MATH 4041 HW 13

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December 13, 2020

Problem 1

First, $(1 \cdot f)(x) = f(1 \cdot x) = f(x)$. Secondly,

$$(h \cdot (g \cdot f))(x) = (g \cdot f)(h \cdot x) = f(g \cdot h \cdot x) = f((gh) \cdot x) = (gh \cdot f)(x)$$

so it does not define a group action, but if we take

$$(g \cdot f)(x) = f(g^{-1} \cdot x)$$

instead, we get

$$(h \cdot (g \cdot f))(x) = (g \cdot f)(h^{-1} \cdot x) = f(g^{-1} \cdot h^{-1} \cdot x) = f((hg)^{-1} \cdot x) = (hg \cdot f)(x)$$

which gives us a group action.

Problem 2

Since we have that $20 = 2^2 \cdot 5$, the order of a 2-Sylow subgroup is 4, and a 5-Sylow subgroup has order 5.

The divisors of 20 are 1,2,4,5,10,20.

The possible amount of 2-Sylow subgroups is 1 or 5. The possible amount of 5-Sylow subgroups is 1.

Problem 3

Since we have that $36 = 2^2 \cdot 3^2$, the order of a 2-Sylow subgroup is 4, and a 3-Sylow subgroup has order 9.

The divisors of 36 are 1,2,3,4,6,9,12,18,36.

The possible amount of 2-Sylow subgroups is 1, 3, 9. The possible amount of 3-Sylow subgroups is 1 or 4.

Problem 4

i

Taking the hint in the problem set, a 2-Sylow subgroup H of G has order 2, and thus if $H = \{1, h\}$, since it is unique, it is normal, so $g^{-1}hg = 1$ or $g^{-1}hg = h$ for $g \in G$; then, if $g^{-1}hg = 1$, then $hg = g \implies h = 1$, but by assumption $h \neq 1$, so we instead have that $g^{-1}hgh = h \implies h \in Z(G)$, so $H \leq Z(G)$.

Then, by an earlier homework, we get that G/H is cyclic since it is of prime order (I can't actually remember if this was on an earlier homework now that I think about it, so suppose $k \neq 1$ is an element of some group K with prime order, and thus $\langle k \rangle \leq K$ but by Lagrange, $|\langle k \rangle| = |K|$ since K has prime order, and thus $\langle k \rangle = K$ since K is finite), and from another (different) earlier homework, we have that this implies G to be abelian.

Now, we have a subgroup of order 2 H and a subgroup of order 3 H', both necessarily cyclic (since they are of prime order). Let g_1 be a generator of H, and g_2 a generator of H'. Then, we have that since G is abelian, and $2, 3$ are coprime, that the order of $g_1 g_2$ is 6, so G is generated by $g_1 g_2$.

ii

Taking the group action $g \cdot H_i = g^{-1} H_i g$ for $g \in G, H_i \in X$, we have that G is transitive on X by the Sylow theorem, since each 2-Sylow subgroup is conjugate to each other. Then, we get that for any $H_i \in X$, $X \cong G/G_{H_i}$, so $|G|/|G_{H_i}| = 3$, so the stabilizers are of order 2. But then, we have that clearly for any $h \in H_i$, $h \cdot H_i = H_i$, so the stabilizer of H_i is exactly H_i itself. Then, if we define $f : G \to S_X$ to be the homomorphism induced by the group action, we have that $g \in \ker(f)$ satisfies that $g \cdot H_i = H_i$ for any $H_i \in X$; from earlier, this means that $g \in H_1 \cap H_2 \cap H_3$, so $g = 1$ and the kernel is therefore trivial, so f is an isomorphism giving $G \cong S_X$, and taking $H_i \mapsto i$ gives $S_X \cong S_3$.

Problem 5

i

Since we have that $|G : H| = 2$, H must be normal, and by Sylow has a subgroup of order 3 as well. Further, since the divisors of 6 are 1,2,3,6, there can only be one subgroup of order 3 of H, so this subgroup must be normal in H. Call this subgroup K. Now, we have that since H is normal, $g^{-1}Kg$ is contained in H, but in particular, this is still a subgroup: $g^{-1}1g = 1$, $g^{-1}h_1 g g^{-1} h_2 h = g^{-1}h_1 h_2 g$, and $g^{-1}hgg^{-1}h^{-1}g = 1$, so we get the indentity, closure, and inverses, and the mapping $h \mapsto g^{-1}hg$ is injective. However, since K is the unique subgroup of order 3, $g^{-1}Kg = K$.

