ECE 250 Midterm **SOLUTION**
November 7, 2014

1. Let $X_k, k=1,2,...,$ be independent and identically-distributed with common density

$$f_X(x)=\begin{cases} e^{-x}, & x \geq 0 \\ 0, & x<0 \end{cases} \qquad E[X_k] = \int_0^\infty x e^{-x} dx = 1$$

Consider the sum $Z_n = \sum_{k=1}^{n} X_k$. Prove that for any $\varepsilon > 0$, $\lim_{n\to\infty} P(|\frac{Z_n}{n}-1| \geq \varepsilon)=0$.

An answer not supported by appropriate reasoning will not receive any credit.

$$E[Z_n] = E\left[\sum_{k=1}^{n} X_k\right] = \sum_{k=1}^{n} E[X_k] = n$$

$$\therefore \quad E\left[\frac{Z_n}{n}-1\right] = 0 \qquad Var\left[\left(\frac{Z_n}{n}-1\right)\right] = Var\left[\frac{1}{n}\sum_{k=1}^{n}(X_k-1)\right]$$

$$= \frac{1}{n^2}\sum_{k=1}^{n} Var[X_k-1]$$

$$= \frac{1}{n} Var[X_k-1]$$

$$= \frac{1}{n} E[(X_k-1)^2]$$

Chebyshov Inequality

$$P\left(\left|\frac{Z_n}{n}-1\right| \geq \epsilon\right) \leq \frac{Var\left[\frac{Z_n}{n}-1\right]}{\epsilon^2} = \frac{E[(X_k-1)^2]}{n\epsilon^2} \xrightarrow[n\to\infty]{} 0$$

The terms $E[(X_k-1)^2]$ are identical for each k and need not be evaluated

However $E[(X_k-1)^2] = 1$

this establishes the desired result

SOLUTION

2. The random variables X and Y are independent with probabilities $P(X = n) = (\frac{1}{2})^{n+1}$ and $P(Y = m) = (\frac{1}{2})^{m+1}$, m, n=0, 1… Evaluate the probability P(X = Y).

An answer not supported by appropriate reasoning will not receive any credit.

$$P(X = Y) = \sum_{n=0}^{\infty} P(X = n, Y = n)$$

$$= \sum_{n=0}^{\infty} P(X = n)\, P(Y = n)$$

X and Y are independent

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} \left(\frac{1}{2}\right)^{n+1}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n+2} = \left(\frac{1}{2}\right)^{2} \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n}$$

$$= \frac{1}{3}$$

"useful formulas"

P(X = Y) = $\frac{1}{3}$

SOLUTION

3. The random variables X and Y are jointly Gaussian with E(X) = E(Y) = 0, Var(X) = σ_X^2, Var(Y) = σ_Y^2, and correlation coefficient ρ. Evaluate P(X ≥ Y).

An answer not supported by appropriate reasoning will not receive any credit.

$P(X \geq Y) = P(X - Y \geq 0)$

Define $Z = X - Y$ and evaluate $P(Z \geq 0)$

a linear combination of jointly Gaussian variables is a Gaussian variable.

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\,\sigma_Z} e^{-\frac{(z-m_Z)^2}{2\sigma_Z^2}}$$

$$m_Z = E[X-Y] = E[X] - E[Y] = 0$$

$$\sigma_Z^2 = \mathrm{Var}[X-Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] - 2E[XY]$$

$$E[XY] = \sigma_X \sigma_Y \rho$$

$$\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 - 2\sigma_X\sigma_Y\rho$$

$$P(Z \geq 0) = \int_0^\infty f_Z(z)\,dz = \frac{1}{\sqrt{2\pi}\,\sigma_Z}\int_0^\infty e^{-\frac{z^2}{2\sigma_Z^2}}\,dz = \frac{1}{2}$$

It is also sufficient to observe that Z is a zero-mean Gaussian variable and is an even function of z. Therefore $P(Z \geq 0) = \frac{1}{2}$

P(X ≥ Y) = $\frac{1}{2}$

250 Final Exam
December 18, 2014

SOLUTION

1. The random variable X has the density

$$f_X(x) = \begin{cases} \alpha e^{-\alpha x} & x \geq 0 \\ 0, & x < 0. \end{cases}$$

A new random variable, Y, is given by Y = e^{-X}.

Obtain an expression for the n-th moment of Y.

An answer not supported by appropriate reasoning will not receive any credit.

$$E[Y^n] = E[e^{-nX}] = \int_0^\infty e^{-nx} \alpha e^{-\alpha x} dx$$

$$\beta = (\alpha+n)x \qquad = \int_0^\infty e^{-(\alpha+n)x} \alpha \, dx$$

$$= \frac{\alpha}{\alpha+n} \int_0^\infty e^{-\beta} d\beta$$

$$= \frac{\alpha}{\alpha+n}$$

$$E[Y^n] = \frac{\alpha}{\alpha+n}$$

SOLUTION

2. Consider the independent, binomial random variables X and Y with probabilities

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k},\ k = 0, 1, \ldots, n.$$

$$P(Y = r) = \binom{m}{r} p^r (1-p)^{m-r},\ r = 0, 1, \ldots, m$$

With $0 < p < 1$ and n and m different, positive integers. Evaluate the probability $P(X + Y = q)$. Clearly indicate the values that q may have. [HINT: You may find the characteristic function useful.]

An answer not supported by appropriate reasoning will not receive any credit.

$$\Phi_X(u) = E[e^{iuX}] = \sum_{k=0}^{n} \binom{n}{k} e^{iuk} p^k (1-p)^{n-k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} (pe^{iu})^k (1-p)^{n-k}$$

(*) (*) (*) $\Phi_X(u) = [1 - p(1-e^{iu})]^n$

Similarly $\longrightarrow \Phi_Y(u) = [1 - p(1-e^{iu})]^m$

Since X and Y are independent

$$\Phi_{X+Y}(u) = \Phi_X(u)\Phi_Y(u) = [1 - p(1-e^{iu})]^{n+m}$$

Now using the transform pair (*) we conclude

$$P(X+Y=q) = \binom{n+m}{q} p^q (1-p)^{n+m-q}$$
$$q = 0, 1, \ldots, n+m$$

$$\boxed{P(X+Y=q) = \binom{n+m}{q} p^q (1-p)^{n+m-q},\ q = 0, 1, \ldots, n+m}$$

SOLUTION

3. The two classic Poisson processes $N_1(t)$ and $N_2(t)$ are independent and have a common rate λ. Let $M(t) = N_1(t) - N_2(t)$. Obtain a closed form expression for $P(M(t) = m)$.

An answer not supported by appropriate reasoning will not receive any credit.

$$\Phi_{M(t)}(u) = E\left[e^{iuM(t)}\right] = E\left[e^{iuN_1(t)} e^{-iuN_2(t)}\right]$$

(independence of $N_1(t)$ and $N_2(t)$)

$$= E\left[e^{iuN_1(t)}\right] E\left[e^{-iuN_2(t)}\right]$$

$$= e^{\lambda t(e^{iu}-1)} e^{\lambda t(e^{-iu}-1)}$$

$$= e^{2\lambda t(\cos u - 1)}$$

This has the same form as problem 4 on problem set 1.

$$\therefore \quad P(M(t) = m) = \frac{1}{2\pi}\int_0^{2\pi} \Phi_{M(t)}(u)\, e^{-ium}\, du$$

$$= e^{-2\lambda t} \frac{1}{2\pi}\int_0^{2\pi} e^{2\lambda t \cos u}\, e^{-ium}\, du$$

(from solution to problem 4 on problem set 1)

$$= I_m(2\lambda t)\, e^{-2\lambda t}$$

$P(M(t) = m) = I_m(2\lambda t)\, e^{-2\lambda t}$

SOLUTION

This problem was suggested in class

4. Let X(t) be a possibly non-wide sense stationary process and let $\overline{R}_X(\tau)$ be the "averaged" correlation function of X(t). You may assume $E[|X(t)|^2] \leq M < \infty$. Prove that

$$|\overline{R}_X(\tau)| \leq \overline{R}_X(0)$$

An answer not supported by appropriate reasoning will not receive any credit.

$$|\overline{R}_X(\tau)| = \left| \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_X(t+\tau, t)\,dt \right| \leq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |R_X(t+\tau,t)|\,dt$$

Schwarz Inequality $\longrightarrow$ Now $|R_X(t+\tau,t)| \leq \left(R_X(t+\tau, t+\tau)\right)^{1/2}\left(R_X(t,t)\right)^{1/2}$

$$|\overline{R}_X(\tau)| \leq \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(R_X(t+\tau,t+\tau)\right)^{1/2}\left(R_X(t,t)\right)^{1/2} dt$$

Schwarz again for integrals $\longrightarrow$ $\leq \left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_X(t+\tau,t+\tau)\,dt\right)^{1/2} \underbrace{\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_X(t,t)\,dt\right)^{1/2}}_{(\overline{R}_X(0))^{1/2}}$

$\alpha = t+\tau$

$$\text{Now } \frac{1}{2T}\int_{-T}^{T} R_X(t+\tau,t+\tau)\,dt = \frac{1}{2T}\int_{-T+\tau}^{T+\tau} R_X(\alpha,\alpha)\,d\alpha$$

$$= \frac{1}{2T}\int_{-T}^{T} R_X(\alpha,\alpha)\,d\alpha - \frac{1}{2T}\int_{-T}^{-T+\tau} R_X(\alpha,\alpha)\,d\alpha + \frac{1}{2T}\int_{T}^{T+\tau} R_X(\alpha,\alpha)\,d\alpha$$

$$\text{Now } \left|\frac{1}{2T}\int_{-T}^{-T+\tau} R_X(\alpha,\alpha)\,d\alpha\right| \leq \frac{M}{2T}\int_{-T}^{-T+\tau} d\alpha = \frac{M\tau}{2T} \xrightarrow[T\to\infty]{} 0$$

$R_X(\alpha,\alpha) \leq M$

$$\text{Similarly } \left|\frac{1}{2T}\int_{T}^{T+\tau} R_X(\alpha,\alpha)\,d\alpha\right| \xrightarrow[T\to\infty]{} 0$$

$$\text{and } \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_X(t+\tau,t+\tau)\,dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_X(\alpha,\alpha)\,d\alpha = \overline{R}_X(0)$$

$\therefore$ the proposition is proved ∎

SOLUTION

5. Consider the shot process $X(t)$

$$X(t) = \sum_{t_k} h(t - t_k).$$

The t_ks are the event times of a classical Poisson process with rate λ and $h(t)$ is absolutely integrable on $(-\infty,\infty)$. Prove that $X(t)$ is strictly stationary.

An answer not supported by appropriate reasoning will not receive any credit.

Consider the characteristic function of $X(t_1), X(t_2), \ldots, X(t_n)$

$$(*) \quad \Phi_{X(t_1),\ldots,X(t_n)}(u_1,\ldots,u_n) = e^{\lambda \int_{-\infty}^{\infty} \left(e^{i\sum_{k=1}^{n} u_k h(t_k - \tau)} - 1 \right) d\tau}$$

from class notes

Now shift all times by the same amount T

$$\Phi_{X(t_1+T),\ldots,X(t_n+T)}(u_1,\ldots,u_n) = e^{\lambda \int_{-\infty}^{\infty} \left(e^{i\sum_{k=1}^{n} u_k h(t_k + T - \tau)} - 1 \right) d\tau}$$

change integration variable $\tau = \beta + T$, $d\tau = d\beta$

then

$$(**) \quad \Phi_{X(t_1+T),\ldots,X(t_n+T)}(u_1,\ldots,u_n) = e^{\lambda \int_{-\infty}^{\infty} \left(e^{i\sum_{k=1}^{n} u_k h(t_k - \beta)} - 1 \right) d\beta}$$

The characteristic functions $(*)$ and $(**)$ are identical, so then are there associated joint distributions (i.e. invariant under a time shift). This is the definition of strictly stationary ∎

SOLUTION

6. Consider the shot process

$$X(t) = \sum_{t_n} h(t - t_n)$$

The process is clearly wide sense stationary

where the t_ns are the event times of a Poisson process with constant rate λ. The function h(t) is given by

$$h(t) = \begin{cases} t^{-1/4}, & 0 \le t \le 1 \\ 0, & otherwise. \end{cases}$$

IsX(t) mean square differentiable?

An answer not supported by appropriate reasoning will not receive any credit.

$$R_X(\tau) = E[X(t+\tau)X(t)]$$

from class notes

$$= \lambda \int_{-\infty}^{\infty} h(t+\tau)h(t)dt + \left(\lambda \int_{-\infty}^{\infty} h(t)dt\right)^2$$

The process will be m.s. differentiable iff $R''(0)$ is well defined. Putting in the given h(t)

interchange of $\frac{d^2}{d\tau^2}$ and integral is valid

$$R_X''(\tau) = \lambda \int_{-\infty}^{\infty} h''(t+\tau)h(t)dt$$

General expression

$$R_X''(0) = \lambda \int_{-\infty}^{\infty} h''(t)h(t)dt = \lambda \int_0^1 \left(\frac{5}{16}t^{-9/4}\right)t^{-1/4}dt$$

$$= \lambda \frac{5}{16} \int_0^1 t^{-5/2} dt$$

$$= \lambda \frac{5}{16} t^{-3/2} \Big|_0^1$$

not defined at $t=0$, so $X(t)$ is NOT m.s. differentiable

Differentiable? (Circle One)	Yes	No

ECE 250 Midterm
February 12, 2014

SOLUTION

1. The independent random variables X_1 and X_2 have the densities

$$f_{X_1}(x) = \begin{cases} \lambda_1 e^{-\lambda_1 x}, & x \geq 0 \\ 0 \quad , & x < 0 \end{cases} \qquad f_{X_2}(x) = \begin{cases} \lambda_2 e^{-\lambda_2 x}, & x \geq 0 \\ 0 \quad , & x < 0 \end{cases}$$

with $\lambda_1 \neq \lambda_2$. Evaluate the probability $P(X_1 > X_2)$.

An answer not supported by appropriate reasoning will not receive any credit.

$$P(X_1 > X_2) = \iint_{x_1 > x_2} f_{X_1}(x_1) f_{X_2}(x_2) \, dx_1 dx_2$$

X_1 and X_2 are independent

General expression

$$= \int_{-\infty}^{\infty} f_{X_1}(x_1) \, dx_1 \int_{-\infty}^{x_1} f_{X_2}(x_2) \, dx_2$$

$x_1 \geq 0$ and $x_2 \geq 0$

$$= \int_{0}^{\infty} f_{X_1}(x_1) \, dx_1 \int_{0}^{x_1} f_{X_2}(x_2) \, dx_2$$

$$= \int_{0}^{\infty} \lambda_1 e^{-\lambda_1 x_1} \, dx_1 \int_{0}^{x_1} \lambda_2 e^{-\lambda_2 x_2} \, dx_2$$

$$= 1 - \frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

$$\boxed{P(X_1 > X_2) = 1 - \frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{\lambda_2}{\lambda_1 + \lambda_2}}$$

SOLUTION

2. Let X_k, $k = 1,2,\ldots,$ be independent, identically-distributed random variables with common density

$$f_X(x) = \begin{cases} 2e^{-2x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

Consider the new random variable $Z_n = \prod_{k=1}^{n} X_k$. Show that, for $\varepsilon > 0$, $\lim_{n\to\infty} P(Z_n \ge \varepsilon) = 0$

An answer not supported by appropriate reasoning will not receive any credit.

clearly $Z_n \ge 0$ so that using the Markov inequality

$$P(Z_n \ge \epsilon) \le \frac{E[Z_n]}{\epsilon}$$

but $E[Z_n] = E[\prod_{k=1}^{n} X_k] = (E[X_k])^n$ ← X_ks are i.i.d.

$$E[X_k] = 2\int_0^\infty x e^{-2x}\,dx = \frac{1}{2}$$

$\therefore$ $E[Z_n] = \left(\frac{1}{2}\right)^n$

and $P(Z_n \ge \epsilon) \le \frac{1}{\epsilon}\left(\frac{1}{2}\right)^n \xrightarrow[n\to\infty]{} 0$

SOLUTION

3. The process Z(t) is given by $Z(t) = X(t)\cos\omega_0 t + Y(t)\sin\omega_0 t$ where X(t) and Y(t) are independent, zero-mean, Gaussian, W.S.S. processes with identical correlation functions R(τ). Is Z(t) wide sense stationary? Is it strictly stationary? clearly $E[Z(t)] = 0$

A proof not supported by appropriate reasoning will not receive any credit.

$$E[Z(t)Z(s)] = E\left[\left(X(t)\cos\omega_0 t + Y(t)\sin\omega_0 t\right)\left(X(s)\cos\omega_0 s + Y(s)\sin\omega_0 s\right)\right]$$

X(t) and Y(t) are indep. with zero mean

$$\rightarrow = E[X(t)X(s)]\cos\omega_0 t\cos\omega_0 s + E[Y(t)Y(s)]\sin\omega_0 t\sin\omega_0 s$$

but $E[X(t)X(s)] = E[Y(t)Y(s)] = R(t-s)$

and $\cos\omega_0 t\cos\omega_0 s + \sin\omega_0 t\sin\omega_0 s = \cos\omega_0(t-s)$

$\therefore E[Z(t)Z(s)] = R(t-s)\cos\omega_0(t-s)$

so that Z(t) is W.S.S.

Note that Z(t) is a linear combination of Gaussian processes and is, itself, Gaussian. For a Gaussian process, wide sense stationarity implies strict sense stationarity. (The joint density or characteristic function depends only on the correlation function.)

W.S.S.? (circle one)	Yes	No	Strictly Stationary? (circle one)	Yes	No

ECE 250 Final Exam. SOLUTION
March 17, 2014

1. Can the following function be the characteristic function of a real random variable?

$$\Phi(u) = \exp\left[e^{-u^2/2} - 1\right]$$

An answer not supported by appropriate reasoning will not receive any credit.

$$\Phi(u) = e^{-1} e^{e^{-u^2/2}} = e^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} e^{-nu^2/2}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(u) e^{-iux}\, du = e^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-nu^2/2} e^{-iux}\, du$$

Note: $e^{-nu^2/2}$ is the Ch.f. of a zero-mean Gaussian variable with variance $= n$

$$\therefore \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-nu^2/2} e^{-iux}\, du \equiv f_n(x) = \frac{1}{2\pi\sqrt{n}} e^{-\frac{x^2}{2n}}$$

So that $\frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(u) e^{-iux}\, du = e^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} f_n(x)$

(1) clearly $f_n(x) \geq 0$ and $\int_{-\infty}^{\infty} f_n(x)dx = 1$

so the

(2) $\int_{-\infty}^{\infty} e^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} f_n(x) dx = e^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} = 1$

Finally the inverse transfom of $\Phi(u)$ is non-negative and integrates to 1.

$\Phi(u)$ is the ch.f. of a real random variable

Valid Ch.F.? (circle one)	Yes	No

SOLUTION

2. Let the discrete-valued process N(t) be characterized by the probabilities

$$P(N(t)=n)=\frac{t^n}{(1+t)^{n+1}}, \quad n=0,1,\ldots \quad t\geq 0.$$

Define T_m as the time to m - th occurrence (event) with $T_o = 0$. Obtain an expression for the probability density of T_m.

An answer not supported by appropriate reasoning will not receive any credit.

$$P(T_m \leq t) = \sum_{k=m}^{\infty} P(N(t)=k)$$

$$= \sum_{k=m}^{\infty} \frac{t^k}{(1+t)^{k+1}}$$

$$f_{T_m}(t) = \frac{d}{dt}P(T_m\leq t) = \sum_{k=m}^{\infty} k\frac{t^{k-1}}{(1+t)^{k+1}} - \sum_{k=m}^{\infty}(k+1)\frac{t^k}{(1+t)^{k+2}}$$

$$k=l \qquad\qquad k+1=l$$

$$= \sum_{l=m}^{\infty} l\frac{t^{l-1}}{(1+t)^{l+1}} - \sum_{l=m+1}^{\infty} l\frac{t^{l-1}}{(1+t)^{l+1}}$$

$$m\frac{t^{m-1}}{(1+t)^{m+1}}$$

$f_{Tm}(t) =$ $m\dfrac{t^{m-1}}{(1+t)^{m+1}}$

SOLUTION

3. It is desired to estimate the rate, λ, of a classical Poisson process N(t). Prove that

$$\underset{T\to\infty}{l.i.m.}\ \frac{N(T)}{T} = \lambda.$$

An answer not supported by appropriate reasoning will not receive any credit.

to show $\underset{T\to\infty}{l.i.m.}\ \frac{N(T)}{T} = \lambda$ it is necessary to show

$$\lim_{T\to\infty} E\left[\left(\frac{N(T)}{T} - \lambda\right)^2\right] = 0$$

$$\boxed{\begin{aligned} E[N(T)] &= \lambda T \\ E[N^2(T)] &= \lambda T + (\lambda T)^2 \end{aligned}}$$

that is

$$\lim_{T\to\infty} \left\{ \frac{E[N^2(T)]}{T^2} - 2\frac{\lambda}{T}E[N(T)] + \lambda^2 \right\} = 0$$

$$\lim_{T\to\infty} \left\{ \frac{\lambda T + (\lambda T)^2}{T^2} - \frac{2\lambda(\lambda T)}{T} + \lambda^2 \right\} = 0$$

$$\lim_{T\to\infty} \left\{ \frac{\lambda}{T} \right\} = 0$$ end of proof

No credit given for simply observing

$$\frac{E[N(T)]}{T} = \lambda$$

SOLUTION

4. Let X(t) be the shot process

$$X(t) = \sum_{t_k} h(t - t_k)$$

where the t_ks are the "event" times of a classical Poisson process with constant rate λ, and

$$h(t) = \begin{cases} e^{-t} - 2e^{-2t}, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

Is X(t) mean square differentiable?

An answer not supported by appropriate reasoning will not receive any credit.

From class notes $X(t)$ is stationary and

$$R_X(\tau) = \lambda \int_{-\infty}^{\infty} h(t+\tau)h(t)dt + \left(\lambda \int_{-\infty}^{\infty} h(t)dt\right)^2$$

note $\int_{-\infty}^{\infty} h(t)dt = 0$

$$\therefore R_X(\tau) = \lambda \int_{-\infty}^{\infty} h(t+\tau)h(t)dt$$

and

$$S_X(\omega) = \lambda |H(i\omega)|^2 \quad \text{with } H(i\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt = \frac{-i\omega}{(1+i\omega)(2+i\omega)}$$

Now $X(t)$ is m.s. differentiable iff

$$\int_{-\infty}^{\infty} \omega^2 |H(i\omega)|^2 d\omega < \infty \qquad |H(i\omega)|^2 = \frac{\omega^2}{(1+\omega^2)(4+\omega^2)}$$

but

$$\int_{-\infty}^{\infty} \frac{\omega^4}{(1+\omega^2)(4+\omega^2)} d\omega \not< \infty$$

and $X(t)$ is NOT differentiable

Differentiable? (circle one)	Yes	No

SOLUTION

5. Consider the Brownian motion process X(t). It is desired to obtain an estimate of X(t) as a linear combination of three earlier estimates. That is

$$\hat{X}(t) = AX(t-t_1) + BX(t-t_2) + CX(t-t_3)$$

where $0 < t_1 < t_2 < t_3 < t$. Determine the values of A, B, C that minimize

$$\mathcal{E} = E\left[(X(t) - \hat{X}(t))^2 \right].$$

An answer not supported by appropriate reasoning will not receive any credit

Using independent increments and $E[(X(t)-X(s))^2] = k|t-s|$ we have $R_X(t,s) = k\min[t,s]$, $t,s \geq 0$

positive constant

The orthogonality principle states that

$E[(X(t)-\hat{X}(t))\hat{X}(t)] = 0$ for minimum $\mathcal{E}$

1) $E[(X(t)-\hat{X}(t))X(t-t_1)] = 0$

2) $E[(X(t)-\hat{X}(t))X(t-t_2)] = 0$

3) $E[(X(t)-\hat{X}(t))X(t-t_3)] = 0$

1) $E[(X(t)-AX(t-t_1)-BX(t-t_2)-CX(t-t_3))X(t-t_1)] = 0$

2) $E[(X(t)-AX(t-t_1)-BX(t-t_2)-CX(t-t_3))X(t-t_2)] = 0$

3) $E[X(t)-AX(t-t_1)-BX(t-t_2)-CX(t-t_3))X(t-t_3)] = 0$

1) $(t-t_1) - A(t-t_1) - B(t-t_2) - C(t-t_3) = 0$

2) $(t-t_2) - A(t-t_2) - B(t-t_2) - C(t-t_3) = 0$

3) $(t-t_3) - A(t-t_3) - B(t-t_3) - C(t-t_3) = 0$

Solution is $A = 1$, $B = 0$, $C = 0$

A = 1	B = 0	C = 0

ECE 250 Midterm Exam
November 6, 2013

SOLUTION

1. Consider the independent random variables X and Y with densities

$$f_X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & otherwise \end{cases} \qquad f_Y(y) = \begin{cases} -\ln y, & 0 \le y \le 1 \\ 0, & otherwise. \end{cases}$$

Evaluate the probability density of the product Z = X Y.

An answer not supported by appropriate reasoning will not receive any credit.

General Expression

X and Y independent

$$F_Z(z) = P(XY \le z) = \iint_{xy \le z} f_{X,Y}(x,y)\,dx\,dy = \int_{-\infty}^{\infty} f_Y(y)\,dy \int_{-\infty}^{z/y} f_X(x)\,dx$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz}\int_{-\infty}^{\infty} f_Y(y)\,dy \int_{-\infty}^{z/y} f_X(x)\,dx = \int_{-\infty}^{\infty} f_Y(y) f_X\left(\frac{z}{y}\right)\frac{dy}{y}$$

$0 \le y \le 1$, $0 \le x \le 1$ $\Downarrow$ $0 \le \frac{z}{y} \le 1$ $\Downarrow$ $y \ge z$

$$= \int_z^1 f_Y(y) f_X\left(\frac{z}{y}\right)\frac{dy}{y} = \int_z^1 -\ln y \,\frac{dy}{y} = \frac{1}{2}(\ln z)^2$$

$f_Z(z) = \frac{1}{2}(\ln z)^2$

SOLUTION

2. Can the following function be the characteristic function of a probability density?

$$\Phi(u) = \frac{1+iu}{1+4u^2}$$ Clearly $\Phi(0)=1$ and $|\Phi(u)| \le 1$

An answer not supported by appropriate reasoning will not receive any credit.

Find inverse transform

$$\Phi(u) = \frac{1+iu}{1+4u^2} = \frac{1}{8}\frac{1}{\frac{1}{2}+iu} + \frac{3}{8}\frac{1}{\frac{1}{2}-iu}$$

From Transform pairs: $\frac{1}{\frac{1}{2}+iu} \longrightarrow e^{x/2}u(-x)$

$$\frac{1}{\frac{1}{2}-iu} \longrightarrow e^{-x/2}u(x)$$

$\therefore$ the density (inverse transform) of $\Phi(u)$ is

$$f(x) = \begin{cases} \frac{1}{8}e^{x/2}, & x<0 \\ \frac{3}{8}e^{-x/2}, & x \ge 0 \end{cases}$$ this is clearly non-negative and a valid density

Ch.F. (circle one)	Yes	No

ECE 250 Final Exam.
December 12, 2013

SOLUTION

1. Prove that if

$$Z = \prod_{k=1}^{n} X_k$$

where the $X_k s$ are i.i.d. with common density

$$f_X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & otherwise \end{cases},$$

then $f_Z(z) = \frac{1}{(n-1)!}(-\ln z)^{n-1}, \quad n = 2,3,\ldots \quad 0 \le z \le 1$

An answer not supported by appropriate reasoning will not receive any credit.

General Expression
$Z = X \cdot Y$ (independent)

(*) $f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y\left(\frac{z}{x}\right) \frac{dx}{x}$

proved in class

let $n = 2$

$$f_Z(z) = \int_z^1 f_X(x) f_X\left(\frac{z}{x}\right)\frac{dx}{x} = \int_z^1 \frac{dx}{x} = -\ln z$$

Also proved in class

let the proposition hold for $n \geq 2$ and show it holds for $n+1$

From (*) with $f_X(x) = \frac{1}{(n-1)!}(-\ln x)^{n-1}$ and $f_Y(y) = \begin{cases} 1, & 0 \le y \le 1 \\ 0, & other \end{cases}$

$\therefore$ for $n+1$ $\quad f_Z(z) = \int_z^1 \frac{1}{(n-1)!}(-\ln x)^{n-1}\frac{dx}{x}$

$$= \frac{-1}{(n-1)!}\int_z^1 (-\ln x)^{n-1}(d{-\ln x}) = \frac{1}{n!}(-\ln z)^n$$

end of proof

SOLUTION

2. A random process has the correlation function

$$R_x(t,s) = \sum_{n=1}^{\infty} \lambda_n \cos n\omega_0 t \cos n\omega_0 s \quad with\ \lambda_n = 4\left(\frac{\sin(n\pi/2)}{(n\pi/2)}\right)^2$$

Evaluate the average correlation function $\bar{R}_X(\tau)$.

