PhD Math Camp: Multivariable Calculus

Shangze Rudy Dai

Food and Resource Economics Department
University of Florida

Aug 5, 2025

Contents

Unless otherwise specified, all functions are continuous and do not consider complex numbers.

Section 1

Review: Limits, Differentiation, and Integration

- What is a number?
    - Natural numbers: $1, 2, 3, \ldots$
    - Integers: $\ldots, -2, -1, 0, 1, 2, \ldots$
    - Rational numbers: $\frac{1}{2}, \frac{3}{4}, \ldots$
    - Irrational numbers: $\sqrt{2}, \pi, e$

Formal Definition of Numbers*

- **Natural Numbers ($\mathbb{N}$)**: Defined using the **Peano axioms**:
  - 0 is a natural number.
  - Every natural number has a unique natural number **successor**: $S(n)$.
  - No number is the successor of 0.
  - a= b iff S(a) = S(b).
  - Induction holds.
- **Integers ($\mathbb{Z}$)**: Extend $\mathbb{N}$ to include negatives: e.g., $(-3, -2, -1, 0, 1, 2, 3)$.
- **Rational Numbers ($\mathbb{Q}$)**: Pairs of integers: $\frac{a}{b}$, where $b \neq 0$.
- **Real Numbers ($\mathbb{R}$)**: Fill the gaps using **Dedekind cuts** or **Cauchy sequences**.
- **Irrational Numbers**: Real numbers that cannot be written as fractions, e.g., $\sqrt{2}, \pi, e$.

Limits

- What's the biggest number?
  - 10, 100, 10^{100}, googol? Infinity?
- What's the smallest positive number?
  - 0.1, 0.0001, 10^{-100}, ...? Can we ever reach 0?

Limits

- We want to understand:
  - What happens when x gets very large? $\to \infty$
  - What happens when x gets very close to 0?

- But we can't plug in "infinity"into a function...

- **Limits** help us make sense of this!

- For example:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$

PS: In mathematics, a function from a set X to a set Y is a rule that assigns to each element of X exactly one element of Y. The set X is called the domain of the function, and the set Y is called the codomain.

Formal Definition of Limit*

1. Finite Limit at a Point:

$$\lim_{x \to a} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0$$

such that $0 < |\text{x - a}| < \delta \Rightarrow |f(x) - L| < \varepsilon$

2. Infinite Limit at Infinity:

$$\lim_{x \to \infty} f(x) = a \iff \forall \varepsilon > 0,\ \exists M > 0 \text{ such that } x > M \Rightarrow |f(x) - a| < \varepsilon$$

Limits describe what happens "near"a point, not necessarily at the point!

Formal Definition of Limit with Examples*

1. Finite Limit at a Point:

$$\lim_{x \to 2} (3x + 1) = 7$$

$$\text{Let } \varepsilon = 0.1, \text{ then choose } \delta = \frac{0.1}{3} = 0.033\overline{3}$$

$$\text{If } 0 < |x - 2| < \delta, \text{ then } |(3x + 1) - 7| = 3|x - 2| < 0.1$$

2. Finite Limit at Infinity:

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\text{Let } \varepsilon = 0.1, \text{ then choose } M = \frac{1}{0.1} = 10$$

$$\text{If } x > 10, \text{ then } \left| \frac{1}{x} - 0 \right| = \frac{1}{x} < 0.1$$

Differentiation

- Limits allow us to ask:

  What happens to $f(x)$ when x changes just a little?

- This is the idea of a **marginal effect** in economics.
- **Example:** Suppose the price of eggs E depends on the price of chicken feed F:
$$E = f(F)$$
- If chicken feed gets slightly more expensive, how much more will eggs cost?
- This is captured by:
$$\lim_{\Delta F \to 0} \frac{\Delta E}{\Delta F} = \frac{dE}{dF}$$
- The limit turns a small change into a powerful tool — the **derivative**.

Formal Definition of Derivative

Definition:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- Measures the **instantaneous rate of change** of $f(x)$ at point $x = a$
- Also known as the slope of the **tangent line** to the graph of $f(x)$ at $x = a$
- In economics: tells us the **marginal effect** — how one variable responds to a tiny change in another

If the limit exists, the function is said to be differentiable at a.

Think Like an Economist: Marginal Analysis

A Walmart employee notices that the price of eggs E depends on the price of chicken feed F according to the function:

$$E = F^2$$

Question:

- Use the formal definition of the derivative to find the rate at which the egg price changes when the feed price changes.
- In other words, compute:
$$\frac{dE}{dF}$$

Hint: Use the definition:

$$\frac{dE}{dF} = \lim_{h \to 0} \frac{(F+h)^2 - F^2}{h}$$

Common Derivatives of Basic Functions

Function	**Notation**	**Derivative**
Constant	c	0
Power rule	x^n	nx^{n-1}
Reciprocal	$\frac{1}{x}$	$-\frac{1}{x^2}$
Square root	$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
Exponential	e^x	e^x
Logarithm	$\ln x$	$\frac{1}{x}$
Sine	$\sin x$	$\cos x$
Cosine	$\cos x$	$-\sin x$
Tangent	$\tan x$	$\sec^2 x$

Tip: Learn these by heart — they're everywhere in economics!

