MATH 4061 HW 3

David Chen, dc3451

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Ch 2, Q7

a

First, we can show that $(E \cup F)' = E' \cup F'$. In the case that $p \in E'$ or $p \in F'$, then we have that for any $r > 0$, $(E \cup F) \cap B_r^\circ(b) = (E \cap B_r^\circ(b)) \cup (F \cap B_r^\circ(b))$, but the right hand side cannot be empty, or else $p \notin E', F'$. Then, we have that $p \in (E \cup F)'$. Thus, we have that $E' \cup F' \subseteq (E \cup F)'$.

To show that $(E \cup F)' \subseteq E' \cup F'$, suppose we have some $p \in (E \cup F)'$, but $p \notin E', p \notin F'$. Then, we have some $r_E > 0$ such that $E \cup B_{r_E}^\circ(p) \setminus \{p\} = \emptyset$ and some $r_F > 0$ such that $E \cup B_{r_F}^\circ(p) \setminus \{p\} = 0$ (otherwise, p would be a limit point of E or F). Then, we have that $r = \min(r_E, r_F)$, such that $(E \cup F) \cap B_r^\circ(p) \setminus \{p\} = (E \cap B_r^\circ(p) \setminus \{p\}) \cup (F \cap B_r^\circ(p) \setminus \{p\}) = \emptyset \cup \emptyset = \emptyset$. Then, clearly $p \notin (E \cup F)'$, $\Rightarrow\Leftarrow$, and so $(E \cup F)' \subseteq E' \cup F'$.

We can now show that $\overline{E \cup F} = \overline{E} \cup \overline{F}$, as we have that $\overline{E \cup F} = (E \cup F) \cup (E \cup F)' = (E \cup F) \cup (E' \cup F') = (E \cup E') \cup (F \cup F') = \overline{E} \cup \overline{F}$.

Then, we can induct to show that $\overline{B} = \overline{\bigcup_{i=1}^n A_i} = \bigcup_{i=1}^n \overline{A}_i$. $n = 1$ is trivial. Then, if it holds for n, we have that

$$\overline{\bigcup_{i=1}^{n+1} A_i} = \overline{\bigcup_{i=1}^{n} A_i \cup A_{n+1}}$$

By the first statement,

$$= \overline{\bigcup_{i=1}^{n} A_i} \cup \overline{A}_{n+1}$$

By inductive hypothesis,

$$= \bigcup_{i=1}^{n} \overline{A}_i \cup \overline{A}_{n+1}$$
$$= \bigcup_{i=1}^{n+1`} \overline{A}_i$$

so we have that for any $n, \overline{B} = \overline{\bigcup_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A}_i$

b

If $p \in \bigcup_{i=1}^{\infty} \overline{A}_i$, then we have that we have some A_j such that $p \in \overline{A}_j$. Then, we have two cases: if $p \in A_j$, then $p \in \bigcup_{i=1}^{\infty} A_i = B$. The second case is that $p \in A_i'$; then, for any $r > 0$, we have that $A_j \cap B_r^\circ(p) \setminus \{p\} \neq \emptyset$, which gives that as $A_j \subset B$, $B \cap B_r^\circ(p) \setminus \{p\} \neq \emptyset$, so $p \in B'$ as well, so $p \in \overline{B}$. In both cases, $p \in \overline{B}$, so $B \supset \bigcup_{i=1}^{\infty} A_i$.

Ch 2, Q9

d

We want to show that $(E^\circ)^c = \overline{E^c}$.

First, for any p, we have two cases. If $p \notin E$, then we have that since $E^\circ \subset E$, $p \notin E^\circ \implies p \in (E^\circ)^c$. Similarly, we have that $\overline{E^c} = E^c \cup (E^c)'$, so $p \in \overline{E^c}$, so we have that for $p \notin E$, $p \in (E^\circ)^c \iff p \in \overline{E^c}$. Then, the only points which remain are the ones which are in E.

Now, for $p \in E$, if $p \in (E^\circ)^c$, then we have that p is not an interior point of E; that is, for every $r > 0$, $B_r^\circ(p) \not\subset E$, so $B_r^\circ(p) \setminus \{p\}$ contains some point not in E, which then is in E^c. Then, for any $r > 0$, we have that $E^c \cap B_r^\circ(p) \setminus \{p\} \neq \emptyset$, so $p \in (E^c)' \implies p \in \overline{E^c}$.

