PhD Math Camp: Static Optimization

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Contents

Section 1

Introduction to Optimization

Motivating Example: A Farmer's Egg Production Problem

A farmer wants to maximize revenue by producing eggs using:

- Chicken feed (x_1) and
- Chicken coops (x_2)

The production function follows a **Cobb-Douglas form**:

$$q = f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}, \quad 0 < \alpha < 1$$

Eggs sell at an exogenous price P per unit. The farmer faces input costs:

- Feed costs p_1 per unit
- Coop costs p_2 per unit

The government can support the farmer at most I.
Question: How should the farmer allocate spending between x_1 and x_2 to maximize revenue?

Problem Setup:

The farmer wants to maximize revenue from egg production:

$$\max_{x_1, x_2} \quad P \cdot x_1^{\alpha} x_2^{1-\alpha}$$

Subject to:

$$p_1 x_1 + p_2 x_2 \leq I \quad \text{(budget constraint)}$$

$$x_1 \geq 0, \quad x_2 \geq 0 \quad \text{(non-negativity)}$$

Goal: Choose (x_1, x_2) to maximize production given the cost constraint.

From Economics to Optimization

This is a classic example of an optimization problem:

- Choosing inputs x_1 and x_2 to maximize output
- Subject to constraints (budget, non-negativity)

General Form of an Optimization Problem:

$$\min_{\mathbf{x}\in S} f(\mathbf{x})$$

- $\mathbf{x} \in \mathbb{R}^n$ is the **decision variable vector**, e.g., $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\top$
- $f(\mathbf{x})$ is the **objective function** (to minimize or maximize)
- $S \subseteq \mathbb{R}^n$ is the **feasible region**, defined by constraints (e.g., budget, non-negativity), it can also just be $\mathbb{R}^n$, meaning no constrant.

Optimization = Find the best $\mathbf{x} \in S$ that gives the lowest (or highest) value of $f(\mathbf{x})$

The **feasible region** S is the set of all values of $\mathbf{x}$ that satisfy the constraints. Types of constraints that define S:

- **Equality constraints:** $g_i(\mathbf{x}) = 0$
- **Inequality constraints:** $h_j(\mathbf{x}) \leq 0$
- **Variable type restrictions:**
  - Non-negativity: $x_k \geq 0$
  - Integer constraints: $x_k \in \mathbb{Z}$ or $x_k \in \mathbb{Z}_+$
  - Binary decisions: $x_k \in \{0, 1\}$

Example:

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^2} \quad & f(x_1, x_2) = x_1^2 + x_2^2 \\ \text{s.t.} \quad & x_1 + x_2 = 10 \quad \text{(equality constraint)} \\ & x_1, x_2 \geq 0 \quad \text{(non-negativity)} \\ & x_1 \in \mathbb{Z}_+ \quad \text{(integer constraint)} \end{aligned}$$

Only the integer points along the line $x_1 + x_2 = 10$ in the first quadrant are feasible!

What if there's more than one objective?

Example: A Farmer's Trade-off

- Objective 1: Maximize egg production

$$f_1(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$$

- Objective 2: Minimize working time

$$f_2(x_1, x_2) = \text{labor cost}(x_1, x_2)$$

These goals may conflict: more feed or coops may increase production but also require more labor.

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These goals may conflict: more feed or coops may increase production but also require more labor.

Multi-objective optimization:

$$\min_{\mathbf{x} \in S} \big(f_1(\mathbf{x}), f_2(\mathbf{x})\big)$$

Key idea: No single "best"solution in general $\Rightarrow$ We need optimality concepts, such as defining a utility function with components of labor input and money made.

Convex Problem

A set $S \subseteq \mathbb{R}^n$ is **convex** if for any $\mathbf{x}, \mathbf{y} \in S$ and any $\lambda \in [0, 1]$,

$$\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in S$$

(Line segment between any two points stays inside the set)

A function $f : \mathbb{R}^n \to \mathbb{R}$ is **convex** if for any $\mathbf{x}, \mathbf{y}$ and $\lambda \in [0, 1]$,

$$f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1 - \lambda) f(\mathbf{y})$$

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Convex Optimization Problem

An optimization problem is **convex** if:

- The objective function $f(\mathbf{x})$ is convex over S
- The feasible set S is convex (i.e., constraints are convex)

Why do we care*?

- Every local minimum is a global minimum
- Efficient algorithms (e.g., gradient methods) exist
- Strong duality often holds

Examples of Convex Sets

Example 1: Convex Set
The set

$$S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 \leq 1,\ x_1 \geq 0,\ x_2 \geq 0\}$$

is convex. *Why?* It's the intersection of half-spaces, and all half-spaces are convex sets.

