CS559 Machine Learning
Probabilistic Graphical Model, Bayesian Network

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Week 11

Outline

- Introduction to Graphical Model
- Bayesian Network

Introduction to Graphical Model

Probabilistic Graphical Models

- A simple way to visualize the structure of a probabilistic model

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- Insight into the properities of the model
- Complex computations can be expressed in terms of graphical manipulations
- Two types:
  - **Directed** graphical models: Bayesian Networks, latent Variable Model, etc
  - **Undirected** graphical models: Markov network (Markov Random Field), Energy-based Model etc

Graphical Models

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- These factorizations are typically equivalent to *conditional independence* statement amongst (sets of) variables in the distributions.

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- The product rule of probability over three variables a, b, c: (different ordering has different factorization)

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- A directed graphical model representing the joint probability distribution over a, b, c:

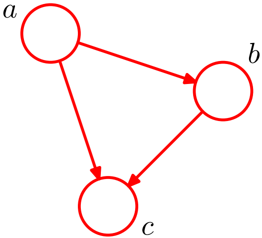

Figure: [C. Bishop, PRML]

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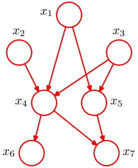

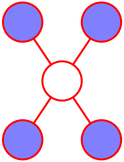

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- Graphs can be represented using: the edge list, the adjacency matrix.

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$$P(X_i|\mathcal{M})$$

- Learning
  - What model is "right" for my data?

$$\mathcal{M} = \text{arg max}_{\mathcal{M} \in M} F(D; \mathcal{M})$$

Basic Probability Concepts

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  - How many state configurations?
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- Inference: If not all variables are observable, how to compute the conditional distribution of latent variables given evidence?
- Learning: where do we get all this probabilities?
  - Maximal-likelihood estimation? but how many data do we need?
  - Are there other est. principles?
  - Where do we put domain knowledge in terms of plausible relationships between variables, and plausible values of the probabilities?

Two Important Rules

1. Chain rule or Product rule: Let $S_1, S_2, ..., S_n$ be events, and $p(S_i) > 0$, then:

$$p(S_1, S_2, ..., S_n) = p(S_1)p(S_2|S_1) \cdots p(S_n|S_{n-1}, ..., S_1)$$

   Let X, Y be two variables, then

$$p(X, Y) = p(X|Y)p(Y)$$

2. Sum rule: Let X, Y be two variables, then:

$$p(X) = \sum_{Y} p(X, Y)$$

Marginalization

- Suppose X and Y are random variables with distribution $p(X, Y)$
  X: Intelligence, Val(X) = {"Very High", "High"}
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- $p(Y = a) =? \ 0.85$
- More generally, suppose we have a joint distribution $p(X_1, ..., X_n)$. Then,

$$p(X_i = x_i) = \sum_{x_1} \sum_{x_2} \cdots \sum_{x_{i-1}} \sum_{x_{i+1}} \cdots \sum_{x_n} p(x_1, ..., x_n)$$

Conditioning

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Conditioning

- Suppose X and Y are random variables with distribution $p(X, Y)$
  X: Intelligence, Val(X) = {"Very High", "High"}
  Y : Grade, Val(Y) = {"a", "b"}
- Can compute the conditional probability:

$$
\begin{aligned}
p(Y = a|X = vh) &= \frac{p(Y = a, X = vh)}{p(X = vh)} \\
&= \frac{p(Y = a, X = vh)}{p(Y = a, X = vh) + p(Y = b, X = vh)} \\
&= \frac{0.7}{0.7 + 0.1} = 0.875
\end{aligned}
$$

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- One famous model, Quick Medical Reference (QMR-DT), has 600 diseases and 4000 findings.

