Chapter 1

Natural Deduction

1.1 Proof diagrams

Natural deduction is a proof system in which an inference is represented as a two-dimensional diagram in the shape of a *tree*, called a *proof diagram*. Let us take a simple example of a syllogism that deduces a conclusion B from two assumptions A and $A \to B$, by using the $\to$-*Elimination Rule* (which is a name of an inference rule in natural deduction which corresponds to *modus ponens*). This inference is depicted by the following tree-style inference diagram.

(1) $$\frac{A \to B \quad A}{B}\,{\scriptstyle (\to E)}$$

The above tree is annotated by the symbol $(\to E)$, which states which inference rule ($\to$-Elimination Rule in this case) is used there. Some reader may feel uncomfortable with calling the fractional diagram above as "a tree", but it can be regarded as corresponding to a tree structure as below, whose root is B and whose leaves are $A \to B$ and A.

(2) $$\begin{tikzcd}[column sep=small] A \to B \arrow[dr, dash] & & A \arrow[dl, dash] \\ & B & \end{tikzcd}$$

A number of assumptions above the vertical line varies, according to the rule that applies. When A is an axiom, a proof diagram of A has no assumptions, as depicted in (3).

(3) $$\frac{}{A}$$

In general, an inference of A with no assumption is called a proof of A. In the same way, an inference diagram with no assumption is called a *proof diagram*.

One advantage of employing tree structures for inference is that it enables us to *graft* two trees together when the conclusion of one tree serves as one of the assumptions of another tree. For example, consider the two inference diagrams in (4) for the inferences from $A, A \to B$ to B on one hand, and the inference from $B, B \to C$ to C on the other hand.

(4) $$\frac{A \quad A \to B}{B}\,{\scriptstyle (\to E)} \qquad \frac{B \quad B \to C}{C}\,{\scriptstyle (\to E)}$$

An inference diagram from $A, A \to B, B \to C$ to C obtains by grafting the left tree into the right tree.

(5)
$$\dfrac{\dfrac{A \quad A \to B}{B}\,(\to E) \qquad B \to C}{C}\,(\to E)$$

In the inference diagram (5), B is deduced from the assumption A and $A \to B$ by the rule $(\to E)$, and C is deduced from the assumption B and $B \to C$ by $(\to E)$. The grafting in (5) makes it clear that the conclusion B of the left diagram in (4) proves an assumption B in the right diagram in (4). The inference diagram (5) shows that B is not an assumption any more for (5): A, $A \to B$ and $B \to C$ are the only assumptions for (5) from which C is deduced.

In natural deduction, the ordering among assumptions is permutable. For instance, we do not distinguish between the inference diagram (5) from (6).

(6)
$$\dfrac{B \to C \qquad \dfrac{A \to B \quad A}{B}\,(\to E)}{C}\,(\to E)$$

1.2 Logical languages

A proof system consists of a logical language and a set of inference rules. A logical language, then, consists of a set of propositions, which is (in most logical languages) a countably infinite set. One of the most commonly used methods for defining countable infinite sets is to describe its grammar in the *BNF* (*Backus-Naur form*) notation, the technique well known to specify a recursive structure by context-free grammar. The following is an example of defining a set of propositions in propositional logic in BNF.

Definition 1 (Propositions)**.** The collection of *propositions* in propositional logic (notation: $\mathcal{F}$) is defined by the following BNF.

$$\mathcal{F} ::= \gamma \mid \bot \mid \mathcal{F} \to \mathcal{F} \mid \mathcal{F} \wedge \mathcal{F} \mid \mathcal{F} \vee \mathcal{F}$$

where γ is a set of propositional letters. e.g. $\gamma \stackrel{def}{\equiv} \{P, Q, R\}$

The definition of the collection $\mathcal{F}$ is specified in BNF, which is equivalent to the following context-free grammar, but is in one line.

$$\begin{array}{lclclcl}
\mathcal{F} & \to & \gamma & \qquad & \gamma & \to & P \\
\mathcal{F} & \to & \bot & & \gamma & \to & Q \\
\mathcal{F} & \to & \mathcal{F} \to \mathcal{F} & & \gamma & \to & R \\
\mathcal{F} & \to & \mathcal{F} \wedge \mathcal{F} & & & & \\
\mathcal{F} & \to & \mathcal{F} \vee \mathcal{F} & & & &
\end{array}$$

1.3 Provability

For a single logical language, there are various ways of providing a set of inference rules, which give rise to different proof systems. Let $\mathcal{S}$ be a natural deduction system, Γ be a set of formula and A be a formula. The notions of deducibility and provability are defined as follows.

