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Homework 8

Apostol p.246 no.5

a

Claim. $\emptyset$ is open.

Proof. The empty set is open is a vacuously true statement; that is, the statement

$$\forall x \in \emptyset, \exists \epsilon > 0 \mid B_\epsilon \subseteq \emptyset$$

is vacuously true. □

b

Claim. $\mathbb{R}^n$ is open.

Proof. For any $x \in \mathbb{R}^n$, take $\epsilon = 1$. Then, $B_\epsilon(x) \subseteq \mathbb{R}^n$, as the ball is defined to consist of elements of $\mathbb{R}^n$ that satisfy a certain condition. □

c

Claim. The union of any collection of open sets is open.

Proof. If an element x is in the union $\cup_{i=1} A_i$, then $\exists i$ such that $x \in A_j$. Then, since A_j is assumed to be open, $\exists \epsilon \mid B_\epsilon(x) \subseteq A_j$. Then, by the definition of union, $y \in B_\epsilon(x) \implies y \in \cup A_i \implies B_\epsilon(x) \subseteq \cup A_i$. □

d

Claim. The intersection of a finite collection of open sets is open.

Proof. If an element x is in the intersection $\cap_{i=1}^n A_i$, then $x \in A_1 \wedge x \in A_2 \wedge \cdots \wedge x \in A_n$.
Now, since each A_j is assumed to be open, we have that $\exists \epsilon_j$ such that $B_{\epsilon_j}(x) \in A_j$.
Further, for two open balls $B_{\epsilon_1}(x)$, $B_{\epsilon_2}(x)$,

$$\epsilon_2 < \epsilon_1 \implies B_{\epsilon_2}(x) \subset B_{\epsilon_1}(x)$$

as if $y \in B_{\epsilon_2}(x)$, we have that $||x - y|| < \epsilon_2 < \epsilon_1 \implies y \in B_{\epsilon_1}(x)$.
Then, take $\epsilon = \min(\epsilon_j)$. We have that $B_\epsilon(x) \subseteq B_{\epsilon_j}(x) \implies B_\epsilon(x) \subseteq A_j$ for any j in the range 1 to n. Thus,

$$B_\epsilon(x) \subseteq \cap_{i=1}^n A_i$$

□

e

Consider
$$S = \bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n})$$
Suppose that $x < 0 \in S$. Then, the archimedian property furnishes some n such that $-x > \frac{1}{n}$. Then, $x \notin (-\frac{1}{n}, \frac{1}{n}) \implies x \notin S$.

Similarly, for $x > 0 \in S$, we have some $n \mid x > \frac{1}{n}$, and so $x \notin S$.

Thus, $S = \{0\} = [0, 0]$, which is closed as seen by the fact that $[a, b]$ in general is closed, as stated in class.

Apostol p.252 no.6

Along $y = mx$, we have that $f(x, y)$ becomes
$$f(x) = \frac{x^2(1 - m^2)}{x^2(1 + m^2)}$$
Then,
$$\lim_{x \to 0} f(x) = \frac{1 - m^2}{1 + m^2} \lim_{x \to 0} \frac{x^2}{x^2} = \frac{1 - m^2}{1 + m^2}$$
There is then no so such definition of $f(0, 0)$ such that $f(x, y)$ is continuous at the origin; in order for it to be continuous at the origin, the limits along $y = mx$ must all coincide, but they clearly don't (take for example $m = 0, 1$, which see the limit being $1, 0$ respectively).

Apostol p.256 no.20

a

Claim. If $f'(x; y) = 0$ for any $x \in B_\epsilon(a)$ and any vector y, then f is constant on $B(a)$.

Proof. The mean value theorem yields that for any $y \in B(a)$ and for one $0 < \theta < 1$, $f'(a + \theta(y - a); y - a) = f(y) - f(a) = 0 \implies f(y) = f(a)$. However, since this holds for any y, f is constant on $B(a)$. □

b

It need not be constant: consider
$$F(x_1, x_2) = x_1^2$$

Then, we have that

$$D_2F = 0$$

everywhere, but obviously $F(x_1, x_2)$ is nonconstant.

We can conclude at the very least that F is periodic, as the mean value theorem yields that for $x, x+ty \in B(n)$ and some $0 < \theta < 1$, $f'(x + ty\theta; ty) = 0 = f(x+ty) - f(x) \implies f(x+ty) = f(x)$, where t is a scalar; we know that $f(x; ty) = tf(x; y) = 0$. Then, f is constant in the direction of y, or that f takes the same value on the line $x + ty$.

