MATH 4061 HW 5

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7

We have that $(\sqrt{a_n} - \frac{1}{n})^2 = a_n + \frac{1}{n^2} - \frac{\sqrt{a_n}}{n} \geq 0$. Then, we have that $a_n + \frac{1}{n^2} \geq \frac{\sqrt{a_n}}{n}$, so if the first series $\sum(a_n + \frac{1}{n^2})$ converges, then the series that we want converges as well. Then, we have that if $\sum a_n$ converges, which it does by assumption, and $\sum \frac{1}{n^2}$ converges, which it does since it is a p-series with exponent -2, then $\sum(a_n + \frac{1}{n^2})$ converges as well, and so $\sum \frac{\sqrt{a_n}}{n}$ converges.

9

a

Ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{(n+1)^3 z^{n+1}}{n^3 z^n} \right| &= \lim_{n\to\infty} \frac{n^3 + 3n^2 + 3n + 1}{n^3} |z| \\
&= \lim_{n\to\infty} \left(1 + 3\frac{1}{n} + 3\frac{1}{n^2} + \frac{1}{n^3} \right) |z| \\
&= |z|
\end{aligned}$$

which the ratio test states converges if $|z| < 1$.

b

Ratio test:

$$\lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+1)!} z^{n+1}}{\frac{2^n}{n!} z^n} \right| = \lim_{n\to\infty} \frac{2}{n+1} |z| = 0|z| = 0$$

so the this series converges everywhere and thus has a radius of convergence of ∞.

c

Ratio test:

$$\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{(n+1)^3}z^{n+1}}{\frac{2^n}{n^3}z^n}\right| = \lim_{n\to\infty} 2\frac{n^3}{(n+1)^3}|z| = 2|z|\lim_{n\to\infty}\frac{n^3}{(n+1)^3}$$

but we already showed that $\lim_{n\to\infty}\frac{(n+1)^3}{n^3} = 1$ in an earlier part, so $\lim_{n\to\infty}\frac{n^3}{(n+1)^3} = 1$, and so the this series converges when $2|z| < 1$, so the radius of convergence is $\frac{1}{2}$.

d

Ratio test:

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)^3}{3^{n+1}}z^{n+1}}{\frac{n^3}{3^n}z^n}\right| = \lim_{n\to\infty}\frac{(n+1)^3}{3n^3}|z| = \frac{|z|}{3}\lim_{n\to\infty}\frac{(n+1)^3}{n^3} = \frac{|z|}{3}$$

so this converges where $\frac{|z|}{3} < 1$, so the radius of convergence is 3.

10

We use the root test. We want to show that $\frac{1}{R} = \limsup_{n\to\infty}\sqrt[n]{|a_n|} \geq 1$. The contrapositive of theorem 3.17 in Rudin (also shown in class) states that for some real x, if there is no integer N such that $n \geq N \implies \sqrt[n]{|a_n|} < x$, then $x \leq \limsup_{n\to\infty}\sqrt[n]{|a_n|}$. However, it is given that an infinite amount of a_n must be distinct from zero. Then, for any N, we have that there is some $n > N$ such that $a_n \neq 0 \implies |a_n| \geq 1$, since there is no integer between 0 and 1. If there were no such n, then there would be at most N a_n distinct from 0, so there is always some $n \geq N$ for which $a_n \neq 0$. Then, according to the earlier statement, $|a_n| \geq 1 \implies \sqrt[n]{|a_n|} \geq 1$ as a basic fact about n^{th} roots, so by the earlier statement, $\limsup_{n\to\infty}\sqrt[n]{|a_n|} \geq 1$, so $\frac{1}{R} \geq 1 \implies R \leq 1$ which is what we wanted.

13

Put $A_n = \sum_{k=0}^n a_k$, $B_n = \sum_{k=0}^n b_k$, $C_n = \sum_{k=0}^n |c_k| = \sum_{k=0}^n \left|\sum_{m=0}^k a_m b_{k-m}\right|$. Then, if both $\sum a_n$, $\sum b_n$ converge absolutely, we will show that $\sum |c_n|$ converge absolutely.
We have that

$$C_n = \sum_{k=0}^n\left|\sum_{m=0}^k a_m b_{k-m}\right| \leq \sum_{k=0}^n\sum_{m=0}^k |a_m b_{k-m}| = \sum_{k=0}^n\sum_{m=0}^k |a_m||b_{k-m}|$$

However, we have that since $\sum |a_m|$ and $\sum |b_m|$ converge (and converge absolutely), then we have that the right hand side as their Cauchy product converges, as shown in Rudin, so C_n is bounded and monotonic, since $C_{n+1} - C_n = \left|\sum_{m=0}^{n+1} a_m b_{k-m}\right| \geq 0$, so C_n converges and thus $\sum c_n$ converges absolutely.

