9

We want to show that given $f : X \to Y$, f is uniformly continuous if and only if it satisfies the property given in the question.

($\implies$) Uniform continuity gives that for any $\epsilon > 0$, $\exists \delta > 0$ such that $d_X(p,q) < \delta \implies d_Y(f(p), f(q)) < \epsilon$ for any $p, q \in X$. Then, if we have any set E such that $\operatorname{diam} E < \delta/2$, then for any two points p, q in E, we have that $d_X(p,q) \leq \delta/2 < \delta$, so for any two points $f(p), f(q) \in f(E)$, we have that $d_Y(f(p), f(q)) < \epsilon$, so $\operatorname{diam} f(E) < \epsilon$, which was what we wanted.

($\impliedby$) Fix $\epsilon > 0$, and take any $p, q \in X$ such that $d_X(p,q) < \delta/2$ where δ is the associated δ to the fixed ϵ given by the property from the question. Then, $q \in B^\circ_{\delta/2}(p)$, and consider that $\operatorname{diam} B^\circ_{\delta/2}(p) \leq \delta/2 < \delta$, so if we have the property from the question, then $\operatorname{diam} f(B^\circ_{\delta/2}(p)) < \epsilon$. Since we have that $q \in B^\circ_{\delta/2}(p) \implies f(q) \in f(B^\circ_{\delta/2}(p)) \implies d_Y(f(q), f(p)) < \epsilon$, we are done.

10

Theorem 4.19 is that if $f : X \to Y$, X compact, f continuous, then f is uniformly continuous. The alternative proof in the problem is as follows:

Suppose that f is not uniformly continuous. Then, for some $\epsilon > 0$ and every $\delta > 0$, there are $p_\delta, q_\delta \in X$ such that $d_X(p_\delta, q_\delta) < \delta$ and $d_Y(f(p_\delta), f(q_\delta)) > \epsilon$. For positive integers n, set p_n, q_n to the associated $p_{1/n}, q_{1/n}$ from before, such that $d_Y(f(p_n), f(q_n)) > \epsilon$ and $d_X(p_n, q_n) < 1/n$. Then, we have that since X is compact, that there are convergent subsequences of p_n, q_n, say p_{n_i}, q_{n_i}, such that $p_{n_i} \to p, q_{n_i} \to q$ as $i \to \infty$, $p, q \in X$.

However, since we have that $d(q, p_{n_i}) \leq d(q, q_{n_i}) + d(q_{n_i}, p_{n_i}) < d(q, q_{n_i}) + 1/n_i$, but both terms on the right $\to 0$ as $i \to \infty$, since $q_{n_i} \to q$ by definition and $n_i \to \infty$, so we get that $d(q, p_{n_i}) \to 0$, so $p = q$. Then, since f is continuous, we have that as $i \to \infty$, $f(p_{n_i}) \to f(p) = f(q)$ and $f(q_{n_i}) \to f(q) = f(p)$, and so both $d(f(p_{n_i}), p)$ and $d(p, f(q_{n_i})) \to 0$ as $i \to \infty$. However, we have that $d(f(p_{n_i}), f(q_{n_i})) = d(f(p_{n_i}), p) + d(p, f(q_{n_i})) > \epsilon$ by construction, so as $i \to \infty$, we get that $0 > \epsilon$, so $\Rightarrow\Leftarrow$. Thus, f must be uniformly continuous.

2

To see that it is strictly increasing, pick any c, d such that $a < c < d < b$. Then, we have that f is continuous on $[c, d]$ since it is differentiable on $(a, b) \supset [c, d]$. Next, the mean value theorem gives that there is some point p on (c, d) such that $f'(p) = \frac{f(d) - f(c)}{d - c}$, and since we have that $d > c \implies d - c > 0$, and the derivative is positive by assumption, then $f(d) - f(c) > 0$ as well, so f must strictly increase.

Then,

$$g'(f(x)) = \lim_{f(t)\to f(x)} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} = \lim_{f(t)\to f(x)} \frac{t-x}{f(t) - f(x)}$$

but since f is continuous,

$$\lim_{f(t)\to f(x)} \frac{t-x}{f(t)-f(x)} = \lim_{t\to x} \frac{1}{\frac{f(t)-f(x)}{t-x}} = \frac{1}{f'(x)}$$

where this last equality comes from the fact that for any $\delta' > 0$, there is some $\delta > 0$ such that if $|t-x| < \delta$, then $|f(t) - f(x)| < \delta'$, so for any $\epsilon > 0$, if we need that $|f(t)-f(x)| < \delta'$ such that $\left|\frac{t-x}{f(t)-f(x)}\right| < \epsilon$, we can also just require $|t-x| < \delta$, and so the limits are equal.

6

We have that g is differentiable for $x > 0$, and by product rule we get

$$g'(x) = \frac{f'(x)}{x} - \frac{f(x)}{x^2} = \frac{xf'(x) - f(x)}{x^2}$$

but by the mean value theorem we get that there is some point $p \in (0, x)$ such that $f(x) - f(0) = (x-0)f'(p) \implies f(x) = xf'(p)$, so

$$g'(x) = \frac{x(f'(x) - f'(p))}{x^2}$$

but since f' is increasing, we have that $f(x) > f(p)$, so $g'(x) > 0$ for $x > 0$ so g is strictly increasing by the last problem.

8

f' is continuous on a compact set (namely $[a,b]$) so f' is uniformly continuous. This gives that for every $\epsilon > 0$, there is some $\delta > 0$ such that if $|y-x| < \delta$, then $|f'(y) - f'(x)| < \epsilon$. Then, the mean value theorem gives for any $t, x \in [a,b]$ some p between x and t such that $f'(p) = \frac{f(t)-f(x)}{t-x}$, but since p is between x and t, $|p-x| < |t-x|$, so if $|t-x| < \delta$, then $|p-x| < \delta$ and $|f'(p) - f'(x)| = \left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| < \epsilon$.

For a vector-valued function, note that it has continuous derivative if and only if each component of the derivative is continuous, in which case each component is uniformly continuous; apply the above, and each component f_i satisfies that it is uniformly differentiable. Then, for $|t-x| < \delta$,

$$\left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| = \left(\sum_{i=1}^{n} \left|\frac{f(t)-f(x)}{t-x} - f'(x)\right|^2\right)^{1/2} < (n\epsilon^2)^{1/2} < \epsilon\sqrt{n}$$

so we have that it holds for vector-valued functions as well.