Application for Derivative 1: L'Hôpital's Rule

Problem: Some limits give indeterminate forms like:

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}$$

L'Hôpital's Rule: If

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

and f and g are differentiable near a, with $g'(x) \neq 0$, then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Example:

$$\lim_{x \to 0} \frac{\sin x}{x} \quad \text{(Apply L'Hôpital's Rule)}$$

Proof of L'Hôpital's Rule ($\frac{0}{0}$ case)*

Theorem: Let $f(x), g(x)$ be differentiable near a, with $f(a) = g(a) = 0$, and $g'(x) \neq 0$. If $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L$$

Proof Idea: (Using Cauchy's Mean Value Theorem)

By Cauchy's MVT, for x close to a, there exists $c \in (a, x)$ such that:

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g'(c)}$$

Since $f(a) = g(a) = 0$, this simplifies to:

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

As $x \to a$, then $c \to a$, so:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{c\to a} \frac{f'(c)}{g'(c)} = L$$

Application for Derivative 2: Taylor Series Expansion

Goal: Approximate a function using a polynomial near a point.

Taylor Series: If f is infinitely differentiable at $x = a$, then:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

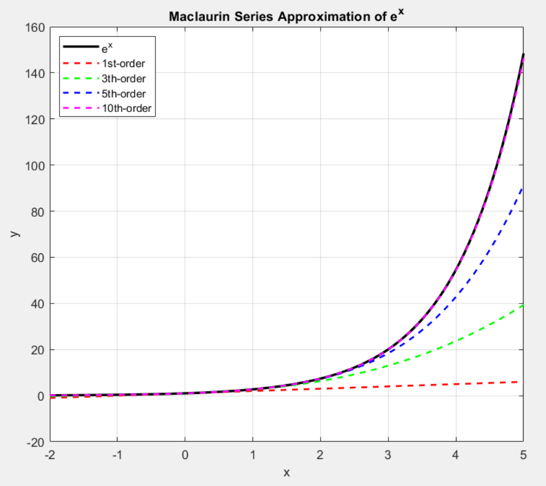
Special case (Maclaurin Series):

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

Example:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Application for Derivative 2: Taylor Series Expansion



Chain Rule (with Informal Proof)

Suppose:

$$y = f(u), \quad u = g(x) \quad \Rightarrow \quad y = f(g(x))$$

Chain Rule:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

Proof Sketch: (suppose the existance of g' and f')

$$\begin{aligned}
\frac{f(g(x+h)) - f(g(x))}{h} &\approx \frac{f(g(x) + g'(x)h) - f(g(x))}{h} \\
&= \frac{f(g(x) + g'(x)h) - f(g(x))}{g'(x)h} \cdot g'(x) \\
&= \frac{f(g(x) + \mu) - f(g(x))}{\mu} \cdot g'(x) \quad (\mu = g'(x)h \to 0) \\
&\to f'(g(x)) \cdot g'(x) \quad \text{as } h \to 0
\end{aligned}$$

Try to prove (f(x)g(x))' = f'(x)g(x) + f(x)g'(x)

We used a linear approximation based on Taylor for $g(x+h) \approx g(x) + g'(x)h$.

Differentiation as Linear Approximation

What is differentiation?
It tells us how a small change in x leads to a small change in $f(x)$.

Differential form:

$$dy = f'(x) \cdot dx$$

This expresses a small change in the output (dy) as approximately proportional to a small change in the input (dx), scaled by the slope $f'(x)$.

Interpretation:

- dx: a small input change
- dy: the corresponding output change
- $f'(x)$: the rate of change — what we have learnt before

This linear approximation is the foundation of marginal analysis in economics.

From Derivative to Integral

A Walmart employee observes:

"For every \$x increase in chicken feed price, egg price increases by \$2x.

That is, the rate of change of egg price with respect to feed price is:

$$\frac{dE}{dF} = 2x \quad \Rightarrow \quad E'(F) = 2x$$

Question: Can we recover the relationship between egg price E and feed price F? That is, what is the original function $E(F)$?

From Derivative to Integral

Answer: Integration!

$$f(x) = \int f'(x)\,dx + C$$

Integration "undoes"differentiation — it adds up small changes to find the total change.

Formal Definition of Integral*

Let $f(x)$ be a real-valued function defined on the interval $[a, b]$.
Step 1: Partition the interval in any way

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b \quad \text{with} \quad \Delta x_i = x_i - x_{i-1}$$

Step 2: Choose sample points Pick $c_i \in [x_{i-1}, x_i]$
Step 3: Form the Riemann sum

$$\sum_{i=1}^{n} f(c_i) \cdot \Delta x_i$$

Step 4: Take the limit

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(c_i) \cdot \Delta x_i$$

The integral is the limit of the sum of rectangles as they become infinitely thin.