Apostol p.256 no.22

a

Claim. There is no $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(a; y) > 0$ for fixed a and any nonzero y.

Proof. Suppose that $f'(a; y) > 0$. Now consider $f'(a; -y)$. Putting $t_2 = -t_1$ we have that

$$f'(a; -y) = \lim_{t_1 \to 0} \frac{F(x - t_1 y) - F(x)}{t_1} = \lim_{t_2 \to 0} \frac{F(x + t_2 y) - F(x)}{-t_2} = -f'(a; y)$$

Thus we have that $f'(a; -y) < 0$, but by assumption, we have that $f'(a; -y) > 0$. $\Rightarrow\!\Leftarrow$. $\square$

b

Consider

$$F(x_1, x_2) = x_1$$

Then,

$$D_1F = 1$$

and therefore $D_1F = F'(x, e_1) = 1 > 0$ for any $x \in \mathbb{R}^2$.

Apostol p.281 no.2

$$
\begin{aligned}
D_1 f(0,0) &= \lim_{t\to 0} \frac{f((0,0)+(t,0)) - f(0,0)}{t} \\
&= \lim_{t\to 0} \frac{0}{y} = 0 \\
D_2 f(0,0) &= \lim_{t\to 0} \frac{f((0,0)+(0,t)) - f(0,0)}{t} \\
&= \lim_{t\to 0} \frac{-t}{t} = -1 \\
D_2 f(t_1,0) &= \lim_{t_2\to 0} \frac{f(t_1,t_2) - f(t_1,0)}{t_2} \\
&= \lim_{t_2\to 0} \frac{t_2 \frac{t_1^2 - t_2^2}{t_1^2 + t_2^2}}{t_2} \\
&= 1 \\
D_{12} f(0,0) &= \lim_{t_1\to 0} \frac{D_2 f(t_1,0) - D_2 f(0,0)}{t_1} \\
&= \lim_{t_1\to 0} \frac{D_2 f(t_1,0) + 1}{t_1} \\
&= \lim_{t_1\to 0} \frac{1+1}{t_1} \text{ which does not exist.} \\
D_1 f(0,t_2) &= \lim_{t_1\to 0} \frac{f(t_1,t_2) - f(0,t_2)}{t_1} \\
&= \lim_{t_1\to 0} \frac{t_2 \frac{t_1^2 - t_2^2}{t_1^2 + t_2^2} - t_2 \frac{-t_2^2}{t_2^2}}{t_1} \\
&= \lim_{t_1\to 0} t_2 \frac{\frac{t_1^2 - t_2^2}{t_1^2 + t_2^2} + 1}{t_1} \\
&= \lim_{t_1\to 0} t_2 \frac{2t_1^2}{t_1} \\
&= 0 \\
D_{21} f(0,0) &= \lim_{t_2\to 0} \frac{D_1 f(0,t_2) - D_1 f(0,0)}{t_2} \\
&= \lim_{t_2\to 0} \frac{D_1 f(0,t_2)}{t_2} \\
&= \lim_{t_2\to 0} \frac{0}{t_2} \\
&= 0
\end{aligned}
$$

Apostol p.281 no.3

a

$$
\begin{aligned}
f'(0;(x,y)) &= \lim_{t\to 0} \frac{F(tx,ty)-F(0,0)}{t} \\
&= \lim_{t\to 0} \frac{F(tx,ty)}{t} \\
&= \lim_{t\to 0} \frac{xy^3}{x^3+t^3y^6} \\
&= \frac{y^3}{x^2}
\end{aligned}
$$

However, if $x=0$, then we have that

$$
\begin{aligned}
f'(0;(0,y)) &= \lim_{t\to 0} \frac{F(0,ty)}{t} \\
&= \lim_{t\to 0} \frac{0}{t} = 0
\end{aligned}
$$

b

Consider the limit along $x=-y^2$. Then,

$$\lim_{y\to 0} \frac{-y^5}{-y^6+y^6} = \lim_{y\to 0} \frac{-y^5}{0}$$

This limit doesn't exist, and this $f(x,y)$ is not continuous at the origin.

Problem 1

Claim. The product $S=(a_1,b_1)\times\cdots\times(a_n,b_n)$ is an open set in $\mathbb{R}^n$.

