MATH 4065 HW 2

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We can just bash this one out in terms of real partial derivatives (there's probably an easier way than this, but this is the obvious route). Let

$f(x+iy) = f_1(x,y) + if_2(x,y), g(x+iy) = g_1(x,y) + ig_2(x,y), h(x+iy) = h_1(x,y) + ih_2(x,y).$

Note that from the chain rule in real multivariable calculus, and treating $a+bi$ as (a,b), we have that

$$\begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{bmatrix} = Dh = Dg \cdot Df = \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

This gives

$$\begin{aligned} \frac{\partial h}{\partial z} =& \frac{1}{2}\left(\frac{\partial h}{\partial x} + \frac{1}{i}\frac{\partial h}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial h_1}{\partial x} + \frac{\partial h_2}{\partial y} + i\left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y}\right)\right) \\ \frac{\partial h}{\partial \bar{z}} =& \frac{1}{2}\left(\frac{\partial h}{\partial x} - \frac{1}{i}\frac{\partial h}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial h_1}{\partial x} - \frac{\partial h_2}{\partial y} + i\left(\frac{\partial h_2}{\partial x} + \frac{\partial h_1}{\partial y}\right)\right) \end{aligned}$$

We can now compute, from the above matrix,

$$\begin{aligned} \frac{\partial h}{\partial z} =& \frac{1}{2}\left(\left(\frac{\partial g_1}{\partial x}\frac{\partial f_1}{\partial x} + \frac{\partial g_1}{\partial y}\frac{\partial f_2}{\partial x}\right) + \left(\frac{\partial g_2}{\partial x}\frac{\partial f_1}{\partial y} + \frac{\partial g_2}{\partial y}\frac{\partial f_2}{\partial y}\right)\right) \\ &+ \frac{i}{2}\left(\left(\frac{\partial g_2}{\partial x}\frac{\partial f_1}{\partial x} + \frac{\partial g_2}{\partial y}\frac{\partial f_2}{\partial x}\right) - \left(\frac{\partial g_1}{\partial y}\frac{\partial f_1}{\partial y} + \frac{\partial g_1}{\partial x}\frac{\partial f_2}{\partial y}\right)\right) \end{aligned}$$

Which factors nicely into the obvious guess, given what we are trying to prove (it is tedious, but easy to check that the following results in the last expression)

$$\begin{aligned} =& \frac{1}{4}\left(\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + i\left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right)\right)\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + i\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\right) \\ &+ \frac{1}{4}\left(\frac{\partial g_1}{\partial x} - \frac{\partial g_2}{\partial y} + i\left(\frac{\partial g_2}{\partial x} + \frac{\partial g_1}{\partial y}\right)\right)\left(\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y} + i\left(-\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\right) \end{aligned}$$

From the first line showing how $\frac{\partial h}{\partial z}$ decomposes, but applied to f, g, we have that this simplies to

$$=\frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial z}$$

Similarly,

$$\begin{aligned}\frac{\partial h}{\partial \bar{z}} =& \frac{1}{2}\left(\left(\frac{\partial g_1}{\partial x}\frac{\partial f_1}{\partial x} + \frac{\partial g_1}{\partial y}\frac{\partial f_2}{\partial x}\right) - \left(\frac{\partial g_2}{\partial x}\frac{\partial f_1}{\partial y} + \frac{\partial g_2}{\partial y}\frac{\partial f_2}{\partial y}\right)\right)\\ &+ \frac{i}{2}\left(\left(\frac{\partial g_2}{\partial x}\frac{\partial f_1}{\partial x} + \frac{\partial g_2}{\partial y}\frac{\partial f_2}{\partial x}\right) + \left(\frac{\partial g_1}{\partial y}\frac{\partial f_1}{\partial y} + \frac{\partial g_1}{\partial x}\frac{\partial f_2}{\partial y}\right)\right)\\ =& \frac{1}{4}\left(\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + i\left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right)\right)\left(\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y} + i\left(\frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial y}\right)\right)\\ &+ \frac{1}{4}\left(\frac{\partial g_1}{\partial x} - \frac{\partial g_2}{\partial y} + i\left(\frac{\partial g_2}{\partial x} + \frac{\partial g_1}{\partial y}\right)\right)\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + i\left(-\frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial y}\right)\right)\\ =& \frac{\partial g}{\partial z}\frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial \bar{z}}\end{aligned}$$

