MATH 4065 HW 4

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October 16, 2020

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We can reduce this to $\varphi(0) = 0$ and $\varphi'(0) = 1$. In particular, if some function $\phi : \Omega \to \Omega$ is a holomorphic function satisfying $\phi(z_0) = z_0$ and $\phi'(z_0) = 1$, we can define $\Omega' = \{z \mid z+z_0 \in \Omega\}$ and $\varphi : \Omega' \to \Omega'$ by taking $\varphi(z) = \phi(z + z_0) - z_0$. Then, we have that $\varphi(0) = \phi(z_0) - z_0 = 0$ and $\varphi'(0) = \phi'(z_0) = 1$. However, if we have that $\varphi(z)$ is linear, such that $\varphi(z) = z$, then we have that $\phi(z + z_0) = z + z_0$ is also linear, so it is sufficient to show that $\varphi(z) = z$.

From here, since φ is holomorphic in Ω containing 0, we have that $\varphi(h) = \sum_{k=0}^{\infty} a_k h^k$ in some neighborhood of 0, say $D_r(0)$. In particular, we have that $a_0 = \varphi(0) = 0$ and $a_1 = \varphi'(0) = 1$. Then, let n be the first index greater than 0 such that $a_n \neq 0$, such that $\varphi(h) = h + a_n h^n + O(h^{n+1})$.

Now define $\varphi_k = \varphi \circ \varphi \circ \cdots \circ \varphi$ as the composition of φ with itself $k-$times. We will show with induction that $\varphi_k(h) = h + k a_n h^n + O(h^{n+1})$ in some neighborhood of 0. This clearly holds for $k = 1$ by the earlier power series expansion. If it holds for k, then $\varphi_{k+1} = \varphi \circ \varphi_k$, and

$$\begin{aligned}
(\varphi_k(h))^m &= (h + k a_n h^n + O(h^{n+1}))^m \\
&= \sum_{i=0}^{m} \binom{m}{i} h^{m-i} (k a_n h^n + O(h^{n+1}))^i \\
&= h^m + \sum_{i=1}^{m} \binom{m}{i} h^{m-i} (k a_n h^n + O(h^{n+1}))^i \\
&= h^m + \sum_{i=1}^{m} \binom{m}{i} h^{m-i} h^{ni} (k a_n + O(h))^i \\
&= h^m + \sum_{i=1}^{m} \binom{m}{i} h^{m+(n-1)i} (k a_n + O(h))^i
\end{aligned}$$

Since $n > 1$, if $m > 1$ then

$$
\begin{aligned}
&= h^m + \sum_{i=1}^{m} \binom{m}{i} O(h^{m+1}) O(h)^i \\
&= h^m + O(h^{m+1})
\end{aligned}
$$

So

$$
\begin{aligned}
\varphi_{k+1}(h) &= \varphi_k(h) + a_n(\varphi_k(h))^n + O((\varphi_k(h))^{n+1}) \\
&= h + ka_n h^n + O(h^{n+1}) + a_n(h^n + O(h^{n+1})) + O(h^{n+1} + O(h^{n+2})) \\
&= h + (k+1)a_n h^n + O(h^{n+1})
\end{aligned}
$$

for some neighborhood of 0, as we have that since φ_k as the composition of analytic functions is analytic, then $|\varphi_k(h) - 0| = |\varphi_k(h)| < r \implies \varphi_l \in D_r(0)$ by continuity on some neighborhood of 0, so on that neighborhood the above relation holds for $k+1$, and thus, all $k \geq 1$ by induction.

Let r be the radius of some disc whose closure is contained in Ω' (which we know exists since Ω' is open). Cauchy inequality gives that $|\varphi_k^{(n)}(0)| = ka_n \leq \frac{n! \sup_{C_r(0)} |\varphi_k|}{r^n}$ for any $k \geq 1$. However, the right hand side is bounded, since we know that Ω' is the codomain of φ (and thus φ_k), and Ω' is a bounded set. Therefore, $k \leq \frac{n! \sup_{C_r(0)} |\varphi_k|}{r^n a_n}$ means that the integers are bounded, as all $k \geq 1$ satisfy being less than some finite number. $\Rightarrow\!\Leftarrow$, so no $a_n \neq 0$ for $n \geq 1$, so $\varphi(h) = h$, which converges everywhere in $\mathbb{C}$, and so φ is linear.

