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Assignment 3

STAT 4001
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3.1.2

$$\sum_{x=1}^{5} cx = 1 \implies c\frac{5(5+1)}{2} = 15c = 1 \implies c = \frac{1}{15}$$

3.1.5

The odds of selecting $n \in \mathbb{Z}$ balls is $\frac{\binom{7}{n}\binom{3}{5-n}}{\binom{10}{5}}$, as there are $\binom{7}{n}$ ways to choose n red balls, $\binom{3}{5-n}$ ways to choose the remaining balls, and $\binom{10}{5}$ ways to choose overall.

3.1.6

$$\sum_{x=0}^{5} \binom{15}{x} 0.5^x (1-0.5)^{15-x} = 0.1508$$

3.2.4

a

$$\int_{-\infty}^{\infty} f(x)dx = \int_{1}^{2} cx^2 dx = \frac{8c}{3} - \frac{c}{3} = 1 \implies c = \frac{3}{7}$$

See end for sketches.

b

$$P(X > \frac{3}{2}) = \int_{\frac{3}{2}}^{\infty} f(x)dx = \int_{\frac{3}{2}}^{2} \frac{3}{7}x^2 dx = \frac{1}{7}x^3\Big|_{\frac{3}{2}}^{2} = \frac{37}{56}$$

3.2.8

a

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} ce^{-2x} dx = \frac{c}{2} = 1 \implies c = 2$$

See end for sketches.

b

$$P(1 < X < 2) = \int_{1}^{2} 2e^{-2x} = -e^{-2x}\Big|_{1}^{2} = e^{-2} - e^{-4}$$

3.3.1

See end for sketches.

3.5.9

(It's out of order on the syllabus too!)

a

Note that this area is a rectangle of area 6.

$$f(x,y) = \begin{cases} \frac{1}{6} & (x,y) \in S \\ 0 & \text{otherwise} \end{cases}$$
$$f_1(x) = \int_1^4 \frac{1}{6} dy = \frac{1}{2}$$
$$f_2(y) = \int_0^2 \frac{1}{6} dy = \frac{1}{3}$$

Both the marginal pdfs are 0 for points not in S.

b

Yes; $f(x,y) = f_1(x)f_2(y)$.

3.4.1

a

$$\int_0^1 \int_0^2 c dx dy = 2c = 1 \implies c = \frac{1}{2}$$

b

$$P(X \geq Y) = \iint_A \frac{1}{2} dx dy = \frac{3}{4}$$

The area A is the trapezoid formed by the given rectangle below $y = x$.

3.4.6

a

Every point is equally likely, so the pdf is constant on the given area, which is a triangle with height and base $1, 4$, such that

$$P(x, y) = \frac{1}{2}$$

b

$$P((x, y) \in S_0) = \iint_{S_0} \frac{1}{2} dx dy = \frac{1}{2}\alpha$$

3.4.10

a

Note that the Taylor expansion of e^{2y} yields

$$e^{2y} = \sum_{x=0}^{\infty} \frac{(2y)^x}{x!}$$

Then,

$$\begin{aligned} \int_0^\infty (\sum_{x=0}^{\infty} \frac{(2y)^x}{x!} e^{-3y}) dy &= \int_0^\infty e^{2y} e^{-3y} dy \\ &= \int_0^\infty e^{-y} dy \\ &= -0 + e^0 = 1 \end{aligned}$$

It is in fact a joint pdf / pf.

b

$$P(X = 0) = \int_0^\infty \frac{(2y)^0}{0!} e^{-3y} dy = \frac{1}{3} e^0 = \frac{1}{3}$$

3.5.2

a

For $x = 0, 1, 2$, we have

$$f_1(x) = \sum_{y=0}^{3} \frac{1}{30}(x+y) = \frac{1}{30}(4x+6) = \frac{2x+3}{15}$$

For $y = 0, 1, 2, 3$, we have

$$f_2(y) = \sum_{x=0}^{2} \frac{1}{30}(x+y) = \frac{1}{30}(3y+3) = \frac{y+1}{10}$$

b

$$f_1(x)f_2(y) = \frac{2xy+3y+2x+3}{150} \neq f(x,y)$$

Thus, X, Y are not independent.

