Învățare automată - Temă S10

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Problema 0.164A / pag. 254

Cunoaștem f funcție derivabilă, $f : \mathbb{R}^d \to \mathbb{R}$.
Regula de actualizare este: $x_i^{(k+1)} \leftarrow x_i^{(k)} - learningRate \cdot \frac{\partial}{\partial x_i} f(x^{(k)})$

a.

$f(x) = 4x^2 - 2x + 1$
$learningRate = 0.1$

$\frac{\partial}{\partial x} f(x) = \frac{\partial}{\partial x}(4x^2 - 2x + 1) = 8x - 2$
$x^{(1)} = 1$

Pentru $k = 0$:
$\frac{\partial}{\partial x} f(x^{(0)}) = 6$
$x^{(1)} = x^{(0)} - learningRate \cdot 6 = 1 - 0.1 \cdot 6 = 0.4$
$f(x^{(1)}) = f(0.4) = 0.84$

Pentru $k = 1$:
$\frac{\partial}{\partial x} f(x^{(1)}) = 1.2$
$x^{(2)} = x^{(1)} - learningRate \cdot 1.2 = 0.4 - 0.1 \cdot 1.2 = 0.28$
$f(x^{(2)}) = f(0.28) = 0.7536$

Pentru $k = 2$:
$\frac{\partial}{\partial x} f(x^{(2)}) = 0.24$
$x^{(3)} = x^{(2)} - learningRate \cdot 0.24 = 0.28 - 0.1 \cdot 0.24 = 0.256$
$f(x^{(3)}) = f(0.256) = 0.7501$

b.

$f(x_1, x_2) = x_1^2 + sin(x_1 + x_2) + x_2^2$
$x^{(k)} = (x_1^{(k)}, x_2^{(k)})$

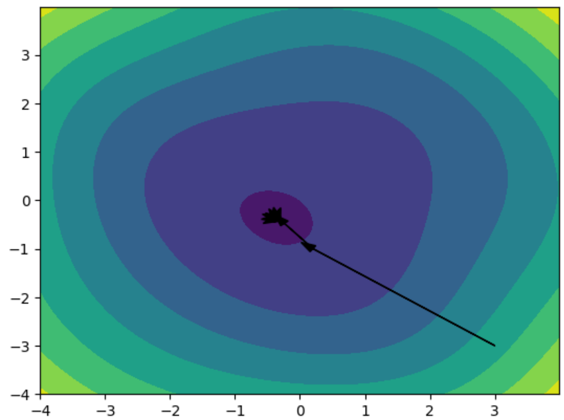
$x_1^{(k+1)} \leftarrow x_1^{(k)} - learningRate \cdot \frac{\partial}{\partial x_1} f(x_1^{(k)}, x_2^{(k)})$

$x_2^{(k+1)} \leftarrow x_2^{(k)} - learningRate \cdot \frac{\partial}{\partial x_2} f(x_1^{(k)}, x_2^{(k)})$

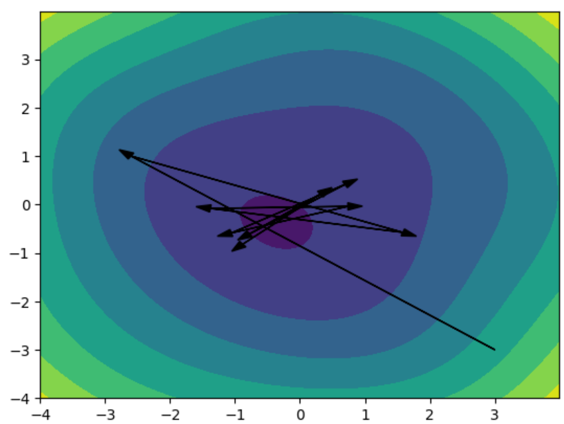
$\frac{\partial}{\partial x_1} f(x_1^{(k)}, x_2^{(k)}) = 2x_1 + cos(x_1 + x_2)$
$\frac{\partial}{\partial x_2} f(x_1^{(k)}, x_2^{(k)}) = 2x_2 + cos(x_1 + x_2)$

$f : [-4, 4] \rightarrow [-4, 4]$

i. Primii 10 pași făcuți de **GD** începând cu $(x_1^{(0)}, x_2^{(0)}) = (3, -3)$ și $learningRate = 0.4$:



ii. Primii 10 pași făcuți de **GD** începând cu $(x_1^{(0)}, x_2^{(0)}) = (3, -3)$ și $learningRate = 0.8$:



Se observă faptul că punctele converg către o valoare de minim. Mai mult, *learningRate*-ul configurează dimensiunea "pașilor" cu care convergența are loc.

