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Homework 11

MATH 1208
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Apostol p.362 no.1

$$\begin{aligned}
\int_0^1 \int_0^1 xy(x+y)dydx &= \int_0^1 \int_0^1 (x^2y + xy^2)dydx \\
&= \int_0^1 \left[\frac{1}{2}x^2y^2 + \frac{1}{3}xy^3\right]\Big|_{y=0}^{y=1} dx \\
&= \int_0^1 (\frac{1}{2}x^2 + \frac{1}{3}x)dx \\
&= \left[\frac{1}{6}x^3 + \frac{1}{6}x^2\right]\Big|_{x=0}^{x=1} \\
&= \frac{1}{3}
\end{aligned}$$

Apostol p.362 no.9

Claim. If $Q = [a_1, b_1] \times [a_2, b_2]$ is a rectangle and all of $\iint_Q fg, \int f, \int g$ exist, then

$$\iint\limits_Q f(x)g(y)dxdy = \left(\int f(x)dx\right)\left(\int g(y)dy\right)$$

Proof. Put

$$\int_{a_1}^{b_1} f(x)dx = F, \int_{a_2}^{b_2} g(y)dy = G$$

First, note that

$$\int_{a_1}^{b_1} f(x)g(y)dx = g(y)\int_{a_1}^{b_1} f(x)dx = F \cdot g(y)$$

$$\begin{aligned}
\iint\limits_Q f(x)g(y)dxdy &= \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x)g(y)dxdy \\
&= \int_{a_2}^{b_2} Fg(y)dy \\
&= F\int_{a_2}^{b_2} g(y)dy \\
&= FG
\end{aligned}$$

$\square$

Apostol p.363 no.14

Claim. Let $f : Q \to \mathbb{R}$ where $Q = [0,1] \times [0,1]$ be given by

$$f(x,y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

Then, $\iint_Q f = 0$.

Proof. To show that the integral exists, for any $\epsilon > 0$, let $\epsilon' = \frac{p}{q}$ where $p, q \in \mathbb{Z}_{>0}$ be $0 < \epsilon' < \epsilon$. Now, consider a partition of Q into q^2 subrectangles, where $R_{i,j} = [\frac{i-1}{q}, \frac{i}{q}] \times [\frac{j-1}{q}, \frac{j}{q}]$ where $1 \leq i, j \leq q$. Then, let

$$s(x,y) = 0 \text{ and } t(x,y) = \begin{cases} 1 & (x,y) \in R_{i,j}, i = j \\ 0 & \text{otherwise} \end{cases}$$

Note that if $x = y$, then $(x,y) \in \mathbb{R}_{c,c}$ where $c = \lfloor xq \rfloor = \lfloor yq \rfloor$. Then, we have that $s \leq f \leq t$, and that

$$\iint_Q (t - s) = \iint_Q t = \frac{1}{q^2} = \frac{1}{q} < \frac{p}{q} = \epsilon' = \epsilon$$

Thus, the integral exists; further, we have that $\iint_Q s = 0$, $\iint_Q t < \epsilon$ for any $\epsilon > 0$ which yields that

$$\iint_Q s \leq \iint_Q f \leq \iint_Q t \implies 0 \leq \iint_Q f \leq \epsilon \implies \iint_Q f = 0$$

$\square$

Apostol p.372 no.1

For drawings, see attached picture at end.

$$\begin{aligned}
\iint_S x\cos(x+y)dxdy &= \int_0^\pi \int_0^x x\cos(x+y)dydx \\
&= \int_0^\pi [-x\sin(x+y)] \Big|_{y=0}^{y=x} dx \\
&= \int_0^\pi x(\sin(x) - \sin(2x))dx \\
&= \left[-x\cos(x) + \frac{1}{2}x\cos(x) + \sin(x) - \frac{1}{4}\sin(2x)\right] \Big|_{x=0}^{x=\pi} \\
&= \frac{3\pi}{2}
\end{aligned}$$

Apostol p.372 no.7

$$\begin{aligned}\int_1^3 \int_{-x}^x (x^2 - y^2) dy dx &= \int_1^3 \left[x^2 y - \frac{1}{3} y^3 \right] \Big|_{y=-x}^{y=x} dx \\ &= \int_1^3 \frac{4}{3} x^3 dx \\ &= \left[\frac{1}{4} x^4 \right] \Big|_{x=1}^{x=3} \\ &= \frac{80}{3}\end{aligned}$$

Apostol p.372 no.14

$$\int_0^1 \int_0^{\log x} f(x,y) dy dx = \int_0^1 \int_{e^y}^e f(x,y) dx dy$$

Problem 1

Fix $\epsilon > 0$, and suppose that $\exists \delta > 0 \mid |x - y| < \delta \implies |g(x,t) - g(y,t)| < \epsilon$ for all $t \in (0,1)$. Consider $t = 1 - \frac{\delta}{4\epsilon}$ and $y = x + \frac{\delta}{2}$. Then,

$$|g(x,t) - g(y,t)| = \left| \frac{x}{1-t} - \frac{y}{1-t} \right| = \frac{4\frac{\delta}{2}}{\delta} \epsilon = 2\epsilon > \epsilon$$

This gives $\Rightarrow\Leftarrow$, so the lemma on uniform continuity needs not hold on open Q.

