On pro-étale cohomology of p-adic analytic spaces

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Lecture 1

K p-adic field w/ discrete val, k_K perfect. $C = \widehat{\overline{K}}$.

Y/K dagger var (overconvergent affinoids,
(+ smooth   analytification of alg vars,
coverings of upper-half plane / Drinfeld spaces,
Stein spaces).

Thm (Colmez – Nizioł, Basic comparison theorem)
For $n \le r$, have

$$\cdots \to X^{r,n-1} \to DR^{r,n-1} \to H^n_{\text{proét}}(Y_C, \mathbb{Q}_p(r)) \to X^{r,n} \to DR^{r,n} \to \cdots$$

where $X^{r,i} = (H^i_{HK}(Y_C) \otimes B^+_{st})^{N=0,\, \varphi = p^r}$

$$DR^{r,i} = H^i(B^+_{dR} \otimes_K R\Gamma_{dR}(Y) / Fil^r).$$

<u>Notations</u> · $H^i_{HK}(Y_C)$ is a C^{ur}-mod. (Hyodo-Kato Cohom)

$$C^{ur} = \bigcup_{[L:K]<\infty} W(k_L)[\tfrac{1}{p}]$$

· φ, N s.t. $N\varphi = p\cdot\varphi N$, $\mathrm{Gal}(\overline{K}/K) = G_K$

· $\iota_{HK}: B^+_{dR} \otimes_{C^{ur}} H^i_{HK} \simeq B^+_{dR} \otimes_K H^i_{HK}$.

Fact Topologically projective limit of fin dim C^{ur}-mods admit smooth G_K-actions.

<u>Examples</u> (1) Y_K proper (sm dagger var)

$$\rightsquigarrow \quad 0 \to H^n_{\text{proet}} \to X^{r,n} \to DR^{r,n} \to 0$$

- Same as in the algebraic case
- H^n_{proet} f.d. $/\mathbb{Q}_p$.

  H^n_{HK} finite rk (think of H^n_{dR}).

(2) $Y_K = \mathbb{A}^d_K$

$$\rightsquigarrow \quad H^n_{\text{proet}}(Y_C, \mathbb{Q}_p(r)) \simeq (\Omega^{n-1}(Y_C)/\ker d)(r-n)$$

$$\simeq \Omega^n(C)^{d=0}(r-n)$$

$$\& \quad H^n_{\text{et}}(Y_C, \mathbb{Q}_p(n)) = 0.$$

(3) More generally, Y_K Stein space:

$$0 \to \Omega^{n-1}(Y_C)/\ker d \to H^n_{\text{proet}}(Y_C, \mathbb{Q}_p(n)) \to X^{n,n} \to 0.$$

<u>Big Conjecture</u> (C_{st}) Let $n \le r$.

(1) Have an exact sequence

$$0 \to H^n_{\text{proet}}(Y_C, \mathbb{Q}_p(r)) \to X^{r,n} \oplus H^n(\mathrm{Fil}^r(B^+_{dR} \otimes R\Gamma_{dR}))$$

$$\to B^+_{dR} \otimes H^n_{dR} \to 0.$$

(2) As K-filtered modules,

$$\mathrm{Hom}_{G_K}(H^n_{\text{proet}}(Y_C, \mathbb{Q}_p), B_{dR}) \simeq H^n_{dR}(Y_K)^*.$$

As (φ, N, G_K) - C^{unr} - mods,

$$\varinjlim_{[L:K]<\infty} \mathrm{Hom}_{G_L}(H^n_{\text{proet}}(Y_C, \mathbb{Q}_p), B_{st}) \simeq H^n_{HK}(Y)^*.$$

<u>Thm</u> (1) $\Rightarrow$ (2).

(1) is true if Y_K is proper, Stein,

+ analytification of alg var

(alternatively, proper minus a small tube of a smooth subvar, use Grosse-Klönne).

Thm (Geometrization) In basic comparison thm, the long exact seq can be shefified on Perf$_C$.

Precisely, S perfectoid alg $/C$,

$$S \longmapsto H^n_{\text{proet}}(Y_S, \mathbb{Q}_p(r))$$

$$S \longmapsto X^{r,n}(S) := \text{replace } \mathbb{B}^+_{st} \text{ by } \mathbb{B}^+_{st}(S)$$

$$S \longmapsto DR^{r,n}(S) := \text{replace } \mathbb{B}^+_{dR} \text{ by } \mathbb{B}^+_{dR}(S).$$

In proper case,

- by Scholze, $H^n_{\text{proet}}(Y, \mathbb{Q}_p(r))$ has fin dim $/\mathbb{Q}_p$.
- H^n_{HK}, H^n_{dR} fin dim
- $DR^{n,r}$: $\mathbb{B}^+_{dR}$-mod lattice,

  inductive lim of $\underbrace{\mathbb{B}^+_{dR}/t^n\text{-mod}}_{\text{succesive ext'n of } \mathbb{G}_a\text{'s}}$.

Now, look at $\text{Hom}(\mathbb{G}_a, \mathbb{Q}_p) = 0$

$\rightsquigarrow$ get exact seq

$$0 \longrightarrow H^n_{\text{proet}}(Y_C, \mathbb{Q}_p(r)) \longrightarrow X^{r,n} \longrightarrow DR^{r,n} \longrightarrow 0.$$

$$0 \longrightarrow (H^n_{dR} \otimes \mathbb{B}^+_{dR})/\text{Fil}^n \longrightarrow DR^{r,n} \longrightarrow \underset{\substack{\| \\ 0}}{H^{n+1}}(\text{Fil}^r)_{\mathbb{B}^+_{dR}\text{-lattice}} \longrightarrow 0$$

$\Rightarrow$ (H^n_{HK}, H^n_{dR}) is a weakly admissible filtered (φ, N, G_K)-mod w/ slope $n \leq r$.

