MATH 4065 HW 7

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($\implies$) Since we have that $f : U \to V$ is a local bijection on U, we have that for any $z \in U$, there is some small neighborhood D centered at z where $f : D \to f(D)$ is holomorphic and bijective; the proposition in the book then gives that $f'(w) \neq 0$ for every $w \in D$, in particular that $f'(z) \neq 0$. Since this holds for any $z \in U$, then we get that $f'(z) \neq 0$ for all $z \in U$.

($\impliedby$) Fix any $z_0 \in U$, and take a small enough neighborhood such that

$$f(z) = f(z_0) + a(z - z_0) + O((z - z_0)^2) \implies f(z) - f(z_0) = a(z - z_0) + O((z - z_0)^2)$$

which then gives that on a (possibly) smaller neighborhood, $|a(z - z_0)| > |O((z - z_0)^2)|$, so we get that $f(z) - f(z_0)$ has the same amount of roots as $a(z - z_0)$, which is exactly one: $z = z_0$ since $a = f'(z_0) \neq 0$. Then, we have that $f(z) \neq f(z_0)$ on the smaller neighborhood for $z \neq z_0$. Then, if we halve the radius of the neighborhood, we get some $r > 0$ such that $|z - z_0| \leq r \implies f(z) - f(z_0) \neq 0$ for $z \neq z_0$. Let m be the minimal value of $|f(z) - f(z_0)|$, which must be positive and attained since $|f(z) - f(z_0)|$ attains its minimum as a continuous function on a compact set and $f(z) - f(z_0) \neq 0$ for $|z - z_0| = r$. Now for $w \in V$ satisfying $|w - f(z_0)| < m$, we get that

$$|(f(z) - w) - (f(z) - f(z_0))| = |f(z_0) - w| < m \leq |f(z) - f(z_0)|$$

on the boundary $|z - z_0| = r$. Then, Rouche's theorem gives that $f(z) - w = (f(z) - w - (f(z) - f(z_0))) + f(z) - f(z_0)$ must have the same amount of zeros as $f(z) - f(z_0)$ on $|z - z_0| < r$, namely one by the earlier fact that $f(z) \neq f(z_0)$ for $z \neq z_0$. This means that for any w within m of w_0, $f(z)$ assumes the value of w exactly once. Then, since f is uniformly continuous on $|z - z_0| \leq r$, we can shrink the neighborhood until the image of the neighborhood has diameter $< m$, such that $|f(z) - f(z_0)| < m$ for every z in the neighborhood. Now since this neighborhood is contained inside $|z - z_0| \leq r$, we get that $f(z)$ assumes each output value at most once, and is thus injective. On this small neighborhood (call it D), we get what we want: $f : D \to f(D)$ is injective from above, and is trivially surjective, so it is a bijection.

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First, we consider that from the chapter on logarithms, that for any non-vanishing holomorphic function in a simply connected region, $f(z)$, that there is some function holomorphic function $g(z)$ satisfying $f(z) = e^{g(z)}$. In particular, this lets us do the following: since $F(z) = F'(z) = 0$, we have that

$$F(z) = 0 + 0(z - z_0) + a_0(z - z_0)^2 + a_1(z - z_0)^3 + \cdots = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+2} = (z - z_0)^2 f(z)$$

where $f(z)$ is some non-vanishing holomorphic function in a neighborhood of (z_0) (in particular, it takes $f(z_0) = a_0 = F''(z) \neq 0$, and we can shrink our neighborhood until 0 is no longer in the image of f). Then, we have that $f(z) = e^{h(z)}$ for some holomorphic $h(z)$, so we get that $F(z) = ((z - z_0)e^{h(z)})^2$.

