An Analysis of the Richardson Arms Race Model

David Bigelow

November 25, 2003

Abstract

According to Lewis Richardson, the likelihood of two nations engaging in conflict can be determined by a set of differential equations. Richardson conjectured that a nation's probability of entering into an aggressive war is based upon its stockpiles of available weaponry, and tempered by the resistance of the citizenry. In the Richardson model, the likelihood of a small dispute erupting into a full war is based upon these variables, and the current conditions. The model has three basic cases: both nations tending towards disarmament, both nations tending towards a runaway arms race, and both nations tending towards a stable equilibrium point.

Lewis Fry Richardson (1881-1953) was a British mathematician and physicist who made several advances in the fields of geophysics, scaling, and fractality. His 1922 book, "Weather Pioneering by Numerical Processes," was the first to suggest techniques of numerical integration which could be applied to atmospheric motion. Richardson worked at the United Kingdom Meteorological Office until 1920, when it was incorporated into the Air Ministry. He then left to teach at a British university until his retirement in 1940. Richardson, a man devoted to the ideals of peace, attempted to come up with mathematical expressions for war. He toiled with this work from 1940 until his death, and the results were published posthumously in 1963. One of the conclusions of his research was a system of differential equations modelling arms races. This system has become known as the **Richardson Arms Race Model**, and has been subject to analysis and refinement since being published. The basic system is:

$$\frac{dx}{dt} = ay - mx + r$$
$$\frac{dy}{dt} = bx - ny + s$$

This system can be expanded to include any number of equations, each one representing a single country. For example, a system of three equations would be expressed as:

$$\frac{dx}{dt} = ay + bz - mx + q$$
$$\frac{dy}{dt} = cx + dz - ny + r$$
$$\frac{dz}{dt} = ex + fy - oz + s$$

In the interests of studying a simpler model, the two-equation system will be focused on. Each differential equation represents the rate of change of arms buildup for a particular country. "X" represents the amount of weaponry that country one has at time t. "Y" is the same for the second country. Each constant has a specific meaning, and vary from system to system.

The constants "a" and "b" are known alternatively as "fear" or "reaction" constants. They represent the desire of a country to increase arms at a rate proportional to the amount of arms that their opponent possesses. The constants "m" and "n" are known as either the "restraint" or "fatigue" factors. They represent the desire of a nation to reduce arms stockpiles at a rate directly proportional to what they possess. Finally, "r" and "s" are the grievance constants, and represent the "leftovers." These constants can contain ambition, external pressure, a revenge motive, and other factors not directly related to arms stockpiles.

It is important to note that this system of equations has meaningful answers *only* in the first quadrant. That is, weapons stockpiles can only be positive. It is also worth noting that only the constants "r" and "s" can be negative (or zero). A negative value for "a" and "b" would imply negative fear, and negative "m" and "n" would imply a country wishes to build up its weaponry at ever-increasing rates.

The premise behind this system is based in the likelihood of a small conflict turning into a full scale war between two nations. As weapons stockpiles group, Richardson assumes that the willingness to *use* these stockpiles grows proportionally.

It is immediately obvious that there are four possible outcomes for this system of differential equations. These four outcomes are:

- All trajectories approach an equilibrium point
- All trajectories go to infinity (a runaway arms race)
- All trajectories go to zero (mutual disarmament)
- The path of each trajectory depends on the initial point

The first case describes a situation in which all possible starting conditions lead both nations towards an equilibrium point. The likelihood of a conflict turning into a war would be an unchanging percentage chance once equilibrium was achieved. However, it is reasonable to assume that small conflicts would change the fear constants appropriately, thus shifting the equilibrium point further and further out. In the long run, with an increasing number of conflicts, this may turn into a runaway arms race. At certain points during the Cold War, the United States and the Soviet Union resembled this model, although it never fully escalated into a runaway arms race.

The next two situations are similar. In the second case, two nations are sufficiently hostile towards each other such that they will enter a runaway arms race regardless of starting conditions. Speaking in worldly terms, this is an impossible model because of the budget constraint. Therefore, such a race will either bankrupt one nation, or lead to an all-out war. The third case is precisely the opposite: two nations are sufficiently friendly towards each other so that they will never enter an arms race. Canada and the United States, when considered as a pair, would be an example of this third case.

