Estimating Fractional Brownian Motion with Noise

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Suppose we have some fBM with stochastic volatility, X_t, and some noisy process that we are able to observe: $Y_t = X_t + \rho Z_t$ where $\rho > 0$ and $(Z_t)_{t\geq 0}$ are i.i.d. $N(0,1)$ variables. We would like to be able to extract the signal from the noise, and to somehow estimate both H and σ. To do so, we will consider weighted averages of Y_t at different time-points to eliminate the noise. In particular, we will take the following:

Definition. For $g : [0,1] \to \mathbb{R}$ and some stochastic process X_t, put

$$k_n = \frac{n^\kappa}{\theta} \tag{1}$$

$$g_j^n = g\left(\frac{j}{k_n}\right) \tag{2}$$

$$\overline{Y}(g)_i^n = \sum_{j=1}^{k_n-1} g_j^n \left(Y_{\frac{i+j-1}{n}} - Y_{\frac{i+j-1}{n}}\right) = \sum_{j=1}^{k_n-1} g_j^n \Delta_{i+j-1}^n Y \tag{3}$$

$$\widehat{Y}(g)_i^n = \sum_{j=1}^{k_n} \left(g_j^n - g_{j-1}^n\right)^2 \left(\Delta_{i+j-1}^n Y\right)^2 = \sum_{j=1}^{k_n} \left(\Delta g_j^n\right)^2 \left(\Delta_{i+j-1}^n Y\right)^2 \tag{4}$$

We will be interested in the following variation,

$$V(g)_T^{n,f}(Y) = \frac{1}{n} \sum_{i=1}^{\lfloor nT \rfloor} f\left(\frac{\overline{Y}(g)_i^n}{(k_n/n)^H}, \frac{\widehat{Y}(g)_i^n}{(k_n/n)^{2H}}\right) \tag{5}$$

where f is some function of our choosing.

Our final goal is the next theorem:

Theorem. *Given some fBM with measurement error,*

$$Y_t = X_t + \rho Z_t \tag{6}$$

where

$$X_t = X_0 + \int_0^t b_s ds + K_H^{-\frac{1}{2}} \int_0^t (t-s)^{H-\frac{1}{2}} \sigma_s dB_s \tag{7}$$

if we have the following conditions,

1. $f : \mathbb{R}^L \to \mathbb{R}$ *is* C^2 *with all partial derivatives up to order 2 of at most polynomial growth.*

2. b, σ *is of size 1 (that is,* $||b_s||_{L_p}, ||\sigma_s||_{L_p}$ *are bounded) and adapted.*

3. σ *is* L^2*-continuous.*

the we get the following convergence: TODO

DRAFTS AND SNIPPETS: NOT FINAL OR EVEN COHERENT!

Truncation is taken as follows:

$$\Delta^{n,\epsilon}_{i+j-1} Y = \int_{\frac{i}{n}-\epsilon}^{\frac{i+j-1}{n}} \left[\left(\frac{i+j-1}{n} - s \right)^{H-\frac{1}{2}} - \left(\frac{i+j-1}{n} - s \right)^{H-\frac{1}{2}} \right] \sigma_s dB_s + \rho \Delta^n_{i+j-1} Z \tag{8}$$

At some point, I may refer to the fBM part here as X.

Similarly,

$$\overline{Y}(g)^{n,\epsilon}_i = \sum_{j=1}^{k_n-1} g^n_j \Delta^{n,\epsilon}_{i+j-1} Y \tag{9}$$

$$\widehat{Y}(g)^{n,\epsilon}_i = \sum_{j=1}^{k_n-1} (\Delta g^n_j)^2 (\Delta^{n,\epsilon}_{i+j-1} Y)^2 \tag{10}$$

$$V(g)^{n,f}_T(Y) = \frac{1}{n} \sum_{i=1}^{\lfloor nT \rfloor} f\left(\frac{\overline{Y}(g)^n_i}{\nu_H}, \frac{\widehat{Y}(g)^n_i}{\nu_H} \right) \tag{11}$$

where ν_H is the appropriate normalization term, dependent on H.

