Chapter 4

Curry-Howard Correspondence

Here is a set of typing rules for STLC introduced in the last chapter.

Simply-typed lambda calculus:

$$\dfrac{\begin{array}{c}\overline{x : A}^{\,i} \\ \vdots \\ M : B\end{array}}{\lambda x.M : A \to B}\,(LAM),i \qquad \dfrac{M : A \to B \quad N : A}{MN : B}\,(APP)$$

$$\dfrac{M : A \quad N : B}{(M, N) : A \times B}\,(PAIR) \quad \dfrac{M : A \times B}{\pi_1(M) : A}\,(PROJ) \quad \dfrac{M : A \times B}{\pi_2(M) : B}\,(PROJ)$$

If we strip the all terms off from the above rules and regard the product type operator $\times$ and the functional type operator $\to$ as the logical operators $\wedge$ and $\to$, these rules are exactly the same as the $\wedge\!\to$-fragment[*1] of the inference rules of natural deduction.

Natural Deduction:

$$\dfrac{\begin{array}{c}\overline{A}^{\,i} \\ \vdots \\ B\end{array}}{A \to B}\,(\to I),i \qquad \dfrac{A \to B \quad A}{B}\,(\to E)$$

$$\dfrac{A \quad B}{A \wedge B}\,(\wedge I) \quad \dfrac{A \wedge B}{A}\,(\wedge E) \quad \dfrac{A \wedge B}{B}\,(\wedge E)$$

This correspondence between STLC and natural deduction, or more generally, the correspondence between type theories and logical proof systems, is called *Curry-Howard correspondence* (or more strongly *Curry-Howard isomorphism*).

If types in type theory *are* propositions in logic, then what are terms in logic? A term M of type A (in type theory), viewed from the perspective of Curry-Howard correspondence, is regarded as an *encoding* of a proof diagram that derives A. For example, consider inferences for $A \wedge B,\ A \to B \vdash B$ in natural deduction. There are at least following two distinct diagrams for this infereces.

[*1] (to be written: on fragment

$$\dfrac{A\wedge B}{B}\,{\scriptstyle(\wedge E)} \qquad\qquad \dfrac{\dfrac{A\wedge B}{A}\,{\scriptstyle(\wedge E)} \qquad A\to B}{B}\,{\scriptstyle(\to E)}$$

In the type rules of STLC, the above two proof diagrams are rewritten as follows, with each node annotated with a term.

$$\dfrac{x : A\times B}{\pi_2(x) : B}\,{\scriptstyle(PROJ)} \qquad\qquad \dfrac{\dfrac{x : A\times B}{\pi_1(x) : A}\,{\scriptstyle(PROJ)} \qquad y : A\to B}{y(\pi_1(x)) : B}\,{\scriptstyle(APP)}$$

In their last lines, there appear two distinct terms, $\pi_2(x)$ and $y(\pi_1(x))$. Each term corresponds exactly to each proof diagram of B, in the sense that 1) each term is uniquely determined by a given proof diagram, and 2) each proof diagram can be uniquely recovered from the structure of a given term: for example, $\pi_2(x)$ should be derived by the application of the $(\wedge E)$ rule from the term x that corresponds to the assumption $A\wedge B$. $y(\pi_1(x))$ should be derived by the application of the $(\to E)$ rule from the functional term y, that corresponds to the assumption $A\to B$, and the term $\pi_1(x)$, which in turn should be derived by the application of the $(\wedge E)$ rule from the term x.

This is the reason why a term can be regarded as the encoding of a proof diagram. In this sense, a term is also called a *proof term*.

4.1 Implication

Let us investigate how we can re-interpret the typing rules of STLC as the rules of natural deduction, in which $\to$ and $\times$ are now interpretd as implication and conjunction. First, $\to$-Introduction/Elimination rules accompany the following proof terms.