The final case is the most unusual and most improbable in the real world. Putting it into real-world terms, two nations will either mutually disarm or enter a runaway arms race, depending on the starting points of their arms stockpiles. There are four straightline solutions that lead to an equilibrium, but the chances of being on one of these lines is unlikely when compared to the probability of being anywhere else. There is no practical model to describe this situation in the real world, yet it is the most interesting to analyze.

The first step in a qualitative analysis of this system is to identify the equilibrium point (there is only a single equilibrium point, given the restraints on the constants). This is done by setting both parts of the system to zero:

$$ay - mx + r = 0$$

$$bx - ny + s = 0$$

To find the equilibrium point, the system of equations must be solved for both x and y, which yields a coordinate. Thus, for all systems of this form, the equilibrium point can be found at:

$$\left(\frac{rn + as}{mn - ab}, \frac{sm + br}{mn - ab}\right)$$

$$where \ \ (mn - ab) \neq 0$$

By plugging in values for the constants, one can determine where the equilibrium point is for any particular case. If the equilibrium point is not in the first quadrant, it can be disregarded and treated as if no equilibrium point exists. In the case of $mn - ab = 0$, the system has *no* equilibrium points and is either a runaway arms race or mutual disarmament. In the Richardson Arms Race Model, an equilibrium point in the first quadrant can mean one of two things. First, the equilibrium point may be a *sink*, which represents the case in which all trajectories lead to an equilibrium solution. Second, the equilibrium point may be a *saddle*, which divides the graph into pieces, as per the fourth standard case.

The *nullclines* of a system provide a good way to determine what type of equilibrium point exists. Or, if no equilibrium point exists, the nullclines can be used to determine whether the model will lead to an arms race or a mutual disarmament. Nullclines can be found by setting equations of the system to zero, then solving for y to obtain the line. Thus, the x-nullcline is described by the line:

$$y = \frac{mx - r}{a}$$

and the y-nullcline is described by the line:

$$y = \frac{bx + s}{n}$$

On the x-nullcline, the rate of change for x is equal to zero. Similarly, the y-nullcline has a rate of change of zero for y. Thus, by solving the differential equation for a point on each nullcline, directional vectors can be computed. Using the following constants,

$$a = 2, b = 2, m = 5, n = 5, r = 5, s = 5$$

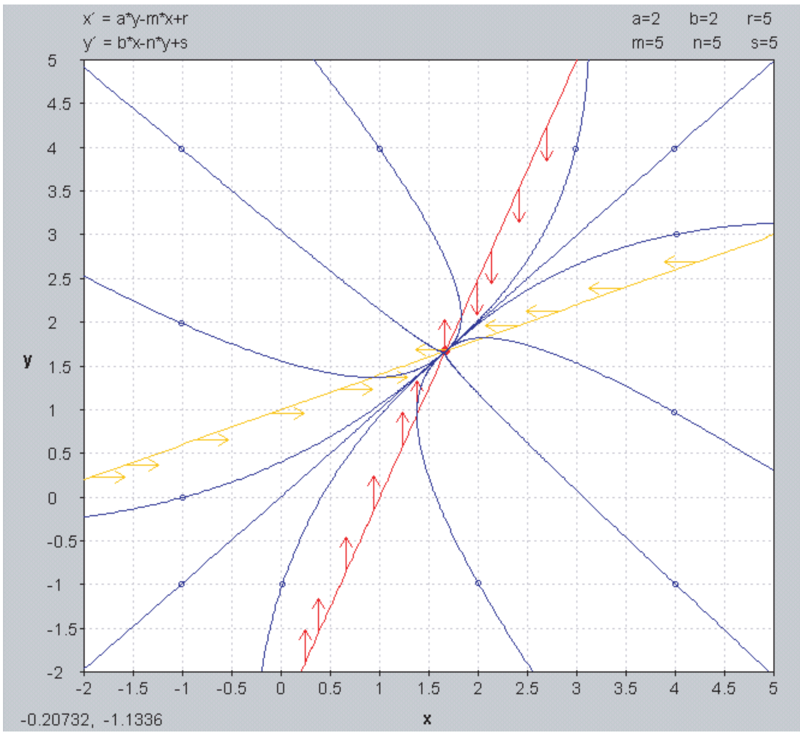
the equilibrium point is at:

$$\left(\frac{(5)(5) + (5)(2)}{(5)(5) - (2)(2)}, \frac{(5)(5)(5) + (5)(5)(2)}{((5)(5) - (2)(2))(2)} - \frac{5}{2}\right) = \left(\frac{5}{3}, \frac{5}{3}\right)$$

with the x and y nullclines defined as:

$$y = \frac{5x-5}{2} \;\; and \;\; y = \frac{2x+5}{5}$$

This yields the following graph (which includes selected trajectories from various initial points):