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Via a theorem from class, we have that for any power series

$$f(z_0 + h) = \sum_{n=0}^{\infty} a_n h^n, \text{ we have } f'(z_0 + h) = \sum_{n=1}^{\infty} n a_n h^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} h^n$$

In general, we have that $f^{(n)}(z_0 + h) = \sum_{n=0}^{\infty} \frac{(n+m)!}{n!} a_{n+m} h^n$. We can induct, as the earlier formula gives the $m = 1$ case; if it holds for m, then applying the formula in the base case of $f'(z_0 + h)$, which follows the same pattern:

$$f^{(m+1)} = \sum_{n=1}^{\infty} n\left(\frac{(n+m)!}{n!}\right) a_{n+m} h^{n-1} = \sum_{n=0}^{\infty} \frac{(n+1+m)!}{n!} a_{n+m+1} h^n$$

so it holds for $m + 1$.

Then, if a function is given by a power series on some region centered at z_0, $f^{(m)}(z_0) = \sum_{n=0}^{\infty} \frac{(n+m)!}{n!} a_{n+m} 0^n = m! a_m$, so $a_m = \frac{f^{(m)}(z_0)}{m!}$.

Then, to expand $f_m(z) = (1-z)^{-m}$ to a power series, we need to find, for the n^{th} term a_n, $\frac{f^n(z_0)}{n}$. We have that since $1 - z$ is holomorphic, $(1-z)^{-m}$ is holomorphic everywhere but $z = 1$, and so on the appropriate domain, $f_m'(z) = \frac{d}{dz}(1-z)^{-m}$. By the power rule (and the fact given in class that $\frac{d}{dz}$ works as expected here),

$$f_m^{(n)}(z) = \frac{(n+m-1)!}{(m-1)!(1-z)^{n+m}}$$

Note that this for $n = 1$ is just a simple power rule application, and if it holds for k, then $f_m^{(k+1)}(z) = \frac{(k+m-1)!}{(m-1)!}\left(\frac{k+m}{(1-z)^{k+1+m}}\right) = \frac{((k+1)+m-1)!}{(m-1)!(1-z)^{n+k+1}}$, so it holds for $k+1$.

Then, we have that, centered at z_0, the coefficients of the power series are

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{(n+m-1)!}{n!(m-1)!(1-z_0)^{n+1}} = \binom{n+m-1}{m-1}\frac{1}{(1-z_0)^{n+1}}$$

Thus, we have that

$$f_m(z_0+h) = \frac{1}{(1-(z_0+h))^m} = \sum_{n=0}^{\infty}\binom{n+m-1}{m-1}\frac{h^n}{(1-z_0)^{n+1}}$$

Further,

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{n+m}{n+1}\frac{1}{|1-z_0|} = \frac{1}{|1-z_0|}$$

so we have a radius of convergence of $|1-z_0|$.

Everything above only applies for $z_0 \neq -1$, as f_m is not defined at -1.

For the second half of the problem, note that we take $z_0 = 0$, and

$$\begin{aligned}\lim_{n\to\infty}\frac{\binom{n+m-1}{m-1}}{\frac{1}{(m-1)!}n^{m-1}} &= \lim_{n\to\infty}\frac{\binom{n+m-1}{m-1}}{\frac{1}{(m-1)!}n^{m-1}} \\ &= \lim_{n\to\infty}\frac{(n+m-1)!}{n^{m-1}n!} \\ &= \lim_{n\to\infty}\frac{\prod_{i=1}^{m-1}(n+i)}{n^{m-1}} \\ &= \lim_{n\to\infty}\prod_{i=1}^{m-1}\frac{n+i}{n} \\ &= \prod_{i=1}^{m-1}\lim_{n\to\infty}\frac{n+i}{n} = 1\end{aligned}$$

So we have that $a_n \sim \frac{1}{(m-1)!}n^{m-1}$.

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We can compute the partial sums; if these partial sums converge to $z/(1-z)$, then we are done. In particular, we can induct on n to show that $\sum_{i=0}^{n}\frac{z^{2^i}}{1-z^{2^{i+1}}} = \frac{\sum_{i=1}^{2^{n+1}-1}z^i}{1-z^{2^{n+1}}}$. In particular, for $n=0$, the sum is only one term, which is $\frac{z}{1-z^2}$, which is of the correct form. Then, if it holds for n, we can see that