Example $\int_0^1 2x\,dx$

Let $f(x) = 2x$ on $[0,1]$. Partition $[0,1]$ into n equal parts:

$$\Delta x = \frac{1}{n}, \quad x_i = \frac{i}{n}$$

Use right endpoints as sample points:

$$c_i = \frac{i}{n}, \quad \text{so } f(c_i) = 2 \cdot \frac{i}{n}$$

Form the Riemann sum:

$$\sum_{i=1}^{n} f(c_i) \cdot \Delta x = \sum_{i=1}^{n} 2 \cdot \frac{i}{n} \cdot \frac{1}{n} = \frac{2}{n^2} \sum_{i=1}^{n} i$$

Use the formula $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$:

$$= \frac{2}{n^2} \cdot \frac{n(n+1)}{2} = \frac{n+1}{n}$$

Take the limit as $n \to \infty$:

$$\int_0^1 2x\,dx = \lim_{n\to\infty} \frac{n+1}{n} = 1$$

Fundamental Theorem of Calculus (Newton–Leibniz)

Let f be a continuous function on $[a, b]$, and let

$$F(x) = \int_a^x f(t)\,dt$$

Then:

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x) \quad \text{(FTC Part I)}$$

And if F is any antiderivative of f, that is $F'(x) = f(x)$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{(FTC Part II)}$$

Integration and differentiation are inverse operations.

Proof Sketch of the Fundamental Theorem*

Let:

$$F(x) = \int_a^x f(t)\,dt \quad \text{(Define the accumulated area from } a \text{ to } x\text{)}$$

Step: Compute $F'(x)$ using the definition of derivative:

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{1}{h} \left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \right]$$

By the additivity of integrals:

$$= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt$$

If f is continuous, then:

$$\int_x^{x+h} f(t)\,dt \approx f(x) \cdot h \Rightarrow F'(x) = f(x)$$

So the derivative of the accumulated area is just the height at that point.

Common Indefinite Integrals (Antiderivatives)

Function	**Indefinite Integral**	**Comment**
x^n	$\frac{x^{n+1}}{n+1} + C$	$n \neq -1$
$\frac{1}{x}$	$\ln \lvert x \rvert + C$	$x \neq 0$
e^x	$e^x + C$	Exponential
a^x	$\frac{a^x}{\ln a} + C$	$a > 0,\ a \neq 1$
$\sin x$	$-\cos x + C$	Trigonometric
$\cos x$	$\sin x + C$	Trigonometric
$\sec^2 x$	$\tan x + C$	Useful
$\frac{1}{1+x^2}$	$\tan^{-1} x + C$	Inverse trig

These are essential for solving applied problems in economics.

Exercise

A consumer faces an exponential demand curve of orange:

$$P(x) = e^{-x} + 1$$

where $P(x)$ is the maximum price a consumer is willing to pay for quantity $x \in [0, 1]$.

The market price is fixed at $P = 1$.

Question: What is the **consumer surplus** when 1 unit is purchased?

Hint: Consumer surplus is the area between the demand curve and the price line:

$$\text{CS} = \int_0^1 P(x)\,dx - (1 \cdot 1) = \int_0^1 e^{-x} + 1\,dx - 1$$

Interpret: This is the total willingness to pay minus total expenditure.

Section 2

Vector Differential Calculus

Functions of Several Variables

A 4th-year PhD student in FRE is writing a dissertation on the **price of goat milk**.

He finds that the price of goat milk depends not only on supply and demand, but also on the prices of:

- **Substitute good:** cow milk
- **Complementary good:** cereal

Therefore: The goat milk price can be modeled as a function of multiple variables:

$$P = f(x, y) \quad \text{where } x = \text{price of cow milk}, \quad y = \text{price of cereal}$$

*To analyze this relationship, we need **multivariable calculus**.*

Example: Cobb–Douglas Production Function

In economics, output often depends on multiple inputs. One classic model is the **Cobb–Douglas production function**:

$$Q = f(K, L) = AK^{\alpha}L^{\beta}$$

where:

- Q = total output
- K = capital input
- L = labor input
- A, α, β = positive constants

This is a **function of two variables** — capital and labor.

Questions we can now ask:

- How does output change if we increase labor?
- What if capital stays fixed but labor changes?
- What is the marginal product of labor or capital?

Limit of a Multivariable Function

Sometimes, we consider the limit by letting only one variable change, while holding the others fixed.

This reduces to a **single-variable limit**, just like in Calculus I.

Example: Let $f(x, y) = x^2 + y^2$. What is $\lim_{x \to 0} f(x, 2)$?

We treat $y = 2$ as constant:

$$f(x, 2) = x^2 + 4 \quad \Rightarrow \quad \lim_{x \to 0} f(x, 2) = 0^2 + 4 = 4$$

But to evaluate the full limit $\lim_{(x,y) \to (a,b)} f(x, y)$*, we must consider* ***all directions****.*

Limit of a Multivariable Function

Let $f(x, y)$ be a function of two variables. We say the limit of $f(x, y)$ as $(x, y) \to (a, b)$ is L if:

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \quad \text{means} \quad \text{for every } \varepsilon > 0,\ \exists\delta > 0$$

$$\text{such that } \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x, y) - L| < \varepsilon$$

Key Point: The limit must be the same no matter how you approach (a, b)!

