MATH 4061 HW 5

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We can induct to show that $s_{2m} = \frac{1}{2} - \frac{1}{2^m}$ and $s_{2m+1} = 1 - \frac{1}{2^m}$. The base case we can compute directly: $s_2 = 0 = \frac{1}{2} - \frac{1}{2}, s_3 = \frac{1}{2} = 1 - \frac{1}{2}$ for $m = 1$, which is as desired. Then, if this holds for m, then $s_{2(m+1)} = s_{(2m+1)+1} = \frac{s_{2m+1}}{2} = \frac{1}{2}\left(1 - \frac{1}{2^m}\right) = \frac{1}{2} - \frac{1}{2^{m+1}}$. Further, $s_{2(m+1)+1} = s_{2m+3} = \frac{1}{2} + s_{2m+2} = \frac{1}{2} + \frac{1}{2} - \frac{1}{2^{m+1}} = 1 - \frac{1}{2^{m+1}}$, again as desired, so the claim holds for $m+1$ as well, and thus for all positive integers.

To see that $\limsup_{n\to\infty} s_n = 1$, we have that if $n_k = 2k+1$,

$$\lim_{k\to\infty} s_{n_k} = \lim_{k\to\infty} s_{2k+1} = \lim_{k\to\infty} 1 - \frac{1}{2^k} = 1$$

Then, if $x > 1$, we have that $s_n < 1$ for all n, which is apparent since either $s_n < 1$ if n is odd, and $s_n < \frac{1}{2} < 1$ if n is even. Then, by the theorem (3.17) in Rudin, also proved in class, we have that $1 = \limsup_{n\to\infty} s_n$.

Similarly, to see that $\liminf_{n\to\infty} s_n = \frac{1}{2}$, we have that if $n_k = 2k$,

$$\lim_{k\to\infty} s_{n_k} = \lim_{k\to\infty} s_{2k} = \lim_{k\to\infty} \frac{1}{2} - \frac{1}{2^k} = \frac{1}{2}$$

Then, if $x < \frac{1}{2}$, we have that since $\lim_{k\to\infty} s_{n_k} = \frac{1}{2}$, for $\epsilon = \frac{1}{2} - x$, there is some K such that if $k > K$, $|s_{n_k} - \frac{1}{2}| < \epsilon \implies s_{n_k} > x$. This shows that $s_n > x$ for even n greater than N, and for odd $n > N$, we have that $s_n = 1 - \frac{1}{2^{(n-1)/2}} > \frac{1}{2}$ as well. Then, by theorem 3.17, we have that $\liminf_{n\to\infty} s_n = \frac{1}{2}$.

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a

This sum telescopes: we can show that $\sum_{i=1}^n a_i = \sqrt{n+1} - 1$, since for $n = 1$, $\sum_{i=1}^n a_i = a_1 = \sqrt{2} - 1$, as desired, and if it holds for n, then $\sum_{i=1}^{n+1} a_i = (\sqrt{n+1} - 1) + (\sqrt{n+2} -$

$\sqrt{n+1}) = \sqrt{n+2} - 1$ so it holds for $n+1$, and from induction holds for all n. Then, $\lim_{n\to\infty} \sqrt{n+2} - 1 = \infty$, so the sum diverges.

b

$$\frac{\sqrt{n+1}-\sqrt{n}}{n}\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} < \frac{1}{n(\sqrt{n}+\sqrt{n})} = \frac{1}{n^{\frac{3}{2}}}$$

Then, by comparison, $\sum a_n$ converges since it is strictly positive (as $n+1 > n \implies \sqrt{n+1} > \sqrt{n}$) and strictly less than a convergence series $\sum \frac{1}{n^{\frac{3}{2}}}$.

c

The root test gives that, since $\sqrt[n]{n} > 1$ for $n > 1$,

$$\lim_{n\to\infty} |(\sqrt[n]{n}-1)^n|^{1/n} = \lim_{n\to\infty} ((\sqrt[n]{n}-1)^n)^{1/n} = \lim_{n\to\infty} \sqrt[n]{n} - 1 = 1 - 1 = 0 < 1$$

the limit $\lim_{n\to\infty} \sqrt[n]{n}$ is shown to be one in Rudin. This series converges.

d

Consider $|z| \le 1$, which gives $|z|^n \le 1$ as well. Then, $\left|\frac{1}{1+z^n}\right| = \frac{1}{|1+z^n|}$, which by the triangle inequality $\frac{1}{|1+z^n|} \ge \frac{1}{1+|z^n|} \ge \frac{1}{1+1}$. However, this clearly bounds $a_n = \frac{1}{1+z^n}$ away from zero, since $|a_n| \ge \frac{1}{2}$, so this series cannot converge.
On the other hand, if $|z| > 1$, then we have that $\left|\frac{1}{1+z^n}\right| = \frac{1}{|1+z^n|}$. Further, we have that

$$\lim_{n\to\infty} \frac{1+z^n}{z^n} = \lim_{n\to\infty} 1 + \frac{1}{z^n} = 1$$

which gives that there is some N such that if $n > N$, $\left|\frac{1+z^n}{z^n}\right| > 1 - \epsilon$, and

$$\frac{1}{|1+z^n|}\frac{|1+z^n|}{|z^n|} = \frac{1}{|z|^n} > \frac{1-\epsilon}{|1+z^n|}$$

