura data $(\widetilde{H},\{h_{\widetilde{H}}\})$ and $(\widetilde{G},\{h_{\widetilde{G}}\})$, cf. [Rap20, §3]. This variant has the nice feature that the Shimura varieties are of PEL type, i.e., the canonical models are related to moduli problems of abelian varieties with polarizations, endomorphisms, and level structures, cf. [Rap20, §4–§5].

For suitable Hermitian spaces and a special level structure $K^\circ_{\widetilde{G}}\subset\widetilde{G}(\mathbb{A}_f)$, we can even define *smooth* integral models (over the ring of integers of the reflex field) of the Shimura variety $\mathrm{Sh}_{K^\circ_{\widetilde{G}}}(\widetilde{G})$. For a general CM extension F'/F, it is rather involved to state this level structure [Rap20, Remark 6.19] and define the integral models [Rap20, §5]. For simplicity, from now on we consider a special case, when $F=\mathbb{Q}$ and $F'=F[\varpi]$ is an imaginary quadratic field. We further assume that the prime 2 is split in F'. We choose $\varpi\in F'$ such that $(\varpi)\subset\mathcal{O}_{F'}$ is the product of all ramified ideals in $\mathcal{O}_{F'}$.

We first define an auxiliary moduli functor $\mathcal{M}_{(r,s)}$ over $\mathrm{Spec}\,\mathcal{O}_{F'}$ for $r+s=n$ (similar to [KR14b, §3.1]). For a locally noetherian scheme S over $\mathrm{Spec}\,\mathcal{O}_{F'}$, $\mathcal{M}_{(r,s)}(S)$ is the groupoid of triples (A,ι,λ) where

- (A,ι) is an abelian scheme over S, with $\mathcal{O}_{F'}$-action $\iota:\mathcal{O}_{F'}\to\mathrm{End}_S(A)$ satisfying the Kottwitz determinant condition of signature (r,s), and
- $\lambda:A\to A^\vee$ is a polarization whose Rosati involution induces on $\mathcal{O}_{F'}$ the non-trivial Galois automorphism of F'/F, and such that $\ker(\lambda)$ is contained in $A[\iota(\varpi)]$ of rank $\#(\mathcal{O}_{F'}/(\varpi))^n$ (resp. $\#(\mathcal{O}_{F'}/(\varpi))^{n-1}$) when $n=r+s$ is even (resp. odd). In particular, we have $\ker(\lambda)=A[\iota(\varpi)]$ if $n=r+s$ is even.

Now we assume that $(r,s)=(1,n-1)$ or $(n-1,1)$. We further impose the *wedge condition* and the *(refined) spin condition*, cf. [Rap20, §4.4]. The functor is represented by a Deligne–Mumford stack again denoted by $\mathcal{M}_{(r,s)}$. It is *smooth* over $\operatorname{Spec}\mathcal{O}_{F'}$, despite the ramification of the field extension $F'/\mathbb{Q}$, cf. [Rap20, §4.4]. Then we have an integral model of copies of the Shimura variety $\mathrm{Sh}_{K^\circ_{\widetilde G}}(\widetilde G)$ defined by

$$\mathcal{M}_{K^\circ_{\widetilde G}}(\widetilde G)=\mathcal{M}_{(0,1)}\times_{\operatorname{Spec}\mathcal{O}_{F'}}\mathcal{M}_{(1,n-2)}\times_{\operatorname{Spec}\mathcal{O}_{F'}}\mathcal{M}_{(1,n-1)}.$$

(In [Rap20, §5.1] we do cut out the desired Shimura variety with the help of a sign invariant. Here, implicitly we need to replace this space by its toroidal compactification.)

We now describe the arithmetic diagonal cycle (or rather, its integral model) for the level $K^\circ_{\widetilde H}=K^\circ_{\widetilde G}\cap\widetilde H(\mathbb{A}_f)$. When n is odd (so $n-1$ is even), we define

$$\mathcal{M}_{K^\circ_{\widetilde H}}(\widetilde H)=\mathcal{M}_{(0,1)}\times_{\operatorname{Spec}\mathcal{O}_{F'}}\mathcal{M}_{(1,n-2)},$$

and we can define an embedding explicitly by "taking products" (one sees easily that the conditions on the kernels of polarizations are satisfied):

$$\begin{array}{ccc}\mathcal{M}_{K^\circ_{\widetilde H}}(\widetilde H) & \longrightarrow & \mathcal{M}_{K^\circ_{\widetilde G}}(\widetilde G)\\ (A_0,\iota_0,\lambda_0,A^\flat,\iota^\flat,\lambda^\flat) & \longmapsto & (A_0,\iota_0,\lambda_0,A^\flat,\iota^\flat,\lambda^\flat,A^\flat\times A_0,\iota^\flat\times\iota_0,\lambda^\flat\times\lambda_0).\end{array}$$

When n is even, the situation is more subtle; see [Rap20, §4.4].

With the smooth integral model, we have an unconditionally defined height pairing (4) on $X=\mathrm{Sh}_{K^\circ_{\widetilde G}}(\widetilde G)$. Now we again apply a suitable Hecke–Künneth projector to the cycle z_K for $K=K^\circ_{\widetilde G}$, and we obtain a cohomologically trivial cycle $z_{K,0}\in\mathrm{Ch}(\mathrm{Sh}_{K^\circ_{\widetilde G}}(\widetilde G))_0$. Then the pairing (5) is well-defined

$$\mathrm{Int}_{\mathrm{BB}}(f)=(R(f)*z_{K,0},z_{K,0})_{\mathrm{BB}},\quad f\in\mathscr{H}(\widetilde G,K^\circ_{\widetilde G}).\tag{8}$$

We can restate Conjecture 3.3 as follows.

Conjecture 3.4. *There is a decomposition*

$$\mathrm{Int}_{\mathrm{BB}}(f)=\sum_\pi\mathrm{Int}_\pi(f),$$

for all $f\in\mathscr{H}(\widetilde G,K^\circ_{\widetilde G})$, such that

(1) *the sum runs over all cohomological (with respect to the trivial coefficient system) automorphic representations of $\widetilde G(\mathbb{A})$ with non-trivial $K^\circ_{\widetilde G}$-invariants, that appear in the middle degree cohomology $H^{2n-3}(\mathrm{Sh}_{K^\circ_{\widetilde G}}(\widetilde G))$ and are trivial on $Z^{\mathbb{Q}}(\mathbb{A})$ (hence such π essentially comes from an automorphic representation of $\mathrm{U}(W_{n-1})\times\mathrm{U}(W_n)$);*

(2) Int_π *is an eigen-distribution for the spherical Hecke algebra $\mathscr{H}(\widetilde G,K^\circ_{\widetilde G})$ with eigen-character λ_π in the sense that for all $f\in\mathscr{H}(\widetilde G,K^\circ_{\widetilde G})$,*

$$\mathrm{Int}_\pi(f)=\lambda_\pi(f)\cdot\mathrm{Int}_\pi(1_{K^\circ_{\widetilde G}}),\tag{9}$$

where λ_π is the "eigen-character" of $\mathscr{H}(\widetilde G,K^\circ_{\widetilde G})$ associated to π.

Moreover, if such a representation π is tempered, then

$$\mathrm{Int}_\pi(f)=2^{-\beta_\pi}\mathscr{L}'(1/2,\pi)\prod_{v<\infty}\mathbb{I}_{\pi_v}(f_v).$$

3.4. **The case of regular integral models with parahoric levels.** We could also state an alternative version of the conjecture using the arithmetic intersection theory of Arakelov and Gillet–Soulé [GS90, §4.2.10] on arithmetic Chow groups. The statement is more straightforward but there is an ambiguity in the choice of Green currents.

In fact, we will now introduce a certain parahoric level structure at a finite set S_{in} of *inert* places and modify the level structure at a subset S_{ram} of ramified places. We refer to [Rap20, §5] for the detailed definition for the cases where we have a regular integral model $\mathcal{M}_K(\widetilde G)$ (resp. $\mathcal{M}_{K_{\widetilde H}}(\widetilde H)$) of the Shimura variety $\mathrm{Sh}_K(\widetilde G)$ (resp. $\mathrm{Sh}_{K_{\widetilde H}}(\widetilde H)$) for an appropriate compact open subgroup $K\subset\widetilde G(\mathbb{A}_f)$ (resp. $K_{\widetilde H}\subset\widetilde H(\mathbb{A}_f)$).