Basic geometrization goes through comparison with syntomic cohom

Thm $\tau_{\leq r} R\Gamma_{syn}(Y_C, r) \simeq \tau_{\leq r} R\Gamma_{proét}(Y_C, \mathbb{Q}_p(r))$.

<u>Note about $R\Gamma_{syn}$</u>

$$R\Gamma_{syn}(Y_C, r) = [F^r R\Gamma_{cris}(\mathcal{X}) \xrightarrow{\varphi - p^r} R\Gamma_{cris}(\mathcal{X})]$$

$$\simeq [R\Gamma_{cris}(\mathcal{X})^{\varphi = p^r} \longrightarrow B_{dR}^+ \otimes R\Gamma_{dR} / Fil^r].$$

Thm $R\Gamma_{cris}(\mathcal{X})^{\varphi=p^r} \simeq (R\Gamma_{HK}(\mathcal{X}) \otimes B_{st}^+)^{N=0, \varphi=p^r}$.

<u>Local argument</u>

$R_\square^+ = \mathcal{O}_C \langle x_1^{\pm 1}, \dots, x_a^{\pm 1}, y_0, \dots, y_b \rangle / (y_0 \cdots y_b - \omega)$, $\omega \in \mathfrak{m}_C$.

$R^+ =$ étale completion over $R_\square^+$.

$R = R^+[\frac{1}{p}]$, $X = (\mathrm{Spf}\, R^+)_\eta$.

$\bar{R}$ maximal étale ext'n of R.

$\bar{R} \supset R_\infty = R[x_i^{p^{-\infty}}, y_j^{p^{-\infty}}]$

Then $\hat{\bar{R}}, \hat{R}_\infty$ are perfectoid algebras $/C$,

$$1 \to H_R \to G_R = \mathrm{Aut}(\bar{R}/R) \to \Gamma_R = \mathrm{Aut}(R_\infty / R) \underset{\mathbb{Z}_p^{a+b}}{\overset{}{\cong}}.$$

For $\tilde{R}_\square^+ = A_{inf} \langle x_1^{\pm 1}, \dots, x_a^{\pm 1}, y_0, \dots, y_b \rangle / (y_0 \cdots y_b - [\omega^\flat])$,

$$\begin{tikzcd} \tilde{R}_\square^+ \arrow[r, "\theta"] \arrow[d] & R_\square^+ \arrow[d] \\ \tilde{R}^+ \arrow[r, "\theta"] & R^+ \end{tikzcd}$$

$\varphi \curvearrowright \tilde{R}_\square^+ : \varphi(x_i) = x_i^p,\ \varphi(y_j) = y_j^p$

extends uniquely to $\varphi \curvearrowright \tilde{R}^+$.

Consider $\iota : \tilde{R}_\square^+ \hookrightarrow W_a(R^\flat)$, $[x_i] \longmapsto [x_i^\flat]$ commuting w/ φ

(inverting a set Q of places containing p)

This extends uniquely to $\iota : \widetilde{R}^+ \hookrightarrow W_Q(\bar{R}^b)$.

$(W_Q(\bar{R}^b)^{H_R} = W_Q(R^b_\infty))$.

<u>Finally</u> $R\Gamma_{proet}(X, \mathbb{Q}_p(r)) \simeq R\Gamma(G_R, \mathbb{Q}_p(r))$ (Scholze)

$0 \to \mathbb{Q}_p(r) \to W_Q(\bar{R}^b(r)) \xrightarrow{\varphi-1} W_Q(\bar{R}^b(r)) \to 0$.

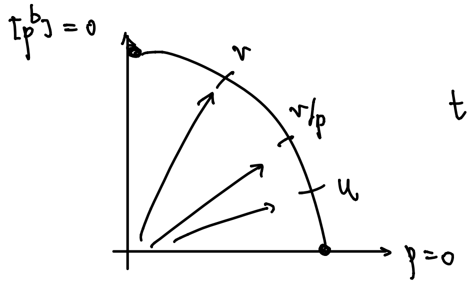
$\simeq [R\Gamma(G_R, W_Q(\bar{R}^b)(r)) \xrightarrow{\varphi-1} R\Gamma(G_R, W_Q(\bar{R}^b)(r))]$

$\simeq [R\Gamma(\Gamma_R, W_Q(R^b_\square)(r)) \xrightarrow{\varphi-1} R\Gamma(\Gamma_R, W_Q(R^b_\square)(r))]$

$W_Q(R^b_\infty) = W_Q(C^b) \hat{\otimes}_{A_{inf}} \widetilde{R}_\infty$

$= [R\Gamma(\Gamma_R, W_Q(C^b) \otimes \hat{R}(r)) \xrightarrow{\varphi-1} R\Gamma(\Gamma_R, W_Q(C^b) \otimes \hat{R}(r))]$

Now use classical (φ, Γ)-module techniques to change $W_Q(C^b)$ into $B^{[u,v]}$ or $B^{[u,v/p]}$.



t exactly has one zero in $[u,v]$ but none in $[u, v/p]$.