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Weierstrass's theorem gives that for any function $f : [0,1] \to \mathbb{R}$ and $\epsilon > 0$, we have that there is some sequence of polynomials p_n such that $\{p_n\}_{n=1}^{\infty}$ converge uniformly to f.

This cannot (always) happen in the complex case. In particular, we already have from the book and class that if $\{f_n\}_{n=1}^{\infty}$ a sequence of holomorphic functions converges uniformly to f, we have that f is also holomorphic. In particular, this gives that if f_n are all polynomials, they are all entire, and thus f must be holomorphic on the open unit disc. As a result, if we can find some continuous function on the open unit disc that is not holomorphic, we are done.

In particular, consider $f(z) = \overline{z}$. This clearly isn't holomorphic as $\frac{\partial f}{\partial \overline{z}} = 1 \neq 0$, so it doesn't obey the Cauchy-Riemann equations. However, it is continuous, as we have that for any w in the unit disc, we know that for any $\epsilon > 0$ and z in the unit disc, picking $\delta = \epsilon$ gives $|z - w| = |x_z + iy_z - (x_w + iy_w)| < \delta \implies |\overline{z} - \overline{w}| = |x_z - iy_z - (x_w - iy_w)| = \sqrt{(x_z - x_w)^2 + (y_z - y_w)^2} = |x_z + iy_z - (x_w + iy_w)| < \delta = \epsilon$.

Thus, we have that $f(z) = \overline{z}$ is not uniformly approximated by polynomials in the unit disc.

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In particular, we can show that for some index n, we have that $c_n = 0$ for an uncountable amount of $w \in \mathbb{C}$, where $f(z) = \sum_{i=n}^{\infty} c_n(z-w)^n$. Suppose that $c_n = 0$ for at most a countable amount of w for every n. Then, consider the set of ordered pairs $S = \{(n, w) \mid c_n = 0, f(z) = \sum_{i=0}^{\infty} c_i(z-w)^i\}$ Then, since we have that $c_n = 0$ for a countable amount of w for any given n, we have that $S = \bigcap_{i=0}^{\infty}\{(i, w) \mid c_i = 0, f(z) = \sum_{j=0}^{\infty} c_j(z-w)^j\}$ is the countable union of countable sets, and is therefore countable itself.

However, we can explicitly see that S must be uncountable, as for every $w \in \mathbb{C}$, there is at least one n such that $c_n = 0$, so we have that there is a bijection given by $(n, w) \mapsto w$ from a subset of S to $\mathbb{C}$, so S contains an uncountable subset, and thus cannot be countable. $\Rightarrow\Leftarrow$, so $c_n = 0$ for an uncountable amount of $w \in \mathbb{C}$ for some index n.

Further, any uncountable subset E of $\mathbb{R}^n$ has a limit point in $\mathbb{R}^n$. To see this, if we partition R^n into unit $n-$cells of the form $\prod_{i=1}^{n}[k_i, k_i+1]$ for $k_i \in \mathbb{Z}$, we have a countable amount of $n-$cells (as we can form a bijection by taking $\prod_{i=1}^{n}[k_i, k_i+1] \mapsto (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$, which is countable). Then, if there were a finite amount of points of E in every cell, we have that E must be countable as it would be the countable union of finite sets. In particular, if K_i is an enumeration of the $n-$cells, $K = \bigcup_{i=1}^{\infty} K_i \cap E$, in which each intersection is finite by assumption. $\Rightarrow\Leftarrow$, so there is some cell with an infinite amount of points in E. Take i such that $K_i \cap E$ is infinite.

