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Homework 10

MATH 1207
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Apostol p.236-237 no.17

We proceed with integration by parts, which can be found in the Apostol reading, to proceed.

$$\begin{aligned}
u &= \log^2(x),\ du = \frac{2\log(x)dx}{x} \\
v &= x,\ dv = dx \\
\implies \int \log^2(x)dx &= x\log^2(x) - 2\int \log(x)dx \\
u &= \log(x),\ du = \frac{dx}{x} \\
v &= x,\ dv = dx \\
\implies \int \log^2(x)dx &= x\log^2(x) - 2(x\log(x) - \int dx) \\
&= x\log^2(x) - 2x\log(x) + 2x + C
\end{aligned}$$

Apostol p.236-237 no.19

We again proceed with integration by parts:

$$\begin{aligned}
u &= \log^2(x),\ du = \frac{2\log(x)dx}{x} \\
v &= \frac{1}{2}x^2,\ dv = xdx \\
\implies \int x\log^2(x)dx &= \frac{1}{2}x^2\log^2(x) - \int x\log(x)dx \\
u &= \log(x),\ du = \frac{dx}{x} \\
v &= \frac{1}{2}x^2,\ dv = xdx \\
\implies \int x\log^2(x)dx &= \frac{1}{2}x^2\log^2(x) - (\frac{1}{2}x^2\log(x) - \frac{1}{2}\int xdx) \\
&= \frac{1}{2}x^2\log^2(x) - \frac{1}{2}x^2\log(x) + \frac{1}{4}x^2 + C
\end{aligned}$$

Apostol p.236-237 no.30

Claim.

$$\forall r \in \mathbb{Q}, a > 0, \log(a^r) = r\log(a)$$

Proof. Let $r = \frac{m}{n}, m, n \in \mathbb{Z}, n \geq 1$. It has been shown in the Apostol reading that for $n \in \mathbb{Z}, n \geq 1, \log(a^n) = n\log(a)$. Then, we have that

$$\begin{aligned}\log(a^r) &= \log(a^{\frac{m}{n}}) \\ &= \log((a^{\frac{1}{n}})^m) \\ &= m\log(a^{\frac{1}{n}})\end{aligned}$$

Further, we have that $n\log(a^{\frac{1}{n}}) = \log((a^{\frac{1}{n}})^n) = \log(a)$. Thus,

$$\log(a^r) = m\log(a^{\frac{1}{n}}) = \frac{m}{n}n\log(a^{\frac{1}{n}}) = \frac{m}{n}\log(a) = r\log(a)$$

□

Apostol p.250 no.39

Claim. Let f be a function defined everywhere on the real axis, with a derivative f' that satisfies the equation

$$\forall x, f'(x) = cf(x)$$

, where c is a constant. There is a constant K such that $f(x) = Ke^{cx}$ for every x.

Proof. Consider $g(x) = f(x)e^{-cx}$. We will show that this is constant.

$$\begin{aligned}g'(x) &= f'(x)e^{-cx} + -cf(x)e^{-cx} \\ &= cf(x)e^{-cx} - cf(x)e^{-cx} \\ &= 0\end{aligned}$$

Thus, we have that $g(x) = K$ for some constant K, and then $K = f(x)e^{-cx} \implies f(x) = Ke^{cx}$. □

Apostol p.382 no.29

Claim. A series cannot converge to two different limits.

Proof. Suppose that a series $\{a_n\}$ converges to L, K, and that $L \neq K$. Then, we have that for any $\frac{\epsilon}{2} > 0, \exists N_L \in \mathbb{Z}_{>0} \mid \forall n > N_L, |L - a_n| < \frac{\epsilon}{2}$. Similarly, $\exists N_K \in \mathbb{Z}_{>0} \mid \forall n > N_K, |K - a_n| < \frac{\epsilon}{2} \implies |L - K| = |(L - a_n) - (K - a_n)| \leq |L - a_n| + |K - a_n| < \epsilon$. Then, we have that since $\forall \epsilon, |L - K| < \epsilon$, $L = K$. (To see this last claim, $L \neq K \implies |L - K| \neq 0 \implies |L - K| > 0$. Then, take $\epsilon = \frac{|L-K|}{2}$, and $\Rightarrow\!\Leftarrow$.) □