- From the left, right, above, below
- Along a line, a curve, a spiral, etc.

If the limit depends on the path, then the limit does not exist.

Examples: Does the Limit Exist?

Example 1: Limit Exists
Let

$$f(x, y) = x^2 + y^2$$

Then:

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0$$

Why? No matter how we approach $(0, 0)$, the value tends to 0.

Example 2: Limit Does Not Exist
Let

$$f(x, y) = \frac{2xy}{x^2 + y^2}, \quad f(0, 0) \text{ undefined}$$

Try two paths:

$$\text{Along } y = x : \quad f(x, x) = \frac{2x^2}{2x^2} = 1$$

$$\text{Along } y = -x : \quad f(x, -x) = \frac{-2x^2}{2x^2} = -1$$

Conclusion: Limit does not exist at $(0, 0)$

Partial Derivatives: Definition

Let $f(x, y)$ be a function of two variables.

We define the **partial derivative with respect to** x as:

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

This means: Treat y as a constant, and take the derivative of f with respect to x only.

Similarly, the **partial derivative with respect to** y is:

$$\frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Partial derivatives measure the rate of change in one direction, holding the other constant.

Partial Derivatives: Example

Consider a Cobb–Douglas production function:

$$Q(K, L) = AK^{\alpha}L^{\beta}$$

Partial derivative with respect to capital:

$$\frac{\partial Q}{\partial K} = A\alpha K^{\alpha-1}L^{\beta} \quad \Rightarrow \text{Marginal Product of Capital (MPK)}$$

Partial derivative with respect to labor:

$$\frac{\partial Q}{\partial L} = \textit{have a try} \quad \Rightarrow \text{Marginal Product of Labor (MPL)}$$

Tips: just regard another variable(s) as constant.

Partial derivatives represent marginal effects: how output responds to changes in one input while holding the other fixed.

Directional Derivative*

Question: Can we measure the rate of change of a function along any direction — not just along x- or y-axis?

Yes! This is called the **Directional Derivative**.

Let $f(x, y)$ be a differentiable function and let $\vec{v} = \langle a, b \rangle$ be a unit vector. The **directional derivative** of f at point (x_0, y_0) in the direction of $\vec{v}$ is:

$$D_{\vec{v}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ah,\, y_0 + bh) - f(x_0, y_0)}{h}$$

Interpretation: It tells us how fast f changes at a point if we move in direction $\vec{v}$.

If $\vec{v}$ points uphill, the directional derivative is large. If $\vec{v}$ is tangent to a level curve, the derivative is zero.

Directional Derivative: An Example*

Consider the production function:

$$Q(K, L) = AK^{\alpha}L^{\beta} \quad \text{with } A = 1,\ \alpha = 0.4,\ \beta = 0.6$$

A social planner plans to hire 2 units of labor and add 1 unit of capital at the same time. This corresponds to the direction vector:

$$\vec{v} = \langle 1, 2 \rangle$$

Let's evaluate the rate of output change at the point $(K, L) = (10, 20)$ in this direction.

Step 1: Compute gradient of Q

$$\nabla Q = \left\langle \frac{\partial Q}{\partial K}, \frac{\partial Q}{\partial L} \right\rangle = \left\langle 0.4K^{-0.6}L^{0.6},\ 0.6K^{0.4}L^{-0.4} \right\rangle$$

At $K = 10,\ L = 20$:

$$\nabla Q(10, 20) \approx \langle \text{value}_K,\ \text{value}_L \rangle$$

Directional Derivative: An Example*

Step 2: Compute unit direction vector

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1,\ 2 \rangle}{\sqrt{1^2 + 2^2}} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

Step 3: Directional derivative

$$D_{\vec{u}}Q = \nabla Q \cdot \vec{u} = 0.677 \cdot \frac{1}{\sqrt{5}} + 0.599 \cdot \frac{2}{\sqrt{5}} = \frac{0.677 + 1.198}{\sqrt{5}} \approx \frac{1.875}{2.236} \approx 0.839$$

Conclusion: Output increases at a rate of approximately 0.839 units per step in that direction.

Is There a L'Hôpital's Rule for Multivariable Functions?

Short Answer: No — there is no general multivariable version of L'Hôpital's Rule.

Why not?

- Multivariable limits must be approached from infinitely many directions.
- The limit may depend on the path — so the 1D logic of L'Hôpital fails.

Multivariable Taylor Series Expansion

Let $f(x, y)$ be a function with continuous partial derivatives. The Taylor expansion of f near point (a, b) is:

First-order (linear approximation):

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Second-order (quadratic approximation):

$$\begin{aligned} f(x, y) \approx f(a, b) &+ f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ &+ \frac{1}{2} f_{xx}(a, b)(x - a)^2 + \frac{1}{2} f_{yy}(a, b)(y - b)^2 \\ &+ f_{xy}(a, b)(x - a)(y - b) \end{aligned}$$

The Taylor expansion helps us approximate nonlinear surfaces with polynomials near a point.

