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Assignment 2

2.1.1

Since we have that $A \subset B$, $A \cap B = A$. Then,

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A)}{\Pr(B)}$$

2.1.4

Odds are parenthetical. He must start with A ($\frac{1}{2}$), then proceed to stay ($\frac{1}{3}$), switch ($\frac{2}{3}$), and stay ($\frac{1}{3}$). This leaves us with

$$\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{1}{27}$$

2.1.11

Note that since $A \cap B$ and $A^C \cap B$ are disjoint, as no x can be both in A and A^C, $\Pr(A \cap B) + \Pr\left(A^C \cap B\right) = \Pr\left((A \cap B) \cup (A^C \cap B)\right) = \Pr(B)$.

$$\Pr\left(A^C \mid B\right) = \frac{\Pr\left(A^C \cap B\right)}{\Pr(B)} = \frac{\Pr(B) - \Pr(A \cap B)}{\Pr(B)} = 1 - \frac{\Pr(A \cap B)}{\Pr(B)} = 1 - \Pr(A \mid B)$$

2.2.2

We have that $\Pr\left(A^C \cap B^C\right) = \Pr\left((A \cup B)^C\right) = 1 - \Pr(A \cup B)$ from De Morgan's, which expands to

$$\begin{aligned} 1 - (\Pr(A) + \Pr(B) - \Pr(A \cap B)) &= 1 - \Pr(A) - \Pr(B) + \Pr(A)\Pr(B) \\ &= (1 - \Pr(A))(1 - \Pr(B)) \\ &= \Pr\left(A^C\right)\Pr\left(B^C\right) = \Pr\left(A^C \cap B^C\right) \end{aligned}$$

This shows that A^C, B^C are independent.

2.2.12a

From before, we have that complements are also independent if the original events were independent; similarly, we have that if A, B are independent, as $A \cap B$ and $A \cap B^C$ are disjoint we have $\Pr\left(A \cap B^C\right) = \Pr(A) - \Pr(A \cap B) = \Pr(A)(1 - \Pr(B)) = \Pr(A)\Pr\left(B^C\right)$, so A, B^C are also independent.

The odds that none of the three occur:

$$(1 - \frac{1}{4})(1 - \frac{1}{3})(1 - \frac{1}{2}) = \frac{1}{4}$$

2.2.12b

The odds that exactly one occurs:

$$\frac{1}{4}(1-\frac{1}{3})(1-\frac{1}{2})+\frac{1}{3}(1-\frac{1}{4})(1-\frac{1}{2})+\frac{1}{2}(1-\frac{1}{4})(1-\frac{1}{3})=\frac{1}{12}+\frac{1}{8}+\frac{1}{4}=\frac{11}{24}$$

2.2.22

Claim. Suppose that A_1, A_2, B are events such that $\Pr(A_1 \cap B) > 0$. Then A_1, A_2 are conditionally independent given $B \iff \Pr(A_2 \mid A_1 \cap B) = \Pr(A_2|B)$.

Proof. ($\implies$) We already have that

$$\Pr(A_1 \cap A_2 \mid B) = \Pr(A_1 \mid B)\Pr(A_2 \mid B)$$

as we assume that A_1, A_2 are conditionally independent.
Expanding,

$$\begin{aligned}
&\Pr(A_1 \cap A_2 \mid B) = \Pr(A_1 \mid B)\Pr(A_2 \mid B)\\
\implies &\frac{\Pr(A_1 \cap A_2 \cap B)}{\Pr(B)} = \frac{\Pr(A_1 \cap B)}{\Pr(B)} \cdot \frac{\Pr(A_2 \cap B)}{\Pr(B)}\\
\implies &\frac{\Pr(A_1 \cap A_2 \cap B)}{\Pr(A_1 \cap B)} = \frac{\Pr(A_2 \cap B)}{\Pr(B)}\\
\implies &\Pr(A_2 \mid A_1 \cap B) = \Pr(A_2 \mid B)
\end{aligned}$$

($\impliedby$) We have that $\Pr(A_2 \mid A_1 \cap B) = \Pr(A_2 \mid B) \implies \frac{\Pr(A_1 \cap A_2 \cap B)}{\Pr(A_1 \cap B)} = \frac{\Pr(A_2 \cap B)}{\Pr(B)}$.