Abbreviate $(z - z_0)e^{h(z)} = g(z)$, and note that g is holomorphic. Then, we get that $F(z) = (g(z))^2$, so $F(z_0) = 0 \implies g(z_0) = 0$ and differentiating,

$$F''(z_0) = 2g''(z_0)g(z_0) + 2(g'(z_0))^2 = g(z_0)^2 \neq 0 \implies g'(z_0) \neq 0$$

Then, on some small neighborhood of z_0 (again, g' is continuous so just keep shrinking the neighborhood until it no longer contains 0) we get that $g'(z) \neq 0$. Then, by the last problem, g is a local bijection on this neighborhood, such that on a small neighborhood D of z_0, g is conformal and admits a holomorphic inverse g^{-1}. Let r be the radius of a neighborhood of $g(z_0) = 0$ which is contained in the image of D under g (this exists, since g is holomorphic and thus an open mapping). Then, consider the paths given as follows:

$$\begin{cases} \Gamma_1 = g^{-1}(tr) & -0.5 \leq t \leq 0.5 \\ \Gamma_2 = g^{-1}(itr) & -0.5 \leq t \leq 0.5 \end{cases}$$

such that $F(\gamma_1) = (tr)^2 = r^2t^2$ which since t, r are both real is obviously minimal at $t = 0$ and is real. Similarly, $F(\gamma_2) = (itr)^2 = -r^2t^2$ which is again real and clearly maximal at $t = 0$. Furthermore, since g is conformal, it preserves angles, so since $\gamma_1 = tr$ and $\gamma_2 = itr$ are orthogonal (the first travels the real line and the second the imaginary line, so they are perpendicular), Γ_1 and Γ_2 must be orthogonal at $g^{-1}(0) = z_0$ as well.

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Yes, we can define one explicitly. Consider first the conformal mapping $\mathbb{D} \to \mathbb{H}$ given in the chapter, $G(w) = i\frac{1-w}{1+w}$, which reduces the question to finding a holomorphic surjection $\mathbb{H} \to \mathbb{C}$. In particular, if we consider $H(z) = (z - i)^2$, we can see that $H \circ G$ is holomorphic since H is entire and G is holomorphic on $\mathbb{D}$, and thus $H \circ G$ is holomorphic on $\mathbb{D}$.

To see that $H \circ G$ is surjective, we first want to see that any complex number has a preimage under H in the upper half plane. In particular, take any $z = re^{i\theta} \in \mathbb{C}$, with $0 \le \theta < 2\pi$. Then, we have that $\sqrt{r}e^{i\theta/2} + i$ is a preimage of z under H, since $(\sqrt{r}e^{i\theta/2} + i - i)^2 = (\sqrt{r}e^{i\theta/2})^2 = re^{i\theta}$ as desired. Then, we have that $0 \le \theta/2 < \pi$ which gives that $\mathrm{Im}(\sqrt{r}e^{i\theta/2}) \ge 0$, so $\mathrm{Im}(\sqrt{r}e^{i\theta/2} + i) \ge 1 > 0$, so any $z \in \mathbb{C}$ has a preimage under f in the upper half plane.

Then, for any $z \in \mathbb{C}$, if $H(w) = z$, $w \in \mathbb{H}$, then we have that w has a preimage under G, since G is a conformal mapping between $\mathbb{D}$ and $\mathbb{H}$, and so this gives a preimage $G^{-1}(w)$ for z under $H \circ G$, so it is surjective and we get what we want.

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It is clearly holomorphic, since z is entire and $1/z$ is holomorphic on $\mathbb{C} \setminus \{0\}$, so $z + 1/z$ is holomorphic on all of $\mathbb{C}$ except the origin. Since the origin is not contained in the half disc, f is holomorphic.

Now,

$$f(x+yi) = -\frac{1}{2}\left(x + yi + \frac{1}{x+yi}\right) = -\frac{1}{2}\left(\frac{(x+yi)(x^2+y^2) + (x-yi)}{x^2+y^2}\right)$$

so $\mathrm{Im}(f(x+yi)) = -\frac{y}{2(x^2+y^2)}(x^2+y^2-1)$, but since $x^2+y^2 = |x+iy|^2 < 1$, we have that $\mathrm{Im}(f(x+yi)) > 0$, so this takes values only in the upper half plane.

To see that f is bijective, consider that if we want $f(z) = w$, we arrive at

$$-\frac{1}{2}\left(z + \frac{1}{2}\right) = w \implies z^2 + 2wz + 1 = 0$$

which has solutions $z = -w \pm \sqrt{w^2 - 1}$ (choose the branch of the square root arbitrarily). Now the discriminant is nonzero for $w \ne \pm 1$, so for any $w \in \mathbb{H}$, we have that there are two distinct solutions $w \pm \sqrt{w^2-1}$. We want to show that exactly one always lies inside the half disc, which will give both injectivity and surjectivity.

