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Assignment 4

4.1.1

$$E(X) = \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a}(\frac{1}{2}(b^2 - a^2)) = \frac{a+b}{2}$$

4.1.11

We first compute the pdf's of Y_1, Y_n. For Y_1, we have that

$$G_1(y) = \int \cdots \int_{S_1} dx_1, \dots, dx_n$$

where S_1 is the $n-$dimensional cube that stretches from $(1, \dots, 1)$ to the point $(y, \dots, y)$. Then, this has area $(1-y)^n$. Similarly,

$$G_n(y) = \int \cdots \int_{S_n} dx_1, \dots, dx_n$$

has S_n the $n-$dimensional cube from the origin to $(y, \dots, y)$. This has area y^n.

Then, on $0 < y < 1$,

$$g_1(y) = n(1-y)^{n-1}, g_2(y) = ny^{n-1}$$

and

$$E(Y_1) = n \int_0^1 y(1-y)^{n-1} dy = n(-(\frac{y(1-y)^n}{n})\Big|_0^1 - \int_0^1 \frac{(1-y)^n dy}{n+1}) = n(-\frac{(1-y)^{n+1}}{n(n+1)}\Big|_0^1) = \frac{1}{n+1}$$

$$E(Y_2) = n \int_0^1 y^n dy = \frac{n}{n+1}$$

4.1.12

This is exactly the same problem as 4.1.11, as we have that the probability integral transform that each $F(X_i)$ is the uniform distribution on the unit interval; furthermore, since $F(Y_1), F(Y_2)$ are just the minimal and maximal elements of these, then

$$E(F(Y_1)) = \frac{1}{n+1}, E(F(Y_2)) = \frac{n}{n+1}$$

4.2.2

We have that

$$E(2X_1 - 3X_2 + X_3 - 4) = 2E(X_1) - 3E(X_2) + E(X_3) - 4 = 10 - 15 + 5 - 4 = -4$$

Assignment 4

4.2.8

Put

$$X_i = \begin{cases} 1 & \text{the } n^{th} \text{ student is a boy} \\ 0 & \text{otherwise} \end{cases}$$

and Y_i similarly.

We have that $E(X_1) = \frac{10}{25}$ Then, we wish to calculate $X = E(\sum_{i=1}^{8} X_i) = 8\frac{10}{25} = \frac{16}{5} = 3.2$, and $Y = E(\sum_{i=1}^{8} Y_i) = 8\frac{15}{25} = \frac{24}{5} = 4.8$. Then, $E(X-Y) = -1.6$.

4.3.1

$$\mathrm{Var}\,(X) = \int_0^1 (x - \frac{1}{2})^2 dx = \int_0^1 x^2 - x^1 + \frac{1}{4} dx = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$$

4.3.6

$$E((X-Y)^2) = E(X^2 - 2XY + Y^2) = E(X^2) - 2E(XY) + E(Y^2)$$

Since they are independent, we have that $2E(XY) = E(X)E(Y) + E(X)E(Y)$, and since $E(X) = E(Y)$, $E(X)E(Y) + E(X)E(Y) = E(X)^2 + E(Y)^2$. Then,

$$E(X^2) - 2E(XY) + E(Y^2) = E(X^2) - (E(X))^2 + E(Y^2) - (E(Y))^2 = \mathrm{Var}\,(X) - \mathrm{Var}\,(Y)$$

4.3.9

$$\begin{aligned} \mathrm{Var}\,(X) &= E(X^2) - E(X)^2 \\ &= \frac{1}{n}\sum_{i=1}^{n} i^2 - (\frac{1}{n}\sum_{i=1}^{n} i)^2 \\ &= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} \\ &= \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12} \end{aligned}$$

4.4.3

$$E((X-\mu)^3) = E(X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3) = E(X^3) - 3E(X^2) + 3E(X) - 1 = 1$$

4.4.7

(4.3.7 is listed in the syllabus, instead of this problem. A similar thing was on problem set number 3 as well, where 3.5.9 is listed in the syllabus and 3.3.9 is in the problem set!)

The mean is simply $\psi'(0) = \frac{1}{4}(3-1) = \frac{1}{2}$, and the second moment is $\psi''(0) = \frac{1}{4}(3+1) = 1$ and the variance is $E((X-\mu))^2 = E(X^2) - \mu^2 = 1^2 - (\frac{1}{2})^2 = \frac{3}{4}$.

4.4.10

(Same as above.)

Put $Z' = 2X - 3Y = Z - 4$. Then,

$$\psi_{Z'} = \psi_{2X}(t)\psi_{-3Y}(t) = \psi(2t)\psi(-3t)$$

We have that $\psi_Z(t) = e^{4t}\psi_{Z'}(t) = e^{4t}\psi(2t)\psi(-3t) = e^{4t}e^{4t^2+6t}e^{9t^2-9t} = e^{13t^2+t}$

4.5.3

We have that

$$\int_0^m e^{-x}dx = 1 - e^{-m} = \frac{1}{2} \implies m = \log(2)$$

4.5.11

The smaller $M.S.E.$ is just the one with the lower variance; this can be computed to be, for a binomial distrubution X, to be

$$\mathrm{Var}\,(X) = \sum_{i=1}^{n} \mathrm{Var}\,(X_i) = \sum_{i=1}^{n}(E(X_i^2) - (E(X_i))^2) = \sum_{i=1}^{n}(p - p^2) = np(1-p)$$

For $n = 7, p = 1/4$, we have that the variance is $\frac{21}{16}$, for $n = 5, p = 1/2$, we have that the variance is $\frac{5}{4} < \frac{21}{16}$. Thus, the MSE can be predicited smaller for $n = 5, p = 1/2$.

