Intership - Sparse Coding

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Abstract

Little document to summarize sparse coding. Mainly based on the Hugo Larochelle courses.

This document is a draft

Contents

1 Sparse coding

1.1 Introduction

Idea Sparse dictionary learning is a representation learning method which aims at finding a sparse representation of the input data (in form of a linear combination of basic elements (called Atoms). The idea of using learned dictionary instead of a predefined one is based on wavelets. Thie sparse learned models has recently led to state-of-the-art result for denoising, classification,...

Unsupervised learning Only use the inputs $x^{(t)}$ ($X = [x_1, ..., x_n]$ in $\mathbb{R}^{m \times n}$) for learning. Automatically extract meaningful features of our data, leverage the availability of unlabeled data and add a data-dependent regularize to trainings.

Sparse coding is one of the neural networks used for unsupervised learning (like restricted boltzmann machines and autoencoders).
The idea behind sparse coding is: For each x^t find a latent representation h^t such that:

- It is sparese: the vector h^t has many zeros (only few nonzero elements)
- We can reconstruct the original input $x^{(t)}$ as well as possible.

That mean, more formally:

$$\min_D \frac{1}{T} \sum_{t=1}^{T} \min_{h^{(t)}} \frac{1}{2} \|x^{(t)} - D\ h^{(t)}\|_2^2 + \lambda \|h^{(t)}\|_1$$

- D is a matrix of weights, usually refer to that matrix as a dictionary matrix (containt atoms) with $D \in \mathbb{R}^{m \times k}$ (k the number of atoms)
- $\|x^{(t)} - D\ h^{(t)}\|_2^2$ is the reconstruction error

- $D\ h^{(t)}$ is the reconstruction of $\hat{x}^{(t)}$
- $\|h^{(t)}\|_1$ is the sparsity penalty (more 0 in h we have, better it is)

This two objectives fight each other. But it still a optimization problem (cf min), and we'll try to optimize it for each training example $x^{(t)}$. This is why we have a sum over all the training examples.

We also constrain the columns of D to be of norm 1 (otherwise, D could grow big while $h^{(t)}$ becomes small to satisfy the prior). And sometimes the columns are constrained to be no greather than 1.

However, $h^{(t)}$ is now a complicated function of $x^{(t)}$:
Encoder is the minimization $h(x^{(t)}) = arg \min_{h^{(t)}} = \frac{1}{2}\|x^{(t)} - D\ h^{(t)}\|_2^2 + \lambda\|h^{(t)}\|_1$, so the optimization problem is more complicated than a simple non linear problem.
The idea to solve this minimization problem is [Lee et al., 2007] :

```
while D not_converged :
    Fix D
    Minimize h          (1)
    Fix h
    Minimize D          (2)
```

Dictionary We can also write $\hat{x}^{(t)} = D\ h(x^{(t)}) \sum_{\substack{k s.t. \\ h(x^{(t)})_k \neq 0}} D_{\cdot,k}\, h(x^{(t)})_k$

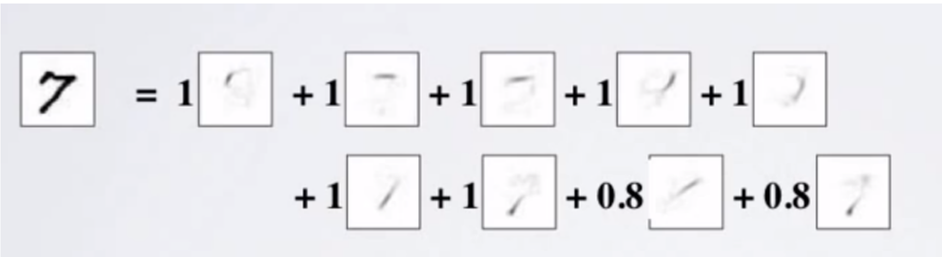


Figure 1: Example of reconstruction using sparse coding

The images refer to $D_{\cdot,k}$ (columns of D wich are not equals to 0) and the factor (1 or 0.8 in this case) refer to $h(x^{(t)})_k$
We also refer to D as the dictionary:

- in certain applications, we know what dictionary matrix to use
- often however, we have to learn it

In general we have $k << n$. But we can use an overcomplete dictionary with $k > m$.

