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Homework 11

Apostol p.391 no.3

$$\begin{aligned}\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n^2-1} &= \sum_{n=2}^{\infty} \frac{1}{(n-1)(n+1)} \\ &= \sum_{n=2}^{\infty} \frac{1}{2}(\frac{1}{n-1} - \frac{1}{n+1}) \\ &= \sum_{n=2}^{\infty} \frac{1}{2}((\frac{1}{n-1} + \frac{1}{n}) - (\frac{1}{n} + \frac{1}{n+1}))\end{aligned}$$

Note that we now have that if we take $b_n = -\frac{1}{2}(\frac{1}{n} + \frac{1}{n+1})$, we have then $a_n = b_n - b_{n-1}$. Then, from class, $\sum_{n=2}^{m} a_n = b_m - b_1$. Then, $\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \lim_{m\to\infty}(b_m - b_1) = \lim_{m\to\infty}(-\frac{1}{2}(\frac{1}{m} + \frac{1}{m+1}) + (\frac{1}{1} + \frac{1}{2})) = \frac{3}{4}$.

Apostol p.391 no.4

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2^n + 3^n}{6^n} &= \sum_{n=1}^{\infty} \frac{2^n}{6^n} + \frac{3^n}{6^n} \\ &= \sum_{n=1}^{\infty} (\frac{1}{3})^n + (\frac{1}{2})^n \\ &= \sum_{n=1}^{\infty} (\frac{1}{3})^n + \sum_{n=1}^{\infty} (\frac{1}{2})^n \\ &= \frac{\frac{1}{3}}{1-\frac{1}{3}} + \frac{\frac{1}{2}}{1-\frac{1}{2}} \\ &= \frac{1}{2} + 1 = \frac{3}{2}\end{aligned}$$

Note that the formula for geometric series was derived in class and that the splitting of the sum is also justified in class as well in the proof of the propositions about absolute convergence.

Apostol p.411 no.50

Claim. If $\sum |a_n|$ converges, then $\sum a_n^2$ also converges.

Proof. This proof will use freely the result from part C that $\sum_{i=0}^{\infty} a_i$ converges if and only if $\sum_{i=l}^{\infty} a_i$ converges, as that does not depend on this result, as well as the result, also from part C, that $\sum_{i=1}^{\infty} a_i$ converging implies that $\lim_{i\to\infty} a_i = 0$.

Since we have that $\sum |a_n|$ converges, we know that $\{|a_n|\}$ converges to 0. Then $\exists N \in \mathbb{Z}_{>0} \mid n \geq N \implies ||a_n| - 0| = |a_n| < 1 \implies |a_n|^2 = a_n^2 < 1$. Consider now the sums $\sum_{i=N}^{\infty} |a_n|, \sum_{i=N}^{\infty} a_n^2$. We have by the comparison test, since $|a_n| > a_n^2$, that $\sum_{i=N}^{\infty} a_n^2$ must also converge, and so $\sum a_n^2$ in general also converges. $\square$

The converse must be false. Consider the harmonic series; $\sum_{i=1}^{\infty} |a_n| = \sum_{i=1}^{\infty} \frac{1}{n}$ diverges, but $\sum_{i=1}^{\infty} a_n^2 = \sum_{i=1}^{\infty} \frac{1}{n^2}$ converges.

Problem 1

Claim. If $\{a_n\}$ is indexed from 0, then for any $k \in \mathbb{Z}_{\geq 0}$, $\sum_{i=0}^{\infty} a_n$ converges if and only if $\sum_{i=k}^{\infty} a_n := \sum_{n=0}^{\infty} a_{i+k}$ also converges.