Proof. For any element $x=(x_1,x_2,\dots,x_n)\in S$, take

$$\epsilon = \min(x_1-a_1, b_1-x_1, x_2-a_2, b_2-x_2, \dots, x_n-a_n, b_n-x_n)$$

That is, $\epsilon=\min(x_i-a_i, b_i-x_i)$ for $1\le i\le n$.
Now, for any $y=(y_1,\dots,y_n)\in B_\epsilon(x)$, we have that

$$||y-x||<\epsilon \implies \sum_{i=1}^{n}(y_i-x_i)^2<\epsilon^2 \implies (y_i-x_i)^2<\epsilon^2 \implies |y_i-x_i|<\epsilon$$

Then, if $y_i - x_i > 0$, then

$$y_i < x_i + \epsilon \leq x_i + (b_i - x_i) = b_i$$

and if $y_i - x_i < 0$, then

$$y_i > x_i - \epsilon \geq x_i - (x_i - a_i) = a_i$$

Thus, we have that for every $1 \leq i \leq n$, $y_i \in (a_i, b_i) \implies y \in S \implies B_\epsilon(x) \in S$. □

Problem 2

a

Claim. Homogeneous functions of degree 1 have that $F(0) = 0$.

Proof. Since it is continuous, we have that

$$\lim_{x \to 0} F(x) = F(0)$$

Now, suppose that $F(0) = c \neq 0$. Then, since $\lim_{x \to 0} F(x) = c$, we have that $|F(x) - c| < \frac{|c|}{2}$ on some $B_\delta(0)$. Now, consider the quantity $\frac{c}{2F(e_1)}$. We have by the archimedian property some $n \geq 1 \mid n\delta > |\frac{c}{2F(e_1)}| \implies \delta > |\frac{c}{2nF(e_1)}| \implies \frac{c}{2nF(e_1)}e_1 \in B_\delta(0)$, as $||\frac{c}{2nF(e_1)}e_1|| = |\frac{c}{2nF(e_1)}| < \delta$. However, $F(\frac{c}{2nF(e_1)}e_1) = \frac{c}{2nF(e_1)}F(e_1) = \frac{c}{2n}$. Then,

$$|F(\frac{c}{2nF(e_1)}e_1) - c| = |\frac{c}{2n} - c| = |c(1 - \frac{1}{2n})|$$

Further, since $n \geq 1$, $1 - \frac{1}{2n} > 0 \implies |c(1 - \frac{1}{2n})| = |c|(1 - \frac{1}{2n})$.
By assumption, we have that $|F(\frac{c}{2nF(e_1)}e_1) - c| = |c|(1 - \frac{1}{2n}) < |\frac{c}{2}| \implies 1 - \frac{1}{2n} < \frac{1}{2} \implies 1 < \frac{1}{2}(1 + \frac{1}{n}) \leq \frac{1}{2}(1 + 1) = 1$. Thus, we have that $1 < 1$. $\Rightarrow\Leftarrow$, so $F(0) = 0$. □

The above ought to work, but there is a much simpler argument that I realized only after doing it: by continuity, we have that

$$\begin{aligned} F(0) &= \lim_{x \to 0} F(x) \\ F(0) &= \lim_{x \to 0} F(2x) \\ &= 2 \lim_{x \to 0} F(x) \end{aligned}$$

Thus, $F(0) = 2F(0) \implies F(0) = 0$.

b

$$\begin{aligned} F'(0;y) &= \lim_{t\to 0} \frac{F(0+ty)-F(0)}{t} \\ &= \lim_{t\to 0} \frac{F(ty)}{t} \\ &= \lim_{t\to 0} \frac{tF(y)}{t} \\ &= F(y) \end{aligned}$$

Problem 3

Claim. If $F : \mathbb{R}^n \to \mathbb{R}$ satisfies $|F(x)| \leq c||x||^2$ for some $c \in \mathbb{R}$ and all $x \in \mathbb{R}^n$, then for any $y \in \mathbb{R}^n$,

$$F'(0;y) = 0$$

Proof. Note that by taking $x = 0$, we have that $|F(0)| \leq c||0||^2 = 0 \implies |F(0)| = 0 \implies F(0) = 0$.

$$F'(0;y) = \lim_{t\to 0} \frac{F(0+yt)-F(0)}{t} = \lim_{t\to 0} \frac{F(yt)}{t}$$

Further, for any $\epsilon > 0$ take $\delta = \frac{\epsilon}{|c|||y||^2}$. Then, for $0 < |t| < \delta$

$$\left|\frac{F(ty)}{t}\right| \leq \frac{|c|||yt||^2}{|t|} = \frac{|c||t|^2||y||^2}{|t|} = |c||t|||y||^2 < \epsilon$$

Thus, we have that $F'(0;y) = 0$.

□