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Homework 10

Apostol p.328 no.7

Put $\gamma(t) = (t, 2t, 4t)$, for $0 \leq t \leq 1$. Then,

$$\begin{aligned}\int f \cdot d\gamma &= \int_0^1 (t, 2t, 4t^2 - 2t) \cdot (1, 2, 4) dt \\ &= \int_0^1 (16t^2 - 3t) dt \\ &= \frac{16}{3} - \frac{3}{2} = \frac{23}{6}\end{aligned}$$

Apostol p.337 no.5a

Claim. $f(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z)) = (y, x, x)$ is not conservative.

Proof. Using the result of the previous problem, if f were conservative, then

$$\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$$

However, we easily see that the left hand side is 0, and the right hand side is 1. Thus, f is not conservative.

Further, consider the closed path $\gamma(t) = (\cos(t), 0, \sin(t))$, for $0 \leq t \leq 2\pi$.

$$\begin{aligned}\oint f \cdot \gamma &= \int_0^{2\pi} (0, \cos(t), \cos(t)) \cdot (-\sin(t), 0, \cos(t)) dt \\ &= \int_0^{2\pi} \cos^2(t) dt > 0\end{aligned}$$

□

Apostol p.345 no.3

$$\begin{aligned}\varphi(x, y) &= \int (2xe^y + y) dx + A(y) = x^2 e^y + xy + A(y) \\ \varphi(x, y) &= \int (x^2 e^y + x - 2y) dy + B(x) = x^2 e^y + xy - y^2 + A(y)\end{aligned}$$

Then, we can take $A(y) = -y^2$, $B(x) = 0$, and arrive at $\varphi = x^2 e^y + xy - y^2 + C$, where $C \in \mathbb{R}$ is a constant.

Apostol p.345 no.9

This is not a gradient; from the earlier results of an Apostol problem,

$$D_1 f_2 = 12x^2z^2 \neq D_2 f_1 = 12y^3z^2$$

shows that this is not a conservative vector field.

Apostol p.345 no.14

Claim. If φ, ψ are potential functions for a continuous vector field f on an open connected set $S \subset \mathbb{R}^n$, then $\varphi - \psi$ is constant on S.

Proof. Note that $\nabla(\varphi - \psi) = \nabla\varphi - \nabla\psi = f - f = 0$.
Then, this implies that all directional derivatives $(\varphi - \psi)'(x; y)$ are zero (they also exist, as they are assumed differentiable), and by an earlier homework problem $\varphi - \psi$ must be constant on S. □

Apostol p.346 no.18

a

Claim. Put $T = \mathbb{R}^2 \setminus \{(x, y) \mid y = 0, x \leq 0\}$, and

$$x = r\cos(\theta), y = r\sin(\theta)$$

where $-\pi < \theta < \pi$, and $r > 0$. Then,

$$\theta = \begin{cases} \arctan(\frac{y}{x}) & x > 0 \\ \frac{\pi}{2} & x = 0 \\ \arctan(\frac{y}{x}) + \pi & x < 0 \end{cases}$$

Proof. If $x > 0$, $\frac{y}{x} = \tan(\theta) \implies \theta = \arctan(\frac{y}{x})$, and we know that $\cos(\theta) > 0 \implies -\frac{\pi}{2} < \theta < \frac{\pi}{2}$, which is what we want.
If $x = 0$, then $\cos(\theta) = 0$, so $\theta = \frac{\pi}{2}$.
If $x < 0$, we must have that $\cos(\theta) < 0 \implies \frac{\pi}{2} < \theta < \pi$ or $-\pi < \theta < -\frac{\pi}{2}$. Apostol seems here to allow, in the third quadrant, that $\pi < \theta < \frac{3\pi}{2}$, but these are the same angles as $-\pi < \theta < -\frac{\pi}{2}$, as shifting by 2π doesn't change the angle. □

b

Claim. θ is a potential function for f on T.

Proof. From a formula for the derivative of arctan in Apostol, we compute the following:

$$\frac{\partial \theta}{\partial x} = -y\frac{1}{x^2}\frac{1}{1+(\frac{y}{x})^2} = -\frac{y}{x^2+y^2}$$
$$\frac{\partial \theta}{\partial y} = \frac{1}{x}\frac{1}{1+(\frac{y}{x})^2} = \frac{x}{x^2+y^2}$$

Note that even with the piecewise definition of θ, we know that $\lim_{x\to 0}\theta = \frac{\pi}{2}$, and thus θ is continuous; however, for both $x < 0, x > 0$, we have that $D_1(\theta) = \frac{\partial}{\partial x}\arctan(\frac{y}{x})$ and $D_2(\theta) = \frac{\partial}{\partial y}\arctan(\frac{y}{x})$, as desired.
This gives θ as a potential for f. $\square$

Problem 1

Claim. Suppose that $F : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$ is the vector field

$$F(x,y) = \left(\frac{x+y}{x^2+y^2}, \frac{y-x}{x^2+y^2}\right)$$

Then, $D_1F_2(x,y) = D_2F_1(x,y)$ on the entire domain, but F is not conservative.

