ECE 250 Midterm November 3, 2017

SOLUTION

1. The random variables X and Y have the joint density

$$f_{X,Y}(x,y) = \frac{1}{2}(x+y)e^{-(x+y)}, 0 \le x, y < \infty.$$

Determine the density of $Z = X + Y$.

An answer not supported by appropriate reasoning will not receive any credit.

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\,dx$$

Now $0 \le x < \infty$ and $0 \le z - x < \infty$

$$\therefore f_Z(z) = \int_0^z f_{X,Y}(x, z-x)\,dx$$

but

$$f_{X,Y}(x, z-x) = \frac{1}{2}(x + [z-x])e^{-(x+[z-x])} = \frac{1}{2} z e^{-z}$$

so that

$$f_Z(z) = \int_0^z \frac{1}{2} z e^{-z}\,dx = \frac{1}{2} z^2 e^{-z}, z \ge 0$$

Aside

$$F_Z(z) = P(X+Y \le z) = \iint_{x+y \le z} f_{X,Y}(x,y)\,dx\,dy = \int_{-\infty}^{\infty} dx \int_{-\infty}^{z-x} f_{X,Y}(x,y)\,dy$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\,dx$$

This is the general formula

$$\boxed{f_Z(z) = \frac{1}{2} z^2 e^{-z}, z \ge 0}$$

SOLUTION

2. The joint characteristic function of the discrete, non-negative random variables N and M is given by

$$\Phi_{N,M}(u,v) = \exp\left[\lambda\left(e^{i(u+v)} + e^{iu} - 2\right)\right].$$

Determine the probabilities of the difference $D = N - M$.

An answer not supported by appropriate reasoning will not receive any credit.

Use characteristic function

$$\Phi_D(u) = E[e^{iuD}] = E[e^{iuN}e^{-iuM}] = \Phi_{N,M}(u,-u)$$

$$= e^{\lambda(e^{iu}-1)}$$

This is just the characteristic function of a Poisson variable with associated probabilities

$$P(D=k) = \frac{\lambda^k}{k!}e^{-\lambda}, \quad k = 0, 1, \cdots$$

$$\boxed{P(D=k) = \frac{\lambda^k}{k!}e^{-\lambda}, \quad k=0,1,\cdots}$$

SOLUTION

3. Let X_k, $k = 1, 2, \cdots$ be independent, identically-distributed random variables with common density

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 \le x \le 1 \\ 0, & otherwise. \end{cases}$$

$$E[X_k] = \frac{1}{2}\int_{-1}^{1} x\,dx = 0$$

$$E[X_k^2] = \frac{1}{2}\int_{-1}^{1} x^2\,dx = \frac{1}{3}$$

Consider the new random variable

$$Z_n = \prod_{k=1}^{n} X_k.$$

$$E[Z_n] = E[\prod_{k=1}^{n} X_k] = \prod_{k=1}^{n} E[X_k] = 0$$

Independence

$$E[Z_n^2] = E[(\prod_{k=1}^{n} X_k)^2] = \prod_{k=1}^{n} E[X_k^2] = \left(\frac{1}{3}\right)^n$$

Show that for $\epsilon > 0$,

$$\lim_{n\to\infty} P(|Z_n| \ge \epsilon) = 0.$$

$$E[|Z_n|] = \prod_{k=1}^{n} E[|X_k|] = \left(\frac{1}{2}\right)^n$$

An answer not supported by appropriate reasoning will not receive any credit.

Using Markov Inequality

$$P(|Z_n| \ge \epsilon) \le \frac{E[|Z_n|]}{\epsilon} \le \frac{1}{\epsilon}\left(\frac{1}{2}\right)^n$$

Using Chebyshev Inequality

$$P(|Z_n - E[Z_n]| \ge \epsilon) \le \frac{Var[Z_n]}{\epsilon^2}$$

Note: $E[Z_n] = 0$

$$\therefore P(|Z_n| \ge \epsilon) \le \frac{Var[Z_n]}{\epsilon^2}$$

$$Var[Z_n] = E[Z_n^2] - (E[Z_n])^2 = \left(\frac{1}{3}\right)^n$$

and finally

$$P(|Z_n| \ge \epsilon) \le \frac{1}{\epsilon^2}\left(\frac{1}{3}\right)^n$$

In either case $\lim_{n\to\infty} P(|Z_n| \ge \epsilon) = 0$ ∎

ECE 250 Final Exam December 14,2017

SOLUTION

1. Can the following function be the characteristic function of a valid probability density function?

$$G(u) = \frac{4+5u^2}{4+5u^2+u^4} = \frac{4+5u^2}{(4+u^2)(1+u^2)}$$

An answer not supported by appropriate reasoning will not receive any credit.