$$\begin{aligned}
\Pr(A_1 \mid B)\Pr(A_2 \mid B) &= \frac{\Pr(A_1 \cap B)}{\Pr(B)} \cdot \frac{\Pr(A_1 \cap B)}{\Pr(B)}\\
&= \frac{\Pr(A_1 \cap B)}{\Pr(B)} \cdot \frac{\Pr(A_2 \cap A_2 \cap B)}{\Pr(A_1 \cap B)}\\
&= \frac{\Pr(A_1 \cap A_2 \cap B)}{\Pr(B)}\\
&= \Pr(A_1 \cap A_2 \mid B)
\end{aligned}$$

□

2.3.4

Let E_1 = sick, E_2 = not sick, and A = a positive test.

$$\Pr(E_1 \mid A) = \frac{\Pr(E_1)\Pr(A \mid E_1)}{\Pr(E_1)\Pr(A \mid E_1) + \Pr(E_2)\Pr(A \mid E_2)} = \frac{0.00001 \cdot 0.95}{0.99999 \cdot 0.05} = 0.00019$$

2.3.8a

The odds of arriving at that configuration with coin n is $p_n(1-p_n)^3$, and let the event of having coin n have odds $\Pr(n) = \frac{1}{5}$. Then, Bayes' Theorem tells us that the posterior odds of having coin n are

$$\frac{\Pr(n)\, p_n(1-p_n)^3}{\sum_{i=1}^5 \Pr(n)\, p_i(1-p_i)^3} = \frac{p_n(1-p_n)^3}{\sum_{i=1}^5 p_i(1-p_i)^5}$$

Actually computing, we have that the posterior odds are $\Pr(1 \mid TTTH) = 0$, $\Pr(2 \mid TTTH) = 0.587$, $\Pr(3 \mid TTTH) = 0.347$, $\Pr(4 \mid TTTH) = 0.065$, and $\Pr(5 \mid TTTH) = 0$.

2.3.8a

This can be computed as $\sum_{i=1}^5 \Pr(i \mid TTTH)\, p_i(1-p_i)^2$, which comes out to be 0.129.

2.5.1

$$\begin{aligned}
& \frac{\Pr(A \cap D)}{\Pr(D)} \geq \frac{\Pr(B \cap D)}{\Pr(D)} \\
\implies & \Pr(A \cap D) \geq \Pr(B \cap D) \\
& \frac{\Pr\left(A \cap D^C\right)}{\Pr(D^C)} \geq \frac{\Pr\left(B \cap D^C\right)}{\Pr(D^C)} \\
\implies & \Pr\left(A \cap D^C\right) \geq \Pr\left(B \cap D^C\right) \\
\implies & \Pr(A \cap D) + \Pr\left(A \cap D^C\right) \geq \Pr(B \cap D) + \Pr\left(B \cap D^C\right) \\
\implies & \Pr(A) \geq \Pr(B)
\end{aligned}$$

2.5.23

Let E_1 = statistician, E_2 = economist, A = shy.

$$\Pr\left(E_1 \mid A\right) = \frac{\Pr\left(E_1\right)\Pr\left(A \mid E_1\right)}{\Pr\left(E_1\right)\Pr\left(A \mid E_1\right) + \Pr\left(E_2\right)\Pr\left(A \mid E_2\right)} = \frac{0.8 * 0.1}{0.8 * 0.1 + 0.15 * 0.9} = 0.372$$

2.5.29a

There are $\binom{52}{13}$ ways to choose 13 cards; $\binom{4}{n}\binom{48}{13-n}$ ways to choose 12 cards and n aces.

$$1 - \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13} - \binom{48}{13}} = 0.369$$

2.5.29b

Let $H =$ picking the ace of hearts, and $A =$ the number of aces. Then, $\Pr\left(H\right) = \frac{13}{52}$.

$$\begin{aligned}
\Pr\left(A \geq 2 \mid H\right) &= \frac{\Pr\left(A \geq 2 \cap H\right)}{\Pr\left(H\right)} \\
&= \frac{\Pr\left(H\right) - \Pr\left(A < 2 \cap H\right)}{\Pr\left(H\right)} \\
&= \frac{\Pr\left(H\right) - \Pr\left(A = 1 \cap H\right)}{\Pr\left(H\right)} \\
&= \frac{\Pr\left(H\right) - \frac{\binom{48}{12}}{\binom{52}{13}}}{\Pr\left(H\right)} = 0.561
\end{aligned}$$