I can't remember if we proved this in class, so note that we can do the following sum rearrangement:

$$\begin{aligned}\sum_{k=0}^{n} \sum_{m=0}^{k} |a_m||b_{k-m}| &= |a_0||b_0| + (|a_1||b_0| + |a_0||b_1|) + \cdots + (|a_0||b_n| + |a_1||b_{n-1}| + \cdots + |a_n||b_0|) \\ &= |a_0|(|b_0| + |b_1| + \cdots + |b_n|) + |a_1|(|b_0| + |b_1| + \cdots + b_{n-1}) + \cdots + |a_n||b_0|\end{aligned}$$

Since $\sum |a_n|$ and $\sum |b_n|$ converge, and each term is positive,

$$\leq \sum_{k=0}^{n} |a_k|(\sum_{m=0}^{\infty} |b_m|) < \left(\sum_{k=0}^{\infty} |a_k|\right)\left(\sum_{m=0}^{\infty} |b_m|\right)$$

so C_n is bounded and monotonic, so it converges.

22

We will show at the end that for a dense set G_n and some nonempty open set E, that $E \cap G_n$ is necessarily nonempty. If we have this, pick any open subset E of X, so $E \cap G_1 \neq \emptyset$. Then, this is the intersection of open sets, and is thus open itself. Pick $x_1 \in E \cap G_1$, such that $B_{r_1}(x_1) \subset E \cap G_1$ for some $r_1 < 1$. Then, we have that $\overline{B_{r_1/2}(x_1)} \subset B_r(x_1) \subset E \cap G_1$, and further that $B_{r_1/2}(x_1) \cap G_2 \neq \emptyset$ as well. Again, we have some $x_2 \in B_{r_1/2}(x_1) \cap G_2$, and some $r_2 < \min(d(x_2, x_1), r_1/2 - d(x_2, x_1), \frac{1}{2})$ such that $\overline{B_{r_2/2}(x_2)} \subset B_{r_2}(x_2) \subset B_{r_1/2}(x_1) \cap G_2 \subset E \cap G_1 \cap G_2$.

Continue this inductively, such that for any n, if we have that $B_{r_{n-1}/2}(x_{n-1}) \cap G_n$ is the intersection of nonempty open sets, and is open and nonempty itself since G_n is dense, and so we pick $x_n \in B_{r_{n-1}/2}(x_{n-1}) \cap G_n$ and $r_n < \min(d(x_{n-1}, x_n), r_{n-1}/2 - d(x_{n-1}, x_n), \frac{1}{n})$ such that $\overline{B_{r_n/2}(x_n)} \subset B_{r_n}(x_n) \subset B_{r_{n-1}/2}(x_{n-1}) \cap G_n \subset E \cap (\bigcap_{i=1}^{n} G_i)$, since $B_{r_{n-1}/2}(x_{n-1})$ by the inductive construction is a subset of $E \cap \left(\bigcap_{i=1}^{n-1} G_i\right)$.

Then, this sequence $\{\overline{B_{r_n/2}(x_n)}\}_{n=1}^{\infty}$ is a sequence of closed and bounded sets. We have that $\lim_{n\to\infty} \operatorname{diam}(\overline{B_{r_n/2}(x_n)}) = 0$, since $\overline{B_{r_n/2}(x_n)} \subseteq \overline{B_{1/2n}(x_n)}$ by construction, and so $\operatorname{diam}(\overline{B_{r_n/2}(x_n)}) \leq \operatorname{diam}(\overline{B_{1/2n}(x_n)})$. Then,

$$0 \leq \lim_{n\to\infty} \operatorname{diam}(\overline{B_{r_n/2}(x_n)}) \leq \lim_{n\to\infty} \operatorname{diam}(\overline{B_{1/2n}(x_n)}) = \lim_{n\to\infty} \frac{1}{2n} = 0$$

Thus a past homework problem gives that since X is complete, the infinite intersection $\bigcap_{i=1}^{\infty} \overline{B_{r_i/2}(x_i)}$is still nonempty (Q21).

Then, we have that there is some x such that $x \in \bigcap_{i=1}^{\infty} \overline{B_{r_i/2}(x_i)}$. Further, $x \in \overline{B_{r_n/2}(x_n)}$ gives by construction that $x \in E \cap (\bigcap_{i=1}^{n} G_n)$, so since x is in every $\overline{B_{r_n/2}(x_n)}$, $x \in E \cap (\bigcap_{i=1}^{\infty} G_n)$, and we have that $\bigcap_{i=1}^{\infty} G_n \neq \emptyset$.

Finally, since this holds for any open set E, pick any $x \in X$ and $\epsilon > 0$ and consider $B_\epsilon(x) \cap (\bigcap_{i=1}^{\infty} G_n)$, which now must be nonempty. Then, x is either a limit point of $(\bigcap_{i=1}^{\infty} G_n)$, or it is contained in $(\bigcap_{i=1}^{\infty} G_n)$, so $(\bigcap_{i=1}^{\infty} G_n)$ is dense.

The only thing left is to show that for a dense set G_n and some open set E, that $E \cap G_n$ is necessarily nonempty. To see this, suppose that $G_n \cap E$ was in fact empty: then, for some $x \in E$, $G_n \cap B_\epsilon(x) = \emptyset$, so $x \notin G_n$ and x cannot be a limit point of G_n, so G_n cannot be dense, and so we are done.