Definition 2 (Deducibility)**.** A is *deducible* from Γ in $\mathcal{S}$ (notation: $\Gamma \vdash_{\mathcal{S}} A$) iff there is a proof diagram of $\mathcal{S}$ from the assumptions Γ to A.

Definition 3 (Provability)**.** A is *provable* in $\mathcal{S}$ iff A is deducible from no assumption (notation: $\vdash_{\mathcal{S}} A$).

Definition 4 (Proof-theoretic equivalence)**.** Any two propositions A and B are *mutually deducible*, or *proof-theoretically equivalent* under $\mathcal{S}$ iff $A \vdash_{\mathcal{S}} B$ and $B \vdash_{\mathcal{S}} A$ (notation: $A \dashv\vdash_{\mathcal{S}} B$).

In each of the above definitions, when $\mathcal{S}$ is obvious from the context of the discussion, it is often omitted and written as $\Gamma \vdash A, \vdash A$ and $A \dashv\vdash B$. These proof-theoretic terms will be used throughout this book.

1.4 Minimal Propositional Logic

From this point on, we will introduce three proof systems for propositional logic. The first one is **NM**, the simplest of the three. **NM** is an acronym for a class of logic called *minimal logic*, minimal in the sense that it has a minimal number of inference rules and thus minimal in the hierarchy of logical systems based on the inclusion relation between sets of theorems that they prove.

The **NM** inference rules consist of rules for conjunction, implication, and disjunction, which I will explain in order.

1.4.1 Conjunction

The following are the inference rules for conjunctions in **NM**.

Definition 5 (Conjunction-Introduction/Elimination Rules)**.** Suppose that A and B are any propositions.

$$\frac{A \quad B}{A \wedge B}\,(\wedge I) \qquad \frac{A \wedge B}{A}\,(\wedge E) \qquad \frac{A \wedge B}{B}\,(\wedge E)$$

The $(\wedge I)$ (read as $\wedge$-introduction) rule specifies what are necessary to verify $A \wedge B$, which are the proofs of A and B in this case, and also they suffice to prove $A \wedge B$. The $(\wedge E)$ (read as $\wedge$-elimination) rules has two instances: each of which specifies what can be proved by using $A \wedge B$, in this case, A and B, respectively.

Remark 6. In each of the inference rules above, A and B are meta variables. In other words, substituting A and B with any propositions yields a valid inference step in proof diagrams. For example, since $\neg P$ and $Q \to R$ are proposition, the use of the following node is valid in every proof diagram.

$$\frac{\neg P \quad Q \to R}{\neg P \wedge (Q \to R)}\,(\wedge I) \qquad \frac{(Q \to R) \wedge \neg P}{Q \to R}\,(\wedge E)$$

Remark 7. The name $\wedge$-introduction rule means that it *introduces* a proposition of the form $A \wedge B$ (not present in the premise part) so that appears in the conclusion part. The name $\wedge$-elimination rule, on the other hand, means that it *eliminates* a proposition of the form $A \to B$ (appearing in the premise part) so that it dissapears in the conclusion part.

In natural deduction, for each truth function, there is a pair of introduction rule and elimination rule.[*1]. The former represents the way in which the proposition is verified (also called verificationist meaning), and the latter represents the way in which the proposition is used (also called pragmatist meaning), respectively.

Theorem 8. Suppose that A, B, C are any propositions.

$$\begin{array}{lrcl} \text{(Associativity)} & A \wedge (B \wedge C) & \dashv\vdash_{\mathbf{NM}} & (A \wedge B) \wedge C \\ \text{(Symmetry)} & A \wedge B & \dashv\vdash_{\mathbf{NM}} & B \wedge A \\ \text{(Idempotency)} & A & \dashv\vdash_{\mathbf{NM}} & A \wedge A \end{array}$$

Proof. The associativity of conjunction is proved by the following proof diagrams of **NM**, each of which shows $A \wedge B \vdash_{\mathbf{NM}} B \wedge A$ and $B \wedge A \vdash_{\mathbf{NM}} A \wedge B$.

$$\dfrac{\dfrac{A \wedge B}{B}\,(\wedge E) \quad \dfrac{A \wedge B}{A}\,(\wedge E)}{B \wedge A}\,(\wedge I) \qquad \dfrac{\dfrac{B \wedge A}{A}\,(\wedge E) \quad \dfrac{B \wedge A}{B}\,(\wedge E)}{A \wedge B}\,(\wedge I)$$

□

Exercise 9. Prove the rest of Theorem 8.

