4.8

(a) 二维极坐标下 Laplace 方程的解为

$$\Phi(\rho,\phi)=c_0+d_0\ln\rho+\sum_{n=1}^{\infty}(a_n\cos n\phi+b_n\sin n\phi)(c_n\rho^n+d_n\rho^{-n}),\tag{1.1}$$

由对称性，$\Phi(\rho,-\phi)=\Phi(\rho,\phi)$，故所有的正弦项系数 $b_n\equiv 0$，

$\rho<a$ 时，

$$\Phi_1(\rho,\phi)=a_0+\sum_{n=1}^{\infty}a_n\rho^n\cos n\phi,\tag{1.2}$$

$a<\rho<b$ 时，

$$\Phi_2(\rho,\phi)=b_0+c_0\ln\rho+\sum_{n=1}^{\infty}(b_n\rho^n+c_n\rho^{-n})\cos n\phi,\tag{1.3}$$

$\rho>b$ 时，由 $\rho\to\infty$ 时，$\Phi=-E_0\rho\cos\phi$

$$\Phi_3(\rho,\phi)=-E_0\rho\cos\phi+\sum_{n=1}^{\infty}d_n\rho^{-n}\cos n\phi,\tag{1.4}$$

$\rho=a,b$ 时边界条件

$$\begin{cases}-\varepsilon\left.\dfrac{\partial\Phi_2}{\partial\rho}\right|_{\rho=a}=-\varepsilon_0\left.\dfrac{\partial\Phi_1}{\partial\rho}\right|_{\rho=a}\\ -\dfrac{1}{a}\left.\dfrac{\partial\Phi_2}{\partial\phi}\right|_{\rho=a}=-\dfrac{1}{a}\left.\dfrac{\partial\Phi_1}{\partial\phi}\right|_{\rho=a}\end{cases}\tag{1.5}$$

$$\begin{cases}-\varepsilon\left.\dfrac{\partial\Phi_2}{\partial\rho}\right|_{\rho=b}=-\varepsilon_0\left.\dfrac{\partial\Phi_3}{\partial\rho}\right|_{\rho=b}\\ -\dfrac{1}{b}\left.\dfrac{\partial\Phi_2}{\partial\phi}\right|_{\rho=b}=-\dfrac{1}{b}\left.\dfrac{\partial\Phi_3}{\partial\phi}\right|_{\rho=b}\end{cases}\tag{1.6}$$

即

$$\begin{cases}\dfrac{c_0}{a}+\displaystyle\sum_{n=1}^{\infty}n(b_na^{n-1}-c_na^{-n-1})\cos n\phi=\dfrac{\varepsilon_0}{\varepsilon}\sum_{n=1}^{\infty}na_na^{n-1}\cos n\phi\\ -\displaystyle\sum_{n=1}^{\infty}n(b_na^n+c_na^{-n})\sin n\phi=-\sum_{n=1}^{\infty}na_na^n\sin n\phi\\ \dfrac{c_0}{b}+\displaystyle\sum_{n=1}^{\infty}n(b_nb^{n-1}-c_nb^{-n-1})\cos n\phi=\dfrac{\varepsilon_0}{\varepsilon}\left(-E_0\cos\phi-\sum_{n=1}^{\infty}nd_nb^{-n-1}\cos n\phi\right)\\ -\displaystyle\sum_{n=1}^{\infty}n(b_nb^n+c_nb^{-n})\sin n\phi=E_0b\sin\phi-\sum_{n=1}^{\infty}nd_nb^{-n}\sin n\phi\end{cases}\tag{1.7}$$

可得 $c_0 = 0$，以及

$$\begin{bmatrix} a^{2n} & -\varepsilon_\mathrm{r} a^{2n} & \varepsilon_\mathrm{r} & \\ a^{2n} & -a^{2n} & -1 & \\ & \varepsilon_\mathrm{r} b^{2n} & -\varepsilon_\mathrm{r} & 1 \\ & b^{2n} & 1 & -1 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \\ c_n \\ d_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -E_0 b^2\, \delta_{1n} \\ -E_0 b^2\, \delta_{1n} \end{bmatrix} \tag{1.8}$$

