MATH 4061 HW 9

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On $x < 0$, we have that $f(x) = -x^3$, and on $x > 0$, we have that $f(x) = x^3$. Then, for $x < 0$, since there derivative is defined as

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

we have that for t on sufficiently small neighborhoods of x, $t < 0$ so

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{t \to x} \frac{-t^3 + x^3}{t - x} = \lim_{t \to x}(-t^2 - tx - x^2) = -3x^2$$

Similarly, for $x > 0$, for t on sufficiently small neighborhoods of x, $t > 0$, so

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{t \to x} \frac{t^3 - x^3}{t - x} = \lim_{t \to x}(t^2 + tx + x^2) = 3x^2$$

and note that if a derivative exists everywhere in a neighborhood of a point x except for at x, then by L'Hopital, if the last limit here exists,

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{t \to x} \frac{f'(t)}{1} = \lim_{t \to x} f'(t)$$

and we clearly have that $\lim_{x \to 0^-} -3x^2 = \lim_{x \to 0^+} 3x^2 = 0$, so the derivative at 0 is 0, and we get

$$f'(x) = \begin{cases} -3x^2 & x < 0 \\ 3x^2 & x \geq 0 \end{cases}$$

Similarly, we get that for $x < 0$, $f''(x) = -6x$ as the derivative of $-3x^2$, and for $x > 0$, $f''(x) = 6x$, and at zero, again,

$$\lim_{x \to 0^-} -6x = \lim_{x \to 0^+} 6x = 0$$

so

$$f''(x) = \begin{cases} -6x & x < 0 \\ 6x & x \geq 0 \end{cases}$$

but now, $f^{(3)}(0) = \lim_{t\to 0} \frac{f''(t)-f''(0)}{t-0} = \lim_{t\to 0} \frac{f''(t)}{t}$, but here,

$$\lim_{t\to 0^-} \frac{f''(t)}{t} = \lim_{t\to 0^-} \frac{-6t}{t} = -6$$

but

$$\lim_{t\to 0^+} \frac{f''(t)}{t} = \lim_{t\to 0^+} \frac{6t}{t} = 6$$

so the left and right limits take different values, and thus the limit doesn't exist.

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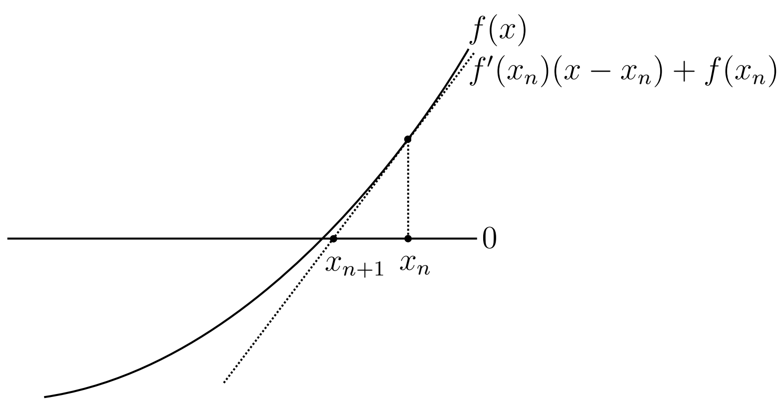
a



If we have some x_n, the next step is to create a tangent line to $f(x)$ at x_n, which is given by $g(x) = f'(x_n)(x - x_n) + f(x_n)$; then, we set x_{n+1} to be the zero of g, which occurs at $x = x_n - \frac{f'(x_n)}{f(x_n)}$.

b

As a note, the problem isn't true as stated? Taking $f(x) = x, [a, b] = [1, 1]$, we get that $f'(x) = 1, f''(x) = 0$, which satisfies the desired properties, but clearly if we take $x_1 \in (0, 1)$ we get $x_2 = x_1 - x_1 = 0$, and $x_3 = x_2 - x_2 = 0 = x_2$, so $x_{n+1} \leq x_n$ is the correct relation. Piazza strengthens the problem to $x_n \neq \xi$, so we'll use that.

We want to show that $f(x_n) > 0$ for all x_n, which will give us that $\frac{f(x_n)}{f'(x_n)} > 0 \implies x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} < x_n$, and since f is increasing, that $f(x_n) > f(\xi) \implies x_n > \xi$. This is true for x_1 since $x_1 \in (\xi, b)$ by construction, and the fact that the function has $f'(x) > 0$ shows that it is monotone increasing and thus $f(\xi) = 0 < f(x_1)$. Now assume that $f(x_n) > 0$ by induction. Then, we have that $x_{n+1} = x_n - \frac{f'(x_n)}{f(x_n)}$, where $\frac{f'(x_n)}{f(x_n)}$ must be positive by assumption, so $x_n - x_{n+1} > 0$.

Now, the mean value theorem gives us that there is some $c \in (x_{n+1}, x_n)$ such that $f'(c) = \frac{f(x_n)-f(x_{n+1})}{x_n - x_{n+1}}$, but we have that by construction, $f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}} = f'(c) - \frac{f(x_{n+1})}{x_n - x_{n+1}}$. Since the second derivative is nonnegative, we have that $f'(c) \leq f'(x_n)$, so we get that (since $x_n - x_{n+1} > 0$) $f(x_{n+1}) \geq 0$. In particular, we assume that $x_n \neq \xi$ for any n, so since ξ is the unique zero of f, $x_{n+1} \neq \xi$ and thus $f(x_{n+1}) \neq 0 \implies f(x_{n+1}) > 0$, which gives us that $\xi < x_{n+1} < x_n$ for all n.

