Apostol p.263 no.10

a

Claim. If $\nabla f(x) = 0$ for every x in $B(a)$, then $f(x)$ is constant on $B(a)$.

Proof. We have that $f'(x; y) = \nabla f(x) \cdot y = 0$, so $f'(x; y) = 0$ on $B(a)$ for any $x \in B(a)$. The mean value theorem then yields for some $0 < \theta < 1$ that on any point $x \in B(a)$,

$$f(a) - f(x) = f'(x + \theta(a - x); a - x) = 0 \implies f(x) = f(a)$$

Thus, $f(x)$ is constant everywhere in $B(a)$. □

b

Claim. If $f(x) \leq f(a)$ for all $x \in B(a)$, then $\nabla f(a) = 0$.

Proof. Suppose that $\nabla f(a) \neq 0$. Then, we have that

$$f'(a; \nabla f(a)) = \nabla f(a) \cdot \nabla f(a) > 0$$

Then, let

$$f'(a; \nabla f(a)) = \lim_{t \to 0} \frac{f(a + t\nabla f(a)) - f(a)}{t} = c > 0$$

In particular, this limit implies that $\exists \delta > 0$ such that $0 < |t| < \delta \implies |\frac{f(a+t\nabla f(a))-f(a)}{t} - c| < \frac{c}{2} \implies \frac{f(a+t\nabla f(a))-f(a)}{t} > 0$. Then, $f(a + t\nabla f(a)) > f(a)$, which $\Rightarrow\!\Leftarrow$, as we can always pick this t such that $0 < |t| < \delta$ and $a + t\nabla f(a) \in B(a)$ (if $B(a)$ has radius δ', then pick $t \mid |t| < \min(\delta, \frac{\delta'}{\nabla f(a)})$).
Thus, we have that $\nabla f(x) = 0$. □

Apostol p.269 no.12

Claim. If $\nabla f(x, y, z)$ is always parallel to (x, y, z), then f must assume equal values at the points $(0, 0, a)$ and $(0, 0, -a)$.

Proof. Consider

$$g(x) = f((0, a\cos(x), a\sin(x)))$$

Then, from the chain rule in Apostol,

$$g'(x) = \nabla f(0, a\cos(x), a\sin(x))\cdot(0, -a\sin(x), a\cos(x)) = -\alpha a\cos(x)\sin(x) + \alpha a\sin(x)\cos(x) = 0$$

where $\nabla f(0, a\cos(x), a\sin(x)) = \alpha(0, a\cos(x), a\sin(x))$. Then, we have that g is constant and that $f(0, 0, a) = g(\frac{\pi}{2}) = g(\frac{-\pi}{2}) = f(0, 0, -a)$.

□

Apostol p.276 no.3ab

a

From the chain rule given in Apostol that applies to scalar fields,

$$\frac{\partial F}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial X}{\partial s} + \frac{\partial f}{\partial s}\frac{\partial Y}{\partial s}$$
$$\frac{\partial F}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial X}{\partial t} + \frac{\partial f}{\partial s}\frac{\partial Y}{\partial t}$$

b

Applying the chain rule to the above,

$$\begin{aligned}\frac{\partial^2 F}{\partial s^2} &= \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x}\right)\frac{\partial X}{\partial s} + \frac{\partial f}{\partial x}\frac{\partial}{\partial s}\left(\frac{\partial X}{\partial s}\right) + \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial y}\right)\frac{\partial Y}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial}{\partial s}\left(\frac{\partial Y}{\partial s}\right)\\ &= \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x}\right)\frac{\partial X}{\partial s} + \frac{\partial f}{\partial x}\frac{\partial^2 X}{\partial s^2} + \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial y}\right)\frac{\partial Y}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial^2 Y}{\partial s^2}\end{aligned}$$

Computing $\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x}\right)$ with the chain rule,

$$\begin{aligned}\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial D_1 f}{\partial x}\frac{\partial X}{\partial s} + \frac{\partial D_1 f}{\partial y}\frac{\partial Y}{\partial s}\\ &= \frac{\partial^2 f}{\partial x^2}\frac{\partial X}{\partial s} + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial Y}{\partial s}\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial D_2 f}{\partial x}\frac{\partial X}{\partial s} + \frac{\partial D_2 f}{\partial y}\frac{\partial Y}{\partial s}\\ &= \frac{\partial^2 f}{\partial x \partial y}\frac{\partial X}{\partial s} + \frac{\partial^2 f}{\partial y^2}\frac{\partial Y}{\partial s}\end{aligned}$$