Further, for $p \in E$, if $p \in \overline{E^c}$, we have that for any $r > 0$, $E^c \cap B_r^\circ(p) \setminus \{p\} \neq 0$. This means that $B_r^\circ(p) \not\subset E$, as it contains some point in E^c, and so p is not an interior point of E, so $p \notin E^\circ \implies p \in (E^\circ)^c$.

Since in both cases $p \in E$ and $p \notin E$, $p \in (E^\circ)^c \iff p \in \overline{E^c}$, the two sets are equal.

e

No: consider something like $E = \mathbb{Q} \subset \mathbb{R}$. Then, the closure of E is $\mathbb{R}$ as $\mathbb{Q}$ is dense in the reals, but the interior of $\mathbb{Q}$ is empty, and the interior of $\mathbb{R}$ is $\mathbb{R}$ itself, which is definitely *not* empty.

f

Again no: consider the same example. $E = \mathbb{Q}$ has interior $\emptyset$, the closure of which is $\emptyset$, but the closure of $\mathbb{Q}$ is $\mathbb{R}$.

Ch 2, Q10

For any $p, q \in X$, we have that if $p \neq q$, $d(p, q) = 1 > 0$ and if $p = q$, $d(p, q) = 0$ so it satisfies the property of a metric.

Further,

$$d(q,p) = \begin{cases} 1 & q \neq p \\ 0 & q = p \end{cases} = \begin{cases} 1 & p \neq q \\ 0 & p = q \end{cases} = d(p,q)$$

so d is symmetric.

Lastly, $d(p, r) + d(r, q) \geq d(p, q)$ can be verified in a few cases.

1. $d(p, q) = 0$. Since $d(p, r) \geq 0$, $d(r, q) \geq 0$ by the first property, we have that $d(p, r) + d(r, q) \geq 0 = d(p, q)$.
2. $d(p, q) = 1$. Then, we have that $p \neq q$, so we cannot have that both $r = p$ and $r = q$. In that case, we have at least one of $r \neq p$ or $r \neq q$, so $d(p, r) + d(r, q) \geq 1 = d(p, q)$.

Thus, d is a metric.

Every set is open. First, note that for any $p \in X$, $B^{\circ}_{\frac{1}{2}}(p) = \{q \in X \mid d(p, q) < \frac{1}{2}\}$. However, $d(p, q) < \frac{1}{2} \implies d(p, q) = 0 \implies p = q$, so $B^{\circ}_{\frac{1}{2}}(p) = \{p\}$. Now, consider $E \subseteq X$. Then, for any point $p \in E$, we have that $B^{\circ}_{\frac{1}{2}}(p) = \{p\} \subset E$, so E is open.

Every set is also closed. First, note that since every set is open, any $E \subset X$ has that E^c is open, and so $E = (E^c)^c$ is the complement of an open set, and is thus closed. Alternatively, note that there are no limit points, since for any $p \in E$, $B^{\circ}_{\frac{1}{2}}(p) \cap (E \setminus \{p\}) = \{p\} \cap (E \setminus \{p\}) = \emptyset$. Thus, every set contains all of its (none) limit points, and is therefore closed.

Only sets of finite order are compact. To see this, we can clearly give a finite subcover of any finite set. Let $E = \bigcup_{\alpha \in A} G_\alpha$ for some indexing set A. Then, since E is finite (say with order n), we can select for each $p_i \in E$ that $p_i \in G_{\alpha_i}$ for some $\alpha_i \in A$ (if no such α_i exists, then $\bigcup_{\alpha \in A} G_\alpha$ does not contain p and is thus not an open cover of E). Then, we have that $\bigcup_{i=1}^n G_{\alpha_i}$ contains every $p_i \in E$, so we have a finite subcover of E.

Furthermore, we can give an open cover of any infinite set that contains no finite subcover. In particular, consider for any infinite set $E \subseteq X$ the open cover $\bigcup_{p \in E} \{p\}$. Any finite subcover $\bigcup_{i=1}^n \{p_i\}$ has finite order, as it is the finite union of finite sets, and so E cannot be a subset of $\bigcup_{i=1}^n \{p_i\}$, as E is infinite. Thus, this open cover admits no finite subcover so no infinite set can be compact.

