MATH 4061 HW 1

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Ch 1, Q2

Euclid's lemma is that if $p \mid ab$, then $p \mid a$ or $p \mid b$. In particular, an easy corollary (the contrapositive) is that if $p \nmid a$ and $p \nmid b$, then we have that $p \nmid ab$.

Let $(p/q)^2 = 12$, where p, q are relatively prime, such that $\nexists n \in \mathbb{N}$ such that $n > 1$ and $n \mid p$, $n \mid q$.

Then, we have that $p^2 = 12q^2 = 3(4q^2)$. Clearly $3 \mid p^2$, and so $3 \mid p$ (if $3 \nmid p$, then $3 \nmid p^2$ as well as a consequence of the earlier stated corollary to Euclid's lemma). Then, if $p = 3k$, we have that $9k^2 = 3(4q^2) \implies 3k^2 = 4q^2$. Since $3 \mid 4q^2$, and $3 \nmid 4$, by Euclid's lemma we have that $3 \mid q^2$, and from the same logic as earlier, we have that $3 \mid q$.

Since we have that 3 divides both p and q, we have $\Rightarrow\Leftarrow$ as we earlier assumed that p, q were relatively prime.

Lemma

Now, we first show that for any set $X \subset \mathbb{R}$ bounded above there is an element $\epsilon-$close to $\sup X$ that is contained in X (that is, $\forall \epsilon > 0, \exists x \in X \mid x > \sup X - \epsilon$). To see this, note that if we no such element, then we would have $\sup X - \epsilon$ as a lower upper bound of X.

In particular, this shows that if $\sup(X) > y$, then $\exists x \in X \mid y < x \leq \sup(X)$.

Ch 1, Q6

a

First, for $a, b \geq 0$, we have that since the n^{th} root of a, b is distinct. This shows the following: if $a^n = b^n$, then the n^{th} root of $b^n = a^n$ is a, and the n^{th} root of $a^n = b^n$ is also b. Since these roots are unique, we must have that $a = b$.

For $n, m \in \mathbb{N}$, we see that $x^{nm} = x \cdot x \cdot \cdots \cdot x$ nm times. However, we also have that $(x^n)^m = (x \cdot x \cdot \cdots \cdot x) \cdot \cdots \cdot (x \cdot \cdots \cdot x)$, where each group in side the parentheses contains n x's and there are m groups, for a total of nm x's. However, since multiplication is associative, we have that $(x^n)^m$ is then the same as x^{nm}.

If $n, m \in \mathbb{Z}$, $nm < 0$, we have the same thing as above, but with x replaced by $1/x$, and we still have $(x^n)^m = x^{nm}$. This allows us to justify the following manipulations.

Consider now $((b^m)^{1/n})^{nq} = b^{mq}$ and $((b^p)^{1/q})^{nq} = b^{pn}$. However, we have that $m/n = p/q \implies mq = pn$, so we have that $b^{mq} = b^{pn}$. From the earlier statement about distinct n^{th} roots, this suggests that $(b^m)^{1/n} = (b^p)^{1/q}$.

b

Let $r = p_r/q_r$ and $s = p_s/q_s$. Then, we have that

$$b^{r+s} = b^{\frac{p_r q_s + p_s q_r}{q_s q_r}} = (b^{p_r q_s + p_s q_r})^{1/q_r q_s} = (b^{p_r q_s} b^{p_s q_r})^{1/q_r q_s}$$

The last step follows from multiplication commuting, and the next step distributing the exponent is justified by Rudin in the book:

$$(b^{p_r q_s} b^{p_s q_r})^{1/q_r q_s} = (b^{p_r q_s})^{1/q_r q_s} (b^{p_s q_r})^{1/q_r q_s} = (b^{\frac{p_r q_s}{q_r q_s}})(b^{\frac{p_s q_r}{q_r q_s}}) = b^{\frac{p_r}{q_r}} b^{\frac{p_s}{q_s}} = b^r b^s$$

c

First, we will show that for $b > 1, r > 0$ with $b \in \mathbb{R}, r \in \mathbb{Q}$ that $b^r > 1$. Put that $r = p/q$. Since $b > 1$, we have that $b^p > 1$ (Rudin claims $0 < y_1 < y_2 \implies y_1^n < y_2^n$ earlier).