An answer not supported by appropriate reasoning will not receive any credit.

$$\bar{R}_X(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_X(t+\tau, t)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sum_{n=1}^{\infty} \lambda_n \cos n\omega_0(t+\tau) \cos n\omega_0 t\, dt$$

$$= \lim_{T\to\infty}\left\{\frac{1}{2T}\sum_{n=1}^{\infty}\lambda_n \int_{-T}^{T}\frac{1}{2}\cos n\omega_0\tau\, dt + \frac{1}{2T}\sum_{n=1}^{\infty}\lambda_n \int_{-T}^{T}\frac{1}{2}\cos(2n\omega_0 t + n\omega_0\tau)\,dt\right\}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\lambda_n \cos n\omega_0\tau + \frac{1}{2}\sum_{n=1}^{\infty}\lambda_n \lim_{T\to\infty}\underbrace{\frac{1}{2T}\int_{-T}^{T}\cos(2n\omega_0 t + n\omega_0\tau)\,dt}_{\to 0}$$

$$\bar{R}_X(\tau) = \frac{1}{2}\sum_{n=1}^{\infty}\lambda_n \cos n\omega_0\tau$$

Not part of the problem but this periodic function = $\bar{R}_X(\tau)$

$$\bar{R}_X(\tau) = \frac{1}{2}\sum_{n=1}^{\infty}\lambda_n \cos n\omega_0\tau$$

SOLUTION

3. Let X be the compound Poisson random variable

$$X = \begin{cases} 0, & N=0 \\ \sum_{k=1}^{N} A_k, & N>0 \end{cases}$$

$$E[e^{sX}] = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\left(E[e^{sA_n}]\right)^n e^{-\lambda} = e^{\lambda\left(\frac{1}{1-s}-1\right)}, \quad s<1 \text{ for convergence}$$

where $P(N=n) = \frac{\lambda^n}{n!}e^{-\lambda}$, n = 0, 1, …, and the A_ks are independent, identically distributed and are, themselves, independent of N. The A_ks have common densities

$$f_A(\alpha) = \begin{cases} e^{-\alpha}, & \alpha \ge 0 \\ 0, & \alpha < 0. \end{cases}$$

Obtain an upper bound on the probability $P(X \ge \mu\lambda)$ that decreases at least exponentially as $\mu \to \infty$.

An answer not supported by appropriate reasoning will not receive any credit.

Chernoff Bound $P(X \ge \mu\lambda) \le e^{-s\mu\lambda} E[e^{sX}]$, $s \ge 0$

$$P(X \ge \mu\lambda) \le e^{\lambda\left(-\mu s + \frac{1}{1-s} - 1\right)}$$ choose $s \ge 0$ to minimize exponent

$$0 = \frac{d}{ds}\left(-\mu s + \frac{1}{1-s} - 1\right)\Big|_{s=s_0} = -\mu + \frac{1}{(1-s_0)^2};\ s_0 = 1 \mp \frac{1}{\sqrt{\mu}}$$

must use "−" sign for $s_0 < 1$

$$\therefore P(X \ge \mu\lambda) \le e^{\lambda(-\mu + 2\sqrt{\mu} - 1)} = e^{-\lambda(1-\sqrt{\mu})^2} = e^{-\mu\lambda\left(1-\frac{1}{\sqrt{\mu}}\right)^2}$$

Any of these bounds are acceptable

$$\boxed{P(X \ge \mu\lambda) \le e^{-\mu\lambda\left(1-\frac{1}{\sqrt{\mu}}\right)^2}}$$

SOLUTION

4. A random process, X(t), is defined by X(t) = u(t-T). The random variable T has the distribution $F_T(\tau)$ and

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

It is desired to estimate $X(t_3)$ using two prior observations, $X(t_1)$ and $X(t_2)$, to form the estimate

$$\hat{X}(t_3) = AX(t_1) + BX(t_2), \text{ with } \quad 0 < t_1 < t_2 < t_3.$$

Determine A and B that will minimize the mean square error

$$\varepsilon = E[(X(t_3) - \hat{X}(t_3))^2],$$

and obtain an expression for ε_{min}.

An answer not supported by appropriate reasoning will not receive any credit.

$$R_X(t,s) = E[X(t)X(s)] = \int_{-\infty}^{\infty} u(t-\tau)u(s-\tau)f_T(\tau)d\tau = F_T(\min[s,t])$$

Orthogonality Principle

$$0 = E[(X(t_3)-\hat{X}(t_3))X(t_1)] = F_T(t_1) - AF_T(t_1) - BF_T(t_1)$$

$$0 = E[(X(t_3)-\hat{X}(t_3))X(t_2)] = F_T(t_2) - AF_T(t_1) - BF_T(t_2)$$

Solution : $A = 0$, $B = 1$

$$\varepsilon_{min} = E[(X(t_3)-\hat{X}(t_3))X(t_3)] = E[(X(t_3)-X(t_2))X(t_3)]$$

$$= F_T(t_3) - F_T(t_2) = \int_{t_2}^{t_3} f_T(\tau)d\tau$$

A = 0	B = 1	$\varepsilon_{min} = F_T(t_3) - F_T(t_2)$

SOLUTION

5. A shot process is defined by

$$X(t) = \sum_{t_k} h(t - t_k)$$

where the t_ks are the event times of a classical Poisson process (constant rate λ). The function h(t) is given by

$$h(t) = \begin{cases} e^{-t} - 2e^{-2t}, & t \geq 0. \\ 0, & t < 0, \end{cases}$$

$$H(i\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt = \frac{-i\omega}{(1+i\omega)(2+i\omega)}$$

Is the process X(t) mean square differential?

An answer not supported by appropriate reasoning will not receive any credit.

$$\Phi_{X(t),X(s)}(u,v) = e^{\lambda \int_{-\infty}^{\infty} \left(e^{iu h(t-t') + iv h(s-t')} - 1 \right) dt'}$$

$$R_X(\tau) = E[X(t+\tau)X(t)] = \lambda \int_{-\infty}^{\infty} h(\tau+\alpha) h(\alpha) d\alpha + \lambda^2 \left(\int_{-\infty}^{\infty} h(\alpha) d\alpha \right)^2$$

$$\int_{-\infty}^{\infty} h(\alpha) d\alpha = 0$$

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-i\omega\tau} d\tau = \lambda |H(i\omega)|^2 = \lambda \frac{\omega^2}{(1+\omega^2)(4+\omega^2)}$$

If $\int_{-\infty}^{\infty} \omega^2 S_X(\omega) d\omega < \infty$ then $X(t)$ is differentiable

Now $\int_{-\infty}^{\infty} \omega^2 S_X(\omega) d\omega = \int_{-\infty}^{\infty} \frac{\omega^4}{(1+\omega^2)(4+\omega^2)} d\omega \not< \infty$ $\therefore$ Not Diff.

Differentiable? (circle one)	Yes	No

ECE 250 Midterm Exam.
February 15, 2013

Name: **SOLUTION**

1. The random variable X has the density

Chebyshev Inequality

$$f_X(x) = \begin{cases} \frac{x^n}{n!} e^{-x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

$$P(X \ge \lambda) \le \min_{s \ge 0} e^{-s\lambda} E[e^{sX}]$$

Find an upper bound on the probability $P(X \ge \lambda)$ that decreases exponentially as $\lambda \to \infty$ and state any restrictions on λ.

$$E[e^{sX}] = \Phi_X(-is) = \left(\frac{1}{1-s}\right)^{n+1}$$

From "Useful Formulas" – Integral II

$$\therefore\ P(X \ge \lambda) \le \min_{s \ge 0} e^{-s\lambda}\left(\frac{1}{1-s}\right)^{n+1} = \min_{s \ge 0} e^{-s\lambda - (n+1)\ln(1-s)}$$

let s_0 be the minimum so that

$$\frac{d}{ds}\Big[-s\lambda - (n+1)\ln(1-s)\Big]_{s=s_0} = -\lambda + (n+1)\frac{1}{1-s_0} \stackrel{\text{set}}{=} 0$$

$\therefore\ 1 - s_0 = \frac{n+1}{\lambda}$ equivalently $s_0 = 1 - \frac{n+1}{\lambda}$

For $s_0 \ge 0$ we must have $\frac{n+1}{\lambda} \le 1$ $(\lambda \ge (n+1))$

Finally $P(X \ge \lambda) \le e^{-\lambda\left(1 - \frac{n+1}{\lambda}\right) - (n+1)\ln\left(\frac{n+1}{\lambda}\right)}$

$$\le e^{-\lambda\left\{1 - \left(\frac{n+1}{\lambda}\right)\left[1 - \ln\left(\frac{n+1}{\lambda}\right)\right]\right\}}$$

Full credit for any form equivalent to these

Not required as part of the problem, but if $\left(\frac{n+1}{\lambda}\right) < 1$ then $\left(\frac{n+1}{\lambda}\right)\left[1 - \ln\left(\frac{n+1}{\lambda}\right)\right] < 1$

$$\boxed{P(X \ge \lambda) \le e^{-\lambda\left\{1 - \left(\frac{n+1}{\lambda}\right)\left[1 - \ln\left(\frac{n+1}{\lambda}\right)\right]\right\}}, \quad \lambda \ge n+1}$$

Name: **SOLUTION**

2. A Poisson process is observed over an interval of random length T. For T fixed at τ, the probability of k events in the interval is

$$P(N = k \mid T = \tau) = \frac{(\lambda\tau)^k}{k!}\, e^{-\lambda\tau},\ k = 0, 1, \ldots$$

The probability density of T is

$$f_T(\tau) = \begin{cases} \sigma e^{-\sigma\tau}, & \tau \geq 0 \\ 0, & \tau < 0. \end{cases}$$

Find the probability $P(N \geq m)$.

$$P(N=k) = \int_0^\infty P(N=k \mid T=\tau) f_T(\tau)\, d\tau$$

$$= \int_0^\infty \frac{(\lambda\tau)^k}{k!} e^{-\lambda\tau} \sigma e^{-\sigma\tau}\, d\tau$$

$$= \frac{\sigma}{(\lambda+\sigma)} \left(\frac{\lambda}{\lambda+\sigma}\right)^k$$

From "Useful Formulas" Equation II with slight modification

Now

$$P(N \geq m) = \sum_{k=m}^{\infty} P(N=k)$$

$$= \sum_{k=m}^{\infty} \frac{\sigma}{(\lambda+\sigma)} \left(\frac{\lambda}{\lambda+\sigma}\right)^k$$

From "Useful Formulas" summations

$$= \left(\frac{\lambda}{\lambda+\sigma}\right)^m$$

$P(N \geq m) = \left(\frac{\lambda}{\lambda+\sigma}\right)^m$

Name: **SOLUTION**

3. A random process is defined by X(t) = u(t-T) where

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

and T is a continuous random variable with $E\left[|T|\right] < \infty$. The process is not wide sense stationary. Evaluate the averaged correlation function

$$\overline{R}_X(\tau) = \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} R_X(t+\tau, t)\,dt.$$

$$R_X(t+\tau,t) = E\left[X(t+\tau-T)\,X(t-T)\right] = \begin{cases} \int_{-\infty}^{t} f_T(t')\,dt' = F_T(t), & \tau \geq 0 \\ \int_{-\infty}^{t+\tau} f_T(t')\,dt' = F_T(t+\tau), & \tau < 0 \end{cases}$$

$\underline{\underline{\tau \geq 0}}$

$$\overline{R}_X(\tau) = \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} F_T(t)\,dt = \lim_{A\to\infty} \frac{1}{2A} \left\{ t F_T(t) \Big|_{t=-A}^{t=A} - \int_{-A}^{A} t f_T(t)\,dt \right\}$$

$$= \lim_{A\to\infty} \left\{ \frac{1}{2} F_T(A) - \frac{1}{2} F_T(-A) \right\} - \lim_{A\to\infty} \frac{1}{2A} \left\{ \int_{-A}^{A} t f_T(t)\,dt \right\}$$

$$= \frac{1}{2} F_T(\infty) - \frac{1}{2} F_T(-\infty) - \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} t f_T(t)\,dt$$

Note: $\left| \int_{-A}^{A} t f_T(t)\,dt \right| \leq \int_{-\infty}^{\infty} |t| f_T(t)\,dt = E[|T|]$

$\therefore$

$$\overline{R}_X(\tau) = \frac{1}{2}, \quad \tau \geq 0$$

Similarly $\overline{R}_X(\tau) = \frac{1}{2}, \quad \tau < 0$

$\overline{R}_X(\tau) = \frac{1}{2}$

ECE 250 Final Exam.

March 18, 2013

Name: **SOLUTION**

1. The random variables X_1, X_2, X_3, and X_4 are independent, identically distributed, Gaussian and have zero mean and unit variance. Evaluate the density of $Z = X_1X_2 + X_3X_4$. [HINT: You may find the characteristic function useful.]

Clearly X_1X_2 and X_3X_4 are independent and identically distributed

$$U \triangleq X_1X_2 \qquad V \triangleq X_3X_4$$

$$\Phi_U(u) = E[e^{iuX_1X_2}] = E_{X_2}[E[e^{iuX_1} | X_2 = x_2]]$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} f_{X_2}(x_2)\,dx_2 \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{iux_1x_2} f_{X_1}(x_1)\,dx_1}_{= \Phi_{X_1}(ux_2) = e^{-\frac{1}{2}u^2x_2^2}}$$

$$\Phi_U(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2x_2^2} e^{-\frac{1}{2}x_2^2}\,dx_2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(u^2+1)x_2^2}\,dx_2$$

$$= \frac{1}{\sqrt{1+u^2}}$$

Similarly

$$\Phi_V(v) = \frac{1}{\sqrt{1+v^2}}$$

$$\therefore \Phi_Z(u) = E[e^{iu(U+V)}] = E[e^{iuU}]E[e^{iuV}]$$

$$= \frac{1}{1+u^2}$$

From Fourier Transform Pairs $f_Z(z) = \frac{1}{2}e^{-|z|}$

$f_Z(z) = \frac{1}{2}e^{-|z|}$

Name: **SOLUTION**

2. The process X(t) is defined by $X(t) = \sum_{n=-\infty}^{\infty} a_n g(t - n\pi)$

In general, such processes are neither stationary nor wide sense stationary. For this problem, the a_ns are independent, identically distributed with zero mean and variance σ^2 and $g(t) = \sin t/t$. Evaluate the mean and correlation function of X(t). [HINT: You may find the following relationship helpful

$$\sum_{n=-\infty}^{\infty} g(t-nT_0)g(s-nT_0) = \frac{1}{2\pi T_0}\int_{-\infty}^{\infty} e^{i\omega(t-s)}G(i\omega)\sum_{n=-\infty}^{\infty} e^{in2\pi s/T_0}G(i\omega - in2\pi/T_0)d\omega$$

where $G(i\omega)$ is the Fourier transform of g(t).]

$$E[X(t)] = E\left[\sum_{n=-\infty}^{\infty} a_n g(t-n\pi)\right] = \sum_{n=-\infty}^{\infty} E[a_n]g(t-n\pi) = 0$$

$$E[X(t)X(s)] = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} E[a_n a_m]\, g(t-n\pi)g(s-m\pi)$$

$$= \sigma^2 \sum_{n=-\infty}^{\infty} g(t-n\pi)g(s-n\pi)$$

$$E[a_n a_m] = \begin{cases} \sigma^2, & n=m \\ 0, & n\neq m \end{cases}$$

Using hint (with $T_0 = \pi$)

$$E[X(t)X(s)] = \frac{\sigma^2}{2\pi^2}\int_{-\infty}^{\infty} e^{i\omega(t-s)}G(i\omega)\sum_{n=-\infty}^{\infty} e^{i2ns}G(i\omega - i2n)\,d\omega$$

From Transform Pairs $\longrightarrow G(i\omega) = \begin{cases} \pi, & -1 \le \omega \le 1 \\ 0, & \text{otherwise} \end{cases}$

Now $G(i\omega)\sum_{n=-\infty}^{\infty} e^{i2ns}G(i\omega - i2n) = G^2(i\omega)$

Because the product is zero for all summands except $n=0$

And
$$E[X(t)X(s)] = \frac{\sigma^2}{2\pi^2}\int_{-\infty}^{\infty} e^{i\omega(t-s)}G^2(i\omega)\,d\omega$$

$$= \frac{\sigma^2}{2}\int_{-1}^{1} e^{i\omega(t-s)}\,d\omega$$

$$= \sigma^2 \frac{\sin(t-s)}{(t-s)}$$

$E[X(t)] = 0$ $\qquad$ $E[X(t)X(s)] = \sigma^2 \frac{\sin(t-s)}{(t-s)}$

Name: **SOLUTION**

3. Let N(t) be a classical Poisson process (N (0) = 0, independent increments, constant rate λ). Define T_M as the time to the M-th event (occurrence). Determine the average and variance of T_M.

$$F_{T_M}(\tau) = P(T_M \le \tau) = \sum_{n=M}^{\infty} P(N(\tau)=n)$$

$$= \sum_{n=M}^{\infty} \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau}$$

Problem suggested in lecture

$$f_{T_M}(\tau) = \frac{d}{d\tau} F_{T_M}(\tau) = \lambda \frac{(\lambda\tau)^{M-1}}{(M-1)!} e^{-\lambda\tau}, \quad \tau \ge 0$$

$$\Phi_{T_M}(u) = \left(\frac{\lambda}{\lambda - iu}\right)^M$$

Useful Formulas relation II

$$E[T_M] = \left(\frac{1}{i}\right)\frac{d}{du}\Phi_{T_M}(u)\Big|_{u=0} = \frac{M}{\lambda}$$

$$E[T_M^2] = \left(\frac{1}{i}\right)^2\frac{d^2}{du^2}\Phi_{T_M}(u)\Big|_{u=0} = \frac{M(M+1)}{\lambda^2}$$

$$Var[T_M] = \frac{M}{\lambda^2}$$

$E[T_M] = \frac{M}{\lambda}$	$Var[T_M] = \frac{M}{\lambda^2}$

Name: **SOLUTION**

4. Let X(t) be a real, zero mean, wide sense stationary Gaussian process. It is desired to estimate X(t) using a single earlier observation $X(t-t_o)$. The estimate is

$\hat{X}(t) = AX(t-t_o) + BX^2(t-t_o)$ and is not necessarily a linear estimate.

Determine the values of A and B that minimize $\varepsilon = E[(X(t)-\hat{X}(t))^2]$ and evaluate the minimum value of ε. [HINT: If X_1, X_2, X_3 are jointly Gaussian with zero mean, $E(X_1X_2X_3) = 0$.]

$$E[X(t)X(s)] = R_X(t-s)$$

$$\varepsilon = E\left[\left(X(t) - AX(t-t_0) - BX^2(t-t_0)\right)^2\right]$$

$$= E[X^2(t)] + A^2E[X^2(t-t_0)] + B^2E[X^4(t-t_0)]$$
$$- 2AE[X(t)X(t-t_0)] - 2BE[X(t)X^2(t-t_0)] + 2ABE[X^3(t-t_0)]$$

$$E[X^4(t-t_0)] = 3\left(E[X^2(t-t_0)]\right)^2$$ From homework problem

$$E[X(t)X^2(t-t_0)] = E[X^3(t-t_0)] = 0$$ From above hint

$$\therefore \varepsilon = R_X(0) + A^2R_X(0) + 3B^2R_X^2(0) - 2AR_X(t_0)$$

$$\frac{\partial\varepsilon}{\partial A} = 2AR_X(0) - 2R_X(t_0) \overset{set}{=} 0$$

$$\frac{\partial\varepsilon}{\partial B} = 6BR_X^2(0) \overset{set}{=} 0$$

$$\therefore A = \frac{R_X(t_0)}{R_X(0)} \qquad B = 0$$

$$\varepsilon_{min} = \frac{1}{R_X(0)}\left\{R_X^2(0) - R_X^2(t_0)\right\}$$

$A = \dfrac{R_X(t_0)}{R_X(0)}$ B = 0 $\varepsilon_{min} = \dfrac{1}{R_X(0)}\left\{R_X^2(0) - R_X^2(t_0)\right\}$

Name: **SOLUTION**

5. A random process X(t) is defined by

$$X(t) = \sum_n A_n h(t - t_n)$$

where the delays, t_n, are event times of a classical Poisson process (constant rate λ), the $A_n s$ are independent, identically distributed random variables and are independent of the underlying Poisson process. The density of the $A_n s$ is

$$f_A(\alpha) = \begin{cases} 2e^{-2\alpha}, & \alpha \geq 0 \\ 0, & \alpha < 0 \end{cases}$$

$$E[A] = \frac{1}{2}$$
$$E[A^2] = \frac{1}{2}$$

and $h(t) = \begin{cases} 1-(1/2)|t|, & |t| \leq 2 \\ 0, & \textit{otherwise.} \end{cases}$

Is X(t) mean square differentiable? An answer not supported by appropriate reasoning will receive no credit.

From Class $\rightarrow$ $E[X(t)X(s)] = \lambda E[A^2]\int_{-\infty}^{\infty} h(t-s+\alpha)h(\alpha)d\alpha + \left(\lambda E[A]\int_{-\infty}^{\infty} h(\alpha)d\alpha\right)^2$

$$= \frac{\lambda}{2}\int_{-\infty}^{\infty} h(t-s+\alpha)h(\alpha)d\alpha + \left(\frac{\lambda}{2}\int_{-\infty}^{\infty} h(\alpha)d\alpha\right)^2$$

Also $E[X(t)] = \lambda\int_{-\infty}^{\infty} h(\alpha)d\alpha = 2\lambda$

$\therefore$ $X(t)$ is W.S.S.

Substituting τ for $t-s$ the power spectral density is

$$S_X(\omega) = \lambda|H(i\omega)|^2 + 8\pi\lambda^2\delta(\omega)$$

$$H(i\omega) = \int_{-\infty}^{\infty} h(\alpha)e^{-i\omega\alpha}d\alpha = 2\left(\frac{\sin\omega}{\omega}\right)^2$$

The process will be m-s. differentiable if

$$\int_{-\infty}^{\infty} \omega^2 S_X(\omega)d\omega < \infty$$

$$\int_{-\infty}^{\infty} \omega^2 S_X(\omega)d\omega = 4\lambda\int_{-\infty}^{\infty} \omega^2\left(\frac{\sin\omega}{\omega}\right)^4 d\omega + 8\pi\lambda^2\int_{-\infty}^{\infty} \omega^2\delta(\omega)d\omega$$

$$< \infty$$

Differentiable? (circle one)	Yes	No

ECE 250 Midterm Exam.
November 5, 2012

Name: **SOLUTION**

1. Is the following function a valid characteristic function of a real random variable?

$$\Phi(u) = 2\frac{2-u^2}{4+5u^2+u^4}$$

Clearly (i) $\Phi(0) = 1$
(ii) $|\Phi(u)| \le 1$

An answer not supported by appropriate reasoning will not receive any credit.

Make partial fraction expansion

$$\Phi(u) = 2\frac{2-u^2}{(4+u^2)(1+u^2)} = \frac{2}{1+u^2} - \frac{4}{4+u^2}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(u)\, e^{-iux}\, du$$

From Fourier Transform Pairs

$$= e^{-|x|} - e^{-2|x|}$$

$$= e^{-|x|}\left(1 - e^{-|x|}\right)$$

obviously $\int_{-\infty}^{\infty} f(x)\,dx = 1$

and $f(x) \ge 0$

$\therefore$ $f(x)$ is a valid density

Valid Characteristic Function? (circle one)	Yes	No

Name: **SOLUTION**

2. Let X_1 and X_2 be independent, identically-distributed, Gaussian random variables with zero mean and unit variance. Evaluate the probability density of $Z = X_1^2 + X_2^2$. [HINT: You may find the characteristic function a useful tool.]

$$f_{X_1}(x) = f_{X_2}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\Phi_Z(u) = E[e^{iuZ}] = E[e^{iu(X_1^2+X_2^2)}] = E[e^{iuX_1^2}]E[e^{iuX_2^2}]$$

$$E[e^{iuX_1^2}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iux^2} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(1-2iu)x^2} dx$$

$\alpha = (\sqrt{1-2iu})\cdot x$

$$= \frac{1}{\sqrt{1-2iu}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\alpha^2} d\alpha = \frac{1}{\sqrt{1-2iu}}$$

Similarly

$$E[e^{iuX_2^2}] = \frac{1}{\sqrt{1-2iu}}$$

$$\therefore \Phi_Z(u) = \frac{1}{1-2iu} = \frac{1}{2}\frac{1}{\frac{1}{2}-iu}$$

From Fourier Transform Pairs

$$f_Z(z) = \begin{cases} \frac{1}{2}e^{-z/2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$f_Z(z) = \begin{cases} \frac{1}{2}e^{-z/2}, & z \geq 0 \\ 0, & z < 0 \end{cases}$

Alternate Solution
(Probably More Straightforward)

Name: **SOLUTION**

2. Let X_1 and X_2 be independent, identically-distributed, Gaussian random variables with zero mean and unit variance. Evaluate the probability density of $Z = X_1^2 + X_2^2$. [HINT: You may find the characteristic function a useful tool.]

$$F_Z(z) = P(Z \le z) = P(X_1^2 + X_2^2 \le z)$$

Clearly $z \ge 0$

$$= \iint_{x_1^2+x_2^2 \le z} f_{X_1}(x_1) f_{X_2}(x_2)\, dx_1 dx_2$$

$$= \iint_{x_1^2+x_2^2 \le z} \frac{1}{2\pi} e^{-\frac{1}{2}(x_1^2+x_2^2)} dx_1 dx_2$$

$x_1 = r\sin\theta$

$x_2 = r\cos\theta$

$$F_Z(z) = \frac{1}{2\pi}\iint_{r^2 \le z} e^{-\frac{1}{2}r^2} r\,dr\,d\theta \qquad dx_1 dx_2 = r\,dr\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_{r^2 \le z} e^{-\frac{1}{2}r^2} r\,dr$$

$$= -\int_{r^2 \le z} d e^{-\frac{1}{2}r^2} = -\int_0^{\sqrt{z}} d e^{-\frac{1}{2}r^2}$$

$$= 1 - e^{-\frac{1}{2}z}$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{1}{2} e^{-\frac{1}{2}z}; \quad f_Z(z) = 0 \quad z < 0$$

$$f_Z(z) = \begin{cases} \frac{1}{2} e^{-z/2}, & z \ge 0 \\ 0, & z < 0 \end{cases}$$

Name: **SOLUTION**

3. The random variable Θ is uniformly distributed on $[0, 2\pi]$.

$$f_\Theta(\Theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \Theta \le 2\pi \\ 0, & \textit{otherwise.} \end{cases}$$

Consider the auxiliary random variables

$X = \cos\Theta \qquad Y = \sin\Theta.$

Show that X and Y are uncorrelated but not independent.

If X and Y are independent, then

$$E[X^n Y^m] = E[X^n]E[Y^m]$$

$n, m = 0, 1, 2, \ldots$

$$E[X] = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta \, d\theta = 0 \qquad E[Y] = \frac{1}{2\pi}\int_0^{2\pi} \sin\theta \, d\theta = 0$$

Useful Formulas

$$E[XY] = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta \sin\theta \, d\theta = \frac{1}{4\pi}\int_0^{2\pi} \sin 2\theta \, d\theta = 0$$

$$\therefore \boxed{E[XY] = E[X]\cdot E[Y] \Rightarrow \text{uncorrelated}}$$

Now Consider

$$E[X^2Y^2] = \frac{1}{2\pi}\int_0^{2\pi} \cos^2\theta \sin^2\theta \, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1}{2}\sin 2\theta\right)^2 d\theta$$

Useful Formulas

$$= \frac{1}{16\pi}\int_0^{2\pi} [1 - \cos 4\theta] \, d\theta = \frac{1}{8}$$

Also $E[X^2] = \frac{1}{2\pi}\int_0^{2\pi} \cos^2\theta \, d\theta = \frac{1}{4\pi}\int_0^{2\pi} [1 + \cos 2\theta] \, d\theta$

$$= \frac{1}{2}$$

$$E[Y^2] = \frac{1}{2\pi}\int_0^{2\pi} \sin^2\theta \, d\theta = \frac{1}{4\pi}\int_0^{2\pi} [1 - \cos 2\theta] \, d\theta$$

$$= \frac{1}{2}$$

$$\boxed{\therefore E[X^2Y^2] = \frac{1}{8} \neq E[X^2]E[Y^2] \quad \text{not independent}}$$

ECE 250 Final Exam.