$$G(u) \overset{set}{=} \frac{A}{4+u^2} + \frac{B}{1+u^2}$$

solve for A and B

$$A = \frac{16}{3} \; ; \; B = \frac{-1}{3}$$

$$G(u) = \frac{4}{3}\,\frac{4}{4+u^2} - \frac{1}{6}\,\frac{2}{1+u^2}$$

$$g(x) = \frac{4}{3}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{4+u^2}e^{-iux}\,du - \frac{1}{6}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2}{1+u^2}e^{-iux}\,du$$

$$= \frac{4}{3}e^{-2|x|} - \frac{1}{6}e^{-|x|}$$

$$= \frac{4}{3}e^{-|x|}\left(e^{-|x|} - \frac{1}{8}\right)$$

Now $\int_{-\infty}^{\infty} g(x)\,dx = 1$

But when $|x| > \ln 8$ $g(x) < 0$ and cannot be a probability density

Valid Characteristic Function? (circle one)	Yes	No

SOLUTION

2. The joint density of the random variables X and Y is

$$f_{X,Y}(x,y) = xe^{-(1+y)x}, x \geq 0, y \geq 0.$$

Determine the density of the product $Z = X \cdot Y$.

An answer not supported by appropriate reasoning will not receive any credit.

$$F_Z(z) = P(XY \leq z) = \iint_{xy \leq z} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{z/x} f_{X,Y}(x,y)\,dy$$

General Expressions

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}\left(x, \frac{z}{x}\right) \frac{dx}{x}$$

Note that $x \geq 0$ and $\frac{z}{x} (= y) \geq 0$

$$= \int_0^{\infty} x e^{-(1+\frac{z}{x})x} \frac{dx}{x}$$

$$= e^{-z}, \quad z \geq 0$$

$f_Z(z) =$ $e^{-z}, \quad z \geq 0$

SOLUTION

3. It is desired to estimate the rate of a Poisson process $N(t)$ ($N(0) = 0$, independent increments, constant rate λ). Prove that, for $\epsilon > 0$,

$$\lim_{T\to\infty} P\left(\left|\frac{N(T)}{T} - \lambda\right| \geq \epsilon\right) = 0.$$

An answer not supported by appropriate reasoning will not receive any credit.

Chebyshev inequality

$$E\left[\frac{N(T)}{T}\right] = \lambda$$

$$P\left(\left|\frac{N(T)}{T} - \lambda\right| \geq \epsilon\right) \leq \frac{Var\left[\frac{N(T)}{T}\right]}{\epsilon^2}$$

$$Var\left[\frac{N(T)}{T}\right] = E\left[\left(\frac{N(T)}{T} - \lambda\right)^2\right] = \frac{E[N^2(T)]}{T^2} - \lambda^2$$

$$\therefore Var\left[\frac{N(T)}{T}\right] = \frac{\lambda T + \lambda^2 T^2}{T^2} - \lambda^2 = \frac{\lambda}{T}$$

$$\therefore \lim_{T\to\infty} \frac{Var\left[\frac{N(T)}{T}\right]}{\epsilon^2} = 0$$

$$\Phi_{N(T)}(u) = E\left[e^{iuN(T)}\right] = e^{\lambda T(e^{iu}-1)}$$

$$E[N(T)] = \lambda T$$

$$E[N^2(T)] = \lambda T + (\lambda T)^2$$

moments can be derived from Ch.F., but it is acceptable to write these from memory

SOLUTION

4. $N(t)$ is a Poisson process ($N(0) = 0$, independent increments, constant rate λ). Define T_M as the time to the M-th event. Evaluate the mean and variance of T_M

An answer not supported by appropriate reasoning will not receive any credit

$$F_{T_M}(\tau) = P(T_M \le \tau) = \sum_{n=M}^{\infty} P(N(\tau)=n)$$

$$= \sum_{n=M}^{\infty} \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau} \qquad \text{clearly } \tau \ge 0$$

$$f_{T_M}(\tau) = \frac{d}{d\tau} F_{T_M}(\tau) = \sum_{n=M}^{\infty} \frac{n\lambda^n \tau^{n-1}}{n!} e^{-\lambda\tau} - \lambda \sum_{n=M}^{\infty} \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau}$$

$$= \lambda \frac{(\lambda\tau)^{M-1}}{(M-1)!} e^{-\lambda\tau}$$

$$\Phi_{T_M}(u) = \int_0^{\infty} e^{iu\tau} f_{T_M}(\tau)\, d\tau = \lambda \int_0^{\infty} e^{iu\tau} \frac{(\lambda\tau)^{M-1}}{(M-1)!} e^{-\lambda\tau}\, d\tau$$

$$= \left(\frac{\lambda}{\lambda - iu}\right)^M$$ from "Useful Formulas" relationship II

