1/15/2017

Fourier Series

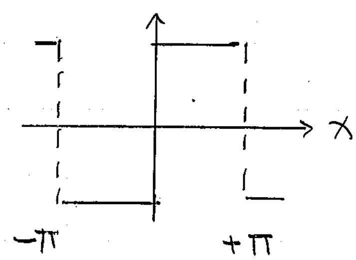
Periodic function (2π)

$$f(x) = \frac{1}{2} a_0 + \sum_{m=1}^{\infty} a_m \cos mx + b_m \sin mx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\alpha)\, d\alpha$$

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\alpha) \cos m\alpha\, d\alpha, \quad b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\alpha) \sin m\alpha\, d\alpha$$

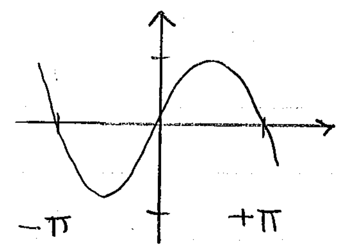
Ex $f(x)$ (square wave)



$\cos(x) = \cos(-x)$

$\sin(-x) = -\sin(x)$

$f_1(x)$

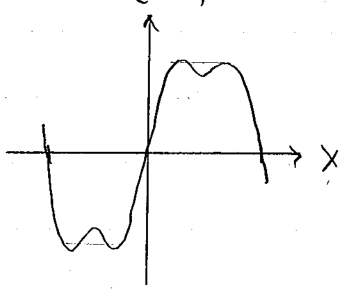


$a_0 = 0$

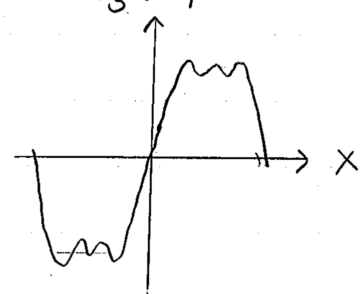
$a_1 = 0$

$b_1 = 4/\pi$

$f_2(x)$



$f_3(x)$



Gibbs ringing remains at discontinuities for finite m

(Explains MRI artifacts - more later)

$$b_1 = \frac{1}{\pi} \int_{-\pi}^{0} f(\alpha) \sin \alpha\, d\alpha + \frac{1}{\pi} \int_{0}^{\pi} f(\alpha) \sin \alpha\, d\alpha$$

$$= \frac{2}{\pi} \int_{0}^{\pi} f(\alpha) \sin \alpha\, d\alpha = \frac{2}{\pi} \left[-\cos \alpha \right]_0^{\pi}$$

Expansion yields

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\,d\alpha + \frac{1}{\pi}\sum_{m=1}^{\infty}\int_{-\pi}^{\pi} f(\alpha)\cos m(x-\alpha)\,d\alpha$$

$$\cos mx \cos m\alpha + \sin mx \sin m\alpha$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\pi}^{\pi} f(\alpha)\cos m(x-\alpha)\,d\alpha$$

Last step follows from $\cos 0 = 1$, $\cos(x) = \cos(-x)$.

$$f(x) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\pi}^{\pi} f(\alpha)\left[\cos m(x-\alpha) + j\sin m(x-\alpha)\right]d\alpha$$

This follows from $\sin(-x) = -\sin(x)$ which implies added sum of integrals equals ϕ due to cancelling terms

Euler notation for complex exponential:

$$e^{j\theta} = \cos\theta + j\sin\theta, \qquad e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\Downarrow$$

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right), \qquad \sin\theta = -\frac{j}{2}\left(e^{j\theta} - e^{-j\theta}\right)$$

Last step follows from $j^2 = -1 \Rightarrow j = -\frac{1}{j}$

Thus,

$$f(x) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\pi}^{\pi} f(\alpha)\, e^{jm(x-\alpha)}\, d\alpha$$

$$= \sum_{m=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\alpha)\, e^{-jm\alpha}\, d\alpha \right] e^{jmx}$$

$\Downarrow$

Complex Fourier Series

$$f(x) = \sum_{m=-\infty}^{\infty} F(m)\, e^{jmx}, \quad F(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\alpha)\, e^{-jm\alpha}\, d\alpha$$

For arbitrary period T and $t = x\frac{2\pi}{T}$,

$$f(t) = \sum_{m=-\infty}^{\infty} F(m)\, e^{j2\pi mt/T}, \quad F(m) = \frac{1}{T} \int_{-T/2}^{T/2} f(\alpha)\, e^{-j2\pi m\alpha/T}\, d\alpha$$

Let $\Delta u = 1/T$

$$f(t) = \sum_{m=-\infty}^{\infty} \Delta u \int_{-T/2}^{T/2} f(\alpha)\, e^{-j2\pi m\Delta u\alpha}\, d\alpha \;\; e^{j2\pi m\Delta u t}$$

