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Assignment 8

STAT 4001
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9.1.1

a

$$\pi(\beta \mid \delta) = P(X \geq 1 \mid \beta) = 1 - (1 - e^{-\beta}) = e^{-\beta}$$

b

The size of the test is

$$\sup_{\beta \geq 1} \left\{ e^{-\beta} \right\} = e^{-1}$$

9.1.3

$$\pi(p \mid \delta) = P(Y \geq 7) + P(Y \leq 1) = \sum_{i=7}^{20} \binom{20}{i} p^i (1-p)^{20-i} + (1-p)^{20} + p(1-p)^{19}$$

Computing, we arrive at

p	$\pi(p \mid \delta)$
0	1
0.1	0.394
0.2	0.158
0.3	0.399
0.4	0.751
0.5	0.942
0.6	0.993
0.7	0.999
0.8	0.999
0.9	0.999
1	1

9.1.20

Let $\theta_0 \in \Omega$ be a random element of the sample space. Then, $g(\theta_0) = g_0 \in \omega(x)$ if δ_{g_0} does not reject the null $H_{0,g_0} : g(\theta) \leq g_0$ when $X = x$ is observed. Since the level of the test is $\alpha_0 = 1 - \gamma$, the probability that δ_{g_0} does not reject is at least $1 - \alpha_0 = \gamma$.

9.2.2

a

From the chapter, we ought to reject if $f_0(x) < 2f_1(x) \implies 1 < 2(2x) \implies x > \frac{1}{4}$. Thus, the procedure is to reject the null if $x > \frac{1}{4}$, and to not reject the null otherwise.

b

The minimum value is

$$2\int_0^{\frac{1}{4}} f_1(x)dx + \int_{\frac{1}{4}}^1 f_0(x)dx = \frac{1}{8} + \frac{3}{4} = \frac{7}{8}$$

9.2.11

We reject if $f_1(x) > f_0(x)$, where

$$f_0(x) = \prod_{i=1}^n \frac{1}{\sqrt{8\pi}} e^{-\frac{(x_i+1)^2}{8}}$$
$$f_1(x) = \prod_{i=1}^n \frac{1}{\sqrt{8\pi}} e^{-\frac{(x_i-1)^2}{8}}$$

Then,

$$f_1(x) - f_0(x) = C(e^{-\sum_{i=1}^n \frac{(x_i+1)^2}{8}} - e^{-\sum_{i=1}^n \frac{(x_i-1)^2}{8}}) > 0 \implies \sum (x_i - 1)^2 < \sum (x_i + 1)^2$$

for some constant C; since $\sum(x_i - 1)^2 - \sum(x_i + 1)^2 = -4\sum x_i$, we have that we reject if $\overline{x_i} > 0$.

Further, $\overline{x_i}$ will be normally distributed, such that $\sqrt{n}(\overline{x_i} + \mu)/2$ will be standard normal. Then, $\alpha(\delta) = 1 - \Phi(\sqrt{n}/2)$, $\beta(\delta) = 1 - \Phi(\sqrt{n}/2)$, such that

$$\alpha(\delta) + \beta(\delta) = 2(1 - \Phi(\sqrt{n}/2))$$

Computing, we get the following:

n	$\alpha(\delta) + \beta(\delta)$
1	0.617
4	0.317
16	0.045
36	0.003

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9.3.1

Let $y = \sum_{i=1}^{n} x_i$. We have that

$$f_n(x \mid \lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \frac{\lambda^y e^{-n\lambda}}{\prod x_i!}$$

Then,

$$\frac{f_n(x \mid \lambda_2)}{f_n(x \mid \lambda_1)} = \frac{\lambda_2^y e^{-n\lambda_2}}{\lambda_1^y e^{-n\lambda_1}} = \left(\frac{\lambda_2}{\lambda_1}\right)^y \frac{e^{-n\lambda_2}}{e^{-n\lambda_1}}$$

which is a stictly increasing function of y.