For a place $\nu : E \hookrightarrow \mathbb{C}$, we fix a choice of an admissible Green current g_ν for the (unmodified) cycle z_K on the complex analytic space $\mathcal{M}_K(\widetilde{G}) \times_{\operatorname{Spec} E,\nu} \operatorname{Spec}\mathbb{C}$. We let $\tilde{z}_K$ denote the element in the arithmetic Chow group $\widehat{\mathrm{Ch}}^{n-1}(\mathcal{M}_K(\widetilde{G}))$ defined by the integral model $\mathcal{M}_{K_{\widetilde{H}}}(\widetilde{H})$ and the Green currents $(g_\nu)_{\nu\in\mathrm{Hom}_{\mathbb{Q}}(E,\mathbb{C})}$. Let $\mathscr{H}^{\mathrm{spl}}(\widetilde{G},K)$ be the spherical Hecke algebra at all *split* places of F'/F. Then in [Rap20, §4] we defined a family of Hecke correspondence $\hat{R}(f)$ on the integral models, for every $f \in \mathscr{H}^{\mathrm{spl}}(\widetilde{G},K)$. Define

$$\mathrm{Int}_{\mathrm{GS}}(f) = (\hat{R}(f) * \hat{z}_K, \hat{z}_K)_{\mathrm{GS}}, \quad f \in \mathscr{H}^{\mathrm{spl}}(\widetilde{G},K), \tag{10}$$

where "GS" indicates the arithmetic intersection theory in Gillet–Soulé [GS90].

Conjecture 3.5. *There exists a choice of admissible Green currents such that there is a decomposition*

$$\mathrm{Int}_{\mathrm{GS}}(f) = \sum_\pi \mathrm{Int}_\pi(f),$$

for all $f \in \mathscr{H}^{\mathrm{spl}}(\widetilde{G},K)$, characterized by the following properties:

(1) *the sum runs over all cohomological (with respect to the trivial coefficient system) automorphic representations of $\widetilde{G}(\mathbb{A})$ with non-trivial K^S-invariants for $S = S_{\mathrm{in}} \cup S_{\mathrm{ram}}$, that are trivial on $Z^{\mathbb{Q}}(\mathbb{A})$;*

(2) $\mathrm{Int}_\pi(f)$ *is an eigen-distribution for the spherical Hecke algebra $\mathscr{H}^{\mathrm{spl}}(\widetilde{G})$ with eigen-character $\lambda_{\pi^{\mathrm{spl}}}$ similarly defined as (9).*

Moreover, if such a representation π is tempered and appears in the middle degree cohomology $H^{2n-3}(\mathrm{Sh}_{K^\circ_{\widetilde{G}}}(\widetilde{G}))$, then

$$\mathrm{Int}_\pi(f) = 2^{-\beta_\pi}\mathscr{L}'(1/2,\pi)\prod_{v<\infty}\mathbb{I}_{\pi_v}(f_v).$$

Remark 3.6. One could formulate a version of the conjecture over a general totally real field, based on [Rap20, Remark 5.3, §8].

Remark 3.7. In [Rap20, §8.2] we also allowed non-hyperspecial level structure at some *split* places and define an integral model using Drinfeld level structure. Though the product Shimura variety may not be regular, we have an ad-hoc definition of the intersection number $\mathrm{Int}(f)$ when f has regular support.

Remark 3.8. Guided by the more recent thesis of Zhiyu Zhang [Zhab], we may also consider a version of the above conjecture for more general maximal parahoric level structure at *inert* places than those considered in [Rap20].

3.5. **The relative trace formula approach.** Next we move to the relative trace formula approach towards the proof of the conjectures above. See [Cho18] for more background.

We first recall the RTF constructed by Jacquet and Rallis [JR11] to attack the Gan–Gross–Prasad conjecture in the hermitian case. It is associated to the triple (G', H_1', H_2') where

$$G' = \mathrm{Res}_{F'/F}(\mathrm{GL}_{n-1} \times \mathrm{GL}_n),$$

and

$$H_1' = \mathrm{Res}_{F'/F}\mathrm{GL}_{n-1}, \qquad H_2' = \mathrm{GL}_{n-1}\times\mathrm{GL}_n,$$

where (H_1', G') is the Rankin–Selberg pair, and (H_2', G') the Flicker–Rallis pair. Moreover it is necessary to insert a quadratic character of $H_2'(\mathbb{A})$:

$$\eta = \eta_{n-1,n} : (h_{n-1}, h_n) \in H_2'(\mathbb{A}) \longmapsto \eta^{n-2}_{F'/F}(\det(h_{n-1}))\eta^{n-1}_{F'/F}(\det(h_n)),$$

where $\eta_{F'/F} : F^\times\backslash\mathbb{A}^\times \to \{\pm1\}$ is the quadratic character associated to F'/F by class field theory.

We introduce (cf. [Zha12, §3.1]) the global distribution on $G'(\mathbb{A})$ parameterized by a complex variable $s \in \mathbb{C}$,

$$\mathbb{J}(f', s) = \int_{[H_1']}\int_{[H_2']} K_{f'}(h_1,h_2)|\det(h_1)|^s\eta(h_2)dh_1dh_2, \quad f' \in \mathscr{C}_c^\infty(G'(\mathbb{A})). \tag{11}$$

We set its value at $s = 0$

$$\mathbb{J}(f') = \mathbb{J}(f', 0),$$

and its value of the first derivative at $s=0$

$$\partial\mathbb{J}(f') = \frac{d}{ds}\Big|_{s=0} \mathbb{J}(f',s).$$

Then the idea is that, in analogy to the usual comparison of two RTFs, we hope to compare the intersection pairing $\mathrm{Int}(f)$ in (8) or (10) and $\partial\mathbb{J}(f')$ for $f \in \mathscr{H}(\widetilde{G}, K^{\circ}_{\widetilde{G}})$ and any transfer $f' \in \mathscr{C}_c^\infty(G'(\mathbb{A}))$.

Remark 3.9. The idea behind the construction of Jacquet–Rallis is that, by the Flicker–Rallis pair (H_2', G'), the cuspidal part of the spectral side in $\mathrm{RTF}_{(G',H_1',H_2')}$ only contains those automorphic representations that are in the image of the quadratic base change from unitary groups.

For $\gamma \in G'(F_v)_{\mathrm{reg}}$, $f' \in \mathscr{C}_c^\infty(G'(F_v))$, and $s \in \mathbb{C}$, we introduce the (weighted) orbital integral

$$\mathrm{Orb}(\gamma, f', s) = \int_{H'_{1,2}(F_v)} f'(h_1^{-1}\gamma h_2)|\det(h_1)|^s \eta(h_2)dh_1 dh_2. \tag{12}$$

We set

$$\mathrm{Orb}(\gamma, f') := \mathrm{Orb}(\gamma, f', 0), \qquad \partial\mathrm{Orb}(\gamma, f') := \frac{d}{ds}\Big|_{s=0} \mathrm{Orb}(\gamma, f', s). \tag{13}$$

Let $f' = \bigotimes_v f_v'$ be a pure tensor with *regular support* in the sense that there is a place u_0 of F where f'_{u_0} has support in the regular semi-simple locus $G'(F_{u_0})_{\mathrm{reg}}$. Then we have a decomposition of (11) into a sum over the set of regular semisimple orbits $[G'(F)_{\mathrm{reg}}]$,

$$\mathbb{J}(f', s) = \sum_{\gamma \in [G'(F)_{\mathrm{reg}}]} \mathrm{Orb}(\gamma, f', s), \tag{14}$$

where each term is a product of local orbital integrals (12),

$$\mathrm{Orb}(\gamma, f', s) = \prod_v \mathrm{Orb}(\gamma, f_v', s).$$

We have a similar definition of orbital integral on the unitary side. For $g \in G(F_v)_{\mathrm{reg}}$ and $f \in \mathscr{C}_c^\infty(G(F_v))$ we introduce the orbital integral

$$\mathrm{Orb}(g, f) = \int_{H_{1,2}(F_v)} f(h_1^{-1} g h_2) dh_1 dh_2. \tag{15}$$

We now recall the notion of (smooth) transfer of test functions. At a place v, we say that $f \in \mathscr{C}_c^\infty(G(F_v))$ and $f' \in \mathscr{C}_c^\infty(G'(F_v))$ are transfer of each other if for every $\gamma \in G'(F_v)_{\mathrm{reg}}$ we have

$$\omega(\gamma)\mathrm{Orb}(\gamma, f') = \begin{cases} \mathrm{Orb}(g, f), & \text{if } \gamma \text{ matches } g \in G(F_v)_{\mathrm{reg}}, \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

The existence of transfer is known for p-adic places by [Zha14b].

