MATH 4061 HW 10

David Chen, dc3451

December 9, 2020

1

We can give a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ for any positive ϵ. In particular, note that the continuity of α at x_0 gives some $\delta > 0$ such that $|x - x_0| < \delta \implies |\alpha(x) - \alpha(x_0)| < \epsilon/2$. Then, consider $P = \{a, x_\ell, x_r, b\}$ such that $x_0 - \delta < x_\ell < x_0 < x_r < x + \delta$ (for instance, $x_\ell, x_r = x_0 \pm \delta/2$). Then,

$$L(P, f, \alpha) = 0(\alpha(a) - \alpha(x_\ell)) + 0(\alpha(x_r) - \alpha(x_\ell)) + 0(\alpha(b) - \alpha(x_r)) = 0$$

and since $|\alpha(x_r) - \alpha(x_\ell)| \leq |\alpha(x_0) - \alpha(x_\ell)| + |\alpha(x_0) - \alpha(x_r)| < \epsilon$,

$$U(P, f, \alpha) = 0(\alpha(a) - \alpha(x_\ell)) + 1(\alpha(x_r) - \alpha(x_\ell)) + 0(\alpha(b) - \alpha(x_r)) = \alpha(x_r) - \alpha(x_\ell) < \epsilon$$

so by the earlier theorem in the book, we get that $f \in \mathscr{R}(\alpha)$ and, since we can give some $U(P, f, \alpha) < \epsilon$, $\inf(U(P, f, \alpha)) \leq 0$, so we get that the integral is exactly 0 since $U(P, f, \alpha) \geq L(P, f, \alpha)$, but the latter can be explicitly 0, so

$$\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha = 0$$

2

Suppose that f is not indentically 0, such that $\exists x_0$ such that $f(x_0) > 0$. Then, since f is continuous, there is some neighborhood of x_0 where f is positive: $\exists \delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < f(x_0) \implies f(x) > 0$.

Then, consider the partition $P = \{a, x_\ell, x_r, b\}$ such that

$$L(P, f, \alpha) = \inf(f([a, x_\ell]))(x_\ell - a) + \inf(f([x_\ell, x_r]))(x_r - x_\ell) + \inf(f([x_r, b]))(b - x_r)$$

Now, $f([a, b]) \geq 0$ so all the infimums are nonnegative and we have that f continuous on the compact set $[x_\ell, x_r]$ realizes a minimum, so $\inf(f([x_\ell, x_r])) > 0$. Thus,

$$L(P, f, \alpha) \geq \inf(f([x_\ell, x_r]))(x_r - x_\ell) > 0$$

so the integral is > 0. $\Rightarrow\Leftarrow$, so f is 0 everywhere.

7

a

Not very parsimonious, but this is immediate from the fact (given by the fundamental theorem of calculus) that $F(c) = \int_c^1 f(x)dx$ is a continuous function of c taking $[0,1] \to \mathbb{R}$ since f is integrable on $[0,1]$. Then, continuity gives us that $\lim_{c\to 0} F(c) = \lim_{c\to 0}\int_c^1 f(x)dx = F(0) = \int_0^1 f(x)dx$.

b

The hint on piazza consists of making a graph of isoceles triangles of area $(-1)^n/n$ on the intervals $[1/(n+1), 1/n]$, which is continuous and thus integrable. Giving an explicit construction is a bit annoying, so we can also make this slightly more simple while keeping the same idea: consider the function

$$f(x) = (-1)^n(n+1)$$

where n is given by $\lfloor 1/x \rfloor$ (that is, $x \in (1/(n+1), 1/n,])$. Then, this is clearly discontinuous at exactly $x = 1/m$ for $m \in J$. This is integrable on any interval $[c,1]$ for $c > 0$. In particular, let $n_c = \lfloor 1/c \rfloor$, such that $c \in [1/(n_c+1), 1/n_c]$. Then, f is a step function with finitely many discontinuities: $\{1/n_c, 1/(n_c-1), \ldots, 1/2\}$ since there are a finite amount of reciprocals of integers between $c > 0$ and 1. Then, f is integrable, and

$$\int_c^1 f(x)dx = \left(\frac{1}{n_c} - c\right)(-1)^{n_c}(n_c+1) + \sum_{n=1}^{n_c-1}\frac{(-1)^n}{n}$$

since we can clearly give $P = \{c, 1/n_c, 1/(n_c-1), \ldots, 1/2, 1\}$ such that

$$\begin{aligned}
U(P,f) = L(P,f) &= \left(\frac{1}{n_c} - c\right)(-1)^{n_c}(n_c+1) + \sum_{n=1}^{n_c-1}\left(\frac{1}{n} - \frac{1}{n+1}\right)(-1)^n(n+1) \\
&= \left(\frac{1}{n_c} - c\right)(-1)^{n_c}(n_c+1) + \sum_{n=1}^{n_c-1}\frac{(-1)^n}{n}
\end{aligned}$$

