MATH 4061 HW 2

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Ch 2, Q9

a

We have that any $p \in E^\circ$ is an interior point of E, by the definition of E°. By the definition of interior point, there is some $r > 0$ such that $B_r^\circ(p) \subset E$. Take any point q in $B_r^\circ(p)$, and note that we have $B_{r-d(p,q)}^\circ(q) \subset B_r^\circ(p)$, as any $x \in B_{r-d(p,q)}^\circ(q)$ satisfies that $d(x,q) < r - d(p,q) \implies d(x,q) + d(q,p) < r \implies d(x,p) < r \implies x \in B_r^\circ(p)$.

Then, we have that q is also an interior point of E, so $q \in E^\circ$, so $B_r^\circ(p) \subset E^\circ$ so E° is open as every point in E° is an interior point of E°.

b

This is simply the definition of open sets; recall that E is defined to be open if all of its points are internal points, so we have ($\implies$) immediately from the definition, as E open $\implies \forall p \in E$, p is interior $\implies p \in E^\circ$, so $E^\circ \subset E$ and $E \subset E^\circ$, so $E = E^\circ$.

If $E^\circ = E$, then no point of E is not an interior point by defintition of (E°), so all $p \in E$ are interior points, so E is open, so we get ($\impliedby$).

c

Since $G \subset E$, if $B_r^\circ(p) \subset G$, then $B_r^\circ(p) \subset E$, so any interior point of G is an interior point of E and therefore a member of E° (so $G^\circ \subset E^\circ$). But since G is open, we have by the last part that $G = G^\circ$, so we have $G = G^\circ \subset E$.

Ch 2, Q11

$d_1(-1,1) = 2^2 > 2 = d_1(-1,0) + d_1(0,1)$, so d_1 fails the triangle inequality.

For the rest of the problems, note that

$$|x-y| = \begin{cases} -(x-y) & x-y<0 \\ x-y & x-y>0 \\ 0 & x-y=0 \end{cases} = \begin{cases} y-x & y-x>0 \\ -(y-x) & y-x<0 \\ 0 & y-x=0 \end{cases} = |y-x|$$

This also shows $|x-y|$ satisfies $|x-y|=0$ if $x=y$ and $|x-y|>0$ otherwise.

This immediately gives that d_2 is symmetric, and also $d_2(x,y)>0$ for $x\neq y$ and $d_2(x,x)=0$.

Then, we can check that the triangle inequality holds since

$$\sqrt{|x-y|} \leq \sqrt{|x-r|} + \sqrt{|r-y|} \iff |x-y| \leq |x-r| + |r-y| + 2\sqrt{|x-r||r-y}$$

However, we know that $2\sqrt{|x-r||r-y|} \geq 0$, and Rudin shows that $|x-y|$ satisfies the triangle inequality earlier in the book, so the right hand side holds, so the left hand side holds, so d_2 is a valid metric.

For d_3, note that $d_3(1,-1) = |1^2-(-1)^2| = 0$, so d_3 is not a metric.

For d_4, note that $d_4(1,1/2) = |1-1| = 0$, so d_4 is not a metric.

Since we know that $|x-y|$ is symmetric, we have that $\frac{|x-y|}{1+|x-y|} = \frac{|y-x|}{1+|y-x|}$, and since $|x-y|>0$ for $x\neq y$, and $1+|x-y|>1$, $d_5(x,y)>0$ for $x\neq y$. Similarly, we have that $d_5(x,x) = 0/1 = 0$.

The last thing to check is the triangle inequality: we have that in general, for some metric $d(x,y)$, we have that

$$\frac{d(x,y)}{1+d(x,y)} \leq \frac{d(x,r)}{1+d(x,r)} + \frac{d(r,y)}{1+d(r,y)}$$

Multiplying by $(1+d(x,y))(1+d(x,r))(1+d(t,y))>0$,

$$d(x,y)(1+d(x,r))(1+d(r,y)) \leq d(x,r)(1+d(x,y))(1+d(r,y)) + d(r,y)(1+d(x,y))(1+d(x,r))$$

Expanding,

$$\begin{aligned} d(x,y) &+ d(x,y)d(x,r) + d(x,y)d(r,y) \\ &+d(x,y)d(x,r)d(r,y) \leq d(x,r) + d(x,y)d(x,r) + d(x,r)d(r,y) + d(x,y)d(x,r)d(r,y) \\ &\qquad + d(r,y) + d(x,y)d(r,y) + d(x,r)d(r,y) + d(x,y)d(x,r)d(r,y) \end{aligned}$$

Which finally leaves us with

$$d(x,y) \leq d(x,r) + d(r,y) + 2d(x,r)d(r,y) + 2d(x,y)d(x,r)d(r,y)$$

Since we have that $d(x,y) \leq d(x,r)+d(r,y)$ since $d(x,y)$ is a metric, and $d(x,r), d(r,y)$ are positive, so the last inequality holds we have that $\frac{d(x,y)}{1+d(x,y)}$ obeys the triangle inequality. In particular, taking $d(x,y)=|x-y|$, which was shown to be an metric earlier in the book, shows that d_5 obeys the triangle inequality and is thus a metric.