ii

Suppose that A_6 had such a subgroup. Then, by the last part, A_6 has a normal subgroup of order 3, call it H; since it is of order 3, it is necessarily cyclic. Then, consider the possible even permutations of $\{1,2,3,4\}$. In particular, any such (non-identity) permutation must either move 3 or 4 elements, since if it moved only two, it would be a transposition. If it moves 4 elemements, then writing it as a product of disjoint cycles, it must either be of the form (a,b,c,d) or $(a,b)(c,d)$ (to see this, note that the size of the support is the sum of the length of the cycles *except* the 1-cycles, and we can only write $4 = 1+3 = 2+2$, so either it is the composition of 2 2-cycles or just one 1 4-cycle), but $(a,b,c,d) = (a,b)(b,c)(c,d)$, so it is not even and thus must be of the form $(a,b)(c,d)$. However, every non-identity element in H must be a generator since it must be of order 3, but $(a,b)(c,d)(a,b)(c,d) = 1$, so every element in H must be a 3-cycle (or the identity). Then, if $(a,b,c) \in H$, then $H = \langle (a,b,c) \rangle = \{1, (a,b,c), (a,b,c)^2\}$, but $(a,b,c)^2 = (c,b,a)$. Then, if you just pick something like $(a,b)(c,d)$ which is its own inverse, we get that

$$(a,b)(c,d)(a,b,c)(a,b)(c,d)$$

sends $a \mapsto d$, so it is not in H, and thus H cannot be normal.

Problem 6

There is exactly one p-Sylow subgroup of G. In particular, the possible amounts of p-Sylow subgroups must all be 1 modulo p. However, this amount must also divide $p^r m$, which has divisors $1, p, p^2, \ldots, p^r, m, pm, \ldots, p^r m$. Clearly no p^i for $1 < i \leq r$ is 1 mod p, and neither are $p^i m$, since these all vanish modulo p. Then, the only choices are 1 and m, but by assumption, $1 < m < p$, so $m \neq 1$ mod p. Then, the only choice left is 1, so the unique p-Sylow subgroup of G is normal and nontrivial since it is of order p^r for $r > 0$.

Problem 7

i

We have that $\ker(f)$ is a normal subgroup of G. Since G is simple and by assumption $\ker(f) \neq G$, we have that the kernel is trivial and thus f is injective.

Now consider $g \in f^{-1}(A_n)$. If we have any $h \in G$, $f(h^{-1}gh) = f(h^{-1})f(g)f(h)$, but $f(n) \in A_n$, a normal subgroup of S_n since $\ker(\varepsilon) = A_n$, so we get that $f(h^{-1}gh) \in A_n$, so $h^{-1}gh \in f^{-1}(A_n)$. Then, the preimage of A_n is normal and thus either exactly $\{1\}$ or all of G. However, if it were exactly $\{1\}$, then G has at most 3 elements, since if there are 3 distinct non-identity elements, there must be a pair g, h such that $gh \neq 1$ (otherwise, if the three elements are g, h, k, we get that $gh = kh = 1 \implies g = k$). Then, G has order exactly 3, since by assumption $|G| > 2$; however, if $G = \{1, g, g^2\}$, we have that $f(g^2) = (f(g))^2$ which is even, but $g^2 \notin f^{-1}(A_n)$ so $\Rightarrow\Leftarrow$ and thus the preimage of A_n must be a nontrivial normal subgroup of G, namely G itself.

ii

The divisors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60. The only ones that are 1 modulo 2 (that is, odd) are 1, 3, 5, 15. Thus, these are the only possibilities.

Now if there is only 1 2-Sylow subgroup, we get that we have a normal subgroup of order 2, $\Rightarrow\Leftarrow$ since by assumption G is simple.

If there are 3 2-Sylow subgroups H_1, H_2, H_3, we get that the group action of G on $X = \{H_1, H_2, H_3\}$ given by $g \cdot H_i = g^{-1} H_i g$ induces a homomorphism $f : G \to S_X$. In particular, this group action is transitive, such that there is some $g \in G$ such that $g^{-1} H_1 g = g^{-1} H_2 g$, so $f(g) \neq \text{id}$, and this homormophism is not trivial and therefore injective. However, $|G| = 60, |S_X| = 3! = 6$, so f cannot be injective. $\Rightarrow\Leftarrow$

If there are 5 2-Sylow subgroups, we consider the same group action as before, which now induces a homomorphism $f : G \to S_X$ for X the set of 2-Sylow subgroups. Then, we have that $f(G) \subseteq A_5$, and f is injective, such that $|G| = |f(G)| = 60$. Then, a subgroup of order 60 in A_5 which itself has order $6!/2 = 60$ must be all of A_5, so $f(G) = A_5$, and thus f is an isomorphism, and we have what we wanted.