1.4.2 Implication

In order to deduce $A \to B$ from the assumptions Γ, it suffices to deduce B from the assumption Γ plus an extra assumption A. Let us consider how to represent this 'natural' proof strategy in mathematics in terms of natural deduction. First, suppose that B is deduced by letting A be an assumption. This means that $A \to B$ is proven, but it is not appropriate to depict it as (7).

(7)
$$\dfrac{\begin{array}{c} A \\ \vdots \\ B \end{array}}{A \to B}\,??$$

The inference diagram (7) implies that A still remain as an assumption for this inference, which is not what we intend since $A \to B$ is deduced only from Γ. In other words, A is an assumption in the deduction of B from Γ and A, while A is not an assumption any more in the deduction of $A \to B$ from Γ.

[*1] To be exact, $(\wedge E)$ rule above consists of two rules, but they are counted as one, since they can be uniformly defined as follows.

$$\dfrac{A_1 \wedge A_2}{A_i}\,(\wedge E) \quad \textit{where } i \in \{1, 2\}\,.$$

For this purpose, natural deduction employs the *discharging* operation. When $A \to B$ is deduced from the fact that B is deduced from A, we draw a horizontal line above A (*discharging* A) indicating that A is not an assumption in the subsequent deduction.

This is reflected in the $\to$-introduction Rule below. It is paired with the $\to$-elimination Rule.

Definition 10 (Implication-Introduction/Elimination Rules)**.** Suppose that A and B are propositions.

$$\dfrac{\begin{matrix}\overline{A}^{\,i}\\ \vdots \\ B\end{matrix}}{A \to B}\,{\scriptstyle(\to I),i} \qquad\qquad \dfrac{A \to B \quad A}{B}\,{\scriptstyle(\to E)}$$

In the $(\to I)$ rule, the horizontal line above A is annotated with the same index i (written on its right side) as the horizontal line above $A \to B$, to show that the assumption A is discharged when deducing $A \to B$. This co-indexing controls the use of discharging, making it clear that which assumptions are discharged by applications of which rules.

The $(\to E)$ rule, also known as modus ponens rule, shows what can be proved from $A \to B$. If we have a proof for A, then we can conclude that, together with a proof of $A \to B$, we have a proof of B.

Example 11. The following proof diagram of **NM** shows that $\vdash_{\mathbf{NM}} (A \to (B \to C)) \to ((A \to B) \to (A \to C))$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A}^{1} \quad \overline{A \to B}^{2}}{B}\,{\scriptstyle(\to E)} \quad \dfrac{\overline{A}^{1} \quad \overline{A \to B \to C}^{3}}{B \to C}\,{\scriptstyle(\to E)}}{C}\,{\scriptstyle(\to E)}}{A \to C}\,{\scriptstyle(\to I),1}}{(A \to B) \to (A \to C)}\,{\scriptstyle(\to I),2}}{(A \to B \to C) \to (A \to B) \to (A \to C)}\,{\scriptstyle(\to I),3}$$

Theorem 12.

$$\begin{array}{ll} \vdash_{\mathbf{NM}} & (B \to C) \to ((A \to B) \to (A \to C)) \\ \vdash_{\mathbf{NM}} & (A \to (A \to B)) \to (A \to B) \\ \vdash_{\mathbf{NM}} & (A \to (B \to C)) \to (B \to (A \to C)) \end{array}$$

Exercise 13. Prove Theorem 12.

Exercise 14. Prove the inference: $A \to (B \to C) \dashv\vdash_{\mathbf{NM}} (A \wedge B) \to C$

1.4.3 Weakening

The following is a derived rule called *weakening*, which just discard one of the two premisses.

