Projective 2D geometry

Lecture 3

A Hierarchy of Transformations

We would like to examine the following hierarchy of transformations:

$$\text{Euclidean} \subset \text{Similarity} \subset \text{Affine} \subset \text{Projective}$$

Projective Transformations

We have seen that projective transformations under composition form a group. This group is called the *projective linear group*.

In the projective plane $\mathbb{P}^2$, projective linear group is denoted as $PL(3)$. The elements of this group are non-singular 3×3 matrices with real entries.

An element of $PL(3)$ is of the form

$$H_P = \begin{pmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ v_1 & v_2 & v \end{pmatrix}.$$

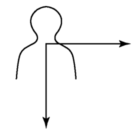
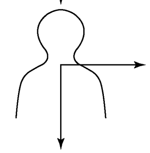
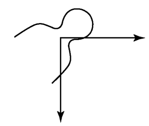
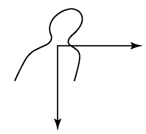
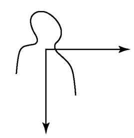
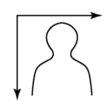
In block matrix form H_P can be written as

$$H_P = \begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^T & v \end{pmatrix}$$

where $\mathbf{A}$ is a 2×2 non-singular matrix, $\mathbf{t}$ a translation 2-vector, and $\mathbf{v} = (v_1, v_2)^T$.

- The most fundamental projective invariant is the cross ratio of four collinear points.
- A projective transformation H_P has 8 degrees of freedom.
- H_P can be computed from 4 point correspondence.

- **Transforms:** rotation, scaling, translation, shear, prospective projection.

Type	Affine Matrix, T	Coordinate Equations	Diagram
Identity	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x = w$ $y = z$	
Scaling	$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x = s_x w$ $y = s_y z$	
Rotation	$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x = w\cos\theta - z\sin\theta$ $y = w\sin\theta + z\cos\theta$	
Shear (horizontal)	$\begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x = w + \alpha z$ $y = z$	
Shear (vertical)	$\begin{bmatrix} 1 & \beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x = w$ $y = \beta w + z$	
Translation	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \delta_x & \delta_y & 1 \end{bmatrix}$	$x = w + \delta_x$ $y = z + \delta_y$	

Special Cases of a Projective Transformation

Affine Transformations

A subgroup of $PL(3)$ consisting of matrices having last row $(0, 0, 1)$ is called an affine group. An element of an affine group is the form

$$H_A = \begin{pmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{pmatrix}.$$

In block matrix form H_A can be written as

$$H_A = \begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix}$$

where $\mathbf{A}$ is a 2×2 non-singular matrix, $\mathbf{t}$ a translation 2-vector, and $\mathbf{0}$ a null 2-vector. The matrix A can be decomposed as $\mathbf{A} = R(\theta)\, R(-\phi)\, D\, R(\phi)$ where $R(\theta)$ and $R(\phi)$ are rotations by θ and ϕ respectively, and D is a diagonal matrix

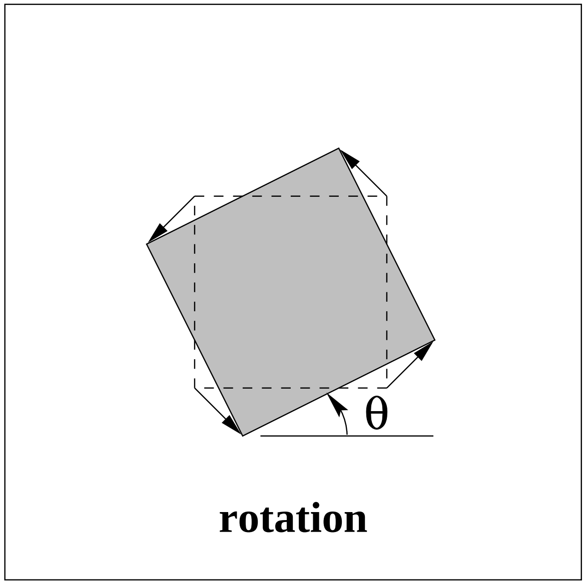
$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

