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Homework 8

Apostol p.155 no.8

Claim. If f is continuous on $[a, b]$, and $\int_a^b f(x)g(x)dx = 0$ for every function g continuous on $[a, b]$, then $f(x) = 0$.

Proof. Take $g(x) = f(x)$. We have that $\int_a^b (f(x))^2 dx = 0$. However, we have that $f(x)^2 \geq 0$. Now, suppose that $f(y)^2 > 0$. Then, we have that, as f is continuous, that $\exists \delta > 0 \mid 0 < |x - y| < \delta \implies |f(x)^2 - f(y)^2| < \frac{1}{2} f(y)^2 \implies f(x)^2 > \frac{1}{2} f(y)^2 > 0$. Thus,

$$\begin{aligned}
\int_a^b f(x)^2 dx &= \int_a^{y-\delta} f(x)^2 dx + \int_{y-\delta}^{y+\delta} f(x)^2 dx + \int_{y+\delta}^b f(x)^2 dx \\
&\geq \int_{y-\delta}^{y+\delta} f(x)^2 dx \\
&\geq \int_{y-\delta}^{y+\delta} f(y)^2 dx \\
&= 2\delta f(y)^2 > 0
\end{aligned}$$

$\Rightarrow\Leftarrow$, so $f(x)^2 = 0 \implies \forall x \in [a, b], f(x) = 0$. $\square$

Apostol p.168 no.22

We first show the power rule for rational exponents. We already have that power rule for integral exponents, and for $f(x) = x^{\frac{p}{q}}$, $f(x)^q = x^p$. Further, we have that the chain rule yields $(f(x)^q)' = (qf(x)^{q-1})(f'(x))$. Thus, $(qf(x)^{q-1})(f'(x)) = px^{p-1} \implies f'(x) = \frac{px^{p-1}}{qf(x)^{q-1}} = \frac{p}{q}\frac{x^{p-1}}{x^{\frac{p(q-1)}{q}}} = \frac{p}{q}x^{p-1-p+\frac{p}{q}} = \frac{p}{q}x^{\frac{p}{q}-1}$.

Then, we have that $(\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2\sqrt{x}}$. Further, $(1 + x)' = (1)' + (x)' = 0 + 1 = 1$.

The quotient rule then yields that

$$f'(x) = \frac{\frac{1}{2\sqrt{x}}(1 + x) - \sqrt{x}}{(1 + x)^2} = \frac{1}{2\sqrt{x}(1 + x)} - \frac{\sqrt{x}}{(1 + x)^2} = \frac{1 - x}{2\sqrt{x}(1 + x)^2}$$

Apostol p.168 no.24

Claim.

$$g' = f_1'(f_2 f_3 ... f_n) + f_2'(f_1 f_3 ... f_n) + ... + f_n'(f_1 f_2 ... f_{n-1}) = \sum_{i=1}^{n}(f_i' \prod_{j=1}^{i-1} f_j \prod_{j=i+1}^{n} f_j)$$

Proof. Take the base case of $n = 1$, or $g = f_1$. Then, $g' = f_1'$, as given by the formula, as the prodcuts are empty.

Assume that the formula holds for $n = k$, and put $g_k = \prod_{i=1}^{k} f_i$. Then,

$$\begin{aligned} g'_{k+1} &= g'_k f_{k+1} + f'_{k+1} g_k \\ &= (\sum_{i=1}^{k} (f'_i \prod_{j=1}^{i-1} f_j \prod_{j=i+1}^{k} f_j)) f_{k+1} + f'_{k+1} (\prod_{i=1}^{k} f_i) \\ &= \sum_{i=1}^{k} (f'_i \prod_{j=1}^{i-1} f_j \prod_{j=i+1}^{k+1} f_j) + f'_{k+1} (\prod_{i=1}^{k} f_i) \\ &= \sum_{i=1}^{k+1} (f'_i \prod_{j=1}^{i-1} f_j \prod_{j=i+1}^{k+1} f_j) \end{aligned}$$

□

Claim.

$$\frac{g'}{g} = \sum_{i=1}^{n} \frac{f'_i}{f_i}$$

Proof.

$$\begin{aligned} g' &= \sum_{i=1}^{n} (f'_i \prod_{j=1}^{i-1} f_j \prod_{j=i+1}^{n} f_j) \\ &= \sum_{i=1}^{n} \frac{f'_i \prod_{j=1}^{n} f_j}{f_i} \\ \implies \frac{g'}{g} &= \frac{\sum_{i=1}^{n} \frac{f'_i \prod_{j=1}^{n} f_j}{f_i}}{\prod_{j=1}^{n} f_j} \\ &= \frac{\sum_{i=1}^{n} f'_i}{f_i} = \sum_{i=1}^{n} \frac{f'_i}{f_i} \end{aligned}$$