Theorem 15 (Weakening)**.**

$$\dfrac{A \quad B}{A}\,(WK)$$

Proof.

$$\dfrac{\dfrac{A \quad B}{A \wedge B}\,(\wedge I)}{A}\,(\wedge E)$$

□

Example 16. The inference $A, B, C \vdash_{\mathbf{NM}} B$ is deducible by only weakening as follows.

$$\dfrac{A \quad \dfrac{B \quad C}{B}\,(WK)}{B}\,(WK)$$

Remark 17. Recall that the order of propositions in the premise part of inference rules can be permuted. Thus the following rule schema is also an instance of the weakening rule.

$$\dfrac{B \quad A}{A}\,(WK)$$

Example 18. $\vdash_{\mathbf{NM}} A \to (B \to A)$

$$\dfrac{\dfrac{\dfrac{\overline{A}^{2} \quad \overline{B}^{1}}{A}\,(WK)}{B \to A}\,(\to I),1}{A \to (B \to A)}\,(\to I),2$$

Remark 19. Weakening is shown as a derived rule as above in systems with conjunctions. In systems without conjunctions, however, such as the $\to$-fragment of **NM**, $A \to B \to A$ is not a theorem unless it is assumed as a rule on its own.

1.4.4 Disjunction

The following are the inference rules for disjunctions in **NM**.

Definition 20 (Disjunction-Introduction/Elimination Rules)**.** Suppose that A,B and C are propositions.

$$\dfrac{A}{A \vee B}\,(\vee I) \qquad \dfrac{B}{A \vee B}\,(\vee I) \qquad \dfrac{A \vee B \quad \begin{matrix}\overline{A}^{i} \\ \vdots \\ C\end{matrix} \quad \begin{matrix}\overline{B}^{i} \\ \vdots \\ C\end{matrix}}{C}\,(\vee E),i$$

The $(\vee I)$ rules specifies what is necessary to verify $A \vee B$, which is either a proof of

A or that of B[*2]. The $(\vee E)$ rule specifies what can be proved by using $A \vee B$. This rule is equivalent to the case division in a proof in mathematics. When $A \vee B$ holds (=has a proof), if we can divide the cases into those in which A holds and those in which B holds, and prove that C holds in both cases, we can prove that C can be derived from $A \vee B$.

Note that A and B are discharged in this step. This is because A and B are no longer assumed in the whole deduction of C from $A \vee B$.

Theorem 21. Suppose that A, B, C are any propositions.

$$\begin{array}{llcl}
\text{(Associativity)} & A \vee (B \vee C) & \dashv\vdash_{\mathbf{NM}} & (A \vee B) \vee C \\
\text{(Symmetry)} & A \vee B & \dashv\vdash_{\mathbf{NM}} & B \vee A \\
\text{(Idempotency)} & A & \dashv\vdash_{\mathbf{NM}} & A \vee A
\end{array}$$

Proof. Proof of $A \vee B \vdash_{\mathbf{NM}} B \vee A$.

$$\dfrac{A \vee B \quad \dfrac{\overline{A}^{1}}{B \vee A}\,(\vee I) \quad \dfrac{\overline{B}^{1}}{B \vee A}\,(\vee I)}{B \vee A}\,(\vee E), 1$$

□

Exercise 22. Prove the rest of Theorem 21.

Exercise 23. Prove the following inferences.

$$\begin{array}{rcl}
A & \dashv\vdash_{\mathbf{NM}} & A \wedge (A \vee B) \\
A & \dashv\vdash_{\mathbf{NM}} & A \vee (A \wedge B) \\
A \vee (B \wedge C) & \dashv\vdash_{\mathbf{NM}} & (A \vee B) \wedge (A \vee C) \\
A \wedge (B \vee C) & \dashv\vdash_{\mathbf{NM}} & (A \wedge B) \vee (A \wedge C)
\end{array}$$

Exercise 24. Prove the following inferences.

$$\begin{array}{rcl}
A \to C, B \to C & \vdash_{\mathbf{NM}} & (A \vee B) \to C \\
(A \vee B) \to C & \vdash_{\mathbf{NM}} & A \to C
\end{array}$$

1.4.5 Negation

Negation is defined via implication and absurdity, as follows:[*3]

Definition 25. $\neg A \stackrel{def}{\equiv} A \to \bot$

From Definition 25, the introduction rule and the elimination rule for negation is derived.

[*2] The $(\vee I)$ rules can be packed into one rule as follows.

$$\dfrac{A_i}{A_1 \wedge A_2}\,(\vee I) \quad where\ i \in \{1, 2\}.$$

[*3] The notation $X \stackrel{def}{\equiv} Y$ means that a new, undefined expression X is defined as Y, which is an expression already defined. Thus X and Y is *syntactically* equivalent.

