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Homework 1

MATH 1208
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Apostol p.451 no.6

a

The components of D are $(x+z, x+y+z, y+z)$

b

If $D=0$, then $x+z=0$, and $x+y+z=0$. Then, $y=0$, and since $y+z=0$, we have that $z=0$ and finally $x=0$ as $x+z=0$.

c

Take $x=-1, y=1, z=2$.

Apostol p.451 no.7

a

The components of D are $(x+2z, x+y+z, x+y+z)$.

b

Take $x=-2, y=1, z=1$.

c

$D=(1,2,3) \implies x+y+z=2$ and $x+y+z=3$. $\Rightarrow\Leftarrow$, so no such picks of x, B, z have $D=(1,2,3)$.

Apostol p.456 no.4

Claim. $\forall B \in \mathbb{R}^N, A \cdot B = 0 \implies A = 0$.

Proof. Let the i^{th} component of A be a_i. Taking $B=A$, we have that $A \cdot A = 0 \implies \sum_{i=1}^n a_i^2 = 0 \implies \forall i, a_i = 0 \implies A = 0$. $\square$

Apostol p.456 no.19

Claim. $||A+B||^2 - ||A-B||^2 = 4A \cdot B$

Proof. Let the i^{th} component of A be a_i, and the corresponding component of B be b_i.

$$\begin{aligned}
||A+B||^2 - ||A-B||^2 &= \sum_{i=1}^{n}(a_i+b_i)^2 - \sum_{i=1}^{n}(a_i-b_i)^2 \\
&= \sum_{i=1}^{n}(a_i+b_i)^2 - (a_i-b_i)^2 \\
&= \sum_{i=1}^{n} a_i^2 + 2a_ib_i + b_i^2 - a_i^2 + 2a_ib_i - b_i^2 \\
&= \sum_{i=1}^{n} 4a_ib_i \\
&= 4\sum_{i=1}^{n} a_ib_i \\
&= 4A \cdot B
\end{aligned}$$

□

Apostol p.555 no.9

Odd functions are indeed a vector space; since they are a subset of $\mathcal{F}(\mathbb{R}, \mathbb{R})$, we have that we only need to show closure under scalar multiplication and vector addition.

$f : \mathbb{R} \to \mathbb{R}$ is odd $\iff$ $f(x) = -f(-x)$, and so $2f(x) = -2f(-x)$, and so it is closed under vector multiplication. Similarly, we have that if $g, f : \mathbb{R} \to \mathbb{R}$ are odd, then $(f+g)(x) = f(x) + g(x) = -f(-x) - g(-x) = -(f+g)(-x)$, and it is closed under vector addition.

Apostol p.555 no.10

This is also a vector space; we prove the same thing as above; for any function f bounded by M, and scalar multiple cf is bounded by cM. Similarly, two functions bounded by M_1, M_2 has their sum bounded by $M_1 + M_2$.

Apostol p.555 no.11

This is not a vector space, being not closed under scalar multiplication; any function f that is increasing has additive inverse $-f$ deceasing instead.

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Homework 1

Apostol p.555 no.21

We know that the space of all series is a vector space over $\mathbb{R}$.

This set in particular is also a vector space. It is closed as we have that if $\{a_n\}$ converges to L, then $\{ca_n\}$ converges to $\{cL\}$, and if $\{a_n\}, \{b_n\}$ converge to K, L, then $\{a_n + b_n\}$ converges to $K + L$.

Apostol p.556 no.29

We denote $\cdot$ and exponentiation in the usual way, and xy will denote the product of x, y in V.

Commutativity:

$$x + y = x \cdot y = y \cdot x = y + x$$

Associativity:

$$(x + y) + z = (x \cdot y) \cdot z = x \cdot (y \cdot z) = x + (y + z)$$

$$(cd)x = x^{cd} = (x^d)^c = c(dx)$$

Distributivity:

$$c(x + y) = (x \cdot y)^c = (x^c) \cdot (y^c) = cx + cy$$

$$(c + d)x = x^{c+d} = x^c \cdot x^d = cx + cd$$

Identity:

$$x + 0 = x \cdot 1 = x$$

$$1x = x^1 = x$$

Inverse:

$$cx + c^{-1}x = x^c \cdot x^{-c} = 0$$

Closure:

$$x, y \in \mathbb{R}_{>0} \implies x + y = (x \cdot y) > 0 \implies x + y \in \mathbb{R}_{>0}$$

$$x \in \mathbb{R}_{>0} \implies cx = (x^c) > 0 \implies xc \in \mathbb{R}_{>0}$$

Problem 1

Claim. For vector spaces U, V over F, a function $f : U \to V$ is linear iff

$$f(\sum_{i=1}^{n} c_i X_i) = \sum_{i=1}^{n} c_i f(X_i).$$

Proof. ($\implies$) Proceed with induction on n. The base case $n = 1$ follows immediately from the linearity of f: $f(c_i X_i) = c_i f(x_i)$.

