EECS 127 Homework 1

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Exercise 1

Let x_1 be the number of Pear, and x_2 be the number of Apricon.

$$\begin{aligned} \min \quad & 350x_1 + 800x_2 \\ \text{s.t.} \quad & 2x_1 + 5x_2 - 3000 \leq 0 \\ & x_1 + 2x_2 - 1200 \leq 0 \end{aligned}$$

Exercise 2

Let x_{ij} be the number of officers who are scheduled to not work on days i and j where $i, j \in \{1, 2, \ldots, 7\}$ and $i < j$.

Minimizing the number of officers whose days off are not consecutive is the same as maximizing the number of officers whose days off are consecutive.

$$
\begin{aligned}
\max \quad & x_{12} + x_{23} + x_{34} + x_{45} + x_{56} + x_{67} + x_{71} \\
\text{s.t.} \quad & 15 - \sum_{\substack{i,j \\ i \neq 2, j \neq 2}} x_{ij} \leq 0 \\
& 25 - \sum_{\substack{i,j \\ i \neq 3, j \neq 3}} x_{ij} \leq 0 \\
& 26 - \sum_{\substack{i,j \\ i \neq 4, j \neq 4}} x_{ij} \leq 0 \\
& 18 - \sum_{\substack{i,j \\ i \neq 5, j \neq 5}} x_{ij} \leq 0 \\
& 22 - \sum_{\substack{i,j \\ i \neq 6, j \neq 6}} x_{ij} \leq 0 \\
& 24 - \sum_{\substack{i,j \\ i \neq 7, j \neq 7}} x_{ij} \leq 0 \\
& 13 - \sum_{\substack{i,j \\ i \neq 8, j \neq 8}} x_{ij} \leq 0
\end{aligned}
$$

Exercise 3

Let $x_{ij} \in \{0, 1\}$ be the decision of whether to assign taxi i to customer j.

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} * d_{ij} \\
\text{s.t.} \quad & \text{Each taxi only picks up 1 customer} \\
& \sum_{j=1}^{n} x_{1j} - 1 \leq 0 \\
& 1 - \sum_{j=1}^{n} x_{1j} \leq 0 \\
& \dots \\
& \sum_{j=1}^{n} x_{nj} - 1 \leq 0 \\
& 1 - \sum_{j=1}^{n} x_{nj} \leq 0 \\
& \text{Each customer only has 1 taxi} \\
& \sum_{i=1}^{n} x_{i1} - 1 \leq 0 \\
& 1 - \sum_{i=1}^{n} x_{i1} \leq 0 \\
& \dots \\
& \sum_{i=1}^{n} x_{in} - 1 \leq 0 \\
& 1 - \sum_{i=1}^{n} x_{in} \leq 0
\end{aligned}
$$

Exercise 4

Let define the following matrix

$$
A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix} \tag{1}
$$

Set S contains all vectors x where $A^T x = 0$.

1. x = [0, 0, 0] is in this space.

2. Let say we have 2 vectors u and v in S, so $A^T u = 0$ and $A^T v = 0$.

$$A^T(u+v) = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}^T \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} u_1 + 2u_2 + 3u_3 + v_1 + 2v_2 + 3v_3 \\ 3u_1 + 2u_2 + u_3 + 3v_1 + 2v_2 + v_3 \end{bmatrix} \tag{3}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4}$$

This means u + v is also in S.

3. Let say we have a vector cu where c is a scalar.

$$A^T cu = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}^T \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} c(u_1 + 2u_2 + 3u_3) \\ c(3u_1 + 2u_2 + u_3) \end{bmatrix} \tag{6}$$

$$= c \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

This means cu is also in S.

As a result, S is a subspace. To find basis for this matrix, we use RREF.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix} \tag{8}$$

$$\sim \begin{bmatrix} 1 & 3 \\ 0 & -4 \\ 0 & -8 \end{bmatrix} \quad \text{(R2 $\rightarrow$ R2 - 2R1, R3 $\rightarrow$ R3 - 3R1)} \tag{9}$$

$$\sim \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{(R2 $\rightarrow$ -$\frac{1}{4}$R2, R3 $\rightarrow$ R3 - 2R2)} \tag{10}$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{(R1 $\rightarrow$ R1 - 3R2)} \tag{11}$$

The basis for the column space is given by the columns corresponding to pivot columns in the RREF:

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\} \tag{12}$$

Dim(A) = 2.

Exercise 5

Let define the following matrix

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{13}$$

and P contains all x where $A^T x = [1]$.

Let $u, v \in P$ and $x = ku + (1-k)v$ where k is a scalar.

$$\begin{aligned} A^T x &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T \begin{bmatrix} ku_1 + (1-k)v_1 \\ ku_2 + (1-k)v_2 \\ ku_3 + (1-k)v_3 \end{bmatrix} && (14) \\ &= \begin{bmatrix} ku_1 + (1-k)v_1 + 2(ku_2 + (1-k)v_2) + +3(ku_3 + (1-k)v_3) \end{bmatrix} && (15) \\ &= \begin{bmatrix} k(u_1 + 2u_2 + 3u_3) + (1-k)(v_1 + 2v_2 + 3v_3) \end{bmatrix} && (16) \\ &= \begin{bmatrix} k + 1 - v \end{bmatrix} && (17) \\ &= \begin{bmatrix} 1 \end{bmatrix} && (18) \end{aligned}$$

This means P is an affine set. Since the null space of A is 1, the dimension of A is 3 - 1 = 2.