We want to show that

$$\limsup_{n\to\infty} \mathbb{E}\left[\left| V(g)^{n,f}_T(Y) - \frac{1}{n} \sum_{i=\lfloor n\epsilon \rfloor+1}^{\lfloor nT \rfloor} f\left(\frac{\overline{Y}(g)^{n,\epsilon}_i}{\nu_H}, \frac{\widehat{Y}(g)^{n,\epsilon}_i}{\nu_H} \right) \right| \right] < C\epsilon$$

The difference is composed of two parts:

$$\frac{1}{n} \sum_{i=1}^{\lfloor n\epsilon \rfloor} f\left(\frac{\overline{Y}(g)^n_i}{\nu_H}, \frac{\widehat{Y}(g)^n_i}{\nu_H} \right) \tag{12}$$

$$\frac{1}{n} \sum_{i=\lfloor n\epsilon \rfloor+1}^{\lfloor nT \rfloor} \left[f\left(\frac{\overline{Y}(g)^n_i}{\nu_H}, \frac{\widehat{Y}(g)^n_i}{\nu_H} \right) - f\left(\frac{\overline{Y}(g)^{n,\epsilon}_i}{\nu_H}, \frac{\widehat{Y}(g)^{n,\epsilon}_i}{\nu_H} \right) \right] \tag{13}$$

(5) has that $f(\cdot)$ is of size 1, since f is assumbed to be of polynomial growth and we showed in "step 0" that the arguments are at worst of size 1. Then, $\exists C > 0$ such that (5) $< C\epsilon$. Then, we want to show that as $n \to \infty$, (6) $\to 0$.

MVT reduces (6) to

$$\frac{1}{n} \sum_{\lfloor n\epsilon \rfloor+1}^{\lfloor nT \rfloor} \begin{pmatrix} \partial_1 f(\xi^n_{1,i}) & \partial_2 f(\xi^n_{2,i}) \end{pmatrix} \cdot \begin{pmatrix} \frac{\overline{Y}(g)^{n,\epsilon}_i - \overline{Y}(g)^n_i}{\nu_H} & \frac{\widehat{Y}(g)^{n,\epsilon}_i - \widehat{Y}(g)^n_i}{\nu_H} \end{pmatrix} \tag{14}$$

In particular, we have that the partial derivatives are of size 1, so all that matters is that the differences on the right $\to 0$.

For reference/clarity,

$$\frac{\overline{Y}(g)_i^{n,\epsilon} - \overline{Y}(g)_i^n}{\nu_H} \tag{15}$$

$$\frac{\widehat{Y}(g)_i^{n,\epsilon} - \widehat{Y}(g)_i^n}{\nu_H} \tag{16}$$

$$\overline{Y}(g)_i^n - \overline{Y}(g)_i^{n,\epsilon} = \sum_{j=1}^{k_n-1} g_i^n (\Delta_{i+j-1}^n Y - \Delta_{i+j-1}^{n,\epsilon} Y) \tag{17}$$

$$= \sum_{j=1}^{k_n-1} g_i^n \int_0^{\frac{i}{n}-\epsilon} \left[\left(\frac{i+j-1}{n} - s \right)^{H-\frac{1}{2}} - \left(\frac{i+j-1}{n} - s \right)^{H-\frac{1}{2}} \right] \sigma_s dB_s \tag{18}$$

BDG:

$$||\overline{Y}(g)_i^n - \overline{Y}(g)_i^{n,\epsilon}||_{L_p} \leq \mathbb{E}\left[\left(\int_0^{\frac{i}{n}-\epsilon} [\cdots]^2 \sigma_s^2 ds \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} \tag{19}$$

$$\leq \left(\int_0^{\frac{i}{n}-\epsilon} [\cdots]^2 ||\sigma_s^2||_{L_{p/2}} ds \right)^{\frac{1}{2}} \tag{20}$$

From fBM proof:

Since σ is of size 1,

$$\leq C_p \left(\int_0^{\frac{i}{n}-\epsilon} \left(\left(\frac{i+l}{n} - s \right)^{H-\frac{1}{2}} - \left(\frac{i+l-1}{n} - s \right)^{H-\frac{1}{2}} \right)^2 ds \right)^{\frac{1}{2}} \tag{21}$$

Substituting $r = \frac{i+l-1}{n} - s$,

$$= C_p \left(\int_{\epsilon+\frac{l-1}{n}}^{\frac{i+l-1}{n}} \left(\left(r + \frac{1}{n} \right)^{H-\frac{1}{2}} - r^{H-\frac{1}{2}} \right)^2 dr \right)^{\frac{1}{2}} \tag{22}$$

$$\leq C_p \left(\int_\epsilon^\infty \left(\left(r + \frac{1}{n} \right)^{H-\frac{1}{2}} - r^{H-\frac{1}{2}} \right)^2 dr \right)^{\frac{1}{2}} \tag{23}$$

