Algorithm-hw1

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1 Asymptotic Notation (40 points)

For each of the following statements explain if it true or false and prove your answer. The base of log is 2 unless otherwise specified, and ln is $\log_e$.

1. $100(n \log^4 n + \frac{1}{2}n^2) = \Theta(n^2)$

   Answer:

   If $100(n \log^4 n + \frac{1}{2}n^2) = \Theta(n^2)$, then:
   $100(n \log^4 n + \frac{1}{2}n^2) = O(n^2)$ and $100(n \log^4 n + \frac{1}{2}n^2) = \Omega(n^2)$.

   which means(according to CLRS, p44):
   $\Theta(f(n)) = \{g(n)$: there exist positive constants c_1, c_2, and n_0 such that $0 \leq c_1 f(n) \leq g(n) \leq c_2 f(n)$ for all $n \geq n_0\}$.

   For question 1, we want to prove that: $c_1 n^2 \leq 100(n \log^4 n + \frac{1}{2}n^2) \leq c_2 n^2$ for $n \geq n_0$.(n_0 does not need to be the same for lower and upper bounds).

   Simplized like:

   $$c_1 \leq \frac{100 \log^4 n}{n} + 50 \leq c_2$$
   $$\frac{c_1 - 50}{100} \leq \frac{\log^4 n}{n} \leq \frac{c_2 - 50}{100}$$

   Obviously,

   $\lim_{n\to\infty} \frac{\log^4 n}{n} = 0,$
   $\lim_{n\to\infty} (\frac{\log^4 n}{n})' = \frac{(4\log^3 n \frac{1}{n \ln 2}) \times n - \log^4 n \times 1}{n^2} = \frac{\log^3 n (\frac{4}{\ln 2} - \log n)}{n^2} = \frac{\log^3 n}{n^2}(\frac{4}{\ln 2} - \log n) < 0.$

   We can conclude that : $\frac{\log^4 n}{n}$ has upper and lower bounds when $n > n_0$,

to some $n_0 > 0$.
Let $c_1 = 50, c_2 = 150, n_0 = 2$:
When $c = c_1 = 50$, for all $n > n_0 = 2$,
$100(n \log^4 n + \frac{1}{2}n^2) \geq 50 \times n^2 \Rightarrow 100(n \log^4 n + \frac{1}{2}n^2) = \Omega(n^2)$.

When $c = c_2 = 150$, for all $n > n_0 = 2$,
$100(n \log^4 n + \frac{1}{2}n^2) \leq 150 \times n^2 \Rightarrow 100(n \log^4 n + \frac{1}{2}n^2) = O(n^2)$.

As a result, $100(n \log^4 n + \frac{1}{2}n^2) = \Theta(n^2)$.

2. $2^n = \Omega(2^{(n/2)})$
   Answer: $2^n = \Omega(2^{(n/2)}) \Leftrightarrow$ there exist constants $c, n_0 > 0$ such that $2^n \geq c \times 2^{(n/2)}$ for all $n > n_0$.

   $2^n \geq c \times 2^{(n/2)} \Leftrightarrow c \leq \dfrac{2^n}{2^{\frac{n}{2}}} \Leftrightarrow c \leq 2^{\frac{n}{2}}$.
   Since 2^x is monotonous ascending exponential,
   We set $c = 1, n_0 = 1$,
   $2^n \geq 2^{(n/2)}$ for all $n > 1 \Rightarrow 2^n = \Omega(2^{(n/2)})$.

3. $\log(n^{6 \log n}) = \Theta\left(\left(\log n^{1/3}\right)^2\right)$
   Answer : According to CLRS, p44,
   $\Theta(f(n)) = \{g(n)$: there exist positive constants c_1, c_2, and n_0 such that $0 \leq c_1 f(n) \leq g(n) \leq c_2 f(n)$ for all $n \geq n_0\}$.
   So, we need to find c_1, c_2 that satisfy $c_1 \times \left(\log n^{1/3}\right)^2 \leq \log(n^{6 \log n}) \leq c_2 \times \left(\log n^{1/3}\right)^2$ for all $n \geq n_0$ (n_0 does not need to be the same for lower and upper bounds).
   Simplized as : $c_1 \times \dfrac{1}{9} \log^2 n \leq 6 \log n \log n \leq c_2 \times \dfrac{1}{9} \log^2 n \Rightarrow c_1 \leq 54 \leq c_2$.
   So we set $c_1 = 9, c_2 = 63, n_0 = 1$,
   $\Rightarrow \log(n^{6 \log n}) = \Theta\left(\left(\log n^{1/3}\right)^2\right)$.

