Mirabolic special cycles and twisted AFL

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§1 Non-reductive geometry

$G/\mathbb{Q}$ red grp. (G, X) Shimura datum
$\qquad\qquad\qquad\quad \overset{\shortparallel}{G(\mathbb{R})/K_\infty}$.

$\rightsquigarrow$ Shimura varieties $\{Sh_K(G,X)\}_{K \le G(\mathbb{A}_f)}$ / E = reflex field.

$\mathbb{C}$-uniformization: $Sh_K(G,X)(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K$.

E.g. modular curve $Y_0(1) = SL_2(\mathbb{Z}) \backslash \mathbb{H} =$  $\mathbb{H}$.

Keynote Geometry of Sh $\rightsquigarrow$ Construct Langlands corr.

$\rightsquigarrow$ construct p-adic L-functions

$\rightsquigarrow$ study arithmetic & standard conjs of relative motives.

Question $\exists$ Sh of linear alg grp G?

Motivation Rep theory of red grps uses lots of non-red grps.

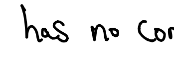
Evidences: (1) Fourier expansion $N_2 = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \subset SL_2$

(2) Parabolic induction $P_2 = \begin{pmatrix} * & * \\ & * \end{pmatrix} \subset SL_2$

(3) Mirabolic subgrp $M_n = \left(\begin{array}{c|c} 1 & * \\ \hline 0 & * \end{array}\right) \subset GL_n$

Answer No / Unknown in most cases $/\mathbb{R}$.

E.g. (1) $N_2(\mathbb{Q}) \backslash N_2(\mathbb{A}) \longrightarrow SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A})$

$\qquad\qquad \overset{\shortparallel}{S^1} \longrightarrow$  Riemann surface.

In general, $G(\mathbb{R})/K_\infty$ has no complex str.

<u>Today</u> Over $\mathbb{Q}_p$, (1)(2)(3) make sense with arith applications.

<u>Recall</u> Global $Sh/\mathbb{Q} \leftrightsquigarrow$ local $Sh/\mathbb{Q}_p$, $K_p \leq G(\mathbb{Q}_p)$

$\leftrightsquigarrow$ local R-Z space $/\mathbb{Z}_p$, $K_p \leq G(\mathbb{Q}_p)$ parahoric.

Define local Shimura datum $(G, b, \{\mu\})$

- $G/\mathbb{Q}_p$ red grp, $b \in G(\breve{\mathbb{Q}}_p)$ σ-conj class,
- $\{\mu\}$ conj class of minuscule char $\mu : \mathbb{G}_m \to G_{\breve{\mathbb{Q}}_p}$.

Assume $p>2$, b basic.

$(G, b, \{\mu\})$ $\begin{cases} \text{Hodge type. } (G,\mu) \hookrightarrow (GL_n, \mu_d) \text{ w/ } \mu_d = (1^d, 0^{n-d}). \\ \text{unramified, } \mathcal{G}/\mathbb{Z}_p \text{ red, } b \text{ comes from a } p\text{-div grp } \mathbb{X} \text{ w/ } G\text{-str} \end{cases}$

<u>Thm</u> (Howard-Pappas, William)

$\exists$ R-Z space $\mathcal{N}(G,b,\{\mu\}) \longrightarrow \mathrm{Spf}\, \breve{\mathbb{Z}}_p$ formal sch

formally sm w/ $J_b(\mathbb{Q}_p)$-action,

where J_b = inner form of G.

Fix $H \leq G$ parabolic w/ Levi M.

Assume $b \in M(\breve{\mathbb{Q}}_p)$, $\mu : \mathbb{G}_m \to M_{\breve{\mathbb{Q}}_p}$.

<u>Thm 1</u> (Zhang, in progress)

$\exists$ formal sch

$$\begin{array}{ccc} \mathcal{N}(H,b,\mu) & \longrightarrow & \mathcal{N}(G,b,\mu) \ / \ \mathrm{Spf}\, \breve{\mathbb{Z}}_p \\ \downarrow & & \\ \mathcal{N}(M,b,\mu) & & \text{formally sm } / \breve{\mathbb{Z}}_p. \end{array}$$

<u>Idea</u> In PEL case, W. Zhang: moduli of filtered p-div grps.

§2 Mirabolic special cycles

$p>2$. Denote V_n split (i.e. has a self-dual lattice) $\mathbb{Q}_{p^2}/\mathbb{Q}_p$-herm space of dim n,

$\mathbb{V}_n$ non-split.