Now, we have that $n-$cells in $\mathbb{R}^n$ are compact. If no point of K_i is a limit point of $K_i \cap E$, then $\forall p \in K_i$, $D_{r_p}(p)$ contains at most one element of $K_i \cap E$ for some $r_p > 0$. Then, $\{D_{r_p}(p)\}_{p \in K_i}$ is an open cover of K_i, but no finite collection of $D_{r_p}(p)$ can cover of all of $K_i \cap E$, as this collection would have a finite amount of points of $K_i \cap E$, which is infinite, so there is no finite subcover of K_i. Then, K_i, a $n-$cell, cannot be compact and $\Rightarrow\Leftarrow$ so some point of K_i is a limit point of $K_i \cap E$ and thus a limit point of E.

Finally, we can finish: since $c_n = 0$ for an uncountable amount of $w \in \mathbb{C}$ for some index n, we have that $f(z) = \sum_{i=0}^{\infty} c_i(z-w)^i \implies c_n = \frac{f^{(n)}(w)}{n!}$, so $f^{(n)}(w) = c_n$ vanishes on an uncountable subset of $\mathbb{C}$ and thus vanishes on some sequence with a limit point in $\mathbb{C}$, and thus $f^{(n)}$ must vanish entirely on $\mathbb{C}$. Then, if $f^{(n)}$ vanishes at every point in $\mathbb{C}$, we have that $f^{(m)}$ vanishes on the entire plane for any $m \geq n$, so $c_m = 0$ as well. Then, the power series expansion of f around 0 gives $f(z) = \sum_{i=0}^{n-1} c_i z^i$, which converges everywhere, so f is a polynomial.

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We extend $f(z) = 1/\overline{f(1/\overline{z})}$ for $z \in \mathbb{C}$, $|z| > 1$. Then, we have that on the open set $\overline{\mathbb{D}}^c$ where this extension is defined, $f(z) = 1/\overline{f(1/\overline{z})}$ is continuous at $z \in \overline{\mathbb{D}}^c$ as $z \in \overline{\mathbb{D}}^c \implies |z| > 1 \implies |1/z| < 1 \implies 1/z \in \mathbb{D} \implies \overline{1/z} = 1/\overline{z} \in \mathbb{D}$, so $f(1/\overline{z})$ is continuous at $1/\overline{z} \implies 1/\overline{f(1/\overline{z})}$ is continuous.

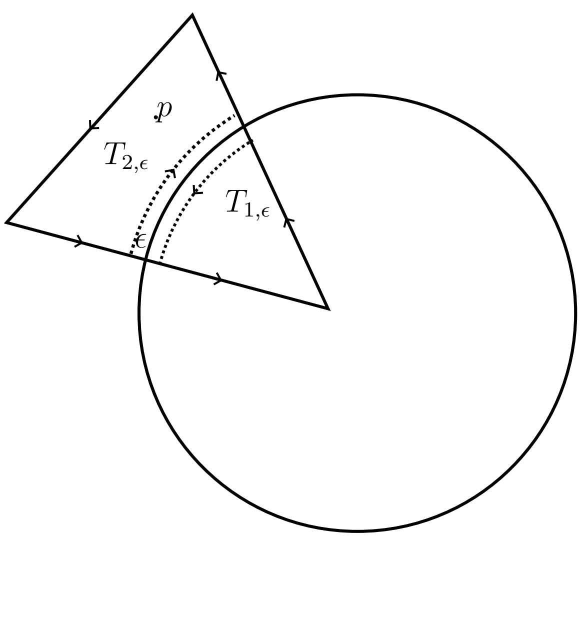
Thus, we have that f with this extension is continuous on the open sets $\overline{\mathbb{D}}^c$ and on $\mathbb{D}$, so if we can show that for $|z| = 1$, $\lim_{w\to z,|w|>1} 1/\overline{f(1/\overline{z})} = \lim_{w\to z,|w|<1} f(z) = f(z)$, we have that f is continuous on all of $\mathbb{C}$. Fortunately, we have on the unit circle that $\overline{z} = 1/z$, so if we extend $1/\overline{f(1/\overline{z})}$ continually to $|z| = 1$, we have that $1/\overline{f(1/\overline{z})} = 1/\overline{f(z)} = f(z)$, so we have that $\lim_{w\to z} f(w) = f(z)$, so f is continuous on all of $\mathbb{C}$ with the above extension.