(to be written: each rule, from the perspective of term as proof, verification condition)

Definition 115 (Implication-Introduction/Elimination Rules)**.**

$$\dfrac{\begin{array}{c}\overline{x : A}^{\,i}\\ \vdots\\ M : B\end{array}}{\lambda x.M : A\to B}\,{\scriptstyle(\to I),i} \qquad\qquad \dfrac{M : A\to B \quad N : A}{MN : B}\,{\scriptstyle(\to E)}$$

The $(\to I)$ rule saids that $A\to B$ is deduced from a deduction of B from an assumption A. Assuming A in natural deduction corresponds to assuming that A has a proof term in STLC. Let the proof term be x, by means of which B is proven, and its proof term is M. If x is used in the construction of M, x occurs free in M. This means that A is based on an *open* assumption.

The proof term of $A\to B$ is $\lambda x.M$, which is constructed from M. Note that A is not an assumption any more in this proof term. This is reflected in the fact that x is not a free variable in $\lambda x.M$, since the λ-operator binds free occurences of x withinh M, and x also serves as an index to mark which assumptions are discharged.

On the other hand, the $(\to E)$ rule saids that a proof term of a functional type $A\to B$ is a function that takes a proof of A and returns a proof of B.

The proof diagram below is an example of the derivation of $\vdash (A\to B\to C)\to$

$(A \to B) \to (A \to C)$, which demonstrates the use of the $(\to I)$ rule and the $(\to E)$ rule.

(20)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{x : A}^{1} \quad \overline{g : A \to B}^{2}}{gx : B}\,{\scriptstyle(\to E)} \quad \dfrac{\overline{x : A}^{1} \quad \overline{f : A \to B \to C}^{3}}{fx : B \to C}\,{\scriptstyle(\to E)}}{fx(gx) : C}\,{\scriptstyle(\to E)}}{\lambda x.fx(gx) : A \to C}\,{\scriptstyle(\to I),1}}{\lambda g.\lambda x.fx(gx) : (A \to B) \to (A \to C)}\,{\scriptstyle(\to I),2}}{\lambda f.\lambda g.\lambda x.fx(gx) : (A \to B \to C) \to (A \to B) \to (A \to C)}\,{\scriptstyle(\to I),3}$$

Exercise 116. Prove the following inferences with proof terms.

$$\vdash \quad ((A \to B) \to A) \to (A \to A \to B) \to A$$
$$\vdash \quad ((A \to B) \to A) \to (A \to A \to B) \to B$$

4.2 Conjunction

$\times$-Introduction/Elimination rules accompany the following proof terms.

Definition 117 (Conjunction-Introduction/Elimination Rules)**.**

$$\dfrac{M : A \quad N : B}{(M, N) : A \times B}\,{\scriptstyle(\times I)} \qquad \dfrac{M : A_1 \times A_2}{\pi_i(M) : A_i}\,{\scriptstyle(\times E)} \qquad \textit{where } i \in \{1, 2\}\,.$$

A pair (M, N), which is the proof term that the $(\times I)$ rule introduces, encodes a proof of a proposition of the form $A \times B$, that consists of M (a proof term of A) and N (a proof term of B).

$\pi_1(M)$ and $\pi_2(M)$, which are the proof terms that the $(\times E)$ rules introduce, apply to a proof term of type $A \times B$, and returns a proof of A and a proof of B, respectively. The operators π_1 and π_2 are called the *first projection* and the *second projection* of M.

Let us assign proof terms to the proof diagram of $A \times B \vdash B \times A$ in Subsection 1.4.1.

(21)
$$\dfrac{\dfrac{x : A \times B}{\pi_2(x) : B}\,{\scriptstyle(\times E)} \quad \dfrac{x : A \times B}{\pi_1(x) : A}\,{\scriptstyle(\times E)}}{(\pi_2(x), \pi_1(x)) : B \times A}\,{\scriptstyle(\times I)}$$

Proof terms for assumptions (two occurences of $A \times B$ above) are variables such as $x, y, z, \ldots$. As will be discussed in Section 4.5, grafting of a proof diagram is regarded as a substitution for a variable with a proof term.