By MVT,

$$\leq C_p \left(\int_\epsilon^\infty \left(\frac{1}{n} \left(H - \frac{1}{2} \right) r^{H-\frac{3}{2}} \right)^2 dr \right)^{\frac{1}{2}} \tag{24}$$

$$\leq C_p \frac{1}{n} \left| H - \frac{1}{2} \right| \left(\int_\epsilon^\infty \left(r^{H-\frac{3}{2}} \right)^2 dr \right)^{\frac{1}{2}} \tag{25}$$

$$= C_\epsilon \frac{1}{n} \tag{26}$$

Then, considering the size of (10), it is $\sim k_n \cdot 1 \cdot n^{-1}$.

Next,

$$\widehat{Y}(g)_i^n - \widehat{Y}(g)_i^{n,\epsilon} = \sum_{j=1}^{k_n-1} (\Delta g_i^n)^2 ((\Delta_{i+j-1}^n Y)^2 - (\Delta_{i+j-1}^{n,\epsilon} Y^2) \tag{27}$$

$$= \sum_{j=1}^{k_n-1} (\Delta g_i^n)^2 (\Delta_{i+j-1}^n Y - \Delta_{i+j-1}^{n,\epsilon} Y)(\Delta_{i+j-1}^n Y + \Delta_{i+j-1}^{n,\epsilon} Y) \tag{28}$$

$$\sim k_n \cdot \frac{1}{k_n^2} \cdot n^{-1} \cdot 1 = \frac{1}{k_n \cdot n} \tag{29}$$

Now we bring back the normalization term ν_H. Recall:

We will see that we actually need strict inequalities for the upper bounds.

If $H \in (0, \frac{1}{2})$: (8) has size $k_n \cdot \frac{1}{n} \cdot \frac{n^H}{\sqrt{k_n}} = \sqrt{k_n} \cdot n^{H-1} = n^{H-1+\frac{\kappa}{2}}$. Since $\kappa < 2 - 2H$, we have that $H - 1 + \frac{\kappa}{2} < 0$.

Simiarly, (9) has size $\frac{1}{k_n \cdot n} \cdot \frac{n^H}{\sqrt{k_n}} = \frac{n^{H-1}}{k_n^{\frac{3}{2}}} = n^{H-1-\frac{3}{2}\kappa}$ which is $o(1)$.

If $H \in (\frac{1}{2}, 1)$: (8) has size $k_n \cdot \frac{1}{n} \cdot \frac{n^H}{k_n^H} = k_n^{1-H} \cdot n^{H-1} = n^{(1-\kappa)(H-1)}$. Since $\kappa < 1$, we have that $1 - \kappa > 0, H - 1 < 0$.

Simiarly, (9) has size $\frac{1}{k_n \cdot n} \cdot \frac{n^H}{k_n^H} = \frac{n^{H-1}}{k_n^{H+1}} = n^{H-1-(H+1)\kappa}$ which is $o(1)$.

i am hungry :(

Next step is to discretize σ and remove the fBM portion from the second argument; we wish to show that

$$\mathbb{E} \left[\left| \frac{1}{n} \sum_{i=\lfloor n\epsilon \rfloor + 1}^{\lfloor nT \rfloor} \left[f\left(\frac{\overline{Y}(g)_i^{n,\epsilon}}{\nu_H}, \frac{\widehat{Y}(g)_i^{n,\epsilon}}{\nu_H} \right) - f\left(\frac{\sigma_{\frac{i}{n}-\epsilon} \overline{B^H}(g)_i^{n,\epsilon} + \rho \overline{Z}(g)_i^n}{\nu_H}, \frac{\rho^2 \widehat{Z}(g)_i^n}{\nu_H} \right) \right] \right| \right] \tag{30}$$

is bounded by some continuity condition on σ, probably dependent on ϵ.
Via MVT, the difference becomes

$$\frac{1}{n}\sum_{i=\lfloor n\epsilon\rfloor+1}^{\lfloor nT\rfloor}\left(\partial_1 f(\xi_{1,i}^n) \quad \partial_2 f(\xi_{2,i}^n)\right)\cdot\left(\frac{\overline{X}(g)_i^{n,\epsilon}-\sigma_{\frac{i}{n}-\epsilon}\overline{B^H}(g)_i^{n,\epsilon}}{\nu_H} \quad \frac{\widehat{Y}(g)_i^{n,\epsilon}-\rho^2\widehat{Z}(g)_i^n}{\nu_H}\right) \tag{31}$$