□

Apostol p.174 no.15

a

Claim.

$$f'(a) = \lim_{h \to 0} \frac{f(h) - f(a)}{h - a}$$

Put $h = k + a$.

$$\begin{aligned} f'(a) &= \lim_{k\to 0} \frac{f(a+k) - f(a)}{k} \\ &= \lim_{h\to a} \frac{f(a+(h-a)) - f(a)}{h-a} \\ &= \lim_{h\to a} \frac{f(h) - f(a)}{h-a} \end{aligned}$$

b

Claim.

$$f'(a) = \lim_{h\to 0} \frac{f(a) - f(a-h)}{h}$$

Proof. Put $h = -k$.

$$\begin{aligned} f'(a) &= \lim_{k\to 0} \frac{f(a+k) - f(a)}{k} \\ &= \lim_{h\to 0} \frac{f(a-h) - f(a)}{-h} \\ &= \lim_{h\to 0} \frac{f(a) - f(a-h)}{h} \end{aligned}$$

□

c

False. Consider $f(x) = x$. Then, $f'(a) = 1$. However,

$$\lim_{t\to 0} \frac{f(a+2t) - f(a)}{t} = \lim_{t\to 0} \frac{2t}{t} = 2$$

In general, we have that since $\lim_{t\to 0} f = \lim_{ct\to 0} f, c \neq 0$,

$$\lim_{t\to 0} \frac{f(a+2t) - f(a)}{t} = 2 \lim_{2t\to 0} \frac{f(a+2t) - f(a)}{2t} = 2 \lim_{t\to 0} \frac{f(a+t) - f(a)}{t} = 2f'(a)$$

d

False. Consider $f(x) = x$. Then, $f'(a) = 1$. However,

$$\lim_{t\to 0} \frac{f(a+2t) - f(a+t)}{2t} = \lim_{t\to 0} \frac{t}{2t} = \frac{1}{2}$$

In general, we have that

$$\begin{aligned}\lim_{t\to 0}\frac{f(a+2t)-f(a+t)}{2t} &= \frac{1}{2}(\lim_{t\to 0}\frac{f(a+2t)-f(a)}{t}-\lim_{t\to 0}\frac{f(a+t)-f(a)}{t})\\ &= \frac{1}{2}(2f'(a)-f'(a))\\ &= \frac{1}{2}f'(a)\end{aligned}$$

Problem 1

Claim. Let $f:[a,b]\to\mathbb{R}$ be integrable. Then, $\exists c\in[a,b]$ such that

$$\int_a^c f(x)dx = \frac{1}{2}\int_a^c f(x)dx$$

.

Proof. First, we extend the Intermediate Value Theorem to be slightly stronger. The original statement is that for $f:[a,b]\to\mathbb{R}$ continuous, if $f(a)<K<f(b)$ then $\exists c\in[a,b]\mid f(c)=K$. Further, we will show that if $f(b)<K<f(a)$ then $\exists c\in[a,b]\mid f(c)=K$. To see this, consider $g(x)-f(x)$. We have that $g(a)<-K<g(b)$, so $\exists c\in[a,b]\mid g(c)=-K \implies f(c)=K$.

Consider $g(x):[a,b]\to\mathbb{R}, g(x)=\int_a^x g(t)dt$. Then, we have that $g(a)=\int_a^a g(t)dt = 0, g(b)=\int_a^b g(t)dt$. Further, $\frac{1}{2}\int_a^b f(x)dx = \frac{g(b)}{2} = \frac{g(a)+g(b)}{2}$, and if $g(b)>0=g(a)$, then $g(a)<\frac{g(a)+g(b)}{2}<g(b)$, and if $g(b)<0=g(a)$, then $g(b)<\frac{g(a)+g(b)}{2}<g(a)$, and so by the Intermediate Value Theorem, $\exists c\in[a,b]\mid g(c)=\frac{g(a)+g(b)}{2} \implies \int_a^c f(x)dx = \frac{1}{2}\int_a^b f(x)dx$. $\square$

Problem 2

Claim. f is continuous on $[0,1]$, and has $f(0)=f(1)$. $\forall n\in\mathbb{Z}_{>0},\exists x\in[0,1-1/n]\mid f(x)=f(x+1/n)$.