Ch 2, Q22

We can show that $\mathbb{Q}^k \subset \mathbb{R}^k$ is dense. In particular, we already know this for $k = 1$, and we can use that to bootstrap to k dimensions. For any $r > 0$ and $p = (p_1, p_2, \dots, p_k) \in \mathbb{R}^k$, we can see that for any point $q = (q_1, q_2, \dots, q_k) \in B_r^\circ(p)$, q satisfies the condition that

$$\left(\sum_{i=1}^{k}(p_i - q_i)^2\right)^{\frac{1}{2}} < r \iff \sum_{i=1}^{k}(p_i - q_i)^2 < r^2$$

Then, we have that we if can take q_i such that $(p_i - q_i)^2 < \frac{r^2}{k}$ for each $1 \leq i \leq k$, then $q \in B_r^\circ(p)$. However, since there is a rational between any two real numbers, as shown in chapter one, we have that $\exists q_i$ rational in the interval $(p_i - r/\sqrt{k}, p_i + r/\sqrt{k})$, which then satisfies $(p_i - q_i)^2 < \frac{r^2}{k}$, so these q_i satisfy $q \in B_r^\circ(p)$, and $q \in \mathbb{Q}^k$, so $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$.

We know that this is countable since the is the finite Cartesian product of a countable set, which was shown to be countable in class.

Ch 2, Q23

Since X is seperable, we know that it contains a dense subset. Call this dense subset E. Consider the set

$$\{V_\alpha\} := \{B_r^\circ(p) \mid r \in \mathbb{Q} \setminus \{0\}, p \in E\}$$

First, we can show that this is countable; consider the function that maps $f : \{V_\alpha\} \to \mathbb{Q} \setminus \{0\} \times E$ defined by $B_r^\circ(p) \mapsto (r, p)$. This is easily a bijection; if $f(B_r^\circ(p)) = f(B_{r'}^\circ(p')) \implies r = r', p = p'$, then $B_r^\circ(p) = B_{r'}^\circ(p)$ as sets; similarly, every (r, p) is hit by the open set $B_r^\circ(p)$, so f is surjective and injective. Since $\{V_\alpha\}$ is bijective to the Cartesian product of countable sets, it itself is countable.

To see that it is a base, consider any open $G \subset X$. Then, pick any $x \in G$. In particular, since we have that G is open, for some real $r > 0$, $B_r^\circ(x) \subset G$ as x is an interior point. Further, since E is a dense subset of X, x is a limit point of E so by definition, $B_{r/2}^\circ(x)$ contains some $p \in E$. Then, since there is a rational between any two real numbers, we have some r' rational in the interval $(d(x, p), r/2)$, so $B_{r'}^\circ(p)$ contains x. Then, we also have that $B_{r'}^\circ(p) \subset B_r^\circ(x)$, as for any y, $d(x, y) \leq d(x, p) + d(p, y) < r/2 + r/2 = r$. Since we have that $B_{r'}^\circ(p)$ is in $\{V_\alpha\}$, we have that $\{V_\alpha\}$ is a countable base for X.

Ch 2, Q24

We can construct a countable dense subset of X. First, fix some $\delta > 0$, and pick any arbitrary $x_1 \in X$. Then, given $x_1, \dots, x_j$, we choose x_{j+1} such that $d(x_{j+1}, x_i) \geq \delta$ for $1 \leq i \leq j$, until no such element exists anymore. This cannot go on forever; if it did, we would have an infinite subset of X, $X' = \{x_1, x_2, \dots\}$ such that for any $p \in X$, $B^\circ_{\delta/2}(p) \cap (X' \setminus \{p\})$ is either $\emptyset$ or $\{x_i\}$ for some $i \in J$. If there are two distinct elements $x_{i,1}, x_{i,2} \in B^\circ_{\delta/2}(p) \cap (X' \setminus \{p\})$, then $d(x_{i,1}, x_{i,2}) \leq d(x_{i,1}, p) + d(p, x_{i,2}) < \delta$, so $\Rightarrow\!\Leftarrow$. In the first case that $B^\circ_{\delta/2}(p) \cap (X' \setminus \{p\}) = \emptyset$, then p is not a limit point of X'. In the second case that $B^\circ_{\delta/2}(p) \cap (X' \setminus \{p\}) = \{x_i\}$, then we have that $B^\circ_{d(p,x_i)}(p) \cap (X' \setminus \{p\}) = \emptyset$, which still shows that p is not a limit point of X'. Either way, no point can be a limit point of X', so the sequence cannot go on forever.

Now, for any given $\delta > 0$, denote a sequence constructed by the above process as X_δ, and consider the set

$$E = \bigcup_{n=1}^{\infty} X_{1/n}$$

As the countable union of finite sets, we have that E itself is countable, as shown in class. Now, given any $x \in X$ and $r > 0$, we have that there is some rational p/q such that $0 < p/q < r$. Then, consider $X_{1/q} \subset E$. The intersection $B^\circ_r(x) \cap (X_{1/q} \setminus \{x\})$ must be nonempty. If it were empty, then we have that there are no elements of $X_{1/q}$ within $r > 1/q$ of x; however, since the construction of $X_{1/q}$ only terminates once we have no choices of x_{j+1} such that $d(x_{j+1}, x_i)$ for $1 \leq i \leq j$, and this x would be a suitable choice of x_{j+1}, we have $\Rightarrow\!\Leftarrow$, and so the intersection is empty. This gives that any point of X is a limit point of E, so E is dense in X.