The arithmetic fundamental lemma and fine Deligne-Lusztig varieties

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§1 Global motivation: AGGP Conj

Gross-Zagier: $E/\mathbb{Q}$ elliptic curve,

$K/\mathbb{Q}$ im quad.

$\rightsquigarrow$ $\underbrace{L'(E/K, 1)}_{\text{analytic}} \sim \langle P_{E,K}, P_{E,K}\rangle_{NT}$.

$P_{E,K}$: $\underbrace{\text{Heegner pts} \in E(K)}_{\text{arith-geom}}$

AGGP = vast generalization to higher dim Shimura var.

$H = SO(2, n-2) \hookrightarrow G = SO(2, n-1)$

or $H = U(1, n-2) \hookrightarrow G = U(1, n-1)$.

$\rightsquigarrow$ $\delta : Sh_H \hookrightarrow Sh_G$ ambient Sh vars

dim $n-2$ $\quad$ $n-1$ $\quad$ (subvar : codim 1)

$\Delta = (1, \delta) : Sh_H \hookrightarrow Sh_{H\times G}$

dim $n-2$ $\quad$ $2n-3$ $\quad$ $(n-2 = \lfloor \frac{2n-3}{2} \rfloor.)$

$\rightsquigarrow$ $\Delta = \Delta(Sh_H)$ $\quad$ (arith middle dim).

Δ: called GGP cycle (dim $2n-4$, codim 1).

AGGP Let π temp cusp auto rep appearing in $H^{2n-1}(Sh_{H\times G})$

Then $L'(\frac{1}{2}, \pi) \sim \langle \Delta_\pi, \Delta_\pi \rangle_{BB}$

Ex $n=2$. $H \times G = SO(2) \times SO(3)$

Δ = Heegner pt on modular curve by Gross-Zagier.

(by GZ, YZZ).

Ex $n=3$, $H\times G = SO(3)\times SO(4)$

Δ = Gross-Schoen cycle on $(\text{curve})^3$. (by YZZ).

Ex (Widely open).

- $SO(5)$ or $GSpin_5$ (essentially GSp_4)

  $\rightsquigarrow$ $Sh_{H\times G}$ of abelian surfaces
- $SO(6) \rightsquigarrow$ abelian 4-fold w/ CM.
- $SO(8) \rightsquigarrow$ abelian 8-fold w/ quaternion multiplication.
- $SO(21) \rightsquigarrow$ polarized K3 surfaces.

Rmk AGGP is still widely open for $n\geq 4$.

§2 Unitary AGGP via RTF approach

Unitary GGP Jacquet-Rallis: Compare two RTFs

Why unitary: E/F quad ext

$$\begin{array}{c} \Delta(U_{n-1})\backslash U_{n-1}\times U_n/\Delta(U_{n-1}) \\ \updownarrow \\ \Delta(GL_{n-1})_E\backslash (GL_{n-1}\times GL_n)_E/(GL_{n-1}\times GL_n)_F. \end{array}$$

Spectrally $|\mathcal{P}_{\Delta(U_{n-1})}(\varphi)|^2 \sim L(\tfrac{1}{2}, \pi_E)$.

Geometrically $\underset{\substack{\uparrow \\ \text{conj action}}}{U_{n-1}\backslash\backslash U_n} \longleftrightarrow (GL_{n-1})_F\backslash \underset{\substack{\uparrow \\ \text{sym space}}}{S_n}$

where $S_n(F) = \{g\in GL_n(E): g\bar{g}=1\}$.

Matching of orbits (E/F unram quad ext of p-adic fields):

$$[U(V_n)]_{rs} \amalg [U(C_n)]_{rs} \simeq [S_n(F)]_{rs}$$

- V_n non-split, C_n split (Herm sps of dim n)

JRFL If $g \in U(V_n)(F)_{rs} \longleftrightarrow \gamma \in S_n(F)_{rs}$

Then $\mathrm{Orb}(g, \mathbb{1}_{U(V_n)(\mathcal{O}_F)}) = \pm \mathrm{Orb}(\gamma, \mathbb{1}_{S_n(\mathcal{O}_F)})$

