Estimating Fractional Brownian Motion with Noise

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Suppose we have some fBM with stochastic volatility, X_t, and some noisy process that we are able to observe: $Y_t = X_t + \rho Z_t$ where $\rho > 0$ and $(Z_t)_{t\geq 0}$ are i.i.d. $N(0,1)$ variables. We would like to be able to extract the signal from the noise, and to somehow estimate both H and σ. To do so, we will consider weighted averages of Y_t at different time-points to eliminate the noise. In particular, we will take the following:

Definition. For $g : [0,1] \to \mathbb{R}$, some constant θ, and some stochastic process W_t, put

$$k_n = \frac{n^\kappa}{\theta} \tag{1}$$

$$g_j^n = g\left(\frac{j}{k_n}\right) \tag{2}$$

$$\overline{W}(g)_i^n = \sum_{j=1}^{k_n-1} g_j^n \left(W_{\frac{i+j-1}{n}} - W_{\frac{i+j-2}{n}}\right) = \sum_{j=1}^{k_n-1} g_j^n \Delta_{i+j-1}^n W \tag{3}$$

$$\widehat{W}(g)_i^n = \sum_{j=1}^{k_n} \left(g_j^n - g_{j-1}^n\right)^2 \left(\Delta_{i+j-1}^n W\right)^2 = \sum_{j=1}^{k_n} \left(\Delta g_j^n\right)^2 \left(\Delta_{i+j-1}^n W\right)^2 \tag{4}$$

Morally, the latter two expressions are *averages* of the increments of the stochastic process, considered to try and smooth out the random noise.

Finally, let the autocovariance of increments of fBM (called B_t^H) be asymptotically

$$\mathbb{E}\left[\left((B_{t+h}^H - B_t^H)(B_{t+(r+1)h}^H - B_{t+rh}^H)\right)\right] \sim h^{2H} K_H(r) \tag{5}$$

and take

$$\Gamma_r^H = \frac{K_H(r)}{K_H} = \begin{cases} 1 & r = 0 \\ \frac{1}{2}\left|(r+1)^{2H} - 2r^{2H} + (r-1)^{2H}\right| & r \geq 1 \end{cases} \tag{6}$$

where $K_H(0)$, the variance of an increment, will be abbreviated to K_H.

Notationally, if we say that a random variable has some size $f(n)$, then all L^p norms are bounded by $f(n)$.

We will ultimately show the following theorem:

Theorem. *Given some fBM with measurement error,*

$$Y_t = X_t + \rho Z_t \tag{7}$$

where

$$X_t = X_0 + A_t + B_t^H = X_0 + \int_0^t b_s ds + K_H^{-\frac{1}{2}} \int_0^t (t-s)^{H-\frac{1}{2}} \sigma_s dB_s \tag{8}$$

if we have the following conditions,

1. $f : \mathbb{R}^L \to \mathbb{R}$ *is* C^2 *with all partial derivatives up to order 2 of at most polynomial growth.*
2. $g : [0,1] \to \mathbb{R}$ *is* C^2.
3. b, σ *are of size 1 (that is,* $||b_s||_{L_p}, ||\sigma_s||_{L_p}$ *are bounded) and adapted.*
4. σ *is* L^2*-continuous.*
5. $\kappa \in \left(\frac{2H}{2H+1}, 1\right)$.

then we get the following convergence:

$$V(g)_T^{n,f}(Y) = \frac{1}{n} \sum_{i=1}^{\lfloor nT \rfloor} f\left(\frac{\overline{Y}(g)_i^n}{(k_n/n)^H}, \frac{\widehat{Y}(g)_i^n}{(k_n/n)^{2H}} \right) \xrightarrow{\mathbb{P}} \int_0^T \mu_f \left(\sigma_s^2 \eta(g), \Theta^2 \rho^2 \int_0^1 g'(r)^2 dr \right) ds \tag{9}$$

where

$$\Theta = \begin{cases} 0 & \kappa \neq \frac{2H}{2H+1} \\ \theta & \kappa = \frac{2H}{2H+1} \end{cases} \tag{10}$$

$$\eta(g) = 2H \int_0^1 g(x) \left(g(1)(1-x)^{2H-1} dx + \int_x^1 (y-x)^{2H-1} g'(y) dy \right) dx \tag{11}$$

Unless otherwise mentioned, the above conditions are assumed in the following lemmas and propositions.

To motivate the normalization term in $V(g)_T^{n,f}(Y)$, we will first compute a bound on the second moment of an averaged fBM with Hurst parameter H:

Proposition 1. *For* $B_t^H = \int_0^t (t-s)^{H-\frac{1}{2}} \sigma_s dB_s$,

$$\mathbb{E}\left[\left(\overline{B^H}(g)_i^n \right)^2 \right]^{\frac{1}{2}} \leq C \left(\frac{k_n}{n} \right)^H \tag{12}$$

Proof. This is a relatively straightforward computation:

$$\begin{aligned}
\mathbb{E}\left[\left(\overline{B^H}(g)_i^n\right)^2\right] &= \mathbb{E}\left[\left(\sum_{j=1}^{k_n-1} g_j^n \Delta_{i+j-1}^n B^H\right)^2\right] \\
&= \mathbb{E}\left[\sum_{j=1}^{k_n-1} (g_j^n \Delta_{i+j-1}^n B^H)^2 + 2\sum_{j=1}^{k_n-2}\sum_{l=j+1}^{k_n-1} g_j^n g_l^n \Delta_{i+j-1}^n B^H \Delta_{i+l-1}^n B^H\right]
\end{aligned} \tag{13}$$