Thus, $x_n \to \zeta$ for some $\zeta \geq \xi$ since x_n is monotone and bounded. However, we have that for sufficiently large n, $|x_{n+1} - x_n| = |\frac{f(x_n)}{f'(x)}| < \epsilon$ for any epsilon. In particular, $f'(x)$ is bounded on $[a, b]$ since it is continuous and $[a, b]$ is compact, so $|f(x_n)| < \epsilon$ for any epsilon for again, sufficiently large n. Then, we clearly have that $f(x_n) \to 0$ since it is monotonic (since x_n is monotone, and f is monotonic since it has positive derivative everywhere, so $x_{n+1} < x_n \implies f(x_{n+1}) < f(x_n)$) and bounded (since $x_n > \xi \implies f(x_n) > 0$) so it is convergent, but the limit is smaller than any positive real number, and thus must be 0. But then, this gives that since f is continuous, $f(\zeta) = 0$, so $\zeta = \epsilon$.

c

Taylor's theorem yields the following estimate when expanded around $x = x_n$:

$$f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2$$

for some $t_n \in (\xi, x_n)$. Then, since $f(\xi) = 0$, we get, after dividing by $f'(x_n)$ which is nonzero,

$$-(\xi - x_n) - \frac{f(x_n)}{f'(x_n)} = \frac{f''(t_n)}{2f'(x_n)}(\xi - x_n)^2$$

which simplifies to (since $x_n - \frac{f(x_n)}{f'(x_n)} = x_{n+1}$)

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

as desired.

d

Note that since $f'(x) > \delta$ and $0 \leq f''(x) \leq M$ for all $x \in [a, b]$, we have that the last part gives us that for any n,

$$0 \leq x_{n+1} - \xi \leq \frac{M}{2\delta}(x_n - \xi)^2 = \frac{1}{A}(A(x_n - \xi))^2$$

and the nonnegativity is from the earlier part where $x_n \geq \xi$ is shown.

This gives us what we want for $n = 1$, since the above inequality translates directly in this case to

$$0 \leq x_2 - \xi \leq A(x_1 - \xi)^2 = \frac{1}{A}(A(x_1 - \xi))^2$$

Then, if we assume $0 \leq x_{n+1} - \xi \leq \frac{1}{A}(A(x_1 - \xi))^{2^n}$ for n, then applying the first inequality we get

$$0 \leq x_{n+2} - \xi \leq \frac{1}{A}(A(x_{n+1} - \xi))^2 \leq \frac{1}{A}\left(A\left(\frac{1}{A}(A(x_1 - \xi))^{2^n}\right)\right)^2 = \frac{1}{A}(A(x_1 - \xi))^{2^n}$$

as desired.

e

We showed earlier that Newton's method generates a sequence converging to ξ such that $f(\xi) = 0$, and it is clear that any fixed point of $g(x) = x$ gives that $f(x)/f'(x) = 0 \implies f(x) = 0$, and that any root of f is a fixed point of g since $f(x) = 0 \implies g(x) = x + \frac{0}{f'(x)} = x$.

Computing the derivative, we get that $g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}$. Then, if $x \to \xi$, $f'(x)$ is bounded away from 0 and $f''(x)$ is bounded above, so $g'(x) \to 0$ as well.

f

The problem here is that $f'(0)$ is not defined, since we have that $\lim_{t \to 0} \frac{t^{1/3}}{t} = \lim_{t \to 0} t^{-2/3} = \infty$, which isn't real, and so Newton's method wont work for finding a root.

In particular, we have that $x_{n+1} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{2/3}} = -2x_n$, so we get that $|x_n| = 2^{n-1}x_1 \to \infty$ as $n \to \infty$ when we pick $x_1 \neq 0$. In fact, it oscillates, since terms of one parity $\to \infty$ and the rest $\to -\infty$.

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We take the hint given in the book.

Defining $M_0 = \sup |f(x)|$ and $M_1 = \sup |f'(x)|$, the asumption that $|f'(x)| < A|f(x)| \implies \sup |f'(x)| \leq \sup |Af(x)| = A \sup |f(x)| \implies M_1 \leq AM_0$. The mean value theorem yields

$$|f'(x)| = \left|\frac{f(x) - f(a)}{x - a}\right| \leq M_1 \implies |f(x)| \leq M_1(x-a) \leq AM_0(x-a)$$

for $x > a$ since $f(a) = 0$. Now, pick $x \in [a, a + \frac{1}{2A}]$. Then, $A(x-a) < A\frac{1}{2A} = \frac{1}{2} < 1$, such that we get $|f(x)| \leq \frac{1}{2}M_0$ and thus $M_0 \leq \frac{1}{2}M_0$, so since $M_0 = \sup |f(x)| \geq 0$, $M_0 = 0$, so $|f(x)| = 0 \implies f(x) = 0$ on $[a, a + \frac{1}{2A}]$.

Now, we can repeat this process on the interval $[a + \frac{1}{2A}, a + \frac{1}{A}]$ to show that f vanishes there as well. This covers all of $[a, b]$ in at most $\lceil 2A(b-a) \rceil$ steps.