(proved by Yun, Gordan, Beuzart-Plessis)

purely local

Unitary AGGP Some modifications input:

(1) $(L(\frac{1}{2}, \pi) \rightsquigarrow L'(\frac{1}{2}, \pi))$.

replace $\mathrm{Orb}(\gamma, \mathbb{1}_{S_n(\mathcal{O}_F)})$ by $\partial\mathrm{Orb}(\gamma)$,

$$\partial\mathrm{Orb}(\gamma) := \frac{\partial}{\partial s}\Big|_{s=0} \left(\int_{GL_{n-1}(F)} \mathbb{1}_{S_n(\mathcal{O}_F)}(h^{-1}\gamma h)\, \eta_{E/F}(\det h) |\det h|^s \, dh \right).$$

(2) $(|P_{A(U_{n-1})}(\varphi)|^2 \rightsquigarrow \langle \Delta_\pi, \Delta_\pi \rangle_{BB})$.

replace $\mathrm{Orb}(g)$ by $\mathrm{Int}(g)$: arith intersection number on a local Shimura var (RZ space).

AFL Conj (Zhang) If $g \in U(C_n)(F)_{rs} \longleftrightarrow \gamma \in S_n(F)_{rs}$

Then $\mathrm{Int}(g) = \pm \partial\mathrm{Orb}(\gamma) \underbrace{/\log q}_{\text{to get an integer.}} \in \mathbb{Z}$

Rmk Zhang proved AFL for $n \leq 3$ (global case).

Q Can we generalize it to higher dim'l Sh vars?

§3 Main result

Thm A (He-Li-Zhu, for unitary grp)

AFL holds when g is minuscule

Rmk In the case $F = \mathbb{Q}_p$, $p > \frac{n+1}{2}$,

Thm A first proved by Rapoport-Terstiege-Zhang (2013).

HLZ's proof is different.

Def of "Int(g)":

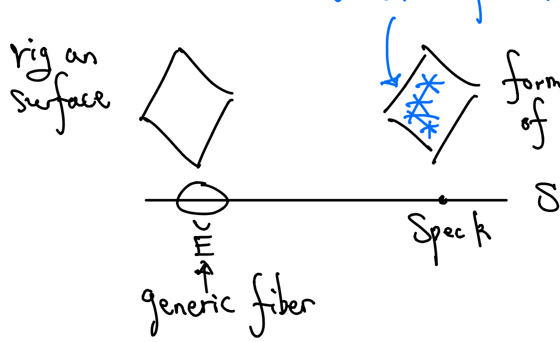
<u>Def</u> Let $\mathcal{N}_n$ be a unitary RZ space

- $\mathcal{N}_n$ is a formal sch $/ \operatorname{Spf} \mathcal{O}_{\breve{E}}$, $\breve{E} = \hat{E}^{ur}$
- parametrizing deformations of unitary p-div grps $X_0 / k = \bar{\mathbb{F}}_q$ of dim n & height 2n, signature $(1, n-1)$.
- p-adic uniformization of $Sh^{ss}_{U(1,n-1)}/k$ (supersingular locus).
- $\dim \mathcal{N}_n / S = n-1$, $S = \operatorname{Spf} \mathcal{O}_{\breve{E}}$
- $\dim \mathcal{N}_n^{red} = \lfloor \frac{n-1}{2} \rfloor$. ($\mathcal{N}_n$ formal sch $\Rightarrow$ highly non-reduced.)

(structure of $\mathcal{N}_n^{red}$ studied by Vollard-Wedhorn (2011).)

<u>Ex</u> $n=3$.

red locus, component $\simeq \{ x^{q+1} + y^{q+1} + z^{q+1} = 0 \} \subseteq \mathbb{P}^2$
Fermat equation



($\leadsto$ red dim $= 1$).

<u>Def</u> Assume $C_n = C_{n-1} \oplus Eu$, $(u,u) = 1$.

$\delta : U(C_{n-1}) \hookrightarrow U(C_n)$ (relative case) <u>note</u> $U(C_{n-1}) \curvearrowright \mathcal{N}_{n-1}$.