We can then state an arithmetic intersection conjecture for the arithmetic diagonal cycle on the global integral model $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ [Rap20, §8.1].

Conjecture 3.10. *Let $f \in \mathscr{H}^{\mathrm{spl}}(\widetilde{G}, K_{\widetilde{G}})$, and let $f' \in \mathscr{C}_c^\infty(G'(\mathbb{A}))$ be a Gaussian transfer of f. Then*

$$\mathrm{Int}(f) = -\partial\mathbb{J}(f') - \mathbb{J}(f'_{\mathrm{corr}}), \tag{17}$$

where $f'_{\mathrm{corr}} \in \mathscr{C}_c^\infty(G'(\mathbb{A}))$ is a correction function. Furthermore, we may choose f' such that $f'_{\mathrm{corr}} = 0$.

Here $f' = \bigotimes_{v \leqslant \infty} f_v' \in \mathscr{C}_c^\infty(G'(\mathbb{A}))$ is a Gaussian transfer of $f = \bigotimes_{v \nmid \infty} f_v \in \mathscr{H}^{\mathrm{spl}}(\widetilde{G}, K_{\widetilde{G}})$ if f_v' is a transfer of f_v (resp. the constant function $\mathbf{1}$ on the *compact* unitary group) for every $v \nmid \infty$ (resp. for every $v \mid \infty$). The existence of Gaussian transfer at $v \mid \infty$ is proved in [BPLZZ21].

This conjecture implies Conjecture 3.5 at least for a certain class of Hecke functions f. In fact, thanks to the technique of isolating cuspidal spectrum in [BPLZZ21], one can produce many f with transfers f' such that $\mathbb{J}(f', s)$ has a spectral decomposition as a sum over cuspidal automorphic representation of $G'(\mathbb{A})$ Then the desired assertion follows from the local spherical character identities at all places.

3.6. **Hecke functions with regular supports.** The comparison in the equation (17) can be localized for a large class of test functions f and f'. Let $f = \bigotimes_v f_v$ be a pure tensor with *regular support* in the sense that there is a place u_0 of F where f_{u_0} has support in the regular semisimple locus $\widetilde{G}(F_{u_0})_{\mathrm{reg}}$.[1] Then the cycles $\hat{R}(f)*\hat{z}_K$ and $\hat{z}_K$ do not meet in the generic fiber $\mathrm{Sh}_{K_{\widetilde{G}}}(\widetilde{G})$. The arithmetic intersection pairing then localizes as a sum over all places w of the reflex field E

$$\mathrm{Int}(f) = \sum_w \mathrm{Int}_w(f).$$

Here for a non-archimedean place w, the local intersection pairing $\mathrm{Int}_w(f)$ is defined through the Euler–Poincaré characteristic of a derived tensor product on $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G}) \otimes_{\mathcal{O}_E} \mathcal{O}_{E,w}$, cf. [GS90, 4.3.8(iv)].

Similarly, on the GL-side, for $f' = \bigotimes_v f'_v$ be a pure tensor with *regular support*, by (14) the first derivative $\partial\mathbb{J}(f')$ then localizes as a sum over places v of F,

$$\partial\mathbb{J}(f') = \sum_v \partial\mathbb{J}_v(f'),$$

where the summand $\partial\mathbb{J}_v(f')$ takes the derivative of the local orbital integral (cf. (13)) at the place v,

$$\partial\mathbb{J}_v(f') = \sum_{\gamma\in[G'(F)_{\mathrm{reg}}]} \partial\mathrm{Orb}(\gamma, f'_v) \cdot \prod_{u\neq v} \mathrm{Orb}(\gamma, f'_u). \tag{18}$$

It is then natural to expect a place-by-place comparison between $\partial\mathbb{J}_v(f')$ and

$$\mathrm{Int}_v(f) := \sum_{w|v} \mathrm{Int}_w(f)$$

over the places w of E lying above v.

If a place v of F splits in F' (and under the above regularity condition on the support of f and of f'), we have [Rap20, Thm. 1.3]

$$\mathrm{Int}_v(f) = \partial\mathbb{J}_v(f') = 0.$$

For every place w of E above a non-split place v of F,[2] we have a smooth integral model $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G}) \otimes_{\mathcal{O}_E} \mathcal{O}_{E,w}$ when $K_{\widetilde{G},v}$ is a hyperspecial compact open subgroup $\widetilde{G}(\mathcal{O}_{F,v})$ (resp. a special parahoric subgroup $K^\circ_{\widetilde{G},v}$ introduced before) for inert v (resp. ramified v). For such places v, we have a "semi-global" conjecture [Rap20, §8].

Conjecture 3.11. *Let $f \in \mathscr{H}^{\mathrm{spl}}(\widetilde{G}, K_{\widetilde{G}})$, and let $f' \in \mathscr{C}_c^\infty(G'(\mathbb{A}))$ be a Gaussian transfer of f. Then*

$$\mathrm{Int}_v(f) = -\partial\mathbb{J}_v(f') - \mathbb{J}(f'_{\mathrm{corr}}),$$

where $f'_{\mathrm{corr}} \in \mathscr{C}_c^\infty(G'(\mathbb{A}))$ is a correction function. Furthermore, we may choose f' such that $f'_{\mathrm{corr}} = 0$.

Theorem 3.12. *Conjecture 3.11 holds for an inert v with a hyperspecial level provided the residue characteristic is odd.*

We sketch the proof. For $i \in \{n-1, n\}$, let $W_i[v]$ be the pair of nearby Hermitian spaces, i.e., the Hermitian space (with respect to F'/F) that is totally negative at archimedean places, and is not equivalent to W_i at v_0. Let $f = \bigotimes_u f_u$ be a pure tensor such that

(1) $f_v = \mathbf{1}_{\widetilde{G}(\mathcal{O}_{F,v})}$, and

(2) there is a place u_0 of F where f_{u_0} has support in the regular semisimple locus $\widetilde{G}(F_{u_0})_{\mathrm{reg}}$.

Then we have a sum over orbits:

$$\mathrm{Int}_v(f) = 2\log q_v \sum_{g\in\widetilde{G}[v](F)_{\mathrm{reg}}} \mathrm{Int}_v(g) \cdot \prod_{u\neq v} \mathrm{Orb}(g, f_u), \tag{19}$$

where the local intersection number $\mathrm{Int}_v(g)$ is defined by (29) in the next section, and q_v is the residue cardinality of $\mathcal{O}_F$ at v. For $F = \mathbb{Q}$, this is proved [Zha12, Thm. 3.9]. The general case is established in (the proof of) [Rap20, Thm. 8.15].

[1] Strictly speaking here we need to bring in non-trivial level structure at split places; see Remark 3.7. Otherwise we do not know how to produce examples of functions with regular semisimple support.

[2] There is also a requirement on the CM type chosen to define the RSZ moduli functor; let us ignore the subtlety in this notes, and we refer to [Rap20] for interested readers.

By the formulas (18) and (19), the comparison between $\mathrm{Int}_{v_0}(f)$ and $\partial\mathbb{J}_{v_0}(f')$ is then reduced to a local conjecture that we will consider in the next section, the *arithmetic fundamental lemma*:

$$2\mathrm{Int}_v(g)\log q_v = -\omega_v(\gamma)\cdot\partial\mathrm{Orb}(\gamma, \mathbf{1}_{\widetilde{G}(\mathcal{O}_{F,v})}) \tag{20}$$

whenever g and γ match. (Here $\omega_v(\gamma)\in\{\pm1\}$ is a certain transfer factor.) Now Theorem 3.12 follows from Theorem 4.1 in the next section.

Remark 3.13. For a ramified place v_0, the analogous question is also reduced to the local *arithmetic transfer* conjecture formulated by Rapoport, Smithling, and W.Zhang in [RSZ17, RSZ18]. We have a result similar to (19) [Rap20, Thm. 8.15].