Abbreviating $t = \frac{i+j-2}{n}, h = \frac{1}{n}, r = l - j$,

$$\begin{aligned}
&= \sum_{j=1}^{k_n-1} (g_j^n)^2 \mathbb{E}\left[\int_0^t \left[(t+h-s)^{H-\frac{1}{2}} - (t-s)_+^{H-\frac{1}{2}}\right]\sigma_s dB_s\right]^2 \\
&\quad + 2\sum_{j=1}^{k_n-2}\sum_{l=j+1}^{k_n-1} g_j^n g_l^n \mathbb{E}\left[\int_0^t \left[(t+h-s)^{H-\frac{1}{2}} - (t-s)_+^{H-\frac{1}{2}}\right]\sigma_s dB_s\right. \\
&\qquad\qquad \left.\cdot \int_0^t \left[(t+(r+1)h-s)^{H-\frac{1}{2}} - (t+rh-s)_+^{H-\frac{1}{2}}\right]\sigma_s dB_s\right]
\end{aligned} \tag{14}$$

Consider the expectations in a single summand, which are just the autocovariances of fBM increments, known asymptotically to be $\sim h^{2H} K_H(r)$ (in the case of the square, take $r = 0$), so

$$\begin{aligned}
&\sim \sum_{j=1}^{k_n-1} (g_j^n)^2 h^{2H} K_H + 2\sum_{j=1}^{k_n-2}\sum_{l=j+1}^{k_n-1} g_j^n g_l^n h^{2H} K_H \Gamma_r^H \\
&= h^{2H} K_H \left[\sum_{j=1}^{k_n-1} (g_j^n)^2 + 2\sum_{j=1}^{k_n-2}\sum_{l=j+1}^{k_n-1} g_j^n g_l^n \Gamma_r^H\right]
\end{aligned} \tag{15}$$

Recall that g is bounded, so this is bounded as

$$\leq Ch^{2H}\left[\sum_{j=1}^{k_n-1} 1 + \sum_{j=1}^{k_n-2}\sum_{l=j+1}^{k_n-1} \left((r+1)^{2H} - 2r^{2H} + (r-1)^{2H}\right)\right] \tag{16}$$

Reindexing, and recalling $h = \frac{1}{n}, r = l - j$,

$$= Cn^{-2H}\left[k_n - 1 + \sum_{j=1}^{k_n-2}\sum_{r=1}^{k_n-1-j} \left((r+1)^{2H} - 2r^{2H} + (r-1)^{2H}\right)\right] \tag{17}$$

The inner sum telescopes to $-1 - (k_n - 1 - j)^{2H} + (k_n - j)^{2H}$, which means the entire sum telescopes to

$$\begin{aligned}
&= Cn^{-2H}\left[k_n - 1 + -(k_n - 2) + (k_n - 1)^{2H} - 1^{2H}\right] \\
&= Cn^{-2H}(k_n - 1)^{2H}
\end{aligned} \tag{18}$$

Taking square roots gets us what we wanted. $\square$

The next thing to consider is the drift in the fBM: $X_0 + A_0 = X_0 + \int_0^t b_s ds$; we will show that in the end this does not matter since b_s is bounded, so we can safely work with only the fBM part of X_t, as long as k_n, the amount of elements we are averaging over, does not grow too fast.

Definition. U_t will be Y_t with the drift removed, namely

$$U_t = Y_t - X_0 - A_0 = B_t^H + \rho Z_t \tag{19}$$

Lemma 1. *If* $\frac{2H}{2H+1} < \kappa < 1$,

$$\lim_{n\to\infty} \mathbb{E}\left[\left|V(g)_T^{n,f}(Y) - V(g)_T^{n,f}(U)\right|\right] = 0 \tag{20}$$

Proof. Recalling the definition of the variation, the difference is really

$$\mathbb{E}\left[\left|\frac{1}{n}\sum_{i=1}^{\lfloor nT\rfloor}\left[f\left(\frac{\overline{Y}(g)_i^n}{(k_n/n)^H}, \frac{\widehat{Y}(g)_i^n}{(k_n/n)^{2H}}\right) - f\left(\frac{\overline{U}(g)_i^n}{(k_n/n)^H}, \frac{\widehat{U}(g)_i^n}{(k_n/n)^{2H}}\right)\right]\right|\right] \tag{21}$$

Applying MVT to the difference, we have that this becomes

$$\frac{1}{n}\sum_{i=1}^{\lfloor nT\rfloor}\begin{pmatrix}\partial_1 f(\xi_{1,i}^n) & \partial_2 f(\xi_{2,i}^n)\end{pmatrix}\cdot\begin{pmatrix}\frac{\overline{Y}(g)_i^n - \overline{U}(g)_i^n}{(k_n/n)^H} & \frac{\widehat{Y}(g)_i^n - \widehat{U}(g)_i^n}{(k_n/n)^{2H}}\end{pmatrix} \tag{22}$$

Now, by assumption each partial is bounded here, so we only care about the differences. Further, $\mathbb{E}[|\Delta_{i+j-1}^n A|] = \mathbb{E}\left[\left|\int_{\frac{i+j-2}{n}}^{\frac{i+j-1}{n}} b_s ds\right|\right] \le Cn^{-1}$ by assumption, so $\mathbb{E}[|\overline{A}(g)_i^n|] \le \sum_{j=1}^{k_n-1} Cn^{-1} \le \frac{k_n}{n}$; thus,

$$\mathbb{E}\left[\left|\frac{\overline{Y}(g)_i^n - \overline{U}(g)_i^n}{(k_n/n)^H}\right|\right] = \mathbb{E}\left[\left|\frac{\overline{A}(g)_i^n}{(k_n/n)^H}\right|\right] \le \left(\frac{k_n}{n}\right)^{1-H} \tag{23}$$

which $\to 0$ as $\kappa < 1$. The other difference is a little trickier:

$$\mathbb{E}\left[\left|\frac{\widehat{Y}(g)_i^n - \widehat{U}(g)_i^n}{(k_n/n)^{2H}}\right|\right] = \mathbb{E}\left[\left|\frac{n^{2H}}{k_n^{2H}}\sum_{j=1}^{k_n}(\Delta g_i^n)^2\left((\Delta_{i+j-1}^n Y)^2 - (\Delta_{i+j-1}^n U)^2\right)\right|\right] \tag{24}$$

$$\le \frac{n^{2H}}{k_n^{2H}}\sum_{j=1}^{k_n}(\Delta g_i^n)^2\left(\mathbb{E}[|(\Delta_{i+j-1}^n A)^2|] + 2\mathbb{E}[|\Delta_{i+j-1}^n A \Delta_{i+j-1}^n U|]\right) \tag{25}$$