Substituting, we arrive at

$$\begin{aligned}\frac{\partial^2 F}{\partial s^2} &= \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial X}{\partial s}\right)^2 + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial Y}{\partial s}\frac{\partial X}{\partial s} + \frac{\partial f}{\partial x}\frac{\partial^2 X}{\partial s^2} + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial X}{\partial s}\frac{\partial Y}{\partial s} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial Y}{\partial s}\right)^2 + \frac{\partial f}{\partial y}\frac{\partial^2 Y}{\partial s^2}\\ &= \frac{\partial f}{\partial x}\frac{\partial^2 X}{\partial s^2} + \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial X}{\partial s}\right)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial Y}{\partial s}\frac{\partial X}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial^2 Y}{\partial s^2} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial Y}{\partial s}\right)^2\end{aligned}$$

Apostol p.276 no.14

a

$$\begin{aligned} Df &= \begin{bmatrix} D_1 f_1 & D_2 f_1 \\ D_1 f_2 & D_2 f_2 \end{bmatrix} \\ &= \begin{bmatrix} e^{x+2y} & 2e^{x+2y} \\ 2\cos(y+2x) & \cos(y+2x) \end{bmatrix} \\ Dg &= \begin{bmatrix} D_1 g_1 & D_2 g_1 & D_3 g_1 \\ D_1 g_2 & D_2 g_2 & D_3 g_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4v & 9w^2 \\ -2u & 2 & 0 \end{bmatrix} \end{aligned}$$

b

$$h(u,w,v) = e^{u+v^2+3w^3+4v-2u^2} i + \sin(2v - u^2 + 2u + 4v^2 + 6w^3) j$$

c

$$Dh(1,-1,1) = \begin{bmatrix} 1 & 2 \\ 2\cos(9) & \cos(9) \end{bmatrix} \begin{bmatrix} 1 & -4 & 9 \\ -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 9 \\ 0 & -6\cos(9) & 9\cos(9) \end{bmatrix}$$

Apostol p.281 no.1

Consider

$$f(x,y) = \begin{cases} 0 & xy = 0 \\ \frac{3}{2}(x+y) & \text{otherwise} \end{cases}$$

Then,

$$\begin{aligned} D_1 f(0,0) &= \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = 0 \\ D_2 f(0,0) &= \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = 0 \\ f((0,0);(1,1)) &= \lim_{t\to 0} \frac{f(t,t) - f(0,0)}{t} = \lim_{t\to 0} \frac{3t}{t} = 3 \end{aligned}$$

It cannot be differentiable as if it were, we would know that $f'((0,0);(1,1)) = D_1 f(0,0) + D_2 f(0,0) = 0$.

Apostol p.281-282 no.12a

We have that $||\mathbf{r}|| = r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Then, for $r \neq 0$,

$$\begin{aligned}
A \cdot \nabla \left(\frac{1}{r}\right) &= A \cdot \nabla \left((x^2 + y^2 + z^2)^{-\frac{1}{2}}\right) \\
&= A \cdot \left(-x(x^2 + y^2 + z^2)^{-\frac{3}{2}}, -y(x^2 + y^2 + z^2)^{-\frac{3}{2}}, -z(x^2 + y^2 + z^2)^{-\frac{3}{2}}\right) \\
&= A \cdot \left(-\frac{1}{r^3}\,(x, y, z)\right) \\
&= -\frac{A \cdot \mathbf{r}}{r^3}
\end{aligned}$$

Problem 1

Claim. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be linear. The derivative of F at any $x \in \mathbb{R}^n$ is just F itself.

Proof. We want for a total derivative T that

$$\begin{aligned}
\lim_{H \to 0} \frac{||F(x + H) - F(x) - T(H)||}{||H||} &= \lim_{H \to 0} \frac{||F(x) + F(H) - F(x) - T(H)||}{||H||} \\
&= \lim_{H \to 0} \frac{||F(H) - T(H)||}{||H||} = 0
\end{aligned}$$

Then, we see that taking $T = F$ clearly works, and since we proved the uniqueness of total derivatives in class, we are done. □

Problem 2

Claim. Let $G : \mathbb{R}^n \to \mathbb{R}$ be homogeneous and continuous. G is differentiable $\iff$ G is linear.