December 13, 2012

Name: **SOLUTION**

PID: ______________________

1. The random variable X has a continuous density. The median of X is the value M that satisfies

$$F_X(M) = 1 - F_X(M).$$

Let X_o be the value that minimizes

$$E[|X - X_o|].$$

Show that $X_o = M$.

Set $\mathcal{E} = E[|X - X_o|]$

$$\mathcal{E} = -\int_{-\infty}^{X_o} (x - X_o) f_X(x)\,dx + \int_{X_o}^{\infty} (x - X_o) f_X(x)\,dx$$

$$0 \overset{\text{set}}{=} \frac{d\mathcal{E}}{dX_o} = -(x - X_o) f_X(x)\Big|_{x=X_o} + \int_{-\infty}^{X_o} f_X(x)\,dx$$

$$- (x - X_o) f_X(x)\Big|_{x=X_o} - \int_{X_o}^{\infty} f_X(x)\,dx$$

$$\therefore \quad 0 = F_X(X_o) - (1 - F_X(X_o))$$

clearly X_o satisfies the definition of a median so that

$$X_o = M$$

Name: **SOLUTION**

PID: ____________

2. A random process is defined by $X(t) = u(t-T)$ where

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

and T is a random variable with density

$$f_T(t') = \tfrac{1}{2} e^{-|t'|}.$$

$$F_T(t') = \begin{cases} \frac{1}{2} e^{t'}, & t < 0 \\ 1 - \frac{1}{2} e^{-t'}, & t \geq 0 \end{cases}$$

The process X(t) is not wide sense stationary. Evaluate the averaged correlation function

$$\overline{R}_X(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} R_X(t+\tau, t)\,dt.$$

$$R_X(t+\tau, t) = E[X(t+\tau-T)X(t-T)] = \int_{-\infty}^{\infty} u(t+\tau-t')u(t-t')f_T(t')dt'$$

$$= \int_{-\infty}^{t} f_T(t')dt' = F_T(t) \qquad \tau \geq 0$$

$$= \int_{-\infty}^{t+\tau} f_T(t')dt' = F(t+\tau), \qquad \tau < 0$$

$$= F_T(\min[t+\tau, t]) \qquad -\infty < \tau < \infty$$

Assume $\tau \geq 0$

$$\overline{R}_X(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_X(t+\tau, t)dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F_T(\min[t+\tau, t])dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F_T(t)dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{0} \frac{1}{2} e^{t} dt + \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{T} [1 - \tfrac{1}{2} e^{-t}]dt$$

$$= \lim_{T\to\infty} \frac{1-e^{-T}}{4T} + \lim_{T\to\infty} \frac{1}{2T} \left\{ T - \tfrac{1}{2}(1 - e^{-T}) \right\} = \frac{1}{2}$$

since $\overline{R}_X(-\tau) = R_X(\tau)$, it follows that

$$\overline{R}_X(\tau) = \frac{1}{2}$$

$$\boxed{\overline{R}_X(\tau) = \frac{1}{2}}$$

Name: **SOLUTION**

PID: ____________________

3. Can the function

$$m(t) = \begin{cases} 8t - 3 + 3e^{-t}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

From LaPlace Trans. pairs

$$M(s) = \frac{8}{s^2} - \frac{3}{s} + \frac{3}{s+1} = \frac{5s+8}{s^2(s+1)}$$

be the mean value of a renewal process N(t)? An answer not supported by appropriate reasoning will not receive any credit.

From notes, the transform of the interevent times is

$$F_\Delta(s) = \frac{sM(s)}{1+sM(s)} = \frac{5s+8}{s^2+6s+8} = \frac{5s+8}{(s+4)(s+2)}$$

$$= \frac{6}{s+4} - \frac{1}{s+2}$$

$$\therefore\ f_\Delta(t) = \begin{cases} 6e^{-4t} - e^{-2t}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

From LaPlace trans. pairs

$$= \begin{cases} e^{-2t}\left[6e^{-2t} - 1\right], & t \ge 0 \\ 0, & t < 0 \end{cases}$$

If t is sufficently large, $f_\Delta(t) < 0$ which cannot happen for a valid probability density

Mean of Renewal Process? (circle one)	Yes	No

Name: **SOLUTION**

PID: ____________________

4. Consider the random process

$$X(t) = \cos(\pi N(t))$$

Where N(t) is a classical Poisson process (N(0) = 0, constant rate λ). Evaluate the correlation function $R_X(t, s)$.

$$R_X(t,s) = E[X(t)X(s)] = E[\cos(\pi N(t))\cos(\pi N(s))]$$

let $s \le t$

$$\cos(\pi N(t))\cos(\pi N(s)) = \frac{1}{2}\cos(\pi(N(t)-N(s))) + \frac{1}{2}\cos(\pi(N(t)+N(s)))$$

Note that $N(t)+N(s) = N(t)-N(s)+2N(s)$

and $\cos(\pi(N(t)+N(s))) = \cos(\pi(N(t)-N(s))+2\pi N(s))$

$\cos(2\pi N(s)) = 1$
$\sin(2\pi N(s)) = 0$

$$= \cos(\pi(N(t)-N(s)))$$

$$\therefore R_X(t,s) = E[\cos(\pi(N(t)-N(s)))] = \frac{1}{2}E[e^{i\pi(N(t)-N(s))}] + \frac{1}{2}E[e^{i\pi(N(t)-N(s))}]$$

$$\text{Now } E[e^{\pm i\pi(N(t)-N(s))}] = \Phi_{N(t)-N(s)}(\pm\pi) = e^{-2\lambda(t-s)}$$

From class notes

$$\therefore R_X(t,s) = e^{-2\lambda(t-s)}, \quad s \le t$$

similarly $R_X(t,s) = e^{-2\lambda(s-t)}, \quad t < s$

$$\therefore R_X(t,s) = e^{-2\lambda|t-s|}$$

$R_X(t, s) =$ $e^{-2\lambda|t-s|}$

Name: **SOLUTION**

PID: ____________

5. A random signal X(t) has zero mean and correlation function $R_X(\tau) = e^{-|\tau|}$. During transmission, this signal is corrupted by additive noise N(t). The noise is a shot process

$$N(t) = \sum_{t_K} A_K h(t - t_K)$$

where the $t_K s$ are the "event" times of a classical Poisson process with a constant rate λ and

$$h(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

$E[A_k] = 0$

The $A_K s$ are i.i.d., independent of the $t_K s$ and have density

$E[A_k^2] = 2$

$$f_A(\alpha) = \frac{1}{2} e^{-|\alpha|}, \quad -\infty < \alpha < \infty.$$

The sum X(t) + N(t) is the input to a linear, time invariant filter with impulse response g(t) whose output $\hat{X}(t)$ is intended to be an estimate of X(t). Determine the impulse response $g_0(t)$ that will minimize the mean square error $\mathcal{E} = E[(X(t) - \hat{X}(t))^2]$.

We know (from class notes) that N(t) is stationary and

$$R_N(\tau) = \lambda E[A^2] \int_{-\infty}^{\infty} h(\tau+\beta)\, h(\beta)\, d\beta + \left(E[A] \int_{-\infty}^{\infty} h(\beta)\, d\beta\right)^2$$

$$= \lambda e^{-|\tau|}$$

From transform tables → Thus $S_N(\omega) = \dfrac{2\lambda}{1+\omega^2}$ and $S_X(\omega) = \dfrac{2}{1+\omega^2}$

Now from class the optimum (possibly unrealizable) transfer function is

$$G_o(i\omega) = \frac{S_X(\omega)}{S_X(\omega) + S_N(\omega)} = \frac{1}{1+\lambda}$$

From transform tables → $$g_o(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+\lambda} e^{i\omega t}\, d\omega = \frac{1}{1+\lambda} \delta(t)$$

$$\boxed{g_o(t) = \frac{1}{1+\lambda}\, \delta(t)}$$

ECE 250 Midterm
February 10, 2012

Name: **Solution**

1. The random variable Θ is uniformly distributed on $[0, 2\pi)$. Consider the related random variables

$$X = \cos^2\Theta \qquad Y = \sin^2\Theta$$

Are X and Y independent? An answer without appropriate reasoning will not receive any credit.

If X and Y are independent we <u>must</u> have $E[XY] = E[X]\cdot E[Y]$ (necessary not sufficient)

Trig. identities from handout

$$E[X] = E[\cos^2\theta] = E\left[\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right]$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right]d\theta = \frac{1}{2}$$

$$E[Y] = E[\sin^2\theta] = E\left[\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right]$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right]d\theta = \frac{1}{2}$$

$$E[XY] = E\left[(\sin\theta\cos\theta)^2\right] = E\left[\left(\tfrac{1}{2}\sin 2\theta\right)^2\right]$$
$$= \frac{1}{4}E[\sin^2 2\theta] = \frac{1}{4}E\left[\tfrac{1}{2} - \tfrac{1}{2}\cos 4\theta\right]$$
$$= \frac{1}{4}\frac{1}{2\pi}\int_0^{2\pi}\left[\tfrac{1}{2} - \tfrac{1}{2}\cos 4\theta\right]d\theta = \frac{1}{8}$$

$$\therefore E[XY] \neq E[X]E[Y]$$

NOT INDEDENDENT

Independent (circle one)	Yes	No

Name: **Solution**

2. The characteristic function of a real random variable X is given by

$$\Phi(u) = (\cos u)^{10}.$$

Determine those integer values for which the probability of X is <u>NOT</u> zero and evaluate those probabilities.

$$\Phi(u) = \left(\frac{e^{iu} + e^{-iu}}{2}\right)^{10} = \left(\frac{1}{2}\right)^{10} \sum_{k=0}^{10} \binom{10}{k} e^{iuk} e^{-iu(10-k)}$$

$$= \left(\frac{1}{2}\right)^{10} \sum_{k=0}^{10} \binom{10}{k} e^{iu(2k-10)}$$

Letting $n = 2k-10$, the probability of $X = n$ is the coefficient of $k = \frac{n}{2} + 5$

$$P(X=n) = \left(\frac{1}{2}\right)^{10} \binom{10}{\frac{n}{2}+5}$$

(i) noting that k is an integer, n must be even.

(ii) $k = 0, 1, \cdots, 10$ implies $n = -10, -8, \cdots, 0, 2, \cdots, 10$

so that

$$P(X=n) = \begin{cases} \left(\frac{1}{2}\right)^{10} \binom{10}{\frac{n}{2}+5}, & n = 0, \pm 2, \pm 4, \cdots, \pm 10. \\ 0, & \text{otherwise} \end{cases}$$

$$\boxed{P(X=n) = \begin{cases} \left(\frac{1}{2}\right)^{10} \binom{10}{n/2+5}, & n = 0, \pm 2, \pm 4, \cdots, \pm 10 \\ 0, & \text{otherwise} \end{cases}}$$

Name: **Solution**

3. A real, zero-mean, wide sense stationary Gaussian process X(t) has the power spectral density

$$S_X(\omega) = \frac{2}{1+\omega^2}.$$

$$E[Y(t)] = E[X^2(t)] = R_X(0) = \text{constant}$$

Define a new process $Y(t) = X^2(t)$. Evaluate the power spectral density of Y(t).

$$R_Y(\tau) = E[Y(t+\tau)Y(t)] = E[X^2(t+\tau)X^2(t)]$$

Use $E[X_1X_2X_3X_4] = E[X_1X_2]E[X_3X_4] + E[X_1X_3]E[X_2X_4] + E[X_1X_4]E[X_2X_3]$

with $X_1 = X_2 = X(t+\tau)$ and $X_3 = X_4 = X(t)$

So that

$$R_Y(\tau) = R_X^2(0) + 2R_X^2(\tau)$$

from handout

$$S_X(\omega) = \frac{2}{1+\omega^2} \leftrightarrow R_X(\tau) = e^{-|\tau|}$$

$\therefore$

$$R_Y(\tau) = 1 + 2e^{-2|\tau|}$$

and

$$S_Y(\omega) = \int_{-\infty}^{\infty}\left[1+2e^{-2|\tau|}\right]e^{-i\omega\tau}\,d\tau$$

$$= 2\pi\delta(\omega) + \frac{8}{4+\omega^2}$$

$S_Y(\omega) = \boxed{2\pi\delta(\omega) + \frac{8}{4+\omega^2}}$

ECE 250 Final
March 19, 2012

Name: **Solution**

1. Let N(t) be a classical Poisson process (N (0) = 0, constant rate λ). It is desired to estimate the rate by observing a realization of the process over a large time interval. Consider the estimate

$$M(T) = \frac{1}{T}\int_0^T \frac{N(t)}{t}\,dt.$$

Show that for $\epsilon > 0$

$$\lim_{T\to\infty} P(|M(T)-\lambda| \geq \epsilon) = 0$$

From class notes

$$E[N(t)] = \lambda t$$

$$E[N^2(t)] = \lambda t + \lambda^2 t^2$$

$$E[N(t)N(s)] = \lambda \min[s,t] + \lambda^2 ts$$

HINT: You may find the equivalent limits $\lim_{\epsilon\to 0} \epsilon \ln \epsilon = 0$ and $\lim_{T\to\infty} (\ln T)/T = 0$ useful.

$$P(|M(T)-\lambda| \geq \epsilon) \leq \frac{E[(M(T)-\lambda)^2]}{\epsilon^2}$$ (Chebyshev Inequality)

$$E[M(T)] = \frac{1}{T}\int_0^T \frac{1}{t} E[N(t)]\,dt = \lambda$$

$$\therefore \quad M(T)-\lambda = \frac{1}{T}\int_0^T (N(t)-\lambda t)\frac{dt}{t}$$

$$E[(M(T)-\lambda)^2] = \frac{1}{T^2}\int_0^T dt \int_0^T ds\, E[(N(t)-\lambda t)(N(s)-\lambda s)]\left(\frac{1}{ts}\right)$$

$$= \frac{\lambda}{T^2}\int_0^T dt \int_0^T \frac{\min[s,t]}{st}\,ds$$

$$= \frac{\lambda}{T^2}\int_0^T \frac{dt}{t}\int_0^t ds + \frac{\lambda}{T^2}\int_0^T dt \int_t^T \frac{ds}{s}$$

$$= \frac{\lambda}{T^2}\int_0^T dt + \frac{\lambda \ln T}{T^2}\int_0^T dt - \frac{\lambda}{T^2}\int_0^T \ln t\,dt$$

$$= 2\frac{\lambda}{T}$$

Since $\lim_{T\to\infty} E[(M(T)-\lambda)^2] = 0$ it follows that $\lim_{T\to\infty} P(|M(T)-\lambda| \geq \epsilon) = 0$

Name: **Solution**

2. The input, X(t), and output, Y(t), of a linear system are related via the differential equation

$$\frac{d^2}{dt^2}Y(t)+2\frac{d}{dt}Y(t)+Y(t)=2\frac{d}{dt}X(t)+2X(t). \quad H(i\omega)=\frac{2+2(i\omega)}{1+2(i\omega)+(i\omega)^2}$$

If the input is a zero-mean, wide sense stationary process with correlation

$$R_X(\tau)=\frac{3}{4}e^{-2|\tau|}, \qquad S_X(\omega)=\frac{3}{4+\omega^2} \leftarrow \text{From handout}$$

determine the correlation function of Y(t).

$$H(i\omega)=\frac{2}{(1+i\omega)} \qquad |H(i\omega)|^2=\frac{4}{1+\omega^2}$$

$$S_Y(\omega)=|H(i\omega)|^2 S_X(\omega)=\frac{12}{(1+\omega^2)(4+\omega^2)}$$

partial fraction $\rightarrow$ $$=\frac{4}{1+\omega^2}-\frac{4}{4+\omega^2}$$

$$\frac{4}{1+\omega^2} \updownarrow 2e^{-|\tau|} \qquad \frac{4}{4+\omega^2} \updownarrow e^{-2|\tau|} \leftarrow \text{From handout}$$

$$\therefore R_Y(\tau)=2e^{-|\tau|}-e^{-2|\tau|}$$

$$\boxed{R_Y(\tau)=2e^{-|\tau|}-e^{-2|\tau|}}$$

Name: **Solution**

3. The random process X(t) is given by X(t) = u (t – W). Here u(t) is the unit step

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

and W is a random variable with a continuous density and finite first and second moments. The process X(t) is not wide sense stationary. Evaluate the average correlation function $\bar{R}_X(\tau)$.

$$R_X(t+\tau, t) = E[X(t+\tau)X(t)] = E[u(t+\tau-W)\,u(t-W)]$$

$$= \int_{-\infty}^{\infty} u(t+\tau-w)\,u(t-w)\,f_W(w)\,dw$$

$$= \int_{-\infty}^{\min[t,t+\tau]} f_W(w)\,dw$$

$$= F_W(\min[t,t+\tau])$$

$$\bar{R}_X(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_X(t+\tau,t)\,dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} F_W(\min[t,t+\tau])\,dt$$

$\underline{\underline{\text{let } \tau \geq 0}}$ $\bar{R}_X(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} F_W(t)\,dt$

integrate by parts $\longrightarrow$ $= \lim_{T\to\infty} \frac{1}{2T}\left\{ T F_W(T) - (-T)F_W(-T) - \int_{-T}^{T} t f_W(t)\,dt \right\}$

E[W] finite $\longrightarrow$ $= \frac{1}{2}F_W(\infty) + \frac{1}{2}F_W(-\infty) - \lim_{T\to\infty}\frac{1}{2T}E[W]$

$= \frac{1}{2}$ (Similarly for $\tau < 0$)

$\boxed{\bar{R}_X(\tau) = \frac{1}{2}}$

Name: **Solution**

4. Consider the shot process

$$X(t) = \sum_{t_n} h(t-t_n) \qquad \Phi_{X(t),X(s)}(u,v) = e^{\lambda \int_{-\infty}^{\infty} \left(e^{iuh(t-\alpha)+ivh(s-\alpha)} - \right) d\alpha}$$

where the t_ns are occurrence times of a classical Poisson process defined on $(-\infty,\infty)$ with constant rate λ. Here h(t) is given by

$$h(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

Is X(t) mean square differentiable? An answer not supported by appropriate reasoning will not receive any credit.

$$R_X(\tau) = E[X(t+\tau)X(t)] = \left(\frac{1}{i}\right)^2 \frac{\partial^2}{\partial u \partial v} \Phi_{X(t+\tau),X(t)}(u,v) \Big|_{u=v=0}$$

$$= \lambda \int_{-\infty}^{\infty} h(\tau+\beta)h(\beta)\,d\beta + \lambda^2 \left(\int_{-\infty}^{\infty} h(\beta)\,d\beta \right)^2$$

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-i\omega\tau}\,d\tau = \lambda |H(i\omega)|^2 + 2\pi |H(i0)|^2 \lambda^2 \delta(\omega)$$

with $H(i\omega) = \int_{-\infty}^{\infty} h(\beta) e^{-i\omega\beta}\,d\beta$

$$H(i\omega) = \frac{1}{a+i\omega} \qquad \left(H(i0) = \frac{1}{a} \right)$$

$$\therefore S_X(\omega) = \lambda \frac{1}{a^2+\omega^2} + 2\pi \frac{\lambda^2}{a^2} \delta(\omega)$$

$X(t)$ will be m.s. differentiable iff

$$\int_{-\infty}^{\infty} \omega^2 S_X(\omega)\,d\omega < \infty \text{ but } \int_{-\infty}^{\infty} \lambda \frac{\omega^2}{a^2+\omega^2}\,d\omega \text{ not } < \infty$$

$\therefore$ $X(t)$ is NOT differentiable

Differentiable? (circle one)	Yes	No

Name: **Solution**

5. The process $X(t)$ is a real, zero-mean, wide sense stationary, Gaussian process with correlation $R(\tau)$. Define $Y(t) = X^3(t)$. It is desired to estimate $Y(t)$ with an earlier observation of $X(t)$. For a fixed value of $t_o > 0$, determine the value of α that minimizes the error

$$\varepsilon = E[(Y(t) - \alpha X(t-t_o))^2]$$

and evaluate the minimum value of ε. You may find the following relationship for a zero-mean Gaussian variable X useful

$$E[X^n] = \begin{cases} 0, & n \text{ odd} \\ 1\cdot 3\cdot 5\cdots(n-1)(E[X^2])^{n/2}, & n \text{ even.} \end{cases}$$

orthogonality principle

$$0 = E[(Y(t) - \alpha X(t-t_o))X(t-t_o)]$$

$$0 = E[Y(t)X(t-t_o)] - \alpha E[X^2(t-t_o)]$$

$$\therefore \alpha = \frac{E[Y(t)X(t-t_o)]}{E[X^2(t-t_o)]} = \frac{E[X^3(t)X(t-t_o)]}{E[X^2(t-t_o)]}$$

HW # 2 Prob. 1

$$E[X^3(t)X(t-t_o)] = 3R(0)R(t_o) \; ; \; E[X^2(t-t_o)] = R(0)$$

$$\therefore \boxed{\alpha = 3R(t_o)}$$

$$\varepsilon_{min} = E[(X^3(t) - 3R(t_o)X(t-t_o))^2]$$

$$= E[X^6(t)] - 6R(t_o)E[X^3(t)X(t-t_o)] + 9R^2(t_o)E[X^2(t-t_o)]$$

$$= 15R^3(0) - 18R(0)R^2(t_o) + 9R(0)R^2(t_o)$$

$$= 15R^3(0) - 9R(0)R^2(t_o)$$

$$= 3R(0)\{5R^2(0) - 3R^2(t_o)\}$$

$\alpha = 3R(t_o)$; $\varepsilon_{min} = 3R(0)\{5R^2(0) - 3R^2(t_o)\}$

ECE 250 Midterm
October 24, 2011

Name: **Solution**

1. The characteristic function of a random variable X is given by

$$\Phi_X(u) = 2e^{-|u|} - e^{-2|u|}.$$

$\Phi_X(u)$ is a valid characteristic function

Evaluate the first two moments of X.

$$\Phi_X(u) = 2\sum_{n=0}^{\infty}\frac{(-|u|)^n}{n!} - \sum_{n=0}^{\infty}\frac{(-2|u|)^n}{n!}$$

$$= 1 - u^2 + (\text{terms in } |u|^3 \text{ and higher powers of } |u|)$$

$\therefore$ $\Phi_X(u)$ has 1st and 2nd derivatives

$$E[X] = \frac{1}{i}\frac{d}{du}\Phi_X(u)\Big|_{u=0} = 0$$

$$E[X^2] = \left(\frac{1}{i}\right)^2\frac{d^2}{du^2}\Phi_X(x)\Big|_{u=0} = 2$$

While not required as part of the problem,

$$f_X(x) = \frac{6}{\pi}\frac{1}{(1+x^2)(4+x^2)}$$

E[X] = 0	E[X²] = 2

Name: **Solution**

2. Let the random variables X and Y be independent and have the densities

$$f_X(x) = \begin{cases} e^{-x}, & x \ge 0 \\ 0, & x < 0 \end{cases} \qquad f_Y(y) = \begin{cases} e^{-y}, & y \ge 0 \\ 0, & y < 0. \end{cases}$$

Determine the probability density of $Z = |X - Y|$. Consider $W = X - Y$

$$\Phi_W(u) = E[e^{iuW}] = E[e^{iuX} e^{-iuY}]$$

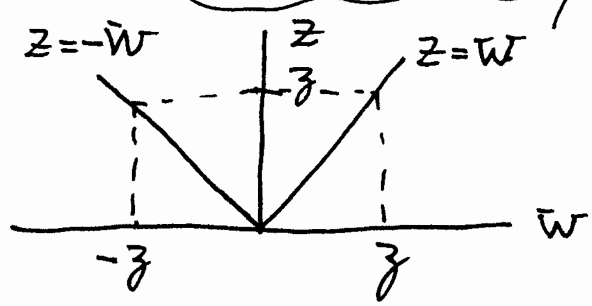
indep. $$= E[e^{iuX}]\,E[e^{-iuY}] = \frac{1}{1-iu} \cdot \frac{1}{1+iu}$$

$$= \frac{1}{1+u^2}$$

$$f_W(w) = \frac{1}{2} e^{-|w|}$$

From Transform table

Now $Z = |W|$



$$F_Z(z) = \int_{-z}^{z} \frac{1}{2} e^{-|w|}\,dw$$

$$= \int_0^z e^{-w}\,dw$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = e^{-z}, \quad z \ge 0$$

obviously

$$f_Z(z) = 0, \quad z < 0$$

$$f_Z(z) = \begin{cases} e^{-z}, & z \ge 0 \\ 0, & z < 0 \end{cases}$$

Name: **Solution**

3. The random variables $X_1, X_2, \ldots, X_{10}$ are independent and identically distributed with probabilities

$$P(X_k = 1) = P(X_k = -1) = ½, \qquad k = 1, 2, \ldots, 10.$$

Consider the sum

$$E[e^{iuX_k}] = \tfrac{1}{2}(e^{iu} + e^{-iu}) \qquad k = 1, 2, \cdots, n$$

$$Y_n = \sum_{k=1}^{10} X_k.$$

Evaluate the probabilities $P(Y_n = 6)$ and $P(Y_n = 7)$. [HINT: You may find the characteristic function helpful.]

$$\Phi_{Y_n}(u) = E[e^{iuY_n}] = E[e^{iu\sum_{k=1}^{10} X_k}] = E[\prod_{k=1}^{10} e^{iuX_k}]$$

X_ks are indep. $\longrightarrow$ $= \frac{1}{2^{10}}(e^{iu} + e^{-iu})^{10}$

Binomial Thm. $\longrightarrow$ $= \frac{1}{2^{10}} \sum_{l=1}^{10} \binom{10}{l} e^{iul} e^{-iu(10-l)}$

$$= \frac{1}{2^{10}} \sum_{l=1}^{10} \binom{10}{l} e^{iu(2l-10)}$$

The probability that $Y_n = q$ is the coefficient of e^{iuq}

$$P(Y_n = 6) = \frac{1}{2^{10}}\binom{10}{8} = \frac{45}{2^{10}}$$

occurs when $l = 8$

$$P(Y_n = 7) = 0$$

occurs when $l = \frac{17}{2}$; not possible

$2^{10} = 1024$

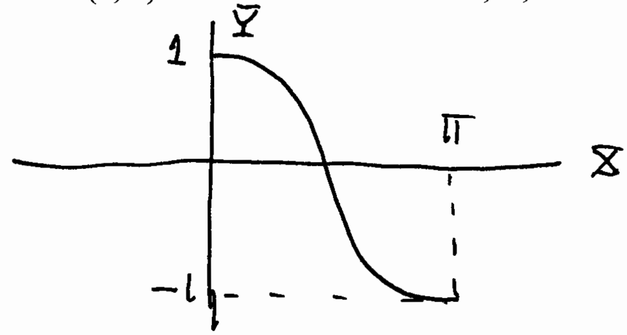
$P(Y_n = 6) = \frac{1}{2^{10}}\binom{10}{8} = \frac{45}{2^{10}}$	$P(Y_n = 7) = 0$

ECE 250 Final Exam
December 8, 2011

Name: **Solution**

1. The random variable X is uniformly distributed on $(0, \pi)$. A new random variable, Y, is formed from X as follows

$$Y = \cos X.$$



Determine the density of Y.