Call the two roots z_1, z_2. Then, we have that $z_1 + z_2 = -2w$ and $z_1 z_2 = 1$, so $|z_1| = 1/|z_2|$, so if one of z_1, z_2 lie on the unit circle, then both lie on the unit circle; in particular, in this case they are inverses, so $z_1 = \overline{z_2} \implies z_1 + z_2$ is real, but $-2w$ has negative imaginary part since $w \in \mathbb{H}$, so $\Rightarrow\Leftarrow$, so neither can lie on the unit circle. Then, exactly one of z_1 is in the unit disc, and the other is outside of it; consider now that one also must be in the upper half plane and the other the lower half plane, since if $z_1 = re^{i\theta}$, then $z_2 = 1/z_1 = r^{-1}e^{-i\theta}$, so if $\theta \in (0, \pi)$, then $z_1 \in \mathbb{H}$ and $z_2 \in -\mathbb{H}$, and if $\theta \in (\pi, 2\pi)$, then $z_1 \in -\mathbb{H}$ and $z_2 \in \mathbb{H}$. Note that neither can be real, since then the other would be real and their sum could not have nonzero imaginary part. Then, since $z_1 + z_2 \in -\mathbb{H}$, the larger root must lie in $-\mathbb{H}$ and the smaller root must lie in $\mathbb{H}$ (since $\mathrm{Im}(z_1) + \mathrm{Im}(z_2) = \sin(\theta)(r - 1/r)$, which is negative if and only if either $r > 1/r$ when $z_1 \in \mathbb{H}$ or $r < 1/r$ when $z_1 \in -\mathbb{H}$, and in both cases, the smaller root is contained in $\mathbb{H}$) and since it has modulus < 1, lies in the upper half disc as desired.

In particular, if we choose the square root to be the root with positive real part, I think that $-w + \sqrt{w^2 - 1}$ is the solution we want, but I can't explicitly show it. Thankfully, I don't need to!

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We have that $G(z) = i\frac{1-z}{1+z}$ is a conformal mapping $\mathbb{D} \to \mathbb{H}$ with inverse $\frac{i-z}{i+z}$. Then, we have that $F \circ G$ is a holomorphic mapping from $\mathbb{D} \to \overline{\mathbb{D}}$, since $|F(z)| \leq 1$. However, we cannot have that $|F(z)| = 1$ since this would violate the maximum modulus principle, since F would attain a maximum somewhere on $\mathbb{H}$, so $F \circ G$ takes $\mathbb{D} \to \mathbb{D}$. Further, $(F \circ G)(0) = F(i) = 0$, so we can apply the Schwarz lemma: $|(F \circ G)(z)| \leq |z|$, and this gives is what we want:

$$|F(z)| = |(F \circ G \circ G^{-1})(z)| = \left|(F \circ G)\left(\frac{i-z}{i+z}\right)\right| \leq \left|\frac{i-z}{i+z}\right| = \left|\frac{z-i}{z+i}\right|$$

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We will first show this for $M = R = 1$. First, note that since f has domain $\mathbb{D}$, an open set, $|f(z)| \neq 1$, or else f would be maximal on $\mathbb{D}$, so $f : \mathbb{D} \to \mathbb{D}$ is holomorphic. Consider the Möbius transformation $\psi(z) = \frac{f(0)-z}{1-\overline{f(0)}z}$. Then, $\psi \circ f$ is $\mathbb{D} \to \mathbb{D}$ holomorphic and satisfies $\psi(0) = \frac{f(0)-f(0)}{1-\overline{f(0)}f(0)} = 0$, so applying the Schwarz lemma,

$$|(\psi \circ f)(z)| = \left|\frac{f(0) - f(z)}{1 - \overline{f(0)}f(z)}\right| = \left|\frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)}\right| \leq |z|$$