4.5.13

If X is symmetric around X, then we have that $P(X \leq m) = P(X \geq m)$, (as in general we have that $P(X \leq m - x) = P(X \geq m - x)$ for any x from the definition of symmetric distrubutions). Then, since the probability of the entire space is 1, $P(X \leq m) + P(X \geq m) = 1 \implies P(X \leq m) = P(X \geq m) = 1/2$,, and so m is a median.

4.6.12

$$
\begin{aligned}
E(X) &= \int_0^1 \int_0^2 x\frac{1}{3}(x+y)dydx = \frac{5}{9} \\
E(Y) &= \int_0^1 \int_0^2 y\frac{1}{3}(x+y)dydx = \frac{11}{9} \\
E(X^2) &= \int_0^1 \int_0^2 x^2\frac{1}{3}(x+y)dydx = \frac{7}{18} \\
E(Y^2) &= \int_0^1 \int_0^2 y^2\frac{1}{3}(x+y)dydx = \frac{16}{9} \\
E(XY) &= \int_0^1 \int_0^2 xy\frac{1}{3}(x+y)dydx = \frac{2}{3} \\
\operatorname{Var}(X) &= \frac{7}{18} - \frac{25}{81} = \frac{13}{162} \\
\operatorname{Var}(Y) &= \frac{16}{9} - \frac{121}{81} = \frac{46}{162} \\
\operatorname{Cov}(X,Y) &= \frac{2}{3} - (\frac{5}{9})(\frac{11}{9}) = -\frac{2}{162} \\
\operatorname{Var}(2X-3Y+8) &= 4\operatorname{Var}(X) + 9\operatorname{Var}(Y) - 12\operatorname{Cov}(()\,X,Y) = \frac{490}{182} = \frac{245}{81}
\end{aligned}
$$

4.6.15

$$
\begin{aligned}
\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) &= \sum_{i=1}^{n}\operatorname{Var}(X_i) + \sum_{i=1}^{n}\sum_{j=1,j\neq 1}^{n}\operatorname{Cov}(()\,X_i,X_j) \\
&= n + \frac{1}{4}(n(n-1)) = \frac{n^2+3n}{4}
\end{aligned}
$$

4.7.3

Let $E(X \mid Y) = c$. Then, $E(E(X \mid Y)) = E(X) = c$. Since $\operatorname{Cov}(X,Y) = E(XY) - E(X)E(Y)$, and $E(XY) = E(E(XY \mid Y)) = E(YE(X \mid Y)) = E(cY) = cE(Y)$, we have that $\operatorname{Cov}(X,Y) = cE(Y) - cE(Y) = 0$. Then, since $\rho_{X,Y} = \operatorname{Cov}(()\,X,Y)/(\sigma_X\sigma_Y)$, we have that $\rho_{X,Y} = 0$.

4.7.7

The marginal pdf of X is

$$
f(x) = \int_0^1 (x+y)dy = x + \frac{1}{2}
$$

Then, we have that the conditional pdf of Y is

$$g(y \mid x) = \frac{f(x,y)}{f(x)} = \frac{x+y}{x+\frac{1}{2}} = \frac{2(x+y)}{2x+1}$$

Then, we have that

$$E(Y \mid X) = \int_0^1 2y\frac{x+y}{2x+1}dy = \frac{3x+2}{6x+3}$$

$$E(Y^2 \mid X) = \int_0^1 2y^2\frac{x+y}{2x+1}dy = \frac{4x+3}{12x+6}$$

$$\operatorname{Var}(Y \mid X) = \frac{4x+3}{12x+6} - (\frac{3x+2}{6x+3})^2 = \frac{6x^2-6x+1}{18(2x+1)^2}$$

4.7.11

Since $\operatorname{Var}(Y \mid X) = E(Y^2 \mid X) - E(Y \mid X)^2$, we have that

$E(\operatorname{Var}(Y \mid X)) = E(E(Y^2 \mid X) - E(Y \mid X)^2) = E(E(Y^2 \mid X)) - E(E(Y \mid X)^2) = E(Y^2) - E(E(Y \mid X)^2)$

Further,

$$\operatorname{Var}(E(Y \mid X)) = E(E(Y \mid X)^2) - E(E(Y \mid X))^2 = E(E(Y \mid X)^2) - E(Y)^2$$

Then, summing, we have that

$$E(\operatorname{Var}(Y \mid X)) + \operatorname{Var}(E(Y \mid X)) = E(Y^2) - E(Y)^2 = \operatorname{Var}(Y)$$