4. $3^n = \Theta((3.1)^n)$
   Answer : According to CLRS, p44,
   $\Theta(f(n)) = \{g(n)$: there exist positive constants c_1, c_2, and n_0 such that $0 \leq c_1 f(n) \leq g(n) \leq c_2 f(n)$ for all $n \geq n_0\}$.

   So, we need to find c_1, c_2 that satisfy $c_1 \times 3^n \leq (3.1)^n \leq c_2 \times 3^n$ for all $n \geq n_0$ (n_0 does not need to be the same for lower and upper bounds).
   Simplized as : $c_1 \leq (\dfrac{3.1}{3})^n \leq c_2$.

   However, $(\frac{3.1}{3})^n$ is a monotonous ascending exponential without upper bound, so there is no c_2 satisfy that.
   As a result, $3^n \neq \Theta((3.1)^n)$.

5. $\sqrt{n+\cos n} = O(\sqrt{n})$
Answer: $\sqrt{n+\cos n} = O(\sqrt{n}) \Leftrightarrow$ there exist constants $c, n_0 > 0$ such that $\sqrt{n+\cos n} \leq c\sqrt{n}$ for all $n > n_0$.
$\sqrt{n+\cos n} \leq c\sqrt{n} \Rightarrow (n+\cos n) \leq c^2 n \Rightarrow \dfrac{\cos n}{n} \leq c^2 - 1.$
Since $\cos n \in [-1,1], |\frac{\cos n}{n}| \leq |\frac{1}{n}|$, $\frac{\cos n}{n}$ is monotonous descending, so it has a lower bound for $n > n_0$.

Let's set $c = 3, n_0 = 2\pi$,
$\sqrt{n+\cos n} \leq 3\sqrt{n}$ for all $n > 2\pi \Rightarrow \sqrt{n+\cos n} = O(\sqrt{n})$.

6. Let f, g be positive functions. Then $f(n)+g(n) = O(\max(f(n), g(n)))$

Answer: $f(n) \leq \max(f(n), g(n)), g(n) \leq \max(f(n), g(n))$, So for all $f(n), g(n)$, $f(n)+g(n) \leq 2\max(f(n), g(n))$.
$f(n)+g(n) \leq 2\max(f(n), g(n))$ for all $n > n_0 \Rightarrow f(n)+g(n) = O(\max(f(n), g(n)))$.

7. Let f, g be positive functions, and let $g(n) = \omega(f(n))$. Then $f(n)+g(n) = \Theta(g(n))$
Answer: $g(n) = \omega(f(n)) \Rightarrow g(n) > cf(n)$ for all $n > n_0$ for any constant $c > 0$.
$f(n)+g(n) = \Theta(g(n)) \Leftrightarrow$ there exist positive constants c_1, c_2, and n_0 such that $0 \leq c_1(f(n)+g(n)) \leq g(n) \leq c_2(f(n)+g(n))$ for all $n \geq n_0$.
Since $g(n) > cf(n)$ for any constant $c > 0 \Rightarrow g(n) > f(n)$,

set $c_1 = \frac{1}{2}, c_2 = 1$,
$\frac{1}{2}(f(n)+g(n)) \leq g(n) \Rightarrow f(n)+g(n) = O(g(n))$,
$f(n)+g(n) \geq g(n) \Rightarrow f(n)+g(n) = \Omega(g(n))$,
$\Rightarrow f(n)+g(n) = \Theta(g(n))$.

