MATH 4061 HW 4

David Chen, dc3451

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1

Note the reverse triangle inequality: $||x| - |y|| \leq |x - y|$, which follows immediately from the normal triangle inequality with $z = x - y$:

$$|z + y| \leq |z| + |y| \implies |x| \leq |x - y| - |y| \implies |x| - |y| \leq |x - y|$$

Then, we have that if a sequence converges normally to s, then for every $\epsilon > 0$ there is some N that for any $n > 0$, $|s_n - s| < \epsilon$, but the reverse triangle inequality then gives $||s_n| - |s|| < |s_n - s| < \epsilon$, so $\{|s_n|\}$ must also converge to $|s|$.

The converse is not true; $\{(-1)^n\}_{n=1}^{\infty}$ does not converge (this is apparent with $\epsilon = 1$), but $\{|(-1)^n|\}_{n=1}^{\infty} = \{1\}_{n=1}^{\infty}$ does converge to 1.

3

We will use that $\sqrt{x} > \sqrt{y} \iff x > y$ for positive x, y, which is something shown way earlier in the course.

We can actually show that the sequence $\{s_n\}_{n=1}^{\infty}$ is monotonic and bounded. In particular, we can show for any n that $0 < s_n < s_{n+1} < 2$. To see this, induct on n; the base case $n = 1$ is obvious, as $0 < \sqrt{2} < \sqrt{2 + \sqrt{2}} < \sqrt{4}$, as we have that $0 < \sqrt{2} < 2$. Then, if this holds for some n, we have that we have that

$$0 < s_n < s_{n+1} < 2 \implies 0 < 2 + \sqrt{s_n} < 2 + \sqrt{s_{n+1}} < 2 + 2 \implies 0 < s_{n+1}^2 < s_{n+2}^2 < 2^2$$

which finally gives that $0 < s_{n+1} < s_{n+2} < 2$ since every term of the sequence is positive as the square root of a real number, which shows the inequality for $n + 1$. By induction, this shows it for all integers ≥ 1, and since every monotonic and bounded sequence converges, this sequence also converges and is bounded above by 2.

16

a

Consider that

$$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - x_n = \frac{1}{2}\left(\frac{\alpha}{x_n} - x_n\right) = \frac{\alpha - x_n^2}{2x_n}$$

but $x_n > \sqrt{\alpha} \implies x_n^2 > \alpha$, so we get that $\frac{\alpha - x_n^2}{2x_n} = x_{n+1} - x_n < 0$, so x_n monotonically decreases. Furthermore, have that

$$x_{n+1} - \sqrt{\alpha} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \alpha = \frac{1}{2}\left(x_n - 2\sqrt{\alpha} + \frac{\alpha}{x_n}\right) = \frac{1}{2}\left(\sqrt{x_n} - \frac{\sqrt{\alpha}}{\sqrt{x_n}}\right)^2 > 0$$

so every term satisfies $x_n > \sqrt{\alpha}$ (the only one this does not cover is x_1, but this is explicitly chosen greater than $\sqrt{\alpha}$).

Then this sequence converges to some limit as it is monotone and bounded. Fix some $\epsilon > 0$. Since the sequence converges, say to L, there is some N such that for $n \geq N$, $|x_n - L| < \epsilon/2$. Then, we have that

$$|x_{n+1} - x_n + (L - L)| \leq |x_n - L| + |x_{n+1} - L| < \epsilon$$

so we have that $x_{n+1} - x_n$ converges to $L - L = 0$. Then, since $x_{n+1} - x_n = \frac{1}{2}\left(\frac{\alpha}{x_n} - x_n\right)$ as computed above, we have that

$$0 = \lim_{n\to\infty} \frac{1}{2}\left(\frac{\alpha}{x_n} - x_n\right) = \frac{1}{2}\left(\lim_{n\to\infty} \frac{\alpha}{x_n} - \lim_{n\to\infty} x_n\right) = \frac{1}{2}\left(\frac{\alpha}{L} - L\right)$$

so $L = \frac{\alpha}{L} \implies L^2 = \alpha \implies L = \sqrt{\alpha}$ since we have that $L > 0$ as the limit of a strictly positive sequence.

