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Homework 3

MATH 1208
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Problem 1

a

Claim.

$$W = \left\{ (x_1, ..., x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 0 \right\}$$

is a subspace of $\mathbb{R}^n$.

Proof. We need to show closure under scalar multiplication and vector addition.
Scalar multiplication:

$$c(x_1, ..., x_n) = (cx_1, ..., cx_n)$$
$$\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i = c(0) = 0$$

Vector addition:

$$(x_1, ..., x_n) + (y_1, ..., y_n) = (x_1 + y_1, ..., x_n + y_n)$$
$$\sum_{i=1}^{n} x_i + y_i = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i = 0 + 0 = 0$$

□

b

Claim.

$$\dim W = n - 1$$

Proof. Put $e = (0, 0, ..., 0, -1) \in \mathbb{R}^n$ (more specifically, put x_i for the i^{th} component of e. Then $x_i = -1 \iff i = n$ and $x_i = 0$ otherwise). W is spanned by $\{e_i + e \mid i \in [n-1]\}$, where $[n-1] = 1, 2, ..., n-1$.

To see this, we will first show that this is a linearly independent set. Put $s_i = e_i + e$, and let

$$\sum_{i=1}^{n-1} c_i s_i = (c_1, c_2, ..., c_{n-1}, -\sum_{i=1}^{n-1} c_i) = 0$$

In order for this to hold, $c_i = 0$, so the above set is linearly independent.

To see that this also spans W, let any element $w \in W$ have i^{th} component w_i. Then, $\sum_{i=1}^{n} w_i = 0 \implies w_n = -\sum_{i=1}^{n-1} w_i$. Since

$$\sum_{i=1}^{n-1} w_i s_i = (w_1, w_2, ..., w_{n-1}, -\sum_{i=1}^{n-1} w_i) = w$$

we have that the above is a basis for W, showing that $\dim W = n - 1$.

Alternatively, we have that taking $T : \mathbb{R}^n \to \mathbb{R}$ such that $T((x_1, x_2, ..., x_n)) = \sum_{i=1}^{n} x_i$ is a linear map with $\ker T = W$; rank-nullity has that $\dim(W) = n - \dim(\text{im}(T)) = n - 1$. $\square$

Problem 2

The matrix representative of Id_V must send each component to itself. Thus, each basis vector v_i has that $\text{Id}_V(v_i) = v_i$, so that $m(\text{Id}_V) \in M_{n\times n}(F)$ has that $a_{ij} = 1 \iff i = j$ and $a_{ij} = 0$ otherwise.

$$m(\text{Id}_V) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

However, if we were to choose a different basis (say $\{-e_i \mid i \in [n]\}$ in the case of $\mathbb{R}^n$) for the codomain, then this no longer holds. In the previously mentioned case of choosing $V = \mathbb{R}^n$, and the basis of the domain to be the standard basis $\{e_i\}$ and the basis of the codomain to be $\{-e_i\}$, we have that $\text{Id}_V(e_i) = e_i = -(-e_i)$, so that now

$$m(\text{Id}_V) = \begin{bmatrix} -1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{bmatrix}$$

where $m(\text{Id}_V) \in M_{n\times n}(F)$ has that $a_{ij} = 1 \iff i = j$ and $a_{ij} = 0$ otherwise.

Problem 3

Put $A \in M_{m\times n}, B \in M_{n\times p}$.

a

Claim. If some row of A is zero then some row of AB will also be zero.

Proof. Some row of some matrix $M \in M_{m\times n}$ being zero is equivalent to the statement that for some i and $j \in [n]$, $M_{ij} = 0$. Suppose that the i^{th} row of A is zero. Since we have that

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

Since all $A_{ik} = 0$ by assumption,

$$= \sum_{k=1}^{n} 0 = 0$$

which implies that the i^{th} row of AB must also be zero. □

b

Consider the counterexample

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

c

Claim. If some column of B is zero then some column of AB will also be zero.

Proof. Some column of some matrix $M \in M_{n\times p}$ being zero is equivalent to the statement that for $i \in [n]$ and some and j, $M_{ij} = 0$. Suppose that the j^{th} column of B is zero. Since we have that

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

Since all $B_{kj} = 0$ by assumption,

$$= \sum_{k=1}^{n} 0 = 0$$

which implies that the j^{th} column of AB must also be zero. □

d

Claim. If two columns of B are identical, then two columns of AB will also be identical.

