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Homework 4

MATH 1208
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Apostol p.80 no.2

Put $A = \begin{bmatrix} x & y & z \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$.

a

If $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, then this asks to compute $\det(\begin{bmatrix} 2a_1 \\ \frac{1}{2}a_2 \\ a_3 \end{bmatrix}) = 2 \cdot \frac{1}{2} \cdot \det(A) = \det(A) = 1$.

b

$$\det(\begin{bmatrix} a_1 \\ 3a_1 + a_2 \\ a_1 + a_3 \end{bmatrix}) = \det(A) = 1$$

c

$$\det(\begin{bmatrix} a_1 - a_3 \\ a_2 + a_3 \\ a_3 \end{bmatrix}) = \det(A) = 1$$

Apostol p.94 no.3

a

$$\begin{aligned} \det\left(\begin{bmatrix} \lambda & -3 \\ -2 & \lambda + 1 \end{bmatrix}\right) &= \lambda^2 + \lambda - 6 \\ &= (\lambda + 3)(\lambda - 2) \end{aligned}$$

The matrix is singular for $\lambda = -3, 2$

b

$$\begin{aligned} \det\left(\begin{bmatrix} \lambda - 1 & 0 & -2 \\ 0 & \lambda + 1 & 2 \\ -2 & 2 & \lambda \end{bmatrix}\right) &= (\lambda - 1)(\lambda(\lambda + 1) - 4) - 2(2\lambda + 2) \\ &= \lambda^3 - 9\lambda \\ &= \lambda(\lambda + 3)(\lambda - 3) \end{aligned}$$

The matrix is singular for $\lambda = 0, \pm 3$

c

$$\det\left(\begin{bmatrix}\lambda-11 & 2 & -8\\ -19 & \lambda+3 & -14\\ 8 & -2 & \lambda+5\end{bmatrix}\right) = (\lambda-11)((\lambda+3)(\lambda+5)-28)$$
$$-2(-19(\lambda+5)+112)-8(-38-8(\lambda+3))$$
$$=\lambda^3-3\lambda^2+\lambda-3$$
$$=(\lambda-3)(\lambda^2+1)$$

The matrix is singular for $\lambda = 3$. If we take matrix over the complex numbers, then it is also singular for $\lambda = \pm i$.

Apostol p.101 no.1

a

Claim. T has eigenvalue $\lambda \implies aT$ has eigenvalue $a\lambda$.

Proof. T has eigenvalue $\lambda \iff \det(T-\lambda I)=0$. Using the properties of the determinant, where $n = \dim(T)$,

$$0 = \det(T-\lambda I) = a^n \det(T-\lambda I) = \det(a(T-\lambda I)) = \det(aT - a\lambda I)$$

□

b

Claim. If x is an eigenvector for both T_1, T_2, then x is a eigenvector for $aT_1 + bT_2$.

Proof. Consider that x is an eigenvector for $T_1, T_2 \implies T_1x = \lambda_1 x, T_2 x = \lambda_2 x$. Then, $(aT_1+bT_2)x = (aT_1)x + (bT_2)x = a(T_1x) + b(T_2x) = a\lambda_1 x + b\lambda_2 x = (a\lambda_1 + b\lambda_2)x$.
Thus, we have that $a\lambda_1 + b\lambda_2$ is an eigenvalue for aT_1+bT_2, with eigenvector x. □

Apostol p.101 no.2

Claim. $T : V \to V$ has an eigenvector x belonging to eigenvalue λ. Then, $P(T)$ has the same eigenvector belonging to eigenvalue λ.

Proof. First show that T^n has eigenvector x with eigenvalue λ^n with induction.
The base case of $n=1$ is trivial. Then, assume that the above holds for $n=k$.

$$T^{k+1}x = (TT^k)x = T(T^kx) = T(\lambda^k x) = \lambda^k(Tx) = \lambda^k(\lambda x) = \lambda^{k+1}x$$

Then, let $P(z) = \sum_{i=0}^{n} a_i z^i$. Induct on n.

Note that the case $n = 0$ has $P(T) = a_0 T^0 = a_0 I_n$, which then has eigenvalue $a_0 = \sum_{i=0}^{0} a_i \lambda^i$ for any vector x, as I_n has eigenvalue 1 for any vector.

