# MATH 4065 HW 9

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December 11, 2020

## 14

Let $f : \mathbb{H} \to \mathbb{D}$ be a conformal mapping. Then, we have that $G(z) = i\frac{1-z}{1+z}$ is a conformal mapping $\mathbb{D} \to \mathbb{H}$, so $(f \circ G)(z) = f\left(i\frac{1-z}{1+z}\right)$ is an automorphism of the disc, and thus takes the form $e^{i\theta}\psi_\alpha(z) = e^{i\theta}\frac{\alpha - z}{1-\overline{\alpha}z}$. But then, $G^{-1}(z) = \frac{i-z}{i+z}$, so

$$\begin{aligned}
f(z) &= e^{i\theta}\frac{\alpha - \frac{i-z}{i+z}}{1 - \overline{\alpha}\frac{i-z}{i+z}} \\
&= e^{i\theta}\frac{\alpha(i+z) - (i-z)}{(i+z) - \overline{\alpha}(i-z)} \\
&= e^{i\theta}\frac{(\alpha+1)z + (\alpha-1)i}{(\overline{\alpha}+1)z - (\overline{\alpha}-1)i} \\
&= e^{i\theta}\frac{\alpha+1}{\overline{\alpha}+1}\frac{z + \frac{\alpha-1}{\alpha+1}i}{z - \frac{\overline{\alpha}-1}{\overline{\alpha}+1}i}
\end{aligned}$$

but we have that $\left|\frac{\alpha+1}{\overline{\alpha}+1}\right| = \sqrt{\frac{(\alpha+1)(\overline{\alpha}+1)}{(\overline{\alpha}+1)(\alpha+1)}} = 1$, thus it is some rotation, so putting $\beta = -\frac{\alpha-1}{\alpha+1}i$,

$$= e^{i\varphi}\frac{z - \beta}{z - \overline{\beta}}$$

The last thing to check is that $\beta$ has positive imaginary part, which is the same as $\frac{\alpha-1}{\alpha+1}$ having negative real part. Let $\alpha = a + bi$:

$$\operatorname{Re}\left(\frac{a-1+bi}{a+1+bi}\right) = \frac{1}{(a+1)^2 + b^2}\operatorname{Re}((a-1+bi)(a+1+bi))$$

but we get

$$\operatorname{Re}((a-1+bi)(a+1+bi)) = \operatorname{Re}(a^2+b^2-1) < 0$$

since $a^2 + b^2 = |\alpha|^2 < 1$, so we get what we wanted.

# 15

## a

$\Phi(z) = \frac{az+b}{cz+d}$ for real $a, b, c, d$ with $ad - bc \neq 0$. Then, on the real line, $\Phi(x) = x$ should have 3 solutions since there are three distinct fixed points which must occur at $x \neq -d/c$, since $\Phi(-d/c) = \infty$. However,

$$\Phi(x) = x \implies x(cx + d) - ax + b = 0$$

which is a quadratic in $x$ when $c \neq 0$, and thus cannot have three roots. Thus, $c = 0 \implies xd - ax + b = 0$, which is linear when $d - a \neq 0$ and cannot have three roots. Then, we get that $a = d, c = 0 \implies b = 0$, so finally

$$\Phi(z) = \frac{az}{d} = z$$

## b

The easy part is uniqueness. If there were two such automorphisms $\Phi_1, \Phi_2$, then we have that $\Phi_1^{-1} \circ \Phi_2$ must take $x_i \mapsto y_i \mapsto x_i$, so $x_1, x_2, x_3$ are fixed points; from earlier, this gives

$$\Phi_1^{-1} \circ \Phi_2 = \text{id} \implies \Phi_2 = \Phi_1$$

To actually construct this, we will actually create two different maps: the first will take $x_1 \mapsto 0$, $x_2 \mapsto 1$ and $x_3 \mapsto \infty$, and the second will take $0 \mapsto y_1$, $1 \mapsto y_2$, and $\infty \mapsto y_3$. The first one is