From above, we have that $\mathbb{E}[|(\Delta^n_{i+j-1}A)^2|] \leq Cn^{-2}$. Remembering that $\Delta^n_{i+j-1}\rho Z \sim N(0,\rho^2)$, Hölder and Minkowski yield that that

$$\mathbb{E}[|\Delta^n_{i+j-1}A\Delta^n_{i+j-1}U|] \leq \mathbb{E}[|\Delta^n_{i+j-1}A|^2]^{\frac{1}{2}}(\mathbb{E}[|\Delta^n_{i+j-1}B^H|^2]^{\frac{1}{2}} + \mathbb{E}[|\Delta^n_{i+j-1}\rho Z|^2]^{\frac{1}{2}}) \tag{26}$$

$$\leq C_1 n^{-1}\left(\left(\frac{k_n}{n}\right)^H + C_2\right) \leq Cn^{-1} \tag{27}$$

For sufficiently large n, since $\kappa < 1$. Applying MVT and noting that g' is bounded by assumption, we have that the expectation is bounded as follows:

$$\mathbb{E}\left[\left|\frac{\widehat{Y}(g)^n_i - \widehat{U}(g)^n_i}{(k_n/n)^{2H}}\right|\right] \leq \frac{n^{2H}}{k_n^{2H}}\sum_{j=1}^{k_n}\left(\frac{g'(\xi^n_i)}{k_n}\right)^2 Cn^{-1} \tag{28}$$

$$\leq \frac{n^{2H-1}}{k_n^{2H+1}} \tag{29}$$

which, since $\kappa > \frac{2H}{2H+1} > \frac{2H-1}{2H+1}$, $\to 0$ as $n \to \infty$. To conclude, the original expectation is bounded by

$$\frac{1}{n}\sum_{i=1}^{\lfloor nT \rfloor} C\left(\mathbb{E}\left[\left|\frac{\overline{Y}(g)^n_i - \overline{U}(g)^n_i}{(k_n/n)^H}\right|\right] + \mathbb{E}\left[\left|\frac{\widehat{Y}(g)^n_i - \widehat{U}(g)^n_i}{(k_n/n)^{2H}}\right|\right]\right) \to 0 \tag{30}$$

□

Note that the lower bound on κ is not the most parsimonious choice possible just examining the previous proof; the next step we take is to justify a lower bound of $\frac{2H}{2H+1}$ rather than $\frac{2H-1}{2H+1}$. Considering $\widehat{Y}(g)^n_i$ and the normalization we have selected, we can now restrict the growth rate of k_n from below to keep $\widehat{Y}(g)^n_i$ finite.

Lemma 2. *If $k \geq \frac{2H}{2H+1}$, $\limsup_{n\to\infty}\left|\left|\frac{\widehat{U}(g)^n_i}{(k_n/n)^{2H}}\right|\right|_{L^p} < \infty$ as $n \to \infty$. Further, if $k > \frac{2H}{2H+1}$, $\left|\left|\frac{\widehat{U}(g)^n_i}{(k_n/n)^{2H}}\right|\right|_{L^p} \to 0$.*

Proof. First consider the quantity $\widehat{U}(g)^n_i = \widehat{B^H}(g)^n_i + \widehat{\rho Z}(g)^n_i$. Then, Minkowski yields that

$$\left|\left|\left|\widehat{B^H}(g)^n_i\right|\right|_{L^p} - \left|\left|\widehat{\rho Z}(g)^n_i\right|\right|_{L^p}\right| \leq \left|\left|\widehat{U}(g)^n_i\right|\right|_{L^p} \leq \left|\left|\widehat{B^H}(g)^n_i\right|\right|_{L^p} + \left|\left|\widehat{\rho Z}(g)^n_i\right|\right|_{L^p} \tag{31}$$

However, as $n \to \infty$, $\left|\left|\widehat{B^H}(g)^n_i\right|\right|_{L^p} \leq \sum_{j=1}^{k_n}|\Delta g^n_j|\,\left|\left|(\Delta^n_{i+j-1}B^H)^2\right|\right|_{L^p} \to 0$ since fBM is almost surely continuous, so in the limit, we are only worried about $\left|\left|\widehat{\rho Z}(g)^n_i\right|\right|_{L^p}$. However,

$$\left|\left|\frac{\widehat{\rho Z}(g)^n_i}{(k_n/n)^{2H}}\right|\right|_{L^p} \leq \frac{n^{2H}}{k_n^{2H}}\sum_{j=1}^{k_n}\left(\Delta g^n_j\right)^2\left|\left|(\Delta^n_{i+j-1}\rho Z)^2\right|\right|_{L^p} \tag{32}$$

$$= \frac{n^{2H}}{k_n^{2H}} \sum_{j=1}^{k_n} \left(\frac{g'\left(\xi_i^n\right)}{k_n} \right)^2 \left|\left|(\Delta_{i+j-1}^n \rho Z)^2\right|\right|_{L^p} \tag{33}$$

where the last equality comes from applying MVT for some $\xi_i^n \in \left(\frac{j-1}{k_n}, \frac{j}{k_n}\right)$. Now, since $\Delta_{i+j-1}^n \rho Z \sim N(0, \rho^2)$, the norm is some constant varying with p, but not n, so the above is bounded by

$$\leq C_p \frac{n^{2H}}{k_n^{2H}} \sum_{j=1}^{k_n} \left(\frac{g'(\xi_i^n)}{k_n} \right)^2 \tag{34}$$

$$\leq C_p \frac{n^{2H}}{k_n^{2H}} \sum_{j=1}^{k_n} \left(\frac{C}{k_n} \right)^2 = C_p \frac{n^{2H}}{k_n^{2H+1}} \tag{35}$$

Since $k_n^{2H+1} = \frac{n^{\kappa(2H+1)}}{\theta^{2H+1}}$, if $\kappa > \frac{2H}{2H+1}$, the above bound $\to 0$ as $n \to \infty$. If $k_n = \frac{2H}{2H+1}$, exactly, this bound remains as C_p. $\square$