8. $2^{\frac{\log n}{2}} = \Theta(n)$
Answer:
$2^{\frac{\log n}{2}} = \Theta(n) \Leftrightarrow$ there exist positive constants c_1, c_2, and n_0 such that $0 \leq c_1 n \leq 2^{\frac{\log n}{2}} \leq c_2 n$ for all $n \geq n_0 \Leftrightarrow c_1 \leq \frac{2^{\frac{\log n}{2}}}{n} \leq c_2 \Leftrightarrow c_1 \leq \frac{1}{n^{\frac{1}{2}}} \leq c_2 \Leftrightarrow c_1 \leq \frac{1}{\sqrt{n}} \leq c_2$.
Since $\frac{1}{\sqrt{n}}$ is monotonous descending when $n > n_0$, $\lim_{n\to\infty}(\dfrac{1}{\sqrt{n}})' = 0$.
We can't find c_1 to satisfy the lower bound. So $2^{\frac{\log n}{2}} \neq \Theta(n)$.

2 Recurrences (35 pts)

Solve the following recurrences, giving your answer in Θ notation. For each of them you may assume $T(x) = 1$ for $x \leq 5$ (or if it makes the base case easier you may assume $T(x)$ is any other constant for $x \leq 5$). Justify your answer (formal proof not necessary, but recommended).

1. $T(n) = 3T(n-2)$
   Answer:
   $T(n) = 3T(n-2)$,
   $T(n-2) = 3T(n-4)$,
   $T(n-4) = 3T(n-6)$,
   ...
   When n is even,$T(4) = 3T(2)$ is the end;
   When n is odd, $T(3) = 3T(1)$ is the end.

   Obviously, there are $\frac{n}{2}$ recursive levels, and for each level with base 3 multiplication. So $T(n) = \Theta(3^{\frac{n}{2}})$.

2. $T(n) = n^{1/3}T(n^{2/3}) + n$
   Answer:
   $T(n) = n^{1/3}T(n^{2/3}) + n$, $T(n^{2/3}) = n^{\frac{1}{3}\times\frac{2}{3}}T(n^{(\frac{2}{3})^2}) + n^{\frac{2}{3}}$,
   So, $T(n) = n^{\frac{1}{3}}n^{\frac{1}{3}\times\frac{2}{3}}T(n^{(\frac{2}{3})^2}) + 2n$.

   $T(n^{(\frac{2}{3})^2}) = n^{(\frac{2}{3})^2\frac{1}{3}}T(n^{(\frac{2}{3})^3}) + n^{(\frac{2}{3})^2}$.
   $T(n) = n^{\frac{1}{3}}n^{\frac{1}{3}\times\frac{2}{3}}(n^{(\frac{2}{3})^2\frac{1}{3}}T(n^{(\frac{2}{3})^3}) + n^{(\frac{2}{3})^2}) + 2n$;
   ...
   $T(n) = n^{1-(\frac{2}{3})^k}T(n^{(\frac{2}{3})^k}) + kn$

   When $n^{(\frac{2}{3})^k} \leq 5, T(n^{(\frac{2}{3})^k}) = 1 \Rightarrow k \geq \log_{\frac{2}{3}}\log_n 5 \Rightarrow T(n) = \dfrac{n}{5} + (\log_{\frac{2}{3}}\log_n 5)n \Rightarrow T(n) = \Theta(\log_{\frac{2}{3}}\log_n 5)n$.

3. $T(n) = 8T(n/4) + n$
   Answer: based on case 3 of Master theorem, $a = 8, b = 4, c = 1, k = 1$,
   $8 > 4^1 \Rightarrow a > b^k \Rightarrow T(n) = \Theta(n^{\log_b a}) \Rightarrow T(n) = \Theta(n^{\log_4 8}) \Rightarrow T(n) = \Theta(n^{\frac{3}{2}})$.

4. $T(n) = T(n-3) + 5$
   Answer:
   $T(n) = T(n-3) + 5$,
   $T(n-3) = T(n-6) + 5$,
   $T(n-6) = T(n-9) + 5$,
   ... $T(4) = T(1) + 5$, $T(5) = T(2) + 5$,$T(6) = T(3) + 5$ can be there ends.
   $T(n) = 1 + \frac{n}{3} \times 5 \Rightarrow T(n) = \Theta(n)$.

