ECE 250 Homework 2 Solutions

Prob. 1

$$P(X=n) = \sum_{m=-\infty}^{\infty} P(X=n, Y=m)$$

$$= \sum_{m=0}^{\infty} \frac{\alpha^m \beta^{n-m}}{m!\,(n-m)!} e^{-(\alpha+\beta)}$$

If $m>n$, $\frac{1}{(n-m)!}=0$ so the sum will only contribute if $m \le n$

$$= \sum_{m=0}^{n} \frac{\alpha^m \beta^{n-m}}{m!\,(n-m)!} e^{-(\alpha+\beta)}$$

$$= \frac{1}{n!}\sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} e^{-(\alpha+\beta)}$$

$$\therefore \quad \boxed{P(X=n) = \frac{(\alpha+\beta)^n}{n!} e^{-(\alpha+\beta)}, \quad n=0,1,\ldots}$$

$$P(Y=m) = \sum_{n=-\infty}^{\infty} P(X=n, Y=m)$$

$$= \sum_{n=m}^{\infty} \frac{\alpha^m \beta^{n-m}}{m!\,(n-m)!} e^{-(\alpha+\beta)}$$

change variables $n-m=k$

$$= \frac{\alpha^m}{m!} \sum_{k=0}^{\infty} \frac{\beta^k}{k!} e^{-(\alpha+\beta)}$$

$$\therefore \quad \boxed{P(Y=m) = \frac{\alpha^m}{m!} e^{-\alpha}, \quad m=0,1,\ldots}$$

Obviously

$$P(X=n, Y=m) \neq P(X=n)\,P(Y=m)$$

So that X and Y are NOT independent

Prob. 2

$$P(X=1) = \sum_{m=0}^{1} P[X=1, Y=m)$$

$$= \frac{5}{6}$$

it follows that $P(X=0) = \frac{1}{6}$

$$P(Y=1) = \sum_{n=0}^{1} P(X=n, Y=1)$$

$$= \frac{2}{3}$$

so that $P(Y=0) = \frac{1}{3}$

$$E[X] = 1 \cdot P(X=1) + 0 = \frac{5}{6}$$

$$E[Y] = 1 \cdot P(Y=1) + 0 = \frac{2}{3}$$

$$E[XY] = \sum_{n=0}^{1} \sum_{m=0}^{1} n\,m\,P(X=n, Y=m)$$

$$= 1 \cdot P(X=1, Y=1) = \frac{1}{2}$$

$$\boxed{E[XY] = \frac{1}{2}}$$

$$\mathrm{Cov}[X,Y] = E[XY] - E[X] \cdot E[Y]$$

$$= \frac{1}{2} - \left(\frac{5}{6}\right)\left(\frac{2}{3}\right)$$

$$\boxed{\mathrm{Cov}[X,Y] = -\frac{1}{18}}$$

Prob. 3

$$E[X] = \int_{-\infty}^{\infty} \cos\theta f_{\Theta}(\theta) d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cos\theta \, d\theta = 0$$

similarly $\quad E[Y] = 0$

$$E[XY] = \frac{1}{2\pi} \int_0^{2\pi} \sin\theta \cos\theta \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \sin 2\theta \, d\theta = 0$$

$\therefore$ $\boxed{E[XY] = E[X]E[Y] \quad X \text{ and } Y \text{ are uncorrelated}}$

If X and Y are <u>independent</u> then

$$E[X^n Y^m] = E[X^n] E[Y^m]$$

provided all indicated moments exist

Let $n = 2$ and $m = 2$

$$E[X^2] = \frac{1}{2\pi} \int_0^{2\pi} \cos^2\theta \, d\theta = \frac{1}{2}$$

similarly

$$E[Y^2] = \frac{1}{2\pi} \int_0^{2\pi} \sin^2\theta \, d\theta = \frac{1}{2}$$

After using successive Trig. identies

$$E[X^2 Y^2] = \frac{1}{8} \neq E[X^2] \cdot E[Y^2]$$

$\therefore$ $\boxed{X \text{ and } Y \text{ are NOT independent}}$

Prob. 4 $f_{X,Y}(x,y) = \begin{cases} \frac{1}{8}(x+y), & 0 \le x \le 2, 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$

$$f_X(x) = \int_0^2 f_{X,Y}(x,y)\,dy = \frac{1}{4}(1+x), \; 0 \le x \le 2$$

$$f_Y(y) = \int_0^2 f_{X,Y}(x,y)\,dx = \frac{1}{4}(1+y), \; 0 \le y \le 2$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{1}{8}(x+y)}{\frac{1}{4}(1+y)}$$

$$\boxed{f_{X|Y}(x|y) = \frac{1}{2}\frac{(x+y)}{(1+y)}, \quad 0 \le x \le 2, \; 0 \le y \le 2}$$

Similarly

$$\boxed{f_{Y|X}(y|x) = \frac{1}{2}\frac{(y+x)}{(1+x)}, \quad 0 \le x \le 2, \; 0 \le y \le 2}$$

Prob. 5 $P(X=n) = \left(\frac{1}{2}\right)^{n+1}; \quad P(Y=m) = \left(\frac{1}{2}\right)^{m+1}, \; n, m = 0, 1, \dots$

$$P(X=Y) = \sum_{k=0}^{\infty} P(X=Y \mid Y=k)\,P(Y=k)$$

Total probability
X and Y indep.

$$= \sum_{k=0}^{\infty} P(X=k)\,P(Y=k)$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k+1} \left(\frac{1}{2}\right)^{k+1} = \frac{1}{3}$$

From useful formulas

$$\boxed{P(X=Y) = \frac{1}{3}}$$

$\boxed{\text{Prob. 6}}$ $\quad \Phi_Z(u) = E[e^{iuZ}] = E[e^{iuX} e^{iuY}]$

X and Y are indep. $\longrightarrow \quad = \Phi_X(u)\, \Phi_Y(u)$

$$\Phi_X(u) = E[e^{iuX}] = \int_0^\infty \lambda e^{-\lambda x} e^{iux}\, dx = \frac{\lambda}{\lambda - iu}$$

similarly

$$\Phi_Y(u) = \frac{\mu}{\mu - iu}$$

from Fourier Transform pairs

$$\therefore \quad \Phi_Z(u) = \frac{\lambda\mu}{(\lambda - iu)(\mu - iu)}$$

$\underline{\mu \neq \lambda}$ $\quad \Phi_Z(u) = \frac{\mu}{\mu-\lambda} \frac{\lambda}{\lambda - iu} - \frac{\lambda}{\mu - \lambda} \frac{\mu}{\mu - iu}$

$$\boxed{\begin{array}{l} f_Z(z) = \frac{\mu}{\mu-\lambda} \lambda e^{-\lambda z} u(z) - \frac{\lambda}{\mu-\lambda} \mu e^{-\mu z} u(z) \\ \mu \neq \lambda \qquad u(z) = \begin{cases} 1, & z \geq 0 \\ 0, & z < 0 \end{cases} \end{array}}$$

$\underline{\mu = \lambda}$ $\quad \boxed{\Phi_Z(u) = \frac{\lambda^2}{(\lambda - iu)^2} \longleftrightarrow \lambda^2 z e^{-\lambda z} u(z)}$

from Fourier transform pairs