Remark 3.14. One may also enlarge the scope of the conjecture by replacing $\mathscr{H}^{\mathrm{spl}}(\widetilde{G}, K_{\widetilde{G}})$ by a bigger (spherical) Hecke algebra, e.g., to include the spherical Hecke algebra at all the inert places. For the purpose of separating spectrum in $\mathbb{J}(f', s)$, Ramakrishnan's density theorem shows that we do not gain more information by doing so. However, besides being a natural and interesting question, there is also another reason to investigate such a question, mainly from the perspective of p-adic height pairing (see the next subsection). We hope to pursue this direction in [LR].

3.7. **p-adic height pairing.** Nekovář defined a p-adic analog of the $\mathbb{R}$-valued height pairing (4), depending on a suitable splitting of the Hodge filtration:

$$(\cdot,\cdot) : \mathrm{Ch}^i(X)_{\mathbb{Q},0}\times \mathrm{Ch}^j(X)_{\mathbb{Q},0}\longrightarrow \mathbb{Q}_p, \quad i+j-1=d=\dim X, \tag{21}$$

which is conditional on more accessible conjectures than (4) (essentially the weight-monodromy conjecture at all places), see [Nek93]. In fact assuming the weight-monodromy conjecture (including the p-adic analog) from now on, the p-adic Abel–Jacobi map lands in the Bloch–Kato Selmer group (a finite dimensional $\mathbb{Q}_p$-vector space) [Nek00]:

$$\mathrm{Ch}^i(X)_0 \longrightarrow H^1_f(E, H^{2i-1}(X_{\overline{E}}, \mathbb{Q}_p(i))).$$

The p-adic height pairing (21) then factors through a pairing on Selmer groups

$$(\cdot,\cdot) : H^1_f(E, H^{2i-1}(X_{\overline{E}}, \mathbb{Q}_p(i)))\times H^1_f(E, H^{2j-1}(X_{\overline{E}}, \mathbb{Q}_p(j)))\longrightarrow \mathbb{Q}_p. \tag{22}$$

Therefore the p-adic height pairing depends only on the absolute cohomological class of cycles, and hence it is more accessible than the $\mathbb{R}$-valued height pairing (4).

Back to the setting of AGGP, we assume that the middle degree cohomology decomposes as a $\mathscr{H}(G,K)\times\mathrm{Gal}(\overline{F}/F')$-module (see [KSZ] for recent progress towards this problem)

$$H^{2n-3}(\mathrm{Sh}_K(G)_{\overline{F}}, \overline{\mathbb{Q}}_p(n-1)) = \bigoplus_{\pi}\pi_f^K\boxtimes\rho_\pi^\vee \tag{23}$$

where π runs over all cohomological automorphic representations of G and ρ_π is a representation of $\mathrm{Gal}(\overline{F}/F')$ with $\overline{\mathbb{Q}}_p$-coefficient, which up to semisimplification is the (normalized) global Langlands correspondence of π at least for stably cuspidal π (note that G is a product of two unitary groups and ρ_π is understood as the tensor product of two Galois representations). Here we fix an isomorphism $\overline{\mathbb{Q}}_p\simeq\mathbb{C}$ to transport $\mathbb{C}$-valued automorphic representations to $\overline{\mathbb{Q}}_p$-valued. For $\varphi^\vee\in(\pi_f^\vee)^K\simeq(\pi_f^K)^*$ (here $\pi_f^\vee$ denotes the contragredient and $(\pi_f^K)^*$ denotes the usual linear dual), we have a linear map $\varphi^\vee(\cdot):\pi_f^K\to\overline{\mathbb{Q}}_p$ and an induced linear map

$$H^1_f(F', H^{2n-3}(\mathrm{Sh}_K(G)_{\overline{F}}, \overline{\mathbb{Q}}_p(n-1)))\xrightarrow{\ \sim\ }\bigoplus_{\pi}\pi_f^K\otimes H^1_f(F', \rho_\pi^\vee)\xrightarrow{\varphi^\vee(\cdot)\otimes\mathrm{id}} H^1_f(F', \rho_\pi^\vee).$$

In particular, we apply the construction to the class of the modified arithmetic diagonal cycle $z_{K,0}$ and obtain an element which we denote by

$$\int_{z_{K,0}}\varphi^\vee\in H^1_f(F', \rho_\pi^\vee). \tag{24}$$

As the notation suggests, this may be viewed as an analog of the automorphic period integral (1). In a work in progress of Daniel Disegni and Wei Zhang [DZ], they attempt to formulate a p-adic analog of the refined GGP conjecture in the style of Ichino–Ikeda, i.e., relating the p-adic height pairing

$$\left(\int_{z_{K,0}}\varphi, \int_{z_{K,0}}\varphi^\vee\right)_p, \quad \varphi\in\pi^K,\ \varphi^\vee\in(\pi^\vee)^K$$

for tempered cuspidal π, to the first derivative of a suitable p-adic L-function $\mathscr{L}'_p(s,\pi)$ times the local canonical invariant forms given by Ichino–Ikeda,

$$2^{-\beta_\pi}\mathscr{L}'_p(1/2,\pi)\prod_{v<\infty}\alpha_v(\varphi_v,\varphi_v^\vee),$$

possibly up to certain local modification at places above p. In particular, this formula would imply that the class $\int_{z_{K,0}}\varphi^\vee\in H^1_f(F,\rho_\pi^\vee)$ in Selmer group does not vanish for some $\varphi\in\pi_f$ if and only if $\mathrm{ord}\mathscr{L}_p(s,\pi)=1$ and the local canonical invariant forms do not vanish.

In [DZ] we hope to prove this conjectural formula when all the p-adic places are split in F'/F, at least for π with mild ramifications and with good ordinary reduction at all p-adic places. We study the p-adic analog of (5)

$$\mathrm{Int}_p(f)=(R(f)*z_{K,0},z_{K,0})_p,\quad f\in\mathscr{H}(G,K) \tag{25}$$

and a p-adic interpolation of the Jacquet–Rallis RTF in §3.5. Assume that the cycles have disjoint support. Then the p-adic height pairing also localized. For almost all places away from p, the local pairing is essentially the same as the intersection pairing in Theorem 3.12 (except replacing $\log q_v$ in (19) by a p-adic $\log_p q_v$). For the remain places away from p and mostly the places above p, substantial work is needed and is beyond the scope of this notes. In some way, the difficulty for the local p-adic height at p-adic places resembles that for the local $\mathbb{R}$-valued height (4) at archimedean places.

4. Local Shimura varieties, arithmetic fundamental lemma and arithmetic transfer

In this section we present more detail on the two local questions, namely the arithmetic analog of fundamental lemma and of (smooth) transfer, which are called the arithmetic fundamental lemma (AFL) and arithmetic transfer (AT).

4.1. **Unitary Rapoport–Zink spaces.** In this section we recall the Rapoport–Zink (RZ) spaces of unitary type, the local analog of CM cycles and local KR divisors, and the statement of the AFL conjecture.

Let F'/F be an unramified quadratic extension of p-adic local fields with p *odd* and let $n\geqslant 1$. Denote by q the residue cardinality of $\mathcal{O}_F$. In this section, we recall the definition of the Rapoport–Zink formal moduli scheme $\mathcal{N}_n=\mathcal{N}_{n,F'/F}$ associated to the unitary group for the quasi-split n-dimensional Hermitian F'-vector space, cf. [RSZ18, §4] and [Zha21, §3].

Let $\breve F$ denote the completion of a maximal unramified extension of F. For $\mathrm{Spf}\,\mathcal{O}_{\breve F}$-schemes S (i.e. an $\mathcal{O}_{\breve F}$-scheme on which p is locally nilpotent), we consider triples (X,ι,λ), where

(1) X is a p-divisible group of absolute height $2nd$ and dimension n over S, where $d:=[F:\mathbb{Q}_p]$.
(2) ι is an $\mathcal{O}_{F'}$-action such that the induced action of $\mathcal{O}_F$ on $\mathrm{Lie}\,X$ is via the structure morphism $\mathcal{O}_F\to\mathcal{O}_S$, and
(3) $\lambda: X\to X^\vee$ is a principal ($\mathcal{O}_F$-relative) polarization.

