MATH 4065 HW 1

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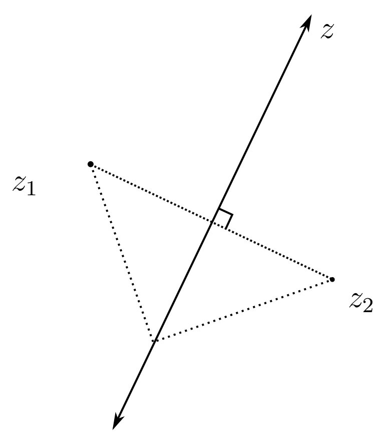
September 25, 2020

1a

Note that if $z_1 = z_2 = z'$, any $z \in \mathbb{C}$ trivially satisfies the desired property $|z - z'| = |z - z'|$. When $z_1 \neq z_2$, the set of all z satisfying the desired property describes a line in the complex plane: in particular, it contains the point $(z_1+z_2)/2$ and is perpendicular to the line containing both z_1 and z_2.



1e

We have that $\mathrm{Re}(az + b) = \mathrm{Re}(az) + \mathrm{Re}(b) = \mathrm{Re}(a)\,\mathrm{Re}(z) - \mathrm{Im}(a)\,\mathrm{Im}(z) + \mathrm{Re}(b)$. Then, if $\mathrm{Re}(z) \neq 0$ and $\mathrm{Im}(z) \neq 0$, we want that z is either above or below (depending on the sign of $\mathrm{Im}(a)$) the line

$$\mathrm{Im}(z) = \frac{\mathrm{Re}(a)}{\mathrm{Im}(a)}\,\mathrm{Re}(z) - \frac{\mathrm{Re}(b)}{\mathrm{Im}(a)}$$

In particular, this becomes more clear if we write $\mathrm{Im}(z) = y, \mathrm{Re}(z) = x$, such that if $\mathrm{Im}(a) < 0$, we want points in the plane that satisfy

$$y > \frac{\mathrm{Re}(a)}{\mathrm{Im}(a)}x - \frac{\mathrm{Re}(b)}{\mathrm{Im}(a)}$$

Similarly, if we have $\operatorname{Im}(a) > 0$, we want points in the plane that satisfy

$$y < \frac{\operatorname{Re}(a)}{\operatorname{Im}(a)}x - \frac{\operatorname{Re}(b)}{\operatorname{Im}(a)}$$

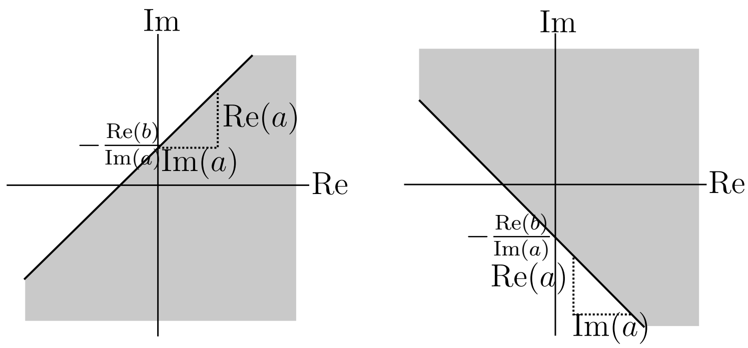


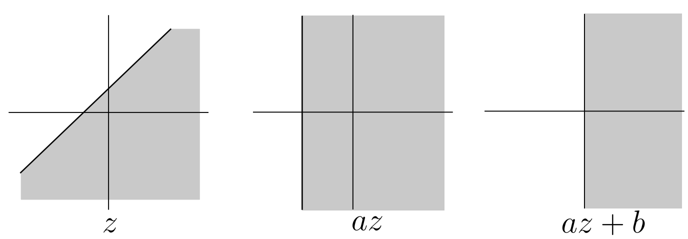
Figure 1: The left is for some a in the first quadrant. z can be anywhere in the gray. The right is for some a in the fourth quadrant.

Now, the expression is easier if a is either real or imaginary; in the case that a is real and positive, we want z such that $\operatorname{Re}(z) > -\frac{\operatorname{Re}(b)}{\operatorname{Re}(a)}$. In particular, this is anything to the right of the vertical line $x = -\frac{\operatorname{Re}(b)}{\operatorname{Re}(a)}$. If a is real and negative, z can be anything to the left of that line.

Similarly, if a is imaginary with positive imaginary part, then we want z such that $\operatorname{Im}(z) < \frac{\operatorname{Re}(b)}{\operatorname{Im}(a)}$. This is anything below the horizontal line $y = \frac{\operatorname{Re}(b)}{\operatorname{Im}(a)}$. If a is imaginary with negative imaginary part, z can be anything above that line.

