MATH 4041 HW 1

David Chen, dc3451

September 10, 2020

Problem 1

In class it was stated that $X \subseteq Y \iff \forall x \in X, x \in Y$.

i

Consider any $x_1 \in X_1$. Then, since $X_1 \cup X_2$ is defined to be $\{x \mid x \in X_1 \vee x \in X_2\}$, we have that $x_1 \in X_1 \cup X_2$. From the definition of $\subseteq$, we have that $X_1 \subseteq X_1 \cup X_2$.

Consider any $x_2 \in X_2$. Then, since $X_1 \cup X_2$ is defined to be $\{x \mid x \in X_1 \vee x \in X_2\}$, we have that $x_2 \in X_1 \cup X_2$. From the definition of $\subseteq$, we have that $X_2 \subseteq X_1 \cup X_2$.

To show that $X_1 \cup X_2$ is the smallest set containing X_1, X_2, consider any $x \in X_1 \cup X_2$. In that case, via the definition of the union of two sets, we have that $x \in X_1$ or $x \in X_2$ (or both). In either case, we have that since $X_1 \subseteq Y \implies \forall x_1 \in X_1, x_1 \in Y$ and $X_2 \subseteq Y \implies \forall x_2 \in X_2, x_2 \in Y$, that $x \in Y$. From the definition of $\subseteq$, we have that $(X_1 \cup X_2) \subseteq Y$.

ii

Consider any $x \in X_1 \cap X_2$. Then, since $X_1 \cap X_2$ is defined to be $\{x \mid x \in X_1 \wedge x \in X_2\}$, we have that $x \in X_1 \cap X_2 \implies x \in X_1$ and $x \in X_2$. From the definition of $\subseteq$, we have that $X_1 \cap X_2 \subseteq X_1, X_1 \cap X_2 \subseteq X_2$.

To show that $X_1 \cap X_2$ is the largest set contained in X_1, X_2, consider any $y \in Y$. Then, we have that $Y \subseteq X_1 \implies y \in X_1$ and $Y \subseteq X_2 \implies y \in X_2$. Since $X_1 \cap X_2$ is defined to be $\{x \mid x \in X_1 \wedge x \in X_2\}$, we have that $y \in X_1 \cap X_2$. From the definition of $\subseteq$, we have that $Y \subseteq (X_1 \cap X_2)$.

Problem 2

Let x_1, x_2 be any two distinct elements of X. Then, $(x_1, x_2) \notin \Delta_X$, (if it were in fact in the diagonal, we would have that $x_1 = x_2$, which contradicts the earlier assumption).

However, we have that $(x_1, x_1), (x_2, x_2) \in \Delta_X$. Now, suppose that $\Delta_X = A \times B$ for some $A, B \subseteq X$. Then, since $A \times B$ is defined to be $\{(a, b) \mid a \in A, b \in B\}$, and $(x_1, x_1) \in A \times B$, we must have that $x_1 \in A, x_1 \in B$. Similarly, since $(x_2, x_2) \in A \times B$, we must have that $x_2 \in A, x_2 \in B$.

Then, again recalling that $A \times B = \{(a, b) \mid a \in A, b \in B\}$, we have that $(x_1, x_2) \in A \times B$. However, we showed earlier that $(x_1, x_2) \notin \Delta_X$, so $\Rightarrow\Leftarrow$, and $\Delta_X \neq A \times B$ for any $A, B \subseteq X$.

Problem 3

i

$g(x) = e^x$ is injective, as the inverse function $g^{-1}(x) = \ln(x)$ shows. It is not surjective, as $e^x > 0$ on the real line. Since it is not surjective, it is not a bijection.

The image of g is the positive reals $\mathbb{R}^+$.

ii

$g(x) = 5x - 12$ is a bijection: for injectivity, we see that if we have x, y such that $g(x) = g(y)$, $5x - 12 = 5y - 12 \implies 5x = 5y \implies x = y$.

For surjectivity, we have that $f^{-1}(x) = \frac{x+12}{5}$ gives a preimage to any given $x \in \mathbb{R}$.

The image of g is the entire real line $\mathbb{R}$.

iii

$g(x) = x^3$ is also a bijection: for injectivity, we see that if we have x, y such that $g(x) = g(y)$, $x^3 = y^3 \implies x = y$. For surjectivity, we have that $g^{-1}(x) = x^{\frac{1}{3}}$ furnishes a preimage for any given input x. The image of g is the entire real line $\mathbb{R}$.

iv

$g(x) = x^3 - 3x$ is not a bijection: it is surjective, but not injective. To see that it is surjective, consider for any $y \in \mathbb{R}$, $g(|y| + 2) = |y|^3 + 2|y|^2 + |y| + 8 \geq |y|$ and $g(-|y| - 2) = -|y|^3 - 2|y|^2 - |y| - 8 \leq -|y|$. Intermediate value theorem gives some preimage of y between those $|y| + 2$ and $-|y| - 2$, since $-|y| \leq y \leq |y|$.