The problematic part of working with fBM here is that the domain of integration for intervals in the stochastic integral stretches all the way from 0, since the integrand is dependent on the upper bound of integration. In normal BM (i.e. with $H = \frac{1}{2}$), we have that an increment $B_{t_1} - B_{t_2} = \int_{t_2}^{t_1} dB_s$, but in fBM,

$$B_{t_1}^H - B_{t_2}^H = \int_0^{t_1} \left[(t_1 - s)^{H-\frac{1}{2}} - (t_2 - s)_+^{H-\frac{1}{2}} \right] dB_s$$

What we would like to be able to do is to truncate this interval arbitrarily close to where we begin averaging, namely at $\frac{i}{n}$ in the case of $\overline{U}(g)_i^n$. To this purpose, we make the following definitions and prove the following lemma:

Definition. A *truncated* increment of a fBM B_t^H is defined as

$$\Delta_{i+j-1}^{n,\epsilon} B_t^H = \int_{\frac{i}{n}-\epsilon}^{\frac{i+j-1}{n}} \left[\left(\frac{i+j-1}{n} - s \right)^{H-\frac{1}{2}} - \left(\frac{i+j-1}{n} - s \right)^{H-\frac{1}{2}} \right] \sigma_s dB_s \tag{36}$$

Similarly, we truncate U_t as follows:

$$\Delta_{i+j-1}^{n,\epsilon} U_t = \Delta_{i+j-1}^{n,\epsilon} B_t^H + \Delta_{i+j-1}^n \rho Z \tag{37}$$

In order for this truncation to be a reasonable endeavor, we need the error we are making to be small; the amount which we discard must vanish in the limit if we take $\epsilon \to 0$.

Lemma 3.

$$\limsup_{n\to\infty} \mathbb{E} \left[\left| V(g)_T^{n,f}(U) - \frac{1}{n} \sum_{i=\lfloor n\epsilon \rfloor+1}^{\lfloor nT \rfloor} f\left(\frac{\overline{U}(g)_i^{n,\epsilon}}{(k_n/n)^H}, \frac{\widehat{U}(g)_i^{n,\epsilon}}{(k_n/n)^{2H}} \right) \right| \right] < C\epsilon \tag{38}$$

Proof. The difference is composed of two parts:

$$\frac{1}{n}\sum_{i=1}^{\lfloor n\epsilon\rfloor} f\left(\frac{\overline{U}(g)_i^n}{(k_n/n)^H}, \frac{\widehat{U}(g)_i^n}{(k_n/n)^H}\right) \tag{39}$$

$$\frac{1}{n}\sum_{i=\lfloor n\epsilon\rfloor+1}^{\lfloor nT\rfloor} \left[f\left(\frac{\overline{U}(g)_i^n}{(k_n/n)^H}, \frac{\widehat{U}(g)_i^n}{(k_n/n)^H}\right) - f\left(\frac{\overline{U}(g)_i^{n,\epsilon}}{(k_n/n)^H}, \frac{\widehat{U}(g)_i^{n,\epsilon}}{(k_n/n)^H}\right)\right] \tag{40}$$

We have that $f(\cdot)$ is of size 1, since f is assumed to be of polynomial growth and we showed in that the arguments are at worst of size 1. Then, $\exists C > 0$ such that (39) $< C\epsilon$. Then, we want to show that as $n \to \infty$, (40) $\to 0$.

MVT reduces (40) to

$$\frac{1}{n}\sum_{\lfloor n\epsilon\rfloor+1}^{\lfloor nT\rfloor} \begin{pmatrix}\partial_1 f(\xi_{1,i}^n) & \partial_2 f(\xi_{2,i}^n)\end{pmatrix} \cdot \begin{pmatrix}\frac{\overline{U}(g)_i^{n,\epsilon}-\overline{U}(g)_i^n}{(k_n/n)^H} & \frac{\widehat{U}(g)_i^{n,\epsilon}-\widehat{U}(g)_i^n}{(k_n/n)^H}\end{pmatrix} \tag{41}$$

In particular, we have that the partial derivatives are of size 1, so all that matters is that the differences on the right $\to 0$.

Explicitly,

$$\overline{U}(g)_i^n - \overline{U}(g)_i^{n,\epsilon} = \sum_{j=1}^{k_n-1} g_i^n(\Delta_{i+j-1}^n U - \Delta_{i+j-1}^{n,\epsilon} U) \tag{42}$$

$$= \sum_{j=1}^{k_n-1} g_i^n \int_0^{\frac{i}{n}-\epsilon}\left[\left(\frac{i+j-1}{n}-s\right)^{H-\frac{1}{2}} - \left(\frac{i+j-1}{n}-s\right)^{H-\frac{1}{2}}\right]\sigma_s dB_s \tag{43}$$

By BDG,

$$||\overline{U}(g)_i^n - \overline{U}(g)_i^{n,\epsilon}||_{L_p} \leq \mathbb{E}\left[\left(\int_0^{\frac{i}{n}-\epsilon}\left[\left(\frac{i+j-1}{n}-s\right)^{H-\frac{1}{2}} - \left(\frac{i+j-1}{n}-s\right)^{H-\frac{1}{2}}\right]^2 \sigma_s^2 ds\right)^{\frac{p}{2}}\right]^{\frac{1}{p}} \tag{44}$$

$$\leq \left(\int_0^{\frac{i}{n}-\epsilon}\left[\left(\frac{i+j-1}{n}-s\right)^{H-\frac{1}{2}} - \left(\frac{i+j-1}{n}-s\right)^{H-\frac{1}{2}}\right]^2 ||\sigma_s^2||_{L_{p/2}} ds\right)^{\frac{1}{2}} \tag{45}$$

Since σ is of size 1,

$$\leq C_p\left(\int_0^{\frac{i}{n}-\epsilon}\left(\left(\frac{i+l}{n}-s\right)^{H-\frac{1}{2}} - \left(\frac{i+l-1}{n}-s\right)^{H-\frac{1}{2}}\right)^2 ds\right)^{\frac{1}{2}} \tag{46}$$