The transformation is one-to-one so that

$$f_Y(y) = \frac{1}{\left|\frac{dy}{dx}\right|} f_X(x) \Big|_{x=\cos^{-1} y}$$

$$\frac{dy}{dx} = -\sin x$$

$$f_Y(y) = \frac{1}{|\sin x|} f_X(x) \Big|_{x=\cos^{-1} y} \cdot \quad \begin{matrix} -1 \le y \le 1 \\ (0 \le x \le \pi) \end{matrix}$$

$$= \frac{1}{\pi} \frac{1}{|\sin x|} = \frac{1}{\pi} \frac{1}{\sqrt{1-\cos^2 x}} = \frac{1}{\pi\sqrt{1-y^2}}, \; -1 \le y \le 1$$

clearly $f_Y(y) = 0$ if $|y| > 1$

$$f_Y(y) = \begin{cases} \frac{1}{\pi\sqrt{1-y^2}}, & -1 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Name: **Solution**

2. Let X_k, $k = 1, 2, \ldots$ be independent, identically distributed random variables with $P(X_k = 1) = \frac{1}{2} = P(X_k = 0)$. Consider the sum

$$Y_n = \sum_{k=1}^{n} X_k.$$

$$\Phi_X(u) = E\left[e^{iuX_k}\right] = \frac{1}{2}\left(1+e^{iu}\right)$$

Find the Edgeworth expansion of the centered, normalized random variable $\hat{Y}_n$ where

$$\hat{Y}_n = \frac{Y_n - E[Y_n]}{\sqrt{Var[Y_n]}}.$$

$$E[Y_n] = \sum_{k=1}^{n} E[X_k] = \frac{n}{2}$$

$$Var[Y_n] = \sum_{k=1}^{n} Var[X_k] = \frac{n}{4}$$

Keep correction terms up to and including those that decrease as $\frac{1}{n}$. You do not need to explicitly evaluate any polynomials involved.

$$\Phi_{\hat{Y}_n}(u) = E\left[e^{iu\frac{Y_n - E[Y_n]}{\sqrt{Var[Y_n]}}}\right] = e^{-iu\frac{E[Y_n]}{\sqrt{Var[Y_n]}}} \cdot E\left[e^{iu\frac{Y_n}{\sqrt{Var[Y_n]}}}\right]$$

$$= e^{-iu\sqrt{n}} \prod_{k=1}^{n} E\left[e^{i\frac{2u}{\sqrt{n}}X_k}\right] = e^{-iu\sqrt{n}}\left(\Phi_X\left(\frac{2u}{\sqrt{n}}\right)\right)^n$$

$$= \left(\frac{e^{i\frac{u}{\sqrt{n}}} + e^{-i\frac{u}{\sqrt{n}}}}{2}\right)^n = e^{n\psi\left(\frac{u}{\sqrt{n}}\right)} \qquad \psi\left(\frac{u}{\sqrt{n}}\right) = \ln\left(\frac{e^{i\frac{u}{\sqrt{n}}} + e^{-i\frac{u}{\sqrt{n}}}}{2}\right)$$

$$\psi\left(\frac{u}{\sqrt{n}}\right) = \ln\left(1 + \frac{(iu)^2}{2n} + \frac{1}{4!}\frac{(iu)^4}{n^2} + \cdots\right)$$

$$= -\frac{u^2}{2n} - \frac{1}{12}\frac{u^4}{n^2} + O\left(\frac{1}{n^3}\right)$$

$$\Phi_{\hat{Y}_n}(u) = e^{\sum_{\ell=0}^{\infty}\frac{(iu)^\ell}{\ell!}\hat{\lambda}_\ell} \qquad \hat{\lambda}_0 = 0;\ \hat{\lambda}_1 = 0;\ \hat{\lambda}_2 = 1;\ \hat{\lambda}_3 = 0;\ \hat{\lambda}_4 = -\frac{2}{n} \ldots\ldots$$

$$\therefore f_{\hat{Y}_n}(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\left\{1 - \frac{1}{12n}H_4(y) + \cdots\right\}$$

Hermite Polynomial

$$\boxed{f_{\hat{Y}_n}(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\left\{1 - \frac{1}{12n}H_4(y) + \cdots\right\}}$$

Name: **Solution**

3. Let N(t) be a classical Poisson process (N(0) = 0, independent increments, constant rate λ). Define T_M as the time to the M-th event. Determine the mean value of T_M.

$$f_{T_M}(\tau) = \frac{(\lambda\tau)^{M-1}}{(M-1)!}\lambda e^{-\lambda\tau}, \quad \tau \geq 0$$

Suggested problem in class

Prob. 3 Midterm Feb. 12, 2007

METHOD I

$$\Phi_{T_M}(u) = \left(\frac{\lambda}{\lambda - iu}\right)^M$$

from tables

$$E[T_M] = \frac{1}{i}\frac{d}{du}\Phi_{T_M}(u)\Big|_{u=0} = \frac{M\lambda^M}{(\lambda - iu)^{M+1}}\Big|_{u=0} = \frac{M}{\lambda}$$

METHOD II

$$E[T_M] = \int_0^\infty \tau f_{T_M}(\tau)\,d\tau = \int_0^\infty \tau \frac{(\lambda\tau)^{M-1}}{(M-1)!}\lambda e^{-\lambda\tau}\,d\tau$$

$\lambda\tau \equiv \alpha$

$$= \frac{1}{\lambda(M-1)!}\underbrace{\int_0^\infty \alpha^M e^{-\alpha}\,d\alpha}_{= M!}$$

from 'special' page of useful formulas

$$= \frac{M!}{\lambda(M-1)!} = \frac{M}{\lambda}$$

$E(T_M) = \frac{M}{\lambda}$

Proof by Contradiction Name: **Solution**

4. The probability density of the random variable Z is continuous, non-negative (for $|z|<\infty$) and contains no unit impulse. The random process X(t) is defined by X(t) = u(t – Z) where

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

Does X(t) have independent increments? An answer that is not supported by an appropriate argument will receive no credit.

Let $s \leq t$ then $E[X(t)X(s)] = E[u(t-z)u(s-z)] = \int_{-\infty}^{s} f_Z(z)\,dz$
$= F_Z(s)$

Let $s \leq u \leq t$

$$E[(X(t)-X(u))(X(u)-X(s))] = F_Z(u) - F_Z(s) - F_Z(u) + F_Z(s) = 0$$

But

$$E[X(t)-X(u)] = \int_u^t f_Z(z)\,dz = F_Z(t) - F_Z(u)$$

$$E[X(u)-X(s)] = \int_s^u f_Z(z)\,dz = F_Z(u) - F_Z(s)$$

Clearly if $(X(t)-X(u))$ and $(X(u)-X(s))$ are independent, we must have

$$E[(X(t)-X(u))(X(u)-X(s))] = E[X(t)-X(u)]\cdot E[X(u)-X(s)]$$

$$\underset{0}{\shortparallel} \qquad\qquad \underset{(F_Z(t)-F_Z(u))\cdot(F_Z(u)-F_Z(s))}{\shortparallel}$$

this is not zero
so increments are not independent

Independent Increments? (circle one)	Yes	No

Direct Proof Name: **Solution**

4. The probability density of the random variable Z is continuous, non-negative (for $|z| < \infty$) and contains no unit impulse. The random process X(t) is defined by X(t) = u(t – Z) where

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

Does X(t) have independent increments? An answer that is not supported by an appropriate argument will receive no credit.

Let $t_1 \leq t_2 \leq t_3 \leq \cdots t_{n+1}$

Define $D_k = X(t_{k+1}) - X(t_k) = u(t_{k+1} - Z) - u(t_k - Z)$

$$= \begin{cases} 1, & t_k < Z \leq t_{k+1} \\ 0, & \text{otherwise} \end{cases}$$

Define $I_k = (t_k, t_{k+1}]$

$$P(D_k = 1) = P(Z \in I_k) = \int_{t_k}^{t_{k+1}} f_Z(z)\,dz = F_Z(t_{k+1}) - F_Z(t_k)$$

$$P(Z \in I_k) = \begin{cases} >0, & t_{k+1} > t_k \\ 0, & t_{k+1} = t_k \end{cases}$$ because $f_Z(z) > 0$

Examine $P(D_1 = 1, D_2 = 1, \cdots D_n = 1) =$

$$= P(\underbrace{Z \in I_1, Z \in I_2, \cdots, Z \in I_n}_{\text{these are mutually exclusive events so}})$$

$= 0$

But $P(D_k = 1) = F_Z(t_{k+1}) - F_Z(t_k) > 0$

$\therefore \prod_{k=1}^{n} P(D_k = 1) > 0$ and $D_1, D_2, \cdots D_n$ are not independent

* Full credit if the above argument is presented for only two increments

Independent Increments? (circle one)	Yes	No

Name: **Solution**

5. Consider the shot process

$$X(t) = \sum_{t_n} h(t - t_n)$$

where the events t_n are determined by a classical Poisson process ($N(0) = 0$, constant rate λ). The function $h(t)$ is given by

$$h(t) = \begin{cases} -\ln t, & 0 \le t \le 1 \\ 0, & \textit{otherwise.} \end{cases}$$

Obtain an explicit, closed-form expression for the probability density of X(t).

$$\Phi_{X(t)}(u) = E\left[e^{iuX(t)}\right] = e^{\lambda \int_{-\infty}^{\infty} \left(e^{iuh(\tau)} - 1\right) d\tau}$$

$$= e^{\lambda \int_0^1 \left(e^{-iu \ln \tau} - 1\right) d\tau}$$

$$= e^{\lambda \int_0^1 \left(\tau^{-iu} - 1\right) d\tau}$$

this ch.f. was examined in detail in class

$$= e^{\lambda \left(\frac{1}{1-iu} - 1\right)} = e^{-\lambda} e^{\lambda \frac{1}{1-iu}}$$

$$f_{X(t)}(x) = e^{-\lambda} \left\{ \delta(x) + \sqrt{\frac{\lambda}{x}}\, I_1\left(2\sqrt{\lambda x}\right) e^{-x} \right\}$$

from class notes

modified Bessel function

$$f_{X(t)}(x) = \boxed{e^{-\lambda} \left\{ \delta(x) + \sqrt{\frac{\lambda}{x}}\, I_1\left(2\sqrt{\lambda x}\right) \right\}}$$

ECE 250 MIDTERM

February 18, 2011

Name: **SOLUTION**

1. The Poisson processes $N_1(t)$ and $N_2(t)$ are independent and identically distributed with probabilities

$$P[N_1(t) = m] = P[N_2(t) = m] = \frac{(t/2)^m}{m!} e^{-t/2}.$$

Obtain a closed-form expression for the probability

$$P[N_1(t) - N_2(t) = 0].$$

$$P(N_1(t) - N_2(t) = 0) = P(N_1(t) = N_2(t))$$

$N_1(t)$ & $N_2(t)$ are indep.

$$= \sum_{m=0}^{\infty} P(N_1(t) = m) P(N_2(t) = m)$$

$$= \sum_{m=0}^{\infty} \frac{(t/2)^m}{m!} e^{-(t/2)} \frac{(t/2)^m}{m!} e^{-(t/2)}$$

$$= \sum_{m=0}^{\infty} \frac{(t^2/4)^m}{m!\,m!} e^{-t}$$

From notes & solutions Prob. 4 HW#1

$$= I_0(t) e^{-t}$$

$$\boxed{P[N_1(t) - N_2(t) = 0] = I_0(t) e^{-t}}$$

Name: **SOLUTION**

2. Let $\{X_k\}$ be a collection of independent identically distributed random variables with common probabilities

$$P(X_k = p) = \frac{\nu^p}{p!} e^{-\nu}, \quad k = 1, 2, \dots$$

$$E[e^{iuX_k}] = e^{\nu(e^{iu}-1)}$$

$$E[Y_n] = n E[X_k] = n\cdot\nu$$

$$Var[Y_n] = n Var[X_k] = n\cdot\nu$$

Consider the sum

$$Yn = \sum_{k=1}^{n} X_k.$$

$$\hat{Y}_n = \frac{Y_n - n\nu}{\sqrt{n\nu}}$$

The normalized centered sum is

$$E[e^{iuY_n}] = \left(E[e^{iuX_k}]\right)^n = e^{n\nu(e^{iu}-1)}$$

$$\hat{Y}_n = \frac{Yn - E[Yn]}{\sqrt{Var[Yn]}}.$$

Obtain the Edgeworth expansion of the density of $\hat{Y}_n$ and include correction terms up to order n^{-1}. You do not need to explicitly evaluate any polynomials.

$$E[e^{iu\hat{Y}_n}] = e^{n\nu\left(e^{i\left(\frac{u}{\sqrt{n\nu}}\right)} - i\left(\frac{u}{\sqrt{n\nu}}\right) - 1\right)}$$

$$= e^{n\nu\sum_{l=2}^{\infty}\frac{1}{l!}\frac{(iu)^l}{(n\nu)^{l/2}}} = e^{\sum_{l=2}^{\infty}\frac{(iu)^l}{l!}(n\nu)^{\frac{2-l}{2}}}$$

$$= e^{\sum_{l=2}^{\infty}\frac{(iu)^l}{l!}\lambda_l}$$

clearly $\lambda_0 = 0$; $\lambda_1 = 0$; $\lambda_2 = 1$; $\lambda_3 = \frac{1}{\sqrt{n\nu}}$; $\lambda_4 = \frac{1}{n\nu}$

From class notes

$$f_{\hat{Y}_n}(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\left\{1 + \frac{1}{3!\sqrt{n\nu}} H_3(y) + \frac{1}{4!}\frac{1}{n\nu} H_4(y) + \frac{1}{2}\frac{1}{(3!)^2 n\nu} H_6(y) \dots\right\}$$

$$\boxed{f_{\hat{Y}_n}(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\left\{1 + \frac{1}{\sqrt{n\nu}}\frac{1}{3!}H_3(y) + \frac{1}{n\nu}\left[\frac{1}{4!}H_4(y) + \frac{1}{2}\left(\frac{1}{3!}\right)^2 H_6(y)\right]\dots\right\}}$$

Name: **SOLUTION**

3. Consider the process X(t) = AcosΩt. Here A is a positive constant and Ω is a Gaussian variable with zero mean and unit variance. This process is not wide sense stationary. Evaluate the averaged correlation function $\overline{R}(\tau)$.

$$R(t+\tau, t) = E\left[A\cos\Omega(t+\tau)\,A\cos\Omega t\right]$$
$$= \frac{A^2}{2}E\left[\cos\Omega\tau\right] + \frac{A^2}{2}E\left[\cos\Omega(2t+\tau)\right]$$

$$f_\Omega(w) = \frac{1}{\sqrt{2\pi}}e^{-\frac{w^2}{2}}$$

$$E\left[\cos\Omega\tau\right] = \frac{1}{2}E\left[e^{i\Omega\tau}\right] + \frac{1}{2}E\left[e^{-i\Omega\tau}\right] = e^{-\frac{1}{2}\tau^2}$$

similarly

$$E\left[\cos\Omega(2t+\tau)\right] = e^{-\frac{1}{2}(2t+\tau)^2}$$

$$\overline{R}(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}R(t+\tau,t)\,dt$$
$$= \frac{A^2}{2}e^{-\frac{1}{2}\tau^2} + \frac{A^2}{2}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^{-\frac{1}{2}(2t+\tau)^2}\,dt$$

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^{-\frac{1}{2}(2t+\tau)^2}\,dt = \lim_{T\to\infty}\frac{1}{4T}\int_{-2T+\tau}^{2T+\tau}e^{-\frac{1}{2}\alpha^2}\,d\alpha$$

$\alpha = 2t+\tau$

Now
$$\int_{-2T+\tau}^{2T+\tau}e^{-\frac{1}{2}\alpha^2}\,d\alpha \le \int_{-\infty}^{\infty}e^{-\frac{1}{2}\alpha^2}\,d\alpha = \sqrt{2\pi}$$

$$\therefore \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^{-\frac{1}{2}(2t+\tau)}\,dt \le \lim_{T\to\infty}\frac{\sqrt{2\pi}}{4T} = 0$$

$$\boxed{\overline{R}(\tau) = \frac{A^2}{2}e^{-\frac{1}{2}\tau^2}}$$

ECE 250 Final Exam

March 14, 2011

Name: **SOLUTION**

1. A discrete random variable has the probabilities

$$P(X = m) = \frac{\beta}{(1+\beta)^{m+1}}, \quad m = 0, 1, \ldots$$

with $\beta > 0$. Evaluate the mean and variance of X.

$$\Phi_X(u) = \sum_{m=0}^{\infty} \frac{\beta}{(1+\beta)^{m+1}} e^{ium} = \frac{\beta}{1+\beta-e^{iu}}$$

$$E[X] = \left(\frac{1}{i}\right)\frac{d}{du}\Phi_X(u)\Big|_{u=0} = \frac{\beta e^{iu}}{(1+\beta-e^{iu})^2}\Big|_{u=0} = \frac{1}{\beta}$$

$$E[X^2] = \left(\frac{1}{i}\right)^2\frac{d^2}{du^2}\Phi_X(u)\Big|_{u=0} = \frac{\beta e^{iu}(1+\beta+e^{iu})}{(1+\beta-e^{iu})^2}\Big|_{u=0} = \frac{2+\beta}{\beta^2}$$

$$Var[X] = E[X^2] - (E[X])^2 = \frac{1+\beta}{\beta^2}$$

$m = E[X] = \frac{1}{\beta}$	$\sigma^2 = Var[X] = \frac{1+\beta}{\beta^2}$

Name: **SOLUTION**

2. If α is a real constant, prove that

$$\Phi(u) = e^{-u^2+\alpha u^3}$$

is the characteristic function of a real random variable only if $\alpha = 0$. [HINT: Consider the conditions that a valid characteristic function must satisfy.]

It is necessary (but not sufficient) that a characteristic function satisfy

(1) $\Phi(0) = 1$

(2) $|\Phi(u)| \le 1$

Clearly the above ch.f. satisfies (1)

In order that it satisfy (2) it is necessary that

$$-u^2 + \alpha u^3 \le 0$$

or

$$(*) \quad \alpha u^3 \le u^2$$

Now if $\alpha u > 1$, we will contradict (*)

If $\alpha < 0$ choose $u = -\frac{2}{|\alpha|}$ and $\alpha u = 2 > 1$

If $\alpha > 0$ choose $u = \frac{2}{\alpha}$ and $\alpha u = 2 > 1$

$\therefore$ (*) can be satified only if $\alpha = 0$

Name: **SOLUTION**

3. Let N be a Poisson variable with the probabilities

$$P(N=n) = \frac{2^n}{n!} e^{-2}, \quad n = 0, 1, \ldots$$

Consider the new random variable

$$M = \begin{cases} 0, & N = 0 \\ \sum_{k=1}^{N} a_k & N > 0 \end{cases}$$

$$E\left[e^{iu\sum_{k=1}^{n} a_k}\right] = E\left[\prod_{k=1}^{n} e^{iua_k}\right] = \prod_{k=1}^{n} E\left[e^{iua_k}\right] = \left[\frac{\cos u + 1}{2}\right]^n$$

Here the a_ks are i.i.d. with probabilities

$$P(a_k = 1) = P(a_k = -1) = 1/4. \quad P(a_k = 0) = 1/2.$$

Determine the probabilities P(M = m) for all integers.

$$\Phi_M(u) = P(N=0) + \sum_{n=1}^{\infty} P(N=n) E\left[e^{iu\sum_{k=1}^{n} a_k}\right]$$

$$= e^{-2} + \sum_{k=1}^{\infty} \frac{2^n}{n!}\left(\frac{\cos u + 1}{2}\right)^n e^{-2}$$

$$= e^{(\cos u - 1)}$$

This is the ch.f. on HW #1

$$\therefore P(M=m) = e^{-1} I_m(1)$$

Directly from solution on HW #1

Bessel Function

$$\boxed{P(M=m) = e^{-1} I_m(1)}$$

Name: **SOLUTION**

4. Let X(t) be the random telegraph signal

$$X(t) = X(0)\,(-1)^{N(t)}$$

where P [X(0) = 1] = 1/2. N(t) is a classical Poisson process (constant rate ν) and is independent of X(0). Consider the moving average process

$$Y(t) = \frac{1}{T}\int_{t-T}^{t} X(\tau)\,d\tau.$$

From class notes

$$E[X(t)] = 0$$

$$R_X(\tau) = e^{-2\nu|\tau|}$$

$$S_X(\omega) = \frac{4\nu}{4\nu^2+\omega^2}$$

Is Y(t) mean square differentiable? An answer not supported by appropriate reasoning will receive no credit. [HINT: Write Y(t) as a convolution of X(t) and an appropriate impulse response.]

$$Y(t) = \frac{1}{T}\int_{t-T}^{t} X(\tau)\,d\tau \underset{\tau = t-\alpha}{=} \frac{1}{T}\int_{0}^{T} X(t-\alpha)\,d\alpha = \int_{-\infty}^{\infty} h(\alpha)\,X(t-\alpha)\,d\alpha$$

where $h(\alpha) = \begin{cases} \frac{1}{T}, & 0 \le \alpha \le T \\ 0, & \text{otherwise} \end{cases}$

Clearly $Y(t)$ is W.S.S. with zero mean

$Y(t)$ will be m.s. differentiable if

$$\int_{-\infty}^{\infty} \omega^2 S_Y(\omega)\,d\omega = \int_{-\infty}^{\infty} \omega^2 |H(i\omega)|^2 S_X(\omega)\,d\omega < \infty$$

$$H(i\omega) = \int_{-\infty}^{\infty} h(\alpha)\,e^{-i\omega\alpha}\,d\alpha = \frac{1-e^{-i\omega T}}{i\omega T}$$

$$\int_{-\infty}^{\infty} \omega^2 S_Y(\omega)\,d\omega = \int_{-\infty}^{\infty} \omega^2 \frac{4\nu}{4\nu^2+\omega^2}\,\frac{|1-e^{-i\omega T}|^2}{\omega^2 T^2}\,d\omega \le \frac{16\nu}{T^2}\int_{-\infty}^{\infty} \frac{1}{4\nu^2+\omega^2}\,d\omega$$

$$< \infty$$

M.S. Differentiable? (Circle One)	Yes	No

Name: **SOLUTION**

5. Consider the shot process

$$X(t) = \sum A_K h(t - t_K).$$

$$f_A(\alpha) = \frac{1}{2}\delta(\alpha-1) + \frac{1}{2}\delta(\alpha+1)$$

Here the $t_K s$ are the event times of a Poisson process with constant rate $\nu = 2$, the $A_K s$ are independent of the event times with probabilities $P(A_K = 1) = 1/2$ and $P(A_K = -1) = 1/2$. The function h(t) is given by

$$h(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

$$H(i\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t}\, dt = \frac{1}{1+i\omega}$$

In transmission this process is corrupted by additive Gaussian noise, N(t), with zero mean and correlation

$$R_n(\tau) = e^{-4|\tau|}.$$

$$S_N(\omega) = \frac{8}{16+\omega^2}$$

Determine the impulse response, $h_o(t)$, of the optimum (possibly non-realizable) linear filter to obtain the best mean square estimate of X(t) from the sum X(t) + N(t).

$$\Phi_{X(t+\tau),X(\tau)}(u,v) = e^{2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(e^{iu\alpha h(t+\tau-t') + iv\alpha h(t-t')} - 1\right) f_A(\alpha)\, d\alpha\, dt'}$$

From this $E[X(t)] = 0$; $R_X(\tau) = 2\int_{-\infty}^{\infty} h(\tau + t')h(t')\,dt'$

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-i\omega\tau}\, d\tau = 2|H(i\omega)|^2 = \frac{2}{1+\omega^2}$$

$$H_o(i\omega) = \frac{S_X(\omega)}{S_X(\omega) + S_N(\omega)} = \frac{1}{5}\,\frac{16+\omega^2}{4+\omega^2} = \frac{1}{5} + \frac{3}{5}\,\frac{2\cdot 2}{2^2+\omega^2}$$

$$\updownarrow$$

$$h_o(t) = \frac{1}{5}\delta(t) + \frac{3}{5} e^{-2|t|}$$

$$\boxed{h_o(t) = \frac{1}{5}\delta(t) + \frac{3}{5} e^{-2|t|}}$$

ECE 250 Midterm

October 27, 2010

Name: **SOLUTION**

1. The Cauchy random variable X has density

$$f_X(x) = \frac{1}{\pi(1+x^2)}.$$

A new random variable Y is obtained from X via $Y = u(X)$

where

$$u(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0. \end{cases}$$

Evaluate the mean and variance of Y.

$$E[Y] = E[u(X)] = \int_{-\infty}^{\infty} u(x) f_X(x)\,dx = \int_0^{\infty} \frac{1}{\pi(1+x^2)}\,dx = \frac{1}{2}$$

$$E[Y^2] = E[u^2(X)] = E[u(X)] = \frac{1}{2}$$

$$Var[Y] = E[Y^2] - (E[Y])^2 = \frac{1}{4}$$

$E[Y] = \frac{1}{2}$	$Var[Y] = \frac{1}{4}$

Name: **SOLUTION**

2. Can the function $\Phi(u) = 1 - \sin^2 u$ be the characteristic function of a real random variable? Your answer must be supported by appropriate reasoning. A simple <u>yes</u> or <u>no</u> without a valid argument will receive no credit.

$$\Phi(u) = 1 - \sin^2 u = 1 - \left(\frac{e^{iu} - e^{-iu}}{2i}\right)^2$$

$$= \frac{1}{2} + \frac{1}{4}e^{i2u} + \frac{1}{4}e^{-i2u}$$

Examine if the density is non-negative and integrates to 1.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(u) e^{-iux}\,du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{\frac{1}{2} + \frac{1}{4}e^{i2u} + \frac{1}{4}e^{-i2u}\right\} e^{-iux}\,du$$

$$= \frac{1}{2}\delta(x) + \frac{1}{4}\delta(x-2) + \frac{1}{4}\delta(x+2)$$

Denoting the associated random variable by $\tilde{X}$ we have

$$P(\tilde{X} = 0) = \frac{1}{2}$$

$$P(\tilde{X} = 2) = \frac{1}{4}$$

$$P(\tilde{X} = -2) = \frac{1}{4}$$

Clearly $\Phi(u)$ is a valid Ch.f.

Valid Ch.F.? (Circle One)	Yes	No

Name: **SOLUTION**

3. Let X and Y be independent, identically-distributed Poisson variables with parameter (1/2). That is

$$P(X = n) = P(Y = n) = \frac{(1/2)^n}{n!} e^{-(1/2)}, n = 0,1,2,\ldots$$

Define Z = X – Y. Find the probability that Z = 3.

independence

$$\Phi_Z(u) = E[e^{iuZ}] = E[e^{iu(X-Y)}] = E[e^{iuX}]E[e^{-iuY}]$$

$$E[e^{iuX}] = e^{\frac{1}{2}(e^{iu}-1)}$$

$$E[e^{-iuY}] = e^{\frac{1}{2}(e^{-iu}-1)}$$

$$\therefore \quad \Phi_Z(u) = e^{(\cos u - 1)}$$

$$= \sum_{n=-\infty}^{\infty} P(Z=n) e^{iun}$$

$$P(Z=3) = \text{coefficient of } e^{iu3}$$

$$P(Z=3) = \frac{1}{2\pi}\int_0^{2\pi} e^{-iu3} e^{(\cos u - 1)} du$$

$$= e^{-1}\frac{1}{2\pi}\int_0^{2\pi} e^{-iu3} e^{\cos u} du$$

From H.W.#1

$$= e^{-1} I_3(1)$$

P(Z = 3) = $e^{-1} I_3(1)$

ECE 250 Final Exam

December 9, 2010

Name: **Solution**

1. Let $\Psi(u)$ be the characteristic function of a real random variable. Show that

$$\Phi(u) = \frac{1}{2-\Psi(u)}$$

is also a valid characteristic function.

from "Useful Formulas" noting $\left|\frac{1}{2}\Psi(u)\right| \le \frac{1}{2}$

$$\Phi(u) = \frac{1}{2}\,\frac{1}{1-\frac{1}{2}\Psi(u)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n[\Psi(u)]^n$$

Show $\Phi(u)$ is the transform of a density – $f(x)$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(u)e^{-iux}\,du = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\frac{1}{2\pi}\int_{-\infty}^{\infty}[\Psi(u)]^n e^{-iux}\,du$$

Define $p(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi(u)e^{-iux}\,du$

clearly $p(x)$ is a valid probability density

Also $[\Psi(u)]^n$ is the ch.f. of $p(x)$ convolved with itself n times. Define

valid probability density

$$p_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\Psi(u)]^n e^{-iux}\,du$$

$$\therefore\ f(x) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n p_n(x)$$

$p_0(x) \equiv \delta(x)$

Clearly each $p_n(x)$ is non-negative and integrates to 1. So $f(x)$ is non-negative and

$$\int_{-\infty}^{\infty}f(x)\,dx = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\int_{-\infty}^{\infty}p_n(x)\,dx = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 1$$

Thus $\Phi(u)$ is a valid characteristic function

Name: **Solution**

2. Let $\{X_n\}$ be i.i.d. random variables with density

$$f_{X_n}(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

$E[X_n] = 1 \qquad E[X_n^2] = 2$

$Var[X_n] = 1$

$E[Y_n] = \sum_{k=1}^{n} E[X_k] = n$

$Var[Y_n] = \sum_{k=1}^{n} Var[X_k] = n$

Consider the sum $Y_n = \sum_{k=1}^{n} X_k$.