Definition 26 (Negation-Introduction/Elimination Rules)**.**

$$\dfrac{\begin{matrix}\overline{A}^{\,i}\\ \vdots\\ \bot\end{matrix}}{\neg A}\,{\scriptstyle(\neg I),i} \qquad \dfrac{A \quad \neg A}{\bot}\,{\scriptstyle(\neg E)}$$

Exercise 27. Derive Definition 26 from Definition 10 and Definition 25.

Exercise 28 (De Morgan's Law)**.** Prove the following inferences.

$$\begin{array}{rcl}\neg A \land \neg B & \dashv\vdash_{\mathbf{NM}} & \neg(A \lor B)\\ \neg A \lor \neg B & \vdash_{\mathbf{NM}} & \neg(A \land B)\\ \neg\neg(A \land B) & \dashv\vdash_{\mathbf{NM}} & \neg\neg A \land \neg\neg B\end{array}$$

Exercise 29. Prove the following inferences.

$$\begin{array}{rcl} & \vdash_{\mathbf{NM}} & (A \to \neg A) \to \neg A\\ & \vdash_{\mathbf{NM}} & A \to \neg\neg A\\ A \to B & \vdash_{\mathbf{NM}} & \neg B \to \neg A\\ A \to \neg B & \vdash_{\mathbf{NM}} & B \to \neg A\end{array}$$

1.5 Intuitionistic Propositional Logic

The system **NJ** is an extension of **NM** obtained by adding the rule (EFQ) below. By (EFQ), any proposition is deduced from the *contradiction* $\bot$.

Definition 30 (Ex Falso Quodlibet)**.**

$$\dfrac{\bot}{A}\,{\scriptstyle(EFQ)}$$

The system **NJ** is known as a system for the *intuitionistic logic*. Our theory of natural language semantics, which will be introduced in the next chapter, is an extention of **NJ**, in which sense **NJ** provides very basic notions which are required to represent meanings.

Remark 31. A typical example of reasoning that cannot be proven with NM, but can be proven with NJ (i.e., only with EFQ), is modus tollendo ponens, shown below.

$$A \lor B, \neg A \quad \vdash_{\mathbf{NJ}} \quad B$$

This inference is proved in **NJ** as follows.

$$\dfrac{A \lor B \qquad \dfrac{\dfrac{\overline{A}^{\,1} \quad \neg A}{\bot}\,{\scriptstyle(\neg E)}}{B}\,{\scriptstyle(EFQ)} \qquad \overline{B}^{\,1}}{B}\,{\scriptstyle(\lor E),1}$$

In this proof, we divide the cases into those in which A holds and those in which B holds. In the mathematical proof, since $\neg A$ holds now, the case in which A holds is contradictory and therefore should not be considered. In natural deduction, this is expressed by saying that if A holds, then $\bot$ is derived, and therefore anything can be derived from $\bot$ by (EFQ).

Exercise 32. Prove the following.

$$\begin{array}{rcl} \neg A \vee B & \vdash_{\mathbf{NJ}} & A \to B \\ \neg\neg A \to \neg\neg B & \vdash_{\mathbf{NJ}} & \neg\neg(A \to B) \end{array}$$

1.6 Classical Propositional Logic

The system **NK** is an extention of **NM** obtained by adding the rule (DNE) below. The system **NK** is a system for the *classical logic*.

Definition 33 (Double Negation Elimination)**.**

$$\dfrac{\begin{array}{c}\overline{\neg A}^{\,i} \\ \vdots \\ \bot\end{array}}{A}\,{\scriptstyle (DNE),i}$$

(DNE) is another way of expressing *reductio ad absurdum* in that it deduces A from the deduction of contradiction by assuming $\neg A$.

Exercise 34. Prove the following inferences.

$$\begin{array}{rcl} & \vdash_{\mathbf{NK}} & A \vee \neg A \\ & \vdash_{\mathbf{NK}} & ((A \to B) \to A) \to A \\ & \vdash_{\mathbf{NK}} & (\neg A \to A) \to A \\ \neg A \to B & \vdash_{\mathbf{NK}} & \neg B \to A \\ \neg A \to \neg B & \vdash_{\mathbf{NK}} & B \to A \\ A \to B,\ \neg A \to B & \vdash_{\mathbf{NK}} & B \\ \neg(A \wedge B) & \vdash_{\mathbf{NK}} & \neg A \vee \neg B \end{array}$$

Remark 35. An inference rule R in $\mathcal{S}$ can be used as an inference rule in the other system $\mathcal{S}'$, when R can be replaced by the inference diagram of $\mathcal{S}'$ (i.e. there exists an inference diagram of $\mathcal{S}'$ that has the same set of assumptions and the conclusion as R). We call such rule R as a *derived rule* in $\mathcal{S}'$.

