# MATH 4062 HW 1

David Chen, dc3451

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## 2

Let $\{f_n\}, \{g_n\}$ converge uniformly to $f$ and $g$ respectively (on some set $E$). Then, we have that $\exists N_f$ such that $n > N_f \implies |f_n(x) - f(x)| < \epsilon/2$ for any $\epsilon > 0$ and any $x \in E$; similarly, there is some $N_g$ for which $n > N_g \implies |g_n(x) - f(x)| < \epsilon/2$ as well; then, for $n > \max\{N_f, N_g\}$, we have have

$$|f(x) + g(x) - f_n(x) - g_n(x)| \leq |f(x) - f_n(x)| + |g(x) - g_n(x)| < \epsilon$$

so $\{f_n + g_n\}$ converges uniformly to $f + g$.

First, if we have some sequence of bounded functions $\{f_n\}$ (let $f_n$ be bounded by $M_n$) converging uniformly on $E$ to $f$, then $\exists N$ such that $m, n > N \implies |f_m(x) - f_n(x)| < 1$ and $|f_n - f(x)| < 1$, then picking $n = N + 1$ we have that $f(x) < 1 + M_{N+1}$, and in particular also that $M_m < 1 + M_{N+1}$ for all $m > N$. Thus, taking $M = \max\{M_1, \ldots, M_N, 1 + M_{N+1}\}$, we get that $f(x) < M$ for all $x \in E$ (and also that $f_n(x) < M$ for any $n$)

Now, if $\{f_n\}, \{g_n\}$ are sequences of bounded functions, let them be uniformly bounded by $M_f$ and $M_g$, and take $M = \max M_f, M_g$, and $N$ sufficiently large that for $n > N$, $|f(x) - f_n(x)| < \epsilon/2M$ and $|g(x) - g_n(x)| < \epsilon/2M$. Then, for $n > N$,

$$\begin{aligned}
|f(x)g(x) - f_n(x)g_n(x)| &= |f(x)g(x) - f(x)g_n(x) + f(x)g_n(x) - f_n(x)g_n(x)| \\
&\leq |f(x)g(x) - f(x)g_n(x)| + |g_n(x)f(x) - g_n(x)f_n(x)| \\
&= f(x)|g(x) - g_n(x)| + g_n(x)|f(x) - f_n(x)| \\
&= M(|g(x) - g_n(x)| + |f(x) - f_n(x)|) < \epsilon
\end{aligned}$$

## 3

Let $f_n(x) = x$ and $g_n(x) = x + \frac{1}{n}$ on all of $\mathbb{R}$. Clearly $f_n(x)$ converges uniformly to $x$ (take $N$ to be your favorite positive integer and for all $n > N$, $|f_n - f| = 0 < \epsilon$), and $g_n(x)$ to $x$

as well (fixing $\epsilon > 0$, take $N > 1/\epsilon$, and we have $n > N \implies |g_n(x) - g(x)| = |1/n| < \epsilon$). Then, $f_n(x)g_n(x) = x^2 + \frac{x}{n}$; now, we have that for any $n$, $|f_n(x)g_n(x) - f(x)g(x)| = |x/n|$, so for any $\epsilon > 0$, we can take $x > n\epsilon$ such that $|f_n(x)g_n(x) - f(x)g(x)| > \epsilon$. Thus, there cannot be some $N$ such that $n > N \implies |f_n(x)g_n(x) - f(x)g(x)| < \epsilon$ for all $\epsilon > 0$ (in particular, consider $\epsilon = 1$ and $x = n = N + 1$).

# 4

Put $f_n$ for the $n^{th}$ partial sum.

First, we rule out $x = -\frac{1}{n^2}$ for $n \in \mathbb{Z}_{>0}$, since in these cases the $n^{th}$ term is not defined. Otherwise, $f(x)$ converges absolutely for $x \neq 0$ since in that case $\frac{1}{1+n^2x} < \frac{1}{xn^2}$ which converges aboslutely. This gives that $f$ cannot converge uniformly on any interval containing 0, and also is unbounded at the origin.