Substituting $r = \frac{i+l-1}{n} - s$,

$$= C_p \left(\int_{\epsilon + \frac{l-1}{n}}^{\frac{i+l-1}{n}} \left(\left(r + \frac{1}{n} \right)^{H-\frac{1}{2}} - r^{H-\frac{1}{2}} \right)^2 dr \right)^{\frac{1}{2}} \tag{47}$$

$$\leq C_p \left(\int_{\epsilon}^{\infty} \left(\left(r + \frac{1}{n} \right)^{H-\frac{1}{2}} - r^{H-\frac{1}{2}} \right)^2 dr \right)^{\frac{1}{2}} \tag{48}$$

By MVT,

$$\leq C_p \left(\int_{\epsilon}^{\infty} \left(\frac{1}{n} \left(H - \frac{1}{2} \right) r^{H-\frac{3}{2}} \right)^2 dr \right)^{\frac{1}{2}} \tag{49}$$

$$\leq C_p n^{-1} \left| H - \frac{1}{2} \right| \left(\int_{\epsilon}^{\infty} \left(r^{H-\frac{3}{2}} \right)^2 dr \right)^{\frac{1}{2}} \tag{50}$$

$$= C_p n^{-1} \tag{51}$$

Normalizing by $(k_n/n)^H$, we get that $\frac{||\overline{U}(g)_i^n - \overline{U}(g)_i^{n,\epsilon}||_{L_p}}{(k_n/n)^H} \leq C_p n^{(\kappa-1)H-1}$ which $\to 0$ as $n \to \infty$.
Next,

$$\widehat{U}(g)_i^n - \widehat{U}(g)_i^{n,\epsilon} = \sum_{j=1}^{k_n-1} (\Delta g_i^n)^2 ((\Delta_{i+j-1}^n U)^2 - (\Delta_{i+j-1}^{n,\epsilon} U^2) \tag{52}$$

$$= \sum_{j=1}^{k_n-1} (\Delta g_i^n)^2 (\Delta_{i+j-1}^n U - \Delta_{i+j-1}^{n,\epsilon} U)(\Delta_{i+j-1}^n U + \Delta_{i+j-1}^{n,\epsilon} U) \tag{53}$$

Which asymptotically is

$$\sim k_n \cdot k_n^{-2} \cdot n^{-1} \cdot 1 = k_n^{-1} n^{-1} \tag{54}$$

and which is $\sim k_n^{-1-H} n^{1-H} \to 0$ as $n \to \infty$. $\square$

Next step is to discretize σ and remove the fBM portion from the second argument:

Lemma 4.

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \mathbb{E} \left[\left| \frac{1}{n} \sum_{i=\lfloor n\epsilon \rfloor + 1}^{\lfloor nT \rfloor} \left[f\left(\frac{\overline{U}(g)_i^{n,\epsilon}}{(k_n/n)^H}, \frac{\widehat{U}(g)_i^{n,\epsilon}}{(k_n/n)^{2H}} \right) - f\left(\frac{\sigma_{\frac{i}{n}-\epsilon} \overline{B'^H}(g)_i^{n,\epsilon} + \rho \overline{Z}(g)_i^n}{(k_n/n)^H}, \frac{\rho^2 \widehat{Z}(g)_i^n}{(k_n/n)^{2H}} \right) \right] \right| \right] = 0 \tag{55}$$

where B'^H is fractional brownian motion with unit volatility and Hurst index H (we use B^H as a part of U).

Proof. Via MVT, the difference becomes

$$\frac{1}{n}\sum_{i=\lfloor n\epsilon\rfloor+1}^{\lfloor nT\rfloor}\left(\partial_1 f(\xi_{1,i}^n) \quad \partial_2 f(\xi_{2,i}^n)\right)\cdot\begin{pmatrix}\frac{\overline{U}(g)_i^{n,\epsilon}-\sigma_{\frac{i}{n}-\epsilon}\overline{B^H}(g)_i^{n,\epsilon}}{(k_n/n)^H} & \frac{\widehat{Y}(g)_i^{n,\epsilon}-\rho^2\widehat{Z}(g)_i^n}{(k_n/n)^H}\end{pmatrix} \tag{56}$$

Again, all we care about are the differences:

$$\frac{\overline{B^H}(g)_i^{n,\epsilon}-\sigma_{\frac{i}{n}-\epsilon}\overline{B'^H}(g)_i^{n,\epsilon}}{(k_n/n)^H} \tag{57}$$

$$\frac{\widehat{B^H}(g)_i^{n,\epsilon}-\rho^2\widehat{Z}(g)_i^n}{(k_n/n)^H} \tag{58}$$

Handling (57) first, we have that $\mathbb{E}\left[\left(\overline{B^H}(g)i^{n,\epsilon}-\sigma_{\frac{i}{n}-\epsilon}\overline{B'^H}(g)_i^{n,\epsilon}\right)^2\right]$ becomes the following:

$$\sum_{j=1}^{k_n} g_j^n \sum_{l=1}^{k_n} g_j^n \mathbb{E}\left[\left(\Delta_{i+j-1}^{n,\epsilon}B^H-\sigma_{\frac{i}{n}-\epsilon}\Delta_{i+j-1}^{n,\epsilon}B'^H\right)\left(\Delta_{i+l-1}^{n,\epsilon}B^H-\sigma_{\frac{i}{n}-\epsilon}\Delta_{i+l-1}^{n,\epsilon}B'^H\right)\right] \tag{59}$$

