MATH 4065 HW 5

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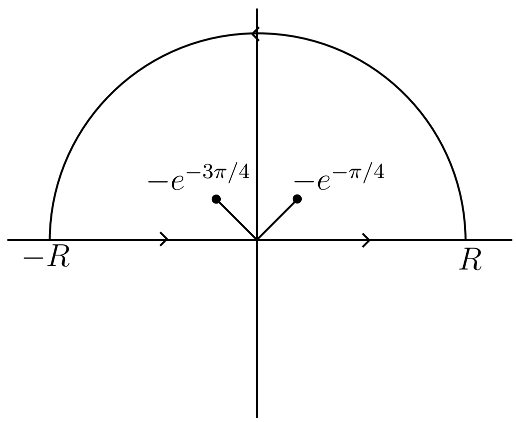

We note that $\frac{1}{1+z^4}$ has poles when $z^4 = -1$, which happens when $z^2 = \pm i$, so the poles occur at $\{e^{-\pi i/4}, e^{-3\pi i/4}, e^{-5\pi i/4}, e^{-7\pi i/4}\}$. In particular, we have that

$$\begin{aligned} \frac{1}{1+z^4} = \frac{1}{(z^2-i)(z^2+i)} &= \frac{1}{(z-e^{-\pi i/4})(z+e^{-\pi i/4})(z-e^{-3\pi i/4})(z+e^{-3\pi i/4})} \\ &= \frac{1}{(z-e^{-\pi i/4})(z-e^{-5\pi i/4})(z-e^{-3\pi i/4})(z-e^{-7\pi i/4})} \end{aligned}$$

so we can see that all of the poles are simple, as for any root w, we have that the right hand side with the $z - w$ term removed in the denominator is a holomorphic function that doesn't vanish in a neighborhood of w.

For example, we have that at $e^{-\pi i/4}$,

$$\frac{1}{1+z^4} = (z-e^{-\pi i/4})\frac{1}{(z-e^{-3\pi i/4})(z-e^{-5\pi i/4})(z-e^{-7\pi i/4})}$$

and $\frac{1}{(z-e^{-3\pi i/4})(z-e^{-5\pi i/4})(z-e^{-7\pi i/4})}$ is clearly holomorphic near $e^{-\pi i/4}$, since the denominator is nonzero, and also clearly does not vanish, since the numerator is nonzero.

Similarly,

$$\frac{1}{1+z^4} = (z - e^{-3\pi i/4})\frac{1}{(z-e^{-\pi i/4})(z-e^{-5\pi i/4})(z-e^{-7\pi i/4})}$$

and $\frac{1}{(z-e^{-\pi i/4})(z-e^{-5\pi i/4})(z-e^{-7\pi i/4})}$ is clearly holomorphic near $e^{-3\pi i/4}$, since the denominator is nonzero, and also clearly does not vanish, since the numerator is nonzero.

Integrating over the contour shown in the picture, we have that

$$\int_{-R}^{R} \frac{dx}{1+x^4} + \int_\gamma \frac{dz}{1+z^4} = 2\pi i \left(\operatorname{res}_{e^{-\pi i/4}} \frac{1}{1+z^4} + \operatorname{res}_{e^{-3\pi i/4}} \frac{1}{1+z^4} \right)$$

where γ is the upper semicircle. Computing the residues,

$$\begin{aligned}
\operatorname{res}_{e^{-\pi i/4}} \frac{1}{1+z^4} &= \lim_{z\to e^{-\pi i/4}} (z-e^{-\pi i/4})\frac{1}{(z-e^{-\pi i/4})(z-e^{-3\pi i/4})(z-e^{-5\pi i/4})(z-e^{-7\pi i/4})} \\
&= \lim_{z\to e^{-\pi i/4}} \frac{1}{(z-e^{-3\pi i/4})(z-e^{-5\pi i/4})(z-e^{-7\pi i/4})} \\
&= \frac{1}{(e^{-\pi i/4}-e^{-3\pi i/4})(e^{-\pi i/4}-e^{-5\pi i/4})(e^{-\pi i/4}-e^{-7\pi i/4})} \\
&= \frac{1}{(\sqrt{2})(\sqrt{2}+\sqrt{2}i)(\sqrt{2}i)} = \frac{1}{4e^{3\pi/4}} \\
\operatorname{res}_{e^{-3\pi i/4}} \frac{1}{1+z^4} &= \lim_{z\to e^{-3\pi i/4}} (z-e^{-3\pi i/4})\frac{1}{(z-e^{-\pi i/4})(z-e^{-3\pi i/4})(z-e^{-5\pi i/4})(z-e^{-7\pi i/4})} \\
&= \lim_{z\to e^{-3\pi i/4}} \frac{1}{(z-e^{-\pi i/4})(z-e^{-5\pi i/4})(z-e^{-7\pi i/4})} \\
&= \frac{1}{(e^{-3\pi i/4}-e^{-\pi i/4})(e^{-3\pi i/4}-e^{-5\pi i/4})(e^{-3\pi i/4}-e^{-7\pi i/4})} \\
&= \frac{1}{(-\sqrt{2})(-\sqrt{2}+\sqrt{2}i)(\sqrt{2}i)} = \frac{1}{4e^{\pi/4}}
\end{aligned}$$