Proof. Consider $g(x)=f(x)-f(x+1/n)$. Suppose that $g>0$. Then, we have that $f(1)>f(0)$. To see this, consider the set $\{g(k/n)\mid k\in\mathbb{Z}_{>0},k\le n\}$. We then have that $k>0 \implies f(k/n)>f(0)$, as we can induct on k. If $k=1$, then $g(1/n)>0 \implies f(1/n)-f(0)>0 \implies f(1/n)>f(0)$. Assume that the hypothesis holds for $k<n$. Then, $g(k/n)>0 \implies f(k/n)-f((k+1)/n)>0 \implies f((k+1)/n>f(k/n)>f(0)$. This shows that $f(k/n)>f(0)$ for all $k\in\mathbb{Z}_{>0},k\le n$. Critically, this then means that $f(1)>f(0)$. $\Rightarrow\Leftarrow$

Now suppose that $g<0$. Then, we have that $f(1)<f(0)$. To see this, consider the set $\{g(k/n)\mid k\in\mathbb{Z}_{>0},k\le n\}$. We then have that $k>0 \implies f(k/n)<f(0)$, as we can induct

on k. If $k = 1$, then $g(1/n) < 0 \implies f(1/n) - f(0) < 0 \implies f(1/n) < f(0)$. Assume that the hypothesis holds for $k < n$. Then, $g(k/n) < 0 \implies f(k/n) - f((k+1)/n) < 0 \implies f((k+1)/n < f(k/n) < f(0)$. This shows that $f(k/n) < f(0)$ for all $k \in \mathbb{Z}_{>0}, k \leq n$. Critically, this then means that $f(1) < f(0)$. $\Rightarrow\Leftarrow$.

Thus, we must have that g cannot be positive nor negative everywhere, meaning that $\exists x, y \in [0, 1-1/n] \mid g(x) > 0, g(y) < 0$. By the Intermediate Value Theorem, we have that $\exists z \in [0, 1-1/n] \mid g(z) = 0 \implies \exists z \in [0, 1-1/n] \mid f(z) - f(z+1/n) = 0 \implies f(z) = f(z+1/n)$. $\square$

Problem 4

a)

Consider the counter example of

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \qquad g(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

We have that $f + g = 1$, which is differentiable everywhere $(f+g)' = 0$. However, we have that f, g are nowhere continuous and thus nowhere differentiable (the contrapositive, differentiable $\implies$ continuous was proved in class).

In general, take any function f not differentiable at x. Then, $f + (-f) = 0$ is differentiable at x, but neither $f, -f$ are.

b)

Claim. If $f(x) \neq 0$, then g is differentiable at x.

Proof. We have that the quotient rule states for functions s, t differentiable at x, then if $t(x) \neq 0$, $(\frac{s}{t})' = \frac{s't - st'}{t^2}$ at x. Taking $s = fg, t = f$, we have that $f(x) \neq 0 \implies g'(x)$ exists by the quotient rule. $\square$

Problem 5

a)

Claim. $f(x) = xg(x)$, g continuous at $0 \implies f$ is differentiable at 0.

Proof.

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h}$$

Consider $\lim_{h\to 0}(\frac{f(h)}{h} - g(h))$. For any ϵ, take arbitrary $\delta > 0$. We then have that $0 < |x| < \delta \implies x \neq 0 \implies \frac{f(x)}{x} = g(x) \implies |\frac{f(x)}{x} - g(x)| = 0 < \epsilon$.
Thus, we have that $\lim_{h\to 0}(\frac{f(h)}{h} - g(h)) = 0 \implies \lim_{h\to 0}\frac{f(h)}{h} = \lim_{h\to 0} g(h) = g(0)$, as g is continuous. $\square$

b)

Claim. Suppose that f is differentiable at 0 and $f(0) = 0$. Then, $\exists g(x) \mid f(x) = xg(x), g$ continuous at 0.

Proof. Consider

$$g(x) = \begin{cases} f'(0) & x = 0 \\ \frac{f(x)}{x} & x \neq 0 \end{cases}$$

which is well defined as we have that f is differentiable at 0.
Then, we have that

$$xg(x) = \begin{cases} 0 & x = 0 \\ f(x) & x \neq 0 \end{cases}$$

This is equal to $f(x)$ everywhere.
Now, to prove that $g(x)$ is continuous, note first that we have that $\lim_{h\to 0}\frac{f(h)}{h} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = f'(0)$, as we have that f is differentiable at 0. Further, $\lim_{h\to 0}(g(h) - \frac{f(h)}{h} = 0$, as for any $\epsilon > 0$, take arbitrary $\delta > 0 \mid 0 < |x| < \delta \implies x \neq 0 \implies g(x) = \frac{f(x)}{x} \implies |g(x) - \frac{f(x)}{x} - 0| = 0 < \epsilon$.
Finally, we have that $\lim_{h\to 0} g(h) = \lim_{h\to 0}\frac{f(h)}{h} = f'(0) = g(0)$, so g is continuous. $\square$