Multivariable Taylor Series Expansion of e^{xy}

Let $f(x,y) = e^{xy}$, and expand around point $(0,0)$.

Zeroth order (constant term):

$$f(0,0) = e^0 = 1$$

First-order approximation:

$$f(x,y) \approx 1 + \left.\frac{\partial f}{\partial x}\right|_{(0,0)} x + \left.\frac{\partial f}{\partial y}\right|_{(0,0)} y$$

$$\frac{\partial f}{\partial x} = ye^{xy}, \quad \frac{\partial f}{\partial y} = xe^{xy} \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 0, \quad \left.\frac{\partial f}{\partial y}\right|_{(0,0)} = 0$$

$$\Rightarrow \boxed{f(x,y) \approx 1} \quad \text{(First-order)}$$

Multivariable Taylor Series Expansion (General Form)*

Let $f : \mathbb{R}^d \to \mathbb{R}$ be smooth, and expand around point $\mathbf{a} = (a_1, \ldots, a_d)$. Then:

$$T(x_1, \ldots, x_d) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} \frac{(x_1 - a_1)^{n_1} \cdots (x_d - a_d)^{n_d}}{n_1! \cdots n_d!} \left(\frac{\partial^{n_1 + \cdots + n_d} f}{\partial x_1^{n_1} \cdots \partial x_d^{n_d}} \right) (\mathbf{a})$$

$$= f(\mathbf{a}) + \sum_{j=1}^{d} \frac{\partial f(\mathbf{a})}{\partial x_j}(x_j - a_j) + \frac{1}{2!} \sum_{j=1}^{d} \sum_{k=1}^{d} \frac{\partial^2 f(\mathbf{a})}{\partial x_j \partial x_k}(x_j - a_j)(x_k - a_k)$$

$$+ \frac{1}{3!} \sum_{j=1}^{d} \sum_{k=1}^{d} \sum_{l=1}^{d} \frac{\partial^3 f(\mathbf{a})}{\partial x_j \partial x_k \partial x_l}(x_j - a_j)(x_k - a_k)(x_l - a_l) + \cdots$$

This series captures the behavior of f around $\mathbf{a}$ using all mixed partial derivatives.

Multivariable Taylor Series Expansion of e^{xy}

Second-order approximation:

$$f(x, y) \approx 1 + \frac{1}{2} f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \frac{1}{2} f_{yy}(0,0)y^2$$

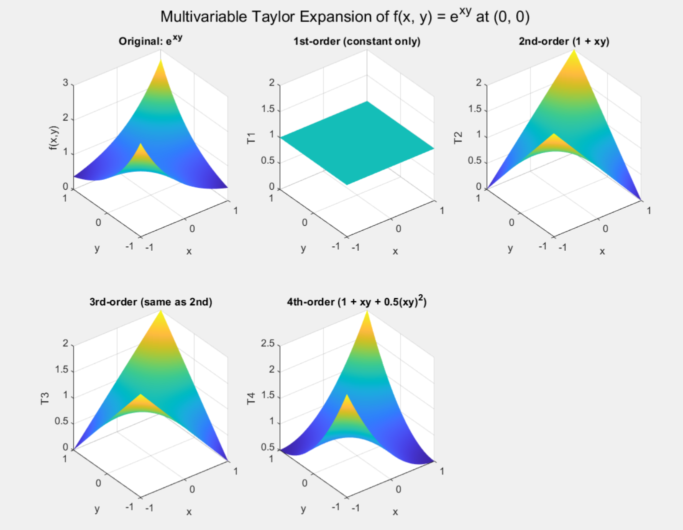
Compute second derivatives:

$$f_{xx} = y^2 e^{xy}, \quad f_{yy} = x^2 e^{xy}, \quad f_{xy} = (1 + xy)e^{xy}$$

$\Rightarrow f_{xx}(0,0) = 0,\ f_{yy}(0,0) = 0,\ f_{xy}(0,0) = 1$

$$\Rightarrow \boxed{f(x, y) \approx 1 + xy} \quad \text{(Second-order)}$$

Multivariable Taylor Series Expansion



Parametric equation

Suppose you have a scalar function of three variables:

$$f(x, y, z)$$

and each of x, y, z depends on parameters u and v:

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

Then the composed function is:

$$F(u, v) = f(x(u, v),\ y(u, v),\ z(u, v))$$

Chain Rule:

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} \quad \text{(similar for } \partial F/\partial v\text{)}$$

Example:
Let

$$f(x, y, z) = x + y^2 + z^3, \quad x = u^2,\ y = uv,\ z = \sin v$$

Then

$$F(u, v) = f(u^2,\ uv,\ \sin v) = u^2 + (uv)^2 + \sin^3 v$$

This structure appears in surface integrals, parameterized geometry, and transformation rules in economics and physics.

Implicit Function

In a perfectly competitive market for eggs, profit is zero. Suppose the market equilibrium is governed by the equation:

$$A(E,F) \cdot D(A(E,F)) - C_E(E) - C_F(F) = 0$$

Where:

- $A(E,F)$: Egg production as a function of feed input E and barn capital F
- $D(A)$: Demand curve (price as a function of quantity)
- $C_E(E), C_F(F)$: Cost functions of feed and barn input

Question: If we slightly increase feed E, can we decrease barn capital F while maintaining equilibrium?