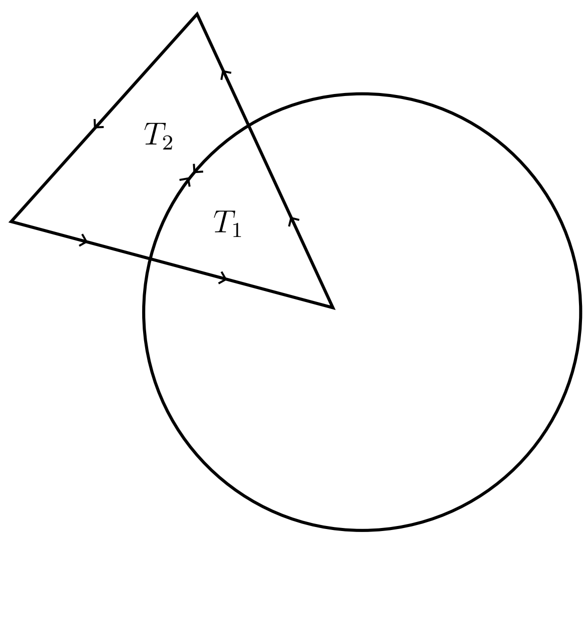
Now, we wish to show that f is entire. We have that f is holomorphic in $\mathbb{D}$ by assumption, and we have that if $f(z) = 1/\overline{f(1/\overline{z})}$ for $|z| > 1$, we have that $|z|, |z_0| > 1 \implies |1/\overline{z}|, |1/\overline{z_0}| < 1$, so expanding around $1/\overline{z_0}$,

$$\begin{aligned}
f\left(\frac{1}{\overline{z}}\right) &= \sum_{n=0}^{\infty} a_n \left(\frac{1}{\overline{z}} - \frac{1}{\overline{z_0}}\right)^n \\
f(z) &= \frac{1}{\overline{f\left(\frac{1}{\overline{z}}\right)}} \\
&= \frac{1}{\overline{\sum_{n=0}^{\infty} a_n \left(\frac{1}{\overline{z}} - \frac{1}{\overline{z_0}}\right)^n}} \\
&= \frac{1}{\sum_{n=0}^{\infty} \overline{a_n} \left(\frac{1}{z} - \frac{1}{z_0}\right)^n}
\end{aligned}$$

but the bottom is then an analytic function expanded around $\frac{1}{z_0}$ and composed with $\frac{1}{z}$, so we have that $f(z)$ is analytic for $|z| > 1$.

