MATH 4041 HW 13

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Problem 1

We have that

$$g_1 \cdot (g_2 \cdot (x_1, x_2)) = g_1 \cdot (g_2 \cdot x_1, g_2 \cdot x_2) = (g_1 \cdot (g_2 \cdot x_1), g_1 \cdot (g_1 \cdot x_2))$$

and since X_1, X_2 are $G-$sets, we have that this becomes

$$(g_1 \cdot (g_2 \cdot x_1), g_1 \cdot (g_1 \cdot x_2)) = ((g_1 g_2) \cdot x_1, (g_1 g_2) \cdot x_2) = (g_1 g_2) \cdot (x_1, x_2)$$

which is what we want.
Checking the identity,

$$1 \cdot (x_1, x_2) = (1 \cdot x_1, 1 \cdot x_2) = (x_1, x_2)$$

Any element that fixes (x_1, x_2) under the group action must fix both x_1 and x_2, since

$$g \cdot (x_1, x_2) = (g \cdot x_1, g \cdot x_2) = (x_1, x_2) \implies g \cdot x_1 = x_1, g \cdot x_2 = x_2$$

so $g \in G_{x_1}, G_{x_2}$. Clearly, if $g \in G_{x_1}, G_{x_2}$ then we get that

$$g \cdot (x_1, x_2) = (g \cdot x_1, g \cdot x_2) = (x_1, x_2)$$

so the stabilizer of (x_1, x_2) under G is exactly $G_{x_1} \cap G_{x_2}$.

Problem 2

1. We can just compute these directly; after all, there are only 6 elements in $|S_3|$.

$$S_3 \cdot (1, 2, 3) = \{(1, 2, 3), (1, 3, 2), (2, 1, 3), (3, 1, 2), (2, 3, 1), (3, 2, 1)\}$$

which has order 6, which divides $|S_3| = 6$ as well. Any element in the stabilizer of $(1, 2, 3)$ needs $(\sigma(1), \sigma(2), \sigma(3)) = (1, 2, 3)$, so $\sigma = 1$ and the stabilizer is trivial.

2.
$$S_3 \cdot (1,1,2) = \{(1,1,2),(1,1,3),(2,2,1),(2,2,3),(3,3,1),(3,3,2)\}$$
which has order 6 dividing $|S_3|$, and any element in the stabilizer has $\sigma(1) = 1, \sigma(2) = 2 \implies \sigma(3) = 3$ and so must be the identity, so the stabilizer is trivial.

3.
$$S_3 \cdot (1,1,1) = \{(1,1,1),(2,2,2),(3,3,3)\}$$
which has order 3 dividing $|S_3| = 6$, and the stabilizer must satisfy that $\sigma(1) = 1$, so
$$(S_3)_{(1,1,1)} = \{1,(2,3)\}$$

Problem 3

i

We need to check that

$$\tau\cdot(\sigma\cdot(x_1,x_2,x_3)) = P(\tau)(P(\sigma)(x_1e_1+x_2e_2+x_3e_3)) = P(\tau)P(\sigma)(x_1e_1+x_2e_2+x_3e_3) = (\tau\sigma)\cdot(x_1,x_2,x_3)$$

but that $P(\tau)P(\sigma) = P(\tau\sigma)$ is a basic fact of permutation matrices, and associativity of matrix multiplication gives the rest of what we want. More explicitly,

$$\begin{aligned} P(\tau)(P(\sigma)(x_1e_1+x_2e_2+x_3e_3)) &= P(\tau)(x_1e_{\sigma(1)}+x_2e_{\sigma(2)}+x_3e_{\sigma(3)})) \\ &= x_1e_{\tau(\sigma(1))}+x_2e_{\tau(\sigma(2))}+x_3e_{\tau(\sigma(3))} \\ &= P(\tau\sigma)(x_1e_1+x_2e_2+x_3e_3) \end{aligned}$$

which was what we wanted. Then, the identity permutation clearly maps $(x_1,x_2,x_3) \mapsto (x_1,x_2,x_3)$ since the preimage of any element is the element itself.

1. We can again just compute these directly.
$$S_3 \cdot (1,2,3) = \{(1,2,3),(1,3,2),(2,1,3),(3,1,2),(2,3,1),(3,2,1)\}$$
which has order 6 dividing $|S_3| = 6$. The stabilizer is then of order 1, and thus only the identity.

2.
$$S_3 \cdot (1,1,2) = \{(1,1,2),(1,2,1),(2,1,1)\}$$
which has order 3 dividing $|S_3| = 6$. The stabilizer then satisfies that $\sigma^{-1}(1), \sigma^{-1}(2) \in \{1,2\}$ and $\sigma^{-1}(3) = 3$. This is then two elements of S_3, and
$$(S_3)_{(1,1,2)} = \{1,(1,2)\}$$

3.

$$S_3 \cdot (1,1,1) = \{1,1,1\}$$

which has order 1 dividing $|S_3| = 6$. The stabilizer must be of order 6 and since it is a subgroup of S_3, a finite group, must be all of S_3 (which is also apparent since every permutation in S_3 fixes $(1,1,1)$, since the orbit is extactly itself).

