一维热传导方程（隐格式）

对于一维热传导方程 $\dfrac{\partial T(x,t)}{\partial t}=D\dfrac{\partial^2 T(x,t)}{\partial x^2}$

其显式差分格式为 $\dfrac{T_{i+1}^{j}-T_{i}^{j}}{\Delta t}=D\dfrac{T_{i}^{j+1}-2T_{i}^{j}+T_{i}^{j-1}}{(\Delta x)^2}$

这种格式计算可能会出现不稳定，所以一般采用隐格式

区别主要在 $\left.\dfrac{\partial^2 T(x,t)}{\partial x^2}\right|_{(x,t)=(x_j,t_i)}=\dfrac{T_{i+1}^{j+1}-2T_{i+1}^{j}+T_{i+1}^{j-1}}{(\Delta x)^2}$ 带入计算

即 $\dfrac{T_{i+1}^{j}-T_{i}^{j}}{\Delta t}=D\dfrac{T_{i+1}^{j+1}-2T_{i+1}^{j}+T_{i+1}^{j-1}}{(\Delta x)^2}$

整理得到 $T_i^j=-rT_{i+1}^{j+1}+(1+2r)T_{i+1}^{j}-rT_{i+1}^{j-1}=(-r\quad 1+2r\quad -r)\begin{pmatrix}T_{i+1}^{j+1}\\T_{i+1}^{j}\\T_{i+1}^{j-1}\end{pmatrix}$

其中 $r=D\dfrac{\Delta t}{(\Delta x)^2}$

所以

$$T_i^{m-1}=(-r\quad 1+2r\quad -r)\begin{pmatrix}T_{i+1}^{m}\\T_{i+1}^{m-1}\\T_{i+1}^{m-2}\end{pmatrix}$$

$$T_i^{m-2}=(-r\quad 1+2r\quad -r)\begin{pmatrix}T_{i+1}^{m-1}\\T_{i+1}^{m-2}\\T_{i+1}^{m-3}\end{pmatrix}$$

$$\vdots$$

$$T_i^{j}=(-r\quad 1+2r\quad -r)\begin{pmatrix}T_{i+1}^{j+1}\\T_{i+1}^{j}\\T_{i+1}^{j-1}\end{pmatrix}$$

$$\vdots$$

$$T_i^{3}=(-r\quad 1+2r\quad -r)\begin{pmatrix}T_{i+1}^{4}\\T_{i+1}^{3}\\T_{i+1}^{2}\end{pmatrix}$$

$$T_i^{2}=(-r\quad 1+2r\quad -r)\begin{pmatrix}T_{i+1}^{3}\\T_{i+1}^{2}\\T_{i+1}^{1}\end{pmatrix}$$

经过整理

$$\begin{pmatrix} T_i^{m-1} \\ T_i^{m-2} \\ \vdots \\ T_i^{j} \\ \vdots \\ T_i^{3} \\ T_i^{2} \end{pmatrix}_{(m-2)\times 1} = \begin{pmatrix} -r & 1+2r & -r & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -r & 1+2r & -r & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -r & 1+2r & -r & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -r & 1+2r & -r & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -r & 1+2r & -r & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -r & 1+2r & -r \end{pmatrix}_{(m-2)\times m} \begin{pmatrix} T_{i+1}^{m} \\ T_{i+1}^{m-1} \\ T_{i+1}^{m-2} \\ \vdots \\ T_{i+1}^{j} \\ \vdots \\ T_{i+1}^{3} \\ T_{i+1}^{2} \\ T_{i+1}^{1} \end{pmatrix}_{m\times 1}$$

加入边界条件，需要看边界条件的形式

例一维热传导方程$\begin{cases} \dfrac{\partial T}{\partial t} = \dfrac{\partial^2 T}{\partial x^2}, 0 \le x \le 1 \\ T(x,0) = 25 \\ T(0,t) = 50 - 25e^{-t}, \left.\dfrac{\partial T}{\partial x}\right|_{x=1} = -(T-50) \end{cases}$

上边界条件：

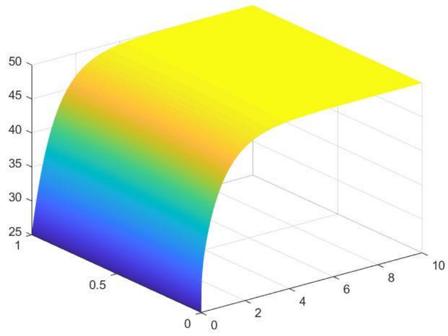
$$(1+q)T_{i+1}^{m} - T_{i+1}^{m-1} = 50q$$

加入上边界条件后

$$\begin{pmatrix} 50q \\ T_i^{m-1} \\ T_i^{m-2} \\ \vdots \\ T_i^{j} \\ \vdots \\ T_i^{3} \\ T_i^{2} \end{pmatrix}_{(m-1)\times 1} = \begin{pmatrix} 1+q & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -r & 1+2r & -r & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -r & 1+2r & -r & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -r & 1+2r & -r & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -r & 1+2r & -r & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -r & 1+2r & -r & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -r & 1+2r & -r \end{pmatrix}_{(m-1)\times m} \begin{pmatrix} T_{i+1}^{m} \\ T_{i+1}^{m-1} \\ T_{i+1}^{m-2} \\ \vdots \\ T_{i+1}^{j} \\ \vdots \\ T_{i+1}^{3} \\ T_{i+1}^{2} \\ T_{i+1}^{1} \end{pmatrix}_{m\times 1}$$

下边界条件：$T_{i+1}^{1} = 50 - 25e^{-t_{i+1}}$

$$\begin{pmatrix} 50q \\ T_i^{m-1} \\ T_i^{m-2} \\ \vdots \\ T_i^{j} \\ \vdots \\ T_i^{3} \\ T_i^{2} \\ 50-25e^{-t_{i+1}} \end{pmatrix}_{m\times 1} = \begin{pmatrix} 1+q & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -r & 1+2r & -r & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -r & 1+2r & -r & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -r & 1+2r & -r & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -r & 1+2r & -r & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -r & 1+2r & -r & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -r & 1+2r & -r \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}_{m\times m} \begin{pmatrix} T_{i+1}^{m} \\ T_{i+1}^{m-1} \\ T_{i+1}^{m-2} \\ \vdots \\ T_{i+1}^{j} \\ \vdots \\ T_{i+1}^{3} \\ T_{i+1}^{2} \\ T_{i+1}^{1} \end{pmatrix}_{m\times 1}$$



隐式差分数值解