Apostol p.125 no.21

Claim. Let f, g be functions that are integrable on every interval and satisfying the following: f is odd, g is even, $f(5) = 7, f(0) = 0, g(x) = f(x+5), f(x) = \int_0^x g(t)dt$ for all x. Then (a) $\forall x, f(x-5) = -g(x)$; (b) $\int_0^5 f(x)dx = 7$; (c) $\int_0^x f(t)dt = g(0) - g(x)$.

Proof. (a)

$$\begin{aligned}
& g(x) = f(x+5) \\
\implies \quad & g(x) = \int_0^{x+5} g(t)dt \\
& g(x) = g(-x) = \int_0^{-x+5} g(t)dt \\
\implies \quad & g(x) = f(-x+5) = -f(x-5) \\
\implies & f(x-5) = -g(x)
\end{aligned}$$

(b) Note that since f, g are integrable on every interval, then we have that $g(x) = f(x+5) \implies g(y-5) = f(y)$ by simply taking $y = x+5$. Since the choice of variables is arbitrary, in general, we have that $g(x-5) = f(x)$.

$$\begin{aligned}
\int_0^5 f(t)dt &= \int_0^5 g(x-5)dx \\
&= \int_{-5}^0 g(t)dt \\
&= \int_0^5 g(-t)dt \\
&= \int_0^5 g(t)dt \\
&= f(5) = 7
\end{aligned}$$

(c) Similarly to above,

$$\begin{aligned}
\int_0^x f(t)dt &= \int_0^x g(t-5)dt \\
&= \int_{-5}^{x-5} g(t)dt \\
&= \int_{-x+5}^{5} g(-t)dt \\
&= \int_{-x+5}^{5} g(t)dt \\
&= \int_{-x+5}^{0} g(t)dt + \int_0^5 g(t)dt \\
&= \int_0^{x-5} g(t)dt + f(5) \\
&= f(x-5) + f(5) \\
&= -g(x) + g(0)
\end{aligned}$$

$\square$

Apostol p.138-139 no.5

Lemma.

$$\lim_{x \to 0} \frac{x}{x} = 1$$

Proof. For any $\epsilon > 0$, taking $\delta = 1224323121$, we have that $0 < |x - 0| < 1224323121 \implies x \neq 0 \implies |\frac{x}{x} - 1| = 0 < \epsilon$. Thus, the limit is then just 1. $\square$

$$
\begin{aligned}
\lim_{h\to 0}\frac{(t+h)^2-t^2}{h} &= \lim_{h\to 0}\frac{t^2+2th+h^2-t^2}{h} \\
&= \lim_{h\to 0}\frac{2th+h^2}{h} \\
&= \lim_{h\to 0}\frac{2th}{h}+\frac{h^2}{h} \\
&= \lim_{h\to 0}\frac{h}{h}(2t+h) \\
&= \lim_{h\to 0}\frac{h}{h}\cdot\lim_{h\to 0}2t+h && \text{(multiplicativity of limits)} \\
&= \lim_{h\to 0}2t+h && \text{(lemma)} \\
&= 2t && \text{(see below)}
\end{aligned}
$$

To evaluate this last limit, consider that for any $\epsilon > 0$, taking $\delta = \frac{\epsilon}{2} \implies \forall x, 0 < |x-0| < \frac{\epsilon}{2} \implies |2t + \frac{\epsilon}{2} - 2t| = \frac{\epsilon}{2} < \epsilon$.

Apostol p.138-139 no.8

$$
\begin{aligned}
\lim_{x\to a}\frac{x^2-a^2}{x^2+2ax+a^2} &= \lim_{x\to a}\frac{(x-a)(x+a)}{(x+a)^2} \\
&= \lim_{x\to a}\frac{x-a}{x+a} && \text{(since } x+a\neq 0 \text{ when } x=a\text{)} \\
&= \frac{\lim_{x\to a}x-a}{\lim_{x\to a}x+a} && \text{(multiplicativity of limits)} \\
&= \frac{0}{2a} = 0
\end{aligned}
$$

We can see that $\lim_{x\to a} x - a = 0$ as we have that for $\epsilon > 0$, $\delta = \epsilon \implies \forall x, 0 < |x-a| < \epsilon$. Similarly, we see that $\lim_{x\to a} x + a = 2a$ as we have that for $\epsilon > 0$, $\delta = \epsilon \implies \forall x, 0 < |x-a| < \epsilon \implies |x+a-2a| = |x-a| < \epsilon$.