To see that it is not injective, note that $g(-\sqrt{3}) = g(0) = g(\sqrt{3}) = 0$.
The image of g is the entire real line $\mathbb{R}$.

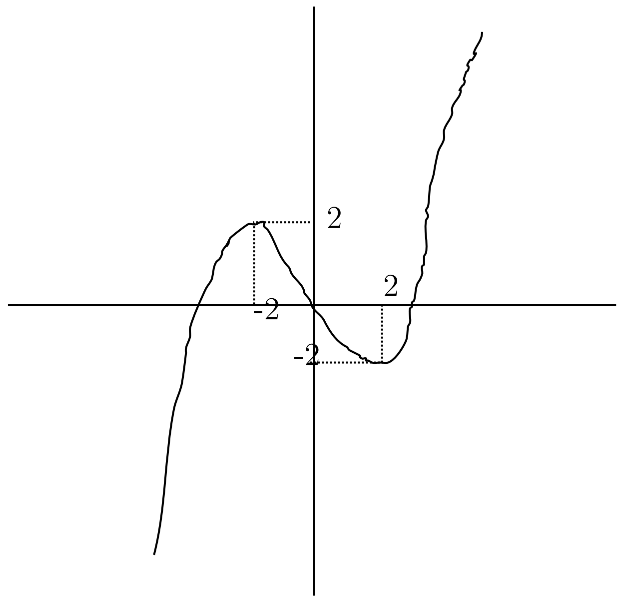

Figure 1: $g(x) = x^3 - 3x$

Problem 4

f is surjective, but not injective. Note that for any n, $n+1$ is a preimage of n, but $f(1) = f(31) = 30$.

a

$$f(\{1, 2, 3, 4, 5\}) = \{30, 1, 2, 3, 4\}$$

b

$$f(\{1, 31\}) = \{30\}$$

c

$$f^{-1}(1) = \{2\}$$

d

$$f^{-1}(\{1,2,3\}) = \{2,3,4\}$$

e

$$f^{-1}(30) = \{1,31\}$$

f

$$f^{-1}(\{1,30\}) = \{1,2,31\}$$

Problem 5

If X is nonempty:

A constant function $f : X \to Y$ is surjective only if Y is a singleton set $\{c\}$ (otherwise, if Y has more than one element, then those other elements have no preimage).

f is injective only if X is a singleton set as well (otherwise, if X has more than one element, then for two distinct elements x_1, x_2, we have that $f(x_1) = f(x_2)$). Combining the two above conditions, we see that $f : X \to Y$ is bijective between two nonempty sets if X, Y both consist of a single element.

If $X = \emptyset$, then the empty function $f = \emptyset$ is constant irregardless of Y so long as Y is nonempty. Picking any arbitrary element of Y, we would have that $\forall x \in X, f(x) = c$.

Then, the empty function is always injective, as $\nexists x_1, x_2 \in \emptyset$ such that $f(x_1) = f(x_2)$, but never surjective (as long as Y is nonempty) as no element in Y can have a preimage in the empty set. Consequentially, the empty function is never a bijection from the empty set to any nonempty range.

Problem 6

Recall the definition of a function $f : X \to Y$ as a subset G of $X \times Y$ such that $\forall x \in X, \exists! y \in Y$ such that $(x, y) \in G$.

Now consider the empty function with graph $\emptyset$. Then, the above condition with quantifier $\forall x \in X$ holds vacuously (note that $\emptyset \times Z = Z \times \emptyset = \emptyset$ for any set Z), and as such the empty function is a function.

As mentioned earlier, the empty function is always injective, but only surjective if Y is also the empty set. Then, the empty function $f : \emptyset \to Y$ is bijective only if $Y = \emptyset$.

For a function $f : X \to \emptyset$, note that if $X = \emptyset$, we showed earlier that the empty function is such a function $\emptyset \to \emptyset$.

Suppose that we have some function $f : X \to \emptyset$ and that X is nonempty. Then we have that for any given $x \in X$, there must be some $(x, y) \in G_f \subset X \times \emptyset$. However, since $X \times \emptyset = \emptyset$, we have that $\Rightarrow\Leftarrow$, so no such function can exist.

Problem 7

If either X or Y is the empty set, we have that $X \times Y = \emptyset$. Then, from earlier, we showed that $\pi_1 : X \times Y \to X$ is well-defined only if $X = \emptyset$ (in which case it is a bijection), and that similarly, π_2 is well-defined only if $Y = \emptyset$ (in which case it is a bijection).

Now, if neither is empty:

π_1 is injective when Y is a singleton set; otherwise, let $y_1, y_2 \in Y$ be distinct. Then, $\pi_1(x, y_1) = \pi_1(x, y_2) = x$.

