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Assignment 5

STAT 4001
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4.9.2

Integration by parts with $u = 1 - F(x), dv = 1$ yields that

$$\int_0^\infty (1 - F(x))dx = (uv|_0^\infty) - \int_0^\infty -xf(x)dx = \int_0^\infty xf(x)dx = E(X)$$

4.9.22

They exhibit a negative linear relationship: that is, $X = Y - 3$ where X, Y are the lengths of the shorter and longer pieces. Thus, $\rho = -1$.

4.9.26

$$\begin{aligned}
\operatorname{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\
&= E(E(XY \mid Z)) - E(E(X)(E(Y)) \mid Z) \\
&= E(\operatorname{Cov}(X, Y \mid Z) + E(X \mid Z)E(Y \mid Z)) - E(E(X) \mid Z)E(E(Y) \mid Z) \\
&= E(\operatorname{Cov}(X, Y \mid Z)) + E(E(X \mid Z)E(Y \mid Z)) - E(E(X) \mid Z)E(E(Y) \mid Z) \\
&= E(\operatorname{Cov}(X, Y \mid Z)) + \operatorname{Cov}(E(X \mid Z), E(Y \mid Z))
\end{aligned}$$

5.2.5

Let X be the amount of heads.

$$P(X = 0) + P(X = 2) + P(X = 4) + P(X = 6) + P(X = 8) = 0.5$$

5.2.9

Baye's Theorem yields that

$$P(X_1 = 1 \mid \sum_{i=1}^n X_i = k) = \frac{P(\sum_{i=1}^n X_i = k \mid X_1 = 1)P(X_1 = 1)}{P(\sum_{i=1}^n X_i = k)} = \frac{P(\sum_{i=2}^n X_i = k - 1)P(X_1 = 1)}{P(\sum_{i=1}^n X_i = k)}$$

We have that $P(\sum_{i=1}^n X_i = k) = \binom{k}{n} p^k (1 - p)^{n-k}$; similarly, $P(\sum_{i=2}^n X_i = k - 1) = \binom{k-1}{n-1} p^{k-1} (1 - p)^{n-k}$, and finally $P(X_1 = 1) = p$.
The above expression then comes out to

$$\frac{k}{n}$$

5.2.11

We have that

$$\sum_{x=2}^{n} x(x-1)\binom{n}{x}p^x(1-p)^{n-x} = \sum_{x=2}^{n} x^2\binom{n}{x}p^x(1-p)^{n-x} - \sum_{x=2}^{n} x\binom{n}{x}p^x(1-p)^{n-x}$$

We can start the sum from $x = 0$, as for $x = 0, 1$ we have that the term is simply zero. Then, the expression works out to be, with X a binomial distribution,

$$E(X^2)-E(X) = \operatorname{Var}(X)+E(X)^2-E(X) = np(1-p)+(np)^2-np = (np)^2-np^2 = (n^2-n)p^2$$

5.3.1

$$\frac{\binom{10}{10}\binom{24}{1}}{\binom{34}{11}} = 8.39 \cdot 10^{-8}$$

5.4.1

We have that this is a Poisson distribution with mean $0.2 \cdot 0.1 \cdot 100 = 2$, such that

$$P(X \geq 2) = 1 - P(X = 1) - P(X = 0) = 1 - 2e^{-2} - e^{-2} = 0.594$$

5.4.6

We have that this is a Poisson distribution with mean $3\frac{6}{5} = 3.6$. The probability of no defects is $e^{-3.6} = 0.027$.