Problema 1.34 / pag. 332

Vom schimba setul de date inițial cu cel de la **Problema 4.6 / pag. 491**.

Astfel, vom face următoarele translări pentru a ne ajuta în rezolvarea problemei curent:

- Clasa: $X_1 \in \{0, 1\}$, unde I = 0, Inferioară = 1;
- Sexul: $X_2 \in \{0, 1\}$, unde Masculin = 0, Feminin = 1;
- Vârsta: $X_3 \in \{0, 1\}$, unde Copil = 0, Adult = 1;
- Supraviețuitor: $Y \in \{0, 1\}$, unde Nu = 0, Da = 1.

Setul de date devine:

Indecși	Număr	X_1	X_2	X_3	Y
$[1, 5]$	5	0	0	0	1
$[6, 123]$	118	0	0	1	0
$[124, 180]$	57	0	0	1	1
$[181, 181]$	1	0	1	0	1
$[182, 185]$	4	0	1	1	0
$[186, 325]$	140	0	1	1	1
$[326, 360]$	35	1	0	0	0
$[361, 384]$	24	1	0	0	1
$[385, 1595]$	1211	1	0	1	0
$[1596, 1876]$	281	1	0	1	1
$[1877, 1893]$	17	1	1	0	0
$[1894, 1920]$	27	1	1	0	1
$[1921, 2025]$	105	1	1	1	0
$[2026, 2201]$	176	1	1	1	1

a.

$l(w) = \sum_{i=1}^{2201} (y^{(i)} \ln \sigma(w \cdot x^{(i)}) + (1 - y^{(i)}) \ln(1 - \sigma(w \cdot x^{(i)})))$
$= 5 \cdot 1 \cdot \ln \sigma(w \cdot (1, 0, 0, 0)^T) + 118 \cdot (1 - 0) \cdot \ln(1 - \sigma(w \cdot (1, 0, 0, 1)^T)) + 57 \cdot 1 \cdot \ln \sigma(w \cdot (1, 0, 0, 1)^T) + 1 \cdot 1 \cdot \ln \sigma(w \cdot (1, 0, 1, 0)^T) + 4 \cdot (1 - 0) \cdot \ln(1 - \sigma(w \cdot (1, 0, 1, 1)^T)) + 140 \cdot 1 \cdot \ln \sigma(w \cdot (1, 0, 1, 1)^T) + 35 \cdot (1 - 0) \cdot \ln(1 - \sigma(w \cdot (1, 1, 0, 0)^T)) + 24 \cdot 1 \cdot \ln \sigma(w \cdot (1, 1, 0, 0)^T) + 1211 \cdot (1 - 0) \cdot \ln(1 - \sigma(w \cdot (1, 1, 0, 1)^T)) + 281 \cdot 1 \cdot \ln \sigma(w \cdot (1, 1, 0, 1)^T) + 17 \cdot (1 - 0) \cdot \ln(1 - \sigma(w \cdot (1, 1, 1, 0)^T)) + 27 \cdot 1 \cdot \ln \sigma(w \cdot (1, 1, 1, 0)^T) + 105 \cdot (1 - 0) \cdot \ln(1 - \sigma(w \cdot (1, 1, 1, 1)^T)) + 176 \cdot 1 \cdot \ln \sigma(w \cdot (1, 1, 1, 1)^T)$
$= 5 \ln \sigma(w_0) + 118 \ln(1 - \sigma(w_0 + w_3)) + 57 \ln \sigma(w_0 + w_3) + \ln \sigma(w_0 + w_2) + 4 \ln(1 - \sigma(w_0 + w_2 + w_3)) + 140 \ln \sigma(w_0 + w_2 + w_3) + 35 \ln(1 - \sigma(w_0 + w_1)) + 24 \ln \sigma(w_0 + w_1) + 1211 \ln(1 - \sigma(w_0 + w_1 + w_3)) + 281 \ln \sigma(w_0 + w_1 + w_3) + 17 \ln(1 - \sigma(w_0 + w_1 + w_2)) + 27 \ln \sigma(w_0 + w_1 + w_2) + 105 \ln(1 - \sigma(w_0 + w_1 + w_2 + w_3)) + 176 \ln \sigma(w_0 + w_1 + w_2 + w_3)$

b.

i.

