A locally analytic p-adic Langlands Correspondence for $GL_n(\mathbb{Q}_p)$ in the crystalline case

Yiwen Ding

<u>Background</u> $E/\mathbb{Q}_p$ finite.

$$\left\{\begin{matrix}n\text{-dim'l WD rep}\\ \text{of } \mathbb{Q}_p \text{ on } E\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{certain sm rep}\\ \text{of } GL_n(\mathbb{Q}_p) \text{ on } E\end{matrix}\right\}$$

Fontaine $\Big\Downarrow$ + Hodge fil'n $\qquad\qquad \cap$

$$\left\{\begin{matrix}n\text{-dim'l de Rham}\\ \text{rep of } \mathrm{Gal}_{\mathbb{Q}_p}\end{matrix}\right\} \overset{?}{\longleftrightarrow} \left\{\begin{matrix}\text{certain loc an}\\ \text{rep of } GL_n(\mathbb{Q}_p)\end{matrix}\right\}.$$

<u>note</u> Known for $GL_2(\mathbb{Q}_p)$.

<u>Q</u> How to translate info of Hodge fil'n to autom side?

<u>Ex</u> $\rho: \mathrm{Gal}_{\mathbb{Q}_p} \longrightarrow GL_2(E)$ crystalline rep of wt $(0,1)$.

$$\rho \rightsquigarrow D_{cris}(\rho) \circlearrowleft \varphi, \ \circlearrowleft \mathrm{Fil}$$

α_1, α_2 eigenvals, $\alpha_1\alpha_2^{-1} \neq 1$ or $p^{\pm 1}$ (generic).

$\hookrightarrow e_1, e_2 \in D_{cris}(\rho)$.

$$\rho \rightsquigarrow r \cong \mathrm{unr}(\alpha_1) \oplus \mathrm{unr}(\alpha_2)$$

$\quad |$

WD rep

$$\rightsquigarrow \pi_{sm}(r) = \left(\mathrm{Ind}_{B^-}^{GL_2} \mathrm{unr}(\alpha_1 p) \otimes \mathrm{unr}(\alpha_2)\right)^{\infty}$$

$$\cong \left(\mathrm{Ind}_{B^-}^{GL_2} \mathrm{unr}(p\alpha_2) \otimes \mathrm{unr}(\alpha_1)\right)^{\infty}.$$

$$\mathrm{Fil}^i D_{cris}(\rho) = \begin{cases} D_{cris}(\rho), & i \le 0 \\ \mathcal{L}_\rho, & i = 1 \\ 0, & i > 1. \end{cases}$$

$\mathcal{L}_\rho$ is at most of cases $\begin{cases} \langle e_1 + e_2 \rangle & \text{— non-critical.} \\ \langle e_1 \rangle & \\ \langle e_2 \rangle & \end{cases}$ $\Big\rangle$ critical

$\rho \rightsquigarrow \pi(\rho)$ la rep.

$$PS(\alpha_1, \alpha_2) := \left(\mathrm{Ind}_{B^-}^{GL_2} \mathrm{unr}(\alpha_1 p) \oplus \mathrm{unr}(\alpha_2)\right)^{la}$$

$$= [\pi_{sm}(r) - \mathcal{C}_1].$$

$$\rightsquigarrow \quad \pi(\rho) \cong \begin{cases} PS(\alpha_1, \alpha_2) \oplus_{\pi_{sm}(r)} PS(\alpha_2, \alpha_1), & \pi_{sm}(r) \begin{smallmatrix} \nearrow \mathcal{C}_1 \\ \searrow \mathcal{C}_2 \end{smallmatrix} \quad \text{non-critical} \\ PS(\alpha_1, \alpha_2) \oplus \mathcal{C}_2, & \mathcal{L}_\rho = \langle e_1 \rangle \quad \text{critical} \end{cases}$$

similar for $\mathcal{L}_\rho = \langle e_2 \rangle$.

