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MATH 1207

Homework 12

December 4, 2019

Apostol p.278 no.5

We have from Apostol that if we have that if $f(x) = P_n(x) + x^n g(x)$, where $P_n(x)$ is a polynomial of degree n, and $\lim_{x\to 0} g(x) = 0$, then we have that $P_n(x)$ is the Taylor polynomial of f of degree n.

It is proved in class that

$$(1-x)\sum_{k=0}^{n} x^k = 1 - x^{n+1}$$

Substituting x^2 for x,

$$\begin{aligned}
\implies (1-x^2)\sum_{k=0}^{n} x^{2k} &= 1 - x^{2n+2} \\
\implies \qquad \frac{1}{1-x^2} &= \sum_{k=0}^{n} x^{2k} + \frac{x^{2n+2}}{1-x^2} \\
\implies \qquad \frac{x}{1-x^2} &= \sum_{k=0}^{n} x^{2k+1} + x^{2n+2}\frac{x}{1-x^2} \\
&= P_{2n+1}(x) + x^{2n+2}\frac{x}{1-x^2}
\end{aligned}$$

Since we have that $\frac{x}{1=x^2}$ is continuous, we have that it approaches zero when x approaches zero.

From the above, we have that

$$P_{2n+1} = T_{2n+1}\left(\frac{x}{1-x^2}\right) = \sum_{k=0}^{n} x^{2k+1}$$

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Apostol p.430 no.17

Claim. Let $f_n(x) = nxe^{-nx^2}$ for $x \in \mathbb{R}$, $n \in \mathbb{Z}_{>0}$.

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)dx$$

Proof. Let us compute the left side first.

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 nxe^{-nx^2} &= \lim_{n\to\infty}\left(-\frac{1}{2}e^{-nx^2}\Big|_0^1\right)\\ &= \lim_{n\to\infty}\left(\frac{1}{2}(1-e^{-n})\right)\\ &= \lim_{n\to\infty}\frac{1}{2}-\lim_{n\to\infty}\frac{1}{2}e^{-n}\\ &= \frac{1}{2}\end{aligned}$$

Taking the limit $\lim_{n\to\infty} f_n(x)$, we see that $\lim_{n\to\infty} nxe^{-nx^2}=0$, as for all $\epsilon>0$, pick

$$\int_0^1 \lim_{n\to\infty}\left(nxe^{-nx^2}\right)=\int_0^1 0=0$$

The limit is proved in the Apostol reading as Theorem 7.11 (and is also entirely believable). $\square$

Problem 1

Claim. Let $f_n:[a,b]\to\mathbb{R}$ be a sequence of integrable functions converging uniformly to $f:[a,b]\to\mathbb{R}$. f is integrable and

$$\lim_{n\to\infty}\int_a^b f_n(x)dx=\int_a^b f(x)dx$$

Proof. A function f is only integrable if any ϵ there exist step functions s,t such that $s\le f\le t$ and $\int_a^b(t-s)<\epsilon$.

Let $\epsilon'=\frac{\epsilon}{3(b-a)}$. Since f_n uniformly converges to f, we have that $\exists N \mid n>N \implies \forall x\in[a,b], |f(x)-f_n(x)|<\epsilon'$. Then, since f_n is integrable, we have s_n,t_n such that $\int_a^b(t_n-s_n)<(b-a)\epsilon'$. Now, consider $\int_a^b((t_n+\epsilon')-(s_n-\epsilon'))=\int_a^b(t_n-s_n)+2(b-a)\epsilon'<3(b-a)\epsilon'=\epsilon$. We have that $s_n-\epsilon'\le f_n(x)-\epsilon'\le f(x)\le f_n(x)+\epsilon'\le t_n$.

We see that there exist step functions $t=t_n+\epsilon', s=s_n-\epsilon'$ such that $\int_a^b(t-s)<\epsilon$ for any $\epsilon>0$.

To show the given identity, for any $\epsilon>0$, let $\epsilon'=\frac{\epsilon}{b-a}$. Then, $\exists N\mid n>N \implies \forall x\in[a,b], |f(x)-f_n(x)|<\epsilon' \implies \int_a^b|f(x)-f_n(x)|dx<(b-a)\epsilon'=\epsilon$.