Compare cohom of Γ_R with cohom of its Lie alg

$\rightsquigarrow$ get the de Rham complex w/ diff d replaced by $t \cdot d$.

$G_R \mapsto G_{R \hat{\otimes} S}$ $(R \mapsto \overline{R \otimes S})$ for $S \in \mathrm{Perf}_C$.

Lecture 2: The C_{st} conjecture

Y_K dagger var.

<u>Recall</u> $\tau_{\leq r} R\Gamma_{proet}(Y_C, \mathbb{Q}_p(r)) \simeq \tau_{\leq r} R\Gamma_{syn}(Y_C, r)$.

$R\Gamma_{syn}$ lives in dist triangle

$$\begin{tikzcd}
R\Gamma_{syn}(Y_C, r) \arrow[r] \arrow[d, dashed] & (B_{st}^+ \otimes R\Gamma_{HK})^{N=0, \varphi=p^r} \arrow[r] \arrow[d] & B_{dR}^+ \otimes R\Gamma_{dR} / Fil^r \arrow[d, "="] \\
Fil^r(B_{dR}^+ \otimes R\Gamma_{dR}) \arrow[r] & B_{dR}^+ \otimes R\Gamma_{dR} \arrow[r] & B_{dR}^+ \otimes R\Gamma_{dR} / Fil^r
\end{tikzcd}$$

Define $F^{r,n} := H^n(Fil^r(B_{dR}^+ \otimes R\Gamma_{dR}))$.

$X^{r,n} := (H^n_{HK}(Y_C) \otimes B_{st}^+)^{N=0, \varphi=p^r}$.

$DR^{r,n} := H^n(B_{dR}^+ \otimes_K R\Gamma_{dR}(X) / Fil^r)$.

$$\rightsquigarrow \begin{tikzcd}
\cdots \arrow[r] & DR^{r,n-1} \arrow[r] \arrow[d, equal] & H^n_{proet}(Y_C, \mathbb{Q}_p(r)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square\,?"] & X^{r,n} \arrow[r] \arrow[d] & DR^{r,n} \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & DR^{r,n-1} \arrow[r] & F^{n,r} \arrow[r] & B_{dR}^+ \otimes H^n_{dR} \arrow[r] & DR^{r,n} \arrow[r] & 0
\end{tikzcd}$$

Conj (C_{st}) The middle diagram is bi-Cartesian,

($\Leftrightarrow$ exact seq of last line

$\Leftrightarrow$ images in $DR^{r,n}$ & in $DR^{r,n-1}$ are the same).

Note
- If Y_K is proper, maps to $DR^{r,n}$ are surjective.
- If Y_K is Stein, true for some $n=m \Rightarrow$ true for $n+1$.
  (So suffices to check for $n=m$)

Recall Geometrization $\Rightarrow$ everything is a sheaf / Perf$_C$

Consider TVS = sheaves of top $\mathbb{Q}_p$-vector spaces.

$\mathbb{W} : S \longmapsto \mathbb{W}(S) \in Mod^{Cond}_{\mathbb{Q}_p}$.

<u>Example</u> $\underline{\mathbb{Q}_p} : S \longmapsto \mathcal{C}(\pi_0(S), \mathbb{Q}_p)$

$\mathbb{G}_a : S \longmapsto S$.

$\rightsquigarrow$ Banach-Colmez spaces are $\mathbb{G}_a^{\oplus d}$ up to $\underline{\mathbb{Q}_p}^{\oplus h}$

<u>Rmk</u> (Le Bras) $\exists$ smallest sub-abelian cat of TVS containing $\underline{\mathbb{Q}_p}$, $\mathbb{G}_a$ and is stable under ext'ns.

<u>Fact</u>

Hom	$\underline{\mathbb{Q}_p}$	$\mathbb{G}_a$
$\underline{\mathbb{Q}_p}$	$\underline{\mathbb{Q}_p}$	C
$\mathbb{G}_a$	0	C

Ext^1	$\underline{\mathbb{Q}_p}$	$\mathbb{G}_a$
$\underline{\mathbb{Q}_p}$	0	0
$\mathbb{G}_a$	C	C

$(\mathrm{Ext}^{\geq 2} = 0)$

note: $0 \to \underline{\mathbb{Q}_p} \xrightarrow{t} (B_{cris}^+)^{\varphi = p} \xrightarrow{\theta} \mathbb{G}_a \to 0$

$0 \to \mathbb{G}_a \xrightarrow{t} B_{dR}^+/t^2 \xrightarrow{\theta} \mathbb{G}_a \to 0$.

<u>Thm</u> $\exists$ additive functors dim & ht on BC s.t. $\dim \underline{\mathbb{Q}_p} = 0$, $\dim \mathbb{G}_a = 1$, $\mathrm{ht}\, \underline{\mathbb{Q}_p} = 1$, $\mathrm{ht}\, \mathbb{G}_a = 0$.

<u>Example</u> $\lambda = d/h$, $(d,h) = 1$, $d \geq 0$, $h \neq 0$.

Take
$$\mathcal{U}_\lambda := \begin{cases} (B_{cris}^+)^{\varphi^h = p^d}, & \text{if } \lambda \geq 0, \\ B_{dR}^+ / t^d B_{dR}^+ + \mathbb{Q}_p^{-h}, & \text{if } \lambda < 0. \end{cases}$$

$\Rightarrow$ $\mathrm{ht}\, \mathcal{U}_\lambda = h$, $\dim \mathcal{U}_\lambda = d$.