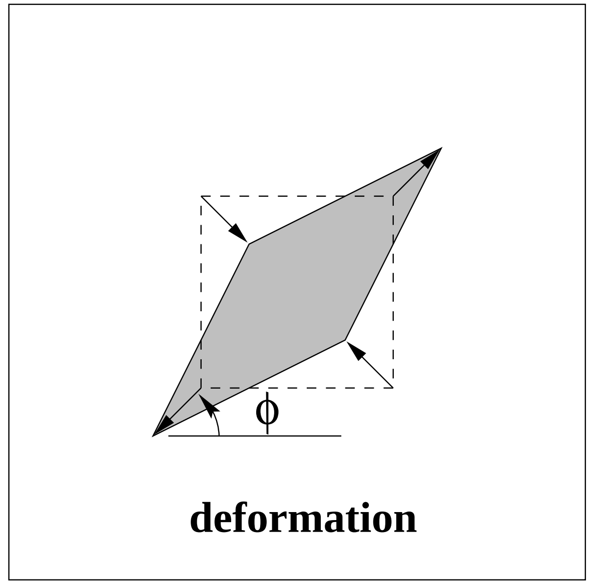
- Parallel lines, ratio of lengths of parallel line segments, and ratio of areas are three important invariant under affine transformation.

- An affine transformation H_A has 6 degrees of freedom.

- H_A can be computed from 3 point correspondence.

- **Transforms:** rotation, scaling, translation, shear.





Distortions from a affine transformation.

Rotation by $R(\theta)$, and a deformation by

$R(-\phi)\,D\,R(\phi)$.

Similarity Transformations

Similarity transformations are a subset of affine transformations. A similarity transformation is the form

$$H_S = \begin{pmatrix} s\ \mathsf{cos}(\theta) & -s\ \mathsf{sin}(\theta) & t_x \\ s\ \mathsf{sin}(\theta) & s\ \mathsf{cos}(\theta) & t_y \\ 0 & 0 & 1 \end{pmatrix}.$$

In block matrix form H_S can be written as

$$H_S = \begin{pmatrix} s\,\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix}$$

where $\mathbf{R}$ is a 2×2 non-singular matrix called a rotation matrix, s an isotropic scaling, $\mathbf{t}$ a translation 2-vector, and $\mathbf{0}$ a null 2-vector. The matrix R is given by

$$\mathbf{R} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

- Angles between lines, ratio of two lengths, ratio of areas are invariant under similarity transformation.

- A similarity transformation H_S has 4 degrees of freedom.

- H_S can be computed from 2 point correspondence.

- **Transforms:** rotation, scaling, translation.

Euclidean Transformations

Euclidean transformations are a subset of similarity transformations. A Euclidean transformation is the form

$$H_E = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & t_x \\ \sin(\theta) & \cos(\theta) & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

Euclidean transformations preserve distance and hence are also known as *isometries*.

In block matrix form H_E can be written as

$$H_E = \begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix}$$

where $\mathbf{R}$ is a rotation matrix, $\mathbf{t}$ a translation 2-vector, and $\mathbf{0}$ a null 2-vector.

- Angles between lines, length between two points, and area are invariant under Euclidean transformations.

- A Euclidean transformation H_E has 3 degrees of freedom.

- H_E can be computed from 2 point correspondence.

- **Transforms:** rotation, translation.

Decomposition of H

A projective transformation H can be decomposed into a chain of transformations, where each matrix in the chain represents a transformation higher in the hierarchy than the previous one.

$$H = \begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^T & v \end{pmatrix} = \begin{pmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^T & v \end{pmatrix} = H_S \, H_A \, H_P$$

The decomposition is valid if $v \neq 0$ and unique if $s > 0$.

