Elementary Number Theory

PROBLEM SET 2

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Problem 1. A positive integer is said to be square-free if it is not divisible by the square of any prime.

(a) Prove that every positive integer can be written uniquely as st^2 where $s \in \mathbb{N}$ is square-free and $t \in \mathbb{N}$.

(b) Use this (and not the prime number theorem) to prove that there is a constant $C > 0$ such that
$$\pi(x) \geqslant C \log x$$
for any real number $x \geqslant 2$.

Problem 2. Let p be a prime. For any positive integer n, we can express it in base p as $n = n_0 + n_1 p + \cdots + n_k p^k$ and write $s_p(n) = n_0 + n_1 + \cdots + n_k$. Prove that
$$v_p(n!) = \frac{n - s_p(n)}{p-1}.$$

Problem 3. Suppose a, b, n are positive integers such that $a!b!$ divides $n!$. Prove that
$$a + b \leqslant n + 1 + 2\frac{\log n}{\log 2}.$$

Problem 4. Let m, n be positive integers such that $1 \leqslant m \leqslant n$. Prove that
$$m\binom{n}{m} \mid \operatorname{lcm}(1, \ldots, n).$$

Problem 5. Let n be a positive integer.

(a) Prove that it N is an integer such that $n\binom{2n+1}{n} \mid N$ and $(n+1)\binom{2n+1}{n+1} \mid N$, then
$$n(n+1)\binom{2n+1}{n} \mid N.$$

(b) Prove that
$$n4^n \leqslant \operatorname{lcm}(1, \ldots, 2n+1).$$

(c) Prove that for any integer $m \geqslant 7$,
$$\operatorname{lcm}(1, \ldots, m) \geqslant 2^m.$$

It then follows that for any integer $x \geqslant 2$,

$$\pi(x) \geqslant \frac{x \log 2}{\log x}$$

by checking the smaller values directly.

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