Since $\left|\int_a^b f(x)dx-\int_a^b f_n(x)dx\right| = |\int_a^b(f(x)-f_n(x))dx| \le \int_a^b|f(x)-f_n(x)|dx<\epsilon$ for all $n>N$, we have that $\lim_{n\to\infty}\int_a^b f_n(x)dx=\int_a^b f(x)dx$. $\square$

Problem 2

a

We have that $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{1+nx^2} = 0$, which will be shown in part b as $|f_n(x)| \leq \frac{1}{\sqrt{n}}$, and so $\lim_{n\to\infty} |f_n(x)| \leq \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \implies \lim_{n\to\infty} f_n(x) = 0$.

For $g(x)$, we have that $f_n'(x) = -\frac{2nx^2}{(1+nx^2)^2} + \frac{1}{1+nx^2} = -\frac{1}{n}\frac{2x^2}{\frac{1}{n^2}+\frac{2x^2}{n}+x^4} + \frac{1}{1+nx^2}$. For all $x \neq 0$, we have that $g(x) = \lim_{n\to\infty} f'(x) = 0$ as well; however, at $x = 0$, we have that $f_n'(0) = 1$, and so $\lim_{n\to\infty} f_n'(0) = 1$.

b

We find the local extrema of f by taking the zeros of the derivative of f. We see then that this occurs when $2nx^2 = 1 + nx^2 \implies nx^2 = 1 \implies x = \pm\frac{1}{\sqrt{n}} \implies f(x) = \pm\frac{1}{2\sqrt{n}}$ as the maxima and minima. Since we see that $\lim_{x\to\infty} f(x) = 0, \lim_{x\to-\infty} f(x) = 0$, $|f_n(x)| < \frac{1}{\sqrt{n}}$. This means that for any $\epsilon > 0$, we have that $\forall x \in \mathbb{R}$, $|f_n(x)| < \epsilon$ if $n > N = \left\lceil \frac{1}{\epsilon^2} \right\rceil$ and so f_n converge uniformly to 0.

c

f is constant and so differentiable everywhere. Then, we have that $f'(x) = g'(x) \iff x \neq 0$.

Problem 5

Claim. If f_n, g_n are sequences of bounded functions on an interval I, and $f_n \to f$ and $g_n \to g$ both uniformly, then $f_n + g_n \to f + g$ uniformly.

Proof. Since f_n, g_n uniformly converge to f, g, we have that for $\frac{\epsilon}{2} > 0, \exists N_f, N_g \mid n > N_f \implies |f_n(x) - f(x)| < \frac{\epsilon}{2}, n > N_g \implies |g_n(x) - g(x)| < \frac{\epsilon}{2}$. Then, for $x > \max\{N_f, N_g\}$, we have that $|f_n(x) + g_n(x) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \epsilon$. □

Problem 7

Claim.

$$\sum_{n=0}^{\infty} \frac{1}{2^n + x^{2n}}$$

is everywhere convergent to a continuous function that can be integrated term by term.

Proof. Uniform convergence follows from the Weierstrass M-test. Note that we have that $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x^n)^2 \geq 0 \implies 2^n + x^{2n} \geq 2^n \implies \frac{1}{2^n+x^{2n}} \leq \frac{1}{2^n}$. Thus, let $M_n = \frac{1}{2^n}$. We have that $\sum_{n=0}^{\infty} M_n$ is a convergent geometric series and thus $\sum_{n=0}^{\infty} \frac{1}{2^n+x^{2n}}$ is uniformly convergent everywhere.

The claim about integrability will follow immediately from application of problem 1 as we have that the individual terms of the series are integrable, and we have that each partial sum is then integrable, and thus the function can be integrated term by term. □

Problem 9

a

$g(x)$ is positive on $(0,\infty)$ since we have that $\exp(t)$ is positive, and $g(x) = \exp(\frac{-1}{x^2})$ on $(0,\infty)$. $g(x)$ is clearly smooth for $x < 0$ as it is constant; in fact on this domain the k^{th} derivative is 0.

For $x > 0$, we will show that $g^{(n)}(x)$ is a function of the form $P(\frac{1}{x})g(x)$, where $P(t)$ is a polynomial in t. Consider first the base case of $n = 0$, which has $P(\frac{1}{x}) = 1$. Then, in the inductive case, we have $f^{(k)}(x) = P(\frac{1}{x})e^{-\frac{1}{x^2}} \implies f^{(k+1)}(x) = P'(\frac{1}{x})e^{-\frac{1}{x^2}} + \left(\frac{2}{x^3}\right) P(\frac{1}{x})e^{-\frac{1}{x^2}} = (P'(\frac{1}{x}) + \left(\frac{2}{x^3}\right) P(\frac{1}{x}))e^{-\frac{1}{x^2}}$. However, we have that $P'(\frac{1}{x})$ is still a polynomial in $\frac{1}{x}$ by the power rule, and so $f^{(k+1)}(x) = P(\frac{1}{x})e^{-\frac{1}{x^2}}$ as well.

