Problems adapted from Calafiore and El Ghaoui, Optimization Models Exercises.

Problem 1

You have \$12,000 to invest at the beginning of the year, and three different funds from which to choose. The municipal bond fund has a 7% yearly return, the local bank's Certificates of Deposit (CDs) have an 8% return, and a high-risk account has an expected (hoped-for) 12% return. To minimize risk, you decide not to invest more than \$2,000 in the high-risk account. For tax reasons, you need to invest at least three times as much in the municipal bonds as in the bank CDs.

(a) Assuming the year-end yields are as expected, formulate the optimization problem in standard form for the best diversification strategy to maximize profit in one year.

SOLUTION: Denote the dollar amounts to invest in the municipal bond fund, local bank CDs, and high-risk account as x, y, and z respectively. To maximize total profit in a year subject to the given constraints, we want to select the optimal x, y, and z by solving the optimization problem

$$\begin{aligned} \max_{x,y,z} \quad & 0.07x + 0.08y + 0.12z \\ \text{subject to } & x + y + z \leq 12000 \\ & z \leq 2000 \\ & x \geq 3y. \end{aligned}$$

Which is written in standard form as

$$\begin{aligned} \min_{x,y,z} \quad & -0.07x - 0.08y - 0.12z \\ \text{subject to } & x + y + z - 12000 \leq 0 \\ & z - 2000 \leq 0 \\ & 3y - x \leq 0. \end{aligned}$$

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(b) If instead for tax reasons you need to invest exactly three times as much in municipal bonds as in bank CDs, how does the optimization problem (again in standard form) change?

SOLUTION: To write in standard form, we formulate the equality constraint $x = 3y$ as two inequality constraints $x \leq 3y$ and $x \geq 3y$, yielding the optimization

$$\begin{aligned} \min_{x,y,z} \quad & -0.07x - 0.08y - 0.12z \\ \text{subject to } & x + y + z - 12000 \leq 0 \\ & z - 2000 \leq 0 \\ & 3y - x \leq 0 \\ & x - 3y \leq 0. \end{aligned}$$

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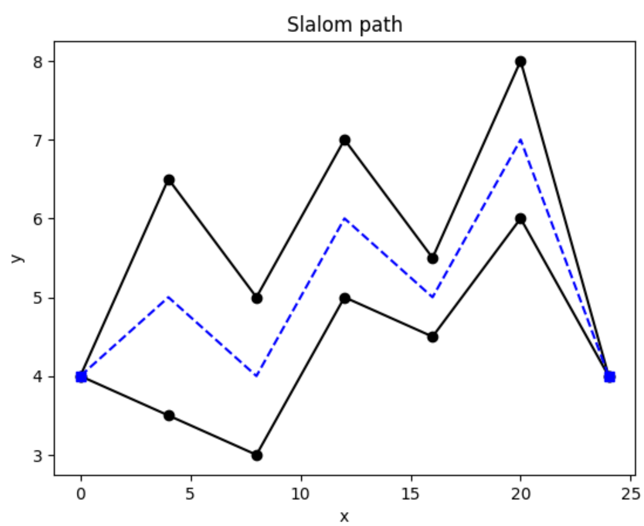
Problem 2

A slalom skier must pass through n parallel gates of known position (x_i, y_i) and width c_i for $i = 1, \ldots, n$. The initial position (x_0, y_0) is given, as well as the final one, (x_{n+1}, y_{n+1}). Before reaching the final position, the skier must go through gate i by passing between the points $(x_i, y_i - c_i/2)$ and $(x_i, y_i + c_i/2)$ for each $i \in \{1, ..., n\}$.

(a) Given the data below, draw a graph to visualize the gate positions and path of the skier.

i	x_i	y_i	c_i
0	0	4	N/A
1	4	5	3
2	8	4	2
3	12	6	2
4	16	5	1
5	20	7	2
6	24	4	N/A

Solution: The constraints are that the skier must pass between the gates, and start and end in the specified positions. The area covered by possible paths is shown within the solid lines. The dashed line represents one intuitive path the skier could take, hitting the midpoints of the slalom gates as well as the start and end positions. However, we can see from visual inspection that shorter paths exist.