$$(S_3)_{(1,1,2)} = S_3$$

ii

1. The orbit of $(1,2,3,4)$ is any $(a,b,c,d) \in \{1,2,3,4\}^4 \subset \mathbb{R}^4$, where $a \neq b \neq c \neq d$. In particular, the element of S_4 which moves $(1,2,3,4)$ to (a,b,c,d) is the inverse to the one taking $1 \mapsto a, 2 \mapsto b, 3 \mapsto c, 4 \mapsto d$, so the orbit is of order 16. Furthermore, any element $\sigma \in S_4$ fixing $(1,2,3,4)$ must satisfy that $\sigma^{-1}(1) = 1, \sigma^{-1}(2) = 2, \sigma^{-1}(3) = 3, \sigma^{-1}(4) = 4$. This clearly shows that $\sigma = 1$, so the stabilizer is exactly the identity element (which gives us another way to compute the orbit's size as $|S_4|/1 = 16$).

2. The stabilizer of $(1,1,2,2)$ is any σ that is the composition of $(1,2)$ and $(3,4)$, since σ can swap the first two elements and the second two elements, but nothing else. This gives a total of 4 elements in the stabilizer and thus $|S_4|/4 = 4$ elements in the orbit (also since the orbit is just all permutations, it is also $\frac{4!}{2!2!} = 4$).

3. The stabilizer of $(1,1,1,1)$ is any $\sigma \in S_4$, since any permutation of $(1,1,1,1)$ is itself $(1,1,1,1)$. This gives a total of 16 elements in the stabilizer and thus $|S_4|/16 = 1$ elements in the orbit, which is just $\{(1,1,1,1)\}$.

Problem 4

Let A be an element of O_n which fixes e_3, and let the element in the i^{th} row and j^{th} column be a_{ij}. Then, basic matrix multiplication gives

$$Ae_n = (a_{1n}, a_{2n}, \dots, a_{nn}) = e_n \implies a_{1n} = a_{2n} = \cdots = a_{(n-1)n} = 0, a_{nn} = 1$$

but since A is orthonormal, $\sum_{i=1}^{n} a_{ni} = a_{nn} + \sum_{i=1}^{n-1} a_{ni} = 1 \implies a_{n1} = a_{n2} = \cdots = a_{n(n-1)} = 1$, so

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & & & 0 \\ & B & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where B is some $n-1 \times n-1$ matrix. Then, we have that B must be orthonormal, since the length of the first $n-1$ row vectors of A are the same as the lengths of the row vectors of B, so B must have row vectors of length 1. Furthermore, the dot product between any of the first $n-1$ row vectors of A is the same as the dot product between the respective row vectors of B, so the row vectors of B must be orthgonal (and from earlier, orthonormal). Then, $B \in O_{n-1}$ gives us our A. Further, any $B \in O_{n-1}$ embeded as above into an $n \times n$ matrix gives the rows as an orthonormal basis since the lengths are all 1 and the dot product of any two distinct rows must be 0.

In particular, A fixes the subspace spanned by e_n, but moves the subspace spanned by $e_1, \ldots, e_{n-1}$ by B, so in some sense it is easily identifiable with an element of O_{n-1}.

Now, the mapping $f : (O_n)_{e_n} \to O_{n-1}$ given by taking A to the $n-1 \times n-1$ submatrix as above is well-defined and bijective; we can also check easily that

$$\begin{bmatrix} & & & 0 \\ & B_1 & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & & 0 \\ & B_2 & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & & & 0 \\ & B_1 B_2 & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so f is an isomorphism between the stabilizer of e_n and O_{n-1}.

Problem 5

Pick some generator $xH \in G/H$, $x \in G$. Then, we have that every element $g \in G$ is contained in some coset $H' \in G/H$, and if $H' = (xH)^n = x^n H$, then $g = x^n h$ for some $h \in H$ and some positive integer n. Now consider any two elements of G, represented as $x^{n_1} h_1$, $x^{n_1} h_2$. Then, since $H \leq Z(G)$, h_1 and h_2 commute with every element of G, so

$$x^{n_1} h_1 x^{n_2} h_2 = x^{n_1} x^{n_2} h_1 h_2 = x^{n_1+n_2} h_1 h_2 = x^{n_2} x^{n_1} h_2 h_1 = x^{n_1} h_2 x^{n_1} h_1$$

so G is abelian.

Problem 6

We have that $kh^{-1}k^{-1} \in H$ since H is normal, and similarly $hkh^{-1} \in K$. Then, $hkh^{-1}k^{-1} = hh' = k'k$ for some $h' \in H, k' \in K$. Then, $hkh^{-1}k^{-1} \in H \cap K \implies hkh^{-1}k^{-1} = 1 \implies (hkh^{-1}k^{-1})kh = hk = kh$.

Problem 7

This is just the right coset Hg. In particular, the it is the set (by definition) $\{h \cdot g \mid h \in H\}$, but the group action is just left multiplication, so it is $\{hg \mid h \in H\}$, which is Hg. The action is transitive when $Hg = G$ for some g, but this means that $1 \in Hg \implies Hg = H = G$, so the action is transitive only when H is the entire group G. The stabilizer of g is trivial; $hg = g \implies h = 1$. This can also be noted for finite groups since $|Hg| = |H|$ since all cosets are the same size, but then $|H_g| = |Hg|/|H| = 1$.