The proof term $(\pi_2(x), \pi_1(x))$ at the bottom of (21) encodes the whole proof diagram that (21) depicts, namely, we first take the second and first projections of the assumption x, and pair them in this order. This construction of the proof term $(\pi_2(x), \pi_1(x))$ corresponds exactly to the construction of the proof diagram of $B \times A$ from $A \times B$, where we first take B and A by applying the $(\times E)$ rule to the assumption $A \times B$, from which we introduce $B \times A$ by the $(\times I)$ rule.

4.3 Disjunction

STLC can be extended with a type which corresponds to disjunction, called the *coproduct type* (notation: $A + B$). The extended syntax of types and terms is as follows.

Definition 118 (Types and Terms in STLC with product and coproduct types)**.**

$$\begin{aligned} \tau &::= \gamma \mid \tau \to \tau \mid \tau \times \tau \mid \tau + \tau \\ \Lambda &::= x \mid c \mid \lambda x.\Lambda \mid \Lambda\Lambda \mid (\Lambda, \Lambda) \mid \pi_i(\Lambda) \mid \iota_i(\Lambda) \mid \mathbf{unpack}^x_\Lambda\,(\Lambda, \Lambda) \end{aligned}$$

where γ is a base type, x is a variable, c is a constant symbol, and $i \in \{1, 2\}$.

Definition 119 (Free variables)**.**

$$\begin{aligned} fv(\iota_1 M) &\stackrel{def}{\equiv} fv(M) \\ fv(\iota_2 M) &\stackrel{def}{\equiv} fv(M) \\ fv(\mathbf{unpack}^x_L\,(M, N)) &\stackrel{def}{\equiv} fv(L) \cup (fv(M) \cup fv(N) - \{x\}) \end{aligned}$$

Definition 120 (Substitution)**.**

$$\begin{aligned} (\iota_1 M)[X/x] &\stackrel{def}{\equiv} \iota_1(M[X/x]) \\ (\iota_2 M)[X/x] &\stackrel{def}{\equiv} \iota_2(M[X/x]) \\ (\mathbf{unpack}^x_L\,(M, N))[L/x] &\stackrel{def}{\equiv} \mathbf{unpack}^x_{L[X/x]}\,(M, N) \\ (\mathbf{unpack}^y_L\,(M, N))[X/x] &\stackrel{def}{\equiv} \mathbf{unpack}^y_{L[X/x]}\,(M[X/x], N[X/x]) \\ &\qquad \textit{where } x \notin fv(L) \cup fv(()M) \cup fv(N) \textit{ or } y \notin fv(X) \end{aligned}$$

Definition 121 (β-redex)**.**

$$\mathbf{unpack}^x_N\,(\iota_1 x, \iota_2 x) \to_\beta M$$

+-Introduction/Elimination rules accompany the following proof terms.

Definition 122 (Disjunction-Introduction/Elimination Rules)**.**

$$\frac{M : A_i}{\iota_i(M) : A_1 + A_2}\,(\oplus I) \quad \textit{where } i \in \{1,2\}. \qquad \frac{L : A + B \quad \begin{array}{c}\overline{x : A}^{\,i}\\ \vdots \\ M : C\end{array} \quad \begin{array}{c}\overline{x : B}^{\,i}\\ \vdots \\ N : C\end{array}}{\mathsf{unpack}^x_L\,(M, N) : C}\,(\oplus E)$$

A proof of $A + B$ obtains either from a proof of A or a proof of B, but the $(+I)$ rule keeps a record about whether the proof of $A + B$ originates from the proof of A or the proof of B. The proof term is $\iota_1(M)$ if M is a proof term of A, and $\iota_2(M)$ if M is a proof term of B.

The $(+E)$ rule assumes that $A + B$ has a proof term L, which means that L is a proof term of either A or B. Thus what we need to verify C from L is to show that C can be proved by assuming either A or B. This corresponds to construct two proof terms, M and N, one proof term of C obtained from a variable x of type A, and another proof term of C obtained from a variable x of type B.