因此，$n > 1$ 时，$a_n, b_n, c_n, d_n \equiv 0$；$n = 1$ 时，

$$\begin{cases} a_1 = -\dfrac{4\varepsilon_\mathrm{r} E_0 b^2}{(\varepsilon_\mathrm{r}+1)^2 b^2 - (\varepsilon_\mathrm{r}-1)^2 a^2} \\ b_1 = -\dfrac{2(\varepsilon_\mathrm{r}+1) E_0 b^2}{(\varepsilon_\mathrm{r}+1)^2 b^2 - (\varepsilon_\mathrm{r}-1)^2 a^2} \\ c_1 = -\dfrac{(\varepsilon_\mathrm{r}-1) E_0 a^2 b^2}{(\varepsilon_\mathrm{r}+1)^2 b^2 - (\varepsilon_\mathrm{r}-1)^2 a^2} \\ d_1 = \dfrac{(\varepsilon_\mathrm{r}^2-1) E_0 (b^2-a^2) b^2}{(\varepsilon_\mathrm{r}+1)^2 b^2 - (\varepsilon_\mathrm{r}-1)^2 a^2} \end{cases} \tag{1.9}$$

从而由边界处连续性知 $a_0 = b_0 = 0$，

$$\Phi = \begin{cases} -\dfrac{4\varepsilon_\mathrm{r} E_0 b^2 \rho\cos\phi}{(\varepsilon_\mathrm{r}+1)^2 b^2 - (\varepsilon_\mathrm{r}-1)^2 a^2}, & \rho < a \\ -\dfrac{E_0 b^2 \left[2(\varepsilon_\mathrm{r}+1)\rho + (\varepsilon_\mathrm{r}-1)a^2/\rho\right]\cos\phi}{(\varepsilon_\mathrm{r}+1)^2 b^2 - (\varepsilon_\mathrm{r}-1)^2 a^2}, & a < \rho < b \\ E_0\left[-\rho + \dfrac{(\varepsilon_\mathrm{r}^2-1)(b^2-a^2)}{(\varepsilon_\mathrm{r}+1)^2 b^2 - (\varepsilon_\mathrm{r}-1)^2 a^2}\dfrac{b^2}{\rho}\right]\cos\phi, & \rho > b \end{cases} \tag{1.10}$$

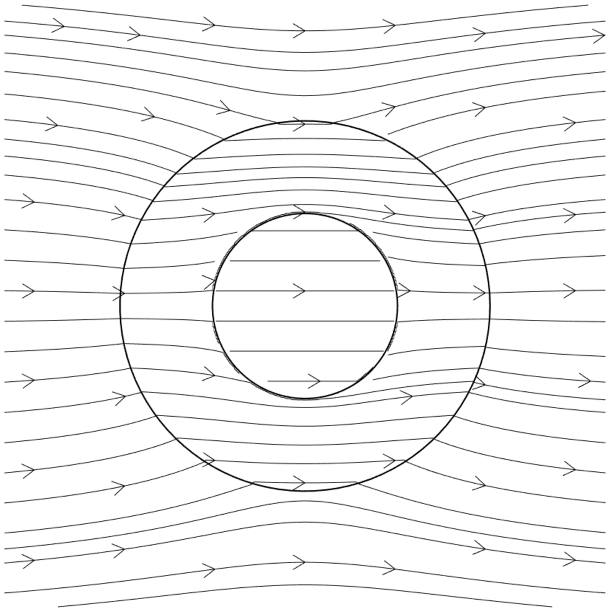
特别的，在空心圆柱内部 $(\rho < a)$ 是匀强电场，

$$E_1 = \frac{4\varepsilon_\mathrm{r} b^2}{(\varepsilon_\mathrm{r}+1)^2 b^2 - (\varepsilon_\mathrm{r}-1)^2 a^2} E_0 \tag{1.11}$$

