dc3451
David Chen

Homework 2

MATH 1207
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Problem 1

Claim. For sets A, B, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Proof.

$$\begin{aligned} x \in \mathcal{P}(A) \cup \mathcal{P}(B) &\implies (x \in \mathcal{P}(A)) \vee (x \in \mathcal{P}(B)) \\ &\implies (x \subseteq A) \vee (x \subseteq B) \\ &\implies (x \subseteq (A \cup B)) \vee (x \subseteq (A \cup B)) \quad (1) \\ &\implies x \subseteq (A \cup B) \\ &\implies x \in \mathcal{P}(A \cup B) \end{aligned}$$

Note that (1) relies that for sets A, B, C, $A \subseteq B$ and $B \subseteq C$ implies that $A \subseteq C$. This follows from $x \in A \implies x \in B \implies x \in C$, so $\forall x \in A, x \in C \implies A \subseteq C$. □

Equality is not true in general. Take $A = \{\emptyset\}, B = \{\{\emptyset\}\}$. $\{\emptyset, \{\emptyset\}\} \in \mathcal{P}(A \cup B)$, but is not in either $\mathcal{P}(\mathcal{A})$ nor $\mathcal{P}(B)$. In fact, equality holds if and only if at least one of A and B is the empty set.

Problem 2

Claim. For sets A, B, $A \neq B \implies \mathcal{P}(A) \neq \mathcal{P}(B)$.

Proof. For sets A, B,

$$\begin{aligned} A \neq B &\iff A \neq B \\ &\iff \neg((A \subseteq B) \wedge (B \subseteq A)) \\ &\iff \neg(A \subseteq B) \vee \neg(B \subseteq A) \\ &\iff \neg(\forall x \in A, x \in B) \vee \neg(\forall x \in B, x \in A) \\ &\iff \exists x | (x \in A) \wedge (x \notin B) \vee \exists x | (x \notin A) \wedge (x \in B). \\ &\iff ((\{x\} \in \mathcal{P}(A)) \wedge (\{x\} \notin \mathcal{P}(B))) \vee ((\{x\} \notin \mathcal{P}(A)) \wedge (\{x\} \in \mathcal{P}(B))) \\ &\iff \mathcal{P}(A) \neq \mathcal{P}(B) \end{aligned}$$

Note that the conclusion of $\mathcal{P}(A) \neq \mathcal{P}(B)$ follows the same procedures as the opening of the proof, but in reverse. □

Problem 3

Claim. For sets A, B, C, $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof.

$$\begin{aligned}
(m,n) \in A \times (B \cap C) &\iff (m \in A) \wedge (n \in (B \cap C)) \\
&\iff (m \in A) \wedge ((n \in B) \wedge (n \in C)) \\
&\iff ((m \in A) \wedge (n \in B)) \wedge ((m \in A) \wedge (n \in C)) \qquad (1) \\
&\iff (m,n) \in A \times (C) \wedge (m,n) \in A \times C \\
&\iff (m,n) \in (A \times B) \cap (A \times C)
\end{aligned}$$

Note that (1) follows from the fact that $P^P = P$ for any statement P as well as the commutativity and associativity of $\wedge$. ☐

Problem 4

Claim. For injective functions $f : S \to T$ and $g : T \to U$, $g \circ f$ is also injective.

Proof.

$$\begin{aligned}
(g \circ f)(x) &= (g \circ f)(x') && (1) \\
\implies \quad g(f(x)) &= g(f(x')) && (2) \\
\implies \quad f(x) &= f(x') && (3) \\
\implies \quad x &= x' && (4)
\end{aligned}$$

Note that we can jump from (2) to (3) and (3) to (4) because g and f are injective, meaning that $\forall x, x', g(x) = g(x') \iff x = x'$. ☐

Problem 5

For functions $f : S \to T$ and $g : T \to S$ such that $g \circ f = id_S$, prove or disprove the following.

a

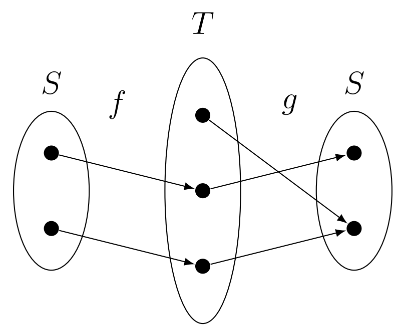
Claim. f is injective.