Apostol p.382 no.30

Claim.

$$\lim_{n\to\infty} a_n = 0 \implies \lim_{n\to\infty} a_n^2 = 0$$

Proof. Since we have that $\lim_{n\to\infty} a_n = 0$, $\forall\epsilon\exists N \in \mathbb{Z}_{>0} \mid \forall n > N, |a_n| < \epsilon$. Further, $\exists N_1 \mid n > N_1 \implies |a_n| < 1$. For any ϵ, take now $\max(N, N_1)$, which means that both $|a_n| < \epsilon$ and $|a_n| < 1$, so $|a_n^2| < |a_n| < 1$, and also $|a_n^2| < |a_n| < \epsilon$ as well for $n > \max(N, N_1)$. Thus, we have that $\lim_{n\to\infty} a_n^2 = 0$. $\square$

Problem 1

Claim. If $\lim_{n\to\infty} a_n = A, \lim_{n\to\infty} b_n = B$, and $a_n \le b_n$ for all n, then $A \le B$.

Proof. Suppose that $A > B$. Then, take $epsilon = \frac{A-B}{3}$. We have that $\exists N_A \in \mathbb{Z}_{>0} \mid n > N_A \implies |a_n - A| < \frac{A-B}{2}$, and also $\exists N_B \in \mathbb{Z}_{>0} \mid n > N_B \implies |b_n - B| < \frac{A-B}{2}$. Consider $N = \max(N_A, N_B)$. Then, $n > N \implies A - \frac{A-B}{2} < a_n < A + \frac{A-B}{2}, B - \frac{A-B}{2} < b_n < B + \frac{A-B}{2} \implies b_n < \frac{A+B}{2} < a_n$. $\Rightarrow\Leftarrow$, so $A \le B$. $\square$

Note that we can't make a stronger claim then $A \le B$ even if $a_n < b_n$. Consider the sequences $a_n = \frac{1}{2^{n+1}}, b_n = \frac{1}{2^n}$. We have that $b_n = 2a_n \implies b_n > a_n$. However, these both converge to 0, as we can take $N = \lceil \log_2(\frac{1}{\epsilon}) \rceil$ and $N = \lceil \log_2(\frac{1}{\epsilon}) \rceil + 1$ respectively.

Problem 2

Claim. Let $\{a_n\}, \{b_n\}$ be sequences such that there exists $M \in \mathbb{Z}_{>0}$ such that $\forall n > M, a_n = b_n$. a_n converges if and only if b_n converges, and if they converge they converge to the same value.

Proof. Suppose that $\{a_n\}$ converges to L. Note that proving this single direction is enough; the opposite direction can be shown by just swapping the names of a_n and b_n with each other. We have that $\forall\epsilon > 0$, $\exists N \in \mathbb{Z}_{>0} \mid n > N \implies |a_n - L| < \epsilon$. Then, consider $N' = \max(N, M)$. We then have that $\forall n > N', a_n = b_n \implies |a_n - L| = |b_n - L| < \epsilon$, and so $\{b_n\}$ converges to L as well. $\square$

Problem 4

Claim. If $\{x_n\}$ converges, then $\{|x_n|\}$ also converges.

Proof. Let $\{x_n\}$ converge to L. Then, $\forall \epsilon > 0, \exists N \in \mathbb{Z}_{>0} \mid n > N \implies |x_n - L| < \epsilon$. Further, we have that $||x_n| - |L|| \leq |x_n - L| < \epsilon$ as a conseqence of the previously proved reverse triangle inequality. Thus, $\{|x_n|\}$ converges to $|L|$. $\square$

Note that the converse isn't true: consider that

$$x_n = \begin{cases} -1 & n \equiv 0 \mod 2 \\ 1 & n \equiv 1 \mod 2 \end{cases}$$

has $\{|x_n|\}$ converge to 1, but $\{x_n\}$ diverges.