Then, we have that

$$\int_{-R}^{R} \frac{dx}{1+x^4} + \int_\gamma \frac{dz}{1+z^4} = 2\pi i \left(\frac{1}{4e^{\pi/4}} + \frac{1}{4e^{3\pi/4}} \right) = \frac{\pi}{2}(e^{-\pi/4} + e^{-3\pi/4}) = \frac{\pi\sqrt{2}}{2}$$

We also have that $\frac{1}{|1+z^4|} \leq \frac{B}{|z^4|}$ for sufficiently large z, where B is a fixed constant. To see this, consider that $\left|\frac{1+z^4}{z^4}\right| = \left|1+\frac{1}{z^4}\right|$. Then, since $1 - \frac{1}{|z^4|} \leq \left|1+\frac{1}{z^4}\right| \leq 1 + \frac{1}{|z^4|}$ by the triangle inequality (and the reverse triangle inequality), so we have that $\lim_{|z|\to\infty} \frac{1}{|z|^4} = 0$, and so for any ϵ, for $|z| > N$ for some N,

$$1-\epsilon \leq \left|\frac{1+z^4}{z^4}\right| \leq 1+\epsilon \implies (1-\epsilon)\left|\frac{1}{1+z^4}\right| \leq \left|\frac{1}{z^4}\right|$$

which gives us what we want. Then, for sufficiently large R, we have that

$$\int_\gamma \frac{dz}{1+z^4} \leq \pi R \frac{B}{R^4} \leq \frac{\pi B}{R^3}$$

so as as $R \to \infty$, the integral $\to 0$. Then,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi\sqrt{2}}{2}$$

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We can see that $f(z) = \frac{e^{-2\pi i z\xi}}{(1+z^2)^2} = \frac{e^{-2\pi i z\xi}}{(z-i)^2(z+i)^2}$. Then, at i, we have that

$$f(z) = (z-i)^{-2} \frac{e^{-2\pi i z\xi}}{(z+i)^2}$$

where $\frac{e^{-2\pi i x\xi}}{(z+i)^2}$ is holomorphic in a neighborhood of i (say $D_1(i)$), since $(z+i)^2$ doesn't vanish on that neighborhood, since $-i \notin D_1(i)$. Similarly, at $-i$,

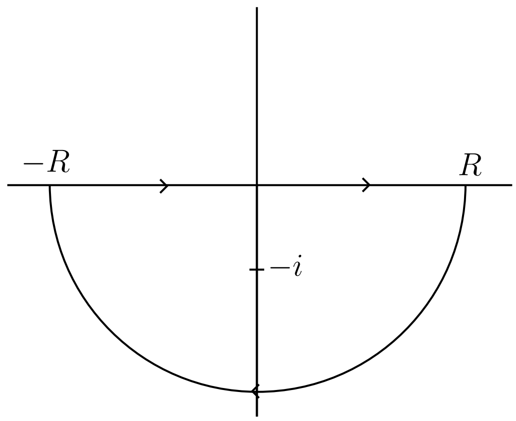
$$f(z) = (z+i)^{-2} \frac{e^{-2\pi i z\xi}}{(z-i)^2}$$

where $\frac{e^{-2\pi i x\xi}}{(z-i)^2}$ is holomorphic in a neighborhood of $-i$ (say $D_1(-i)$), since $(z-i)^2$ doesn't vanish on that neighborhood, since $i \notin D_1(-i)$.

