The Hodge-Tate period map for Shimura varieties of Hodge type

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(Joint work with P. Scholze.)

Applications Already p-adic autom forms,

Gal reps attached to cohom classes, etc.

§1 Hodge theory

X/E proj sm var, $E \hookrightarrow \mathbb{C}$,

X^{an} = complex analytic vector space.

(a) Betti cohom $\quad H^*(X^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p,q} H^q(X^{an}, \Omega^p_{X^{an}})$

holomorphic p-forms

admitting the Hodge decomp.

(b) de Rham cohom $\quad H^*_{dR}(X)/E \cong \mathbb{H}(X, \Omega^\bullet_X)$

algebraic de Rham Complex

admitting Hodge-Tate fil'n on $H^*_{dR}(X)$.

(c) Comparison isom

$$H^*(X^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \simeq H^*_{dR}(X) \otimes_E \mathbb{C}.$$

w/ Hodge fil'n $\mathrm{Fil}^p H^n = \bigoplus_{p' \geq p} H^{p',q'} \quad (p'+q'=n)$.

note $H^*_{dR}(X)$ exists also for adic spaces

e.g. $\mathrm{Spa}(K, \mathcal{O}_K)$ for $K/\mathbb{Q}_p$ finite.

$A/\mathbb{C}$ an AV $\Longleftrightarrow$ HS of type $(-1,0), (0,-1)$.

$$0 \to H^0(A, \Omega^1_A) \to H^1_{dR}(A) \to H^1(A, \mathcal{O}_A) \to 0$$

$\mathcal{O}_A = \Omega^0_A$

Hodge-de Rham fil'n.

§2 Shimura varieties of Hodge type

(G, X) : $G/\mathbb{Q}$ red grp, X = conj class of $h_i : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$

(Tannakian grp for real Hodge strs.)

satisfying certain axioms.

$$X \simeq G(\mathbb{R})/K_\infty^\circ A_\infty^\circ \quad \text{Herm symm domain.}$$

Assume (G, X) is of Hodge type, i.e.

$\exists$ an embedding of Shimura data

$$(G, X) \hookrightarrow (\tilde{G}, \tilde{X})$$

$$\tilde{G} = GSp(V, \psi), \quad \psi \text{ symplectic form.}$$

Borel embedding

h determines Hodge cochar $\mu : \mathbb{G}_m \to G_{\mathbb{C}}$

Tannakian grp for graded vec spaces.

$\rightsquigarrow$ fil'n on $\mathrm{Rep}_{\mathbb{C}}(G)$ (descending)

h is real HS $\Rightarrow$ Fil_μ is a Hodge-de Rham fil'n.

Define $P_\mu^{std} = \{ g \in G \mid \lim_{t \to 0} \mathrm{ad}(\mu(t)) g \text{ exists} \}$

$= $ Stabilizer of G in Fil_μ.

$P_\mu = \{ g \in G \mid \lim_{t \to \infty} \mathrm{ad}(\mu(t)) g \text{ exists} \}$

$= $ Stabilizer of opposite (complex conj) fil'n Fil_μ on $\mathrm{Rep}_{\mathbb{C}}(G)$.

Then

$G(\mathbb{C})/P_\mu^{std}(\mathbb{C}) = \mathcal{F}l_G^{std}$ = moduli of fil'ns conj to Fil_μ.

$G(\mathbb{C})/P_\mu(\mathbb{C}) = \mathcal{F}l_G$ = moduli of fil'ns conj to Fil^μ.

Borel embedding $X \hookrightarrow \mathcal{F}l_G^{std}$ (ensembles $\mathcal{H} \hookrightarrow \mathbb{P}^1(\mathbb{C})$)

$h \longmapsto \mu_h$ holomorphic

(Also $X \hookrightarrow \mathcal{F}l_G$ anti-holomorphic.)

$K \subseteq G(\mathbb{A}_f)$ compact open.

$\rightsquigarrow Sh_K(G,X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K$.

X universal cover of each conn component.

<u>Note</u> Sh_K, $\mathcal{F}l_G^{std}$ have models over E = reflex field

= field of def'n of μ.

