ECE 250 Homework 4 Solutions

Prob. 1 output spectral density $= S_Y(\omega)$

$$S_Y(\omega) = |H(i\omega)|^2 S_X(\omega)$$

From differential equation

$$H(i\omega) = \frac{2(i\omega)+2}{(i\omega)^2+2(i\omega)+1} = \frac{2}{1+i\omega}$$

$$\therefore \quad |H(i\omega)|^2 = \frac{4}{1+\omega^2}$$

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-i\omega\tau} d\tau$$

$$= \frac{3}{4}\int_{-\infty}^{\infty} e^{-2|\tau|} e^{-i\omega\tau}$$

from transform pairs

$$= \frac{3}{4+\omega^2}$$

Partial Fraction

$$\therefore S_Y(\omega) = \frac{12}{(4+\omega^2)(1+\omega^2)} = \frac{4}{1+\omega^2} - \frac{4}{4+\omega^2}$$

$$\frac{4}{1+\omega^2} \leftrightarrow 2e^{-|\tau|} \qquad \frac{4}{4+\omega^2} \leftrightarrow e^{-2|\tau|}$$

transform pairs

$$\therefore \boxed{R_Y(\tau) = 2e^{-|\tau|} - e^{-2|\tau|}}$$

$\boxed{\text{Prob. 2}}$

$$Y(t) = X(t) + \int_{-\infty}^{\infty} h(t-\alpha) X(\alpha)\, d\alpha$$

$$= \int_{-\infty}^{\infty} \delta(t-\alpha) X(\alpha)\, d\alpha + \int_{-\infty}^{\infty} h(t-\alpha) X(\alpha)\, d\alpha$$

$$= \int_{-\infty}^{\infty} \hat{h}(t-\alpha) X(\alpha)\, d\alpha$$

where $\hat{h}(t) = \delta(t) + h(t) = \delta(t) + e^{-t} u(t)$ unit step

Find spectral density of $Y(t)$

$$S_Y(w) = |\hat{H}(iw)|^2 S_X(w)$$

$$\hat{H}(iw) = \int_{-\infty}^{\infty} \hat{h}(t) e^{-iwt}\, dt = 1 + \frac{1}{1+iw} = \frac{2+iw}{1+iw}$$

$$\therefore \quad |H(iw)|^2 = \frac{4+w^2}{1+w^2}$$

$$S_X(w) = \frac{4}{4+w^2} \quad \text{transform pairs}$$

$$\therefore \quad S_Y(w) = \frac{4+w^2}{1+w^2} \cdot \frac{4}{4+w^2} = 2\frac{2}{1+w^2}$$

$$\boxed{R_Y(\tau) = 2e^{-|\tau|}} \quad \text{Transform pairs}$$

$\boxed{\text{Prob. 3}}$ Consider power spectral density

$$\hat{S}(w) = \int_{-\infty}^{\infty} \hat{R}(\tau) e^{-iw\tau} d\tau$$

$$= \int_{-\infty}^{\infty} R(\tau - T_0) e^{-iw\tau} d\tau + \int_{-\infty}^{\infty} R(\tau) e^{-iw\tau} d\tau + \int_{-\infty}^{\infty} R(\tau + T_0) e^{-iw\tau} d\tau$$

denote by $S(w)$ the spectral density of $R(\tau)$

$$S(w) = \int_{-\infty}^{\infty} R(\tau) e^{-iw\tau}$$

then

$$\hat{S}(w) = \left(e^{-iwT_0} + 1 + e^{iwT_0}\right) S(w)$$

$$= (1 + 2\cos wT_0) S(w)$$

but

$$S(w) = \begin{cases} 1, & -\pi \le w \le \pi \\ 0, & \text{otherwise} \end{cases}$$

and finally

$$\hat{S}(w) = \begin{cases} (1 + 2\cos wT_0), & -\pi \le w \le \pi \\ 0, & \text{otherwise} \end{cases}$$

Now if w is such that $\hat{S}(w) < 0$ then $\hat{R}(\tau)$ cannot be a valid correlation function

If $\frac{3\pi}{4T_0} \le w \le \frac{5\pi}{4T_0}$

$\hat{S}(w) < 0$ and $\boxed{\hat{R}(\tau) \text{ can } \underline{\text{NOT}} \text{ be a valid correlation function}}$

$\boxed{\text{Prob. 4}}$ $\quad X(t) = X(0)(-1)^{N(t)}, \quad t \geq 0$

$s \leq t \quad P(X(t) = 1 \mid X(s) = 1) = P(N(t) - N(s) = \text{even})$

From HW Prob. $\quad = \frac{1}{2}\left(1 + e^{-2\lambda(t-s)}\right)$

Similarly $\quad P(X(t) = -1 \mid X(s) = 1) = P(N(t) - N(s) = \text{odd})$

Also From HW Prob. $\quad = \frac{1}{2}\left(1 - e^{-2\lambda(t-s)}\right)$

Now $\quad E[X(t) \mid X(s) = 1] = (1)\, P(X(t) = 1 \mid X(s) = 1) + (-1)\, P(X(t) = -1 \mid X(s) = 1)$