<u>note</u> This phenomenon is special for $GL_2(\mathbb{Q}_p)$.
(can distinguish params by crit / non-crit.)

<u>$GL_n(\mathbb{Q}_p)$ case</u> ρ crys $\rightsquigarrow D_{cris}(\rho) \circlearrowleft \varphi$, $\circlearrowleft \mathrm{Fil}_H$

$\underline{\alpha} = \alpha_1, \cdots, \alpha_n$ (generic).

$w \in S_n \rightsquigarrow w(\underline{\alpha}) \rightsquigarrow \mathrm{Fil}_w$.

<u>Q</u> Weyl flag Fil_w v.s. Fil_H Hodge flag?

$\rightsquigarrow w_{alg}(w) \in S_n$ ($w_{alg}(w) = w_0$ non-crit)

↳ generalized non-crit case.

* <u>Breuil's la soc conj</u>

(Almost known by Breuil-Hellmann-Schraen)

<u>Ex</u> ρ 3-dim'l, $\alpha_1, \alpha_2, \alpha_3$, e_1, e_2, e_3 eigen vects.

(wt 0, 1, 2)

$$\mathrm{Fil}^i D_{cris}(\rho) = \begin{cases} D_{cris}(\rho), & i \leq 0 \\ \langle e_1 + e_2,\ e_1 + a_p e_2 + e_3 \rangle, & i = 1 \\ \langle e_1 + a_p e_2 + e_3 \rangle, & i = 2 \\ 0, & i > 2. \end{cases}$$

New param $a_p \in E$: p-adic Hodge param.

↳ Galois datum.

<u>Def</u> $$\Phi\Gamma_{nc}(\underline{\alpha}) = \left\{ \begin{array}{l} \text{non-crit crys } (\varphi, \Gamma)\text{-mods of} \\ \varphi\text{-eigenvals } (\underline{\alpha}) \text{ of wt } (0,1,2) \end{array} \right\}$$

<u>Thm</u> (Local corresp.)

$D \in \Phi\Gamma_{nc}(\underline{\alpha}) \rightsquigarrow \pi(D) \circlearrowleft GL_n(\mathbb{Q}_p)$.

s.t. $D \simeq D' \iff \pi(D) \simeq \pi(D')$.

<u>Local-global compatibility</u>

G definite unitary grp of rk n, assoc to F/F^+,

- $\exists \wp$, $F^+_\wp \simeq \mathbb{Q}_p$.
- p-adic place of F^+ splits in F + assumption.

- $\mathcal{U}^p \subseteq G(\mathbb{A}_{F^+}^p)$

$\rightsquigarrow \hat{S}(\mathcal{U}^p, E) = \{ f : G(F^+) \backslash G(\mathbb{A}_{F^+}^\infty) / \mathcal{U}^p \to E \text{ cont} \}.$

$\circlearrowleft$

$\mathbb{T}^p \times \mathrm{Gal}(\mathbb{Q}_p)$

Assume ρ is automorphic, i.e.

$\exists\, \tilde{\rho} : \mathrm{Gal}_F \to GL_n(E)$ s.t. $\tilde{\rho}|_{\mathrm{Gal}_{F_{\tilde{p}}}} \simeq \rho$.

& $\tilde{\rho}$ is assoc to an autom rep of G

<u>note</u> $\tilde{\rho} \rightsquigarrow \mathfrak{m}_{\tilde{\rho}} \subseteq \mathbb{T}^p \rightsquigarrow \hat{S}(\mathcal{U}^p, E)[\mathfrak{m}_{\tilde{\rho}}] \circlearrowleft GL_n(\mathbb{Q}_p)$

This is a global setup.

<u>Thm</u> (Local-global compatibility)

$\pi(D) \hookrightarrow \hat{S}(\mathcal{U}^p, E)^{an}[\mathfrak{m}_{\tilde{\rho}}] \iff D \simeq D_{cris}(\rho)$.

