dc3451
David Chen

Homework 1

Apostol p.14 no.24a

Suppose that $\dim(S) > \dim(V)$, and put $\dim(V) = v$. We have from class that this would imply that S contains a set of $v + 1$ independent vectors, by the definition of dimension. However, since $x \in S \implies x \in V$, we have that V contains $v + 1$ independent vectors. $\Rightarrow\!\Leftarrow$, so $\dim(S) \leq \dim(V)$. In particular, since V is finite dimensional, S also must be finite dimensional.

Apostol p.14 no.24b

Claim. $\dim(S) = \dim(V) \iff S = V$

Proof. ($\implies$) Since we have that $S \subseteq V$, let $\{s_1, s_2, ..., s_n\}$ be a basis for S. We can extend this into a basis for V by a proposition proved in class; fortunately, we have to extend this by zero vectors as $\dim(S) = \dim(V)$, so V is also spanned by $\{s_1, s_2, ..., s_n\}$.

This means that any $v \in V, v = \sum_{i=1}^{n} a_i s_i$ is also in S. Since $S \subseteq V, V \subseteq S$, $V = S$.

($\impliedby$) $\dim(V) = \dim(S)$ as $V = S$. □

Apostol p.14 no.24c

By a proposition from class, we have that if we have n independent vectors, where $n \leq \dim(V)$, we can extend this into a basis for V.

Then, the $\dim(S) \leq \dim(V)$ independent vectors that form a basis for S can be extended into a basis for V.

Apostol p.14 no.24d

Take the following counterexample to the claim that "A basis for V contains a basis for S": let $V = \mathbb{R}^2$, $S = \{(0, x) \mid x \in \mathbb{R}\}$. V has as a basis $\{(1, 1), (-1, 1)\}$, but neither $(1, 1)$ nor $(-1, 1)$ fall in S.

Apostol p.35 no.24

$$\begin{aligned}
T(p(x)+q(x)) &= T((p+q)(x)) \\
&= (p+q)(x+1) \\
&= p(x+1)+q(x+1) \\
&= T(p(x))+T(q(x)) \\
T(cp(x)) &= cp(x+1) \\
&= c(p(x+1)) = cT(p(x))
\end{aligned}$$

T is linear. $\ker T = \{0\}$ as well; $T(p(x)) = 0 \implies p(x+1) = 0 \implies \sum_{i=1}^{n} p_i(x+1)^i = 0 \implies p_i = 0 \implies p(x) = 0$.

Thus, the nullity is 0 and the rank is $\dim V = n$.

Problem 1

a

Since we have that $\{v_1, v_2, ..., v_k\} \subseteq \{v_1, ..., v_n\}$ for $n \geq k$, if $v \in V$ has $v = \sum_{i=1}^{k} a_i v_i$, then that same combination is a linear combination of $\{v_1, ..., v_n\}$ that equals v (take $a_{k+1}, ..., a_n = 0$). Since this is not required to be unique, nor with all nonzero coefficients, we are done.

b

Suppose that $\{v_1, ..., v_k\}$ is linearly dependent, such that $\sum_{i=1}^{k} a_i v_i = 0$. Then, since $n \geq k$, we would have that $\{v_1, ..., v_n\}$ is linearly dependent as a linear combination of vectors here sum to zero. $\Rightarrow\Leftarrow$, so $\{v_1, ..., v_k\}$ is linearly independent.

Problem 2

a

We will first show that $Id_V - T$ is linear.

$$\begin{aligned}
(Id_v - T)(cx) &= Id_V(cx) - T(cx) \\
&= cx - cT(x) \\
&= c(Id_V(x)) - cT(x) \\
&= c(Id_V(x) - T(x)) \\
&= c((Id_v - T)(x)) \\
(Id_v - T)(x+y) &= Id_V(x+y) - T(x+y) \\
&= x + y - T(x) - T(y) \\
&= Id_V(x) - T(x) + Id_V(y) - T(y) \\
&= (Id_V - T)(x) + (Id_V - T)(y)
\end{aligned}$$

It has an inverse, namely $Id_v + T + T^2$:

$$\begin{aligned}
(Id_v + T + T^2)((Id_v - T)(x)) &= Id_v(x - T(x)) + T(x - T(x)) + T(T(x - T(x))) \\
&= x - T(x) + T(x) - T(T(x)) + T(T(x)) - T(T(T(x))) \\
&= x
\end{aligned}$$

Via the theorem proved in class, we have that $Id_V - T$ is an isomorphism.

b

If $T^n = 0$ for $n \in \mathbb{Z}_{>0}$ then $T_0 - T$ is still an isomorphism with inverse $\sum_{i=0}^{n-1} T^i$. (Note that if $n = 0$, we have that $Id_V = 0 \implies$ the vector space is trivial, and this still holds trivially with inverse also Id_V)

$$\begin{aligned}(\sum_{i=0}^{n-1} T^i)((T^0 - T)(x)) &= (\sum_{i=0}^{n-1} T^i)(T^0(x) - T(x)) \\ &= \sum_{i=0}^{n-1} T^i(T^0(x) - T(x)) \\ &= \sum_{i=0}^{n-1} T^i(T^0(x)) - \sum_{i=0}^{n-1} T^i(T(x)) \\ &= \sum_{i=0}^{n-1} T^i(x) - \sum_{i=0}^{n-1} T^{i+1}(x) \\ &= \sum_{i=0}^{n-1} T^i(x) - \sum_{i=1}^{n} T^i(x) \\ &= T^0(x) - T^n(x) = Id_V\end{aligned}$$

Problem 3

Claim. $\{\sin(x), \sin(2x), ..., \sin(2^n x), ...\}$ is linearly independent.

