1/19/17

Discrete Fourier Transform

Assume f is sampled:

$$\tilde{f}(k\Delta x) = \delta(x - k\Delta x) f(x), \qquad k = 0, 1, \ldots, N-1$$

$$\tilde{F}(u) = \int_{-\infty}^{\infty} \delta(x - k\Delta x) f(x) e^{j2\pi u x} dx$$

$$= \sum_{k=0}^{N-1} \tilde{f}(k\Delta x) e^{j2\pi u k \Delta x}$$

Assume $\tilde{f}$ is periodic, $T = N\Delta x$. Then $\tilde{F}$ is periodic

Sample $\tilde{F}$ at fundamental freq, $\Delta u = \frac{1}{T} = \frac{1}{N\Delta x}$
(one cycle per sequence)

$$\tilde{F}(\ell \Delta u) = \sum_{k=0}^{N-1} \tilde{f}(k\Delta x) e^{-j2\pi \ell \Delta u k \Delta x}, \qquad \ell = 0, 1, \ldots, N-1$$

$$= \sum_{k=0}^{N-1} \tilde{f}(k\Delta x) e^{-j2\pi \ell k / N}$$

$$\tilde{f}(k\Delta x) = \int_{-\infty}^{\infty} \tilde{F}(u) e^{j2\pi u k \Delta x} du$$

$$= \sum_{\ell=0}^{N-1} \tilde{F}(\ell \Delta u) e^{j2\pi \ell k / N}$$

Turns out above is missing a scale factor. Adding it and changing back to $x \equiv k\Delta x$ and $u \equiv \ell \Delta u$ notation gives us the DFT pair we need.

DFT

$$f(x) = \sum_{u=0}^{N-1} F(u)\, e^{j2\pi ux/N}, \qquad x = 0, 1, \ldots, N-1$$

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x)\, e^{-j2\pi ux/N}, \qquad u = 0, \ldots, N-1$$

To see that the above is correct, show $f = \mathcal{F}^{-1}\mathcal{F}\{f\}$

$$f(x) = \sum_{u=0}^{N-1} \left[\frac{1}{N} \sum_{m=0}^{N-1} f(m)\, e^{-j2\pi um/N} \right] e^{j2\pi ux/N}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} f(m) \left[\sum_{u=0}^{N-1} e^{j2\pi u(x-m)/N} \right]$$

$$= \frac{1}{N} f(x) \left[N \right] = f(x)$$

That is, outer sum only non-zero for $m = x$ in which case inner sum equals N. See sidebar note on next page.

Sidebar

Geometric series $(r \neq 1)$ $\quad \sum_{u=0}^{N-1} r^u = \frac{r^N - 1}{r - 1}$

Substitute $r = e^{j\theta}$ $(\theta \neq 0)$ to obtain

$$\sum_{u=0}^{N-1} e^{ju\theta} = \frac{e^{jN\theta} - 1}{e^{j\theta} - 1}$$

$$= \frac{e^{jN\theta/2}}{e^{j\theta/2}} \times \frac{e^{jN\theta/2} - e^{-jN\theta/2}}{e^{j\theta/2} - e^{-j\theta/2}}$$

$$= e^{j(N-1)\theta/2} \frac{\sin N\theta/2}{\sin \theta/2}$$

Substitute $\theta = 2\pi k/N$ such that $\theta/2 = \pi k/N$ $\quad (k \neq 0)$

$$\sum_{u=0}^{N-1} e^{j2\pi uk/N} = e^{j(N-1)\pi k/N} \frac{\sin \pi k}{\sin \pi k/N} = 0$$

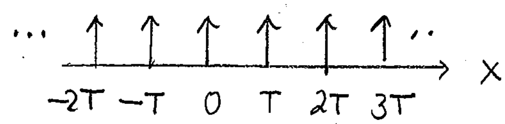
$\Downarrow$

$$\sum_{u=0}^{N-1} e^{j2\pi u(x-m)/N} = \begin{cases} N & \text{if } x = m \\ 0 & \text{otherwise} \end{cases} \qquad (x - m \equiv k)$$

Comb Function

Pulse train

$$f(x) = \sum_{k=-\infty}^{\infty} \delta(x - kT)$$



use "complex Fourier series" to repr. periodic function

$$f(x) = \sum_{m=-\infty}^{\infty} F(m/T)\, e^{j2\pi m x/T}$$

$$F(m/T) = \frac{1}{T}\int_{-T/2}^{T/2} \delta(\alpha)\, e^{-j2\pi m \alpha/T}\, d\alpha = \frac{1}{T} e^{-j0} = \frac{1}{T}$$