Now

$$E[T_M] = \left(\frac{1}{i}\right)\frac{d}{du}\Phi_{T_M}(u)\Big|_{u=0} = \frac{M}{\lambda}$$

$$E[T_M^2] = \left(\frac{1}{i}\right)^2 \frac{d^2}{du^2}\Phi_{T_M}(u)\Big|_{u=0} = \frac{M(M+1)}{\lambda^2}$$

$E[T_M] = \dfrac{M}{\lambda}$ $\qquad$ $Var[T_M] = E[T_M^2] - (E[T_M])^2 = \dfrac{M}{\lambda^2}$

SOLUTION

5. The W.S.S. process $X(t)$ has zero mean and correlation function

$$R_X(\tau) = 2e^{-|\tau|} - e^{-2|\tau|}.$$

This process is passed through a linear time invariant system. The input, $X(t)$, and output, $Y(t)$ are related via the differential equation.

$$\frac{d}{dt}Y(t) + 3Y(t) = \frac{d^2}{dt^2}X(t) + 3\frac{d}{dt}X(t) + 2X(\tau).$$

Determine the correlation function of $Y(t)$.

An answer not supported by appropriate reasoning will not receive any credit

$$H(i\omega) = \frac{(i\omega)^2 + 3(i\omega) + 2}{(i\omega) + 3} = \frac{(2-\omega^2) + i(3\omega)}{3 + i(\omega)}$$

$$|H(i\omega)|^2 = \frac{(2-\omega^2)^2 + 9\omega^2}{9+\omega^2} = \frac{4 + 5\omega^2 + \omega^4}{9+\omega^2}$$

$$S_X(\omega) = \frac{4}{1+\omega^2} - \frac{4}{4+\omega^2} = \frac{12}{(1+\omega^2)(4+\omega^2)} = \frac{12}{4+5\omega^2+\omega^4}$$

from Transform pairs

Now

$$S_Y(\omega) = |H(i\omega)|^2 S_X(\omega) = \frac{12}{9+\omega^2} = 2\,\frac{2\cdot(3)}{(3)^2+\omega^2}$$

$$R_Y(\tau) = 2e^{-3|\tau|}$$

Transform pairs again

$$\boxed{R_Y(\tau) = 2e^{-3|\tau|}}$$

SOLUTION

6. A zero-mean, wide sense stationary process, $X(t)$, is passed through a linear system whose output is given by

$$Y(t) = X(t) + \int_{-\infty}^{\infty} h(t-\propto)X(\propto)d\propto$$

with

$$h(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$S_X(w) = \int_{-\infty}^{\infty} 4\cos 2\tau \, e^{-iw\tau} d\tau = 4\pi\left(\delta(w-2)+\delta(w+2)\right)$$

The correlation function of $X(t)$ is $R_X(\tau) = 4\cos 2\tau$.

Determine the correlation function of $Y(t)$.

$S_Y(w) =$

An answer not supported by appropriate reasoning will not receive any credit

Rewrite $Y(t)$ $\quad Y(t) = \int_{-\infty}^{\infty} \hat{h}(t-\alpha)X(\alpha)d\alpha$

where $\hat{h}(t) = \delta(t) + h(t)$ unit impulse

$$S_Y(w) = |\hat{H}(iw)|^2 S_X(w) \qquad \hat{H}(iw) = \int_{-\infty}^{\infty} \hat{h}(t)e^{-iwt}dt$$

$$= \int_{-\infty}^{\infty} \delta(t)e^{-iwt}dt + \int_{0}^{\infty} e^{-t}e^{-iwt}dt = 1 + \frac{1}{1+iw} = \frac{2+iw}{1+iw}$$

$$|\hat{H}(iw)|^2 = \frac{4+w^2}{1+w^2}$$

$$S_Y(w) = |\hat{H}(iw)|^2 S_X(w)$$

$$\therefore R_Y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(w)e^{iw\tau}dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4+w^2}{1+w^2} 4\pi e^{iw\tau}\delta(w-2)dw + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4+w^2}{1+w^2} 4\pi e^{iw\tau}\delta(w+2)dw$$

$$= \frac{32}{5}\cos 2\tau$$

$$\boxed{R_Y(\tau) = \frac{32}{5}\cos 2\tau}$$

SOLUTION

7. Two shot processes are defined by

$$X_1(t) = \sum_{t_k} h_1(t - t_k) \text{ and } X_2(t) = \sum_{t_k} h_2(t - t_k).$$

Here the delays t_k are governed by a Poisson process (independent increments, $N(0) = 0$ and a constant rate λ). The functions h_1 and h_2 are given by

$$h_1(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & otherwise \end{cases} \quad \text{and} \quad h_2(t) = \begin{cases} 1, & -1 \le t \le 0 \\ 0, & otherwise \end{cases}$$

Are the random variables $X_1(t_0)$ and $X_2(t_0)$ independent?