$\rightsquigarrow$ Unitary R-Z space $\mathcal{N}_n \longrightarrow \mathrm{Spf}\,\breve{\mathbb{Z}}_p$ formally sm of dim $n-1$.

via the moduli description

$$S \longmapsto \{(X, \iota, \lambda, \rho)\}.$$

where
- X p-div grp $/S$ of dim $= n$ & ht $= 2n$.
- ι given by $\mathbb{Z}_{p^2} \circlearrowright X$ of sgn $(1, n-1)$.
- $\lambda: X \xrightarrow{\sim} X^\vee$ polarization
- $\rho: \mathbb{X} \times \bar{S} \longrightarrow X \times \bar{S}$, $\bar{S} = S/p$ reduced framing q-isog of ht 0.

$\mathcal{E}/\mathrm{Spf}\,\breve{\mathbb{Z}}_p$ canonical lifting of $\mathbb{Z}_{p^2}$-Lubin-Tate mod $/\bar{\mathbb{F}}_p$ of ht $=2$, dim $=1$.

$\rightsquigarrow$ $(\mathcal{E}, \lambda_{\mathcal{E}}, \iota_{\mathcal{E}})$, $\lambda_{\mathcal{E}}$ has sgn $(0,1)$ (fix such a choice).

Prop $\mathbb{V}_n = \mathrm{Hom}_{\mathbb{Z}_{p^2}}(\mathcal{E}_{\bar{\mathbb{F}}_p}, \mathbb{X}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

$U(\mathbb{V}_n) = \mathrm{Aut}^\circ(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}}) \circlearrowright \mathcal{N}_n$ by changing $\rho \mapsto \rho \circ g^{-1}$.

Def Kudla-Rapoport cycle $u \in \mathbb{V}_n - 0$

$\mathcal{Z}(u) \longrightarrow \mathcal{N}_n$ closed formal subsch as lifting locus of u.

· $\mathcal{Z}(u) \to \mathcal{N}_n$ (local) relative Cartier divisor

§2 Non-reductive geometry of Weil rep

Application Kudla program $\begin{cases} \text{arithmetic theta lifting (global)} \\ \text{Kudla-Rapoport Conj (local)} \end{cases}$

on $\mathcal{Z}(u_1) \cap \mathcal{Z}(u_2) \cap \cdots \cap \mathcal{Z}(u_n)$.

Mirabolic cycles $\quad \mathcal{N}_n^{GL} \longrightarrow \mathrm{Spf}\, \breve{\mathbb{Z}}_p$ formal sch.

parametrized by $S \longmapsto \{(X, \iota, \rho)\}$,

formally sm of dim $2n-2$.

Map: $\mathcal{N}_n \hookrightarrow \mathcal{N}_n^{GL}$ (forget λ)

Involution $\quad \sigma : \mathcal{N}_n^{GL} \longrightarrow \mathcal{N}_n^{GL} \qquad \sigma = \sigma_{\lambda_*}$.

$$(X, \iota, \rho) \longmapsto (X^\vee, \iota^\vee, (\rho^\vee)^{-1} \circ \lambda_*).$$

Prop $(\mathcal{N}_n^{GL})^{\sigma = id} = \mathcal{N}_n$.

Def'n $u \in \mathbb{V}_n - \{0\}$, $u^* \in \mathbb{V}_n^* - \{0\}$.

$\mathrm{Hom}_{\mathbb{Z}_{p^2}}(\mathbb{X}, \mathcal{E}_{\bar{\mathbb{F}}_p}) \otimes \mathbb{Q}_p$.

$\mathcal{Z}^{GL}(u) \longrightarrow \mathcal{N}_n^{GL}$ lifting locus of u

$\mathcal{Z}^{GL}(u^*) \longrightarrow \mathcal{N}_n^{GL}$ lifting locus of u^*.

Thm (Zhang) $\mathcal{Z}^{GL}(u)$, $\mathcal{Z}^{GL}(u^*)$ are relative Cartier divisors

and (1) $\exists$ Cartesian diagram

$$\begin{tikzcd} \mathcal{Z}(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{N}_n \arrow[d] \\ \mathcal{Z}^{GL}(u) \arrow[r, hook] & \mathcal{N}_n^{GL} \circlearrowleft \sigma \end{tikzcd}$$

(2) if $u^* = \lambda_{\mathbb{E}}^{\vee} \circ u^{\vee} \circ \lambda_{\mathbb{X}}$

then $\sigma(\mathcal{Z}^{GL}(u^*)) = \mathcal{Z}^{GL}(u)$ via duality.