Hence $(X,\iota|_{\mathcal{O}_F})$ is a strict $\mathcal{O}_F$-module of relative height $2n$ and dimension n. We require that the Rosati involution Ros_λ induces on $\mathcal{O}_F$ the non-trivial Galois automorphism in $\mathrm{Gal}(F'/F)$, denoted by $\mathcal{O}_{F'}\ni a\mapsto\overline{a}$, and that the *Kottwitz condition* of signature $(n-1,1)$ is satisfied, i.e.

$$\mathrm{char}(\iota(a)\mid \mathrm{Lie}\,X;T)=(T-a)^{n-1}(T-\overline{a})\in\mathcal{O}_S[T] \tag{26}$$

for all $a\in\mathcal{O}_{F'}$. An isomorphism $(X,\iota,\lambda)\xrightarrow{\sim}(X',\iota',\lambda')$ between two such triples is an $\mathcal{O}_{F'}$-linear isomorphism $\varphi: X\xrightarrow{\sim}X'$ such that $\varphi^*(\lambda')=\lambda$.

Over the residue field $\mathbb{F}$ of $\mathcal{O}_{\breve F}$, there is a triple $(\mathbb{X}_n,\iota_{\mathbb{X}_n},\lambda_{\mathbb{X}_n})$ such that $\mathbb{X}_n$ is supersingular, unique up to $\mathcal{O}_{F'}$-linear quasi-isogeny compatible with the polarization. We fix such a triple which we call the *framing object*. Then $\mathcal{N}_n$ by definition represents the functor over $\mathrm{Spf}\,\mathcal{O}_{\breve F}$ that associates to each S the set of isomorphism classes of quadruples (X,ι,λ,ρ) over S, where the last entry is an $\mathcal{O}_F$-linear quasi-isogeny of height zero defined over the special fiber $\overline{S}:=S\times_{\mathrm{Spf}\,\mathcal{O}_{\breve F}}\mathrm{Spec}\,\mathbb{F}$,

$$\rho: X\times_S\overline{S}\longrightarrow\mathbb{X}_n\times_{\mathrm{Spec}\,\mathbb{F}}\overline{S},$$

such that $\rho^*((\lambda_{\mathbb{X}_n})_{\overline{S}})=\lambda_{\overline{S}}$. The map ρ is called the *framing*. The formal scheme $\mathcal{N}_n$ is formally locally of finite type and formally smooth over $\mathrm{Spf}\,\mathcal{O}_{\breve F}$ of relative dimension $n-1$.

The group of quasi-automorphisms of the framing object is

$$\mathrm{Aut}^\circ(\mathbb{X}_n,\iota_{\mathbb{X}_n},\lambda_{\mathbb{X}_n})=\{g\in\mathrm{End}^\circ_F(\mathbb{X}_n)\mid g^\vee\circ\lambda_{\mathbb{X}_n}\circ g=\lambda_{\mathbb{X}_n}\}.$$

The condition $g^\vee \circ \lambda_{\mathbb{X}_n} \circ g = \lambda_{\mathbb{X}_n}$ may also be formulated as $gg^* = \mathrm{id}$, where $g \mapsto g^* = \mathrm{Ros}_{\lambda_{\mathbb{X}_n}}(g)$ denotes the Rosati involution. Then $\mathrm{Aut}^\circ(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n})$ acts on $\mathcal{N}_n$ by changing the framing:

$$g \cdot (X, \iota, \lambda, \rho) = (X, \iota, \lambda, g \circ \rho).$$

Another description is as follows. Taking $n = 1$, there is a unique triple $(\mathbb{E}, \iota_{\mathbb{E}}, \lambda_{\mathbb{E}})$ over $\mathbb{F}$ with signature $(1, 0)$. Set

$$\mathbb{V}_n := \mathrm{Hom}^\circ_{\mathcal{O}_{F'}}(\mathbb{E}, \mathbb{X}_n), \tag{27}$$

called the space of *special homomorphisms*, which is an n-dimensional Hermitian F'-vector space with respect to the Hermitian form

$$\langle x, y\rangle = \lambda_{\mathbb{E}}^{-1} \circ y^\vee \circ \lambda_{\mathbb{X}_n} \circ x \in \mathrm{End}^\circ_{F'}(\mathbb{E}) \simeq F'.$$

It is the unique (up to isomorphism) n-dimensional Hermitian space that does *not* contain a self-dual $\mathcal{O}_F$-lattice. Then there is a natural isomorphism

$$\mathrm{Aut}^\circ(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n}) \cong \mathrm{U}(\mathbb{V}_n)(F), \tag{28}$$

where $\mathrm{Aut}^\circ(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n})$ acts by composition on $\mathbb{V}_n$.

We also define a broader class of RZ spaces $\mathcal{N}_N^{[t]}$ for an integer $0 \leqslant t \leqslant n$. We only need to replace the principal polarization in the triple (X, ι, λ) by the following variant:

(3′) $\lambda : X \to X^\vee$ is a polarization such that $\ker(\lambda) \subset X[\varpi]$ has height q^{2t}.

In particular, we have $\mathcal{N}_n = \mathcal{N}_N^{[0]}$. The resulting RZ space $\mathcal{N}_n^{[t]}$ is formally locally of finite type and of strictly semi-stable reduction over $\mathrm{Spf}\, \mathcal{O}_{\breve{F}}$ with relative dimension $n - 1$ [Cho18, Gö01]. The case of $t = 1$ is of special interest, termed as the "almost unramified level" in [RSZ18], see also §4.4 below. In general $\mathcal{N}_n^{[t]}$ is attached to a parahoric subgroup fixing a vertex lattice of type t in a Hermitian space V whose discriminant has valuation $\equiv t \bmod 2$. Then the space $\mathbb{V}_n$ (depending on t but we suppress t from the notation) of special homomorphisms defined by (27) has the structure of Hermitian space of discriminant with valuation $\equiv 1 + t \bmod 2$. Moreover, there is a natural isomorphism $\mathcal{N}_n^{[t]} \simeq \mathcal{N}_n^{[n-t]}$ [Zhab, §5.1]. In particular we have $\mathcal{N}_1^{[0]} \simeq \mathcal{N}_1^{[1]}$. When $n = 2$, $\mathcal{N}_2^{[0]} \simeq \mathcal{N}_2^{[2]}$ is isomorphic to the Lubin–Tate space of height two, and $\mathcal{N}_2^{[1]}$ is isomorphic to Deligne's semistable integral model of the Drinfeld half plane $\widehat{\Omega}_2$ [KR14a].

4.2. **The AFL conjecture/theorem.** Let $\mathcal{N}_{n-1,n} = \mathcal{N}_{n-1} \times_{\mathrm{Spf}\, \mathcal{O}_{\breve{F}}} \mathcal{N}_n$. Then $\mathcal{N}_{n-1,n}$ admits an action by $G(F)$ where we denote now $G = \mathrm{U}(\mathbb{V}_{n-1}) \times \mathrm{U}(\mathbb{V}_{n-1})$.

There is a natural closed embedding $\delta : \mathcal{N}_{n-1} \to \mathcal{N}_n$ (a local analog of the closed embedding $\mathcal{M}_{K^\circ_{\widetilde{H}}}(\widetilde{H}) \to \mathcal{M}_{K^\circ_{\widetilde{G}}}(\widetilde{G})$ on page 8). Let

$$\Delta : \mathcal{N}_{n-1} \longrightarrow \mathcal{N}_{n-1,n}$$

be the graph morphism of δ. We denote by $\Delta_{\mathcal{N}_{n-1}}$ the image of Δ. For $g \in G(F)_{\mathrm{reg}}$, we consider the intersection number on $\mathcal{N}_{n-1,n}$ of $\Delta_{\mathcal{N}_{n-1}}$ with its translate $g\Delta_{\mathcal{N}_{n-1}}$, defined through the derived tensor product of the structure sheaves,

$$\mathrm{Int}(g) := (\Delta_{\mathcal{N}_{n-1}}, g \cdot \Delta_{\mathcal{N}_{n-1}})_{\mathcal{N}_{n-1,n}} := \chi(\mathcal{N}_{n-1,n}, \mathcal{O}_{\Delta_{\mathcal{N}_{n-1}}} \otimes^{\mathbb{L}}_{\mathcal{O}_{n-1,n}} \mathcal{O}_{g\cdot\Delta_{\mathcal{N}_{n-1}}}). \tag{29}$$

Here, for a finite complex $\mathcal{F}$ of coherent $\mathcal{O}_{\mathcal{N}_{n-1,n}}$-modules, we define its Euler–Poincaré characteristic as

$$\chi(\mathcal{N}_n, \mathcal{F}) = \sum_{i,j} (-1)^{i+j} \,\mathrm{length}_{\mathcal{O}_{\breve{F}}} H^j(\mathcal{N}_{n-1,n}, H_i(\mathcal{F}))$$

if the lengths are all finite. The regularity of g implies that the (formal) schematic intersection $\Delta_{\mathcal{N}_{n-1}} \cap g \cdot \Delta_{\mathcal{N}_{n-1}}$ is a *proper scheme* over $\mathrm{Spf}\, \mathcal{O}_{\breve{F}}$, and hence the finiteness of the Euler–Poincaré characteristic.

