dc3451 | MATH 1207
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Homework 3

Apostol p.28 no.1

Claim. For $x, y \in \mathbb{R}, x < y \implies \exists z \in \mathbb{R} \mid x < z < y$.

Proof. Consider $z = \frac{x}{2} + \frac{y}{2}$. We have $z \in \mathbb{R}$ as $\mathbb{R}$ is closed under addition and multiplication as it is a field under those operations.

Further, we have that $x \leq y \implies \frac{x}{2} < \frac{y}{2} \implies \frac{x}{2} + \frac{x}{2} < \frac{x}{2} + \frac{y}{2}, \frac{x}{2} + \frac{y}{2} < \frac{y}{2} + \frac{y}{2}$.

As $\forall a \in \mathbb{R}, \frac{a}{2} + \frac{a}{2} = \frac{1}{2}(a + a) = \frac{1}{2}(2a) = a$, we have that $x < \frac{x}{2} + \frac{y}{2} < y$. $\square$

Apostol p.28 no.3

Claim. For $x \in \mathbb{R}, x > 0 \implies \exists n \in \mathbb{Z}_{>0} \mid \frac{1}{n} < x$.

Proof. The Archimedian property of the reals furnishes an $n \in \mathbb{Z}_{>0} \mid nx > 1$. Then, we see that $nx > 1 \implies 1 < nx \implies n^{-1}(1) < n^{-1}(nx) \implies \frac{1}{n} < x$. $\square$

Apostol p.28 no.4

Claim. For $x \in \mathbb{R}, \exists! n \in \mathbb{Z} \mid n < x < n + 1$.

Proof. We will first show existence. Consider $S = \{n \in \mathbb{Z} \mid n \leq x\}$. This must be nonempty, or else x would be a lower bound to $\mathbb{Z}$, as $\neg \exists n \in \mathbb{Z} \mid n \leq x \implies \forall n \in \mathbb{Z}, \neg(n \leq x) \implies \forall n \in \mathbb{Z}, x \leq n$.

Now, note that if x is a lower bound for $\mathbb{Z}$, then $-x$ is an upper bound for $\mathbb{Z}$. This follows as $x \leq n \implies -x \geq -n$, but as $n \in \mathbb{Z} \implies -n \in \mathbb{Z}$, we have that $\forall n \in \mathbb{Z}, x \leq n \implies \forall n \in \mathbb{Z}, x \leq -n \implies \forall n \in \mathbb{Z}, -x \geq -(-n) = n$.

However, we proved that $\mathbb{Z}_{\geq 0} \subset \mathbb{Z}$ has no upper bound, meaning that $-x$ cannot be an upper bound of $\mathbb{Z}$. Thus, S must be nonempty.

Now, the approximation theorem proved in class furnishes $n \in S \mid \sup(S) - 1 < n$. Thus, since we have $\sup(S) - 1 < n \implies \sup(S) < n + 1 \implies n + 1 \notin S$, and by definition of S, $n \in S \implies n \leq x$ and $n + 1 \notin S \implies \neg(n + 1 \leq x) \implies x < n + 1 \implies n \leq x < n + 1$.

We will now show uniqueness: suppose that $\exists n, n' \in \mathbb{Z} \mid n \neq n', n \leq x < n+1, n' \leq x < n'+1$. $n' > n \implies n' \geq n + 1 > x$. However, $n' < n$, then we have that $n \geq n' + 1 > x$. Either way, we have $\Rightarrow\Leftarrow$, so $n = n'$.

The above relies on the fact that $a, b \in \mathbb{Z}, a > b \implies a \geq b + 1$. This follows from $a > b \implies a - b > 0$, and as $a - b \in \mathbb{Z}$, the fact that there is no integer between 0 and 1 (proved in an earlier homework) allows that $a - b = 1$ or $a - b > 1$ by trichotomy. However, this means that $a - b \geq 1 \implies a \geq b + 1$. $\square$

Apostol p.28 no.6

Claim. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. We shall start by proving at for $x, y \in \mathbb{R}, x < y, \exists r \in \mathbb{Q} \mid x < r < y$. The Archimedian property furnishes $n \in \mathbb{Z}_{>0} \mid n(y-x) > 1 \implies ny > nx + 1$. Now consider $[nx]$. We have that $[nx] \leq nx \implies [nx] + 1 \leq nx + 1 < ny$, and also $nx < [nx] + 1$.

