

Multiple View Geometry in Computer Vision

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Basic Information

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Course Information

Textbook: **Multiple View Geometry in Computer Vision**, by R. Hartley and A. Zisserman

Class Room: NS 212F

Scheduled Time: 11:00 - 12:15

Office Hour: 1:30 - 2:30 pm

Prerequisites: **Geometry, Vector Calculus, Abstract Algebra, and Linear Algebra.**

Target Audience

The target audience is:

- (1) EE graduate students that are doing or planning to do research in CVIP lab.
- (2) Mathematics graduate students that are interested in applications of modern geometry and multi-linear algebra to computer vision.

Course Objectives

The objectives of this course are:

- (1) To understand projective geometry underlying multiview image formation.
- (2) To understand the general principles of linear and non-linear parameter estimation methods.
- (3) To be able to apply the state of the art methods for estimating 2D and 3D geometry from images.

Grading Scheme

Grades will be based on 2 tests and Final Project.

The Final Project can be:

- (1) On a significant literature research project (you are welcome to propose your own topic).
- (2) A practical project in computer vision consistent with the course.

Final Project

For the Final Project you will:

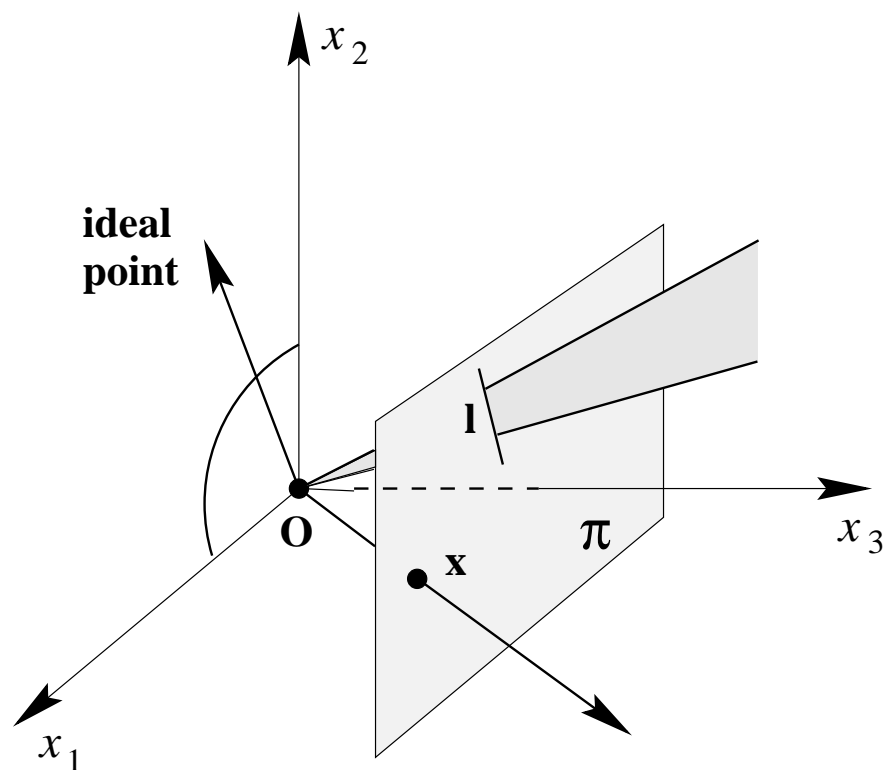
- (1) Hand in a written proposal.
- (2) Prepare one presentation on your topic for the class.
- (3) Hand in a type set final report (about 6-8 pages in the format and style of a research report).

Changes to Syllabus

The instructor reserves the right to make changes in the syllabus when necessary to meet learning objectives, to compensate for missed classes, or for similar reasons.

Projective 2D geometry

Lecture 1



Projective 2D geometry

In Chapter1, we will study:

- Points, lines and conics
- Transformations and invariants
- 1D projective geometry and the Cross-ratio

Notation

A point in the plane may be represented by the pair of coordinates $(x, y) \in \mathbb{R}^2$. Thus it is common to identify the plane with \mathbb{R}^2 .

Since \mathbb{R}^2 is a vector space, the coordinate pair (x, y) is a vector. Hence we identify a point as a vector.