Let $T \to \infty$ such that $\Sigma \sim \int$, $\Delta u \sim du$ and $m\Delta u \sim u$

$$f(t) = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} f(\alpha)\, e^{-j2\pi u\alpha}\, d\alpha \;\; e^{j2\pi ut}$$

$$f(t) = \int_{-\infty}^{\infty} F(u)\, e^{j2\pi ut}\, du, \quad F(u) = \int_{-\infty}^{\infty} f(\alpha)\, e^{-j2\pi u\alpha}\, d\alpha$$

~~Let $x = t$~~ ~~Let $x = \alpha$~~

Fourier transform Pair

$$f(x) = \int_{-\infty}^{\infty} F(u)\, e^{j2\pi ux}\, du, \quad F(u) = \int_{-\infty}^{\infty} f(x)\, e^{-j2\pi ux}\, dx$$

$\mathcal{F}^{-1}$: inverse transform $\qquad$ $\mathcal{F}$: forward transform

PROPERTY Linearity

$$\mathcal{F}\{af(x)+bg(x)\} = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}$$

Easily shown from defn.

Ex

Delta function: $\delta(x) = \begin{cases} +\infty & x=0 \\ 0 & x\neq 0 \end{cases}, \quad \int_{-\infty}^{\infty} \delta(x)\,dx = 1$

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0), \quad \int_{-\infty}^{\infty} \delta(x-x_0) f(x)\, dx = f(x_0)$$

$$\mathcal{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(x)\, e^{-j2\pi ux}\, dx = e^{-j0} = 1$$

$$\mathcal{F}^{-1}\{\delta(u)\} = 1 \Rightarrow \mathcal{F}\{1\} = \delta(u)$$

PROPERTY Local $\leftrightarrow$ Global

disturbance $\quad$ impact

(Ex

$$\mathcal{F}^{-1}\{\delta(u-u_0)\} = e^{j2\pi u_0 x}$$

typo: +

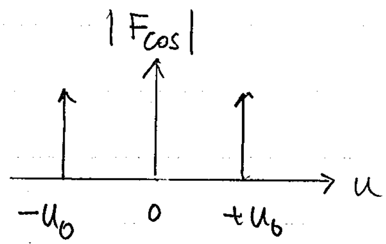
$$\mathcal{F}^{-1}\{\tfrac{1}{2}(\delta(u-u_0)+\delta(u+u_0))\} = \tfrac{1}{2}(e^{j2\pi u_0 x} - e^{-j2\pi u_0 x})$$

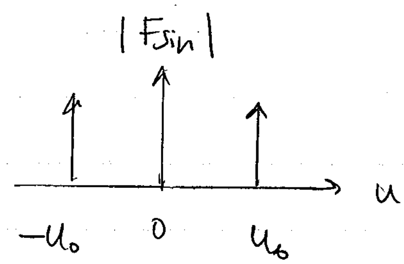
$$= \cos 2\pi u_0 x = \cos \omega_0 x$$

$\Downarrow$

$$\mathcal{F}\{\cos \omega_0 x\} = \tfrac{1}{2}(\delta(u-u_0)+\delta(u+u_0))$$

$$\mathcal{F}\{\sin \omega_0 x\} = -\tfrac{j}{2}(\delta(u-u_0)-\delta(u+u_0))$$



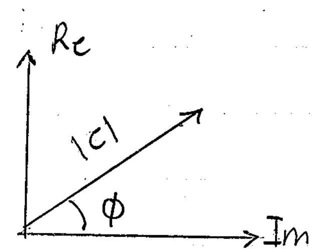


$$\phi_{cos}(u) = \tan^{-1}(0) = 0 \qquad \phi_{sin}(u) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

Complex variable: $C = Re + jIm.$

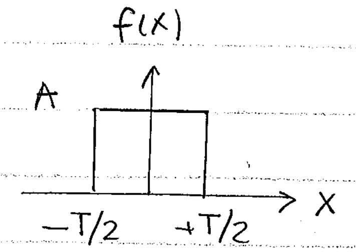
Magnitude $|C| = \sqrt{Re^2 + Im^2}$

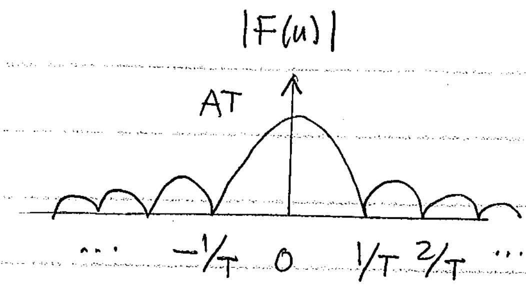
Phase angle $\phi = \tan^{-1}\dfrac{Im}{Re}$



Result for cos, sin expected: $\sin(\omega_0 x + \frac{\pi}{2}) = \cos(\omega_0 x)$