Proof. If $\exists s, s' \in S | f(s) = f(s')$, $g(f(s)) = id_S(s) = s$ and $g(f(s')) = id_S(s') = s'$. We then have that $g(f(s)) = g(f(s')) \implies s = s'$ as $f(s) = f'(s)$. ☐

b, c

Claim. f is surjective and g is injective.

This is false. Consider this counterexample to both:



d

Claim. g is surjective.

Proof. $\forall s \in S, (g \circ f)(s) = id_S(s) = s$. We also have that $(g \circ f)(s) = g(f(s))$, so that $\forall s \in S, \exists t = f(s) \in T | g(t) = s$. $\square$

Problem 6

A function $f : A \to B$ can be expressed as its graph $\Gamma(f)$, which is the just the set $\{(a, f(a)) | a \in A\}$. Then, all functions $f : A \to B$ are sets containing elements of the form $(a, b) \in A \times B$. This implies that $\Gamma(f) \in \mathcal{P}(A \times B)$, which exists by the axiom of power sets.

Note that we put (a, b) as shorthand for a set $\{\{a\}, \{a, b\}\}$, and that products such as $A \times B$ are defined as all such $(a, b), a \in A, b \in B$.

Then, by the axiom of specification, we pull out only the elements of $\mathcal{P}(A \times B)$ where it is a valid function graph.

$$\{\Gamma \in \mathcal{P}(A \times B) | (\forall a \in A, \exists (x, y) \in \Gamma \text{ s.t. } a = x) \wedge (\neg \exists a \in A \text{ s.t. } (a, y), (a, y') \in \Gamma, y \neq y')\}.$$

The set above is exactly B^A if we equate functions and their graphs.

Problem 7

There are m^n elements in B^A, as each element of A must be sent to an element in B, of which there are m. This is equivalent to making choosing 1 from m n times, so that there are m^n total choices for the function.

Problem 8

Proof. To show that W is surjective, we can furnish a function $f : A \to B$ such that $W(f) = A'$ for any set $A' \in \mathcal{P}(A)$. This function's graph is given exactly by

$$\Gamma(f) = \{(a,1)|a \in A'\} \cup \{(a,0)|a \in A \setminus A'\}.$$

In other words, for a function $f : A \to B$ such that

$$f(a) = \begin{cases} 1 & a \in A' \\ 0 & a \in A \setminus A' \end{cases}$$

To show that W is injective, we will show that if $f, g \in B^A$ satisfy $W(f) = W(g) = A' \in \mathcal{P}(A)$ then $f = g$. To show that $f = g$, we need to demonstrate that $\forall a \in A, f(a) = g(a)$. Since we have that $A' \cup A \setminus A' = A$ and $A' \cap A \setminus A' = \emptyset$, we have two cases for a.

Firstly, if $a \in A'$, then $a \in f^{-1}(\{1\}) \implies f(a) = f(f^{-1}(1)) = 1$. Similarly, $a \in g^{-1}(\{1\}) \implies g(a) = 1$ such that $f(a) = g(a)$.

Secondly, if $a \in A \setminus A'$, then $a \notin f^{-1}(\{1\}) \implies f(a) \neq f(f^{-1}(1)) \neq 1$. Similarly, $a \notin g^{-1}(\{1\}) \implies g(a) \neq 1$. However, both $f(a)$ and $g(a)$ are still members of B, meaning that they must be both equal to 0.

Thus, we have that $\forall a \in A, f(a) = g(a) \implies f = g$ and so W is injective. W is now shown to be both injective and surjective, so it must be bijective. $\square$