Recall from (13) the derivative of the orbital integral.

Theorem 4.1 (Arithmetic Fundamental Lemma (AFL))**.** *Let $\gamma \in G'(F)_{\mathrm{reg}}$ match an element $g \in G(F)_{\mathrm{reg}}$. Then*

$$\omega(\gamma) \cdot \partial\mathrm{Orb}(\gamma, \mathbf{1}_{G'(\mathcal{O}_F)}) = -2\mathrm{Int}(g) \cdot \log q.$$

Here $\omega(\gamma) \in \{\pm 1\}$ is an explicit transfer factor (see [RSZ18]).

This statement was conjectured in [Zha12], where the low rank case of $n = 2, 3$ was proved (a simplified proof was given by Mihatsch in [Mih17]). Rapoport, Terstiege, and Zhang in [RTZ13] proved a special case, the so-called minuscule case, for p large comparing to n (a simplified proof was given by Li and Zhu in [LZ]).

For $F = \mathbb{Q}_p$, the theorem is proved [Zha21] in large residue characteristic. The general case is established in [MZ] in large residue characteristic and in [Zhab] in small (and *odd*) residue characteristic.

Remark 4.2. Note that in the formulation of AFL we have restricted ourselves to odd residue characteristic p. This is caused by the same assumption in the theory of RZ spaces [RZ96]. We expect the statement in the AFL conjecture to also hold when $p = 2$.

We refer to [RSZ18, §4] for some other equivalent formulations of the AFL conjectures. Moreover, in [Zhaa] we also formulated an AFL conjecture for the general Bessel cycles, generalizing the arithmetic diagonal cycle $\Delta_{\mathcal{N}_{n-1}}$. However, there are no global analogs of these local cycles due to the lack of relevant Shimura varieties. The corresponding FL conjecture was formulated by Yifeng Liu in [Liu14], which is still unproven.

4.3. **The AFL conjecture of Liu in the Fourier–Jacobi $\mathrm{U}(n) \times \mathrm{U}(n)$ case.** A closely related statement is the AFL conjecture in the Fourier–Jacobi $\mathrm{U}(n) \times \mathrm{U}(n)$ case, formulated by Yifeng Liu [Liu21]. This statement is called the semi-Lie version of AFL conjecture in [Zha21, MZ, Zhab] and is shown to be equivalent to the AFL in Theorem 4.1. In fact the interplay between these two statements is crucial to the inductive proof of both simutanously!

In [KR11], Kudla and Rapoport have defined for every non-zero $u \in \mathbb{V}_n$ a special divisor $\mathcal{Z}(u)$ on $\mathcal{N}_n$. For its definition, note that $\mathcal{N}_1 \cong \operatorname{Spf} \mathcal{O}_{\breve{F}}$, so $(\mathbb{E}, \iota_{\mathbb{E}}, \lambda_{\mathbb{E}})$ deforms to a unique triple $(\mathcal{E}, \iota_{\mathcal{E}}, \lambda_{\mathcal{E}})$ over $\mathcal{O}_{\breve{F}}$, called its *canonical lift*. (This is the universal object over $\mathcal{N}_1$ with Galois conjugated $\mathcal{O}_{F'}$-action.) Then $\mathcal{Z}(u)$ is defined as the locus where the quasi-homomorphism $u : \mathbb{E} \to \mathbb{X}_n$ lifts to a homomorphism from $\mathcal{E}$ to the universal object over $\mathcal{N}_n$. By [KR11, Prop. 3.5], $\mathcal{Z}(u)$ is a relative Cartier divisor (or empty). It follows from the definition that if $g \in \mathrm{U}(\mathbb{V}_n)(F)$, then

$$g\mathcal{Z}(u) = \mathcal{Z}(gu).$$

For simplicity we will write $\mathcal{N}_n \times \mathcal{N}_n$ for the fiber product $\mathcal{N}_n \times_{\operatorname{Spf} \mathcal{O}_{\breve{F}}} \mathcal{N}_n$. For $g \in \mathrm{U}(\mathbb{V}_n)(F)$, let $\Gamma_g \subset \mathcal{N}_n \times \mathcal{N}_n$ be the graph of the automorphism of $\mathcal{N}_n$ induced by g. The fixed point locus of g is defined as the intersection

$$\mathcal{N}_n^g := \Gamma_g \cap \Delta_{\mathcal{N}_n},$$

viewed as a closed formal subscheme of $\mathcal{N}_n$. We also form the *derived fixed point locus*, denoted by ${}^{\mathbb{L}}\mathcal{N}_n^g$, i.e. the derived tensor product

$${}^{\mathbb{L}}\mathcal{N}_n^g := \Gamma_g \cap^{\mathbb{L}} \Delta_{\mathcal{N}_n} := \mathcal{O}_{\Gamma_g} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_n \times \mathcal{N}_n}} \mathcal{O}_{\Delta_{\mathcal{N}_n}}$$

viewed as an element in the K-group $K_0^{\mathcal{N}_n^g}(\mathcal{N}_n)$ with support, cf. [Zha21, Appendix B].

For a pair $(g, u) \in (\mathrm{U}(\mathbb{V}_n) \times \mathbb{V}_n)(F)_{\mathrm{reg}}$ (i.e. $\{g^i u\}_{i=1}^{n-1}$ form a basis of $\mathbb{V}_n$), we now set

$$\operatorname{Int}(g, u) := \chi(\mathcal{N}_n, \mathcal{O}_{\mathcal{Z}(u)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_n}} {}^{\mathbb{L}}\mathcal{N}_n^g). \tag{30}$$

When (g, u) is regular semi-simple, the intersection $\mathcal{Z}(u) \cap \mathcal{N}_n^g$ is a proper *scheme* over $\operatorname{Spf} \mathcal{O}_{\breve{F}}$ and hence the right-hand side of (30) is finite. The number $\operatorname{Int}(g, u)$ depends only on the $\mathrm{U}(\mathbb{V}_n)(F)$-orbit of $(g, u) \in (\mathrm{U}(\mathbb{V}_n) \times \mathbb{V}_n)(F)$. There is an equivalent definition that does not involve the derived fixed point locus ${}^{\mathbb{L}}\mathcal{N}_n^g$ (cf. [Zha21, Rem. 3.1]),

$$\operatorname{Int}(g, u) = \chi(\mathcal{N}_n \times \mathcal{N}_n, \mathcal{O}_{\Gamma_g} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_n \times \mathcal{N}_n}} \mathcal{O}_{\Delta_{\mathcal{Z}(u)}}). \tag{31}$$

To define the analytic side, we consider the symmetric space

$$S_n := \{\gamma \in \operatorname{Res}_{F'/F} \mathrm{GL}_n \mid \gamma\overline{\gamma} = 1_n\},$$

and the F-vector space

$$V_n' = F^n \times (F^n)^*,$$

where $(F^n)^* = \operatorname{Hom}_F(F^n, F)$ denotes the F-linear dual space. For convenience we will identify F^n (resp. $(F^n)^*$) with the space of column vectors (resp. row vectors). With the tautological pairing, we will view V_n' as an $(F \times F)/F$-hermitian space. Let $\mathrm{GL}_{n,F}$ act (diagonally) on the product $S_n \times V_n'$,

$$h \cdot (\gamma, (u_1, u_2)) = (h^{-1}\gamma h, (h^{-1}u_1, u_2 h)).$$

For $(\gamma,u')\in(S_n\times V_n')(F)_{\mathrm{reg}}$, $\Phi'\in\mathscr{C}_c^\infty((S_n\times V_n')(F))$, and $s\in\mathbb{C}$, we define the orbital integral

$$\mathrm{Orb}((\gamma,u'),\Phi',s):=\int_{\mathrm{GL}_n(F)}\Phi'(h\cdot(\gamma,u'))|\det h|^s\eta(h)dh,$$

with special values

$$\mathrm{Orb}((\gamma,u'),\Phi'):=\mathrm{Orb}((\gamma,u'),\Phi',0),$$
$$\partial\mathrm{Orb}((\gamma,u'),\Phi'):=\frac{d}{ds}\Big|_{s=0}\mathrm{Orb}((\gamma,u'),\Phi',s).$$