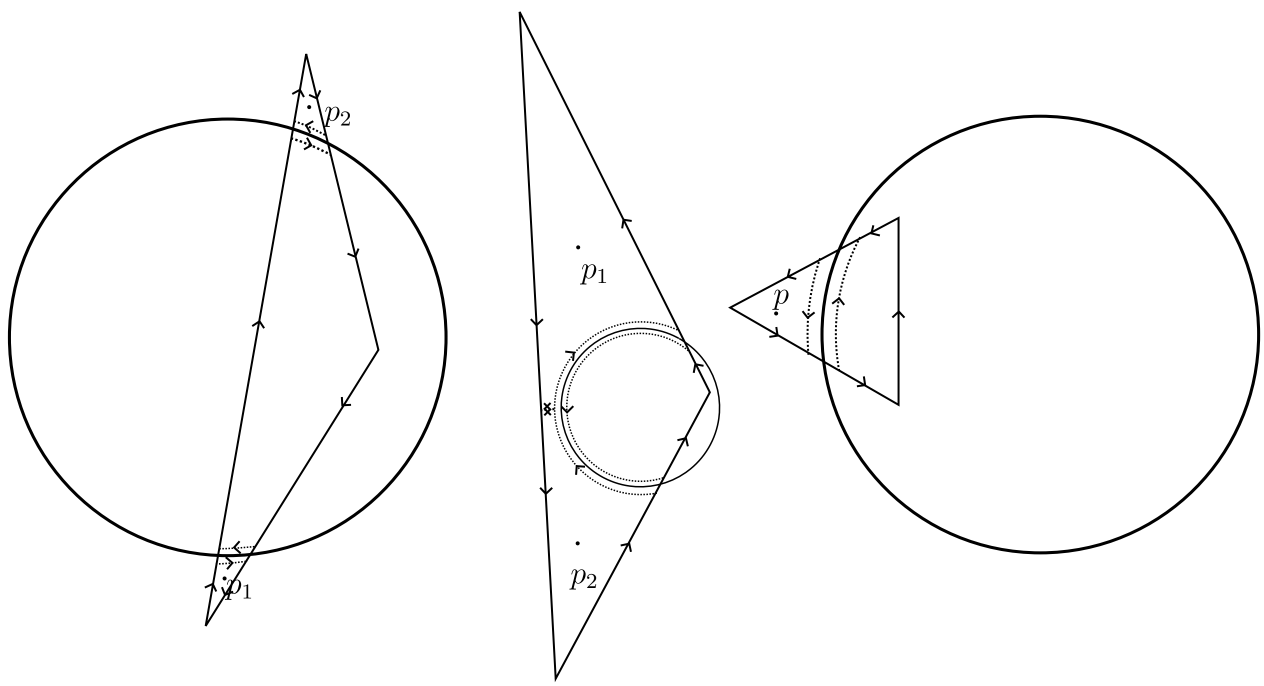
Now, to show that f is analytic on the unit circle, consider any triangle T in $\mathbb{C}$. If the triangle does not intersect the unit circle, either $T \subset \mathbb{D}$ or $T \subset \overline{\mathbb{D}}^c$. In either case, f is holomorphic in those open sets, so $\int_T f = 0$. In the case that T intersects the unit circle, we can split T into two contours, as shown on the right:





We have that $\int_T f = \int_{T_1} f + \int_{T_2} f = \lim_{\epsilon\to 0}\int_{T_{1,\epsilon}} f + \lim_{\epsilon\to 0}\int_{T_{2,\epsilon}} f$ by continuity. However, $T_{1,\epsilon}, T_{2,\epsilon}$ lie in $\mathbb{D}$ and $\overline{\mathbb{D}}^c$ respectively. Cauchy's theorem on discs gives that $\int_{T_{1,\epsilon}} f = 0$ since f is holomorphic on $\mathbb{D}$, and taking Cauchy's theorem on star-shaped domains with a point inside the region such as p witnessing star-shaped condition gives that $\int_{T_{2,\epsilon}} f = 0$ as well, so we can conclude $\int_T f = 0$.

In particular, the way to split the triangle may vary, but we can always find star-shaped regions and paths to do the same argument:



In all of these cases, we have that $\int_T f = \sum_{i=1}^n \lim_{\epsilon\to 0}\int_{T_{i,\epsilon}} f$ where $T_{i,\epsilon}$ is a path in a star-shaped region either completely inside or outside the disc, since we can split the triangle into subsets that are either inside or outside of $\mathbb{D}$, and take T_i along those subsets and the portion of the unit circle connecting them, so $\int_T f = 0$. Finally, by Morera's theorem, f must be holomorphic on the unit disc as well.

At last, we have that f under the given extension is entire. Further, since f is continuous in $\overline{D}$, we have that $|f|$ must be bounded in $\mathbb{D}$ since $|f|$ is continuous on a compact set. This also means that $|f|$ must also attain its minimal value in $\overline{\mathbb{D}}$. Furthermore, since $|f(z)| = 1/|f(1/\overline{z})|$ for z with $|z| > 1$, we have that since $1/\overline{z} \in \mathbb{D}$, and f does not vanish in $\mathbb{D}$ and $|f|$ attains its minimal value, say m, $1/|f(1/\overline{z})|$ must be bounded by $1/|m|$. Therefore, f is entire and bounded, and by Liouville's is a constant function.

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Note that since $1 \leq d(n) \leq n$, $\limsup_{n\to\infty} |d(n)|^{1/n} = 1$.