Find the Edgeworth expansion of the centered, normalized variable

$$\hat{Y}_n = (Y_n - E[Y_n]) / (Var[Y_n])^{1/2}. \quad = \frac{Y_n - n}{\sqrt{n}}$$

Keep only the first non-zero correction term. It is not required to evaluate any polynomials appearing in the expansion.

$$\phi(u) = E\left[e^{iuX_k}\right] = \frac{1}{1-iu}$$

$$\Phi_{\hat{Y}_n}(u) = E\left[e^{iu\left(\frac{Y_n - n}{\sqrt{n}}\right)}\right]$$

$$= e^{-i\left(\frac{u}{\sqrt{n}}\right)n} E\left[e^{i\left(\frac{u}{\sqrt{n}}\right)\sum_{k=1}^{n} X_k}\right] = e^{-i\left(\frac{u}{\sqrt{n}}\right)n}\left(\frac{1}{1-i\left(\frac{u}{\sqrt{n}}\right)}\right)^n$$

$$= e^{n\psi(iu)} \quad \text{where} \quad \psi(iu) = -i\left(\frac{u}{\sqrt{n}}\right) - \ln\left(1 - i\left(\frac{u}{\sqrt{n}}\right)\right)$$

expand $\psi(iu)$ in a power series

$$\psi(iu) = -\frac{u^2}{2n} - \frac{iu^3}{3n^{3/2}} + \frac{u^4}{4n^2} + \cdots$$

$$\therefore \quad \Phi_{\hat{Y}_n}(u) = e^{-\frac{u^2}{2}} e^{\frac{-iu^3}{3\sqrt{n}} + \frac{u^4}{4n} + O\left(\frac{1}{n^{3/2}}\right)}$$

$$= e^{-\frac{u^2}{2}}\left\{1 + \frac{(iu)^3}{3\sqrt{n}} + O\left(\frac{1}{n}\right)\right\}$$

$$f_{\hat{Y}_n}(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuy} e^{-\frac{u^2}{2}}\left\{1 + \frac{(iu)^3}{3\sqrt{n}} + O\left(\frac{1}{n}\right)\right\} du =$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\left\{1 + \frac{1}{3\sqrt{n}} H_3(y) + O\left(\frac{1}{n}\right)\right\}$$ From class notes

$$\boxed{f_{\hat{Y}_n}(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\left\{1 + \frac{1}{3\sqrt{n}} H_3(y) + O\left(\frac{1}{n}\right)\right\}}$$

Name: **Solution**

3. A Gaussian process X(t) is wide-sense stationary with zero mean, unit variance and correlation function R(τ). The process Y(t) is defined by

$$Y(t) = g[X(t)]$$

Denote transform of $g(\cdot)$ by $G(i\omega)$

where g(•) is a function with a well-defined Fourier transform G(iω). Is Y(t) wide sense stationary? A simple "yes" or "no" answer without the appropriate reasoning will not receive any credit. [Hint: You may want to evaluate and use the joint characteristic function of X(t) and X(s).]

$$Y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(i\omega)\, e^{i\omega X(t)}\, d\omega$$

ch. funct. of zero-mean, unit variance Gaussian variable

$$E[Y(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(i\omega)\, E\left[e^{i\omega X(t)}\right] d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(i\omega)\, e^{-\frac{\omega^2}{2}}\, d\omega = \text{const.}$$

$$E[Y(t)Y(s)] = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} G(i\omega)\,d\omega \int_{-\infty}^{\infty} G(i\nu)\,d\nu\; E\left[e^{i\omega X(t) + i\nu X(s)}\right]$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} G(i\omega)\,d\omega \int_{-\infty}^{\infty} G(i\nu)\,d\nu\; e^{-\frac{1}{2}\left[\omega^2 + \nu^2 + 2\omega\nu R(t-s)\right]}$$

clearly Y(t) has a constant mean and a correlation function is a function of only (t−s)

Y(t) W.S.S.? (Circle One)	Yes	No

Name: **Solution**

4. A wide sense stationary process X(t) with correlation function $R_X(\tau) = 1/(1+\tau^2)$ is passed through a linear, time-invariant filter with impulse response $3\sin\pi t/\pi t$. Denote the output by Y(t). Obtain an explicit expression for the output correlation.

Transfer function $H(i\omega) = \begin{cases} 3, & -\pi \le \omega \le \pi \\ 0, & \text{otherwise} \end{cases}$

From Transform Tables

Now $S_Y(\omega) = |H(i\omega)|^2 S_X(\omega)$

$$S_X(\omega) = \pi e^{-|\omega|}$$

$$\therefore\ S_Y(\omega) = \begin{cases} 9\pi e^{-|\omega|}, & -\pi \le \omega \le \pi \\ 0, & \text{otherwise} \end{cases}$$

$$R_Y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega) e^{i\omega\tau}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} 3\pi e^{-|\omega|} e^{i\omega\tau}\, d\omega$$

$$= \frac{9}{2}\int_{-\pi}^{0} e^{\omega} e^{i\omega\tau}\, d\omega + \frac{3}{2}\int_{0}^{\pi} e^{-\omega} e^{i\omega\tau}\, d\omega$$

$$= \frac{9}{2}\left\{ \frac{1-e^{-\pi(1+i\tau)}}{1+i\tau} + \frac{e^{\pi(-1+i\tau)}-1}{-1+i\tau} \right\}$$

$$= \frac{9}{2}\left[\frac{1}{1+i\tau} + \frac{1}{1-i\tau}\right] - \frac{3}{2}e^{-\pi}\left[\frac{e^{-i\pi\tau}}{1+i\tau} + \frac{e^{i\pi\tau}}{1-i\tau}\right]$$

$$= \frac{9}{1+\tau^2}\left\{1 + (\tau\sin\pi\tau - \cos\pi\tau)e^{-\pi}\right\}$$

$$\boxed{R_Y(\tau) = \frac{9}{1+\tau^2}\left\{1 + (\tau\sin\pi\tau - \cos\pi\tau)e^{-\pi}\right\}}$$

Name: **Solution**

5. Consider a random telegraph signal X(t). The underlying Poisson process has rate $\lambda = 1$ and $P[X(0) = 1] = P[X(0) = -1] = 1/2$. During transmission this signal is corrupted by additive noise N(t) which is wide-sense stationary with zero mean and correlation function

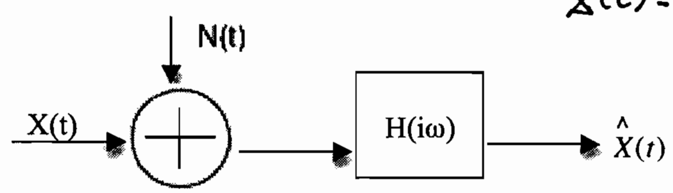
$$R_N(\tau) = e^{-4|\tau|}. \qquad R_X(\tau) = e^{-2|\tau|} \leftarrow \text{From class notes}$$

$$\text{From tables} \rightarrow S_N(\omega) = \frac{8}{16+\omega^2} \qquad S_X(\omega) = \frac{4}{4+\omega^2} \leftarrow \text{From tables}$$

The signal and noise are independent processes, and their sum is passed through an ideal low-pass filter with (one-sided) bandwidth Ω. That is

$$H(i\omega) = \begin{cases} 1, & -\Omega \le \omega \le \Omega \\ 0, & \text{otherwise} \end{cases}.$$

$$\hat{X}(t) = \int_{-\infty}^{\infty} h(t')\left[X(t-t') + N(t-t')\right]dt'$$



Determine the bandwidth that minimizes the difference

$$\varepsilon = E[(X(t) - \hat{X}(t))^2].$$

[NOTE: This is not a problem to determine the optimum filter, but only to find the optimum bandwidth Ω.]

$$E[X(t)\hat{X}(t)] = \int_{-\infty}^{\infty} h(t') R_X(t')dt'$$

$$E[\hat{X}^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t')h(t'')\left[R_X(t'-t'') + R_N(t'-t'')\right]dt'dt''$$

Express correlations by inverse transform of spectral density

$$\therefore \varepsilon = R_X(0) - \frac{2}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)H^*(i\omega)d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(i\omega)|^2\left[S_X(\omega) + S_N(\omega)\right]d\omega$$

$$= R_X(0) + \frac{1}{2\pi}\int_{-\Omega}^{\Omega}\left[S_N(\omega) - S_X(\omega)\right]d\omega = R_X(0) + \frac{1}{\pi}\int_0^{\Omega}\left[S_N(\omega) - S_X(\omega)\right]d\omega$$

$$\frac{d\varepsilon}{d\Omega} \overset{\text{set}}{=} 0 \Rightarrow S_N(\Omega) = S_X(\Omega) \Rightarrow \frac{8}{16+\Omega^2} = \frac{4}{4+\Omega^2} \Rightarrow \Omega = 2\sqrt{2}$$

$\Omega = 2\sqrt{2}$

ECE 250 Midterm
November 6, 2009

Name: **SOLUTION**

1. The characteristic function of a random variable, X, is

$$\Phi_X(u) = \frac{e^{iu}}{2-e^{iu}}. = \frac{1}{2} e^{iu} \frac{1}{1-\frac{1}{2}e^{iu}}$$

Determine P(X = 0) and P(X = 2).

$$\Phi_X(u) = \frac{1}{2} e^{iu} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{iun} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} e^{iu(n+1)}$$

change summation variable

$$n+1 = k \quad (n = k-1)$$

$$\Phi_X(u) = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k e^{iuk}$$

$P(X=0)$ = coefficient of $e^{iu \cdot 0}$ $= 0$

$P(X=2)$ = coefficient of e^{iu2} $= \frac{1}{4}$

P(X = 0) = 0	P(X = 2) = $\frac{1}{4}$

Name: **SOLUTION**

2. It is observed that for all real numbers α_1 and α_2 the sum $Y = \alpha_1 X_1 + \alpha_2 X_2$ is a zero mean Gaussian random variable. Show that X_1 and X_2 are jointly Gaussian, zero mean random variables. [Hint: You may find the characteristic function useful.]

Define

$$m_1 = E[X_1]; \quad m_2 = E[X_2]; \quad \sigma_1^2 = Var[X_1]; \quad \sigma_2^2 = Var[X_2]$$

$$\rho = \frac{E[X_1 X_2] - m_1 m_2}{\sigma_1 \sigma_2}$$

We know that a linear combination of Gaussian variables is a Gaussian variable and

$$E[e^{iY}] = e^{iE[Y] - \frac{1}{2}Var[Y]}$$

$$E[Y] = \alpha_1 m_1 + \alpha_2 m_2 = 0$$

Set $\alpha_1 = 0$, then $m_2 = 0$
Set $\alpha_2 = 0$, then $m_1 = 0$

$$Var[Y] = \alpha_1^2 \sigma_1^2 + 2\alpha_1\alpha_2\sigma_1\sigma_2\rho + \alpha_2^2\sigma_2^2$$

and

$$(*) \quad E[e^{iY}] = e^{-\frac{1}{2}(\alpha_1^2\sigma_1^2 + 2\alpha_1\alpha_2\sigma_1\sigma_2\rho + \alpha_2^2\sigma_2^2)}$$

Now consider

$$\Phi_{X_1,X_2}(u_1,u_2) \triangleq E\left[e^{i(u_1 X_1 + u_2 X_2)}\right]$$

If we define $\alpha_1 = u_1$ and $\alpha_2 = u_2$, the joint characteristic function is just $E[e^{iY}]$ as in $(*)$ so that

$$\Phi_{X_1,X_2}(u_1,u_2) = e^{-\frac{1}{2}(u_1^2\sigma_1^2 + 2u_1u_2\sigma_1\sigma_2\rho + u_2^2\sigma_2^2)}$$

This is precisely the joint characteristic function of two zero-mean Gaussian variables. ∎

Name: **SOLUTION**

3. Let $R(\tau)$ be the correlation function of a real zero-mean, wide sense stationary process. For $t_0 > 0$, consider the function $\hat{R}(\tau) = 2R(\tau) - R(\tau - t_0) - R(\tau + t_0)$. Can $\hat{R}(\tau)$ be the correlation function of a real, wide-sense stationary process? Your answer must be supported by appropriate reasoning. A simple "yes" or "no" will not receive any credit.

Show that $\hat{R}(\tau)$ is the inverse transform of a non-negative, integrable function.

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

Now examine

$$\int_{-\infty}^{\infty} R(\tau - t_0) e^{-i\omega\tau} d\tau = e^{-i\omega t_0} \int_{-\infty}^{\infty} R(\alpha) e^{-i\omega\alpha} d\alpha = e^{-i\omega t_0} S(\omega)$$

and

$$\int_{-\infty}^{\infty} R(\tau + t_0) e^{-i\omega\tau} d\tau = e^{i\omega t_0} S(\omega)$$

Now examine

$$\hat{S}(\omega) = \int_{-\infty}^{\infty} \hat{R}(\tau) e^{-i\omega\tau} d\tau = 2S(\omega) - e^{-i\omega t_0} S(\omega) - e^{i\omega t_0} S(\omega)$$

$$= 2[1 - \cos\omega t_0] S(\omega)$$

Clearly $\hat{S}(\omega) \geq 0$ and integrable (because $S(\omega)$ is a power spectral density)

$\hat{R}(\tau)$ W.S.S. Correlation? (circle one)	Yes	No

ECE 250 Final Examination
December 7, 2009

Name: **SOLUTION**

1. A random variable is defined by the probabilities

$$P\ (X = k) = \frac{\alpha^k}{(1+\alpha)^{k+1}}, \quad \begin{array}{l}\alpha > 0 \\ k = 0,1,\ldots\end{array} .$$

For $n \geq 1$, find an exact expression for the probability $P\ (X \geq n)$. Show that this probability decreases exponentially as n increases. Evaluate any constants used.

$$P(X \geq n) = \sum_{k=n}^{\infty} P(X=k) = \sum_{k=n}^{\infty} \frac{\alpha^k}{(1+\alpha)^{k+1}}$$

$k = l + n$

$$= \sum_{l=0}^{\infty} \left(\frac{\alpha}{1+\alpha}\right)^{l+n} \frac{1}{(1+\alpha)}$$

$$= \left(\frac{\alpha}{1+\alpha}\right)^n \frac{1}{1+\alpha} \underbrace{\sum_{l=0}^{\infty} \left(\frac{\alpha}{1+\alpha}\right)^l}_{\frac{1}{1-\frac{\alpha}{1+\alpha}}}$$

from handouts

$$P(X \geq n) = \left(\frac{\alpha}{1+\alpha}\right)^n$$

$$\frac{\alpha}{1+\alpha} \overset{set}{=} e^{-\beta}$$

note: $\frac{\alpha}{1+\alpha} < 1$

$$\therefore \quad P(X \geq n) = e^{-\beta n}$$

$P\ (X \geq n) =$ $e^{-\beta n}$ $\qquad \beta = \ln\left(\frac{1+\alpha}{\alpha}\right)$

Name: **SOLUTION**

2. Let X(t) be a zero-mean, Gaussian, white noise process with power spectral density $S_X(\omega) = P$. Show that

$$Y(t) = \int_0^t X(\tau)d\tau \,, \quad t \geq 0 \qquad E[X(\alpha)X(\beta)] = P_0\delta(\alpha-\beta)$$

is a Brownian motion process. [This relationship is why Gaussian white noise is often regarded as the derivative of the Brownian motion process.]

Clearly $Y(t)$ is Gaussian and $Y(0) = 0$

It remains to show that it has independent, stationary increments

$$t_1 < t_2 < t_3 < t_4$$

Independent Increments

$$E[(X(t_2)-X(t_1))(X(t_4)-X(t_3))]$$
$$= E\left[\int_{t_1}^{t_2} X(\alpha)d\alpha \int_{t_3}^{t_4} X(\beta)d\beta\right] = P_0\int_{t_1}^{t_2} d\alpha \int_{t_3}^{t_4} \delta(\alpha-\beta)d\beta = 0$$

Obviously $E[X(t_2)-X(t_1)] = E[X(t_4)-X(t_3)] = 0$

$\therefore$ increments are independent (Gauss. + uncorrelated)

Stationary Increments

Because the increments are independent, we only need show

$$E[(X(t_2)-X(t_1))^2] = E[(X(t_2+t_0)-X(t_1+t_0))^2]$$

$$E\left[\int_{t_1}^{t_2} X(\alpha)d\alpha \int_{t_1}^{t_2} X(\beta)d\beta\right] \qquad E\left[\int_{t_1+t_0}^{t_2+t_0} X(\alpha)d\alpha \int_{t_1+t_0}^{t_2+t_0} X(\beta)d\beta\right]$$

$$P_0\int_{t_1}^{t_2} d\alpha \int_{t_1}^{t_2} \delta(\alpha-\beta)d\beta \qquad P_0\int_{t_1+t_0}^{t_2+t_0} d\alpha \int_{t_1+t_0}^{t_2+t_0} \delta(\alpha-\beta)d\beta$$

$$P_0(t_2-t_1) = P_0(t_2-t_1)$$

Name: **SOLUTION**

3. Let N(t) be a classical Poisson process with constant rate λ. A new process M(t) is formed as follows: event times from N(t) are independently and randomly chosen as event times of M(t) with probability p ($0 < p < 1$). [This procedure is sometimes referred to as "thinning" of a Poisson process.] We may write for M(t)

$$M(t) = \begin{cases} 0, & N(t) = 0 \\ \sum_{k=1}^{N(t)} a_k, & N(t) > 0 \end{cases} \qquad P(N(t)=n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

where the random variables a_k are independent and identically distributed with probabilities

$$P(a_k = 1) = p. \quad P(a_k = 0) = 1 - p, \; k = 1, 2, \ldots \; .$$

Show that M(t) is a Poisson process and determine its rate.

Determine the ch.f. of M(t) using an argument similar to one presented for shot noise

$$E\left[e^{iuM(t)}\right] = 1 \cdot P(N(t)=0) + E_{N(t)}\left(E_{a_k}\left[e^{iu\sum_{k=1}^{N(t)} a_k} \,\middle|\, N(t)=n \right] \right)$$

$$= e^{-\lambda t} + \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \, E_{a_k}\left[e^{iu\sum_{k=1}^{n} a_k} \right]$$

a_k same iid

$$E_{a_k}\left[e^{iu\sum_{k=1}^{n} a_k} \right] = \left(E\left[e^{iua_k}\right]\right)^n = \left(pe^{iu} + (1-p)\right)^n$$

$$E\left[e^{iuM(t)}\right] = e^{-\lambda t} + \sum_{n=1}^{\infty} \frac{\left[\lambda t(pe^{iu}+1-p)\right]^n}{n!} e^{-\lambda t}$$

$$= e^{p\lambda t(e^{iu}-1)}$$

ch.f. of Poisson process with rate $p\lambda$

Rate of M(t) = $p\lambda$

Name: **SOLUTION**

4. The input, X(t), and output, Y(t), of a system are related via the differential equation

$$2\frac{d^2}{dt^2}Y(t) - 2\frac{d}{dt}Y(t) + Y(t) = \frac{d^2}{dt^2}X(t) + \frac{d}{dt}X(t) + \frac{1}{2}X(t).$$

Here X(t) is a wide sense stationary process with correlation

$$R_X(\tau) = \begin{cases} 1-|\tau|, & -1 \le \tau \le 1 \\ 0, & \textit{otherwise.} \end{cases} \qquad S_X(\omega) = \left(\frac{\sin(\omega/2)}{(\omega/2)}\right)^2$$

Determine the correlation function of Y(t).

$$H(i\omega) = \frac{(i\omega)^2 + (i\omega) + \frac{1}{2}}{2(i\omega)^2 - 2(i\omega) + 1} = \frac{1}{2}\,\frac{(i\omega)^2 + (i\omega) + \frac{1}{2}}{(i\omega)^2 - (i\omega) + \frac{1}{2}}$$

$$= \frac{1}{2}\,\frac{\left(\frac{1}{2} - \omega^2\right) + i\omega}{\left(\frac{1}{2} - \omega^2\right) - i\omega}$$

$$|H(i\omega)|^2 = \frac{1}{4}$$

$$S_Y(\omega) = |H(i\omega)|^2 S_X(\omega) = \frac{1}{4} S_X(\omega)$$

$$\therefore \quad R_Y(\tau) = \frac{1}{4} R_X(\tau) = \begin{cases} \frac{1}{4}(1-|\tau|), & -1 \le \tau \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$R_Y(\tau) = \frac{1}{4} R_X(\tau) = \begin{cases} \frac{1}{4}(1-|\tau|), & -1 \le \tau \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Name: **SOLUTION**

5. Consider the random telegraph signal

$$X(t) = X(0)\,(-1)^{N(t)}.$$

Here P [X(0) = 1] = P[X(0) = -1] = ½. N(t) is a classical Poisson process with constant rate λ and is independent of X(0). X(t) is passed through a linear, time-invariant filter to produce the process Y(t)

$$Y(t) = \int_{-\infty}^{\infty} h(\alpha) X(t-\alpha)\,d\alpha$$

From class notes

$$\left.\begin{aligned} E[X(t)] &= 0 \\ E[X(t)X(s)] &= e^{-2\lambda|t-s|} \end{aligned}\right) \text{ } X(t) \text{ is w.s.s}$$

$$S_X(\omega) = \frac{4\lambda}{4\lambda^2+\omega^2}$$

with $h(t) = \left(\frac{\sin t}{t}\right)^2$. Is Y(t) mean square differentiable? An answer not supported by appropriate reasoning will receive no credit.

Y(t) will be m.s. differentiable iff

$$\int_{-\infty}^{\infty} \omega^2 S_Y(\omega)\,d\omega < \infty$$

$$H(i\omega) = \begin{cases} \pi\left(1-\frac{|\omega|}{2}\right), & -2 \le \omega \le 2 \\ 0, & \text{otherwise} \end{cases}$$

$$|H(i\omega)|^2 = \begin{cases} \pi^2\left(1-\frac{|\omega|}{2}\right)^2, & -2 \le \omega \le 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore \int_{-\infty}^{\infty} \omega^2 S_Y(\omega)\,d\omega = \pi^2 \int_{-2}^{2} \frac{4\lambda\omega^2}{4\lambda^2+\omega^2}\left(1-\frac{|\omega|}{2}\right)^2 d\omega$$

$$\frac{4\lambda\omega^2}{4\lambda^2+\omega^2} \le 4\lambda \text{ ; } \left(1+\frac{|\omega|}{2}\right)^2 \le 1$$

$$\int_{-\infty}^{\infty} \omega^2 S_Y(\omega)\,d\omega \le 4\pi^2\lambda \int_{-2}^{2} d\omega = 16\pi^2\lambda < \infty$$

Y(t) m.s. differentiable? (Circle One)	Yes	No

ECE 250 Midterm Examination
February 15, 2008 Name: SOLUTION

1. Which of the following can be moments of a real random variable?

I. $E[X^n] = 1, \quad n = 0, 1, \ldots$
II. $E[X^n] = n, \quad n = 0, 1, \ldots$
III. $E[X^n] = \frac{n}{n+1}, \quad n = 0, 1, \ldots$

Form characteristic function

I. $\Phi_X(u) = \sum_{n=0}^{\infty} \frac{(iu)^n}{n!} E[X^n] = \sum_{n=0}^{\infty} \frac{(iu)^n}{n!} = e^{iu}$

$\Updownarrow$

$f_X(x) = \delta(x-1)$ $\boxed{\text{Yes}}$

II. $\Phi_X(u) = \sum_{n=0}^{\infty} \frac{(iu)^n}{n!} E[X^n] = \sum_{n=0}^{\infty} \frac{(iu)^n}{n!} n = (iu)e^{iu}$

not a characteristic function because $\Phi_X(0) = 0$ (not 1) $\boxed{\text{No}}$

III. $\Phi_X(u) = \sum_{n=0}^{\infty} \frac{(iu)^n}{n!} E[X^n] = \sum_{n=0}^{\infty} \frac{(iu)^n}{n!} \frac{n}{n+1}$

$= \sum_{n=0}^{\infty} \frac{(iu)^n}{(n+1)!} n = \sum_{n=0}^{\infty} \frac{(iu)^n}{n!} - \sum_{n=0}^{\infty} \frac{(iu)^n}{(n+1)!}$

$\left\{ n = p-1 \right.$ $\left\{ \frac{n}{n+1} = 1 - \frac{1}{n+1} \right.$

$= e^{iu} - \frac{1}{iu} \sum_{p=1}^{\infty} \frac{(iu)^p}{p!}$

$= e^{iu} - \frac{e^{iu} - 1}{(iu)}$

Again $\Phi_X(0) = 0$ $\boxed{\text{No}}$

Valid Moments? (circle one)	I yes no	II yes no	III yes no

Name: SOLUTION

2. The random variables X and Y are independent with probabilities

$$P(X = 1) = ½, \; P(X = -1) = ½, \; P(Y = 1) = ½, \; P(Y = -1) = ½.$$

Prove that X and Z = XY are independent.

Show that $\Phi_{X,Z}(u,v) = \Phi_X(u)\Phi_Z(v)$

$$\Phi_{X,Z}(u,v) = E\left[e^{iuX + ivXY}\right]$$

independence of X & Y

$$= \frac{1}{4}e^{iu+iv} + \frac{1}{4}e^{iu-iv} + \frac{1}{4}e^{-iu-iv} + \frac{1}{4}e^{-iu+iv}$$

$$= \frac{1}{2}\cos(u+v) + \frac{1}{2}\cos(u-v) = \cos u \cos v$$

Now note that

$$P(Z=1) = P(X=1, Y=1) + P(X=-1, Y=-1) = \frac{1}{2}$$

similarly $P(Z=-1) = \frac{1}{2}$

Now $\Phi_X(u) = \frac{1}{2}e^{iu} + \frac{1}{2}e^{-iu} = \cos u$

$\Phi_Z(v) = \frac{1}{2}e^{iv} + \frac{1}{2}e^{-iv} = \cos v$

Clearly

$$\Phi_{X,Z}(u,v) = \Phi_X(u)\Phi_Z(v)$$

and X and Z are independent

Name: SOLUTION

3. Consider the process X(t)

$$X(t) = A\cos(\omega t + \Phi).$$

Here A and Φ are real constants and the radian frequency ω is a Gaussian random variable with zero mean and unit variance. This process is not W.S.S. Determine the averaged correlation

$$\bar{R}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E\,[X(t+\tau)\,X(t)]\,dt.$$

$$E[X(t+\tau)X(t)] = A^2 E[\cos(\omega t + \omega\tau + \Phi)\cos(\omega t + \Phi)]$$
$$= \frac{A^2}{2} E[\cos\omega\tau] + \frac{A^2}{2} E[\cos(\omega 2t + \omega\tau + 2\Phi)]$$

$\boxed{\text{From HW No. 1}}$

$$E[\cos\omega\tau] = e^{-\tau^2/2}; \quad E[\cos(\omega 2t + \omega\tau + 2\Phi)] = \cos 2\Phi\, e^{-(2t+\tau)^2/2}$$

Clearly $\displaystyle \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-\tau^2/2}\,dt = e^{-\tau^2/2}$

$\boxed{2t+\tau = \alpha}$

Now $\displaystyle \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos 2\Phi\, e^{-(2t+\tau)^2/2}\,dt = \cos 2\Phi \lim_{T\to\infty} \frac{1}{2T}\int_{-2T+\tau}^{2T+\tau} e^{-\alpha^2/2}\,\frac{d\alpha}{2}$

but $\displaystyle 0 \le \int_{-2T+\tau}^{2T+\tau} e^{-\alpha^2/2}\,d\alpha/2 \le \int_{-\infty}^{\infty} e^{-\alpha^2/2}\,d\alpha/2 = \sqrt{\frac{\pi}{2}}$

$\boxed{\text{Gaussian density}}$

$\therefore \displaystyle \cos 2\Phi \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-(2t+\tau)^2/2}\,dt \le \cos 2\Phi \lim_{T\to\infty} \frac{\sqrt{\pi/2}}{2T} = 0$

$$\boxed{\bar{R}(\tau) = \frac{A^2}{2} e^{-\tau^2/2}}$$

ECE 250 FINAL EXAMINATION
March 17, 2008

Name: SOLUTION

1. The random variables X and Y are independent with densities

$$f_X(x) = \begin{cases} e^{-x}, & x \ge 0 \\ 0, & x < 0 \end{cases} \qquad f_Y(y) = \begin{cases} e^{-y}, & y \ge 0 \\ 0, & y < 0. \end{cases}$$

Find the probability density of

$$Z = \frac{X}{X+Y}.$$

clearly $0 \le Z \le 1 \therefore f_Z(z) = 0, z < 0$

$f_Z(z) = 0, z > 1$

$$F_Z(z) = P(Z \le z) = P\left(\frac{X}{X+Y} \le z\right)$$

$$= P\left(X \le \frac{z}{1-z} Y\right) \qquad 0 \le z \le 1$$

$$= \iint\limits_{x \le \frac{z}{1-z} y} f_X(x) f_Y(y)\, dx\, dy$$

$$= \int_0^\infty e^{-y} dy \int_0^{\frac{z}{1-z} y} e^{-x} dx$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{1}{(1-z)^2} \int_0^\infty y e^{-y} e^{-\frac{z}{1-z} y} dy$$

$$= 1, \quad 0 \le z \le 1$$

$$f_Z(z) = \begin{cases} 1, & 0 \le z \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Name: SOLUTION

2. The random process X(t) is given by X(t) = u(t-T) where

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

and

$$f_T(\tau) = 1/2\, e^{-|\tau|}.$$

Evaluate the correlation

$$R_X(t, s) = E[X(t)X(s)].$$

$\underline{\underline{s \leq t}}$ $R_X(t,s) = E[u(t-T)u(s-T)] = \int_{-\infty}^{s} f_T(\tau)\,d\tau = F_T(s)$

$u(t-\tau)u(s-\tau) = u(s-\tau)$

$u(t-\tau)u(s-\tau) = u(t-\tau)$

$\underline{\underline{t < s}}$ $R_X(t,s) = E[u(t-T)u(s-T)] = \int_{-\infty}^{t} f_T(\tau)\,d\tau = F_T(t)$

$\therefore$ $R_X(t,s) = F_T(\min[s,t])$

$$= \begin{cases} \frac{1}{2} e^{\min[s,t]}, & \min[s,t] < 0 \\ 1 - \frac{1}{2} e^{-\min[s,t]}, & \min[s,t] \geq 0 \end{cases}$$

$$\boxed{R_X(t, s) = F_T(\min[s,t]) = \begin{cases} \frac{1}{2} e^{\min[s,t]}, & \min[s,t] < 0 \\ 1 - \frac{1}{2} e^{-\min[s,t]}, & \min[s,t] \geq 0 \end{cases}}$$

Name: SOLUTION

3. An integer-valued random process M(t) has independent increments, M(0) = 0 and the probability of an increment is given by

$$P\,(M(t) - M(s) = n) = \frac{(t-s)^n}{[1+(t-s)]^{n+1}}, \quad n = 0,1,\ldots \quad s \le t.$$

A new process is formed from M(t) as follows

$$X(t) = X(0)\,(-1)^{M(t)}$$

Here X(0) is independent of M(t) and assumes the values +1 and -1 with equal probability. Obtain an expression for the correlation function $R_X(t, s) = E[X(t)X(s)]$.