$\Downarrow$

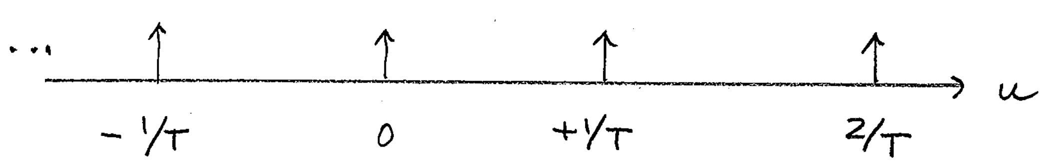
$$f(x) = \frac{1}{T}\sum_{m=-\infty}^{\infty} e^{j2\pi m x/T}$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty} \cos 2\pi m x/T + j \sin 2\pi m x/T$$

$$= \frac{1}{T}\left(1 + 2\sum_{m=1}^{\infty} \cos 2\pi m x/T\right)$$

Since $\mathcal{F}\{1\} = \delta(u)$ and $\mathcal{F}\{\cos w_0 x\} = \frac{1}{2}(\delta(u-u_0) + \delta(u+u_0))$

$$F(u) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \delta(u - k/T)$$



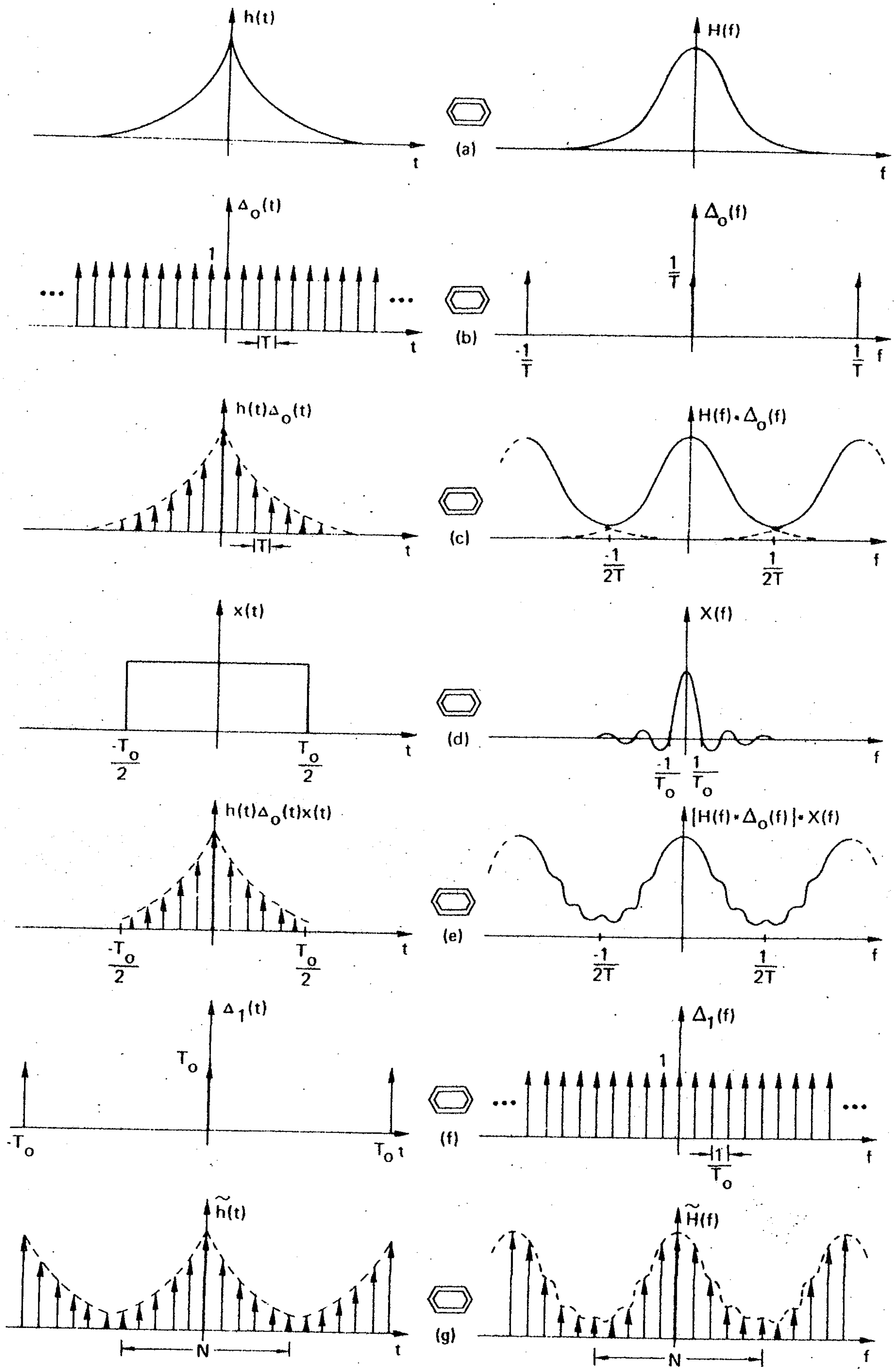


Figure 6-1. Graphical development of the discrete Fourier transform.

