# MATH 4065 HW 7

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## 18

Let $C$ be centered at $z_0$ with radius $R$, and the interior disc be $D$.

We have that $f$ is holomorphic in a region containing $C$ and the interior of $C$, so we have that we can explicitly define $\gamma_s(\theta) = (z_0 + s(z - z_0)) + (1-s)Re^{i\theta}$ for $s \in [0,1]$ and $z$ any point inside $C$. Then, for any given $s$, $\gamma_s$ is the circle centered at $z + s(z - z_0)$, and in particular, we have that any point $\gamma_s(\theta)$ satisfies that

$$|\gamma_s(\theta) - z_0| = |s(z - z_0) + (1-s)Re^{i\theta}| \leq s|z - z_0| + (1-s)R$$

and since $z$ is contained inside $C$, $|z - z_0| < R$, so $|\gamma_s(\theta) - z_0| \leq sR + (1-s)R = R$ so $\gamma_s$ is always contained inside of $C$ for $s > 0$, and is $C$ for $s = 0$, so we have that $\gamma_s$ is always contained in the region where $f$ is holomorphic. In particular, we have that we can consider $\Omega$ to be a disc slightly larger that $C$, containing $C$, which is then simply connected, and note that we have $\gamma_s$ is contained in $\Omega$ for any $s \in [0,1]$ so since each $\gamma_s$ are homotopic to each other, we get that

$$\int_C \frac{f(w) - f(z)}{w - z} dw = \int_{\gamma_0} \frac{f(w) - f(z)}{w - z} dw = \int_{\gamma_{1-\epsilon}} \frac{f(w) - f(z)}{w - z} dw$$

where explicitly, $\gamma_{1-\epsilon}(\theta) = z - \epsilon(z - z_0) + \epsilon Re^{i\theta}$.

$$\int_{\gamma_{1-\epsilon}} \frac{f(w) - f(z)}{w - z} dw \leq 2\pi\epsilon \sup \left| \frac{f(w) - f(z)}{w - z} \right|$$

so in the limit,

$$\int_C \frac{f(w) - f(z)}{w - z} dw = \int_{\gamma_{1-\epsilon}} \frac{f(w) - f(z)}{w - z} dw \leq \lim_{\epsilon \to 0} 2\pi\epsilon \sup \left| \frac{f(w) - f(z)}{w - z} \right|$$

but as $\epsilon \to 0$, since $w = z + \epsilon(z - z_0) + \epsilon Re^{i\theta}$ for some $\theta$, $w \to z$, so $\frac{f(w)-f(z)}{w-z} = f'(z)$ since $f$ is holomorphic at $z \in D$, so $\lim_{\epsilon \to 0} 2\pi\epsilon \sup \left| \frac{f(w)-f(z)}{w-z} \right| = 0$.

Then,

$$\int_C \frac{f(w)-f(z)}{w-z}dw = 0 \implies \int_C \frac{f(w)}{w-z}dw = \int_C \frac{f(z)}{w-z}dw = f(z)\int_C \frac{dw}{w-z}$$

but we showed that this last integral, equivalent to $\int_{C-z} \frac{dw}{w}$, is just $2\pi i$, so we arrive at

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw$$

# 20

I'll take the hint in the book (even admitting the hint given in the assignment which seems to just follow from Cauchy-Schwartz inequality, I don't see how the desired result follows) to use the mean value property, which follows from the Cauchy integral formula immediately anyway, since under the parameterization of $C = z + re^{i\theta}$,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z+re^{i\theta})}{z+re^{i\theta}-z}ire^{i\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})d\theta$$

which then gives

$$|f(z)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})d\theta\right| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z+re^{i\theta})|d\theta$$

so we have that in the context of the problem, we apply this to $f^2(z)$, such that

$$|f(z)|^2 = |f^2(z)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f^2(z+re^{i\theta})|d\theta$$

and we multiply by $r$ (so we can convert to rectangular coordinates) and integrate from $r = 0$ to $r = t$ to get that

$$\int_0^t |f(z)|^2 r dr \leq \frac{1}{2\pi}\int_0^t \int_0^{2\pi} |f^2(z+re^{i\theta})| r d\theta dr \implies \frac{t^2}{2}|f(z)|^2 \leq \frac{1}{2\pi}\iint_{D_t(z)} |f^2(z)|dxdy$$

which finally gives that

$$|f(z)|^2 \leq \frac{1}{t^2\pi}\iint_{D_t(z)} |f(z)|^2 dxdy$$

for any disc $D_t(z)$ in which $f$ is holomorphic. But then, we have that for any $z \in D_s$,

$$|f(z)|^2 \leq \frac{1}{(r-s)^2\pi}\iint_{D_{r-s}(z)} |f(z)|^2 dxdy$$

but we have that $D_{r-s}(z)$ must be contained inside of $D_r(z_0)$, since $|z - z_0| < s$ and for $w \in D_{r-s}(z)$, $|w - z| < r - s \implies |w - z_0| < s + r - s = r$, and since $|f(z)|^2$ is clearly nonnegative,

