Optimization Models in Engineering—Homework 1 Solution

1. The Fruit Computer company produces two types of computers: Pear computers and Apricot computers. The following table shows the number of hours and the number of chips needed to make a computer as well as the equipment cost and selling price:

Computer	Labor	Chips	Equipment cost per unit ($)	Selling price ($)
Pear	1 hour	2	50	400
Apricot	2 hours	5	100	900

A total of 3000 chips and 1200 hours of labor are available. The company needs to decide how many computers from each type should be made in order to maximize the profit.

Formulate this problem as an optimization problem.

Solution 1.

Denote the number of Pear computers as x and the number of Apricot computers as y. The per-product profit is $400 - 50 = 350$ for the Pear computer and $900 - 100 = 800$ for the Apricot computer. To maximize total profit, we want to select the optimal x and y by solving the optimization problem

$$\begin{aligned} \max_{x,y} \quad & 350x + 800y, \\ \text{subject to } & 2x + 5y \leq 3000, \\ & x + 2y \leq 1200. \end{aligned}$$

2. The Paradise City Police Department employs 28 police officers. Each officer works 5 days per week. The crime rate fluctuates with the day of the week, so the minimum number of police officers required each day depends on which day of the week it is: Saturday, 24; Sunday, 13; Monday, 15; Tuesday, 25; Wednesday, 26; Thursday, 18; Friday, 22. The police department wants to schedule police officers to minimize the number of officers whose days off are not consecutive.

   Formulate this problem as an optimization problem (hint: define the variable x_{ij}, where $i, j \in \{1, 2, \dots, 7\}$ and $i \neq j$, to be the number of officers who are scheduled to not work on days i and j).

   Solution 2.

   Following the hint, we define the variable x_{ij}, where $i, j \in \{1, 2, \dots, 7\}$ and $i \neq j$, to be the number of officers who are scheduled to not work on days i and j. Note that by definition, $x_{ij} = x_{ji}$ and $x_{ij} \in \mathbb{N}$ for all $i \neq j$ where $\mathbb{N} := \{0, 1, 2, \dots\}$ is the set of natural numbers. Furthermore, because we need to make sure that all 28 police officers get their arrangements, we must satisfy $\sum_{i=1}^{7} \sum_{j=1}^{7} x_{ij}/2 = 28$. Here, we divide by two because $x_{ij} = x_{ji}$ for each i, j pair so each officer is counted twice.

   Since each office has two days off, the number of officers off on the i^{th} day of the week is $\sum_{j \in \{1,2,\dots,7\}, j \neq i} x_{ij}$. Since we know the maximum number of off-duty officers for each day, we have now formulated seven additional constraints. For example, for $i = 1$, the corresponding constraint is $x_{12}+x_{13}+x_{14}+x_{15}+x_{16}+x_{17} \leq 28-24$.

   The goal of our problem is to minimize the number of officers whose days off are not consecutive. Hence, we want to maximize x_{ij}'s where i and j are consecutive. Note that since weekdays are cyclic, we should treat 1 and 7 as consecutive. Specifically, we need to maximize $x_{12} + x_{23} + x_{34} + x_{45} + x_{56} + x_{67} + x_{71}$.

