ECE 250 Homework 3 Solutions

Prob. 1

$$E[Z(t)Z(s)] = E\left[(X\cos\omega t + Y\sin\omega t)(X\cos\omega s + Y\sin\omega s)\right]$$

$$= E[X^2]\cos\omega t\cos\omega s + E[Y^2]\sin\omega t\sin\omega s + E[XY]\cos\omega t\sin\omega s + E[XY]\sin\omega t\cos\omega s$$

$$E[X^2] = E[Y^2] = \int_{-1}^{1}\alpha^2(1-|\alpha|)\,d\alpha = \frac{1}{6}$$

X and Y are indep $\longrightarrow E[XY] = E[X]E[Y] = 0$

$\therefore$

$$E[Z(t)Z(s)] = \frac{1}{6}\{\cos\omega t\cos\omega s + \sin\omega t\sin\omega s\}$$

$$\boxed{\begin{array}{c} E[Z(t)Z(s)] = \frac{1}{6}\cos\omega(t-s) \\ Z(t) \text{ is W.S.S.} \end{array}}$$

After much careful work

$$E[Z^4(t)] = \frac{1}{15}(\cos^4\omega t + \sin^4\omega t) + \frac{1}{6}\cos^2\omega t\cdot\sin^2\omega t$$

$\therefore$ $E[Z^4(t)]$ is a function of t. But if $Z(t)$ is strictly stationary, all moments must be constant (i.e. not a function of t).

$\boxed{\text{NOT strictly stationary}}$

Prob. 2

$$Y_n = \sum_{k=1}^{n} X_k$$

$$E[Y_n Y_m] = E\left[\sum_{k=1}^{n} X_k \sum_{l=1}^{m} X_l\right]$$

$$= \sum_{k=1}^{n} \sum_{l=1}^{m} \delta_{k,l}$$

$$E[X_k X_l] = \delta_{k,l} = \begin{cases} 1, & k=l \\ 0, & k \neq l \end{cases}$$

$\underline{m \leq n}$ $$= \sum_{k=1}^{m} 1 = m$$

$\underline{n < m}$ $$= \sum_{k=1}^{n} 1 = n$$

$$\therefore \boxed{E[Y_n Y_m] = \min[n, m]}$$

Not W.S.S.

$$\boxed{E[Y_n^2] = n}$$

Not strict sense stationary

Prob. 3

$$R_X(t,s) = \min[s,t]$$

$$R_Y(t,s) = E\left[e^{t} X(e^{-2t}) e^{s} X(e^{-2s})\right]$$

$$= e^{(t+s)} E[X(e^{-2t}) X(e^{-2s})]$$

$\underline{s \leq t}$ $$E[X(e^{-2t}) X(e^{-2s})] = e^{-2t}$$

$-2s \geq -2t$

$\boxed{\text{Prob. 3 Cont.}}$

so that $R_Y(t,s) = e^{-(t-s)}$, $s \le t$

similarly $R_Y(t,s) = e^{-(s-t)}$, $s > t$

In all cases this can be written as

$$\boxed{R_Y(t,s) = e^{-|t-s|}}$$

$\boxed{\text{Prob. 4}}$ $X(t) = u(t-T)$

The random variable T has distribution and density $F_T(\tau)$ and $f_T(\tau)$

$$E[X(t)] = \int_{-\infty}^{\infty} X(\tau) f_T(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} u(t-\tau) f_T(\tau)\, d\tau$$

because $u(t-\tau)=0$ if $\tau \ge t$ $\longrightarrow$ $= \int_{-\infty}^{t} f_T(\tau)\, d\tau = F_T(t)$

$$\boxed{E[X(t)] = F_T(t)}$$

Prob. 4 Cont.