An answer not supported by appropriate reasoning will not receive any credit

$$\Phi_{X_1(t_0), X_2(t_0)}(u,v) = E\left[e^{iuX_1(t_0) + ivX_2(t_0)}\right]$$

$$(*) \quad = e^{\lambda \int_{-\infty}^{\infty} \left[e^{iuh_1(\tau) + ivh_2(\tau)} - 1\right] d\tau}$$

$$= e^{\lambda \int_0^1 \left[e^{iuh_1(\tau)} - 1\right] d\tau + \lambda \int_{-1}^{0} \left[e^{ivh_2(\tau)} - 1\right] d\tau}$$

$h_2 = 0$ $\quad$ $h_1 = 0$

$$= e^{\lambda \int_0^1 (e^{iu} - 1) d\tau + \lambda \int_{-1}^{0} (e^{iv} - 1) d\tau}$$

$$= e^{\lambda \{(e^{iu} - 1) + (e^{iv} - 1)\}}$$

$$= e^{\lambda(e^{iu} - 1)} \cdot e^{\lambda(e^{iv} - 1)}$$

but from (*) $\Phi_{X_1(t_0)}(u) = e^{\lambda(e^{iu} - 1)}$ and $\Phi_{X_2(t_0)}(v) = e^{\lambda(e^{iv} - 1)}$

set $v = 0$ $\quad$ set $u = 0$

Independent? (circle one)	Yes	No

Clearly $\Phi_{X_1(t_0), X_2(t_0)}(u,v) = \Phi_{X_1(t_0)}(u) \cdot \Phi_{X_2(t_0)}(v)$ INDEP.

SOLUTION

8. It is desired to estimate the value of a wide sense stationary process, $X(t)$, at time t by a linear combination of three earlier observations. The correlation function of the process is

$$R(\tau) = e^{-|\tau|}$$

and the estimate is

$$\tilde{X}(t) = A_1X(t-t_1) + A_2X(t-t_2) + A_3X(t-t_3)$$

with

$$t_1 = t_0, t_2 = 2t_0, t_3 = 3t_0 \text{ and } t_0 > 0.$$

The coefficients A_1, A_2, A_3 are to be chosen to minimize the mean square error

$$\mathcal{E} = E\left[\left(X(t) - \tilde{X}(t)\right)^2\right].$$

Determine the values of the coefficients that will minimize $\mathcal{E}$.

An answer not supported by appropriate reasoning will not receive any credit

Orthogonality Principle

$$(1)\quad 0 = E[(X(t)-\tilde{X}(t))X(t-t_1)]$$

$$(2)\quad 0 = E[(X(t)-\tilde{X}(t))X(t-t_2)]$$

$$(3)\quad 0 = E[(X(t)-\tilde{X}(t))X(t-t_3)]$$

$$(1')\quad 0 = E[(X(t)-A_1X(t-t_1)-A_2X(t-t_2)-A_3X(t-t_3))X(t-t_1)]$$

$$(2')\quad 0 = E[(X(t)-A_1X(t-t_1)-A_2X(t-t_2)-A_3X(t-t_3))X(t-t_2)]$$

$$(3')\quad 0 = E[(X(t)-A_1X(t-t_1)-A_2X(t-t_2)-A_3X(t-t_3))X(t-t_3)]$$

continued on next page

$A_1 = e^{-t_0}$	$A_2 = 0$	$A_3 = 0$

SOLUTION

Problem 8 continued

Now $E[X(t)X(s)] = R(t-s) = e^{-|t-s|}$

$\therefore$

$$(1'')\ 0 = R(t_0) - A_1R(0) - A_2R(t_0) - A_3R(2t_0)$$

$$(2'')\ 0 = R(2t_0) - A_1R(t_0) - A_2R(0) - A_3R(t_0)$$

$$(3'')\ 0 = R(3t_0) - A_1R(2t_0) - A_2R(t_0) - A_3R(0)$$

Substituting $R(\tau) = e^{-|\tau|}$

$$(1''')\ 0 = e^{-t_0} - A_1 - A_2e^{-t_0} - A_3e^{-2t_0}$$

$$(2''')\ 0 = e^{-2t_0} - A_1e^{-t_0} - A_2 - A_3e^{-t_0}$$

$$(3''')\ 0 = e^{-3t_0} - A_1e^{-2t_0} - A_2e^{-t_0} - A_3$$

The solutions are (obtain using any method)

$$A_1 = e^{-t_0}$$

$$A_2 = 0$$

$$A_3 = 0$$

ECE 250 Midterm
October 28, 2016

SOLUTION

1. The discrete random variable X has the probabilities

   P(X = -2) = P(X = -1) = P(X = 1) = P(X = 2) = 1/4.