$= (1)\, P(N(t) - N(s) = \text{even}) + (-1)\, P(N(t) - N(s) = \text{odd})$

$$\boxed{E[X(t) \mid X(s) = 1] = e^{-2\lambda(t-s)}}$$

$\boxed{\text{Prob. 5}}$ Note: If $X(t) = 1$

$$\cos(\omega_0 t + \phi - \tfrac{\pi}{2}) = \sin(\omega_0 t + \phi)$$

If $X(t) = -1$

$$\cos(\omega_0 t + \phi - \tfrac{\pi}{2}) = -\sin(\omega_0 t + \phi)$$

$$\therefore \quad Y(t) = X(t)\sin(\omega_0 t + \phi)$$

$$E[Y(t)Y(s)] = E[X(t)X(s)] \cdot E_\phi[\sin(\omega_0 t + \phi)\sin(\omega_0 s + \phi)]$$

$$E[X(t)X(s)] = E\left[X^2(0)(-1)^{N(t)+N(s)}\right]$$

$X^2(0) = 1$

$(-1)^{N(s)} = (-1)^{-N(s)}$

$$\longrightarrow = E\left[(-1)^{N(t)-N(s)}\right]$$

$$= E\left[e^{i\pi(N(t)-N(s))}\right]$$

$$\boxed{E[X(t)X(s)] = e^{-2\lambda|t-s|}}$$

$$E_\phi[\sin(\omega_0 t + \phi)\sin(\omega_0 s + \phi)] = E_\phi\left[\tfrac{1}{2}\cos\omega_0(t-s)\right] - E_\phi\left[\tfrac{1}{2}\cos[\omega_0(t+s)+2\phi]\right]$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \tfrac{1}{2}\cos\omega_0(t-s)\, d\phi - \frac{1}{2\pi}\int_0^{2\pi} \tfrac{1}{2}\cos(\omega_0 t + \omega_0 s + 2\phi)\, d\phi$$

$$\boxed{E_\phi[\sin(\omega_0 t + \phi)\sin(\omega_0 s + \phi)] = \tfrac{1}{2}\cos\omega_0(t-s)}$$

$\boxed{\text{Prob. 5 Cont.}}$ setting $\tau = t - s$

$$\boxed{R_Y(\tau) = \frac{1}{2}\cos\omega_0\tau\, e^{-2\lambda|\tau|}}$$

Clearly $Y(t)$ is w.s.s.

$$S_Y(\omega) = \int_{-\infty}^{\infty} R_Y(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} e^{-2\lambda|\tau|}\, e^{-i(\omega-\omega_0)\tau}\, d\tau$$

$$+ \frac{1}{4}\int_{-\infty}^{\infty} e^{-2\lambda|\tau|}\, e^{-i(\omega+\omega_0)\tau}\, d\tau$$

Define

$$S_X(\omega) = \int_{-\infty}^{\infty} E[X(t+\tau)X(t)]\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2\lambda|\tau|}\, e^{-i\omega\tau}\, d\tau$$

$$\boxed{S_X(\omega) = \frac{4\lambda}{4\lambda^2+\omega^2}}$$

Finally

$$\boxed{S_Y(\omega) = \frac{1}{4}\left\{S_X(\omega-\omega_0) + S_X(\omega+\omega_0)\right\}}$$

$\boxed{\text{Prob. 6}}$ Using the orthogonality principle we must find the value of λ that satisfies

$$0 = E\left[\{X(t_0) - \lambda X(t)\} X(t)\right]$$

or
$$\lambda = \frac{E[X(t_0)X(t)]}{E[X^2(t)]} \qquad t_0 > t > 0$$

$$E[X(t)X(s)] = \int_{-\infty}^{\infty} u(z-t)\,u(z-s)\,f_Z(z)\,dz$$

$z - t \geq 0$ and $z - s \geq 0$

$$= \int_{\max[s,t]}^{\infty} f_Z(z)\,dz = 1 - \int_{-\infty}^{\max[s,t]} f_Z(z)$$

$$= 1 - F_Z(\max[s,t])$$

$$\therefore \quad \lambda = \frac{1 - F_Z(t_0)}{1 - F_Z(t)} = \frac{1 - \{1 - e^{-t_0}\}}{1 - \{1 - e^{-t}\}}$$

$$\therefore \quad \boxed{\lambda = e^{-(t_0 - t)}}$$