Proof. ($\implies$) Now, consider that since G is differentiable, we have that for any fixed y, putting T as the derivative,

$$\lim_{t \to 0} \frac{G(x + ty) - G(x) - T(ty)}{||ty||} = 0$$

In particular, if we select $y = x$,

$$\begin{aligned}
\lim_{t \to 0} \frac{G(x+tx) - G(x) - T(tx)}{||tx||} &= \lim_{t \to 0} \frac{(t+1)G(x) - G(x) - tT(x)}{t||x||} \\
&= \lim_{t \to 0} \frac{t(G(x) - T(x))}{t||x||} \\
&= \frac{G(x) - T(x)}{||x||} = 0
\end{aligned}$$

Then we have that $G = T$ for any x.

The mean value theorem yields that for any $a, b \in \mathbb{R}^n$ and some $0 < \theta < 1$,

$$G(a) - G(b) = G'(b + \theta(a-b); a-b)$$

However, we have that $G'(b + \theta(a-b); a-b) = (G'(b + \theta(a-b)))(a-b)$, but the total derivative anywhere is just G itself, and thus

$$G(a) - G(b) = G(a-b)$$

So we finally have that $G(a) - G(b) = G(a-b)$; replacing b with $-b$, since this holds for any b, we have that $G(a+b) = G(a) + G(b)$, and that $G(tx) = tG(x)$, and so G is linear.

($\Longleftarrow$) As in the first problem, we have that the derivative of G at any point is G itself, and is thus differentiable. $\square$

Problem 3

Claim. Suppose that $G : \mathbb{R}^n \to \mathbb{R}^m$, with U open in $\mathbb{R}^n$, is written with coordinates as

$$F = (F_1, F_2, \ldots, F_m)$$

where $F_i : U \to \mathbb{R}$ for $1 \leq i \leq m$. Then, for any point $x \in U$, F is differentiable at $x \iff F_i$ is differentiable at x for all $1 \leq i \leq m$.

Proof. ($\Longrightarrow$) The total derivative T satisfies that

$$\lim_{H \to 0} \frac{||F(x+H) - F(x) - T(x)||}{||H||} = 0 \iff \lim_{H \to 0} \frac{F(x+H) - F(x) - T(x)}{||H||} = 0$$

In particular, we have that this means that each individual component must approach 0, i.e. for some $X = (x_1, x_2, \ldots, x_n)$, $\lim_{H \to 0} X = 0 \iff \lim_{H \to 0} x_i = 0$. This was shown in class.

Then, putting $T(x) = (T_1(x), T_2(x), \ldots, T_m(x))$ in the same way as F,

$$
\begin{aligned}
&\lim_{H\to 0} \frac{F(x+H) - F(x) - T(H)}{||H||} \\
=&\lim_{H\to 0} \frac{(F_1(x+H), \ldots, F_m(x+H)) - (F_1(x), \ldots, F_m(x)) - (T_1(H), \ldots, T_m(H))}{||H||} \\
=&\lim_{H\to 0} \left(\frac{F_1(x+H) - F_1(x) - T_1(H)}{||H||}, \ldots, \frac{F_m(x+H) - F_m(x) - T_m(H)}{||H||} \right) \\
=&0
\end{aligned}
$$

From above, we have that for $1 \le i \le m$,

$$\lim_{H\to 0} \frac{F_i(x+H) - F_i(x) - T_i(H)}{||H||} = 0$$

Then, we must have that F_i is differentiable with derivative T_i.
($\Longleftarrow$) Consider $T = (T_1, T_2, \ldots, T_n)$, where F_i has derivative T_i. Then, the above relation

$$\lim_{H\to 0} \frac{F(x+H) - F(x) - T(H)}{||H||} = \lim_{H\to 0} \left(\frac{F_1(x+H) - F_1(x) - T_1(H)}{||H||}, \ldots, \frac{F_m(x+H) - F_m(x) - T_m(H)}{||H||} \right)$$

still holds, and we have that T is the total derivative of F as each component of $\frac{F(x+H)-F(x)-T(x)}{||H||}$ is 0 in the limit, and thus $\frac{F(x+H)-F(x)-T(x)}{||H||}$ must then be zero itself as $H \to 0$. $\square$

Problem 4

Claim. If $F : U \to \mathbb{R}^m$ is a function such that all of the entries of its Jacobian matrix are well-defined and continuous, then F is differentiable.

Proof. We have that each F_i is differentiable, as all of its partial derivatives, which is the i^{th} row of the Jacobian, are continuous, and thus F_i is C^1 on U by assumption. Since we showed in class that F_i being C^1 on $U \implies F$ is differentiable on U, from the above problem, F itself is differentiable. $\square$