1.2 Inderence of Sparse code

1.2.1 Compute h

Idea Here we develop the (1) computation from the initial idea.
Assume we are given a dictionary matrix D, how do we compute $h(x^{(t)})$. We have to optimize:

$$l(x^{(t)}) = \frac{1}{2}\|x^t - Dh^{(t)}\|_2^2 + \lambda\|h^{(t)}\|_1 w.r.t. h^{(t)}$$

We could use a gradient descent method:

$$\Delta_{h^{(t)}} l(x^{(t)}) = D^T(Dh^{(t)} - x^{(t)}) + \lambda sign(h^{(t)})$$

The issue is L1 norm is not differentiable at 0. The solution is : if $h^(t)$ changes sign because of L1 norm gradient then clamp to 0.That mean :

$h_k^{(t)} = h_k^{(t)} - \alpha(D_{.,k})^T(Dh^{(t)} - x^{(t)})$
if $\text{sign}(h_k^{(t)}) \neq \text{sign}(h_k^{(t)} - \alpha\lambda\ \text{sign}(h_k^{(t)}))$ then: $h_k^{(t)} = 0$
else $h_k^{(t)} = h_k^{(t)} - \alpha\lambda\ \text{sign}(h_k^{(t)})$

ISTA (iterative Shrinkage and Thresholding Algorithm :

```
initialize h
while h not_converged:
    for each h_k in h:
        h_k = h_k -alpha * transpose(D[:,k]) * (D*h - x)
        h_k = shrink(h_k, alpha*lambda_coef)
return h
```

Here **shrink(a,b)** = [..., sign(a_i) max($|a_i| - b_i$, 0), ...]

1.2.2 Compute D

There are three algorithms used for dictionary update.

Algorithm 1: A gradient descent method Our original problem is:

$$\min_D \frac{1}{T}\sum_{t=1}^T \min_{h^{(t)}} \frac{1}{2}\|x^{(t)} - D\ h^{(t)}\|_2^2 + \lambda\|h^{(t)}\|_1$$

But here we assume $h(x^{(t)})$ doesn't depend on D. So we must minimize:

$$\min_D \frac{1}{T}\sum_{t=1}^T \min_{h^{(t)}} \frac{1}{2}\|x^{(t)} - D\ h^{(t)}\|_2^2$$

```
while D not_converged:
    # Perform gradient update of D
    D = D - alpha * (1/T)* sum((x - D h)* tranpose(h))
    # Renormalize the columns of D
    for each column D[:,j]:
        D[:,j] = (D[:,j] / norm(D[:,j]))

return D
```

Algorithm 2: Block-coordinate descent We must minimize:

$$\min_D \frac{1}{T}\sum_{t=1}^T \min_{h^{(t)}} \frac{1}{2}\|x^{(t)} - D\ h^{(t)}\|_2^2$$

The idea is to solve for each column $D_{.,j}$ in cycle (that mean to optimize in one direction at time). For that we must set the gradient for $D_{.,j}$ to zero.
We have:

$$0 = \frac{1}{T}\sum_{t=1}^T (x^{(t)} - Dh(x^{(t)}))\ h_j^{(t)}$$

We separe $D_{.,j}$ from the rest of D:

$$0 = \frac{1}{T}\sum_{t=1}^T (x^{(t)} - (\sum_{i\neq j} D_{.,i}\ h(x^{(t)})_i\ ) - (D_{.,j}\ h(x^{(t)})_j)\ )\ h_j^{(t)}$$

Our aim is to find the value of $D_{.,j}$, we must isolate $D_{.,j}$:

$$0 = \frac{1}{T}\sum_{t=1}^T (x^{(t)}h_j^{(t)} - (\sum_{i\neq j} D_{.,i}\ h_i^{(t)}\ h_j^t) - (D_{.,j}\ h_j^{(t)2}))$$

$$0 = (\sum_{t=1}^T (x^{(t)}h_j^{(t)} - (\sum_{i\neq j} D_{.,i}\ h_i^{(t)}\ h_j^t)\ ) - (\sum_{t=1}^T (D_{.,j}\ h_j^{(t)2})))$$

$$\sum_{t=1}^T (D_{.,j}\ h_j^{(t)2}) = \sum_{t=1}^T (x^{(t)}h_j^{(t)} - (\sum_{i\neq j} D_{.,i}\ h_i^{(t)}\ h_j^t)\ )$$

$$D_{.,j}\sum_{t=1}^{T} h_j^{(t)2} = \sum_{t=1}^{T}(x^{(t)}h_j^{(t)} - (\sum_{i\neq j} D_{.,i}\ h_i^{(t)}\ h_j^{t})\ )$$

