MATH 4041 HW 2

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Problem 1

Let g be a left inverse of $f : X \to Y$, and take any x_1, x_2 in X such that $f(x_1) = f(x_2)$. Then, we have that $g(f(x_1)) = g(f(x_2)) \implies x_1 = x_2$, and so f is injective.

Fixing any arbitrary element $x' \in X$ (since $X \neq \emptyset$), we can define $g : Y \to X$ as follows:

$$g(y) = \begin{cases} x & \exists x \mid f(x) = y \\ x' & \text{otherwise} \end{cases}$$

To check that this is well defined for every element in Y, we can partition Y into two groups. If y is in the image of f, then we have that $y = f(x)$ for exactly one x, as otherwise f would not be injective, which means that $g(y) = x$. If y is not in the image of f, then we have that $g(y) = x'$.

Note that only the empty function exists that takes $\emptyset \to Y$, and is injective, but since a left inverse would take $Y \to \emptyset$, which does not exist for nonempty Y, no left inverse can exist.

To check that this is a left inverse, pick any $x \in X$. Then, $(g \circ f)(x) = g(f(x)) = x$, as we see that x clearly satisfies the first condition in the definition of g.

Now, if $f : X \to Y$ has a right inverse g, then we can explicitly give a preimage under f in X to every $y \in Y$; namely, $g(y)$, as we have by the definition of right inverse that for all $y \in Y$, $f(g(y)) = y$, meaning that $g(y)$ is a preimage of y.

Problem 2

i

To show that h is a right inverse, take any natural number n, and note that $h(n) = 1 \implies 1 = n + 1 \implies n < 1$, but 1 is the least natural number, so $\Rightarrow\Leftarrow$ and $\nexists n \in \mathbb{N}$ such that

$h(n) = 1$. Then,

$$f(h(n)) = \begin{cases} 30 & h(n) = 1 \\ h(n) - 1 & \text{otherwise} \end{cases}$$

Since the first case is impossible,

$$\begin{aligned} &= h(n) - 1 \\ &= (n+1) - 1 = n \end{aligned}$$

We have that any right inverse h of f satisfies that $f(h(n)) = n$, which yields

$$f(h(n)) = \begin{cases} 30 & h(n) = 1 \\ h(n) - 1 & \text{otherwise} \end{cases}$$

which means that we need at the least that if 1 is not in the range of h, then $f(h(n)) = h(n) - 1 = n$, which gives $h(n) = n + 1$ which is the right inverse from earlier. Now, if 1 is in the range of h (let $h(m) = 1$), then we have that $30 = f(1) = f(h(m)) = m$, so we have that

$$h(n) = \begin{cases} 1 & n = 30 \\ n + 1 & \text{otherwise} \end{cases}$$

is a right inverse, as

$$f(h(n)) = \begin{cases} 30 & h(n) = 1 \\ h(n) - 1 & \text{otherwise} \end{cases} = \begin{cases} 30 & n = 30 \\ h(n) - 1 & \text{otherwise} \end{cases} = n$$

This gives all two possible right inverses for f.

ii

Let f be a left inverse of h. Then, let any $n \in \mathbb{N}, n > 1$ be written as $n = m + 1$. Then, we have that $f(h(m)) = m \implies f(m+1) = m \implies f(n) = m$. This completely fixes f for any input $n > 1$. However, note that we can freely pick $f(1)$ to be any value we want, since (as proved in the first part) 1 is not in the image of h. This creates an infinite amount of left inverses of h.

Checking that any such f is indeed an inverse, pick any $m \in \mathbb{N}$:

$$f(h(n)) = \begin{cases} m & h(n) = 1 \\ h(n) - 1 & \text{otherwise} \end{cases}$$