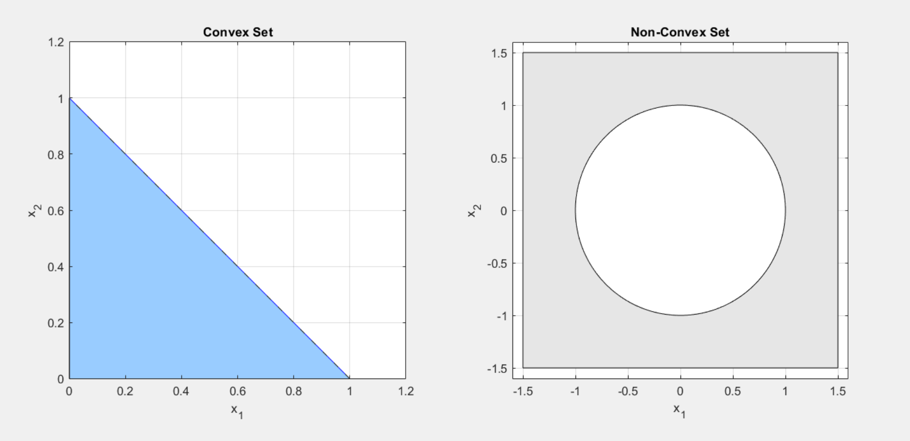
Example 2: Non-convex Set (for contrast)
The set

$$S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \geq 1\}$$

is not convex: a line between two boundary points may cut through the hole inside.

A set that is not convex is called a non-convex set.

Examples of Convex Sets



Examples of Convex Functions

Example 1: $f(x) = x^2$

Let $f(x) = x^2$, and take any $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$.

Then:

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= (\lambda x + (1-\lambda)y)^2 \\ &= \lambda^2 x^2 + 2\lambda(1-\lambda)xy + (1-\lambda)^2 y^2 \end{aligned}$$

Compare with:

$$\lambda f(x) + (1-\lambda)f(y) = \lambda x^2 + (1-\lambda)y^2$$

Taking the difference:

$$\lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y) = \lambda(1-\lambda)(x-y)^2 \geq 0$$

$\Rightarrow$ So $f(x) = x^2$ is convex.

Shortcut: Second derivative test

$$f''(x) = 2 > 0 \quad \Rightarrow \text{convex on } \mathbb{R}$$

Convexity of Multi-variable Functions

Example: Let

$$f(x_1, x_2) = x_1^2 + 3x_2^2$$

To check convexity, we compute the **Hessian matrix** $H_f(\mathbf{x})$:

$$H_f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$$

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Criterion: A twice-differentiable function is convex on a convex domain if its Hessian is positive semi-definite (PSD).

$$\text{Since } H_f(\mathbf{x}) \succeq 0 \text{ (all eigenvalues} > 0), \Rightarrow f \text{ is convex on } \mathbb{R}^2.$$

Reminder: A symmetric matrix H is positive semi-definite if for all vectors $z \in \mathbb{R}^n$,

$$z^\top H z \geq 0$$

Concavity: The Opposite of Convexity

Definition: A function $f : \mathbb{R}^n \to \mathbb{R}$ is **concave** if for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,

$$f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \geq \lambda f(\mathbf{x}) + (1 - \lambda) f(\mathbf{y})$$

Interpretation: The line segment lies below the graph of the function.

Examples:

- $f(x) = \log(x)$, defined on $(0, \infty)$
- $f(x) = \sqrt{x}$, defined on $[0, \infty)$
- $f(x) = -x^2$, defined on $\mathbb{R}$

Quick check:

- Single variable: $f''(x) \leq 0 \Rightarrow f$ is concave
- Multivariable: Hessian matrix $H_f(\mathbf{x}) \preceq 0 \Rightarrow f$ is concave

Section 2

Continuous Optimization without Constraint

Continuous Case Without Constraint: Single Variable

Problem:

$$\min_{x \in \mathbb{R}} \quad f(x)$$

First-order necessary condition (FONC):

$$f'(x^*) = 0$$

Second-order condition (SONC):

- $f''(x^*) \geq 0$ x^* is a local minimum (necessary)
- $f''(x^*) > 0$ x^* is a strict local minimum (sufficient)

Example:

$$f(x) = x^2 + 3x + 5 \quad \Rightarrow \quad f'(x) = 2x + 3 = 0 \Rightarrow x^* = -\frac{3}{2}$$

$$f''(x) = 2 > 0 \quad \Rightarrow \text{Strict local minimum}$$

Why? It remains unchanged at a certain point (FOC) and has an upward growth trend (SOC).