3.5.10

a

The circle has area π, such that the joint pdf f is

$$f(x,y) = \begin{cases} \frac{1}{\pi} & (x,y) \in S \\ 0 & \text{otherwise} \end{cases}$$

For $0 \leq x \leq 1$, $f(x,y) \neq 0 \iff -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$, so

$$f_1(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2}{\pi}\sqrt{1-x^2}$$

and is zero everywhere else.
Similarly,

$$f_2(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi} dx = \frac{2}{\pi}\sqrt{1-y^2}$$

on $-1 \leq y \leq 1$ and vanishes everywhere else.

b

$f_1(x)f_2(y) \neq f(x,y)$, so they are not independent.

3.6.3

a

Note that the area described is a circle of radius 3. Then,

$$\begin{aligned}
P(x,y) &= \begin{cases} \frac{1}{9\pi} & (x,y) \in S \\ 0 & \text{otherwise} \end{cases} \\
f_1(x) &= \int_{-2-\sqrt{9-(x-1)^2}}^{-2+\sqrt{9-(x-1)^2}} \frac{dy}{9\pi} \\
&= \frac{2\sqrt{9-(x-1)^2}}{9\pi} \\
g_2(y \mid x) &= \frac{f(x,y)}{f_1(x)} \\
&= \begin{cases} \frac{1}{2\sqrt{9-(x-1)^2}} & (x,y) \in S \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

b

$$g_2(y \mid x=2) = \begin{cases} \frac{2}{\sqrt{8}} & -2-\sqrt{8} < y < -2+\sqrt{8} \\ 0 & \text{otherwise} \end{cases}$$

3.6.8

a

$$f_1(x) = \int_0^1 \frac{2}{5}(2x+3y)dy = \frac{2}{5}(2x+\frac{3}{2}) \implies P(X > 0.8) = \int_{0.8}^1 \frac{2}{5}(2x+\frac{3}{2}) = 0.264$$

b

$$
\begin{aligned}
f_2(y) &= \int_0^1 \frac{2}{5}(2x+3y)dx = \frac{2}{5}(1+3y) \\
g_1(x \mid y) &= \frac{f(x,y)}{f_2(y)} \\
&= \frac{2x+3y}{1+3y} \\
g(x > 0.8 \mid y = 0.3) &= \int_{0.8}^1 \frac{2x+0.9}{1+0.9}dx = 0.284
\end{aligned}
$$

c

$$
\begin{aligned}
g_2(y \mid x) &= \frac{f(x,y)}{f_1(x)} \\
&= \frac{2x+3y}{2x+1.5} \\
g(y > 0.8 \mid x = 0.3) &= \int_{0.8}^1 \frac{0.6+3x}{2.1}dx = 0.314
\end{aligned}
$$

3.7.3

a

$$\int_0^\infty \int_0^\infty \int_0^\infty ce^{-(x_1+2x_2+3x_3)}dx_1dx_2dx_3 = c\int_0^\infty e^{-3x_3}\int_0^\infty e^{-2x_2}\int_0^\infty e^{-x_1}dx_1dx_2dx_3 = \frac{1}{6}c$$

Thus, $\frac{1}{6}c = 1 \implies c = 6$.

b

$$f_{13}(x_1, x_3) = \int_0^\infty 6e^{-(x_1+2x_2+3x_3)}dx_2 = 3e^{-(x_1+3x_3)}$$

c

$$
\begin{aligned}
f_{23}(x_2, x_3) &= \int_0^\infty 6e^{-(x_1+2x_2+3x_3)}dx_1 = 6e^{-(2x_2+3x_3)} \\
g_1(x_1 \mid x_2, x_3) &= \frac{f(x_1, x_2, x_3)}{f_{23}(x_2, x_3)} \\
&= e^{-x_1}
\end{aligned}
$$

$$\int_0^1 e^{-x_1} = 1 - e^{-1}$$

3.7.12

$$
\begin{aligned}
g_1(y, z \mid w) &= \frac{f(y, z, w)}{f_w(w)} \\
&= \frac{f(y, z, w)}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y, z, w)dydz} \\
g_2(y \mid w) &= \frac{f_{y,w}(y, w)}{f_w(w)} \\
&= \frac{\int_{\infty}^{\infty} f(y, z, w)dz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y, z, w)dydz}
\end{aligned}
$$