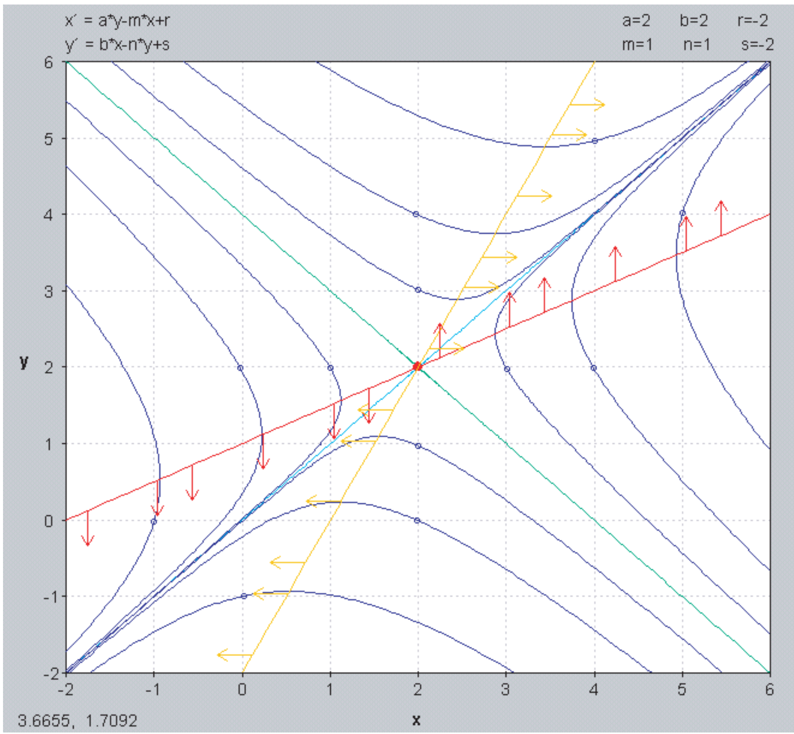


As shown on the graph, every trajectory leads to the equilibrium point $(\frac{5}{3}, \frac{5}{3})$, regardless of the starting conditions. Looking at the directional vectors on each nullcline, it should be immediately visible that "all roads lead to equilibrium." In practical terms, the graph indicates that this particular arms race will stabilize to a constant stockpile of arms for each side.

With different constants for the variables, the system may be a runaway arms race (all trajectories to infinity) or disarmament (all trajectories to zero). In both of these cases, the graph is not particularly interesting, as there are no "points of interest" in which special behavior can be observed. The nullclines for the runaway arms race point outward (away from zero) and the nullclines for the disarmament point inward (towards zero).

The last case, however, is the most interesting to analyze. The equilibrium point for this system is a *saddle* point, and neatly divides the first quadrant into two halves. Given the following initial starting conditions, a graph of this sort is produced:

$$a = 2, b = 2, m = 1, n = 1, r = -2, s = -2$$



There are three areas of interest on this graph. As shown, curves either go towards mutual disarmament $(0,0)$, or towards a runaway arms race (∞, ∞). There are, however, four "special" curves that tend towards the equilibrium point at $(2,2)$. These are the straightline solutions, and will only happen on the lines $y = c_1 x + c_2$, where c_1 and c_2 are as-of-yet undetermined constants. Only an initial point on this line will go towards the equilibrium. In this case, the straightline solutions are:

$$y = x \quad \text{and} \quad y = -x + 4$$

In this particular system of equations, the starting point is critical to determining the outcome, moreso than any other case. In the first three cases, the initial starting conditions do not have any bearing on the *type* of finishing condition. Here, especially around the straightline solutions, the starting conditions are critical to determining what will happen. Unfortunately, as noted earlier, this situation is quite improbably in the real world. However, in a fictional world with only two global superpowers, such a model is likely (they will either wish to build up and go to war, or devote resources to other things).