$$D_{.,j} = \frac{1}{\sum_{t=1}^{T} h_j^{(t)2}}\sum_{t=1}^{T}(x^{(t)}h_j^{(t)} - (\sum_{i\neq j} D_{.,i}\ h_i^{(t)}\ h_j^{t})\ )$$

$$D_{.,j} = \underbrace{\frac{1}{\sum_{t=1}^{T} h_j^{(t)2}}}_{A_{j,j}}\underbrace{\sum_{t=1}^{T}(x^{(t)}h_j^{(t)})}_{B_{.,j}} - \sum_{i\neq j} D_{.,i}(\underbrace{\sum_{t=1}^{T} h_i^{(t)}h_j^{t})}_{A_{i,j}}$$

$$D_{.,j} = \frac{1}{A_{j,j}}(B_{.,j} - DA_{.,j} + D_{.,j}A_{j,j})$$

```
while D not_converged:
    # For each column D[:,j] perform updates
    for each column D[:,j]:
        D[:,j] = (1/A[j, j])*(B[:, j] - D A[:, j] + D[:, j] A[j, j])
        # Normalization
        D[:,j] = D[:,j]/norm(D[:,j])

return D
```

Algorithm 3: Online learning algorithm For large datasets we want to update D after visiting each $x^{(t)}$. The solution is for each $x^{(t)}$ [Mairal et al., 2009] :

- Perform inference of $h(x^{(t)})$ after visiting each $x^{(t)}$
- Update running averages of the quantities required to update D:
  - B = $\beta B + (1-\beta)x^{(t)}h(x^{(t)})^T$
  - A = $\beta A + (1-\beta)h(x^{(t)})h(x^{(t)})^T$
- Use current value of D as " warm start" to block-coordinate descent (warm start $\iff$ With the previous value of D)

(We have to specifie β like a learning rate α in the gradient descent)

```
Initialize D # Not to 0 ! (To respect the constraint we define before)
while D not_converged:
    for each x:
        Infer code h
        #Update dictionary
        A = A +  h * transpose(h)
        B = B + x * transpose(h)
        #Batch upgrade
        #A = beta * A + ( 1 - beta ) * h * transpose(h)
        #B = beta * B + ( 1 - beta ) * x * transpose(h)
        while D not_converged:
            for each column D[:,j]:
                D[:,j] = (1/A[j,j])*(B[:,j] - D A[:,j] + D[:,j] A[j,j])
                # Normalization
                D[:,j] = D[:,j]/norm(D[:,j])
```

Optimizing the Algorithm In practice, it's possible to improve the convergence speed of this algorithm by using a Mini-batch extension: By drawing $\eta > 1$ signals at each iteration instead of a single one.

$$\begin{cases} A_t = \beta A_{t-1} + \sum_{i=1}^{\eta}\alpha_{t,i}\alpha_{t,i}^T \\ B_t = \beta B_{t-1} + \sum_{i=1}^{\eta} x\alpha_{t,i}^T \end{cases}$$

Then $\beta = \frac{\theta+1-\eta}{\theta+1}$, where $\theta = t\eta$ if $t < \eta$ and $\eta^2 + t - \eta$ if $t \geq \eta$

1.3 Other algorithms

There are other algorithms like Efficient Shift-Invariant Dictionary learning which refers to the problem of discovering a set of latent basis vectors that capture informative *local patterns* at different locations of the input sequences and not in all input sequences [Zheng et al., 2016]. But in our problem we used dictionary learning on windows of 10ms, with approach we already capture informative local patterns, we don't need to used Shift-Invariant dictionary learning.

1.4 Application for MNIST dataset

The MNIST database of handwritten digits, available from Yann Lecun's website. MNIST has a training set of 60,000 examples, and a test set of 10,000 examples. It is a subset of a larger set available from NIST. The digits have been size-normalized and centered in a fixed-size image. In my test I'll use 55000 examples from the training set (using Tensorflow datasets). These are 28×28 images.