Since we have that the denominator is independent of z,

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{f(y, z, w)}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y, z, w)dydz}dz \\
&= \int_{-\infty}^{\infty} g(y, z \mid w)dz
\end{aligned}
$$

3.8.1

Note that on the interval $0 < x < 1$, $y = 1 - x^2$ is injective, such that the cdf of Y can be given by

$$P(Y \leq y) = P(1 - x^2 \leq y) = P(1 - y \leq x^2) = \int_{(1-y)^{\frac{1}{2}}}^{1} 3x^2dx = 1 - (1 - y)^{\frac{3}{2}}$$

Then, differentiating, the pdf is

$$g(y) = \frac{3}{2}(1 - y)^{\frac{1}{2}}$$

for $0 < y < 1$.

3.8.14

Put $Y = cX + d$. We compute cdf

$$P(Y \leq y) = P(cX + d \leq y) = P(x \leq \frac{y-d}{c}) = \int_a^{\frac{y-d}{c}} \frac{dx}{b-a} = \frac{1}{b-a}(\frac{y-d}{c} - a)$$

Differentiating,

$$g(y) = \frac{1}{c(b-a)}$$

for $ca + d \leq y \leq cb + d$, as this is where $cX + d$ maps the interval $[a, b]$.

3.9.6

Start by computing the cdf. On $0 \leq z \leq 1$, we have that

$$G(z) = \int_0^{\frac{z}{2}} \int_x^{z-x} 2(x+y)dydx = \frac{z^3}{3}$$

Thus, we have that $g(z) = \frac{dG}{dz} = z^2$ on $0 \leq z \leq 1$. Then, on $1 \leq z \leq 2$, we have that

$$G(z) = 1 - \int_{\frac{z}{2}}^1 \int_{z-y}^y 2(x+y)dxdy = z^2 - \frac{z^3}{3}$$

Thus, we have that $g(z) = 2z - z^2$ for $1 \leq z \leq 2$.

3.9.16

As a corollary of the factorization, we must have that $f_{12}(x_1, x_2) = \lambda g(x_1, x_2)$, $f_{345}(x_3, x_4, x_5) = \lambda^{-1} h(x_3, x_4, x_5)$, where $\lambda = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x_3, x_4, x_5)dx_3dx_4dx_5$.
Then, we have that $f(x_1, x_2, x_3, x_4, x_5) = f_{12}(x_1, x_2)f_{345}(x_3, x_4, x_5)$.

$$\begin{aligned}
P(Y_1 \in A_1 \wedge Y_2 \in A_2) &= \int\int\int\int\int_{r_1(x_1,x_2)\in A_1 \wedge r_2(x_3,x_4,x_5)\in A_2} f(x_1 \ldots x_5)dx_1 \ldots dx_5 \\
&= \int\int_{r_1(x_1,x_2)\in A_1} f_{12}(x_1, x_2)dx_1dx_2 \int\int\int_{r_2(x_1,x_2,x_3)\in A_2} f_{345}f(x_3, x_4, x_5)dx_3dx_4dx_5 \\
&= P(Y_1 \in A_1)P(Y_2 \in A_2)
\end{aligned}$$

3.9.19

We have by the theorem on convolutions that the pdf is

$$g(y) = \int_{-\infty}^{\infty} f(y-z)f(z)dz = \int_0^y e^{z-y}e^{-z}dz = \int_0^y e^{-y}dz = ye^{-y}$$

For $y < 0$, we have that $g(y) = 0$, so

$$g(y) = \begin{cases} ye^{-y} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$