Apostol p.138-139 no.21

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\sqrt{1-x^2}}{x^2} &= \lim_{x\to 0}\frac{1-\sqrt{1-x^2}}{x^2}\left(\frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right) \\
&= \lim_{x\to 0}\frac{1-(1-x^2)}{x^2(1+\sqrt{1-x^2})} \\
&= \lim_{x\to 0}\frac{x^2}{x^2}\lim_{x\to 0}\frac{1}{1+\sqrt{1-x^2}} && \text{(multiplicativity of limits)} \\
&= \lim_{x\to 0}\frac{1}{1+\sqrt{1-x^2}} && \text{(lemma)} \\
&= \frac{\lim_{x\to 0} 1}{\lim_{x\to 0} 1+\lim_{x\to 0}\sqrt{1-x^2}} && \text{(additivity, multiplicativity of limits)} \\
&= \frac{1}{1+\lim_{x\to 0}\sqrt{1-x^2}} \\
&= \frac{1}{2}
\end{aligned}$$

To compute the last limit, note that we have $(\lim_{x\to 0}\sqrt{1-x^2})^2 = \lim_{x\to 0} 1-x^2 = 1$, as for any $\epsilon > 0$, take $\delta = \sqrt{\epsilon} \implies 0 < |x| < \sqrt{\epsilon} \implies |1-x^2-1| = |x^2| = |x| < \epsilon$. Thus, $\lim_{x\to 0}\sqrt{1-x^2} = \sqrt{1} = 1$.

Apostol p.138-139 no.31

Consider

$$f(x) = \begin{cases} x & x \in \mathbb{R}\setminus\mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$$

Claim. This is continuous at 0, but nowhere else.

Proof. At 0, we observe that for any $\epsilon > 0$, the density of $\mathbb{R}\setminus\mathbb{Q}$ in $\mathbb{R}$ furnishes $\delta \in \mathbb{R}\setminus\mathbb{Q}$ such that $0 < \delta < \epsilon$. Then, $\forall x, 0 < |x| < \delta \implies |f(x)| < \delta$, as either $x \in \mathbb{R}\setminus\mathbb{Q} \implies |f(x)| = |x| < \delta$ or $x \in \mathbb{Q} \implies |f(x)| = 0 < \delta$. Thus, we have that $\lim_{x\to 0} f(x) = 0 = f(0)$.

However, for any $c \in \mathbb{R}, c \neq 0$, we have that $\lim_{x\to c} f(x)$ does not exist. Suppose that it did, and it had value K. Take $\epsilon = |\frac{c}{2}| > 0$. For any δ, pick $x_1 \in \mathbb{Q}, x_2 \in \mathbb{R}\setminus\mathbb{Q}, 0 < |x_1 - c| < \delta, 0 < |x_2 - c| < \delta$, and consider that the existence of the limit has $|f(x_0) - K| = |K| < |\frac{c}{2}|, |f(x_1) - K| = |x_1 - K| = |x_1 - K + c - c| \leq |x_1 - c| + |K - c| < |\frac{c}{2}|$. However, $|K - c| > |\frac{c}{2}|$ as $|K| < |\frac{c}{2}|$, and so we have that both $|K - c| < |\frac{c}{2}|$ and $|K - c| > |\frac{c}{2}|$. $\Rightarrow\Leftarrow$, so the limit does not exist and f cannot be continuous for $x \neq 0$. $\square$

Problem 1

Claim. f is integrable $\implies |f|$ is integrable.

Proof. Let f be integrable over $[a, b]$.
We have from the triangle inequality that $|a-b| \leq |a| + |b| \implies |a-b| - |b| \leq |a|$. Replacing a with $x - y$ and b with $-y$, we have that $|x| - |y| \leq |x - y|$.
Then, over any partition $P = \{x_0, ..., x_n\}$ of $[a, b]$, we have that on a open subinterval of the partition (x_i, x_{i+1}), the theorem on approximation gives $x_1, x_2 \in (x_i, x_{i=1}), \epsilon > 0 \mid \sup(|f|) - \inf(|f|) - 2\epsilon < |f(x_1)| - |f(x_2)| \leq |f(x_1 - f(x_2)| < |\sup(f) - \inf(f)| = \sup(f) - \inf(f)$.
Then, for that partition and putting $\inf_I, \sup_I$ for the infimum and supremum over I, we have that

