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Homework 9

Apostol p.180 no.19bcd

b

We use the chain rule to compute that

$$\begin{aligned} g'(x) &= (\sin^2(x))' f'(\sin^2(x)) + (\cos^2(x))' f(\cos^2(x)) \\ &= \cos(x)(2\sin(x)) f'(\sin^2(x)) - \sin(x)(2\cos(x)) f'(\cos^2(x)) \\ &= 2\sin(x)\cos(x)(f'(\sin^2(x)) - f'(\cos^2(x))) \\ &= \sin(2x)(f'(\sin^2(x)) - f'(\cos^2(x))) \end{aligned}$$

c

We use the chain rule to compute that

$$g'(x) = f'(f(x))f'(x)$$

d

We use the chain rule to compute that

$$g'(x) = f'(f(f(x)))(f(f(x)))' = f'(f(f(x)))f'(f(x))f'(x)$$

Apostol p.186-187 no.7

a

Claim. If f has r zeros, counting multiplicity, then the k-th derivative $f^k(x)$ has at least $r-k$ zeros counting multiplicity.

Proof. Assuming that any real zero a of multiplicity m of f satisfy $f(x) = (x-a)^m g(x)$ for some polynomial $g(x) \mid g(r) \neq 0$, as given in the problem. Further, let the function f have distinct zeros $a_1, a_2, ..., a_k$ with respective multiplicities $m_1, m_2, ..., m_k$. We have that if $f(x) = (x-a)^m g(x)$, then $f'(x) = m(x-a)^{m-1}g(x) + (x-a)^m g'(x) = (x-a)^{m-1}(mg(x) + (x-a)g'(x))$, and thus $f'(x)$ has a as a root of multiplicity $m-1$ (note that if $m-1=0$, then a is no longer a root, but this still works if we define the number of roots to be $r = \sum_{i=1}^{k} m_i$, as adding $m_i = 0$ counts no new roots).

Further, for each distinct root, we have that the Mean Value Theorem gives some $b_i \in (a_i, a_{i+1})$ such that $f'(b) = \frac{f(a_{i+1}) - f(a_i)}{a_{i+1} - a_i} = 0$. These roots have at least multiplicity 1, as

$f(b) = 0$ and there are $k - 1$ such intervals, and none were old roots. The total amount of zeros is then at least $\sum_{i=1}^{k}(m_i - 1) + k - 1 = \sum_{i=1}^{k} m_i - 1 = r - 1$.

To extend this to the k-th derivative, we induct on k. The base case was just proved for $k = 1$. Assume that for $k = n$, the claim holds. Then, we have that $f^{n+1}(x)$ has at least $(r - n) - 1 = r - (n + 1)$ zeros by the same reasoning as before (simply put $f(x) = f^n(x)$). The claim then holds for $k = n + 1$. □

b

If the k-th derivative has exactly r zeros in $[a, b]$, then we can conclude that $f^{k-1}(x)$ had at most $r+1$ zeros in $[a, b]$ (as if $f^{k-1}(x)$ has r_{k-1} zeros, then $f^k(x)$ has at least $r_{k-1} - 1 \leq r$ zeros, so $r_{k-1} \leq r + 1$. Similarly, we have that $f^{k-i}(x)$ has at most $r + i$ zeros, for $i \in \{1, 2, ..., k\}$.

Apostol p.186-187 no.8b

Claim. For $0 < y \leq x, n \in \mathbb{Z}_{>0}$,

$$ny^{n-1}(x - y) \leq x^n - y^n \leq nx^{n-1}(x - y)$$

Proof. First, if $x = y$, then we have that the inequality is $0 \leq 0 \leq 0$, which is true. We then just have to consider the case that $y < x$.