Theorem 36. (EFQ) is a derived rule in **NK**.

Proof.

$$\dfrac{\dfrac{\overline{\neg A}^{\,1} \quad \bot}{\bot}\,{\scriptstyle (w)}}{A}\,{\scriptstyle (DNE),1}$$

□

Thus, any theorem in **NJ** is a theorem in **NK**.

1.7 General elimination rule

Conjunction has the following alternative elimination rule in place of the $(\land E)$ rules described in the previous section.

$$\dfrac{A \land B \qquad \begin{array}{c} \overline{A}^{\,i} \quad \overline{B}^{\,i} \\ \vdots \\ C \end{array}}{C}\,(\land E'),i$$

Intuitively, the implicature of this rule is this: given a proof of $A \land B$, it suffices as a proof of C if it can be shown that a proof of C can be obtained from assuming proofs of A and B. Although this rule (and what it implies) appears more complicated than $(\land E)$, on a system excluding $(\land E)$ from **NM**, the two rules are equivalent.

Proof. Case of $\underline{(\land E) \Rightarrow (\land E')}$

$$\begin{array}{c} \dfrac{A \land B}{A}\,(\land E) \qquad \dfrac{A \land B}{B}\,(\land E) \\ \vdots \\ C \end{array}$$

Case of $\underline{(\land E') \Rightarrow (\land E)}$

$$\dfrac{A \land B \qquad \dfrac{\overline{A}^{\,1} \quad \overline{B}^{\,1}}{A}\,(WK)}{A}\,(\land E'),1 \qquad\qquad \dfrac{A \land B \qquad \dfrac{\overline{A}^{\,1} \quad \overline{B}^{\,1}}{B}\,(WK)}{B}\,(\land E'),1$$

□

In fact, the latter rule is derived from a general form called a *general elimination rule*. The general elimination rule says that given a set of introduction rules for each type, the elimination rule is automatically derived.

Definition 37 (General elimination rule: A preliminary form)**.** Suppose that $\heartsuit$ has the following set of n introduction rules.

$$\dfrac{B_{11} \quad \ldots \quad B_{1m_1}}{A}\,(\heartsuit I) \quad \ldots \quad \dfrac{B_{n1} \quad \ldots \quad B_{nm_n}}{A}\,(\heartsuit I)$$

Then the general elimination rule for $\heartsuit$ is the following form.

$$\dfrac{A \qquad \begin{array}{c} \overline{B_{11}}^{\,i} \cdots \overline{B_{1m_1}}^{\,i} \\ \vdots \\ C \end{array} \qquad \ldots \qquad \begin{array}{c} \overline{B_{1n}}^{\,i} \cdots \overline{B_{nm_n}}^{\,i} \\ \vdots \\ C \end{array}}{C}\,(\heartsuit E),i$$

The reason for this rule is this: There are n-ways (and no other way) to construct a proof of type A (This is an implicit assumption in natural deduction, which is presupposed by the general elimination rule). In the first way of constructing a proof of A, A is proved from $B_{11}, \cdots, B_{1m_1}$. Thus, if C is shown from $B_{11}, \cdots, B_{1m_1}$, then if A is shown in the first way, then C is proved from that A. Similarly, if we consider up to the n-th way of constructing a proof of A, whatever the construction of A is, C is proved from A.

The previous conjunction example is the case where $n = 1$ and $m_1 = 2$. That is, there is only one way to verify $A \wedge B$, where $A \wedge B$ is proved from A, B. Thus, if C is proved from A, B, then C is also proved from $A \wedge B$.

Let us consider the general elimination rule as applied to a disjunction. There are two introduction rules for disjunction as the following scheme shows.

$$\frac{A}{A \vee B}\,(\vee I) \qquad \frac{B}{A \vee B}\,(\vee I)$$

Thus, this is the case where $n = 2$ and $m_1 = m_2 = 1$. That is, there are only two ways to verify $A \vee B$, in the first, it is verified from A, and in the second, it is verified from B. Thus, if C is proved from A and also from B, then C is proved from $A \vee B$. And this is the (∨E) rule itself.

$$\frac{A \vee B \quad \begin{matrix} \overline{A}^{\,i} \\ \vdots \\ C \end{matrix} \quad \begin{matrix} \overline{B}^{\,i} \\ \vdots \\ C \end{matrix}}{C}\,(\vee E),i$$

At first glance, the elimination rule for $\vee$ seems exceptional because its form departs from the rules for $\to$ and $\wedge$. However, it turned out that this rule can be read off from the set of introduction rules of $\vee$.