   Putting everything together, we arrive at the optimization problem

$$
\begin{aligned}
\max_{\substack{x_{ij} \\ i,j \in \{1,2,\dots,7\}, i \neq j}} \quad & x_{12} + x_{23} + x_{34} + x_{45} + x_{56} + x_{67} + x_{71}, \\
\text{subject to} \quad & \frac{1}{2} \sum_{i=1}^{7} \sum_{j=1}^{7} x_{ij} = 28, \\
& x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} \leq 28 - 24, \\
& x_{21} + x_{23} + x_{24} + x_{25} + x_{26} + x_{27} \leq 28 - 13, \\
& x_{31} + x_{32} + x_{34} + x_{35} + x_{36} + x_{37} \leq 28 - 15, \\
& x_{41} + x_{42} + x_{43} + x_{45} + x_{46} + x_{47} \leq 28 - 25, \\
& x_{51} + x_{52} + x_{53} + x_{54} + x_{56} + x_{57} \leq 28 - 26, \\
& x_{61} + x_{62} + x_{63} + x_{64} + x_{65} + x_{67} \leq 28 - 18, \\
& x_{71} + x_{72} + x_{73} + x_{74} + x_{75} + x_{76} \leq 28 - 22, \\
& x_{ij} \in \mathbb{N}, \quad \forall i, j \in \{1, 2, \dots, 7\}, \\
& x_{ij} = x_{ji}, \quad \forall i, j \in \{1, 2, \dots, 7\},\ i < j.
\end{aligned}
$$

3. A taxi company has n taxis available and n customers to be picked up as soon as possible. For every $i, j \in \{1, \ldots, n\}$, if taxi i decides to pick up customer j, the amount of time (delay) to pick up the customer is d_{ij}. Each taxi is allowed to pick up only one customer. The goal is to assign each customer to a taxi so that the total delay (i.e., sum of the delays for all customers) is minimized.

   Formulate this assignment problem as an optimization problem.

   Solution 3.

   Define binary variables x_{ij} for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. We let x_{ij} be 1 if taxi i picks up customer j and 0 otherwise.

   Since each taxi must pick up one customer, $\sum_{j=1}^{n} x_{ij}$ must be 1 for all $i = 1, 2, \ldots, n$. Similarly, since each customer must be picked up by one taxi, $\sum_{i=1}^{n} x_{ij}$ must be 1 for all $j = 1, 2, \ldots, n$.

   Note that the total delay is $\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} d_{ij}$. As we want to minimize the total delay, we want to minimize this quantity.

   Putting everything together, we arrive at the optimization problem

$$
\begin{aligned}
\min_{\substack{x_{ij} \\ i,j \in \{1,2,\ldots,n\}}} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} d_{ij} \\
\text{subject to} \quad & \sum_{j=1}^{n} x_{ij} = 1, \quad \forall i \in \{1, 2, \ldots, n\}, \\
& \sum_{i=1}^{n} x_{ij} = 1, \quad \forall j \in \{1, 2, \ldots, n\} \\
& x_{ij} \in \{0, 1\}, \quad \forall i, j \in \{1, 2, \ldots, n\}.
\end{aligned}
$$

4. Consider the set $\mathcal{S}$ defined as

$$\mathcal{S} = \{x \in \mathbb{R}^3 \mid x_1 + 2x_2 + 3x_3 = 0,\ 3x_1 + 2x_2 + x_3 = 0\} \tag{1}$$

Show that $\mathcal{S}$ is a subspace. Determine its dimension and find a basis for it.

Solution 4.

Since the origin $(0,0,0)$ is in $\mathcal{S}$, the set $\mathcal{S}$ is non-empty. To show that it is a subspace, we simply need to show that for all $x, y \in \mathcal{S}$, $\alpha x + \beta y$ also belongs to $\mathcal{S}$ for all $\alpha, \beta \in \mathbb{R}$.

To show this, we denote a vector z as $\alpha x + \beta y$ and show that $z \in \mathcal{S}$. Since $x, y \in \mathcal{S}$, it holds that $x_1 + 2x_2 + 3x_3 = 0$ and $y_1 + 2y_2 + 3y_3 = 0$. Therefore,

$$\begin{aligned} z_1 + 2z_2 + 3z_3 =& (\alpha x_1 + \beta y_1) + 2(\alpha x_2 + \beta y_2) + 3(\alpha x_3 + \beta y_3) \\ =& \alpha(x_1 + 2x_2 + 3x_3) + \beta(y_1 + 2y_2 + 3y_3) \\ =& \alpha \cdot 0 + \beta \cdot 0 = 0. \end{aligned}$$

Similarly, it holds that

$$\begin{aligned} 3z_1 + 2z_2 + z_3 =& 3(\alpha x_1 + \beta y_1) + 2(\alpha x_2 + \beta y_2) + (\alpha x_3 + \beta y_3) \\ =& \alpha(3x_1 + 2x_2 + x_3) + \beta(3y_1 + 2y_2 + y_3) \\ =& \alpha \cdot 0 + \beta \cdot 0 = 0. \end{aligned}$$

Hence, we have proved that $z \in \mathcal{S}$, and thus $\mathcal{S}$ is a subspace.

