Dual abelian variety

Def: $\mathrm{Pic}^0(X) = \{ L$ line bundle $|$ $\phi_L(x) = 0$ for any $x \} \subseteq \mathrm{Pic}(X)$.

$\phi_L(x) = T_x^* L - L$

By thm of the square (§6, Cor 4, P57)

$T_{x+y}^* L \otimes L \cong T_x^* L \otimes T_y^* L$

We have exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{Hom}(X, \mathrm{Pic}^0(X))$$
$$L \longmapsto \phi_L$$

Goal: Construct $\hat{X} \cong \mathrm{Pic}^0(X)$

Preparations/propositions:

(i) $L \in \mathrm{Pic}^0(X) \Leftrightarrow T_x^* L \cong L$ for all $x \in X$
$\Leftrightarrow m^* L \cong p_1^* L \otimes p_2^* L$ on $X \times X$

(ii) $L \in \mathrm{Pic}^0(X)$ $f, g : S \to X$. then
$(f+g)^* L \cong f^* L \otimes g^* L$

(iii) $L \in \mathrm{Pic}^0(X)$, $n_X^* L \cong L^n$

(iv) $L \in \mathrm{Pic}(X)$, $n_X^* L \cong L^{n^2} \otimes$ (sth. in $\mathrm{Pic}^0(X)$)

(v) L has finite order, then $L \in \mathrm{Pic}^0(X)$

(vi) S variety. L line bundles on $X \times S$. $L_s = L|_{X \times \{s\}}$

then $L_{s_1} \otimes L_{s_0}^{-1} \in Pic^0(X)$

(vii) $L \in Pic^0(X)$ non-trivial. then $H^i(X, L) = 0$. for all i.

Prove one by one:

(i). first "$\Leftrightarrow$": Definition

second "$\Leftrightarrow$": See-saw thm (§5. Cor 6. P51)

$m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} \triangleq T$

T trivial $\Leftrightarrow$ $T|_{X \times \{a\}}$. $T|_{\{0\} \times X}$ is trivial for all a

But $T|_{\{0\} \times X} = \mathcal{O}_{\{0\} \times X}$

$T|_{X \times \{a\}} = T_a^* L \times L^{-1}$

then T trivial $\Leftrightarrow$ $T_a^* L = L$ for all a

(ii) consider $S \xrightarrow{(f,g)} X \times X \xrightarrow{m} X$

then $(f+g)^* L = (f,g)^* \circ m^* L$

$= (f,g)^* (p_1^* L \otimes p_2^* L)$

$= f^* L \otimes g^* L$

(iii) By (ii), trivial

(iv) By (§6, App 2, P59), $n_X^* L \triangleq L^{\frac{n(n+1)}{2}} \otimes (-1_X)^* L^{\frac{n(n-1)}{2}}$

$= L^{n^2} \otimes (L \otimes (-1_X)^* L)^{-\frac{n(n-1)}{2}}$

It suffices to show: $L \otimes (-1_X)^* L \in Pic^0(X)$

$T_x^* (L \otimes (-1_X)^* L) = T_x^* L \otimes (-1_X)^* T_{-x}^* L^{-1}$

$= T_x^* L \otimes (-1_X)^* L^{-1} \otimes (-1_X)^* (L \otimes T_{-x}^* L^{-1})$

$$= T_x^* L \otimes (-1_x)^* L^{-1} \otimes (L^{-1} \otimes T_{-x}^* L)$$

(This step is by $L \otimes T_{-x}^* L^{-1} \in Pic^0(X)$ and (iii))

$$= L \otimes (-1_X)^* L^{-1}$$

(This step is by thm of the square)

Then $L \otimes (-1_x)^* L^{-1} \in Pic^0(X)$

(v) L^n is trivial, $0 = \phi_{L^n}(X) = n\phi_L(X) = \phi_L(nx)$

X is divisible, then n_X is surjective $\Rightarrow \phi_L = 0$.

