Range, null space, and rank of a matrix

Recall some definitions. The **range** (or **column space**) of a matrix $A \in \mathbb{R}^{m \times n}$ is the set of all possible linear combinations of its column vectors.

$$\text{Range}(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$$

In other words, it is the span of the columns of A. The **null space** of a matrix $A \in \mathbb{R}^{m \times n}$ is the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{0}$.

$$\text{Null}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$$

The **rank** of a matrix $A \in \mathbb{R}^{m \times n}$ is the dimension of its range (or column space). It represents the number of linearly independent columns in A.

$$\text{rank}(A) = \dim(\text{Range}(A))$$

We are going to prove later in the class that, the rank of A is also the dimension of the orthogonal complement of the null space of A^T:

$$\text{rank}(A) = \dim(\text{Null}(A^T)^\perp)$$

Problem 1: Range of a Matrix

Let $A \in \mathbb{R}^{3 \times 2}$ be defined as:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

1. Find the range of the matrix A.
2. Express the range in terms of a span of vectors.

SOLUTION: To find the range of the matrix A, we consider its columns. The matrix A is:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

The columns of A are:

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

The range of A, denoted as Range(A), is the set of all linear combinations of its columns:

$$\text{Range}(A) = \left\{\mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y} = A\mathbf{x},\ \mathbf{x} \in \mathbb{R}^2\right\} = \{\mathbf{y} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 \mid x_1, x_2 \in \mathbb{R}\}$$

To express the range as a span, we need to find a basis for the column space of A. We can check if the columns $\mathbf{a}_1$ and $\mathbf{a}_2$ are **linearly independent**. In this case it is easy to check by inspection that the two columns are linearly independent but in a more general case we should proceed as follow.

Suppose there exist scalars c_1, c_2, not both zero, such that:

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 = \mathbf{0}$$

This gives:

$$c_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which leads to the system of equations:

$$\begin{aligned} c_1 + 2c_2 = 0 \quad (1) \\ 3c_1 + 4c_2 = 0 \quad (2) \\ 5c_1 + 6c_2 = 0 \quad (3) \end{aligned}$$

From equation (1): $c_1 = -2c_2$. Substituting into equation (2):

$$3(-2c_2) + 4c_2 = -6c_2 + 4c_2 = -2c_2 = 0$$

Therefore:

$$-2c_2 = 0 \implies c_2 = 0$$

Then:

$$c_1 = -2c_2 = 0$$

Since $c_1 = c_2 = 0$, the only solution is the trivial one. Thus, the columns $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly independent.

Therefore, the range of A is the span of $\mathbf{a}_1$ and $\mathbf{a}_2$:

$$\text{Range}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}$$

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Problem 2: Null Space of a Matrix

Let $B \in \mathbb{R}^{3\times 3}$ be defined as:

$$B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \end{bmatrix}$$

1. Find the null space of the matrix B.
2. Find a basis for the null space and determine its dimension.

SOLUTION: To find the null space of the matrix B, we need to solve the system $B\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. The system $B\mathbf{x} = \mathbf{0}$ is:

$$\begin{cases} 1x_1 - 1x_3 = 0 & (1) \\ 1x_2 - 1x_3 = 0 & (2) \\ 1x_1 + 1x_2 - 2x_3 = 0 & (3) \end{cases}$$

From the system $x_1 = x_3$ and also $x_2 = x_3$.

Now, substitute $x_1 = x_3$ and $x_2 = x_3$ into equation (3):

$$x_1 + x_2 - 2x_3 = 0$$

Substitute the expressions for x_1 and x_2:

$$x_3 + x_3 - 2x_3 = 0$$

This confirms that the equations are consistent. Therefore, the solution to the system is:

$$x_1 = x_2 = x_3$$

Let $x_3 = t$, where t is a real parameter. Then the general solution is:

$$\mathbf{x} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Answers:

(a) The null space of B is:

$$\text{Null}(B) = \left\{ t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}$$

This means that any vector $\mathbf{x}$ in the null space is a scalar multiple of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(b) A basis for the null space is:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

The dimension of the null space is the number of vectors in the basis, which is:

$$\dim(\text{Null}(B)) = 1$$

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Problem 3: Rank of a Matrix

Let $C \in \mathbb{R}^{4\times 3}$ be defined as:

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \\ 4 & 8 & 12 \end{bmatrix}$$

Questions:

- Find the rank of the matrix C.
- Verify the rank using different methods (e.g., row reduction, rank-nullity theorem).