We will denote a point \mathbf{x} as column vector and its transpose \mathbf{x}^T will denote the row vector. Hence we write

$$\mathbf{x} = (x, y)^T,$$

both sides of this equation representing column vectors.

Homogeneous Representation of Lines

The equation of a line in the plane is given by

$$ax + by + c = 0.$$

The different choices of a , b and c give rise to different lines. Hence one can represent a line by a 3-vector

$$(a, b, c)^T.$$

The correspondence between lines and vectors $(a, b, c)^T$ is not one-to-one since lines

$$ax + by + c = 0 \quad \text{and} \quad kax + kby + kc = 0$$

are same for any non-zero constant k .

Thus the vectors

$$(a, b, c)^T \quad \text{and} \quad k(a, b, c)^T$$

represent the same line for any non-zero constant k .

In this case, we say that the vectors $(a, b, c)^T$ and $k(a, b, c)^T$ are same (or equivalent). We denote this by writing

$$(a, b, c)^T \sim k(a, b, c)^T$$

An equivalence class of vectors under this equivalence relationship is known as a homogeneous vector. The set of equivalence classes of vectors in $\mathbb{R}^3 \setminus (0, 0, 0)^T$ forms the projective space \mathbb{P} .

Homogeneous Representation of Pints

A point $\mathbf{x} = (x, y)^T$ lies on the line $\ell = (a, b, c)^T$ if and only if $ax + by + c = 0$. Note that

$$ax + by + c = (x, y, 1)(a, b, c)^T = (x, y, 1)\ell = 0.$$

This suggests that the point $\mathbf{x} = (x, y)^T$ can be represented as a 3-vector $(x, y, 1)^T$.

For any non-zero constant k and line ℓ

$$(kx, ky, k) \ell = 0 \quad \text{if and only if} \quad (x, y, 1) \ell.$$

Hence the set of vectors $(kx, ky, k)^T$ for varying values of k is a representation of the point $(x, y)^T$ in \mathbb{R}^2 .

The arbitrary homogeneous vector representation of a point is of the form $\mathbf{x} = (x_1, x_2, x_3)^T$, representing the point $\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right)^T$ in \mathbb{R}^2 . Here obviously $x_3 \neq 0$.

Points and lines as homogeneous 3-vectors are elements of the projective space \mathbb{P}^2 .

When does a point lies on a line?

A point $\mathbf{x} = (x, y, 1)^T$ lies on a line $\ell = (a, b, c)^T$ if the following equation holds:

$$ax + by + c = 0.$$

But the last equation can be written as $\mathbf{x}^T \ell = 0$, where $\mathbf{x}^T \ell$ denotes the inner product of the vectors \mathbf{x} and ℓ .

Hence we have the following results:

Result 1.1. The point \mathbf{x} lies on the line ℓ if and only if $\mathbf{x}^T \ell = 0$.

A point and a line both have 2 degrees of freedom.

What is the intersection of two lines?

Let ℓ_1 and ℓ_2 be two given lines. Since $\ell_1 \times \ell_2$ is orthogonal to both the vectors ℓ_1 and ℓ_2 , therefore $(\ell_1 \times \ell_2)^T \ell_i = 0$ for $i = 1, 2$. If we let $\mathbf{x} = \ell_1 \times \ell_2$ then we have $\mathbf{x}^T \ell_i = 0$ for $i = 1, 2$. This is in the form of the equation for a line – in particular, this expresses the set of all points \mathbf{x} in the plane that have zero distance from the lines ℓ_1 and ℓ_2 .

If x has zero distance from both lines, then it must lie at their intersection.

Hence we have the following result:

Result 1.2. The intersection of two lines ℓ_1 and ℓ_2 is the point $x = \ell_1 \times \ell_2$.

Two distinct points x_1 and x_2 lie on a single line ℓ and this line can be determined as follows:

Result 1.3. The line ℓ through two points x_1 and x_2 is point $\ell = x_1 \times x_2$.

Some Examples

Example 1. Find the intersection of the lines $x = 1$ and $y = 1$.