Proof. Suppose that we have some linear combination $\sum_{i=0}^n a_i sin(2^i x) = 0$. Consider $x = \frac{\pi}{2^{k+1}}$, where k is the least integer such that $a_k \neq 0$.

Then, we have that $\sin(\frac{2^i \pi}{2^{k+1}}) = \sin(2^{i-k-1}\pi) = 0$ for any $i > k$; for any $i < k$, we have that $a_i = 0$; for $i = k$, we have that $\sin(\frac{2^k \pi}{2^{k+1}}) = \sin(\frac{\pi}{2}) = 1$. Thus, no nontrivial combination sums to 0. □

Problem 4

Claim. $\{1, 1+x, 1+x+x^2, ..., 1+x+x^2+...+x^n, ...\}$ is linearly independent.

Proof. We will show that $\sum_{i=0}^n a_i \sum_{j=0}^i x^j = \sum_{i=0}^n (x^i \sum_{j=i}^n a_j)$ through induction on n. The base case, which has $n = 0$, follows immediately as $\sum_{i=0}^0 (a_i \sum_{j=0}^i x^j) = a_0$.

Now assume the above hypothesis for $n = k$. Then,

$$\begin{aligned}
\sum_{i=0}^{k+1}(a_i \sum_{j=0}^{i} x^j) &= \sum_{i=0}^{k}(a_i \sum_{j=0}^{i} x^j) + a_{k+1}\sum_{j=0}^{k+1} x^j \\
&= \sum_{i=0}^{k}(x^i \sum_{j=i}^{k} a_j) + a_{k+1}\sum_{j=0}^{k+1} x^j \\
&= \sum_{i=0}^{k}(x^i \sum_{j=i}^{k+1} a_j) + a_{k+1} \\
&= \sum_{i=0}^{k+1}(x^i \sum_{j=i}^{k+1} a_j)
\end{aligned}$$

Since we have from earlier that a polynomial $\sum_{i=0}^{n}(x^i \sum_{j=i}^{n} a_j)$ is zero everywhere if and only if all of its coefficients are zero, we have that all of $\sum_{j=i}^{k+1} a_j$ must be zero. Since i ranges from 0 to $k+1$ inclusive, we can show that these are all 0 if and only if all $a_j = 0$.

Taking $i = k+1$, we have that $a_{k+1} = 0$. If $a_{k+1}, a_k, ..., a_l = 0$, we can induct backwards on l until $l = 0$. Taking $i = l - 1$ shows that $a_{l-1} = 0$.

Thus, the only linear combination of the original set that vanishes is the trivial one. □

Problem 5

a

Let the base be also be the same as V, W.

Commutativity:

$$(v, w) + (v', w') = (v + v', w + w') = (v' + v, w' + w) = (v', w') + (v, w)$$

Associativity:

$$\begin{aligned}
(v, w) + ((v', w') + (v'', w'')) &= (v, w) + (v' + v'', w' + w'') \\
&= (v + v' + v'', w + w' + w'') \\
&= (v + v', w + w') + (v'', w'') \\
&= ((v, w) + (v', w')) + (v'', w'')
\end{aligned}$$

$$(cd)(v,w) = ((cd)v,(cd)w) = (c(dv),c(dw)) = c(dv,dw) = c(d(v,w))$$

Distributivity:

$$\begin{aligned} c((v,w)+(v',w')) &= c(v+v',w+w') \\ &= (c(v+v'),c(w+w')) \\ &= (cv+cv',cw+cw') \\ &= (cv,cw)+(cv',cw') = c(v,w)+c(v',w') \end{aligned}$$

$$\begin{aligned} (c+d)(v,w) &= ((c+d)v,(c+d)w) \\ &= (cv+dv,cw+dw) \\ &= (cv,cw)+(dv,dw) \\ &= c(v,w)+d(v,w) \end{aligned}$$

Identity:

$$(v,w)+(0_V,0_W) = (v+0_V,w+0_W) = (v,w)$$

$$1(v,w) = (1v,1w) = (v,w)$$

Inverse:

$$(v,w)+(-v,-w) = (v-v,w-w) = (0,0)$$

Closure:

Since $(v,w)+(v',w') = (v+v',w+w')$ and $v+v' \in V, w+w' \in W$, we have that $(v,w)+(v',w') \in V \oplus W$.