Unfolding definitions and using Ito's isometry, the expectation becomes, where we take $j \le l$, and the subscript $+$ denotes that the expression vanishes outside of the appropriate domain:

$$\begin{aligned}&= \mathbb{E}\Bigg[\Bigg(\int_{\frac{i}{n}-\epsilon}^{\frac{i+l-1}{n}}\left(\frac{i+j-1}{n}-s\right)_+^{H-\frac{1}{2}}-\left(\frac{i+j-2}{n}-s\right)_+^{H-\frac{1}{2}}\Bigg) \\ &\qquad \cdot\left(\left(\frac{i+j-1}{n}-s\right)_+^{H-\frac{1}{2}}-\left(\frac{i+j-2}{n}-s\right)_+^{H-\frac{1}{2}}\right)(\sigma_s-\sigma_{\frac{i}{n}-\epsilon})^2 ds\Bigg]\end{aligned} \tag{60}$$

$$\begin{aligned}&\le \sup_{s\in(\frac{i}{n}-\epsilon,\frac{i+k_n}{n})}\mathbb{E}\left[(\sigma_s-\sigma_{\frac{i}{n}-\epsilon})^2\right]\mathbb{E}\Bigg[\Bigg(\int_{\frac{i}{n}-\epsilon}^{\frac{i+l-1}{n}}\left(\frac{i+j-1}{n}-s\right)_+^{H-\frac{1}{2}}-\left(\frac{i+j-2}{n}-s\right)_+^{H-\frac{1}{2}}\Bigg) \\ &\qquad \cdot\left(\left(\frac{i+j-1}{n}-s\right)_+^{H-\frac{1}{2}}-\left(\frac{i+j-2}{n}-s\right)_+^{H-\frac{1}{2}}\right)ds\Bigg]\end{aligned} \tag{61}$$

Taking $u=\frac{i+j-1}{n}-s, h=\frac{1}{n}, r=l-j$, and $S_{n,e}$ as the above supremum,

$$= S_{n,\epsilon}\int_0^{t+h}\left(u_+^{H-\frac{1}{2}}-(u-h)_+^{H-\frac{1}{2}}\right)\left((u+rh)_+^{H-\frac{1}{2}}-(u+(r-1)h)_+^{H-\frac{1}{2}}\right)du \tag{62}$$

Let $u=vh$:

$$= S_{n,\epsilon}h^{2H}\int_0^{\frac{t}{h}+1}\left(b_+^{H-\frac{1}{2}}-(v-1)_+^{H-\frac{1}{2}}\right)\left((v+r)_+^{H-\frac{1}{2}}-(v+r-1)_+^{H-\frac{1}{2}}\right)du \tag{63}$$

Which becomes, as $n \to \infty$ and $h \to 0$, asymptotically

$$\sim S_{n,\epsilon} h^{2H} K_H \Gamma_r^H \tag{64}$$

Now, we can reconsider the original summation, which is bounded by (recall that g is bounded by some constant C_0):

$$C_0 S_{n,\epsilon} h^{2H} \sum_{j,l=1}^{k_n} K_H \Gamma_r^H = C_0 S_{n,\epsilon} h^{2H} \left(\sum_{j=l=1}^{k_n} K_H + 2 \sum_{1 \le j < l \le k_n} \frac{1}{2} \left((r+1)^{2H} - 2r^{2H} + (r-1)^{2H} \right) \right) \tag{65}$$

Telescoping,

$$= C_0 S_{n,\epsilon} h^{2H} K_H ((k_n - 1)^{2H} + 3 + 4^H) \tag{66}$$

which goes $\to 0$ as $n \to \infty$ and $\epsilon \to 0$ by the L_2-continuity of σ after normalizing by $(k_n/n)^{2H}$. $\square$

Now, we are able to center to the conditional expectation, with the ultimate goal of computing an explicit form:

Lemma 5.

$$\begin{aligned} \frac{1}{n} \sum_{i=\lfloor n\epsilon \rfloor + 1}^{\lfloor nT \rfloor} \Bigg[f \left(\frac{\sigma_{\frac{i}{n}-\epsilon} \overline{B^H}(g)_i^{n,\epsilon} + \rho \overline{Z}(g)_i^n}{(k_n/n)^{2H}}, \frac{\rho^2 \widehat{Z}(g)_i^n}{(k_n/n)^H} \right) \\ - \mathbb{E}\left[f \left(\frac{\sigma_{\frac{i}{n}-\epsilon} \overline{B^H}(g)_i^{n,\epsilon} + \rho \overline{Z}(g)_i^n}{(k_n/n)^H}, \frac{\rho^2 \widehat{Z}(g)_i^n}{(k_n/n)^{2H}} \right) \Big| \mathcal{F}_{\frac{i-1}{n}} \right] \Bigg] \xrightarrow{\mathbb{P}} 0 \end{aligned} \tag{67}$$

Proof. Abbreviate $f\left(\frac{\sigma_{\frac{i}{n}-\epsilon} \overline{B^H}(g)_i^{n,\epsilon} + \rho \overline{Z}(g)_i^n}{(k_n/n)^H}, \frac{\rho^2 \widehat{Z}(g)_i^n}{(k_n/n)^{2H}} \right)$ to f_i. We will show that

$$\mathbb{E}\left[\left(\sum_{i=1}^{\lfloor nT \rfloor} f_i - \mathbb{E}\left[f_i \mid \mathcal{F}_{\frac{i-1}{n}} \right] \right)^2 \right] \to 0 \tag{68}$$

which gives us convergence in probability. Developing the square, this becomes

$$\frac{1}{n^2} \sum_{i,j=\lfloor n\epsilon \rfloor + 1}^{\lfloor nT \rfloor} \mathbb{E}\left[\left\{ f_i - \mathbb{E}[f_i \mid \mathcal{F}_{\frac{i-1}{n}}] \right\} \left\{ f_j - \mathbb{E}[f_j \mid \mathcal{F}_{\frac{j-1}{n}}] \right\} \right] \tag{69}$$