We have by the Mean Value Theorem that

$$\exists z \in (x, y) \mid nz^{n-1} = \frac{x^n - y^n}{x - y} \implies nz^{n-1}(x - y) = x^n - y^n$$

Further, we have that $0 < y < z < x, 0 < n, \implies ny^{n-1}(x-y) < nz^{n-1}(x-y) < nx^{n-1}(x-y)$. However, we have that $nz^{n-1}(x-y) = x^n-y^n \implies ny^{n-1}(x-y) < x^n-y^n < nx^{n-1}(x-y)$. □

Apostol p.186-187 no.9

Claim. f has second derivative f'' defined on $[a, b]$. The line segment connecting $(a, f(a))$ and $(b, f(b))$ intersects f at $a < c < b$. Then, $\exists t \in [a, b] \mid f''(t) = 0$.

Proof. Let $g(x)$ be the linear function containing that line segment (specifically, $g(x) = \frac{f(b)-f(a)}{b-a}(x-a)+f(a)$). Then, define $h(x) = f(x)-g(x)$, so that $h(a) = 0, h(b) = 0, h(c) = 0$. This means by the Mean Value Theorem that $\exists c_1 \in (a, c), c_2 \in (c, b) \mid h'(c_1) = h'(c_2) = 0$. Further, we have again by the Mean Value Theorem that $t \in (c_1, c_2) \mid h''(t) = 0$. However, we also have that $g''(x) = 0 \implies h''(t) = f''(t) - g''(t) = f''(t) \implies f''(t) = 0$. □

Apostol p.209 no.19

$$\begin{aligned}
f(x) &= \frac{1}{2}\int_0^x (x-t)^2 g(t)dt \\
&= \frac{1}{2}\int_0^x (x^2-2xt+t^2)g(t)dt \\
&= \frac{1}{2}\int_0^x x^2 g(t)dt - \frac{1}{2}\int_0^x 2xtg(t)dt + \frac{1}{2}\int_0^x t^2 g(t)dt \\
&= \frac{x^2}{2}\int_0^x g(t)dt - x\int_0^x tg(t)dt + \frac{1}{2}\int_0^x t^2 g(t)dt \\
\implies f'(x) &= (x\int_0^x g(t)dt + \frac{x^2}{2}g(x)) - (\int_0^x tg(t)dt + x(xg(x))) + (\frac{1}{2}x^2 g(x)) \\
&= x\int_0^x g(t)dt - \int_0^x tg(t)dt \\
\implies f''(x) &= (\int_0^x g(t)dt + xg(x)) - xg(x) \\
&= \int_0^x g(t)dt \\
\implies f'''(x) &= g(x) \\
\implies f''(1) &= \int_0^1 g(t)dt = 2, f'''(1) = g(1) = 5
\end{aligned}$$

We have that $\frac{1}{2}\int_0^x x^2 g(t)dt = \frac{x^2}{2}\int_0^x g(t)dt, \frac{1}{2}\int_0^x 2xtg(t)dt = x\int_0^x tg(t)$ as $x^2, 2x$ are independent of t and can be pulled out from the integral.

Further, the derivative is computed with the product rule and the Fundamental Theorem of Calculus.

Apostol p.209 no.20

In general, we have that if $g(x) = \int_0^x (1+t^2)^{-3}dt$, then $g'(x) = (1+x^2)^{-3}$ by the Fundamental Theorem of Calculus.

a

We have that

$$f(x) = g(x) \implies f'(x) = g'(x) = (1+x^2)^{-3}$$

b

We have that
$$f(x) = g(x^2) \implies f'(x) = 2xg'(x^2) = 2x(1+x^4)^{-3}$$

c

We have that
$$\begin{aligned} f(x) &= \int_{x^3}^{x^2} (1+t^2)^{-3} dt \\ &= \int_{x^3}^{0} (1+t^2)^{-3} dt + \int_{0}^{x^2} (1+t^2)^{-3} dt \\ &= -g(x^3) + g(x^2) \\ \implies f'(x) &= -3x^2 g'(x^3) + 2xg'(x^2) \\ &= -3x^2(1+x^6)^{-3} + 2x(1+x^4)^{-3} \end{aligned}$$