These together yield that

$$\begin{aligned} &nx < [nx] + 1 \leq nx + 1 < ny \\ \implies &n^{-1}(nx) < n^{-1}([nx] + 1) \leq n^{-1}(nx + 1) < n^{-1}(ny) \\ \implies &x < \frac{[nx] + 1}{n} < y \end{aligned}$$

Critically, $[nx] \in \mathbb{Z}$, meaning that as $[nx] + 1, n \in \mathbb{Z}$, we have $\frac{[nx]+1}{n} \in \mathbb{Q}$.

Now that we have one such r, we can construct infinitely many: simply use the above process to find r' such that $r < r' < y$. This can be repeated ad infinitum. □

Apostol p.64 no.4b

Claim.

$$[-x] = \begin{cases} -[x] & x \in \mathbb{Z} \\ -[x] - 1 & x \notin \mathbb{Z} \end{cases}$$

Proof. Note that if we find one such a such that $a \leq x < a + 1$, we have that $[x] = a$ as we have shown previously that such an a must be unique.

Suppose that $x \in \mathbb{Z}$. Then we have that $-x \leq -x < -x + 1$, and so $[-x] = -x$.

Otherwise, we have that $[x] \leq x$. However, we have that as $[x] \in \mathbb{Z}, x \notin \mathbb{Z}$, $[x] < x$. This then provides that $-[x] > -x$. Further, $x < [x] + 1 \implies -x > -([x] + 1) = -[x] - 1$.

These together give us $-[x] - 1 < -x < -[x] \implies -[x] - 1 \leq -x < -[x] \implies [-x] = -[x] - 1$. □

Apostol p.64 no.4d

Claim. $[2x] = [x] + [x + \frac{1}{2}]$

Proof. Consider $[x] + \frac{1}{2}$. We have that by trichotomy, exactly one of $x < y, x = y, x > y$ is true.

If $x < [x]+\frac{1}{2}$, then $[x] \le x < [x]+\frac{1}{2} \implies [x] < x+\frac{1}{2} < [x]+\frac{1}{2}+\frac{1}{2} = [x]+1 \implies [x+\frac{1}{2}] = [x]$. Further, $2[x] \le 2x < 2([x]+\frac{1}{2}) = 2[x]+1 \implies [2x] = 2[x] = [x]+[x] = [x]+[x+\frac{1}{2}]$.

If $x = [x]+\frac{1}{2}$, then $x+\frac{1}{2} = [x]+\frac{1}{2}+\frac{1}{2} = [x]+1$, and $[x]+1 \in Z \implies [x]+1 \le [x]+1 < [x]+2 \implies [x+\frac{1}{2}] = [[x]+1] = [x]+1$. Further, $2x = 2([x]+\frac{1}{2}) = 2[x]+1 = [x]+[x]+1 = [x]+[x+\frac{1}{2}]$.

If $x > [x]+\frac{1}{2}$, then $[x]+\frac{1}{2} \le x < [x]+1 \implies [x]+\frac{1}{2}+\frac{1}{2} \le x+\frac{1}{2} < [x]+1+\frac{1}{2} \implies [x]+1 \le x+\frac{1}{2} < [x]+2 \implies [x+\frac{1}{2}] = [x]+1$. Further, $2x > 2([x]+\frac{1}{2}) = 2[x]+1$, and $x < [x]+1 \implies 2x < 2[x]+2 \implies 2[x]+1 < 2x < 2[x]+2 \implies [2x] = 2[x]+1 = [x]+[x]+1 = [x]+[x+\frac{1}{2}]$.

(What an awful proof) □

Problem 1

Suppose $S \subseteq \mathbb{R}, c \in \mathbb{R}$. Let $cS = \{cx \mid x \in S\}$.

a)

Claim. If $c > 0$ and S is bounded above, then cS is also bounded above.

Proof. Let $r \in \mathbb{R}$ be an upper bound of S. Then $\forall s \in S, s \le r \implies \forall s \in S, cs \le cr$. However, for any element $t \in cS$, we have that $\exists s \in S \mid t = cs$. This means that for any element $t \in cS$, we have that $t = cs \le cr$, so cr is an upper bound on cS. □

b)

Claim. If $c > 0$, then $\sup(cS) = c\sup(S)$.

Proof. (I use problem 2 in this proof freely, as that proof does not rely on this one.)