<u>Application</u> Kudla-Rapoport (w/ levels) of $\mathcal{N}_{\mathcal{U}_n^{ram}} \hookrightarrow \mathcal{N}_n^{GL}$.

<u>§3 Twisted AFL</u>

$$\mathcal{U}(\mathbb{V}_n) \circlearrowright \mathcal{N}_n \longrightarrow \mathcal{N}_n^{GL} \circlearrowleft GL(\mathbb{V}_n)$$

dim $\quad n \quad\quad 2n-1$

$\mathcal{Z}^{GL}(u)$ dim $2n-2$

$g \in GL(\mathbb{V}_n)/\mathcal{U}(\mathbb{V}_n)$, $u \in \mathbb{V}_n - \{0\}$.

<u>Def'n</u> $\mathrm{Int}(g,u) = \mathcal{N}_n \overset{\mathbb{L}}{\cap} g\mathcal{N}_n \overset{\mathbb{L}}{\cap} \mathcal{Z}^{GL}(u)$.

$\rightsquigarrow$ $\mathrm{Int}(g,u)$ only depends on

$[g,u] \in \mathcal{U}(\mathbb{V}_n) \backslash [GL(\mathbb{V}_n)/\mathcal{U}(\mathbb{V}_n) \times \mathbb{V}_n]$.

<u>Prop</u> If (g,u) is regular semisimple,

then $\mathrm{Int}(g,u) \in \mathbb{Z}$ well-def'd.

<u>Prop</u> (Zhang) $\exists$ matching of orbits

$[GL_n(\mathbb{Q}_p) \backslash GL_n(\mathbb{Q}_p) \times \mathbb{Q}_p^n \times (\mathbb{Q}_p^n)^*]_{rs}$

$\longleftrightarrow \mathcal{U}(V_n) \backslash [GL(V_n)/\mathcal{U}(V_n) \times V_n]_{rs}$.

Set $F_{std}^{1} = 1_{GL_n(\mathbb{Z}_p)} \times 1_{\mathbb{Z}_p^n} \times 1_{(\mathbb{Z}_p^n)^*}$,

$x = (\gamma, u_1, u_2) \in GL_n(\mathbb{Q}_p) \times \mathbb{Q}_p^n \times (\mathbb{Q}_p^n)^*$

Define $\quad \mathrm{Orb}(F_{std}^{1}, x, s) = \int_{h \in GL_n(\mathbb{Q}_p)} F_{std}(h \cdot (\gamma, u_1, u_2)) \cdot |\det h|^s \cdot \eta(h)\, dh$, $\forall s \in \mathbb{C}$

where $\eta : \mathbb{Q}_p^\times \to \{\pm 1\}$ the char given by LCFT for $\mathbb{Q}_{p^2}/\mathbb{Q}_p$

Also define the derived orbital integral

$$\partial\, \mathrm{Orb}(x) := \underset{\{\pm 1\}}{\omega(x)} \frac{d^2}{ds}\Big|_{s=0} \mathrm{Orb}(F_{std}^{1}, x, s).$$

__Thm__ (Daniel Wang)

$\forall$ x regular semi-simple,

$$\mathrm{Orb}(x) = \begin{cases} 0, & \text{if } x \text{ matches } y \in \big(U(\mathbb{V}_n)\backslash[GL(\mathbb{V}_n)/U(\mathbb{V}_n) \times \mathbb{V}_n]\big)_{rs} \\ \mathrm{Orb}(y, 1_{U_n(\mathbb{Z}_p)} \times 1_{(\mathbb{Z}_{p^2})^n}), & \text{if } x \text{ matches } y \in \big(U(V_n)\backslash[GL(V_n)/U(V_n) \times V_n]\big)_{rs} \end{cases}$$

__Application__ Twisted GGP conj.

__Thm__ (Zhang, twisted AFL)

$p > 2$. If $x = (\gamma, u_1, u_2)$ matches $y \in (g, u) \in [GL(\mathbb{V}) \times \mathbb{V}]_{rs}$

then $\quad \partial\mathrm{Orb}(x) = -\mathrm{Int}(g, u) \cdot \log p.$

__Key__ Inductive structure : If $(u, u^*) = 1$.

then $\mathcal{Z}(u) \overset{\shortparallel}{\cap} \mathcal{Z}(u^*) \simeq \mathcal{N}_n^{GL}$.