dc3451
David Chen

Homework 5

Apostol p.70-71 no.3

Claim.

$$\int_a^b [x]dx + \int_a^b [-x]dx = a - b$$

Proof. We first show that $\forall x \in \mathbb{R} \setminus \mathbb{Z}, [x] + [-x] = -1$. If for $n \in \mathbb{Z}, n < x < n+1 \implies -(n+1) < -x < -n \implies [x] = n, [-x] = -(n+1) \implies [x] + [-x] = n + -(n+1) = -1$. Should $x \in \mathbb{Z}$, we have that $[x] = x, [-x] = -x \implies [x] + [-x] = 0$.

We also will eventually need that for $\{x_0, x_1, ..., x_n \mid x_i \in \mathbb{R}\}, \sum_{i=1}^n -(x_i - x_{i-1}) = x_0 - x_n$. We induct on n, and the base case $n = 1$, and $\sum_{i=1}^1 -(x_i - x_{i-1}) = x_0 - x_1$. For the inductive case, assume the statement holds for $n = k$. Then,

$$\begin{aligned}\sum_{i=1}^{k+1} -(x_i - x_{i-1}) &= \sum_{i=1}^{k} -(x_i - x_{i-1}) + -(x_{k+1} - x_k) \\ &= x_0 - x_k - x_{k+1} + x_k \\ &= x_0 - x_{k+1}\end{aligned}$$

and we are done.

We have proved additivty of integrals in class, so we have that $\int_a^b [x]dx + \int_a^b [-x]dx = \int_a^b ([x] + [-x])dx$. However, consider that $\forall x \in \mathbb{R} \setminus \mathbb{Z}, [x] + [-x] = 1$, so $[x] + [-x]$, restricted to $[a, b]$, is then a step function with partition $\{a, b\} \cup \{n \in \mathbb{Z} \mid a < n < b\} = \{a = x_0, x_1, x_2, ..., x_k = b\}$ and constant values all equal to -1 (On any interval (x_{i-1}, x_i), we have that there are no integers and thus must that $f|_{(x_{i-1}, x_i)} = -1$). This then leaves us with

$$\int_a^b ([x] + [-x])dx = \sum_{i=1}^{k} (-1)(x_i - x_{i-1}) = \sum_{i=1}^{k} -(x_1 - x_{i-1}) = x_0 - x_k = a - b$$

□

Apostol p.70-71 no.5a

Claim.

$$\int_0^2 [t^2]dt = 5 - \sqrt{2} - \sqrt{3}$$

Proof. First, note that $0 \leq a < b \implies a(a) < b(a), a(b) < b(b) \implies a^2 < b^2$, so that $a < t < b \implies a^2 < t^2 < b^2$.

It is easy to see that $[t^2]$ on the open subintervals of the partition $P = \{0, 1, \sqrt{2}, \sqrt{3}, 2\}$, $[t^2]$ is constant. To be clear,

$$\begin{aligned}
0 < t < 1 &\implies 0 < t^2 < 1 \implies [t^2] = 0\\
1 < t < \sqrt{2} &\implies 1 < t^2 < 2 \implies [t^2] = 1\\
\sqrt{2} < t < \sqrt{3} &\implies 2 < t^2 < 3 \implies [t^2] = 2\\
\sqrt{3} < t < 2 &\implies 3 < t^2 < 4 \implies [t^2] = 3
\end{aligned}$$

Then, we have that

$$\begin{aligned}
\int_0^2 [t^2]dt &= \sum_{i=1}^{4} c_i(x_i - x_{i-1})\\
&= 0(1-0) + 1(\sqrt{2}-1) + 2(\sqrt{3}-\sqrt{2}) + 3(2-\sqrt{3})\\
&= 5 - \sqrt{2} - \sqrt{3}
\end{aligned}$$

