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Homework 4

MATH 1208
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Apostol pg.68 no.11

Claim. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\iff ad - bc \neq 0$.

Proof. ($\Longleftarrow$)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & -ab + ab \\ cd - dc & da - cb \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - cb \end{bmatrix} = (ad - bc)I$$

Then, we have that by the properties of matrix multiplication shown in class (i.e. that $A(cB) = c(AB)$), $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that the inverse is $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (and this is the only inverse since matrix inverses are unique).

($\Longrightarrow$) Suppose that $ad - bc = 0$. We have that then $a = \frac{b}{d}c, b = \frac{a}{c}d$. However, we have that $ad = cb \implies \frac{b}{d} = \frac{a}{c}$. Call this k, so that the matrix becomes

$$\begin{bmatrix} kc & kd \\ c & d \end{bmatrix}$$

which does not have full row rank as $(kc, kd) = k(c, d)$ and so the matrix is not invertible. $\Rightarrow\!\Leftarrow$, so then $ad - bc \neq 0$. $\square$

Apostol pg.68 no.12

$$\left[\begin{array}{cccccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & \frac{3}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 0 & -2 & -3 & -1 & 1 & 0 \\ 0 & \frac{5}{2} & 4 & \frac{1}{2} & 0 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{cccccc} 1 & 0 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} & -\frac{3}{4} & \frac{5}{4} & 1 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 5 & -8 & -6 \\ 0 & 0 & 1 & -3 & 5 & 4 \end{array}\right]$$

The inverse is then

$$\begin{bmatrix} -1 & 2 & 1 \\ 5 & -8 & -6 \\ -3 & 5 & 4 \end{bmatrix}$$

Apostol pg.68 no.2a

True:

$$\begin{aligned}
AABBB &= A(AB)BB \\
&= A(-BA)BB \\
&= -ABABB \\
&= -(AB)ABB \\
&= (BA)ABB \\
&= BA(-BA)B \\
&= B(BA)AB \\
&= BBAAB \\
&= BBA(-BA) \\
&= BB(BA)A \\
&= BBBAA
\end{aligned}$$

Apostol pg.68 no.2b

False: take any nonsingular matrix A, and its additive inverse $-A$. Then, both A and $-A$ are invertible, but $A + -A = O$ is not invertible.

Apostol pg.68 no.2c

True: note that $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$. This means that the inverse exists and is equal to $B^{-1}A^{-1}$.

Apostol pg.68 no.2f

True: From Problem 4 on Part C: $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B)) \implies \text{rank}(\prod A_i) \leq \min(\text{rank}(A_i))$. Now suppose that A_k is nonsingular, and that they are all of dimension n; then, we have that $\text{rank}(A_k) \leq n \implies \text{rank}(\prod A_i) \leq n$ and so $\prod A_i$ is not invertible. $\Rightarrow\!\Leftarrow$, so each individual matrix must also be invertible.

Apostol pg.69 no.7

Let $A \in M_{m\times n}, B \in M_{n\times p}$.

Apostol pg.69 no.7b

$$(A+B)^T_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = A^T_{ij} + B^T_{ij}$$

Apostol pg.69 no.7d

$$
\begin{aligned}
(B^T A^T)_{ij} &= \sum_{k=1}^{n} B^T_{ik} A^T_{kj} \\
&= \sum_{k=1}^{n} A_{jk} B_{ki} \\
&= (AB)_{ji} \\
&= (AB)^T_{ij}
\end{aligned}
$$

Apostol pg.69 no.7e

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I, (A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

Thus, we have that $(A^T)^{-1} = (A^{-1})^T$ be the definition and uniqueness of inverses.