SOLUTION: To find the rank of the matrix C, we will use two methods:

(a) Row Reduction

(b) Analyzing Linear Dependence of Columns and Rows

Method 1 - Row Reduction We can notice that the rows are linearly dependent, in particular the 2nd, 3rd and 4th row are multiple of the first one. In general we should perform elementary row operations to reduce C to its simplest form and count the number of non-zero rows or

$$\text{Step 1:} \quad \text{Subtract } 2 \times \text{Row 1 from Row 2}$$
$$\text{Row 2} \leftarrow \text{Row 2} - 2 \times \text{Row 1}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2-2(1) & 4-2(2) & 6-2(3) \\ 3 & 6 & 9 \\ 4 & 8 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 3 & 6 & 9 \\ 4 & 8 & 12 \end{bmatrix}$$

Step 2: Subtract $3 \times$ Row 1 from Row 3

$$\text{Row } 3 \leftarrow \text{Row } 3 - 3 \times \text{Row } 1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 3-3(1) & 6-3(2) & 9-3(3) \\ 4 & 8 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 8 & 12 \end{bmatrix}$$

Step 3: Subtract $4 \times$ Row 1 from Row 4

$$\text{Row } 4 \leftarrow \text{Row } 4 - 4 \times \text{Row } 1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4-4(1) & 8-4(2) & 12-4(3) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, the matrix is in a *simple* form with only one non-zero row.

$$\text{rank}(C) = \text{Number of non-zero rows} = 1$$

Method 2 - Analyzing Linear Dependence of Columns and Rows Consider the columns of C:

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \\ 12 \end{bmatrix}$$

Observe that: $\mathbf{c}_2 = 2\mathbf{c}_1$ and $\mathbf{c}_3 = 3\mathbf{c}_1$.

This means all columns are scalar multiples of $\mathbf{c}_1$ and are therefore linearly dependent.

Consider the rows of C and observe that: $\mathbf{r}_2 = 2\mathbf{r}_1$, $\mathbf{r}_3 = 3\mathbf{r}_1$, $\mathbf{r}_4 = 4\mathbf{r}_1$

This means all rows are scalar multiples of $\mathbf{r}_1$ and are therefore linearly dependent.

Conclusion:

Since there is only one linearly independent column and one linearly independent row:

$$\text{rank}(C) = 1$$

Verification Using the Rank-Nullity Theorem

The rank-nullity theorem states:

$$\text{rank}(C) + \text{nullity}(C) = n$$

Where n is the number of columns of C. Here, $n = 3$.

We have:

$$\text{rank}(C) = 1 \implies \text{nullity}(C) = n - \text{rank}(C) = 3 - 1 = 2$$

To confirm, we can find the dimension of the null space of C.

Finding the Null Space of C:

We need to solve $C\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

The system of equations is:

$$\begin{cases} 1x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 + 4x_2 + 6x_3 = 0 \\ 3x_1 + 6x_2 + 9x_3 = 0 \\ 4x_1 + 8x_2 + 12x_3 = 0 \end{cases}$$

Notice that all equations are multiples of the first equation. Therefore, we effectively have one unique equation:

$$x_1 + 2x_2 + 3x_3 = 0$$

This single equation in three variables implies that the solution space (null space) is two-dimensional.

Conclusion:

The dimension of the null space is:

$$\dim(\text{Null}(C)) = 2$$

Which confirms:

$$\text{rank}(C) + \dim(\text{Null}(C)) = 1 + 2 = 3 = n$$

Thus, using different methods, we have verified that:

$$\text{rank}(C) = 1$$

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Problem 4: Rank, Range, and Null Space Relations

Let $D \in \mathbb{R}^{4\times 4}$ be defined as:

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Questions:

- Determine the rank of the matrix D.
- Find the null space of D.
- Describe the relationship between the rank and the null space of D.

SOLUTION: The matrix D is in row echelon form. The non-zero rows are linearly independent, so:

$$\text{rank}(D) = 3$$

To find the null space, solve $D\mathbf{x} = \mathbf{0}$ where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives:

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0$$

Thus, the null space is spanned by:

$$\text{Null}(D) = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The rank-nullity theorem confirms that:

$$\text{rank}(D) + \text{nullity}(D) = n$$

where $n = 4$. Hence:

$$\text{nullity}(D) = 4 - \text{rank}(D) = 1$$

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MATLAB CVX - More examples and plots

Building up on the previous discussion here two other interesting problems and with visualizations! Recall that convex functions are the ones that are "bowl" shaped with positive curvature and convex optimization problems are problems whose objective function to minimize is convex, and whose constraints are defined in terms of inequalities and equalities of convex and affine (linear) functions.