Suppose that the x^{th} and y^{th} columns of B are identical (that is, $B_{ix} = B_{iy}$ for $i \in [n]$).

$$\begin{aligned}(AB)_{ix} &= \sum_{k=1}^{n} A_{ik}B_{kx} \\ &= \sum_{k=1}^{n} A_{ik}B_{ky} \\ &= (AB)_{iy}\end{aligned}$$

Thus, the x^{th} and y^{th} columns of AB are also identical.

Problem 4

a

Claim. For P_n, the set of all polynomials $\mathbb{R} \to \mathbb{R}$ of degree $\leq n$, the map $G : P_n \to \mathbb{R}^k$ where $G(f) = (f(1), f(2), ..., f(k))$ is linear, and is also surjective when $k \leq n+1$.

Proof. Put the i^{th} component of any vector $v \in \mathbb{R}^k$ as v_i.

$$\begin{aligned} G(f+g)_i &= f(i) + g(i) \\ &= G(f)_i + G(g)_i \\ \implies G(f+g) &= G(f) + G(g) \\ G(cf)_i &= (cf)(i) \\ &= cf(i) \\ &= cG(f)_i \\ \implies \quad G(cf) &= cG(f) \end{aligned}$$

The above shows that G is indeed linear.

To show surjectivity, consider the set of polynomials

$$p_i(x) = \prod_{j=1, j\neq i}^{k} \frac{x-j}{i-j}$$

Since we have that $k \leq n+1$, we have that $p_i(x) \in P_n$.

The above has that $p_i(i) = \prod_{j=1,j\neq i}^{k} \frac{i-j}{i-j} = 1$. Further, for $l \in \{1, 2, ..., \hat{i}, ..., k\}$, we have that $p_i(l) = \prod_{j=1,j\neq i}^{k} \frac{l-j}{i-j} = 0$.

Now for any element $y = (y_1, y_2, ..., y_k) \in \mathbb{R}^k$, we have that

$$f(x) = \sum_{i=1}^{k} y_i p_i(x) \implies G(f) = y$$

as for $l \in [k]$, $f(l) = \sum_{i=1}^{k} y_i p_i(l) = y_l p_l(l) = y_l$. $\square$

b

This follows directly from rank-nullity. The desired quanitity is the dimension of the subspace that is killed by G, which is exactly $\dim(\ker(G)) = \dim(P_n) - \dim(\operatorname{im}(G)) = n + 1 - k$.

Problem 5

a

Claim. There is a linear map $T : V \to F$ such that $T(v) = 1$ for $v \neq 0$.

Proof. Suppose that for some basis $v_1, ..., v_n$ of V, $v = \sum_{i=1}^{n} c_i v_i$. Then, let c_j be the first nonzero coefficient, and consider the transformation

$$T(\sum_{i=1}^{n} a_i v_i) = c_j^{-1} a_j$$

This sends $v \mapsto c_j^{-1} c_j = 1$, and is linear:

$$\begin{aligned}
T(\sum_{i=1}^{n} a_i v_i + \sum_{i=1}^{n} b_i v_i) &= T(\sum_{i=1}^{n} (a_i + b_i) v_i) \\
&= a_j + b_j \\
&= T(\sum_{i=1}^{n} a_i v_i) + T(\sum_{i=1}^{n} b_i v_i) \\
T(c\sum_{i=1}^{n} a_i v_i) &= T(\sum_{i=1}^{n} c a_i v_i) \\
&= c a_j \\
&= cT(\sum_{i=1}^{n} a_i v_i)
\end{aligned}$$

$\square$

b

Claim. There is a linear map $T : V \to F$ such that $\ker(T) = W$ for W some subspace of dimension $n-1$.