$$\sum_{i=0}^{n} (\inf_{(x_i, x_{i+1})}(f))(x_{i+1} - x_i) \in \underline{I}(f)$$
$$\sum_{i=0}^{n} (\sup_{(x_i, x_{i+1})}(f))(x_{i+1} - x_i) \in \overline{I}(f)$$
$$\sum_{i=0}^{n} (\inf_{(x_i, x_{i+1})}(|f|))(x_{i+1} - x_i) \in \underline{I}(|f|)$$
$$\sum_{i=0}^{n} (\sup_{(x_i, x_{i+1})}(|f|))(x_{i+1} - x_i) \in \overline{I}(|f|)$$

By properties of sums proved on previous homework, we then have that

$$\begin{aligned}
\sum_{i=0}^{n} (\sup_{(x_i, x_{i+1})}(|f|))(x_{i+1} - x_i) - \sum_{i=0}^{n} (\inf_{(x_i, x_{i+1})}(|f|))(x_{i+1} - x_i) \\
&= \sum_{i=0}^{n} (\sup_{(x_i, x_{i+1})}(|f|) - \inf_{(x_i, x_{i+1})}(|f|))(x_{i+1} - x_i) \\
&\leq \sum_{i=0}^{n} (|\sup_{(x_i, x_{i+1})}(f) - \inf_{(x_i, x_{i+1})}(f)|)(x_{i+1} - x_i) \\
&= \sum_{i=0}^{n} (\sup_{(x_i, x_{i+1})}(f) - \inf_{(x_i, x_{i+1})}(f))(x_{i+1} - x_i) \\
&= \sum_{i=0}^{n} (\sup_{(x_i, x_{i+1})}(f))(x_{i+1} - x_i) - \sum_{i=0}^{n} (\inf_{(x_i, x_{i+1})}(f))(x_{i+1} - x_i)
\end{aligned}$$

Finally, since f is integrable, we have that we can pick a partition such that the $RHS < \epsilon$ for any $\epsilon > 0$. Thus, we can find $x \in \overline{S}(|f|), y \in \underline{S}(|f|)$ such that $0 \leq x - y < \epsilon$, and so $\overline{I}(|f|) - \underline{I}(|f|) = 0$ by a previous homework result. $\square$

Problem 2

a

Claim. $x^n, n \in \mathbb{Z}_{\geq 0}$ is monotonic on both $(-\infty, 0]$ and $[0, \infty)$.

Proof. We induct on n. The base case is $n = 0 \implies x^n = 1$, and so for any x, y in $(-\infty, 0]$ or x, y in $[0, \infty)$, we have that $x > y \implies x^0 = y^0 = 1$.

For the inductive case, suppose that the claim holds for $n = k$. Then, $x^{k+1} = x^k \cdot x$.

For any x, y in $(-\infty, 0]$, we have that if x^k is monotonically increasing, then $x > y \implies x^k \geq y^k \implies x^k \cdot x < y^k \cdot y$, as in general we have shown $a \geq b, 0 \geq c > d \implies ac < bd$ as a property of the ordering. Similarly, if x^k is monotonically decreasing, then $x > y \implies x^k \geq y^k \implies x^k \cdot x > y^k \cdot y$.

For any x, y in $[0, \infty)$, we have that if x^k is monotonically increasing, then $x > y \implies x^k \geq y^k \implies x^k \cdot x > y^k \cdot y$. Similarly, if x^k is monotonically decreasing, then $x > y \implies x^k \geq y^k \implies x^k \cdot x < y^k \cdot y$.

Thus, x^n is monotonic on both $(-\infty, 0]$ and $[0, \infty)$. $\square$

b

Claim. All monomials are integrable on any closed interval.

Proof. In the cases that $[a, b] \subseteq (-\infty, 0]$, or $[a, b] \subseteq [0, \infty)$, we have that the function is monotonic and bounded (in general, x^n bounded by $\max(a^n, b^n)$ over the interval $[a, b]$). This has been shown to be integrable in class.