$$|f(z)|^2 \leq \frac{1}{(r-s)^2\pi} \iint_{D_{r-s}(z)} |f(z)|^2 dxdy \leq \frac{1}{(r-s)^2\pi} \iint_{D_r(z_0)} |f(z)|^2 dxdy$$

and taking the square root gives

$$|f(z)| \leq \frac{1}{(r-s)\sqrt{\pi}} \left( \iint_{D_r(z_0)} |f(z)|^2 dxdy \right)^{1/2} = C||f||_{L^2(D_r(z_0))}$$

so $C||f||_{L^2(D_r(z_0))}$ is an upper bound on $|f(z)|$ for $z \in D_s(z_0)$, so we get that

$$\sup_{z \in D_s(z_0)} |f(z)| = ||f||_{L^\infty(U)} \leq C||f||_{L^2(D_r(z_0))}$$

as desired.

To show the second part, first we have that for any subset $E$ of $U$, we have that

$$\iint_U |f(z)|^2 dxdy = \int_E |f(z)|^2 dzdy + \int_{U \setminus E} |f(z)|^2 dxdy$$

but both integrands on the RHS are nonnegative, so

$$||f||_{L^2(U)} \geq ||f||_{L^2(E)}$$

Now pick any compact subset $K$ of $U$. We will always have that the distance between $K$ and $\mathbb{C} \setminus U$ is positive, since $K$ is closed, and $\mathbb{C} \setminus U$ is the complement of an open set and thus closed as well, and the two sets have no overlap. Then, if the distance is 0, we must have that for any $n \in \mathbb{N}$, there are $z_n \in K$, $w_n \in \mathbb{C} \setminus U$, such that $|z_n - w_n| < \frac{1}{n}$. Then, since $K$ is compact, we have that there is some convergent subsequence $z_{n_m}$ such that $\{z_{n_m}\}_{m=1}^\infty$ converge to some point $z$ of $K$. Then, we have that $|z - w_{n_m}| < |z - z_{n_m}| + |z_{n_m} - w_{n_m}| < |z - z_{n_m}| + \frac{1}{n_m}$. Then, since in the limit both terms in the RHS vanish, we get that $z$ is a limit point on $C \setminus U$, and so $z \in C \setminus U$, a closed set. $\Rightarrow\Leftarrow$, so the distance is positive.

Now take $2d$ to be that distance, so $D_{2d}(z) \subset U$ for any $z \in K$; then, $K$ is covered by a finite amount of these discs $D_d(z)$, such that $K \subset \bigcup_{i=1}^n D_d(z_i)$. Then, on each of these discs,

$$||f||_{L^\infty(D_d(z_i))} \leq C_i ||f||_{L^2(D_{2d}(z_i))} \leq C||f||_{L^2(U)}$$

where $C = \sup C_i$ for any holomorphic function $f$.

Now, fix an $\epsilon > 0$. This gives that if for $m, n > N$, $||f_n - f_m||_{L^2(U)} < \epsilon$, then on $K$, with $z \in K$,

$$|f_n(z) - f_m(z)| \leq \sup_{z \in K} |f_n(z) - f_m(z)| = ||f_n - f_m||_{L^\infty(K)} = || < C||f_n - f_m||_{L^2(U)} < C\epsilon$$

so we get uniform convergence of $f_n$ on $K$ since uniform Cauchy convergence and uniform convergence are equivalent in functions from $\mathbb{R}^n \to \mathbb{R}^m$. Then, by a theorem of Weierstrass from a chapter back, we get that since $f_n$ are holomorphic, $f_n$ converge to some holomorphic function in particular, which gives us what we want.

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We have that since $f$ is holomorphic on the unit disc, which is clearly simply connected, so for $\gamma_\epsilon(\theta) = (1-\epsilon)e^{i\theta}$ the circle centered at 0 with radius $1-\epsilon$, for $\epsilon > 0$, $1-\epsilon < 1$, so $\gamma_\epsilon \in \mathbb{D}$, and so

$$\int_{\gamma_\epsilon} f(z)dz = 0$$

but since the continuation to $\partial D$ is continuous, we have that

$$\int_C f(z)dz = \lim_{\epsilon \to 0^+} \int_{\gamma_\epsilon} f(z)dz = \lim_{\epsilon \to 0^+} 0 = 0$$

but on the unit circle, $f(z) = 1/z$, so

$$\int_C f(z)dz = \int_C \frac{dz}{z} = 2\pi i$$

so $\Rightarrow\Leftarrow$, and $f$ cannot exist.

## 2

We know that $T(z) = \frac{z_0 - z}{1-\overline{z_0}z}$ is a Blaschke factor, and thus (from the question in the first problem set) it is holomorphic on the unit disc and takes $\mathbb{D}$ to $\mathbb{D}$ bijectively.