Now, consider any interval with 0 as a limit point. If it lies in the positive half line, then it contains some subinterval $(0, \delta)$, on which we have $\sum_{i=1}^{n} \frac{1}{1+nx^2} < \sum_{i=1}^{n} \frac{1}{1} = n$, so each partial sum is bounded, and from the earlier problem, if $f$ converges abosultely on this interval, $f$ must be bounded as well; $\Rightarrow\Leftarrow$.

For intervals on the negative half line, it contains some subinterval $(-\delta, 0)$; however, consider the Cauchy criterion. We have that for $x = -\frac{1}{2n^2}$, $|f_n(x) - f_{n-1}(x)| = |\frac{1}{1-\frac{1}{2}}| = 2$, so for large $n$, $x = -\frac{1}{2n^2} \in (-\delta, 0)$, and so $f$ cannot be uniformly convergent on these intervals.

Now, for intervals bounded away from 0, we have that they must be subintervals of either $[\delta, \infty)$ or $(-\infty, -\delta]$. In the first case, the Weierstrass M-test yields that since $\frac{1}{1+n^2x} < \frac{1}{1+\delta n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{1+\delta n^2}$ is convergent, $f$ is uniformly convergent on $[\delta, \infty)$. Similarly, for $(-\infty, -\delta]$, we have (for $n > \sqrt{1/\delta}$)

$$\left|\frac{1}{1+n^2x}\right| = \frac{1}{n^2x - 1} < \frac{1}{n^2\delta - 1}$$

which has $\sum_{n}^{\infty} \frac{1}{n^2\delta - 1}$ convergent, so $f$ is uniformly convergent here as well.

Uniform convergence gives continuity where $f$ is defined (that is, $x \neq -1/n^2, 0$).

# 6

First, to see that it does not converge absolutely for any value of $x$, consider that

$$\frac{x^2+n}{n^2} \geq \frac{n}{n^2} = \frac{1}{n}$$

so by comparison to the harmonic series, $\sum_{n=1}^{\infty} \frac{x^2+n}{n^2}$ diverges.

Pick any bounded interval with end points $a, b$, with $a < b$. Then, we have that we can split the sum as

$$\sum_{n=1}^{\infty}(-1)^n \frac{x^2}{n^2} + \sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2}$$

where the left hand side converges uniformly via the Weierstrass M-test, with $\frac{x^2}{n^2} < \frac{\max\{|a|,|b|\}^2}{n^2}$, and $\sum_{n=1}^{\infty} \frac{\max\{|a|,|b|\}^2}{n^2}$ converges. Then, $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ is just some constant, so the entire sum converges uniformly (via the first problem on this HW).

# 7

The point here is that $f_n \to 0$ uniformly. In particular, we have that

$$\frac{d}{dx}\frac{x}{1+nx^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

so the maximal value of $|f_n|$ is at $1/\sqrt{n}$, so $|f_n| < \frac{1}{2\sqrt{n}}$, so for any $\epsilon > 0$, pick $N > \frac{1}{4\epsilon^2}$ and all $n > N$ satisfies that $|f_n(x) - 0| < \epsilon$ for any $x \in \mathbb{R}$. Then, we have that for $x \neq 0$,

$$\frac{1-nx^2}{(1+nx^2)^2} \to 0$$

as $n \to \infty$, so $f'(x) = 0 = \lim_{n\to\infty} f_n'(x)$, but for $x = 0$, $f_n'(0) = 1 \neq 0$.

# 9

Fix some $\epsilon > 0$. Then there is some $N_1$ such that $n > N_1$ yields $|f_n(x_n) - f(x_n)| < \epsilon/2$, and since $f$ is continuous at $x$, we have that for some $N_2$ such that $n > N_2$ yields $|f(x_n) - f(x)| < \epsilon/2$; then, for $n > \max\{N_1, N_2\}$,

$$|f_n(x_n) - f(x)| = |f_n(x_n) - f(x_n) + f(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \epsilon$$