$$\begin{aligned}
\sum_{i=0}^{n+1} \frac{z^{2^i}}{1-z^{2^{i+1}}} &= \frac{\sum_{i=1}^{2^{n+1}-1} z^i}{1-z^{2^{n+1}}} + \frac{z^{2^{n+1}}}{1-z^{2^{n+2}}} \\
&= \frac{(1+z^{2^{n+1}})\sum_{i=1}^{2^{n+1}-1} z^i}{(1-z^{2^{n+1}})(1+z^{2^{n+1}})} + \frac{z^{2^{n+1}}}{1-z^{2^{n+2}}} \\
&= \frac{(1+z^{2^{n+1}})\sum_{i=1}^{2^{n+1}-1} z^i + z^{2^{n+1}}}{1-z^{2^{n+2}}} \\
&= \frac{\sum_{i=1}^{2^{n+1}-1} z^i + z^{2^{n+1}} + \sum_{i=1}^{2^{n+1}-1} z^{i+2^{n+1}}}{1-z^{2^{n+2}}} \\
&= \frac{\sum_{i=1}^{2^{n+2}-1} z^i}{1-z^{2^{n+2}}}
\end{aligned}$$

We have that for $|z|<1$,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{2^{n+2}-1} z^i}{1-z^{2^{n+2}}} = \frac{\lim_{n\to\infty}\sum_{i=1}^{2^{n+2}-1} z^i}{\lim_{n\to\infty} 1-z^{2^{n+2}}}$$

as long as the numerator and denominator are both finite (and the denominator is non-zero). Then, we have that the top is $z(\sum_{i=0}^{\infty} z^i) = z\frac{1}{1-z}$, as the radius of convergence for the power series $\sum_{i=0}^{\infty} z^i$ associated to $\frac{1}{1-z}$ centered at 0 being 1 as a special case of last problem's result and allows us to evaluate the numerator limit. Further, since we have that $|z|<1 \implies \lim_{n\to\infty} |z|^n = 0 \implies \lim_{n\to\infty} z^n = 0$, so the limit in the denominator goes to 1, and we know that

$$\sum_{i=0}^{\infty} \frac{z^{2^i}}{1-z^{2^{i+1}}} = \lim_{n\to\infty} \frac{\sum_{i=1}^{2^{n+2}-1} z^i}{1-z^{2^{n+2}}} = \frac{\lim_{n\to\infty}\sum_{i=1}^{2^{n+2}-1} z^i}{\lim_{n\to\infty} 1-z^{2^{n+2}}} = \frac{z}{1-z}$$

To see the second one, consider the following:

$$
\begin{aligned}
\frac{2^n z^{2^n}}{1+z^{2^n}} &= \frac{2^n z^{2^n}(1-z^{2^n})}{1-z^{2^{n+1}}} \\
&= \frac{2^n z^{2^n} - 2^n z^{2^{n+1}}}{1-z^{2^{n+1}}} \\
&= \frac{2^n z^{2^n} + 2^{n+1} z^{2^{n+1}} - 2^n z^{2^{n+1}}}{1-z^{2^{n+1}}} \\
&= \frac{2^n z^{2^n}(1-z^{2^n}) + 2^{n+1} z^{2^{n+1}}}{1-z^{2^{n+1}}} \\
&= \frac{2^n z^{2^n}(1-z^{2^n})}{1-z^{2^{n+1}}} + \frac{2^{n+1} z^{2^{n+1}}}{1-z^{2^{n+1}}} \\
&= \frac{2^n z^{2^n}}{1-z^{2^n}} + \frac{2^{n+1} z^{2^{n+1}}}{1-z^{2^{n+1}}}
\end{aligned}
$$

This means that $\sum_{i=1}^{n} \frac{2^i z^{2^i}}{1+z^{2^i}}$ can be seen to be $\frac{z}{1-z} - \frac{2^{n+1} z^{2^{n+1}}}{1-z^{2^{n+1}}}$, as the sum telescopes; in particular, we can see the base case of $n = 0$ from just apply $n = 0$ to the above identity. Then, if it holds for n, then

$$
\sum_{i=1}^{n+1} \frac{2^i z^{2^i}}{1+z^{2^i}} = \frac{z}{1-z} - \frac{2^{n+1} z^{2^{n+1}}}{1-z^{2^{n+1}}} + \frac{2^{n+1} z^{2^{n+1}}}{1-z^{2^{n+1}}} = \frac{z}{1-z} + \frac{2^{n+1} z^{2^{n+1}}}{1-z^{2^{n+1}}}
$$

so it holds for $n + 1$.