By conditioning, we can see that if $\left(\frac{i}{n} - \epsilon, \frac{i+k_n}{n}\right)$ and $\left(\frac{i}{n} - \epsilon, \frac{i+k_n}{n}\right)$ are disjoint, then this covariance is actually 0 since all of the i terms will be $\mathcal{F}_{\frac{j-1}{n}}$-measurable. In this case,

$$\mathbb{E}\left[\mathbb{E}\left[\left\{ f_i - \mathbb{E}[f_i \mid \mathcal{F}_{\frac{i-1}{n}}] \right\} \left\{ f_j - \mathbb{E}[f_j \mid \mathcal{F}_{\frac{j-1}{n}}] \right\} \Big| \mathcal{F}_{\frac{j-1}{n}} \right] \right] \tag{70}$$

$$= \mathbb{E}\left[\left(f_i - \mathbb{E}[f_i \mid \mathcal{F}_{\frac{i-1}{n}}]\right)\mathbb{E}\left[\left\{f_j - \mathbb{E}[f_j \mid \mathcal{F}_{\frac{j-1}{n}}]\right\}\Big|\mathcal{F}_{\frac{j-1}{n}}\right]\right] \tag{71}$$

$$= \mathbb{E}\left[\left(f_i - \mathbb{E}[f_i \mid \mathcal{F}_{\frac{i-1}{n}}]\right)\mathbb{E}\left[f_j - f_j\Big|\mathcal{F}_{\frac{j-1}{n}}\right]\right] = 0 \tag{72}$$

So in (69), we can perform the following size estimate:

$$\left(\frac{1}{n^2}\right) \times \lfloor nT \rfloor \times (2k_n + n\epsilon) \times C \sim C\epsilon \to 0 \tag{73}$$

as $\epsilon \to 0$. The above estimate holds since for each i index, there will be at $\sim 2k_n + n\epsilon$ j indices for which the corresponding expectation is not zero. Each of these terms will be of size one due to our conditions on f. □

We now define an auxilliary function μ_f to simplify future expressions.

Definition. Define

$$\mu_f(v_1, v_2) = \mathbb{E}\left[f(\sqrt{v_1}Z_1 + \sqrt{v_2}Z_2, 2v)\right] \tag{74}$$

where $Z_1, Z_2 \sim N(0,1)$ and are i.i.d.

Lemma 6.

$$\lim_{n\to\infty} \mathbb{E}\left[\sum_{i=\lfloor n\epsilon\rfloor+1}^{\lfloor nT\rfloor} f\left(\frac{\sigma_{\frac{i}{n}-\epsilon}\overline{B^H}(g)_i^{n,\epsilon} + \rho\overline{Z}(g)_i^n}{(k_n/n)^{2H}}, \frac{\rho^2\widehat{Z}(g)_i^n}{(k_n/n)^{2H}}\right) - \mu_f\left(\frac{\sigma^2_{\frac{i}{n}-\epsilon}}{(k_n/n)^{2H}}\sum_{j,l=1}^{k_n-1} g_j^n g_l^n \Gamma^H_{|j-l|}, \frac{\rho^2}{(k_n/n)^{2H}}\sum_{j=\lfloor n\epsilon\rfloor+1}^{k_n}(\Delta g_j^n)^2\right)\Big|\mathcal{F}_{\frac{i-1}{n}}\right] \to 0 \tag{75}$$

Proof. The second argument will be handled first; we wish to show that

$$\lim_{n\to\infty} \mathbb{E}\left[\sum_{i=\lfloor n\epsilon\rfloor+1}^{\lfloor nT\rfloor} f\left(\frac{\sigma_{\frac{i}{n}-\epsilon}\overline{B^H}(g)_i^{n,\epsilon} + \rho\overline{Z}(g)_i^n}{(k_n/n)^{2H}}, \frac{\rho^2\widehat{Z}(g)_i^n}{(k_n/n)^{2H}}\right) - f\left(\frac{\sigma_{\frac{i}{n}-\epsilon}\overline{B^H}(g)_i^{n,\epsilon} + \rho\overline{Z}(g)_i^n}{(k_n/n)^{2H}}, \frac{2\rho^2}{(k_n/n)^{2H}}\sum_{j=1}^{k_n}(\Delta g_j^n)^2\right)\Big|\mathcal{F}_{\frac{i-1}{n}}\right] \to 0 \tag{76}$$

Usage of MVT as above yields that it is sufficient to show that it is enough to show that as $n \to \infty$,

$$\mathbb{E}\left[\frac{\rho^2\widehat{Z}(g)_i^n}{(k_n/n)^H} - \frac{2\rho^2}{(k_n/n)^{2H}}\sum_{j=1}^{k_n}(\Delta g_j^n)^2\right] \to 0 \tag{77}$$

We will handle the second moment instead, which bounds the first moment from above. The square of the expectation becomes

$$\frac{\rho^4 n^{4H}}{k_n^{4H}} \sum_{j,l=1}^{k_n} (\Delta g_j^n)^2 (\Delta g_l^n)^2 \mathbb{E}\left[(\Delta_{i+j-1}^n Z^2 - 2)(\Delta_{i+l-1}^n Z^2 - 2)\right] \leq \frac{\rho^4 n^{4H}}{k_n^{4H}} 3 k_n k_n^{-4} C$$

Note that the covariance in the sum is 0 unless $|l-j| \leq 1$, since otherwise the two distributions $\Delta_{i+j-1}^n Z^2$ and $\Delta_{i+l-1}^n Z^2$ are independent. Then, since the covariance is bounded and $\Delta g_i^n \sim k_n^{-1}$, the above inequality follows for some constant C. Taking the square root, we have that the desired expectation is bound by

$$C\rho^2 \frac{n^{2H}}{k_n^{2H+\frac{3}{2}}} \to 0 \tag{78}$$

To handle the first argument, we will want to compute the variance v_1 of the random variable $\frac{\sigma_{\frac{i}{n}-\epsilon} \overline{B^H}(g)_i^{n,\epsilon}}{(k_n/n)^H}$; in particular, this is explicitly

$$\operatorname{Var}\left(\frac{\sigma_{\frac{i}{n}-\epsilon}}{(k_n/n)^H} \sum_{j=\lfloor n\epsilon \rfloor+1}^{k_n-1} g_j^n \Delta_{i+h-1}^{n,\epsilon} B^H\right) = \sum_{j,l=\lfloor n\epsilon \rfloor+1}^{k_n-1} g_j^n g_l^n \operatorname{Cov}\left(\Delta_{i+j-1}^{n,\epsilon} B^H, \Delta_{i+l-1}^{n,\epsilon} B^H\right) \tag{79}$$