Suppose that the above holds for $n = k$. Then,

$$f(\sum_{i=1}^{k+1} c_i X_i) = f(\sum_{i=1}^{k} c_i X_i) + f(c_{k+1} X_{k+1}) = \sum_{i=1}^{k} c_i f(X_i) + c_{k+1} f(X_{k+1}) = \sum_{i=1}^{k+1} c_i f(X_i).$$

The identity holds for all $n \in \mathbb{Z}_{>0}$.

($\impliedby$) Take $n = 1$. Then, $f(c_1 X_1) = c_1 f(X_1)$. When $n = 2, c_1 = c_2 = 1$, $f(c_1 X_1 + c_2 X_2) = f(X_1 + X_2) = f(X_1) + f(X_2)$, and so f is linear. $\square$

Problem 2

Claim. For vector spaces U, V over F, a linear map $f : U \to V$ is injective iff $\ker f = \{0\}$.

Proof. First we will show that $0 \in \ker f$ in every case of f is linear, as $f(0) = f(0(0)) = 0f(0) = 0$.

($\implies$) Since f is injective, we have that only one member of the domain is mapped to 0. Since $f(0) = 0$ always, $\ker f = \{0\}$.

($\impliedby$) Suppose that f is not injective. Then, we must have that $\exists x, y \mid f(x) = f(y), x \neq y$. Then, $f(x) - f(y) = f(x - y) = 0 \implies x - y \in \ker f$, which means that $\ker f \neq \{0\}$, and $x - y \neq 0$. $\square$

Problem 4

Claim. If $f : U \to V$ and $g : V \to W$ are linear maps, then $g \circ f$ is also linear.

Proof.

$$\begin{aligned}(g \circ f)(cu) &= g(f(cu)) \\ &= g(cf(u)) \\ &= cg(f(u)) \\ &= c(g \circ f)(u) \\ (g \circ f)(u_1 + u_2) &= g(f(u_1 + u_2) \\ &= g(f(u_1) + f(u_2)) \\ &= g(f(u_1)) + g(f(u_2)) \\ &= (g \circ f)(u_1) + (g \circ f)(u_2)\end{aligned}$$

□

Problem 5

a

Claim. With $g : V \to W$ is a fixed linear map, then $L_g : \mathcal{L}(U, V) \to \mathcal{L}(V, W)$ is linear where $L_g(f) = g \circ f$.

Proof.

$$\begin{aligned}(L_g(cf))(u) &= (g \circ (cf))u) \\ &= g(cf(u)) \\ &= cg(f(u)) \\ &= (c(g \circ f))(u) \\ &= (cL_g(f))(u) \\ (L_g(f_1 + f_2))(u) &= (g \circ (f_1 + f_2))(u) \\ &= g((f_1 + f_2)(u)) \\ &= g(f_1(u) + f_2(u)) \\ &= g(f_1(u)) + g(f_2(u)) \\ &= (L_g(f_1))(u) + (L_g(f_2))(u)\end{aligned}$$

We then have that $L_g(cf) = cL_g(f)$ and $L_g(f_1 + f_2) = L_g(f_1) + L_g(f_2)$. □

b

Claim. With $f : U \to V$ is a fixed linear map, then $R_g : \mathcal{L}(V, W) \to \mathcal{L}(U, W)$ is linear where $R_f(g) = g \circ f$.

Proof.

$$\begin{aligned}
(R_f(cg))(u) &= ((cg) \circ f)u) \\
&= cg(f(u)) \\
&= (c(g \circ f))(u) \\
&= (cR_f(f))(u) \\
(R_f(g_1 + g_2))(u) &= ((g_1 + g_2) \circ f)(u) \\
&= ((g_1 + g_2)(f(u)) \\
&= g_1(f(u)) + g_2(f(u)) \\
&= (R_f(g_1))(u) + (R_f(g_2))(u)
\end{aligned}$$

We then have that $R_f(cg) = cR_f(g)$ and $R_f(g_1 + g_2) = R_f(g_1) + R_f(g_2)$. $\square$