Then, we have that the limit of the partial sums is

$$
\lim_{n\to\infty} \frac{z}{1-z} + \frac{2^{n+1} z^{2^{n+1}}}{1-z^{2^{n+1}}} = \frac{z}{1-z} + \lim_{n\to\infty} \frac{2^{n+1} z^{2^{n+1}}}{1-z^{2^{n+1}}}
$$

Then, if we can show that $\lim_{n\to\infty} \frac{2^{n+1} z^{2^{n+1}}}{1-z^{2^{n+1}}} = 0$, we are done. Since we have that if the numerator and the denominator are finite, and the denominator is nonzero that $\lim_{n\to\infty} \frac{2^{n+1} z^{2^{n+1}}}{1-z^{2^{n+1}}} = \frac{\lim_{n\to\infty} 2^{n+1} z^{2^{n+1}}}{\lim_{n\to\infty} 1-z^{2^{n+1}}}$, and we saw the same denominator go to one in the first part of the problem, we have that if we can show that $\lim_{n\to\infty} 2^{n+1} z^{2^{n+1}} = 0$, we are done. In particular, we have that if $\lim_{n\to\infty} |2^{n+1} z^{2^{n+1}}| = 0$, we will be done.

We can further reduce this limit, as if $\lim_{m\to\infty} |mz^m| = 0$, then for any $\epsilon > 0$, $\exists M \mid \forall m > M$, $|mz^m| < \epsilon$; then, we have that for all $n > \log_2(M)$, $2^{n+1} > M \implies |2^{n+1} z^{2^{n+1}}| < \epsilon$. However, this is fairly easy to see: we have that

$$
\lim_{m\to\infty} |mz^m| = \lim_{m\to\infty} |m||z|^m = \lim_{m\to\infty} \frac{|m|}{\left(\frac{1}{|z|}\right)^m} = \lim_{m\to\infty} \frac{m}{\left(\frac{1}{|z|}\right)^m}
$$

since it is enough to show this for $m > 0$. Then, L'Hopital gives that

$$\lim_{m\to\infty} \frac{m}{\left(\frac{1}{|z|}\right)^m} = \lim_{m\to\infty} \frac{1}{\log(\frac{1}{|z|})(\frac{1}{|z|})^m} = 0$$

as the denominator approaches ∞ as $\frac{1}{|z|} > 1$.
This finally gives us what the limit of the partial sums is $\frac{z}{1-z}$, and we are done.

25b

Let γ have radius r and be centered at z such that $|z| < r$. Then, we can parameterize $\gamma(\theta) = re^{i\theta} + z$, and so

$$\int_\gamma z^n dz = \int_0^{2\pi} (re^{i\theta} + z)^n ire^{i\theta} d\theta$$

If $n \geq 0$, with the convention that $0^0 = 1$,

$$\begin{aligned}
&= \int_0^{2\pi} \left(ire^{i\theta} \sum_{j=0}^{n} \binom{n}{j} r^j e^{ij\theta} z^{n-j} \right) d\theta \\
&= \sum_{j=0}^{n} \binom{n}{i} \int_0^{2\pi} r^{j+1} ie^{i(j+1)\theta} z^{n-j} d\theta \\
&= \sum_{j=0}^{n} \binom{n}{i} \int_0^{2\pi} r^{j+1} ie^{i(j+1)\theta} z^{n-j} d\theta \\
&= \sum_{j=0}^{n} \binom{n}{i} \int_0^{2\pi} r^{j+1} i(\cos((j+1)\theta) - i\sin((j+1)\theta)) z^{n-j} d\theta
\end{aligned}$$

However, we have that $\int_0^{2\pi} \cos((j+1)\theta)d\theta = \int_0^{2\pi} \sin((j+1)\theta)d\theta = 0$ for integer $j \neq 0$,

$$= \sum_{j=0}^{n} \binom{n}{i} 0 = 0$$

In particular, we actually have that any circle, not necessarily containing the origin, satisfies that $\int_\gamma z^n = 0$ for nonnegative n.

