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Homework 7

MATH 1207
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Apostol p.139-140 no.33.a

Suppose that we have $f : [a,b] \to \mathbb{R}, |f(u) - f(v)| \leq |u - v|$.

Claim. f is continuous on $[a,b]$.

Proof. For any $c \in [a,b]$, f must be continuous: consider that for any $\epsilon > 0$, we take $\delta = \epsilon$ and then have that $0 < |x-c| < \epsilon \implies |f(x)-f(c)| < |x-c| < \epsilon$. Thus, $\lim_{x\to c} f(x) = f(c)$. □

Apostol p.139-140 no.33.b

Claim.

$$\left|\int_a^b f(x)dx - (b-a)f(a)\right| \leq \frac{(b-a)^2}{2}$$

Proof. Note that $\int_a^b f(a)dx = (b-a)f(a)$, as $f(a)$ is just a step function with partition $\{a,b\}$. Then, the claim becomes

$$\begin{aligned}
\left|\int_a^b f(x)dx - \int_a^b f(a)dx\right| &= \left|\int_a^b (f(x) - f(a))dx\right| \\
&\leq \int_a^b |(f(x) - f(a))| \\
&\leq \int_a^b |x-a|dx \\
&= \int_a^b (x-a)dx \\
&= \int_a^b xdx - \int_a^b adx \\
&= \int_a^b xdx - (b-a)a \\
&= \frac{b^2}{2} - \frac{a^2}{2} - (b-a)a \\
&= \frac{b^2 + a^2 - 2ab}{2} \\
&= \frac{(b-a)^2}{2}
\end{aligned}$$

Note that $\int_a^b xdx$ was shown on a past homework. □

Apostol p.139-140 no.33.c

Claim.

$$\left|\int_a^b f(x)dx - (b-a)f(c)\right| \leq \frac{(b-a)^2}{2}$$

Proof.

$$\begin{aligned}
\left|\int_a^b f(x)dx - \int_a^b f(c)dx\right| &= \left|\int_a^b (f(x)-f(c))dx\right| \\
&\leq \int_a^b |(f(x)-f(c))| \\
&\leq \int_a^b |x-c|dx \\
&= \int_a^c (x-c)dx + \int_c^b -(x-c)dx
\end{aligned}$$

This last step is justified by the fact that if $x \in (a,c) \implies x < c \implies |x-c| = -(x-c)$.

$$\begin{aligned}
&= -\int_a^c (x-c)dx + \int_c^b (x-c)dx \\
&= -(\int_a^c xdx - \int_a^c cdx) + (\int_c^b xdx - \int_c^b cdx) \\
&= -(\int_a^c xdx - (c-a)c) + (\int_c^b xdx - (b-c)c) \\
&= -\int_a^c xdx + \int_c^b xdx - (a+b)c \\
&= -(\frac{c^2}{2} - \frac{a^2}{2}) + (\frac{b^2}{2} - \frac{c^2}{2}) - (a+b)c + 2c^2 \\
&= \frac{a^2+b^2-2c^2}{2} - (a+b)c + 2c^2 \\
&= \frac{a^2+b^2}{2} + c^2 - (a+b)c \\
&\leq \frac{a^2+b^2}{2} + b^2 - (a+b)b \\
&= \frac{a^2+b^2}{2} - ab \\
&= \frac{(b-a)^2}{2}
\end{aligned}$$

The last inequality holds, as we have that $(b^2-(a+b)b)-(c^2-(a+b)c) = b^2-c^2-(a+b)(b-c) = (b-c)(b+c-(a+b) = (b-c)(c-a) > 0$, as $a < c < b$. $\square$

Apostol p.142 no.11

$$\begin{aligned}
\lim_{x\to -2} \frac{x^3+8}{x^2-4} &= \lim_{x\to -2} (\frac{x+2}{x+2})(\frac{x^2-2x+4}{x-2}) \\
&= (\lim_{x\to -2} \frac{x+2}{x+2})(\lim_{x\to -2} \frac{x^2-2x+4}{x-2}) \\
&= \lim_{x\to -2} \frac{x^2-2x+4}{x-2}
\end{aligned}$$

We have proved $\lim_{h\to 0} \frac{h}{h} = 0$ on a past homework (for any $\epsilon > 0$, $\delta = 1231231231231$ works). Since we have that polynomials are continuous, and that $x-2$ at $x=-2$ is nonzero, we have that $\frac{x^2-2x+4}{x-2}$ is continuous, and so $\lim_{x\to -2} \frac{x^2-2x+4}{x-2} = \frac{4+4+4}{-4} = -3$.

Apostol p.142 no.12

We have that $\lim_{x\to 4} \sqrt{1+\sqrt{x}}$. We have that $f : x \mapsto \sqrt{x}$ is continuous, as is $g : x \mapsto 1+x$, and thus $\sqrt{1+\sqrt{x}} = (f \circ g \circ f)$ is also continuous as the composition of continuous functions. Thus, $\lim_{x\to 4} = \sqrt{1+\sqrt{4}} = \sqrt{3}$.
To see that $\sqrt{x}$ is continuous, consider that for any $\epsilon > 0$, take $\delta = \epsilon^2$, such that $0 < |x-c| < \delta, y = |x-c| \implies |\sqrt{x} - \sqrt{x+y}| < |\sqrt{x} - \sqrt{x+2\sqrt{xy}+y}| = |\sqrt{x} - (\sqrt{x}+\sqrt{y})| = |\sqrt{y}| < \sqrt{\epsilon^2} < \epsilon$. Thus, $\lim_{y\to x} \sqrt{y} = \sqrt{x}$.