We will first show that if one exists only if the other exists. Suppose that $\sup(cS)$ exists. Then, we can see that for any element $t \in cS$ we have $t = cs$ for some $s \in S$, meaning that $\forall t \in cS, \sup(cS) \ge t \implies \forall s \in S, \sup(cS) \ge cs \implies \forall s \in S, c^{-1}\sup(cS) \ge s$, so $c^{-1}\sup(cS)$ in particular is an upper bound for S.

Suppose now $\sup(S)$ exists. Then we can see for any $t \in cS$, we have $t = cs$ for some $s \in S$, such that $\forall s \in S, \sup(S) \ge s \implies \forall s \in S, c\sup(S) \ge cs \implies \forall t \in cS, c\sup(S) \ge t$, so $c\sup(S)$ in particular is an upper bound for cS.

We will now show that $\sup(cS)$ is exactly $c\sup(S)$. Now suppose that $c\sup(S)$ is not the least upper bound of cS, meaning that $\exists \epsilon > 0 \mid \sup(cS) + \epsilon < c\sup(S)$. By approximation theorem, we have another $\epsilon' > 0 \mid s + \epsilon > \sup(S)$ for some $s \in S$. Then, we have that $c(s+\epsilon') = cs + c\epsilon' > c\sup(S)$. However, $\sup(cS) > cs$, as $cs \in cS$, so we have that $\sup(cS) + c\epsilon' > cs + c\epsilon > c\sup(S)$. However, since we can take ϵ, ϵ' to be any two positive reals, we can take $\epsilon' = c^{-1}\epsilon$, such that we have $\sup(cS)+\epsilon > c\sup(S)$ as well as $\sup(cS)+\epsilon < c\sup(S)$. This violates trichotomy, so $\Rightarrow\Leftarrow$ and thus $\sup(cS) = c\sup(S)$. $\square$

c)

Take $c = -1, S = (0,1)$. Clearly $cS = (-1,0), \sup(S) = 1, \sup(cS) = 0$, and so $c\sup(S) = -1(1) = -1 \neq \sup(cS)$. In fact, if $c < 0$, then $\sup(cS) = c\inf(S)$. This is most clearly seen by noticing that multiplying by $c < 0$ swaps the order, so the infimum gets mapped to the supremum of the new set and vice versa.

Problem 2

Claim. Suppose $S \subseteq \mathbb{R}, t \in \mathbb{R}$. $t = \sup(S) \iff \forall s \in S, t \geq s$, and $\forall \epsilon > 0, \exists x \in S \mid x > t - \epsilon$.

Proof. ($\implies$) $t = \sup(S)$ implies that t is an upper bound of S, as $\sup(S)$ is an upper bound of S by definition. The rest follows from the approximation theorem exactly, which can be proved as follows:

We proceed via contradiction. Suppose that $\exists\epsilon \mid \forall x \sup(S) - \epsilon \geq x$. Then $\sup(S) - \epsilon$ is an upper bound for S. By definition of sup, we have the statement $\sup(S) < \sup(S) - \epsilon$ but as $\epsilon > 0$, $\Rightarrow\Leftarrow$

($\impliedby$) The first half establishes t as an upper bound. Further, suppose that t is not the least upper bound, such that $\exists t' \in \mathbb{R} \mid \forall s \in S, t' \geq s, t' < t$. However, $t' < t \implies t - t' > 0$, meaning that we have for $\epsilon = \frac{t-t'}{2}$, we have that $\forall x \in S, t - \frac{t-t'}{2} = \frac{t}{2} - \frac{t'}{2} > \frac{t'}{2} + \frac{t'}{2} = t' > x$. $\Rightarrow\Leftarrow$, as thus there is no $x \in S$ that can satisfy the premise without violating trichotomy, so t is the least upper bound. $\square$

Problem 3

Claim. Suppose $S, T \subseteq R$, both nonempty and bounded above, with a bijective function $f : S \to T$ such that $\forall x \in S, x \geq f(x)$. Then $\sup(S) \geq \sup(T)$.

