Fall 2020: Topics in Number Theory: Fermat's Last Theorem

Instructor: Liang Xiao (肖梁)
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Meeting time: Monday 1–2, and Wednesday 7–8(even week)
Lecture room: Lecture Building #3, Room 206 (三教 206)
Office hours: Friday 10–11am, at 78101–1
Webpage: `http://bicmr.pku.edu.cn/~lxiao/20fall/index.htm`

Goal of this course: The focus will be on understanding the Langlands correspondence and the Taylor-Wiles method. The course consists of five parts:

(1) We review basic facts on structures of Galois group of local and global number fields, and discuss Galois cohomology and duality theory.
(2) We give a light introduction to p-adic Hodge theory.
(3) We discuss deformation of Galois representations, local and global.
(4) We shift to discuss modular forms and associated Galois representations.
(5) We prove a modularity lifting theorem, through which we deduce Fermat's Last Theorem.

Our main reference includes various lecture notes by Richard Taylor, and a lecture note by Sug Woo Shin at Berkeley. For modularity lifting theorems, we will partially follow Toby Gee's notes at Arizona Winter School.

Prerequisite:

- Class field theory, especially descent understanding of Galois cohomology (but will recall some in first several lectures).
- Homology theory (in topology).
- Algebraic geometry (Hartshorne Chap 2, can concurrently learning Chap 3)
- Modular forms.
- Very basic knowledge of elliptic curves.

Grade Distribution:

Homeworks: 50%, every two-weeks starting week 4, in total 7 times
Other means: to be announced.

Homework: *Homework problems are posted on the course webpage, and are usually due on the Wednesday of odd weeks.* You are welcome and encouraged to work with other students on the problems, but you should write up your homework independently.

Syllabus (Tentative)

Lecture	Dates	Content
1	9/21	Introduction and Background in Number Theory I: structure of local Galois group, ℓ-adic representations of local Galois group, Grothendieck's ℓ-adic monodromy theorem, Weil–Deligne representations.
2	9/28	Background in Number Theory II: Higher ramification groups, local and global class field theory in terms of Artin maps, ℓ-adic representations of global Galois group.
3	9/30	Background in Number Theory III: Galois cohomology, local duality theorems for Galois cohomology, first touch on Selmer complex.
		Happy National's Day and Mid-Autumn Festival!
4	10/12	Background in Number Theory IV: Global duality theorems for Galois cohomology, first touch on Selmer complex. (HW 1 due)
5	10/14	p-adic Hodge theory I: definition of $\mathbb{B}_{\mathrm{dR}}$ and $\mathbb{B}_{\mathrm{cris}}$, Hodge filtration, weakly admissible and admissible filtered ϕ-modules.
6	10/19	p-adic Hodge theory II: Galois representations coming from geometry, Grothendieck's mysterious functor question, Galois cohomology for p-adic representation of $\mathrm{Gal}_{\mathbb{Q}_p}$.
7	10/26	p-adic Hodge Theory III: Fontaine–Laffaille theory, Galois cohomology associated to Fontaine–Laffaile modules. (HW 2 due)
8	10/28	Galois deformation I: Framed deformation, computation of tangent space, and criterion of smoothness, lots of examples.
9	11/2	Galois deformation II: Schlessinger's criterion.
10	11/9	Galois deformation III: Galois deformation with local conditions, computation of tangent space and Euler characteristics via Selmer complexes. (HW 3 due)
11	11/11	Galois deformation IV: Explicit computation of some local deformation rings (including Fontaine-Laffaille deformation and Taylor's Ihara avoidance deformation)
	11/16	Overflow

12	11/23	Automorphic forms I: Review modular forms, Hecke operators, relation to automorphic forms. (HW 4 due)
13	11/25	Automorphic forms II: Basic representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$, smooth, admissible representations, principal series, unramified principal series, interpretation of new/old form theory in terms of local representation theory.
14	11/30	Automorphic forms III: Introduction to local Langlands for GL_2.
15	12/7	Geometric modular forms I: Modular curves and geometric realization of modular forms, Jacobian of modular curves, extension of Hecke action. (HW 5 due)
16	12/9	Geometric modular forms II: Attaching Galois representations to modular forms of weight 2, Eichler–Shimura relations.
17	12/14	Geometric modular forms III: de Rham local systems and automorphic line bundles, Eichler–Shimura isomorphisms.
18	12/21	Geometric modular forms IV: Attaching Galois representations to modular forms of higher weight (using étale cohomology as a black box). (HW 6 due)
19	12/23	Modularity lifting theorem I: Logic background (ultra filter), setup for the theorem, choice of Taylor–Wiles prime.
20	12/28	Modularity lifting theorem II: The main argument.
21	1/4	Modularity lifting theorem III: Wiles' 3-5 trick and finish of the proof (HW 7 due)
22	1/6	Review / Overflow
	TBA	**Final Exam**