$\nabla_w l(w) = \sum_{i=1}^{2201} [y^{(i)} - \sigma(w \cdot x^{(i)})] x^{(i)}$
$= 5[1 - \sigma(w_0)](1, 0, 0, 0)^T - 118\sigma(w_0 + w_3)(1, 0, 0, 1)^T + 57[1 - \sigma(w_0 + w_3)](1, 0, 0, 1)^T + [1 - \sigma(w_0 + w_2)](1, 0, 1, 0)^T - 4\sigma(w_0 + w_2 + w_3)(1, 0, 1, 1)^T + 140[1 - \sigma(w_0 + w_2 + w_3)](1, 0, 1, 1)^T - 35\sigma(w_0 + w_1)(1, 1, 0, 0)^T + 24[1 - \sigma(w_0 + w_1)](1, 1, 0, 0)^T - 1211\sigma(w_0 + w_1 + w_3)(1, 1, 0, 1)^T + 281[1 - \sigma(w_0 + w_1 + w_3)](1, 1, 0, 1)^T - 17\sigma(w_0 + w_1 + w_2)(1, 1, 1, 0)^T + 27[1 - \sigma(w_0 + w_1 + w_2)](1, 1, 1, 0)^T - 105\sigma(w_0 + w_1 + w_2 + w_3)(1, 1, 1, 1)^T + 176[1 - \sigma(w_0 + w_1 + w_2 + w_3)](1, 1, 1, 1)^T$

$= (711 - 5\sigma(w_0) - 175\sigma(w_0+w_3) - \sigma(w_0+w_2) - 144\sigma(w_0+w_2+w_3) - 59\sigma(w_0+w_1) - 1492\sigma(w_0+w_1+w_3) - 44\sigma(w_0+w_1+w_2) - 281\sigma(w_0+w_1+w_2+w_3),$
$508 - 59\sigma(w_0+w_1) - 1492\sigma(w_0+w_1+w_3) - 44\sigma(w_0+w_1+w_2) - 281\sigma(w_0+w_1+w_2+w_3),$
$344 - \sigma(w_0+w_2) - 144\sigma(w_0+w_2+w_3) - 44\sigma(w_0+w_1+w_2) - 281\sigma(w_0+w_1+w_2+w_3),$
$654 - 175\sigma(w_0+w_3) - 144\sigma(w_0+w_2+w_3) - 1492\sigma(w_0+w_1+w_3) - 281\sigma(w_0+w_1+w_2+w_3))^T$

ii.

Alegem $j = 2$:

$\frac{\partial}{\partial w_2} l(w) = \frac{\partial}{\partial x_2} \ln\sigma(w_0+w_2) + \frac{\partial}{\partial x_2} 4\ln(1-\sigma(w_0+w_2+w_3)) + \frac{\partial}{\partial x_2} 140\ln\sigma(w_0+w_2+w_3) + \frac{\partial}{\partial x_2} 17\ln(1-\sigma(w_0+w_1+w_2)) + \frac{\partial}{\partial x_2} 27\ln\sigma(w_0+w_1+w_2) + \frac{\partial}{\partial x_2} 105\ln(1-\sigma(w_0+w_1+w_2+w_3)) + \frac{\partial}{\partial x_2} 176\ln\sigma(w_0+w_1+w_2+w_3)$
$= 344 - \sigma(w_0+w_2) - 144\sigma(w_0+w_2+w_3) - 44\sigma(w_0+w_1+w_2) - 281\sigma(w_0+w_1+w_2+w_3)$

Se poate ușor observa că rezultatul coincide cu poziția 2 din vectorul gradient calculat la **i**.

c.

$w = 0 \in \mathbb{R}^4$, deci $w_i = 0, i \in \{0,1,2,3\}$

$\sigma(z) = \frac{1}{1+e^{-z}} \implies \sigma(0) = \frac{1}{1+e^0} = 0.5$

$\nabla_w l(w) = (711 - 5\cdot 0.5 - 175\cdot 0.5 - 0.5 - 144\cdot 0.5 - 59\cdot 0.5 - 1492\cdot 0.5 - 44\cdot 0.5 - 281\cdot 0.5, 508 - 59\cdot 0.5 - 1492\cdot 0.5 - 44\cdot 0.5 - 281\cdot 0.5, 344 - 0.5 - 144\cdot 0.5 - 44\cdot 0.5 - 281\cdot 0.5, 654 - 175\cdot 0.5 - 144\cdot 0.5 - 1492\cdot 0.5 - 281\cdot 0.5)^T$
$= (-389.5, -430, 109, -392)^T$

Aplicam rata de învățare: $learningRate = 0.1$

$0.1 \cdot (-389.5, -430, 109, -392)^T = (-38.95, -43, 10.9, -39.2)^T$

d.