To find a basis of this subspace, we need to find a set of linearly independent vectors whose linear combinations can represent all vectors in $\mathcal{S}$.

In order for a vector x to be in $\mathcal{S}$, it must satisfy $x_1 + 2x_2 + 3x_3 = 0$ and $3x_1 + 2x_2 + x_3 = 0$. Subtracting the second equation from the first one, we have $-2x_1 + 2x_3 = 0$, and hence $x_1 = x_3$. Plugging this back into the first equation gives us $2x_2 + 4x_3 = 0$, and hence $x_2 = -2x_3$. Therefore, we can pick an arbitrary x_3, and then x_2 will be found from the relationship $x_2 = -2x_3$, with x_1 determined via $x_1 = x_3$.

This implies that the subspace can be characterized by only one free parameter, implying that its dimension is one. More precisely, all vectors in $\mathcal{S}$ can be written as $\gamma \cdot (1, -2, 1)$ for some $\gamma \in \mathbb{R}$, and therefore $\{(1, -2, 1)\}$ forms a basis for the subspace $\mathcal{S}$. Since this basis only consists of one vector, its dimension is 1.

5. Consider the set $\mathcal{P}$ defined as

$$\mathcal{P} = \{x \in \mathbb{R}^3 \mid x_1 + 2x_2 + 3x_3 = 1\} \tag{2}$$

Show that $\mathcal{P}$ is an affine set of dimension 2. To this end, express it as $x^{(0)} + \text{span}(x^{(1)}, x^{(2)})$, where $x^{(0)} \in \mathcal{P}$ and $x^{(1)}, x^{(2)}$ are linearly independent vectors.

Solution 5.

To show that $\mathcal{P}$ is an affine set, we need to show that all vectors in $\mathcal{P}$ can be written in the form of $x^{(0)} + \text{span}(x^{(1)}, x^{(2)})$ for some $x^{(0)} \in \mathcal{P}$ and linearly independent $x^{(1)}, x^{(2)}$.

For $x^{(0)}$, we can select an arbitrary vector in $\mathcal{P}$. One option is $x^{(0)} = (1, 0, 0)$.

Now, we just need to show that the set

$$\tilde{\mathcal{P}} := \{x \in \mathbb{R}^3 \mid x + x^{(0)} \in \mathcal{P}\} = \{x \in \mathbb{R}^3 \mid x_1 + 2x_2 + 3x_3 = 0\}$$

is a two-dimensional subspace, and one way to show this is to find a set of basis for $\tilde{\mathcal{P}}$.

Intuitively, we can select x_2 and x_3 arbitrarily, and then x_1 will be obtained automatically through the equation. In other words, this subspace can be characterized by two free parameters, implying its dimension to be two. To show this, consider the vectors $x^{(1)} = (1, -2, 1)$ and $x^{(2)} = (2, -1, 0)$. Clearly, both vectors satisfy $x_1 + 2x_2 + 3x_3 = 0$ and they are linearly independent. Hence, they form a basis for $\tilde{\mathcal{P}}$.

Since we have found a set of $x^{(0)}$, $x^{(1)}$, and $x^{(2)}$ such that $\mathcal{P}$ can be expressed as $x^{(0)} + \text{span}(x^{(1)}, x^{(2)})$, we have shown that $\mathcal{P}$ is an affine set of dimension 2.