Answer: The line $x = 1$ can be written as $-x + 1 = 0$. Hence this line can be written in homogeneous coordinates as $\ell_1 = (-1, 0, 1)^T$. Similarly, the line $y = 1$ can be written as $\ell_2 = (0, -1, 1)^T$.

The point of intersection of these two lines is given by

$$\mathbf{x} = \ell_1 \times \ell_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} = (1, 1, 1)^T$$

which is the inhomogeneous point $(1, 1)^T$ in \mathbb{R}^2 .

Example 2. Find the intersection of the parallel lines $\ell_1 = (a, b, c)^T$ and $\ell_2 = (a, b, d)^T$.

Answer: The point of intersection of these two lines is given by

$$\begin{aligned}\mathbf{x} = \ell_1 \times \ell_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ a & b & d \end{vmatrix} = \mathbf{i} b(d - c) - \mathbf{j} c(d - c) + \mathbf{k} 0 \\ &= (d - c) (b, -c, 0)^T\end{aligned}$$

Ignoring the scaling factor $d - c$, we get $\mathbf{x} = (b, -c, 0)^T$.

Remark 1. The inhomogeneous representation of the point $\mathbf{x} = (b, -c, 0)^T$ is

$$\left(\frac{b}{0}, -\frac{b}{0}\right)^T$$

which makes no sense. But it suggests that the point of intersection has infinitely large coordinates. This observation agrees with the usual notion that parallel lines meet at infinity.

Ideal Points

In general, points with homogeneous coordinates

$$(x, y, 0)^T$$

do not correspond to any point in \mathbb{R}^2 . The points whose last coordinate is zero are called *ideal points* or *points at infinity*.

Line at Infinity

The set of ideal points lies on a single line $(0, 0, 1)^T$ called the *line at infinity*. We denote this line by ℓ_∞ .

Hence

$$\ell_\infty = (0, 0, 1)^T$$

Projective Plane

The projective plane \mathbb{P}^2 can be defined as

$$\mathbb{P}^2 = \mathbb{R}^2 \cup \ell_\infty.$$

Hence the projective plane \mathbb{P}^2 consists of all the points in \mathbb{R}^2 together with all the ideal points.

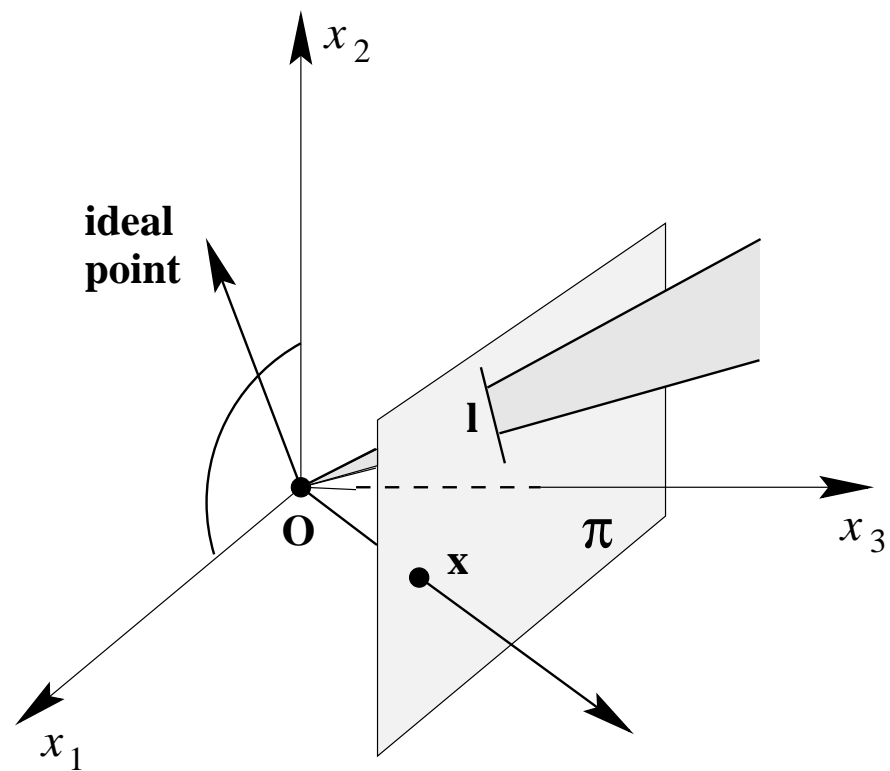
In projective plane \mathbb{P}^2

- (1) two distinct lines meet in a single point, and
- (2) two distinct points lie on a single line.