(vi) Replacing S by an open covering, we may assume

$L|_{\{0\} \times S}$ is trivial

Replacing L by $L \otimes p_1^*(L_{s_0}^{-1})$, we may assume

$L_{s_0} = L|_{X \times \{s_0\}}$ is trivial

It suffices to show $L_s \in Pic^0(X)$ for all $s \in S$

Consider line bundle M on $X \times X \times S$

$M = \mu^* L \otimes p_{13}^* L^{-1} \otimes p_{23}^* L^{-1}$

$\mu(x, y, s) = (x+y, s)$

$p_{13}(x, y, s) = (x, s)$

$p_{23}(x, y, s) = (y, s)$

then $M|_{X \times \{0\} \times S}$, $M|_{\{0\} \times X \times S}$, $M|_{X \times X \times \{s_0\}}$ are trivial

By thm of the cube (§6. P52), M is trivial then $\mathcal{M}|_{X \times X \times \{s\}} = m^*(L_s) \otimes p_1^*(L_s^{-1}) \otimes p_2^*(L_s^{-1})$ is trivial

$L_s \in Pic^0(X)$

Remark*: General definition of $Pic^0(U)$ for any non-singular projective variety U:

Let C an irreducible curve (or ns variety).

Δ a divisor on $C \times V$ without vertical component

then $Pic^0(V)$ is the subgp of $Pic(V)$ generated by all $V_\lambda - V_\mu$. $\lambda, \mu \in C$ (for any choice of C, Δ, λ, μ)

(Serre. Lectures on the Mordell-Weil Theorem. P_{25})

(vii) (Abuse the notations of sheaves and line bundles)

① proof of $H^0(X, L) = 0$:

If $H^0(X, L) \neq 0$. $L = \mathcal{O}_X(D)$ for some effective D

then $L^{-1} = (-1_X)^* L = \mathcal{O}_X((-1_X)^* D)$

Hence $\mathcal{O}_X \cong L \otimes L^{-1} \cong \mathcal{O}_X(D + (-1_X)^* D)$

$\Rightarrow D + (-1_X)^* D = 0$

$\Rightarrow D = 0$ (effectiveness). Contradiction!

② By induction. Assume $H^i(X, L) = 0$ for $i < k$ $(k > 1)$

consider the maps

$$X \xrightarrow{s_1} X \times X \xrightarrow{m} X$$

$$x \longmapsto (x, 0)$$

$m \circ s_1 = id$

then it induces the diagram

$$\begin{array}{ccccc} H^k(X,L) & \xleftarrow{s_i^*} & H^k(X\times X, m^*L) & \xleftarrow{m^*} & H^k(X,L) \\ & & \| & & \\ & & H^k(X\times X, p_1^*L\otimes p_2^*L) & & \\ & & \| & & \\ & & \prod_{i+j=k} H^i(X,L)\otimes H^j(X,L)=0 & & \end{array}$$

(Künneth formula)

Because $s_i^*\circ m^* = id$, $\therefore H^k(X,L)=0$

Conclusion holds. $\square$

Thm 1. Let L ample and $M\in Pic^0(X)$ then for some $x\in X$.

$M = T_x^*L\otimes L^{-1}$

i.e. the map $\phi_L : X\to Pic^0(X)$ is surjective

We need some preparations about Grothendieck spectral sequence to complete the proof

Preparation: A very brief introduction of spectral sequence

(a) $\mathcal{C}$ abelian category, $E_r^{p,q}\in\mathcal{C}$ for $p,q\in\mathbb{Z}$, $r\geq r_0$

(sometimes we need $E_r^{p,q}=0$ for all $p<0$ or $q<0$, and $r_0=1$ or 2)

(b) $d_r^{p,q}: E_r^{p,q}\longrightarrow E_r^{p+r,q-r+1}$, $d_r\circ d_r=0$

(c) $E_{r+1}^{p,q}\cong\dfrac{\ker(d_r^{p,q})}{\mathrm{Im}(d_r^{p-r,q+r-1})}$

(d) ① Filtered objects $E^n \in \mathcal{C}$. $\cdots \supseteq F^{-1}(E^n) \supseteq F^0(E^n) \supseteq F^1(E^n) \supseteq \cdots$

