

## Projective Morphisms (II)

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Eisenbud-Harris is particularly recommended in this section.

### S1 Relative Proj

X sch.  $S = \bigoplus_{n=0}^{\infty} S_n$  graded qcoh  $\mathcal{O}_X$ -alg.

$\forall U \in X, \text{Proj}(S|_U) \longrightarrow U$

$\hookrightarrow$  glue to give  $\underbrace{\text{Proj } S}_{\text{relative proj of } S} \longrightarrow X$ .

Assume  $S_1$  f.g. (local notion)

&  $S = S_0[S_1]$  (loc) gen'd as  $S_0$ -alg.

Also assume:  $S_0 = \mathcal{O}_X/?$  quotient.

(Hartshorne: assume in fact  $S_0 = \mathcal{O}_X$ ).

Pick open affine  $U \subset \mathcal{O}_U^{\oplus(m+1)} \longrightarrow S_1|_U$

$\hookrightarrow \text{Sym}(\mathcal{O}_U^{\oplus(m+1)}) \longrightarrow S|_U$ .

$\bigoplus_{n=0}^{\infty} \text{Sym}^n(\mathcal{O}_U^{\oplus(m+1)})$ .

$\hookrightarrow \text{Proj } S \longrightarrow \underbrace{\text{Proj } \text{Sym}(\mathcal{O}_U^{\oplus(m+1)})}_{= \mathbb{P}_U^m} \Rightarrow \text{Proj } S \longrightarrow X \text{ loc proj.}$

Def'n  $f: Y \rightarrow X$  proj iff it occurs as a relative Proj.

i.e.  $Y = \text{Proj } S$  for some  $S$ .

note: globally proj  $\Rightarrow$  this proj  $\Rightarrow$  loc proj  
but no coincidence.

Following EGA:  $\mathbb{P}(\mathcal{F}) = \text{Proj } \text{Sym } \mathcal{F}, \mathcal{F} \in \mathbf{Qcoh}(\mathbf{Mod}_{\mathcal{O}_X})$  f.g.

## §2 Very Ample Sheaves

Def'n An immersion  $f: Y \rightarrow X$  is

on Top:  $Y \approx V \overset{\text{closed}}{\subseteq} U \overset{\text{open}}{\subseteq} X$  ( $V$  loc closed)

s.t.  $\forall y \in Y$ ,  $y \mapsto x$ ,  $f^*: \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,y}$  surj.

Note (1) Any composition of closed imm & open imm is an imm  
 (2) Any imm factors as (1): open  $\circ$  closed.

Def'n  $f: Y \rightarrow X$ ,  $\mathcal{F} \in \mathbf{Qcoh}(Y)$  very ample rel to  $f$

if  $\exists Y \xrightarrow{g} \mathbb{P}(\mathcal{S}_1)$  imm for  $\mathcal{S}_1 \in \mathbf{Qcoh}(\mathrm{Mod} \mathcal{O}_X)$  f.g.

under which  $\mathcal{F} \cong g^*(\mathcal{O}_1)$ .

Philosophy saying " $\mathcal{F}$  very ample rel  $f: Y \rightarrow X$ "

= saying all rel information of  $Y$  is recorded in  $\mathcal{F}$   
 namely, one can recover  $Y$  from  $X$  via  $\mathcal{F}$ .

Rmk Unlike projectivity, this notion is indeed base on the base.

Lemma  $f: Y \rightarrow X$  proj.  $\Leftrightarrow f$  proper &  $\exists \mathcal{F}$  very ample rel to  $f$ .

Proof. See Hartshorne Rmk II.5.16.1.  $\mathbf{Qcoh}(Y)$ .

Note  $\mathcal{F} = j^* \mathcal{O}_U$  retrieve  $Y \rightarrow \mathbb{P}(\mathcal{S}_1)$ .

namely,  $\mathcal{S}_1$  glob f.g.,  $\mathcal{S}_1 = (\bigoplus \widetilde{A} s_i)$

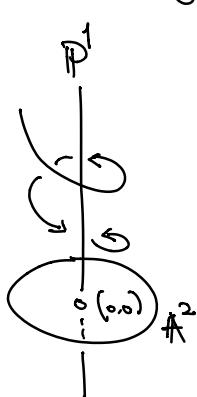
$\downarrow$        $\left. \begin{array}{l} \text{pull back} \\ \text{sections of } j^* \mathcal{O}_U. \end{array} \right\}$

morphism to  $\mathbb{P}_X^n$ . (see Hartshorne II.7.1).

### §3 Blowups

A neat class of rel Proj:

If.g. ideal  $\mathcal{I} \in \text{Sh}(X)$ ,  $Y = \mathbb{P}(\mathcal{I})$  = blowup of  $X$  along  $\mathcal{I}$ .