Answer: Use implicit differentiation:
Let $\Phi(E,F) = A(E,F)D(A(E,F)) - C_E(E) - C_F(F)$
Then at equilibrium:

$$\frac{dF}{dE} = -\frac{\frac{\partial \Phi}{\partial E}}{\frac{\partial \Phi}{\partial F}}$$

If $\frac{dF}{dE} < 0$, then more feed allows for less barn input — they are substitutable.

Implicit Function

Consider an equation:

$$F(x, y) = 0$$

Implicit Function Theorem (2D Case):
If F is continuously differentiable near (x_0, y_0), and

$$F(x_0, y_0) = 0, \quad \frac{\partial F}{\partial y}(x_0, y_0) \neq 0$$

then there exists a function $y = f(x)$, defined near x_0, such that:

$$F(x, f(x)) = 0 \quad \text{and} \quad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

This allows us to differentiate implicitly defined relationships.

Why Does It Work? Total Differentiation!

Suppose we have an equation involving two variables:

$$F(x, y) = 0 \quad \text{(defines } y \text{ implicitly as a function of } x\text{)}$$

We apply **total differentiation** to both sides:

$$dF = F_x \, dx + F_y \, dy = 0$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Conclusion: Even if y is not given explicitly, we can study how it changes with x by differentiating both sides!

This is the foundation for many economic comparative statics and equilibrium sensitivity analyses. It will occur in your core exam in 99% probability.

Example: Feed vs. Barn — Total Differentiation

Equilibrium Condition:

$$\Phi(E,F) = A(E,F) \cdot D(A(E,F)) - p_E E - p_F F = 0$$

Assume:

$$A(E,F) = E^{0.5}F^{0.5}, \quad D(A) = e^{-A}$$

Then:

$$\Phi(E,F) = \underbrace{E^{0.5}F^{0.5} \cdot e^{-E^{0.5}F^{0.5}}}_{\text{Revenue}} - p_E E - p_F F$$

Take total differential:

$$d\Phi = \frac{\partial \Phi}{\partial E} dE + \frac{\partial \Phi}{\partial F} dF = 0 \Rightarrow \frac{dF}{dE} = -\frac{\Phi_E}{\Phi_F}$$

Interpretation: This tells us: how much can we decrease barn input F if we increase feed E, while keeping equilibrium unchanged?

The sign of $\frac{dF}{dE}$ reveals whether E and F are substitutes or compliment.

Substitution Between Feed and Barn: Analytical Result

Recall the equilibrium condition:

$$\Phi(E,F) = E^{0.5}F^{0.5} \cdot e^{-E^{0.5}F^{0.5}} - p_E E - p_F F$$

Let $A = \sqrt{EF}$, then:

$$\frac{\partial \Phi}{\partial E} = (1-A)e^{-A} \cdot \frac{F^{0.5}}{2E^{0.5}} - p_E$$

$$\frac{\partial \Phi}{\partial F} = (1-A)e^{-A} \cdot \frac{E^{0.5}}{2F^{0.5}} - p_F$$

By total differentiation:

$$\frac{dF}{dE} = -\frac{\Phi_E}{\Phi_F} = -\frac{(1-A)\,e^{-A} \cdot \frac{F^{0.5}}{2E^{0.5}} - p_E}{(1-A)\,e^{-A} \cdot \frac{E^{0.5}}{2F^{0.5}} - p_F}$$

Gradient and Divergence

Gradient (∇f): Measures the direction and rate of steepest increase of a scalar function $f(x, y, z)$.

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

- Points in the direction of greatest increase of f. - Magnitude tells how fast f increases in that direction.

Divergence ($\nabla \cdot \vec{F}$): Applies to a vector field $\vec{F} = (F_1, F_2, F_3)$. Measures how much the field spreads out from a point.

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

- Positive divergence: source (outflow) - Negative divergence: sink (inflow)

Section 3

Vector Integral Calculus

Citrus Farm Questions

Story: Dr. Smith is an Extension Specialist at UF's Citrus Research Center. One day, a farmer comes to him with a few questions:

- His orange grove lies on a sloped hillside.
- He wants to **build a fence** around it — but how long should it be?
- He wants to know the **area** of the grove.
- He also stores supplies in a **warehouse** — what is its volume?

Three Questions' Answers — Three Integrals

Dr. Smith realizes that each question requires a different type of integral:

- **Fence length** → *Line integral* (length along a curve)
- **Land area** → *Surface (area) integral*
- **Warehouse volume** → *Triple integral (volume)*

One powerful idea: Integrals help us measure length, area, and volume — even in curved, sloped, or irregular regions.

And there's more:

- What if the fence has varying **density**? → Line integrals with a *weight function* (e.g., mass or cost per unit length)
- What if the goods in the warehouse have **varying density**? → Volume integrals of scalar fields (e.g., he wants to compute the *total weight*)

Let's explore them one by one.