<u>What does $\pi(D)$ look like?</u>

$\pi_{sm}(r) \cong \left(\mathrm{Ind}_{B^-}^{GL_n} \mathrm{unr}(w(\underline{\alpha}))\, \eta\right)^\infty, \quad \eta = 1 \cdot 1^{1-n} \otimes 1 \otimes \cdots \otimes 1.$

$\downarrow$

$PS(w) := \left(\mathrm{Ind}_{B^-}^{GL_n} \mathrm{unr}(w(\underline{\alpha}))\, \eta\right)^{an}$

$\bigoplus_{\pi_{sm}(r)} PS(w) \twoheadrightarrow \pi(\underline{\alpha})$

quotient of socle of $\pi_{sm}(\underline{\alpha})$

only depends on WD rep r.

$\pi(D)$ is an ext'n of $\pi_{sm}(\underline{\alpha})^{\oplus\left(2^n - \frac{n(n+1)}{2} - 1\right)}$ by $\pi(\underline{\alpha})$.

exotic phenomenon :

This multiplicity is

$$\begin{array}{ll} \binom{n}{3} + \binom{n}{4} + \cdots + \binom{n}{n} & 0, \quad n=2 \\ \| & 1, \quad n=3 \\ 2^n - \frac{n(n+1)}{2} - 1. & 5, \quad n=4 \end{array}$$

<u>Gal side</u> $D \in \Phi\Gamma_{nc}(\underline{\alpha})$, w

$\rightsquigarrow \kappa_w : \mathrm{Ext}^1_w(D,D) \to \mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\delta_w, \delta_w) \xrightarrow{\sim} \mathrm{Hom}(T(\mathbb{Q}_p), E).$

$\uparrow$ trianguline param

<u>Fact</u> $\ker \kappa_w$ indep of w.

$\| \; \mathrm{Ext}^1_0(D,D)$

$\rightsquigarrow \kappa_w : \overline{\mathrm{Ext}}^1_w(D,D) \xrightarrow{\sim} \mathrm{Hom}(T(\mathbb{Q}_p), E)$

modulo kernel.

<u>Autom side</u> $\mathrm{Hom}(T(\mathbb{Q}_p), E) \xrightarrow{\sim} \mathrm{Ext}^1_{T(\mathbb{Q}_p)}(\delta'_w, \delta'_w)$

$\longrightarrow \mathrm{Ext}^1_{GL_n(\mathbb{Q}_p)}(\pi_{sm}(\underline{\alpha}), PS(w(\underline{\alpha})))$

$\hookrightarrow \mathrm{Ext}^1_{GL_n(\mathbb{Q}_p)}(\pi_{sm}(\underline{\alpha}), \pi(\underline{\alpha}))$

where $\delta'_w := \mathrm{unr}(w(\underline{\alpha}))\eta$.

<u>Thm</u>

$$\begin{tikzcd} \bigoplus_w \overline{\mathrm{Ext}}^1_w(D,D) \arrow[rr, "\sim", "\text{known}"'] \arrow[d, "\text{clear}"', "\kappa"] & & \bigoplus_w \mathrm{Ext}^1(\pi_{sm}(\underline{\alpha}), PS(w(\underline{\alpha}))) \arrow[d, "\text{clear}"] \\ \overline{\mathrm{Ext}}^1(D,D) & & \mathrm{Ext}^1(\pi_{sm}(\underline{\alpha}), \pi(\underline{\alpha})) \arrow[ll, dashed, "\exists!\, t_D"'] \end{tikzcd}$$

& $\dim \ker t_D = 2^n - \frac{n(n+1)}{2} - 1$.

$\ker t_D \rightsquigarrow \pi(D)$.

N.B. Thm $\rightsquigarrow \pi(D) \hookrightarrow \hat{S}(\mathcal{U}^p, E)[m_{\tilde{p}}]$

by global triangularization & adjunction.

Thm $\ker t_D$ determines D ($\rightsquigarrow \ker \kappa$ determines D).

This carries info of Hodge fil'n.