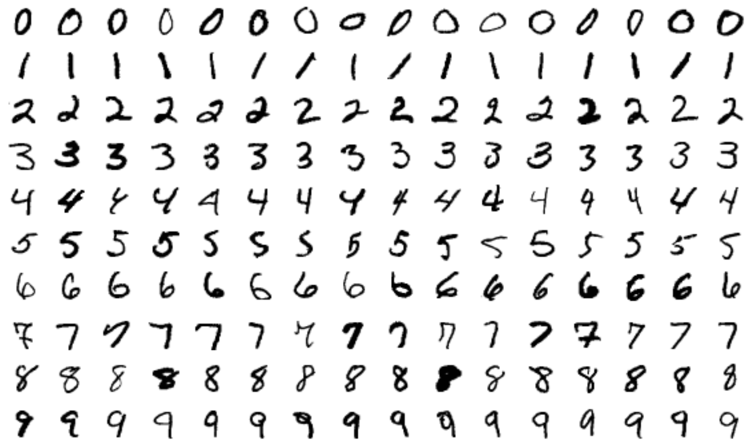


Figure 2: Example of MNIST's handwitten digits

1.4.1 Prototype

My first task is to realise a Sparse Coding prototype to compute Sparse Coding on this dataset, using Python. The aim here, is to understand the underlying principles of this method, you can found this prototype in `Code directory` of this repository as `SparseCoding.py`.
These are some results of this prototype: For time saving I used only 100 digits as input.

1.4.2 SPAMS

SPAMS (SPArse Modeling Software) is an optimization toolbox for solving various sparse estimation problems.

- Dictionary learning and matrix factorization (NMF, sparse PCA, ...)
- Solving sparse decomposition problems with LARS, coordinate descent, OMP, SOMP, proximal methods
- Solving structured sparse decomposition problems (l1/l2, l1/linf, sparse group lasso, tree-structured regularization, structured sparsity with overlapping groups,...).

It is developed and maintained by Julien Mairal (Inria), and contains sparse estimation methods resulting from collaborations with various people: notably, Francis Bach, Jean Ponce, Guillermo Sapiro, Rodolphe Jenatton and Guillaume Obozinski.
There are some results of my first SPAMS tests:

Test 1 In the first test I used 256 atoms, 2 000 iterations and $\lambda = 0.015$ to learn the dictionary and the sparse coefficients.

Results Some results:

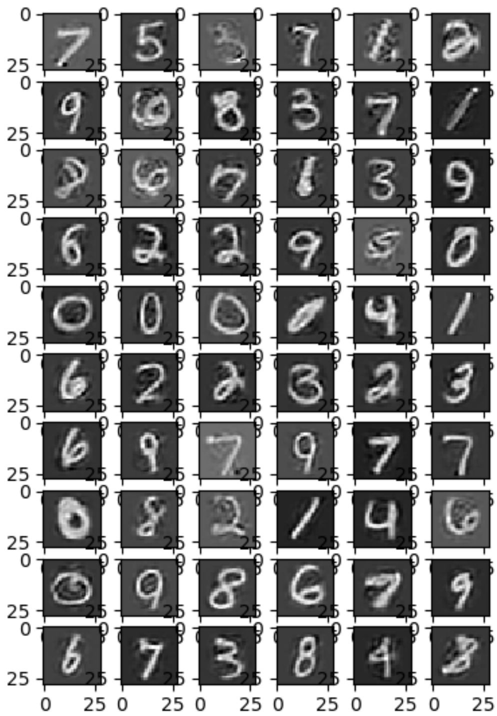
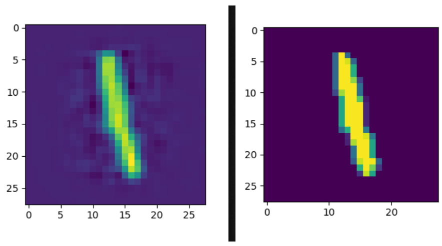
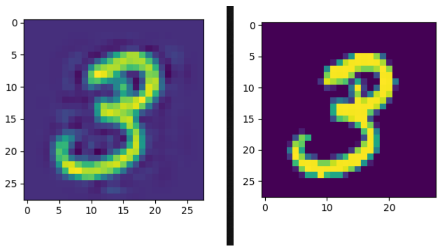


Figure 3: Few atoms of D



(a) Reconstructed 1 vs Original 1



(b) Reconstructed 3 vs Original 3

Interpretation blabla

Test 2 In the second test I used 1024 atoms, 1 000 iterations and $\lambda = \frac{1.2}{\sqrt{m}}$ [Mairal et al., 2009] *(In my case* ≈ 0.0042857*)*.

Result

Interpretation

Test 3 In the third test I used 1024 atoms, 1 000 iterations and $\lambda = 5$.