(Ex





$$F(u) = A \int_{-T/2}^{T/2} e^{-j2\pi ux}\, dx \qquad z = -j2\pi ux$$

$$\Downarrow \quad dx = \frac{j}{2\pi u}\, dz$$

$$= j\frac{A}{2\pi u} \int_{j\pi uT}^{-j\pi uT} e^{z}\, dz$$

$$= -j\frac{A}{2\pi u} \int_{-j\pi uT}^{j\pi uT} e^{z}\, dz \qquad \text{switch limits}$$

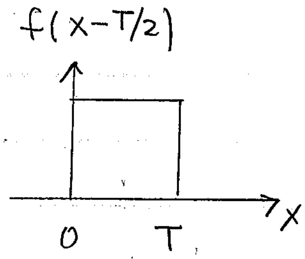
$$= -j\frac{A}{2\pi u}\left[e^{z}\right]_{-j\pi uT}^{j\pi uT}$$

$$= \frac{A}{\pi u}\left(-\frac{j}{2}\left(e^{j\pi uT} - e^{-j\pi uT}\right)\right)$$

$$= \frac{A}{\pi u}\sin \pi uT = \frac{AT}{\pi uT}\sin \pi uT$$

$$= AT\ \mathrm{sinc}\,(uT)$$

Ex



$$F(u) = AT\,\mathrm{sinc}(uT)\; e^{-j\pi u T}$$

<u>PROPERTY</u> $f \leftrightarrow F$

f	F	examples
real, even	real, even	cos, centered box
real, odd	imag, odd	sin
real	real, even imag, odd	offset box

<u>PROPERTY</u> Time / Frequency shifting

$$f(x \pm x_0) \longleftrightarrow e^{\pm j 2\pi u x_0} F(u)$$

$$e^{\pm j 2\pi u_0 x} f(x) \longleftrightarrow F(u \mp u_0)$$

<u>PROPERTY</u> Scaling

$$f(ax) \longleftrightarrow \frac{1}{|a|} F\left(\frac{u}{a}\right)$$

PROPERTY Integration

$$F(0) = \int_{-\infty}^{\infty} f(x)\,dx$$ "DC" component of signal

PROPERTY Convolution theorem

$$\mathcal{F}\{f * h\} = F(u)H(u), \quad f(x)h(x) = \mathcal{F}^{-1}\{F * H\}$$

$$F(u)H(u) = \int_{-\infty}^{\infty} f(x)\, e^{-j2\pi ux}\,dx \int_{-\infty}^{\infty} h(\alpha)\, e^{-j2\pi u\alpha}\,d\alpha$$

$$= \int_{-\infty}^{\infty} f(x)\left[\int_{-\infty}^{\infty} h(\alpha)\, e^{-j2\pi u\alpha}\,d\alpha\right] e^{-j2\pi ux}\,dx$$

$$= \int_{-\infty}^{\infty} f(x)\left[\int_{-\infty}^{\infty} h(\alpha)\, e^{-j2\pi u(\alpha+x)}\,d\alpha\right] dx$$

$$= \int_{-\infty}^{\infty} f(x)\left[\int_{-\infty}^{\infty} h(w-x)\, e^{-j2\pi uw}\,dw\right] dx$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x)\,h(w-x)\,dx\right] e^{-j2\pi uw}\,dw$$

2D Fourier Transform

$$f(x,y) = \iint_{-\infty}^{\infty} F(u,v)\, e^{j2\pi(ux+vy)}\, du\, dv$$

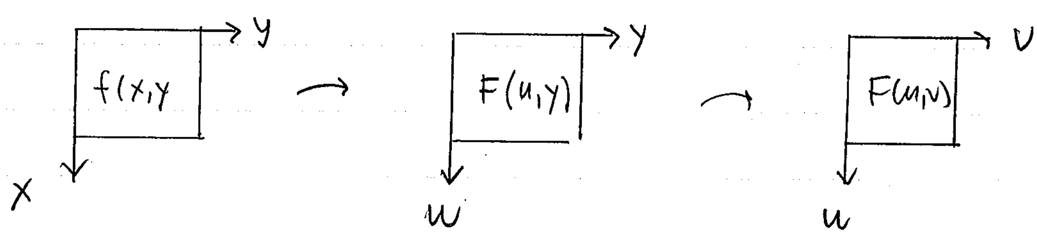
$$F(u,v) = \iint_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi(ux+vy)}\, dx\, dy$$

PROPERTY Separability

$$F(u,v) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi ux}\, dx \right] e^{-j2\pi vy}\, dy$$

$$= \int_{-\infty}^{\infty} F(u,y)\, e^{-j2\pi vy}\, dy$$

Thus, compute 2D FT from two 1D FTs



Fourier transform one dimension

Then the other dimension