Estimating Fractional Brownian Motion With Noise

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Introduction

Suppose we have some unknown fBM with stochastic volatility, X_t, and some process with added noise that we are able to observe: $Y_t = X_t + \rho Z_t$. We would like to be able to extract the signal from the noise, and to somehow estimate both H and σ. To do so, we will consider weighted averages of Y_t at different time-points to eliminate the noise. Estimators of H and the integrated volatility have been previously shown for fBM without noise [2] and without stochastic volatility [3]; the main result of this project is to combine both generalizations.

Background and Setup

- The Riemann-Liouville fractional Brownian motion with Hurst Index H is the process given by the stochastic integral
$$B_t^H := \int_0^t (t-s)^{H-\frac{1}{2}}\, dB_s.$$
- Measurement noise will be considered as random normal variables with variance ρ^2.
- We consider the following process:
$$Y_t = X_0 + \int_0^t b_s ds + K_H^{-\frac{1}{2}} B_t^H + \rho Z_t,$$
where $\int_0^t b_s ds$ is some drift process, $K_H^{-\frac{1}{2}}$ is a normalizing constant, and $(Z_t)_{t\geq 0}$ are i.i.d. $N(0,1)$ variables.

Pre-Averaging

We attempt to remove the error from the process by considering weighted averages of the increments of Y_t to try and smooth out the random noise. The variation from these *pre-averages* are then used to construct estimators for H and $\int_0^T \sigma^2 ds$. Formally, for $g : [0,1] \to \mathbb{R}$, some constant θ, and some stochastic process W_t, put

$$k_n = \frac{n^\kappa}{\theta},\ g_j^n = g\left(\frac{j}{k_n}\right)$$

$$\overline{W}(g)_i^n = \sum_{j=1}^{k_n - 1} g_j^n \left(W_{\frac{i+j-1}{n}} - W_{\frac{i+j-2}{n}}\right) = \sum_{j=1}^{k_n-1} g_j^n \Delta_{i+j-1}^n W$$

$$\widehat{W}(g)_i^n = \sum_{j=1}^{k_n} \left(g_j^n - g_{j-1}^n\right)^2 \left(\Delta_{i+j-1}^n W\right)^2 = \sum_{j=1}^{k_n} \left(\Delta g_j^n\right)^2 \left(\Delta_{i+j-1}^n W\right)^2.$$

Main Theorem

Given some fBM with measurement error,

$$Y_t = X_t + \rho Z_t$$

if we have the following conditions,

1. $f : \mathbb{R}^L \to \mathbb{R}$ is C^2 with all partial derivatives up to order 2 of at most polynomial growth.
2. $g : [0,1] \to \mathbb{R}$ is C^2.
3. b, σ are of size 1 (that is, $||b_s||_{L_p}, ||\sigma_s||_{L_p}$ are bounded) and adapted.
4. σ is L^2-continuous.
5. $\kappa \in \left(\frac{2H}{2H+1}, 1\right)$.

then we get the following convergence:

$$V(g)_T^{n,f}(Y) = \frac{1}{n}\sum_{i=1}^{\lfloor nT \rfloor} f\left(\frac{\overline{Y}(g)_i^n}{(k_n/n)^H}, \frac{\widehat{Y}(g)_i^n}{(k_n/n)^{2H}}\right) \xrightarrow{\mathbb{P}} \int_0^T \mu_f\left(\sigma_s^2 \eta^H(g), \Theta^2\rho^2 \int_0^1 g'(r)^2 dr\right) ds$$

where

$$\mu_f(v_1, v_2) = \mathbb{E}\left[f\left(v_1 Z_2 + v_2 Z_2, 2v_2\right)\right],\ Z_1, Z_2 \sim N(0,1) \text{ i.i.d.}$$

$$\Theta = \begin{cases} 0 & \kappa \neq \frac{2H}{2H+1} \\ \theta & \kappa = \frac{2H}{2H+1} \end{cases}$$

$$\eta^H(g) = 2H \int_0^1 g(x)\left(g(1)(1-x)^{2H-1} + \int_x^1 (y-x)^{2H-1} g'(y) dy\right) dx$$

Sketch of the Proof

We make a series of adjustments to $V(g)_T^{n,f}(Y)$ to get closer to our final result. We first remove the drift: if $U_t = K_H^{-\frac{1}{2}} B_t^H + \rho Z_t$,

$$\lim_{n\to\infty} \mathbb{E}\left[\left|V(g)_T^{n,f}(Y) - V(g)_T^{n,f}(U)\right|\right] = 0.$$