Now suppose that the above claim holds for $n = k$. Then,

$$\sum_{i=1}^{k+1} a_i T^i = \sum_{i+1}^{k} a_i T^i + a_{k+1} T^{k+1}$$

From the inductive hypothesis we have that $\sum_{i=1}^{k} a_i T^i$ has eigenvector x belonging to eigenvalue $\sum_{i=1}^{k} a_i \lambda^i$, and from the earlier problem and claim $a_{k+1} T^{k+1}$ has eigenvalue $a_{k+1}\lambda^{k+1}$ with eigenvector x.

Using the second half of the last problem, we have that $\sum_{i=1}^{k+1} a_i T^i$ has eigenvector x and eigenvalue $\sum_{i=1}^{k} a_i \lambda^i + a_{k+1}\lambda^{k+1} = \sum_{i=1}^{k+1} a_i \lambda^i$. □

Apostol p.101 no.4

Claim. If $T : V \to V$ has that T^2 has eigenvalue λ^2, at least one of λ and $-\lambda$ is an eigenvalue for T.

Proof.

$$\begin{aligned} \det(T^2 - \lambda^2 I) &= \det((T - \lambda I)(T + \lambda I)) \\ &= \det(T - \lambda I)\det(T + \lambda I) \end{aligned}$$

However, we know that $ab = 0 \iff a = 0$ or $b = 0$ in a field, so one of $\det(T-\lambda I), \det(T+\lambda I)$ must be 0 and thus T must have at least one of $\pm\lambda$ as an eigenvalue. □

Apostol p.101 no.6

Claim. If V is the vector space of all real polynomials of degree $\leq n$, and $q = T(p) \iff q(t) = p(t+1)$ for all real t, then T has only the eigenvalue 1.

Proof. Let T have eigenvector p belonging to eigenvalue λ. Then, we have that $T(p) = \lambda p \implies p(t+1) = \lambda p(t)$ for all $t \in \mathbb{R}$, where p is a nonzero polynomial of degree n.

More specifically, we have $\sum_{i=0}^{n} a_i (t+1)^i = \lambda \sum_{i=0}^{n} a_i t^i$, where $a_n \neq 0$. Taking a look at the highest degree term, $a_i t^i = \lambda a_i t^i \implies \lambda = 1$.

We have that the corresponding eigenspace includes the constant functions, as $p(x) = c = p(x+1)$.

To prove that these are all possible eigenvectors, note that $p(r) = 0$ for some $r \in \mathbb{R}$ implies that $p(r \pm n) = 0$, where $n \in \mathbb{N}$. This generates an infinite amount of zeros, and so p must not have any zeros (or be the zero polynomial).

Furthermore, $p'(t) = p(t) - y$ for any $y \in \mathbb{R}$ must also have no zeros; consider that $p'(t+1) = p(t+1) - y = p(t) - y = p'(t)$, and so $p'(t)$ is also an eigenvector and must have no zeros (or be the zero polynomial).

Take $y = -p(x)$ for some arbitrarily selected $x \in \mathbb{R}$, and so $p'(x)$ has a zero, and thus $p'(x) = 0$, in which case $p(t) = y$, and it is constant.

Thus, the only eigenvectors are the constant functions (without the zero polynomial).

□

Apostol p.108 no.11

Claim. If $A, B \in M_{n\times n}$, with A nonsingular, then AB, BA have the same set of eigenvalues.

Proof. It is shown in Apostol that similar matrices have the same eigenvalues. Then,

$$BA = (A^{-1}A)BA = A^{-1}(AB)A$$

Since AB, BA are similar, they have the same eigenvalues. □

Apostol p.113 no.7

a

Claim. A square matrix A is nonsingular $\iff$ 0 is not an eigenvalue of A

Proof. ($\implies$) This is the same as proving the contrapositive that if 0 is an eigenvalue, A is singular.

Then, if 0 is an eigenvalue, for some $x \neq 0$, $Ax = 0x = 0$. This would mean that $\ker(A) \neq \{0\}$, and thus A is singular.

($\impliedby$) 0 is not an eigenvalue of $A \iff \det(A - 0I) \neq 0$. Then $\det(A) \neq 0$, so A is nonsingular. □

b

Claim. If A is nonsingular, then the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A.