Theorem 4.3 (AFL, semi-Lie algebra version)**.** *Suppose that $(\gamma,u')\in(S_n\times V_n')(F)_{\mathrm{reg}}$ matches the element $(g,u)\in(\mathrm{U}(\mathbb{V}_n)\times\mathbb{V}_n)(F)_{\mathrm{reg}}$. Then*

$$\omega(\gamma,u')\cdot\partial\mathrm{Orb}((\gamma,u'),\mathbf{1}_{(S_n\times V_n')(\mathcal{O}_F)})=-\mathrm{Int}(g,u)\cdot\log q.$$

Here, $\omega(\gamma,u')\in\{\pm1\}$ is the transfer factor [Zha21, (2.13)].

The case of a fixed u with $(u,u)\in\mathcal{O}_F^\times$ is equivalent to the AFL conjecture in Theorem 4.1, cf. also [Zha21, Conj. 3.2]. In that case there is a natural isomorphism

$$\mathcal{Z}(u)\simeq\mathcal{N}_{n-1}.$$

By [Zha21, Prop. 4.12], the intersection number (31) is equal to the one considered in [Zha12],

$$\chi(\mathcal{N}_{n-1,n},\mathcal{O}_{(1,g)\cdot\Delta_{\mathcal{N}_{n-1}}}\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_{n-1,n}}}\mathcal{O}_{\Delta_{\mathcal{N}_{n-1}}}),$$

and the orbital integral reduces to the one in [MZ] as well, and hence Theorem 4.3 implies Theorem 4.1. It is less trivial to show in [Zha21, MZ, Zhab] that that Theorem 4.1 when $n+1$ replacing n implies Theorem 4.3.

Remark 4.4. The strategy of the proof in [Zha21, MZ, Zhab] is of global nature, crucially relying on the modularity of generating series of special divisors (in [Zha21] W. Zhang uses a theorem of Bruinier–Howard–Kudla–Rapoport–Yang [BHK+20], and in [MZ, Zhab] the authors uses a theorem in [YZZ09]). On the other hand, Beuzart-Plessis has found a local proof of the Jacquet–Rallis fundamental lemma [BP21]. Is there a local proof of the AFL conjecture?

A survey on the proof of AFL can be found in [Zha19].

4.4. **Arithmetic transfer conjectures.** We finally discuss analogs of the smooth transfer in the arithmetic context over a p-adic field. Let F'/F be as above. Fix an integer $0\leqslant t\leqslant n-1$ and $e\in\{0,1\}$. Then there is a natural embedding $\delta:\mathcal{N}_{n-1}^{[t]}\to\mathcal{N}_n^{[t+e]}$ and let $\Delta:\mathcal{N}_{n-1}^{[t]}\to\mathcal{N}_{n-1,n}$ be the graph morphism of δ.

Consider the product $\mathcal{N}_{n-1,n}=\mathcal{N}_{n-1}^{[t]}\times_{\mathrm{Spf}\,\mathcal{O}_{\breve F}}\mathcal{N}_n^{[t+e]}$. It is regular precisely when (at least) one of the two factors is formally smooth over $\mathrm{Spf}\,\mathcal{O}_{\breve F}$; there cases were considered in [RSZ18]. Otherwise, in [Zhab] Z. Zhang resolves the singularity by blowing up a certain Weil divisor of $\mathcal{N}_{n-1,n}$, the resulting space is regular and denoted by $\widetilde{\mathcal{N}}_{n-1,n}$. The action of $G(F)=(\mathrm{U}(\mathbb{V}_{n-1})\times\mathrm{U}(\mathbb{V}_n))(F)$ lifts to one on $\widetilde{\mathcal{N}}_{n-1,n}$. Moreover, the strict transform of $\Delta_{\mathcal{N}_{n-1}^{[t]}}$ does not change and hence the morphism Δ lifts to $\widetilde{\Delta}:\mathcal{N}_{n-1}^{[t]}\to\widetilde{\mathcal{N}}_{n-1,n}$. Similar to (29) we may then define for $g\in G(F)_{\mathrm{reg}}$,

$$\mathrm{Int}(g):=\chi(\widetilde{\mathcal{N}}_{n-1,n},\mathcal{O}_{\widetilde{\Delta}_{\mathcal{N}_{n-1}^{[t]}}}\otimes^{\mathbb{L}}_{\mathcal{O}_{\widetilde{\mathcal{N}}_{n-1,n}}}\mathcal{O}_{g\cdot\widetilde{\Delta}_{\mathcal{N}_{n-1}^{[t]}}}).$$

Let $\Lambda^\flat$ be a Hermitian lattice of type t (i.e., $\Lambda^\flat\subset(\Lambda^\flat)^\vee$ is of colength t, where $(\Lambda^\flat)^\vee$ is the dual lattice of $\Lambda^\flat$). It corresponds to a vertex on the Bruhat–Tits building of the unitary group $\mathrm{U}(V^\flat)(F)$, where $V^\flat=\Lambda^\flat\otimes_{\mathcal{O}_F}F$, and the stabilizer of this vertex is a maximal parahoric subgroup. Let $\Lambda=\Lambda^\flat\oplus\mathcal{O}_{F'}u$ be an orthogonal sum for a vector u whose norm (u,u) has valuation $e\in\{0,1\}$. Then Λ is a Hermitian lattice of type $t+e$. Choose an orthogonal basis $\{u_1,\dots,u_{n-1}\}$ of $\Lambda^\flat$ such that $\mathrm{val}(u_i,u_i)\in\{0,1\}$. Using this basis to identify $\Lambda^\flat\simeq\mathcal{O}_{F'}^{n-1}\subset F'^{n-1}$. The lattice chain $\{\Lambda^\flat,(\Lambda^\flat)^\vee\}$ defines a parahoric integral model of $\mathrm{Res}_{F'/F}\mathrm{GL}_{n-1}(F)$. Note that we have a "good-position" condition $\mathrm{Res}_{F'/F}\mathrm{GL}_{n-1}(\mathcal{O}_F)\subset\mathrm{Res}_{F'/F}\mathrm{GL}_n(\mathcal{O}_F)$. Let $G'(\mathcal{O}_F)$ denote the product of the two parahoric subgroups.

In [Zhab] Z. Zhang proved:

Theorem 4.5 (Arithmetic Transfer Conjecture (ATC) at maximal parahoric levels)**.** *Assume that F is unramified over $\mathbb{Q}_p$ when $0 < t < n-1$. Let $\gamma \in G'(F)_{\mathrm{reg}}$ match an element $g \in G(F)_{\mathrm{reg}}$. Then*

$$\omega(\gamma) \cdot \partial\mathrm{Orb}(\gamma, c \cdot \mathbf{1}_{G'(\mathcal{O}_F)}) = -2\mathrm{Int}(g) \cdot \log q.$$

Here $\omega(\gamma) \in \{\pm 1\}$ is an explicit transfer factor (see [Zhab]*), and c is an explicit constant (depending on n, t, e).*

Remark 4.6.

(1) In a work in progress [LRZ], Li, Rapoport, and W. Zhang are formulating more AT conjectures, including new examples involving the space $\mathcal{N}_{n-1}^{[1]} \times_{\mathrm{Spf}\,\mathcal{O}_{\breve{F}}} \mathcal{N}_n^{[0]}$ which does not appear in the above theorem of Z. Zhang.

(2) When F'/F is a *ramified* quadratic extension of p-adic fields (for odd p), Rapoport, Smithling, and W. Zhang proposed an AT conjecture in [RSZ17, RSZ18]. They proved the conjecture for $n = 2, 3$. However, the higher rank case remains open.