Continuous Case Without Constraint: Multivariable

Problem:

$$\min_{\mathbf{x}\in\mathbb{R}^n} \quad f(\mathbf{x})$$

First-order necessary condition (FONC):

$$\nabla f(\mathbf{x}^*) = 0$$

Second-order condition (SONC):

- Let $H_f(\mathbf{x}^*)$ be the Hessian matrix of f
- If $H_f(\mathbf{x}^*) \succeq 0$ (positive semi-definite), then $\mathbf{x}^*$ is a local minimum (necessary)
- If $H_f(\mathbf{x}^*) \succ 0$ (positive definite), then $\mathbf{x}^*$ is a strict local minimum (sufficient)

Example:

$$f(x,y) = x^2 + 3y^2 \Rightarrow \nabla f = \begin{bmatrix} 2x \\ 6y \end{bmatrix},\ H_f = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\Rightarrow \nabla f = 0 \text{ at } (0,0),\ H_f \succ 0 \Rightarrow \text{strict local minimum}$$

Example: Unconstrained Optimization – Farmer's Profit

A farmer produces eggs using:

- E: chicken feed
- F: chicken coops

The production function is:

$$q = (E \cdot F)^{0.4}$$

Eggs sell at 2 USD per pound.

The cost:

$$\text{Feed: } 1 \text{ USD/unit}, \quad \text{Coop: } 0.5 \text{ USD/unit}$$

Profit function:

$$\pi(E, F) = 2 \cdot (EF)^{0.4} - E - 0.5F$$

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Step 1: First-order conditions (FONC):

$$\frac{\partial \pi}{\partial E} = 0.8F^{0.4}E^{-0.6} - 1 = 0$$

$$\frac{\partial \pi}{\partial F} = 0.8E^{0.4}F^{-0.6} - 0.5 = 0$$

Example: Unconstrained Optimization – Farmer's Profit

Step 2: Solve the system
From FONC:

$$\frac{F^{0.4}}{E^{0.6}} = \frac{1}{0.8}, \quad \frac{E^{0.4}}{F^{0.6}} = \frac{0.5}{0.8}$$

Multiply both sides:

$$\left(\frac{F^{0.4}}{E^{0.6}}\right) \cdot \left(\frac{E^{0.4}}{F^{0.6}}\right) \Rightarrow (EF)^{0.2} = \frac{0.64}{0.5} = 1.28 \Rightarrow EF = (1.28)^5 \approx 3.43$$

From first FONC:

$$\frac{F^{0.4}}{E^{0.6}} = 1.25 \Rightarrow \frac{F}{E^{1.5}} = 1.25^{2.5} \approx 1.74 \Rightarrow F = 1.74 \cdot E^{1.5}$$

Substitute into $EF = 3.43$, solve:

$$E \cdot (1.74 \cdot E^{1.5}) = 3.43 \Rightarrow E^* \approx 1.31, \quad F^* \approx 2.62$$

Example: Unconstrained Optimization – Farmer's Profit

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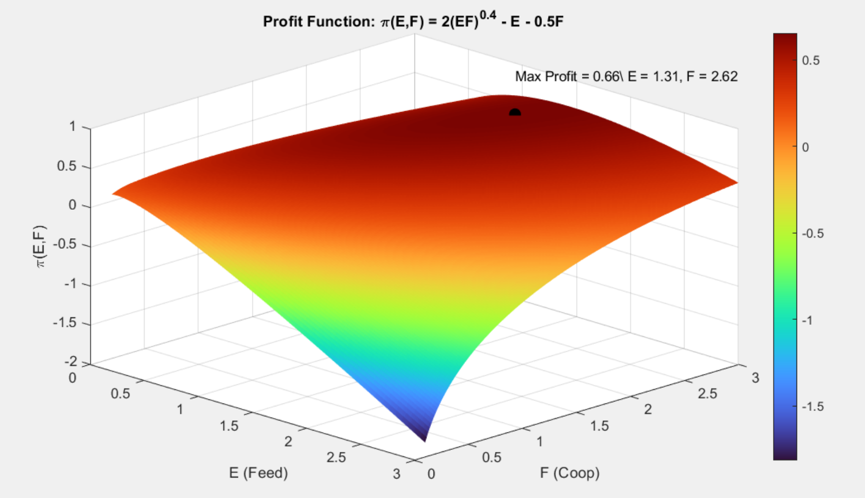
$$E \cdot (1.74 \cdot E^{1.5}) = 3.43 \Rightarrow E^* \approx 1.31, \quad F^* \approx 2.62$$

Conclusion:

$$\pi^* = 2(EF)^{0.4} - E - 0.5F \approx 0.65$$

The profit-maximizing input levels exist and are interior.