The projective geometry is the study of the geometry of the projective plane \mathbb{P}^2 .

A Model for Projective Plane

- The rays in \mathbb{R}^3 are points in \mathbb{P}^2 .
- The planes through the origin are the lines in \mathbb{P}^2 .



Duality Principle

The role of points and lines may be interchanged in the statements concerning the properties of lines and points.

Duality Principle. To any theorem of 2-D projective geometry there corresponds a dual theorem, which may be obtained by interchanging the roles of points and lines in the original theorem.

Duality Principle

$$\mathbf{x} \longleftrightarrow \ell$$

$$\mathbf{x}^T \ell = 0 \longleftrightarrow \ell^T \mathbf{x} = 0$$

$$\mathbf{x} = \ell \times \ell' \longleftrightarrow \ell = \mathbf{x} \times \mathbf{x}'$$

Projective 2D geometry

Lecture 2

Conics and Dual Conics

Conics are curves described by 2nd-degree equation:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

Letting $x \mapsto \frac{x_1}{x_3}$ and $y \mapsto \frac{x_2}{x_3}$, we have

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0.$$

In matrix form a conic can be represented as

$$\mathbf{x}^T \mathbf{C} \mathbf{x} = 0$$

where

$$\mathbf{C} = \begin{pmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{pmatrix}$$

- The matrix \mathbf{C} represents a conic.
- The matrix \mathbf{C} is a symmetric matrix.

Since multiplying \mathbf{C} by a non-zero scalar does not affect the equation $\mathbf{x}^T \mathbf{C} \mathbf{x} = 0$, therefore \mathbf{C} is a homogeneous representation of a conic.

- The conic has 5 degrees of freedom.

Since a conic has 5 degrees of freedom, one needs only five points to uniquely determine a conic.

Let the conic passes through a five points (x_i, y_i) for $i = 1, 2, \dots, 5$. Then

$$ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0,$$

that is

$$(x_i^2, x_iy_i, y_i^2, x_i, y_i, 1) \mathbf{c} = 0$$

where $\mathbf{c} = (a, b, c, d, e, f)^T$.

Stacking the constraints from 5 points, we get

$$\begin{pmatrix} x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4 y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5 y_5 & y_5^2 & x_5 & y_5 & 1 \end{pmatrix} \mathbf{c} = 0.$$

Tangent to a Conic

The line ℓ tangent to a conic at a point \mathbf{x} has a very simple form in homogeneous coordinates.