9.3.7

The power is the likelihood of rejection; thus $\pi(\theta \mid \delta) = 0.05$.

9.3.11

Let $y = \sum_{i=1}^{n} x_i$. From the chapter, we have that the UMP test occurs when we have some c such that $P(y > c|\theta = 1) = 0.0143$, and we reject if $y > c$. In particular, we have that y has Poisson distribution with parameter $n\lambda = 10$. Computing, we have that we need $c = 18$.

9.4.12

a

We must have that

$$P(x \leq c_1 \mid \beta = 1) + P(x \geq c_2 \mid \beta = 1) = 1 - e^{-c_1} + e^{-c_2} = \alpha_0$$

b

For any c_1, c_2, we can have $c_1 = -\log(1 - \alpha_0/2), c_2 = -\log(\alpha_0/2)$, such that $c_1 = 0.051, c_2 = 3$.

9.5.1

a

The test statistic is

$$U = \sqrt{10}\frac{1.379 - 1.2}{0.3277} = 1.73$$

Then, we reject H_0 at $\alpha_0 = 0.05$ if $U \geq 1.833$, which it is not. Thus, we do not reject H_0 at level 0.05.

b

The p-value is, where Φ is the cdf of a $t-$distribution with 9 degrees of freedom,

$$p = 1 - \Phi(1.73) = 0.0589$$

9.5.2

a

First we compute $U = \sqrt{9}\frac{2}{3} = 2$. Then, we reject if $H_0 \geq 1.86$, so we do reject.

b

We reject if $|U| \geq 2.306$, so we do not reject.

c

We have the confidence interval with bounds $22 \pm 2.306 = (19.694, 24.306)$.

9.5.18

We first identify $\Omega_0 = \{(\mu, \sigma^2) \mid \mu \geq \mu_0\}, \Omega = \{(\mu, \sigma^2)\}$. Then, if $\overline{x_n} \geq \mu_0$, $\Lambda(x) = 1$. Otherwise, we have from the chapter that $\Lambda(x) = \left(\frac{\sigma^2}{\sigma_0^2}\right)^{\frac{n}{2}}$. Thus, Λ is a nondecreasing function of u, and for $k < 1$, we have that $\Lambda(x) \leq k \iff U \leq c$ via the same algebra as the book.

9.6.2

We have hypotheses:

$$H_0 : \mu_1 \geq \mu_2 H_1 : \mu_1 < \mu_2$$

where the subscripts $1, 2$ denote the mean concentrations of drugs A and B respectively. Then, we compute

$$U = \frac{\sqrt{m+n-2}(\overline{x_n} - \overline{y_n})}{\sqrt{\frac{1}{m} + \frac{1}{n}}\sqrt{S_X^2 + S_Y^2}} = -1.692$$

We reject if $U < -1.356$; thus, we reject the null.

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9.6.8

We now have that $\psi = \left(\frac{1}{m} + \frac{1}{n}\right)^{-\frac{1}{2}} = 2.108$. Computing, we get that

$$\pi(\mu_1, \mu_2, \sigma^2 \mid \delta) = T_{16}(-2.921) + 1 - T_{16}(2.921) = 0.247$$

9.9.2

For some real value μ', $Z = 100(\overline{X_n} - \mu')$ has the standard normal distribution, and c must be the 97.5^{th} quintile of the normal distribution, 1.96, such that

$$\begin{aligned} P(|\overline{X}| \geq c \mid \mu = \mu') &= P(100\overline{X} \leq -1.96, 1.96 \leq 100\overline{X} \mid \mu = \mu') \\ &= P(Z \leq -1.96 - 100\mu', 1.96 - 100\mu' \leq Z) \\ &= \Phi(-1.96 - 100\mu') + 1 - \Phi(1.96 - 100\mu') \end{aligned}$$

a

Computing, we get the probability of rejection to be 0.17.

b

Computing, we get the probability of rejection to be 0.516.