Then, we have poles of order 2 at $\pm i$. Computing the residues,

$$\begin{aligned}
\operatorname{res}_i f &= \lim_{z\to i} \frac{d}{dz} \frac{e^{-2\pi i z\xi}}{(z+i)^2} \\
&= \lim_{z\to i} \left(-2(z+i)^{-3} e^{-2\pi i z\xi} - 2\pi i\xi (z+i)^{-2} e^{-2\pi i z\xi}\right) \\
&= \left(\frac{1}{4i} + \frac{\pi i\xi}{2}\right) e^{2\pi\xi} \\
&= \left(\frac{\pi i\xi}{2} - \frac{i}{4}\right) e^{2\pi\xi} \\
\operatorname{res}_{-i} f &= \lim_{z\to -i} \frac{d}{dz} \frac{e^{-2\pi i z\xi}}{(z-i)^2} \\
&= \lim_{z\to -i} \left(-2(z-i)^{-3} e^{-2\pi i z\xi} - 2\pi i\xi (z-i)^{-2} e^{-2\pi i z\xi}\right) \\
&= \left(-\frac{1}{4i} + \frac{\pi i\xi}{2}\right) e^{-2\pi\xi} \\
&= \left(\frac{\pi i\xi}{2} + \frac{i}{4}\right) e^{-2\pi\xi}
\end{aligned}$$

We handle the case where $\xi \geq 0$ first, where $\xi = |\xi|$. Then, we integrate over the semicircle in the negative half-plane, as shown:



In this case, we have that, where γ is the semicircle,

$$\int_{-R}^{R} \frac{e^{-2\pi i x\xi}}{(1+x^2)^2} dx + \int_{\gamma} \frac{e^{-2\pi i z\xi}}{(1+z^2)^2} dz = -2\pi i \operatorname{res}_{-i} f = \frac{\pi}{2}(1+2\pi\xi)e^{-2\pi\xi} = \frac{\pi}{2}(1+2\pi|\xi|)e^{-2\pi|\xi|}$$

where it is $-2\pi i \operatorname{res}_{-1} f$ since the path is negatively oriented. Then, all we have to do is to show that the second integral vanishes:

$$\begin{aligned}
\int_{\gamma} \frac{e^{-2\pi i z\xi}}{(1+z^2)^2} dz &\leq (\pi R) \sup \left(\frac{|e^{-2\pi i z\xi}|}{|(1+z^2)^2|} \right) \\
&= (\pi R) \sup \left(\frac{e^{\operatorname{Re}(-2\pi i z\xi)}}{|1+z^2|^2} \right) \\
&= (\pi R) \sup \left(\frac{e^{2\pi \operatorname{Im}(z)\xi}}{|1+z^2|^2} \right)
\end{aligned}$$

Since $\xi \geq 0$ and $\operatorname{Im}(z) \leq 0$, we have that $2\pi \operatorname{Im}(z)\xi \leq 0$, so

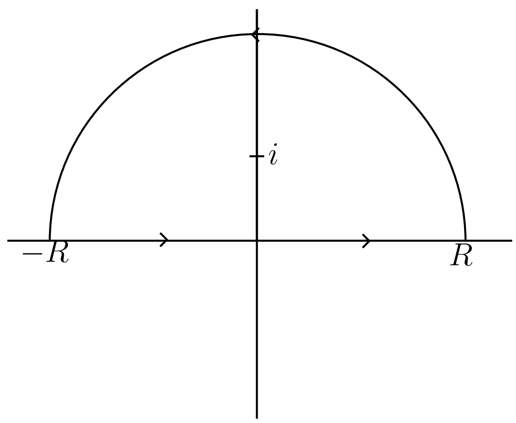
$$\begin{aligned}
&\leq (\pi R) \sup \left(\frac{1}{|1+z^2|^2} \right) \\
&= \sup \frac{\pi R}{|1+z^2|^2}
\end{aligned}$$

We have by the reverse triangle inequality (should've used this in the first problem), $|1+z^2| \geq |z^2| - 1$, so

$$\begin{aligned} &\leq \sup \frac{\pi R}{(|z^2|-1)^2} \\ &= \frac{\pi R}{(R^2-1)^2} \end{aligned}$$

Taking $R \to \infty$, we have that in the limit $\int_\gamma \frac{e^{-2\pi i z\xi}}{(1+z^2)^2} dz = 0$, and this gives us what we want for $\xi \geq 0$.