Proof. Let $A \in M_{n\times n}$ have eigenvalue λ. Then,

$$\begin{aligned}
&\det(A-\lambda I) = 0\\
\implies\quad &(\frac{1}{\lambda})^n \det(A-\lambda I) = (\frac{1}{\lambda})^n 0\\
\implies\quad &\det(\frac{1}{\lambda}A - I) = 0\\
\implies\quad &\det(A^{-1})\det(\frac{1}{\lambda}A - I) = \det(A^{-1})0\\
\implies\quad &\det(\frac{1}{\lambda}I - A^{-1}) = 0\\
\implies\quad &\det(A^{-1} - \frac{1}{\lambda}I) = (-1)^n 0 = 0
\end{aligned}$$

Thus, A^{-1} has eigenvalue $\frac{1}{\lambda}$.
Note that this shows that A has eigenvalue $\lambda \implies A^{-1}$ has eigenvalue $\frac{1}{\lambda}$, and substituting $A = A^{-1}, \lambda = \frac{1}{\lambda}$ at the beginning shows that A^{-1} has eigenvalue $\frac{1}{\lambda} \implies A$ has eigenvalue λ. $\square$

Apostol p.113 no.8

We have $A^2 = -I$.

a

Claim. A is nonsingular.

Proof. Consider $-A$. $A(-A) = -(AA) = -(-I) = I$, and $(-A)A = -(AA) = I$. Thus, $A^{-1} = -A$. $\square$

b

Claim. $\dim(A) = n$ is even.

Proof.

$$\begin{aligned}
A^2 &= -I\\
\det(A^2) &= \det(-I)\\
\det(A)^2 &= (-1)^n \det(I)\\
\det(A)^2 &= (-1)^n
\end{aligned}$$

Since we have that $\det(A) \in \mathbb{R}$, $(-1)^n \geq 0$, so n must be even. $\square$

c

Claim. A has no real eigenvalues.

Proof. Suppose that $Ax = \lambda x$, i.e. A has some real eigenvalue λ with eigenvector x. Then, $-x = -Ix = A^2x = \lambda^2 x$, which would means that $\lambda^2 = -1$, which has no real solutions. $\Rightarrow\!\Leftarrow$, A has no real eigenvalues. $\square$

d

Proof. Consider $\det(t\lambda - A)$. We have that this, the characteristic polynomial of A, has roots exactly the complex eigenvalues of A. In particular,

$$\det(t\lambda - A) = \prod_{i=1}^{n}(t - \lambda_i)$$

where λ_i are the eigenvalues of A. We know that the characteristic polynomial splits completely over $\mathbb{C}$.

The sign of the leading coefficient can be seen by the permutation formula; the only way to get t^n from $\prod_{i=1}^{n} a_{i\sigma(i)}\mathrm{sgn}(\sigma)$ is to have the identity permutation, thus taking the product over the diagonal. Then, $\mathrm{sgn}(\mathrm{Id}) = 1$, so the leading coefficient is positive.

Taking $t = 0$,

$$det(-A) = (-1)^n \det(A) = \det(A) = \prod_{i=1}^{n}(-1)^n(\lambda_i) = \prod_{i=1}^{n}\lambda_i$$

However, we have that since $\lambda^2 = -1$ from above, and that the coefficients of the characteristic polynomial must be real, $\lambda_i = \pm i$ and come in conjugate pairs of $i, \pm i$. Thus, $\prod_{i=1}^{n}\lambda_i = \prod_{i=1}^{n/2}(i \cdot -i) = 1$. $\square$

Problem 1

Claim. Let A, B be, respectively, column and row vectors of dimension n.

$$\det(AB) = 0$$

Proof. We have that the determinant is linear on each row; further, if any of the rows of its input are identical, then the determinant is 0.

Compute:

$$(AB)_{ij} = \sum_{k=1}^{1} A_{ik}B_{kj} = A_{i1}B_{1j}$$

However, note that the i^{th} row is

$$\begin{bmatrix} A_{i1}B_{11} & A_{i1}B_{12} & \dots & A_{i1}B_{1n} \end{bmatrix} = A_{i1} \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1n} \end{bmatrix}$$

This means that $\det(AB) = \prod_{i=1}^{n} A_{i1} \det(B')$ where

$$B' = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1n} \\ B_{11} & B_{12} & \dots & B_{1n} \\ \vdots & & & \vdots \\ B_{11} & B_{12} & \dots & B_{1n} \end{bmatrix}$$

However, since each row of B' are identical, then $\det(B') = 0 \implies \det(AB) = 0$. $\square$

Problem 3

For each, we need to find the eigenvectors and eigenvalues of the given matrix.

a

$$\begin{aligned} \det(A - \lambda I_n) &= \det\left(\begin{bmatrix} 20-\lambda & -9 \\ 30 & -13-\lambda \end{bmatrix}\right) \\ &= (20-\lambda)(-13-\lambda) + 270 \\ &= \lambda^2 - 7\lambda + 10 \\ &= (\lambda - 5)(\lambda - 2) \end{aligned}$$

Since we want the determinant to be 0,

$$\lambda = 2, 5$$

Finding the eigenvectors, we have that

Take $\lambda = 2$

$$\begin{bmatrix} 18 & -9 \\ 30 & -15 \end{bmatrix} \to \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

Then, this has eigenvector $(1, 2)$. Now take $\lambda = 5$

$$\begin{bmatrix} 15 & -9 \\ 30 & -18 \end{bmatrix} \to \begin{bmatrix} 5 & -3 \\ 0 & 0 \end{bmatrix}$$

Then, this has eigenvector $(3, 5)$.

