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Assignment 7

8.1.1

We have from the previous chapter that the MLE of such a random sample is exactly $\max\{X_1, \dots, X_n\}$. Define $F(t \mid \theta)$ as follows:

$$\begin{aligned} F(t \mid \theta) &= P(\hat{\theta} \leq t \mid \theta) \\ &= P(\max\{X_1, \dots, X_n\} \leq t \mid \theta) \\ &= \left(\frac{t}{\theta}\right)^n \end{aligned}$$

Then, we are seeking to compute

$$P(|\hat{\theta} - \theta| \leq 0.1\theta) = F(1.1\theta) - F(0.9\theta) = 1 - F(0.9\theta) = 1 - (0.9)^n \geq 0.95$$

Computing, we get $n \geq 29$.

8.2.4

Here, we have that $\overline{X}_n$ is normally distributed with mean θ and variance $2/n$. Then, we have that $Z = \sqrt{n}(\overline{X}_n - \theta)/2$.

$$P(|\overline{X}_n - \theta| \leq 0.1) = P(|2Z/\sqrt{n}| \leq 0.1) = P(|Z| \leq 0.05\sqrt{n}) \geq 0.95$$

Computing, we arrive at $n = 1537$.

8.2.6

Consider the vairable $D^2 = X^2 + Y^2 + Z^2$. Then, we re looking for $P(D^2 \leq 16\sigma^2)$, but $D' = D^2/2\sigma^2$ is the χ^2 distribution with 3 degrees of freedom, so we arrive at $P(D' \leq 8) = 0.95$.

8.3.6

a

Let X have χ^2 distribution with 16 degrees of freedom. Then, we wish to find

$$P(n/2 \leq X \leq 2n) = P(8 \leq X \leq 32) = 0.94$$

b

Let X have χ^2 distribution with $16 - 1$ degrees of freedom. Then, we wish to find

$$P(n/2 \leq X \leq 2x) = 0.91$$

8.3.8

Consider X as $\sum_{i=1}^{200} X_i$, where each X_i is a χ^2 distribution with 1 degree of freedom, but the central limit theorem yields that X is normally distributed, with mean $\mu = 200, \sigma^2 = 400$. Then, $P(160 < X < 240) \approx P(-2 < Z < 2) = 0.954$ where Z is the standard normal distribution.

8.4.2

Put T as a variable with the $t-$distribution with $17 - 1 = 16$ degrees of freedom.

$$P(\hat{\mu} > \mu + k\hat{\sigma}) = P(\frac{\hat{X} - \mu}{\hat{\sigma}} > k) = P(T > k\sqrt{n-1}) = P(T > 4k) > 0.95$$

Computing, we get $k = -0.436$.

8.4.6

As above, we are looking, in terms of T with $t-$distribution with $20 - 1 = 19$ degrees of freedom,

$$P(\hat{\mu} > \mu + c\sigma') = P(T > c\sqrt{20})$$

Then, we want $c\sqrt{20} = 1.729 \implies c = 0.387$.

8.5.1

$$\begin{aligned}
&P\left(\overline{X}_n - \Phi^{-1}\left(\frac{1+\gamma}{2}\right)\frac{\sigma}{n^{1/2}} < \mu < \overline{X}_n + \Phi^{-1}\left(\frac{1+\gamma}{2}\right)\frac{\sigma}{n^{1/2}}\right)\\
=&P\left(-\Phi^{-1}\left(\frac{1+\gamma}{2}\right) < \frac{n^{1/2}(\mu - \overline{X}_n)}{\sigma} < \overline{X}_n + \Phi^{-1}\left(\frac{1+\gamma}{2}\right)\right)
\end{aligned}$$

However, the middle term is the standard normal distribution; therefore, the probability is then

$$\frac{1+\gamma}{2} - (1 - \frac{1+\gamma}{2}) = \gamma$$

8.5.5

We have that $\frac{\sum(X_i - \overline{X}_n)^2}{\sigma^2}$ has a χ^2 distribution with $n-1$ degrees of freedom, let $c_1 = C^{-1}(-\frac{1+\gamma}{2}), c_2 = C^{-1}(\frac{1+\gamma}{2})$, where C^{-1} is the inverse of the cdf of the corresponding χ^2

distribution. Then, the confidence interval

$$(\frac{\sum(x_i - \overline{X}_n)^2}{c_1}, \frac{\sum(x_i - \overline{X}_n)^2}{c_2})$$

will suffice.

8.6.1

Y must be the normal distribution with mean $a\mu + b$ and variance $\frac{a^2}{\tau}$, which is the same as a precision of $\frac{\tau}{a^2}$.

8.6.2

In chapter seven, we get the following relationships:

$$\begin{aligned}
\mu_1 &= \frac{(1/\tau)\mu_0 + n(1/\lambda_0)\overline{x}_n}{(1/\tau) + n(1/\lambda_0)} = \frac{\lambda_0\mu + n\tau\overline{x}_n}{\lambda_0 + n\tau} \\
v_1^2 &= \frac{(1/\tau)(1/\lambda_0)}{(1/\tau) + n(1/\lambda_0)} = \frac{1}{\lambda_0 + n\tau} \\
\tau_1 &= \frac{1}{v^2} = \lambda_0 + n\tau
\end{aligned}$$

8.6.3

With the theorems regarding the joint pdf, as well as the prior and posteriors in the text, we get that

$$\begin{aligned}
\xi(\tau \mid \mathbf{x}) &\propto f_n(\mathbf{x} \mid \tau)\xi\tau \\
&\propto \tau^{n/2} e^{-\frac{\tau}{2}\sum(x_i-\mu)^2} \tau^{\alpha_0 - 1} e^{-\beta_0\tau} \\
&= \tau^{\alpha_0 + n/2 - 1} e^{-tau(\beta_0 + \frac{1}{2}\sum(x_i-\mu)^2)}
\end{aligned}$$

which is the desired gamma distribution.