Proof. ($\implies$) Let $b_n = \sum_{i=0}^{n} a_i$, and let $c_n = \sum_{i=0}^{n} a_{i+k} = \sum_{i=k}^{k+n} a_i$. Then, we have that for $n \geq k$, $b_n = \sum_{i=0}^{n} a_i = \sum_{i=0}^{k-1} a_i + \sum_{i=k}^{n} a_i = b_{k-1} + c_{n-k}$. Reindexing, we have that $c_n = b_{n+k} - b_{k-1}$. Then, since we have that $\lim_{n\to\infty} b_n$ exists, for any $\epsilon > 0 \exists N \in \mathbb{Z}_{\geq 0} \mid n > N \implies |b_n - L| < \epsilon$. Then, we must also have that $\lim_{n\to\infty} b_{k-1} + c_n$ exists, as for ϵ, consider take $N' = \max(k, N)$. Then, $n > N' \implies |c_n - (L - b_{k-1})| = |b_{n+k} - b_{k-1} - (L - b_{k-1})| = |b_{n+k} - L| < \epsilon$.

($\impliedby$) Let b_n, c_n be as before, such that $b_n = b_{k-1} + c_{n-k}$. Then, we have that $\lim_{n\to\infty} c_n = L$, so for any $\epsilon > 0, \exists N \mid n > N \implies |c_n - L| < \epsilon$. Then, $\lim_{n\to\infty} b_n$ must also converge, as for ϵ take $N' = \max(k, N)$, so $n > N' \implies |b_n - (L + b_{k-1})| = |b_{k-1} + c_n - (L + b_{k-1})| = |c_n - L| < \epsilon$. $\square$

Note that this statement immediately generalizes to sums not indexed from 0; for any sequence $\{a_n\}$ indexed from m, let $a_0, a_1, ..., a_m = 0$. Then, we have that $\sum_{i=m}^{k} a_i = \sum_{i=0}^{k}$, and the result shows that $\sum_{i=m}^{k} a_i$ converges if and only if $\sum_{i=m'}^{k} a_i$ converges for $m' \geq m$.

Problem 2

Claim. For sequences $\{a_n\}, \{b_n\}$ and a number $M \in \mathbb{Z}_{>0}$ such that $n \geq M \implies a_n = b_n$, prove that the series $\sum_{i=1}^{\infty} a_n$ converges if and only if $\sum_{i=1}^{\infty} b_n$ does.

Proof. $\sum_{i=1}^{\infty} a_n$ converges if and only if, as above, $\sum_{i=M}^{\infty} a_n = \sum_{i=M}^{\infty} b_n$ also converges, which converges if and only if $\sum_{i=1}^{\infty} b_n$ converge. $\square$

Problem 3

Claim. If $\{a_n\}$ is a sequence and the sum $\sum_{i=1}^{\infty} a_i$ converges, then the sequence converges to 0.

Proof. Let $b_n = \sum_{i=1}^{n} a_i$, and $\lim_{n\to\infty} b_n = L$. Then, for any $\frac{\epsilon}{2} > 0, \exists N \in \mathbb{Z}_{>0} \mid n > N \implies |b_n - L| < \frac{\epsilon}{2} \implies |b_{n+1} - L| + |b_n - L| = |b_{n+1} - L| + |L - b_n| < \epsilon \implies |b_{n+1} - L + (L - b_n)| = |b_{n+1} - b_n| = |a_{n+1} - 0| < \epsilon$. Then, we have that $\forall \epsilon > 0$, $\exists N \in \mathbb{Z}_{>0} \mid n > N \implies |a_n - 0| < \epsilon$, so $\{a_n\}$ must converge to 0. □

Problem 4

Claim. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if whenever $\{x_n\}$ converges with $\lim_{n\to\infty} x_n = x$, then $\{f(x_n)\}$ converges with $\lim_{n\to\infty} f(x_n) = f(x)$.

Proof. ($\implies$) f is continuous means that for any $\epsilon > 0$, $\exists \delta \mid |x_n - x| < \delta \implies |f(x_n) - f(x)| < \epsilon$. Then, since $x_n \to x$, for $\delta \exists N \in \mathbb{Z}_{>0} \mid n > N \implies |x_n - x| < \delta \implies |f(x_n) - f(x)| < \epsilon$.