Line Integrals

What is a Line Integral? A line integral adds up values of a function along a curve — for example, computing cost, work, or length.

Scalar Line Integral: If a scalar function $f(x, y)$ (you can extend into whatever dimensions) is defined along a smooth curve C, then

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \cdot \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Vector Line Integral (Extension)*:

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(x(t), y(t)) \cdot \vec{r}\,'(t)\, dt$$

Where $\vec{F} = (F_1, F_2)$ and $\vec{r}(t) = (x(t), y(t))$

Steps to Compute:

1. Parameterize the curve: $\vec{r}(t) = (x(t), y(t))$, $t \in [a, b]$
2. Compute $\vec{r}\,'(t)$ and evaluate $f(x(t), y(t))$
3. Plug into the integral formula and integrate over t

Line Integral: Example

Curve: Let the path be parameterized by

$$\vec{r}(t) = (\sin t, \cos t, t), \quad t \in [0, \pi]$$

Question: What is the length of this curve? That is, compute:

$$\int_C ds = \int_0^\pi \|\vec{r}\,'(t)\| \, dt$$

Step 1: Compute derivative

$$\vec{r}\,'(t) = (\cos t, -\sin t, 1)$$

Step 2: Compute magnitude

$$\|\vec{r}\,'(t)\| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{1+1} = \sqrt{2}$$

Step 3: Integrate

$$\int_0^\pi \sqrt{2} \, dt = \sqrt{2} \cdot \pi$$

Answer: The length of the curve is $\boxed{\sqrt{2} \cdot \pi}$

It allows us to compute the total value of a quantity (like area, mass, or flux) over a curved surface.

Suppose: A surface is parameterized by

$$\vec{r}(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$$

Then: For a scalar field $f(x,y,z)$, the surface integral is:

$$\iint_S f(x,y,z)\,dS = \iint_D f(x(u,v), y(u,v), z(u,v)) \cdot \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| du\,dv$$

How to compute the cross product:

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \quad \Rightarrow \quad \text{take the norm: } \|\cdot\|$$

Each small patch has a vector area — weighted by f

Surface Integral: Example

Problem: What is the actual surface area of the grove?

$$z = 4 - x^2 - y^2, \quad \text{with boundary } x = \cos t,\ y = \sin t,\ t \in [0, 2\pi]$$

Step 1: Parameterize the surface:

$$\vec{r}(r,t) = \langle r\cos t,\ r\sin t,\ 4 - r^2\rangle, \quad r \in [0,1],\ t \in [0, 2\pi]$$

Compute partial derivatives:

$$\vec{r}_r = \langle \cos t,\ \sin t,\ -2r\rangle, \quad \vec{r}_t = \langle -r\sin t,\ r\cos t,\ 0\rangle$$

Step 2: Cross product:

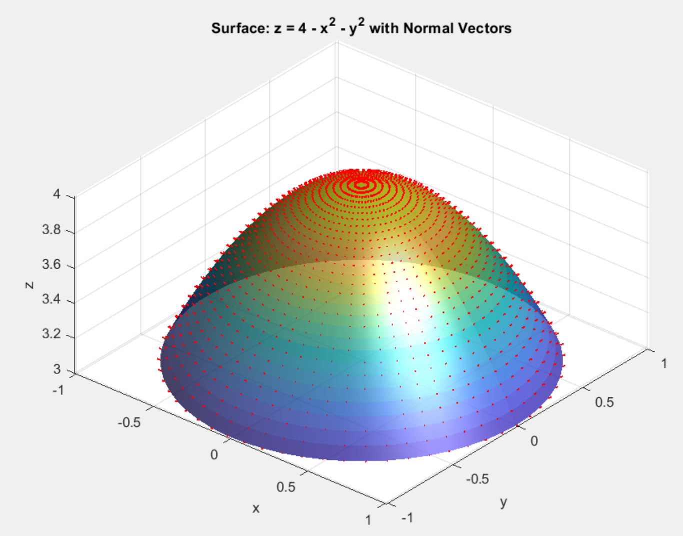
$$\vec{r}_r \times \vec{r}_t = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos t & \sin t & -2r \\ -r\sin t & r\cos t & 0 \end{vmatrix} = \langle 2r^2\cos t,\ 2r^2\sin t,\ r\rangle$$

$$\|\vec{r}_r \times \vec{r}_t\| = \sqrt{4r^4 + r^2} = r\sqrt{4r^2 + 1}$$

Step 3: Surface area integral (Try):

$$A = \int_0^{2\pi} \int_0^1 r\sqrt{4r^2+1}\, dr\, dt$$

Surface Integral: Example



Triple Integrals

If we only want to know the volumne.
Idea: Break the volume into tiny boxes (like 3D rectangles), sum them up, and take the limit:

$$V = \iiint_D 1\, dV$$

More generally, to integrate a scalar field (i.e. density) $f(x, y, z)$ over a 3D region D:

$$\iiint_D f(x, y, z)\, dx\, dy\, dz$$

Interpretation:

- If $f(x, y, z) = 1$: you're computing volume.
- If f is density: you're computing total mass.