$\underline{s \le t}$

$$R_X(t,s) = E[X(t)X(s)] = E\left[X(0)(-1)^{M(t)}X(0)(-1)^{M(s)}\right]$$

$$= E\left[(-1)^{M(t)-M(s)}\right] = E\left[e^{i\pi[M(t)-M(s)]}\right]$$

$$= \Phi_{M(t)-M(s)}(\pi)$$

$$\Phi_{M(t)-M(s)}(u) = \sum_{n=0}^{\infty} \frac{(t-s)^n}{[1+(t-s)]^{n+1}} e^{+inu}$$

$$= \frac{1}{1+(t-s)} \sum_{n=0}^{\infty} \left(\frac{(t-s)e^{+iu}}{1+(t-s)}\right)^n$$

From "useful formulas"

$$= \frac{1}{1+(t-s)(1-e^{iu})}$$

$$\Phi_{M(t)-M(s)}(\pi) = \frac{1}{1+2(t-s)}$$

Similarly for $t < s$ so that

$$R_X(t,s) = \frac{1}{1+2|t-s|}$$

$$\boxed{R_X(t,s) = \frac{1}{1+2|t-s|}}$$

Name: SOLUTION

4. A random process is defined by the sum

$$X(t) = \sum_{n=-\infty}^{\infty} X_n\, g(t-nT_0).$$

Here g(t) is a real, non-random function that is square integrable over $(-\infty,\infty)$. The random variables X_n are zero mean and satisfy

$$E[X_n X_m] = \begin{cases} \sigma^2, & n = m \\ 0, & n \neq m. \end{cases}$$

In general, processes of this sort are neither stationary nor wide sense stationary. The process is, however, wide sense cyclostationary, that is

$$E[X(t+T_0)] = E[X(t)] \quad \text{and} \quad E[X(t+T_0)X(s+T_0)] = E[X(t)X(s)].$$

Define a new process by $Y(t) = X(t+W)$

where W is a random variable that is independent of the X_n s with density

$$f_W(w) = \begin{cases} \frac{1}{T_0}, & 0 \le w \le T_0 \\ 0, & \text{otherwise.} \end{cases}$$

$$E[X(t+\tau)|W] = \sum_{n=-\infty}^{\infty} E[X_n]\, g(t-nT_0) = 0$$

$$\therefore E[Y(t)] = 0$$

Show that Y(t) is wide sense stationary.

$$E[Y(t+\tau)Y(t)|W] = E\left[\sum_{n=-\infty}^{\infty} X_n\, g(t+\tau+W-nT_0) \sum_{m=-\infty}^{\infty} X_m\, g(t+W-mT_0)\right]$$

$$= \sigma^2 \sum_{n=-\infty}^{\infty} g(t+\tau+W-nT_0)\, g(t+W-nT_0)$$

$$R_Y(t+\tau, t) = \frac{1}{T_0}\int_0^{T_0} \sigma^2 \sum_{n=-\infty}^{\infty} g(t+\tau+w-nT_0)\, g(t+w-nT_0)\, dw$$

$\boxed{\text{define } \alpha = t+w-nT_0}$

$$= \frac{\sigma^2}{T_0} \sum_{n=-\infty}^{\infty} \int_{t-nT_0}^{t-nT_0+T_0} g(\tau+\alpha)\, g(\alpha)\, d\alpha$$

The sum of the individual integrals is equivalent to an integral over $(-\infty,\infty)$

$$= \frac{\sigma^2}{T_0} \int_{-\infty}^{\infty} g(\tau+\alpha)\, g(\alpha)\, d\alpha$$

clearly just a function of τ

Name: SOLUTION

5. Consider the shot process (shot noise)

$$X(t) = \sum_k h(t-t_k).$$

Here the t_k s are governed by a classical Poisson process (constant rate λ, $N(0) = 0$) and $h(t)$ is the non-random function

$$h(t) = e^{-|t|}$$

Examine the correlation function $R_X(t,s)$

Is X(t) m.s. differentiable?

$$\Phi_{X(t),X(u)}(u,v) = e^{\lambda \int_{-\infty}^{\infty} \left[e^{iuh(t-\alpha)+ivh(s-\alpha)} - 1 \right] d\alpha}$$

$$R_X(t,s) = \left(\frac{1}{i}\right)^2 \frac{\partial^2}{\partial u \partial v} \Phi_{X(t),X(s)}(u,v) \Big|_{u=v=0}$$

$$(*) \qquad = \lambda \int_{-\infty}^{\infty} h(t-s+\alpha) h(\alpha) + \left(\lambda \int_{-\infty}^{\infty} h(\alpha) d\alpha\right)^2$$

$\therefore$ $X(t)$ is w.s.s.

It will be m.s. differentiable if

$$(**) \qquad \int_{-\infty}^{\infty} \omega^2 S_X(\omega) d\omega < \infty$$

From $(*)$ $\quad S_X(\omega) = \lambda |H(i\omega)|^2 + \lambda^2 \left(\int_{-\infty}^{\infty} h(\alpha) d\alpha\right)^2 2\pi \delta(\omega)$

but from Fourier Transform tables

$$H(i\omega) = \frac{2}{1+\omega^2}$$

Thus $(**)$ appears as

$$\int_{-\infty}^{\infty} \omega^2 S_X(\omega) = 4\lambda \int_{-\infty}^{\infty} \frac{\omega^2}{(1+\omega^2)^2} d\omega + 2\pi \lambda^2 \left(\int_{-\infty}^{\infty} h(\alpha) d\alpha\right)^2 \int_{-\infty}^{\infty} \omega^2 \delta(\omega) d\omega$$

$$= 4\lambda \int_{-\infty}^{\infty} \frac{\omega^2}{(1+\omega^2)^2} d\omega \quad < \infty$$

m.s. differentiable?	Yes	No

ECE 250 MIDTERM

February 12, 2007

Name: SOLUTION

1. The random variable X is uniformly distributed on [0,1). Define

$$Y = X - X^2.$$

Prove that X and Y are uncorrelated but not independent.

$$E[X] = \int_0^1 x\,dx = \frac{1}{2}$$

$$E[Y] = \int_0^1 (x - x^2)\,dx = \frac{1}{6}$$

$$E[XY] = \int_0^1 (x^2 - x^3)\,dx = \frac{1}{12}$$

Clearly $E[XY] = E[X]\,E[Y]$ and so are uncorrelated

If X and Y are independent

$E[X^n Y^m] = E[X^n] \cdot E[Y^m]$ for any n, m

Try $n = 2,\ m = 1$

$E[X^2] = \frac{1}{3}$; $E[Y] = \frac{1}{6}$ (from above)

$$E[X^2 Y] = \int_0^1 (x^3 - x^4)\,dx = \frac{1}{20}$$

clearly

$$E[X^2 Y] \neq E[X^2]\,E[Y]$$

and X and Y are NOT independent

Name: SOLUTION

2. The random variables $X_1, X_2, \ldots, X_n$ are independent with common density

$$f_{X_k}(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x \geq 0 \end{cases} \qquad k = 1, 2, \ldots, n.$$

For fixed n form the sum

$$Y = \sum_{k=1}^{n} X_k$$

Find a bound on the probability $P(Y \geq \lambda)$ that decreases at least exponentially as $\lambda \to \infty$.

$$P(Y \geq \lambda) \leq e^{-s\lambda} E[e^{sY}], \quad s>0, \text{ Chernoff bound}$$

$$E[e^{sY}] = E[e^{s\sum_{k=1}^{n} X_k}] \underset{\uparrow}{=} \prod_{k=1}^{n} E[e^{sX_k}]$$

independence

$$= \left(\int_0^\infty e^{-x} e^{sx}\, dx\right)^n$$

$$= \left(\frac{1}{1-s}\right)^n \quad s \geq 0 \text{ but also for convergence we must have } s<1$$

$$\therefore P(Y \geq \lambda) \leq e^{-s\lambda}\left(\frac{1}{1-s}\right)^n = e^{-s\lambda - n\ln(1-s)}$$

minimize exponent – denote minimums by s_0

$$\frac{d}{ds}\left(-s\lambda - n\ln(1-s)\right)\Big|_{s=s_0} \overset{\text{set}}{=} 0 = -\lambda + \frac{n}{1-s_0}$$

$$s_0 = 1 - \frac{n}{\lambda} \longleftrightarrow \text{for } s_0<1 \text{ we must have } \lambda > n$$

Finally $P(Y \geq \lambda) \leq e^{-\lambda(1-\frac{n}{\lambda}) - n\ln(n/\lambda)}$

$$= e^{-\lambda\left(1 - \frac{n}{\lambda^2} - \frac{n\ln n}{\lambda} + \frac{n\ln\lambda}{\lambda}\right)}$$

$$\boxed{P(Y \geq \lambda) \leq e^{-\lambda(1-\frac{n}{\lambda}) - n\ln(n/\lambda)} = e^{-\lambda\left(1 - \frac{n}{\lambda^2} - \frac{n\ln n}{\lambda} + \frac{n\ln\lambda}{\lambda}\right)}}$$

Name: SOLUTION

3. The probability of k "events" of a Poisson process in the interval [0, t) is given by

$$P(N(t)=k)=\frac{(\lambda t)^k}{k!}e^{-\lambda t}, k=0,1,2,\ldots$$

with $\lambda > 0$. Define T_n as the time to the n-th "event." Determine the probability density of T_n. [Hint: relate the <u>distribution</u> of T_n to a probability involving N(t).]

$$F_{T_n}(t) = P(T_n \le t) = P(N(t) \ge n)$$

$$= \sum_{k=n}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

$$f_{T_n}(t) = \frac{d}{dt} F_{T_n}(t) = \sum_{k=n}^{\infty} \lambda k \frac{(\lambda t)^{k-1}}{k!} e^{-\lambda t} - \lambda \sum_{k=n}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

$k = p+1$ $\qquad$ $k = p$

$$= \lambda \sum_{p=n-1}^{\infty} \frac{(\lambda t)^p}{p!} e^{-\lambda t} - \sum_{p=n}^{\infty} \frac{(\lambda t)^p}{p!} e^{-\lambda t}$$

$$= \lambda \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t}$$

$$\boxed{f_{T_n}(t) = \lambda \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t}}$$

ECE 250 FINAL EXAMINATION

March 19, 2007

Name: SOLUTION

1. Let X(t) be a zero-mean, wide sense stationary, Gaussian process with correlation function $R(\tau) = \cos(\tau)$. A new process, Y(t) is obtained by squaring X(t)

$$Y(t) = X^2(t).$$

The process Y(t) is passed through a linear, time invariant filter with impulse response

$$h(t) = \begin{cases} e^{-t}, t \geq 0 \\ 0, t < 0. \end{cases}$$

Find the correlation function of the output of this filter, Z(t).

$$E[Y(t)] = E[X^2(t)] = R_X(0) = 1$$

$$E[Y(t)Y(s)] = E[X(t)X(t)X(s)X(s)] = R_X^2(0) + 2R_X^2(t-s)$$

From Prob. Set 2

$\therefore$ Y(t) is W.S.S.

$$S_Y(\omega) = \int_{-\infty}^{\infty} [R_X^2(0) + 2R_X^2(\tau)] e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} [1 + 2\cos^2\tau] e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} [2 + \tfrac{1}{2} e^{i2\tau} + \tfrac{1}{2} e^{-i2\tau}] e^{-i\omega\tau} d\tau$$

$$= 4\pi\delta(\omega) + \pi\delta(\omega-2) + \pi\delta(\omega+2)$$

$$H(i\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt = \int_0^{\infty} e^{-t} e^{-i\omega t} dt = \frac{1}{1+i\omega}$$

$\therefore$

$$S_Z(\omega) = |H(i\omega)|^2 S_Y(\omega) = \frac{1}{1+\omega^2}[4\pi\delta(\omega) + \pi\delta(\omega-2) + \pi\delta(\omega+2)]$$

Z(t) is also W.S.S.

$$R_Z(t,s) = R_Z(t-s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Z(\omega) e^{i\omega\tau} d\omega$$

$$= 2 + \frac{1}{5}\cos 2\tau$$

$$\boxed{R_Z(t,s) = 2 + \frac{1}{5}\cos 2\tau}$$

Name: SOLUTION

2. The random processes X(t) and Y(t) are independent and wide sense stationary with

$$E[X(t)] = E[Y(t)] = m \neq 0$$

$$R_X(\tau) = R_Y(\tau) = R(\tau).$$

$Z(t)$ is NOT W.S.S.

A new process is formed by

$$Z(t) = X(t)\cos 2\pi t + Y(t)\sin 2\pi t.$$

This new process is passed through a linear, time-invariant filter with impulse response

$$h(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Denote output by

$$W(t) = \int_{-\infty}^{\infty} Z(t-\alpha)h(\alpha)\,d\alpha = \int_0^1 Z(t-\alpha)\,d\alpha$$

Is the filter output wide sense stationary? No ~~credit will be given for an answer that is not~~ supported by appropriate reasoning.

(using appropriate trig. identities)

$$R_Z(t,s) = E[Z(t)Z(s)] = R(t-s)\cos 2\pi(t-s) + m^2 \sin 2\pi(t+s)$$

Also

$$E[Z(t)] = m\{\cos 2\pi t + \sin 2\pi t\}$$

Now

$$E[W(t)] = m\int_0^1 \{\cos 2\pi(t-\alpha) + \sin 2\pi(t-\alpha)\}\,d\alpha = 0$$

$$E[W(t)W(s)] = \int_0^1\int_0^1 R_Z(t-\alpha, s-\beta)\,d\alpha\, d\beta$$

$$= \int_0^1\int_0^1 R(t-s+\beta-\alpha)\cos 2\pi(t-s+\beta-\alpha)\,d\alpha\, d\beta$$

$$+ m^2\int_0^1\int_0^1 \sin 2\pi(t+s-\beta-\alpha)\,d\alpha\, d\beta$$

This integral = 0 because the integration is over an integer number of periods

clearly a function of $t-s$

$$\therefore E[W(t)W(s)] = \int_0^1\int_0^1 R(t-s+\beta-\alpha)\cos 2\pi(t-s+\beta-\alpha)\,d\alpha\, d\beta$$

Output W.S.S.? (circle one)	Yes	No

Name: SOLUTION

3. The characteristic function of a real random variable satisfies the relationship

$$\Phi(\beta\upsilon) = [\Phi(\upsilon)]^{|\beta|} \quad -\infty < \beta < \infty$$

Clearly $\Phi(\upsilon) = 1$ satisfies this relationship. In addition to $\Phi(\upsilon) = 1$, what other non-trivial characteristic functions satisfy this relationship and what is the functional form of the associated probability density, f(x).

The proof follows the outline of the similar problem done in class

Obviously $\Phi(-\nu) = \Phi^*(\nu)$

letting $\beta = -1$

$$\Phi(-\nu) = \Phi(\nu) = \Phi^*(\nu) \quad \therefore \Phi(\nu) \text{ is real}$$

let $\nu = 1$

$$\Phi(\beta) = [\Phi(1)]^{|\beta|}$$ this holds for all real β

$\Phi(1)$ is real, non-negative and < 1

If $\Phi(1) < 0$, then $\Phi(\tfrac{1}{2}) = \sqrt{\Phi(1)}$ which cannot happen because $\Phi(\nu)$ is real

If $\Phi(1) = 1$, then $\Phi(\nu) = 1$ for all ν which is not the solution we seek

$\therefore \Phi(\beta) = e^{-k|\beta|}$ for some $k > 0$ and all β

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k|\beta|} e^{-i\beta x}\, d\beta = \frac{k}{\pi(k^2+x^2)}$$ From class notes

$$f(x) = \frac{k}{\pi(k^2+x^2)} \qquad k > 0$$

Name: SOLUTION

4. The wide sense stationary process X(t) is bound limited (i.e., its power spectral density vanishes for $|\omega| \geq \Omega$). Show that X(t) has mean square derivatives of all orders.

$X^{(1)}(t)$ exists iff $R_X^{(2)}(0)$ is well-defined

Now note $E[X^{(1)}(t)X^{(1)}(s)] = \frac{\partial^2}{\partial t \partial s} E[X(t)X(s)] = -R_X^{(2)}(t-s)$

It follows that

$X^{(2)}(t)$ exists iff $R_X^{(4)}(0)$ is well-defined

Using this same reasoning, we conclude that $X^{(n)}(t)$ exists iff $R_X^{(2n)}(0)$ is well-defined

Now

$$R_X^{(2n)}(0) = \frac{(-1)^n}{2\pi} \int_{-\infty}^{\infty} \omega^{2n} S_X(\omega)\, d\omega$$

but $S_X(\omega) = 0$ when $|\omega| \geq \Omega$ so that

$$R_X^{(2n)}(0) = \frac{(-1)^n}{2\pi} \int_{-\Omega}^{\Omega} \omega^{2n} S_X(\omega)\, d\omega$$

the integrand is bounded by $\Omega^{2n} S(\omega)$

(i.e. $\omega^{2n} S_X(\omega) \leq \Omega^{2n} S_X(\omega)$)

$$\therefore \quad R_X^{(2n)}(0) \leq \frac{\Omega^{2n}}{2\pi} \int_{-\Omega}^{\Omega} S_X(\omega)\, d\omega = \frac{\Omega^{2n}}{2\pi} R_X(0) < \infty$$

Clearly, then, $X^{(n)}(t)$ exists for all n.

Name: SOLUTION

5. Let N(t) be a classical Poisson process with constant rate ν.

$$P(N(t)=k)=\frac{(\nu t)^k}{k!}e^{-\nu t}.$$

Let X_n, n=1, 2, … be a sequence of independent, identically-distributed random variables with probabilities

$$P(X_n = 1) = \rho \qquad P(X_n = 0) = 1 - \rho$$

with $0 < \rho < 1$. N(t) and the X_ns are independent. For fixed t form the process

$$Y(t) = \begin{cases} 0, & N(t) = 0 \\ \sum_{k=1}^{N(t)} X_k, & N(t) > 0. \end{cases}$$

Find an upper bound on the probability

$$P(Y(t) \geq \lambda)$$

that decreases more rapidly than $e^{-k\lambda}$ as $\lambda \to \infty$ (k is any positive constant). [Hint: You may find the use of conditional expectations helpful in deriving the bound.]

Using the Chernoff bound

$$P(Y(t) \geq \lambda) \leq e^{-s\lambda}E[e^{sY(t)}], \qquad s \geq 0$$

$$E[e^{sY(t)}] = P(N(t)=0) + \sum_{n=1}^{\infty} P(N(t)=n)\, E[e^{s\sum_{k=1}^{n} X_k}]$$

$$E[e^{s\sum_{k=1}^{n} X_k}] = E[\prod_{k=1}^{n} e^{sX_k}] = (E[e^{sX_k}])^n$$

X_ks are i.i.d.

$$= (1-\rho+\rho e^{s})^n$$

$$\boxed{P(Y(t) \geq \lambda) \leq e^{-\lambda \ln\left(\frac{\lambda}{\rho\nu t}\right) - \rho\nu t + \lambda} = e^{-\lambda \ln\lambda\left[1 - \frac{\ln \rho\nu t}{\ln\lambda} + \frac{\rho\nu t}{\lambda\ln\lambda} - \frac{1}{\ln\lambda}\right]}}$$

$$\boxed{\lambda \geq \rho\nu t}$$

Problem 5 continued

$$\therefore \quad E[e^{s\tilde{Y}(t)}] = P(N(t)=0) + \sum_{n=1}^{\infty} P(N(t)=n)(1-\rho+\rho e^{s})^{n}$$

$$= \sum_{n=0}^{\infty} \frac{(\nu t)^n (1-\rho+\rho e^s)^n}{n!} e^{-\nu t}$$

$$= e^{\rho\nu t(e^s - 1)}$$

So that for any $s \geq 0$

$$(*) \quad P(\tilde{Y}(t) \geq \lambda) \leq e^{-s\lambda + \rho\nu t(e^s-1)}$$

$$\leq \min_{s \geq 0} e^{-s\lambda + \rho\nu t(e^s - 1)}$$

minimize this exponent (denote by s_0)

s_0 is the solution of

$$0 = \rho\nu t e^{s_0} - \lambda$$

or

$$s_0 = \ln \frac{\lambda}{\rho\nu t}$$

clearly for $s_0 \geq 0$ we must have $\lambda \geq \rho\nu t$

Substituting $s = s_0$ into (*) above yields

$$P(\tilde{Y}(t) \geq \lambda) \leq e^{-\lambda \ln\left(\frac{\lambda}{\rho\nu t}\right) - \rho\nu t + \lambda}$$

$$= e^{-\lambda \ln\lambda \left[1 - \frac{\ln \rho\nu t}{\ln\lambda} + \frac{\rho\nu t}{\lambda \ln\lambda} - \frac{1}{\ln\lambda}\right]}$$

$$\lambda \geq \rho\nu t$$

ECE 250 MIDTERM EXAMINATION

February 22, 2006 Name: SOLUTION

1. The random variable X and Y are independent and they are Gaussian with zero mean and unit variance. Define

$$V = \max[X, Y], \qquad W = \min[X, Y].$$

Find the probability density of $Z = V - W$.
[Hint: express Z in terms of the original variables X and Y.]

If $X \geq Y$ $\quad V = X, \quad W = Y; \quad Z = X - Y$

If $Y > X$ $\quad V = Y, \quad W = X; \quad Z = Y - X$

$\therefore \quad Z = |X - Y|$ in all cases

But $X - Y$ is Gaussian with

$E[X - Y] = 0$

$\mathrm{Var}[X - Y] = 2$

Define $D = X - Y$ and $f_D(\delta) = \frac{1}{2\sqrt{\pi}} e^{-\frac{\delta^2}{4}}$

Finally

$$F_Z(z) = P(Z \leq z) = P(-z < D \leq z)$$

$$= \int_{-z}^{z} f_D(\delta)\, d\delta = \frac{1}{\sqrt{\pi}} \int_0^{z} e^{-\frac{\delta^2}{4}} d\delta$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{1}{\sqrt{\pi}} e^{-\frac{z^2}{4}}, \quad z \geq 0$$

$$\boxed{f_Z(z) = \begin{cases} 0, & z < 0 \\ \frac{1}{\sqrt{\pi}} e^{-\frac{z^2}{4}}, & z \geq 0 \end{cases}}$$

Name: SOLUTION

2. The process $X(t)$ is a real, second order process. Which of the following can be valid autocorrelations of this process?

(I) $R_x(\tau) = e^{-\cos\tau - 1}$

(II) $R_x(\tau) = e^{\cos\tau - 1}$

(III) $R_x(\tau) = 2e^{-3|\tau|} - e^{-|\tau|}$

A simple "yes" or "no" will receive no credit unless supported by valid reasoning.

Case (I) $R_X(0) = e^{-2}$; $R_X(\pi) = e^{0} = 1$

$\therefore$ $|R_X(\pi)| > R_X(0)$ $\therefore$ NOT an autocorrelation

Case (II) $e^{\cos\tau - 1}$ has the same form as the characteristic function $[e^{\cos u - 1}]$ on the homework. Therefore $R_X(\tau)$ is non-negative definite and IS an autocorrelation.