Notation: $\forall\, x \in X_{FF}$, $\mathbb{B}_{m,x} := B_{dR}^+ / t_x^m$.

Then
$$\mathcal{U}_\lambda = \begin{cases} H^0(X_{FF}, \mathcal{O}(\lambda)), & \text{if } \lambda \geq 0 \quad (H^1 = 0) \\ H^1(X_{FF}, \mathcal{O}(\lambda)), & \text{if } \lambda < 0 \quad (H^0 = 0). \end{cases}$$

& $\mathbb{B}_{m,x} = H^0(X_{FF}, \mathcal{O}_{X_{FF},x} / t_x^m)$.

Thm (Le Bras) Any BC space is of form

$$W = \bigoplus_{\lambda} \mathcal{U}_\lambda^{\oplus m_\lambda} \oplus \bigoplus_{x \in X_{FF}} \bigoplus_{m} \mathbb{B}_{m,x}^{\oplus n_{m,x}}$$

This is non-canonical. But H-N fil'n is canonical,

$$\mathrm{slope}(V_\lambda) = -1/\lambda \quad (= +\infty \text{ if } \lambda = 0)$$

$$\mathrm{slope}(\mathbb{B}_{m,x}) = 0 \qquad (+\infty\,?)$$

<u>Curvature filtration</u> $W \supset W_{\geq 0} \supset W_{>0} \supset 0$.

Here $W_{>0} = \bigoplus_{\lambda<0} \mathcal{U}_\lambda^{m_\lambda} \oplus \bigoplus_{x \neq \infty} \bigoplus_{m} \mathbb{B}_{m,x}^{n_{m,x}}$

$W_{=0} := W_{\geq 0} / W_{>0} = \bigoplus_{m} \mathbb{B}_{m,\infty}^{n_{m,\infty}}$ ($\mathbb{B}_{dR}^+$-mod)

$W_{<0} := W / W_{\geq 0} = \bigoplus_{\lambda > 0} \mathcal{U}_\lambda^{m_\lambda}$ (free $\mathbb{B}_{dR}^+$-mod)

$W / W_{>0}$ highest fil'n ($\mathbb{B}_{dR}^+$-mod).

Def A qBC (quasi-Banach-Colmez) space W is a TVS

+ a fil'n $W \supset W_{\geq 0} \supset W_{>0} \supset 0$.

- $W_{>0}$ is a BC w/ curvature >0
- $W_{<0}$ is a BC w/ curvature <0
- $W_{=0}$ is a $\mathbb{B}_{dR}^+$-pair (W_1, W_2):

$\exists m$ s.t. $W_{=0}(S) = \big(\mathbb{B}_m(S) \hat{\otimes}_{\mathbb{B}_m} W_2 \to \mathbb{B}_m(S) \hat{\otimes}_{\mathbb{B}_m} W_1\big)$,

induced from $W_2 \to W_1$

where $\mathbb{B}_m := \mathbb{B}_{m,\infty}$, W_1, W_2 $\underbrace{\text{top } \mathbb{B}_m\text{-mods}}_{\text{a very "rigid" cat.}}$ (separated)

Thm qBC's form an abelian cat on which ht is additive.

In particular, $\mathrm{ht}(W) = \mathrm{ht}(W_{>0}) + \mathrm{ht}(W_{<0})$.

Now Consider $H^*(X_{FF})$:

$$0 \to H^0(X_{FF}, \mathcal{E}) \to H^0(X_{FF} \setminus \infty, \mathcal{E})$$
$$\to (\mathrm{Fr}\, \underbrace{\widehat{\mathcal{O}}_{X_{FF},\infty}}_{\substack{\| \\ B_{dR}}} \otimes \mathcal{E}) / \underbrace{\widehat{\mathcal{O}}_{X_{FF},\infty}}_{\substack{\| \\ B_{dR}^+}} \otimes \mathcal{E}) \longrightarrow H^1(X_{FF}, \mathcal{E}) \to 0.$$

Define $\quad h^j(H^i_{HK}, r) := H^j(X_{FF}, \mathcal{E}(H^i_{HK}(Y_C)\{-r\}, F^r(B_{dR} \otimes H^i_{dR})))$.

Then $\quad h^0(H^i_{HK}, r) = \ker(X^{r,i} \to (B_{dR}^+ \otimes H^i_{dR}) / \mathrm{Fil}^r)$,

$\quad\quad\quad h^1(H^i_{HK}, r) = \mathrm{coker}(X^{r,i} \to (B_{dR}^+ \otimes H^i_{dR}) / \mathrm{Fil}^r)$.

Prop (1) $H^n_{proet}(\mathbb{Q}_p(r)) =: H^{r,n}$ is a qBC space

& $\quad H^{r,n}_{>0} = h^1(H^{n-1}_{HK}, r)$

$\quad\quad H^{r,n}_{=0} = F^{r,n}_{B_{dR}^+} = \ker(F^{r,n} \to B_{dR}^+ \otimes H^n_{dR})$.

$\quad\quad H^{r,n}_{<0} = h^0(H^n_{HK}, r)$.