Apostol p.217 no.23

Claim.
$$\int_x^1 \frac{dt}{1+t^2} = \int_1^{\frac{1}{x}} \frac{dt}{1+t^2}$$

Proof. Let $u = \frac{1}{t}, u' = -\frac{1}{t^2}, f(x) = \frac{1}{1+x^2}$. Then, change of variables lets us have that

$$\begin{aligned} \int_x^1 f(t)dt &= \int_{\frac{1}{x}}^1 f(u(t))u'(t)dt \\ \implies \int_x^1 \frac{dt}{1+t^2} &= \int_{\frac{1}{x}}^1 \frac{-\frac{1}{u^2}du}{1+(\frac{1}{u})^2} \\ &= \int_1^{\frac{1}{x}} (\frac{1}{u^2(1+\frac{1}{u^2})})du \\ &= \int_1^{\frac{1}{x}} (\frac{1}{u^2+1})du \\ &= \int_1^{\frac{1}{x}} \frac{1}{1+t^2}dt \end{aligned}$$

The last change of variables is justified since we can freely rename the variable of integration. $\square$

Apostol p.217 no.24

Claim.

$$\int_0^1 x^m(1-x)^n dx = \int_0^1 x^n(1-x)^m dx$$

Proof. Let $u = 1 - x, u' = -1$. Then, change of variables has that

$$\int_0^1 x^m(1-x)^n dx = \int_1^0 (1-u)^m u^n(-1)du = \int_0^1 (1-u)^m u^n du = \int_0^1 x^n(1-x)^m dx$$

□

Apostol p.222-223 no.5ab

We have $f, g \mid f' = g, g' = -f, f(0) = 0, g(0) = 1$.

a

Claim.

$$f^2(x) + g^2(x) = 1$$

Proof.

$$\begin{aligned}(f^2(x) + g^2(x))' &= 2f(x)f'(x) + 2g(x)g'(x) \\ &= 2(f(x)g(x) + g(x)(-f(x))) \\ &= 2(f(x)g(x) - f(x)g(x)) = 0\end{aligned}$$

Thus, $f^2(x)+g^2(x)$ must be constant, and we know that $f^2(0)+g^2(0) = 1$, so $f^2(x)+g^2(x) = 1$. □

b

Claim. If F, G also satisfy these conditions, then $F = f, G = g$.

Proof. Consider $h(x) = (F(x) - f(x))^2 + (G(x) - g(x))^2$. Then we have that

$$\begin{aligned}h'(x) &= 2(F(x) - f(x))(F'(x) - f'(x)) + 2(G(x) - g(x))(G'(x) - g'(x)) \\ &= 2(F(x) - f(x))(G(x) - g(x)) + 2(G(x) - g(x))(-F(x) + f(x)) \\ &= 2(F(x) - f(x))(G(x) - g(x)) - 2(F(x) - f(x))(G(x) - g(x)) \\ &= 0\end{aligned}$$

Thus, we have that $h(x)$ is constant. However, $h(1) = 0$, so we have that $h(x) = 0$. However, since $(F(x)-f(x))^2, (G(x)-g(x))^2 \geq 0$, we must have that $(F(x)-f(x))^2 = (G(x)-g(x))^2 = 0 \implies F(x) - f(x) = G(x) - g(x) = 0$, and so $F = f, G = g$ in the interval where these properties are satisfied. □

Apostol p.222-223 no.7

We have that $(g(x^2))' = 2xg'(x^2) = 2x(x^3) = 2x^4$.

$$\begin{aligned} \int_1^y (g(x^2)'dx &= \int_1^y 2x^4 dx \\ \implies g(y^2) - g(1) &= \frac{2}{5}y^5 - \frac{2}{5} \\ \implies \qquad g(y^2) &= \frac{2}{5}y^5 - \frac{2}{5} + g(1) \\ &= \frac{2}{5}y^5 + \frac{3}{5} \end{aligned}$$

Then, $g(4) = g(2^2) = \frac{2}{5}2^5 + \frac{3}{5} = \frac{67}{5}$.