Similarly, for $\xi < 0$, we integrate over the top half plane, as shown:



which gives

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{(1+x^2)^2} dx + \int_\gamma \frac{e^{-2\pi i z\xi}}{(1+z^2)^2} dz = 2\pi i \operatorname{res}_{-i} f = \frac{\pi}{2}(1-2\pi\xi)e^{2\pi\xi} = \frac{\pi}{2}(1+2\pi|\xi|)e^{-2\pi|\xi|}$$

since $\xi < 0 \implies -\xi = |\xi|$.

Then, again we have that

$$\int_\gamma \frac{e^{-2\pi i z\xi}}{(1+z^2)^2} dz \leq (\pi R) \sup \left(\frac{e^{2\pi \operatorname{Im}(z)\xi}}{|1+z^2|^2} \right)$$

Since $\xi < 0$ and $\operatorname{Im}(z) \geq 0$, we have that $2\pi \operatorname{Im}(z)\xi \leq 0$, so

$$\begin{aligned} &\leq (\pi R) \sup \left(\frac{1}{|1+z^2|^2} \right) \\ &\leq \frac{\pi R}{(R^2-1)^2} \end{aligned}$$

which again gives that in the limit $\int_\gamma \frac{e^{-2\pi i z\xi}}{(1+z^2)^2} dz = 0$, and this gives us what we want for $\xi < 0$.

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We have poles of order $n+1$ at $\pm i$. In particular,

$$\frac{1}{(1+z^2)^{n+1}} = \frac{1}{((z-i)(z+i))^{n+1}} = \frac{1}{(z-i)^{n+1}(z+i)^{n+1}} = (z-i)^{-(n+1)}(z+i)^{-(n+1)}$$

and clearly $(z-i)^{-(n+1)}$ and $(z+i)^{-(n+1)}$ are holomorphic in a neighborhood around $-i$ and i respectively (for a concrete one, say $D_1(-i)$ and $D_1(i)$, which do not contain i and $-i$ respectively).

Computing the residues, we have that

$$\begin{aligned}\operatorname{res}_i \frac{1}{(1+z^2)^{n+1}} &= \lim_{z\to i} \frac{1}{n!}\left(\frac{d}{dz}\right)^n (z-i)^{n+1}\frac{1}{(1+z^2)^{n+1}} \\ &= \lim_{z\to i} \frac{1}{n!}\left(\frac{d}{dz}\right)^n (z+i)^{-(n+1)}\end{aligned}$$

We can show that $\frac{d^n}{dz^n}(z+i)^{-m} = (-1)^n \left(\prod_{i=0}^{n-1}(m+i)\right)(z+i)^{-m-n}$ for $n, m \geq 0$. In particular, inducting on n, the base case $n=1$ is easy:

$$\frac{d}{dz}(z \pm i)^{-1} = -(z \pm i)^{-2}$$

as desired. Then, if it holds for n,

$$\begin{aligned}\frac{d^{n+1}}{dz^{n+1}}(z\pm i)^{-m} &= \frac{d}{dz}\left(\frac{d^n}{dz^n}(z\pm i)^{-m}\right) \\ &= \frac{d}{dz}\left((-1)^n\left(\prod_{j=1}^{n-1}(m+j)\right)(z\pm i)^{-m-n}\right) \\ &= -(m-n)\left((-1)^n\left(\prod_{j=0}^{n-1}(m+j)\right)(z\pm i)^{-m-(n+1)}\right) \\ &= (-1)^{n+1}\left(\prod_{j=0}^{n}(m+j)\right)(z\pm i)^{-m-(n+1)}\end{aligned}$$

so it holds for $n+1$, and so it holds in general for any $n \geq 1$. In particular, for $m = n+1$,

$$\frac{d^n}{dz^n}(z\pm i)^{-(n+1)} = (-1)^n\left(\prod_{j=0}^{n-1}(n+1+j)\right)(z\pm i)^{-(2n+1)}$$

Then,

$$\begin{aligned}
\operatorname{res}_i \frac{1}{(1+z^2)^{n+1}} &= \lim_{z\to i} \frac{1}{n!}(-1)^n \left(\prod_{j=1}^{n}(n+j)\right)(z+i)^{-(2n+1)} \\
&= \frac{i^{2n}}{2i^{2n+1}} \left(\frac{\prod_{j=1}^{2n} j}{2^{2n}(\prod_{j=1}^{n} j)(\prod_{j=1}^{n} j)}\right) \\
&= \frac{1}{2i} \left(\frac{\prod_{j=1}^{n}(n+j)}{(\prod_{j=1}^{n} 2j)(\prod_{j=1}^{n} 2j)}\right)
\end{aligned}$$