Example The projective transformation

$$H = \begin{pmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{pmatrix}$$

may be decomposed as

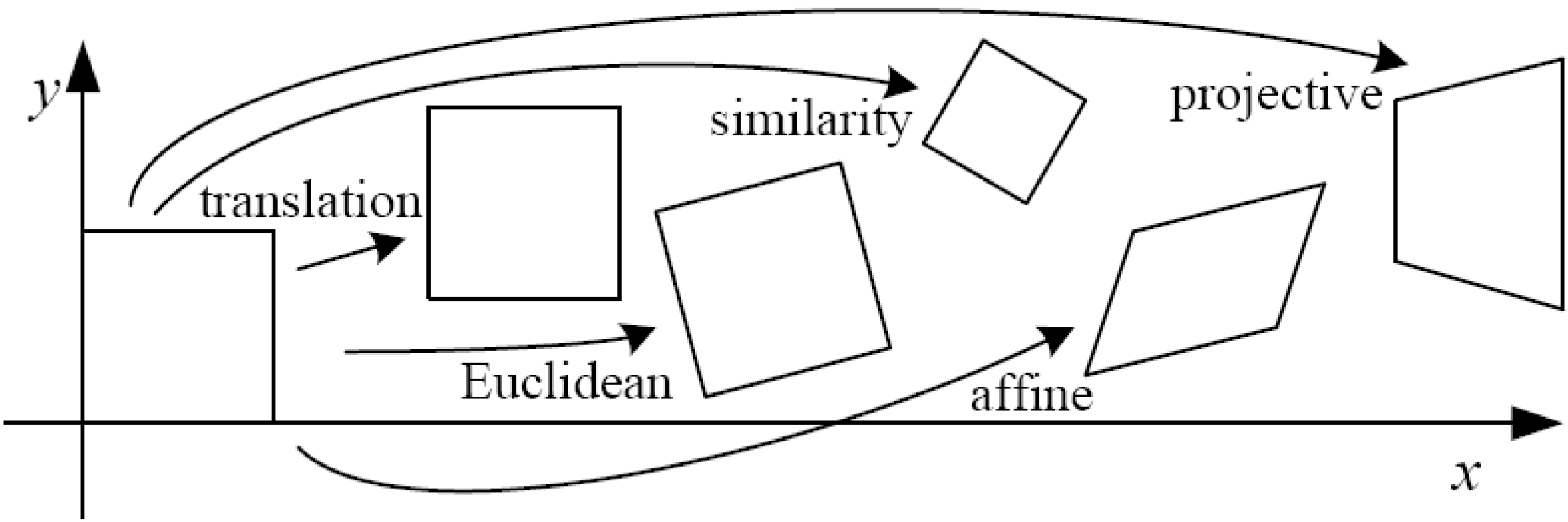
$$H = \begin{pmatrix} 2\cos 45^o & -2\sin 45^o & 1 \\ 2\sin 45^o & 2\cos 45^o & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

Number of Invariants

Result 1.7. The number of functionally independent invariants is equal to or greater than the number of degrees of freedom of the configuration minus the number of degrees of freedom of the transformation.

Example 1. The number of invariants for a configuration of 4 points under an affine transformation is at least $8 - 6 = 2$.

Overview of 2D transformations



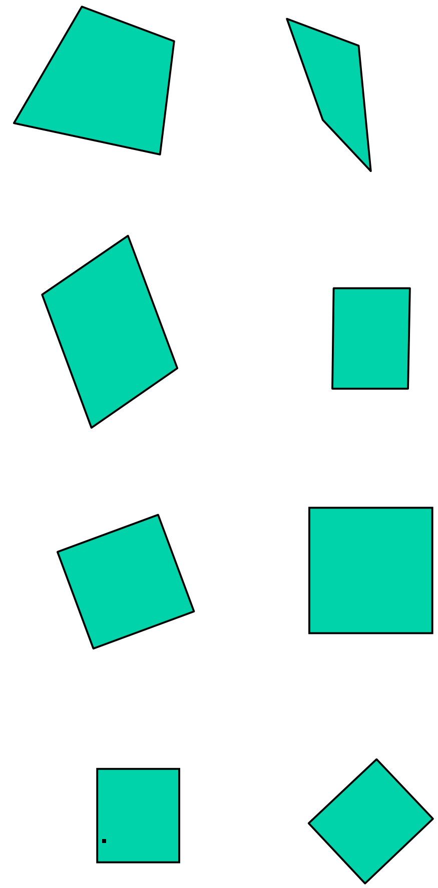
Overview of transformations

Projective
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

Affine
$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Similarity
$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Euclidean
$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Projective Geometry of 1D

The projective geometry of a line, $\mathbb{P}$, can be developed similar to $\mathbb{P}^2$. A point x on the line is represented by homogeneous coordinates $(x_1, x_2)^T$. A point for which $x_2 = 0$ is called an ideal point or point at infinity. The point $(x_1, x_2)^T$ will be denoted by the 2-vector $\bar{\mathbf{x}}$.

- A projective transformation on a line is represented by a 2×2 homogeneous matrix $H_{2\times2}$ and has 3 degrees of freedom.
- A projective transformation on a line can be determined from a three points correspondence.

