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Homework 3

MATH 1207
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Apostol p.28 no.1

Claim. For $x, y \in \mathbb{R}, x < y \implies \exists z \in \mathbb{R} \mid x < z < y$.

Proof. Consider $z = \frac{x}{2} + \frac{y}{2}$. We have $z \in \mathbb{R}$ as $\mathbb{R}$ is closed under addition and multiplication as it is a field under those operations.

Further, we have that $x \leq y \implies \frac{x}{2} < \frac{y}{2} \implies \frac{x}{2} + \frac{x}{2} < \frac{x}{2} + \frac{y}{2}, \frac{x}{2} + \frac{y}{2} < \frac{y}{2} + \frac{y}{2}$.

As $\forall a \in \mathbb{R}, \frac{a}{2} + \frac{a}{2} = \frac{1}{2}(a + a) = \frac{1}{2}(2a) = a$, we have that $x < \frac{x}{2} + \frac{y}{2} < y$. $\square$

Apostol p.28 no.3

Claim. For $x \in \mathbb{R}, x > 0 \implies \exists n \in \mathbb{Z}_{>0} \mid \frac{1}{n} < x$.

Proof. The Archimedian property of the reals furnishes an $n \in \mathbb{Z}_{>0} \mid nx > 1$. Then, we see that $nx > 1 \implies 1 < nx \implies n^{-1}(1) < n^{-1}(nx) \implies \frac{1}{n} < x$. $\square$

Apostol p.28 no.4

Claim. For $x \in \mathbb{R}, \exists! n \in \mathbb{Z} \mid n < x < n + 1$.

Proof. We will first show existence. Consider $S := \{n \in \mathbb{Z} \mid n \leq x\}$. This must be nonempty, or else x would be a lower bound to $\mathbb{Z}$, as $\neg \exists n \in \mathbb{Z} \mid n \leq x \implies \forall n \in \mathbb{Z}, \neg(n \leq x) \implies \forall n \in \mathbb{Z}, x \leq n$.

Now, note that if x is a lower bound for $\mathbb{Z}$, then $-x$ is an upper bound for $\mathbb{Z}$. This follows as $x \leq n \implies -x \geq -n$, but as $n \in \mathbb{Z} \implies -n \in \mathbb{Z}$, we have that $\forall n \in \mathbb{Z}, x \leq n \implies \forall n \in \mathbb{Z}, x \leq -n \implies \forall n \in \mathbb{Z}, -x \geq -(-n) = n$.

However, we proved that $\mathbb{Z}_{\geq 0} \subset \mathbb{Z}$ has no upper bound, meaning that $-x$ cannot be an upper bound of $\mathbb{Z}$. Thus, S must be nonempty.

Now, the approximation theorem proved in class furnishes $n \in S \mid \sup(S) - 1 < n$. Thus, since we have $\sup(S) - 1 < n \implies \sup(S) < n + 1 \implies n + 1 \notin S$, and by definition of S, $n \in S \implies n \leq x$ and $n + 1 \notin S \implies \neg(n + 1 \leq x) \implies x < n + 1 \implies n \leq x < n + 1$.

We will now show uniqueness: suppose that $\exists n, n' \in \mathbb{Z} \mid n \neq n', n \leq x < n+1, n' \leq x < n'+1$. $n' > n \implies n' \geq n + 1 > x$. However, $n' < n$, then we have that $n \geq n' + 1 > x$. Either way, we have $\Rightarrow\Leftarrow$, so $n = n'$.

The above relies on the fact that $a, b \in Z, a > b \implies a \geq b + 1$. This follows from $a > b \implies a - b > 0$, and as $a - b \in \mathbb{Z}$, the fact that there is no integer between 0 and 1 (proved in an earlier homework) allows that $a - b = 1$ or $a - b > 1$ by trichotomy. However, this means that $a - b \geq 1 \implies a \geq b + 1$. $\square$

Apostol p.28 no.6

Claim. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. We shall start by proving at for $x, y \in \mathbb{R}, x < y, \exists r \in \mathbb{Q} \mid x < r < y$. The Archimedian property furnishes $n \in \mathbb{Z}_{>0} \mid n(y - x) > 1 \implies ny > nx + 1$. Now consider $[nx]$. We have that $[nx] \leq nx \implies [nx] + 1 \leq nx + 1 < ny$, and also $nx < [nx] + 1$.