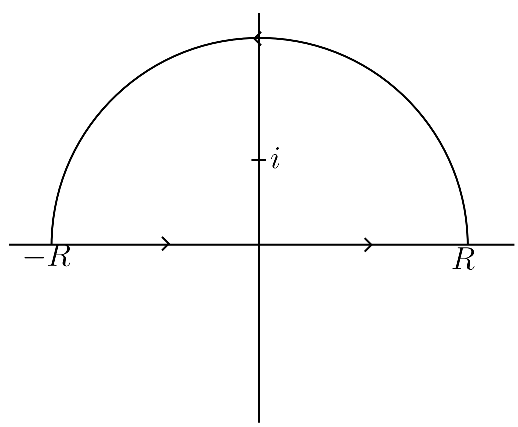
which is what we wanted, since

$$\begin{aligned}
\frac{\prod_{j=1}^{2n} j}{(\prod_{j=1}^{n} 2j)(\prod_{j=1}^{n} 2j)} &= \frac{\prod_{j=1}^{n} 2j \prod_{j=1}^{n}(2j-1)}{(\prod_{j=1}^{n} 2j)(\prod_{j=1}^{n} 2j)} \\
&= \frac{\prod_{j=1}^{n}(2j-1)}{\prod_{j=1}^{n} 2j}
\end{aligned}$$

Alternatively,

$$\begin{aligned}
\frac{\prod_{j=1}^{2n} j}{(\prod_{j=1}^{n} 2j)(\prod_{j=1}^{n} 2j)} &= \frac{1\cdot 2\cdot 3\cdots 2n}{(2\cdot 4\cdot 6\cdots 2n)(2\cdot 4\cdot 6\cdots 2n)} \\
&= \frac{1\cdot 3\cdot 5\cdots (2n-1)}{2\cdot 4\cdot 6\cdots 2n}
\end{aligned}$$

Integrating over the semicircle in the positive half plane,



Thus, we have that, where γ is the arc of the semicircle,

$$\begin{aligned}\int_{-R}^{R} \frac{dx}{(1+x^2)^{n+1}} + \int_\gamma \frac{dz}{(1+z^2)^{n+1}} &= 2\pi i \operatorname{res}_i \frac{1}{(1+z^2)^{n+1}} \\ &= 2\pi i \left(\frac{1}{2i} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \right) \\ &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \pi\end{aligned}$$

The only left is to show that $\int_\gamma \frac{dz}{(1+z^2)^{n+1}}$ vanishes in the limit. In particular, we have that

$$\begin{aligned}\int_\gamma \frac{dz}{(1+z^2)^{n+1}} &\leq \pi R \sup \left(\frac{1}{|(1+z^2)^{n+1}|} \right) \\ &= \pi R \sup \left(\frac{1}{|1+z^2|^{n+1}} \right)\end{aligned}$$

By the reverse triangle inequality, $|1+z^2| \geq |z^2| - 1$, so

$$\begin{aligned}&\leq \pi R \sup \left(\frac{1}{(|z|^2-1)^{n+1}} \right) \\ &= \frac{\pi R}{(R^2-1)^{n+1}}\end{aligned}$$

Since we have that $n \geq 1$, this vanishes in the limit and we are left with the desired result.

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We will show that if $\lim_{z \to z_0} (z-z_0) f(z) = 0$, then f has a removable singularity at z_0. If we have this, then for f bounded as in the problem,

$$0 \leq \lim_{z \to z_0} |(z-z_0) f(z)| \leq \lim_{z \to z_0} A|(z-z_0)(z-z_0)^{-1+\epsilon}| = \lim_{z \to z_0} A|(z-z_0)^\epsilon| = A(z_0-z_0)^\epsilon = 0$$

so $\lim_{z \to z_0} (z-z_0) f(z) = 0$.

To see the claim about removable singularities, we have that if we take

$$g(z) = \begin{cases} (z-z_0)^2 f(z) & z \in D_r(z_0) \setminus \{z_0\} \\ 0 & z = z_0 \end{cases}$$

then $g(z)$ is holomorphic on $D_r(z_0) \setminus \{z_0\}$ as the product of two holomorphic functions. Further, we have that

$$g'(z_0) = \lim_{z \to z_0} \frac{(z-z_0)^2 f(z) - g(z_0)}{z - z_0} = \lim_{z \to z_0} (z - z_0) f(z) = 0$$