Proof. Now, suppose that $\sup(S) < \sup(T)$. Approximation furnishes $t \in T$ such that for arbitrary $\epsilon > 0, \sup(T) - \epsilon < t$. Further, we have that $\sup(S) < \sup(T) \implies \sup(T) - \sup(S) > 0$. Taking $\epsilon = \sup(T) - \sup(S)$, we see that $\exists t \in T \mid t > \sup(T) - (\sup(T) - \sup(S)) = \sup(S)$. Thus, we have, as f is surjective, that $\exists s \in S \mid t = f(s) > \sup(S) \geq s$. However, we have that $\forall s \in S, s \geq f(s)$. $\Rightarrow\Leftarrow$, so $\sup(S) > \sup(T)$. $\square$

We can't conclude that $x > f(x) \implies \sup(S) > \sup(T)$, as the above reasoning fails in that we can only say, after assuming the opposite of $\sup(S) \geq \sup(T)$, that $\sup(T) - \sup(S) \geq 0$. Then, we cannot reason with $\epsilon = \sup(T) - \sup(S) > 0$, as it is possible $\sup(T) - \sup(S) = 0$.

An actual example is that $f : (\frac{1}{2}, 1) \to (0, 1)$, where $f(x) = 2x - 1$ is a bijection such that $\forall x \in (\frac{1}{2}, 1), x > f(x)$, as $x > 2x - 1 \iff x < 1$. However, $\sup((\frac{1}{2}, 1)) = \sup((0, 1)) = 1$.

Claim. However, we can claim that $\sup(T) \notin T$.

Proof. Note that otherwise we could take the above $\epsilon = 0$, as $\sup(T)$ is exactly the element in T such that $\sup(T) - 0 = \sup(T) \in T$. Then, moving in the same line of reasoning as the original proof, we have that $\forall s \in S, s < f(s)$ and $\exists s \in S \mid f(s) \geq s$. $\Rightarrow\Leftarrow$. $\square$

Problem 4

a)

Claim. For $S = \{\frac{n-1}{n} \mid n \in \mathbb{Z}_{>0}\}$, $\sup(S) = 1$.

First, we have that $\frac{n-1}{n} < \frac{n-1}{n} + \frac{1}{n} = \frac{n-1+1}{n} = 1$. This holds as $\frac{1}{n} > 0 \impliedby n \in \mathbb{Z}_{>0}$. Thus, 1 is an upper bound on S.

Now, for any $\epsilon > 0$, we have that the Archimedian property furnishes an $n \in \mathbb{Z}_{>0}$ such that $n\epsilon > 1$. Then, $\exists n \in \mathbb{Z}_{>0} \mid \epsilon > \frac{1}{n} \implies -\epsilon < -\frac{1}{n} \implies 1 - \epsilon < 1 - \frac{1}{n} = \frac{n-1}{n} \in S$.

We use problem 2 here to show that since 1 is an upper bound for $S = \{\frac{n-1}{n} \mid n \in \mathbb{Z}_{>0}\}$ and $\forall \epsilon > 0, \exists x \in S \mid x > 1 - \epsilon$, we can conclude $\sup(S) = 1$.

b)

Claim. For $S = \{\frac{n+1}{n} \mid n \in \mathbb{Z}_{>0}\}$, $\inf(S) = 1$.

Proof. We first must show that for $S \subseteq \mathbb{R}, t \in \mathbb{R}$, $\forall s \in S, t \leq s$, and $\forall \epsilon > 0, \exists x \in S \mid x > t + \epsilon \implies t = \inf(S)$.

The first half establishes t as an lower bound. Further, suppose that t is not the greatest lower bound, such that $\exists t' \in \mathbb{R} \mid \forall s \in S, t' \geq s, t' > t$. However, $t' > t \implies t' - t > 0$, meaning that we have for $\epsilon = \frac{t'-t}{2}$, we have that $\forall x \in S, t + \frac{t'-t}{2} = \frac{t}{2} + \frac{t'}{2} < \frac{t'}{2} + \frac{t'}{2} = t' < x$. $\Rightarrow\Leftarrow$, thus there is no $x \in S$ that can satisfy the premise without violating trichotomy, so t is the greatest lower bound.

First, we have that $\frac{n+1}{n} > \frac{n+1}{n} - \frac{1}{n} = \frac{n-1+1}{n} = 1$. This holds as $-\frac{1}{n} < 0 \impliedby -n < 0 \impliedby n \in \mathbb{Z}_{>0}$. Thus, 1 is an lower bound on S.

Now, for any $\epsilon > 0$, we have that the Archimedian property furnishes an $n \in \mathbb{Z}_{>0}$ such that $n\epsilon > 1$. Then, $\exists n \in \mathbb{Z}_{>0} \mid \epsilon > \frac{1}{n} \implies 1 + \epsilon < 1 + \frac{1}{n} = \frac{n+1}{n} \in S$.

Thus, 1 must be the greatest lower bound of S. $\square$