If $a = 0$, the choice of b will fix that either z can be anything in $\mathbb{C}$ (if $\operatorname{Re}(b) > 0$) or nothing.

Geometrically, this is reasonable as the transformation $az+b$ does the following to the shaded region:



11

First, we'll do 10, which is that $4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = \Delta$. This is true at least under the assumption that the partial derivatives are continuous, which allows that $\frac{\partial}{\partial x}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}$.

Then,

$$
\begin{aligned}
\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \bar{z}}\right) &= \frac{1}{2}\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial \bar{z}}\right)\right) + \frac{1}{2i}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial \bar{z}}\right)\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x}\left(\frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y}\right)\right) + \frac{1}{2i}\left(\frac{\partial}{\partial y}\left(\frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y}\right)\right) \\
&= \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{4i}\left(\frac{\partial}{\partial y}\frac{\partial}{\partial x} + \frac{\partial}{\partial x}\frac{\partial}{\partial y}\right) \\
&= \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) = \frac{1}{4}\Delta
\end{aligned}
$$

Now, we have that if f is holomorphic, then f obeys the Cauchy-Riemann equations, and thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, where $f(x+yi) = u(x,y) + v(x,y)i$.
However, this means that

$$
\begin{aligned}
\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right) \\
&= \frac{1}{2}\left(\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) - \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right) \\
&= \frac{1}{2}\left(\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} - \frac{1}{i^2}\frac{\partial u}{\partial y}\right)\right) \\
&= \frac{1}{2}\left(\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right) \\
&= \frac{1}{2}(0+0) = 0
\end{aligned}
$$

Then, this gives us what we wanted: $\Delta f = \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} f = \frac{\partial}{\partial z}(0) = 0$.

13b

Again, we have the Cauchy-Riemann equations, which give that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. However, if $f(x+yi) = u(x,y) + v(x,y)i$ has that $\mathrm{Im}(f)$ is constant, then $v(x,y)$ is constant which gives that $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$. The Cauchy-Riemann equations then tell us that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$.
However, we have that since limit definition of the complex derivative allows us to approach the point from any direction, we can take that (as f is holomorphic) $f'(x+yi) = \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = 0 + 0$. By a proposition proved in class, f is constant on Ω (which is taken to be an open connected set).

14

The relationship does not hold if $N < M$.

For the rest of this problem, the empty sum is 0, as it is in the book.

First, we check the degenerate case that $N = M$, in which case

$$\begin{aligned}
a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1}(a_{n+1} - a_n)B_n &= a_M B_M - a_M B_{M-1} - \sum_{n=M}^{M-1}(a_{n+1} - a_n)B_n \\
&= a_M(B_M - B_{M-1}) \\
&= a_M b_M \\
&= \sum_{n=M}^{M} a_n b_n = \sum_{n=M}^{N} a_n b_n
\end{aligned}$$

We have the following if $N > M$:

$$\sum_{n=M}^{N-1}(a_{n+1} - a_n)B_n = \sum_{n=M}^{N-1} a_{n+1}B_n - \sum_{n=M}^{N-1} a_n B_n$$

Reindexing,

$$= \sum_{n=M+1}^{N} a_n B_{n-1} - \sum_{n=M}^{N-1} a_n B_n$$

This step is why we need $N > M$ to pull out the terms, otherwise both sums are empty:

$$\begin{aligned}
&= a_N B_{N-1} - a_M B_M + \sum_{n=M+1}^{N-1} a_n B_{n-1} - \sum_{n=M+1}^{N-1} a_n B_n \\
&= a_N B_{N-1} - a_M B_M - \sum_{n=M+1}^{N-1} a_n(B_n - B_{n-1}) \\
&= a_N B_{N-1} - a_M B_M - \sum_{n=M+1}^{N-1} a_n b_n
\end{aligned}$$

Then, we have that

$$\begin{aligned}
a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n &= a_N B_N - a_M B_{M-1} - a_N B_{N-1} + a_M B_M + \sum_{n=M+1}^{N-1} a_n b_n \\
&= a_N (B_N - B_{N-1}) + a_M (B_M - B_{M-1}) + \sum_{n=M+1}^{N-1} a_n b_n \\
&= a_N b_N + a_M b_M + \sum_{n=M+1}^{N-1} a_n b_n \\
&= \sum_{n=M}^{N} a_n b_n
\end{aligned}$$

which was what we wanted.