Example: Unconstrained Optimization – Farmer's Profit



Example: Unconstrained Optimization – Farmer's Profit

Problem: Unconstrained Optimization with Increasing Returns
Let's change the production function to:

$$q = (E \cdot F)^{0.6}$$

Profit function:

$$\pi(E, F) = 2 \cdot (EF)^{0.6} - E - 0.5F$$

Question: Does this problem have a finite solution? Try solving the first-order conditions.

Section 3

Continuous Optimization with Constraint: KKT Conditions

Continuous Optimization Problems

In this section, we focus on optimization problems where the decision variables can take **any real value**:

$$\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}) \quad \text{subject to} \quad \mathbf{x} \in S$$

- No integer, binary, or discrete restrictions
- Feasible region $S \subseteq \mathbb{R}^n$ is typically defined by continuous constraints
- Tools from calculus, linear algebra, and convex analysis are applicable

Karush–Kuhn–Tucker (KKT) Conditions can help with most of the problem in the form of above.

The Lagrangian Function

General constrained optimization problem:

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & g_j(\mathbf{x}) \le 0, \quad j = 1, \ldots, m \\ & h_k(\mathbf{x}) = 0, \quad k = 1, \ldots, p \end{aligned}$$

Lagrangian function:

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\beta}) = f(\mathbf{x}) + \sum_{j=1}^{m} \lambda_j g_j(\mathbf{x}) + \sum_{k=1}^{p} \beta_k h_k(\mathbf{x})$$

- $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$: multipliers for inequality constraints
- $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)$: multipliers for equality constraints

Idea: Convert a constrained problem into an unconstrained one $\rightarrow$ Critical points of $\mathcal{L}$ under KKT conditions are candidate optima.

Karush–Kuhn–Tucker (KKT) Conditions

Consider the problem:

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & g_j(\mathbf{x}) \le 0, \quad j = 1, \ldots, m \\ & h_k(\mathbf{x}) = 0, \quad k = 1, \ldots, p \end{aligned}$$

Then the **KKT conditions** are:

$$\nabla f(\mathbf{x}) + \sum_{j=1}^{m} \lambda_j \nabla g_j(\mathbf{x}) - \sum_{k=1}^{p} \beta_k \nabla h_k(\mathbf{x}) = 0$$

$$\lambda_j g_j(\mathbf{x}) = 0, \quad j = 1, \ldots, m \quad \text{(Complementary slackness)}$$

$$g_j(\mathbf{x}) \le 0, \quad \lambda_j \ge 0 \quad \text{(Primal and dual feasibility)}$$

$$h_k(\mathbf{x}) = 0, \quad k = 1, \ldots, p \quad \text{(Equality constraints)}$$

Interpretation: These are first-order necessary conditions for optimality under constraints.

KKT Example: Minimizing a Nonlinear Function

Problem:

$$\min_{x,y} \quad \frac{e^x}{y^2}$$

Subject to:

$$h(x,y) = x - 2y = 0 \quad \text{(equality constraint)}$$
$$g(x,y) = 1 - x \leq 0 \quad \text{(inequality constraint: } x \geq 1)$$

Step 1: Lagrangian

$$\mathcal{L}(x, y, \lambda, \beta) = \frac{e^x}{y^2} + \beta(x - 2y) + \lambda(1 - x)$$

Step 2: KKT conditions

- Stationarity:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{e^x}{y^2} - \lambda + \beta = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = -\frac{2e^x}{y^3} - 2\beta = 0$$

- With $x = 2y$, and $\lambda(1 - x) = 0$

Solving the KKT System

KKT system:

$$\begin{cases} \dfrac{e^x}{y^2} - \lambda + \beta = 0 & \text{(stationarity in } x) \\ -\dfrac{2e^x}{y^3} - 2\beta = 0 & \text{(stationarity in } y) \\ x - 2y = 0 & \text{(equality constraint)} \\ \lambda(1-x) = 0 & \text{(complementary slackness)} \\ x \geq 1, \quad \lambda \geq 0 & \text{(primal and dual feasibility)} \end{cases}$$