Cross Ratio

Given four points $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$, $\overline{\mathbf{x}}_3$ and $\overline{\mathbf{x}}_4$, the cross ratio is defined as

$$Cross(\overline{\mathbf{x}}_1, \overline{\mathbf{x}}_2, \overline{\mathbf{x}}_3, \overline{\mathbf{x}}_4) = \frac{|\overline{\mathbf{x}}_1\,\overline{\mathbf{x}}_2|\;\;|\overline{\mathbf{x}}_3\,\overline{\mathbf{x}}_4|}{|\overline{\mathbf{x}}_1\,\overline{\mathbf{x}}_3|\;\;|\overline{\mathbf{x}}_2\,\overline{\mathbf{x}}_4|}$$

where

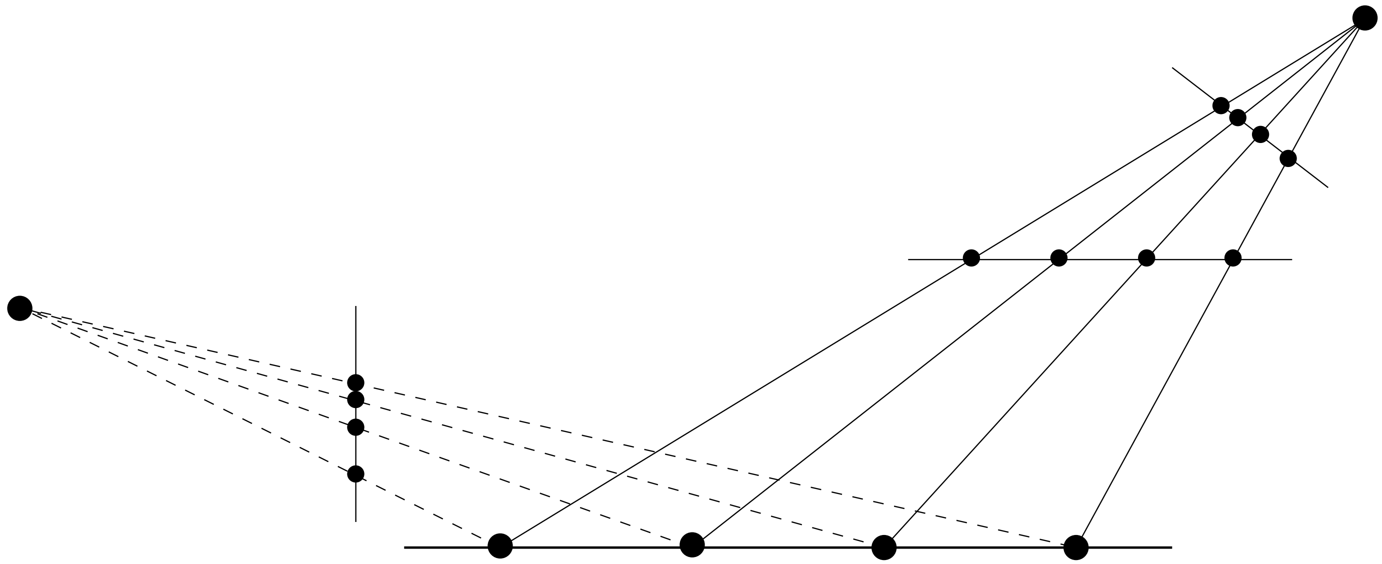
$$|\overline{\mathbf{x}}_i\,\overline{\mathbf{x}}_j| = \begin{vmatrix} x_{i1} & x_{j1} \\ x_{i2} & x_{j2} \end{vmatrix}.$$

Cross ratio simplifies to

$$
\begin{aligned}
&Cross(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4) \\
&\qquad = \frac{(x_{11}\,x_{22} - x_{21}\,x_{12})(x_{31}\,x_{42} - x_{41}\,x_{32})}{(x_{11}\,x_{32} - x_{31}\,x_{12})(x_{21}\,x_{42} - x_{41}\,x_{22})} \\
&\qquad = \frac{(x_{12}\,x_{22})(x_{32}\,x_{42})}{(x_{12}\,x_{32})(x_{22}\,x_{42})} \frac{\left(\frac{x_{11}}{x_{12}} - \frac{x_{21}}{x_{22}}\right)\left(\frac{x_{31}}{x_{32}} - \frac{x_{41}}{x_{42}}\right)}{\left(\frac{x_{11}}{x_{12}} - \frac{x_{31}}{x_{32}}\right)\left(\frac{x_{21}}{x_{22}} - \frac{x_{41}}{x_{42}}\right)} \\
&\qquad = \frac{\left(\frac{x_{11}}{x_{12}} - \frac{x_{21}}{x_{22}}\right)\left(\frac{x_{31}}{x_{32}} - \frac{x_{41}}{x_{42}}\right)}{\left(\frac{x_{11}}{x_{12}} - \frac{x_{31}}{x_{32}}\right)\left(\frac{x_{21}}{x_{22}} - \frac{x_{41}}{x_{42}}\right)} \\
&\qquad = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_3)(x_2 - x_4)}.
\end{aligned}
$$