We will now proceed to show that we can really use the covariance of the untruncated fBM, i.e. that as $n \to \infty$,

$$\sup_{j,l} \sup_{i \geq \lfloor n\epsilon \rfloor} \left|\left|\operatorname{Cov}\left(\Delta_{i+j-1}^{n,\epsilon} B^H, \Delta_{i+l-1}^{n,\epsilon} B^H\right) - \operatorname{Cov}\left(\Delta_{i+j-1}^{n} B^H, \Delta_{i+l-1}^{n} B^H\right)\right|\right|_{L^p} \to 0 \tag{80}$$

The difference between the two covariances is exactly

$$\begin{aligned} &\int_0^{\frac{i}{n}-\epsilon} \left[\left(\frac{i+j-1}{n} - s\right)^{H-\frac{1}{2}} - \left(\frac{i+j-2}{n} - s\right)^{H-\frac{1}{2}}\right] \\ &\qquad \cdot \left[\left(\frac{i+l-1}{n} - s\right)^{H-\frac{1}{2}} - \left(\frac{i+l-2}{n} - s\right)^{H-\frac{1}{2}}\right] ds \end{aligned} \tag{81}$$

Which is bounded by MVT to

$$\leq \frac{(H-\frac{1}{2})^2}{n^2} \int_0^{\frac{i}{n}-\epsilon} \left(\left(\frac{i+j-2}{n} - s\right)\left(\frac{i+l-2}{n} - s\right)\right)^{H-\frac{3}{2}} ds \tag{82}$$

Using the general inequality $xy \leq x^2 + y^2$, this is bounded by

$$\leq \frac{(H-\frac{1}{2})^2}{n^2} \left[\int_0^{\frac{i}{n}-\epsilon} \left(\frac{i+j-2}{n} - s\right)^{2H-3} ds + \int_0^{\frac{i}{n}-\epsilon} \left(\frac{i+l-2}{n} - s\right)^{2H-3} ds\right] \tag{83}$$

Taking a change of variables $r_j = \frac{i+j-2}{n} - s, r_l = \frac{i+l-2}{n} - s$ for clarity,

$$= \frac{(H-\frac{1}{2})^2}{n^2}\left[\int_{\frac{j-2}{n}+\epsilon}^{\frac{i+j-2}{n}} r_j^{2H-3}ds + \int_{\frac{l-2}{n}+\epsilon}^{\frac{i+l-2}{n}} r_l^{2H-3}ds\right] \tag{84}$$

$$\leq \frac{(H-\frac{1}{2})^2}{n^2}\left[\int_{\frac{j-2}{n}+\epsilon}^{\infty} r_j^{2H-3}ds + \int_{\frac{l-2}{n}+\epsilon}^{\infty} r_l^{2H-3}ds\right] \tag{85}$$

$$\leq C\frac{(H-\frac{1}{2})^2}{n^2} \tag{86}$$

which $\to 0$ as desired.

Then, we have that as $n \to \infty$, the desired variance $\to \sigma^2_{\frac{i}{n}-\epsilon} \sum_{j,l=}$ □

Lemma 7.

$$\lim_{\epsilon\to 0}\mathbb{E}\left[\left|\frac{1}{n}\sum_{i=\lfloor n\epsilon\rfloor+1}^{\lfloor nT\rfloor}\mu_f\left(\frac{\sigma^2_{\frac{i}{n}-\epsilon}}{(k_n/n)^{2H}}\sum_{j,l=1}^{k_n-1}g_j^n g_l^n \Gamma^H_{|j-l|}, \frac{\rho^2}{(k_n/n)^{2H}}\sum_{j=1}^{k_n}(\Delta g_j^n)^2\right)\right.\right.$$
$$\left.\left. - \int_0^T \mu_f\left(\sigma_s^2 \mu_f(\nu(g), \Theta^2\rho^2\int_0^1 g'(r)^2 dr)\right)\right|\right] \xrightarrow{\mathbb{P}} 0 \tag{87}$$

Proof. □

The resulting theorem is just putting steps together. Restating it from before,

Theorem. *Given some fBM with measurement error,*

$$Y_t = X_t + \rho Z_t \tag{88}$$

where

$$X_t = X_0 + A_t + B_t^H = X_0 + \int_0^t b_s ds + K_H^{-\frac{1}{2}}\int_0^t (t-s)^{H-\frac{1}{2}}\sigma_s dB_s \tag{89}$$

if we have the following conditions,

1. *$f: \mathbb{R}^L \to \mathbb{R}$ is C^2 with all partial derivatives up to order 2 of at most polynomial growth.*
2. *$g: [0,1] \to \mathbb{R}$ is piecewise C^2.*
3. *b, σ are of size 1 (that is, $||b_s||_{L_p}, ||\sigma_s||_{L_p}$ are bounded) and adapted.*
4. *σ is L^2-continuous.*
5. *$\kappa \in \left(\frac{2H}{2H+1}, 1\right)$.*

then we get the following convergence:

$$V(g)^{n,f}_T(Y) = \frac{1}{n}\sum_{i=1}^{\lfloor nT \rfloor} f\left(\frac{\overline{Y}(g)^n_i}{(k_n/n)^H}, \frac{\widehat{Y}(g)^n_i}{(k_n/n)^{2H}}\right) \xrightarrow{\mathbb{P}} \int_0^T \mu_f\left(\sigma_s^2\eta\left(g\right), \Theta^2\rho^2\int_0^1 g'(r)^2dr\right)ds \tag{90}$$

where

$$\Theta = \begin{cases} 0 & \kappa \neq \frac{2H}{2H+1} \\ \theta & \kappa = \frac{2H}{2H+1} \end{cases} \tag{91}$$

$$\eta(g) = 2H\int_0^1 g(x)\left(g(1)(1-x)^{2H-1}dx + \int_x^1 (y-x)^{2H-1}g'(y)dy\right)dx \tag{92}$$

Proof. □