Since $c(v,w) = (cv,cw)$ and $cv \in V, cw \in W$, we have that $c(v,w) \in V \oplus W$.

b

Claim.

$$\dim(V \oplus W) = \dim(V) + \dim(W)$$

This is actually a special case of Problem 7, part d, where $V \cap W = \{0\}$. The above follows.

This isn't obvious by the given definition of the direct product, so here is a more direct proof: consider the set $\{(v_1, 0), (v_2, 0), ..., (v_m, 0), (0, w_1), (0, w_2), ..., (0, w_n)\}$, where $\{v_1, v_2, ..., v_m\}$ and $\{w_1, w_2, ..., w_n\}$ are bases for V and W respectively.

Any $(v, w) \in V \oplus W$ has:

$$\begin{aligned}(v, w) &= (\sum_{i=1}^{m} a_i v_i, \sum_{i=1}^{n} b_i w_i) \\ &= \sum_{i=1}^{n} (a_i v_i, 0) + \sum_{i=1}^{n} (0, b_i w_i) \\ &= \sum_{i=1}^{n} a_i (v_i, 0) + \sum_{i=1}^{n} b_i (0, w_i)\end{aligned}$$

which yields a basis of size $\dim(V) + \dim(W)$ for $V \oplus W$.

Problem 6

a

Suppose that

$$\sum_{i=1}^{n} a_i f_{s_i} = 0$$

where the s_i are a finite collection of n distinct elements of S. For any k where $1 \leq k \leq n$, we have that $0 = (\sum_{i=1}^{n} a_i f_{s_i})(s_k) = a_k f_{s_k}(s_k) = a_k$. Thus, only the trivial solution exists to $\sum_{i=1}^{n} a_i f_{s_i} = 0$.

Suppose that $\mathcal{F}(\mathbb{Z}_{>0}, \mathbb{R}), \mathcal{F}(\mathbb{R}, \mathbb{R})$ are n-dimensional. Then, we can take $n + 1$ independent vectors in the vector space by just considering $C = \{f_s \mid s \in [n + 1]\}$, where $[n + 1] = 1, 2, ..., n + 1$. $\Rightarrow\Leftarrow$, since all bases are the same size and contain the maximum amount of linearly independent vectors, so they are not finite dimensional.

b

Consider $f : S \to F$ such that $f(s) = 1$. Then, take any finite linear combination $\sum_{i=1}^{n} a_i f_{s_i}$ from C, and take an element from S, s, such that $s \neq s_k$ for any k that has $1 \leq k \leq n$, which is always possible since S is infinite. Then, $(\sum_{i=1}^{n} a_i f_{s_i})(s) = \sum_{i=1}^{n} a_i f_{s_i}(s) = \sum_{i=1}^{n} 0 = 0$. Thus, C does not span $\mathcal{F}(S, F)$, and is therefore not a basis.

Problem 7

a

Let set $\{v_1, v_2, ..., v_k\}$ be a basis for $U \cap V$, where $u_1 = v_1, u_2 = v_2, ..., u_k = v_k$. This set is then linearly independent, and therefore can then be extended to bases for U and V via a theorem proved and used in class.

Further, we have that for $i > k$, $v_i \neq u_i$, as these by the invoked thereom are linearly independent of $\{v_1, ..., v_k\}$ and being indentical would form a basis of size $k+1$ for $U \cap V$, which would violate another theorem used in class.

b

The span of $U \cup V$ can be written as $\{u + v \mid u \in U, v \in V\}$. Then, we have that any such vector in that span is expressed

$$\begin{aligned} u + v &= \sum_{i=1}^{m} a_i u_i + \sum_{i=1} b_i v^n \\ &= \sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{k} b_i v_i + \sum_{i=k+1}^{n} b_i v_i \\ &= \sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{k} b_i u_i + \sum_{i=k+1}^{n} b_i v_i \end{aligned}$$

which is a finite linear combination of $u_1, ..., u_m, v_{k+1}, ..., v_n$.

Further, this must be linearly independent, as we have that $v_{k+1}, ..., v_n$ are not in the span of $u_1, ..., u_m$ as well as that $u_1, ..., u_m$ and $v_{k+1}, ..., v_n$ are all linearly independent within those collections as they are bases for vector spaces by assumption.

c

We have an explicit basis: $u_1, ..., u_m, v_{k+1}, ..., v_n$. Since all bases are the same size for any given vector space, there are $m+n-k$ elements in the basis and so $\dim(U+V) = m+n-k = \dim(U) + \dim(V) - \dim(U \cap V)$ by the definitions of m, n and k.

d

Consider $U = \{(x, y, 0) \mid x, y \in \mathbb{R}\}, V = \{(0, y, z) \mid y, z \in \mathbb{R}\}$. This has $U \cap V = \{(0, y, 0) \mid y \in \mathbb{R}\}$ such that $\dim(U) = \dim(V) = 2, \dim(U \cap V) = 1$, and $U + V = \mathbb{R}^3$. Thus,

$\dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap V)$.