ChatGPT 4o

共享 IA

proof f(x)=1+x+⋯+x^n/n!  is irreducible over q(x)

To prove that the polynomial $f(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ is irreducible over $\mathbb{Q}(x)$, we can use **Eisenstein's Criterion** by substituting $x$ with $x+1$ in the polynomial, as this often simplifies handling the constant term and the powers of the coefficients.

Here's the proof in detail:

1. **Rewrite $f(x)$:** Define a polynomial $g(x)$ by setting $f(x) = g(x+1)$, where

$$g(x) = f(x+1) = 1 + (x+1) + \frac{(x+1)^2}{2!} + \cdots + \frac{(x+1)^n}{n!}.$$

Expanding each term $\frac{(x+1)^k}{k!}$ in terms of powers of $x$ will create a polynomial in $\mathbb{Q}[x]$ where the coefficients of $g(x)$ can pot ially be analyzed for divisibility properties.

ChatGPT 4o

共享 IA

2. **Apply Eisenstein's Criterion:** Eisenstein's Criterion states that a polynomial $h(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$ if there exists a prime $p$ such that:
   - $p$ divides all coefficients $a_0, a_1, \ldots, a_{n-1}$,
   - $p$ does not divide $a_n$,
   - $p^2$ does not divide $a_0$.

   By careful selection of a suitable prime $p = n$, we examine the divisibility of coefficients in $g(x)$:
   - For terms involving $\frac{x^k}{k!}$ where $k < n$, the coefficients involve factorials in the denominator, so $n$ divides the coefficients of these terms.
   - For the highest-degree term $\frac{x^n}{n!}$, $n$ does not divide the coefficient since $\frac{1}{n!}$ is not divisible by $n$.
   - Furthermore, $n^2$ does not divide the constant term 1.

3. **Conclusion:** By Eisenstein's Criterion, $g(x)$ is irreducible over $\mathbb{Q}$, and therefore $f(x) = g(x+1)$ is also irreducible over $\mathbb{Q}$.

Thus, $f(x)$ is irreducible over $\mathbb{Q}(x)$.