is defined, so g is holomorphic at z_0 as well. Thus, g is holomorphic on all of $D_r(z_0)$, and we have that $g(z_0) = g'(z_0) = 0$, such that on $D_r(z_0)$,

$$g(z) = g(0) + g'(0)(z - z_0) + \sum_{n=2}^{\infty} a_n (z - z_0)^n = \sum_{n=2}^{\infty} a_n (z - z_0)^n$$

and

$$\frac{g(z)}{(z - z_0)^2} = \sum_{n=0}^{\infty} a_{n+2} (z - z_0)^n$$

satisfies that on $D_r(z_0) \setminus \{z_0\}$, $\frac{g(z)}{(z-z_0)^2} = \frac{(z-z_0)^2 f(z)}{(z-z_0)^2} = f(z)$, but is holomorphic on all of $D_r(z_0)$, so taking $f(z_0) = a_2$ extends f to $D_r(z_0)$, and so the singularity is removable.

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Assume $b \neq 0$; otherwise this integral collapes to $\int_0^{2\pi} \frac{d\theta}{a} = \frac{\theta}{a}\Big|_0^{2\pi} = \frac{2\pi}{a}$, which is what we wanted.

We will first transform this into a complex integral on the unit circle C. In particular, under the parameterization $\gamma(\theta) = e^{i\theta}$ where $\theta \in [0, 2\pi]$, we have that

$$\int_C f(z) dz = \int_0^{2\pi} f(e^{i\theta}) i e^{i\theta} d\theta$$

so if we take $f(z) = \left(a + b\frac{z+z^{-1}}{2}\right)^{-1} (iz)^{-1} = -\frac{2i}{bz^2 + 2az + b}$, we have that

$$\int_C f(z) dz = \int_0^{2\pi} \frac{i e^{i\theta} d\theta}{\left(a + b\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)\right)(ie^{i\theta})} = \int_0^{2\pi} \frac{d\theta}{a + b\cos(\theta)}$$

Put $r_1 = \frac{a + \sqrt{a^2 - b^2}}{b}$, $r_2 = \frac{a - \sqrt{a^2 - b^2}}{b}$; these are the roots to $bz^2 + 2az + b = 0$, and thus f has a simple pole at both r_1 and r_2 when they differ (in particular, they differ in this problem since $a > |b| \implies a^2 - b^2 > 0$), since $f(z) = \frac{-2i}{b(z-r_1)(z-r_2)}$. Note that both r_1, r_2 are real since $a, b \in \mathbb{R}$ and $a^2 - b^2 > 0 \implies \sqrt{a^2 - b^2} \in \mathbb{R}$.

Then, we have that $|r_1| = \frac{|a + \sqrt{a^2 - b^2}|}{|b|}$, but we have that $a > |b| > 0$ and $\sqrt{a^2 - b^2} > 0$ so $|a + \sqrt{a^2 - b^2}| = a + \sqrt{a^2 - b^2} \geq a$, and so $|r_1| \geq \frac{a}{|b|} > 1$. To see that the other one lies in the unit circle, note that we have that $|b|^2 + (\sqrt{a^2 - b^2})^2 = a^2$, and so $(|b| + \sqrt{a^2 - b^2})^2 = a^2 + 2|b|\sqrt{a^2 - b^2}$, and since (remember that we take $b \neq 0$) $|b|, \sqrt{a^2 - b^2} > 0$, we have that $a < |b| + \sqrt{a^2 - b^2} \implies a - \sqrt{a^2 - b^2} < |b|$. Lastly, since $a = \sqrt{a^2} > \sqrt{a^2 - b^2} > 0$, we have that $a - \sqrt{a^2 - b^2} = |a - \sqrt{a^2 - b^2}| < |b|$, so $|r_2| < 1$.

Computing the residue,

$$\begin{aligned}
\operatorname{res}_{r_2} f &= \lim_{z \to r_2} (z - r_2) \frac{-2i}{b(z - r_1)(z - r_2)} \\
&= \lim_{z \to r_2} \frac{-2i}{b(z - r_1)} \\
&= \frac{-2i}{b(r_2 - r_1)} \\
&= \frac{-2i}{2\sqrt{a^2 - b^2}} \\
&= \frac{1}{i\sqrt{a^2 - b^2}}
\end{aligned}$$

Then,

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos(\theta)} = \int_C f(z)dz = 2\pi i \operatorname{res}_{r_2} f = 2\pi i \frac{1}{i\sqrt{a^2 - b^2}} = \frac{2\pi}{a^2 - b^2}$$

as desired.