$$F_1(z) = \frac{(z - x_1)(x_2 - x_3)}{(z - x_3)(x_2 - x_1)}$$

and the second

$$F_2(z) = \frac{y_3 z + \frac{y_3 - y_2}{y_2 - y_1} y_1}{z + \frac{y_3 - y_2}{y_2 - y_1}} = \frac{y_3(y_2 - y_1)z + y_1(y_3 - y_2)}{(y_2 - y_1)z + (y_3 - y_2)}$$

where their composition is an automorphism of $\mathbb{H}$ with the correct values, since the determinants of both mappings' associated matrix are positive (the first has $ad - bc = (x_1 - x_3)(x_2 - x_3)(x_2 - x_1) = - \cdot - \cdot + > 0$ and the second $ad - bc = (y_3 - y_2)(y_2 - y_1)(y_3 - y_1) = + \cdot + \cdot + > 0$). After a tedious manipulation to check, we get the composition as

$$\frac{(y_3(y_2 - y_1)(x_2 - x_3) + y_1(y_3 - y_2)(x_2 - x_1))z - x_1 y_3(y_2 - y_1)(x_2 - x_3) - x_3 y_1(y_3 - y_2)(x_2 - x_1)}{((y_2 - y_1)(x_2 - x_3) + (y_3 - y_2)(x_2 - x_1))z - x_1(y_2 - y_1)(x_2 - x_3) - x_3(y_3 - y_2)(x_2 - x_1)}$$

which recovers the desired values when substituted.

## 17

Since we only have to do the left equation, this is relatively fast. Consider that the area of the unit disc is $\pi$, so $\frac{1}{\pi}\iint_{\mathbb{D}} dxdy = 1$. Then, under the change of variables $(x, y) = \psi(u, v) = x(u, v) + y(u, v)$ (where $\psi(u + vi) = \psi(u, v)$), the determinant of the Jacobian (proved back in chapter 1, it follows immediately from Cauchy-Rieman and the definition of the Wirtinger derivatives, though here the role of $x, y$ and $u, v$ are swapped from the book) is

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2 = \left|2\frac{\partial u}{\partial z}\right|^2 = |\psi'(z)|^2$$

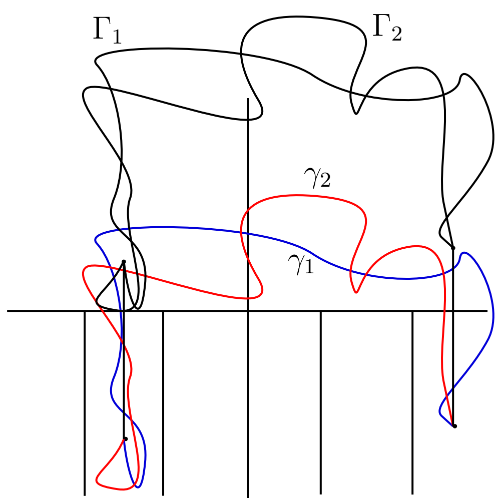
but $\psi$ is an automorphism of $\mathbb{D}$, so the area of integration stays the same, so

$$\frac{1}{\pi}\iint_{\mathbb{D}} dxdy = \frac{1}{\pi}\iint_{\mathbb{D}} |\psi'(z)|^2 dudv = 1$$

## 19

Call the slit plane in the problem $S$. (I assume that the $A_k$ in the problem are real numbers?) Suppose we have two points $z_1, z_2 \in S$ with paths $\gamma_1$ and $\gamma_2$ connecting them. Then, note that if $z \in S$, then $z + yi \in S$ for any $y \geq 0$, since if $z + yi \notin S$, then $z + yi = A_k + yi'$ for $y' \leq 0 \implies z = A_k + (y' - y)i \notin S$.

The goal here is to raise the paths and use that the upper half plane is simply connected.




