ECE 250 Midterm
November 5, 2018

SOLUTION

1. The random variables X and Y have the joint characteristic function

$$\Phi_{X,Y}(u,v) = \frac{1}{4}\,\frac{1+e^{iu}+e^{iv}+e^{i(u+v)}}{1-iu-iv-uv}.$$

Are X and Y independent?

An answer not supported by appropriate reasoning will not receive any credit.

(1) $\Phi_X(u) = \Phi_{X,Y}(u,0) = \frac{1}{2}\,\frac{1+e^{iu}}{1-iu}$

(2) $\Phi_Y(v) = \Phi_{X,Y}(0,v) = \frac{1}{2}\,\frac{1+e^{iv}}{1-iv}$

X and Y will be independent iff

$$\Phi_{X,Y}(u,v) = \Phi_X(u)\cdot\Phi_Y(v)$$

but from (1) and (2)

$$\Phi_X(u)\,\Phi_Y(v) = \frac{1}{4}\,\frac{1+e^{iu}+e^{iv}+e^{i(u+v)}}{1-iu-iv-uv}$$

$\therefore$ X and Y <u>are</u> independent

Independent? (circle one)	Yes	No

SOLUTION

2. An integer-valued random variable has the probabilities

$$P(X = n) = \left(\frac{1}{2}\right)^{n+1}, n = 0, 1, \cdots.$$

Evaluate the mean and variance of X.
[Hint: You may find the characteristic function useful.]

An answer not supported by appropriate reasoning will not receive any credit.

Using the hint

$$\Phi_X(u) = E[e^{iuX}] = \sum_{n=0}^{\infty} e^{iun}\left(\frac{1}{2}\right)^{n+1} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{iu}\right)^n = \frac{1}{2-e^{iu}}$$

from useful formulas

$$E[X] = \left(\frac{1}{i}\right)\frac{d}{du}\Phi_X(u)\Big|_{u=0} = \frac{e^{iu}}{(2-e^{iu})^2}\Big|_{u=0} = 1$$

$$E[X^2] = \left(\frac{1}{i}\right)^2\frac{d^2}{du^2}\Phi_X(u)\Big|_{u=0} = \frac{e^{iu}(2+e^{iu})}{(2-e^{iu})^3}\Big|_{u=0} = 3$$

$$Var[X] = E[X^2] - (E[X])^2 = 3 - 1 = 2$$

$E[X] =$ 1	$Var[X] =$ 2

SOLUTION

3. The random variables X and Y have the joint density

$$f_{X,Y}(x,y) = \begin{cases} y\, e^{-(1+x)y}, & 0 \le x < \infty, 0 \le y < \infty \\ 0, & otherwise \end{cases}$$

Evaluate the correlation $E[X\,Y]$.

An answer not supported by appropriate reasoning will not receive any credit.

$$E[X\,Y] = \int_0^\infty \int_0^\infty x\,y\, f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^\infty \int_0^\infty x\,y\,y\,e^{-(1+x)y}\,dx\,dy$$

$$= \int_0^\infty y^2 e^{-y}\,dy \underbrace{\int_0^\infty x\,e^{-xy}\,dx}_{\shortparallel}$$

define $\alpha = xy$ then

$$= \int_0^\infty y^2 e^{-y}\,dy \int_0^\infty \frac{1}{y^2}\,\alpha\, e^{-\alpha}\,d\alpha$$

$$= \underbrace{\int_0^\infty e^{-y}\,dy}_{\shortparallel \atop 1} \underbrace{\int_0^\infty \alpha\, e^{-\alpha}\,d\alpha}_{\shortparallel \atop 1}$$

$$= 1$$

$E[X\,Y] =$ 1