16a

We use the ratio test for radius of convergence, as given in class (and proved later in the problem set).

$$\begin{aligned}
\lim_{n\to\infty} \frac{|(\log(n+1))^2|}{|(\log(n))^2|} &= \lim_{n\to\infty} \left| \frac{(\log(n+1))^2}{(\log(n))^2} \right| \\
&= \lim_{n\to\infty} \left| \left(\frac{\log(n+1)}{\log(n)} \right)^2 \right| \\
&= \lim_{n\to\infty} \left(\frac{\log(n+1)}{\log(n)} \right)^2
\end{aligned}$$

From L'Hopital's, we get

$$\begin{aligned}
&= \lim_{n\to\infty} \left(\frac{\frac{2\log(n+1)}{n+1}}{\frac{2\log(n)}{n}} \right) \\
&= \lim_{n\to\infty} \left(\frac{\log(n+1)}{\log(n)} \right) \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1} \right) \\
&= 1
\end{aligned}$$

which gives us a radius of convergence of 1.

16c

$$\lim_{n\to\infty}\frac{\left|\frac{(n+1)^2}{4^{n+1}+3(n+1)}\right|}{\left|\frac{n^2}{4^n+3n}\right|} = \lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}\frac{4^n+3n}{4^{n+1}+3(n+1)}\right|$$
$$= \lim_{n\to\infty}\frac{(n+1)^2}{n^2}\left|\frac{4^n+3n}{4^{n+1}+3(n+1)}\right|$$
$$= \lim_{n\to\infty}\left|\frac{4^n+3n}{4^{n+1}+3(n+1)}\right|$$

We can take n large enough such that everything inside the modulus is positive, so

$$= \lim_{n\to\infty}\frac{4^n+3n}{4^{n+1}+3(n+1)}$$

From L'Hopital,

$$= \lim_{n\to\infty}\frac{4^x\log(4)+3}{4^{x+1}\log(4)+3}$$
$$= \lim_{n\to\infty}\frac{4^x\log(4)^2}{4^{x+1}\log(4)^2}$$
$$= \lim_{n\to\infty}\frac{1}{4}\frac{4^x}{4^x}$$
$$= \frac{1}{4}$$

which gives us a radius of convergence of 4.

16e

The constant term of the series does not affect convergence. In particular, if we take the empty product to be 1, we have that F can be reindexed to start from $n=0$.

$$\lim_{n\to\infty}\frac{\left|\frac{\alpha(\alpha+1)\cdots(\alpha+n)\beta(\beta+1)\cdots(\beta+n)}{(n+1)!\gamma(\gamma+1)\cdots(\gamma+n)}\right|}{\left|\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)}\right|} = \lim_{n\to\infty}\left|\frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)}\right| = 1$$

which gives us a radius of convergence of 1. The limit follows from the numerator and the denominator both being monic quadratics.

17

We have that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$ yields some N such that $\forall n \geq N$, $\left|\frac{|a_{n+1}|}{|a_n|} - L\right| < \epsilon$ for any positive $\epsilon \in \mathbb{R}$. Then, we have that $L - \epsilon < \frac{|a_{n+1}|}{|a_n|} < L + \epsilon$. Since this holds for any $n \geq N$, and $|a_{n+1}|, |a_n| > 0 \implies L > 0$ (we can then pick only $0 < \epsilon < L$, which is good enough) we have that

$$(L-\epsilon)^{n-N} < \prod_{i=N}^{n-1} \frac{|a_{n+1}|}{|a_n|} = \frac{|a_n|}{|a_N|} < (L+\epsilon)^{n-N}$$

for $n > N$.

Rearranging, we have that

$$(L-\epsilon)^n (L-\epsilon)^{-N} |a_N| < |a_n| < (L+\epsilon)^n (L+\epsilon)^{-N} |a_N|$$

which gives

$$(L-\epsilon)((L-\epsilon)^{-N}|a_N|)^{1/n} < |a_n|^{1/n} < (L+\epsilon)((L+\epsilon)^{-N}|a_N|)^{1/n}$$

Since we know that $\lim_{n\to\infty} b^{1/n} = 1$ (for $b > 0$), we have that $\exists N'$ such that for any ϵ', $1 - \epsilon' < b^{1/n} < 1 + \epsilon'$. Applying this to $b = (L-\epsilon)^{-N}|a_N|$ and $\epsilon' = \frac{\epsilon}{L-\epsilon}$, we get that $(L-\epsilon)(1-\epsilon') = L - 2\epsilon < (L-\epsilon)((L-\epsilon)^{-N}|a_N|)^{1/n}$, and similarly for $b = (L+\epsilon)^{-N}|a_N|$ and $\epsilon' = \frac{\epsilon}{L+\epsilon}$, we get $(L+\epsilon)((L+\epsilon)^{-N}|a_N|)^{1/n} < (L+\epsilon)(1+\epsilon') = L + 2\epsilon$ for sufficiently large n, i.e. $n > \max N, N'$.

