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# Homework 8



## Apostol p.155 no.8

## Apostol p.168 no.22

## Apostol p.168 no.24

## Apostol p.174 no.14

# Problem 1

**Claim.** Let $f : [a, b] \to \mathbb{R}$ be integrable. Then, $\exists c \in [a, b]$ such that

$$\int_a^c f(x)dx = \frac{1}{2}\int_a^c f(x)dx$$

.

*Proof.* First, we extend the Intermediate Value Theorem to be slightly stronger. The original statement is that for $f : [a, b] \to \mathbb{R}$ continuous, if $f(a) < K < f(b)$ then $\exists c \in [a, b] \mid f(c) = K$. Further, we will show that if $f(b) < K < f(a)$ then $\exists c \in [a, b] \mid f(c) = K$. To see this, consider $g(x) - f(x)$. We have that $g(a) < -K < g(b)$, so $\exists c \in [a, b] \mid g(c) = -K \implies f(c) = K$.

Consider $g(x) : [a, b] \to \mathbb{R}, g(x) = \int_a^x g(t)dt$. Then, we have that $g(a) = \int_a^a g(t)dt = 0, g(b) = \int_a^b g(t)dt$. Further, $\frac{1}{2}\int_a^b f(x)dx = \frac{g(b)}{2} = \frac{g(a)+g(b)}{2}$, and if $g(b) > 0 = g(a)$, then $g(a) < \frac{g(a)+g(b)}{2} < g(b)$, and if $g(b) < 0 = g(a)$, then $g(b) < \frac{g(a)+g(b)}{2} < g(a)$, and so by the Intermediate Value Theorem, $\exists c \in [a, b] \mid g(c) = \frac{g(a)+g(b)}{2} \implies \int_a^c f(x)dx = \frac{1}{2}\int_a^b f(x)dx$. $\square$

# Problem 2

**Claim.** $f$ is continuous on $[0, 1]$, and has $f(0) = f(1)$. $\forall n \in \mathbb{Z}_{>0}, \exists x \in [0, 1 - 1/n] \mid f(x) = f(x + 1/n)$.

*Proof.* Consider $g(x) = f(x) - f(x + 1/n)$. Suppose that $g > 0$. Then, we have that $f(1) < f(0)$. To see this, consider the set $\{g(k/n) \mid k \in \mathbb{Z}_{>0}, k \leq n\}$. We then have that $k > 0 \implies f(k/n) < f(0)$, as we can induct on $k$. If $k = 1$, then $g(1/n) > 0 \implies f(1/n) - f(0) < 0 \implies f(1/n) < f(0)$. Assume that the hypothesis holds for $k < n$. Then, $g(k/n) > 0 \implies f(k/n) - f((k + 1)/n) > 0 \implies f((k + 1)/n < f(k/n) < f(0)$. This shows that $f(k/n) < f(0)$ for all $k \in \mathbb{Z}_{>0}, k \leq n$. Critically, this then means that $f(1) < f(0)$.

Now suppose that $g < 0$. Then, we have that $f(1) > f(0)$. To see this, consider the set $\{g(k/n) \mid k \in \mathbb{Z}_{>0}, k \leq n\}$. We then have that $k > 0 \implies f(k/n) > f(0)$, as we can induct on $k$. If $k = 1$, then $g(1/n) < 0 \implies f(1/n) - f(0) < 0 \implies f(1/n) < f(0)$. Assume that the hypothesis holds for $k < n$. Then, $g(k/n) > 0 \implies f(k/n) - f((k+1)/n) > 0 \implies f((k+1)/n < f(k/n) < f(0)$. This shows that $f(k/n) < f(0)$ for all $k \in \mathbb{Z}_{>0}, k \leq n$. Critically, this then means that $f(1) < f(0)$.

□

## Problem 4

### a)

Consider the coutner example of

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \qquad g(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

We have that $f + g = 1$, which is differentiable everywhere $(f + g)' = 0$. However, we have that $f, g$ are nowhere continuous and thus nowhere differentiable (the contrapositive, differentiable $\implies$ continuous was proved in class).

In general, take any function $f$ not differentiable at $x$. Then, $f + (-f) = 0$ is differentiable at $x$, but neither $f, -f$ are.

### b)

**Claim.** If $f(x) \neq 0$, then $g$ is differentiable at $x$.

*Proof.* We have that the quotient rule states for functions $s, t$ differentiable at $x$, then if $t(x) \neq 0$, $(\frac{s}{t})' = \frac{s't - st'}{t^2}$ at $x$. Taking $s = fg, t = f$, we have that $f(x) \neq 0 \implies g'(x)$ exists by the quotient rule. □

## Problem 5

### a)

**Claim.** $f(x) = xg(x)$, $g$ continuous at $0 \implies f$ is differentiable at 0.

*Proof.*

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h}$$

Consider $\lim_{h \to 0}(\frac{f(h)}{h} - g(h))$. For any $\epsilon$, take arbitrary $\delta > 0$. We then have that $0 < |x| < \delta \implies x \neq 0 \implies \frac{f(x)}{x} = g(x) \implies |\frac{f(x)}{x} - g(x)| = 0 < \epsilon$.

Thus, we have that $\lim_{h\to 0}(\frac{f(h)}{h} - g(h)) = 0 \implies \lim_{h\to 0}\frac{f(h)}{h} = \lim_{h\to 0} g(h) = g(0)$, as $g$ is continuous. $\square$

## b)

**Claim.** Suppose that $f$ is differentiable at 0 and $f(0) = 0$. Then, $\exists g(x) \mid f(x) = xg(x), g$ continuous at 0.

*Proof.* Consider

$$g(x) = \begin{cases} f'(0) & x = 0 \\ \frac{f(x)}{x} & x \neq 0 \end{cases}$$

which is well defined as we have that $f$ is differentiable at 0.
Then, we have that

$$xg(x) = \begin{cases} 0 & x = 0 \\ f(x) & x \neq 0 \end{cases}$$

This is equal to $f(x)$ everywhere.
Now, to prove that $g(x)$ is continuous, note first that we have that $\lim_{h\to 0}\frac{f(h)}{h} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = f'(0)$, as we have that $f$ is differentiable at 0. Further, $\lim_{h\to 0}(g(h) - \frac{f(h)}{h} = 0$, as for any $\epsilon > 0$, take arbitrary $\delta > 0 \mid 0 < |x| < \delta \implies x \neq 0 \implies g(x) = \frac{f(x)}{x} \implies |g(x) - \frac{f(x)}{x} - 0| = 0 < \epsilon$.
Finally, we have that $\lim_{h\to 0} g(h) = \lim_{h\to 0}\frac{f(h)}{h} = f'(0) = g(0)$, so $g$ is continuous. $\square$