Similarly, π_2 is injective when X is a singleton set.

π_1 is always surjective, as x has the preimage $\{(x, y) \mid y \in Y\}$ which is nonempty for nonempty Y.

Similarly, π_2 is also always surjective.

Problem 8

Remember that $(x, y) \in X \times Y$ for *any* $x \in X, y \in Y$. In particular, note that this suggests that $(x, y) \in X \times Y \implies x \in X, y \in Y \implies (y, x) \in Y \times X$.

To show surjectivity, note that for any element in the range $(y, x) \in Y \times X$, we have that $(x, y) \in X \times Y$ satisfies $t(x, y) = (y, x)$.

For injectivity, we will need to show that if $t(x_1, y_1) = t(x_2, y_2)$ then $(x_1, y_1) = (x_2, y_2)$. Suppose that $t(x_1, y_1) = t(x_2, y_2)$. Then, we have that $(y_1, x_1) = (y_2, x_2)$. By the definition of ordered pairs we have that $y_1 = y_2$ and $x_1 = x_2$, which also gives that $(x_1, y_1) = (x_2, y_2)$, which is what we wanted. Hence, t is a bijection.

Problem 9

i

We have that $\chi_A^{-1}(0) = \{x \in X \mid \chi_A(x) = 0\}$. However, from the definition of χ_A, we have that $\chi_A(x) = 0 \iff x \notin A$. Then, we have that $\chi_A^{-1}(0) = \{x \in X \mid x \notin A\}$, which is exactly $X \setminus A$.

We have that $\chi_A^{-1}(1) = \{x \in X \mid \chi_A(x) = 1\}$. However, from the definition of χ_A, we have that $\chi_A(x) = 1 \iff x \in A$. Then, we have that $\chi_A^{-1}(1) = \{x \in X \mid x \in A\} = \{x \in A\}$, which is exactly A.

χ_A is a constant function when X is empty, or when $A = X$ or $A = \emptyset$. We showed earlier that $\chi_A : \emptyset \to \{0, 1\}$ is a constant function.

If X is nonempty and A is a proper nonempty subset of X, then let $a \in A$ and $x \in X \setminus A$. Then, $\chi_A(a) = 1, \chi_A(x) = 0$, and so χ_A is nonconstant.

Now we also verify that χ_A is constant for $A = X$ and $A = \emptyset$. In the first case, $a \in X \implies a \in A$, so $\chi_A(x) = 1$ for every $x \in X$ if $A = X$. Similarly, no $x \in X$ has $x \in \emptyset$, so $\chi_A(x) = 0$ for every $x \in X$ if $A = \emptyset$.

Again, if $X = \emptyset$ then the function is injective but not surjective, as shown earlier. Otherwise, χ_A is injective in the following cases:

1. X is a singleton. In this case, there is only one element of the domain, so the preimage of both 0 and 1 has at most one element, irregardless of A.

2. X has two elements, and A has one element. Then, for the unique $a \in A$, we have that $\chi_A(a) = \{1\}$ and the preimage of the other element in X is $\{0\}$.

If X has more than 2 elements, then one of the two sets A and $X \setminus A$ has at least 2 elements. Then, letting b_1, b_2 be two distinct elements of the larger of A and $X \setminus A$, we have that $\chi_A(b_1) = \chi_A(b_2)$.

X is surjective only if $A \neq \emptyset$ and $A \neq X$. To show this, we have that A is nonempty, and that there exists some $x \in X$ that also has $x \notin A$. Then, we have that $\chi_A(a) = 1$ for some $a \in A$, and that $\chi_A(x) = 0$, which covers the entire range.

Now, if X is a singleton, then the only subsets of X are $\emptyset$ and X itself, so χ_A cannot be surjective. Then, for χ_A to be surjective, we must have that X contains two elements, and that A must contain exactly one element. This satisfies the conditions for being both injective and surjective, such that χ_A is a bijection.

ii

Take any $x \in X$. Then, we have two cases: $x \in S_f$ or $x \notin S_f$. In the first case, we have that $x \in S_f \implies x \in f^{-1}(1) \implies f(x) = 1 = \chi_{S_f}(x)$. In the second case, we have that $x \notin S_f \implies x \notin f^{-1}(1)$; however, since every element of the domain gets mapped to some element of the range, we have that $f(x) = 0 = \chi_{S_f}(x)$.

Thus, for every $x \in X$, we have that $f(x) = \chi_{S_f}(x)$.

iii

We have already shown that $X_{S_f} = f$ in the last part.

The first part (i) showed that $\chi_A^{-1}(1) = A$, and the second part (ii) has by definition of S_{χ_A} that $\chi_A^{-1}(1) = S_{\chi_A}$. Then, we have that

$$A = \chi_A^{-1}(1) = S_{\chi_A}.$$