   The random variable Y is defined by Y = $|X|$. Are X and Y uncorrelated? Are they independent? Independent iff $P(X=n, Y=m) = P(X=n)P(Y=m)$ for ALL n, m

   An answer not supported by appropriate reasoning will not receive any credit.

X	$Y=\|X\|$	XY	Prob.
-2	2	-4	$1/4$
-1	1	-1	$1/4$
1	1	1	$1/4$
2	2	4	$1/4$

$$E[XY] = (-4)\times P(X=-2) + (-1)\times P(X=1) + (1)\times P(X=1) + (4)\times P(X=2) = 0$$

$$E[X] = (-2)P(X=-2) + (-1)P(X=-1) + (1)P(X=1) + (2)P(X=2) = 0$$

$$E[Y] = (2)P(X=-2) + (1)P(X=-1) + (1)P(X=1) + 2P(X=2) = \frac{3}{2} \quad \therefore E[XY] = E[X]E[Y]$$

$$P(X=-2, Y=2) = \frac{1}{4}; \quad P(X=-2) = \frac{1}{4}; \quad P(Y=2) = \frac{1}{4}$$

$$P(X=-2)P(Y=2) = \frac{1}{16} \neq P(X=-2, Y=2)$$

Sufficient to use any pair of variables X and Y

Uncorrelated?	Yes	No
Independent? (Circle One)	Yes	No

SOLUTION

2. The random variables X and Y have the joint characteristic function

$$\Phi_{X,Y}(u,v) = \frac{1}{2}\cos(u-v) + \frac{1}{2}\cos(u+v).$$

Are X and Y independent?

An answer not supported by appropriate reasoning will not receive any credit.

$$\Phi_X(u) = \Phi_{X,Y}(u,0) = \cos u$$

$$\Phi_Y(v) = \Phi_{X,Y}(0,v) = \cos v$$

$$\Phi_X(u)\,\Phi_X(v) = \cos u \cos v = \frac{1}{2}\cos(u-v) + \frac{1}{2}\cos(u+v)$$

$$= \Phi_{X,Y}(u,v)$$

from "Useful Formulas"

$\therefore$ X and Y are independent

Independent? (Circle One)	Yes	No

SOLUTION

3. The random variables X and Y are independent and identically distributed with probabilities

$$P(X = n) = \left(\frac{1}{2}\right)^n \text{ and } P(Y = m) = \left(\frac{1}{2}\right)^m, \quad n, m = 1, 2, \ldots .$$

Evaluate the probability $P(X \geq Y)$.

An answer not supported by appropriate reasoning will not receive any credit.

$$P(X \geq Y) = \sum_{n \geq m}\sum P(X = n, Y = m)$$

X and Y are indep.

$$= \sum_{n=1}^{\infty} P(X = n) \sum_{m=1}^{n} P(Y = m)$$

$$\sum_{m=1}^{n} P(Y = m) = \sum_{m=1}^{n} \left(\frac{1}{2}\right)^m = 1 - \left(\frac{1}{2}\right)^n$$

from "Useful Formulas"

$$\therefore P(X \geq Y) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \left[1 - \left(\frac{1}{2}\right)^n\right]$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n - \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$$

$$= \frac{2}{3}$$

$P(X \geq Y) =$

SOLUTION

4. The random variables Xand Y have the joint density

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 \le x \le y \le 1 \\ 0, & \textit{otherwise.} \end{cases}$$

clearly $0 \le X+Y \le 2$

Determine the probability density of Z = X + Y. You must provide values of the density for <u>all</u> possible values of z.

An answer not supported by appropriate reasoning will not receive any credit.

$$F_Z(z) = P(Z \le z) = P(X+Y \le z) = \iint_{x+y \le z} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{z-x} f_{X,Y}(x,y)\,dy$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\,dx$$ General expression

Using conditions for x and y

$$0 \le x \le z-x \le 1$$

$$0 \le x \le \frac{z}{2} \qquad z-1 \le x \le \frac{z}{2}$$

we must consider two regions

$\underline{\underline{0 \le z \le 1}}$

$$f_Z(z) = 2\int_0^{z/2} dx = z$$

$\underline{\underline{1 \le z \le 2}}$

$$2\int_{z-1}^{z/2} dx = 2-z$$

$f_Z(z) = 0$ if $z < 0$ or if $z > 2$

$$f_Z(z) = \begin{cases} z, & 0 \le z \le 1 \\ 2-z, & 1 \le z \le 2 \\ 0, & \text{otherwise} \end{cases}$$

ECE 250 Final Exam
December 8, 2016

SOLUTION

1. Consider the random variables

$$X_1 = (-1)^{N_1} \quad \text{and} \quad X_2 = (-1)^{N_2}.$$

Here N_1 and N_2 are independent Poisson variables

$$P(N_1 = n) = P(N_2 = n) = \frac{\lambda^n}{n!} e^{-\lambda}, \quad n = 0, 1, \ldots$$

Define $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.