② For any fixed (p,q), then exists r (If $E_r^{p,q}$ trivial for $p<0$ or $q<0$, then for $r > \max\{p+1, q+1\}$ $E_r^{p,q} = E_{r+1}^{p,q} = \cdots = E_\infty^{p,q}$)

$$E_r^{p,q} = E_{r+1}^{p,q} = E_{r+2}^{p,q} = \cdots \triangleq E_\infty^{p,q}$$

③ For any fixed n, then $F^p(E^n) = E^n$ for sufficiently small p

$F^p(E^n) = 0$ for sufficiently large p

(e) $E_\infty^{p,q} \cong G^p(E^{p+q})$ (Notation: $G^n(A) = \dfrac{F^n(A)}{F^{n+1}(A)}$)

Grothendieck spectral sequence

$F: \mathcal{A} \longrightarrow \mathcal{B}$ $G: \mathcal{B} \longrightarrow \mathcal{C}$. Left exact

$\mathcal{A}, \mathcal{B}, \mathcal{C}$ have enough injectives

F maps injectives to acyclics.

Then for $A \in \mathcal{A}$, $\exists$ spectral sequence $\{E_r^{p,q}(A)\}$

$$E_2^{p,q}(A) = R^pG \circ R^qF(A)$$

$$E_\infty^{p,q}(A) \cong R^{p+q}(G \circ F)(A)$$

Back to the proof:

$$K = m^*L \otimes p_1^*L^{-1} \otimes p_2^*(L^{-1} \otimes M^{-1})$$

$$Sh(X \times X) \xrightarrow{p_{1,*}} Sh(X) \xrightarrow{\Gamma} Ab$$

$$E_2^{k,l} = H^l(X, R^k p_{1,*}(K)) \Rightarrow H^{k+l}(X \times X, K) = E_\infty^{k,l}$$

(Notation: "$\Rightarrow$" means "converge", i.e. $E_\infty^{*,*}$)

Similarly, $H^l(X, R^k p_{2,*}(K)) \Rightarrow H^{k+l}(X\times X, K)$

Notice $K|_{\{x\}\times X} = T_x^* L \otimes L^{-1} \otimes M^{-1}$

$K|_{X\times\{x\}} = T_x^* L \otimes L^{-1}$

if $M \neq T_x^* L \otimes L^{-1}$ for any x, $K|_{\{x\}\times X}$ is non-trivial

for any x. by (vii). $H^k(X, K|_{\{x\}\times X}) = 0$ for all k, x

By (§5, Cor 2. P48) for any $x \in X$. the fiber is $\{x\}\times X$

we have $\dim_{k(x)} H^p(\{x\}\times X, K|_{\{x\}\times X}) = 0$. for all p

then $R^{p-1} p_{1,*}(K) = 0$

Therefore $H^k(X\times X, K) = 0$.

The other spectral sequence:

For $x \notin K(L) = \ker(\varphi_L)$, $T_x^* L \otimes L^{-1}$ non-trivial

and in $\mathrm{Pic}^0(X)$. Use (§5, Cor 2) again

we get $\mathrm{supp}(R^k p_{2,*}(K)) \subset K(L)$. which is a finite set

Then the spectral sequence degenerates to

$$\bigoplus_{x\in K(L)} R^k p_{2,*}(K)_x \cong H^k(X\times X, K) = 0$$

so $R^k p_{2,*}(K) = 0$

By (§5, Cor 4, P51), $H^k(X, K|_{X\times\{x\}}) = 0$ for all $x \in X$, $k \geq 0$

But $K|_{X\times\{x\}}$ is trivial for $x \in K(L)$

$H^0(K|_{X\times\{x\}}) \neq 0$. Contradiction!

$\square$

We've got a group structure $X/K(L) \cong \mathrm{Pic}^\circ(X)$!

Fix L ample on X, by (§7, P63), exists $\hat{X}$ s.t.

$$0 \to K(L) \to X \to \hat{X} \to 0$$ exact as groups.