Result

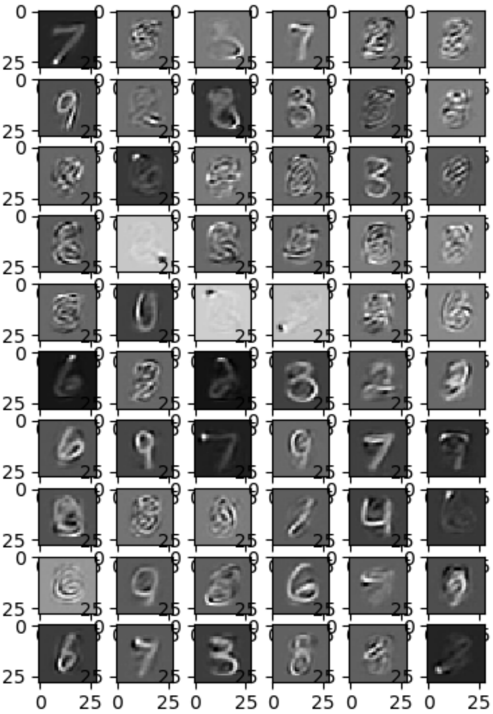
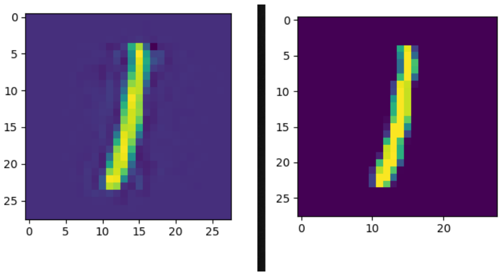
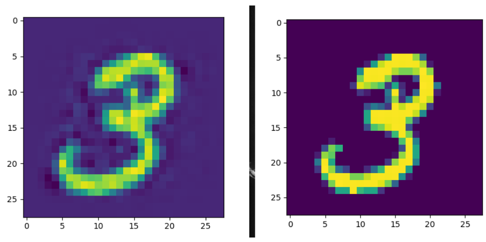


Figure 5: Few atoms of D



(a) Reconstructed 1 vs Original 1



(b) Reconstructed 3 vs Original 3

Interpretation

Test 4 In the fourth test I used 2048 atoms, 1 000 iterations and $\lambda = \frac{1.2}{\sqrt{m}}$

Results

Interpretation

1.5 Application for Lenna

2 Sparse Coding for speech recognition

In this part of the paper we will see novel feature exraction technique based on the principles of sparse coding [S. V. S. Sivaram et al., 2010]. Sparse codigin deals with the problem of how represent a given audio input as a linear combination of a minimum number of basis function. The weights of the linear

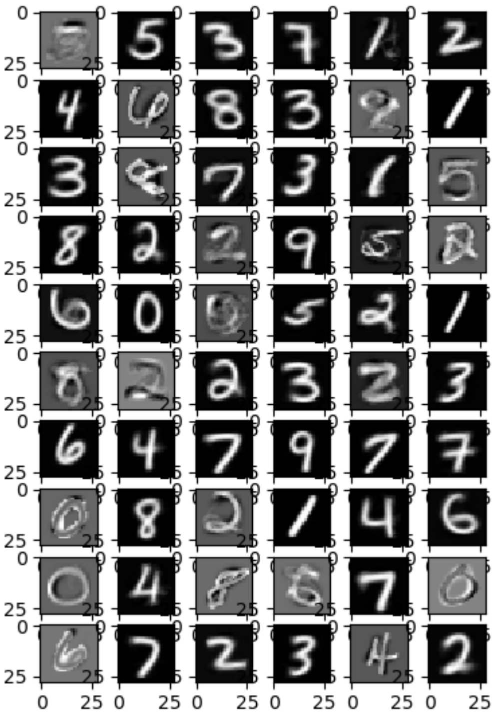
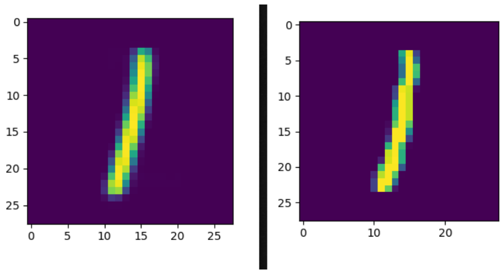
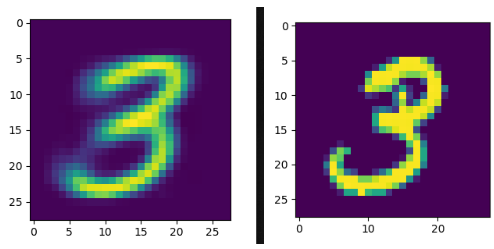