Being able to construct both M and N means that, a proof of C is obtained regardless of whether L is a proof term of type A or B, thus we obtain a proof of C as a whole (and the proof term that encode all this argument is $\mathsf{unpack}_L\,(M, N)$).

Notice that $\mathsf{unpack}_L\,(M, N)$ is a single proof term. The proof diagram (22) is an example of an inference $A + B \vdash B + A$.

(22)
$$\frac{y : A \oplus B \quad \dfrac{\overline{x : A}^{\,1}}{\iota_2(x) : B + A}\,(+I) \quad \dfrac{\overline{x : B}^{\,1}}{\iota_1(x) : B + A}\,(+I)}{\mathsf{unpack}^x_y\,(\iota_2(x), \iota_1(x)) : B + A}\,(+E),1$$

Exercise 123. Prove the following with proof terms.

$$A + (B \times C) \vdash (A + B) \times (A + C)$$
$$A \times (B + C) \vdash (A \times B) + (A \times C)$$

4.4 Weakening

Weakening rule is given as follows, that is, to select one of the premises as the conclusion and discard the others.

Definition 124 (Weakening rule)**.**

$$\frac{M : A \quad N : B}{M : A}\,(WK)$$

This version of the Weakening rule is in some respects contrary to the policy that the proof diagram can be uniquely recovered from the proof term, because the use of the Weakening rule is not recorded in the proof term. Thus, the proof term of the proof diagram containing Weakening cannot identify where the Weakening was used.

To prevent this, some systems allow Weakening to have its own proof constructor. For example, one method is to define a combinator called K as $\lambda x.\lambda y.x$, and the proof term of the weakening rule as x as follows.

$$\frac{M : A \quad N : B}{\mathrm{K}MN : A}\,(w)$$

4.5 Cut and Substitution

Grafting of proof diagrams that we discussed in Section 1.1 corresponds to a substitution of a variable in a proof term with free variables with another proof term of the other diagram.

Consider the following two proof diagrams in (23). The left diagram is a proof of B. The right diagram assumes B twice. Thus there are two options with regard to the ways of grafting the left diagram into the right diagram.

(23)
$$\frac{x : A \quad f : A \to B}{fx : B}\,(\to E) \qquad \frac{z : B \quad \dfrac{y : B \quad g : B \to B \to C}{gy : B \to C}\,(\to E)}{(gy)z : C}\,(\to E)$$

The first option, where the left diagram is grafted to the point represented by the variable y, yields a proof diagram (24).

(24)
$$\frac{z : B \quad \dfrac{\dfrac{x : A \quad f : A \to B}{fx : B}\,(\to E) \quad g : B \to B \to C}{g(fx) : B \to C}\,(\to E)}{g(fx)z : C}\,(\to E)$$

In (24), every occurrence of y in the right diagram is substituted by fx, which is the proof diagram of the left diagram in (23).

The second option, where the left diagram is grafted to the point represented by the variable z, yields a proof diagram (25).

(25)
$$\frac{\dfrac{x : A \quad f : A \to B}{fx : B}\,(\to E) \quad \dfrac{y : B \quad g : B \to B \to C}{gy : B \to C}\,(\to E)}{gy(fx) : C}\,(\to E)$$

In (25), every occurrence of z in the right diagram is substituted by fx, which was the proof term of the left diagram in (23). In this way, adopting variables as proof terms for assumptions allows us to keep track of the point of grafting, and from the perspective of Curry-Howard correspondence, this is regarded as a substitution of a variable with a proof term, which provides us another view about what the cut operation in natural deduction means.