Just like double integrals — but now in 3D!

Triple Integral: Warehouse Volume

Scenario: Dr. Smith finds the farmer's warehouse sits beneath a dome-shaped hill:

$$z = 4 - x^2 - y^2, \quad \text{with } x^2 + y^2 \le 1$$

Step 1: Set up the triple integral

$$V = \iiint_D 1\, dz\, dx\, dy \quad \text{where } D = \{(x, y, z) : x^2 + y^2 \le 1,\ 0 \le z \le 4 - x^2 - y^2\}$$

Step 2: Switch to cylindrical coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \quad r \in [0, 1],\ \theta \in [0, 2\pi]$$

$$\Rightarrow V = \int_0^{2\pi} \int_0^1 \int_0^{4-r^2} r\, dz\, dr\, d\theta$$

Step 3: Compute inner integral:

$$V = \int_0^{2\pi} \int_0^1 [rz]_0^{4-r^2}\, dr\, d\theta = \int_0^{2\pi} \int_0^1 r(4 - r^2)\, dr\, d\theta$$

Three Theorems That Simplify Integrals*

These integrals can not be solved easily.
However, there are three powerful theorems that **may** help:

- **Green's Theorem** (planar region with boundary curve)
- **Stokes' Theorem** (surface with boundary curve)
- **Gauss' Divergence Theorem** (volume with surface)

Each theorem converts a "big" integral into an "edge" integral — often making calculations easier.

Green's Theorem*

Statement: Let C be a positively oriented, simple closed curve in the plane, and R the region it encloses. If $\vec{F} = (P(x, y), Q(x, y))$ is continuously differentiable:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

Interpretation: Circulation around the boundary C equals the total "microscopic rotation"inside R.

Applications:

- Compute work done around a closed path
- Transform line integrals into double integrals

Example: Green's Theorem

Problem: Evaluate the line integral

$$\oint_C (x^2 - y)\,dx + (x + y^2)\,dy$$

where C is the positively oriented unit circle $x^2 + y^2 = 1$.

Solution via Green's Theorem:

$$P = x^2 - y, \quad Q = x + y^2 \Rightarrow \frac{\partial Q}{\partial x} = 1, \quad \frac{\partial P}{\partial y} = -1$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (1 - (-1))\; dx\,dy = 2 \cdot \text{Area of Unit Disk} = 2\pi$$

Answer: $\boxed{2\pi}$

Stokes' Theorem*

Statement: Let S be a smooth, oriented surface with boundary curve $C = \partial S$, and $\vec{F}$ a vector field:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$

Here, $\hat{n}$ is a unit normal vector of S; ∇ means to compute the grad.

Interpretation: Circulation around the boundary curve equals the total curl over the surface.

Applications:

- Generalizes Green's Theorem to 3D
- Used in fluid flow, electromagnetism

Example: Stokes' Theorem

Problem: Let $\vec{F} = (-y, x, 0)$. Let C be the unit circle in the xy-plane:

$$C : \vec{r}(t) = (\cos t, \sin t, 0), \quad t \in [0, 2\pi]$$

Compute the line integral:

$$\oint_C \vec{F} \cdot d\vec{r}$$

Stokes says:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$

Compute the curl:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \left(0, 0, \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right) = (0, 0, 1 + 1) = (0, 0, 2)$$

Over the flat disk (0,0,1) being the $\hat{n}$:

$$\iint_S 2\, dS = 2 \cdot \pi = \boxed{2\pi}$$

Gauss's Theorem (Divergence Theorem)*

Statement: Let V be a solid region bounded by surface S (with outward normal). For a vector field $\vec{F}$:

$$\iiint_V (\nabla \cdot \vec{F})\, dV = \iint_S \vec{F} \cdot \hat{n}\, dS$$

Interpretation: Total divergence (outflow) inside a region equals the flux through the boundary surface.

Applications:

- Compute flux without surface integration
- Widely used in physics (e.g., Gauss's Law)

Example: Gauss's Theorem

Problem: Let $\vec{F} = (x, y, z)$, and let V be the solid unit ball: $x^2 + y^2 + z^2 \leq 1$. Find the flux $\iint_S \vec{F} \cdot \hat{n}\, dS$, where $S = \partial V$.

Solution via Gauss's Theorem:

$$\nabla \cdot \vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3 \Rightarrow \iiint_V 3\, dV = 3 \cdot \text{Volume of unit ball} = 3 \cdot \frac{4}{3}\pi = \boxed{4\pi}$$

Summary and What's Next

In this session, we:

- Reviewed key concepts in **single-variable calculus**
- Explored tools of **multivariable calculus**:

Up next: *Static Optimization* — finding maximum and minimum values of functions under constraints. A key tool in economics and decision-making.

See you next time!

References

- Kreyszig, E. (2011). *Advanced Engineering Mathematics* (10th ed.).
- Wikipedia contributors. Various articles. Retrieved from `https://en.wikipedia.org`
- Images and figures generated using:
  - MATLAB R2024a
  - OpenAI's ChatGPT-4o image generation