$\square$

Apostol p.70-71 no.5b

Note that $[t^2] = [(-t)^2]$, so by problem 1 we have that $\int_{-3}^{3}[t^2]dt = \int_{-3}^{0}[t^2]dt + \int_0^3[t^2]dt = 2\int_0^3[t^2]dt = 2(\int_0^2[t^2]dt + \int_2^3[t^2]dt) = 2((5-\sqrt{2}-\sqrt{3}) + \int_2^3[t^2]dt)$.
We can then check that we have the following:

$$\begin{aligned}
2 < t < \sqrt{5} &\implies 4 < t^2 < 5 \implies [t^2] = 4\\
\sqrt{5} < t < \sqrt{6} &\implies 5 < t^2 < 6 \implies [t^2] = 5\\
\sqrt{6} < t < \sqrt{7} &\implies 6 < t^2 < 7 \implies [t^2] = 6\\
\sqrt{7} < t < \sqrt{8} &\implies 7 < t^2 < 8 \implies [t^2] = 7\\
\sqrt{8} < t < 3 &\implies 8 < t^2 < 9 \implies [t^2] = 8
\end{aligned}$$

Thus, we have that

$$\int_2^3 [t^2] = 4(\sqrt{5}-2) + 5(\sqrt{6}-\sqrt{5}) + 6(\sqrt{7}-\sqrt{6}) + 7(\sqrt{8}-\sqrt{7}) + 8(3-\sqrt{8}))$$

Then, we finally get that $\int_{-3}^{3}[t^2]dt = 2(5-\sqrt{2}-\sqrt{3}+(4(\sqrt{5}-2)+5(\sqrt{6}-\sqrt{5})+6(\sqrt{7}-\sqrt{6})+7(\sqrt{8}-\sqrt{7})+8(3-\sqrt{8}))) = 2(21-\sqrt{2}-\sqrt{3}-\sqrt{5}-\sqrt{6}-\sqrt{7}-\sqrt{8})$.
Generally, $\int_0^n[t^2] = (n^2-1)n - \sum_{i=1}^{n^2-1}\sqrt{i}$.

Apostol p.70-71 no.11

Unless specified otherwise, all functions are assumed step functions.

Apostol p.70-71 no.11a

Claim.

$$\int_a^b s + \int_b^c s = \int_a^c s$$

Proof. True; we have that $P = P_1 \cup P_2 = \{x_0, ..., x_{n+m}\}$, where $P_1 = \{x_0, ..., x_n\}, P_2 = \{x_n, ..., x_m\}$ are partitions of s on $[a, b]$ and $[b, c]$, is then a partition of s on $[a, c]$.

$$\int_a^b s + \int_b^c s = \sum_{i=n}^{m} s_i^3(x_i - x_{i-1}) + \sum_{i=1}^{n} s_i^3(x_i - x_{i-1}) = \sum_{i=1}^{n+m} s^3(x_i - x_{i-1}) = \int_a^c s$$

□

Apostol p.70-71 no.11b

Claim.

$$\int_a^b (s + t) = \int_a^b s + \int_a^b t$$

This is false, consider that $\int_a^b 2 = 2^3(b - a) \neq \int_a^b 1 + \int_a^b 1 = 1^3(b - a) + 1^3(b - a) = 2(b - a)$.

Apostol p.70-71 no.11c

Claim.

$$c \int_a^b s = \int_a^b c \cdot s$$

This is also false, consider again $\int_a^b 2 = 2^3(b - a) \neq 2 \int_a^b 1 = 2(b - a)$.

Apostol p.70-71 no.11d

Claim.

$$\int_{a+c}^{b+c} s(x)dx = \int_a^b s(x + c)dx$$

Proof. True; we have that for a partition $P = \{x_0, x_1, ..., x_n\}$ of $s(x)$ over $[a+c, b+c]$, then $P_c = \{x_0 - c, x_1 - c, ..., x_n - c\}$ is a partition of $s(x+c)$ over $[a+c, b+c]$. Then, we have that

$$\int_{a+c}^{b+c} s(x)dx = \sum_{i=1}^{n} c_i^3(x_i - x_{i-1}) = \sum_{i=1}^{n} c_n^3((x_i - c) - (x_{i-1} - c)) = \int_a^b s(x+c)dx$$

□

Apostol p.70-71 no.11e

Claim. $\forall x \in [a,b], s(x) < t(x) \implies \int_a^b s < \int_a^b t.$

Proof. Note that for $a, b \in \mathbb{R}, a < b \implies a^3 < b^3$. This follows from casework on a, b: if $a = 0$, then $ab^2 < b^3 \implies 0 = a^3 < b^3$. If $b = 0$, then $a^3 < ba^2 = 0 = b^3$. If $a, b < 0$, then $a < b \implies a^2 > b^2 \implies a^3 < b^3$. If $a < 0, b > 0$, then $a < b \implies a^3 < 0 < b^3$. If $a > 0, b > 0$, then $a < b \implies a^2 < b^2 \implies a^3 < b^3$.