(2) Have exact seq

$$0 \to h^1(H^{n-1}_{HK}, r) \to H^{r,n} \to F^{r,n} \oplus X^{r,n}$$
$$\to B_{dR}^+ \otimes H^n_{dR} \to h^1(H^n_{HK}, r) \to 0.$$

Thm TFAE (when H^*_{dR} fin dim):

(i) Cst conj for $H^n_{proet}(-, \mathbb{Q}_p(r))$

(ii) $h^1(H^{n-1}_{HK}, r) = h^1(H^n_{HK}, r) = 0 \Rightarrow$

$\Uparrow$ H^1_{HK} weakly admissible

$\quad\quad\quad\quad\quad\quad\quad \Downarrow$

$\mathcal{E}(H^1_{HK}, H^1_{dR})$ is acyclic.

(iii) $\mathrm{ht}(H^n_{proet}) = \dim H^n_{dR}$

(if $h^1(H^1_{HK}, r) \neq 0$, then its $\mathrm{ht} < 0$).

Lecture 3: Duality theorems (I)

(Colmez - Gilles - Nizioł)

Outline p-adic Hodge theory

$\Rightarrow$ (I) arithmetic duality for curves

(II) geom duality for any dim.

Let $K/\mathbb{Q}_p$ finite, $\mathcal{O}_K \twoheadrightarrow k$, $\mathcal{G}_K := \mathrm{Gal}(\bar{K}/K)$, $C := \hat{\bar{K}}$.

Arithmetic duality for curves

Thm: X smooth, geom irred, dagger var of dim 1 / K.

Then (1) $\exists$ natural trace map of solid $\mathbb{Q}_p$-v.s.

$$\mathrm{Tr}_X : H^4_{\text{proét},c}(X, \mathbb{Q}_p(2)) \xrightarrow{\sim} \mathbb{Q}_p.$$

(2) The pairing

$$H^i(X, \mathbb{Q}_p(j)) \otimes^{\blacksquare}_{\mathbb{Q}_p} H_c^{4-i}(X, \mathbb{Q}_p(2-j)) \xrightarrow{\cup} H_c^4(X, \mathbb{Q}_p(2)) \xrightarrow[\sim]{\mathrm{Tr}_X} \mathbb{Q}_p$$

is a perfect duality, i.e.

$$\gamma_{X,i} : H^i(X, \mathbb{Q}_p(j)) \xrightarrow{\sim} H_c^{4-i}(X, \mathbb{Q}_p(2-j))^*,$$

$$\gamma^\circ_{X,i} : H_c^i(X, \mathbb{Q}_p(j)) \xrightarrow{\sim} H^{4-i}(X, \mathbb{Q}_p(2-j))^*.$$

Here $(-)^* := \underline{\mathrm{Hom}}_{\mathbb{Q}_p}(-, \mathbb{Q}_p)$.

Rmk (1) X partially proper. $U_n \Subset U_{n+1}$ (U_i qc)

$R\Gamma_{\text{proét},c}(X, \mathbb{Q}_p) := R\Gamma_{\text{ét},c,\mathrm{Hu}}(X, \mathbb{Q}_p)$

in Huber's sense.

This a priori works for all X rigid.

But when X Stein, can compute by mapping fibre

$$R\Gamma_{\text{proét},c}(X,\mathbb{Q}_p) = [\,R\Gamma_{\text{proét}}(X,\mathbb{Q}_p) \to R\Gamma_{\text{proét}}(\partial X,\mathbb{Q}_p)\,]$$

where $R\Gamma_{\text{proét}}(\partial X,\mathbb{Q}_p) := \varinjlim_{\substack{\mathcal{U}_n \,\text{qc} \\ \text{colim } n}} R\Gamma_{\text{proét}}(X \setminus \mathcal{U}_n, \mathbb{Q}_p)$

(2) <u>Topology</u> Assume X dim 1, proper affinoid + Stein

- X proper: all cohom grps are finite
- X Stein (resp. affinoid)

  $H^i(X,\mathbb{Q}_p(j))$ nuclear Fréchet (resp. of cpt type)

  $H^i_c(X,\mathbb{Q}_p(j))$ of cpt type (resp. nuclear Fréchet).

(3) Assume X dim 1, partially proper.

$\exists$ derived duality in $\mathcal{D}(\mathbb{Q}_{p,\square})$:

$$\gamma_X : R\Gamma(X,\mathbb{Q}_p(j)) \xrightarrow{\sim} \mathbb{D}(R\Gamma_c(X,\mathbb{Q}_p(2-j))[1])$$

$$\mathbb{D}(-) := R\underline{\mathrm{Hom}}_{\mathbb{Q}_p}(-,\mathbb{Q}_p)$$

Note: $\forall\, i \geq 1$, $\mathrm{Ext}^i_{\mathbb{Q}_p}(\underbrace{H^j_c(X,\mathbb{Q}_p(j))}_{\text{of cpt type}},\mathbb{Q}_p) = 0 \quad \rightsquigarrow \gamma_{X,i}$.

$\rightsquigarrow \gamma^c_{X,i}$ b/c $H^j_c(X,\mathbb{Q}_p)$ reflexive.

(4) <u>Conj</u> X smooth, Stein $/K$, geom irred, dim d. Then

(1) $H^i(X,\mathbb{Q}_p(j))$, $H^i_c(X,\mathbb{Q}_p(j))$

are nuclear Fréchet & of cpt type

(2) We have quasi-isom in $\mathcal{D}(\mathbb{Q}_{p,\square})$:

$$R\Gamma(X,\mathbb{Q}_p(j)) \simeq \mathbb{D}(R\Gamma_c(X,\mathbb{Q}_p(d+1-j))[2d+2])$$

$$H^i(X,\mathbb{Q}_p(j)) \simeq H^{2d+2-i}_c(X,\mathbb{Q}_p(d+1-j))^*,$$

$$H^i_c(X, \mathbb{Q}_p(j)) \simeq H^{2d+2-i}(X, \mathbb{Q}_p(d+1-j))^*.$$

Rmk for (4) (a) proved by Zhenghui Li in his thesis

(b) project of Anschütz - Le Bras - Mann on 6 functor formalism for $\mathbb{Q}_p$-cohom

$\rightsquigarrow \exists$ analogy of this duality.