4.6 Normalization and Reduction

In the following pairs of proof diagrams, the left one contains a *detour* which allows us to reduce the structure of diagram into the right diagram that is simpler.

$$\frac{\dfrac{\begin{matrix}\overline{A}^{\,i} \\ \mathcal{D}_2 \\ B\end{matrix}}{A \to B}\,(\to I), i \quad \begin{matrix}\mathcal{D}_1 \\ A\end{matrix}}{B}\,(\to E) \quad \Longrightarrow \quad \begin{matrix}\mathcal{D}_1 \\ A \\ \mathcal{D}_2 \\ B\end{matrix}$$

Remark 125. $\overset{\mathcal{D}_1}{N : A}$ という記法は、$N : A$ という type assignment を最下段に持つ証

明図を指す。$\mathcal{D}$ 単独で証明図全体を指す。一方、$\begin{array}{c} x:A \\ \mathcal{D}_2 \\ M:B \end{array}$ という記法は、$x:A$ という type assignment を仮定した、$M:B$ を最下段に持つ証明図を指す。これを $\begin{array}{c} \mathcal{D}_1 \\ N:A \\ D_2[N/x] \\ M[N/x]:B[N/x] \end{array}$ のように縦にならべた場合は、$\mathcal{D}_1$ の結論である N を $\mathcal{D}_2$ の証明図の中の x に代入したことを表す。

$$
\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ A_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ A_2\end{array}}{A_1 \times A_2}\,(\times I)}{A_i}\,(\times E) \quad \Longrightarrow \quad \begin{array}{c}\mathcal{D}_i\\ A_i\end{array} \qquad (i \in \{1,2\})
$$

$$
\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A_i\end{array}}{A_1 + A_2}\,(+I) \quad \begin{array}{c}\overline{A_1}^{\,i}\\ \mathcal{D}_1\\ B\end{array} \quad \begin{array}{c}\overline{A_2}^{\,i}\\ \mathcal{D}_2\\ B\end{array}}{B}\,(+E),i \quad \Longrightarrow \quad \begin{array}{c}\mathcal{D}\\ A_i\\ \mathcal{D}_i\\ B\end{array} \qquad (i \in \{1,2\})
$$

These detours have a common feature that the elimination rule is used just after the introduction rule of the same logical operator. The operation that removes such detours from a proof diagram as above is called *conversion*, and removing all the detours by reduction is called *normalization* of proof.

Curry-Howard correspondence gives normalization a different perspective. Detours in a diagram corresponds to β-redex, and reduction corresponds to β-reduction of proof terms.

$$
\dfrac{\dfrac{\begin{array}{c}\overline{x:A}^{\,i}\\ \mathcal{D}_2\\ M:B\end{array}}{\lambda x.M : A \to B}\,(\to I),i \quad \begin{array}{c}\mathcal{D}_1\\ N:A\end{array}}{(\lambda x.M)N : B}\,(\to E) \quad \Longrightarrow \quad \begin{array}{c}\mathcal{D}_1\\ x[N/x]:A\\ \mathcal{D}_2\\ M[N/x]:B\end{array}
$$

$$
\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ M_1:A_1\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ M_2:A_2\end{array}}{(M_1, M_2) : A_1 \times A_2}\,(\times I)}{\pi_i(M_1, M_2) : A_i}\,(\times E) \quad \Longrightarrow \quad \begin{array}{c}\mathcal{D}_i\\ M_i : A_i\end{array} \qquad (i \in \{1,2\})
$$

$$
\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ x:A_i\end{array}}{\iota_i(x) : A_1 + A_2}\,(+I) \quad \begin{array}{c}\overline{y:A_1}^{\,i}\\ \mathcal{D}_1\\ M_1:B\end{array} \quad \begin{array}{c}\overline{y:A_2}^{\,i}\\ \mathcal{D}_2\\ M_2:B\end{array}}{\mathrm{unpack}^{y}_{\iota_i x}(M_1, M_2) : B}\,(+E),i \quad \Longrightarrow \quad \begin{array}{c}\mathcal{D}\\ y[x/y]:A_i\\ \mathcal{D}_i\\ M_i[x/y]:B\end{array} \qquad (i \in \{1,2\})
$$

On the other hands, the redex of η-reduction corresponds to the proof diagram in which the introduction rule is applied just after the elimination rule.