However, to apply the general elimination concept to $\to$, the Definition 37 must be extended. Recall that the introduction rule for $\to$ is as follows.

$$\frac{\begin{matrix} \overline{A}^{\,i} \\ \vdots \\ B \end{matrix}}{A \to B}\,(\to I),i$$

This is the case where $n = 1$ and $m_1 = 1$, but since A is a discharged assumpotion, the Definition 37 does not apply as is. Generalizing the form of the introduction rule to include the case with discharged assumptions, we obtain the following.

Definition 38 (General elimination rule: the general form)**.**

$$\dfrac{\begin{matrix}\overline{D_{11}}^{\,i} & & \overline{D_{1m_1}}^{\,i}\\ \vdots & & \vdots \\ B_{11} & \dots & B_{1m_1}\end{matrix}}{A}\,(\heartsuit I),i \qquad \dots \qquad \dfrac{\begin{matrix}\overline{D_{n1}}^{\,i} & & \overline{D_{nm_n}}^{\,i}\\ \vdots & & \vdots \\ B_{n1} & \dots & B_{nm_n}\end{matrix}}{A}\,(\heartsuit I),i$$

then the general elimination rule is as follows:

$$\dfrac{A \quad D_{11} \quad \dots \quad D_{1m_1} \quad \begin{matrix}\overline{B_{11}}^{\,i}\dots\overline{B_{1m_1}}^{\,i}\\ \vdots \\ C\end{matrix} \quad \dots \quad D_{n1} \quad \dots \quad D_{nm_n} \quad \begin{matrix}\overline{B_{1n}}^{\,i}\dots\overline{B_{nm_n}}^{\,i}\\ \vdots \\ C\end{matrix}}{C}\,(\heartsuit E),i$$

That is, if there exists a discharged assumption $D_{i1}, \dots, D_{im_i}$ for each of $B_{i1}, \dots, B_{im_i}$ assumed in the ith introduction rule, the general elimination rule assumes them directly. Applying this to $\to$, we obtain the following.

$$\dfrac{A \to B \quad A \quad \begin{matrix}\overline{B}^{\,i}\\ \vdots \\ C\end{matrix}}{C}\,(\to E'),i$$

That is, since there exists a discharged assumption A for B assumed in the first (and only) introduction rule, if C is proved from B, assuming that A in the general elimination rule, it is sufficient as a proof of C.

Exercise 39. Prove the equivalence between $(\to E)$ and $(\to E')$ on the system NM-($\to$E).

Remark 40. In Remark 7, we stated that the behavior of each proposition is defined by a pair of introduction and elimination rule, and the former represents the way it is proved and the latter the way it is used. However, given the discussion in this section, the introduction rule is primary and the elimination rule is secondary, because the elimination rule is automatically derived via the general elimination rule given the set of introduction rules. From here, a view emerges that the meaning of a proposition is given by its set of introduction rules. In this view, the meaning of a proposition is defined as its verificationist conditions, which is called the *verificationist view*.

History and further reading

The history of natural deduction originates back to Gentzen (1935). Instead of Hilbert's axiomatic proof systems, he proposed natural deduction as a system with a form more similar to actual mathematical proofs. The idea that the behavior of a logical operator is given by a pair of introduction and elimination rules, and that the introduction rule serves as the definition of the logical operator, was presented in this revolutionary paper. Prawitz (1965) is an important milestone for natural deduction, which proposed many key notions of natural deduction such as the tree notation, the general elimination rule, among others.

There have been published many textbooks which include an introduction to natural deduction: Bostock (1997) introduces various kinds of proof systems, in a very plain style. Troelstra and Schwichtenberg (2000) is a standard textbook of proof theory. Restall (2005) is a recent textbook on a modern view of proof theory, focusing on natural deduction. Sørensen and Urzyczyn (2006) is one of a few textbooks about Curry=Howard isomorphism/correspondence, whose chapter 2 is devoted to natural deduction.