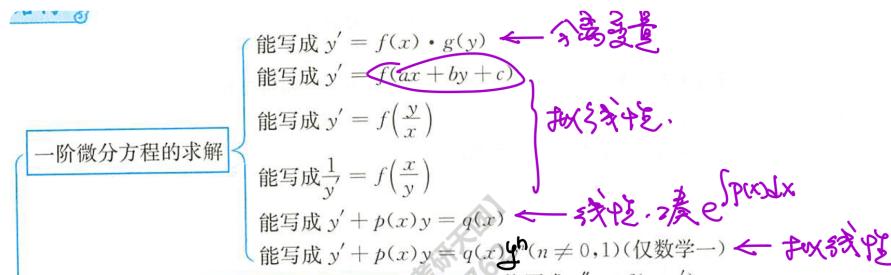


$$\begin{aligned}
 & - \int \left( \frac{3}{z-2} - \frac{2}{z-1} \right) dz \\
 &= \ln \frac{(z-1)^2}{|z-2|^3} = \ln |\ln| + C \\
 &\quad \left. \begin{array}{l} u = v+z = x^2+2 \\ z = \frac{m}{n} = \frac{u+1}{v+2} = \frac{y^2+1}{x^2+2} \end{array} \right\} \\
 & \ln(x^2+2) + C = \ln(\dots).
 \end{aligned}$$

$$\begin{aligned}
 4. \quad & \frac{dy}{dx} = \frac{2x^3+3xy^2-7x}{3x^2y+2y^3-8y} \\
 & \Leftrightarrow \frac{y dy}{x dx} = \frac{2x^2+3y^2-7}{3x^2+2y^2-8}. \quad \left\{ \begin{array}{l} u=y^2, v=x^2 \\ du=2y dy, dv=2x dx \end{array} \right. \\
 & \Rightarrow \frac{du}{dv} = \frac{2v+3u-7}{3v+2u-8}. \quad \text{令} \left\{ \begin{array}{l} 2v+3u-7=0 \\ 3v+2u-8=0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} u=1 \\ v=2 \end{array} \right. \\
 & \left\{ \begin{array}{l} m=u-1, n=v-2. \quad \text{即} \\ \frac{dm}{dn} = \frac{2n+3m}{2m+3n} \end{array} \right. \quad \left\{ \begin{array}{l} m=2n \\ u=1 \end{array} \right. \\
 & \Rightarrow \text{分离变量为 } \left( \frac{3+2z}{z-2z^2} \right) dz = \frac{dn}{n} \\
 & \Rightarrow \frac{3}{4} \ln \left| \frac{1+z}{1-z} \right| - \frac{1}{2} \ln |1-z^2| = \ln n + C \\
 & \Rightarrow (x^2-y^2-1)^{\frac{3}{4}} = C(x^2+y^2-3).
 \end{aligned}$$



## 差分方程

差分定义：离散版微分

若  $f(x) = y$  只在  $x_0, x_1, \dots$  上定义.

$x_k$  到  $x_{k+1}$  时， $y = f(x)$  的近似值是  $\Delta y_k = f(x_{k+1}) - f(x_k)$

$\Rightarrow \Delta y_k = y_{k+1} - y_k$ ,  $y_k = f(x_k)$  - 价差分.

更精确版本  $\Delta y = \min_{\Delta x > 0} \frac{y_{k+\Delta k} - y_k}{\Delta k} = y_{k+1} - y_k (\Delta k=1)$ .

微分： $y' = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

本质是让  $\Delta x$  充分小（以直代曲误差充分小）.

二阶差分： $\Delta^2 y_k = \Delta y_{k+1} - \Delta y_k$ .

$n$  阶差分： $\Delta^n y_k = \Delta^{n-1} y_{k+1} - \Delta^{n-1} y_k$ .

差分运算规则.

(1)  $\Delta C = 0$  ( $C$  常数)

(2)  $\Delta(Cy_k) = C \Delta y_k$

(3)  $\Delta(C_1 y_k \pm C_2 z_k) = C_1 \Delta y_k \pm C_2 \Delta z_k$

(4)  $\Delta(y_k \cdot z_k) = y_{k+1} \cdot \Delta z_k + z_k \cdot \Delta y_k = y_k \cdot \Delta z_k + z_{k+1} \cdot \Delta y_k$  □

(5)  $\Delta\left(\frac{y_k}{z_k}\right) = \frac{z_k \Delta y_k - y_k \Delta z_k}{z_k z_{k+1}} = \frac{z_{k+1} \Delta y_k - y_{k+1} \Delta z_k}{z_k z_{k+1}}$

$$\text{例： } \Delta^4 y_x - 6y_{x+2} + 4y_{x+1} - y_{x+1} = 0$$

$$\begin{aligned} \Delta^4 y_x &= \Delta^3 y_{x+1} - \Delta^3 y_x \\ &= \Delta^2 y_{x+2} - \Delta^2 y_{x+1} - \Delta^2 y_{x+1} + \Delta^2 y_x \end{aligned}$$

$$= \Delta y_{x+3} - \Delta y_{x+2} - \Delta y_{x+2} + \Delta y_{x+1} - \Delta y_{x+2} + \Delta y_{x+1} + \Delta y_{x+1} - \Delta y_x$$

$$\begin{aligned}
&= \Delta y_{x+3} - 3\Delta y_{x+2} + 3\Delta y_{x+1} - \Delta y_x \\
&= y_{x+4} - y_{x+3} - 3(y_{x+3} - y_{x+2}) + 3(y_{x+2} - y_{x+1}) - (y_{x+1} - y_x) \\
&= y_{x+4} - 4y_{x+3} + 6y_{x+2} - 4y_{x+1} + y_x \\
\Rightarrow &\quad \Delta^4 y_x - 6y_{x+2} + 4y_{x+1} - y_{x+1} \\
&= y_{x+4} - 4y_{x+3} + 1 = 0 \\
\text{假设 } & u = x+3 \Rightarrow y_{u+1} - 4y_u + 1 = 0.
\end{aligned}$$