Inverting the column matrix $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, we arrive at $\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$

Then, we have that

$$\begin{bmatrix} 20 & -9 \\ 30 & -13 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

b

This is not diagonalizable.

$$\begin{aligned} \det(A - \lambda I_n) &= \det\left(\begin{bmatrix} 8-\lambda & 4 \\ -9 & -4-\lambda \end{bmatrix}\right) \\ &= (8-\lambda)(-4-\lambda) + 36 \\ &= \lambda^2 - 4\lambda + 4 \\ &= (\lambda - 2)^2 \end{aligned}$$

Since we want the determinant to be 0,

$$\lambda = 2$$

However, we have that with $\lambda = 2$,

$$\begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix} \to \begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}$$

The eigenspace is spanned by $(2, -3)$, and since if we want a matrix to be diagonalizable the geometric multiplicity must be the algebraic multiplicity, the matrix is not diagonalizable.

c

$$\begin{aligned} \det(A - \lambda I_n) &= \det\left(\begin{bmatrix} -1-\lambda & 4 & 4 \\ 0 & -5-\lambda & -4 \\ 0 & 8 & 7-\lambda \end{bmatrix}\right) \\ &= (-1-\lambda)((-5-\lambda)(7-\lambda) + 32) \\ &= (-1-\lambda)(\lambda^2 - 2\lambda - 3) \\ &= (-1-\lambda)(\lambda - 3)(\lambda + 1) \end{aligned}$$

Since we want the determinant to be 0,

$$\lambda = -1, 3$$

Finding the eigenvectors, we have that

Take $\lambda = -1$

$$\begin{bmatrix} 0 & 4 & 4 \\ 0 & -4 & -4 \\ 0 & 8 & 8 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then, this has eigenspace spanned by $(1, 0, 0), (0, -1, 1)$. Now take $\lambda = 3$

$$\begin{bmatrix} -4 & 4 & 4 \\ 0 & -8 & -4 \\ 0 & 8 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Then, this has eigenspace spanned by $(1, -1, 2)$.

Inverting the column vector matrix,

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Then, we have that

$$\begin{bmatrix} -1 & 4 & 4 \\ 0 & -5 & -4 \\ 0 & 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

Problem 4

Claim. If A is an upper triangular matrix, then the eigenvalues of A are exactly its diagonal entries.

Proof. We first show that the determinant of a diagonal matrix is the product of its diagonal entries.

Let the dimension of A be n, and let the elements be a_{ij} and the corresponding matrix with row i and column j removed be A_{ij}. Induct on n.

If $n = 1$, then $A = [a_{11}] \implies \det(A) = a_{11}$, which holds. Now assume the hypothesis for $n = k$, such that $A' \in M_{k\times k}, A \in M_{(k+1)\times(k+1)}$ where $A_k = A_{(k+1)(k+1)}$:

$$\det(A') = \prod_{i=1}^{k} a_{ii}$$

Then, from class, the cofactor formula for the determinant has that

$$\det(A) = \sum_{j=1}^{k+1} a_{(k+1)j} \det(A_{(k+1)j})$$

Since we have that A', A are upper triangular, $a_{(k+1)j} = 0$ for $j < k + 1$. Then,

$$\det(A) = a_{(k+1)(k+1)} \det(A_{(k+1)(k+1)}) = a_{(k+1)(k+1)} \det(A') = \prod_{i=1}^{k+1} a_{ii}$$

Now, we have that $A - \lambda I$ is still diagonal, as any element $(A - \lambda I)_{ij}$ for $i > j$ is still simply $0 + \lambda(0) = 0$.

Then,

$$\det(A - \lambda I) = \prod_{i=1}^{n} (a_{ii} - \lambda)$$

The above product is zero $\iff \lambda = a_{ii}$ for some $i \in [1, n]$. Thus, the eigenvalues are exactly all of the a_{ii}. $\square$