Then, for $n < 0$, we have to use the earlier problem; we can consider the expansion of $\frac{1}{1-z}$ centered at $z_0 = 0$, which gives

$$(1-z)^{-m} = \sum_{n=0}^{\infty} \binom{n+m-1}{m-1} z^n$$

Let us actually rename the circle to have center $-z$, such that $\gamma(\theta) = re^{i\theta} - z$ instead. Then,

$$\begin{aligned}\int_\gamma z^n dz &= \int_0^{2\pi} (re^{i\theta} - z)^n ire^{i\theta} d\theta = \int_0^{2\pi} (re^{i\theta})^n \left(1 - \frac{ze^{-i\theta}}{r}\right)^n ire^{i\theta} d\theta \\ &= ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} \left(1 - \frac{ze^{-i\theta}}{r}\right)^n d\theta\end{aligned}$$

We have that the circle encloses the origin, so we have that $|ze^{-i\theta}| = |z| < r$, so we can use the earlier expansion but confusingly, with m, n swapped since I'm bad at planning indices:

$$\begin{aligned}&= ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} \sum_{m=0}^{\infty} \binom{m-n-1}{-n-1} \left(\frac{ze^{-i\theta}}{r}\right)^m d\theta \\ &= ir^{n+1} \sum_{m=0}^{\infty} \binom{m-n-1}{-n-1} \left(\frac{z}{r}\right)^m \int_0^{2\pi} e^{i(n+1-m)\theta} d\theta \\ &= ir^{n+1} \sum_{m=0}^{\infty} \binom{m-n-1}{-n-1} \left(\frac{z}{r}\right)^m \int_0^{2\pi} (\cos((n+1-m)\theta) + i\sin((n+1-m)\theta)) d\theta\end{aligned}$$

But, we know, as in the case shown in class, that the integral vanishes if $n+1-m \neq 0$, as $\int_0^{2\pi} \cos(k\theta) d\theta = \int_0^{2\pi} \sin(k\theta) d\theta = 0$ for integral $k \neq 0$. Finally, $n+1-m < 0$ for $n < -1$, $m \geq 0$, so in these cases,

$$= 0$$

Thus, we have that for any $n \neq -1$, $\int_\gamma z^n dz = 0$. In the case of $n = -1$, we have that

$$\begin{aligned}&= ir^{n+1} \sum_{m=0}^{\infty} \binom{m-n-1}{-n-1} \left(\frac{z}{r}\right)^m \int_0^{2\pi} (\cos((n+1-m)\theta) + i\sin((n+1-m)\theta)) d\theta \\ &= ir^{-1+1} \sum_{m=0}^{\infty} \binom{m-n-1}{-n-1} \left(\frac{z}{r}\right)^m \int_0^{2\pi} (\cos((-m)\theta) + i\sin((-m)\theta)) d\theta\end{aligned}$$

Again, this integral is only nonzero when $m = 0$, so

$$\begin{aligned}&= i\binom{0}{0} \left(\frac{z}{r}\right)^0 \int_0^{2\pi} (\cos(0) + i\sin(0)) d\theta \\ &= 2\pi i\end{aligned}$$

We have finally reached the conclusion:

$$\int_\gamma z^n dz = \begin{cases} 0 & n \neq -1 \\ 2i\pi & n = -1 \end{cases}$$

25c

Taking the parameterization $\gamma(\theta) = re^{i\theta}$, we have that

$$\begin{aligned}
\int_\gamma \frac{1}{(z-a)(z-b)} dz &= \int_0^{2\pi} \frac{1}{(re^{i\theta}-a)(re^{i\theta}-b)} rie^{i\theta} d\theta \\
&= \frac{i}{a-b} \int_0^{2\pi} \left(\frac{a}{re^{i\theta}-a} - \frac{b}{re^{i\theta}-b} \right) d\theta \\
&= \frac{i}{a-b} \int_0^{2\pi} \frac{a}{re^{i\theta}-a} d\theta - \frac{i}{a-b} \int_0^{2\pi} \frac{b}{re^{i\theta}-b} d\theta \\
&= \frac{1}{a-b} \int_0^{2\pi} \left(\frac{rie^{i\theta}}{re^{i\theta}-a} - 1 \right) d\theta - \frac{1}{a-b} \int_0^{2\pi} \left(\frac{rie^{i\theta}}{re^{i\theta}-b} - 1 \right) d\theta \\
&= \frac{1}{a-b} \int_0^{2\pi} \frac{rie^{i\theta}}{re^{i\theta}-a} d\theta - \frac{1}{a-b} \int_0^{2\pi} \frac{rie^{i\theta}}{re^{i\theta}-b} d\theta
\end{aligned}$$

However, consider $\gamma_1(\theta) = re^{i\theta} - a$ and $\gamma_2(\theta) = re^{i\theta} - b$; we have that

$$= \frac{1}{a-b} \left(\int_{\gamma_1} z^{-1} dz - \int_{\gamma_2} z^{-1} dz \right)$$