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- Estimation of joint distribution would require a huge amount of data
- Inference of conditional probabilities, e.g. $p(\text{common cold} = 1 | \text{cough} = 1, \text{fever} = 1, \text{vomiting} = 0)$ would require summing over exponentially many variables values

Structure through independence

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- However, this is not a very useful model–observing a variable X_i cannot influence our predictions of X_j
- If $X_1, ..., X_n$ are conditionally independent given Y, denoted as $X_i \perp X_{-i} | Y$ then:

$$\begin{aligned} p(y, x_1, ..., x_n) &= p(y)p(x_1|y)\prod_{i=2}^{n} p(x_i|x_1, ..., x_{i-1}, y) \\ &= p(y)p(x_1|y)\prod_{i=2}^{n} p(x_i|y) \end{aligned}$$

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- Simple, yet powerful

Example: naive Bayes

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$$p(Y = 1|x_1, ..., x_n) = \frac{p(Y = 1) \prod_{i=1}^{n} p(x_i|Y = 1)}{\sum_{y=\{0,1\}} p(Y = y) \prod_{i=1}^{n} p(x_i|Y = y)}$$

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- Philosophy: Nearly all probabilistic models are wrong, but many are nonetheless useful

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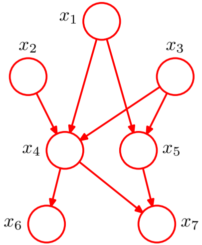

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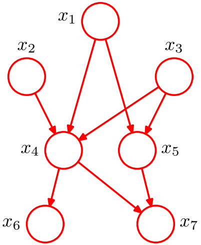

- $p(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = p(x_1)p(x_2)p(x_3)p(x_4|x_1, x_2, x_3)p(x_5|x_1, x_3)p(x_6|x_4)p(x_7|x_4, x_5)$

Bayesian Networks

- A graph with K nodes, the joint distribution is given by:

$$p(\mathbf{x}) = \prod_{k=1}^{K} p(x_k | pa_k)$$

  where pa_k denotes the set of parents of x_k and $\mathbf{x} = \{x_1, ..., x_K\}$

- This key equation expresses the *factorization* properties of the joint distribution for a directed graphical model.

Alarm Example

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- Without loss of generality, we can write:

$$\begin{aligned} p(A,R,E,B) &= p(A|R,E,B)P(R,E,B) \\ &= p(A|R,E,B)p(R|E,B)p(E,B) \\ &= p(A|R,E,B)p(R|E,B)p(E|B)p(B) \end{aligned}$$

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- Burglaries don't directly "cause" earthquakes $p(E|B) = P(E)$
- Therefore, we have:

$$p(A, R, E, B) = p(A|E, B)p(R|E)p(E)p(B)$$

Alarm Example - probability table

- DAG for $p(A, R, E, B) = p(A|E, B)p(R|E)p(E)p(B)$

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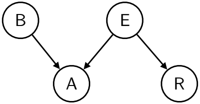

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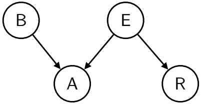

- Probability table for $p(A|B, E)$:

Alarm $= 1$	Burglar	Earthquake
0.9999	1	1
0.99	1	0
0.99	0	1
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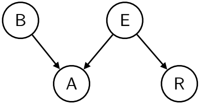

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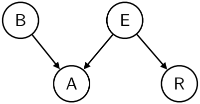

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- Probability table for $p(R|E)$:

Radio = 1	Earthquake=1
1	1
0	0

- $p(B = 1) = 0.01$ and $p(E = 1) = 0.000001$

Alarm Example - probability table

- Initial evidence: the alarm is sounding

$$
\begin{aligned}
p(B=1|A=1) &= \frac{\sum_{E,R} p(B=1,E,A=1,R)}{\sum_{B,E,R} P(B,E,A=1,R)} \\
&= \frac{\sum_{E,R} p(A=1|B=1,E)p(B=1)p(R|E)p(E)}{\sum_{B,E,R} p(A=1|B,E)p(B)p(E)p(R|E)} \\
&\approx 0.99
\end{aligned}
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- A similar calculation gives: $p(B = 1|A = 1, R = 1) \approx 0.01$
- Initially, because the alarm sounds, Sally thinks that she's been burgled. However, this probability drops dramatically when she hears that there has been an earthquake.
- The earthquake "explains away" to an extent the fact that the alarm is ringing

Wet grass example

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- Is it due to overnight Rain (R) or did she forget to turn off the sprinkler (S) last night?
- Next she notices that the grass of her neighbor, Jack, is also wet (J). This explains away to some extent the possibility that her sprinkler was left on, and she concludes therefore that it has probably been raining.