Proof. Let W have basis $w_1, w_2, ..., w_{n-1}$. Extend this basis by one more vector to get a basis $v_1 = w_1, v_2 = w_2, ..., v_{n-1} = w_{n-1}, v_n$ for V. Then, consider the transformation from above that sends $v_n \mapsto 1$, i.e.

$$T(\sum_{i=1}^{n} a_i v_i) = a_n$$

This kills any vector in W, as

$$T(\sum_{i=1}^{n-1} a_i w_i) = T(\sum_{i=1}^{n-1} a_i v_i + 0v_n) = 0$$

but is still linear as proved above. $\square$

c

Claim. There is a linear map $T : V \to F$ such that $\ker(T) = W$ for W some subspace of V.

Proof. The approach is the same as above: let W have basis $w_1, w_2, ..., w_{n-k}$ and V basis $v_1 = w_1, v_2 = w_2, ..., v_{n-k} = w_{n-k}, v_{n-k+1}, ..., v_n$. Then, consider

$$T(\sum_{i=1}^{n} a_i v_i) = (a_{n-k+1}, a_{n-k+2}, ..., a_n)$$

or more specifically, $T(\sum_{i=1}^{n} a_i v_i) = f$, where the i^{th} component of f is a_{n-k+f}. Any vector in W is killed:

$$T(\sum_{i=1}^{n-k} a_i w_i) = T(\sum_{i=1}^{n-k} a_i v_i + \sum_{i=n-k+1}^{n} 0v_i) = (0, 0, ..., 0)$$

This is also linear:

$$
\begin{aligned}
T(\sum_{i=1}^{n} a_i v_i + \sum_{i=1}^{n} b_i v_i)_j &= a_j + b_j \\
&= T(\sum_{i=1}^{n} a_i v_i)_j + T(\sum_{i=1}^{n} b_i v_i)_j \\
T(c\sum_{i=1}^{n} a_i v_i)_j &= T(\sum_{i=1}^{n} c a_i v_i)_j \\
&= c a_j \\
&= cT(\sum_{i=1}^{n} a_i v_i)_j
\end{aligned}
$$

We now have a linear map $T : V \to F^k$ such that $\ker(T) = W$. □

Problem 7

a

Claim. $\exists T : U \to V$ and T surjective and linear $\implies \dim(U) = m \geq \dim(V) = n$.

Proof. We have by surjectivity that $\mathrm{im}(T) = V$. Rank nullity has that $\dim(U) = \dim(\ker(T)) + \dim(\mathrm{im}(T)) \implies m = \dim(\ker(T)) + n$, and since dimension is nonnegative, $m \geq n$. □

b

Claim. $\exists T : U \to V$ and T injective and linear $\implies \dim(U) = m \leq \dim(V) = n$.

Proof. We have by injectivity that $\ker(T) = 0$. Rank nullity has that $\dim(U) = \dim(\ker(T)) + \dim(\mathrm{im}(T)) \implies m = \dim(\mathrm{im}(T))$. However, we have that $\mathrm{im}(T)$ is a subspace of $V \implies \dim(\mathrm{im}(T)) \leq \dim(V)$, so we have that $m = \dim(\mathrm{im}(T)) \leq n$. □

Problem 8

Claim. Suppose $A \in M_{n\times m}, B \in M_{m\times n}$ are matrices such that $AB = I_n$. Then ,$m \geq n$.

Proof. First we will show that proving the corresponding claim for linear maps: suppose that for $T_A : F^m \to F^n, T_B : F^n \to F^m$, we have $T_A \circ T_B = \mathrm{Id}_{F^m}$.

Then, we have that since T is linear, we can pick the standard basis for □

Problem 9

Claim. Let $A, B \in M_{m\times n}, C \in M_{n\times p}$. Then,

$$(A+B)C = AC + BC$$

Proof.

$$\begin{aligned}
((A+B)C)_{ij} &= \sum_{k=1}^{n}(A+B)_{ik}C_{kj} \\
&= \sum_{k=1}^{n}(A_{ik}+B_{ik})C_{kj} \\
&= \sum_{k=1}^{n} A_{ik}C_{kj} + B_{ik}C_{kj} \\
&= \sum_{k=1}^{n} A_{ik}C_{kj} + \sum_{k=1}^{n} B_{ik}C_{kj} \\
&= (AC)_{ij} + (BC)_{ij} \\
&= (AC+BC)_{ij}
\end{aligned}$$

□