Are Y_1 and Y_2 uncorrelated? Are they independent?

$$E[X_1] = \sum_{n=0}^{\infty} (-1)^n P(N_1 = n) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} e^{-\lambda} = e^{-2\lambda}$$

Similarly $E[X_2] = e^{-2\lambda}$

X_1 and X_2 are indep.

An answer not supported by appropriate reasoning will not receive any credit.

$$Y_1 Y_2 = (X_1 + X_2)(X_1 - X_2) = X_1^2 - X_2^2 = 0$$

$$E[Y_1] = E[X_1] + E[X_2] = 2e^{-2\lambda} \qquad E[Y_2] = E[X_1] - E[X_2] = 0$$

$$\therefore E[Y_1 Y_2] = E[Y_1] E[Y_2] \quad \therefore \text{UNCORRELATED}$$

$$\Phi_{Y_1,Y_2}(u,v) = E[e^{iuY_1 + ivY_2}] = E[e^{i(u+v)X_1}] E[e^{i(u-v)X_2}]$$

(X_1 and X_2 indep)

$$E[e^{i(u+v)X_1}] = \sum_{n=0}^{\infty} e^{i(u+v)n} \frac{\lambda^n}{n!} e^{-\lambda} = e^{\lambda(e^{i(u+v)} - 1)}$$

similarly

$$E[e^{i(u-v)X_2}] = e^{\lambda(e^{i(u-v)} - 1)}$$

$$\therefore \Phi_{Y_1,Y_2}(u+v) = e^{\lambda\{e^{i(u+v)} + e^{i(u-v)} - 2\}}$$

$$\Phi_{Y_1}(u) = \Phi_{Y_1,Y_2}(u,0) = e^{2\lambda(e^{iu} - 1)}; \quad \Phi_{Y_2}(v) = \Phi_{Y_1,Y_2}(0,v) = e^{\lambda(e^{iv} + e^{-iv} - 2)}$$

$$\boxed{\text{Obviously } \Phi_{Y_1,Y_2}(u,v) \neq \Phi_{Y_1}(u)\Phi_{Y_2}(v) \text{ NOT INDEP}}$$

Uncorrelated?	Yes	No
Independent? (Circle One)	Yes	No

SOLUTION

2. The random variable X has a continuous probability density. The median of X is the value M that satisfies

   $F_X(M) = 1 - F_X(M)$.

   Let W be the value that minimizes $E[|X-W|]$. Prove that W = M.

   An answer not supported by appropriate reasoning will not receive any credit.

$$E[|X-W|] = \int_{-\infty}^{\infty} |x-W| f_X(x)\,dx$$

$$= \int_{-\infty}^{W} (W-x) f_X(x)\,dx + \int_{W}^{\infty} (x-W) f_X(x)\,dx$$

$$= W\int_{-\infty}^{W} f_X(x)\,dx - \int_{-\infty}^{W} x f_X(x)\,dx + \int_{W}^{\infty} x f_X(x)\,dx - W\int_{W}^{\infty} f_X(x)\,dx$$

$$\frac{d}{dW} E[|X-W|] = \int_{-\infty}^{W} f_X(x)\,dx + W f_X(W) - W f_X(W) - W f_X(W) - \int_{W}^{\infty} f_X(x)\,dx + W f_X(W)$$

$$= \int_{-\infty}^{W} f_X(x)\,dx - \int_{W}^{\infty} f_X(x)\,dx$$

$$= F_X(W) - (1 - F_X(W)) \overset{set}{=} 0$$

$\therefore$ W satisfies the definition of a median

SOLUTION

3. N(t) is a Poisson process (independent increments, N(0) = 0, constant rate λ). Show that, for $\varepsilon > 0$,

Chebyshev Inequality

$$\lim_{t \to \infty} P\left(\left|\frac{N(t)}{t} - \lambda\right| \geq \varepsilon\right) = 0. \qquad P\left(\left|\frac{N(t)}{t} - \lambda\right| \geq \epsilon\right) \leq \frac{Var\left[\frac{N(t)}{t} - \lambda\right]}{\epsilon^2}$$

[HINT: Begin by evaluating the moments of N(t).]