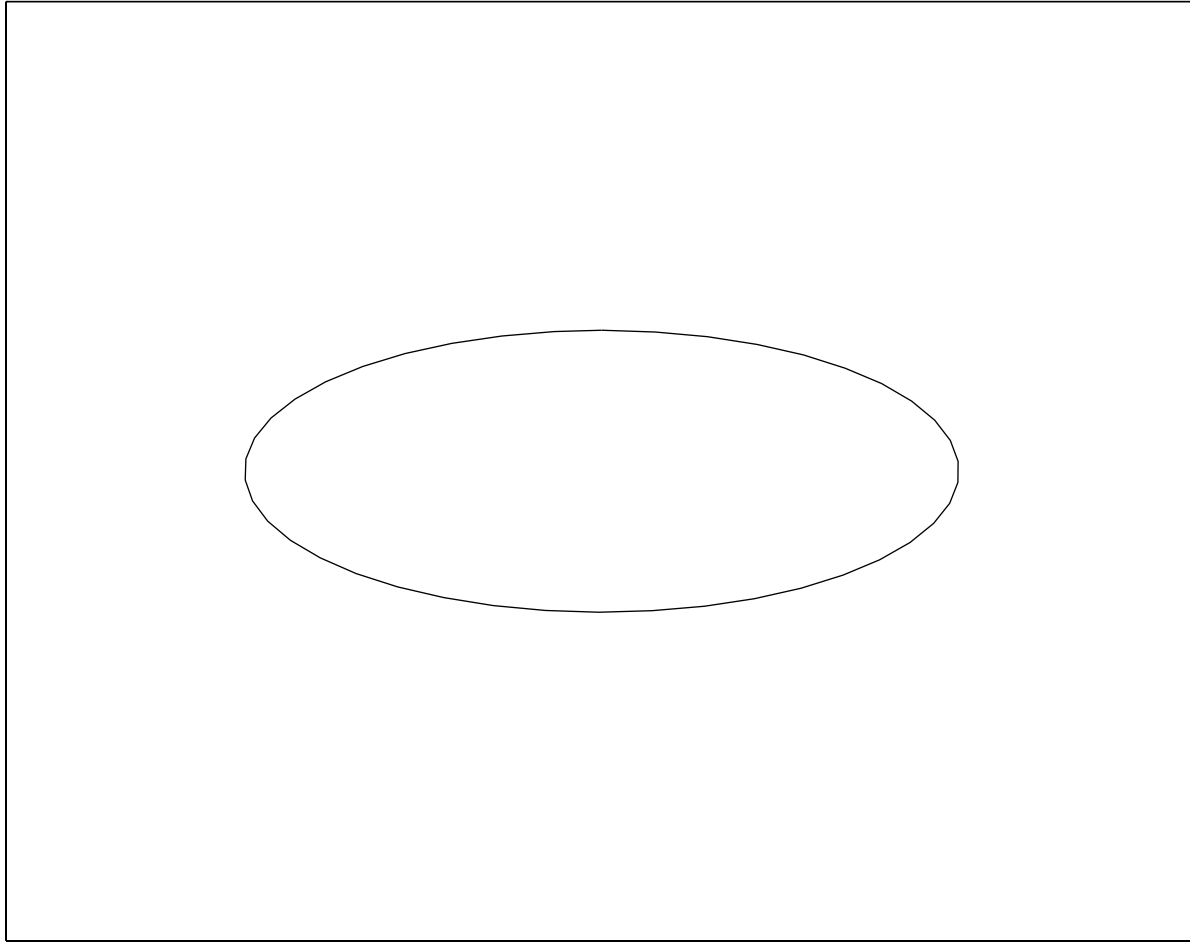
Result 1.4. The line ℓ tangent to the conic \mathbf{C} at a point \mathbf{x} is given by $\ell = \mathbf{C} \mathbf{x}$.