Problem 2

a

Claim.

$$f(x,y) = \chi_{\mathbb{Q}}(xy)$$

where $\chi_{\mathbb{Q}}$ is the indicator function on $\mathbb{Q}$, is not (Riemann) integrable on $S = [0,1] \times [0,1]$.

Proof. Consider any step function $s \leq f$. Suppose that $s > 0$ on some subrectangle $R = [x, x + \epsilon_1] \times [y, y + \epsilon_2] \subset S$. Then, from the density of the irrationals in $\mathbb{R}$, we have that $\exists x', y' \notin \mathbb{Q} \mid x < x' < x + \epsilon_1$ and $y < y' < y + \epsilon_2$. Then, $f(x', y') = 0$ and so $s(x', y') > f(x', y')$.

Similarly, for any step function $t \geq f$, suppose $t < 1$ on some subrectangle $R = [x, x+\epsilon_1] \times [y, y+\epsilon_2] \subset S$. Then, from the density of $\mathbb{Q}$ in $\mathbb{R}$, we have that $\exists x', y' \in \mathbb{Q} \mid x < x' < x + \epsilon_1$ and $y < y' < y + \epsilon_2$. Then, $f(x', y') = 1$ and so $t(x', y') < f(x', y')$.

Thus, from the comparison theorem $\int_S s \leq \int_S 0 = 0$, and $\int_S t \geq \int_S 1 = 1$, such that $\underline{I}(f) = 0 \neq \overline{I}(f) = 1$. Thus, f is not integrable. □

b

Claim.

$$g(x,y) = \begin{cases} 0 & y < \frac{1}{2}, y > \frac{1}{2} \\ \chi_{\mathbb{Q}}(x) & y = \frac{1}{2} \end{cases}$$

where $\chi_{\mathbb{Q}}$ is the indicator function on $\mathbb{Q}$, has that $\int_0^1 g(x,y)dy$ exists for any fixed x, but $\int_0^1 g(x,y)dx$ does not for $y = \frac{1}{2}$.

Proof.

$$\int_0^1 g(x, \frac{1}{2})dx = \int_0^1 \chi_{\mathbb{Q}}(x)$$

which is not integrable as shown in the first semester.

For $\int_0^1 g(x,y)dy$, we can directly compute the integral to be 0; for any $\epsilon > 0$, consider that we can get the step functions

$$s(y) = \begin{cases} -\frac{\epsilon}{3} & y < \frac{1}{2}, y > \frac{1}{2} \\ 0 & y = \frac{1}{2} \end{cases} \text{ and } t(y) = \begin{cases} \frac{\epsilon}{3} & y < \frac{1}{2}, y > \frac{1}{2} \\ 1 & y = \frac{1}{2} \end{cases}$$

such that $s \leq f \leq t$. Then, we have that

$$\int (t - s) = \frac{2\epsilon}{3} < \epsilon$$

which implies that f is integrable. □

c

Claim.

$$h(x,y) = \chi_{\mathbb{Q}}(x)$$

where $\chi_{\mathbb{Q}}$ is the indicator function on $\mathbb{Q}$, has that $\int_0^1 h(x,y)dy$ exists for any fixed x, but $\int_0^1 \int_0^1 h(x,y)dxdy$ does not exist.

Proof. Note that

$$\int_0^1 h(x,y)dy = \begin{cases} \int_0^1 1dy & x \in \mathbb{Q} \\ \int_0^1 0dy & x \notin \mathbb{Q} \end{cases} = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} = \chi_{\mathbb{Q}}(x)$$

So the integral $\int_0^1 h(x,y)dy$ exists. $\int_0^1\int_0^1 h(x,y)dxdy$ does not exist by reasoning similar to part a of this question.

Consider any step function $s \leq h$. Suppose that $s > 0$ on some subrectangle $R = [x, x+\epsilon_1] \times [y, y+\epsilon_2] \subset S$. Then, from the density of the irrationals in $\mathbb{R}$, we have that $\exists x' \notin \mathbb{Q} \mid x < x' < x + \epsilon_2$. Then, $h(x', y+\frac{\epsilon_2}{2}) = 0$ and so $s(x', y+\frac{\epsilon_2}{2}) > h(x', y+\frac{\epsilon_2}{2})$.

Similarly, for any step function $t \geq h$, suppose $t < 1$ on some subrectangle $R = [x, x+\epsilon_1] \times [y, y+\epsilon_2] \subset S$. Then, from the density of $\mathbb{Q}$ in $\mathbb{R}$, we have that $\exists x' \in \mathbb{Q} \mid x < x' < x+\epsilon_1$ Then, $h(x', y+\frac{\epsilon_2}{2}) = 1$ and so $t(x', y+\frac{\epsilon_2}{2}) < h(x', y+\frac{\epsilon_2}{2})$.