Wet grass example (cont)

- $R \in \{0, 1\}, R = 1$ means that it has been raining, and 0 otherwise

Wet grass example (cont)

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- $S \in \{0, 1\}, S = 1$ means that Tracey has forgotten to turn off the sprinkler, and 0 otherwise

Wet grass example (cont)

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- $J \in \{0, 1\}, J = 1$ means that Jack's grass is wet, and 0 otherwise
- Joint probability

$$\begin{aligned} p(T, J, R, S) &= p(T|J, R, S)p(J, R, S) \\ &= p(T|J, R, S)p(J|R, S)p(R, S) \\ &= p(T|J, R, S)p(J|R, S)p(R|S)p(S) \end{aligned}$$

Wet grass example (cont)

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Wet grass example (cont)

- Original equation:

$$\begin{aligned} p(T, J, R, S) &= p(T|J, R, S)p(J, R, S) \\ &= p(T|J, R, S)p(J|R, S)p(R, S) \\ &= p(T|J, R, S)p(J|R, S)p(R|S)p(S) \end{aligned}$$

Wet grass example (cont)

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- Now it becomes:

$$p(T, J, R, S) = p(T|R, S)p(J|R)p(R)p(S)$$

Wet grass example - probability table

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Wet grass example - probability table

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- $p(J = 1|R = 1) = 1$, $p(J = 1|R = 0) = 0.2$

Wet grass example - probability table

- $p(R=1)=0.2$ and $p(S=1)=0.1$
- $p(J=1|R=1)=1$, $p(J=1|R=0)=0.2$
- Probability table for $p(T|R,S)$:

T = 1	R	S
1	1	0
1	1	1
0.9	0	1
0	0	0

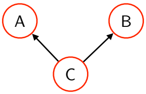
Wet grass example - inference

$$
\begin{aligned}
p(S=1|T=1) &= \frac{p(S=1,T=1)}{p(T=1)} = \frac{\sum_{J,R} p(T=1,J,R,S=1)}{\sum_{J,R,S} p(T=1,J,R,S)} \\
&= \frac{\sum_{J,R} p(T=1|R,S=1)p(J|R)p(R)p(S=1)}{\sum_{J,R,S} p(T=1|R,S)p(J|R)p(R)p(S)} \\
&= \frac{\sum_{R} p(T=1|R,S=1)p(R)p(S=1)}{\sum_{R,S} p(T=1|R,S)p(R)p(S)} \\
&= \frac{0.9\cdot 0.8\cdot 0.1+0.2\cdot 0.1}{0.9\cdot 0.8\cdot 0.1+1\cdot 0.2\cdot 0.1+0\cdot 0.8\cdot 0.9+1\cdot 0.2\cdot 0.9} \approx 0.3382
\end{aligned}
$$

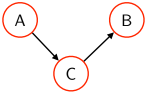
Wet grass example - inference

$$
\begin{aligned}
p(S=1|T=1,J=1) &= \frac{p(S=1,T=1,J=1)}{p(T=1,J=1)} \\
&= \frac{\sum_R p(T=1,J=1,R,S=1)}{\sum_{R,S} p(T=1,J=1,R,S)} \\
&= \frac{\sum_R p(J=1|R)p(T=1|R,S=1)p(R)p(S=1)}{\sum_{R,S} p(J=1|R)p(T=1|R,S)p(R)p(S)} \\
&= \frac{0.0344}{0.2144} \approx 0.1604
\end{aligned}
$$

Independence in Bayesian Network - Marginal independence



A,B are marginally dependent: $A \not\perp B$
$p(A,B) = \sum_C p(A|C)P(B|C)P(C) \neq p(A)p(B)$



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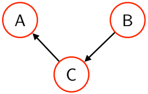

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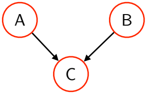
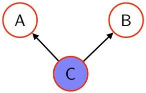


A,B are marginally independent: $A \perp B$
$p(A,B,C) = p(A)p(B)p(C|A,B) \rightarrow p(A,B) = p(A)p(B)$

Independence in Bayesian Network - Conditional independence

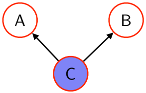


A,B are conditionally independent given C

$p(A, B|C) = \frac{p(A,B,C)}{p(C)} = \frac{p(A|C)p(B|C)p(C)}{p(C)} = p(A|C)p(B|C)$

Node C is said to be *tail-to-tail*.