Since we have that $|a| < r < |b|$, γ_1 encloses the origin, but γ_2 does not. By the last problem we have that

$$= \frac{1}{a-b} \left(2\pi i - \int_{\gamma_2} z^{-1} dz \right) = \frac{2\pi i}{a-b} - \frac{1}{a-b} \int_{\gamma_2} z^{-1} dz$$

Further, we have that $1/z$ as a quotient of two entire functions, is holomorphic wherever $z \neq 0$; however, we have that on and inside γ_2, $z \neq 0$ as γ_2 does not enclose the origin. In particular, we actually have that $1/z$ is holomorphic on $D_{|b|}(b)$, which does not include the origin, but does contain γ_2. Then (I am slightly cheating, using a theorem from chapter 2), we can find some F holomorphic that satisfies $F' = f$. Thus, we have $\int_{\gamma_2} z^{-1} dz = 0$, and

$$= \frac{2\pi i}{a-b}$$

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Take $|z| < 1$.

First, we will show that we can encode some sequence $\{a, a+d, a+2d, \dots\}$ as $\frac{z^a}{1-z^d}$. We have that $\frac{1}{1-z} = \sum_{i=0}^\infty z^i$. Then, replacing z with z^d, we get $\frac{1}{1-z^d} = \sum_{i=0}^\infty z^{id}$; further multiplying by z^a gives us

$$\frac{z^a}{1-z^d} = \sum_{i=0}^\infty z^{a+id}$$

This gives us a way to represent any arithmetic progression with start a and step d by $\frac{z^a}{1-z^d}$. In particular, some subset $E \subseteq \mathbb{N}$ is represented by a power series $\sum_{i=0}^{\infty} a_i z^i$ if for any $i \in \mathbb{N}$, $i \in E \iff a_i = 1$, where $a_i \in \{0, 1\}$. This suggests that $\mathbb{N}$ is represented by $\frac{z}{1-z}$.

Further, we can represent the union of two disjoint sets by the sum of their representative functions. To see this, let S_1 be represented by $\sum_{i=0}^{\infty} a_{i,1} z^i$ and S_2 by $\sum_{i=0}^{\infty} a_{i,2} z^i$, and finally $S_1 \cup S_2$ by $\sum_{i=0}^{\infty} a_i z^i$. We will show that taking $a_i = a_{i,1} + a_{i,2}$ is sufficient.

We have that $a_i = 1 \implies i \in S_1 \cup S_2$; further, since i is in the union of two disjoint sets, i is in exactly one of S_1 or S_2, so $a_{i,1} + a_{i,2} = a_i$. Similarly, $a_i = 0 \implies i \notin S_1, \notin S_2 \implies a_{i,1} + a_{i,2} = 0 + 0 = a_i$. Thus, at every index, $a_i = a_{i,1} + a_{i,2}$, which was what we wanted.

Now, suppose that $\mathbb{N}$ could be partitioned into a finite number, say n, of arithmetic progressions with distinct steps; in particular, assign a random order and let the i^{th} progression be denoted S_i have step d_i and start a_i. Since this is a partition, we have that all such S_i are disjoint, and their union must be $\mathbb{N}$. Then, we have the earlier properties of the representative functions,

$$\sum_{i=1}^{n} \frac{z^{a_i}}{1 - z^{d_i}} = \frac{z}{1 - z}$$

The above statement holds in each respective ball of convergence, so $|z|^{d_i} < 1 \implies |z| < 1$. Now, since we disallowed the trivial case $a = d = 1$, $n \geq 2$. In that case, let d be the maximal step of all the S_i, and μ be a primitive d^{th} root of unity. Then, since we know that μ lies on $|z| = 1$ and is a limit point of $|z| < 1$, we can see that as $z \to \mu$, $\sum_{i=1}^{n} \frac{z^{a_i}}{1-z^{d_i}} \to \infty$, but $\frac{z}{1-z} \to \frac{\mu}{1-\mu}$, which satisfies $|\frac{\mu}{1-\mu}| = \frac{|\mu|}{|1-\mu|} = \frac{1}{|1-\mu|}$. Further, since μ is a primitive d^{th} root of unity for $d > 1$, $\mu \neq 1$, so $|\frac{\mu}{1-\mu}|$ is bounded. Then, the left hand side and the right hand side go to different limits, so $\Rightarrow\Leftarrow$. $\mathbb{N}$ thus cannot be partitioned in such a way.

In particular, we need the largest step, otherwise two of the $\frac{z^{a_i}}{1-z^{d_i}}$ could blow up, and the behaviour here is unclear. But the largest step works, so we are done!