(5) Example $X = D$ open unit disc.

Then
$$H^1(X, \mathbb{Q}_p(1)) \simeq (\mathcal{O}(D)/K) \oplus H^1(\mathcal{G}_K, \mathbb{Q}_p(1))$$
$$H^3_c(X, \mathbb{Q}_p(1)) \simeq \mathcal{O}(\partial D)/\mathcal{O}(D) \oplus H^1(\mathcal{G}_K, \mathbb{Q}_p)$$

$\Rightarrow$ Galois duality $H^i(\mathcal{G}_K, \mathbb{Q}_p) \simeq H^{2-i}(\mathcal{G}_K, \mathbb{Q}_p(1))^*$.

note: $\mathcal{O}(D)/C = H^0(D, \Omega^1_D) \simeq \Omega^1(D)$
$\simeq H^1_c(D, \mathcal{O})^*$ } Serre duality (van der Put)

$$\mathcal{O}(\partial D)/\mathcal{O}(D) \simeq H^1_c(D, \mathcal{O}_D).$$

(6) Solid formalism. Had to pass to solid universe from LCVS.

b/c (a) need well-behaved derived functors

(b) existence of HS spectral seq.

Pf of Thm 1

Example (true) X sm proper, dim d.

Pf · $\exists$ geom Poincaré duality: Zavyalov, Gabber, Mann
$$H^i(X_C, \mathbb{Q}_p(j)) \simeq H^{2d-i}(X_C, \mathbb{Q}_p(d-j)).$$

· Hoschild - Serre spectral seq, used to descend:
$$E_2^{a,b} := H^a(\mathcal{G}_K, H^b(X_C, \mathbb{Q}_p(j))) \Rightarrow H^{a+b}(X, \mathbb{Q}_p(j))$$

(degenerate at E_3).

- Tate duality:

$$H^i(\mathcal{G}_K, V) \simeq H^{2-i}(\mathcal{G}_K, V^*(1))^*.$$

Now let X a Stein curve.

<u>Step 1</u> Geometric comparison thms (CDN, CN, AGN):

(1) Vanishing thms: $H^i(X_C, \mathbb{Q}_p) = 0, \quad \forall\, i \neq 0,1$

$$H^i_c(X_C, \mathbb{Q}_p) = 0, \quad \forall\, i \neq 1,2.$$

(2) Isomorphisms:

$$H^0(X_C, \mathbb{Q}_p) \simeq \mathbb{Q}_p$$

$$H^1_c(X_C, \mathbb{Q}_p) \simeq HK^1_c(X_C, 1)$$

where $HK^j_*(X_C, i) := (H^j_{HK,*}(X_C) \hat{\otimes}^{\square}_{\breve{C}} \hat{B}^+_{st})^{N=0,\varphi=p^i}$.

(3) Exact sequences:

Comparison $\Rightarrow$ $\begin{cases} 0 \to \mathcal{O}(X_C)/C \to H^1(X_C, \mathbb{Q}_p(1)) \to HK^1(X_C, 1) \to 0 \\ HK^1_c(X_C, 2) \to H^1 \underbrace{DR_c(X_C, 2)} \to H^2_c(X_C, \mathbb{Q}_p(2)) \xrightarrow{Tr_{X_C}} \mathbb{Q}_p(1) \to 0 \end{cases}$

$$DR_c(X_C, i) := [H^1_c(X, \mathcal{O}) \otimes^{\square}_K (B^+_{dR}/F^i) \xrightarrow{d} H^1_c(X, \Omega^1) \otimes^{\square}_K (B^+_{dR}/F^{i-1})][-1]$$

(deg 1.

$\rightsquigarrow Tr_X : H^4_c(X, \mathbb{Q}_p(2)) \simeq H^2(\mathcal{G}_K, H^2_c(X_C, \mathbb{Q}_p(2)))$

$$\xrightarrow{Tr_X(1)} H^2(\mathcal{G}_K, \mathbb{Q}_p(1)) \xrightarrow{Tr_X} \mathbb{Q}_p.$$

<u>Step 2</u> Galois descent. (HS + Step 1.)

$$H^a(\mathcal{G}_K, H^b(X_C, \mathbb{Q}_p(j))) = E_2^{a,b} \text{ of HS}$$

is the Galois cohom of

$$H^0(X, \Omega) \otimes^{\square}_K C(j), \quad H^1_c(X, \mathcal{O}) \otimes^{\square}_K C(j) \otimes \mathbb{Q}_p, \quad HK^1_c(X_C, 1)(s).$$

Recall: Tate

$$\rightsquigarrow H^i(\mathcal{G}_K, C(j)) \simeq \begin{cases} K, & \text{if } j=0 \ \& \ i=0,1 \\ 0, & \text{otherwise.} \end{cases}$$

$\Rightarrow$ (a) H^i nuclear Fréchet

H_c^i of cpt type

<u>Claim</u> If $|H^1_{HK}| < \infty$, then

$E_2^{a,b}$: $H^0(X, \Omega)$, $H^1_c(X, \mathcal{O})$, $H^i(\mathcal{G}_K, V)$

Does not quite work.