By (§7, Prop 2, P66), we get P s.t.

$$(1_X \times \pi)^* P = K = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$$

(In details, $\ker(1_X \times \pi) = (0) \times K(L)$,

for any $a \in K(L)$, we have

$$T_{(0,a)}^* K = T_{(0,a)}^* m^* L \otimes T_{(0,a)}^* p_1^* L^{-1} \otimes T_{(0,a)}^* p_2^* L^{-1}$$

$$= m^* T_a^* L \otimes p_1^* L^{-1} \otimes p_2^* T_a^* L^{-1}$$

$$= m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} = K\ )$$

Our goals:

(a) ① $\hat{X} \xrightarrow{\sim} \mathrm{Pic}^\circ(X)$

$y \longmapsto P|_{X\times\{y\}}$

② $P|_{\{0\}\times\hat{X}}$ is trivial

(b) for any normal scheme S. and line bundle K on $X\times S$

s.t. ① $K_s = K|_{X\times\{s\}} \in \mathrm{Pic}^0(X)$ for some $s\in S$

(In fact, if ① holds for one $s\in S$, then it holds for all $s\in S$, by (vi) which we have proved before)

② $K|_{\{0\}\times S}$ is trivial

Then exists unique $f: S\to\hat{X}$

s.t. $K_s \cong P_{f(s)}$

and $K \cong (1_X\times f)^* P$

Remark: In other words, $(\hat{X}, P)$ represents the functor

$S \longmapsto \{K\in \mathrm{Pic}(X\times S) \mid K_s\in \mathrm{Pic}^0(X)$ for all $s\in S$,

and $K|_{\{0\}\times S}$ is trivial $\}$

P is called Poincaré bundle

Proof:

(a) ① $\pi: X\to \hat{X} \simeq \mathrm{Pic}^0(X)$. for $y\in\hat{X}$. $\exists\, \pi(x)=y$

then $P|_{X\times\{y\}} = (1_X\times \pi)^* P_{X\times\{x\}} = K|_{X\times\{x\}} = T_x^* L\otimes L^{-1}$

$= \phi_L(x) \longleftrightarrow y$

(Because of commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\phi_L"'] \arrow[dr, "\pi"] & \\ \mathrm{Pic}^0(X) \arrow[rr, no head, "\sim"] & & \hat{X} \end{tikzcd}$$

)

② Because $K|_{\{0\}\times X} = L \otimes L^{-1} = \mathcal{O}_X$,

$P|_{\{0\}\times \hat{X}}$ is trivial as well

(b) For given S and K, consider line bundle on $X \times S \times \hat{X}$

$E = p_{12}^*(K) \otimes p_{13}^*(P^{-1})$

Then $E|_{X\times\{(s,\alpha)\}} = K_s \otimes P_\alpha^{-1}$

By see-saw thm (§5, Cor 6, P51)

$T = \{(s,\alpha) \mid E|_{X\times\{(s,\alpha)\}}$ is trivial$\}$ is Zariski-closed

$E|_{X\times\{(s,\alpha)\}}$ is trivial iff $K_s \simeq P_\alpha$. $K_s \in \mathrm{Pic}^0(X)$

Then for any K_s, exists unique P_α. s.t. $K_s \simeq P_\alpha$

It follows set-theoretic map $\varphi: T \to S$

is bijective.

Since char $k=0$, φ is bijective

And since S is normal, by Zariski Main Thm

(GTM52, III. Cor 11.4, P280), $\varphi_* \mathcal{O}_T = \mathcal{O}_S$, $T \simeq S$.

On the other hand, the second projection gives

$T \hookrightarrow S \times \hat{X} \to \hat{X}$

so we get $S \xrightarrow{\sim} T \to \hat{X}$.