Since we have this for $M = R = 1$, for the general case of $f : D(0, R) \to \mathbb{C}$ and $|f(z)| \leq M$, we can write define some $g(z) = \frac{f(Rz)}{M}$, which now takes $g : \mathbb{D} \to \mathbb{C}$ and satisfies $g(z) \leq 1$. Then,

$$\left|\frac{f(z) - f(0)}{M^2 - \overline{f(0)}f(z)}\right| = \left|\frac{Mg\left(\frac{z}{R}\right) - Mg(0)}{M^2 - \overline{Mg(0)}\left(Mg\left(\frac{z}{R}\right)\right)}\right| = \frac{1}{M}\left|\frac{g(w) - g(0)}{1 - \overline{g(0)}g(w)}\right|$$

where $w = z/R$. Then, from earlier,

$$\left|\frac{g(w) - g(0)}{1 - \overline{g(0)}g(w)}\right| \leq |w| \implies \left|\frac{f(z) - f(0)}{M^2 - \overline{f(0)}f(z)}\right| = \frac{1}{M}\left|\frac{g(w) - g(0)}{1 - \overline{g(0)}g(w)}\right| \leq \frac{|w|}{M} = \frac{|z|}{MR}$$

which was what we wanted.

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a

Let the two fixed points of f be z_1, z_2, and consider the Möbius transformation $\psi(z) = \frac{z_1 - z}{1 - \overline{z_1} z}$ and the composition $\psi \circ f \circ \psi$, which takes $\mathbb{D} \to \mathbb{D}$ holomorphic. Then, we have that

$$(\psi \circ f \circ \psi)(0) = (\psi \circ f)(z_1) = \psi(z_1) = 0$$

so we can apply the Schwarz lemma.

Then, note that since ψ is an automorphism of the disc, there is some preimage of z_2, say z_2' distinct from 0 since ψ is a bijection, such that

$$(\psi \circ f \circ \psi)(z_2') = (\psi \circ f)(z_2) = \psi(z_2) = z_2'$$

since ψ is its own inverse.

Then, this gives us that $|(\psi \circ f \circ \psi)(z_2')| = |z_2'|$, so by the Schwarz lemma, $\psi \circ f \circ \psi$ is a rotation, say $(\psi \circ f \circ \psi)(z) = e^{i\theta} z$. But since we have that $(\psi \circ f \circ \psi)(z_2') = e^{i\theta} z_2' = z_2'$, we have that $\theta = 0$ and $\psi \circ f \circ \psi$ is the identity mapping. Then, we get that $\psi \circ \psi \circ f \circ \psi = \psi \implies f \circ \psi = \psi \implies f = \text{id}$.

b

Not every holomorphic function $\mathbb{D} \to \mathbb{D}$ has a fixed point. Consider $F^{-1} \circ g \circ F$ where F is a conformal mapping $\mathbb{D} \to \mathbb{H}$ taking $z \mapsto \frac{i-z}{i+z}$ and g takes $z \mapsto z+1$. In particular, we already have what we want: $(F^1 \circ g \circ F)(z) = z \implies (g \circ F)(z) = F(z)$ but this would require g to have a fixed point $F(z)$, but g has no fixed point since $z \neq z+1$ for any $z \in \mathbb{H}$. $\Rightarrow\!\Leftarrow$, so $F^{-1} \circ g \circ F$ is a holomorphic map $\mathbb{D} \to \mathbb{D}$ with no fixed points.

Explicitly, we get that

$$(F^{-1} \circ g \circ F)(z) = F^{-1}\left(\frac{i-z}{i+z} + 1\right) = F^{-1}\left(\frac{2i}{i+z}\right) = i\frac{1 - \frac{2i}{i+z}}{1 + \frac{2i}{i+z}} = i\frac{z-i}{z+3i}$$

which has no fixed points in the unit disc ($i\frac{z-i}{z+3i} = z \implies (z-i)^2 = 0 \implies z = i \notin \mathbb{D}$) while being holomorphic in $\mathbb{D}$ and taking values only in $\mathbb{D}$ (since the range of F^{-1} is $\mathbb{D}$).

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If we can get $F_1, \ldots, F_5$ as in the book, then the function $\frac{1}{\pi}\arg(z) \circ F_5 \circ F_4 \circ F_3 \circ F_2 \circ F_1$ is (by a lemma from the book) a harmonic function on open first quadrant with the desired boundary conditions. The actual hard part is finding such conformal mappings.