b

If we put $\epsilon_n = x_n - \sqrt{a}$, we computed in the last part that $x_{n+1} - \sqrt{\alpha} = \frac{1}{2}\left(\sqrt{x_n} - \frac{\sqrt{\alpha}}{\sqrt{x_n}}\right)^2$, so

$$e_{n+1} = x_{n+1} - \sqrt{\alpha} = \frac{1}{2}\left(\sqrt{x_n} - \frac{\sqrt{\alpha}}{\sqrt{x_n}}\right)^2 = \frac{1}{2}\left(\frac{x_n - \sqrt{\alpha}}{\sqrt{x_n}}\right)^2 = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{a}}$$

Then, inducting on n, we can show that $e_{n+1} < \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^n}$ where $\beta = 2\sqrt{\alpha}$. In particular, for $n = 1$, we have that $\epsilon_2 < \frac{\epsilon_1^2}{\beta}$, which is the same statement as in the last step. Then, if this holds for some n, then

$$\epsilon_{n+2} < \frac{\epsilon_{n+1}^2}{\beta} < \frac{(\beta\,(\epsilon_1/\beta)^{2^n})^2}{\beta} = \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^{n+1}}$$

which was what we wanted.

c

As before, we have that $\epsilon_1/\beta = (2-\sqrt{3})/(2\sqrt{3}) = \frac{1}{\sqrt{3}} - \frac{1}{2} = \frac{2\sqrt{3}-3}{6}$. However, we have that $1.5 < \sqrt{3} < 1.8$ since $1.5^2 < 3 < 1.8^2$, so $0 < \epsilon_1/\beta < \frac{0.6}{6} = \frac{1}{10}$. Thus, since $\beta = 2\sqrt{3}$ and $\sqrt{3} < 2$, $\beta < 4$, so we have that

$$\epsilon_5 < \beta 10^{-2^4} < 4 \cdot 10^{-16}$$
$$\epsilon_6 < \beta 10^{-2^5} < 4 \cdot 10^{-32}$$

20

Fix $\epsilon > 0$. The sequence is Cauchy, so we have that for $\epsilon/2 > 0$, there is some N such that $m, n \geq N \implies d(x_m, x_n) < \epsilon/2$. Similarly, we have that since the subsequence $\{p_{n_i}\}$ converges, $d(p, p_{n_k}) < \epsilon/2$ for $k \geq K$ for some K. Let p_{n_K} be the M term in the original sequence. Then, take $N' = \max(N, M)$, so we have that for $n > N'$,

$$d(p, p_n) \leq d(p, p_M) + d(p_M, p_n) < \epsilon/2 + \epsilon/2 = \epsilon$$

21

Consider the following sequence: pick p_n such that $p_n \in E_n$. Then, we have that each E_n contains infinite points in the sequence, namely the subsequence starting that the n^{th} index $p_n, p_{n+1}, \ldots$, which was shown earlier in the book to be Cauchy if and only if $\lim_{n\to\infty} \operatorname{diam}\{p_i\}_{i=n}^{\infty} = 0$. In particular, since we have that $\{p_i\}_{i=1}^{\infty} \subset E_n$, we have that $\operatorname{diam}\{p_i\}_{i=n}^{\infty} \leq \operatorname{diam} E_n$, so

$$\lim_{n\to\infty} \operatorname{diam} E_n = 0 \implies \lim_{n\to\infty} \operatorname{diam}\{p_i\}_{i=n}^{\infty} = 0$$

so the sequence is Cauchy. Then, since the metric space is complete, this is convergent to some point p, but since each E_n contains a subsequence $\{p_i\}_{i=n}^{\infty}$ which converges to p *and* E_n is closed and nonempty, we have that p is a limit point of E_n, and so $p \in E_n$ for every n. Thus, the intersection $\bigcap_{i=1}^{\infty} E_i$ is nonempty, as it contains at least p.

Then, if $E = \bigcap_{i=1}^{\infty} E_i$ contains more than one distinct point, say $p_1, p_2 \in E$ and $p_1 \neq p_2$, then $\operatorname{diam} E \geq d(p_1, p_2) > 0$. But since $E \subset E_n$ for any E_n, this means that $\lim_{n\to\infty} \operatorname{diam} E_n \geq \operatorname{diam} E \geq d(p_1, p_2) > 0$, so $\Rightarrow\!\Leftarrow$. Thus, the intersection contains exactly one point.