Suppose that s, t have partitions P_s, P_t. Then $P = P_s \cup P_t = \{x_0, ...x_n\}$ is a partition for both, where s has constant values s_i and t has constant values t_i, where i ranges from 1 to n. Now, $\int_a^b s = \sum_{i=1}^n s_i^3(x_i - x_{i-1}) < \sum_{i=}^n t_i^3(x_i - x_{i-1}) = \int_a^b t$. We can show that the inequality holds by inducting on n.

In general, we wish to show that if for $i = 1, 2, ..., n$, we have that $a_i < b_i$, then $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$. We induct on n. If $n = 1$, then we have that $a_1 < b_1$, which is true by the premise. For the inductive step, suppose that the inequality holds for $n = k$. Then, $\sum_{i=1}^{k+1} a_i = \sum_{i=1}^k a_i + a_{k+1} < \sum_{i=1}^k b_i + a_{k+1} < \sum_{i=1}^k b_i + b_{k+1} < \sum_{i=1}^{k+1} b_i$.

Now, we know that as $s(x) < t(x)$ for $x \in [a,b]$, we have that $s_i < t_i \implies s_i^3 < t_i^3$. Applying this to the sum, we see that the inequality holds.

□

Apostol p.70-71 no.15

Claim. For step functions s, t, we have that $\forall x \in [a,b], s(x) < t(x) \implies \int_a^b s < \int_a^b t.$

Proof. From above, we have that if for $i = 1, 2, ..., n$, we have that if $a_i < b_i$, then $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$.

Suppose that s, t have partitions P_s, P_t. Then $P = P_s \cup P_t = \{x_0, ...x_n\}$ is a partition for both, where s has constant values s_i and t has constant values t_i, where i ranges from 1 to n. Now, $\int_a^b s = \sum_{i=1}^n s_i(x_i - x_{i-1}) < \sum_{i=}^n t_i(x_i - x_{i-1}) = \int_a^b t$.

The middle inequality holds as we have that for any interval (x_{i-1}, x_i), we have that $x \in (x_{i-1}, x_i) \implies s(x) = s_i < t(x) = t_i \implies s_i(x_i - x_{i-1}) < t_i(x_i - x_{i-1})$ as we have that $x_i > x_{i-1} \implies x_i - x_{i-1} > 0$.

$\square$

Apostol p.83 no.25a

Claim. If $f : [a, b] \to \mathbb{R}$ is an even function, $\int_{-b}^{b} f = 2\int_0^b f$.

Proof. From problem 1, we have that $\int_{-b}^{b} f = \int_{-b}^{0} f(x)dx + \int_0^b f(x)dx = \int_0^b f(-x) + \int_0^b f(x) = \int_0^b f(x) + \int_0^b f(x) = 2\int_0^b f$ $\square$

Apostol p.83 no.25b

Claim. If $f : [a, b] \to \mathbb{R}$ is an odd function, $\int_{-b}^{b} f = 0$.

Proof. From problem 1, we have that $\int_{-b}^{b} f = \int_{-b}^{0} f(x)dx + \int_0^b f(x)dx = \int_0^b f(-x) + \int_0^b f(x) = -\int_0^b f(x) + \int_0^b f(x) = 0$ $\square$

Problem 1

This does not rely on any Apostol problem which relies on this.