Example 1. (Constrained least-squares) In this problem we are going to find the minimum of a bowl shape function and visualize the result in 3D. The function $f(\boldsymbol{x}) : \mathbb{R}^2 \to \mathbb{R}$ and it is defined to be $f(\boldsymbol{x}) = \|A\boldsymbol{x} - b\|_2$. We are looking for a solution in $[0, 1]$.

```
% Generate problem data
rng(1); % Set random seed for reproducibility
n = 2; % Number of variables (for 3D visualization)
m = 20; % Number of observations

% Create random matrices and vectors
A = randn(m, n);
b = randn(m, 1);

sign = 1; % Change with -1 and change minimize with maximize
% Define the CVX problem
cvx_begin
    % Define optimization variable
    variable x(n)

    % Define the least-squares objective
    minimize(sign*norm(A * x - b, 2))

    % Add constraints: 0 <= x <= 1
    subject to
        0 <= x <= 1
cvx_end

% Display the optimal value of x
disp('Optimal value of x:');
disp(x);

% Visualization
% Create a meshgrid for the domain
[x1, x2] = meshgrid(linspace(-1, 2, 30), linspace(-1, 2, 30));
x_grid = [x1(:), x2(:)];

% Compute the objective function values over the grid
objective_values = zeros(size(x_grid, 1), 1);
for i = 1:size(x_grid, 1)
    x_temp = x_grid(i, :)';
    objective_values(i) = sign * norm(A * x_temp - b, 2);
end

% Reshape for surface plot
objective_values = reshape(objective_values, size(x1));

% Plot the 3D surface
```

```
figure;
surf(x1, x2, objective_values, 'EdgeColor', 'none');
colorbar;
title('Objective Function: Least-Squares Error');
xlabel('x_1');
ylabel('x_2');
zlabel('Objective Value');
grid on;
hold on

% Plot the optimal value
hold on;
plot3(x(1), x(2), sign * norm(A * x - b, 2), 'ro', 'MarkerSize', 10, 'LineWidth', 2);

% Boundary points
z = [[0, 0]; [1,1]; [0,1]; [1,0]];

for i = 1:4
    plot3(z(i,1), z(i,2), sign * norm(A * z(i,:)' - b, 2), 'go', 'MarkerSize', 10, '
    LineWidth', 2);
end

legend('Objective Function', 'Optimal Value', 'Boundary points');
hold off;
```

NOTE: does the example still work if we put a minus in front of the norm and change minimize with maximize? If yes, why?

Example 2. (How would you solve this problem by hand?)

In this problem we are going to find again the minimum of a function. $f(\boldsymbol{x}) : \mathbb{R}^2 \to \mathbb{R}$ and it is defined to be $f(\boldsymbol{x}) = log(e^{x_1} + e^{x_2})$. We are looking again for a solution in $[0, 1]$.

```
% Define and solve the CVX problem
cvx_begin
    % Define optimization variables
    variable x(2)

    % Define the logarithmic objective
    minimize ((log(exp(x(1)) + exp(x(2)))))

    % Add constraints: 0 <= x <= 1
    subject to
        0 <= x(1) <= 1
        0 <= x(2) <= 1
cvx_end

% Display the optimal value of x
disp('Optimal value of x:');
disp(x);

% Create a meshgrid for visualization
[x1, x2] = meshgrid(linspace(-1, 2, 30), linspace(-1, 2, 30));
objective_values = log(exp(x1) + exp(x2));

% Plot the 3D surface
```

```
figure;
surf(x1, x2, objective_values, 'EdgeColor', 'none');
colorbar;
hold on;

% Plot the optimal value
plot3(x(1), x(2), log(exp(x(1)) + exp(x(2))), 'ro', 'MarkerSize', 10, 'LineWidth', 2)
    ;

% Labels and title
title('Logarithmic Objective Function');
xlabel('x_1');
ylabel('x_2');
zlabel('Objective Value');
legend('Objective Function', 'Optimal Value');
grid on;
hold off;
```