(b) Apply some duality + Galois duality

+ "percolate" it through HS seq.

Need: Compatibility of étale & coherent products.

(Reciprocity laws: Bloch-Kato about Tamagawa number, Benois, Ribeiro.)

<u>Step 3</u> Reductions.

Stein $\overset{\text{limit}}{\rightsquigarrow}$ wide opens

i.e. $\overline{X} = X \cup \bigcup_{i=1}^{m} D_i$, D_i open disjoint discs.

+ Mayer-Vietoris seq.

Cohom of proper + $\underbrace{\text{open disc + open annuli}}$

reduces to the "ghost circle" ∂D.

this is the "real curve".

<u>Thm 0</u> (Duality for ghost circle)

(1) $Tr_X : H^3(Y, \mathbb{Q}_p(2)) \xrightarrow{\sim} \mathbb{Q}_p$

(2) $\gamma_X : H^i(Y, \mathbb{Q}_p(j)) \xrightarrow{\sim} H^{3-i}(Y, \mathbb{Q}_p(2-j))^*$.

Lecture 4: Duality theorems (II)

p prime. $K/\mathbb{Q}_p$ finite. $\mathcal{O}_K \twoheadrightarrow k$, $C := \hat{\bar{K}}$.

Thm 1 X/K smooth, partially proper, dim d.

Then in $D(\mathbb{Q}_{p,\square})$:

$$R\Gamma_{\text{proét}}(X_C, \mathbb{Q}_p) \simeq R\mathrm{Hom}_{TVS}(R\Gamma_{\text{proét},c}(X_C, \mathbb{Q}_p(d))[2d], \mathbb{Q}_p).$$

Rmks (1) TVS = Cat of top v.s. (topologically enriched v.s.): sheaves on $\mathrm{Perf}_{C,v}$ with values in $D(\mathbb{Q}_{p,\square})$.

(need enriched Yoneda Lemma).

(2) $R\Gamma_{\text{proét},c}(X_C, \mathbb{Q}_p) := R\Gamma_{\text{ét},\text{Hu},c}(X_C, \mathbb{Q}_p)$.

Explicitly, $X \supset \{U_n\}$, U_i qc, $U_n \Subset U_{n+1}$

$\rightsquigarrow$ $R\Gamma_{\text{proét},c}(X_C, \mathbb{Q}_p) := [R\Gamma_{\text{proét}}(X_C, \mathbb{Q}_p) \to R\Gamma_{\text{proét}}(\partial X_C, \mathbb{Q}_p)]$.

(3) (2) $\rightsquigarrow$ $R\Gamma_*(X_C, \mathbb{Q}_p) \in TVS$.

$\mathrm{Perf}_C \ni S \longmapsto R\Gamma_*(X_S, \mathbb{Q}_p)$ + canonical top.

Related work Anschütz - Le Bras - Mann, 6 functor formalism. for p-adic pro-ét coh: announced in Thm 1.

Motivating example

D/C open unit disc of dim 1, pHT ($\Rightarrow$ nonzero).

$$H^0(D, \mathbb{Q}_p(j)) \simeq \mathbb{Q}_p(j), \quad H^1(D, \mathbb{Q}_p(j)) \simeq \mathcal{O}(D)/C(j-1)$$

$$H^2_c(D, \mathbb{Q}_p(j)) \simeq \mathbb{Q}_p(j-1) \oplus \mathcal{O}(\partial D)/\mathcal{O}(D)(j-1).$$

$\mathcal{O}(D)/\mathbb{C} \simeq \Omega^1(D)$. $\quad \mathcal{O}(\partial D)/\mathcal{O}(D) \simeq H^1_c(D, \mathcal{O})$.

Apply Serre duality: $H^1_c(D, \mathcal{O})^* \simeq \Omega^1(D)$.

Issue Numerology is wrong $\rightsquigarrow$ naive Poincaré duality ($R\mathrm{Hom}(-, --)$) does not work here.

In TVS $(\mathbb{G}_a, \mathbb{Q}_p)$:

$$\mathrm{Hom}_{TVS}(\mathbb{Q}_p, \mathbb{Q}_p(1)) = \mathbb{Q}_p(1), \quad \mathrm{Ext}^1_{TVS}(\mathbb{Q}_p, \mathbb{Q}_p(1)) = 0$$

$$\mathrm{Hom}_{TVS}(\mathbb{G}_a, \mathbb{Q}_p(1)) = 0, \quad \mathrm{Ext}^1_{TVS}(\mathbb{G}_a, \mathbb{Q}_p(1)) \simeq C \simeq \mathbb{G}_a$$

$$0 \to \mathbb{Q}_p(1) \to \mathbb{B}^{+, \varphi = p}_{cris} \to \mathbb{G}_a \to 0.$$

But $\mathrm{Ext}^{\geq 2}_{TVS} = 0$.

$\rightsquigarrow$ Get $H^1(D, \mathbb{Q}_p(j)) \simeq \mathrm{Ext}^1_{TVS}(H^2_c(D, \mathbb{Q}_p(2-j)), \mathbb{Q}_p)$.