Thus, from the comparison theorem $\int_S s \leq \int_S 0 = 0$, and $\int_S t \geq \int_S 1 = 1$, such that $\underline{I}(f) = 0 \neq \overline{I}(f) = 1$. Thus, f is not integrable. $\square$

Problem 4

a

Claim. For S, T disjoint bounded subsets of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$, if $\int_S f$ and $\int_T f$ exist, then $\int_{S \cup T} f = \int_S f + \int_T f$.

Proof. Let $S \subset Q_S, T \subset Q_T$ where Q_S, Q_T are closed rectangles. Then, if $Q_S = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and $Q_T = [c_1, d_1] \times \cdots \times [c_n, d_n]$, consider

$$Q = [\min\{a_1, c_1\}, \max\{b_1, d_1\}] \times \cdots [\min\{a_b, c_b\}, \max\{b_b, d_b\}]$$

Then, Q is a closed rectangle such that $Q_s, Q_T \subset Q$.

Now consider the extension by zero functions $f_S, f_T, f_U : Q \to \mathbb{R}$.

$$f_S(x) = \begin{cases} f(x) & x \in S \\ 0 & \text{otherwise} \end{cases}$$

$$f_T(x) = \begin{cases} f(x) & x \in T \\ 0 & \text{otherwise} \end{cases}$$

$$f_U(x) = \begin{cases} f(x) & x \in S \cup T \\ 0 & \text{otherwise} \end{cases}$$

Then by definition, $\int_S f = \int_Q f_S$, $\int_T f = \int_Q f_T, \int_{S \cup T} f = \int_Q f_U$.

Note that

$$f_S(x)+f_T(x) = \begin{cases} f_S(x)+f_T(x) & x \in S \\ f_S(x)+f_T(x) & x \in T \\ f_S(x)+f_T(x) & \text{otherwise} \end{cases} = \begin{cases} f(x)+0 & x \in S \\ 0+f(x) & x \in T \\ 0+0 & \text{otherwise} \end{cases} = f_U(x)$$

as S, T are disjoint. Then, additivity of the integral yields $\int_Q (f_S + f_T) = \int_Q (f_U)$, which gives us the desired result. ☐

b

Claim. For S, T bounded subsets of $\mathbb{R}^n$ which intersect in a hyperplane $x_i = c$ and $f : \mathbb{R}^n \to \mathbb{R}$, if $\int_S f$ and $\int_T f$ exist, then $\int_{S \cup T} f = \int_S f + \int_T f$.

Proof. Take the same definition for Q as above. We will prove that $\int_Q g = 0$ where

$$g(x) = \begin{cases} f(x) & x \in S \cap T \\ 0 & \text{otherwise} \end{cases}$$

where $S \cap T$ is a subset of the hyperplane $x_i = c$.

Consider $s \le g \le t$ where s, t are step functions $Q \to \mathbb{R}$.

Then, $s \le 0$. Consider any subrectangle $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$. In particular, if $c \notin [a_i, b_i]$, then we have that $f = 0$ on R; if $c \in [a_i, b_i]$, then $x = (\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \ldots, c+\epsilon, \ldots, \frac{a_{n-1}+b_{n-1}}{2}, \frac{a_n+b_n}{2}) \in R$, where $c + \epsilon \in [a_i, b_i], \epsilon \ne 0$ and we have that $f(x) = 0, s(x) \le 0$ and thus $s \le 0$ on any subrectangle R.

Further, $t \le 0$. Consider any subrectangle $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$. In particular, if $c \notin [a_i, b_i]$, then we have that $f = 0$ on R; if $c \in [a_i, b_i]$, then $x = (\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \ldots, c+\epsilon, \ldots, \frac{a_{n-1}+b_{n-1}}{2}, \frac{a_n+b_n}{2}) \in R$ where $c + \epsilon \in [a_i, b_i], \epsilon \ne 0$, and we have that $f(x) = 0, t(x) \ge 0$ and thus $t \ge 0$ on any subrectangle R.

We give now s, t such that $\int_Q s = \int_Q t = 0$. If f is bounded by K on Q,

$$Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

define

$$Q_s = [a_1, b_i] \times \cdots \times [a_i, c] \times \ldots [a_n, b_n]$$

$$Q_t = [a_1, b_i] \times \cdots \times [c, b_i] \times \ldots [a_n, b_n]$$

and

$$s = \begin{cases} 0 & x \in Q_s \\ 0 & x \in Q_t \\ -K & x_i = c \end{cases}$$

$$t = \begin{cases} 0 & x \in Q_s \\ 0 & x \in Q_t \\ K & x_i = c \end{cases}$$

Then, we have that $\underline{I}(g) = \overline{I}(g) = 0$.

Take the same definitions for Q, f_S, f_T, f_U as part a. Then, we have that $f_S + f_T - g = f_U$, which then gives that

$$\int_Q (f_S + f_T - g) = \int_Q f_S + \int_Q f_T - 0 = \int_Q f_U \implies \int_S f + \int_T f = \int_{S \cup T} f$$

$\square$