Independence in Bayesian Network - Conditional independence



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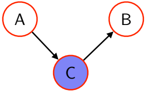

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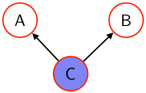

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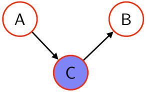

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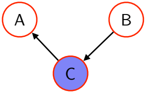

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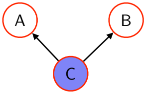

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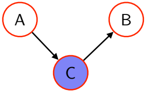

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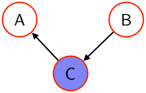

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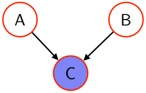

A,B are conditionally dependent given C
$p(A,B|C) \propto p(C|A,B)p(A)p(B) \neq p(A|C)p(B|C)$
Node C is said to be *head-to-head*.

Example: Fuel system on the car

- $B \in \{0, 1\}$: representing the state of a battery that is either charged $(B = 1)$ or flat $(B = 0)$.
- $F \in \{0, 1\}$: representing the state of the fuel tank that is either full of fuel $(F = 1)$ or empty $(F = 0)$.
- $G \in \{0, 1\}$: representing the state of an electric fuel gauge and which indicates either full $(G = 1)$ or empty $(G = 0)$.

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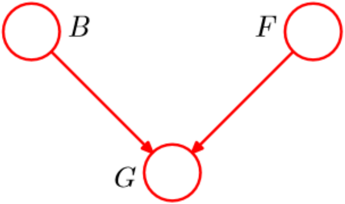


Figure: [C. Bishop, PRML]

Example: Fuel system on the car

Probability table: $p(G, B, F) = p(B)p(F)p(G|B, F)$

$$\begin{aligned} p(B = 1) &= 0.9 \\ p(F = 1) &= 0.9 \end{aligned}$$

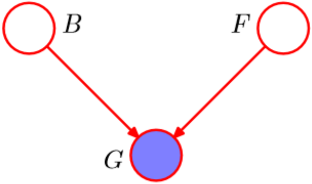
G = 1	B	F
0.8	1	1
0.2	1	0
0.2	0	1
0.1	0	0

Example: Fuel system on the car

- $p(F = 0) = 0.1$: before observe any data, the prior probability of the fuel tank being empty.

Example: Fuel system on the car

- $p(F = 0) = 0.1$: before observe any data, the prior probability of the fuel tank being empty.
- $p(F = 0|G = 0)$: suppose we observe the fuel gauge and discover it reads empty, i.e., $G = 0$.



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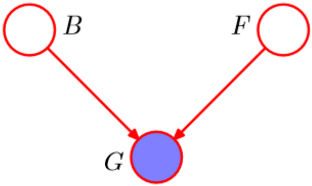


Figure: [C. Bishop, PRML]

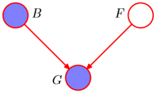
- $p(F=0|G=0) > p(F=0)$ observing that the gauge reads empty makes it more likely that the tank is indeed empty.

$$p(F=0|G=0)=\frac{p(G=0|F=0)p(F=0)}{p(G=0)}\approx 0.257$$

Example: Fuel system on the car

- Now observed the states of both the fuel gauge and the battery:

$$p(F = 0 | G = 0, B = 0) \approx 0.111$$



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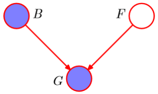

Figure: [C. Bishop, PRML]

- The probability that the tank is empty has *decreased*(from 0.257 to 0.111) as a result of the observation of the state of the battery. Finding out that the battery is flat *explains away* the observation that the fuel gauge reads empty.

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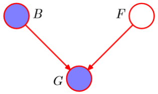

Figure: [C. Bishop, PRML]

- The probability that the tank is empty has *decreased*(from 0.257 to 0.111) as a result of the observation of the state of the battery. Finding out that the battery is flat *explains away* the observation that the fuel gauge reads empty.
- The state of the fuel tank (F) and that of the battery (B) have indeed become dependent on each other as a result of observing the reading on the fuel gauge (G).

D-separation ("directed separated") in Bayesian networks

- Algorithm to find out whether $A \perp B | \mathbf{C}$ is implied by a given directed acyclic graph.