PROPERTY: Periodicity (implied by DFT)

$$f(x) = f(x \pm kN), \qquad F(u) = F(u \pm kN)$$

$$F(u \pm kN) = \frac{1}{N} \sum_{x=0}^{N-1} f(x)\, e^{-j2\pi(u \pm kN)x/N}$$

$$= \frac{1}{N} \sum_{x=0}^{N-1} f(x)\, e^{-j2\pi ux/N} \underbrace{e^{\pm j2\pi kx}}_{\cos 2\pi kx \pm j \sin 2\pi kx = 1}$$

$$= F(u)$$

PROPERTY Time/freq shifting

$$f(x \pm x_0) \leftrightarrow e^{\pm j2\pi u x_0/N} F(u)$$

$$e^{\pm j2\pi u_0 x/N} f(x) \leftrightarrow F(u \mp u_0)$$

Special case of interest : $u_0 = N/2$

$$e^{j2\pi Nx/2N} = e^{j\pi x}$$

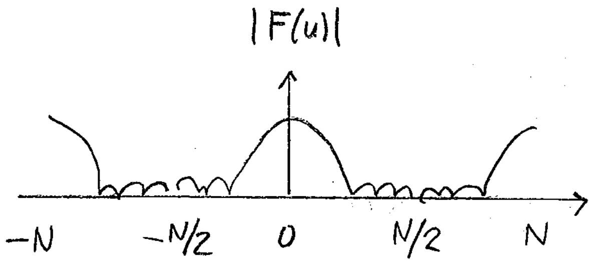
$$= \cos \pi x + j \sin \pi x$$

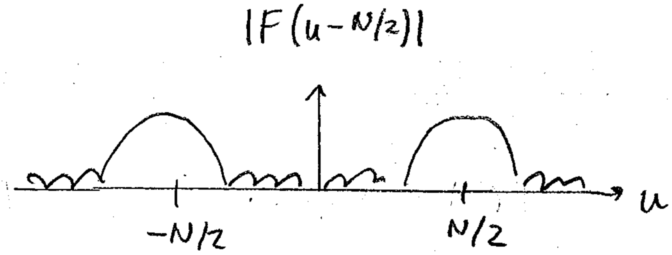
$$= \cos \pi x$$

$$= (-1)^x$$

$$(-1)^x f(x) \leftrightarrow F(u - N/2)$$

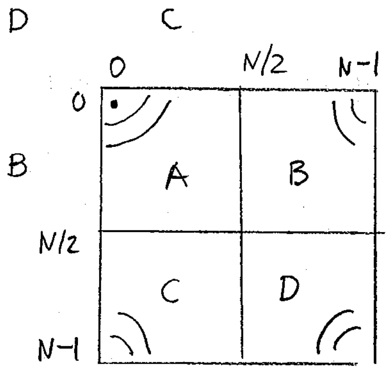
shifts 0 to N/2

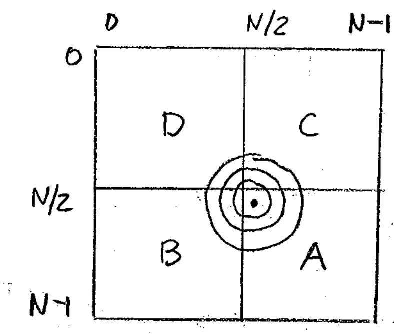




$$(-1)^{(x+y)} f(x,y) \leftrightarrow F(u-N/2,\ v-N/2)$$

shift (0,0) to (N/2, N/2)





Extension to 2D

$$f(x,y) = \sum_{u=0}^{N_1-1} \sum_{v=0}^{N_2-1} F(u,v)\, e^{j2\pi (ux/N_1 + vy/N_2)}$$

$$F(u,v) = \frac{1}{N_1 N_2} \sum_{x=0}^{N_1-1} \sum_{y=0}^{N_2-1} f(x,y)\, e^{-j2\pi (ux/N_1 + vy/N_2)}$$

Again, $\Delta_u = \frac{1}{N_1 \Delta x}$ and $\Delta_v = \frac{1}{N_2 \Delta y}$

WARNING Convolution theorem

$$\mathcal{F}\{f * h\} = F(u)H(u), \quad f(x)h(x) = \mathcal{F}^{-1}\{F * H\}$$

When computed via DFT, the implied periodicity may introduce overlap of signals

$$f(x) * h(x) = \sum_{s=0}^{N-1} f(s)\, h(x-s)$$

(Ex $f = [\,1\ 2\ 5\ 2\ 4\,]$ $h = [1\ -1]$ finite-diff approx of derivative

$1\ 1\ 3\ -3\ 2\ -4$

Note $\underset{A}{|f| = 5}$, $\underset{B}{|h| = 2}$, $|f * h| = \underset{A+B-1}{5+2-1} = 6$

That is, zero pad f and h to be $N \geq A+B-1$ long

Then

$$\hat{f} * \hat{h} = \mathcal{F}^{-1}\{\mathcal{F}\{\hat{f}\}\,\mathcal{F}\{\hat{h}\}\}$$

Truncate to desired signal length