Some Facts about Cross Ratio

- The value of cross ratio is independent of which homogeneous coordinates is used.
- If the second coordinate is 1 for each of the point $\overline{\mathbf{x}}_i$ and $\overline{\mathbf{x}}_j$, then $|\overline{\mathbf{x}}_i \, \overline{\mathbf{x}}_j|$ represents the signed distance from $\overline{\mathbf{x}}_i$ and $\overline{\mathbf{x}}_j$.
- The value of the cross ratio is invariant under any projective transformation of the line.



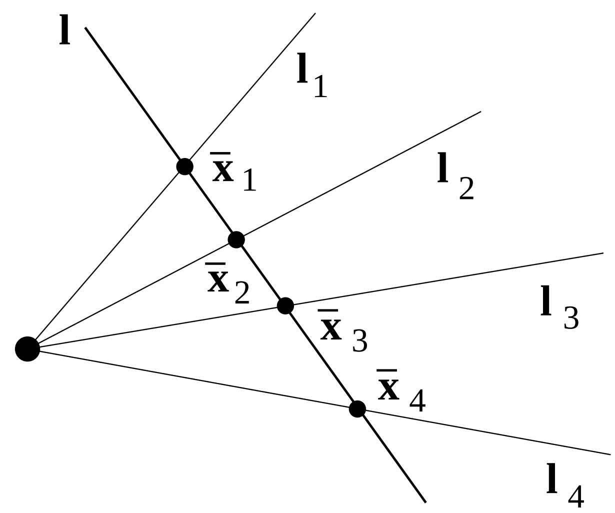
Projective transformations between lines

Each set of four collinear points is related to the others by a line-to-line projectivity, and the value of the cross ratio of each set is same.

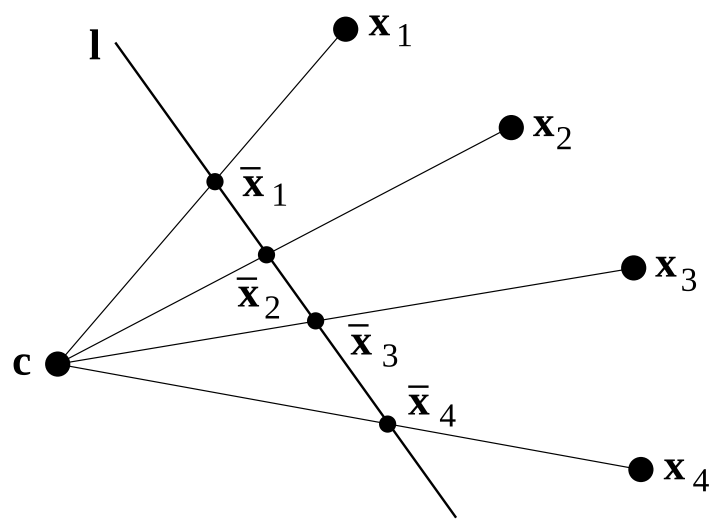
Concurrent lines

In projective projective plane $\mathbb{P}^2$ we have seen that lines are dual to points.

In projective line $\mathbb{P}$, lines are also dual to points.



A configuration of concurrent lines is a dual to collinear points on a line. Hence the value of the cross ratio of these lines is given by the cross ratio of the points.



If $\mathbf{c}$ represents a camera center, and the line ℓ represents an image line, then the points $\overline{\mathbf{x}}_i$ are the projections of points $\mathbf{x}_i$ into the image line.