Valorile optime pentru weight-uri sunt următoarele:

$w = (0.3614, -1.1881, 2.1105, -0.6651)$

Vom clasifica următoarele instanțe:

Nume	X_1	X_2	X_3	Y
U	0	1	1	?
V	1	1	0	?
W	1	0	1	?

Clasificăm instanța U:
$w_0 + w_1 \cdot x_1 + w_2 \cdot x_2 + w_3 \cdot x_3$
$= 0.3614 + 0 \cdot -1.1881 + 1 \cdot 2.1105 + 1 \cdot -0.6651$
$= 1.8068 > 0 \implies Y_U^{prediction} = 1$

Clasificăm instanța V:

$w_0 + w_1 \cdot x_1 + w_2 \cdot x_2 + w_3 \cdot x_3$
$= 0.3614 + 1 \cdot -1.1881 + 1 \cdot 2.1105 + 0 \cdot -0.6651$
$= 1.2830 > 0 \implies Y_V^{prediction} = 1$

Clasificăm instanța W:

$w_0 + w_1 \cdot x_1 + w_2 \cdot x_2 + w_3 \cdot x_3$
$= 0.3614 + 1 \cdot -1.1881 + 0 \cdot 2.1105 + 1 \cdot -0.6651$
$= -1.4918 < 0 \implies Y_W^{prediction} = 0$

e.

i.

$H_w = -\sum_{i=1}^{n} h(x^{(i)})(1 - h(x^{(i)}))x^{(i)}(x^{(i)})^T$

$H_w = -\{5 \cdot \sigma(w_0)(1 - \sigma(w_0))(1, 0, 0, 0)^T(1, 0, 0, 0)$
$+ 175 \cdot \sigma(w_0 + w_3)(1 - \sigma(w_0 + w_3))(1, 0, 0, 1)^T(1, 0, 0, 1)$
$+ \sigma(w_0 + w_2)(1 - \sigma(w_0 + w_2))(1, 0, 1, 0)^T(1, 0, 1, 0)$
$+ 144 \cdot \sigma(w_0 + w_2 + w_3)(1 - \sigma(w_0 + w_2 + w_3))(1, 0, 1, 1)^T(1, 0, 1, 1)$
$+ 59 \cdot \sigma(w_0 + w_1)(1 - \sigma(w_0 + w_1))(1, 1, 0, 0)^T Z(1, 1, 0, 0)$
$+ 1492 \cdot \sigma(w_0 + w_1 + w_3)(1 - \sigma(w_0 + w_1 + w_3))(1, 1, 0, 1)^T(1, 1, 0, 1)$
$+ 44 \cdot \sigma(w_0 + w_1 + w_2)(1 - \sigma(w_0 + w_1 + w_2))(1, 1, 1, 0)^T(1, 1, 1, 0)$
$+ 281 \cdot \sigma(w_0 + w_1 + w_2 + w_3)(1 - \sigma(w_0 + w_1 + w_2 + w_3))(1, 1, 1, 1)^T(1, 1, 1, 1)\}$

ii.

Alegem $j = 2$ și $k = 1$:

$H_w(1, 2) = \frac{\partial^2}{\partial w_1 \partial w_2} l(w) = \frac{\partial}{\partial w_1}(\frac{\partial}{\partial w_2} l(w))$
$= \frac{\partial}{\partial w_1}(344 - \sigma(w_0 + w_2) - 144\sigma(w_0 + w_2 + w_3) - 44\sigma(w_0 + w_1 + w_2) - 281\sigma(w_0 + w_1 + w_2 + w3))$
$= -44\sigma(w_0 + w_1 + w_2) \cdot (1 - \sigma(w_0 + w_1 + w_2)) - 281\sigma(w_0 + w_1 + w_2 + w_3) \cdot (1 - \sigma(w_0 + w_1 + w_2 + w_3))$

Se poate vedea că rezultatul obținut coincide cu cel de pe linia și coloana corespunzătoare din hessiană.

iii.

Vom calcula $H(2, 1)$ și verifica egalitatea cu $H(1, 2)$.

