Optimization Models in Engineering—Homework 2

1. Consider the subspace $\mathcal{S} = \text{span}(x^{(1)}, x^{(2)}, x^{(3)})$, where

$$x^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x^{(2)} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad x^{(3)} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \tag{1}$$

   i) Find the dimension of $\mathcal{S}$.

   ii) Calculate the projection of the point $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ on $\mathcal{S}$.

Solution:

i)

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \tag{2}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \quad \text{(R2} \to \text{R2 - R1, R3} \to \text{R3 - R1)} \tag{3}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(R3} \to \text{R3 - 2R2)} \tag{4}$$

$$\tag{5}$$

We've now reached row echelon form. The rank of the matrix is equal to the number of non-zero rows in this form, which is 2. Therefore, dim(A) = 2.

ii)

1.1. Apply Gram-Schmidt process
$\vec{u}_1 = \vec{x}_1 = \begin{bmatrix} 1 \ 1 \ 1 \end{bmatrix}$
$\vec{u}_2 = \vec{x}_2 - \text{proj}\vec{u}_1(\vec{x}_2) = \begin{bmatrix} -1 \ 0 \ 1 \end{bmatrix} - \frac{\vec{x}_2 \cdot \vec{u}_1}{|\vec{u}_1|^2}\vec{u}_1 = \begin{bmatrix} -1 \ 0 \ 1 \end{bmatrix} - 0\vec{u}_1 = \begin{bmatrix} -1 \ 0 \ 1 \end{bmatrix}$
$\vec{u}_3 = \vec{v}3 - \text{proj}\vec{u}_1(\vec{x}_3) - \text{proj}\vec{u}_2(\vec{x}_3) = \begin{bmatrix} 0 \ 1 \ 2 \end{bmatrix} - \frac{\vec{x}_3 \cdot \vec{u}_1}{|\vec{u}_1|^2}\vec{u}_1 - \frac{\vec{x}_3 \cdot \vec{u}_2}{|\vec{u}_2|^2}\vec{u}_2 = \begin{bmatrix} 0 \ 1 \ 2 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \ 1 \ 1 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} -1 \ 0 \ 1 \end{bmatrix} = \begin{bmatrix} 0 \ 1 \ 2 \end{bmatrix} - \begin{bmatrix} 1 \ 1 \ 1 \end{bmatrix} - \begin{bmatrix} -1 \ 0 \ 1 \end{bmatrix} = \begin{bmatrix} 0 \ 0 \ 0 \end{bmatrix}$

1.2. Normalize the basis vectors
$\vec{v}_1 = \frac{\vec{u}_1}{|\vec{u}_1|} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \ 1 \ 1 \end{bmatrix}$
$\vec{v}_2 = \frac{\vec{u}_2}{|\vec{u}_2|} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \ 0 \ 1 \end{bmatrix}$
Note that $\vec{u}_3 = \vec{0}$, so we only have two orthonormal basis vectors.
Now we can compute the projection using the formula: $\text{proj}_S(\vec{b}) = \langle \vec{b}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{b}, \vec{v}_2 \rangle \vec{v}_2$

1.3. Calculate dot products
$\langle \vec{b}, \vec{v}_1 \rangle = \begin{bmatrix} 1 \ 2 \ 4 \end{bmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \ 1 \ 1 \end{bmatrix} = \frac{1}{\sqrt{3}}(1 + 2 + 4) = \frac{7}{\sqrt{3}}$

$\langle \vec{b}, \vec{v}_2 \rangle = \begin{bmatrix} 1 \; 2 \; 4 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \; 0 \; 1 \end{bmatrix} = \frac{1}{\sqrt{2}}(-1+0+4) = \frac{3}{\sqrt{2}}$

1.4. Calculate the projection
$\text{proj}_S(\vec{b}) = \frac{7}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \; 1 \; 1 \end{bmatrix} + \frac{3}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \; 0 \; 1 \end{bmatrix} = \frac{7}{3} \begin{bmatrix} 1 \; 1 \; 1 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} -1 \; 0 \; 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \; \frac{7}{3} \; \frac{7}{3} \end{bmatrix} + \begin{bmatrix} -\frac{3}{2} \; 0 \; \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{7}{3} - \frac{3}{2} \; \frac{7}{3} \; \frac{7}{3} + \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{29}{6} \; \frac{31}{6} \; \frac{34}{6} \end{bmatrix}$

2. Consider the box $\mathcal{S}_1$ and ball $\mathcal{S}_2$ defined as

$$\mathcal{S}_1 = \{x \in \mathbb{R}^2 \mid -2 \le x_1 \le 2, \; -0.5 \le x_2 \le 0.5\}, \quad \mathcal{S}_2 = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \le 1\} \tag{6}$$

Given a point $z \in \mathbb{R}^2$, write an optimization problem in a standard form that finds the projection of z onto the set $\mathcal{S}_1 \cap \mathcal{S}_2$ (i.e., the solution of the optimization problem should correspond to the closest point in $\mathcal{S}_1 \cap \mathcal{S}_2$ to z; note that you do not need to solve the optimization problem).