Since the first case is impossible,

$$\begin{aligned} &= h(n) - 1 \\ &= (n+1) - 1 = n \end{aligned}$$

Problem 3

i

There are two elements, namely the two given by

$$f_1(x) = x, f_2(x) = \begin{cases} 1 & x = 2 \\ 2 & x = 1 \end{cases}$$

We can just check that $f_1 \circ f_2 = f_2 \circ f_1$ directly by computing both:

$$\begin{aligned}
(f_1 \circ f_2)(1) &= f_1(f_2(1)) \\
&= f_1(2) = 2 \\
(f_1 \circ f_2)(2) &= f_1(f_2(2)) \\
&= f_1(1) = 1 \\
(f_2 \circ f_1)(1) &= f_2(f_1(1)) \\
&= f_2(1) \\
&= 2 = (f_1 \circ f_2)(1) \\
(f_2 \circ f_1)(2) &= f_2(f_1(2)) \\
&= f_2(2) \\
&= 1 = (f_1 \circ f_2)(2)
\end{aligned}$$

ii

There are 6 elements of S_3, as there are 3 choices for the image of 1, 2 choices (since injectivity) for the image of 2, and only one remaining choice for the image of 3. Note that choosing this way also hits every element in the codomain, as we selected 3 distinct elements in the codomain, which contains exactly 3 elements.

Take

$$f(x) = \begin{cases} 2 & x = 1 \\ 3 & x = 2 \\ 1 & x = 3 \end{cases}, g(x) = \begin{cases} 3 & x = 1 \\ 2 & x = 2 \\ 1 & x = 3 \end{cases}$$

Then, $(f \circ g)(1) = 1$, but $(g \circ f)(1) = 2$, so $f \circ g \neq g \circ f$.

Problem 4

We compute all $e^{i\frac{2k\pi}{n}}$ for $k \in \{1, 2, \ldots, n\}$ to get μ_k:

μ_3	μ_4	μ_8
1	1	1
		$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}i}{2}$
	i	i
$-\frac{1}{2} + \frac{\sqrt{3}i}{2}$		
		$-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}i}{2}$
	-1	-1
		$-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}i}{2}$
$-\frac{1}{2} - \frac{\sqrt{3}i}{2}$		
	$-i$	$-i$
		$\frac{\sqrt{2}}{2} - \frac{\sqrt{2}i}{2}$

Problem 5

i

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+2 & -1+2 \\ 3+4 & -3+4 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 7 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1-3 & 2-4 \\ 1+3 & 2+4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 4 & 6 \end{bmatrix}$$

This is an example of matrix multiplication not commuting.

ii

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 6 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} -2+2 & 6-6 \\ -2+2 & 6-6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Neither matrix is invertible; this is easily seen by the fact that they both have determinant 0. But in general for A, B in some ring (in this case, $\mathbb{M}_2(\mathbb{Z})$), if $AB = 0$ and $A, B \neq 0$ then neither A nor B is invertible. To see this suppose that A^{-1} is an inverse to A: then, we have that $A^{-1}AB = A^{-1}0 \implies B = 0$, $\Rightarrow\Leftarrow$. Similarly, if B^{-1} is a suitable inverse for B, then $ABB^{-1} = 0B^{-1} \implies A = 0$, $\Rightarrow\Leftarrow$.

Problem 6

Only properties of the inner product (bilinearity, symmetry) will be used.

a

First compute that the $(\alpha,1)^{th}$ entry of Ae_j is $(Ae_j)_{\alpha 1} = \sum_{k=1}^{n} a_{\alpha k}(e_j)_{k1} = a_{\alpha j}$, and similarly for ${}^tAe_i$, $({}^tA)_\alpha = \sum_{k=1}^{m} a_{k\alpha}(e_i)_{k1} = a_{i\alpha}$.
Then,

$$Ae_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} = \sum_{k=1}^{m} a_{kj}e_k \text{ and } {}^tAe_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix} = \sum_{k=1}^{n} a_{ik}e_k$$

which gives that

$$\begin{aligned} \langle e_i, Ae_i\rangle &= \left\langle e_i, \sum_{k=1}^{m} a_{kj}e_k \right\rangle \\ &= \sum_{k=1}^{m} \langle e_i, a_{kj}e_k\rangle \\ &= \sum_{k=1}^{m} a_{kj} \langle e_i, e_k\rangle \end{aligned}$$