$H(2, 1) = \frac{\partial^2}{\partial w_2 \partial w_1} l(w) = \frac{\partial}{\partial w_2}(\frac{\partial}{\partial w_1} l(w))$
$= \frac{\partial}{\partial w_2}(35 \ln(1 - \sigma(w_0 + w_1)) + 24 \ln \sigma(w_0 + w_1) + 1211 \ln(1 - \sigma(w_0 + w_1 + w_3)) + 281 \ln \sigma(w_0 + w_1 + w_3) + 17 \ln(1 - \sigma(w_0 + w_1 + w_2)) + 27 \ln \sigma(w_0 + w_1 + w_2) + 105 \ln(1 - \sigma(w_0 + w_1 + w_2 + w_3)) + 176 \ln \sigma(w_0 + w_1 + w_2 + w_3))$
$= \frac{\partial}{\partial w_2}(508 - 59\sigma(w_0 + w_1) - 1492\sigma(w_0 + w_1 + w_3) - 44\sigma(w_0 + w_1 + w_2) - 281\sigma(w_0 + w_1 + w_2 + w_3))$
$= -44\sigma(w_0 + w_1 + w_2)(1 - \sigma(w_0 + w_1 + w_2)) - 281\sigma(w_0 + w_1 + w_2 + w_3)(1 - \sigma(w_0 + w_1 + w_2 + w_3))$

Asadar, egalitatea cerută se respectă.

f.

Valorile optime obținute pentru w sunt următoarele:

$w = (0.5231, -1.1903, 2.1018, -0.6594)$

Modelul astfel obținut va clasifica identic instanțele de test, întrucât weight-urile converg către aceleași puncte.

Metoda gradientului necesita un număr de iterații semnificativ mai mare decât Newton-Raphson, întrucât cel din urma converge cu rată de ordin pătratic. În schimb, timpul de execuție este mult mai mic in cazul metodei gradientului, deoarece Newton-Raphson necesită calcularea matricii hessiene.

g.

$J(w) = \frac{1}{n}\sum_{i=1}^{n} \ln(1 + exp(-y'^{(i)} \cdot w \cdot x^{(i)}))$
$= \frac{1}{2201}(5\ln(1+exp(-w_0)) + 118\ln(1+exp(w_0+w_3)) + 57\ln(1+exp(-w_0-w_3)) + \ln(1+exp(-w_0-w_2)) + 4\ln(1+exp(w_0+w_2+w_3)) + 140\ln(1+exp(-w_0-w_2-w_3)) + 35\ln(1+exp(w_0+w_1)) + 24\ln(1+exp(-w_0-w_1)) + 1211\ln(1+exp(w_0+w_1+w_3)) + 281\ln(1+exp(-w_0-w_1-w_2)) + 105\ln(1+exp(w_0+w_1+w_2+w_3)) + 176\ln(1+exp(-w_0-w_1-w_2-w_3)))$

Știm că $1 - \sigma(z) = \sigma(-z)$, unde $z = w \cdot x$.

$\sum_{i=1}^{n} \ln(1 + exp(-y^{(i)} \cdot w \cdot x^{(i)})) = \ln\prod_{i=1}^{n}(1 + exp(-y^{(i)} \cdot w \cdot x^{(i)}))$

$$1 + exp(-y^{(i)} \cdot w \cdot x^{(i)}) = \begin{cases} 1 + e^{-w \cdot x^{(i)}} = (\frac{1}{1+e^{-w \cdot x^{(i)}}})^{-1} = (\sigma(w \cdot x^{(i)}))^{(-1)y'^{(i)}}, & \text{dacă } y'^{(i)} = 1, \\ 1 + e^{w \cdot x^{(i)}} = (\frac{1}{1+e^{w \cdot x^{(i)}}})^{-1} = (\sigma(-w \cdot x^{(i)}))^{(-1)y'^{(i)}}, & \text{dacă } y'^{(i)} = 0. \end{cases}$$

Deci, pentru fiecare $y^{(i)}$:

$$\ln(1 + exp(-y^{(i)} \cdot w \cdot x^{(i)})) = \begin{cases} -\ln\sigma(w \cdot x^{(i)}), & \text{dacă } y^{(i)} = 1, \\ -\ln(1 - \sigma(w \cdot x^{(i)})), & \text{dacă } y^{(i)} = 0. \end{cases}$$

Astfel, putem rescrie formula lui $J(w)$, utilizând proprietățile logaritmului și definiția precendentă:

$\ln\prod_{i=1}^{n}(\sigma(w \cdot x^{(i)}))^{-y^{(i)}} \cdot (1 - \sigma(w \cdot x^{(i)}))^{-(1-y^{(i)})} = -\sum_{i=1}^{n}(y^{(i)}\ln\sigma(w \cdot x^{(i)}) + (1 - y^{(i)})\ln(1 - \sigma(w \cdot x^{(i)})))$

De unde reiese relația $J(w) = -\frac{1}{n}l(w)$.