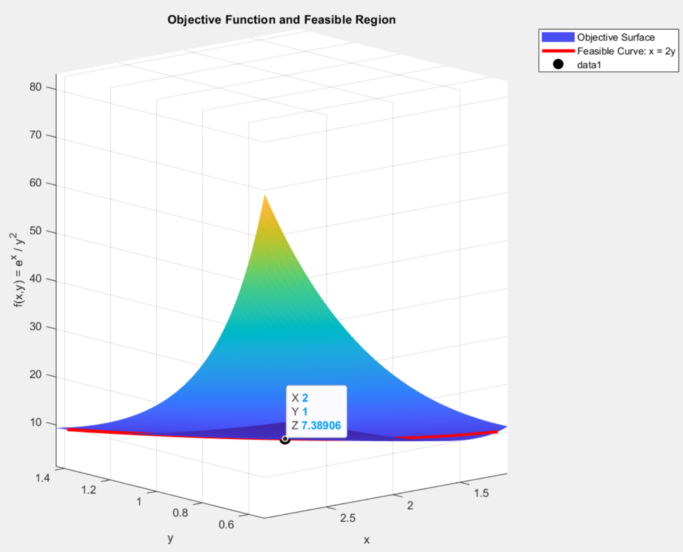
Then, we can use Matlab (or other ways) to solve.

```
syms x y lambda beta
eq1 = exp(x)/y^2 + lambda - beta == 0;
eq2 = -2*exp(x)/y^3 - 2*lambda == 0;
eq3 = x - 2*y == 0;
eq4 = beta*(1 - x) == 0;

S = solve([eq1, eq2, eq3, eq4, x >= 1, beta >= 0], ...
          [x, y, lambda, beta], 'Real', true);
double([S.x, S.y, S.lambda, S.beta])
```

Solution: $x = 2$, $y = 1$, $\beta = -7.39$, $\lambda = 0$

Solving the KKT System



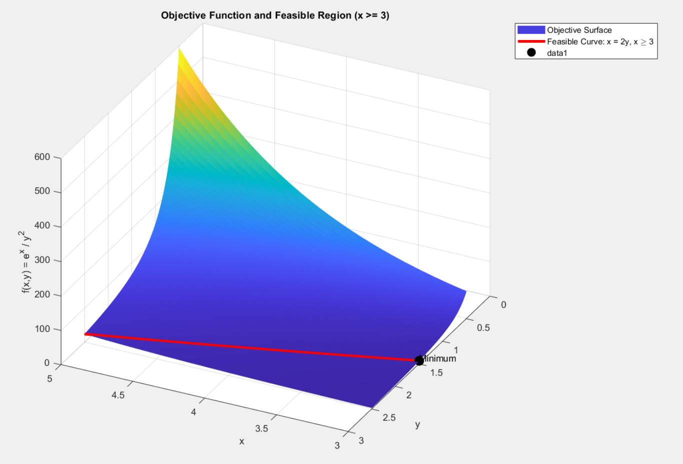
Boundary Solution Case

KKT system:

$$\begin{cases} \dfrac{e^x}{y^2} - \lambda + \beta = 0 & \text{(stationarity in } x\text{)} \\ -\dfrac{2e^x}{y^3} - 2\beta = 0 & \text{(stationarity in } y\text{)} \\ x - 2y = 0 & \text{(equality constraint)} \\ \lambda(3 - x) = 0 & \text{(complementary slackness)} \\ x \geq 3, \quad \lambda \geq 0 & \text{(primal and dual feasibility)} \end{cases}$$

Solution: $x = 3$, $y = 1.5$, $\beta = -5.95$, $\lambda = 2.98$

Boundary Solution Case



Lagrange Multiplier β in Equality Constraints

Problem family:

$$\min_{x,y} \quad 2x^2 + y^2 \quad \text{subject to } x + y - b = 0$$

Step 1: Solve using Lagrangian

Lagrangian:

$$\mathcal{L}(x, y, \beta) = 2x^2 + y^2 + \beta(b - x - y)$$

First-order conditions:

$$\frac{\partial \mathcal{L}}{\partial x} = 4x - \beta = 0 \Rightarrow x = \frac{\beta}{4}$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2y - \beta = 0 \Rightarrow y = \frac{\beta}{2}$$

Constraint:

$$x + y = b \Rightarrow \frac{\beta}{4} + \frac{\beta}{2} = b \Rightarrow \frac{3\beta}{4} = b \Rightarrow \beta = \frac{4b}{3}$$

Then:

$$x = \frac{b}{3}, \quad y = \frac{2b}{3}$$

Lagrange Multiplier β in Equality Constraints

Step 2: Compute optimal value and interpret β

$$f^*(b) = 2x^2 + y^2 = 2\left(\frac{b^2}{9}\right) + \left(\frac{4b^2}{9}\right) = \frac{6b^2}{9} = \frac{2b^2}{3}$$

$$\frac{df^*}{db} = \frac{4b}{3} = \beta \Rightarrow \boxed{\beta = \frac{df^*}{db}}$$

Interpretation: In equality constraints, β measures the *rate of change of the optimal value* with respect to the right-hand side of the constraint.