Solution:

$$\begin{aligned} \min \quad & \|z - z^*\|_2 \\ \text{s.t.} \quad & {z_1^*}^2 + {z_2^*}^2 \le 1 \\ & z_1^* \le 2 \\ & -z_1^* \le 2 \\ & z_2^* \le 0.5 \\ & -z_2^* \le 0.5 \end{aligned}$$

3. A company has n factories. Factory i (for $i = 1, 2, ..., n$) is located at point (a_i, b_i) in the two-dimensional plane $\mathbb{R}^2$. The company wants to locate a warehouse at a point (x_1, x_2) that minimizes

$$\sum_{i=1}^{n} (\text{distance from factory } i \text{ to the warehouse})^2$$

Find all possible values of the point (x_1^*, x_2^*) that satisfy the necessary condition for local optimality.

Solution:
Let formulate the minimization above as follows:

$$\sum_{i=1}^{n} (\text{distance from factory } i \text{ to the warehouse})^2 = \sum_{i=1}^{n} [(a_i - x_1)^2 + (b_i - x_2)^2] \tag{7}$$

$$\tag{8}$$

We need to solve for the necessary condition of local optimality where $\nabla f(\mathbf{x}) = \begin{bmatrix} 2nx_1 - 2\sum_{i=1}^{n} a_i \\ 2nx_2 - 2\sum_{i=1}^{n} b_i \end{bmatrix} = 0$

This means $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{\sum_{i=1}^{n} a_i}{n} \\ \frac{\sum_{i=1}^{n} b_i}{n} \end{bmatrix}$

4. Given a natural number $k \in \{1, 2, \ldots\}$, a symmetric matrix $P \in \mathbb{R}^{n \times n}$, a vector $q \in \mathbb{R}^n$ and a scalar $r \in \mathbb{R}$, consider the optimization problem

$$\min_{x \in \mathbb{R}^n} \quad (x^\top P x)^k + q^\top x + r \tag{9}$$

Assume that q is a nonzero vector.

i) Calculate the gradient of the function $q^\top x$.

ii) Calculate the gradient of the function $x^\top Px$.

iii) Calculate the gradient of the objective function of the optimization problem (9).

iv) Given a point x^*, write the necessary optimality condition for x^* to be a local minimum of the optimization problem (9).

v) Assume that q is not in the range of P. Prove that the optimization problem (9) cannot have any local minimum (hint: show that the necessary optimality condition has no solution).

vi) Assume that P is invertible. Given a local minimum x^* of the optimization problem (9), show that there is a scalar α such that $x^* = \alpha P^{-1}q$.

vii) Again assume that P is invertible. Solve for α in Part (vi) and calculate it in terms of only the known parameters P, q, r, k (hint: Substitute the formula $x^* = \alpha P^{-1}q$ into the optimality condition and write it in terms of α).

Solution:

i) $\nabla q^\top x = q$

ii) $\nabla x^\top Px$

Step 1: Expand the quadratic form
First, let's write out what $x^\top Px$ means: x$^\top Px = \sum_{i=1}^n \sum_{j=1}^n x_i p_{ij} x_j$ where p_{ij} is the element in the i-th row and j-th column of P.

Step 2: Calculate the partial derivatives
To find the gradient, we need to calculate the partial derivative with respect to each x_k: $\frac{\partial}{\partial x_k}(x^\top Px) = \frac{\partial}{\partial x_k}\sum_{i=1}^n \sum_{j=1}^n x_i p_{ij} x_j$
Using the product rule and noting that p_{ij} is constant: $\frac{\partial}{\partial x_k}(x^\top Px) = \sum_{i=1}^n p_{ik}x_i + \sum_{j=1}^n p_{kj}x_j$

Step 3: Simplify using matrix notation
We can write this more compactly as: $\frac{\partial}{\partial x_k}(x^\top Px) = (Px)_k + (P^\top x)_k$ where $(Px)_k$ denotes the k-th element of the vector Px.

Step 4: Combine results for all k
Combining the results for all k, we get the gradient vector: $\nabla x^\top Px = Px + P^\top x = 2Px$ as P is symmetric.

iii) $\nabla(x^\top Px)^k + q^\top x + r = 2k(x^\top Px)^{k-1}Px + q$

iv) $2k(x^{*\top} Px^*)^{k-1}Px^* + q = 0$

v)

Let assume that (9) has a local minimum. This means (iv) has a solution.

This means there exist a x^* st. $2k(x^{*\top} Px^*)^{k-1}Px^* + q = 0 => -2k(x^{*\top} Px^*)^{k-1}Px^* = q$
This means that q is in the range of P which is a contradiction.

vi)
$2k(x^{*\top} Px^*)^{k-1}Px^* + q = 0 => -2k(x^{*\top} Px^*)^{k-1}Px^* = q$
Since P is invertible, we have $x^* = \frac{1}{-2k(x^{*\top} Px^*)^{k-1}}P^{-1}q$
$\alpha = \frac{1}{-2k(x^{*\top} Px^*)^{k-1}}$