Ch 2, Q12

Suppose that we have an open cover such that $K \subset \bigcup_{\alpha \in A} G_\alpha$ for some indexing set A. Then, there is some G_{α_0} for $\alpha_0 \in A$ such that $0 \in G_{\alpha_0}$. Then, since G_{α_0} is open, we have that there is some $r > 0$ such that $B_r^\circ(0) \subset G_{\alpha_0}$.

Then, for $n \in \mathbb{N}$, $n > 1/r \implies 1/n < r \implies 1/n \in G_{\alpha_0}$. Let N be the greatest natural such that $N \leq 1/r$. Then, for $1 \leq n \leq N$, we know that $\exists$ some G_{α_n} such that $\alpha_n \in A$ and $1/n \in G_{\alpha_n}$, since $\bigcup_{\alpha \in A} G_\alpha$ is an open cover of a set containing $1/n$. Then, we have that $\bigcup_{i=0}^N G_{\alpha_i}$ is a finite subcover of K, as $0 \in G_{\alpha_0}$, and for any $n \in N$, we have that either $n > 1/r \implies 1/n \in G_{\alpha_0}$ or $n \leq 1/r \implies 1/n \in G_{\alpha_n}$.

Ch 2, Q25

For every $n \in \mathbb{N}$, consider the open cover of K, $\bigcup_{p \in K} B_{\frac{1}{n}}^\circ(p)$. Then, since K is compact, there is some finite collection $\{p_{n_i}\}$ such that $\bigcup_{i=1}^{m_n} B_{\frac{1}{n}}^\circ(p_{n_i})$ is an open cover of K. Then, we have for each n some finite associated collection of sets $E_n = \{B_{\frac{1}{n}}^\circ(p_{n_i})\}_{i=1}^{m_n}$. We know that as the countable union of finite sets, $\bigcup_{n=1}^\infty E_n$ is countable itself.

We can check that $\bigcup_{n=1}^\infty E_n$ is a base for K: for every $p \in K$ and open $G \subset K$ that contains p, we have that since G is open, there is some $r > 0$ such that $B_r^\circ(p) \subset G$. Then, there is some $n > 2/r \implies 1/n < \frac{r}{2}$; consider now E_n, which is an open cover of K, and thus also an open cover of $B_r^\circ(p)$. Therefore, we must have some $B_{\frac{1}{n}}^\circ(p_{n_i}) \in E_n$ such that $p \in B_{\frac{1}{n}}^\circ(p_{n_i})$. However, for any $q \in B_{\frac{1}{n}}^\circ(p_i)$, we have that $d(p,q) \leq d(p, p_{n_i}) + d(p_{n_i}, q) \leq \frac{1}{n} + \frac{1}{n} < r$, so $p \in B_{\frac{1}{n}}^\circ(p_{n_i}) \subset B_r^\circ(p) \subset G$, so we have that E_n is a countable base for K.

To conclude that K is separable, note that having a countable basis is sufficient to show that a set is separable. To see this, we can construct a countable dense subset of K from a countable basis $\{V_\alpha\}_{\alpha \in A}$ where A is some countable index set. In particular, select one point p_α from every V_α. The set of all $\{p_\alpha\}_{\alpha \in A}$ is a countable dense subset; to see this, we need to show that every point $p \in K$ is a limit point of $E = \{p_\alpha\}_{\alpha \in A}$ or in E.

We have two choices for $p \in K$. Either $p \in E$ or $p \notin E$; in the first case, we are immediately done. In the second case, for any $r > 0$, we have that $B_r^\circ(p)$ contains some $p_\alpha \in E$ where $p_\alpha \neq p$, as $V_\alpha \subset B_r$ for some α by the definition of base (as $B_r^\circ(p)$ is a nonempty open set), and $p_\alpha \in V_\alpha$ by construction. Then, $B_r^\circ(p) \cap (E \setminus p) \neq \emptyset$ and p is a limit point of E, so $\overline{E} = K$, so E is a countable dense subset of K.