Homework 6

Apostol p.30 no.2

a

Proceed with Gram-Schmidt:

$$
\begin{aligned}
u_1 &= (1,1,0,0) \\
e_1 &= \frac{1}{\sqrt{2}}(1,1,0,0) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0) \\
u_2 &= (0,1,1,0) - \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0) = (-\frac{1}{2}, \frac{1}{2}, 1, 0) \\
e_2 &= \sqrt{\frac{2}{3}}(-\frac{1}{2}, \frac{1}{2}, 1, 0) = (-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}, 0) \\
u_3 &= (0,0,1,1) - \sqrt{\frac{2}{3}}(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{3}, 0) = (\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, 1) \\
e_3 &= \frac{\sqrt{3}}{2}(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, 1) = (\frac{1}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2})
\end{aligned}
$$

Note that $(1,0,0,1) = (1,1,0,0) + (0,0,1,1) - (0,1,1,0)$, so it is linearly dependent. Thus, e_1, e_2, e_3 form an orthonormal basis.

b

Proceed with Gram-Schmidt:

$$
\begin{aligned}
u_1 &= (1,1,0,1) \\
e_1 &= \frac{1}{\sqrt{3}}(1,1,0,1) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}) \\
u_2 &= (1,0,2,1) - \frac{2}{\sqrt{3}}(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}) = (\frac{1}{3}, -\frac{2}{3}, 2, \frac{1}{3}) \\
e_2 &= \sqrt{\frac{3}{14}}(\frac{1}{3}, -\frac{2}{3}, 2, \frac{1}{3}) = (\frac{1}{\sqrt{42}}, -\frac{2}{\sqrt{42}}, \sqrt{\frac{6}{7}}, \frac{1}{\sqrt{42}})
\end{aligned}
$$

Note that $(1,2,-2,1) = 2(1,1,0,1) - (1,0,2,1)$, so e_1, e_2 form an orthonormal basis.

Apostol p.30 no.5

First we will show that $\int_0^\infty e^{-t}t^n dt = n!$ for $n \in \mathbb{Z}_{>0}$.

With integration by parts, $\int_0^\infty e^{-t}t^{n+1}dt = [-e^{-t}t^{n+1}]\Big|_0^\infty - n\int_0^\infty e^{-t}t^n dt = n\int_0^\infty -e^{-t}t^n dt$.

Inducting on n, for $n = 0$ we have that $\int_0^\infty = e^{-t}dt = [-e^{-t}]\Big|_0^\infty = 1$. Then, assuming the above for $n = k$, $\int_0^\infty e^{-t}t^{k+1}dt = (k+1)\int_0^\infty e^{-t}t^k dt = (k+1)(k!) = (k+1)!$.

$$
\begin{aligned}
y_0 &= 1 \\
y_1 &= t - \int_0^\infty e^{-t}tdt \\
&= t - 1 \\
y_2 &= t^2 - \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle} - \frac{\langle t^2, t-1\rangle}{\langle t-1, t-1\rangle}(t-1) \\
&= t^2 - 2! - \frac{3! - 2!}{2! - 2! + 1!}(t-1) = t^2 - 4t + 2 \\
y_3 &= t^3 - \frac{\langle t^3, 1\rangle}{\langle 1, 1\rangle} - \frac{\langle t^3, t-1\rangle}{\langle t-1, t-1\rangle}(t-1) - \frac{\langle t^3, t^2-4t+2\rangle}{\langle t^2-4t+2\rangle}(t^2-4t+2) \\
&= t^3 - 3! - (4! - 3!)(t-1) - \frac{5! - 4(4!) + 2(3!)}{4! - 8(3!) - 20(2!) - 16 + 4}(t^2-4t+2) \\
&= t^3 - 6 - 18(t-1) - 9(t^2-4t+2) \\
&= t^3 - 9t^2 + 18t - 6
\end{aligned}
$$

Problem 1

Claim. Let V be a finite dimensional inner product space, and $U \subseteq V$ any subspace. Then, $\dim(U) + \dim(U^\perp) = V$.

Proof. We will show sometihng stronger:

$$U + U^\perp = V$$

where $U + U^\perp = \{u + v \mid u \in U, v \in U^\perp\}$.

We have from class that for any vector $v \in V$, $\exists x, x^\perp \mid x + x^\perp = v$ where $x \in U, x^\perp \in U^\perp$, so $V \subseteq U + U^\perp$. Further, since $U, U^\perp$ are subspaces of V, if $u \in U, v \in V^\perp, u, v \in V \implies u + v \in V$, so $U + U^\perp \subseteq V$.

Then, $U + U^\perp = V$, and from an ealier homework, $\dim(U) + \dim(U^\perp) - \dim(U \cap U^\perp) = V$. However, if $x \in U, U^\perp, \langle x, x\rangle = 0 \implies x = 0$. Thus, $\dim(U \cap U^\perp) = 0$, and the initial claim follows. $\square$

Problem 3

Let $S : U \to V$ be a linear map with adjoint S^*.

a

Claim. S^* has an adjoint and $(S^*)^* = S$.