Apostol p.222-223 no.10

Let

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

and

$$Q(h) = \frac{f(h)}{h} \text{ if } h \neq 0$$

a

Claim.

$$\lim_{h \to 0} Q(h) = 0$$

Proof. For any $\epsilon > 0$, take $\delta = \epsilon$. Then, $0 < |h - 0| < \delta \implies$

$$|Q(h) - 0| = \begin{cases} |\frac{h^2}{h}| & h \in Q \\ 0 & h \notin \mathbb{Q} \end{cases} = \begin{cases} |h| & h \in \mathbb{Q} \\ 0 & h \notin \mathbb{Q} \end{cases} < \delta = \epsilon$$

Thus, we have that $\lim_{h \to 0} Q = 0$. □

b

Claim. f is differentiable at 0.

Proof. The result follows from part a, as $f(0) = 0$.

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h} = 0.$$

□

Problem 1

Claim. If $|f|$ is differentiable at x, and f is continuous at x, then f is differentiable at x.

Proof. Suppose that $f(x) > 0$. Then, we have that continuity of $f(x)$ implies that for $\epsilon = \frac{f(x)}{2}$, we have $\delta > 0 \mid 0 < |y - x| < \delta \implies |f(y) - f(x)| < \frac{f(x)}{2} \implies f(y) > \frac{f(x)}{2} > 0$. Then, on $(x - \delta, x + \delta)$, we have that $f > 0 \implies f = |f|$.

Similarly, if $f(x) < 0$, we have that continuity of f implies that for $\epsilon = \frac{f(x)}{2}$, we have $\delta > 0$ such that on $(x - \delta, x + \delta)$, $f < 0 \implies f = -|f|$.

In either case, we have that since on a δ neighborhood of x, $f = |f|$ or $f = -|f|$, and $|f|, -|f|$ are differentiable at x, so f must be differentiable at x (for any $\epsilon > 0$, one can take $0 < \delta' < \delta$ such that $||f| - |f|'(x)| < \epsilon \implies | \pm f - |f|'(x)| < \epsilon$).

The only remaining case is that $f(x) = 0$. In this case, we must have that $|f|'(x) = 0$. Suppose that $|f|'(x) \neq 0$. Then, if $|f|'(x) > 0$, we have that similarly to above, $\exists \delta$ such that on $(x - \delta, x + \delta)$, $|f|'(x) > 0$. Then, this means that $\int_{x-\delta}^{x} |f|'(x) dx = |f|(x) - |f|(x - \delta) > 0 \implies |f|(x - \delta) < 0$. $\Rightarrow\Leftarrow$. Similarly, if we have that $|f|'(x) < 0$, we have that $\exists \delta$ such that on $(x - \delta, x + \delta)$, $|f| < 0$. Then, $\int_{x}^{x+\delta} |f|'(x) dx = |f|(x + \delta) - |f|(x) < 0 \implies |f|(x + \delta) < 0$. $\Rightarrow\Leftarrow$. Thus, $|f|'(x) = 0$.

Since we have that $|f|'(x) = 0$, we know that $\lim_{h \to 0} \frac{|f(x+h)| - |f(x)|}{h} = 0 \implies \lim_{h \to 0} \frac{|f(x+h)|}{h} = 0 \implies \forall \epsilon > 0, \exists \delta \mid 0 < |h| < \delta \implies \frac{|f(x+h)|}{|h|} < \epsilon$. Now, consider $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. For any ϵ, take δ to be the same δ for the corresponding ϵ for $|f|$. Then, we have that $0 < |h| < \delta \implies |\frac{f(x+h) - f(x)}{h}| = \frac{|f(x+h)|}{|h|} < \epsilon$. Thus, we have that $f'(x) = 0$. □