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Homework 1

Problem 1

Problem 2

a

We will first show that $Id_V - T$ is linear.

$$\begin{aligned}
(Id_v - T)(cx) &= Id_V(cx) - T(cx) \\
&= cx - cT(x) \\
&= c(Id_V(x)) - cT(x) \\
&= c(Id_V(x) - T(x)) \\
&= c((Id_v - T)(x)) \\
(Id_v - T)(x+y) &= Id_V(x+y) - T(x+y) \\
&= x + y - T(x) - T(y) \\
&= Id_V(x) - T(x) + Id_V(y) - T(y) \\
&= (Id_V - T)(x) + (Id_V - T)(y)
\end{aligned}$$

It has an inverse, namely $Id_v + T + T^2$:

$$\begin{aligned}
(Id_v + T + T^2)((Id_v - T)(x)) &= Id_v(x - T(x)) + T(x - T(x)) + T(T(x - T(x))) \\
&= x - T(x) + T(x) - T(T(x)) + T(T(x)) - T(T(T(x))) \\
&= x
\end{aligned}$$

Via the theorem proved in class, we have that $Id_V - T$ is an isomorphism.

Problem 3

Claim. $\{\sin(x), \sin(2x), ..., \sin(2^n x), ...\}$ is linearly independent.

Proof. Suppose that we have some linear combination $\sum_{i=0}^{n} a_i sin(2^i x) = 0$. Consider $x = \frac{\pi}{2^{k+1}}$, where k is the least integer such that $a_k \neq 0$.

Then, we have that $\sin(\frac{2^i \pi}{2^{k+1}}) = \sin(2^{i-k-1}\pi) = 0$ for any $i > k$; for any $i < k$, we have that $a_i = 0$; for $i = k$, we have that $\sin(\frac{2^k \pi}{2^{k+1}}) = \sin(\frac{\pi}{2}) = 1$. $\square$

Problem 4

Claim. $\{1, 1+x, 1+x+x^2, ..., 1+x+x^2+...+x^n, ...\}$ is linearly independent.

Proof. We will show that $\sum_{i=0}^{n} a_i \sum_{j=0}^{i} x^j = \sum_{i=0}^{n}(x^i \sum_{j=i}^{n} a_j)$ through induction on n. The base case, which has $n = 0$, follows immediately as $\sum_{i=0}^{0}(a_i \sum_{j=0}^{i} x^j) = a_0$.
Now assume the above hypothesis for $n = k$. Then,

$$\begin{aligned}
\sum_{i=0}^{k+1}(a_i \sum_{j=0}^{i} x^j) &= \sum_{i=0}^{k}(a_i \sum_{j=0}^{i} x^j) + a_{k+1} \sum_{j=0}^{k+1} x^j \\
&= \sum_{i=0}^{k}(x^i \sum_{j=i}^{k} a_j) + a_{k+1} \sum_{j=0}^{k+1} x^j \\
&= \sum_{i=0}^{k}(x^i \sum_{j=i}^{k+1} a_j) + a_{k+1} \\
&= \sum_{i=0}^{k+1}(x^i \sum_{j=i}^{k+1} a_j)
\end{aligned}$$

Since we have from earlier that a polynomial is zero everywhere if and only if all of its coefficients are zero, we have that all of $\sum_{j=i}^{k+1} a_j$ are zero for $0 \leq i \leq n$. $\square$