Claim. If f is integrable on $[a, b]$, then

$$\int_{-b}^{-a} f(-x)dx = \int_a^b f(x)dx$$

Proof. We will first show this for step functions. If $P = \{x_0, ..., x_n\}$ is a partition for $f(x)$ over $[a, b]$ with constant values $\{c_1, ..., c_n\}$, then we have that $-P = \{-x_n, ..., -x_0\}$ is a partition for $g(x) : [-b, -a] \to \mathbb{R} : x \mapsto f(-x)$ over $[-b, -a]$ with the same constant values (though in reverse order). This can be seen as we have that for any open subinterval of $-P$, $(-x_i, -x_{i-1})$, $\forall x \in \mathbb{R} - x_{i-1} < x < -x_i \implies x_{i-1} < -x < x_i \implies g(x) = f(-x) = c_i$.
Thus, we have that $\int_{-b}^{-a} g(x)dx = \int_{-b}^{-a} f(-x)dx = \sum_{i=1}^{n} c_i(-x_{i-1} - (-x_i)) = \sum_{i=1}^{n} c_i(x_i - x_{i-1}) = \int_a^b f(x)dx$.
Now to show the general case, we will show that for f, g we have that $\underline{I}(f) = \underline{I}(g)$ and $\overline{I}(f) = \overline{I}(g)$.
Suppose that $\int_a^b s(x)dx \in \underline{I}(f)$. Then we have that $s(x) < f(x) \implies s(-x) < f(-x) = g(x) \implies \int_{-b}^{-a} s(-x)dx = \int_a^b s(x)dx \in \underline{I}(g)$.
Suppose that $\int_a^b s(x)dx \in \underline{I}(g)$. Then we have that $s(x) < g(x) \implies s(-x) < g(-x) = f(x) \implies \int_{-b}^{-a} s(-x)dx = \int_a^b s(x)dx \in \underline{I}(f)$.

Suppose that $\int_a^b s(x)dx \in \overline{I}(f)$. Then we have that $s(x) > f(x) \implies s(-x) > f(-x) = g(x) \implies \int_{-b}^{-a} s(-x)dx = \int_a^b s(x)dx \in \overline{I}(g)$.

Suppose that $\int_a^b s(x)dx \in \overline{I}(g)$. Then we have that $s(x) > g(x) \implies s(-x) > g(-x) = f(x) \implies \int_{-b}^{-a} s(-x)dx = \int_a^b s(x)dx \in \overline{I}(f)$.

Thus, we have that $\underline{I}(f) = \underline{I}(g), \overline{I}(f) = \overline{I}(g)$ and so $\int_a^b f(x)dx = \int_{-b}^{-a} g(x)dx = \int_{-b}^{-a} f(-x)dx$. □

Problem 2

Claim.

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

Proof. We first need to show that for any integrable functions f, g on $[a, b]$, we have that $f \leq g \implies \int_a^b f \leq \int_a^b g$.

In the case that $f < g$, we have that $\int_a^b f < \int_a^b g$ holds for step functions (as shown in Apostol pg.70-71 no.15). The same proof holds for $\leq$ instead of $<$ as well.

In the general case, we have that $f \leq g \implies \underline{I}(f) \subseteq \underline{I}(g), \overline{I}(g) \subseteq \overline{I}(f)$, as $s < f \implies s < g$, and $s > g \implies s > f$. Thus, we have that $\sup(\underline{I}(f)) \leq \sup(\underline{I}(g)), \inf(\overline{I}(g)) \geq \inf(\overline{I}(g))$ as properties of $\inf, \sup$. These lead to the conclusion that $\sup(\underline{I}(f)) = \inf(\overline{I}(f)) \leq \inf(\overline{I}(g)) = \sup(\overline{I}(g)) \implies \int_a^b f \leq \int_a^b g$.

Note that we have that $-|f(x)| \leq f(x) \leq |f(x)|$ as a general property of the absolute value. Further, we have then $\int_a^b -|f(x)|dx = -\int_a^b |f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx$. The linearity of the integral was proved in class.

In general, we have that $-a \leq x \leq a \implies |x| \leq a$, as we have three cases: $x = 0$, which follows as $-a \leq 0 \leq a \implies 0 \leq a$, $x < 0$, which follows as $-a \leq x \implies |x| = -x \leq -(-a) = a$, and $x > 0$, which follows as $x \leq a \implies |x| = x \leq a$.

We can now conclude that $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$. □

Problem 3

The function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is not, as we showed in class, integrable on any interval $[a, b], a \neq b$. However, we have that

$$\begin{aligned} |f(x)| &= \begin{cases} |1| & x \in \mathbb{Q} \\ |-1| & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \\ &= \begin{cases} 1 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \\ &= 1 \end{aligned}$$

which is integrable over $[a, b]$. Namely, since it is equivalent to a step function with partition $\{a, b\}$, it evaluates to $b - a$.

Problem 4

a

Claim. The only function both odd and even is $f(x) = 0$.