Then by comparison to the geometric series $\frac{1}{|z|} < 1$ (the inequality holds since $|z| > 1$), $\sum \frac{1-\epsilon}{|1+z^n|}$ converges, so $\sum \frac{1}{1+z^n}$ converges absolutely for $|z| > 1$.

8

We can show that the partial sums are Cauchy.

Now, let $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} b_n = b$ (the second limit converges since it is monotonic and bounded, and thus convergent *and* Cauchy). Then, we have that $\lim_{n\to\infty} A_n b_n = Ab$ as well as the product of two convergent sequences. Since A_n converges, it is also bounded; let B the the larger of the two bounds of A_n and b_n.

Fix $\epsilon > 0$. Then, we have that there is N_1 such that $n > N_1 \implies |A_n b_n - Ab| < \frac{\epsilon}{3}$, and since b_n is Cauchy, we have that there is N_2 such that $m, n > N_2 \implies |b_m - b_n| < \frac{\epsilon}{3B}$. Then, for $N = \max(N_1, N_2)$, and $N < p - 1 < p \le q$, the difference between the p, q partial sums is

$$\begin{aligned}
\left|\sum_{n=p}^{q} a_n b_n\right| &= \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right| \\
&= \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1})\right| + |A_q b_q - A_{p-1} b_p| \\
&\le \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1})\right| + |A_q b_q - Ab| + |Ab - A_{p-1} b_p| \\
&< \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1})\right| + \frac{2\epsilon}{3} \\
&\le \sum_{n=p}^{q-1} |A_n(b_n - b_{n+1})| + \frac{2\epsilon}{3} \\
&\le B \sum_{n=p}^{q-1} |b_n - b_{n+1}| + \frac{2\epsilon}{3}
\end{aligned}$$

Then, since we have that the sequence is monotonic, we have that for any given n, $|b_n - b_{n+1}| = b_n - b_{n+1}$, or that for any given n, $|b_n - b_{n+1}| = b_{n+1} - b_n$. Either way, the sum telescopes, and we have that

$$= B(b_p - b_q) + \frac{2\epsilon}{3}$$

or

$$= B(b_q - b_p) + \frac{2\epsilon}{3}$$

In either case, the term in the parentheses is positive, so

$$\begin{aligned}
&= B|b_q - b_p| + \frac{2\epsilon}{3} \\
&< B\frac{\epsilon}{3B} + \frac{2\epsilon}{3} \\
&< \epsilon
\end{aligned}$$

Thus, the partial sums are Cauchy, and thus converge.

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a

We can see that since $r_m - r_n = \sum_{i=m}^{\infty} a_i - \sum_{i=n}^{\infty} a_i$, and the sum converges absolutely, we can cancel the terms such that $r_m - r_n = \sum_{i=m}^{n-1} a_i$, which is a finite sum of positive terms and is therefore positive itself. Then, we have that $m < n \implies r_n < r_m$ and $\frac{r_n}{r_m} < 1$; further, the RHS becomes

$$1 - \frac{r_n}{r_m} = \frac{r_m - r_n}{r_m} = \frac{\sum_{i=m}^{n-1} a_i}{r_m}$$

And the LHS satisfies for each term that $\frac{r_t}{r_t} > \frac{r_t}{r_n}$ for any $m \leq t < n$. Then,

$$\sum_{i=m}^{n} \frac{a_i}{r_i} > \sum_{i=m}^{n} \frac{a_i}{r_m} = \frac{\sum_{i=m}^{n} a_i}{r_m} > \frac{\sum_{i=m}^{n-1} a_i}{r_m} = 1 - \frac{r_n}{r_m}$$

where the last inequality follows from the fact that $a_n > 0$.

Then, we have that the partial sums of $\sum \frac{a_n}{r_n}$ are not Cauchy, as for any N, fix $m = N$ and take n large enough that $r_n < r_m/2$, such that $|\sum_{i=1}^{n} \frac{a_i}{r_i} - \sum_{i=1}^{m-1} \frac{a_i}{r_i}| > 1 - \frac{r_n}{r_m} > 1 - \frac{r_m/2}{r_m} = \frac{1}{2}$. Then, fixing $\epsilon > 1/2$, we have that there is no N such that all $n, m \geq N$ satisfy $|\sum_{i=1}^{n} \frac{a_i}{r_i} - \sum_{i=1}^{m} \frac{a_i}{r_i}| < \epsilon$. The reason that we can take n large enough that $r_n < r_m/2$ is that once we fix r_m, we have that $r_m/2$ is a fixed real number, and since $\lim_{n\to\infty} r_n = 0$ since the series converges, we have that there is some large N' such that $r_n < r_m/2$ for $n > N'$.