(b) 特别的，当 $b = 2a$ 时，化简为

$$\Phi = \begin{cases} -\dfrac{16\varepsilon_\mathrm{r} E_0 \rho\cos\phi}{3\varepsilon_\mathrm{r}^2 + 10\varepsilon_\mathrm{r} + 3}, & \rho < a \\ -\dfrac{4E_0 \left[2(\varepsilon_\mathrm{r}+1)\rho + (\varepsilon_\mathrm{r}-1)a^2/\rho\right]\cos\phi}{3\varepsilon_\mathrm{r}^2 + 10\varepsilon_\mathrm{r} + 3}, & a < \rho < 2a \\ E_0\left[-\rho + \dfrac{12(\varepsilon_\mathrm{r}^2-1)}{3\varepsilon_\mathrm{r}^2 + 10\varepsilon_\mathrm{r} + 3}\dfrac{a^2}{\rho}\right]\cos\phi, & \rho > 2a \end{cases} \tag{1.12}$$

由 MATLAB 画出电力线如图



(c) $a \to 0$ 对应实心圆柱，

$$\Phi = \begin{cases} -\dfrac{2E_0}{\varepsilon_{\mathrm{r}}+1}\rho\cos\phi, & \rho < b \\ E_0\left(-\rho + \dfrac{\varepsilon_{\mathrm{r}}-1}{\varepsilon_{\mathrm{r}}+1}\dfrac{b^2}{\rho}\right)\cos\phi, & \rho > b \end{cases} \tag{1.13}$$

$b \to \infty$ 对应圆柱腔

$$\Phi = \begin{cases} -\dfrac{4\varepsilon_{\mathrm{r}}E_0}{(\varepsilon_{\mathrm{r}}+1)^2}\rho\cos\phi, & \rho < a \\ -\dfrac{E_0\left[2(\varepsilon_{\mathrm{r}}+1)\rho + (\varepsilon_{\mathrm{r}}-1)a^2/\rho\right]\cos\phi}{(\varepsilon_{\mathrm{r}}+1)^2}, & \rho > a \end{cases} \tag{1.14}$$

4.10

(a) 分析知，假设电场沿径向分量可满足边界条件，由唯一性定理，真空电场 $\boldsymbol{E}_1$ 与介质电场 $\boldsymbol{E}_2$ 均沿径向，故由 Gauss 定律，

$$(D_1 + D_2)\cdot 2\pi r^2 = Q, \quad D_1 = \varepsilon_0 E_1,\ D_2 = \varepsilon E_2, \tag{2.15}$$

又边界处切向电场连续，故

$$E_1 = E_2, \tag{2.16}$$

因此 $\boldsymbol{E}_1 = \boldsymbol{E}_2 =: \boldsymbol{E}$

$$\boldsymbol{E} = \frac{Q}{2\pi(\varepsilon+\varepsilon_0)}\frac{\boldsymbol{r}}{r^3}. \tag{2.17}$$

(b) 内球内部无电场，$\boldsymbol{E} = \boldsymbol{P} = \boldsymbol{D} = \boldsymbol{0}$，由

$$\boldsymbol{D}\cdot\hat{\boldsymbol{n}} = \sigma, \tag{2.18}$$

知，真空-真空内边界上电荷密度

$$\sigma_1 = D_1 = \varepsilon_0 E_1 = \frac{\varepsilon_0}{\varepsilon+\varepsilon_0}\frac{Q}{2\pi r^2}; \tag{2.19}$$

电介质-真空内边界上电荷密度

$$\sigma_2 = D_2 = \varepsilon E_2 = \frac{\varepsilon}{\varepsilon+\varepsilon_0}\frac{Q}{2\pi r^2}. \tag{2.20}$$

(c) 由

$$\boldsymbol{P}\cdot\hat{\boldsymbol{n}} = -\sigma_{\text{pol}}, \tag{2.21}$$

真空-真空内边界上，$\boldsymbol{P}_1 = \boldsymbol{0}$，没有极化电荷；

电介质-真空内边界上极化电荷密度

$$\sigma_{\text{pol}} = -P_2 = -(D_2 - \varepsilon_0 E_2) = -\frac{\varepsilon-\varepsilon_0}{\varepsilon+\varepsilon_0}\frac{Q}{2\pi r^2}. \tag{2.22}$$