However, since $i \neq j \implies \langle e_i, e_j\rangle = 0$,

$$= a_{ij}\langle e_i, e_i\rangle = a_{ij}$$

Similarly,

$$\begin{aligned} \langle {}^tAe_i, e_i\rangle &= \left\langle \sum_{k=1}^{n} a_{ik}e_k, e_j \right\rangle \\ &= \sum_{k=1}^{n} \langle a_{ik}e_k, e_j\rangle \\ &= \sum_{k=1}^{n} a_{ik} \langle e_k, e_j\rangle \\ &= a_{ij}\langle e_j, e_j\rangle = a_{ij} \end{aligned}$$

b

For any $v \in \mathbb{R}^m, w \in \mathbb{R}^n$, we have that the standard basis vectors give some representations

$$v = \sum_{i=1}^{m} v_ie_i, w = \sum_{j=1}^{n} w_je_j$$

Then, as matrix multiplication distributes, that

$$Aw = A\sum_{j=1}^{n} w_j e_j = \sum_{j=1}^{n} Aw_j e_j$$

Similarly, ${}^tAv = \sum_{i=1}^{m} {}^tAv_i e_i$. Then,

$$\begin{aligned}
\langle v, Aw\rangle &= \left\langle \sum_{i=1}^{m} v_i e_i, \sum_{j=1}^{n} Aw_j e_j \right\rangle \\
&= \sum_{i=1}^{m} v_i \left\langle e_i, \sum_{j=1}^{n} Aw_j e_j \right\rangle \\
&= \sum_{i=1}^{m} v_i \left(\sum_{j=1}^{n} w_j \langle e_i, Ae_j\rangle \right) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} v_i w_j \langle e_i, Ae_j\rangle \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} v_i w_j \langle {}^tAe_i, e_j\rangle \\
&= \sum_{i=1}^{m} v_i \left(\sum_{j=1}^{n} w_j \langle {}^tAe_i, e_j\rangle \right) \\
&= \sum_{i=1}^{m} v_i \left\langle {}^tAe_i, \sum_{j=1}^{n} w_j e_j \right\rangle \\
&= \left\langle \sum_{i=1}^{m} {}^tAv_i e_i, \sum_{j=1}^{n} w_j e_j \right\rangle \\
&= \langle {}^tAv, w\rangle
\end{aligned}$$

To show that ${}^tA$ is the unique matrix that has this property, let $B = {}^tA + C$. Then, if we have that if $\langle v, Aw\rangle = \langle Bv, w\rangle$,

$$\langle Bv, w\rangle = \langle ({}^tA + C)v, w\rangle = \langle {}^tAv, w\rangle + \langle Cv, w\rangle$$

Since we assumed earlier that $\langle {}^tAv, w\rangle = \langle v, Aw\rangle = \langle Bv, w\rangle$, we have that $\langle Cv, w\rangle = 0$.

From the fact given on the homework, we have that $Cv = 0$ for all $v \in \mathbb{R}^m$. However, note that if we consider $v = e_i$, we show that $Cv = {}^t\begin{bmatrix} c_{1i} & c_{2i} & \dots & c_{ni} \end{bmatrix} = 0$, which suggests that every entry of C, c_{ij} must be zero as $Ce_j = 0$.

Then, this gives that $B = {}^tA$, which was what we wanted.

c

We have that

$$\langle {}^tB{}^tAv, w\rangle = \langle {}^tB({}^tAv), w\rangle = \langle {}^tAv, Bw\rangle = \langle v, ABw\rangle$$

However, we have from part b that the only such matrix C that satisfies $\langle Cv, w\rangle = \langle v, ABw\rangle$ is ${}^t(AB)$, so we have that ${}^tB{}^tA = {}^t(AB)$.