Again, all we care about are the differences:

$$\frac{\overline{X}(g)_i^{n,\epsilon}-\sigma_{\frac{i}{n}-\epsilon}\overline{B^H}(g)_i^{n,\epsilon}}{\nu_H} \tag{32}$$

$$\frac{\widehat{Y}(g)_i^{n,\epsilon}-\rho^2\widehat{Z}(g)_i^n}{\nu_H} \tag{33}$$

Handling (25) first, we have that

$$\overline{X}(g)_i^{n,\epsilon}-\sigma_{\frac{i}{n}-\epsilon}\overline{B^H}(g)_i^{n,\epsilon}=\sum_{j=1}^{k_n-1}g_i^n\left(\Delta_{i+j-1}^{n,\epsilon}X-\sigma_{\frac{i}{n}-\epsilon}\Delta_{i+j-1}^{n,\epsilon}B^H\right) \tag{34}$$

$$\left|\left|\Delta_{i+l}^{n,\epsilon}X-\sigma_{\frac{i}{n}-\epsilon}\Delta_{i+l}^{n,\epsilon}B^H\right|\right|_{L_2}=\left|\left|\int_{\frac{i}{n}-\epsilon}^{\frac{i+l}{n}}\left(\left(\frac{i+l}{n}-s\right)^{H-\frac{1}{2}}-\left(\frac{i+l-1}{n}-s\right)^{H-\frac{1}{2}}\right)(\sigma_s-\sigma_{\frac{i}{n}-\epsilon})dB_s\right|\right|_{L_2} \tag{35}$$

By Ito's,

$$=\left(\int_{\frac{i}{n}-\epsilon}^{\frac{i+l}{n}}\mathbb{E}\left[\left(\left(\frac{i+l}{n}-s\right)^{H-\frac{1}{2}}-\left(\frac{i+l-1}{n}-s\right)^{H-\frac{1}{2}}\right)^2(\sigma_s-\sigma_{\frac{i}{n}-\epsilon})^2\right]ds\right)^{\frac{1}{2}} \tag{36}$$

$$=\left(\int_{\frac{i}{n}-\epsilon}^{\frac{i+l}{n}}\left(\left(\frac{i+l}{n}-s\right)^{H-\frac{1}{2}}-\left(\frac{i+l-1}{n}-s\right)^{H-\frac{1}{2}}\right)^2\mathbb{E}\left[(\sigma_s-\sigma_{\frac{i}{n}-\epsilon})^2\right]ds\right)^{\frac{1}{2}} \tag{37}$$

Note that for sufficiently large n, $\frac{l}{n}<\epsilon$, so we have that

$$\mathbb{E}\left[(\sigma_s-\sigma_{\frac{i}{n}-\epsilon})^2\right]\leq\sup_{\substack{0\leq r,s\leq T\\|r-s|\leq 2\epsilon}}\mathbb{E}\left[(\sigma_s-\sigma_r)^2\right] \tag{38}$$

Abbreviate this last supremum to S and take $r=\frac{i+l}{n}-s$:

$$\leq S^{\frac{1}{2}}\left(\int_0^{\epsilon+\frac{l}{n}}\left(\left(r-\frac{1}{n}\right)^{H-\frac{1}{2}}-r^{H-\frac{1}{2}}\right)^2dr\right)^{\frac{1}{2}} \tag{39}$$

With $u = nr$,

$$= S^{\frac{1}{2}} \left(\int_0^{n\epsilon+1} \left(\left(\frac{u}{n} - \frac{1}{n} \right)^{H-\frac{1}{2}} - \left(\frac{u}{n} \right)^{H-\frac{1}{2}} \right)^2 n^{-1} du \right)^{\frac{1}{2}} \tag{40}$$

$$= S^{\frac{1}{2}} n^{-H} \left(\int_0^{n\epsilon+1} \left((u-1)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}} \right)^2 du \right)^{\frac{1}{2}} \tag{41}$$

$$\leq S^{\frac{1}{2}} n^{-H} \left(\int_0^{\infty} \left((u-1)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}} \right)^2 du \right)^{\frac{1}{2}} \tag{42}$$

$$= C S^{\frac{1}{2}} n^{-H} \tag{43}$$

PROBLEM! Returning to (27), we get that the L_2 norm will be bounded by $\sum^{k_n-1} C S^{\frac{1}{2}} n^{-H} \cdot \nu_H^{-1}$, which $\to \infty$ as $n \to \infty$ if you fix ϵ.