($\impliedby$) Suppose that f is not continuous. Then $\exists \epsilon > 0 \mid \forall \delta > 0, \exists x' \mid |x' - x| < \delta$ and $|f(x') - f(x)| \geq \epsilon$. We define x_n as follows: let $\delta = \frac{1}{n}$. Then, $\exists x' \mid |x' - x| < \frac{1}{n}$ and $|f(x') - f(x)| \geq \epsilon$ since f is discontinuous at x. Let $x_n = x'$.

Now, $\forall n, |f(x_n) - f(x)| \geq \epsilon$, so $\lim_{n\to\infty} f(x_n) \neq f(x)$. Further, for any $\epsilon > 0$, let $N = \lceil \frac{1}{\epsilon} \rceil$. Then, we have that $|x_n - x| < \frac{1}{n}$, so $n > N \implies |x_n - x| < \frac{1}{N} = \frac{1}{\lceil \frac{1}{\epsilon} \rceil} < \frac{1}{\frac{1}{\epsilon}} = \epsilon$, so $\lim_{n\to\infty} x_n = x$. $\Rightarrow\Leftarrow$, so f must be continuous. □

Problem 5

a

Claim. Let $\{a_n\}$ be a sequence. Suppose that for all $c \in \mathbb{R}$, $\exists N \in \mathbb{Z}_{>0} \mid \forall n \in \mathbb{Z}_{>0}, n \geq N \implies |a_n| > c$. $\{a_n\}$ is divergent.

Proof. Any convergent series $\{a_n\}$ must be bounded above. Let $\lim_{n\to\infty} a_n = a$. To see why, pick $\epsilon = 1$, such that $\exists N \mid n \geq N \implies |a_n - a| < 1$. Then, let M_1 be the maximum of $\{a_1, a_2, ...a_N\}$. Then, we see that $\forall n$, $a_n \leq \max(M_1, a + 1)$.

Now suppose that the initial hypothesis holds, and $\{|a_n|\}$ is bounded by M. However, for $M \in \mathbb{R}$, $\exists N \in \mathbb{Z}_{>0} \mid n \geq N \implies |a_n| > M$. $\Rightarrow\Leftarrow$, so $\{|a_n|\}$ is not bounded, and thus diverges. Further, since $\{|a_n|\}$ diverges then $\{a_n\}$ must also diverge as shown on an earlier homework. □

b

Claim. $|x| > 1 \implies \{x^n\}$ is divergent.

Proof. We will show that for all $c \in \mathbb{R}$, $\exists N \in \mathbb{Z}_{>0} \mid \forall n \in \mathbb{Z}_{>0}, n \geq N \implies |a_n| > c$.
First, we have that $|x| > 1 \implies |x^{n+1}| = |x^n \cdot x| = |x^n||x| > |x^n|$, so $a > b \implies |x^a| > |x^b|$.
If $c \leq 1$, we have that $|x^1| = |x| > c$, so $n > 1 \implies |x^n| > c$.
For $c \geq 1$, consider that $x^y = e^{y \log(x)}$. We know that that $y \log(x) : \mathbb{R} \to \mathbb{R}$, where $\log(x)$ is a constant, is surjective and monotonically increasing, and also that $e^y : \mathbb{R} \to \mathbb{R}_{>0}$ is also surjective and monotonically increasing, so $x^y : \mathbb{R} \to \mathbb{R}_{>0}$ must be surjective and monotonically increasing, and so $\exists y' \mid x^{y'} = c$. Then, let $N = \lceil y' \rceil$. We have that $n > N \implies x^n > x^N \geq x^{y'} = c$, as x^y is monotonically increasing. Thus, $\{x^n\}$ must diverge. ☐

c

Claim. $|x| > 1 \implies \sum_{n=0}^{\infty} x^n$ is divergent.

Proof. Suppose $\sum_{n=0}^{\infty} x^n$ converges. Then, we must have that $\lim_{n \to \infty} x^n$ converges to 0. However, $\lim_{n \to \infty} x^n$ diverges, so the sum must diverge. ☐