Cor of Thm 1 $\quad X$ Stein, $\exists$ short exact seq

$$0 \to \mathrm{Ext}^1_{TVS}(H^{2d-i+1}_c(X_C, \mathbb{Q}_p(d)), \mathbb{Q}_p) \to H^i(X_C, \mathbb{Q}_p)$$
$$\to \mathrm{Hom}_{TVS}(H^{2d-i}_c(X_C, \mathbb{Q}_p(d)), \mathbb{Q}_p) \to 0.$$

Use spectral seq:

$$E_2^{ab} := \mathrm{Ext}^a_{TVS}(H^{-b}_c(X_C, \mathbb{Q}_p), \mathbb{Q}_p) \Rightarrow H^{a+b}(R\mathrm{Hom}_{TVS}).$$

It suffices that $\mathrm{Ext}^a_{TVS}(H^{-b}_c(X_C, \mathbb{Q}_p), \mathbb{Q}_p) = 0$, $a \geq 2$.

Idea: reduce to

Prop $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{BC}$ (TVS generated by $\mathbb{Q}_p$, $\mathbb{G}_a$ + ext).

$\rightsquigarrow$ $\mathrm{Ext}^a_{TVS}(\mathcal{F}_1, \mathcal{F}_2) = 0$, $a \geq 2$.

pf (1) $\mathrm{Ext}^a_{TVS}(\mathcal{F}_1, \mathcal{F}_2) \simeq \mathrm{Ext}^a_{VS}(\mathcal{F}_1, \mathcal{F}_2)$

LHS : rep'd by complex of Frechet spaces

(top exact $\Leftrightarrow$ alg exact, for Fréchet spaces)

(2) Apply Anschütz - Le Bras's VS version. $\square$

Back to duality: X Stein. In TVS, want

$$\mathbb{R}(X_C, \mathbb{Q}_p) \xrightarrow{\sim} R\mathcal{H}om_{VS}(\mathbb{R}_c(X_C, \mathbb{Q}_p(d))[2d], \mathbb{Q}_p)$$

represent $R\Gamma_{pro\acute{e}t}$ in TVS.

<u>Step 1</u> (Duality on $X_{FF, C^\flat}$).

Represent $R\Gamma_{pro\acute{e}t}$ by solid qcoh sheaves (Andreychev) on X_{FF}:

$$\mathcal{E}_{pro\acute{e}t,*}(X_C, \mathbb{Q}_p(r)) \in QCoh(X_{FF})$$

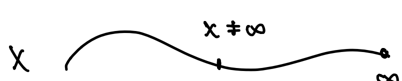
s.t. $R\Gamma(X_{FF}, \mathcal{E}_{pro\acute{e}t,*}(X_C, \mathbb{Q}_p(r))) \simeq R\Gamma(X_C, \mathbb{Q}_p(r)) \in D(\mathbb{Q}_{p,\square})$.

<u>Thm 0</u> (Duality on X_{FF})

(*) $\mathcal{E}_{pro\acute{e}t}(X_C, \mathbb{Q}_p) \xrightarrow{\sim} R\mathcal{H}om_{QCoh(X_{FF})}(\mathcal{E}_{pro\acute{e}t,c}(X_C, \mathbb{Q}_p(d))[2d], \mathcal{O})$.

pf Pass to syntomic cohom (HK-twisted $\mathcal{E}_{HK}$

dR-twisted $\mathcal{E}_{dR}$).



$R\Gamma_{HK}$-twisted by B^+_{cris}

$\rightsquigarrow$ reduce to HK duality.

$\rightsquigarrow$ reduce to classical Poincaré duality for $\mathcal{E}_{dR}$

Period isom (CDN, CN, AGN): for $r \geq 2d$,

$$R\Gamma_*(X_C, \mathbb{Q}_p(r)) \overset{r \geq 2d}{\simeq} R\Gamma_{syn,*}(X_C, \mathbb{Q}_p(r)).$$

Have to geometrize: $C \mapsto S \in Perf_C$.

$$\mathrm{RHS} \simeq \left[\left(R\Gamma_{HK,*}(X_C) \otimes^{\mathbb{L},\square}_{C,\square} B^+_{st}(S) \right)^{N=0,\varphi=p^r} \xrightarrow{\iota_{HK}} R\Gamma_{dR,*}(X_C / B^+_{dR}(S)) / F^r \right]. \qquad \square$$

Cor (of computations on X_{FF})

For X, Y Stein / K, have Künneth formula

$$\mathcal{E}_{proét}(X, \mathbb{Q}_p) \otimes^{\mathbb{L}}_{\mathcal{O}} \mathcal{E}_{proét}(Y, \mathbb{Q}_p) \xrightarrow{\sim} \mathcal{E}_{proét}(X \times Y, \mathbb{Q}_p).$$

($\leadsto$ Syn $\leadsto$ (HK, B^+_{dR}),

but no idea to descend to TVS).

Step 2 (Descend from X_{FF} to TVS).

$$R\tau_* : \mathcal{F} \in QCoh(X_{FF}) \longmapsto (S \longmapsto R\Gamma(X_{FF,S}, \mathcal{F}|_{X_{FF,S}})).$$

$(*)$ (in Thm 0) $\overset{R\tau_*}{\leadsto}$ $R\tau_* \mathcal{E}_{syn,*} \simeq \mathbb{R}_{syn,*}$.

Both LHS, RHS reduce to $\mathbb{R}_{syn}$.

Suffices: $\quad R\tau_* R\mathcal{H}om(\mathcal{E}_{syn}, \mathcal{O}) \overset{?}{\simeq} R\mathcal{H}om_{TVS}(R\tau_* \mathcal{E}_{syn}, \mathbb{Q}_p).$