For F_1, we want a transformation that sends $\infty \to 1, 0 \to -1, 1 \to 0$. Then, we consider the Möbius transformation $F_1 : z \mapsto \frac{z-1}{z+1}$. Now, the book already showed that $w \mapsto \frac{w+1}{w-1}$ takes the upper half disc to the first quadrant conformally, so we get compute the inverse to be exactly F_1, which takes the first quadrant to the upper half disc conformally. We only need to check the boundary conditions are correct:

1. The imaginary axis is taken to the semicircle:
$$\left|\frac{ai-1}{ai+1}\right| = \left|\frac{(ai-1)^2}{a^2+1}\right| = \frac{1}{a^2+1}|ai-1|^2 = 1$$
2. $[0,1]$ is taken to $[-1,0]$: $x \in [0,1]$ satisfies that $x-1 \leq 0$, $x+1 > 0$, and $|x-1| < |x+1|$, so $\frac{x-1}{x+1} \in [0,1]$.
3. $(1,\infty)$ is taken to $(0,1)$, since $x \in (0,1)$ satisfies that $x-1 > 0, x+1 > 0$, and $|x-1| < |x+1|$, so $\frac{x-1}{x+1} \in (0,1)$.

For F_2, consider the principle branch of $\log(z)$, which takes the closed upper half disc (minus 0) to the pictured region $\{z \mid \text{Re}(z) \leq 0, 0 \leq \text{Im}(z) \leq 1\}$. The book already showed that this is a conformal mapping between the open upper half disc and the interior of the above region, so we only need to check the boundaries.

1. The semicircle is taken to the line $\text{Re}(z) = 0$, since $\text{Re}(\log(z)) = \text{Re}(\log(|z|) + i\arg(z)) = \text{Re}(i\arg(z)) = 0$.
2. $(0,1)$ is taken to the line $\text{Im}(z) = 0$, since $\arg(z) = 0$ for $z \in (0,1)$.
3. $(-1,0)$ is taken to line $\text{Im}(z) = \pi$, since $\arg(z) = \pi$ for $z \in (-1,0)$.

For F_3, the mapping is $z \mapsto -iz$, which gives us what we want on inspection.

For F_4, the mapping is $z \mapsto \sin(z)$, as shown in the book to be conformal on this strip.

1. The half line $\text{Re}(z) = -\pi/2, \text{Im}(z) > 0$ gets taken to $(-\infty,-1)$, since we have that
$$\sin(z) = \frac{e^{-i\pi/2-\text{Im}(z)} - e^{i\pi/2+\text{Im}(z)}}{2i} = \frac{-i(e^{-\text{Im}(z)} + e^{\text{Im}(z)})}{2i} = -\frac{e^{-\text{Im}(z)} + e^{\text{Im}(z)}}{2} < -1$$
since $e^{-\text{Im}(z)} + e^{\text{Im}(z)} > 2$ since $\text{Im}(z) > 0$ (note that $e^x + e^{-x}$ achieves a minimum of 2 at $x = 0$).
2. The half line $\text{Re}(z) = \pi/2, \text{Im}(z) > 0$ gets taken to $(1,\infty)$, since we have that
$$\sin(z) = \frac{e^{i\pi/2-\text{Im}(z)} - e^{i\pi/2+\text{Im}(z)}}{2i} = \frac{i(e^{-\text{Im}(z)} + e^{\text{Im}(z)})}{2i} = \frac{e^{-\text{Im}(z)} + e^{\text{Im}(z)}}{2} > 1$$

3. The interval $[\pi/2, \pi/2]$ gets taken to $[-1, 1]$.

For F_5, the mappings is $z \mapsto z - 1$, which is clearly what we want.

Now, we have that $\frac{1}{\pi}\arg(z)$ on the upper half plane is harmonic from the book, and takes values on the boundaries as follows: $x < 0 \implies \frac{1}{\pi}\arg(x) = 1$ and $x > 0 \implies \frac{1}{\pi}\arg(x) = 0$. Then, we have that under the series of mappings, the half lines $\{y = 0, x > 1\}$ and $\{x = 0, y > 0\}$ are taken to $(-\infty, -2)$ and $(-2, 0)$ respectively, and the interval $\{0 < x < 1, y = 0\}$ is taken to $(0, \infty)$, so we have that $f = u \circ F_5 \circ F_4 \circ F_3 \circ F_2 \circ F_1$ satisfies that f is harmonic on the first quadrant and has the desired boundaries.