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(b) Write an optimization problem that minimizes the total length of the path of the skier in terms of the variables $\{(x_i, y_i, c_i)\}_{i=0}^{n+1}$.

Solution: Assuming that (x_i, z_i) is the crossing point of gate i, the path length minimization problem can be formulated as

$$\begin{aligned} \min_{z} \quad & \sum_{i=1}^{n+1} \left\| \begin{bmatrix} x_i \\ z_i \end{bmatrix} - \begin{bmatrix} x_{i-1} \\ z_{i-1} \end{bmatrix} \right\|_2 \\ \text{subject to } & y_i - c_i/2 \leq z_i \leq y_i + c_i/2, \text{ for } i = 1, \ldots, n \\ & z_0 = y_0, z_{n+1} = y_{n+1}. \end{aligned}$$

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Problem 3

We are given a set of $n = 3$ types of food, each of which has the per serving nutritional characteristics described below. Formulate an optimization to determine a breakfast having minimum cost, total calories between 2000 and 2250, total amount of vitamin between 5000 and 10000, and sugar level no larger than 1000, assuming that the maximum number of servings is 10.

Food	Cost	Vitamin	Sugar	Calories
Corn	0.15	107	45	70
Milk	0.25	500	40	121
Bread	0.05	0	60	65

SOLUTION: Let the variables c, m and b denote the number of servings of corn, milk, and bread respectively.

$$\begin{aligned} & \min_{c,m,b} 0.15c + 0.25m + 0.05b \\ \text{subject to } & 2000 \leq 70c + 121m + 65b \leq 2250 \\ & 5000 \leq 107c + 500m \leq 10000 \\ & 45c + 40m + 60b \leq 1000 \\ & c + m + b \leq 10 \\ & c, m, b \geq 0. \end{aligned}$$

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Problem 4

You'd like to model traffic in a small city. You know the road network as well as the historical average of flows on each road segment.

(a) We call q_i the flow of vehicles on each road segment $i \in I$. Write a linear equation corresponding to the conservation of vehicles at each intersection $j \in J$. Assume each road segment flows into, flows out of, or does not touch each intersection. *Hint: think about how you might represent the road network and directional traffic flows in terms of matrices, vectors, etc.*

SOLUTION: Intuitively, this network can be modeled as a graph in which the nodes represent each intersection j and the edges represent each road segment i with flow q_i. To formalize the connectivity and flow directions of the network, we have incidence matrix M for which

- $M_{ji} = 1$ if road segment i flows into intersection j
- $M_{ji} =$ -1 if road segment i flows out of intersection j
- $M_{ji} = 0$ if road segment i does not touch intersection j

With this formulation, the conservation of vehicles at each intersection can be formalized as the linear constraint

$$Mq = 0$$

where M has dimension $|J| \times |I|$ and q is a vector of dimension $|I|$. ■

(b) The goal of the estimation is to estimate the traffic flow on each of the road segments. The flow estimates should satisfy the conservation of vehicles exactly at each intersection. Among the solutions that satisfy this constraint, we are searching for the estimate $\hat{q}$ that is the closest to the historical average, $\bar{q}$, in the l_2-norm sense. The vector $\bar{q}$ has size I and the i-th element represents the average for the road segment i. Formulate the optimization problem using an l_2-norm objective and the equality constraint from above.

SOLUTION: To minimize the difference between $\hat{q}$ and $\bar{q}$ in the l_2-norm sense, subject to the flow conservation from above, we have the optimization

$$\min_{\hat{q}} ||\hat{q} - \bar{q}||_2^2$$
$$\text{subject to } M\hat{q} = 0.$$

This is a least-squares optimization with an affine (specifically, linear) equality constraint. ■