In the last case that $[a, b] \nsubseteq (-\infty, 0]$ or $[0, \infty)$, and since $[a, b] \subseteq (-\infty, 0] \cup [0, \infty) = \mathbb{R}$, then $[a, b] = [a, 0] \cup [0, b]$. This can bee seen by the fact that there must be some element in $[a, b]$ that is greater than zero, and one that is less than zero. Further, since a, b are bounds on the interval, they must be less and and greater than zero each.

Then, $\int_a^0 x^n dx + \int_0^b x^n dx = \int_a^b x^n dx$, as the two parts on the LHS are integrable as they are bounded and monotonic, and the RHS is then also integrable. $\square$

c

Claim.

$$g(x) = \sum_{i=0}^{n} c_i x^i$$

is integrable.

Proof. Note that we have already proved that $f(x) = c$ is integrable, and that the sum and product of integrable functions is itself integrable in class.

We induct on n. The base case, $n = 0$, has $g(x) = c_0$, which is integrable. Then, if the claim for $n = k$ holds, then $g_{k+1}(x) = \sum_{i=0}^{k+1} c_i x^i = \sum_{i=0}^{k} c_i x^i + c_{k+1}x^{k+1}$. We have that $\sum_{i=0}^{k} c_i x^i$ is integrable by the inductive premise, and that c_{k+1} and x^{k+1} are both integrable as well. Thus, $\sum_{i=0}^{k} c_i x^i + c_{k+1}x^{k+1} = g_{k+1}(x)$ is then integrable.

Induction then yields that $g(x)$ is integrable for any n. $\square$

Problem 3

$$\begin{aligned} &f : (a, b) \to \mathbb{R}, x \in (a, b) \\ a)\ &\lim_{h\to 0} |f(x + h) - f(x)| = 0 \\ b)\ &\lim_{h\to 0} |f(x + h) - f(x - h)| = 0 \end{aligned}$$

Claim. $a) \implies b)$.

Proof. We have that for any $\epsilon > 0$, $\exists\delta \mid 0 < |h| < \delta \implies |f(x + h) - f(x)| < \epsilon$.

We want to show that also $\lim_{h\to 0} |f(x + h) - f(x)| = 0 \implies \lim_{h\to 0} |f(x - h) - f(x)| = 0$. Consider that for $\epsilon > 0$, we can take the same δ as before. For $0 < |h| < \delta$ note that for any given h, we have that $f(x - h) = f(x + (-h))$, but $|-h| < \delta \implies |f(x + (-h)) - f(x)| < \epsilon$.

Further, for any $\epsilon' = 2\epsilon > 0$, we have that $|f(x+h)-f(x-h)| = |f(x+h)-f(x-h)+f(x)-f(x)| \leq |f(x+h)-f(x)|+|f(x-h)-f(x)| < 2\epsilon = \epsilon'$. Thus, $\lim_{h\to 0} |f(x+h)-f(x-h)| = 0$. $\square$

We do not in fact have that $b) \implies a)$. Consider the following function $f : (-1, 1) \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then, take $x = 0$. $\lim_{h\to 0} |f(h) - f(-h)| = 0$, as we have that for $\forall\epsilon > 0, \delta = 1 \implies \forall h, 0 < |h| < 1 \implies |f(h) - f(-h) - 0| = |1 - 1 - 0| = 0 < \epsilon$.

However, we have that $\lim_{h\to 0}|f(h) - f(0)| = \lim_{h\to 0}|f(h)| = 1$, as $\forall\epsilon > 0, \delta = 1 \implies \forall h, 0 < |h| < 1 \implies |f(h) - 1| = 0 < \epsilon$. Thus, $\lim_{h\to 0}|f(0+h) - f(0)| \neq 0$.

Problem 4

$$f(x) = \begin{cases} 1 & x \in \mathbb{R}\setminus\mathbb{Q} \\ \frac{1}{n} & x = \frac{m}{n},\ m, n \in \mathbb{Z}_{>0}, (m,n) = 1 \end{cases}$$

a

Claim. f is continuous at x if and only if x is irrational.

