证明递推式

$$P'_{\ell+1} - P'_{\ell-1} - (2\ell+1)P_\ell = 0,$$

证明： 由 Rodrigues 公式

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}(x^2-1)^\ell. \tag{0.1}$$

可得

$$\begin{aligned} P'_\ell &= \frac{1}{2^\ell \ell!}\frac{\mathrm{d}^{\ell+1}}{\mathrm{d}x^{\ell+1}}(x^2-1)^\ell = \frac{1}{2^{\ell-1}(\ell-1)!}\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\left[x(x^2-1)^{\ell-1}\right] \\ &= \frac{1}{2^{\ell-1}(\ell-1)!}\frac{\mathrm{d}^{\ell-1}}{\mathrm{d}x^{\ell-1}}\left[\big((2\ell-1)x^2-1\big)(x^2-1)^{\ell-2}\right]. \end{aligned}$$

从而

$$\begin{aligned} P'_{\ell+1} - P'_{\ell-1} = \frac{1}{2^\ell \ell!}\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}&\left[\big((2\ell+1)x^2-1\big)(x^2-1)^{\ell-1}\right] - \\ &\frac{1}{2^{\ell-1}(\ell-1)!}\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}(x^2-1)^{\ell-1} \\ = \frac{1}{2^\ell \ell^2}\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}&\left[(2\ell+1)(x^2-1)^\ell\right] = (2\ell+1)P_\ell. \end{aligned}$$

□