The final case is that g must be infinitely differentiable at $x = 0$. To do this, we will show that $g^{(n)}(x) = 0$ for all n. We first show that $\lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^m} = 0$ for all $m > 0$. Putting t to $\frac{1}{x^2}$, the limit becomes $\lim_{t\to\infty} \frac{t^{\frac{m}{2}}}{e^t}$, which is shown to be 0 in Apostol (as mentioned above). Extending, we see that for any polynomial $P(\frac{1}{x})$ we have that $\lim_{x\to 0} P(\frac{1}{x})e^{-\frac{1}{x^2}} = 0$, as the sum of limits is the limit of the sum.

We now proceed with induction. The base case is $n = 0$, and we see that $g(0) = 0$ by definition. We compute the inductive step as follows: assume that $g^{(k)}(0) = 0$. Then, $g^{(k+1)}(0) = \lim_{h\to 0} \frac{g^{(k)}(h)}{h} = \lim_{h\to 0} \frac{g(h)P(\frac{1}{x})}{h} = \lim_{h\to 0} e^{-\frac{1}{h^2}} P(\frac{1}{x}) = 0$, which holds since for $h < 0$, we have that the derivative is also 0. The derivative is then 0 at $x = 0$, and the function is then smooth.

b

We will show here that the product of smooth functions is itself smooth. Let a, b be smooth functions on an interval I. Then the n^{th} derivative is given by $\sum_{k=0}^{n} \binom{n}{k} a^{(k)} b^{(n-k)}$. To show this identity, we can induct on n, as the base case holds with n = 0. Then, consider that if $(ab)^{(k)} = \sum_{k=0}^{n} \binom{n}{k}(a^{(k+1)}b^{(n-k)} + a^{(k)}b^{(n-k+1)}) = \sum_{k=1}^{n}(\binom{n}{k-1} + \binom{n}{k})a^{(k)}b^{(n+1-k)} + a^{(n+1)}b +$

$ab^{(n+1)} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{(k)} b^{n+1-k}$. These all exist by the fact that a, b are themselves smooth.
We have that $h(x) = g(1-x)g(1+x)$ as the product of smooth functions is itself smooth, and since $F'(x) = \frac{1}{A}h(x)$ by the fundamental theorem of calculus, we have that F also must be smooth.

Further, we have that since $h(x) = 0$ for $x < -1$ as $g(1+x) = 0$ and $h(x) = 0$ for $x > 1$ as $g(1-x) = 0$, for $x < -1$, $\int_{-1}^{x} h(t)dt = -\int_{x}^{-1} h(t)dt = \int_{x}^{-1} 0dt = 0$. Similarly, we have that for $x > 1$, $\int_{-1}^{x} h(t)dt = \int_{-1}^{1} h(t)dt + \int_{1}^{x} h(t)dt = A + \int_{1}^{x} 0dt = A$.

This gives us that for $x < -1$, $F(x) = 0$, for $x > 1$, $F(x) = \frac{1}{A}A = 1$.

This gives us that for $x < -1$, $F(x) = 0$, for $x > 1$, $F(x) = \frac{1}{A}A = 1$.

c

Consider

$$\phi(x) = F(2-x)F(2+x)$$

For $x \in \mathbb{R}$, we have that $0 \leq F(2-x), |F(2+x) \leq 1 \implies 0 \leq \phi(x) \leq 1$..

For $x < -3$, $F(2+x) = 0 \implies \phi(x) = 0$. For $x > 0$, $F(2-x) = 0 \implies \phi(x) = 0$.

For $x \in (-1,1), F(2-x) = F(2+x) = 1 \implies \phi(x) = 1$.

ϕ is smooth as the product of smooth functions.