We have that on the unit disc via power series expansion, $\frac{1}{1-z} = \sum_{m=0}^{\infty} z^n$, so

$$\frac{z^n}{1-z^n} = z^n \sum_{m=0}^{\infty} (z^n)^m = \sum_{m=1}^{\infty} z^{nm} = \sum_{i=1}^{\infty} a_{n,i} z^i$$

where $a_{n,i} = 1$ if $i = nm$ for some m and $a_{n,i} = 0$ otherwise. In particular, this means that $a_{n,i} = 1$ if and only if $n|i$.

Then, we have that $\sum_{n=1}^{\infty} \frac{z^n}{1-z^n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} z^{nm} = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} a_{n,i} z^i = \sum_{i=1}^{\infty} (\sum_{n=1}^{\infty} a_{n,i}) z^i = \sum_{i=1}^{\infty} (\sum_{n=1}^{i} a_{n,i}) z^i$, where the last equality holds since $n > i \implies n \nmid i \implies a_{n,i} = 0$. However, since $d(i)$ counts the amount of divisors of i, we have that $\sum_{n=1}^{i} a_{n,i} z^i = d(i)$, as $a_{n,i} = 1$ for each divisor of i and 0 otherwise, so we have that $\sum_{n=1}^{\infty} \frac{z^n}{1-z^n} = \sum_{i=1}^{\infty} (\sum_{n=1}^{i} a_{n,i}) z^i = \sum_{i=1}^{\infty} d(i) z^i$. Then, since this converges absolutely for $|z| < 1$, we have that all the interchanges of the sums were justified.

Note that if we define the step function $k(x)$ such that for $n \geq 0, n \in \mathbb{Z}$, on the interval $(n, n+1)$, $k(x) = r^n/(1-r^n)$, we have that since $r^x/(1-r^x)$ is decreasing for $r < 1$, as

$$\frac{d}{dx}\frac{r^x}{1-r^x} = \frac{r^{2x}\log(r)}{(1-r^x)^2} + \frac{r^x \log(r)}{(1-r^x)} = \log(r)\left(\frac{r^{2x}}{(1-r^x)^2} + \frac{r^x}{1-r^x}\right) < 0 \text{ as } \log(r) < 0$$

Then, we have that $r^n/(1-r^n) \leq r^x/(1-r^x)$ for $x \in (n, n+1)$, so $r^x/(1-r^x) \leq k(x)$. Then we have that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{r^n}{1-r^n} &= \int_1^{\infty} k(x)dx \\ &\leq \int_1^{\infty} \frac{r^x}{1-r^x} dx\end{aligned}$$

u-sub with $u = 1 - r^x$ gives

$$\begin{aligned}&= -\int_{1-r}^{1} \frac{1}{u\log(r)} du \\ &= -\frac{1}{\log(r)} \log(u)\Big|_{1-r}^{1} \\ &= \frac{1}{\log(r)} \log(1-r) \\ &= -\frac{1}{\log(r)} \log\left(\frac{1}{1-r}\right)\end{aligned}$$

Then, since we have that Taylor expanding $\log(r)$ around $r = 1$ gives that $\log(r) = (r-1) + O((r-1)^2) \implies \log(r) \leq (r-1) + c'(r-1) = \frac{1}{c}(r-1)$ as $r \to 1$ for constants c, c', so

$$\leq c\frac{1}{r-1}\log\left(\frac{1}{1-r}\right)$$

so we have that $|F(r)| \geq c\frac{1}{r-1}\log\left(\frac{1}{1-r}\right)$ as desired.

To start the next part, we have that

$$\begin{aligned}
|F(re^{i\theta})| &= \left|\sum_{n=1}^{\infty}\frac{r^n e^{2\pi pn/q}}{1-r^n e^{2\pi pn/q}}\right| \\
&= \left|\sum_{n=1,q|n}^{\infty}\frac{r^n e^{2\pi pn/q}}{1-r^n e^{2\pi pn/q}} + \sum_{n=1,q\nmid n}^{\infty}\frac{r^n e^{2\pi pn/q}}{1-r^n e^{2\pi pn/q}}\right| \\
&= \left|\sum_{n=1}^{\infty}\frac{r^{nq}}{1-r^{nq}} + \sum_{n=1,q\nmid n}^{\infty}\frac{r^n e^{2\pi pn/q}}{1-r^n e^{2\pi pn/q}}\right|
\end{aligned}$$