Problem 7

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$$

Problem 8

We can compute the inverse of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to be $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, which yields that

$$A_\theta^{-1} = \frac{1}{\cos^2(\theta) + \sin^2(\theta)} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = A_{-\theta}$$

$$\begin{aligned} A_{\theta_1}A_{\theta_2} &= \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \cdot \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) & -(\sin(\theta_2)\cos(\theta_1) + \sin(\theta_1)\cos(\theta_2)) \\ \sin(\theta_1)\cos(\theta_2) + \sin(\theta_2)\cos(\theta_1) & \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \end{bmatrix} \end{aligned}$$

With angle addition formulas,

$$\begin{aligned} &= \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix} \\ &= A_{\theta_1 + \theta_2} \end{aligned}$$

Then, taking $\theta = \theta_1 = \theta_2$, we get that

$$A_\theta^2 = A_\theta A_\theta = \begin{bmatrix} \cos(\theta + \theta) & -\sin(\theta + \theta) \\ \sin(\theta + \theta) & \cos(\theta + \theta) \end{bmatrix} = A_{2\theta}$$

$$
\begin{aligned}
B_\theta^2 &= \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \\
&= \begin{bmatrix} \cos^2(\theta) + \sin^2(\theta) & \sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta) & \sin^2(\theta) + \cos^2(\theta) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I
\end{aligned}
$$

This immediately suggests that $B_\theta B_\theta = I \implies B_\theta^{-1} = B_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$.

$$
\begin{aligned}
A_\theta R &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1(\cos(\theta)) + 0(-\sin(\theta)) & 0(\cos(\theta)) + (-1)(-\sin(\theta)) \\ 1(\sin(\theta)) + 0(\cos(\theta)) & 0(\sin(\theta)) + (-1)(\cos(\theta)) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} = B_\theta
\end{aligned}
$$

$$
R^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1^2 + 0^2 & 1(0) + 0(-1) \\ 0(1) + -1(0) & 0^2 + (-1)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I
$$

We have by matrix associativity that $R^{-1}A_\theta R = R^{-1}(A_\theta R)$, and the earlier identity gives $R^{-1}(A_\theta R) = R^{-1}B_\theta$, and since $R^2 = I$, $R^{-1}B_\theta = RB_\theta$. This gives us the first half of the desired chain of equalities, and we can see from the earlier identities that

$$
A_\theta R B_\theta = (A_\theta R)B_\theta = B_\theta B_\theta = I
$$

which gives us that RB_θ is the multiplicative inverse to A_θ. Then, since we showed earlier that $A_\theta^{-1} = A_{-\theta}$, we have that $R^{-1}A_\theta R = RB_\theta = A_\theta^{-1} = A_{-\theta}$.

I am not quite sure what is meant by "another computation" in the problem set. I guess that we have $A_\theta R A_\theta R = A_\theta(R^{-1}A_\theta R) = A_\theta A_{-\theta} = A_{\theta-\theta} = A_0 = \begin{bmatrix} \cos(0) & -\sin(0) \\ \sin(0) & \cos(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$?

We showed $A_{\theta_1}A_{\theta_2} = A_{\theta_1+\theta_2}$ earlier.

$$
B_{\theta_1}B_{\theta_2} = A_{\theta_1}RA_{\theta_2}R = A_{\theta_1}(R^{-1}A_{\theta_2}R) = A_{\theta_1}A_{-\theta_2} = A_{\theta_1-\theta_2}
$$

$$
A_{\theta_1}B_{\theta_2} = A_{\theta_1}(A_{\theta_2}R) = (A_{\theta_1}A_{\theta_2})R = A_{\theta_1+\theta_2}R = B_{\theta_1+\theta_2}
$$

$$B_{\theta_1} A_{\theta_2} = (A_{\theta_1} R) A_{\theta_2} I = A_{\theta_1} R A_{\theta_2} R R^{-1}$$

Since $R = R^{-1}$,

$$A_{\theta_1} R A_{\theta_2} R R^{-1} = A_{\theta_1} (R^{-1} A_{\theta_2} R) R = A_{\theta_1} A_{-\theta_2} R = A_{\theta_1 - \theta_2} R = B_{\theta_1 - \theta_2}$$

We have the final identity $A_\theta R A_\theta^{-1} = A_\theta R A_{-\theta}$ by the earlier computation that gave $A_\theta^{-1} = A_{-\theta}$. Then, we have that $B_\theta = A_\theta R \implies A_\theta R A_{-\theta} = B_\theta A_{-\theta} = B_{\theta-(-\theta)} = B_{2\theta}$, which was what we wanted.