Then, since we have that b^r is q^{th} root of b^p, and we know that for $0 < y \leq 1$ that $y^n \leq 1$, we have that $b^r > 1$ (otherwise, we would have that $(b^r)^q < 1$, but we have that $(b^r)^q = b^p > 1$).

Now, consider $r, s \in \mathbb{Q}$, and $r > s$. Then, we have that $b^r - b^s = b^s(b^{r-s} - 1)$. Since we have that $b^s > 0$, and that $b^{r-s} > 1$, $b^r - b^s > 0$, and so $b^r > b^s$.

We will show that b^r is an upper bound of $B(r)$ first. For any $b^t \in B(r)$, we have that $t \leq r \implies b^t \leq b^r$. Then, b^r is an upper bound of $B(r)$. To see that it is the least upper bound, note that $b^r \in B(r)$, such that any real number $< b^r$ is not an upper bound for $B(r)$.

d

We have that $b^{x+y} = \sup B(x+y)$. We will show that $b^x b^y = \sup B(x) \sup B(y)$ is the least upper bound of $B(x+y)$.

First, to see that it is in fact an upper bound, consider any $t \leq x+y$. We want to construct two rationals r, s such that $r \leq x$, $s \leq y$ and $r+s=t$.

I realized when I'm about to submit that there is an easier construction than what I originally did: pick some rational r in the interval $[t-y, x]$ (since we have $t - y \leq x$ and one such rational is guaranteed to exist by the density $\mathbb{Q}$ as shown by Rudin), and take $s = t - r$. Then, we have that $r < x$ and $s = t - r \leq t - (t-y) = y$ which gives r, s as desired. The boxed construction *should also* work, but it's irrelevant.

To do this, consider that the denseness of $\mathbb{Q}$ in $\mathbb{R}$ (as shown earlier by Rudin) gives some rational x' in the interval $[\frac{x-y+t}{2}, x]$ and some other rational y' in $[\frac{y-x+t}{2}, y]$. Note that $x' + y' \geq \frac{x-y+t}{2} + \frac{y-x+t}{2} = t$. Then, consider the quantities

$$r = x' - \frac{x'+y'-t}{2}, s = y' - \frac{x'+y'-t}{2}$$

Note that $r+s=t$, and since $r \leq x' \leq x, s \leq y' \leq y$, we have our desired r, s.

Now, we have that $b^t = b^r b^s$, with $b^t \in B(x+y), b^r \in B(x), b^s \in B(y)$. Now, we have that $\sup B(x) \geq b^r, \sup B(y) \geq b^s$, such that $\sup B(x) \sup B(y) \geq b^r b^s = b^t$. This shows that $\sup B(x) \sup B(y)$ is a upper bound.

Now consider any real a that is less than $\sup B(x) \sup B(y)$. Then, we have that $a < \sup B(x) \sup B(y)$. Then, by the earlier lemma, we have some $b^\beta \in B(x)$ such that $\alpha / \sup B(y) < b^\beta$, and now some $b^\gamma \in B(y)$ such that $\alpha / b^\beta < b^\gamma$ with β, γ rational. Now, we have that $\alpha < b^\beta b^\gamma = b^{\beta+\gamma}$. However, since we have that $\beta \leq x$ and $\gamma \leq y$ from the definitions of $B(x), B(y)$, we have that $\beta + \gamma \leq x + y \implies b^{\beta\gamma} \in B(x+y)$. Then, we have that a cannot be an upper bound for $\sup B(x+y)$, and thus $\sup B(x) \sup B(y) = \sup B(x+y)$.

Ch1, Q5

We will show first that $-\sup(-A)$ is a lower bound of A. In particular, for any element $a \in A$, we have that $-a \in -A$, and by definition $-a \leq \sup(-A)$. Then, $-(-a) = a \geq -\sup(-A)$.

To see that it is the greatest lower bound, consider any $x > -\sup(-A)$. Then, we would have that $-x < \sup(-A)$. Now, in an earlier problem we showed that there is some element $a \in -A$ such that $\sup(-A) - a < \sup(-A) - (-x) \implies -a < x$. However, since we have that $-A$ has the form of $\{-\alpha \mid \alpha \in A\}$, we have that any $a \in -A \implies -a \in A$. However, since $-a < x$, x cannot be a lower bound for A.

Thus, we have that $-\sup(-A)$ is the greatest lower bound of A, $\inf A$.