Lagrange Multipliers in Inequality Constraints*

Now consider a problem with an **inequality constraint**:

$$\min_{\mathbf{x}} \; f(\mathbf{x}) \quad \text{subject to } g(\mathbf{x}) \leq 0$$

The KKT condition includes a multiplier $\lambda \geq 0$, such that:

$$\nabla f(\mathbf{x}^*) + \lambda \nabla g(\mathbf{x}^*) = 0$$

$$\lambda \cdot g(\mathbf{x}^*) = 0 \quad \text{(Complementary slackness)}$$

Interpretation of λ:

- If the constraint is binding, then $\lambda > 0$
- If the constraint is non-binding, then $\lambda = 0$
- λ represents the **shadow price** of the constraint: the marginal value of relaxing the constraint by 1 unit.

$$\frac{df^*}{db} = -\lambda \quad \text{where } g(\mathbf{x}) \leq b$$

Economic meaning: How much the optimal objective would improve if we were allowed to "spend" 1 more unit of the constrained resource.

Lagrange Multipliers in Inequality Constraints*

Problem:

$$\min_{x,y} \quad 2x^2 + y^2 \quad \text{s.t.} \quad 0 \geq 1 - x - y$$

Solution:

- Optimal point: $x^* = \frac{1}{3},\ y^* = \frac{2}{3}$
- Optimal value: $f^* = \frac{2}{3}$
- Lagrange multiplier : $\lambda = \frac{4}{3}$

Lagrange Multipliers in Inequality Constraints*

Relaxed problem:

$$\min_{x,y} \quad 2x^2 + y^2 \quad \text{s.t.} \quad 0.01 \geq 1 - x - y$$

Compare optimal values:

- Last slide: $f^* = \frac{2}{3}, \ \lambda = \frac{4}{3}$
- This slide: $f^* \approx 0.6534$,
- Difference in objective: $\Delta f^* \approx 0.6667 - 0.6534 = 0.0133$
- Approximated by shadow price: $\lambda \cdot \Delta b = 1.33 \cdot 0.01 = 0.0133$

Conclusion: The shadow price λ accurately predicts how much the objective improves when the constraint is relaxed.

Why Are KKT Conditions Valid?*

Problem:

$$\min_{\mathbf{x}\in\mathbb{R}^n} \; f(\mathbf{x}) \quad \text{subject to:} \quad g_j(\mathbf{x}) \leq 0,\ j=1,\ldots,m, \quad h_k(\mathbf{x})=0,\ k=1,\ldots,p$$

Goal: Find necessary conditions for $\mathbf{x}^*$ to be a local minimum.

Idea: Use a first-order Taylor expansion of the objective and constraints to characterize directions of descent.
Let d be a feasible direction from $\mathbf{x}^*$, i.e.,

$$\nabla h_k(\mathbf{x}^*)^\top d = 0, \quad \nabla g_j(\mathbf{x}^*)^\top d \leq 0 \text{ for active } j$$

Then local optimality implies:

$$\nabla f(\mathbf{x}^*)^\top d \geq 0 \quad \text{(cannot decrease objective)}$$

Why Are KKT Conditions Valid?*

This leads to a **convex cone** of feasible directions and a **separation theorem:** If no feasible d can improve the objective, then there exist multipliers $\lambda_j \geq 0$, $\beta_k \in \mathbb{R}$ such that:

$$\nabla f(\mathbf{x}^*) + \sum_{j=1}^{m} \lambda_j \nabla g_j(\mathbf{x}^*) + \sum_{k=1}^{p} \beta_k \nabla h_k(\mathbf{x}^*) = 0$$

Together with:

$$\lambda_j \geq 0, \quad g_j(\mathbf{x}^*) \leq 0, \quad \lambda_j g_j(\mathbf{x}^*) = 0$$

These are the KKT conditions.

Example

A poor farmer produces eggs using two inputs: **F** (feed) and **E** (chicken coop), with production function:

$$q = (F \cdot E)^{0.5}$$

The price of eggs is 2 dollar per pound. The prices of inputs are:

$$p_F = 1, \quad p_E = 3$$

The government provides a maximum subsidy (free of charge) of 10 dollar — i.e., total spending on inputs must be no more than 10.