Problem 1

Consider the following augmented matrix:

$$
\begin{bmatrix} 2 & 4 & 8 & 6 & a \\ 5 & 6 & 8 & 7 & b \\ 6 & 7 & 9 & 8 & c \\ 5 & 4 & 2 & 3 & d \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 & 3 & \frac{a}{2} \\ 5 & 6 & 8 & 7 & b \\ 6 & 7 & 9 & 8 & c \\ 5 & 4 & 2 & 3 & d \end{bmatrix}
$$
$$
\to \begin{bmatrix} 1 & 2 & 4 & 3 & \frac{a}{2} \\ 0 & -4 & -12 & -8 & b - \frac{5a}{2} \\ 0 & -5 & -15 & -10 & c - 3a \\ 0 & -6 & -18 & -12 & d - \frac{5a}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 & 3 & \frac{a}{2} \\ 0 & 1 & 4 & 2 & -\frac{b}{4} + \frac{5a}{8} \\ 0 & 1 & 3 & 2 & -\frac{c}{5} + \frac{3a}{5} \\ 0 & 1 & 3 & 2 & -\frac{d}{6} + \frac{5a}{12} \end{bmatrix}
$$
$$
\to \begin{bmatrix} 1 & 0 & -2 & -1 & -\frac{3a}{4} + \frac{b}{2} \\ 0 & 1 & 3 & 2 & -\frac{b}{4} + \frac{5a}{8} \\ 0 & 0 & 0 & 0 & -\frac{c}{5} + \frac{3a}{5} + \frac{b}{4} - \frac{5a}{8} \\ 0 & 0 & 0 & 0 & -\frac{d}{6} + \frac{5a}{12} + \frac{b}{4} - \frac{5a}{8} \end{bmatrix}
$$

a

Taking $a = b = c = d = 0$, we have that x_3, x_4 are free; we have then that $x_2 = -3x_3 - 2x_4$, and that $x_1 = 2x_3 + x_4$.

b

Taking $a = 2, b = 9, c = d = 11$, we have that $x_2 = -1 - 3x_3 - 2x_4$, and $x_1 = 3 + 2x_3 + x_4$.

c

Taking $a = 2, b = 9, c = 6, d = 11$, we have that the system is inconsistent as we have that $-\frac{c}{5} + \frac{3a}{5} + \frac{b}{4} - \frac{5a}{8} \neq 0$. This would imply that $0x_1 + 0x_2 + 0x_3 + 0x_4 \neq 0$, which is obviously $\Rightarrow\!\Leftarrow$.

Problem 2

a

Claim. A is symmetric and invertible $\implies A^{-1}$ is symmetric.

Proof. We have that since $AA^{-1} = A^{-1}A = I$, and since I is obviously symmetric such that $I_{ij} = I_{ji}$,

$$I_{ij=}\sum_{k=1}^{n} A_{ik}A^{-1}_{kj} = \sum_{k=1}^{n} A^{-1}_{jk}A_{ki} = I_{ji} \implies \sum_{k=1}^{n} A_{ik}A^{-1}_{kj} = \sum_{k=1}^{n} A^{-1}_{jk}A_{ik} \implies \sum_{k=1}^{n} A_{ik}(A^{-1}_{kj} - A^{-1}_{jk}) = 0$$

for any $1 \leq i \leq n$. However, this would mean that

$$A \begin{bmatrix} A^{-1}_{k1} - A^{-1}_{1k} \\ A^{-1}_{k2} - A^{-1}_{2k} \\ \vdots \\ A^{-1}_{kn} - A^{-1}_{2n} \end{bmatrix} = 0$$

Since we have that A is invertible, the only solution to the above equation, as proved in class, is $\vec{0}$ as the kernel of the map must be zero dimensional (from rank-nullity, as we have that the matrix is of full rank due to the invertible nature of A).
Thus, $A^{-1}_{ki} - A^{-1}_{ik} = 0 \implies A^{-1}_{ki} = A^{-1}_{ik}$ and A^{-1} is symmetrical. □

The following example holds, where the inverse is computed as

$$\frac{1}{ad - bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

b

Claim. A is skew-symmetric and invertible $\implies A^{-1}$ is skew-symmetric.