Example  $X = \text{Spec } k[x,y]$ ,  $k$  any field.

$\mathcal{I} = (x,y)$  ideal sheaf.  $U = D(x) \cup D(y)$

$\hookrightarrow \mathcal{O}_U \cong \mathcal{O}_X|_U = \mathcal{O}_U$ .  $Y = \mathbb{P}(\mathcal{I})$ .

$\hookrightarrow Y \times_X U \xrightarrow{\cong} U$

But the fibre of  $y \rightarrow x$  over  $(0,0)$  looks like  $\mathbb{P}^1$   
(with homo coord  $x,y$ ).

Note The blowup  $\mathbb{P}(\mathcal{I})$  carries less information than  $\mathcal{I}$ .

e.g.  $\mathcal{I}$  loc principal,  $\mathbb{P}(\mathcal{I}) = X$ .

Special Property For  $f: Y \rightarrow X$  &  $\mathcal{I} \in \text{Sh}_{\text{ideal}}(X)$

may compose  $\mathcal{I} \rightarrow \mathcal{O}_X$  with  $f^*: \mathcal{O}_X \rightarrow f_* \mathcal{O}_Y$

$\hookrightarrow f^* \mathcal{I} \rightarrow \mathcal{O}_Y$  by performing adjunction.

$\text{im} = \text{ideal sheaf on } Y$

"inverse image ideal sheaf of  $\mathcal{I}$  under  $f$ ".

Then  $f: Y \rightarrow X$  blowup def'd by f.g.  $\mathcal{I}$  on  $X$ .

$\Rightarrow f^{-1}\mathcal{I} = \text{im}(f^*\mathcal{I} \rightarrow \mathcal{O}_Y)$  loc principal

( $\Rightarrow$  has trivial blowups)

Proof.  $Y = \text{Proj } S$ ,  $S_n = \text{Sym}^n \mathcal{I}$

$\hookrightarrow f^{-1}\mathcal{I} = \mathcal{O}_{Y(1)}$  loc free.  $\square$

Fact (Univ property of blowups)

$f: Y \rightarrow X$  blowup.  $\forall Z \rightarrow X$ , s.t.  $g^* I$  loc principal on  $Z$ :

$$\begin{array}{ccc} g^* I & Z & \xrightarrow{\exists!} Y & f^* I \\ & \searrow g & \downarrow f & \\ & & X & I \end{array}$$

(Hartshorne Prop II.7.17).

More concrete description of the standard example:

blowup of  $\text{Spec } k[x,y]$  at  $(x,y)$ :

$Y = \mathbb{P}(I)$  covered by  $\text{Spec } k[x, \frac{y}{x}]$  &  $\text{Spec } k[y, \frac{x}{y}]$   
glued along  $\text{Spec } k[x, y, \frac{y}{x}, \frac{x}{y}]$ .

In fact: any blowup can be described analogously:

$\mathbb{P}(I) \rightarrow \text{Spec } A$ ,  $I \mapsto I = (r_0, \dots, r_m)$

$\stackrel{f}{\text{covered by }} m+1$  charts, a typical one of which is  
 $\text{Spec } A[t_1, \dots, t_m]/(t_0 r_0 - r_1, \dots, t_m r_0 - r_m)$ .

Upshot  $f^* I$  is supposed to be loc principal.

$\Rightarrow$  must force  $r_i/r_j$

different choices  $\Rightarrow$  different charts.

Blowup = special e.g. of modification

$f: Y \rightarrow X$  proper, surj, birational

(i.e.  $f \circ \text{isom}$ ,  $\forall$  open dense  $U \subseteq Y$ ).

(Hartshorne Thm II.7.17: under certain circumstance,

every modif can be a blowup.)

Catch  $\mathcal{I}$  is not unique, e.g.  $P(x,y) \rightarrow \text{Spec} k[x,y]$  {  
 $P(x^2, xy, y^2) \rightarrow \text{Spec} k[x,y]$   
defines the same blowup.

### §4 Chow's Lemma

Modification: good at proper  $\xrightarrow{\text{turn into}}$  proper sch / some base.

Thm (Chow's Lemma)

$f: X \rightarrow S$  finite type.

Assume: Either  $S$  noe

or  $S$  q-cpt &  $X$  fin. many irred components.

$\Rightarrow \exists$  quasi-proj  $X'/S$  &  $f: X' \rightarrow X$  proj surj  
s.t.  $f|_U$  isom &  $f^{-1}(U) \cong U$  dense in  $X'$ .

Moreover,  $X$  red/irred/int  $\Rightarrow$  so is  $X'$ .

Reference: EGA 2, Lemma 5.6.1 (weaker result).

Hartshorne Ex II.4.10.