These together yield that

$$\begin{aligned} &nx < [nx] + 1 \leq nx + 1 < ny \\ \implies &n^{-1}(nx) < n^{-1}([nx] + 1) \leq n^{-1}(nx + 1) < n^{-1}(ny) \\ \implies &x < \frac{[nx] + 1}{n} < y \end{aligned}$$

Critically, $[nx] \in \mathbb{Z}$, meaning that as $[nx] + 1, n \in \mathbb{Z}$, we have $\frac{[nx]+1}{n} \in \mathbb{Q}$.

Now that we have one such r, we can construct infinitely many: simply use the above process to find r' such that $r < r' < y$. This can be repeated ad infinitum. $\square$

Apostol p.64 no.4b

Claim.

$$[-x] = \begin{cases} -[x] & x \in \mathbb{Z} \\ -[x] - 1 & x \notin \mathbb{Z} \end{cases}$$

Proof. Note that if we find one such a such that $a \leq x < a + 1$, we have that $[x] = a$ as we have shown previously that such an a must be unique.

Suppose that $x \in \mathbb{Z}$. Then we have that $-x \leq -x < -x + 1$, and so $[-x] = -x$.

Otherwise, we have that $[x] \leq x$. However, we have that as $[x] \in \mathbb{Z}, x \notin \mathbb{Z}$, $[x] < x$. This then provides that $-[x] > -x$. Further, $x < [x] + 1 \implies -x > -([x] + 1) = -[x] - 1$.

These together give us $-[x] - 1 < -x < -[x] \implies -[x] - 1 \leq -x < -[x] \implies [-x] = -[x] - 1$. $\square$

Apostol p.64 no.4d

Claim. $[2x] = [x] + [x + \frac{1}{2}]$

Proof. Consider $[x] + \frac{1}{2}$. We have that by trichotomy, exactly one of $x < y, x = y, x > y$ is true.

If $x < [x]+\frac{1}{2}$, then $[x] \leq x < [x]+\frac{1}{2} \implies [x] < x+\frac{1}{2} < [x]+\frac{1}{2}+\frac{1}{2} = [x]+1 \implies [x+\frac{1}{2}] = [x]$. Further, $2[x] \leq 2x < 2([x]+\frac{1}{2}) = 2[x]+1 \implies [2x] = 2[x] = [x]+[x] = [x]+[x+\frac{1}{2}]$.

If $x = [x]+\frac{1}{2}$, then $x+\frac{1}{2} = [x]+\frac{1}{2}+\frac{1}{2} = [x]+1$, and $[x]+1 \in Z \implies [x]+1 \leq [x]+1 < [x]+2 \implies [x+\frac{1}{2}] = [[x]+1] = [x]+1$. Further, $2x = 2([x]+\frac{1}{2}) = 2[x]+1 = [x]+[x]+1 = [x]+[x+\frac{1}{2}]$.

If $x > [x]+\frac{1}{2}$, then $[x]+\frac{1}{2} \leq x < [x]+1 \implies [x]+\frac{1}{2}+\frac{1}{2} \leq x+\frac{1}{2} < [x]+1+\frac{1}{2} \implies [x]+1 \leq x+\frac{1}{2} < [x]+2 \implies [x+\frac{1}{2}] = [x]+1$. Further, $2x > 2([x]+\frac{1}{2}) = 2[x]+1$, and $x < [x]+1 \implies 2x < 2[x]+2 \implies 2[x]+1 < 2x < 2[x]+2 \implies [2x] = 2[x]+1 = [x]+[x]+1 = [x]+[x+\frac{1}{2}]$. $\square$

Problem 1

Problem 2

Problem 3

Problem 4