Proof.

$$\langle S(u), v\rangle = \langle u, S^*(v)\rangle \implies \overline{\langle v, S(u)\rangle} = \overline{\langle S^*(v), u\rangle} \implies \langle v, S(u)\rangle = \langle S^*(v), u\rangle$$

The last part holds by taking the complex conjugate of both sides, which is a bijective operation as shown on an earlier homework. Further, since adjoints are unique, $(S^*)^* = S$. □

b

Claim.

$$\ker(S^*) = (\mathrm{im}S)^\perp$$

Proof. If $x \in \ker(S^*)$, then we have that

$$\langle S(u), x\rangle = \langle u, S(x)\rangle = \langle u, 0\rangle = 0$$

Then $x \in (\mathrm{im}S)^\perp$.
Similarly, if $x \in (\mathrm{im}S)^\perp$ then

$$0 = \langle S(u), x\rangle = \langle u, S^*(x)\rangle$$

Then, one of $u, S^*(x) = 0$. Since this holds for any u, we have that $S^*(x) = 0$.
Thus, $x \in \ker(S^*) \iff x \in (\mathrm{im}S)^\perp$. □

c

Claim.

$$\ker(S) = (\mathrm{im}S^*)^\perp$$

Proof. This follows from part a and b immediately, as we simply take part b and apply it to S^*. Then, since we have that $(S^*)^* = S$, part b gives that

$$\ker((S^*)^*) = \ker(S) = (\mathrm{im}S^*)^\perp$$

□

d

Claim. If S is invertible, then S^* is invertible, S^{-1} has an adjoint, and $(S^*)^{-1} = (S^{-1})^*$.

Proof. We know that the adjoint exists, as it is equivalent to the conjugate transpose of the original matrix representing the linear transformation.

Further, since $I^T = I$, and I is a real matrix, then the adjoint of I is still I.

In general, we have that

$$\langle u, v\rangle = \langle u, \mathrm{Id}^*(v)\rangle$$

we easily see that $\mathrm{Id}^* = \mathrm{Id}$ works, and so $\mathrm{Id}^* = \mathrm{Id}$ as adjoints are unique.

Now, from problem 2, we have that

$$(T \circ S)^* = S^* \circ T^*$$

Taking $T = S^{-1}$, we have that $\mathrm{Id}^* = S^* \circ (S^{-1})^*$. By the definition of inverse, we have that $(S^{-1})^* = (S^*)^{-1}$. ☐

Problem 4

Claim. Let $S : U \to V$ be a linear map between finite inner product spaces. Then,

$$\dim(\mathrm{im} S^*) = \dim(\mathrm{im} S).$$

Proof. We have that as $\ker(S^*) = (\mathrm{im}(S))^\perp$, $\dim(\ker(S^*)) = \dim((\mathrm{im}(S))^\perp)$. From problem 1, we have that $\dim((\mathrm{im}(S))^\perp) = \dim(V) - \dim(\mathrm{im}(S))$, and so

$$\dim(\ker(S^*)) = \dim(V) - \dim(\mathrm{im}(S))$$

From rank-nullity, we have that

$$\dim(V) = \dim(\mathrm{im}(S^*)) + \dim(\ker(S^*))$$

Combining the two, we get that $\dim(\mathrm{im}(S^*)) = \dim(\mathrm{im}(S))$.

Now, we have from the email that S^* is the complex conjugate of the transpose of A, where A is the matrix representing S for a choice of basis. Further, put $\overline{A}$ for the matrix with entries the complex conjugate of A.

Then, $\dim(\mathrm{im}(S^*)) = \dim(\mathrm{im}(S))$, where the image of S is simply the column space of A, and the image of S^* is then the column space of $\overline{A^T}$ (as this is a property of column spaces shown in class).

Now, we have that the dimension of the span of a set of vectors $v_1, \ldots v_n$ is the same as the dimension of the space of $\overline{v_1}, \ldots, \overline{v_n}$.

To see this, note that for $x = \sum_{i=1}^n c_i v_i$, $\overline{x} = \sum_{i=1}^n c_i \overline{v_i}$, by a property proved on a past homework.

Then, any basis of $v_1, \dots v_n$, say $x_1, \dots, x_k$, has a corresponding basis $\overline{x_1}, \dots \overline{x_n}$ for $\overline{v_1}, \dots \overline{v_n}$, and so they must be of the same dimension.

Then, the column space of A^T has the same dimension as the column space of $\overline{A^T}$; since the column rank of $\overline{A^T} = \dim(\operatorname{im}(S^*))$, we have that $\dim(\operatorname{im}(S^*)) =$ the column rank of A^T.

The column rank of A^T, however, is also the row rank of A, by the definition of transposes, and so the row rank of $A = \dim(\operatorname{im}(S^*)) = \dim(\operatorname{im}(S)) =$ the column rank of A.

Thus, we have that row and column rank are the same. $\square$