- The cross ratio of the points $\overline{\mathbf{x}}_i$ characterizes the the projective configuration of these image points $\overline{\mathbf{x}}_i$.
- The actual position of the image line is not relevant as far as the projective configuration of the four image points $\overline{\mathbf{x}}_i$ is concerned.
- The projective geometry of concurrent lines is important to the understanding of the projective geometry of epipolar lines (in Chapter 8).

Recovering affine and metric properties from images

- The main affine properties are **parallelism of lines**, and **ratio of areas**.

- The main metric properties (that is up to similarity properties) are **angles between lines**, and the **ratio of two lengths**.

A projective transformation has 8 degrees of freedom.

A similarity transformation has 4 degrees of freedom.

So we need only four (that is, $8-4=4$) degrees of freedom to determine metric properties. In projective geometry these 4 degrees of freedom are given by the **line at infinity** ℓ_∞, and the **two circular points**.

The projective distortion may be removed once the image of ℓ_∞ is specified. Similarly, the affine distortion may be removed once the image of the circular points is specified.

Thus we examine the image of the line at infinity, and then the image of two circular points.

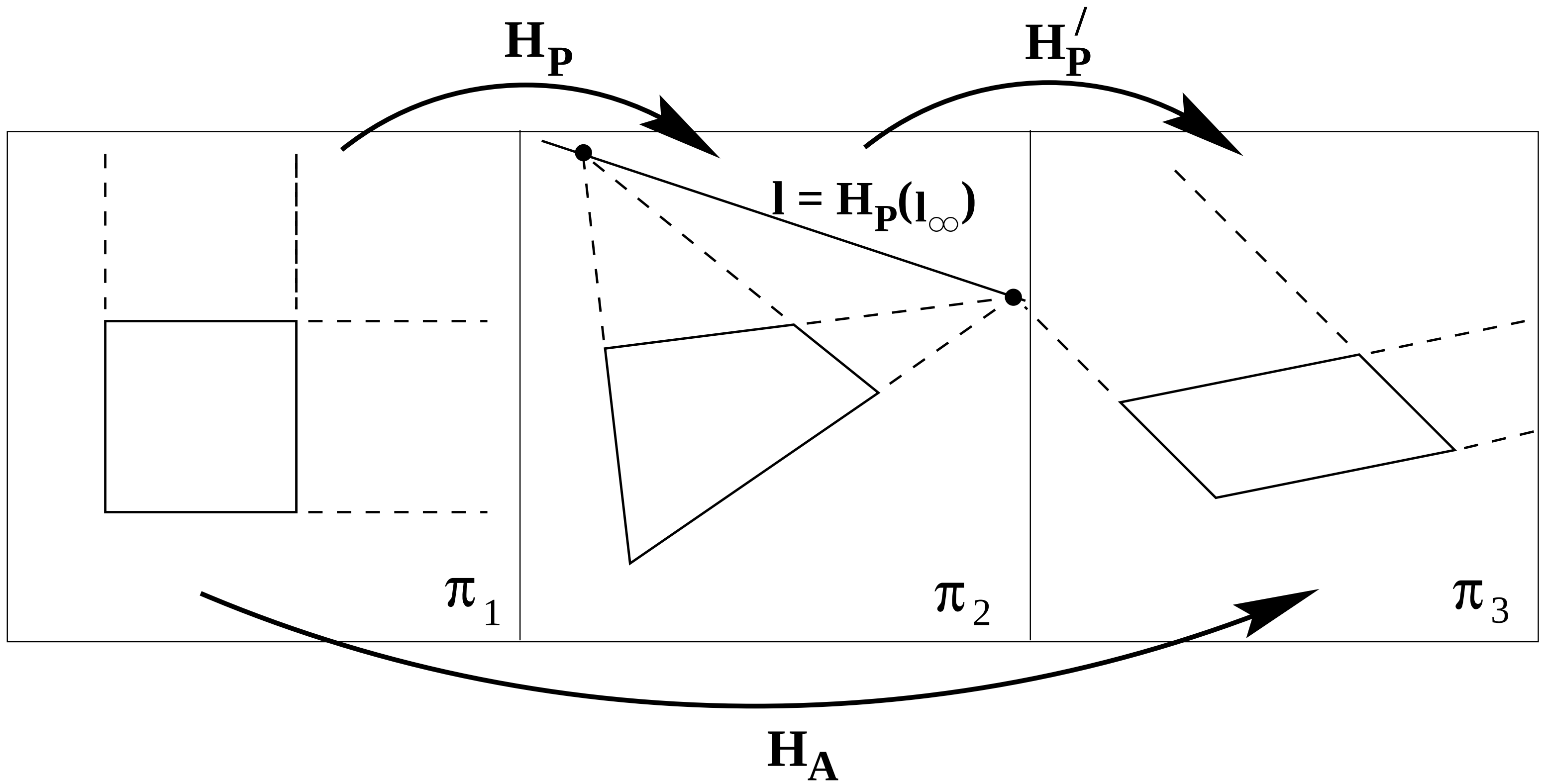
The line at infinity

Under a projective transformation the line at infinity ℓ_∞ is mapped to a finite line. However, if the transformation is an affine, then ℓ_∞ is mapped to ℓ_∞. This follows from the following:

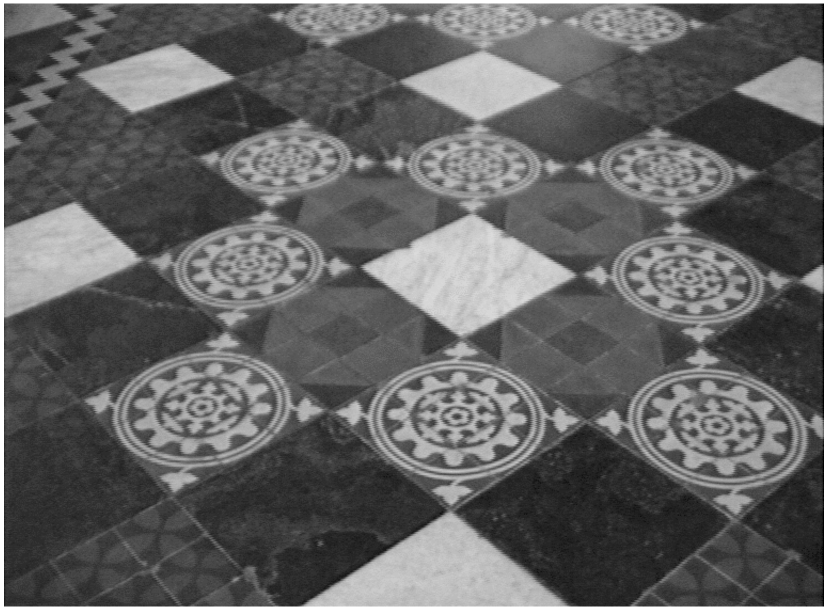
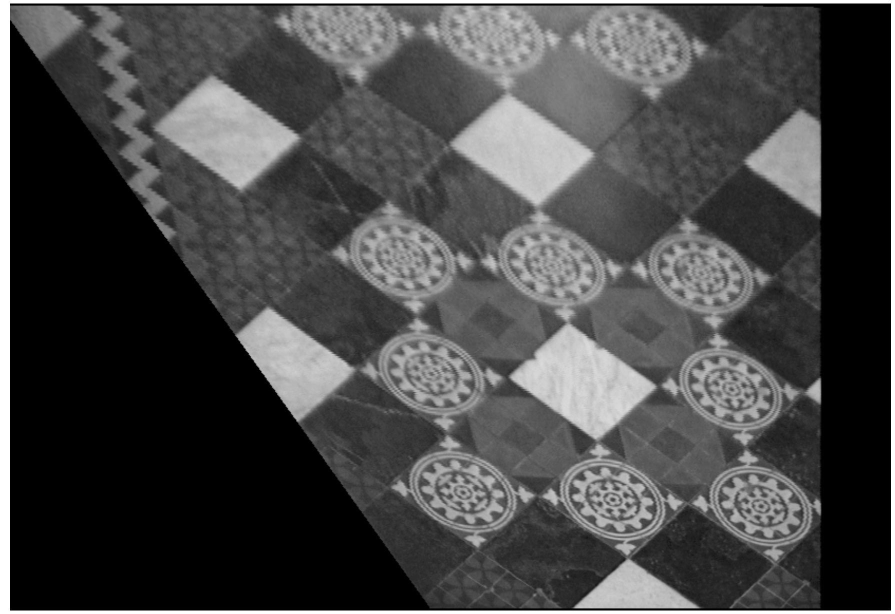
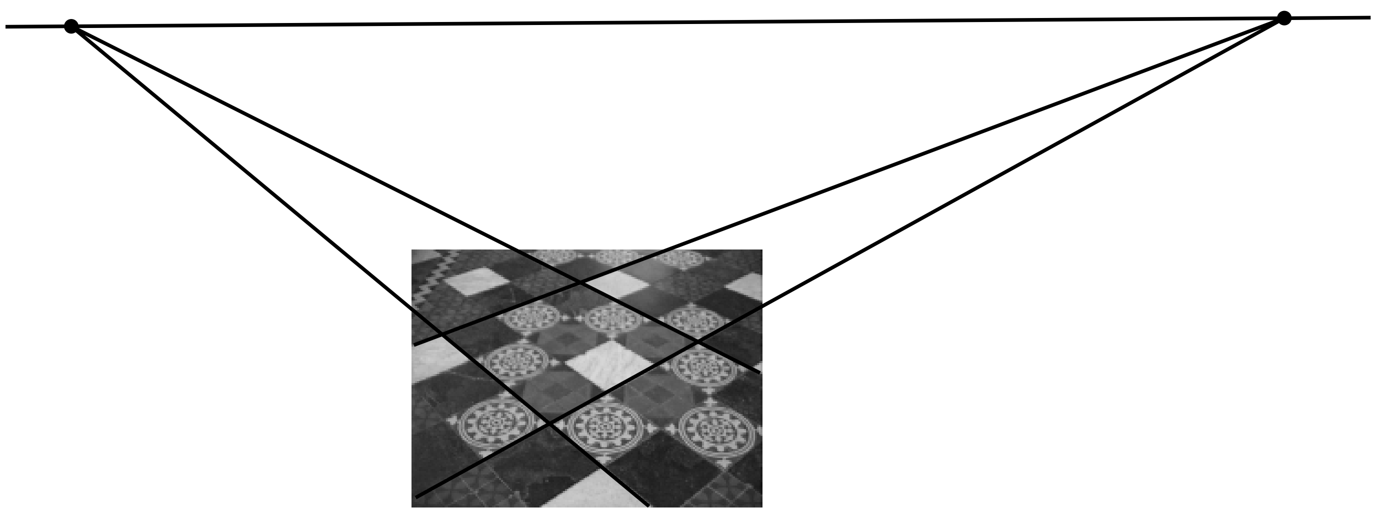
$$\ell'_\infty = \mathbf{H}_A^{-T}\,\ell_\infty = \begin{pmatrix} \mathbf{A}^{-T} & \mathbf{0} \\ -\mathbf{t}^{-T}\mathbf{A}^{-T} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \ell_\infty.$$

