Problem 1

1

We have the following maximization problem on households:

$$\max_{C_1, C_2}\{\log(C_1) + \log(C_2)\} \mid C_1 + \frac{C_2}{1+r} = Y_1 - T_1 + \frac{Y_2 - T_2}{1+r}$$

In particular, we have that $G_1 = 2, G_2 = 0, Y_1 = 1, T_1 + \frac{T_2}{1+r} = G_1 = 2$. Further, for the given technology we have the following optimality condition for investment:

$$\frac{5}{\sqrt{I_1}} = 1 + r \implies I_1 = \left(\frac{5}{1+r}\right)^2$$

Then,

$$\begin{aligned}
\mathcal{L}(C_1, C_2, \lambda) &= \log(C_1) + \log(C_2) - \lambda(C_1 + \frac{C_2}{1+r} - 8 - \frac{\Pi_2}{1+r}) \\
\frac{\partial \mathcal{L}}{\partial C_1} &= \frac{1}{C_1} - \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial C_1} &= \frac{1}{C_2} - \frac{\lambda}{1+r} = 0
\end{aligned}$$

The above two conditions yield that $(1+r)C_1 = C_2$. Then,

$$\frac{\partial \mathcal{L}}{\partial \lambda} = C_1 + \frac{C_2}{1+r} - 8 - \frac{\Pi_2}{1+r} = 0$$

Here, we have that $C_1 = 4 + \frac{\Pi_2}{2(1+r)}, C_2 = C_1(1+r)$. Furthermore, we have that private savings is

$$\begin{aligned}
S^p &= Y_1 - T_1 - C_1 \\
&= 5 - \frac{\Pi_2}{2(1+r)}
\end{aligned}$$

With government savings $T_1 - G_1 = -1$, we have total savings

$$S = 4 - \frac{\Pi_2}{2(1+r)}$$

Since in equilibrium we have that savings is identical to investment,

$$\left(\frac{5}{1+r}\right)^2 = 4 - \frac{25}{2(1+r)^2}$$

The equilibrium interest rate is then

$$r = 2.06 = 206\%$$

With corresponding investment and consumption

$$I_1 = 2.67, C_1 = 5.33$$

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We take the same process as before, and note that we now have $T_1 + \frac{T_2}{1+r} = G_1 = 2.2$, and that

$$S = 3.9 - \frac{\Pi_2}{2(1+r)}$$
$$\left(\frac{5}{1+r}\right)^2 = 3.9 - \frac{25}{2(1+r)^2}$$

and arrive at equilibrium values

$$r = 2.10 = 210\%, I_1 = 2.60, C = 5.2$$

We have essentially lower investment (as there are less savings due to government deficit spending), which creates a situation of higher equilibrium interest rates and also lower consumption in period one (in line with higher interest rates).

Problem 2

1

In period one, households face the following

$$Y_1 - T_1 = C_1 + S$$

In period two, they face the following

$$Y_2 - T_2 + S(1+r) = C_2$$

The intertemporal budget constraint is then

$$C_1 + \frac{C_2}{1+r} = Y_1 - T_1 + \frac{Y_2 - T_2}{1+r}$$

2

If firms borrow I_1 in period one, then

$$\frac{T_2}{1+r} = \tau^I(1+r)I_1 \implies T_2 = \tau^I(1+r)^2 I_1$$

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$$\max_{I_1}\{2\sqrt{I_1} - (1-\tau^I)(1+r)I_1\}$$

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The firm must have that

$$f'(I_1) = \frac{1}{\sqrt{I_1}} = (1-\tau^I)(1+r) \implies I_1 = \frac{1}{(1-\tau^I)^2(1+r)^2}$$

Note that this means that the optimal level of investment is a strictly decreasing function of r, but a strictly increasing function of τ^I for $0 < \tau^I < 1$, as

$$\frac{\partial I_1}{\partial \tau^I} = \frac{2}{(1+r)^2(1-\tau^I)^3} > 0$$

This makes sense: borrowing is more expensive at higher interest rates, but cheaper the larger the subsidy.

For $\tau \geq 1$, we have negative costs to borrowing: they do it infinitely in this case.

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If the subsidy is zero, then all taxes and spending vanish, such that equilibrium investment follows

$$I_1 = \frac{1}{(1+r)^2}, \Pi = \frac{1}{1+r}$$

and has savings

$$S = Y_1 - C_1 = Y_1 - \frac{1}{2}(Y_1 + \frac{\Pi}{1+r}) = \frac{1}{2} - \frac{1}{2(1+r)^2}$$

In equilibrium, we have that the above quantities are equal, such that

$$r = 0.732 = 73.2\%, I_1 = 0.33, C_1 = 0.67, C_2 = 1.15$$

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In the case of $\tau^I = 0.5$, we have that

$$I_1 = \frac{1}{(1-\tau^I)^2(1+r)^2}, \Pi = \frac{1}{(1-\tau^I)(1+r)}$$

For government spending, we have, when firms profit maximize

$$T_2 = \tau^I(1+r)^2 I_1 = \frac{\tau^I}{(1-\tau^I)^2}$$

Then, we have savings

$$S = Y_1 - G_1 - C_1 = 1 - \frac{\tau^I}{(1+r)(1-\tau^I)^2} - \frac{1}{2}(1 + \frac{\Pi - \frac{\tau^I}{(1-\tau^I)^2}}{1+r}) = \frac{1}{2} - \frac{1+\tau^I r}{2(1-\tau^I)(1+r)^2}$$