If we put $w_p = e^{2\pi p/q}$, since $w_p^n = w_p^{n+q}$this reduces to

$$\begin{aligned}
&= \left|\sum_{n=1}^{\infty}\frac{r^{nq}}{1-r^{nq}} + \sum_{p=1}^{q-1}\sum_{n=1}^{\infty}\frac{r^{qn+p}w_p}{1-r^{qn+p}w_p}\right| \\
&\geq \left|\sum_{n=1}^{\infty}\frac{r^{nq}}{1-r^{nq}}\right| - \left|\sum_{p=1}^{q-1}\sum_{n=1}^{\infty}\frac{r^{qn+p}w_p}{1-r^{qn+p}w_p}\right| \\
&\geq c\frac{1}{1-r^q}\log\left(\frac{1}{1-r^q}\right) - \left|\sum_{p=1}^{q-1}\sum_{n=1}^{\infty}\frac{r^{qn+p}w_p}{1-r^{qn+p}w_p}\right|
\end{aligned}$$

Let $c' = \min(1 - r^n w_p)$ for $r \in \mathbb{R}$, $1 \leq p \leq q-1$. Then, we have that this is nonzero, as the set of all points $1 - r^n w_p$ is a collection of lines in the complex plane, none passing through zero, as w_p has nonzero imaginary part. Then,

$$
\begin{aligned}
&\geq c\frac{1}{1-r^q}\log\left(\frac{1}{1-r^q}\right) - \left|\sum_{p=1}^{q-1}\sum_{n=1}^{\infty}\frac{r^{qn+p}w_p}{1-r^{qn+p}w_p}\right| \\
&\geq c\frac{1}{1-r^q}\log\left(\frac{1}{1-r^q}\right) - \sum_{p=1}^{q-1}\sum_{n=1}^{\infty}\left|\frac{r^{qn+p}w_p}{1-r^{qn+p}w_p}\right| \\
&\geq c\frac{1}{1-r^q}\log\left(\frac{1}{1-r^q}\right) - \sum_{p=1}^{q-1}\sum_{n=1}^{\infty}\frac{1}{c'}\left|r^{qn+p}w_p\right| \\
&\geq c\frac{1}{1-r^q}\log\left(\frac{1}{1-r^q}\right) - \sum_{p=1}^{q-1}\sum_{n=1}^{\infty}\frac{1}{c}r^{qn+p} \\
&= c\frac{1}{1-r^q}\log\left(\frac{1}{1-r^q}\right) - \frac{1}{c'}\sum_{n=1}^{\infty}r^n \\
&= c\frac{1}{1-r^q}\log\left(\frac{1}{1-r^q}\right) - \frac{1}{c'}\sum_{n=1}^{\infty}r^n \\
&= c\frac{1}{1-r^q}\log\left(\frac{1}{1-r^q}\right) - \frac{1}{c'}\frac{1}{1-r}
\end{aligned}
$$

From L'Hopital twice applied to $\frac{1-r^q}{1-r}$ and $\frac{\log(r)}{\log(r^q)}$, $\lim_{r\to 1}\frac{1}{1-r}\log\left(\frac{1}{r}\right)/\frac{1}{1-r^q}\log\left(\frac{1}{r^q}\right) = q$. Then, since this ratio is bounded near $r = 1$, we can get new constants c'', c''' such that

$$
\begin{aligned}
&\geq c''\frac{1}{1-r}\log\left(\frac{1}{1-r}\right) - \frac{1}{c'}\frac{1}{1-r} \\
&= c''\frac{1}{1-r}\left(\log\left(\frac{1}{1-r}\right) - \frac{1}{c'''}\right)
\end{aligned}
$$

Then, as $r \to 1$, we have that $\log\left(\frac{1}{1-r}\right) > \frac{1}{c'''}$, so we can conclude

$$
\geq c''\frac{1}{1-r}\log\left(\frac{1}{1-r}\right)
$$