We first show that $\gamma_1$ and $\gamma_2$ are homotopic to two different curves, $\Gamma_1$ and $\Gamma_2$ respectively. Since both $\gamma_1, \gamma_2$ are bounded, let $M = \max\{\sup|\gamma_1|, \sup|\gamma_2|\} + 1$ and define

$$\Gamma_i(t) = \begin{cases} \gamma_i(0) + 3Mit & 0 \leq t < \frac{1}{3} \\ \gamma_i(3t-1) + Mi & \frac{1}{3} \leq t < \frac{2}{3} \\ \gamma_i(1) + 3Mi(1-t) & \frac{2}{3} \leq t \leq 1 \end{cases}$$

Then, we have that the mappings

$$h_i(s,t) = \gamma_{i,s}(t) = \begin{cases} \gamma_i(0) + 3Mit & 0 \le t < \frac{s}{3} \\ \gamma_i\left(\frac{3t-s}{3t-2s}\right) + sMi & \frac{s}{3} \le t < 1 - \frac{s}{3} \\ \gamma_i(1) + 3Mi(1-t) & 1 - \frac{s}{3} \le t \le 1 \end{cases}$$

gives us a homotopy between $\gamma_1$ and $\Gamma_1$ as well as $\gamma_2$ and $\Gamma_2$. Then, the paths $\Gamma_{1,r} = \Gamma_1(t)|_{[\frac{1}{3},\frac{2}{3}]}$ and $\Gamma_{2,r} = \Gamma_2(t)|_{[\frac{1}{3},\frac{2}{3}]}$ lie entirely in the upper half plane (that is, the restrictions of $\Gamma_1$ and $\Gamma_2$), since $|\operatorname{Im}(\gamma_i(t))| < M$, and are thus homotopic. Furthermore, $\Gamma_1$ and $\Gamma_2$ coincide on $t \in [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Then, there is some continuous mapping $H_r(s,t)$ which satisfies $H_r(0,t) = \Gamma_{1,r}(t)$ and $H(1,t) = \Gamma_{2,r}(t)$, such that we can give

$$H(s,t) = \begin{cases} \Gamma_1(t) & 0 \le t < \frac{1}{3} \\ H_r(s, 3t-1) & \frac{1}{3} \le t < \frac{2}{3} \\ \Gamma_1(t) & \frac{2}{3} \le t \le 1 \end{cases}$$

which gives us a homotopy between $\Gamma_1$ and $\Gamma_2$. Then, we have that $\gamma_1$ and $\gamma_2$ are homotopic.

# 2

## a

Earlier, the book gives that the angle between two complex numbers $z, w$ is determined by the two quantities

$$\frac{(z,w)}{|z||w|} \text{ and } \frac{(z,-iw)}{|z||w|}$$

Then, we just directly these quantities for $(f \circ \gamma)'(t_0)$ and $(f \circ \eta)'(t_0)$, putting $\gamma(t_0) = \eta(t_0) = z_0$:

$$\frac{((f \circ \gamma)'(t_0), (f \circ \eta)'(t_0))}{|(f \circ \gamma)'(t_0)||(f \circ \eta)'(t_0)|} = \frac{(f'(z_0)\gamma'(t_0), f'(z_0)\eta'(t_0))}{|f'(z_0)\gamma'(t_0)||f'(z_0)\eta'(t_0)|}$$

and by the conjugate linearity of the inner product,

$$\frac{(f'(z_0)\gamma'(t_0), f'(z_0)\eta'(t_0))}{|f'(z_0)\gamma'(t_0)||f'(z_0)\eta'(t_0)|} = \frac{f'(z_0)\overline{f'(z_0)}(\gamma'(t_0), \eta'(t_0))}{|f'(z_0)||\gamma'(t_0)||f'(z_0)||\eta'(t_0)|} = \frac{(\gamma'(t_0), \eta'(t_0))}{|\gamma'(t_0)||\eta'(t_0)|}$$