See-saw thm also gives $K = (1\times f)^* P$. □

Remarks:

(1) For every line bundle L on X, $\phi_L: X \to \hat{X}$ is a morphism

Proof: Let $S = X$, $K = m^*(L) \otimes p_1^*(L^{-1}) \otimes p_2^*(L^{-1})$

By universal property (b). We get ϕ_L is a morphism

(2) If $f: X \to Y$ homomorphism between two abelian varieties

f induces $Pic^0(Y) \to Pic^0(X)$, then we get $\hat{f}: \hat{Y} \to \hat{X}$

Proof: Q is the Poincaré on $Y \times \hat{Y}$

$(f \times 1)^* Q$ is a line bundle on $X \times \hat{Y}$.

for $\hat{y} \in \hat{Y}$, $((f \times 1_{\hat{Y}})^* Q)|_{X \times \{\hat{y}\}} \in Pic^0(X)$

and $(f \times 1_{\hat{Y}})^* Q|_{0 \times \hat{Y}}$ is trivial

By universal property, we get $\hat{f}: \hat{Y} \to \hat{X}$

(3) $f: X \to Y$ is an isogeny, then so is $\hat{f}$

and there is a canonical duality between

$\ker f$ and $\ker \hat{f}$

Proof: By ([§7], Cor 2, P_{70}), $\ker f$ and $\ker f^*$

$(f^*: Pic(Y) \to Pic(X))$

if $\hat{y} \in Pic(Y)$ s.t. $f^*(\hat{y}) = 0$, then $\hat{y}$ has finite order hence $\mathcal{Y} \in \mathrm{Pic}^0(Y)$. Therefore, $\ker f^* = \ker f$

Finally, $\dim \hat{X} = \dim X = \dim Y = \dim \hat{Y} \Rightarrow \hat{f}$ is isogeny.

$\square$

Our next goal: symmetric property between X and $\hat{X}$

Def: X, Y abelian varieties. A divisorial correspondence between X and Y is a line bundle on $X \times Y$ whose restrictions to $\{0\} \times Y$ and $X \times \{0\}$ are trivial

Pro 2. X, Y abelian varieties of the same dimension

Q divisorial correspondence between X and Y.

then (1) If $Q|_{\{x\} \times Y}$ is trivial, then $x = 0$

(2) If $Q|_{X \times \{y\}}$ is trivial, then $y = 0$

are equivalent

If these hold, then $X \cong \hat{Y}$, and Q isomorphic to the Poincaré bundle of P_Y of Y, so do X and P_X

Proof: (1) $\Rightarrow$ (2):

By definition of Q, $Q|_{X \times \{0\}}$, $Q|_{\{0\} \times Y}$ are trivial.

then we get $\phi: X \longrightarrow \hat{Y}$, $Q = (\phi \times 1_Y)^* P_Y$ from (b)

Because of (1), ϕ must be injective

On the other hand, $\dim X = \dim Y = \dim \hat{Y}$, ϕ is surjective.

then ϕ is isomorphic, $X \simeq \hat{Y}$

Let $\varphi: Y \to \hat{X}$, $(1_X \times \varphi)^* P_X = Q$.

To prove (2), we only need to prove φ is injective

If not, there exists $\{0\} \neq K \subset \ker\varphi$

φ factorizes as $Y \xrightarrow{\eta} Y/K \xrightarrow{\tilde{\varphi}} \hat{X}$

η is the natural morphism (still by §7. P63, we also need K finite)

Let $L = (1_X \times \tilde{\varphi})^* P_X$. line bundle on $X \times (Y/K)$

then we get $Q \cong (1_X \times \eta)^* L$

L induces a morphism $\alpha: X \to \widehat{(Y/K)}$

and the composition $X \xrightarrow{\alpha} \widehat{(Y/K)} \xrightarrow{\hat{\eta}} \hat{Y}$ is precisely

induced by Q, then it is an isomorphism

thus α is injective. $\dim X = \dim Y = \dim(Y/K)$.

then α is surjective, so $\alpha, \hat{\eta}$ are isomorphisms

But because η is isogeny, $\ker(\hat{\eta})$, $\ker(\eta)$ are non-trivial

Contradiction! Then φ is injective. $\square$

Remark: By the proposition, we get known about the symmetric property between X and $\hat{X}$ from its Poincaré bundles, and $X \cong \hat{\hat{X}}$, that's why it's called "Dual Variety"