To define automorphic vec bundles $/Sh_K$:

(algebraic, def'd $/E$)

$\mathrm{Rep}\, P_\mu^{std} \simeq \{ G(\mathbb{C})\text{-equivariant vec bundles on } \mathcal{F}l_G^{std} \}$

$\longrightarrow \{ G(\mathbb{R})\text{-equivariant vec bundles on } X \}$

descend to autom v.b. on Sh_K.

<u>Idea</u> Consider $(G,X) \hookrightarrow (\tilde{G},\tilde{X})$ a Siegel embedding

$\exists\, (s_\alpha) \in V^\otimes$ (finitely many) s.t. $G = \mathrm{Stab}(s_\alpha)$.

$Sh_K \hookrightarrow \tilde{Sh}_{\tilde{K}} \times_{\mathbb{Q}} E$ closed embedding.

Restriction $\pi : A \longrightarrow Sh_K$ gives an AV

$\mathcal{V}_{dR} := R^1 \pi_{dR,*} \mathcal{O}_A \quad / Sh_K$.

$\mathcal{G}_{dR} = G\text{-torsor} \ /Sh_K$

$= \{ \beta : \mathcal{V}_{dR} \longrightarrow V_\mu \otimes \mathcal{O},\ \beta(s_{\alpha,dR}) = s_\alpha \}$

de Rham realization of s_α

(horizontal sections of $\mathcal{V}_{dR}^\otimes$).

$$\begin{tikzcd} & \mathcal{G}_{dR} \arrow[dl, "\kappa"'] \arrow[dr, "j"] & \\ Sh_K & & \mathcal{F}l_G^{std} \end{tikzcd}$$

$\mathcal{G}_{dR}$ has a P_μ^{std}-structure

(obtained from P_μ^{std}-torsor $\mathcal{P}_{dR}$ via $P_\mu^{std} \to G$).

$P_{dR} = \{ \beta \in \mathcal{G}_{dR} : \beta(\mathrm{Fil}^0 V_{dR}) = \text{Hodge-de Rham fil'n} \}$.

Get autom vbs on Sh_K by first pulling back along j and then descending via π.

§3 p-adic picture

$C = \widehat{\overline{\mathbb{Q}}}_p$. $\varprojlim_{K_p} (Sh_{K^pK_p} \times_E C)^{ad} \sim \mathcal{S}_{K^p}$

$\uparrow$ perfectoid Shimura variety.

There is a map of adic spaces

$$\begin{array}{ccc} \pi'_{HT} : \mathcal{S}_{K^p} & \longrightarrow & \mathcal{F}\ell_{\tilde{G}}^{ad} \circlearrowleft \tilde{G}(\mathbb{Q}_p) \\ \circlearrowleft & & \parallel \\ G(\mathbb{Q}_p) & & (\mathcal{F}\ell_{\tilde{G}} \times C)^{ad} \end{array}$$

$((G, X) \hookrightarrow (\tilde{G}, \tilde{X}))$

π_{HT} is equivariant for the $G(\mathbb{Q}_p)$-action, and (varying K^p in a tower) for the $G(\mathbb{A}_f^p)$-action (w/ trivial action on $\mathcal{F}\ell_{\tilde{G}}$).

If A/C is an AV of dim g,

$A/C \rightsquigarrow$ pt of $\tilde{\mathcal{S}}_{\tilde{K}^p}$

$H^1_{et}(A, \mathbb{Q}_p) \xrightarrow{\sim} \mathbb{Q}_p^{2g}$

w/ Hodge-Tate fil'n

$$0 \to H^1(A, \mathcal{O}_A) \to H^1_{et}(A, \mathbb{Q}_p) \otimes C \to H^0(A, \Omega^1_A)(-1) \to 0$$

* This gives a pt in $\mathcal{F}\ell_{\tilde{G}}$.

Thm (Caraiani, Scholze)

(1) $\exists$ $G(\mathbb{Q}_p)$-equiv map of adic spaces

$$\pi_{HT} : \mathcal{S}_{K^p} \longrightarrow \mathcal{F}\ell_G^{ad} = (\mathcal{F}\ell_G \times_E C)^{ad}$$

compatible w/ π'_{HT}.