Figure 7: Few atoms of D



(a) Reconstructed 1 vs Original 1



(b) Reconstructed 3 vs Original 3

combination are used as feature for speech recognition (acoustic modeling). Note the input dimensionality is typically **much** less than the number of atoms in the dictionary *i.e.* we use overcomplete dictionary. We use Sparse Coding algorithm as describe before and we get the dictionary D and the matrix of sparse coefficients h.

Reflection path In [S. V. S. Sivaram et al., 2010] they used spectro-temporal speech domain wich is obtained by performing a short time Fourier transform (STFT) with an analysis window of length 25 ms and a frameshit of 10 ms on the input signal. Log critical band energies are subsequently obtained by projecting the magnitude square values of the STFT output on a set of frequency weights, which are equally spaced on the Bark frequence scale, and then applying a logarithm on the output projections.

References

[Lee et al., 2007] Lee, H., Battle, A., Raina, R., and Ng, A. Y. (2007). Efficient sparse coding algorithms. In Schölkopf, B., Platt, J. C., and Hoffman, T., editors, *Advances in Neural Information Processing*

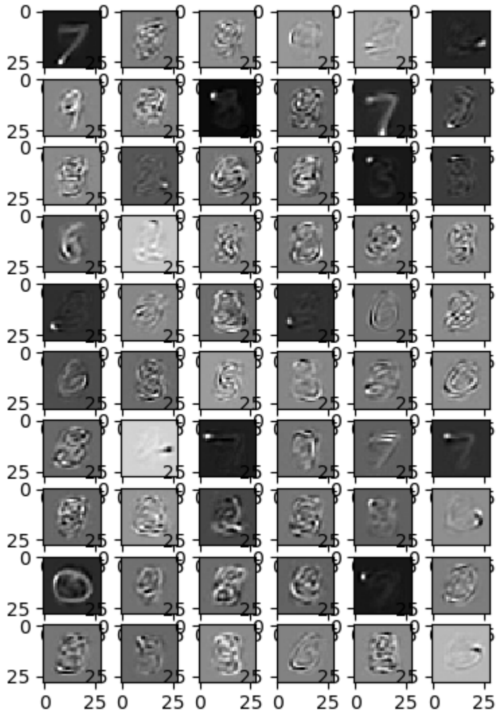
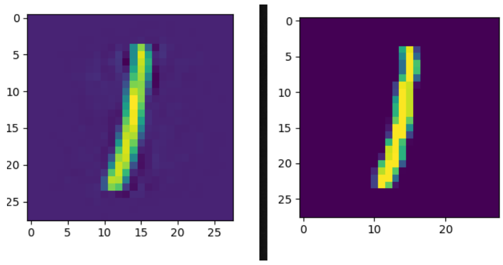
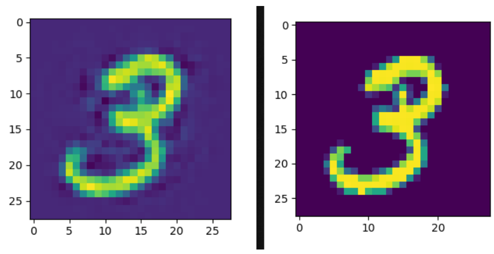


Figure 9: Few atoms of D



(a) Reconstructed 1 vs Original 1



(b) Reconstructed 3 vs Original 3

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[Mairal et al., 2009] Mairal, J., Bach, F., Ponce, J., and Sapiro, G. (2009). Online dictionary learning for sparse coding. In *Proceedings of the 26th Annual International Conference on Machine Learning*, ICML '09, pages 689–696, New York, NY, USA. ACM.

[S. V. S. Sivaram et al., 2010] S. V. S. Sivaram, G., Krishna Nemala, S., Elhilali, M., D. Tran, T., and Hermansky, H. (2010). Sparse coding for speech recognition.

[Zheng et al., 2016] Zheng, G., Yang, Y., and Carbonell, J. (2016). Efficient shift-invariant dictionary learning. In *Proceedings of the 22Nd ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, KDD '16, pages 2095–2104, New York, NY, USA. ACM.