Then, $\lim_{c\to 0}\int_c^1 f(x)dx = -\ln(2)$, since we get that as $c \to 0$, $n_c \to \infty$ and

$$\left|\left(\frac{1}{n_c} - c\right)(-1)^{n_c}(n_c+1)\right| < \left|\left(\frac{1}{n_c} - \frac{1}{n_c+1}\right)(n_c+1)\right| = \left|\frac{1}{n_c}\right| \to 0$$

so we get that

$$\lim_{c\to 0}\int_c^1 f(x)dx = \left(\frac{1}{n_c} - c\right)(-1)^{n_c}(n_c+1) + \sum_{n=1}^{n_c-1}\frac{(-1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -\ln(2)$$

but

$$\lim_{c\to 0}\left|\int_c^1 f(x)dx\right| = \left(\frac{1}{n_c} - c\right)(n_c+1) + \sum_{n=1}^{n_c-1}\frac{1}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$$

which diverges.

8

Consider the following picture:

($\implies$) Note that as $x \to 0$, if $f(x) \to m > 0$, then $f(x) \geq m > 0$ for all $x \in [1, \infty,)$, and so $\lim_{b\to\infty}\int_1^b f(x)dx > \lim_{b\to\infty}\int_1^b m dx = (b-1)m = \infty$. $\Rightarrow\Leftarrow$

Then, consider the new function $g(x) = f(\lfloor x+1 \rfloor)$, such that for integral n, $g(n) = f(n+1)$ and thus $\sum_{n=1}^{\infty} g(n) = \sum_{n=2}^{\infty} f(n)$. In particular, on any finite interval $[1, b]$, g is discontinuous at a subset of $\{2, 3, \ldots, \lfloor b \rfloor\}$, and satisfies that $g(x) = f(\lfloor x+1 \rfloor) \leq f(x)$, so g is integrable and $\int_1^b g \leq \int_1^b f$. But, we get that with the partition $\{1, 2, 3, \ldots, \lfloor b \rfloor, b\}$,

$$\int_1^b g = (b - \lfloor b \rfloor)g(\lfloor b \rfloor) + \sum_{n=1}^{\lfloor b \rfloor} g(n)$$

but $0 \leq b - \lfloor b \rfloor < 1$, and $g(\lfloor b \rfloor) \to 0$ as $b \to 0$ since $f \to 0$ as $x \to 0$. Then,

$$\lim_{b\to\infty}\int_1^b g = \sum_{n=1}^{\infty} g(n) = \sum_{n=2}^{\infty} f(n) \leq \lim_{b\to\infty}\int_1^b f$$

so $\sum_{n=1}^{\infty} f(n) \leq f(1) + \int_1^{\infty} f$, so the partial sums are monotonic and bounded, and thus converges.

($\impliedby$) Since $\int_1^b f$ is a continuous increasing function of b, if it is bounded above, we get that the limit as $b \to \infty$ converges. Then, consider that $f(\lfloor x \rfloor) \geq f(x)$ since $\lfloor x \rfloor \leq x$, and on any finite interval $[1, b]$, we get that there are finite discontinuities $\{2, 3, \ldots, \lfloor b \rfloor\}$ just as before; then $f(\lfloor x \rfloor)$ is integrable, and thus gives us

$$\int_1^b f(\lfloor x \rfloor)dx = (b - \lfloor b \rfloor)f(\lfloor b \rfloor) + \sum_{n=1}^{\lfloor b \rfloor} f(n) \geq \int_1^b f(x)dx$$

but as $b \to \infty$, $f(\lfloor b \rfloor) \to 0$ and $b - \lfloor b \rfloor \leq 1$, so

$$\sum_{n=1}^{\infty} f(n) \geq \int_1^b f$$

where the LHS converges by assumption so we get what we want.