$$E[X(t)X(s)] = \int_{-\infty}^{\infty} u(t-\tau)u(s-\tau)f_T(\tau)d\tau$$

$$\underline{s \le t} \quad = \int_{-\infty}^{s} f_T(\tau)d\tau = F_T(s)$$

$$t < s \quad = \int_{-\infty}^{t} f_T(\tau)d\tau = F_T(t)$$

Note that $u(t-\tau)u(s-\tau)$ requires $\tau \le t$ and $\tau \le s$ $\therefore$ the limit is the smaller of s or t

$$\boxed{E[X(t)X(s)] = F_T(\min[s,t])}$$

Prob. 5

$$\underline{s \le t} \quad E[N(t)N(s)] = E[(N(t)-N(0))(N(s)-N(0))]$$

Using the facts that
(1) $s \le t$
(2) non-overlapping increments are independent
(3) $N(0)=0$

$$= E[\{(N(t)-N(s))+(N(s)-N(0))\} \cdot \{N(s)-N(0)\}]$$

$$= E[\{N(t)-N(s)\}\{N(s)-N(0)\}] + E[\{N(s)-N(0)\}^2]$$

$$= E[N(t)-N(s)]E[N(s)-N(0)] + E[(N(s)-N(0))^2]$$

$\boxed{\text{Prob. 5 Cont.}}$ Evaluate $E[N(t)-N(s)]$ and $E[(N(s)-N(0))^2]$ using characteristic function

$s \le t$

$$\Phi_{N(t)-N(s)}(u) = \sum_{n=0}^{\infty} e^{iun} P(N(t)-N(s)=n)$$

$$= e^{\lambda\cdot(t-s)(e^{iu}-1)}$$

(*) $E[N(t)-N(s)] = \frac{1}{i}\frac{d}{du}\Phi_{N(t)-N(s)}(u)\Big|_{u=0} = \lambda\cdot(t-s)$

$E[(N(s)-N(0))^2] = \left(\frac{1}{i}\right)^2 \frac{d^2}{du^2}\Phi_{N(s)-N(0)}(u)\Big|_{u=0} = \lambda s + \lambda^2 s^2$

$\therefore$

if $s \le t$ $\quad E[N(t)N(s)] = \lambda s + \lambda^2 st$

Similarly if $t < s$ $\quad E[N(t)N(s)] = \lambda t + \lambda^2 st$

more consicely

$$E(N(t)N(t)) = \lambda \min[s,t] + \lambda^2 st$$

From (*) above $E[N(t)] = \lambda t$ and $E[N(s)] = \lambda s$

now $\quad Cov[N(t),N(s)] = E[N(t)N(s)] - E[N(t)]E[N(s)]$

$$\boxed{Cov[N(t),N(s)] = \lambda \min[s,t]}$$

$\boxed{\text{Prob. 6}}$

$$M_T = \frac{2}{T^2}\int_0^T N(t)dt$$

$$E[M_T] = \frac{2}{T^2}\int_0^T E[N(t)]dt = \frac{2}{T^2}\int_0^T \lambda t\,dt = \lambda$$

Chebyshev inequality

$$P(|M_T - \lambda| \geq \epsilon) = P(|M_T - E[M_T]| \geq \epsilon) \leq \frac{E[(M_T - E[M_T])^2]}{\epsilon^2}$$

$$M_T - E[M_T] = \frac{2}{T^2}\int_0^T N(t)dt - \frac{2}{T^2}\int_0^T \lambda t\,dt$$

$$= \frac{2}{T^2}\int_0^T (N(t) - \lambda t)dt$$

$$E[(M_T - E[M_T])^2] = \frac{4}{T^4}\int_0^T dt\int_0^T ds\, E[(N(t)-\lambda t)(N(s)-\lambda s)]$$

From Prob. 5

$$= \frac{4}{T^4}\int_0^T dt\int_0^T ds\, Cov[N(t), N(s)]$$

$$= \frac{4}{T^4}\int_0^T\int_0^T \lambda \min[s, t]\,dt\,ds$$

$$= \frac{4}{T^4}\left\{\lambda\int_0^T dt\int_0^t s\,ds + \lambda\int_0^T dt\int_t^T t\,ds\right\}$$

$$= \frac{4}{T^4}\left\{\frac{\lambda T^3}{3}\right\} = \frac{4\lambda}{3T}$$

$$\therefore \boxed{P(|M_T - \lambda| \geq \epsilon) \leq \frac{4\lambda}{3\epsilon^2 T} \xrightarrow[T \to 0]{} 0} \quad \epsilon > 0$$