8.7.1

a

The desired variance of a Poisson distribution is also the mean of the distribution. Thus,

$$\sigma^2 = g(\theta) = \theta$$

b

The MLE is then derived as follows:

$$\begin{aligned}
f(x \mid \theta) &= \prod_{i=1}^{n} \frac{\theta^{x_i} e^{-\theta}}{x_i!} \\
\log(f) &= \sum_{i=1}^{n} (x_i \log(\theta) - \theta - \log(x_i!)) \\
\frac{df}{d\theta} &= \sum_{i=1}^{n} (\frac{x_i}{\theta} - 1) \\
&= \frac{\sum_{i=1}^{n} x_i}{\theta} - n = 0 \\
\theta &= \frac{\sum_{i=1}^{n} x_i}{n} = \overline{x}_n
\end{aligned}$$

Further, $E(\overline{x}_n) = \theta = E(X_i) = g(\theta)$, so it is unbiased.

8.8.1

$$\begin{aligned}
f(x \mid \mu) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \\
f'(x \mid \mu) &= -\frac{1}{\sqrt{2\pi\sigma^2}} \frac{x-\mu}{\sigma^2} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = \frac{x-\mu}{\sigma^2} f(x \mid \mu) \\
f''(x \mid \mu) &= \frac{1}{\sigma^2} f(x \mid \mu) + \frac{x-\mu}{\sigma^2} f'(x \mid \mu) = \left(\frac{1}{\sigma^2} - \left(\frac{x-\mu}{\sigma} \right)^2 \right) f(x \mid \mu) \\
\int_{-\infty}^{\infty} f'(x \mid \mu) dx &= \frac{1}{\sigma^2} \int_{-\infty}^{\infty} (x-\mu) f(x \mid \mu) dx = \frac{1}{\sigma^2} E(x-\mu) = \frac{1}{\sigma^2}(\mu - \mu) = 0 \\
\int_{-\infty}^{\infty} f''(x \mid \mu) dx &= \int_{-\infty}^{\infty} \frac{f(x \mid u)}{\sigma^2} dx - \int_{-\infty}^{\infty} \frac{(x-\mu)^2}{\sigma^4} f(x \mid \mu) = \frac{1}{\sigma^2} - \frac{E((x-\mu)^2)}{\sigma^4} = \frac{1}{\sigma^2} - \frac{1}{\sigma^2} = 0
\end{aligned}$$

8.8.3

$$\begin{aligned}
\lambda(x \mid \theta) &= x \log(\theta) - \theta - \log(x!) \\
\lambda''(x \mid \theta) &= -\frac{x}{\theta^2} \\
I(\theta) &= -E(\lambda''(X \mid \theta)) = \frac{E(X)}{\theta^2} = \frac{1}{\theta}
\end{aligned}$$

8.8.7

We first compute $I(P)$

$$\begin{aligned}
\lambda(x \mid P) &= x\log(p) + (1-x)\log(1-p) \\
\lambda''(x \mid P) &= -\frac{x}{p^2} - \frac{1-x}{(1-p)^2} \\
-E(\lambda''(X \mid \theta)) &= \frac{E(X)}{p^2} + \frac{1-E(X)}{1-p^2} \\
&= \frac{1}{p(1-p)}
\end{aligned}$$

Then, we know that $E(\overline{X_n}) = p, \mathrm{Var}\left(\overline{X_n}\right) = \frac{p(1-p)}{n}$. Then, we have that $\mathrm{Var}\left(\overline{X_n}\right) = \frac{1}{nI(P)}$ and is thus an efficient estimator.

8.8.17

$$\begin{aligned}
\lambda(x \mid P) &= \log(\binom{n}{x}) + x\log(p) + (n-x)\log(1-p) \\
\lambda''(x \mid P) &= -\frac{x}{p^2} - \frac{n-x}{(1-p)^2} \\
-E(\lambda''(X \mid \theta)) &= \frac{E(X)}{p^2} + \frac{n-E(X)}{1-p^2} \\
&= \frac{n}{p} + \frac{n-np}{(1-p)^2} \\
&= \frac{n}{p(1-p)}
\end{aligned}$$

8.9.15

$$\begin{aligned}
\lambda(x \mid \theta) &= \log(\theta) + (\theta-1)\log(x) \\
\lambda''(x \mid \theta) &= -\frac{1}{\theta^2} \\
I(\theta) &= \frac{1}{\theta^2}
\end{aligned}$$

Then the theorem on asymptotic normality yields that

$$\sqrt{nI(\theta)}(\hat{\theta}_n - \theta) = \frac{\sqrt{n}}{\theta}(\hat{\theta}_n - \theta)$$

is standard normal, and thus $\hat{\theta_n}$ is normal with mean θ and variance θ^2/n.

8.9.16

$$
\begin{aligned}
\lambda(x \mid \theta) &= -\log(\theta) - \frac{x}{\theta} \\
\lambda''(x \mid \theta) &= \frac{1}{\theta^2} - \frac{2x}{\theta^3} \\
I(\theta) &= -\left(\frac{1}{\theta^2} - \frac{2}{\theta^2}\right) = \frac{1}{\theta^2}
\end{aligned}
$$