Problem 10

Consider

$$f(x) = \phi(\frac{6}{|c-d|}(x-c))g(x) + \phi(\frac{6}{|c-d|}(x-d))h(x)$$

For $x \in (c - \frac{|c-d|}{6}, c + \frac{|c-d|}{6})$, we have that $\frac{6}{|c-d|}(x-c) \in (-1,1)$ and for $x \notin (c - \frac{|c-d|}{2}, c + \frac{|c-d|}{2})$, we have that $\frac{6}{|c-d|}(x-c) \notin (-3,3)$ and $f(x) = \phi(\frac{6}{|c-d|}(x-d))h(x)$.

For $x \in (d - \frac{|c-d|}{6}, d + \frac{|c-d|}{6})$, we have that $\frac{6}{|c-d|}(x-d) \in (-1,1)$ and for $x \notin (d - \frac{|c-d|}{2}, d + \frac{|c-d|}{2})$, we have that $\frac{6}{|c-d|}(x-d) \notin (-3,3)$ and $f(x) = \phi(\frac{6}{|c-d|}(x-c))g(x)$.

Furthermore, we have that $(d - \frac{|c-d|}{2}, d + \frac{|c-d|}{2}) \cup (c - \frac{|c-d|}{2}, c + \frac{|c-d|}{2}) = \emptyset$, so within $\delta = \frac{|c-d|}{6}$ of c, we have that $f(x) = g(x)$ and within δ of d, we have that $f(x) = h(x)$.

Problem 11

We will first prove a couple lemmas:

Lemma. Let $U = \{x \in [c, b) \mid \forall y \in [c, x), f(y) = g(y)\}$. Then the supremum u of this set exists and $\exists u' > u \mid f(x) = g(x)$ on $[c, \min(b, u'))$.

Proof. We know that $c + \delta \in U$, and the supremum exists as it is bounded above by b. Thus, $\sup(U) = u$ exists.

Consider that $f - g$ is also analytic at u. Thus, we have that for $R > 0$, within $(u - R, u + R)$, $f - g$ is equal to some power series $\sum_{n=0}^{\infty} a_n(x - u)^n$.

Now we show that if a power series f is convergent on $(c - R, c + R)$ is zero on a sequence $\{x_n\}$ that converges to c on that interval, then the power series must be the zero function. Similarly, since $f - g$ is analytic and thus smooth, we must have that $f^{(k)}(x)$ is continuous, and also has as $\lim_{n\to\infty} f^{(k)}(x_n) = f^{(k)}(c) = 0$ by the definition of sequential continuity.

We now apply the above fact to the power series centered at u, which now must vanish as the approximation property furnishes such a sequence converging to u, as for any $\epsilon > 0, \exists x \in U \mid u - x < \epsilon$ and $(f - g)(x) = 0$. Thus, we have that on $(u - R, u + R), f - g = 0$. The lemma is satisfied with $u' = u + R$. □

Lemma. Let $V = \{x \in (a, c] \mid \forall y \in (x, c], f(y) = g(y)\}$. Then the infimum v of this set exists and $\exists v' < v \mid f(x) = g(x)$ on $(\max(a, v'), c]$.

Proof. We know that $c - \delta \in V$, and the infimum exists as it is bounded below by b. Thus, $\inf(V) = v$ exists.

Consider that $f - g$ is also analytic at v. Thus, we have that for $R > 0$, within $(v - R, v + R)$, $f - g$ is equal to some power series $\sum_{n=0}^{\infty} a_n(x - u)^n$.

Since we have a sequence within V converging to v from the approximation property, we have that that power series is 0 and thus the lemma is satisfied with $v' = v - R$. □

Claim. Let $f, g : (a, b) \to \mathbb{R}$ be analytic functions. $\exists c \in (a, b), \delta > 0 \implies |x - c| < \delta \implies f(x) = g(x)$. Then $f = g$.

Proof. Let U, V, u, v, u', v' be as above. We know that $u \leq b$. Suppose that $u < b$. Then we have from the first lemma that $f = g$ on $[c, \min(b, u'))$, and so $\min(b, u') \in U$ but also $\min(b, u') > \sup(U)$ as well. $\Rightarrow\Leftarrow$, so $u = b$.

Similarly, we know that $v \geq a$. Suppose that $v > a$. Then we have from the second lemma that $f = g$ on $(\max(a, v'), c]$, and so $\max(a, v') \in V$ but also $\min(a, v') < \inf(V)$ as well. $\Rightarrow\Leftarrow$, so $v = a$.

Thus, we have that $f = g$ on (a, b). □