5.4.16

a

Let $B = (\{X = k\} \cap (\cup_{i=1}^{n} A_i))$. Then, we have that $(W_n = k) = (\{X = k\} \cap (\cup_{i=1}^{n} A_i)^C)$, and as the sum of disjoint unions, we have that

$$P(B) + P(W_n = k) = P(X = k)$$

b

We have that as the time intervals do not overlap, the A_i are independent. Then,

$$\begin{aligned} P((\cup_{i=1}^n A_i)^C) &= P(\cap_{i=1}^n A_i^C) \\ &= (1 - A_i)^n \\ &= (1 - o(\frac{t}{n}))^n \end{aligned}$$

where the last line follows from the second assumption.
Then, we have that

$$\lim_{n\to\infty} P(\cup_{i=1}^n A_i) = \lim_{n\to\infty} 1 - (1 - o(\frac{t}{n}))^n = 1 - 1^n = 0$$

c

$$\begin{aligned} P(W_n = k) &= \binom{n}{k}(\frac{\lambda t}{n} + o(\frac{t}{n}))^k(1 - (\frac{\lambda t}{n} + o(\frac{t}{n})))^{n-k} \\ &= (\frac{1}{k!})(\frac{n!}{(n-k)!n^k})((\frac{\lambda t}{n} + o(\frac{t}{n}))n)^k(1 - (\frac{\lambda t}{n} + o(\frac{t}{n})))^{n-k} \end{aligned}$$

Considering the factors of the product here, we have:

$$\lim_{n\to\infty} \frac{n!}{(n-k)!n^k} = 1 \tag{1}$$

$$\lim_{n\to\infty} (1 - (\frac{\lambda t}{n} + o(\frac{t}{n})))^{n-k} = e^{-\lambda t} \tag{2}$$

$$\tag{3}$$

The above are given in the book. Further,

$$\begin{aligned} \lim_{n\to\infty} ((\frac{\lambda t}{n} + o(\frac{t}{n}))n)^k &= \lim_{n\to\infty} n^k \sum_{i=0}^{k} \binom{k}{i}(\frac{\lambda t}{n})^i(o(\frac{t}{n}))^{k-i} \\ &= \sum_{i=0}^{k} \binom{k}{i}(\lambda t)^i(\frac{o(\frac{t}{n})}{\frac{1}{n}})^{k-i} \\ &= \binom{k}{k}(\lambda t)^k \end{aligned}$$

We finally arrive at

$$\lim_{n\to\infty} P(W_n = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

d

Since we have that
$$P(X = k) = P(W_n = k) + P(B)$$
we have that
$$P(X = k) = \lim_{n\to\infty} P(W_n = k) + \lim_{n\to\infty} P(B) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

5.5.5

We do this by checking that the moment generating functions are the same. For $\sum_{i=1}^k X_i$, we have

$$\begin{aligned}\psi(t) &= \prod_{i=1}^k \psi_i(t)\\ &= \prod_{i=1}^k \left(\frac{p}{1-(1-p)e^t}\right)^{r_i}\\ &= \left(\frac{p}{1-(1-p)e^t}\right)^{\sum_{i=1}^k r_i}\end{aligned}$$

which is the mgf of the desired negative binomial distribution.

5.6.1

From the table at the end of the book, we have the following:

0.5	0
0.75	0.675
0.25	-0.675 = 0.325
0.9	1.285
0.1	-1.285

5.6.11

We have that $\overline{X_n} - \mu$ is normal with mean 0 and variance $\frac{4}{n}$. Then, we have that with the standard normal distribution Z,

$$\overline{X_n} - \mu = \frac{2}{\sqrt{n}} Z \implies |\overline{X_n} - \mu| < 0.1 \iff |Z| < 0.05\sqrt{n}$$

Looking for the 95^{th} quantile, we have that $0.05\sqrt{n} = 1.645 \implies n = 1082.41$. Rounding up,

$$n = 1083$$

5.6.14

Put

$$X = \frac{1}{2}(X_A + X_A) - \frac{1}{3}(X_B + X_B + X_B)$$

X is normally distributed with mean $\mu = \mu_A - \mu_B = 25$ and variance $\sigma^2 = \frac{1}{2}\sigma_A^2 + \frac{1}{3}\sigma_B^2 = 100$. Then, we have that

$$X = 10Z + 25 > 0 \iff Z > -2.5$$

Then, $P(X > 0) = P(Z < 2.5) = 0.9938$.