A similar manipulation gives

$$\frac{((f \circ \gamma)'(t_0), -i(f \circ \eta)'(t_0))}{|(f \circ \gamma)'(t_0)||(f \circ \eta)'(t_0)|} = \frac{(f'(z_0)\gamma'(t_0), -if'(z_0)\eta'(t_0))}{|f'(z_0)\gamma'(t_0)||f'(z_0)\eta'(t_0)|}$$

and

$$\frac{(f'(z_0)\gamma'(t_0), -if'(z_0)\eta'(t_0))}{|f'(z_0)\gamma'(t_0)||f'(z_0)\eta'(t_0)|} = \frac{f'(z_0)\overline{f'(z_0)}(\gamma'(t_0), -i\eta'(t_0))}{|f'(z_0)||\gamma'(t_0)||f'(z_0)||\eta'(t_0)|} = \frac{(\gamma'(t_0), -i\eta'(t_0))}{|\gamma'(t_0)||\eta'(t_0)|}$$

so the angle between $(f \circ \gamma)'(t)$ and $(f \circ \eta)'(t)$ is the same as the angle between $\gamma'(t)$ and $\eta'(t)$.

## b

We just do the same calculation, but instead viewing $f$ as taking $\Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$. In particular, let $f(x,y) = (u(x,y), v(x,y))$ and $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, such that we get that $J_{f\circ\gamma} = J_f \cdot J_\gamma$ and similar for $\eta$. Then, we have that since $f$ preserves angles,

$$\frac{(J_{f\circ\gamma}, J_{f\circ\eta})}{|J_{f\circ\gamma}||J_{f\circ\eta}|} = \frac{(J_\gamma, J_\eta)}{|J_\gamma||J_\eta|} \implies (J_f \cdot J_\gamma, J_f \cdot J_\eta) = C(J_\gamma, J_\eta)$$

where $c = \frac{|J_{f\circ\gamma}||J_{f\circ\eta}|}{|J_\gamma||J_\eta|}$. Then, we have that by a easy property of the inner product,

$$(J_f \cdot J_\gamma, J_f \cdot J_\eta) = (J_\gamma, (J_f^T J_f) \cdot J_\eta) = C(J_\gamma, J_\eta) = (J_\gamma, CI \cdot J_\eta).$$

However, this holds for all paths $\gamma$ and $\eta$; thus, we can show that $J_f^T J_f = cI$. The ultimate goal is to show that $J_f$ is a rotation matrix. To see this, let $A = J_f^T J_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and take $\gamma(t) = z_0 + t$, $\eta(t) = z_0 + it$, such that $J_\gamma = \begin{bmatrix}1, 0\end{bmatrix}, J_\eta = \begin{bmatrix}0, 1\end{bmatrix}$ and we get that $(J_\gamma, AJ_\eta) = b = c(J_\gamma, J_\eta) = 0$. Taking this with $\gamma, \eta$ switched gives $c = 0$, and taking $\gamma(t) = \eta(t) = z_0 + t$ and $z_0 + it$ gives $a = d = C$ respectively. Then, we get that if $J_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we get

$$J_f^T J_f = \begin{bmatrix} a^2 + c^2 & ad + bc \\ ad + bc & b^2 + d^2 \end{bmatrix}$$

and since $a^2 + c^2 = C$ and $ad + bc = 0$, substituting we get $c^2(\frac{d^2}{b^2} + 1) = C\frac{c^2}{b^2} = C$ so $b = \pm c$ and $a = \mp d$. Now, we also have that (note multiplying by $-i$ is a rotation of $-90°$):

$$\frac{(J_{f\circ\gamma}, RJ_{f\circ\eta})}{|J_{f\circ\gamma}||J_{f\circ\eta}|} = \frac{(J_\gamma, RJ_\eta)}{|J_\gamma||J_\eta|} \implies (J_f \cdot J_\gamma, RJ_f \cdot J_\eta) = C(J_\gamma, RJ_\eta)$$

where $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, so by a similar argument to above we get that $J_f^T RJ_f = CR$, so then if $J_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we get that $ad - bc = C > 0$, but since $a = \pm d$ and $b = \pm c$, exactly one is positive (one has that both are of the same sign, the other that they are opposite), so we need $ad > 0$ and $bc < 0$, so $a = d$ and $b = -c$, so the Jacobian of $f$ has the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

which is exactly the Cauchy-Riemann equations, and we are done. Further, $|f'(z_0)|^2 = |J_f(z_0)|$ as seen earlier in the problem set, and $|J_f(z_0)| = a^2 + b^2 = C > 0$, so the derivative is also nonzero.