结论：高阶差可以化为低阶。

### 差分方程通解结构。

如果  $y_x^{(1)}, y_x^{(2)}, \dots, y_x^{(n)}$  是  $n$  阶齐次常系数差分方程

$$a_n \Delta^n y_x + a_{n-1} \Delta^{n-1} y_x + \dots + a_0 y_x = 0$$

$$\Leftrightarrow b_n y_{x+n} + b_{n-1} y_{x+(n-1)} + \dots + b_1 y_{x+1} + b_0 y_x = 0$$

那么通解为

$$Y_x = C_1 y_x^{(1)} + C_2 y_x^{(2)} + \dots + C_n y_x^{(n)}.$$

### 齐次差分方程通解。

$$\textcircled{1} \quad y_{x+1} + m y_x = 0$$

$$\text{通解 } y_c(x) = C \cdot (-m)^x.$$

$$\text{验证: } y_c(x+1) = (-m) \cdot C \cdot (-m)^x = -m y_c(x).$$

$$\textcircled{2} \quad y_{x+1} + m y_x = f(x)$$

特解，通解 = 特解 + 齐次通解

(i)  $f(x)$  是  $n$  次多项式  $P_n(x)$

$$m \neq -1: y_x^* = Q_n(x) \quad (\text{多项式不是 } P_n \text{ 的子集})$$

( $n$  次项不会被消掉)

$m = -1$ :  $n > 2$  會被消掉

$$y_x^* = x \cdot Q_n(x) \quad (\text{若 } n > 2)$$

原因: 因  $Q_n(x) = B_n x^n + B_{n-1} x^{n-1} + \dots + B_0$

$$Q_n(x+1) = B_n (x+1)^n + B_{n-1} (x+1)^{n-1} + \dots + B_0$$

$$= B_n x^n + C_{n-1} x^{n-1} + \dots + C_0 \quad (C_{n-1} \neq B_{n-1})$$

$$\Rightarrow Q_n(x+1) - Q_n(x) = (C_{n-1} - B_{n-1}) x^{n-1} + \dots + C_0 - B_0$$

$\begin{cases} \rightarrow n-1 \\ \text{次} \end{cases}$  次

$\# m = -1$ ,  $y_{x+1} + m y_x = y_{x+1} - y_x = n-1 \text{ 次} \neq P_n(x)$ .

(所以左邊必須補-1次).

(ii)  $f(x) = d^x \cdot P_n(x)$ ,  $d \neq 0$  為常數

$$m + d \neq 0, y_x^* = d^x \cdot Q_n(x).$$

檢查:  $y_{x+1} = d^{x+1} \cdot Q_n(x+1)$

$$\Rightarrow y_{x+1} + m y_x = d \cdot d^x \cdot Q_n(x+1) + m d^x \cdot Q_n(x) = d^x \cdot P_n(x)$$

$$\Rightarrow d \cdot Q_n(x+1) + m \cdot Q_n(x) = P_n(x).$$

( $\# d + m = 0$ ,  $n > 2$  會被消掉).

$$m + d = 0, y_x^* = x \cdot d^x \cdot Q_n(x) \quad (\text{補-1次})$$

(iii)  $f(x) = b_1 \cos \omega x + b_2 \sin \omega x \quad (\omega \neq 0, b_1, b_2 \text{ 不妨設 } = 0)$

$$D = \begin{vmatrix} m + \omega \sin \omega & \sin \omega \\ -\sin \omega & m + \cos \omega \end{vmatrix} \neq 0,$$

$$y_x^* = \alpha \omega \sin \omega x + \beta \cos \omega x, \quad \alpha, \beta \text{ 未定}.$$

$$D = 0, \quad y_x^* = x(\alpha \cos \omega x + \beta \sin \omega x).$$

## 差分方程的例子

Hansen-Samuelson 國民收入分析模型

$Y_t$ -國民收入,  $C_t$ -消費,  $I_t$ -投資,  $G_t$ -政府支出總額

作为最终  $\rightarrow Y_t = C_t + I_t + G_0 \leftarrow$  方便起见,  $G_t = G_0$  为常数

差分方程的未知量  $I_t = \beta(C_t - C_{t-1})$ ,  $\beta > 0$ .

$$Y(0) = Y_0, Y_1 = Y_1$$

$$C_t = \alpha Y_{t-1}, 0 < \alpha < 1$$

$$\begin{aligned} \Rightarrow Y_t &= \alpha Y_{t-1} + \beta(C_t - C_{t-1}) + G_0 \\ &= \alpha Y_{t-1} + \beta(\alpha Y_{t-1} - \alpha Y_{t-2}) + G_0 \end{aligned}$$

$$\Rightarrow Y_t - (1+\beta)\alpha Y_{t-1} + \alpha\beta Y_{t-2} - G_0 = 0.$$

= 阶梯式双非齐次差分方程.