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- Look at all possible paths from A to B, any such path is said to be **blocked** if it includes a node such that either
    1. the arrows on the path meet either head-to-tail or tail-to-tail at the node, and the node is in the set C
    2. the arrows meet head-to-head at the node, and neither the node, nor any of its descendants, is in the set C.

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- If all such paths are blocked, then A is said to be *d-separated* from B by C. And the joint distribution for all variables in the graph will satisfy $A \perp B | \mathbf{C}$.

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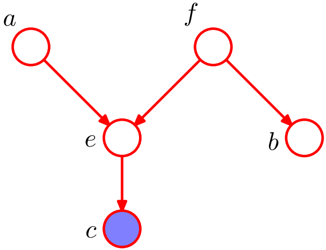
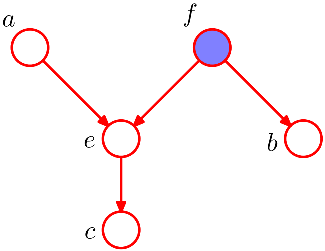
- If all such paths are blocked, then A is said to be *d-separated* from B by C. And the joint distribution for all variables in the graph will satisfy $A \perp B | \mathbf{C}$.
- d-separation reduces statistical independencies (hard) to connectivity in graphs (easy)
- Important because it allows us to quickly prune the Bayesian network, finding just the relevant variables for answering a query

D-separation examples



(a) $a \not\perp b | c$

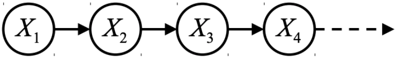


(b) $a \perp b | f$

Some frequently used graphical models

Markov Models

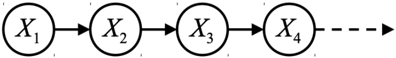
- A Markov model is a chain-structured BN
  - Each node is identically distributed (stationarity)
  - Value of X at a given time is called the state
  - As a BN:



$$P(X_1) \qquad P(X_2|X_1) \qquad P(X_T|X_{T-1})$$

  - Parameters: called transition probabilities or dynamics, specify how the state evolves over time (also, initial probs)

Markov Models - Conditional Independence



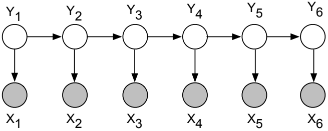
$P(X_1)$ $\quad P(X_2|X_1)$ $\quad P(X_T|X_{T-1})$

- Basic conditional independence
  - Past and future independent of the present
  - Each time step only depends on the previous
  - This is called the (first order) Markov property
- Note that the chain is just a (growing) BN
  - We can always use generic BN reasoning on it (if we truncate the chain)

Hidden Markov models

- Markov chains not so useful for most agents
  - Eventually you dont know anything anymore
  - Prediction needs previous observations.

- Hidden Markov Models



  - Frequently used for speech recognition and video modelling
  - Underlying Markov chain over states S
  - Joint distribution factors as:

$$p(\mathbf{y}, \mathbf{x}) = p(y_1)p(x_1|y_1)\prod_{t=2}^{T} p(y_t|y_{t-1})p(x_t|y_t)$$

L.R. Rabiner.A tutorial on hidden Markov models and selected applications in speech recognition. Proceedings of the IEEE. Volume: 77 , Issue: 2 , Feb 1989.

Hidden Markov models

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$$p(\mathbf{y}, \mathbf{x}) = p(y_1)p(x_1|y_1) \prod_{t=2}^{T} p(y_t|y_{t-1})p(x_t|y_t)$$

  - $p(y_1)$ is the distribution for the starting state
  - $p(y_t|y_{t-1})$ is the transition probability between any two states
  - $p(x_t|y_t)$ is the emission probability
- What are the conditional independencies here? e.g. $Y_1 \perp \{Y_3, ..., Y_6\} | Y_2$
- Markov assumptions:
  - The current state is conditionally independent of all the past states given the states in the previous time step.
  - The current evidence is only dependent on the current state.

Acknowledgement and Further Reading

Slides are adapted from Dr. Y. Ning's Spring 19 offering of CS-559.

Further Reading:
Chapter 8.1 8.2 of *Pattern Recognition and Machine Learning* by C. Bishop.