An answer not supported by appropriate reasoning will not receive any credit.

$$\Phi_{N(t)}(u) = E\left[e^{iuN(t)}\right] = \sum_{n=0}^{\infty} e^{iun} P(N(t) = n)$$

$$= \sum_{n=0}^{\infty} e^{iun} \frac{(\lambda t)^n}{n!} e^{-\lambda t} = e^{\lambda t(e^{iu} - 1)}$$

$$E[N(t)] = \frac{1}{i}\frac{d}{du}\Phi_{N(t)}(u)\Big|_{u=0} = \lambda t$$

$$E[N^2(t)] = \left(\frac{1}{i}\right)^2 \frac{d^2}{du^2}\Phi_{N(t)}(u)\Big|_{u=0} = \lambda t + \lambda^2 t^2$$

$$E\left[\left(\frac{N(t)}{t} - \lambda\right)\right] = 0$$

$$\therefore \quad Var\left[\frac{N(t)}{t} - \lambda\right] = E\left[\left(\frac{N(t)}{t} - \lambda\right)^2\right]$$

$$= E\left[\frac{N^2(t)}{t^2} - 2\lambda\frac{N(t)}{t} + \lambda^2\right]$$

$$= \frac{\lambda}{t}$$

$$\therefore$$

$$\lim_{t \to \infty} P\left(\left|\frac{N(t)}{t} - \lambda\right| \geq \epsilon\right) \leq \lim_{t \to \infty} \frac{\lambda}{t} = 0$$

SOLUTION

4. The non-random function g(t) is periodic with period T. That is g(t + T) = g(t). The random variable A is uniformly distributed on the interval [0, T]

$$f_A(\alpha) = \begin{cases} \dfrac{1}{T}, & 0 \le \alpha \le T \\ 0, & otherwise. \end{cases}$$

Consider the random process X(t) = g(t + A). Is X(t) wide sense stationary?

An answer not supported by appropriate reasoning will not receive any credit.

$$E[X(t)] = \int_{-\infty}^{\infty} g(t+\alpha) f_A(\alpha)\,d\alpha = \int_0^T \frac{1}{T} g(t+\alpha)\,d\alpha$$

set $\beta = t+\alpha$

The integral of a periodic function over one period is const.

$$= \frac{1}{T}\int_t^{t+T} g(\beta)\,d\beta = \text{const.}$$

$$E[X(t+\tau)X(t)] = \frac{1}{T}\int_{-\infty}^{\infty} g(t+\tau+\alpha) g(t+\alpha) f_A(\alpha)\,d\alpha$$

$$= \frac{1}{T}\int_0^T g(t+\tau+\alpha) g(t+\alpha)\,d\alpha$$

set $\beta = t+\alpha$

$$= \frac{1}{T}\int_t^{t+T} g(\tau+\beta) g(\beta)\,d\beta$$

$g(\tau+\beta)g(\beta)$ is a periodic function with period T

may integrate over any full period = T

$$= \frac{1}{T}\int_0^T g(\tau+\beta) g(\beta)\,d\beta$$

= function only of τ

W.S.S.? (Circle One)	Yes	No

SOLUTION

5. Consider the Poisson process N(t) (independent increments, N(0) = 0, constant rate λ). Denote by Δk the time between the (k – 1)st event and the k + n event. Let T_M be the time until the M-th event

$$T_M = \sum_{k=1}^{M} \Delta k$$

The times between adjacent events are independent

Evaluate the mean and variance of T_M.

An answer not supported by appropriate reasoning will not receive any credit.

$$E[T_M] = E\left[\sum_{k=1}^{M} \Delta_k\right] = \sum_{k=1}^{M} E[\Delta_k]$$

$$Var[T_M] = Var\left[\sum_{k=1}^{M} \Delta_k\right] = \sum_{k=1}^{M} Var[\Delta_k]$$

If events are indep., the variance of a sum is the sum of the variances

We know (from class) that

$$f_{\Delta_k}(\tau) = \begin{cases} \lambda e^{-\lambda \tau}, & \tau \geq 0 \\ 0, & \tau < 0 \end{cases} \quad k = 1, 2, \cdots, M$$

$$\therefore E[\Delta_k] = \int_0^{\infty} \tau \lambda e^{-\lambda \tau} d\tau = \frac{1}{\lambda}$$

$$Var[\Delta_k] = E[\Delta_k^2] - (E[\Delta_k])^2 = \int_0^{\infty} \tau^2 \lambda e^{-\lambda \tau} d\tau - \frac{1}{\lambda^2}$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\therefore E[T_M] = \sum_{k=1}^{M} \frac{1}{\lambda} = \frac{M}{\lambda}$$

$$Var[T_M] = \sum_{k=1}^{M} \frac{1}{\lambda^2} = \frac{M}{\lambda^2}$$

$E[T_M] = \dfrac{M}{\lambda}$	$Var[T_M] = \dfrac{M}{\lambda^2}$

SOLUTION

6. A zero-mean, white Gaussian process with constant power spectral density P_0 is passed through a linear, time-invariant filter

$$Y(t) = \int_{-\infty}^{\infty} h(\alpha)X(t-\alpha)\,d\alpha.$$

Here $h(t) = \dfrac{\sin \pi t}{\pi t}, \quad -\infty < t < \infty.$

Evaluate the probability density of Y(t).