Proof. $f(x) = f(-x) = -f(x)$. In general, $a \in \mathbb{R}, a = -a \implies a = 0$. This is because $a = -a \implies a + a = 0 \implies 2a = 0 \implies a = (2^{-1})0 = 0$. Thus, $f(x) = -f(x) \implies f(x) = 0$. $\square$

b

Claim. Let f be integrable on every closed interval $[a, b]$, and let $g(x) = \int_0^x f(t)dt$. If f is odd, then g is even, and if f is even, then g is odd.

Proof. Consider that we have that $\int_a^b f = -\int_b^a f$, as we have that $\int_a^b f + \int_b^c f = \int_a^c f$. Taking $c = a$, we see that $\int_a^b f + \int_b^a f = \int_a^a f = 0 \implies \int_a^b f = -\int_b^a f$.
Further, we have that $\int_0^x g(x)dx = \int_{-x}^0 g(-x)dx$ by problem 1.
We will first show that f odd $\implies g$ even.

$$\begin{aligned}
g(-x) &= \int_0^{-x} f(x)dx \\
&= \int_x^0 f(-x)dx \\
&= \int_x^0 -f(x)dx \\
&= -\int_x^0 f(x)dx \\
&= \int_0^x f(x)dx = g(x)
\end{aligned}$$

Now, we handle that f even $\implies g$ odd.

$$\begin{aligned}
g(-x) &= \int_0^{-x} f(x)dx \\
&= \int_x^0 f(-x)dx \\
&= \int_x^0 f(x)dx \\
&= \int_x^0 f(x)dx \\
&= -\int_0^x f(x)dx = -g(x)
\end{aligned}$$

□

Problem 5

Claim. $f : [a, b] \to \mathbb{R}$ is integrable iff $\forall \epsilon > 0, \exists s, t$ step functions such that $s \le f \le t$ and $\int_a^b (t - s)dx < \epsilon$.

Proof. ($\implies$) f is integrable $\implies \sup(\underline{I}(f)) = \inf(\overline{I}(f))$. This means that the approximation theorem furnishes $x = \int_a^b s \in \underline{I}(f) \mid \sup(\underline{I}(f)) - \frac{\epsilon}{2} < x \implies -\int_a^b s = \int_a^b -s < -\sup(\underline{I}(f)) + \frac{\epsilon}{2}$ for any $\epsilon > 0$. Similarly, we can get $y = \int_a^b t \in \overline{I}(f) \mid \inf(\underline{I}(f)) + \frac{\epsilon}{2} > y = \int_a^b t$. Adding the two inequalities, we have that

$$\int_a^b t + \int_a^b -s = \int_a^b (t - s) < \inf(\overline{I}(f)) - \sup(\underline{I}(f)) + \epsilon = \epsilon$$

$(\Longleftarrow)$ We have that $\forall \epsilon > 0, \exists s, t \mid s \leq f \leq t, \int_a^b (t-s) < \epsilon \implies \int_a^b t - \int_a^b s < \epsilon \implies \int_a^b t < \int_a^b s + \epsilon$.

We have in general that if for any $a \in A, b \in B, \forall \epsilon > 0, b < a + \epsilon \implies \sup(A) \geq \inf(B)$. To see this, consider that if $\sup(A) < \inf(B)$, we would have $\epsilon = \frac{\inf(B)-\sup(A)}{2}$ such that $a + \epsilon \leq \sup A + \epsilon = \frac{\inf(B)+\sup(A)}{2} < \inf(B) \leq b$. $\Rightarrow\!\Leftarrow$.

Also, we have that if $\forall a \in A, \forall b \in B, a \leq b \implies \sup(A) \leq \inf(B)$. To see this, consider that if $\sup(A) > \inf(B), \exists \epsilon > 0, a \in A, b \in B \mid \sup(A) - \epsilon < a, \inf(B) + \epsilon > b$ If we take $\epsilon = \frac{\sup(A)-\inf(B)}{2}$, then we have that $b < \frac{\sup(A)+\inf(B)}{2} < a$. $\Rightarrow\!\Leftarrow$.

The premise then gives us the fact that $\sup(\underline{I}(f)) \geq \inf(\overline{U}(f))$, as well as that since $s \leq t \implies \int_a^b s \leq \int_a^b t$, $\sup(\underline{I}(f)) \leq \inf(\overline{I}(f))$. Thus, to not violate trichotomy, we must have $\sup(\underline{I}(f)) = \inf(\overline{I}(f))$, and thus f is integrable. $\square$