b

Consider that for any two positive distinct reals x, y, we have that $(x+y) - 2\sqrt{xy} = (\sqrt{x} - \sqrt{y})^2 > 0 \implies \frac{x+y}{2} > \sqrt{xy}$ (this is AMGM for $n = 2$). Then, since $r_n - r_{n+1} = a_n$, we have that these are both distinct positive reals (as they converge absolutely and have all positive terms), and so satisfy $\frac{r_n + r_{n+1}}{2} > \sqrt{r_n r_{n+1}}$.

Then, we have that

$$2(r_n - \sqrt{r_n r_{n+1}}) > 2\left(r_n - \frac{r_n + r_{n+1}}{2}\right) = 2\left(\frac{r_n - r_{n+1}}{2}\right) = r_n - r_{n+1} = a_n$$

Rearranging,

$$a_n < 2(r_n - \sqrt{r_n r_{n+1}}) \implies \frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

Then, we have that $\sum_{i=1}^{n} 2(\sqrt{r_i} - \sqrt{r_{i+1}}) = 2(\sqrt{r_1} - \sqrt{r_{n+1}})$ since the sum telescopes, and so $\sum_{i=1}^{\infty} 2(\sqrt{r_i} - \sqrt{r_{i+1}}) = \lim_{n\to\infty} 2(\sqrt{r_1} - \sqrt{r_{n+1}})$. Since we showed in part a that $\lim_{n\to\infty} r_{n+1} =$

0, we have that fixing $\epsilon > 0$, there is some N such that $r_n < \epsilon^2$ for $n > N$, and so $\sqrt{r_n} < \epsilon$, and so $\lim_{n\to\infty} \sqrt{r_{n+1}} = 0$, and the sum $\sum_{i=1}^{\infty} 2(\sqrt{r_i} - \sqrt{r_{i+1}}) = 2\sqrt{r_1}$ is convergent. Then. by comparison, $\sum \frac{a_n}{r_n}$ is convergent.

5

We will show that there is a subsequence of a_n that converges to some a and a subsequence of b_n that converges to some b such that $a+b = \limsup_{n\to\infty}(a_n+b_n)$; then, we have that since a is the limit of some subsequence of $\{a_n\}$ (and similarly for b_n), we have that $a \leq \limsup_{n\to\infty} a_n$ and $b \leq \limsup_{n\to\infty} b_n$, and $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$ follows.

Note that we have that the limit superior of a set is always achieved by some subsequence; then, let $\{a_{n_k} + b_{n_k}\}_{k=1}^{\infty}$ be a subsequence converging to $\limsup_{n\to\infty}(a_n + b_n)$.

Then, consider the sequence $\{a_{n_k}\}_{k=1}^{\infty}$ and $\{b_{n_k}\}_{k=1}^{\infty}$. We have that there is some further subsequence $\{a_{n_{k_l}}\}_{l=1}^{\infty}$ such that $\lim_{l\to\infty} a_{n_{k_l}} = \limsup_{k\to\infty} a_{n_k}$. Call this last quantity a. Then, we have that $\lim_{l\to\infty} b_{n_{k_l}} = \lim_{l\to\infty}((a_{n_{k_l}} + b_{n_{k_l}}) - a_{n_{k_l}}) = \lim_{l\to\infty}(a_{n_{k_l}} + b_{n_{k_l}}) - a$. We can always do this cancellation when the a_n and b_n are bounded above (and hence $a_n + b_n$ is bounded above), since then $\lim_{l\to\infty} a_{n_{k_l}} = \limsup_{k\to\infty} a_{n_k}$ is real and so is $\lim_{l\to\infty}(a_{n_{k_l}} + b_{n_{k_l}}) = \lim_{k\to\infty}(a_{n_k} + b_{n_k}) = \limsup_{n\to\infty}(a_n + b_n)$ is real, so we can add the limits.

Call this last quantity $\lim_{l\to\infty} b_{n_{k_l}} = b$, such that $a + b = \lim_{l\to\infty}(a_{n_{k_l}} + b_{n_{k_l}}) = \lim_{k\to\infty}(a_{n_k} + b_{n_k}) = \limsup_{n\to\infty}(a_n + b_n)$, as desired. By the earlier logic, we are done.

We still have to handle the case that one or more of the a_n, b_n are unbounded above. If one is unbounded, say WLOG that a_n is unbounded, we have that $\lim_{k\to\infty} a_{n_k} = \infty$; then, we have that $\lim_{k\to\infty}(a_{n_k} + b_{n_k}) = \infty$ as well, so $\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty} a_{n_k} + \limsup_{n\to\infty} b_{n_k} = \infty$.