$\leadsto \delta : \mathcal{N}_{n-1} \hookrightarrow \mathcal{N}_n$.

$\leadsto$ local GGP cycle $\Delta(\mathcal{N}_{n-1}) \subseteq \mathcal{N}_{n-1} \times_S \mathcal{N}_n$.

Def $g \in U(C_n)(F) \circlearrowright \mathcal{N}_n$.

$$\mathrm{Int}(g) = \langle \Delta, (1\times g)\Delta \rangle$$
$$:= \chi(\mathcal{N}_{n-1} \times_S \mathcal{N}_n, \ \mathcal{O}_\Delta \overset{\mathbb{L}}{\otimes} \mathcal{O}_{(1\times g)\Delta})$$

outside derived cat : int'n can be non-proper, not transversal.

(to deal improper intersection)

Def of "minuscule":

Def Say g is <u>regular semisimple</u> if $L(g) = \mathcal{O}_E u + \mathcal{O}_E g u + \cdots + \mathcal{O}_E g^{n-1} u \subseteq C_n$ is full rank $\mathcal{O}_E$-lattice.

Say g is <u>minuscule</u> if
$$\varpi L(g)^\vee \subseteq L(g) \subseteq L(g)^\vee.$$

Rmk g minuscule

$\Rightarrow$ (1) $\partial\mathrm{Orb}(\gamma)$ is easy to compute

(2) $\mathrm{Int}(g) = \mathrm{length}_k(\underbrace{\Delta \cap (1\times g)}_{\text{Artinian sch}/k}\Delta)$

Thm A reduces to a specific formula as follows.

§4 Explicit formula for Int(g) (g minuscule)

Def Let $V = L(g)^\vee / L(g)$ (hermitian space / $\mathbb{F}_{q^2}$).

g stabilize $L(g)$ & $L(g)^\vee$

$\rightsquigarrow \bar{g} \in U(W)(\mathbb{F}_q)$.

Let f = char poly of $\bar{g}$, $f \in \mathbb{F}_{q^2}[T]$.

Def For any $Q \in \mathbb{F}_{q^2}[T]$

write $Q = \prod_i (T - \lambda_i)$, $\lambda_i \in \bar{\mathbb{F}}_q$.

Define Q^* (its reciprocal)

$$Q^* := \prod_i (T - 1/\lambda_i^q)$$

Then $f^* = f$.

Def $\forall$ irred $Q \mid f$, write m_Q its multiplicity.

Thm B (HLZ) Assume $\mathrm{Int}(g) \neq 0$. Then

$\exists!$ Irr $Q_0 \mid f$ s.t. $Q_0^* = Q$ (self-reciprocal)

& m_{Q_0} odd.

Also, $$\mathrm{Int}(g) = \frac{m_{Q_0}+1}{2} \deg Q_0 \cdot \prod_{\substack{(Q,Q^*) \\ Q \neq Q^*}} (1 + m_Q).$$

§5 Two ingredients in proving Thm B

(A) Thm (Li-Zhu)

$$\Delta \cap (1 \times g)(\Delta) \simeq \boxed{X^{\bar{g}}}$$

e.g. of closed fine DL var assoc to some partial flag var of finite unitary.

Sm proj var $/k$

Cor $\mathrm{Int}(g) = \mathrm{Tr}(\bar{g}, H^*(X, \mathbb{Q}_\ell))$.

(by Lefschetz trace formula).

(B) Thm (HLZ)

$\exists$ explicit char formula for closed fine DL var

(including all those fine DLV appearing

in basic loci of Sh var of Coxeter type.)

Idea of pf: $X = \coprod X_i$ loc closed strata w/ $X_i = \mathrm{Ind}_{P_i}^{U_n} DL_i$

DL_i = classical DL var.

<u>Ex</u> X = Fermat curve

$$X = X_1 \sqcup X_2$$

$$\begin{array}{ccc} & \| & \| \\ (q^3+1 \text{ pts}) & X(\mathbb{F}_{q^2}) & DL_2 \\ & \| & \\ & \mathrm{Ind}_{P_1}^{U_3} \underbrace{DL_1}_{\text{pts (0-dim'l)}} & \end{array}$$