A linear operation on a Gaussian process results in another Gaussian process. $\therefore$ $Y(t)$ is a Gauss. variable

An answer not supported by appropriate reasoning will not receive any credit.

$$f_{Y(t)}(y) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(y-m)^2}{2\sigma^2}} \qquad m = E[Y(t)], \quad \sigma^2 = Var[Y(t)]$$

$$R_Y(\tau) \longleftrightarrow S_Y(w)$$

from transform pairs

$$S_Y(w) = |H(iw)|^2 S_X(w)$$

$$H(iw) = rect\left(\frac{w}{2\pi}\right) = \begin{cases} 1, & -\pi \le w \le \pi \\ 0, & \text{otherwise} \end{cases}$$

$$S_X(w) = P_0$$

from transform pairs

$$= P_0\, rect\left(\frac{w}{2\pi}\right) = \begin{cases} P_0, & -\pi \le w \le \pi \\ 0, & \text{otherwise} \end{cases}$$

$$* \quad R_Y(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_0 e^{iw\tau}\,dw = P_0 \frac{\sin \pi\tau}{\pi\tau}$$

$$E[Y(t)] = \int_{-\infty}^{\infty} h(\alpha) X(t-\alpha)\,d\alpha = 0 \qquad E[X(t)] = 0$$

$$* \quad \therefore \quad m = 0 \text{ and } \sigma^2 = R_Y(0) = P_0$$

finally from (*) and (*) $\quad f_{Y(t)}(y) = \dfrac{1}{\sqrt{2\pi P_0}} e^{-\frac{y^2}{2P_0}}$

$$\boxed{f_{Y(t)}(x) = \frac{1}{\sqrt{2\pi P_0}} e^{-\frac{y^2}{2P_0}}}$$

SOLUTION

7. N(t) is a Poisson process (independent increments, N(0) = 0, constant rate λ).

$$P(N(t)-N(s)=n)=\frac{[\lambda(t-s)]^n}{n!}e^{-\lambda(t-s)},\qquad \begin{matrix} n=0,1,\ldots. \\ t\geq s. \end{matrix}$$

Evaluate the conditional expectation

$$E[N(t+\tau)|N(t)=m],\quad \tau\geq 0,\quad m\geq 0.$$

An answer not supported by appropriate reasoning will not receive any credit.

$$P(N(t+\tau)=n\,|\,N(t)=m)=P(\{N(t+\tau)-N(t)\}+N(t)=n\,|\,N(t)=m)$$

$$=P(\{N(t+\tau)-N(t)\}=n-m\,|\,N(t)=m)$$

$\{N(t+\tau)-N(t)\}$ and $N(t)$ are independent - non-overlapping intervals

$$=\frac{P(\{N(t+\tau)-N(t)\}=n-m,\ N(t)=m)}{P(N(t)=m)}$$

$$=P(N(t+\tau)-N(t)=n-m)$$

Now

$$E[N(t+\tau)|N(t)=m]=\sum_{n=m}^{\infty}n\,P(N(t+\tau)=n\,|\,N(t)=m)$$

clearly $n\geq m$

$$=\sum_{n=m}^{\infty}n\,P(N(t+\tau)-N(t)=n-m)$$

$n-m=k$

$$=\sum_{k=0}^{\infty}(k+m)P(N(t+\tau)-N(t)=k)$$

$$=\sum_{k=0}^{\infty}(k+m)\frac{(\lambda\tau)^k}{k!}e^{-\lambda\tau}$$

$$=m+\lambda\tau$$

$E[N(t+\tau)|N(t)=m]=$ $m+\lambda\tau$

SOLUTION

8. The input, X(t), and output, Y(t), of a linear system are related via the differential equation

$$\frac{d^2}{dt^2}Y(t)+2\frac{d}{dt}Y(t)+Y(t)=2\frac{d}{dt}X(t)+2X(t).$$

$$H(i\omega) = \frac{2+2(i\omega)}{1+2(i\omega)+(i\omega)^2} = \frac{2}{(1+i\omega)}$$

If the input is a zero-mean, wide sense stationary process with correlation

$R_X(\tau) = (3/4)e^{-2|\tau|}$, $S_X(\omega) = \frac{3}{4+\omega^2}$ ← Fourier Trans. pairs

determine the correlation function of Y(t).

An answer not supported by appropriate reasoning will not receive any credit.

$$S_Y(\omega) = |H(i\omega)|^2 S_X(\omega) = \frac{3}{4+\omega^2}\cdot\frac{4}{1+\omega^2}$$

partial fraction → $$= \frac{4}{1+\omega^2} - \frac{4}{4+\omega^2}$$

$$\therefore\ R_Y(\tau) = 2e^{-|\tau|} - e^{-2|\tau|}$$

$R_Y(\tau) = 2e^{-|\tau|} - e^{-2|\tau|}$