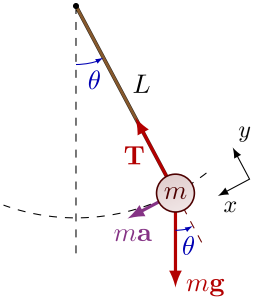
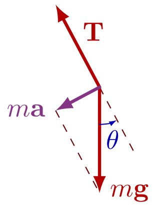
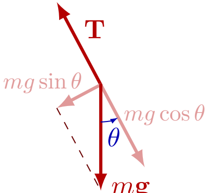


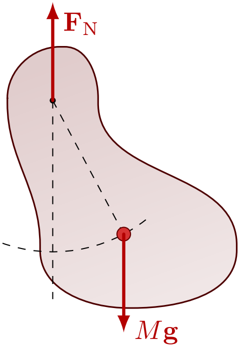
θ
L
y
$\mathbf{T}$
m
x
$m\mathbf{a}$
θ
$m\mathbf{g}$



T
ma
θ
mg



T
$mg\sin\theta$
$mg\cos\theta$
θ
$m\mathbf{g}$


$\mathbf{F}_{\mathrm{N}}$
$M\mathbf{g}$

The differential equation for a simple pendulum is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega_0^2 \sin\theta = 0.$$

For the initial conditions

$$\begin{cases} \theta(0) = \theta_0 \\ \left.\dfrac{\mathrm{d}\theta}{\mathrm{d}t}\right|_0 = 0 \end{cases}$$

with $0 < \theta_0 < \pi$, the bounded solution is given by

$$\theta(t) = 2\arcsin\left[k\,\mathrm{sn}\left(\frac{\omega_0 T}{4} - \omega_0 t, k^2\right)\right] \le \theta_0,$$

where

$$T = \frac{4K(k^2)}{\omega_0}, \quad k = \sin\frac{\theta_0}{2}$$

and where sn is the Jacobi elliptic functions, and K is complete elliptic integral of the first kind. The first derivative is

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{2k\omega_0}{\sqrt{1-(k\,\mathrm{sn}(\alpha,k^2))^2}}\,\mathrm{cn}\big(\alpha,k^2\big)\,\mathrm{dn}\big(\alpha,k^2\big)$$

with $\alpha = \omega_0 T/4 - \omega_0 t$. The maximum angular velocity is

$$\Omega_{\max} = \left.\frac{\mathrm{d}\theta}{\mathrm{d}t}\right|_{T/2} = 2k\omega_0.$$

Therefore, the total energy is

$$E = \frac{mg^2\Omega_{\max}^2}{2\omega_0^4} = 2\frac{mg^2}{\omega_0^2}\sin^2\frac{\theta_0}{2}.$$

with kinetic and potential energy

$$K = \frac{mg^2}{2\omega_0^4}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = 2\frac{mg^2}{\omega_0^2}\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right)$$

$$U = 2\frac{mg^2}{\omega_0^2}\sin^2\frac{\theta}{2}.$$

The differential equation for a simple pendulum is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega_0^2 \sin\theta = 0.$$

For the initial conditions

$$\begin{cases} \theta(0) = 0 \\ \left.\dfrac{\mathrm{d}\theta}{\mathrm{d}t}\right|_0 = \Omega_0 \end{cases}$$

with $\Omega_0 > 2\omega_0$, the unbounded solution is given by

$$\theta(t) = 2\arcsin\left[\operatorname{sn}\left(\frac{\Omega_0}{2}t, \frac{1}{k^2}\right)\right] \geq 0,$$

where

$$T = \frac{4K(k^2)}{\Omega_0}, \quad k = \frac{\Omega_0}{2\omega_0} > 2$$

and where sn is the Jacobi elliptic functions, and K is complete elliptic integral of the first kind. Note that the solution $\theta(t)$ is continuous and increase monotonically, so if one assumes $|\arcsin x| \leq \pi/2$ for each real x,

$$\theta(t) = 2\pi n + (-1)^n 2\arcsin\left[\operatorname{sn}\left(\frac{\Omega_0}{2}t, \frac{1}{k^2}\right)\right],$$

where $n = \lfloor (t + T/2)/T \rfloor = \lfloor t/T + 1/2 \rfloor$.
The first derivative is

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 2\Omega_0 \operatorname{dn}\left(\frac{\Omega_0}{2}t, \frac{1}{k^2}\right),$$

The maximum angular velocity $\Omega_{\max} = \Omega_0$, and the minimum is

$$\Omega_{\min} = 2\omega_0\sqrt{k^2 - 1}.$$