A *true* real-life model is far more complicated, given the number of factors involved. There could be close to 200 equations in the system, one for each country in the world, as well as other major political groups. However, the simple model can still have some relevance when applied to certain pairs of countries.

India and Pakistan, for example, could have a fairly accurate representation through this model.

Finding the general solution to this set of equations is somewhat more difficult, given the number of variables involved. The first step in finding the general solution is transforming the equation to depend on the variables u and v, with no constant. Therefore, the following definitions are made:

$$\frac{du}{dt} = -mu + av$$
$$\frac{dv}{dt} = bu - nv$$

where

$$u = x - c \quad and \quad v = y - d$$

The next step is determining what the parameters c and d are, which can be done by substitution. For example, the first equation can be expanded to $u' = -mx + ay + (mc - ad)$, and therefore $(mc - ad) = r$. Combined with the second equation, a solution for c and d can be found in terms of the constants a, b, m, n, r, s. Therefore, the equations for u and v can be rewritten as:

$$u = \left(x - \frac{rn + as}{mn - ab}\right)$$
$$v = \left(y - \frac{sm + br}{mn - ab}\right)$$

With the new system of differential equations, a general solution can be computed. To find a general solution, one must first find the eigenvalues, which can be calculated with a determinate matrix. Solving the determinate matrix for λ yields two eigenvalues, given by:

$$\lambda_1 = \frac{-(m+n) + \sqrt{m^2 + n^2 - 2mn + 4ab}}{2}$$
$$\lambda_2 = \frac{-(m+n) - \sqrt{m^2 + n^2 - 2mn + 4ab}}{2}$$

If the value $m^2 + n^2 - 2mn + 4ab$ can be anything other than positive, there will be a bifurcation at certain values of a, b, m, n. However, recall that these four parameters can only have a *positive* value, *not* negative or equal to zero. Thus, it can be shown that this whole can never be less than or equal to zero. Because of this, the general solution will be of the following form:

$$Y(t) = k_1 e^{\lambda_1 t}(V_1) + k_2 e^{\lambda_2 t}(V_2) + (C_1)$$

Therefore, the only steps left are to discover what the two vectors (V_1 and V_2) are, and add the constant (C_1), which accounts for the transformation of xy space into uv space. The vectors can be found by solving for u in terms of v with the following equations:

$$\frac{du}{dt} = -mu + av = \lambda_1 u$$
$$\frac{du}{dt} = -mu + av = \lambda_2 u$$

Numbers are now chosen to substitute for the variables in the vectors. Any vector is permissable, as long as it satisfies the above conditions. In addition, (C_1) must be added in. The two components of (C_1) were discovered above by the solving for c and d. Thus, the entire general solution is:

$$Y(t) = k_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ \frac{\lambda_1+m}{a} \end{pmatrix} + k_2 e^{\lambda_2 t} \begin{pmatrix} 1 \\ \frac{\lambda_2+m}{a} \end{pmatrix} + \begin{pmatrix} \frac{rn+as}{mn-ab} \\ \frac{sm+br}{nm-ab} \end{pmatrix}$$

where λ_1 and λ_2 are defined as above

Thus, using this general solution, the number of arms at time t can be known, given initial starting conditions.

The qualitative analysis of the Richardson Arms Race Model is most useful for determining the long run behavior of a system. Using equilibrium points and nullclines, the behavior of any system can be easily predicted. The general solution is best for determining the amount of weaponry present at a time t. Overall, the Richardson Arms Race Model is overly simplistic to be entirely accurate, but can show the trends of a system with some precision.

References

[1] Dunne, Paul; Nikolaidou, Eftychia; and Smith, Ron. **Arms Race Models and Econometric Applications**. 1999. Available: http://mubs.mdx.ac.uk/research/Discussion_Papers/Economics/dpap_%20econ_no74.pdf

[2] Fox, William P. **Arms Control and Warfare**. 1999. Available: http://www.dean.usma.edu/math/pubs/mmm99/DDS1.HTM

[3] Sandler, Todd and Hartley, Keith. **The economies of defense**. Cambridge University Press. Cambridge, Great Britain:1995

[4] Williams, Adam P. **Richardson Arms Race Model**. 2002. Available: http://shakti.trincoll.edu/ pbrown/armsrace.html