Result 1.8. The line at infinity, ℓ_∞, is a fixed line under the projective transformation $\mathbf{H}$ if and only if $\mathbf{H}$ is an affine transformation.

The identification of ℓ_∞ allows one to recover the affine properties (parallellism, ratio of areas).



A projective transformation $\mathbf{H}_P$ maps ℓ_∞ on π_1 to a finite line ℓ on π_2. A projective transformation $\mathbf{H}'_P$ is constructed such that ℓ is mapped back to ℓ_∞.

Circular points and their dual

Under any similarity transformation there are two points on ℓ_∞ which are fixed. These are the circular points $\mathbf{I}, \mathbf{J}$ with canonical coordinates

$$\mathbf{I} = (1, i, 0)^T \qquad \text{and} \qquad \mathbf{J} = (1, -i, 0)^T.$$

These circular points are fixed under an orientation-preserving similarity transformation:

$$\mathbf{I}' = \mathbf{H}_s \, \mathbf{I} = \begin{pmatrix} s\,\cos(\theta) & -s\,\sin(\theta) & t_x \\ s\,\sin(\theta) & s\,\cos(\theta) & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = s\, e^{-i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \sim \mathbf{I}.$$

Result 1.9. The circular points $\mathbf{I}, \mathbf{J}$ are fixed points under the projective transformation $\mathbf{H}$ if and only if $\mathbf{H}$ is a similarity.

$\mathbf{I}$ and $\mathbf{J}$ are called circular points because every circle intersect ℓ_∞ at the circular points.

The equation of a circle is

$$x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0.$$

The ℓ_∞ intersects the circle if $x_3 = 0$. Hence we get $x_1^2 + x_2^2 = 0$ with solution $\mathbf{I} = (1, i, 0)^T$, $\mathbf{J} = (1, -i, 0)^T$.

Hence every circle intersects ℓ_∞ at circular points.