5. Other related works

5.1. **Some other generalizations of Gross–Zagier formula.** Besides the orthogonal and unitary AGGP conjecture in §3.2.2, there are several other conjectural generalizations of Gross–Zagier formula, corresponding to different constructions of algebraic cycles (and possibly different families of L-functions). Shouwu Zhang summarized some of them in his notes [Zha10] and since then there have been a couple of new additions.

The first one is the arithmetic Rallis inner product for both orthogonal and unitary groups, which involves the standard L-function of an automorphic representation of $\mathrm{SO}(2n+1)$ and $\mathrm{U}(n)$. By the doubling method, this is largely tied with Kudla's program on the arithmetic analog of Siegel–Weil formula. We refer to Chao Li's notes [Li22] for more detail on the history and the recent progress.

Another one is formulated by Yifeng Liu [Liu21], which is the arithmetic analog of the $\mathrm{U}(n) \times \mathrm{U}(n)$ Fourier–Jacobi case of GGP conjecture. As already mentioned in §4.3, the local questions are closely related to the $\mathrm{U}(n) \times \mathrm{U}(n+1)$-case. Note that there is no known conjectural generalization of Gross–Zagier formula for the Rankin–Selberg L-function for other pairs $\mathrm{U}(m) \times \mathrm{U}(n)$ (a very curious coincidence: the Rankin–Selberg theory takes the simplest form for $\mathrm{GL}(n) \times \mathrm{GL}(m)$ when $|n-m| \in \{0, 1\}$). Moreover, we do not know whether Liu's construction has an analog for orthogonal groups. In fact, all the known "arithmetic conjectures" (including the one in the next paragraph) involve "*an incoherent group*" G satisfying the following condition (cf. §1)

- ◦ *$G(F \otimes_{\mathbb{Q}} R)$ is compact modulo its center,*

a condition labeled as $(*)$ in Gan–Gross–Prasad [GGP12, §27]. Here, following Kudlas' terminology, by "an incoherent group" we refer to the isometric group associated to an incoherent collection of quadratic or Hermitian spaces.

More recently, there is another construction of special cycles based on a symmetric pair (H, G), a (not necessarily inner) form of the pair $(\mathrm{GL}_n \times \mathrm{GL}_n, \mathrm{GL}_{2n})$, for example $(\mathrm{U}(n) \times \mathrm{U}(n), \mathrm{U}(2n))$. See the introduction of [Li] for the global Shimura varieties and subvarieties. We obtain new algebraic cycles on Shimura varieties associated to unitary groups which may locally be the unit group of any central simple algebra, a feature not seen in the AGGP conjecture or Kudla's program, see the paper by Q. Li [Li] and the work in progress by Li–Mihatsch [LM]. For the $(\mathrm{U}(n) \times \mathrm{U}(n), \mathrm{U}(2n))$ automorphic period integral ("unitary Friedberg–Jacquet period"), a relative trace formula approach is proposed and the stable FL on Lie algebra is established in [XZ]; the relative endoscopic FL on Lie algebra is established by Leslie [Les].

On the more speculative side, from [Zhaa] we see that there may also be "exceptional" examples, for which there is local cycle (i.e., on local Shimura variety) but there is no global Shimura variety. Besides the general Bessel case $\mathrm{SO}(m) \times \mathrm{SO}(m+2r+1)$, there is also the (local) cycle for the Ginzburg–Rallis (H, G), which is related to the exterior cube L-function of automorphic representation on GL_6. We will have to wait for the invention of new "global Shimura varieties" to formulate the corresponding Gross–Zagier problem!

5.2. **Some arithmetic applications: the Beilinson–Bloch–Kato conjecture.** In Introduction §1, we have mentioned that Waldspurger's formula was a crucial ingredient in the proof of the Birch–Swinnerton-Dyer conjecture for (modular) elliptic curves over a totally real number field in the analytic rank zero case, by Bertolini–Darmon's variant of Kolyvagin system argument for Heegner points on a

Shimura curve. In a higher dimensional (than curve) case, Yifeng Liu first proved the rank zero analog for the "twisted triple product" automorphic motives in the product of a Shimura curve and a Hilbert–Blumenthal modular surface [Liu16], which may be placed in the framework of the triple product L-function and the Gross–Kudla–Schoen diagonal cycle (or equivalently the $\mathrm{SO}(3)\times\mathrm{SO}(4)$-case of the Gross–Prasad construction); see also [Liu19, LT20, Wan] for other triple product L-functions.

In [LTX+22] we extend this paradigm to the Rankin–Selberg motives in the context of the Gan–Gross–Prasad construction for unitary groups $G = \mathrm{U}(n-1)\times\mathrm{U}(n)$. Let π be a cohomological stably cuspidal tempered automorphic representation of $G(\mathbb{A})$ appearing in the cohomology (23). Under certain technical conditions (mostly on the image of the residue Galois representation), we show that

$$L(1/2,\pi)\neq 0 \implies H^1_f(F',\rho_\pi)=0.$$

This confirm the corresponding Beilinson–Bloch–Kato conjecture in the analytic rank zero case.

We also obtain a partial result towards the rank one case: under certain technical conditions on π, we show that

- ⋄ *if the class* $\int_{z_{K,0}}\varphi\in H^1(F',\rho_\pi)$ *in Selmer group (24) is non-zero, then* $\dim H^1_f(F',\rho_\pi)=1$.

This is a direct generalization of Kolyvagin's theorem for Heegner points. Using a different Euler system argument (more like Kolyvagin's Heegner point Euler system), Jetchev, Nekovář, and Skinner also obtain a similar result (under different conditions) as well as one divisibility in a generalized Iwasawa Main Conjecture. Combining the above result in the rank one case with the p-adic version of AGGP in §3.7, we would obtain a p-adic version of the Beilinson–Bloch–Kato conjecture, formulated by Fontaine–Perrin-Riou that (under technical conditions on π)

$$\mathrm{ord}L_p(1/2,\pi)=1 \implies \dim H^1_f(F',\rho_\pi)=1.$$

We would like to mention a series of recent remarkable work by Loeffler and Zerbes [LPSZ21, LSZ22, LZ], partly joint with Pilloni and Skinner, in which they utilize the Gross–Prasad construction for orthogonal groups $\mathrm{SO}(4)\times\mathrm{SO}(5)$ and the exceptional isogeny $\mathrm{SO}(4)\sim\mathrm{SL}(2)\times\mathrm{SL}(2)$ and $\mathrm{SO}(5)\sim\mathrm{Sp}(4)$. As one of the applications, Loeffler and Zerbes [LZ] proved that, under certain technical hypotheses, for a modular abelian surface A over $\mathbb{Q}$,

$$\mathrm{ord}_{s=1}L(s,H^1(A))=0 \implies \mathrm{rank}\,A(\mathbb{Q})=0,$$

as predicted by (a generalization of) Birch–Swinnerton-Dyer conjecture! Rather than using Chow cycles as in the above work on Rankin–Selberg motives, their work makes use of "higher Chow cycles" (elements in higher Chow groups), similar to Kato's Euler system argument using Siegel's units on modular curves (this seems one of the reasons why the result is currently only known when the base field is $\mathbb{Q}$). One of the difficulties is that the Tate module of an abelian surfaces has non-regular Hodge–Tate weights, hence does not appear directly in the (étale) cohomology of any Shimura varieties. To overcome this difficulty, a limit argument is necessary to extrapolate the Iwasawa theoretical Euler system from the regular cases to non-regular cases.

For another example of employing "higher Chow cycles", Zhou [Zho] proved a level-raising result using higher cycle classes on the special fiber of quaternionic Shimura varieties attached to an inner form of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ for a totally real field F. It would be interesting to know whether the local classes constructed in such a way can be lifted to global ones.

5.3. **Functional field analogs.** When the global field F is a function field, there is an even further upgrade of $[H]\subset[G]$ to one for moduli spaces of shtukas [Yun18, Yun]. In [YZ17, YZ19], in the case (2),

$$H=\mathrm{Res}_{F'/F}\,\mathbb{G}_m,\qquad G=B^\times,$$

we express arbitrary higher order derivatives of L-functions of mildly ramified π in terms of the intersection numbers of π-isotypical components of a special cycle associated to H. See also the related work by Howard and Shnidman [HS19, Shn], and by C. Qiu [Qiu19]. In recent preprints in [FYZa, FYZb, FYZc], Feng, Yun, and W. Zhang investigated a function field analog of Kudla's program on generating series of special cycles and the arithmetic Siegel–Weil formula.

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