5. $T(n) = 3T(n/3) + n\log_3 n$
   Proof. let assume that $n = 3^k$ for some integer k.

$$\begin{aligned} T(n) &= 3T(\frac{n}{3}) + n\log_3 n \\ &= 3(3T(\frac{n}{3^2}) + \frac{n}{3}\log_3\frac{n}{3}) + n\log_3 n \\ &= 3^2T(\frac{n}{3^2}) + n\log_3\frac{n}{3} + n\log_3 n \\ &= ... \end{aligned}$$

$$
\begin{aligned}
&= 3^k T(1) + n \sum_{i=1}^{k} \log_3 \frac{n}{3^{k-i}} \\
&= nT(1) + n \sum_{i=1}^{k} \log_3 3^i \\
&= nT(1) + n \sum_{i=1}^{k} i \\
&= nT(1) + n \frac{k(k+1)}{2}
\end{aligned}
$$

As we known, $k = \log_3 n$, we have $T(n) = \Theta(nk^2) = \Theta(n(\log_3^2 n))$.
When $n \neq 3k$. It will not affect our result when n does not have the form of 3k. we have $T(3k-1) \leq T(n) \leq T(3k)$, which still gives us $T(n) = \Theta(nlog_3^2 n)$.

3 Basic Proofs (25 pts)

1. Prove that $\sum_{k=1}^{2n} (-1)^{k+1} \frac{1}{k} = \sum_{k=n+1}^{2n} \frac{1}{k}$ for all $n \geq 1$. Hint use induction.
   Answer:
   For $n = 1$, $\sum_{k=1}^{2} (-1)^{k+1} \frac{1}{k} = \sum_{k=2}^{2} \frac{1}{k} \Rightarrow 1 - \frac{1}{2} = \frac{1}{2}$, true.
   For $n = 2$, $\sum_{1}^{2\times 2} (-1)^{k+1} \frac{1}{k} = \sum_{3}^{2\times 2} \frac{1}{k} \Rightarrow 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{1}{3} + \frac{1}{4}$, true.
   ...
   We suppose, $n = m$ satisfies the equation $\Rightarrow \sum_{1}^{2m} (-1)^{k+1} \frac{1}{k} = \sum_{m+1}^{2m} \frac{1}{k}$.

   When $n = m+1$, left $= \sum_{1}^{2m+2} (-1)^{k+1} \frac{1}{k}$, right $= \sum_{m+2}^{2m+2} \frac{1}{k}$.

   left' $= \sum_{1}^{2m+2} (-1)^{k+1} \frac{1}{k} - \sum_{1}^{2m} (-1)^{k+1} \frac{1}{k} = \frac{1}{2m+1} - \frac{1}{2m+2}$;
   right' $= \sum_{m+2}^{2m+2} \frac{1}{k} - \sum_{m+1}^{2m} \frac{1}{k} = \frac{1}{2m+2} + \frac{1}{2m+1} - \frac{1}{m+1} = \frac{1}{2m+1} - \frac{1}{2m+2} =$ left'.
   When n=m satisfies the equation, n=m+1 also satisfies.
   With this induction, $\sum_{k=1}^{2n} (-1)^{k+1} \frac{1}{k} = \sum_{k=n+1}^{2n} \frac{1}{k}$ for all $n \geq 1$.

2. There are 9 course assistants for this class. Let us assume that 92 students submit their assignments for this problem set, and each submission is graded by one course assistant. Prove that there is some course assistant who grades at least 11 submissions.
   Answer:
   I would like to use contradiction. Suppose that there is at most 10 submissions per 1 course assistant. Because there are 9 course assistants for this class, the maximum of the assignments that can be reviewed is $9 \times 10 = 90$, which is smaller than 92. Because there are 92 students who submit their assignments, all of the submissions are graded,so it cannot be at most 10

submissions per 1 course assistant. So, there is some course assistant who grades at least 11 submissions.

3. Let $x_1, x_2, \dots x_n$ be real numbers. Prove that for any $1 \leq k \leq n$, $\sum_{i=k}^{n} x_i \leq n \cdot \max_{i=1}^{n}\{x_i\} - \sum_{j=1}^{k-1} x_j$

   Answer:
   $\sum_{i=k}^{n} x_i + \sum_{j=1}^{k-1} x_j = \sum_{i=1}^{n} x_i \leq n \cdot \max_{i=1}^{n}\{x_i\} \Rightarrow \sum_{i=k}^{n} x_i \leq n \cdot \max_{i=1}^{n}\{x_i\} - \sum_{j=1}^{k-1} x_j$.