5.6.16

Note that

$$\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dx = \frac{1}{2\pi} e^{-\frac{y^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

Thus, X and Y are both the standard normal distribution, and $X + Y = \sqrt{2}Z$. This then means that

$$P(-\sqrt{2} < X + Y < 2\sqrt{2}) = P(-1 < Z < 2) = 0.8186$$

5.6.5

This is the complement of non lasting 290 hours. Note that $X_i = 10Z + 300$, such that

$$P(X_i < 290) = P(Z < -1) = 0.1587$$

The desired probability is then

$$1 - 0.1587^3 = 0.996$$

5.6.9

We have the overall length is normally distributed with mean 56 and variance 0.09. Then, we see that

$$P(55.7 < X < 56.3) = P(-1 < Z < 1) = 0.6827$$

5.6.13

The distribution X of the difference between the hole diameters is normal with mean 0.02 and variance 0.0025, such that $X = 0.05Z + 0.02$. Thus, the desired probability is

$$P(0 < X < 0.05) = P(-0.4 < Z < 0.6) = 0.3812$$

5.8.5

$$\begin{aligned} E[X^r(1-X)^s] &= \int_0^1 x^r(1-x)^s \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx \\ &= \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+r)\Gamma(\beta+s)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+r+s)} \end{aligned}$$

5.10.1

The conditional distribution of the height of the wife is normal with mean $66.8 + 0.68 \cdot 2 \cdot \frac{72-70}{2} = 68.16$ and variance $(1-0.68)^2 2^2 = 2.15$. The 0.95 quantile is then

$$68.16 + 2.15\frac{1}{2}(1.65) = 70.6$$

5.10.11

We have from the book that $X_1 + X_2, X_1 - X_2$ are bivariate normally distributed. Thus, we need only to show that the covariance vanishes:

$$\begin{aligned} \mathrm{Cov}\,(X_1+X_2, X_1-X_2) &= E((X_1+X_2)(X_1-X_2)) - E(X_1+X_2)E(X_1-X_2) \\ &= E(X_1^2 - X_2^2) - (E(X_1)+E(X_2))(E(X_1)-E(X_2)) \\ &= E(X_1^2) - E(X_1)^2 - E(X_2^2) + E(X_2)^2 \\ &= \mathrm{Var}\,(X_1) - \mathrm{Var}\,(X_2) = 0 \end{aligned}$$

Thus, they are independent.

5.10.13

The bivariate normal pdf has the form, for a suitable constant C,

$$Ce^{-\frac{1}{2(1-\rho^2)}((\frac{x_1-\mu_1}{\sigma_1})^2 - 2\rho(\frac{x_1-\mu_1}{\sigma_1})(\frac{x_2-\mu_2}{\sigma_2}) + (\frac{x_2-\mu_2}{\sigma_2})^2)}$$

This then yeilds

$$\begin{aligned}
a &= \frac{1}{2\sigma_1(1-\rho^2)} \\
b &= \frac{1}{2\sigma_2(1-\rho^2)} \\
c &= -\frac{\rho}{\sigma_1\sigma_2(1-\rho^2)} \\
e &= -\frac{\mu_1}{\sigma_1^2(1-\rho^2)} + \frac{\mu_2\rho}{\sigma_1\sigma_2(1-\rho^2)} \\
g &= -\frac{\mu_2}{\sigma_2^2(1-\rho^2)} + \frac{\mu_1\rho}{\sigma_1\sigma_2(1-\rho^2)} \\
\rho &= -\frac{c}{2\sqrt{ab}} \\
\sigma_1^2 &= \frac{1}{2a - \frac{c^2}{2b}} \\
\sigma_2^2 &= \frac{1}{2b - \frac{c^2}{2a}} \\
\mu_1 &= \frac{cg - 2be}{4ab - c^2} \\
\mu_2 &= \frac{ce - 2ag}{4ab - c^2}
\end{aligned}$$

The only conditions placed on these coefficients are that $a, b > 0$ and that the correlation is $-1 < \rho < 1$. Luckily, we have exactly that: $ab > (\frac{c}{2})^2 \implies |\rho| < \frac{2c^2}{2c^2} = 1$.