Ch1, Q7

a

We have the identity that for any positive integer n, $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + y^{n-1})$ as used by Rudin earlier. Then, taking $x = b, y = 1$, we have that $b^n - 1 = (b-1)(b^{n-1} + b^{n-2} + \cdots + 1)$. In particular, since $b > 1, b^n > 1$ (for n any positive integer) and taking the convention that $b^0 = 1$, we have that

$$\frac{b^n - 1}{b - 1} = \frac{(b-1)\left(\sum_{i=0}^{n-1} b^i\right)}{b-1} = \sum_{i=0}^{n-1} b^i \geq \sum_{i=0}^{n-1} 1 = n$$

Then, this gives us what we wanted, as $\frac{b^n-1}{b-1} \geq n \implies b^n - 1 \geq n(b-1)$.

b

We have that the above holds for any $b > 1$. In particular, if we wish to show that $b - 1 \geq n(b^{1/n} - 1)$, then we simply take part a and apply it to $b^{1/n}$, which gives us what we wanted. The only thing to check is that for any $b > 1$, we still have that $b^{1/n} > 1$. Suppose otherwise; then we what that $b^{1/n} \leq 1 \implies (b^{1/n})^n \leq 1^n = 1$, $\Rightarrow\!\Leftarrow$.

c

Applying the above, we have

$$n > \frac{b-1}{t-1} \implies n(t-1) > b - 1 \geq n(b^{1/n} - 1) \implies t - 1 > b^{1/n} - 1 \implies t > b^{1/n}$$

d

As Rudin suggests, applying the above with $t = yb^{-w}$ yields that for any positive integer n, we have that

$$n > \frac{b-1}{t-1} \implies yb^{-w} > b^{1/n} \implies yb^{-w}b^w > b^{1/n}b^w \implies y > b^{w+1/n}$$

In particular, since b, y, w are fixed (meaning that $(b-1)/(t-1)$ is some fixed real) and that the integers have no greatest element, there exists some positive integer $n > (b-1)/(t-1)$. The only thing left to check is that $t = yb^{-w} > 1$, but this is directly given by $b^w > y \implies b^w b^{-w} = 1 > yb^{-w}$.

e

Using the same trick as before but with $t = b^w/y$ (note that $b^w > y \implies b^w/y > 1$), we have that for any positive integer n,

$$n > \frac{b-1}{t-1} \implies \frac{b^w}{y} > b^{1/n} \implies b^w b^{-1/n} > b^{-1/n} b^{1/n} y \implies b^{w-1/n} > y$$

Again, we have some fixed bound $(b-1)/(t-1)$ above which $b^{w-1/n} > y$.

f,g

We will first show part g.

If $x > 0$, we have that $b^x > 1$ as there is some rational $r = p/q$ in the interval $(0, x)$ such that $r \in B(x)$, and we show in Q5, part c that $b^r > 0$. Then, since $b^x = \sup B(x)$, we have that $b^x \geq b^r > 1$. Correspondingly, $b^x > 1 \implies b^x b^{-x} = 1 < b^{-x}$. Now, if $b^x = y$ and $x' > x$, $b^{x'} = b^x b^{x'-x} = y b^{x'-x} > y$. Similarly, if $x' < x$, we have that $b^{x'} = b^x b^{x'-x} = y b^{-(x'-x)} < y$. Then, $b^x = b^{x'} \iff x = x'$.

First, we have to see that A is nonempty: for any $y > 1$, $0 \in A$. Otherwise, $\exists n$ such that $b^n > 1/y$; in particular, we know that n such that $n(b-1) > 1/y$ from the Archimedean property, which gives some $b^n \geq n(b-1) > 1/y$. Then, we have that $b^{-n} < y$.

Now, if $b^x > y$, then we have that $\exists n \in \mathbb{Z}^+$ such that $b^{x-1/n} > y > b^w \implies x - 1/n > w$ for any $w \in A$ via the proof of part g, and so $x - 1/n$ is a lower upper bound of A. $\Rightarrow\Leftarrow$

If $b^x < y$, then we have that $\exists n \in \mathbb{Z}^+$ such that $b^{x+1/n} < y$, which means that $x + 1/n \in A$, and x cannot be an upper bound of A. $\Rightarrow\Leftarrow$

This only leaves $b^x = y$.