Dual Conic

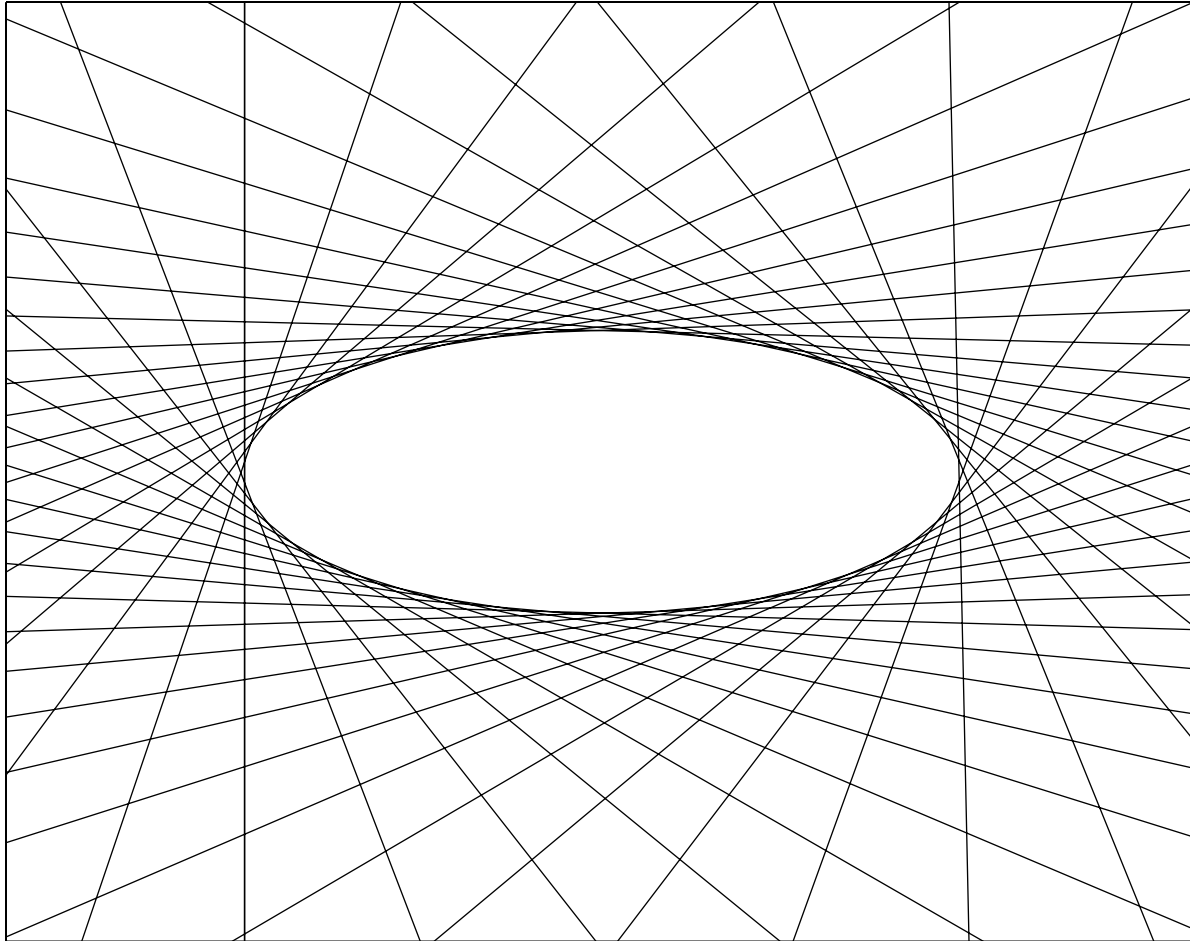
The conic defined above is called a point conic, since it defines an equations on points.

Because of duality principle one can define a conic using an equation on lines.

This dual conic is represented by a 3×3 matrix denoted by C^* . Indeed C^* is the adjoint of C .



Point conic



A dual conic is also called a conic envelope.

Degenerate Conics

- If the matrix C is not of full rank, then C is called a degenerate conic.
- If the rank of the matrix C is 2, then the conic C includes two lines.
- If the rank of the matrix C is 1, then the conic C has a repeated line.

Projective Transformations

2D projective geometry is the study of properties of the projective plane \mathbb{P}^2 that are invariant under a group of transformations known as *projectivities*.

Definition. A projectivity is an invertible mapping $h : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ such that three points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ lie on a line if and only if $h(\mathbf{x}_1), h(\mathbf{x}_2), h(\mathbf{x}_3)$ lie on a line.

- The set of all projectivities forms a group under composition.
- A projectivity is also known as a
 - *collineation*
 - *homography*
 - *projective transformation*

Theorem 1.1. A mapping $h : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is a projectivity if and only if there exists a non-singular 3×3 matrix \mathbf{H} such that for any point in \mathbb{P}^2 represented by a vector \mathbf{x} it is true that $h(\mathbf{x}) = \mathbf{H}\mathbf{x}$.

Definition. A planner projective transformation is a linear transformation on homogeneous 3-vectors represented by a non-singular 3×3 matrix:

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

or more briefly as

$$\mathbf{x}' = \mathbf{H} \mathbf{x}.$$

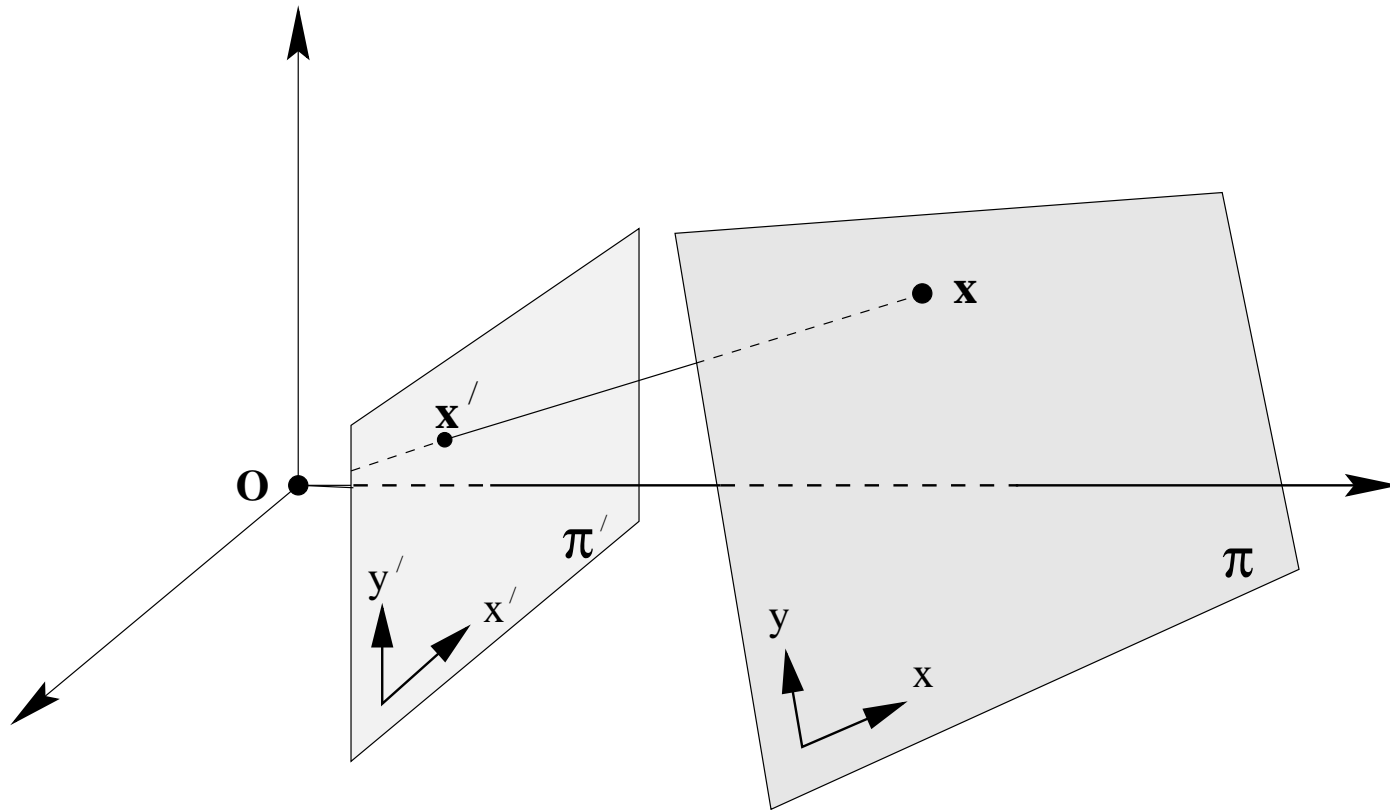
Remark 2. Like \mathbf{C} , the matrix \mathbf{H} is a homogeneous matrix.

Remark 3. Projectivity transformation \mathbf{H} has 8 degrees of freedom.

Mappings between planes

Projection along rays through a common point defines a mapping from one plane to another.

If the coordinate system is defined in each plane and points are represented in homogeneous coordinates, then the central projection mapping may be expressed as $\mathbf{x}' = \mathbf{H}\mathbf{x}$, where \mathbf{H} is a non-singular 3×3 matrix.



Central projection maps points on one plane to points on another plane. (An application of Theorem 1.1.)

Removing projective distortion

Shape is distorted under prospective imaging.

In general parallel lines on a scene plane (or world plane) are not parallel in the image but instead converge to a finite point.

We have seen that a central projection image of a plane is related to the original plane through a projective transformation.

So the image is a projective distortion of the original.

It is possible to “undo” this projective transformation by computing the inverse transformation and applying it to the image.

The result will be a synthesized image in which the object in the plane are shown with correct geometric shape.

Select a section of the image corresponding to a planer section of the world. Let $(x, y)^T$ and $(x', y')^T$ be points in the world and image plane.