We show also that one can truncate the the domain of integration to within ϵ of where we first begin averaging, i.e. $\frac{i}{n} - \epsilon$ while making only an error bounded by ϵ. We now *discretize* the volatility by showing that

$$\lim_{\epsilon\to 0}\lim_{n\to\infty} \mathbb{E}\left[\left|\frac{1}{n}\sum_{i=\lfloor n\epsilon \rfloor+1}^{\lfloor nT \rfloor}\left[V(g)_T^{n,f}(Y) - f\left(\frac{\sigma_{\frac{i}{n}-\epsilon}\overline{B^H}(g)_i^{n,\epsilon} + \rho\overline{Z}(g)_i^n}{(k_n/n)^H}, \frac{\rho^2\widehat{Z}(g)_i^n}{(k_n/n)^{2H}}\right)\right]\right|\right] = 0$$

We now center to the conditional expectation:

$$\frac{1}{n}\sum_{i=\lfloor n\epsilon\rfloor+1}^{\lfloor nT\rfloor}\left[f\left(\frac{\sigma_{\frac{i}{n}-\epsilon}\overline{B^H}(g)_i^{n,\epsilon} + \rho\overline{Z}(g)_i^n}{(k_n/n)^{2H}}, \frac{\rho^2\widehat{Z}(g)_i^n}{(k_n/n)^H}\right)\right.$$
$$\left. - \mathbb{E}\left[f\left(\frac{\sigma_{\frac{i}{n}-\epsilon}\overline{B^H}(g)_i^{n,\epsilon} + \rho\overline{Z}(g)_i^n}{(k_n/n)^H}, \frac{\rho^2\widehat{Z}(g)_i^n}{(k_n/n)^{2H}}\right) \Big| \mathcal{F}_{\frac{i-1}{n}}\right]\right] \xrightarrow{\mathbb{P}} 0$$

Once we compute the expectation and take limits ($n \to \infty$, $\epsilon \to 0$), we arrive at what we wanted.

Estimating H

Changing the freqency of the averaging allows us to create an estimator for H; consider taking $f(x,y) = x^2 - \frac{1}{2}y$ and a suitable κ such that $\Theta = 0$, such that

$$V(g)_T^{n,f}(Y),\ V(g)_T^{\frac{n}{2},f}(Y) = \frac{2}{n}\sum_{i=1}^{\lfloor nT/2 \rfloor}\left(\frac{\overline{Y}(g)_i^{n/2}}{(2k_n/n)^H}\right)^2$$

both converge in probability to $\int_0^T \sigma^2 \eta^H(g) ds$, so

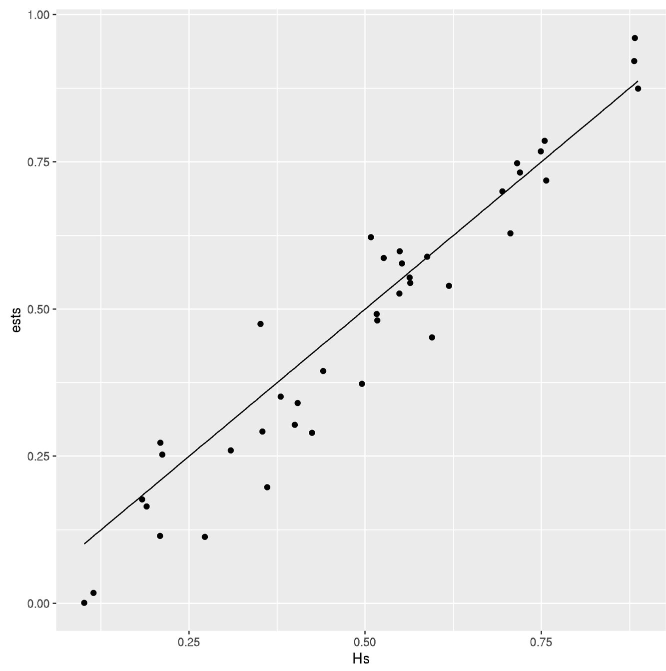
$$\frac{1}{2(1-\kappa)}\log_2\left(\frac{\widetilde{V}(g)_T^{\frac{n}{2},f}(Y)}{\widetilde{V}(g)_T^{n,f}(Y)}\right) \xrightarrow{\mathbb{P}} H.$$

where $\widetilde{V}(g)_T^{n,f}(Y)$ is simply $V(g)_T^{n,f}(Y)$ without the normalizing factor. Furthermore, we can estimate the integrated volatility:

$$\frac{V(g)_T^{n,f}(Y)}{\eta^{\hat{H}_n}(g)} \to \int_0^T \sigma_s^2 ds.$$

In Practice

Estimating simulated datasets with $n = 10^5$ and random Hurst indices, the following is $\hat{H}_b$ plotted against H.



Consistent estimations of the volatility, for which convergence is slow, is an ongoing effort; related code may be found at [4].

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