Proof. This is more or less the same as above, with a few signs changed.
We have that since $AA^{-1} = A^{-1}A = I$, and since I is obviously symmetric such that $I_{ij} = I_{ji}$,

$$I_{ij=} \sum_{k=1}^{n} A_{ik}A_{kj}^{-1} = \sum_{k=1}^{n} A_{jk}^{-1}A_{ki} = I_{ji} \implies \sum_{k=1}^{n} A_{ik}A_{kj}^{-1} = \sum_{k=1}^{n} A_{jk}^{-1}{-A_{ik}} \implies \sum_{k=1}^{n} A_{ik}(A_{kj}^{-1}{+A_{jk}^{-1}}) = 0$$

for any $1 \leq i \leq n$. However, this would mean that

$$A \begin{bmatrix} A_{k1}^{-1} + A_{1k}^{-1} \\ A_{k2}^{-1} + A_{2k}^{-1} \\ \vdots \\ A_{kn}^{-1} + A_{2n}^{-1} \end{bmatrix} = 0$$

Since we have that A is invertible, the only solution to the above homogenous equations, as proved in class, is $\vec{0}$ as the kernel of the map must be zero dimensional (from rank-nullity, as we have that the matrix is of full rank due to the invertible nature of A).
Thus, $A_{ki}^{-1} + A_{ik}^{-1} = 0 \implies A_{ki}^{-1} = -A_{ik}^{-1}$ and A^{-1} is skew-symmetrical. □

The following example holds, where the inverse is computed as

$$\frac{1}{ad - bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem 4

Claim. If A, B are $n \times n$ matrices, then $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$.

Proof. Consider the underlying linear transformations: $AB = m(T_A \circ T_B)$. We have from class that the column space is the image of the corresponding linear transformation.
Rank-nullity gives that for any $T : U \to V$, $\dim(\text{im}(T)) \leq \dim(U)$. Then, we have that $\dim(\text{im}(T_B)) \leq n$, and that when T_A is restricted to $\text{im}(T_B), \dim(T_A|_{\text{im}(T_B)}) \leq \dim(\text{im}(T_B)) \leq n$, and since it is a restriction, that $\dim(T_A|_{\text{im}(T_B)}) \leq \dim(T_A)$. Thus, we have that $\dim(T_A \circ T_B) \leq \dim(T_A), \dim(T_B)$. Thus, from the above correspondence between column space and

the linear transformation, we have that $\text{rank}(AB) \leq \text{rank}(A), \text{rank}(B) \implies \text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$.

Alternatively, with matrices, we have that if $v \in \ker(B)$, then $v \in \ker(AB)$ (as $T_A(T_B(v)) = T_A(0) = 0$). Similarly, if $v \in \ker(A^T) \implies v \in \ker(B^T A^T)$.

Thus, we have that $\ker(B) \subseteq \ker(AB), \ker(A^T) \subseteq \ker((AB)^T) \implies \dim(\ker(B)) \leq \dim(\ker(AB)), \dim(\ker(A^T)) \leq \dim(\ker((AB)^T))$; from rank nullity, we have that for any $M \in M_{n\times n}$, that $\dim(\ker(M)) + \text{rank}(M) = n \implies \dim(\ker(M)) = n - \text{rank}(M)$.

Thus, we have that $-\text{rank}(B) \leq -\text{rank}(AB), -\text{rank}(A^T) \leq -\text{rank}((AB)^T) \implies \text{rank}(AB) \leq \text{rank}(B), \text{rank}((AB^T)) \leq \text{rank}(A^T)$. However, since column and row rank are the same, $\text{rank}(A) = \text{rank}(A^T)$ and $\text{rank}(AB) = \text{rank}((AB)^T)$. Finally,

$$\text{rank}(AB) \leq \text{rank}(A), \text{rank}(AB) \leq \text{rank}(B) \implies \text{rank}(AB) \leq \min(\text{rank}(B), \text{rank}(A))$$

$\square$