Apostol p.142 no.19

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} &= \lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} \\
&= \lim_{x\to 0} \frac{1+x-(1-x)}{x\sqrt{1+x}+\sqrt{1-x}} \\
&= \lim_{x\to 0} \frac{2x}{x\sqrt{1+x}+\sqrt{1-x}} \\
&= \lim_{x\to 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}}
\end{aligned}$$

Given that the quotient, sum, and composition of continuous functions is continuous, we have that $\frac{2}{\sqrt{1+x}+\sqrt{1-x}}$ is then also continuous, and so $\lim_{x\to 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}} = \frac{2}{\sqrt{1}+\sqrt{1}} = 1$.

Apostol p.155 no.7

Claim. Let f be integrable and nonnegative. Then $\int_a^b f(x)dx = 0 \implies f(x) = 0$ at every point of continuity.

Proof. Suppose that $f(x) \neq 0$ and f is continuous at x. Then we have that for $\epsilon = \frac{f(x)}{2}$, $\exists \delta \mid 0 < |y - x| < \delta \implies \frac{f(x)}{2} < f(y) < \frac{3f(x)}{2}$, meaning that for $y \in (x-\delta, x+\delta)$, $f(y) > 0$. Put $\gamma_1 = \max(a, x-\delta), \gamma_2 = \min(b, x+\delta)$. Importantly, we would then have that $\int_{\gamma_1}^{\gamma_2} f(y)dy > \int_{\gamma_1}^{\gamma_2} 0dy = 0$. Thus, we would then have that $\int_a^{\gamma_1} f(y)dy + \int_{\gamma_1}^{\gamma_2} f(y)dy + \int_{\gamma_2}^{b} f(y)dy = \int_a^b f(y)dy > 0$, as f being nonegative means that $\int_a^{\gamma_1} f(y)dy, \int_{\gamma_2}^{b} f(y)dy \geq 0$. $\Rightarrow\Leftarrow$, so $f(x) = 0$. $\square$

Problem 1

a

$$f(x) = \begin{cases} x-1 & x < 0 \\ x & x \geq 0 \end{cases}$$

This is monotonic: consider that for any $a, b \in \mathbb{R}$, suppose that $b > a$. If $a, b < 0$, then $f(b) - f(a) = (b-1) - (a-1) = b - a > 0$. If $a < 0, b \geq 0$, then $f(b) - f(a) = (b) - (a-1) = b - a + 1 > 0$. If $a, b > 0$, then $f(b) - f(a) = (b) - (a) = b - a > 0$. Thus, the function is monotonic.

However, it is not continuous: consider that $\lim_{x\to 0} f(x)$ does not exist. Suppose that $\lim_{x\to 0} f(x) = K$. Then, for any $\epsilon > 0$, we must have $\delta > 0 \mid 0 < |x| < \delta \implies |f(x) - f(0)| = |f(x)| < \epsilon$. However, we have that for $x \in (-\delta, 0)$, $f(x) = x - 1 < -1 \implies |f(x)| > 1$. Thus, for $\epsilon < 1$, no such δ exists, and f is discontinuous at 1.

b

Claim. Any monotonic surjective function $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Proof. We will show that f must be continuous at any $c \in \mathbb{R}$. Consider that for any $\epsilon > 0$, $f(c) + \frac{\epsilon}{2}$ must be achieved by some $b \in \mathbb{R}$, as f is surjective and also as f is monotonic, $f(b) > f(c) \implies b > c$. Similarly, $f(c) - \frac{\epsilon}{2}$ must be achieved by some $a < c$. Take $\delta = \min(|b - c|, |a - c|)$.

Since f is monotonic, we have that $a < d < c < e < b \implies f(a) \leq f(d) \leq f(c) \leq f(e) \leq f(b) \implies f(a) - f(c) \leq f(d) - f(c) \leq 0 \leq f(e) - f(c) \leq f(b) - f(c) \implies -\frac{\epsilon}{2} \leq f(d) - f(c) < f(e) - f(c) \leq \frac{\epsilon}{2} \implies |f(d) - f(c)|, |f(e) - f(c)| \leq \frac{\epsilon}{2} < \epsilon$. Thus, $0 < |x-c| < \delta \implies a < x < b \implies |f(x)-f(c)| < \epsilon$. Thus, we have that $\lim_{x\to c} f(x) = f(c)$, and f is then continuous at c. $\square$

Problem 2

Claim. For $f, g : [a, b] \to \mathbb{R}$, both continuous, if $f(x) = g(x)$ whenever $x \in \mathbb{Q}$, then $f = g$.

Proof. Suppose that for some x, $f(x) \neq g(x)$. Without loss of generality, take $g(x) > f(x)$. Then, x must be irrational. However, we have that since f is continuous, for any $\epsilon > 0$, we have δ such that $0 < |y-x| < \delta_f \implies |f(x)-f(y)| < \frac{\epsilon}{2}$. Similarly, we have that $0 < |y-x| < \delta_g \implies |g(x)-g(y)| < \frac{\epsilon}{2}$. Taking $\delta = \min(\delta_f, \delta_g)$, we have that $|g(x)-g(y)|, |f(x)-f(y)| < \frac{\epsilon}{2}$. Then, we have that $|g(x) - g(y) - f(x) + f(y)| < |g(x) - g(y)| + |f(x) - f(y)| = \epsilon$ by the triangle inequality, but since $\mathbb{Q}$ is dense, we can always find $y \in \mathbb{Q} \mid 0 < |x - y| < \delta$. In that case, we have that $f(y) = g(y)$, so $|g(x) - g(y) - f(x) + f(y)| = |g(x) - f(x)| < \epsilon$ for any ϵ, especially for $\epsilon = g(x) - f(x) > 0$. $\Rightarrow\Leftarrow$, so $f(x) = g(x)$ everywhere.

$\square$