Case (III) $R_X(\tau) = 2e^{-3|\tau|} - e^{-|\tau|} \leftrightarrow S_X(\omega) = \frac{12}{9+\omega^2} - \frac{2}{1+\omega^2}$

$$S_X(\omega) = \frac{-6 + 10\omega^2}{(9+\omega^2)(1+\omega^2)}$$

If $|\omega| < \sqrt{\frac{3}{5}}$ $S_X(\omega) < 0$ $\therefore$ NOT an autocorrelation

Valid autocorrelation? (circle one)	(I)	(II)	(III)
	yes no	yes no	yes no

Name: SOLUTION

3. Let $N(t)$ be a classical Poisson process (stationary, independent increments, $N(0) = 0$; constant rate ν). Consider the new process

$$P(N(t)=k) = \frac{(\nu t)^k}{k!} e^{-\nu t}$$

$$M(t) = \begin{cases} \frac{N(t)}{2}, & N(t) = \textit{even} \\ 0, & N(t) = \textit{odd} \end{cases}$$

from class notes

Find the expected value of $M(t)$.

$$P(M(t)=0) = P(N(t)=\text{odd}) = \tfrac{1}{2}(1-e^{-2\nu t})$$

$$P(M(t)=\ell) = P(N(t)=2\ell) = \frac{(\nu t)^{2\ell}}{(2\ell)!} e^{-\nu t}$$

$$E[M(t)] = \sum_{\ell=0}^{\infty} \ell\, P(M(t)=\ell) = \sum_{\ell=1}^{\infty} \ell \frac{(\nu t)^{2\ell}}{(2\ell)!} e^{-\nu t}$$

$$= \frac{1}{2}\sum_{\ell=1}^{\infty} 2\ell \frac{(\nu t)^{2\ell}}{(2\ell)!} e^{-\nu t} = \frac{1}{2}\sum_{\ell=1}^{\infty} \frac{(\nu t)^{2\ell}}{(2\ell-1)!} e^{-\nu t}$$

$2\ell-1=p$

$$= \left(\frac{\nu t}{2}\right) \sum_{\substack{p=1 \\ p\ \text{odd}}}^{\infty} \frac{(\nu t)^p}{p!} e^{-\nu t}$$

$$= \left(\frac{\nu t}{2}\right)\left(\frac{1}{2}\right)\left(1-e^{-2\nu t}\right)$$

$$\boxed{E[M(t)] = \frac{\nu t}{4}\left(1-e^{-2\nu t}\right)}$$

ECE 250 FINAL EXAMINATION

March 20, 2006 Name: SOLUTION

1. A random process is defined by

$$X(t) = A + Bt, \ t \geq 0$$

with

$$f_{A,B}(\alpha,\beta) = \begin{cases} e^{-\alpha}, & 0 \leq \beta \leq \alpha < \infty \\ 0, & \text{otherwise.} \end{cases}$$

clearly $X(t) \geq 0$ so that $f_{X(t)}(x) = 0$ when $x < 0$

Determine the first-order probability density of $X(t)$ as a function of t. That is, determine $f_{X(t)}(x)$.

$$F_{X(t)}(x) = P(X(t) \leq x) = P(A + Bt \leq x)$$

$$= \iint_{\alpha+\beta t \leq x} f_{A,B}(\alpha,\beta)\, d\alpha\, d\beta = \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{x-\beta t} f_{A,B}(\alpha,\beta)\, d\alpha$$

$$f_{X(t)}(x) = \frac{d}{dx} F_{X(t)}(x) = \int_{-\infty}^{\infty} f_{A,B}(x-\beta t, \beta)\, d\beta$$

Note that $f_{A,B}(x-\beta t,\beta) = \begin{cases} e^{-(x-\beta t)}, & 0 \leq \beta \leq (x-\beta t) \\ 0, & \text{otherwise} \end{cases}$

$$f_{X(t)}(x) = \int_0^{\frac{x}{1+t}} e^{-(x-\beta t)}\, d\beta = e^{-x} \frac{1}{t}\left(e^{x\frac{t}{1+t}} - 1\right)$$

$$= \frac{1}{t}\left(e^{-\frac{x}{1+t}} - e^{-x}\right)$$

$$\boxed{f_{X(t)}(x) = \begin{cases} \frac{1}{t}\left(e^{-\frac{x}{1+t}} - e^{-x}\right), & x \geq 0 \\ 0, & x < 0 \end{cases}}$$

Name: SOLUTION

2. Let $N(t)$ be a classical Poisson process with rate λ. Define the sequence of times t_n by

$$t_n = 2\frac{n}{n+1}, \quad n = 1, 2, \ldots,$$

Consider the new random variable

$$M = \sum_{n=1}^{\infty} N(t_{n+1}) - N(t_n)$$

Obtain an explicit expression for

$$P(M = m) \quad m = 0, 1, 2, \ldots$$

$$M = [N(t_2) - N(t_1)] + [N(t_3) - N(t_2)] + [N(t_4) - N(t_3)] + \cdots$$

$$M = \lim_{n \to \infty} N(t_n) - N(t_1) = N(2) - N(1)$$

$$P(M = m) = P(N(2) - N(1) = m)$$

$$= \frac{\lambda^m}{m!} e^{-\lambda}$$

$$\boxed{P(M = m) = \frac{\lambda^m}{m!} e^{-\lambda}, \quad m = 0, 1, \ldots}$$

Name: SOLUTION

3. Let $X(t)$ be the random telegraph signal

$$X(t) = X(0)\,(-1)^{N(t)}$$

where $N(t)$ is a Poisson process with constant rate ν and the random variable $X(0)$ is independent of $N(t)$ with $P(X(0) = +1) = \frac{1}{4}$ and $P(X(0) = -1) = \frac{3}{4}$. The time average of $X(t)$ over an interval $(0, T)$ is given by

$$M(T) = \frac{1}{T}\int_0^T X(t)\,dt.$$

From class

$$E[X(t)X(s)] = e^{-2\nu|t-s|}$$

Show that for $\varepsilon > 0$

$$\lim_{T\to\infty} P(|M(T)| > \varepsilon) = 0$$

[Hint: Chebyshev Inequality]

$$P(|M(T)| > \epsilon) \le \frac{1}{\epsilon^2}E[|M(T)|^2] = \frac{1}{\epsilon^2}E\left[\left(\frac{1}{T}\int_0^T X(t)\,dt\right)\left(\frac{1}{T}\int_0^T X(s)\,ds\right)\right]$$

$$\le \frac{1}{\epsilon^2T^2}\int_0^T\int_0^T E[X(t)X(s)]\,dt\,ds$$

$$\le \frac{1}{\epsilon^2T^2}\left(\int_0^T\int_0^T e^{-2\nu|t-s|}\,dt\,ds\right)$$

From HW #3

$$\le \frac{1}{\epsilon^2T^2}\int_{-T}^{T}[T-|\tau|]\,e^{-2\nu|\tau|}\,d\tau$$

$$\le \frac{2}{\epsilon^2T^2}\int_0^T[T-\tau]\,e^{-2\nu\tau}\,d\tau$$

Symmetry of integrand

Note that if $0 \le \tau \le T$ then $0 \le T-\tau \le T$

$$\le \frac{2}{\epsilon^2T}\int_0^T e^{-2\nu\tau}\,d\tau = \frac{2}{\epsilon^2}\,\frac{1-e^{-2\nu T}}{2\nu T}$$

Clearly then $\lim_{T\to\infty} P(|M(T)| > \epsilon) \le \lim_{T\to\infty}\frac{2}{\epsilon^2}\left(\frac{1-e^{-2\nu T}}{2\nu T}\right) = 0$

Name: SOLUTION

4. Consider the random process $X(t) = \sum_{n=1}^{\infty} Xn \cos n\pi t + Yn \sin n\pi t$.

The coefficients Xn and Yn are i.i.d. sequences with densities

$$f_{Xn}(x) = \frac{n}{\sqrt{2\pi}} e^{-\frac{n^2x^2}{2}}; \; f_{Yn}(y) = \frac{n}{\sqrt{2\pi}} e^{-\frac{n^2y^2}{2}}.$$

In addition the sequences are independent (i.e., Xn and Yn are independent for any choice of n and m). Is $X(t)$ mean square differentiable? You must prove your answer. A simple "yes" or "no" unaccompanied by appropriate reasoning will not receive any credit.

$$E[X(t)X(s)] = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E\left[(X_n \cos n\pi t + Y_n \sin n\pi t)(X_m \cos m\pi s + Y_m \sin m\pi s)\right]$$

Note that: $E[X_n X_m] = E[Y_n Y_m] = \frac{1}{n^2}\delta_{n,m}$

$E[X_n Y_m] = E[Y_n X_m] = 0$

$$\therefore \; E[X(t)X(s)] = \sum_{n=1}^{\infty} \frac{1}{n^2}(\cos n\pi t \cos n\pi s + \sin n\pi t \sin n\pi s)$$

$$R(t,s) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi (t-s)$$

Clearly $X(t)$ is W.S.S.

$$R(\tau) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi\tau$$

$X(t)$ will be m.s. differentiable iff $\frac{d^2}{d\tau^2} R(\tau)\Big|_{\tau=0}$ exists

$$-\frac{d^2}{d\tau^2} R(\tau) = \pi^2 \sum_{n=1}^{\infty} \cos n\pi\tau \Rightarrow \frac{d^2}{d\tau^2} R(\tau)\Big|_{\tau=0} = -\pi^2 \sum_{n=1}^{\infty} 1$$

$\therefore \; \frac{d^2}{d\tau^2} R(\tau)\Big|_{\tau=0}$ is not well-defined and $X(t)$ is NOT m.s. differentiable

Not required as part of the problem: $R(\tau+2) = R(\tau)$ and $R(\tau) = \frac{\pi^2}{2}\left[(2/3) - |t|(2-|t|)\right], \; -1 \le t \le 1$

Differentiable? (circle one)	Yes	No

Name: SOLUTION

5. Consider the process $X(t) = \sum_{n=-\infty}^{\infty} X_n g(t - nTo)$.

Such processes commonly arise in digital transmission schemes. For this problem let the $X_n s$ be independent and identically distributed with zero mean, variance σ^2 and let $g(t)$ be band limited with bandwidth Ω (i.e., its Fourier transform, $G(i\omega)$, vanishes if $|\omega| \geq \Omega$). In general such processes are neither stationary nor wide sense stationary. Show that in the special case for which $\Omega < \pi/To$, the process $X(t)$ will be wide sense stationary. Hint: you may find the following, formal relationship useful

$$\sum_{n=-\infty}^{\infty} e^{-iunTo} = \frac{2\pi}{To} \sum_{n=-\infty}^{\infty} \delta\left(u - n\frac{2\pi}{To}\right).$$

Clearly $E[X(t)] = 0$ and $X_n s$ are real

$$E[X(t)X^*(s)] = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} E[X_n X_m]\, g(t-nTo)\, g^*(s-nTo)$$

$$= \sigma^2 \sum_{n=-\infty}^{\infty} g(t-nTo)\, g^*(s-nTo)$$

$$= \sigma^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} G(i\omega) e^{i\omega t}\, d\omega\, \frac{1}{2\pi}\int_{-\infty}^{\infty} G^*(i\nu) e^{-i\nu s}\, d\nu \sum_{n=-\infty}^{\infty} e^{-i(\omega-\nu)nTo}$$

use hint with $u = \omega - \nu$

$$= \frac{\sigma^2}{2\pi To}\int_{-\infty}^{\infty} G(i\omega) e^{i\omega t}\, d\omega \int_{-\infty}^{\infty} G^*(i\nu) e^{-i\nu s} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \nu - \frac{n2\pi}{To}\right)$$

$$= \frac{\sigma^2}{2\pi To}\int_{-\infty}^{\infty} G(i\omega) e^{i\omega(t-s)}\, d\omega \sum_{n=-\infty}^{\infty} e^{in\frac{2\pi}{To}s}\, G^*\left(i\omega - in\frac{2\pi}{To}\right)$$

$G(i\omega) = 0$ if $|\omega| \geq \Omega$

$$= \frac{\sigma^2}{2\pi To}\int_{-\Omega}^{\Omega} G(i\omega) e^{i\omega(t-s)}\, d\omega \sum_{n=-\infty}^{\infty} e^{in\frac{2\pi}{To}s}\, G^*\left(i\omega - in\frac{2\pi}{To}\right)$$

Now note that if $\Omega < \pi/To$ all terms in the above infinite sum, except the $n=0$ term, will be shifted outside the interval $(-\Omega, \Omega)$.

$$\therefore\ E[X(t)X(s)] = \frac{\sigma^2}{2\pi To}\int_{-\Omega}^{\Omega} |G(i\omega)|^2 e^{i\omega(t-s)}\, d\omega$$

Clearly W.S.S.

ECE 250 Midterm

October 29, 2004

Name SOLUTION

1. Let X be a real random variable with a finite second moment. Can the following be characteristic functions for X ?

(I) $\Phi(u) = \frac{\sin 2u}{u} e^{iu}$

(II) $\Phi(u) = e^{-u^2 + u^3}$

(III) $\Phi(u) = e^{iu^2}$

No credit will be given for an answer not supported by a correct argument.

(I) $\Phi(0) = 2$, but a characteristic function must satisfy $\Phi(0) = 1$. Ans: NO

(II) $\Phi(2) = e^4 > 1$, but a characteristic function must satisfy $|\Phi(u)| \le \Phi(0) = 1$. Ans: NO

(III) $E[X^2] = \frac{1}{i^2}\frac{d^2}{du^2}\Phi(u)\Big|_{u=0} = -2i$, but the second moment of a real random variable must be real and non-negative. Ans: NO

Valid Characteristic Function?	(I)	(II)	(III)
	Yes No	Yes No	Yes No

Name SOLUTION

2. Let X and Y be independent, identically-distributed random variables. Form the new random variables.

$$Z_1 = \frac{X}{Y} \qquad Z_2 = \frac{Y}{X}.$$

Using a symmetry argument, it is reasonable to conjecture that

(I) $f_{Z_1}(z) = f_{Z_2}(z)$

On the other hand, since $Z_2 = 1/Z_1$, it is also reasonable to conjecture that

(II) $f_{Z_1}(z) \neq f_{Z_2}(z)$

Are either or both of these conjectures true? No credit will be given for an answer not supported by a correct argument.

Consider Z_1. Fix $Y = y$ and examine the conditional density $f_{Z_1}(z \mid Y = y)$

In this case $Z_1 = \left(\frac{1}{y}\right)X$, a constant times the variable X

$\therefore \quad f_{Z_1}(z \mid Y = y) = \frac{1}{\left|\frac{dz}{dx}\right|} f_X(x)\Big|_{x = zy} = |y| f_X(zy)$

Averaging over Y we have

$f_{Z_1}(z) = \int_{-\infty}^{\infty} f_Y(y) f_X(zy) |y| \, dy$

Similarly

$f_{Z_2}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(zx) |x| \, dx$

These densities are identical (change the integration variable from x to y in the expression for $f_{Z_2}(z)$.

Conjecture (I)	Conjecture (II)
True False	True False

Name SOLUTION

3. A complex random process is defined by

$$X(t) = A e^{i(Vt+\theta)}$$

where A, V and θ are independent random variables with densities

Clearly $E[X(t)]=0$ [Because $E[e^{i\theta}]=0$]

$$f_A(a) = \begin{cases} 0 & , a<0; \\ e^{-a} & , a \ge 0 \end{cases} \quad f_V(v) = \frac{1}{\sqrt{2\pi}} e^{\frac{-v^2}{2}}; \quad f_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & , 0 \le \theta \le 2\pi \\ 0 & , \text{otherwise}. \end{cases}$$

Show that X (t) is wide sense stationary and find its power spectral density.

$$R_X(t+\tau,t) = E[X(t+\tau)X^*(t)] = E[A e^{i(Vt+V\tau+\theta)} A e^{-i(Vt+\theta)}]$$

$$= E[A^2 e^{iV\tau}] = \int_0^\infty a^2 e^{-a} da \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{iv\tau} e^{-\frac{v^2}{2}} dv$$

$$\therefore \quad R_X(\tau) = 2 e^{-\frac{1}{2}\tau^2}$$

$$S_X(\omega) = \int_{-\infty}^\infty e^{-i\omega\tau} R_X(\tau) d\tau = 2\int_{-\infty}^\infty e^{-i\omega\tau} e^{-\frac{\tau^2}{2}} d\tau$$

$$= 2\sqrt{2\pi} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-i\omega\tau} e^{-\frac{\tau^2}{2}} d\tau \right\}$$

$$= 2\sqrt{2\pi}\, e^{-\frac{1}{2}\omega^2}$$

$$\boxed{S_X(\omega) = 2\sqrt{2\pi}\, e^{-\frac{1}{2}\omega^2}}$$

ECE 250 FINAL EXAMINATION

December 8, 2004 Name SOLUTION

1. A random process is defined by

$$X(t) = u(t-T)$$

where $u(t)$ is the unit step

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

and T is a random variable with a continuous probability density $f_T(\tau)$. Does $X(t)$ have orthogonal increments? No credit will be given for an answer that is not supported by an appropriate argument.

Let $t_1 \leq t_2 \leq t_3 \leq t_4$

$$X(t_2) - X(t_1) = u(t_2 - T) - u(t_1 - T) = \begin{cases} 1, & t_1 < T \leq t_2 \\ 0, & \text{otherwise} \end{cases}$$

$$X(t_4) - X(t_3) = u(t_4 - T) - u(t_3 - T) = \begin{cases} 1, & t_3 < T \leq t_4 \\ 0, & \text{otherwise} \end{cases}$$

$$\{X(t_4) - X(t_3)\} \cdot \{X(t_2) - X(t_1)\} = 0$$

$\therefore$

$$E[\{X(t_4) - X(t_3)\} \cdot \{X(t_2) - X(t_1)\}] = 0$$

the increments are orthogonal

Orthogonal Increments?	Yes	No

Name SOLUTION

2. In many digital communication systems data is transmitted via a pulse train appearing as

$$\sum_{n=-\infty}^{\infty} X_n\, g(t-nT_0)$$

where g(t) is a real, non-random pulse, the X_n s are real, independent, identically-distributed random variables (binary, quarternary, etc.) and T_0 is the spacing between adjacent pulses. For this problem assume that

$$E[X_n]=0; \qquad E[X_nX_m]=\begin{cases}\sigma^2, & n=m\\ 0, & n\neq m\end{cases}$$

in practice timing errors often result in random spacing between adjacent pulses. Consider the following pulse train.

$$X(t)=\sum_{n=-\infty}^{\infty} X_n g(t-nT_0+\tau_n)$$

where g(t) and the X_n s are as above and the τ_n s are independent of the X_n s and are themselves independent and identically-distributed with common probability density

$$f_T(\tau)=\begin{cases}\frac{1}{T_0}, & -\frac{T_0}{2}\le\tau\le\frac{T_0}{2}\\[4pt] 0, & \text{otherwise}.\end{cases}$$

Show that X(t) is wide sense stationary and find its power spectral density.

$$(1)\quad E[X(t)] = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} d\tau_n \sum_{n=-\infty}^{\infty} E[X_n]\, g(t-nT_0+\tau_n)=0$$

$$R_X(t+\tau,t)=\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} d\tau_n\,\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} d\tau_m$$

$$\cdot\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} E[X_nX_m]\,g(t+\tau-nT_0+\tau_n)\,g(t-mT_0+\tau_m)$$

$$=\sigma^2\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} d\tau_n\,\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} d\tau_m\sum_{n=-\infty}^{\infty} g(t+\tau-nT_0+\tau_n)\,g(t-nT_0+\tau_n)$$

$$=\sigma^2\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} d\tau_n\sum_{n=-\infty}^{\infty} g(t+\tau-nT_0+\tau_n)\,g(t-nT_0+\tau_n)$$

$$\boxed{S_X(\omega)=\frac{\sigma^2}{T_0}\,|G(i\omega)|^2}$$

Problem 2 (continued)

Interchange summation and the integral

$$R_X(t+\tau,t)=\sigma^2\sum_{n=-\infty}^{\infty}\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} d\tau_n\, g(t+\tau-nT_0+\tau_n)\,g(t-nT_0+\tau_n)$$

define $\alpha = t-nT_0+\tau_n$

$$=\sigma^2\sum_{n=-\infty}^{\infty}\frac{1}{T_0}\int_{t-nT_0-T_0/2}^{t-nT_0+T_0/2} g(\alpha+\tau)\,g(\alpha)\,d\alpha$$

The sum of each integral adds up to an integral from $(-\infty,\infty)$

finally

$$(2)\quad R_X(\tau)=\frac{\sigma^2}{T_0}\int_{-\infty}^{\infty} g(\alpha)\,g(\alpha+\tau)\,d\alpha$$

From (1) and (2) it is clear that $X(t)$ is W.S.S.

$$S_X(\omega)=\int_{-\infty}^{\infty} R_X(\tau)\,e^{-i\omega\tau}\,d\tau$$

$$=\frac{\sigma^2}{T_0}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} g(\alpha)\,g(\alpha+\tau)\,d\alpha\right\}e^{-i\omega\tau}\,d\tau$$

$$=\frac{\sigma^2}{T_0}\,|G(i\omega)|^2$$

where $G(i\omega)=\int_{-\infty}^{\infty} g(t)\,e^{-i\omega t}\,dt$

Name SOLUTION

3. Let $X(t)$ be a shot process given by

$$X(t)=\sum X_n u(t-t_n)$$

where the t_ns are the event times of a Poisson process with constant rate λ. $u(t)$ is the unit step

$$u(t)=\begin{cases}1, & t\ge 0\\ 0, & t<0,\end{cases}$$

and the X_ns are independent and identically distributed with common probabilities

$$P(X_n=1)=p,\ P(X_n=0)=(1-p),\ 0<p<1.$$

The random variables $\{X_n\}$ and the event times $\{t_n\}$ are independent of each other. Find an explicit, closed-form expression for the probability

$$P(X(t)-X(s)=m)\quad, s\le t$$

where m is an integer.

$$X(t)-X(s)=\sum_n X_n[u(t-t_n)-u(s-t_n)]$$

From class notes:

$$\Phi_{X(t)-X(s)}(u)=E\left[e^{iu[X(t)-X(s)]}\right]$$

$$=e^{\lambda\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[e^{iux[u(t-\tau)-u(s-\tau)]}-1\right]f_X(x)\,dx\,d\tau}$$

where $f_X(x)=(1-p)\delta(x)+p\delta(x-1)$

$$=e^{p\lambda(t-s)[e^{iu}-1]}$$

This is just a classical Poisson process with const. rate $p\cdot\lambda$

$$\therefore P(X(t)-X(s)=m)=\frac{[p\lambda(t-s)]^m}{m!}e^{-p\lambda(t-s)}$$

$$\boxed{P(X(t)-X(s)=m)=\frac{[p\cdot\lambda\cdot(t-s)]^m}{m!}e^{-p\cdot\lambda\cdot(t-s)}}$$

Name SOLUTION

4. A real random signal $A(t)$ is corrupted by a real additive noise $N(t)$. The processes $A(t)$ and $N(t)$ are orthogonal and wide sense stationary with respective correlation functions

$$R_A(\tau)=2\cos\omega_0\tau\,\frac{\sin\omega_0\tau}{\pi\tau};\qquad R_N(\tau)=2\,\frac{\sin\frac{1}{2}\omega_0\tau}{\pi\tau}.$$

the sum $A(t)+N(t)$ is passed through a linear, time-invariant filter that is chosen to minimize the mean square error between the filter output, $Y_0(t)$, and $A(t)$

$$E[|Y_0(t)-A(t)|^2].$$

Determine the impulse response of the optimum filter (i.e., the one that minimizes the above mean square error).

We need the transform relationship

$$\int_{-\infty}^{\infty}\frac{\sin\Omega\tau}{\Omega\tau}e^{-i\omega\tau}\,d\tau=\begin{cases}\frac{\pi}{\Omega}, & |\omega|\le\Omega\\ 0, & \text{otherwise}\end{cases}$$

From class notes the optimum transfer function is given by

$$H(i\omega)=\frac{S_A(\omega)}{S_A(\omega)+S_N(\omega)}\qquad S_A(\omega)=\begin{cases}1, & |\omega|\le 2\omega_0\\ 0, & \text{otherwise}\end{cases}\qquad S_N(\omega)=\begin{cases}2, & |\omega|\le\frac{\omega_0}{2}\\ 0, & \text{otherwise}\end{cases}$$

$$=\begin{cases}\frac{1}{3}, & 0\le|\omega|\le\frac{\omega_0}{2}\\ 1, & \frac{\omega_0}{2}<|\omega|\le 2\omega_0\\ 0, & \text{otherwise}\end{cases}$$

$$h(t)=\frac{1}{2\pi}\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}}\frac{1}{3}e^{i\omega t}\,d\omega+\frac{1}{2\pi}\int_{\frac{\omega_0}{2}<|\omega|\le 2\omega_0}1\,e^{i\omega t}\,dt$$

$$=\frac{\sin 2\omega_0 t}{\pi t}-\frac{2}{3}\,\frac{\sin\frac{\omega_0}{2}t}{\pi t}$$

$$\boxed{h(t)=\frac{\sin 2\omega_0 t}{\pi t}-\frac{2}{3}\,\frac{\sin\frac{\omega_0}{2}t}{\pi t}}$$

Name SOLUTION

5. A wide sense stationary process has the correlation function

$$R_x(\tau) = \frac{2}{3}e^{-|\tau|} - \frac{1}{3}e^{-2|\tau|}.$$

Is this process mean square differentiable? No credit will be given for an answer not supported by appropriate reasoning.

Make power series expansion for $R_X(\tau)$

$$R_X(\tau) = \frac{2}{3}\left\{1 + (-|\tau|) + \frac{1}{2}(-|\tau|)^2 + \frac{1}{6}(-|\tau|)^3 \cdots\right\}$$
$$- \frac{1}{3}\left\{1 + (-2|\tau|) + \frac{1}{2}(-2|\tau|)^2 + \frac{1}{6}(-2|\tau|)^3 \cdots\right\}$$

$$= \frac{1}{3}\left\{1 - \tau^2 + |\tau|^3 \cdots\right\}$$

Clearly $R''_X(\tau)\big|_{\tau=0}$ is well-defined $\left(= -\frac{2}{3}\right)$ and so $X(t)$ is mean square differentiable

Alternatively

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-i\omega\tau} d\tau$$

$$= \frac{4}{(1+\omega^2)(4+\omega^2)}$$

Obviously

$$\int_{-\infty}^{\infty} \omega^2 S_X(\omega) d\omega = \int_{-\infty}^{\infty} \frac{4\omega^2}{(1+\omega^2)(4+\omega^2)} d\omega < \infty$$

Differentiable? (Circle one)	Yes	No

Name SOLUTION

ECE 250 Quiz No. 1 October 17, 2003

Let X and Y be independent random variables with probability densities

$$f_X(x) = \begin{cases} e^{-x}, & x \ge 0 \\ 0, & x < 0 \end{cases} \qquad f_Y(y) = \begin{cases} e^{-y}, & y \ge 0 \\ 0, & y < 0. \end{cases}$$

Find the probability density of

$$Z = \frac{X}{X+Y}$$

Clearly $0 \le z \le 1$

$$F_Z(z) = P\left(\frac{X}{X+Y} \le z\right) = P\left(X \le \left(\frac{z}{1-z}\right)Y\right)$$

$$= \iint_{x \le \left(\frac{z}{1-z}\right)y} f_{X,Y}(x,y)\,dx\,dy = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\left(\frac{z}{1-z}\right)y} f_{X,Y}(x,y)\,dx$$

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \frac{1}{(1-z)^2}\int_{-\infty}^{\infty} f_{X,Y}\left(\frac{z}{1-z}y, y\right) y\,dy$$

because of independence

$$= \frac{1}{(1-z)^2}\int_{-\infty}^{\infty} f_X\left(\frac{z}{1-z}y\right) f_Y(y)\, y\,dy$$

$$= \frac{1}{(1-z)^2}\int_0^{\infty} e^{-\frac{y}{1-z}}\, y\,dy = 1$$

$$\therefore f_Z(z) = \begin{cases} 1, & 0 \le z \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\boxed{f_Z(z) = \begin{cases} 1, & 0 \le z \le 1 \\ 0, & \text{otherwise} \end{cases}}$$

Name Solution

ECE 250 Quiz No. 2 October 31, 2003

The real, second-order process X (t) is wide-sense stationary and its correlation function, $R_X(\tau)$, is differentiable. Show that for the process to have orthogonal increments it is necessary and sufficient that $R_X(\tau)$ = constant. [HINT. For necessity, use the assumed orthogonality to examine the derivative of $R_X(\tau)$ and its relationship to the derivative of $R_X(\tau)$ at the origin.]

<u>Sufficiency</u> Let $R_X(\tau) = C$ = constant and let $s \le u \le t$

$$(*)\; E\left[\{X(t)-X(u)\}\{X(u)-X(s)\}\right] = R_X(t-u) - R_X(t-s) - R_X(0) + R_X(u-s)$$

$$\text{If } R_X(\tau) = C \quad = C - C - C + C = 0$$

<u>Necessity</u> Let $X(t)$ have orthogonal increments and set $\alpha = t-u$; $\beta = u-s$. Then from $(*)$

$$0 = R_X(\alpha) - R_X(\alpha+\beta) - R_X(0) + R_X(\beta)$$

equivalently

$$R_X(\alpha+\beta) - R_X(\alpha) = R_X(\beta) - R_X(0)$$

divide by β and let $\beta \to 0$

$$(**)\quad R_X'(\alpha) = \lim_{\beta\to 0}\frac{R_X(\alpha+\beta)-R_X(\alpha)}{\beta} = \lim_{\beta\to 0}\frac{R_X(\beta)-R_X(0)}{\beta} = R_X'(0)$$

But since $X(t)$ is real $R_X(-\tau) = R_X(\tau)$, which implies (because $|R_X(\tau)| \le R_X(0)$) that $R_X'(0) = 0$.