Question: What is the maximum profit the farmer can earn under this budget constraint?

Standard Formulation

Standard minimization form:

$$\min_{(F,E)\in S} \quad f(F,E) = -2 \cdot (FE)^{0.5}$$

Feasible set S:

$$S = \left\{(F,E) \in \mathbb{R}^2_+ \;\middle|\; F + 3E - 10 \leq 0\right\}$$

Interpretation:

- Decision variables: F, E
- Objective: Minimize cost minus revenue (i.e., negative profit)
- Constraint: Total input cost cannot exceed subsidy budget

Section 4

Discrete Optimization

Motivation for Discrete Optimization

Real-World Motivation:

In many real-world problems, the decision variables cannot take arbitrary real values. For example:

- A firm cannot buy **2.5 machines**—machine units must be whole numbers.
- A factory decides whether to **open or close** a production line—decisions are binary.
- A delivery company needs to **assign trucks to routes**—choices are discrete.

Motivation for Discrete Optimization

This leads us to:

Integer Optimization

Some or all decision variables are required to take integer values:

$$x \in \mathbb{Z}^n \quad \text{or} \quad x_i \in \{0, 1\}$$

This section will focus on integer optimization, and other discrete cases can be expanded from this.

Solving Discrete Optimization Problems

Why are discrete problems hard?

- The feasible region is **non-convex**, often **finite but very large**.
- Standard calculus-based methods don't apply.

Naive approach: Try every possible combination — *exhaustive search* $\Rightarrow$ quickly becomes computationally infeasible as problem size grows.

Three smarter strategies:

1. **Graphical Method:** For small problems in 2 or 3 dimensions — plot all integer points in feasible region and evaluate objective function.
2. **Branch and Bound (B&B)*:** Recursive divide-and-conquer approach:
   - Solve relaxed problem (e.g., continuous version),
   - Use bounds to eliminate subregions,
   - Systematically explore feasible solutions.
3. **Binary Reformulation:** Convert 0-1 or integer variables into continuous variables with additional constraints.

Example: Graphical Method for Integer Programming

Problem: Minimize the function

$$\max_{x,y} \quad \frac{1}{xy}$$

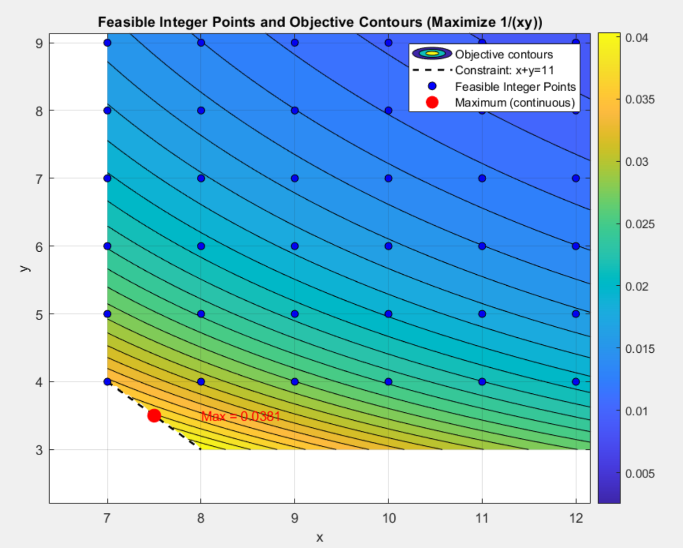
subject to:

$$x + y \geq 11$$
$$x \geq 6.5$$
$$y \geq 3.5$$
$$x, y \in \mathbb{Z}_+$$

Observation:

- Objective function is non-linear, but well-defined and decreasing in both x and y.
- Feasible region consists of finitely many integer lattice points satisfying all constraints.

Example: Graphical Method for Integer Programming



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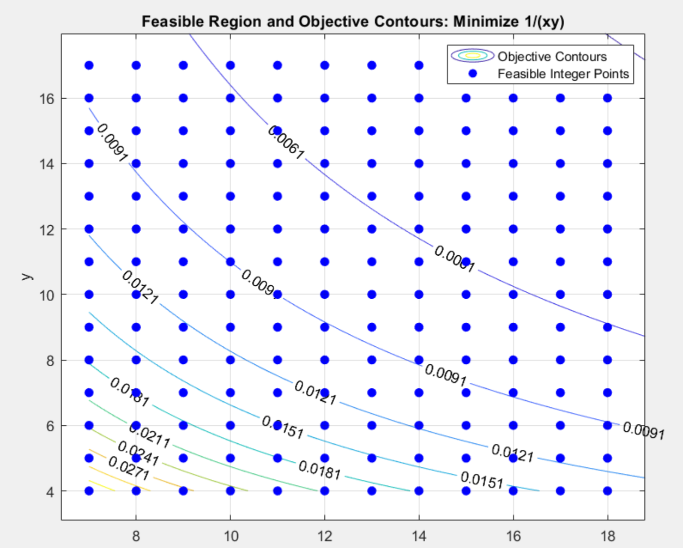

Branch and Bound*

Key Idea: Solve the relaxed (continuous) version of the integer program, then progressively narrow down the feasible space to find the optimal integer solution.