Proof. ($\implies$) Suppose that $x = \frac{m}{n}, (m,n) = 1$ is not irrational. Then, since f is continuous at x, we must have that for any $\epsilon > 0$, $\exists\delta \mid 0 < |y-x| < \delta \implies |f(y)-f(x)| = |f(y)-\frac{1}{n}| < \epsilon$. However, since $\mathbb{R}\setminus\mathbb{Q}$ is dense in $\mathbb{R}$, we see that $\exists y \in \mathbb{R}\setminus\mathbb{Q} \mid 0 < |y-x| < \delta$, and we have that $|f(y) - \frac{1}{n}| = \frac{1}{n}$, and, for example, there is no δ that satisfies the condition with $\epsilon = \frac{1}{n+1}$. Thus, x must be irrational.

($\impliedby$) If x is irrational, then we have that $f(x) = 0$. For any $\epsilon > 0$, take $n \in \mathbb{Z}$, such that $\frac{1}{n} < \epsilon$, which is possible due to the Archimedian property of $\mathbb{R}$. For any q in $\mathbb{Z}$, $\exists p \in \mathbb{Z} \mid x \in (\frac{p}{q}, \frac{p+1}{q})$. Then, let $d_q = \min(|\frac{p}{q} - x|, |\frac{p+1}{q} - x|)$. Take $\delta = \min(\delta_1, \delta_2, ..., \delta_n)$, so that $\delta < \frac{1}{n}$. $\forall y, 0 < |y-x| < \delta$, we have that $|f(y) - f(x)| = |f(y)|$. If y is irrational, then $|f(y)| = 0 < \epsilon$ and we are done. Otherwise, the way that δ is constructed means that if $y \in \mathbb{Q}, 0 < |y-x| < \delta$, then $y = \frac{m'}{n'}, n' > n$, so $|f(y)| = \frac{1}{n'} < \frac{1}{n} < \epsilon$. To see why that is true, if $n' \leq n$, then we would have that $y - x > \delta_{n'}$, but $\delta \leq \delta_{n'}$. Thus, $\lim_{y\to x} f(y) = f(x)$ if x is irrational. □

b

Claim.

$$\int_0^1 f = 0$$

Proof. We begin by showing that $\underline{I}(f) = 0$. To calculate the lower integral, let s be a step function with $s \leq f$. By definition, $\exists$ partition $P = \{x_0, ...x_n\}$ with $s|_{(x_{i-1},x_i)}$ constant. However, $\exists y \in \mathbb{R}\setminus\mathbb{Q}$ with $x_{i-1} < y < x_i \implies f(y) = 0 \implies s(y) \leq 0 \implies s|_{(x_{i-1},x_i)} \leq 0 \implies s \leq 0$ except at points of P. Hence, $\int_0^1 s \leq 0 \implies \forall x \in \underline{I}(f), x \leq 0$. Further, since we can for any $\epsilon > 0$, we can show that the step function s with partition $P = \{0, 1\}$, with constant value $\frac{-\epsilon}{2}$ has $\int_0^1 s = \frac{-\epsilon}{2} \in \underline{S}(f)$, so that $\exists x \in \underline{S}(f) \mid 0 - \epsilon < x$, so $\underline{I}(f) = 0$.

Finding $\overline{I}(f)$ is harder. For $N \in \mathbb{Z}_{>0}$, take the partition $P_N = \{\frac{p}{q} \leq 1 \mid p, q \in \mathbb{Z}_{\geq 0}, (p,q) = 1, q < N\} = \{x_0 = 0, ...x_n = 1\}$. The step function s_N such that for $x_i = \frac{p}{q}, s|_{(x_i,x_{i+1})} =$

$\frac{1}{N}$, $s(x_i) = \frac{1}{q}$ satisfies that $s \leq f$ on $[0, 1]$. Further, since the actual values of the step function on the points of the partition don't matter in our formulation, we have that

$$\begin{aligned}
\int_0^1 s_N &= \sum_{i=0}^{n} \frac{1}{N}(x_{i+1} - x_i) \\
&= \frac{1}{N} \sum_{i=0}^{n} (x_{i+1} - x_i) \\
&= \frac{1}{N}(x_n - x_0) = \frac{1}{N}(1 - 0) = \frac{1}{N}
\end{aligned}$$

The telescoping was proved on an earlier homework.

Thus, for any $\epsilon > 0$, we can take $\frac{1}{N} < \epsilon$, and then $\int_0^1 s_N < \epsilon \implies \overline{I}(f) = 0$, so we conclude $\int_0^1 f = \overline{I}(f) = \underline{I}(f) = 0$. $\square$