Thus from $(**)$ we have

$$R_X'(\alpha) = 0$$

$$\Rightarrow \quad R_X(\tau) = \text{constant}$$

Name SOLUTION

ECE 250 Quiz No. 3 November 19, 2003

Let $X_1(t)$ and $X_2(t)$ be independent random telegraph signals whose underlying classical Poisson processes have the same constant rate. Also let

$$P(X_1(0)=1) = P(X_2(0)=1) = p, 0 < p < 1.$$

Let $F(t)$ be the fraction of time that $X_1(t)=X_2(t)$ in the interval $[0,t)$. Find the expected Value of $F(t)$. [Hint. You may find it convenient to define a new process $Y(t)$ by setting $Y(t)=1$ when $X_1(t)=X_2(t)$ and $Y(t)=0$ when $X_1(t) \neq X_2(t)$.]

$$Y(t) = \begin{cases} 1, & X_1(t) = X_2(t) \\ 0, & \text{otherwise} \end{cases} \qquad \text{Rate} = \lambda$$

$$P(Y(t)=1) = P(X_1(t)=1, X_2(t)=1) + P(X_1(t)=-1, X_2(t)=-1)$$

clearly both $X_1(t)$ & $X_2(t)$ have identical distribution, and, from class $P(X_1(t)=1) = \frac{1}{2} + \left(\frac{2p-1}{2}\right) e^{-2\lambda t}$

$$\therefore P(Y(t)=1) = \frac{1}{2} + \frac{1}{2}(2p-1)^2 e^{-4\lambda t}$$

Now $F(t) = \frac{1}{t}\int_0^t Y(t')dt'$ and $E[F(t)] = \frac{1}{t}\int_0^t E[Y(t')]dt'$

with $E[Y(t')] = P(Y(t)=1) = \frac{1}{2} + \frac{1}{2}(2p-1)^2 e^{-4\lambda t}$

and finally $E[F(t)] = \frac{1}{2} + \frac{(2p-1)^2}{2}\left(\frac{1-e^{-4\lambda t}}{4\lambda t}\right)$

$$\boxed{E[F(t)] = \frac{1}{2} + \frac{(2p-1)^2}{2}\left(\frac{1-e^{-4\lambda t}}{4\lambda t}\right)}$$

ECE 250   FINAL EXAMINATION

December 10, 2003

Name: SOLUTION

1. A random variable, X, has a continuous probability density $f_X(x)$ and corresponding, distribution $F_X(x)$. Define a nonlinear function $G(w)$ by

$$G(w)=F_X(w).$$

Clearly $0\le Z\le 1$

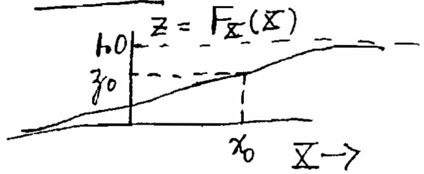
$$\therefore F_Z(z)=\begin{cases}0, & z<0\\ 1, & z>1.\end{cases}$$

Consider the new random variable

$$Z=G(X).$$

Derive a general expression for the probability density of Z.

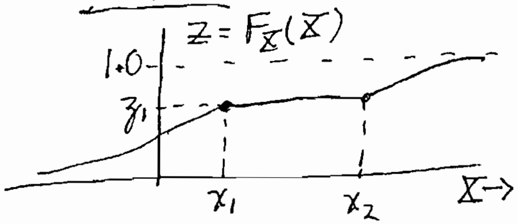
Case I $F_X(x)$ is monotone increasing



pick a point z_0 and the corresponding point x_0 on the X axis

$$F_Z(z_0)=P(Z\le z_0)=P(X\le x_0)=F_X(x_0)=z_0,\quad 0\le z_0\le 1$$

Case II $F_X(x)$ is constant over a region



Let z_1 lie in a region where $F_X(x)$ is constant.

$$F_Z(z_1)=P(Z\le z_1)=P(X\le x_2)=z_1,\quad 0\le z_1\le 1$$

In either case $F_Z(z)=\begin{cases}0; & z<0\\ z; & 0\le z\le 1\\ 1; & 1<z\end{cases}$

so that $f_Z(z)=1$

$$\boxed{f_Z(z)=\begin{cases}0, & z<0\\ 1, & 0\le z\le 1\\ 0, & 1<z\end{cases}}$$

Name: SOLUTION

2. A process $X(t)$ has monotone increasing sample functions. At intervals spaced T_0 seconds apart, the process may increase by one unit with probability p or it may remain at the same value with probability $(1-p)$. The process is given by;

$$X(t)=\sum_{n=1}^{\infty}a_n\cdot u(t-nT_0),\ t\ge 0$$

Here the a_ns are i. i. d. with

$P(a_n=1)=p$ $P(a_n=0)=(1-p),\ 0<p<1$

$$u(t)=\begin{cases}1, & t\ge 0\\ 0, & t<0\end{cases}$$

$E[a_n]=p$

$E[a_n^2]=p$

$E[a_na_m]=\begin{cases}p, & n=m\\ p^2, & n\ne m\end{cases}$

Find the correlation function

$$R_X(t,s)=E[X(t)X(s)],\quad t,s\ge 0$$

$$R_X(t,s)=E\Big[\sum_{n=1}^{\infty}a_n u(t-nT_0)\sum_{m=1}^{\infty}a_m u(s-mT_0)\Big]$$

$$=p(1-p)\sum_{n=1}^{\infty}u(t-nT_0)u(s-nT_0)+p^2\Big(\sum_{n=1}^{\infty}u(t-nT_0)\Big)\Big(\sum_{m=1}^{\infty}u(s-mT_0)\Big)$$

Now $\displaystyle\sum_{n=1}^{\infty}u(t-nT_0)=\sum_{n=1}^{[\![t/T_0]\!]}1=[\![\tfrac{t}{T_0}]\!]$. $[\![\tfrac{t}{T_0}]\!]\equiv$ integer part of $\tfrac{t}{T_0}$

$$\sum_{m=1}^{\infty}u(s-mT_0)=[\![\tfrac{s}{T_0}]\!]$$

$$\sum_{n=1}^{\infty}u(t-nT_0)u(s-nT_0)=\min\Big[[\![\tfrac{t}{T_0}]\!],[\![\tfrac{s}{T_0}]\!]\Big]$$

$$\therefore R_X(t,s)=p(1-p)\min\Big[[\![\tfrac{t}{T_0}]\!],[\![\tfrac{s}{T_0}]\!]\Big]+p^2[\![\tfrac{t}{T_0}]\!][\![\tfrac{s}{T_0}]\!]$$

$$\boxed{R_X(t,s)=p(1-p)\min\Big[[\![\tfrac{t}{T_0}]\!],[\![\tfrac{s}{T_0}]\!]\Big]+p^2[\![\tfrac{t}{T_0}]\!][\![\tfrac{s}{T_0}]\!]}$$

Name: SOLUTION

3. Let the random variable X have the density

$$f_X(x) = \frac{1}{2}e^{-|x|} \qquad -\infty < x < \infty$$

and let $u(w)$ be the unit step

$$u(w) = \begin{cases} 1, & w \ge 0 \\ 0, & w \ge 0 . \end{cases}$$

Is the process $Y(t) = u(t+x), -\infty < t < \infty$ mean square continuous?
You must support your answer with appropriate reasoning. A simple "yes" or "no" will receive no credit.

Must show $\lim_{\epsilon \to 0} E[|Y(t+\epsilon) - Y(t)|^2] = 0$

Because $\epsilon \to 0$, it is not a restriction to assume $0 < |\epsilon| < |t|$ when $|t| > 0$.

$$Y(t+\epsilon) - Y(t) = u(t+\epsilon+x) - u(t+x)$$

$$E[|Y(t+\epsilon) - Y(t)|^2] = \frac{1}{2}\int_{-\infty}^{\infty} |u(t+\epsilon+x) - u(t+x)|^2 e^{-|x|}\,dx$$

$\epsilon > 0$

$$|u(t+\epsilon+x) - u(t+x)| = \begin{cases} 1, & -t-\epsilon \le x < -t \\ 0, & \text{otherwise} \end{cases}$$

$$E[|Y(t+\epsilon) - Y(t)|^2] = \frac{1}{2}\int_{-t-\epsilon}^{-t} e^{-|x|}\,dx = \begin{cases} \frac{1}{2}(1-e^{-\epsilon})e^{-t}, & t \ge 0 \\ \frac{1}{2}(e^{\epsilon}-1)e^{t}, & t < 0 \end{cases}$$

$\epsilon < 0$

$$|u(t+\epsilon+x) - u(t+x)| = \begin{cases} 1, & -t \le x < -t-\epsilon \\ 0, & \text{otherwise} \end{cases}$$

$$E[|Y(t+\epsilon) - Y(t)|^2] = \frac{1}{2}\int_{-t}^{-t-\epsilon} e^{-|x|}\,dx = \begin{cases} \frac{1}{2}(e^{-\epsilon}-1)e^{-t}, & t > 0 \\ \frac{1}{2}(1-e^{\epsilon})e^{t}, & t \le 0 \end{cases}$$

In all cases

$$\lim_{\epsilon \to 0} E[|Y(t+\epsilon) - Y(t)|^2] = 0$$

Mean Square Continuous? Circle one	Yes	No

Name: SOLUTION

4. Let $X(t)$ be a real, wide sense stationary process. This process is passed through a linear, time-invariant system with impulse response $h(t)$. The output is $Y(t)$

$$Y(t) = \int_{-\infty}^{\infty} h(t-t')\,X(t')\,dt'$$

Show that a sufficient condition for $Y(t)$ to be mean square differentiable is

$$|H(i\omega)| \le \frac{1}{1+|\omega|}$$

where $H(i\omega)$ is the transfer function of the linear system. Do not assume $X(t)$ is mean square differentiable.

$Y(t)$ is w.s.s. $\quad S_Y(\omega) = |H(i\omega)|^2 S_X(\omega) \le \frac{1}{(1+|\omega|)^2} S_X(\omega)$

For m.s. differentiablity we must have

$$\int_{-\infty}^{\infty} \omega^2 S_Y(\omega)\,d\omega < \infty$$

$$\le 0 \frac{\omega^2}{(1+|\omega|)^2} \le 1$$

But this integral is

$$\int_{-\infty}^{\infty} \omega^2 S_Y(\omega)\,d\omega \le \int_{-\infty}^{\infty} \frac{\omega^2}{(1+|\omega|)^2} S_X(\omega)\,d\omega \le \int_{-\infty}^{\infty} S_X(\omega)\,d\omega < \infty$$

Name: S.LUTION

5. The process X(t) is real and has zero mean.

$$R_X(t,s)=E\,[X(t)X(s)]=\frac{1}{2}A(t-s)+\frac{1}{2}A(t+s)$$

where $A(\tau)$ is a symmetric function, i.e. $A(-\tau) = A(\tau)$, which is real and periodic with period T_0. Find a series expansion for X(t) having orthogonal random coefficients and demonstrate the orthogonality of these coefficients.

Express $A(\tau)$ as a Fourier series

$$A(\tau)=\sum_{n=0}^{\infty}\lambda_n\cos n\omega_0\tau \qquad \omega_0=\frac{2\pi}{T_0} \qquad \lambda_0=\frac{1}{T_0}\int_{-T_0/2}^{T_0/2}A(\tau)d\tau$$

$$\lambda_n=\frac{2}{T_0}\int_{-T_0/2}^{T_0/2}A(\tau)\cos n\omega_0\tau\, d\tau,\ n=1,2,\cdots$$

Then

$$(1)\quad R_X(t,s)=\frac{1}{2}\sum_{n=0}^{\infty}\lambda_n\cos n\omega_0(t-s)+\frac{1}{2}\sum_{n=0}^{\infty}\lambda_n\cos n\omega_0(t+s)=\sum_{n=0}^{\infty}\lambda_n\cos n\omega_0 t\cos n\omega_0 s$$

Note the orthogonality relationship

$$\frac{2}{T_0}\int_{-T_0/2}^{T_0/2}\cos n\omega_0 t\cos m\omega_0 t\,dt=\begin{cases}0, & n\neq m\\ 1, & n=m,\ n\neq 0,\ m\neq 0\\ 2, & n=m=0\end{cases}$$

We now have the integral equation

$$(2)\quad \frac{2}{T_0}\int_{-T_0/2}^{T_0/2}R_X(t,s)\cos n\omega_0 s\,ds=\begin{cases}\lambda_n\cos n\omega_0 t, & n=1,2,\cdots\\ 2\lambda_0, & n=0\end{cases}$$

From (1) and (2) we will have the K-L expansion

$$X(t)=\sum_{n=0}^{\infty}X_n\cos n\omega_0 t \text{ with } X_n=\begin{cases}\frac{1}{T_0}\int_{-T_0/2}^{T_0/2}X(t)dt, & n=0\\ \frac{2}{T_0}\int_{-T_0/2}^{T_0/2}X(t)\cos n\omega_0 t\,dt, & n=1,2,\cdots\end{cases}$$

For orthogonality of the coefficients

$$E[X_nX_m]=\frac{2}{T_0}\int_{-T_0/2}^{T_0/2}\cos n\omega_0 t\,dt\cdot\frac{2}{T_0}\int_{-T_0/2}^{T_0/2}R_X(t,s)\cos m\omega_0 s\,ds=\frac{2}{T_0}$$

$$=\frac{2}{T_0}\int_{-T_0/2}^{T_0/2}\cos n\omega_0 t\begin{Bmatrix}\lambda_m\cos m\omega_0 t, & m\neq 0\\ 2\lambda_0, & m=0\end{Bmatrix}dt$$

$$=0 \text{ if } n\neq m$$

ECE 250A MIDTERM

November 4, 2002

Name: SOLUTION

1. The random variables X and Y are independent and jointly Gaussian with zero mean and unit variance. Find the probability density of
Find the probability density of $Z = X^2/Y^2$.

$$F_Z(z) = P(Z \le z) = P\left(\frac{X^2}{Y^2} \le z\right) = P(X^2 \le z Y^2)$$

$$= \iint_{x^2 \le z y^2} f_{X,Y}(x,y)\,dx\,dy$$

Clearly $z \ge 0$

independence of X & Y

$$= \int_{-\infty}^{\infty} dy\, f_Y(y) \int_{x^2 \le z y^2} f_X(x)\,dx = \int_{-\infty}^{\infty} f_Y(y)\,dy \int_{-\sqrt{z}|y|}^{\sqrt{z}|y|} f_X(x)\,dx$$

symmetry

$$= 4 \int_0^{\infty} f_Y(y)\,dy \int_0^{\sqrt{z}y} f_X(x)\,dx$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{1}{\pi\sqrt{z}} \int_0^{\infty} e^{-\frac{1}{2}(1+z)y^2}\, y\,dy$$

$$= \frac{1}{\pi\sqrt{z}(1+z)}, \quad z \ge 0$$

$$\boxed{f_Z(z) = \frac{1}{\pi\sqrt{z}(1+z)}, \quad z \ge 0}$$

Name: SOLUTION

2. Let X and Y be jointly Gaussian random variables with zero mean, unit variance and covariance coefficient $\rho_{X,Y}$. Do not assume $\rho_{X,Y} = 0$.
Consider the complex random variables

$$U = e^{iX} \qquad V = e^{iY}$$

Express $\rho_{U,V}$ (the covariance coefficient of U and V in terms of $\rho_{X,Y}$.

$$\left.\begin{aligned} E[U] &= E[e^{iX}] = e^{-\frac{1}{2}} \\ E[|U|^2] &= 1 \end{aligned}\right\} \Rightarrow Var[U] = 1 - e^{-1}$$

$$\left.\begin{aligned} E[V] &= E[e^{iY}] = e^{-\frac{1}{2}} \\ E[|V|^2] &= 1 \end{aligned}\right\} \Rightarrow Var[V] = 1 - e^{-1}$$

$$E[UV^*] = E[e^{i(X-Y)}] = e^{-(1-\rho_{X,Y})}$$

$X - Y$ is Gaussian with zero mean and variance $= 2(1-\rho_{X,Y})$

$$\rho_{U,V} = \frac{E[UV^*] - E[U]E[V^*]}{\sqrt{Var[U]\cdot Var[V]}}$$

$$= \frac{e^{\rho_{X,Y}} - 1}{e^1 - 1}$$

$$\boxed{\rho_{U,V} = \frac{e^{\rho_{X,Y}} - 1}{e^1 - 1}}$$

Name: SOLUTION

3. The process X(t) is real, mean square continuous and has zero mean. In addition it has orthogonal, stationary increments with

$$E\left[|X(t)-X(s)|^2\right]=|t-s|. \qquad \text{and} \quad X(0)=0$$

A new process is defined by

$$Y(t)=e^{-bt}X(e^{+2bt})$$

with $b>0$. Determine the correlation function of $Y(t)$.

$\underline{s \le t}$

$$E[Y(t)Y(s)] = E[e^{-bt}X(e^{2bt})e^{-bs}X(e^{2bs})]$$

$$= e^{-b(t+s)}E[X(e^{2bt})X(e^{2bs})]$$

$e^{2bt} \ge e^{2bs}$

orthogonal increments and $X(0)=0$

$$= e^{-b(t+s)}\Big(E\big[\{X(e^{2bt})-X(e^{2bs})\}\{X(e^{2bs})-X(0)\}\big] + E\big[\{X(e^{2bs})-X(0)\}^2\big]\Big)$$

$$= e^{-b(t+s)}e^{2bs} = e^{-b(t-s)}$$

Similarly for $t \le s$

$$E[Y(t)Y(s)] = e^{-b(s-t)}$$

$$\therefore \quad E[Y(t)Y(s)] = e^{-b|t-s|}$$

$$\boxed{E[Y(t)Y(s)] = e^{-b|t-s|}}$$

ECE 250 FINAL EXAMINATION

December 9, 2002

Name: SOLUTION

1. The random variables $X_1, X_2, \cdots$ are independent and identically distributed with densities

$$f_{X_n}(x) = \frac{1}{\pi(1+x^2)} \qquad n = 1,2,\cdots$$

$$\Phi_{X_n}(u) = E[e^{iuX_n}] = e^{-|u|}$$

Consider the new random variable

$$Y = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n X_n .$$

Find the probability density of Y. [Hint: the characteristic function may be a useful tool.]

$$\Phi_Y(u) = E[e^{iuY}] = E\left[e^{iu\sum_{n=1}^{\infty}(\frac{1}{2})^n X_n}\right]$$

$$\xrightarrow{\text{independence of } X_n\text{s}} = \prod_{n=1}^{\infty} E\left[e^{iu(\frac{1}{2})^n X_n}\right]$$

$$= \prod_{n=1}^{\infty} e^{-(\frac{1}{2})^n|u|}$$

$$= e^{-|u|\sum_{n=1}^{\infty}(\frac{1}{2})^n}$$

$$\xrightarrow{\sum_{n=1}^{\infty}(\frac{1}{2})^n = 1} = e^{-|u|}$$

$$f_Y(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|u|} e^{-iuy}\, du$$

$$= \frac{1}{\pi(1+y^2)}$$

$$\boxed{f_Y(y) = \frac{1}{\pi(1+y^2)}}$$

Name: SOLUTION

2. The random variables X and Y are jointly Gaussian with zero mean, variances σ_X^2 and σ_Y^2 and correlation coefficient $\rho_{X,Y}$. Find the correlation coefficient of X^2 and Y^2.

$$\rho_{X^2,Y^2} = \frac{E[X^2Y^2] - E[X^2]E[Y^2]}{\sqrt{Var[X^2]\,Var[Y^2]}}$$

$$E[X^2] = \sigma_X^2 \qquad E[Y^2] = \sigma_Y^2$$

$$Var[X^2] = E[X^4] - (E[X^2])^2 = 3\sigma_X^4 - (\sigma_X^2)^2 = 2\sigma_X^4$$

$$Var[Y^2] = E[Y^4] - (E[Y^2])^2 = 3\sigma_Y^4 - (\sigma_Y^2)^2 = 2\sigma_Y^4$$

$$E[X^2Y^2] \overset{\text{using the relationship of Prob. 1 in HW \#2}}{=} E[X^2]E[Y^2] + 2(E[XY])^2$$

$$E[XY] = \rho_{X,Y}\sigma_X\sigma_Y$$

$$= \sigma_X^2\sigma_Y^2 + 2\rho_{X,Y}^2\sigma_X^2\sigma_Y^2$$

$$= (1 + 2\rho_{X,Y}^2)\sigma_X^2\sigma_Y^2$$

$$\therefore \quad \rho_{X^2,Y^2} = \frac{2\rho_{X,Y}^2\sigma_X^2\sigma_Y^2}{\sqrt{(2\sigma_X^4)(2\sigma_Y^4)}} = \rho_{X,Y}^2$$

It is customary to use only the positive square root of the denominator, but an acceptable answer is $\pm\rho_{X,Y}^2$

$$\boxed{\rho_{X^2,Y^2} = \rho_{X,Y}^2}$$

Name: SOLUTION

3. Let $X(t)$ be a real, classical Brownian motion process (independent, stationary, zero-mean, Gaussian increments, $X(0)=0$) with

$$E\left[|X(t)-X(s)|^2\right] = K\cdot|t-s| .$$

Define a new process by

$$Y(t) = X(t)-X(t-T_0) \ , T_0>0 .$$

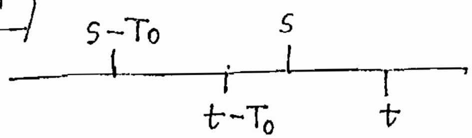
Find the correlation function of $Y(t)$.

Consider the case where $s \le t$

$\boxed{t-s>T_0}$ $E[Y(t)Y(s)] = E\left[\{X(t)-X(t-T_0)\}\{X(s)-X(s-T_0)\}\right]$

indep. increments and zero mean $\rightarrow = 0$

$\boxed{0 \le t-s \le T_0}$



$$X(t)-X(t-T_0) = \{X(t)-X(s)\} + \{X(s)-X(t-T_0)\}$$
$$X(s)-X(s-T_0) = \{X(s)-X(t-T_0)\} + \{X(t-T_0)-X(s-T_0)\}$$

Noting that non-overlapping intervals have independent increments

$$E[Y(t)Y(s)] = E\left[|X(s)-X(t-T_0)|^2\right] = K\{T_0-(t-s)\}$$

Similarly for $-T_0 \le t-s < 0$ and $t-s < -T_0$ so that

$$E[Y(t)Y(s)] = \begin{cases} K\{T_0-|t-s|\}, & |t-s| \le T_0 \\ 0, & |t-s| > T_0 \end{cases}$$

$$\boxed{E[Y(t+\tau)Y(t)] = \begin{cases} K\{T_0-|\tau|\}, & |\tau| \le T_0 \\ 0, & |\tau| > T_0 \end{cases}}$$

Name: SOLUTION

4. The variables $X_1, X_2, \cdots$ are real, independent and identically distributed with mean and variance m and σ^2 respectively. Consider the sum

$$S_n = \sum_{k=1}^{n} a_K X_K .$$

The coefficients $\{a_K\}$ are real and non-random. Find necessary and sufficient conditions on the coefficients for the sum S_n to converge in the mean.

It is necessary and sufficient that $E[S_n S_m]$ converge to a finite limit as $n, m \to \infty$

This ensures that $\{S_n\}$ is a Cauchy sequence

$$E[S_n S_m] = E\left[\sum_{k=1}^{n} a_k X_k \sum_{\ell=1}^{m} a_\ell X_\ell\right] = \sum_{k=1}^{n}\sum_{\ell=1}^{m} a_k a_\ell E[X_k X_\ell]$$

$$E[X_k X_\ell] = m^2 + \sigma^2 \delta_{k,\ell}$$

$$\therefore \ E[S_n S_m] = m^2\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{\ell=1}^{m} a_\ell\right) + \sigma^2 \sum_{k=1}^{n}\sum_{\ell=1}^{m} \delta_{k,\ell} a_k a_\ell$$
$$= m^2\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{\ell=1}^{m} a_\ell\right) + \sigma^2 \sum_{k=1}^{\min[n,m]} a_k^2$$

Clearly if both $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$ and $\lim_{n\to\infty} \sum_{k=1}^{n} a_k^2$ exist, then $\lim_{n,m\to\infty} E[S_n S_m]$ exists

$\boxed{\text{Conditions: Both } \sum_{k=1}^{n} a_k \text{ and } \sum_{k=1}^{n} a_k^2 \text{ converge as } n\to\infty}$

Name: SOLUTION

5. Let X(t) be a real random process satisfying

Define

$$E[X(t)] = t$$

$$R_X(t,s) = E[X(t)X(s)]$$

$$E[X(t)X(s)] = st + \min[s,t] .$$

This process is passed through a linear, time-invariant filter with impulse response

$$h(t) = u(t) - u(t-1) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwise} . \end{cases}$$

Denoting the output process by Y(t), determine if Y(t) is mean square differentiable. A simple "yes" or "no" will not receive any credit. You must justify your answer.

$$Y(t) = \int_{-\infty}^{\infty} h(u-\alpha) X(\alpha)\, d\alpha = \int_{u-1}^{u} X(\alpha)\, d\alpha$$

If, for all t, the second partial

$$\left.\frac{\partial^2}{\partial u \partial v} E[Y(u)Y(v)]\right|_{u=v=t}$$ exists, the process $Y(t)$ is differentiable

$$\frac{\partial^2}{\partial u \partial v} E[Y(u)Y(v)] = \frac{\partial^2}{\partial u \partial v} \int_{u-1}^{u} \int_{v-1}^{v} R_X(\alpha,\beta)\, d\alpha\, d\beta$$

$$= R_X(u,v) - R_X(u-1,v) - R_X(u,v-1) + R_X(u-1,v-1)$$

evaluating this at $u=v=t$ we have

$$\left.\frac{\partial^2}{\partial u \partial v} E[Y(u)Y(v)]\right|_{u=v=t} = 2$$

Clearly $Y(t)$ is differentiable

Differentiable? (circle one)	Yes	No

(over)

Name: SOLUTION

6. The real process X(t) has zero mean and correlation function

$$E[X(t)\,X(s)] = \frac{1}{2} e^{-|t-s|}$$

$$R_X(\tau) = \frac{1}{2} e^{-|\tau|}$$

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-i\omega\tau} d\tau = \frac{1}{1+\omega^2}$$

This process is used to modulate a sinusoidal carrier

$$Y(t) = X(t)\cos(\omega_0 t).$$

In general Y(t) is neither stationary nor wide sense stationary. Useful information about the spectral content of Y(t) can often be obtained from the averaged correlation

$$\bar{R}_Y(\tau) = \lim_{A\to\infty} \frac{1}{2A}\int_{-A}^{A} E[Y(t+\tau)\,Y(t)]\,dt.$$

In effect this averaging randomizes the time origin of the process. Obtain explicit expressions for $\bar{R}_Y(\tau)$ and the associated power spectral density

$$\bar{S}_Y(\omega) = \int_{-\infty}^{\infty} \bar{R}_Y(\tau)\, e^{-i\omega\tau} d\tau.$$

$$E[Y(t+\tau)Y(t)] = R_X(\tau)\cos\omega_0(t+\tau)\cos\omega_0 t$$

$\cos A \cos B = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$

$$= \frac{1}{2}R_X(\tau)\cos\omega_0\tau + \frac{1}{2}R_X(\tau)\cos\omega_0(2t+\tau)$$

$$\bar{R}_Y(\tau) = \lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A} E[Y(t+\tau)Y(t)]\,dt$$

$$= \frac{1}{2}R_X(\tau)\cos\omega_0\tau + \frac{1}{2}R_X(\tau)\lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A}\cos\omega_0(2t+\tau)\,dt$$

$s = 2t$

$$= \frac{1}{2}R_X(\tau)\cos\omega_0\tau + \frac{1}{2}R_X(\tau)\lim_{A\to\infty}\frac{1}{4A}\int_{-2A}^{2A}\cos\omega_0(s+\tau)\,ds$$

$$\boxed{\bar{R}_Y(\tau) = \frac{1}{2}R_X(\tau)\cos\omega_0\tau \qquad \bar{S}_Y(\omega) = \frac{1}{4}\left\{\frac{1}{1+(\omega-\omega_0)^2} + \frac{1}{1+(\omega+\omega_0)^2}\right\}}$$

Now $\displaystyle\lim_{A\to\infty}\frac{1}{4A}\int_{-2A}^{2A}\cos\omega_0(s+\tau)\,ds = 0$

This can be seen by noting that the integral $\displaystyle\int_{-2A}^{2A}\cos\omega_0(s+\tau)\,ds$ extends over an integer number of periods $\left(=\frac{2\pi}{\omega_0}\right)$ of the cosine with an additional extent that cannot exceed two periods. Moreover, noting that $|\cos\omega_0(s+\tau)| \le 1$, we have

$$\left|\int_{-2A}^{2A}\cos\omega_0(s+\tau)\,ds\right| \le \frac{4\pi}{\omega_0}$$

$$\therefore \lim_{A\to\infty}\frac{1}{4A}\left|\int_{-2A}^{2A}\cos\omega_0(s+\tau)\,ds\right| \le \lim_{A\to\infty}\frac{\pi}{\omega_0 A} = 0$$

$$\therefore \boxed{\bar{R}_Y(\tau) = \frac{1}{2}R_X(\tau)\cos\omega_0\tau}$$

It follows that

$$\bar{S}_Y(\omega) = \int_{-\infty}^{\infty}\frac{1}{2}R_X(\tau)\cos\omega_0\tau\, e^{-i\omega\tau}d\tau = \frac{1}{4}\left\{S_X(\omega-\omega_0) + S_X(\omega+\omega_0)\right\}$$

$$\boxed{\bar{S}_Y(\omega) = \frac{1}{4}\left\{\frac{1}{1+(\omega-\omega_0)^2} + \frac{1}{1+(\omega+\omega_0)^2}\right\}}$$