Steps:

1. **Relaxation:** Solve the problem without the integer constraints (continuous case). This gives an *upper bound* (for maximization).
2. **Branching:** If the solution is not integer, divide the feasible region into two (or more) sub-regions. For example, if $x = 3.7$, create two branches: $x \leq 3$ and $x \geq 4$.
3. **Bounding:** For each sub-region, solve the relaxed problem again. Use the solution as an upper bound, and compare with the best known **feasible integer solution** (lower bound).
4. **Pruning:** Discard sub-regions if:
   - Their upper bound is worse than the current best integer solution.
   - They are infeasible.
5. **Repeat:** Continue branching and bounding until all regions are either pruned or yield integer solutions.

Goal: Find the integer solution with the best (optimal) objective value.

Example*

$$\min_{x \in \mathbb{Z}_{\geq 0}} \quad f(x) = e^x - 2x$$

Step 1: Relax the problem Allow $x \in \mathbb{R}_{\geq 0}$. Then minimize:

$$f(x) = e^x - 2x$$

First-order condition:

$$f'(x) = e^x - 2 = 0 \Rightarrow x^* = \ln(2) \approx 0.693$$

$$f(x^*) = e^{\ln 2} - 2 \cdot \ln 2 \approx 2 - 1.386 \approx 0.614$$

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Step 2: Branch Try integer candidates (Given the convexity of the function):

$$x = 0 \Rightarrow f(0) = 1, \quad x = 1 \Rightarrow f(1) = e - 2 \approx 0.718$$

Step 3: Compare Relaxed minimum is ≈ 0.614, not feasible. Try nearby integers: Best integer feasible value:

$$\boxed{x = 1}, \quad \boxed{f(1) = 0.718}$$

Binary Reformulation: A Farmer's Dilemma

A farmer owns a plot of **100 square meters**. He wants to raise animals, but each animal requires space:

- **Cow**: 43 m^2, annual profit = \$50
- **Sheep**: 35 m^2, annual profit = \$40
- **Goose**: 23 m^2, annual profit = \$30
- **Chicken**: 18 m^2, annual profit = \$20

It can be found that the land can hold at most only 3 animals at once. The farmer decides to give **one or two animal to a neighbor**.

Goal: Choose **which animal(s) to give away** to maximize total annual income, while respecting the **land constraint of 100 m^2**.

Binary Integer Programming Formulation

Let $y_i \in \{0, 1\}$ indicate whether animal i is kept:

- y_1: Cow (\$50, 43 m²)
- y_2: Sheep (\$40, 35 m²)
- y_3: Goose (\$30, 23 m²)
- y_4: Chicken (\$20, 18 m²)

Objective (maximize total income):

$$\max \ 50y_1 + 40y_2 + 30y_3 + 20y_4$$

Subject to:

$$43y_1 + 35y_2 + 23y_3 + 18y_4 \leq 100 \quad \text{(land constraint)}$$

$$y_i(1 - y_i) = 0, \quad \text{for } i = 1, 2, 3, 4 \quad \text{(binary constraint)}$$

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This is a classic small-scale **0-1 knapsack** with cardinality constraint, which has been converted into the standard continuous constraint optimization.

Wrap-Up: Optimization Techniques

This lecture covered:

- Unconstrained optimization (single- and multi-variable)
- Constrained optimization using:
  - Lagrange multipliers
  - Karush–Kuhn–Tucker (KKT) conditions
  - Integer programming (discrete optimization)
- Visual and computational interpretation of optimality

Key Takeaway: KKT conditions will 100% appear on the PhD Core Exam.

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Key Takeaway: KKT conditions will 100% appear on the PhD Core Exam.

Next class: Dynamic Optimization, Modern Method

These topics (except for comparative statics analysis) will not be tested in the PhD Core Exam, but will be **critical for second and third-year field courses**, such as:

- Resource & Environmental Economics
- Bioeconomic models (e.g., fisheries, livestock)
- Machine Learning for Econometrics

References

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