This bounds

$$L - 2\epsilon < |a_n|^{1/n} < L + 2\epsilon$$

for $n > \max N, N'$. This then gives that $\lim_{n\to\infty} |a_n|^{1/n} = L$, which was what we wanted.

7

a

Since $\left|\frac{w-z}{1-\bar{w}z}\right| < 1 \iff \left|\frac{w-z}{1-\bar{w}z}\right|^2 < 1^2$ and $\left|\frac{w-z}{1-\bar{w}z}\right| = 1 \iff \left|\frac{w-z}{1-\bar{w}z}\right|^2 = 1^2$ as the modulus is nonnegative, we can compute

$$\frac{w-z}{1-\bar{w}z}\overline{\frac{w-z}{1-\bar{w}z}} = \frac{|w|^2 - \bar{z}w - z\bar{w} + |z|^2}{1 - \bar{z}w - z\bar{w} + |wz|^2}$$

This means that we want to show

$$|w|^2 - \bar{z}w - z\bar{w} + |z|^2 \leq 1 - \bar{z}w - z\bar{w} + |wz|^2$$

which reduces to $0 \leq 1 + |w|^2|z|^2 - |w|^2 - |z|^2 = (1-|w|^2)(1-|z|^2)$. Since we have that if $|w|, |z| < 1$ that the right hand is positive, we have that $\left|\frac{w-z}{1-\bar{w}z}\right| < 1$ in that case. If either of $|w|, |z| = 1$, then the right hand vanishes, and $\left|\frac{w-z}{1-\bar{w}z}\right| = 0$.

b

i

We have that if f, g are holomorphic, then fg, $f+g$, and f/g are all holomorphic from class. This gives that $w - z$ is holomorphic, as is $1 - \bar{w}z$ since w is fixed, and these are linear functions in z. Since f/g is holomorphic, we have that $F = (w-z)/(1-\bar{w}z)$ is itself holomorphic.

To show that it takes $\mathbb{D} \to \mathbb{D}$, consider that since w is fixed in $\mathbb{D}$, we have that $|w| < 1$, and since $z \in \mathbb{D}$, that also $|z| < 1$. From part a, we have that $|F(z)| < 1 \implies F(z) \in \mathbb{D}$.

To show that F is actually a bijection, see part iv.

ii

We can just compute this directly:

$$F(0) = \frac{w - 0}{1 - \bar{w}0} = \frac{w}{1} = w$$

$$F(w) = \frac{w - w}{1 - \bar{w}w} = \frac{0}{1 - |w|^2} = 0$$

We can take $\frac{0}{1-|w|^2} = 0$ since we have that w is in the unit disc, and thus satisfies that $|w| < 1 \implies |w|^2 < 1 \implies 1 - |w|^2 > 0$.

iii

We have that $|F(z)| = \left|\frac{w-z}{1-\overline{w}z}\right|$, which was shown to be 1 when $|z| = 1$ in part a.

iv

We will take the book's hint and compute

$$\begin{aligned} F(F(z)) &= \frac{w - \frac{w-z}{1-\overline{w}z}}{1 - \overline{w}\frac{w-z}{1-\overline{w}z}} \\ &= \frac{\frac{w(1-\overline{w}z)-(w-z)}{1-\overline{w}z}}{\frac{1-\overline{w}z-\overline{w}(w-z)}{1-\overline{w}z}} \end{aligned}$$

Since we have that $w, z \in \mathbb{D}, |w|, |z| < 1 \implies |\overline{w}z| < 1 \implies 1 \neq \overline{w}z$, we can cancel:

$$
\begin{aligned}
&= \frac{w(1 - \overline{w}z) - (w - z)}{1 - \overline{w}z - \overline{w}(w - z)} \\
&= \frac{-|w|^2 z + z}{1 - |w|^2} \\
&= \frac{z(1 - |w|^2)}{1 - |w|^2}
\end{aligned}
$$

Again, we have that $|w| < 1 \implies 1 - |w|^2 > 0$, so

$$= z$$

We can now find a preimage for any $z \in \mathbb{D}$, which is $F(z)$, as we have that $F(F(z)) = z$, so F is surjective. Further, suppose we have $z_1, z_2 \in \mathbb{D}$ and $F(z_1) = F(z_2)$. Then, $F(z_1) = F(z_2) \implies F(F(z_1)) = F(F(z_2)) \implies z_1 = z_2$, so F is also injective, and therefore an injection.