(2) There is an isom of tensor functors given by

$$\begin{tikzcd} \mathrm{Rep}\, M_\mu \arrow[r] \arrow[d] & \left\{ \begin{array}{c} G(\mathbb{Q}_p)\text{-equiv vec} \\ \text{bundles on } \mathcal{F}\ell_G^{ad} \end{array} \right\} \arrow[d, "\pi_{HT}^*"] \\ \left\{ \begin{array}{c} \text{Autom vb} \\ \text{on } \mathcal{S}_{K^p} \end{array} \right\} \arrow[r] & \left\{ \begin{array}{c} G(\mathbb{Q}_p)\text{-equiv vec} \\ \text{bundles on } \mathcal{S}_{K^p} \end{array} \right\} \end{tikzcd}$$

where $M_\mu = \mathrm{Cent}(\mu)$ common Levi of P_μ, P_μ^{std}.

Rmk Can also write in K^p-tower with $G(\mathbb{A}_f)$-action, (being trivial on $\mathcal{F}\ell_G^{ad}$).

Ideas (1) p-adic de Rham comparison in families (Scholze) induces HT fil'n.

(2) $V_p = R^1\pi_*\mathbb{Q}_p$, $\pi : A \to \mathcal{S}_K$.

construct M_μ, P_μ, G-torsors trivializing $V_p \otimes \widehat{\mathcal{O}}_{\mathcal{S}_K}$.

(1) Comparison: $V_p \otimes \widehat{\mathcal{O}}_{\mathcal{S}_K} \xrightarrow{\sim} R^1\pi_*\widehat{\mathcal{O}}_A$

(X adic space $\rightsquigarrow$ $\widehat{\mathcal{O}}_X$ sheaf on $(X)_{\text{proét}}$).

First step in relative Hodge-Tate fil'n:

$$0 \to R^1\pi_*\mathcal{O}_A \otimes_{\mathcal{O}_{\mathcal{S}_K}} \widehat{\mathcal{O}}_{\mathcal{S}_K} \to R^1\pi_*\widehat{\mathcal{O}}_A.$$

Another way: p-adic de Rham comparison

$\mathbb{B}_{dR,A}^{(+)}$ relative period sheaf on $(A)_{\text{proét}}$

"$(W(\mathcal{O}_A^{+\flat})[\frac{1}{p}])^\wedge_{\ker\theta}$ where $\theta : W(\mathcal{O}_A^{+\flat}) \twoheadrightarrow \mathcal{O}_A^+$.

equipped w/ fil'n $\mathrm{Gr}^0\mathbb{B}_{dR,A}^{(+)} = \widehat{\mathcal{O}}_A$

$\mathcal{O}\mathbb{B}_{dR}^{(+)}$ structural period sheaf

$$R^1\pi_*\mathbb{B}_{dR,A}^{(+)} \otimes_{\mathbb{B}_{dR,\mathcal{S}_K}^+} \mathbb{B}_{dR,\mathcal{S}_K} \simeq R^1\pi_{dR,*}\mathcal{O}_A \otimes_{\mathcal{O}_{\mathcal{S}_K}} \mathbb{B}_{dR,\mathcal{S}_K}.$$

Compatible w/ fil'n and connection

$$R^1\pi_*\widehat{\mathcal{O}}_A = Gr^0(LHS)^{\nabla=0}.$$

- $M = R^1\pi_*\mathbb{B}_{dR,A}^{(+)}$ local system on $\mathcal{S}_K$

  (corresponding to étale cohom)

  Hodge-Tate fil'n

- $M_0 = (R^1\pi_{dR,*}\mathcal{O}_A \otimes_{\mathcal{O}_A} \mathcal{O}\mathbb{B}_{dR}^{(+)})^{\nabla=0}$

  local system corresponding to de Rham cohomology

  Hodge-de Rham fil'n.

Just like Scholze's talk,

two $\mathbb{B}_{dR,\mathcal{S}_K}^{(+)}$ local systems sitting inside same $\mathbb{B}_{dR,\mathcal{S}_K}$ local system

(on points, gives 2 lattices.)

This gives you a Hodge-de Rham fil'n on M_0

& a Hodge-Tate fil'n on M

(related by comparison isom.)

- $\mathbb{B}_{dR}^+$-structure on M_0 induces HT fil'n on $R^1\pi_*\widehat{\mathcal{O}}_A = Gr^0(LHS)^{\nabla=0}$

  and vice versa to get Hodge-de Rham fil'n.

(2) Check that tensors preserve HT fil'n

using this comparison + preservation of Hodge fil'n.