Then, if $u(z) = \text{Re}(F(z))$ for some holomorphic function $F$ as shown in class, we have that $u(T(z)) = \text{Re}(F(T(z)))$, which is the real part of a holomorphic function and thus harmonic as shown both in earlier HW and in class.

Applying the mean value theorem for $u_0(z) = u(T(z))$, we get that

$$u_0(0) = \frac{1}{2\pi}\int_0^{2\pi} u_0(e^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi} u\left(\frac{z_0 - e^{i\theta}}{1-\overline{z_0}e^{i\theta}}\right)d\theta$$

We now consider

$$e^{i\varphi} = \frac{z_0 - e^{i\theta}}{1-\overline{z_0}e^{i\theta}}$$

which gives us

$$\begin{aligned}
ie^{i\varphi}d\varphi &= \frac{(1-\overline{z_0}e^{i\theta})(-ie^{i\theta}) - (z_0 - e^{i\theta})(-\overline{z_0}ie^{i\theta})}{(1-\overline{z_0}e^{i\theta})^2}d\theta \\
&= -ie^{i\theta}\frac{1-\overline{z_0}e^{i\theta} - |z_0|^2 + \overline{z_0}e^{i\theta}}{(1-\overline{z_0}e^{i\theta})^2}d\theta \\
&= -ie^{i\theta}\frac{1-|z_0|^2}{(1-\overline{z_0}e^{i\theta})^2}d\theta
\end{aligned}$$

We proved back in the first problem set that $T(T(z)) = z$, so we get that $e^{i\theta} = \frac{z_0 - e^{i\varphi}}{1-\overline{z_0}e^{i\varphi}}$, giving us that

$$\begin{aligned}
d\varphi &= -e^{-i\varphi}\left(\frac{z_0 - e^{i\varphi}}{1-\overline{z_0}e^{i\varphi}}\right)(1-|z_0|^2)\left(1-\overline{z_0}\frac{z_0 - e^{i\varphi}}{1-\overline{z_0}e^{i\varphi}}\right)^{-2} d\theta \\
&= -e^{-i\varphi}\left(\frac{z_0 - e^{i\varphi}}{1-\overline{z_0}e^{i\varphi}}\right)(d\theta 1-|z_0|^2)\left(\frac{1-|z_0|^2}{1-\overline{z_0}e^{i\varphi}}\right)^{-2} d\theta \\
&= -e^{-i\varphi}\frac{(z_0 - e^{i\varphi})(1-\overline{z_0}e^{i\varphi})}{1-|z_0|^2} d\theta \\
&= \frac{(z_0 - e^{i\varphi})(\overline{z_0} - e^{-i\varphi})}{1-|z_0|^2} d\theta \\
&= \frac{(z_0 - e^{i\varphi})\overline{(z_0 - e^{i\varphi})}}{1-|z_0|^2} = \frac{|z_0 - e^{i\varphi}|^2}{1-|z_0|^2} d\theta
\end{aligned}$$

which gives us in the integral that since $T(0) = \frac{z_0 - 0}{1-\overline{z_0}0}$,

$$u(z_0) = u_0(0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(\frac{z_0 - e^{i\theta}}{1-\overline{z_0}e^{i\theta}}\right) d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z_0|^2}{|e^{i\varphi} - z_0|^2} u(e^{i\varphi}) d\varphi$$

as desired.

Now, if we take and rename the variable of integration back to $\theta$ and take $z_0 = re^{i\varphi}$, we have that

$$\begin{aligned}
\frac{1-|z_0|^2}{|e^{i\theta} - z_0|^2} &= \frac{1-|re^{i\varphi}|^2}{|e^{i\theta} - re^{i\varphi}|^2} \\
&= \frac{1-r^2|e^{i\varphi}|^2}{|e^{i\theta} - re^{i\varphi}|^2} \\
&= \frac{1-r^2}{|e^{i\theta} - re^{i\varphi}|^2}
\end{aligned}$$

In the denominator,

$$\begin{aligned}
|e^{i\theta} - re^{i\varphi}|^2 &= (e^{i\theta} - re^{i\varphi})\overline{(e^{i\theta} - re^{i\varphi})} \\
&= (e^{i\theta} - re^{i\varphi})(e^{-i\theta} - re^{-i\varphi}) \\
&= 1 + r^2 - r(e^{i(\theta-\varphi)} + e^{i(\varphi-\theta)}) \\
&= 1 + r^2 - 2r\frac{e^{i(\theta-\varphi)} + e^{-i(\theta-\varphi)}}{2} \\
&= 1 + r^2 - 2r\cos(\theta - \varphi)
\end{aligned}$$

so we get that

$$\frac{1-|z_0|^2}{|e^{i\theta} - z_0|^2} = \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}$$

as desired.