$$
\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ M : A \to B\end{array} \quad \overline{x:A}^{\,i}}{Mx : B}\,(\to E)}{\lambda x.Mx : A \to B}\,(\to I),i \quad \Longrightarrow \quad \begin{array}{c}\mathcal{D}\\ M : A \to B\end{array}
$$

$$\dfrac{\dfrac{\overset{\mathcal{D}}{M : A_1 \times A_2}}{\pi_1 M : A_1}\,(\times E) \qquad \dfrac{\overset{\mathcal{D}}{M : A_1 \times A_2}}{\pi_2 M : A_2}\,(\times E)}{(\pi_1 M, \pi_2 M) : A_1 \times A_2}\,(\times I) \quad \Longrightarrow \quad \overset{\mathcal{D}}{M : A_1 \times A_2}$$

$$\dfrac{\overset{\mathcal{D}}{M : A_1 + A_2} \qquad \dfrac{\overline{x : A_1}^{\,i}}{\iota_1 x : A_1 + A_2}\,(+I) \qquad \dfrac{\overline{x : A_2}^{\,i}}{\iota_2 x : A_1 + A_2}\,(+I)}{\mathrm{unpack}^x_M\,(\iota_1 x, \iota_2 x) : A_1 + A_2}\,(+E),i \quad \Longrightarrow \quad \overset{\mathcal{D}}{M : A_1 + A_2}$$

Exercise 126. Assign proof terms to the following proof diagrams, and normalize them.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \times B}^{\,1}}{B}\,(\times E) \qquad \overline{B \to C}^{\,2}}{C}\,(\to E) \qquad \dfrac{\dfrac{\overline{C}^{\,3} \qquad \dfrac{\overline{A \times B}^{\,1}}{A}\,(\times E)}{A}\,(w)}{C \to A}\,(\to I),3}{A}\,(\to E)}{(B \to C) \to A}\,(\to I),2}{(A \times B) \to (B \to C) \to A}\,(\to I),1$$

$$\dfrac{\dfrac{\dfrac{\overline{\neg(A+B)}^{\,4} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A}^{\,1}}{A+B}\,(+I) \qquad \overline{\neg(A+B)}^{\,3}}{\bot}\,(\neg E)}{\neg A}\,(\neg I),1 \qquad \dfrac{\dfrac{\dfrac{\overline{B}^{\,2}}{A+B}\,(+I) \qquad \overline{\neg(A+B)}^{\,3}}{\bot}\,(\neg E)}{\neg B}\,(\neg I),2}{\neg A \times \neg B}}{\neg(A+B) \to \neg A \times \neg B}\,(\to I),3}{\neg A \times \neg B}\,(\to E)}{\neg A}\,(\times E)}{\neg(A+B) \to \neg A}\,(\to I),$$

There is another kind of conversion which is called a *permutation conversion* for disjunction.

$$\dfrac{\dfrac{\overset{\mathcal{D}}{L : A + B} \qquad \overset{\overline{x : A}^{\,i}}{\overset{\mathcal{D}_1}{M : C}} \qquad \overset{\overline{x : B}^{\,i}}{\overset{\mathcal{D}_2}{N : C}}}{\mathrm{unpack}_L\,(M, N) : C}\,(+E),i \qquad f : D'}{f(\mathrm{unpack}_L\,(M, N) : C) : D}\,(\spadesuit E)$$

$$\Longrightarrow \quad \dfrac{\overset{\mathcal{D}}{L : A + B} \qquad \dfrac{\overset{\overline{x : A}^{\,i}}{\overset{\mathcal{D}_1}{M : C}} \quad f : D'}{fM : D}\,(\spadesuit E) \qquad \dfrac{\overset{\overline{x : B}^{\,i}}{\overset{\mathcal{D}_2}{N : C}} \quad f : D'}{fN : D}\,(\spadesuit E)}{\mathrm{unpack}_L\,(fM, fN) : C}\,(+E),i$$

History and Further Reading

Curry and Feys (1958) for propositional logic

Howard (1969) for first-order logic

Sørensen and Urzyczyn (2006), Geuver