Proof.

$$D_1F_2(x,y) = \frac{-(x^2+y^2) - 2x(y-x)}{(x^2+y^2)^2} = \frac{x^2-2xy-y^2}{(x^2+y^2)^2}$$
$$D_2F_1(x,y) = \frac{(x^2+y^2) - 2y(x+y)}{(x^2+y^2)^2} = \frac{x^2-2xy-y^2}{(x^2+y^2)^2}$$

Thus, we have that $D_1F_2(x,y) = D_2F_1(x,y)$. To show that F is not conservative, consider the integral counterclockwise on the unit circle, where $\gamma(t) = (\cos(t), \sin(t))$. Then, $\gamma'(t) =$

$(-\sin(t), \cos(t))$.

$$\begin{aligned}\oint F \cdot d\gamma &= \int_0^{2\pi} (\cos(t)+\sin(t), \sin(t)-\cos(t)) \cdot (-\sin(t), \cos(t))dt \\ &= \int_0^{2\pi} (-\sin(t)\cos(t) - \sin^2(t) + \sin(t)\cos(t) - \cos^2(t))dt \\ &= \int_0^{2\pi} -1dt \\ &= -2\pi \neq 0\end{aligned}$$

Thus, we have by the theorem proved in class, that F is not conservative. □

Problem 2

Claim. Suppose that $F : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ is a vector field that can be expressed as $F(x) = f(||x||)\frac{x}{||x||}$, where $f : (0, \infty) \to \mathbb{R}$ is a continuously differentiable function. F is conservative.

Proof. First we will show that for some $U \subset \mathbb{R}^n$, if $g : U \to \mathbb{R}, h : \mathbb{R} \to \mathbb{R}$,

$$\nabla(h \circ g) = (h' \circ g)\nabla g$$

Consider the i^{th} component; the chain rule yields:

$$D_i(h \circ g) = (D_i g)(h' \circ g)$$

Then,

$$\nabla(h\circ g) = ((D_1 g)(h'\circ g), (D_2 g)(h'\circ g), \ldots, (D_n g)(h'\circ g)) = (h'\circ g)(D_1 g, \ldots, D_n g) = (h'\circ g)\nabla g$$

Next, we compute the j^{th} component of $\nabla r = ||x|| = (\sum_{i=1}^n x_i^2)^{1/2}$ to be

$$x_j \left(\sum_{i=1}^n x_i^2\right)^{-1/2}$$

such that

$$\nabla r = \frac{x}{||x||}$$

Now consider $\varphi = \int_0^x f(t)dt$ and $r = ||x||$, such that

$$\nabla(\varphi \circ r) = (\varphi' \circ r)\nabla r = (f \circ r)\nabla r = f(||x||)\frac{x}{||x||}$$

We now have a potential $\varphi \circ r$, and so F is conservative. □

Problem 3

Let $F : (0, \infty) \times (0, \infty) \to \mathbb{R}^2$ be defined by

$$F(x, y) = \left(\frac{y+1}{x^2 y}, \frac{x+1}{xy^2} \right)$$

Claim. F is conservative, and the potential φ satisfies

$$\varphi(x, y) = -\frac{x+y+1}{xy} + C$$

where C is some constant.

Proof. We have that $(0, \infty) \times (0, \infty)$ is an open set (in particular for any point $a = (x, y)$, take $\epsilon = \min\{x, y\}$, and $B_\epsilon(a) \subset (0, \infty) \times (0, \infty)$), as well as star-shaped, as any point can be the center point, as for two points $(x_1, y_1), (x_2, y_2)$, any point on the connecting line $(x_1, y_1) + t((x_2 - x_1, y_2 - x_1)) = (x_2 + (1-t)x_1, y_2 + (1-t)y_1)$ for $0 < t < 1$ has that both coordinates are > 0, and thus in the domain of F.

Computing,

$$D_1 F_2(x, y) = \frac{xy^2 - y^2(x+1)}{x^2 y^4} = \frac{-y^2}{x^2 y^4} = -\frac{1}{x^2 y^2}$$
$$D_2 F_1(x, y) = \frac{x^2 y - x^2(y+1)}{x^4 y^2} = \frac{-x^2}{x^4 y^2} = -\frac{1}{x^2 y^2}$$

and we have from class that F is closed on a star-shaped and open domain $\implies$ F is conservative.

The potential φ can be found with indefinite integration as specified in Apostol:

$$\varphi(x, y) = \int \frac{y+1}{x^2 y} dx + A(y) = -\frac{y+1}{xy} + A(y)$$
$$\varphi(x, y) = \int \frac{x+1}{xy^2} dy + A(y) = -\frac{x+1}{xy} + B(x)$$

Setting the two equal, we have $A(y) - B(x) = -\frac{x-y}{xy} = \frac{1}{x} - \frac{1}{y}$, yielding $A(y) = -\frac{1}{y}, B(x) = -\frac{1}{x} \implies \varphi(x, y) = -\frac{x+y+1}{xy}$. We can arbitraily add some constant $C \in \mathbb{R}$ that vanishes upon differentiation. $\square$