The projective transformation can be written in inhomogeneous form as

$$x' = \frac{x'_1}{x'_1} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}$$

$$y' = \frac{x'_2}{x'_1} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}.$$

Hence we have

$$x' (h_{31}x + h_{32}y + h_{33}) = h_{11}x + h_{12}y + h_{13}$$

$$x' (h_{31}x + h_{32}y + h_{33}) = h_{21}x + h_{22}y + h_{23}.$$

These equations are linear in the elements of \mathbf{H} . Four points correspondence lead to eight such linear equations in the entries of \mathbf{H} . Hence we can solve for \mathbf{H} up to a multiplicative constant.

Once \mathbf{H} is computed we can compute \mathbf{H}^{-1} and apply it to the whole image to “undo” the distortion.

An example of removing perspective distortion is shown in next two illustrations. In the original image the lines of the windows clearly converge at a finite point.





Transformation of lines

Suppose the points \mathbf{x}_i transform to points \mathbf{x}'_i according to $\mathbf{x}'_i = \mathbf{H} \mathbf{x}_i$. We know from Theorem 1.1 that if the points \mathbf{x}_i lie on a line ℓ , then the points \mathbf{x}'_i lie on a transformed line ℓ' . We like to find this ℓ' .

Since the points \mathbf{x}_i lies on the line ℓ , we have

$$\begin{aligned} 0 &= \ell^T \mathbf{x}_i \\ &= \ell^T \mathbf{H}^{-1} \mathbf{H} \mathbf{x}_i \\ &= \ell^T \mathbf{H}^{-1} \mathbf{x}'_i \quad (\text{since } \mathbf{x}'_i = \mathbf{H} \mathbf{x}_i) \\ &= \left(\mathbf{H}^{-T} \ell \right)^T \mathbf{x}'_i \\ &= \ell'^T \mathbf{x}'_i \end{aligned}$$

where $\ell' = \mathbf{H}^{-T} \ell$ (or $\ell'^T = \ell^T \mathbf{H}^{-1}$).

Remark 4.

Points transform according to \mathbf{H} .

Lines transform according to \mathbf{H}^{-1} .

Transformation of conics

Result 1.5. Under a point transformation $\mathbf{x}'_i = \mathbf{H} \mathbf{x}_i$, a conic \mathbf{C} transforms to $\mathbf{C}' = \mathbf{H}^{-T} \mathbf{C} \mathbf{H}^{-1}$.

Proof: Under a point transformation $\mathbf{x}'_i = \mathbf{H} \mathbf{x}_i$, the equation $\mathbf{x}^T \mathbf{C} \mathbf{x} = 0$ yields

$$\begin{aligned}
0 &= \mathbf{x}^T \mathbf{C} \mathbf{x} \\
&= (\mathbf{H}^{-1} \mathbf{x}')^T \mathbf{C} \mathbf{H}^{-1} \mathbf{x}' \\
&= \mathbf{x}'^T (\mathbf{H}^{-1})^T \mathbf{C} \mathbf{H}^{-1} \mathbf{x}' \\
&= \mathbf{x}'^T \mathbf{H}^{-T} \mathbf{C} \mathbf{H}^{-1} \mathbf{x}' \\
&= \mathbf{x}'^T \mathbf{C}' \mathbf{x}'
\end{aligned}$$

where $\mathbf{C}' = \mathbf{H}^{-T} \mathbf{C} \mathbf{H}^{-1}$.

Transformation of dual conics

Result 1.6. Under a point transformation $\mathbf{x}'_i = \mathbf{H} \mathbf{x}_i$, a dual conic \mathbf{C}^* transforms to $\mathbf{C}^{*'} = \mathbf{H} \mathbf{C}^* \mathbf{H}^T$.

Note that \mathbf{C}^* is the adjoint of the matrix \mathbf{C} .

Transformation of lines and conics

Summary

- Transformation for points: $\mathbf{x}'_i = \mathbf{H} \mathbf{x}_i$
- Transformation for lines: $\ell' = \mathbf{H}^{-T} \ell$
- Transformation for conics: $\mathbf{C}' = \mathbf{H}^{-T} \mathbf{C} \mathbf{H}^{-1}$
- Transformation for dual conics: $\mathbf{C}^{\star'} = \mathbf{H} \mathbf{C}^{\star} \mathbf{H}^T$.